Differertiation for Functions of Several Variables

Definition 1. *$f(x,y)$ is called to have a **limit**, L, at (a,b) if the value of $f(x,y)$ can approach L arbitrarily while (x,y) is near (a,b) enough. This means:*

$$\forall \varepsilon>0, \exists \delta>0 \text{ such that } |f(x,y)-f(a,b)|<\varepsilon \text{ if } \|(x,y)-(a,b)\|<\delta$$

where $\|(x,y)-(a,b)\| = \sqrt{(x-a)^2+(y-b)^2}$.
*$f(x,y)$ is called **continuous** at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y)=f(a,b)$.*

Example 2. The following limit

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$

fails to exist since the limits from different directions are not the same:
Approach $(0,0)$ along X-axis $\neq$ Approach $(0,0)$ along Y-axis:

$$1=\lim_{\substack{(x,y)\to(0,0)\\ y=0,x\to 0}} \frac{x-y}{x+y} \neq \lim_{\substack{(x,y)\to(0,0)\\ x=0,y\to 0}} \frac{x-y}{x+y}=-1$$

Example 3. The limit of the function

$$f(x,y)=\begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0.0) \\ 0 & \text{if } (x,y)=(0.0) \end{cases}$$

exists and $f(x,y)$ is continuous at $(0,0)$ since:

$$\left|\frac{x^2y}{x^2+y^2}\right| \leqslant \left|\frac{(x^2+y^2)(x^2+y^2)^{1/2}}{x^2+y^2}\right| \leqslant (x^2+y^2)^{1/2} \xrightarrow{(x,y)\to(0,0)} 0=f(0,0)$$

Definition 4. *The partial derivative with respect to x^i at (x_0^i) is defined as*

$$f_i=\frac{\partial f}{\partial x^i}=\lim_{k\to 0}\frac{f(x_0^1,\cdots,x_0^{i-1},x_0^i+k,x_0^{i+1},\cdots,x_0^n)-f(x_0^1,x_0^2,\cdots,x_0^n)}{k}$$

The gradient of $f(\vec{x})$, denoted as ∇f, is defined as $(f_1,\cdots,f_n)$.

Theorem 5. (Chain Rule)

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$$

we also have the similar result for multivariate functions:

$$\left(\frac{\partial z}{\partial t^i}\right)=\left(\frac{\partial z}{\partial x^j}\right)\left(\frac{\partial x^j}{\partial t^i}\right)$$

where

$$\begin{aligned}\left(\frac{\partial z}{\partial t^i}\right) &= \left(\frac{\partial z}{\partial t^1},\frac{\partial z}{\partial t^2},\cdots,\frac{\partial z}{\partial t^n}\right)\\ \left(\frac{\partial x^j}{\partial t^i}\right) &= \begin{pmatrix}\frac{\partial x^1}{\partial t^1} & \cdots & \frac{\partial x^1}{\partial t^n}\\ \vdots & \ddots & \vdots \\ \frac{\partial x^m}{\partial t^1} & \cdots & \frac{\partial x^m}{\partial t^n}\end{pmatrix}\end{aligned}$$

Example 6. Suppose that

$$\begin{aligned} z &= f(x,y)=\sin(x+y^2)\\ (x,y) &= (st, s^2+t^2)\end{aligned}$$

Then

$$\begin{aligned}\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right) &= \left(\cos(x+y^2) \quad 2y\cos(x+y^2)\right)\\ \begin{pmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{pmatrix} &= \begin{pmatrix} t & s\\ 2t & 2s\end{pmatrix}\\ \left(\frac{\partial z}{\partial s},\frac{\partial z}{\partial t}\right) &= \left(\cos(x+y^2) \quad 2y\cos(x+y^2)\right)\begin{pmatrix} t & s\\ 2s & 2t\end{pmatrix}\\ &= \cos((t^2+s^2)^2+st)\cdot((4s(t^2+s^2)+t),(4t(t^2+s^2)+s))\end{aligned}$$

Definition 7. The directional derivative in the direction, $\vec{e}=(e^1,\cdots,e^n)$ is:

$$D_{\vec{e}}f=\nabla f\cdot\vec{e}/\|\vec{e}\|$$

Theorem 8. *Directional derivative will attain its maximum(minimum) if*

$$\vec{e}=\nabla f/\|\nabla f\| \quad (-\nabla f/\|\nabla f\|)$$

Example 9. The directional derivative of $f(x,y)=\sqrt{x}+\sqrt{y}$ at $(x,y)=(1,1)$ in the $(3,4)$ direction is calculated as:

$$\begin{aligned}(3,4) &\to \frac{1}{5}(3,4)=\vec{e}\\ D_{\vec{e}}f(1,1) &= \nabla f(1,1)\cdot\vec{e}\\ &= \frac{1}{2}(1,1)\cdot\frac{1}{5}(3,4)\\ &= \frac{7}{10}\end{aligned}$$

The maximum of directional derivative will attain in the direction and have the maximal value L:

$$\begin{aligned}\vec{e} &= \frac{\nabla f(1,1)}{\|\nabla f(1,1)\|}=\frac{\frac{1}{2}(1,1)}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}=\frac{(1,1)}{\sqrt{2}}\\ D_{\vec{e}}f(1,1) &= \|\nabla f(1,1)\|=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{\sqrt{2}}\end{aligned}$$

Optimization for general domain:

Theorem 10. *Let $A=\frac{\partial^2 f}{\partial x^2}(x_0,y_0)$, $B=\frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)=\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)$, $C=\frac{\partial^2 f}{\partial y^2}(x_0,y_0)$, $D=AC-B^2$ and (x_0,y_0) is the critical point of f, , then*

i. *if $D>0$ and $A<0$, $f(x_0,y_0)$ is a relative maximum,*
ii. *if $D>0$ and $A>0$, $f(x_0,y_0)$ is a relative minimum,*
iii. *if $D<0$, $(x_0,y_0,f(x_0,y_0))$ is asaddle point,*
iv. *if $D=0$, no conclusion.*

Example 11. $f(x,y)=x^4+y^4-4xy$, $f(x,y)\nearrow\infty$ (no Maximum),$f(x,y)\cancel{\searrow}-\infty$ (Min exists)
a) find the critical values:

$$\begin{aligned} & f_1=4x^3-4y=0 \text{ and } f_2=4y^3-4x=0\\ \Longrightarrow\quad & x=y^3 \text{ and } y=x^3 \ (i.e.\, x=x^9)\\ \Longrightarrow\quad & (x,y)=(0,0) \text{ or } (\pm1,\pm1)\end{aligned}$$

b) Evaluate extremum

$$H=\begin{pmatrix} 12x^2 & -4\\ -4 & 12y^2\end{pmatrix}$$

i. $(x,y)=(0,0)$: saddle point, since

$$D=\left|\begin{pmatrix} 0 & -4\\ -4 & 0\end{pmatrix}\right|=-16<0$$

ii. at $(x,y)=(\pm1,\pm1)$:

$$D=\left|\begin{pmatrix} 12 & -4\\ -4 & 12\end{pmatrix}\right|=128>0$$

with $f_{11}(\pm1,\pm1)=12>0$. Then $f(-1,-1)=-2$ is a relative minimum (but not minimum) and $f(1,1)=-2$ is also a relative minimum (minimum).

Optimization in bounded Domain:

Example 12. $f(x,y)=x^2-xy+y^2-x+y-6$ for $(x,y)\in\{x^2+y^2\leqslant 1\}$:

a) critical value:

$$\begin{aligned}\vec{0} &= (2x-y-1, 2y-x+1)\\ \Longrightarrow\quad & x=1/3 \text{ and } y=-1/3\\ f(1/3,-1/3) &= -6\frac{1}{3}\end{aligned}$$

b) On the boundary, $\partial\Omega=\{(x,y)\,|\,x^2+y^2=1\}$, i.e. $x=\cos\theta$ and $y=\sin\theta$, $0\leqslant\theta\leqslant 2\pi$,

$$\begin{aligned} f(x,y) &= -\sin\theta\cos\theta-\cos\theta+\sin\theta-5\\ \frac{df}{d\theta}=0 &\implies (\sin\theta+\cos\theta)(\sin\theta-\cos\theta+1)=0\end{aligned}$$

i. $\sin\theta+\cos\theta=0$: $\tilde{\theta}=3\pi/4$ and $7\pi/4$, this implies

$$f(\cos\tilde{\theta},\sin\tilde{\theta}) = \sqrt{2}-4\frac{1}{2}$$

ii. $\sin\theta-\cos\theta+1=0$: $\hat{\theta}=0$ or $\frac{3\pi}{2}$, this implies

$$f(\cos\hat{\theta},\sin\hat{\theta}) = -6$$

Maximum is $\sqrt{2}-4\frac{1}{2}$ and minimum is $-6\frac{1}{3}$ at $(x,y)=(1/3,-1/3)$.

Optimization with constraints:

Theorem 13. *If a relative extrema of $f(\vec{x})$ and $g^i(\vec{x})=0$ occurs at $\vec{x}_0$, then there exist (λ^i) for which $(\vec{x}_0,\lambda)$ is the critical point of $L=f(\vec{x})+\sum_i\lambda^i g^i(\vec{x})$.*

Example 14. Extremum of $100x^{1/4}y^{3/4}$ with $x+2y=8$.

a) Lagrangian function:

$$L(x,y,\lambda)=\ln(100x^{1/4}y^{3/4})+\lambda(8-2x-1y)$$

b) critical point(s):

$$\begin{aligned}\vec{0} &= \nabla L(x,y,\lambda)\\ \Longrightarrow\quad & x=3 \text{ and } y=2\end{aligned}$$

$(x,y)=(3,3)$ is the only one critical point. Since $0\leqslant x,y\leqslant 8$, $P(x,y)$ has to be a maximum in such closed region. Therefore, maximum is equal to $100\cdot 3^{1/4}2^{3/4}$ at (3, 2).

Example 15. Find extrema of $f(x,y,z)=x^2-xy+y^2-z^2+1$ subjust to $x^2+y^2=1$ and $z^2=xy$.

a) Lagrangian:

$$L(x,y,z;\lambda,\mu)=x^2-xy+y^2-z^2+1+\lambda(1-x^2-y^2)+\mu(z^2-xy)$$

b) critical values:

$$\begin{aligned}\nabla L=\vec{0} &\Rightarrow (2x-y-2\lambda x-\mu y, 2x-y-2\lambda x-\mu y, -2z+2\mu z)=\vec{0}\\ -2z+2\mu z=0 &\Rightarrow z=0 \text{ or } \mu=1\\ \begin{cases} z=0\\ \mu=1\end{cases} &\Rightarrow \begin{cases} x^2+y^2=1, xy=0\\ x/y=y/x=\frac{1}{1-\lambda}\to y=x\end{cases}\\ &\Rightarrow \begin{cases}(x,y,z)=(\pm1,0,0),(0,\pm1,0)\\ (x,y,z)=\pm\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)\end{cases}\end{aligned}$$

c) Extrema exist since $|x|,|y|\leqslant 1$ and maxima=2 and minimum=1

$$\begin{aligned} f(\pm1,0,0)=f(0,\pm1,0) &= 2\\ f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)=f\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right) &= 1\end{aligned}$$