The semantic lax descent factorization of a functor

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In the classical context of [5, 6]: assuming that C has pullbacks, if $\mathcal{A} : \mathsf{C}^{\mathrm{op}} \to \mathsf{Cat}$ is an indexed category, the descent category $\mathrm{Desc}_{\mathcal{A}}(p)$ is the category of actions of the internal groupoid/equivalence induced by the kernel pair of p. The celebrated Bénabou-Roubaud Theorem [1] shows that $\mathrm{Desc}_{\mathcal{A}}(p)$ is equivalent to the category of algebras induced by $\mathcal{A}(p)! \dashv \mathcal{A}(p)$ in the classical context of [5], under the so-called Beck-Chevalley condition.

In [2], we started investigating whether commuting properties of higher dimensional limits are useful in proving classical and new theorems of Grothendieck descent theory. Exploiting this perspective, we were able to give a generalization of the Bénabou-Roubaud Theorem in terms of commuting properties of bilimits in [2, Theorem 7.4 and Theorem 8.5] (see, also, [4]). More than that, further investigation of commuting properties yields, in particular, to the main result of [3] which can be seen as a counterpart to the Bénabou-Roubaud Theorem, giving a characterization of monadic functors in terms of an exact condition, and showing that descent data generalizes both "algebraic structure and coalgebraic structure", as we explain below.

Every functor that has a left adjoint has two well-known factorizations. The first one is through the category of Eilenberg-Moore algebras of the induced monad, while the second one is the factorization through the category of free coalgebras (co-Kelsili category) of the induced comonad. As usual in category theory, we also have the dual cases: a functor that has a right adjoint has a factorization through the category of Eilenberg-Moore coalgebras of the induced comonad, and other factorization through the Kleisli category of the induced monad.

In [3], we first observe that, given a 2-category A satisfying suitable hypothesis, every morphism inside a 2-category with opcomma objects (and pushouts) has a 2-dimensional cokernel diagram which, in the presence of the descent objects, induces a factorization of the morphism – that we called therein "the semantic lax descent factorization of p". Intuitively, this is a two-dimensional (lax) version/categorification of the usual factorization of a morphism coming from the equalizer of its cokernel diagram (the (co)image factorization in case of the category of sets Set).

Our main results of [3] shows that the semantic lax descent factorization of a morphism p generalizes the usual (co)Eilenberg-Moore and Kleisli factorizations. More precisely, whenever "the semantic lax descent factorization of p" coincides with the (co)Eilenberg-Moore factorization whenever p has a (right) left adjoint (or even in more lenient hypotheses).

The result is novel even in the case $A = \mathsf{Cat}$, the usual 2-categories of categories. In this case, we have that every functor has a factorization through the category of descent data of its 2-dimensional cokernel diagram. We show that, if a functor F has a left adjoint, this descent factorization coincides with the factorization through the category of algebras. Dually, if F has a right adjoint, this descent factorization coincides with the factorization through the category of coalgebras. This specializes in a new connection between monadicity and descent theory. It also leads in particular to a (formal) monadicity theorem.

In this talk, we present some aspects of this work in descent theory and, if time allows, we talk about ongoing work.

References

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