1. 1 Multi-variable C

In [1]:

```
from IPython.core.display import HTML
css_file = 'css/ngcmstyle.css'
HTML(open(css_file, "r").read())
```

Out[1]:

In [1]:

```
%matplotlib inline

#rcParams['figure.figsize'] = (10,3) #wide graphs by defaul
import scipy
import numpy as np
import time
from sympy import symbols,diff,pprint,sqrt,exp,sin,cos,log,


from mpl_toolkits.mplot3d import Axes3D
from IPython.display import clear_output,display,Math
import matplotlib.pylab as plt
```

1.1 Partial Differentiation

1.2 Definition

Suppose that (x_0, y_0) is in the domain of $z = f(x, y)$ **1.** the p
the limit

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x_0}, \mathbf{y_0}) = \lim_{\mathbf{h} \to \mathbf{0}} \frac{\mathbf{f}(\mathbf{x_0} + \mathbf{h}, \mathbf{y_0}) - \mathbf{f}(\mathbf{x_0}, \mathbf{y_0})}{\mathbf{h}}$$

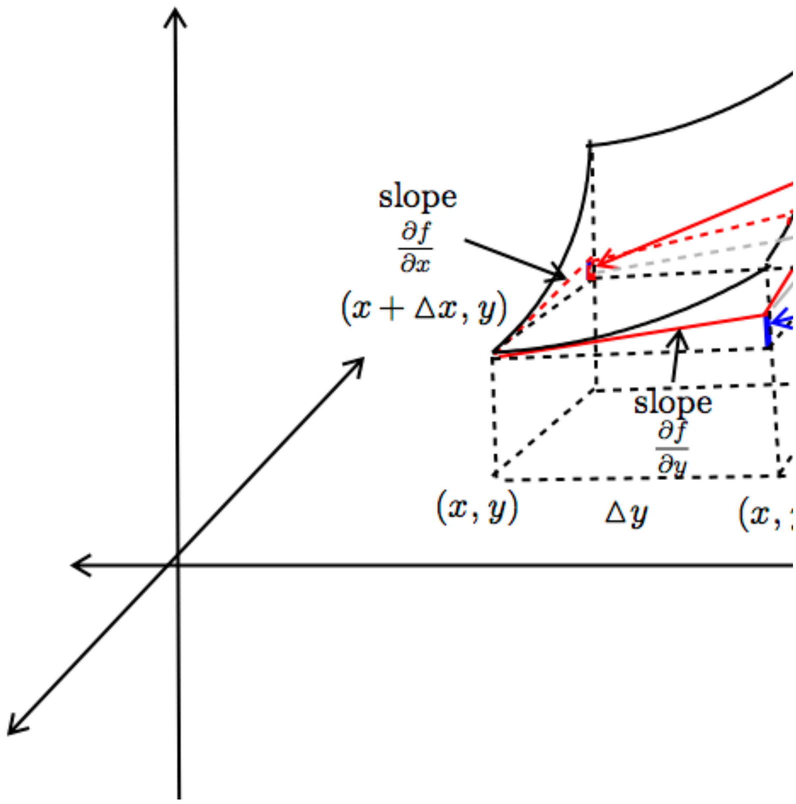
Geometrically, the value of this limit is the slope of the tanger
this quantity is the rate of change of $f(x, y)$ at (x_0, y_0) along

2. the partial derivative with respect to y at (x_0, y_0) is the lim

$$\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{x_0}, \mathbf{y_0}) = \lim_{\mathbf{k} \to \mathbf{0}} \frac{\mathbf{f}(\mathbf{x_0}, \mathbf{y_0} + \mathbf{k}) - \mathbf{f}(\mathbf{x_0}, \mathbf{y_0})}{\mathbf{k}}$$

Geometrically, the value of this limit is the slope of the tanger
this quantity is the rate of change of $f(x, y)$ at (x_0, y_0) along

Here is a geometric meaning about partial derivative:



In [2]:

```
import plotly.graph_objs as go

import plotly
from plotly.offline import init_notebook_mode,iplot
init_notebook_mode()
```

In [18]:

```
from numpy import sqrt
X = np.arange(.2, 1, 0.02)
Y = np.arange(0.2, 1, 0.02)
t = np.arange(-0.2, 1.2, 0.02)
s = np.arange(0.4, 0.8, 0.01)
X,Y = np.meshgrid(X,Y)
f= sqrt(X*X + Y*Y)
z0=sqrt(0.4**2+0.4**2)
u=np.arange(0., z0, 0.01)

surface = go.Surface(x=X, y=Y, z=f,colorscale=0.5)
Xaxis = go.Scatter3d(x=t, y=0*t, z=0*t,
            mode = "lines",
            line = dict(
                        color='black',
                        width = 5
            )
         )
Yaxis = go.Scatter3d(x=0*t, y=t, z=0*t,
            mode = "lines",
            line = dict(
                        color='black',
                        width = 5
            )
         )
X0 = go.Scatter3d(x=s, y=0.4+0*s, z=0*s,
            mode = "lines",
            line = dict(
                        color='brown',
                        width = 5
            )
     )
X01= go.Scatter3d(x=0.4+0*u, y=0.4+0*u, z=u,
            mode = "lines",
            line = dict(
                        color='blue',
                        width = 5
            )
     )
Y0 = go.Scatter3d(y=s, x=0.4+0*s, z=0*s,
            mode = "lines",
            line = dict(
                        color='orange',
                        width = 5
            )
     )
#Line2 = go.Scatter3d(x=0*t, y=t, z=0*t)
#Line3 = go.Scatter3d(x=t, y=t, z=np.ones(len(t))/2)
#Line4 = go.Scatter3d(x=t, y=-t, z=-np.ones(len(t))/2)
data = [surface,Xaxis,Yaxis,X0,Y0,X01]

fig = go.Figure(data=data)
iplot(fig)
```

The same definition can be also applied to the functions with

1.3 Definition

Suppose that $(x_0^i) = (x_0^1, x_0^2, \cdots, x_0^n)$ in the domain of $f(\mathbf{x})$
respect to x^i at (x_0^i) is defined as

$$\mathbf{f_i(x_0) = \frac{\partial f}{\partial x^i}(x)\Big|_{x=x_0} = \lim_{k\to 0} \frac{f(x_0^1, \cdots, x_0^{i-1}, {\color{red}x_0^i + k}, x_0^{i+1},}{}}$$

1.4 Definition

${\color{red}(f_1, \cdots, f_n)}$ is called gradient of $f(\vec{x})$, denoted as ${\color{red}\nabla f}$.

1.5 Example

Find the $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}(1,3)$, $\frac{\partial f}{\partial y}(2,-4)$ if $f(x,y) = x^3 + 4x^2y$

In [34]:
```python
from sympy import symbols, diff,pprint,sqrt
x,y=symbols('x y')
f=x**3+4*x*x*y**3+y*y

grad = lambda func, vars :[diff(func,var) for var in vars]

df=grad(f,[x,y])
pprint(df)
```

```
⎡   2        3      2  2      ⎤
⎣3⋅x  + 8⋅x⋅y , 12⋅x ⋅y  + 2⋅y⎦
```

In [4]:
```python
def df_val(f,val):
    return [ff.subs({x:val[0],y:val[1]}) for ff in f]
df_val(df,[1,3])
```

Out[4]: `[-9*sin(9) + cos(9), -6*sin(9)]`

In [3]:
```python
df_val(df,[2,-4])
```

Out[3]: `[-1012, 760]`

1.6 Example

Find the $\frac{\partial f}{\partial x}$ if $f(x,y) = x^3 + 4x^2y^3 + y^2$.

In [5]:
```python
f=2*x**2*y**3-3*x*y**2+2*x**2+3*y*y*1

grad = lambda func, vars :[diff(func,var) for var in vars]

df=grad(f,[x,y])
pprint(df)
```

```
⎡     3            2      2  2              ⎤
⎣4⋅x⋅y  + 4⋅x - 3⋅y , 6⋅x ⋅y  - 6⋅x⋅y + 6⋅y⎦
```

1.7 Example

Find the $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ if $f(x, y) = x \cos xy^2$.

In [2]:
```
from sympy import cos
x,y=symbols("x y")
f=x*cos(x*y**2)

grad = lambda func, vars :[diff(func,var) for var in vars]

df=grad(f,[x,y])
pprint(df)
```

```
⎡     2    ⎛   2⎞       ⎛   2⎞        2      ⎛   2⎞⎤
⎣- x⋅y ⋅sin⎝x⋅y ⎠ + cos⎝x⋅y ⎠, -2⋅x ⋅y⋅sin⎝x⋅y ⎠⎦
```

1.8 Example

Find all the first partial derivatives of Cobb-Douglas function
$f(x_1, \cdots, x_n) = Ax_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $A > 0, 0 < \alpha_1, \cdots \alpha_n <$
Sol:

$$\begin{aligned}
\frac{\partial f}{\partial x_i} &= Ax_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} \alpha_i x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} \\
&= A\alpha_i x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} / x_i \\
&= \alpha_i \frac{f(x_1, \cdots, x_n)}{x_i} \quad \text{for } i = 1, \cdots, n
\end{aligned}$$

1.9 Eexample

A factory produces two kinds of machine parts, says A and E
x hundred units of A and y hundred units of B is:
$C(x, y) = 200 + 10x + 20y - \sqrt{x + y}$

```
In [13]: C = 200+10*x+20*y-sqrt(x+y)
         Cxy=grad(C,[x,y])
         df_val(Cxy,[5,6])
```

```
Out[13]: [-sqrt(11)/22 + 10, -sqrt(11)/22 + 20]
```

$\frac{\partial C}{\partial x}(5,6) = 10 - \frac{1}{22}\sqrt{11}$,i.e. an increase for x from 5 to 6 w
cost function approximately 9.85. And $\frac{\partial C}{\partial y}(5,6) = 20 - \sqrt{1}$
kept at 5 will result in an increase in daily cost function appro

1.10 Example

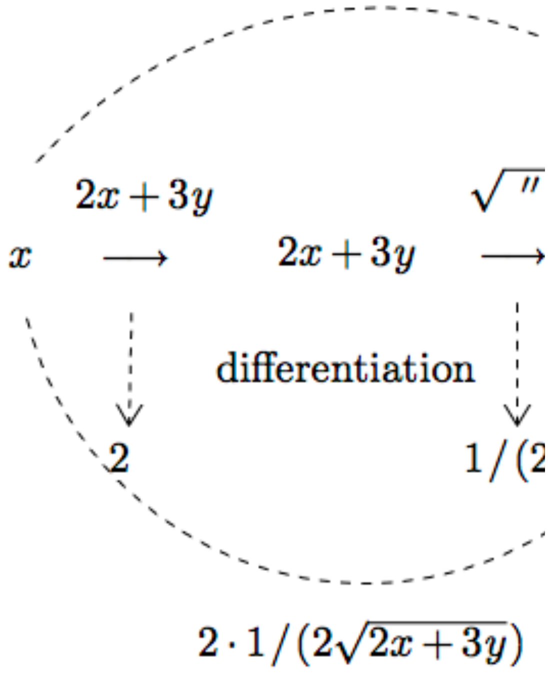
If $f(x,y) = x^2e^{y^3} + \sqrt{2x+3y}$, $\frac{\partial f}{\partial x} = 2xe^{y^3} + (2x+3y)^{-1/}$

1.11 Solution

Since the partial differentiation only works for the defaulted v
treated as constants in such operation. Therefore

$$\begin{aligned}
\frac{\partial}{\partial x}\left(x^2e^{y^3} + \sqrt{2x+3y}\right) &= e^{y^3}\frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial x}\sqrt{2x+3y} \\
&= e^{y^3}\cdot 2x + 2\cdot\frac{1}{2\sqrt{2x+3y}} \\
&= 2xe^{y^3} + \frac{1}{\sqrt{2x+3y}}
\end{aligned}$$

Note that the last result comes from the {\tmstrong{Chain Ru



As the same reason, we also have the result for partial deriva

$$\frac{\partial}{\partial y}\left(x^2 e^{y^3} + \sqrt{2x+3y}\right) = x^2 \frac{\partial}{\partial y} e^{y^3} + \frac{\partial}{\partial y}\sqrt{2x+3y}$$
$$= 3x^2 y^2 e^{y^3} + \frac{3}{2\sqrt{2x+3y}}$$

1.12 Example

Find out the first order derivatives for $f(x, y) = x\sqrt{y} - y\sqrt{x}$

In [19]:
```
f=x*sqrt(y)-y*sqrt(x)
grad(f,[x,y])
```

Out[19]:
```
[sqrt(y) - y/(2*sqrt(x)), -sqrt(x) + x/(2*sqrt(y))]
```

1.13 Example

For Cobb-Douglas production function, $f(K, L) = 20K^{1/4}L^{3}$

1. **The marginal productivity of capital** when $K = 16$ and
   in K from 16 to 17 will result in an increase of approxima

2. **The marginal productivity of labor** when $K = 16$ and
   L from 81 to 82 will result in an increase of approximatel

Description

Note the partial derivatives are as follows:

$$\frac{\partial f}{\partial K} = 5\left(\frac{L}{K}\right)^{3/4}$$
$$\frac{\partial f}{\partial L} = 15\left(\frac{K}{L}\right)^{1/4}$$

1.14 Example

For Cobb-Douglas production function, $f(K, L) = 20K^{2/3}L^{1}$

1. **The marginal productivity of capital** when $K = 125$ ar
   in K from 125 to 126 will result in an increase of approxi
2. **The marginal productivity of labor** when $K = 125$ and
   increase in L from 27 to 28 will result in an increase of ap

Description

Note the partial derivatives are as follows:

$$\frac{\partial f}{\partial K} = \frac{40}{3}\left(\frac{L}{K}\right)^{1/3}$$
$$\frac{\partial f}{\partial L} = \frac{20}{3}\left(\frac{K}{L}\right)^{2/3}$$

In [ ]:

Two products are said to be **competitive** with each other if a
decrease in demand for the other. **Complementary** products
Suppose that $f(p, q)$ and $g(p, q)$ are the demand for produc
have

1. $\frac{\partial f}{\partial p} < 0$ and $\frac{\partial g}{\partial q} < 0$ sine raising price always results in a
2. If $\frac{\partial f}{\partial q} > 0$ and $\frac{\partial g}{\partial p} > 0$ Then A and B are in **competitive**
3. If $\frac{\partial f}{\partial q} < 0$ and $\frac{\partial g}{\partial p} < 0$ Then A and B are in **complemen**

1.15 Example

If $f(p, q) = 400 - 5p^2 + 16q$ and $g(p, q) = 600 + 12p -$

$$\frac{\partial f}{\partial q} = 16 > 0$$
$$\frac{\partial g}{\partial p} = 12 > 0$$

1.16 Example

If $f(p, q) = \frac{30p}{2p+3q}$ and $g(p, q) = \frac{10q}{p+4q}$, then A and B are co

$$\begin{aligned}\frac{\partial f}{\partial q} &= \frac{\partial}{\partial q}\frac{30p}{2p+3q} \\ &= \frac{-90p}{(2p+3q)^2} < 0 \\ \frac{\partial g}{\partial p} &= \frac{\partial}{\partial p}\frac{10q}{p+4q} \\ &= \frac{-10q}{(p+4q)^2} < 0\end{aligned}$$

1.17 Implicit Differentiation

Suppose that z is differentiable and defined implicitly as follo
$x^2 + y^3 - z + 2yz^2 = 5.$

In [9]:
```
from sympy import Function,solve
x,y  = symbols('x y')
z = Function('z')(x,y)
```

In [12]:
```
eq= x**2+y**3-z+2*y*z**2-5
gradv=grad(eq,[x,y])
gradv
```

Out[12]:
```
[2*x + 4*y*z(x, y)*Derivative(z(x, y), x) - Derivative(z(
 3*y**2 + 4*y*z(x, y)*Derivative(z(x, y), y) + 2*z(x, y)*
```

In [17]:
```
pprint("dz/dx = %s" %solve(gradv[0],diff(z, x))[0])
```

```
dz/dx = -2*x/(4*y*z(x, y) - 1)
```

In [22]:
```
pprint("dz/dy = %s" %solve(gradv[1],diff(z, y))[0])
```

```
dz/dy = -(3*y**2 + 2*z(x, y)**2)/(4*y*z(x, y) - 1)
```

1.18 Example

a). If $f(x, y, z) = x^2y + y^2z + zx$, then $f_x = 2xy + z$;
b). If $h(x, y, zw) = \frac{xw^2}{y+\sin zw}$, then $h_w =$

In [24]:
```
from sympy import sin
x,y,z,w=symbols(" x y z w")
f=x*w**2/(y+sin(z*w))
pprint(diff(f,w))
```

```
   2
  w ·x·z·cos(w·z)       2·w·x
- ─────────────── + ────────────
                2   y + sin(w·z)
  (y + sin(w·z))
```

1.19 Note

Higher order partial derivative. As the functions of single varia
derivatives of functions with multiple variables as follows:

1. Two variables: Suppose that $f(x, y)$ is smooth enough,

Partial derivatives for $f(x, y)$

Order	Partial Deriva
1st	$\mathbf{f_1} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}, \mathbf{f_2} =$
2nd	$\mathbf{f_{ij}} =$
More	$\mathbf{f_{\dots i}} =$

2. More than two variables: Suppose that $f(\vec{x}) = f(x^1, \cdots, x$

Order	partial
1st	$\mathbf{f_i} = \frac{\partial \mathbf{f}}{\partial \mathbf{x^i}}$, for i
2nd	$\mathbf{f_{ij}} = \frac{\partial^2 \mathbf{f}}{\partial \mathbf{x^i} \partial \mathbf{x^j}}$, for 1
More	$\mathbf{f_{\dots i}} = \frac{\partial}{\partial \mathbf{x^i}} \mathbf{f_{\dots}}$, for i

1.20 Example

Find the second order derivatives of $f(x, y) = x^2y^3 + e^{4x}$ ln

All the 1st order partial derivatives are as follows:

$$\frac{\partial f}{\partial x} = 2x \cdot y^3 + 4e^{4x} \cdot \ln y\text{and } \frac{\partial f}{\partial y} = x^2 \cdot 3y^2 + e^{4x} \cdot \frac{1}{y}$$

And all the 2nd order of partial derivatives are as follows:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}(2xy^3 + 4e^{4x} \ln y) \\
&= 2 \cdot y^3 + 16e^{4x} \cdot \ln y \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(3x^2y^2 + e^{4x}\frac{1}{y}\right) \\
&= 6x^2y - e^{4x}\frac{1}{y^2} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}(2xy^3 + 4e^{4x} \ln y) \\
&= 2x \cdot 3y^2 + 4e^{4x} \cdot \frac{1}{y} \\
\frac{\partial^2 f}{\partial x \partial y} &= 6xy^2 + 4e^{4x}/y
\end{aligned}$$

In [15]:

```
from sympy import exp,log

f=x*x*y**3+exp(4*x)*log(y)
X=[x,y]
g=grad(f,X)
g
```

Out[15]:

```
[2*x*y**3 + 4*exp(4*x)*log(y), 3*x**2*y**2 + exp(4*x)/y]
```

In [18]:
```
pprint([[diff(gg,var) for gg in g] for var in X])
```

```
⎡⎡                                                    4⋅x⎤   ⎡
⎢⎢   3        4⋅x                    2    4⋅e         ⎥   ⎢      2   4
⎢⎢2⋅y  + 16⋅e    ⋅log(y), 6⋅x⋅y  + ───────⎥,  ⎢6⋅x⋅y  + ───
⎢⎣                                              y     ⎦   ⎢
⎣                                                          ⎣
```

1.21 Example

Find the first three order derivatives of $f(x, y) = 4x^2 - 6xy^3$

All the 1st order partial derivatives are as follows:

$$\frac{\partial f}{\partial x} = 8x - 6y^3 \text{and } \frac{\partial f}{\partial y} = 0 - 6x \cdot 3y^2$$

And the 2nd order of partial derivatives are as follows:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}(8x - 6y^3) \\
&= 8 \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}(-18xy^2) \\
&= -36xy \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}(8x - 6y^3) \\
&= 0 - 12y^2 \\
\frac{\partial^2 f}{\partial x \partial y} &= -12xy^2
\end{aligned}$$

And 3rd order of partial derivatives are as follows:

$$\begin{aligned}
f_{111} &= \frac{\partial}{\partial x}(8) = 0 \\
f_{222} &= \frac{\partial}{\partial y}(-36xy) = -36x \\
f_{112} = f_{121} = f_{211} &= \frac{\partial}{\partial y}(8 - 6y^3) = -18y^2 \\
f_{122} = f_{212} = f_{221} &= \frac{\partial}{\partial x}(-36xy) = -36x
\end{aligned}$$

In [26]:

```
f=4*x**2-6*x*y**3
diff(f,x,y)==diff(f,y,x)
```

Out[26]: True

1.22 Example

Let $f(x, y, z) = xe^{yz}$, then

1. $f_{xzy} = (1 + yz)e^{yz}$,
2. $f_{yzx} = (1 + yz)e^{yz}$,

They are equal to with respectively.

1.23 Theorem

If $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ are all continuous near (x_0, y_0), then $\frac{\partial}{\partial x}$

1.24 Example

For $f(x, y, z) = xe^{yz}$,

In [28]:
```
from sympy import exp
f=x*exp(y*z)
pprint("fxzy= %s" %diff(f,x,z,y) )
```

```
fxzy= (y*z + 1)*exp(y*z)
```

In [29]:
```
from sympy import exp
f=x*exp(y*z)
pprint("fyxz= %s" %diff(f,y,x,z) )
```

```
fyxz= (y*z + 1)*exp(y*z)
```

```
Example
---

Suppose that
$$ f (x, y) = \left\{ \begin{array}{ll}
       \frac{x^3  y - y^3 x}{x^2 + y^2} & \text{ if } (x, y) \n
       0 & \text{ if } (x, y) = (0.0)
     \end{array} \right. $$


  $f (x, y)$ is continuous at $(0, 0)$.

**1.** first partial derivatives of $f (x.y)$ where $(x, y) \ne

\begin{eqnarray*}
      \frac{\partial f}{\partial x} & = & \frac{(3 x^2 y + y^3)
      2 x (x^3  y + y^3 x)}{(x^2 + y^2)^2}\\
      & = & \frac{- y^5 + 4 x^2 y^3 + x^4 y}{(x^2 + y^2)^2}\\
      \frac{\partial f}{\partial y} & = & \frac{x^5 - 4 y^2 x^3
      + y^2)^2}
    \end{eqnarray*}


**2.** first partial derivatives of $f (x.y)$ at $(x, y) = (0,

\begin{eqnarray*}
      \frac{\partial f}{\partial x} (0, 0) & = & \lim_{h \right
      \frac{f (h, 0) - f (0, 0)}{h}\\
      & = & 0\\
      \frac{\partial f}{\partial x} (0, 0) & = & 0
    \end{eqnarray*}

**3.** Second partial derivatives $\frac{\partial^2 f}{\partial
\partial x}$ of $f (x.y)$ where $(x, y) = (0, 0)$:

\begin{eqnarray*}
      \frac{\partial^2 f}{\partial x^{} \partial y} (0, 0) & =
      \rightarrow 0} \frac{f_y (h, 0) - f_y (0, 0)}{h}\\
      & = & \lim_{h \rightarrow 0} \frac{\frac{- h^5 + 40^2 h^3
      + 0^2)^2} - 0}{h} = - 1\\
      \frac{\partial^2 f}{\partial y^{} \partial x} (0, 0) & =
      \rightarrow 0} \frac{f_x (0, k) - f_x (0, 0)}{k}\\
      & = & \lim_{k \rightarrow 0} \frac{\frac{k^5 - 40^2 k^3 -
      k^2)^2} - 0}{k} = 1
    \end{eqnarray*}

The last result shows  $\color{red}{\frac{\partial^2 f}{\partia
{\partial y \partial x} (0, 0)}$.
```

1.26 Definition

A function, $u(x, y)$,is called harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

1.27 Example

$u = e^x \cos y$ is harmonic,

In [64]:
```
x,y=symbols("x y")
u=exp(x)*cos(y)
diff(u,x,x)+diff(u,y,y)
```

Out[64]: 0

1.28 Exercise p. 1069

In [5]:
```
# 20 log(exp(x)+exp(y))
from sympy import log,exp,cos,sin,diff,integrate,symbols,Ma
z=log(exp(x)+exp(y))
pprint(grad(z,[x,y]))
```

```
⎡    x         y    ⎤
⎢   e         e     ⎥
⎢───────, ───────⎥
⎢ x    y   x    y ⎥
⎣e  + e   e  + e  ⎦
```

In [9]:
```
# 22 partial derivative of int_x^y cos t dt

t=symbols("t")
f=integrate(cos(t),[t,x,y])
pprint(grad(f,[x,y]))
```

```
[-cos(x), cos(y)]
```

In [40]:
```
#32 2cos(x+2y)+sin yz -1=0
z=Function("z")(x,y)
eq= 2*cos(x+2*y)+sin(y*z)-1

gradv=grad(eq,[x,y])

pprint("dz/dx = %s" %solve(gradv[0],diff(z, x))[0])
pprint("dz/dy = %s" %solve(gradv[1],diff(z, y))[0])
```

```
dz/dx = 2*sin(x + 2*y)/(y*cos(y*z(x, y)))
dz/dy = (-z(x, y) + 4*sin(x + 2*y)/cos(y*z(x, y)))/y
```

In [41]:
```
#38 Second derivate of \sqrt(x^2+y^2)
from sympy import sqrt
f= sqrt(x**2+y**2)
pprint("fxx = %s" %diff(f, x,x))
pprint("fxy = fyx = %s" %diff(f, x,y))
pprint("fyy = %s" %diff(f, y,y))
```

```
fxx = (-x**2/(x**2 + y**2) + 1)/sqrt(x**2 + y**2)
fxy = fyx = -x*y/(x**2 + y**2)**(3/2)
fyy = (-y**2/(x**2 + y**2) + 1)/sqrt(x**2 + y**2)
```

In [10]:
```
#46 f=exp(-2x)cos(3y), fxy=fyx
f=exp(-2*x)*sin(3*y)
```

In [6]:
```
def highdiff(f,xy):
    fpart=f
    for x in xy:
        fpart=diff(fpart,x)
    return fpart
```

In [7]:
```
fxy = highdiff(f,[x,y])
fyx = highdiff(f,[y,x])
if (fxy == fyx):
    print("fxy = fyx = %s" %fxy)
else:
    print("fxy ≠ fyx and  fxy= %s, fyx= %s" %(fxy,fyx))
```

```
fxy = fyx = -2*x**2*y**3*cos(x*y**2) - 4*x*y*sin(x*y**2)
```

#90.

Suppose that

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0.0) \\ 0 & \text{if } (x,y) = (0.0) \end{cases}$$

$f(x,y)$ is continuous at $(0,0)$.

a. first partial derivatives of $f(x.y)$ where $(x,y) \neq (0,0)$:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{(3x^2y+y^3)(x^2+y^2)-2x(x^3y+y^3x)}{(x^2+y^2)^2} \\ &= \frac{-y^5+4x^2y^3+x^4y}{(x^2+y^2)^2} \\ \frac{\partial f}{\partial y} &= \frac{x^5-4y^2x^3-y^4x}{(x^2+y^2)^2} \end{aligned}$$

b. first partial derivatives of $f(x.y)$ at $(x,y) = (0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$
$$= 0$$
$$\frac{\partial f}{\partial x}(0,0) = 0$$

c. Second partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ of $f(x.y)$ where (

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{-h^5 + 40^2 h^3 + 0^4 h}{(h^2+0^2)^2} - 0}{h} = -1$$
$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$
$$= \lim_{k \to 0} \frac{\frac{k^5 - 40^2 k^3 - 0^4 k}{(0^2+k^2)^2} - 0}{k} = 1$$

The last result shows $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$. This does not
not continuous at $(0,0)$ since
$\lim_{x=0, y \to 0} f_{x,y}(x,y) \neq f_{x,y}(0,0)$

1.29 Differentials

Let $f(x,y)$, and let $\triangle x$, $\triangle y$ be the increments of x and y re

$$\mathbf{dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = f_x dx + f_y dy}$$

Example Let $f(x,y) = 2x^2 - xy$

In [8]:
```
def differential(func,xy):
    df=""
    for x in xy:
        fx=diff(func,x)
        if df!="":
            df="%s + (%s) d%s" %(df,fx,x)
        else:
            df="(%s) d%s" %(fx,x)
    return df
```

In [9]:
```
f=2*x**2-x*y
df=differential(f,[x,y])
print(df)
```

```
(4*x - y) dx + (-x) dy
```

While (x, y) changes from (1,1) to (0.98,1.03):

1. $dx = 0.98 - 1 = 0.02, dy = 1.03 - 1),$
2. $dz = (4 \times 1 - 1)dx - 1dy = -0.09,$
3. $\triangle z = z(0.98, 1.03) - z(1, 1) \approx -0.0886 \sim dz$

1.30 Example, body mass inde

The body mass index (BMI) or Quetelet index is a value deriv
individual. The BMI is defined as the body mass divided by th
expressed in units of kg/m2,

$$\text{BMI} = \frac{\text{weight}}{\text{height}^2}$$

resulting from mass in kilograms and height in metres.

What's the increase of BMI if one's weight increases from 68
170cm?

Sol. As problem stated, assume

$$\text{BMI}(w, h) = \frac{w}{h^2}$$

where w, h represent one's weight (in kg) and height (in m).

In [7]:
```
w,h=symbols("w h")
BMI= w/h/h
dBMI=grad(BMI,[w,h])
#df_val(BMI,[2,0.01])
dBMI
```
Out[7]:
```
[h**(-2), -2*w/h**3]
```

In [9]:
```
differential(BMI,[w,h])
```
Out[9]:
```
'(h**(-2)) dw + (-2*w/h**3) dh'
```

In [31]:
```
h=1.69
dh=0.01
w=68
dw=2
whh0=w/h/h
whh1=(w+dw)/(h+dh)/(h+dh)
exact=(whh1-whh0)/whh0
dBMIvalpercent=(dw/h/h-2*dh*w/h/h/h)/whh0
print("BMI increases from %5.3f to %5.3f, approximately %4.
      %(whh0,whh1,dBMIvalpercent,exact))
```
```
BMI increases from 23.809 to 24.221, approximately 0.018
```

1.31 p. 1082

In [10]:
```
#16
z=symbols("z")
w=sqrt(x*x+x*y+z**2)
wxyz=differential(w,[x,y,z])
pprint(wxyz)
```
```
((x + y/2)/sqrt(x**2 + x*y + z**2)) dx + (x/(2*sqrt(x**2
(z/sqrt(x**2 + x*y + z**2)) dz
```

1.32 Chain Rule

As result in one-variable function:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

we also have the similar result for multivariate functions:

$$\left(\frac{\partial z}{\partial t^i}\right) = \left(\frac{\partial z}{\partial x^j}\right)\left(\frac{\partial x^j}{\partial t^i}\right)$$

where

$$\left(\frac{\partial z}{\partial t^i}\right) = \left(\frac{\partial z}{\partial t^1}, \frac{\partial z}{\partial t^2}, \cdots, \frac{\partial z}{\partial t^n}\right)$$

$$\left(\frac{\partial x^j}{\partial t^i}\right) = \begin{pmatrix} \frac{\partial x^1}{\partial t^1} & \cdots & \frac{\partial x^1}{\partial t^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^m}{\partial t^1} & \cdots & \frac{\partial x^m}{\partial t^n} \end{pmatrix}$$

In [48]:

```
def ChainRule(func, x,t,xt,output=False):
    z=Matrix([func])
    X=Matrix(x)
    n=len(x)
    Xt=Matrix(x).subs({x[n]:xt[n] for n in range(len(x))})
    T=Matrix(t)
    dzdt=z.jacobian(X).subs({x[n]:xt[n] for n in range(len(
    if len(t)!=1:
        print("∂ %s /∂ %s\n" %(func,t))
        pprint(dzdt)
    else:
        print("d ( %s) /d %s\n" %(func,t))
        pprint(dzdt)
    if output==True:
        return dzdt
```

1.33 Example, $(\mathbb{R}(t) \rightarrow \mathbb{R}^n(\mathbf{x}$

$$\frac{dW}{dt} = \frac{\partial W}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{\partial W}{\partial x^1} & \frac{\partial W}{\partial x^2} & \cdots & \frac{\partial W}{\partial x^n} \end{bmatrix}_{1\times n} \begin{bmatrix} \frac{dx^1}{dt} \\ \frac{dx^2}{dt} \\ \vdots \\ \frac{dx^n}{dt} \end{bmatrix}_{n\times 1}$$

e.g. $n = 2$

$$\frac{dW}{dt} = \frac{\partial W}{\partial(x,y)}\frac{d(x,y)}{dt} = \begin{bmatrix} \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \frac{\partial W}{\partial x}\frac{dx}{dt}$$

Let $w = x^2y - xy^3$,$(x, y) = (\cos t, e^t)$. Find dw/dt and its v

In [44]:
```
x,y,s,t=symbols("x y s t")
w=Matrix([x**2*y-x*y**3])
X=Matrix([cos(t),exp(t)])

pprint(w.jacobian(Matrix([x,y]))*X.jacobian(Matrix([t])))
```

```
⎡⎛ 2          2⎞  t   ⎛            3⎞         ⎤
⎣⎝x  - 3⋅x⋅y ⎠⋅e  - ⎝2⋅x⋅y - y ⎠⋅sin(t)⎦
```

In [49]:
```
ChainRule(x**2*y-x*y**3, [x,y],[t],[cos(t),exp(t)])
```

```
d ( x**2*y - x*y**3) /d [t]

⎡⎛      2⋅t                2    ⎞  t   ⎛   3⋅t       t
⎣⎝- 3⋅e    ⋅cos(t) + cos (t)⎠⋅e  - ⎝- e    + 2⋅e ⋅cos(t)
```

$$\left.\frac{dw}{dt}\right|_{t=0} = (-3+1)\cdot 1 - (-1+2)\cdot 0 = -2$$

1.34 Example, ($\mathbb{R}^m(\mathbf{u}_{1\times m}) \to \mathbb{R}^n(\mathbf{x}_{1\times n}) \to \mathbb{R}($

$$\frac{\partial W}{\partial \mathbf{u}} = \frac{\partial W}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial W}{\partial x^1} & \frac{\partial W}{\partial x^2} & \cdots & \frac{\partial W}{\partial x^n} \end{bmatrix}_{1\times n} \begin{bmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} \\ \frac{\partial x^2}{\partial u^1} & \ddots \\ \vdots & \\ \frac{\partial x^n}{\partial u^1} & \frac{\partial x^n}{\partial u^2} \end{bmatrix}$$

1.35 e.g. $m, n = 2, 2$

$$\frac{\partial W}{\partial(u,v)} = \frac{\partial W}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Let $w = 2x^2y$,$(x, y) = (u^2 + v^2, u^2 - v^2)$. Find $\partial w/\partial u$ and

In [5]:
```
x,y,u,v=symbols("x y u v")
z=[2*x*x*y]
X=Matrix([x,y])
Xt=Matrix([x,y]).subs({x:u*u+v*v,y:u*u-v*v})
T=Matrix([u,v])
pprint(Matrix(z).jacobian(X)*Xt.jacobian(T))
```

```
⎡      2                         2           ⎤
⎣4⋅u⋅x  + 8⋅u⋅x⋅y  - 4⋅v⋅x  + 8⋅v⋅x⋅y⎦
```

In [11]:
```
ChainRule(z, [x,y],[u,v],[u*u+v*v,u*u-v*v])
```

```
∂ [2*x**2*y] /∂ [u, v]

⎡                                                 2
⎢     ⎛ 2    2⎞ ⎛ 2    2⎞         ⎛ 2    2⎞           ⎛ 2
⎣8⋅u⋅⎝u  - v ⎠⋅⎝u  + v ⎠ + 4⋅u⋅⎝u  + v ⎠    8⋅v⋅⎝u  -

    2⎤
  2⎞ ⎥
 v ⎠ ⎦
```

1.36 Example

Suppose that

$$z = f(x,y) = \sin(x+y^2)$$
$$(x,y) = (st, s^2+t^2)$$

Then

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = \left(\cos(x+y^2) \quad 2y\cos(x+y^2)\right)$$
$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} t & s \\ 2t & 2s \end{pmatrix}$$
$$\left(\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}\right) = \left(\cos(x+y^2) \quad 2y\cos(x+y^2)\right)\begin{pmatrix} t & s \\ 2t & 2 \end{pmatrix}$$
$$= \left(t\cos(x+y^2) + 4ty\cos(x+y^2) \quad s\cos(\right.$$
$$= \cos((t^2+s^2)^2+st)\cdot((4s(t^2+s^2)+t), ($$

In [64]:
```
x,y,s,t=symbols("x y s t")
f=sin(x+y*y)
Xt=[s*t,s*s+t*t]
ChainRule(f, [x,y],[s,t],Xt)
```

```
∂ sin(x + y**2) /∂ [s, t]

⎡                             ⎛                   2⎞            ⎛
⎢      ⎛ 2     2⎞             ⎜       ⎛ 2     2⎞   ⎟            ⎜
⎣4⋅s⋅ ⎝s   + t ⎠ ⋅cos ⎝s⋅t +  ⎝s   + t ⎠  ⎠ + t⋅cos ⎝s⋅t +  ⎝

          2⎞                        ⎛                  2⎞ ⎤
2     2⎞   ⎟           ⎛ 2     2⎞   ⎜      ⎛ 2     2⎞   ⎟ ⎥
  + t ⎠   ⎠  + 4⋅t⋅ ⎝s   + t ⎠ ⋅cos ⎝s⋅t + ⎝s   + t ⎠  ⎠ ⎦
```

1.37 Example ($\mathbb{R}^2(r,s) \to \mathbb{R}^3(x,y,z) \to \mathbb{R}(f$

Suppose that

$$w(x,y,z) = x^2y + y^2z^3$$
$$(x,y,z) = (r\cos s, r\sin s, re^s)$$

Find $\partial w/\partial s$ at $(r,s) = (1,0)$

In [59]:
```
x,y,z,r,s=symbols("x y z r s")
w=x*x*y+y*y*z*z*z
Xt=[r*cos(s),r*sin(s),r*exp(s)]
wxyz=ChainRule(w, [x,y,z],[r,s],Xt,output=1)
```

```
∂ x**2*y + y**2*z**3 /∂ [r, s]

⎡   4  3⋅s    2           2             2         ⎛   4  3⋅s
⎣3⋅r ⋅e   ⋅sin (s) + 2⋅r ⋅sin(s)⋅cos (s) + ⎝2⋅r ⋅e   ⋅sin(

          5  3⋅s    2            3    2                  ⎛   4
(s)   3⋅r ⋅e   ⋅sin (s) - 2⋅r ⋅sin (s)⋅cos(s) + r⋅ ⎝2⋅r ⋅e

 ⎞         ⎤
)⎠ ⋅cos(s)⎦
```

In [62]:
```
wxyz.subs({r:1,s:0})[1]
```

Out[62]: 1

And

$$\left.\frac{\partial w}{\partial s}\right|_{(r,s)=(1,0)} = 3\cdot 1\cdot 0 - 2\cdot 1\cdot 0 + 1\cdot(2\cdot 0 + 1\cdot 1) =$$

1.38 Example

Suppose that

$$f(x,y,z) = \frac{1}{x^2+y^2+z^2}$$
$$(x,y,z) = (r\cos t, r\sin t, r)$$

Then

$$\begin{pmatrix} f_x & f_y & f_z \end{pmatrix} = \frac{-2}{(x^2+y^2+z^2)^2}\begin{pmatrix} x & y & z \end{pmatrix}$$
$$\frac{\partial(x,y,z)}{\partial(r,t)} = \begin{pmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} f_r & f_t \end{pmatrix} = \frac{-2}{(x^2+y^2+z^2)^2}\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} \cos t & - \\ \sin t & r \\ 1 & \end{pmatrix}$$
$$= \frac{-2}{(x^2+y^2+z^2)^2}\begin{pmatrix} x\cos t + y\sin t + z \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{2r\sin(t)^2+2r\cos(t)^2+2r}{(r^2\sin(t)^2+r^2\cos(t)^2+r^2)^2} & 0 \end{pmatrix}$$

In [65]:

```python
x,y,z,r,t=symbols("x y z r t")
f=1/(x*x+y*y+z*z)
Xt=[r*cos(t),r*sin(t),r]
ChainRule(f, [x,y,z],[r,t],Xt)
```

```
∂ 1/(x**2 + y**2 + z**2) /∂ [r, t]

⎡                        2                                     2
⎢              2⋅r⋅sin (t)                           2⋅r⋅cos (t
⎢- ─────────────────────────────────────────── - ─────────────────
⎢                                            2
⎢   ⎛ 2    2         2    2         2⎞         ⎛ 2    2         2
⎣   ⎝r ⋅sin (t) + r ⋅cos (t) + r ⎠         ⎝r ⋅sin (t) + r ⋅c

                                    ⎤
         2⋅r                        ⎥
──────────────────────────────── 0 ⎥
                              2     ⎥
2          2     2          2⎞       ⎥
 (t) + r ⋅cos (t) + r ⎠       ⎦
```

1.39 Exercise

1. Suppose that $f(x, y) = x^2 + 3xy + y^2$ and $(x, y) = (st,$
   f with respect to (x, y) and (s, t).

2. Suppose that $f(x^1, x^2, \cdots, x^n) = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$
   equal to power i of t. Find all the first-order partial deriva

1.40 Answer

1.

$$\left(\frac{\partial f}{\partial x^i}\right) = \begin{pmatrix} 2x + 3y & 3x + 2y \end{pmatrix}$$

$$\left(\frac{\partial x^i}{\partial (st)}\right) = \begin{pmatrix} t & s \\ 2st & s^2 \end{pmatrix}$$

$$\left(\frac{\partial f}{\partial (st)}\right) = \begin{pmatrix} 2x + 3y & 3x + 2y \end{pmatrix} \begin{pmatrix} t & s \\ 2st & s^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2st + 3s^2 t & 3st + 2s^2 t \end{pmatrix} \begin{pmatrix} t & s \\ 2st & s^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2st^2 + 9s^2 t^2 + 4s^3 t^2 & 3s^2 t + 6s^3 t + 2s^4 t \end{pmatrix}$$

2.

$$(\partial f / \partial x^i) = \left(\frac{x^i}{\sqrt{(x^1)^2 + \cdots + (x^n)^2}}\right)$$

$$(\partial f / \partial t) = \left(\frac{x^i}{\sqrt{(x^1)^2 + \cdots + (x^n)^2}}\right) (\partial x^i / \partial t)$$

$$= \sum_{i=1}^{n} \frac{i x^i (t)^{i-1}}{\sqrt{(x^1)^2 + \cdots + (x^n)^2}}$$

$$= \sum_{i=1}^{n} \frac{i (t)^{2i-1}}{\sqrt{(t)^2 + \cdots + (t)^{2n}}}$$

1.41 p. 1093

4. $w = \ln(x + y^2), (x, y) = (\tan t, \sec t)$

In [23]:
```
from sympy import tan,sec
x,y,t=symbols("x y  t")
w=log(x+y**2)
Xt=[tan(t),sec(t)]

ChainRule(w, [x,y],[t],Xt)
```

```
d ( log(x + y**2)) /d [t]
```

$$\left[\frac{\tan^2(t) + 1}{\tan(t) + \sec^2(t)} + \frac{2\cdot\tan(t)\cdot\sec^2(t)}{\tan(t) + \sec^2(t)} \right]$$

8. $w = x\sqrt{y^2 + z^2}, (x, y, z) = (1/t, e^{-t}\cos t, e^{-t}\sin t)$

In [24]:
```
x,y,z,t=symbols("x y z t")
w=[x*sqrt(y*y+z*z)]
Xt=[1/t,exp(-t)*cos(t),exp(-t)*sin(t)]

ChainRule(w, [x,y,z],[t],Xt)
```

```
d ( [x*sqrt(y**2 + z**2)]) /d [t]
```

$$\left[\frac{\left(- e^{-t}\cdot\sin(t) - e^{-t}\cdot\cos(t)\right)\cdot e^{-t}\cdot\cos(t)}{t\cdot\sqrt{e^{-2\cdot t}\cdot\sin^2(t) + e^{-2\cdot t}\cdot\cos^2(t)}} + \frac{\left(- e^{-t}\cdot\sin(t) \ldots\right.}{t\cdot\sqrt{e^{-2\cdot t} \ldots}} \right.$$

$$\left. \ldots t) - \frac{\sqrt{e^{-2\cdot t}\cdot\sin^2(t) + e^{-2\cdot t}\cdot\cos^2(t)}}{t^2} \right]$$

10. $w = \sin xy, (x, y) = \left((u + v)^3, \sqrt{v}\right)$

In [25]:

```
x,y,u,v=symbols("x y u v")
w=sin(x*y)
Xt=[(u+v)**3,sqrt(v)]

ChainRule(w, [x,y],[u,v],Xt)
```

```
∂ sin(x*y) /∂ [u, v]

⎡
⎢                2       ⎛             3⎞                2      ⎛
⎢ 3⋅√v⋅(u + v) ⋅cos⎝√v⋅(u + v) ⎠  3⋅√v⋅(u + v) ⋅cos⎝√v⋅(u
⎣

⎛            3⎞ ⎤
⎝√v⋅(u + v) ⎠ ⎥
─────────────⎥
⋅√v           ⎦
```

20. Let $w = x\sqrt{y} + \sqrt{x}$, $(x, y) = (2s + t, s^2 - 7t)$; evaluat

In [50]:

```
x,y,u,v=symbols("x y u v")
w=x*sqrt(y)+sqrt(x)
Xt=[2*s+t,s*2-7*t]

wst=ChainRule(w, [x,y],[s,t],Xt,output=1)
```

```
∂ sqrt(x) + x*sqrt(y) /∂ [s, t]

⎡       _________            1                2⋅s + t             ______
⎢ 2⋅╲╱  2⋅s - 7⋅t  + ────────────── + ────────────────    ╲╱  2⋅s
⎢                       _________          _________
⎣                     ╲╱  2⋅s + t        ╲╱  2⋅s - 7⋅t

   7⋅(2⋅s + t)    ⎤
- ─────────────── ⎥
       _________  ⎥
  2⋅╲╱  2⋅s - 7⋅t ⎦
```

In [57]:

```
wst.subs({s:4,t:1})[1]
```

Out[57]: -91/3

$$\left.\frac{\partial \mathbf{w}}{\partial \mathbf{t}}\right|_{(\mathbf{s},\mathbf{t})=(\mathbf{4},\mathbf{1})} = 1 + 1/6 - 63/2 = -91/3$$

28. Given $x = (u^2 - v^2)/2$, $y = uv$, find $\partial(x, y)/\partial(u, v)$, $\partial(u,$

In [41]:
```
x,y,u,v = symbols("x y u v")
fxy=Matrix([(u*u-v*v)/2, u*v])
print("∂(x,y)/∂(u,v) = %s" %fxy.jacobian([u,v]))
```

```
∂(x,y)/∂(u,v) = Matrix([[u, -v], [v, u]])
```

From the fact,

$$\left(\frac{\partial(u,v)}{\partial(x,y)}\right) = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1} = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}^{-1} = \frac{1}{u^2+v^2}$$

In [ ]:

Type *Markdown* and LaTeX: α^2

1.42 Tangent Plane

1. Let $P(a,b,c)$ on the surface S, at which satisfies $F(x,y,$
   parallel to $\nabla F(a,b,c)$, i.e.
   $$\frac{x-a}{F_x(a,b,c)} = \frac{y-b}{F_y(a,b,c)} = \frac{z-c}{F_z(a,b,c)}$$
2. Partial derivative represents the ratio of changes in the re
   could exist a tangent plane for $z = f(x,y)$ at certain poir
   vector.

Suppose that the surface in $\mathbb{R}^3$ satisfies:

$$\begin{aligned} z &= f(x,y) \\ &\Downarrow \\ 0 &= F(x,y,z) \\ &= f(x,y) - z \end{aligned}$$

And suppose that all the partial derivatives of $f(x,y)$ are cont
represented as follows:
$(x(t), y(t), z(t))$
Thus we have:

$$
\begin{aligned}
0 &= F(x(t), y(t), z(t)) \\
&\Downarrow \\
0 &= \frac{dF(t)}{dt} \\
&= \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt} \\
(1^\circ) &= \nabla F \cdot \frac{d(x, y, z)}{dt} \\
(2^\circ) &= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + (-1) \cdot \frac{dz}{dt} \\
&= (\nabla f, -1) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)
\end{aligned}
$$

Given that $(x_0, y_0, f(x_0, y_0))$ lies on the surface, and so in th tangent plane, the vector $(x - x_0, y - y_0, z - f(x_0, y_0))$ mus to the normal to the curve and this tangent plane is always in

$$
\begin{aligned}
0 &= (\nabla f, -1) \cdot (x - x_0, y - y_0, z - f(x_0,
\\
&\Downarrow \\
f(x, y) - f(x_0, y_0) &= \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) \\
&\Downarrow \\
f(x, y) &= f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0
\end{aligned}
$$

1.43 Example

The gradient of $f(x, y) = \sqrt{x} + \sqrt{y}$ at $(x, y) = (1, 1)$ is:

$$
\begin{aligned}
\nabla f(1,1) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Bigg|_{(1,1)} \\
&= \left(\frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}}\right)\Bigg|_{(1,1)} \\
&= \left(\frac{1}{2}, \frac{1}{2}\right)
\end{aligned}
$$

Then the normal vector of the tangent plane passing through

$$
0 = \left(\frac{1}{2}, \frac{1}{2}, -1\right) \cdot (x - 1, y - 1, z - 2)
$$

$$\Downarrow$$

$$2z = x + y + 2$$

1.44 Example

The normal line and tangent plane of $4x^2 + y^2 + 4z^2 = 16$

$$\frac{x-1}{8} = \frac{y-2}{4} = \frac{z-\sqrt{2}}{8\sqrt{2}}$$

$$8(x-1) + 4(y-2) + 8\sqrt{2}(z - \sqrt{2}) = 0$$

1.45 Example

The normal line and tangent plane of $f(x, y) = 4x^2 + y^2 + 2$

$$\frac{x-1}{-8} = \frac{y-1}{-2} = \frac{z-7}{1}$$

$$-8(x-1) - 2(y-1) + (z-7) = 0$$

In [4]:
```python
from sympy import log,exp,cos,sin,diff,integrate,symbols,Ma

x,y,z=symbols("x y z")
```

In [37]:
```python
grad = lambda func, vars :[diff(func,var) for var in vars]

def df_val(f,val):
    return [ff.subs({x:val[0],y:val[1],z:val[2]}) for ff in

def tangentplane(f,X,A):
    if len(A)==2:
        A=[A[0],A[1],0]
        A[2]= f.subs({X[0]:A[0],X[1]:A[1]})
    if len(X)==2:
        f=f-z
        X=[X[0],X[1],z]
    df=grad(f,X)
    df0=df_val(df,A)

    print(df0[0]*(X[0]-A[0])+df0[1]*(X[1]-A[1])+df0[2]*(X[2
```

In [10]:
```python
f=4*x**2+y**2+4*z**2-16
tangentplane(f,[x,y,z],[1,2,sqrt(2)])
```

```
8*x + 4*y + 8*sqrt(2)*(z - sqrt(2)) - 16 = 0
```

1.46 p. 1115 Exercise

20. tangent plane of $xyz = -4$ at $(P = (2, -1, 2)$ is

In [7]:
```python
f=x*y*z+4
tangentplane(f,[x,y,z],[2,-1,2])
```

```
-2*x + 4*y - 2*z + 12 = 0
```

26. tangent plane of $z = \exp(x)\sin(\pi y)$ at $(P = (0, 1, 0))$ is

In [36]:
```python
f=exp(x)*sin(pi*y)-z
tangentplane(f,[x,y,z],[0,1,0])
```

```
-z - pi*(y - 1) = 0
```

In [35]:
```python
f=exp(x)*sin(pi*y)
tangentplane(f,[x,y],[0,1])
```

```
-z - pi*(y - 1) = 0
```

The change of f in the other directions different to $x, y, \cdots,$ ca

1.47 Definition

The directional derivative in the unitary direction, $\vec{e} = (e^1$
$D_{\vec{e}} f = \nabla f \cdot \vec{e}$
where $\cdot$ means inner product.

```
In [ ]:
```

1.48 Example

The directional derivative of $f(x, y) = \sqrt{x} + \sqrt{y}$ at $(x, y) =$

$$
\begin{aligned}
(3,4) &\rightarrow \frac{1}{5}(3,4) \\
D_{\vec{e}} f(1,1) &= \nabla f(1,1) \cdot \vec{e} \\
&= \frac{1}{2}(1,1) \cdot \frac{1}{5}(3,4) \\
&= \frac{7}{10}
\end{aligned}
$$

In which direction does the directional derivative attain its ma
$\vec{a}$ and $\vec{b}$ is:
$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$
where θ is the intersection angle between $\vec{a}$ and $\vec{b}$, the direct
and $\vec{e}$ are parallel.

1.49 Example

The directional derivative of $f(x,y) = \exp^x \cos y$ at $(x,y) =$

$$\begin{aligned}
(2,3) &\rightarrow \frac{1}{\sqrt{13}}(2,3) \\
D_{\vec{e}} f(0,\pi/4) &= \nabla f(0,\pi/4)\cdot \vec{e} \\
&= (0,-2)\cdot \frac{1}{\sqrt{13}}(2,3) \\
&= \frac{-6}{\sqrt{13}}
\end{aligned}$$

1.50 Theorem

Directional derivative will attain its maximum (minimum) if
$\vec{e} = \nabla f/\|\nabla f\|(-\nabla f/\nabla f\|)$

1.51 Example

The maximum of directional derivative of $f(x,y) = \sqrt{x} + \sqrt{}$
$\vec{e} = \nabla f/\|\nabla f\| = (1/2,1/2)/\sqrt{(1/2)^2 + (1/2)^2} = (1/\sqrt{2}, 1$
and is equal to:
maximum of $D_{\vec{e}} f(1,1) = (1/2,1/2)\cdot(1/\sqrt{2}, 1/\sqrt{2}) = 1/$

In [15]:
```
from numpy import sqrt
#X = np.arange(.2, 1, 0.02)
#Y = np.arange(0.2, 1, 0.02)
x = np.arange(0, 1.4, 0.1)
y = np.arange(0, 1.4, 0.1)

t = np.arange(-0.2, 1.2, 0.02)
s = np.arange(0.4, 0.8, 0.01)
X,Y = np.meshgrid(x,y)
f= sqrt(X) + sqrt(Y)
z0=sqrt(0.4**2+0.4**2)
u=np.arange(0., z0, 0.01)
```

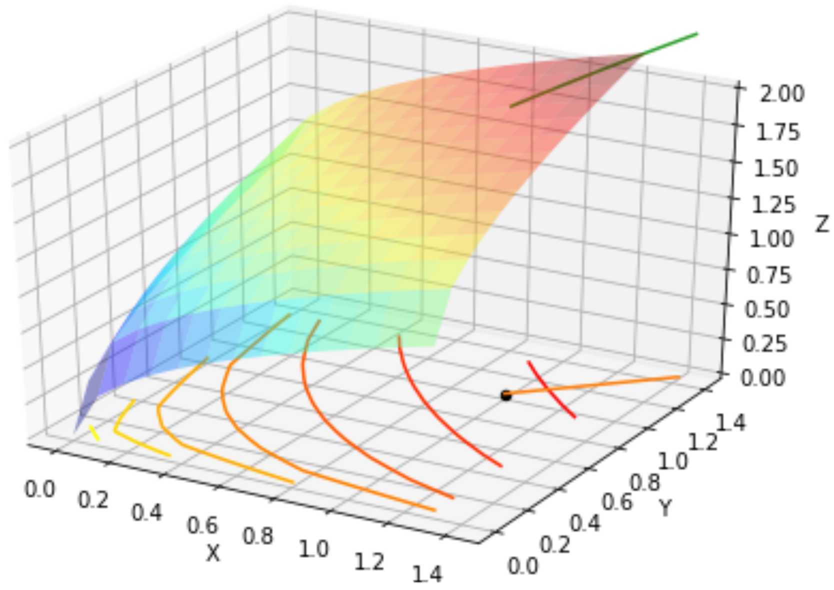
```
Pf=(X-1)/2+(Y-1)/2+2
surface = go.Surface(x=X, y=Y, z=f,opacity=0.95)
P = go.Surface(x=X, y=Y, z=Pf,colorscale=0.1,opacity=1)
Xaxis = go.Scatter3d(x=t, y=0*t, z=0*t,
            mode = "lines",
            line = dict(
                        color='black',
                        width = 5
            )
        )
Yaxis = go.Scatter3d(x=0*t, y=t, z=0*t,
            mode = "lines",
            line = dict(
                        color='black',
                        width = 5
            )
        )
X0 = go.Scatter3d(x=[1,1], y=[1,1], z=[0,2],
            mode = "lines",
            line = dict(
                        color='black',
                        width = 5
            )
    )
XY = go.Scatter3d(x=[1,1+1/2.], y=[1,1+1/2.], z=[0,0],
            mode = "lines",
            line = dict(
                        color='blue',
                        width = 3
            )
    )
N = go.Scatter3d(x=[1,1+1/2.], y=[1,1+1/2.], z=[2,2-1],
            mode = "lines",
            line = dict(
                        color='blue',
                        width = 3
            )
    )
Y0 = go.Scatter3d(y=s, x=0.4+0*s, z=0*s,
            mode = "lines",
            line = dict(
                        color='orange',
                        width = 5
            )
    )
#Line2 = go.Scatter3d(x=0*t, y=t, z=0*t)
#Line3 = go.Scatter3d(x=t, y=t, z=np.ones(len(t))/2)
#Line4 = go.Scatter3d(x=t, y=-t, z=-np.ones(len(t))/2)
data = [surface,Xaxis,Yaxis,X0,XY,N,P]

fig = go.Figure(data=data)
iplot(fig)
```

In [4]:

```
def plot3d(x,y,z):
    fig = plt.figure()
    ax = Axes3D(fig)
    ax.plot_surface(x, y, z, rstride=1, cstride=1, cmap=plt
    ax.contour(x, y, z,  lw=3, cmap="autumn_r", linestyles=
    ax.set_xlabel('X')
    ax.set_ylabel('Y')
    ax.set_zlabel('Z')
    ax.set_zlim(0, 2)
    ax.scatter3D([1],[1],[0],color=(0,0,0));
    ax.arrow(x=1,y=1,dx=0.1,dy=0.1)
    xt=np.linspace(1,1.414,100)
    yt=np.linspace(1,1.414,100)
    zt=np.zeros(100)
    ax.plot3D(xt,yt,zt)
    ax.plot3D(xt,yt,np.sqrt(xt)+np.sqrt(yt))
```

```
In [5]: x = np.arange(0, 1.4, 0.1)
        y = np.arange(0, 1.4, 0.1)
        x,y=np.meshgrid(x,y)
        f= np.sqrt(x)+np.sqrt(y)
        plot3d(x,y,f)
```



From above picture, the value of $f(x, y)$ inscreases fastest al projection on the $X - Y$ plane is orthogonal to the level curv

1.52 Example

Find directional derivative of $f(x, y) = x^2 - 2xy$ at $(x, y) =$ $(2-(-1), 3-2) = (3, 1) \to \frac{1}{\sqrt{10}}(3, 1)$

$$
\begin{aligned}
D_e f(1, -2) &= \nabla f(1, -2) \cdot e^{\rightarrow} \\
&= (6, -2) \cdot \frac{1}{\sqrt{10}}(3, 1) \\
&= \frac{16}{\sqrt{10}}
\end{aligned}
$$

1.53 Example

Suppose that $f(x) = x^2 \sin(\pi y/6)$. **1.** The gradient of $f(x)$ at

$$\nabla f(1,1) = \left(2x \sin(\pi y/6), \pi x^2 \cos(\pi y/6)/6\right)\Big|_{(x,y)=(1,1)}$$

$$= \left(1, \frac{\sqrt{3}\pi}{12}\right)$$

2. The directional derivative at the direction, $\vec{u} = (1,0)$, is:

$$\nabla_u f(1,1) = \left(1, \frac{\sqrt{3}\pi}{12}\right) \cdot (1,0) = 1$$

3. The directional derivative at the direction, $\vec{v} = (1,1)$, is:

$$\vec{v} \Rightarrow (1,1)/\sqrt{1^2+1^2} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\Downarrow$$

$$\nabla_v f(1,1) = \left(1, \frac{\sqrt{3}\pi}{12}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{\sqrt{6}}{24}$$

4. The maximum of the directional derivative is:

$$\|\nabla f(1,1)\| = \sqrt{1^2 + \left(\frac{\sqrt{3}\pi}{12}\right)^2}$$

and in the direction:

$$\vec{e} = \nabla f(1,1)/\|\nabla f(1,1)\|$$

1.54 Example

Suppose that $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$. Find the directional de

directions $\overrightarrow{e}_1 = (2, 1, -2)$, b). Find the direction at which th
what is the maximal rate of increase.

- gradient at P

$$\nabla f(P) = \left(\frac{-x/\sqrt{x^2+y^2}}{(\sqrt{x^2+y^2}+z^2)^2}, \frac{-y/\sqrt{x^2+y^2}}{(\sqrt{x^2+y^2}+z^2)^2}, \frac{-}{(} \right.$$

$$= \left(\frac{-1/\sqrt{5}}{(\sqrt{5}+9)^2}, \frac{-2/\sqrt{5}}{(\sqrt{5}+9)^2}, \frac{-6}{(\sqrt{5}+9)^2} \right)$$

- unit direction:

$$\vec{v} \Rightarrow (2, 1, 2)/\sqrt{2^2+1^2+2^2} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right)$$

- directional derivative:

$$\nabla f(P) \cdot \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right) = \frac{-12}{(\sqrt{5}+9)^2}$$

- At the direction, $\nabla f(P)$, the directional derivative increas

1.55 Exercise

Suppose that $f(x, y) = 3x^2 + 4xy + 5y^2$. Find the direction
a) $\overrightarrow{e}_1 = (3, -4)$, b) $\overrightarrow{e}_2 = (1, 1)$. Find the direction at whic

In [20]:
```
f=3*x*x+4*x*y+5*y*y
df=grad(f,[x,y])
dfv=df_val(df,[1,1])
def df_dir(f,val):
    l=f[0]*val[0]+f[1]*val[1]
    return l/(sqrt(val[0]**2+val[1]**2))
df_dir(dfv,[3,-4])
```

Out[20]: -26/5

In [21]:
```
df_dir(dfv,[1,1])
```
Out[21]:
```
12*sqrt(2)
```

Exercise p. 1105

8. Find the gradient of $f(x, y, z) = (x + y)/(x + z)$ at $(1, 2, 3$

In [2]:
```
x,y,z=symbols("x y z")
```

In [9]:
```
grad = lambda func, vars :[diff(func,var) for var in vars]

def df_valX(f,X,P):
    """
    input
    f: function
    X: [x,y,...], variables
    P: position
    output
    gradient vector at P
    """
    df=grad(f,X)
    return [ff.subs({X[i]:P[i] for i in range(len(X))}) for

def norm(v):
    """norm of v"""
    d=0
    for i in range(len(v)):
        d+=v[i]**2
    return sqrt(d)
def df_dir(f,X,P,vec):
    """
    Input
    f: function
    X: [x,y,...], variables
    P: position
    vec: direction
    output
    directional derivative of f at P in direction vec
    """
    dotsum=0
    dfv=df_valX(f,X,P)
    for i in range(len(dfv)):
        dotsum+=dfv[i]*vec[i]
    return dotsum/norm(vec)
```

In [10]:
```
f=(x+y)/(x+z)
df_valX(f,[x,y,z],[1,2,3])
```
Out[10]:
```
[1/16, 1/4, -3/16]
```

In [5]:
```
grad(f,[x,y,z])
```
Out[5]:
```
[-(x + y)/(x + z)**2 + 1/(x + z), 1/(x + z), -(x + y)/(x
```

12. Find the gradient of $f(x, y, z) = x^3 - y^3$ at $(2, 1)$ in the d

In [11]:
```
f=x**3-y**3
X=[x,y]
P=[2,1]
v=[1,1]

pprint(df_dir(f,X,P,v))
```
$$\frac{9 \cdot \sqrt{2}}{2}$$

20. Find the gradient of $f(x, y, z) = x^2 + 2xy^2 + 2yz^3$ at $(2,$

In [12]:
```
f=x*x+2*x*y*y+2*y*z**3
X=[x,y,z]
P=[2,1,-1]
v=[1,2,2]

df_dir(f,X,P,v)
```
Out[12]:
```
10
```

38. Find the direction at which the directional derivative of $f($

In [8]:
```
f=x*exp(-y**2)

df_valX(f,[x,y],[1,0])
```
Out[8]:
```
[1, 0]
```

This concludes that directional derivative increases rapidly at

In [4]:
```
!jupyter nbconvert --to html 6*Differ*-2.ipynb
```
```
[NbConvertApp] Converting notebook 6 Multi-variable Calcu
[NbConvertApp] Writing 2410764 bytes to 6 Multi-variable
```

In [ ]:

```
In [ ]:
```