

# Monotone weak distributive laws over weakly lifted powerset monads in categories of algebras

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- ▶ True with a *distributive law* (Beck 1969), i.e. a  $\lambda: TS \Rightarrow ST$  s.t.:

$$\begin{array}{ccc}
 & T & \\
 T\eta^S \swarrow & & \searrow \eta^{ST} \\
 TS & \xrightarrow{\lambda} & ST
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 \eta^{TS} \swarrow & & \searrow S\eta^T \\
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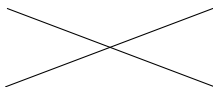
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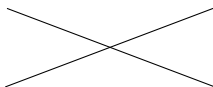
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- A WDL yields a *weak composite monad*  $S \bullet T$

# Examples of weak distributive laws

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The formula is the same... is this just a coincidence?

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	$\beta P \Rightarrow P\beta^{\text{iii}}$		$(V, \eta^V, \mu^V)$ in $\mathbf{EM}(\beta) \cong \mathbf{KHaus}$

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<sup>iii</sup> $\beta$  is the ultrafilter monad

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  - ▶ weakly lifting the construction of  $(\underline{P}, \mu^P)$  on spans

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- ▶ **Examples.** Set is regular,  $\text{Rel}(\text{Set}) = \text{Rel}$ .  $\text{KHaus}$  is regular,  $\text{Rel}(\text{KHaus}) = \text{compact Hausdorff spaces and closed relations}$ .

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- ▶ **Examples.**  $P$  and  $\mu^P$  are nearly cartesian and  $\text{Kl}(P) \cong \text{Rel}$  (Garner 2020).  $V$  and  $\mu^V$  are nearly cartesian and  $\text{Kl}(V) \hookrightarrow \text{Rel}(\text{KHaus})$  (Goy, Petrişan, and Aiguier 2021).

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Moreover:

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  - ▶  $(T, \eta^T, \mu^T)$  has a monotone weak extension to  $\mathbf{Rel} \cong \mathbf{Kl}(P)$ ;
  - ▶ there is a monotone WDL  $TP \Rightarrow PT$ ;
  - ▶  $(P, \eta^P, \mu^P)$  has a weak lifting  $(\bar{P}, \eta^{\bar{P}}, \mu^{\bar{P}})$  to  $\mathbf{EM}(T)$ ;

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- ▶ **Lemma.** Weakly lifting preserves *near cartesianness*.
- ▶ **Example.**  $V$  weak lifting of  $P$  to  $\mathbf{EM}(\beta) \cong \mathbf{KHaus}$ :  
 $P$  and  $\mu^P$  are nearly cartesian hence  $V$  and  $\mu^V$  are as well.

## Weakly lifting the setting for monotone WDLs, part 2

- ▶ Is  $\text{Kl}(\overline{P})$  always a subcategory of  $\text{Rel}(\text{EM}(T))$ ? If so, when does the relational extension of  $\overline{S}$  restrict to  $\text{Kl}(\overline{P})$ ?
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- ▶ **Theorem.** Let  $\overline{S}$  be a weak lifting of a monad  $S$  with a monotone WDL  $SP \Rightarrow PS$ . There is a monotone WDL  $\overline{S}\overline{P} \Rightarrow \overline{P}\overline{S}$  iff  $\overline{S}$  preserves decomposable morphisms

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  - ▶ **Theorem.**  $R$  preserves surjective open maps hence there is a (unique) monotone WDL  $RV_* \Rightarrow V_*R^{\text{iv}}$ . See also (Goubault-Larrecq 2024).

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# Conclusion: no-go theorems for monotone WDLs

$PP \Rightarrow PP$  and  $VV \Rightarrow VV$  look the same... but monotone WDLs over  $\overline{P}$  are quite rare otherwise:

	KHaus		JSL	Conv	Mon				CMon		
	V	R	$\overline{P}$	$\overline{P}$	$\overline{M}$	$\overline{D}$	$\overline{P}$	$\overline{M_S}$	$\overline{M}$	$\overline{D}$	$\overline{P}$
$\overline{P}$	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
$\overline{P}_*$	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗

## ► What's next?

- extending this framework: Pos-regular categories, other monads of relations
- no-go theorems for (all) WDLs
- seeing this in the setting of monoidal topology