Лекц:

Муруй ба гадаргуу
Splines, Curves and Surfaces

Computer Graphics and Imaging

UC Berkeley CS184/284A, Spring 2016

Гөлгөр муруй ба гадаргуу (Smooth Curves and Surfaces)

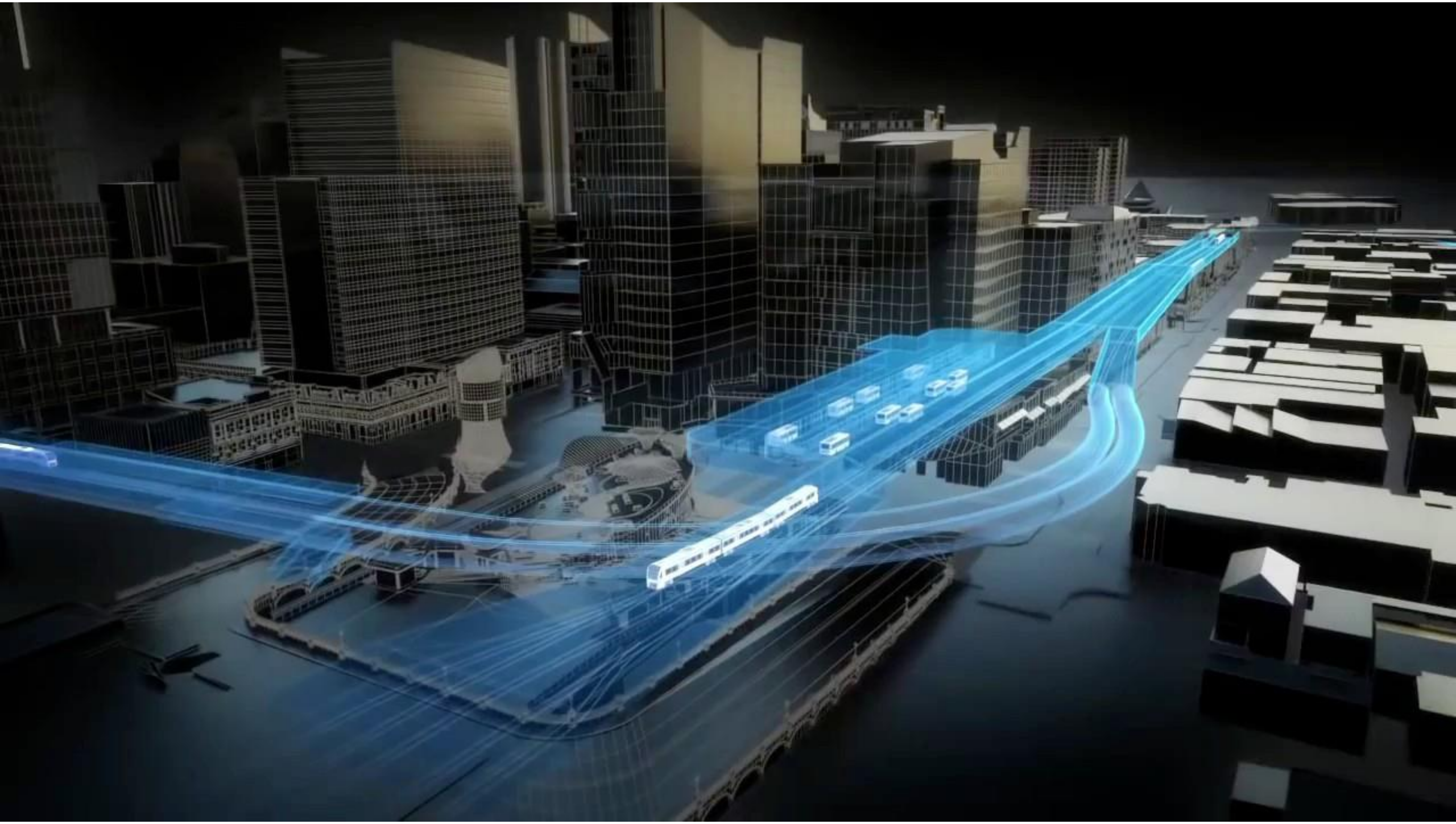
Одоогийн байдлаар бид дараах зүйлсийг хийж чадна:

- Ирмэг, булан (lines, triangles, squares, …)
- Тусгай дүрс (circles, ellipses, …)

Ихэнх аппликейшнууд нь нарийн, smooth хэлбэр шаарддаг.

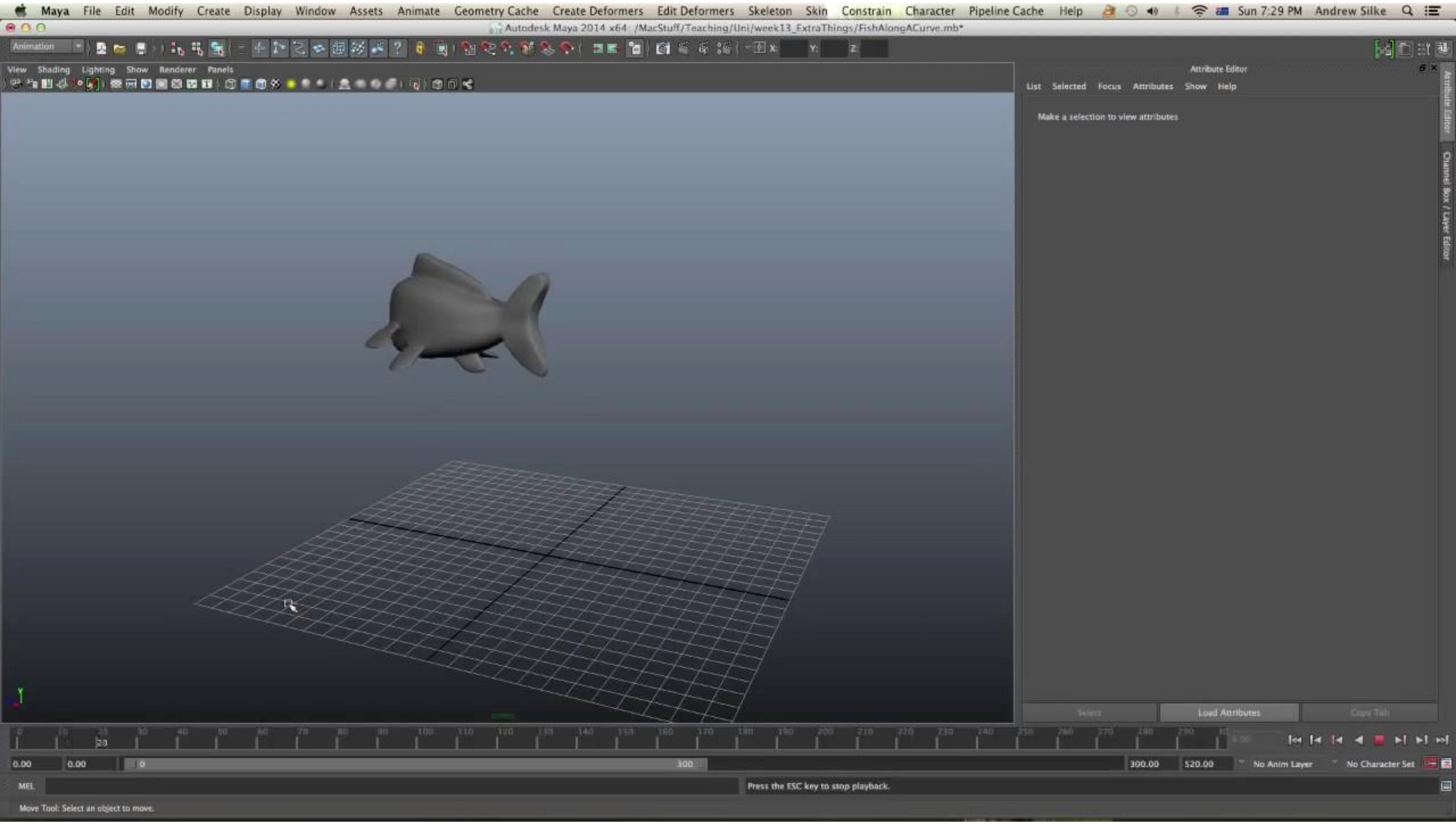
- Camera paths (камерийн зам), vector fonts (вектор фонт), …
- Филтер функцүүдийг дахин тохируулах(filter functions)
- CAD design, object modeling (объект загварчлал), …

Camera Paths/Камерийн зам



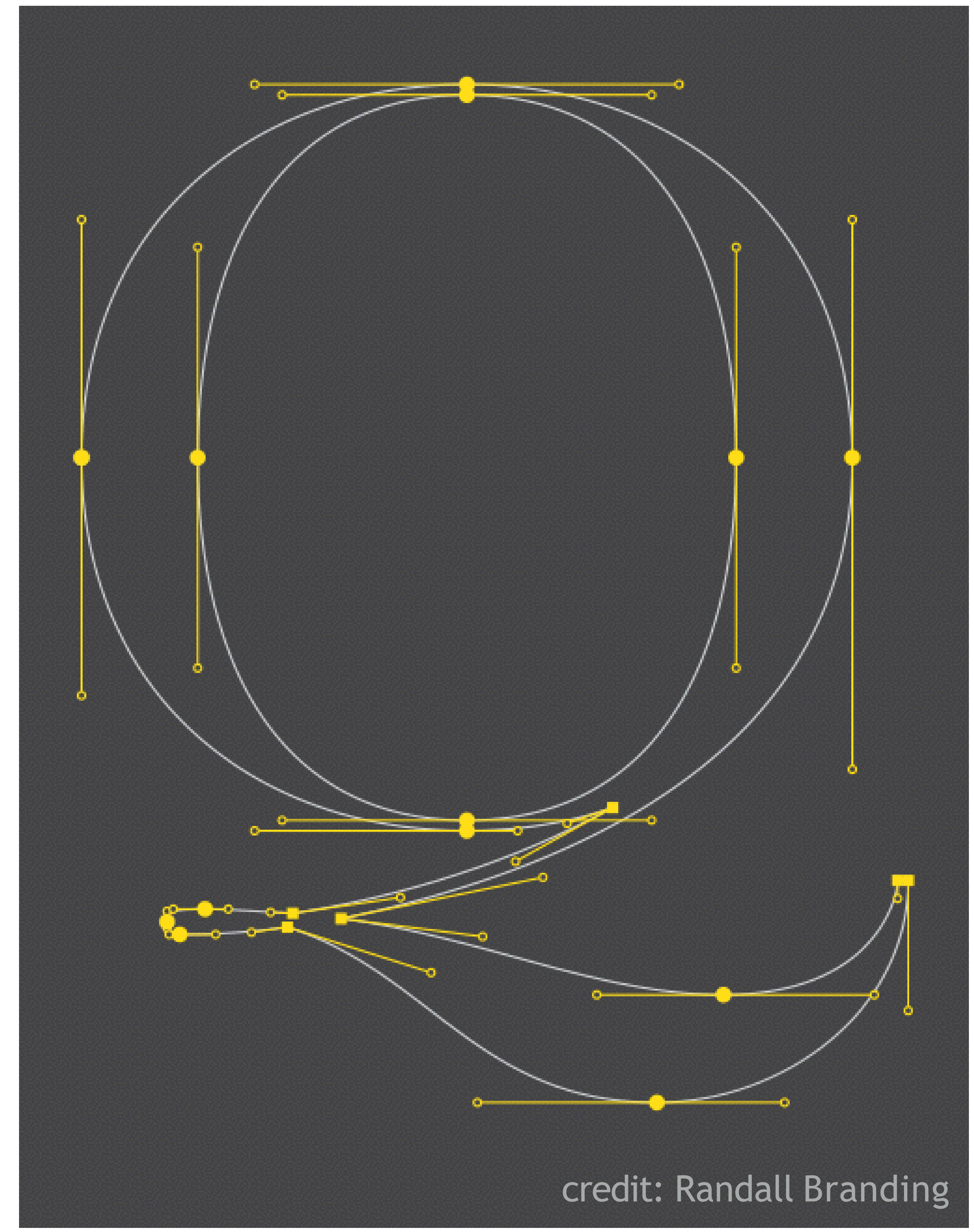
Flythrough of proposed Perth Citylink subway, https://youtu.be/rIJMuQPwr3E

Animation Curves/Хөдөлгөөнт муруй



Maya Animation Tutorial: https://youtu.be/b-o5wtZlJPc

Vector Fonts/Вектор фонт



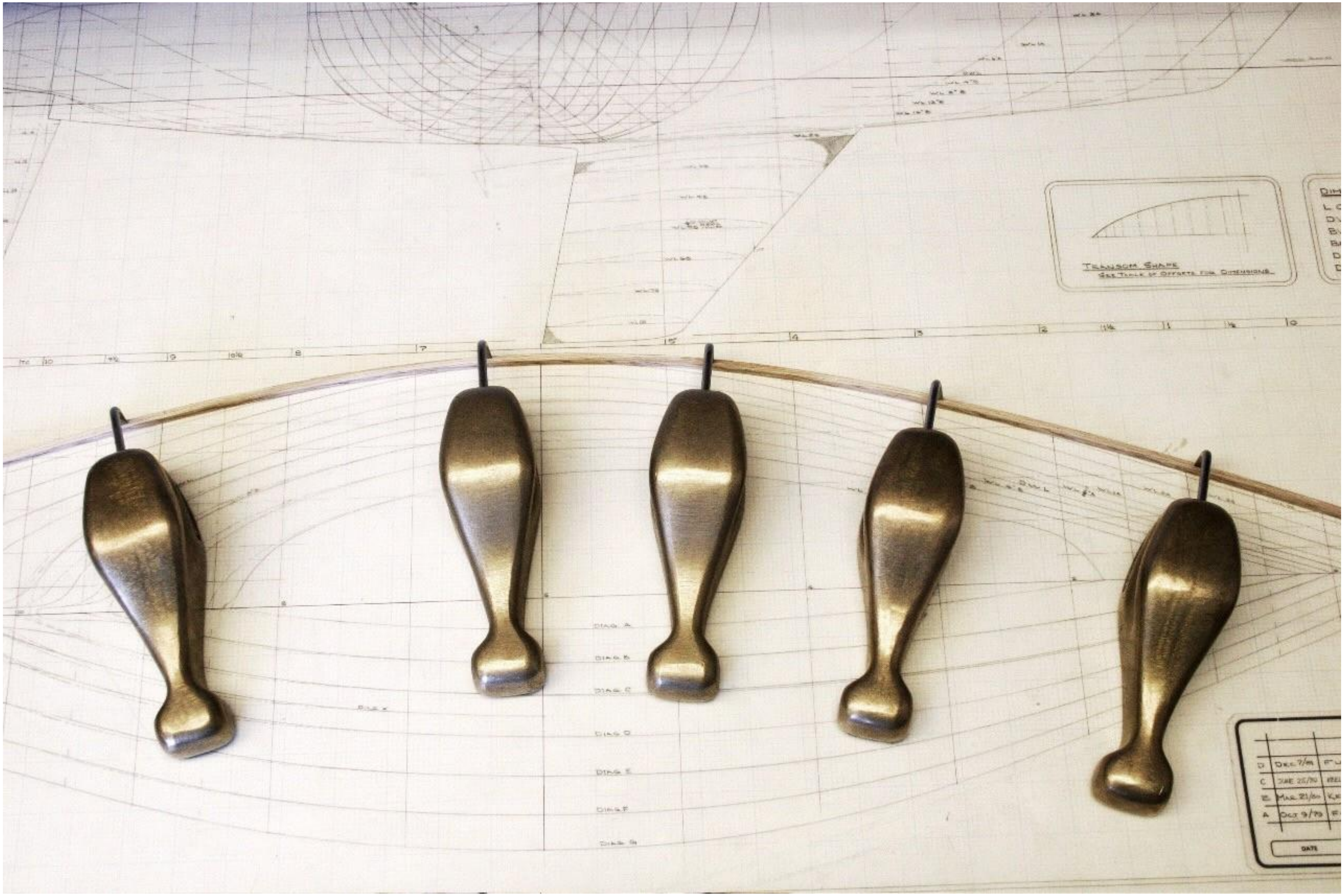
Baskerville font - represented as cubic Bézier splines

CAD Design/CAD дизайн

3D Car Modeling with Rhinoceros

Splines

Бодит зураачийн Spline



http://www.alatown.com/spline-history-architecture/

Spline Topics/Spline сэдвүүд

Интерполяци

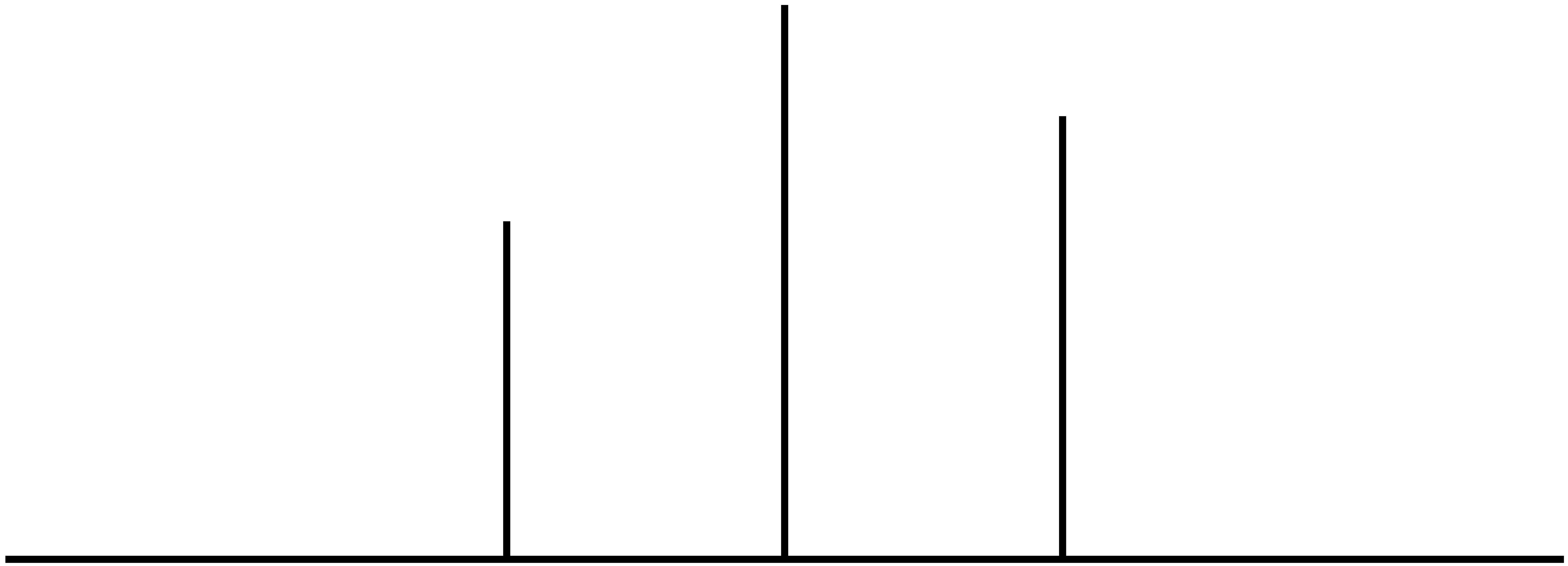
- Cubic Hermite интерполяци
- Catmull-Rom интерполяци

Bezier curves/Bezier муруй

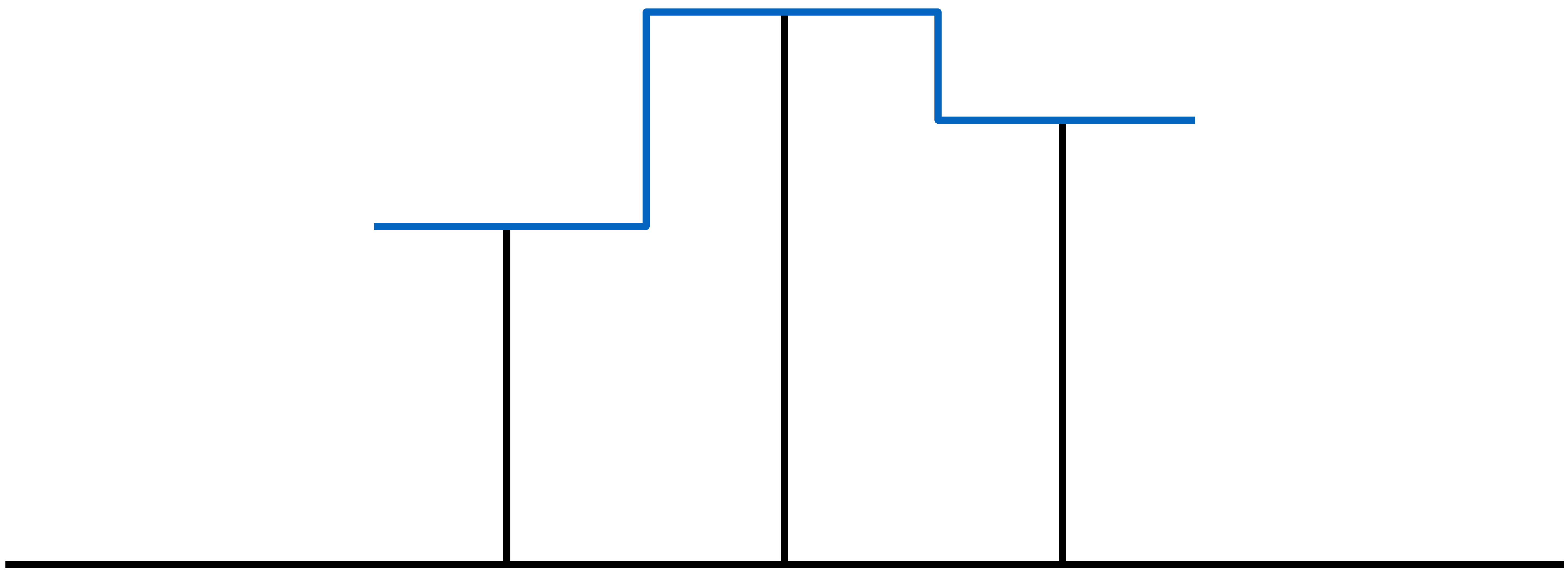
Bezier surfaces/Bezier гадаргуу

Cubic Hermite Interpolation

Зорилго: Утгуудыг интеполяцлах

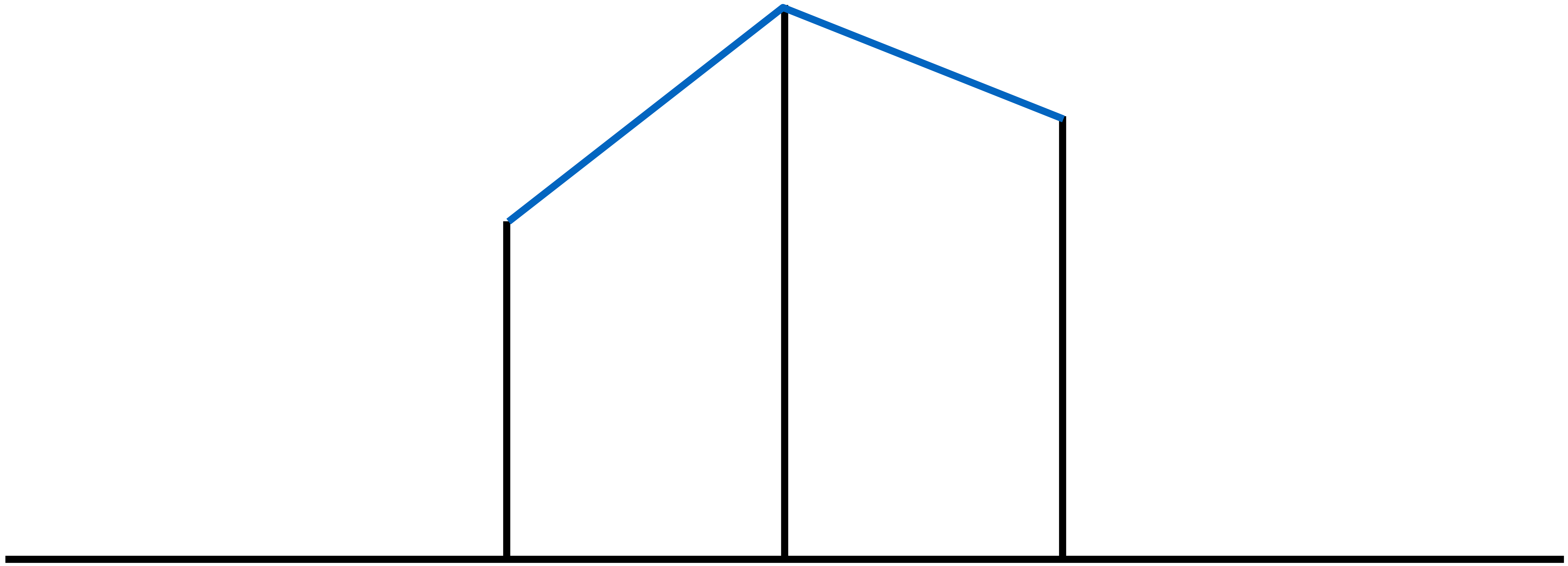


Ойролцоох хөрш интерполяци



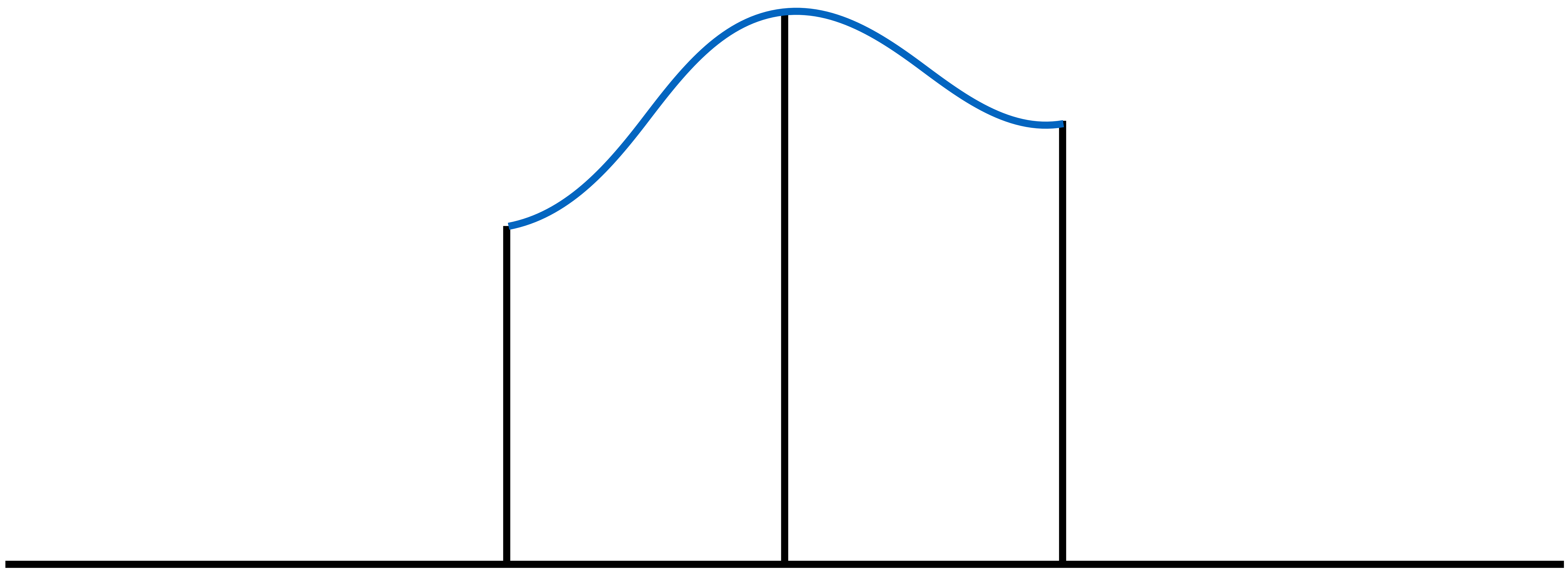
Асуудал: утгууд нь тасралтгүй байна

Linear Interpolation/Шугаман интерполяци

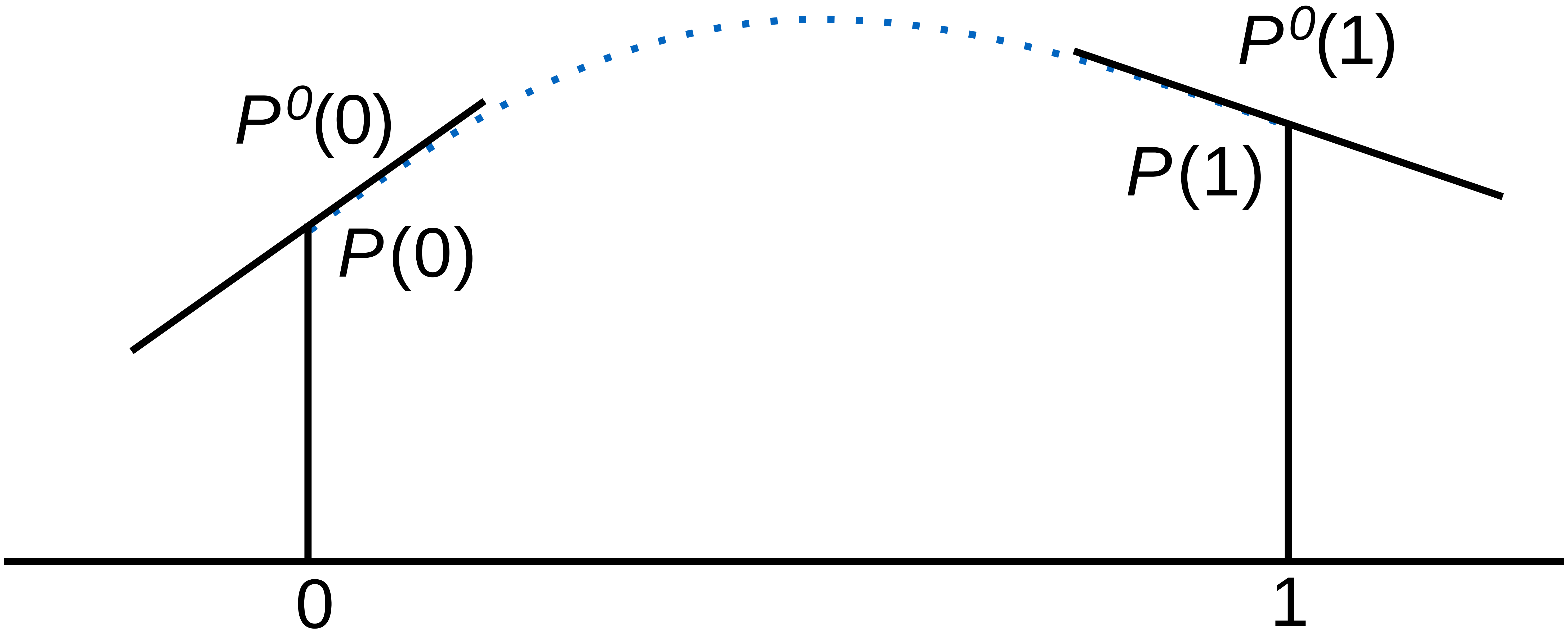


Асуудал: Уламжлал тасралтгүй биш байна

Smooth Interpolation?



Cubic Hermite Interpolation



Ороnт: values and derivatives at endpoints

Cubic Polynomial Interpolation

Cubic polynomial

$$P(t) = a\,t^3 + b\,t^2 + c\,t + d$$

Why cubic?

4 input constraints – need 4 degrees of freedom

$$P(0) = h_0$$

$$P(1) = h_1$$

$$P^{0}(0) = h_2$$

$$P^{0}(1) = h_3$$

Cubic Polynomial Interpolation

Cubic polynomial

$$P(t) = a\,t^3 + b\,t^2 + c\,t + d$$

$$P^{0}(t) = 3a\,t^2 + 2b\,t + c$$

Set up constraint equations

$$P(0) = h_0 = d$$

$$P(1) = h_1 = a + b + c + d$$

$$P^{0}(0) = h_2 = c$$

$$P^{0}(1) = h_3 = 3a + 2b + c$$

Solve for Polynomial Coefficients/Олон гишүүнт коэффициентийн шийдэл

$$h_0 = d$$

$$h_1 = a + b + c + d$$

$$h_2 = c$$

$$h_3 = 3a + 2b + c$$

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Solve for Polynomial Coefficients/Олон гишүүнт коэффициентийн шийдэл

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

(Check that these matrices are inverses)

Hermite функцын матриц хэлбэр

$$
\begin{aligned}
P(t) &= a\,t^3 + b\,t^2 + c\,t + d \\
&= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\
&= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} \\
&= H_0(t)\,h_0 + H_1(t)\,h_1 + H_2(t)\,h_2 + H_3(t)\,h_3
\end{aligned}
$$

Hermite функцын матрицын хэлбэр

$$P(t) = \boxed{a}\, t^3 + b\, t^2 + c\, t + d$$

$$= \begin{bmatrix} \boxed{t^3} & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$= H_0(t)\ h_0 + H_1(t)\ h_1 + H_2(t)\ h_2 + H_3(t)\ h_3$$

Matrix rows = coefficient formulas

Hermite функцын матрицын хэлбэр

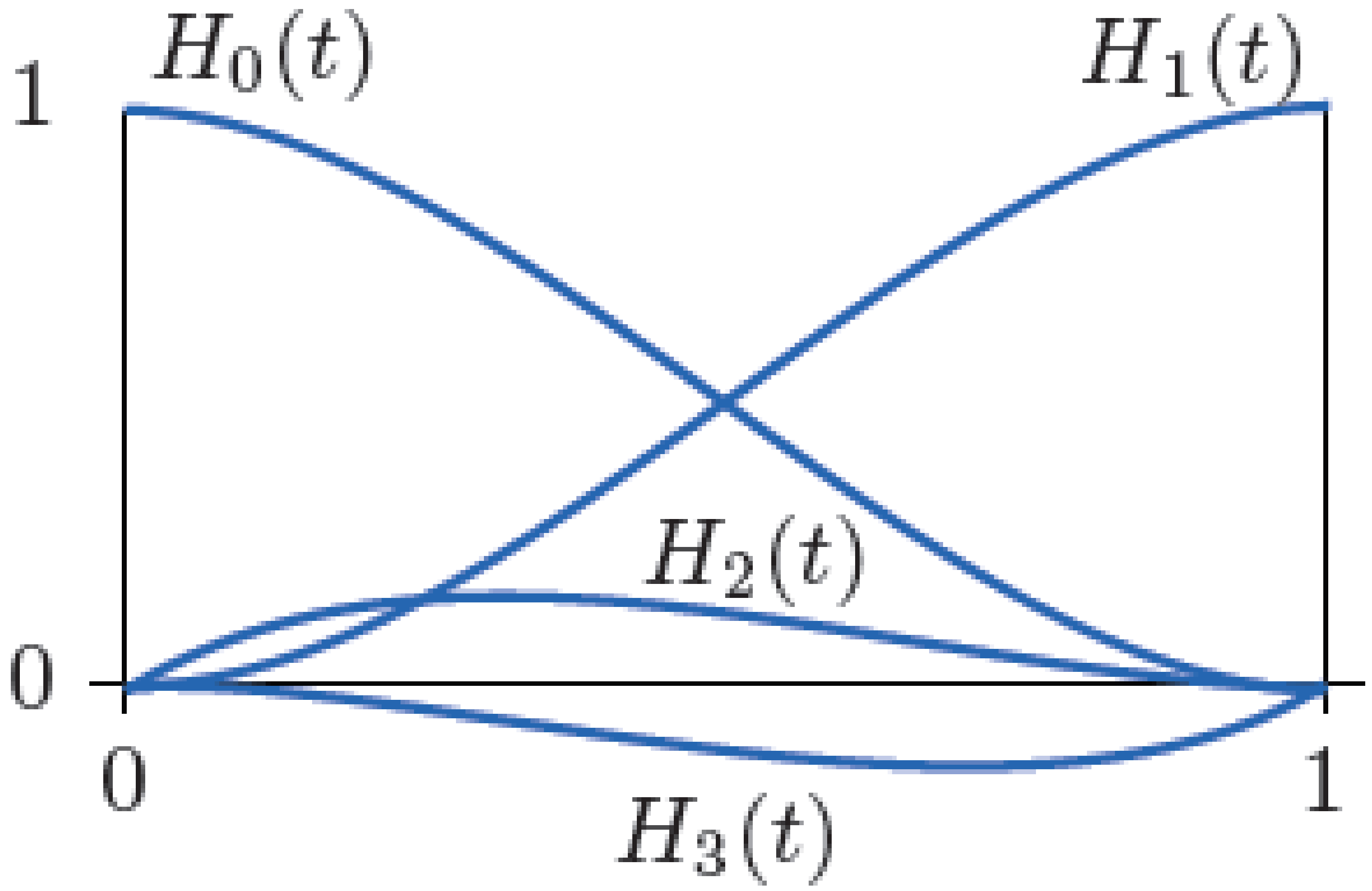
$$P(t) = a\,t^3 + b\,t^2 + c\,t + d$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$= H_0(t)\;h_0 + H_1(t)\;h_1 + H_2(t)\;h_2 + H_3(t)\;h_3$$

Matrix columns = Hermite basis functions
Call this matrix the Hermite basis matrix

Hermite Үндсэн функц

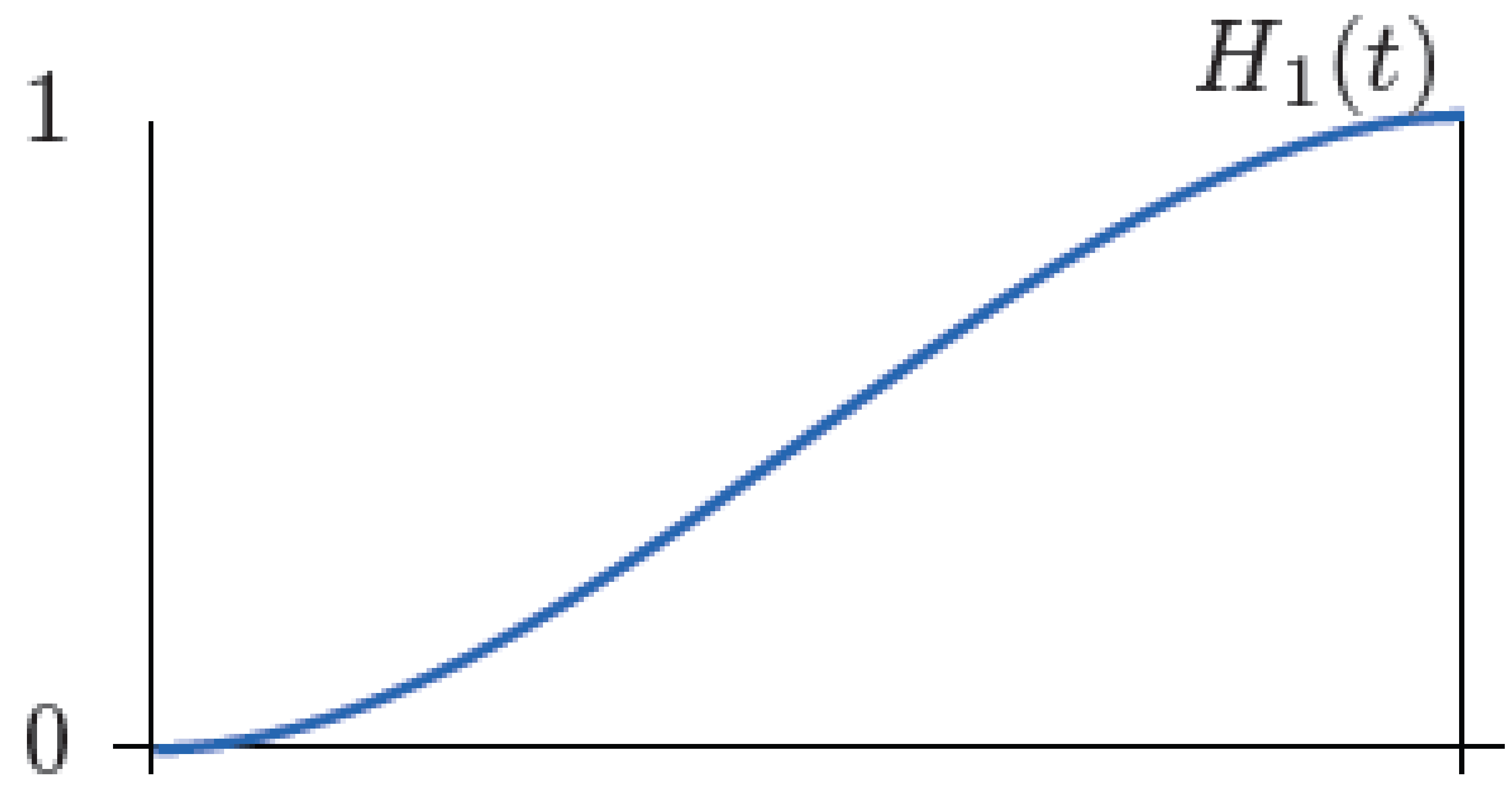


$$H_0(t) = 2t^3 - 3t^2 + 1$$
$$H_1(t) = -2t^3 + 3t^2$$
$$H_2(t) = t^3 - 2t^2 + t$$
$$H_3(t) = t^3 - t^2$$

Хялбар функц

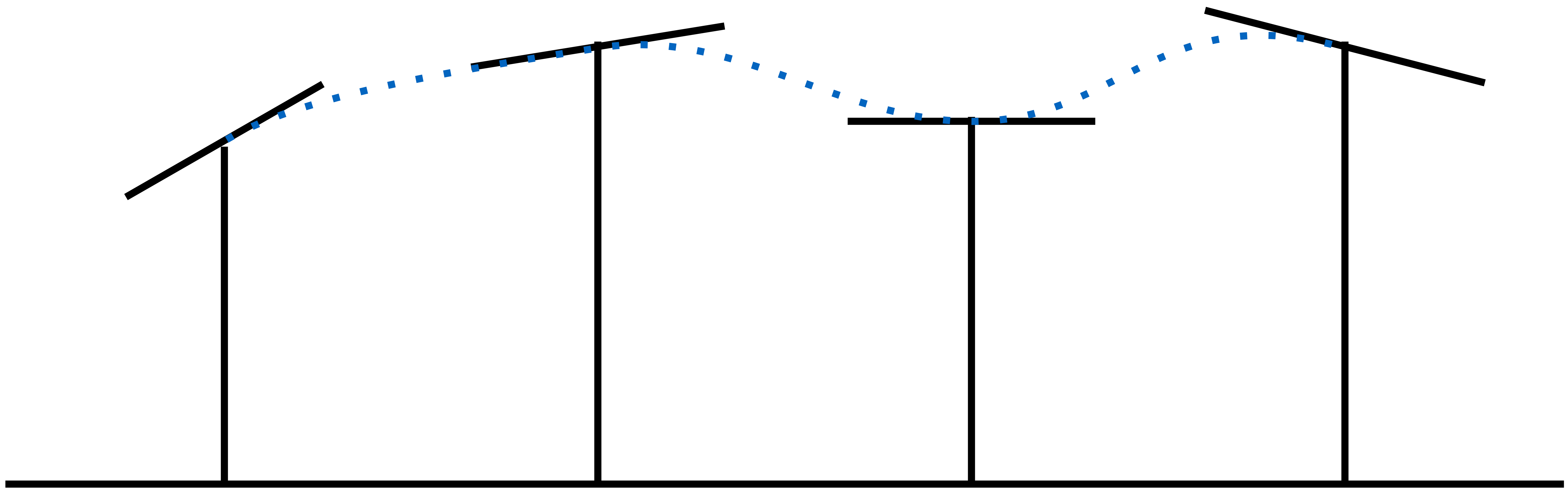
Хамгийн өргөн хэрэглэгддэг функц

Хөдөлгөөнд аажуухан эхлээд аажуухан зогсоох (zero velocity)



$$H_1(t) = -2t^3 + 3t^2 = t^2(3-2t)$$

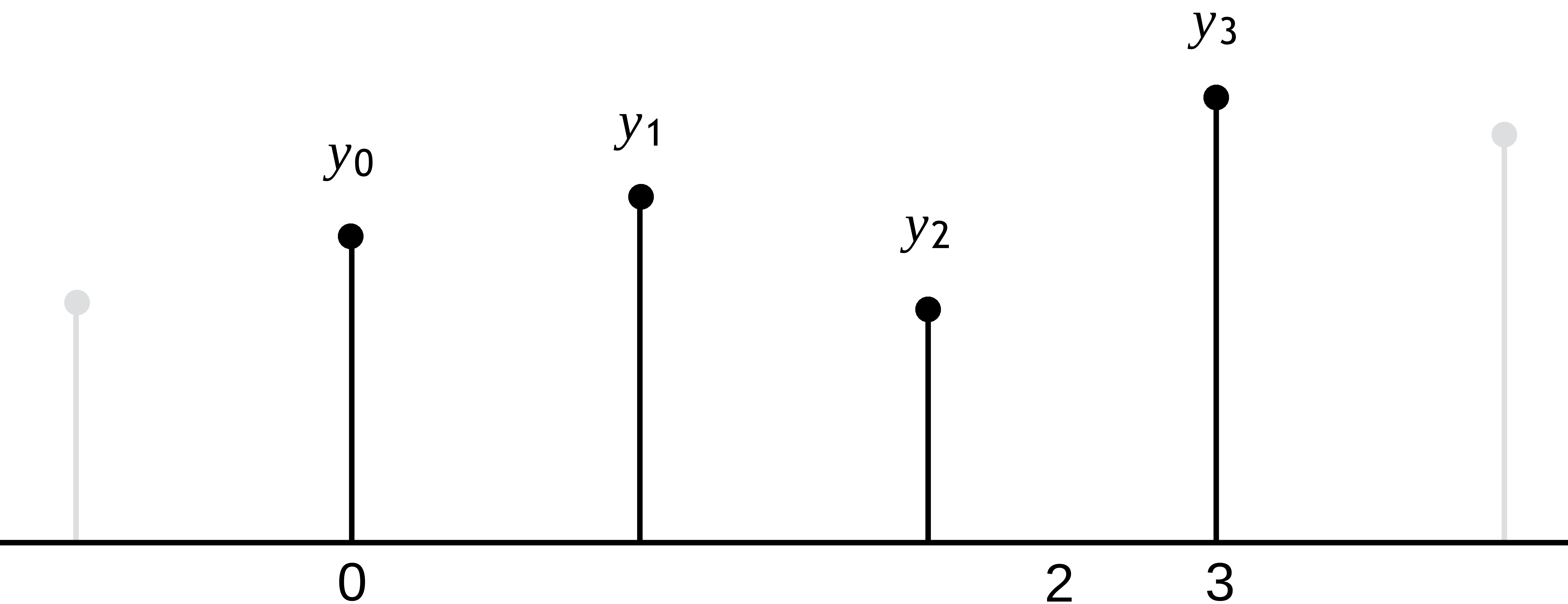
Hermite Spline Интерполяци



Оролт: sequence of values and derivatives

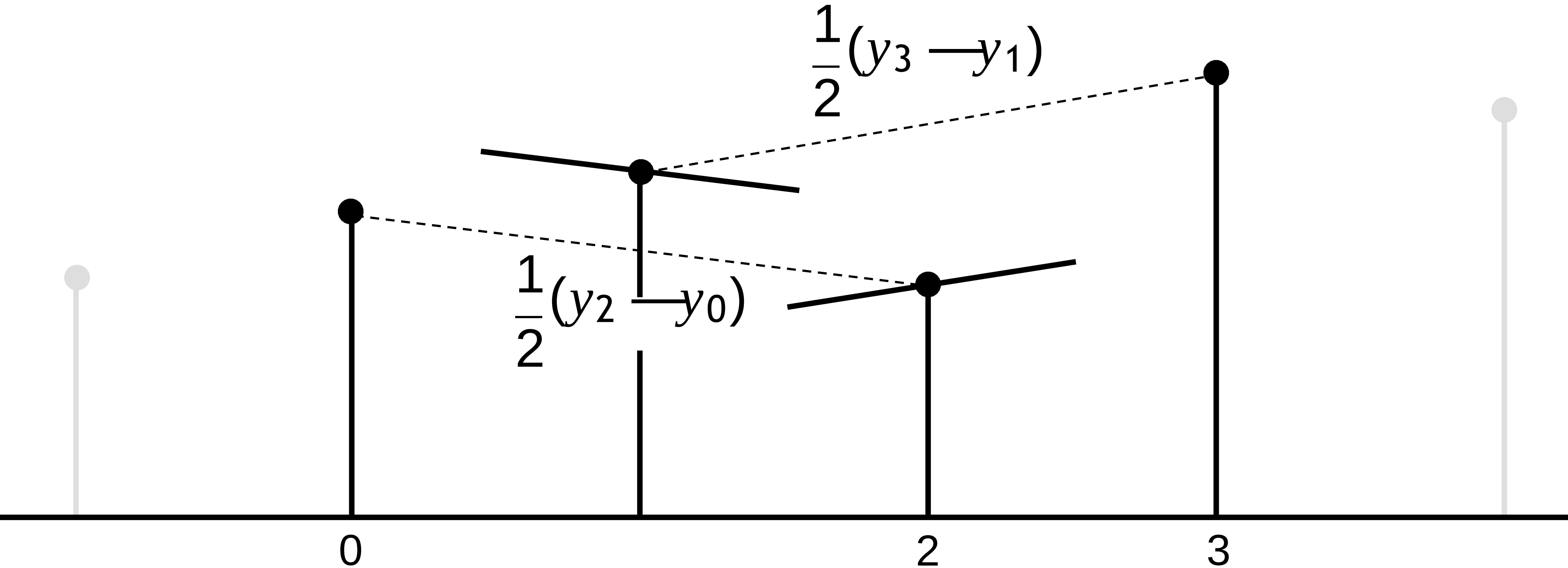
Catmull-Rom Интерполяци

Catmull-Rom Интерполяци



Оролт: sequence of values

Catmull-Rom Интерполяци

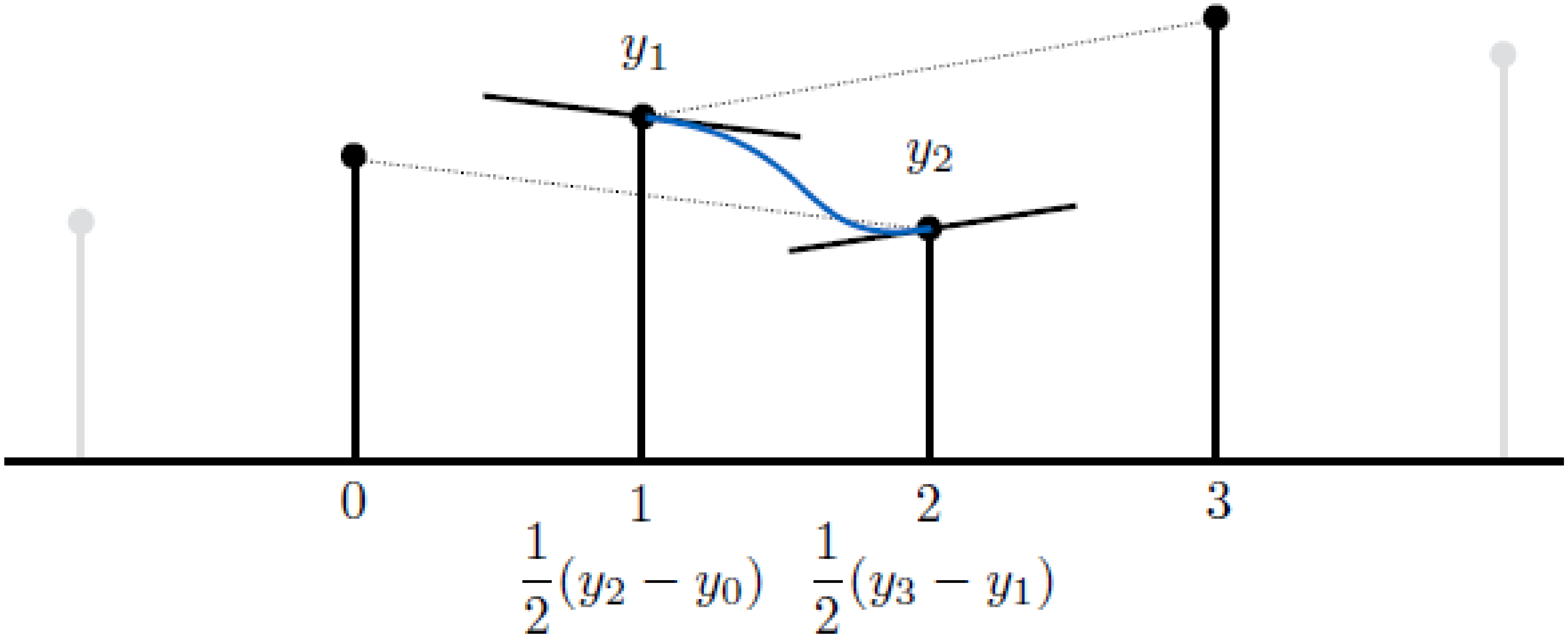


Rule for derivatives:

Match slope between previous and next values

Өмнөх болон дараагийн утгуудын хоорондох налууг тааруулна

Catmull-Rom Interpolation



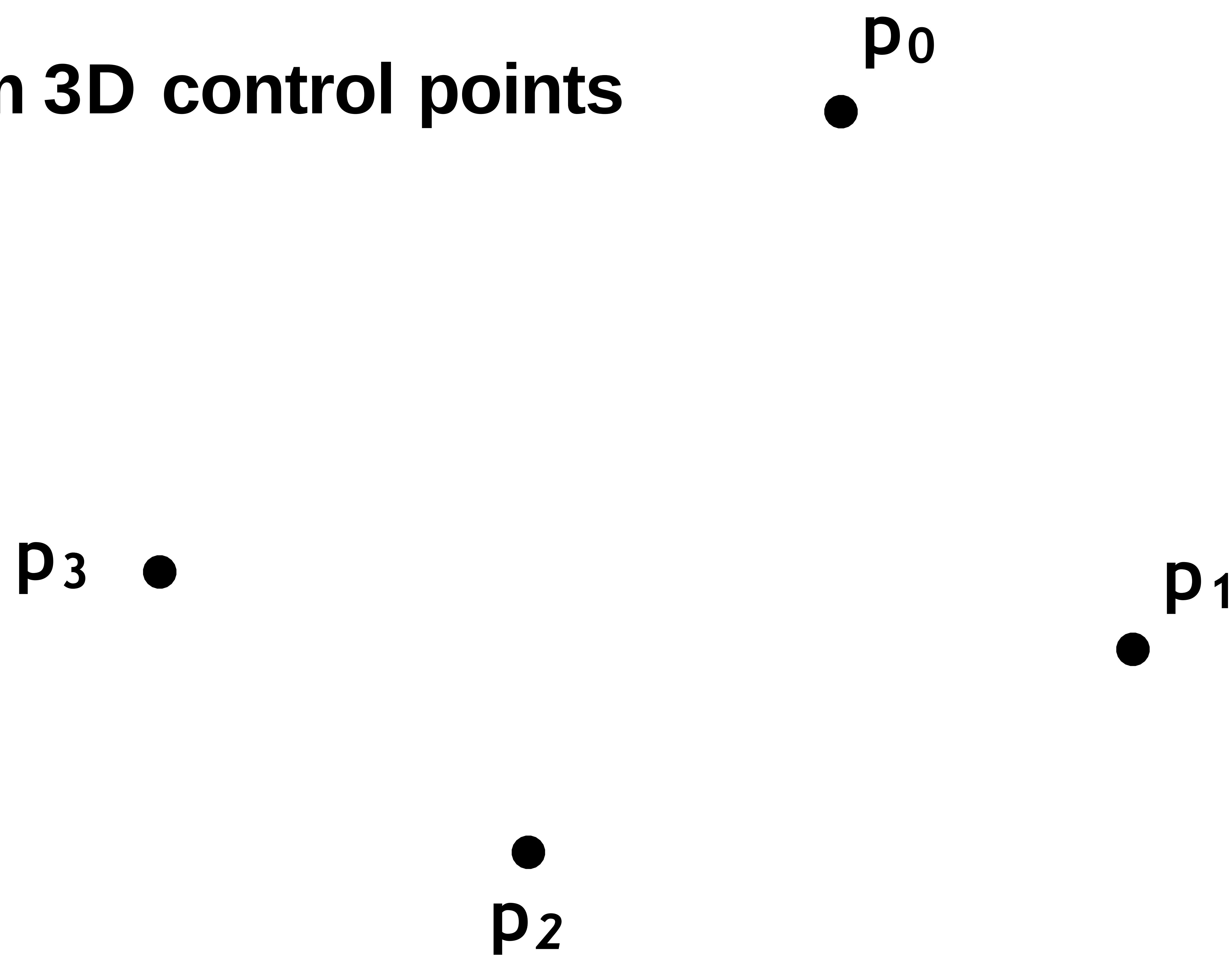
Then use Hermite interpolation

Дараагаар нь Hermite интерполяци ашиглана.

Цэг болон векторуудыг интерполяцлах

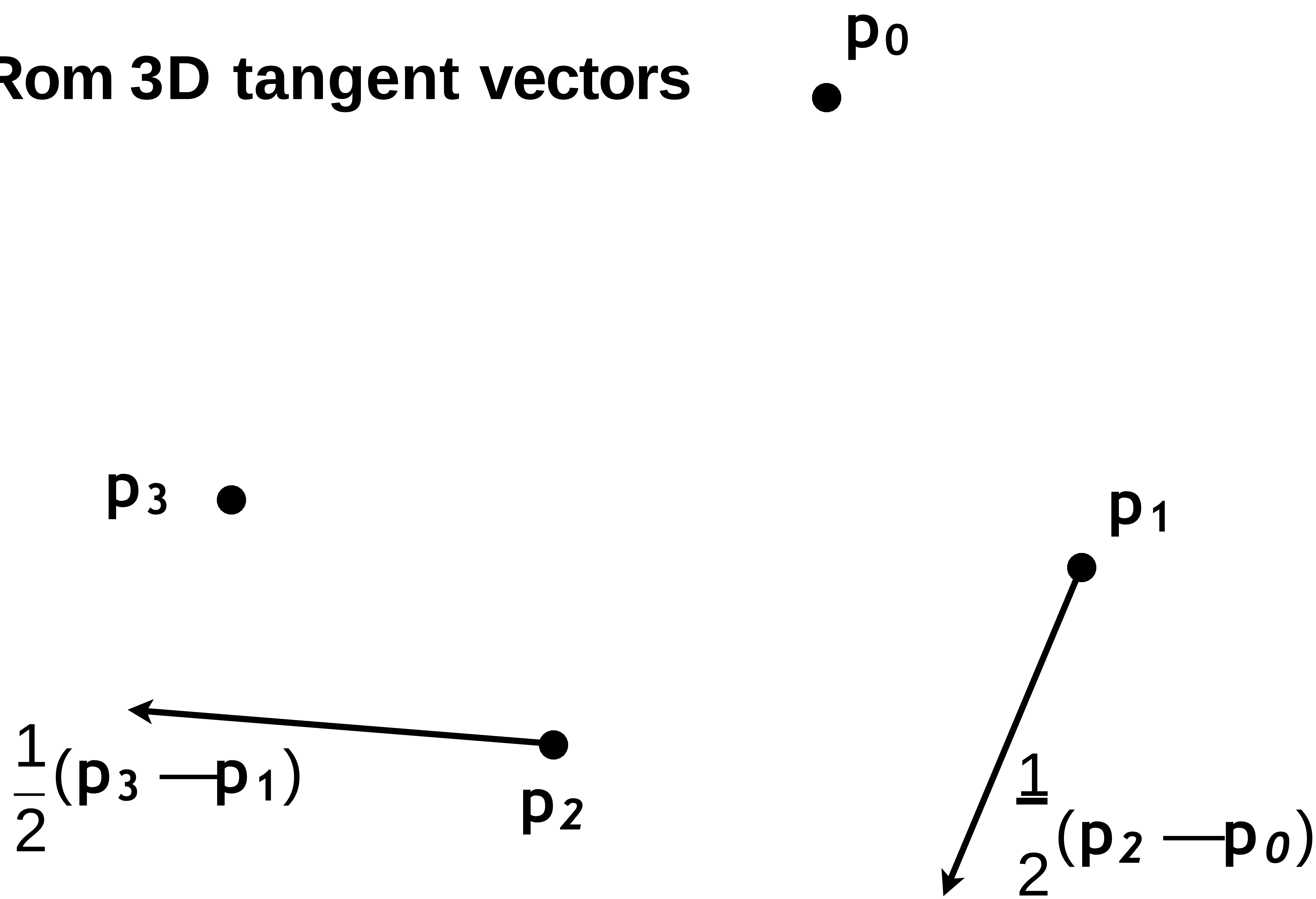
Цэгүүдийг утгуудын адил хялбар интерполяци хийнэ.

Catmull-Rom 3D control points



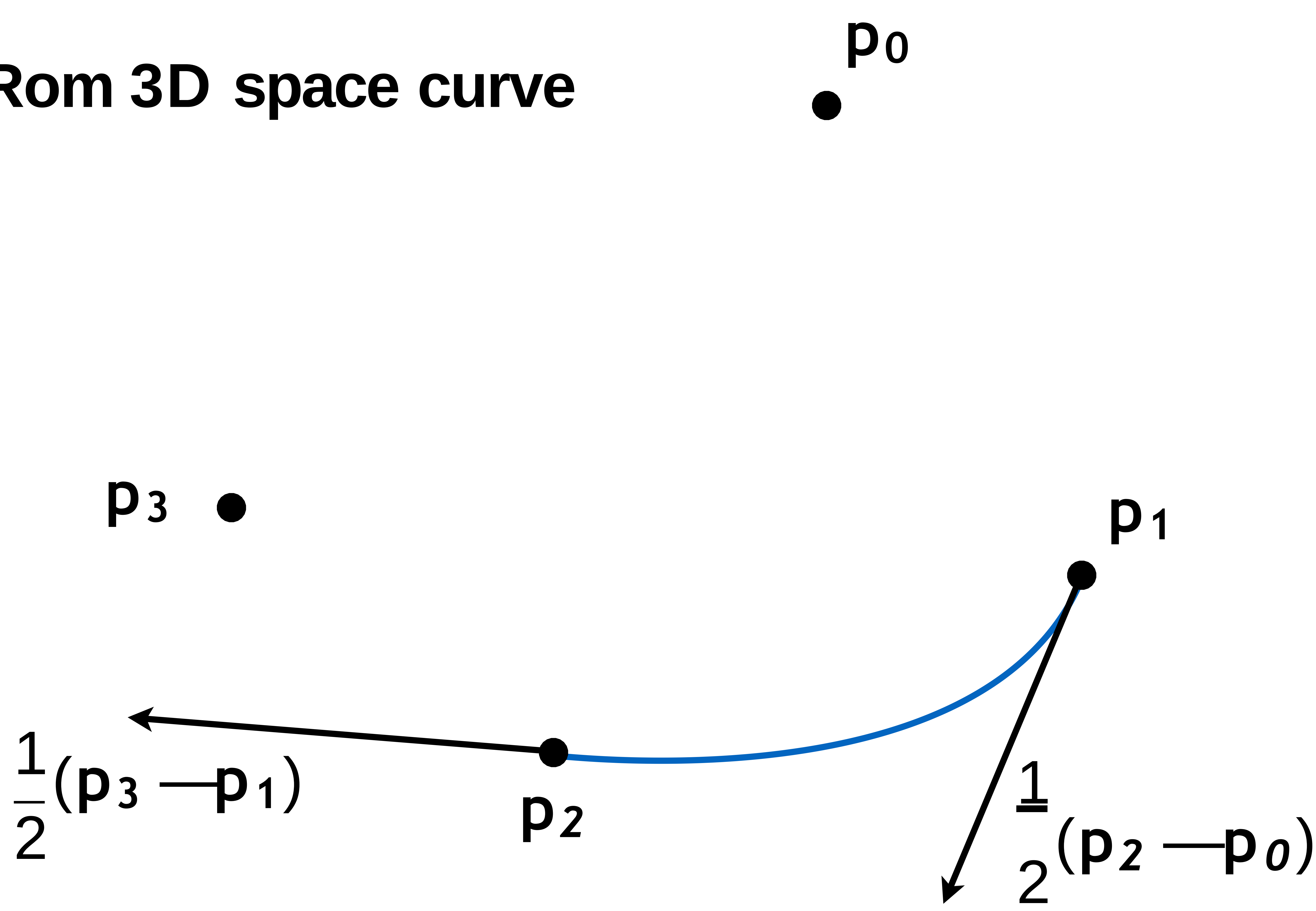
Цэгүүдийг утгуудын адил хялбар интерполяци хийнэ.

Catmull-Rom 3D tangent vectors



Цэгүүдийг утгуудын адил хялбар интерполяци хийнэ.

Catmull-Rom 3D space curve



Муруйг тодорхойлоход үндсэн функцүүд ашиглах нь

Иртерполяцийн ерөнхий томъёо

$$\mathrm{p}(t) = \sum_{i=0}^{n} \mathrm{p}_i F_i(t)$$

$$x(t) = \sum_{i=0}^{n} x_i F_i(t) \qquad y(t) = \sum_{i=0}^{n} y_i F_i(t) \qquad z(t) = \sum_{i=0}^{n} z_i F_i(t)$$

Коэффициент p_i нь цэг & вектор, Зөвхөн Fi (t) утга бус

харин интерполяцийн схемд зориулсан үндсэн

функцүүд байна.

H_i(t) Hermite интерполяцыг бид өмнө үзсэн. Ci (t) Catmull-Rom –ын **C_i(t)** удахгүй үзэх ба Bézier схемийн **B_i(t)** хувьд дараа үзнэ. Үндсэн функц нь интерполяцийн схемийн шинж чанар(properties) юм.

Catmull-Rom муруйн матриц хэлбэр?

Hermite матриц хэлбэрийг ашигладаг.

- **Цэг ба шүргэгч нь Catmull-Rom дүрмээр өгөгдсөн байна.**

Hermite points

$$\mathbf{h}_0 = \mathbf{p}_1$$

$$\mathbf{h}_1 = \mathbf{p}_2$$

Hermite tangents

$$\mathbf{h}_2 = \tfrac{1}{2}(\mathbf{p}_2 - \mathbf{p}_0)$$

$$\mathbf{h}_3 = \tfrac{1}{2}(\mathbf{p}_3 - \mathbf{p}_1)$$

$$P(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

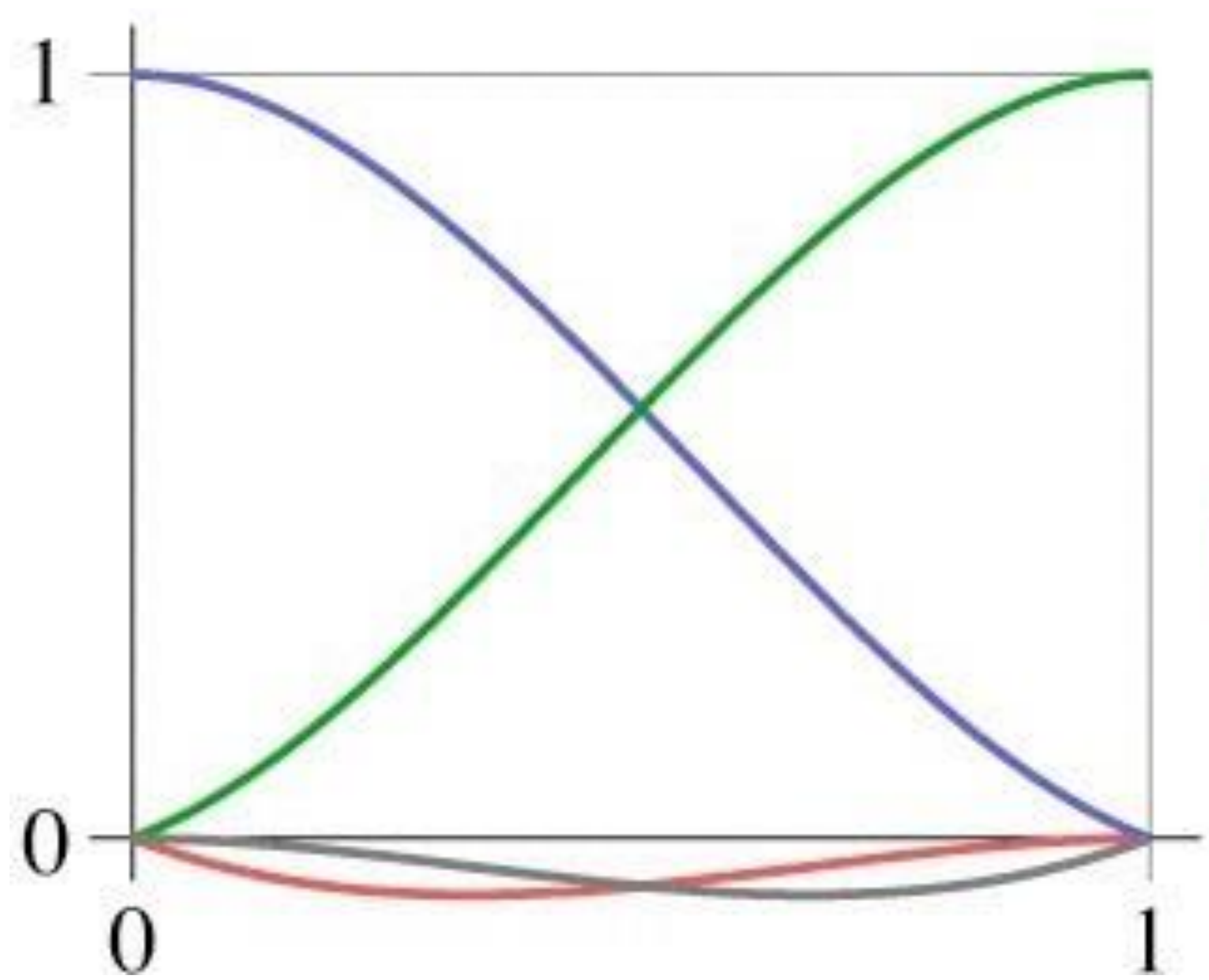
Catmull-Rom муруйн матриц хэлбэр

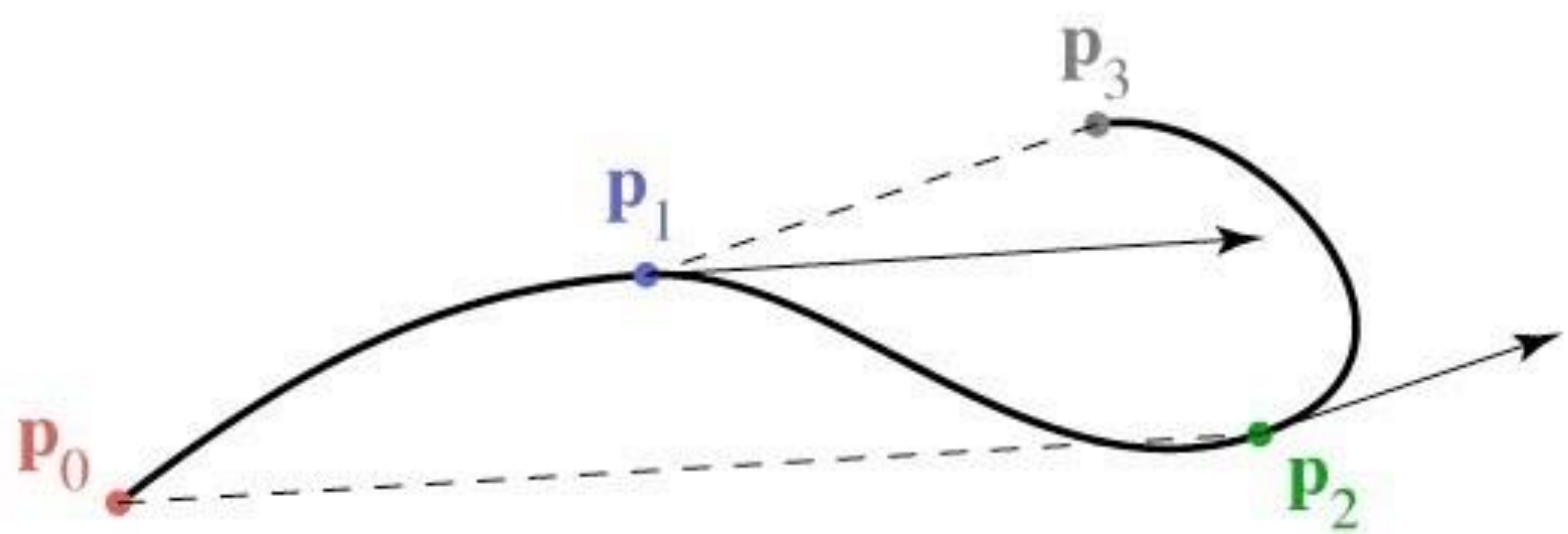
$$P(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \\ 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{p}_0 \\ \mathrm{p}_1 \\ \mathrm{p}_2 \\ \mathrm{p}_3 \end{bmatrix}$$

$$= C_0(t)\ \mathrm{p}_0 + C_1(t)\ \mathrm{p}_1 + C_2(t)\ \mathrm{p}_2 + C_3(t)\ \mathrm{p}_3$$

Matrix columns = Catmull-Rom basis functions

Catmull-Rom Үндсэн Функц

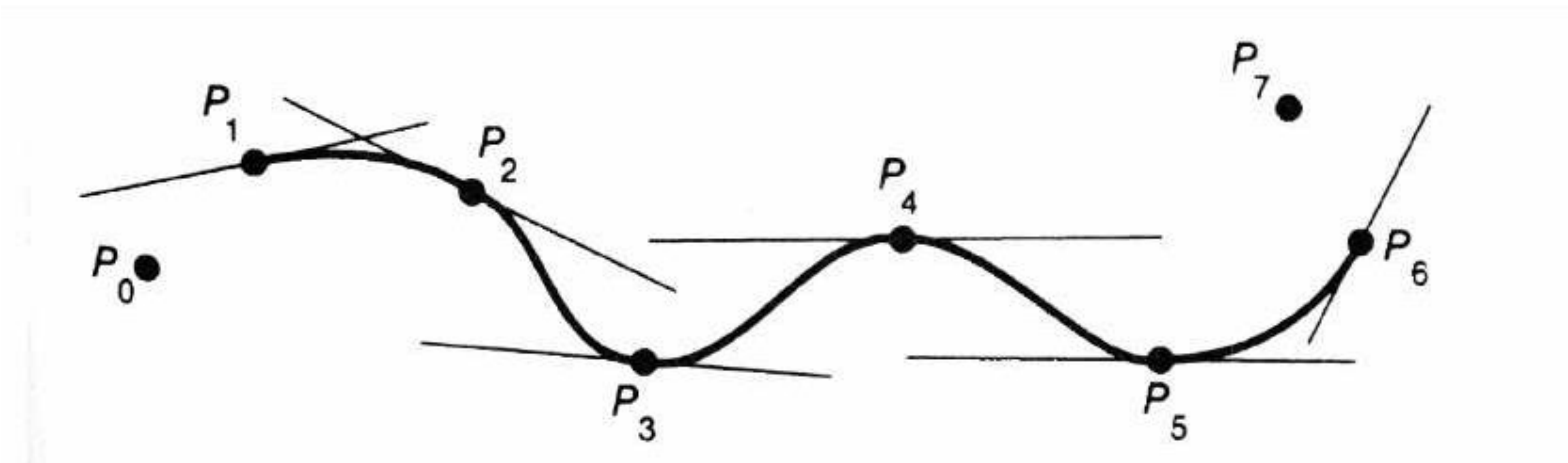




Catmull-Rom Spline

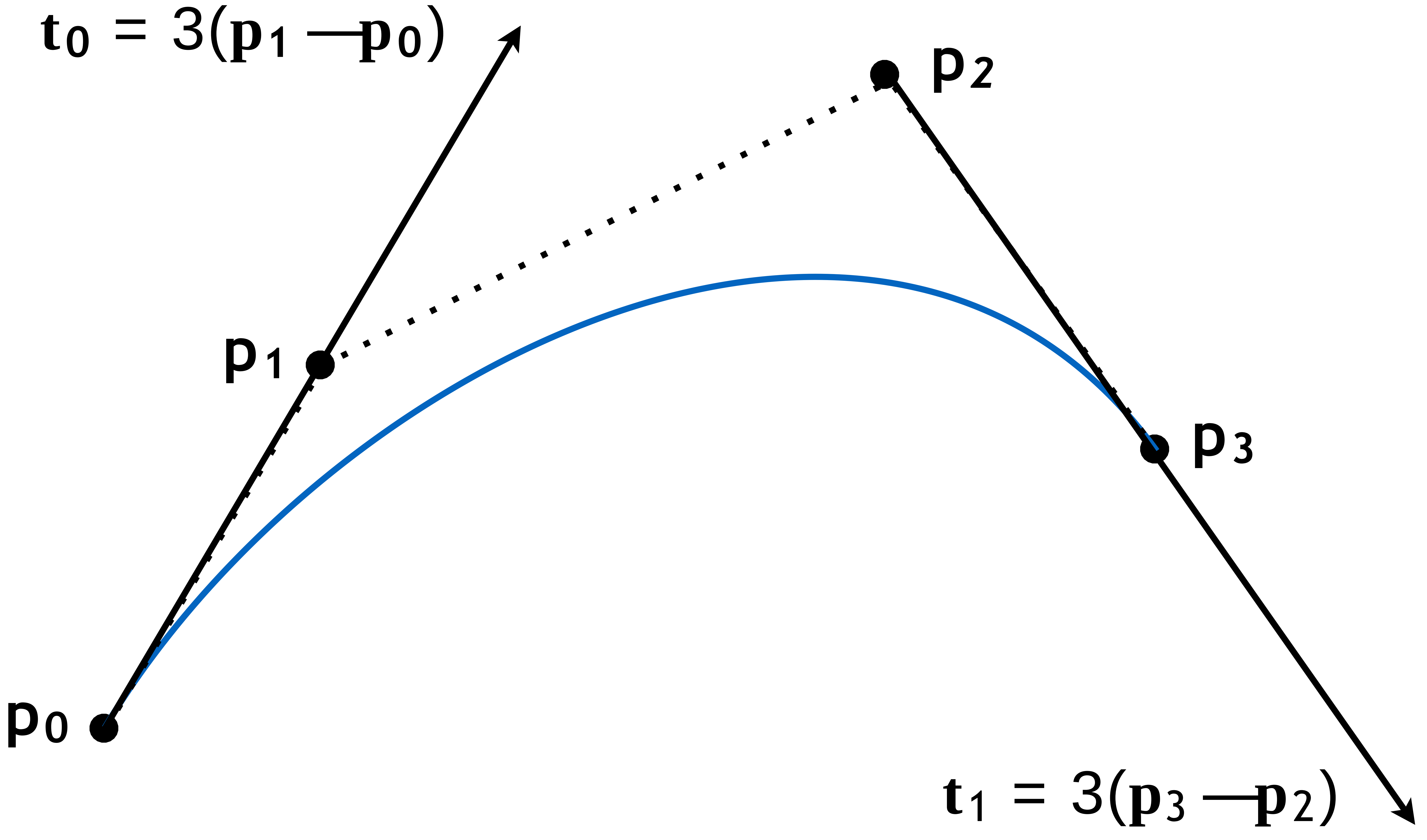
Оролт: sequence of points

Гаралт: spline that interpolates all points with C1 continuity



Bézier Curves/ Bézier муруй

Defining Cubic Bézier Curve With Tangents



Cubic Bézier муруйн матриц хэлбэр?

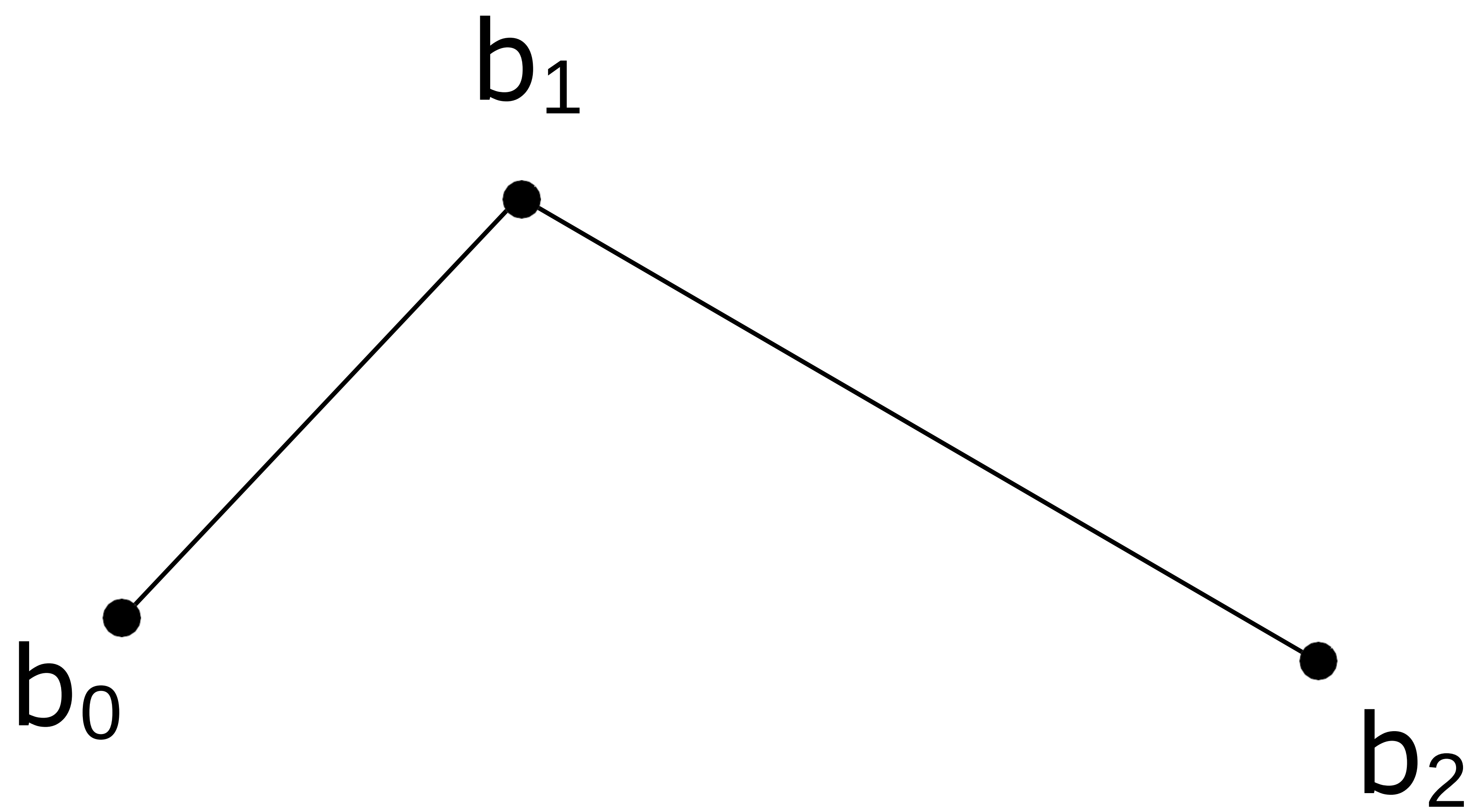
$$P(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} \mathrm{p}_0 \\ \mathrm{p}_1 \\ \mathrm{p}_2 \\ \mathrm{p}_3 \end{bmatrix}$$

$$= B_0^3(t)\ \mathrm{p}_0 + B_1^3(t)\ \mathrm{p}_1 + B_2^3(t)\ \mathrm{p}_2 + B_3^3(t)\ \mathrm{p}_3$$

Good exercise to derive this matrix yourself.
One way: use Hermite matrix equation again.
What are the points and tangents?

Bézier Curves – de Casteljau Алгоритм

3 цэг авч үзнэ.

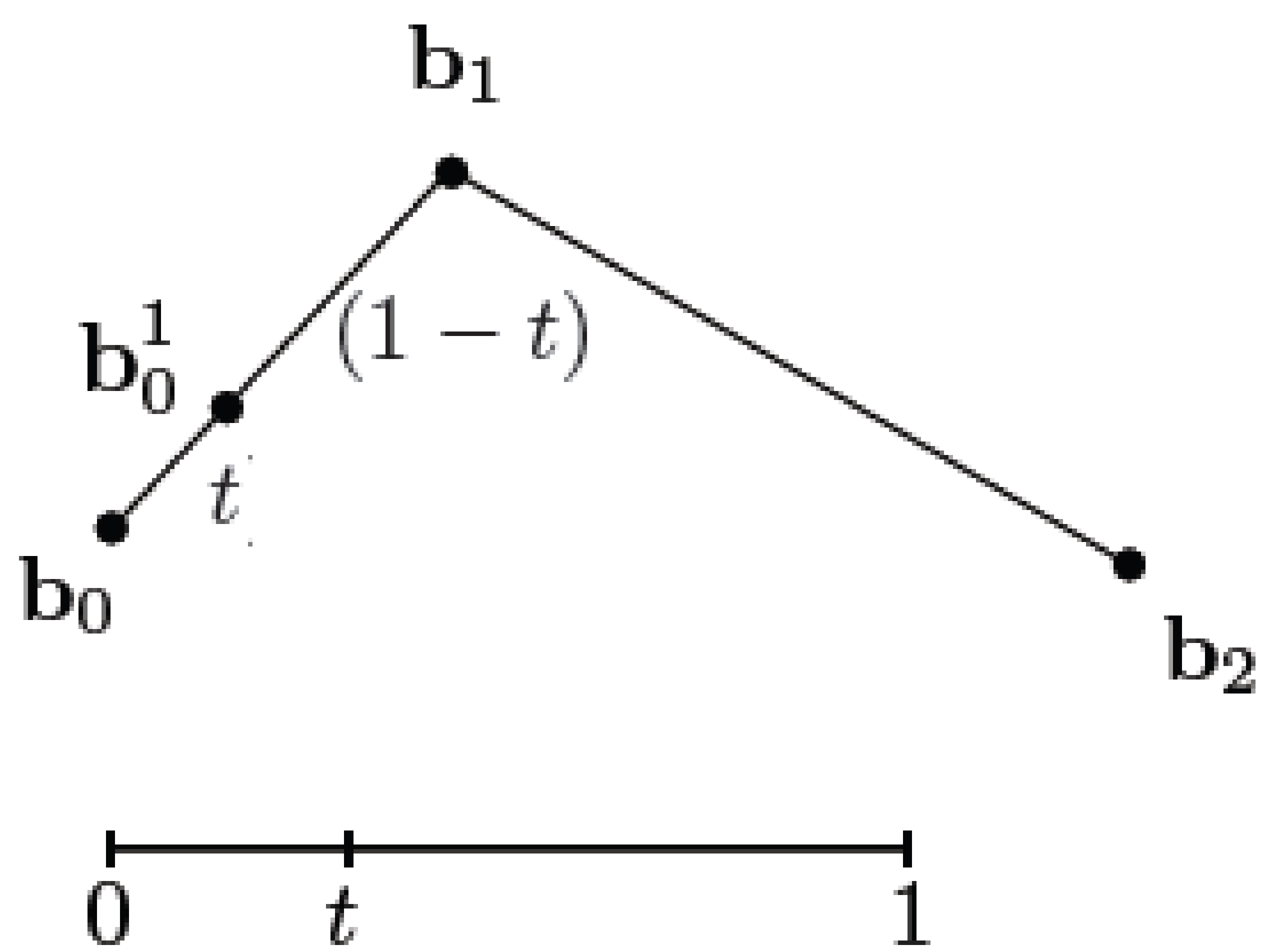


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Алгоритм

Шугаман интерполяц ашиглан цэг оруулах

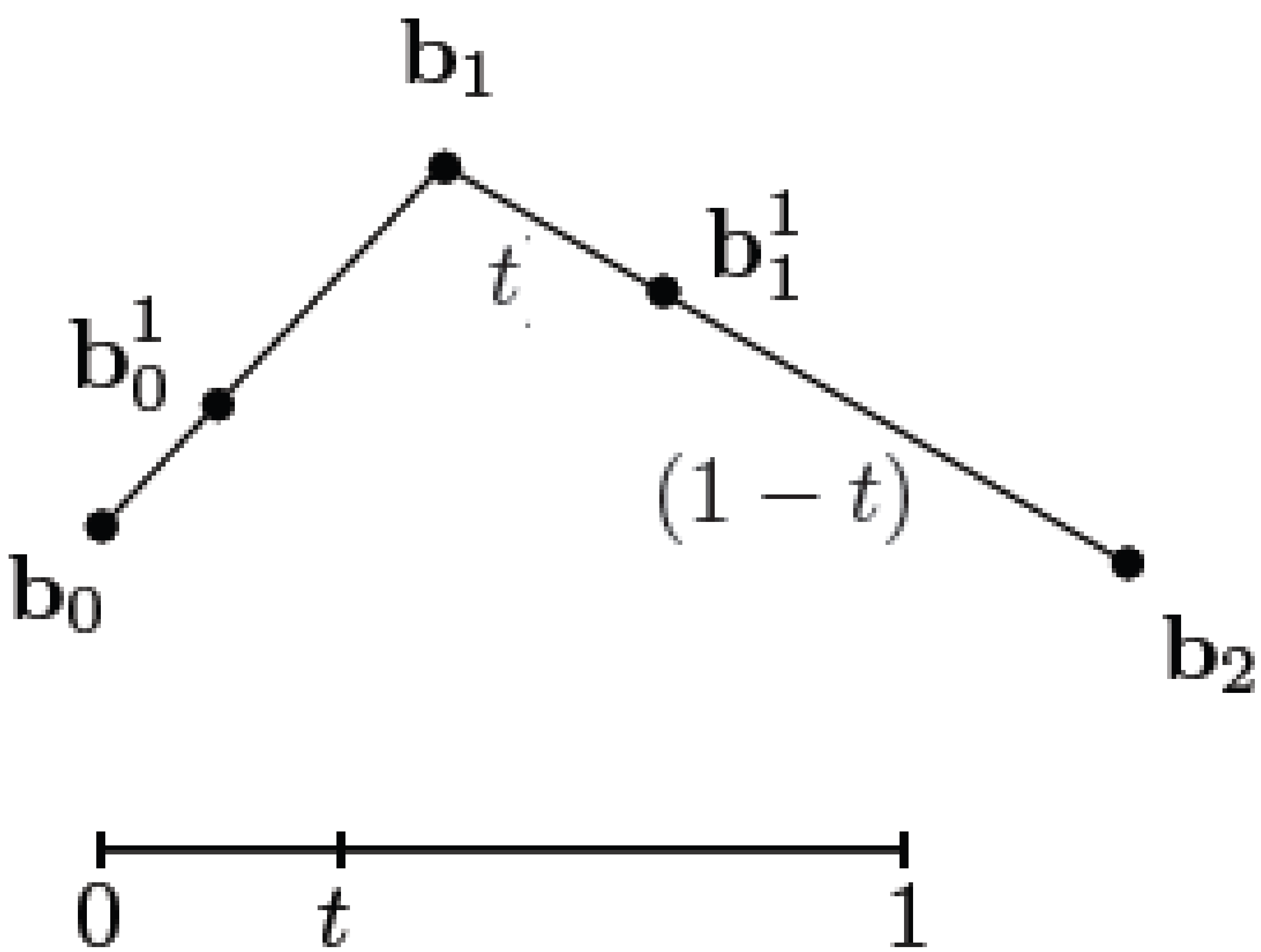


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Алгоритм

2 ирмэгийг оруулна.

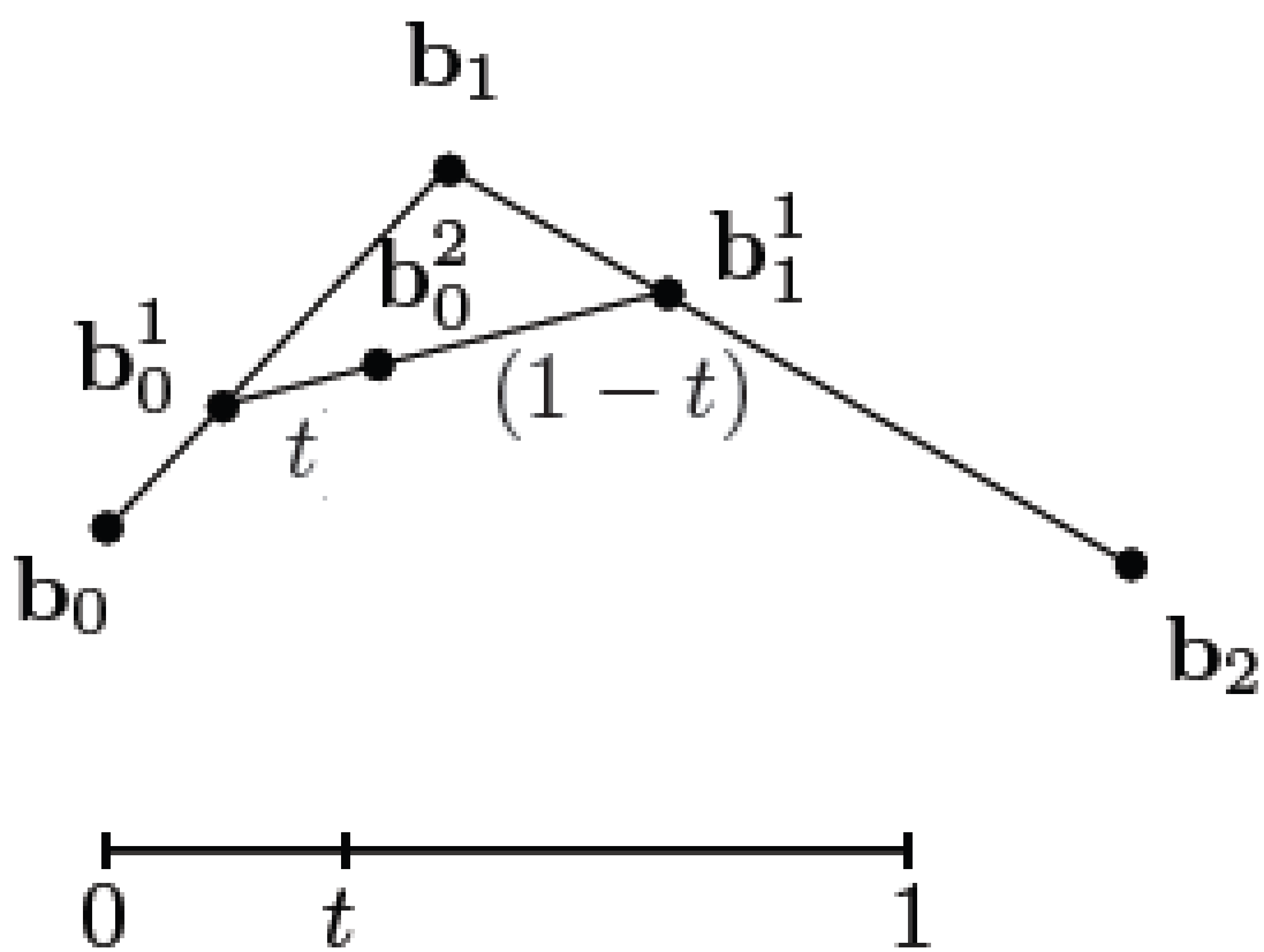


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Алгоритм

Рекурсив давталт

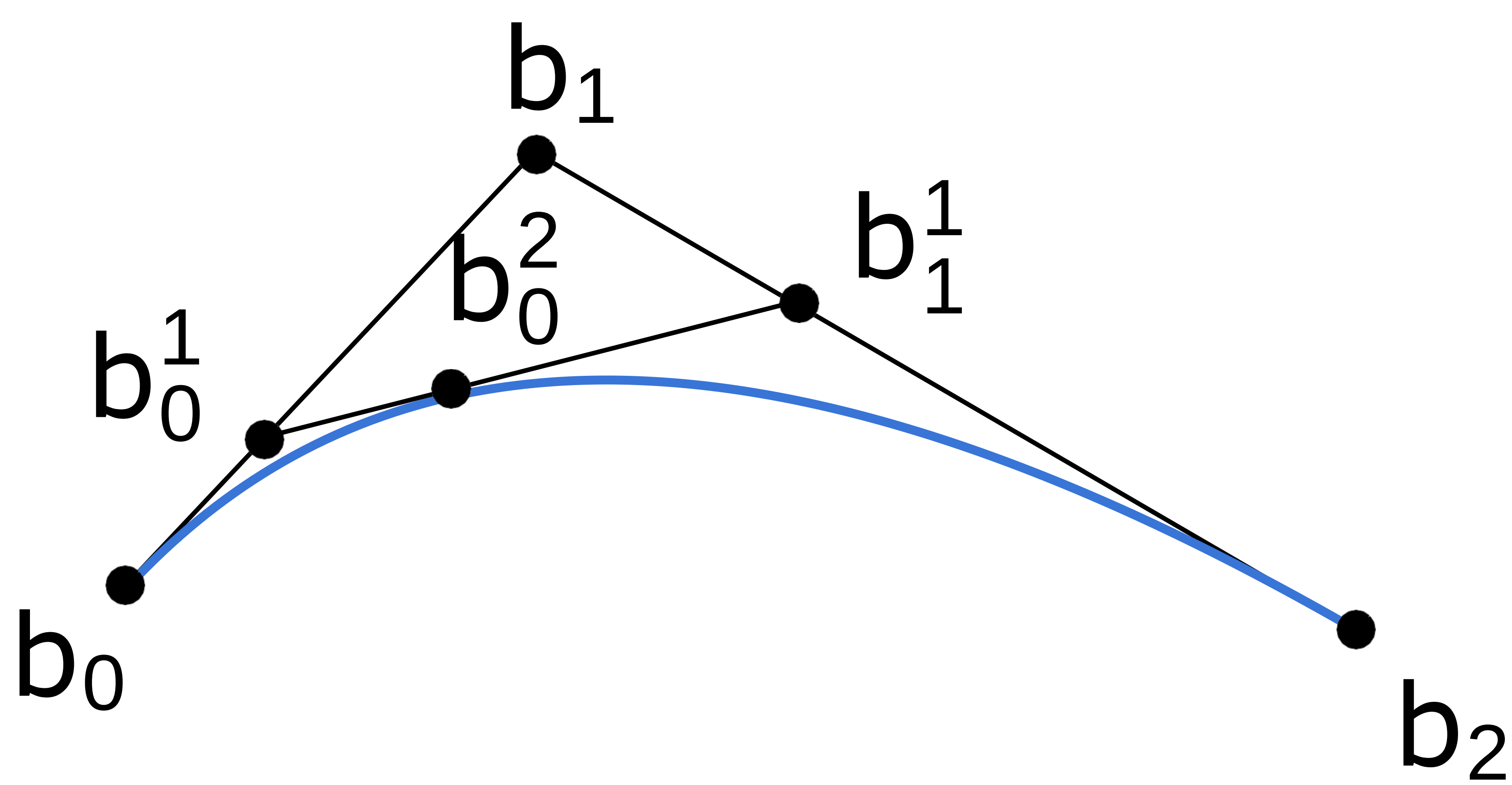


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Алгоритм

Муруйг тодорхойлох алгоритм

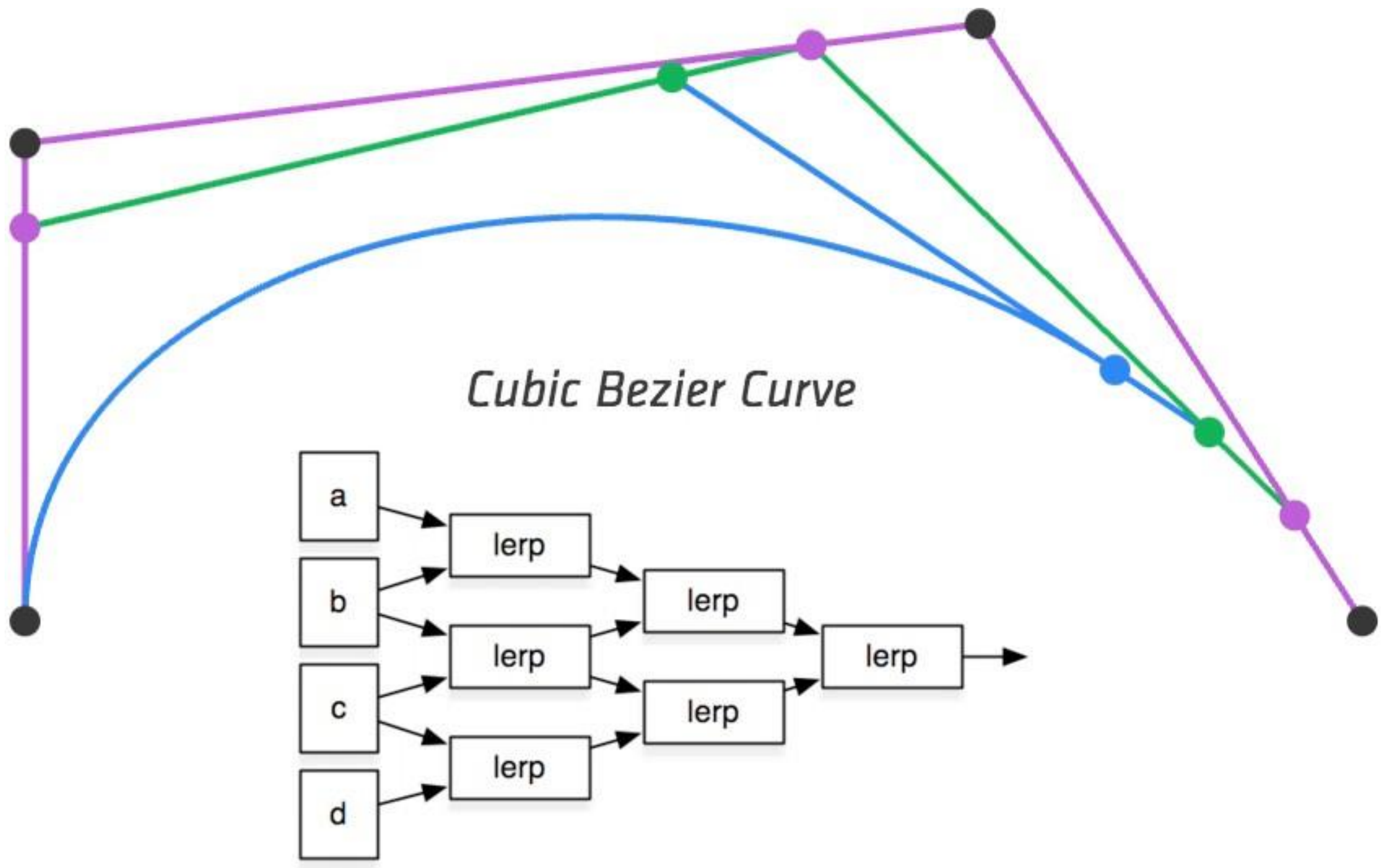


"Corner cutting" recursive subdivision

Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Visualizing de Casteljau алгоритмийг дүрслэх

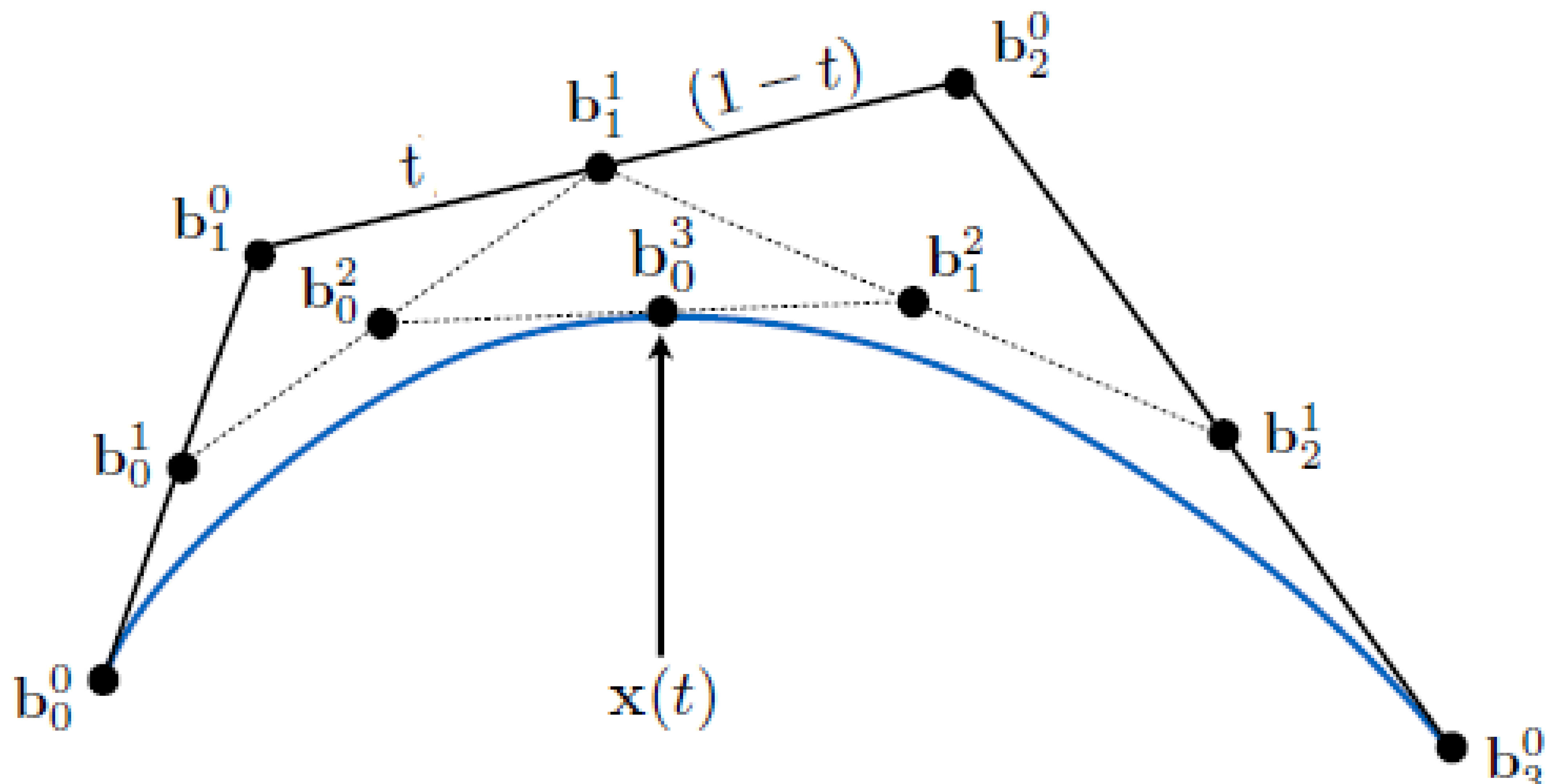


Animation: Steven Wittens, Making Things with Maths, http://acko.net

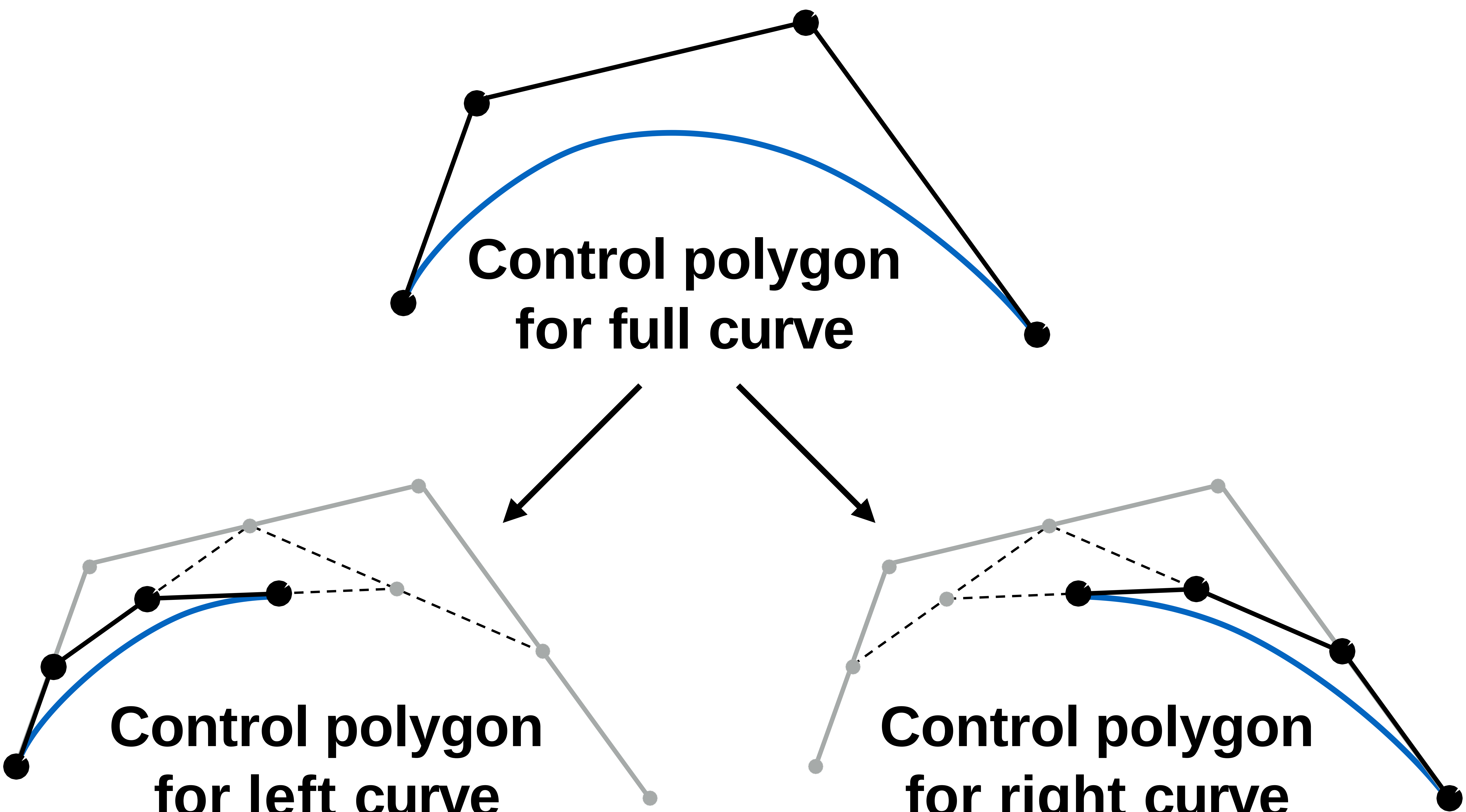
Cubic Bézier Curve – de Casteljau

4 цэг авч үзнэ.

Рекурсив шугаман интерполяци ижил байна



de Casteljau algorithm Subdivides Curve
Муруйг дэд хэсгүүдэд хуваах de Casteljau Алгоритм



Bézier муруйн шинж чанар

Төгсгөлийн цэг интерполяци

- **cubic Bézier-ын хувьд:** $\mathrm{b}(0) = \mathrm{b}_0; \quad \mathrm{b}(1) = \mathrm{b}_3$

Tangent to end segments

- **Cubic case:** $\mathrm{b}'(0) = 3(\mathrm{b}_1 - \mathrm{b}_0); \quad \mathrm{b}'(1) = 3(\mathrm{b}_3 - \mathrm{b}_2)$

Affine transformation property

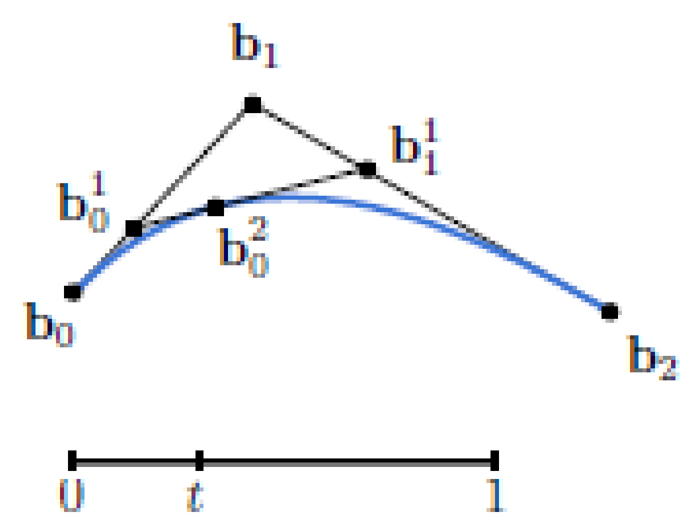
- **Transform curve by transforming control points**

Convex hull property

- **Curve is within convex hull of control points**

Bézier Curve – Алгебрийн томъёо

Гурван цэгийн квадрат Bézier муруй



$$\mathbf{b}_0^1(t) = (1-t)\mathbf{b}_0 + t\mathbf{b}_1$$

$$\mathbf{b}_1^1(t) = (1-t)\mathbf{b}_1 + t\mathbf{b}_2$$

$$\mathbf{b}_0^2(t) = (1-t)\mathbf{b}_0^1 + t\mathbf{b}_1^1$$

$$\mathbf{b}_0^2(t) = (1-t)^2\mathbf{b}_0 + 2t(1-t)\mathbf{b}_1 + t^2\mathbf{b}_2$$

Bézier Curve – Ерөнхий Алгебрийн томъёо

N дарааллын Bézier муруйн Bernstein хэлбэр

$$\mathrm{b}^n(t) = \mathrm{b}_0^n(t) = \sum_{j=0}^{n} \mathrm{b}_j B_j^n(t)$$

Bézier удирдлагын цэг

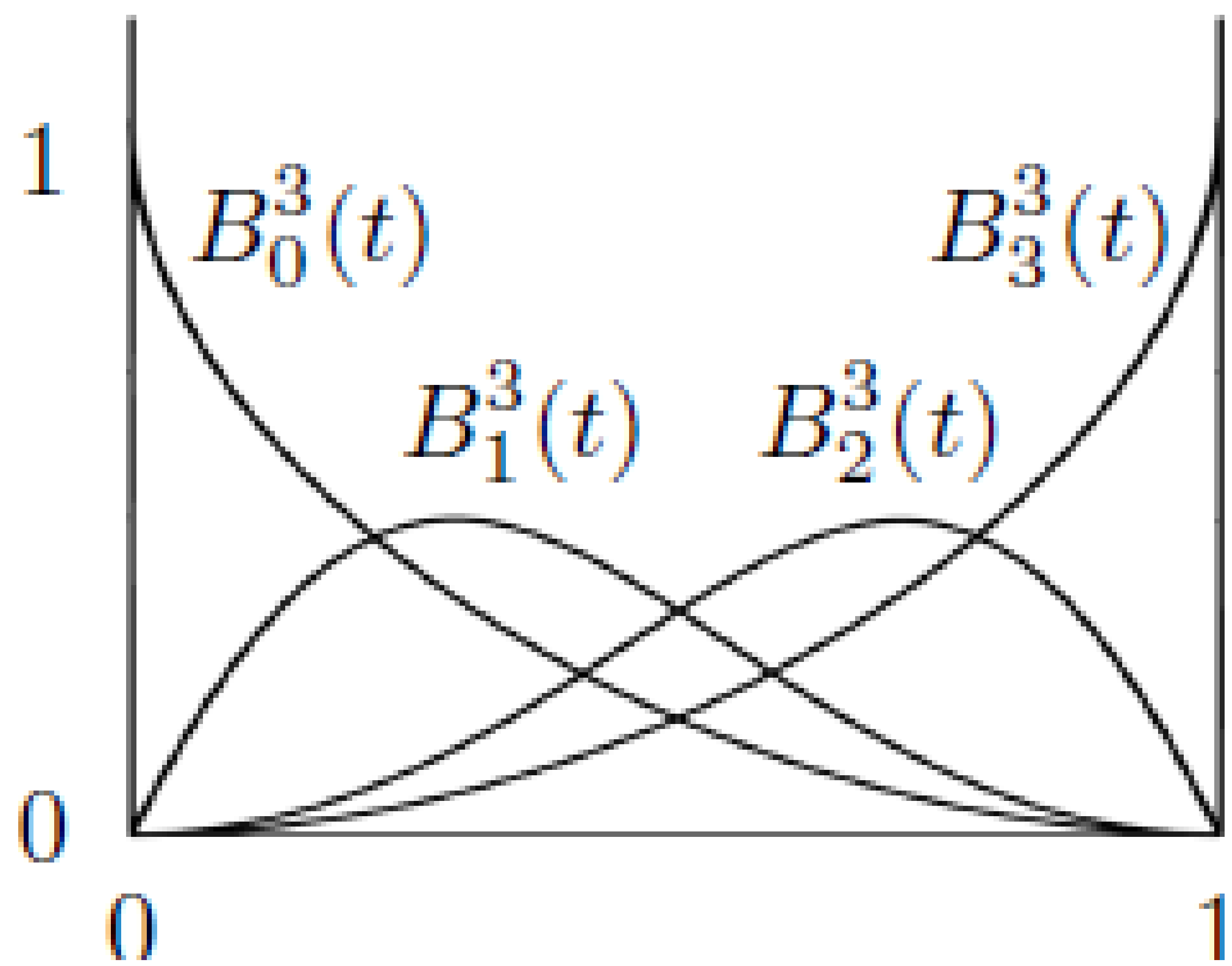
Bernstein олон гишүүн:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Cubic Bézier Үндсэн Функц

Bernstein олон гишүүн:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

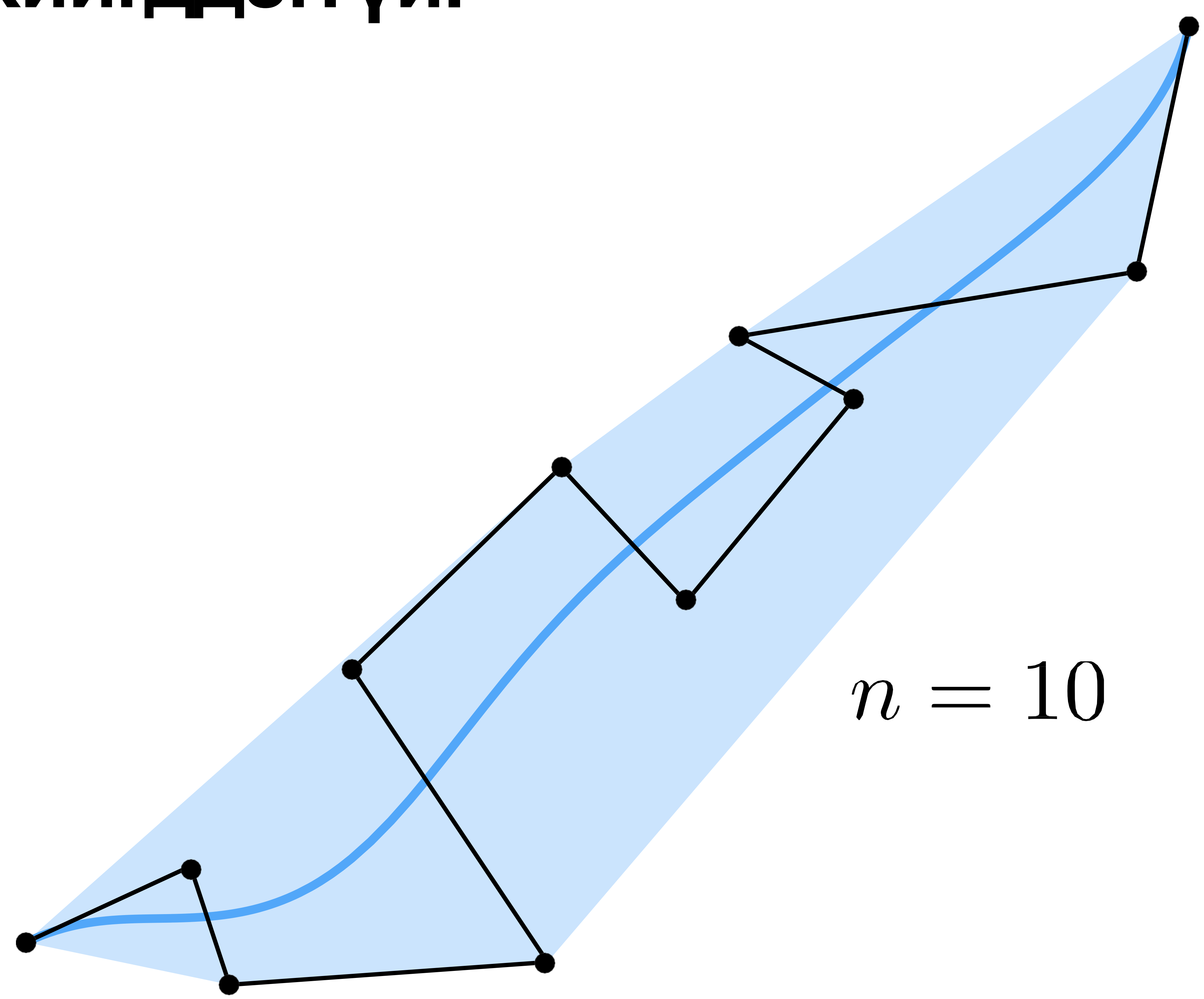


Sergei N. Bernstein
1880 – 1968

Piecewise Bézier Curves
(Bézier Spline)

Higher-Order Bézier Curves?

Higher-Order Bernstein олон гишүүн интерполяци сайн хийгддэггүй.



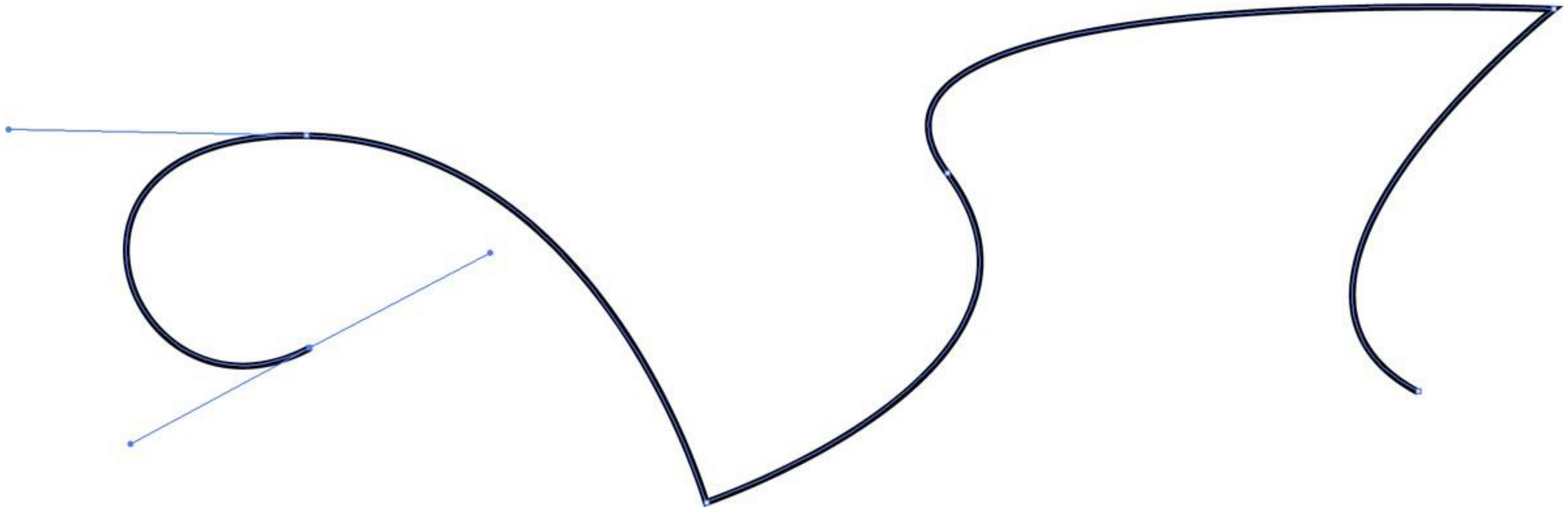
$$n = 10$$

Удирдахад хэцүү!

Нийтлэг бус

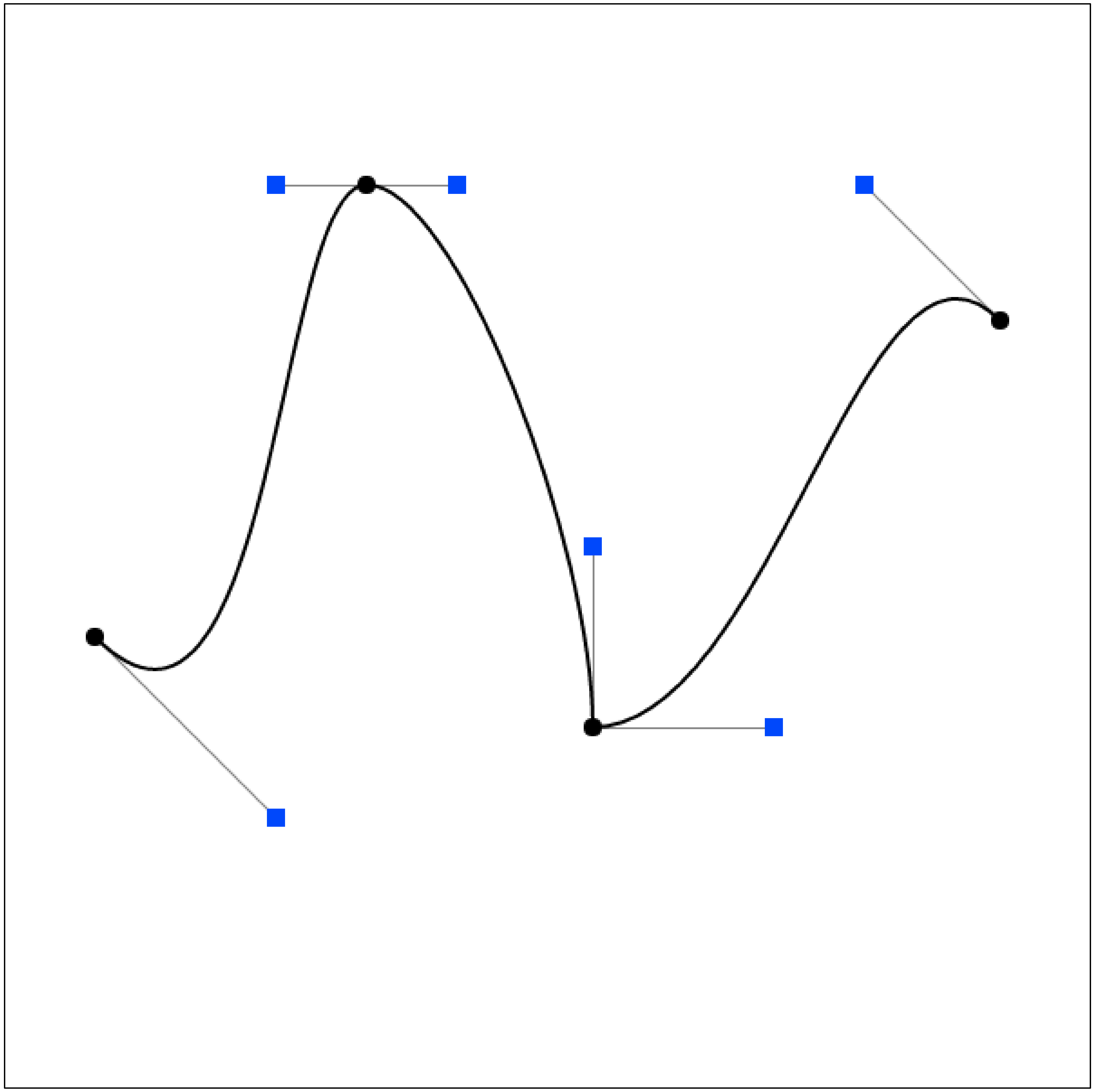
Piecewise Bézier Муруй

Piecewise cubic Bézier хамгийн өргөн хэрэглэгддэг арга



Хамгийн өргөн хэрэгдэгддэг (fonts, paths, Illustrator, Keynote, …)

Demo – Piecewise Bézier Curve



David Eck, http://math.hws.edu/eck/cs424/notes2013/canvas/bezier.html

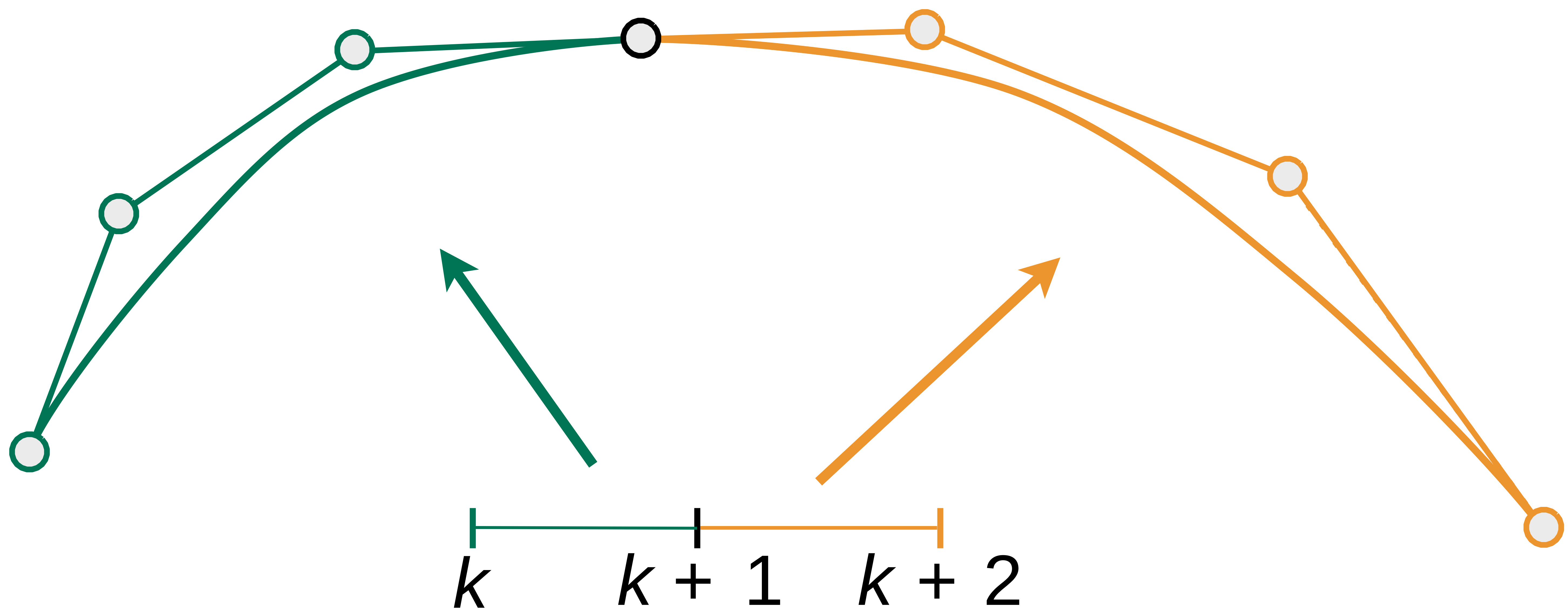
Piecewise Bézier Curve – Тасралтгүй

2 Bézier муруй

$$\mathbf{a} : [k, k+1] \to \mathbb{R}^N$$

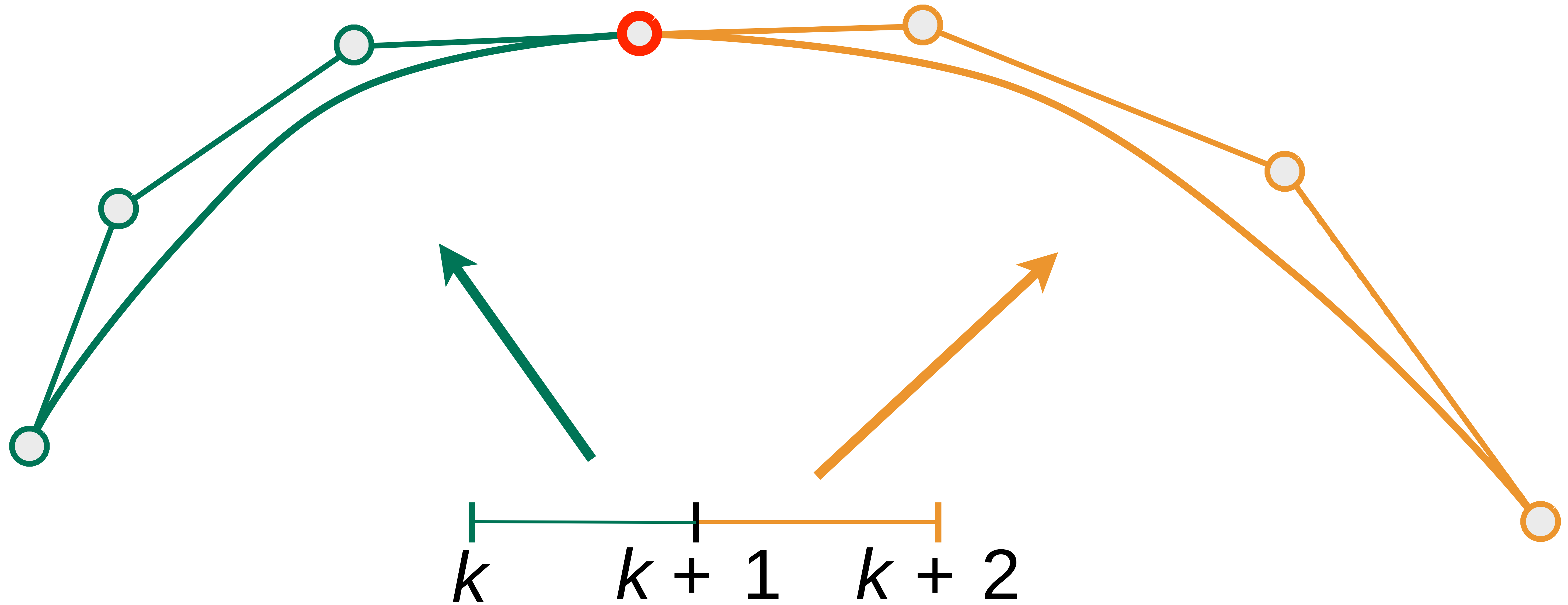
$$\mathbf{b} : [k+1, k+2] \to \mathbb{R}^N$$

Assuming integer partitions here, can generalize



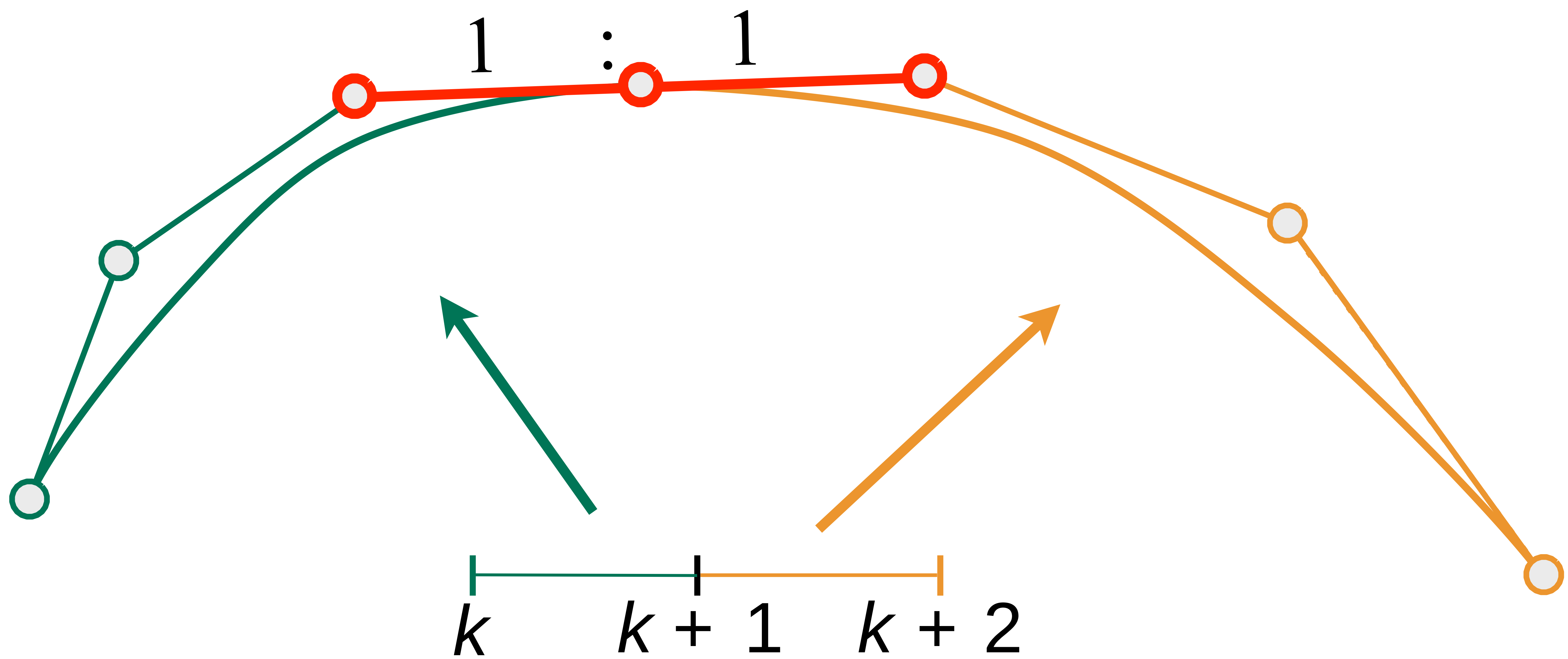
Piecewise Bézier Curve – Тасралтгүй

C^0 continuity: $a_n = b_0$



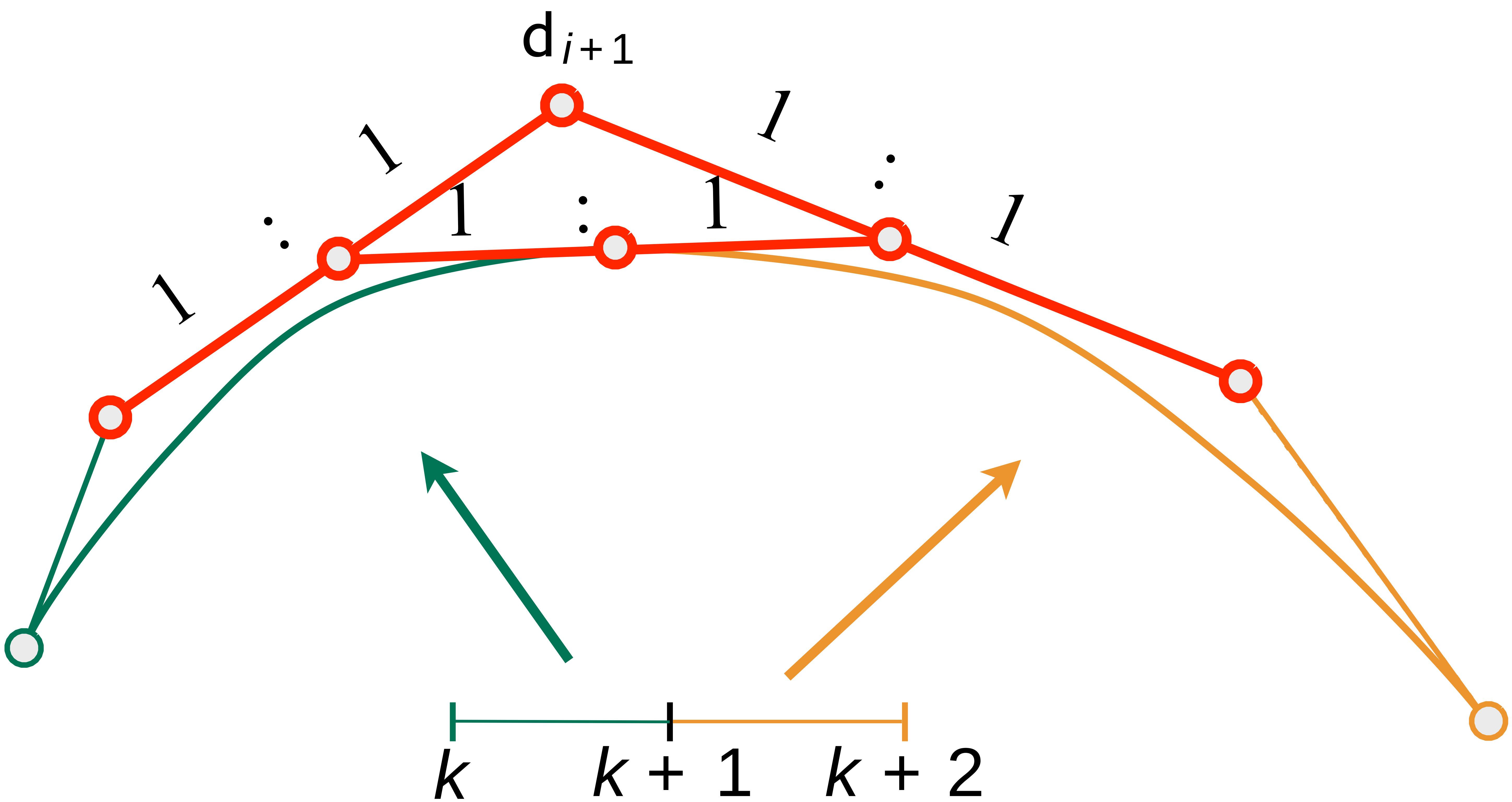
Piecewise Bézier Curve – Continuity

C^1 continuity: $$a_n = b_0 = \frac{1}{2}(a_{n-1} + b_1)$$



Piecewise Bézier Curve – Continuity

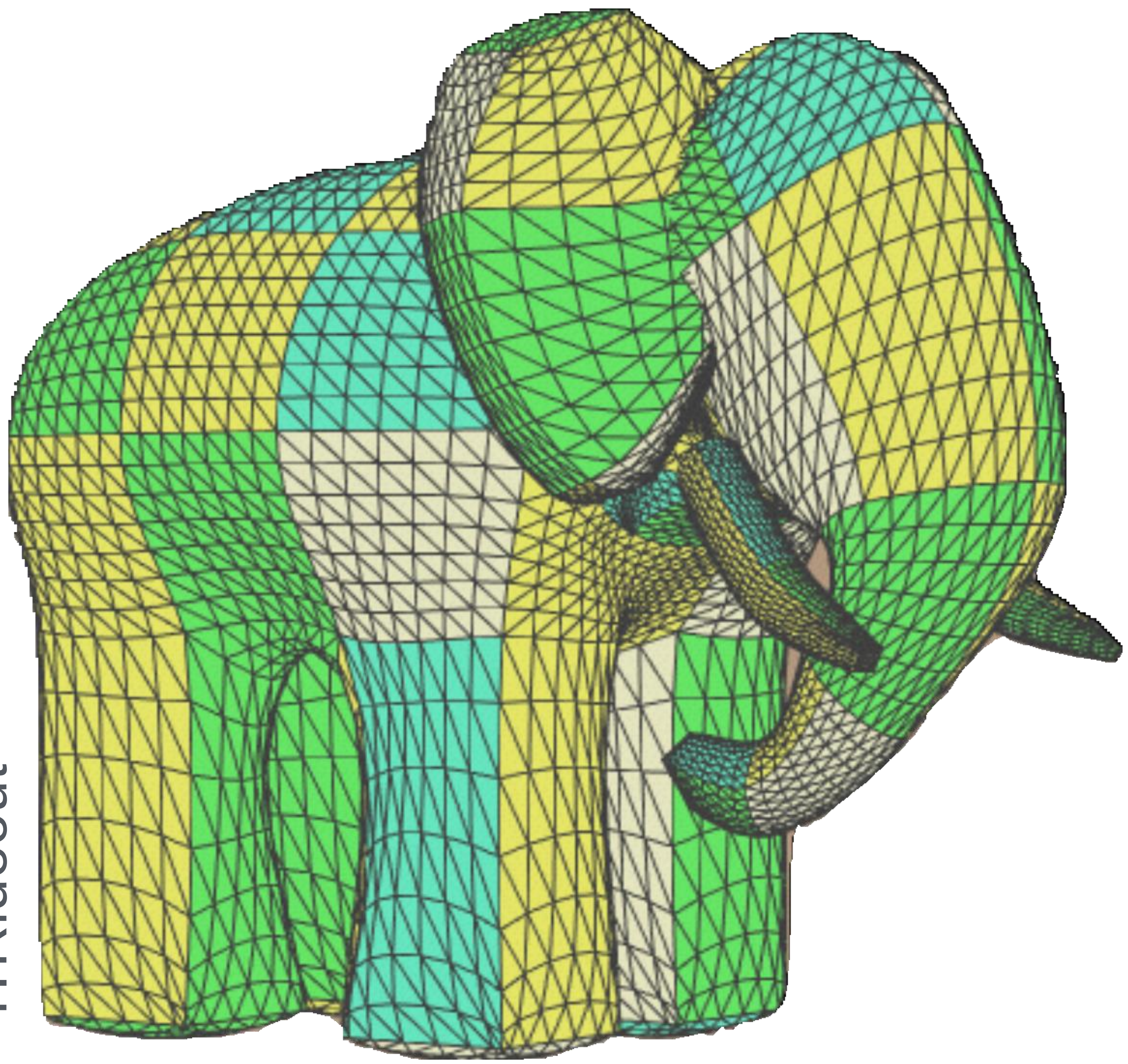
C^2 continuity: "A-frame" construction



Bézier Surfaces

Bézier Surfaces

Extend Bézier curves to surfaces



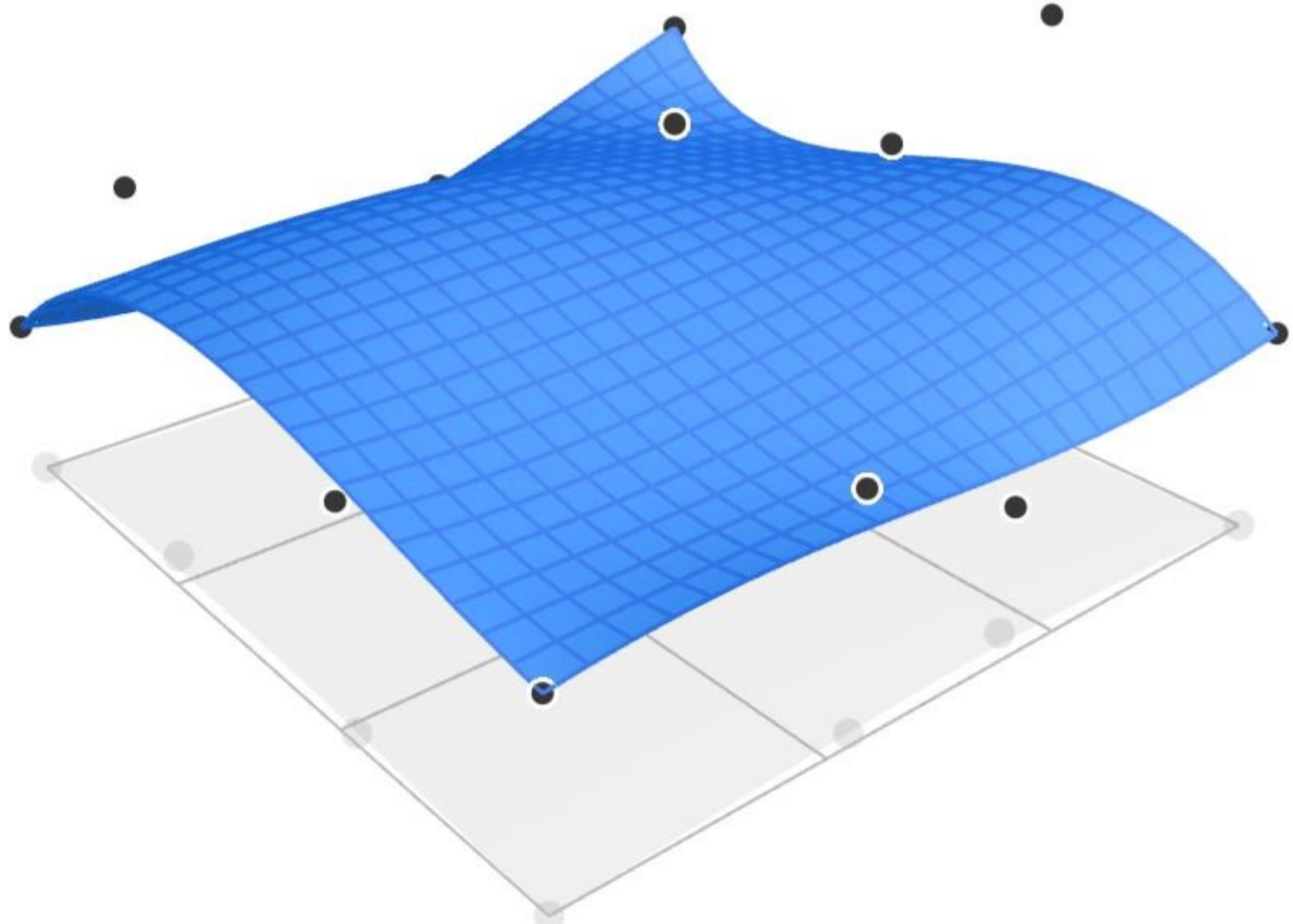
P.Rideout

Ed Catmull's "Gumbo" model

renderspirit.com

Utah Teapot

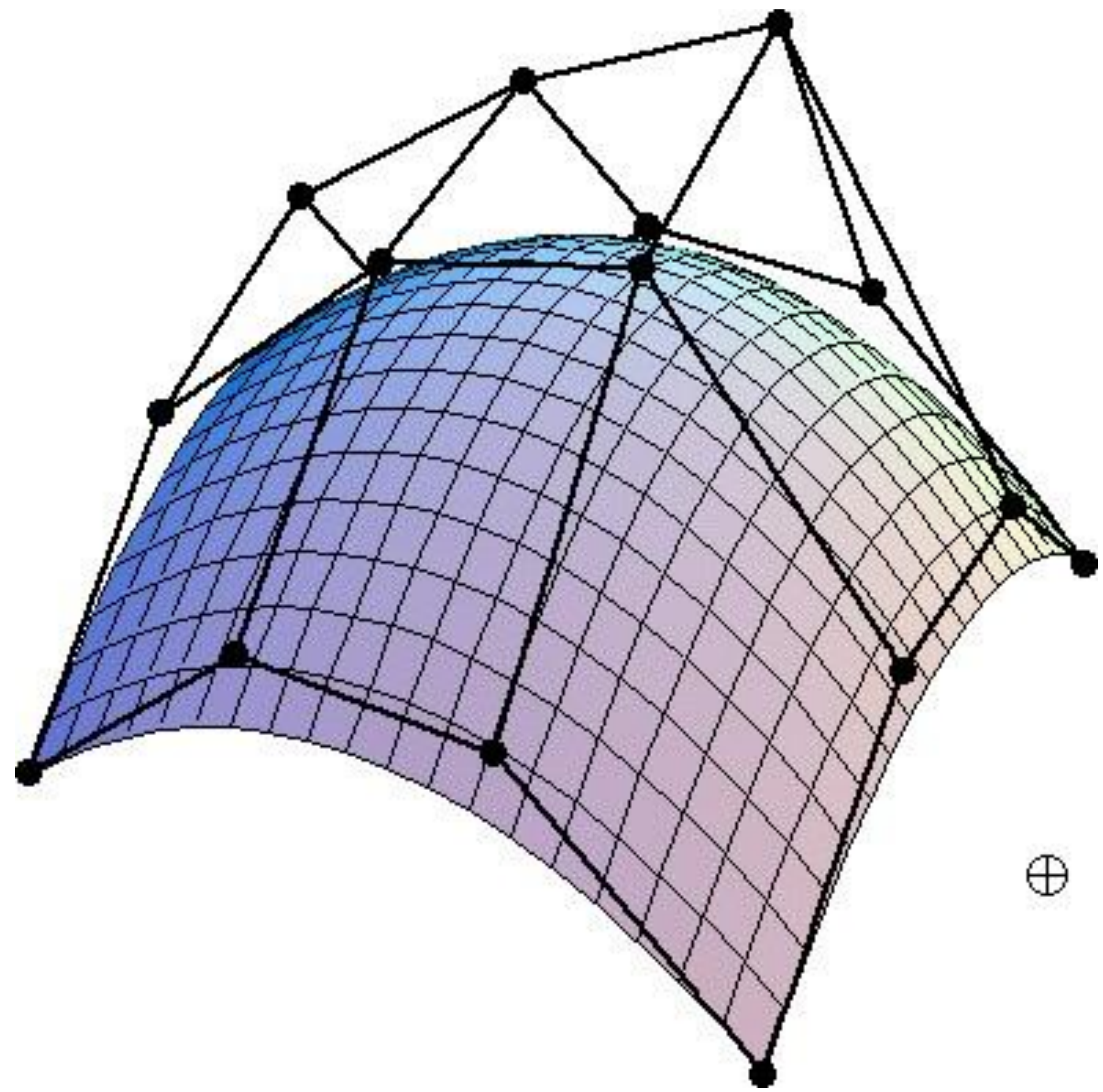
Visualizing Bicubic Bézier Surface Patch



Bicubic Bezier Surface

Animation: Steven Wittens, Making Things with Maths, http://acko.net

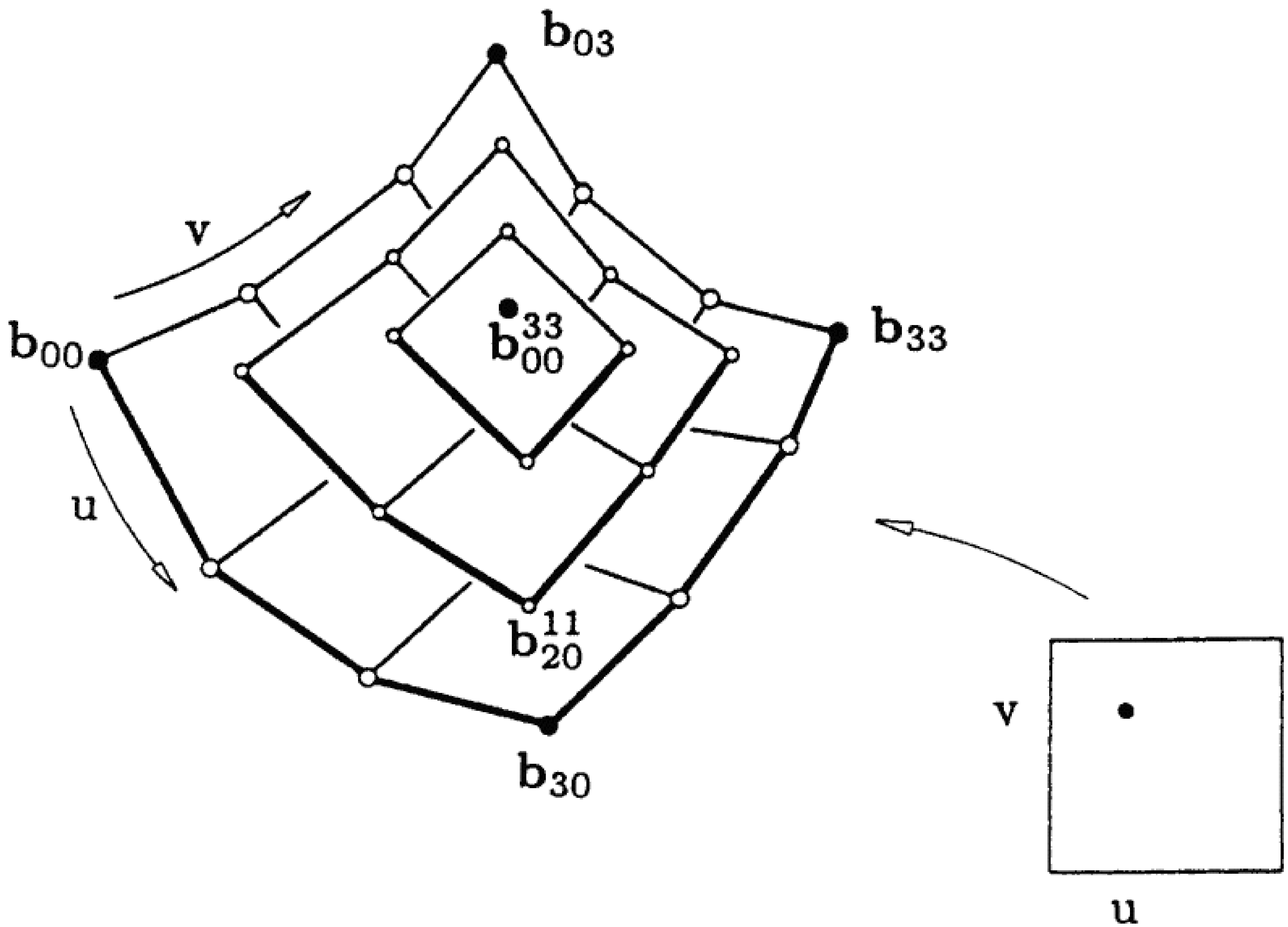
Bicubic Bézier Surface Patch



Bezier surface ба удирдлагын цэгийн 4 x 4 массив

2D de Casteljau Алгоритм

Repeated application of bilinear interpolation



2D de Casteljau Алгоритм

Жишээ: $(u, v) = \left(\frac{1}{2}, \frac{1}{2}\right)$

$$
\underset{r=1}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 2 \\ 0 \\ \hline 0 \\ 4 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \\ 0 \\ \hline 2 \\ 2 \\ 0 \\ \hline 2 \\ 4 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 0 \\ 0 \\ \hline 4 \\ 2 \\ 2 \\ \hline 4 \\ 4 \\ 4 \end{bmatrix}} \longrightarrow \underset{r=2}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ \hline 1 \\ 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 1 \\ 0.5 \\ \hline 3 \\ 3 \\ 2.5 \end{bmatrix}} \longrightarrow \underset{r=3}{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}
$$

2D de Casteljau Алгритм

Өгөгдөл:

- **Удирдлагын цэгийн 2D массив** $\mathrm{b}_{i,j} = \mathrm{b}_{i,j}^{0,0},\ 0 \leq i, j \leq n$
- **Параметр утгууд** (u, v)

Recursive bilinear interpolation

$$\mathrm{b}_{i,j}^{r,r} = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathrm{b}_{i,j}^{r-1,r-1} & \mathrm{b}_{i,j+1}^{r-1,r-1} \\ \mathrm{b}_{i+1,j}^{r-1,r-1} & \mathrm{b}_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}$$

$$r = 1, \ldots, n$$

$$i, j = 0, \ldots, n-r$$

Bézier Patch – A Tensor Product Surface

Хөдөлгөөний муруй Bézier муруйн m зэрэг байг

$$\mathrm{b}^m(u) = \sum_{i=0}^{m} \mathrm{b}_i B_i^m(u)$$

Хяналтын цэг b_i бүр нь Bézier муруйн n зэрэг дагуу хөдөлж байг.

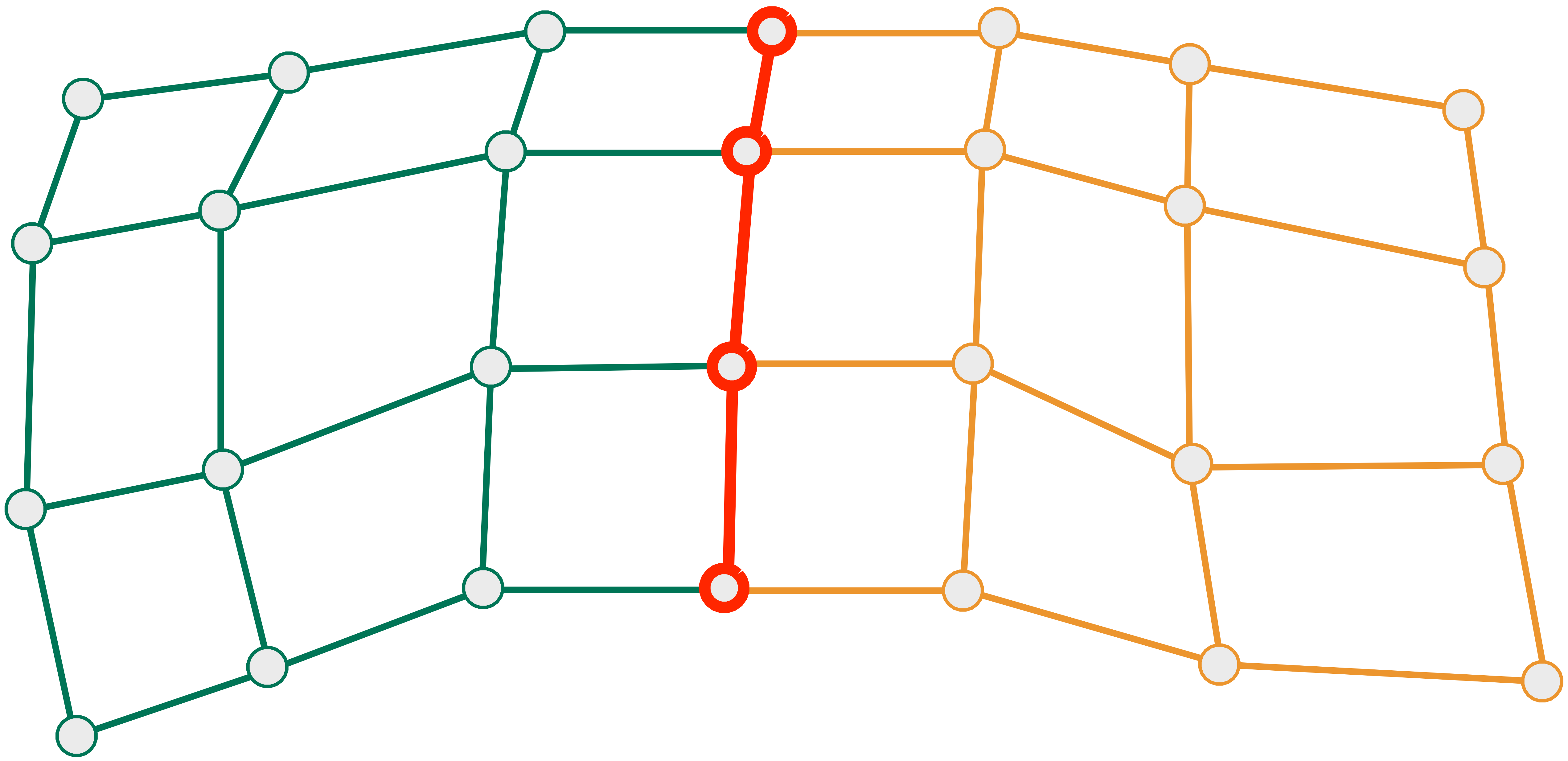
$$\mathrm{b}_i = \mathrm{b}_i(v) = \sum_{j=0}^{n} \mathrm{b}_{i,j} B_j^n(v)$$

Tensor product Bézier patch

$$\mathrm{b}^{m,n}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathrm{b}_{i,j} B_i^m(u) B_j^n(v)$$

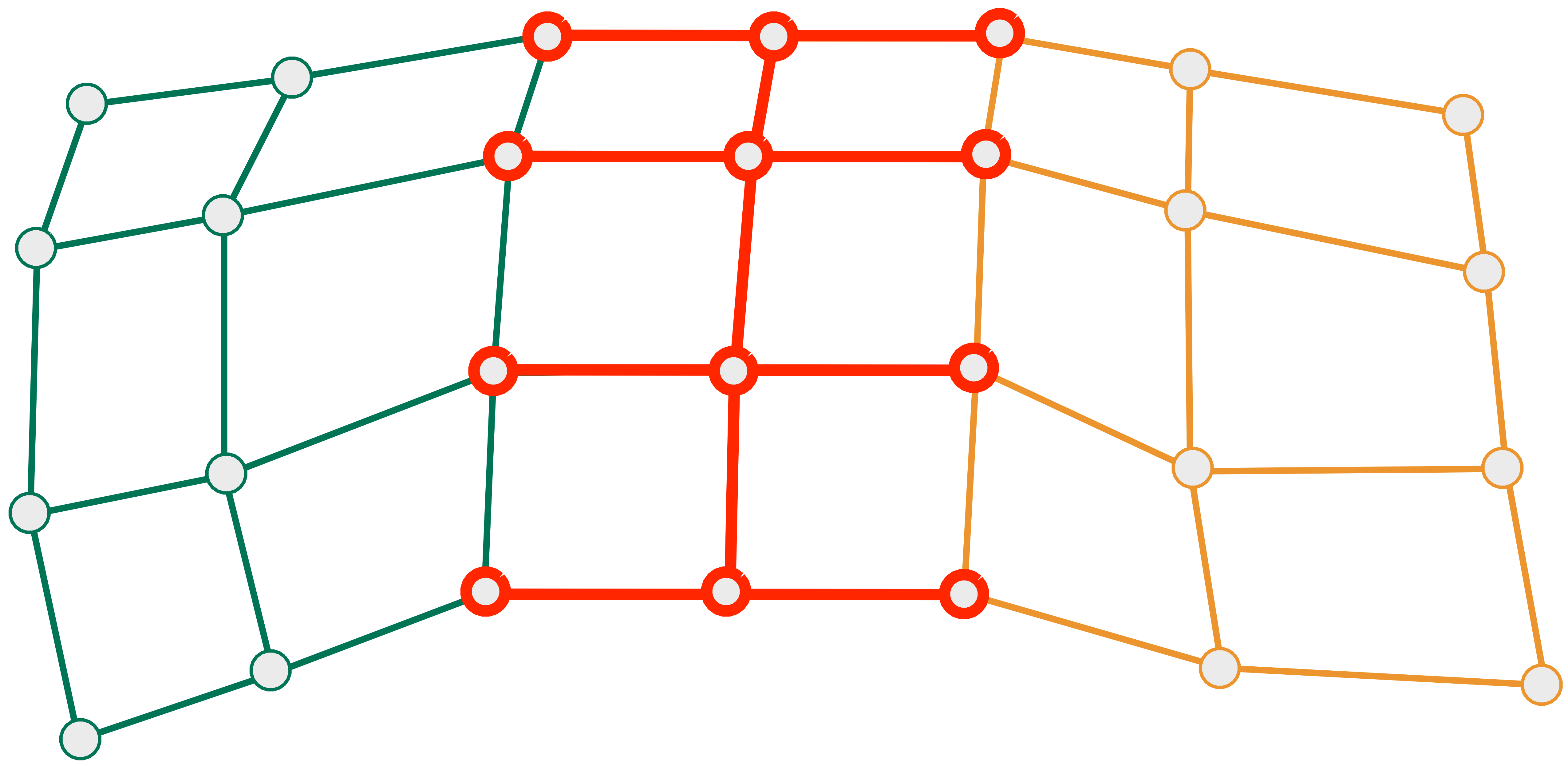
Piecewise Bézier Surfaces

C^0 тасралтгүй: Boundary curves/хилийн муруй



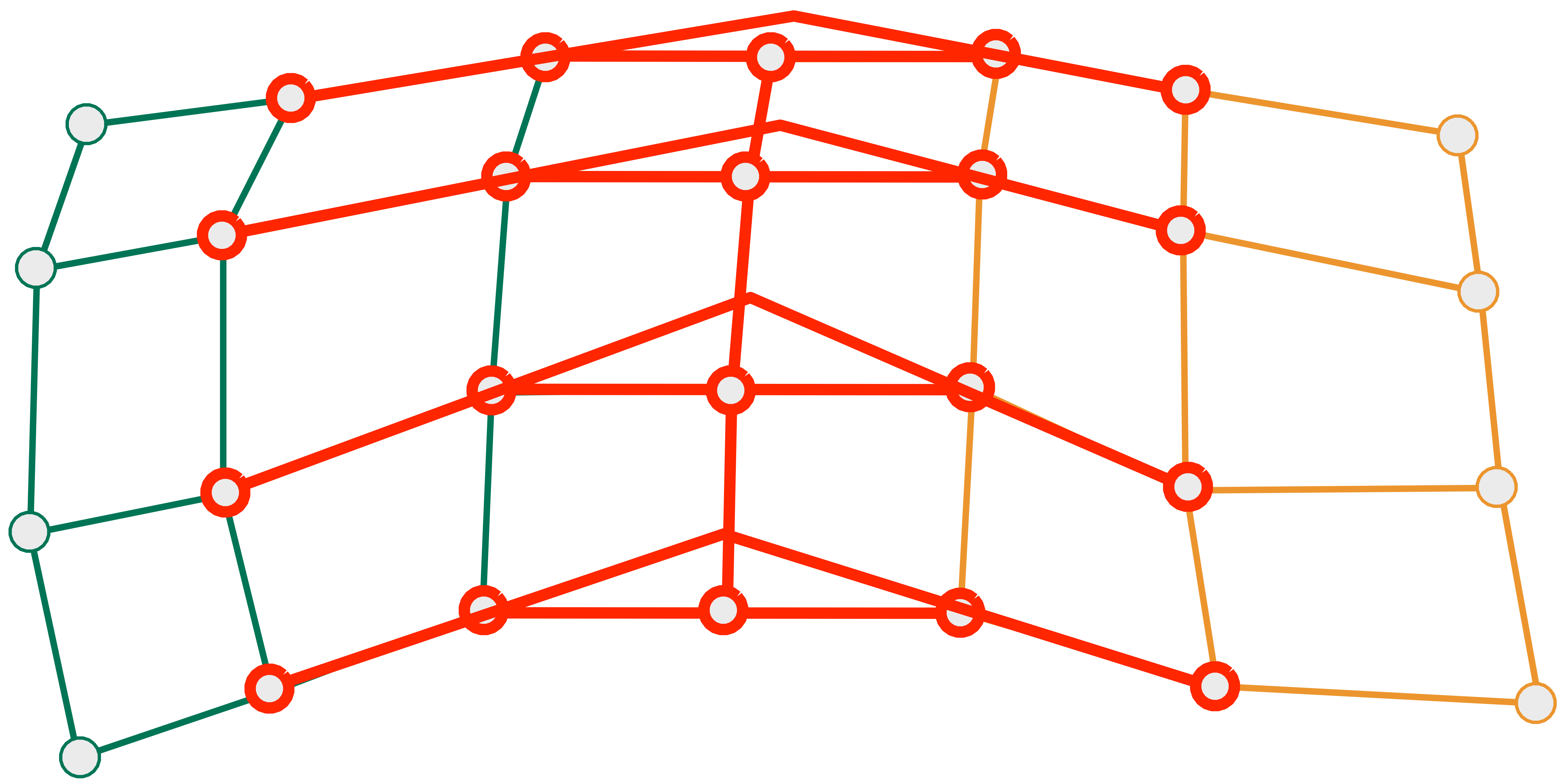
Piecewise Bézier Surfaces

C^1 тасралтгүй: Collinearity/нэг шулуун дээр байрлах



Piecewise Bézier Surfaces

C^2 тасралтгүй: A-frames



Санах зүйлс

Splines

- Cubic Hermite and Catmull-Rom interpolation
- Matrix representation of cubic polynomials

Bézier curves

- Easy to control spline
- Recursive linear interpolation – de Casteljau algorithm
- Properties of Bézier curves
- Piecewise Bézier curve – continuity types and how to achieve

Bézier surfaces

- Bicubic Bézier patches – tensor product surface
- 2D de Casteljau algorithm

Acknowledgments

Thanks to Pat Hanrahan, Mark Pauly and Steve Marschner for presentation resources.