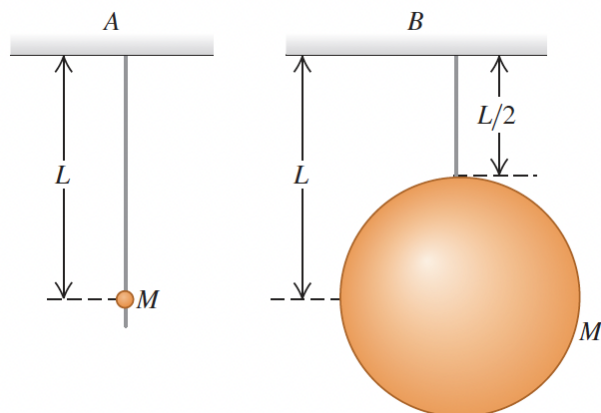# Opgaver til forelæsning 26

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Opg. 14.53

Figur 1:

The two pendulums shown in **Figur 1** each consist of a uniform solid ball of mass M supported by a rigid massless rod, but the ball for pendulum A is very tiny while the ball for pendulum B is much larger. Find the period of each pendulum for small displacements. Which ball takes longer to complete a swing?

For små oscillationer gælder at perioden for et simpelt pendul generelt kan findes som

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

Altså har vi at

$$T_A = 2\pi\sqrt{\frac{L}{g}}.$$

Det store pendul modelleres som et fysisk pendul og dets periode kan derfor findes som

$$T = 2\pi\sqrt{\frac{I}{mgd}}.$$

Idet inertimomentet af en solid sfære kan findes som

$$I = \frac{2}{5}MR^2 \implies I_{B_{\text{center}}} = \frac{2}{5}ML^2.$$

Og vi derfor kan finde inertimomentet omkring den nye omdrejningsakse som

$$I_B = \frac{2}{5}M\frac{L^2}{4} + ML^2 = \frac{11}{10}ML^2.$$

Har vi derfor at

$$T_B = 2\pi\sqrt{\frac{11ML^2}{10MgL}}$$
$$= 2\pi\sqrt{\frac{11L}{10g}}.$$

Opg. 14.56

A 50,0 g hard-boiled egg moves on the end of a spring with force constant $k = 25{,}0\,\text{N/m}$. Its initial displacement is 0,300 m. A damping force $F_x = -bv_x$ acts on the egg, and the amplitude of the motion decreases to 0,100 m in 5,00 s. Calculate the magnitude of the damping constant b.

Det gælder generelt at for en dæmpet oscillation kan den nye amplitude A_2 findes som

$$A_2 = A_1 e^{-\frac{b}{2m}t}.$$

Hvor b er dæmpningskonstanten, m er massen og t er tiden. Udtrykket fra ovenfor kan omskrives til

$$A_2 e^{\frac{b}{2m}t} = A_1$$
$$\frac{b}{2m}t = \ln\frac{A_1}{A_2}$$
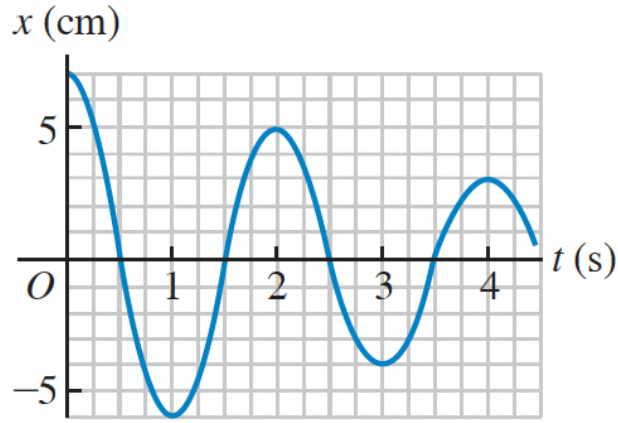$$b = \frac{2m}{t}\ln\frac{A_1}{A_2}.$$

Sættes kendte størrelser ind fås

$$b = \frac{2 \cdot 50{,}0\,\text{g}}{5{,}00\,\text{s}}\ln\frac{0{,}300\,\text{m}}{0{,}100\,\text{m}} = 0{,}0220\,\frac{\text{kg}}{\text{s}}.$$

Opg. 14.58

A mass is vibrating at the end of a spring of force constant $225\,\text{N/m}$. **Figur 2** shows a graph of its position x as a function of time t.

Figur 2:

(a)

At what times is the mass not moving?

Massen har en hastighed på netop $v = 0$ når positionen som funktion af tid ikke ændrer sig. Altså når $\dot{x} = 0$. Dette sker for $t = 0\,\text{s}, 1\,\text{s}, 2\,\text{s}, 3\,\text{s}, 4\,\text{s}$.

(b)

How much energy did the system originally contain?

Den totale energi, E_f, i en fjeder er generelt givet som

$$E_f = \frac{1}{2}kA^2.$$

Sættes kendte størrelser ind fås

$$E_f = \frac{1}{2} \cdot 225\,\frac{\text{N}}{\text{m}} \cdot (7\,\text{cm})^2 = 0{,}551\,\text{J}.$$

(c)

How much energy did the system lose between $t = 1{,}00\,\text{s}$ and $t = 4{,}00\,\text{s}$? Where did this energy go?
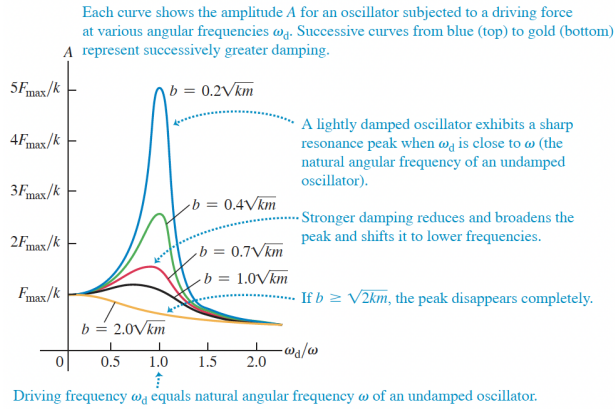
Til tiden $t = 1{,}00\,\text{s}$ er amplituden $A_{t_1} = 6{,}00\,\text{cm}$ og til tiden $t = 4{,}00\,\text{s}$ er amplituden $A_{t_2} = 3{,}00\,\text{cm}$. Altså er den samlede tabte energi

$$\begin{aligned}
\Delta E &= E_{t_1} - E_{t_2} \\
&= \frac{1}{2} \cdot 225\,\frac{\text{N}}{\text{m}} \cdot \left((6{,}00\,\text{cm})^2 - (3{,}00\,\text{cm})^2\right) \\
&= 0{,}304\,\text{J}.
\end{aligned}$$

Opg. 14.60

$$A = \frac{F_{\max}}{\sqrt{\left(k - m\omega_d^2\right)^2 + b^2\omega_d^2}} \tag{1}$$

Figur 3:

Each curve shows the amplitude A for an oscillator subjected to a driving force at various angular frequencies ω_d. Successive curves from blue (top) to gold (bottom) represent successively greater damping.

$b = 0.2\sqrt{km}$

A lightly damped oscillator exhibits a sharp resonance peak when ω_d is close to ω (the natural angular frequency of an undamped oscillator).

$b = 0.4\sqrt{km}$

$b = 0.7\sqrt{km}$

$b = 1.0\sqrt{km}$

Stronger damping reduces and broadens the peak and shifts it to lower frequencies.

$b = 2.0\sqrt{km}$

If $b \geq \sqrt{2km}$, the peak disappears completely.

Driving frequency ω_d equals natural angular frequency ω of an undamped oscillator.

Ligning 1 and **Figur 3** describe a damped and driven oscillator.

(a)

For a damping constant $b = 0{,}20\sqrt{km}$ confirm that the amplitude A is $5F_{\max}/k$ when $\omega_d = \omega$, where $\omega = \sqrt{k/m}$ is the natural angular frequency.

Sættes de givne størrelser ind i **Ligning 1** fås

$$
\begin{aligned}
A &= \frac{F_{\max}}{\sqrt{\left(k - m \cdot \sqrt{\frac{k}{m}}^2\right) + \left(0{,}20\sqrt{km}\right)^2 \cdot \sqrt{\frac{k}{m}}^2}} \\
&= \frac{F_{\max}}{\sqrt{k - m\frac{k}{m} + 0{,}04km \cdot \frac{k}{m}}} \\
&= \frac{F_{\max}}{\sqrt{k - k + 0{,}04k^2}} \\
&= \frac{F_{\max}}{\sqrt{0{,}04k^2}} \\
&= \frac{F_{\max}}{0{,}2k} \\
&= 5\frac{F_{\max}}{k}.
\end{aligned}
$$

(b)

Repeat part (a) for $b = 0{,}40\sqrt{km}$ and confirm that the amplitude A is $2{,}5F_{\max}/k$ when $\omega_d = \omega$.

Sættes de givne størrelser ind i **Ligning 1** fås

$$A = \frac{F_{\max}}{\sqrt{\left(k - m\sqrt{\frac{k}{m}}^2 + \left(0{,}40\sqrt{km}\right)^2 \cdot \sqrt{\frac{k}{m}}^2\right)}}$$

$$= \frac{F_{\max}}{\sqrt{0{,}16k^2}}$$

$$= 2{,}5\frac{F_{\max}}{k}.$$

(c)

As a measure of the width of the resonance peak, calculate A when $\omega_d = \omega/2$ for $b = 0{,}20\sqrt{km}$ and for $b = 0{,}40\sqrt{km}$. In each case, what is the ratio of the amplitude for $\omega_d = \omega$ to the amplitude for $\omega_d = \omega/2$? For which value of the damping constant does the amplitude increase by the larger factor?

For $b = 0{,}20\sqrt{km}$ indsættes kendte størrelser i **Ligning 1** for at få

$$A_{\frac{\omega}{2}} = \frac{F_{\max}}{\sqrt{\left(k - m\left(0{,}5\sqrt{\frac{k}{m}}\right)^2\right)^2 + \left(0{,}20\sqrt{km}\right)^2 \cdot \left(0{,}5\sqrt{\frac{k}{m}}\right)^2}}$$

$$= \frac{F_{\max}}{\sqrt{\left(k - \frac{1}{4}k\right)^2 + 0{,}04k \cdot \frac{1}{4}k}}$$

$$= \frac{F_{\max}}{\sqrt{\frac{9}{16}k^2 + 0{,}01\,k^2}}$$

$$= 1{,}32\frac{F_{\max}}{k}.$$

Og for $b = 0{,}40\sqrt{km}$ har vi at

$$A_{\frac{\omega}{2}} = \frac{F_{\max}}{\sqrt{\left(k - m\left(0{,}5\sqrt{\frac{k}{m}}\right)^2\right)^2 + \left(0{,}40\sqrt{km}\right)^2 \cdot \left(0{,}5\sqrt{\frac{k}{m}}\right)^2}}$$

$$= \frac{F_{\max}}{\sqrt{\frac{9}{16}k^2 + \frac{4}{25}k \cdot \frac{1}{4}k}}$$

$$= \frac{F_{\max}}{\sqrt{\frac{9}{16}k^2 + \frac{4}{100}k^2}}$$

$$= 1{,}29\frac{F_{\max}}{k}.$$

Dermed har vi for $b = 0{,}20\sqrt{km}$ at forholdet mellem amplituderne er

$$\frac{A}{A_{\frac{\omega}{2}}} = \frac{5F_{\max}}{k} \cdot \frac{k}{1{,}32F_{\max}} = \frac{5}{1{,}32} = 3{,}79.$$

Og for $b = 0{,}40\sqrt{km}$ har vi at forholdet mellem amplituderne er

$$\frac{A}{A_{\frac{\omega}{2}}} = \frac{2{,}5F_{\max}}{k} \cdot \frac{k}{1{,}29F_{\max}} = \frac{2{,}5}{1{,}29} = 1{,}94.$$

Altså stiger ampltituden mest for den mindste dæmpning.

Opg. 14.83

A rifle bullet with mass $8{,}00\,\mathrm{g}$ and initial horizontal velocity $280\,\mathrm{m/s}$ strikes and embeds itself in a block with mass $0{,}992\,\mathrm{kg}$ that rests on a frictionless surface and is attached to one end of an ideal spring. The other end of the spring is attached to the wall. The impact compresses the spring a maximum distance of $15{,}0\,\mathrm{cm}$. After the impact, the block moves in SHM. Calculate the period of this motion.

Idet de to objekter sidder sammen efter kollisionen må det gælde at kollisionen er komplet uelastisk. For inelastiske kollisioner har vi at

$$m_A v_{A1} + m_B v_{B1} = (m_A + m_B)v_2.$$

Sættes kendte størrelser ind kan hastigheden efter kollisionen findes som

$$(0{,}992\,\mathrm{kg} + 0{,}008\,00\,\mathrm{kg})v_2 = 0{,}008\,00\,\mathrm{kg} \cdot 280\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$v_2 = \frac{0{,}008\,00\,\mathrm{kg}}{1{,}000\,\mathrm{kg}} \cdot 280\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$= 2{,}24\,\frac{\mathrm{m}}{\mathrm{s}}.$$

Før fjederen skubber massen tilbage skal al den kinetiske energi omdannes til potentiel energi i fjederen. Altså må det både gælde at amplituden for svingningen (idet fjederen er ideel) er 15,0 cm og at den kinetiske energi efter kollisionen netop svarer til energien i fjederen. Altså har vi at

$$E = \frac{1}{2}Mv_2^2 = \frac{1}{2}kA^2.$$

Heri kan fjederkonstanten isoleres og findes som

$$k = \frac{Mv_2^2}{A^2} = \frac{1{,}000\,\text{kg} \cdot \left(2{,}24\,\frac{\text{m}}{\text{s}}\right)^2}{\left(15{,}0\,\text{cm}\right)^2} = 223\,\frac{\text{N}}{\text{m}}.$$

Perioden for et objekt i SHM kan findes som

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

Sættes kendte størrelser ind fås

$$T = 2\pi\sqrt{\frac{1{,}00\,\text{kg}}{223\,\frac{\text{N}}{\text{m}}}} = 0{,}421\,\text{s}.$$

Opg. 14.86

The Silently Ringing Bell. A large, 34,0 kg bell is hung from a wooden beam so it can swing back and forth with negligible friction. The bell's center of mass is 0,60 m below the pivot. The bell's moment of inertia about an axis at the pivot is 18,0 kg m². The clapper is a small, 1,8 kg mass attached to one end of a slender rod of length L and negligible mass. The other end of the rod is attached to the inside of the bell; the rod can swing freely about the same axis as the bell. What should be the length L of the clapper rod for the bell to ring silently – that is, for the period of oscillation for the bell to equal that of the clapper?
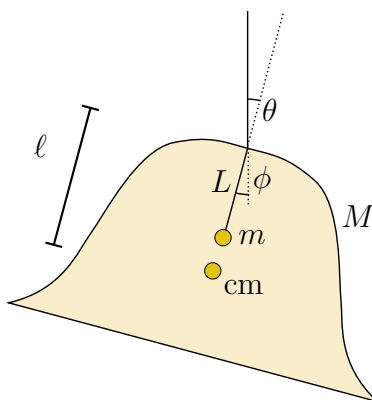
Figur 4: Fritlegemediagram

Vi har i opgaven givet at $M = 34{,}0\,\text{kg}$, $\ell = 0{,}60\,\text{m}$, $I_b = 18{,}0\,\text{kg}\,\text{m}^2$ og $m = 1{,}8\,\text{kg}$. Det gælder generelt at perioden for et fysisk pendul kan findes som

$$T = 2\pi\sqrt{\frac{I}{mgd}}.$$

Vi har altså for klokken at

$$T_b = 2\pi\sqrt{\frac{I_b}{Mg\ell}}.$$

Idet *clapperen* modelleres som en punktmasse kommer hele dennes inertimoment fra forskydningen af omdrejningsaksen. Vi har fra parallel-akse-teoremet at *clapperens* inertimoment derfor bliver

$$I_c = mL^2.$$

Dermed kan perioden for *clapperen* T_c findes som

$$T_c = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}.$$

Vi ønsker at finde L for $T_b = T_c$ og altså sættes de to størrelser lig hinanden som

$$T_c = T_b$$
$$2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{I_b}{Mg\ell}}$$
$$\frac{L}{g} = \frac{I_b}{Mg\ell}$$
$$L = \frac{I_b}{M\ell}.$$

Sættes kendte størrelser ind fås

$$L = \frac{18{,}0\,\text{kg}\,\text{m}^2}{34{,}0\,\text{kg} \cdot 0{,}60\,\text{m}} = 0{,}88\,\text{m}.$$

Opg. 14.87

A slender, uniform, metal rod with mass M is pivoted without friction about an axis through its midpoint and perpendicular to the rod. A horizontal spring with force constant k is attached to the lower end of the rod, with the other end of the spring attached to a rigid support. If the rod is displaced by a small angle θ from the vertical (**Figur 5**) and released, show that it moves in angular SHM and calculate the period. (*Hint*: Assume that the angle θ is small enough for the approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ to be valid. The motion is simple harmonic if $\frac{d^2\theta}{dt^2} = -\omega^2\theta$, and the period is then $T = 2\pi/\omega$.)

Figur 5:
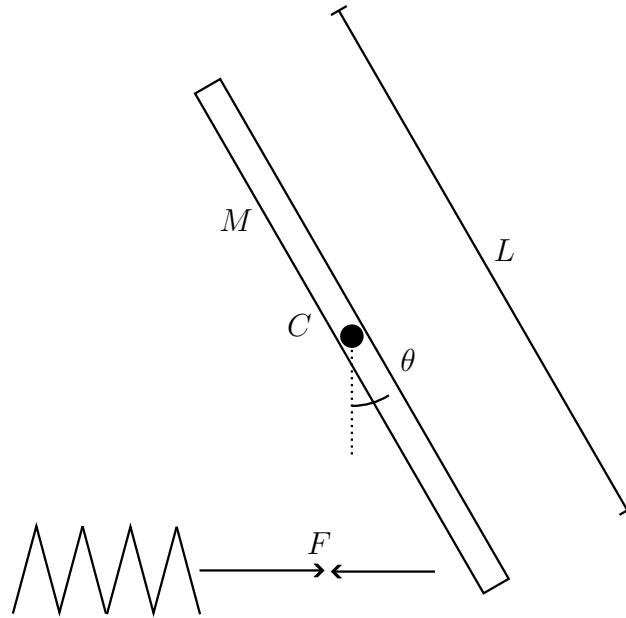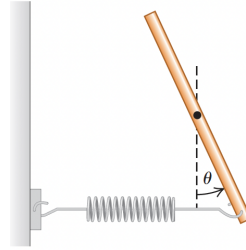
Figur 6: Fritlegemediagram

Hvis der ses bort fra at tyngdekraften aftager en smule jo længere man bevæger sig fra jordens overflade vil det samlede kraftmoment fra tyngdekraften på stangen være 0, idet tyngdekraften "trækker" lige hårdt i begge ender af stangen. Forskydningen i fjederen idet stangen slippes kan approksimeres som $x \approx -\frac{L}{2}\theta$ og dermed bliver kraften

$$F = k\frac{L}{2}\theta.$$

Bevægelsesligningen bliver da

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -k\frac{L}{2} \cdot \frac{L}{2}\theta.$$

Inertimomentet for en stang der roterer om sit endepunkt er

$$\frac{1}{12}mL^2.$$

Dermed har vi at

$$\frac{1}{12}mL^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -k\frac{L^2}{4}\theta$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -3\frac{k}{m}\theta$$

$$\omega^2 = 3\frac{k}{m}$$

$$\omega = \sqrt{3\frac{k}{m}}.$$

Og dermed kan perioden $T = \frac{2\pi}{\omega}$ findes som

$$T = 2\pi\sqrt{\frac{m}{3k}}.$$