PSSL 109

Abstract Submission

- Title: Causal Coverage in Ordered Locales
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Abstract. In this talk we discuss ongoing work on *causal coverages* defined in ordered locales, introduced in the author's PhD thesis [vdS24]. Informally, we say a region A *covers* U *from the past* if: "all information reaching U along causal paths has to have come from A." This is supposed to generalise the canonical (Grothendieck) coverage of a locale that takes additional causal structure into account. This work is deeply inspired by a similar notion of coverage first introduced in [CC05]. Our framework provides an improvement in that the new notion of causal coverage satisfies a suitable pullback stability axiom, as expected of a coverage, whereas theirs does not [CC05, Remark 2.13]. Indeed, when made precise, the above informal idea leads to a coverage relation $\mathrm{Cov}^-_{\trianglelefteq}$ that satisfies properties closely resembling the axioms of a Grothendieck topology: identity, pullback stability, and local characteristic.

As in [CC05], our motivation stems from considerations on the mathematical foundations of spacetimes. Towards applications, we will show how causal coverages naturally give rise to an abstract notion of *domain of dependence*, an important concept in mathematical relativity theory [Ger70]. Towards abstraction, we will sketch how causal coverages can be seen as a generalised notion of Grothendieck topology "taking values in a monad." The sheaf condition with respect to causal coverages then encodes a type of deterministic evolution. An explicit example is the sheaf of solutions to the wave equation on Minkowski space.

Details. We use the framework of ordered locales as introduced in [HS24] to provide a point-free analogue of ordered topological spaces (which in turn we think of as abstract models of spacetimes). There, an ordered locale $(X, \trianglelefteq)$ consists of a locale X together with a preorder $\trianglelefteq$ on the frame of opens $\mathcal{O}X$, satisfying suitable axioms. Equivalently, one can view an ordered locale as a pair of operators $\Uparrow$ and $\Downarrow$ on $\mathcal{O}X$, where for a region $U \in \mathcal{O}X$ the new region $\Uparrow U$ is called the *future cone* and $\Downarrow U$ is called the *past cone* of U. We think of these as the regions of future and past influence of U, and they provide point-free analogues of the familiar lightcones in spacetimes.

Thinking of $(X, \trianglelefteq)$ as an arena in which information can "flow" along causal paths, we say informally that A *causally covers* U if: all information reaching U from the past along causal paths has to have come from A. This is made precise as follows. First, by a *path* we mean a finite chain of non-empty regions $p_0 \trianglelefteq \cdots \trianglelefteq p_N$. The region p_N is called the *endpoint* of p. We say another path q *refines* p if every component of p contains some component of q. Informally, a *local refinement* of p consists of a family $(q^i)_{i\in I}$ of paths that suitably refine p and such that the join of the endpoints of the q^i cover the endpoint of p. We say A covers U from the past if: every path that lands in U admits a local refinement $(q^i)_{i\in I}$ such that each q^i inhabits A.

The main result is that the resulting notion of coverage, written $A \in \mathrm{Cov}^-_{\trianglelefteq}(U)$, has the following properties that resemble the axioms of a Grothendieck topology:

(i) $U \in \mathrm{Cov}^-_{\trianglelefteq}(U)$ and $\Downarrow U \in \mathrm{Cov}^-_{\trianglelefteq}(U)$; (*identity*)

(ii) if $A \in \mathrm{Cov}^-_{\trianglelefteq}(U)$ and $W \sqsubseteq U$, then $A \wedge \Downarrow W \in \mathrm{Cov}^-_{\trianglelefteq}(W)$; (*pullback stability*)

(iii) if $B \in \mathrm{Cov}^-_{\trianglelefteq}(A)$ and $A \in \mathrm{Cov}^-_{\trianglelefteq}(U)$, then $B \in \mathrm{Cov}^-_{\trianglelefteq}(U)$. (*local characteristic*)

Using these properties, it can be shown that $\mathrm{Cov}^-_{\trianglelefteq}$ defines a generalised type of Grothendieck topology J^- on $\mathcal{O}X$ "taking values" in the monad $\Downarrow$. Namely, for a sieve R on $\Downarrow U$ we say

$$R \in J^-(U) \qquad \text{if and only if} \qquad \bigvee R \in \mathrm{Cov}^-_{\trianglelefteq}(U).$$

(Based on joint work with Chris Heunen.)

References

[CC05] J. D. Christensen and L. Crane. “Causal sites as quantum geometry”. In: *Journal of Mathematical Physics* 46.12 (Dec. 2005) (cit. on p. 1).

[Ger70] R. Geroch. “Domain of Dependence”. In: *Journal of Mathematical Physics* 11.2 (Oct. 1970), pp. 437–449 (cit. on p. 1).

[HS24] C. Heunen and N. van der Schaaf. “Ordered Locales”. In: *Journal of Pure and Applied Algebra* 228.7 (2024), p. 107654 (cit. on p. 1).

[vdS24] N. van der Schaaf. *Towards Point-Free Spacetimes*. PhD thesis. 2024. arXiv: 2406.15406 (cit. on p. 1).