Table of Contents

7. Functions of Several Variables (Integration)

In [3]:
```
from IPython.core.display import HTML
css_file = 'css/ngcmstyle.css'
HTML(open(css_file, "r").read())
```

Out[3]:

In [4]:
```
%matplotlib inline

#rcParams['figure.figsize'] = (10,3) #wide graphs by default
import scipy
import numpy as np
import time
from matplotlib import cm
from mpl_toolkits.mplot3d import Axes3D
from IPython.display import clear_output,display
import matplotlib.pylab as plt
```

In [4]:
```
from mpld3 import enable_notebook
#enable_notebook()
```

In [3]:
```
from sympy import symbols,pprint,integrate,pi,sqrt,sin,cos,diff
x,y=symbols("x y")
def doubleInt(f,X,xr,yr):
    Iy=integrate(f,[X[1],yr[0],yr[1]])
    I=integrate(Iy,[X[0],xr[0],xr[1]])
    print("the double integral of %s over [%s<%s<%s,%s<%s<%s] is %s" %(f,xr[0],X[0],xr[1],yr[0],X[1],yr[1
],I))
    return I
```

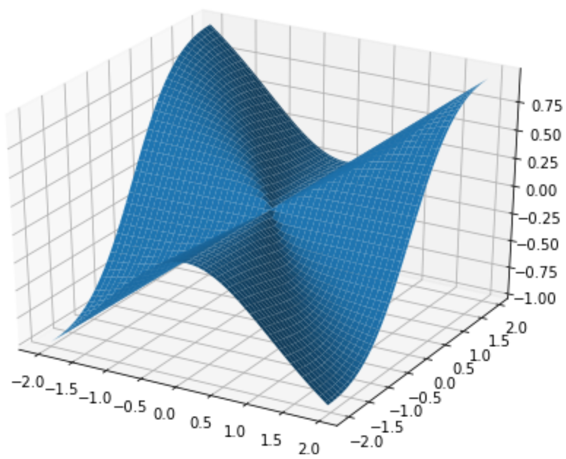
In [2]:
```
W  = '\033[0m'  # white (normal)
K  = '\033[30m' # black
R  = '\033[31m' # red
G  = '\033[32m' # green
O  = '\033[1;33m' # orange
B  = '\033[34m' # blue
P  = '\033[35m' # purple
T =  '\033[1;33;47m' #Title
```

In [12]:
```
def doubleInt3(f,X,xr,yr):
    Iy=integrate(f,[X[1],yr[0],yr[1]])
    I=integrate(Iy,[X[0],xr[0],xr[1]])
    #print("  %s \t  %s" %(xr[1],yr[1]))
    yrs=str(yr[1])
    xrs=' '+str(xr[1])
    print(xrs.ljust(9,' ')+yrs)
    print("∫   d",R+'{}'.format(X[0]),K+"∫ ",B+"{}".format(f),K+" d",R+"{}".format(X[1]),K+" = ","{}".for
mat(I))

    yrs0=str(yr[0])
    xrs0=str(xr[0])
    print(xrs0.ljust(8,' ')+yrs0)
    return I
```

In [4]:
```
def tripleInt3(f,X,xr,yr,zr):
    Iz=integrate(f,[X[2],zr[0],zr[1]])
    Iy=integrate(Iz,[X[1],yr[0],yr[1]])
    Ix=integrate(Iy,[X[0],xr[0],xr[1]])
    zrs=' '+str(zr[1])
    yrs=str(yr[1])
    xrs=' '+str(xr[1])
    print(xrs.ljust(9,' ')+O+yrs.ljust(7,' ')+K+zrs)
    #print("  %s \t  %s  \t  %s" %(xr[1],yr[1],zr[1]))
    print("∫   d",R+'{}'.format(X[0]),
        K+"∫   d",R+'{}'.format(X[1]),
        K+"∫ ",B+"{}".format(f),K+"  d",R+"{}".format(X[2]),K+" = ","{}".format(Ix))
    zrs1=' '+str(zr[0])
    yrs1=str(yr[0])
    xrs1=' '+str(xr[0])
    print(xrs1.ljust(8,' ')+O+yrs1.ljust(7,' ')+K+zrs1)
    #print(" %s \t %s  \t %s" %(xr[0],yr[0],zr[0]))
    return Ix
```

In [8]:
```
fig = plt.figure(figsize=(8,6))
ax = fig.gca(projection='3d')
X = np.arange(-2, 2, 0.04)
Y = np.arange(-2, 2, 0.04)
X, Y = np.meshgrid(X, Y)
f= X*X*Y/(X*X+Y*Y)
ax.plot_surface(X, Y, f)
```

Out[8]: <mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x108560358>



There are plenty of visualization packages in Python, MayaVi is one of them which provides advanced handy utilies, animation, interaction etc. Here, after struggle installztion of VTK library, MayaVi with its plugin works on notebook environment:

In [ ]:
```
import numpy as np
import mayavi.mlab as mlab
#import  moviepy.editor as mpy
```

In [ ]:
```
mlab.init_notebook(backend='x3d', local=True)
```

In [ ]:
```
duration = 2 # duration of the animation in seconds (it will loop)

# MAKE A FIGURE WITH MAYAVI

fig = mlab.figure(size=(200, 200), bgcolor=(1,1,1))

#u = np.linspace(0,2*np.pi,100)
X = np.arange(-2, 2, 0.1)
Y = np.arange(-2, 2, 0.1)
X, Y = np.meshgrid(X, Y)
f= X*X*Y/(X*X+Y*Y)
#xx,yy,zz = np.cos(u), np.sin(3*u), np.sin(u) # Points
#l = mlab.plot3d(xx,yy,zz, representation="wireframe", tube_sides=5,
mlab.surf(f, warp_scale='auto')
#mlab.plot3d(xx,yy,f, representation="wireframe",tube_sides=5,line_width=.5, tube_radius=0.2, figure=fig)
```

In [ ]:
```
def plot3d(x,y,z,contour=False):
    fig = plt.figure()
    ax = Axes3D(fig)
    ax.plot_surface(x, y, z, rstride=1, cstride=1, cmap=cm.jet,alpha=0.6)
    if contour==True:
       ax.contour(x, y, z,  lw=3, cmap="autumn_r", linestyles="solid", zdir='z',offset=-2)
    ax.set_xlabel('X')
    ax.set_ylabel('Y')
    ax.set_zlabel('Z')
    ax.set_zlim(-2, 1)
```

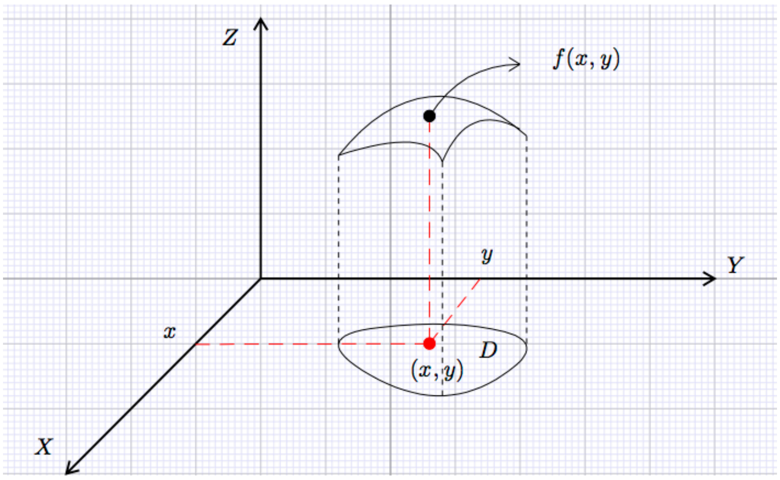
In [ ]:
```
x = np.arange(-2, 2, 0.1)
y = np.arange(-2, 2, 0.1)
x,y=np.meshgrid(x,y)
f= x**2*y/(x**2+y**2)
plot3d(x,y,f)
```

Double Integrals

If $z = f(x, y)$ is continuous and $f(x, y)$ is nonnegative for all (x, y) in a region D on X-Y plane, then the volume of solid under the graph of $f(x, y)$ and above $X - Y$ plane by the region D is

$$V = \iint_D f(x, y)dA$$

where $dA = dxdy$ is the element of area and V is called the double integral of $f(x, y)$ over D.



Theorem (Fubini's Theorem)

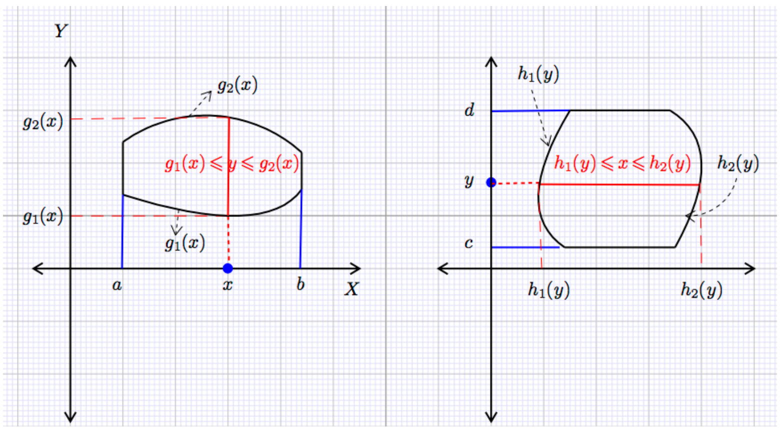
If $f(x, y)$ is continuous over D,

1. and D=$\{(x, y)|a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\}$,

$$\iint_D \mathbf{f(x, y)dA} = \int_a^b \mathbf{dx} \int_{g_1(x)}^{g_2(x)} \mathbf{f(x, y)dy}$$

2. and D=$\{(x, y)|c \leqslant y \leqslant d, h_1(y) \leqslant x \leqslant h_2(y)\}$,

$$\iint_D \mathbf{f(x, y)dA} = \int_c^d \mathbf{dy} \int_{h_1(y)}^{h_2(y)} \mathbf{f(x, y)dx}$$

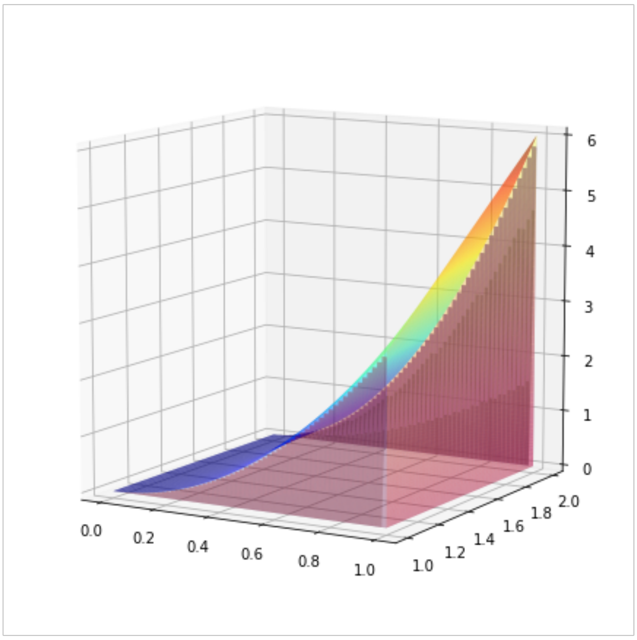


Example

Evaluate the double integral of $f(x) = 3x^2y$ over square region $D = \{1 \le y \le 2, 0 \le x \le 1\}$:

$$\iint\limits_{\{1 \le y \le 2, 0 \le x \le 1\}} 3x^2ydA$$

In [12]:
```
fig = plt.figure(figsize=(6,6))
ax = Axes3D(fig)

X = np.linspace(0, 1, 60)
X1 = np.linspace(0, 1, 60)
Y = np.linspace(1, 2, 60)
Z1 =  3*X1*X1*1
Z2 = 3*X1*X1*2
X,Y=np.meshgrid(X,Y)
func= 3*X*X*Y
base=0*X

ax.plot_surface(0*X+1,Y, func, rstride=1, cstride=1, cmap=cm.autumn,alpha=0.3);
ax.plot_surface(X,Y, base, rstride=1, cstride=1, cmap=cm.jet,alpha=0.2)
ax.plot_surface(X,Y, func, rstride=1, cstride=1, cmap=cm.jet,alpha=0.6)

Xs,Zs=np.meshgrid(X1,Z1)

Zs[Zs>3*Xs*Xs]=0

X2s,Z2s=np.meshgrid(X1,Z2)

Z2s[Z2s>3*X2s*X2s*2]=0


ax.plot_surface(Xs,0*Xs+1,Zs, rstride=1, cstride=1,alpha=0.2)
ax.plot_surface(X2s,0*X2s+2,Z2s, rstride=1, cstride=1,alpha=0.6)


ax.view_init(elev=10, azim=-60)
```



In [13]:
```
I=doubleInt3(3*x*x*y,[y,x],[1,2],[0,1])
```

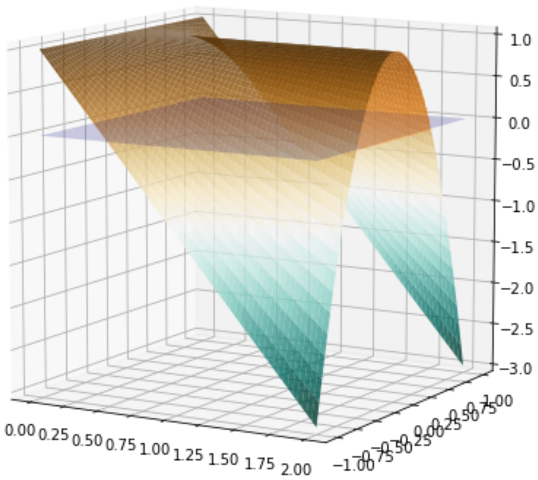
```
 2         1
∫    d y ∫  3*x**2*y   d x  =  3/2
1         0
```

Example

Evaluate the following double integral:

$$\iint\limits_{\{0\leq x\leq 2,-1\leq y\leq 1\}} (1-2xy^2)dA$$

In [7]:
```
fig = plt.figure(figsize=(6,6))
ax = Axes3D(fig)

a,b,c,d=0,2,-1,1

X = np.linspace(a, b, 60)
X1 = np.linspace(a, b, 60)
Y = np.linspace(c, d, 60)
Z1 =  1-2*X1*c*c
Z2 = 1-2*3*X1*d*d
X,Y=np.meshgrid(X,Y)
func= 1-2*X*Y*Y
func2= 1-2*b*Y*Y
base=0*X

Y0=np.linspace(-0.5,0.5,30)
for y in Y0:
    zs=np.linspace(0,1-2*2*y**2,30)
    ax.plot(2+0*zs,y+0*zs,zs,color='C1',alpha=0.4)


#ax.plot_surface(0*X+b,Y, func2, rstride=1, cstride=1, cmap=cm.autumn,alpha=0.3);
ax.plot_surface(X,Y, base, rstride=1, cstride=1, cmap=cm.jet,alpha=0.2)
ax.plot_surface(X,Y, func, rstride=1, cstride=1, cmap=cm.BrBG_r,alpha=0.8)

#Xs,Zs=np.meshgrid(X1,Z1)

#Zs[Zs>1-2*Xs*c*c]=0

#X2s,Z2s=np.meshgrid(X1,Z2)

#Z2s[Z2s>1-2*X2s*d*d]=0


#ax.plot_surface(Xs,0*Xs+c,Zs, rstride=1, cstride=1,alpha=0.2)
#ax.plot_surface(X2s,0*X2s+d,Z2s, rstride=1, cstride=1,alpha=0.6)

ax.view_init(elev=10, azim=-60)
```



In [ ]:
```
I=doubleInt3(1-2*x*y*y,[x,y],[0,2],[-1,1])
```

Properties of Double Integrals

Suppose that both $\iint_D f(x,y)dA$ and $\iint_D g(x,y)dA$ exist and $c \in \mathbb{R}$. Then

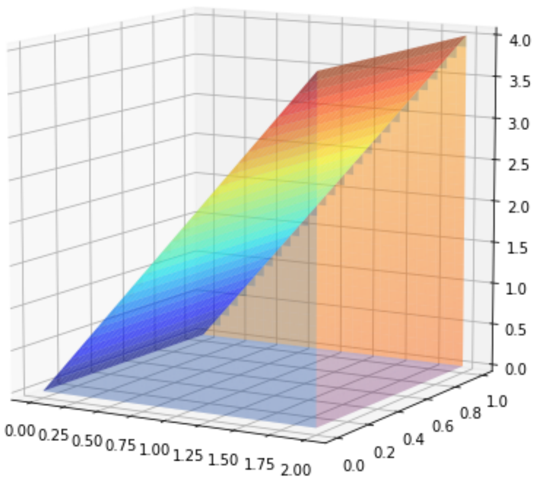
1. $\iint_D cf(x,y)dA = c\iint_D f(x,y)dA$,
2. $\iint_D [f(x,y) \pm g(x,y)]dA = \iint_D f(x,y)dA \pm \iint_D g(x,y)dA$,
3. If $f(x,y) \geq 0$, then $\iint_D f(x,y)dA \geq 0$,
4. If $f(x,y) \geq g(x,y)$, then $\iint_D f(x,y)dA \geq \iint_D f(x,y)dA$.
5. $\iint_{D=D_1\cup D_2} f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$ where $D_1 \cap D_2 = \emptyset$.

Exercise, p.1155

14.

$\iint_{0\leq x\leq 2, 0\leq y\leq 1} 2xdA$ is the solid under surface $f(x,y) = 2x$ and above the rectangle, $\{0 \leq x \leq 2, 0 \leq y \leq 1\}$.

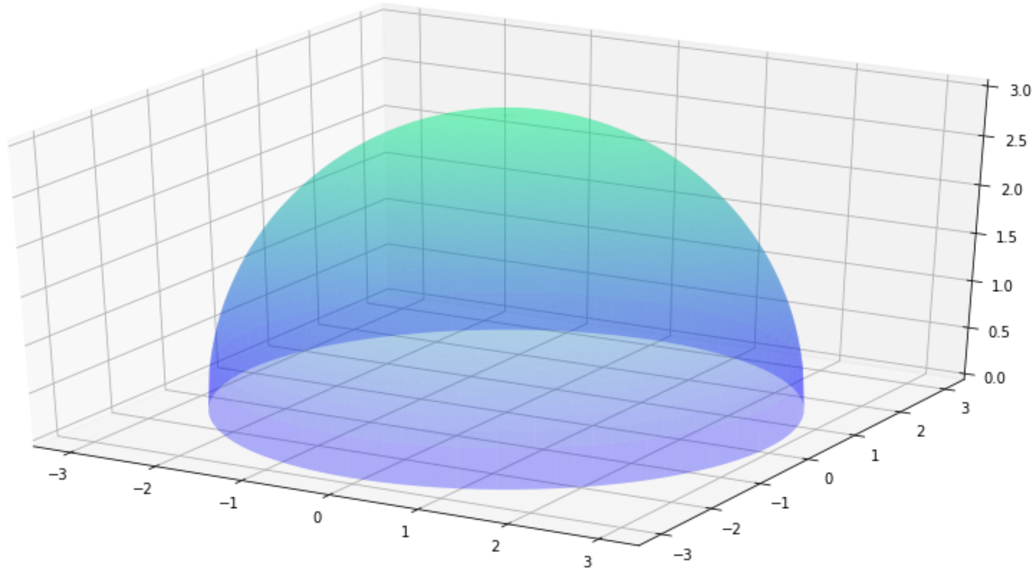
In [15]:
```
fig = plt.figure(figsize=(6,6))
ax = Axes3D(fig)

X = np.linspace(0, 2, 30)
X1 = np.linspace(0, 2, 30)
Y = np.linspace(0, 1, 30)
Zs =  2*X1
X,Y=np.meshgrid(X,Y)
func= 2*X
base=0*X

Xs,Zs=np.meshgrid(X1,Zs)

for i in range(len(Xs)):
    Xs[i][:i]=0
    Zs[i][:i]=0
ax.plot_surface(Xs,0*Xs,Zs, rstride=1, cstride=1,alpha=0.2)
ax.plot_surface(Xs,0*Xs+1,Zs, rstride=1, cstride=1, alpha=0.4)

ax.plot_surface(X,Y, func, rstride=1, cstride=1, cmap=cm.jet,alpha=0.6)
ax.plot_surface(X,Y, base, rstride=1, cstride=1, cmap=cm.jet,alpha=0.2)
ax.plot_surface(0*X+2,Y, func, rstride=1, cstride=1, cmap=cm.autumn,alpha=0.1);
ax.view_init(elev=10, azim=-60)
```



14.

$\iint\limits_{0\le x,y,\ x^2+y^2\le 9} \sqrt{9-x^2-y^2}dA$ is half upper sphere centred at $(0,0,0)$ with radius 3.

In [16]:
```
fig = plt.figure(figsize=(12,6))
ax = Axes3D(fig)
r = np.linspace(0, 3, 100)
t = np.linspace(0, 2*np.pi, 100)
r,t=np.meshgrid(r,t)
Xr= r*np.cos(t)
Yr= r*np.sin(t)
func=np.sqrt(9-r*r)
ax.plot_surface(Xr,Yr, func, rstride=1, cstride=1, cmap=cm.winter,alpha=0.3);
#plot3d(X,Y,0*func)
```



In [17]:
```
I=doubleInt3(x+2*y,[x,y],[0,2],[0,3])
```

```
 2         3
∫   d x ∫  x + 2*y  d y  =  24
0         0
```

In [18]:
```
# 21 problem changed
I=doubleInt3(x*y*sqrt(1+x**2+y**2),[x,y],[0,1],[0,1])
```

```
 1         1
∫   d x ∫  x*y*sqrt(x**2 + y**2 + 1)  d y  =  -8*sqrt(2)/15 + 1/15 + 3*sqrt(3)/5
0         0
```

In [19]:
```
# 22 changed
from sympy import exp
I=doubleInt3(x/(exp(x*y)),[x,y],[0,1],[1,3])
```

```
 1         3
∫   d x ∫  x*exp(-x*y)  d y  =  -exp(-1) + exp(-3)/3 + 2/3
0         1
```

Example

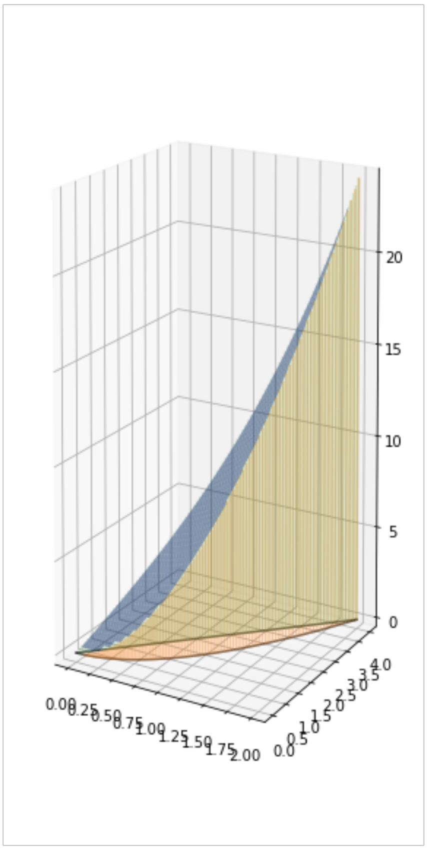
Find the volume of solid under the surface of $z = x^3 + 4y$ and over the region, R, which bounded by $y = 2x$ and $y = x^2$.

$$V = \iint_{0\le x\le 2, x^2\le y\le 2x} z dA = \int_0^2 dx \int_{x^2}^{2x} (x^3+4y)dy = \int_0^4 dy \int_{y/2}^{\sqrt{y}} (x^3+4y)dx$$

In [8]:

```
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
ax.set_aspect('equal','box')


X0 = np.linspace(0, 2, 60)
#X1 = np.linspace(0, 2, 30)
Y0 = np.linspace(0, 4, 60)
#Zs =  2*X1
X,Y=np.meshgrid(X0,Y0)
func= X**3+4*Y
base=0*X

t= np.linspace(0, 2, 60)
Xt=t
Yt=Xt**2
Xt,Yt=np.meshgrid(Xt,Yt)
ft=np.ones_like(X**3+4*Yt)

#Xs,Zs=np.meshgrid(X1,Zs)

#for i in range(len(Xs)):


#    Xs[i][:i]=0
#    Zs[i][:i]=0
#ax.plot_surface(Xs,0*Xs,Zs, rstride=1, cstride=1,alpha=0.2)
#ax.plot_surface(Xs,0*Xs+1,Zs, rstride=1, cstride=1, alpha=0.4)

R1=np.where(Y<=2*X,func,np.nan)

R=np.where(X**2<=Y,R1,np.nan)

S1=np.where(Y!=2*X,func,np.nan)
S2=np.where(Y!=X*X,func,np.nan)


X00=np.linspace(0,2,100)
l1x=X00
l2y=2*X00
l1x,l2y=np.meshgrid(l1x,l2y)

Z0=np.linspace(0,24,60)
Xs,Zs=np.meshgrid(X0,Z0)
S1=(Zs+Xs**3)/4


ax.plot_surface(X,Y, R, rstride=1, cstride=1, cmap=cm.jet,alpha=0.4)
ax.plot_surface(l1x,2*l1x,l1x*0, rstride=1, cstride=1, cmap=cm.jet,alpha=0.7)
ax.plot(X00,2*X00,0*X00,color="k",alpha=0.6)
ax.plot(X00,X00**2,0*X00,color="k",alpha=0.6)
for xs in X00:
    #ax.plot(X00,X00*X00,0*X00,color="k",alpha=0.6)
    ys=xs**2
    zs=np.linspace(0,xs**3+4*ys,50)
    ax.plot(xs+0*zs,ys+0*zs,zs,color='C1',alpha=0.4)
for xs in X00:
    #ax.plot(X00,X00*X00,0*X00,color="k",alpha=0.6)
    ys=2*xs
    zs=np.linspace(0,xs**3+4*ys,50)
    ax.plot(xs+0*zs,ys+0*zs,zs,color='C2',alpha=0.2)

#ax.plot_surface(Xs,S1, Zs, rstride=1, cstride=1, cmap=cm.jet,alpha=0.4)
#ax.plot_surface(X,Y, base, rstride=1, cstride=1, cmap=cm.jet,alpha=0.2)
#ax.plot_surface(X,X*X, S2, rstride=1, cstride=1, cmap=cm.jet,alpha=0.1)
#ax.plot_surface(X,Y, S1, rstride=1, cstride=1, cmap=cm.jet,alpha=0.1)

#ax.plot_surface(0*X+2,Y, func, rstride=1, cstride=1, cmap=cm.autumn,alpha=0.1);
#ax.plot_surface(Xt,Yt, ft, rstride=1, cstride=1, cmap=cm.jet,alpha=0.4)

ax.view_init(elev=10, azim=-60)
```

```
/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/matplotlib/colors.py:496: RuntimeWarning: in
valid value encountered in less
  cbook._putmask(xa, xa < 0.0, -1)
```



In [46]:
```
doubleInt3(x**3+4*y,[x,y],[0,2],[x*x,2*x])
```

```
 2       2*x
∫   d x ∫  x**3 + 4*y  d y  =  32/3
0       x**2
```

Out[46]: 32/3

In [47]:
```
doubleInt3(x**3+4*y,[y,x],[0,4],[y/2,sqrt(y)])
```

```
 4       sqrt(y)
∫   d y ∫  x**3 + 4*y  d x  =  32/3
0       y/2
```

Out[47]: 32/3

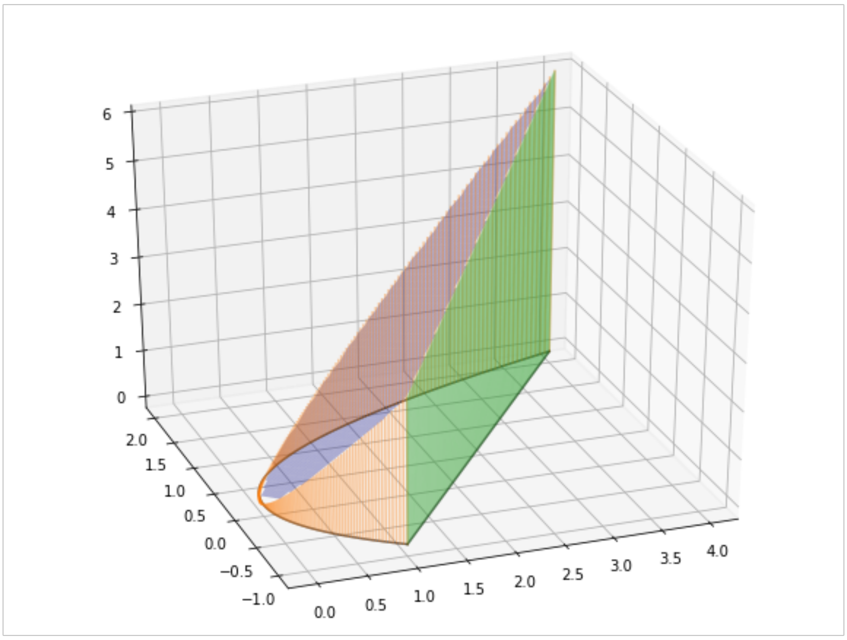
Example

Find the volume of solid under the surface of $z = 2x - y$ and over the region, R, which bounded by $x = y^2$ and $x - y = 2$.

$$V = \iint_{0 \le x \le 2, x^2 \le y \le 2x} z dA == \int_{-1}^{2} dy \int_{y^2}^{y+2} (2x - y) dx$$

In [151]:
```
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
ax.set_aspect('equal','box')


X0 = np.linspace(0, 4, 60)

Y0 = np.linspace(-1, 2, 60)
#Zs =  2*X1
X,Y=np.meshgrid(X0,Y0)
func= 2*X-Y
base=0*X

# surfsce of function
R1=np.where(Y*Y<=X,func,np.nan)
R=np.where(X<=Y+2,R1,np.nan)
ax.plot_surface(X,Y, R, rstride=1, cstride=1, cmap=cm.jet,alpha=0.3)

# boundary of base
ax.plot(Y0**2,Y0,0*Y0,color="k",alpha=0.6)
ax.plot(Y0+2,Y0,0*Y0,color="k",alpha=0.6)

Y00=np.linspace(-1,2,200)

# site surface
for ys in Y00:
    xs=ys**2
    zs=np.linspace(0,2*xs-ys,10)
    ax.plot(xs+0*zs,ys+0*zs,zs,color='C1',alpha=0.4)
for ys in Y00:
    xs=ys+2
    zs=np.linspace(0,2*xs-ys,10)
    ax.plot(xs+0*zs,ys+0*zs,zs,color='C2',alpha=0.2)


ax.view_init(elev=30, azim=-110)
```

```
/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/matplotlib/colors.py:496: RuntimeWarning: in
valid value encountered in less
  cbook._putmask(xa, xa < 0.0, -1)
```



In [48]:
```
doubleInt3(2*x-y,[y,x],[-1,2],[y*y,y+2])
```

```
 2        y + 2
∫   d y ∫  2*x - y  d x  =  243/20
-1      y**2
```

Out[48]: 243/20

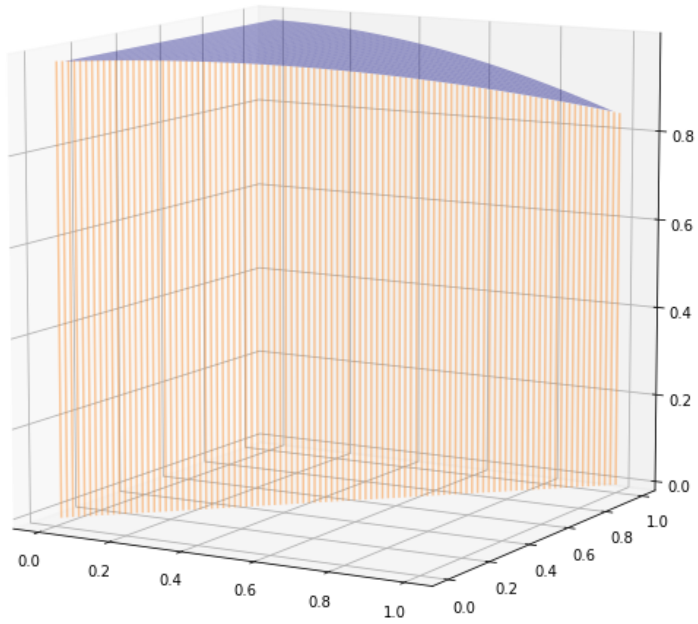
Example

Evaluate

$$\iint_{0\le y\le 1, y\le x\le 1} \frac{\sin x}{x} dA = \int_0^1 dx \int_0^x \frac{\sin x}{x} dy$$

In [16]:
```python
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
ax.set_aspect('equal','box')


X0 = np.linspace(0, 1, 60)
Y0 = np.linspace(0, 1, 60)
#Zs =  2*X1
X,Y=np.meshgrid(X0,Y0)
func= np.sin(X)/X
base=0*X

R=np.where(X<Y,func,np.nan)

ax.plot_surface(X,Y, R, rstride=1, cstride=1, cmap=cm.jet,alpha=0.4)

X00=np.linspace(0,1,100)
for xs in X00:
    #ax.plot(X00,X00*X00,0*X00,color="k",alpha=0.6)
    ys=xs
    zs=np.linspace(0,np.sin(xs)/xs,100)
    ax.plot(xs+0*zs,ys+0*zs,zs,color='C1',alpha=0.4)


ax.view_init(elev=10, azim=-60)
```

```
/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/ipykernel_launcher.py:10: RuntimeWarning: in
valid value encountered in true_divide
  # Remove the CWD from sys.path while we load stuff.
/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/ipykernel_launcher.py:21: RuntimeWarning: in
valid value encountered in double_scalars
/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/matplotlib/colors.py:496: RuntimeWarning: in
valid value encountered in less
  cbook._putmask(xa, xa < 0.0, -1)
```



In [49]:
```
doubleInt3(sin(x)/x,[x,y],[0,1],[0,x])
```

```
 1       x
∫   d x ∫  sin(x)/x  d y  =  -cos(1) + 1
0       0
```

Out[49]: -cos(1) + 1

Exercise p.1165

In [65]:
```
# 6
from sympy import pi,sin,cos,exp,log
doubleInt3(exp(-x)*sin(y),[y,x],[0,pi/2],[0,log(2)])
```

```
 pi/2    log(2)
∫   d y ∫  exp(-x)*sin(y)  d x  =  1/2
0       0
```

Out[65]: 1/2

In [51]:
```
# 8

doubleInt3(2*x*y,[x,y],[0,1/2],[0,sqrt(1-x)])
```

```
 0.5     sqrt(-x + 1)
∫   d x ∫  2*x*y  d y  =  0.0833333333333333
0       0
```

Out[51]: 0.0833333333333333

12.

Evaluate

$$\int_0^{\pi} dx \int_{\exp(-2x)}^{\exp(\cos x)} \frac{\ln y}{y} dy = \int_0^{\pi} dx \int_{-2x}^{\cos x} u du$$

by substitution, $y = \exp(u)$:

In [52]:
```
u=symbols("u")
I=doubleInt3(u,[x,u],[0,pi],[-2*x,cos(x)])
```

```
 pi      cos(x)
∫   d x ∫  u  d u  =  -2*pi**3/3 + pi/4
0       -2*x
```

18.

$$\iint_{0 \le x \le 1, 0 \le y \le x} \sqrt{1 - x^2} dA$$

In [53]:
```
f=sqrt(1-x**2)
I=doubleInt3(f,[x,y],[0,1],[0,x])
```

```
 1       x
∫   d x ∫  sqrt(-x**2 + 1)  d y  =  1/3
0       0
```

22.

$$\iint_{0 \le y \le 1, -y-1 \le x \le y-1} (x^2 + y^2) dA$$

In [54]:
```
f=x**2+y**2
I=doubleInt3(f,[y,x],[0,1],[-y-1,y-1])
```

```
 1       y - 1
∫   d y ∫  x**2 + y**2  d x  =  5/3
0       -y - 1
```

24.

$$\iint_{1 \le y \le e, y \le x \le y^2} \frac{1}{xy} dA$$

In [59]:
```
from mpmath import e
f=1/x/y
I=doubleInt3(f,[y,x],[1,e],[y,y*y])
```

```
 2.71828182845905y**2
∫   d y ∫  1/(x*y)  d x  =  0.500000000000000
1       y
```

28.

$$\iint_R (x^2 + y) dA$$

where the region, R, is bounded by $y = x^2 + 2, x = 0, x = 1, y = 0$.

In [60]:
```
f=x*x+y
I=doubleInt3(f,[x,y],[0,1],[0,x*x+2])
```

```
 1        x**2 + 2
∫   d x ∫  x**2 + y  d y  =  109/30
0       0
```

32.

$$\iint_R y dA$$

where the region, R, is bounded by $x^2 + y^2 \leq 1, y \geq 0.$

In [62]:
```
f=y
I=doubleInt3(f,[x,y],[-1,1],[0,sqrt(1-x*x)])
```

```
 1        sqrt(-x**2 + 1)
∫   d x ∫  y  d y  =  2/3
-1      0
```

56.

Evaluate

$$\int_0^2 dx \int_{x^2}^4 x \cos y^2 dy = \int_0^4 dy \int_0^{\sqrt{y}} x \cos y^2 dx$$

In [63]:
```
f=x*cos(y**2)
I=doubleInt3(f,[y,x],[0,4],[0,sqrt(y)])
```

```
 4        sqrt(y)
∫   d y ∫  x*cos(y**2)  d x  =  sin(16)/4
0       0
```

In [64]:
```
# 70
f=x*y/sqrt(x**2+y**2)
I=doubleInt3(f,[x,y],[0,1],[1,2])
```

```
 1        2
∫   d x ∫  x*y/sqrt(x**2 + y**2)  d y  =  -7/3 - 2*sqrt(2)/3 + 5*sqrt(5)/3
0       1
```

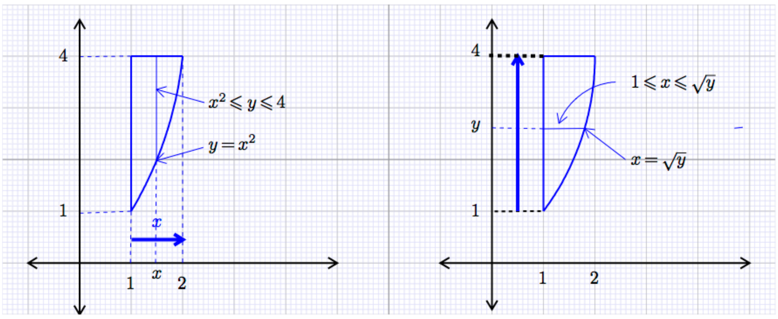
Example

Evaluate the following double integral:

$$\iint\limits_D (x+y)dA$$

where

$$\begin{aligned} D &= \left\{(x,y)\middle| 1 \leqslant x \leqslant \sqrt{y}, 1 \leqslant y \leqslant 4\right\} \\ &= \left\{(x,y)\middle| x^2 \leqslant y \leqslant 4, 1 \leqslant x \leqslant 2\right\} \end{aligned}$$



Reference the above graph, we can calculate the double integral with two different ways:

1. Along X-axis:

$$\begin{aligned} \iint\limits_D (x+y)dA &= \int_1^4 dy \int_1^{\sqrt{y}} (x+y)dx \\ &= \int_1^4 \left(\frac{x^2}{2} + xy\right)\Big|_1^{\sqrt{y}} dy \\ &= \int_1^4 \left(y^{3/2} - \frac{1}{2} - \frac{y}{2}\right) dy \\ &= \left(\frac{2y^{5/2}}{5} - \frac{y}{2} - \frac{y^2}{4}\right)\Big|_1^4 \\ &= 61\frac{3}{20} \end{aligned}$$

2. Along Y-axis:

$$\begin{aligned} \iint\limits_D (x+y)dA &= \int_1^2 dx \int_{x^2}^{4} (x+y)dy \\ &= \cdots \\ &= 61\frac{3}{20} \end{aligned}$$

In [ ]:
```
X = np.arange(-2, 2, 0.1)
Y = np.arange(-2, 2, 0.1)
X,Y=np.meshgrid(X,Y)
f=X+Y
plot3d(X, Y,f)
```

In [ ]:
```
from sympy import symbols,pprint,integrate,pi,sqrt,sin,cos,diff
x,y=symbols("x y")
def doubleInt(f,X,xr,yr):
    Iy=integrate(f,[X[1],yr[0],yr[1]])
    I=integrate(Iy,[X[0],xr[0],xr[1]])
    print("the double integral of %s over [%s<%s<%s,%s<%s<%s] is %s" %(f,xr[0],X[0],xr[1],yr[0],X[1],yr[1
],I))
    return I
```

In [ ]:
```
W  = '\033[0m'  # white (normal)
K  = '\033[30m' # black
R  = '\033[31m' # red
G  = '\033[32m' # green
O  = '\033[1;33m' # orange
B  = '\033[34m' # blue
P  = '\033[35m' # purple
T =  '\033[1;33;47m' #Title

def doubleInt2(f,X,xr,yr):
    Iy=integrate(f,[X[1],yr[0],yr[1]])
    I=integrate(Iy,[X[0],xr[0],xr[1]])
    #print("  %s \t  %s" %(xr[1],yr[1]))
    print(' ','{}'.format(xr[1]),'     ','{}'.format(yr[1]))
    print("∫   d",R+'{}'.format(X[0]),K+"∫ ",B+"{}".format(f),K+" d",R+"{}".format(X[1]),K+" = ","{}".for
mat(I))
    print(" %s \t %s" %(xr[0],yr[0]))
    return I
```

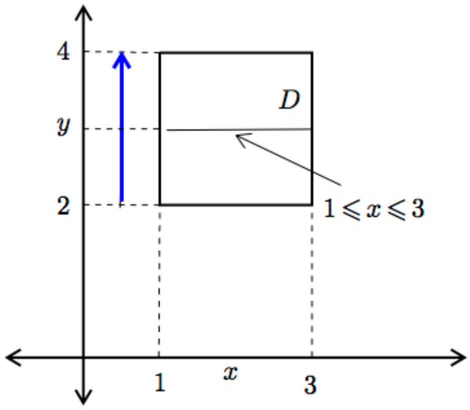
In [ ]:
```
def doubleInt3(f,X,xr,yr):
    Iy=integrate(f,[X[1],yr[0],yr[1]])
    I=integrate(Iy,[X[0],xr[0],xr[1]])
    #print("  %s \t  %s" %(xr[1],yr[1]))
    yrs=str(yr[1])
    xrs=' '+str(xr[1])
    print(xrs.ljust(9,' ')+yrs)
    print("∫   d",R+'{}'.format(X[0]),K+"∫ ",B+"{}".format(f),K+" d",R+"{}".format(X[1]),K+" = ","{}".for
mat(I))

    yrs0=str(yr[0])
    xrs0=str(xr[0])
    print(xrs0.ljust(8,' ')+yrs0)
    return I
```

In [ ]:
```
I=doubleInt3(y/x,[x,y],[1,3],[2,4])
```

In [ ]:
```
I=doubleInt3(x,[x,y],[-2,2],[0,sqrt(4-x**2)])
```

Example

If $D = \{(x, y) | 1 \leqslant x \leqslant 3, 2 \leqslant y \leqslant 4\}$,



By Fubini's theorem, we have

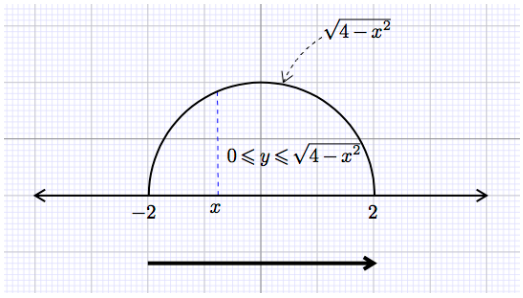
$$\iint_D \frac{y}{x} dA = \int_2^4 dy \int_1^3 \frac{y}{x} dx = 6 \ln 3$$

In [ ]:
```
I=doubleInt(y/x,[x,y],[1,3],[2,4])
```

In [ ]:
```
I=doubleInt2(y/x,[x,y],[1,3],[2,4])
```

Example

If $D = \left\{(x, y) | -2 \leqslant x \leqslant 2, 0 \leqslant y \leqslant \sqrt{4 - x^2}\right\}$,



then

$$\begin{aligned} \iint_D 1 dA &= \int_{-2}^{2} dx \int_0^{\sqrt{4-x^2}} dy \\ &= \int_{-2}^{2} \sqrt{4 - x^2} dx \\ &= \frac{1}{2} 2^2 \pi = 2\pi \end{aligned}$$

i.e. the area of half circle, D, is 2π.

In [ ]:
```
I=doubleInt(1,[x,y],[-2,2],[0,sqrt(4-x**2)])
```

In [ ]:
```
I=doubleInt2(1,[x,y],[-2,2],[0,sqrt(4-x**2)])
```

Suppose that all the points (x, y)in D can be transformed as:

$$x = \phi(u, v), y = \psi(u, v).$$

Then the double integral can be evaluated as followed:

$$\iint\limits_D f(x, y)dA = \iint\limits_D f(\phi(u, v), \psi(u, v))|J|dudv$$

where J is called the Jacobian of (x, y) and equal to:

$$\begin{aligned} J &= \left|\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| \\ &= \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u} \end{aligned}$$

Especially, as in polar coordinate system, we have

$$x = r\cos\theta, y = r\sin\theta$$

where r is the distance between (x, y) and origin and θ is the angle between the line, connecting (x, y) and origin, and X-axis. In this case,

$$\frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial y}{\partial r} = \sin\theta \text{ and } \frac{\partial y}{\partial \theta} = r\cos\theta$$

and $|J| = r$ since

$$\begin{aligned} J &= \left|\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}\right| \\ &= \cos\theta \cdot r\cos\theta - (-r\sin\theta) \cdot \sin\theta \\ &= r \end{aligned}$$

Example

If $D = \left\{(x, y) | -2 \le x \le 2, 0 \le y \le \sqrt{4 - x^2}\right\}$, then

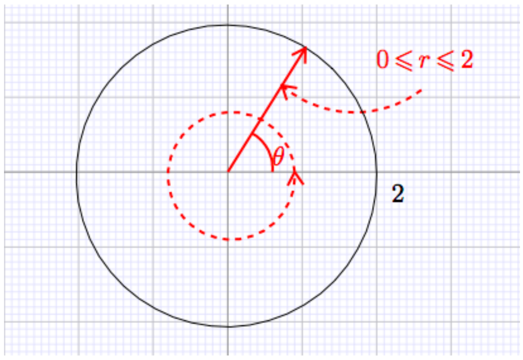
$$\begin{aligned} \iint\limits_D 1dA &= \int_{-2}^{2} dx \int_0^{\sqrt{4-x^2}} dy \\ &= \int_{-2}^{2} \sqrt{4 - x^2}dx \\ &= \frac{1}{2}2^2\pi = 2\pi \end{aligned}$$

i.e. the area of half circle, D, is 2π.

Find the volume of the semi-sphere above X-Y plane with radius 2, i.e.

$$\iint\limits_{\{(x,y)|x^2+y^2\leqslant 4\}} \sqrt{4 - x^2 - y^2}dA$$

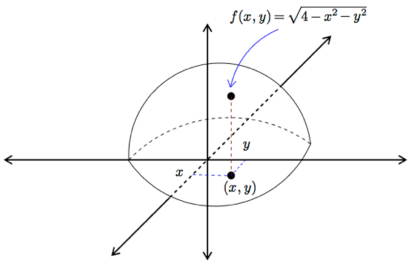
Sol: Since $\{(x, y)|x^2 + y^2 \leqslant 4\} = \{(r, \theta)|0 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant 2\pi\}$



and $|J| = r$. Then

$$\iint\limits_{\{(x,y)|x^2+y^2\le 4\}} \sqrt{4-x^2-y^2}dA = \iint\limits_{\{(r,\theta)|0\le r\le 2, 0\leqslant\theta\le 2\pi\}} \sqrt{4-r^2}\cdot rdrd\theta$$
$$= \int_0^{2\pi} d\theta \int_0^2 r\sqrt{4-r^2}dr$$
$$= \int_0^{2\pi} \left(-\frac{1}{3}(4-r^2)^{3/2}\right)\Big|_0^2 d\theta$$
$$= \int_0^{2\pi} \frac{8}{3}d\theta$$
$$= \frac{16\pi}{3}$$

i.e. half of volume of ball with radius 2, reference the following:




```
In [ ]: r,t=symbols("r t")
        I=doubleInt3(r*sqrt(4-r**2),[r,t],[0,2],[0,2*pi])
```

Example

If $R = \left\{(x, y) | 1 \le x^2 + y^2 \le 4, 0 \le x, y\right\}$, then

$$\iint_{\{(x,y)|1\le x^2+y^2\le 4,0\le x,y\}} (2x+3y)dA = \iint_{\{(r,\theta)|1\le r\le 2,0\le\theta\le 2\pi\}} r(2\cos\theta + 3\sin\theta)\cdot r dr d\theta$$
$$= \frac{35}{3}$$

In [70]:
```
r,t=symbols("r t")
I=doubleInt3(r**2*(2*cos(t)+3*sin(t)),[r,t],[1,2],[0,pi/2])
```

```
 2       pi/2
∫   d r ∫  r**2*(3*sin(t) + 2*cos(t))  d t  =  35/3
1       0
```

Example

Find the volume of solid, S, lies below $z = \sqrt{9 - x^2 - y^2}$ and above XY-plane inside $x^2 + y^2 = 1$.

$$V = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{9 - r^2} dr = \frac{2\pi}{3}(27 - 16\sqrt{2})$$

In [71]:
```
I=doubleInt3(r*sqrt(9-r*r),[r,t],[0,1],[0,2*pi])
```

```
 1       2*pi
∫   d r ∫  r*sqrt(-r**2 + 9)  d t  =  -32*sqrt(2)*pi/3 + 18*pi
0       0
```

Example

Find the volume of solid, S, lies below $z = 4 - x^2 - y^2$ and above XY-plane inside $(x-1)^2 + y^2 = 1$.

$$V = \int_{-\pi/2}^{2\pi} d\theta \int_0^{2\cos\theta} r(4 - r^2)dr = \frac{5\pi}{2}$$

In [74]:
```
I=doubleInt3(r*(4-r*r),[t,r],[-pi/2,pi/2],[0,2*cos(t)])
```

```
 pi/2    2*cos(t)
∫   d t ∫  r*(-r**2 + 4)  d r  =  5*pi/2
-pi/2   0
```

Example

Evaluate the integral $\int_0^\infty e^{-x^2} dx$.

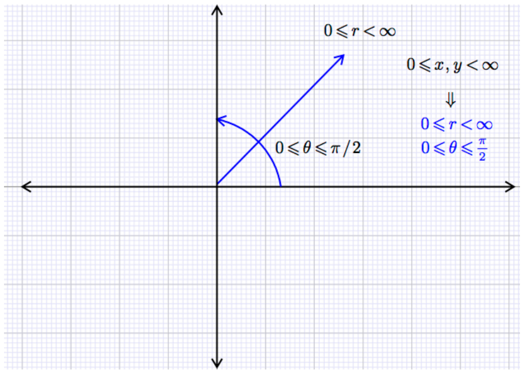
Sol: Let $I = \int_0^\infty e^{-x^2} dx$. Then $I = \int_0^\infty e^{-y^2} dy$ by changing the dummy variable x into y. Consider the product:

$$\begin{aligned} I^2 &= I \cdot I \\ &= \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy \\ &= \int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy \\ &= \int_0^{\pi/2} \int_0^\infty e^{-r^2} \cdot rdrd\theta \\ &= \int_0^{\pi/2} \frac{1}{2} d\theta = \frac{\pi}{4} \end{aligned}$$

In the third and forth equalities, the domain is as follows:

$$\begin{aligned} D &= \{(x, y) | 0 \leqslant x, y < \infty\} \\ &= \{(r, \theta) | 0 \leqslant r < \infty, 0 \leqslant \theta \leqslant \pi/2\} \end{aligned}$$

reference the following:



i.e. the whole first quadrant. This implies $I = \frac{\sqrt{\pi}}{2}$.

Note: The related formula are listed:

1. By symmetry, we have

$$\int_{-\infty}^{\infty} e^{-x^2}dx = 2\int_{0}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

2. To prove

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = 1$$

change the variable by substitution of $t = \frac{x-\mu}{\sqrt{2}\sigma}$ and $dt = \frac{dx}{\sqrt{2}\sigma}$. Also

$$x\Big|_{-\infty}^{\infty} \Longrightarrow t = \frac{x-\mu}{\sqrt{2}\sigma}\Big|_{-\infty}^{\infty}$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2} dt \\ &= \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1\end{aligned}$$

3. As the similar procedure, we can also calculate $\int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \mu$. By using the same substitution in ii), $t = \frac{x-\mu}{\sqrt{2}\sigma}$, we have:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx &= \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \frac{x-\mu}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \\ &= \mu + \int_{-\infty}^{\infty} \frac{t}{\sqrt{\pi}} e^{-t^2} dt \\ &= \mu\end{aligned}$$

The last result holds since the definite integral of odd function over interval symmetry with respect to 0.

We can also describe the result by the graphs of such functions.

In [ ]:
```
x=np.linspace(-10,10,201)
def expf(x,mu=0,sigma=1):
    return np.exp(-(x-mu)**2/2/sigma**2)/(np.sqrt(2*np.pi*sigma))
plt.plot(x,expf(x))
plt.plot([2,2],[0,0.4],'r--')
plt.plot(x,expf(x,mu=2))
```

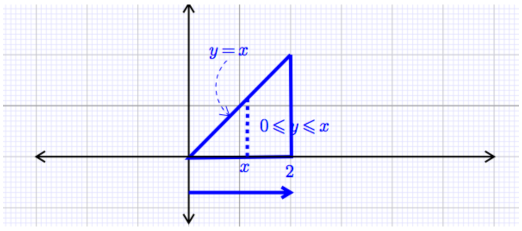
Obviously, the latter is as the same \ as the former but forward 2 units. Since the limits of both are the same, from $-\infty$ to ∞, it is no doubt that both the integrals for

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{ and } \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}}$$

are the same.

Exercise

Integrate $y\sqrt{x^3+1}$ over D:



Then

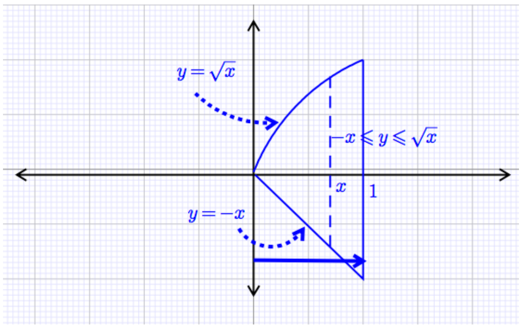
$$
\begin{aligned}
\iint_D y\sqrt{x^3+1}dA &= \int_0^2\int_0^x y\sqrt{x^3+1}dydx \\
&= \int_0^2 \sqrt{x^3+1}\frac{y^2}{2}\Big|_0^x dx \\
&= \int_0^2 \frac{x^2}{2}\sqrt{x^3+1}dx \\
&= \frac{1}{6}\int_0^2 \sqrt{x^3+1}d(x^3+1) \\
&= \frac{1}{6}\cdot\frac{2}{3}\cdot(x^3+1)^{3/2}\Big|_0^2 \\
&= \frac{26}{9}
\end{aligned}
$$

In [ ]:
```
X = np.arange(0, 2, 0.1)
Y = np.arange(0, 2, 0.1)
X,Y=np.meshgrid(X,Y)
f=Y*np.sqrt(X**3+1)
plot3d(X, Y,f)
```

In [ ]:
```
I=doubleInt(y*sqrt(1+x**3),[x,y],[0,2],[0,x])
```

Exercise

Integrate $f(x, y) = y/(1 + x)$ over D:



Then

$$
\begin{aligned}
\iint_D \frac{y}{1+x} dA &= \int_0^1 \int_{-x}^{\sqrt{x}} \frac{y}{1+x} dy dx \\
&= \int_0^1 \frac{1}{1+x} \frac{y^2}{2} \Big|_{-x}^{\sqrt{x}} dx \\
&= \frac{1}{2} \int_0^1 \frac{x - x^2}{1+x} dx \\
&= \frac{1}{2} \int_0^1 \left(-x + 2 - \frac{2}{1+x} \right) dx \\
&= \frac{1}{2} \left(-\frac{x^2}{2} + 2x - 2 \ln |1+x| \right) \Big|_0^1 \\
&= \frac{1}{2} (3/2 - 2 \ln 2)
\end{aligned}
$$

In [ ]:

```
X = np.arange(0, 1, 0.1)
Y = np.arange(-1, 1, 0.1)
X,Y=np.meshgrid(X,Y)
f=Y/(1+X)
plot3d(X, Y,f)
```

In [ ]:

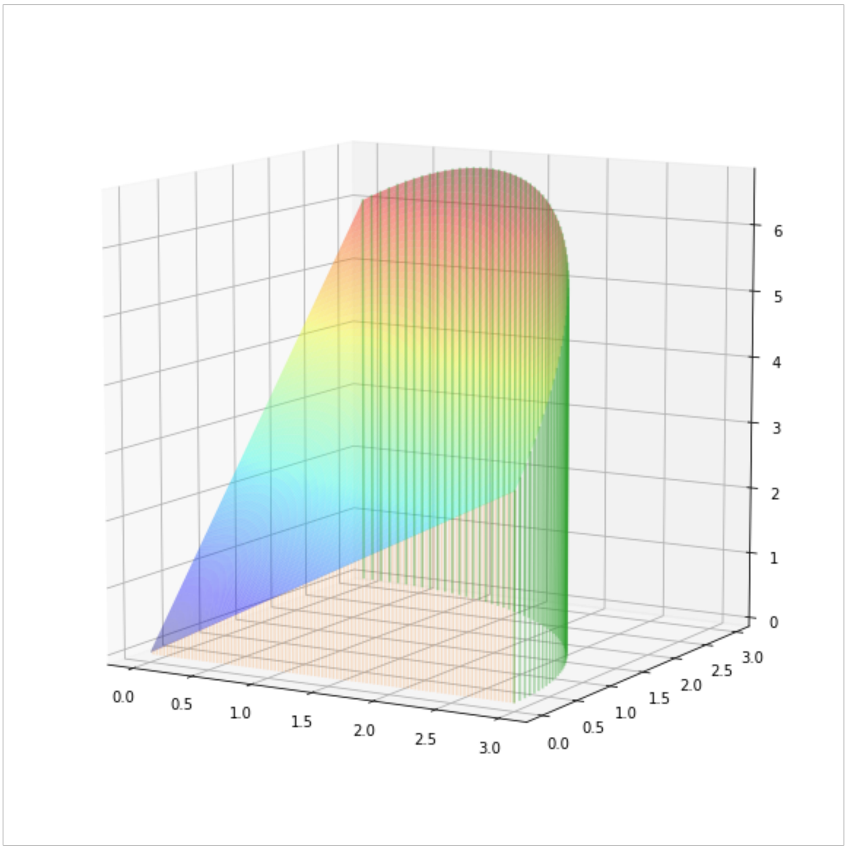
```
I=doubleInt(y/(1+x),[x,y],[0,1],[-x,sqrt(x)])
```

p.1173 Exercise

10.

$$\iint_{\{x^2+y^2 \leq 9, x, y \geq 0\}} (x + 2y) dA$$

In [23]:
```
from numpy import cos,sin,pi
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
ax.set_aspect('equal','box')


r0 = np.linspace(0, 3, 60)
t0 = np.linspace(0, np.pi/2, 60)
#Zs =  2*X1
R,T=np.meshgrid(r0,t0)
func= R*cos(T)+2*R*sin(T)

#R=np.where(X<Y,func,np.nan)
X=R*cos(T)
Y=R*sin(T)
ax.plot_surface(X,Y, func, rstride=1, cstride=1, cmap=cm.jet,alpha=0.4)

X00=np.linspace(0,3,100)
for xs in X00:
    ys=0*xs
    zs=np.linspace(0,xs+0,100)
    ax.plot(xs+0*zs,ys+0*zs,zs,color='C1',alpha=0.2)
for ts in t0:
    zs=np.linspace(0,3*(cos(ts)+2*sin(ts)),100)
    ax.plot(3*cos(ts)+0*zs,3*sin(ts)+0*zs,zs,color='C2',alpha=0.4)
ax.view_init(elev=10, azim=-60)
```



In [77]:
```
I=doubleInt3(r*(2*r*sin(t)+r*cos(t)),[r,t],[0,3],[0,pi/2])
```

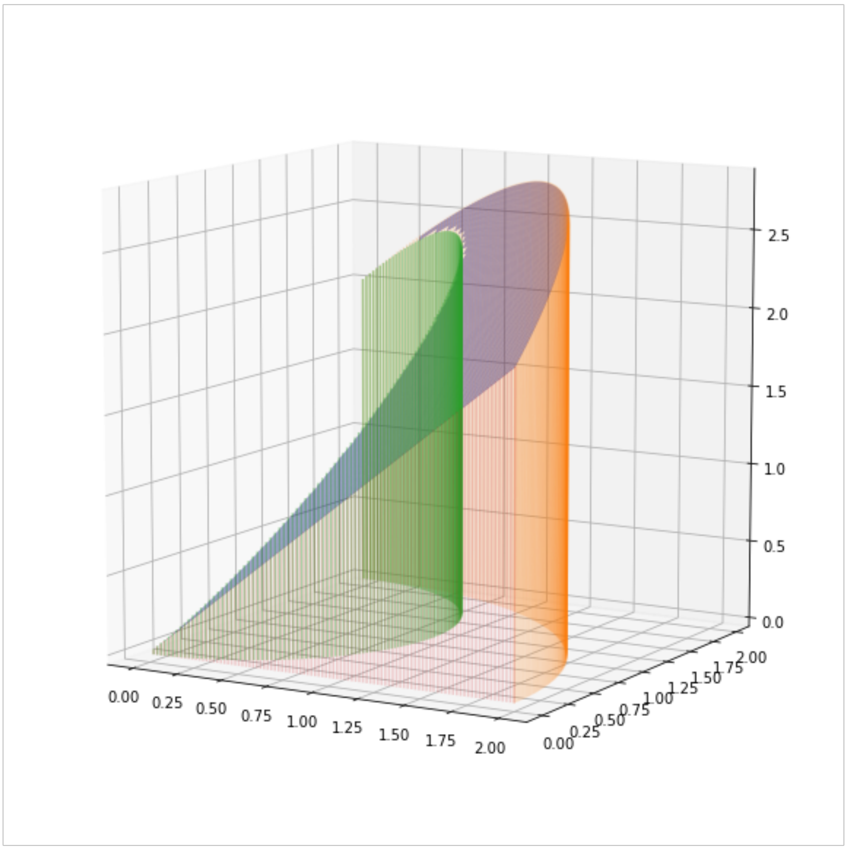
$$\int_0^3 dr \int_0^{\pi/2} r*(2*r*\sin(t) + r*\cos(t))\, dt = 27$$

16.

$$\iint_{\{x^2+y^2\le 4, x^2+(y-1)^2\ge 1, x,y\ge 0\}} (x+y)dA$$

In [43]:
```
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
ax.set_aspect('equal','box')


r0 = np.linspace(0, 2, 60)
t0 = np.linspace(0, np.pi/2, 150)
#Zs =  2*X1
R,T=np.meshgrid(r0,t0)
func= R*cos(T)+R*sin(T)

Res=np.where(2*sin(T)<=R,func,np.nan)
X=R*cos(T)
Y=R*sin(T)
ax.plot_surface(X,Y, Res, rstride=1, cstride=1, cmap=cm.jet,alpha=0.4)

#X00=np.linspace(0,3,100)
for ts in t0:
    zs=np.linspace(0,2*(cos(ts)+sin(ts))+0,100)
    xs=zs*0+2*cos(ts)
    ys=zs*0+2*sin(ts)
    ax.plot(xs,ys,zs,color='C1',alpha=0.2)
t0 = np.linspace(0, np.pi/2, 150)
for ts in t0:
    r1=2*sin(ts)
    x1=r1*cos(ts)
    y1=r1*sin(ts)
    z1=np.linspace(0,x1+y1,100)
#    zs=np.linspace(0,3*(cos(ts)+2*sin(ts)),100)
    ax.plot(x1+0*zs,y1+0*zs,z1,color='C2',alpha=0.4)
X0=np.linspace(0,2,100)
for x0 in X0:
    zs=np.linspace(0,x0,30)
    ax.plot(x0+0*zs,0*zs,zs,color='C3',alpha=0.2)
ax.view_init(elev=10, azim=-60)
```

```
/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/matplotlib/colors.py:496: RuntimeWarning: in
valid value encountered in less
  cbook._putmask(xa, xa < 0.0, -1)
```



In [80]:
```
I=doubleInt3(r*(r*sin(t)+r*cos(t)),[t,r],[0,pi/2],[2*sin(t),2])
```

```
 pi/2     2
∫    d t ∫  r*(r*sin(t) + r*cos(t))  d r  =  -pi/2 + 14/3
0       2*sin(t)
```

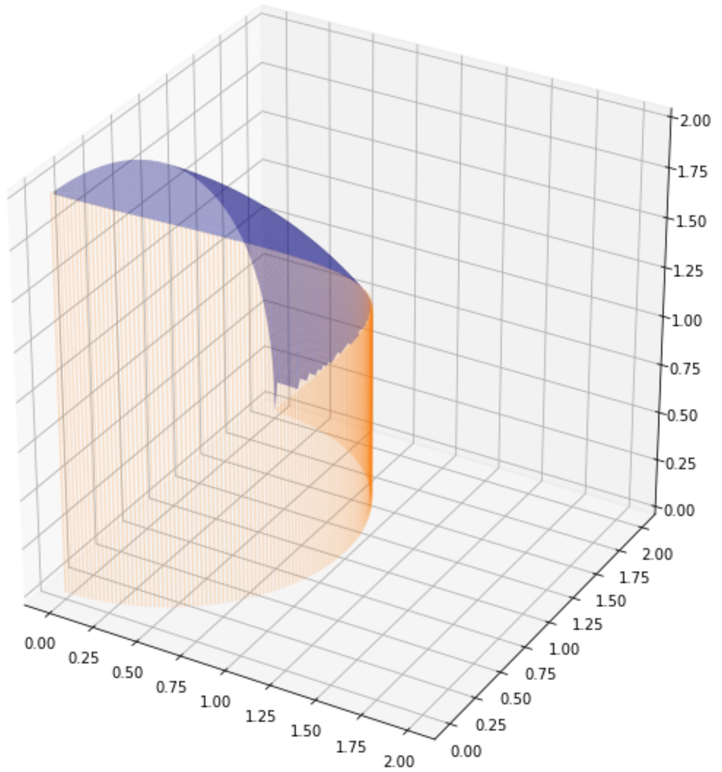
26. Volume of solid, T, which inside $x^2 + y^2 + z^2 = 4$ and inside $x^2 + y^2 = 2y$ is

$$\iint_{\{r \leq 2\sin\theta\}} r\sqrt{4 - r^2}\,dr d\theta = \frac{8\pi}{3} - \frac{32}{9}$$

In [4]:

```
from numpy import sqrt,exp,sin,cos
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
ax.set_aspect('equal','box')

r0 = np.linspace(0, 2, 300)
t0 = np.linspace(0, np.pi/2, 150)
#Zs =  2*X1
R,T=np.meshgrid(r0,t0)
func= sqrt(4-R**2)

Res=np.where(2*sin(T)>=R,func,np.nan)
X=R*cos(T)
Y=R*sin(T)
ax.plot_surface(X,Y, Res, rstride=1, cstride=1, cmap=cm.jet,alpha=0.4)

#X00=np.linspace(0,3,100)
for ts in t0:
    r1=2*sin(ts)
    zs=np.linspace(0,sqrt(4-r1**2),100)
    xs=zs*0+r1*cos(ts)
    ys=zs*0+r1*sin(ts)
    ax.plot(xs,ys,zs,color='C1',alpha=0.2)
#t0 = np.linspace(0, np.pi/2, 150)
#for ts in t0:
#    r1=2*sin(ts)
#    x1=r1*cos(ts)
#    y1=r1*sin(ts)
#    z1=np.linspace(0,sqrt(r1**1),100)
#    zs=np.linspace(0,3*(cos(ts)+2*sin(ts)),100)
#    ax.plot(x1+0*zs,y1+0*zs,z1,color='C2',alpha=0.4)
#X0=np.linspace(0,2,100)
#for x0 in X0:
#    zs=np.linspace(0,x0,30)
#    ax.plot(x0+0*zs,0*zs,zs,color='C3',alpha=0.2)
#ax.view_init(elev=10, azim=-60)
```

```
/Users/cch/anaconda36/anaconda/lib/python3.6/site-packages/matplotlib/colors.py:496: RuntimeWarning: in
valid value encountered in less
  cbook._putmask(xa, xa < 0.0, -1)
```



In [89]:
```
f=2*r*sqrt(4-r**2)

doubleInt3(f,[t,r],[0,pi/2],[0,2*sin(t)])
```

```
 pi/2    2*sin(t)
∫   d t ∫   2*r*sqrt(-r**2 + 4)  d r  =  16*(Integral(-sqrt(-sin(t)**2 + 1), (t, 0, pi/2)) + Integral(sq
rt(-sin(t)**2 + 1)*sin(t)**2, (t, 0, pi/2)) + Integral(1, (t, 0, pi/2)))/3
0        0
```

Out[89]:
```
16*(Integral(-sqrt(-sin(t)**2 + 1), (t, 0, pi/2)) + Integral(sqrt(-sin(t)**2 + 1)*sin(t)**2, (t, 0, pi/
2)) + Integral(1, (t, 0, pi/2)))/3
```

In [92]:
```
from sympy import simplify
pprint(simplify(integrate(f,(r,0,2*sin(t)))))
```

```
                            3/2
          ⎛       2        ⎞
    16⋅⎝- sin (t) + 1⎠      16
  - ─────────────────────── + ──
                3              3
```

Above equal to $\frac{16}{3}(1 - \cos^3\theta)$

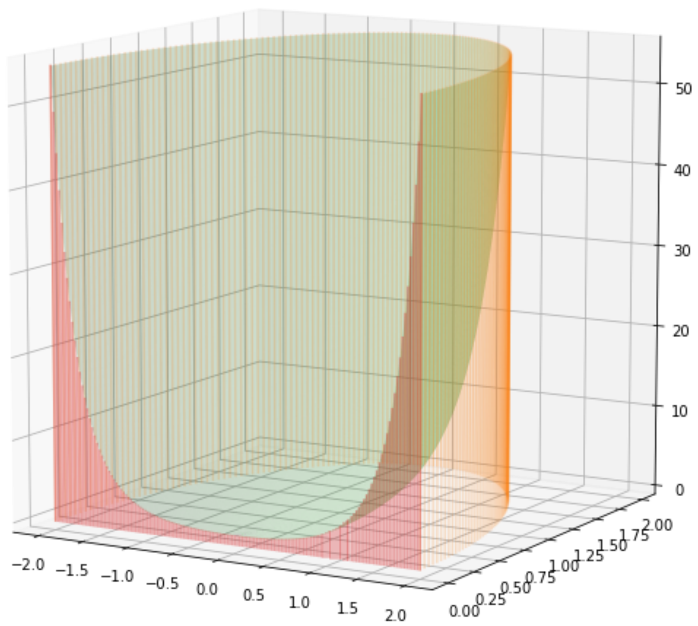
In [95]:
```
f1=16*(1-cos(t)**3)/3
integrate(f1,(t,0,pi/2))
```

Out[95]:
```
-32/9 + 8*pi/3
```

37.

$$\int_{-2}^{2} dx \int_{0}^{\sqrt{4-x^2}} e^{x^2+y^2} dy = \int_{0}^{\pi} d\theta \int_{0}^{2} re^{r^2} dr$$

In [73]:
```
from numpy import exp
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
#ax.set_aspect('equal','box')

r0 = np.linspace(0, 2, 50)
t0 = np.linspace(0, np.pi, 150)
#Zs =  2*X1
R,T=np.meshgrid(r0,t0)
func= exp(R**2)

#Res=np.where(2*sin(T)>=R,func,np.nan)
X=R*cos(T)
Y=R*sin(T)
ax.plot_surface(X,Y, func, rstride=1, cstride=1, color='C2',alpha=0.2)

#X00=np.linspace(0,3,100)
for ts in t0:
    zs=np.linspace(0,exp(4),100)
    xs=zs*0+2*cos(ts)
    ys=zs*0+2*sin(ts)
    ax.plot(xs,ys,zs,color='C1',alpha=0.2)

x0 = np.linspace(-2,2, 150)
for xs in x0:
    zs=np.linspace(0,exp(xs**2),100)
    ax.plot(xs+0*zs,0*zs,zs,color='C3',alpha=0.4)
#X0=np.linspace(0,2,100)
#for x0 in X0:
#    zs=np.linspace(0,x0,30)
#    ax.plot(x0+0*zs,0*zs,zs,color='C3',alpha=0.2)
ax.view_init(elev=10, azim=-60)
```



In [84]:
```
f=r*exp(r**2)

I=doubleInt3(f,[t,r],[0,pi],[0,2])
```

```
 pi      2
∫   d t ∫  r*exp(r**2)  d r  =  pi*(-1/2 + exp(4)/2)
0       0
```

Applications for Changing variables

In probability and statistic, the techniques of change of variables are usually used to find the probability density functions (abbr,. as p.d.f.) of new random variables.

Example

Suppose that one variable, x, is chosen randomly and uniformly from $[0, 1]$, and another variable, y, is also in such similar condition. What is the probability that $x \leqslant y$?

Sol: Let D the domain that $x \leqslant y$. Then The answer of this problem can be calculated by the following double integral:

$$\begin{aligned}\iint_D 1dxdy &= \iint_{\{0\leqslant x\leqslant y\leqslant 1\}} 1dxdy \\ &= \int_0^1 dy \int_0^y dx \\ &= \int_0^1 ydy \\ &= \frac{1}{2}\end{aligned}$$

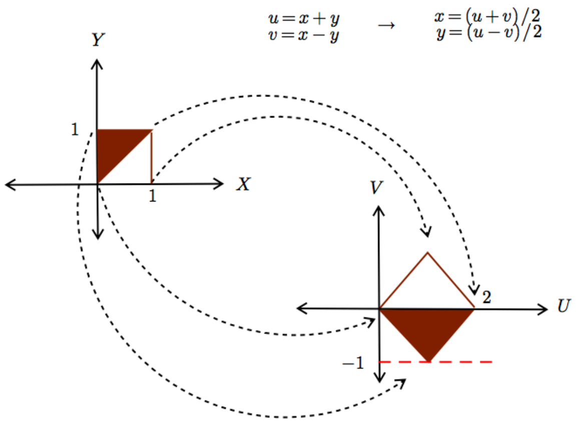
Another method is by changing variables from (x, y) to (u, v) where $u = x + y$ and $v = x - y$. In this case, the double integral has to be changed as:

1. variables change:

$$u = x + y, v = x - y \Rightarrow x = \frac{u+v}{2} \text{ and } y = \frac{u-v}{2}$$

$$|J| = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \frac{1}{2}$$

2. Domain change: reference the following picture



3. The double integral is:

$$\begin{aligned}\iint_D 1dxdy &= \int_{-1}^0 dv \int_{-v}^{2+v} \frac{1}{2}du \\ &= \frac{1}{2}\end{aligned}$$

Example

Change the following double integral in (X, Y) into (U, V):

$$\int_0^\infty \int_0^\infty \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} y^{\beta-1} e^{-x-y} dxdy, 0 \leqslant x, y \text{ and } 0 < u, v$$

where

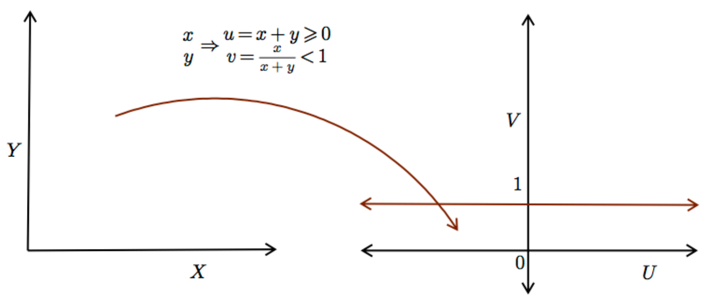
$$u = x + y \text{ and } v = \frac{x}{x+y}$$

Ans:

$$\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1}(1-v)^{\beta-1} dv \int_0^\infty \frac{1}{\Gamma(\alpha+\beta)} u^{\alpha+\beta-1} e^{-u} du$$

Note that

$$0 \leqslant x, y \Rightarrow 0 \leqslant u \text{ and } 0 \leqslant v < 1$$



Example

Change the following double integral in (X, Y) into (U, V):

$$\iint\limits_{\{0 \leqslant x,y\}} \frac{1}{4} e^{-\frac{x+y}{2}} dxdy$$

where $u = x + y$ and $v = y$.

Note that

$$\begin{aligned} \begin{matrix} 0 \leqslant x \\ 0 \leqslant y \end{matrix} &\Rightarrow \begin{matrix} 0 \leqslant x = u - v \\ 0 \leqslant y = v \end{matrix} \\ &\Rightarrow \begin{matrix} 0 \leqslant v \\ v \leqslant u \end{matrix} \end{aligned}$$

and

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right| = 1$$

then

$$\begin{aligned} \iint\limits_{\{0 \leqslant x,y\}} \frac{1}{4} e^{-\frac{x+y}{2}} dxdy &= \int_0^\infty du \int_0^u \frac{1}{4} e^{-u/2} dv \\ &= \int_0^\infty \frac{1}{4} u e^{-u/2} du \end{aligned}$$

i.e. sum of two independent χ_2^2 is χ_4^2.

In Monte-Carlo simulation, the data generating by normal density are usually used. But how can they be generated? The answer is very simple: they can be generated by the data comes from uniform distribution on $[0, 1]$.

Example (Monte-Carlo Simulation, Normal Data)

Change the following double integral in (X, Y) into (U, V):

$$\iint_{\{0<x,y<1\}} 1 dxdy$$

where $u = (-2\ln x)^{1/2}\cos 2\pi y$ and $v = (-2\ln x)^{1/2}\sin 2\pi y$.

1. Since $0 < x, y < 1$, we have

\begin{eqnarray*} \begin{array}{c}

```
    - 2 \ln x \in (0, \infty)\\
    2 \pi y \in (0, 2 \pi)
  \end{array} \begin{array}{c}
    \\
  \end{array} & \Rightarrow & \begin{array}{c}
    u, v \in \mathbb{R}
  \end{array}
\end{eqnarray*}
```

2. change the variable-pair, from (x, y) to (u, v):

$$\begin{array}{c} u = (-2\ln x)^{1/2}\cos 2\pi y \\ v = (-2\ln x)^{1/2}\sin 2\pi y \end{array} \Rightarrow u^2 + v^2 = -2\ln x, \frac{v}{u} = \tan 2\pi y$$
$$\Rightarrow x = \exp(-(u^2+v^2)/2), y = \frac{1}{2\pi}\tan^{-1}\frac{v}{u}$$

3. evaluate the Jacobian:

\begin{eqnarray*} J & = & \left| \left(\begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array}\right) \right|\ & = & \mathbf{\left| \left(\begin{array}{cc}

```
    - u e^{- (u^2 + v^2) / 2} & - v e^{- (u^2 + v^2) / 2}\\
    \frac{- v}{2 \pi (u^2 + v^2)} & \frac{- u}{2 \pi (u^2 + v^2)}
  \end{array}\right) \right|}\\
  & = & \frac{(u^2 + v^2)}{2 \pi (u^2 + v^2)} e^{- (u^2 + v^2) / 2}\\
  & = & \frac{1}{\sqrt{2 \pi}} e^{- u^2 / 2} \cdot \frac{1}{\sqrt{2 \pi}}
  e^{- v^2 / 2}
\end{eqnarray*}
```

4. change the double integral with (x, y)-pair to (u, v)-pair

$$\iint_{\{0<x,y<1\}} 1 dxdy = \iint_{\{(u,v)\in\mathbb{R}^2\}} J dudv$$
$$= \int_{\{u\in\mathbb{R}\}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \int_{\{v\in\mathbb{R}\}} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$$

this means that U, V are standard normal random variables and is independent, since integrand is in the form $f_U(u) g_V(v)$.

During the simulation, some few data in front are always to be discarded for the randomcy.

Example (t-distribution data)

Change the following double integral in (X, Y) into (T, V):

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \int_0^{\infty} \frac{y^{r/2-1} e^{-y/2}}{\Gamma(r/2) 2^{r/2}} dy$$

where

$$t = \frac{x}{\sqrt{\frac{y}{r}}} \text{ and } v = y$$

Moreover, we have

$$\begin{aligned} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{y^{r/2-1} e^{-y/2}}{\Gamma(r/2) 2^{r/2}} dy &= \frac{\Gamma((r+1)/2)}{\sqrt{\pi r}\Gamma(r/2)} \frac{1}{(1+u^2/2)^{(1+r)/2}} \\ &= f_T(t) \text{ where } -\infty < t < \infty \end{aligned}$$

This is called the p.d.f of t-distribution.

Example (F-distribution data)

Change the following double integral in (X, Y) into (F, V)

$$\iint_{\{0<x,y\}} \frac{x^{r/2-1} y^{s/2-1} e^{-(x+y)/2}}{\Gamma(r/2)\Gamma(s/2) 2^{(r+s)/2}} dxdy$$

where

$$f = \frac{x/r}{y/s} \text{ and } v = y$$

Moreover, we have

$$\begin{aligned} \int_{\{0<y\}} \frac{x^{r/2-1} y^{s/2-1} e^{-(x+y)/2}}{\Gamma(r/2)\Gamma(s/2) 2^{(r+s)/2}} dxdy &= f_F(f) \\ &= \frac{\Gamma((r+s)/2)(r/s)^{r/2}}{\Gamma(r/2)\Gamma(s/2)(1+rf/s)^{(r+s)/2}} f^{r/2-1}, \text{ where } 0 < f \end{aligned}$$

Exercise

Suppose that one variable, x, is chosen randomly and uniformly from $[0, 1]$, and another variable, y, is also in such similar condition. What is the probability that $x \leqslant 2y$, i.e. the value of x is less than twice of value of y?

This case is evaluated as follows:

$$\begin{aligned}\wp(0 \leqslant x \leqslant 2y \leqslant 1) &= \iint_D 1dxdy \\ &= \int_0^1 dx \int_{y/2}^1 1dx \\ &= \int_0^1 (1 - y/2)dy \\ &= 3/4\end{aligned}$$

Triple Integrals

Similar to last section, we can consider the multiple integrations for functions with three variables. If $w = f(x, y, z)$ is continuous and $f(x, y, z)$ is nonnegative for all (x, y, z) in a solid region R of subset in $\mathbb{R}^3$, then the triple integral of $f(x, y, z)$ and above $X - Y$ over R is defined as

$$\iiint_R f(x, y, z)dV = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$$

where $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$, $\Delta *_i$ being the length of the partition subinterval in each direction respectively, is the element of volume and $\|\Delta\|$ is the longest length among $\Delta *_i$'s. Fubini's theorem can be used to evaluate the triple integrals:

Theorem (Fibini's Theorem)

If $f(x, y, z)$ is continuous over V and

$$R=\{(x, y, z)| a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x), h_1(x, y) \leqslant z \leqslant h_2(x, y)\},$$

then

$$\iiint\limits_{\mathbf{R}} \mathbf{f(x, y, z)dV} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{dx} \int_{\mathbf{g_1(x)}}^{\mathbf{g_2(x)}} \mathbf{dy} \int_{\mathbf{h_1(x,y)}}^{\mathbf{h_2(x,y)}} \mathbf{f(x, y, z)dz}$$

Certainly, the order of integrations can be changed as double integrals if necessary. Note that if $f(x, y, z) \equiv 1$ then the value of triple integral is equal to the volume of R.

Example

Evaluate the following triple integral

$$\iiint\limits_{-1\leq x\leq 1, 0\leq y\leq 3, 1\leq z\leq 2} \mathbf{(x^2y + yz^2)dV = 24}$$

In [12]:
```
x,y,z = symbols("x y z")
I=tripleInt3(x**2*y+y*z**2,[x,y,z],[-1,1],[0,3],[1,2])
```

```
  1       3        2
 ∫   d x ∫   d y ∫  x**2*y + y*z**2   d z  =  24
 -1       0        1
```

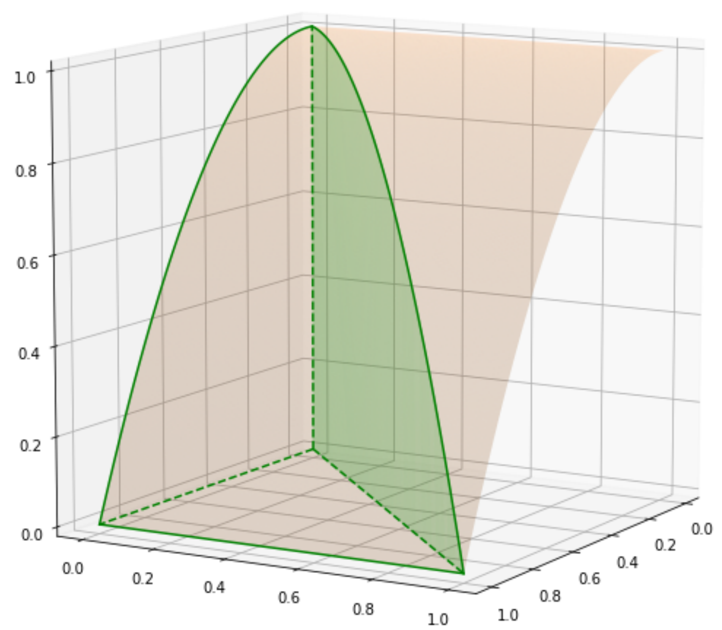
Example

Evaluate the following triple integral

$$\iiint_{T} z\,dV = \frac{1}{12}$$

where T is the solid in the first octant and bounded by $z = 1 - x^2$ and $y = x$.

In [82]:
```
from numpy import sqrt,exp,sin,cos
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
ax.set_aspect('equal','box')

x0 = np.linspace(0, 1, 100)
y0 = np.linspace(0, 1, 100)
X,Y=np.meshgrid(x0,y0)
func= 1-X*X

ax.plot_surface(X,Y, func, rstride=1, cstride=1, color='C1',alpha=0.2)
t0 = np.linspace(0,1, 100)
for xs in t0:
    zs=np.linspace(0,1-xs**2,30)
    ax.plot(xs+0*zs,xs+0*zs,zs,color='C2',alpha=0.2)
    #ax.plot(xs+0*zs,0*zs,zs,color='C2',alpha=0.4)
ax.plot(x0,x0,1-x0*x0,color='g')
ax.plot(x0,0*x0,1-x0*x0,color='g')
ax.plot(1+0*x0,x0,0*x0,color='g')
ax.plot(x0,0*x0,0*x0,'g--')
ax.plot(x0,x0,0*x0,'g--')
ax.plot(0*x0,0*x0,x0,'g--')
ax.view_init(elev=10, azim=30)
```



In [14]:
```
x,y,z = symbols("x y z")
I=tripleInt3(z,[x,y,z],[0,1],[0,x],[0,1-x**2])
```

```
 1         x          -x**2 + 1
 ∫   d x ∫    d y ∫   z    d z  =  1/12
 0         0          0
```

Example

Evaluate the following triple integral

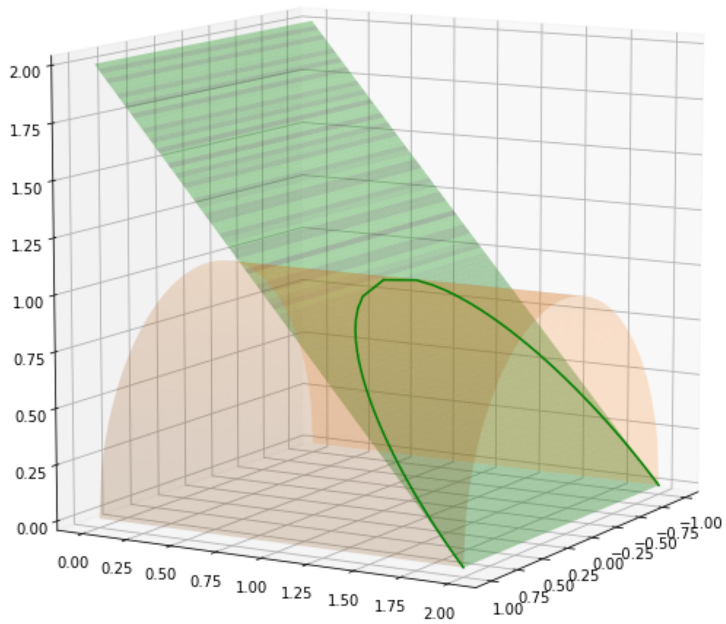
$$\iiint\limits_{\mathbf{T}} \sqrt{\mathbf{x}^2 + \mathbf{z}^2}\,\mathbf{dV} = \frac{\mathbf{4\pi}}{\mathbf{3}}$$

where T is the solid, bounded by $x^2 + z^2 = 1, y + z = 2$ and $y = 0$.

In this case, we seperate the triple integral into 2 part, single-variable integral of y, from 0 to $2 - z$, the other, double integral for x, z, in $R = \{x^2 + z^2 \leq 1\}$; use integration in polor coordinate to integrate the latter integral:

$$\iiint\limits_{\mathbf{T}} \sqrt{\mathbf{x}^2 + \mathbf{z}^2}\,\mathbf{dV} = \iint\limits_{\mathbf{x^2+z^2\leq 1}} \mathbf{dA} \int_{\mathbf{0}}^{\mathbf{2-z}} \sqrt{\mathbf{x}^2 + \mathbf{z}^2}\,\mathbf{dy}$$

In [35]:
```
from numpy import sqrt,exp,sin,cos,pi
fig = plt.figure(figsize=(8,8))

ax = Axes3D(fig)
ax.set_aspect('equal','box')

x0 = np.linspace(-1, 1, 100)
y0 = np.linspace(0, 2, 100)
X,Y=np.meshgrid(x0,y0)

ax.plot_surface(X,Y, sqrt(1-X**2), rstride=1, cstride=1, color='C1',alpha=0.2)
ax.plot_surface(X,Y, 2-Y, rstride=1, cstride=1, color='C2',alpha=0.4)

z0=np.linspace(0,1,30)
ax.plot(sqrt(1-z0*z0),2-z0,z0,color="g")
ax.plot(-sqrt(1-z0*z0),2-z0,z0,color="g")
#t0 = np.linspace(0,1, 100)
#for xs in t0:
#    zs=np.linspace(0,1-xs**2,30)
#    ax.plot(xs+0*zs,zs,zs,color='C2',alpha=0.4)
#    ax.plot(xs+0*zs,0*zs,zs,color='C2',alpha=0.4)

ax.view_init(elev=10, azim=30)
```



In [ ]:
```
# not solable
x,y,z = symbols("x y z")
I=tripleInt3(sqrt(x**2+z**2),[x,z,y],[-1,1],[-sqrt(1-x**2),sqrt(1-x**2)],[0,2-z])
```

In [17]:
```
r,t,z = symbols("r theta z")
I=tripleInt3(r*r,[r,t,y],[0,1],[0,2*pi],[0,2-r*sin(t)])
```

```
 1        2*pi     -r*sin(theta) + 2
 ∫   d r ∫    d theta ∫   r**2   d y  =  4*pi/3
 0        0         0
```

While $f(x, y, z) = 1$, the triple integral is the volume of T which is the domain of $f(x, y, z)$:

$$\iiint_{\mathbf{R}} \mathbf{1dV = volume(T)}$$

Example

If $R = \{(x, y) | 1 \leqslant x \leqslant 3, 2 \leqslant y \leqslant 4, 0 \leqslant z \leqslant 2\}$, then

$$\begin{aligned} \iiint_R 1 dV &= \int_2^4 dy \int_1^3 dx \int_0^2 dz \\ &= 2 \cdot 2 \cdot 2 \end{aligned}$$

This result is equal to the volume of cubic solid.

Example

Evaluate the triple integral

$$\begin{aligned} & \int_1^2 \int_x^{x^2} \int_0^{x+y} (x+1)(y+z) dV \\ = & \int_1^2 dx \int_x^{x^2} (x+1) \left(\frac{3y^2 + 4xy + x^2}{2} \right) dy \\ = & \int_1^2 (x+1) \cdot \frac{x^6 + 2x^5 + x^4 - 4x^3}{2} dx \\ = & \frac{23577}{560} \end{aligned}$$

Example

Suppose that The solid region R is given by

$$R=\left\{(x,y,z)\in\mathbb{R}^3\middle|0\leqslant x\leqslant\sqrt{\frac{\pi}{2}},x\leqslant y\leqslant\sqrt{\frac{\pi}{2}},0\leqslant z\leqslant 2\right\}$$

Evaluate the triple integral

$$\iiint\limits_R \sin(y^2)dV$$

Sol:

As mentioned in the section of integration technique, $\sin(y^2)$ can not be integrated directly by any method. Therefore we have to arrange the orders of integration carefully. Note that

$$R=\left\{(x,y,z)\in\mathbb{R}^3\middle|\,0\leqslant y\leqslant\sqrt{\frac{\pi}{2}},0\leq x\leq y,0\leqslant z\leqslant 2\right\}$$

then by Fubini's theorem, the triple integral is evaluated as:

$$\begin{aligned}
&\iiint_R \sin(y^2)dV\\
=&\int_0^{\sqrt{\frac{\pi}{2}}}dy\int_0^y dx\int_0^2 \sin(y^2)dz\\
=&\int_0^{\sqrt{\frac{\pi}{2}}}2\cdot y\cdot\sin(y^2)dy\\
=&\quad 1
\end{aligned}$$

In [15]:
```
from sympy import symbols,pprint,integrate,pi,sqrt,sin,cos,diff

def tripleInt(f,X,xr,yr,zr):
    Iz=integrate(f,[X[2],zr[0],zr[1]])
    Iy=integrate(Iz,[X[1],yr[0],yr[1]])
    Ix=integrate(Iy,[X[0],xr[0],xr[1]])
    return Ix
```

In [ ]:
```
x,y,z =symbols('x y z')
tripleInt(sin(y**2),[y,x,z],[0,sqrt(pi/2)],[0,y],[0,2])
```

In [ ]:
```
def tripleInt2(f,X,xr,yr,zr):
    Iz=integrate(f,[X[2],zr[0],zr[1]])
    Iy=integrate(Iz,[X[1],yr[0],yr[1]])
    Ix=integrate(Iy,[X[0],xr[0],xr[1]])
    print("  %s \t  %s  \t  %s" %(xr[1],yr[1],zr[1]))
    print("∫   d",R+'{}'.format(X[0]),
        K+"∫   d",R+'{}'.format(X[1]),
        K+"∫ ",B+"{}".format(f),K+" d",R+"{}".format(X[2]),K+" = ","{}".format(Ix))
    print(" %s \t %s  \t %s" %(xr[0],yr[0],zr[0]))
    return Ix
```

In [16]:
```
def tripleInt3(f,X,xr,yr,zr):
    Iz=integrate(f,[X[2],zr[0],zr[1]])
    Iy=integrate(Iz,[X[1],yr[0],yr[1]])
    Ix=integrate(Iy,[X[0],xr[0],xr[1]])
    zrs=' '+str(zr[1])
    yrs=str(yr[1])
    xrs=' '+str(xr[1])
    print(xrs.ljust(9,' ')+O+yrs.ljust(7,' ')+K+zrs)
    #print("  %s \t  %s  \t  %s" %(xr[1],yr[1],zr[1]))
    print("∫   d",R+'{}'.format(X[0]),
        K+"∫   d",R+'{}'.format(X[1]),
        K+"∫ ",B+"{}".format(f),K+"  d",R+"{}".format(X[2]),K+" = ","{}".format(Ix))
    zrs1=' '+str(zr[0])
    yrs1=str(yr[0])
    xrs1=' '+str(xr[0])
    print(xrs1.ljust(8,' ')+O+yrs1.ljust(7,' ')+K+zrs1)
    #print(" %s \t %s  \t %s" %(xr[0],yr[0],zr[0]))
    return Ix
```

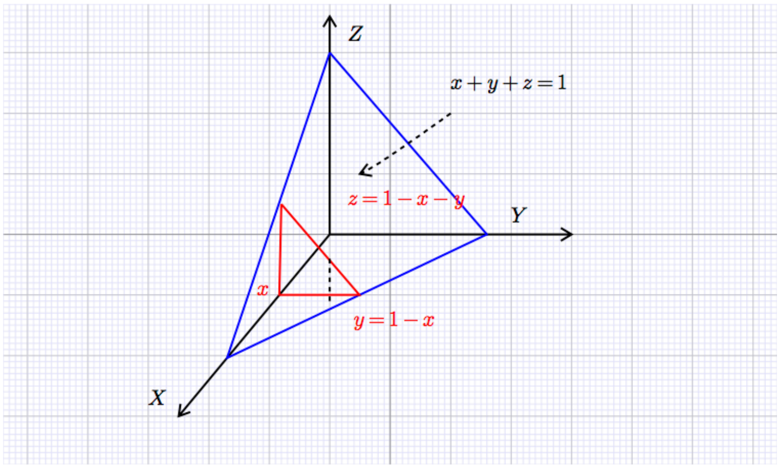
In [ ]:
```
I=tripleInt3(x**2*y,[x,y,z],[0,2],[0,2],[0,2])
```

Example

Evaluate the following triple integral:

$$\iiint_V \frac{dV}{(1+x+y+z)^{3/2}}$$

where V is the domain bounded by the plane, $x+y+z=1$, in the first octant.



$$\begin{aligned}
I &= \iiint\limits_V \frac{dV}{(1+x+y+z)^{3/2}} \\
&= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(1+x+y+z)^{3/2}} \\
&= \int_0^1 dx \int_0^{1-x} \left.\frac{-2}{(1+x+y+z)^{1/2}}\right|_{z=0}^{z=1-x-y} dy \\
&= \int_0^1 dx \int_0^{1-x} \left(\frac{2}{(1+x+y)^{1/2}} - \sqrt{2}\right) dy \\
&= \int_0^1 4\sqrt{1+x+y} - \sqrt{2}y\Big|_{y=0}^{y=1-x} dx \\
&= \int_0^1 (4\sqrt{2} - \sqrt{2}(1-x) - 4\sqrt{1+x})dx \\
&= 4\sqrt{2} - \frac{\sqrt{2}}{2} - \frac{8}{3}(2^{3/2}-1) = \frac{8}{3} - \frac{11}{6}\sqrt{2}
\end{aligned}$$

In [ ]:
```
I=tripleInt2(x**2*y,[x,y,z],[0,2],[0,2],[0,2])
```

In [ ]:
```
I=tripleInt2(sqrt(1+x+y+z)**(-3),[x,y,z],[0,1],[0,1-x],[0,1-y-x])
```

Exercise

Evaluate the following triple integrals:

1. $\iiint\limits_V x^2 y dV$ where $V = \{(x,y,z) \in \mathbb{R}^3 | 0 \leqslant x, y, z \leqslant 2\}$;
2. $\iiint\limits_V x^2 y dV$ where $V = \{(x,y,z) \in \mathbb{R}^3 | 0 \leqslant x \leqslant y \leqslant z \leqslant 2\}$;
3. $\iiint\limits_V \frac{y}{x} dV$ where $V = \{(x,y,z) \in \mathbb{R}^3 | 1 \leqslant x \leqslant y \leqslant z \leqslant 2\}$;

In [ ]:
```
tripleInt3(x*x*y,[x,y,z],[0,2],[x,2],[y,2])
```

In [ ]:
```
tripleInt2(x**2*y,[z,y,x],[0,2],[0,z],[0,y])
```

In [ ]:
```
tripleInt2(y/x,[z,y,x],[1,2],[1,z],[1,y])
```

p.1199 Exercise

In [18]:
```
# 6
tripleInt3(2*x*z,[z,y,x],[0,1],[0,z],[0,y])
```

```
 1       z       y
∫   d z ∫   d y ∫  2*x*z   d x  =  1/15
 0       0       0
```

Out[18]: 1/15

In [20]:
```
#10
from sympy import exp,log
tripleInt3(2*log(y),[x,y,z],[1,exp(1)],[1,x],[0,1/x/y])
```

```
 E       x       1/(x*y)
∫   d x ∫   d y ∫  2*log(y)   d z  =  1/3
 1       1       0
```

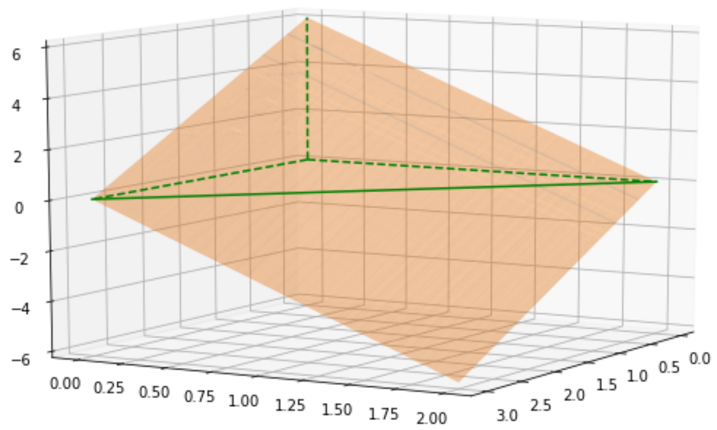
Out[20]: 1/3

In [21]:
```
# 16 volume of T bounded by x=y=z=0,2x+3y+z=6
tripleInt3(1,[x,y,z],[0,3],[0,2-2*x/3],[0,6-2*x-3*y])
```

```
 3       -2*x/3 + 2 -2*x - 3*y + 6
∫   d x ∫   d y ∫  1   d z  =  6
 0       0       0
```

Out[21]: 6

In [47]:
```
from numpy import sqrt,exp,sin,cos
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
ax.set_aspect('equal','box')

x0 = np.linspace(0, 3, 100)
y0 = np.linspace(0, 2, 100)
X,Y=np.meshgrid(x0,y0)
func= 6-2*X-3*Y

ax.plot_surface(X,Y, func, rstride=1, cstride=1, color='C1',alpha=0.4)

ax.plot(x0,2-2*x0/3, 0*x0,  color='g')
ax.plot(0*x0,0*x0, 2*x0,'g--')
ax.plot(x0,0*x0, 0*x0, 'g--')
ax.plot(0*x0,y0, 0*x0, 'g--')
ax.view_init(elev=10, azim=30)
```

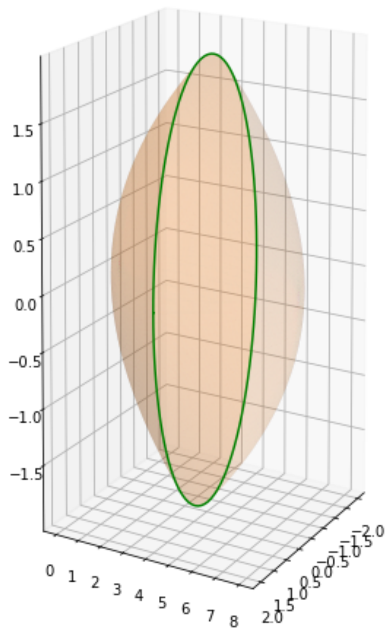


In [23]:
```
# 22  f(x,y,z)=sqrt(x^2+z^2) on T bounded by y=x^2+z^2, y=8-x^2-z^2
tripleInt3(r*r,[r,t,y],[0,2],[0,2*pi],[r,8-r**2])
```

```
 2       2*pi    -r**2 + 8
∫   d r ∫   d theta ∫  r**2    d y   =   328*pi/15
 0       0        r
```

Out[23]: 328*pi/15

In [55]:
```
from numpy import sqrt,exp,sin,cos
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
ax.set_aspect('equal','box')

x0 = np.linspace(-2, 2, 100)
z0 = np.linspace(-2, 2, 100)
X,Z=np.meshgrid(x0,z0)
r0 = np.linspace(0, 2, 100)
t0 = np.linspace(0, 2*pi, 100)
R,T=np.meshgrid(r0,t0)
#func1= X*X+Z*Z
#func2=8-X*X-Z*Z
#ax.plot_surface(X,func1,Z, rstride=1, cstride=1, color='C1',alpha=0.2)
#ax.plot_surface(X,func2,Z, rstride=1, cstride=1, color='C2',alpha=0.2)
func1=R*R
func2=8-R*R
ax.plot_surface(R*cos(T),func1,R*sin(T), rstride=1, cstride=1, color='C1',alpha=0.2)
ax.plot_surface(R*cos(T),func2,R*sin(T), rstride=1, cstride=1, color='C1',alpha=0.1)


r0 = np.linspace(0, 2, 100)
t0 = np.linspace(0, 2*pi, 100)
ax.plot(2*cos(t0),4+0*t0,2*sin(t0),'g')
ax.view_init(elev=10, azim=30)
```

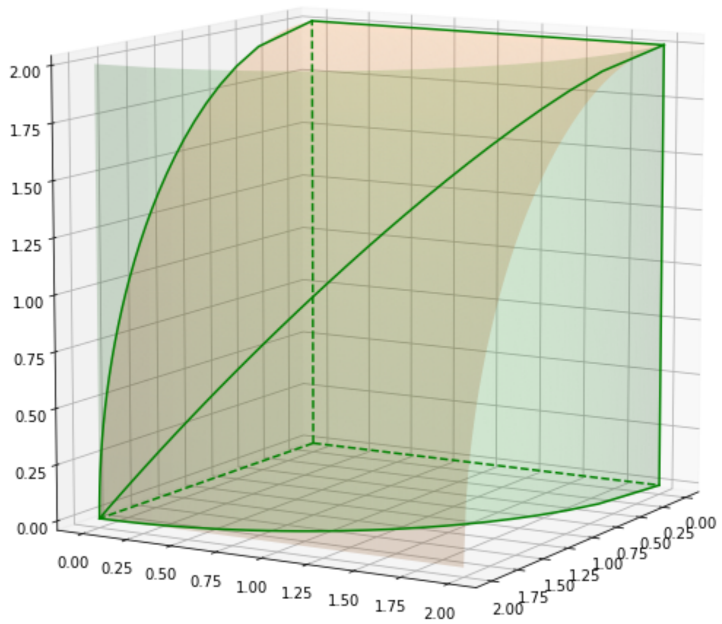


28, T: bounded by $x^2 + z^2 = 4$ and $y^2 + z^2 = 4$.

Move the cross-session, $A(z)$, parallell to $X - Y$ plane, along Z-axis vertically; it implies $0 \leq z \leq 2$ and the area of $A(z)$, is $(2\sqrt{(4-z^2)})$. Therefor the volume of solid is:

$$2\int_0^2 \left(2\sqrt{4-z^2}\right)^2 dz = \frac{1024}{3}$$

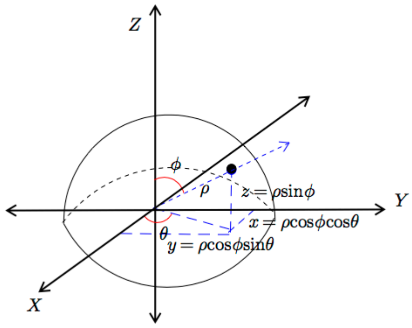
In [69]:
```
from numpy import sqrt,exp,sin,cos
fig = plt.figure(figsize=(8,8))
ax = Axes3D(fig)
ax.set_aspect('equal','box')

x0 = np.linspace(0, 2, 100)
z0 = np.linspace(0, 2, 100)
y0 = np.linspace(0, 2, 100)

X,Y=np.meshgrid(x0,y0)
r0 = np.linspace(0, 2, 100)
t0 = np.linspace(0, pi/2, 100)
R,T=np.meshgrid(r0,t0)
func=sqrt(4-X*X)

ax.plot_surface(X,Y,func, rstride=1, cstride=1, color='C1',alpha=0.2)#ax.plot_surface(X,func2,Z rstride=1
, cstride=1, color='C2',alpha=0.2)

ax.plot_surface(X,func,Y, rstride=1, cstride=1, color='C2',alpha=0.2)

t0 = np.linspace(0, 2, 30)
ax.plot(sqrt(4-t0*t0),t0,t0,'g')
ax.plot(sqrt(4-t0*t0),0*t0,t0,'g')
ax.plot(sqrt(4-t0*t0),t0,0*t0,'g')
ax.plot(0*t0,0*t0,t0,'g--')
ax.plot(0*t0,t0,0*t0,'g--')
ax.plot(t0,0*t0,0*t0,'g--')
ax.plot(0*t0,t0,2+0*t0,'g')
ax.plot(0*t0,2+0*t0,t0,'g')
ax.view_init(elev=10, azim=30)
```



Triple integrals in other coordinates

Recall that the relations between Cartesian coordinates, (x, y, z), and cylindrical coordinates, (r, θ, z), are given by:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}
$$



And the relations between Cartesian coordinates, (x, y, z), and cylindrical coordinates{\index{cylindrical coordinates}}, (ρ, θ, ϕ), are given by:

$$
\begin{aligned}
x &= \rho\cos\theta\cos\phi \\
y &= \rho\sin\theta\cos\phi \\
z &= \rho\sin\phi
\end{aligned}
$$

Since the Jacobian matrix, J, between two different coordinates is defined as

$$\frac{\partial(x^i)}{\partial(u^j)} = \left(\frac{\partial x^i}{\partial u^j}\right)_{i,j}$$

we have the following \ integration rules:

Theorem

$$\iiint\limits_{\mathbf{R}} \mathbf{f(x, y, z)dV} = \iiint\limits_{\mathbf{R}} \mathbf{f(x(u, v, w), y(u, v, w), z(u, v, w))|J|dudvdw}$$

where $|J|$ is the absolute value of determinant of J. In cylindrical coordinates, we have

$$\iiint\limits_{\mathbf{R}} \mathbf{f(x, y, z)dV} = \iiint\limits_{\mathbf{R}} \mathbf{f(r}\cos\theta\mathbf{, r}\sin\theta\mathbf{, z)rdrd}\theta\mathbf{dz}$$

In spherical coordinates{\index{spherical coordinates}}, we have

$$\iiint\limits_{\mathbf{R}} \mathbf{f(x, y, z)dV} = \iiint\limits_{\mathbf{R}} \mathbf{f(}\rho\cos\theta\cos\phi\mathbf{,}\,\rho\sin\theta\cos\phi\mathbf{,}\,\rho\sin\phi\mathbf{)}\rho^2\sin\phi\mathbf{d}\rho\mathbf{d}\theta\mathbf{d}\phi$$

Example

Evaluate triple integral of $f(x, y, z) = \sqrt{x^2 + y^2}$ on the T, bounded by $z = \sqrt{x^2 + y^2}$ and $z = 2$.

The solid region can be represented in cylindrical coordinates as:

$$r \leqslant z \leqslant 2$$
$$0 \leqslant r \leqslant 2$$
$$0 \leqslant \theta \leqslant 2\pi$$

Then the volume of the solid is equal to

$$\iiint_R \sqrt{x^2 + y^2} dV$$
$$= \int_0^2 r dr \int_0^{2\pi} d\theta \int_r^2 r dz$$
$$= \frac{8\pi}{3}$$

In [25]:
```
tripleInt3(r**2,[r,t,z],[0,2],[0,2*pi],[r,2])
```

```
 2        2*pi    2
∫   d r ∫    d theta ∫  r**2   d z  =  8*pi/3
 0      0        r
```

Out[25]: `8*pi/3`

Example

Volume of hemisphere with radius a is $\frac{2}{3}\pi a^3$.

In [26]:
```
a=symbols("a")
tripleInt3(r,[r,t,z],[0,a],[0,2*pi],[0,sqrt(a**2-r**2)])
```

```
 a        2*pi    sqrt(a**2 - r**2)
∫   d r ∫    d theta ∫  r   d z  =  2*pi*a**2*sqrt(a**2)/3
 0      0        0
```

Out[26]: `2*pi*a**2*sqrt(a**2)/3`

In [29]:
```
r,t,p=symbols("r theta phi")
tripleInt3(r**2*sin(p),[r,t,p],[0,a],[0,2*pi],[0,pi/2])
```

```
 a        2*pi    pi/2
∫   d r ∫    d theta ∫  r**2*sin(phi)   d phi  =  2*pi*a**3/3
 0      0        0
```

Out[29]: `2*pi*a**3/3`

Example

Evaluate the triple integral:

$$\iiint_{T=\{x^2+y^2+z^2\leq 1, x,y,z\geq 0\}} x dV = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \int_0^1 \rho\cos\theta\sin\phi\cdot\rho^2\sin\phi d\rho = \frac{\pi}{16}$$

In [19]:
```
r,t,p=symbols("rho theta phi")
tripleInt3(r**3*cos(t)*sin(p)*sin(p),[r,t,p],[0,1],[0,pi/2],[0,pi/2])
```

```
 1        pi/2    pi/2
∫   d rho ∫   d theta ∫  rho**3*sin(phi)**2*cos(theta)   d phi  =  pi/16
 0      0       0
```

Out[19]: pi/16

Example

Evaluate the triple integral:

$$\iiint_{T=\{\sqrt{x^2+y^2}\leq z\leq x^2+y^2+z^2\}} 1dV = \int_0^{\pi/4} d\phi \int_0^{2\pi} d\theta \int_0^{\cos\phi} \rho^2 \sin\phi d\rho = \frac{\pi}{8}$$

In [20]:
```
r,t,p=symbols("rho theta phi")
tripleInt3(r**2*sin(p),[p,t,r],[0,pi/4],[0,2*pi],[0,cos(p)])
```

```
 pi/4     2*pi    cos(phi)
∫   d phi ∫   d theta ∫  rho**2*sin(phi)   d rho  =  pi/8
 0      0       0
```

Out[20]: pi/8

Example

Find the volume of the solid bounded by $z = x^2 + y^2$ and $z = 4$.

The solid region can be represented in cylindrical coordinates as:

$$\begin{gathered} 0 \leqslant z \leqslant 4 \\ 0 \leqslant r \leqslant \sqrt{z} \\ 0 \leqslant \theta \leqslant 2\pi \end{gathered}$$

Then the volume of the solid is equal to

$$\begin{aligned} & \iiint_R 1dV \\ = & \int_0^4 dz \int_0^{\sqrt{z}} rdr \int_0^{2\pi} d\theta \\ = & \int_0^4 \pi z dz \\ = & 8\pi \end{aligned}$$

In [ ]:
```
r,t,p=symbols("r t p")
tripleInt3(r,[z,t,r],[0,4],[0,2*pi],[0,sqrt(z)])
```

Example

Find the volume of the solid bounded by $z^2 = x^2 + y^2, x^2 + y^2 + z^2 = 4$ and above $X - Y$ plane.

Sol:

In spherical coordinates, the solid is represented as

$$
\begin{aligned}
0 \leqslant \rho \leqslant 2 \\
0 \leqslant \phi \leqslant \tfrac{\pi}{4} \\
0 \leqslant \theta \leqslant 2\pi
\end{aligned}
$$

Then the volume of the solid is equal to

$$
\begin{aligned}
& \iiint_R 1 dV \\
= & \int_0^2 \rho^2 d\rho \int_0^{\frac{\pi}{4}} \sin \phi d\phi \int_0^{2\pi} d\theta \\
= & \quad 2\pi \int_0^2 \rho^2 (1 - \frac{1}{\sqrt{2}}) d\rho \\
= & \quad \frac{16}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\pi
\end{aligned}
$$

In [ ]:
```
I=tripleInt3(r*r*sin(p),[r,p,t],[0,2],[0,pi/4],[0,2*pi])
```

Exercise, p.1207

#6

$$\iiint_{T=\{x^2+y^2\leq 4, 0\leq z\leq 4\}} \exp(x^2+y^2) dV = \int_0^{2\pi} d\theta \int_0^2 dr \int_0^4 r\exp(r^2) dz = 4\pi(\exp(4)-1)$$

In [7]:
```
from sympy import exp
r,t,z=symbols("r theta z")
I=tripleInt3(r*exp(r*r),[r,t,z],[0,2],[0,2*pi],[0,4])
```

```
 2        2*pi    4
∫   d r ∫   d theta ∫  r*exp(r**2)   d z  =  -4*pi + 4*pi*exp(4)
 0       0        0
```

#10

$$\iiint_{T=\{x^2+y^2\leq 1, 0\leq z\leq 2x^2+2y^2\}} y^2 dV = \int_0^{2\pi} d\theta \int_0^1 dr \int_0^{2r^2} r^2 \sin^2 \cdot r\theta dz = \frac{\pi}{3}$$

In [11]:
```
I=tripleInt3(r*r*r*sin(t)**2,[r,t,z],[0,1],[0,2*pi],[0,2*r**2])
```

```
 1        2*pi    2*r**2
∫   d r ∫   d theta ∫  r**3*sin(theta)**2   d z  =  pi/3
 0       0        0
```

#20

$$\iiint\limits_{T=\{x^2+y^2+z^2\leq 1, 0\leq z,x,y\}} \exp(x^2+y^2+z^2)^{3/2}dV = \int_0^1 d\rho \int_0^{\pi/2} d\theta \int_0^{\pi/2} \rho^2 \exp(\rho^3)\sin\phi d\phi = \pi\frac{e-1}{6}$$

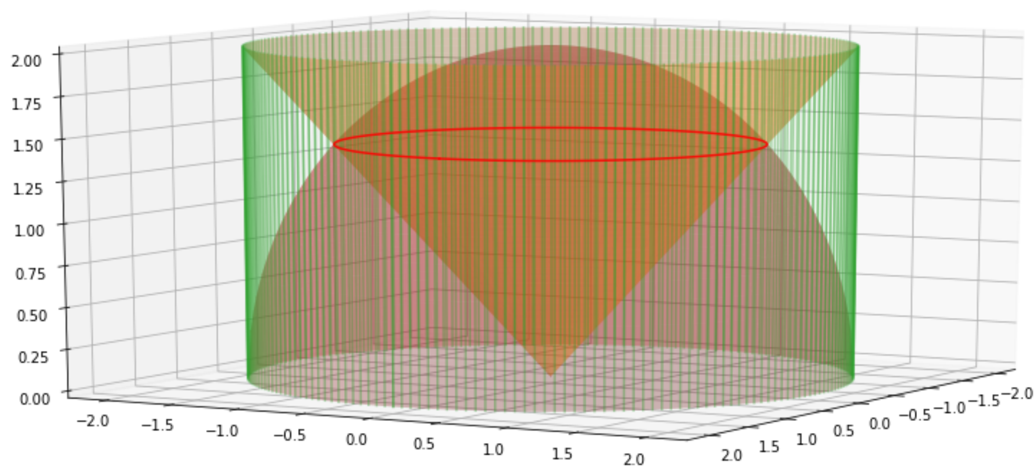
In [28]:
```
from sympy import exp
r=symbols("rho")
I=tripleInt3(r**2*sin(p)*exp(r**3),[r,t,p],[0,1],[0,pi/2],[0,pi/2])
```

```
 1        pi/2    pi/2
∫   d rho ∫   d theta ∫  rho**2*exp(rho**3)*sin(phi)   d phi  =  -pi/6 + E*pi/6
 0       0       0
```

#24 T is the solid bounded above by $x^2+y^2+z^2=4$ and bounded below $z=\sqrt{x^2+y^2}$:

$$\iiint\limits_T zdV = \int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} d\phi \int_2^{2/\sin\phi} \rho^3 \sin\phi\cos\phi d\phi = 2\pi$$

In [53]:
```
fig = plt.figure(figsize=(12,6))
ax = Axes3D(fig)
r0 = np.linspace(0, 2, 100)
t0 = np.linspace(0, 2*np.pi, 100)
r0,t0=np.meshgrid(r0,t0)
Xr= r0*np.cos(t0)
Yr= r0*np.sin(t0)
func=np.sqrt(4-r0*r0)
cone=np.sqrt(r0*r0)
ax.plot_surface(Xr,Yr, func, color="C3",alpha=0.3);
ax.plot_surface(Xr,Yr, cone, color="C1",alpha=0.3);


t000=np.linspace(0, 2*np.pi, 200)
for t00 in t000:
    zs=np.linspace(0,2,150)
    ax.plot(2*np.cos(t00)+0*zs,2*np.sin(t00)+0*zs,zs,color='C2',alpha=0.4)

x0=  np.linspace(0, 2*np.pi, 100)
ax.plot(np.sqrt(2)*np.cos(x0),np.sqrt(2)*np.sin(x0),np.sqrt(2)+0*x0,'red',alpha=0.9);
ax.view_init(elev=10, azim=30)
#plot3d(X,Y,0*func)
```



```
In [33]: I=tripleInt3(r**3*sin(p)*cos(p),[t,p,r],[0,2*pi],[pi/4,pi/2],[2,2/sin(p)])
```

```
 2*pi     pi/2     2/sin(phi)
∫    d theta ∫    d phi ∫   rho**3*sin(phi)*cos(phi)    d rho   =   2*pi
 0      pi/4     2
```

Exercise

Resolve the last problem with the cylindrical coordinates. **Sol:** Since the points in the solid are satisfied the following inequalities:

$$\begin{aligned} z^2 &\leqslant x^2 + y^2 \\ x^2 + y^2 + z^2 &\leqslant 4 \\ z &\geqslant 0 \end{aligned}$$

the ranges for \ (r, θ, z) are:

$$\begin{aligned} r \leqslant z &\leqslant \sqrt{4 - r^2} \\ 0 \leqslant r &\leqslant \sqrt{2} \\ 0 \leqslant \theta &\leqslant 2\pi \end{aligned}$$

Therefore the volume is

$$\begin{aligned} &\iiint_R 1dV \\ = &\int_0^{\sqrt{2}} rdr \int_0^{2\pi} d\theta \int_r^{\sqrt{4-r^2}} dz \\ = &\ 2\pi \int_0^{\sqrt{2}} (\sqrt{4 - r^2} - r)rdr \\ = &\ \frac{16}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\pi \end{aligned}$$

}

Exercise

Evaluate the following triple integrals:

1.

$$ \int^2{- 2} \int^{\sqrt{4 - x^2}}{- \sqrt{4 - x^2}} \int_0^{\sqrt{4

```
    - x^2 - y^2}} (y^2 + z^2) z d z d y d x $$
Hint: the domain is half upper ball.
```

2.

$$\int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{2x^2+y^2}^{4-y^2} ydzdydx$$

Hint: by cylindrical coordinates, $x = r\cos\theta$ and $y = r\sin\theta$.

1. by spherical integration

$$
\begin{aligned}
I &= \int_0^2 \rho^5 d\rho \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} (\sin^2\phi + \sin^2\theta\cos^2\phi)\sin^2\phi d\phi \\
&= \frac{32}{3} \cdot \int_0^{2\pi} \left(\frac{3\pi}{8}\sin^2\theta + \frac{\pi}{8}\right) d\theta \\
&= 8\pi^2
\end{aligned}
$$

2. by cylindrical integration

$$
\begin{aligned}
I &= \int_0^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (4 - 2x^2 - 2y^2) y dy dx \\
&= \int_0^{\sqrt{2}} dr \int_0^{\frac{\pi}{2}} (4 - 2r^2) r\sin\theta \cdot r d\theta \\
&= \int_0^{\sqrt{2}} (4r^2 - 2r^4) dr \\
&= \frac{16\sqrt{2}}{15}
\end{aligned}
$$

Exercise

Find the volume of solid bounded by

$$
V : x^{2/3} + y^{2/3} + z^{2/3} \leqslant 2^2
$$

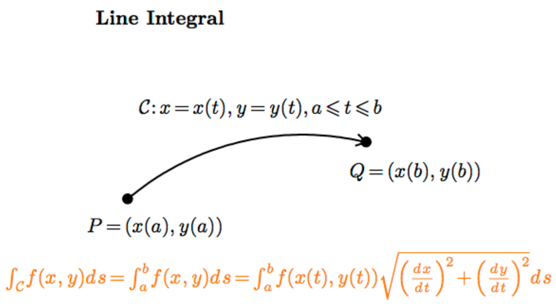
This volume is equal to the following triple integral:

$$
\begin{aligned}
I &= \iiint_V 1 dV \\
&\Downarrow \left(x = X^3, y = Y^3, z = Z^3, J = \left(\frac{\partial x^i}{\partial X^j}\right) = 27X^2Y^2Z^2\right) \\
&= \iiint_{X^2+Y^2+Z^2 \leqslant 2^2} 27X^2Y^2Z^2 dXdYdZ \\
&= 27 \int_{-2}^{2} r^8 dr \int_0^{2\pi} \sin^5\theta\cos^2\theta d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\phi\cos^2\phi d\phi \\
&= \frac{2048}{35}\pi
\end{aligned}
$$

Line Integral

Suppose that a plane curve C is given by the following parametric equations:

$$x = x(t), y = y(t) \text{where} a \leqslant t \leqslant b$$



Definition

If f is defines on a smooth curve C, then the line integral of f along C is:

$$\oint_C \mathbf{f(x,y)ds} = \lim_{\mathbf{n} \to \infty} \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{f(x_i^*, y_i^*)\Delta s_i}$$

where Δs_i is line elemnet if limit exists.

Suppose that the point (x, y) on curve C can be represented as $x = x(t)$ and $y = y(t)$ for $a \leqslant t \leqslant b$. If $x(t), y(t)$ have continuous derivatives, then the line integral can be calulated as follows: \

Theorem

$$\oint_C \mathbf{f(x,y)ds} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x(t),y(t))}\sqrt{\left(\frac{\mathbf{dx}}{\mathbf{dt}}\right)^2 + \left(\frac{\mathbf{dy}}{\mathbf{dt}}\right)^2} \mathbf{dt}$$

Example

Evaluate the line integral

$$\oint_C x^2y^2ds$$

where C is the move along unit circle countclockwise and starting from $(0, 0)$ and end at the same position.

Here

$$C : (x, y) = (\cos t, \sin t), 0 \leqslant t \leqslant 2\pi$$

and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t + \cos^2 t = 1$$

Then

$$\oint_C f(x,y)ds = \int_0^{2\pi} \cos^2 t \sin^2 t dt$$
$$= \pi/4$$

```
In [ ]: integrate(cos(t)**2*sin(t)**2,(t,0,2*pi))
```

In [ ]:
```
def line_int(func,parameters,t,t0,t1):
    [p0,p1]=parameters
    f=func.subs({x:p0,y:p1})
    ds=sqrt(diff(p0,t)**2+diff(p1,t)**2)
    integrand=f*ds
    I=integrate(integrand,(t,t0,t1))
    print("Line integral of %s along C=(%s,%s) is %s" %(func, p0,p1,I))
    return integrate(integrand,(t,t0,t1))
```

In [ ]:
```
def line_intS(func,parameters,t,t0,t1):
    [p0,p1]=parameters
    f=func.subs({x:p0,y:p1})
    ds=sqrt(diff(p0,t)**2+diff(p1,t)**2)
    integrand=f*ds
    I=integrate(integrand,(t,t0,t1))
    print("Line integral of %s along C=(%s,%s):" %(func, p0,p1))
    print("∮ %s ds = %s" %(func,I))
    print(" C")
    return integrate(integrand,(t,t0,t1))
```

In [ ]:
```
f = x**2*y**2
t0=0;t1=2*pi
parameters=[cos(t),sin(t)]
line_int(f,parameters,t,t0,t1)
```

In [ ]:
```
from sympy import pi,sin,cos
t=symbols("t")
f = x**2*y**2
t0=0;t1=2*pi
parameters=[cos(t),sin(t)]
I=line_intS(f,parameters,t,t0,t1)
```

Note

Sometimes, we can consider the following sum of line integrals:

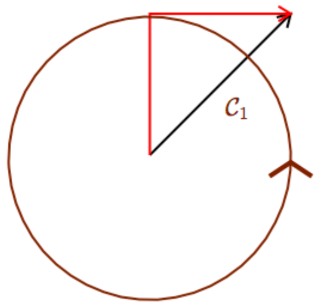
1. $\Delta s = \Delta x$: $\oint_C P(x,y)ds = \oint_C P(x,y)dx$
2. $\Delta s = \Delta y$: $\oint_C Q(x,y)ds = \oint_C Q(x,y)dy$ along the same path as:

$$\oint_C P(x,y)dx + \oint_C Q(x,y)dy$$

Example

Evaluate $\oint_C ((x-y)dx + (x+y)dy)$ along

1. line from $(0,0)$ to $(1,1)$;
2. line from $(0,0)$ to $(0,1)$ and turn right to $(1,1)$;
3. along unit circle countclockwise and starting from $(1,0)$ ending at $(1,0)$.



Solve: 1. $C:(x,y)=(t,t), 0\leqslant t\leqslant 1$

$$
\begin{aligned}
I &= \int 0dt + \int_0^1 (t+t)\frac{dt}{dt}dt \\
&= 1
\end{aligned}
$$

2. $C = C_1 \cup C_2, C_1 : (x,y)=(0,t), 0\leqslant t\leqslant 1; C_2 = (t,1), 0\leqslant t\leqslant 1$

$$
\begin{aligned}
I &= \oint_{C_1} + \oint_{C_2} \\
&= \int_0^1 (0+t)dt + \int_0^1 (t-1)dt \\
&= 0
\end{aligned}
$$

3. $C:(x,y)=(\cos t, \sin t), 0\leqslant t\leqslant 2\pi$

$$
\begin{aligned}
I &= \int_0^{2\pi} (\mathrm{cost} - \sin t)(-\sin t)dt \\
&\quad + \int_0^{2\pi} (\mathrm{cost} + \sin t)\cos t dt \\
&= 2\pi
\end{aligned}
$$

In [ ]:
```
def line_int2(P,Q,x,y,t,t0,t1):
    integrand=P*diff(x,t)+Q*diff(y,t)
    return integrate(integrand,(t,t0,t1))
```

In [ ]:
```
x,y=cos(t),sin(t)
P,Q=x-y,x+y
t0,t1=0,2*pi
line_int2(P,Q,x,y,t,t0,t1)
```

Theorem (Green's Theorem)

Suppose that C is a positive oriented,smooth and simple planar curve and D is the region bounded by C. If P and Q have continuous partial derivatives on interior od D. Then

$$\oint_C \mathbf{P(x,y)dx + Q(x,y)dy} = \iint\limits_{\mathbf{D}} \left(\frac{\partial \mathbf{Q}}{\partial \mathbf{x}} - \frac{\partial \mathbf{P}}{\partial \mathbf{y}} \right) \mathbf{dA}$$

Example

Along unit circle countclockwise and starting from $(0,0)$ ending at $(0,0)$, evaluate the following:

1. $\oint_C (x-y)dx + (x+y)dy$
2. $\oint_C \frac{ydx - xdy}{(x+y)^2}$

Solve:

1.

$$\begin{aligned} I &= \iint_{x^2+y^2 \leqslant 1} \left(\frac{\partial(x-y)}{\partial x} + \frac{\partial(x+y)}{\partial y} \right) dA \\ &= \iint_{x^2+y^2 \leqslant 1} 2dA = 2\pi \end{aligned}$$

2.

\begin{eqnarray*} I & = & \iint_{x^2 + y^2 \leqslant 1} \left(\frac{x - y}{(x + y)^3}

```
  - \frac{x - y}{(x + y)^3} \right) d A = 0
\end{eqnarray*}
```

Here, modify above python code to calulate the line integral:

Exercise

Evaluate $\oint_C (2x-y)dx + (x+y)dy$ along

1. line from $(0,0)$ to $(3,4)$;
2. line from $(0,0)$ to $(0,4)$ and turn right to $(3,4)$;
3. along circle, $x^2 + (y-1)^2 = 1$, countclockwise and starting from $(1,1)$ ending at $(0,2)$.

Sol:

1. $C : (x,y) \text{with} y = \frac{4}{3}x$ and let $x = t, y = 4t/3$ where $0 \leqslant t \leqslant 3$:

$$\begin{aligned} \oint_C (2x-y)dx + (x+y)dy &= \int_0^3 \left(\frac{2}{3}t + \frac{4}{3} \cdot \frac{7}{3}t \right) dt \\ &= \int_0^3 \frac{34}{9} t dt = 17 \end{aligned}$$

2.

$$
\begin{aligned}
I &= \int_{(0,0)\to(0,4)} + \int_{(0,4)\to(3,4)} \\
&= \int_0^4 (0+y)dy + \int_0^3 (2x-4)dx \\
&= 5
\end{aligned}
$$

3. Since $(x, y) = (\sin t, 1 - \cos t)$ with $0 \leqslant t \leqslant \pi$ for (x, y) in C:

$$
\begin{aligned}
I = & \int_0^\pi (\cancel{2}\sin t - \cancel{2} + 2\cos t)\cos \mathrm{td}t \\
& + \int_0^\pi (1 + \sin t - \cancel{c}\mathrm{os}\, t)\sin t dt \\
= & \int_0^\pi (1 + \sin^2 t + 2\cos^2 t)dt = 5\pi/2
\end{aligned}
$$

Exercise

Along unit circle countclockwise and starting from $(1, 0)$ ending at $(1, 0)$, evaluate the following:

1. $\oint_C ydx - xdy$
2. $\oint_C (y + x^3 y)dx + (x - y^3 x)dy$

Answer

1. by Green's theorem:

$$
\oint_C ydx - xdy = \iint_{x^2+y^2 \leqslant 1} (-1-1)dA = -4\pi
$$

2. also by Green's theorem,

\begin{eqnarray*} \oint{*\mathcal{C}} (y + x^3 y) d x + (x - y^3 x) d y & = & \iint*{x^2

```
+ y^2 \leqslant 1} (1 - \color{red}{\not{y^3}} - 1 - \color{red}{\not{x^3}}
) d A = 0
```

\end{eqnarray*}

Exercise

Compute line integral

$$
\oint_C (2x - y)dx + (x + y)dy
$$

where C is the path from $(1, 1)$ to $(2, 2)$ along $(x-1)^2 + (y-2)^2 = 1$ counterclockwise.

1. $(x-1)^2 + (y-2)^2 = 1 \Rightarrow x = 1 + \sin t, y = 2 - \cos t$;

2.

$$
\begin{aligned}
I &= \int_0^{\square} \left((\square)\cos t + (\square)\sin t \right) dt \\
&= \int_0^{\square} \left(1 + \square \sin t \cos t + \square \sin t \right) dt \\
&= \square \pi + \square
\end{aligned}
$$

Surface Integral

Suppose that $f(x, y, z)$ is defined on the smooth sruface $S \in \mathbb{R}^3$. Suppose that $S = \bigcup_i \Delta S_i$ with $\|S\| = \max_i \|S_i\| \to 0$. The **surface integral** of $f(x, y, z)$ on S is defined as the follwing limit:

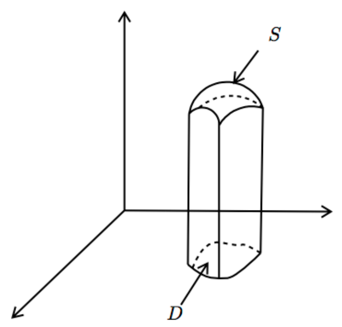
$$\mathbf{\iint_S f(x,y,z)dS = \lim_{\|\Delta S_i\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta S_i}$$

Theorem

Suppose that $f(x, y, z) = f(x, y, z(x, y))$ for (x, y, z) on S with $\|S\| \to 0$, then

$$\mathbf{\iint\limits_S f(x,y,z)dS = \iint\limits_D f(x,y,z(x,y))\sqrt{1 + z_x^2 + z_y^2}dA}$$

where D is the projection of S on $X - Y$ plane.

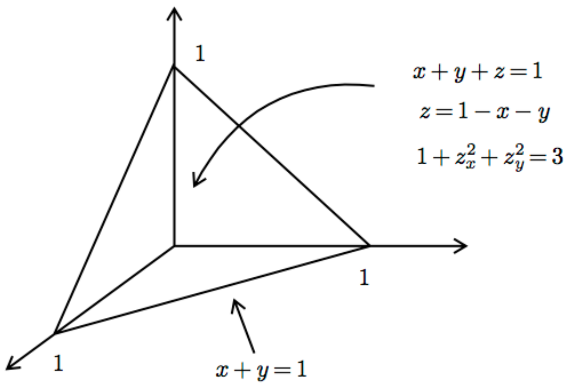


Example

Compute the surface integral:

$$\iint_S (xy + 2z) dS$$

where $S = \{(x, y, z) | x + y + z = 1\}$ in the first octant.



$$\begin{aligned}\iint_S (xy + 2z) dS &= \iint_{\{x+y\leqslant 1, x,y\geqslant 0\}} (xy + 2(1 - x - y))\sqrt{3} dA \\ &= \sqrt{3}\int_0^1 dx \int_0^{1-x} (2 - 2x - 2y - xy) dy \\ &= \frac{7\sqrt{3}}{24}\end{aligned}$$

In [ ]:
```
doubleInt3(2-2*x-2*y-x*y,[x,y],[0,1],[0,1-x])
```

Example

Evaluate the surface integral on the surface of the upper half unit sphere

$S = \{(x, y, z) | x^2 + y^2 + z^2 = 1, z \geqslant 0\}$

\begin{eqnarray*} \iintS *\left((x^2 + y^2 + (z - 1)^2\right) d S & = & \iint*{x^2

```
+ y^2 \leqslant 1}  \left( (x^2 + y^2 + (z - 1)^2)  \right)\frac{1}{\sqrt{1
- x^2
- y^2}} d x d y \\
& = & 2 \iint_{x^2 + y^2 \leqslant 1} \frac{1 - \sqrt{1 - x^2 -
y^2}}{\sqrt{1 - x^2 - y^2}} d x d y\\
& = & 2 \int^{2 \pi}_0 d \theta \int^1_0 \frac{r}{\sqrt{1 - r^2}} d r - 2
\pi\\
& = & 2 \pi
```

\end{eqnarray*}

In [ ]:
```
doubleInt3(r/sqrt(1-r**2),[t,r],[0,2*pi],[0,1])
```

Suppose that the point $r = (x, y, z) \in S$ can be represented as the parametric form, $r(u, v) = (x(u, v), y(u, v), z(u, v))$. Then

Theorem

$$\iint\limits_{\mathbf{S}} \mathbf{f(x, y, z)dS} = \iint\limits_{\mathbf{D}} \mathbf{f(x(u, v), y(u, v), z(u, v))} \left| \frac{\partial \mathbf{r}}{\partial \mathbf{u}} \times \frac{\partial \mathbf{r}}{\partial \mathbf{v}} \right| \mathbf{dA}$$

where $\cdot \times \cdot$ means exterior product.

Example

As the last example, we have:

$$r = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \phi \leqslant \pi/2$$

Then

\begin{eqnarray*} \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} & = & \left(\begin{array}{ccc} i & j & k\

```
  - \sin \phi \sin \theta & \sin \phi \cos \theta & 0\\
  \cos \phi \cos \theta & \cos \phi \sin \theta & - \sin \phi
\end{array}\right)\\
& = & (- \sin^2 \phi \cos \theta, \sin^2 \phi \cos \theta, - 4 \sin \phi
\cos \phi)\\
& \Downarrow & \\
\color{red}{\left| \frac{\partial r}{\partial u} \times \frac{\partial
r}{\partial v} \right| } & = & \sin \phi
```

\end{eqnarray*}

where $i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$.

$$\begin{aligned} \iint_S ((x^2 + y^2 + (z-1)^2) dS &= \int_0^{2\pi} d\theta \int_0^{\pi/2} (2 - 2\cos\phi)\sin\phi d\phi \\ &= 2\pi \end{aligned}$$

Exercise

As the last example, evaluate the following integral:

$$\iint_S (x^2 + y^2) dS$$

Exercise

Suppose that S is the portion of the cylinder $x^2 + y^2 = 4$ that lines between $z = 0$ and $z = 4$. Evaluate the following integral:

$$\iint_S z dS$$

Hint: $r = (2\cos\theta, 2\sin\theta, z), 0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant z \leqslant 4$.

\begin{eqnarray*} \frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial z} & = & \left(\begin{array}{ccc} i & j & k\

```
      - 2 \sin \theta & \begin{array}{|l|}
        \hline
        \\
        \hline
      \end{array} & 0\\
      \begin{array}{|l|}
        \hline
        \\
        \hline
      \end{array} & 0 & \begin{array}{|l|}
        \hline
        \\
        \hline
      \end{array}
    \end{array}\right)\\
    & = & (\begin{array}{|l|}
      \hline
      \\
      \hline
    \end{array} \cos \theta, \begin{array}{|l|}
      \hline
      \\
      \hline
    \end{array} \sin \theta, 0)\\
    & \Downarrow & \\
    \color{red}{\left| \frac{\partial r}{\partial u} \times \frac{\partial
    r}{\partial v} \right| } & = & \begin{array}{|l|}
      \hline
      \\
      \hline
    \end{array}
```

\end{eqnarray*}

Then

$$\begin{aligned}\iint_S z dS &= \int_0^{2\pi} d\theta \int_0^4 \square \, dz \\ &= \square\end{aligned}$$