Opgaver til forelæsning 18

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Opg. 9.13

Show that the angle α between a plumb line and the direction of the earth's center is well approximated by $\tan\alpha = \frac{R_c\Omega^2 \sin 2\theta}{2g}$, where g is the observed free-fall acceleration and we assume the earth is perfectly spherically symmetric. Estimate the maximum and minimum values of the magnitude of α.

Vi har en effektiv tyngdeacceleration på jorden givet ved

$$g_{eff} = g_0 + \Omega^2 R \sin\theta \hat{\rho}.$$

Denne har en radiel komponent, g_{rad}, og en tangentiel komponent, g_{tan}, der svarer til hhv.

$$g_{rad} = g_0 - \Omega^2 R \sin^2\theta$$

og

$$g_{tan} = \Omega^2 R \sin\theta\cos\theta.$$

Dermed bliver

$$\begin{aligned}
\tan\alpha &= \frac{g_{tan}}{g_{rad}} \\
&= \frac{\Omega^2 R \sin\theta\cos\theta}{g_0 - \Omega^2 R \sin^2\theta} \\
&\approx \frac{\Omega^2 R \sin\theta\cos\theta}{g_0} \\
&= \frac{\Omega^2 R}{2g_0}\sin 2\theta.
\end{aligned}$$

Hvilket skulle vises. α har et minimum for $\sin 2\theta = 0 \implies \theta = 0$, hvor $\alpha = 0$. α's maximum er ved $\sin 2\theta = 1 \implies \theta = \frac{\pi}{4} = 45°$, hvo $\alpha = \tan^{-1}\frac{\Omega^2 R}{2g_0}$

Opg. 9.25

A high-speed train is travelling at a constant 150 m/s (about 300 mph) on a straight, horizontal track across the South Pole. Find the angle between a plumb line suspended from the ceiling inside the train and another inside a hut on the ground. In what direction is the plumb line on the train deflected.

Vi har $\dot{y} = v_{y0}$ og $\theta = \pi$, da vi er ved sydpolen. Dermed bliver bevægelsesligningerne i alle retninger (9.53) i notatet

$$\begin{aligned}\ddot{x} &= 2\Omega v_{y0}\cos\theta = -2\Omega v_{y0}\\ \ddot{y} &= 0\\ \ddot{z} &= -g.\end{aligned}$$

Da bliver $\tan(\alpha)$

$$\tan\alpha = \frac{2\Omega v_{y0}}{g}.$$

Denne vinkel, α, er vinklen som loddet på toget laver til et tilsvarende lod i et stillestående system (huset). Loddet i toget bliver peger desuden lidt mod “højre” grundet corioliskraften pr. højrehåndreglen.

Opg. 9.26

In Section 9.8, we used a method of successive approximations to find the orbit of an object that is dropped from rest, correct to first order in the earth's angular velocity Ω. Show in the same way that if an object is thrown with initial velocity, v_0 from a point O on the earth's surface at colatitude θ, that to first order in Ω its orbit is

$$\begin{aligned}x &= v_{x0}t + \Omega(v_{y0}\cos\theta - v_{z0}\sin\theta)t^2 + \frac{1}{3}\Omega g t^3\sin\theta\\ y &= v_{y0}t - \Omega(v_{x0}\cos\theta)t^2\\ z &= v_{z0}t - \frac{1}{2}gt^2 + \Omega(v_{x0}\sin\theta)t^2.\end{aligned}$$

[First solve the equations of motion in zeroth order, that is, ignoring Ω entirely. Substitute your zeroth-order solution for $\dot{x}$, $\dot{y}$, and $\dot{z}$ into the right side of the equations above and integrate to give the next approximation. Assume that v_0 is small enough that air resistance is negligible and that g is a constant throughout the flight.]

Givet $\dot{r} = (\dot{x}, \dot{y}, \dot{z}) = (v_{x0}, v_{y0}, v_{z0})$ har vi

$$\begin{aligned}\ddot{x} &= 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta)\\ \ddot{y} &= -2\Omega\dot{x}\cos\theta\\ \ddot{z} &= -g + 2\Omega\dot{x}\sin\theta.\end{aligned}$$

0.-ordens approksimationen bliver da

$$\begin{array}{lll} \ddot{x} = 0, & \ddot{y} = 0, & \ddot{z} = -g \\ & \Downarrow & \\ \dot{x} = v_{x0}, & \ddot{y} = v_{y0}, & \ddot{z} = -gt + v_{z0} \\ & \Downarrow & \\ x = v_{x0}t, & y = v_{y0}t, & z = h - gt^2 + v_{z0}t. \end{array}$$

Disse kan indsættes i vores bevægelsesligninger som

$$\begin{aligned} \ddot{x} &= 2\Omega(v_{y0}\cos\theta - (v_{z0} - gt)\sin\theta) \\ \ddot{y} &= -2\Omega v_{x0}\cos\theta \\ \ddot{z} &= -g + 2\Omega v_{x0}\sin\theta \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \dot{x} &= 2\Omega(v_{y0}t\cos\theta - (v_{z0}t - \frac{1}{2}gt^2)\sin\theta) + v_{x0} \\ \dot{y} &= -2\Omega v_{x0}t\cos\theta + v_{y0} \\ \dot{z} &= -gt + 2\Omega v_{x0}t\sin\theta + v_{z0} \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} x &= v_{x0}t + 2\Omega\left(\frac{1}{2}t^2 v_{y0}\cos\theta - \left(\frac{1}{2}v_{z0}t^2 - \frac{1}{6}gt^3\right)\sin\theta\right) \\ &= v_{x0}t + \Omega\left(t^2\left(v_{y0}\cos\theta - v_{z0}\sin\theta\right) + \frac{1}{3}gt^3\sin\theta\right) \\ y &= v_{y0}t - \Omega v_{x0}t^2\cos\theta \\ z &= v_{z0}t - \frac{1}{2}gt^2 + \Omega v_{x0}t^2\sin\theta. \end{aligned}$$

Hvilket skulle vises.