# 1

# Calculus, 2016-2-IE-1

### 1.0.1

**Name:** **Sequence Number:**

**1° ).** Determine whether the following sequences $\{\mathbf{a_n}\}$ are convergent or divergent? Why? (total 10%, each 5%)

a°). $a_n = 1 - \cos n\pi$ b°). $\mathbf{a_n} = \frac{\mathbf{1\cdot3\cdot5\cdots(2n-3)\cdot(2n-1)}}{\mathbf{n!}}$

**2° ).** Determine whether the following series are convergent or divergent? Why? (total 30%, each 5% (×6))

a°). $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ b°). $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ c°). $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$ d°). $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

e°). $\sum_{n=1}^{\infty} (-1)^n 3^n$ f°). $\sum_{n=1}^{\infty} \frac{2^{-n}}{n^2+n}$

**3° ).** Find the Taylor series of $f(x)$ below and its convergent radius (10%):

$$f(x) = \frac{1}{1+x} \text{ at } c = 2$$

**Hint:**

$$\frac{1}{1+x} = \frac{1}{3}\frac{1}{1+(x-2)/3} =$$

**4° ).** (10%) Suppose that

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0.0) \\ 0 & \text{if } (x,y) = (0.0) \end{cases}$$

Is $f(x)$ is continuous at $x = 0$?

**5°).** (20%) Suppose that $(\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^n)$.
a°). (5%) The gradient, $\nabla \mathbf{f}$ , of $\mathbf{f}(\mathbf{x})$ is defined as vector of its all partial derivatives ,i.e.:

$$\nabla \mathbf{f} = ?$$

b°). (total 10%, each 5%) Find all the partial derivatives of $\mathbf{f}(\mathbf{x})$:
i°). $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \ln(\exp(-\mathbf{x}) + \exp(\mathbf{y}))$,  ii°).
$\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x}^2\mathbf{y} + \mathbf{y}^2\mathbf{z} + \mathbf{z}\mathbf{x}$,
c°). (5%) $\mathbf{u} = \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{e}^{\mathbf{x}} \cos \mathbf{y}$. Evaluate $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = ?$.

**6°).** (total 20%, each 10%)
a°). Suppose that $w = x\sqrt{y^2 + z}$, $(x, y, z) = (1/t, e^{-t} \cos t, e^{-t} \sin t)$. Evaluate $\frac{\mathbf{dw}}{\mathbf{dt}}$
b°). Suppose that $\mathbf{w} = \sin \mathbf{xy}$, $(\mathbf{x}, \mathbf{y}) = \left((\mathbf{u} + \mathbf{v})^3, \sqrt{\mathbf{v}}\right)$. Evaluate $\frac{\partial \mathbf{w}}{\partial (\mathbf{u}, \mathbf{v})}$

**Ans:**
**1.** a°). $a_n = 1 - \cos n\pi$ divergent, since $a_n = 1 - (-1)^n$.

b°).

$$\begin{aligned}
a_n &= \frac{(2n)!}{n!2 \cdot 4 \cdot 6 \cdots (2n-2) \cdot 2n} \\
&= \frac{(n+1) \cdot (n+2) \cdots (2n-1) \cdot (2n)}{2 \cdot 4 \cdot 6 \cdots (2n-2) \cdot 2n} \\
&= \frac{n+1}{2} \cdot \frac{n+2}{4} \cdots \cdot \frac{2n-1}{2n-2} \cdot \frac{2n}{2n} \\
&\geq \frac{n+1}{2} \to \infty
\end{aligned}$$

**Ans:**

**2.** a°). convergent, since $4/3 > 1$.

b°). divergent since

$$f(n) = \frac{1}{n\ln n} \Rightarrow f(x) = \frac{1}{x\ln x}, f \geq 0, f \searrow$$
$$\int_2^\infty \frac{1}{x\ln x}dx = \int_2^\infty \frac{1}{\ln x}d\ln x$$
$$= \ln|\ln x||_2^\infty = \infty$$

c°). convergent, since $0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \to 0$

d°). convergent since $a_n \leq 1/(n^2)$.

e°). $3^n \not\to 0$, divergent; ($n$-term test)

f°). $(\frac{2^{-n-1}}{(n+1)^2+n+1})/(\frac{2^{-n}}{n^2+n}) \to 1/2 < 1$, convergent, (ratio test)

**Ans:**

**3.**

$$\frac{1}{1+x} = \frac{1}{3}\frac{1}{1+(x-2)/3} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-2}{3}\right)^n$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-2)^n$$

where $|(x-2)/3| < 1$, ($|x-2| < 3$) i.e. convergent radius: $3$.

**Ans:**

**4.** Since the limit of $f(x,y)$ at $(0,0)$ is equal to $0$ ( $= f(0,0)$ ):

$$|f(x,y)| \leq 2\frac{(x^2+y^2)^{3/2}}{x^2+y^2} = 2\sqrt{x^2+y^2} \to 0$$

$f(x,y)$ is continuous at $(0,0)$.

**5.**

**a°).** (5%)

$$\nabla \mathbf{f} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}^1}, \cdots, \frac{\partial \mathbf{f}}{\partial \mathbf{x}^n}\right)$$

**b°).** (total 10%, each 5%) Find all the partial derivatives of $\mathbf{f}(\mathbf{x})$:

i°). $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \ln(\exp(-\mathbf{x}) + \exp(\mathbf{y}))$, ii°). $\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x}^2\mathbf{y} + \mathbf{y}^2\mathbf{z} + \mathbf{z}\mathbf{x}$,

i°).

$$\nabla \mathbf{f} = \left(\frac{-\exp(-\mathbf{x})}{\exp(-\mathbf{x}) + \exp(\mathbf{y})}, \frac{-\exp(\mathbf{y})}{\exp(-\mathbf{x}) + \exp(\mathbf{y})}\right)$$

ii°).

$$\nabla \mathbf{f} = (\mathbf{2xy} + \mathbf{z}, \mathbf{x}^2 + \mathbf{2yz}, \mathbf{y}^2 + \mathbf{x})$$

c°). $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0}.$

**Ans:**

**6.**

a°).

$$\left(\sqrt{\mathbf{y}^2 + \mathbf{z}}, \frac{\mathbf{xyz}}{\sqrt{\mathbf{y}^2 + \mathbf{z}}}, \frac{\mathbf{x}}{2\sqrt{\mathbf{y}^2 + \mathbf{z}}}\right) \cdot (-\mathbf{1}/\mathbf{t}^2, -\mathbf{e}^{-\mathbf{t}} \sin \mathbf{t} - \mathbf{e}^{-\mathbf{t}} \cos \mathbf{t}, -\mathbf{e}^{-\mathbf{t}} \sin \mathbf{t} + \mathbf{e}^-$$

b°).

$$\begin{bmatrix} \mathbf{y} \cos \mathbf{xy} & x \cos xy \end{bmatrix} \begin{bmatrix} \mathbf{3(u + v)^2} & 3(u + v)^2 \\ 0 & -1/(2\sqrt{v}) \end{bmatrix}$$

In [2]:

```
!jupyter nbconvert --to html 2016-2-ie-1.ipynb
```

```
[NbConvertApp] Converting notebook 2016-2-ie-1.ipynb
to html
[NbConvertApp] Writing 258306 bytes to 2016-2-ie-1.ht
ml
```

In [ ]:

```

```