Opgaver til forelæsning 20

Noah Rahbek Bigum Hansen

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Opg. 13.31

The star Rho^1 Cancri is 57 light-years from the earth and has a mass 0,85 times that of our sun. A planet has been detected in a circular orbit around Rho^1 with an orbital radius equal to 0,11 times the radius of the earth's orbit around the sun. What are

(a)

The orbital speed of the planet of Rho^1 Cancri?

Vi har formlen for hastigheden, v, i en jævn tyngdebundet cirkelbevægelse som

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{G \cdot 0{,}85 \cdot 1{,}99 \cdot 10^{30}\,\mathrm{kg}}{0{,}11 \cdot 1{,}5 \cdot 10^{8}\,\mathrm{km}}} = 82\,720\,\frac{\mathrm{m}}{\mathrm{s}}.$$

(b)

The orbital period of the planet of Rho^1 Cancri?

Formlen for perioden for et objekt i en tyngdebundet jævn cirkelbevægelse er

$$T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}} = \frac{2\pi (0{,}11 \cdot 1{,}5 \cdot 10^{8}\,\mathrm{km})^{\frac{3}{2}}}{\sqrt{G \cdot 0{,}85 \cdot 1{,}99 \cdot 10^{30}\,\mathrm{kg}}} = 1{,}254 \cdot 10^{6}\,\mathrm{s} \cong 14{,}5\,\mathrm{days}.$$

Opg. 13.35

A thin spherical shell has radius $r_A = 4{,}00\,\mathrm{m}$ and mass $m_A = 20{,}0\,\mathrm{kg}$. It is concentric with a second thin spherical shell that has radius $r_B = 6{,}00\,\mathrm{m}$ and mass $m_B = 40{,}0\,\mathrm{kg}$. What is the net gravitational force that the two shells exert on a point mass of 0,0200 kg that is a distance r from the common center of the two shells.

(a)

$r = 2{,}00\,\mathrm{m}$ (inside both shells)

Fra Newtons skalteorem har vi at tyngdefeltet indeni en hul skal er 0. I dette tilfælde er massen placeret indeni begge skaller og derfor er det effektive tyngdefelt 0.

(b)

$r = 5{,}00\,\mathrm{m}$ (in the space between the two shells

Her er det kun den inderste masse der bidrager til tyngdefeltet. Vi får

$$F_g = \frac{GMm}{r^2} = \frac{G \cdot 20{,}0\,\mathrm{kg} \cdot 0{,}0200\,\mathrm{kg}}{(5{,}00\,\mathrm{m})^2} = 1{,}07 \cdot 10^{-12}\,\mathrm{N}.$$

(c)

$r = 8{,}00\,\mathrm{m}$ (outside both shells)

Her bidrager den yderste skal også så vi får

$$F_g = \frac{Gm}{r^2}(m_a + m_b) = \frac{G \cdot 0{,}0200\,\mathrm{kg}}{(8{,}00\,\mathrm{m})^2}(20{,}0\,\mathrm{kg} + 40{,}0\,\mathrm{kg}) = 1{,}25 \cdot 10^{-12}\,\mathrm{N}.$$

Opg. 13.37

A uniform, solid, 1000,0 kg sphere has a radius of 5,00 m.

(a)

Find the gravitational force the sphere exerts on a 2,00 kg point mass placed at the following distances from the center of the sphere

(i)

5,01 m

Vi benytter gravitationsloven som

$$F_g = \frac{GMm}{r^2} = 5{,}34 \cdot 10^{-9}\,\mathrm{N}.$$

(ii)

2,50 m

Fra Eksempel 13.10 har vi gravitationskraften, F_g, som

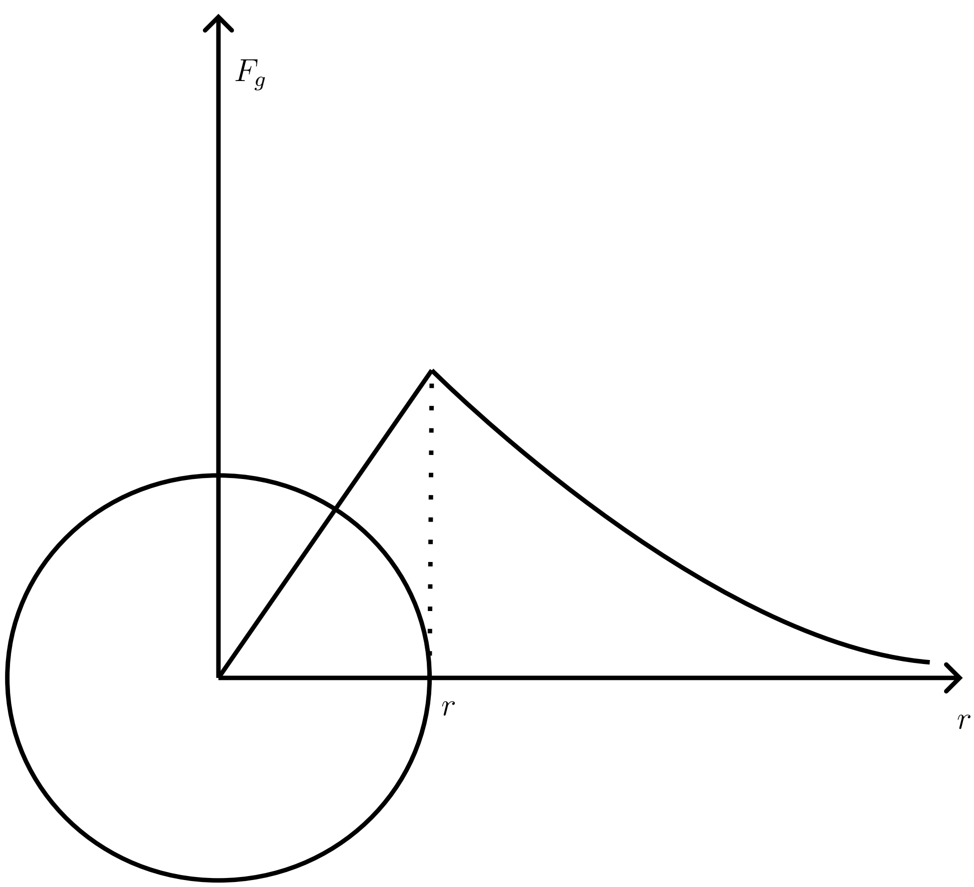
$$F_g = \frac{GMm}{R^3} r,$$

hvor R er radius af objektet og r er afstanden til objektets centrum. Vi har altså

$$F_g = \frac{G \cdot 1000{,}0\,\mathrm{kg} \cdot 2{,}00\,\mathrm{kg}}{(5{,}00\,\mathrm{m})^3} \cdot 2{,}50\,\mathrm{m} = 2{,}67 \cdot 10^{-9}\,\mathrm{N}.$$

(b)

Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \to \infty$.

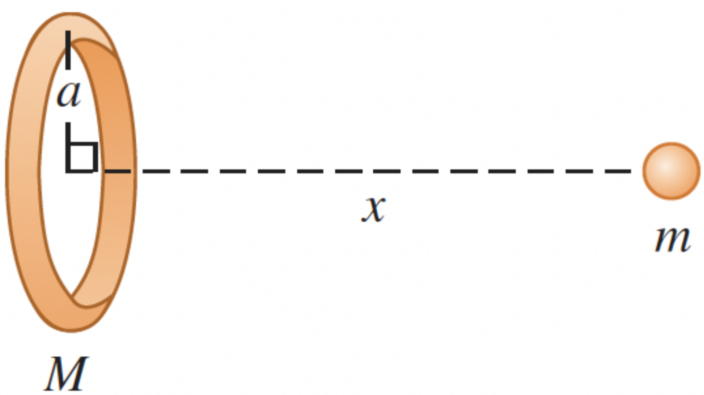


Figur 1: Tegning af situationen

Se **Figur 1**

Opg. 13.39

Figur 2:



Consider the ringshaped object in **Figur 2**. A particle with mass m is placed a distance x from the center of the ring, along the line through the center of the ring and perpendicular to its plane.

(a)

Calculate the gravitational potential energy U of this system. Take the potential energy to be zero when the two objects are far apart.

Afstanden fra m til et punkt på cirkelperiferien $s = \sqrt{x^2 + a^2}$ er konstant for alle punkter på cirkelperiferien. Derudover sørger ringens symmetri for at enhver del af massen der ikke trækker parallelt i m ift. x har en masse netop modsat midten der trækker ligeså meget i den anden retning og derfor er den potentielle gravitationelle energi

$$U_g = -\frac{GMm}{\sqrt{x^2 + a^2}}.$$

(b)

Show that your answer to part (a) reduces to the expected result when x is much larger than the radius a of the ring.

For $x \to \infty$ er $a \ll x$ og derfor fås at

$$\lim_{x \to \infty} U_g = -\frac{GMm}{x}.$$

(c)

Use $F_x = -\frac{\mathrm{d}U}{\mathrm{d}x}$ to find the magnitude and direction of the force on the particle (see Section 7.4).

Vi løser so så.

$$F_x = -\frac{\mathrm{d}U}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{GMm}{\sqrt{x^2+a^2}} = -\frac{GMmx}{(x^2+a^2)^{\frac{3}{2}}}.$$

(d)

Show that your answer to part (c) reduces to the expected result when x is much larger than a.

$$\lim_{x\to\infty} F_x = -\frac{GMmx}{x^3} = -\frac{GMm}{x^2}.$$

(e)

What are the values of U and F_x when $x = 0$? Explain why these results make sense.

For $x = 0$ har vi

$$U_g = -\frac{GMm}{a}$$
$$F_x = 0.$$

Opg. 13.71

Planets are not uniform inside. Normally, they are densest at the center and have decreasing density outward toward the surface. Model a spherically symmetric planet, with the same radius as the earth, as having a density that decreases linearly with distance from the center. Let the density be $15{,}0 \cdot 10^3\,\mathrm{kg/m^3}$ at the center and $2{,}0 \cdot 10^3\,\mathrm{kg/m^3}$ at the surface. What is the acceleration due to gravity at the surface of this planet?

Vi kan opstille følgende funktion for densiteten af planeten som funktion af afstanden til centrum r.

$$\rho(r) = (\rho_R - \rho_0)\frac{r}{R} + \rho_0.$$

Den gravitationelle acceleration ved overfladen kun af planetens radius og af dens masse. Planetens radius er kendt og derfor skal kun dens masse findes. Dette gøres ved at inddele planeten i en række tynde skaller, der hver har volumenet

$$\mathrm{d}V = 4\pi r^2\,\mathrm{d}r.$$

Vi får altså

$$\begin{aligned}
M &= \int_0^{R_E} 4\pi r^2 \rho(r)\,\mathrm{d}r \\
&= 4\pi\left(\rho_0 \int_0^{R_E} r^2\,\mathrm{d}r - \frac{\rho_0-\rho_R}{R}\int_0^{R_E} r^3\,\mathrm{d}r\right) \\
&= 4\pi\left(\rho_0\frac{R^3}{3} - \frac{\rho_0-\rho_R}{R}\frac{R^4}{4}\right) \\
&= 4\pi R^3\left(\frac{\rho_0}{3} - \frac{\rho_0-\rho_R}{4}\right) \\
&= \pi R^3\left(\rho_R + \frac{1}{3}\rho_0\right).
\end{aligned}$$

Vi kan dermed benytte formlen for tyngdeacceleration ved overfladen af en planet

$$g = \frac{GM}{R^2} = GR\pi\left(\rho_R + \frac{1}{3}\rho_0\right) = \pi G \cdot 6378\,\mathrm{km}\left(2{,}0\cdot 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3} + \frac{1}{3}\cdot 15{,}0\cdot 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3}\right) = 9{,}361\,\frac{\mathrm{m}}{\mathrm{s}^2}.$$

Opg. 13.73

An object in the shape of a thin ring has radius a and mass M. A uniform sphere with mass m and radius R is placed with its center at a distance x to the right of the center of the ring, along a line through the center of the ring, and perpendicular to its plane (see Figur 2). What is the gravitational force that the sphere exerts on the ring-shaped object? Show that your result reduces to the expected result when x is much larger than a.

Se Opg. 13.39