

Presentations of Topological Modalities in Homotopy Type Theory

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Homotopy type theory (HoTT) is an internal language for $(\infty, 1)$ -topoi [3] and as such we are able to right down the notion of a subtopos of the ambient universe. In particular we can describe those which come from a topology: the so called topological modalities [4].

In this work we introduce a notion of presentation for a topological modality, to more directly encode the choice of Grothendieck topology. Using these, and exploiting homotopy theoretic ideas, we are able to recover internal n -dimensional sheaf conditions for n -truncated types. For each external natural number we are able to show this corresponds to the usual sheaf condition. By placing the additional hypothesis that all objects in the presentation are projective we are further able to describe explicit sheafification of -1 -truncated types and can recover some Kripke-Joyal style reasoning. We derive a “local choice” axiom and obtain results about how cohomology in the subtopos relates to cohomology in the base universe.

We apply these results to an extension of HoTT axiomatising the internal logic of the classifying topos of an algebraic theory. In this setting we can derive the axioms for synthetic algebraic geometry [1] and simplify a proof that the interval is simplicial in synthetic category theory [2].

References

- [1] Felix Cherubini, Thierry Coquand, and Matthias Hutzler. *A Foundation for Synthetic Algebraic Geometry*. 2023. arXiv: [2307.00073 \[math.AG\]](#).
- [2] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. *Directed univalence in simplicial homotopy type theory*. 2024. arXiv: [2407.09146 \[cs.LG\]](#). URL: <https://arxiv.org/abs/2407.09146>.
- [3] Michael Shulman. *All $(\infty, 1)$ -toposes have strict univalent universes*. 2019. arXiv: [1904.07004 \[math.AT\]](#).
- [4] Bas Spitters, Michael Shulman, and Egbert Rijke. “Modalities in homotopy type theory”. In: *Logical Methods in Computer Science* 16 (2020).