




The details of non-maximal symmetry breaking patterns of $\mathfrak{su}(8) \rightarrow \mathfrak{g}_{531}/\mathfrak{g}_{351}$

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Abstract

We present the details of the non-maximal symmetry breaking patterns of $\mathfrak{su}(8) \rightarrow \mathfrak{g}_{531}/\mathfrak{g}_{351}$.

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1 The intermediate stages of the SSW symmetry breaking pattern

Schematically, we assign the Higgs VEVs according to the symmetry breaking pattern as follows:

$$\mathfrak{g}_{531} \rightarrow \mathfrak{g}_{431} : \quad \langle (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \text{IV}} \rangle \equiv \frac{1}{\sqrt{2}} W_{\bar{\mathbf{5}}, \text{IV}}, \quad (1a)$$

$$\mathfrak{g}_{431} \rightarrow \mathfrak{g}_{331} : \quad \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \text{V}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\bar{\mathbf{4}}, \text{V}}, \quad \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \text{i, VII}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\bar{\mathbf{4}}, \text{i, VII}}, \quad (1b)$$

$$\begin{aligned} \mathfrak{g}_{331} \rightarrow \mathfrak{g}_{\text{SM}} : \quad \langle (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{3, VI}} \rangle &\equiv \frac{1}{\sqrt{2}} V_{\bar{\mathbf{3}}, \text{3, VI}}, \\ \langle (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{VIII}} \rangle &\equiv \frac{1}{\sqrt{2}} V_{\bar{\mathbf{3}}, \text{VIII}}, \quad \langle (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})'_{\mathbf{H}, \text{2, IX}} \rangle \equiv \frac{1}{\sqrt{2}} V'_{\bar{\mathbf{3}}, \text{2, IX}}, \end{aligned} \quad (1c)$$

$$\text{EWSB} : \quad \langle (\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{H}} \rangle \equiv \frac{1}{\sqrt{2}} v_{\text{EW}}. \quad (1d)$$

For our later convenience, we also parametrize different symmetry breaking VEVs in terms of the following dimensionless quantities:

$$\begin{aligned} \zeta_1 &\equiv \frac{W_{\bar{\mathbf{5}}, \text{IV}}}{M_{\text{pl}}}, \quad \zeta_2 \equiv \frac{w_{\bar{\mathbf{4}}, \text{V}}}{M_{\text{pl}}}, \quad \dot{\zeta}_2 \equiv \frac{w_{\bar{\mathbf{4}}, \text{i, VII}}}{M_{\text{pl}}}, \\ \zeta_3 &\equiv \frac{V_{\bar{\mathbf{3}}, \text{3, VI}}}{M_{\text{pl}}}, \quad \dot{\zeta}_3' \equiv \frac{V'_{\bar{\mathbf{3}}, \text{2, IX}}}{M_{\text{pl}}}, \quad \dot{\zeta}_3 \equiv \frac{V_{\bar{\mathbf{3}}, \text{VIII}}}{M_{\text{pl}}}, \\ \zeta_0 &\gg \zeta_1 \gg \zeta_2 \sim \dot{\zeta}_2 \gg \zeta_3 \sim \dot{\zeta}_3' \sim \dot{\zeta}_3, \quad \zeta_{ij} \equiv \frac{\zeta_j}{\zeta_i}, \quad (i < j). \end{aligned} \quad (2)$$

In Table 1, we summarize all vectorlike fermions that become massive during different stages of the SSW symmetry breaking pattern.

stages	$Q_e = -\frac{1}{3}$	$Q_e = +\frac{2}{3}$	$Q_e = -1$	$Q_e = 0$
v_{531} $\{\Omega\}$	\mathfrak{D}' IV	-	$(\mathfrak{e}', \mathfrak{n}')$ IV	$\{\check{\mathfrak{n}}, \check{\mathfrak{n}}''\}$ $\{\text{IV}'', \text{IV}'\}$
v_{431} $\{\Omega\}$	$\mathfrak{d}, \{\mathfrak{D}'', \mathfrak{D}'''\}$ $\{\text{V}, \text{VII}\}$	$\mathfrak{u}, \mathfrak{d}$	$\mathfrak{E}, (\mathfrak{e}'', \mathfrak{n}''), (\mathfrak{e}''''', \mathfrak{n}''''')$ $\{\text{V}, \text{VII}\}$	$\{\check{\mathfrak{n}}', \check{\mathfrak{n}}'''\}$ $\{\text{V}'', \text{VII}\}$
v_{331} $\{\Omega\}$	$\{\mathfrak{D}, \mathfrak{D}''''', \mathfrak{D}''''''\}$ $\{\text{VI}, \text{VIII}, \text{IX}\}$	-	$(\mathfrak{e}, \mathfrak{n}), (\mathfrak{e}''''', \mathfrak{n}'''''), (\mathfrak{e}''''', \mathfrak{n}''''')$ $\{\text{VI}, \text{VIII}, \text{IX}\}$	-

Table 1: The vectorlike fermions at different intermediate symmetry breaking stages along the SSW symmetry breaking pattern of the $\mathfrak{su}(8)$ theory.

1.1 The first stage

The first symmetry breaking stage of $\mathfrak{g}_{531} \rightarrow \mathfrak{g}_{431}$ is achieved by $(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \text{IV}} \subset \overline{\mathbf{8}}_{\mathbf{H}, \text{IV}}$ in the rank-two sector, according to the $\widetilde{\text{U}}(1)_{T'}$ -neutral components in Table 2. Accordingly, the term of

$\overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega}$	
\mathcal{T}'	$-4t$	$-\frac{4}{3}t$	
$\mathbf{28}_{\mathbf{F}}$	$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}$	$(\overline{\mathbf{5}}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}$
\mathcal{T}'	$-\frac{4}{3}t$	$\frac{4}{3}t$	$4t$
$\mathbf{56}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}'$	$(\overline{\mathbf{5}}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}}$	$(\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}}$
\mathcal{T}'	$-4t$	$-\frac{4}{3}t$	$4t$
			$\frac{4}{3}t$
$\overline{\mathbf{8}}_{\mathbf{H},\omega}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega}$	$(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega}$	
\mathcal{T}'	$\frac{8}{3}t$	0	
$\mathbf{28}_{\mathbf{H},\dot{\omega}}$	$(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H},\dot{\omega}}$	$(\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H},\dot{\omega}}$	$(\overline{\mathbf{10}}, \mathbf{1}, +\frac{2}{5})_{\mathbf{H},\dot{\omega}}$
\mathcal{T}'	$\frac{16}{3}t$	$\frac{8}{3}t$	0
$\mathbf{70}_{\mathbf{H}}$	$(\mathbf{10}, \mathbf{3}, +\frac{4}{15})_{\mathbf{H}}$	$(\overline{\mathbf{10}}, \mathbf{3}, -\frac{4}{15})_{\mathbf{H}}$	
\mathcal{T}'	$-\frac{16}{3}t$	$-\frac{8}{3}t$	

Table 2: The $\tilde{\mathbf{U}}(1)_{T'}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{531} theory.

$Y_{\mathcal{B}}\overline{\mathbf{8}}_{\mathbf{F}}^{\text{IV}}\mathbf{28}_{\mathbf{F}}\overline{\mathbf{8}}_{\mathbf{H},\text{IV}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}', \mathfrak{n}'', \mathfrak{e}', \mathfrak{n}', \mathfrak{n})$ as follows:

$$\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8}}_{\mathbf{F}}^{\text{IV}}\mathbf{28}_{\mathbf{F}}\overline{\mathbf{8}}_{\mathbf{H},\text{IV}} + H.c. \\
& \supset Y_{\mathcal{B}}\left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\text{IV}} \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{IV}} \otimes (\overline{\mathbf{5}}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}}\right] \otimes \langle (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\text{IV}} \rangle + H.c. \\
& \Rightarrow \frac{1}{\sqrt{2}}Y_{\mathcal{B}}(\mathfrak{D}'_L\mathfrak{D}'_R{}^c + \mathfrak{n}''_L\mathfrak{n}''_R{}^c + \mathfrak{e}'_L\mathfrak{e}'_R{}^c - \mathfrak{n}'_L\mathfrak{n}'_R{}^c + \mathfrak{n}_L\mathfrak{n}_R{}^c)W_{\overline{\mathbf{5}},\text{IV}} + H.c..
\end{aligned} \tag{3}$$

After this stage, the remaining massless \mathfrak{g}_{431} fermions are the following:

$$\begin{aligned}
& \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'}\right] \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus \overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}, \\
& \Omega = (\omega, \dot{\omega}), \quad \omega = (3, \text{V}, \text{VI}), \quad \dot{\omega} = (\dot{\mathbf{1}}, \dot{\mathbf{2}}, \text{VII}, \text{VIII}, \text{IX}), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}'} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\text{IV}}, \\
& (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \oplus \left[(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}' \oplus (\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}}\right] \oplus \left[(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}\right] \subset \mathbf{28}_{\mathbf{F}}, \\
& (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}' \oplus \left[(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}' \oplus (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{F}}\right] \oplus \left[(\overline{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}'\right] \\
& \oplus \left[(\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}}' \oplus (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}}\right] \subset \mathbf{56}_{\mathbf{F}}.
\end{aligned} \tag{4}$$

1.2 The second stage

The second symmetry breaking stage of $\mathfrak{g}_{431} \rightarrow \mathfrak{g}_{331}$ is achieved by $(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H},\text{V}} \subset \overline{\mathbf{8}}_{\mathbf{H},\text{V}}$ in the rank-two sector, $(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H},\text{VII}} \subset \overline{\mathbf{28}}_{\mathbf{H},\text{VII}}$ and $(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{H}}^{\text{VII}} \subset \mathbf{28}_{\mathbf{H}}^{\text{VII}}$ in the rank-three sector, according to the $\tilde{\mathbf{U}}(1)_{T''}$ -neutral components in Table 3. The term of $Y_{\mathcal{B}}\overline{\mathbf{8}}_{\mathbf{F}}^{\text{V}}\mathbf{28}_{\mathbf{F}}\overline{\mathbf{8}}_{\mathbf{H},\text{V}} + H.c.$ leads to

$\overline{8_F}^\Omega$	$(\overline{4}, 1, +\frac{1}{4})_F^\Omega$	$(1, 1, 0)_F^{\Omega'}$	$(1, \overline{3}, -\frac{1}{3})_F^\Omega$	
\mathcal{T}''	$-4t$	$-4t$	$-\frac{4}{3}t$	
28_F	$(6, 1, -\frac{1}{2})_F$	$(4, 3, +\frac{1}{12})_F$	$(1, \overline{3}, +\frac{2}{3})_F$	
\mathcal{T}''	$4t$	$\frac{4}{3}t$	$-\frac{4}{3}t$	
56_F	$(1, 1, +1)_F'$	$(1, \overline{3}, +\frac{2}{3})_F'$	$(4, \overline{3}, +\frac{5}{12})_F$	
\mathcal{T}''	$-4t$	$-\frac{4}{3}t$	$-\frac{4}{3}t$	
	$(\overline{4}, 1, -\frac{3}{4})_F$	$(6, 1, -\frac{1}{2})_F'$	$(4, 3, +\frac{1}{12})_F'$	$(6, 3, -\frac{1}{6})_F$
\mathcal{T}''	$4t$	$4t$	$\frac{4}{3}t$	$\frac{4}{3}t$
$\overline{8_{H,\omega}}$	$(1, \overline{3}, -\frac{1}{3})_{H,\omega}$	$(\overline{4}, 1, +\frac{1}{4})_{H,\omega}$		
\mathcal{T}''	$\frac{8}{3}t$	0		
$28_{H,\omega}$	$(1, 3, -\frac{2}{3})_{H,\omega}$	$(1, \overline{3}, -\frac{1}{3})_{H,\omega}$	$(\overline{4}, \overline{3}, -\frac{1}{12})_{H,\omega}$	$(\overline{4}, 1, +\frac{1}{4})_{H,\omega}$
\mathcal{T}''	$\frac{16}{3}t$	$\frac{8}{3}t$	$\frac{8}{3}t$	0
70_H	$(4, \overline{3}, +\frac{5}{12})_H$	$(\overline{4}, 3, -\frac{5}{12})_H$		
\mathcal{T}''	$-\frac{16}{3}t$	$-\frac{8}{3}t$		

Table 3: The $\widetilde{U}(1)_{T''}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{431} theory.

the vectorial masses of $(\mathfrak{D}'', \mathfrak{e}'', \mathfrak{n}'', \mathfrak{n}')$ as follows:

$$\begin{aligned}
& Y_B \overline{8_F}^V 28_F \overline{8_H}_{V} + H.c. \\
\supset & Y_B \left[(\overline{5}, 1, +\frac{1}{5})_F^V \otimes (10, 1, -\frac{2}{5})_F \oplus (1, \overline{3}, -\frac{1}{3})_F^V \otimes (5, 3, +\frac{2}{15})_F \right] \otimes (\overline{5}, 1, +\frac{1}{5})_{H,V} + H.c. \\
\supset & Y_B \left[(\overline{4}, 1, +\frac{1}{4})_F^V \otimes (6, 1, -\frac{1}{2})_F \oplus (1, 1, 0)_F^V \otimes (4, 1, -\frac{1}{4})_F \right. \\
& \oplus (1, \overline{3}, -\frac{1}{3})_F^V \otimes (4, 3, +\frac{1}{12})_F \left. \right] \otimes \langle (\overline{4}, 1, +\frac{1}{4})_{H,V} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_B \left(\mathfrak{D}_L'' \mathfrak{D}_R''^c + \tilde{\mathcal{N}}_L^{V'} \mathfrak{n}_R''^c + \mathfrak{e}_L'' \mathfrak{e}_R''^c - \mathfrak{n}_L'' \mathfrak{n}_R''^c + \mathfrak{n}_L' \mathfrak{n}_R'^c \right) w_{\overline{4},V} + H.c.. \tag{5}
\end{aligned}$$

The term of $Y_D \overline{8_F}^{V_{II}} 56_F \overline{28_H}_{V_{II}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}''', \mathfrak{e}'''' , \mathfrak{n}'''' , \mathfrak{n}''')$ as follows:

$$\begin{aligned}
& Y_D \overline{8_F}^{V_{II}} 56_F \overline{28_H}_{V_{II}} + H.c. \\
\supset & Y_D \left[(\overline{5}, 1, +\frac{1}{5})_F^{V_{II}} \otimes (\overline{10}, 1, -\frac{3}{5})_F \oplus (1, \overline{3}, -\frac{1}{3})_F^{V_{II}} \otimes (10, 3, -\frac{1}{15})_F \right] \otimes (\overline{10}, 1, +\frac{2}{5})_{H,V_{II}} + H.c. \\
\supset & Y_D \left[(\overline{4}, 1, +\frac{1}{4})_F^{V_{II}} \otimes (6, 1, -\frac{1}{2})_F' \oplus (1, \overline{3}, -\frac{1}{3})_F^{V_{II}} \otimes (4, 3, +\frac{1}{12})_F' \right] \otimes \langle (\overline{4}, 1, +\frac{1}{4})_{H,V_{II}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_D \left(\mathfrak{D}_L''' \mathfrak{D}_R'''^c + \mathfrak{e}_L'''' \mathfrak{e}_R''''^c - \mathfrak{n}_L'''' \mathfrak{n}_R''''^c + \mathfrak{n}_L''' \mathfrak{n}_R'''^c \right) w_{\overline{4},V_{II}} + H.c.. \tag{6}
\end{aligned}$$

The Yukawa coupling between two $\mathbf{56_F}$'s by the following $d = 5$ operator

$$\begin{aligned}
& \frac{c_4}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\text{VII}} + H.c. \\
\supset & c_4 \frac{v_U}{M_{\text{pl}}} \left[(\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}' \oplus (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \otimes (\mathbf{5}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \right] \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{H}}^{\text{VII}} + H.c. \\
\supset & c_4 \zeta_0 \left[(\overline{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}} \otimes (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}' \oplus (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}} \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{F}} \right] \otimes \langle (\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{H}}^{\text{VII}} \rangle + H.c. \\
\Rightarrow & \frac{c_4}{\sqrt{2}} \zeta_0 (\mathfrak{E}_L \mathfrak{E}_R^c + \mathfrak{U}_L \mathfrak{U}_R^c + \mathfrak{D}_L \mathfrak{D}_R^c - \mathfrak{u}_L \mathfrak{u}_R^c) w_4^{\text{VII}} + H.c., \tag{7}
\end{aligned}$$

leads to massive vectorlike fermions of $(\mathfrak{E}, \mathfrak{U}, \mathfrak{u}, \mathfrak{D})$.

After integrating out the massive fermions, the remaining massless fermions expressed in terms of the \mathfrak{g}_{331} IRs are the following:

$$\begin{aligned}
& \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega} \right] \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega} \subset \overline{\mathbf{8_F}}^{\Omega}, \\
& \Omega = (\omega, \dot{\omega}), \quad \omega = (3, \text{VI}), \quad \dot{\omega} = (\dot{1}, \dot{2}, \text{VIII}, \text{IX}), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}'} \subset \overline{\mathbf{8_F}}^{\text{IV}}, \quad (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}'} \subset \overline{\mathbf{8_F}}^{\text{V}}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VII}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VII}'} \subset \overline{\mathbf{8_F}}^{\text{VII}}, \\
& (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \oplus \cancel{(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}} \oplus \left[\cancel{(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}}'' \oplus (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \right] \\
\oplus & \left[\cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}}'' \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}'' \right] \oplus \left[\cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}}'' \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \right] \subset \mathbf{28_F}, \\
& \cancel{(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}}'' \oplus (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}' \oplus \left[(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}'' \oplus \cancel{(\mathbf{3}, \overline{\mathbf{3}}, +\frac{1}{3})_{\mathbf{F}}} \right] \\
\oplus & \left[\cancel{(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}' \right] \oplus \left[\cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}}''' \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}'' \right] \\
\oplus & \left[\cancel{(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}}''' \oplus (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}' \right] \oplus \left[(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}'' \oplus \cancel{(\mathbf{3}, \mathbf{3}, -\frac{1}{3})_{\mathbf{F}}} \right] \subset \mathbf{56_F}. \tag{8}
\end{aligned}$$

1.3 The third stage

The third symmetry breaking stage of $\mathfrak{g}_{331} \rightarrow \mathfrak{g}_{\text{SM}}$ is achieved by $(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{VI}} \subset \overline{\mathbf{8_H}}_{\text{VI}}$ in the rank-two sector and $(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{VIII}} \subset \mathbf{28_H}_{\text{VIII}}$, $(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})'_{\mathbf{H}, \text{IX}} \subset \mathbf{28_H}_{\text{IX}}$ in the rank-three sector, according to the $\widetilde{\text{U}}(1)_{T''''}$ -neutral components in Table 4. The term of $Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\text{VI}} \mathbf{28_F} \overline{\mathbf{8_H}}_{\text{VI}} + H.c.$ leads

$\overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega}$
\mathcal{T}'''	$-\frac{4}{3}t$	$-4t$	$-4t$	$-4t$
$\mathbf{28}_{\mathbf{F}}$	$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$	
\mathcal{T}'''	$4t$	$\frac{4}{3}t$	$-\frac{4}{3}t$	
$\mathbf{56}_{\mathbf{F}}$	$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})'_{\mathbf{F}}$	$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})''_{\mathbf{F}}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}}$	
\mathcal{T}'''	$4t$	$4t$	$-\frac{4}{3}t$	
	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{3}, 0)'_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{3}, 0)''_{\mathbf{F}}$	
\mathcal{T}'''	$-\frac{4}{3}t$	$\frac{4}{3}t$	$\frac{4}{3}t$	
$\overline{\mathbf{8}}_{\mathbf{H},\omega}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega}$			
\mathcal{T}'''	0			
$\mathbf{28}_{\mathbf{H},\dot{\omega}}$	$(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H},\dot{\omega}}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\dot{\omega}}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})'_{\mathbf{H},\dot{\omega}}$	
\mathcal{T}'''	0	0	0	
$\mathbf{70}_{\mathbf{H}}$	$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}}$	$(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H}}$		
\mathcal{T}'''	0	$-8t$		

Table 4: The $\widetilde{\mathbf{U}}(1)_{T'''}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{331} theory.

to the vectorial masses of $(\mathfrak{D}, \mathfrak{e}, \mathfrak{n})$ as follows:

$$\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8}}_{\mathbf{F}}^{\text{VI}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}}_{\mathbf{H},\text{VI}} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \oplus (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\text{VI}} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VI}'} \otimes (\mathbf{1}, \mathbf{3}, +\frac{1}{3})'_{\mathbf{F}} \right. \\
& \oplus \left. (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\text{VI}} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VI}'} \otimes (\mathbf{1}, \mathbf{3}, +\frac{1}{3})'_{\mathbf{F}} \right. \\
& \oplus \left. (\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{1}, \mathbf{3}, +\frac{1}{3})'_{\mathbf{F}} \right] \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\text{VI}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{B}} \left(\mathfrak{n}_L \mathfrak{n}_R^c - \mathfrak{e}_L \mathfrak{e}_R^c + \tilde{\mathcal{N}}_L^{\text{VI}'} \tilde{\mathfrak{n}}_R^c + \tilde{\mathcal{N}}_L^{\text{VI}} \tilde{\mathfrak{n}}_R'^c + \mathfrak{D}_L \mathfrak{D}_R^c \right) V_{\overline{\mathbf{3}},\text{VI}} + H.c.. \tag{9}
\end{aligned}$$

The term of $Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\text{VIII}, \text{IX}} \mathbf{56_F} \overline{\mathbf{28_H}}_{\text{VIII}, \text{IX}} + H.c.$ leads to the vectorial masses of $(\mathcal{D}'''' , \epsilon''' , n''' , \mathcal{D}'''' , \epsilon'''' , n'''')$ as follows:

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\text{VIII}} \mathbf{56_F} \overline{\mathbf{28_H}}_{\text{VIII}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\text{VIII}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{VIII}} \otimes (\mathbf{5}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \text{VIII}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\text{VIII}} \otimes (\mathbf{4}, \mathbf{3}, +\frac{1}{12})'_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{VIII}} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})'_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{VIII}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{VIII}} \otimes (\mathbf{3}, \mathbf{3}, 0)'_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VIII}} \otimes (\mathbf{1}, \mathbf{3}, +\frac{1}{3})'''_{\mathbf{F}} \right. \\
& \left. \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{VIII}} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})'_{\mathbf{F}} \right] \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{VIII}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} \left(\mathcal{D}_L'''' \mathcal{D}_R''' c + \tilde{\mathcal{N}}_L^{\text{VIII}} \tilde{n}_R''' c - \epsilon_L''' \epsilon_R''' c + n_L''' n_R''' c \right) V_{\overline{\mathbf{3}}, \text{VIII}} + H.c. , \tag{10}
\end{aligned}$$

and

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\text{IX}} \mathbf{56_F} \overline{\mathbf{28_H}}_{\text{IX}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{5}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \text{IX}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}'} \otimes (\mathbf{4}, \mathbf{3}, +\frac{1}{12})'_{\mathbf{F}} \right. \\
& \left. \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \text{IX}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{3}, \mathbf{3}, 0)''_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}'} \otimes (\mathbf{1}, \mathbf{3}, +\frac{1}{3})'''_{\mathbf{F}} \right. \\
& \left. \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})''_{\mathbf{F}} \right] \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})'_{\mathbf{H}, \text{IX}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} \left(\mathcal{D}_L'''' \mathcal{D}_R''' c + \tilde{\mathcal{N}}_L^{\text{IX}'} \tilde{n}_R''' c - \epsilon_L''' \epsilon_R''' c + n_L''' n_R''' c \right) V'_{\overline{\mathbf{3}}, \text{IX}} + H.c. . \tag{11}
\end{aligned}$$

The remaining massless fermions of the \mathfrak{g}_{SM} are listed as follows:

$$\begin{aligned}
& \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'} \right] \oplus \left[(\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} \right] \subset \overline{\mathbf{8_F}}^{\Omega}, \quad \Omega = (\dot{1}, \dot{2}, 3), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}} \oplus \dots \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}} \subset \overline{\mathbf{8_F}}^{\Omega}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}'} \oplus \dots \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}'} \subset \overline{\mathbf{8_F}}^{\Omega'}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VI}''} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VIII}''} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}''} \subset \overline{\mathbf{8_F}}^{\Omega''}, \\
& \left[(\overline{\mathbf{1}}, \overline{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}} \right] \oplus \left[(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})'_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})''_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)'_{\mathbf{F}} \right. \\
\oplus & \left. (\overline{\mathbf{3}}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)''_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{1}, -\frac{1}{3})''_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \right] \subset \mathbf{28_F}, \\
& (\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \oplus \left[(\overline{\mathbf{1}}, \overline{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, +1)''_{\mathbf{F}} \oplus (\overline{\mathbf{1}}, \overline{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, +1)'''_{\mathbf{F}} \right. \\
\oplus & \left. (\mathbf{3}, \mathbf{1}, +\frac{2}{3})_{\mathbf{F}} \oplus (\mathbf{3}, \overline{\mathbf{2}}, +\frac{1}{6})_{\mathbf{F}} \right] \oplus \left[(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}} \right] \\
\oplus & \left[(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)'''_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{2}, +\frac{1}{6})'_{\mathbf{F}} \right. \\
\oplus & \left. (\overline{\mathbf{3}}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{2}, +\frac{1}{6})''_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{2}, -\frac{1}{6})_{\mathbf{F}} \right] \subset \mathbf{56_F}. \tag{12}
\end{aligned}$$

1.4 The $d = 5$ bi-linear fermion operators

The SM up-type quark masses are due to two $d = 5$ operators of

$$c_4 \mathcal{O}_{\mathcal{F}}^{(4,1)} = c_4 \mathbf{56_F} \mathbf{56_F} \cdot \overline{\mathbf{28_H}}_{,\dot{\omega}} \cdot \mathbf{70_H}, \quad (13a)$$

$$c_5 \mathcal{O}_{\mathcal{F}}^{(5,1)} = c_5 \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{,\omega} \cdot \mathbf{70_H}. \quad (13b)$$

For all down-type quarks and charged leptons, we conjecture two sets of Higgs mixing terms

$$d_{\mathcal{A}} \mathcal{O}_{\mathcal{A}}^{d=5} \equiv d_{\mathcal{A}} \epsilon_{\omega_1 \omega_2 \omega_3 \omega_4} \overline{\mathbf{8_H}}_{,\omega_1}^{\dagger} \overline{\mathbf{8_H}}_{,\omega_2}^{\dagger} \overline{\mathbf{8_H}}_{,\omega_3}^{\dagger} \overline{\mathbf{8_H}}_{,\omega_4}^{\dagger} \mathbf{70_H}^{\dagger}, \quad \mathcal{PQ} = 2(2p + 3q_2) \neq 0, \quad (14a)$$

$$d_{\mathcal{B}} \mathcal{O}_{\mathcal{B}}^{d=5} \equiv d_{\mathcal{B}} (\overline{\mathbf{28_H}}_{,\dot{\kappa}_1}^{\dagger} \overline{\mathbf{28_H}}_{,\dot{\kappa}_2}^{\dagger}) \cdot \overline{\mathbf{28_H}}_{,\dot{\omega}_1}^{\dagger} \overline{\mathbf{28_H}}_{,\dot{\omega}_2}^{\dagger} \mathbf{70_H}^{\dagger}, \quad \mathcal{PQ} = 2(p + q_2 + q_3), \\ \text{with } \dot{\kappa}_2 \neq (\dot{\kappa}_1, \dot{\omega}_1, \dot{\omega}_2), \quad (14b)$$

For the operator of $\mathcal{O}_{\mathcal{F}}^{(4,1)}$ in Eq. (13a), it is decomposed as

$$\begin{aligned} & \frac{c_4}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \cdot \overline{\mathbf{28_H}}_{,\dot{\omega}} \cdot \mathbf{70_H} + H.c. \\ \supset & \frac{c_4}{M_{\text{pl}}} (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, +\frac{2}{5})_{\mathbf{H}, \dot{\omega}} \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}} + H.c. \\ \supset & \frac{c_4}{M_{\text{pl}}} \left[(\overline{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}} \otimes (\mathbf{4}, \mathbf{3}, +\frac{1}{12})'_{\mathbf{F}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}} \right] \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \dot{\omega}} \rangle \\ \otimes & (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}} + H.c. \\ \supset & c_4 \frac{w_{\overline{\mathbf{4}}, \text{VII}}}{\sqrt{2} M_{\text{pl}}} \left[(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{3}, 0)'_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{3}, 0)''_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}} + H.c. \\ \Rightarrow & \frac{c_4}{2} \zeta_2 (u_L c_R^c + c_L u_R^c) v_{\text{EW}} + H.c.. \end{aligned} \quad (15)$$

For the operator of $\mathcal{O}_{\mathcal{F}}^{(5,1)}$ in Eq. (13b), it is decomposed as

$$\begin{aligned} & \frac{c_5}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{,\omega} \cdot \mathbf{70_H} + H.c. \\ \supset & \frac{c_5}{M_{\text{pl}}} \left[(\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \right] \otimes \langle (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \omega} \rangle \\ \otimes & (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}} + H.c. \\ \supset & c_5 \frac{W_{\overline{\mathbf{5}}, \text{IV}}}{\sqrt{2} M_{\text{pl}}} \left[(\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{4}, \mathbf{3}, +\frac{1}{12})'_{\mathbf{F}} \right] \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}} + H.c. \\ \supset & c_5 \frac{W_{\overline{\mathbf{5}}, \text{IV}}}{\sqrt{2} M_{\text{pl}}} \left[(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \otimes (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{3}, 0)'_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}} + H.c. \\ \Rightarrow & \frac{c_5}{2} \zeta_1 (t_L u_R^c + u_L t_R^c) v_{\text{EW}} + H.c., \end{aligned} \quad (16)$$

and

$$\begin{aligned}
& \frac{c_5}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} \left[(\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega} \\
& \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} \left[(\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}} \otimes (\overline{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}} \right] \\
& \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H},\omega} \rangle \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}} + H.c. \\
\supset & c_5 \frac{w_{\overline{\mathbf{4}},\text{V}}}{\sqrt{2}M_{\text{pl}}} \left[(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \otimes (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{3}, 0)''_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}} + H.c. \\
\Rightarrow & \frac{c_5}{2} \zeta_2 (t_L c_R^c + c_L t_R^c) v_{\text{EW}} + H.c. . \tag{17}
\end{aligned}$$

1.5 The $d = 5$ irreducible Higgs mixing operators

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1}$, we find the mass terms of

$$\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1} \times \frac{d_{\mathcal{A}}}{M_{\text{pl}}} \epsilon_{\omega_1 \omega_2 \omega_3 \omega_4} \overline{\mathbf{8_H}}_{,\omega_1}^\dagger \overline{\mathbf{8_H}}_{,\omega_2}^\dagger \overline{\mathbf{8_H}}_{,\omega_3}^\dagger \overline{\mathbf{8_H}}_{,\omega_4}^\dagger \mathbf{70_H}^\dagger + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega_1} \\
& \times \frac{d_{\mathcal{A}}}{M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega_1}^\dagger \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega_2}^\dagger \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega_3}^\dagger \otimes \langle (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega_4}^\dagger \rangle \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega_1} \\
& \times d_{\mathcal{A}} \frac{W_{\overline{\mathbf{5}},\text{IV}}}{\sqrt{2}M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega_1}^\dagger \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega_2}^\dagger \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H},\omega_3}^\dagger \rangle \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega_1} \\
& \times d_{\mathcal{A}} \frac{W_{\overline{\mathbf{5}},\text{IV}} w_{\overline{\mathbf{4}},\text{V}}}{2M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega_1}^\dagger \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega_2}^\dagger \rangle \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{B}} d_{\mathcal{A}}}{4} \frac{W_{\overline{\mathbf{5}},\text{IV}} w_{\overline{\mathbf{4}},\text{V}} V_{\overline{\mathbf{3}},\text{VI}}}{M_{\text{pl}} m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},3}}^2} (b_L b_R^c + \tau_L \tau_R^c) v_{\text{EW}} + H.c. , \tag{18}
\end{aligned}$$

via the Higgs mixing operator of $\mathcal{O}_{\mathcal{A}}^{d=5}$ in Eq. (14a).

For the Yukawa coupling of $\overline{\mathbf{8}_F^{\dot{\omega}_1}} \mathbf{56}_F \overline{\mathbf{28}_H^{\dot{\omega}_1}}$, we find the mass terms of

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8}_F^{\dot{\omega}_1}} \mathbf{56}_F \overline{\mathbf{28}_H^{\dot{\omega}_1}} \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}}^\dagger \mathbf{70}_H^\dagger \left(\overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \right) + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{5}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1} \\
\times & \frac{d_{\mathcal{B}}}{M_{\text{pl}}} (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}}^\dagger \langle \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \rangle + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{\mathbf{4}}, \mathbf{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{2M_{\text{pl}}} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_1} \\
\times & (\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{\mathbf{4}}, \mathbf{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{2M_{\text{pl}} m_{(\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_1}}^2} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}'' \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}'' \right] \\
\otimes & \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_2}^\dagger \rangle \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{D}} d_{\mathcal{B}}}{4} \zeta_3 \left[\frac{w_{\overline{\mathbf{4}}, \mathbf{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_1}}^2} (s_L d_R^c + e_L \mu_R^c) + \frac{w_{\overline{\mathbf{4}}, \mathbf{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_2}}^2} (s_L s_R^c + \mu_L \mu_R^c) \right] v_{\text{EW}} + H.c., \quad (19)
\end{aligned}$$

and

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8}_F^{\dot{\omega}_1}} \mathbf{56}_F \overline{\mathbf{28}_H^{\dot{\omega}_1}} \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}}^\dagger \mathbf{70}_H^\dagger \left(\overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \right) + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{5}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1} \\
\times & \frac{d_{\mathcal{B}}}{M_{\text{pl}}} (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}}^\dagger \langle \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \rangle + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{\mathbf{4}}, \mathbf{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{2M_{\text{pl}}} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}}' \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}' \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_1} \\
\times & (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{\mathbf{4}}, \mathbf{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{2M_{\text{pl}} m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_1}}^2} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}' \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}' \right] \\
\otimes & \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_2}^\dagger \rangle \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{D}} d_{\mathcal{B}}}{4} \zeta_3' \left[\frac{w_{\overline{\mathbf{4}}, \mathbf{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_1}}^2} (d_L d_R^c + e_L e_R^c) + \frac{w_{\overline{\mathbf{4}}, \mathbf{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_2}}^2} (d_L s_R^c + \mu_L e_R^c) \right] v_{\text{EW}} + H.c., \quad (20)
\end{aligned}$$

via the Higgs mixing operator of $\mathcal{O}_{\mathcal{B}}^{d=5}$ in Eq. (14b). For convenience, we parametrize the following ratios of

$$\Delta_{\dot{\omega}} \equiv \frac{w_{\overline{\mathbf{4}}, \mathbf{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}}}^2}, \quad \Delta'_{\dot{\omega}} \equiv \frac{w_{\overline{\mathbf{4}}, \mathbf{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}}}^2}. \quad (21)$$

With the Higgs VEVs assignments in Eq. (1), we can obtain the following propagator masses of

$$\begin{aligned}
m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_1}} & \sim m_{(\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_1}} \sim \mathcal{O}(v_{431}), \\
m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_2}} & \sim m_{(\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_2}} \sim \mathcal{O}(v_{331}). \quad (22)
\end{aligned}$$

1.6 The SM quark/lepton masses and the benchmark

For all up-type quarks with $Q_e = +\frac{2}{3}$, we write down the following tree-level masses from both the renormalizable Yukawa couplings and the gravity-induced terms in the basis of $\mathcal{U} \equiv (u, c, t)$

$$\mathcal{M}_{\mathcal{U}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & c_4 \dot{\zeta}_2 / \sqrt{2} & c_5 \dot{\zeta}_1 / \sqrt{2} \\ c_4 \dot{\zeta}_2 / \sqrt{2} & 0 & c_5 \dot{\zeta}_2 / \sqrt{2} \\ c_5 \dot{\zeta}_1 / \sqrt{2} & c_5 \dot{\zeta}_2 / \sqrt{2} & Y_{\mathcal{T}} \end{pmatrix} v_{\text{EW}}. \quad (23)$$

For all down-type quarks with $Q_e = -\frac{1}{3}$, we find the following tree-level mass matrix in the basis of $\mathcal{D} \equiv (d, s, b)$

$$\mathcal{M}_{\mathcal{D}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1 \dot{\zeta}'_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2 \dot{\zeta}'_3 & 0 \\ Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1' \dot{\zeta}_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2' \dot{\zeta}_3 & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathcal{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\text{EW}}, \quad (24)$$

where we interpret the ratio between two \mathbf{g}_{331} VEVs as the Cabibbo angle of $\tan \lambda \equiv \frac{\dot{\zeta}'_3}{\dot{\zeta}_3}$. For all charged leptons with $Q_e = -1$, their tree-level mass matrix is the transposition of the down-type quark mass matrix and expressed as follows:

$$\mathcal{M}_{\mathcal{L}} = \mathcal{M}_{\mathcal{D}}^T \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1 \dot{\zeta}'_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1' \dot{\zeta}_3 & 0 \\ Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2 \dot{\zeta}'_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2' \dot{\zeta}_3 & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathcal{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\text{EW}}. \quad (25)$$

Based on the above SM quark/lepton mass matrices, we find the following benchmark point of

$$v_{531} \simeq 1.4 \times 10^{17} \text{ GeV}, \quad v_{431} \simeq 4.8 \times 10^{15} \text{ GeV}, \quad v_{331} \simeq 4.8 \times 10^{13} \text{ GeV}, \quad (26)$$

to reproduce the observed hierarchical masses as well as the CKM mixing pattern, as summarized in Table 5.

ζ_1	ζ_2	ζ_3	$Y_{\mathcal{D}}$	$Y_{\mathcal{B}}$	$Y_{\mathcal{T}}$
6.0×10^{-2}	2.0×10^{-3}	2.0×10^{-5}	0.5	0.5	0.8
λ	c_4	c_5	$d_{\mathcal{A}}$	$d_{\mathcal{B}}$	
0.22	0.2	1.0	0.01	0.01	
m_u	m_c	m_t	$m_d = m_e$	$m_s = m_{\mu}$	$m_b = m_{\tau}$
1.6×10^{-3}	0.6	139.2	0.5×10^{-3}	6.4×10^{-2}	1.5
$ V_{ud} $	$ V_{us} $	$ V_{ub} $			
0.98	0.22	2.1×10^{-3}			
$ V_{cd} $	$ V_{cs} $	$ V_{cb} $			
0.22	0.98	5.3×10^{-2}			
$ V_{td} $	$ V_{ts} $	$ V_{tb} $			
0.013	5.3×10^{-2}	1			

Table 5: The parameters of the $\mathbf{su}(8)$ benchmark point and the predicted SM quark/lepton masses (in unit of GeV) as well as the CKM mixings.

2 The intermediate stages of the SWS symmetry breaking pattern

Schematically, we assign the Higgs VEVs according to the symmetry breaking pattern as follows:

$$\mathfrak{g}_{531} \rightarrow \mathfrak{g}_{431} : \quad \langle (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \text{IV}} \rangle \equiv \frac{1}{\sqrt{2}} W_{\bar{\mathbf{5}}, \text{IV}}, \quad (27a)$$

$$\mathfrak{g}_{431} \rightarrow \mathfrak{g}_{421} : \quad \langle (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{V}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\bar{\mathbf{3}}, \text{V}}, \quad \langle (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{i, VII}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\bar{\mathbf{3}}, \text{i, VII}}, \quad (27b)$$

$$\begin{aligned} \mathfrak{g}_{421} \rightarrow \mathfrak{g}_{\text{SM}} : \quad & \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \text{3, VI}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\bar{\mathbf{4}}, \text{3, VI}}, \\ & \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \text{VIII}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\bar{\mathbf{4}}, \text{VIII}}, \quad \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})'_{\mathbf{H}, \text{2, IX}} \rangle \equiv \frac{1}{\sqrt{2}} V'_{\bar{\mathbf{4}}, \text{2, IX}}, \end{aligned} \quad (27c)$$

$$\text{EWSB} : \quad \langle (\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{H}} \rangle \equiv \frac{1}{\sqrt{2}} v_{\text{EW}}. \quad (27d)$$

For our later convenience, we also parametrize different symmetry breaking VEVs in terms of the following dimensionless quantities:

$$\begin{aligned} \zeta_1 &\equiv \frac{W_{\bar{\mathbf{5}}, \text{IV}}}{M_{\text{pl}}}, \quad \zeta_2 \equiv \frac{w_{\bar{\mathbf{3}}, \text{V}}}{M_{\text{pl}}}, \quad \dot{\zeta}_2 \equiv \frac{w_{\bar{\mathbf{3}}, \text{i, VII}}}{M_{\text{pl}}}, \\ \zeta_3 &\equiv \frac{V_{\bar{\mathbf{4}}, \text{3, VI}}}{M_{\text{pl}}}, \quad \dot{\zeta}_3' \equiv \frac{V'_{\bar{\mathbf{4}}, \text{2, IX}}}{M_{\text{pl}}}, \quad \dot{\zeta}_3 \equiv \frac{V_{\bar{\mathbf{4}}, \text{VIII}}}{M_{\text{pl}}}, \\ \zeta_0 &\gg \zeta_1 \gg \zeta_2 \sim \dot{\zeta}_2 \gg \zeta_3 \sim \dot{\zeta}_3' \sim \dot{\zeta}_3, \quad \zeta_{ij} \equiv \frac{\zeta_j}{\zeta_i}, \quad (i < j). \end{aligned} \quad (28)$$

In Table 6, we summarize all vectorlike fermions that become massive during different stages of the SWS symmetry breaking pattern.

stages	$Q_e = -\frac{1}{3}$	$Q_e = +\frac{2}{3}$	$Q_e = -1$	$Q_e = 0$
v_{531} $\{\Omega\}$	\mathfrak{D}' IV	-	$(\mathfrak{e}', \mathfrak{n}')$ IV	$\{\tilde{\mathfrak{n}}, \tilde{\mathfrak{n}}''\}$ $\{\text{IV}'', \text{IV}\}$
v_{431} $\{\Omega\}$	$\{\mathfrak{D}, \mathfrak{D}''''\}$ $\{\text{V}, \text{VII}\}$	\mathfrak{U}	$\mathfrak{E}, (\mathfrak{e}, \mathfrak{n}), (\mathfrak{e}''', \mathfrak{n}''')$ $\{\text{V}, \text{VII}\}$	$\{\tilde{\mathfrak{n}}', \tilde{\mathfrak{n}}'''\}$ $\{\text{V}, \text{VII}\}$
v_{421} $\{\Omega\}$	$\mathfrak{D}, \{\mathfrak{D}'', \mathfrak{D}''', \mathfrak{D}'''''\}$ $\{\text{VI}, \text{IX}, \text{VIII}\}$	\mathfrak{u}	$(\mathfrak{e}'', \mathfrak{n}''), (\mathfrak{e}''''', \mathfrak{n}'''''), (\mathfrak{e}''''', \mathfrak{n}''''')$ $\{\text{VI}, \text{VIII}, \text{IX}\}$	-

Table 6: The vectorlike fermions at different intermediate symmetry breaking stages along the SWS symmetry breaking pattern of the $\mathfrak{su}(8)$ theory.

2.1 The first stage

The first symmetry breaking stage of $\mathfrak{g}_{531} \rightarrow \mathfrak{g}_{431}$ is achieved by $(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \text{IV}} \subset \overline{\mathbf{8}}_{\mathbf{H}, \text{IV}}$ in the rank-two sector, according to the $\widetilde{\text{U}}(1)_{T'}$ -neutral components in Table 7. Accordingly, the term of

$\overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega}$	
\mathcal{T}'	$-4t$	$-\frac{4}{3}t$	
$\mathbf{28}_{\mathbf{F}}$	$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}$	$(\overline{\mathbf{5}}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}$
\mathcal{T}'	$-\frac{4}{3}t$	$+\frac{4}{3}t$	$+4t$
$\mathbf{56}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}'$	$(\overline{\mathbf{5}}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}}$	$(\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}}$ $(\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}}$
\mathcal{T}'	$-4t$	$-\frac{4}{3}t$	$+4t$ $+\frac{4}{3}t$
$\overline{\mathbf{8}}_{\mathbf{H},\omega}$	$(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega}$	
\mathcal{T}'	0	$+\frac{8}{3}t$	
$\mathbf{28}_{\mathbf{H},\dot{\omega}}$	$(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H},\dot{\omega}}$	$(\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H},\dot{\omega}}$	$(\overline{\mathbf{10}}, \mathbf{1}, +\frac{2}{5})_{\mathbf{H},\dot{\omega}}$
\mathcal{T}'	$+\frac{16}{3}t$	$+\frac{8}{3}t$	0
$\mathbf{70}_{\mathbf{H}}$	$(\mathbf{10}, \mathbf{3}, +\frac{4}{15})_{\mathbf{H}}$		
\mathcal{T}'	$-\frac{16}{3}t$		

Table 7: The $\tilde{\mathbf{U}}(1)_{T'}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{531} theory.

$Y_{\mathcal{B}}\overline{\mathbf{8}}_{\mathbf{F}}^{\text{IV}}\mathbf{28}_{\mathbf{F}}\overline{\mathbf{8}}_{\mathbf{H},\text{IV}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}', \mathfrak{n}'', \mathfrak{e}', \mathfrak{n}', \mathfrak{n})$ as follows:

$$\begin{aligned}
& Y_{\mathcal{B}}\overline{\mathbf{8}}_{\mathbf{F}}^{\text{IV}}\mathbf{28}_{\mathbf{F}}\overline{\mathbf{8}}_{\mathbf{H},\text{IV}} + H.c. \\
& \supset Y_{\mathcal{B}}\left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\text{IV}} \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{IV}} \otimes (\overline{\mathbf{5}}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}}\right] \otimes \langle (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\text{IV}} \rangle + H.c. \\
& \Rightarrow \frac{1}{\sqrt{2}}Y_{\mathcal{B}}(\mathfrak{D}'_L\mathfrak{D}'_R{}^c + \mathfrak{n}''_L\mathfrak{n}''_R{}^c + \mathfrak{e}'_L\mathfrak{e}'_R{}^c - \mathfrak{n}'_L\mathfrak{n}'_R{}^c + \mathfrak{n}_L\mathfrak{n}_R{}^c)W_{\overline{\mathbf{5}},\text{IV}} + H.c..
\end{aligned} \tag{29}$$

After this stage, the remaining massless fermions expressed in terms of the \mathfrak{g}_{431} IRs are the following:

$$\begin{aligned}
& \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'}\right] \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}, \\
& \Omega = (\omega, \dot{\omega}), \quad \omega = (3, \text{V}, \text{VI}), \quad \dot{\omega} = (\dot{\mathbf{1}}, \dot{\mathbf{2}}, \text{VII}, \text{VIII}, \text{IX}), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}'} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\text{IV}}, \\
& (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \oplus \left[(\mathbf{1}, \overline{\mathbf{3}}, +\frac{1}{3})_{\mathbf{F}}' \oplus (\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}}\right] \oplus \left[(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}}' \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}\right] \subset \mathbf{28}_{\mathbf{F}}, \\
& (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}' \oplus \left[(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}' \oplus (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{F}}\right] \\
& \oplus \left[(\overline{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}'\right] \oplus \left[(\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}}' \oplus (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}}\right] \subset \mathbf{56}_{\mathbf{F}}.
\end{aligned} \tag{30}$$

2.2 The second stage

The second symmetry breaking stage of $\mathfrak{g}_{431} \rightarrow \mathfrak{g}_{421}$ is achieved by $(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\text{V}} \subset \overline{\mathbf{8}}_{\mathbf{H},\text{V}}$ in the rank-two sector, $(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\text{VII}} \subset \overline{\mathbf{28}}_{\mathbf{H},\text{VII}}$ and $(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{H}}^{\text{VII}} \subset \mathbf{28}_{\mathbf{H}}^{\text{VII}}$ in the rank-three sector, according to the $\tilde{\mathbf{U}}(1)_{T''}$ -neutral components in Table 8. The terms of $Y_{\mathcal{B}}\overline{\mathbf{8}}_{\mathbf{F}}^{\text{V}}\mathbf{28}_{\mathbf{F}}\overline{\mathbf{8}}_{\mathbf{H},\text{V}} + H.c.$ leads to

$\overline{8_F}^\Omega$	$(\overline{4}, 1, +\frac{1}{4})_F^\Omega$	$(1, 1, 0)_F^{\Omega'}$	$(1, \overline{3}, -\frac{1}{3})_F^\Omega$
\mathcal{T}''	$-2t$	$-4t$	$-4t$
28_F	$(1, \overline{3}, +\frac{2}{3})_F$	$(4, 3, +\frac{1}{12})_F$	$(6, 1, -\frac{1}{2})_F$
\mathcal{T}''	$+4t$	$+2t$	0
56_F	$(1, 1, +1)_F'$	$(1, \overline{3}, +\frac{2}{3})_F'$	$(4, \overline{3}, +\frac{5}{12})_F$
\mathcal{T}''	$+4t$	$+4t$	$+2t$
	$(\overline{4}, 1, -\frac{3}{4})_F$	$(6, 1, -\frac{1}{2})_F'$	$(4, 3, +\frac{1}{12})_F'$
\mathcal{T}''	$-2t$	0	$+2t$
			0
$\overline{8_{H,\omega}}$	$(\overline{4}, 1, +\frac{1}{4})_{H,\omega}$	$(1, \overline{3}, -\frac{1}{3})_{H,\omega}$	
\mathcal{T}''	$+2t$	0	
$28_{H,\dot{\omega}}$	$(1, 3, -\frac{2}{3})_{H,\dot{\omega}}$	$(1, \overline{3}, -\frac{1}{3})_{H,\dot{\omega}}$	$(\overline{4}, \overline{3}, -\frac{1}{12})_{H,\dot{\omega}}$
\mathcal{T}''	0	0	$+2t$
70_H	$(4, \overline{3}, +\frac{5}{12})_H$		$+2t$
\mathcal{T}''	$-2t$		

Table 8: The $\widetilde{U}(1)_{T''}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{431} theory.

the vectorial masses of $(\mathfrak{D}, \mathfrak{e}, \mathfrak{n}, \check{\mathfrak{n}}')$ as follows:

$$\begin{aligned}
& Y_B \overline{8_F}^V 28_F \overline{8_H}_V + H.c. \\
\supset & Y_B \left[(\overline{5}, 1, +\frac{1}{5})_F^V \otimes (5, 3, +\frac{2}{15})_F \oplus (1, \overline{3}, -\frac{1}{3})_F^V \otimes (1, \overline{3}, +\frac{2}{3})_F \right] \otimes (1, \overline{3}, -\frac{1}{3})_{H,V} + H.c. \\
\supset & Y_B \left[(\overline{4}, 1, +\frac{1}{4})_F^V \otimes (4, 3, +\frac{1}{12})_F \oplus (1, 1, 0)_F^{V'} \otimes (1, 3, +\frac{1}{3})_F' \right. \\
& \oplus \left. (1, \overline{3}, -\frac{1}{3})_F^V \otimes (1, \overline{3}, +\frac{2}{3})_F \right] \otimes \langle (1, \overline{3}, -\frac{1}{3})_{H,V} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_B \left(\mathcal{D}_L \mathcal{D}_R^c + \check{\mathfrak{n}}'_L \check{\mathfrak{n}}'^c_R - \mathfrak{e}_L \mathfrak{e}_R^c + \mathfrak{n}_L \mathfrak{n}_R^c + \check{\mathcal{N}}_L^{V'} \check{\mathfrak{n}}_R^c \right) w_{\overline{3},V} + H.c.. \tag{31}
\end{aligned}$$

The terms of $Y_D \overline{8_F}^{VII} 56_F 28_H_{VII} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}''', \mathfrak{e}''', \mathfrak{n}''', \check{\mathfrak{n}}''')$ as follows:

$$\begin{aligned}
& Y_D \overline{8_F}^{VII} 56_F 28_H_{VII} + H.c. \\
\supset & Y_D \left[(\overline{5}, 1, +\frac{1}{5})_F^{VII} \otimes (10, 3, -\frac{1}{15})_F \oplus (1, \overline{3}, -\frac{1}{3})_F^{VII} \otimes (5, \overline{3}, +\frac{7}{15})_F \right] \otimes (\overline{5}, \overline{3}, -\frac{2}{15})_{H,VII} + H.c. \\
\supset & Y_D \left[(\overline{4}, 1, +\frac{1}{4})_F^{VII} \otimes (4, 3, +\frac{1}{12})_F' \oplus (1, \overline{3}, -\frac{1}{3})_F^{VII} \otimes (1, \overline{3}, +\frac{2}{3})_F' \right] \otimes \langle (1, \overline{3}, -\frac{1}{3})_{H,VII} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_D \left(\mathcal{D}_L''' \mathcal{D}_R'''^c - \mathfrak{e}_L''' \mathfrak{e}_R'''^c + \mathfrak{n}_L''' \mathfrak{n}_R'''^c + \check{\mathfrak{n}}_L''' \check{\mathfrak{n}}_R'''^c \right) w_{\overline{3},VII} + H.c.. \tag{32}
\end{aligned}$$

The Yukawa coupling between two $\mathbf{56_F}$ s by the following $d = 5$ operator

$$\begin{aligned}
& \frac{c_4}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\text{VII}} + H.c. \\
\supset & \ c_4 \frac{v_U}{M_{\text{pl}}} \left[(\mathbf{5}, \mathbf{\bar{3}}, +\frac{7}{15})_{\mathbf{F}} \otimes (\mathbf{\bar{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus \cancel{(\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}}} \right] \\
& \otimes (\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{H}}^{\text{VII}} + H.c. \\
\supset & \ c_4 \zeta_0 (\mathbf{4}, \mathbf{\bar{3}}, +\frac{5}{12})_{\mathbf{F}} \otimes (\mathbf{\bar{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}} \otimes \langle (\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{H}}^{\text{VII}} \rangle + H.c. \\
\Rightarrow & \ \frac{c_4}{\sqrt{2}} \zeta_0 (\mathfrak{E}_L \mathfrak{E}_R^c + \mathfrak{U}_L \mathfrak{U}_R^c) w_{\mathbf{3}}^{\text{VII}} + H.c., \tag{33}
\end{aligned}$$

leads to massive vectorlike fermions of $(\mathfrak{E}, \mathfrak{U})$.

After integrating out the massive fermions, the remaining massless fermions expressed in terms of the \mathfrak{g}_{421} IRs are the following:

$$\begin{aligned}
& (\mathbf{\bar{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'} \oplus (\mathbf{1}, \mathbf{\bar{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} \subset \mathbf{\bar{8}_F}^{\Omega}, \\
& \Omega = (\omega, \dot{\omega}), \quad \omega = (3, \text{VI}), \quad \dot{\omega} = (\dot{1}, \dot{2}, \text{VIII}, \text{IX}), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}'} \subset \mathbf{\bar{8}_F}^{\text{IV}}, \quad (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}'} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}''} \subset \mathbf{\bar{8}_F}^{\text{V}}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VII}'} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VII}''} \subset \mathbf{\bar{8}_F}^{\text{VII}}, \\
& \left[\cancel{(\mathbf{1}, \mathbf{\bar{2}}, +\frac{1}{2})_{\mathbf{F}}} \oplus (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}} \right] \oplus \left[\cancel{(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}} \right] \oplus \left[\cancel{(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}}} \oplus (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}} \right] \\
\oplus & \left[\cancel{(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}} \right] \subset \mathbf{28_F}, \\
& (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}' \oplus \left[\cancel{(\mathbf{1}, \mathbf{\bar{2}}, +\frac{1}{2})_{\mathbf{F}}}''' \oplus (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}'' \right] \oplus \left[\cancel{(\mathbf{4}, \mathbf{1}, +\frac{3}{4})_{\mathbf{F}}} \oplus (\mathbf{4}, \mathbf{\bar{2}}, +\frac{1}{4})_{\mathbf{F}} \right] \oplus \left[\cancel{(\mathbf{4}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}}} \right] \\
\oplus & (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}' \oplus \left[\cancel{(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}}}'' \oplus (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}}' \right] \oplus \left[(\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}'' \oplus (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \right] \subset \mathbf{56_F}. \tag{34}
\end{aligned}$$

2.3 The third stage

The third symmetry breaking stage of $\mathfrak{g}_{421} \rightarrow \mathfrak{g}_{\text{SM}}$ is achieved by $(\mathbf{\bar{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \text{VI}} \subset \mathbf{\bar{8}_H, \text{VI}}$ in the rank-two sector and $(\mathbf{\bar{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \text{VIII}} \subset \mathbf{\bar{28}_H, \text{VIII}}$, $(\mathbf{\bar{4}}, \mathbf{1}, +\frac{1}{4})'_{\mathbf{H}, \text{IX}} \subset \mathbf{\bar{28}_H, \text{IX}}$, $(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{H}}^{\text{IX}'} \subset \mathbf{28_H, IX'}$ in the rank-three sector, according to the $\tilde{\text{U}}(1)_{T'''}\text{-neutral}$ components in Table 9. The term of

$\overline{8}_{\mathbf{F}}^{\Omega}$	$(\overline{4}, 1, +\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(1, 1, 0)_{\mathbf{F}}^{\Omega'}$	$(1, \overline{2}, -\frac{1}{2})_{\mathbf{F}}^{\Omega}$	$(1, 1, 0)_{\mathbf{F}}^{\Omega''}$
\mathcal{T}'''	$-4t$	$-4t$	0	$-4t$
$28_{\mathbf{F}}$	$(1, 1, +1)_{\mathbf{F}}$	$(4, 2, +\frac{1}{4})_{\mathbf{F}}$	$(6, 1, -\frac{1}{2})_{\mathbf{F}}$	
\mathcal{T}'''	$-4t$	0	$+4t$	
$56_{\mathbf{F}}$	$(1, 1, +1)_{\mathbf{F}}'$	$(1, 1, +1)_{\mathbf{F}}''$	$(4, \overline{2}, +\frac{1}{4})_{\mathbf{F}}$	$(6, 1, -\frac{1}{2})_{\mathbf{F}}'$
\mathcal{T}'''	$-4t$	$-4t$	0	$+4t$
	$(4, 2, +\frac{1}{4})_{\mathbf{F}}'$	$(6, 1, -\frac{1}{2})_{\mathbf{F}}''$	$(6, 2, 0)_{\mathbf{F}}$	
\mathcal{T}'''	0	$+4t$	0	
$\overline{8}_{\mathbf{H}, \omega}$	$(\overline{4}, 1, +\frac{1}{4})_{\mathbf{H}, \omega}$	$(1, \overline{2}, -\frac{1}{2})_{\mathbf{H}, \omega}$		
\mathcal{T}'''	0	$+4t$		
$28_{\mathbf{H}, \omega}$	$(1, 2, -\frac{1}{2})_{\mathbf{H}, \omega}$	$(\overline{4}, \overline{2}, -\frac{1}{4})_{\mathbf{H}, \omega}$	$(\overline{4}, 1, +\frac{1}{4})_{\mathbf{H}, \omega}$	$(\overline{4}, 1, +\frac{1}{4})_{\mathbf{H}, \omega}'$
\mathcal{T}'''	$+4t$	$+4t$	0	0
	$(1, \overline{2}, -\frac{1}{2})_{\mathbf{H}, \omega}'$			
\mathcal{T}'''	$+4t$			
$70_{\mathbf{H}}$	$(4, \overline{2}, +\frac{1}{4})_{\mathbf{H}}$			
\mathcal{T}'''	$-4t$			

Table 9: The $\widetilde{\mathbf{U}}(1)_{T'''}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{421} theory.

$Y_B \overline{8_F}^{VI} \overline{28_F 8_H}_{VI} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}'', \mathfrak{e}'', \mathfrak{n}'')$ as follows:

$$\begin{aligned}
& Y_B \overline{8_F}^{VI} \overline{28_F 8_H}_{VI} + H.c. \\
\supset & Y_B \left[(\overline{5}, 1, +\frac{1}{5})_F^{VI} \otimes (10, 1, -\frac{2}{5})_F \oplus (1, \overline{3}, -\frac{1}{3})_F^{VI} \otimes (5, 3, +\frac{2}{15})_F \right] \otimes (\overline{5}, 1, +\frac{1}{5})_{H,VI} + H.c. \\
\supset & Y_B \left[(\overline{4}, 1, +\frac{1}{4})_F^{VI} \otimes (6, 1, -\frac{1}{2})_F \oplus (1, 1, 0)_F^{VI'} \otimes (4, 1, -\frac{1}{4})_F' \right. \\
& \oplus (1, \overline{3}, -\frac{1}{3})_F^{VI} \otimes (4, 3, +\frac{1}{12})_F \left. \right] \otimes (\overline{4}, 1, +\frac{1}{4})_{H,VI} + H.c. \\
\supset & Y_B \left[(\overline{4}, 1, +\frac{1}{4})_F^{VI} \otimes (6, 1, -\frac{1}{2})_F \oplus (1, 1, 0)_F^{VI'} \otimes (4, 1, -\frac{1}{4})_F' \right. \\
& \oplus (1, \overline{2}, -\frac{1}{2})_F^{VI} \otimes (4, 2, +\frac{1}{4})_F \oplus (1, 1, 0)_F^{VI''} \otimes (4, 1, -\frac{1}{4})_F \left. \right] \otimes \langle (\overline{4}, 1, +\frac{1}{4})_{H,VI} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_B \left(\mathfrak{e}_L'' \mathfrak{e}_R''^c - \mathfrak{n}_L'' \mathfrak{n}_R''^c + \tilde{\mathcal{N}}_L^{VI'} \mathfrak{n}_R''^c + \tilde{\mathcal{N}}_L^{VI''} \mathfrak{n}_R'^c + \mathfrak{D}_L'' \mathfrak{D}_R''^c \right) V_{4,VI} + H.c. .
\end{aligned} \tag{35}$$

The term of $Y_D \overline{8_F}^{VIII,IX} \overline{56_F 28_H}_{VIII,IX} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}''''', \mathfrak{e}''''', \mathfrak{n}''''', \mathfrak{D}''', \mathfrak{e}''''', \mathfrak{n}''''')$ as follows:

$$\begin{aligned}
& Y_D \overline{8_F}^{VIII} \overline{56_F 28_H}_{VIII} + H.c. \\
\supset & Y_D \left[(\overline{5}, 1, +\frac{1}{5})_F^{VIII} \otimes (10, 3, -\frac{1}{15})_F \oplus (1, \overline{3}, -\frac{1}{3})_F^{VIII} \otimes (5, \overline{3}, +\frac{7}{15})_F \right] \otimes (\overline{5}, \overline{3}, -\frac{2}{15})_{H,VIII} + H.c. \\
\supset & Y_D \left[(\overline{4}, 1, +\frac{1}{4})_F^{VIII} \otimes (6, 3, -\frac{1}{6})_F \oplus (1, 1, 0)_F^{VIII'} \otimes (4, 3, +\frac{1}{12})_F' \oplus (1, \overline{3}, -\frac{1}{3})_F^{VIII} \otimes (4, \overline{3}, +\frac{5}{12})_F \right] \\
& \otimes (\overline{4}, \overline{3}, -\frac{1}{12})_{H,VIII} + H.c. \\
\supset & Y_D \left[(\overline{4}, 1, +\frac{1}{4})_F^{VIII} \otimes (6, 1, -\frac{1}{2})_F'' \oplus (1, 1, 0)_F^{VIII'} \otimes (4, 1, -\frac{1}{4})_F'' \oplus (1, \overline{2}, -\frac{1}{2})_F^{VIII} \otimes (4, \overline{2}, +\frac{1}{4})_F \right] \\
& \otimes \langle (\overline{4}, 1, +\frac{1}{4})_{H,VIII} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_D \left(\mathfrak{D}_L'''' \mathfrak{D}_R''''^c + \tilde{\mathcal{N}}_L^{VIII'} \mathfrak{n}_R''''^c - \mathfrak{e}_L'''' \mathfrak{e}_R''''^c + \mathfrak{n}_L'''' \mathfrak{n}_R''''^c \right) V_{4,VIII} + H.c. ,
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
& Y_D \overline{8_F}^{IX} \overline{56_F 28_H}_{IX} + H.c. \\
\supset & Y_D \left[(\overline{5}, 1, +\frac{1}{5})_F^{IX} \otimes (\overline{10}, 1, -\frac{3}{5})_F \oplus (1, \overline{3}, -\frac{1}{3})_F^{IX} \otimes (10, 3, -\frac{1}{15})_F \right] \otimes (\overline{10}, 1, +\frac{2}{5})_{H,IX} + H.c. \\
\supset & Y_D \left[(\overline{4}, 1, +\frac{1}{4})_F^{IX} \otimes (6, 1, -\frac{1}{2})_F' \oplus (1, \overline{3}, -\frac{1}{3})_F^{IX} \otimes (4, 3, +\frac{1}{12})_F' \right] \otimes (\overline{4}, 1, +\frac{1}{4})_{H,IX} + H.c. \\
\supset & Y_D \left[(\overline{4}, 1, +\frac{1}{4})_F^{IX} \otimes (6, 1, -\frac{1}{2})_F' \oplus (1, \overline{2}, -\frac{1}{2})_F^{IX} \otimes (4, 2, +\frac{1}{4})_F' \right. \\
& \oplus (1, 1, 0)_F^{IX''} \otimes (4, 1, -\frac{1}{4})_F'' \left. \right] \otimes \langle (\overline{4}, 1, +\frac{1}{4})_{H,IX} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_D \left(\mathfrak{D}_L''' \mathfrak{D}_R'''^c + \tilde{\mathcal{N}}_L^{IX''} \mathfrak{n}_R'''^c + \mathfrak{e}_L''' \mathfrak{e}_R'''^c - \mathfrak{n}_L''' \mathfrak{n}_R'''^c \right) V_{4,IX}' + H.c. .
\end{aligned} \tag{37}$$

The Yukawa coupling between two $\mathbf{56_F}$ s by the following $d = 5$ operator

$$\begin{aligned}
& \frac{c_4}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\text{IX}} + H.c. \\
\supset & c_4 \zeta_0 \left[(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{5}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \right] \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{H}}^{\text{IX}} + H.c. \\
\supset & c_4 \zeta_0 \left[(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \otimes (\overline{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}} \oplus (\overline{\mathbf{4}}, \mathbf{3}, +\frac{5}{12})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}} \right] \otimes (\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{H}}^{\text{IX}} + H.c. \\
\supset & c_4 \zeta_0 \left[(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \otimes (\overline{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{4}, \mathbf{1}, +\frac{3}{4})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{2})''_{\mathbf{F}} \oplus (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{F}} \right. \\
& \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \left. \right] \otimes \langle (\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{H}}^{\text{IX}'} \rangle + H.c. \\
\Rightarrow & c_4 \zeta_0 (\mathfrak{E}_L \mu_R^c + \mathfrak{U}_L c_R^c + \mathfrak{u}_L \mathfrak{u}_R^c - \mathfrak{d}_L \mathfrak{d}_R^c) V_4^{\text{IX}} + H.c., \tag{38}
\end{aligned}$$

further leads to massive vectorlike fermions of $(\mathfrak{u}, \mathfrak{d})$.

The remaining massless fermions of the \mathfrak{g}_{SM} are listed as follows:

$$\begin{aligned}
& \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega} \right] \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'} \oplus \left[(\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} \right] \subset \overline{\mathbf{8_F}}^{\Omega}, \quad \Omega = (\dot{1}, \dot{2}, 3), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VI}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VIII}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}} \subset \overline{\mathbf{8_F}}^{\Omega}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}'} \oplus \dots \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}'} \subset \overline{\mathbf{8_F}}^{\Omega'}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}''} \oplus \dots \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}''} \subset \overline{\mathbf{8_F}}^{\Omega''}, \\
& \left[(\overline{\mathbf{1}}, \overline{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}} \right] \oplus \left[(\mathbf{1}, \mathbf{2}, +\frac{1}{2})'_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)'_{\mathbf{F}} \right] \\
\oplus & \left[(\overline{\mathbf{1}}, \overline{\mathbf{2}}, +\frac{1}{2})''_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)''_{\mathbf{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''_{\mathbf{F}} \right. \\
\oplus & \left. (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \right] \subset \mathbf{28_F}, \\
& (\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \oplus \left[(\overline{\mathbf{1}}, \overline{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, +1)''_{\mathbf{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{1}, +\frac{2}{3})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, +1)'''_{\mathbf{F}} \right] \\
\oplus & \left[(\overline{\mathbf{3}}, \overline{\mathbf{2}}, +\frac{1}{6})_{\mathbf{F}} \oplus (\overline{\mathbf{1}}, \overline{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{F}} \right] \oplus \left[(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}} \right] \oplus \left[(\overline{\mathbf{3}}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}} \right] \\
\oplus & \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)'''_{\mathbf{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{2}, +\frac{1}{6})'_{\mathbf{F}} \oplus (\overline{\mathbf{1}}, \overline{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{F}} \right] \oplus \left[(\overline{\mathbf{3}}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}} \right] \\
\oplus & \left[(\mathbf{3}, \mathbf{2}, +\frac{1}{6})''_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \overline{\mathbf{2}}, +\frac{1}{6})_{\mathbf{F}} \right] \subset \mathbf{56_F}. \tag{39}
\end{aligned}$$

2.4 The $d = 5$ bi-linear fermion operators

For the operator of $\mathcal{O}_{\mathcal{F}}^{(4,1)}$ in Eq. (13a), it is decomposed as

$$\begin{aligned}
& \frac{c_4}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \cdot \overline{\mathbf{28_H}}_{,\dot{\omega}} \cdot \mathbf{70_H} + H.c. \\
\supset & \frac{c_4}{M_{\text{pl}}} (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, +\frac{2}{5})_{\mathbf{H},\dot{\omega}} \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_4}{M_{\text{pl}}} (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}} \otimes (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H},\dot{\omega}} \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_4}{M_{\text{pl}}} (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})'_{\mathbf{H},\dot{\omega}} \rangle \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow & \frac{c_4}{2} \zeta'_3 (c_L u_R^c) v_{\text{EW}} + H.c. .
\end{aligned} \tag{40}$$

For the operator of $\mathcal{O}_{\mathcal{F}}^{(5,1)}$ in Eq. (13b), it is decomposed as

$$\begin{aligned}
& \frac{c_5}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} \left[(\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \right] \otimes \langle (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega} \rangle \\
& \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset & c_5 \frac{W_{\overline{\mathbf{5}},\text{IV}}}{\sqrt{2}M_{\text{pl}}} \left[(\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{4}, \mathbf{3}, +\frac{1}{12})'_{\mathbf{F}} \right] \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}} + H.c. \\
\supset & c_5 \frac{W_{\overline{\mathbf{5}},\text{IV}}}{\sqrt{2}M_{\text{pl}}} \left[(\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{4}, \mathbf{2}, +\frac{1}{4})'_{\mathbf{F}} \right] \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow & \frac{c_5}{2} \zeta_1 (t_L u_R^c + u_L t_R^c) v_{\text{EW}} + H.c. ,
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
& \frac{c_5}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega} \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}} \otimes (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H},\omega} \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H},\omega} \rangle \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow & \frac{c_5}{2} \zeta_3 (c_L t_R^c) v_{\text{EW}} + H.c. ,
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
& \frac{c_5}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} (\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\text{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\text{F}} \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega} \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\text{H}} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} (\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\text{F}} \otimes (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\text{F}} \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega} \rangle \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\text{H}} + H.c. \\
\supset & c_5 \frac{w_{\overline{\mathbf{3}},\text{V}}}{\sqrt{2}M_{\text{pl}}} (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\text{F}} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\text{F}}'' \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\text{H}} + H.c. \\
\Rightarrow & \frac{c_5}{2} \zeta_2 (t_L c_R^c) v_{\text{EW}} + H.c. .
\end{aligned} \tag{43}$$

2.5 The $d = 5$ irreducible Higgs mixing operators

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1}$, we find the mass terms of

$$\begin{aligned}
& Y_{\text{B}} \overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1} \times \frac{d_{\mathcal{A}}}{M_{\text{pl}}} \epsilon_{\omega_1 \omega_2 \omega_3 \omega_4} \overline{\mathbf{8_H}}_{,\omega_1}^\dagger \overline{\mathbf{8_H}}_{,\omega_2}^\dagger \overline{\mathbf{8_H}}_{,\omega_3}^\dagger \overline{\mathbf{8_H}}_{,\omega_4}^\dagger \mathbf{70_H}^\dagger + H.c. \\
\supset & Y_{\text{B}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\text{F}}^{\omega_1} \otimes (\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\text{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{F}}^{\omega_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\text{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega_1} \\
\times & \frac{d_{\mathcal{A}}}{M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega_1}^\dagger \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega_2}^\dagger \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\text{H},\omega_3}^\dagger \otimes \langle (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\text{H},\omega_4}^\dagger \rangle \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\text{H}}^\dagger + H.c. \\
\supset & Y_{\text{B}} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\text{F}}^{\omega_1} \otimes (\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\text{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{F}}^{\omega_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\text{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega_1} \\
\times & d_{\mathcal{A}} \frac{W_{\overline{\mathbf{5}},\text{IV}}}{\sqrt{2}M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega_1}^\dagger \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega_2}^\dagger \rangle \otimes (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\text{H},\omega_3}^\dagger \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\text{H}}^\dagger + H.c. \\
\supset & Y_{\text{B}} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\text{F}}^{\omega_1} \otimes (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\text{F}} \oplus (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\text{F}}^{\omega_1} \otimes (\mathbf{1}, \mathbf{1}, +1)_{\text{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\text{H},\omega_1} \\
\times & d_{\mathcal{A}} \frac{W_{\overline{\mathbf{5}},\text{IV}} w_{\overline{\mathbf{3}},\text{V}}}{2M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\text{H},\omega_1}^\dagger \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\text{H},\omega_3}^\dagger \rangle \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\text{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\text{B}} d_{\mathcal{A}}}{4} \frac{W_{\overline{\mathbf{5}},\text{IV}} w_{\overline{\mathbf{3}},\text{V}} V_{\overline{\mathbf{4}},\text{VI}}}{M_{\text{pl}} m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},3}}^2} (b_L b_R^c + \tau_L \tau_R^c) v_{\text{EW}} + H.c. ,
\end{aligned} \tag{44}$$

via the Higgs mixing operator of $\mathcal{O}_{\mathcal{A}}^{d=5}$ in Eq. (14a).

For the Yukawa coupling of $\overline{\mathbf{8}_F^{\dot{\omega}_1}} \mathbf{56}_F \overline{\mathbf{28}_H^{\dot{\omega}_1}}$, we find the mass terms of

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8}_F^{\dot{\omega}_1}} \mathbf{56}_F \overline{\mathbf{28}_H^{\dot{\omega}_1}} \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}}^\dagger \mathbf{70}_H^\dagger \left(\overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \right) + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{5}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1} \\
\times & \frac{d_{\mathcal{B}}}{M_{\text{pl}}} (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}}^\dagger \langle \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \rangle + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{\mathbf{3}}, \mathbf{i}} w_{\overline{\mathbf{3}}, \text{VII}}}{2M_{\text{pl}}} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{4}, \mathbf{3}, +\frac{1}{12})_{\mathbf{F}}' \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}' \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_1} \\
\times & (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{\mathbf{3}}, \mathbf{i}} w_{\overline{\mathbf{3}}, \text{VII}}}{2M_{\text{pl}} m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_1}}^2} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}}' \oplus (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}'' \right] \\
\otimes & \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \dot{\omega}_2}^\dagger \rangle \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{D}} d_{\mathcal{B}}}{4} \dot{\zeta}_3 \left[\frac{w_{\overline{\mathbf{3}}, \mathbf{i}} w_{\overline{\mathbf{3}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \mathbf{i}}}^2} (d_L d_R^c + e_L e_R^c) + \frac{w_{\overline{\mathbf{3}}, \mathbf{i}} w_{\overline{\mathbf{3}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{2}}}^2} (d_L s_R^c + \mu_L e_R^c) \right] v_{\text{EW}} + H.c., \quad (45)
\end{aligned}$$

and

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8}_F^{\dot{\omega}_1}} \mathbf{56}_F \overline{\mathbf{28}_H^{\dot{\omega}_1}} \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}}^\dagger \mathbf{70}_H^\dagger \left(\overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \right) + H.c. \\
\supset & Y_{\mathcal{D}} (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1} \\
\times & \frac{d_{\mathcal{B}}}{M_{\text{pl}}} (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}}^\dagger \langle \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \rangle + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{\mathbf{3}}, \mathbf{i}} w_{\overline{\mathbf{3}}, \text{VII}}}{2M_{\text{pl}}} (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{6}, \mathbf{3}, -\frac{1}{6})_{\mathbf{F}} \otimes (\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_1} \\
\times & (\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_2}^\dagger \rangle \otimes (\mathbf{4}, \overline{\mathbf{3}}, +\frac{5}{12})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{\mathbf{3}}, \mathbf{i}} w_{\overline{\mathbf{3}}, \text{VII}} w_{\overline{\mathbf{3}}, \text{VII}}}{2\sqrt{2}M_{\text{pl}} m_{(\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}_1}}^2} (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{D}} d_{\mathcal{B}}}{4} \dot{\zeta}_2 \left[\frac{w_{\overline{\mathbf{3}}, \mathbf{i}} w_{\overline{\mathbf{3}}, \text{VII}}}{m_{(\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \mathbf{i}}}^2} (s_L d_R^c) + \frac{w_{\overline{\mathbf{3}}, \mathbf{i}} w_{\overline{\mathbf{3}}, \text{VII}}}{m_{(\overline{\mathbf{4}}, \overline{\mathbf{3}}, -\frac{1}{12})_{\mathbf{H}, \dot{2}}}^2} (s_L s_R^c) \right] v_{\text{EW}} + H.c., \quad (46)
\end{aligned}$$

and

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{8_F^{\dot{\omega}_1}} 56_F \overline{28_H^{\dot{\omega}_1}} \times \frac{d_{\mathcal{B}}}{M_{\text{Pl}}} \overline{28_H^{\dot{\omega}_1}}^\dagger \overline{28_H^{\dot{\omega}_2}}^\dagger 70_H^\dagger \left(\overline{28_H^{\dot{\omega}_1}}^\dagger \overline{28_H^{\dot{\omega}_2}} \right) + H.c. \\
\supset & Y_{\mathcal{D}} (1, \overline{3}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (1, 1, +1)_{\mathbf{F}}' \otimes (1, 3, -\frac{2}{3})_{\mathbf{H}, \dot{\omega}_1} \\
& \times \frac{d_{\mathcal{B}}}{M_{\text{Pl}}} (1, 3, -\frac{2}{3})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{10}, 1, +\frac{2}{5})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (10, \overline{3}, +\frac{4}{15})_{\mathbf{H}}^\dagger \langle \overline{28_H^{\dot{\omega}_1}}^\dagger \overline{28_H^{\dot{\omega}_2}} \rangle + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{3}, i} w_{\overline{3}, \text{VII}}}{2M_{\text{Pl}}} (1, \overline{3}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (1, 1, +1)_{\mathbf{F}}' \otimes (1, 3, -\frac{2}{3})_{\mathbf{H}, \dot{\omega}_1} \\
& \times (1, 3, -\frac{2}{3})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{4}, 1, +\frac{1}{4})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (4, \overline{3}, +\frac{5}{12})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{3}, i} w_{\overline{3}, \text{VII}}}{2M_{\text{Pl}} m_{(1, 3, -\frac{2}{3})_{\mathbf{H}, \dot{\omega}_1}}^2} (1, \overline{2}, -\frac{1}{2})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (1, 1, +1)_{\mathbf{F}}' \otimes \langle (\overline{4}, 1, +\frac{1}{4})_{\mathbf{H}, \dot{\omega}_2}^\dagger \rangle \otimes (4, \overline{2}, +\frac{1}{4})_{\mathbf{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{D}} d_{\mathcal{B}}}{4} \dot{\zeta}_3 \left[\frac{w_{\overline{3}, i} w_{\overline{3}, \text{VII}}}{m_{(1, 3, -\frac{2}{3})_{\mathbf{H}, i}}^2} (e_L \mu_R^c) + \frac{w_{\overline{3}, i} w_{\overline{3}, \text{VII}}}{m_{(1, 3, -\frac{2}{3})_{\mathbf{H}, \dot{2}}}^2} (\mu_L \mu_R^c) \right] v_{\text{EW}} + H.c., \tag{47}
\end{aligned}$$

via the Higgs mixing operator of $\mathcal{O}_{\mathcal{B}}^{d=5}$ in Eq. (14b). For convenience, we parametrize the following ratios of

$$\Delta_{\dot{\omega}} \equiv \frac{w_{\overline{3}, i} w_{\overline{3}, \text{VII}}}{m_{(1, \overline{3}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}}}^2}, \quad \Delta'_{\dot{\omega}} \equiv \frac{w_{\overline{3}, i} w_{\overline{3}, \text{VII}}}{m_{(\overline{4}, \overline{3}, -\frac{1}{12})_{\mathbf{H}, \dot{\omega}}}^2}, \quad \Delta''_{\dot{\omega}} \equiv \frac{w_{\overline{3}, i} w_{\overline{3}, \text{VII}}}{m_{(1, 3, -\frac{2}{3})_{\mathbf{H}, \dot{\omega}}}^2}. \tag{48}$$

With the Higgs VEVs assignments in Eq. (27), we can obtain the following propagator masses of

$$\begin{aligned}
m_{(1, \overline{3}, -\frac{1}{3})_{\mathbf{H}, i}} & \sim m_{(\overline{4}, \overline{3}, -\frac{1}{12})_{\mathbf{H}, i}} \sim m_{(1, 3, -\frac{2}{3})_{\mathbf{H}, i}} \sim \mathcal{O}(v_{431}), \\
m_{(1, \overline{3}, -\frac{1}{3})_{\mathbf{H}, \dot{2}}} & \sim m_{(\overline{4}, \overline{3}, -\frac{1}{12})_{\mathbf{H}, \dot{2}}} \sim m_{(1, 3, -\frac{2}{3})_{\mathbf{H}, \dot{2}}} \sim \mathcal{O}(v_{421}). \tag{49}
\end{aligned}$$

2.6 The SM quark/lepton masses and the benchmark

For all up-type quarks with $Q_e = +\frac{2}{3}$, we write down the following tree-level masses from both the renormalizable Yukawa couplings and the gravity-induced terms in the basis of $\mathcal{U} \equiv (u, c, t)$

$$\mathcal{M}_{\mathcal{U}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & c_5 \zeta_1 / \sqrt{2} \\ c_4 \dot{\zeta}_3' / \sqrt{2} & 0 & c_5 \zeta_3 / \sqrt{2} \\ c_5 \zeta_1 / \sqrt{2} & c_5 \zeta_2 / \sqrt{2} & Y_{\mathcal{T}} \end{pmatrix} v_{\text{EW}}. \tag{50}$$

For all down-type quarks with $Q_e = -\frac{1}{3}$, we find the following tree-level mass matrix in the basis of $\mathcal{D} \equiv (d, s, b)$

$$\mathcal{M}_{\mathcal{D}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1 \dot{\zeta}_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2 \dot{\zeta}_3 & 0 \\ Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1' \dot{\zeta}_2 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2' \dot{\zeta}_2 & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathcal{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\text{EW}}. \tag{51}$$

For all charged leptons with $Q_e = -1$, their tree-level mass matrix is expressed as follows:

$$\mathcal{M}_{\mathcal{L}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1 \dot{\zeta}_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1'' \dot{\zeta}_3' & 0 \\ Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2 \dot{\zeta}_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2'' \dot{\zeta}_3' & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathcal{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\text{EW}}, \tag{52}$$

in the basis of $\mathcal{L} \equiv (e, \mu, \tau)$. Based on the above SM quark/lepton mass matrices, we find the following benchmark point of

$$v_{531} \simeq 1.4 \times 10^{17} \text{ GeV}, \quad v_{431} \simeq 4.8 \times 10^{15} \text{ GeV}, \quad v_{421} \simeq 1.1 \times 10^{15} \text{ GeV}, \quad (53)$$

to reproduce the observed hierarchical masses as well as the CKM mixing pattern, as summarized in Table 10.

ζ_1	ζ_2	ζ_3	$Y_{\mathcal{D}}$	$Y_{\mathcal{B}}$	$Y_{\mathcal{T}}$
6.0×10^{-2}	2.0×10^{-3}	4.4×10^{-4}	0.5	0.5	1.0
$\lambda = \zeta_{23}$	c_4	c_5	$d_{\mathcal{A}}$	$d_{\mathcal{B}}$	m_{μ}
0.22	2.5	1.0	0.2	0.1	0.1
m_u	m_c	m_t	$m_d \approx m_e$	m_s	$m_b \approx m_{\tau}$
2.3×10^{-3}	0.3	174.2	0.7×10^{-3}	0.6×10^{-1}	1.7
$ V_{ud} $	$ V_{us} $	$ V_{ub} $			
0.98	0.22	0.1×10^{-3}			
$ V_{cd} $	$ V_{cs} $	$ V_{cb} $			
0.22	0.98	4.2×10^{-2}			
$ V_{td} $	$ V_{ts} $	$ V_{tb} $			
9.4×10^{-2}	4.2×10^{-2}	1			

Table 10: The parameters of the $\mathfrak{su}(8)$ benchmark point and the predicted SM quark/lepton masses (in unit of GeV) as well as the CKM mixings.

3 The intermediate stages of the WSS symmetry breaking pattern

Schematically, we assign the Higgs VEVs according to the symmetry breaking pattern as follows:

$$\mathfrak{g}_{531} \rightarrow \mathfrak{g}_{521} \quad : \quad \langle (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{IV}} \rangle \equiv \frac{1}{\sqrt{2}} W_{\bar{\mathbf{3}}, \text{IV}}, \quad (54a)$$

$$\mathfrak{g}_{521} \rightarrow \mathfrak{g}_{421} \quad : \quad \langle (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \text{V}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\bar{\mathbf{5}}, \text{V}}, \quad \langle (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \text{i}, \text{VII}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\bar{\mathbf{5}}, \text{i}, \text{VII}}, \quad (54b)$$

$$\begin{aligned} \mathfrak{g}_{421} \rightarrow \mathfrak{g}_{\text{SM}} \quad : \quad & \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \text{3}, \text{VI}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\bar{\mathbf{4}}, \text{3}, \text{VI}}, \\ & \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \text{VIII}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\bar{\mathbf{4}}, \text{VIII}}, \quad \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})'_{\mathbf{H}, \text{2}, \text{IX}} \rangle \equiv \frac{1}{\sqrt{2}} V'_{\bar{\mathbf{4}}, \text{2}, \text{IX}}, \end{aligned} \quad (54c)$$

$$\text{EWSB} \quad : \quad \langle (\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{H}} \rangle \equiv \frac{1}{\sqrt{2}} v_{\text{EW}}. \quad (54d)$$

For later convenience, we also parametrize different symmetry breaking VEVs in terms of the following dimensionless quantities:

$$\begin{aligned}\zeta_1 &\equiv \frac{W_{\bar{\mathbf{3}},\text{IV}}}{M_{\text{pl}}}, \quad \zeta_2 \equiv \frac{w_{\bar{\mathbf{5}},\text{V}}}{M_{\text{pl}}}, \quad \dot{\zeta}_2 \equiv \frac{w_{\bar{\mathbf{5}},\text{i},\text{VII}}}{M_{\text{pl}}}, \\ \zeta_3 &\equiv \frac{V_{\bar{\mathbf{4}},\text{3},\text{VI}}}{M_{\text{pl}}}, \quad \dot{\zeta}_3' \equiv \frac{V_{\bar{\mathbf{4}},\dot{\mathbf{2}},\text{IX}}}{M_{\text{pl}}}, \quad \dot{\zeta}_3 \equiv \frac{V_{\bar{\mathbf{4}},\text{VIII}}}{M_{\text{pl}}}, \\ \zeta_0 &\gg \zeta_1 \gg \zeta_2 \sim \dot{\zeta}_2 \gg \zeta_3 \sim \dot{\zeta}_3' \sim \dot{\zeta}_3, \quad \zeta_{ij} \equiv \frac{\zeta_j}{\zeta_i}, \quad (i < j).\end{aligned}\tag{55}$$

In Table 11, we summarize all vectorlike fermions that become massive during different stages of the WSS symmetry breaking pattern.

stages	$Q_e = -\frac{1}{3}$	$Q_e = +\frac{2}{3}$	$Q_e = -1$	$Q_e = 0$
v_{531} $\{\Omega\}$	\mathfrak{D} IV	-	$(\mathfrak{e}, \mathfrak{n})$ IV	$\{\check{\mathfrak{n}}, \check{\mathfrak{n}}'\}$ $\{\text{IV}, \text{IV}'\}$
v_{521} $\{\Omega\}$	$\{\mathfrak{D}', \mathfrak{D}''''\}$ $\{\text{V}, \text{VII}\}$	\mathfrak{L}	$\mathfrak{E}, (\mathfrak{e}', \mathfrak{n}'), (\mathfrak{e}''', \mathfrak{n}''')$ $\{\text{V}, \text{VII}\}$	$\{\check{\mathfrak{n}}'', \check{\mathfrak{n}}'''\}$ $\{\text{V}, \text{VII}\}$
v_{421} $\{\Omega\}$	$\mathfrak{d}, \{\mathfrak{D}'', \mathfrak{D}''', \mathfrak{D}''''\}$ $\{\text{VI}, \text{IX}, \text{VIII}\}$	\mathfrak{u}	$(\mathfrak{e}'', \mathfrak{n}''), (\mathfrak{e}''', \mathfrak{n}'''), (\mathfrak{e}'''', \mathfrak{n}''''')$ $\{\text{VI}, \text{VIII}, \text{IX}\}$	-

Table 11: The vectorlike fermions at different intermediate symmetry breaking stages along the WSS symmetry breaking pattern of the $\mathfrak{su}(8)$ theory.

3.1 The first stage

$\overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}$	$(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega}$		
\mathcal{T}'	$-\frac{12}{5}t$	$-4t$		
$\mathbf{28}_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{3}}, -\frac{2}{3})_{\mathbf{F}}$	$(\bar{\mathbf{5}}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}$	
\mathcal{T}'	$+4t$	$+\frac{12}{5}t$	$+\frac{4}{5}t$	
$\mathbf{56}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}'$	$(\bar{\mathbf{5}}, \bar{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}}$	$(\mathbf{10}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}}$	$(\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}}$
\mathcal{T}'	$+4t$	$+\frac{12}{5}t$	$-\frac{4}{5}t$	$+\frac{4}{5}t$
$\overline{\mathbf{8}}_{\mathbf{H},\omega}$	$(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega}$	$(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega}$		
\mathcal{T}'	$+\frac{8}{5}t$	0		
$\mathbf{28}_{\mathbf{H},\dot{\omega}}$	$(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H},\dot{\omega}}$	$(\bar{\mathbf{5}}, \bar{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H},\dot{\omega}}$	$(\mathbf{10}, \mathbf{1}, +\frac{2}{5})_{\mathbf{H},\dot{\omega}}$	
\mathcal{T}'	0	$+\frac{8}{5}t$	$+\frac{16}{5}t$	
$\mathbf{70}_{\mathbf{H}}$	$(\mathbf{10}, \bar{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}}$			
\mathcal{T}'	$-\frac{16}{5}t$			

Table 12: The $\tilde{\mathbf{U}}(1)_{T'}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{531} theory.

The first symmetry breaking stage of $\mathfrak{g}_{531} \rightarrow \mathfrak{g}_{521}$ is achieved by $(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{IV}} \subset \overline{\mathbf{8}_{\mathbf{H}, \text{IV}}}$ in the rank-two sector, according to the $\tilde{\text{U}}(1)_{T'}$ -neutral components in Table 12. Accordingly, the term of $Y_B \overline{\mathbf{8}_{\mathbf{F}}}^{\text{IV}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}_{\mathbf{H}, \text{IV}}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}, \mathfrak{N}', \mathfrak{e}, \mathfrak{n}, \mathfrak{n})$ as follows:

$$\begin{aligned}
& Y_B \overline{\mathbf{8}_{\mathbf{F}}}^{\text{IV}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}_{\mathbf{H}, \text{IV}}} + H.c. \\
\supset & Y_B \left[(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\text{IV}} \otimes (\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{IV}} \otimes (\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \right] \otimes \langle (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{IV}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_B (\mathfrak{D}_L \mathfrak{D}_R^c + \mathfrak{n}_L \mathfrak{n}_R^c - \mathfrak{e}_L \mathfrak{e}_R^c + \mathfrak{n}_L \mathfrak{n}_R^c + \mathfrak{n}'_L \mathfrak{n}'_R^c) W_{\bar{\mathbf{3}}, \text{IV}} + H.c. .
\end{aligned} \tag{56}$$

After this stage, the remaining massless fermions expressed in terms of the \mathfrak{g}_{521} IRs are the following:

$$\begin{aligned}
& (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\Omega} \oplus [(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} \oplus (\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega}] \subset \overline{\mathbf{8}_{\mathbf{F}}}^{\Omega} , \\
& \Omega = (\omega, \dot{\omega}), \quad \omega = (3, \text{V}, \text{VI}), \quad \dot{\omega} = (\dot{1}, \dot{2}, \text{VII}, \text{VIII}, \text{IX}), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}''} \subset \overline{\mathbf{8}_{\mathbf{F}}}^{\text{IV}}, \\
& \left[\cancel{(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}} \oplus (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}} \right] \oplus \left[\cancel{(\mathbf{5}, \mathbf{1}, -\frac{1}{5})_{\mathbf{F}}} \oplus (\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\mathbf{F}} \right] \oplus (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \subset \mathbf{28}_{\mathbf{F}}, \\
& (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}' \oplus \left[(\mathbf{5}, \mathbf{1}, +\frac{4}{5})_{\mathbf{F}} \oplus (\mathbf{5}, \bar{\mathbf{2}}, +\frac{3}{10})_{\mathbf{F}} \right] \oplus (\bar{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus \left[(\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}' \right. \\
& \oplus \left. (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \right] \subset \mathbf{56}_{\mathbf{F}} .
\end{aligned} \tag{57}$$

3.2 The second stage

$\overline{\mathbf{8}_{\mathbf{F}}}^{\Omega}$	$(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''}$
\mathcal{T}''	$-4t$	0	$-4t$
$\mathbf{28}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}$	$(\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\mathbf{F}}$	$(\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}$
\mathcal{T}''	$-4t$	0	$+4t$
$\mathbf{56}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}'$	$(\mathbf{5}, \mathbf{1}, +\frac{4}{5})_{\mathbf{F}}$	$(\mathbf{5}, \bar{\mathbf{2}}, +\frac{3}{10})_{\mathbf{F}}$
\mathcal{T}''	$-4t$	$-4t$	0
	$(\bar{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}}$	$(\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}'$	$(\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}}$
\mathcal{T}''	$+4t$	$+4t$	0
$\overline{\mathbf{8}_{\mathbf{H}, \omega}}$	$(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \omega}$	$(\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{H}, \omega}$	
\mathcal{T}''	0	$+4t$	
$\mathbf{28}_{\mathbf{H}, \dot{\omega}}$	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}}$	$(\bar{\mathbf{5}}, \bar{\mathbf{2}}, -\frac{3}{10})_{\mathbf{H}, \dot{\omega}}$	$(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \dot{\omega}}$
\mathcal{T}''	$+4t$	$+4t$	0
$\mathbf{70}_{\mathbf{H}}$	$(\mathbf{10}, \bar{\mathbf{2}}, +\frac{1}{10})_{\mathbf{H}}$		
\mathcal{T}''	$-4t$		

Table 13: The $\tilde{\text{U}}(1)_{T'}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{521} theory.

The second symmetry breaking stage of $\mathfrak{g}_{521} \rightarrow \mathfrak{g}_{421}$ is achieved by $(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \mathbf{V}} \subset \overline{\mathbf{8}_{\mathbf{H}, \mathbf{V}}}$ in the rank-two sector, $(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \mathbf{VII}} \subset \overline{\mathbf{28}_{\mathbf{H}, \mathbf{VII}}}$ and $(\mathbf{5}, \mathbf{1}, -\frac{1}{5})_{\mathbf{H}}^{\mathbf{VII}} \subset \mathbf{28}_{\mathbf{H}}^{\mathbf{VII}}$ in the rank-three sector, according to the $\tilde{\mathbf{U}}(1)_{T''}$ -neutral components in Table 13. The term of $Y_B \bar{\mathbf{8}}_{\mathbf{F}}^{\mathbf{V}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}_{\mathbf{H}, \mathbf{V}}}$ + *H.c.* leads to the vectorial masses of $(\mathfrak{D}', \mathfrak{e}', \mathfrak{n}', \mathfrak{n}'')$ as follows:

$$\begin{aligned}
& Y_B \bar{\mathbf{8}}_{\mathbf{F}}^{\mathbf{V}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}_{\mathbf{H}, \mathbf{V}}} + H.c. \\
\supset & Y_B \left[(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\mathbf{V}} \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\mathbf{V}} \otimes (\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}} \right] \otimes (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \mathbf{V}} + H.c. \\
\supset & Y_B \left[(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\mathbf{V}} \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\mathbf{V}} \otimes (\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\mathbf{F}} \right. \\
& \left. \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\mathbf{V}''} \otimes (\mathbf{5}, \mathbf{1}, -\frac{1}{5})_{\mathbf{F}} \right] \otimes \langle (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \mathbf{V}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_B \left(\mathfrak{D}'_L \mathfrak{D}'^c_R + \mathfrak{n}''_L \mathfrak{n}''^c_R + \mathfrak{e}'_L \mathfrak{e}'^c_R - \mathfrak{n}'_L \mathfrak{n}'^c_R + \tilde{\mathcal{N}}_L^{\mathbf{V}''} \mathfrak{n}'^c_R \right) w_{\bar{\mathbf{5}}, \mathbf{V}} + H.c. .
\end{aligned} \tag{58}$$

The term of $Y_D \bar{\mathbf{8}}_{\mathbf{F}}^{\mathbf{VII}} \mathbf{56}_{\mathbf{F}} \overline{\mathbf{28}_{\mathbf{H}, \mathbf{VII}}}$ + *H.c.* leads to the vectorial masses of $(\mathfrak{D}''', \mathfrak{e}''', \mathfrak{n}''', \mathfrak{n}''')$ as follows:

$$\begin{aligned}
& Y_D \bar{\mathbf{8}}_{\mathbf{F}}^{\mathbf{VII}} \mathbf{56}_{\mathbf{F}} \overline{\mathbf{28}_{\mathbf{H}, \mathbf{VII}}} + H.c. \\
\supset & Y_D \left[(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\mathbf{VII}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\mathbf{VII}} \otimes (\mathbf{5}, \bar{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \right] \otimes (\bar{\mathbf{5}}, \bar{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \mathbf{VII}} + H.c. \\
\supset & Y_D \left[(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\mathbf{VII}} \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}' \oplus (\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\mathbf{VII}} \otimes (\mathbf{5}, \bar{\mathbf{2}}, +\frac{3}{10})_{\mathbf{F}} \right] \otimes \langle (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \mathbf{VII}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_D \left(\mathfrak{D}'''_L \mathfrak{D}'''^c_R - \mathfrak{e}'''_L \mathfrak{e}'''^c_R + \mathfrak{n}'''_L \mathfrak{n}'''^c_R + \mathfrak{n}'''_L \mathfrak{n}'''^c_R \right) w_{\bar{\mathbf{5}}, \mathbf{VII}} + H.c. .
\end{aligned} \tag{59}$$

The Yukawa coupling between two $\mathbf{56}_{\mathbf{F}}$'s by the following $d = 5$ operator

$$\begin{aligned}
& \frac{c_4}{M_{\text{pl}}} \mathbf{56}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \mathbf{63}_{\mathbf{H}} \mathbf{28}_{\mathbf{H}}^{\mathbf{VII}} + H.c. \\
\supset & c_4 \frac{v_U}{M_{\text{pl}}} \left[(\mathbf{5}, \bar{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \otimes (\bar{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \right] \\
& \otimes (\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{H}}^{\mathbf{VII}} + H.c. \\
\supset & c_4 \zeta_0 (\mathbf{5}, \mathbf{1}, +\frac{4}{5})_{\mathbf{F}} \otimes (\bar{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \otimes \langle (\mathbf{5}, \mathbf{1}, -\frac{1}{5})_{\mathbf{H}}^{\mathbf{VII}} \rangle + H.c. \\
\Rightarrow & \frac{c_4}{\sqrt{2}} \zeta_0 (\mathfrak{E}_L \mathfrak{E}_R^c + \mathfrak{U}_L \mathfrak{U}_R^c) w_{\mathbf{5}}^{\mathbf{VII}} + H.c. ,
\end{aligned} \tag{60}$$

leads to massive vectorlike fermions of $(\mathfrak{E}, \mathfrak{U})$.

After integrating out the massive fermions, the remaining massless fermions expressed in terms of the \mathfrak{g}_{421} IRs are the following:

$$\begin{aligned}
& \left[(\bar{4}, 1, +\frac{1}{4})_{\mathbf{F}}^{\Omega} \oplus (1, 1, 0)_{\mathbf{F}}^{\Omega'} \right] \oplus (1, \bar{2}, -\frac{1}{2})_{\mathbf{F}}^{\Omega} \oplus (1, 1, 0)_{\mathbf{F}}^{\Omega''} \subset \overline{8}_{\mathbf{F}}^{\Omega}, \\
& \Omega = (\omega, \dot{\omega}), \quad \omega = (3, \text{VI}), \quad \dot{\omega} = (\dot{1}, \dot{2}, \text{VIII}, \text{IX}), \\
& (1, 1, 0)_{\mathbf{F}}^{\text{IV}'''} \subset \overline{8}_{\mathbf{F}}^{\text{IV}}, \quad (1, 1, 0)_{\mathbf{F}}^{\text{V}'} \oplus (1, 1, 0)_{\mathbf{F}}^{\text{V}'''} \subset \overline{8}_{\mathbf{F}}^{\text{V}}, \\
& (1, 1, 0)_{\mathbf{F}}^{\text{VII}'} \oplus (1, 1, 0)_{\mathbf{F}}^{\text{VII}'''} \subset \overline{8}_{\mathbf{F}}^{\text{VII}}, \\
& \cancel{(1, \bar{2}, +\frac{1}{2})_{\mathbf{F}}} \oplus (1, 1, +1)_{\mathbf{F}} \oplus \left[\cancel{(4, 1, -\frac{1}{4})_{\mathbf{F}}} \oplus (1, 1, 0)_{\mathbf{F}}' \right] \oplus \left[\cancel{(1, \bar{2}, +\frac{1}{2})_{\mathbf{F}}} \oplus (4, 2, +\frac{1}{4})_{\mathbf{F}} \right] \\
& \oplus \left[\cancel{(4, 1, -\frac{1}{4})_{\mathbf{F}}} \oplus (6, 1, -\frac{1}{2})_{\mathbf{F}} \right] \subset 28_{\mathbf{F}}, \\
& (1, 1, +1)_{\mathbf{F}}' \oplus \left[(1, 1, +1)_{\mathbf{F}}'' \oplus \cancel{(4, 1, +\frac{3}{4})_{\mathbf{F}}} \right] \oplus \left[\cancel{(1, \bar{2}, +\frac{1}{2})_{\mathbf{F}}}''' \oplus (4, \bar{2}, +\frac{1}{4})_{\mathbf{F}} \right] \oplus \left[\cancel{(4, 1, -\frac{3}{4})_{\mathbf{F}}} \right. \\
& \oplus \left. (6, 1, -\frac{1}{2})_{\mathbf{F}}' \right] \oplus \left[\cancel{(4, 1, -\frac{1}{4})_{\mathbf{F}}}'' \oplus (6, 1, -\frac{1}{2})_{\mathbf{F}}'' \right] \oplus \left[(4, 2, +\frac{1}{4})_{\mathbf{F}}' \oplus (6, 2, 0)_{\mathbf{F}} \right] \subset 56_{\mathbf{F}}. \quad (61)
\end{aligned}$$

3.3 The third stage

$\overline{8}_{\mathbf{F}}^{\Omega}$	$(\bar{4}, 1, +\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(1, 1, 0)_{\mathbf{F}}^{\Omega'}$	$(1, \bar{2}, -\frac{1}{2})_{\mathbf{F}}^{\Omega}$	$(1, 1, 0)_{\mathbf{F}}^{\Omega''}$
\mathcal{T}'''	$-4t$	$-4t$	0	$-4t$
$28_{\mathbf{F}}$	$(1, 1, +1)_{\mathbf{F}}$	$(4, 2, +\frac{1}{4})_{\mathbf{F}}$	$(6, 1, -\frac{1}{2})_{\mathbf{F}}$	
\mathcal{T}'''	$-4t$	0	$+4t$	
$56_{\mathbf{F}}$	$(1, 1, +1)_{\mathbf{F}}'$	$(1, 1, +1)_{\mathbf{F}}''$	$(4, \bar{2}, +\frac{1}{4})_{\mathbf{F}}$	$(6, 1, -\frac{1}{2})_{\mathbf{F}}'$
\mathcal{T}'''	$-4t$	$-4t$	0	$+4t$
	$(6, 1, -\frac{1}{2})_{\mathbf{F}}''$	$(4, 2, +\frac{1}{4})_{\mathbf{F}}'$	$(6, 2, 0)_{\mathbf{F}}$	
\mathcal{T}'''	$+4t$	0	0	
$\overline{8}_{\mathbf{H}, \omega}$	$(\bar{4}, 1, +\frac{1}{4})_{\mathbf{H}, \omega}$	$(1, \bar{2}, -\frac{1}{2})_{\mathbf{H}, \omega}$		
\mathcal{T}'''	0	$+4t$		
$28_{\mathbf{H}, \dot{\omega}}$	$(1, \bar{2}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}}$	$(\bar{4}, \bar{2}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}}$	$(\bar{4}, 1, +\frac{1}{4})_{\mathbf{H}, \dot{\omega}}$	$(\bar{4}, 1, +\frac{1}{4})_{\mathbf{H}, \dot{\omega}}'$
\mathcal{T}'''	$+4t$	$+4t$	0	0
	$(1, \bar{2}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}}'$			
\mathcal{T}'''	$+4t$			
$70_{\mathbf{H}}$	$(4, \bar{2}, +\frac{1}{4})_{\mathbf{H}}$			
\mathcal{T}'''	$-4t$			

Table 14: The $\widetilde{\text{U}}(1)_{\mathcal{T}'''}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{421} theory.

The third symmetry breaking stage of $\mathfrak{g}_{421} \rightarrow \mathfrak{g}_{\text{SM}}$ is achieved by $(\bar{4}, 1, +\frac{1}{4})_{\mathbf{H}, \text{VI}} \subset \overline{8}_{\mathbf{H}, \text{VI}}$ in the rank-two sector and $(\bar{4}, 1, +\frac{1}{4})_{\mathbf{H}, \text{VIII}} \subset \overline{28}_{\mathbf{H}, \text{VIII}}$, $(\bar{4}, 1, +\frac{1}{4})_{\mathbf{H}, \text{IX}}' \subset \overline{28}_{\mathbf{H}, \text{IX}}$, $(4, 1, -\frac{1}{4})_{\mathbf{H}}^{\text{IX}'} \subset \overline{28}_{\mathbf{H}}^{\text{IX}}$

in the rank-three sector, according to the $\tilde{U}(1)_{T''''}$ -neutral components in Table 14. The term of $Y_{\mathcal{B}} \overline{8_F^{VI}} 28_F \overline{8_H}_{VI} + H.c.$ leads to the vectorial masses of $(\mathcal{D}'', \epsilon'', n'')$ as follows:

$$\begin{aligned}
& Y_{\mathcal{B}} \overline{8_F^{VI}} 28_F \overline{8_H}_{VI} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\overline{5}, 1, +\frac{1}{5})_{\mathbf{F}}^{VI} \otimes (10, 1, -\frac{2}{5})_{\mathbf{F}} \oplus (1, \overline{3}, -\frac{1}{3})_{\mathbf{F}}^{VI} \otimes (5, 3, +\frac{2}{15})_{\mathbf{F}} \right] \otimes (\overline{5}, 1, +\frac{1}{5})_{\mathbf{H}, VI} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\overline{5}, 1, +\frac{1}{5})_{\mathbf{F}}^{VI} \otimes (10, 1, -\frac{2}{5})_{\mathbf{F}} \right. \\
& \oplus (1, \overline{2}, -\frac{1}{2})_{\mathbf{F}}^{VI} \otimes (5, 2, +\frac{3}{10})_{\mathbf{F}} \oplus (1, 1, 0)_{\mathbf{F}}^{VI''} \otimes (5, 1, -\frac{1}{5})_{\mathbf{F}} \left. \right] \otimes (\overline{5}, 1, +\frac{1}{5})_{\mathbf{H}, VI} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\overline{4}, 1, +\frac{1}{4})_{\mathbf{F}}^{VI} \otimes (6, 1, -\frac{1}{2})_{\mathbf{F}} \oplus (1, 1, 0)_{\mathbf{F}}^{VI'} \otimes (4, 1, -\frac{1}{4})_{\mathbf{F}}' \right. \\
& \oplus (1, \overline{2}, -\frac{1}{2})_{\mathbf{F}}^{VI} \otimes (4, 2, +\frac{1}{4})_{\mathbf{F}} \oplus (1, 1, 0)_{\mathbf{F}}^{VI''} \otimes (4, 1, -\frac{1}{4})_{\mathbf{F}} \left. \right] \otimes (\overline{4}, 1, +\frac{1}{4})_{\mathbf{H}, VI} + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{B}} \left(\epsilon_L'' \epsilon_R''^c - n_L'' n_R''^c + \tilde{\mathcal{N}}_L^{VI'} \check{n}_R''^c + \tilde{\mathcal{N}}_L^{VI''} \check{n}_R^c + \mathcal{D}_L'' \mathcal{D}_R''^c \right) V_{4, VI} + H.c.. \tag{62}
\end{aligned}$$

The term of $Y_{\mathcal{D}} \overline{8_F^{VIII, IX}} 56_F \overline{28_H}_{VIII, IX} + H.c.$ leads to the vectorial masses of $(\mathcal{D}''''', \epsilon''''', n''''', \mathcal{D}''', \epsilon''''', n''''')$ as follows:

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{8_F^{VIII}} 56_F \overline{28_H}_{VIII} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{5}, 1, +\frac{1}{5})_{\mathbf{F}}^{VIII} \otimes (10, 3, -\frac{1}{15})_{\mathbf{F}} \oplus (1, \overline{3}, -\frac{1}{3})_{\mathbf{F}}^{VIII} \otimes (5, \overline{3}, +\frac{7}{15})_{\mathbf{F}} \right] \otimes (\overline{5}, \overline{3}, -\frac{2}{15})_{\mathbf{H}, VIII} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{5}, 1, +\frac{1}{5})_{\mathbf{F}}^{VIII} \otimes (10, 1, -\frac{2}{5})_{\mathbf{F}}' \oplus (1, \overline{2}, -\frac{1}{2})_{\mathbf{F}}^{VIII} \otimes (5, \overline{2}, +\frac{3}{10})_{\mathbf{F}} \right] \otimes (\overline{5}, 1, +\frac{1}{5})_{\mathbf{H}, VIII} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{4}, 1, +\frac{1}{4})_{\mathbf{F}}^{VIII} \otimes (6, 1, -\frac{1}{2})_{\mathbf{F}}'' \oplus (1, 1, 0)_{\mathbf{F}}^{VIII'} \otimes (4, 1, -\frac{1}{4})_{\mathbf{F}}'' \oplus (1, \overline{2}, -\frac{1}{2})_{\mathbf{F}}^{VIII} \otimes (4, \overline{2}, +\frac{1}{4})_{\mathbf{F}} \right] \\
& \otimes (\overline{4}, 1, +\frac{1}{4})_{\mathbf{H}, VIII} + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} \left(\mathcal{D}_L'''' \mathcal{D}_R''''^c + \tilde{\mathcal{N}}_L^{VIII'} \check{n}_R''''^c - \epsilon_L'''' \epsilon_R''''^c + n_L'''' n_R''''^c \right) V_{4, VIII} + H.c., \tag{63}
\end{aligned}$$

and

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{8_F^{IX}} 56_F \overline{28_H}_{IX} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{5}, 1, +\frac{1}{5})_{\mathbf{F}}^{IX} \otimes (\overline{10}, 1, -\frac{3}{5})_{\mathbf{F}} \oplus (1, \overline{3}, -\frac{1}{3})_{\mathbf{F}}^{IX} \otimes (10, 3, -\frac{1}{15})_{\mathbf{F}} \right] \otimes (\overline{10}, 1, +\frac{2}{5})_{\mathbf{H}, IX} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{5}, 1, +\frac{1}{5})_{\mathbf{F}}^{IX} \otimes (\overline{10}, 1, -\frac{3}{5})_{\mathbf{F}} \oplus (1, \overline{2}, -\frac{1}{2})_{\mathbf{F}}^{IX} \otimes (10, 2, +\frac{1}{10})_{\mathbf{F}} \oplus (1, 1, 0)_{\mathbf{F}}^{IX''} \otimes (10, 1, -\frac{2}{5})_{\mathbf{F}}' \right] \\
& \otimes (\overline{10}, 1, +\frac{2}{5})_{\mathbf{H}, IX} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{4}, 1, +\frac{1}{4})_{\mathbf{F}}^{IX} \otimes (6, 1, -\frac{1}{2})_{\mathbf{F}}' \oplus (1, \overline{2}, -\frac{1}{2})_{\mathbf{F}}^{IX} \otimes (4, 2, +\frac{1}{4})_{\mathbf{F}}' \oplus (1, 1, 0)_{\mathbf{F}}^{IX''} \otimes (4, 1, -\frac{1}{4})_{\mathbf{F}}'' \right] \\
& \otimes (\overline{4}, 1, +\frac{1}{4})_{\mathbf{H}, IX} + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} \left(\mathcal{D}_L''' \mathcal{D}_R'''^c + \tilde{\mathcal{N}}_L^{IX''} \check{n}_R'''^c + \epsilon_L'''' \epsilon_R''''^c - n_L'''' n_R''''^c \right) V_{4, IX}' + H.c.. \tag{64}
\end{aligned}$$

The Yukawa coupling between two $\mathbf{56_F}$'s by the following $d = 5$ operator

$$\begin{aligned}
& \frac{c_4}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\text{IX}} + H.c. \\
\supset & c_4 \frac{v_U}{M_{\text{pl}}} \left[(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{5}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \right] \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{H}}^{\text{IX}} + H.c. \\
\supset & c_4 \zeta_0 \left[(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{5}, \mathbf{1}, +\frac{4}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})'_{\mathbf{F}} \oplus (\mathbf{5}, \overline{\mathbf{2}}, +\frac{3}{10})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \right] \\
& \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{H}}^{\text{IX}} + H.c. \\
\supset & c_4 \frac{v_U}{M_{\text{pl}}} \left[(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \otimes (\overline{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{4}, \mathbf{1}, +\frac{3}{4})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{2})''_{\mathbf{F}} \oplus (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{F}} \right. \\
& \left. \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \right] \otimes \langle (\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{H}}^{\text{IX}'} \rangle + H.c. \\
\Rightarrow & \frac{c_4}{\sqrt{2}} \zeta_0 (\mathfrak{E}_L \mu_R^c + \mathfrak{U}_L c_R^c + \mathfrak{u}_L u_R^c - \mathfrak{d}_L \mathfrak{d}_R^c) V_4^{\text{IX}'} + H.c., \tag{65}
\end{aligned}$$

further leads to massive vectorlike fermions of $(\mathfrak{u}, \mathfrak{d})$.

The remaining massless fermions of the \mathfrak{g}_{SM} are listed as follows:

$$\begin{aligned}
& \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega} \right] \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'} \oplus (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} \subset \overline{\mathbf{8_F}}^{\Omega}, \quad \Omega = (\dot{1}, \dot{2}, 3), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VI}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VIII}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}} \subset \overline{\mathbf{8_F}}^{\Omega}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}'} \oplus \dots \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}'} \subset \overline{\mathbf{8_F}}^{\Omega'}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}''} \oplus \dots \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}''} \subset \overline{\mathbf{8_F}}^{\Omega''}, \\
& \cancel{(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}} \oplus (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}} \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}} \right] \oplus (\mathbf{1}, \mathbf{1}, 0)'_{\mathbf{F}} \\
\oplus & \cancel{(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})'_{\mathbf{F}}} \oplus \left[\cancel{(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})''_{\mathbf{F}}} \oplus (\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)''_{\mathbf{F}} \right] \\
\oplus & \left[\cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''_{\mathbf{F}}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \right] \subset \mathbf{28_F}, \\
& (\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, +1)''_{\mathbf{F}} \oplus \left[(\mathbf{3}, \mathbf{1}, +\frac{2}{3})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, +1)'''_{\mathbf{F}} \right] \oplus \cancel{(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{F}}} \\
\oplus & \left[\cancel{(\mathbf{3}, \overline{\mathbf{2}}, +\frac{1}{6})_{\mathbf{F}}} \oplus \cancel{(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{F}}} \right] \oplus \left[(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}} \right] \oplus \left[\cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}} \right] \\
\oplus & \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)'''_{\mathbf{F}} \right] \oplus \left[\cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''''_{\mathbf{F}}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{2}, +\frac{1}{6})'_{\mathbf{F}} \oplus \cancel{(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})''''_{\mathbf{F}}} \right] \\
\oplus & \left[(\mathbf{3}, \mathbf{2}, +\frac{1}{6})''_{\mathbf{F}} \oplus \cancel{(\mathbf{3}, \overline{\mathbf{2}}, +\frac{1}{6})_{\mathbf{F}}} \right] \subset \mathbf{56_F}. \tag{66}
\end{aligned}$$

3.4 The $d = 5$ bi-linear fermion operators

For the operator of $\mathcal{O}_{\mathcal{F}}^{(4,1)}$ in Eq. (13a), it is decomposed as

$$\begin{aligned}
& \frac{c_4}{M_{\text{pl}}} \mathbf{56}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \cdot \overline{\mathbf{28}_{\mathbf{H},\dot{\omega}}} \cdot \mathbf{70}_{\mathbf{H}} + H.c. \\
\supset & \frac{c_4}{M_{\text{pl}}} (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, +\frac{2}{5})_{\mathbf{H},\dot{\omega}} \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_4}{M_{\text{pl}}} (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, +\frac{2}{5})_{\mathbf{H},\dot{\omega}} \otimes (\mathbf{10}, \overline{\mathbf{2}}, +\frac{1}{10})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_4}{M_{\text{pl}}} (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})'_{\mathbf{H},\dot{\omega}} \rangle \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow & \frac{c_4}{2} \zeta'_3 (c_L u_R^c) v_{\text{EW}} + H.c. .
\end{aligned} \tag{67}$$

For the operator of $\mathcal{O}_{\mathcal{F}}^{(5,1)}$ in Eq. (13b), it is decomposed as

$$\begin{aligned}
& \frac{c_5}{M_{\text{pl}}} \mathbf{28}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \cdot \overline{\mathbf{8}_{\mathbf{H},\omega}} \cdot \mathbf{70}_{\mathbf{H}} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} \left[(\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega} \\
\otimes & (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} \left[(\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\mathbf{F}} \otimes (\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}} \oplus (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \right] \otimes \langle (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega} \rangle \\
\otimes & (\mathbf{10}, \overline{\mathbf{2}}, +\frac{1}{10})_{\mathbf{H}} + H.c. \\
\supset & c_5 \frac{w_{\overline{\mathbf{5}},\text{V}}}{\sqrt{2} M_{\text{pl}}} \left[(\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{4}, \mathbf{2}, +\frac{1}{4})'_{\mathbf{F}} \right] \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow & \frac{c_5}{2} \zeta_2 (t_L u_R^c + u_L t_R^c) v_{\text{EW}} + H.c. ,
\end{aligned} \tag{68}$$

and

$$\begin{aligned}
& \frac{c_5}{M_{\text{pl}}} \mathbf{28}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \cdot \overline{\mathbf{8}_{\mathbf{H},\omega}} \cdot \mathbf{70}_{\mathbf{H}} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega} \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega} \otimes (\mathbf{10}, \overline{\mathbf{2}}, +\frac{1}{10})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H},\omega} \rangle \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{H}} + H.c. \\
\Rightarrow & \frac{c_5}{2} \zeta_3 (c_L t_R^c) v_{\text{EW}} + H.c. ,
\end{aligned} \tag{69}$$

and

$$\begin{aligned}
& \frac{c_5}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_H}}_{,\omega} \cdot \mathbf{70_H} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} (\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\text{F}} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\text{F}} \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega} \rangle \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\text{H}} + H.c. \\
\supset & c_5 \frac{W_{\overline{\mathbf{3}},\text{IV}}}{\sqrt{2}M_{\text{pl}}} (\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\text{F}} \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})'_{\text{F}} \otimes (\mathbf{10}, \overline{\mathbf{2}}, +\frac{1}{10})_{\text{H}} + H.c. \\
\supset & c_5 \frac{W_{\overline{\mathbf{3}},\text{IV}}}{\sqrt{2}M_{\text{pl}}} (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\text{F}} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{2})''_{\text{F}} \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\text{H}} + H.c. \\
\Rightarrow & \frac{c_5}{2} \zeta_1 (t_L c_R^c) v_{\text{EW}} + H.c.. \tag{70}
\end{aligned}$$

3.5 The $d = 5$ irreducible Higgs mixing operators

For the Yukawa coupling of $\overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1}$, we find the mass terms of

$$\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8_F}}^{\omega_1} \mathbf{28_F} \overline{\mathbf{8_H}}_{,\omega_1} \times \frac{d_{\mathcal{A}}}{M_{\text{pl}}} \epsilon_{\omega_1 \omega_2 \omega_3 \omega_4} \overline{\mathbf{8_H}}_{,\omega_1}^\dagger \overline{\mathbf{8_H}}_{,\omega_2}^\dagger \overline{\mathbf{8_H}}_{,\omega_3}^\dagger \overline{\mathbf{8_H}}_{,\omega_4}^\dagger \mathbf{70_H}^\dagger + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\text{F}}^{\omega_1} \otimes (\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\text{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{F}}^{\omega_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\text{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega_1} \\
\times & \frac{d_{\mathcal{A}}}{M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega_1}^\dagger \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\text{H},\omega_2}^\dagger \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\text{H},\omega_3}^\dagger \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},\omega_4}^\dagger \rangle \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\text{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\text{F}}^{\omega_1} \otimes (\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\text{F}} \oplus (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\text{F}}^{\omega_1} \otimes (\mathbf{1}, \mathbf{1}, +1)_{\text{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\text{H},\omega_1} \\
\times & d_{\mathcal{A}} \frac{W_{\overline{\mathbf{3}},\text{IV}}}{\sqrt{2}M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\text{H},\omega_1}^\dagger \otimes \langle (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\text{H},\omega_2}^\dagger \rangle \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\text{H},\omega_3}^\dagger \otimes (\mathbf{10}, \overline{\mathbf{2}}, +\frac{1}{10})_{\text{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\text{F}}^{\omega_1} \otimes (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\text{F}} \oplus (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\text{F}}^{\omega_1} \otimes (\mathbf{1}, \mathbf{1}, +1)_{\text{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\text{H},\omega_1} \\
\times & d_{\mathcal{A}} \frac{W_{\overline{\mathbf{3}},\text{IV}} w_{\overline{\mathbf{5}},\text{V}}}{2M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\text{H},\omega_1}^\dagger \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\text{H},\omega_3}^\dagger \rangle \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\text{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{B}} d_{\mathcal{A}}}{4} \frac{W_{\overline{\mathbf{3}},\text{IV}} w_{\overline{\mathbf{5}},\text{V}} V_{\overline{\mathbf{4}},\text{VI}}}{M_{\text{pl}} m^2_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\text{H},3}}} (b_L b_R^c + \tau_L \tau_R^c) v_{\text{EW}} + H.c., \tag{71}
\end{aligned}$$

via the Higgs mixing operator of $\mathcal{O}_{\mathcal{A}}^{d=5}$ in Eq. (14a).

For the Yukawa coupling of $\overline{\mathbf{8}_F^{\dot{\omega}_1}} \mathbf{56}_F \overline{\mathbf{28}_H^{\dot{\omega}_1}}$, we find the mass terms of

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8}_F^{\dot{\omega}_1}} \mathbf{56}_F \overline{\mathbf{28}_H^{\dot{\omega}_1}} \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}}^\dagger \mathbf{70}_H^\dagger \left(\overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \right) + H.c. \\
\supset & Y_{\mathcal{D}} (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}' \otimes (\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H}, \dot{\omega}_1} \\
& \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} (\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{\mathbf{10}}, \mathbf{1}, +\frac{2}{5})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}}^\dagger \langle \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \rangle + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{\mathbf{5}}, \mathbf{i}} w_{\overline{\mathbf{5}}, \text{VII}}}{2M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}' \otimes (\mathbf{1}, \mathbf{2}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}_1} \\
& \times (\mathbf{1}, \mathbf{2}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{\mathbf{10}}, \mathbf{1}, +\frac{2}{5})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{10}, \overline{\mathbf{2}}, +\frac{1}{10})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{D}} d_{\mathcal{B}} \frac{w_{\overline{\mathbf{5}}, \mathbf{i}} w_{\overline{\mathbf{5}}, \text{VII}}}{2M_{\text{pl}} m_{(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H}, \dot{\omega}_1}}^2} (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}' \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \dot{\omega}_2}^\dagger \rangle \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{D}} d_{\mathcal{B}}}{4} \dot{\zeta}_3 \left[\frac{w_{\overline{\mathbf{5}}, \mathbf{i}} w_{\overline{\mathbf{5}}, \text{VII}}}{m_{(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H}, \mathbf{i}}}^2} (e_L \mu_R^c) + \frac{w_{\overline{\mathbf{5}}, \mathbf{i}} w_{\overline{\mathbf{5}}, \text{VII}}}{m_{(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H}, \dot{2}}}^2} (\mu_L \mu_R^c) \right] v_{\text{EW}} + H.c., \tag{72}
\end{aligned}$$

and

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8}_F^{\dot{\omega}_1}} \mathbf{56}_F \overline{\mathbf{28}_H^{\dot{\omega}_1}} \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}}^\dagger \mathbf{70}_H^\dagger \left(\overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \right) + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{5}, \overline{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1} \\
& \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{\mathbf{5}}, \overline{\mathbf{3}}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{10}, \overline{\mathbf{3}}, +\frac{4}{15})_{\mathbf{H}}^\dagger \langle \overline{\mathbf{28}_H^{\dot{\omega}_1}}^\dagger \overline{\mathbf{28}_H^{\dot{\omega}_2}} \rangle + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{5}, \mathbf{1}, +\frac{4}{5})_{\mathbf{F}} \right] \otimes (\overline{\mathbf{5}}, \overline{\mathbf{2}}, -\frac{3}{10})_{\mathbf{H}, \dot{\omega}_1} \\
& \times \frac{d_{\mathcal{B}} w_{\overline{\mathbf{5}}, \mathbf{i}} w_{\overline{\mathbf{5}}, \text{VII}}}{2M_{\text{pl}}} (\overline{\mathbf{5}}, \overline{\mathbf{2}}, -\frac{3}{10})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{10}, \overline{\mathbf{2}}, +\frac{1}{10})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}}' \oplus (\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}'' \right] \\
& \times \frac{d_{\mathcal{B}} w_{\overline{\mathbf{5}}, \mathbf{i}} w_{\overline{\mathbf{5}}, \text{VII}}}{2M_{\text{pl}} m_{(\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}_1}}^2} \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \dot{\omega}_2}^\dagger \rangle \otimes (\mathbf{4}, \overline{\mathbf{2}}, +\frac{1}{4})_{\mathbf{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{D}} d_{\mathcal{B}}}{4} \dot{\zeta}_3 \left[\frac{w_{\overline{\mathbf{5}}, \mathbf{i}} w_{\overline{\mathbf{5}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{H}, \mathbf{i}}}^2} (d_L d_R^c + e_L e_R^c) \right. \\
& \left. + \frac{w_{\overline{\mathbf{5}}, \mathbf{i}} w_{\overline{\mathbf{5}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{H}, \dot{2}}}^2} (d_L s_R^c + \mu_L e_R^c) \right] v_{\text{EW}} + H.c., \tag{73}
\end{aligned}$$

and

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{8_F}^{\dot{\omega}_1} \mathbf{56_F} \overline{28_H}^{\dot{\omega}_1} \times \frac{d_{\mathcal{B}}}{M_{\text{Pl}}} \overline{28_H}^{\dot{\omega}_1} \overline{28_H}^{\dot{\omega}_2} \mathbf{70_H}^{\dagger} \left(\overline{28_H}^{\dagger} \overline{28_H}^{\text{VII}} \right) + H.c. \\
\supset & Y_{\mathcal{D}} (\overline{5}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}} \otimes (\overline{5}, \overline{3}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1} \\
& \times \frac{d_{\mathcal{B}}}{M_{\text{Pl}}} (\overline{5}, \overline{3}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_1}^{\dagger} \otimes (\overline{5}, \overline{3}, -\frac{2}{15})_{\mathbf{H}, \dot{\omega}_2}^{\dagger} \otimes (\mathbf{10}, \overline{3}, +\frac{4}{15})_{\mathbf{H}}^{\dagger} \otimes \langle \overline{28_H}^{\dagger} \overline{28_H}^{\text{VII}} \rangle + H.c. \\
\supset & Y_{\mathcal{D}} (\overline{5}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \otimes (\overline{5}, \overline{2}, -\frac{3}{10})_{\mathbf{H}, \dot{\omega}_1} \\
& \times \frac{d_{\mathcal{B}} w_{\overline{5}, \mathbf{i}} w_{\overline{5}, \text{VII}}}{2M_{\text{Pl}}} (\overline{5}, \overline{2}, -\frac{3}{10})_{\mathbf{H}, \dot{\omega}_1}^{\dagger} \otimes \langle (\overline{5}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \dot{\omega}_2}^{\dagger} \rangle \otimes (\mathbf{10}, \overline{2}, -\frac{1}{10})_{\mathbf{H}}^{\dagger} + H.c. \\
\supset & \frac{Y_{\mathcal{D}} d_{\mathcal{B}} w_{\overline{5}, \mathbf{i}} w_{\overline{5}, \text{VII}}}{2\sqrt{2}m_{(\overline{4}, \overline{2}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}_1}}^2} \dot{\zeta}_2 (\overline{4}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \otimes (\mathbf{4}, \overline{2}, +\frac{1}{4})_{\mathbf{H}}^{\dagger} + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{D}} d_{\mathcal{B}}}{4} \dot{\zeta}_2 \left[\frac{w_{\overline{5}, \mathbf{i}} w_{\overline{5}, \text{VII}}}{m_{(\overline{4}, \overline{2}, -\frac{1}{4})_{\mathbf{H}, \mathbf{i}}}^2} (s_L d_R^c) + \frac{w_{\overline{5}, \mathbf{i}} w_{\overline{5}, \text{VII}}}{m_{(\overline{4}, \overline{2}, -\frac{1}{4})_{\mathbf{H}, \dot{2}}}^2} (s_L s_R^c) \right] v_{\text{EW}} + H.c., \tag{74}
\end{aligned}$$

via the Higgs mixing operator of $\mathcal{O}_{\mathcal{B}}^{d=5}$ in Eq. (14b). For convenience, we parametrize the following ratios of

$$\Delta_{\dot{\omega}} \equiv \frac{w_{\overline{5}, \mathbf{i}} w_{\overline{5}, \text{VII}}}{m_{(\mathbf{1}, \overline{2}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}}}^2}, \quad \Delta'_{\dot{\omega}} \equiv \frac{w_{\overline{5}, \mathbf{i}} w_{\overline{5}, \text{VII}}}{m_{(\overline{4}, \overline{2}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}}}^2}, \quad \Delta''_{\dot{\omega}} \equiv \frac{w_{\overline{5}, \mathbf{i}} w_{\overline{5}, \text{VII}}}{m_{(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H}, \dot{\omega}}}^2}. \tag{75}$$

With the Higgs VEVs assignments in Eq. (54), we can obtain the following propagator masses of

$$\begin{aligned}
m_{(\mathbf{1}, \overline{2}, -\frac{1}{2})_{\mathbf{H}, \mathbf{i}}} & \sim m_{(\overline{4}, \overline{2}, -\frac{1}{4})_{\mathbf{H}, \mathbf{i}}} \sim m_{(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H}, \mathbf{i}}} \sim \mathcal{O}(v_{431}), \\
m_{(\mathbf{1}, \overline{2}, -\frac{1}{2})_{\mathbf{H}, \dot{2}}} & \sim m_{(\overline{4}, \overline{2}, -\frac{1}{4})_{\mathbf{H}, \dot{2}}} \sim m_{(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H}, \dot{2}}} \sim \mathcal{O}(v_{421}). \tag{76}
\end{aligned}$$

3.6 The SM quark/lepton masses and the benchmark

For all up-type quarks with $Q_e = +\frac{2}{3}$, we write down the following tree-level masses from both the renormalizable Yukawa couplings and the gravity-induced terms in the basis of $\mathcal{U} \equiv (u, c, t)$

$$\mathcal{M}_{\mathcal{U}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & c_5 \zeta_2 / \sqrt{2} \\ c_4 \dot{\zeta}_3' / \sqrt{2} & 0 & c_5 \zeta_3 / \sqrt{2} \\ c_5 \zeta_2 / \sqrt{2} & c_5 \zeta_1 / \sqrt{2} & Y_{\mathcal{T}} \end{pmatrix} v_{\text{EW}}. \tag{77}$$

For all down-type quarks with $Q_e = -\frac{1}{3}$, we find the following tree-level mass matrix in the basis of $\mathcal{D} \equiv (d, s, b)$

$$\mathcal{M}_{\mathcal{D}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1 \dot{\zeta}_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2 \dot{\zeta}_3 & 0 \\ Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1' \dot{\zeta}_2 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2' \dot{\zeta}_2 & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathcal{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\text{EW}}. \tag{78}$$

For all charged leptons with $Q_e = -1$, their tree-level mass matrix is expressed as follows:

$$\mathcal{M}_{\mathcal{L}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1 \dot{\zeta}_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1'' \dot{\zeta}_3' & 0 \\ Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2 \dot{\zeta}_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2'' \dot{\zeta}_3' & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathcal{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\text{EW}}, \tag{79}$$

in the basis of $\mathcal{L} \equiv (e, \mu, \tau)$. Based on the above SM quark/lepton mass matrices, we find the following benchmark point of

$$v_{531} \simeq 1.4 \times 10^{17} \text{ GeV}, \quad v_{521} \simeq 4.8 \times 10^{15} \text{ GeV}, \quad v_{421} \simeq 1.1 \times 10^{15} \text{ GeV}, \quad (80)$$

to reproduce the observed hierarchical masses as well as the CKM mixing pattern, as summarized in Table 15.

ζ_1	ζ_2	ζ_3	$Y_{\mathcal{D}}$	$Y_{\mathcal{B}}$	$Y_{\mathcal{T}}$
6.0×10^{-2}	2.0×10^{-3}	4.4×10^{-4}	0.5	0.5	1.0
$\lambda = \zeta_{23}$	c_4	c_5	$d_{\mathcal{A}}$	$d_{\mathcal{B}}$	m_{μ}
0.22	5	0.6	0.2	0.1	0.1
m_u	m_c	m_t	$m_d \approx m_e$	m_s	$m_b \approx m_{\tau}$
3.7×10^{-3}	0.3	174.2	0.7×10^{-3}	0.6×10^{-1}	1.7
$ V_{ud} $	$ V_{us} $	$ V_{ub} $			
0.98	0.22	0.1×10^{-3}			
$ V_{cd} $	$ V_{cs} $	$ V_{cb} $			
0.22	0.98	2.5×10^{-2}			
$ V_{td} $	$ V_{ts} $	$ V_{tb} $			
0.6×10^{-2}	2.5×10^{-2}	1			

Table 15: The parameters of the $\mathfrak{su}(8)$ benchmark point and the predicted SM quark/lepton masses (in unit of GeV) as well as the CKM mixings.

4 The intermediate stages of the WWW symmetry breaking pattern

Schematically, we assign the Higgs VEVs according to the symmetry breaking pattern as follows:

$$\mathfrak{g}_{351} \rightarrow \mathfrak{g}_{341} \quad : \quad \langle (\mathbf{1}, \bar{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H}, \text{IV}} \rangle \equiv \frac{1}{\sqrt{2}} W_{\bar{\mathbf{5}}, \text{IV}}, \quad (81a)$$

$$\mathfrak{g}_{341} \rightarrow \mathfrak{g}_{331} \quad : \quad \langle (\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \text{V}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\bar{\mathbf{4}}, \text{V}}, \quad \langle (\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \text{i, VII}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\bar{\mathbf{4}}, \text{i, VII}}, \quad (81b)$$

$$\begin{aligned} \mathfrak{g}_{331} \rightarrow \mathfrak{g}_{\text{SM}} \quad : \quad & \langle (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{3, VI}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\bar{\mathbf{3}}, \text{3, VI}}, \\ & \langle (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{VIII}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\bar{\mathbf{3}}, \text{VIII}}, \quad \langle (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})'_{\mathbf{H}, \text{2, IX}} \rangle \equiv \frac{1}{\sqrt{2}} V'_{\bar{\mathbf{3}}, \text{2, IX}}, \end{aligned} \quad (81c)$$

$$\text{EWSB} \quad : \quad \langle (\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{H}} \rangle \equiv \frac{1}{\sqrt{2}} v_{\text{EW}}. \quad (81d)$$

For later convenience, we also parametrize different symmetry breaking VEVs in terms of the following dimensionless quantities:

$$\begin{aligned}
\zeta_1 &\equiv \frac{W_{\bar{\mathbf{5}},\text{IV}}}{M_{\text{pl}}}, & \zeta_2 &\equiv \frac{w_{\bar{\mathbf{4}},\text{V}}}{M_{\text{pl}}}, & \dot{\zeta}_2 &\equiv \frac{w_{\bar{\mathbf{4}},\text{i},\text{VII}}}{M_{\text{pl}}}, \\
\zeta_3 &\equiv \frac{V_{\bar{\mathbf{3}},\text{3},\text{VI}}}{M_{\text{pl}}}, & \dot{\zeta}_3 &\equiv \frac{V'_{\bar{\mathbf{3}},\dot{\mathbf{2}},\text{IX}}}{M_{\text{pl}}}, & \dot{\zeta}_3 &\equiv \frac{V_{\bar{\mathbf{3}},\text{VIII}}}{M_{\text{pl}}}, \\
\zeta_0 &\gg \zeta_1 \gg \zeta_2 \sim \dot{\zeta}_2 \gg \zeta_3 \sim \dot{\zeta}_3' \sim \dot{\zeta}_3, & \zeta_{ij} &\equiv \frac{\zeta_j}{\zeta_i}, & (i < j).
\end{aligned} \tag{82}$$

In Table 16, we summarize all vectorlike fermions that become massive during different stages of the WWW symmetry breaking pattern.

stages	$Q_e = -\frac{1}{3}$	$Q_e = +\frac{2}{3}$	$Q_e = -1$	$Q_e = 0$
v_{351} $\{\Omega\}$	\mathfrak{D} IV	-	$(\mathfrak{e}'', \mathfrak{n}'')$ IV	$\{\check{\mathfrak{n}}', \check{\mathfrak{n}}''\}$ $\{\text{IV}', \text{IV}''\}$
v_{341} $\{\Omega\}$	$\mathfrak{d}, \{\mathfrak{D}'', \mathfrak{D}''''\}$ $\{\text{V}, \text{VII}\}$	$\mathfrak{u}, \mathfrak{u}$	$\mathfrak{E}, (\mathfrak{e}, \mathfrak{n}), (\mathfrak{e}''', \mathfrak{n}''')$ $\{\text{V}, \text{VII}\}$	$\{\check{\mathfrak{n}}, \check{\mathfrak{n}}'''\}$ $\{\text{V}', \text{VII}'\}$
v_{331} $\{\Omega\}$	$\{\mathfrak{D}', \mathfrak{D}''', \mathfrak{D}''''\}$ $\{\text{VI}, \text{VIII}, \text{IX}\}$	-	$(\mathfrak{e}', \mathfrak{n}'), (\mathfrak{e}''', \mathfrak{n}'''), (\mathfrak{e}''''', \mathfrak{n}''''')$ $\{\text{VI}, \text{VIII}, \text{IX}\}$	-

Table 16: The vectorlike fermions at different intermediate symmetry breaking stages along the WWW symmetry breaking pattern of the $\mathfrak{su}(8)$ theory.

4.1 The first stage

The first symmetry breaking stage of $\mathfrak{g}_{351} \rightarrow \mathfrak{g}_{341}$ is achieved by $(\mathbf{1}, \bar{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H},\text{IV}} \subset \overline{\mathbf{8}}_{\mathbf{H},\text{IV}}$ in the rank-two sector, according to the $\widetilde{\text{U}}(1)_{T'}$ -neutral components in Table 17. Accordingly, the term of $Y_{\mathcal{B}} \overline{\mathbf{8}}_{\mathbf{F}}^{\text{IV}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}}_{\mathbf{H},\text{IV}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}, \check{\mathfrak{n}}'', \mathfrak{e}'', \mathfrak{n}'', \check{\mathfrak{n}}')$ as follows:

$$\begin{aligned}
&Y_{\mathcal{B}} \overline{\mathbf{8}}_{\mathbf{F}}^{\text{IV}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}}_{\mathbf{H},\text{IV}} + H.c. \\
&\supset Y_{\mathcal{B}} \left[(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{IV}} \otimes (\mathbf{3}, \mathbf{5}, -\frac{2}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{5}}, -\frac{1}{5})_{\mathbf{F}}^{\text{IV}} \otimes (\mathbf{1}, \mathbf{10}, +\frac{2}{5})_{\mathbf{F}} \right] \otimes \langle (\mathbf{1}, \bar{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H},\text{IV}} \rangle + H.c. \\
&\Rightarrow \frac{1}{\sqrt{2}} Y_{\mathcal{B}} (\mathfrak{D}_L \mathfrak{D}_R^c + \check{\mathfrak{n}}_L' \check{\mathfrak{n}}_R''^c + \mathfrak{e}_L'' \mathfrak{e}_R''^c - \mathfrak{n}_L'' \mathfrak{n}_R''^c + \check{\mathfrak{n}}_L' \check{\mathfrak{n}}_R'^c) W_{\bar{\mathbf{5}},\text{IV}} + H.c..
\end{aligned} \tag{83}$$

$\overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{F}}^{\Omega}$		
\mathcal{T}'	$-\frac{4}{3}t$	$-4t$		
$\mathbf{28}_{\mathbf{F}}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{3}, \overline{\mathbf{5}}, -\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{10}, +\frac{2}{5})_{\mathbf{F}}$	
\mathcal{T}'	$-\frac{4}{3}t$	$+\frac{4}{3}t$	$4t$	
$\mathbf{56}_{\mathbf{F}}$	$(\overline{\mathbf{3}}, \mathbf{5}, -\frac{7}{15})_{\mathbf{F}}$	$(\mathbf{1}, \overline{\mathbf{10}}, +\frac{3}{5})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{10}, +\frac{1}{15})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}}$
\mathcal{T}'	$-\frac{4}{3}t$	$4t$	$\frac{4}{3}t$	$-4t$
$\overline{\mathbf{8}}_{\mathbf{H},\omega}$	$(\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H},\omega}$			
\mathcal{T}'	0			
$\mathbf{28}_{\mathbf{H},\dot{\omega}}$	$(\mathbf{1}, \overline{\mathbf{10}}, -\frac{2}{5})_{\mathbf{H},\dot{\omega}}$			
\mathcal{T}'	0			
$\mathbf{70}_{\mathbf{H}}$	$(\mathbf{1}, \overline{\mathbf{5}}, +\frac{4}{5})_{\mathbf{H}}$	$(\mathbf{1}, \mathbf{5}, -\frac{4}{5})_{\mathbf{H}}$		
\mathcal{T}'	0	$-8t$		

Table 17: The $\tilde{\mathbf{U}}(1)_{T'}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{351} theory.

After this stage, the remaining massless fermions expressed in terms of the \mathfrak{g}_{341} IRs are the following:

$$\begin{aligned}
& \left[(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'} \right] \oplus (\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}, \\
& \Omega = (\omega, \dot{\omega}), \quad \omega = (\mathbf{3}, \mathbf{V}, \mathbf{VI}), \quad \dot{\omega} = (\dot{\mathbf{1}}, \dot{\mathbf{2}}, \mathbf{VII}, \mathbf{VIII}, \mathbf{IX}), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\text{IV}}, \\
& (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \oplus \left[(\mathbf{3}, \mathbf{4}, -\frac{1}{12})_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}} \right] \oplus \left[(\mathbf{1}, \mathbf{4}, +\frac{1}{4})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}} \right] \subset \mathbf{28}_{\mathbf{F}}, \\
& (\overline{\mathbf{3}}, \mathbf{4}, -\frac{5}{12})_{\mathbf{F}} \oplus (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}''' \oplus (\mathbf{1}, \overline{\mathbf{4}}, +\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}}' \oplus (\mathbf{3}, \mathbf{4}, -\frac{1}{12})_{\mathbf{F}}' \\
& \oplus (\mathbf{3}, \mathbf{6}, +\frac{1}{6})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}} \subset \mathbf{56}_{\mathbf{F}}.
\end{aligned} \tag{84}$$

4.2 The second stage

The second symmetry breaking stage of $\mathfrak{g}_{341} \rightarrow \mathfrak{g}_{331}$ is achieved by $(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H},\mathbf{V}} \subset \overline{\mathbf{8}}_{\mathbf{H},\mathbf{V}}$ in the rank-two sector, $(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H},\mathbf{VII}} \subset \mathbf{28}_{\mathbf{H},\mathbf{VII}}$ and $(\mathbf{1}, \mathbf{4}, +\frac{1}{4})_{\mathbf{H}}^{\text{VII}} \subset \mathbf{28}_{\mathbf{H}}^{\text{VII}}$ in the rank-three sector, according to the $\tilde{\mathbf{U}}(1)_{T''}$ -neutral components in Table 18. The term of $Y_{\mathbf{B}} \overline{\mathbf{8}}_{\mathbf{F}}^{\mathbf{V}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}}_{\mathbf{H},\mathbf{V}} + H.c.$ leads to

$\overline{8_F}^\Omega$	$(\overline{3}, 1, +\frac{1}{3})_F^\Omega$	$(1, 1, 0)_F^\Omega$	$(1, \overline{4}, -\frac{1}{4})_F^\Omega$
\mathcal{T}''	$-\frac{4}{3}t$	$-4t$	$-4t$
28_F	$(\overline{3}, 1, -\frac{2}{3})_F$	$(3, 4, -\frac{1}{12})_F$	$(1, 6, +\frac{1}{2})_F$
\mathcal{T}''	$-\frac{4}{3}t$	$+\frac{4}{3}t$	$+4t$
56_F	$(\overline{3}, 4, -\frac{5}{12})_F$	$(\overline{3}, 1, -\frac{2}{3})_F'''$	$(1, \overline{4}, +\frac{3}{4})_F$
\mathcal{T}''	$-\frac{4}{3}t$	$-\frac{4}{3}t$	$4t$
	$(1, 6, +\frac{1}{2})_F'$	$(3, 4, -\frac{1}{12})_F'$	$(3, 6, +\frac{1}{6})_F$ $(1, 1, -1)_F$
\mathcal{T}''	$4t$	$+\frac{4}{3}t$	$+\frac{4}{3}t$ $-4t$
$\overline{8_{H,\omega}}$	$(1, \overline{4}, -\frac{1}{4})_{H,\omega}$		
\mathcal{T}''	0		
$28_{H,\dot{\omega}}$	$(1, \overline{4}, -\frac{1}{4})_{H,\dot{\omega}}$	$(1, 6, -\frac{1}{2})_{H,\dot{\omega}}$	
\mathcal{T}''	0	0	
70_H	$(1, \overline{4}, +\frac{3}{4})_H$		
\mathcal{T}''	0		

Table 18: The $\widetilde{U}(1)_{T''}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{341} theory.

the vectorial masses of $(\mathfrak{D}'', \mathfrak{e}, \mathfrak{n}, \check{\mathfrak{n}})$ as follows:

$$\begin{aligned}
& Y_B \overline{8_F}^V 28_F \overline{8_{H,V}} + H.c. \\
\supset & Y_B \left[(\overline{3}, 1, +\frac{1}{3})_F^V \otimes (3, 5, -\frac{2}{15})_F \oplus (1, \overline{5}, -\frac{1}{5})_F^V \otimes (1, 10, +\frac{2}{5})_F \right] \otimes (1, \overline{5}, -\frac{1}{5})_{H,V} + H.c. \\
\supset & Y_B \left[(\overline{3}, 1, +\frac{1}{3})_F^V \otimes (3, 4, -\frac{1}{12})_F \oplus (1, \overline{4}, -\frac{1}{4})_F^V \otimes (1, 6, +\frac{1}{2})_F \right. \\
& \oplus (1, 1, 0)_F^V \otimes (1, 4, +\frac{1}{4})_F \left. \right] \otimes \langle (1, \overline{4}, -\frac{1}{4})_{H,V} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_B (\mathfrak{D}_L'' \mathfrak{D}_R''^c + \check{\mathcal{N}}_L^V \check{\mathfrak{n}}_R''^c + \mathfrak{e}_L \mathfrak{e}_R^c - \mathfrak{n}_L \mathfrak{n}_R^c + \check{\mathfrak{n}}_L \check{\mathfrak{n}}_R^c) w_{\overline{4},V} + H.c.. \tag{85}
\end{aligned}$$

The term of $Y_D \overline{8_F}^{\text{VI}\text{II}} 56_F \overline{28_{H,\text{VI}\text{II}}}$ + H.c. leads to the vectorial masses of $(\mathfrak{D}''''', \mathfrak{e}''''', \mathfrak{n}''''', \check{\mathfrak{n}}''''')$ as follows:

$$\begin{aligned}
& Y_D \overline{8_F}^{\text{VI}\text{II}} 56_F \overline{28_{H,\text{VI}\text{II}}} + H.c. \\
\supset & Y_D \left[(\overline{3}, 1, +\frac{1}{3})_F^{\text{VI}\text{II}} \otimes (3, 10, +\frac{1}{15})_F \oplus (1, \overline{5}, -\frac{1}{5})_F^{\text{VI}\text{II}} \otimes (1, \overline{10}, +\frac{3}{15})_F \right] \otimes (1, \overline{10}, -\frac{2}{5})_{H,\text{VI}\text{II}} + H.c. \\
\supset & Y_D \left[(\overline{3}, 1, +\frac{1}{3})_F^{\text{VI}\text{II}} \otimes (3, 4, -\frac{1}{12})_F' \oplus (1, \overline{4}, -\frac{1}{4})_F^{\text{VI}\text{II}} \otimes (1, 6, +\frac{1}{2})_F' \right] \otimes \langle (1, \overline{4}, -\frac{1}{4})_{H,\text{VI}\text{II}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_D (\mathfrak{D}_L'''' \mathfrak{D}_R''''^c + \mathfrak{e}_L'''' \mathfrak{e}_R''''^c - \mathfrak{n}_L'''' \mathfrak{n}_R''''^c + \check{\mathfrak{n}}_L'''' \check{\mathfrak{n}}_R''''^c) w_{\overline{4},\text{VI}\text{II}} + H.c.. \tag{86}
\end{aligned}$$

The Yukawa coupling between two $\mathbf{56_F}$'s by the following $d = 5$ operator

$$\begin{aligned}
& \frac{c_4}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\text{VII}} + H.c. \\
\supset & c_4 \frac{v_U}{M_{\text{pl}}} \left[(\bar{\mathbf{3}}, \mathbf{5}, -\frac{7}{15})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{10}, +\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{10}}, +\frac{3}{5})_{\mathbf{F}} \otimes (\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}} \right] \otimes (\mathbf{1}, \mathbf{10}, +\frac{2}{5})_{\mathbf{H}}^{\text{VII}} + H.c. \\
\supset & c_4 \zeta_0 \left[(\bar{\mathbf{3}}, \mathbf{4}, -\frac{5}{12})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{6}, +\frac{1}{6})_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4})_{\mathbf{F}} \otimes (\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}} \right] \otimes \langle (\mathbf{1}, \mathbf{4}, +\frac{1}{4})_{\mathbf{H}}^{\text{VII}} \rangle + H.c. \\
\Rightarrow & \frac{c_4}{\sqrt{2}} \zeta_0 (\mathfrak{E}_L \mathfrak{E}_R^c + \mathfrak{U}_L \mathfrak{U}_R^c + \mathfrak{D}_L \mathfrak{D}_R^c - \mathfrak{u}_L \mathfrak{u}_R^c) w_4^{\text{VII}} + H.c., \tag{87}
\end{aligned}$$

leads to massive vectorlike fermions of $(\mathfrak{E}, \mathfrak{U}, \mathfrak{u}, \mathfrak{D})$.

After integrating out the massive fermions, the remaining massless fermions expressed in terms of the \mathfrak{g}_{331} IRs are the following

$$\begin{aligned}
& (\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega} \subset \bar{\mathbf{8_F}}^{\Omega}, \\
& \Omega = (\omega, \dot{\omega}), \quad \omega = (3, \text{VI}), \quad \dot{\omega} = (\dot{1}, \dot{2}, \text{VIII}, \text{IX}), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}} \subset \bar{\mathbf{8_F}}^{\text{IV}}, \quad (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}''} \subset \bar{\mathbf{8_F}}^{\text{V}}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VII}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VII}''} \subset \bar{\mathbf{8_F}}^{\text{VII}}, \\
& (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \oplus \left[(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}'' \right] \oplus (\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}} \\
\oplus & \left[(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}'' \right] \oplus \left[(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}' \oplus (\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \right] \subset \mathbf{28_F}, \\
& \left[(\mathbf{3}, \mathbf{3}, -\frac{1}{3})_{\mathbf{F}} \oplus (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}'' \right] \oplus (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}''' \oplus \left[(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}'' \oplus (\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}' \right] \\
\oplus & \left[(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}'' \oplus (\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}'' \right] \oplus \left[(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}' \oplus (\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}'''' \right] \oplus \left[(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \right. \\
\oplus & \left. (\mathbf{3}, \bar{\mathbf{3}}, +\frac{1}{3})_{\mathbf{F}} \right] \oplus (\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}} \subset \mathbf{56_F}. \tag{88}
\end{aligned}$$

4.3 The third stage

The third symmetry breaking stage of $\mathfrak{g}_{331} \rightarrow \mathfrak{g}_{\text{SM}}$ is achieved by $(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{VI}} \subset \bar{\mathbf{8_H}}_{\text{VI}}$ in the rank-two sector and $(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{VIII}} \subset \mathbf{28_H}_{\text{VIII}}$, $(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})'_{\mathbf{H}, \text{IX}} \subset \mathbf{28_H}_{\text{IX}}$ in the rank-three sector, according to the $\tilde{\text{U}}(1)_{T''''}$ -neutral components in Table 19. The term of $Y_B \bar{\mathbf{8_F}}^{\text{VI}} \mathbf{28_F} \bar{\mathbf{8_H}}_{\text{VI}} + H.c.$ leads

$\overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''}$
\mathcal{T}'''	$-\frac{4}{3}t$	$-4t$	$-4t$	$-4t$
$\mathbf{28}_{\mathbf{F}}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}$	$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}$	
\mathcal{T}'''	$-\frac{4}{3}t$	$+\frac{4}{3}t$	$+4t$	
$\mathbf{56}_{\mathbf{F}}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}''$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}'''$	$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}'$	
\mathcal{T}'''	$-\frac{4}{3}t$	$-\frac{4}{3}t$	$4t$	
	$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}''$	$(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}'$	$(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}''$	$(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}}$
\mathcal{T}'''	$4t$	$+\frac{4}{3}t$	$+\frac{4}{3}t$	$-4t$
$\overline{\mathbf{8}}_{\mathbf{H},\omega}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\omega}$			
\mathcal{T}'''	0			
$\mathbf{28}_{\mathbf{H},\dot{\omega}}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\dot{\omega}}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\dot{\omega}}'$	$(\mathbf{1}, \mathbf{3}, -\frac{2}{3})_{\mathbf{H},\dot{\omega}}$	
\mathcal{T}'''	0	0	0	
$\mathbf{70}_{\mathbf{H}}$	$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}}$			
\mathcal{T}'''	0			

Table 19: The $\widetilde{\mathbf{U}}(1)_{\mathcal{T}'''}$ charges for massless fermions and possible symmetry breaking Higgs components in the \mathfrak{g}_{331} theory.

to the vectorial masses of $(\mathfrak{D}', \mathfrak{e}', \mathfrak{n}')$ as follows:

$$\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8}}_{\mathbf{F}}^{\text{VI}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}}_{\mathbf{H},\text{VI}} + H.c. \\
& \supset Y_{\mathcal{B}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{3}, \mathbf{5}, -\frac{2}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{1}, \mathbf{10}, +\frac{2}{5})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H},\text{VI}} + H.c. \\
& \supset Y_{\mathcal{B}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{3}, \mathbf{4}, -\frac{1}{12})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}} \right. \\
& \oplus \left. (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{1}, \mathbf{4}, +\frac{1}{4})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H},\text{VI}} + H.c. \\
& \supset Y_{\mathcal{B}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \right. \\
& \oplus \left. (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VI}''} \otimes (\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}' \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VI}} \otimes (\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}} \right] \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H},\text{VI}} \rangle + H.c. \\
& \Rightarrow \frac{1}{\sqrt{2}} Y_{\mathcal{B}} \left(\mathfrak{n}'_L \mathfrak{n}'_R{}^c - \mathfrak{e}'_L \mathfrak{e}'_R{}^c + \tilde{\mathcal{N}}_L^{\text{VI}''} \check{\mathfrak{n}}_R^c + \tilde{\mathcal{N}}_L^{\text{VI}} \check{\mathfrak{n}}_R'^c + \mathfrak{D}'_L \mathfrak{D}'_R{}^c \right) V_{\overline{\mathbf{3}},\text{VI}} + H.c.. \tag{89}
\end{aligned}$$

The term of $Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\text{vIII}, \text{IX}} \mathbf{56_F} \overline{\mathbf{28_H}} + H.c.$ leads to the vectorial masses of $(\mathfrak{D}'''' , \mathfrak{e}'''' , \mathfrak{n}'''' , \mathfrak{D}''' , \mathfrak{e}''' , \mathfrak{n}''')$ as follows:

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\text{IX}} \mathbf{56_F} \overline{\mathbf{28_H}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{3}, \mathbf{10}, +\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{1}, \overline{\mathbf{10}}, +\frac{3}{15})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{10}}, -\frac{2}{5})_{\mathbf{H}, \text{IX}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{3}, \mathbf{4}, -\frac{1}{12})'_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{1}, \mathbf{6}, +\frac{1}{2})'_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \text{IX}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{3}, \mathbf{3}, 0)'_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})''_{\mathbf{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}} \otimes (\mathbf{1}, \mathbf{3}, +\frac{1}{3})''_{\mathbf{F}} \right] \\
& \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})'_{\mathbf{H}, \text{IX}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} \left(\mathfrak{D}_L'''' \mathfrak{D}_R''' c + \tilde{\mathcal{N}}_L^{\text{IX}''} \check{\mathfrak{n}}_R''' c - \mathfrak{e}_L'''' \mathfrak{e}_R''' c + \mathfrak{n}_L'''' \mathfrak{n}_R''' c \right) V_{\overline{\mathbf{3}}, \text{IX}}' + H.c., \tag{90}
\end{aligned}$$

and

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8_F}}^{\text{vIII}} \mathbf{56_F} \overline{\mathbf{28_H}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{vIII}} \otimes (\mathbf{3}, \mathbf{10}, +\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{F}}^{\text{vIII}} \otimes (\mathbf{1}, \overline{\mathbf{10}}, +\frac{3}{15})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{10}}, -\frac{2}{5})_{\mathbf{H}, \text{vIII}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{vIII}} \otimes (\mathbf{3}, \mathbf{6}, +\frac{1}{6})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\text{vIII}} \otimes (\mathbf{1}, \overline{\mathbf{4}}, +\frac{3}{4})_{\mathbf{F}} \right. \\
& \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{vIII}} \otimes (\mathbf{1}, \mathbf{6}, +\frac{1}{2})'_{\mathbf{F}} \left. \right] \otimes (\mathbf{1}, \mathbf{6}, -\frac{1}{2})_{\mathbf{H}, \text{vIII}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\text{vIII}} \otimes (\mathbf{3}, \mathbf{3}, 0)''_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\text{vIII}} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})'_{\mathbf{F}} \right. \\
& \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{vIII}} \otimes (\mathbf{1}, \mathbf{3}, +\frac{1}{3})''_{\mathbf{F}} \left. \right] \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \text{vIII}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} \left(\mathfrak{D}_L''' \mathfrak{D}_R''' c + \tilde{\mathcal{N}}_L^{\text{vIII}} \check{\mathfrak{n}}_R''' c - \mathfrak{e}_L''' \mathfrak{e}_R''' c + \mathfrak{n}_L''' \mathfrak{n}_R''' c \right) V_{\overline{\mathbf{3}}, \text{vIII}} + H.c.. \tag{91}
\end{aligned}$$

The remaining massless fermions of the \mathfrak{g}_{SM} are listed as follows:

$$\begin{aligned}
& (\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega} \oplus \left[(\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'} \right] \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}, \quad \Omega = (\dot{1}, \dot{2}, 3), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}} \oplus \dots \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VI}'} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VIII}'} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}'} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\Omega'}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}''} \oplus \dots \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IX}''} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\Omega''}, \\
& (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \oplus \left[\cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}'} \oplus (\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} \right] \oplus \cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}''} \oplus \cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}'} \\
& \oplus \left[(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}'' \oplus \cancel{(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}'} \right] \oplus \cancel{(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}''} \oplus \left[\cancel{(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}} \oplus \cancel{(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}} \right] \\
& \oplus \left[(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}} \oplus \cancel{(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}'} \right] \subset \mathbf{28}_{\mathbf{F}}, \\
& \left[\cancel{(\bar{\mathbf{3}}, \mathbf{2}, -\frac{1}{6})_{\mathbf{F}}} \oplus \cancel{(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}'} \right] \oplus (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}'' \oplus (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}''' \oplus \left[\cancel{(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}} \oplus (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}' \right] \\
& \oplus \cancel{(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}''} \oplus \left[\cancel{(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}''''} \oplus \cancel{(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}''''} \right] \oplus \left[\cancel{(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}''''} \oplus (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}''' \right] \oplus \left[\cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}''''} \right] \\
& \oplus (\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}}' \oplus (\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}'''' \oplus \left[\cancel{(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}'''} \oplus (\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}}'' \right] \\
& \oplus \left[\cancel{(\mathbf{3}, \mathbf{1}, +\frac{2}{3})_{\mathbf{F}}} \oplus \cancel{(\mathbf{3}, \bar{\mathbf{2}}, +\frac{1}{6})_{\mathbf{F}}''} \right] \oplus (\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}} \subset \mathbf{56}_{\mathbf{F}}. \tag{92}
\end{aligned}$$

4.4 The $d = 5$ bi-linear fermion operators

For the operator of $\mathcal{O}_{\mathcal{F}}^{(4,1)}$ in Eq. (13a), it is decomposed as

$$\begin{aligned}
& \frac{c_4}{M_{\text{pl}}} \mathbf{56}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \cdot \overline{\mathbf{28}}_{\mathbf{H}, \dot{\omega}} \cdot \mathbf{70}_{\mathbf{H}} + H.c. \\
& \supset \frac{c_4}{M_{\text{pl}}} (\bar{\mathbf{3}}, \mathbf{5}, -\frac{7}{15})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{10}, +\frac{1}{15})_{\mathbf{F}} \otimes (\mathbf{1}, \bar{\mathbf{10}}, -\frac{2}{5})_{\mathbf{H}, \dot{\omega}} \otimes (\mathbf{1}, \bar{\mathbf{5}}, +\frac{4}{5})_{\mathbf{H}} + H.c. \\
& \supset \frac{c_4}{M_{\text{pl}}} \left[(\bar{\mathbf{3}}, \mathbf{4}, -\frac{5}{12})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{4}, -\frac{1}{12})_{\mathbf{F}}' \oplus (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}''' \otimes (\mathbf{3}, \mathbf{6}, +\frac{1}{6})_{\mathbf{F}} \right] \otimes \langle (\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}} \rangle \otimes (\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4})_{\mathbf{H}} + H.c. \\
& \supset \frac{c_4 w_{\bar{\mathbf{4}}, \text{VII}}}{\sqrt{2} M_{\text{pl}}} \left[(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}'' \otimes (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}' \otimes (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}''' \otimes (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}'' \right] \otimes (\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}} + H.c. \\
& \Rightarrow \frac{c_4}{2} \zeta_2 (u_L c_R^c + c_L u_R^c) v_{\text{EW}} + H.c.. \tag{93}
\end{aligned}$$

For the operator of $\mathcal{O}_{\mathcal{F}}^{(5,1)}$ in Eq. (13b), it is decomposed as

$$\begin{aligned}
& \frac{c_5}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_{H,\omega}}} \cdot \mathbf{70_H} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} \left[(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{10}, +\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{5}, -\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{3}}, \mathbf{5}, -\frac{7}{15})_{\mathbf{F}} \right] \otimes \langle (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H},\omega} \rangle \\
& \otimes (\mathbf{1}, \overline{\mathbf{5}}, +\frac{4}{5})_{\mathbf{H}} + H.c. \\
\supset & c_5 \frac{W_{\overline{\mathbf{5}},\text{IV}}}{\sqrt{2}M_{\text{pl}}} \left[(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{4}, -\frac{1}{12})'_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{4}, -\frac{1}{12})_{\mathbf{F}} \otimes (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{4}}, +\frac{3}{4})_{\mathbf{H}} + H.c. \\
\supset & c_5 \frac{W_{\overline{\mathbf{5}},\text{IV}}}{\sqrt{2}M_{\text{pl}}} \left[(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{3}, 0)'_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \otimes (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}} + H.c. \\
\Rightarrow & \frac{c_5}{2} \zeta_1 (u_L t_R^c + t_L u_R^c) v_{\text{EW}} + H.c., \tag{94}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{c_5}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \cdot \overline{\mathbf{8_{H,\omega}}} \cdot \mathbf{70_H} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} \left[(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{10}, +\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{5}, -\frac{2}{15})_{\mathbf{F}} \otimes (\overline{\mathbf{3}}, \mathbf{5}, -\frac{7}{15})_{\mathbf{F}} \right] \\
& \otimes (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H},\omega} \otimes (\mathbf{1}, \overline{\mathbf{5}}, +\frac{4}{5})_{\mathbf{H}} + H.c. \\
\supset & \frac{c_5}{M_{\text{pl}}} \left[(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{6}, +\frac{1}{6})_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{4}, -\frac{1}{12})_{\mathbf{F}} \otimes (\overline{\mathbf{3}}, \mathbf{4}, -\frac{5}{12})_{\mathbf{F}} \right] \\
& \otimes \langle (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H},\omega} \rangle \otimes (\mathbf{1}, \overline{\mathbf{4}}, +\frac{3}{4})_{\mathbf{H}} + H.c. \\
\supset & c_5 \frac{w_{\overline{\mathbf{4}},\text{V}}}{\sqrt{2}M_{\text{pl}}} \left[(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \otimes (\mathbf{3}, \mathbf{3}, 0)''_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \otimes (\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}} + H.c. \\
\Rightarrow & \frac{c_5}{2} \zeta_2 (c_L t_R^c + t_L c_R^c) v_{\text{EW}} + H.c. \tag{95}
\end{aligned}$$

4.5 The $d = 5$ irreducible Higgs mixing operators

For the Yukawa coupling of $\overline{\mathbf{8}_F}^{\omega_1} \mathbf{28}_F \overline{\mathbf{8}_H}_{\omega_1}$, we find the mass terms of

$$\begin{aligned}
& Y_B \overline{\mathbf{8}_F}^{\omega_1} \mathbf{28}_F \overline{\mathbf{8}_H}_{\omega_1} \times \frac{d_{\mathcal{A}}}{M_{\text{pl}}} \epsilon_{\omega_1 \omega_2 \omega_3 \omega_4} \overline{\mathbf{8}_H}_{\omega_1}^\dagger \overline{\mathbf{8}_H}_{\omega_2}^\dagger \overline{\mathbf{8}_H}_{\omega_3}^\dagger \overline{\mathbf{8}_H}_{\omega_4}^\dagger \mathbf{70}_H^\dagger + H.c. \\
\supset & Y_B \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{3}, \mathbf{5}, -\frac{2}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}, \mathbf{10}, +\frac{2}{5})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H}, \omega_1} \\
\times & \frac{d_{\mathcal{A}}}{M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H}, \omega_1}^\dagger \otimes (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H}, \omega_2}^\dagger \otimes (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H}, \omega_3}^\dagger \otimes \langle (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{H}, \omega_4}^\dagger \rangle \otimes (\mathbf{1}, \overline{\mathbf{5}}, +\frac{4}{5})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_B \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{3}, \mathbf{4}, -\frac{1}{12})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \omega_1} \\
\times & \frac{d_{\mathcal{A}} W_{\overline{\mathbf{5}}, \text{IV}}}{\sqrt{2} M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \omega_1}^\dagger \otimes \langle (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \omega_2}^\dagger \rangle \otimes (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \omega_3}^\dagger \otimes (\mathbf{1}, \overline{\mathbf{4}}, +\frac{3}{4})_{\mathbf{H}}^\dagger + H.c. \\
\supset & Y_B \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\omega_1} \otimes (\mathbf{1}, \mathbf{3}, +\frac{2}{3})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \omega_1} \\
\times & d_{\mathcal{A}} \frac{W_{\overline{\mathbf{5}}, \text{IV}} w_{\overline{\mathbf{4}}, \text{V}}}{2 M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \omega_1}^\dagger \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \omega_3}^\dagger \rangle \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_B d_{\mathcal{A}}}{4} \frac{W_{\overline{\mathbf{5}}, \text{IV}} w_{\overline{\mathbf{4}}, \text{V}} V_{\overline{\mathbf{3}}, \text{VI}}}{M_{\text{pl}} m_{(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, 3}}^2} (b_L b_R^c + \tau_L \tau_R^c) v_{\text{EW}} + H.c., \tag{96}
\end{aligned}$$

via the Higgs mixing operator of $\mathcal{O}_{\mathcal{A}}^{d=5}$ in Eq. (14a).

For the Yukawa coupling of $\overline{\mathbf{8}_F}^{\dot{\omega}_1} \mathbf{56}_F \overline{\mathbf{28}_H}_{\dot{\omega}_1}$, we find the mass terms of

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8}_F}^{\dot{\omega}_1} \mathbf{56}_F \overline{\mathbf{28}_H}_{\dot{\omega}_1} \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} \overline{\mathbf{28}_H}_{\dot{\omega}_1}^\dagger \overline{\mathbf{28}_H}_{\dot{\omega}_2}^\dagger \mathbf{70}_H^\dagger \left(\overline{\mathbf{28}_H}_{\dot{\omega}_1}^\dagger \overline{\mathbf{28}_H}_{\text{VII}} \right) + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{3}, \mathbf{10}, +\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \overline{\mathbf{10}}, +\frac{3}{5})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{10}}, -\frac{2}{5})_{\mathbf{H}, \dot{\omega}_1} \\
\times & \frac{d_{\mathcal{B}}}{M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{10}}, -\frac{2}{5})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\mathbf{1}, \overline{\mathbf{10}}, -\frac{2}{5})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{1}, \overline{\mathbf{5}}, +\frac{4}{5})_{\mathbf{H}}^\dagger \otimes \langle \overline{\mathbf{28}_H}_{\dot{\omega}_1}^\dagger \overline{\mathbf{28}_H}_{\text{VII}} \rangle + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{3}, \mathbf{4}, -\frac{1}{12})'_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \overline{\mathbf{6}}, +\frac{1}{2})'_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}_1} \\
\times & \frac{d_{\mathcal{B}} w_{\overline{\mathbf{4}}, \text{I}} w_{\overline{\mathbf{4}}, \text{VII}}}{2 M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}_1}^\dagger \otimes (\mathbf{1}, \mathbf{6}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}_2}^\dagger \otimes (\mathbf{1}, \overline{\mathbf{4}}, +\frac{3}{4})_{\mathbf{H}}^\dagger + H.c. \\
\supset & \frac{Y_{\mathcal{D}} d_{\mathcal{B}} w_{\overline{\mathbf{4}}, \text{I}} w_{\overline{\mathbf{4}}, \text{VII}}}{2 M_{\text{pl}} m_{(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}_1}}^2} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{3}, \mathbf{3}, 0)'_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \right] \\
\otimes & \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{\omega}_2}^\dagger \rangle \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}}^\dagger + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{D}} d_{\mathcal{B}}}{4} \zeta_3 \left[\frac{w_{\overline{\mathbf{4}}, \text{I}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}_1}}^2} (d_L d_R^c + e_L e_R^c) \right. \\
& \left. + \frac{w_{\overline{\mathbf{4}}, \text{I}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}_2}}^2} (d_L s_R^c + \mu_L e_R^c) \right] v_{\text{EW}} + H.c., \tag{97}
\end{aligned}$$

and

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8}_F}^{\dot{\omega}_1} \mathbf{56}_F \overline{\mathbf{28}_H}^{\dot{\omega}_1} \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} \overline{\mathbf{28}_H}^{\dot{\omega}_1} \overline{\mathbf{28}_H}^{\dot{\omega}_2} \mathbf{70}_H^{\dagger} \left(\overline{\mathbf{28}_H}^{\dagger} \overline{\mathbf{28}_H}^{\text{VII}} \right) + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{3}, \mathbf{10}, +\frac{1}{15})_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{5}}, -\frac{1}{5})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \mathbf{10}, +\frac{3}{5})_{\mathbf{F}} \right] \otimes (\mathbf{1}, \overline{\mathbf{10}}, -\frac{2}{5})_{\mathbf{H}, \dot{\omega}_1} \\
& \times \frac{d_{\mathcal{B}}}{M_{\text{pl}}} (\mathbf{1}, \overline{\mathbf{10}}, -\frac{2}{5})_{\mathbf{H}, \dot{\omega}_1}^{\dagger} \otimes (\mathbf{1}, \overline{\mathbf{10}}, -\frac{2}{5})_{\mathbf{H}, \dot{\omega}_2}^{\dagger} \otimes (\mathbf{1}, \overline{\mathbf{5}}, +\frac{4}{5})_{\mathbf{H}}^{\dagger} \otimes \langle \overline{\mathbf{28}_H}^{\dagger}, \overline{\mathbf{28}_H}^{\text{VII}} \rangle + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{3}, \mathbf{6}, +\frac{1}{6})'_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \mathbf{4}, +\frac{3}{4})'_{\mathbf{F}} \right] \otimes (\mathbf{1}, \mathbf{6}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}_1} \\
& \times \frac{d_{\mathcal{B}} w_{\overline{\mathbf{4}}, \text{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{2M_{\text{pl}}} (\mathbf{1}, \mathbf{6}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}_1}^{\dagger} \otimes (\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}_2}^{\dagger} \otimes (\mathbf{1}, \overline{\mathbf{4}}, +\frac{3}{4})_{\mathbf{H}}^{\dagger} + H.c. \\
\supset & \frac{Y_{\mathcal{D}} d_{\mathcal{B}} w_{\overline{\mathbf{4}}, \text{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{2M_{\text{pl}} m_{(\mathbf{1}, \overline{\mathbf{6}}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}_1}}^2} \left[(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{3}, \mathbf{3}, 0)''_{\mathbf{F}} \oplus (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\dot{\omega}_1} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{1}{3})'_{\mathbf{F}} \right] \\
& \otimes \langle (\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})'_{\mathbf{H}, \dot{\omega}_2} \rangle^{\dagger} \otimes (\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}}^{\dagger} + H.c. \\
\Rightarrow & \frac{Y_{\mathcal{D}} d_{\mathcal{B}}}{4} \zeta_3' \left[\frac{w_{\overline{\mathbf{4}}, \text{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{6}}, -\frac{1}{2})_{\mathbf{H}, \text{i}}}^2} (s_L d_R^c + e_L \mu_R^c) + \frac{w_{\overline{\mathbf{4}}, \text{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{6}}, -\frac{1}{2})_{\mathbf{H}, \dot{2}}}^2} (s_L s_R^c + \mu_L \mu_R^c) \right] v_{\text{EW}} + H.c., \quad (98)
\end{aligned}$$

via the Higgs mixing operator of $\mathcal{O}_{\mathcal{B}}^{d=5}$ in Eq. (14b). For convenience, we parametrize the following ratios of

$$\Delta_{\dot{\omega}} \equiv \frac{w_{\overline{\mathbf{4}}, \text{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \dot{\omega}}}^2}, \quad \Delta'_{\dot{\omega}} \equiv \frac{w_{\overline{\mathbf{4}}, \text{i}} w_{\overline{\mathbf{4}}, \text{VII}}}{m_{(\mathbf{1}, \overline{\mathbf{6}}, -\frac{1}{2})_{\mathbf{H}, \dot{\omega}}}^2}. \quad (99)$$

With the Higgs VEVs assignments in Eq. (81), we can obtain the following propagator masses of

$$\begin{aligned}
m_{(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \text{i}}} & \sim m_{(\mathbf{1}, \overline{\mathbf{6}}, -\frac{1}{2})_{\mathbf{H}, \text{i}}} \sim \mathcal{O}(v_{341}), \\
m_{(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \dot{2}}} & \sim m_{(\mathbf{1}, \overline{\mathbf{6}}, -\frac{1}{2})_{\mathbf{H}, \dot{2}}} \sim \mathcal{O}(v_{331}). \quad (100)
\end{aligned}$$

4.6 The SM quark/lepton masses and the benchmark

For all up-type quarks with $Q_e = +\frac{2}{3}$, we write down the following tree-level masses from both the renormalizable Yukawa couplings and the gravity-induced terms in the basis of $\mathcal{U} \equiv (u, c, t)$

$$\mathcal{M}_{\mathcal{U}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & c_4 \dot{\zeta}_2 / \sqrt{2} & c_5 \zeta_1 / \sqrt{2} \\ c_4 \dot{\zeta}_2 / \sqrt{2} & 0 & c_5 \zeta_2 / \sqrt{2} \\ c_5 \zeta_1 / \sqrt{2} & c_5 \zeta_2 / \sqrt{2} & Y_{\mathcal{T}} \end{pmatrix} v_{\text{EW}}. \quad (101)$$

For all down-type quarks with $Q_e = -\frac{1}{3}$, we find the following tree-level mass matrix in the basis of $\mathcal{D} \equiv (d, s, b)$

$$\mathcal{M}_{\mathcal{D}} \approx \frac{1}{4} \begin{pmatrix} Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1 \dot{\zeta}_3 & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2 \dot{\zeta}_3 & 0 \\ Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_1' \dot{\zeta}_3' & Y_{\mathcal{D}} d_{\mathcal{B}} \Delta_2' \dot{\zeta}_3' & 0 \\ 0 & 0 & Y_{\mathcal{B}} d_{\mathcal{A}} \zeta_{23}^{-1} \zeta_1 \end{pmatrix} v_{\text{EW}}, \quad (102)$$

where we interpret the ratio between two \mathbf{g}_{331} VEVs as the Cabibbo angle of $\tan \lambda \equiv \frac{\dot{\zeta}_3'}{\zeta_3}$. For all charged leptons with $Q_e = -1$, their tree-level mass matrix is related to the down-type quark mass matrix as

$\mathcal{M}_{\mathcal{L}} = \mathcal{M}_{\mathcal{D}}^T$. Based on the above SM quark/lepton mass matrices, we find the following benchmark point of

$$v_{351} \simeq 1.4 \times 10^{17} \text{ GeV}, \quad v_{341} \simeq 4.8 \times 10^{15} \text{ GeV}, \quad v_{331} \simeq 4.8 \times 10^{13} \text{ GeV}, \quad (103)$$

to reproduce the observed hierarchical masses as well as the CKM mixing pattern, as summarized in Table 20.

ζ_1	ζ_2	ζ'_3	$Y_{\mathcal{D}}$	$Y_{\mathcal{B}}$	$Y_{\mathcal{T}}$
6.0×10^{-2}	2.0×10^{-3}	2.0×10^{-5}	0.5	0.5	0.8
λ	c_4	c_5	$d_{\mathcal{A}}$	$d_{\mathcal{B}}$	
0.22	0.2	1.0	0.01	0.01	
m_u	m_c	m_t	$m_d = m_e$	$m_s = m_\mu$	$m_b = m_\tau$
1.6×10^{-3}	0.6	139.2	0.5×10^{-3}	6.4×10^{-2}	1.5
$ V_{ud} $	$ V_{us} $	$ V_{ub} $			
0.98	0.22	2.1×10^{-3}			
$ V_{cd} $	$ V_{cs} $	$ V_{cb} $			
0.22	0.98	5.3×10^{-2}			
$ V_{td} $	$ V_{ts} $	$ V_{tb} $			
0.013	5.3×10^{-2}	1			

Table 20: The parameters of the $\mathfrak{su}(8)$ benchmark point and the predicted SM quark/lepton masses (in unit of GeV) as well as the CKM mixings.

5 The intermediate stages of the SSS symmetry breaking pattern

Schematically, we assign the Higgs VEVs according to the symmetry breaking pattern as follows:

$$\mathfrak{g}_{621} \rightarrow \mathfrak{g}_{521} \quad : \quad \langle (\bar{\mathbf{6}}, \mathbf{1}, +\frac{1}{6})_{\mathbf{H}, \text{IV}} \rangle \equiv \frac{1}{\sqrt{2}} W_{\bar{\mathbf{6}}, \text{IV}}, \quad (104a)$$

$$\mathfrak{g}_{521} \rightarrow \mathfrak{g}_{421} \quad : \quad \langle (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \text{V}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\bar{\mathbf{5}}, \text{V}}, \quad \langle (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \text{i}, \text{VII}} \rangle \equiv \frac{1}{\sqrt{2}} w_{\bar{\mathbf{5}}, \text{i}, \text{VII}}, \quad (104b)$$

$$\begin{aligned} \mathfrak{g}_{421} \rightarrow \mathfrak{g}_{\text{SM}} \quad : \quad & \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \text{3}, \text{VI}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\bar{\mathbf{4}}, \text{3}, \text{VI}}, \\ & \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H}, \text{VIII}} \rangle \equiv \frac{1}{\sqrt{2}} V_{\bar{\mathbf{4}}, \text{VIII}}, \quad \langle (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})'_{\mathbf{H}, \text{2}, \text{IX}} \rangle \equiv \frac{1}{\sqrt{2}} V'_{\bar{\mathbf{4}}, \text{2}, \text{IX}}, \end{aligned} \quad (104c)$$

$$\text{EWSB} \quad : \quad \langle (\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{H}} \rangle \equiv \frac{1}{\sqrt{2}} v_{\text{EW}}. \quad (104d)$$

For later convenience, we also parametrize different symmetry breaking VEVs in terms of the following dimensionless quantities:

$$\begin{aligned}\zeta_1 &\equiv \frac{W_{\bar{\mathbf{6}},\text{IV}}}{M_{\text{pl}}}, \quad \zeta_2 \equiv \frac{w_{\bar{\mathbf{5}},\text{V}}}{M_{\text{pl}}}, \quad \dot{\zeta}_2 \equiv \frac{w_{\bar{\mathbf{5}},\text{i},\text{VII}}}{M_{\text{pl}}}, \\ \zeta_3 &\equiv \frac{V_{\bar{\mathbf{4}},3,\text{VI}}}{M_{\text{pl}}}, \quad \dot{\zeta}_3 \equiv \frac{V'_{\bar{\mathbf{4}},\dot{2},\text{IX}}}{M_{\text{pl}}}, \quad \zeta_3 \equiv \frac{V_{\bar{\mathbf{4}},\text{VIII}}}{M_{\text{pl}}}, \\ \zeta_0 &\gg \zeta_1 \gg \zeta_2 \sim \dot{\zeta}_2 \gg \zeta_3 \sim \dot{\zeta}_3 \sim \zeta_3, \quad \zeta_{ij} \equiv \frac{\zeta_j}{\zeta_i}, \quad (i < j).\end{aligned}\tag{105}$$

In Table 21, we summarize all vectorlike fermions that become massive during different stages of the SSS symmetry breaking pattern.

stages	$Q_e = -\frac{1}{3}$	$Q_e = +\frac{2}{3}$	$Q_e = -1$	$Q_e = 0$
v_{621} $\{\Omega\}$	\mathfrak{D}'' IV	-	$(\mathfrak{e}'', \mathfrak{n}'')$ IV	$\{\check{\mathfrak{n}}', \check{\mathfrak{n}}''\}$ $\{\text{IV}, \text{IV}'\}$
v_{521} $\{\Omega\}$	$\{\mathfrak{D}', \mathfrak{D}'''''\}$ $\{\text{V}, \text{VII}\}$	-	$(\mathfrak{e}', \mathfrak{n}'), (\mathfrak{e}''''', \mathfrak{n}''''')$ $\{\text{V}, \text{VII}\}$	$\{\check{\mathfrak{n}}, \check{\mathfrak{n}}'''\}$ $\{\text{V}, \text{VII}\}$
v_{421} $\{\Omega\}$	$\{\mathfrak{D}, \mathfrak{D}''''', \mathfrak{D}'''''\}$ $\{\text{VI}, \text{VIII}, \text{IX}\}$	\mathfrak{U}	$\mathfrak{E}, (\mathfrak{e}, \mathfrak{n}), (\mathfrak{e}''''', \mathfrak{n}'''''), (\mathfrak{e}''''', \mathfrak{n}''''')$ $\{\text{VI}, \text{VIII}, \text{IX}\}$	-

Table 21: The vectorlike fermions at different intermediate symmetry breaking stages along the SSS symmetry breaking pattern of the $\mathfrak{su}(8)$ theory.

5.1 The first stage

The first symmetry breaking stage of $\mathfrak{g}_{621} \rightarrow \mathfrak{g}_{521}$ is achieved by $(\bar{\mathbf{6}}, \mathbf{1}, +\frac{1}{6})_{\text{H},\text{IV}} \subset \overline{\mathbf{8}}_{\text{H},\text{IV}}$ in the rank-two sector.

$$\begin{aligned}& Y_B \overline{\mathbf{8}}_{\text{F}}^{\text{IV}} \mathbf{28}_{\text{F}} \overline{\mathbf{8}}_{\text{H},\text{IV}} + H.c. \\ \supset & Y_B \left[(\bar{\mathbf{6}}, \mathbf{1}, +\frac{1}{6})_{\text{F}}^{\text{IV}} \otimes (\mathbf{15}, \mathbf{1}, -\frac{1}{3})_{\text{F}} \oplus (\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\text{F}}^{\text{IV}} \otimes (\mathbf{6}, \mathbf{2}, +\frac{1}{3})_{\text{F}} \right] \otimes \langle (\bar{\mathbf{6}}, \mathbf{1}, +\frac{1}{6})_{\text{H},\text{IV}} \rangle + H.c. \\ \Rightarrow & \frac{1}{\sqrt{2}} Y_B (\mathfrak{D}_L'' \mathfrak{D}_R''^c + \check{\mathfrak{n}}_L' \check{\mathfrak{n}}_R'^c + \check{\mathfrak{n}}_L'' \check{\mathfrak{n}}_R''^c + \mathfrak{e}_L'' \mathfrak{e}_R''^c - \mathfrak{n}_L'' \mathfrak{n}_R''^c) W_{\bar{\mathbf{6}},\text{IV}} + H.c.. \end{aligned}\tag{106}$$

After this stage, the remaining massless fermions expressed in terms of the \mathfrak{g}_{521} IRs are the following:

$$\begin{aligned}& (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\text{F}}^{\Omega} \oplus \left[(\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\Omega''} \oplus (\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\text{F}}^{\Omega} \right] \subset \overline{\mathbf{8}}_{\text{F}}^{\Omega}, \\ & \Omega = (\omega, \dot{\omega}), \quad \omega = (3, \text{V}, \text{VI}), \quad \dot{\omega} = (\dot{1}, \dot{2}, \text{VII}, \text{VIII}, \text{IX}), \\ & (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\text{IV}''} \subset \overline{\mathbf{8}}_{\text{F}}^{\text{IV}}, \\ & (\mathbf{1}, \mathbf{1}, +1)_{\text{F}} \oplus \left[(\mathbf{10}, \mathbf{1}, -\frac{2}{5})'_{\text{F}} \oplus (\mathbf{5}, \mathbf{1}, -\frac{1}{5})_{\text{F}} \right] \oplus \left[(\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\text{F}} \oplus (\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\text{F}}'' \right] \subset \mathbf{28}_{\text{F}}, \\ & \left[(\mathbf{10}, \mathbf{1}, -\frac{3}{5})_{\text{F}} \oplus (\mathbf{10}, \mathbf{1}, -\frac{2}{5})'_{\text{F}} \right] \oplus \left[(\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\text{F}} \oplus (\mathbf{5}, \bar{\mathbf{2}}, +\frac{3}{10})'_{\text{F}} \right] \oplus \left[(\mathbf{5}, \mathbf{1}, +\frac{4}{5})_{\text{F}} \right. \\ & \left. \oplus (\mathbf{1}, \mathbf{1}, +1)_{\text{F}}' \right] \subset \mathbf{56}_{\text{F}}. \end{aligned}\tag{107}$$

5.2 The second stage

The second symmetry breaking stage of $\mathfrak{g}_{521} \rightarrow \mathfrak{g}_{421}$ is achieved by $(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \mathbf{V}} \subset \overline{\mathbf{8}}_{\mathbf{H}, \mathbf{V}}$ in the rank-two sector, $(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \mathbf{VII}} \subset \overline{\mathbf{28}}_{\mathbf{H}, \mathbf{VII}}$ and $(\mathbf{5}, \mathbf{1}, -\frac{1}{5})_{\mathbf{H}}^{\mathbf{VII}} \subset \mathbf{28}_{\mathbf{H}}^{\mathbf{VII}}$ in the rank-three sector.

$$\begin{aligned}
& Y_{\mathcal{B}} \overline{\mathbf{8}}_{\mathbf{F}}^{\mathbf{V}} \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}}_{\mathbf{H}, \mathbf{V}} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\bar{\mathbf{6}}, \mathbf{1}, +\frac{1}{6})_{\mathbf{F}}^{\mathbf{V}} \otimes (\mathbf{15}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\mathbf{V}} \otimes (\mathbf{6}, \mathbf{2}, +\frac{1}{3})_{\mathbf{F}} \right] \otimes (\bar{\mathbf{6}}, \mathbf{1}, +\frac{1}{6})_{\mathbf{H}, \mathbf{V}} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\mathbf{V}} \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})'_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\mathbf{V}} \otimes (\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\mathbf{F}} \right] \otimes \langle (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \mathbf{V}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{B}} (\mathcal{D}'_L \mathcal{D}'_R{}^c + \check{\mathbf{n}}_L \check{\mathbf{n}}_R{}^c + \epsilon'_L \epsilon'_R{}^c - \mathbf{n}'_L \mathbf{n}'_R{}^c) w_{\bar{\mathbf{5}}, \mathbf{V}} + H.c.. \tag{108}
\end{aligned}$$

The term of $Y_{\mathcal{D}} \overline{\mathbf{8}}_{\mathbf{F}}^{\mathbf{VII}} \mathbf{56}_{\mathbf{F}} \overline{\mathbf{28}}_{\mathbf{H}, \mathbf{VII}} + H.c.$ leads to the vectorial masses of $(\mathcal{D}''''', \epsilon''''', \mathbf{n}''''', \check{\mathbf{n}}''''')$ as follows:

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{\mathbf{8}}_{\mathbf{F}}^{\mathbf{VII}} \mathbf{56}_{\mathbf{F}} \overline{\mathbf{28}}_{\mathbf{H}, \mathbf{VII}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\bar{\mathbf{6}}, \mathbf{1}, +\frac{1}{6})_{\mathbf{F}}^{\mathbf{VII}} \otimes (\mathbf{20}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\mathbf{VII}} \otimes (\mathbf{15}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} \right] \otimes (\bar{\mathbf{15}}, \bar{\mathbf{1}}, +\frac{1}{3})_{\mathbf{H}, \mathbf{VII}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{F}}^{\mathbf{VII}} \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})'_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\mathbf{VII}} \otimes (\mathbf{5}, \mathbf{2}, +\frac{3}{10})'_{\mathbf{F}} \right] \otimes \langle (\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H}, \mathbf{VII}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} (\mathcal{D}''''_L \mathcal{D}''''_R{}^c + \check{\mathbf{n}}''''_L \check{\mathbf{n}}''''_R{}^c - \epsilon''''_L \epsilon''''_R{}^c + \mathbf{n}''''_L \mathbf{n}''''_R{}^c) w_{\bar{\mathbf{5}}, \mathbf{VII}} + H.c.. \tag{109}
\end{aligned}$$

The Yukawa coupling between two $\mathbf{56}_{\mathbf{F}}$'s by the following $d = 5$ operator $\frac{c_4}{M_{\text{pl}}} \mathbf{56}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \mathbf{63}_{\mathbf{H}} \mathbf{28}_{\mathbf{H}}^{\mathbf{VII}}$

$$\begin{aligned}
& \frac{1}{M_{\text{pl}}} \mathbf{56}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \langle \mathbf{63}_{\mathbf{H}} \rangle \mathbf{28}_{\mathbf{H}}^{\dot{\omega}} \\
\supset & \zeta_0 (\mathbf{15}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} \otimes (\mathbf{15}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} \otimes (\mathbf{15}, \mathbf{1}, -\frac{1}{3})_{\mathbf{H}}^{\dot{\omega}} \\
\supset & \zeta_0 (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \otimes \langle (\mathbf{5}, \mathbf{1}, -\frac{1}{5})_{\mathbf{H}}^{\dot{\omega}} \rangle, \tag{110}
\end{aligned}$$

has already been proven in the text to vanished.

After integrating out the massive fermions, the remaining massless fermions expressed in terms of the \mathfrak{g}_{421} IRs are the following:

$$\begin{aligned}
& \left[(\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega} \right] \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} \oplus (\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}, \\
& \Omega = (\omega, \dot{\omega}), \quad \omega = (3, \text{VI}), \quad \dot{\omega} = (\dot{\mathbf{i}}, \dot{\mathbf{2}}, \text{VIII}, \text{IX}), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}''} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\text{IV}}, \quad (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}'} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}''} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\text{V}}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VII}'} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VII}''} \subset \overline{\mathbf{8}}_{\mathbf{F}}^{\text{VII}}, \\
& (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}} \oplus \left[(\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \oplus \cancel{(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}}} \right] \oplus \left[\cancel{(\mathbf{4}, \mathbf{1}, -\frac{1}{4})'_{\mathbf{F}}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}} \right] \\
\oplus & \left[(\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}} \oplus \cancel{(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}} \right] \oplus \cancel{(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}''} \subset \mathbf{28}_{\mathbf{F}}, \\
& \left[(\bar{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}} \oplus (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \right] \oplus \left[(\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}'' \oplus \cancel{(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}}''} \right] \oplus \left[(\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \oplus (\mathbf{4}, \mathbf{2}, +\frac{1}{4})'_{\mathbf{F}} \right] \\
\oplus & \left[\cancel{(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}''''} \oplus (\mathbf{4}, \bar{\mathbf{2}}, +\frac{1}{4})_{\mathbf{F}}'' \right] \oplus \left[(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}'' \oplus \cancel{(\mathbf{4}, \mathbf{1}, +\frac{3}{4})_{\mathbf{F}}} \right] \oplus (\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}''' \subset \mathbf{56}_{\mathbf{F}}. \tag{111}
\end{aligned}$$

5.3 The third stage

The third symmetry breaking stage of $\mathfrak{g}_{421} \rightarrow \mathfrak{g}_{\text{SM}}$ is achieved by $(\bar{4}, 1, +\frac{1}{4})_{\text{H,VI}} \subset \overline{8}_{\text{H,VI}}$ in the rank-two sector and $(\bar{4}, 1, +\frac{1}{4})_{\text{H,VIII}} \subset \overline{28}_{\text{H,VIII}}$, $(\bar{4}, 1, +\frac{1}{4})'_{\text{H,IX}} \subset \overline{28}_{\text{H,IX}}$, $(4, 1, -\frac{1}{4})_{\text{H}}^{\text{IX}'} \subset \overline{28}_{\text{H}}^{\text{IX}}$ in the rank-three sector.

$$\begin{aligned}
& Y_{\mathcal{B}} \overline{8}_{\text{F}}^{\text{VI}} \overline{28}_{\text{F}} \overline{8}_{\text{H,VI}} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\bar{6}, 1, +\frac{1}{6})_{\text{F}}^{\text{VI}} \otimes (15, 1, -\frac{1}{3})_{\text{F}} \oplus (1, \bar{2}, -\frac{1}{2})_{\text{F}}^{\text{VI}} \otimes (6, 2, +\frac{1}{3})_{\text{F}} \right] \otimes (\bar{6}, 1, +\frac{1}{6})_{\text{H,VI}} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\bar{5}, 1, +\frac{1}{5})_{\text{F}}^{\text{VI}} \otimes (10, 1, -\frac{2}{5})'_{\text{F}} \oplus (1, \bar{2}, -\frac{1}{2})_{\text{F}}^{\text{VI}} \otimes (5, 2, +\frac{3}{10})_{\text{F}} \right] \otimes (\bar{5}, 1, +\frac{1}{5})_{\text{H,VI}} + H.c. \\
\supset & Y_{\mathcal{B}} \left[(\bar{4}, 1, +\frac{1}{4})_{\text{F}}^{\text{VI}} \otimes (6, 1, -\frac{1}{2})'_{\text{F}} \oplus (1, \bar{2}, -\frac{1}{2})_{\text{F}}^{\text{VI}} \otimes (4, 2, +\frac{1}{4})_{\text{F}} \right] \otimes \langle (\bar{4}, 1, +\frac{1}{4})_{\text{H,VI}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{B}} (\mathcal{D}_L \mathcal{D}_R^c + \epsilon_L \epsilon_R^c - \mathbf{n}_L \mathbf{n}_R^c) V_{\bar{4},\text{VI}} + H.c. .
\end{aligned} \tag{112}$$

The term of $Y_{\mathcal{D}} \overline{8}_{\text{F}}^{\text{VIII,IX}} \overline{56}_{\text{F}} \overline{28}_{\text{H,VIII,IX}} + H.c.$ leads to the vectorial masses of $(\mathcal{D}''''', \epsilon''''', \mathbf{n}''''', \mathcal{D}''', \epsilon''', \mathbf{n}''')$ as follows:

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{8}_{\text{F}}^{\text{VIII}} \overline{56}_{\text{F}} \overline{28}_{\text{H,VIII}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\bar{6}, 1, +\frac{1}{6})_{\text{F}}^{\text{VIII}} \otimes (20, 1, -\frac{1}{2})_{\text{F}} \oplus (1, \bar{2}, -\frac{1}{2})_{\text{F}}^{\text{VIII}} \otimes (15, \bar{2}, +\frac{1}{6})_{\text{F}} \right] \otimes (\bar{15}, 1, +\frac{1}{3})_{\text{H,VIII}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\bar{5}, 1, +\frac{1}{5})_{\text{F}}^{\text{VIII}} \otimes (10, 1, -\frac{2}{5})'_{\text{F}} \oplus (1, \bar{2}, -\frac{1}{2})_{\text{F}}^{\text{VIII}} \otimes (5, 2, +\frac{3}{10})'_{\text{F}} \right] \otimes (\bar{5}, 1, +\frac{1}{5})_{\text{H,VIII}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\bar{4}, 1, +\frac{1}{4})_{\text{F}}^{\text{VIII}} \otimes (6, 1, -\frac{1}{2})''_{\text{F}} \oplus (1, \bar{2}, -\frac{1}{2})_{\text{F}}^{\text{VIII}} \otimes (4, 2, +\frac{1}{4})_{\text{F}} \right] \otimes \langle (\bar{4}, 1, +\frac{1}{4})_{\text{H,VIII}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} (\mathcal{D}_L'''' \mathcal{D}_R''''^c + \epsilon_L'''' \epsilon_R''''^c - \mathbf{n}_L'''' \mathbf{n}_R''''^c) V_{\bar{4},\text{VIII}} + H.c. ,
\end{aligned} \tag{113}$$

and

$$\begin{aligned}
& Y_{\mathcal{D}} \overline{8}_{\text{F}}^{\text{IX}} \overline{56}_{\text{F}} \overline{28}_{\text{H,IX}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\bar{6}, 1, +\frac{1}{6})_{\text{F}}^{\text{IX}} \otimes (20, 1, -\frac{1}{2})_{\text{F}} \oplus (1, \bar{2}, -\frac{1}{2})_{\text{F}}^{\text{IX}} \otimes (15, 2, +\frac{1}{6})_{\text{F}} \right] \otimes (\bar{15}, 1, +\frac{1}{3})_{\text{H,IX}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\bar{5}, 1, +\frac{1}{5})_{\text{F}}^{\text{IX}} \otimes (\bar{10}, 1, -\frac{3}{5})_{\text{F}} \oplus (1, \bar{2}, -\frac{1}{2})_{\text{F}}^{\text{IX}} \otimes (10, 2, +\frac{1}{10})_{\text{F}} \right] \otimes (\bar{10}, 1, +\frac{2}{5})_{\text{H,IX}} + H.c. \\
\supset & Y_{\mathcal{D}} \left[(\bar{4}, 1, +\frac{1}{4})_{\text{F}}^{\text{IX}} \otimes (6, 1, -\frac{1}{2})'_{\text{F}} \oplus (1, \bar{2}, -\frac{1}{2})_{\text{F}}^{\text{IX}} \otimes (4, 2, +\frac{1}{4})'_{\text{F}} \right] \otimes \langle (\bar{4}, 1, +\frac{1}{4})'_{\text{H,IX}} \rangle + H.c. \\
\Rightarrow & \frac{1}{\sqrt{2}} Y_{\mathcal{D}} (\mathcal{D}_L''' \mathcal{D}_R'''^c + \epsilon_L''' \epsilon_R'''^c - \mathbf{n}_L''' \mathbf{n}_R'''^c) V_{\bar{4},\text{IX}}' + H.c. .
\end{aligned} \tag{114}$$

The Yukawa coupling between two $\mathbf{56_F}$'s by the following $d = 5$ operator

$$\begin{aligned}
& \frac{c_4}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \mathbf{63_H} \mathbf{28_H}^{\text{IX}} + H.c. \\
\supset & c_4 \frac{v_U}{M_{\text{pl}}} \left[(\mathbf{6}, \mathbf{1}, +\frac{5}{6})_{\text{F}} \otimes (\mathbf{20}, \mathbf{1}, -\frac{1}{2})_{\text{F}} \oplus (\mathbf{15}, \mathbf{2}, +\frac{1}{6})_{\text{F}} \otimes (\mathbf{15}, \mathbf{2}, +\frac{1}{6})_{\text{F}} \right] \otimes (\mathbf{15}, \mathbf{1}, -\frac{1}{3})_{\text{H}}^{\text{IX}} + H.c. \\
\supset & c_4 \zeta_0 \left[(\mathbf{1}, \mathbf{1}, +1)_{\text{F}}''' \otimes (\mathbf{10}, \mathbf{1}, -\frac{3}{5})_{\text{F}} \oplus (\mathbf{5}, \mathbf{1}, +\frac{4}{5})_{\text{F}} \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})'_{\text{F}} \right] \otimes (\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\text{H}}^{\text{IX}} + H.c. \\
\supset & c_4 \frac{v_U}{M_{\text{pl}}} \left[(\mathbf{1}, \mathbf{1}, +1)_{\text{F}}''' \otimes (\mathbf{4}, \mathbf{1}, -\frac{3}{4})_{\text{F}} \oplus (\mathbf{4}, \mathbf{1}, +\frac{3}{4})_{\text{F}} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\text{F}} \right] \otimes \langle (\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\text{H}}^{\text{IX}'} \rangle + H.c. \\
\Rightarrow & \frac{c_4}{\sqrt{2}} \zeta_0 (\mathfrak{E}_L \mathfrak{E}_R^c + \mathfrak{U}_L \mathfrak{U}_R^c) V_4^{\text{IX}'} + H.c., \tag{115}
\end{aligned}$$

further leads to massive vectorlike fermions of $(\mathfrak{E}, \mathfrak{U})$.

The remaining massless fermions of the \mathfrak{g}_{SM} are listed as follows:

$$\begin{aligned}
& \left[(\mathbf{3}, \mathbf{1}, +\frac{1}{3})_{\text{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\Omega} \right] \oplus (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\Omega'} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\Omega''} \oplus (\mathbf{1}, \mathbf{2}, -\frac{1}{2})_{\text{F}}^{\Omega} \subset \overline{\mathbf{8_F}}^{\Omega}, \quad \Omega = (\dot{1}, \dot{2}, 3), \\
& (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\text{VI}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\text{VIII}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\text{IX}} \subset \overline{\mathbf{8_F}}^{\Omega}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\text{V}'} \oplus \dots \oplus (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\text{IX}'} \subset \overline{\mathbf{8_F}}^{\Omega'}, \\
& (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\text{IV}''} \oplus \dots \oplus (\mathbf{1}, \mathbf{1}, 0)_{\text{F}}^{\text{IX}''} \subset \overline{\mathbf{8_F}}^{\Omega''}, \\
& (\mathbf{1}, \mathbf{1}, +1)_{\text{F}} \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\text{F}} \oplus (\mathbf{3}, \mathbf{1}, -\frac{2}{3})'_{\text{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'_{\text{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\text{F}} \right] \\
\oplus & \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''_{\text{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)'_{\text{F}} \right] \oplus (\mathbf{1}, \mathbf{1}, 0)''_{\text{F}} \left[(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\text{F}} \oplus (\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\text{F}} \right] \oplus \\
\oplus & (\mathbf{1}, \mathbf{2}, +\frac{1}{2})'_{\text{F}} \oplus (\mathbf{1}, \mathbf{2}, +\frac{1}{2})''_{\text{F}} \subset \mathbf{28_F}, \\
& \left[(\mathbf{1}, \mathbf{1}, -1)_{\text{F}} \oplus (\mathbf{3}, \mathbf{1}, -\frac{2}{3})'_{\text{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\text{F}} \oplus (\mathbf{3}, \mathbf{1}, -\frac{2}{3})''_{\text{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\text{F}} \oplus (\mathbf{3}, \mathbf{1}, -\frac{2}{3})'''_{\text{F}} \right] \\
\oplus & \left[(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\text{F}} \oplus (\mathbf{1}, \mathbf{1}, 0)'''_{\text{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{2}, +\frac{1}{6})'_{\text{F}} \oplus (\mathbf{3}, \mathbf{2}, -\frac{1}{6})_{\text{F}} \right] \oplus \left[(\mathbf{3}, \mathbf{2}, +\frac{1}{6})''_{\text{F}} \oplus (\mathbf{1}, \mathbf{2}, +\frac{1}{2})'''_{\text{F}} \right] \\
\oplus & \left[(\mathbf{3}, \mathbf{2}, +\frac{1}{6})'''_{\text{F}} \oplus (\mathbf{1}, \mathbf{2}, +\frac{1}{2})'''_{\text{F}} \right] \oplus (\mathbf{1}, \mathbf{2}, +\frac{1}{2})'''_{\text{F}} \oplus \left[(\mathbf{3}, \mathbf{1}, -\frac{2}{3})_{\text{F}} \oplus (\mathbf{1}, \mathbf{1}, +1)_{\text{F}} \right] \oplus (\mathbf{1}, \mathbf{1}, +1)''_{\text{F}} \\
\oplus & (\mathbf{1}, \mathbf{1}, +1)'''_{\text{F}} \subset \mathbf{56_F}. \tag{116}
\end{aligned}$$

5.4 The $d = 5$ operators

In this section, we write down all the operators of $d = 5$ that seem likely to contribute to the mass of \mathfrak{u} and \mathfrak{d} . First, one can certainly consider an alternative $d = 5$ operator of $\frac{1}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \mathbf{8_H}^{\omega_1} \mathbf{8_H}^{\omega_2}$, we can carry a similar analysis to the $\frac{1}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \langle \mathbf{63_H} \rangle \mathbf{28_H}^{\dot{\omega}}$ operator as follows:

$$\begin{aligned}
& \frac{1}{M_{\text{pl}}} \mathbf{56_F} \mathbf{56_F} \mathbf{8_H}^{\omega_1} \mathbf{8_H}^{\omega_2} \\
\supset & \frac{1}{M_{\text{pl}}} (\mathbf{15}, \mathbf{2}, +\frac{1}{6})_{\text{F}} \otimes (\mathbf{15}, \mathbf{2}, +\frac{1}{6})_{\text{F}} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{6})_{\text{H}}^{\omega_1} \otimes (\mathbf{6}, \mathbf{1}, -\frac{1}{6})_{\text{H}}^{\omega_2} \\
\supset & \zeta_1 (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\text{F}} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\text{F}} \otimes \langle (\mathbf{5}, \mathbf{1}, -\frac{1}{5})_{\text{H}}^{\omega_1} \rangle. \tag{117}
\end{aligned}$$

It is clear that this operator also must be vanishing due to the antisymmetric properties. Essentially, no matter what the Higgs fields are assumed to compose a higher-dimensional gauge-invariant bilinear fermionic operator involving two $\mathbf{56_F}$, the vectorlike (\mathbf{u} , \mathbf{d}) quarks always belong to the representations of $\mathbf{15}, \mathbf{2}, +\frac{1}{6}$ $_{\mathbf{F}}$ for the $\mathfrak{su}(8) \rightarrow \mathfrak{g}_{621}$ symmetry breaking pattern. The above bi-linear fermionic product term is thus inevitable and must be vanishing due to the antisymmetric properties.

For the higher-dimensional gauge-invariant operators between $\mathbf{28_F}$ and $\mathbf{56_F}$, one can consider the $d = 5$ operator $\frac{1}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \mathbf{8_H}^\omega \mathbf{28_H}^{\dot{\omega}}$ and $\frac{1}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \overline{\mathbf{8_H}}_{,\omega} \mathbf{70_H}$ as follows:

$$\begin{aligned}
& \frac{1}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \mathbf{8_H}^\omega \mathbf{28_H}^{\dot{\omega}} \\
\supset & \frac{1}{M_{\text{pl}}} (\mathbf{6}, \mathbf{2}, +\frac{1}{3})_{\mathbf{F}} \otimes (\mathbf{15}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} \otimes \langle (\mathbf{6}, \mathbf{1}, -\frac{1}{6})_{\mathbf{H}}^\omega \rangle \otimes (\mathbf{15}, \mathbf{1}, -\frac{1}{3})_{\mathbf{H}}^{\dot{\omega}} \\
\supset & \zeta_1 (\mathbf{5}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \otimes (\mathbf{15}, \mathbf{1}, -\frac{1}{3})_{\mathbf{H}}^{\dot{\omega}} \\
\supset & \zeta_1 (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \otimes \langle (\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{H}}^{\dot{\omega}} \rangle \\
\Rightarrow & \frac{\zeta_1}{\sqrt{2}} V_{421} (\mathbf{u}_R^c t_L + \mathbf{d}_R^c b_L), \tag{118}
\end{aligned}$$

and for the $\frac{1}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \overline{\mathbf{8_H}}_{,\omega} \mathbf{70_H}$ operator

$$\begin{aligned}
& \frac{1}{M_{\text{pl}}} \mathbf{28_F} \mathbf{56_F} \overline{\mathbf{8_H}}_{,\omega} \mathbf{70_H} \\
\supset & \frac{1}{M_{\text{pl}}} (\mathbf{15}, \mathbf{1}, -\frac{1}{3})'_{\mathbf{F}} \otimes (\mathbf{15}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} \otimes \langle (\overline{\mathbf{6}}, \mathbf{1}, +\frac{1}{6})_{\mathbf{H},\omega} \rangle \otimes (\mathbf{20}, \mathbf{2}, 0)_{\mathbf{H}} \\
\supset & \zeta_1 (\mathbf{5}, \mathbf{1}, -\frac{1}{5})'_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{H}} \\
& \oplus \frac{1}{M_{\text{pl}}} (\mathbf{10}, \mathbf{1}, -\frac{2}{5})'_{\mathbf{F}} \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}} \otimes \langle (\overline{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})_{\mathbf{H},\omega} \rangle \otimes (\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{H}} \\
\supset & \zeta_1 (\mathbf{4}, \mathbf{1}, -\frac{1}{4})'_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \otimes (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{H}} \\
& \oplus \zeta_2 (\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \otimes (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{H}} \\
& \oplus \frac{1}{M_{\text{pl}}} (\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}} \otimes (\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}} \otimes \langle (\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{H},\omega} \rangle \otimes (\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{H}} \\
\Rightarrow & \frac{1}{\sqrt{2}} (\zeta_1 \mathfrak{D}_L'' \mathfrak{d}_R^c + \zeta_2 \mathfrak{D}_L' \mathfrak{d}_R^c + \zeta_3 \mathfrak{D}_L \mathfrak{d}_R^c) v_{\text{EW}}. \tag{119}
\end{aligned}$$

We can find that even though there is a mass mixing term for righthanded \mathbf{u} and \mathbf{d} with lefthanded (t , b) quarks and vectorlike fermions (\mathfrak{D}_L , \mathfrak{D}_L' , \mathfrak{D}_L''), there is no corresponding mass term for lefthanded \mathbf{u} and \mathbf{d} in the $d = 5$ operators between $\mathbf{28_F}$ and $\mathbf{56_F}$. The conclusion that the \mathbf{u} and \mathbf{d} quarks remain massless is unchanged. Hence, this symmetry breaking pattern cannot be a realistic one.