1

Calculus, 2017-1-IE-1

1.0.1

Name: **Sequence Number:**

1° °). Find the following limits: (total 10%, each 5%)

a° °). $\lim\limits_{\theta\to 0}\frac{\sin 2\theta}{\theta}$ $\lim\limits_{\theta\to 0}\frac{\sin 2\theta}{\theta}$

b° °). $\lim\limits_{x\to\pi/4}\frac{\sin x-\frac{\sqrt{2}}{2}}{x-\frac{\pi}{4}}$ $\lim\limits_{x\to\pi/4}\frac{\sin x-\frac{\sqrt{2}}{2}}{x-\frac{\pi}{4}}$ (Hint: Consider the derivative of $\sin x$ $\sin x$).

2° °). Evaluate the following derivatives of given functions: (total 30%, each 5% ($\times 6$ $\times$ 6))

a° °). $\left[\left(\frac{x-x^{-1}}{x}\right)^2\right]'$ $\left[\left(\frac{x-x^{-1}}{x}\right)^2\right]'$ b° °). $\left[\sqrt{\left(\frac{1}{x}\right)}\right]'$ $\left[\sqrt{\left(\frac{1}{x}\right)}\right]'$ c° °).

$\left[x^2\cos x\right]'$ $\left[x^2\cos x\right]'$ d° °). $\left[\frac{2x-x^2}{x-1}\right]'$ $\left[\frac{2x-x^2}{x-1}\right]'$

e° °). $\frac{d}{dx}(x^4-x^3-4x+6)\Big|_{x=1}$ $\frac{d}{dx}(x^4-x^3-4x+6)\Big|_{x=1}$ f° °).

$\frac{d}{dx}\left(\frac{1+\sin x}{\cos x}\right)$ $\frac{d}{dx}\left(\frac{1+\sin x}{\cos x}\right)$ g° °). $D_x\left(\sin x^3\right)$ $D_x\left(\sin x^3\right)$

3° °). Find the values of A, B such that the following function is continuous and differentiable at $x = 1$: (total 10%)

$$f(x) = \begin{cases} x^2, \text{ if } x \le 1, \\ Ax + B, \text{ if } x > 1 \end{cases}$$

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4° °). (total 10%)

Suppose that

$$f(x) = x^2 \sin(-x)$$

$$f(x) = x^2 \sin(-x)$$

Find the third derivative of $f(x)$, $f'''(x)$.

5° °). (total 10%)
Suppose that $x, y \in \mathbb{R}$ satisfy:

$$\frac{x+y}{x-y} = y^2 + 1$$

$$\frac{x+y}{x-y} = y^2 + 1$$

Find the derivative of y (10%)

6° °). (20%) Suppose that $f(x) = x - 3x^{1/3}$ for all $x \in \mathbb{R}$.
a° °). (5%) Find all the critical values of $f(x)$.
b° °). (5%) Determine the interval at which $f(x)$ is increasing and concave downward.
c° °). (5%) Find all the relative extreme values of $f(x)$ if any.
d° °). (5%) Plot the graph of $f(x)$.

7° °). (total 10%, each 5%) Describe the following Theorems:
a° °). Rolle's Theorem
b° °). Mean Value Theorem.

1. a) 2 b) $\frac{\sqrt{2}}{2}$

2. a) $\frac{4\cdot\left(1-\frac{1}{x^{2}}\right)}{x^{3}}$

   b) $-0.5\cdot x^{-1.5}$

   c) $-x^{2}\cdot\sin(x)+2\cdot x\cdot\cos(x)$

   d) $\frac{-2\cdot x+2}{x-1}-\frac{-x^{2}+2\cdot x}{(x-1)^{2}}$

   e) 5

   f) $(1+\cos x)/\cos^{2} x$

   g) $3\cdot x^{2}\cdot\cos\left(x^{3}\right)$

3. A=2,B=-1
4. $x^{2}\cdot\cos(x)+6\cdot x\cdot\sin(x)-6\cdot\cos(x)$

5. y/(2y-x)

6. a) 0, 1,-1 b) x<-1 c)relative maximum f(-1)=2, relative minimum f(1)=-2

1.1 Answer

In [1]:

```
from sympy import symbols, pprint,limit,diff,sin,tan,sqrt,cos,pi
```

In [7]:

```
x,t =symbols("x t")
```

In [3]:

```
# 1. a)
print("1. a) The limit of sin(2t)/(t)) at t=0, is:")
pprint(limit(sin(2*t)/(t),t,0))
```

```
1. a) The limit of sin(2t)/(t)) at t=0, is:
2
```

In [6]:

```
#1. b)
print("1. b) The derivative of sin(x) at x=pi/4,, is:")
pprint(limit((sin(x)-sqrt(2)/2)/(x-pi/4),x,pi/4))
```

```
1. b) The derivative of sin(x) at x=pi/4,, is:
√2
──
2
```

In [14]:

```
#2. a)
print("2. a) The derivatice of (x^2-x^{-2})^2/x, is:")
pprint(diff((1-1/x**2)**2,x))
```

```
2. a) The derivatice of (x^2-x^{-2})^2/x, is:
  ⎛    1 ⎞
4⋅⎜1 - ──⎟
  ⎜     2⎟
  ⎝    x ⎠
──────────
     3
    x
```

In [21]:

```
#2. b)
print("2. b) The derivatice of x^(-1/2), is:")
pprint(diff(1/x**(1/2),x))
```

```
2. b) The derivatice of x^(-1/2), is:
      -1.5
-0.5⋅x
```

In [22]:

```
#2. c)
print("2. c) The derivatice of x^2 cos x, is:")
pprint(diff(x**2*cos(x),x))
```

```
2. c) The derivatice of x^2 cos x, is:
   2
- x ⋅sin(x) + 2⋅x⋅cos(x)
```

In [23]:

```
#2. d)
print("2. d) The derivatice of (2x-x^2)/(x-1), is:")
pprint(diff((2*x-x*x)/(x-1),x))
```

```
2. d) The derivatice of (2x-x^2)/(x-1), is:
                 2
-2⋅x + 2    - x   + 2⋅x
──────── - ────────────
 x - 1               2
              (x - 1)
```

In [24]:

```
#2. e)
print("2. e) The derivatice of -x^4-x^3+4x+6 at x=-1, is:")
pprint(diff(-x**4-x**3+4*x+6,x).subs({x:-1}))
```

```
2. e) The derivatice of -x^4-x^3+4x+6 at x=-1, is:
5
```

In [25]:

```
#2. f)
print("2. f) The derivatice of (1+sin x)/cos x, is:")
pprint(diff(sqrt(1+sin(x)/(cos(x))),x))
```

```
2. f) The derivatice of (1+sin x)/cos x, is:
      2
  sin (x)     1
 ───────── + ─
       2      2
 2⋅cos (x)
────────────────
      ___________
     ╱ sin(x)
    ╱  ────── + 1
  ╲╱   cos(x)
```

In [27]:

```
#2. g)
print("2. g) The derivatice of sin x^3, is:")
pprint(diff(sin(x**3),x))
```

```
2. g) The derivatice of sin x^3, is:
   2     ⎛ 3⎞
3⋅x ⋅cos⎝x ⎠
```

1.2 Note

above equal to the simple form, $\frac{1}{1+\cos x}\,\frac{1}{1+\cos x}$.

In [15]:

```
#2. g)
print("2. g) The derivatice of sin^3(x), is:")
pprint(diff(sin(x)**3,x))
```

```
2. g) The derivatice of sin^3(x), is:
     2
3⋅sin (x)⋅cos(x)
```

2. h)

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{h\sin(1/h)}{h} \\ &= \lim_{h\to 0} \sin(1/h) \end{aligned}$$

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{h\sin(1/h)}{h} \\ &= \lim_{h\to 0} \sin(1/h) \end{aligned}$$

This concludes that the limit fails to exist.

In [36]:

```
#3. a)
print("3. The third-order derivatice of x^2 sin x, is:")
pprint(diff(-x**2*sin(x),x,3))
```

```
3. The third-order derivatice of x^2 sin x, is:
 2
x ⋅cos(x) + 6⋅x⋅sin(x) - 6⋅cos(x)
```

In [34]:

```
def ImplicitDiff(express):
    l=diff(express,x);
    print("y'(x) =",solve(l,Derivative(y,x))[0])
```

In [37]:

```
y=Function("y")
y=y(x)
print("2. f) The derivatice of (x+y)/(x-y)=y^2+1, is:")
pprint(diff((x+y)/(x-y)-y*y-1,x))
```

```
2. f) The derivatice of (x+y)/(x-y)=y^2+1, is:
                    d                          ⎛d
⎞
                    ──(y(x)) + 1    (x + y(x))⋅⎜──(y(
x)) - 1⎟
         d          dx                         ⎝dx
⎠
- 2⋅y(x)⋅──(y(x)) + ──────────── + ──────────────────
──────
         dx           x - y(x)                        2
                                            (x - y(x))
```

If $y'(x) = 0y'(x) = 0$, it implies $x = \pm 1x = \pm 1$. Then $(\pm 1)^3 + y^3 = 3 \cdot (\pm 1) \rightarrow \mathbf{y = \pm 2^{1/3}}$

4). a)

$$\left|\frac{\cos a - \cos b}{a-b}\right| = |\sin x_0| \leq 1$$
$$\implies \quad |\cos a - \cos b| \leq |a-b|$$

b) $\mathbf{x} \in \mathbb{R}$

5). **a)** critical values:

$$f'(t) = \left(4t^{\frac{1}{3}} + 3t^{\frac{4}{3}}\right)' = \frac{4}{3}\left(t^{-\frac{2}{3}} + 3t^{\frac{1}{3}}\right) = \frac{4(1+3t)}{3t^{\frac{2}{3}}}$$

i). If $f'(t) = 0 \Rightarrow 1 + 3t = 0 \rightarrow t = -1/3$
ii). If $f'(t) = 0$ fails to exist, then the denominator is zero, $t^{2/3} = 0 \rightarrow t = 0$

b). Since

$$\lim_{t \to \pm\infty} f(t) = +\infty,$$

$f(t)$ can only attain its absolute minimum. The minimum is $f(-1/3)$, which is smaller than 0, since it is smaller than $f(0) = 0$.

6) .
a). Rolle's theorem, Assume that $f(x)$ is continuous on $[a, b]$ and differentiable in (a, b). If $f(a) = f(b)$, there exists at least $c \in (a, b)$ such that $f'(c) = 0$.
b). Mean Value Theorem, MVT, Assume that $f(x)$ is continuous on $[a, b]$ and differentiable in (a, b). There exists at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In [28]:

```
!jupyter nbconvert --to html 2016-1-me-1.ipynb
```

```
[NbConvertApp] Converting notebook 2016-1-me-1.ipynb
to html
[NbConvertApp] Writing 274903 bytes to 2016-1-me-1.ht
ml
```

In [ ]:

```

```

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2^\circ). Evaluate the following derivatives of given functions: (total 30%, each 5% (\color{brown}{\times6}))
a^\circ). \mathbf{\left[\left(\frac{x-x^{-1}}{x}\right)^{2} \right]' } b^\circ). \mathbf{\left[\sqrt{\left(\frac{1}{x}\right) } \right]'} c^\circ). \mathbf{\left[x^2 \cos x \right]'}
d^\circ). \mathbf{\left[\frac{2x-x^2}{x-1} \right]'}
e^\circ). \mathbf{\left.\frac{d }{dx}(x^4-x^3-4x+6)\right|_{x=1} } f^\circ). \mathbf{\frac{d }{dx}\left(\frac{1+\sin x}{\cos x}\right)} g^\circ). \mathbf{D_x\left(\sin x^3\right)}

3^\circ). Find the values of \mathbf{A,B} such that the following function is continuous and differentiable at \mathbf{x=1}: (total 10%)

\mathbf{f (x) = \left\{ \begin{array}{l} x^2 \text{, if } x \le 1,\\ Ax+B, \text{ if } x > 1 \end{array} \right. }

4^\circ). (total 10%)

Suppose that \mathbf{f(x)=x^2\sin (-x)} Find the third derivative of \mathbf{f(x)}, \mathbf{f'''(x)}.

5^\circ). (total 10%)
Suppose that x,y\in\mathbb{R} satisfy: \mathbf{ \frac{x+y}{x-y}=y^2+1} Find the derivative of y (10%)

6^\circ). (20%) Suppose that \mathbf{f (x) =x-3x^{1/3}} for all \mathbf{x\in \mathbb{R}}.
a^\circ). (5%) Find all the critical values of \mathbf{f(x)}.
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d^\circ). (5%) Plot the graph of \mathbf{f (x)}.

7^\circ). (total 10%, each 5%) Describe the following Theorems:
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Answer

In [1]:

In [7]:

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```
1. a) The limit of sin(2t)/(t)) at t=0, is:
2
```

In [6]:

1. b) The derivative of sin(x) at x=pi/4,, is:

$$\frac{\sqrt{2}}{2}$$

In [14]:

2. a) The derivatice of (x^2-x^{-2})^2/x, is:

$$\frac{4\cdot\left(1-\frac{1}{x^{2}}\right)}{x^{3}}$$

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2. b) The derivatice of x^(-1/2), is:

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$$\frac{-2 \cdot x + 2}{x - 1} - \frac{- x^{2} + 2 \cdot x}{(x - 1)^{2}}$$

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2. e) The derivatice of -x^4-x^3+4x+6 at x=-1, is:
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In [25]:

2. f) The derivatice of (1+sin x)/cos x, is:

$$\frac{\frac{\sin^{2}(x)}{2 \cdot \cos^{2}(x)} + \frac{1}{2}}{\sqrt{\frac{\sin(x)}{\cos(x)} + 1}}$$

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2. g) The derivatice of sin x^3, is:
   2     ⎛ 3⎞
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Note

above equal to the simple form, \mathbf{\frac{1}{1+\cos x}}.

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2. g) The derivatice of sin^3(x), is:
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2. h)

\begin{eqnarray*} f' (0) & = & \lim_{h \rightarrow 0} \frac{f (h) - f (0)}{h}\\ & = & \lim_{h \rightarrow 0} \frac{h \sin (1 / h)}{h}\\ & = & \lim_{h \rightarrow 0} \sin (1 / h) \end{eqnarray*} This concludes that the limit fails to exist.

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2. f) The derivatice of (x+y)/(x-y)=y^2+1, is:
```

$$- 2\cdot y(x)\cdot\frac{d}{dx}(y(x)) + \frac{\frac{d}{dx}(y(x)) + 1}{x - y(x)} + \frac{(x + y(x))\cdot\left(\frac{d}{dx}(y(x)) - 1\right)}{(x - y(x))^2}$$

If $y'(x)=0$, it implies $x=\pm1$. Then $(\pm1)^3+y^3=3\cdot(\pm1)\to \mathbf{y=\pm2^{1/3}}$

4). a)

$$\begin{aligned} \left|\frac{\cos a-\cos b}{a-b}\right|&= \left|\sin x_0\right|\le1\\ \Longrightarrow&\left|\cos a-\cos b\right|\le \left|a-b\right| \end{aligned}$$

b) $\mathbf{x\in\mathbb{R}}$

5). **a)** critical values: $f'(t)=\left(4t^{\frac{1}{3}}+3t^{\frac{4}{3}}\right)'=\frac{4}{3}\left(t^{-\frac{2}{3}}+3t^{\frac{1}{3}}\right) =\frac{4(1+3t)}{3t^\frac{2}{3}}$

i). If $f'(t)=0\Rightarrow1+3t=0\to t=-1/3$
ii). If $f'(t)=0$ fails to exist, then the denominator is zero, $t^{2/3}=0\to t=0$

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In [28]:

```
[NbConvertApp] Converting notebook 2016-1-me-1.ipynb to html
[NbConvertApp] Writing 274903 bytes to 2016-1-me-1.html
```

In [ ]: