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(Integration)

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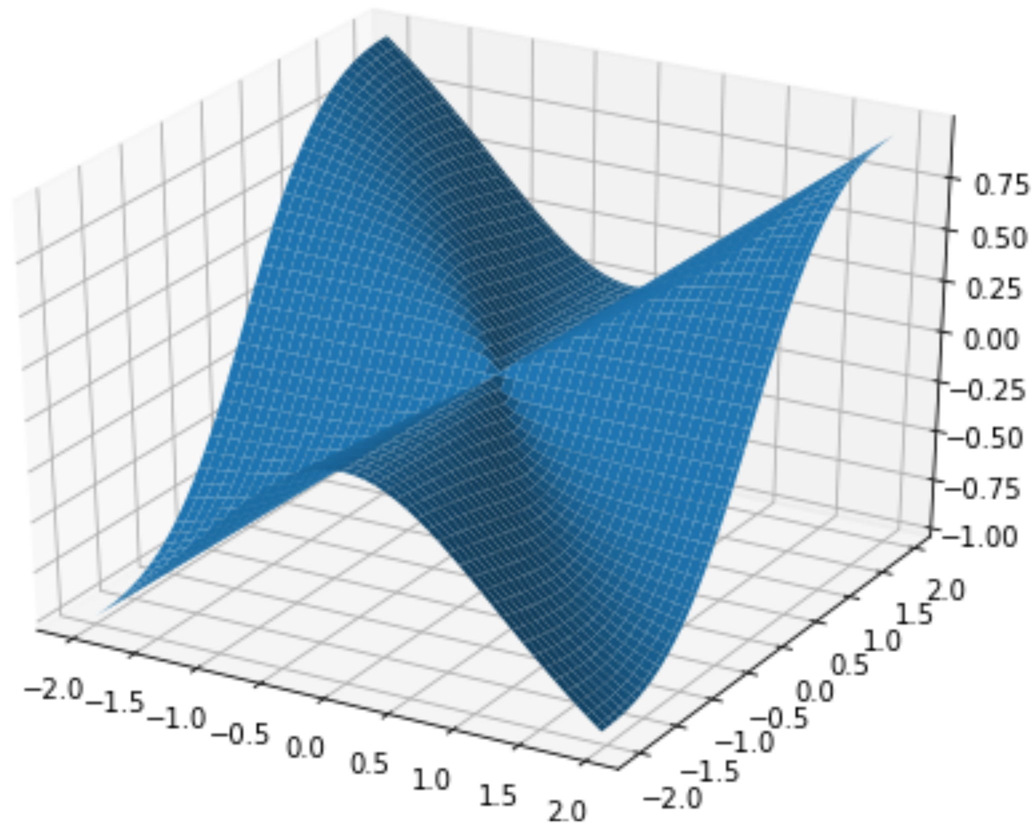
```
<mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x111c29cc0>
```



There are plenty of visualization packages in Python, MayaVi is one
utilies, animation, interaction etc. Here, after struggle installztion of
on notebook environment:

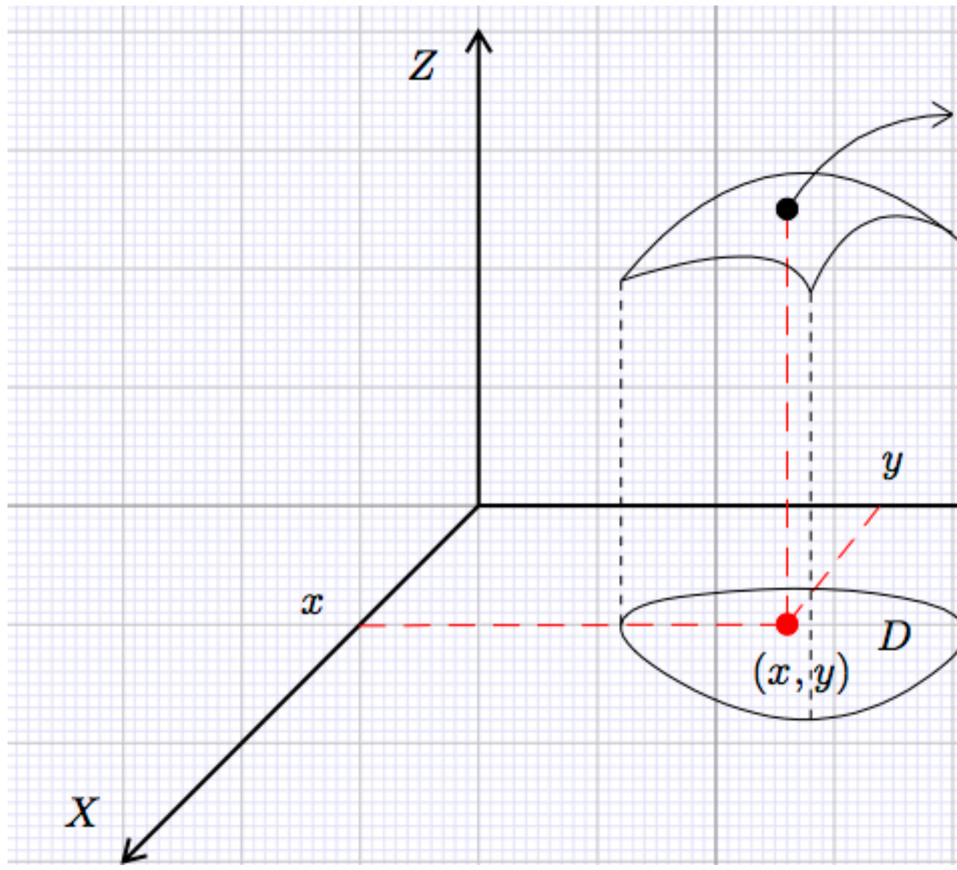
```
---------------------------------------------------------------------------
ModuleNotFoundError                       Traceback (most recent cal
~/anaconda36/anaconda/lib/python3.6/site-packages/tvtk/tvtk_classes.
t_module(fname)
     22         mod = __import__('tvtk.custom.%s'%fname,
---> 23                          globals(), locals(), [fname])
     24     except ImportError:

ModuleNotFoundError: No module named 'tvtk.custom.light'

During handling of the above exception, another exception occurred:

ModuleNotFoundError                       Traceback (most recent cal
<ipython-input-9-83de05f3965f> in <module>()
      1 import numpy as np
----> 2 import mayavi.mlab as mlab
      3 #import  moviepy.editor as mpy

~/anaconda36/anaconda/lib/python3.6/site-packages/mayavi/mlab.py in
```

1.1 Double Integrals

If $z = f(x, y)$ is continuous and $f(x, y)$ is nonnegative for all (x, y)
volume of solid under the graph of $f(x, y)$ and above $X - Y$ plane

$$V = \iint_D f(x, y)dA$$

where $dA = dxdy$ is the element of area and V is called the doub



1.2 Theorem (Fubini's Theorem)

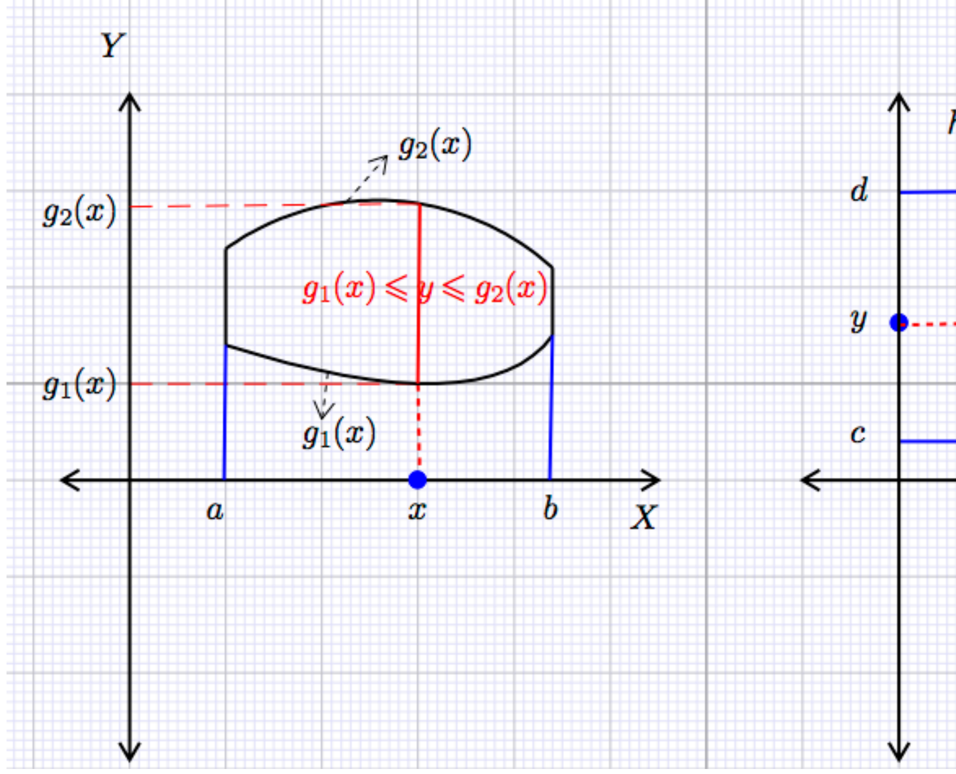
If $f(x, y)$ is continuous over D,

1. and D=$\{(x, y)|a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\}$,

$$\iint_D \mathbf{f(x, y)dA} = \int_a^b \mathbf{dx} \int_{g_1(x)}^{g_2(x)}$$

2. and D=$\{(x, y)|c \leqslant y \leqslant d, h_1(y) \leqslant x \leqslant h_2(y)\}$,

$$\iint_D \mathbf{f(x, y)dA} = \int_c^d \mathbf{dy} \int_{h_1(y)}^{h_2(y)}$$



1.3 Example

Evaluate the double integral of $f(x) = 3x^2y$ over square region D

$$\iint_{\{1 \le y \le 2, 0 \le x \le 1\}} 3x^2 y dA$$

1.4 Example

Evaluate the following double integral:

$$\iint_{\{0 \le x \le 2, -1 \le y \le 1\}} (1 - 2xy^2)d$$

1.5 Properties of Double Integral

Suppose that both $\iint\limits_D f(x,y)dA$ and $\iint\limits_D g(x,y)dA$ exist and $c \in \mathbb{R}$

1. $\iint\limits_D cf(x,y)dA = c\iint\limits_D f(x,y)dA$,
2. $\iint\limits_D [f(x,y) \pm g(x,y)]dA = \iint\limits_D f(x,y)dA \pm \iint\limits_D g(x,y)dA$,
3. If $f(x,y) \geq 0$, then $\iint\limits_D f(x,y)dA \geq 0$,
4. If $f(x,y) \geq g(x,y)$, then $\iint\limits_D f(x,y)dA \geq \iint\limits_D f(x,y)dA$.
5. $\iint\limits_{D=D_1\cup D_2} f(x,y)dA = \iint\limits_{D_1} f(x,y)dA + \iint\limits_{D_2} f(x,y)dA$ where D_1 ∩

1.6 Exercise, p.1155

14.

$\iint\limits_{0\leq x\leq 2, 0\leq y\leq 1} 2xdA$ is the solid under surface $f(x,y) = 2x$ and abov

14.

$\iint\limits_{0\leq x,y,\ x^2+y^2\leq 9} \sqrt{9-x^2-y^2}dA$ is half upper sphere centred at $(0,$

1.7 Example

Find the volume of solid under the surface of $z = x^3 + 4y$ and ove
and $y = x^2$.

$$V = \iint_{0 \le x \le 2, x^2 \le y \le 2x} z dA = \int_0^2 dx \int_{x^2}^{2x} (x^3 + 4y) dy$$

1.8 Example

Find the volume of solid under the surface of $z = 2x - y$ and over
and $x - y = 2$.

$$V = \iint_{0 \le x \le 2, x^2 \le y \le 2x} z dA == \int_{-1}^2 dy \int_y$$

1.9 Example

Evaluate

$$\iint_{0 \le y \le 1, y \le x \le 1} \frac{\sin x}{x} dA = \int_0^1 dx \int_0$$

1.10 Exercise p.1165

1.11 12.

Evaluate

$$\int_0^{\pi} dx \int_{\exp(-2x)}^{\exp(\cos x)} \frac{\ln y}{y} dy = \int_0^{\pi} d$$

by substitution, $y = \exp(u)$:

18.

$$\iint_{0 \le x \le 1, 0 \le y \le x} \sqrt{1 - x^2} dA$$

22.

$$\iint_{0 \le y \le 1, -y-1 \le x \le y-1} (x^2 + y^2)$$

24.

$$\iint_{1 \le y \le e, y \le x \le y^2} \frac{1}{xy} dA$$

28.

$$\iint_R (x^2 + y) dA$$

where the region, R, is bounded by $y = x^2 + 2, x = 0, x = 1, y =$

32.

$$\iint_R y dA$$

where the region, R, is bounded by $x^2 + y^2 \leq 1, y \geq 0$.

56.

Evaluate

$$\int_0^2 dx \int_{x^2}^4 x \cos y^2 dy = \int_0^4 dy \int_0$$

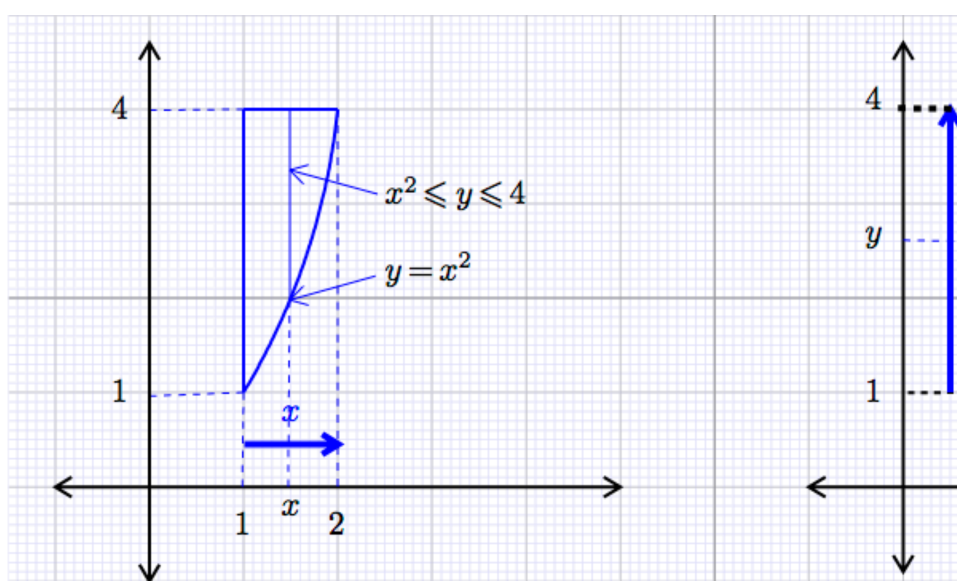
1.12 Example

Evaluate the following double integral:

$$\iint_D (x+y)dA$$

where

$$D = \left\{(x,y)\middle|1 \leqslant x \leqslant \sqrt{y}, 1 \leqslant\right.$$
$$= \left\{(x,y)\middle|x^2 \leqslant y \leqslant 4, 1 \leqslant\right.$$



Reference the above graph, we can calculate the double integral w
1. Along X-axis:

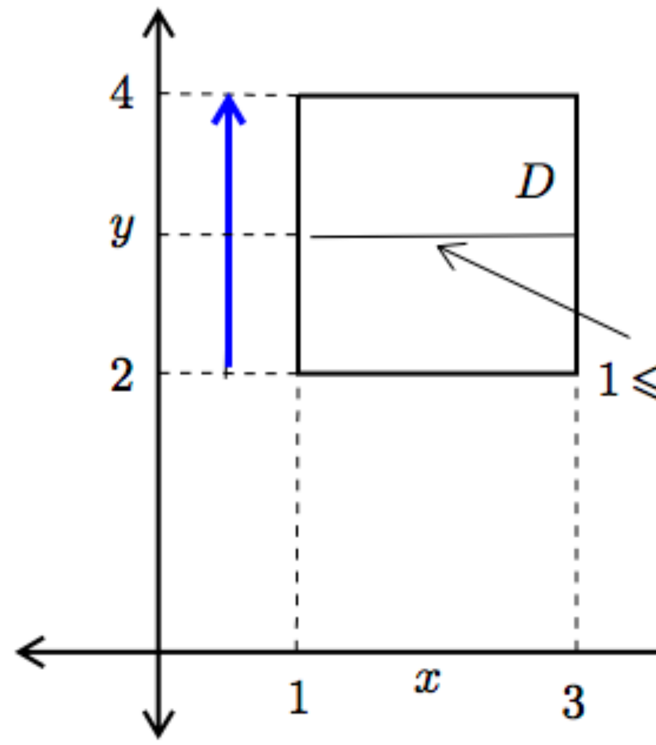
$$
\begin{aligned}
\iint_D (x+y)dA &= \int_1^4 dy \int_1^{\sqrt{y}} (x \\
&= \int_1^4 \left(\frac{x^2}{2} + xy\right)\Big|_1 \\
&= \int_1^4 \left(y^{3/2} - \frac{1}{2} \right. \\
&= \left(\frac{2y^{5/2}}{5} - \frac{y}{2} - \right. \\
&= 61\frac{3}{20}
\end{aligned}
$$

2. Along Y-axis:

$$
\begin{aligned}
\iint_D (x+y)dA &= \int_1^2 dx \int_{x^2}^4 (\\
&= \cdots \\
&= 61\frac{3}{20}
\end{aligned}
$$

1.13 Example

If $D = \{(x, y) | 1 \leqslant x \leqslant 3, 2 \leqslant y \leqslant 4\}$,

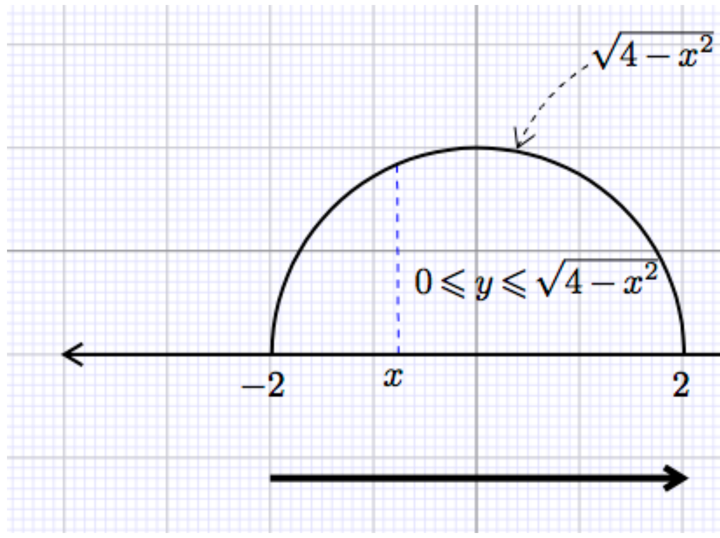


By Fubini's theorem, we have

$$\iint\limits_D \frac{y}{x} dA = \int_2^4 dy \int_1^3 \frac{y}{x} dx$$

1.14 Example

If $D = \left\{ (x, y) | -2 \leqslant x \leqslant 2, 0 \leqslant y \leqslant \sqrt{4 - x^2} \right\}$,



then

$$\iint_D 1 dA = \int_{-2}^{2} dx \int_0^{\sqrt{4-}}$$

$$= \int_{-2}^{2} \sqrt{4 - x^2} d$$

$$= \frac{1}{2} 2^2 \pi = 2\pi$$

i.e. the area of half circle, D, is 2π.

Suppose that all the points (x, y)in D can be transformed as:

$$x = \phi(u, v), y = \psi(u, v$$

Then the double integral can be evaluated as followed:

$$\iint_D f(x, y)dA = \iint_D f(\phi(u, v), \psi(u$$

where J is called the Jacobian of (x, y) and equal to:

$$J = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

$$= \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial}{\partial}$$

Especially, as in polar coordinate system, we have

$$x = r\cos\theta, y = r\sin$$

where r is the distance between (x, y) and origin and θ is the angl origin, and X-axis. In this case,

$$\frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial y}{\partial r} = \sin$$

and $|J| = r$ since

$$J = \left| \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \right|$$

$$= \cos\theta \cdot r\cos\theta - (-r\sin$$

$$= r$$

1.15 Example

If $D = \left\{ (x, y) | -2 \le x \le 2, 0 \le y \le \sqrt{4 - x^2} \right\}$, then

$$\iint_D 1 dA = \int_{-2}^{2} dx \int_0^{\sqrt{4-}}$$

$$= \int_{-2}^{2} \sqrt{4-x^2}d$$

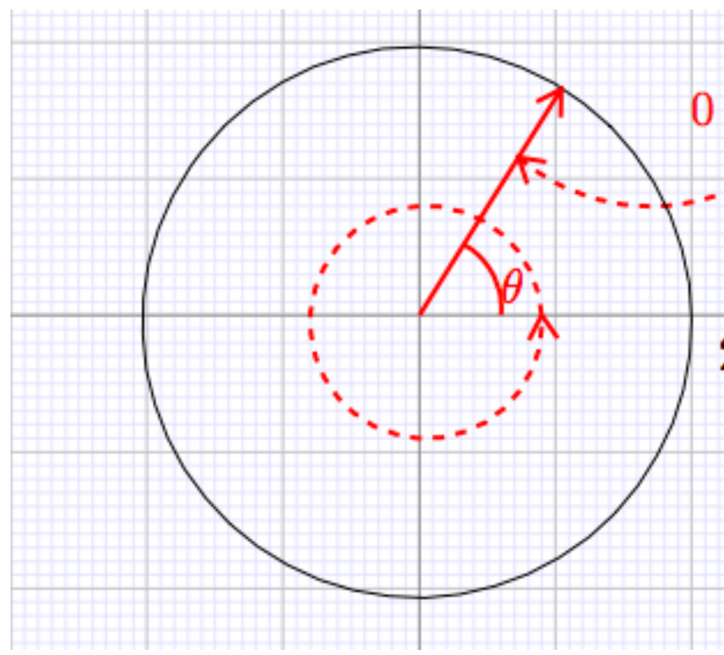
$$= \frac{1}{2}2^2\pi = 2\pi$$

i.e. the area of half circle, D, is 2π.

Find the volume of the semi-sphere above X-Y plane with radius 2

$$\iint_{\{(x,y)|x^2+y^2\leqslant 4\}} \sqrt{4-x^2-y}$$

Sol: Since $\{(x,y)|x^2+y^2 \leqslant 4\} = \{(r,\theta)|0 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant 2$



and $|J| = r$. Then

$$\iint_{\{(x,y)|x^2+y^2\leq 4\}} \sqrt{4-x^2-y^2}dA = \iint_{\{(r,\theta)|0\leq r\leq 2,0\leq}$$

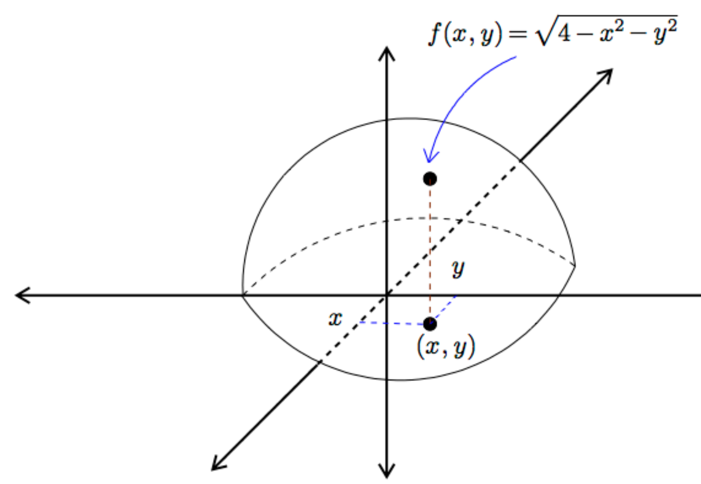
$$= \int_0^{2\pi} d\theta \int_0$$

$$= \int_0^{2\pi} \left(-\frac{1}{3}\right.$$

$$= \int_0^{2\pi} \frac{8}{3}d\theta$$

$$= \frac{16\pi}{3}$$

i.e. half of volume of ball with radius 2, reference the following:



1.16 Example

If $R = \left\{(x,y)|1 \leq x^2+y^2 \leq 4, 0 \leq x, y\right\}$, then

$$\iint_{\{(x,y)|1\leq x^2+y^2\leq 4,0\leq x,y\}} (2x+3y)dA = \iint_{\{(r,\theta)|1\leq r\leq 2,0\leq\theta\leq 2\pi\}}$$

$$= \frac{35}{3}$$

1.17 Example

Find the volume of solid, S, lies below $z = \sqrt{9 - x^2 - y^2}$ and abo

$$V = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{9 - r^2} dr = \frac{2}{3}$$

1.18 Example

Find the volume of solid, S, lies below $z = 4 - x^2 - y^2$ and above

$$V = \int_{-\pi/2}^{2\pi} d\theta \int_0^{2\cos\theta} r(4 - r^2$$

1.19 Example

Evaluate the integral $\int_0^\infty e^{-x^2} dx$.

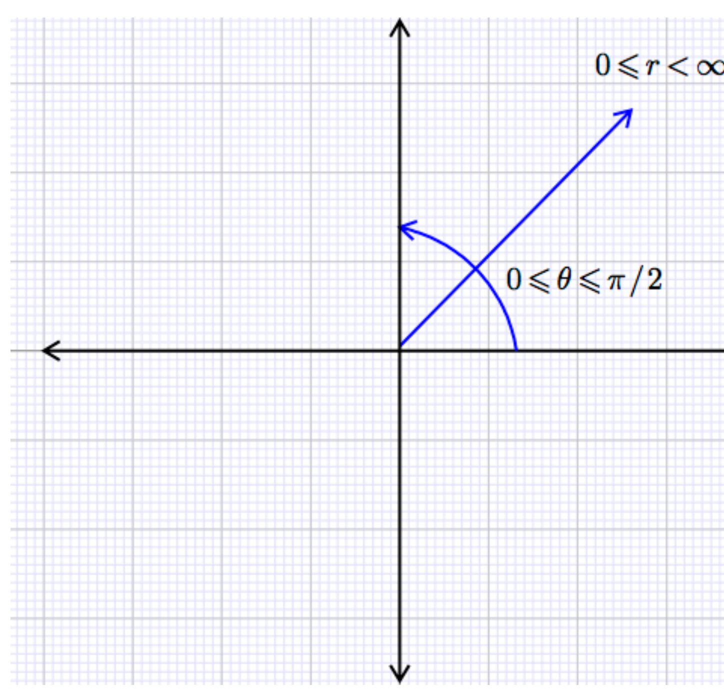
Sol: Let $I = \int_0^\infty e^{-x^2} dx$. Then $I = \int_0^\infty e^{-y^2} dy$ by changing the c
product:

$$
\begin{aligned}
I^2 &= I \cdot I \\
&= \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e \\
&= \int_0^\infty \int_0^\infty e^{-x^2-y^2} dx \\
&= \int_0^{\pi/2} \int_0^\infty e^{-r^2} \cdot rdr \\
&= \int_0^{\pi/2} \frac{1}{2} d\theta = \frac{\pi}{4}
\end{aligned}
$$

In the third and forth equalities, the domain is as follows:

$$
\begin{aligned}
D &= \{(x, y) | 0 \leqslant x, y < \infty\} \\
&= \{(r, \theta) | 0 \leqslant r < \infty, 0 \leqslant
\end{aligned}
$$

reference the following:



i.e. the whole first quadrant. This implies $I = \frac{\sqrt{\pi}}{2}$.

Note: The related formula are listed:

1. By symmetry, we have

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_{0}^{\infty} e^{-x^2} dx$$

2. To prove

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx =$$

change the variable by substitution of $t = \frac{x-\mu}{\sqrt{2}\sigma}$ and $dt = \frac{dx}{\sqrt{2}\sigma}$. Al

$$x\Big|_{-\infty}^{\infty} \Longrightarrow t = \frac{x-\mu}{\sqrt{2}\sigma}\Big|$$

Then

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{\pi}} \cdot$$

3. As the similar procedure, we can also calculate $\int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)}{2\sigma^2}}$

ii), $t = \frac{x-\mu}{\sqrt{2}\sigma}$, we have:

$$\int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} d$$

$$= \mu + \int_{-\infty}^{\infty} \frac{t}{\sqrt{\pi}} e^{-t^2} dt$$

$$= \mu$$

The last result holds since the definite integral of odd function over

We can also describe the result by the graphs of such functions.

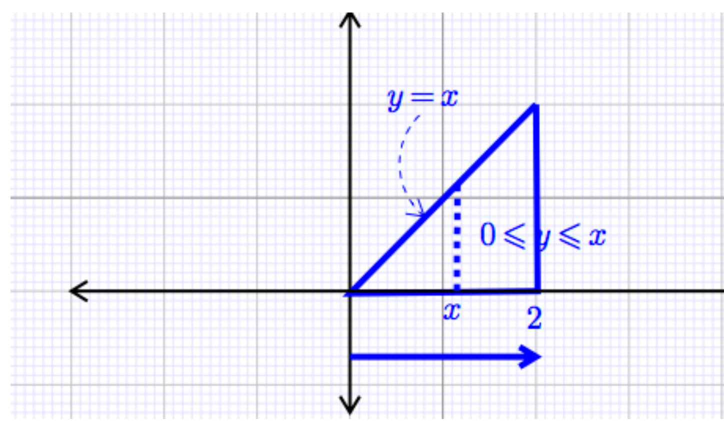
Obviously, the latter is as the same \ as the former but forward 2 u
from $-\infty$ to ∞, it is no doubt that both the integrals for

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{ and } \frac{1}{\sqrt{2\pi}} e^{-}$$

are the same.

1.20 Exercise

Integrate $y\sqrt{x^3+1}$ over D:

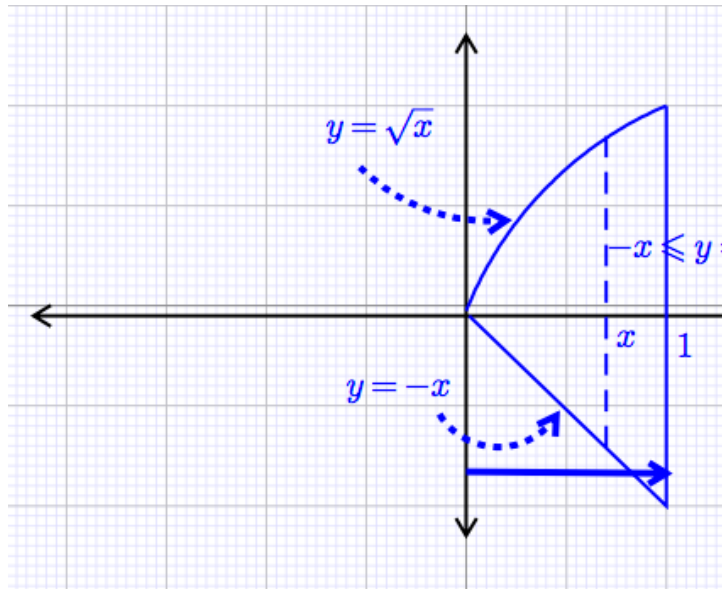


Then

$$\begin{aligned}
\iint\limits_D y\sqrt{x^3+1}\,dA &= \int_0^2 \int_0^x y\sqrt{} \\
&= \int_0^2 \sqrt{x^3+1} \\
&= \int_0^2 \frac{x^2}{2}\sqrt{x^3} \\
&= \frac{1}{6}\int_0^2 \sqrt{x^3+} \\
&= \frac{1}{6}\cdot\frac{2}{3}\cdot(x^3+ \\
&= \frac{26}{9}
\end{aligned}$$

1.21 Exercise

Integrate $f(x, y) = y/(1 + x)$ over D:



Then

$$\begin{aligned}
\iint_D \frac{y}{1+x} dA &= \int_0^1 \int_{-x}^{\sqrt{x}} \frac{y}{1+x} \\
&= \int_0^1 \frac{1}{1+x} \frac{y^2}{2}\Big|_{-x}^{\sqrt{x}} \\
&= \frac{1}{2} \int_0^1 \frac{x-x^2}{1+x} dx \\
&= \frac{1}{2} \int_0^1 \Big(-x + 2 \\
&= \frac{1}{2} \Big(-\frac{x^2}{2} + 2x - \\
&= \frac{1}{2} (3/2 - 2 \ln 2)
\end{aligned}$$

1.22 **p.1173** Exercise

10.

$$\iint_{\{x^2+y^2\leq 9, x,y\geq 0\}} (x+2y)dA$$

16.

$$\iint_{\{x^2+y^2\leq 4, x^2+(y-1)^2\geq 1, x,y\geq 0\}} (x+$$

26. Volume of solid, T, which inside $x^2 + y^2 + z^2 = 4$ and inside x

$$\iint_{\{r\leq 2\sin\theta\}} r\sqrt{4-r^2}drd\theta =$$

Above equal to $\frac{16}{3}(1-\cos^3\theta)$

37.

$$\int_{-2}^{2} dx \int_{0}^{\sqrt{4-x^2}} e^{x^2+y^2} dy = \int_{0}^{\pi} d$$

1.23 Applications for Changing v

In probability and statistic, the techniques of change of variables a
density functions (abbr,. as p.d.f.) of new random variables.

1.24 Example

Suppose that one variable, x, is chosen randomly and uniformly fro
such similar condition. What is the probability that $x \leqslant y$?

Sol: Let D the domain that $x \leqslant y$. Then The answer of this problen
integral:

$$\iint\limits_{D} 1dxdy = \iint\limits_{\{0 \leqslant x \leqslant y \leqslant 1\}} 1d$$

$$= \int_0^1 dy \int_0$$

$$= \int_0^1 ydy$$

$$= \frac{1}{2}$$

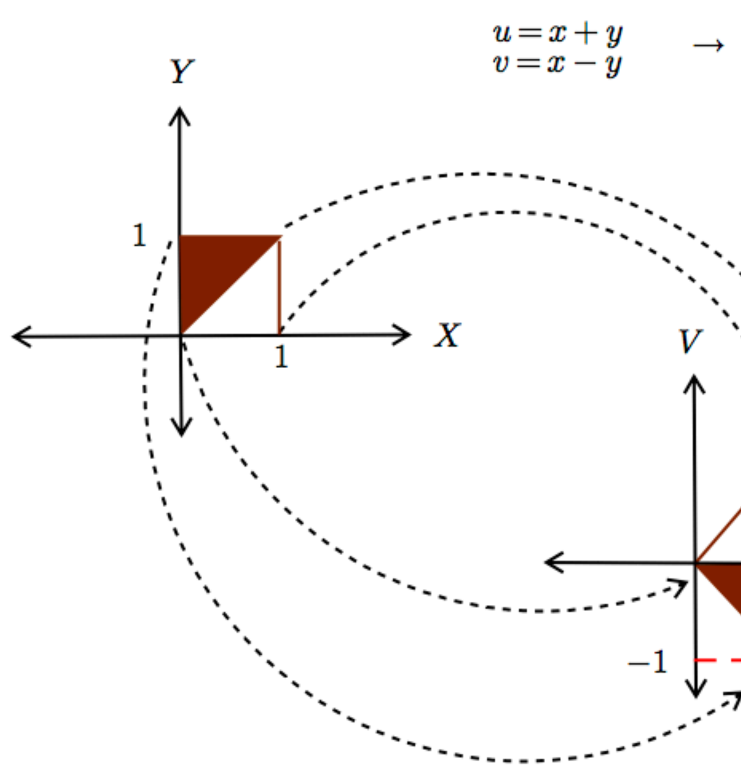
Another method is by changing variables from (x, y) to (u, v) wher
the double integral has to be changed as:

1. variables change:

$$u = x + y, v = x - y \Rightarrow x = \frac{u+v}{2}$$

$$|J| = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| =$$

2. Domain change: reference the following picture



3. The double integral is:

$$\iint_D 1dxdy = \int_{-1}^{0} dv \int_{-v}^{2+v}$$
$$= \frac{1}{2}$$

1.25 Example

Change the following double integral in (X, Y) into (U, V):

$$\int_0^\infty \int_0^\infty \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} y^{\beta-1} e^{-x-y} dxdy, 0$$

where

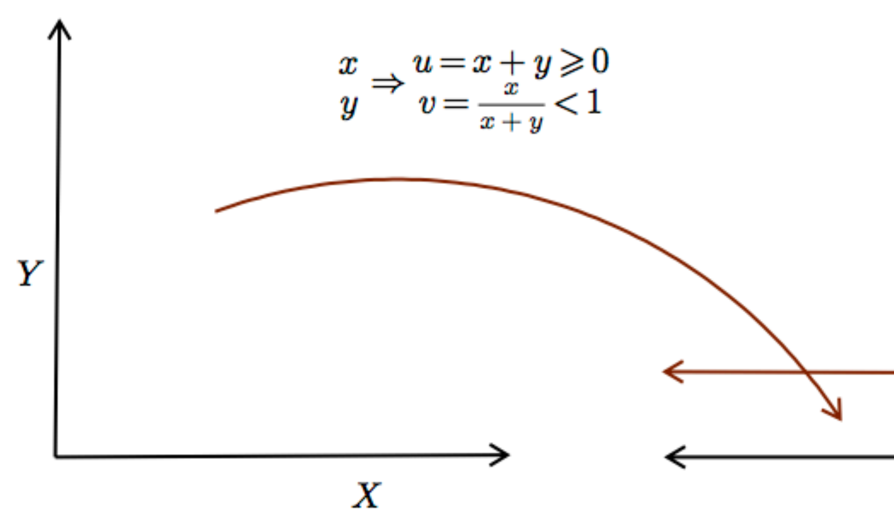
$$u = x + y \text{ and } v = \frac{x}{x +}$$

Ans:

$$\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1}(1-v)^{\beta-1} dv \int_0^\infty \frac{}{\Gamma(}$$

Note that

$$0 \leqslant x, y \Rightarrow 0 \leqslant u \text{ and } 0 \leqslant$$



1.26 Example

Change the following double integral in (X, Y) into (U, V):

$$\iint_{\{0\leqslant x,y\}} \frac{1}{4}e^{-\frac{x+y}{2}}dxdy$$

where $u = x + y$ and $v = y$.

Note that

$$\begin{array}{l} 0 \leqslant x \\ 0 \leqslant y \end{array} \Rightarrow \begin{array}{r} 0 \leqslant x = u - \\ 0 \leqslant y = v \end{array}$$

$$\Rightarrow \begin{array}{l} 0 \leqslant v \\ v \leqslant u \end{array}$$

and

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \left|\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right|$$

then

$$\iint_{\{0\leqslant x,y\}} \frac{1}{4}e^{-\frac{x+y}{2}}dxdy = \int_0^\infty du \int_0$$

$$= \int_0^\infty \frac{1}{4}ue^{-}$$

i.e. sum of two independent χ_2^2 is χ_4^2.

In Monte-Carlo simulation, the data generating by normal density a
generated? The answer is very simple: they can be generated by th
$[0, 1]$.

1.27 Example (Monte-Carlo Sim
Data)

Change the following double integral in (X, Y) into (U, V):

$$\iint_{\{0<x,y<1\}} 1dxdy$$

where $u = (-2\ln x)^{1/2}\cos 2\pi y$ and $v = (-2\ln x)^{1/2}\sin 2\pi y$.

1. Since $0 < x, y < 1$, we have

$$\begin{aligned} -2\ln x \in (0,\infty) \\ 2\pi y \in (0, 2\pi) \end{aligned} \quad \Rightarrow \ u, v$$

2. change the variable-pair, from (x, y) to (u, v):

$$\begin{aligned} u = (-2\ln x)^{1/2}\cos 2\pi y \\ v = (-2\ln x)^{1/2}\sin 2\pi y \end{aligned} \Rightarrow u^2 + v^2 = -2\ln$$

$$\Rightarrow x = \exp(-(u^2 +$$

3. evaluate the Jacobian:

$$\begin{aligned} J &= \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} \mathbf{-ue^{-(u^2+v^2)/2}} & -ve^{-(u} \\ \frac{-v}{2\pi(u^2+v^2)} & \frac{-}{2\pi(u^2} \end{pmatrix} \right. \\ &= \frac{(u^2+v^2)}{2\pi(u^2+v^2)} e^{-(u^2+v^2)/2} \\ &= \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-v^2/2} \end{aligned}$$

4. change the double integral with (x, y)-pair to (u, v)-pair

$$\begin{aligned} \iint_{\{0<x,y<1\}} 1dxdy &= \iint_{\{(u,v)\in\mathbb{R}^2\}} Jdudv \\ &= \int_{\{u\in\mathbb{R}\}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \int \end{aligned}$$

this means that U, V are standard normal random variables and is
$f_U(u)g_V(v)$.

During the simulation, some few data in front are always to be disc

1.28 Example (t-distribution data

Change the following double integral in (X, Y) into (T, V):

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \int_0^{\infty} \frac{y^{r/2-1}}{\Gamma(r/2}$$

where

$$t = \frac{x}{\sqrt{\frac{y}{r}}} \text{ and } v = y$$

Moreover, we have

$$\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{y^{r/2-1} e^{-y/2}}{\Gamma(r/2)2^{r/2}} dy = \frac{\Gamma((r +}{\sqrt{\pi r}\Gamma(}$$
$$= f_T(t) \text{ wh}$$

This is called the p.d.f of t-distribution.

1.29 Example (F-distribution dat

Change the following double integral in (X, Y) into (F, V)

$$\iint_{\{0<x,y\}} \frac{x^{r/2-1} y^{s/2-1} e^{-(x+y)/2}}{\Gamma(r/2)\Gamma(s/2)2^{(r+s)/2}}$$

where

$$f = \frac{x/r}{y/s} \text{ and } v = y$$

Moreover, we have

$$\int_{\{0<y\}} \frac{x^{r/2-1} y^{s/2-1} e^{-(x+y)/2}}{\Gamma(r/2)\Gamma(s/2)2^{(r+s)/2}} dxdy = f_F(f)$$

$$= \frac{\Gamma((r+s)/2)(}{\Gamma(r/2)\Gamma(s/2)(1+}$$

1.30 Exercise

Suppose that one variable, x, is chosen randomly and uniformly fro
such similar condition. What is the probability that $x \leqslant 2y$, i.e. the

This case is evaluated as follows:

$$\begin{aligned} \wp(0 \leqslant x \leqslant 2y \leqslant 1) &= \iint_D 1d \\ &= \int_0^1 dx \\ &= \int_0^1 (1 \\ &= 3/4 \end{aligned}$$

1.31 Triple Integrals

Similar to last section, we can consider the multiple integrations fo
$w = f(x, y, z)$ is continuous and $f(x, y, z)$ is nonnegative for all $(x$
then the triple integral of $f(x, y, z)$ and above $X - Y$ over R is defi

$$\iiint_R f(x, y, z)dV = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x$$

where $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$, $\Delta *_i$ being the length of the partition sub
the element of volume and $\|\Delta\|$ is the longest length among $\Delta *_i$'s
the triple integrals:

1.32 Theorem (Fibini's Theorem

If $f(x, y, z)$ is continuous over V and

$$R = \{(x, y, z) | a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x), h$$

then

$$\iiint_{\mathbf{R}} \mathbf{f(x, y, z)dV} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{dx} \int_{\mathbf{g_1(x)}}^{\mathbf{g_{2(x)}}} \mathbf{dy} \int$$

Certainly, the order of integrations can be changed as double integ
then the value of triple integral is equal to the volume of R.

1.33 Example

Evaluate the following triple integral

$$\iiint_{\mathbf{-1\leq x\leq 1, 0\leq y\leq 3, 1\leq z\leq 2}} \mathbf{(x^2y + yz^2)d}$$

1.34 Example

Evaluate the following triple integral

$$\iiint_{T} \mathbf{zdV} = \frac{\mathbf{1}}{\mathbf{12}}$$

where T is the solid in the first octant and bounded by $z = 1 - x^2$

1.35 Example

Evaluate the following triple integral

$$\iiint_{T} \sqrt{\mathbf{x^2 + z^2}}\mathbf{dV} = \frac{\mathbf{4}}{\mathbf{3}}$$

where T is the solid, bounded by $x^2 + z^2 = 1, y + z = 2$ and $y =$

In this case, we seperate the triple integral into 2 part, single-variab
double integral for x, z, in $R = \{x^2 + z^2 \leq 1\}$; use integration in p
integral:

$$\iiint_{T} \sqrt{\mathbf{x^2 + z^2}}\mathbf{dV} = \iint_{\mathbf{x^2+z^2 \leq 1}} \mathbf{dA} \int_{\mathbf{0}}^{\mathbf{2-}}$$

While $f(x, y, z) = 1$, the triple integral is the volume of T which is t

$$\iiint_{R} \mathbf{1dV} = \mathbf{volume(T}$$

1.36 Example

If $R = \{(x, y)|1 \leqslant x \leqslant 3, 2 \leqslant y \leqslant 4, 0 \leqslant z \leqslant 2\}$, then

$$\iiint\limits_R 1dV = \int_2^4 dy \int_1^3 dx$$

$$= 2 \cdot 2 \cdot 2$$

This result is equal to the volume of cubic solid.

1.37 Example

Evaluate the triple integral

$$\int_1^2 \int_x^{x^2} \int_0^{x+y} (x+1)(y$$

$$= \int_1^2 dx \int_x^{x^2} (x+1) \left(\frac{3y^2 + 4x}{2} \right.$$

$$= \int_1^2 (x+1) \cdot \frac{x^6 + 2x^5 + x^4}{2}$$

$$= \frac{23577}{560}$$

1.38 Example

Suppose that The solid region R is given by

$$R = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| 0 \leqslant x \leqslant \sqrt{\frac{\pi}{2}}, x \leqslant y$$

Evaluate the triple integral

$$\iiint_R \sin(y^2) dV$$

Sol:

As mentioned in the section of integration technique, $\sin(y^2)$ can r
Therefore we have to arrange the orders of integration carefully. No

$$R = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| 0 \leqslant y \leqslant \sqrt{\frac{\pi}{2}}, 0 \leq$$

then by Fubini's theorem, the triple integral is evaluated as:

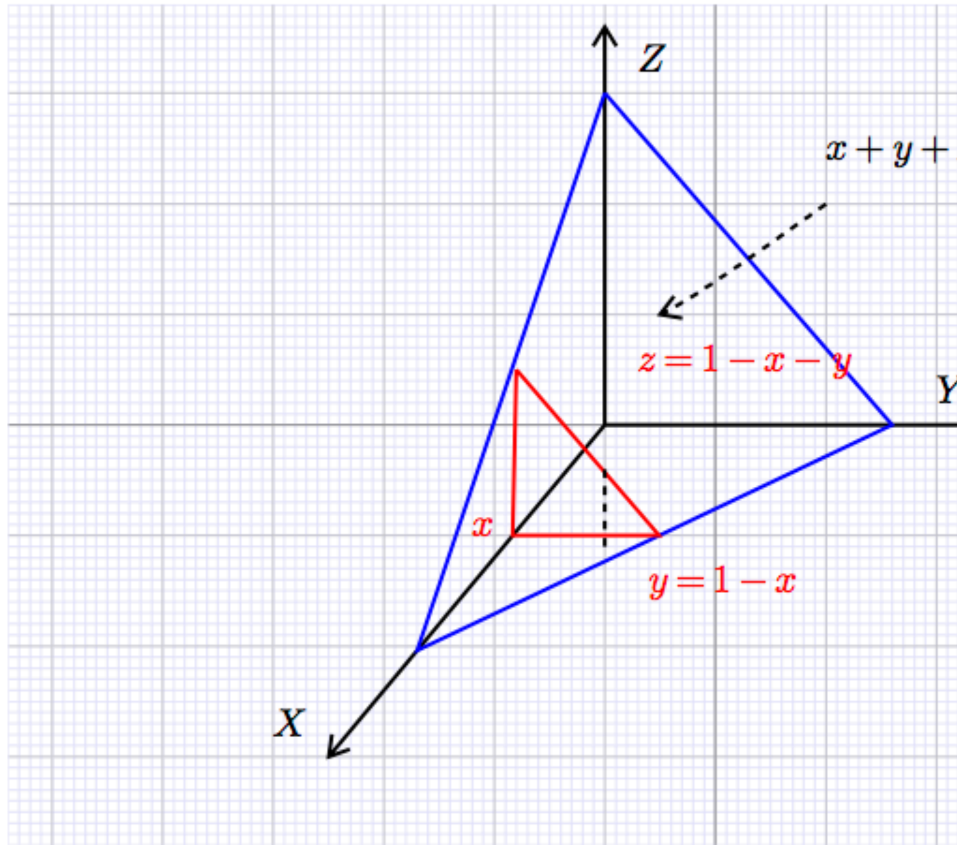
$$\begin{aligned} &\iiint_R \sin(y^2) dV \\ = &\int_0^{\sqrt{\frac{\pi}{2}}} dy \int_0^y dx \int_0^2 \sin \\ = &\int_0^{\sqrt{\frac{\pi}{2}}} 2 \cdot y \cdot \sin(y^2) d \\ = &\quad 1 \end{aligned}$$

1.39 Example

Evaluate the following triple integral:

$$\iiint_V \frac{dV}{(1 + x + y + z)^{3/2}}$$

where V is the domain bounded by the plane, $x + y + z = 1$, in th



$$I = \iiint\limits_V \frac{dV}{(1 + x + y + z)^{3/2}}$$

$$= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{}{(1 +}$$

$$= \int_0^1 dx \int_0^{1-x} \frac{-2}{(1 + x + y + z)^{1}}$$

$$= \int_0^1 dx \int_0^{1-x} \left(\frac{2}{(1 + x + y)^{1/2}} \right.$$

$$= \int_0^1 4\sqrt{1 + x + y} - \sqrt{2}y \Big|_{y=0}^{y=1-x}$$

$$= \int_0^1 (4\sqrt{2} - \sqrt{2}(1 - x) - 4\sqrt{}$$

$$= 4\sqrt{2} - \frac{\sqrt{2}}{2} - \frac{8}{3}(2^{3/2} - 1) =$$

1.40 Exercise

Evaluate the following triple integrals:

1. $\iiint\limits_V x^2 y dV$ where $V = \{(x, y, z) \in \mathbb{R}^3 | 0 \leqslant x, y, z \leqslant 2\}$;
2. $\iiint\limits_V x^2 y dV$ where $V = \{(x, y, z) \in \mathbb{R}^3 | 0 \leqslant x \leqslant y \leqslant z \leqslant 2\}$
3. $\iiint\limits_V \frac{y}{x} dV$ where $V = \{(x, y, z) \in \mathbb{R}^3 | 1 \leqslant x \leqslant y \leqslant z \leqslant 2\}$;

p.1199 Exercise

28, T: bounded by $x^2 + z^2 = 4$ and $y^2 + z^2 = 4$.

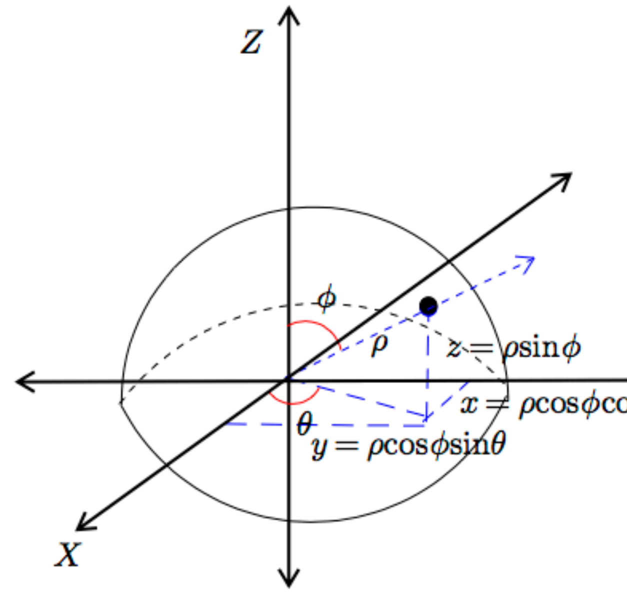
Move the cross-session, $A(z)$, parallell to $X - Y$ plane, along Z-a
area of $A(z)$, is $(2\sqrt{(4 - z^2)})$. Therefor the volume of solid is:

$$\mathbf{2 \int_0^2 \left(2\sqrt{4 - z^2}\right)^2 dz =}$$

1.41 Triple integrals in other coo

Recall that the relations between Cartesian coordinates, (x, y, z), a
given by:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$



And the relations between Cartesian coordinates, (x, y, z), and cyl
coordinates}}, (ρ, θ, ϕ), are given by:

$$x = \rho\cos\theta\cos\phi$$
$$y = \rho\sin\theta\cos\phi$$
$$z = \rho\sin\phi$$

Since the Jacobian matrix, J, between two different coordinates is

$$\frac{\partial(x^i)}{\partial(u^j)} = \left(\frac{\partial x^i}{\partial u^j} \right)_{i,j}$$

we have the following \ integration rules:

1.42 Theorem

$$\iiint\limits_{\mathbf{R}} \mathbf{f(x, y, z)dV} = \iiint\limits_{\mathbf{R}} \mathbf{f(x(u, v, w), y(u, v, w}$$

where $|J|$ is the absolute value of determinant of J. In cylindrical co

$$\iiint\limits_{\mathbf{R}} \mathbf{f(x, y, z)dV} = \iiint\limits_{\mathbf{R}} \mathbf{f(r}\cos\theta, \mathbf{r}\ \mathrm{s}$$

In spherical coordinates{\index{spherical coordinates}}, we have

$$\iiint\limits_{\mathbf{R}} \mathbf{f(x, y, z)dV} = \iiint\limits_{\mathbf{R}} \mathbf{f(}\rho\cos\theta\cos\phi, \rho\sin\theta\,\mathrm{co}$$

1.43 Example

Evaluate triple integral of $f(x, y, z) = \sqrt{x^2 + y^2}$ on the T, bounded

The solid region can be represented in cylindrical coordinates as:

$$\begin{aligned} r \leqslant z \leqslant 2 \\ 0 \leqslant r \leqslant 2 \\ 0 \leqslant \theta \leqslant 2\pi \end{aligned}$$

Then the volume of the solid is equal to

$$\begin{aligned} & \iiint_R \sqrt{x^2 + y^2} dV \\ = & \int_0^2 rdr \int_0^{2\pi} d\theta \int_r^2 r \\ = & \frac{8\pi}{3} \end{aligned}$$

1.44 Example

Volume of hemisphere with radius a is $\frac{2}{3}\pi a^3$.

1.45 Example

Evaluate the triple integral:

$$\iiint_{T=\{x^2+y^2+z^2 \leq 1, x, y, z \geq 0\}} xdV = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \int_0^1 \rho$$

1.46 Example

Evaluate the triple integral:

$$\iiint_{T=\{\sqrt{x^2+y^2}\le z\le x^2+y^2+z^2\}} 1dV = \int_0^{\pi/4} d\phi \int_0^{2\pi} d\theta$$

1.47 Example

Find the volume of the solid bounded by $z = x^2 + y^2$ and $z = 4$.

The solid region can be represented in cylindrical coordinates as:

$$0 \leqslant z \leqslant 4$$
$$0 \leqslant r \leqslant \sqrt{z}$$
$$0 \leqslant \theta \leqslant 2\pi$$

Then the volume of the solid is equal to

$$\begin{aligned} & \iiint_R 1dV \\ = & \int_0^4 dz \int_0^{\sqrt{z}} rdr \int_0^{2\pi} \\ = & \int_0^4 \pi z dz \\ = & 8\pi \end{aligned}$$

1.48 Example

Find the volume of the solid bounded by $z^2 = x^2 + y^2, x^2 + y^2 +$

Sol:

In spherical coordinates, the solid is represented as

$$0 \leqslant \rho \leqslant 2$$
$$0 \leqslant \phi \leqslant \tfrac{\pi}{4}$$
$$0 \leqslant \theta \leqslant 2\pi$$

Then the volume of the solid is equal to

$$\begin{aligned} & \iiint_R 1 dV \\ = & \int_0^2 \rho^2 d\rho \int_0^{\frac{\pi}{4}} \sin\phi d\phi \int_0 \\ = & \quad 2\pi \int_0^2 \rho^2 (1 - \frac{1}{\sqrt{2}}) d \\ = & \quad \frac{16}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\pi \end{aligned}$$

1.49 Exercise, p.1207

#6

$$\iiint_{T=\{x^2+y^2\leq 4, 0\leq z\leq 4\}} \exp(x^2+y^2) dV = \int_0^{2\pi} d\theta \int_0^2 dr \int_0$$

#10

$$\iiint_{T=\{x^2+y^2\le 1, 0\le z\le 2x^2+2y^2\}} y^2 dV = \int_0^{2\pi} d\theta \int_0^1 dr$$

#20

$$\iiint_{T=\{x^2+y^2+z^2\le 1, 0\le z,x,y\}} \exp(x^2+y^2+z^2)^{3/2} dV = \int_0^1 d\rho \int_0^{\pi/2}$$

#24 T is the solid bounded above by $x^2 + y^2 + z^2 = 4$ and boun

$$\iiint_T z dV = \int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} d\phi \int_2^{2/\sin\phi} \rho^3$$

1.50 Exercise

Resolve the last problem with the cylindrical coordinates. **Sol:** Sinc
following inequalities:

$$z^2 \leqslant x^2 + y^2$$
$$x^2 + y^2 + z^2 \leqslant 4$$
$$z \geqslant 0$$

the ranges for \ (r, θ, z) are:

$$r \leqslant z \leqslant \sqrt{4 - r^2}$$
$$0 \leqslant r \leqslant \sqrt{2}$$
$$0 \leqslant \theta \leqslant 2\pi$$

Therefore the volume is

$$\iiint_R 1 dV$$
$$= \int_0^{\sqrt{2}} r dr \int_0^{2\pi} d\theta \int_r^{\sqrt{4-}}$$
$$= \ 2\pi \int_0^{\sqrt{2}} (\sqrt{4 - r^2} - r)$$
$$= \frac{16}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\pi$$

}

1.51 Exercise

Evaluate the following triple integrals:

1.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} (y^2 + z$$

Hint: the domain is half upper ball.

2.

$$\int_{0}^{\sqrt{2}}\int_{0}^{\sqrt{2-x^2}}\int_{2x^2+y^2}^{4-y^2} y dz$$

Hint: by cylindrical coordinates, $x = r\cos\theta$ and $y = r\sin\theta$.

1. by spherical integration

$$
\begin{aligned}
I =& \int_0^2 \rho^5 d\rho \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} (\sin^2 \phi + \sin \\
=& \frac{32}{3} \cdot \int_0^{2\pi} \left(\frac{3\pi}{8} \sin^2 \theta + \frac{\pi}{8} \right) d\theta \\
=& \, 8\pi^2
\end{aligned}
$$

2. by cylindrical integration

$$
\begin{aligned}
I =& \int_0^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (4 - 2x^2 - \\
=& \int_0^{\sqrt{2}} dr \int_0^{\frac{\pi}{2}} (4 - 2r^2) r \, \mathrm{si} \\
=& \int_0^{\sqrt{2}} (4r^2 - 2r^4) dr \\
=& \frac{16\sqrt{2}}{15}
\end{aligned}
$$

1.52 Exercise

Find the volume of solid bounded by

$$V : x^{2/3} + y^{2/3} + z^{2/3} \leqslant$$

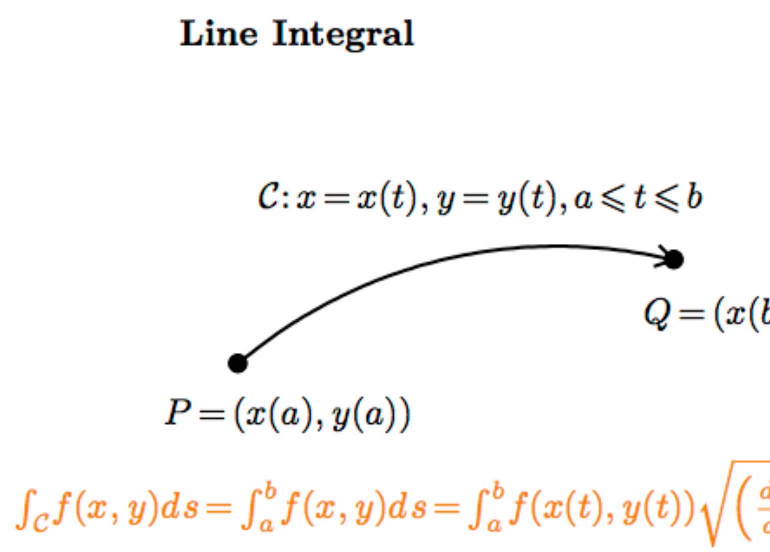
This volume is equal to the following triple integral:

$$\begin{aligned}
I &= \iiint_V 1 dV \\
&\Downarrow (x = X^3, y = Y^3, z = Z^3, J = \left(\frac{\partial x}{\partial X} \right. \\
&= \iiint_{X^2+Y^2+Z^2 \leqslant 2^2} 27X^2Y^2Z^2 dXdYdZ \\
&= 27 \int_{-2}^{2} r^8 dr \int_0^{2\pi} \sin^5\theta \cos^2\theta d\theta \int \\
&= \frac{2048}{35}\pi
\end{aligned}$$

1.53 Line Integral

Suppose that a plane curve C is given by the following parametric

$$x = x(t), y = y(t) \text{where} a \leqslant$$



1.54 Definition

If f is defines on a smooth curve C, then the line integral of f along

$$\oint_C \mathbf{f(x,y)ds} = \lim_{\mathbf{n \to \infty}} \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{f(x_i^*,}$$

where Δs_i is line elemnet if limit exists.

Suppose that the point (x, y) on curve C can be represented as x
$x(t), y(t)$ have continuous derivatives, then the line integral can be

1.55 Theorem

$$\oint_C \mathbf{f(x,y)ds} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x(t), y(t))} \sqrt{\Big(\frac{\mathbf{dx}}{\mathbf{dt}}}$$

1.56 Example

Evaluate the line integral

$$\oint_C x^2y^2ds$$

where C is the move along unit circle countclockwise and starting

Here

$$C : (x, y) = (\cos t, \sin t), 0 \leqslant$$

and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t + c$$

Then

$$\oint_C f(x, y)ds = \int_0^{2\pi} \cos^2 t\, s$$
$$= \pi/4$$

1.57 Note

Sometimes, we can consider the following sum of line integrals:

1. $\Delta s = \Delta x$: $\oint_C P(x, y)ds = \oint_C P(x, y)dx$
2. $\Delta s = \Delta y$: $\oint_C Q(x, y)ds = \oint_C Q(x, y)dy$ along the same path

$$\oint_C P(x, y)dx + \oint_C Q(x$$

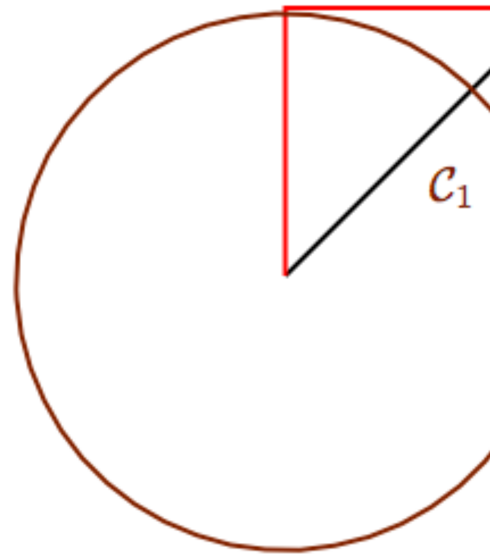
1.58 Example

Evaluate $\oint_C ((x - y)dx + (x + y)dy)$ along

1. line from $(0, 0)$ to $(1, 1)$;

2. line from $(0,0)$ to $(0,1)$ and turn right to $(1,1)$;

3. along unit circle countclockwise and starting from $(1,0)$ endin



Solve: 1. $C:(x,y)=(t,t), 0 \leqslant t \leqslant 1$

$$I = \int 0dt + \int_0^1 (t+t)$$
$$= 1$$

2. $C = C_1 \cup C_2, C_1 : (x,y) = (0,t), 0 \leqslant t \leqslant 1; C_2 = (t,1), 0 \leqslant$

$$I = \oint_{C_1} + \oint_{C_2}$$
$$= \int_0^1 (0+t)dt + \int_0^1 (t$$
$$= 0$$

3. $C:(x,y)=(\cos t, \sin t), 0 \leqslant t \leqslant 2\pi$

$$I = \int_0^{2\pi} (\mathrm{cost} - \sin t)(-\mathrm{s}$$
$$+ \int_0^{2\pi} (\mathrm{cost} + \sin t)\,\mathrm{c}$$
$$= 2\pi$$

1.59 Theorem (Green's Theorem

Suppose that C is a positive oriented,smooth and simple planar cu
and Q have continuous partial derivatives on interior od D. Then

$$\oint_C \mathbf{P(x, y)dx + Q(x, y)dy} = \iint_{\mathbf{D}} \Bigg($$

1.60 Example

Along unit circle countclockwise and starting from $(0, 0)$ ending at

1. $\oint_C (x - y)dx + (x + y)dy$
2. $\oint_C \frac{ydx - xdy}{(x+y)^2}$

Solve:

1.

$$\begin{aligned} I &= \iint_{x^2+y^2\leqslant 1} \left(\frac{\partial(x-y)}{\partial x} + \frac{\partial(x\right. \\ &= \iint_{x^2+y^2\leqslant 1} 2dA = 2\pi \end{aligned}$$

2.

$$I = \iint_{x^2+y^2\leqslant 1} \left(\frac{x-y}{(x+y)^3} - \frac{x-}{(x+y\right.$$

Here, modify above python code to calulate the line integral:

1.61 Exercise

Evaluate $\oint_C (2x - y)dx + (x + y)dy$ along

1. line from $(0,0)$ to $(3,4)$;
2. line from $(0,0)$ to $(0,4)$ and turn right to $(3,4)$;
3. along circle, $x^2 + (y-1)^2 = 1$, countclockwise and starting

Sol:

1. $C : (x, y)\text{with}y = \frac{4}{3}x$ and let $x = t, y = 4t/3$ where $0 \leqslant t \leqslant$

$$\oint_C (2x - y)dx + (x + y)dy = \int_0^3 \Big(\frac{2}{3}$$

$$= \int_0^3 \frac{34}{9}$$

2.

$$I = \int_{(0,0)\to(0,4)} + \int_{(0,4)\to(3,4)}$$

$$= \int_0^4 (0 + y)dy + \int_0^3 (2x$$

$$= 5$$

3. Since $(x, y) = (\sin t, 1 - \cos t)$ with $0 \leqslant t \leqslant \pi$ for (x, y) in C:

$$I = \int_0^\pi (\cancel{2}\sin t - \cancel{2} + 2\cos t$$

$$+ \int_0^\pi (1 + \sin t - \cancel{\cos t})\, s$$

$$= \int_0^\pi (1 + \sin^2 t + 2\cos^2 t)$$

1.62 Exercise

Along unit circle countclockwise and starting from $(1, 0)$ ending at

1. $\oint_C ydx - xdy$
2. $\oint_C (y + x^3y)dx + (x - y^3x)dy$

Answer

1. by Green's theorem:

$$\oint_C ydx - xdy = \iint_{x^2+y^2\leqslant 1} (-1 - $$

2. also by Green's theorem,

$$\oint_C (y + x^3 y)dx + (x - y^3 x)dy = \iint_{x^2+y^2\leqslant 1} (1-$$

1.63 Exercise

Compute line integral

$$\oint_C (2x - y)dx + (x + y)$$

where C is the path from $(1, 1)$ to $(2, 2)$ along $(x - 1)^2 + (y - 2)$

1. $(x - 1)^2 + (y - 2)^2 = 1 \Rightarrow x = 1 + \sin t, y = 2 - \cos t;$

2.

$$\begin{aligned} I &= \int_0^{\square} \left((\square) \cos t + (\square) \text{si} \right. \\ &= \int_0^{\square} \left(1 + \square \sin t \cos t + \right. \\ &= \square \pi + \square \end{aligned}$$

1.64 Surface Integral

Suppose that $f(x, y, z)$ is defined on the smooth sruface $S \in \mathbb{R}^3$.

$\|S\| = \max_i \|S_i\| \to 0$. The **surface integral** of $f(x, y, z)$ on S is

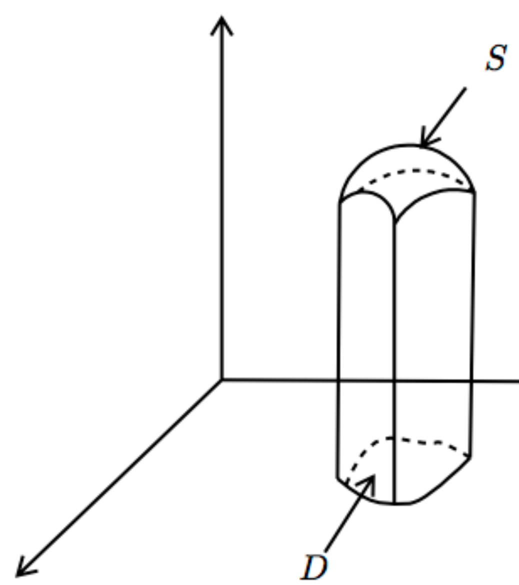
$$\iint_S \mathbf{f(x, y, z)dS} = \lim_{\|\Delta S_i\| \to 0} \sum_{i=1}^{n} \mathbf{f(x}$$

1.65 Theorem

Suppose that $f(x, y, z) = f(x, y, z(x, y))$ for (x, y, z) on S with $\|S$

$$\iint_S \mathbf{f(x, y, z)dS} = \iint_D \mathbf{f(x, y, z(x, y))}$$

where D is the projection of S on $X - Y$ plane.

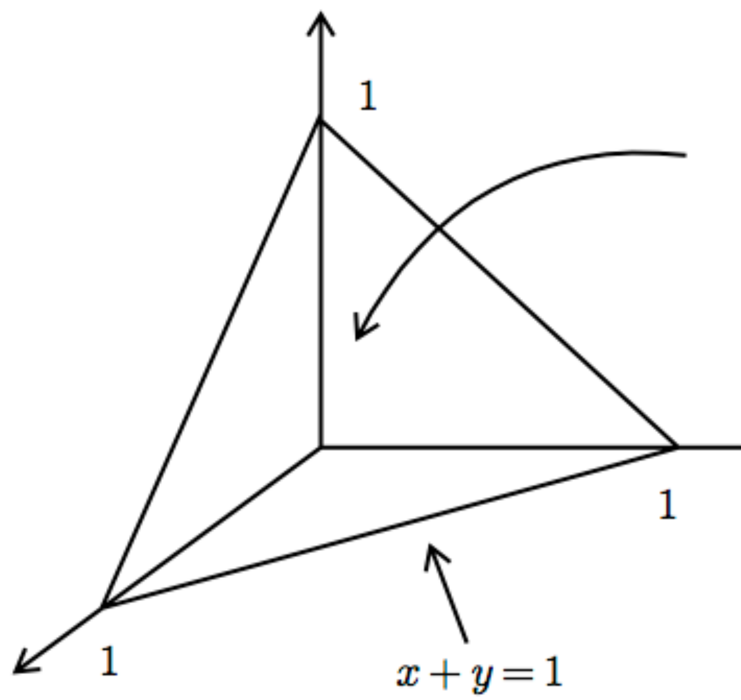


1.66 Example

Compute the surface integral:

$$\iint_S (xy + 2z)dS$$

where $S = \{(x, y, z) | x + y + z = 1\}$ in the first octant.



$$\begin{aligned}\iint_S (xy + 2z)dS &= \iint_{\{x+y \leqslant 1, x,y \geqslant 0\}} (xy + 2 \\ &= \sqrt{3} \int_0^1 dx \int_0^{1-x} (2 - \\ &= \frac{7\sqrt{3}}{24}\end{aligned}$$

1.67 Example

Evaluate the surface integral on the surface of the upper half unit s

$S = \{(x, y, z) | x^2 + y^2 + z^2 = 1, z \geqslant 0\}$

$$\iint_S \left((x^2 + y^2 + (z-1)^2\right) dS = \iint_{x^2+y^2\leqslant 1} \left((x^2 + y^2 +\right.$$
$$= 2 \iint_{x^2+y^2\leqslant 1} \frac{1 - \sqrt{1 -}}{\sqrt{1 - x}}$$
$$= 2 \int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1 -}}$$
$$= 2\pi$$

Suppose that the point $r = (x, y, z) \in S$ can be represented as th
$r(u, v) = (x(u, v), y(u, v), z(u, v))$. Then

1.68 Theorem

$$\iint_{\mathbf{S}} \mathbf{f(x, y, z)dS} = \iint_{\mathbf{D}} \mathbf{f(x(u, v), y(u, v), z(}$$

where $\cdot \times \cdot$ means exterior product.

1.69 Example

As the last example, we have:

$$r = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) 0 \leqslant \theta$$

Then

$$\begin{aligned}\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} &= \begin{pmatrix} i & j \\ -\sin\phi\sin\theta & \sin\phi\cos \\ \cos\phi\cos\theta & \cos\phi\sin \end{pmatrix} \\ &= (-\sin^2\phi\cos\theta, \sin^2\phi\cos \\ &\Downarrow \\ \left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right| &= \sin\phi\end{aligned}$$

where $i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$.

$$\begin{aligned}\iint_S ((x^2 + y^2 + (z-1)^2) dS &= \int_0^{2\pi} d\theta \int_0^{\pi/} \\ &= 2\pi\end{aligned}$$

1.70 Exercise

As the last example, evaluate the following integral:

$$\iint_S (x^2 + y^2) dS$$

1.71 Exercise

Suppose that S is the portion of the cylinder $x^2 + y^2 = 4$ that line
following integral:

$$\iint_S z dS$$

Hint: $r = (2\cos\theta, 2\sin\theta, z), 0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant z \leqslant 4$.

$$\begin{aligned}
\frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial z} &= \begin{pmatrix} i & j \\ -2\sin\theta & \square \\ \square & 0 \end{pmatrix} \\
&= (\square \cos\theta, \square \text{ s}
\end{aligned}$$

$$\Downarrow$$

$$\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| = \square$$

Then

$$\begin{aligned}
\iint_S z dS &= \int_0^{2\pi} d\theta \int_0^4 \square \\
&= \square
\end{aligned}$$