6. Infinite Series

Prelude

莊子天下篇 (33)
『一尺之捶， 日取其半，萬世不竭。』

What does this mean?

$$1, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \cdots, \left(\frac{1}{2}\right)^n, \cdots$$

None of above is equal to **zero**!

In [3]:

```
%matplotlib inline

#rcParams['figure.figsize'] = (10,3) #wide graphs by default
import scipy
import numpy as np
import time
from IPython.display import clear_output,display
import matplotlib.pylab as plt
```

Infinite Sequences

An infinite sequence is a set of indexed numbers, terms, denoted as $\{a_1, a_2, a_3, \cdots\}$ or $\{a_n\}_{n=1}^{\infty}$:

1. $\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$
2. $\{e^{-\lambda}, e^{-\lambda}\lambda, e^{-\lambda}\lambda^2/2, \cdots, e^{-\lambda}\lambda^n/n!, \cdots\}$, where $\lambda > 0$.

Limit of Sequence

L is said to be the limit of $\{a_n\}_{n=1}^{\infty}$ if

$$\lim_{n\to\infty} a_n = L$$

If L exists, sequences is called converent orthwised divergent.

$\{\sin(\frac{\pi}{n})\}_{n=1}^{\infty}$ is convergent since

$$\lim\ {}_{n\to\infty}\sin\left(\frac{\pi}{n}\right) = \sin(0) = 0$$

$\left\{\sin(\frac{n\pi}{2})\right\}_{n=1}^{\infty}$ is divergent since

$$\begin{array}{c} \sin\left(\frac{n\pi}{2}\right) \overset{n=1,2,\cdots}{\longrightarrow} \quad 1, 0, -1, 0, \cdots \\ \Downarrow \\ \lim\ {}_{n\to\infty}\sin\left(\frac{n\pi}{2}\right) \quad \neq \quad L \in \mathrm{R} \end{array}$$

Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent, then

1. $\lim\ {}_{n\to\infty}\{a_n \pm b_n\}_{n=1}^{\infty} = \lim\ {}_{n\to\infty}\{a_n\}_{n=1}^{\infty} \pm \lim\ {}_{n\to\infty}\{b_n\}_{n=1}^{\infty}$
2. $\lim\ {}_{n\to\infty}\{ca_n\}_{n=1}^{\infty} = c\ \lim\ {}_{n\to\infty}\{a_n\}_{n=1}^{\infty}$
3. $\lim\ {}_{n\to\infty}\{\frac{a_n}{b_n}\}_{n=1}^{\infty} = \frac{\lim_{n\to\infty}\{a_n\}_{n=1}^{\infty}}{\lim_{n\to\infty}\{b_n\}_{n=1}^{\infty}}$ provided $\lim\ {}_{n\to\infty}\{b_n\}_{n=1}^{\infty} \neq 0$.

Note

$$\lim_{n\to\infty}\frac{1}{n} = 0$$

Example

The limit of $\left\{\frac{6n^3+5n^2+7}{4n^3-2n+2}\right\}_{n=1}^{\infty}$ is:

$$\begin{aligned} \lim_{n\to\infty}\frac{6n^3+5n^2+7}{4n^3-2n+2} &= \lim_{n\to\infty}\frac{6+\frac{5}{n}+\frac{7}{n^3}}{4-\frac{2}{n^2}+\frac{2}{n^3}} \\ &= \frac{3}{2} \end{aligned}$$

$\lim_{n\to\infty}\ \{\ln(n+4) - \ln(n)\} = 0$ since

$$\begin{aligned}\ln(n+4) - \ln(n) &= \ln\left(\frac{n+4}{n}\right)\\ &= \ln\left(\frac{1+\frac{4}{n}}{1}\right)\\ &\to \ln(1) = 0\end{aligned}$$

Example

Certain company inspects its products with size n randomly. If $\alpha\% = 2\%$ of the entire lot is defective, then the the probability of finding no defectives products of size n is $\left(\frac{100-\alpha}{100}\right)^n$.

And the probability of finding at least one defective product is $P(n) = 1 - \left(\frac{100-\alpha}{100}\right)^n$.

Since $\lim_{n\to\infty} P(n) = 1$,the probability of finding at least one defective product is 100% as the size increases largely.

Theorem

Bounded, Monotonic implies Convergent.

A bounded and monotone sequence is convergent, increasing with bounded above or decreating with bounded below, implies $\lim a_n$ exists!

By completeness Axiom.

For instance:

1. $1, \frac{1}{2}, \frac{1}{3}, \cdots \searrow$ and $\leqslant 1$, convergent;
2. $\ln 1, \ln 2, \ln 3, \cdots \nearrow$ \ but no upper bound, $\not\Rightarrow$ convergent.
3. $a_1 = 1/2$, $a_n = (1 + a_{n-1})/2$ for $n = 2, 3, \cdots$. Since $a_n \leqslant 1$ and $a_n \nearrow$, $\{a_n\}$ is convergent and limit is equal to 1.
4. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
5. $\lim_{n\to\infty} \frac{e^n}{n^2} = \infty$

Example

$\left\{\frac{\mathbf{2^n}}{\mathbf{n!}}\right\}$ is convergent to zero since it is bounded below and decreasing.

Example

Suppose that $f(x) = 2/x, x \neq 0$.

1. Define a sequence, $\{a_n\}_{n=0}^{\infty}$ as follows:

$$a_n = \begin{cases} a & \text{if } n = 0, \\ f(a_{n-1}) & \text{if } n \neq 0. \end{cases}$$

Since $f(a) = 2/a$ and $f(2/a) = a$ the sequence is also listed as follows:

$$\{a, 2/a, a, 2/a, \cdots\}$$

Note that $f^2(x) = f(f(x)) = x$, x is also called periodic point with cycle of period 2.

2. Suppose that the sequence is convergent, its limit can be found out by following s:

$$\begin{aligned} L &= \lim_{n\to\infty} f^n(x) \\ &= \lim_{n\to\infty} f^{n+1}(x) = f(\lim_{n\to\infty} f^n(x)) \\ &= f(L) = 2/L \\ &\Rightarrow L^2 = 2 \\ &\Rightarrow L = \pm\sqrt{2} \end{aligned}$$

i.e. limit is $\pm\sqrt{2}$ depend on the sign of initial value, a.

3. How to find the value of $\sqrt{2}$ only by additions, multiplication and divisions?

If $a_0 = 1$, the sequence is $\{1, 2, 1, 2, \cdots\}$. In other words, it is not convergent. Redefine this sequence as the follows:

$$a_0 = 1, a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \text{ if } n \neq 0$$

Following gives the first ten results:

Exercise (p743)

26. $a_n = \cos n\pi + 2$ divergent, since $a_n = (-1)^n + 2$.

32. $a_n = \frac{\ln n^2}{\sqrt{n}}$ convergent since $\lim_{n\to\infty} \frac{2\ln n}{n^{1/2}} \to \infty$

39. $a_n = \frac{\sin^2 n}{\sqrt{n}}$ convergent since $|a_n| \leq \frac{1}{\sqrt{n}} \to 0$

40. $a_n = \frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{n!}$ divergent

$$\begin{aligned} a_n &= \frac{(2n)!}{n!2\cdot 4\cdot 6\cdots(2n-2)\cdot 2n} \\ &= \frac{(n+1)\cdot(n+2)\cdots(2n-1)\cdot(2n)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot 2n} \\ &= \frac{n+1}{2}\cdot\frac{n+2}{4}\cdots\cdot\frac{2n-1}{2n-2}\cdot\frac{2n}{2n} \\ &\geq \frac{n+1}{2}\to\infty \end{aligned}$$

50. $\lim\limits_{n\to\infty} n\left(1-\sqrt[7]{1-\frac{1}{n}}\right)=\frac{1}{7}$

$$\begin{aligned} \frac{1-\sqrt[7]{1-\frac{1}{n}}}{1/n} &= \frac{1-\sqrt[7]{1-x}}{x} \text{ where } x=1/n \\ &= \frac{1-(1-x)}{x}\cdot\frac{1}{\sqrt[7]{(1-x)^6}+\sqrt[7]{(1-x)^5}+\cdots+1}\to\frac{1}{7} \end{aligned}$$

54. $a_n=2+\frac{(-1)^n}{n}$ is bounded but not monotonic. $\{a_n\}$ is convergent since $\frac{(-1)^n}{n}\to 0$.

68. $\lim \sqrt{2\sqrt{2\sqrt{2\sqrt{\cdots}}}}=2$

- let $a_0=\sqrt{2}, a_n=\sqrt{a_{n-1}}$
- $a_n\leq 4$ and a_n is increasing; theirfore $L=\lim a_n$ exists.
- $L=\sqrt{2L}$ implies $L=2$.

74. For any $a>0$, there exist one n, large enough such that $1/n<a<n$, therefore $\lim\limits_{n\to\infty}\sqrt[n]{a}=1$

In [2]:

```
import numpy as np
x=1
for n in range(10):
  x=(x+2/x)/2
  print(x)
```

```
1.5
1.4166666666666665
1.4142156862745097
1.4142135623746899
1.414213562373095
1.414213562373095
1.414213562373095
1.414213562373095
1.414213562373095
1.414213562373095
```

Suppose that a sequence, $\{a_n\}_{n=0}^{\infty}$, satisfies the following recursive dormula:

$$a_{n+1}=f(a_n)$$

for some function $f(x)$ and $n = 1, 2, , 3, \cdots$. Every element in the sequence can be represented in the form of composed function f^n as following:

$$\begin{aligned} a_n &= f(a_{n-1}) \\ &= f(f(a_{n-2})) = f^2(a_{n-2}) \\ &= \cdots \\ &= f^n(a_0) \end{aligned}$$

Definition

The first term a_0 in $\{a_n\}_{n=0}^{\infty}$, with $a_{n+1} = f(a_n)$, is called periodic point with cycle of period m if there exists $m \in \mathrm{N}$ such that

$$f(a_m) = a_0$$

a_0 is called fixed point if $f(a_0) = a_0$.

Theorem

Suppose that sequence $\{a_n\}_{n=0}^{\infty}$, with $a_n = f(a_{n-1})$ and f being differentiable, is period m. Then

$$f'(a_{n-1}) = \prod_{k=0}^{n-1} f'(a_k)$$

This is trivial since

$$\begin{aligned} (a_n)' &= (f^n(a_0))' \\ &= (f(f^{n-1}(a_0)))' \\ &= f'(f^{n-1}(a_0)) \cdot (f^{n-1}(a_0))' \\ &= f'(a_{n-1}) \cdot (f^{n-1}(a_0))' \\ &= f'(a_{n-1}) \cdot f'(a_{n-2}) \cdot (f^{n-2}(a_0))' \\ &= \cdots \\ &= \prod_{k=0}^{n-1} f'(a_k) \end{aligned}$$

Example

Find the periodic point(s) with period 2 for the sequence:

$$a_{n+1} = 4a_n(1 - a_n)$$

i.e. $f(x) = 4x(1 - x)$

Suppose that $a_0 = a$. Then we want find a such that $a = f^2(a)$. It shows that $a = 3/4$ and 0 are fixed points. Only $-\frac{\sqrt{5}-5}{8}$ and $\frac{\sqrt{5}+5}{8}$ are points with cycle of period 2. Also we have $\left(f^2\left(-\frac{\sqrt{5}-5}{8}\right)\right)' = -3.9999999666151753$ and it is equal to

$$f'\left(-\frac{\sqrt{5}-5}{8}\right)f'\left(\frac{\sqrt{5}+5}{8}\right)$$

In [ ]:

```

```

Example

Consider the following function:

$$h(x) = 2 - |x - 1|$$

1. It is trivial:

$$h(x) = \begin{cases} 3 - x, & \text{if } x > 1; \\ 1 + x, & \text{if } x \leqslant 1. \end{cases}$$

2. The graphs about $h(x)$ and its composed functions, $g^n(x)$ are as follows pictures.

$h(x)$ is a wedge with straight lines and one vertex on the line $y = 2$, (the blue line). $h^2(x)$ is a wedge with three vertices on the line $y = 2$, (the green line). The solution of equation is the part which these two curves intersect with each other, (the line with thicker segment and in red), i.e. $x \in [1, 2]$. It is trivial to prove that any point within this range is fixed point only!

In [4]:

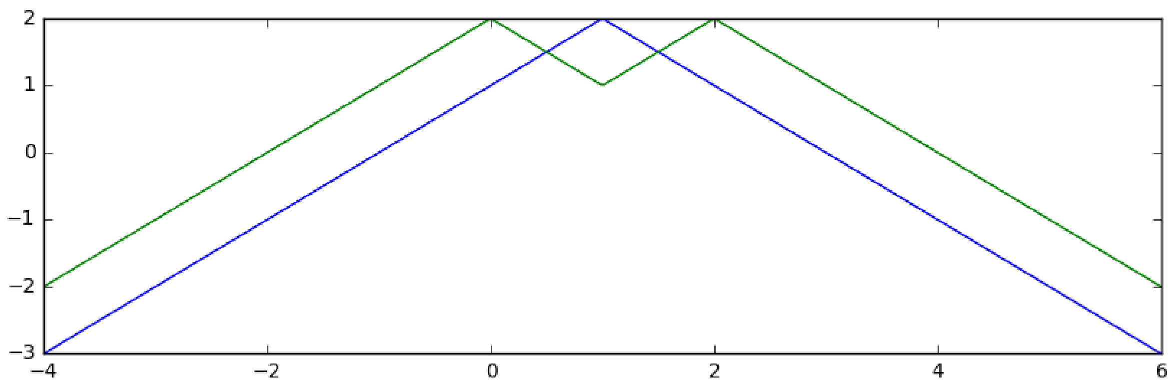
```
plt.rcParams['figure.figsize'] = (10,3) #wide graphs by default
def h(x):
    return 2-abs(x-1)
x=np.linspace(-4,6,400)
```

In [15]:

```
plt.plot(x,h(x),x,h(h(x)))
```

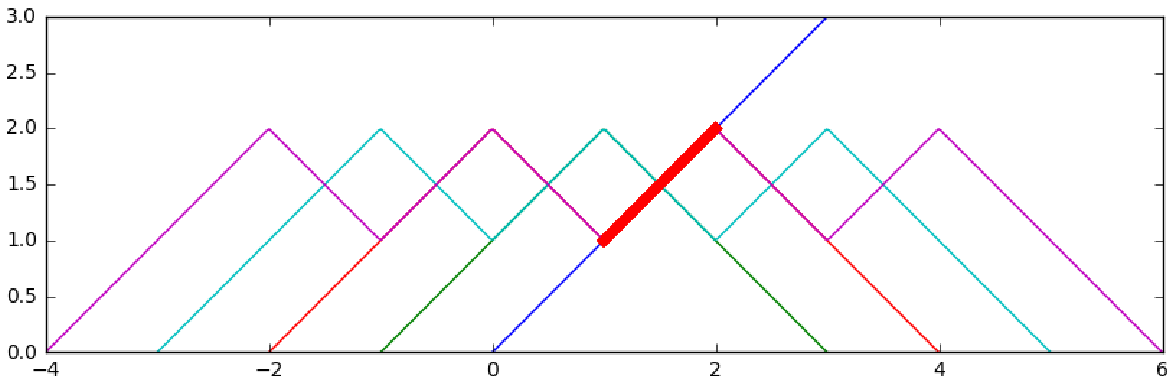
Out[15]:

```
[<matplotlib.lines.Line2D at 0x1085cc0b8>,
 <matplotlib.lines.Line2D at 0x109288e48>]
```



In [13]:

```
f, ax = plt.subplots()
ax.set_xlim([-4,6]);ax.set_ylim([0,3])

x=np.linspace(-4,6,400);y=x
for i in np.arange(5):
    plt.plot(x,y)
    plt.plot((1,2),(1,2),'r-',linewidth=5)
    y=h(y)

    time.sleep(2)
    clear_output()
    display(f)
plt.close()
```



Eexercise

Find the limits of the following sequences if exist:

1. $\{(-1)^n\}_{n=1}^{\infty}$
2. $\{1-\cos(n\pi)\}_{n=1}^{\infty}$
3. $\left\{\frac{1+2^n}{3^{n+1}-3\cdot 2^n}\right\}_{n=1}^{\infty}$

Answer

1. $\{(-1)^{\mathbf{n}}\}_{\mathbf{n=1}}^{\infty}=\{-\mathbf{1},\mathbf{1},-\mathbf{1},\mathbf{1},\cdots\}$: divergent;
2. $\{\mathbf{1}-\cos(\mathbf{n}\pi)\}_{\mathbf{n=1}}^{\infty}=\{\mathbf{2},\mathbf{0},\mathbf{2},\mathbf{0},\cdots\}$: divergent;
3. $\frac{\mathbf{1+2^n}}{\mathbf{3^{n+1}-3\cdot 2^n}}=\frac{\mathbf{(1/3)^n+(2/3)^n}}{\mathbf{3-3\cdot(2/3)^n}}\to\mathbf{0}$: convergent.

Exercise

Determine which one(s) in the following sequences is/are convergent:

1. $\{\frac{\ln n}{n}\}_{n=1}^{\infty}$
2. $\{\frac{2^n}{3^n+1}\}_{n=1}^{\infty}$

3. $\{\frac{n-(-1)^n}{n+(-1)^n}\}_{n=1}^{\infty}$
4. $\{\cos(n\pi/4)\}_{n=1}^{\infty}$ Divergent, since $\lim_{n\to\infty}\cos\left(\frac{n\pi}{4}\right)$ diverges.
5. $a_1 = 25$, and $a_n = \frac{1}{2}(50 + a_{n-1})$ for $n = 2, 3, \cdots$. Find $\lim n\to_{\infty} a_n$ if exists. (Hint: Increasing bounded sequence is convergent)

Answer

1. $\frac{\ln n}{n} \to 0$ Convergent, $L = 0$.
2. $\frac{2^n}{3^n+1} \to 0$ is convergent, $L = 0$.
3. $\frac{n-(-1)^n}{n+(-1)^n} \to 1$ convergent, $L = 1$.
4. $\{\cos(n\pi/4)\}_{n=1}^{\infty} = \{\sqrt{2}/2, 0, -\sqrt{2}/2, -1, -\sqrt{2}/2, 0, \sqrt{2}/2, 1, \cdots\}$ divergent.
5. For $a_0 \geq 0$ and $a_n = \frac{1}{2}(a_{n-1} + 50, n = 1, 2, \cdots$. Then $\{a_n\}$ is convergent. Trivially by mathmatical induction, the sequence is increasing and bounded above:

   - a_n is bounded by 50: if $a_k \leqslant 50$

$$a_{k+1} = \frac{1}{2}(50 + a_k) \leq 50$$

   - a_n is increasing: suppose that $a_{k-1} \leqslant a_k$:

$$\begin{aligned} a_{k+1} &= \frac{1}{2}(50 + a_k) \\ &\geq \frac{1}{2}(50 + a_{k-1}) = a_k \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{2}(50 + a_{n-1}) \\ &\Downarrow \\ L &= \frac{1}{2}(50 + L) \\ &\Downarrow \\ L &= 50 \end{aligned}$$

Infinite Series

The sum of infinite constants is called infinite series, like as:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

Which is(are) convergent of the following:

1. $1 + 1 + 1 + \cdots,$
2. $1 + \frac{1}{2} + \frac{1}{3} + \cdots,$
3. $1 - 1 + 1 - 1 + \cdots,$

4. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$

Convergence of Infinite Series

Limit of Partial Sum Sequence

Define a new partial sum sequence, $\{S_n\}_{n=1}^{\infty}$, as

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= S_1 + a_2 = a_1 + a_2 \\ &\vdots \\ S_n &= S_{n-1} + a_n \\ &= \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \end{aligned}$$

$$\lim_{n\to\infty} S_n \text{ convergent } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergent}$$

Example

1. the infinite series $1 + 1 + 1 + \cdots$ is divergent since

$$\begin{aligned} S_n &= 1 + 1 + \cdots + 1 \\ &= n \\ &\to \infty \end{aligned}$$

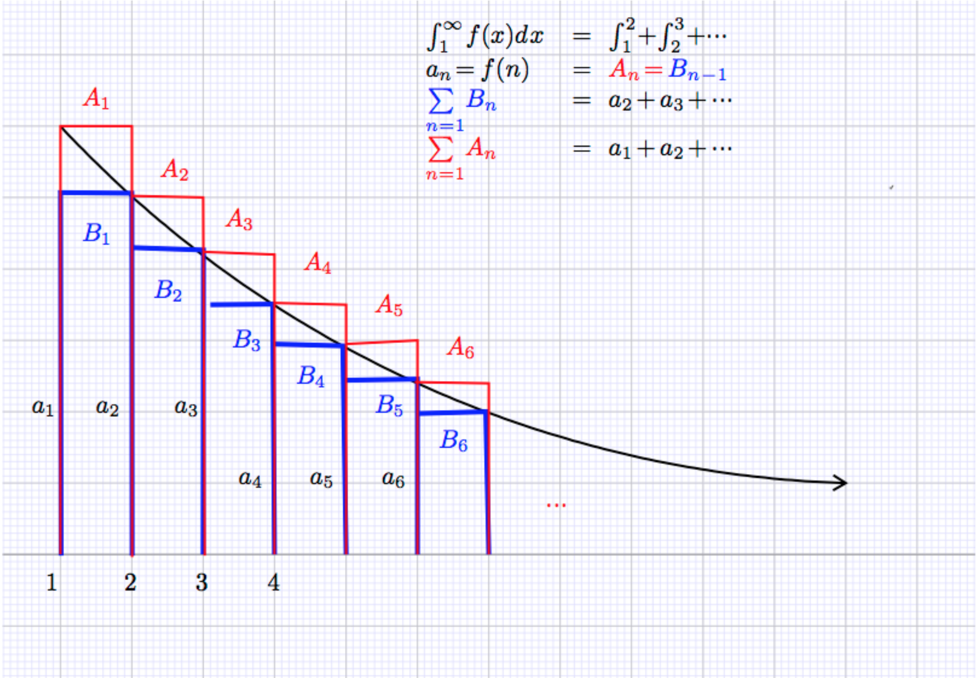
2. $1 - 1 + 1 - 1 + \cdots$ is divergent since

$$\begin{aligned} S_1, S_3, S_5, \cdots &= 1 \\ S_2, S_4, S_6, \cdots &= 0 \\ \{S_n\} &= \{1, 0, 1, 0, \cdots\} \\ &\not\longrightarrow \text{ convergent} \end{aligned}$$

Integral test

Suppose that $a_n \geq 0$ for each n, and there exists a continuous function $f(x) \geq 0$ and $f \searrow 0$ for $x \in [1, \infty)$, such that $f(n) = a_n$. Then

$$\int_1^{\infty} f(x)dx \text{ is convergent if and only if} \sum_{n=1}^{\infty} a_n \text{is convergent.}$$



$$B_n \leqslant \int_n^{n+1} f(x)dx \leqslant A_n$$
$$\Downarrow$$
$$\sum_{n=1}^{\infty} B_n \leqslant \int_1^\infty f(x)dx \leqslant \sum_{n=1}^{\infty} A_n$$
$$\Downarrow$$
$$\left(\sum_{n=1}^{\infty} a_n - a_1\right) = \sum_{n=2}^{\infty} a_n \leqslant \int_1^\infty f(x)dx \leqslant \sum_{n=1}^{\infty} a_n$$
$$\Downarrow$$
$$\int_1^\infty f(x)dx \text{ convergent} \quad \text{if and only if} \quad \sum_{n=1}^{\infty} a_n \text{ convergent}$$

Conclusion

$$\int_1^\infty f(x)dx \leqslant \sum_{n=1}^{\infty} a_n \leqslant \int_1^\infty f(x)dx + a_1$$

Example

1. Harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, is divergent since

$$f(n) = \frac{1}{n} \Rightarrow f(x) = \frac{1}{x}, f \searrow, f \to 0$$
$$\int_1^\infty \frac{1}{x} dx = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{divergent}$$

2. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ divergent since

$$
\begin{aligned}
f(n) = \frac{1}{n\ln n} &\Rightarrow f(x) = \frac{1}{x\ln x}, f \geq 0, f \searrow \\
\int_2^\infty \frac{1}{x\ln x} dx &= \int_2^\infty \frac{1}{\ln x} d\ln x \\
&= \ln|\ln x||_2^\infty = \infty
\end{aligned}
$$

3. p-Series, Determine the range of $p > 0$, such that the following p-series is convergent

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

Let $f(x) = x^{-p}$

- $0 < p < 1$ case:

$$\int_1^\infty x^{-p} dx = \frac{1}{1-p} x^{1-p} \Big|_1^\infty = \infty$$

- $p = 1$ case:

$$\int_1^\infty x^{-1} dx = \ln|x| \Big|_1^\infty = \infty$$

- $p > 1$ case:

$$\int_1^\infty x^{-p} dx = \frac{1}{1-p} \frac{1}{x^{p-1}} \Big|_1^\infty = \frac{1}{p-1} < \infty$$

  Then $p > 1 \Rightarrow \sum_{n=1}^\infty \frac{1}{n^p}$ convergent.

  - $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ is divergent since $p = 1/2 \leqslant 1$;
  - $\sum_{n=1}^\infty \frac{1}{n^2}$ is convergent since $p = 2 > 1$;

4. $\sum_{n=1}^\infty \frac{1}{1+n^2}$ convergent since $\int_1^\infty \frac{dx}{1+x^2} < \infty$.

Exercise p760

10. $\sum_{n=1}^\infty \frac{1}{n^{2/3}}$ divergent, since $2/3 < 1$.

16. $\sum_{n=1}^\infty \frac{n}{\sqrt{2n^2+1}}$ divergent $\int_1 \frac{xdx}{\sqrt{2x^2+1}} = \infty$.

22. $\sum_{n=2}^\infty \frac{\ln n}{n^2}$ convergent since $\int_2 \frac{\ln xdx}{x^2} < \infty$.

28. $\sum_{n=1}^\infty \frac{n}{2^n}$ convergent, since $\int_1 \frac{xdx}{2^x} < \infty$.

34. $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$ convergent since $\int_2 \frac{\ln x dx}{x^p} < \infty$ if $p > 1$.

58. (True or False) If positive series $\sum_{n=1}^{\infty} a_n$ convergent, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ is also convergent. (False, for instance, $a_n = 1/n^2$)

Alternating Series

$\sum_{n=0}^{\infty} a_n = a_0 - a_1 + a_2 - a_3 \cdots$ is called an alternating series where $a_0, a_1, a_2, \cdots \geq 0$.

A convergent test for alternating series is:

Theorem (Alternating Test)

An alternating series converges if and only if $\lim\ _{n \to \infty} a_n = 0$

Example, Alternating Harmonic Series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent since $1/n \to 0$ as $n \to \infty$. This test is not right for the positive series.

More Details

$$\begin{aligned}
S_1 &= 1 \\
S_3 &= 1 - \frac{1}{2} + \frac{1}{3} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) < 1 = S_1 \\
S_5 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} < S_3 \\
&\vdots \text{ by Mathematical Induction} \\
S_{2k+1} &= 1 - \frac{1}{2} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \frac{1}{2k+1} \\
&= S_{2k-1} - \frac{1}{2k} + \frac{1}{2k+1} < S_{2k-1} \\
&\Downarrow \\
S_1, S_3, S_5, \cdots & \quad \searrow
\end{aligned}$$

$$\begin{aligned}
S_2 &= 1 - \frac{1}{2} \\
S_4 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} > S_2 \\
&\vdots \\
S_{2k+2} &= 1 - \frac{1}{2} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \frac{1}{2k+1} - \frac{1}{2k+2} \\
&= S_{2k} + \frac{1}{2k+1} - \frac{1}{2k+2} > S_{2k} \\
S_2, S_4, S_6, \cdots & \quad \nearrow
\end{aligned}$$

$$S_{2k} \nearrow \text{ and } S_{2k+1} \searrow$$

$$S_{2k+1} = \left(1 - \frac{1}{2}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) + \frac{1}{2k+1} \geqslant 0$$

$$S_{2k+2} = 1\left(-\frac{1}{2} + \frac{1}{3}\right)\cdots + \left(-\frac{1}{2k-2} + \frac{1}{2k-1}\right) + \left(-\frac{1}{2k} + \frac{1}{2k+1}\right) - \frac{1}{2k+2} \leqslant 1$$

$$\Downarrow$$

$$\lim_{k\to\infty} S_{2k+1} \text{ convergent} \qquad \lim_{k\to\infty} S_{2k+2} \text{ convergent}$$

Above also implies the convergence of S_n:

$$S_{2k+2} - S_{2k+1} = -\frac{1}{2k+2}$$
$$\Downarrow$$
$$\lim_{k\to\infty} (S_{2k+2} - S_{2k+1}) = 0$$
$$\Downarrow$$
$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k+2}$$
$$\Downarrow$$
$$\lim_{k\to\infty} S_k \quad \text{convergent}$$

Example

1. $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n-1}$ divergent, since $\frac{2n}{4n-1} \to 1/2 \neq 0$
2. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3n}{4n^2-1}$ convergent, since $\frac{3n}{4n^2-1} \to 0$
3. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!}$ convergent, since $\frac{1}{n!} \to 0$

Exercise (p.773)

9. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{\ln n}$ divergent, since $\frac{n}{\ln n} \to \infty \neq 0$

14. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ convergent, since $\frac{1}{n} \to 0$

20. $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$ convergent, since $0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \to 0$

24. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt[n]{n}}$ divergent, since $\frac{1}{\sqrt[n]{n}} \to 1 \neq 0$

28 $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^s}$ is convergent for $s > 0$.

Comparison Test

If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are two positive infinite series with $0 \leqslant a_n \leqslant b_n$ for each n, then

1. $\sum\limits_{n=1}^{\infty} b_n$ is convergent $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ is convergent
2. $\sum\limits_{n=1}^{\infty} a_n$ is divergent $\Rightarrow \sum\limits_{n=1}^{\infty} b_n$ is divergent

Example

1. Harmonic series being divergent implies $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1/2}}$ divergent, since

$$\frac{1}{n} \leqslant \frac{1}{\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ divergent} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ divergent}$$

2. $\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ being convergent implies $\sum\limits_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ is convergent since

$$\frac{1}{n^2} \geqslant \frac{1}{n^2 \ln n} \text{ and } \sum_{n=2}^{\infty} \frac{1}{n^2} \text{ convergent} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2 \ln n} \text{ convergent}$$

and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+2}$ convergent too.

3. $\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ being convergent cannot imply $\sum\limits_{n=2}^{\infty} \frac{1}{n^2-1}$ since

$$\frac{1}{n^2} \leqslant \frac{1}{n^2-1} \text{ but } \sum_{n=2}^{\infty} \frac{1}{n^2} \text{ convergent} \not\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2-1} \text{ convergent}$$

4. $\sum\limits_{n=1}^{\infty} \frac{1}{3+2^n}$ convergent since $\frac{1}{3+2^n} \leq \frac{1}{2^n}$
5. $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$ is divergent since $\sqrt{n}-1 < \sqrt{n}$.

Limit Comparison Test

Assume that $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ satisfies $0 \leqslant a_n, b_n$ for any n. If $r = \lim_{n\to\infty} \frac{a_n}{b_n}$ and $0 < r < \infty$, then both the series are convergent or divergent.

Example

1. $\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ being convergent implies $\sum\limits_{n=2}^{\infty} \frac{1}{n^2-n+1}$ convergent since

$$\lim_{n\to\infty}\frac{\frac{1}{n^2}}{\frac{1}{n^2-n+1}} = 1 \text{ and } \sum_{n=2}^{\infty}\frac{1}{n^2} \text{ convergent}$$

2. $\sum_{n=1}^{\infty}\frac{1}{n}$ being divergent implies $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ divergent since

$$\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{2n+1}} = 2 \text{ and } \sum_{n=1}^{\infty}\frac{1}{n} \text{ divergent}$$

3. $\sum_{n=2}^{\infty}\frac{1}{3^n}$ being convergent implies $\sum_{n=2}^{\infty}\frac{1}{3^n-2^n}$ convergent since

$$\lim_{n\to\infty}\left|\frac{\frac{1}{3^n}}{\frac{1}{3^n-2^n}}\right| = 1 \text{ and } \sum_{n=2}^{\infty}\frac{1}{3^n} \text{ convergent}$$

4 $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1}$ is divergent since $\frac{\sqrt{n}+1}{\sqrt{n}} \to 1.$

5 $\sum_{n=1}^{\infty}\frac{2n^2+n}{\sqrt{4n^7+3}}$ is convergent since $\frac{\frac{2n^2+n}{\sqrt{4n^7+3}}}{\sqrt{2}n^{3/2}} \to 1.$

6 $\sum_{n=1}^{\infty}\frac{\sqrt{n}+\ln n}{n^2+1}$ is convergent since $\frac{\frac{\sqrt{n}+\ln n}{n^2+1}}{n^{3/2}} \to 1.$

Exercises (p.767)

10. $\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^2}$ is convergent since $a_n \le 1/(n^2).$

22. $\sum_{n=1}^{\infty}\frac{\ln n}{n^3-1}$ is convergent since $a_n \le n/(n^3).$

24. $\sum_{n=1}^{\infty}\tan\frac{1}{n}$ is cdivergent since $a_n/(1/n) \to 1.$

32. $\sum_{n=1}^{\infty}\frac{\tan^{-1} n}{n^3+1}$ is convergent since $a_n \le 1/(n^2) \to 0.$

Convergent Problem

The condition of geometric series

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots \text{ where } a \ne 0$$

is well-known as:

$$\begin{aligned} |r| < 1 &\Rightarrow S \text{ convergent} \\ |r| > 1 &\Rightarrow S \text{ divergent} \\ |r| = 1 &\Rightarrow S \text{ divergent} \end{aligned}$$

But what can we say about:

$$\tilde{S} = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

It is **NOT** geometric!

But note that

$$r = \frac{\cancel{a}r}{\cancel{a}} = \frac{\cancel{a}r^{\cancel{2}}}{\cancel{a}\cancel{r}} = \cdots = \frac{\cancel{a}r^{\cancel{n}+1}}{\cancel{a}\cancel{r}^n} = \cdots$$

Absolute Convergence

1. $\sum_{n=1}^{\infty} a_n$ is calle absoulute convergent if $\sum_{n=1}^{\infty} |a_n|$ convergent.
2. $\sum_{n=1}^{\infty} a_n$ is calle conditional convergent if $\sum_{n=1}^{\infty} a_n$ convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Example

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolute convergent.
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditional convergent.
3. $\sum_{n=1}^{\infty} (-1)^n n$ is divergent.
4. $\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$ is absolute convergent.

Ratio Test

Suppost that

$$S = a_1 + a_2 + \cdots + a_n + \cdots$$

and let

$$r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then

1. $r < 1$ implies $\sum_{n=1}^{\infty} a_n$ is convergent;
2. $r > 1$ implies $\sum_{n=1}^{\infty} a_n$ is divergent;
3. $r = 1$ no conclusion.

Root Test

Suppost that

$$S = a_1 + a_2 + \cdots + a_n + \cdots$$

and let

$$r = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

Then

1. $r < 1$ implies $\sum_{n=1}^{\infty} a_n$ is convergent;
2. $r > 1$ implies $\sum_{n=1}^{\infty} a_n$ is divergent;
3. $r = 1$ no conclusion.

Example

Determine whether the infinite series are convergent or divergent.

1. $\sum_{n=1}^{\infty} n/2^n$ is convergent since

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)/2^{n+1}}{n/2^n}\right| \\ &= \left|\frac{n+1}{2n}\right| \\ &\to \frac{1}{2} < 1 \end{aligned}$$

2. $\sum_{n=1}^{\infty} \frac{4^n}{2^n+3^n}$ is divergent since

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{4^{\cancel{n}+1}/(2^{n+1}+3^{n+1})}{4^{\cancel{n}}/(2^n+3^n)}\right| \\ &= \frac{4}{3}\left|\frac{(2/3)^n+1}{(2/3)^{n+1}+1}\right|\left(\frac{\cdot\ /3^{n+1}}{\cdot\ /3^{n+1}}\right) \\ &\to \frac{4}{3} > 1 \end{aligned}$$

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent but $r = 1$.

4. $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent but $r = 1$.

5. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{2^n}$ is absolute convergent:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{((n+1)^2+1)/2^{n+1}}{(n^2+1)/2^n}\right| \\ &\to \frac{1}{2} < 1\end{aligned}$$

6. $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$ is absolute convergent.

7. $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{3^n}$ is divergent.

8. $\sum_{n=1}^{\infty} (-1)^n \frac{2^{n+3}}{(n+1)^n}$ is absolute convergent.

Exercise

Determine which $p-\text{series}$ is convergent:

1. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$;
2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$.

Answer

1. $p = 3/2 > 1$, convergent;
2. $p = 1/3 < 1$, divergent

Exercise

Which one(s) is(are) convergent?

1. $\sum_{n=1}^{\infty} \cos n\pi$
2. $\sum_{n=1}^{\infty} (-1)^n n^2$
3. $\sum_{n=1}^{\infty} (-1)^n 2^n$
4. $\sum_{n=1}^{\infty} \frac{3n-1}{1+n+n^2}$
5. $\sum_{n=1}^{\infty} \frac{2^n}{n^2+n}$
6. $\sum_{n=1}^{\infty} n^{-(1+1/n)}$

Answer

1. $\cos n\pi = (-1)^{n+1} \not\to 0$, divergent; ($n$-term test)
2. $2^n \not\to 0$, divergent; (n-term test)
3. $(\frac{3n-1}{1+n+n^2})/(1/n) \to 3$, divergent (limit comparison test)
4. $(\frac{2^{n+1}}{(n+1)^2+n+1})/(\frac{2^n}{n^2+n}) \to 2 > 1$, divergent, (ratio test)
5. $n^{-(1+1/n)}/n^{-1} = n^{-1/n} \to 1$, divergent, (limit comparison test)

Taylor Series

As well-known:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \text{ for } |x| < 1$$

Problem

Can we expect any function, $f(x)$, to be expanded as above?

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

where $a_0, a_1, a_2, \cdots$ are constants?

Defition Power function

A power series in x is a series on the form as follows:

$$\sum_{i=0}^{n} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

Suppose that $f(x)$is smooth enough oven an open interval containing $x = a$ then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for any x within this interval. And this infinite series is called the Taylor series of $f(x)$ at $x = a$. If $a = 0$, then it is also called Maclaurin series of $f(x)$, i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Convergent intervals and Convergent radius

If $f(x) = \sum_{n=0} a_n(x-a)^n$ is convergent for $x \in (a-R, a+R)$, then $(a-R, a+R)$ is called the **convergent interval** and R is the radius of convergent.

R must be one of three cases:

$$R > 0, R = 0 \text{ or } R = +\infty.$$

Examples

Power Series	Convergent Interval	Convergent Radius
$\sum_{n=1}^{\infty} \frac{x^n}{n}$	$[-1, 1]$	1
$\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n n^2}$	$[-1, 5]$	5
$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{\sqrt{n+1}}$	$[-1/2, 1/2]$	$1/2$

Exercise p792

no.	Power Series -------	Convergent Interval	Convergent Radius
8	$\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$	R	∞
14	$\sum_{n=1}^{\infty} \sqrt{n}(2x+3)^n$	$(-2, -1)$	$1/2$
18	$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{(n^2+1)n^2}$	$[-4, 0)$	2
26	$\sum_{n=1}^{\infty} \frac{n^n(3x+5)^n}{(2n)!}$	$\left(-\frac{2/e-5}{3}, \frac{2/e-5}{3}\right)$	$3e/2$

Examples

1. $$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp \cdots + (\mp 1)^n x^n + \cdots = \sum_{n=0}^{\infty} (\mp 1)^n x^n \text{ for } |x| < 1$$

2. $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ for } |x^2| < 1 i.e. |x| < 1$$

3. Find the Taylor's series of e^x at $x = 0$ and $x = a$.

Solution

suppose that

$$
\begin{aligned}
e^x &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\
&\Downarrow \text{ let} x = 0 \\
1 = e^0 &= a_0 + a_1 \cdot 0 + a_2 \cdot 0 + \cdots \\
&\Downarrow \\
a_0 &= 1 \\
e^x = (e^x)' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots \\
&\Downarrow \text{ let} x = 0 \\
1 &= a_1 \\
e^x = (e^x)'' &= 2a_2 + 3 \cdot 2 \cdot a_3 x + \cdots + n(n-1) a_n x^{n-2} + \cdots \\
&\Downarrow \text{ let} x = 0 \\
1 &= 2a_2 \\
&\Downarrow \\
a_2 &= \frac{1}{2!} = \frac{1}{2} \\
e^x = (e^x)''' &= 3 \cdot 2 \cdot 1 \cdot a_3 + \cdots + n(n-1)(n-2) a_n x^{n-3} + \cdots \\
&\Downarrow \text{ let } x = 0 \\
a_3 &= \frac{1}{3!} \\
&\Downarrow \text{ by Mathematical Induction} \\
a_n &= \frac{1}{n!}
\end{aligned}
$$

These conclude:

$$
\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} x^n
\end{aligned}
$$

Since

$$
\begin{aligned}
e^x &= e^{a + (x-a)} = e^a e^{x-a} \\
&= e^a \sum_{n=0}^{\infty} \frac{1}{n!} (x-a)^n \\
&= \sum_{n=0}^{\infty} \frac{e^a}{n!} (x-a)^n
\end{aligned}
$$

Example

For $|x| < 1$:

$$\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{2n(2n-2)!}{(n!2^n)^2} x^n$$

Example Find the Maclaurin's series of $\ln(1+x)$.

Sol: Let $f(x) = \ln(1+x)$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$\frac{1}{1+x}$	1
2	$\frac{-1}{(1+x)^2}$	$(-1)^1 \cdot 1!$
3	$\frac{(-1)^2 2!}{(1+x)^3}$	$(-1)^2 \cdot 2!$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$	$(-1)^{n-1}(n-1)!$
$\vdots$	$\vdots$	$\vdots$

Then

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{n}$$

and $a_0 = \ln 1 = 0$, and

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

Example

For all $x \in \mathrm{R}$,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

The following function can be expanded as:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

only for $|x| < 1$. The expansion fails as $|x| \geq 1$, for example:

$$0 > -1 = \frac{1}{1-2} \neq 1 + 2 + 2^2 + \cdots > 1$$

The problem arises:

```
At what range would Taylor's series be convergent?
```

Convergence for Taylor's series

(with the help of Ratio test)

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, if

$$R = \lim n \to_\infty \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n} \right| = \lim n \to_\infty \left| \frac{a_{n+1}}{a_n} \right| |x-a| < 1 \Rightarrow \text{ series is convergent}$$

$$R = \lim n \to_\infty \left| \frac{a_{n+1}}{a_n} \right| |x-a| > 1 \Rightarrow \text{ series is divergent}$$

$$R = \lim n \to_\infty \left| \frac{a_{n+1}}{a_n} \right| |x-a| = 1 \Rightarrow \text{ no conclusion}\}$$

Question

Show

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} a_n$$

convergent for all $x \in \mathrm{R}$.

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\frac{1}{(n+1)!} x^{n+1}}{\frac{1}{n!} x^n} \right| \\ &= \frac{|x|}{n+1} \\ &\xrightarrow{n \to \infty} 0 < 1 \end{aligned}$$

Therefore,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

for any $x \in \mathrm{R}$.

Example, Show $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$.

Since

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(-1)^n}{n+1}x^{n+1}}{\frac{(-1)^{n-1}}{n}x^n}\right| \\
&= \left|\frac{n}{n+1}\right||x| \\
&\xrightarrow{n\to\infty} |x|
\end{aligned}$$

then

$$\begin{aligned}
|x| < 1 &\Rightarrow \text{ convergent} \\
|x| > 1 &\Rightarrow \text{ divergent}
\end{aligned}$$

and

1. for $x = 1$:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}1^n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\cdots\text{convergent}$$

2. for $x = -1$:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(-1)^n = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right) = -\infty$$

Then for $x \in (-1, 1]$, $\ln(1+x)$ is convergent and

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

Taking $x \to 1^-$ gets:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\cdots$$

Example, Find the convergent interval for

$$\sum_{n=0}^{\infty} n!x^n$$

Since

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\
&= (n+1)|x| \\
&\to \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{if } x \neq 0 \end{cases}
\end{aligned}$$

it is convergent only when $x = 0$, i.e.

$$\sum_{n=0}^{\infty} n!x^n \Big|_{x=0} = 1 + x + 2!x^2 + 3!x^3 + 4!x^4|_{x=0}$$
$$= 1 + 0 + 0 + \cdots = 1$$

Theorem (Uniqueness) If

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=0}^{\infty} b_n(x-a)^n$$

then $a_n = b_n$ for for all n.

As usually known:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

Then (\color{brown}{let $x = 1/2$)}

$$2 = \frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots = 2$$

But it is **NOT** right ($x = 2$):

$$-1 = \frac{1}{1-2} \overset{?}{=} 1 + 2 + 2^2 + 2^3 + \cdots = +\infty \longrightarrow \times \longleftarrow$$

This means that the infinite series is not convergent for every $x \in \mathrm{R}$!

Theorem

If $f(x) = \sum_n a_n(x-a)^n$ is convergent for $|x-a| < R$,then

1.

$$f'(x) = \sum^{\infty} n=0(a_n(x-a)^n)' = \cdots = \sum^{\infty} n=0(n+1)a_{n+1}(x-a)^n$$

is convergent for $|x-a| < R$.

2.

$$\int_a^x f(t)dt = \sum^{\infty} n=0\int_a^x a_n(t-a)^n dt = \sum^{\infty} n=0\frac{a_n}{n+1}(x-a)^{n+1}$$

is convergent for $|x-a| < R$.

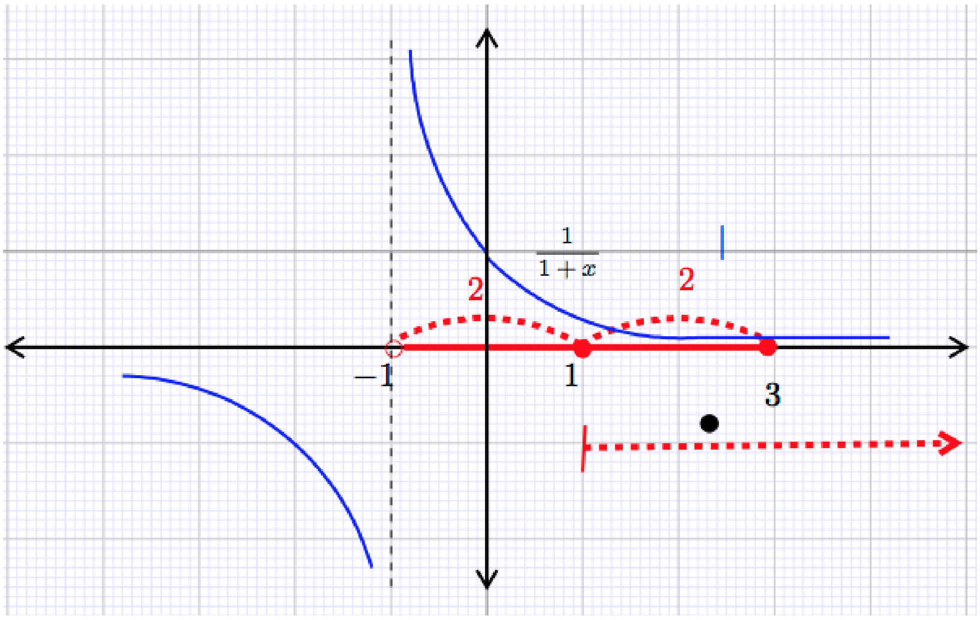
Example

1. $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \Rightarrow R = +\infty$
2. $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n \Rightarrow R = 1$
3. Find Taylor series of $\frac{1}{1+x}$ centered at $x = 1$ and its convergent interval

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{2+(x-1)} \\ &= \frac{1}{2\left[1+\left(\frac{x-1}{2}\right)\right]} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-1}{2}\right)^n \end{aligned}$$

for

$$\left|\frac{x-1}{2}\right| < 1 \Leftrightarrow -1 < x < 3$$



Exercise

From last result, guess the convergent value for the alternating series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + (-1)^{n-1}\frac{1}{n} + \dots$$

Compare

$$\frac{(-1)^{n-1}}{n}x^n \text{and } (-1)^{n-1}\frac{1}{n}$$
$$\Downarrow$$
$$x =$$
$$\Downarrow$$
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n\Bigg|_{x=} = \sum_{n=1}^{\infty}$$
$$||$$
$$f(x)|_{x=} = \ln(1+x)|_{x=} = \ln$$

2. Find the Maclaurin's series of $\frac{x^2}{(1-x)}$. (Hint: $(1-x)^{-1} = 1 + x + x^2 + \cdots$

$$\begin{aligned} (1-x)^{-1} &= 1 + x + x^2 + \cdots \\ &\Downarrow \\ \frac{x^2}{(1-x)} &= x^2 \cdot \\ &= x^2 \cdot (1 + \ \ + \ \ + \cdots) \\ &= x^2 + \ \ + \ \ + \cdots \\ &= \sum_{n=2}^{\infty} x^n \end{aligned}$$

Exercise

Find Taylor series of (a) $\frac{1}{2+x}$ (b) $\frac{1}{1+2x}$ centered at $x = 1$ and their convergent intervals. Also confirm your results with their graphs.

a).

$$\begin{aligned} \frac{1}{2+x} &= \frac{1}{\ \ + (x-1)} \\ &= \frac{1}{\ } \cdot \frac{1}{1 + \left(\frac{x-1}{\ }\right)} \\ &= \frac{1}{\ } \cdot \sum_{n=0}^{\infty} {}^n(\text{-})^n \\ &= \sum_{n=0}^{\infty} \ \ \cdot (x-1)^n \end{aligned}$$

b).

$$
\begin{aligned}
\frac{1}{1+2x} &= \frac{1}{\quad + 2(x-1)} \\
&= \frac{1}{\ } \cdot \frac{1}{1+\left(\frac{x-1}{\ }\right)} \\
&= \frac{1}{\ } \cdot \sum_{n=0}^{\infty} {}^{n}(\text{-})^{n} \\
&= \sum_{n=0}^{\infty} \quad \cdot (x-1)^n
\end{aligned}
$$

Answer

a)

$$
\begin{aligned}
\frac{1}{2+x} &= \frac{1}{3+(x-1)} \\
&= \frac{1}{3} \cdot \frac{1}{1+\left(\frac{x-1}{3}\right)} \\
&= \frac{1}{3} \cdot \sum_{n=0}^{\infty} -1^{n}\left(\frac{x-1}{3}\right)^{n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} \cdot (x-1)^n
\end{aligned}
$$

b).

$$
\begin{aligned}
\frac{1}{1+2x} &= \frac{1}{3+2(x-1)} \\
&= \frac{1}{3} \cdot \frac{1}{1+\left(\frac{x-1}{3/2}\right)} \\
&= \frac{1}{3} \cdot \sum_{n=0}^{\infty} -1^{n}\left(\frac{x-1}{3/2}\right)^{n} \\
&= \sum_{n=0}^{\infty} \frac{(-2)^n}{3^{n+1}} \cdot (x-1)^n
\end{aligned}
$$

Example

Replacing x with x^2 in the geometric series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n |x| < 1$$

gets

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} |x^2| < 1(\Leftrightarrow |x| < 1)$$

And integrating both sides gets

$$\begin{aligned} \int_0^x \frac{dt}{1+t^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \\ \Longrightarrow \quad \tan^{-1} x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \end{aligned}$$

This equation holds for $|x^2| < 1$, i.e. $|x| < 1$. Specially, as $x = 1$,

$$\begin{aligned} \frac{\pi}{4} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \end{aligned}$$

Example

Find the exact form of $f(t) = \sum_{n=0}^{\infty} \frac{e^{tn} e^{-\lambda} \lambda^n}{n!}$, $f'(t)$ and so $f'(0)$.

It is trivial that

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^n e^{tn}}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= e^{\lambda(e^t - 1)} \\ &\Downarrow \\ f'(0) &= e^{\lambda(e^t-1)} e^t \lambda |_{t=0} = \lambda \end{aligned}$$

Example

Find the Maclaurin's series of $(1-x)^{-2}$ and $(1-x)^{-3}$.

Differentiating the both sides :

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

gets

$$\begin{aligned} \frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} n x^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) x^n \\ &\Downarrow \\ \left(\frac{1}{1-x}\right)'' = \frac{2}{(1-x)^3} &= \sum_{n=0}^{\infty} (n+2)(n+1) x^n \end{aligned}$$

Exercise

As the last example, find the exact form of $f''(t)$ and so $f''(0)$:

$$\begin{aligned} f''(t) &= (e^{\lambda(e^t-1)})'' \\ &= \left(\cdot e \ \cdot e^{\lambda(e^t-1)} \right)' \\ &= \ \cdot e \ \cdot e^{\lambda(e^t-1)} + \ \cdot e \ \cdot e^{\lambda(e^t-1)} \\ &\Downarrow \\ f''(0) &= \lambda + \lambda^2 \end{aligned}$$

$$\begin{aligned} f''(t) &= (e^{\lambda(e^t-1)})'' \\ &= \left(\lambda \cdot e^t \cdot e^{\lambda(e^t-1)} \right)' \\ &= \lambda \cdot e^t \cdot e^{\lambda(e^t-1)} + \lambda^2 \cdot e^{2t} \cdot e^{\lambda(e^t-1)} \\ &\Downarrow \\ f''(0) &= \lambda + \lambda^2 \end{aligned}$$

Exercise

Suppose that $M(t) = \sum_{n=0}^{\infty} \frac{t^n e^{-\lambda} \lambda^n}{n!}$, find$M'(t)$ and so $M'(1)$.

$$\begin{aligned} M(t) &= \sum_{n=0}^{\infty} \frac{t^n e^{-\lambda} \lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\ ^n}{n!} \\ \left(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) &= e^{-\lambda} \cdot e \\ &= e \\ M'(1) &= M'(t)|_{t=1} \\ &= \ e|_{t=1} \\ &= \end{aligned}$$

$$\begin{aligned} M(t) &= \sum_{n=0}^{\infty} \frac{t^n e^{-\lambda} \lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{t\lambda^n}{n!} \\ \left(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\right) &= e^{-\lambda} \cdot e^{\lambda t} \\ &= e^{\lambda(t-1)} \\ M'(1) &= M'(t)|_{t=1} \\ &= \lambda e^{\lambda(t-1)} \Big|_{t=1} \\ &= \lambda \end{aligned}$$

Example

Find Maclaurin Series of $\sin x$

We have

$$\begin{aligned} (\sin x)^{(4k)}|_{x=0} &= \sin x|_{x=0} = 0 \\ (\sin x)^{(4k+1)}|_{x=0} &= \cos x|_{x=0} = 1 \\ (\sin x)^{(4k+2)}|_{x=0} &= -\sin x|_{x=0} = 0 \\ (\sin x)^{(4k+3)}|_{x=0} &= -\cos x|_{x=0} = -1 \end{aligned}$$

implies

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!} x^n \\ &= \sum_{k=0}^{\infty} \left[\frac{\sin^{(4k)}(0)}{(4k)!} x^{4k} + \frac{\sin^{(4k+1)}(0)}{(4k+1)!} x^{4k+1} + \frac{\sin^{(4k+2)}(0)}{(4k+2)!} x^{4k+2} + \frac{\sin^{(4k+3)}(0)}{(4k+3)!} x^{4k+3} \right] \\ &= \sum_{k=0}^{\infty} \left[\frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!} \right] \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \end{aligned}$$

Exercise

1.

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{2x}{1-x} &= 2x\sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} x^{n+1} \\
&= \sum_{n=1}^{\infty} x^n \\
\frac{x}{1-3x} &= x \cdot \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} x^{n+1} \\
&= \sum_{n=1}^{\infty} x^n \text{ convergent for } x \in
\end{aligned}$$

2.

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} (n+1)x^n \\
\frac{x}{(1-x)^2} &= \sum_{n=1}^{\infty} n\,x^n \\
\frac{x}{(1-x^2)^2} &= \sum_{n=1}^{\infty} n\,x^{2n-1} \\
\int_0^x \frac{t}{(1-t^2)^2} dt &= \frac{x^2}{2(1-x^2)} \\
&= \sum_{n=1}^{\infty} \frac{1}{2} x^{2n}
\end{aligned}$$

Answer

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{2x}{1-x} &= 2x\sum_{n=0}^{\infty} 1\, x^n \\
&= \sum_{n=0}^{\infty} 2\, x^{n+1} \\
&= \sum_{n=1}^{\infty} 2\, x^n \\
\frac{x}{1-3x} &= x \cdot \sum_{n=0}^{\infty} 3^n x^n \\
&= \sum_{n=0}^{\infty} 3^n x^{n+1} \\
&= \sum_{n=1}^{\infty} 3^{n+1} x^n \text{ convergent for } x \in (-1/3, 1/3)
\end{aligned}
$$

Exercise

As the reason, we also have

$$\cos x = \sum_{n=0}^{\infty} a_{2n} x^{2n}$$

$$
\begin{aligned}
(\cos x)^{(4k)}|_{x=0} &= \cos x|_{x=0} = \\
(\cos x)^{(4k+1)}|_{x=0} &= -\sin x|_{x=0} = \\
(\cos x)^{(4k+2)}|_{x=0} &= -\cos x|_{x=0} = \\
(\cos x)^{(4k+3)}|_{x=0} &= \sin x|_{x=0} =
\end{aligned}
$$

$$
\Rightarrow \begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{\cos^{(n)}(0)}{n!} x^n \\
&= \sum_{k=0}^{\infty} \left[\frac{\cos^{(4k)}(0)}{(4k)!} x^{4k} + \frac{\cos^{(4k+1)}(0)}{(4k+1)!} x^{4k+1} + \frac{\cos^{(4k+2)}(0)}{(4k+2)!} x^{4k+2} + \frac{\cos^{(4k+3)}(0)}{(4k+3)!} x^{4k+3} \right] \\
&= \sum_{k=0}^{\infty} \left[x^{4k} - x^{4k+2} \right] \\
&= \sum_{n=0}^{\infty} x^{2n}
\end{aligned}
$$

$$
\begin{aligned}
(\cos x)^{(4k)}|_{x=0} &= \cos x|_{x=0} = 1 \\
(\cos x)^{(4k+1)}|_{x=0} &= -\sin x|_{x=0} = 0 \\
(\cos x)^{(4k+2)}|_{x=0} &= -\cos x|_{x=0} = -1 \\
(\cos x)^{(4k+3)}|_{x=0} &= \sin x|_{x=0} = 0
\end{aligned}
$$

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{\cos^{(n)}(0)}{n!} x^n \\
&= \sum_{k=0}^{\infty} \left[\frac{\cos^{(4k)}(0)}{(4k)!} x^{4k} + \frac{\cos^{(4k+1)}(0)}{(4k+1)!} x^{4k+1} + \frac{\cos^{(4k+2)}(0)}{(4k+2)!} x^{4k+2} + \frac{\cos^{(4k+3)}(0)}{(4k+3)!} x^{4k+3} \right] \\
\Rightarrow \quad &= \sum_{k=0}^{\infty} \left[\frac{1}{(4k)!} x^{4k} - \frac{1}{(4k+2)!} x^{4k+2} \right] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2k)!} x^{2n}
\end{aligned}$$

Proposition

(Binomial series) If $r \neq 0, 1, 2, 3, \cdots$, and $r \in R$, then

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$

where $\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}$

Example

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \begin{pmatrix} \frac{1}{2} \\ n \end{pmatrix} x^n$$

where

$$\begin{pmatrix} \frac{1}{2} \\ n \end{pmatrix} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!}.$$

The coefficient can be simplied as follows:

$$\begin{aligned}
\binom{\frac{1}{2}}{n} &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!} \\
&= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-3}{2})}{n!} \\
&= (-1)^{n-1}\frac{1\cdot 3\cdot \cdots \cdot (2n-3)}{2^n n!} \\
&= (-1)^{n-1}\frac{1\cdot 3\cdot \cdots \cdot (2n-3)}{2^n n!}\cdot\frac{2\cdot 4\cdot \cdots \cdot (2n-2)}{2\cdot 4\cdot \cdots \cdot (2n-2)} \\
&= (-1)^{n-1}\frac{(2n-2)!}{2^n n!}\cdot\frac{n}{2^{n-1}\cdot 1\cdot 2\cdot \cdots \cdot (n-1)\cdot n} \\
&= (-1)^{n-1}\frac{n(2n-2)!}{2^{2n-1}(n!)^2}
\end{aligned}$$

Such that

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{n(2n-2)!}{2^{2n-1}(n!)^2}x^n$$

Example

Integrating both sides of

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n \text{ for } |x| < 1$$

gets

$$\begin{aligned}
\ln|1+x| &= \int_0^x \frac{1}{1+t}dt \\
&= \int_0^x \sum_{n=0}^{\infty}(-1)^n t^n dt \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{t^{n+1}}{n+1}\Bigg|_0^x \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}
\end{aligned}$$

Exercise

Find the binomal series of $(1+x)^{-1/2}$ and also simply the binomial coefficients.

$$\begin{aligned}
\frac{1}{(1+x)^{-1/2}} &= \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} x^m \\
&= \sum_{m=0}^{\infty} \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-m+1)}{m!} x^m \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{1\cdot 3\cdot \cdots \cdot (2m-1)}{2^m m!} x^m \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{1\cdot 3\cdot \cdots \cdot (2m-1)}{2^m m!} \cdot \frac{2\cdot 4\cdot \cdots \cdot 2m}{2\cdot 4\cdot \cdots \cdot 2m} x^m \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{(2m)!}{2^{2m}(m!)^2} x^m
\end{aligned}$$

Exercise, p.805

Find the Taylor series of $f(x)$ below and its convergent radius

4. $f(x) = e^{-2x}, c = 3$

$$\begin{aligned}
f(x) &= e^{-4} e^{-2(x-2)} \\
&= \sum_{n=0}^{\infty} \frac{e^{-4}(-2)^n}{n!} (x-2)^n
\end{aligned}$$

where $x \in \mathrm{R}$, i.e. convergent radius: ∞.

14. $f(x) = \frac{1}{1+3x}$ at $c = 2$

$$\begin{aligned}
f(x) &= \frac{1}{1+3x} = \frac{1}{7}\frac{1}{1+3(x-2)/7} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{7^{n+1}} (x-2)^n
\end{aligned}$$

where $|3(x-2)/7| < 1$, $\left(|x-2| < \frac{7}{3}\right)$, i.e. convergent radius: $7/3$.

22. $f(x) = \sin^2 x = (1 - \cos 2x)/2$ at $c = 0$

$$\begin{aligned}
f(x) &= (1 - \cos 2x)/2 \\
&= \frac{1}{2} - \sum_{n=0}^{\infty} \frac{(-1)^{-n} 2^{2n}}{2n!} x^{2n}
\end{aligned}$$

where $x \in \mathrm{R}$, i.e. convergent radius: ∞.

32. Find the convergent radius of the following binomial sequence:

$$f(x) = \frac{1}{\sqrt[3]{8+x}} = \frac{1}{2\sqrt[3]{1+x/8}} = \frac{1}{2}\left(1 + \frac{x}{8}\right)^{-1/3}$$

is a binomial series which is convergent for $\left|\frac{x}{8}\right| < 1, (|x| < 8)$, i.e. convergent interval is 8.

50. The power series of

$$\int \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int \frac{x^{2n+1}}{x} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}$$

60. Find the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!2^{2n+1}} \pi^{2n+1} = \sin\frac{\pi}{2} = 1$$

66. Evaluate the following limit:

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}}{x-1} = 1$$

Fourier Series

Suppose that $f(t)$ is piecewise smooth and continuous, periodic with period T where $T = t_2 - t_1$ and

$$\int_{t_1}^{t_2} |f(t)|^2 dt < \infty$$

Then the Fourier series expansion of $f(t)$ is:

$$\mathbf{f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n\cos(\omega_n t) + b_n\sin(\omega_n t))}$$

where

- $\omega_{\mathbf{n}} = \mathbf{\frac{2n\pi}{T}, n = 1, 2, \cdots}$
- $\mathbf{a_n = \frac{2}{T}\int_{t_1}^{t_2} f(t)\cos(\omega_n t)dt}$
- $\mathbf{b_n = \frac{2}{T}\int_{t_1}^{t_2} f(t)\sin(\omega_n t)dt}$

Note

1. For $m \neq n$,

$$\int_0^T \sin\omega_n t \sin\omega_m t dt = \int_0^T \cos\omega_n t \cos\omega_m t dt = 0$$

And $\int_0^T \sin\omega_n t \cos\omega_m t dt = 0$ for any $m, n \in \mathrm{N}$.

Example

For $m = 2, n = 3$ and $T = 2\pi$

$$\int_0^{2\pi} \cos\frac{2 \cdot 2 \cdot \pi t}{2\pi} \cos\frac{2 \cdot 3 \cdot \pi t}{2\pi} dt = \frac{1}{2}\int_0^{2\pi} (\cos 5t + \cos t) dt = 0$$

2. For $m = n$,

$$\int_0^T \sin\omega_n t \sin\omega_n t dt = \int_0^T \cos\omega_n t \cos\omega_n t dt = \frac{T}{2}$$

Example

For $m = 2, n = 2$ and $T = 2\pi$

$$\begin{aligned}\int_0^T \cos\frac{2 \cdot 2 \cdot \pi t}{2\pi} \cos\frac{2 \cdot 2 \cdot \pi t}{2\pi} dt &= \int_0^{2\pi} \left(\cos^2 2t\right) dt \\ &= \frac{1}{2}\int_0^{2\pi} (1 + \cos 4t) dt = \pi\end{aligned}$$

3. Coefficients of Fourier series could be gotten by integration as follows:

- To get the value of a_n, integrate both sides of $f(t)$ and its Fourier series with product with $\cos\omega_n$ as follows:

$$\begin{aligned}\int_0^T f(t)\cos(\omega_n t) dt &= \int_0^T \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(\omega_n t) + b_n \sin(\omega_n t))\right)\cos(\omega_n t) dt \\ &= a_n \frac{T}{2} \\ \Rightarrow a_n &= \frac{2}{T}\int_0^T f(t)\cos(\omega_n t) dt\end{aligned}$$

As the same procedure, we also have:

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(\omega_n t) dt$$

Example

Consider

$$\begin{aligned}f(x) &= x \text{ for } -\pi < x < \pi \\ f(x + 2\pi) &= f(x) \text{ for } x \in \mathrm{R}\end{aligned}$$

reference the following picture ouptut.

Then $T = 2\pi$

$$
\begin{aligned}
a_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} x\cos(nx)dx \\
&= 0 \text{ (since odd)} \\
b_n &= \frac{\cancel{2}}{\cancel{2}\pi}\int_{-\pi}^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\left(\frac{-x\cos nx}{n}\bigg|_0^{\pi} + \int_0^{\pi}\frac{\cos n\pi}{n}dx\right) \\
&= 2\frac{(-1)^{n+1}}{n}
\end{aligned}
$$

And

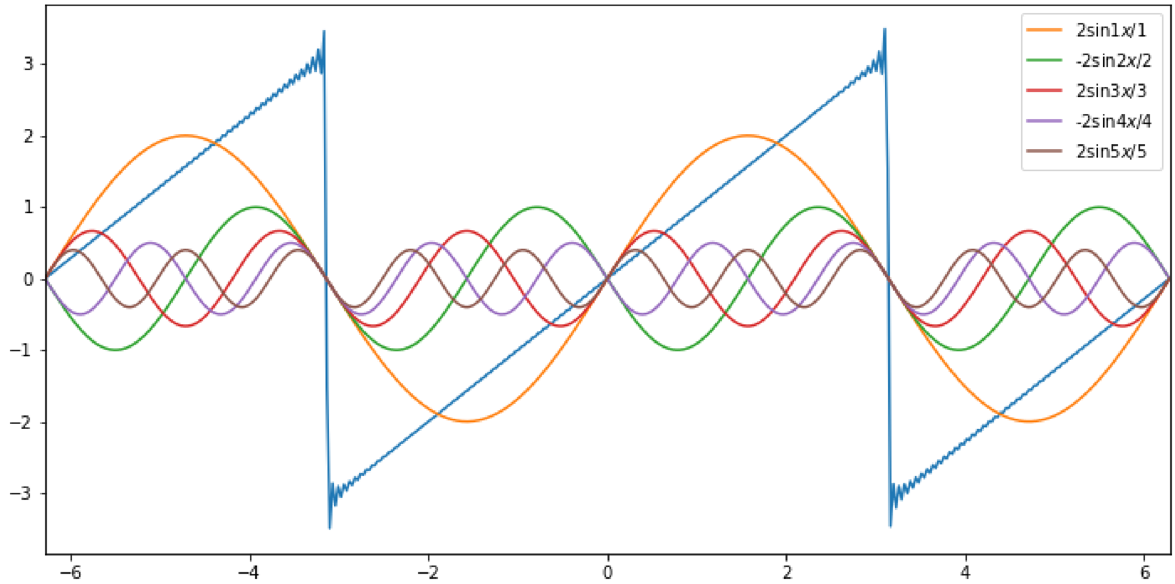
$$
\begin{aligned}
f(x) &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) \\
&= 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)
\end{aligned}
$$

In [4]:

```
def sawf(x,n):
    f=np.zeros(len(x))
    for i in range(1,n+1):
        f+=2*(-1)**(i+1)*sin(i*x)/i
    return f
```

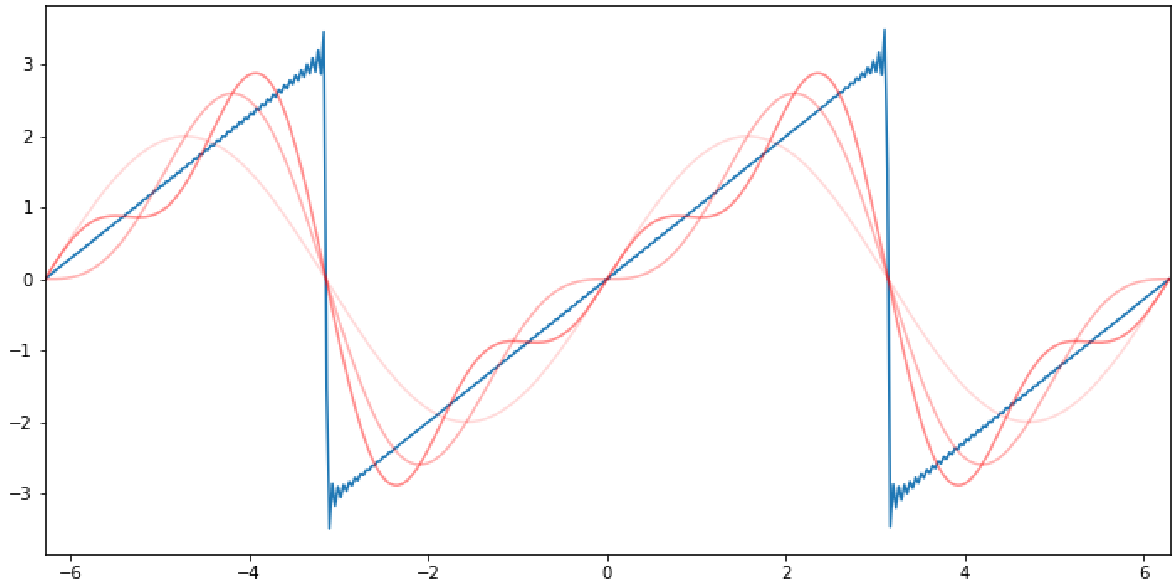
In [6]:

```
from numpy import pi,sin,cos
x=np.linspace(-2*pi,2*pi,400)
plt.figure(figsize=(12,6))
plt.xlim([-2*pi,2*pi])

plt.plot(x,sawf(x,100))
for i in range(1,6):
    plt.plot(x,2*(-1)**(i-1)*sin(i*x)/i,label="%s$\sin$%s$x$/%s" %((-1)**(i+1)*2,i,i
plt.legend();
```



In [14]:

```
from numpy import pi,sin,cos
x=np.linspace(-2*pi,2*pi,400)
plt.figure(figsize=(12,6))
plt.xlim([-2*pi,2*pi])

plt.plot(x,sawf(x,100))
f=np.zeros(len(x))
for i in range(1,4):
    f=f+2*(-1)**(i-1)*sin(i*x)/i
    plt.plot(x,f,color="red",alpha=0.15*i)
```



Example

(Square Wave)

$$f(x) = \begin{cases} 1 & \text{for} 0 \leqslant x < \pi \\ -1 & \text{for} -\pi \leqslant x < 0 \end{cases}$$
$$f(x+2\pi) = f(x) \text{ for } x \in \mathrm{R}$$

reference the following output picture.

Then ($T = 2\pi$)

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \cos(nx)dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos(nx)dx - \frac{1}{\pi}\int_{-\pi}^{0} 1 \cdot \cos(nx)dx \\ &= 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin(nx)dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)dx - \frac{1}{\pi}\int_{-\pi}^{0} \sin(nx)dx \\ &= \frac{4}{n\pi} \text{ if } n \text{ is odd.} \end{aligned}$$

This implies

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(nx)\left(1-(-1)^n\right)$$
$$= \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right)$$

In [15]:

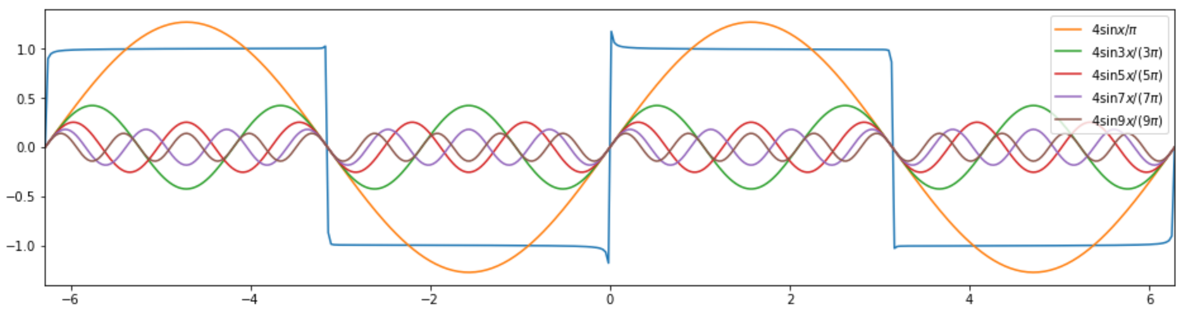
```
def squaref(x,n):
    f=np.zeros(len(x))
    for i in range(1,n):
        f+=4*sin((2*i-1)*x)/(2*i-1)/pi
    return f
```

In [16]:

```
from numpy import pi,sin,cos
x=np.linspace(-2*pi,2*pi,400)
plt.figure(figsize=(16,4))
plt.xlim([-2*pi,2*pi])

plt.plot(x,squaref(x,100))

plt.plot(x,4*sin(x)/pi,label="4sin$x$/$\pi$")
for i in range(2,6):
    plt.plot(x,4*sin((2*i-1)*x)/(2*i-1)/pi,label="4sin%s$x$/(%s$\pi$)" %(2*i-1,2*i-
plt.legend()
```

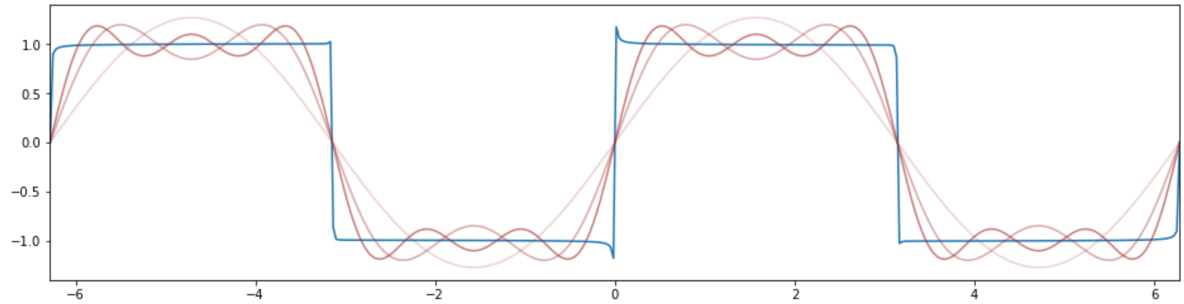
Out[16]:

<matplotlib.legend.Legend at 0x10c113400>



In [21]:

```
from numpy import pi,sin,cos
x=np.linspace(-2*pi,2*pi,400)
plt.figure(figsize=(16,4))
plt.xlim([-2*pi,2*pi])

plt.plot(x,squaref(x,100))

f=np.zeros(len(x))
for i in range(1,4):
    f=f+4*sin((2*i-1)*x)/(2*i-1)/pi
    plt.plot(x,f,color="brown",alpha=0.2*i)
```



In [ ]:

```

```

Example (Sum of series with term, $1/n^2$)

Suppose that

$$f(x) = \begin{cases} x & \text{for} 0 \leqslant x < \pi \\ -x & \text{for} -\pi \leqslant x < 0 \end{cases}$$
$$f(x+2\pi) = f(x) \text{ for } x \in \mathrm{R}$$

Since $f(x)$ is even, $b_n = 0$ and

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{2}{\pi}\int_0^{\pi} x dx = \pi \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\cdot\cos(nx)dx \\ &= \frac{2}{n^2\pi}\Big((-1)^n - 1\Big) \end{aligned}$$

Therefore

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)\right)$$

By the way, we can can calculate the value of

$$L = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

Take the value of $f(x)$ at $x = 0$,

$$\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi^2}{8}$$

$$\begin{aligned} L = & \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) \\ = & \frac{\pi^2}{8} + \frac{L}{4} \\ \Downarrow & \\ L = & \frac{\pi^2}{6} \end{aligned}$$

Example (Function neither Even nor Odd with Period T)

With $T \in [-2, 2]$, the function is defined as follows:

$$f(x) = \begin{cases} 0 & \text{for } -2 \leqslant x < 0 \\ x & \text{for } 0 < x \leqslant 2 \end{cases}$$
$$f(x+4) = f(x) \text{ for } x \in \mathrm{R}$$

Then period is $(T = 4)$, and

$$f(x) = \frac{a_0}{2} + \sum_{i=1} a_n \cos\omega_n x + b_n \sin\omega x$$

- $\frac{2}{T} = \frac{2}{4} = \frac{1}{2}$,
- $\omega_n = 2n\pi/T = n\pi/2$

$$\begin{aligned} a_0 = & \frac{2}{T}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_0^2 xdx = 1 \\ a_n = & \frac{2}{T}\int_{-2}^{2} f(x)\cos\omega_n xdx \\ = & \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx = \frac{1}{2}\cdot\left(\frac{2}{n\pi}\right)\left(\cancel{x}\sin\frac{n\pi x}{2}\Big|_0^2 - \int_0^2 \sin\frac{n\pi x}{2}dx\right) \\ = & \frac{1}{2}\left(\frac{2}{n\pi}\right)^2\left((-1)^n - 1\right) \\ b_n = & \frac{2}{n\pi}(-1)^{n+1} \end{aligned}$$

And

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}\left(\frac{2}{n\pi}\right)^2((-1)^n - 1)\cos\frac{n\pi x}{2} + \frac{2}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{2}\right]$$

In [38]:

```
import sympy as sp
from sympy import Symbol,symbols,integrate,pi,sin,cos
```

In [45]:

```
n,x =symbols("n x")

integrate(x*cos(n*pi*x/2)/2,(x,0,2))
```

Out[45]:

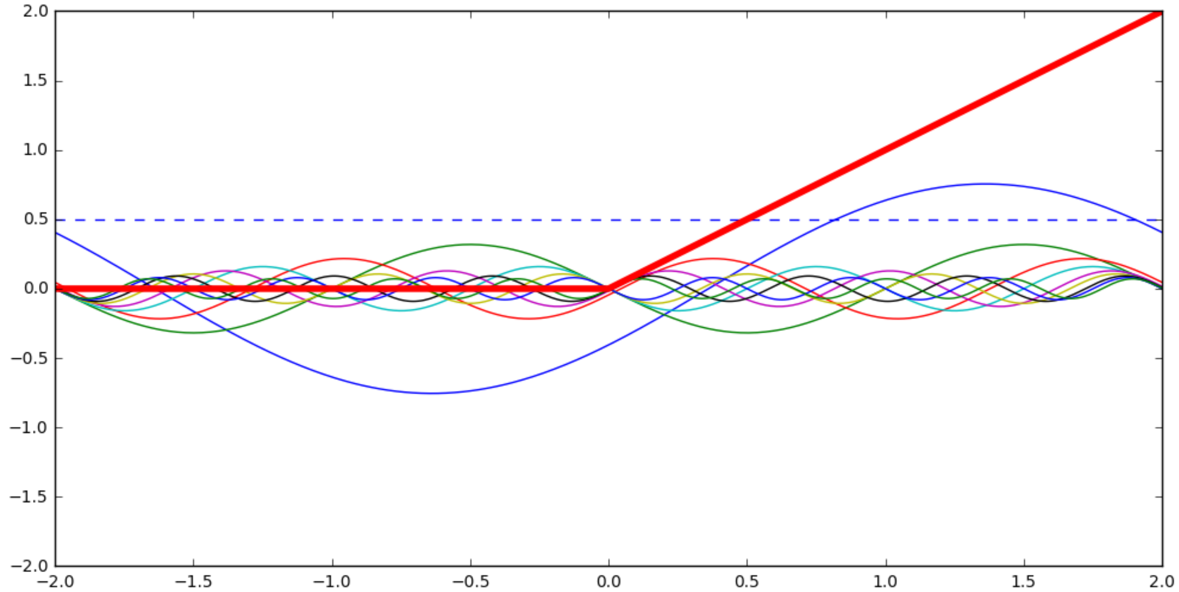
```
-Piecewise((0, Eq(n, 0)), (4/(pi**2*n**2), True))/2 + Piecewise((2, Eq
(n, 0)), (4*sin(pi*n)/(pi*n) + 4*cos(pi*n)/(pi**2*n**2), True))/2
```

In [56]:

```
from numpy import pi,sin,cos
t=np.linspace(-2,2,400)
plt.figure(figsize=(12,6))
plt.xlim([-2,2])
plt.ylim([-2,2])
n=31
f=np.ones(len(t))/2
plt.plot(t,f,'b--')
for i in np.arange(1,10):
    f=((2/i/pi)**2/2*((-1)**i-1)*cos(i*pi*t/2)+2/i/pi*(-1)**(i+1)*sin(i*pi*t/2))
    plt.plot(t,f)
plt.plot([-2,0,2],[0,0,2],'r',lw=4)
```

Out[56]:

```
[<matplotlib.lines.Line2D at 0x1122e56d8>]
```

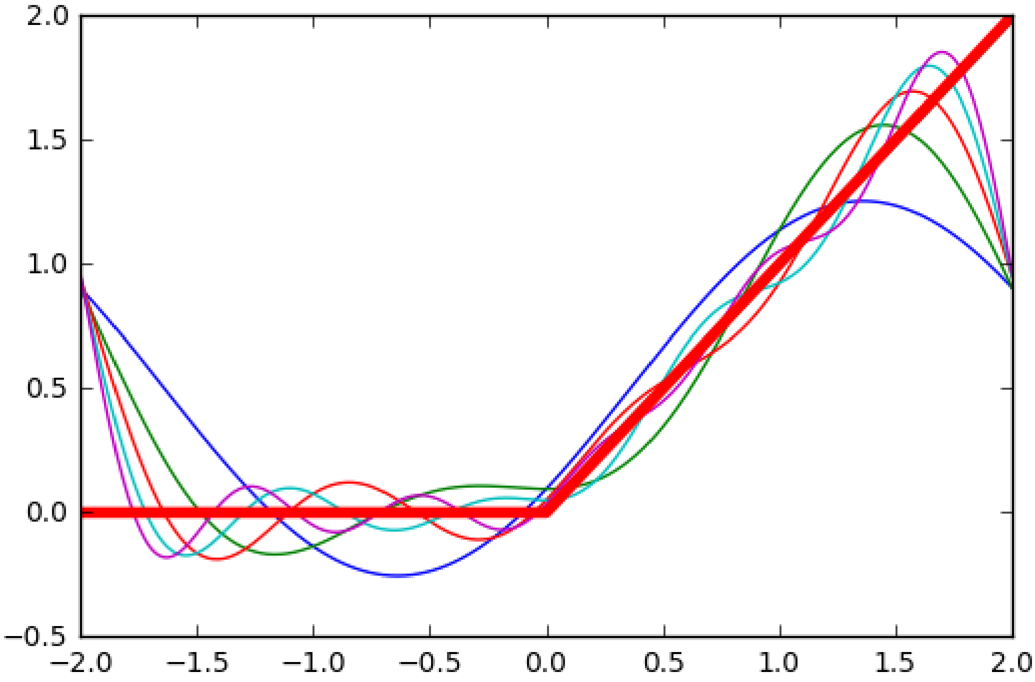


In [54]:

```
f=np.ones(len(t))/2
for i in np.arange(1,6):
    f+=1/1*((2/i/pi)**2/2*((-1)**i-1)*cos(i*pi*t/2)+2/i/pi*(-1)**(i+1)*sin(i*pi*t/2
    plt.plot(t,f)
plt.plot([-2,0,2],[0,0,2],'r',lw=4)
```

Out[54]:

```
[<matplotlib.lines.Line2D at 0x112028128>]
```

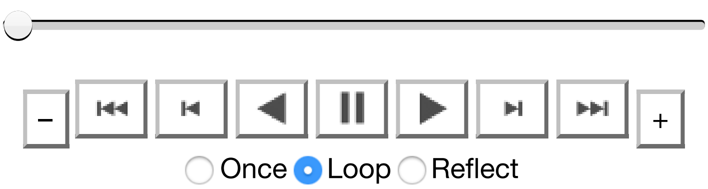


Jupyter provides functions capable of animating the data, for instance JSAnimation, pymovie.py, which could

In [101]:

```
from matplotlib import animation
from JSAnimation import IPython_display
from numpy import sin,cos,pi

fig = plt.figure(figsize=(4,4))
ax = plt.axes(xlim=(-2, 2), ylim=(-2, 2))
line, = ax.plot([], [], lw=2)
plt.title('General Fourier Series')
x = np.linspace(-2., 2., 400)
#plt.text(1,1,'$x I_{0<x<2}$')

def fn(n):
    f=np.ones(len(x))/2.
    for i in range(1,n+1):
        f+=((2/i/pi)**2/2*((-1)**i-1)*cos(i*pi*x/2)+2/i/pi*(-1)**(i+1)*sin(i*pi*x/2
    return f

def init():
    line.set_data([], [])
    return line,
def animate(i):
    y=fn(i)
    line.set_data(x, y)
    return line,

animation.FuncAnimation(fig, animate, init_func=init, frames=60, interval=30)
```

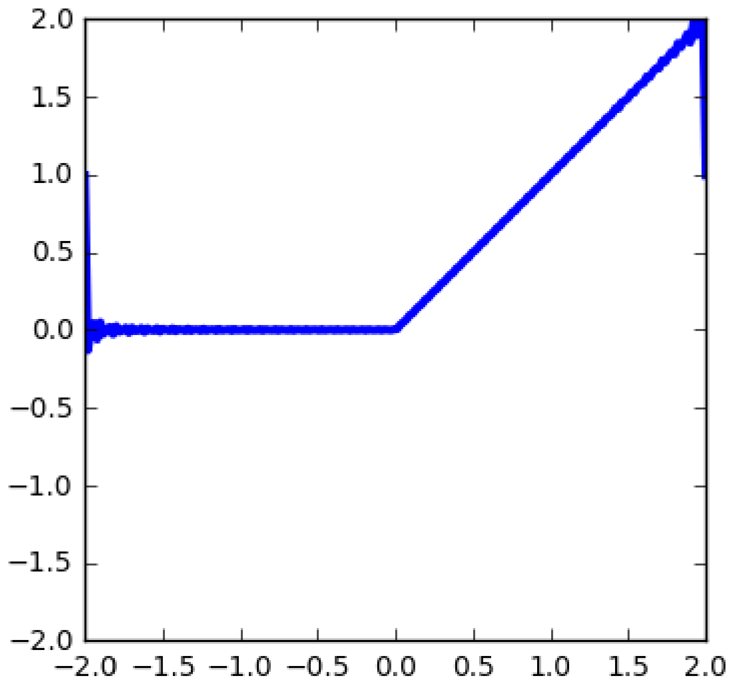
Out[101]:

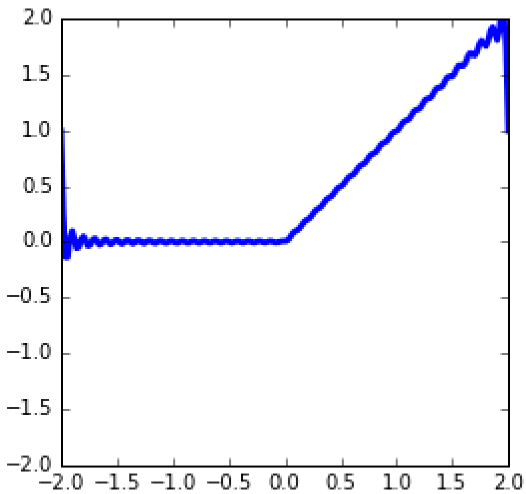


In [98]:

```
from moviepy.video.io.bindings import mplfig_to_npimage
import moviepy.editor as mpy

# DRAW A FIGURE WITH MATPLOTLIB

fig, ax = plt.subplots(1,figsize=(4,4), facecolor='white')
x = np.linspace(-2,2,200) # the x vector

def fn(t):
    n=int(40*t)
    f=np.ones(len(x))/2.
    for i in range(1,n+1):
        f+=((2/i/pi)**2/2*((-1)**i-1)*cos(i*pi*x/2)+2/i/pi*(-1)**(i+1)*sin(i*pi*x/2
    return f

ax.set_ylim(-2,2)
line, = ax.plot(x, fn(0), lw=3)
duration=1

# ANIMATE WITH MOVIEPY (UPDATE THE CURVE FOR EACH t). MAKE A GIF.

def make_frame(t):
    # defaulted time 1 second/40 frames
    line.set_ydata( fn(t))  # <= Update the curve
    return mplfig_to_npimage(fig) # RGB image of the figure

animation =mpy.VideoClip(make_frame, duration=duration)
animation.write_gif("FourierSeries.gif", fps=20)
```

```
[MoviePy] Building file ForuierSeries.gif with imageio


  0%|          | 0/41 [00:00<?, ?it/s]
  7%|▋         | 3/41 [00:00<00:01, 29.86it/s]
 15%|█▍        | 6/41 [00:00<00:01, 29.79it/s]
 22%|██▏       | 9/41 [00:00<00:01, 29.07it/s]
 29%|██▉       | 12/41 [00:00<00:01, 28.43it/s]
 37%|███▋      | 15/41 [00:00<00:00, 26.61it/s]
 44%|████▍     | 18/41 [00:00<00:00, 26.33it/s]
 51%|█████     | 21/41 [00:00<00:00, 26.01it/s]
 59%|█████▊    | 24/41 [00:00<00:00, 25.65it/s]
 66%|██████▌   | 27/41 [00:01<00:00, 25.92it/s]
 73%|███████▎  | 30/41 [00:01<00:00, 25.69it/s]
 80%|████████  | 33/41 [00:01<00:00, 25.95it/s]
 88%|████████▊ | 36/41 [00:01<00:00, 25.91it/s]
 95%|█████████▌| 39/41 [00:01<00:00, 26.00it/s]
 98%|█████████▊| 40/41 [00:01<00:00, 26.23it/s]
```





In [22]:

```
!jupyter nbconvert 5*ipynb
```

```
[NbConvertApp] Converting notebook 5 Infinite Series.ipynb to html
[NbConvertApp] Writing 2375886 bytes to 5 Infinite Series.html
```

In [ ]:

```

```