# Multicategorical Meta-Theorem and Completeness of Restricted Algebraic Deduction Systems

David Forsman

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#### Outline

Introduction

Context Structures and Structure Categories

Universal Algebra in Categorical Logic

Using logic to automatically generalize results from **Set** to different settings.

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 Eckmann-Hilton argument from sets to symmetric monoidal categories:

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$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k, n \in \mathbb{N},$$

from commutative semirings in sets to commutative semirings in symmetric diagonal multicategories.

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Let  $\sigma$  be an R-signature with an R-theory  $E \cup \{\phi\}$ . Assume that R is either a balanced modelable or the cartesian context structure. Then for any  $\Delta_R$ -multicategory C,

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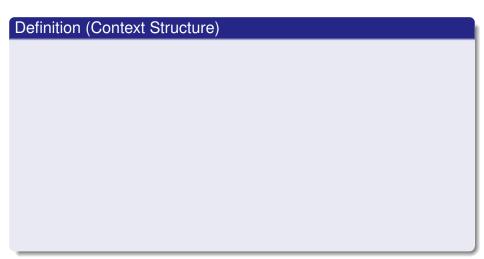
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There are exactly 8 different modelable context structures!



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$$\theta'_{k_1,\dots,k_n} \colon [L_m] \to [K_n], L_{i-1} + x \mapsto K_{\theta(i)-1} + x, \text{ for } x \in [k_{\theta(i)}]$$

$$L_i = k_{\theta(1)} + \cdots + k_{\theta(i)}$$
 and  $K_j = k_1 + \cdots + k_j$ 

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# Structure Monoid *I* and Constructing Structure Categories

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 $\Delta_J \subset \Delta \subset \Delta^J$  for any structure category  $\Delta$ , J the set of cardinalities of the fibers of functions in  $\Delta$ .

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- $\Delta^{\{1\}}$ : bijections
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Structure monoids  $I_N = \{1, n \mid n \geq N\}$  for  $N \in \mathbb{N}$  induce infinitely many structure categories  $\Delta^I$ .

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$$E \vdash_R t_1 \approx_{\nu_1 \cdots \nu_n} t_2 \Rightarrow E \vdash_R s_1(t_1) \approx_w s_2(t_2)$$

for type-preserving  $s_1, s_2 \colon \text{Var}(v) \to \textit{Term}$ , where  $wRw_1 \cdots w_n$  and  $E \vdash_R s_1(v_i) \approx_{w_i} s_2(v_i)$  for  $i \leq n$ .

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- A composition operation

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Let  $\Delta$  be a structure category. Then C is called a  $\Delta$ -multicategory if, for each morphism  $\theta \colon [m] \to [n]$  in  $\Delta$ , there exists an action map:

$$heta^*_{a_1\cdots a_n,b}\colon \mathit{C}(a_{ heta(1)}\cdots a_{ heta(m)},b) o \mathit{C}(a_1\cdots a_n,b),$$

which respects the  $\Delta$ -structure and satisfies the  $\Delta$ -multicategory action axioms.

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Composition with Action 1:

$$g \circ (\sigma_1^* f_1, \ldots, \sigma_n^* f_n) = (\sigma_1 + \cdots + \sigma_n)_{a,c}^* (g \circ (f_1, \ldots, f_n)).$$

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$$(\tau \circ \sigma)_{a,b}^* = \tau_{a,b}^* \circ \sigma_{a_{\tau(1)}\cdots a_{\tau(m)},b}^*.$$

Composition with Action 1:

$$g \circ (\sigma_1^* f_1, \ldots, \sigma_n^* f_n) = (\sigma_1 + \cdots + \sigma_n)_{a,c}^* (g \circ (f_1, \ldots, f_n)).$$

Composition with Action 2:

$$\tau_{b,c}^*(g) \circ (f_1,\ldots,f_n) = (\tau'_{k_1,\ldots,k_n})_{a,c}^* (g \circ (f_{\tau(1)},\ldots,f_{\tau(m)})),$$



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$$m_{v}(t) = \begin{cases} m_{v,()}^{*}(m(c)), & \text{if } t = c \text{ is a constant} \\ m_{v,v_{i}}^{*}(id), & \text{if } t = v_{i} \\ m_{v,w_{1}\cdots w_{n}}^{*}(m(f)(m_{w_{1}}(t_{1}),\ldots,m_{w_{n}}(t_{n}))), & \text{if } t = f(t_{1},\ldots,t_{n}). \end{cases}$$

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- Soundness:  $E \vdash_R t_1 \approx_{V} t_2 \Rightarrow E \vDash_C t_1 \approx t_2$  for all  $\Delta_{B}$ -multicategories C.
- **Set-Completeness:** If *R* balanced or cartesian,  $E \models_{\mathbf{Sot}} t_1 \approx_{\mathsf{V}} t_2 \Rightarrow E \vdash_{\mathsf{R}} t_1 \approx_{\mathsf{V}} t_2$



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- A  $\Delta_R$ -multicategory provides a **semantic universe** for usual equational reasoning for modelable R.
- Multicategorical Meta-Theorem: Transfers properties from the cartesian multicategory of sets to Δ-multicategories for six different Δ.
- Two-dimensional generalization: Potential to generalize equational results (equations of 2-cells) from Cat to any 2, Δ<sub>R</sub>-multicategory C for modelable balanced or cartesian R.

# Acknowledgment and References

# Thank you!

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#### David Forsman.

On the multicategorical meta-theorem and the completeness of restricted algebraic deduction systems, 2024.



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