Lax comma categories: descent and exponentiability

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- objects: morphisms $f\colon B \to X$ of $\mathbb{A}$ with B in $\mathbb{B}$,
- morphisms $f \to g$: 2-cells θ of $\mathbb{A}$ of the form

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\overset{\theta}{\Longrightarrow}"] & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

  where b is a morphism in $\mathbb{B}$.

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that is, set-indexed families $(\theta_a\colon X_a \to Y_{f(a)})_{a\in A}$ of morphisms in $\mathcal{X}$.

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Thus, $\mathsf{Set} \Downarrow \mathcal{X} = \mathsf{Fam}(\mathcal{X})$.

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  that is, $\alpha(a) \leqslant \beta(f(a))$ for all $a \in A$.

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that is, we have a fibration $\mathbb{A} \Downarrow X \to \mathbb{A}$.

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If $\mathcal{X}$ is cartesian closed, then so is $\mathsf{Cat} \Downarrow \mathcal{X}$.

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We say p is an *effective descent morphism* (descent morphism) if p^* is monadic (premonadic).

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Theorem (Lucatelli Nunes, P. 2024)
If X satisfies mild conditions, then

$$\mathbb{A} \Downarrow X \to \mathbb{A}$$

preserves effective descent morphisms.

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- $f\colon (\alpha(a))_{a\leqslant a'} \to (\beta(b))_{b\leqslant b'}$ is a descent morphism in $\mathsf{Fam}(X)$.

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- Characterization of effective descent morphisms in $\mathsf{Cat} \Downarrow X$.

Dank wel!