Multicategorical Meta-Theorem and the Completeness of Restricted Algebraic Logics

D. Forsman

David Forsman (david.forsman@uclouvain.be)
Université catholique de Louvain

Abstract.

Consider the classical Eckmann-Hilton argument: if a set carries two unital magma structures that distribute over each other, then these structures coincide and determine a single commutative monoid. This argument extends from sets to symmetric multicategories, but why? More generally, when do equational proofs in sets lift to categorical settings?

We answer this question through the Multicategorical Meta-Theorem. We identify precisely eight subcategories Δ of **FinOrd** of finite ordinals and functions, each corresponding to a coherent structure that can be added to a monoidal category (like symmetry, deletion, or duplication). For six of these structures — including symmetric multicategories and cartesian multicategories — we prove that if $E \cup \{\phi\}$ is an Δ–theory and the equation ϕ follows from the set of equations E in **Set**, then ϕ follows from E in any model in any Δ-multicategory.

We prove the Multicategorical Meta-Theorem by restricting the equational deduction system $\vdash$ into $\vdash_\Delta$ for eight different structures Δ and prove categorical soundness and completeness theorems. Exactly six of the eight deduction systems $\vdash_\Delta$ have completeness in sets. Completeness in set-based models and soundness in all models yields the meta-theorem.

In addition, a corollary of our result is that if $T \cup \{\phi\}$ is an algebraic Δ-theory and Δ is one of the six structures, then $T \vdash \phi$ if and only if $T \vdash_\Delta \phi$. This shows that in certain cases one may simplify the proof search without sacrificing the completeness.