Opgaver til forelæsning uge 19

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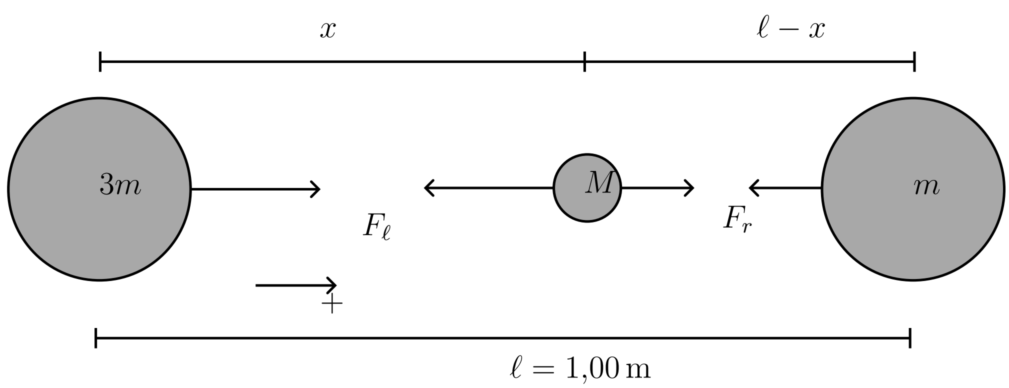
Opg. 13.9

A particle of mass $3m$ is located $1{,}00\,\mathrm{m}$ from a particle of mass m.

(a)

Where should you put a third mass M so that the net gravitational force on M due to the two masses is exactly zero?



Figur 1: Fritlegemediagram for de tre masser

For at der er statik må det gælde at $\sum F_{xM} = 0$. Vi har desuden Newtons tyngdelov der siger at

$$F_g = \frac{GMm}{r^2}.$$

Altså kan vi opstille følgende lighed

$$F_\ell + F_r = 0.$$

Vi får altså

$$
\begin{aligned}
\frac{GM3m}{x^2} &= \frac{GMm}{(l-x)^2}\\
x^2 &= 3(\ell-x)^2\\
&= 3x^2+3\ell^2-6\ell x\\
0 &= 2x^2+3\ell^2-6\ell x\\
&= x^2-3\ell x+\frac{3}{2}\ell^2\\
x &= \frac{3\ell\pm\sqrt{9\ell^2-6\ell^2}}{2}\\
\frac{x}{\ell} &= \frac{3\pm\sqrt{3}}{2}.
\end{aligned}
$$

For $\ell = 1{,}00\,\mathrm{m}$ får vi $x = 1{,}5 \pm \frac{\sqrt{3}}{2}$. For at $x < \ell$ kan vi forkaste løsningen hvor $\pm = +$ og får dermed $x = 1{,}5 - \frac{\sqrt{3}}{2}$.

(b)

Is the equilibrium of M at this point stable or unstable

(i)

for points along the line connecting m and $3m$, and

Vi har generelt fra perturbationsteori, at en ligevægt er stabil, hvis $\frac{\mathrm{d}^2U}{\mathrm{d}x^2} > 0$ eftersom enhver lille forskydning af massen $\mathrm{d}x$ i dette tilfælde medfører en restaurerende kraft og vice versa. Vi regner den potentielle gravitationelle energi som

$$U(x) = -\frac{GMm}{x} - \frac{3GMm}{1-x}.$$

Og vi differentierer to gange for at få

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = -\frac{2GMm}{x^3} - \frac{6GMm}{(1-x)^3}.$$

Hvis vi sætter en positiv værdi ind på x's plads bliver nævnerne hhv.

$$
\begin{aligned}
n_1\left(1{,}5-\frac{\sqrt{3}}{2}\right) &= \left(1{,}5-\frac{\sqrt{3}}{2}\right)\\
n_2\left(1{,}5-\frac{\sqrt{3}}{2}\right) &= \left(1-1{,}5+\frac{\sqrt{3}}{2}\right).
\end{aligned}
$$

Begge disse er positive og fortegnet for $\frac{\mathrm{d}^2U}{\mathrm{d}x^2}$ bliver derfor negativt og derfor er situationen ustabil.

(ii)

for points along the line passing through M and perpendicular to the line connecting m and $3m$?

I dette tilfælde vil M bevæge sig væk fra den rette linje mellem $3m$ og m og derfor vil gravitationskraften fra disse to virke restaurerende og derfor er denne situation mere stabil end før, omend situationen ikke er stabil som så, idet situationen bliver ustabil så snart M kommer ned til linjen mellem $3m$ og m.

Opg. 13.27

Two satellites are in circular orbits around a planet that has radius $9{,}00 \cdot 10^6\,\mathrm{m}$. One satellite has mass $68{,}0\,\mathrm{kg}$, orbital radius $7{,}00 \cdot 10^7\,\mathrm{m}$, and orbital speed $4800\,\mathrm{m/s}$. The second satellite has mass $84{,}0\,\mathrm{kg}$ and orbital radius $3{,}00 \cdot 10^7\,\mathrm{m}$. What is the orbital speed of this second satellite?

Vi har oplyst følgende

$$
\begin{aligned}
m_1 &= 68{,}0\,\mathrm{kg}, & v_1 &= 4800\,\frac{\mathrm{m}}{\mathrm{s}}, & r_1 &= 7{,}00 \cdot 10^7\,\mathrm{m}, \\
m_2 &= 84{,}0\,\mathrm{kg}, & r_2 &= 3{,}00 \cdot 10^7\,\mathrm{m}, & v_2 &= ??.
\end{aligned}
$$

Vi har den velkendte formel for hastigheden af et objekt i en jævn cirkelbevægelse om et massivt objekt som

$$v^2 = \frac{Gm_p}{r}.$$

Vi har dermed at

$$v_1^2 r_1 = Gm_p = \text{const.} = v_2^2 r_2.$$

Altså fås at

$$v_2 = 4800\,\frac{\mathrm{m}}{\mathrm{s}} \sqrt{\frac{7{,}00 \cdot 10^7\,\mathrm{m}}{3{,}00 \cdot 10^7\,\mathrm{m}}} = 7332\,\frac{\mathrm{m}}{\mathrm{s}}.$$

Opg. 13.47

An experiment is performed in deep space with two uniform spheres, one with mass $50{,}0\,\mathrm{kg}$ and the other with mass $100{,}0\,\mathrm{kg}$. They have equal radii, $r = 0{,}20\,\mathrm{m}$. The spheres are released from rest with their centers $40{,}0\,\mathrm{m}$ apart. They accelerate toward each other because of their mutual gravitational attraction. You can ignore all gravitational forces other than that between the two spheres.

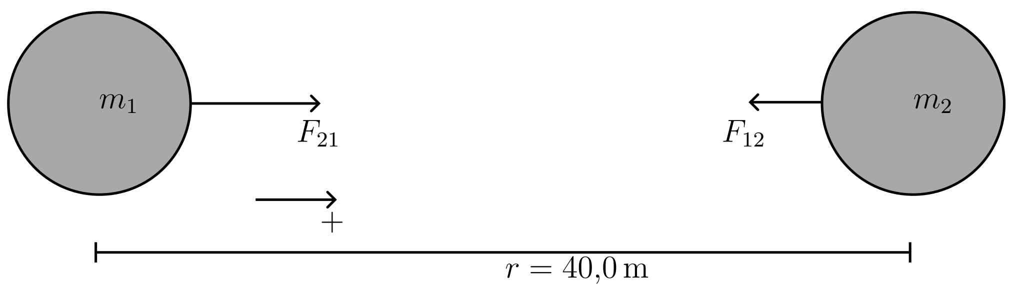
(a)

Explain why linear momentum is conserved.

Eftersom ingen udefrakommende kræfter virker på de to kugler udgør de et lukket system og derfor er deres lineære impuls konserveret.

(b)

When their centers are 20,0 m apart, find



Figur 2: Fritlegemediagram for de to kugler i initialpositionen.

(i)

the speed of each sphere and

Energi er konserveret i lukkede systemer og derfor må det gælde at

$$U_0 - U_1 = +K.$$

Hvor K er den totale kinetiske energi. Den potentielle energi i et tyngdefelt er generelt givet som

$$U_g = -\frac{GMm}{r}.$$

Vi har derfor følgende udtryk

$$-\frac{GMm}{r_0} + \frac{GMm}{r_f} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2.$$

Vi har desuden fra konservation af impuls at

$$m_1 v_1 = m_2 v_2 \implies v_2 = \frac{m_1}{m_2} v_1 = \frac{1}{2} v_1.$$

Altså er hastigheden v_2 halvt så stor som hastigheden v_1. Vi har dermed

$$\begin{aligned}
\frac{GMm}{r_f} - \frac{GMm}{r_0} &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2(0{,}5 \cdot v_1)^2 \\
\frac{G \cdot 50{,}0\,\mathrm{kg} \cdot 100\,\mathrm{kg}}{20{,}0\,\mathrm{m}} - \frac{G \cdot 50{,}0\,\mathrm{kg} \cdot 100\,\mathrm{kg}}{40{,}0\,\mathrm{m}} &= \frac{1}{2} \cdot 50{,}0\,\mathrm{kg} \cdot v_1^2 + \frac{1}{2} \cdot 100{,}0\,\mathrm{kg} \cdot 0{,}25v_1^2 \\
8{,}343 \cdot 10^{-9}\,\mathrm{J} &= 37{,}5\,\mathrm{kg}v_1^2 \\
v_1 &= \sqrt{\frac{8{,}3429 \cdot 10^{-9}\,\mathrm{J}}{37{,}5\,\mathrm{kg}}} \\
v_1 &= 1{,}492 \cdot 10^{-5}\,\frac{\mathrm{m}}{\mathrm{s}} \\
\implies v_2 &= \frac{1}{2} \cdot 1{,}492 \cdot 10^{-5}\,\frac{\mathrm{m}}{\mathrm{s}} = 7{,}458 \cdot 10^{-6}\,\frac{\mathrm{m}}{\mathrm{s}}.
\end{aligned}$$

Altså er de to hastigheder fundet til den angivne seperation.

(ii)

the magnitude of the relative velocity with which one sphere is approaching the other.

Deres relative indbyrdes hastighed er

$$|v_1| + |v_2| = 3v_2 = 2{,}24 \cdot 10^{-5}\,\frac{\mathrm{m}}{\mathrm{s}}.$$

(c)

How far from the initial position of the center of the 50,0 kg sphere do the surfaces of the two spheres collide?

Konservation af momentum giver at

$$\begin{aligned}
m_1v_1 + m_2v_2 &= 0 \\
\implies m_1x_1 + m_2x_2 &= 0 \\
\implies m_1x_1 &= m_2x_2.
\end{aligned}$$

Vi har desuden, at

$$x_1 + x_2 = 40\,\mathrm{m} \implies x_2 = 40\,\mathrm{m} - x_1.$$

Vi har altså, at

$$\frac{m_1}{m_2}x_1 = \frac{1}{2}x_1 = 40\,\mathrm{m} - x_1.$$

Og vi kan slutteligt løse for x_1 som

$$\begin{aligned}
\frac{1}{2}x_1 &= 40\,\mathrm{m} - x_1 \\
\frac{3}{2}x_1 &= 40\,\mathrm{m} \\
x_1 &= \frac{2}{3}40\,\mathrm{m} \approx 26{,}67\,\mathrm{m}.
\end{aligned}$$

Opg. 13.61

Falling hammer. A hammer with mass m is dropped from rest from a height h above the earth's surface. This height is not necessarily small compared with the radius R_E of the earth. Ignoring air resistance, derive an expression for the speed v of the hammer when it reaches the earth's surface. Your expression should involve h, R_E, and m_E (the earth's mass).

Den potentielle energi til en given højde, h er givet som

$$U = -\frac{GMm}{r}.$$

Idet energi er konserveret har vi at

$$U_0 + k_0 = U_1 + k_1.$$

Dette kan omskrives til

$$\begin{aligned}
-\frac{Gmm_E}{R_E} + \frac{1}{2}mv^2 &= -\frac{Gmm_E}{R_E + h} + 0 \\
\frac{1}{2}mv^2 &= \frac{Gmm_E}{R_E} - \frac{Gmm_E}{R_E + h} \\
&= \frac{Gmm_E}{R_E}\left(1 - \frac{R_E}{R_E + h}\right) \\
&= \frac{Gmm_E}{R_E}\left(\frac{h}{R_E + h}\right) \\
&= \frac{Gmm_e h}{R_E(R_E + h)} \\
v^2 &= \frac{2Gm_e h}{R_E(R_E + h)} \\
v &= \sqrt{\frac{2Gm_e h}{R_E(R_E + h)}}.
\end{aligned}$$

Opg. 13.63

Binary Star-Equal Masses. Two identical stars with mass M orbit around their center of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle.

(a)

Find the gravitational force of one star on the other.

Newtons gravitationslov lyder generelt

$$F_g = \frac{GMm}{r^2}.$$

Sættes symboler fra opgaven ind her fås

$$F_g = \frac{GM^2}{4R^2}.$$

(b)

Find the orbital speed of each star and the period of the orbit.

Fra Newtons 2. lov ($\sum \vec{F} = m\vec{a}$) har vi at

$$\begin{aligned} \frac{Mv^2}{R} &= \frac{GM^2}{4R^2} \\ v &= \sqrt{\frac{GM}{4R}}. \end{aligned}$$

Og idet $T = \frac{2\pi r}{v}$ må perioden blive

$$T = \frac{2\pi R}{v} = 2\pi R \sqrt{\frac{4R}{GM}} = 4\pi \sqrt{\frac{R^3}{GM}}.$$

(c)

How much energy would be required to separate the two stars to infinity?

Vi opskriver udtrykket for energikonservation, idet vi sætter energien i sluttilstanden til 0 og tilføjer et udført arbejde, W_{appl}, på initialsiden som

$$\begin{aligned} K_1 + U_1 + W_{appl} &= 0 \\ W_{appl} &= -(K_1 + U_1) \\ &= -\left(2\left(\frac{1}{2}Mv^2\right) - \frac{GM^2}{2R}\right) \\ &= \frac{GM^2}{2R} - Mv^2 \\ &= \frac{GM^2}{2R} - M\frac{GM}{4R} \\ &= \frac{GM^2}{4R}. \end{aligned}$$

Opg. 13.65

Comets travel around the sun in elliptical orbits with large eccentricities. If a comet has speed $2{,}0 \cdot 10^4\,\mathrm{m/s}$ when at a distance of $2{,}5 \cdot 10^{11}\,\mathrm{m}$ from the center of the sun, what is its speed when at a distance of $5{,}0 \cdot 10^{10}\,\mathrm{m}$?

Fra energikonservation har vi at

$$\begin{aligned} U_0 + k_0 &= U_1 + k_1 \\ -\frac{GMm}{r_1} + \frac{1}{2}mv_1^2 &= -\frac{GMm}{r_2} + \frac{1}{2}mv_2^2 \\ \frac{1}{2}v_2^2 &= \frac{1}{2}v_1^2 + GM\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \\ v_2 &= \sqrt{v_1^2 + 2GM\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}. \end{aligned}$$

Sættes kendte størrelser ind fås at

$$v_2 = \sqrt{\left(2{,}0 \cdot 10^4\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + 2G \cdot 1{,}99 \cdot 10^{30}\,\mathrm{kg}\left(\frac{1}{5{,}0 \cdot 10^{10}\,\mathrm{m}} - \frac{1}{2{,}5 \cdot 10^{11}\,\mathrm{m}}\right)} = 68\,190\,\frac{\mathrm{m}}{\mathrm{s}}.$$