Final Quiz – Calculus (2012-2s-ME100)

Class: Seq No: Name:

Problem 1. Find extrema of $f(x,y,z)=1+x^2+y^2-xy-z^2$ subject to $x^2+y^2=1$ and $z^2=xy$. Fill the blanks to solve the problem: (total 30%)

a) Define Lagrangian as follows:(total 6%, each 2%)

$$L(x,y,z;\lambda,\mu)=\_\_\_\_\_\_\_\_\_\_\_+\lambda\_\_\_\_\_\_\_\_\_\_\_\_+\mu\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

b) Find the critical values of $L(x,y,z;\lambda,\mu)$: (total 14%, each 2%, 2%, 2%, 8%)

$$\begin{aligned} \nabla L &= \vec{0} \\ \left(\frac{\partial L}{\partial x},\frac{\partial L}{\partial y},\frac{\partial L}{\partial z}\right) &= (\_\_\_\_\_\_\_\_\_,\_\_\_\_\_\_\_\_\_\_,\_\_\_\_\_\_\_\_\_\_\_) \\ \Rightarrow (x,y,z) &= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{aligned}$$

c) Extrema exist since ____________. (2%)

d) Maxima:__________ and minimum:________________ (total 8%, each 4%)

Problem 2. Describe the Fubini formula for double integration in Cartesian Coordinates under the assumption: suppose that $f(x,y)$ is continuous on $D\subseteq\mathbb{R}^2$ and $D=\{(x,y)\in\mathbb{R}^2|\, c\leqslant y\leqslant d,\, g_1(y)\leqslant x\leqslant g_2(y)\}$. (10%)

Problem 3. Evaluate $\int\int_D x\,dA$ where $D=\{0\leqslant y\leqslant x+1\leqslant 2\}$ (20%)

Problem 4. Evaluate the double integral:

$$\int\int_D e^{-2x^2+2xy-y^2} dA$$

where $D = \{(x, y) \in \mathbb{R}^2 | 0 \leqslant y < \infty\}$ (20%)

Problem 5. Evaluate the following integrals: (20%)

$$\iiint_{\{x^{2/3}+y^{2/3}+z^{2/3} \leqslant 4, \text{ and } x,y,z \geqslant 0\}} 1\, dV$$

1. a) $2-xy-z^2+\lambda(1-x^2-y^2)+\mu(z^2-xy)$

   b) $(-y-2\lambda x-\mu y,-y-2\lambda x-\mu y,-2z+2\mu z)$

   c) $\begin{cases}(x,y,z)=(\pm 1,0,0),(0,\pm 1,0)\\ (x,y,z)=\pm\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)\end{cases}$

   d) x,y and z are bounded.

   e) maximum: 2, minimum: 1

2.

3.

$$\begin{aligned}\iint_D x\,dA &= \int_{-1}^{1} dx\int_0^{x+1} x\,dy\\ &= \int_{-1}^{1} x(x+1)dx = 2/3\end{aligned}$$

4.

$$\begin{aligned}I &= \int_0^{\infty} dy\int_{-\infty}^{\infty} e^{-2(x-\frac{y}{2})^2}e^{-\frac{1}{2}y^2}dx\\ &= \sqrt{\frac{\pi}{2}}\int_0^{\infty} e^{-\frac{1}{2}y^2}dy\\ &= \sqrt{\frac{\pi}{2}}\frac{\sqrt{\pi}}{\sqrt{2}}=\frac{\pi}{2}\end{aligned}$$

5.

$$\begin{aligned}&\iiint_{\{x^{2/3}+y^{2/3}+z^{2/3}\leqslant 4,\text{ and }x,y,z\geqslant 0\}} 1dV\\ =&\iiint_{\{X^2+Y^2+Z^2\leqslant 4,\text{ and }X,Y,Z\geqslant 0\}} 27X^2Y^2Z^2dXdYdZ\\ =&\ 27\int_0^2 d\rho\int_0^{\pi/2} d\theta\int_0^{\pi/2}\rho^8\cos^2\theta\sin^2\theta\sin^5\phi\cos^2\phi d\phi\\ =&\ 96\pi\left(\frac{8}{105}\right)\end{aligned}$$