A study of Kock's fat Delta

PSSL 2024

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The study of higher categories involves many tools based on the simplex category Δ and simplicial methods [8]. Indeed, Δ enables the encoding of coherences in geometric shapes for higher structures [4, 7]. However, the degeneracy maps in Δ encode the identity structure *strictly* in contrast to the associativity structure. The category referred by fat Delta, denoted $\underline{\Delta}$ and first introduced by Kock [5] as the category of relative finite semiordinals (i.e. relative finite ordinals with a total strict order relation), was developed as a means of providing a geometric interpretation of *weak* identity arrows in higher categories. Since then, it has been studied further by Paoli [9] and, in both cases [5, 9], their research was motivated by the investigation of Simpson's conjecture in low dimensions. For different motivations related to Homotopy Type Theory (HoTT), Kraus and Sattler [6] constructed and studied a direct replacement of Δ, that turns out to be a variation of $\underline{\Delta}$. The background is that, internally to a type theory, it is unknown how to represent presheaves on a category in general, but presehaves on a direct category have a natural representation via type families [10].
We present a comprehensive study of $\underline{\Delta}$ mainly via the theory of monads with arities [11, 3], which offers an abstract setting to produce nerve theorems and study Segal conditions. The primary notion in our work is that of a relative semicategory; the latter is similar to the one of relative categories [1] without requiring identity structure, and we denote the category of relative semicategories by $\mathsf{RelSemiCat}$. Our first main result is the nerve theorem for relative semicategories.

Theorem 1 *Let* $\mathsf{RelGraph}$ *denote the category of relative directed graphs and let* $\underline{j} : \underline{\Delta}_0 \hookrightarrow \underline{\Delta}$ *be the inclusion of the wide subcategory of relative semiordinals and relative graph morphisms. The nerve functor* $\underline{\mathcal{N}} : \mathsf{RelSemiCat} \to \widehat{\underline{\Delta}}$ *is fully faithful. The essential image is spanned by the presheaves whose restriction along* $\underline{j}$ *belong to the essential image of* $\underline{\mathcal{N}}_0 : \mathsf{RelGraph} \to \widehat{\underline{\Delta}_0}$.

In particular, this indicates that $\underline{\Delta}$ is for relative semicategories what Δ is for categories. We obtain Theorem 1 by proving that the free relative semicategory monad on $\mathsf{RelGraph}$ is strongly cartesian in the sense of [11]. This is achieved by showing that the monad is induced by the free semicategory monad on directed graphs and that the property of being strongly cartesian is preserved. As a consequence we can construct a category of arities from a category inspired by weak identity coherences, namely the category of alternatingly marked linear graphs, and show that $\underline{\Delta}$ has a special orthogonal factorisation system.

Theorem 2 *The category of arities of the free relative semicategory monad* $\mathfrak{f}^+$ *is isomorphic to* $\underline{\Delta}_0$. *Thus,* $\underline{\Delta}$ *can be recovered as the category of free algebras of* $\mathfrak{f}^+$ *over* $\underline{\Delta}_0$, *and it admits an active/inert factorisation system* $(\underline{\Delta}_a, \underline{\Delta}_0)$.

This active/inert factorisation system allows us to more easily express the Segal condition of [9, Section 4.3]. Additionally, using $(\underline{\Delta}_a, \underline{\Delta}_0)$, we can relate $\underline{\Delta}$ to Berger's theory of moment categories [2]:

Theorem 3 *The category* $\underline{\Delta}$ *has the structure of strongly unital hypermoment category.*

References

[1] Clark Barwick and Daniël M Kan. "Relative categories: another model for the homotopy theory of homotopy theories". In: *Indagationes Mathematicae* 23.1–2 (2012), pp. 42–68. DOI: 10.1016/j.indag.2011.10.002.

[2] Clemens Berger. "Moment categories and operads". In: *Theory and Application of Categories* 38.39 (2022), pp. 1485–1537.

[3] Clemens Berger, Paul-André Melliès, and Mark Weber. "Monads with arities and their associated theories". In: *Journal of Pure and Applied Algebra* 216.8–9 (2012), pp. 2029–2048. DOI: 10.1016/j.jpaa.2012.02.039.

[4] André Joyal. "The Theory of Quasi-Categories and its Applications". Lectures at Advanced Course on Simplicial Methods in Higher Categories, CRM. 2008.

[5] Joachim Kock. "Weak identity arrows in higher categories". In: *International Mathematics Research Papers* 2006 (2006), pp. 1–54. DOI: 10.1155/IMRP/2006/69163.

[6] Nicolai Kraus and Christian Sattler. *Space-Valued Diagrams, Type-Theoretically (Extended Abstract)*. 2017. DOI: 10.48550/arXiv.1704.04543.

[7] Jacob Lurie. *Higher Topos Theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, 2009. DOI: 10.48550/arXiv.math/0608040.

[8] Simona Paoli. *Simplicial Methods for Higher Categories: Segal-type Models of Weak n-Categories*. Vol. 26. Algebra and Applications. Springer, 2019. DOI: 10.1007/978-3-030-05674-2.

[9] Simona Paoli. *Weakly globular double categories and weak units*. 2024. DOI: 10.48550/arXiv:2008.11180.

[10] Michael Shulman. "Univalence for inverse diagrams and homotopy canonicity". In: *Mathematical Structures in Computer Science* 25.5 (2015), 1203–1277. DOI: 10.1017/S0960129514000565.

[11] Mark Weber. "Familial 2-functors and parametric right adjoints". In: *Theory and Applications of Categories* 18.22 (2007), pp. 665–732.