ESE 4481 ASSIGNMENT #2: DYNAMICS

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Due Date: 09/26/23 Electronic submissions of all models, code, and writing should be included in Canvas before the deadline. Late work loses 10 points a day.

1. Academic Integrity

This is a partner assignment. Do not plagiarise. Write your own report.

2. What to turn in

Please submit all answers in a self-contained short report that answers all questions.

3. Assignment

This assignment is to help you grasp dynamics. It walks through the derivation of conservation of translational and angular momentum for the UAV. The dynamics are derived in a frame centered at the c.g. of the UAV and rotating with the aircraft's principle axes, the body frame.

The derivation in this homework assignment makes practical assumptions. A simplified dynamics for the quadcopter may satisfy the following 5 assumptions [1]:

- The body has a fixed mass distribution and constant mass;
- The air is at rest relative to the earth;
- The earth is fixed in inertial space;
- Flight in the earth atmosphere is close to the surface of earth, so the earth surface can be approximated as flat;
- Gravity is uniform so that the aircraft c.g. and center of mass are coincident, and gravitational forces do not change with altitude.

4. Notation

X is a force in the x body axis. Y is a force in the y body axis. Z is a force in the z body axis. L is a moment about the body x axis. M is a moment about body y axis. N is a moment about the body z axis.

Assume the standard aerospace notation discussed in class holds with (Z-Y-X) euler angles. ϕ is the roll euler angle between the earth (NED) frame and the body frame. θ is the pitch euler angle between the earth frame and the body frame. ψ is the yaw euler angle between the earth frame and the body frame.

Recall that a vector in the body frame p^{b} can represented as follows:

$$p^{\mathrm{b}} = R_{\mathrm{E}}^{\mathrm{b}} p^{\mathrm{E}} = R(\phi)R(\theta)R(\psi)p^{\mathrm{E}}, \tag{4.1}$$

where p^E is the vector as measured in the earth frame.

Assume that V^b is the airspeed vector as measured in the body frame with components as follows:

$$V^{\mathrm{b}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

5. Translational and Angular Moment Balance in the Body Frame

The linear momentum of the UAV is $\mathcal{P}$,

$$\mathcal{P} = mV,$$

and the angular momentum of the UAV is

$$\mathcal{H} = J\omega.$$

For the time being, assume that the rotors do not spin and are also fixed so that they do not contribute to angular momentum.

Newton's 2nd law applied to a quadcopter can be written in the earth frame (inertial frame) as follows:

$$\vec{F}^{\mathrm{E}} = \left(\frac{d}{dt}(\mathcal{P})\right)^{\mathrm{E}}, \tag{5.1}$$

$$\vec{M}^{\mathrm{E}} = \left(\frac{d}{dt}(\mathcal{H})\right)^{\mathrm{E}}, \tag{5.2}$$

where m is the mass of the UAV, V is the translational velocity of the body, $\omega_{\mathrm{b/E}}$ is the angular velocity of the body, and $\vec{F}$ and $\vec{M}$ represent applied moments and forces,

$$\vec{F} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \vec{M} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}. \tag{5.3}$$

Rewrite (5.2) in a frame that rotates relative to the earth frame, the body frame. Assume that the centers of the body frame and the earth frame are coincident, but the body frame rotates relative to the earth frame by $\omega_{\mathbf{b/E}}$. These laws now take the form under the assumption of constant mass and mass distribution:

$$\dot{V}^b = \frac{1}{m}\left(\vec{F}^b - \omega_{\mathrm{b/E}} \times mV^b\right), \tag{5.4}$$

$$\dot{\omega}_{\mathrm{b/E}} = J^{-1}\left(\vec{M}^b - \omega_{\mathrm{b/E}} \times J\omega_{\mathrm{b/E}}\right). \tag{5.5}$$

The CrazyFlie 2.1 drone has a maximum mass (with its payload) of 42g and a size of 92x92x29mm. Estimate its moment of inertia matrix and state the assumptions used in the calculations.

6. Rotational Kinematics

Recall that the rate of change of the euler angles $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ is not the same as the body axis rotational rate that would be measured in a gyroscope p, q, and r.

The rate of change of euler angles can be thought of as the magnitude of angular velocity vectors for the Earth axis [2].

$$\dot{\phi}\begin{bmatrix}1\\0\\0\end{bmatrix}, \dot{\theta}\begin{bmatrix}0\\1\\0\end{bmatrix}, \dot{\psi}\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Changing the sum of the vectors rotated to the body axis frame, form the familiar roll rate, pitch rate, and yaw rate. This is done as follows (recalling the Z-Y-X order):

$$\begin{bmatrix}\dot{p}\\\dot{q}\\\dot{r}\end{bmatrix} = R(\phi)\begin{bmatrix}\dot{\phi}\\0\\0\end{bmatrix} + R(\phi)R(\theta)\begin{bmatrix}0\\\dot{\theta}\\0\end{bmatrix} + R(\phi)R(\theta)R(\psi)\begin{bmatrix}0\\0\\\dot{\psi}\end{bmatrix}. \tag{6.1}$$

Let the vector Θ represent the euler angles,

$$\Theta = (\phi, \theta, \psi)^T.$$

This yields the following expression that relates the rate of change of euler angles x to the angular velocities roll rate, pitch rate, and yaw rate $\omega_{\mathrm{b/E}} = (p, q, r)^T$.

$$\omega_{\mathrm{b/E}} = \begin{bmatrix}1 & 0 & -\sin(\theta)\\0 & \cos\phi & \sin\phi\cos\theta\\0 & -\sin\phi & \cos\phi\cos\theta\end{bmatrix}\begin{bmatrix}\dot{\phi}\\\dot{\theta}\\\dot{\psi}\end{bmatrix} \tag{6.2}$$

$$= H(\Theta)\dot{\Theta}, \tag{6.3}$$

where H is the matrix in (6.2). To calculate the rate of change of euler angles in terms of angular rates the equation is inverted as follows

$$\dot{\Theta} = H(\Theta)^{-1}\omega_{\mathrm{b/E}} \tag{6.4}$$

Find the singularities in this equation

7. Model Gravity

Transform the gravity vector from the earth frame to the body frame. Assume that gravity points downwards and has a constant acceleration 9.81 m/s. Write out the gravity vector in body coordinates as a function of ϕ and θ. Explain why the gravity vector resolved in body coordinates does not depend on the yaw angle.

$$F_g^b = R_{\mathrm{E}}^{\mathrm{b}} F_g^{\mathrm{E}} \tag{7.1}$$

8. Propulsive Motor Forces and Moments

The speed of the CrazyFly 2.0 four propellers is n_1, n_2, n_3 , and n_4. There is 92 mm of distance diagonal distance between opposite motors (e.g., between motor 1 and 3). The propellers are 45 mm in diameter.

A table of data of the net thrust from 4 propellers is provided with the following data:

(1) **Estimate the propeller's thrust coefficient. Assume sea level air density.**

Amps	Total Thrust (g)	Voltage	PWM (%)	Average RPM
0.24	0.0	4.01	0	0
0.37	1.6	3.98	6.25	4485
0.56	4.8	3.95	12.5	7570
0.75	7.9	3.92	18.75	9374
0.94	10.9	3.88	25	10885
1.15	13.9	3.84	31.25	12277
1.37	17.3	3.80	37.5	13522
1.59	21.0	3.76	43.25	14691
1.83	24.4	3.71	50	15924
2.11	28.6	3.67	56.25	17174
2.39	32.8	3.65	62.5	18179
2.71	37.3	3.62	68.75	19397
3.06	41.7	3.56	75	20539
3.46	46.0	3.48	81.25	21692
3.88	51.9	3.40	87.5	22598

(2) **Estimate the propeller's power coefficient. Assume sea level air density.**
(3) **Estimate the propeller's torque coefficient. Assume sea level air density.**
(4) **Determine pitching moment produced by each motor as a function of motor speed**
(5) **Determine the rolling moment produced by each motor as a function of motor speed**
(6) **Determine yawing moment produced by each motor as a function of motor speed**
(7) **Estimate the angular velocity of the four propellers (assuming they all maintain the same constant speed) to produce a thrust equal and opposite to gravity (trim to 1g).**

9. Translational Kinemetics

The velocity of the vehicle in an earth frame can be represented as

$$V^{\mathrm{E}} = \begin{bmatrix} V_{\mathrm{N}} \\ V_{\mathrm{E}} \\ V_{\mathrm{D}} \end{bmatrix}.$$

This is the ground speed of the UAV. When there is no wind, this is equivalent to the airpseed of the vehicle. The airspeed in body axis is related to the ground speed in the earth frame as follows:

$$V^{\mathrm{b}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R^{\mathrm{b}}_{\mathrm{E}} V^{\mathrm{E}}.$$

Derive the expression for $R^{\mathbf{E}}_{\mathbf{B}}$ in terms of the euler angles ϕ, θ, and ψ.

10. The Equations of Motion

The combined drone equations of motion neglecting gyroscopic moments and aerodynamics are as follows:

$$\dot{p}^E = V^E \tag{10.1}$$

$$\dot{\Theta} = H(\Theta)^{-1}\omega_{b/E} \tag{10.2}$$

$$\dot{V}^b = \frac{1}{m}\left(F_T^b + F_G^b - \omega_{b/E} \times mV^b\right), \tag{10.3}$$

$$\dot{\omega}_{b/E} = J^{-1}\left(M_T^b - \omega_{b/E} \times J\omega_{b/E}\right). \tag{10.4}$$

Use symbolic matlab to derive the linearization of CrazyFlie around trim condition where the four motors generate thrust equal and opposite to gravity, and the drone is in a hover. Are the nonlinear dynamics locally stable around this reference condition?

References

[1] Morelli, Eugene A., and Vladislav Klein. Aircraft system identification: theory and practice. Vol. 2. Williamsburg, VA: Sunflyte Enterprises, 2016.

[2] Stevens, Brian L., Frank L. Lewis, and Eric N. Johnson. Aircraft control and simulation: dynamics, controls design, and autonomous systems. John Wiley & Sons, 2015.

[3] Brekelmans, G. H. Extended quadrotor dynamics: from simulations to experiments. Technical report, MSc Thesis, Eindhoven University of Technology, Department of Mechanical Engineering, Dynamics and Control Group, Eindhoven, 2019.

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