The details of non-maximal symmetry breaking patterns of $\mathfrak{su}(8) \to \mathfrak{g}_{531}/\mathfrak{g}_{351}/\mathfrak{g}_{621}$

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Abstract

We present the details of the non-maximal symmetry breaking patterns of $\mathfrak{su}(8) \to \mathfrak{g}_{531}/\mathfrak{g}_{351}/\mathfrak{g}_{621}$.

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1 The intermediate stages of the SSW symmetry breaking pattern

Schematically, we assign the Higgs VEVs according to the symmetry breaking pattern as follows:

$$\mathfrak{g}_{531} \to \mathfrak{g}_{431} \quad : \quad \langle (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H},\mathrm{IV}} \rangle \equiv \frac{1}{\sqrt{2}} W_{\overline{\mathbf{5}},\mathrm{IV}}\,, \tag{1a}$$

$$\mathfrak{g}_{431} \to \mathfrak{g}_{331} \quad : \quad \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\mathrm{V}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\overline{\mathbf{4}},\mathrm{V}}\,,\ \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\dot{1},\dot{\mathrm{V}}\mathrm{II}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\overline{\mathbf{4}},\dot{1},\dot{\mathrm{V}}\mathrm{II}}\,, \tag{1b}$$

$$\begin{aligned} \mathfrak{g}_{331} \to \mathfrak{g}_{\mathrm{SM}} \quad : \quad & \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},3,\mathrm{VI}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\overline{\mathbf{3}},3,\mathrm{VI}}\,, \\ & \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\dot{\mathrm{V}}\mathrm{III}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\overline{\mathbf{3}},\dot{\mathrm{V}}\mathrm{III}}\,,\ \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})'_{\mathbf{H},\dot{2},\dot{\mathrm{I}}\mathrm{X}} \rangle \equiv \frac{1}{\sqrt{2}} V'_{\overline{\mathbf{3}},\dot{2},\dot{\mathrm{I}}\mathrm{X}}\,, \end{aligned} \tag{1c}$$

$$\mathrm{EWSB} \quad : \quad \langle (\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{H}} \rangle \equiv \frac{1}{\sqrt{2}} v_{\mathrm{EW}}\,. \tag{1d}$$

For our later convenience, we also parametrize different symmetry breaking VEVs in terms of the following dimensionless quantities:

$$\begin{aligned} & \zeta_1 \equiv \frac{W_{\overline{\mathbf{5}},\mathrm{IV}}}{M_{\mathrm{pl}}}\,, \quad \zeta_2 \equiv \frac{w_{\overline{\mathbf{4}},\mathrm{V}}}{M_{\mathrm{pl}}}\,, \quad \dot{\zeta}_2 \equiv \frac{w_{\overline{\mathbf{4}},\dot{1},\dot{\mathrm{V}}\mathrm{II}}}{M_{\mathrm{pl}}}\,, \\ & \zeta_3 \equiv \frac{V_{\overline{\mathbf{3}},3,\mathrm{VI}}}{M_{\mathrm{pl}}}\,, \quad \dot{\zeta}'_3 \equiv \frac{V'_{\overline{\mathbf{3}},\dot{2},\dot{\mathrm{I}}\mathrm{X}}}{M_{\mathrm{pl}}}\,, \quad \dot{\zeta}_3 \equiv \frac{V_{\overline{\mathbf{3}},\dot{\mathrm{V}}\mathrm{III}}}{M_{\mathrm{pl}}}\,, \\ & \zeta_0 \gg \zeta_1 \gg \zeta_2 \sim \dot{\zeta}_2 \gg \zeta_3 \sim \dot{\zeta}'_3 \sim \dot{\zeta}_3\,, \quad \zeta_{ij} \equiv \frac{\zeta_j}{\zeta_i}\,, \quad (i<j)\,. \end{aligned} \tag{2}$$

In Table 1, we summarize all vectorlike fermions that become massive during different stages of the SSW symmetry breaking pattern.

stages	$Q_e=-\frac{1}{3}$	$Q_e=+\frac{2}{3}$	$Q_e=-1$	$Q_e=0$
v_{531} $\{\Omega\}$	$\mathfrak{D}'$ IV	-	$(\mathfrak{e}'\,,\mathfrak{n}')$ IV	$\{\check{\mathfrak{n}}\,,\check{\mathfrak{n}}''\}$ $\{\mathrm{IV}''\,,\mathrm{IV}'\}$
v_{431} $\{\Omega\}$	$\mathfrak{d}\,,\{\mathfrak{D}''\,,\mathfrak{D}'''\}$ $\{\mathrm{V}\,,\dot{\mathrm{V}}\mathrm{II}\}$	$\mathfrak{u}\,,\mathfrak{U}$	$\mathfrak{E}\,,(\mathfrak{e}''\,,\mathfrak{n}'')\,,(\mathfrak{e}'''''\,,\mathfrak{n}''''')$ $\{\mathrm{V}\,,\dot{\mathrm{V}}\mathrm{II}\}$	$\{\check{\mathfrak{n}}'\,,\check{\mathfrak{n}}'''\}$ $\{\mathrm{V}''\,,\dot{\mathrm{V}}\mathrm{II}\}$
v_{331} $\{\Omega\}$	$\{\mathfrak{D}\,,\mathfrak{D}''''\,,\mathfrak{D}'''''\}$ $\{\mathrm{VI}\,,\dot{\mathrm{V}}\mathrm{III}\,,\dot{\mathrm{I}}\mathrm{X}\}$	-	$(\mathfrak{e}\,,\mathfrak{n})\,,(\mathfrak{e}'''\,,\mathfrak{n}''')\,,(\mathfrak{e}''''\,,\mathfrak{n}'''')$ $\{\mathrm{VI}\,,\dot{\mathrm{V}}\mathrm{III}\,,\dot{\mathrm{I}}\mathrm{X}\}$	-

Table 1: The vectorlike fermions at different intermediate symmetry breaking stages along the SSW symmetry breaking pattern of the $\mathfrak{su}(8)$ theory.

1.1 The first stage

The first symmetry breaking stage of $\mathfrak{g}_{531} \to \mathfrak{g}_{431}$ is achieved by $(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H},\mathrm{IV}} \subset \overline{\mathbf{8}_{\mathbf{H}}}_{,\mathrm{IV}}$ in the rank-two sector, according to the $\widetilde{\mathrm{U}}(1)_{T'}$-neutral components in Table 2. Accordingly, the term of

$\overline{\mathbf{8_F}}^{\Omega}$	$(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\Omega}$		
$\mathcal{T}'$	$-4t$	$-\frac{4}{3}t$		
$\mathbf{28_F}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}$	
$\mathcal{T}'$	$-\frac{4}{3}t$	$\frac{4}{3}t$	$4t$	
$\mathbf{56_F}$	$(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}'$	$(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}}$	$(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}$	$(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}$
$\mathcal{T}'$	$-4t$	$-\frac{4}{3}t$	$4t$	$\frac{4}{3}t$
$\overline{\mathbf{8_H}}_{,\omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega}$	$(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega}$		
$\mathcal{T}'$	$\frac{8}{3}t$	0		
$\overline{\mathbf{28_H}}_{,\dot\omega}$	$(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H}\,,\dot\omega}$	
$\mathcal{T}'$	$\frac{16}{3}t$	$\frac{8}{3}t$	0	
$\mathbf{70_H}$	$(\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}}$	$(\overline{\mathbf{10}}\,,\mathbf{3}\,,-\frac{4}{15})_{\mathbf{H}}$		
$\mathcal{T}'$	$-\frac{16}{3}t$	$-\frac{8}{3}t$		

Table 2: The $\widetilde{\rm U}(1)_{T'}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{531}$ theory.

$Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\rm IV}\mathbf{28_F}\overline{\mathbf{8_H}}_{,{\rm IV}}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}'\,,\check{\mathfrak{n}}''\,,\mathfrak{e}'\,,\mathfrak{n}'\,,\check{\mathfrak{n}})$ as follows:

$$\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\rm IV}\mathbf{28_F}\overline{\mathbf{8_H}}_{,{\rm IV}}+H.c.\\
\supset\;& Y_{\mathcal{B}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\rm IV}\otimes(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\rm IV}\otimes(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}\Big]\otimes\langle(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,{\rm IV}}\rangle+H.c.\\
\Rightarrow\;& \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{D}_L'\mathfrak{D}_R'^{c}+\check{\mathfrak{n}}_L''\check{\mathfrak{n}}_R''^{c}+\mathfrak{e}_L'\mathfrak{e}_R'^{c}-\mathfrak{n}_L'\mathfrak{n}_R'^{c}+\check{\mathfrak{n}}_L\check{\mathfrak{n}}_R^{c}\right)W_{\overline{\mathbf{5}},{\rm IV}}+H.c.\,.
\end{aligned}\tag{3}$$

After this stage, the remaining massless $\mathfrak{g}_{431}$ fermions are the following:

$$\begin{aligned}
& \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\Omega}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega'}\Big]\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\Omega}\oplus\subset\overline{\mathbf{8_F}}^{\Omega}\,,\\
& \Omega=(\omega\,,\dot\omega)\,,\quad \omega=(3\,,{\rm V}\,,{\rm VI})\,,\quad \dot\omega=(\dot{\rm I}\,,\dot{2}\,,\dot{\rm VII}\,,\dot{\rm VIII}\,,\dot{\rm IX})\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm IV'}\subset\overline{\mathbf{8_F}}^{\rm IV}\,,\\
& (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}\oplus\Big[\cancel{(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}'}\oplus(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}\Big]\oplus\Big[\cancel{(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}}\oplus(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}\Big]\subset\mathbf{28_F}\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}'\oplus\Big[(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}'\oplus(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{F}}\Big]\oplus\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}}\oplus(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}'\Big]\\
\oplus\;& \Big[(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}'\oplus(\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}}\Big]\subset\mathbf{56_F}\,.
\end{aligned}\tag{4}$$

1.2 The second stage

The second symmetry breaking stage of $\mathfrak{g}_{431}\to\mathfrak{g}_{331}$ is achieved by $(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,{\rm V}}\subset\overline{\mathbf{8_H}}_{,{\rm V}}$ in the rank-two sector, $(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot{\rm VII}}\subset\overline{\mathbf{28_H}}_{,\dot{\rm VII}}$ and $(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{H}}^{\dot{\rm VII}}\subset\mathbf{28_H}^{\dot{\rm VII}}$ in the rank-three sector, according to the $\widetilde{\rm U}(1)_{T''}$-neutral components in Table 3. The term of $Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\rm V}\mathbf{28_F}\overline{\mathbf{8_H}}_{,{\rm V}}+H.c.$ leads to

$\overline{\mathbf{8_F}}^{\Omega}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega'}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\Omega}$	
$\mathcal{T}''$	$-4t$	$-4t$	$-\frac{4}{3}t$	
$\mathbf{28_F}$	$(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}$	
$\mathcal{T}''$	$4t$	$\frac{4}{3}t$	$-\frac{4}{3}t$	
$\mathbf{56_F}$	$(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}'$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}'$	$(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{F}}$	
$\mathcal{T}''$	$-4t$	$-\frac{4}{3}t$	$-\frac{4}{3}t$	
	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}'$	$(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}'$	$(\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}}$
$\mathcal{T}''$	$4t$	$4t$	$\frac{4}{3}t$	$\frac{4}{3}t$
$\overline{\mathbf{8_H}}_{,\omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\omega}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\omega}$		
$\mathcal{T}''$	$\frac{8}{3}t$	0		
$\overline{\mathbf{28_H}}_{,\dot{\omega}}$	$(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H},\dot{\omega}}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\dot{\omega}}$	$(\overline{\mathbf{4}}\,,\overline{\mathbf{3}}\,,-\frac{1}{12})_{\mathbf{H},\dot{\omega}}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\dot{\omega}}$
$\mathcal{T}''$	$\frac{16}{3}t$	$\frac{8}{3}t$	$\frac{8}{3}t$	0
$\mathbf{70_H}$	$(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{H}}$	$(\overline{\mathbf{4}}\,,\mathbf{3}\,,-\frac{5}{12})_{\mathbf{H}}$		
$\mathcal{T}''$	$-\frac{16}{3}t$	$-\frac{8}{3}t$		

Table 3: The $\widetilde{\mathrm{U}}(1)_{T''}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{431}$ theory.

the vectorial masses of $(\mathfrak{D}''\,,\mathfrak{e}''\,,\mathfrak{n}''\,,\check{\mathfrak{n}}')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{V}}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\mathrm{V}}+H.c.\\
\supset\quad & Y_{\mathcal{B}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{V}}\otimes(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{V}}\otimes(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H},\mathrm{V}}+H.c.\\
\supset\quad & Y_{\mathcal{B}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\mathrm{V}}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V}'}\otimes(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}\\
\oplus\quad & (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{V}}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}\Big]\otimes\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\mathrm{V}}\rangle+H.c.\\
\Rightarrow\quad & \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{D}_L''\mathfrak{D}_R''^{c}+\check{\mathcal{N}}_L^{\mathrm{V}'}\check{\mathfrak{n}}_R''^{c}+\mathfrak{e}_L''\mathfrak{e}_R''^{c}-\mathfrak{n}_L''\mathfrak{n}_R''^{c}+\check{\mathfrak{n}}_L'\check{\mathfrak{n}}_R'^{c}\right)w_{\overline{\mathbf{4}},\mathrm{V}}+H.c.\,.
\end{aligned}
\tag{5}
$$

The term of $Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot{\mathrm{V}}\mathrm{II}}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot{\mathrm{V}}\mathrm{II}}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}'''\,,\mathfrak{e}'''''\,,\mathfrak{n}'''''\,,\check{\mathfrak{n}}''')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot{\mathrm{V}}\mathrm{II}}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot{\mathrm{V}}\mathrm{II}}+H.c.\\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}}\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H},\dot{\mathrm{V}}\mathrm{II}}+H.c.\\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}'\Big]\otimes\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\dot{\mathrm{V}}\mathrm{II}}\rangle+H.c.\\
\Rightarrow\quad & \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}_L'''\mathfrak{D}_R'''^{c}+\mathfrak{e}_L'''''\mathfrak{e}_R'''''^{c}-\mathfrak{n}_L'''''\mathfrak{n}_R'''''^{c}+\check{\mathfrak{n}}_L'''\check{\mathfrak{n}}_R'''^{c}\right)w_{\overline{\mathbf{4}},\dot{\mathrm{V}}\mathrm{II}}+H.c.\,.
\end{aligned}
\tag{6}
$$

The Yukawa coupling between two $\mathbf{56_F}$'s by the following $d=5$ operator

$$
\begin{aligned}
& \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\dot{\rm VII}} + H.c. \\
\supset\quad & c_4 \frac{v_U}{M_{\rm pl}} \Big[(\overline{\mathbf{10}}\,, \mathbf{1}\,, -\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}^{\prime} \oplus (\mathbf{10}\,, \mathbf{3}\,, -\frac{1}{15})_{\mathbf{F}} \otimes (\mathbf{5}\,, \overline{\mathbf{3}}\,, +\frac{7}{15})_{\mathbf{F}} \Big] \otimes (\mathbf{10}\,, \mathbf{1}\,, -\frac{2}{5})_{\mathbf{H}}^{\dot{\rm VII}} + H.c. \\
\supset\quad & c_4 \zeta_0 \Big[(\overline{\mathbf{4}}\,, \mathbf{1}\,, -\frac{3}{4})_{\mathbf{F}} \otimes (\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}^{\prime} \oplus (\mathbf{6}\,, \mathbf{3}\,, -\frac{1}{6})_{\mathbf{F}} \otimes (\mathbf{4}\,, \overline{\mathbf{3}}\,, +\frac{5}{12})_{\mathbf{F}} \Big] \otimes \langle (\mathbf{4}\,, \mathbf{1}\,, -\frac{1}{4})_{\mathbf{H}}^{\dot{\rm VII}} \rangle + H.c. \\
\Rightarrow\quad & \frac{c_4}{\sqrt{2}} \zeta_0 \left(\mathfrak{E}_L \mathfrak{E}_R{}^c + \mathfrak{U}_L \mathfrak{U}_R{}^c + \mathfrak{d}_L \mathfrak{d}_R{}^c - \mathfrak{u}_L \mathfrak{u}_R{}^c \right) w_{\mathbf{4}}^{\dot{\rm VII}} + H.c.\,,
\end{aligned}
\tag{7}
$$

leads to massive vectorlike fermions of $(\mathfrak{E}\,, \mathfrak{U}\,, \mathfrak{u}\,, \mathfrak{d})$.

After integrating out the massive fermions, the remaining massless fermions expressed in terms of the $\mathfrak{g}_{331}$ IRs are the following:

$$
\begin{aligned}
& \Big[(\overline{\mathbf{3}}\,, \mathbf{1}\,, +\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\Omega} \Big] \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\Omega^\prime} \oplus (\mathbf{1}\,, \overline{\mathbf{3}}\,, -\frac{1}{3})_{\mathbf{F}}^{\Omega} \subset \overline{\mathbf{8_F}}^{\Omega}\,, \\
& \Omega = (\omega\,, \dot\omega)\,, \quad \omega = (3\,, {\rm VI})\,, \quad \dot\omega = (\dot 1\,, \dot 2\,, \dot{\rm VIII}\,, \dot{\rm IX})\,, \\
& (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{{\rm IV}^\prime} \subset \overline{\mathbf{8_F}}^{\rm IV}\,, \quad (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\rm V} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{{\rm V}^\prime} \subset \overline{\mathbf{8_F}}^{\rm V}\,, \\
& (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\dot{\rm VII}} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\dot{\rm VII}^\prime} \subset \overline{\mathbf{8_F}}^{\dot{\rm VII}}\,, \\
& (\mathbf{1}\,, \overline{\mathbf{3}}\,, +\frac{2}{3})_{\mathbf{F}} \oplus \cancel{(\mathbf{1}\,, \mathbf{3}\,, +\frac{1}{3})_{\mathbf{F}}^{\prime}} \oplus \Big[\cancel{(\mathbf{1}\,, \mathbf{3}\,, +\frac{1}{3})_{\mathbf{F}}^{\prime\prime}} \oplus (\mathbf{3}\,, \mathbf{3}\,, 0)_{\mathbf{F}} \Big] \\
\oplus\quad & \cancel{\Big[(\mathbf{3}\,, \mathbf{1}\,, -\frac{1}{3})_{\mathbf{F}}^{\prime} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\prime\prime} \Big]} \oplus \Big[\cancel{(\mathbf{3}\,, \mathbf{1}\,, -\frac{1}{3})_{\mathbf{F}}^{\prime\prime}} \oplus (\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \Big] \subset \mathbf{28_F}\,, \\
& \cancel{(\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}^{\prime}} \oplus (\mathbf{1}\,, \overline{\mathbf{3}}\,, +\frac{2}{3})_{\mathbf{F}}^{\prime} \oplus \Big[(\mathbf{1}\,, \overline{\mathbf{3}}\,, +\frac{2}{3})_{\mathbf{F}}^{\prime\prime} \oplus \cancel{(\mathbf{3}\,, \overline{\mathbf{3}}\,, +\frac{1}{3})_{\mathbf{F}}} \Big] \\
\oplus\quad & \Big[\cancel{(\mathbf{1}\,, \mathbf{1}\,, -1)_{\mathbf{F}}} \oplus (\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}}^{\prime} \Big] \oplus \Big[\cancel{(\mathbf{3}\,, \mathbf{1}\,, -\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime}} \oplus (\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}}^{\prime\prime} \Big] \\
\oplus\quad & \Big[\cancel{(\mathbf{1}\,, \mathbf{3}\,, +\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime}} \oplus (\mathbf{3}\,, \mathbf{3}\,, 0)_{\mathbf{F}}^{\prime} \Big] \oplus \Big[(\mathbf{3}\,, \mathbf{3}\,, 0)_{\mathbf{F}}^{\prime\prime} \oplus \cancel{(\overline{\mathbf{3}}\,, \mathbf{3}\,, -\frac{1}{3})_{\mathbf{F}}} \Big] \subset \mathbf{56_F}\,.
\end{aligned}
\tag{8}
$$

1.3 The third stage

The third symmetry breaking stage of $\mathfrak{g}_{331} \to \mathfrak{g}_{\rm SM}$ is achieved by $(\mathbf{1}\,, \overline{\mathbf{3}}\,, -\frac{1}{3})_{\mathbf{H}\,, {\rm VI}} \subset \overline{\mathbf{8_H}}_{\,, {\rm VI}}$ in the rank-two sector and $(\mathbf{1}\,, \overline{\mathbf{3}}\,, -\frac{1}{3})_{\mathbf{H}\,, \dot{\rm VIII}} \subset \overline{\mathbf{28_H}}_{\,, \dot{\rm VIII}}$, $(\mathbf{1}\,, \overline{\mathbf{3}}\,, -\frac{1}{3})_{\mathbf{H}\,, \dot{\rm IX}}^{\prime} \subset \overline{\mathbf{28_H}}_{\,, \dot{\rm IX}}$ in the rank-three sector, according to the $\widetilde{\rm U}(1)_{T^{\prime\prime\prime}}$-neutral components in Table 4. The term of $Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\rm VI} \mathbf{28_F} \overline{\mathbf{8_H}}_{\,, {\rm VI}} + H.c.$ leads

$\overline{\mathbf{8_F}}^{\Omega}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega'}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\Omega}$
$\mathcal{T}'''$	$-\frac{4}{3}t$	$-4t$	$-4t$	$-4t$
$\mathbf{28_F}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}$	
$\mathcal{T}'''$	$4t$	$\frac{4}{3}t$	$-\frac{4}{3}t$	
$\mathbf{56_F}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}'$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}''$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}'$	
$\mathcal{T}'''$	$4t$	$4t$	$-\frac{4}{3}t$	
	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}''$	$(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}'$	$(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}''$	
$\mathcal{T}'''$	$-\frac{4}{3}t$	$\frac{4}{3}t$	$\frac{4}{3}t$	
$\overline{\mathbf{8_H}}_{\,,\omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega}$			
$\mathcal{T}'''$	0			
$\overline{\mathbf{28_H}}_{\,,\dot{\omega}}$	$(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot{\omega}}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\dot{\omega}}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\dot{\omega}}'$	
$\mathcal{T}'''$	0	0	0	
$\mathbf{70_H}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{H}}$	$(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}}$		
$\mathcal{T}'''$	0	$-8t$		

Table 4: The $\widetilde{\mathrm{U}}(1)_{\mathcal{T}'''}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{331}$ theory.

to the vectorial masses of $(\mathfrak{D}\,,\mathfrak{e}\,,\mathfrak{n})$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{VI}}\mathbf{28_F}\overline{\mathbf{8_H}}_{\,,\mathrm{VI}}+H.c.\\
\supset\;& Y_{\mathcal{B}}\Big[(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}\oplus(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{VI}}+H.c.\\
\supset\;& Y_{\mathcal{B}}\Big[(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{VI}'}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}'\\
\oplus\;& (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{VI}}+H.c.\\
\supset\;& Y_{\mathcal{B}}\Big[(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{VI}'}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}'\\
\oplus\;& (\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}''\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{VI}}\rangle+H.c.\\
\Rightarrow\;& \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{n}_L\mathfrak{n}_R{}^c-\mathfrak{e}_L\mathfrak{e}_R{}^c+\check{\mathcal{N}}_L^{\mathrm{VI}'}\check{\mathfrak{n}}_R^c+\check{\mathcal{N}}_L^{\mathrm{VI}}\check{\mathfrak{n}}_R'^{\,c}+\mathfrak{D}_L\mathfrak{D}_R^c\right)V_{\overline{\mathbf{3}},\mathrm{VI}}+H.c.\,. \qquad (9)
\end{aligned}
$$

The term of $Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\mathrm{V\dot{I}II},\mathrm{I\dot{X}}}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\mathrm{V\dot{I}II},\mathrm{I\dot{X}}}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}'''' \,, \mathfrak{e}''' \,, \mathfrak{n}''' \,, \mathfrak{D}''''' \,, \mathfrak{e}'''' \,, \mathfrak{n}'''')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\mathrm{V\dot{I}II}}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\mathrm{V\dot{I}II}} + H.c. \\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H}\,,\mathrm{V\dot{I}II}} + H.c. \\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}'\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{V\dot{I}II}} + H.c. \\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}'\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}''' \\
\oplus\;\; & (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}'\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{V\dot{I}II}}\rangle + H.c. \\
\Rightarrow\;\; & \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}_L''''\mathfrak{D}_R''''^{\,c}+\check{\mathcal{N}}_L^{\mathrm{V\dot{I}II}}\check{\mathfrak{n}}_R'''^{\,c}-\mathfrak{e}_L'''\mathfrak{e}_R'''^{c}+\mathfrak{n}_L'''\mathfrak{n}_R'''^{c}\right)V_{\overline{\mathbf{3}},\mathrm{V\dot{I}II}} + H.c.\,,
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\mathrm{I\dot{X}}}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\mathrm{I\dot{X}}} + H.c. \\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{I\dot{X}}}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{I\dot{X}}}\otimes(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H}\,,\mathrm{I\dot{X}}} + H.c. \\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\mathrm{I\dot{X}}}\otimes(\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{I\dot{X}}'}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}' \\
\oplus\;\; & (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{I\dot{X}}}\otimes(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{4}}\,,\overline{\mathbf{3}}\,,-\frac{1}{12})_{\mathbf{H}\,,\mathrm{I\dot{X}}} + H.c. \\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{I\dot{X}}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}''\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{I\dot{X}}'}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}''' \\
\oplus\;\; & (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{I\dot{X}}}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}''\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{I\dot{X}}}'\rangle + H.c. \\
\Rightarrow\;\; & \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}_L'''''\mathfrak{D}_R'''''^{\,c}+\check{\mathcal{N}}_L^{\mathrm{I\dot{X}}'}\check{\mathfrak{n}}_R'''^{\,c}-\mathfrak{e}_L''''\mathfrak{e}_R''''^{c}+\mathfrak{n}_L''''\mathfrak{n}_R''''^{c}\right)V_{\overline{\mathbf{3}},\mathrm{I\dot{X}}}' + H.c.\,.
\end{aligned}
\tag{11}
$$

The remaining massless fermions of the $\mathfrak{g}_{\mathrm{SM}}$ are listed as follows:

$$
\begin{aligned}
& \Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega'}\Big]\oplus\Big[(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\Omega}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''}\Big]\subset\overline{\mathbf{8_F}}^{\Omega}\,,\quad \Omega=(\dot{1}\,,\dot{2}\,,3)\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V}}\oplus...\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{I\dot{X}}}\subset\overline{\mathbf{8_F}}^{\Omega}\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{IV}'}\oplus...\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{I\dot{X}}'}\subset\overline{\mathbf{8_F}}^{\Omega'}\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{VI}''}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V\dot{I}II}''}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{I\dot{X}}''}\subset\overline{\mathbf{8_F}}^{\Omega''}\,, \\
& \Big[\cancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}}\oplus(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}\Big]\oplus\Big[\cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}}\oplus\cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}''\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}'} \\
\oplus\;\; & \cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}}\oplus(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}\Big]\oplus\Big[\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}''}\oplus\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}''}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}\Big]\subset\mathbf{28_F}\,, \\
& \cancel{(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}'}\oplus\Big[\cancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}'''}\oplus(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}''\oplus\cancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}''''}\oplus(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}''' \\
\oplus\;\; & \cancel{(\mathbf{3}\,,\mathbf{1}\,,+\frac{2}{3})_{\mathbf{F}}\oplus(\mathbf{3}\,,\overline{\mathbf{2}}\,,+\frac{1}{6})_{\mathbf{F}}}\Big]\oplus\Big[\cancel{(\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}}}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}'\oplus\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}'''}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}''\Big] \\
\oplus\;\; & \Big[\cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}'''''\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}'''}\oplus\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}''''}\oplus(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}' \\
\oplus\;\; & \cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}'''''}\oplus(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}''\oplus\cancel{(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}'''\oplus(\overline{\mathbf{3}}\,,\mathbf{2}\,,-\frac{1}{6})_{\mathbf{F}}}\Big]\subset\mathbf{56_F}\,.
\end{aligned}
\tag{12}
$$

1.4 The $d=5$ bi-linear fermion operators

The SM up-type quark masses are due to two $d=5$ operators of

$$c_4\,\mathcal{O}_{\mathcal{F}}^{(4\,,1)} \;=\; c_4\,\mathbf{56_F}\mathbf{56_F}\cdot\overline{\mathbf{28_H}}_{\,,\dot\omega}\cdot\mathbf{70_H}\,, \tag{13a}$$
$$c_5\,\mathcal{O}_{\mathcal{F}}^{(5\,,1)} \;=\; c_5\,\mathbf{28_F}\mathbf{56_F}\cdot\overline{\mathbf{8_H}}_{\,,\omega}\cdot\mathbf{70_H}\,. \tag{13b}$$

For all down-type quarks and charged leptons, we conjecture two sets of Higgs mixing terms

$$d_{\mathscr{A}}\,\mathcal{O}_{\mathscr{A}}^{d=5} \;\equiv\; d_{\mathscr{A}}\,\epsilon_{\omega_1\omega_2\omega_3\omega_4}\overline{\mathbf{8_H}}^{\dagger}_{\,,\omega_1}\overline{\mathbf{8_H}}^{\dagger}_{\,,\omega_2}\overline{\mathbf{8_H}}^{\dagger}_{\,,\omega_3}\overline{\mathbf{8_H}}^{\dagger}_{\,,\omega_4}\mathbf{70_H}^{\dagger}\,,\quad \mathcal{PQ}=2(2p+3q_2)\neq 0\,, \tag{14a}$$
$$d_{\mathscr{B}}\,\mathcal{O}_{\mathscr{B}}^{d=5} \;\equiv\; d_{\mathscr{B}}\,(\overline{\mathbf{28_H}}^{\dagger}_{\,,\dot\kappa_1}\overline{\mathbf{28_H}}_{\,,\dot\kappa_2})\cdot\overline{\mathbf{28_H}}^{\dagger}_{\,,\dot\omega_1}\overline{\mathbf{28_H}}^{\dagger}_{\,,\dot\omega_2}\mathbf{70_H}^{\dagger}\,,\quad \mathcal{PQ}=2(p+q_2+q_3)\,,$$
$$\text{with}\;\; \dot\kappa_2\neq(\dot\kappa_1\,,\dot\omega_1\,,\dot\omega_2)\,, \tag{14b}$$

For the operator of $\mathcal{O}_{\mathcal{F}}^{(4\,,1)}$ in Eq. (13a), it is decomposed as

$$\begin{aligned}
&\frac{c_4}{M_{\rm pl}}\mathbf{56_F}\mathbf{56_F}\cdot\overline{\mathbf{28_H}}_{\,,\dot\omega}\cdot\mathbf{70_H}+H.c.\\
\supset\;&\frac{c_4}{M_{\rm pl}}(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H}\,,\dot\omega}\otimes(\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}}+H.c.\\
\supset\;&\frac{c_4}{M_{\rm pl}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})'_{\mathbf{F}}\oplus(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})'_{\mathbf{F}}\otimes(\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}}\Big]\otimes\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot\omega}\rangle\\
\otimes\;&(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{H}}+H.c.\\
\supset\;&c_4\frac{w_{\overline{\mathbf{4}}\,,\dot{\rm VII}}}{\sqrt{2}M_{\rm pl}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})'_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)'_{\mathbf{F}}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})''_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)''_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{H}}+H.c.\\
\Rightarrow\;&\frac{c_4}{2}\dot\zeta_2\,(u_L{c_R}^c+c_L{u_R}^c)\,v_{\rm EW}+H.c.\,.
\end{aligned}\tag{15}$$

For the operator of $\mathcal{O}_{\mathcal{F}}^{(5\,,1)}$ in Eq. (13b), it is decomposed as

$$\begin{aligned}
&\frac{c_5}{M_{\rm pl}}\mathbf{28_F}\mathbf{56_F}\cdot\overline{\mathbf{8_H}}_{\,,\omega}\cdot\mathbf{70_H}+H.c.\\
\supset\;&\frac{c_5}{M_{\rm pl}}\Big[(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}\oplus(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\Big]\otimes\langle(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega}\rangle\\
\otimes\;&(\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}}+H.c.\\
\supset\;&c_5\frac{W_{\overline{\mathbf{5}}\,,{\rm IV}}}{\sqrt{2}M_{\rm pl}}\Big[(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})'_{\mathbf{F}}\oplus(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})'_{\mathbf{F}}\Big]\otimes(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{H}}+H.c.\\
\supset\;&c_5\frac{W_{\overline{\mathbf{5}}\,,{\rm IV}}}{\sqrt{2}M_{\rm pl}}\Big[(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}\otimes(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})''_{\mathbf{F}}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)'_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{H}}+H.c.\\
\Rightarrow\;&\frac{c_5}{2}\zeta_1\,(t_L{u_R}^c+u_L{t_R}^c)\,v_{\rm EW}+H.c.\,,
\end{aligned}\tag{16}$$

and

$$
\begin{aligned}
& \frac{c_5}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} \Big[(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}} \Big] \otimes (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega} \\
\otimes\ & (\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} \Big[(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}} \otimes (\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}} \Big] \\
\otimes\ & \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\omega} \rangle \otimes (\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{H}} + H.c. \\
\supset\ & c_5 \frac{w_{\overline{\mathbf{4}}\,,\rm V}}{\sqrt{2} M_{\rm pl}} \Big[(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}} \otimes (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}' \oplus (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}'' \Big] \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{c_5}{2} \zeta_2 \left(t_L {c_R}^c + c_L {t_R}^c \right) v_{\rm EW} + H.c.\,.
\end{aligned} \tag{17}
$$

1.5 The $d=5$ irreducible Higgs mixing operators

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1}$, we find the mass terms of

$$
\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1} \times \frac{d_{\mathscr{A}}}{M_{\rm pl}} \epsilon_{\omega_1\omega_2\omega_3\omega_4} \overline{\mathbf{8_H}}^\dagger_{,\omega_1} \overline{\mathbf{8_H}}^\dagger_{,\omega_2} \overline{\mathbf{8_H}}^\dagger_{,\omega_3} \overline{\mathbf{8_H}}^\dagger_{,\omega_4} \mathbf{70_H}^\dagger + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}} \Big] \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega_1} \\
\times\ & \frac{d_{\mathscr{A}}}{M_{\rm pl}} (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H}\,,\omega_1} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H}\,,\omega_2} \otimes (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^\dagger_{\mathbf{H}\,,\omega_3} \otimes \langle (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^\dagger_{\mathbf{H}\,,\omega_4} \rangle \otimes (\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})^\dagger_{\mathbf{H}} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}} \Big] \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega_1} \\
\times\ & d_{\mathscr{A}} \frac{W_{\overline{\mathbf{5}}\,,\rm IV}}{\sqrt{2} M_{\rm pl}} (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H}\,,\omega_1} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H}\,,\omega_2} \otimes \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})^\dagger_{\mathbf{H}\,,\omega_3} \rangle \otimes (\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})^\dagger_{\mathbf{H}} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}} \Big] \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega_1} \\
\times\ & d_{\mathscr{A}} \frac{W_{\overline{\mathbf{5}}\,,\rm IV} w_{\overline{\mathbf{4}}\,,\rm V}}{2 M_{\rm pl}} (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H}\,,\omega_1} \otimes \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H}\,,\omega_2} \rangle \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})^\dagger_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{Y_{\mathcal{B}} d_{\mathscr{A}}}{4} \frac{W_{\overline{\mathbf{5}}\,,\rm IV} w_{\overline{\mathbf{4}}\,,\rm V} V_{\overline{\mathbf{3}}\,,\rm VI}}{M_{\rm pl} m^2_{(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,3}}} \left(b_L {b_R}^c + \tau_L {\tau_R}^c \right) v_{\rm EW} + H.c.\,,
\end{aligned} \tag{18}
$$

via the Higgs mixing operator of $\mathcal{O}_{\mathscr{A}}^{d=5}$ in Eq. (14a).

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}$, we find the mass terms of

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}\times\frac{d_{\mathscr{B}}}{M_{\rm pl}}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_1}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_2}\mathbf{70_H}^\dagger\Big(\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm I}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\Big)+H.c.\\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}},\mathbf{1},+\frac{1}{5})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{10},\mathbf{3},-\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{5},\overline{\mathbf{3}},+\frac{7}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}},\overline{\mathbf{3}},-\frac{2}{15})_{\mathbf{H},\dot\omega_1}\\
\times\;&\frac{d_{\mathscr{B}}}{M_{\rm pl}}(\overline{\mathbf{5}},\overline{\mathbf{3}},-\frac{2}{15})^\dagger_{\mathbf{H},\dot\omega_1}\otimes(\overline{\mathbf{5}},\overline{\mathbf{3}},-\frac{2}{15})^\dagger_{\mathbf{H},\dot\omega_2}\otimes(\mathbf{10},\overline{\mathbf{3}},+\frac{4}{15})^\dagger_{\mathbf{H}}\otimes\langle\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm I}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\rangle+H.c.\\
\supset\;& Y_{\mathcal{D}}d_{\mathscr{B}}\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{2M_{\rm pl}}\Big[(\overline{\mathbf{4}},\mathbf{1},+\frac{1}{4})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{6},\mathbf{3},-\frac{1}{6})_{\mathbf{F}}\oplus(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{4},\overline{\mathbf{3}},+\frac{5}{12})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})_{\mathbf{H},\dot\omega_1}\\
\times\;&(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})^\dagger_{\mathbf{H},\dot\omega_1}\otimes(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})^\dagger_{\mathbf{H},\dot\omega_2}\otimes(\mathbf{4},\overline{\mathbf{3}},+\frac{5}{12})^\dagger_{\mathbf{H}}+H.c.\\
\supset\;& Y_{\mathcal{D}}d_{\mathscr{B}}\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{2M_{\rm pl}m^2_{(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})_{\mathbf{H},\dot\omega_1}}}\Big[(\overline{\mathbf{3}},\mathbf{1},+\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{3},\mathbf{3},0)''_{\mathbf{F}}\oplus(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{1},\overline{\mathbf{3}},+\frac{2}{3})''_{\mathbf{F}}\Big]\\
\otimes\;&\langle(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})^\dagger_{\mathbf{H},\dot\omega_2}\rangle\otimes(\mathbf{1},\overline{\mathbf{3}},+\frac{2}{3})^\dagger_{\mathbf{H}}+H.c.\\
\Rightarrow\;&\frac{Y_{\mathcal{D}}d_{\mathscr{B}}}{4}\dot\zeta_3'\Big[\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})_{\mathbf{H},\dot{\rm I}}}}(s_Ld_R{}^c+e_L\mu_R{}^c)+\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})_{\mathbf{H},\dot 2}}}(s_Ls_R{}^c+\mu_L\mu_R{}^c)\Big]v_{\rm EW}+H.c.\,,
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}\times\frac{d_{\mathscr{B}}}{M_{\rm pl}}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_1}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_2}\mathbf{70_H}^\dagger\Big(\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm I}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\Big)+H.c.\\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}},\mathbf{1},+\frac{1}{5})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{10},\mathbf{3},-\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{5},\overline{\mathbf{3}},+\frac{7}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}},\overline{\mathbf{3}},-\frac{2}{15})_{\mathbf{H},\dot\omega_1}\\
\times\;&\frac{d_{\mathscr{B}}}{M_{\rm pl}}(\overline{\mathbf{5}},\overline{\mathbf{3}},-\frac{2}{15})^\dagger_{\mathbf{H},\dot\omega_1}\otimes(\overline{\mathbf{5}},\overline{\mathbf{3}},-\frac{2}{15})^\dagger_{\mathbf{H},\dot\omega_2}\otimes(\mathbf{10},\overline{\mathbf{3}},+\frac{4}{15})^\dagger_{\mathbf{H}}\otimes\langle\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm I}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\rangle+H.c.\\
\supset\;& Y_{\mathcal{D}}d_{\mathscr{B}}\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{2M_{\rm pl}}\Big[(\overline{\mathbf{4}},\mathbf{1},+\frac{1}{4})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{4},\mathbf{3},+\frac{1}{12})'_{\mathbf{F}}\oplus(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{1},\overline{\mathbf{3}},+\frac{2}{3})'_{\mathbf{F}}\Big]\otimes(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{H},\dot\omega_1}\\
\times\;&(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})^\dagger_{\mathbf{H},\dot\omega_1}\otimes(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})^\dagger_{\mathbf{H},\dot\omega_2}\otimes(\mathbf{4},\overline{\mathbf{3}},+\frac{5}{12})^\dagger_{\mathbf{H}}+H.c.\\
\supset\;& Y_{\mathcal{D}}d_{\mathscr{B}}\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{2M_{\rm pl}m^2_{(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{H},\dot\omega_1}}}\Big[(\overline{\mathbf{3}},\mathbf{1},+\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{3},\mathbf{3},0)'_{\mathbf{F}}\oplus(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{1},\overline{\mathbf{3}},+\frac{2}{3})'_{\mathbf{F}}\Big]\\
\otimes\;&\langle(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})'_{\mathbf{H},\dot\omega_2}\rangle^\dagger\otimes(\mathbf{1},\overline{\mathbf{3}},+\frac{2}{3})^\dagger_{\mathbf{H}}+H.c.\\
\Rightarrow\;&\frac{Y_{\mathcal{D}}d_{\mathscr{B}}}{4}\dot\zeta_3'\Big[\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{H},\dot{\rm I}}}}(d_Ld_R{}^c+e_Le_R{}^c)+\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{H},\dot 2}}}(d_Ls_R{}^c+\mu_Le_R{}^c)\Big]v_{\rm EW}+H.c.\,,
\end{aligned}
\tag{20}
$$

via the Higgs mixing operator of $\mathcal{O}_{\mathscr{B}}^{d=5}$ in Eq. (14b). For convenience, we parametrize the following ratios of

$$
\Delta_{\dot\omega}\equiv\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{H},\dot\omega}}}\,,\quad\Delta'_{\dot\omega}\equiv\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})_{\mathbf{H},\dot\omega}}}\,.
\tag{21}
$$

With the Higgs VEVs assignments in Eq. (1), we can obtain the following propagator masses of

$$
\begin{aligned}
& m_{(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{H},\dot{\rm I}}}\sim m_{(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})_{\mathbf{H},\dot{\rm I}}}\sim\mathcal{O}(v_{431})\,,\\
& m_{(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{H},\dot 2}}\sim m_{(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})_{\mathbf{H},\dot 2}}\sim\mathcal{O}(v_{331})\,.
\end{aligned}
\tag{22}
$$

1.6 The SM quark/lepton masses and the benchmark

For all up-type quarks with $Q_e = +\frac{2}{3}$, we write down the following tree-level masses from both the renormalizable Yukawa couplings and the gravity-induced terms in the basis of $\mathcal{U} \equiv (u\,, c\,, t\,)$

$$\mathcal{M}_{\mathcal{U}} \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & c_4\dot{\zeta}_2/\sqrt{2} & c_5\zeta_1/\sqrt{2} \\ c_4\dot{\zeta}_2/\sqrt{2} & 0 & c_5\zeta_2/\sqrt{2} \\ c_5\zeta_1/\sqrt{2} & c_5\zeta_2/\sqrt{2} & Y_{\mathcal{T}} \end{pmatrix} v_{\rm EW} \,. \tag{23}$$

For all down-type quarks with $Q_e = -\frac{1}{3}$, we find the following tree-level mass matrix in the basis of $\mathcal{D} \equiv (d\,, s\,, b\,)$

$$\mathcal{M}_{\mathcal{D}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot{1}} \dot{\zeta}_3' & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot{2}} \zeta_3' & 0 \\ Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot{1}}' \dot{\zeta}_3 & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot{2}}' \dot{\zeta}_3 & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathscr{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\rm EW} \,, \tag{24}$$

where we interpret the ratio between two $\mathfrak{g}_{331}$ VEVs as the Cabibbo angle of $\tan\lambda \equiv \frac{\dot{\zeta}_3'}{\dot{\zeta}_3}$. For all charged leptons with $Q_e = -1$, their tree-level mass matrix is the transposition of the down-type quark mass matrix and expressed as follows:

$$\mathcal{M}_{\mathcal{L}} = \mathcal{M}_{\mathcal{D}}^T \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot{1}} \dot{\zeta}_3' & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot{1}}' \dot{\zeta}_3 & 0 \\ Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot{2}} \zeta_3' & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot{2}}' \dot{\zeta}_3 & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathscr{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\rm EW} \,. \tag{25}$$

Based on the above SM quark/lepton mass matrices, we find the following benchmark point of

$$v_{531} \simeq 1.4 \times 10^{17}\,{\rm GeV}\,, \quad v_{431} \simeq 4.8 \times 10^{15}\,{\rm GeV}\,, \quad v_{331} \simeq 4.8 \times 10^{13}\,{\rm GeV}\,, \tag{26}$$

to reproduce the observed hierarchical masses as well as the CKM mixing pattern, as summarized in Table 5.

ζ_1	ζ_2	ζ_3	$Y_{\mathcal{D}}$	$Y_{\mathcal{B}}$	$Y_{\mathcal{T}}$
6.0×10^{-2}	2.0×10^{-3}	2.0×10^{-5}	0.5	0.5	0.8
λ	c_4	c_5	$d_{\mathscr{A}}$	$d_{\mathscr{B}}$	
0.22	0.2	1.0	0.01	0.01	
m_u	m_c	m_t	$m_d = m_e$	$m_s = m_\mu$	$m_b = m_\tau$
1.6×10^{-3}	0.6	139.2	0.5×10^{-3}	6.4×10^{-2}	1.5
$\lvert V_{ud}\rvert$	$\lvert V_{us}\rvert$	$\lvert V_{ub}\rvert$			
0.98	0.22	2.1×10^{-3}			
$\lvert V_{cd}\rvert$	$\lvert V_{cs}\rvert$	$\lvert V_{cb}\rvert$			
0.22	0.98	5.3×10^{-2}			
$\lvert V_{td}\rvert$	$\lvert V_{ts}\rvert$	$\lvert V_{tb}\rvert$			
0.013	5.3×10^{-2}	1			

Table 5: The parameters of the $\mathfrak{su}(8)$ benchmark point and the predicted SM quark/lepton masses (in unit of GeV) as well as the CKM mixings.

2 The intermediate stages of the SWS symmetry breaking pattern

Schematically, we assign the Higgs VEVs according to the symmetry breaking pattern as follows:

$$\mathfrak{g}_{531} \to \mathfrak{g}_{431} \quad : \quad \langle (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\mathrm{IV}} \rangle \equiv \frac{1}{\sqrt{2}} W_{\overline{\mathbf{5}}\,,\mathrm{IV}}\,, \tag{27a}$$

$$\mathfrak{g}_{431} \to \mathfrak{g}_{421} \quad : \quad \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{V}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\overline{\mathbf{3}}\,,\mathrm{V}}\,,\ \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\dot{1},\dot{\mathrm{VII}}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\overline{\mathbf{3}}\,,\dot{1},\dot{\mathrm{VII}}}\,, \tag{27b}$$

$$\begin{aligned}\mathfrak{g}_{421} \to \mathfrak{g}_{\mathrm{SM}} \quad : \quad & \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,3,\mathrm{VI}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\overline{\mathbf{4}}\,,3,\mathrm{VI}}\,, \\ & \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot{\mathrm{VIII}}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\overline{\mathbf{4}}\,,\dot{\mathrm{VIII}}}\,,\ \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})'_{\mathbf{H}\,,\dot{2},\dot{\mathrm{IX}}} \rangle \equiv \frac{1}{\sqrt{2}} V'_{\overline{\mathbf{4}}\,,\dot{2},\dot{\mathrm{IX}}}\,,\end{aligned} \tag{27c}$$

$$\mathrm{EWSB} \quad : \quad \langle (\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{H}} \rangle \equiv \frac{1}{\sqrt{2}} v_{\mathrm{EW}}\,. \tag{27d}$$

For our later convenience, we also parametrize different symmetry breaking VEVs in terms of the following dimensionless quantities:

$$\begin{aligned}&\zeta_1 \equiv \frac{W_{\overline{\mathbf{5}}\,,\mathrm{IV}}}{M_{\mathrm{pl}}}\,, \quad \zeta_2 \equiv \frac{w_{\overline{\mathbf{3}}\,,\mathrm{V}}}{M_{\mathrm{pl}}}\,, \quad \dot{\zeta}_2 \equiv \frac{w_{\overline{\mathbf{3}}\,,\dot{1},\dot{\mathrm{VII}}}}{M_{\mathrm{pl}}}\,, \\ &\zeta_3 \equiv \frac{V_{\overline{\mathbf{4}}\,,3,\mathrm{VI}}}{M_{\mathrm{pl}}}\,, \quad \dot{\zeta}_3' \equiv \frac{V'_{\overline{\mathbf{4}}\,,\dot{2},\dot{\mathrm{IX}}}}{M_{\mathrm{pl}}}\,, \quad \dot{\zeta}_3 \equiv \frac{V_{\overline{\mathbf{4}}\,,\dot{\mathrm{VIII}}}}{M_{\mathrm{pl}}}\,, \\ &\zeta_0 \gg \zeta_1 \gg \zeta_2 \sim \dot{\zeta}_2 \gg \zeta_3 \sim \dot{\zeta}_3' \sim \dot{\zeta}_3\,, \quad \zeta_{ij} \equiv \frac{\zeta_j}{\zeta_i}\,, \quad (i<j)\,.\end{aligned} \tag{28}$$

In Table 6, we summarize all vectorlike fermions that become massive during different stages of the SWS symmetry breaking pattern.

stages	$Q_e = -\frac{1}{3}$	$Q_e = +\frac{2}{3}$	$Q_e = -1$	$Q_e = 0$
v_{531} $\{\Omega\}$	$\mathfrak{D}'$ IV	-	$(\mathfrak{e}'\,,\mathfrak{n}')$ IV	$\{\check{\mathfrak{n}}\,,\check{\mathfrak{n}}''\}$ $\{\mathrm{IV}''\,,\mathrm{IV}\}$
v_{431} $\{\Omega\}$	$\{\mathfrak{D}\,,\mathfrak{D}''''\}$ $\{\mathrm{V}\,,\dot{\mathrm{VII}}\}$	$\mathfrak{U}$	$\mathfrak{E}\,,(\mathfrak{e}\,,\mathfrak{n})\,,(\mathfrak{e}'''\,,\mathfrak{n}''')$ $\{\mathrm{V}\,,\dot{\mathrm{VII}}\}$	$\{\check{\mathfrak{n}}'\,,\check{\mathfrak{n}}'''\}$ $\{\mathrm{V}\,,\dot{\mathrm{VII}}\}$
v_{421} $\{\Omega\}$	$\mathfrak{d}\,,\{\mathfrak{D}''\,,\mathfrak{D}'''\,,\mathfrak{D}'''''\}$ $\{\mathrm{VI}\,,\dot{\mathrm{IX}}\,,\dot{\mathrm{VIII}}\}$	$\mathfrak{u}$	$(\mathfrak{e}''\,,\mathfrak{n}'')\,,(\mathfrak{e}''''\,,\mathfrak{n}'''')\,,(\mathfrak{e}'''''\,,\mathfrak{n}''''')$ $\{\mathrm{VI}\,,\dot{\mathrm{VIII}}\,,\dot{\mathrm{IX}}\}$	-

Table 6: The vectorlike fermions at different intermediate symmetry breaking stages along the SWS symmetry breaking pattern of the $\mathfrak{su}(8)$ theory.

2.1 The first stage

The first symmetry breaking stage of $\mathfrak{g}_{531} \to \mathfrak{g}_{431}$ is achieved by $(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\mathrm{IV}} \subset \overline{\mathbf{8}_{\mathbf{H}}}_{\,,\mathrm{IV}}$ in the rank-two sector, according to the $\widetilde{\mathrm{U}}(1)_{T'}$-neutral components in Table 7. Accordingly, the term of

$\overline{\mathbf{8_F}}^\Omega$	$(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^\Omega_{\mathbf{F}}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\Omega_{\mathbf{F}}$		
$\mathcal{T}'$	$-4t$	$-\frac{4}{3}t$		
$\mathbf{28_F}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}$	
$\mathcal{T}'$	$-\frac{4}{3}t$	$+\frac{4}{3}t$	$+4t$	
$\mathbf{56_F}$	$(\mathbf{1}\,,\mathbf{1}\,,+1)'_{\mathbf{F}}$	$(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}}$	$(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}$	$(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}$
$\mathcal{T}'$	$-4t$	$-\frac{4}{3}t$	$+4t$	$+\frac{4}{3}t$
$\overline{\mathbf{8_H}}_{,\omega}$	$(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H},\omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\omega}$		
$\mathcal{T}'$	0	$+\frac{8}{3}t$		
$\overline{\mathbf{28_H}}_{,\dot\omega}$	$(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H},\dot\omega}$	$(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H},\dot\omega}$	$(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H},\dot\omega}$	
$\mathcal{T}'$	$+\frac{16}{3}t$	$+\frac{8}{3}t$	0	
$\mathbf{70_H}$	$(\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}}$			
$\mathcal{T}'$	$-\frac{16}{3}t$			

Table 7: The $\widetilde{\mathrm{U}}(1)_{T'}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{531}$ theory.

$Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{IV}}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\mathrm{IV}}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}'\,,\check{\mathfrak{n}}''\,,\mathfrak{e}'\,,\mathfrak{n}'\,,\check{\mathfrak{n}})$ as follows:

$$\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{IV}}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\mathrm{IV}}+H.c.\\
\supset\;& Y_{\mathcal{B}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^{\mathrm{IV}}_{\mathbf{F}}\otimes(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^{\mathrm{IV}}_{\mathbf{F}}\otimes(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}\Big]\otimes\langle(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H},\mathrm{IV}}\rangle+H.c.\\
\Rightarrow\;& \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{D}_L'\mathfrak{D}_R'^{\,c}+\check{\mathfrak{n}}_L''\check{\mathfrak{n}}_R''^{\,c}+\mathfrak{e}_L'\mathfrak{e}_R'^{\,c}-\mathfrak{n}_L'\mathfrak{n}_R'^{\,c}+\check{\mathfrak{n}}_L\check{\mathfrak{n}}_R^c\right)W_{\mathbf{5},\mathrm{IV}}+H.c.\,.
\end{aligned}\tag{29}$$

After this stage, the remaining massless fermions expressed in terms of the $\mathfrak{g}_{431}$ IRs are the following:

$$\begin{aligned}
& \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})^\Omega_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)^{\Omega'}_{\mathbf{F}}\Big]\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\Omega_{\mathbf{F}}\subset\overline{\mathbf{8_F}}^\Omega\,,\\
& \Omega=(\omega\,,\dot\omega)\,,\quad \omega=(3\,,\mathrm{V}\,,\mathrm{VI})\,,\quad \dot\omega=(\dot1\,,\dot2\,,\dot{\mathrm{VII}}\,,\dot{\mathrm{VIII}}\,,\dot{\mathrm{IX}})\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,0)^{\mathrm{IV}'}_{\mathbf{F}}\subset\overline{\mathbf{8_F}}^{\mathrm{IV}}\,,\\
& (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}\oplus\Big[\cancel{(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})'_{\mathbf{F}}}\oplus(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}\Big]\oplus\Big[\cancel{(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})'_{\mathbf{F}}}\oplus(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}\Big]\subset\mathbf{28_F}\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,+1)'_{\mathbf{F}}\oplus\Big[(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})'_{\mathbf{F}}\oplus(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{F}}\Big]\\
\oplus\;& \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}}\oplus(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})'_{\mathbf{F}}\Big]\oplus\Big[(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})'_{\mathbf{F}}\oplus(\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}}\Big]\subset\mathbf{56_F}\,.
\end{aligned}\tag{30}$$

2.2 The second stage

The second symmetry breaking stage of $\mathfrak{g}_{431}\to\mathfrak{g}_{421}$ is achieved by $(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\mathrm{V}}\subset\overline{\mathbf{8_H}}_{,\mathrm{V}}$ in the rank-two sector, $(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\dot{\mathrm{VII}}}\subset\overline{\mathbf{28_H}}_{,\dot{\mathrm{VII}}}$ and $(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})^{\dot{\mathrm{VII}}}_{\mathbf{H}}\subset\mathbf{28_H}^{\dot{\mathrm{VII}}}$ in the rank-three sector, according to the $\widetilde{\mathrm{U}}(1)_{T''}$-neutral components in Table 8. The terms of $Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{V}}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\mathrm{V}}+H.c.$ leads to

$\overline{\mathbf{8_F}}^{\Omega}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega'}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\Omega}$	
$\mathcal{T}''$	$-2t$	$-4t$	$-4t$	
$\mathbf{28_F}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}$	$(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}$	
$\mathcal{T}''$	$+4t$	$+2t$	0	
$\mathbf{56_F}$	$(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}'$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}'$	$(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{F}}$	
$\mathcal{T}''$	$+4t$	$+4t$	$+2t$	
	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}'$	$(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}'$	$(\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}}$
$\mathcal{T}''$	$-2t$	0	$+2t$	0
$\overline{\mathbf{8_H}}_{,\omega}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega}$		
$\mathcal{T}''$	$+2t$	0		
$\overline{\mathbf{28_H}}_{,\dot\omega}$	$(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot\omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{4}}\,,\overline{\mathbf{3}}\,,-\frac{1}{12})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot\omega}$
$\mathcal{T}''$	0	0	$+2t$	$+2t$
$\mathbf{70_H}$	$(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{H}}$			
$\mathcal{T}''$	$-2t$			

Table 8: The $\widetilde{\rm U}(1)_{T''}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{431}$ theory.

the vectorial masses of $(\mathfrak{D}\,,\mathfrak{e}\,,\mathfrak{n}\,,\check{\mathfrak{n}}')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\rm V}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\rm V}+H.c.\\
\supset\;& Y_{\mathcal{B}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\rm V}\otimes(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\rm V}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\rm V}+H.c.\\
\supset\;& Y_{\mathcal{B}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\rm V}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm V'}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}'\\
\oplus\;& (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\rm V}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\rm V}\rangle+H.c.\\
\Rightarrow\;& \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{D}_L{\mathfrak{D}_R}^c+\check{\mathfrak{n}}_L'\check{\mathfrak{n}}_R'^c-\mathfrak{e}_L{\mathfrak{e}_R}^c+\mathfrak{n}_L{\mathfrak{n}_R}^c+\check{\mathcal{N}}_L^{\rm V'}{\check{\mathfrak{n}}_R}^c\right)w_{\overline{\mathbf{3}},\rm V}+H.c.\,.
\end{aligned}\tag{31}
$$

The terms of $Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot{\rm V}\rm II}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot{\rm V}\rm II}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}'''',\mathfrak{e}''',\mathfrak{n}''',\check{\mathfrak{n}}''')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot{\rm V}\rm II}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot{\rm V}\rm II}+H.c.\\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot{\rm V}\rm II}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\dot{\rm V}\rm II}\otimes(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H}\,,\dot{\rm V}\rm II}+H.c.\\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\dot{\rm V}\rm II}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\dot{\rm V}\rm II}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}'\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\dot{\rm V}\rm II}\rangle+H.c.\\
\Rightarrow\;& \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}_L''''\mathfrak{D}_R''''^c-\mathfrak{e}_L'''\mathfrak{e}_R'''^c+\mathfrak{n}_L'''\mathfrak{n}_R'''^c+\check{\mathfrak{n}}_L'''\check{\mathfrak{n}}_R'''^c\right)w_{\overline{\mathbf{3}},\dot{\rm V}\rm II}+H.c.\,.
\end{aligned}\tag{32}
$$

The Yukawa coupling between two $\mathbf{56_F}$s by the following $d=5$ operator

$$
\begin{aligned}
& \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\dot{\rm VII}} + H.c. \\
\supset\ & c_4 \frac{v_U}{M_{\rm pl}} \Big[(\mathbf{5}\,, \overline{\mathbf{3}}\,, +\frac{7}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}\,, \mathbf{1}\,, -\frac{3}{5})_{\mathbf{F}} \oplus \cancel{(\mathbf{10}\,, \mathbf{3}\,, -\frac{1}{15})_{\mathbf{F}} \otimes (\mathbf{10}\,, \mathbf{3}\,, -\frac{1}{15})_{\mathbf{F}}} \Big] \\
& \otimes (\mathbf{5}\,, \mathbf{3}\,, +\frac{2}{15})_{\mathbf{H}}^{\dot{\rm VII}} + H.c. \\
\supset\ & c_4 \zeta_0 (\mathbf{4}\,, \overline{\mathbf{3}}\,, +\frac{5}{12})_{\mathbf{F}} \otimes (\overline{\mathbf{4}}\,, \mathbf{1}\,, -\frac{3}{4})_{\mathbf{F}} \otimes \langle (\mathbf{1}\,, \mathbf{3}\,, +\frac{1}{3})_{\mathbf{H}}^{\dot{\rm VII}} \rangle + H.c. \\
\Rightarrow\ & \frac{c_4}{\sqrt{2}} \zeta_0 \left(\mathfrak{E}_L {\mathfrak{E}_R}^c + \mathfrak{U}_L {\mathfrak{U}_R}^c \right) w_{\mathbf{3}}^{\dot{\rm VII}} + H.c.\,,
\end{aligned} \tag{33}
$$

leads to massive vectorlike fermions of $(\mathfrak{E}\,, \mathfrak{U})$.

After integrating out the massive fermions, the remaining massless fermions expressed in terms of the $\mathfrak{g}_{421}$ IRs are the following:

$$
\begin{aligned}
& (\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\Omega'} \oplus (\mathbf{1}\,, \overline{\mathbf{2}}\,, -\frac{1}{2})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\Omega''} \subset \overline{\mathbf{8_F}}^{\Omega}\,, \\
& \Omega = (\omega\,, \dot\omega)\,, \quad \omega = (3\,, {\rm VI})\,, \quad \dot\omega = (\dot{1}\,, \dot{2}\,, \dot{\rm VIII}\,, \dot{\rm IX})\,, \\
& (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\rm IV'} \subset \overline{\mathbf{8_F}}^{\rm IV}\,, \quad (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\rm V'} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\rm V''} \subset \overline{\mathbf{8_F}}^{\rm V}\,, \\
& (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\dot{\rm VII}'} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\dot{\rm VII}''} \subset \overline{\mathbf{8_F}}^{\dot{\rm VII}}\,, \\
& \Big[\cancel{(\mathbf{1}\,, \overline{\mathbf{2}}\,, +\frac{1}{2})_{\mathbf{F}}} \oplus (\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}} \Big] \oplus \Big[\cancel{(\mathbf{1}\,, \mathbf{2}\,, +\frac{1}{2})_{\mathbf{F}}' \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}} \Big] \oplus \Big[\cancel{(\mathbf{4}\,, \mathbf{1}\,, -\frac{1}{4})_{\mathbf{F}}} \oplus (\mathbf{4}\,, \mathbf{2}\,, +\frac{1}{4})_{\mathbf{F}} \Big] \\
\oplus\ & \Big[\cancel{(\mathbf{4}\,, \mathbf{1}\,, -\frac{1}{4})_{\mathbf{F}}'} \oplus (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}} \Big] \subset \mathbf{28_F}\,, \\
& (\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}' \oplus \Big[\cancel{(\mathbf{1}\,, \overline{\mathbf{2}}\,, +\frac{1}{2})_{\mathbf{F}}'''} \oplus (\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}'' \Big] \oplus \Big[\cancel{(\mathbf{4}\,, \mathbf{1}\,, +\frac{3}{4})_{\mathbf{F}}} \oplus (\mathbf{4}\,, \overline{\mathbf{2}}\,, +\frac{1}{4})_{\mathbf{F}} \Big] \oplus \Big[\cancel{(\overline{\mathbf{4}}\,, \mathbf{1}\,, -\frac{3}{4})_{\mathbf{F}}} \\
\oplus\ & (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}' \Big] \oplus \Big[\cancel{(\mathbf{4}\,, \mathbf{1}\,, -\frac{1}{4})_{\mathbf{F}}''} \oplus (\mathbf{4}\,, \mathbf{2}\,, +\frac{1}{4})_{\mathbf{F}}' \Big] \oplus \Big[(\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}'' \oplus (\mathbf{6}\,, \mathbf{2}\,, 0)_{\mathbf{F}} \Big] \subset \mathbf{56_F}\,.
\end{aligned} \tag{34}
$$

2.3 The third stage

The third symmetry breaking stage of $\mathfrak{g}_{421} \to \mathfrak{g}_{\rm SM}$ is achieved by $(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,{\rm VI}} \subset \overline{\mathbf{8_H}}_{\,,{\rm VI}}$ in the rank-two sector and $(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\dot{\rm VIII}} \subset \overline{\mathbf{28_H}}_{\,,\dot{\rm VIII}}$, $(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})'_{\mathbf{H}\,,\dot{\rm IX}} \subset \overline{\mathbf{28_H}}_{\,,\dot{\rm IX}}$, $(\mathbf{4}\,, \mathbf{1}\,, -\frac{1}{4})_{\mathbf{H}}^{\dot{\rm IX}'} \subset \mathbf{28_H}^{\dot{\rm IX}}$ in the rank-three sector, according to the $\widetilde{\rm U}(1)_{T'''}$-neutral components in Table 9. The term of

$\overline{\mathbf{8_F}}^{\Omega}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{1}\,,0)^{\Omega'}_{\mathbf{F}}$	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{1}\,,0)^{\Omega''}_{\mathbf{F}}$
$\mathcal{T}'''$	$-4t$	$-4t$	0	$-4t$
$\mathbf{28_F}$	$(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}$	$(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}$	
$\mathcal{T}'''$	$-4t$	0	$+4t$	
$\mathbf{56_F}$	$(\mathbf{1}\,,\mathbf{1}\,,+1)'_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{1}\,,+1)''_{\mathbf{F}}$	$(\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})'_{\mathbf{F}}$
$\mathcal{T}'''$	$-4t$	$-4t$	0	$+4t$
	$(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})'_{\mathbf{F}}$	$(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})''_{\mathbf{F}}$	$(\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}}$	
$\mathcal{T}'''$	0	$+4t$	0	
$\overline{\mathbf{8_H}}_{,\omega}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\omega}$	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\omega}$		
$\mathcal{T}'''$	0	$+4t$		
$\overline{\mathbf{28_H}}_{,\dot{\omega}}$	$(\mathbf{1}\,,\mathbf{2}\,,-\frac{1}{2})_{\mathbf{H}\,,\dot{\omega}}$	$(\overline{\mathbf{4}}\,,\overline{\mathbf{2}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot{\omega}}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot{\omega}}$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})'_{\mathbf{H}\,,\dot{\omega}}$
$\mathcal{T}'''$	$+4t$	$+4t$	0	0
	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})'_{\mathbf{H}\,,\dot{\omega}}$			
$\mathcal{T}'''$	$+4t$			
$\mathbf{70_H}$	$(\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{H}}$			
$\mathcal{T}'''$	$-4t$			

Table 9: The $\widetilde{\mathrm{U}}(1)_{\mathcal{T}'''}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{421}$ theory.

$Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\rm VI}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\rm VI}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}''\,,\mathfrak{e}''\,,\mathfrak{n}'')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\rm VI}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\rm VI}+H.c.\\
\supset\;\; & Y_{\mathcal{B}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\rm VI}\otimes(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\rm VI}\otimes(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H},\rm VI}+H.c.\\
\supset\;\; & Y_{\mathcal{B}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\rm VI}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm VI'}\otimes(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})'_{\mathbf{F}}\\
\oplus\;\; & (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\rm VI}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\rm VI}+H.c.\\
\supset\;\; & Y_{\mathcal{B}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\rm VI}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm VI'}\otimes(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})'_{\mathbf{F}}\\
\oplus\;\; & (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\rm VI}\otimes(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm VI''}\otimes(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}\Big]\otimes\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\rm VI}\rangle+H.c.\\
\Rightarrow\;\; & \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{e}''_L\mathfrak{e}''^{\,c}_R-\mathfrak{n}''_L\mathfrak{n}''^{\,c}_R+\check{\mathcal{N}}_L^{\rm VI'}\check{\mathfrak{n}}''^{\,c}_R+\check{\mathcal{N}}_L^{\rm VI''}\check{\mathfrak{n}}'^{\,c}_R+\mathfrak{D}''_L\mathfrak{D}''^{\,c}_R\right)V_{\overline{\mathbf{4}},\rm VI}+H.c.\,.
\end{aligned}
\tag{35}
$$

The term of $Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\rm V\dot{I}II\,,I\dot{X}}\mathbf{56_F}\overline{\mathbf{28}}_{\mathbf{H}\,,\rm V\dot{I}II\,,I\dot{X}}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}''''\,,\mathfrak{e}''''\,,\mathfrak{n}''''\,,\mathfrak{D}'''\,,\mathfrak{e}'''''\,,\mathfrak{n}''''')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\rm V\dot{I}II}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\rm V\dot{I}II}+H.c.\\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\rm V\dot{I}II}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\rm V\dot{I}II}\otimes(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H}\,,\rm V\dot{I}II}+H.c.\\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\rm V\dot{I}II}\otimes(\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm V\dot{I}II'}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})'_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\rm V\dot{I}II}\otimes(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{F}}\Big]\\
\otimes\;\; & (\overline{\mathbf{4}}\,,\overline{\mathbf{3}}\,,-\frac{1}{12})_{\mathbf{H}\,,\rm V\dot{I}II}+H.c.\\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\rm V\dot{I}II}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})''_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm V\dot{I}II'}\otimes(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})''_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\rm V\dot{I}II}\otimes(\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{F}}\Big]\\
\otimes\;\; & \langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\rm V\dot{I}II}\rangle+H.c.\\
\Rightarrow\;\; & \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}''''_L\mathfrak{D}''''^{\,c}_R+\check{\mathcal{N}}_L^{\rm V\dot{I}II'}\check{\mathfrak{n}}'''^{\,c}_R-\mathfrak{e}''''_L\mathfrak{e}''''^{\,c}_R+\mathfrak{n}''''_L\mathfrak{n}''''^{\,c}_R\right)V_{\overline{\mathbf{4}},\rm V\dot{I}II}+H.c.\,,
\end{aligned}
\tag{36}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\rm I\dot{X}}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\rm I\dot{X}}+H.c.\\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\rm I\dot{X}}\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\rm I\dot{X}}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H}\,,\rm I\dot{X}}+H.c.\\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\rm I\dot{X}}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})'_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\rm I\dot{X}}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})'_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\rm I\dot{X}}+H.c.\\
\supset\;\; & Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\rm I\dot{X}}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})'_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\rm I\dot{X}}\otimes(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})'_{\mathbf{F}}\\
\oplus\;\; & (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm I\dot{X}''}\otimes(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})''_{\mathbf{F}}\Big]\otimes\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})'_{\mathbf{H}\,,\rm I\dot{X}}\rangle+H.c.\\
\Rightarrow\;\; & \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}'''_L\mathfrak{D}'''^{\,c}_R+\check{\mathcal{N}}_L^{\rm I\dot{X}''}\check{\mathfrak{n}}'''^{\,c}_R+\mathfrak{e}'''''_L\mathfrak{e}'''''^{\,c}_R-\mathfrak{n}'''''_L\mathfrak{n}'''''^{\,c}_R\right)V'_{\overline{\mathbf{4}},\rm I\dot{X}}+H.c.\,.
\end{aligned}
\tag{37}
$$

The Yukawa coupling between two $\mathbf{56_F}$s by the following $d=5$ operator

$$
\begin{aligned}
& \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\dot{\rm I}\mathrm{X}} + H.c. \\
\supset\ & c_4 \zeta_0 \Big[(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime} \otimes (\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}} \Big] \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{H}}^{\dot{\rm I}\mathrm{X}} + H.c. \\
\supset\ & c_4 \zeta_0 \Big[(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime} \otimes (\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}} \oplus (\overline{\mathbf{4}}\,,\mathbf{3}\,,+\frac{5}{12})_{\mathbf{F}} \otimes (\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}} \Big] \otimes (\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{H}}^{\dot{\rm I}\mathrm{X}} + H.c. \\
\supset\ & c_4 \zeta_0 \Big[(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime} \otimes (\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{4}\,,\mathbf{1}\,,+\frac{3}{4})_{\mathbf{F}} \otimes (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}^{\prime\prime} \oplus (\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{F}} \\
\otimes\ & (\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}} \Big] \otimes \langle (\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{H}}^{\dot{\rm I}\mathrm{X}^\prime} \rangle + H.c. \\
\Rightarrow\ & c_4 \zeta_0 \, (\mathfrak{E}_L \mu_R^c + \mathfrak{U}_L c_R^c + \mathfrak{u}_L \mathfrak{u}_R{}^c - \mathfrak{d}_L \mathfrak{d}_R{}^c) \, V_{\mathbf{4}}^{\dot{\rm I}\mathrm{X}} + H.c. \,, \qquad (38)
\end{aligned}
$$

further leads to massive vectorlike fermions of $(\mathfrak{u}\,,\mathfrak{d})$.

The remaining massless fermions of the $\mathfrak{g}_{\rm SM}$ are listed as follows:

$$
\begin{aligned}
& \Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega} \Big] \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega^\prime} \oplus \Big[(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega^{\prime\prime}} \Big] \subset \overline{\mathbf{8_F}}^{\Omega} \,, \quad \Omega = (\dot 1\,,\dot 2\,,3) \,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm VI} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm V\dot{I}II} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\dot{\rm I}\mathrm{X}} \subset \overline{\mathbf{8_F}}^{\Omega} \,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm IV^\prime} \oplus ... \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\dot{\rm I}\mathrm{X}^\prime} \subset \overline{\mathbf{8_F}}^{\Omega^\prime} \,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm V^{\prime\prime}} \oplus ... \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\dot{\rm I}\mathrm{X}^{\prime\prime}} \subset \overline{\mathbf{8_F}}^{\Omega^{\prime\prime}} \,, \\
& \Big[\cancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}} \oplus (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}} \Big] \oplus \Big[\cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}} \Big] \oplus \Big[\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\prime}} \Big] \\
\oplus\ & \Big[\xcancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime\prime}} \oplus (\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}} \Big] \oplus \Big[\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}^{\prime} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\prime\prime}} \Big] \oplus \Big[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}^{\prime\prime}} \\
\oplus\ & (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}} \Big] \subset \mathbf{28_F} \,, \\
& (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime} \oplus \Big[\cancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime\prime\prime}} \oplus (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime\prime} \Big] \oplus \Big[\cancel{(\mathbf{3}\,,\mathbf{1}\,,+\frac{2}{3})_{\mathbf{F}} \oplus (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime\prime\prime}} \Big] \\
\oplus\ & \Big[\xcancel{(\mathbf{3}\,,\overline{\mathbf{2}}\,,+\frac{1}{6})_{\mathbf{F}}} \oplus \xcancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime\prime\prime\prime}} \Big] \oplus \Big[\cancel{(\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}} \oplus (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}^{\prime}} \Big] \oplus \Big[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime}} \oplus (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}^{\prime\prime} \Big] \\
\oplus\ & \Big[\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime\prime} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\prime\prime\prime}} \Big] \oplus \Big[(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}^{\prime} \oplus \xcancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime\prime\prime\prime\prime}} \Big] \oplus \Big[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime\prime\prime}} \oplus (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}^{\prime\prime\prime} \Big] \\
\oplus\ & \Big[(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}^{\prime\prime} \oplus \xcancel{(\overline{\mathbf{3}}\,,\mathbf{2}\,,-\frac{1}{6})_{\mathbf{F}}} \Big] \subset \mathbf{56_F} \,. \qquad (39)
\end{aligned}
$$

2.4 The $d = 5$ bi-linear fermion operators

For the operator of $\mathcal{O}_{\mathcal{F}}^{(4\,,1)}$ in Eq. (13a), it is decomposed as

$$
\begin{aligned}
& \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \cdot \overline{\mathbf{28_H}}_{\,,\dot\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\ & \frac{c_4}{M_{\rm pl}} (\overline{\mathbf{10}}\,, \mathbf{1}\,, -\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,, \mathbf{3}\,, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}\,, \mathbf{1}\,, +\frac{2}{5})_{\mathbf{H}\,,\dot\omega} \otimes (\mathbf{10}\,, \overline{\mathbf{3}}\,, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_4}{M_{\rm pl}} (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}' \otimes (\mathbf{6}\,, \mathbf{3}\,, -\frac{1}{6})_{\mathbf{F}} \otimes (\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\dot\omega} \otimes (\mathbf{4}\,, \overline{\mathbf{3}}\,, +\frac{5}{12})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_4}{M_{\rm pl}} (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}' \otimes (\mathbf{6}\,, \mathbf{2}\,, 0)_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\dot\omega}' \rangle \otimes (\mathbf{4}\,, \overline{\mathbf{2}}\,, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{c_4}{2} \dot\zeta_3' \, (c_L {u_R}^c) \, v_{\rm EW} + H.c. \,.
\end{aligned} \tag{40}
$$

For the operator of $\mathcal{O}_{\mathcal{F}}^{(5\,,1)}$ in Eq. (13b), it is decomposed as

$$
\begin{aligned}
& \frac{c_5}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{\,,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} \Big[(\mathbf{5}\,, \mathbf{3}\,, +\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}\,, \mathbf{1}\,, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}\,, \mathbf{1}\,, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,, \mathbf{3}\,, -\frac{1}{15})_{\mathbf{F}} \Big] \otimes \langle (\overline{\mathbf{5}}\,, \mathbf{1}\,, +\frac{1}{5})_{\mathbf{H}\,,\omega} \rangle \\
\otimes\ & (\mathbf{10}\,, \overline{\mathbf{3}}\,, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset\ & c_5 \frac{W_{\overline{\mathbf{5}}\,,{\rm IV}}}{\sqrt{2} M_{\rm pl}} \Big[(\mathbf{4}\,, \mathbf{3}\,, +\frac{1}{12})_{\mathbf{F}} \otimes (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}' \oplus (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{4}\,, \mathbf{3}\,, +\frac{1}{12})_{\mathbf{F}}' \Big] \otimes (\mathbf{4}\,, \overline{\mathbf{3}}\,, +\frac{5}{12})_{\mathbf{H}} + H.c. \\
\supset\ & c_5 \frac{W_{\overline{\mathbf{5}}\,,{\rm IV}}}{\sqrt{2} M_{\rm pl}} \Big[(\mathbf{4}\,, \mathbf{2}\,, +\frac{1}{4})_{\mathbf{F}} \otimes (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}' \oplus (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{4}\,, \mathbf{2}\,, +\frac{1}{4})_{\mathbf{F}}' \Big] \otimes (\mathbf{4}\,, \overline{\mathbf{2}}\,, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{c_5}{2} \zeta_1 \, (t_L {u_R}^c + u_L {t_R}^c) \, v_{\rm EW} + H.c. \,,
\end{aligned} \tag{41}
$$

and

$$
\begin{aligned}
& \frac{c_5}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{\,,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} (\mathbf{10}\,, \mathbf{1}\,, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,, \mathbf{3}\,, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{5}}\,, \mathbf{1}\,, +\frac{1}{5})_{\mathbf{H}\,,\omega} \otimes (\mathbf{10}\,, \overline{\mathbf{3}}\,, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{6}\,, \mathbf{3}\,, -\frac{1}{6})_{\mathbf{F}} \otimes (\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\omega} \otimes (\mathbf{4}\,, \overline{\mathbf{3}}\,, +\frac{5}{12})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{6}\,, \mathbf{2}\,, 0)_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\omega} \rangle \otimes (\mathbf{4}\,, \overline{\mathbf{2}}\,, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{c_5}{2} \zeta_3 \, (c_L {t_R}^c) \, v_{\rm EW} + H.c. \,,
\end{aligned} \tag{42}
$$

and

$$
\begin{aligned}
& \frac{c_5}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} (\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega} \otimes (\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} (\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}} \otimes (\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}} \otimes \langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega}\rangle \otimes (\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{H}} + H.c. \\
\supset\ & c_5 \frac{w_{\overline{\mathbf{3}},\rm V}}{\sqrt{2} M_{\rm pl}} (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}} \otimes (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})^{\prime\prime}_{\mathbf{F}} \otimes (\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{c_5}{2} \zeta_2 \, (t_L {c_R}^c)\, v_{\rm EW} + H.c.\,.
\end{aligned} \tag{43}
$$

2.5 The $d=5$ irreducible Higgs mixing operators

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1}$, we find the mass terms of

$$
\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1} \times \frac{d_{\mathscr{A}}}{M_{\rm pl}} \epsilon_{\omega_1\omega_2\omega_3\omega_4} \overline{\mathbf{8_H}}^\dagger_{,\omega_1} \overline{\mathbf{8_H}}^\dagger_{,\omega_2} \overline{\mathbf{8_H}}^\dagger_{,\omega_3} \overline{\mathbf{8_H}}^\dagger_{,\omega_4} \mathbf{70_H}^\dagger + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^{\omega_1}_{\mathbf{F}} \otimes (\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^{\omega_1}_{\mathbf{F}} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}} \Big] \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega_1} \\
\times\ & \frac{d_{\mathscr{A}}}{M_{\rm pl}} (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H}\,,\omega_1} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H}\,,\omega_2} \otimes (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^\dagger_{\mathbf{H}\,,\omega_3} \otimes \langle(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^\dagger_{\mathbf{H}\,,\omega_4}\rangle \otimes (\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})^\dagger_{\mathbf{H}} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})^{\omega_1}_{\mathbf{F}} \otimes (\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^{\omega_1}_{\mathbf{F}} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}} \Big] \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega_1} \\
\times\ & d_{\mathscr{A}} \frac{W_{\overline{\mathbf{5}},\rm IV}}{\sqrt{2} M_{\rm pl}} (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H}\,,\omega_1} \otimes \langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H}\,,\omega_2}\rangle \otimes (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})^\dagger_{\mathbf{H}\,,\omega_3} \otimes (\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})^\dagger_{\mathbf{H}} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})^{\omega_1}_{\mathbf{F}} \otimes (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})^{\omega_1}_{\mathbf{F}} \otimes (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}} \Big] \otimes (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\omega_1} \\
\times\ & d_{\mathscr{A}} \frac{W_{\overline{\mathbf{5}},\rm IV} w_{\overline{\mathbf{3}},\rm V}}{2 M_{\rm pl}} (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})^\dagger_{\mathbf{H}\,,\omega_1} \otimes \langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})^\dagger_{\mathbf{H}\,,\omega_3}\rangle \otimes (\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})^\dagger_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{Y_{\mathcal{B}} d_{\mathscr{A}}}{4} \frac{W_{\overline{\mathbf{5}},\rm IV} w_{\overline{\mathbf{3}},\rm V} V_{\overline{\mathbf{4}},\rm VI}}{M_{\rm pl} m^2_{(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,3}}} \, (b_L {b_R}^c + \tau_L {\tau_R}^c)\, v_{\rm EW} + H.c.\,,
\end{aligned} \tag{44}
$$

via the Higgs mixing operator of $\mathcal{O}^{d=5}_{\mathscr{A}}$ in Eq. (14a).

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}$, we find the mass terms of

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}\times\frac{d_{\mathscr{B}}}{M_{\rm pl}}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_1}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_2}\mathbf{70_H}^\dagger\Big(\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm i}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\Big)+H.c.\\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H},\dot\omega_1}\\
\times\quad & \frac{d_{\mathscr{B}}}{M_{\rm pl}}(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H},\dot\omega_1}^\dagger\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H},\dot\omega_2}^\dagger\otimes(\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}}^\dagger\otimes\langle\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm i}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\rangle+H.c.\\
\supset\quad & Y_{\mathcal{D}}d_{\mathscr{B}}\frac{w_{\overline{\mathbf{3}},\dot{\rm i}}w_{\overline{\mathbf{3}},\dot{\rm VII}}}{2M_{\rm pl}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{4}\,,\mathbf{3}\,,+\frac{1}{12})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}'\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\dot\omega_1}\\
\times\quad & (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\dot\omega_1}^\dagger\otimes(\overline{\mathbf{4}}\,,\overline{\mathbf{3}}\,,-\frac{1}{12})_{\mathbf{H},\dot\omega_2}^\dagger\otimes(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{H}}^\dagger+H.c.\\
\supset\quad & Y_{\mathcal{D}}d_{\mathscr{B}}\frac{w_{\overline{\mathbf{3}},\dot{\rm i}}w_{\overline{\mathbf{3}},\dot{\rm VII}}}{2M_{\rm pl}m^2_{(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\dot\omega_1}}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}''\Big]\\
\otimes\quad & \langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\dot\omega_2}^\dagger\rangle\otimes(\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{H}}^\dagger+H.c.\\
\Rightarrow\quad & \frac{Y_{\mathcal{D}}d_{\mathscr{B}}}{4}\dot\zeta_3\Big[\frac{w_{\overline{\mathbf{3}},\dot{\rm i}}w_{\overline{\mathbf{3}},\dot{\rm VII}}}{m^2_{(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\dot{\rm i}}}}(d_L{d_R}^c+e_L{e_R}^c)+\frac{w_{\overline{\mathbf{3}},\dot{\rm i}}w_{\overline{\mathbf{3}},\dot{\rm VII}}}{m^2_{(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\dot{2}}}}(d_L{s_R}^c+\mu_L{e_R}^c)\Big]v_{\rm EW}+H.c.\,,
\end{aligned}
\tag{45}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}\times\frac{d_{\mathscr{B}}}{M_{\rm pl}}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_1}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_2}\mathbf{70_H}^\dagger\Big(\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm i}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\Big)+H.c.\\
\supset\quad & Y_{\mathcal{D}}(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H},\dot\omega_1}\\
\times\quad & \frac{d_{\mathscr{B}}}{M_{\rm pl}}(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H},\dot\omega_1}^\dagger\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H},\dot\omega_2}^\dagger\otimes(\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}}^\dagger\otimes\langle\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm i}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\rangle+H.c.\\
\supset\quad & Y_{\mathcal{D}}d_{\mathscr{B}}\frac{w_{\overline{\mathbf{3}},\dot{\rm i}}w_{\overline{\mathbf{3}},\dot{\rm VII}}}{2M_{\rm pl}}(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{6}\,,\mathbf{3}\,,-\frac{1}{6})_{\mathbf{F}}\otimes(\overline{\mathbf{4}}\,,\overline{\mathbf{3}}\,,-\frac{1}{12})_{\mathbf{H},\dot\omega_1}\\
\times\quad & (\overline{\mathbf{4}}\,,\overline{\mathbf{3}}\,,-\frac{1}{12})_{\mathbf{H},\dot\omega_1}^\dagger\otimes\langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\dot\omega_2}^\dagger\rangle\otimes(\mathbf{4}\,,\overline{\mathbf{3}}\,,+\frac{5}{12})_{\mathbf{H}}^\dagger+H.c.\\
\supset\quad & Y_{\mathcal{D}}d_{\mathscr{B}}\frac{w_{\overline{\mathbf{3}},\dot{\rm i}}w_{\overline{\mathbf{3}},\dot{\rm VII}}w_{\overline{\mathbf{3}},\dot{\rm VII}}}{2\sqrt{2}M_{\rm pl}m^2_{(\overline{\mathbf{4}},\overline{\mathbf{3}}\,,-\frac{1}{12})_{\mathbf{H},\dot\omega_1}}}(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\dot\omega_1}\otimes(\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}}\otimes(\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{H}}^\dagger+H.c.\\
\Rightarrow\quad & \frac{Y_{\mathcal{D}}d_{\mathscr{B}}}{4}\dot\zeta_2\Big[\frac{w_{\overline{\mathbf{3}},\dot{\rm i}}w_{\overline{\mathbf{3}},\dot{\rm VII}}}{m^2_{(\overline{\mathbf{4}},\overline{\mathbf{3}}\,,-\frac{1}{12})_{\mathbf{H},\dot{\rm i}}}}(s_L{d_R}^c)+\frac{w_{\overline{\mathbf{3}},\dot{\rm i}}w_{\overline{\mathbf{3}},\dot{\rm VII}}}{m^2_{(\overline{\mathbf{4}},\overline{\mathbf{3}}\,,-\frac{1}{12})_{\mathbf{H},\dot{2}}}}(s_L{s_R}^c)\Big]v_{\rm EW}+H.c.\,,
\end{aligned}
\tag{46}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\dot\omega_1} \mathbf{56_F} \overline{\mathbf{28_H}}_{,\dot\omega_1} \times \frac{d_{\mathscr{B}}}{M_{\rm pl}} \overline{\mathbf{28_H}}^\dagger_{,\dot\omega_1} \overline{\mathbf{28_H}}^\dagger_{,\dot\omega_2} \mathbf{70_H}^\dagger \left(\overline{\mathbf{28_H}}^\dagger_{,\rm i} \overline{\mathbf{28_H}}_{,\dot{\rm VII}} \right) + H.c. \\
\supset\ & Y_{\mathcal{D}} (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1} \otimes (\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \otimes (\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H},\dot\omega_1} \\
\times\ & \frac{d_{\mathscr{B}}}{M_{\rm pl}} (\mathbf{1}, \mathbf{3}, -\frac{2}{3})^\dagger_{\mathbf{H},\dot\omega_1} \otimes (\overline{\mathbf{10}}, \mathbf{1}, +\frac{2}{5})^\dagger_{\mathbf{H},\dot\omega_2} \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})^\dagger_{\mathbf{H}} \otimes \langle \overline{\mathbf{28_H}}^\dagger_{,\rm i} \overline{\mathbf{28_H}}_{,\dot{\rm VII}} \rangle + H.c. \\
\supset\ & Y_{\mathcal{D}} d_{\mathscr{B}} \frac{w_{\overline{\mathbf{3}},\rm i} w_{\overline{\mathbf{3}},\dot{\rm VII}}}{2 M_{\rm pl}} (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot\omega_1} \otimes (\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \otimes (\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H},\dot\omega_1} \\
\times\ & (\mathbf{1}, \mathbf{3}, -\frac{2}{3})^\dagger_{\mathbf{H},\dot\omega_1} \otimes (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})^\dagger_{\mathbf{H},\dot\omega_2} \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})^\dagger_{\mathbf{H}} + H.c. \\
\supset\ & Y_{\mathcal{D}} d_{\mathscr{B}} \frac{w_{\overline{\mathbf{3}},\rm i} w_{\overline{\mathbf{3}},\dot{\rm VII}}}{2 M_{\rm pl} m^2_{(\mathbf{1},\mathbf{3},-\frac{2}{3})_{\mathbf{H},\dot\omega_1}}} (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\dot\omega_1} \otimes (\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})'^\dagger_{\mathbf{H},\dot\omega_2} \rangle \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})^\dagger_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{Y_{\mathcal{D}} d_{\mathscr{B}}}{4} \dot\zeta_3' \Big[\frac{w_{\overline{\mathbf{3}},\rm i} w_{\overline{\mathbf{3}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\mathbf{3},-\frac{2}{3})_{\mathbf{H},\rm i}}} (e_L \mu_R{}^c) + \frac{w_{\overline{\mathbf{3}},\rm i} w_{\overline{\mathbf{3}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\mathbf{3},-\frac{2}{3})_{\mathbf{H},\dot 2}}} (\mu_L \mu_R{}^c) \Big] v_{\rm EW} + H.c. \,,
\end{aligned} \tag{47}
$$

via the Higgs mixing operator of $\mathcal{O}_{\mathscr{B}}^{d=5}$ in Eq. (14b). For convenience, we parametrize the following ratios of

$$
\Delta_{\dot\omega} \equiv \frac{w_{\overline{\mathbf{3}},\rm i} w_{\overline{\mathbf{3}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{H},\dot\omega}}} \,, \quad \Delta'_{\dot\omega} \equiv \frac{w_{\overline{\mathbf{3}},\rm i} w_{\overline{\mathbf{3}},\dot{\rm VII}}}{m^2_{(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})_{\mathbf{H},\dot\omega}}} \,, \quad \Delta''_{\dot\omega} \equiv \frac{w_{\overline{\mathbf{3}},\rm i} w_{\overline{\mathbf{3}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\mathbf{3},-\frac{2}{3})_{\mathbf{H},\dot\omega}}} \,. \tag{48}
$$

With the Higgs VEVs assignments in Eq. (27), we can obtain the following propagator masses of

$$
\begin{aligned}
& m_{(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{H},\rm i}} \sim m_{(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})_{\mathbf{H},\rm i}} \sim m_{(\mathbf{1},\mathbf{3},-\frac{2}{3})_{\mathbf{H},\rm i}} \sim \mathcal{O}(v_{431}) \,, \\
& m_{(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{H},\dot 2}} \sim m_{(\overline{\mathbf{4}},\overline{\mathbf{3}},-\frac{1}{12})_{\mathbf{H},\dot 2}} \sim m_{(\mathbf{1},\mathbf{3},-\frac{2}{3})_{\mathbf{H},\dot 2}} \sim \mathcal{O}(v_{421}) \,.
\end{aligned} \tag{49}
$$

2.6 The SM quark/lepton masses and the benchmark

For all up-type quarks with $Q_e = +\frac{2}{3}$, we write down the following tree-level masses from both the renormalizable Yukawa couplings and the gravity-induced terms in the basis of $\mathcal{U} \equiv (u, c, t)$

$$
\mathcal{M}_{\mathcal{U}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & c_5 \zeta_1/\sqrt{2} \\ c_4 \dot\zeta_3'/\sqrt{2} & 0 & c_5 \zeta_3/\sqrt{2} \\ c_5 \zeta_1/\sqrt{2} & c_5 \zeta_2/\sqrt{2} & Y_{\mathcal{T}} \end{pmatrix} v_{\rm EW} \,. \tag{50}
$$

For all down-type quarks with $Q_e = -\frac{1}{3}$, we find the following tree-level mass matrix in the basis of $\mathcal{D} \equiv (d, s, b)$

$$
\mathcal{M}_{\mathcal{D}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\rm i} \dot\zeta_3 & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot 2} \dot\zeta_3 & 0 \\ Y_{\mathcal{D}} d_{\mathscr{B}} \Delta'_{\rm i} \dot\zeta_2 & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta'_{\dot 2} \dot\zeta_2 & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathscr{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\rm EW} \,. \tag{51}
$$

For all charged leptons with $Q_e = -1$, their tree-level mass matrix is expressed as follows:

$$
\mathcal{M}_{\mathcal{L}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\rm i} \dot\zeta_3 & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta''_{\rm i} \dot\zeta_3' & 0 \\ Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot 2} \dot\zeta_3 & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta''_{\dot 2} \dot\zeta_3' & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathscr{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\rm EW} \,, \tag{52}
$$

in the basis of $\mathcal{L} \equiv (e\,, \mu\,, \tau\,)$. Based on the above SM quark/lepton mass matrices, we find the following benchmark point of

$$v_{531} \simeq 1.4 \times 10^{17}\,\text{GeV}\,, \quad v_{431} \simeq 4.8 \times 10^{15}\,\text{GeV}\,, \quad v_{421} \simeq 1.1 \times 10^{15}\,\text{GeV}\,, \tag{53}$$

to reproduce the observed hierarchical masses as well as the CKM mixing pattern, as summarized in Table 10.

ζ_1	ζ_2	ζ_3	$Y_{\mathcal{D}}$	$Y_{\mathcal{B}}$	$Y_{\mathcal{T}}$
6.0×10^{-2}	2.0×10^{-3}	4.4×10^{-4}	0.5	0.5	1.0
$\lambda = \zeta_{23}$	c_4	c_5	$d_{\mathscr{A}}$	$d_{\mathscr{B}}$	m_μ
0.22	2.5	1.0	0.2	0.1	0.1
m_u	m_c	m_t	$m_d \approx m_e$	m_s	$m_b \approx m_\tau$
2.3×10^{-3}	0.3	174.2	0.7×10^{-3}	0.6×10^{-1}	1.7
$\lvert V_{ud}\rvert$	$\lvert V_{us}\rvert$	$\lvert V_{ub}\rvert$			
0.98	0.22	0.1×10^{-3}			
$\lvert V_{cd}\rvert$	$\lvert V_{cs}\rvert$	$\lvert V_{cb}\rvert$			
0.22	0.98	4.2×10^{-2}			
$\lvert V_{td}\rvert$	$\lvert V_{ts}\rvert$	$\lvert V_{tb}\rvert$			
9.4×10^{-2}	4.2×10^{-2}	1			

Table 10: The parameters of the $\mathfrak{su}(8)$ benchmark point and the predicted SM quark/lepton masses (in unit of GeV) as well as the CKM mixings.

3 The intermediate stages of the WSS symmetry breaking pattern

Schematically, we assign the Higgs VEVs according to the symmetry breaking pattern as follows:

$$\mathfrak{g}_{531} \to \mathfrak{g}_{521} \quad : \quad \langle (\mathbf{1}\,, \overline{\mathbf{3}}\,, -\frac{1}{3})_{\mathbf{H}\,,\mathrm{IV}} \rangle \equiv \frac{1}{\sqrt{2}} W_{\overline{\mathbf{3}}\,,\mathrm{IV}}\,, \tag{54a}$$

$$\mathfrak{g}_{521} \to \mathfrak{g}_{421} \quad : \quad \langle (\overline{\mathbf{5}}\,, \mathbf{1}\,, +\frac{1}{5})_{\mathbf{H}\,,\mathrm{V}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\overline{\mathbf{5}}\,,\mathrm{V}}\,, \ \langle (\overline{\mathbf{5}}\,, \mathbf{1}\,, +\frac{1}{5})_{\mathbf{H}\,,\mathrm{i}\,,\dot{\mathrm{V}}\mathrm{II}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\overline{\mathbf{5}}\,,\mathrm{i}\,,\dot{\mathrm{V}}\mathrm{II}}\,, \tag{54b}$$

$$\begin{aligned} \mathfrak{g}_{421} \to \mathfrak{g}_{\mathrm{SM}} \quad : \quad & \langle (\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,3\,,\mathrm{VI}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\overline{\mathbf{4}}\,,3\,,\mathrm{VI}}\,, \\ & \langle (\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\dot{\mathrm{V}}\mathrm{III}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\overline{\mathbf{4}}\,,\dot{\mathrm{V}}\mathrm{III}}\,, \ \langle (\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})'_{\mathbf{H}\,,\dot{2}\,,\dot{\mathrm{I}}\mathrm{X}} \rangle \equiv \frac{1}{\sqrt{2}} V'_{\overline{\mathbf{4}}\,,\dot{2}\,,\dot{\mathrm{I}}\mathrm{X}}\,, \end{aligned} \tag{54c}$$

$$\mathrm{EWSB} \quad : \quad \langle (\mathbf{1}\,, \overline{\mathbf{2}}\,, +\frac{1}{2})_{\mathbf{H}} \rangle \equiv \frac{1}{\sqrt{2}} v_{\mathrm{EW}}\,. \tag{54d}$$

For later convenience, we also parametrize different symmetry breaking VEVs in terms of the following dimensionless quantities:

$$\zeta_1 \equiv \frac{W_{\overline{\mathbf{3}}\,,\mathrm{IV}}}{M_{\rm pl}}\,, \quad \zeta_2 \equiv \frac{w_{\overline{\mathbf{5}}\,,\mathrm{V}}}{M_{\rm pl}}\,, \quad \dot\zeta_2 \equiv \frac{w_{\overline{\mathbf{5}}\,,\dot 1,\dot{\mathrm{V}}\mathrm{II}}}{M_{\rm pl}}\,,$$
$$\zeta_3 \equiv \frac{V_{\overline{\mathbf{4}}\,,3,\mathrm{VI}}}{M_{\rm pl}}\,, \quad \dot\zeta_3' \equiv \frac{V'_{\overline{\mathbf{4}}\,,\dot 2,\mathrm{I}\dot{\mathrm{X}}}}{M_{\rm pl}}\,, \quad \dot\zeta_3 \equiv \frac{V_{\overline{\mathbf{4}}\,,\mathrm{V}\dot{\mathrm{I}}\mathrm{II}}}{M_{\rm pl}}\,,$$
$$\zeta_0 \gg \zeta_1 \gg \zeta_2 \sim \dot\zeta_2 \gg \zeta_3 \sim \dot\zeta_3' \sim \dot\zeta_3\,, \quad \zeta_{ij} \equiv \frac{\zeta_j}{\zeta_i}\,, \quad (i<j)\,. \tag{55}$$

In Table 11, we summarize all vectorlike fermions that become massive during different stages of the WSS symmetry breaking pattern.

stages	$Q_e=-\frac{1}{3}$	$Q_e=+\frac{2}{3}$	$Q_e=-1$	$Q_e=0$
v_{531} $\{\Omega\}$	$\mathfrak{D}$ IV	-	$(\mathfrak{e}\,,\mathfrak{n})$ IV	$\{\check{\mathfrak{n}}\,,\check{\mathfrak{n}}'\}$ $\{\mathrm{IV}\,,\mathrm{IV}'\}$
v_{521} $\{\Omega\}$	$\{\mathfrak{D}'\,,\mathfrak{D}''''\}$ $\{\mathrm{V}\,,\dot{\mathrm{V}}\mathrm{II}\}$	$\mathfrak{U}$	$\mathfrak{E}\,,(\mathfrak{e}'\,,\mathfrak{n}')\,,(\mathfrak{e}'''\,,\mathfrak{n}''')$ $\{\mathrm{V}\,,\dot{\mathrm{V}}\mathrm{II}\}$	$\{\check{\mathfrak{n}}''\,,\check{\mathfrak{n}}'''\}$ $\{\mathrm{V}\,,\dot{\mathrm{V}}\mathrm{II}\}$
v_{421} $\{\Omega\}$	$\mathfrak{d}\,,\{\mathfrak{D}''\,,\mathfrak{D}'''\,,\mathfrak{D}'''''\}$ $\{\mathrm{VI}\,,\mathrm{I}\dot{\mathrm{X}}\,,\mathrm{V}\dot{\mathrm{I}}\mathrm{II}\}$	$\mathfrak{u}$	$(\mathfrak{e}''\,,\mathfrak{n}'')\,,(\mathfrak{e}''''\,,\mathfrak{n}'''')\,,(\mathfrak{e}'''''\,,\mathfrak{n}''''')$ $\{\mathrm{VI}\,,\mathrm{V}\dot{\mathrm{I}}\mathrm{II}\,,\mathrm{I}\dot{\mathrm{X}}\}$	-

Table 11: The vectorlike fermions at different intermediate symmetry breaking stages along the WSS symmetry breaking pattern of the $\mathfrak{su}(8)$ theory.

3.1 The first stage

$\overline{\mathbf{8_F}}^\Omega$	$(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^\Omega$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^\Omega$		
$\mathcal{T}'$	$-\frac{12}{5}t$	$-4t$		
$\mathbf{28_F}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}$	
$\mathcal{T}'$	$+4t$	$+\frac{12}{5}t$	$+\frac{4}{5}t$	
$\mathbf{56_F}$	$(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}'$	$(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}}$	$(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}$	$(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}$
$\mathcal{T}'$	$+4t$	$+\frac{12}{5}t$	$-\frac{4}{5}t$	$+\frac{4}{5}t$
$\overline{\mathbf{8_H}}_{\,,\omega}$	$(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega}$		
$\mathcal{T}'$	$+\frac{8}{5}t$	0		
$\overline{\mathbf{28_H}}_{\,,\dot\omega}$	$(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H}\,,\dot\omega}$	
$\mathcal{T}'$	0	$+\frac{8}{5}t$	$+\frac{16}{5}t$	
$\mathbf{70_H}$	$(\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}}$			
$\mathcal{T}'$	$-\frac{16}{5}t$			

Table 12: The $\widetilde{\mathrm{U}}(1)_{T'}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{531}$ theory.

The first symmetry breaking stage of $\mathfrak{g}_{531} \to \mathfrak{g}_{521}$ is achieved by $(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{IV}} \subset \overline{\mathbf{8_H}}_{\,,\mathrm{IV}}$ in the rank-two sector, according to the $\widetilde{\mathrm{U}}(1)_{\mathcal{T}'}$-neutral components in Table 12. Accordingly, the term of $Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{IV}}\mathbf{28_F}\overline{\mathbf{8_H}}_{\,,\mathrm{IV}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}\,,\check{\mathfrak{n}}'\,,\mathfrak{e}\,,\mathfrak{n}\,,\check{\mathfrak{n}})$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{IV}}\mathbf{28_F}\overline{\mathbf{8_H}}_{\,,\mathrm{IV}} + H.c. \\
\supset\ & Y_{\mathcal{B}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{IV}} \otimes (\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{IV}} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}\Big] \otimes \langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{IV}}\rangle + H.c. \\
\Rightarrow\ & \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{D}_L\mathfrak{D}_R^c + \check{\mathfrak{n}}_L\check{\mathfrak{n}}_R^c - \mathfrak{e}_L\mathfrak{e}_R^c + \mathfrak{n}_L\mathfrak{n}_R^c + \check{\mathfrak{n}}'_L\check{\mathfrak{n}}'^c_R\right) W_{\overline{\mathbf{3}},\mathrm{IV}} + H.c.\,. \qquad (56)
\end{aligned}
$$

After this stage, the remaining massless fermions expressed in terms of the $\mathfrak{g}_{521}$ IRs are the following:

$$
\begin{aligned}
& (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\Omega} \oplus \Big[(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''} \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\Omega}\Big] \subset \overline{\mathbf{8_F}}^{\Omega}\,, \\
& \Omega = (\omega\,,\dot\omega)\,, \quad \omega = (3\,,\mathrm{V}\,,\mathrm{VI})\,, \quad \dot\omega = (\dot{1}\,,\dot{2}\,,\dot{\mathrm{VII}}\,,\dot{\mathrm{VIII}}\,,\dot{\mathrm{IX}})\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{IV}''} \subset \overline{\mathbf{8_F}}^{\mathrm{IV}}\,, \\
& \Big[\cancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}} \oplus (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}\Big] \oplus \Big[\cancel{(\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{F}}} \oplus (\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}}\Big] \oplus (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}} \subset \mathbf{28_F}\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,+1)'_{\mathbf{F}} \oplus \Big[(\mathbf{5}\,,\mathbf{1}\,,+\frac{4}{5})_{\mathbf{F}} \oplus (\mathbf{5}\,,\overline{\mathbf{2}}\,,+\frac{3}{10})_{\mathbf{F}}\Big] \oplus (\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}} \oplus \Big[(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})'_{\mathbf{F}} \\
\oplus\ & (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}}\Big] \subset \mathbf{56_F}\,. \qquad (57)
\end{aligned}
$$

3.2 The second stage

$\overline{\mathbf{8_F}}^{\Omega}$	$(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''}$	
$\mathcal{T}''$	$-4t$	0	$-4t$	
$\mathbf{28_F}$	$(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}$	$(\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}}$	$(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}$	
$\mathcal{T}''$	$-4t$	0	$+4t$	
$\mathbf{56_F}$	$(\mathbf{1}\,,\mathbf{1}\,,+1)'_{\mathbf{F}}$	$(\mathbf{5}\,,\mathbf{1}\,,+\frac{4}{5})_{\mathbf{F}}$	$(\mathbf{5}\,,\overline{\mathbf{2}}\,,+\frac{3}{10})_{\mathbf{F}}$	
$\mathcal{T}''$	$-4t$	$-4t$	0	
	$(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}$	$(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})'_{\mathbf{F}}$	$(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}}$	
$\mathcal{T}''$	$+4t$	$+4t$	0	
$\overline{\mathbf{8_H}}_{\,,\omega}$	$(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega}$	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\omega}$		
$\mathcal{T}''$	0	$+4t$		
$\overline{\mathbf{28_H}}_{\,,\dot\omega}$	$(\mathbf{1}\,,\mathbf{2}\,,-\frac{1}{2})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{5}}\,,\overline{\mathbf{2}}\,,-\frac{3}{10})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H}\,,\dot\omega}$
$\mathcal{T}''$	$+4t$	$+4t$	0	0
$\mathbf{70_H}$	$(\mathbf{10}\,,\overline{\mathbf{2}}\,,+\frac{1}{10})_{\mathbf{H}}$			
$\mathcal{T}''$	$-4t$			

Table 13: The $\widetilde{\mathrm{U}}(1)_{\mathcal{T}''}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{521}$ theory.

The second symmetry breaking stage of $\mathfrak{g}_{521} \to \mathfrak{g}_{421}$ is achieved by $(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\mathrm{V}} \subset \overline{\mathbf{8_H}}_{\,,\mathrm{V}}$ in the rank-two sector, $(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\dot{\mathrm{V}}\mathrm{II}} \subset \overline{\mathbf{28_H}}_{\,,\dot{\mathrm{V}}\mathrm{II}}$ and $(\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{H}}^{\dot{\mathrm{V}}\mathrm{II}} \subset \mathbf{28_H}^{\dot{\mathrm{V}}\mathrm{II}}$ in the rank-three sector, according to the $\widetilde{\mathrm{U}}(1)_{T''}$-neutral components in Table 13. The term of $Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\mathrm{V}} \mathbf{28_F} \overline{\mathbf{8_H}}_{\,,\mathrm{V}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}'\,,\mathfrak{e}'\,,\mathfrak{n}'\,,\check{\mathfrak{n}}'')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\mathrm{V}} \mathbf{28_F} \overline{\mathbf{8_H}}_{\,,\mathrm{V}} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{V}} \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{V}} \otimes (\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}} \Big] \otimes (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\mathrm{V}} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{V}} \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\mathrm{V}} \otimes (\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}} \\
\oplus\ & (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V}''} \otimes (\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{F}} \Big] \otimes \langle (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\mathrm{V}} \rangle + H.c. \\
\Rightarrow\ & \frac{1}{\sqrt{2}} Y_{\mathcal{B}} \left(\mathfrak{D}_L' \mathfrak{D}_R'^{c} + \check{\mathfrak{n}}_L'' \check{\mathfrak{n}}_R''^{c} + \mathfrak{e}_L' \mathfrak{e}_R'^{\,c} - \mathfrak{n}_L' \mathfrak{n}_R'^{\,c} + \check{\mathcal{N}}_L^{\mathrm{V}''} \check{\mathfrak{n}}_R'^{c} \right) w_{\overline{\mathbf{5}},\mathrm{V}} + H.c.\,.
\end{aligned} \tag{58}
$$

The term of $Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\dot{\mathrm{V}}\mathrm{II}} \mathbf{56_F} \overline{\mathbf{28_H}}_{\,,\dot{\mathrm{V}}\mathrm{II}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}''''\,,\mathfrak{e}'''\,,\mathfrak{n}'''\,,\check{\mathfrak{n}}''')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\dot{\mathrm{V}}\mathrm{II}} \mathbf{56_F} \overline{\mathbf{28_H}}_{\,,\dot{\mathrm{V}}\mathrm{II}} + H.c. \\
\supset\ & Y_{\mathcal{D}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}} \otimes (\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}} \otimes (\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}} \Big] \otimes (\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H}\,,\dot{\mathrm{V}}\mathrm{II}} + H.c. \\
\supset\ & Y_{\mathcal{D}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}} \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}' \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}} \otimes (\mathbf{5}\,,\overline{\mathbf{2}}\,,+\frac{3}{10})_{\mathbf{F}} \Big] \otimes \langle (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\dot{\mathrm{V}}\mathrm{II}} \rangle + H.c. \\
\Rightarrow\ & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} \left(\mathfrak{D}_L'''' \mathfrak{D}_R''''^{c} - \mathfrak{e}_L''' \mathfrak{e}_R'''^{c} + \mathfrak{n}_L''' \mathfrak{n}_R'''^{c} + \check{\mathfrak{n}}_L''' \check{\mathfrak{n}}_R'''^{\,c} \right) w_{\overline{\mathbf{5}},\dot{\mathrm{V}}\mathrm{II}} + H.c.\,.
\end{aligned} \tag{59}
$$

The Yukawa coupling between two $\mathbf{56_F}$'s by the following $d=5$ operator

$$
\begin{aligned}
& \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\dot{\mathrm{V}}\mathrm{II}} + H.c. \\
\supset\ & c_4 \frac{v_U}{M_{\rm pl}} \Big[(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}} \oplus \cancel{(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}} \Big] \\
\otimes\ & (\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{H}}^{\dot{\mathrm{V}}\mathrm{II}} + H.c. \\
\supset\ & c_4 \zeta_0 (\mathbf{5}\,,\mathbf{1}\,,+\frac{4}{5})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}} \otimes \langle (\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{H}}^{\dot{\mathrm{V}}\mathrm{II}} \rangle + H.c. \\
\Rightarrow\ & \frac{c_4}{\sqrt{2}} \zeta_0 \left(\mathfrak{E}_L \mathfrak{E}_R{}^{c} + \mathfrak{U}_L \mathfrak{U}_R{}^{c} \right) w_{\mathbf{5}}^{\dot{\mathrm{V}}\mathrm{II}} + H.c.\,,
\end{aligned} \tag{60}
$$

leads to massive vectorlike fermions of $(\mathfrak{E}\,,\mathfrak{U})$.

After integrating out the massive fermions, the remaining massless fermions expressed in terms of the $\mathfrak{g}_{421}$ IRs are the following:

$$
\begin{aligned}
& \Big[(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\Omega'} \Big] \oplus (\mathbf{1}\,, \overline{\mathbf{2}}\,, -\frac{1}{2})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\Omega''} \subset \overline{\mathbf{8}_{\mathbf{F}}}^{\Omega}\,, \\
& \Omega = (\omega\,, \dot\omega)\,, \quad \omega = (3\,, \mathrm{VI})\,, \quad \dot\omega = (\dot{1}\,, \dot{2}\,, \mathrm{V\dot{I}II}\,, \mathrm{I\dot{X}})\,, \\
& (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\mathrm{IV}''} \subset \overline{\mathbf{8}_{\mathbf{F}}}^{\mathrm{IV}}\,, \quad (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\mathrm{V}'} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\mathrm{V}''} \subset \overline{\mathbf{8}_{\mathbf{F}}}^{\mathrm{V}}\,, \\
& (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\mathrm{V\dot{I}I}'} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\mathrm{V\dot{I}I}''} \subset \overline{\mathbf{8}_{\mathbf{F}}}^{\mathrm{V\dot{I}I}}\,, \\
& \cancel{(\mathbf{1}\,, \overline{\mathbf{2}}\,, +\frac{1}{2})_{\mathbf{F}}} \oplus (\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}} \oplus \Big[\cancel{(\mathbf{4}\,, \mathbf{1}\,, -\frac{1}{4})_{\mathbf{F}} \oplus (\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}'} \Big] \oplus \Big[\cancel{(\mathbf{1}\,, \mathbf{2}\,, +\frac{1}{2})_{\mathbf{F}}{}'} \oplus (\mathbf{4}\,, \mathbf{2}\,, +\frac{1}{4})_{\mathbf{F}} \Big] \\
\oplus \quad & \Big[\cancel{(\mathbf{4}\,, \mathbf{1}\,, -\frac{1}{4})_{\mathbf{F}}'} \oplus (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}} \Big] \subset \mathbf{28}_{\mathbf{F}}\,, \\
& (\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}' \oplus \Big[(\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}'' \oplus \cancel{(\mathbf{4}\,, \mathbf{1}\,, +\frac{3}{4})_{\mathbf{F}}} \Big] \oplus \Big[\cancel{(\mathbf{1}\,, \overline{\mathbf{2}}\,, +\frac{1}{2})_{\mathbf{F}}'''} \oplus (\mathbf{4}\,, \overline{\mathbf{2}}\,, +\frac{1}{4})_{\mathbf{F}} \Big] \oplus \Big[\cancel{(\overline{\mathbf{4}}\,, \mathbf{1}\,, -\frac{3}{4})_{\mathbf{F}}} \\
\oplus \quad & (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}' \Big] \oplus \Big[\cancel{(\mathbf{4}\,, \mathbf{1}\,, -\frac{1}{4})_{\mathbf{F}}''} \oplus (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}'' \Big] \oplus \Big[(\mathbf{4}\,, \mathbf{2}\,, +\frac{1}{4})_{\mathbf{F}}' \oplus (\mathbf{6}\,, \mathbf{2}\,, 0)_{\mathbf{F}} \Big] \subset \mathbf{56}_{\mathbf{F}}\,. \qquad (61)
\end{aligned}
$$

3.3 The third stage

$\overline{\mathbf{8}_{\mathbf{F}}}^{\Omega}$	$(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\Omega'}$	$(\mathbf{1}\,, \overline{\mathbf{2}}\,, -\frac{1}{2})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,, \mathbf{1}\,, 0)_{\mathbf{F}}^{\Omega''}$
$\mathcal{T}'''$	$-4t$	$-4t$	0	$-4t$
$\mathbf{28}_{\mathbf{F}}$	$(\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}$	$(\mathbf{4}\,, \mathbf{2}\,, +\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}$	
$\mathcal{T}'''$	$-4t$	0	$+4t$	
$\mathbf{56}_{\mathbf{F}}$	$(\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}'$	$(\mathbf{1}\,, \mathbf{1}\,, +1)_{\mathbf{F}}''$	$(\mathbf{4}\,, \overline{\mathbf{2}}\,, +\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}'$
$\mathcal{T}'''$	$-4t$	$-4t$	0	$+4t$
	$(\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}''$	$(\mathbf{4}\,, \mathbf{2}\,, +\frac{1}{4})_{\mathbf{F}}'$	$(\mathbf{6}\,, \mathbf{2}\,, 0)_{\mathbf{F}}$	
$\mathcal{T}'''$	$+4t$	0	0	
$\overline{\mathbf{8}_{\mathbf{H}}}_{\,,\omega}$	$(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\omega}$	$(\mathbf{1}\,, \overline{\mathbf{2}}\,, -\frac{1}{2})_{\mathbf{H}\,,\omega}$		
$\mathcal{T}'''$	0	$+4t$		
$\overline{\mathbf{28}_{\mathbf{H}}}_{\,,\dot\omega}$	$(\mathbf{1}\,, \mathbf{2}\,, -\frac{1}{2})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{4}}\,, \overline{\mathbf{2}}\,, -\frac{1}{4})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\dot\omega}$	$(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\dot\omega}'$
$\mathcal{T}'''$	$+4t$	$+4t$	0	0
	$(\mathbf{1}\,, \overline{\mathbf{2}}\,, -\frac{1}{2})_{\mathbf{H}\,,\dot\omega}'$			
$\mathcal{T}'''$	$+4t$			
$\mathbf{70}_{\mathbf{H}}$	$(\mathbf{4}\,, \overline{\mathbf{2}}\,, +\frac{1}{4})_{\mathbf{H}}$			
$\mathcal{T}'''$	$-4t$			

Table 14: The $\widetilde{\mathrm{U}}(1)_{\mathcal{T}'''}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{421}$ theory.

The third symmetry breaking stage of $\mathfrak{g}_{421} \to \mathfrak{g}_{\mathrm{SM}}$ is achieved by $(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\mathrm{VI}} \subset \overline{\mathbf{8}_{\mathbf{H}}}_{\,,\mathrm{VI}}$ in the rank-two sector and $(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\mathrm{V\dot{I}II}} \subset \overline{\mathbf{28}_{\mathbf{H}}}_{\,,\mathrm{V\dot{I}II}}$, $(\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\mathrm{I\dot{X}}}' \subset \overline{\mathbf{28}_{\mathbf{H}}}_{\,,\mathrm{I\dot{X}}}$, $(\mathbf{4}\,, \mathbf{1}\,, -\frac{1}{4})_{\mathbf{H}}^{\mathrm{I\dot{X}}'} \subset \mathbf{28}_{\mathbf{H}}{}^{\mathrm{I\dot{X}}}$

in the rank-three sector, according to the $\widetilde{\mathrm{U}}(1)_{T'''}$-neutral components in Table 14. The term of $Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{VI}}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\mathrm{VI}}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}''\,,\mathfrak{e}''\,,\mathfrak{n}'')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{VI}}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\mathrm{VI}}+H.c. \\
\supset\quad & Y_{\mathcal{B}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H},\mathrm{VI}}+H.c. \\
\supset\quad & Y_{\mathcal{B}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}} \\
\oplus\quad & (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{VI}''}\otimes(\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H},\mathrm{VI}}+H.c. \\
\supset\quad & Y_{\mathcal{B}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{VI}'}\otimes(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}' \\
\oplus\quad & (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{VI}''}\otimes(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}\Big]\otimes\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\mathrm{VI}}\rangle+H.c. \\
\Rightarrow\quad & \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{e}_L''\mathfrak{e}_R''^{\,c}-\mathfrak{n}_L''\mathfrak{n}_R''^{\,c}+\check{\mathcal{N}}_L^{\mathrm{VI}'}\check{\mathfrak{n}}_R''^{\,c}+\check{\mathcal{N}}_L^{\mathrm{VI}''}\check{\mathfrak{n}}_R^{\,c}+\mathfrak{D}_L''\mathfrak{D}_R''^{\,c}\right)V_{\overline{\mathbf{4}},\mathrm{VI}}+H.c.\,.
\end{aligned}
\tag{62}
$$

The term of $Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\mathrm{V\dot{I}II},\mathrm{\dot{I}X}}\mathbf{56_F}\overline{\mathbf{28}}_{\mathbf{H},\mathrm{V\dot{I}II},\mathrm{\dot{I}X}}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}'''''\,,\mathfrak{e}''''\,,\mathfrak{n}''''\,,\mathfrak{D}'''\,,\mathfrak{e}'''''\,,\mathfrak{n}''''')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\mathrm{V\dot{I}II}}\mathbf{56_F}\overline{\mathbf{28}}_{\mathbf{H},\mathrm{V\dot{I}II}}+H.c. \\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H},\mathrm{V\dot{I}II}}+H.c. \\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{5}\,,\overline{\mathbf{2}}\,,+\frac{3}{10})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H},\mathrm{V\dot{I}II}}+H.c. \\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}''\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V\dot{I}II}'}\otimes(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}''\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{F}}\Big] \\
\otimes\quad & \langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\mathrm{V\dot{I}II}}\rangle+H.c. \\
\Rightarrow\quad & \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}_L'''''\mathfrak{D}_R'''''^{\,c}+\check{\mathcal{N}}_L^{\mathrm{V\dot{I}II}'}\check{\mathfrak{n}}_R'''^{\,c}-\mathfrak{e}_L''''\mathfrak{e}_R''''^{c}+\mathfrak{n}_L''''\mathfrak{n}_R''''^{c}\right)V_{\overline{\mathbf{4}},\mathrm{V\dot{I}II}}+H.c.\,,
\end{aligned}
\tag{63}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\mathrm{\dot{I}X}}\mathbf{56_F}\overline{\mathbf{28}}_{\mathbf{H},\mathrm{\dot{I}X}}+H.c. \\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H},\mathrm{\dot{I}X}}+H.c. \\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{\dot{I}X}''}\otimes(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}'\Big] \\
\otimes\quad & (\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H},\mathrm{\dot{I}X}}+H.c. \\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{\dot{I}X}''}\otimes(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}''\Big] \\
\otimes\quad & \langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\mathrm{\dot{I}X}}'\rangle+H.c. \\
\Rightarrow\quad & \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}_L'''\mathfrak{D}_R'''^{\,c}+\check{\mathcal{N}}_L^{\mathrm{\dot{I}X}''}\check{\mathfrak{n}}_R'''^{\,c}+\mathfrak{e}_L'''''\mathfrak{e}_R'''''^{c}-\mathfrak{n}_L'''''\mathfrak{n}_R'''''^{c}\right)V_{\overline{\mathbf{4}},\mathrm{\dot{I}X}}'+H.c.\,.
\end{aligned}
\tag{64}
$$

The Yukawa coupling between two $\mathbf{56_F}$'s by the following $d=5$ operator

$$
\begin{aligned}
& \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\dot{\rm IX}} + H.c. \\
\supset\quad & c_4 \frac{\upsilon_U}{M_{\rm pl}} \Big[(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime} \otimes (\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}} \Big] \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{H}}^{\dot{\rm IX}} + H.c. \\
\supset\quad & c_4 \zeta_0 \Big[(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime} \otimes (\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{5}\,,\mathbf{1}\,,+\frac{4}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}^{\prime} \oplus (\mathbf{5}\,,\overline{\mathbf{2}}\,,+\frac{3}{10})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}} \Big] \\
\otimes\quad & (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{H}}^{\dot{\rm IX}} + H.c. \\
\supset\quad & c_4 \frac{\upsilon_U}{M_{\rm pl}} \Big[(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime} \otimes (\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{4}\,,\mathbf{1}\,,+\frac{3}{4})_{\mathbf{F}} \otimes (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}^{\prime\prime} \oplus (\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{F}} \\
\otimes\quad & (\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}} \Big] \otimes \langle (\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{H}}^{\dot{\rm IX}^\prime} \rangle + H.c. \\
\Rightarrow\quad & \frac{c_4}{\sqrt{2}} \zeta_0 \, (\mathfrak{E}_L {\mu_R}^c + \mathfrak{U}_L {c_R}^c + \mathfrak{u}_L {\mathfrak{u}_R}^c - \mathfrak{d}_L \mathfrak{d}_R^c) \, V_{\mathbf{4}}^{\dot{\rm IX}^\prime} + H.c. \,,
\end{aligned} \tag{65}
$$

further leads to massive vectorlike fermions of $(\mathfrak{u}\,,\mathfrak{d})$.

The remaining massless fermions of the $\mathfrak{g}_{\rm SM}$ are listed as follows:

$$
\begin{aligned}
& \Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega} \Big] \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega^\prime} \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega^{\prime\prime}} \subset \overline{\mathbf{8_F}}^{\Omega} \,, \quad \Omega = (\dot 1\,,\dot 2\,,3) \,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm VI} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\dot{\rm VIII}} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\dot{\rm IX}} \subset \overline{\mathbf{8_F}}^{\Omega} \,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm V^\prime} \oplus ... \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\dot{\rm IX}^\prime} \subset \overline{\mathbf{8_F}}^{\Omega^\prime} \,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm IV^{\prime\prime}} \oplus ... \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\dot{\rm IX}^{\prime\prime}} \subset \overline{\mathbf{8_F}}^{\Omega^{\prime\prime}} \,, \\
& \cancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}} \oplus (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}} \oplus \cancel{\Big[(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}} \Big]} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\prime} \\
\oplus\quad & \cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime}} \oplus \Big[\xcancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime\prime}} \oplus (\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}} \Big] \oplus \cancel{\Big[(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}^{\prime} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\prime\prime} \Big]} \\
\oplus\quad & \Big[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}^{\prime\prime}} \oplus (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}} \Big] \subset \mathbf{28_F} \,, \\
& (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime} \oplus (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime\prime} \oplus \cancel{\Big[(\mathbf{3}\,,\mathbf{1}\,,+\frac{2}{3})_{\mathbf{F}} \oplus (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}^{\prime\prime\prime} \Big]} \oplus \cancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime\prime\prime}} \\
\oplus\quad & \Big[\xcancel{(\mathbf{3}\,,\overline{\mathbf{2}}\,,+\frac{1}{6})_{\mathbf{F}}} \oplus \xcancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime\prime\prime\prime}} \Big] \oplus \cancel{\Big[(\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}} \oplus (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}^{\prime} \Big]} \oplus \Big[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime}} \oplus (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}^{\prime\prime} \Big] \\
\oplus\quad & \cancel{\Big[(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime\prime} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\prime\prime\prime} \Big]} \oplus \Big[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime\prime\prime}} \oplus (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}^{\prime\prime\prime} \Big] \oplus \Big[(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}^{\prime} \oplus \xcancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime\prime\prime\prime\prime}} \Big] \\
\oplus\quad & \Big[(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}^{\prime\prime} \oplus \xcancel{(\overline{\mathbf{3}}\,,\mathbf{2}\,,-\frac{1}{6})_{\mathbf{F}}} \Big] \subset \mathbf{56_F} \,.
\end{aligned} \tag{66}
$$

3.4 The $d=5$ bi-linear fermion operators

For the operator of $\mathcal{O}_{\mathcal{F}}^{(4\,,1)}$ in Eq. (13a), it is decomposed as

$$
\begin{aligned}
& \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \cdot \overline{\mathbf{28_H}}_{\,,\dot\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\ & \frac{c_4}{M_{\rm pl}} (\overline{\mathbf{10}}\,, \mathbf{1}\,, -\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,, \mathbf{3}\,, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}\,, \mathbf{1}\,, +\frac{2}{5})_{\mathbf{H}\,,\dot\omega} \otimes (\mathbf{10}\,, \overline{\mathbf{3}}\,, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_4}{M_{\rm pl}} (\overline{\mathbf{10}}\,, \mathbf{1}\,, -\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,, \mathbf{2}\,, +\frac{1}{10})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}\,, \mathbf{1}\,, +\frac{2}{5})_{\mathbf{H}\,,\dot\omega} \otimes (\mathbf{10}\,, \overline{\mathbf{2}}\,, +\frac{1}{10})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_4}{M_{\rm pl}} (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}' \otimes (\mathbf{6}\,, \mathbf{2}\,, 0)_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\dot\omega}' \rangle \otimes (\mathbf{4}\,, \overline{\mathbf{2}}\,, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{c_4}{2} \dot\zeta_3' \, (c_L {u_R}^c) \, v_{\rm EW} + H.c. \,.
\end{aligned} \tag{67}
$$

For the operator of $\mathcal{O}_{\mathcal{F}}^{(5\,,1)}$ in Eq. (13b), it is decomposed as

$$
\begin{aligned}
& \frac{c_5}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{\,,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} \Big[(\mathbf{5}\,, \mathbf{3}\,, +\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}\,, \mathbf{1}\,, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}\,, \mathbf{1}\,, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,, \mathbf{3}\,, -\frac{1}{15})_{\mathbf{F}} \Big] \otimes (\overline{\mathbf{5}}\,, \mathbf{1}\,, +\frac{1}{5})_{\mathbf{H}\,,\omega} \\
\otimes\ & (\mathbf{10}\,, \overline{\mathbf{3}}\,, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} \Big[(\mathbf{5}\,, \mathbf{2}\,, +\frac{3}{10})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}\,, \mathbf{1}\,, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}\,, \mathbf{1}\,, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,, \mathbf{2}\,, +\frac{1}{10})_{\mathbf{F}} \Big] \otimes \langle (\overline{\mathbf{5}}\,, \mathbf{1}\,, +\frac{1}{5})_{\mathbf{H}\,,\omega} \rangle \\
\otimes\ & (\mathbf{10}\,, \overline{\mathbf{2}}\,, +\frac{1}{10})_{\mathbf{H}} + H.c. \\
\supset\ & c_5 \frac{w_{\overline{\mathbf{5}}\,,\rm V}}{\sqrt{2} M_{\rm pl}} \Big[(\mathbf{4}\,, \mathbf{2}\,, +\frac{1}{4})_{\mathbf{F}} \otimes (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}}' \oplus (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{4}\,, \mathbf{2}\,, +\frac{1}{4})_{\mathbf{F}}' \Big] \otimes (\mathbf{4}\,, \overline{\mathbf{2}}\,, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{c_5}{2} \zeta_2 \, (t_L {u_R}^c + u_L {t_R}^c) \, v_{\rm EW} + H.c. \,,
\end{aligned} \tag{68}
$$

and

$$
\begin{aligned}
& \frac{c_5}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{\,,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} (\mathbf{10}\,, \mathbf{1}\,, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,, \mathbf{3}\,, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{5}}\,, \mathbf{1}\,, +\frac{1}{5})_{\mathbf{H}\,,\omega} \otimes (\mathbf{10}\,, \overline{\mathbf{3}}\,, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} (\mathbf{10}\,, \mathbf{1}\,, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,, \mathbf{2}\,, +\frac{1}{10})_{\mathbf{F}} \otimes (\overline{\mathbf{5}}\,, \mathbf{1}\,, +\frac{1}{5})_{\mathbf{H}\,,\omega} \otimes (\mathbf{10}\,, \overline{\mathbf{2}}\,, +\frac{1}{10})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} (\mathbf{6}\,, \mathbf{1}\,, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{6}\,, \mathbf{2}\,, 0)_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}\,, \mathbf{1}\,, +\frac{1}{4})_{\mathbf{H}\,,\omega} \rangle \otimes (\mathbf{4}\,, \overline{\mathbf{2}}\,, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{c_5}{2} \zeta_3 \, (c_L {t_R}^c) \, v_{\rm EW} + H.c. \,,
\end{aligned} \tag{69}
$$

and

$$
\begin{aligned}
& \frac{c_5}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\quad & \frac{c_5}{M_{\rm pl}} (\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}} \otimes \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega} \rangle \otimes (\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset\quad & c_5 \frac{W_{\overline{\mathbf{3}},\rm IV}}{\sqrt{2} M_{\rm pl}} (\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}^{\prime} \otimes (\mathbf{10}\,,\overline{\mathbf{2}}\,,+\frac{1}{10})_{\mathbf{H}} + H.c. \\
\supset\quad & c_5 \frac{W_{\overline{\mathbf{3}},\rm IV}}{\sqrt{2} M_{\rm pl}} (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}} \otimes (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}^{\prime\prime} \otimes (\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow\quad & \frac{c_5}{2} \zeta_1 \, (t_L {c_R}^c) \, v_{\rm EW} + H.c.\,.
\end{aligned} \tag{70}
$$

3.5 The $d = 5$ irreducible Higgs mixing operators

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1}$, we find the mass terms of

$$
\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1} \times \frac{d_{\mathscr{A}}}{M_{\rm pl}} \epsilon_{\omega_1\omega_2\omega_3\omega_4} \overline{\mathbf{8_H}}^{\dagger}_{,\omega_1} \overline{\mathbf{8_H}}^{\dagger}_{,\omega_2} \overline{\mathbf{8_H}}^{\dagger}_{,\omega_3} \overline{\mathbf{8_H}}^{\dagger}_{,\omega_4} \mathbf{70_H}^{\dagger} + H.c. \\
\supset\quad & Y_{\mathcal{B}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{5}\,,\mathbf{3}\,,+\frac{2}{15})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}} \Big] \otimes (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega_1} \\
\times\quad & \frac{d_{\mathscr{A}}}{M_{\rm pl}} (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega_1}^{\dagger} \otimes (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega_2}^{\dagger} \otimes (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega_3}^{\dagger} \otimes \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega_4}^{\dagger} \rangle \otimes (\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})_{\mathbf{H}}^{\dagger} + H.c. \\
\supset\quad & Y_{\mathcal{B}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}} \Big] \otimes (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\omega_1} \\
\times\quad & d_{\mathscr{A}} \frac{W_{\overline{\mathbf{3}},\rm IV}}{\sqrt{2} M_{\rm pl}} (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\omega_1}^{\dagger} \otimes \langle (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega_2}^{\dagger} \rangle \otimes (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega_3}^{\dagger} \otimes (\mathbf{10}\,,\overline{\mathbf{2}}\,,+\frac{1}{10})_{\mathbf{H}}^{\dagger} + H.c. \\
\supset\quad & Y_{\mathcal{B}} \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}} \Big] \otimes (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\omega_1} \\
\times\quad & d_{\mathscr{A}} \frac{W_{\overline{\mathbf{3}},\rm IV} w_{\overline{\mathbf{5}},\rm V}}{2 M_{\rm pl}} (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\omega_1}^{\dagger} \otimes \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\omega_3}^{\dagger} \rangle \otimes (\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})_{\mathbf{H}}^{\dagger} + H.c. \\
\Rightarrow\quad & \frac{Y_{\mathcal{B}} d_{\mathscr{A}}}{4} \frac{W_{\overline{\mathbf{3}},\rm IV} w_{\overline{\mathbf{5}},\rm V} V_{\overline{\mathbf{4}},\rm VI}}{M_{\rm pl} m^2_{(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,3}}} \, (b_L {b_R}^c + \tau_L {\tau_R}^c) \, v_{\rm EW} + H.c.\,,
\end{aligned} \tag{71}
$$

via the Higgs mixing operator of $\mathcal{O}_{\mathscr{A}}^{d=5}$ in Eq. (14a).

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}$, we find the mass terms of

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}\times\frac{d_{\mathscr{B}}}{M_{\rm pl}}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_1}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_2}\mathbf{70_H}^\dagger\left(\overline{\mathbf{28_H}}^\dagger_{,\rm\dot{I}}\overline{\mathbf{28_H}}_{,\rm\dot{V}II}\right)+H.c.\\
\supset\quad & Y_{\mathcal{D}}(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{1}\,,\mathbf{1}\,,+1)'_{\mathbf{F}}\otimes(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot\omega_1}\\
\times\quad & \frac{d_{\mathscr{B}}}{M_{\rm pl}}(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})^\dagger_{\mathbf{H}\,,\dot\omega_1}\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})^\dagger_{\mathbf{H}\,,\dot\omega_2}\otimes(\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})^\dagger_{\mathbf{H}}\otimes\langle\overline{\mathbf{28_H}}^\dagger_{,\rm\dot{I}}\overline{\mathbf{28_H}}_{,\rm\dot{V}II}\rangle+H.c.\\
\supset\quad & Y_{\mathcal{D}}d_{\mathscr{B}}\frac{w_{\overline{\mathbf{5}},\rm\dot{I}}w_{\overline{\mathbf{5}},\rm\dot{V}II}}{2M_{\rm pl}}(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{1}\,,\mathbf{1}\,,+1)'_{\mathbf{F}}\otimes(\mathbf{1}\,,\mathbf{2}\,,-\frac{1}{2})_{\mathbf{H}\,,\dot\omega_1}\\
\times\quad & (\mathbf{1}\,,\mathbf{2}\,,-\frac{1}{2})^\dagger_{\mathbf{H}\,,\dot\omega_1}\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})^\dagger_{\mathbf{H}\,,\dot\omega_2}\otimes(\mathbf{10}\,,\overline{\mathbf{2}}\,,+\frac{1}{10})^\dagger_{\mathbf{H}}+H.c.\\
\supset\quad & Y_{\mathcal{D}}d_{\mathscr{B}}\frac{w_{\overline{\mathbf{5}},\rm\dot{I}}w_{\overline{\mathbf{5}},\rm\dot{V}II}}{2M_{\rm pl}m^2_{(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot\omega_1}}}(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{1}\,,\mathbf{1}\,,+1)'_{\mathbf{F}}\otimes\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})'^\dagger_{\mathbf{H}\,,\dot\omega_2}\rangle\otimes(\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})^\dagger_{\mathbf{H}}+H.c.\\
\Rightarrow\quad & \frac{Y_{\mathcal{D}}d_{\mathscr{B}}}{4}\dot\zeta_3'\Big[\frac{w_{\overline{\mathbf{5}},\rm\dot{I}}w_{\overline{\mathbf{5}},\rm\dot{V}II}}{m^2_{(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\rm\dot{I}}}}\,(e_L\mu_R{}^c)+\frac{w_{\overline{\mathbf{5}},\rm\dot{I}}w_{\overline{\mathbf{5}},\rm\dot{V}II}}{m^2_{(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot{2}}}}\,(\mu_L\mu_R{}^c)\Big]v_{\rm EW}+H.c.\,,
\end{aligned}\tag{72}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}\times\frac{d_{\mathscr{B}}}{M_{\rm pl}}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_1}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_2}\mathbf{70_H}^\dagger\left(\overline{\mathbf{28_H}}^\dagger_{,\rm\dot{I}}\overline{\mathbf{28_H}}_{,\rm\dot{V}II}\right)+H.c.\\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{5}\,,\overline{\mathbf{3}}\,,+\frac{7}{15})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H}\,,\dot\omega_1}\\
\times\quad & \frac{d_{\mathscr{B}}}{M_{\rm pl}}(\mathbf{5}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})^\dagger_{\mathbf{H}\,,\dot\omega_1}\otimes(\mathbf{5}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})^\dagger_{\mathbf{H}\,,\dot\omega_2}\otimes(\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})^\dagger_{\mathbf{H}}\otimes\langle\overline{\mathbf{28_H}}^\dagger_{,\rm\dot{I}}\overline{\mathbf{28_H}}_{,\rm\dot{V}II}\rangle+H.c.\\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{5}\,,\mathbf{1}\,,+\frac{4}{5})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{5}}\,,\overline{\mathbf{2}}\,,-\frac{3}{10})_{\mathbf{H}\,,\dot\omega_1}\\
\times\quad & \frac{d_{\mathscr{B}}w_{\overline{\mathbf{5}},\rm\dot{I}}w_{\overline{\mathbf{5}},\rm\dot{V}II}}{2M_{\rm pl}}(\mathbf{5}\,,\overline{\mathbf{2}}\,,-\frac{3}{10})^\dagger_{\mathbf{H}\,,\dot\omega_1}\otimes(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^\dagger_{\mathbf{H}\,,\dot\omega_2}\otimes(\mathbf{10}\,,\overline{\mathbf{2}}\,,+\frac{1}{10})^\dagger_{\mathbf{H}}+H.c.\\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})'_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{1}\,,\mathbf{1}\,,+1)''_{\mathbf{F}}\Big]\\
\times\quad & \frac{d_{\mathscr{B}}w_{\overline{\mathbf{5}},\rm\dot{I}}w_{\overline{\mathbf{5}},\rm\dot{V}II}}{2M_{\rm pl}m^2_{(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\dot\omega_1}}}\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})^\dagger_{\mathbf{H}\,,\dot\omega_2}\rangle\otimes(\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})^\dagger_{\mathbf{H}}+H.c.\\
\Rightarrow\quad & \frac{Y_{\mathcal{D}}d_{\mathscr{B}}}{4}\dot\zeta_3\Big[\frac{w_{\overline{\mathbf{5}},\rm\dot{I}}w_{\overline{\mathbf{5}},\rm\dot{V}II}}{m^2_{(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\rm\dot{I}}}}\,(d_Ld_R{}^c+e_Le_R{}^c)\\
+\quad & \frac{w_{\overline{\mathbf{5}},\rm\dot{I}}w_{\overline{\mathbf{5}},\rm\dot{V}II}}{m^2_{(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\dot{2}}}}\,(d_Ls_R{}^c+\mu_Le_R{}^c)\Big]v_{\rm EW}+H.c.\,,
\end{aligned}\tag{73}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\dot\omega_1} \mathbf{56_F} \overline{\mathbf{28_H}}_{,\dot\omega_1} \times \frac{d_{\mathscr{B}}}{M_{\rm pl}} \overline{\mathbf{28_H}}^\dagger_{,\dot\omega_1} \overline{\mathbf{28_H}}^\dagger_{,\dot\omega_2} \mathbf{70_H}^\dagger \left(\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm I}} \overline{\mathbf{28_H}}_{,\dot{\rm VII}} \right) + H.c. \\
\supset\ & Y_{\mathcal{D}} (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot\omega_1} \otimes (\mathbf{10}\,,\mathbf{3}\,,-\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{5}}\,,\overline{\mathbf{3}}\,,-\frac{2}{15})_{\mathbf{H}\,,\dot\omega_1} \\
\times\ & \frac{d_{\mathscr{B}}}{M_{\rm pl}} (\mathbf{5}\,,\mathbf{3}\,,-\frac{2}{15})^\dagger_{\mathbf{H}\,,\dot\omega_1} \otimes (\mathbf{5}\,,\mathbf{3}\,,-\frac{2}{15})^\dagger_{\mathbf{H}\,,\dot\omega_2} \otimes (\mathbf{10}\,,\overline{\mathbf{3}}\,,+\frac{4}{15})^\dagger_{\mathbf{H}} \otimes \langle \overline{\mathbf{28_H}}^\dagger_{,\dot{\rm I}} \overline{\mathbf{28_H}}_{,\dot{\rm VII}} \rangle + H.c. \\
\supset\ & Y_{\mathcal{D}} (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot\omega_1} \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}} \otimes (\overline{\mathbf{5}}\,,\overline{\mathbf{2}}\,,-\frac{3}{10})_{\mathbf{H}\,,\dot\omega_1} \\
\times\ & \frac{d_{\mathscr{B}} w_{\overline{\mathbf{5}}\,,\dot{\rm I}} w_{\overline{\mathbf{5}}\,,\dot{\rm VII}}}{2M_{\rm pl}} (\mathbf{5}\,,\overline{\mathbf{2}}\,,-\frac{3}{10})^\dagger_{\mathbf{H}\,,\dot\omega_1} \otimes \langle (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})^\dagger_{\mathbf{H}\,,\dot\omega_2} \rangle \otimes (\mathbf{10}\,,\overline{\mathbf{2}}\,,-\frac{1}{10})^\dagger_{\mathbf{H}} + H.c. \\
\supset\ & \frac{Y_{\mathcal{D}} d_{\mathscr{B}} w_{\overline{\mathbf{5}}\,,\dot{\rm I}} w_{\overline{\mathbf{5}}\,,\dot{\rm VII}}}{2\sqrt{2} m^2_{(\overline{\mathbf{4}}\,,\overline{\mathbf{2}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot\omega_1}}} \dot\zeta_2 (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\dot\omega_1} \otimes (\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}} \otimes (\mathbf{4}\,,\overline{\mathbf{2}}\,,+\frac{1}{4})^\dagger_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{Y_{\mathcal{D}} d_{\mathscr{B}}}{4} \dot\zeta_2 \Big[\frac{w_{\overline{\mathbf{5}}\,,\dot{\rm I}} w_{\overline{\mathbf{5}}\,,\dot{\rm VII}}}{m^2_{(\overline{\mathbf{4}}\,,\overline{\mathbf{2}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot{\rm I}}}} (s_L {d_R}^c) + \frac{w_{\overline{\mathbf{5}}\,,\dot{\rm I}} w_{\overline{\mathbf{5}}\,,\dot{\rm VII}}}{m^2_{(\overline{\mathbf{4}}\,,\overline{\mathbf{2}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot 2}}} (s_L {s_R}^c) \Big] v_{\rm EW} + H.c.\,, \qquad (74)
\end{aligned}
$$

via the Higgs mixing operator of $\mathcal{O}_{\mathscr{B}}^{d=5}$ in Eq. (14b). For convenience, we parametrize the following ratios of

$$
\Delta_{\dot\omega} \equiv \frac{w_{\overline{\mathbf{5}}\,,\dot{\rm I}} w_{\overline{\mathbf{5}}\,,\dot{\rm VII}}}{m^2_{(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\dot\omega}}}\,, \quad \Delta'_{\dot\omega} \equiv \frac{w_{\overline{\mathbf{5}}\,,\dot{\rm I}} w_{\overline{\mathbf{5}}\,,\dot{\rm VII}}}{m^2_{(\overline{\mathbf{4}}\,,\overline{\mathbf{2}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot\omega}}}\,, \quad \Delta''_{\dot\omega} \equiv \frac{w_{\overline{\mathbf{5}}\,,\dot{\rm I}} w_{\overline{\mathbf{5}}\,,\dot{\rm VII}}}{m^2_{(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot\omega}}}\,. \qquad (75)
$$

With the Higgs VEVs assignments in Eq. (54), we can obtain the following propagator masses of

$$
\begin{aligned}
& m_{(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\dot{\rm I}}} \sim m_{(\overline{\mathbf{4}}\,,\overline{\mathbf{2}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot{\rm I}}} \sim m_{(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot{\rm I}}} \sim \mathcal{O}(v_{431})\,, \\
& m_{(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{H}\,,\dot 2}} \sim m_{(\overline{\mathbf{4}}\,,\overline{\mathbf{2}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot 2}} \sim m_{(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot 2}} \sim \mathcal{O}(v_{421})\,. \qquad (76)
\end{aligned}
$$

3.6 The SM quark/lepton masses and the benchmark

For all up-type quarks with $Q_e = +\frac{2}{3}$, we write down the following tree-level masses from both the renormalizable Yukawa couplings and the gravity-induced terms in the basis of $\mathcal{U} \equiv (u\,,c\,,t\,)$

$$
\mathcal{M}_{\mathcal{U}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & c_5 \zeta_2/\sqrt{2} \\ c_4 \dot\zeta_3'/\sqrt{2} & 0 & c_5 \zeta_3/\sqrt{2} \\ c_5 \zeta_2/\sqrt{2} & c_5 \zeta_1/\sqrt{2} & Y_{\mathcal{T}} \end{pmatrix} v_{\rm EW}\,. \qquad (77)
$$

For all down-type quarks with $Q_e = -\frac{1}{3}$, we find the following tree-level mass matrix in the basis of $\mathcal{D} \equiv (d\,,s\,,b\,)$

$$
\mathcal{M}_{\mathcal{D}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot{\rm I}} \dot\zeta_3 & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot 2} \dot\zeta_3 & 0 \\ Y_{\mathcal{D}} d_{\mathscr{B}} \Delta'_{\dot{\rm I}} \dot\zeta_2 & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta'_{\dot 2} \dot\zeta_2 & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathscr{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\rm EW}\,. \qquad (78)
$$

For all charged leptons with $Q_e = -1$, their tree-level mass matrix is expressed as follows:

$$
\mathcal{M}_{\mathcal{L}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot{\rm I}} \dot\zeta_3 & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta''_{\dot{\rm I}} \dot\zeta_3' & 0 \\ Y_{\mathcal{D}} d_{\mathscr{B}} \Delta_{\dot 2} \dot\zeta_3 & Y_{\mathcal{D}} d_{\mathscr{B}} \Delta''_{\dot 2} \dot\zeta_3' & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathscr{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\rm EW}\,, \qquad (79)
$$

in the basis of $\mathcal{L} \equiv (e\,,\mu\,,\tau\,)$. Based on the above SM quark/lepton mass matrices, we find the following benchmark point of

$$v_{531} \simeq 1.4 \times 10^{17}\,\mathrm{GeV}\,, \quad v_{521} \simeq 4.8 \times 10^{15}\,\mathrm{GeV}\,, \quad v_{421} \simeq 1.1 \times 10^{15}\,\mathrm{GeV}\,, \tag{80}$$

to reproduce the observed hierarchical masses as well as the CKM mixing pattern, as summarized in Table 15.

ζ_1	ζ_2	ζ_3	$Y_{\mathcal{D}}$	$Y_{\mathcal{B}}$	$Y_{\mathcal{T}}$
6.0×10^{-2}	2.0×10^{-3}	4.4×10^{-4}	0.5	0.5	1.0
$\lambda = \zeta_{23}$	c_4	c_5	$d_{\mathscr{A}}$	$d_{\mathscr{B}}$	m_μ
0.22	5	0.6	0.2	0.1	0.1
m_u	m_c	m_t	$m_d \approx m_e$	m_s	$m_b \approx m_\tau$
3.7×10^{-3}	0.3	174.2	0.7×10^{-3}	0.6×10^{-1}	1.7
$\lvert V_{ud}\rvert$	$\lvert V_{us}\rvert$	$\lvert V_{ub}\rvert$			
0.98	0.22	0.1×10^{-3}			
$\lvert V_{cd}\rvert$	$\lvert V_{cs}\rvert$	$\lvert V_{cb}\rvert$			
0.22	0.98	2.5×10^{-2}			
$\lvert V_{td}\rvert$	$\lvert V_{ts}\rvert$	$\lvert V_{tb}\rvert$			
0.6×10^{-2}	2.5×10^{-2}	1			

Table 15: The parameters of the $\mathfrak{su}(8)$ benchmark point and the predicted SM quark/lepton masses (in unit of GeV) as well as the CKM mixings.

4 The intermediate stages of the WWW symmetry breaking pattern

Schematically, we assign the Higgs VEVs according to the symmetry breaking pattern as follows:

$$\mathfrak{g}_{351} \to \mathfrak{g}_{341} \quad : \quad \langle (\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{H}\,,\mathrm{IV}} \rangle \equiv \frac{1}{\sqrt{2}} W_{\overline{\mathbf{5}}\,,\mathrm{IV}}\,, \tag{81a}$$

$$\mathfrak{g}_{341} \to \mathfrak{g}_{331} \quad : \quad \langle (\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\mathrm{V}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\overline{\mathbf{4}}\,,\mathrm{V}}\,,\ \langle (\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\mathrm{i},\dot{\mathrm{V}}\mathrm{II}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\overline{\mathbf{4}}\,,\mathrm{i},\dot{\mathrm{V}}\mathrm{II}}\,, \tag{81b}$$

$$\begin{aligned} \mathfrak{g}_{331} \to \mathfrak{g}_{\mathrm{SM}} \quad : \quad & \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,3,\mathrm{VI}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\overline{\mathbf{3}}\,,3,\mathrm{VI}}\,, \\ & \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\dot{\mathrm{V}}\mathrm{III}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\overline{\mathbf{3}}\,,\dot{\mathrm{V}}\mathrm{III}}\,,\ \langle (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})'_{\mathbf{H}\,,\dot{2},\mathrm{I}\dot{\mathrm{X}}} \rangle \equiv \frac{1}{\sqrt{2}} V'_{\overline{\mathbf{3}}\,,\dot{2},\mathrm{I}\dot{\mathrm{X}}}\,, \end{aligned} \tag{81c}$$

$$\mathrm{EWSB} \quad : \quad \langle (\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{H}} \rangle \equiv \frac{1}{\sqrt{2}} v_{\mathrm{EW}}\,. \tag{81d}$$

For later convenience, we also parametrize different symmetry breaking VEVs in terms of the following dimensionless quantities:

$$
\begin{aligned}
&\zeta_1 \equiv \frac{W_{\overline{\mathbf{5}}\,,\mathrm{IV}}}{M_{\rm pl}}\,, \quad \zeta_2 \equiv \frac{w_{\overline{\mathbf{4}}\,,\mathrm{V}}}{M_{\rm pl}}\,, \quad \dot\zeta_2 \equiv \frac{w_{\overline{\mathbf{4}}\,,\dot 1,\dot{\mathrm{V}}\mathrm{II}}}{M_{\rm pl}}\,,\\
&\zeta_3 \equiv \frac{V_{\overline{\mathbf{3}}\,,3,\mathrm{VI}}}{M_{\rm pl}}\,, \quad \dot\zeta_3' \equiv \frac{V'_{\overline{\mathbf{3}}\,,\dot 2,\dot{\mathrm{I}}\mathrm{X}}}{M_{\rm pl}}\,, \quad \dot\zeta_3 \equiv \frac{V_{\overline{\mathbf{3}}\,,\mathrm{V}\dot{\mathrm{I}}\mathrm{II}}}{M_{\rm pl}}\,,\\
&\zeta_0 \gg \zeta_1 \gg \zeta_2 \sim \dot\zeta_2 \gg \zeta_3 \sim \dot\zeta_3' \sim \dot\zeta_3\,, \quad \zeta_{ij} \equiv \frac{\zeta_j}{\zeta_i}\,, \quad (i<j)\,.
\end{aligned} \tag{82}
$$

In Table 16, we summarize all vectorlike fermions that become massive during different stages of the WWW symmetry breaking pattern.

stages	$Q_e=-\frac{1}{3}$	$Q_e=+\frac{2}{3}$	$Q_e=-1$	$Q_e=0$
v_{351} $\{\Omega\}$	$\mathfrak{D}$ IV	-	$(\mathfrak{e}''\,,\mathfrak{n}'')$ IV	$\{\check{\mathfrak{n}}'\,,\check{\mathfrak{n}}''\}$ $\{\mathrm{IV}'\,,\mathrm{IV}''\}$
v_{341} $\{\Omega\}$	$\mathfrak{d}\,,\{\mathfrak{D}''\,,\mathfrak{D}'''''\}$ $\{\mathrm{V}\,,\dot{\mathrm{V}}\mathrm{II}\}$	$\mathfrak{u}\,,\mathfrak{U}$	$\mathfrak{E}\,,(\mathfrak{e}\,,\mathfrak{n})\,,(\mathfrak{e}''''\,,\mathfrak{n}'''')$ $\{\mathrm{V}\,,\dot{\mathrm{V}}\mathrm{II}\}$	$\{\check{\mathfrak{n}}\,,\check{\mathfrak{n}}'''\}$ $\{\mathrm{V}'\,,\dot{\mathrm{V}}\mathrm{II}'\}$
v_{331} $\{\Omega\}$	$\{\mathfrak{D}'\,,\mathfrak{D}'''\,,\mathfrak{D}''''\}$ $\{\mathrm{VI}\,,\mathrm{V}\dot{\mathrm{I}}\mathrm{II}\,,\dot{\mathrm{I}}\mathrm{X}\}$	-	$(\mathfrak{e}'\,,\mathfrak{n}')\,,(\mathfrak{e}'''\,,\mathfrak{n}''')\,,(\mathfrak{e}'''''\,,\mathfrak{n}''''')$ $\{\mathrm{VI}\,,\mathrm{V}\dot{\mathrm{I}}\mathrm{II}\,,\dot{\mathrm{I}}\mathrm{X}\}$	-

Table 16: The vectorlike fermions at different intermediate symmetry breaking stages along the WWW symmetry breaking pattern of the $\mathfrak{su}(8)$ theory.

4.1 The first stage

The first symmetry breaking stage of $\mathfrak{g}_{351} \to \mathfrak{g}_{341}$ is achieved by $(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{H}\,,\mathrm{IV}} \subset \overline{\mathbf{8_H}}_{\,,\mathrm{IV}}$ in the rank-two sector, according to the $\widetilde{\mathrm{U}}(1)_{T'}$-neutral components in Table 17. Accordingly, the term of $Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{IV}}\mathbf{28_F}\overline{\mathbf{8_H}}_{\,,\mathrm{IV}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}\,,\check{\mathfrak{n}}''\,,\mathfrak{e}''\,,\mathfrak{n}''\,,\check{\mathfrak{n}}')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{IV}}\mathbf{28_F}\overline{\mathbf{8_H}}_{\,,\mathrm{IV}} + H.c.\\
\supset\;& Y_{\mathcal{B}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{IV}}\otimes(\mathbf{3}\,,\mathbf{5}\,,-\frac{2}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{F}}^{\mathrm{IV}}\otimes(\mathbf{1}\,,\mathbf{10}\,,+\frac{2}{5})_{\mathbf{F}}\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{H}\,,\mathrm{IV}}\rangle + H.c.\\
\Rightarrow\;& \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{D}_L\mathfrak{D}_R^c + \check{\mathfrak{n}}_L''\check{\mathfrak{n}}_R''^c + \mathfrak{e}_L''\mathfrak{e}_R''^c - \mathfrak{n}_L''\mathfrak{n}_R''^c + \check{\mathfrak{n}}_L'\check{\mathfrak{n}}_R'^c\right)W_{\overline{\mathbf{5}},\mathrm{IV}} + H.c.\,.
\end{aligned} \tag{83}
$$

$\overline{\mathbf{8_F}}^{\Omega}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{F}}^{\Omega}$		
$\mathcal{T}'$	$-\frac{4}{3}t$	$-4t$		
$\mathbf{28_F}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{5}\,,-\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{10}\,,+\frac{2}{5})_{\mathbf{F}}$	
$\mathcal{T}'$	$-\frac{4}{3}t$	$+\frac{4}{3}t$	$4t$	
$\mathbf{56_F}$	$(\overline{\mathbf{3}}\,,\mathbf{5}\,,-\frac{7}{15})_{\mathbf{F}}$	$(\mathbf{1}\,,\overline{\mathbf{10}}\,,+\frac{3}{5})_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{10}\,,+\frac{1}{15})_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}}$
$\mathcal{T}'$	$-\frac{4}{3}t$	$4t$	$\frac{4}{3}t$	$-4t$
$\overline{\mathbf{8_H}}_{,\omega}$	$(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{H}\,,\omega}$			
$\mathcal{T}'$	0			
$\overline{\mathbf{28_H}}_{,\dot{\omega}}$	$(\mathbf{1}\,,\overline{\mathbf{10}}\,,-\frac{2}{5})_{\mathbf{H}\,,\dot{\omega}}$			
$\mathcal{T}'$	0			
$\mathbf{70_H}$	$(\mathbf{1}\,,\overline{\mathbf{5}}\,,+\frac{4}{5})_{\mathbf{H}}$	$(\mathbf{1}\,,\mathbf{5}\,,-\frac{4}{5})_{\mathbf{H}}$		
$\mathcal{T}'$	0	$-8t$		

Table 17: The $\widetilde{\rm U}(1)_{T'}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{351}$ theory.

After this stage, the remaining massless fermions expressed in terms of the $\mathfrak{g}_{341}$ IRs are the following:

$$
\begin{aligned}
&\Big[(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{F}}^{\Omega}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega'}\Big]\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega}\oplus\subset\overline{\mathbf{8_F}}^{\Omega}\,,\\
&\Omega=(\omega\,,\dot{\omega})\,,\quad \omega=(3\,,{\rm V}\,,{\rm VI})\,,\quad \dot{\omega}=(\dot{1}\,,\dot{2}\,,\dot{\rm VII}\,,\dot{\rm VIII}\,,\dot{\rm IX})\,,\\
&(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm IV}\subset\overline{\mathbf{8_F}}^{\rm IV}\,,\\
&(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}\oplus\Big[(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}\oplus\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}}\Big]\oplus\Big[\cancel{(\mathbf{1}\,,\mathbf{4}\,,+\frac{1}{4})_{\mathbf{F}}}\oplus(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}\Big]\subset\mathbf{28_F}\,,\\
&(\overline{\mathbf{3}}\,,\mathbf{4}\,,-\frac{5}{12})_{\mathbf{F}}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}^{\prime\prime\prime}\oplus(\mathbf{1}\,,\overline{\mathbf{4}}\,,+\frac{3}{4})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}^{\prime}\oplus(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}^{\prime}\\
\oplus\quad&(\mathbf{3}\,,\mathbf{6}\,,+\frac{1}{6})_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}}\subset\mathbf{56_F}\,.
\end{aligned}\tag{84}
$$

4.2 The second stage

The second symmetry breaking stage of $\mathfrak{g}_{341}\to\mathfrak{g}_{331}$ is achieved by $(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,{\rm V}}\subset\overline{\mathbf{8_H}}_{,{\rm V}}$ in the rank-two sector, $(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot{\rm VII}}\subset\overline{\mathbf{28_H}}_{,\dot{\rm VII}}$ and $(\mathbf{1}\,,\mathbf{4}\,,+\frac{1}{4})_{\mathbf{H}}^{\dot{\rm VII}}\subset\mathbf{28_H}^{\dot{\rm VII}}$ in the rank-three sector, according to the $\widetilde{\rm U}(1)_{T''}$-neutral components in Table 18. The term of $Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\rm V}\mathbf{28_F}\overline{\mathbf{8_H}}_{,{\rm V}}+H.c.$ leads to

$\overline{\mathbf{8_F}}^{\Omega}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{F}}^{\Omega}$	
$\mathcal{T}''$	$-\frac{4}{3}t$	$-4t$	$-4t$	
$\mathbf{28_F}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}$	
$\mathcal{T}''$	$-\frac{4}{3}t$	$+\frac{4}{3}t$	$+4t$	
$\mathbf{56_F}$	$(\overline{\mathbf{3}}\,,\mathbf{4}\,,-\frac{5}{12})_{\mathbf{F}}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}'''$	$(\mathbf{1}\,,\overline{\mathbf{4}}\,,+\frac{3}{4})_{\mathbf{F}}$	
$\mathcal{T}''$	$-\frac{4}{3}t$	$-\frac{4}{3}t$	$4t$	
	$(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}'$	$(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}'$	$(\mathbf{3}\,,\mathbf{6}\,,+\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}}$
$\mathcal{T}''$	$4t$	$+\frac{4}{3}t$	$+\frac{4}{3}t$	$-4t$
$\overline{\mathbf{8_H}}_{,\omega}$	$(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\omega}$			
$\mathcal{T}''$	0			
$\overline{\mathbf{28_H}}_{,\dot{\omega}}$	$(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot{\omega}}$	$(\mathbf{1}\,,\mathbf{6}\,,-\frac{1}{2})_{\mathbf{H}\,,\dot{\omega}}$		
$\mathcal{T}''$	0	0		
$\mathbf{70_H}$	$(\mathbf{1}\,,\overline{\mathbf{4}}\,,+\frac{3}{4})_{\mathbf{H}}$			
$\mathcal{T}''$	0			

Table 18: The $\widetilde{\mathrm{U}}(1)_{T''}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{341}$ theory.

the vectorial masses of $(\mathfrak{D}''\,,\mathfrak{e}\,,\mathfrak{n}\,,\check{\mathfrak{n}})$ as follows:

$$\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{V}}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\mathrm{V}}+H.c.\\
\supset\;& Y_{\mathcal{B}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{V}}\otimes(\mathbf{3}\,,\mathbf{5}\,,-\frac{2}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{F}}^{\mathrm{V}}\otimes(\mathbf{1}\,,\mathbf{10}\,,+\frac{2}{5})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{H}\,,\mathrm{V}}+H.c.\\
\supset\;& Y_{\mathcal{B}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{V}}\otimes(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{F}}^{\mathrm{V}}\otimes(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}\\
\oplus\;& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V}}\otimes(\mathbf{1}\,,\mathbf{4}\,,+\frac{1}{4})_{\mathbf{F}}\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\mathrm{V}}\rangle+H.c.\\
\Rightarrow\;& \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{D}_L''\mathfrak{D}_R''^{\,c}+\check{\mathcal{N}}_L^{\mathrm{V}}\check{\mathfrak{n}}_R''^{\,c}+\mathfrak{e}_L\mathfrak{e}_R{}^{c}-\mathfrak{n}_L\mathfrak{n}_R{}^{c}+\check{\mathfrak{n}}_L\check{\mathfrak{n}}_R{}^{c}\right)w_{\overline{\mathbf{4}},\mathrm{V}}+H.c.\,.
\end{aligned} \tag{85}$$

The term of $Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot{\mathrm{V}}\mathrm{II}}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot{\mathrm{V}}\mathrm{II}}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}'''''\,,\mathfrak{e}''''\,,\mathfrak{n}''''\,,\check{\mathfrak{n}}''')$ as follows:

$$\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot{\mathrm{V}}\mathrm{II}}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot{\mathrm{V}}\mathrm{II}}+H.c.\\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}}\otimes(\mathbf{3}\,,\mathbf{10}\,,+\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}}\otimes(\mathbf{1}\,,\overline{\mathbf{10}}\,,+\frac{3}{15})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{10}}\,,-\frac{2}{5})_{\mathbf{H}\,,\dot{\mathrm{V}}\mathrm{II}}+H.c.\\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}}\otimes(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{F}}^{\dot{\mathrm{V}}\mathrm{II}}\otimes(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}'\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot{\mathrm{V}}\mathrm{II}}\rangle+H.c.\\
\Rightarrow\;& \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}_L'''''\mathfrak{D}_R'''''^{\,c}+\mathfrak{e}_L''''\mathfrak{e}_R''''^{\,c}-\mathfrak{n}_L''''\mathfrak{n}_R''''^{\,c}+\check{\mathfrak{n}}_L'''\check{\mathfrak{n}}_R'''^{\,c}\right)w_{\overline{\mathbf{4}}\,,\dot{\mathrm{V}}\mathrm{II}}+H.c.\,.
\end{aligned} \tag{86}$$

The Yukawa coupling between two $\mathbf{56_F}$'s by the following $d = 5$ operator

$$
\begin{aligned}
& \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\mathrm{\dot{V}II}} + H.c. \\
\supset\quad & c_4 \frac{v_U}{M_{\rm pl}} \Big[(\overline{\mathbf{3}}\,,\mathbf{5}\,,-\frac{7}{15})_{\mathbf{F}} \otimes (\mathbf{3}\,,\mathbf{10}\,,+\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{10}}\,,+\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}} \Big] \otimes (\mathbf{1}\,,\mathbf{10}\,,+\frac{2}{5})_{\mathbf{H}}^{\mathrm{\dot{V}II}} + H.c. \\
\supset\quad & c_4 \zeta_0 \Big[(\overline{\mathbf{3}}\,,\mathbf{4}\,,-\frac{5}{12})_{\mathbf{F}} \otimes (\mathbf{3}\,,\mathbf{6}\,,+\frac{1}{6})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{4}}\,,+\frac{3}{4})_{\mathbf{F}} \otimes (\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}} \Big] \otimes \langle (\mathbf{1}\,,\mathbf{4}\,,+\frac{1}{4})_{\mathbf{H}}^{\mathrm{\dot{V}II}} \rangle + H.c. \\
\Rightarrow\quad & \frac{c_4}{\sqrt{2}} \zeta_0 \left(\mathfrak{E}_L \mathfrak{E}_R{}^c + \mathfrak{U}_L \mathfrak{U}_R{}^c + \mathfrak{d}_L \mathfrak{d}_R{}^c - \mathfrak{u}_L \mathfrak{u}_R{}^c \right) w_{\mathbf{4}}^{\mathrm{\dot{V}II}} + H.c.\,,
\end{aligned} \tag{87}
$$

leads to massive vectorlike fermions of $(\mathfrak{E}\,,\mathfrak{U}\,,\mathfrak{u}\,,\mathfrak{d})$.

After integrating out the massive fermions, the remaining massless fermions expressed in terms of the $\mathfrak{g}_{331}$ IRs are the following

$$
\begin{aligned}
& (\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega} \subset \overline{\mathbf{8_F}}^{\Omega}\,, \\
& \Omega = (\omega\,,\dot\omega)\,, \quad \omega = (3\,,\mathrm{VI})\,, \quad \dot\omega = (\dot{1}\,,\dot{2}\,,\mathrm{\dot{V}III}\,,\mathrm{I\dot{X}})\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{IV}} \subset \overline{\mathbf{8_F}}^{\mathrm{IV}}\,, \quad (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V}} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V}''} \subset \overline{\mathbf{8_F}}^{\mathrm{V}}\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{\dot{V}II}} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{\dot{V}II}''} \subset \overline{\mathbf{8_F}}^{\mathrm{\dot{V}II}}\,, \\
& (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}} \oplus \Big[(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}} \oplus \cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}''} \Big] \oplus \cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}} \\
\oplus\quad & \cancel{\Big[(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}} \oplus (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}'' \Big]} \oplus \Big[\cancel{(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}'} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}} \Big] \subset \mathbf{28_F}\,, \\
& \Big[\cancel{(\overline{\mathbf{3}}\,,\mathbf{3}\,,-\frac{1}{3})_{\mathbf{F}}} \oplus (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}'' \Big] \oplus (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}''' \oplus \Big[\cancel{(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}''} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}' \Big] \\
\oplus\quad & \Big[\cancel{(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}''} \oplus (\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}'' \Big] \oplus \Big[(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}' \oplus \cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}'''''} \Big] \oplus \Big[(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}'' \\
\oplus\quad & \cancel{(\mathbf{3}\,,\overline{\mathbf{3}}\,,+\frac{1}{3})_{\mathbf{F}}} \Big] \oplus (\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}} \subset \mathbf{56_F}\,.
\end{aligned} \tag{88}
$$

4.3 The third stage

The third symmetry breaking stage of $\mathfrak{g}_{331} \to \mathfrak{g}_{\rm SM}$ is achieved by $(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{VI}} \subset \overline{\mathbf{8_H}}_{\,,\mathrm{VI}}$ in the rank-two sector and $(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{\dot{V}III}} \subset \overline{\mathbf{28_H}}_{\,,\mathrm{\dot{V}III}}$, $(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})'_{\mathbf{H}\,,\mathrm{I\dot{X}}} \subset \overline{\mathbf{28_H}}_{\,,\mathrm{I\dot{X}}}$ in the rank-three sector, according to the $\widetilde{\rm U}(1)_{T'''}$-neutral components in Table 19. The term of $Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\mathrm{VI}} \mathbf{28_F} \overline{\mathbf{8_H}}_{\,,\mathrm{VI}} + H.c.$ leads

$\overline{\mathbf{8_F}}^{\Omega}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''}$
$\mathcal{T}'''$	$-\frac{4}{3}t$	$-4t$	$-4t$	$-4t$
$\mathbf{28_F}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}$	
$\mathcal{T}'''$	$-\frac{4}{3}t$	$+\frac{4}{3}t$	$+4t$	
$\mathbf{56_F}$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}''$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}'''$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}'$	
$\mathcal{T}'''$	$-\frac{4}{3}t$	$-\frac{4}{3}t$	$4t$	
	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}''$	$(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}'$	$(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}''$	$(\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}}$
$\mathcal{T}'''$	$4t$	$+\frac{4}{3}t$	$+\frac{4}{3}t$	$-4t$
$\overline{\mathbf{8_H}}_{,\omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\omega}$			
$\mathcal{T}'''$	0			
$\overline{\mathbf{28_H}}_{,\dot\omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\dot\omega}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\dot\omega}'$	$(\mathbf{1}\,,\mathbf{3}\,,-\frac{2}{3})_{\mathbf{H}\,,\dot\omega}$	
$\mathcal{T}'''$	0	0	0	
$\mathbf{70_H}$	$(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{H}}$			
$\mathcal{T}'''$	0			

Table 19: The $\widetilde{\mathrm{U}}(1)_{\mathcal{T}'''}$ charges for massless fermions and possible symmetry breaking Higgs components in the $\mathfrak{g}_{331}$ theory.

to the vectorial masses of $(\mathfrak{D}'\,,\mathfrak{e}'\,,\mathfrak{n}')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\mathrm{VI}}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\mathrm{VI}}+H.c.\\
\supset\ & Y_{\mathcal{B}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{3}\,,\mathbf{5}\,,-\frac{2}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{1}\,,\mathbf{10}\,,+\frac{2}{5})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{H}\,,\mathrm{VI}}+H.c.\\
\supset\ & Y_{\mathcal{B}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}\\
\oplus\ & (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{1}\,,\mathbf{4}\,,+\frac{1}{4})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\mathrm{VI}}+H.c.\\
\supset\ & Y_{\mathcal{B}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}\\
\oplus\ & (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{VI}''}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{VI}}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H}\,,\mathrm{VI}}\rangle+H.c.\\
\Rightarrow\ & \frac{1}{\sqrt{2}}Y_{\mathcal{B}}\left(\mathfrak{n}_L'\mathfrak{n}_R'^{\,c}-\mathfrak{e}_L'\mathfrak{e}_R'^{\,c}+\check{\mathcal{N}}_L^{\mathrm{VI}''}\check{\mathfrak{n}}_R^{c}+\check{\mathcal{N}}_L^{\mathrm{VI}}\check{\mathfrak{n}}_R'^{c}+\mathfrak{D}_L'\mathfrak{D}_R'^{c}\right)V_{\overline{\mathbf{3}},\mathrm{VI}}+H.c.\,.
\end{aligned}
\tag{89}
$$

The term of $Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\mathrm{V\dot{I}II},\mathrm{\dot{I}X}}\mathbf{56_F}\overline{\mathbf{28}}_{\mathbf{H},\mathrm{V\dot{I}II},\mathrm{\dot{I}X}}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}'''' \,,\mathfrak{e}'''' \,,\mathfrak{n}'''' \,,\mathfrak{D}''' \,,\mathfrak{e}''' \,,\mathfrak{n}''')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\mathrm{\dot{I}X}}\mathbf{56_F}\overline{\mathbf{28}_{\mathbf{H}}}_{,\mathrm{\dot{I}X}} + H.c. \\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{3}\,,\mathbf{10}\,,+\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{1}\,,\overline{\mathbf{10}}\,,+\frac{3}{15})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{10}}\,,-\frac{2}{5})_{\mathbf{H},\mathrm{\dot{I}X}} + H.c. \\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}'\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H},\mathrm{\dot{I}X}} + H.c. \\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}''\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{\dot{I}X}}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}''\Big] \\
\otimes\;& \langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\mathrm{\dot{I}X}}'\rangle + H.c. \\
\Rightarrow\;& \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}_L''''\mathfrak{D}_R''''^{\,c}+\check{\mathcal{N}}_L^{\mathrm{\dot{I}X}''}\check{\mathfrak{n}}_R'''^{\,c}-\mathfrak{e}_L''''\mathfrak{e}_R''''^{c}+\mathfrak{n}_L''''\mathfrak{n}_R''''^{c}\right)V_{\overline{\mathbf{3}},\mathrm{\dot{I}X}}' + H.c.\,,
\end{aligned}
\tag{90}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\mathrm{V\dot{I}II}}\mathbf{56_F}\overline{\mathbf{28}_{\mathbf{H}}}_{,\mathrm{V\dot{I}II}} + H.c. \\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{3}\,,\mathbf{10}\,,+\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{1}\,,\overline{\mathbf{10}}\,,+\frac{3}{15})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{10}}\,,-\frac{2}{5})_{\mathbf{H},\mathrm{V\dot{I}II}} + H.c. \\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{3}\,,\mathbf{6}\,,+\frac{1}{6})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{1}\,,\overline{\mathbf{4}}\,,+\frac{3}{4})_{\mathbf{F}} \\
\oplus\;& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}'\Big]\otimes(\mathbf{1}\,,\mathbf{6}\,,-\frac{1}{2})_{\mathbf{H},\mathrm{V\dot{I}II}} + H.c. \\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}''\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{F}}' \\
\oplus\;& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\mathrm{V\dot{I}II}}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{1}{3})_{\mathbf{F}}''\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\mathrm{V\dot{I}II}}\rangle + H.c. \\
\Rightarrow\;& \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}_L'''\mathfrak{D}_R'''^{\,c}+\check{\mathcal{N}}_L^{\mathrm{V\dot{I}II}}\check{\mathfrak{n}}_R'''^{\,c}-\mathfrak{e}_L'''\mathfrak{e}_R'''^{c}+\mathfrak{n}_L'''\mathfrak{n}_R'''^{c}\right)V_{\overline{\mathbf{3}},\mathrm{V\dot{I}II}} + H.c.\,.
\end{aligned}
\tag{91}
$$

The remaining massless fermions of the $\mathfrak{g}_{\rm SM}$ are listed as follows:

$$
\begin{aligned}
& (\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega}\oplus\Big[(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\Omega}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega'}\Big]\subset\overline{\mathbf{8}_{\mathbf{F}}}^{\Omega}\,,\quad \Omega=(\dot{1}\,,\dot{2}\,,3)\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm IV}\oplus\ldots\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm \dot{I}X}\subset\overline{\mathbf{8}_{\mathbf{F}}}^{\Omega}\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm VI'}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm V\dot{I}II'}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm \dot{I}X'}\subset\overline{\mathbf{8}_{\mathbf{F}}}^{\Omega'}\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm V''}\oplus\ldots\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\rm \dot{I}X''}\subset\overline{\mathbf{8}_{\mathbf{F}}}^{\Omega''}\,,\\
& (\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}\oplus\Big[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}'}\oplus(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}\Big]\oplus\bcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}''}\oplus\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}}\\
\oplus\quad & \cancel{\Big[(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}''\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}'\Big]}\oplus\cancel{(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}''}\oplus\bcancel{\Big[(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}\Big]}\\
\oplus\quad & \Big[(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}\oplus\xcancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}'}\Big]\subset\mathbf{28}_{\mathbf{F}}\,,\\
& \bcancel{\Big[(\overline{\mathbf{3}}\,,\mathbf{2}\,,-\frac{1}{6})_{\mathbf{F}}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}'\Big]}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}''\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}'''\oplus\Big[\xcancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}}\oplus(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}'\Big]\\
\oplus\quad & \bcancel{(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}''}\oplus\bcancel{\Big[(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}''''\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}'''\Big]}\oplus\Big[\xcancel{(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{F}}'''''}\oplus(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}'''\Big]\oplus\Big[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}''''}\\
\oplus\quad & (\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}'\Big]\oplus\bcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}'''''}\oplus\Big[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}'''}\oplus(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}''\Big]\\
\oplus\quad & \bcancel{\Big[(\mathbf{3}\,,\mathbf{1}\,,+\frac{2}{3})_{\mathbf{F}}\oplus(\mathbf{3}\,,\overline{\mathbf{2}}\,,+\frac{1}{6})_{\mathbf{F}}'''\Big]}\oplus\bcancel{(\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}}}\subset\mathbf{56}_{\mathbf{F}}\,.
\end{aligned}\tag{92}
$$

4.4 The $d=5$ bi-linear fermion operators

For the operator of $\mathcal{O}_{\mathcal{F}}^{(4\,,1)}$ in Eq. (13a), it is decomposed as

$$
\begin{aligned}
& \frac{c_4}{M_{\rm pl}}\mathbf{56}_{\mathbf{F}}\mathbf{56}_{\mathbf{F}}\cdot\overline{\mathbf{28}_{\mathbf{H}}}_{\,,\dot{\omega}}\cdot\mathbf{70}_{\mathbf{H}}+H.c.\\
\supset\quad & \frac{c_4}{M_{\rm pl}}(\overline{\mathbf{3}}\,,\mathbf{5}\,,-\frac{7}{15})_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{10}\,,+\frac{1}{15})_{\mathbf{F}}\otimes(\mathbf{1}\,,\overline{\mathbf{10}}\,,-\frac{2}{5})_{\mathbf{H}\,,\dot{\omega}}\otimes(\mathbf{1}\,,\overline{\mathbf{5}}\,,+\frac{4}{5})_{\mathbf{H}}+H.c.\\
\supset\quad & \frac{c_4}{M_{\rm pl}}\Big[(\overline{\mathbf{3}}\,,\mathbf{4}\,,-\frac{5}{12})_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}'\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}'''\otimes(\mathbf{3}\,,\mathbf{6}\,,+\frac{1}{6})_{\mathbf{F}}\Big]\otimes\langle(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H}\,,\dot{\omega}}\rangle\otimes(\mathbf{1}\,,\overline{\mathbf{4}}\,,+\frac{3}{4})_{\mathbf{H}}+H.c.\\
\supset\quad & \frac{c_4 w_{\overline{\mathbf{4}}\,,\rm V\dot{I}I}}{\sqrt{2}M_{\rm pl}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}''\otimes(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}'\otimes(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}'''\otimes(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}''\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})_{\mathbf{H}}+H.c.\\
\Rightarrow\quad & \frac{c_4}{2}\dot{\zeta}_2\,(u_L {c_R}^c+c_L {u_R}^c)\,v_{\rm EW}+H.c.\,.
\end{aligned}\tag{93}
$$

For the operator of $\mathcal{O}_{\mathcal{F}}^{(5\,,1)}$ in Eq. (13b), it is decomposed as

$$
\begin{aligned}
& \frac{c_5}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{\,,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} \Big[(\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}\,, \mathbf{10}\,, +\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{3}\,, \mathbf{5}\,, -\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{3}}\,, \mathbf{5}\,, -\frac{7}{15})_{\mathbf{F}} \Big] \otimes \langle (\mathbf{1}\,, \overline{\mathbf{5}}\,, -\frac{1}{5})_{\mathbf{H}\,,\omega} \rangle \\
\otimes\ & (\mathbf{1}\,, \overline{\mathbf{5}}\,, +\frac{4}{5})_{\mathbf{H}} + H.c. \\
\supset\ & c_5 \frac{W_{\overline{\mathbf{5}}\,,{\rm IV}}}{\sqrt{2} M_{\rm pl}} \Big[(\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}\,, \mathbf{4}\,, -\frac{1}{12})_{\mathbf{F}}' \oplus (\mathbf{3}\,, \mathbf{4}\,, -\frac{1}{12})_{\mathbf{F}} \otimes (\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}}''' \Big] \otimes (\mathbf{1}\,, \overline{\mathbf{4}}\,, +\frac{3}{4})_{\mathbf{H}} + H.c. \\
\supset\ & c_5 \frac{W_{\overline{\mathbf{5}}\,,{\rm IV}}}{\sqrt{2} M_{\rm pl}} \Big[(\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}\,, \mathbf{3}\,, 0)_{\mathbf{F}}' \oplus (\mathbf{3}\,, \mathbf{3}\,, 0)_{\mathbf{F}} \otimes (\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}}''' \Big] \otimes (\mathbf{1}\,, \overline{\mathbf{3}}\,, +\frac{2}{3})_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{c_5}{2} \zeta_1 \left(u_L {t_R}^c + t_L {u_R}^c \right) v_{\rm EW} + H.c.\,,
\end{aligned} \tag{94}
$$

and

$$
\begin{aligned}
& \frac{c_5}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{\,,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} \Big[(\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}\,, \mathbf{10}\,, +\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{3}\,, \mathbf{5}\,, -\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{3}}\,, \mathbf{5}\,, -\frac{7}{15})_{\mathbf{F}} \Big] \\
\otimes\ & (\mathbf{1}\,, \overline{\mathbf{5}}\,, -\frac{1}{5})_{\mathbf{H}\,,\omega} \otimes (\mathbf{1}\,, \overline{\mathbf{5}}\,, +\frac{4}{5})_{\mathbf{H}} + H.c. \\
\supset\ & \frac{c_5}{M_{\rm pl}} \Big[(\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}\,, \mathbf{6}\,, +\frac{1}{6})_{\mathbf{F}} \oplus (\mathbf{3}\,, \mathbf{4}\,, -\frac{1}{12})_{\mathbf{F}} \otimes (\overline{\mathbf{3}}\,, \mathbf{4}\,, -\frac{5}{12})_{\mathbf{F}} \Big] \\
\otimes\ & \langle (\mathbf{1}\,, \overline{\mathbf{4}}\,, -\frac{1}{4})_{\mathbf{H}\,,\omega} \rangle \otimes (\mathbf{1}\,, \overline{\mathbf{4}}\,, +\frac{3}{4})_{\mathbf{H}} + H.c. \\
\supset\ & c_5 \frac{w_{\overline{\mathbf{4}}\,,{\rm V}}}{\sqrt{2} M_{\rm pl}} \Big[(\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}\,, \mathbf{3}\,, 0)_{\mathbf{F}}'' \oplus (\mathbf{3}\,, \mathbf{3}\,, 0)_{\mathbf{F}} \otimes (\overline{\mathbf{3}}\,, \mathbf{1}\,, -\frac{2}{3})_{\mathbf{F}}'' \Big] \otimes (\mathbf{1}\,, \overline{\mathbf{3}}\,, +\frac{2}{3})_{\mathbf{H}} + H.c. \\
\Rightarrow\ & \frac{c_5}{2} \zeta_2 \left(c_L {t_R}^c + t_L {c_R}^c \right) v_{\rm EW} + H.c.\,.
\end{aligned} \tag{95}
$$

4.5 The $d=5$ irreducible Higgs mixing operators

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\omega_1}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\omega_1}$, we find the mass terms of

$$
\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\omega_1}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\omega_1}\times\frac{d_{\mathscr{A}}}{M_{\rm pl}}\epsilon_{\omega_1\omega_2\omega_3\omega_4}\overline{\mathbf{8_H}}^\dagger_{,\omega_1}\overline{\mathbf{8_H}}^\dagger_{,\omega_2}\overline{\mathbf{8_H}}^\dagger_{,\omega_3}\overline{\mathbf{8_H}}^\dagger_{,\omega_4}\mathbf{70_H}^\dagger+H.c.\\
\supset\ & Y_{\mathcal{B}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})^{\omega_1}_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{5}\,,-\frac{2}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})^{\omega_1}_{\mathbf{F}}\otimes(\mathbf{1}\,,\mathbf{10}\,,+\frac{2}{5})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})_{\mathbf{H},\omega_1}\\
\times\ & \frac{d_{\mathscr{A}}}{M_{\rm pl}}(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})^\dagger_{\mathbf{H},\omega_1}\otimes(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})^\dagger_{\mathbf{H},\omega_2}\otimes(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})^\dagger_{\mathbf{H},\omega_3}\otimes\langle(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})^\dagger_{\mathbf{H},\omega_4}\rangle\otimes(\mathbf{1}\,,\overline{\mathbf{5}}\,,+\frac{4}{5})^\dagger_{\mathbf{H}}+H.c.\\
\supset\ & Y_{\mathcal{B}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})^{\omega_1}_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})^{\omega_1}_{\mathbf{F}}\otimes(\mathbf{1}\,,\mathbf{6}\,,+\frac{1}{2})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H},\omega_1}\\
\times\ & \frac{d_{\mathscr{A}}W_{\overline{\mathbf{5}},\rm IV}}{\sqrt{2}M_{\rm pl}}(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})^\dagger_{\mathbf{H},\omega_1}\otimes\langle(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})^\dagger_{\mathbf{H},\omega_2}\rangle\otimes(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})^\dagger_{\mathbf{H},\omega_3}\otimes(\mathbf{1}\,,\overline{\mathbf{4}}\,,+\frac{3}{4})^\dagger_{\mathbf{H}}+H.c.\\
\supset\ & Y_{\mathcal{B}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})^{\omega_1}_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^{\omega_1}_{\mathbf{F}}\otimes(\mathbf{1}\,,\mathbf{3}\,,+\frac{2}{3})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},\omega_1}\\
\times\ & d_{\mathscr{A}}\frac{W_{\overline{\mathbf{5}},\rm IV}w_{\overline{\mathbf{4}},\rm V}}{2M_{\rm pl}}(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H},\omega_1}\otimes\langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H},\omega_3}\rangle\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})^\dagger_{\mathbf{H}}+H.c.\\
\Rightarrow\ & \frac{Y_{\mathcal{B}}d_{\mathscr{A}}}{4}\frac{W_{\overline{\mathbf{5}},\rm IV}w_{\overline{\mathbf{4}},\rm V}V_{\overline{\mathbf{3}},\rm VI}}{M_{\rm pl}m^2_{(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})_{\mathbf{H},3}}}\,(b_Lb_R{}^c+\tau_L\tau_R{}^c)\,v_{\rm EW}+H.c.\,,
\end{aligned}\tag{96}
$$

via the Higgs mixing operator of $\mathcal{O}^{d=5}_{\mathscr{A}}$ in Eq. (14a).

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}$, we find the mass terms of

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot\omega_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega_1}\times\frac{d_{\mathscr{B}}}{M_{\rm pl}}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_1}\overline{\mathbf{28_H}}^\dagger_{,\dot\omega_2}\mathbf{70_H}^\dagger\left(\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm I}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\right)+H.c.\\
\supset\ & Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{10}\,,+\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{5}}\,,-\frac{1}{5})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{1}\,,\overline{\mathbf{10}}\,,+\frac{3}{5})_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{10}}\,,-\frac{2}{5})_{\mathbf{H},\dot\omega_1}\\
\times\ & \frac{d_{\mathscr{B}}}{M_{\rm pl}}(\mathbf{1}\,,\overline{\mathbf{10}}\,,-\frac{2}{5})^\dagger_{\mathbf{H},\dot\omega_1}\otimes(\mathbf{1}\,,\overline{\mathbf{10}}\,,-\frac{2}{5})^\dagger_{\mathbf{H},\dot\omega_2}\otimes(\mathbf{1}\,,\overline{\mathbf{5}}\,,+\frac{4}{5})^\dagger_{\mathbf{H}}\otimes\langle\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm I}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\rangle+H.c.\\
\supset\ & Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{4}\,,-\frac{1}{12})'_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{1}\,,\overline{\mathbf{6}}\,,+\frac{1}{2})'_{\mathbf{F}}\Big]\otimes(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H},\dot\omega_1}\\
\times\ & \frac{d_{\mathscr{B}}w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{2M_{\rm pl}}(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})^\dagger_{\mathbf{H},\dot\omega_1}\otimes(\mathbf{1}\,,\mathbf{6}\,,-\frac{1}{2})^\dagger_{\mathbf{H},\dot\omega_2}\otimes(\mathbf{1}\,,\overline{\mathbf{4}}\,,+\frac{3}{4})^\dagger_{\mathbf{H}}+H.c.\\
\supset\ & \frac{Y_{\mathcal{D}}d_{\mathscr{B}}w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{2M_{\rm pl}m^2_{(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H},\dot\omega_1}}}\Big[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{3}\,,\mathbf{3}\,,0)'_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^{\dot\omega_1}_{\mathbf{F}}\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})''_{\mathbf{F}}\Big]\\
\otimes\ & \langle(\mathbf{1}\,,\overline{\mathbf{3}}\,,-\frac{1}{3})^\dagger_{\mathbf{H},\dot\omega_2}\rangle\otimes(\mathbf{1}\,,\overline{\mathbf{3}}\,,+\frac{2}{3})^\dagger_{\mathbf{H}}+H.c.\\
\Rightarrow\ & \frac{Y_{\mathcal{D}}d_{\mathscr{B}}}{4}\dot\zeta_3\Big[\frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H},\dot{\rm I}}}}\,(d_Ld_R{}^c+e_Le_R{}^c)\\
+\ & \frac{w_{\overline{\mathbf{4}},\dot{\rm I}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\mathbf{1}\,,\overline{\mathbf{4}}\,,-\frac{1}{4})_{\mathbf{H},\dot 2}}}\,(d_Ls_R{}^c+\mu_Le_R{}^c)\Big]v_{\rm EW}+H.c.\,,
\end{aligned}\tag{97}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot{\omega}_1}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot{\omega}_1}\times\frac{d_{\mathscr{B}}}{M_{\rm pl}}\overline{\mathbf{28_H}}^\dagger_{,\dot{\omega}_1}\overline{\mathbf{28_H}}^\dagger_{,\dot{\omega}_2}\mathbf{70_H}^\dagger\left(\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm 1}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\right)+H.c.\\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}},\mathbf{1},+\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1}\otimes(\mathbf{3},\mathbf{10},+\frac{1}{15})_{\mathbf{F}}\oplus(\mathbf{1},\overline{\mathbf{5}},-\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1}\otimes(\mathbf{1},\mathbf{10},+\frac{3}{5})_{\mathbf{F}}\Big]\otimes(\mathbf{1},\overline{\mathbf{10}},-\frac{2}{5})_{\mathbf{H},\dot{\omega}_1}\\
\times\;& \frac{d_{\mathscr{B}}}{M_{\rm pl}}(\mathbf{1},\overline{\mathbf{10}},-\frac{2}{5})^\dagger_{\mathbf{H},\dot{\omega}_1}\otimes(\mathbf{1},\overline{\mathbf{10}},-\frac{2}{5})^\dagger_{\mathbf{H},\dot{\omega}_2}\otimes(\mathbf{1},\overline{\mathbf{5}},+\frac{4}{5})^\dagger_{\mathbf{H}}\otimes\langle\overline{\mathbf{28_H}}^\dagger_{,\dot{\rm 1}}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}\rangle+H.c.\\
\supset\;& Y_{\mathcal{D}}\Big[(\overline{\mathbf{3}},\mathbf{1},+\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1}\otimes(\mathbf{3},\mathbf{6},+\frac{1}{6})'_{\mathbf{F}}\oplus(\mathbf{1},\overline{\mathbf{4}},-\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1}\otimes(\mathbf{1},\mathbf{4},+\frac{3}{4})'_{\mathbf{F}}\Big]\otimes(\mathbf{1},\mathbf{6},-\frac{1}{2})_{\mathbf{H},\dot{\omega}_1}\\
\times\;& \frac{d_{\mathscr{B}}w_{\overline{\mathbf{4}},\dot{\rm 1}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{2M_{\rm pl}}(\mathbf{1},\mathbf{6},-\frac{1}{2})^\dagger_{\mathbf{H},\dot{\omega}_1}\otimes(\mathbf{1},\overline{\mathbf{4}},-\frac{1}{4})^\dagger_{\mathbf{H},\dot{\omega}_2}\otimes(\mathbf{1},\overline{\mathbf{4}},+\frac{3}{4})^\dagger_{\mathbf{H}}+H.c.\\
\supset\;& \frac{Y_{\mathcal{D}}d_{\mathscr{B}}w_{\overline{\mathbf{4}},\dot{\rm 1}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{2M_{\rm pl}m^2_{(\mathbf{1},\overline{\mathbf{6}},-\frac{1}{2})_{\mathbf{H},\dot{\omega}_1}}}\Big[(\overline{\mathbf{3}},\mathbf{1},+\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1}\otimes(\mathbf{3},\mathbf{3},0)''_{\mathbf{F}}\oplus(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1}\otimes(\mathbf{1},\overline{\mathbf{3}},+\frac{1}{3})'_{\mathbf{F}}\Big]\\
\otimes\;& \langle(\mathbf{1},\overline{\mathbf{3}},-\frac{1}{3})'_{\mathbf{H},\dot{\omega}_2}\rangle^\dagger\otimes(\mathbf{1},\overline{\mathbf{3}},+\frac{2}{3})^\dagger_{\mathbf{H}}+H.c.\\
\Rightarrow\;& \frac{Y_{\mathcal{D}}d_{\mathscr{B}}}{4}\dot{\zeta}_3'\Big[\frac{w_{\overline{\mathbf{4}},\dot{\rm 1}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\overline{\mathbf{6}},-\frac{1}{2})_{\mathbf{H},\dot{\rm 1}}}}(s_L{d_R}^c+e_L\mu_R^c)+\frac{w_{\overline{\mathbf{4}},\dot{\rm 1}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\overline{\mathbf{6}},-\frac{1}{2})_{\mathbf{H},\dot{2}}}}(s_L{s_R}^c+\mu_L\mu_R^c)\Big]v_{\rm EW}+H.c.\,,
\end{aligned}
\tag{98}
$$

via the Higgs mixing operator of $\mathcal{O}_{\mathscr{B}}^{d=5}$ in Eq. (14b). For convenience, we parametrize the following ratios of

$$
\Delta_{\dot{\omega}}\equiv\frac{w_{\overline{\mathbf{4}},\dot{\rm 1}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\overline{\mathbf{4}},-\frac{1}{4})_{\mathbf{H},\dot{\omega}}}}\,,\quad \Delta'_{\dot{\omega}}\equiv\frac{w_{\overline{\mathbf{4}},\dot{\rm 1}}w_{\overline{\mathbf{4}},\dot{\rm VII}}}{m^2_{(\mathbf{1},\overline{\mathbf{6}},-\frac{1}{2})_{\mathbf{H},\dot{\omega}}}}\,.
\tag{99}
$$

With the Higgs VEVs assignments in Eq. (81), we can obtain the following propagator masses of

$$
\begin{aligned}
& m_{(\mathbf{1},\overline{\mathbf{4}},-\frac{1}{4})_{\mathbf{H},\dot{\rm 1}}}\sim m_{(\mathbf{1},\overline{\mathbf{6}},-\frac{1}{2})_{\mathbf{H},\dot{\rm 1}}}\sim\mathcal{O}(v_{341})\,,\\
& m_{(\mathbf{1},\overline{\mathbf{4}},-\frac{1}{4})_{\mathbf{H},\dot{2}}}\sim m_{(\mathbf{1},\overline{\mathbf{6}},-\frac{1}{2})_{\mathbf{H},\dot{2}}}\sim\mathcal{O}(v_{331})\,.
\end{aligned}
\tag{100}
$$

4.6 The SM quark/lepton masses and the benchmark

For all up-type quarks with $Q_e=+\frac{2}{3}$, we write down the following tree-level masses from both the renormalizable Yukawa couplings and the gravity-induced terms in the basis of $\mathcal{U}\equiv(u\,,c\,,t\,)$

$$
\mathcal{M}_{\mathcal{U}}\;=\;\frac{1}{\sqrt{2}}\begin{pmatrix}0 & c_4\dot{\zeta}_2/\sqrt{2} & c_5\zeta_1/\sqrt{2}\\ c_4\dot{\zeta}_2/\sqrt{2} & 0 & c_5\zeta_2/\sqrt{2}\\ c_5\zeta_1/\sqrt{2} & c_5\zeta_2/\sqrt{2} & Y_{\mathcal{T}}\end{pmatrix}v_{\rm EW}\,.
\tag{101}
$$

For all down-type quarks with $Q_e=-\frac{1}{3}$, we find the following tree-level mass matrix in the basis of $\mathcal{D}\equiv(d\,,s\,,b\,)$

$$
\mathcal{M}_{\mathcal{D}}\approx\frac{1}{4}\begin{pmatrix}Y_{\mathcal{D}}d_{\mathscr{B}}\Delta_{\dot{\rm 1}}\dot{\zeta}_3 & Y_{\mathcal{D}}d_{\mathscr{B}}\Delta_{\dot{2}}\dot{\zeta}_3 & 0\\ Y_{\mathcal{D}}d_{\mathscr{B}}\Delta'_{\dot{\rm 1}}\dot{\zeta}_3' & Y_{\mathcal{D}}d_{\mathscr{B}}\Delta'_{\dot{2}}\dot{\zeta}_3' & 0\\ 0 & 0 & Y_{\mathcal{B}}d_{\mathscr{A}}\zeta_{23}^{-1}\zeta_1\end{pmatrix}v_{\rm EW}\,,
\tag{102}
$$

where we interpret the ratio between two $\mathfrak{g}_{331}$ VEVs as the Cabibbo angle of $\tan\lambda\equiv\frac{\dot{\zeta}_3'}{\dot{\zeta}_3}$. For all charged leptons with $Q_e=-1$, their tree-level mass matrix is related to the down-type quark mass matrix as

$\mathcal{M}_{\mathcal{L}} = \mathcal{M}_{\mathcal{D}}^T$. Based on the above SM quark/lepton mass matrices, we find the following benchmark point of

$$v_{351} \simeq 1.4 \times 10^{17}\,\text{GeV}\,, \quad v_{341} \simeq 4.8 \times 10^{15}\,\text{GeV}\,, \quad v_{331} \simeq 4.8 \times 10^{13}\,\text{GeV}\,, \tag{103}$$

to reproduce the observed hierarchical masses as well as the CKM mixing pattern, as summarized in Table 20.

ζ_1	ζ_2	$\dot{\zeta}_3'$	$Y_{\mathcal{D}}$	$Y_{\mathcal{B}}$	$Y_{\mathcal{T}}$
6.0×10^{-2}	2.0×10^{-3}	2.0×10^{-5}	0.5	0.5	0.8
λ	c_4	c_5	$d_{\mathscr{A}}$	$d_{\mathscr{B}}$	
0.22	0.2	1.0	0.01	0.01	
m_u	m_c	m_t	$m_d = m_e$	$m_s = m_\mu$	$m_b = m_\tau$
1.6×10^{-3}	0.6	139.2	0.5×10^{-3}	6.4×10^{-2}	1.5
$\lvert V_{ud}\rvert$	$\lvert V_{us}\rvert$	$\lvert V_{ub}\rvert$			
0.98	0.22	2.1×10^{-3}			
$\lvert V_{cd}\rvert$	$\lvert V_{cs}\rvert$	$\lvert V_{cb}\rvert$			
0.22	0.98	5.3×10^{-2}			
$\lvert V_{td}\rvert$	$\lvert V_{ts}\rvert$	$\lvert V_{tb}\rvert$			
0.013	5.3×10^{-2}	1			

Table 20: The parameters of the $\mathfrak{su}(8)$ benchmark point and the predicted SM quark/lepton masses (in unit of GeV) as well as the CKM mixings.

5 The intermediate stages of the SSS symmetry breaking pattern

$\mathfrak{su}(8)$	$\mathfrak{g}_{621}$	$\mathfrak{g}_{521}$	$\mathfrak{g}_{421}$	$\mathfrak{g}_{\rm SM}$
$\overline{\mathbf{8_F}}^\Omega$	$(\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{F}}^\Omega$	$(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^\Omega$	$(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^\Omega$	$(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^\Omega \ : \ \mathcal{D}_R^{\Omega\,c}$
				$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^\Omega \ : \ \check{\mathcal{N}}_L^\Omega$
			$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega'}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega'} \ : \ \check{\mathcal{N}}_L^{\Omega'}$
		$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''}$	$(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''} \ : \ \check{\mathcal{N}}_L^{\Omega''}$
	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^\Omega$	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^\Omega$	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^\Omega$	$(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^\Omega \ : \ \mathcal{L}_L^\Omega = (\mathcal{E}_L^\Omega\,, -\mathcal{N}_L^\Omega)^T$

Table 21: The $\mathfrak{su}(8)$ fermion representation of $\overline{\mathbf{8_F}}^\Omega$ under the $\mathfrak{g}_{621}\,,\mathfrak{g}_{521}\,,\mathfrak{g}_{421}\,,\mathfrak{g}_{\rm SM}$ subalgebras for the three-generational $\mathfrak{su}(8)$ theory, with $\Omega \equiv (\omega\,,\dot{\omega})$, $\omega = (3, \mathrm{IV}, \mathrm{V}, \mathrm{VI})$, $\dot{\omega} = (\dot{1}, \dot{2}, \dot{\mathrm{VII}}, \dot{\mathrm{VIII}}, \dot{\mathrm{IX}})$.

$\mathfrak{su}(8)$	$\mathfrak{g}_{621}$	$\mathfrak{g}_{521}$	$\mathfrak{g}_{421}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{28_F}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}$	$\underline{(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}} ~:~ \tau_R{}^c}$
	$(\mathbf{15}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}^{\prime}$	$(\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}^{\prime}$	$(\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}^{\prime}$	$\underline{(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}^{\prime} ~:~ t_R{}^c}$
				$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}} ~:~ \mathfrak{D}_L$
			$(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}^{\prime} ~:~ \mathfrak{D}_L^{\prime}$
				$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}} ~:~ \check{\mathfrak{n}}_R^c$
		$(\mathbf{5}, \mathbf{1}, -\frac{1}{5})_{\mathbf{F}}$	$(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}}^{\prime}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}^{\prime\prime} ~:~ \mathfrak{D}_L^{\prime\prime}$
				$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\prime} ~:~ \check{\mathfrak{n}}_R^{\prime c}$
			$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\prime\prime}$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\prime\prime} ~:~ \check{\mathfrak{n}}_R^{\prime\prime c}$
	$(\mathbf{6}, \mathbf{2}, +\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\mathbf{F}}$	$(\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}}$	$\underline{(\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} ~:~ (t_L, b_L)^T}$
				$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}} ~:~ (\mathfrak{e}_R{}^c, \mathfrak{n}_R{}^c)^T$
			$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}^{\prime}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}^{\prime} ~:~ (\mathfrak{e}_R^{\prime}{}^c, \mathfrak{n}_R^{\prime}{}^c)^T$
		$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}^{\prime\prime}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}^{\prime\prime}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}^{\prime\prime} ~:~ (\mathfrak{e}_R^{\prime\prime}{}^c, \mathfrak{n}_R^{\prime\prime}{}^c)^T$

Table 22: The $\mathfrak{su}(8)$ fermion representation of $\mathbf{28_F}$ under the $\mathfrak{g}_{621}, \mathfrak{g}_{521}, \mathfrak{g}_{421}, \mathfrak{g}_{\rm SM}$ subalgebras for the three-generational $\mathfrak{su}(8)$ theory. All IRs for SM fermions are marked with underlines.

$\mathfrak{su}(8)$	$\mathfrak{g}_{621}$	$\mathfrak{g}_{521}$	$\mathfrak{g}_{421}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{56_F}$	$(\mathbf{20}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}$	$(\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}}$	$(\overline{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}}$	$\underline{(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}^{\prime} ~:~ c_R{}^c}$
				$(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}} ~:~ \mathfrak{E}_L$
			$(\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}^{\prime}$	$\underline{(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}^{\prime\prime} ~:~ u_R{}^c}$
				$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime} ~:~ \mathfrak{D}_L^{\prime\prime\prime}$
		$(\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}^{\prime}$	$(\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}^{\prime\prime}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}^{\prime\prime\prime} ~:~ \mathfrak{U}_R{}^c$
				$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime\prime} ~:~ \mathfrak{D}_L^{\prime\prime\prime\prime}$
			$(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}}^{\prime\prime}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}^{\prime\prime\prime\prime\prime} ~:~ \mathfrak{D}_L^{\prime\prime\prime\prime\prime}$
				$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\prime\prime\prime} ~:~ \check{\mathfrak{n}}_R^{\prime\prime\prime\,c}$
	$(\mathbf{15}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}}$	$(\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}}$	$(\overline{\mathbf{3}}, \mathbf{2}, -\frac{1}{6})_{\mathbf{F}} ~:~ (\mathfrak{d}_R{}^c, \mathfrak{u}_R{}^c)^T$
				$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}}^{\prime} ~:~ (\mathfrak{u}_L, \mathfrak{d}_L)^T$
			$(\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}}^{\prime}$	$\underline{(\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}}^{\prime\prime} ~:~ (c_L, s_L)^T}$
				$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}^{\prime\prime\prime} ~:~ (\mathfrak{e}_R^{\prime\prime\prime c}, \mathfrak{n}_R^{\prime\prime\prime c})^T$
		$(\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\mathbf{F}}^{\prime}$	$(\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}}^{\prime\prime}$	$\underline{(\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}}^{\prime\prime\prime} ~:~ (u_L, d_L)^T}$
				$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}^{\prime\prime\prime\prime} ~:~ (\mathfrak{e}_R^{\prime\prime\prime\prime c}, \mathfrak{n}_R^{\prime\prime\prime\prime c})^T$
			$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}^{\prime\prime\prime\prime\prime}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}^{\prime\prime\prime\prime\prime} ~:~ (\mathfrak{e}_R^{\prime\prime\prime\prime\prime c}, \mathfrak{n}_R^{\prime\prime\prime\prime\prime c})^T$
	$(\mathbf{6}, \mathbf{1}, +\frac{5}{6})_{\mathbf{F}}$	$(\mathbf{5}, \mathbf{1}, +\frac{4}{5})_{\mathbf{F}}$	$(\mathbf{4}, \mathbf{1}, +\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, +\frac{2}{3})_{\mathbf{F}} ~:~ \mathfrak{U}_L$
				$\underline{(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}^{\prime} ~:~ e_R{}^c}$
			$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}^{\prime\prime}$	$\underline{(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}^{\prime\prime} ~:~ \mu_R{}^c}$
		$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}^{\prime\prime\prime}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}^{\prime\prime\prime}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}^{\prime\prime\prime} ~:~ \mathfrak{E}_R{}^c$

Table 23: The $\mathfrak{su}(8)$ fermion representation of $\mathbf{56_F}$ under the $\mathfrak{g}_{621}, \mathfrak{g}_{521}, \mathfrak{g}_{421}, \mathfrak{g}_{\rm SM}$ subalgebras. All IRs for SM fermions are marked with underlines.

Schematically, we assign the Higgs VEVs according to the symmetry breaking pattern as follows:

$$\mathfrak{g}_{621}\to\mathfrak{g}_{521}\quad:\quad\langle(\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{H}\,,\mathrm{IV}}\rangle\equiv\frac{1}{\sqrt{2}}W_{\overline{\mathbf{6}}\,,\mathrm{IV}}\,,\tag{104a}$$

$$\mathfrak{g}_{521}\to\mathfrak{g}_{421}\quad:\quad\langle(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\mathrm{V}}\rangle\equiv\frac{1}{\sqrt{2}}w_{\overline{\mathbf{5}}\,,\mathrm{V}}\,,\ \langle(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\dot{1},\dot{\mathrm{V}}\mathrm{II}}\rangle\equiv\frac{1}{\sqrt{2}}w_{\overline{\mathbf{5}}\,,\dot{1},\dot{\mathrm{V}}\mathrm{II}}\,,\tag{104b}$$

$$\begin{aligned}\mathfrak{g}_{421}\to\mathfrak{g}_{\mathrm{SM}}\quad:\quad&\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,3,\mathrm{VI}}\rangle\equiv\frac{1}{\sqrt{2}}V_{\overline{\mathbf{4}}\,,3,\mathrm{VI}}\,,\\&\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot{\mathrm{V}}\mathrm{III}}\rangle\equiv\frac{1}{\sqrt{2}}V_{\overline{\mathbf{4}}\,,\dot{\mathrm{V}}\mathrm{III}}\,,\ \langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})'_{\mathbf{H}\,,\dot{2},\dot{\mathrm{I}}\mathrm{X}}\rangle\equiv\frac{1}{\sqrt{2}}V'_{\overline{\mathbf{4}}\,,\dot{2},\dot{\mathrm{I}}\mathrm{X}}\,,\end{aligned}\tag{104c}$$

$$\mathrm{EWSB}\quad:\quad\langle(\mathbf{1}\,,\overline{\mathbf{2}}\,,+\frac{1}{2})_{\mathbf{H}}\rangle\equiv\frac{1}{\sqrt{2}}v_{\mathrm{EW}}\,.\tag{104d}$$

We also decompose the Higgs fields into components that can be responsible for the sequential symmetry breaking pattern as follows

$$\mathbf{8_H}^{\omega}\supset\langle(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{6})_{\mathbf{H}}^{\omega}\rangle\supset\langle(\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{H}}^{\omega}\rangle\supset\langle(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{H}}^{\omega}\rangle\,,\tag{105}$$

$$\overline{\mathbf{8_H}}_{,\omega}\supset\langle(\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{H}\,,\omega}\rangle\supset\langle(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega}\rangle\supset\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\omega}\rangle\,,\tag{106}$$

$$\begin{aligned}\mathbf{28_H}^{\dot\omega}&\supset\underline{(\mathbf{15}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{H}}^{\dot\omega}}\oplus(\mathbf{6}\,,\mathbf{2}\,,+\frac{1}{3})_{\mathbf{H}}^{\dot\omega}\oplus(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{H}}^{\dot\omega}\\&\supset\langle(\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{H}}^{\dot\omega}\rangle\oplus\underline{(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{H}}^{\dot\omega}}\\&\supset\langle(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{H}}^{\dot\omega}\rangle\oplus(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{H}}^{\dot\omega}\oplus\langle(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{H}}^{\prime\dot\omega}\rangle\,,\end{aligned}\tag{107}$$

$$\begin{aligned}\overline{\mathbf{28_H}}_{,\dot\omega}&\supset\underline{(\overline{\mathbf{15}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{H}\,,\dot\omega}}\oplus(\overline{\mathbf{6}}\,,\overline{\mathbf{2}}\,,-\frac{1}{3})_{\mathbf{H}\,,\dot\omega}\oplus(\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{H}\,,\dot\omega}\\&\supset\langle(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\dot\omega}\rangle\oplus\underline{(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H}\,,\dot\omega}}\\&\supset\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot\omega}\rangle\oplus(\mathbf{6}\,,\mathbf{1}\,,+\frac{1}{2})_{\mathbf{H}\,,\dot\omega}\oplus\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})'_{\mathbf{H}\,,\dot\omega}\rangle\,,\end{aligned}\tag{108}$$

$$\begin{aligned}\mathbf{70_H}&\supset\underline{(\mathbf{20}\,,\mathbf{2}\,,0)_{\mathbf{H}}}\oplus(\mathbf{15}\,,\mathbf{1}\,,+\frac{2}{3})_{\mathbf{H}}\oplus(\overline{\mathbf{15}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{H}}\\&\supset\underline{(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{H}}}\supset\underline{(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{H}}}\supset\langle(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{H}}\rangle\,,\end{aligned}\tag{109}$$

where all components that are likely to develop VEVs are labelled by $\langle...\rangle$. For later convenience, we also parametrize different symmetry breaking VEVs in terms of the following dimensionless quantities:

$$\begin{aligned}&\zeta_1\equiv\frac{W_{\overline{\mathbf{6}}\,,\mathrm{IV}}}{M_{\mathrm{pl}}}\,,\quad\zeta_2\equiv\frac{w_{\overline{\mathbf{5}}\,,\mathrm{V}}}{M_{\mathrm{pl}}}\,,\quad\dot\zeta_2\equiv\frac{w_{\overline{\mathbf{5}}\,,\dot{1},\dot{\mathrm{V}}\mathrm{II}}}{M_{\mathrm{pl}}}\,,\\&\zeta_3\equiv\frac{V_{\overline{\mathbf{4}}\,,3,\mathrm{VI}}}{M_{\mathrm{pl}}}\,,\quad\dot\zeta_3'\equiv\frac{V'_{\overline{\mathbf{4}}\,,\dot{2},\dot{\mathrm{I}}\mathrm{X}}}{M_{\mathrm{pl}}}\,,\quad\dot\zeta_3\equiv\frac{V_{\overline{\mathbf{4}}\,,\dot{\mathrm{V}}\mathrm{III}}}{M_{\mathrm{pl}}}\,,\\&\zeta_0\gg\zeta_1\gg\zeta_2\sim\dot\zeta_2\gg\zeta_3\sim\dot\zeta_3'\sim\dot\zeta_3\,,\quad\zeta_{ij}\equiv\frac{\zeta_j}{\zeta_i}\,,\quad(i<j)\,.\end{aligned}\tag{110}$$

In Table 24, we summarize all vectorlike fermions that become massive during different stages of the SSS symmetry breaking pattern.

stages	$Q_e = -\frac{1}{3}$	$Q_e = +\frac{2}{3}$	$Q_e = -1$	$Q_e = 0$
v_{621} $\{\Omega\}$	$\mathfrak{D}''$ IV	-	$(\mathfrak{e}''\,,\mathfrak{n}'')$ IV	$\{\check{\mathfrak{n}}'\,,\check{\mathfrak{n}}''\}$ $\{\text{IV}\,,\text{IV}'\}$
v_{521} $\{\Omega\}$	$\{\mathfrak{D}'\,,\mathfrak{D}'''''\}$ $\{\text{V}\,,\dot{\text{VII}}\}$	-	$(\mathfrak{e}'\,,\mathfrak{n}')\,,(\mathfrak{e}'''''\,,\mathfrak{n}''''')$ $\{\text{V}\,,\dot{\text{VII}}\}$	$\{\check{\mathfrak{n}}\,,\check{\mathfrak{n}}'''\}$ $\{\text{V}\,,\dot{\text{VII}}\}$
v_{421} $\{\Omega\}$	$\{\mathfrak{D}\,,\mathfrak{D}''''\,,\mathfrak{D}'''\}$ $\{\text{VI}\,,\dot{\text{VIII}}\,,\dot{\text{IX}}\}$	$\mathfrak{U}$	$\mathfrak{E}\,,(\mathfrak{e}\,,\mathfrak{n})\,,(\mathfrak{e}''''\,,\mathfrak{n}'''')\,,(\mathfrak{e}'''\,,\mathfrak{n}''')$ $\{\text{VI}\,,\dot{\text{VIII}}\,,\dot{\text{IX}}\}$	-

Table 24: The vectorlike fermions at different intermediate symmetry breaking stages along the SSS symmetry breaking pattern of the $\mathfrak{su}(8)$ theory.

5.1 The first stage

The first symmetry breaking stage of $\mathfrak{g}_{621} \to \mathfrak{g}_{521}$ is achieved by $(\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{H}\,,\text{IV}} \subset \overline{\mathbf{8}_{\mathbf{H}}}_{\,,\text{IV}}$ in the rank-two sector.

$$
\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8}_{\mathbf{F}}}^{\text{IV}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}_{\mathbf{H}}}_{\,,\text{IV}} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{F}}^{\text{IV}} \otimes (\mathbf{15}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}' \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\text{IV}} \otimes (\mathbf{6}\,,\mathbf{2}\,,+\frac{1}{3})_{\mathbf{F}} \Big] \otimes \langle (\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{H}\,,\text{IV}} \rangle + H.c. \\
\Rightarrow\ & \frac{1}{\sqrt{2}} Y_{\mathcal{B}} \left(\mathfrak{D}_L'' \mathfrak{D}_R''^{\,c} + \check{\mathfrak{n}}_L' \check{\mathfrak{n}}_R'^{\,c} + \check{\mathfrak{n}}_L'' \check{\mathfrak{n}}_R''^{\,c} + \mathfrak{e}_L'' \mathfrak{e}_R''^{\,c} - \mathfrak{n}_L'' \mathfrak{n}_R''^{\,c} \right) W_{\overline{\mathbf{6}},\text{IV}} + H.c.\,.
\end{aligned} \tag{111}
$$

After this stage, the remaining massless fermions expressed in terms of the $\mathfrak{g}_{521}$ IRs are the following:

$$
\begin{aligned}
& (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\Omega} \oplus \Big[(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''} \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\Omega} \Big] \subset \overline{\mathbf{8}_{\mathbf{F}}}^{\Omega}\,, \\
& \Omega = (\omega\,,\dot{\omega})\,, \quad \omega = (3\,,\text{V}\,,\text{VI})\,, \quad \dot{\omega} = (\dot{1}\,,\dot{2}\,,\dot{\text{VII}}\,,\dot{\text{VIII}}\,,\dot{\text{IX}})\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\text{IV}''} \subset \overline{\mathbf{8}_{\mathbf{F}}}^{\text{IV}}\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}} \oplus \Big[(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}' \oplus \cancel{(\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{F}}} \Big] \oplus \Big[(\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}} \oplus \cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}''} \Big] \subset \mathbf{28}_{\mathbf{F}}\,, \\
& \Big[(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}' \Big] \oplus \Big[(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}} \oplus (\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}}' \Big] \oplus \Big[(\mathbf{5}\,,\mathbf{1}\,,+\frac{4}{5})_{\mathbf{F}} \\
\oplus\ & (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}''' \Big] \subset \mathbf{56}_{\mathbf{F}}\,.
\end{aligned} \tag{112}
$$

5.2 The second stage

The second symmetry breaking stage of $\mathfrak{g}_{521} \to \mathfrak{g}_{421}$ is achieved by $(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\text{V}} \subset \overline{\mathbf{8}_{\mathbf{H}}}_{\,,\text{V}}$ in the rank-two sector, $(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\dot{\text{VII}}} \subset \overline{\mathbf{28}_{\mathbf{H}}}_{\,,\dot{\text{VII}}}$ and $(\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{H}}^{\dot{\text{VII}}} \subset \mathbf{28}_{\mathbf{H}}^{\dot{\text{VII}}}$ in the rank-three sector.

$$
\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8}_{\mathbf{F}}}^{\text{V}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}_{\mathbf{H}}}_{\,,\text{V}} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{F}}^{\text{V}} \otimes (\mathbf{15}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}' \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\text{V}} \otimes (\mathbf{6}\,,\mathbf{2}\,,+\frac{1}{3})_{\mathbf{F}} \Big] \otimes (\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{H}\,,\text{V}} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\text{V}} \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}' \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\text{V}} \otimes (\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}} \Big] \otimes \langle (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\text{V}} \rangle + H.c. \\
\Rightarrow\ & \frac{1}{\sqrt{2}} Y_{\mathcal{B}} \left(\mathfrak{D}_L' \mathfrak{D}_R'^{\,c} + \check{\mathfrak{n}}_L \check{\mathfrak{n}}_R^{\,c} + \mathfrak{e}_L' \mathfrak{e}_R'^{\,c} - \mathfrak{n}_L' \mathfrak{n}_R'^{\,c} \right) w_{\overline{\mathbf{5}},\text{V}} + H.c.\,.
\end{aligned} \tag{113}
$$

The term of $Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot{\rm VII}}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot{\rm VII}}+H.c.$ leads to the vectorial masses of $(\mathfrak{D}''''' \,,\mathfrak{e}''''' \,,\mathfrak{n}''''' \,,\check{\mathfrak{n}}''')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot{\rm VII}}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot{\rm VII}} + H.c. \\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{F}}^{\dot{\rm VII}}\otimes(\mathbf{20}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\dot{\rm VII}}\otimes(\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}\Big]\otimes(\overline{\mathbf{15}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{H}\,,\dot{\rm VII}}+H.c. \\
\supset\quad & Y_{\mathcal{D}}\Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot{\rm VII}}\otimes(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\dot{\rm VII}}\otimes(\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}}'\Big]\otimes\langle(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\dot{\rm VII}}\rangle+H.c. \\
\Rightarrow\quad & \frac{1}{\sqrt{2}}Y_{\mathcal{D}}\left(\mathfrak{D}_L'''''\mathfrak{D}_R'''''^{c}+\check{\mathfrak{n}}_L'''\check{\mathfrak{n}}_R'''^{c}+\mathfrak{e}_L'''''\mathfrak{e}_R'''''^{c}-\mathfrak{n}_L'''''\mathfrak{n}_R'''''^{c}\right)w_{\overline{\mathbf{5}}\,,\dot{\rm VII}}+H.c.\,. \qquad (114)
\end{aligned}
$$

The Yukawa coupling between two $\mathbf{56_F}$'s by the following $d=5$ operator $\frac{c_4}{M_{\rm pl}}\mathbf{56_F}\mathbf{56_F}\mathbf{63_H}\mathbf{28_H}^{\dot{\rm VII}}$

$$
\begin{aligned}
& \frac{1}{M_{\rm pl}}\mathbf{56_F}\mathbf{56_F}\langle\mathbf{63_H}\rangle\mathbf{28_H}^{\dot\omega} \\
\supset\quad & \zeta_0(\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}\otimes(\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}\otimes(\mathbf{15}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{H}}^{\dot\omega} \\
\supset\quad & \zeta_0(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}}\otimes(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}}\otimes\langle(\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{H}}^{\dot\omega}\rangle\,, \qquad (115)
\end{aligned}
$$

has been shown to be vanishing in the main context.

After integrating out the massive fermions, the remaining massless fermions expressed in terms of the $\mathfrak{g}_{421}$ IRs are the following:

$$
\begin{aligned}
& \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\Omega}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega}\Big]\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\Omega''}\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\Omega}\subset\overline{\mathbf{8_F}}^{\Omega}\,, \\
& \Omega=(\omega\,,\dot\omega)\,,\quad \omega=(3\,,{\rm VI})\,,\quad \dot\omega=(\dot{1}\,,\dot{2}\,,{\rm V\dot{I}II}\,,{\rm \dot{I}X})\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{{\rm IV}''}\subset\overline{\mathbf{8_F}}^{\rm IV}\,,\quad (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{{\rm V}'}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{{\rm V}''}\subset\overline{\mathbf{8_F}}^{\rm V}\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\dot{\rm VII}'}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}^{\dot{\rm VII}''}\subset\overline{\mathbf{8_F}}^{\dot{\rm VII}}\,, \\
& (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}\oplus\Big[(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}'\oplus\cancel{(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}}\Big]\oplus\Big[\cancel{(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}'\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}''}\Big] \\
\oplus\quad & \Big[(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}}\oplus\cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}{}'}\Big]\oplus\cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}{}''}\subset\mathbf{28_F}\,, \\
& \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}}\oplus(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}'\Big]\oplus\Big[(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}''\oplus\cancel{(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}''}\Big]\oplus\Big[(\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}}\oplus(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}}'\Big] \\
\oplus\quad & \Big[\cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}'''''}\oplus(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}}''\Big]\oplus\Big[(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}''\oplus\cancel{(\mathbf{4}\,,\mathbf{1}\,,+\frac{3}{4})_{\mathbf{F}}}\Big]\oplus(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}'''\subset\mathbf{56_F}\,. \qquad (116)
\end{aligned}
$$

5.3 The third stage

The third symmetry breaking stage of $\mathfrak{g}_{421}\to\mathfrak{g}_{\rm SM}$ is achieved by $(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,{\rm VI}}\subset\overline{\mathbf{8_H}}_{,{\rm VI}}$ in the rank-two sector and $(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,{\rm V\dot{I}II}}\subset\overline{\mathbf{28_H}}_{,{\rm V\dot{I}II}}\,,$ $(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,{\rm \dot{I}X}}'\subset\overline{\mathbf{28_H}}_{,{\rm \dot{I}X}}\,,$ $(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{H}}^{{\rm \dot{I}X}'}\subset\mathbf{28_H}^{\rm \dot{I}X}$ in the rank-three sector.

$$
\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\rm VI} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\rm VI} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{F}}^{\rm VI} \otimes (\mathbf{15}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}' \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\rm VI} \otimes (\mathbf{6}\,,\mathbf{2}\,,+\frac{1}{3})_{\mathbf{F}} \Big] \otimes (\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{H},\rm VI} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\rm VI} \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}' \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\rm VI} \otimes (\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}} \Big] \otimes (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H},\rm VI} + H.c. \\
\supset\ & Y_{\mathcal{B}} \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\rm VI} \otimes (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}' \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\rm VI} \otimes (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}} \Big] \otimes \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H},\rm VI} \rangle + H.c. \\
\Rightarrow\ & \frac{1}{\sqrt{2}} Y_{\mathcal{B}} \left(\mathfrak{D}_L \mathfrak{D}_R^c + \mathfrak{e}_L {\mathfrak{e}_R}^c - \mathfrak{n}_L {\mathfrak{n}_R}^c \right) V_{\overline{\mathbf{4}},\rm VI} + H.c.\,.
\end{aligned} \tag{117}
$$

The term of $Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\dot{\rm VIII}\,,\dot{\rm IX}} \mathbf{56_F} \overline{\mathbf{28}}_{\mathbf{H}\,,\dot{\rm VIII}\,,\dot{\rm IX}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}''''\,,\mathfrak{e}''''\,,\mathfrak{n}''''\,,\mathfrak{D}'''\,,\mathfrak{e}'''\,,\mathfrak{n}''')$ as follows:

$$
\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\dot{\rm VIII}} \mathbf{56_F} \overline{\mathbf{28}}_{\mathbf{H}\,,\dot{\rm VIII}} + H.c. \\
\supset\ & Y_{\mathcal{D}} \Big[(\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{F}}^{\dot{\rm VIII}} \otimes (\mathbf{20}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\dot{\rm VIII}} \otimes (\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}} \Big] \otimes (\overline{\mathbf{15}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{H}\,,\dot{\rm VIII}} + H.c. \\
\supset\ & Y_{\mathcal{D}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot{\rm VIII}} \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}' \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\dot{\rm VIII}} \otimes (\mathbf{5}\,,\mathbf{2}\,,+\frac{3}{10})_{\mathbf{F}}' \Big] \otimes (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\dot{\rm VIII}} + H.c. \\
\supset\ & Y_{\mathcal{D}} \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\dot{\rm VIII}} \otimes (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}'' \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\dot{\rm VIII}} \otimes (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}}'' \Big] \otimes \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot{\rm VIII}} \rangle + H.c. \\
\Rightarrow\ & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} \left(\mathfrak{D}_L'''' \mathfrak{D}_R''''{}^c + \mathfrak{e}_L'''' \mathfrak{e}_R''''{}^c - \mathfrak{n}_L'''' \mathfrak{n}_R''''{}^c \right) V_{\overline{\mathbf{4}},\dot{\rm VIII}} + H.c.\,,
\end{aligned} \tag{118}
$$

and

$$
\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\dot{\rm IX}} \mathbf{56_F} \overline{\mathbf{28}}_{\mathbf{H}\,,\dot{\rm IX}} + H.c. \\
\supset\ & Y_{\mathcal{D}} \Big[(\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{F}}^{\dot{\rm IX}} \otimes (\mathbf{20}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\dot{\rm IX}} \otimes (\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}} \Big] \otimes (\overline{\mathbf{15}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{H}\,,\dot{\rm IX}} + H.c. \\
\supset\ & Y_{\mathcal{D}} \Big[(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{F}}^{\dot{\rm IX}} \otimes (\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\dot{\rm IX}} \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}} \Big] \otimes (\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H}\,,\dot{\rm IX}} + H.c. \\
\supset\ & Y_{\mathcal{D}} \Big[(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{F}}^{\dot{\rm IX}} \otimes (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}' \oplus (\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}^{\dot{\rm IX}} \otimes (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}}' \Big] \otimes \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot{\rm IX}}' \rangle + H.c. \\
\Rightarrow\ & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} \left(\mathfrak{D}_L''' \mathfrak{D}_R'''{}^c + \mathfrak{e}_L''' \mathfrak{e}_R'''{}^c - \mathfrak{n}_L''' \mathfrak{n}_R'''{}^c \right) V_{\overline{\mathbf{4}},\dot{\rm IX}}' + H.c.\,.
\end{aligned} \tag{119}
$$

The Yukawa coupling between two $\mathbf{56_F}$'s by the following $d=5$ operator

$$
\begin{aligned}
& \frac{c_4}{M_{\rm pl}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\dot{\rm IX}} + H.c. \\
\supset\ & c_4 \zeta_0 \Big[(\mathbf{6}\,,\mathbf{1}\,,+\frac{5}{6})_{\mathbf{F}} \otimes (\mathbf{20}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}} \oplus \cancel{(\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}} \otimes (\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}} \Big] \otimes (\mathbf{15}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{H}}^{\dot{\rm IX}} + H.c. \\
\supset\ & c_4 \zeta_0 \Big[(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}''' \otimes (\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{5}\,,\mathbf{1}\,,+\frac{4}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}' \Big] \otimes (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{H}}^{\dot{\rm IX}} + H.c. \\
\supset\ & c_4 \zeta_0 \Big[(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}''' \otimes (\overline{\mathbf{4}}\,,\mathbf{1}\,,-\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{4}\,,\mathbf{1}\,,+\frac{3}{4})_{\mathbf{F}} \otimes (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}'' \Big] \otimes \langle (\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{H}}^{\dot{\rm IX}'} \rangle + H.c. \\
\Rightarrow\ & \frac{c_4}{\sqrt{2}} \zeta_0 \left(\mathfrak{E}_L {\mathfrak{E}_R}^c + \mathfrak{U}_L {\mathfrak{U}_R}^c \right) V_{\mathbf{4}}^{\dot{\rm IX}'} + H.c.\,,
\end{aligned} \tag{120}
$$

further leads to massive vectorlike fermions of $(\mathfrak{E}\,,\mathfrak{U})$.

The remaining massless fermions of the $\mathfrak{g}_{\rm SM}$ are listed as follows:

$$
\begin{aligned}
& \left[(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})^{\Omega}_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)^{\Omega}_{\mathbf{F}}\right]\oplus(\mathbf{1}\,,\mathbf{1}\,,0)^{\Omega'}_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)^{\Omega''}_{\mathbf{F}}\oplus(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})^{\Omega}_{\mathbf{F}}\subset\overline{\mathbf{8_F}}^{\Omega}\,,\quad \Omega=(\dot{1}\,,\dot{2}\,,3)\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,0)^{\rm VI}_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)^{\rm V\dot{I}II}_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)^{\rm I\dot{X}}_{\mathbf{F}}\subset\overline{\mathbf{8_F}}^{\Omega}\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,0)^{\rm V'}_{\mathbf{F}}\oplus...\oplus(\mathbf{1}\,,\mathbf{1}\,,0)^{\rm I\dot{X}'}_{\mathbf{F}}\subset\overline{\mathbf{8_F}}^{\Omega'}\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,0)^{\rm IV''}_{\mathbf{F}}\oplus...\oplus(\mathbf{1}\,,\mathbf{1}\,,0)^{\rm I\dot{X}''}_{\mathbf{F}}\subset\overline{\mathbf{8_F}}^{\Omega''}\,,\\
& (\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}\oplus\left[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})'_{\mathbf{F}}\right]\oplus\left[\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})'_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)_{\mathbf{F}}}\right]\\
\oplus\quad & \left[\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})''_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)'_{\mathbf{F}}}\right]\oplus(\mathbf{1}\,,\mathbf{1}\,,0)''_{\mathbf{F}}\left[\xcancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}}\oplus(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}\right]\oplus\\
\oplus\quad & \xcancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})'_{\mathbf{F}}}\oplus\cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{F}}}{}''\subset\mathbf{28_F}\,,\\
& \left[\xcancel{(\mathbf{1}\,,\mathbf{1}\,,-1)_{\mathbf{F}}}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})'_{\mathbf{F}}\right]\oplus\left[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})'''_{\mathbf{F}}}\oplus(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})''_{\mathbf{F}}\right]\oplus\left[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})''''_{\mathbf{F}}}\oplus\xcancel{(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})'''_{\mathbf{F}}}\right]\\
\oplus\quad & \left[\cancel{(\mathbf{3}\,,\mathbf{1}\,,-\frac{1}{3})'''''_{\mathbf{F}}\oplus(\mathbf{1}\,,\mathbf{1}\,,0)'''_{\mathbf{F}}}\right]\oplus\left[(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})'_{\mathbf{F}}\oplus(\overline{\mathbf{3}}\,,\mathbf{2}\,-\frac{1}{6})_{\mathbf{F}}\right]\oplus\left[(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})''_{\mathbf{F}}\oplus\xcancel{(\mathbf{1}\,,\mathbf{2}\,+\frac{1}{2})'''_{\mathbf{F}}}\right]\\
\oplus\quad & \left[(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})'''_{\mathbf{F}}\oplus\xcancel{(\mathbf{1}\,,\mathbf{2}\,+\frac{1}{2})''''_{\mathbf{F}}}\right]\oplus\cancel{(\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})'''''_{\mathbf{F}}}\oplus\left[\xcancel{(\mathbf{3}\,,\mathbf{1}\,,+\frac{2}{3})_{\mathbf{F}}}\oplus(\mathbf{1}\,,\mathbf{1}\,,+1)'_{\mathbf{F}}\right]\oplus(\mathbf{1}\,,\mathbf{1}\,,+1)''_{\mathbf{F}}\\
\oplus\quad & \xcancel{(\mathbf{1}\,,\mathbf{1}\,,+1)'''_{\mathbf{F}}}\subset\mathbf{56_F}\,.
\end{aligned}
\tag{121}
$$

5.4 The $d=5$ operators

In this section, we write down all the $d=5$ operators that may contribute to the masses of $\mathfrak{u}$ and $\mathfrak{d}$. First, one can certainly consider an alternative $d=5$ operator of the form $\frac{1}{M_{\rm pl}}\mathbf{56_F}\mathbf{56_F}\mathbf{8_H}^{\omega_1}\mathbf{8_H}^{\omega_2}$, we can perform a similar analysis to that of the operator $\frac{1}{M_{\rm pl}}\mathbf{56_F}\mathbf{56_F}\langle\mathbf{63_H}\rangle\mathbf{28_H}^{\dot\omega}$ as follows:

$$
\begin{aligned}
& \frac{1}{M_{\rm pl}}\mathbf{56_F}\mathbf{56_F}\mathbf{8_H}^{\omega_1}\mathbf{8_H}^{\omega_2}\\
\supset\quad & \frac{1}{M_{\rm pl}}(\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}\otimes(\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{6})^{\omega_1}_{\mathbf{H}}\otimes(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{6})^{\omega_2}_{\mathbf{H}}\\
\supset\quad & \zeta_1(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}}\otimes(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}}\otimes\langle(\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})^{\omega_1}_{\mathbf{H}}\rangle\,.
\end{aligned}
\tag{122}
$$

It is clear that this operator must also vanish due to antisymmetric properties. Essentially, regardless of how the Higgs fields are arranged to form a higher-dimensional gauge-invariant bilinear fermionic operator involving two $\mathbf{56_F}$, the vectorlike ($\mathfrak{u}$, $\mathfrak{d}$) quarks always transform as $(\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}$ under the $\mathfrak{su}(8)\to\mathfrak{g}_{621}$ symmetry breaking pattern. Consequently, such a bilinear fermionic term is inevitable and must vanish due to antisymmetric properties.

Let us further consider two $d=5$ gauge-invariant terms of $\frac{1}{M_{\rm pl}}\mathbf{28_F}\mathbf{56_F}\mathbf{8_H}^{\omega}\mathbf{28_H}^{\dot\omega}$ and $\frac{1}{M_{\rm pl}}\mathbf{28_F}\mathbf{56_F}\overline{\mathbf{8_H}}_{\,,\omega}\mathbf{70_H}$, and one $d=6$ term of $\frac{1}{M_{\rm pl}^2}\mathbf{28_F}\mathbf{56_F}\mathbf{70_H}\overline{\mathbf{28_H}}_{\,,\dot\omega}\mathbf{8_H}^{\omega}$. Their contributions from the $\frac{1}{M_{\rm pl}}\mathbf{28_F}\mathbf{56_F}\mathbf{8_H}^{\omega}\mathbf{28_H}^{\dot\omega}$

are explicitly given as follows

$$\begin{aligned}
& \frac{1}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \mathbf{8_H}^{\omega} \mathbf{28_H}^{\dot\omega} \\
\supset\quad & \frac{1}{M_{\rm pl}} (\mathbf{6}\,,\mathbf{2}\,,+\frac{1}{3})_{\mathbf{F}} \otimes (\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}} \otimes \langle (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{6})_{\mathbf{H}}^{\omega} \rangle \otimes (\mathbf{15}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{H}}^{\dot\omega} \\
\supset\quad & \zeta_1 (\mathbf{5}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}} \otimes (\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{H}}^{\dot\omega} \\
\supset\quad & \zeta_1 (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{F}} \otimes (\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}} \otimes \langle (\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{H}}^{\dot\omega} \rangle \\
\Rightarrow\quad & \frac{\zeta_1}{\sqrt{2}} V_{421} (\mathfrak{u}_R{}^c t_L + \mathfrak{d}_R{}^c b_L)\,,
\end{aligned} \tag{123}$$

and for the $\frac{1}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \overline{\mathbf{8_H}}_{,\omega} \mathbf{70_H}$ operator we have

$$\begin{aligned}
& \frac{1}{M_{\rm pl}} \mathbf{28_F} \mathbf{56_F} \overline{\mathbf{8_H}}_{,\omega} \mathbf{70_H} \\
\supset\quad & \frac{1}{M_{\rm pl}} (\mathbf{15}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}' \otimes (\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}} \otimes \langle (\overline{\mathbf{6}}\,,\mathbf{1}\,,+\frac{1}{6})_{\mathbf{H}\,,\omega} \rangle \otimes (\mathbf{20}\,,\mathbf{2}\,,0)_{\mathbf{H}} \\
\supset\quad & \zeta_1 (\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{H}} \\
& \oplus \frac{1}{M_{\rm pl}} (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}' \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}} \otimes \langle (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\omega} \rangle \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{H}} \\
\supset\quad & \zeta_1 (\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}}' \otimes (\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}} \otimes (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{H}} \\
& \oplus \zeta_2 (\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})_{\mathbf{F}} \otimes (\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}} \otimes (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{H}} \\
& \oplus \frac{1}{M_{\rm pl}} (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}' \otimes (\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\omega} \rangle \otimes (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{H}} \\
\Rightarrow\quad & \frac{1}{\sqrt{2}} (\zeta_1 \mathfrak{D}_L'' \mathfrak{d}_R{}^c + \zeta_2 \mathfrak{D}_L' \mathfrak{d}_R{}^c + \zeta_3 \mathfrak{D}_L \mathfrak{d}_R{}^c) v_{\rm EW}\,,
\end{aligned} \tag{124}$$

and for $d=6$ operator $\frac{1}{M_{\rm pl}^2} \mathbf{28_F} \mathbf{56_F} \mathbf{70_H} \overline{\mathbf{28_H}}_{,\dot\omega} \mathbf{8_H}{}^{,\omega}$ operator we find

$$\begin{aligned}
& \frac{1}{M_{\rm pl}^2} \mathbf{28_F} \mathbf{56_F} \mathbf{70_H} \overline{\mathbf{28_H}}_{,\dot\omega} \mathbf{8_H}^{\omega} \\
\supset\quad & \frac{1}{M_{\rm pl}^2} (\mathbf{15}\,,\mathbf{1}\,,-\frac{1}{3})_{\mathbf{F}}' \otimes (\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}} \otimes (\mathbf{20}\,,\mathbf{2}\,,0)_{\mathbf{H}} \otimes (\overline{\mathbf{15}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{H}\,,\dot\omega} \otimes \langle (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{6})_{\mathbf{H}}^{\omega} \rangle \\
\supset\quad & \frac{\zeta_1}{M_{\rm pl}} (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}' \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{H}} \otimes (\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\dot\omega} \\
& \oplus \frac{1}{M_{\rm pl}^2} (\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})_{\mathbf{F}}' \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}} \otimes (\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{H}} \otimes (\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H}\,,\dot\omega} \otimes \langle (\mathbf{5}\,,\mathbf{1}\,,-\frac{1}{5})_{\mathbf{H}}^{\omega} \rangle \\
\supset\quad & \frac{\zeta_1}{M_{\rm pl}} (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}' \otimes (\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}} \otimes (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{H}} \otimes \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot\omega} \rangle \\
& \oplus \frac{\zeta_2}{M_{\rm pl}} (\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}' \otimes (\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}} \otimes (\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{H}} \otimes \langle (\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot\omega}' \rangle \\
\Rightarrow\quad & \frac{1}{\sqrt{2}} (\zeta_1 \zeta_3 + \zeta_2 \zeta_3') \mathfrak{u}_L t_R{}^c v_{\rm EW}\,.
\end{aligned} \tag{125}$$

By collecting the above mass terms, we find the following mass matrices for the up-type and down-type quarks

$$(\mathfrak{u}_L\,, t_L)\cdot\left(\begin{array}{cc} 0 & \frac{1}{\sqrt{2}}\zeta_1\zeta_3 v_{\rm EW} \\ \frac{\zeta_1}{\sqrt{2}}V_{421} & m_t \end{array}\right)\cdot\left(\begin{array}{c} {\mathfrak{u}_R}^c \\ {t_R}^c \end{array}\right)+H.c.\,, \tag{126}$$

$$(\mathfrak{d}_L\,, \mathfrak{D}''_L)\cdot\left(\begin{array}{cc} 0 & 0 \\ \frac{\zeta_1}{\sqrt{2}}v_{\rm EW} & m_{\mathfrak{D}''} \end{array}\right)\cdot\left(\begin{array}{c} {\mathfrak{d}_R}^c \\ {\mathfrak{D}_R}''^c \end{array}\right)+H..c.\,, \tag{127}$$

in terms of flavor eigenstates. The above results suggest a seesaw-type mass mixing for the vectorlike $\mathfrak{u}$ quark. Meanwhile, such operators involving the $\mathbf{28_F}$ and the $\mathbf{56_F}$ generate the mass mixing terms involving only the right-handed components of ${\mathfrak{d}_R}^c$, but not any left-handed components of $\mathfrak{d}_L$. Therefore, the above mass matrix can only lead to zero eigenvalue for the vectorlike $\mathfrak{d}$ quark.

If one further looks for the bilinear terms between two $\mathbf{56_F}$s, such as $\frac{1}{M_{\rm pl}}\mathbf{56_F}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega}\mathbf{70_H}$, we can find the mass terms as follows

$$\begin{aligned}
& \frac{1}{M_{\rm pl}}\mathbf{56_F}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega}\mathbf{70_H} \\
\supset\ & \frac{1}{M_{\rm pl}}(\mathbf{20}\,,\mathbf{1}\,,-\frac{1}{2})_{\mathbf{F}}\otimes(\mathbf{15}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}\otimes(\overline{\mathbf{15}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{H}\,,\dot\omega}\otimes(\mathbf{20}\,,\mathbf{2}\,,0)_{\mathbf{H}} \\
\supset\ & \frac{1}{M_{\rm pl}}(\mathbf{10}\,,\mathbf{1}\,,-\frac{2}{5})'_{\mathbf{F}}\otimes(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}}\otimes\langle(\overline{\mathbf{5}}\,,\mathbf{1}\,,+\frac{1}{5})_{\mathbf{H}\,,\dot\omega}\rangle\otimes(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{H}} \\
& \oplus\frac{1}{M_{\rm pl}}(\overline{\mathbf{10}}\,,\mathbf{1}\,,-\frac{3}{5})_{\mathbf{F}}\otimes(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{F}}\otimes(\overline{\mathbf{10}}\,,\mathbf{1}\,,+\frac{2}{5})_{\mathbf{H}\,,\dot\omega}\otimes(\mathbf{10}\,,\mathbf{2}\,,+\frac{1}{10})_{\mathbf{H}} \\
\supset\ & \zeta_2(\mathbf{4}\,,\mathbf{1}\,,-\frac{1}{4})''_{\mathbf{F}}\otimes(\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}}\otimes(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{H}} \\
& \oplus\frac{1}{M_{\rm pl}}(\mathbf{6}\,,\mathbf{1}\,,-\frac{1}{2})'_{\mathbf{F}}\otimes(\mathbf{6}\,,\mathbf{2}\,,0)_{\mathbf{F}}\otimes\langle(\overline{\mathbf{4}}\,,\mathbf{1}\,,+\frac{1}{4})_{\mathbf{H}\,,\dot\omega}\rangle\otimes(\mathbf{4}\,,\mathbf{2}\,,+\frac{1}{4})_{\mathbf{H}} \\
\Rightarrow\ & \frac{1}{\sqrt{2}}(\zeta_2\mathfrak{D}''''_L{\mathfrak{d}_R}^c+\zeta_3\mathfrak{D}'''_L{\mathfrak{d}_R}^c)v_{\rm EW}\,,
\end{aligned} \tag{128}$$

which are still mass mixing terms only involving the right-handed ${\mathfrak{d}_R}^c$.

We wish to give further remarks on this issue.

- From the above results, one can only generate additional mass mixing terms that only involve the right-handed ${\mathfrak{d}_R}^c$ but not the left-handed $\mathfrak{d}_L$.
- If we look back to the fermion contents in Tabs. 21, 22 and 23 and ask if the left-handed components of $\mathfrak{d}_L$ can mix with any other fermionic components to obtain the masses, one then go through all possible right-handed down-type quarks. Potentially, one can think of an operator of the form

$$\overline{\mathbf{8_F}}^{\Omega}\mathbf{56_F}\mathbf{70_H}\times\mathcal{O}_{\mathbf{H}}\,, \tag{129}$$

where the $\mathcal{O}_{\mathbf{H}}$ stands for any combination of Higgs fields that form a gauge-invariant operator in the above expression. Only the $\overline{\mathbf{8_F}}^{\Omega}$ contain the right-handed down-type quark in the above tables, besides of the $\mathfrak{d}_R{}^c$. Note that a $\mathbf{70_H}$ is necessary, since the $\mathfrak{d}_L$ is within the component of $(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}$. However, when one decomposes the above operator, one can find

$$
\begin{aligned}
& \overline{\mathbf{8_F}}^{\Omega}\mathbf{56_F}\mathbf{70_H} \times \langle \mathcal{O}_{\mathbf{H}} \rangle \\
\supset\ & (\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}^{\Omega} \otimes (\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}' \otimes (\mathbf{1}\,,\mathbf{2}\,,+\frac{1}{2})_{\mathbf{H}} \times \prod_i V_i
\end{aligned} \tag{130}
$$

which is apparently not gauge-invariant. The $\prod_i V_i$ stands for the VEVs contributed from the combined Higgs fields of $\mathcal{O}_{\mathbf{H}}$, which must be singlets of the SM.

Although possible mass mixing terms between the $\mathfrak{u}$ quark and the top quark were found, the $\mathfrak{d}$ quark is still found to be massless by going through the potential mixing terms.