Opgaver til forelæsning uge 24

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Opg. 12.45

A sealed tank containing seawater to a height of 11,0 m also contains air above the water at a gauge pressure of 3,00 atm. Water flows out from the bottom through a small hole. How fast is this water moving?

Vi har Bernoullis ligning for situationen som

$$p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = p_2 + 0 + \frac{1}{2}\rho v_2^2.$$

Fra kontinuitetsligningen har vi

$$A_1 v_1 = A_2 v_2.$$

Så idet det antages, at $A_1 \ll A_2$ fås, at $v_1 \approx 0$. Altså har vi

$$p_1 + \rho \cdot g \cdot y_1 = p_2 + \frac{1}{2}\rho v_2^2.$$

Idet $p = 3{,}00\,\mathrm{atm}$ er et såkaldt *gauge pressure* betyder det at p_1 er 3,00 atm større end p_2 og vi får da

$$\begin{aligned}
\frac{1}{2}\rho v_2^2 &= p + \rho g y_1 \\
v_2^2 &= \frac{2}{\rho}p + 2 g y_1 \\
v_2 &= \sqrt{\frac{2}{\rho}p + 2 g y_1} \\
&= \sqrt{2\frac{3{,}039\,75 \cdot 10^5\,\mathrm{Pa}}{1000\,\frac{\mathrm{kg}}{\mathrm{m}^3}} + 2 \cdot 9{,}80\,\frac{\mathrm{m}}{\mathrm{s}^2} \cdot 11{,}0\,\mathrm{m}} \\
&= 28{,}7\,\frac{\mathrm{m}}{\mathrm{s}}.
\end{aligned}$$

Opg. 12.51

A golf course sprinkler system discharges water from a horizontal pipe at the rate of 7200 $\frac{\text{cm}^3}{\text{s}}$. At one point in the pipe, where the radius is 4,00 cm, the water's absolute pressure is $2{,}40 \cdot 10^5$ Pa. At a second point in the pipe, the water passes through a constriction where the radius is 2,00 cm. What is the water's absolute pressure as it flows through this constriction?

Først findes tværsnitsarealerne af de to punkter i røret som hhv.

$$A_1 = \pi \cdot (4{,}00\,\text{cm})^2 = 50{,}27\,\text{cm}^2$$
$$A_2 = \pi \cdot (2{,}00\,\text{cm})^2 = 12{,}57\,\text{cm}^2.$$

Fra kontinuitetsligningen har vi desuden at

$$7200\,\frac{\text{cm}^3}{\text{s}} = \frac{\mathrm{d}V}{\mathrm{d}t} = A_1 v_1 = A_2 v_2.$$

De to hastigheder kan da findes som

$$v_1 = \frac{1}{A_1}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{50{,}72\,\text{cm}^2} \cdot 7200\,\frac{\text{cm}^3}{\text{s}} = 1{,}432\,\frac{\text{m}}{\text{s}}$$
$$v_2 = \frac{1}{A_2}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{12{,}57\,\text{cm}^2} \cdot 7200\,\frac{\text{cm}^3}{\text{s}} = 5{,}730\,\frac{\text{m}}{\text{s}}.$$

Vi kan da benytte Bernoullis ligning til at finde trykket ved indsnævringen som

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$
$$p_2 = p_1 + \frac{1}{2}\rho\left(v_1^2 - v_2^2\right)$$
$$= 2{,}40 \cdot 10^5\,\text{Pa} + \frac{1000\,\frac{\text{kg}}{\text{m}^3}}{2}\left(\left(1{,}432\,\frac{\text{m}}{\text{s}}\right)^2 - \left(5{,}730\,\frac{\text{m}}{\text{s}}\right)^2\right)$$
$$= 2{,}25 \cdot 10^5\,\text{Pa}.$$

Opg. 12.73

A closed and elevated vertical cylindrical tank with diameter 2,00 m contains water to a depth of 0,800 m. A worker accidently pokes a circular hole with diameter 0,0200 m in the bottom of the tank. As the water drains from the tank, compressed air above the water in the tank maintains a gauge pressure of $5{,}00 \cdot 10^3$ Pa at the surface of the water. Ignore any effects of viscosity.

(a)

Just after the hole is made, what is the speed of the water as it emerges from the hole? What is the ratio of this speed to the efflux speed if the top of the tank is open to the air?

Idet vi har opgivet et *gauge pressure* udgår det atmosfæriske tryk fra Bernoullis ligning så vi får, at

$$p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = \frac{1}{2}\rho v_2^2.$$

Vi har desuden kontinuitetsligningen, der siger, at

$$A_1 v_1 = A_2 v_2 \implies d_1^2 v_1 = d_2^2 v_2 \implies v_1 = \left(\frac{d_2}{d_1}\right)^2 v_2.$$

Vi kan sætte dette ind i Bernoullis ligning som

$$\begin{aligned}
\frac{1}{2}\rho\left(v_2^2 - v_1^2\right) &= p_1 + \rho g y_1 \\
\frac{1}{2}\rho\left(v_2^2 - \left(\frac{d_2^2}{d_1^2}\right)^2 v_2^2\right) &= p_1 + \rho g y_1 \\
\frac{1}{2}\rho\left(1 - \left(\frac{d_2}{d_1}\right)^2\right) v_2^2 &= p_1 + \rho g y_1 \\
v_2 &= \sqrt{\frac{2}{\rho \cdot \left(1 - \left(\frac{d_2}{d_1}\right)^4\right)}(p_1 + \rho g y_1)}.
\end{aligned}$$

Sættes tallene fra opgaven ind fås, at

$$v_2 = \sqrt{\frac{2}{1000\,\frac{\mathrm{kg}}{\mathrm{m}^3}\left(1 - \left(\frac{0{,}0200\,\mathrm{m}}{2{,}00\,\mathrm{m}}\right)^4\right)}\left(5{,}00 \cdot 10^3\,\mathrm{Pa} + 1000\,\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 9{,}80\,\frac{\mathrm{m}}{\mathrm{s}^2} \cdot 0{,}800\,\mathrm{m}\right)} = 5{,}07\,\frac{\mathrm{m}}{\mathrm{s}}.$$

Altså er hastigheden på vandet der fosser ud netop i det der laves hul 5,07 m/s.
Samme beregning kan nu laves uden bidraget fra overtrykket på $5{,}00 \cdot 10^3$ Pa indeni beholderen for at finde hastigheden, hvis der ikke var tryk i beholderen som

$$v_{2_a} = \sqrt{\frac{2 \cdot 9{,}80\,\frac{\mathrm{m}}{\mathrm{s}^2} \cdot 0{,}800\,\mathrm{m}}{1 - \left(\frac{0{,}0200\,\mathrm{m}}{2{,}00\,\mathrm{m}}\right)^4}} = 3{,}96\,\frac{\mathrm{m}}{\mathrm{s}}.$$

Altså bliver forholdet

$$\frac{5{,}07\,\frac{\mathrm{m}}{\mathrm{s}}}{3{,}96\,\frac{\mathrm{m}}{\mathrm{s}}} = 1{,}28.$$

(b)

How much time does it take for all the water to drain from the tank? What is the ratio of this time to the time it takes for the tank to drain if the top of the tank is open to the air?

Vi har at $v_1 = -\frac{\mathrm{d}y}{\mathrm{d}t}$, da hastigheden af vandet i toppen af cylinderen nødvendigvis må tilsvare hastigheden som vandspejlet falder med. I (a) fandt vi et andet udtryk for v_1 og de to udtryk kan nu sættes lig hinanden som

$$v_1 = -\frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{d_2}{d_1}\right)^2 v_2.$$

Vi kan nu sætte dette ind i formlen for v_2 fra (a) som

$$-\frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{d_2}{d_1}\right)^2 \sqrt{\frac{2(p_1 + \rho g y_1)}{\rho\left(1 - \left(\frac{d_2}{d_1}\right)^4\right)}}.$$

Vi kan nu bemærke, at

$$\frac{1}{1 - \left(\frac{d_2}{d_1}\right)^4} = 1{,}000\,000\,01 \approx 1.$$

Denne faktor ignoreres derfor i det følgende. Vi får da at

$$-\frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{d_2}{d_1}\right)^2 \sqrt{\frac{2}{\rho} \cdot (p_1 + \rho g y_1)}.$$

Dette simplificerer til

$$-\frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{d_2}{d_1}\right)^2 \sqrt{2} \cdot \sqrt{\frac{p_1 + \rho g y_1}{\rho}} = \left(\frac{d_2}{d_1}\right)^2 \sqrt{2g} \cdot \sqrt{\frac{p_1}{\rho g} + y_1}.$$

Vi får altså at

$$\frac{1}{\sqrt{\frac{p_1}{\rho g} + y}} \,\mathrm{d}\left(y + \frac{p_1}{\rho g}\right) = -\left(\frac{d_2}{d_1}\right)^2 \sqrt{2g}\,\mathrm{d}t.$$

Integralerne løses som

$$\int_{y_1}^{0} \frac{1}{\sqrt{y + \frac{p_1}{\rho g}}} \,\mathrm{d}\left(y + \frac{p_1}{\rho g}\right) = -\left(\frac{d_2}{d_1}\right)^2 \sqrt{2g} \int_0^t \mathrm{d}t.$$

Vi får da

$$2\sqrt{y + \frac{p_1}{\rho g}}\,\Bigg|_{y_1}^{0} = -\left(\frac{d_2}{d_1}\right)^2 \sqrt{2g}t.$$

Hvilket løses som

$$\begin{aligned} 2\left(\sqrt{y_1 + \frac{p_1}{\rho g}} - \sqrt{\frac{p_1}{\rho g}}\right) &= \left(\frac{d_2}{d_1}\right)^2 \sqrt{2g}t \\ t &= \frac{2}{\sqrt{2g}} \cdot \left(\frac{d_1}{d_2}\right)^2 \left(\sqrt{y_1 + \frac{p_1}{\rho g}} - \sqrt{\frac{p_1}{\rho g}}\right) \\ &= 1944\,\mathrm{s} = 32{,}4\,\mathrm{min} = 0{,}540\,\mathrm{hr}. \end{aligned}$$

Sættes overtrykket i beholderen til 0 fås i stedet

$$t = 4041\,\mathrm{s}.$$

Forholdet er altså

$$\frac{4041\,\mathrm{s}}{1944\,\mathrm{s}} = 2{,}08.$$

Opg. 12.80

A cylindrical bucket, open at the top, is 25,0 cm high and 10,0 cm in diameter. A circular hole with a cross-sectional area $1{,}50\,\mathrm{cm}^2$ is cut in the center of the bottom of the bucket. Water flows into the bucket from a tube above it at the rate of $2{,}40 \cdot 10^{-4}\,\mathrm{m}^3/\mathrm{s}$. How high will the water in the bucket rise?

Vi har i opgaven fået oplyst, at

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2{,}40 \cdot 10^{-4}\,\frac{\mathrm{m}^3}{\mathrm{s}}.$$

Vi betragter nu tilfældet efter at vandniveuaet har stabiliseret sig. Her har vi

$$\frac{\mathrm{d}V}{\mathrm{d}t} = A_2 v_2.$$

Vi kan altså finde udstrømningshastigheden som

$$v_2 = \frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{A_2} = \frac{2{,}4 \cdot 10^{-4}\,\mathrm{m}\frac{3}{\mathrm{s}}}{1{,}50\,\mathrm{cm}^2} = 1{,}60\,\frac{\mathrm{m}}{\mathrm{s}}.$$

Vha. Bernoullis ligning kan der nu opstilles et udtryk for “starthøjden” af væsken som

$$\begin{aligned}
\rho g h &= \frac{1}{2}\rho v_2^2 \\
h &= \frac{v_2^2}{2g} \\
&= \frac{\left(1{,}60\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2}{2 \cdot 9{,}80\,\frac{\mathrm{m}}{\mathrm{s}^2}} && = 0{,}131\,\mathrm{m} = 13{,}1\,\mathrm{cm}.
\end{aligned}$$

Opg. 12.83

Two very large open tanks A and F (**Figur 1**) contain the same liquid. A horizontal pipe BCD, having a constriction at C and open to the air at D, leads out of the bottom of tank A, and a vertical pipe E opens into the constriction at C and dips into the liquid in tank F. Assume streamline flow and no viscosity. If the cross-sectional area at C is one-half the area at D and if D is a distance h_1 below the level of the liquid in A, to what height h_2 will liquid rise in pipe E? Express your answer in terms of h_1.

Figur 1: