1

Calculus, 2017-1-ME-1

1.0.1

Name: **Sequence Number:**

1° °). Find the following limits: (total 10%, each 5%)

a° °). $\lim\limits_{\theta\to 0} \frac{\tan 2\theta}{\theta}$ $\lim\limits_{\theta\to 0} \frac{\tan 2\theta}{\theta}$

b° °). $\lim\limits_{x\to 1} \frac{\sqrt{x+1}-\sqrt{2}}{x-1}$ $\lim\limits_{x\to 1} \frac{\sqrt{x+1}-\sqrt{2}}{x-1}$ (Hint: Consider the derivative of $\sqrt{x+1}$ $\sqrt{x+1}$).

2° °). Evaluate the following derivatives of given functions: (total 30%, each 5% (×6 × 6))

a° °). $\left[\left(\frac{x^2-x^{-2}}{x}\right)^2\right]'$ $\left[\left(\frac{x^2-x^{-2}}{x}\right)^2\right]'$ b° °). $\left[\sqrt{\left(\frac{1}{x}\right)}\right]'$ $\left[\sqrt{\left(\frac{1}{x}\right)}\right]'$ c° °).

$[x\tan x]'$ $[x\tan x]'$ d° °). $\left[\frac{2-x}{2x+1}\right]'$ $\left[\frac{2-x}{2x+1}\right]'$

e° °). $\frac{d}{dx}\left(-\frac{3}{4}x^4-2x^3+5x-6\right)\Big|_{x=-1}$ $\frac{d}{dx}\left(-\frac{3}{4}x^4-2x^3+5x-6\right)\Big|_{x=-1}$ f° °). $\frac{d}{dx}\left(\frac{1-\sin x}{\cos x}\right)$ $\frac{d}{dx}\left(\frac{1-\sin x}{\cos x}\right)$ g° °). $D_x\left((\sin x)^3\right)$ $D_x\left((\sin x)^3\right)$

3° °). Find the values of $\mathbf{A}, \mathbf{B}$ $\mathbf{A}, \mathbf{B}$ such that the following function is continuous and differentiable at $\mathbf{x} = \mathbf{0}$ $\mathbf{x} = \mathbf{0}$: (total 10%)

$$\mathbf{f(x)} = \begin{cases} x^2, \text{ if } x \le 0, \\ Ax + B, \text{ if } x > 0 \end{cases}$$

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4° °). (total 10%)

Suppose that

$$\mathbf{f(x)} = \mathbf{x}^3 \sin \mathbf{2x}$$

$$\mathbf{f(x)} = \mathbf{x}^3 \sin \mathbf{2x}$$

Find the third derivative of $\mathbf{f(x)}$ $\mathbf{f(x)}$, $\mathbf{f'''(x)}$ $\mathbf{f'''(x)}$.

5° °). (total 10%)
Suppose that $x, y \in \mathbb{R}$ $x, y \in \mathrm{R}$ satisfy:

$$\frac{\mathbf{x} - \mathbf{y}}{\mathbf{x} + \mathbf{y}} = \mathbf{x}^2 + \mathbf{1}$$

$$\frac{\mathbf{x} - \mathbf{y}}{\mathbf{x} + \mathbf{y}} = \mathbf{x}^2 + \mathbf{1}$$

Find the derivative of y y (10%)

6° °). (20%) Suppose that $\mathbf{f(t)} = \mathbf{15t^{2/3}} - \mathbf{3t^{5/3}}$ $\mathbf{f(t)} = \mathbf{15t^{2/3}} - \mathbf{3t^{5/3}}$ for all $\mathbf{t} \in \mathbb{R}$ $\mathbf{t} \in \mathrm{R}$.
a° °). (5%) Find all the critical values of $\mathbf{f(t)}$ $\mathbf{f(t)}$.
b° °). (5%) Determine the interval at which $\mathbf{f(t)}$ $\mathbf{f(t)}$ is increasing and concave downward.
c° °). (5%) Find all the relative extreme values of $\mathbf{f(t)}$ $\mathbf{f(t)}$ if any.
d° °). (5%) Plot the graph of $\mathbf{f(t)}$ $\mathbf{f(t)}$.

7° °). (total 10%, each 5%) Describe the following Theorems:
a° °). Rolle's Theorem
b° °). Mean Vaule Theorem.

1. a) 2 b) $\frac{\sqrt{2}}{4}$

2.

a) $\left(2 + \frac{6}{\frac{4}{x}}\right)\cdot\left(x - \frac{1}{\frac{3}{x}}\right)$

b) $x^{-3/2}/2$

c) $x\cdot\sec^{2}(x) + \tan(x)$

d) $-\frac{2\cdot(-x + 2)}{(2\cdot x + 1)^{2}} - \frac{1}{2\cdot x + 1}$

e) 2

f) $(1-\sin x)/\cos^{2} x$

g) $3\cdot\sin^{2}(x)\cdot\cos(x)$

3. A=0, B=0

4. $- 8\cdot x^{3}\cdot\cos(2\cdot x) - 36\cdot x^{2}\cdot\sin(2\cdot x) + 36\cdot x\cdot\cos(2\cdot x) + 6\cdot\sin(2\cdot x)$

5. (y'(x) = -x**2 - 2*x*y(x) - y(x)**2 + y(x))/x

6.

a), 0,2

b) at [0,2]

c) relative minimum f(0)=0, relative maximaum $f(2)=9\cdot 2^{2/3}$

d)

1.1 Answer

In [2]:

```python
from sympy import symbols, pprint,limit,diff,sin,tan,sqrt,cos,so
```

In [4]:

```python
x,t =symbols("x t")
```

In [5]:

```python
# 1. a)
print("1. a) The limit of tan(2t)/(t)) at t=0, is:")
pprint(limit(tan(2*t)/(t),t,0))
```

```
1. a) The limit of tan(2t)/(t)) at t=0, is:
2
```

In [7]:

```python
#1. b)
print("1. b) The derivative of sqrt(x+1) at x=1,, is:")
pprint(limit((sqrt(x+1)-sqrt(2))/(x-1),x,1))
```

```
1. b) The derivative of sqrt(x+1) at x=1,, is:
√2
──
4
```

In [9]:

```python
#2. a)
print("2. a) The derivatice of (x^2-x^{-2})^2/x, is:")
pprint(diff(((-x**(-2)+x*x)/x)**2,x))
```

```
2. a) The derivatice of (x^2-x^{-2})^2/x, is:
                                            2
⎛        4 ⎞   ⎛ 2    1 ⎞         ⎛ 2    1 ⎞
⎜4⋅x + ──⎟⋅⎜x  - ──⎟     2⋅⎜x  - ──⎟
⎜       3 ⎟   ⎜      2⎟         ⎜      2⎟
⎝      x  ⎠   ⎝     x  ⎠         ⎝     x  ⎠
──────────────────── - ────────────
          2                     3
         x                     x
```

In [16]:

```
#2. b)
print("2. b) The derivatice of (2+x^2)^(1/2)/x, is:")
pprint(diff(((2+x*x)/x)**(1/2),x))
```

```
2. b) The derivatice of (2+x^2)^(1/2)/x, is:
              0.5
   ⎛ 2      ⎞       ⎛            ⎛ 2     ⎞ ⎞
   ⎜x   + 2 ⎟       ⎜        0.5⋅⎝x  + 2⎠ ⎟
x⋅ ⎜────────⎟     ⋅ ⎜1.0 - ────────────── ⎟
   ⎝   x    ⎠       ⎜             2       ⎟
                    ⎝            x       ⎠
─────────────────────────────────────────
                   2
                  x  + 2
```

In [10]:

```
#2. c)
print("2. c) The derivatice of x tan x, is:")
pprint(diff(x*tan(x),x))
```

```
2. c) The derivatice of x tan x, is:
  ⎛   2       ⎞
x⋅⎝tan (x) + 1⎠ + tan(x)
```

In [11]:

```
#2. d)
print("2. d) The derivatice of (2-x)/(1+2x), is:")
pprint(diff((2-x)/(1+2*x),x))
```

```
2. d) The derivatice of (2-x)/(1+2x), is:
  2⋅(-x + 2)       1
- ────────── - ───────
           2   2⋅x + 1
  (2⋅x + 1)
```

In [12]:

```
#2. e)
print("2. e) The derivatice of -3x^4/4-2x^3+5x-6 at x=-1, is:")
pprint(diff(-3*x**4/4-2*x**3+5*x-6,x).subs({x:-1}))
```

```
2. e) The derivatice of -3x^4/4-2x^3+5x-6 at x=-1, is
:
2
```

In [13]:

```python
#2. f)
print("2. f) The derivatice of (1-sin x)/cos x, is:")
pprint(diff(sqrt(1-sin(x)/(cos(x))),x))
```

```
2. f) The derivatice of (1-sin x)/cos x, is:
       2
    sin (x)     1
 - ──────── - ─
         2      2
  2⋅cos (x)
──────────────────
    ______________
   ╱   sin(x)
  ╱  - ────── + 1
╲╱     cos(x)
```

1.2 Note

above equal to the simple form, $\frac{1}{1+\cos x}\,\frac{1}{1+\cos x}$.

In [15]:

```python
#2. g)
print("2. g) The derivatice of sin^3(x), is:")
pprint(diff(sin(x)**3,x))
```

```
2. g) The derivatice of sin^3(x), is:
     2
3⋅sin (x)⋅cos(x)
```

2. h)

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{h\sin(1/h)}{h} \\ &= \lim_{h\to 0} \sin(1/h) \end{aligned}$$

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{h\sin(1/h)}{h} \\ &= \lim_{h\to 0} \sin(1/h) \end{aligned}$$

This concludes that the limit fails to exist.

In [25]:

```
#3. a)
print("3. a) The third-order derivatice of x^2 sin x, is:")
pprint(diff(x**2*sin(x),x,3))
```

```
3. a) The third-order derivatice of x^2 sin x, is:
   2
- x ·cos(x) - 6·x·sin(x) + 6·cos(x)
```

In [26]:

```
def ImplicitDiff(express):
    l=diff(express,x);
    print("y'(x) =",solve(l,Derivative(y,x))[0])
```

In [27]:

```
y=Function("y")
y=y(x)
print("3. b) The derivatice of y for x^3+y^3=3x, is:")
ImplicitDiff(x**3+y**3-3*x)
```

```
3. b) The derivatice of y for x^3+y^3=3x, is:
y'(x) = (-x**2 + 1)/y(x)**2
```

If $y'(x) = 0$$y'(x) = 0$, it implies $x = \pm 1$$x = \pm 1$. Then $(\pm 1)^3 + y^3 = 3 \cdot (\pm 1) \rightarrow \mathbf{y} = \pm \mathbf{2^{1/3}}$$(\pm 1)^3 + y^3 = 3 \cdot (\pm 1) \rightarrow \mathbf{y} = \pm \mathbf{2^{1/3}}$

4). a)

$$\left|\frac{\cos a - \cos b}{a-b}\right| = |\sin x_0| \leq 1$$
$$\implies \quad |\cos a - \cos b| \leq |a-b|$$

$$\left|\frac{\cos a - \cos b}{a-b}\right| = |\sin x_0| \leq 1$$
$$\implies \quad |\cos a - \cos b| \leq |a-b|$$

b) $\mathbf{x} \in \mathbb{R}$$\mathbf{x} \in \mathrm{R}$

5). **a)** critical values:

$$f'(t) = \left(4t^{\frac{1}{3}} + 3t^{\frac{4}{3}}\right)' = \frac{4}{3}\left(t^{-\frac{2}{3}} + 3t^{\frac{1}{3}}\right) = \frac{4(1+3t)}{3t^{\frac{2}{3}}}$$

$$f'(t) = \left(4t^{\frac{1}{3}} + 3t^{\frac{4}{3}}\right)' = \frac{4}{3}\left(t^{-\frac{2}{3}} + 3t^{\frac{1}{3}}\right) = \frac{4(1+3t)}{3t^{\frac{2}{3}}}$$

i). If $f'(t) = 0 \Rightarrow 1 + 3t = 0 \rightarrow t = -1/3$
ii). If $f'(t) = 0$ fails to exist, then the denominator is zero, $t^{2/3} = 0 \rightarrow t = 0$

b). Since

$$\lim_{t \to \pm\infty} f(t) = +\infty,$$

$$\lim_{t \to \pm\infty} f(t) = +\infty,$$

$f(t)$ can only attain its absolute minimum. The minimum is $f(-1/3)$, which is smaller than 0, since it is smaller than $f(0) = 0$.

6) .
a). Rolle's theorem, Assume that $f(x)$ is continuous on $[a, b]$ and differentiable in (a, b). If $f(a) = f(b)$, there exists at least $c \in (a, b)$ such that $f'(c) = 0$.
b). Mean Value Theorem, MVT, Assume that $f(x)$ is continuous on $[a, b]$ and differentiable in (a, b). There exists at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In [28]:

```
!jupyter nbconvert --to html 2016-1-me-1.ipynb
```

```
[NbConvertApp] Converting notebook 2016-1-me-1.ipynb
to html
[NbConvertApp] Writing 274903 bytes to 2016-1-me-1.ht
ml
```

In [ ]:

```

```

Calculus, 2017-1-ME-1

Name: **Sequence Number:**

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a°). $\lim_{\theta \to 0} \frac{\tan 2\theta}{\theta}$

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2°). Evaluate the following derivatives of given functions: (total 30%, each 5% ($\times 6$))

a°). $\left[\left(\frac{x^2-x^{-2}}{x}\right)^2\right]'$ b°). $\left[\sqrt{\left(\frac{1}{x}\right)}\right]'$ c°). $[x\tan x]'$ d°). $\left[\frac{2-x}{2x+1}\right]'$

e°). $\left.\frac{d}{dx}\left(-\frac{3}{4}x^4 - 2x^3 + 5x - 6\right)\right|_{x=-1}$ f°). $\frac{d}{dx}\left(\frac{1-\sin x}{\cos x}\right)$ g°). $D_x\left((\sin x)^3\right)$

3°). Find the values of A, B such that the following function is continuous and differentiable at $x = 0$: (total 10%)

$$f(x) = \begin{cases} x^2, \text{ if } x \le 0, \\ Ax + B, \text{ if } x > 0 \end{cases}$$

4°). (total 10%)

Suppose that

$$f(x) = x^3 \sin 2x$$

Find the third derivative of $f(x)$, $f'''(x)$.

5°). (total 10%)
Suppose that $x, y \in R$ satisfy:

$$\frac{\mathbf{x-y}}{\mathbf{x+y}} = \mathbf{x^2+1}$$

Find the derivative of y (10%)

6°). (20%) Suppose that $\mathbf{f(t) = 15t^{2/3} - 3t^{5/3}}$ for all $\mathbf{t} \in R$.
a°). (5%) Find all the critical values of $\mathbf{f(t)}$.
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Answer

In [2]:

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```
1. a) The limit of tan(2t)/(t)) at t=0, is:
2
```

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1. b) The derivative of sqrt(x+1) at x=1,, is:
```

$$\frac{\sqrt{2}}{4}$$

In [9]:

2. a) The derivatice of (x^2-x^{-2})^2/x, is:

$$\frac{\left(4 \cdot x + \frac{4}{x^{3}}\right) \cdot \left(x^{2} - \frac{1}{x^{2}}\right)}{x^{2}} - \frac{2 \cdot \left(x^{2} - \frac{1}{x^{2}}\right)^{2}}{x^{3}}$$

In [16]:

2. b) The derivatice of (2+x^2)^(1/2)/x, is:

$$\frac{x \cdot \left(\frac{x^{2} + 2}{x}\right)^{0.5} \cdot \left(1.0 - \frac{0.5 \cdot \left(x^{2} + 2\right)}{x^{2}}\right)}{x^{2} + 2}$$

In [10]:

2. c) The derivatice of x tan x, is:

$$x \cdot \left(\tan^{2}(x) + 1\right) + \tan(x)$$

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2. d) The derivatice of (2-x)/(1+2x), is:

$$-\frac{2\cdot(-x + 2)}{(2\cdot x + 1)^2} - \frac{1}{2\cdot x + 1}$$

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Note

above equal to the simple form, $\frac{\mathbf{1}}{\mathbf{1+\cos x}}$.

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```

If $y'(x) = 0$, it implies $x = \pm 1$. Then $(\pm 1)^3 + y^3 = 3 \cdot (\pm 1) \rightarrow \mathbf{y} = \pm \mathbf{2^{1/3}}$

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$$\left|\frac{\cos a - \cos b}{a - b}\right| = \left|\sin x_0\right| \leq 1$$
$$\Longrightarrow \quad |\cos a - \cos b| \leq |a - b|$$

b) $\mathbf{x} \in \mathrm{R}$

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```

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