Statics and Strength of Materials

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Lecture 1: Intro + Force % position vectors 25. August 2025

1 Introduction

This is my personal notes for the Statics and strength of materials course taught at Aarhus University by Tito Andriollo and Souhayl Sadik.

1.1 Terminology

Some basic terminology must be defined before we can start.

Definition 1: Particle

A particle is a body whose geometry can be neglected in the problem at hand.

Definition 2: Rigid body

A rigid body is a combination of a large number of particles with a fixed distance from each other. I.e. immovable atoms.

Definition 3: Deformable body

A deformable body is a combination of a large number of particles, but where the distance between the particle can vary subject to forces.

2 Force vectors

2.1 Scalars and vectors

Oftentimes in engineering mechanics one utilizes scalars or vectors to measure physical quantities.

Definition 4: Scalar

A *scalar* is any positive or negative numerical quantity – i.e. any physical quantity that is expressable solely by its *magnitude*. Scalar quantities include length, mass, time, etc.

Definition 5: Vector

A *vector* is any physical quantity that requires both a *magnitude* and a *direction* to be fully described. Vector quantities include force, moment, velocity, etc. In this collection of notes vector quantities will be represented by boldface notation such as $\mathbf{A}$.

2.2 Vector operations

2.2.1 Multiplication and division of a vector by a scalar

When one multiplies or divides a vector quantity by a scalar its magnitude is simply changed by that amount. The direction will remain the same.

2.2.2 Vector addition and subtraction

When adding two vector quantities one must both account for the magnitudes and the directions of the vector quantities. To do this graphically one places the tail end one of one of the vectors at the head end of the other – the endpoint is their sum. Algebraically this corresponds to adding the component vectors pairwise.

For subtraction one can once again use the algebraic method of the pairwise components or graphically one can place the tail ends of the vectors at the same point and find the 'difference vector' that combines their heads.

2.3 Addition of a system of coplanar forces

When a force is resolved into components along the x and y axes, the components are called *rectangular components*. We can represent these either by *scalar notation* or *Cartesian notation*. Only cartesian notation will be covered in these notes.

2.3.1 Cartesian notation

One can represent a force **F** as a sum of the magnitudes of the force F_x and F_y and the cartesian unit vectors **i** and **j**. That is:

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}.$$

2.3.2 Coplanar force resultants

Given three forces:

$$\begin{aligned}
\mathbf{F}_1 &= F_{1x}\mathbf{i} + F_{1y}\mathbf{j} \\
\mathbf{F}_2 &= -F_{2x}\mathbf{i} + F_{2y}\mathbf{j} \\
\mathbf{F}_3 &= F_{3x}\mathbf{i} - F_{3y}\mathbf{j}.
\end{aligned}$$

One can compute the vector resultant as described in subsubsection 2.2.2 as:

$$\begin{aligned}
\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\
&= \left(F_{1x} - F_{2x} + F_{3x}\right)\mathbf{i} + \left(F_{1y} + F_{2y} - F_{3y}\right)\mathbf{j} \\
&= \left(F_r\right)_x \mathbf{i} + \left(F_r\right)_y \mathbf{j}.
\end{aligned}$$

The magnitude of the resultant force can be found from the magnitudes of the resultant component vectors and the Pythagorean theorem as:

$$F_r = \sqrt{\left(F_r\right)_x^2 + \left(F_r\right)_y^2}.$$

The angle which specifies the direction of the resultant force can be determined using trigonometry as:

$$\theta = \tan^{-1}\left|\frac{\left(F_R\right)_y}{\left(F_R\right)_x}\right|.$$

The same principle applies in three dimensions.

2.4 Position vectors

A position vector **r** is defines as a fixed vector which locates a specific point in space relative to another specific point in space. A position vector between two forces with the same tail end point will correspond do the subtracyion of the vectors.

2.4.1 Force vector directed along a line

Oftentimes in three-dimensional statics problems, the direction of a force **F** is specified by two points through which its line of action passes. If we call these two points A and B we can formulate **F** by realizing that it has the same direction as the position vector **r** from point A to point B. The common direction is specified by the unit vector $\mathbf{u} = \mathbf{r}/r$. We get:

$$\mathbf{F} = F\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right) = F\left(\frac{\left(x_B - x_A\right)\mathbf{i} + \left(y_B - y_A\right)\mathbf{j} + \left(z_B - z_A\right)\mathbf{k}}{\sqrt{\left(x_B - x_A\right)^2 + \left(y_B - y_A\right)^2 + \left(z_B - z_A\right)^2}}\right).$$

Lecture 2: Moment of a force

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2.5 Dot product

The dot product of vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as:

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta.$$

This is also called the scalar product of the vectors. The following laws of operation apply:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A} \\ a\left(\mathbf{A} \cdot \mathbf{B}\right) &= \left(a\mathbf{A}\right) \cdot \mathbf{B} = \mathbf{A} \cdot \left(a\mathbf{B}\right) \\ \mathbf{A} \cdot \left(\mathbf{B} + \mathbf{D}\right) &= \left(\mathbf{A} \cdot \mathbf{B}\right) + \left(\mathbf{A} \cdot \mathbf{D}\right). \end{aligned}$$

In Cartesian form the dot product can be expressed as:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z.$$

3 Force system resultants

3.1 Moment of a force – Scalar formulation

A force applied to a body will tend to produce a rotation about a point that is not on the line of action of the force – this phenomenon is called the *moment* or the *torque*.

In general, if we consider the force $\mathbf{F}$ and point O to lie in the shaded plane shown on **Figure 3.1**, then the moment $\mathbf{M}_O$ about the point O, or about an axis through O and perpendicular to the plane is a vector quantity since it has both a magnitude and a direction.

The magnitude of $\mathbf{M}_O$ is

$$M_O = Fd$$

where d is the *moment arm* or the perpendicular distance from the axis through point O to the line of action of the force.

The direction of $\mathbf{M}_O$ can be determined using the right hand rule – curl the fingers in the direction of rotation and the thumb will be pointing in the direction of the moment.

The resultant moment of multiple forces lying in the same plane is simply found by their algebraic sum. I.e.

$$M_{R_O} = \sum Fd.$$

3.2 Principle of moments

A very useful concept often used in mechanics is the principle of moments, which is also known as Varignon's theorem.

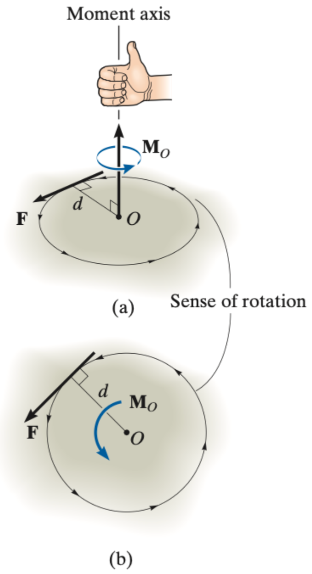

Figure 3.1:

Definition 6: Principle of moments

The moment of a force about a point is equal to the sum of the moments of the components of the force about the point.

This can be applied to the case shown on **Figure 3.2**. Here a force **F** is split into components and the resultant moment can then simply be found as:

$$M_O = F_x y + F_y x.$$

3.3 Cross product

The cross product of the two vectors **A** and **B** yields a vector **C** as:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}.$$

The magnitude of **C** is given by:

$$C = AB \sin\theta.$$

The vector **C** will have a direction perpendicular to the plane containing **A** and **B** according to the right hand rule.

The following laws of operation apply:

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= -\mathbf{B} \times \mathbf{A} \\ a\left(\mathbf{A} \times \mathbf{B}\right) &= \left(a\mathbf{A}\right) \times \mathbf{B} = \mathbf{A} \times \left(a\mathbf{B}\right) \\ \mathbf{A} \times \left(\mathbf{B} + \mathbf{D}\right) &= \left(\mathbf{A} \times \mathbf{B}\right) + \left(\mathbf{A} \times \mathbf{D}\right). \end{aligned}$$

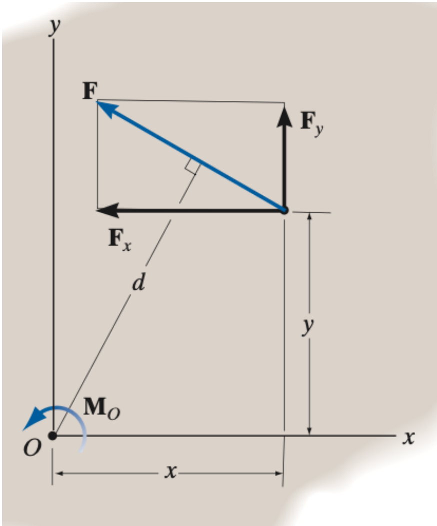

Figure 3.2:

In Cartesian coordinates the cross product may be written as:

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\,\mathbf{i} - (A_x B_z - A_z B_x)\,\mathbf{j} + (A_x B_y - A_y B_x)\,\mathbf{k}.$$

In determinant form it is:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

3.4 Moment of a force – Vector formulation

The moment of a force $\mathbf{F}$ about point O can be expressed using the vector cross product as:

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}.$$

Here $\mathbf{r}$ is a position vector *from O to any point* on the line of action of $\mathbf{F}$. When using this formulation $\mathbf{M}_O$ will have the correct magnitude and direction no matter what position vector is chosen along as the previous rule is respected.

The magnitude of this is:

$$M_O = rF\sin\theta = F\,(r\sin\theta) = Fd$$

and the direction of $\mathbf{M}_O$ is once again determined by the right hand rule.

The cross product is typically used in three dimensions since the perpendicular distance or moment arm from point O to the line of action of the force is not needed. In other words any position vector $\mathbf{r}_i$ pointing from the point O to the line of action of $\mathbf{F}$ is adequate.

If a body is acted upon by a system of forces, the resultant moment of the forces about point O can be determined by vector addition of the moment of each force as:

$$\mathbf{M}_{R_O} = \sum (\mathbf{r} \times \mathbf{F})\,.$$

Lecture 3: Systems of forces and moments 1. September 2025

4 Force System Resultants

4.1 Moment of a couple

A *couple* is defined as two parallel forces that hav equal magnitude but opposite directions and are separated by a distance d as shown on **Figure 4.1**. Since the resultant force of the couple is zero the only effect is to produce a tendency for rotation.

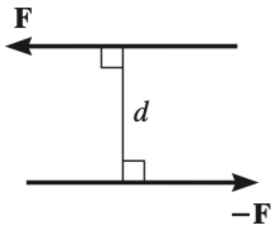

Figure 4.1:

The moment produced by a couple is called a *couple moment* it can be shown that this is equal to:

$$\mathbf{M} = \mathbf{r} \cdot \mathbf{F}.$$

This is a *free vector*, i.e. it can act at *any point* since $\mathbf{M}$ only depends on the position vector r between the forces and not the position vectors from the origin.

4.2 Simplification of a force and couple system

It is often convenient to reduce a complex system of forces and couple moments acting on a body to a simpler form by replacing it with an equivalent system. A system is equivalent if the *external effects* it produces on a body are the same as those caused by the original force and moment system.

Every system of several forces and couple moments can be broken down into a single resultant force acting at a point O and a resultant couple moment. This can be done by adding together the forces in each Cartesian direction as well as the moments each on their own.

4.3 Reduction of a simple distributed loading

Sometimes, a body is subjected to a load distributed over its surface. The pressure caused by this distributed loading on the surface represents the loading intensity and is measured in Pa.

The most common type of distributed loading is that along a single axis. Take for example a beam of constant width b subjected to a pressure loading that varies along the x-axis. The loading is described by the function $p = p(x)$. Since this contains only one variable we can quickly "one-dimensionalize" the problem by $w(x) = p(x)b$.

The magnitude of the resultant force in this case must be found using integration as summing "all the forces" would require an infinte sum. I.e.

$$F_R = \int_L w(x)\,\mathrm{d}x = \int_A \mathrm{d}A = A.$$

The location $\overline{x}$ of $\mathbf{F}_R$ can be determined using static equilibrium and moments as:

$$-\overline{x}F_R = -\int_L xw(x)\,\mathrm{d}x.$$

And solving for $\overline{x}$ we have:

$$\overline{x} = \frac{\int_L xw(x)\,\mathrm{d}x}{\int_L w(x)\,\mathrm{d}x} = \frac{\int_A x\,\mathrm{d}A}{\int_A \mathrm{d}A}.$$

Lecture 4: Center of Gravity 4. September 2025

5 Center of Gravity and Centroid

5.1 Center of Gravity and the centroid of a body

A body can be thought of as consisting of an infinite amount of differential particles (actually bodies consist of discrete atoms, but it is helpful to think of it in the infinite sense). If a body is located in a gravitational field, each of the particles will be subject to a weight $\mathrm{d}W$. These weights will form an approximately parallel system of forces and the resultant of this system is the total weight of the body, which passes through a single point called the center of gravity.

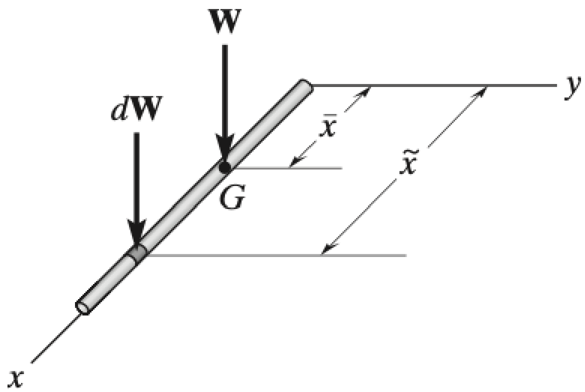

Figure 5.1:

We consider the rod on Figure 5.1, where the segment with weight $\mathrm{d}W$ is located at an arbitrary position $\tilde{x}$. The total weight of the rod is then:

$$W = \int \mathrm{d}W.$$

The location of the canter of gravity, measured from the y-axis is determined by equating the moment of W about the y-axis to the sum of the moments of the weights of all its particles about said axis. I.e.

$$\overline{x}W = \int \overline{x}\,\mathrm{d}W \implies \overline{x} = \frac{\int \overline{x}\,\mathrm{d}W}{\mathrm{d}W}.$$

This same procedure can be applied for all axes, i.e.

$$\overline{x} = \frac{\int \overline{x}\,\mathrm{d}W}{\int \mathrm{d}W}, \quad \overline{y} = \frac{\int \overline{y}\,\mathrm{d}W}{\int \mathrm{d}W}, \quad \overline{z} = \frac{\int \overline{z}\,\mathrm{d}W}{\int \mathrm{d}W}.$$

5.1.1 Centroid of a Volume

We consider the body on Figure 5.2. If it is made from a homogeneous material, then its specific weight γ will be constant. Therefore, a differential element of volume $\mathrm{d}V$ has a weight $\mathrm{d}W = \gamma\,\mathrm{d}V$. Substituting this into the equations from before, and cancelling out ρ we obtain the coordinates of the centroid C as:

$$\overline{x} = \frac{\int_V \tilde{x}\,\mathrm{d}V}{\int_V \mathrm{d}V}, \quad \overline{y} = \frac{\int_V \tilde{y}\,\mathrm{d}V}{\int_V \mathrm{d}V}, \quad \overline{z} = \frac{\int_V \tilde{z}\,\mathrm{d}V}{\int_V \mathrm{d}V}.$$

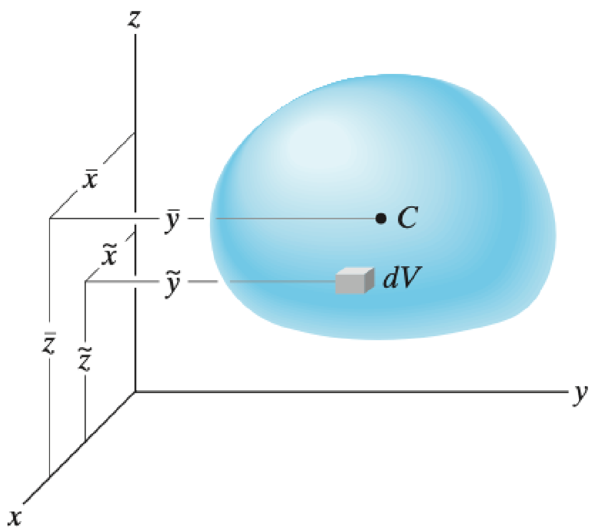

Figure 5.2:

5.1.2 Centroid of an area

Similarly, if an area lies in the x-y-plane and is bounded by the curve $y = f(x)$ then its centroid will be in the same plane and can be determined as:

$$\overline{x} = \frac{\int_A \tilde{x}\,\mathrm{d}A}{\int_A \mathrm{d}A}, \quad \overline{y} = \frac{\int_A \tilde{y}\,\mathrm{d}A}{\int_A \mathrm{d}A}.$$

5.2 Composite Bodies

A composite body consists of a series of connected "simpler" bodies. Provided the weight and location of center of gravity of each part we can eliminate the need for integration as:

$$\overline{x} = \frac{\sum \tilde{x} W}{\sum W}, \quad \overline{y} = \frac{\sum \tilde{y} W}{\sum W}, \quad \overline{z} = \frac{\sum \tilde{z} W}{\sum W}.$$