Math 000 Notes

Full name(s)

(month/day) This is for patch notes.
(12/04) Today's a good day!

These notes are adapted from professor ???'s lecture, FA/SP20XX.

If you spot any error in these notes, please reach out to me via email `???@gmail.com`, or you can direct message me on Discord at `???`.

First Section

Here is a brief summary of this section.

Vocabulary 1.1 (cotangent bundle)**.** Recall that the dual vector space of a finite dimensional vector space V is $V^* := \hom(V, \mathbb{R})$. For any basis $v^1, ..., v^n$ of V, we can find a basis $dv^1, ..., dv^n$ of V^* characterized by

$$dv^j(v^i) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{everything else} \end{cases} = \delta^{ij}$$

For $E = TM$, the dual bundle $(TM)^* = T^*M$ is the cotangent bundle of M. We can apply this to T_pM and T_p^*M. Given local coordinates $x^1, ..., x^n$ on $U \subseteq M$, for each $p \in U$ the coordinate basis $\frac{\partial}{\partial x^1}\big|_p, ..., \frac{\partial}{\partial x^n}\big|_p$ provides a basis for T_pM, which gives rise to a dual basis for T_p^*M: $dx^1|_p, ..., dx^n|_p$. Similarly, it is characterized by

$$dx^j|_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \delta^{ij} \in C^\infty(U)$$

Remark. More generally, for any $\omega \in \Gamma(T^*M) = \mathfrak{X}^*(M)$, we can pair any $X \in \Gamma(TM) = \mathfrak{X}(M)$ with $\omega(X) \in C^\infty(M)$. Recall that $\omega : M \to T^*M$ with coordinate functions $\omega_1, ..., \omega_n$. It follows from the coordinate formula of (co)vector fields that

$$\omega|_U = \omega_i dx^i \quad \text{and} \quad \omega|_U \left(\frac{\partial}{\partial x^j} \right) = \omega_i dx^i \left(\frac{\partial}{\partial x^j} \right) = \omega_i \tag{$\blacksquare$}$$

for some $U \subseteq M$.

Second Section

Here is a brief summary of this section.

Example. Some k-forms

1. $\Omega^1(M) = \Gamma(T^*M)$, i.e. differential 1-form are covectors.
2. $\Omega^0(M) = C^\infty(M)$. This becomes $\Omega^1(M)$ under d. So we have a map $d : \Omega^0(M) \to \Omega^1(M)$. $df(V) = V(f)$ for any $V \in \mathfrak{X}(M)$. Locally,
$$df = \frac{\partial f}{\partial x^i} dx^i$$

Third Section

Here is a brief summary of this section. Maybe this is not that useful for shorter notes. Who knows.

Lemma. Every Cauchy sequence is bounded.

Proof. Let (a_n) be Cauchy. We choose $0 < \epsilon_0$. So for all $n > m \geq N_0$ we have that $|a_n - a_m| < \epsilon_0$. Therefore (a_n) is bounded for all $m \geq N_0$ by ϵ_0. Since $\mathbb{N}_{N_0}$ is finite, it is bounded. So, for all $m < N_0$, (a_n) is bounded. Therefore (a_n) is bounded. $\square$

Theorem 1. $\mathbb{R}^n$ is complete for each $n \in \mathbb{N}$ (with respect to any metric d_p for any $p \in [1, \infty]$)

Proof. Take any Cauchy sequence (x_n) in $\mathbb{R}^n$. It is bounded. So $(x_n)_{n \in \mathbb{N}}$ is contained in some closed ball (with respect to d_p). This ball is closed and bounded in $\mathbb{R}^n$, thus compact. This implies sequentially compact. It follows that there exists a subsequence of (x_n) converging to some x in a ball. This implies (x_n) converges to $x \in \mathbb{R}^n$. $\square$

Proposition 1. Blah blah blah.

Vocabulary 3.1 (product)**.** A diagram $A \xleftarrow{p_1} P \xrightarrow{p_2}$ is a product if for all object Z and a diagram $A \xleftarrow{z_1} Z \xrightarrow{z_2}$, there exists a unique $u : Z \to P$ such that $z_i = p_i \circ u_i$. The coproduct is the dual of product, that is, a diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a coproduct if for all object Z and a diagram $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$, there exists a unique $u : Q \to Z$ such that $z_i = q_i \circ u$.

$$\begin{tikzcd}
& Z \arrow[dl, "z_1"'] \arrow[d, dotted, "u"] \arrow[dr, "z_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& Z & \\
A \arrow[ur, "z_1"] \arrow[r, "q_1"] & Q \arrow[u, dotted, "u"'] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"']
\end{tikzcd}$$

Appendix

Basic commands and symbols they make.

Command	Symbol
`\sub`	$\subset$
`\sube`	$\subseteq$
`\supe`	$\supseteq$
`\nsub`	$\not\subset$
`\nsup`	$\not\supset$
`\nsube`	$\nsubseteq$
`\nsupe`	$\nsupseteq$
`\Ra`	$\Rightarrow$
`\La`	$\Leftarrow$
`\al`	α
`\be`	β
`\ga`	γ
`\de`	δ
`\si`	σ
`\la`	λ
`\< ...\>`	$\langle ... \rangle$
`\es`	$\varnothing$

Index