Math 000 Notes

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(month/day) This is for patch notes. (12/04) Today's a good day!

These notes are adapted from professor ???'s lecture, FA/SP20XX.

If you spot any error in these notes, please reach out to me via email ????@gmail.com, or you can direct message me on Discord at ????.

First Section

Here is a brief summary of this section.

Vocabulary 1.1 (cotangent bundle). Recall that the dual vector space of a finite dimensional vector space V is $V^* := \text{hom}(V, \mathbb{R})$. For any basis $v^1, ..., v^n$ of V, we can find a basis $dv^1, ..., dv^n$ of V^* characterized by

$$dv^{j}(v^{i}) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{everything else} \end{cases} = \delta^{ij}$$

For E=TM, the dual bundle $(TM)^*=T^*M$ is the cotangent bundle of M. We can apply this to T_pM and T_p^*M . Given local coordinates $x^1,...,x^n$ on $U\subseteq M$, for each $p\in U$ the coordinate basis $\frac{\partial}{\partial x^1}\big|_p,...,\frac{\partial}{\partial x^n}\big|_p$ provides a basis for T_pM , which gives rise to a dual basis for T_p^*M : $dx^1|_p,...,dx^n|_p$. Similarly, it is characterized by

$$dx^{j}|_{p}\left(\frac{\partial}{\partial x^{i}}\Big|_{p}\right) = \delta^{ij} \in C^{\infty}(U)$$

Remark. More generally, for any $\omega \in \Gamma(T^*M) = \mathfrak{X}^*(M)$, we can pair any $X \in \Gamma(TM) = \mathfrak{X}(M)$ with $\omega(X) \in C^{\infty}(M)$. Recall that $\omega : M \to T^*M$ with coordinate functions $\omega_1, ..., \omega_n$. It follows from the coordinate formula of (co)vector fields that

$$\omega|_{U} = \omega_{i} dx^{i}$$
 and $\omega|_{U} \left(\frac{\partial}{\partial x^{j}}\right) = \omega_{i} dx^{i} \left(\frac{\partial}{\partial x^{j}}\right) = \omega_{i}$ (\blacksquare)

for some $U \subseteq M$.

Second Section

Here is a brief summary of this section.

Example. Some k-forms

- 1. $\Omega^1(M) = \Gamma(T^*M)$, i.e. differential 1-form are covectors.
- 2. $\Omega^0(M) = C^{\infty}(M)$. This becomes $\Omega^1(M)$ under d. So we have a map $d: \Omega^0(M) \to \Omega^1(M)$. df(V) = V(f) for any $V \in \mathfrak{X}(M)$. Locally,

 $df = \frac{\partial f}{\partial x^i} dx^i$

Third Section

Here is a brief summary of this section. Maybe this is not that useful for shorter notes. Who knows.

Lemma. Every Cauchy sequence is bounded.

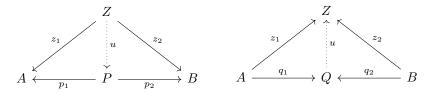
Proof. Let (a_n) be Cauchy. We choose $0 < \epsilon_0$. So for all $n > m \ge N_0$ we have that $|a_n - a_m| < \epsilon_0$. Therefore (a_n) is bounded for all $m \ge N_0$ by ϵ_0 . Since \mathbb{N}_{N_0} is finite, it is bounded. So, for all $m < N_0$, (a_n) is bounded. \square

Theorem 1. \mathbb{R}^n is complete for each $n \in \mathbb{N}$ (with respect to any metric d_p for any $p \in [1, \infty]$)

Proof. Take any Cauchy sequence (x_n) in \mathbb{R}^n . It is bounded. So $(x_n)_{n\in\mathbb{N}}$ is contained in some closed ball (with respect to d_p). This ball is closed and bounded in \mathbb{R}^n , thus compact. This implies sequentially compact. It follows that there exists a subsequence of (x_n) converging to some x in a ball. This implies (x_n) converges to $x \in \mathbb{R}^n$. \square

Proposition 1. Blah blah blah.

Vocabulary 3.1 (product). A diagram $A \stackrel{p_1}{\leftarrow} P \stackrel{p_2}{\rightarrow}$ is a product if for all object Z and a diagram $A \stackrel{z_1}{\leftarrow} Z \stackrel{z_2}{\rightarrow}$, there exists a unique $u: Z \rightarrow P$ such that $z_i = p_i \circ u_i$. The coproduct is the dual of product, that is, a diagram $A \stackrel{q_1}{\rightarrow} Q \stackrel{q_2}{\leftarrow} B$ is a coproduct if for all object Z and a diagram $A \stackrel{z_1}{\rightarrow} Z \stackrel{z_2}{\leftarrow} B$, there exists a unique $u: Q \rightarrow Z$ such that $z_i = q_i \circ u$.



Appendix

Basic commands and symbols they make.

Command	Symbol
\sub	\subset
\sube	⊆ ⊇ ⊄
\supe	\supseteq
\nsub	¢
\nsup	⊅ ⊈ ⊉
\nsube	⊈
\nsupe	⊉
\Ra	\Rightarrow
\La \al	←
	α
\be	β
\ga	γ
\de	$\frac{\gamma}{\delta}$
\si	σ
\la	λ
\<\>	\langle\rangle
\es	Ø

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