Notes on Causal Incompatibility Inequalities

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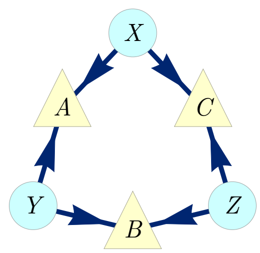
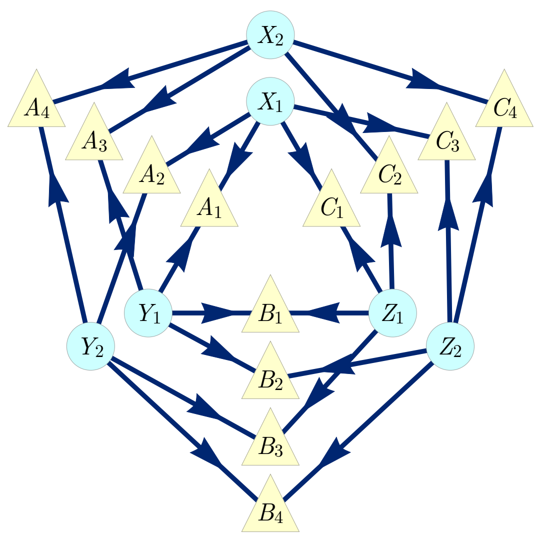
Just trying to flesh out some definitions and ideas regarding causal compatibility inequalities of the Hardy type.

Building a Hypergraph From The Marginal Description Matrix

Beginning with the triangle scenario,



We inflate to a particular inflation:



Then identify the maximal preinjectable sets (with extra notation to denote the factorizations into injectable sets),

$$\{\{A_1, B_1, C_1\}, \{A_4, B_4, C_4\}\}$$
$$\{\{A_1, B_2, C_3\}, \{A_4, B_3, C_2\}\}$$
$$\{\{A_2, B_3, C_1\}, \{A_3, B_2, C_4\}\}$$
$$\{\{A_2, B_4, C_3\}, \{A_3, B_1, C_2\}\}$$
$$\{\{A_1\}, \{B_3\}, \{C_4\}\}$$
$$\{\{A_1\}, \{B_4\}, \{C_2\}\}$$
$$\{\{A_2\}, \{B_1\}, \{C_4\}\}$$
$$\{\{A_2\}, \{B_2\}, \{C_2\}\}$$

$$\{\{A_3\},\{B_3\},\{C_3\}\}$$
$$\{\{A_3\},\{B_4\},\{C_1\}\}$$
$$\{\{A_4\},\{B_1\},\{C_3\}\}$$
$$\{\{A_4\},\{B_2\},\{C_1\}\}$$

The marginal description matrix $\mathcal{D}$ is a matrix that effectively describes how marginal distributions over the preinjectable sets arise from a joint distribution over all of the observable random variables in the causal model above.

Definition 0.1. Outcome Spaces Borrowing the notation from Fritz's BBT2, a random variable v has an outcome space denoted O_v corresponding to the set of all possible outcomes of v. A particular outcome of v can be denoted as $o[v] \in O_v$.

This notation generalizes to set of random variables $V = \{v_1, \dots, v_{|V|}\}$. A *specific* outcome for a set of random variables is denoted

$$o[V] = \left(o[v_1], o[v_2], \dots, o\left[v_{|V|}\right]\right)$$

Whereas the joint outcome space over V is a tensor product of all of the combinations of outcomes,

$$O_V = O_{v_1} \otimes \cdots \otimes O_{v_{|V|}}$$

The rows of $\mathcal{D}$ correspond to the elements of the outcome spaces over the preinjectable sets. Let $P_i \in \mathcal{P}$ be a particular preinjectable set of random variables. Take for example,

$$P_2 = \{A_1, B_2, C_3, A_4, B_3, C_2\}$$

To iterate over the outcome space of P_2, we need to define a canonical ordering and the individual outcome spaces. In our case, all observable outcomes have 4 possible outcomes,

$$O_{A_1} = O_{A_2} = \cdots = O_{C_4} = \{0, 1, 2, 3\}$$

And the canonical ordering is alphanumeric,

$$P_2 = \{A_1, A_4, B_2, B_3, C_2, C_3\}$$

Therefore for P_2 be have $|O_{P_2}| = \prod_{v \in P_2} |O_v| = 4^6 = 4096$ possible outcomes. In total over all the preinjectable sets we have,

$$\sum_{i=1}^{12} |O_{P_i}| = 4 \cdot \left(4^6\right) + 8 \cdot \left(4^3\right) = 16,896$$

Rows in the marginal description matrix $\mathcal{D}$. The columns in the marginal description matrix $\mathcal{D}$ correspond to the outcomes over all of the observable variables,

$$X = (A_1, A_2, A_3, \cdots, C_3, C_4)$$
$$|O_X| = 4^{12} = 16,777,216$$

The entries of $\mathcal{D}$ are either 1 or 0. A 1 is placed whenever an outcome over the preinjectable set is *extendable* to the corresponding outcome over X.
Explicitly,

$$(o[A_1] = 2, o[B_3] = 0, o[C_4] = 0)$$

Is extendable to lots of possible outcomes,

$$4^{|O_X| - |O_{P_5}|} = 262144$$

A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4
-	2	-	-	-	-	0	-	-	-	-	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0	2	1	2	3	2	0	3	2	1	0	0
3	2	2	0	3	2	0	1	2	0	0	0

Deriving Inequalities

In order to identify the contrapositive forms of a logical tautologies over the preinjectable outcomes, we are searching for cases where a particular event E_0 implies the occurrence of at least one of the other set of events. This possibilistic constraint,

$$E_0 \implies E_1 \vee \cdots \vee E_k = \bigvee_{i=1}^{k} E_i$$

Translates to a weaker description in terms of a probabilistic constraint,

$$P\left(E_0\right) \leq P\left(\bigvee_{i=1}^{k} E_i\right) \leq \sum_{i=1}^{k} P\left(E_i\right)$$

The events E_i in this case correspond to joint outcomes of a particular preinjectable set.

$$E_i \longleftrightarrow o\left[P_j\right]$$

To build logical tautologies for a particular $E_0 = o\left[P_2\right]$, one needs to find other events over the rows of $\mathcal{D}$ that cover all of the possible extensions of E_0. To explain this further,

- Assume that the random variables X over the inflation DAG admit a joint distribution
- If say $E_0 = o\left[P_5\right] = \left(o\left[A_1\right] = 2, o\left[B_3\right] = 0, o\left[C_4\right] = 0\right)$ happened to occur, then it must correspond to the fact that at a particular outcome occurred $o\left[X\right]$ that is expendable/compatible with $o\left[P_5\right]$ (under specifying the remaining $X \setminus P_5$ random variables)
- Given this outcome $o\left[X\right]$ occurred, then some other marginal outcomes over other preinjectable sets had to occur $o\left[P_i\right], i \neq 5$
- If you can find a set $\{E_i\}$ of these *possibly implied* outcomes over the preinjectable sets that covers all of the ways to extend E_0, then at least one of the elements of $\{E_i\}$ *had to have occured*.

Finding a set of outcomes over the preinjectable sets that accomplishes this gives us a compatibility inequality I' over the inflation random variables X.

If I' is satisfied, then compatibility between X and the inflation DAG is not ruled out. If I' is violated, then an assumption made above must be wrong. Namely, that there exists a joint distribution over X.

Furthermore, since I' is written in terms of distributions over the preinjectable sets, this translates directly to an analogous incompatibility inequality I over the deflated random variables and the deflated DAG.

Techniques

Definition 0.2. Hypergraph: A *Hypergraph* $\mathcal{H}$ is an ordered tuple $(\mathcal{N}, \mathcal{E})$ of *nodes* and *edges* respectively where the nodes can represent any object and the edges are subsets of nodes.
For convenience of notation, one defines an index set over the nodes and edges of a hypergraph $\mathcal{H}$ denoted $\mathcal{I}_{\mathcal{N}}$ and $\mathcal{I}_{\mathcal{E}}$ respectively.

$$\mathcal{N} = \{n_i \mid i \in \mathcal{I}_{\mathcal{N}}\}$$
$$\mathcal{E} = \{e_i \mid i \in \mathcal{I}_{\mathcal{E}}, e_i \subseteq \mathcal{N}\}$$

Note: Where the index for an edge or node is arbitrary, it will be omitted.

There is a dual correspondence between edges $e \in \mathcal{E}$ and nodes $n \in \mathcal{N}$ in a Hypergraph. An edge e is viewed as a set of nodes $\{n_i\}$, and a node n can be viewed as the set of edges $\{e_i\}$ that contain it.

Definition 0.3. Hypergraph Transversal: A *Transversal* $\mathcal{T}$ of a Hypergraph $\mathcal{H}$ is a set of nodes $\mathcal{T} \subseteq \mathcal{N}$ that intersect with every edge in $\mathcal{E}$.

$$\mathcal{T} = \{n_i \in \mathcal{N} \mid i \in \mathcal{I}_{\mathcal{T}}\} \quad \forall e \in \mathcal{E} : \mathcal{T} \cap e \neq \emptyset$$

Definition 0.4. Weighted Hypergraph: A *Weighted Hypergraph* $\mathcal{H}_{\mathcal{W}}$ is a regular hypergraph equipped with a set of weights $\mathcal{W}$ ascribed to each node such that a weighted hypergraph is written as a triplet $(\mathcal{W}, \mathcal{N}, \mathcal{E})$.

$$\mathcal{W} = \{w_i \mid i \in \mathcal{I}_{\mathcal{N}}, w_i \in \mathbb{R}\}$$

One would say that a particular node n_i carries weight w_i.

Definition 0.5. Weighted Transversal: A *weighted transversal* of a weighted hypergraph $\mathcal{H}_{\mathcal{W}}$ is a transversal $\mathcal{T}$ of the unweighted hypergraph $\mathcal{H}$ and a real number t (denoted $\mathcal{T}_t$) such that the sum of the node weights of the transversal is bounded by t.

$$\mathcal{T}_t = \left\{ n_i \middle| i \in \mathcal{I}_{\mathcal{T}}, \sum_{j \in \mathcal{I}_{\mathcal{T}}} w_j \leq t \right\}$$

Sparse Matrix As A Hypergraph Data Structure

To illustrate how a general hypergraph can be viewed as a matrix, consider the hypergraph,

$$\mathcal{H} = (\mathcal{N}, \mathcal{E})$$
$$\mathcal{N} = \{n_1, n_2, n_3, n_4, n_5\}$$
$$\mathcal{E} = \{e_1 = \{n_1, n_3\}, e_2 = \{n_2\}, e_3 = \{n_5\}, e_4 = \{n_2, n_4, n_5\}, e_5 = \{n_1, n_2\}, e_6 = \{n_1, n_4\}\}$$

That can be casted as a matrix:

$$M_{\mathcal{H}} = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The Dual-dual relation as a matrix.

$$(\mathcal{H}^*)^* \Leftrightarrow \left(M_{\mathcal{H}}^T\right)^T$$

Reducing The Size of The Hypergraph

Beginning with the marginal description matrix $\mathcal{D}$, one obtains a hypergraph $\mathcal{H}$ to transverse for a particular antecedent $E_0 = o[P_i]$ by building a partial hypergraph out of the extendable outcomes (edges) of O_X and the then a sub-hypergraph over the contractable outcomes (nodes) of O_P.

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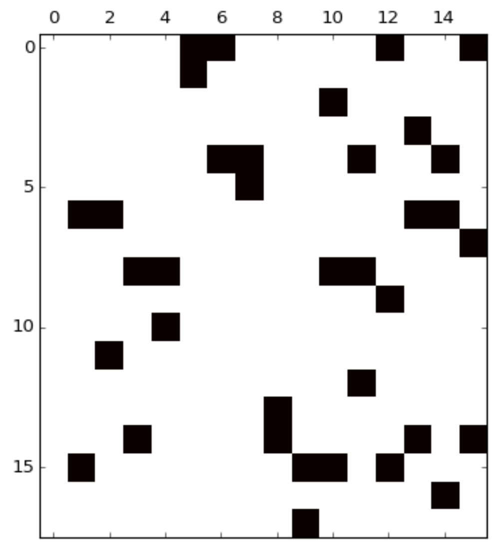
Reduced Hypergraph



Examples

As an example of a derived inequality,

$$P_{ABC}\left(002\right)P_{ABC}\left(003\right) \leq P_A(0)P_B(0)P_C(3) + P_A(0)P_B(1)P_C(3) + P_A(0)P_B(2)P_C(3) + P_A(0)P_B(3)P_C(3)$$

Ends up being fundamentally trivial. O_B is covered on the RHS,

$$P_{ABC}\left(002\right)P_{ABC}\left(003\right) \leq P_A(0)P_C(3)$$

Now notice that marginalization never decreases probabilistic values,

$$P_{ABC}\left(002\right) \leq P_A(0) \qquad P_{ABC}\left(003\right) \leq P_C(3)$$

However, we have derived numerous ($\approx 100,000$) inequalities. Using this technique, we can just keep running the algorithm and find many, many, many more.

Example of A Non-Trivial Inequality

$$P_{ABC} = \frac{1}{12}\left([012] + [201] + [120] + [123] + [312] + [231] + [230] + [023] + [302] + [301] + [130] + [013]\right)$$

$$P(012)P(130) \leq$$

$$\begin{gathered}
2P(000)P(102) + 2P(000)P(112) + 2P(000)P(122) + 2P(000)P(132) + \\
2P(010)P(102) + 2P(010)P(112) + 2P(010)P(122) + 2P(010)P(132) + \\
2P(020)P(102) + 2P(020)P(122) + 2P(020)P(132) + 2P(022)P(110) + \\
2P(032)P(100) + 2P(032)P(110) + 2P(032)P(120) + P(000)P(002) + \\
P(000)P(012) + P(000)P(022) + P(000)P(032) + P(000)P(101) + \\
P(000)P(103) + P(000)P(111) + P(000)P(113) + P(000)P(121) + \\
P(000)P(123) + P(000)P(131) + P(000)P(133) + P(001)P(100) + \\
P(001)P(101) + P(001)P(102) + P(001)P(103) + P(001)P(110) + \\
P(001)P(111) + P(001)P(112) + P(001)P(113) + P(001)P(120) + \\
P(001)P(121) + P(001)P(122) + P(001)P(123) + P(001)P(130) +
\end{gathered}$$

$$P(001)P(131) + P(001)P(132) + P(001)P(133) + P(002)P(010)+$$
$$P(002)P(020) + P(002)P(100) + P(002)P(101) + P(002)P(102)+$$
$$P(002)P(110) + P(002)P(111) + P(002)P(112) + P(002)P(120)+$$
$$P(002)P(121) + P(002)P(122) + P(002)P(131) + P(002)P(132)+$$
$$P(002)P(200) + P(002)P(210) + P(002)P(220) + P(002)P(300)+$$
$$P(002)P(310) + P(002)P(320) + P(003)P(101) + P(003)P(102)+$$
$$P(003)P(111) + P(003)P(112) + P(003)P(121) + P(003)P(122)+$$
$$P(003)P(131) + P(003)P(132) + P(010)P(012) + P(010)P(022)+$$
$$P(010)P(032) + P(010)P(101) + P(010)P(103) + P(010)P(111)+$$
$$P(010)P(113) + P(010)P(121) + P(010)P(123) + P(010)P(131)+$$
$$P(010)P(133) + P(010)P(230) + P(010)P(232) + P(010)P(233)+$$
$$P(010)P(330) + P(010)P(332) + P(010)P(333) + P(011)P(100)+$$
$$P(011)P(101) + P(011)P(102) + P(011)P(103) + P(011)P(110)+$$
$$P(011)P(111) + P(011)P(112) + P(011)P(113) + P(011)P(120)+$$
$$P(011)P(121) + P(011)P(122) + P(011)P(123) + P(011)P(130)+$$
$$P(011)P(131) + P(011)P(132) + P(011)P(133) + P(012)P(100)+$$
$$P(012)P(101) + P(012)P(102) + P(012)P(110) + P(012)P(111)+$$
$$P(012)P(112) + P(012)P(121) + P(012)P(122) + P(012)P(131)+$$
$$P(012)P(132) + P(012)P(200) + P(012)P(210) + P(012)P(300)+$$
$$P(012)P(310) + P(013)P(101) + P(013)P(102) + P(013)P(111)+$$
$$P(013)P(112) + P(013)P(121) + P(013)P(122) + P(013)P(131)+$$
$$P(013)P(132) + P(013)P(232) + P(013)P(233) + P(013)P(332)+$$
$$P(013)P(333) + P(020)P(022) + P(020)P(032) + P(020)P(101)+$$
$$P(020)P(103) + P(020)P(110) + P(020)P(111) + P(020)P(112)+$$
$$P(020)P(113) + P(020)P(121) + P(020)P(123) + P(020)P(131)+$$
$$P(020)P(133) + P(021)P(100) + P(021)P(101) + P(021)P(102)+$$
$$P(021)P(103) + P(021)P(110) + P(021)P(111) + P(021)P(112)+$$
$$P(021)P(113) + P(021)P(120) + P(021)P(121) + P(021)P(122)+$$
$$P(021)P(123) + P(021)P(130) + P(021)P(131) + P(021)P(132)+$$
$$P(021)P(133) + P(022)P(100) + P(022)P(101) + P(022)P(102)+$$
$$P(022)P(111) + P(022)P(112) + P(022)P(113) + P(022)P(120)+$$
$$P(022)P(121) + P(022)P(122) + P(022)P(131) + P(022)P(132)+$$
$$P(022)P(200) + P(022)P(210) + P(022)P(220) + P(022)P(300)+$$
$$P(022)P(310) + P(022)P(320) + P(023)P(101) + P(023)P(102)+$$
$$P(023)P(110) + P(023)P(111) + P(023)P(112) + P(023)P(113)+$$
$$P(023)P(121) + P(023)P(122) + P(023)P(131) + P(023)P(132)+$$
$$P(030)P(100) + P(030)P(101) + P(030)P(102) + P(030)P(103)+$$
$$P(030)P(110) + P(030)P(111) + P(030)P(112) + P(030)P(113)+$$
$$P(030)P(120) + P(030)P(121) + P(030)P(122) + P(030)P(123)+$$
$$P(030)P(130) + P(030)P(131) + P(030)P(132) + P(030)P(133)+$$
$$P(031)P(100) + P(031)P(101) + P(031)P(102) + P(031)P(103)+$$
$$P(031)P(110) + P(031)P(111) + P(031)P(112) + P(031)P(113)+$$
$$P(031)P(120) + P(031)P(121) + P(031)P(122) + P(031)P(123)+$$
$$P(031)P(130) + P(031)P(131) + P(031)P(132) + P(031)P(133)+$$
$$P(032)P(101) + P(032)P(102) + P(032)P(103) + P(032)P(111)+$$
$$P(032)P(112) + P(032)P(113) + P(032)P(121) + P(032)P(122)+$$

$$P(032)P(123) + P(032)P(130) + P(032)P(131) + P(032)P(132)+$$
$$P(032)P(133) + P(032)P(200) + P(032)P(210) + P(032)P(220)+$$
$$P(032)P(300) + P(032)P(310) + P(032)P(320) + P(033)P(100)+$$
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$$P(033)P(111) + P(033)P(112) + P(033)P(113) + P(033)P(120)+$$
$$P(033)P(121) + P(033)P(122) + P(033)P(123) + P(033)P(130)+$$
$$P(033)P(131) + P(033)P(132) + P(033)P(133) + P(100)P(102)+$$
$$P(100)P(112) + P(100)P(122) + P(100)P(132) + P(102)P(110)+$$
$$P(102)P(120) + P(102)P(200) + P(102)P(210) + P(102)P(220)+$$
$$P(102)P(300) + P(102)P(310) + P(102)P(320) + P(110)P(112)+$$
$$P(110)P(122) + P(110)P(132) + P(110)P(230) + P(110)P(232)+$$
$$P(110)P(233) + P(110)P(330) + P(110)P(332) + P(110)P(333)+$$
$$P(112)P(200) + P(112)P(210) + P(112)P(300) + P(112)P(310)+$$
$$P(113)P(232) + P(113)P(233) + P(113)P(332) + P(113)P(333)+$$
$$P(120)P(122) + P(120)P(132) + P(122)P(200) + P(122)P(210)+$$
$$P(122)P(220) + P(122)P(300) + P(122)P(310) + P(122)P(320)+$$
$$P(132)P(200) + P(132)P(210) + P(132)P(220) + P(132)P(300)+$$
$$P(132)P(310) + P(132)P(320) + P(210)P(230) + P(210)P(232)+$$
$$P(210)P(233) + P(210)P(330) + P(210)P(332) + P(210)P(333)+$$
$$P(213)P(232) + P(213)P(233) + P(213)P(332) + P(213)P(333)+$$
$$P(230)P(310) + P(232)P(310) + P(232)P(313) + P(233)P(310)+$$
$$P(233)P(313) + P(310)P(330) + P(310)P(332) + P(310)P(333)+$$
$$P(313)P(332) + P(313)P(333)$$

Example of A Fritz Inequality

$$P \equiv P_{ABC}$$
$$P(000)P(233) \leq \ldots$$

$$2P(001)P(230) + 2P(001)P(232) + 2P(003)P(230) + 2P(003)P(232)+$$
$$2P(020)P(203) + 2P(020)P(213) + 2P(020)P(233) + 2P(030)P(203)+$$
$$2P(030)P(213) + 2P(030)P(223) + 2P(030)P(233) + P(000)P(003)+$$
$$P(000)P(013) + P(000)P(023) + P(000)P(033) + P(000)P(103)+$$
$$P(000)P(113) + P(000)P(123) + P(000)P(133) + P(000)P(202)+$$
$$P(000)P(203) + P(000)P(212) + P(000)P(213) + P(000)P(222)+$$
$$P(000)P(230) + P(000)P(232) + P(000)P(303) + P(000)P(313)+$$
$$P(000)P(330) + P(001)P(200) + P(001)P(201) + P(001)P(202)+$$
$$P(001)P(203) + P(001)P(210) + P(001)P(211) + P(001)P(212)+$$
$$P(001)P(213) + P(001)P(220) + P(001)P(221) + P(001)P(222)+$$
$$P(001)P(223) + P(001)P(231) + P(001)P(233) + P(001)P(330)+$$
$$P(001)P(332) + P(002)P(200) + P(002)P(202) + P(002)P(210)+$$
$$P(002)P(212) + P(002)P(220) + P(002)P(222) + P(002)P(230)+$$
$$P(002)P(232) + P(003)P(010) + P(003)P(020) + P(003)P(030)+$$
$$P(003)P(100) + P(003)P(110) + P(003)P(120) + P(003)P(130)+$$
$$P(003)P(200) + P(003)P(201) + P(003)P(202) + P(003)P(203)+$$
$$P(003)P(210) + P(003)P(211) + P(003)P(212) + P(003)P(213)+$$
$$P(003)P(220) + P(003)P(221) + P(003)P(222) + P(003)P(223)+$$

$P(003)P(231) + P(003)P(233) + P(003)P(330) + P(003)P(332)+$
$P(010)P(013) + P(010)P(023) + P(010)P(103) + P(010)P(113)+$
$P(010)P(123) + P(010)P(202) + P(010)P(203) + P(010)P(212)+$
$P(010)P(213) + P(010)P(222) + P(010)P(232) + P(010)P(303)+$
$P(010)P(313) + P(011)P(200) + P(011)P(201) + P(011)P(202)+$
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$P(011)P(213) + P(011)P(220) + P(011)P(221) + P(011)P(222)+$
$P(011)P(223) + P(011)P(230) + P(011)P(231) + P(011)P(232)+$
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$P(012)P(232) + P(013)P(020) + P(013)P(030) + P(013)P(100)+$
$P(013)P(110) + P(013)P(120) + P(013)P(130) + P(013)P(200)+$
$P(013)P(201) + P(013)P(202) + P(013)P(203) + P(013)P(210)+$
$P(013)P(211) + P(013)P(212) + P(013)P(213) + P(013)P(220)+$
$P(013)P(221) + P(013)P(222) + P(013)P(223) + P(013)P(230)+$
$P(013)P(231) + P(013)P(232) + P(013)P(233) + P(020)P(023)+$
$P(020)P(033) + P(020)P(103) + P(020)P(113) + P(020)P(123)+$
$P(020)P(133) + P(020)P(200) + P(020)P(201) + P(020)P(202)+$
$P(020)P(210) + P(020)P(211) + P(020)P(212) + P(020)P(222)+$
$P(020)P(223) + P(020)P(230) + P(020)P(231) + P(020)P(232)+$
$P(020)P(303) + P(020)P(313) + P(020)P(323) + P(020)P(333)+$
$P(021)P(200) + P(021)P(201) + P(021)P(202) + P(021)P(203)+$
$P(021)P(210) + P(021)P(211) + P(021)P(212) + P(021)P(213)+$
$P(021)P(220) + P(021)P(221) + P(021)P(222) + P(021)P(223)+$
$P(021)P(230) + P(021)P(231) + P(021)P(232) + P(021)P(233)+$
$P(022)P(200) + P(022)P(202) + P(022)P(203) + P(022)P(210)+$
$P(022)P(212) + P(022)P(213) + P(022)P(220) + P(022)P(222)+$
$P(022)P(230) + P(022)P(232) + P(023)P(030) + P(023)P(100)+$
$P(023)P(110) + P(023)P(120) + P(023)P(130) + P(023)P(200)+$
$P(023)P(201) + P(023)P(202) + P(023)P(203) + P(023)P(210)+$
$P(023)P(211) + P(023)P(212) + P(023)P(213) + P(023)P(220)+$
$P(023)P(221) + P(023)P(222) + P(023)P(223) + P(023)P(230)+$
$P(023)P(231) + P(023)P(232) + P(023)P(233) + P(030)P(033)+$
$P(030)P(103) + P(030)P(113) + P(030)P(123) + P(030)P(133)+$
$P(030)P(200) + P(030)P(201) + P(030)P(202) + P(030)P(210)+$
$P(030)P(211) + P(030)P(212) + P(030)P(220) + P(030)P(221)+$
$P(030)P(222) + P(030)P(230) + P(030)P(231) + P(030)P(232)+$
$P(030)P(303) + P(030)P(313) + P(030)P(323) + P(030)P(333)+$
$P(031)P(200) + P(031)P(201) + P(031)P(202) + P(031)P(203)+$
$P(031)P(210) + P(031)P(211) + P(031)P(212) + P(031)P(213)+$
$P(031)P(220) + P(031)P(221) + P(031)P(222) + P(031)P(223)+$
$P(031)P(230) + P(031)P(231) + P(031)P(232) + P(031)P(233)+$
$P(032)P(200) + P(032)P(201) + P(032)P(202) + P(032)P(203)+$
$P(032)P(210) + P(032)P(211) + P(032)P(212) + P(032)P(213)+$
$P(032)P(220) + P(032)P(222) + P(032)P(230) + P(032)P(232)+$
$P(032)P(301) + P(033)P(100) + P(033)P(120) + P(033)P(130)+$

$$
\begin{gathered}
P(033)P(200)+P(033)P(201)+P(033)P(202)+P(033)P(203)+\\
P(033)P(210)+P(033)P(211)+P(033)P(212)+P(033)P(213)+\\
P(033)P(220)+P(033)P(221)+P(033)P(222)+P(033)P(223)+\\
P(033)P(230)+P(033)P(231)+P(033)P(232)+P(033)P(233)+\\
P(100)P(103)+P(100)P(113)+P(100)P(123)+P(100)P(133)+\\
P(100)P(203)+P(100)P(213)+P(100)P(223)+P(100)P(230)+\\
P(100)P(233)+P(100)P(303)+P(100)P(313)+P(100)P(330)+\\
P(101)P(230)+P(101)P(232)+P(101)P(330)+P(101)P(332)+\\
P(103)P(110)+P(103)P(120)+P(103)P(130)+P(103)P(230)+\\
P(103)P(232)+P(103)P(330)+P(103)P(332)+P(110)P(113)+\\
P(110)P(123)+P(110)P(203)+P(110)P(213)+P(110)P(223)+\\
P(110)P(303)+P(110)P(313)+P(113)P(120)+P(113)P(130)+\\
P(120)P(123)+P(120)P(133)+P(120)P(203)+P(120)P(213)+\\
P(120)P(223)+P(120)P(233)+P(120)P(303)+P(120)P(313)+\\
P(120)P(323)+P(123)P(130)+P(130)P(133)+P(130)P(203)+\\
P(130)P(213)+P(130)P(223)+P(130)P(233)+P(130)P(303)+\\
P(130)P(313)+P(132)P(301)+P(132)P(303)+P(200)P(230)+\\
P(200)P(330)+P(201)P(230)+P(201)P(232)+P(201)P(330)+\\
P(201)P(332)+P(203)P(230)+P(203)P(232)+P(203)P(330)+\\
P(203)P(332)+P(230)P(300)+P(230)P(301)+P(230)P(303)+\\
P(232)P(301)+P(232)P(303)+P(300)P(330)+P(301)P(330)+\\
P(301)P(332)+P(303)P(330)+P(303)P(332)
\end{gathered}
$$

Symmetric Inequality

$$2P(000)P(233)+2P(000)P(323)+2P(000)P(332)\leq$$

$$
\begin{gathered}
2P(000)P(003)+2P(000)P(022)+2P(000)P(030)+2P(000)P(033)+2P(000)P(113)+\\
2P(000)P(122)+2P(000)P(131)+2P(000)P(133)+2P(000)P(202)+2P(000)P(212)+\\
2P(000)P(220)+2P(000)P(221)+2P(000)P(223)+2P(000)P(232)+2P(000)P(300)+\\
2P(000)P(303)+2P(000)P(311)+2P(000)P(313)+2P(000)P(322)+2P(000)P(330)+\\
2P(000)P(331)+2P(001)P(220)+2P(001)P(221)+2P(001)P(222)+2P(001)P(223)+\\
2P(001)P(330)+2P(001)P(331)+2P(001)P(332)+2P(002)P(020)+2P(002)P(110)+\\
2P(002)P(200)+2P(002)P(220)+2P(002)P(221)+2P(002)P(222)+2P(002)P(223)+\\
2P(002)P(330)+2P(002)P(331)+2P(002)P(332)+2P(002)P(333)+2P(003)P(030)+\\
2P(003)P(110)+2P(003)P(220)+2P(003)P(221)+2P(003)P(222)+2P(003)P(223)+\\
2P(003)P(300)+2P(003)P(330)+2P(003)P(331)+2P(003)P(332)+2P(003)P(333)+\\
2P(010)P(202)+2P(010)P(212)+2P(010)P(222)+2P(010)P(232)+2P(010)P(303)+\\
2P(010)P(313)+2P(010)P(323)+2P(011)P(200)+2P(011)P(211)+2P(011)P(222)+\\
2P(011)P(233)+2P(011)P(300)+2P(011)P(311)+2P(011)P(322)+2P(012)P(210)+\\
2P(013)P(310)+2P(020)P(101)+2P(020)P(200)+2P(020)P(202)+2P(020)P(212)+\\
2P(020)P(222)+2P(020)P(232)+2P(020)P(303)+2P(020)P(313)+2P(020)P(323)+\\
2P(020)P(333)+2P(021)P(201)+2P(022)P(100)+2P(022)P(200)+2P(022)P(202)+\\
2P(022)P(220)+2P(022)P(222)+2P(022)P(300)+2P(023)P(203)+2P(023)P(320)+\\
2P(030)P(101)+2P(030)P(202)+2P(030)P(212)+2P(030)P(222)+2P(030)P(232)+\\
2P(030)P(300)+2P(030)P(303)+2P(030)P(313)+2P(030)P(323)+2P(030)P(333)+
\end{gathered}
$$

$$2P(031)P(301)+2P(032)P(230)+2P(032)P(302)+2P(033)P(100)+2P(033)P(200)+$$
$$2P(033)P(211)+2P(033)P(222)+2P(033)P(233)+2P(033)P(300)+2P(033)P(303)+$$
$$2P(033)P(322)+2P(033)P(330)+2P(100)P(122)+2P(100)P(133)+2P(100)P(222)+$$
$$2P(100)P(233)+2P(100)P(322)+2P(101)P(121)+2P(101)P(131)+2P(101)P(222)+$$
$$2P(101)P(232)+2P(101)P(323)+2P(102)P(120)+2P(103)P(130)+2P(110)P(112)+$$
$$2P(110)P(113)+2P(110)P(222)+2P(110)P(223)+2P(110)P(332)+2P(112)P(330)+$$
$$2P(121)P(303)+2P(122)P(200)+2P(122)P(300)+2P(133)P(200)+2P(133)P(300)+$$
$$2P(200)P(222)+2P(200)P(233)+2P(200)P(322)+2P(200)P(333)+2P(202)P(220)+$$
$$2P(202)P(222)+2P(203)P(230)+2P(220)P(222)+2P(222)P(300)+2P(222)P(303)+$$
$$2P(222)P(330)+2P(223)P(330)+2P(232)P(303)+2P(233)P(300)+2P(300)P(322)+$$
$$2P(300)P(333)+2P(302)P(320)+2P(303)P(323)+2P(303)P(330)+2P(330)P(332)+$$
$$6P(000)P(222)+P(000)P(013)+P(000)P(023)+P(000)P(031)+P(000)P(032)+$$
$$P(000)P(103)+P(000)P(123)+P(000)P(130)+P(000)P(132)+P(000)P(203)+$$
$$P(000)P(213)+P(000)P(230)+P(000)P(231)+P(000)P(301)+P(000)P(302)+$$
$$P(000)P(310)+P(000)P(312)+P(000)P(320)+P(000)P(321)+P(001)P(020)+$$
$$P(001)P(021)+P(001)P(022)+P(001)P(023)+P(001)P(030)+P(001)P(031)+$$
$$P(001)P(032)+P(001)P(033)+P(001)P(120)+P(001)P(121)+P(001)P(122)+$$
$$P(001)P(123)+P(001)P(130)+P(001)P(131)+P(001)P(132)+P(001)P(133)+$$
$$P(001)P(200)+P(001)P(201)+P(001)P(202)+P(001)P(203)+P(001)P(210)+$$
$$P(001)P(211)+P(001)P(212)+P(001)P(213)+P(001)P(230)+P(001)P(231)+$$
$$P(001)P(232)+P(001)P(233)+P(001)P(300)+P(001)P(301)+P(001)P(302)+$$
$$P(001)P(303)+P(001)P(310)+P(001)P(311)+P(001)P(312)+P(001)P(313)+$$
$$P(001)P(320)+P(001)P(321)+P(001)P(322)+P(001)P(323)+P(002)P(010)+$$
$$P(002)P(021)+P(002)P(022)+P(002)P(023)+P(002)P(030)+P(002)P(031)+$$
$$P(002)P(032)+P(002)P(033)+P(002)P(100)+P(002)P(120)+P(002)P(121)+$$
$$P(002)P(122)+P(002)P(123)+P(002)P(130)+P(002)P(131)+P(002)P(132)+$$
$$P(002)P(133)+P(002)P(201)+P(002)P(202)+P(002)P(203)+P(002)P(210)+$$
$$P(002)P(211)+P(002)P(212)+P(002)P(213)+P(002)P(230)+P(002)P(231)+$$
$$P(002)P(232)+P(002)P(233)+P(002)P(300)+P(002)P(301)+P(002)P(302)+$$
$$P(002)P(303)+P(002)P(310)+P(002)P(311)+P(002)P(312)+P(002)P(313)+$$
$$P(002)P(320)+P(002)P(321)+P(002)P(322)+P(002)P(323)+P(003)P(010)+$$
$$P(003)P(020)+P(003)P(021)+P(003)P(022)+P(003)P(023)+P(003)P(031)+$$
$$P(003)P(032)+P(003)P(033)+P(003)P(100)+P(003)P(120)+P(003)P(121)+$$
$$P(003)P(122)+P(003)P(123)+P(003)P(130)+P(003)P(131)+P(003)P(132)+$$
$$P(003)P(133)+P(003)P(200)+P(003)P(201)+P(003)P(202)+P(003)P(203)+$$
$$P(003)P(210)+P(003)P(211)+P(003)P(212)+P(003)P(213)+P(003)P(230)+$$
$$P(003)P(231)+P(003)P(232)+P(003)P(233)+P(003)P(301)+P(003)P(302)+$$
$$P(003)P(303)+P(003)P(310)+P(003)P(311)+P(003)P(312)+P(003)P(313)+$$
$$P(003)P(320)+P(003)P(321)+P(003)P(322)+P(003)P(323)+P(010)P(012)+$$
$$P(010)P(013)+P(010)P(022)+P(010)P(023)+P(010)P(032)+P(010)P(033)+$$
$$P(010)P(102)+P(010)P(103)+P(010)P(112)+P(010)P(113)+P(010)P(122)+$$
$$P(010)P(123)+P(010)P(132)+P(010)P(133)+P(010)P(200)+P(010)P(201)+$$
$$P(010)P(203)+P(010)P(210)+P(010)P(211)+P(010)P(213)+P(010)P(220)+$$
$$P(010)P(221)+P(010)P(223)+P(010)P(230)+P(010)P(231)+P(010)P(233)+$$
$$P(010)P(300)+P(010)P(301)+P(010)P(302)+P(010)P(310)+P(010)P(311)+$$
$$P(010)P(312)+P(010)P(320)+P(010)P(321)+P(010)P(322)+P(010)P(330)+$$

$P(010)P(331) + P(010)P(332) + P(011)P(201) + P(011)P(202) + P(011)P(203)+$
$P(011)P(210) + P(011)P(212) + P(011)P(213) + P(011)P(220) + P(011)P(221)+$
$P(011)P(223) + P(011)P(230) + P(011)P(231) + P(011)P(232) + P(011)P(301)+$
$P(011)P(302) + P(011)P(303) + P(011)P(310) + P(011)P(312) + P(011)P(313)+$
$P(011)P(320) + P(011)P(321) + P(011)P(323) + P(011)P(330) + P(011)P(331)+$
$P(011)P(332) + P(012)P(020) + P(012)P(030) + P(012)P(100) + P(012)P(110)+$
$P(012)P(120) + P(012)P(130) + P(012)P(200) + P(012)P(201) + P(012)P(202)+$
$P(012)P(203) + P(012)P(211) + P(012)P(212) + P(012)P(213) + P(012)P(220)+$
$P(012)P(221) + P(012)P(222) + P(012)P(223) + P(012)P(230) + P(012)P(231)+$
$P(012)P(232) + P(012)P(233) + P(012)P(300) + P(012)P(301) + P(012)P(302)+$
$P(012)P(303) + P(012)P(310) + P(012)P(311) + P(012)P(312) + P(012)P(313)+$
$P(012)P(320) + P(012)P(321) + P(012)P(322) + P(012)P(323) + P(012)P(330)+$
$P(012)P(331) + P(013)P(020) + P(013)P(030) + P(013)P(100) + P(013)P(110)+$
$P(013)P(120) + P(013)P(130) + P(013)P(200) + P(013)P(201) + P(013)P(202)+$
$P(013)P(203) + P(013)P(210) + P(013)P(211) + P(013)P(212) + P(013)P(213)+$
$P(013)P(220) + P(013)P(221) + P(013)P(222) + P(013)P(223) + P(013)P(230)+$
$P(013)P(231) + P(013)P(232) + P(013)P(233) + P(013)P(300) + P(013)P(301)+$
$P(013)P(302) + P(013)P(303) + P(013)P(311) + P(013)P(312) + P(013)P(313)+$
$P(013)P(320) + P(013)P(321) + P(013)P(322) + P(013)P(323) + P(013)P(330)+$
$P(013)P(331) + P(013)P(332) + P(020)P(022) + P(020)P(023) + P(020)P(032)+$
$P(020)P(033) + P(020)P(100) + P(020)P(102) + P(020)P(103) + P(020)P(112)+$
$P(020)P(113) + P(020)P(122) + P(020)P(123) + P(020)P(132) + P(020)P(133)+$
$P(020)P(201) + P(020)P(203) + P(020)P(210) + P(020)P(211) + P(020)P(213)+$
$P(020)P(220) + P(020)P(221) + P(020)P(223) + P(020)P(230) + P(020)P(231)+$
$P(020)P(233) + P(020)P(300) + P(020)P(301) + P(020)P(302) + P(020)P(310)+$
$P(020)P(311) + P(020)P(312) + P(020)P(320) + P(020)P(321) + P(020)P(322)+$
$P(020)P(330) + P(020)P(331) + P(020)P(332) + P(021)P(100) + P(021)P(101)+$
$P(021)P(102) + P(021)P(103) + P(021)P(200) + P(021)P(202) + P(021)P(203)+$
$P(021)P(210) + P(021)P(211) + P(021)P(212) + P(021)P(213) + P(021)P(220)+$
$P(021)P(221) + P(021)P(222) + P(021)P(223) + P(021)P(230) + P(021)P(231)+$
$P(021)P(232) + P(021)P(233) + P(021)P(300) + P(021)P(301) + P(021)P(302)+$
$P(021)P(303) + P(021)P(310) + P(021)P(311) + P(021)P(312) + P(021)P(313)+$
$P(021)P(320) + P(021)P(321) + P(021)P(322) + P(021)P(330) + P(021)P(331)+$
$P(021)P(332) + P(022)P(030) + P(022)P(101) + P(022)P(102) + P(022)P(103)+$
$P(022)P(110) + P(022)P(120) + P(022)P(130) + P(022)P(201) + P(022)P(203)+$
$P(022)P(210) + P(022)P(212) + P(022)P(213) + P(022)P(221) + P(022)P(223)+$
$P(022)P(230) + P(022)P(231) + P(022)P(232) + P(022)P(301) + P(022)P(302)+$
$P(022)P(303) + P(022)P(310) + P(022)P(320) + P(022)P(330) + P(023)P(030)+$
$P(023)P(100) + P(023)P(101) + P(023)P(102) + P(023)P(103) + P(023)P(110)+$
$P(023)P(120) + P(023)P(130) + P(023)P(200) + P(023)P(201) + P(023)P(202)+$
$P(023)P(210) + P(023)P(211) + P(023)P(212) + P(023)P(213) + P(023)P(220)+$
$P(023)P(221) + P(023)P(222) + P(023)P(223) + P(023)P(230) + P(023)P(231)+$
$P(023)P(232) + P(023)P(233) + P(023)P(300) + P(023)P(301) + P(023)P(302)+$
$P(023)P(303) + P(023)P(310) + P(023)P(322) + P(023)P(330) + P(023)P(332)+$
$P(030)P(032) + P(030)P(033) + P(030)P(100) + P(030)P(102) + P(030)P(103)+$
$P(030)P(112) + P(030)P(113) + P(030)P(122) + P(030)P(123) + P(030)P(132)+$

$P(030)P(133) + P(030)P(200) + P(030)P(201) + P(030)P(203) + P(030)P(210)+$
$P(030)P(211) + P(030)P(213) + P(030)P(220) + P(030)P(221) + P(030)P(223)+$
$P(030)P(230) + P(030)P(231) + P(030)P(233) + P(030)P(301) + P(030)P(302)+$
$P(030)P(310) + P(030)P(311) + P(030)P(312) + P(030)P(320) + P(030)P(321)+$
$P(030)P(322) + P(030)P(330) + P(030)P(331) + P(030)P(332) + P(031)P(100)+$
$P(031)P(101) + P(031)P(102) + P(031)P(103) + P(031)P(200) + P(031)P(201)+$
$P(031)P(202) + P(031)P(203) + P(031)P(210) + P(031)P(211) + P(031)P(212)+$
$P(031)P(213) + P(031)P(220) + P(031)P(221) + P(031)P(222) + P(031)P(223)+$
$P(031)P(230) + P(031)P(231) + P(031)P(232) + P(031)P(233) + P(031)P(300)+$
$P(031)P(302) + P(031)P(303) + P(031)P(310) + P(031)P(311) + P(031)P(312)+$
$P(031)P(313) + P(031)P(320) + P(031)P(321) + P(031)P(322) + P(031)P(323)+$
$P(031)P(330) + P(031)P(331) + P(031)P(332) + P(032)P(100) + P(032)P(101)+$
$P(032)P(102) + P(032)P(103) + P(032)P(110) + P(032)P(120) + P(032)P(130)+$
$P(032)P(200) + P(032)P(201) + P(032)P(202) + P(032)P(203) + P(032)P(210)+$
$P(032)P(211) + P(032)P(212) + P(032)P(213) + P(032)P(220) + P(032)P(221)+$
$P(032)P(222) + P(032)P(223) + P(032)P(231) + P(032)P(232) + P(032)P(233)+$
$P(032)P(300) + P(032)P(301) + P(032)P(303) + P(032)P(310) + P(032)P(320)+$
$P(032)P(322) + P(032)P(323) + P(032)P(330) + P(033)P(101) + P(033)P(102)+$
$P(033)P(103) + P(033)P(110) + P(033)P(120) + P(033)P(130) + P(033)P(201)+$
$P(033)P(202) + P(033)P(203) + P(033)P(210) + P(033)P(212) + P(033)P(213)+$
$P(033)P(220) + P(033)P(221) + P(033)P(223) + P(033)P(230) + P(033)P(231)+$
$P(033)P(232) + P(033)P(301) + P(033)P(302) + P(033)P(310) + P(033)P(312)+$
$P(033)P(320) + P(033)P(321) + P(033)P(323) + P(033)P(332) + P(100)P(102)+$
$P(100)P(103) + P(100)P(112) + P(100)P(113) + P(100)P(120) + P(100)P(121)+$
$P(100)P(123) + P(100)P(130) + P(100)P(131) + P(100)P(132) + P(100)P(202)+$
$P(100)P(203) + P(100)P(212) + P(100)P(213) + P(100)P(220) + P(100)P(221)+$
$P(100)P(223) + P(100)P(230) + P(100)P(231) + P(100)P(232) + P(100)P(302)+$
$P(100)P(303) + P(100)P(312) + P(100)P(313) + P(100)P(320) + P(100)P(321)+$
$P(100)P(323) + P(100)P(330) + P(100)P(331) + P(100)P(332) + P(101)P(120)+$
$P(101)P(122) + P(101)P(123) + P(101)P(130) + P(101)P(132) + P(101)P(133)+$
$P(101)P(220) + P(101)P(221) + P(101)P(223) + P(101)P(230) + P(101)P(231)+$
$P(101)P(233) + P(101)P(320) + P(101)P(321) + P(101)P(322) + P(101)P(330)+$
$P(101)P(331) + P(101)P(332) + P(102)P(110) + P(102)P(121) + P(102)P(122)+$
$P(102)P(123) + P(102)P(130) + P(102)P(131) + P(102)P(132) + P(102)P(133)+$
$P(102)P(200) + P(102)P(210) + P(102)P(220) + P(102)P(221) + P(102)P(222)+$
$P(102)P(223) + P(102)P(230) + P(102)P(231) + P(102)P(232) + P(102)P(233)+$
$P(102)P(300) + P(102)P(310) + P(102)P(320) + P(102)P(321) + P(102)P(322)+$
$P(102)P(323) + P(102)P(330) + P(102)P(331) + P(103)P(110) + P(103)P(120)+$
$P(103)P(121) + P(103)P(122) + P(103)P(123) + P(103)P(131) + P(103)P(132)+$
$P(103)P(133) + P(103)P(200) + P(103)P(210) + P(103)P(220) + P(103)P(221)+$
$P(103)P(222) + P(103)P(223) + P(103)P(230) + P(103)P(231) + P(103)P(232)+$
$P(103)P(233) + P(103)P(300) + P(103)P(310) + P(103)P(320) + P(103)P(321)+$
$P(103)P(322) + P(103)P(323) + P(103)P(330) + P(103)P(331) + P(103)P(332)+$
$P(110)P(122) + P(110)P(123) + P(110)P(132) + P(110)P(133) + P(110)P(202)+$
$P(110)P(203) + P(110)P(212) + P(110)P(213) + P(110)P(232) + P(110)P(233)+$
$P(110)P(302) + P(110)P(303) + P(110)P(312) + P(110)P(313) + P(110)P(322)+$

$$
\begin{aligned}
&P(110)P(323) + P(112)P(120) + P(112)P(130) + P(112)P(200) + P(112)P(210)+\\
&P(112)P(230) + P(112)P(300) + P(112)P(310) + P(112)P(320) + P(113)P(120)+\\
&P(113)P(130) + P(113)P(200) + P(113)P(210) + P(113)P(300) + P(113)P(310)+\\
&P(120)P(122) + P(120)P(123) + P(120)P(132) + P(120)P(133) + P(120)P(200)+\\
&P(120)P(201) + P(120)P(202) + P(120)P(203) + P(120)P(212) + P(120)P(213)+\\
&P(120)P(222) + P(120)P(223) + P(120)P(232) + P(120)P(233) + P(120)P(300)+\\
&P(120)P(301) + P(120)P(302) + P(120)P(303) + P(120)P(312) + P(120)P(313)+\\
&P(120)P(322) + P(120)P(332) + P(121)P(200) + P(121)P(201) + P(121)P(203)+\\
&P(121)P(300) + P(121)P(301) + P(121)P(302) + P(122)P(130) + P(122)P(201)+\\
&P(122)P(202) + P(122)P(203) + P(122)P(210) + P(122)P(220) + P(122)P(230)+\\
&P(122)P(301) + P(122)P(302) + P(122)P(303) + P(122)P(310) + P(122)P(320)+\\
&P(122)P(330) + P(123)P(130) + P(123)P(200) + P(123)P(201) + P(123)P(202)+\\
&P(123)P(203) + P(123)P(210) + P(123)P(300) + P(123)P(301) + P(123)P(302)+\\
&P(123)P(303) + P(123)P(310) + P(123)P(330) + P(130)P(132) + P(130)P(133)+\\
&P(130)P(200) + P(130)P(201) + P(130)P(202) + P(130)P(203) + P(130)P(212)+\\
&P(130)P(213) + P(130)P(222) + P(130)P(223) + P(130)P(232) + P(130)P(233)+\\
&P(130)P(300) + P(130)P(301) + P(130)P(302) + P(130)P(303) + P(130)P(312)+\\
&P(130)P(313) + P(130)P(322) + P(130)P(323) + P(130)P(332) + P(131)P(200)+\\
&P(131)P(201) + P(131)P(300) + P(131)P(301) + P(132)P(200) + P(132)P(201)+\\
&P(132)P(210) + P(132)P(220) + P(132)P(230) + P(132)P(300) + P(132)P(301)+\\
&P(132)P(303) + P(132)P(310) + P(132)P(320) + P(132)P(330) + P(133)P(201)+\\
&P(133)P(210) + P(133)P(301) + P(133)P(310) + P(200)P(202) + P(200)P(203)+\\
&P(200)P(212) + P(200)P(213) + P(200)P(220) + P(200)P(221) + P(200)P(223)+\\
&P(200)P(230) + P(200)P(231) + P(200)P(232) + P(200)P(302) + P(200)P(303)+\\
&P(200)P(312) + P(200)P(313) + P(200)P(320) + P(200)P(321) + P(200)P(323)+\\
&P(200)P(330) + P(200)P(331) + P(200)P(332) + P(201)P(220) + P(201)P(221)+\\
&P(201)P(222) + P(201)P(223) + P(201)P(230) + P(201)P(231) + P(201)P(232)+\\
&P(201)P(320) + P(201)P(321) + P(201)P(322) + P(201)P(323) + P(201)P(330)+\\
&P(201)P(331) + P(201)P(332) + P(202)P(210) + P(202)P(221) + P(202)P(223)+\\
&P(202)P(230) + P(202)P(300) + P(202)P(310) + P(202)P(320) + P(202)P(321)+\\
&P(202)P(322) + P(202)P(330) + P(203)P(210) + P(203)P(220) + P(203)P(221)+\\
&P(203)P(222) + P(203)P(223) + P(203)P(232) + P(203)P(300) + P(203)P(310)+\\
&P(203)P(320) + P(203)P(321) + P(203)P(322) + P(203)P(323) + P(203)P(330)+\\
&P(203)P(332) + P(210)P(212) + P(210)P(213) + P(210)P(222) + P(210)P(223)+\\
&P(210)P(232) + P(210)P(302) + P(210)P(303) + P(210)P(312) + P(210)P(313)+\\
&P(210)P(322) + P(210)P(323) + P(210)P(332) + P(212)P(220) + P(212)P(230)+\\
&P(212)P(300) + P(212)P(310) + P(212)P(320) + P(212)P(330) + P(213)P(300)+\\
&P(213)P(310) + P(213)P(330) + P(220)P(232) + P(220)P(300) + P(220)P(301)+\\
&P(220)P(302) + P(220)P(303) + P(220)P(312) + P(220)P(322) + P(221)P(300)+\\
&P(221)P(301) + P(221)P(302) + P(221)P(303) + P(222)P(230) + P(222)P(301)+\\
&P(222)P(302) + P(222)P(310) + P(222)P(320) + P(223)P(230) + P(223)P(300)+\\
&P(223)P(301) + P(223)P(302) + P(223)P(303) + P(223)P(310) + P(223)P(320)+\\
&P(230)P(232) + P(230)P(300) + P(230)P(301) + P(230)P(302) + P(230)P(303)+\\
&P(230)P(312) + P(230)P(322) + P(230)P(323) + P(230)P(332) + P(231)P(300)+\\
&P(231)P(301) + P(231)P(303) + P(232)P(300) + P(232)P(301) + P(232)P(302)+\\
&P(232)P(310) + P(232)P(320) + P(232)P(330) + P(233)P(301) + P(233)P(302)+
\end{aligned}
$$

$$P(233)P(303) + P(233)P(310) + P(233)P(320) + P(233)P(330) + P(300)P(302)+$$
$$P(300)P(303) + P(300)P(312) + P(300)P(313) + P(300)P(320) + P(300)P(321)+$$
$$P(300)P(323) + P(300)P(330) + P(300)P(331) + P(300)P(332) + P(301)P(320)+$$
$$P(301)P(321) + P(301)P(322) + P(301)P(323) + P(301)P(330) + P(301)P(331)+$$
$$P(301)P(332) + P(302)P(310) + P(302)P(321) + P(302)P(322) + P(302)P(323)+$$
$$P(302)P(330) + P(303)P(310) + P(303)P(320) + P(303)P(321) + P(303)P(322)+$$
$$P(303)P(332) + P(310)P(312) + P(310)P(313) + P(310)P(322) + P(310)P(323)+$$
$$P(310)P(332) + P(312)P(320) + P(312)P(330) + P(320)P(322) + P(320)P(332)+$$
$$P(322)P(330) + P(323)P(330)$$