f(x, y) is continuous:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |f(x,y) - f(a,b)| < \varepsilon$$
$$\text{if } \|(x,y) - (a,b)\| < \delta$$

To toggle on/off the raw Python code, click [here].

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} \quad \text{fails to exist}$$

$$1 = \lim_{\substack{(x,y)\to(0,0)\\ y=0,x\to 0}} \frac{x-y}{x+y} \neq \lim_{\substack{(x,y)\to(0,0)\\ x=0,y\to 0}} \frac{x-y}{x+y} = -1$$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0.0) \\ 0 & \text{if } (x,y) = (0.0) \end{cases} \quad \text{discontinuous at (}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ (x,y)\to(x,ax)}} \frac{xy}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ (x,y)\to(x,ax)}} \frac{ax^2}{x^2+a^2x^2} = \frac{a}{1+a^2} \neq 0 = f(0,$$

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0.0) \\ 0 & \text{if } (x,y) = (0.0) \end{cases} \quad \text{discontinuous at (0,0)}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ (x,y)\to(ay^2,y)}} \frac{xy^2}{x^2+y^4} = \lim_{\substack{(x,y)\to(0,0)\\ (x,y)\to(ay^2,y)}} \frac{a^2y^4}{a^2y^4+y^4} = \frac{a^2}{1+a^2} \neq 0 = f(0,0)$$

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0.0) \\ 0 & \text{if } (x,y) = (0.0) \end{cases} \quad \text{continuous at (0,0)}$$

$$\left|\frac{x^2y}{x^2+y^2}\right| \leqslant \left|\frac{(x^2+y^2)(x^2+y^2)^{1/2}}{x^2+y^2}\right| \leqslant (x^2+y^2)^{1/2} \xrightarrow{(x,y)\to(0,0)} 0 = f(0,0)$$

Partial Derivative:

$$f_i = \frac{\partial f}{\partial x^i} = \lim_{k\to 0} \frac{f(x^1,\cdots,x^{i-1},x^i+k,x^{i+1},\cdots,x^n)-f(x^1,x^2,\cdots,x^n)}{k}$$

Gradient: $\nabla f = (f_1, \cdots, f_n)$.

Chain Rule

$$\frac{\partial \vec{z}}{\partial \vec{t}} = \begin{pmatrix} \partial z^1/\partial x^1 & \cdots & \partial z^1/\partial x^m \\ \vdots & \ddots & \vdots \\ \partial z^k/\partial x^1 & \cdots & \partial z^k/\partial x^m \end{pmatrix} \begin{pmatrix} \partial x^1/\partial t^1 & \cdots & \partial x^1/\partial t^n \\ \vdots & \ddots & \vdots \\ \partial x^m/\partial t^1 & \cdots & \partial x^m/\partial t^n \end{pmatrix}$$

$$z = \sin(x + y^2)$$

$$(x, y) = (st, s^2 + t^2)$$

$$\begin{aligned} \left(\partial z/\partial s, \partial z/\partial t \right) &= \left(\partial z/\partial x, \partial z/\partial y \right) \begin{pmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{pmatrix} \\ &= \left(\cos(x + y^2) \quad 2y\cos(x + y^2) \right) \begin{pmatrix} t & s \\ 2s & 2t \end{pmatrix} \\ &= \cos((t^2 + s^2)^2 + st) \cdot \left(4s(t^2 + s^2) + t, 4t(t^2 + s^2) + s \right) \end{aligned}$$

Directional Derivative: $D_e f = \nabla f \cdot \vec{e}/\|\vec{e}\|$

Maximum, Minimum : $\|\nabla f\|(-\|\nabla f\|)$, at $\vec{e} = \nabla f/\|\nabla f\| \quad (-\nabla f/$

Directional Derivative

$\mathbf{f(x,y) = \sqrt{x} + \sqrt{y}}$ at $\mathbf{(x,y) = (1,1)}$ in the direction $\mathbf{(3,4)}$:

- $\mathbf{(3,4) \to \frac{1}{5}(3,4) = \vec{e}}$
- $\mathbf{D_{\vec{e}}f(1,1) = \nabla f(1,1) \cdot \vec{e} = \frac{1}{2}(1,1) \cdot \frac{1}{5}(3,4) = \frac{7}{10}}$
- Maximum
  - $$\mathbf{\vec{e} = \frac{\nabla f(1,1)}{\|\nabla f(1,1)\|} = \frac{\frac{1}{2}(1,1)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} = \frac{(1,1)}{\sqrt{2}}}$$
  - $$\mathbf{D_{\vec{e}}f(1,1) = \|\nabla f(1,1)\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}}$$

Extrema

Critical point: $\mathbf{(x_0, y_0)}$, $\mathbf{H} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x_0,y_0) & \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0) \\ \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) & \frac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{pmatrix}$

1. **if $|H| > 0$ and $A < 0$: $f(x_0, y_0)$ relative maximum,**
2. **if $|H| > 0$ and $A > 0$: $f(x_0, y_0)$ relative minimum,**
3. **if $|H| < 0$: $(x_0, y_0, f(x_0, y_0))$ saddle point,**

Extrema without boundary

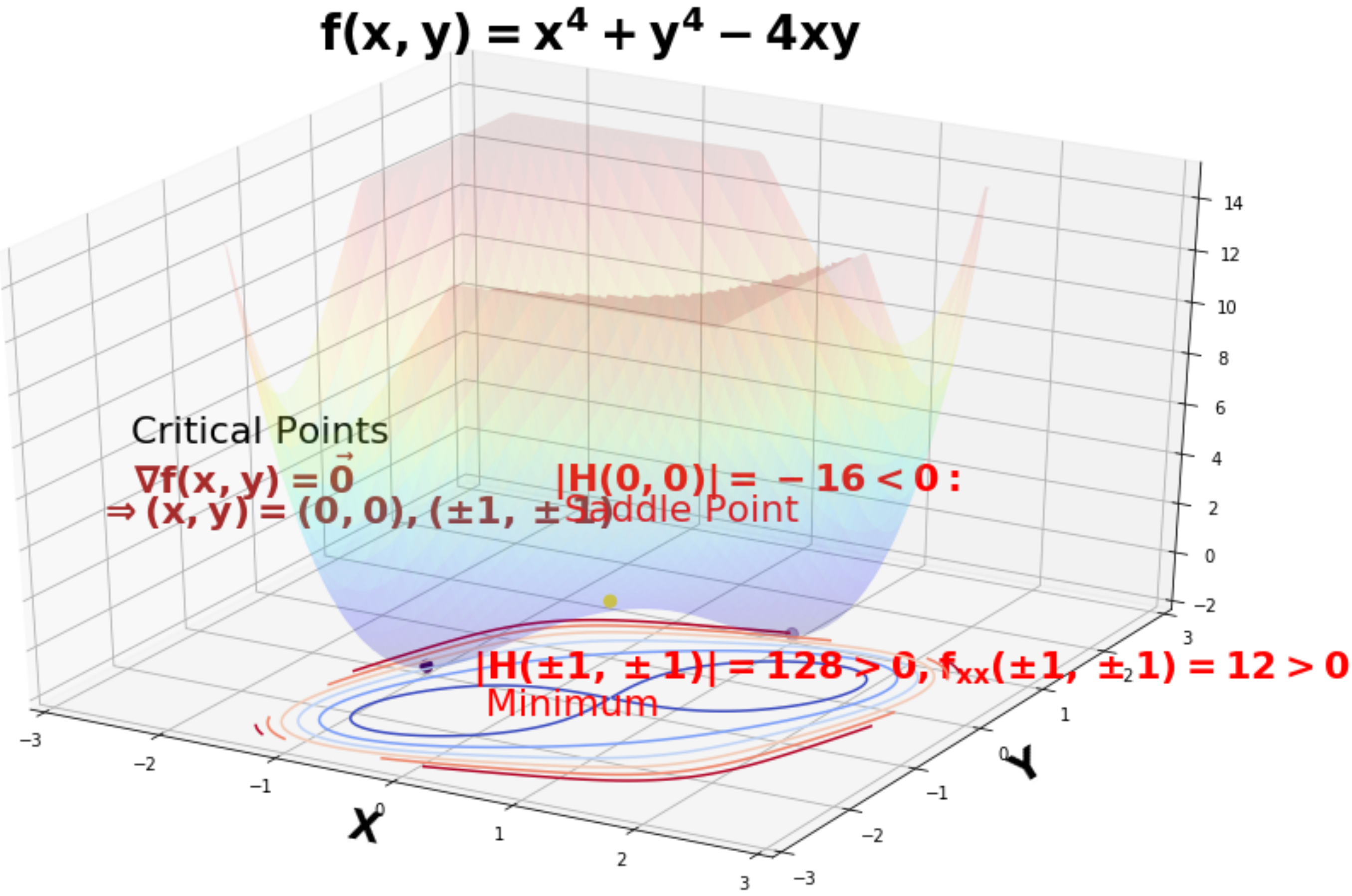
$\mathbf{f(x,y) = x^4 + y^4 - 4xy}$

- $\mathbf{f(x,y)} \nearrow \infty$ (no Maximum), $\mathbf{f(x,y)} \not\searrow -\infty$ (Min exists)
- Find the critical values:

$$\begin{aligned} & \mathbf{f_1 = 4x^3 - 4y = 0 \text{ and } f_2 = 4y^3 - 4x = 0} \\ \Longrightarrow \quad & \mathbf{x = y^3 \text{ and } y = x^3 (i.e.\, x = x^9)} \\ \Longrightarrow \quad & \mathbf{(x,y) = (0,0) \text{ or } (\pm 1, \pm 1)} \end{aligned}$$

- Evaluate Extremum
  - $(x, y) = (0, 0)$: saddle point, since
    $$D = \begin{vmatrix} \mathbf{0} & \mathbf{-4} \\ \mathbf{-4} & \mathbf{0} \end{vmatrix} = -16 < 0$$
  - $\mathbf{(x,y) = (\pm 1, \pm 1)}$:
    $$\mathbf{f_{11}(\pm 1, \pm 1) = 12 > 0}, D = \begin{vmatrix} \mathbf{12} & \mathbf{-4} \\ \mathbf{-4} & \mathbf{12} \end{vmatrix} = 128 > 0$$

    $\mathbf{f(\pm 1, \pm 1) = -2}$ is a relative minimum (but not minimum).



Extrema with boundary

$\mathbf{f(x,y) = x^2 - xy + y^2 - x + y - 6,\ for\ (x,y) \in \{x^2 + y^2 \leqslant 1\}}$

- **Extrema at Interior Critical Values**

$$\left.\begin{array}{ll} f_1 & = 2x - y - 1 = 0 \\ f_2 & = 2y - x + 1 = 0 \end{array}\right\} \Rightarrow (x, y) = (1/3, -1/3)$$

  - $A = 2 > 0, |D| = 2 \cdot 2 - 1 > 0$: $f(1/3, -1/3) = -6\frac{1}{3}$ is minimum.

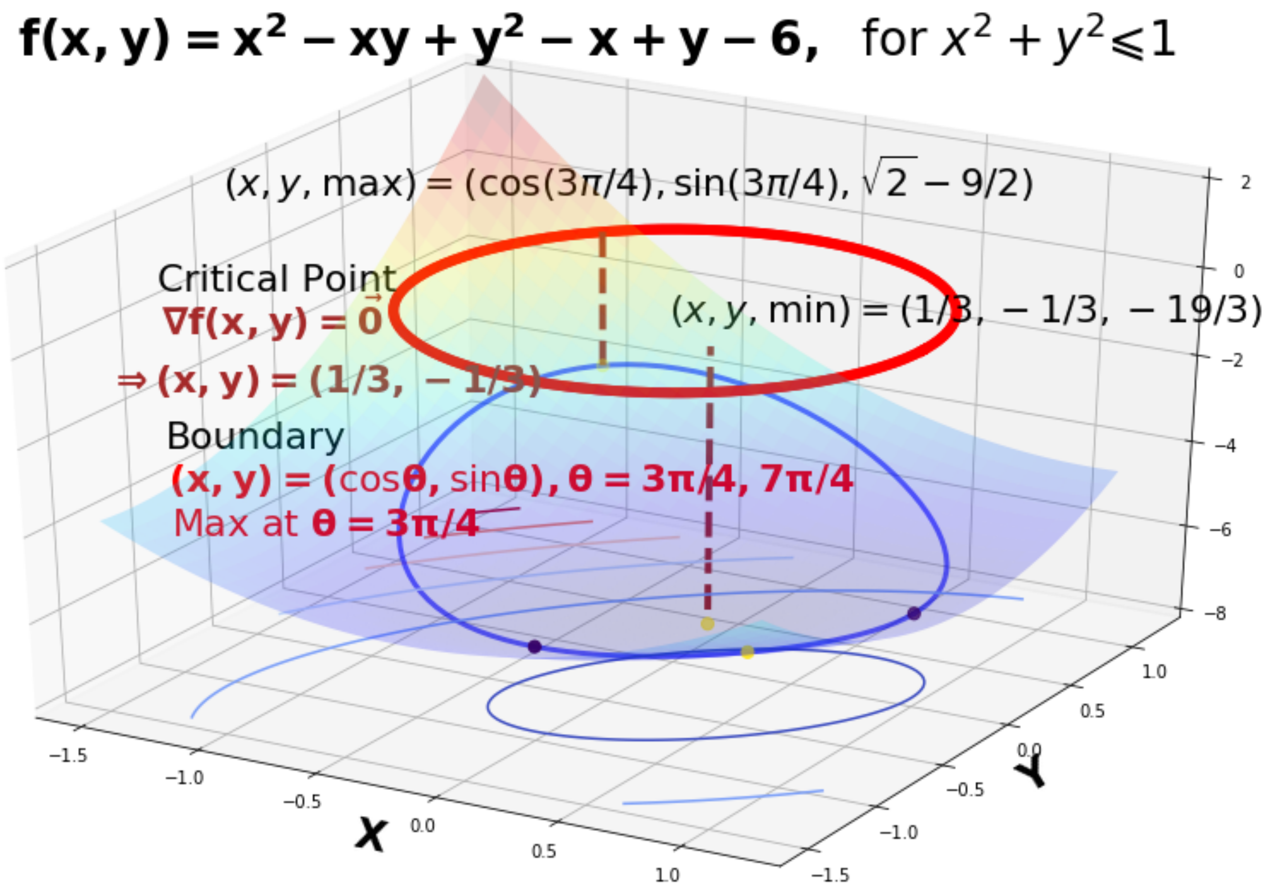
- **Extrema On the Boundary**

$$\partial\Omega = \{(x, y)|x^2 + y^2 = 1\} \longrightarrow (x, y) = (\cos\theta, \sin\theta), 0 \leqslant \theta \leqslant 2\pi$$

$$\begin{aligned} f(x, y) &= -\sin\theta\cos\theta - \cos\theta + \sin\theta - 5 \\ \frac{df}{d\theta} &= 0 \Rightarrow (\sin\theta + \cos\theta)(\sin\theta - \cos\theta + 1) = 0 \end{aligned}$$

  - $\sin\theta + \cos\theta = 0 \Longrightarrow \tilde{\theta} = 3\pi/4, 7\pi/4$:
  $$f(\cos\tilde{\theta}, \sin\tilde{\theta}) = \sqrt{2} - 4\frac{1}{2}, (-\sqrt{2} - 4\frac{1}{2})$$
  - $\sin\theta - \cos\theta + 1 = 0 \Longrightarrow \hat{\theta} = 0, 3\pi/2$:
  $$f(\cos\hat{\theta}, \sin\hat{\theta}) = -6$$

  Maximum: $\sqrt{2} - 4\frac{1}{2}$ at $(-1/\sqrt{2}, 1/\sqrt{2})$, **Minimum**: $-6\frac{1}{3}$ at $(x, y) = (1/3, -1/3)$.



Lagrange's Multipliers

Relative extrema of $\mathbf{f}(\vec{\mathbf{x}})$ with constraints $\mathbf{g^i}(\vec{\mathbf{x}}) = \mathbf{0}$ occurs at the critical point of $\mathbf{f}(\vec{\mathbf{x}}) + \sum_{\mathbf{i}} \lambda^{\mathbf{i}} \mathbf{g^i}(\vec{\mathbf{x}})$.

Extremum of $2x^{1/4}y^{3/4}$ with $2x + y = 8$

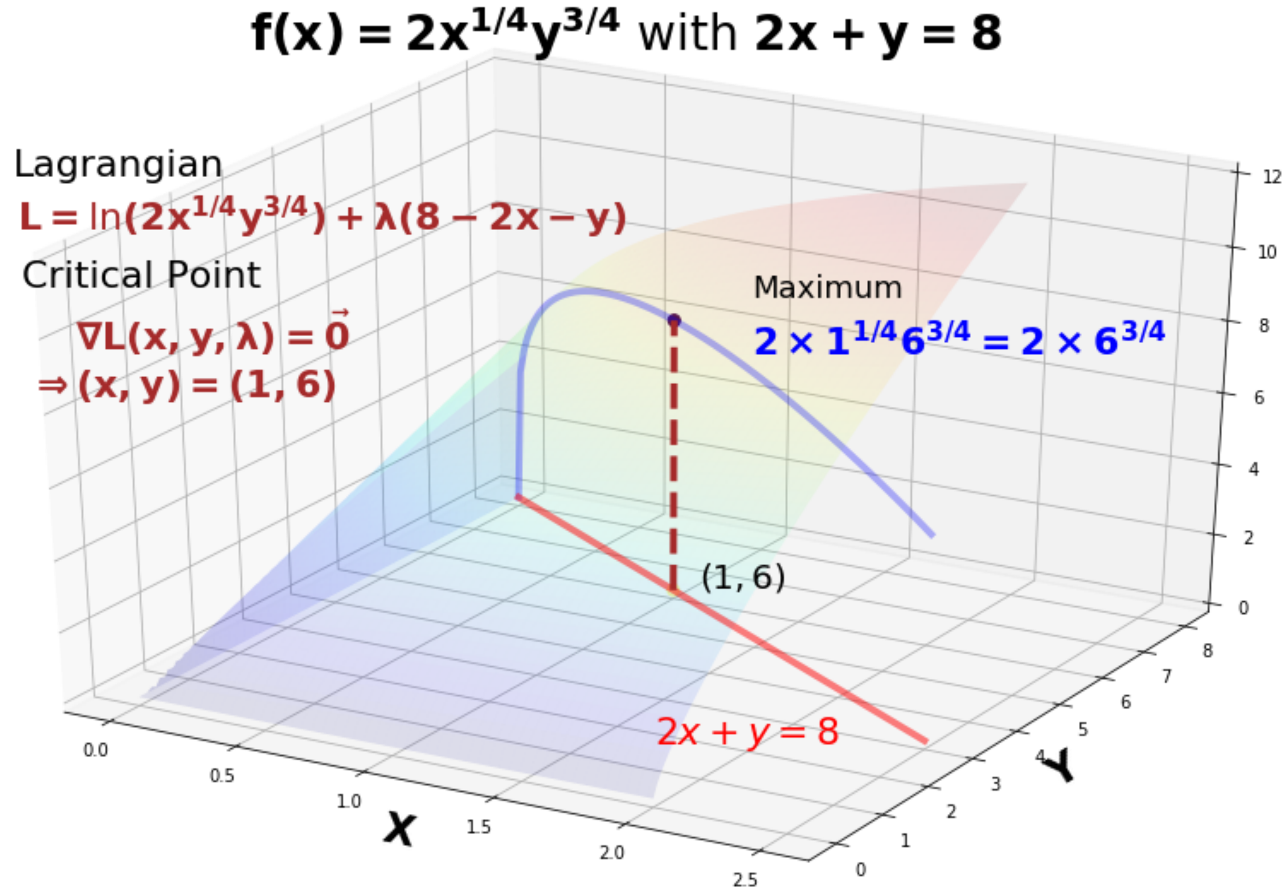
- **Lagrangian function**

$$\mathbf{L(x, y, \lambda) = \ln(2x^{1/4}y^{3/4}) + \lambda(8 - 2x - y)}$$

- **Critical point(s)**

$$\mathbf{\nabla L(x, y, \lambda) = \vec{0} \Longrightarrow x = 1 \text{ and } y = 6}$$

Maximum: $\mathbf{2 \times 1^{1/4} 6^{3/4} = 2 \times 6^{3/4}}$.



Extremum of $f(x, y, z) = x^2 - xy + y^2 - z^2 + 1$ with $x^2 + y^2 = 1, z^2 = xy$

- **Lagrangian function**

$$\mathbf{L(x, y, z; \lambda, \mu) = x^2 - xy + y^2 - z^2 + 1 + \lambda(1 - x^2 - y^2) + \mu(z^2 - xy)}$$

- **Critical point(s)**

$$\nabla \mathbf{L} = \vec{\mathbf{0}} \Rightarrow \mathbf{(2x - y - 2\lambda x - \mu y, 2x - y - 2\lambda x - \mu y, -2z + 2\mu z, 0, 0)} = \vec{\mathbf{0}}$$

$$\mathbf{-2z + 2\mu z = 0 \Rightarrow z = 0 \text{ or } \mu = 1}$$

$$\begin{cases} \mathbf{z = 0} \\ \mathbf{\mu = 1} \end{cases} \Rightarrow \begin{cases} \mathbf{x^2 + y^2 = 1, xy = 0} \\ \mathbf{x/y = y/x = 1/(1 - \lambda) \rightarrow y = x} \end{cases}$$

$$\Rightarrow \begin{cases} \mathbf{(x, y, z) = (\pm 1, 0, 0), (0, \pm 1, 0)} \\ \mathbf{(x, y, z) = \pm\left(1/\sqrt{2}, 1/\sqrt{2}, \pm 1/\sqrt{2}\right)} \end{cases}$$

$|\mathbf{x}|, |\mathbf{y}| \leqslant \mathbf{1} \Rightarrow$ **Extrema exist: Maximum** $= 2$ **and Minimum** $= 1$

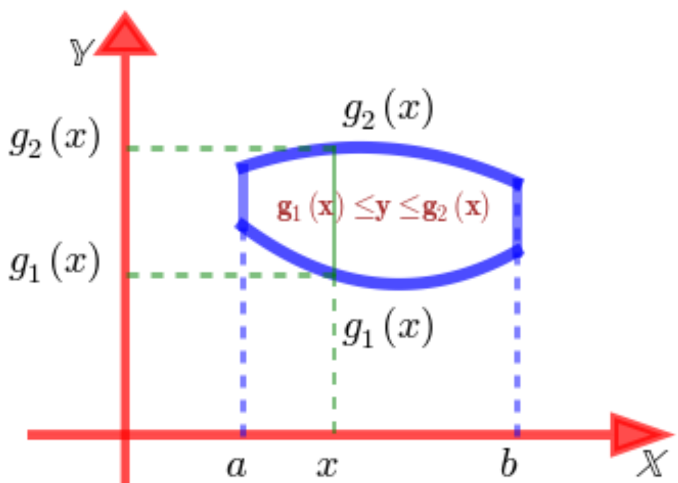
$$\mathbf{f(\pm 1, 0, 0) = f(0, \pm 1, 0)} = 2$$

$$\mathbf{f}\left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}, \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}, \pm\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\right) = \mathbf{f}\left(\frac{\mathbf{-1}}{\sqrt{\mathbf{2}}}, \frac{\mathbf{-1}}{\sqrt{\mathbf{2}}}, \pm\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\right) = 1$$

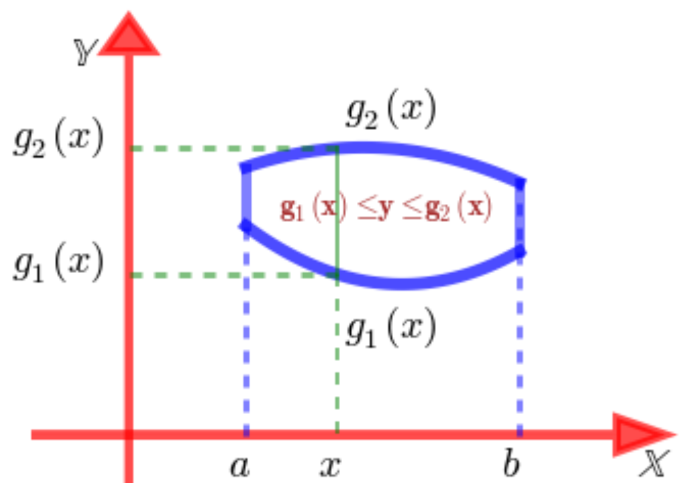
Fubini's Theorem

If $\mathbf{f(x,y)}$ is continuous over $\mathbf{D}$,

- $\mathbf{D} = \{(\mathbf{x},\mathbf{y}) \mid \mathbf{a} \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{g_1(x)} \leqslant \mathbf{y} \leqslant \mathbf{g_2(x)}\}$,
$$\iint_D f(x,y)dA = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)dy$$
- $\mathbf{D} = \{(\mathbf{x},\mathbf{y}) \mid \mathbf{c} \leqslant \mathbf{y} \leqslant \mathbf{d}, \mathbf{h_1(y)} \leqslant \mathbf{x} \leqslant \mathbf{h_2(y)}\}$,
$$\iint_D f(x,y)dA = \int_c^d dy \int_{h_1(y)}^{h_2(y)} f(x,y)dx$$



$$\iint_D f(x,y)dA = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)dy$$

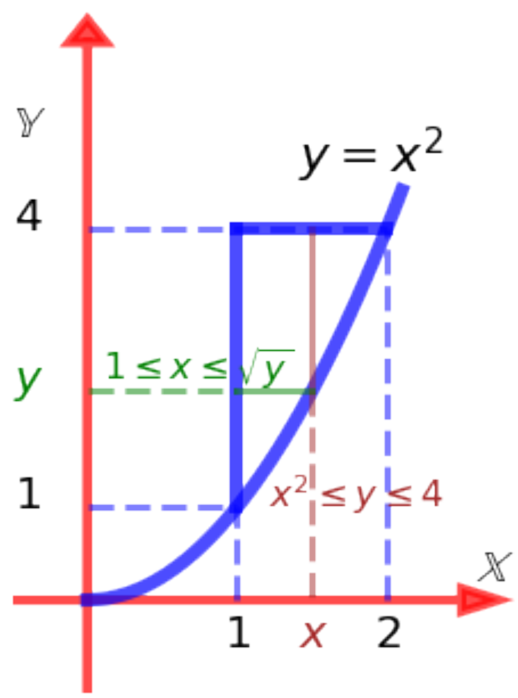


$$\iint_D f(x,y)dA = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)dy$$

$$\iint_{x^2 \leqslant y \leqslant 4, 1 \leqslant x \leqslant 2} (x+y)dA = \int_1^4 dy \int_1^{\sqrt{y}} (x+y)dx$$

$$= \int_1^2 dx \int_{x^2}^4 (x+y)dy = 7\frac{3}{20}$$

```
(-0.5, 8.0, -1.0, 7.0)
```



$$\iint_D (x+y)dA = \int_1^4 dy \int_1^{\sqrt{y}} (x+y)dx$$

$$= \int_1^2 dx \int_{x^2}^4 (x+y)dy = 7\frac{3}{20}$$

For

$$x = \phi(u,v), y = \psi(u,v).$$

$$\iint_D f(x,y)dA = \iint_D f(\phi(u,v),\psi(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$\iint_{\{x^2+y^2\leqslant 4\}} \sqrt{4-x^2-y^2}dA = \iint_{\substack{\{0\leqslant r\leqslant 2, 0\leqslant\theta\leqslant 2\pi\}\\ t=4-r^2}} \sqrt{4-r^2}\cdot rdrd\theta = \frac{16\pi}{3}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}dx = \int_{-\infty \atop t=(x-\mu)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}dt = 1$$

Fibini's Theorem

If $f(x,y,z)$ is continuous over R and

$$R=\{(x,y,z)|a\leqslant x\leqslant b, g_1(x)\leqslant y\leqslant g_2(x), h_1(x,y)\leqslant z\leqslant h_2(x,y)\},$$

then

$$\iiint_R f(x,y,z)dV = \int_a^b dx \int_{g_1(x)}^{g_2(x)} dy \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dz$$

Fibini's Theorem

$$\iiint_R f(x,y,z)dV = \int_a^b dx \int_{g_1(x)}^{g_2(x)} dy \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dz$$

where $\mathbf{R}=\{(x,y,z)|a\leqslant x\leqslant b, g_1(x)\leqslant y\leqslant g_2(x), h_1(x,y)\leqslant z\leqslant h_2(x,y)\}$.

$$\iiint\limits_{\substack{0\leqslant y\leqslant\sqrt{\pi/2}\\0\leqslant x\leqslant y,0\leqslant z\leqslant 2}} \sin(y^2)dV = \int_0^{\sqrt{\pi/2}} dy \int_0^{y} dx \int_0^{2} \sin(y^2)dz = 1$$

$$\iiint\limits_{\substack{0\leqslant y\leqslant\sqrt{\pi/2}\\0\leqslant x\leqslant y,0\leqslant z\leqslant 2}} \sin(y^2)dV = \int_0^{\sqrt{\pi/2}} dy \int_0^{y} dx \int_0^{2} \sin(y^2)dz = \int_0^{\sqrt{\pi/2}} y\cdot 2\sin(y^2)dy =$$

Cylindrical Coordinates (r, θ, z)

$$\iiint_R f(x, y, z)dV = \iiint_R f(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz$$

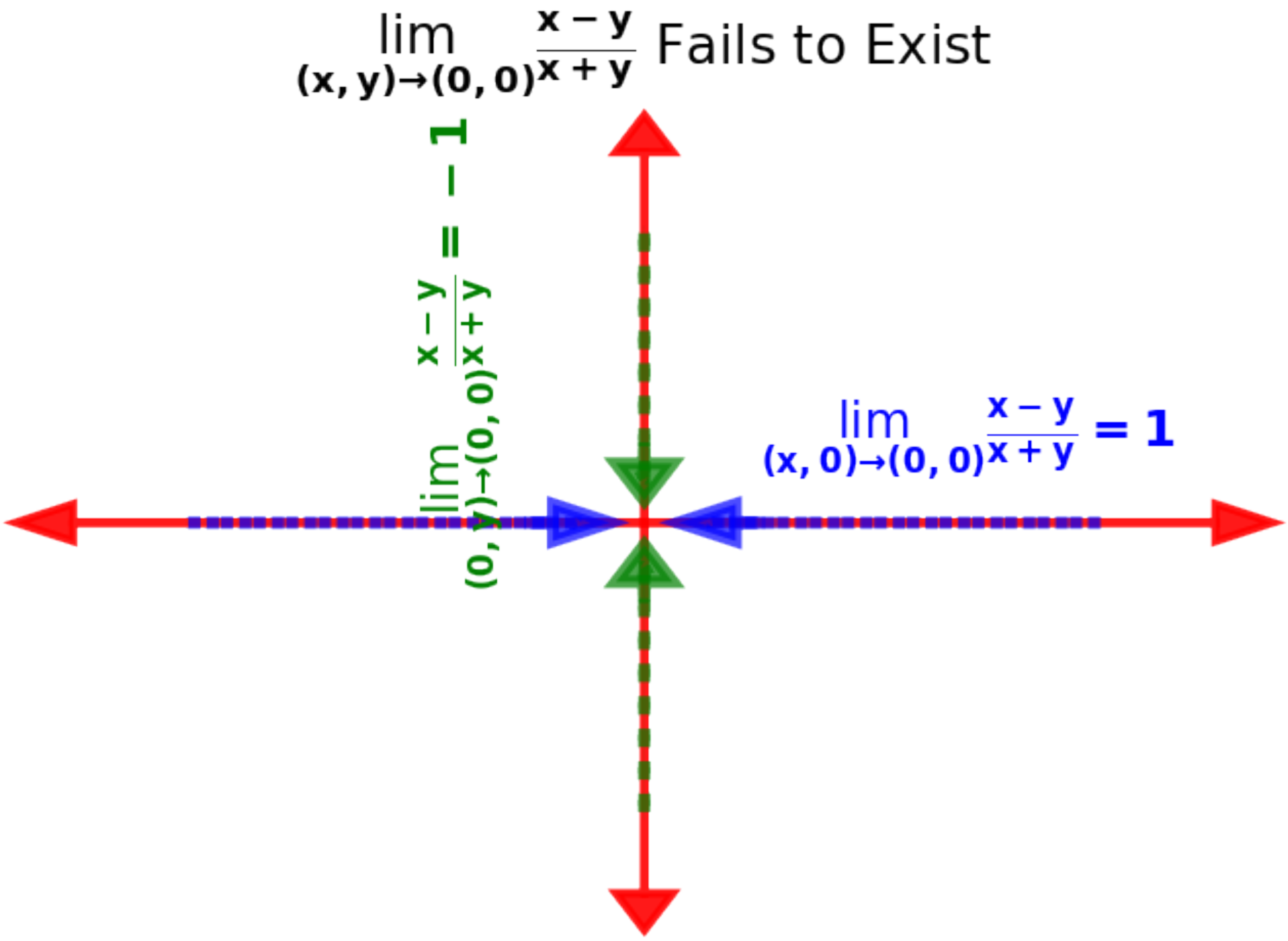
$$\iiiint\limits_{\{x^2+y^2\leq z\leq 4\}} 1dV = \int_0^4 dz \int_0^{\sqrt{z}} r\,dr \int_0^{2\pi} d\theta = 8\pi$$

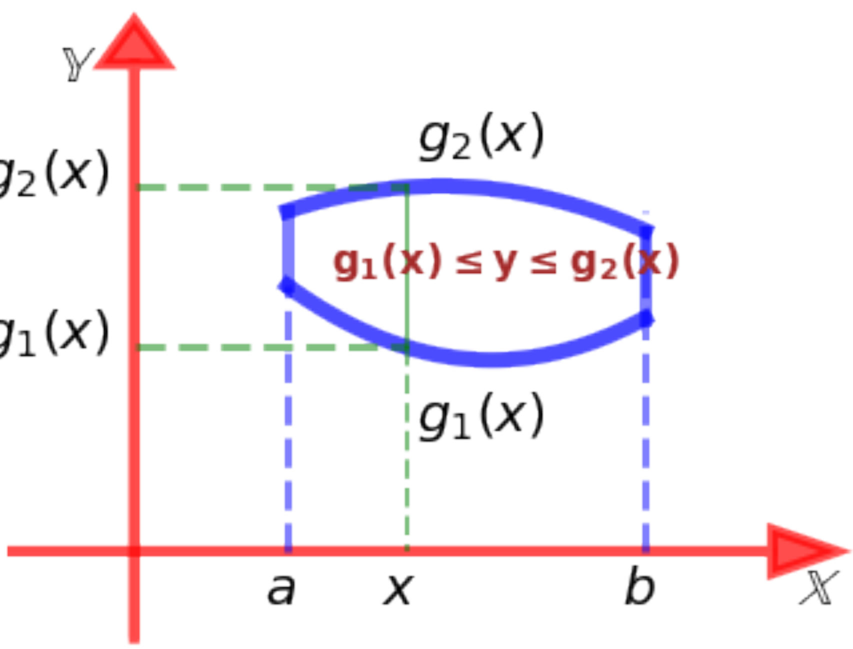
Spherical Coordinates (ρ, θ, ϕ)

$$\iiint_R f(x, y, z)dV = \iiint_R f(\rho\cos\theta\cos\phi, \rho\sin\theta\cos\phi, \rho\sin\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

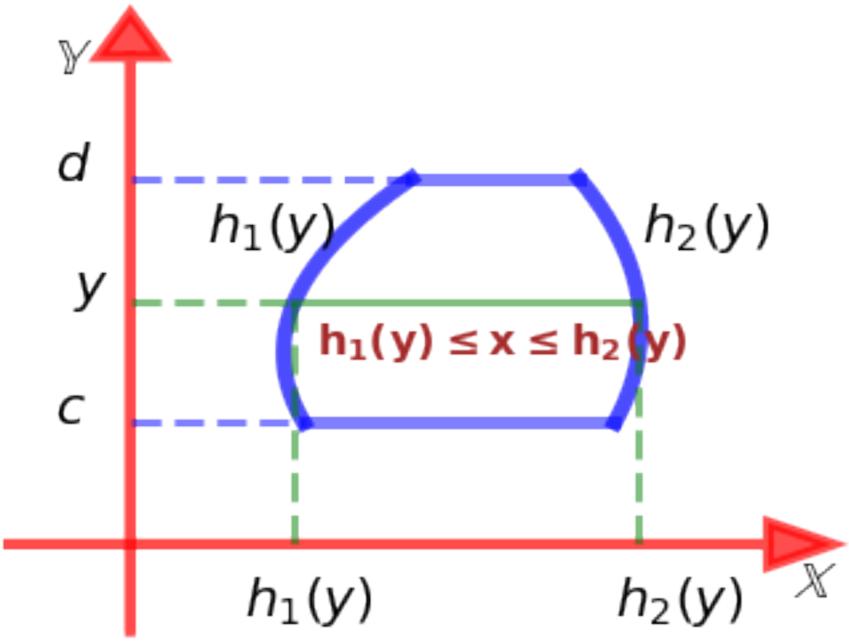
$$\iiint\limits_{\substack{x^2+y^2+z^2\leq 4\\0\leq z,z^2\leq x^2+y^2}} 1dV = \int_0^2 \rho^2 d\rho \int_0^{\pi/4} \sin\phi\,d\phi \int_0^{2\pi} d\theta = \frac{16}{3}\left(1-\frac{1}{\sqrt{2}}\right)\pi$$

(-3.0, 3.0, -3.0, 3.0)



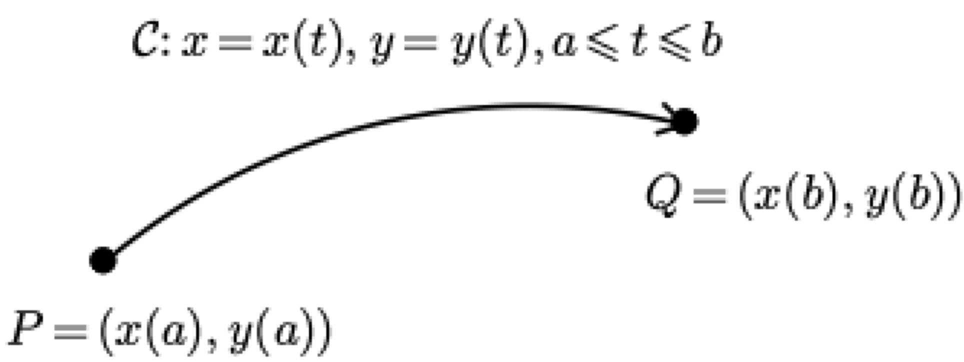


$$\int\int_D f(x,y)dA = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)dy$$

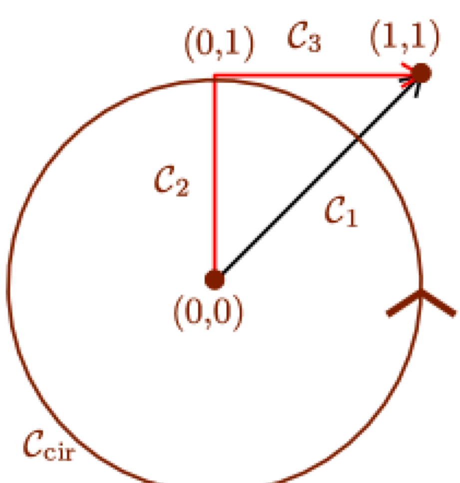


$$\int\int_D \mathbf{f(x, y)dA} = \int_c^d \mathbf{dy} \int_{h_1(y)}^{h_2(y)} \mathbf{f(x, y)dx}$$

Line Integral



$$\int_{\mathcal{C}} f(x, y)ds = \int_a^b f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}ds$$



(a) $\mathcal{C}_1: (x, y) = (t, t),\ 0 \leqslant t \leqslant 1$

$\int_{\mathcal{C}_1} (x - y)dx + (x + y)dy = \int 0\,dt + \int_0^1 (t + t)\frac{dt}{dt}dt = 1$

(b) $\mathcal{C} = \mathcal{C}_2 \cup \mathcal{C}_3$

$\mathcal{C}_1: (x, y) = (0, t),\ 0 \leqslant t \leqslant 1;\ \mathcal{C}_2 = (t, 1),\ 0 \leqslant t \leqslant 1$

$$\int_{\mathcal{C}} (x - y)dx + (x + y)dy = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2}$$
$$= \int_0^1 (0 + t)dt + \int_0^1 (t - 1)dt = 0$$

(c) $\mathcal{C}_{\text{cir}}: (x, y) = (\cos t, \sin t),\ 0 \leqslant t \leqslant 2\pi$

$$\int_C = \int_0^{2\pi} (\cos t - \sin t)(-\sin t)dt + \int_0^{2\pi} (\cos t + \sin t)\cos t\,dt = 2\pi$$

Green's Theorem

Suppose that C is a positive oriented,smooth and simple planar curve and D is the region bounded by C. If P and Q have continuous partial derivatives on interior od D. Then

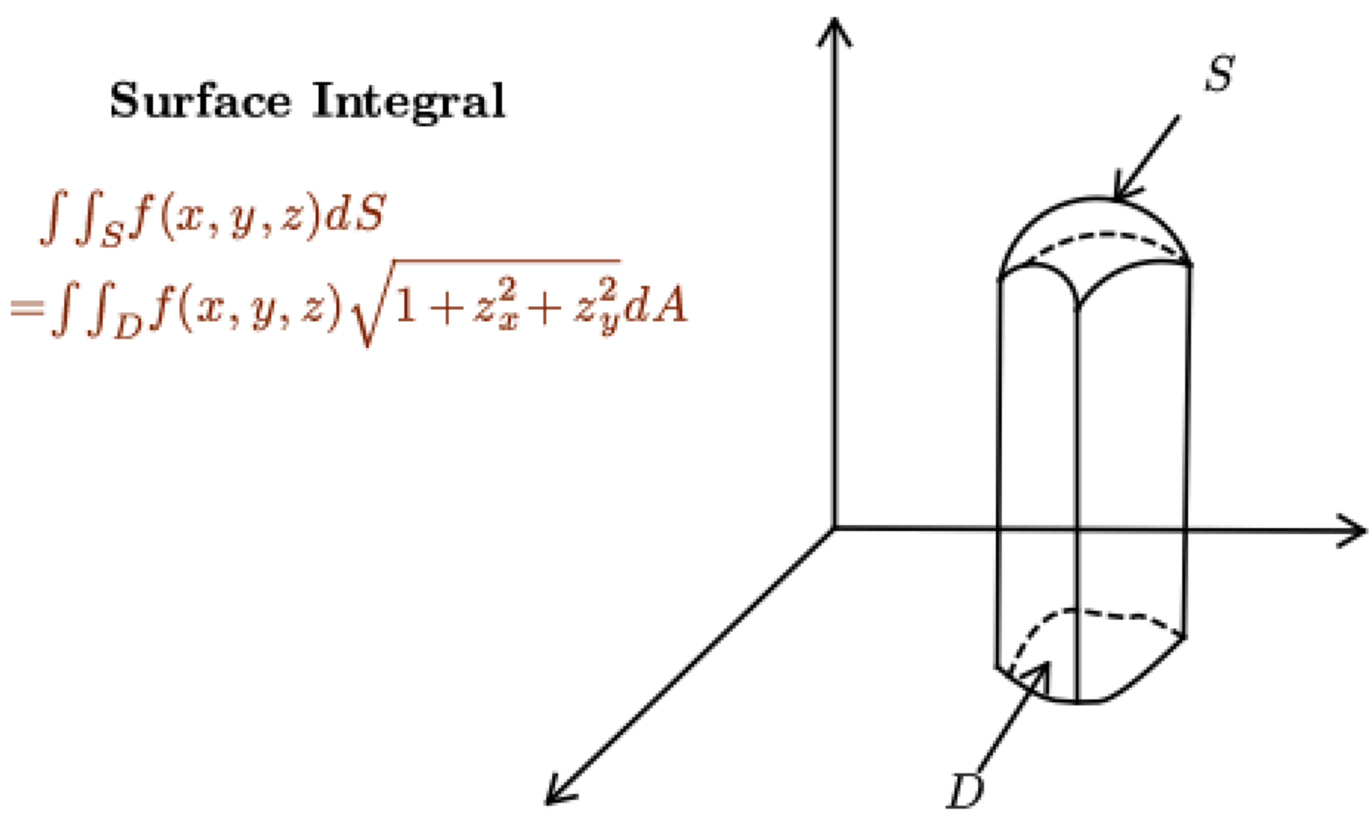
$$\int_C \mathbf{P(x,y)dx + Q(x,y)dy} = \iint_D \left(\frac{\partial \mathbf{Q}}{\partial \mathbf{x}} - \frac{\partial \mathbf{P}}{\partial \mathbf{y}} \right) \mathbf{dA}$$

$$\int_C \mathbf{(x-y)dx + (x+y)dy} = \iint\limits_{\mathbf{x^2+y^2 \leqslant 1}} \left(\frac{\partial \mathbf{(x-y)}}{\partial \mathbf{x}} + \frac{\partial \mathbf{(x+y)}}{\partial \mathbf{y}} \right) \mathbf{dA} = \mathbf{2}\pi$$

Green's Theorem

$$\int_C \mathbf{P(x,y)dx + Q(x,y)dy} = \iint_D \left(\frac{\partial \mathbf{Q}}{\partial \mathbf{x}} - \frac{\partial \mathbf{P}}{\partial \mathbf{y}} \right) \mathbf{dA}$$

$$\int_C \mathbf{(x-y)dx + (x+y)dy} = \iint\limits_{\mathbf{x^2+y^2 \leqslant 1}} \left(\frac{\partial \mathbf{(x-y)}}{\partial \mathbf{x}} + \frac{\partial \mathbf{(x+y)}}{\partial \mathbf{y}} \right) \mathbf{dA} = \mathbf{2}\pi$$

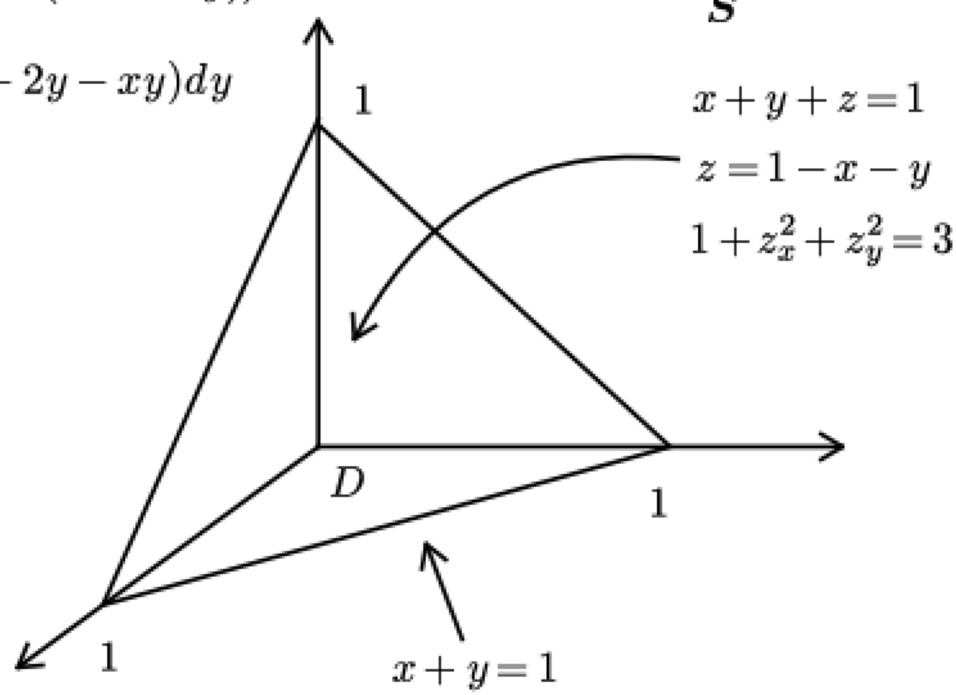


$$\int\int_S (xy+2z)\,dS$$

$$=\int\int_{\{x+y\leqslant 1,x,y\geqslant 0\}}(xy+2(1-x-y))\sqrt{3}\,dA$$

$$=\sqrt{3}\int_0^1 dx\int_0^{1-x}(2-2x-2y-xy)dy$$

$$=\frac{7\sqrt{3}}{24}$$



$$\iint\limits_{\substack{\mathbf{r}\in S\\ \mathbf{r}=(x,y,z)}} \mathbf{f(x,y,z)dS} = \iint\limits_{\substack{D\\ x=x(u,v)\\ \cdots}} \mathbf{f(x(u,v),y(u,v),z(u,v))}\left|\frac{\partial \mathbf{r}}{\partial \mathbf{u}}\times\frac{\partial \mathbf{r}}{\partial \mathbf{v}}\right|\mathbf{d}$$

$$\iint\limits_{\substack{x^2+y^2+z^2=1\\ z\geq 0}} \left((x^2 + y^2 + (z-1)^2\right) dS$$

$$= \iint\limits_{x^2+y^2\leqslant 1} \frac{x^2 + y^2 + (z-1)^2}{\sqrt{1 - x^2 - y^2}} dxdy$$

$$= 2 \iint\limits_{x^2+y^2\leqslant 1} \frac{1 - \sqrt{1 - x^2 - y^2}}{\sqrt{1 - x^2 - y^2}} dxdy$$

$$= 2 \int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1 - r^2}} dr - 2\pi = 2\pi$$

(II) $$\iint\limits_{\substack{x^2+y^2+z^2=1\\ z\geq 0}} \left((x^2 + y^2 + (z-1)^2\right) dS$$

$$= \iint\limits_{x^2+y^{\leq}1} \left((\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + (\cos\phi - 1)^2\right) \cdot \left|\frac{\partial r}{\partial u}\right.$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} (2 - 2\cos\phi)\sin\phi d\phi = 2\pi$$