6. Infinite Series

Prelude

『一尺之捶， 日取其半，萬世不竭。』
莊子天下篇 (33)

What does this mean?

$$1, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \cdots, \left(\frac{1}{2}\right)^n, \cdots$$

None of above is equal to **zero**!

In [2]:

```
%matplotlib inline

#rcParams['figure.figsize'] = (10,3) #wide graphs by default
import scipy
import numpy as np
import time
from IPython.display import clear_output,display
import matplotlib.pylab as plt
```

Infinite Sequences

An infinite sequence is a set of indexed numbers, terms, denoted as $\{a_1, a_2, a_3, \cdots\}$ or $\{a_n\}_{n=1}^{\infty}$:

1. $\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$
2. $\{e^{-\lambda}, e^{-\lambda}\lambda, e^{-\lambda}\lambda^2/2, \cdots, e^{-\lambda}\lambda^n/n!, \cdots\}$, where $\lambda > 0$.

Or sequences could be defined recursively, such as the following example:

Example. Recusive sequences, $\{a_n\}_{n\geq 1}$ is defined as $a_1 = 2, a_2 = 4$ and $a_{n+1} = 2a_n - a_{n-1}$ for $n > 1$.

Solving the difference equation get the result, $a_n = 2n$.

Note.

1. Replace $a_n = ar^n$ into the equation and gets the characteristic value, r, 1 (multiplicity 2);
2. this implies $a_n = a + bn$
3. and $a_n = 2n$ by the initial values.

Limit of Sequence

L is said to be the limit of $\{a_n\}_{n=1}^{\infty}$ if

$$\lim_{n\to\infty} a_n = L$$

If L exists, sequences is called converent orthwised divergent.

$\{\sin(\frac{\pi}{n})\}_{n=1}^{\infty}$ is convergent since

$$\lim\ {n\to\infty}\sin\left(\frac{\pi}{n}\right) = \sin(0) = 0$$

$\left\{\sin(\frac{n\pi}{2})\right\}_{n=1}^{\infty}$ is divergent since

$$\sin\left(\frac{n\pi}{2}\right)\xrightarrow{n=1,2,\cdots} 1, 0, -1, 0, \cdots$$

$$\Downarrow$$

$$\lim\ {n\to\infty}\sin\left(\frac{n\pi}{2}\right) \quad \neq \quad L \in \mathrm{R}$$

Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent, then

1. $\lim\ n \to \infty \{a_n \pm b_n\}_{n=1}^{\infty} = \lim\ n \to \infty \{a_n\}_{n=1}^{\infty} \pm \lim\ n \to \infty \{b_n\}_{n=1}^{\infty}$
2. $\lim\ n \to \infty \{ca_n\}_{n=1}^{\infty} = c \lim\ n \to \infty \{a_n\}_{n=1}^{\infty}$
3. $\lim\ n \to \infty \{\frac{a_n}{b_n}\}_{n=1}^{\infty} = \frac{\lim n \to \infty \{a_n\}_{n=1}^{\infty}}{\lim n \to \infty \{b_n\}_{n=1}^{\infty}}$ provided $\lim\ n \to \infty \{b_n\}_{n=1}^{\infty} \neq 0$.

Note

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

Example

The limit of $\left\{ \frac{6n^3+5n^2+7}{4n^3-2n+2} \right\}_{n=1}^{\infty}$ is:

$$\begin{aligned}\lim_{n \to \infty} \frac{6n^3+5n^2+7}{4n^3-2n+2} &= \lim_{n \to \infty} \frac{6+\frac{5}{n}+\frac{7}{n^3}}{4-\frac{2}{n^2}+\frac{2}{n^3}} \\ &= \frac{3}{2}\end{aligned}$$

$\lim\limits_{n \to \infty} \{\ln(n+4) - \ln(n)\} = 0$ since

$$\begin{aligned}\ln(n+4) - \ln(n) &= \ln\left(\frac{n+4}{n}\right) \\ &= \ln\left(\frac{1+\frac{4}{n}}{1}\right) \\ &\to \ln(1) = 0\end{aligned}$$

Example

Certain company inspects its products with size n randomly. If $\alpha\% = 2\%$ of the entire lot is defective, then the the probability of finding no defectives products of size n is $\left(\frac{100-\alpha}{100}\right)^n$.

And the probability of finding at least one defective product is $P(n) = 1 - \left(\frac{100-\alpha}{100}\right)^n$.

Since $\lim_{n\to\infty} P(n) = 1$,the probability of finding at least one defective product is 100% as the size increases largely.

Theorem

Bounded, Monotonic implies Convergent.

A bounded and monotone sequence is convergent, increasing with bounded above or decreating with bounded below, implies $\lim a_n$ exists!

By completeness Axiom:
Every sequence bounded above has a least upper bound; every sequences bounded below has a largest lower bound.

For instance:

1. $1, \frac{1}{2}, \frac{1}{3}, \cdots \searrow$ and $\leqslant 1$, convergent;
2. $\ln 1, \ln 2, \ln 3, \cdots \nearrow$ \ but no upper bound, $\not\Rightarrow$ convergent.
3. $a_1 = 1/2$, $a_n = (1 + a_{n-1})/2$ for $n = 2, 3, \cdots$. Since $a_n \leqslant 1$ and $a_n \nearrow$, $\{a_n\}$ is convergent and limit is equal to 1.
4. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
5. $\lim_{n\to\infty} \frac{e^n}{n^2} = \infty$

Proof:

Suppose that U is a upper bound of increasing sequences, $\{a_n\}$, then

$$U > L > a_N, \text{ and } a_N > L - \epsilon$$

where L is the least upper bound, N is very large number, and $\epsilon > 0$. This implies

$$0 < L - a_n < L - a_N < \epsilon$$

for any $n > N$. This proves L is the limit of sequences. ■

Example

$\left\{\dfrac{2^n}{n!}\right\}$ is convergent to zero since it is bounded below and decreasing.

Example

Suppose that $f(x) = 2/x,\ x \neq 0$.

1. Define a sequence, $\{a_n\}_{n=0}^{\infty}$ as follows:

$$a_n = \begin{cases} a & \text{if } n = 0, \\ f(a_{n-1}) & \text{if } n \neq 0. \end{cases}$$

Since $f(a) = 2/a$ and $f(2/a) = a$ the sequence is also listed as follows:

$$\{a, 2/a, a, 2/a, \cdots\}$$

Note that $f^2(x) = f(f(x)) = x$, x is also called periodic point with cycle of period 2.

2. Suppose that the sequence is convergent, its limit can be found out by following s:

$$\begin{aligned} L &= \lim_{n\to\infty} f^n(x) \\ &= \lim_{n\to\infty} f^{n+1}(x) = f(\lim_{n\to\infty} f^n(x)) \\ &= f(L) = 2/L \\ &\Rightarrow L^2 = 2 \\ &\Rightarrow L = \pm\sqrt{2} \end{aligned}$$

i.e. limit is $\pm\sqrt{2}$ depend on the sign of initial value, a.

3. How to find the value of $\sqrt{2}$ only by additions, multiplication and divisions?

If $a_0 = 1$, the sequence is $\{1, 2, 1, 2, \cdots\}$. In other words, it is not convergent. Redefine this sequence as the follows:

$$a_0 = 1, a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \text{ if } n \neq 0$$

Following gives the first ten results:

Exercise (p743)

26. $a_n = \cos n\pi + 2$ divergent, since $a_n = (-1)^n + 2$.

32. $a_n = \dfrac{\ln n^2}{\sqrt{n}}$ convergent since $\lim\limits_{n\to\infty} \dfrac{2\ln n}{n^{1/2}} \to \infty$

39. $a_n = \dfrac{\sin^2 n}{\sqrt{n}}$ convergent since $|a_n| \le \dfrac{1}{\sqrt{n}} \to 0$

40. $a_n = \dfrac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{n!}$ divergent

$$\begin{aligned} a_n &= \frac{(2n)!}{n!2\cdot 4\cdot 6\cdots(2n-2)\cdot 2n} \\ &= \frac{(n+1)\cdot(n+2)\cdots(2n-1)\cdot(2n)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot 2n} \\ &= \frac{n+1}{2}\cdot\frac{n+2}{4}\cdots\cdot\frac{2n-1}{2n-2}\cdot\frac{2n}{2n} \\ &\ge \frac{n+1}{2} \to \infty \end{aligned}$$

50. $\lim\limits_{n\to\infty} n\left(1-\sqrt[7]{1-\dfrac{1}{n}}\right) = \dfrac{1}{7}$

$$\begin{aligned} \frac{1-\sqrt[7]{1-\frac{1}{n}}}{1/n} &= \frac{1-\sqrt[7]{1-x}}{x} \text{ where } x = 1/n \\ &= \frac{1-(1-x)}{x}\cdot\frac{1}{\sqrt[7]{(1-x)^6}+\sqrt[7]{(1-x)^5}+\cdots+1} \to \frac{1}{7} \end{aligned}$$

54. $a_n = 2 + \dfrac{(-1)^n}{n}$ is bounded but not monotonic. $\{a_n\}$ is convergent since $\dfrac{(-1)^n}{n} \to 0.$

68. $\lim \sqrt{2\sqrt{2\sqrt{2\sqrt{\cdots}}}} = 2$

a). let $a_0 = \sqrt{2}, a_n = \sqrt{2a_{n-1}}$

b). $a_n \le 4$ and a_n is increasing;
these concludes that $L = \lim a_n$ exists.

c). $L = \sqrt{2L}$ implies $L = 2$.

74. For any $a > 0$, there exist one n, large enough such that $1/n < a < n$, therefore $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$

In [2]:

```
import numpy as np
x=1
for n in range(10):
  x=(x+2/x)/2
  print(x)
```

```
1.5
1.4166666666666665
1.4142156862745097
1.4142135623746899
1.414213562373095
1.414213562373095
1.414213562373095
1.414213562373095
1.414213562373095
1.414213562373095
```

Suppose that a sequence, $\{a_n\}_{n=0}^{\infty}$, satisfies the following recursive dormula:

$$a_{n+1} = f(a_n)$$

for some function $f(x)$ and $n = 1, 2, , 3, \cdots$. Every element in the sequence can be represented in the form of composed function f^n as following:

$$\begin{aligned} a_n &= f(a_{n-1}) \\ &= f(f(a_{n-2})) = f^2(a_{n-2}) \\ &= \cdots \\ &= f^n(a_0) \end{aligned}$$

Definition

The first term a_0 in $\{a_n\}_{n=0}^{\infty}$, with $a_{n+1} = f(a_n)$, is called periodic point with cycle of period m if there exists $m \in \mathrm{N}$ such that

$$f(a_m) = a_0$$

a_0 is called fixed point if $f(a_0) = a_0$.

Theorem

Suppose that sequence $\{a_n\}_{n=0}^{\infty}$, with $a_n = f(a_{n-1})$ and f being differentiable, is period m. Then

$$f'(a_{n-1}) = \prod_{k=0}^{n-1} f'(a_k)$$

This is trivial since

$$\begin{aligned} (a_n)' &= (f^n(a_0))' \\ &= (f(f^{n-1}(a_0)))' \\ &= f'(f^{n-1}(a_0)) \cdot (f^{n-1}(a_0))' \\ &= f'(a_{n-1}) \cdot (f^{n-1}(a_0))' \\ &= f'(a_{n-1}) \cdot f'(a_{n-2}) \cdot (f^{n-2}(a_0))' \\ &= \cdots \\ &= \prod_{k=0}^{n-1} f'(a_k) \end{aligned}$$

Example

Find the periodic point(s) with period 2 for the sequence:

$$a_{n+1} = 4a_n(1 - a_n)$$

i.e. $f(x) = 4x(1 - x)$

Suppose that $a_0 = a$. Then we want find a such that $a = f^2(a)$. It shows that $a = 3/4$ and 0 are fixed points. Only $-\frac{\sqrt{5}-5}{8}$ and $\frac{\sqrt{5}+5}{8}$ are points with cycle of period 2. Also we have $\left(f^2\left(-\frac{\sqrt{5}-5}{8}\right)\right)' = -3.9999999666151753$ and it is equal to

$$f'\left(-\frac{\sqrt{5}-5}{8}\right)f'\left(\frac{\sqrt{5}+5}{8}\right)$$

In [ ]:

```

```

Example

Consider the following function:

$$h(x) = 2 - |x - 1|$$

1. It is trivial:

$$h(x) = \begin{cases} 3 - x, & \text{if } x > 1; \\ 1 + x, & \text{if } x \leqslant 1. \end{cases}$$

2. The graphs about $h(x)$ and its composed functions, $g^n(x)$ are as follows pictures.

$h(x)$ is a wedge with straight lines and one vertex on the line $y = 2$, (the blue line). $h^2(x)$ is a wedge with three vertices on the line $y = 2$, (the green line). The solution of equation is the part which these two curves intersect with each other, (the line with thicker segment and in red), i.e. $x \in [1, 2]$. It is trivial to prove that any point within this range is fixed point only!

In [4]:

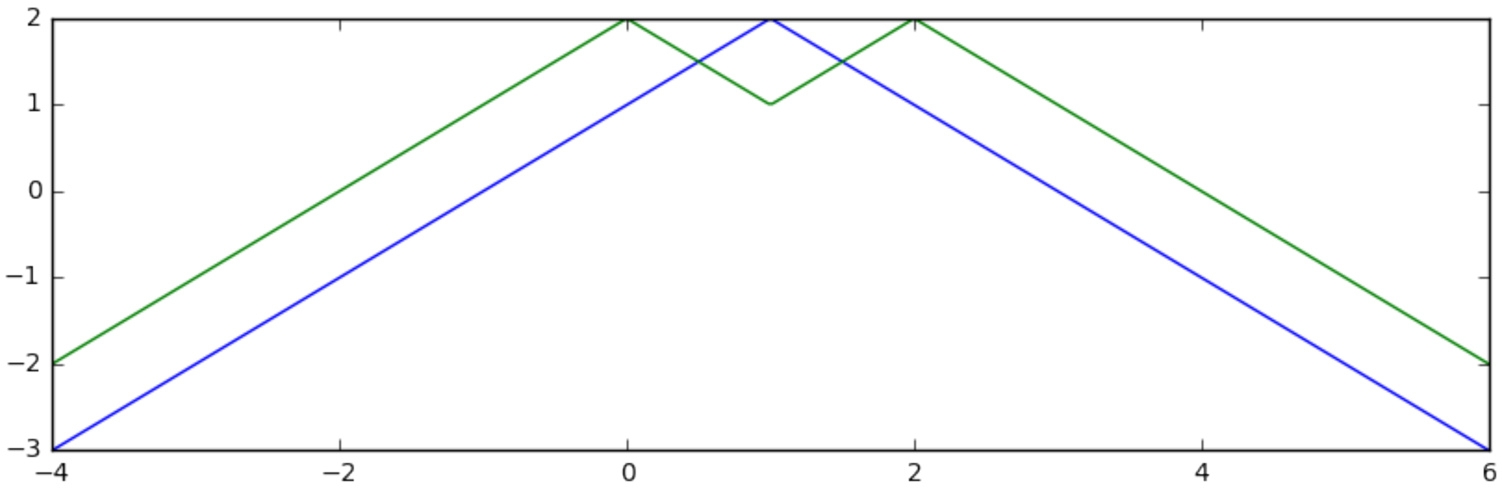
```
plt.rcParams['figure.figsize'] = (10,3) #wide graphs by default
def h(x):
    return 2-abs(x-1)
x=np.linspace(-4,6,400)
```

In [15]:

```
plt.plot(x,h(x),x,h(h(x)))
```

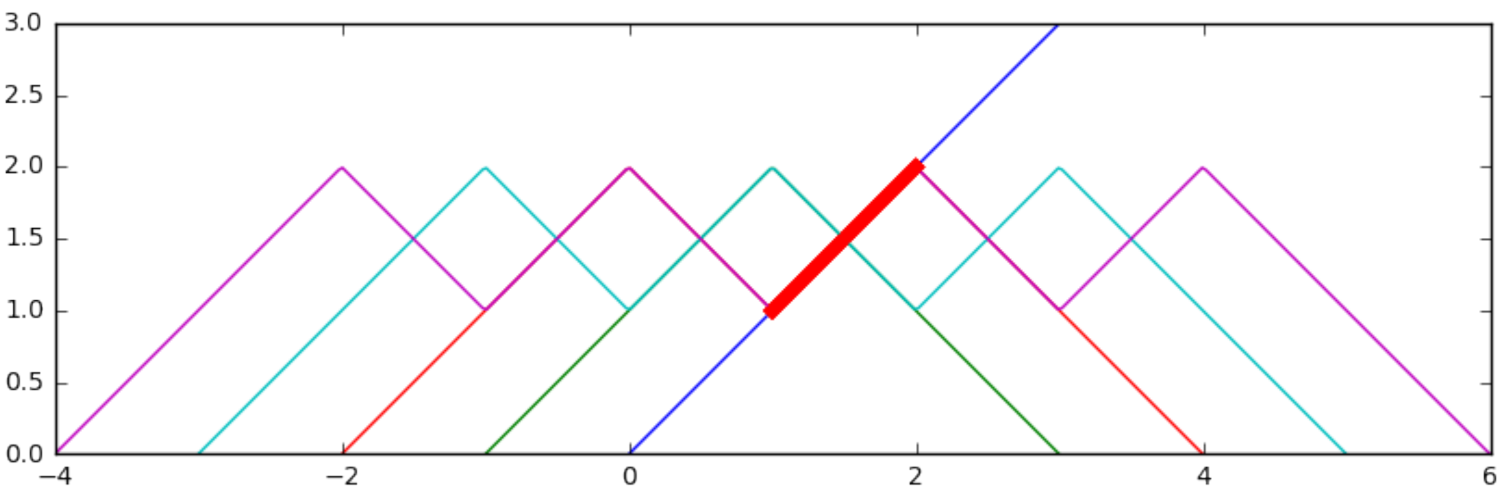
Out[15]:

```
[<matplotlib.lines.Line2D at 0x1085cc0b8>,
 <matplotlib.lines.Line2D at 0x109288e48>]
```



In [13]:

```
f, ax = plt.subplots()
ax.set_xlim([-4,6]);ax.set_ylim([0,3])

x=np.linspace(-4,6,400);y=x
for i in np.arange(5):
    plt.plot(x,y)
    plt.plot((1,2),(1,2),'r-',linewidth=5)
    y=h(y)

    time.sleep(2)
    clear_output()
    display(f)
plt.close()
```



Eexercise

Find the limits of the following sequences if exist:

1. $\{(-1)^n\}_{n=1}^{\infty}$
2. $\{1-\cos(n\pi)\}_{n=1}^{\infty}$
3. $\left\{\frac{1+2^n}{3^{n+1}-3\cdot 2^n}\right\}_{n=1}^{\infty}$

Answer

1. $\{(-1)^{\mathbf{n}}\}_{\mathbf{n=1}}^{\infty} = \{-\mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}, \cdots\}$: divergent;
2. $\{\mathbf{1}-\cos(\mathbf{n}\pi)\}_{\mathbf{n=1}}^{\infty} = \{\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \cdots\}$: divergent;
3. $\frac{\mathbf{1+2^n}}{\mathbf{3^{n+1}-3\cdot 2^n}} = \frac{\mathbf{(1/3)^n+(2/3)^n}}{\mathbf{3-3\cdot(2/3)^n}} \to \mathbf{0}$: convergent.

Exercise

Determine which one(s) in the following sequences is/are convergent:

1. $\{\frac{\ln n}{n}\}_{n=1}^{\infty}$
2. $\{\frac{2^n}{3^n+1}\}_{n=1}^{\infty}$
3. $\{\frac{n-(-1)^n}{n+(-1)^n}\}_{n=1}^{\infty}$
4. $\{\cos(n\pi/4)\}_{n=1}^{\infty}$ Divergent, since $\lim\ n\to\infty \cos\left(\frac{n\pi}{4}\right)$ diverges.
5. $a_1 = 25$, and $a_n = \frac{1}{2}(50 + a_{n-1})$ for $n = 2, 3, \cdots$. Find $\lim n\to_{\infty} a_n$ if exists. (Hint: Increasing bounded sequence is convergent)

Answer

1. $\frac{\ln n}{n} \to 0$ Convergent, $L = 0$.
2. $\frac{2^n}{3^n+1} \to 0$ is convergent, $L = 0$.
3. $\frac{n-(-1)^n}{n+(-1)^n} \to 1$ convergent, $L = 1$.
4. $\{\cos(n\pi/4)\}_{n=1}^{\infty} = \{\sqrt{2}/2, 0, -\sqrt{2}/2, -1, -\sqrt{2}/2, 0, \sqrt{2}/2, 1, \cdots\}$ divergent.
5. For $a_0 \ge 0$ and $a_n = \frac{1}{2}(a_{n-1} + 50, n = 1, 2, \cdots$. Then $\{a_n\}$ is convergent. Trivially by mathmatical induction, the sequence is increasing and bounded above:

   - a_n is bounded by 50: if $a_k \leqslant 50$

   $$a_{k+1} = \frac{1}{2}(50 + a_k) \le 50$$

   - a_n is increasing: suppose that $a_{k-1} \leqslant a_k$:

   $$\begin{aligned} a_{k+1} &= \frac{1}{2}(50 + a_k) \\ &\ge \frac{1}{2}(50 + a_{k-1}) = a_k \end{aligned}$$

   $$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{2}(50 + a_{n-1}) \\ &\Downarrow \\ L &= \frac{1}{2}(50 + L) \\ &\Downarrow \\ L &= 50 \end{aligned}$$

Infinite Series

The sum of infinite constants is called infinite series, like as:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

Im mathematics, convergence of sum of infinite sereis is different from that of sum of finter terms. We can not conclude the convergent of infinite seris even then any sum of terms in front of series are convergent!

Example. Determine following series which are covergent or divergent:

1. $1 + 1 + 1 + \cdots$, ✘
2. $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, ✘
3. $1 - 1 + 1 - 1 + \cdots$, ✘
4. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, ✓
   But why?

Convergence of Infinite Series

Limit of Partial Sum Sequence

Define a new partial sum sequence, $\{S_n\}_{n=1}^{\infty}$, as

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= S_1 + a_2 = a_1 + a_2 \\ &\vdots \\ S_n &= S_{n-1} + a_n \\ &= \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \end{aligned}$$

$$\lim_{n \to \infty} S_n \text{ convergent } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergent}$$

Example

1. the infinite series $1 + 1 + 1 + \cdots$ is divergent since

$$\begin{aligned} S_n &= 1 + 1 + \cdots + 1 \\ &= n \\ &\to \infty \end{aligned}$$

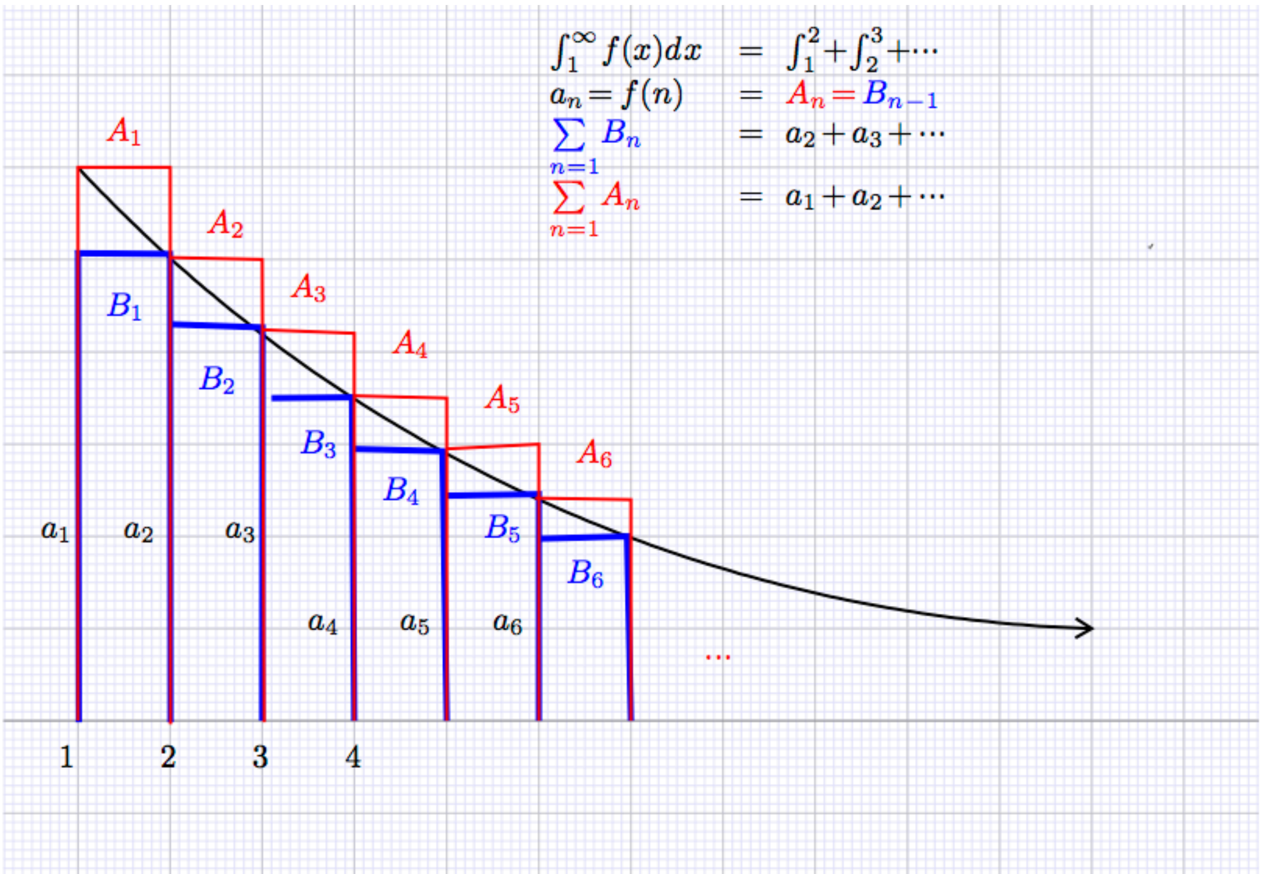
2. $1 - 1 + 1 - 1 + \cdots$ is divergent since

$$\begin{aligned} S_1, S_3, S_5, \cdots &= 1 \\ S_2, S_4, S_6, \cdots &= 0 \\ \{S_n\} &= \{1, 0, 1, 0, \cdots\} \\ &\not\longrightarrow \text{convergent} \end{aligned}$$

Integral test

Suppose that $a_n \geq 0$ for each n, and there exists a continuous function $f(x) \geq 0$ and $f \searrow 0$ for $x \in [1, \infty)$, such that $f(n) = a_n$. Then

$$\int_1^\infty f(x)dx \text{ is convergent if and only if} \sum_{n=1}^\infty a_n \text{is convergent.}$$



$$B_n \leqslant \int_n^{n+1} f(x)dx \leqslant A_n$$

$$\Downarrow$$

$$\sum_{n=1}^\infty B_n \leqslant \int_1^\infty f(x)dx \leqslant \sum_{n=1}^\infty A_n$$

$$\Downarrow$$

$$\left(\sum_{n=1}^\infty a_n - a_1\right) = \sum_{n=2}^\infty a_n \leqslant \int_1^\infty f(x)dx \leqslant \sum_{n=1}^\infty a_n$$

$$\Downarrow$$

$$\int_1^\infty f(x)dx \text{ convergent} \quad \text{if and only if} \quad \sum_{n=1}^\infty a_n \text{ convergent}$$

Conclusion

$$\int_1^\infty f(x)dx \leqslant \sum_{n=1}^\infty a_n \leqslant \int_1^\infty f(x)dx + a_1$$

Example

1. Harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, is divergent since

$$f(n) = \frac{1}{n} \Rightarrow f(x) = \frac{1}{x}, f \searrow, f \to 0$$

$$\int_1^{\infty} \frac{1}{x} dx = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{divergent}$$

2. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ divergent since

$$f(n) = \frac{1}{n\ln n} \Rightarrow f(x) = \frac{1}{x\ln x}, f \geq 0, f \searrow$$

$$\begin{aligned} \int_2^{\infty} \frac{1}{x\ln x} dx &= \int_2^{\infty} \frac{1}{\ln x} d\ln x \\ &= \ln|\ln x||_2^{\infty} = \infty \end{aligned}$$

3. p-Series, Determine the range of $p > 0$, such that the following p-series is convergent

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Let $f(x) = x^{-p}$

- $0 < p < 1$ case:

$$\int_1^{\infty} x^{-p} dx = \frac{1}{1-p} x^{1-p} \Bigg|_1^{\infty} = \infty$$

- $p = 1$ case:

$$\int_1^{\infty} x^{-1} dx = \ln|x| \Bigg|_1^{\infty} = \infty$$

- $p > 1$ case:

$$\int_1^{\infty} x^{-p} dx = \frac{1}{1-p} \frac{1}{x^{p-1}} \Bigg|_1^{\infty} = \frac{1}{p-1} < \infty$$

  Then $p > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent.

  - $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent since $p = 1/2 \leqslant 1$;
  - $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent since $p = 2 > 1$;

4. $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ convergent since $\int_1^{\infty} \frac{dx}{1+x^2} < \infty$.

Exercise p760

10. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ divergent, since $2/3 < 1$.

16. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{2n^2+1}}$ divergent $\int_1 \frac{xdx}{\sqrt{2x^2+1}} = \infty$.

22. $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ convergent since $\int_2 \frac{\ln xdx}{x^2} < \infty$.

28. $\sum_{n=1}^{\infty} \frac{n}{2^n}$ convergent, since $\int_1 \frac{xdx}{2^x} < \infty$.

34. $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$ convergent since $\int_2 \frac{\ln xdx}{x^p} < \infty$ if $p > 1$.

58. (True or False) If positive series $\sum_{n=1}^{\infty} a_n$ convergent, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ is also convergent.

Alternating Series

$\sum_{n=0}^{\infty} a_n = a_0 - a_1 + a_2 - a_3 \cdots$ is called an alternating series where $a_0, a_1, a_2, \cdots \geq 0$.

A convergent test for alternating series is:

Theorem (Alternating Test)

An alternating series converges if and only if $\lim_{n\to\infty} a_n = 0$

Example, Alternating Harmonic Series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent since $1/n \to 0$ as $n \to \infty$. This test is not right for the positive series.

More Details

$$
\begin{aligned}
S_1 &= 1 \\
S_3 &= 1-\frac{1}{2}+\frac{1}{3}=1-\left(\frac{1}{2}-\frac{1}{3}\right)<1=S_1 \\
S_5 &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}<S_3 \\
&\vdots \ \text{by Mathematical Induction} \\
S_{2k+1} &= 1-\frac{1}{2}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}+\frac{1}{2k+1} \\
&= S_{2k-1}-\frac{1}{2k}+\frac{1}{2k+1}<S_{2k-1}
\end{aligned}
$$

$$\Downarrow$$

$$(a).\ S_1, S_3, S_5, \cdots \quad \searrow$$

$$
\begin{aligned}
S_2 &= 1-\frac{1}{2} \\
S_4 &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}>S_2 \\
&\vdots \\
S_{2k+2} &= 1-\frac{1}{2}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2} \\
&= S_{2k}+\frac{1}{2k+1}-\frac{1}{2k+2}>S_{2k}
\end{aligned}
$$

$$(b).\ S_2, S_4, S_6, \cdots \quad \nearrow$$

$$S_{2k} \nearrow \ \text{and } S_{2k+1} \searrow$$

$$
\begin{aligned}
S_{2k+1} &= \left(1-\frac{1}{2}\right)+\cdots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)+\frac{1}{2k+1} \geqslant 0 \\
S_{2k+2} &= 1\left(-\frac{1}{2}+\frac{1}{3}\right)\cdots+\left(-\frac{1}{2k-2}+\frac{1}{2k-1}\right)+\left(-\frac{1}{2k}+\frac{1}{2k+1}\right)-\frac{1}{2k+2} \leqslant 1
\end{aligned}
$$

$$\Downarrow$$

$$\lim_{k\to\infty} S_{2k+1} \text{ convergent} \qquad \lim_{k\to\infty} S_{2k+2} \text{ convergent}$$

Above also implies the convergence of S_n:

$$S_{2k+2}-S_{2k+1} = -\frac{1}{2k+2}$$

$$\Downarrow$$

$$\lim_{k\to\infty}(S_{2k+2}-S_{2k+1}) = 0$$

$$\Downarrow$$

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k+2}$$

$$\Downarrow$$

$$\lim_{k\to\infty} S_k \quad \text{convergent}$$

Example

1. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{2n}{4n-1}$ divergent, since $\frac{2n}{4n-1} \to 1/2 \neq 0$
2. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{3n}{4n^2-1}$ convergent, since $\frac{3n}{4n^2-1} \to 0$
3. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n!}$ convergent, since $\frac{1}{n!} \to 0$

Exercise (p.773)

9. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{n}{\ln n}$ divergent, since $\frac{n}{\ln n} \to \infty \neq 0$

14. $\sum\limits_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ convergent, since $\frac{1}{n} \to 0$

20. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$ convergent, since $0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \to 0$

24. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt[n]{n}}$ divergent, since $\frac{1}{\sqrt[n]{n}} \to 1 \neq 0$

28 $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n^s}$ is convergent for $s > 0$.

Comparison Test

If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are two positive infinite series with $0 \leqslant a_n \leqslant b_n$ for each n, then

1. $\sum\limits_{n=1}^{\infty} b_n$ is convergent $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ is convergent
2. $\sum\limits_{n=1}^{\infty} a_n$ is divergent $\Rightarrow \sum\limits_{n=1}^{\infty} b_n$ is divergent

Example

1. Harmonic series being divergent implies $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ divergent, since

$$\frac{1}{n} \leqslant \frac{1}{\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ divergent} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ divergent}$$

2. $\sum_{n=2}^{\infty} \frac{1}{n^2}$ being convergent implies $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ is convergent since

$$\frac{1}{n^2} \geqslant \frac{1}{n^2 \ln n} \text{ and } \sum_{n=2}^{\infty} \frac{1}{n^2} \text{ convergent} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2 \ln n} \text{ convergent}$$

   and $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ convergent too.

3. $\sum_{n=2}^{\infty} \frac{1}{n^2}$ being convergent cannot imply $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ since

$$\frac{1}{n^2} \leqslant \frac{1}{n^2-1} \text{ but } \sum_{n=2}^{\infty} \frac{1}{n^2} \text{ convergent} \not\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2-1} \text{ convergent}$$

4. $\sum_{n=1}^{\infty} \frac{1}{3+2^n}$ convergent since $\frac{1}{3+2^n} \leq \frac{1}{2^n}$
5. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$ is divergent since $\sqrt{n} - 1 < \sqrt{n}$.

Limit Comparison Test

Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ satisfies $0 \leqslant a_n, b_n$ for any n. If $r = \lim_{n\to\infty} \frac{a_n}{b_n}$ and $0 < r < \infty$, then both the series are convergent or divergent.

Example

1. $\sum_{n=2}^{\infty}\frac{1}{n^2}$ being convergent implies $\sum_{n=2}^{\infty}\frac{1}{n^2-n+1}$ convergent since

$$\lim_{n\to\infty}\frac{\frac{1}{n^2}}{\frac{1}{n^2-n+1}}=1 \text{ and } \sum_{n=2}^{\infty}\frac{1}{n^2} \text{ convergent}$$

2. $\sum_{n=1}^{\infty}\frac{1}{n}$ being divergent implies $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ divergent since

$$\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{2n+1}}=2 \text{ and } \sum_{n=1}^{\infty}\frac{1}{n} \text{ divergent}$$

3. $\sum_{n=2}^{\infty}\frac{1}{3^n}$ being convergent implies $\sum_{n=2}^{\infty}\frac{1}{3^n-2^n}$ convergent since

$$\lim_{n\to\infty}\left|\frac{\frac{1}{3^n}}{\frac{1}{3^n-2^n}}\right|=1 \text{ and } \sum_{n=2}^{\infty}\frac{1}{3^n} \text{ convergent}$$

4 $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1}$ is divergent since $\frac{\sqrt{n}+1}{\sqrt{n}}\to 1.$

5 $\sum_{n=1}^{\infty}\frac{2n^2+n}{\sqrt{4n^7+3}}$ is convergent since $\frac{\frac{2n^2+n}{\sqrt{4n^7+3}}}{\sqrt{2}n^{3/2}}\to 1.$

6 $\sum_{n=1}^{\infty}\frac{\sqrt{n}+\ln n}{n^2+1}$ is convergent since $\frac{\frac{\sqrt{n}+\ln n}{n^2+1}}{n^{3/2}}\to 1.$

Exercises (p.767)

10. $\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^2}$ is convergent since $a_n\le 1/(n^2).$

22. $\sum_{n=1}^{\infty}\frac{\ln n}{n^3-1}$ is convergent since $a_n\le n/(n^3).$

24. $\sum_{n=1}^{\infty}\tan\frac{1}{n}$ is divergent since $a_n/(1/n)\to 1.$

32. $\sum_{n=1}^{\infty}\frac{\tan^{-1} n}{n^3+1}$ is convergent since $a_n\le 1/(n^2)\to 0.$

Convergent Problem

The condition of geometric series

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots \text{ where } a \neq 0$$

is well-known as:

$$\begin{aligned} |r| < 1 &\Rightarrow S \text{ convergent} \\ |r| > 1 &\Rightarrow S \text{ divergent} \\ |r| = 1 &\Rightarrow S \text{ divergent} \end{aligned}$$

But what can we say about:

$$\tilde{S} = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

It is **NOT** geometric!

But note that

$$r = \frac{\cancel{a}r}{\cancel{a}} = \frac{\cancel{a}r^{\cancel{2}}}{\cancel{a}\cancel{r}} = \cdots = \frac{\cancel{a}r^{\cancel{n}+1}}{\cancel{a}\cancel{r}^n} = \cdots$$

Absolute Convergence

1. $\sum_{n=1}^{\infty} a_n$ is calle absoulute convergent if $\sum_{n=1}^{\infty} |a_n|$ convergent.
2. $\sum_{n=1}^{\infty} a_n$ is calle conditional convergent if $\sum_{n=1}^{\infty} a_n$ convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Example

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolute convergent.
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditional convergent.
3. $\sum_{n=1}^{\infty} (-1)^n n$ is divergent.
4. $\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$ is absolute convergent.

Ratio Test

Suppost that

$$S = a_1 + a_2 + \cdots + a_n + \cdots$$

and let

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then

1. $r < 1$ implies $\sum_{n=1}^{\infty} a_n$ is convergent;
2. $r > 1$ implies $\sum_{n=1}^{\infty} a_n$ is divergent;
3. $r = 1$ no conclusion.

Root Test

Suppost that

$$S = a_1 + a_2 + \cdots + a_n + \cdots$$

and let

$$r = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

Then

1. $r < 1$ implies $\sum_{n=1}^{\infty} a_n$ is convergent;
2. $r > 1$ implies $\sum_{n=1}^{\infty} a_n$ is divergent;
3. $r = 1$ no conclusion.

Example

Determine whether the infinite series are convergent or divergent.

1. $\sum_{n=1}^{\infty} n/2^n$ is convergent since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)/2^{n+1}}{n/2^n}\right|$$

$$= \left|\frac{n+1}{2n}\right|$$

$$\to \frac{1}{2} < 1$$

2. $\sum_{n=1}^{\infty} \frac{4^n}{2^n+3^n}$ is divergent since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{4^{\cancel{n}+1}/(2^{n+1}+3^{n+1})}{4^{\cancel{n}}/(2^n+3^n)}\right|$$

$$= \frac{4}{3}\left|\frac{(2/3)^n+1}{(2/3)^{n+1}+1}\right|\left(\frac{\cdot\ /3^{n+1}}{\cdot\ /3^{n+1}}\right)$$

$$\to \frac{4}{3} > 1$$

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent but $r = 1$.

4. $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent but $r = 1$.

5. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{2^n}$ is absolute convergent:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)^2+1)/2^{n+1}}{(n^2+1)/2^n}\right|$$

$$\to \frac{1}{2} < 1$$

6. $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$ is absolute convergent.

7. $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{3^n}$ is divergent.

8. $\sum_{n=1}^{\infty} (-1)^n \frac{2^{n+3}}{(n+1)^n}$ is absolute convergent.

Exercise

Determine which $p-\text{series}$ is convergent:

1. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$;
2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$.

Answer

1. $p = 3/2 > 1$, convergent;
2. $p = 1/3 < 1$, divergent

Exercise

Which one(s) is(are) convergent?

1. $\sum_{n=1}^{\infty} \cos n\pi$
2. $\sum_{n=1}^{\infty} (-1)^n n^2$
3. $\sum_{n=1}^{\infty} (-1)^n 2^n$
4. $\sum_{n=1}^{\infty} \frac{3n-1}{1+n+n^2}$
5. $\sum_{n=1}^{\infty} \frac{2^n}{n^2+n}$
6. $\sum_{n=1}^{\infty} n^{-(1+1/n)}$

Answer

1. $\cos n\pi = (-1)^{n+1} \not\to 0$, divergent; ($n$-term test)
2. $2^n \not\to 0$, divergent; (n-term test)
3. $(\frac{3n-1}{1+n+n^2})/(1/n) \to 3$, divergent (limit comparison test)
4. $(\frac{2^{n+1}}{(n+1)^2+n+1})/(\frac{2^n}{n^2+n}) \to 2 > 1$, divergent, (ratio test)
5. $n^{-(1+1/n)}/n^{-1} = n^{-1/n} \to 1$, divergent, (limit comparison test)

Taylor Series

As well-known:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \text{ for } |x| < 1$$

Problem

Can we expect any function, $f(x)$, to be expanded as above?

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where $a_0, a_1, a_2, \cdots$ are constants?

Defition Power function

A power series in x is a series on the form as follows:

$$\sum_{i=0}^{n} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Suppose that $f(x)$is smooth enough oven an open interval containing $x = a$ then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for any x within this interval. And this infinite series is called the Taylor series of $f(x)$ at $x = a$. If $a = 0$, then it is also called Maclaurin series of $f(x)$, i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Convergent intervals and Convergent radius

If $f(x) = \sum_{n=0} a_n (x-a)^n$ is convergent for $x \in (a-R, a+R)$, then $(a-R, a+R)$ is called the **convergent interval** and R is the radius of convergent.

R must be one of three cases:

$$R > 0, R = 0 \text{ or } R = +\infty.$$

Examples

Power Series	Convergent Interval	Convergent Radius
$\sum_{n=1}^{\infty} \frac{x^n}{n}$	$[-1, 1]$	1
$\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n n^2}$	$[-1, 5]$	5
$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{\sqrt{n+1}}$	$[-1/2, 1/2]$	$1/2$

Exercise p792

no.	Power Series -------	Convergent Interval	Convergent Radius
8	$\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$	R	∞
14	$\sum_{n=1}^{\infty} \sqrt{n}(2x+3)^n$	$(-2, -1)$	$1/2$
18	$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{(n^2+1)n^2}$	$[-4, 0)$	2
26	$\sum_{n=1}^{\infty} \frac{n^n(3x+5)^n}{(2n)!}$	$\left(-\frac{2/e-5}{3}, \frac{2/e-5}{3}\right)$	$3e/2$

Examples

1. $$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp \cdots + (\mp 1)^n x^n + \cdots = \sum_{n=0}^{\infty} (\mp 1)^n x^n \text{ for } |x| < 1$$

2. $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ for } |x^2| < 1 i.e. |x| < 1$$

3. Find the Taylor's series of e^x at $x = 0$ and $x = a$.

Solution

suppose that

$$
\begin{aligned}
e^x &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\
&\Downarrow \text{ let} x = 0 \\
1 = e^0 &= a_0 + a_1 \cdot 0 + a_2 \cdot 0 + \cdots \\
&\Downarrow \\
a_0 &= 1 \\
e^x = (e^x)' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots \\
&\Downarrow \text{ let} x = 0 \\
1 &= a_1 \\
e^x = (e^x)'' &= 2a_2 + 3 \cdot 2 \cdot a_3 x + \cdots + n(n-1) a_n x^{n-2} + \cdots \\
&\Downarrow \text{ let} x = 0 \\
1 &= 2a_2 \\
&\Downarrow \\
a_2 &= \frac{1}{2!} = \frac{1}{2} \\
e^x = (e^x)''' &= 3 \cdot 2 \cdot 1 \cdot a_3 + \cdots + n(n-1)(n-2) a_n x^{n-3} + \cdots \\
&\Downarrow \text{ let } x = 0 \\
a_3 &= \frac{1}{3!} \\
&\Downarrow \text{ by Mathematical Induction} \\
a_n &= \frac{1}{n!}
\end{aligned}
$$

These conclude:

$$
\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} x^n
\end{aligned}
$$

Since

$$
\begin{aligned}
e^x &= e^{a + (x-a)} = e^a e^{x-a} \\
&= e^a \sum_{n=0}^{\infty} \frac{1}{n!} (x-a)^n \\
&= \sum_{n=0}^{\infty} \frac{e^a}{n!} (x-a)^n
\end{aligned}
$$

Example

For $|x| < 1$:

$$
\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{2n(2n-2)!}{(n!2^n)^2} x^n
$$

Example Find the Maclaurin's series of $\ln(1+x)$.

Sol: Let $f(x) = \ln(1+x)$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$\frac{1}{1+x}$	1
2	$\frac{-1}{(1+x)^2}$	$(-1)^1 \cdot 1!$
3	$\frac{(-1)^2 2!}{(1+x)^3}$	$(-1)^2 \cdot 2!$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$	$(-1)^{n-1}(n-1)!$
$\vdots$	$\vdots$	$\vdots$

Then

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{n}$$

and $a_0 = \ln 1 = 0$, and

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

Example

For all $x \in \mathrm{R}$,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

The following function can be expanded as:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

only for $|x| < 1$. The expansion fails as $|x| \geq 1$, for example:

$$0 > -1 = \frac{1}{1-2} \neq 1 + 2 + 2^2 + \cdots > 1$$

The problem arises:

```
At what range would Taylor's series be convergent?
```

Convergence for Taylor's series

(with the help of Ratio test)

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, if

$$R = \operatorname{lim}n \to \infty \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n} \right| = \operatorname{lim}n \to \infty \left| \frac{a_{n+1}}{a_n} \right| |x-a| < 1 \Rightarrow \text{ series is convergent}$$

$$R = \operatorname{lim}n \to \infty \left| \frac{a_{n+1}}{a_n} \right| |x-a| > 1 \Rightarrow \text{ series is divergent}$$

$$R = \operatorname{lim}n \to \infty \left| \frac{a_{n+1}}{a_n} \right| |x-a| = 1 \Rightarrow \text{ no conclusion\}}$$

Question

Show

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} a_n$$

convergent for all $x \in \mathrm{R}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)!}x^{n+1}}{\frac{1}{n!}x^n}\right|$$

$$= \frac{|x|}{n+1}$$

$$\xrightarrow{n\to\infty} 0 < 1$$

Therefore,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

for any $x \in \mathrm{R}$.

Example, Show $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$.

Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^n}{n+1} x^{n+1}}{\frac{(-1)^{n-1}}{n} x^n}\right|$$

$$= \left|\frac{n}{n+1}\right| |x|$$

$$\xrightarrow{n\to\infty} |x|$$

then

$$|x| < 1 \Rightarrow \text{ convergent}$$
$$|x| > 1 \Rightarrow \text{ divergent}$$

and

1. for $x = 1$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 1^n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots \text{convergent}$$

2. for $x = -1$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-1)^n = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right) = -\infty$$

Then for $x \in (-1, 1]$, $\ln(1+x)$ is convergent and

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

Taking $x \to 1^-$ gets:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

Example, Find the convergent interval for

$$\sum_{n=0}^{\infty} n!x^n$$

Since

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= (n+1)|x| \\ &\to \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{if } x \neq 0 \end{cases}\end{aligned}$$

it is convergent only when $x = 0$, i.e.

$$\begin{aligned}\left.\sum_{n=0}^{\infty} n!x^n\right|_{x=0} &= 1 + x + 2!x^2 + 3!x^3 + 4!x^4|_{x=0} \\ &= 1 + 0 + 0 + \cdots = 1\end{aligned}$$

Theorem (Uniqueness) If

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=0}^{\infty} b_n(x-a)^n$$

then $a_n = b_n$ for for all n.

As usually known:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

Then (\color{brown}{let $x = 1/2$)}

$$2 = \frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots = 2$$

But it is **NOT** right $(x = 2)$:

$$-1 = \frac{1}{1-2} \overset{?}{=} 1 + 2 + 2^2 + 2^3 + \cdots = +\infty \longrightarrow \times \longleftarrow$$

This means that the infinite series is not convergent for every $x \in \mathrm{R}$!

Theorem

If $f(x) = \sum_n a_n (x-a)^n$ is convergent for $|x-a| < R$,then

1.

$$f'(x) = \sum^{\infty} {}_{n=0} (a_n (x-a)^n)' = \cdots = \sum^{\infty} {}_{n=0} (n+1) a_{n+1} (x-a)^n$$

is convergent for $|x-a| < R$.

2.

$$\int_a^x f(t)dt = \sum^{\infty} {}_{n=0} \int_a^x a_n (t-a)^n dt = \sum^{\infty} {}_{n=0} \frac{a_n}{n+1} (x-a)^{n+1}$$

is convergent for $|x-a| < R$.

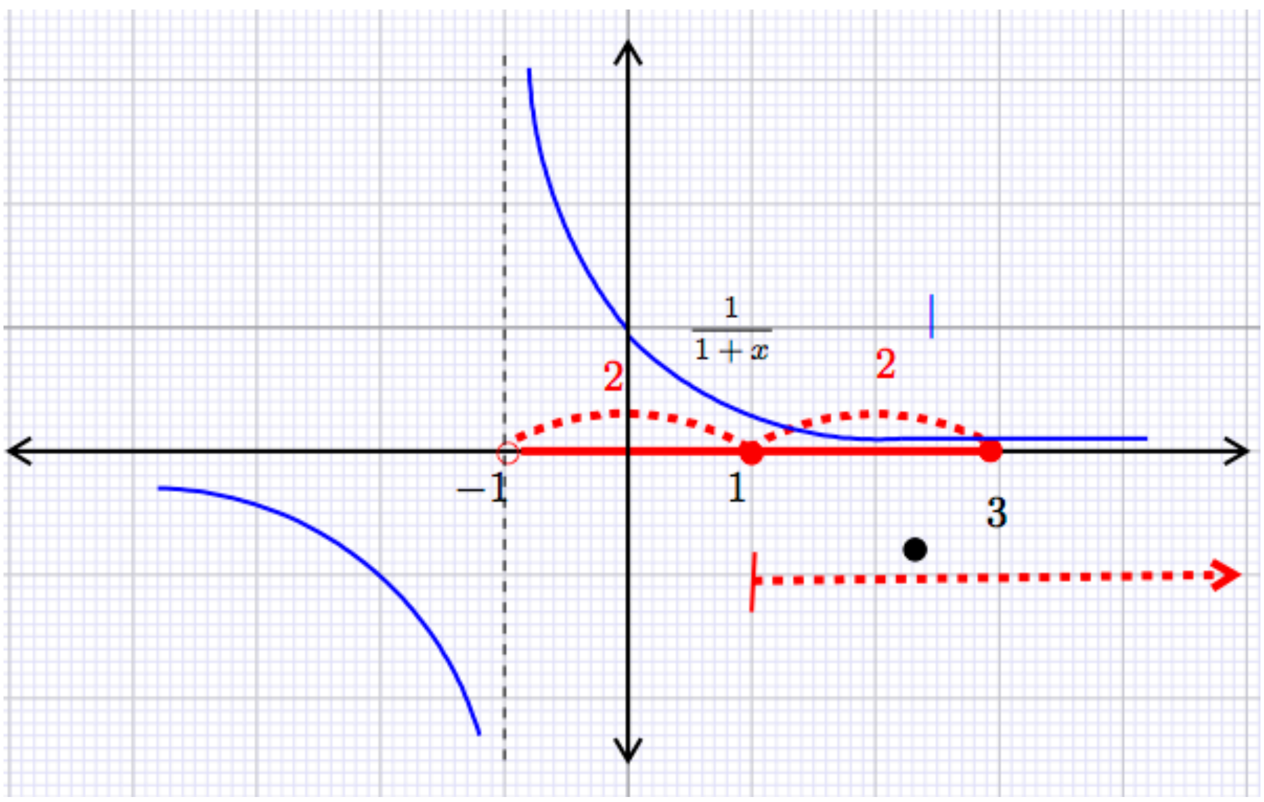
Example

1. $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \Rightarrow R = +\infty$
2. $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n \Rightarrow R = 1$
3. Find Taylor series of $\frac{1}{1+x}$ centered at $x = 1$ and its convergent interval

$$\begin{aligned}\frac{1}{1+x} &= \frac{1}{2+(x-1)} \\ &= \frac{1}{2\left[1+\left(\frac{x-1}{2}\right)\right]} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-1}{2}\right)^n\end{aligned}$$

for

$$\left|\frac{x-1}{2}\right| < 1 \Leftrightarrow -1 < x < 3$$



Exercise

From last result, guess the convergent value for the alternating series:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n-1}\frac{1}{n}+\cdots$$

Compare

$$\frac{(-1)^{n-1}}{n}x^n \text{and } (-1)^{n-1}\frac{1}{n}$$
$$\Downarrow$$
$$x =$$
$$\Downarrow$$
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n\bigg|_{x=} = \sum_{n=1}^{\infty}$$
$$\|$$
$$f(x)|_{x=} = \ln(1+x)|_{x=} = \ln$$

2. Find the Maclaurin's series of $\frac{x^2}{(1-x)}$. (Hint: $(1-x)^{-1}=1+x+x^2+\cdots$

$$\begin{aligned}(1-x)^{-1} &= 1+x+x^2+\cdots\\ &\Downarrow\\ \frac{x^2}{(1-x)} &= x^2\cdot\\ &= x^2\cdot(1+\quad+\quad+\cdots)\\ &= x^2+\quad+\quad+\cdots\\ &= \sum_{n=2}^{\infty}x^n\end{aligned}$$

Exercise

Find Taylor series of (a) $\frac{1}{2+x}$ (b) $\frac{1}{1+2x}$ centered at $x = 1$ and their convergent intervals. Also confirm your results with their graphs.

a).

$$
\begin{aligned}
\frac{1}{2+x} &= \frac{1}{\ + (x-1)} \\
&= \frac{1}{\ } \cdot \frac{1}{1+\left(\frac{x-1}{\ }\right)} \\
&= \frac{1}{\ } \cdot \sum_{n=0}^{\infty} {}^{n}(-)^{n} \\
&= \sum_{n=0}^{\infty} \ \cdot (x-1)^{n}
\end{aligned}
$$

b).

$$
\begin{aligned}
\frac{1}{1+2x} &= \frac{1}{\ + 2(x-1)} \\
&= \frac{1}{\ } \cdot \frac{1}{1+\left(\frac{x-1}{\ }\right)} \\
&= \frac{1}{\ } \cdot \sum_{n=0}^{\infty} {}^{n}(-)^{n} \\
&= \sum_{n=0}^{\infty} \ \cdot (x-1)^{n}
\end{aligned}
$$

Answer

a)

$$\begin{aligned}
\frac{1}{2+x} &= \frac{1}{3+(x-1)} \\
&= \frac{1}{3} \cdot \frac{1}{1+\left(\frac{x-1}{3}\right)} \\
&= \frac{1}{3} \cdot \sum_{n=0}^{\infty} -1^n \left(\frac{x-1}{3}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} \cdot (x-1)^n
\end{aligned}$$

b).

$$\begin{aligned}
\frac{1}{1+2x} &= \frac{1}{3+2(x-1)} \\
&= \frac{1}{3} \cdot \frac{1}{1+\left(\frac{x-1}{3/2}\right)} \\
&= \frac{1}{3} \cdot \sum_{n=0}^{\infty} -1^n \left(\frac{x-1}{3/2}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(-2)^n}{3^{n+1}} \cdot (x-1)^n
\end{aligned}$$

Example

Replacing x with x^2 in the geometric series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \, |x| < 1$$

gets

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \, |x^2| < 1 (\Leftrightarrow |x| < 1)$$

And integrating both sides gets

$$\begin{aligned} \int_0^x \frac{dt}{1+t^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \\ \Longrightarrow \tan^{-1} x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \end{aligned}$$

This equation holds for $|x^2| < 1$, i.e. $|x| < 1$. Specially, as $x = 1$,

$$\begin{aligned} \frac{\pi}{4} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \end{aligned}$$

Example

Find the exact form of $f(t) = \sum_{n=0}^{\infty} \frac{e^{tn} e^{-\lambda} \lambda^n}{n!}$, $f'(t)$ and so $f'(0)$.

It is trivial that

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^n e^{tn}}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= e^{\lambda(e^t - 1)} \\ &\Downarrow \\ f'(0) &= e^{\lambda(e^t-1)} e^t \lambda |_{t=0} = \lambda \end{aligned}$$

Example

Find the Maclaurin's series of $(1-x)^{-2}$ and $(1-x)^{-3}$.

Differentiating the both sides :

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

gets

$$\begin{aligned} \frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1)x^n \\ &\Downarrow \end{aligned}$$

$$\left(\frac{1}{1-x}\right)'' = \frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} (n+2)(n+1)x^n$$

Exercise

As the last example, find the exact form of $f''(t)$ and so $f''(0)$:

$$\begin{aligned} f''(t) &= \left(e^{\lambda(e^t-1)}\right)'' \\ &= \left(\quad \cdot e \quad \cdot e^{\lambda(e^t-1)} \right)' \\ &= \quad \cdot e \quad \cdot e^{\lambda(e^t-1)} + \quad \cdot e \quad \cdot e^{\lambda(e^t-1)} \\ &\Downarrow \\ f''(0) &= \lambda + \lambda^2 \end{aligned}$$

$$\begin{aligned} f''(t) &= \left(e^{\lambda(e^t-1)}\right)'' \\ &= \left(\lambda \cdot e^t \cdot e^{\lambda(e^t-1)}\right)' \\ &= \lambda \cdot e^t \cdot e^{\lambda(e^t-1)} + \lambda^2 \cdot e^{2t} \cdot e^{\lambda(e^t-1)} \\ &\Downarrow \\ f''(0) &= \lambda + \lambda^2 \end{aligned}$$

Exercise

Suppose that $M(t) = \sum_{n=0}^{\infty} \frac{t^n e^{-\lambda} \lambda^n}{n!}$, find$M'(t)$ and so $M'(1)$.

$$\begin{aligned}
M(t) &= \sum_{n=0}^{\infty} \frac{t^n e^{-\lambda} \lambda^n}{n!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{n}{n!} \\
\left(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) &= e^{-\lambda} \cdot e \\
&= e \\
M'(1) &= M'(t)|_{t=1} \\
&= e\,|_{t=1} \\
&=
\end{aligned}$$

$$\begin{aligned}
M(t) &= \sum_{n=0}^{\infty} \frac{t^n e^{-\lambda} \lambda^n}{n!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{t\lambda^n}{n!} \\
\left(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) &= e^{-\lambda} \cdot e^{\lambda t} \\
&= e^{\lambda(t-1)} \\
M'(1) &= M'(t)|_{t=1} \\
&= \lambda e^{\lambda(t-1)} \Big|_{t=1} \\
&= \lambda
\end{aligned}$$

Example

Find Maclaurin Series of $\sin x$

We have

$$\begin{aligned}
(\sin x)^{(4k)}|_{x=0} &= \sin x|_{x=0} = 0 \\
(\sin x)^{(4k+1)}|_{x=0} &= \cos x|_{x=0} = 1 \\
(\sin x)^{(4k+2)}|_{x=0} &= -\sin x|_{x=0} = 0 \\
(\sin x)^{(4k+3)}|_{x=0} &= -\cos x|_{x=0} = -1
\end{aligned}$$

implies

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!} x^n \\
&= \sum_{k=0}^{\infty} \left[\frac{\sin^{(4k)}(0)}{(4k)!} x^{4k} + \frac{\sin^{(4k+1)}(0)}{(4k+1)!} x^{4k+1} + \frac{\sin^{(4k+2)}(0)}{(4k+2)!} x^{4k+2} + \frac{\sin^{(4k+3)}(0)}{(4k+3)!} x^{4k+3} \right] \\
&= \sum_{k=0}^{\infty} \left[\frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!} \right] \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots
\end{aligned}$$

Exercise

1.

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{2x}{1-x} &= 2x\sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} x^{n+1} \\
&= \sum_{n=1}^{\infty} x^n \\
\frac{x}{1-3x} &= x \cdot \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} x^{n+1} \\
&= \sum_{n=1}^{\infty} x^n \text{ convergent for } x \in
\end{aligned}$$

2.

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} (n+1)x^n \\
\frac{x}{(1-x)^2} &= \sum_{n=1}^{\infty} n x^n \\
\frac{x}{(1-x^2)^2} &= \sum_{n=1}^{\infty} n x^{2n-1} \\
\int_0^x \frac{t}{(1-t^2)^2} dt &= \frac{x^2}{2(1-x^2)} \\
&= \sum_{n=1}^{\infty} \frac{1}{2} x^{2n}
\end{aligned}$$

Answer

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{2x}{1-x} &= 2x\sum_{n=0}^{\infty} 1\,x^n \\
&= \sum_{n=0}^{\infty} 2\,x^{n+1} \\
&= \sum_{n=1}^{\infty} 2\,x^n \\
\frac{x}{1-3x} &= x \cdot \sum_{n=0}^{\infty} 3^n x^n \\
&= \sum_{n=0}^{\infty} 3^n x^{n+1} \\
&= \sum_{n=1}^{\infty} 3^{n+1} x^n \text{ convergent for } x \in (-1/3,\, 1/3)
\end{aligned}$$

Exercise

As the reason, we also have

$$\cos x = \sum_{n=0}^{\infty} a_{2n} x^{2n}$$

$$\begin{aligned}
(\cos x)^{(4k)}|_{x=0} &= \cos x|_{x=0} = \\
(\cos x)^{(4k+1)}|_{x=0} &= -\sin x|_{x=0} = \\
(\cos x)^{(4k+2)}|_{x=0} &= -\cos x|_{x=0} = \\
(\cos x)^{(4k+3)}|_{x=0} &= \sin x|_{x=0} =
\end{aligned}$$

$$\Rightarrow \begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{\cos^{(n)}(0)}{n!} x^n \\
&= \sum_{k=0}^{\infty} \left[\frac{\cos^{(4k)}(0)}{(4k)!} x^{4k} + \frac{\cos^{(4k+1)}(0)}{(4k+1)!} x^{4k+1} + \frac{\cos^{(4k+2)}(0)}{(4k+2)!} x^{4k+2} + \frac{\cos^{(4k+3)}(0)}{(4k+3)!} x^{4k+3} \right] \\
&= \sum_{k=0}^{\infty} \left[x^{4k} - x^{4k+2} \right] \\
&= \sum_{n=0}^{\infty} x^{2n}
\end{aligned}$$

$$\begin{aligned}
(\cos x)^{(4k)}|_{x=0} &= \cos x|_{x=0} = 1 \\
(\cos x)^{(4k+1)}|_{x=0} &= -\sin x|_{x=0} = 0 \\
(\cos x)^{(4k+2)}|_{x=0} &= -\cos x|_{x=0} = -1 \\
(\cos x)^{(4k+3)}|_{x=0} &= \sin x|_{x=0} = 0
\end{aligned}$$

$$\Rightarrow \begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{\cos^{(n)}(0)}{n!} x^n \\
&= \sum_{k=0}^{\infty} \left[\frac{\cos^{(4k)}(0)}{(4k)!} x^{4k} + \frac{\cos^{(4k+1)}(0)}{(4k+1)!} x^{4k+1} + \frac{\cos^{(4k+2)}(0)}{(4k+2)!} x^{4k+2} + \frac{\cos^{(4k+3)}(0)}{(4k+3)!} x^{4k+3} \right] \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{(4k)!} x^{4k} - \frac{1}{(4k+2)!} x^{4k+2} \right] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2k)!} x^{2n}
\end{aligned}$$

Proposition

(Binomial series) If $r \neq 0, 1, 2, 3, \cdots$, and $r \in R$, then

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$

where $\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}$

Example

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \begin{pmatrix} \frac{1}{2} \\ n \end{pmatrix} x^n$$

where

$$\begin{pmatrix} \frac{1}{2} \\ n \end{pmatrix} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!}.$$

The coefficient can be simplied as follows:

$$\begin{aligned}
\begin{pmatrix} \frac{1}{2} \\ n \end{pmatrix} &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!} \\
&= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-3}{2})}{n!} \\
&= (-1)^{n-1}\frac{1\cdot 3\cdot \cdots \cdot (2n-3)}{2^n n!} \\
&= (-1)^{n-1}\frac{1\cdot 3\cdot \cdots \cdot (2n-3)}{2^n n!}\cdot\frac{2\cdot 4\cdot \cdots \cdot (2n-2)}{2\cdot 4\cdot \cdots \cdot (2n-2)} \\
&= (-1)^{n-1}\frac{(2n-2)!}{2^n n!}\cdot\frac{n}{2^{n-1}\cdot 1\cdot 2\cdot \cdots \cdot (n-1)\cdot n} \\
&= (-1)^{n-1}\frac{n(2n-2)!}{2^{2n-1}(n!)^2}
\end{aligned}$$

Such that

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} (-1)^{n-1}\frac{n(2n-2)!}{2^{2n-1}(n!)^2}x^n$$

Example

Integrating both sides of

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n \text{ for } |x| < 1$$

gets

$$\begin{aligned}
\ln|1+x| &= \int_0^x \frac{1}{1+t}dt \\
&= \int_0^x \sum_{n=0}^{\infty}(-1)^n t^n dt \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{t^{n+1}}{n+1}\Bigg|_0^x \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1} x^n}{n}
\end{aligned}$$

Exercise

Find the binomal series of $(1+x)^{-1/2}$ and also simply the binomial coefficients.

$$\begin{aligned}
\frac{1}{(1+x)^{-1/2}} &= \sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}x^m \\
&= \sum_{m=0}^{\infty}\frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-m+1)}{m!}x^m \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{1\cdot 3\cdot \cdots \cdot(2m-1)}{2^m m!}x^m \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{1\cdot 3\cdot \cdots \cdot(2m-1)}{2^m m!}\cdot\frac{2\cdot 4\cdot \cdots \cdot 2m}{2\cdot 4\cdot \cdots \cdot 2m}x^m \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{(2m)!}{2^{2m}(m!)^2}x^m
\end{aligned}$$

Exercise, p.805

Find the Taylor series of $f(x)$ below and its convergent radius

4. $f(x) = e^{-2x}, c = 3$

$$
\begin{aligned}
f(x) &= e^{-4}e^{-2(x-2)} \\
&= \sum_{n=0}^{\infty} \frac{e^{-4}(-2)^n}{n!}(x-2)^n
\end{aligned}
$$

where $x \in \mathrm{R}$, i.e. convergent radius: ∞.

14. $f(x) = \frac{1}{1+3x}$ at $c = 2$

$$
\begin{aligned}
f(x) &= \frac{1}{1+3x} = \frac{1}{7}\frac{1}{1+3(x-2)/7} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{7^{n+1}}(x-2)^n
\end{aligned}
$$

where $|3(x-2)/7| < 1, \left(|x-2| < \frac{7}{3}\right)$, i.e. convergent radius: $7/3$.

22. $f(x) = \sin^2 x = (1 - \cos 2x)/2$ at $c = 0$

$$
\begin{aligned}
f(x) &= (1 - \cos 2x)/2 \\
&= \frac{1}{2} - \sum_{n=0}^{\infty} \frac{(-1)^{-n}2^{2n}}{2n!}x^{2n}
\end{aligned}
$$

where $x \in \mathrm{R}$, i.e. convergent radius: ∞.

32. Find the convergent radius of the following binomial sequence:

$$
f(x) = \frac{1}{\sqrt[3]{8+x}} = \frac{1}{2\sqrt[3]{1+x/8}} = \frac{1}{2}\left(1+\frac{x}{8}\right)^{-1/3}
$$

is a binomial series which is convergent for $\left|\frac{x}{8}\right| < 1, (|x| < 8)$, i.e. convergent interval is 8.

50. The power series of

$$
\int \frac{\sin x}{x}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\int \frac{x^{2n+1}}{x}dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}x^{2n+1}
$$

60. Find the sum of the following series:

$$
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!2^{2n+1}}\pi^{2n+1} = \sin\frac{\pi}{2} = 1
$$

66. Evaluate the following limit:

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}}{x-1} = 1$$

Fourier Series

Suppose that $f(t)$ is piecewise smooth and continuous, periodic with period T where $T = t_2 - t_1$ and

$$\int_{t_1}^{t_2}|f(t)|^2 dt < \infty$$

Then the Fourier series expansion of $f(t)$ is:

$$\mathbf{f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(\omega_n t) + b_n\sin(\omega_n t))}$$

where

- $\omega_\mathbf{n} = \mathbf{\frac{2n\pi}{T}, n = 0, 1, 2, \cdots}$
- $\mathbf{a_n = \frac{2}{T}\int_{t_1}^{t_2}f(t)\cos(\omega_n t)dt}$
- $\mathbf{b_n = \frac{2}{T}\int_{t_1}^{t_2}f(t)\sin(\omega_n t)dt}$

Note

1. For $m \neq n$,

$$\int_0^T \sin\omega_n t\sin\omega_m t dt = \int_0^T \cos\omega_n t\cos\omega_m t dt = 0$$

And $\int_0^T \sin\omega_n t\cos\omega_m t dt = 0$ for any $m, n \in \mathrm{N}$.

Example

For $m = 2, n = 3$ and $T = 2\pi$

$$\int_0^{2\pi}\cos\frac{2\cdot 2\cdot \pi t}{2\pi}\cos\frac{2\cdot 3\cdot \pi t}{2\pi}dt = \frac{1}{2}\int_0^{2\pi}(\cos 5t + \cos t)dt = 0$$

2. For $m = n$,

$$\int_0^T \sin\omega_n t \sin\omega_n t dt = \int_0^T \cos\omega_n t \cos\omega_n t dt = \frac{T}{2}$$

Example

For $m = 2, n = 2$ and $T = 2\pi$

$$\begin{aligned}\int_0^T \cos\frac{2\cdot 2\cdot \pi t}{2\pi}\cos\frac{2\cdot 2\cdot \pi t}{2\pi}dt &= \int_0^{2\pi}\left(\cos^2 2t\right)dt \\ &= \frac{1}{2}\int_0^{2\pi}(1+\cos 4t)dt = \pi\end{aligned}$$

3. Coefficients of Fourier series could be gotten by integration as follows:

- To get the value of a_n, integrate both sides of $f(t)$ and its Fourier series with product with $\cos\omega_n$ as follows:

$$\begin{aligned}\int_0^T f(t)\cos(\omega_n t)dt &= \int_0^T\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(\omega_n t) + b_n\sin(\omega_n t))\right)\cos(\omega_n t)dt \\ &= a_n\frac{T}{2} \\ \Rightarrow a_n &= \frac{2}{T}\int_0^T f(t)\cos(\omega_n t)dt\end{aligned}$$

As the same procedure, we also have:

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(\omega_n t)dt$$

Example

Consider

$$
\begin{aligned}
f(x) &= x \text{ for } -\pi < x < \pi \\
f(x+2\pi) &= f(x) \text{ for } x \in \mathrm{R}
\end{aligned}
$$

reference the following picture ouptut.

Then $T = 2\pi$

$$
\begin{aligned}
a_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} x\cos(nx)dx \\
&= 0 \text{ (since odd)} \\
b_n &= \frac{\cancel{2}}{\cancel{2}\pi}\int_{-\pi}^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\left(\frac{-x\cos nx}{n}\Bigg|_0^{\pi} + \int_0^{\pi}\frac{\cos n\pi}{n}dx\right) \\
&= 2\frac{(-1)^{n+1}}{n}
\end{aligned}
$$

And

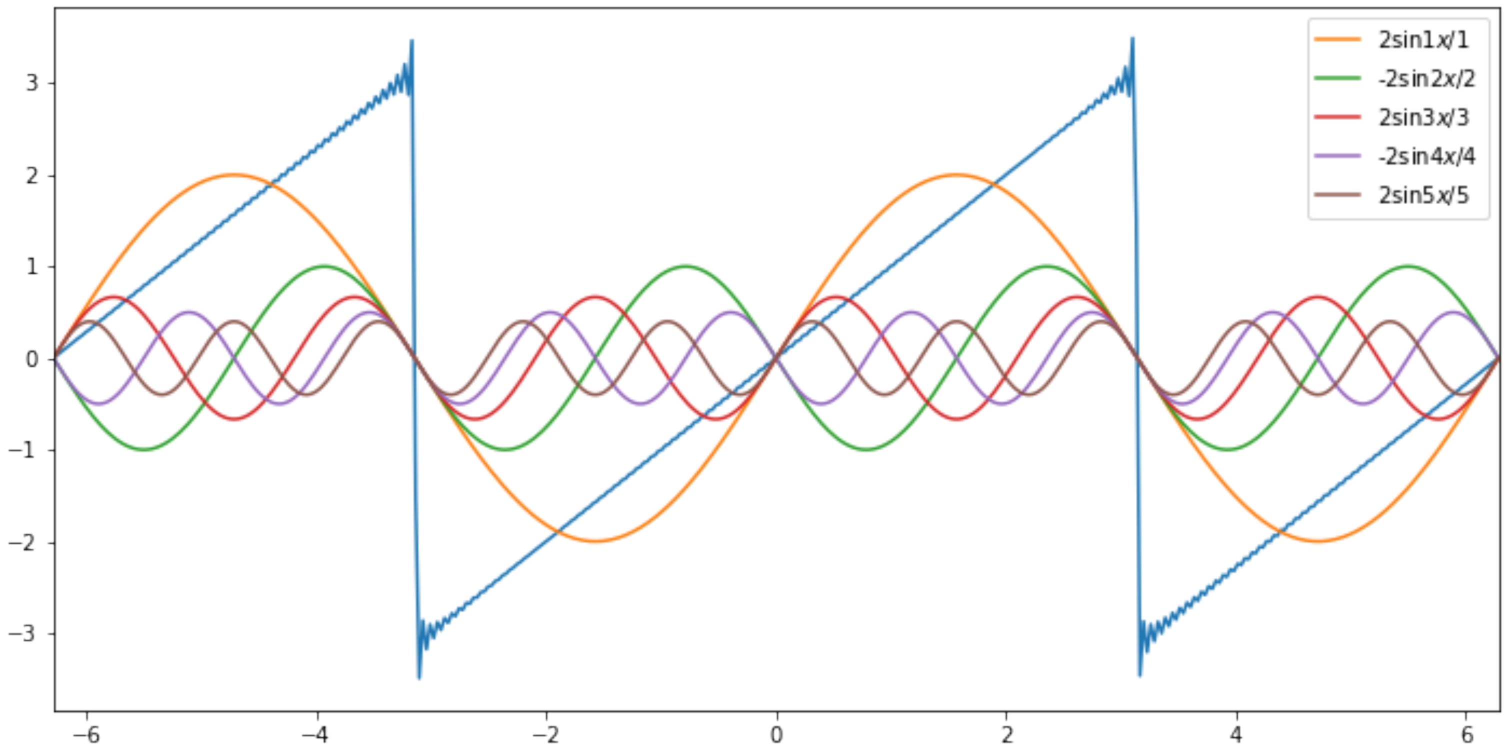
$$
\begin{aligned}
f(x) &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) \\
&= 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right)
\end{aligned}
$$

In [4]:

```
def sawf(x,n):
    f=np.zeros(len(x))
    for i in range(1,n+1):
        f+=2*(-1)**(i+1)*sin(i*x)/i
    return f
```

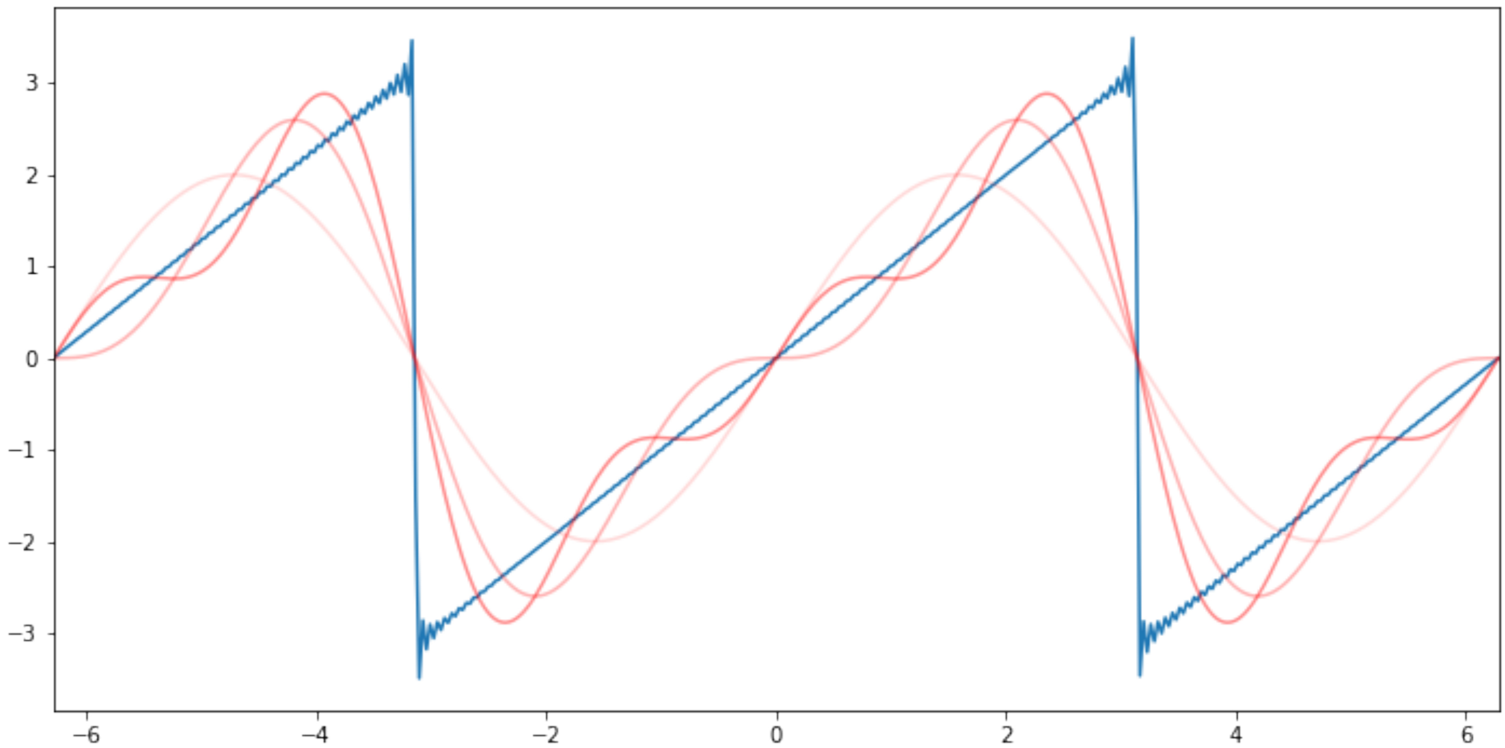
In [6]:

```
from numpy import pi,sin,cos
x=np.linspace(-2*pi,2*pi,400)
plt.figure(figsize=(12,6))
plt.xlim([-2*pi,2*pi])

plt.plot(x,sawf(x,100))
for i in range(1,6):
    plt.plot(x,2*(-1)**(i-1)*sin(i*x)/i,label="%s$\sin$%s$x$/%s" %((-1)**(i+1)*2,
plt.legend();
```



In [14]:

```
from numpy import pi,sin,cos
x=np.linspace(-2*pi,2*pi,400)
plt.figure(figsize=(12,6))
plt.xlim([-2*pi,2*pi])

plt.plot(x,sawf(x,100))
f=np.zeros(len(x))
for i in range(1,4):
    f=f+2*(-1)**(i-1)*sin(i*x)/i
    plt.plot(x,f,color="red",alpha=0.15*i)
```



Example

(Square Wave)

$$f(x) = \begin{cases} 1 & \text{for} 0 \leqslant x < \pi \\ -1 & \text{for} -\pi \leqslant x < 0 \end{cases}$$
$$f(x+2\pi) = f(x) \text{ for } x \in \mathrm{R}$$

reference the following output picture.

Then ($T = 2\pi$)

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \cos(nx)dx \\ &= \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)dx - \frac{1}{\pi}\int_{-\pi}^{0} 1 \cdot \cos(nx)dx \\ &= 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin(nx)dx \\ &= \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)dx - \frac{1}{\pi}\int_{-\pi}^{0}\sin(nx)dx \\ &= \frac{4}{n\pi} \text{ if } n \text{ is odd.} \end{aligned}$$

This implies

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(nx)\left(1-(-1)^n\right) \\ &= \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right) \end{aligned}$$

In [15]:

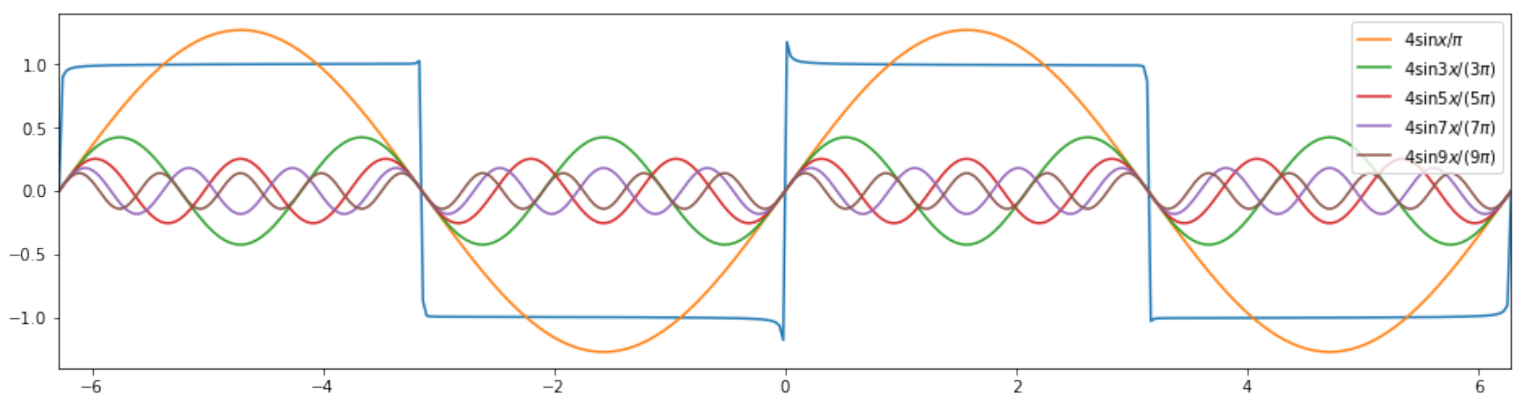
```
def squaref(x,n):
    f=np.zeros(len(x))
    for i in range(1,n):
        f+=4*sin((2*i-1)*x)/(2*i-1)/pi
    return f
```

In [16]:

```python
from numpy import pi,sin,cos
x=np.linspace(-2*pi,2*pi,400)
plt.figure(figsize=(16,4))
plt.xlim([-2*pi,2*pi])

plt.plot(x,squaref(x,100))

plt.plot(x,4*sin(x)/pi,label="4sin$x$/$\pi$")
for i in range(2,6):
    plt.plot(x,4*sin((2*i-1)*x)/(2*i-1)/pi,label="4sin%s$x$/(%s$\pi$)" %(2*i-1,2*
plt.legend()
```

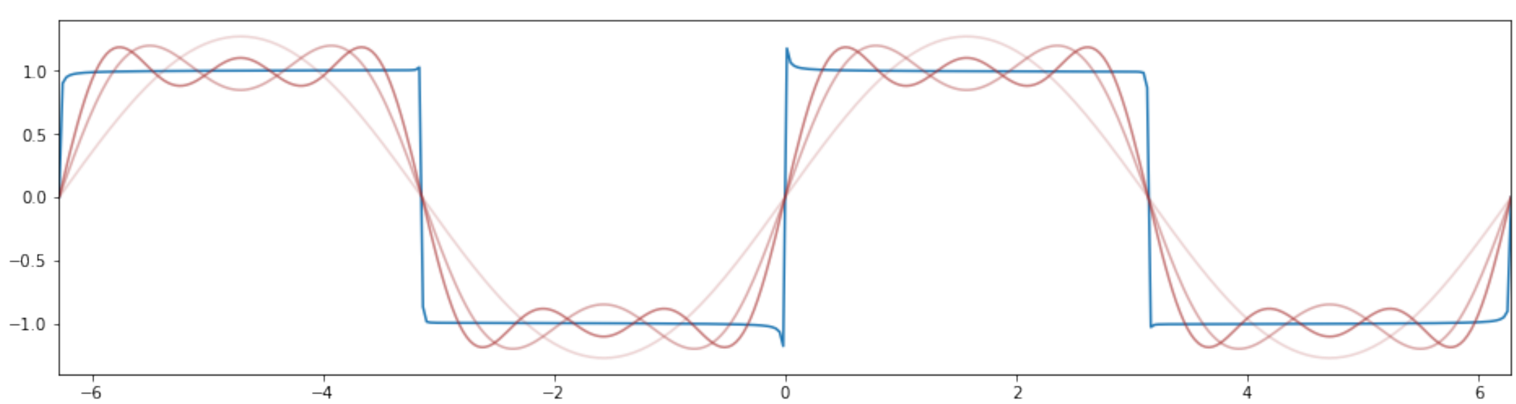
Out[16]:

```
<matplotlib.legend.Legend at 0x10c113400>
```



In [21]:

```python
from numpy import pi,sin,cos
x=np.linspace(-2*pi,2*pi,400)
plt.figure(figsize=(16,4))
plt.xlim([-2*pi,2*pi])

plt.plot(x,squaref(x,100))

f=np.zeros(len(x))
for i in range(1,4):
    f=f+4*sin((2*i-1)*x)/(2*i-1)/pi
    plt.plot(x,f,color="brown",alpha=0.2*i)
```



In [ ]:

Example (Sum of series with term, $1/n^2$)

Suppose that

$$f(x) = \begin{cases} x & \text{for} 0 \leqslant x < \pi \\ -x & \text{for } -\pi \leqslant x < 0 \end{cases}$$

$$f(x+2\pi) = f(x) \text{ for } x \in \mathrm{R}$$

Since $f(x)$ is even, $b_n = 0$ and

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{2}{\pi}\int_0^{\pi} xdx = \pi \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\cdot\cos(nx)dx \\ &= \frac{2}{n^2\pi}\left((-1)^n - 1\right) \end{aligned}$$

Therefore

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)\right)$$

By the way, we can can calculate the value of

$$L = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

Take the value of $f(x)$ at $x = 0$,

$$\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi^2}{8}$$

$$\begin{aligned} L &= \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) \\ &= \frac{\pi^2}{8} + \frac{L}{4} \\ &\Downarrow \\ L &= \frac{\pi^2}{6} \end{aligned}$$

Example (Function neither Even nor Odd with Period T)

With $T \in [-2, 2]$, the function is defined as follows:

$$f(x) = \begin{cases} 0 & \text{for } -2 \leqslant x < 0 \\ x & \text{for } 0 < x \leqslant 2 \end{cases}$$
$$f(x+4) = f(x) \text{ for } x \in \mathrm{R}$$

Then period is $(T = 4)$, and

$$f(x) = \frac{a_0}{2} + \sum_{i=1} a_n \cos\omega_n x + b_n \sin\omega x$$

- $\frac{2}{T} = \frac{2}{4} = \frac{1}{2}$,
- $\omega_n = 2n\pi/T = n\pi/2$

$$\begin{aligned}
a_0 &= \frac{2}{T}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_0^2 x dx = 1 \\
a_n &= \frac{2}{T}\int_{-2}^{2} f(x)\cos\omega_n x dx \\
&= \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx = \frac{1}{2}\cdot\left(\frac{2}{n\pi}\right)\left(\cancel{x}\sin\frac{n\pi x}{2}\Big|_0^2 - \int_0^2 \sin\frac{n\pi x}{2}dx\right) \\
&= \frac{1}{2}\left(\frac{2}{n\pi}\right)^2\left((-1)^n - 1\right) \\
b_n &= \frac{2}{n\pi}(-1)^{n+1}
\end{aligned}$$

And

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}\left(\frac{2}{n\pi}\right)^2((-1)^n - 1)\cos\frac{n\pi x}{2} + \frac{2}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{2}\right]$$

In [38]:

```
import sympy as sp
from sympy import Symbol,symbols,integrate,pi,sin,cos
```

In [45]:

```
n,x =symbols("n x")

integrate(x*cos(n*pi*x/2)/2,(x,0,2))
```

Out[45]:

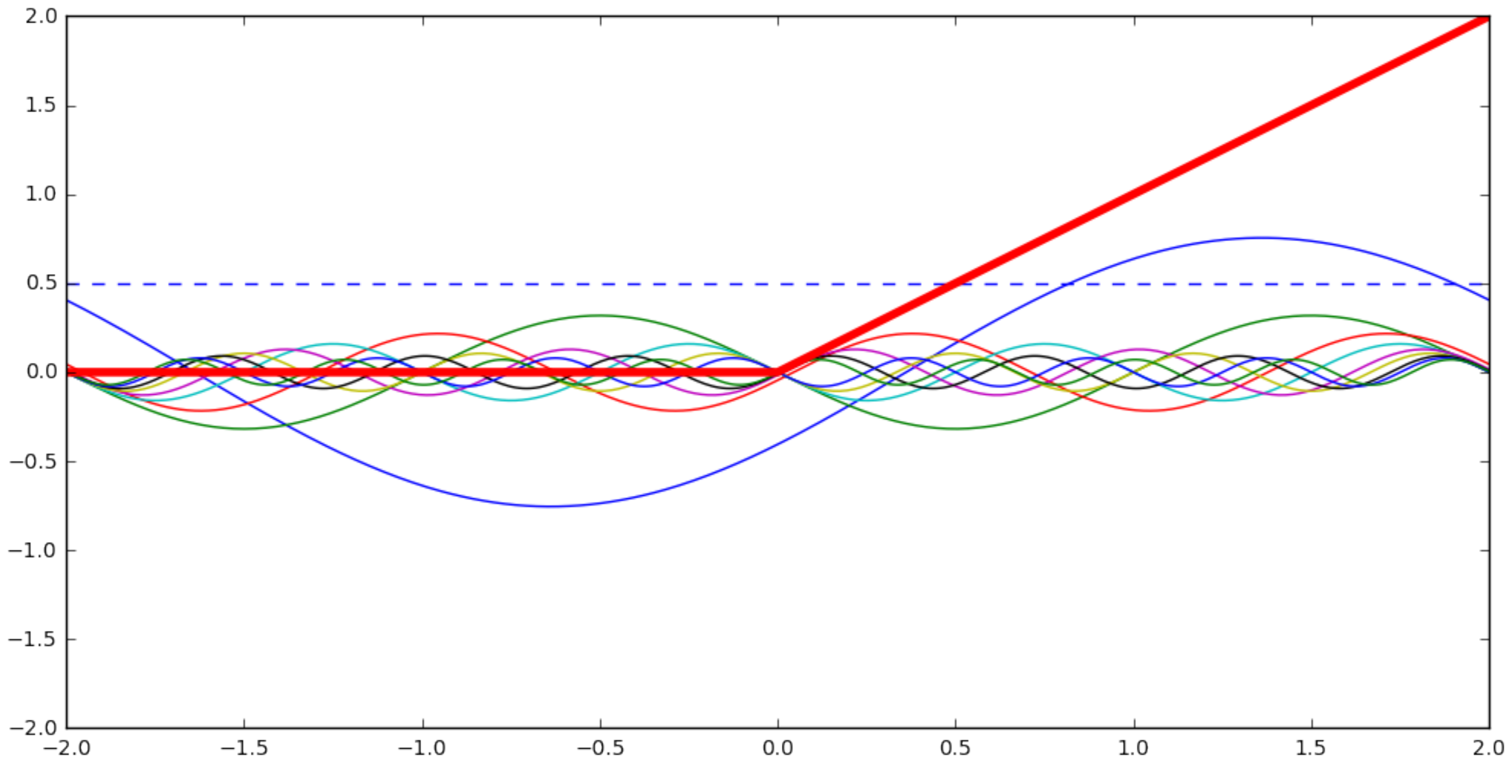
```
-Piecewise((0, Eq(n, 0)), (4/(pi**2*n**2), True))/2 + Piecewise((2, Eq(n, 0)), (4*sin(pi*n)/(pi*n) + 4*cos(pi*n)/(pi**2*n**2), True))/2
```

In [56]:

```
from numpy import pi,sin,cos
t=np.linspace(-2,2,400)
plt.figure(figsize=(12,6))
plt.xlim([-2,2])
plt.ylim([-2,2])
n=31
f=np.ones(len(t))/2
plt.plot(t,f,'b--')
for i in np.arange(1,10):
    f=((2/i/pi)**2/2*((-1)**i-1)*cos(i*pi*t/2)+2/i/pi*(-1)**(i+1)*sin(i*pi*t/2))
    plt.plot(t,f)
plt.plot([-2,0,2],[0,0,2],'r',lw=4)
```

Out[56]:

```
[<matplotlib.lines.Line2D at 0x1122e56d8>]
```

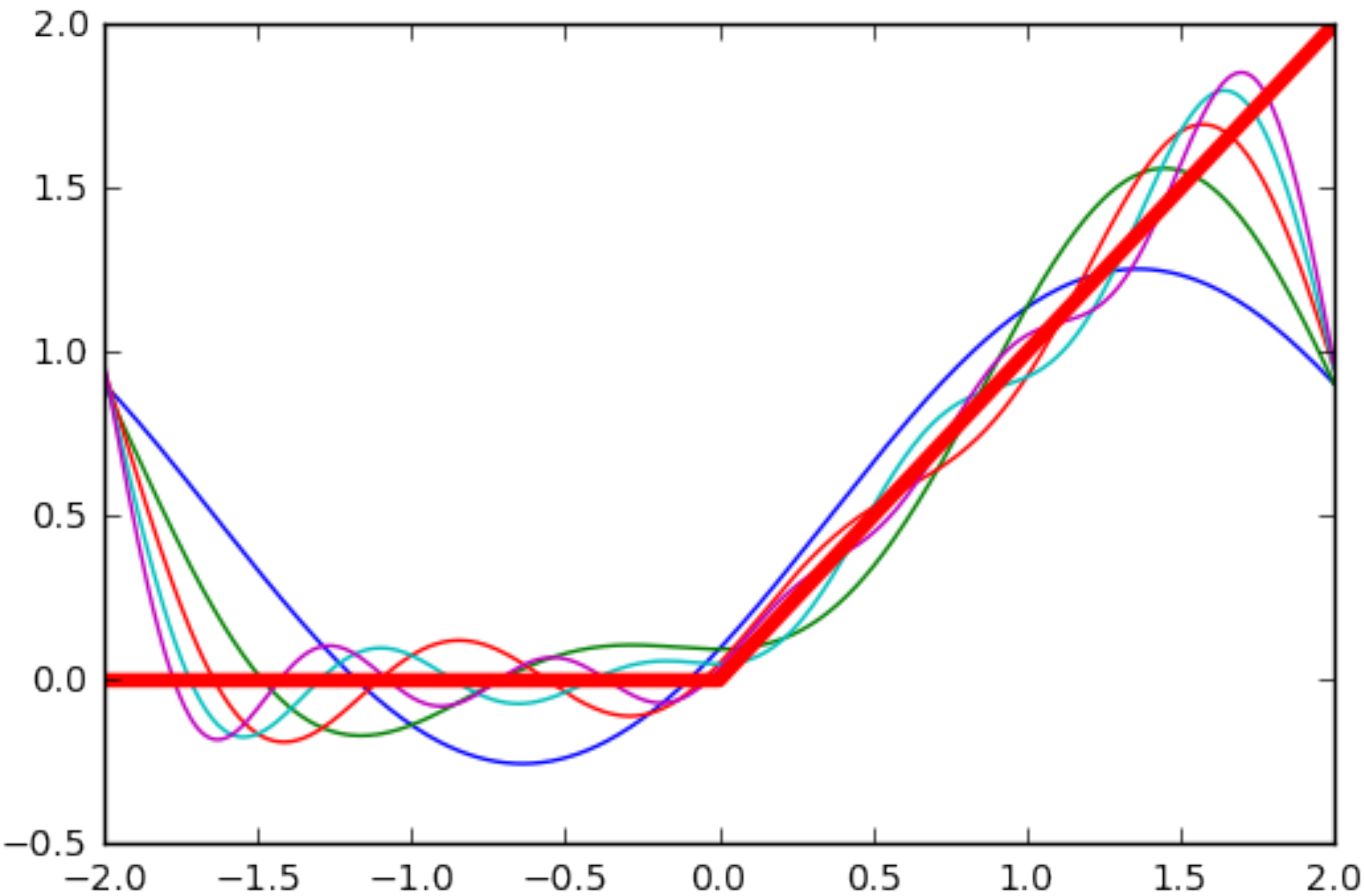


In [54]:

```
f=np.ones(len(t))/2
for i in np.arange(1,6):
    f+=1/1*((2/i/pi)**2/2*((-1)**i-1)*cos(i*pi*t/2)+2/i/pi*(-1)**(i+1)*sin(i*pi*t
    plt.plot(t,f)
plt.plot([-2,0,2],[0,0,2],'r',lw=4)
```

Out[54]:

```
[<matplotlib.lines.Line2D at 0x112028128>]
```

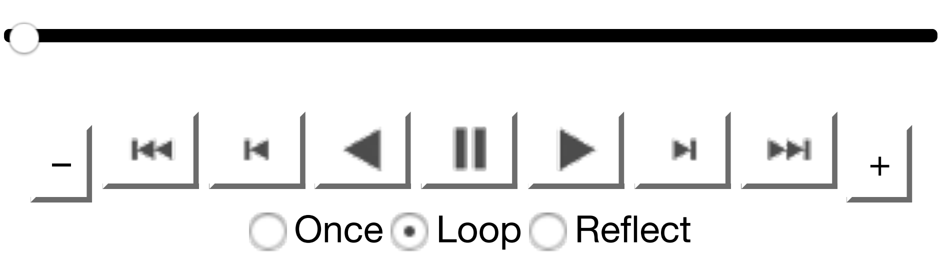


Jupyter provides functions capable of animating the data, for instance JSAnimation, pymovie.py, which could

In [101]:

```
from matplotlib import animation
from JSAnimation import IPython_display
from numpy import sin,cos,pi

fig = plt.figure(figsize=(4,4))
ax = plt.axes(xlim=(-2, 2), ylim=(-2, 2))
line, = ax.plot([], [], lw=2)
plt.title('General Fourier Series')
x = np.linspace(-2., 2., 400)
#plt.text(1,1,'$x I_{0<x<2}$')

def fn(n):
    f=np.ones(len(x))/2.
    for i in range(1,n+1):
        f+=((2/i/pi)**2/2*((-1)**i-1)*cos(i*pi*x/2)+2/i/pi*(-1)**(i+1)*sin(i*pi*x
    return f

def init():
    line.set_data([], [])
    return line,
def animate(i):
    y=fn(i)
    line.set_data(x, y)
    return line,

animation.FuncAnimation(fig, animate, init_func=init, frames=60, interval=30)
```

Out[101]:

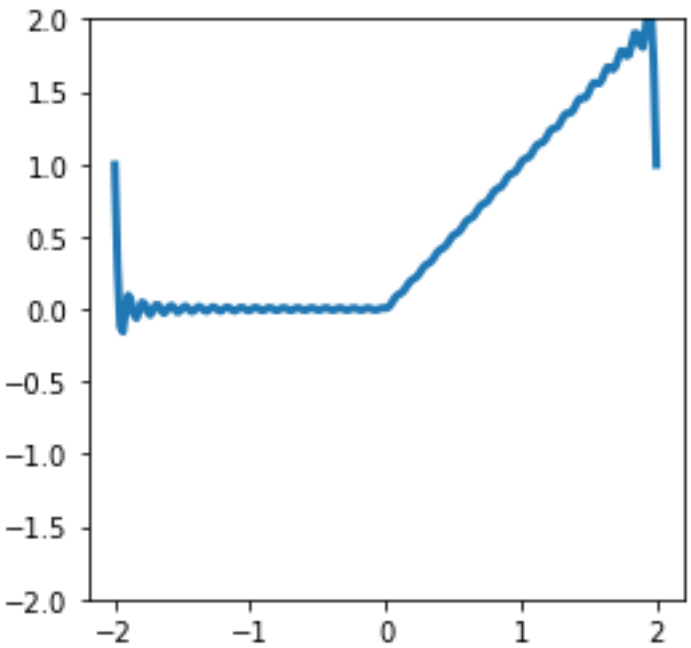


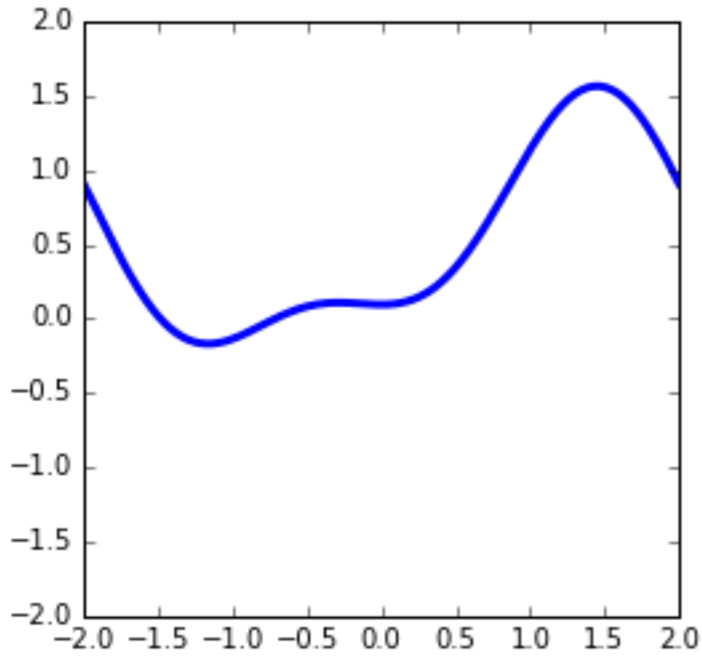
In [5]:

```
from moviepy.video.io.bindings import mplfig_to_npimage
import moviepy.editor as mpy
from numpy import pi,cos,sin

# DRAW A FIGURE WITH MATPLOTLIB

fig, ax = plt.subplots(1,figsize=(4,4), facecolor='white')
x = np.linspace(-2,2,200) # the x vector

def fn(t):
    n=int(40*t)
    f=np.ones(len(x))/2.
    for i in range(1,n+1):
        f+=((2/i/pi)**2/2*((-1)**i-1)*cos(i*pi*x/2)+2/i/pi*(-1)**(i+1)*sin(i*pi*x
    return f

ax.set_ylim(-2,2)
```

```
line, = ax.plot(x, fn(0), lw=3)
duration=1

# ANIMATE WITH MOVIEPY (UPDATE THE CURVE FOR EACH t). MAKE A GIF.

def make_frame(t):
    # defaulted time 1 second/40 frames
    line.set_ydata( fn(t))  # <= Update the curve
    return mplfig_to_npimage(fig) # RGB image of the figure

animation =mpy.VideoClip(make_frame, duration=duration)
animation.write_gif("FourierSeries.gif", fps=20)
```

```
[MoviePy] Building file FourierSeries.gif with imageio

  0%|          | 0/21 [00:00<?, ?it/s]
 14%|█         | 3/21 [00:00<00:00, 26.39it/s]
 29%|██        | 6/21 [00:00<00:00, 25.92it/s]
 43%|███       | 9/21 [00:00<00:00, 25.53it/s]
 57%|████      | 12/21 [00:00<00:00, 25.33it/s]
 71%|█████     | 15/21 [00:00<00:00, 25.18it/s]
 86%|██████    | 18/21 [00:00<00:00, 24.91it/s]
 95%|███████   | 20/21 [00:00<00:00, 24.76it/s]
```





In [4]:

```
!jupyter nbconvert 5_Infinite_Series.ipynb
```

```
Traceback (most recent call last):
  File "/Users/cch/anaconda36/anaconda/bin/jupyter-nbconvert", line
11, in <module>
    sys.exit(main())
  File "/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/j
upyter_core/application.py", line 266, in launch_instance
    return super(JupyterApp, cls).launch_instance(argv=argv, **kwarg
s)
  File "/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/t
raitlets/config/application.py", line 658, in launch_instance
    app.start()
  File "/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/n
bconvert/nbconvertapp.py", line 325, in start
    self.convert_notebooks()
  File "/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/n
bconvert/nbconvertapp.py", line 482, in convert_notebooks
    cls = get_exporter(self.export_format)
  File "/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/n
bconvert/exporters/base.py", line 110, in get_exporter
    % (name, ', '.join(get_export_names())))
ValueError: Unknown exporter "html", did you mean one of: html_ch, h
tml_embed, html_toc, html_with_lenvs, html_with_toclenvs, latex_with
_lenvs, selectLanguage?
```

In [ ]:

```
```