Fluid Mechanics

Noah Rahbek Bigum Hansen

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Contents

Lecture 1: Introduction to Fluid Mechanics 25. August 2025

1 Fundamental Concepts

1.1 Fluid as a Continuum

Fluids are normally experienced as being continuous or "smooth" when percieved in the macroscopic world. If one looks at a fluid microscopically one would start to see that the fluid is not continuous at all but instead composed of distinct particles. We wish to determine the minimum volume, $\partial V'$ that a point C must be such that we can talk about the fluid being continous at this point. In other words, under what circumstances can a fluid be treated as a continuum, for which, by definition, properties vary smoothly over all points.

We define the mass ∂m as the instantaneous number of molecules in ∂V and the mass of each of these molecules. The average density inside volume ∂V is hence given by $\rho = \frac{\partial m}{\partial V}$. It is important to note that this necessarily is an *average value* as the number of molecules in ∂V and hence the density fluctuates. Due to the law of large numbers one will experience that for very small volume the density will fluctuate greatly over time, however at a certain volume $\partial V'$, the density becomes stable and will not fluctuate greatly over time. For $\partial V = 0{,}001\,\mathrm{mm}^3$ (about the size of a grain of sand) there will already be an average of $2{,}5 \cdot 10^{13}$ molecules present. Hence a liquid can be treated as a continuous medium as long as we consider a "point" to be no smaller than about this size – at least this is sufficiently precise for most engineering appplications.

This continuum hypothesis is an integral part of fluid mechanics. It is valid when treating the behaviour of fluids under normal conditions and only breaks down when the mean free path of the molecules becomes the same order of magnitude as the smallest characteristic dimension of the problem.

As a consequence of the continuum assumption, each property of the fluid is assumed to have a definite value at each point in space. I.e. density, temperature, velocity, and so on are each continuous functions of both position and time. We now have a definition of the density at a point:

$$\rho \equiv \lim_{\partial V \to \partial V'} \frac{\partial m}{\partial V}.$$

As point C was chosen arbitrarily the density at any other point in the liquid can be determined in the same manner. If one measures this simultaneously for all points in the fluid, an expression for the density distribution as a function of the space coordinates $\rho = \rho(x, y, z)$ can be found.

The density at a specific point may also vary with time. Therefore the complete representation of density, the so-called *field representation*, can be written as:

$$\rho = \rho(x, y, z, t) \tag{1}$$

As density is a scalar quantity the field given by Equation 1 is a scalar field.

The density can also be expressed as the specific gravity, i.e. the weith compared to the maximum density of water, $\rho_{\mathrm{H_2O}} = 1000\,\frac{\mathrm{kg}}{\mathrm{m}^3}$ at $4\,^\circ\mathrm{C}$. Thus the specific gravity of a substance can be found as:

$$\mathrm{SG} = \frac{\rho}{\rho_{\mathrm{H_2O}}}.$$

Another useful property is the *specific weight*, γ, of a substance. It is defined as the weight of a substance per unit volume, i.e.:

$$\gamma = \frac{mg}{V} \implies \gamma = \rho g.$$

1.2 Velocity field

A very important property defined by a field is the velocity field, given by:

$$\mathbf{V} = \mathbf{V}(x, y, z, t) \tag{2}$$

As velocity is a vector quantity, the field given in Equation 2 is a vector field.

The velocity vector, $\mathbf{V}$, can also be written in terms of its scalar components. Denoting the components in the x-, y-, and z-directions by u, v, and w, respectively we get:

$$\mathbf{V} = u\hat{i} + v\hat{j} + w\hat{k}.$$

Here, each component, u, v, and w, will generally be functions of x, y, z and t.

We also need to be make sure to remember that $\mathbf{V}(x, y, z, t)$ represents the velocity of a fluid particle passing through the point (x, y, z) at time t. Therefore $\mathbf{V}(x, y, z, t)$ should be thought of as the velocity field of the entire fluid and not of an individual particle.

If properties at every point in a flow field are constant with respect to time, the flow is termed *steady*. This is defined mathematically as:

$$\frac{\partial \eta}{\partial t} = 0.$$

Where η is any fluid property. Hence, for steady flow:

$$\frac{\partial \rho}{\partial t} = 0 \text{ or } \rho = \rho(x, y, z)$$

and

$$\frac{\partial \mathbf{V}}{\partial t} = 0 \text{ or } \mathbf{V} = \mathbf{V}(x, y, z).$$

As such any property may vary from point to point in the field but remain constant with time at every point for steady flow.

1.3 One-, Two-, and Three-Dimensional flows

A flow is either one-, two-, or three-dimensional depending on the amount of spatial coordinates required to specify the velocity field. Equation 2 shows that the velocity field in some cases is a function of three spatial coordinates and time – in this case it is a three-dimensional flow.

Almost all flows are three-dimensional in nature – however, analysis based on fewer dimensions is often sufficient. The complexity of analysis increases sharply as more dimensions are added, and oftentimes in engineering a one-dimensional analysis is sufficient.

To not break the continuum assumption all fluids must have zero velocity at any solid surface, e.g. the inner side of a pipe. Due to this all flow in pipes is inherently three-dimensional. To simplify the analysis one often uses the notion of *uniform flow* at a given cross-section. In a flow that is uniform at a given cross-section, the velocity is constant across any section normal to the flow as shown in **Figure 1.1**. Under this assumption, the flow simplifies to be simply a function of x alone. The term *uniform flow field* is used to describe a flow in which the velocity is constant throughout the entire field.

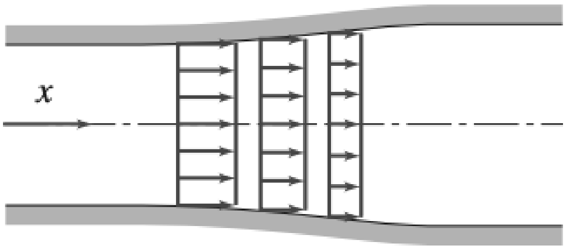

Figure 1.1: *Uniform flow* at a given cross-section.

Definition 1: Uniform flow at a cross section

An assumption that states that a fluid has the same velocity everywhere in a given cross section as shown on **Figure 1.1**. This actually breaks the continuum hypothesis but it makes a lot of calculations easier.

Definition 2: Uniform flow field

A flow field where the velocity is constant everywhere throughout the flow field.

1.4 Timelines, Pathlines, Streaklines, and Streamlines

Wind tunnels have traditionally often utilized to visualize flow fields. In modern times the advent of computer simulations has meant that these have become much more prevalent recently. A few different terms must be defined for proper understanding of these.

Definition 3: Timeline

A *timeline* is produced by marking adjacent fluid particles in a flow field at a given instant. Subsequent observation of the timeline can give insights into the flow field.

Definition 4: Pathline

A *pathline* is the trajectory traced out by a moving fluid particle. These can be visualized by marking a fluid particle at a given instant, e.g. with dye or smoke, and then tracing the path of this particle as it moves through the field.

Definition 5: Streakline

A *streakline* is the line traced out by marking the fluid particles at a fixed point in space, e.g. with smoke or dye. This is therefore a way to see how fluid particles that passed through a specific point behave afterwards.

Definition 6: Streamlines

A *streamline* are lines drawn in a flow field such that at any given instant they are tangent to the velocity vector at every point in the flow. This means there can be no flow across a streamline. These

> are the most commonly used visualization technique

We can use the velocity field to derive the shapes of streaklines, pathlines and streamlines. As the streamlines are parallel to the velocity vector, for a two dimensional flow field, we can write:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\text{streamline}} = \frac{v(x,y)}{u(x,y)} \tag{3}$$

Note that these are obtained at a given instant in time. If the flow is unsteady, time t is held constant in **Equation 3**. Solution of the equation gives $y = y(x)$, with an undetermined integration constant, the value of which depends on the particular streamline.

For pathlines, we let $x = x_p(t)$ and $y = y_p(t)$ where $x_p(t)$ and $y_p(t)$ are the instantaneous coordinates of a specific particle. In this case we get

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\text{particle}} = u(x,y,t) \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{\text{particle}} = v(x,y,t) \tag{4}$$

The simultaneous solution of these equations gives the path of a particle in parametric form, $x_p(t), y_p(t)$.

For streaklines, the first step is to compute the pathline of a particle with Equation 4 that was released from the streak source at x_0, y_0 at time t_0, in the form

$$x_{\text{particle}}(t) = x(t, x_0, y_0, t_0) \qquad y_{\text{particle}}(t) = y(t, x_0, y_0, t_0).$$

Then now, instead of interpreting this as the position of a particle over time, we instead write the equations as:

$$x_{\text{streakline}}\left(t_0\right) = x\left(t, x_0, y_0, t_0\right) \qquad y_{\text{streakline}}\left(t_0\right) = y\left(t, x_0, y_0, t_0\right) \tag{5}$$

Equation 5 gives the line generated (by time t) from a streak source placed at (x_0, y_0). In these equations t_0 is varied from 0 to t to show the *instantaneous* positions of all particles released up to time t.

1.5 Stress Field

To understand the behaviour of fluids one must first understand the nature of the forces that act upon fluid particles. A fluid particle can experience either:

- *Surface forces*, e.g. pressure or friction, that are generated due to contact with other particles or surfaces
- *Body forces*, e.g. gravity and electromagnetic, that are experienced throughout the particle.

Surface forces on a fluid particle leads to *stresses*. Stress is an important concept when describing how forces acting on the boundaries of a medium are transmitted throughout the medium.

We consider the surface of a particle in contact with other fluid particles and the contact force being generated between these. Let $\delta\mathbf{A}$ be a portion of the surface at some point C. The orientation of $\delta\mathbf{A}$ is given by the unit vector $\hat{n}$, which is perpendicular to the surface as seen in **Figure 1.2**.

The force, $\delta\mathbf{F}$, acting on the surface portion $\delta\mathbf{A}$ may be split into two components – a *normal stress* σ_n normal to the surface and a *shear stress* τ_n tangential to the surface, defined as:

$$\sigma_n = \lim_{\sigma A_n \to 0} \frac{\delta F_n}{\delta A_n}$$
$$\tau_n = \lim_{\delta A_n \to 0} \frac{\delta F_t}{\delta A_n}.$$

The subscript n on the stress is a reminder that the stresses are associated with the surface portion $\delta\mathbf{A}$ through C, which has an outward normal in the $\hat{n}$ direction.

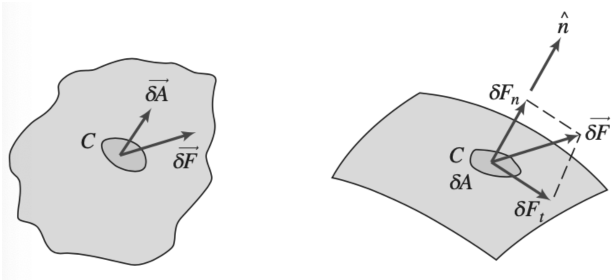

Figure 1.2: Stress in a continuum.

We consider the stress on the element δA_x whose normal is in the $+x$-direction. We can then split the force acting upon this point $\delta\mathbf{F}$ into components along each coordinate direction. By dividing the magnitude of each force component by the area δA_x and taking the limit as δA_x approaches zero we define three stress components as:

$$\sigma_{xx} = \lim_{\delta A_x \to 0} \frac{\delta F_x}{\delta A_x}$$
$$\tau_{xy} = \lim_{\delta A_x \to 0} \frac{\delta F_y}{\delta A_x}$$
$$\tau_{xz} = \lim_{\delta A_x \to 0} \frac{\delta F_z}{\delta A_x}.$$

This is also shown graphically in **Figure 1.3**.

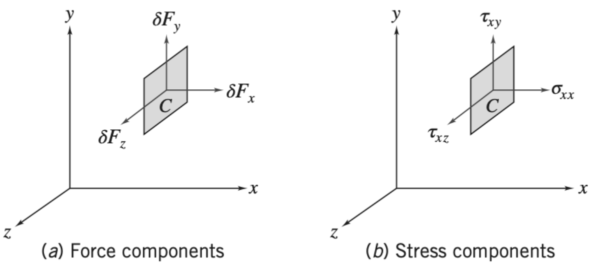

Figure 1.3: Force and stresses at surface element δA_x

Here the first subscript (x) indicates the *plane* on which the stress acts, in this case a surface perpendicular to the x-axis. The second direction indicates the *direction* in which the stress acts. I.e. consideration of the element δA_y would lead to the stresses σ_{yy}, τ_{yx} and τ_{yz} and similarly for δA_z.

As the coordinate system was chosen arbitrarily it is easily realized that one can define an infinite amount of stresses through a point C depending on how the axes are placed. Luckily, the state of stress at any point is completely described by the stresses acting in any three mutually perpendicular planes through the point. Therefore the stress at a point is specified by the nine components:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}.$$

Planes are normally named for the direction in which their normal vector is pointing. Also we normally define a stress component to be positive when the stress component and the plane on which it acts are either both positive or negative.

1.6 Viscosity

For a solid stresses develop when the material is elastically deformed or strained; for a fluid, shear stresses instead arise due to viscosity. For a fluid at rest there will be no shear stresses.

We consider the behaviour of a fluid element between two infinite planes sketched on **Figure 1.4**. The rectangular fluid element is initially at rest at time t. We now suppose a constant rightward force δF_x is applied to the upper plate such that it is dragged across the fluid at constant velocity δu. The shearing action of the plates produces a shear stress τ_{yx}, which acts on the fluid element, and is given by:

$$\tau_{yx} = \lim_{\delta A_y \to 0} \frac{\delta F_x}{\delta A_y} = \frac{\mathrm{d}F_x}{\mathrm{d}A_y}$$

where δA_y is the contact area between the fluid element between the plade and δF_x is the force exerted by the plate on the element.

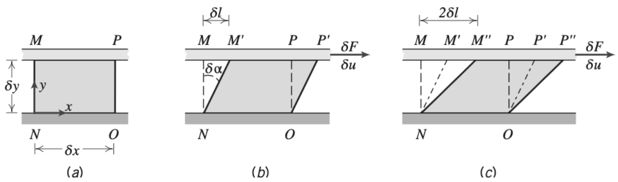

Figure 1.4: (a) Fluid element at time t, (b) deformation of fluid element at time $t + \delta t$, (c) deformation of fluid element at time $t + 2\delta t$

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Focusing on the time interval δt (**Figure 1.4b**) the deformation of the fluid is given by:

$$\text{deformation rate} = \lim_{\delta t \to 0} \frac{\delta \alpha}{\delta t} = \frac{\mathrm{d}\alpha}{\mathrm{d}t}.$$

We will now seek to express $\frac{\mathrm{d}\alpha}{\mathrm{d}t}$ in terms of measurable quantities. The distance, δl, between the points M and M' is given by:

$$\delta l = \delta u \delta t.$$

For small angles this simplifies to:

$$\delta l = \delta y \delta \alpha.$$

Equating these two expressions gives:

$$\frac{\delta\alpha}{\delta t} = \frac{\delta u}{\delta y}.$$

By taking the limits of both sides of this, we get:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}y}.$$

Thus, the fluid element will, when subjected to a shear stress τ_{yx}, experience a rate of deformation given by $\frac{\mathrm{d}u}{\mathrm{d}y}$. We have thus established that any fluid will flow and have a shear rate when subjected to a shear stress. Fluids in which the shear stress is proportional to the deformation rate are termed *Newtonian* and fluids for which this is not the case are termed *non-Newtonian*.

1.7 Newtonian Fluid

Most common fluids are Newtonian under normal conditions. If the fluid on **Figure 1.4** is Newtonian, then:

$$\tau_{yx} \propto \frac{\mathrm{d}u}{\mathrm{d}y}.$$

The constant of proportionality between these two quantities is the *absolute viscosity* also called the dynamic viscosity, μ. Thus in terms of the coordinates on **Figure 1.4** Newton's law of viscosity for one-dimensional flow is:

$$\tau_{yx} = \mu \frac{\mathrm{d}u}{\mathrm{d}y} \tag{6}$$

By dimensional analysis on **Equation 6** it can be seen that the units of viscosity are $\mathrm{kg}\,/\,\mathrm{m\,s}$ or $\mathrm{Pa\,s}$.

In fluid mechanics the ratio of absolute viscosity μ to density ρ often arises. This ratio is called the *kinematic viscosity* and is represented by ν. The units for this is $1\,\mathrm{stoke} \equiv 1\,\mathrm{cm}^2\,/\,\mathrm{s}$.

1.8 Non-Newtonian Fluids

Many empirical equations have been proposed to model the observed relations between τ_{yx} and $\frac{\mathrm{d}u}{\mathrm{d}y}$ for time-independent non-Newtonian fluids. For many engineering applications the power law model is sufficient, for which one-dimensional flow becomes:

$$\tau_{yx} = k\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^n.$$

Here n is called the *flow behaviour index* and k is called the *consistency index*. Newton's law of viscosity is a special case of this with $n = 1$ and $k = \mu$. To ensure that τ_{yx} has the same sign as $\frac{\mathrm{d}u}{\mathrm{d}y}$ the equation is rewritten as:

$$\tau_{yx} = k\left|\frac{\mathrm{d}u}{\mathrm{d}y}\right|^{n-1}\frac{\mathrm{d}u}{\mathrm{d}y} = \eta\frac{\mathrm{d}u}{\mathrm{d}y} \tag{7}$$

Here the term $\eta = k\left|\frac{\mathrm{d}u}{\mathrm{d}y}\right|^{n-1}$ is referred to as the *apparent viscosity*. When using **Equation 7** we end up with a viscosity η that is used in a formula in the same form as **Equation 6**. The big difference between these two is that while μ is constant (at a constant temperature), η depends on the shear rate.

Fluids for which the apparent viscosity decreases with increasing deformation rate ($n < 1$) are called *pseudoplastic* or shear thinning fluids. Most non-Newtonian fluids are of this type. If the apparent viscosity increases with deformation rate ($n > 1$) the fluid is termed *dilatant* or shear thickening.

A "fluid" that behaves as a solid until a minimum yield stress τ_y is exceeded and subsequently exhibits a linear relation between stress rate and deformation is termed an ideal or *Bingham plastic*. The shear stress model for a Bingham plastic is:

$$\tau_{yx} = \tau_y + \mu_p \frac{\mathrm{d}u}{\mathrm{d}y}.$$

The apparent viscosity may also be time-dependent. *Thixotropic* fluids show a decrease in η with time at a constant stress and *Rheopectic* fluids show an increase in η with time. Some fluids partially return to their original shape when the stress is removed – these are called *viscoelastic*.

1.9 Surface Tension

Droplets can either "flatten out" or remain as little drops when dropped on a surface. We define a liquid as *wetting* a surface when the contact angle is $< 90°$. This is shown on **Figure 1.5**.

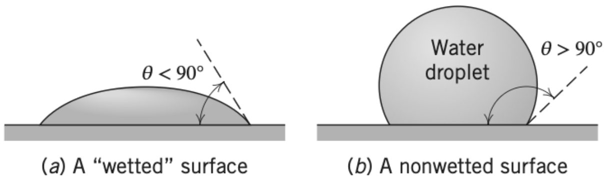

Figure 1.5: Surface tension effects on water droplets.

Surface tension arises at the interface between the liquid and a solid – this interface acts like a stretched elastic membrane in turn creating surface tension. This membrane is completely described by two features: the contact angle θ and the magnitude of the surface tension σ [N/m]. Both of these depend on both the type of liquid and the characteristics of the surface with which it shares an interface.

1.10 Description and Classification of Fluid Motions

The two most difficult aspects of a fluid mechanics analysis to deal with are: (1) the fluid s viscous nature and (2) its compressibility. When fluid mechanics first was developed it was occupied with a frictionless and incompressible fluid. Whilst extremely elegant this leads to the infamous result called d'Alembert's paradox, which states that all bodies experience zero drag as they move through a liquid – which of course is not consistent with observations.

1.10.1 Viscous and Inviscid Flows

Any object moving through a fluid will experience gravity and an aerodynamic drag force. This drag force is in part due to viscous friction and in part due to pressure differences being produced as the liquid moves out of the way of the object. We can estimate whether or not viscous forces are negligible compared to pressure forces by computing the Reynolds number:

$$\mathrm{Re} = \rho \frac{VL}{\mu}.$$

where ρ and μ are the density and viscosity of the fluid, respectively, and V and L are the "characteristic" velocity and size scale of the flow, respectively. If the Reynolds number is "large" viscous effects will be small; however, if the Reynolds number is neither large nor small, both are important.

The idealized notion of frictionless flow is called *inviscid flow*. It predicts streamlines as shown in **Figure 1.6a**. These streamlines are symmetric front to back. As flow between two streamlines is constant the velocity in the vicinity of points A and C must be relatively low compared to the velocity at point B. Hence, points A and C have large pressures whereas B will be a point of low pressure. In fact, the pressure distribution around the sphere is symmetric and there is therefore no net drag force due to pressure. As we are assuming inviscid flow there will be no drag force due to friction either – this is the d'Alembert paradox of 1752.

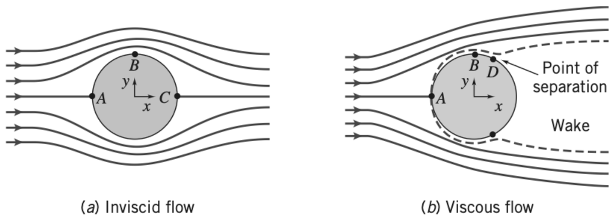

Figure 1.6: Incompressible flow over a sphere

The answer to this was obtained by Prandtl in 1904. The no-slip condition requires that the velocity everywhere at the surface of the ball must be zero, but inviscid theory states that it is high at point B. Prandtl suggested that even though friction is negligible for high-Heynolds number flows, there will always be a thin boundary layer, in which friction is significant and over which the velocity will increase rapidly from zero at the surface to the value predicted by inviscid theory at the edge of the boundary layer.

This thin boundary layer explains why drag arises. The boundary layer however also has another important consequence. It often leads to bodies moving through a fluid having a *wake* as shown on **Figure 1.6b**. Here point D is termed a *separation point*, where fluid particles are pushed of the object thus creating the wake. It turns out that this wake always will have low pressure compared to the front of the sphere, hence the sphere now experiences quite a large *pressure* or *form drag*.

We can now also begin to see how *streamlining* works. The main drag force in most aerodynamics is due to the low pressure wake. If we can reduce or even eliminate this wake the drag will be greatly reduced. Imagine that the sphere was instead teardrop shaped – the pressure gradient will not be changing as quickly along the back half of the object and thus the wake will be smaller leading to less drag. This illustrates the *very* significant different between inviscid flow ($\mu = 0$) and flows where the viscosity can be assumed negligible but not zero ($\mu \to 0$).

1.10.2 Laminar and Turbulent flows

A faucet turned on at a very low flow rate will lead to water running out smoothly – if you increase the flow rate the water will exit in a much more chaotic manner. In fluid dynamics these are termed either *laminar flow* or *turbulent flow*. Oftentimes turbulence is unwanted but unavoidable.

The velocity of laminar flow is simply u. The velocity of turbulent flow is given by the mean velocity $\overline{u}$ plus the three components of randomly fluctuating velocity u', v', and w'. Many turbulent flows may be steady in the mean, but the presence of these random velocity fluctuations makes analysis of turbulent flows extremely difficult. In one-dimensional laminar flow, the shear stress is related to the velocity gradient by the simple relation:

$$\tau_{yx} = \mu \frac{\mathrm{d}u}{\mathrm{d}y}.$$

For a turbulent flow, no such simple relation is valid. In turbulent flow momentum can be transported across streamlines and therefore there us no universal relationship between the stress field and the mean-velocity field. This means we often have to rely on semi-empirical theories for analyzing turbulent flow.

1.10.3 Compressible and Incompressible Flows

A flow in which the variation in density is negligible is termed *incompressible* and those in which density variations are not negligible are called *compressible*. Gasses are normally treated as compressible, whereas liquids are normally treated as incompressible. At high temperatures, however, compressibility effects can start to become important. Pressure and density changes in liquids are related by the *bulk compressibility modulus* or the modulus of elasticity,

$$E_v \equiv \frac{\mathrm{d}p}{\frac{\mathrm{d}\rho}{\rho}}.$$

If the bulk modulus is independent of temperature, then density is only a function of pressure (the fluid is said to be *barotropic*).

Gas flows with negligible heat transfer can be considered incompressible so long as the flow speeds are small relative to the speed of sound. The ratio of flow speed, V, to the local speed of sound, c, in the gas is defined as the Mach number:

$$M \equiv \frac{V}{c}.$$

For $M < 0{,}3$ the maximum variation in density is less than 5%. Thus gas flows with $M < 0{,}3$ can be treated as being incompressible.

1.10.4 Internal and External Flows

Flows completely bounded by solid surfaces are called either *internal*, *pipe*, or *duct flows*. Flows over bodies immersed in an unbounded fluid are termed *external flows*.

An example of an internal flow is that of water in a pipe. The Reynolds number for pipe flows is defined as $\mathrm{Re} = \rho \overline{V} D/\mu$, where $\overline{V}$ is the average flow velocity and D is the pipe diameter. Based on this one can predict if the flow will be laminar or turbulent. For $\mathrm{Re} < 2300$ the flow will generally by laminar and turbulent for larger values. Flow in a pipe of constant diameter will always be entirely laminar or entirely turbulent, depending on the value of the velocity $\overline{V}$.

Lecture 2: Basic equations of fluid statics and buoyancy, surface tension 27. August 2025

2 Fluid Statics

2.1 The basic equations of fluid statics

Consider a differential fluid element of mass $\mathrm{d}m = \rho\,\mathrm{d}V$ with sides $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ as shown on **Figure 2.1**. The fluid element is stationary relative to the coordinate system shown.

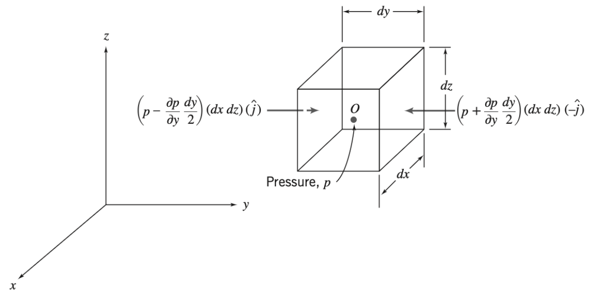

Figure 2.1: Differential fluid element with pressure forces shown in the y-direction

It has previously been mentioned that only *body forces* and *surface forces* may be applied to a fluid. In this course the body forces caused by electric or magnetic fields is assumed negligible and only the gravitational force must therefore be accounted for.

For the differential fluid element the (gravitational) body force is:

$$\mathrm{d}\mathbf{F}_B = \mathbf{g}\,\mathrm{d}m = \mathbf{g}\rho\,\mathrm{d}V.$$

where $\mathbf{g}$ denotes the local gravity vector, ρ is the density and $\mathrm{d}V$ is the volume of the fluid element. This can also be expressed as:

$$\mathrm{d}\mathbf{F}_B = \rho\mathbf{g}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Per definition there can be no shear stresses in a static fluid, so the only surface force is that due to pressure, which is a scalar field, $p = p(x, y, z)$. The net pressure force can be found by summing the forces on each of the six faces of the fluid element. The pressure at the center O is denoted p. The pressure at each face of the element will be found using a Taylor series expansion of the pressure about point O. The pressure on the left side of the fluid element is:

$$p_L = p + \frac{\partial p}{\partial y}(y_L - y) = p + \frac{\partial p}{\partial y}\left(-\frac{\mathrm{d}y}{2}\right) = p - \frac{\partial p}{\partial y}\frac{\mathrm{d}y}{2}.$$

Terms of higher order are omitted as they will vanish at a later step anyways. Similarly, the pressure on the right face of the differential element is:

$$p_R = p\frac{\partial p}{\partial y}(y_R - y) = p + \frac{\partial p}{\partial y}\frac{\mathrm{d}y}{2}.$$

The pressure *forces* produced by this pressure is shown on **Figure 2.1**. These consist of three factors. Namely the magnitude of the pressure, the area of the face and a unit vector to indicate direction. A positive pressure here is defined as a compressive pressure on the differential element. The pressure forces in the other directions can be obtained in the same way giving:

$$\begin{aligned}\mathrm{d}\mathbf{F}_S = &\left(p - \frac{\partial p}{\partial x}\frac{\mathrm{d}x}{2}\right)(\mathrm{d}y\,\mathrm{d}z)\left(\hat{\mathbf{i}}\right) + \left(p + \frac{\partial p}{\partial x}\frac{\mathrm{d}x}{2}\right)(\mathrm{d}y\,\mathrm{d}z)\left(-\hat{\mathbf{i}}\right)\\ &+ \left(p - \frac{\partial p}{\partial y}\frac{\mathrm{d}y}{2}\right)(\mathrm{d}x\,\mathrm{d}z)\left(\hat{\mathbf{j}}\right) + \left(p + \frac{\partial p}{\partial y}\frac{\mathrm{d}y}{2}\right)(\mathrm{d}x\,\mathrm{d}z)\left(-\hat{\mathbf{j}}\right)\\ &+ \left(p - \frac{\partial p}{\partial z}\frac{\mathrm{d}z}{2}\right)(\mathrm{d}x\,\mathrm{d}y)\left(\hat{\mathbf{k}}\right) + \left(p + \frac{\partial p}{\partial z}\frac{\mathrm{d}z}{2}\right)(\mathrm{d}x\,\mathrm{d}y)\left(-\hat{\mathbf{k}}\right).\end{aligned}$$

Simplifying this, we obtain

$$\mathrm{d}\mathbf{F}_S = -\left(\frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

We can now recognize the term in the parentheses as the gradient of the pressure which may be written as either grad$p$ or ∇p. This is:

$$\text{grad}p \equiv \nabla p \equiv \left(\hat{\mathbf{i}}\frac{\partial p}{\partial x} + \hat{\mathbf{j}}\frac{\partial p}{\partial y} + \hat{\mathbf{k}}\frac{\partial p}{\partial z}\right) \equiv \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) p.$$

Using this notation we can rewrite the expression as:

$$\mathrm{d}\mathbf{F}_S = -\text{grad}p(\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z) = -\nabla p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Note that the pressure magnitude is not relevant when computing the net pressure force – instead only the rate of change of pressure with distance, the pressure gradient, is needed.

We can now combine the expressions for the surface and body forces on the fluid element.

$$\mathrm{d}\mathbf{F} = \mathrm{d}\mathbf{F}_S + \mathrm{d}\mathbf{F}_B = (-\nabla p + \rho\mathbf{g})\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = (-\nabla p + \rho\mathbf{g})\,\mathrm{d}V$$

which can also be stated as the force acting per unit volume as

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}V} = -\nabla p + \rho\mathbf{g}.$$

Applying Newton's second law to a static fluid particle gives

$$\mathbf{F} = \mathbf{a}\,\mathrm{d}m = \mathbf{0}\,\mathrm{d}m = \mathbf{0}.$$

Thus

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}V} = \rho\mathbf{a} = \mathbf{0}.$$

Combining these two gives:

$$-\nabla p + \rho\mathbf{g} = \mathbf{0}.$$

This can be stated in words as the sum of the net pressure force per unit volume at a point and the body force per unit volume at a point is zero. As this is a three-dimensional vector equation it is comprised of three individual equations that must each be satisfied individually.

$$-\frac{\partial p}{\partial x} + \rho g_x = 0$$
$$-\frac{\partial p}{\partial y} + \rho g_y = 0$$
$$-\frac{\partial p}{\partial z} + \rho g_z = 0.$$

It is now convenient to choose a coordinate system such that the gravity vector is aligned with one of the axes. If we place the coordinate system "normally" we would get $g_x = 0$, $g_y = 0$, and $g_z = -g$. The component equations then become

$$\frac{\partial p}{\partial x} = 0$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial p}{\partial z} = -\rho g.$$

This means that the pressure depends on one variable only and the total derivative may therefore be employed instead of the partial as

$$\frac{\partial p}{\partial z} = -\rho g \equiv -\gamma.$$

This is the basic pressure-height relation of fluid statics.

2.2 Pressure variation in a static fluid

2.2.1 Incompressible fluids

For an incompressible fluid $\rho =$ constant. If we assume the gravity to be constant with elevation then we get

$$\frac{\partial p}{\partial z} = -\rho g = \text{constant}.$$

Now we can easily integrate this as

$$\int_{p_0}^{p} \mathrm{d}p = -\int_{z_0}^{z} \rho g \,\mathrm{d}z.$$

The height difference is defined as $h = z_0 - z$ and thus we upon integration obtain

$$p - p_0 = \Delta p = \rho g h.$$

To find the pressure difference Δp between two points separated by a series of fluid, we can compute the change in pressure as

$$\Delta p = g \sum_i \rho_i h_i.$$

2.2.2 Gases

Pressure variation in a compressible fluid must be found by integration of the previous result. First, however, an expression for the density as a function of either p or z must be found. To do this we employ the ideal gas law:

$$p = \rho R T.$$

Here the temperature T is introduced as an additional variable. In the U.S. Standard Atmosphere, the temperature decreases linearly with altitude up to an elevation of 11,0 km. Therefore the temperature can be expressed as:

$$T = T_0 - mz.$$

We therefore get

$$\mathrm{d}p = -\rho g \,\mathrm{d}z = -\frac{pg}{RT}\,\mathrm{d}z = -\frac{pg}{R\,(T_0 - mz)}\,\mathrm{d}z.$$

By separation of variables we obtain:

$$\int_{p_0}^{p} \frac{\mathrm{d}p}{p} = -\int_{0}^{z} \frac{g\,\mathrm{d}z}{R\,(T_0 - mz)}.$$

Which gives

$$\ln\frac{p}{p_0} = \frac{g}{mR}\ln\left(\frac{T_0 - mz}{T_0}\right) = \frac{g}{mR}\ln\left(1 - \frac{mz}{T_0}\right).$$

Which can be solved for p as

$$p = p_0\left(1 - \frac{mz}{T_0}\right)^{\frac{g}{mR}} = p_0\left(\frac{T}{T_0}\right)^{\frac{g}{mR}}.$$

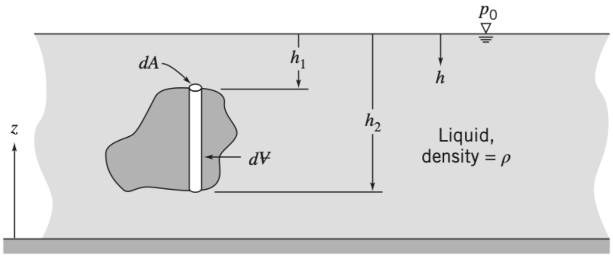

Figure 2.2: Immersed body in a static liquid.

2.3 Buoyancy

Any object submerged in a liquid will experience a vertical force acting on it due to liquid pressure – this is termed *buoyancy*. Consider the object on **Figure 2.2** which is split into cylindrical volume elements.

We have previously derived an equation for the pressure p at depth h in a liquid:

$$p = p_0 + \rho g h.$$

The net vertical force on the element is thus

$$\mathrm{d}F_z = (p_0 + \rho g h_2)\ \mathrm{d}A - (p_0 + \rho g h_1)\ \mathrm{d}A = \rho g\,(h_2 - h_1)\ \mathrm{d}A.$$

As $(h_2 - h_1)\ \mathrm{d}A = \mathrm{d}V$ we get

$$F_z = \int \mathrm{d}F_z = \int_V \rho g\,\mathrm{d}V = \rho g V.$$

I.e. for a submerged body, the buoyancy force is equal to the weight of the displaced fluid.

Lecture 3: Control volume analysis I: Basic laws and mass conservation 1. September 2025

3 Basic Equations in Integral Form for a Control Volume

In general, there are two approaches to study flowing fluids. Either one can study how an individual particle or group of particles move through space, this is often called the *system approach.* This often leds to one needing to solve a set of partial differential equations.

One can also choose to study a region of space as fluid flows through it; this is the *control volume* approach. This has widespread applications, e.g. in aerodynamics where the focus is often on the lift and drag on a wing rather than the individual fluid particles.

3.1 Basic Laws for a System

A few basic laws will be applied; these are conservation of mass, Newton's second law, the angular-momentum principle, and the first and second laws of thermodynamics. It turns out that to convert these system equations to equivalent control volume formulas we will express the properties of the system in terms of the rates of flow in and out, hence these equations are termed *rate equations.*

3.1.1 Conservation of Mass

For a system we have the simple result that $M =$ constant. To express this law as a rate equation we write:

$$\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right)_{\text{system}} = 0$$

where

$$M_{\text{system}} = \int_{M(\text{system})} \mathrm{d}m = \int_{V(\text{system})} \rho \, \mathrm{d}V.$$

3.1.2 Newton's Second Law

For a system (the fluid) moving relative to an inertial reference frame (the control volume), Newton's second law states that the sum of all external forces acting on the system is equal to the time rate of change of the linear momentum of the system,

$$\mathbf{F} = \left.\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}\right)_{\text{system}}$$

where the linear momentum of the system if given by

$$\mathbf{P}_{\text{system}} = \int_{M(\text{system})} \mathbf{V} \, \mathrm{d}m = \int_{V(\text{system})} \mathbf{V}\rho \, \mathrm{d}V.$$

3.1.3 The Angular-Momentum Principle

The angular-momentum principle for a system states that the rate change of angular momentum is equal to the sum of all torques acting on the system:

$$\mathbf{T} = \left.\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}\right)_{\text{system}}$$

where the angular momentum of the system is given by

$$\mathbf{H}_{M(\text{system})} \mathbf{r} \times \mathbf{V} \, \mathrm{d}m = \int_{V(\text{system})} \mathbf{r} \times \mathbf{V}\rho \, \mathrm{d}V.$$

Torque can be produced both by surface and body forces and also by shafts that cross the system boundary,

$$\mathbf{T} = \mathbf{r} \times \mathbf{F}_s + \int_{M(\text{system})} \mathbf{r} \times \mathbf{g} \, \mathrm{d}m + \mathbf{T}_{\text{shaft}}.$$

3.1.4 The First Law of Thermodynamics

The first law of thermodynamics is a statement of conservation of energy for a system,

$$\delta Q - \delta W = \mathrm{d}E.$$

Which in rate form is

$$\dot{Q} - \dot{W} = \left.\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{system}}$$

where the total energy of the system is

$$E_{\text{system}} = \int_{M(\text{system})} e\,\mathrm{d}m = \int_{V(\text{system})} e\rho\,\mathrm{d}V$$

and

$$e = u + \frac{V^2}{2} + gz.$$

Here $\dot{Q}$ is positive when heat is added to the system; $\dot{W}$ is positive when work is done by the system on its surroundings; u is the specific internal energy, V the speed, and z the height relative to a particle of substance with mass $\mathrm{d}m$.

3.1.5 The Second Law of Thermodynamics

When an amount of heat, δQ, is transferred to a system at temperature T, the second law of thermodynamics states that the change in entropy, $\mathrm{d}S$, of the system satisfies,

$$\mathrm{d}S \geq \frac{\delta Q}{T}.$$

On a rate basis this is

$$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right)_{\text{system}} \geq \frac{1}{T}\dot{Q}$$

where the total entropy of the system is given by

$$S_{\text{system}} = \int_{M(\text{system})} s\,\mathrm{d}m = \int_{V(\text{system})} s\rho\,\mathrm{d}V.$$

3.1.6 Relation of System derivatives to the control volume formulation

Let any of the parameters $M, \mathbf{P}, \mathbf{H}, E$, or S be represented by the symbol N. Corresponding to the extensive property (N) that we are trying to find we, will need the intensive (i.e., per unit mass) property η. Thus:

$$N_{\text{system}} = \int_{M(\text{system})} \eta\,\mathrm{d}m = \int_{V(\text{system})} \eta\rho\,\mathrm{d}V.$$

I.e. if:

$$\begin{aligned}
N &= M, && \text{then } \eta = 1 \\
N &= \mathbf{P}, && \text{then } \eta = \mathbf{V} \\
N &= \mathbf{H}, && \text{then } \eta = \mathbf{r} \times \mathbf{V} \\
N &= E, && \text{then } \eta = e \\
N &= S, && \text{then } \eta = s.
\end{aligned}$$

3.1.7 Derivation

Let t represent the initial time for the system and $t+\Delta t$ represent a small time increment later. Our objective here is to relate any arbitrary extensive property, N, of the system to quantities associated with the control

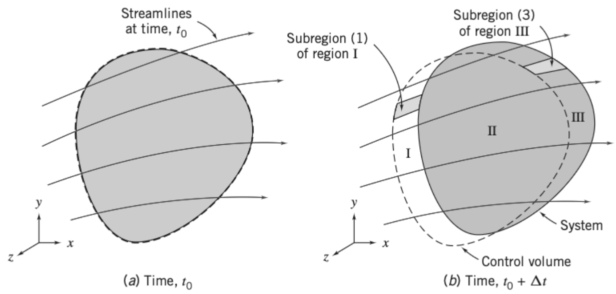

Figure 3.1: Configuration of the system and control volume.

volume. From the definition of the derivative, the rate of change of N_{system} is

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{system}} \equiv \lim_{\Delta t\to 0} \frac{N_s)_{t_0+\Delta t} - N_s)_{t_0}}{\Delta t} \tag{8}$$

Here subscript s is used to denote the system.

From the geometry of **Figure 3.1**,

$$N_s)_{t_0+\Delta t} = (N_{\text{II}} + N_{\text{III}})_{t_0+\Delta t} = (N_{\text{CV}} - N_{\text{I}} + N_{\text{III}})_{t_0+\Delta t}$$

and

$$N_s)_{t_0} = (N_{\text{CV}})_{t_0}\,.$$

Substituting these into the definition of the system derivative in **Equation 8** we get

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_s = \lim_{\Delta t\to 0} \frac{(N_{\text{CV}} - N_{\text{I}} + N_{\text{III}})_{t_0+\Delta t} - N_{\text{CV}})_{t_0}}{\Delta t}.$$

Which simplifies to

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_s = \lim_{\Delta t\to 0} \frac{N_{\text{CV}})_{t_0+\Delta t} - N_{\text{CV}})_{t_0}}{\Delta t} + \lim_{\Delta t\to 0} \frac{N_{\text{III}})_{t_0+\Delta t}}{\Delta t} - \lim_{\Delta t\to 0} \frac{N_{\text{I}})_{t_0+\Delta t}}{\Delta t}.$$

Each of the three terms can now be evaluated individually. We start with term 1, which simplifies to:

$$\lim_{\Delta t\to 0} \frac{N_{\text{CV}})_{t_0+\Delta t} - N_{\text{CV}})_{t_0}}{\Delta t} = \frac{\partial N_{\text{CV}}}{\partial t} = \frac{\partial}{\partial} \int_{\text{XC}} \eta\rho\,\mathrm{d}V.$$

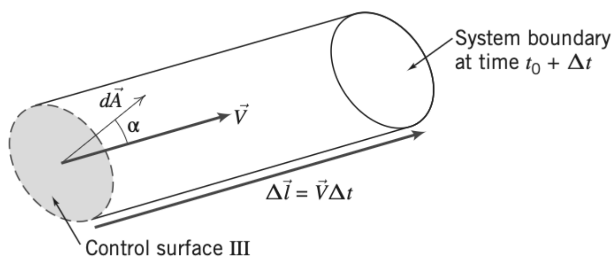

Figure 3.2: Enlarged view of subregion 3 from **Figure 3.1**

To evaluate term 2 we first need develop an expression for $N_{\mathrm{III}})_{t_0+\Delta t}$ by looking at the subregion 3 of region III shown on **Figure 3.2**. The vector area element $\mathrm{d}\mathbf{A}$ of the control surface has magnitude $\mathrm{d}A$ and its direction is normal outward of the area element. The velocity vector $\mathbf{V}$ will be at some angle α with respect to $\mathrm{d}\mathbf{A}$

For this subregion we have

$$\mathrm{d}N_{\mathrm{III}})_{t_0+\Delta t} = (\eta\rho\mathrm{d}V)_{t_0+\Delta t}\,.$$

To obtain an expression for the volume $\mathrm{d}V$ of this cylindrical element we first note that its vector length is given by $\Delta\mathbf{l} = \mathbf{V}\Delta t$. Furthermore, the volume of a prismatic cylinder, whose area $\mathrm{d}\mathbf{A}$ is at an angle α to its length $\Delta\mathbf{l}$, is given by $\mathrm{d}V = \Delta l\,\mathrm{d}A\cos\alpha = \Delta\mathbf{l}\cdot\mathrm{d}\mathbf{A}\Delta t$.

Then for the entire region III we can integrate and obtain the second term as:

$$\lim_{\Delta t\to 0}\frac{N_{\mathrm{III}})_{t_0+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\int_{\mathrm{CS}_{\mathrm{III}}}\mathrm{d}N_{\mathrm{III}})_{t_0+\Delta t}}{\Delta t} = \int_{\mathrm{CS}_{\mathrm{III}}}\eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

We can perform a similar analysis for subregion 1 of region I and obtain term 3 in the equation as:

$$\lim_{\Delta t\to 0}\frac{N_{\mathrm{I}})_{t_0+\Delta t}}{\Delta t} = -\int_{\mathrm{CS}_{\mathrm{I}}}\eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

For subregion 1 the velocity vector acts into the control volume hence producing a negative scalar product. Hence the minus sign is needed to cancel the negative result of the scalar product.

We can now substitute in the three terms we have found as:

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\mathrm{system}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}}\eta\rho\,\mathrm{d}V + \int_{\mathrm{CS}_{\mathrm{I}}}\eta\rho\mathrm{V}\cdot\mathrm{d}\mathbf{A} + \int_{\mathrm{CS}_{\mathrm{III}}}\eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

As CS_{I} and $\mathrm{CS}_{\mathrm{III}}$ constitute the entire control surface the last two integrals can be combined as:

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\mathrm{system}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}}\eta\rho\,\mathrm{d}V + \int_{\mathrm{CS}}\eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A} \tag{9}$$

Equation 9 is the fundamental relation between the rate of change of any extensive property N of a system and the variations of this property associated with a control volume. This is also called the *Reynolds Transport Theorem.*

Here it is important to note that the system is defined as the matter that happens to be passing through the chosen control volume at the instant we chose. I.e.

- $\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\mathrm{system}}$ is the rate of change of the system extensive property N, e.g. if $N = \mathbf{P}$ we obtain the rate of change of momentum
- $\frac{\partial}{\partial t}\int_{\mathrm{CV}}\eta\rho\,\mathrm{d}V$ is the rate of change of N in the control volume.
- $\int_{\mathrm{CS}}\eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}$ is the rate at which N is exiting the surface of the control volume.

3.2 Conservation of Mass

We remember from subsubsection 3.1.1 that the mass of the system remains constant,

$$\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right)_{\mathrm{system}} = 0.$$

If we set $N = M$ and $\eta = 1$ and plug it into Equation 9 we obtain

$$\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \rho \,\mathrm{d}V + \int_{\text{CS}} \rho \mathbf{V} \cdot \mathrm{d}\mathbf{A}.$$

And combining these two gives the control volume formulation of conservation of mass:

$$\frac{\partial}{\partial t}\int_{\text{CV}} \rho \,\mathrm{d}V + \int_{\text{CS}} \rho \mathbf{V} \cdot \mathrm{d}\mathbf{A} = 0 \tag{10}$$

3.2.1 Special cases

In some special cases Equation 10 can be simplified. Consider first the case of an incompressible fluid, i.e. ρ is constant in space and time. Consequently Equation 10 can be written as:

$$\rho\frac{\partial}{\partial t}\int_{\text{CV}} \mathrm{d}V + \rho\int_{\text{CS}} \mathbf{V} \cdot \mathrm{d}A = 0.$$

The integral of $\mathrm{d}V$ over the control volume is simply the volume, this

$$\frac{\partial V}{\partial t} + \int_{\text{CS}} \mathbf{V} \cdot \mathrm{d}\mathbf{A} = 0.$$

For a control volume of fixed size and shape, $V =$ constant. The conservation of mass for incompressible flow through a fixed volume becomes:

$$\int_{\text{CS}} \mathbf{V} \cdot \mathrm{d}\mathbf{A} = 0.$$

A useful special case is when we have uniform velocity at each inlet and exit. In this case it simplifies to

$$\sum_{\text{CS}} \mathbf{V} \cdot \mathbf{A} = 0.$$

We now consider the case of *steady*, *compressible flow* through a fixed control volume. Since the flow is steady no fluid property varies with time. Consequently the first term of Equation 10 must be zero and hence for steady flow the statement of conservation of mass becomes:

$$\int_{\text{CS}} \rho\mathbf{V} \cdot \mathrm{d}\mathbf{A} = 0.$$

When we have uniform velocity at each inlet and exit this simplifies to:

$$\sum_{\text{CS}} \rho\mathbf{V} \cdot \mathbf{A} = 0.$$

Lecture 4: Control volume analysis II: Momentum & energy equation 3. September 2025

3.3 Momentum Equation for Inertial Control Volume

We will now find a control volume form of Newton's second law. Note that in the following the coordinates (with respect to which velocities are meassured) are inertial, i.e. either at rest or moving at a constant speed with respect to an "absolute" set of coordinates.

We have previously defined Newton's second law for a system moving relative to an inertial coordinate system in subsubsection 3.1.2 as:

$$\mathbf{F} = \left.\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}\right)_{\text{system}}$$

where the linear momentum of the system is given by

$$\mathbf{P}_{\text{system}} = \int_{M(\text{system})} \mathbf{V}\,\mathrm{d}m = \int_{V(\text{system})} \mathbf{V}\rho\,\mathrm{d}V$$

and the resultant force $\mathbf{F}$ includes all surface and body forces acting on the system

$$\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B.$$

We have previously derived the relation between the system and control volume formulations in Equation 9 as:

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\rho\,\mathrm{d}V + \int_{\text{CS}} \eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

In this we now set $N = \mathbf{P}$ and $\eta = \mathbf{V}$. We thus obtain:

$$\left.\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{V}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

Since in deriving Equation 9 the system and control volume conincided at t_0, then

$$\mathbf{F})_{\text{on system}} = \mathbf{F})_{\text{on control volume}} = \mathbf{F}_S + \mathbf{F}_B.$$

Now these can be combined as:

$$\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{V}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A} \tag{11}$$

For cases with uniform flow at each inlet and exit this becomes:

$$\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}\rho\,\mathrm{d}V + \sum_{\text{CS}} \mathbf{V}\rho\mathbf{V}\cdot\mathbf{A}.$$

When applying Equation 11 we need to be a little careful. First we must choose a control volume and its control surface such that the volume integral and surface integral can be evaluated. In fluid mechanics the body force is usually gravity, so

$$\mathbf{F}_B = \int_{\text{CV}} \rho\mathbf{g}\,\mathrm{d}V = \mathbf{W}_{\text{CW}} = M\mathbf{g}.$$

In many applications the surface force is due to pressure,

$$\mathbf{F}_S = \int_A -p\,\mathrm{d}\mathbf{A}.$$

The momentum equation in Equation 11 is a vector equation therefore it can be written in three scalar components as:

$$\begin{aligned}
F_x = F_{S_x} + F_{B_x} &= \frac{\partial}{\partial t}\int_{\text{CV}} u\rho\,\mathrm{d}V + \int_{\text{CS}} u\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}\\
F_y = F_{S_y} + F_{B_y} &= \frac{\partial}{\partial t}\int_{\text{CV}} v\rho\,\mathrm{d}V + \int_{\text{CS}} v\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}\\
F_z = F_{S_z} + F_{B_z} &= \frac{\partial}{\partial t}\int_{\text{CV}} w\rho\,\mathrm{d}V + \int_{\text{CS}} w\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.
\end{aligned}$$

Or in the case of uniform flow at each inlet and exit as

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} u\rho \,\mathrm{d}V + \sum_{\mathrm{CS}} u\rho \mathbf{V}\cdot\mathbf{A}$$
$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} v\rho \,\mathrm{d}V + \sum_{\mathrm{CS}} v\rho \mathbf{V}\cdot\mathbf{A}$$
$$F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} w\rho \,\mathrm{d}V + \sum_{\mathrm{CS}} w\rho \mathbf{V}\cdot\mathbf{A}.$$

3.3.1 Differential Control Volume Analysis

The control volume approach that has been presented in the above is useful when applied to a finite region.

If we instead apply the approach to a differential control volume, we can obtain differential equations describing a flow field. Let us apply the continuity and momentum equations to a steady incompressible flow without friction as on Figure 3.3. The control volume is fixed in space and bounded by flow streamlines, and is thus an element of a stream tube. The length of the control volume is $\mathrm{d}s$.

As the control volume is bounded by streamlines, flow across the boundaries of the control volume only happens at the end sections located at coordinates s and $s + \mathrm{d}s$. Properties at the inlet sections are assigned arbitrary symbolic values and the properties at the outlet section are assumed to increase by differential amounts.

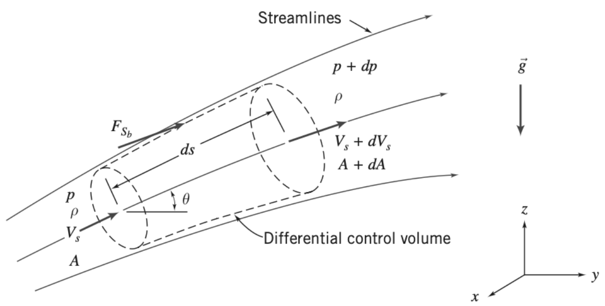

Figure 3.3: Differential control volume through a stream tube.

Now we can apply the continuity equation,

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \,\mathrm{d}V + \int_{\mathrm{CS}} \rho \mathbf{V}\cdot \mathrm{d}\mathbf{A} = 0.$$

Which under assumptions of steady flow, no flow across the bounding streamlines and that the flow is incompressible ($\rho =$ constant), reduces to:

$$(-\rho V_s A) + (\rho\,(V_s + \mathrm{d}V_s)\,(A + \mathrm{d}A)) = 0 \implies \rho\,(V_s + \mathrm{d}V_s)\,(A + \mathrm{d}A) = \rho V_s A.$$

Simplifying we obtain

$$V_s\,\mathrm{d}A + A\,\mathrm{d}V_s + \mathrm{d}A\,\mathrm{d}V_s = 0.$$

The product of the differentials $\mathrm{d}A\,\mathrm{d}V_s \approx 0$ compared to the other terms so this can be neglected, leaving us with:

$$V_s\,\mathrm{d}A + A\,\mathrm{d}V_s = 0.$$

Using the streamwise component of the momentum equation we start with:

$$F_{S_s} + F_{B_s} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} u_s \rho \,\mathrm{d}V + \int_{\mathrm{CS}} u_s \rho \mathbf{V}\cdot \mathrm{d}\mathbf{A}.$$

As we assume no friction F_{S_s} is due to pressure forces only, and will thus have three terms:

$$F_{S_s} = pA - (p + \mathrm{d}p)(A + \mathrm{d}A) + \left(p + \frac{\mathrm{d}p}{2}\right)\mathrm{d}A.$$

The first and second terms in this are the pressure forces on the end faces of the control surface. The third term is F_{s_b}, the pressure force acting in the s direction on the bounding stream surface. The above equation simplifies to

$$F_{S_s} = -A\,\mathrm{d}p - \frac{1}{2}\,\mathrm{d}p\,\mathrm{d}A.$$

The body force component in the s direction is

$$F_{B_s} = \rho g_s \,\mathrm{d}V = \rho\left(-g\sin\theta\right)\left(A + \frac{\mathrm{d}A}{2}\right)\mathrm{d}s.$$

But $\sin\theta\,\mathrm{d}s = \mathrm{d}z$ so

$$F_{B_s} = -\rho g\left(A + \frac{\mathrm{d}A}{2}\right)\mathrm{d}z.$$

The momentum flux will be

$$\int_{\mathrm{CS}} u_s \rho \mathbf{V}\cdot \mathrm{d}\mathbf{A} = V_s\left(-\rho V_s A\right) + \left(V_s + \mathrm{d}V_s\right)\left(\rho\left(V_s + \mathrm{d}V_s\right)\left(A + \mathrm{d}A\right)\right).$$

The two mass flux factors are equal from continuity so

$$\int_{\mathrm{CS}} u_s \rho \mathbf{V}\cdot \mathrm{d}\mathbf{A} = V_s\left(-\rho V_s A\right) + \left(V_s + \mathrm{d}V_s\right)\left(\rho V_s A\right) = \rho V_s A \mathrm{d}V_s.$$

Substituting all of these into the momentum equation yields

$$-A\,\mathrm{d}p - \frac{1}{2}\,\mathrm{d}p\,\mathrm{d}A - \rho g A\,\mathrm{d}z - \frac{1}{2}\rho g\,\mathrm{d}A\,\mathrm{d}z = \rho V_s A \mathrm{d}V_s.$$

Dividing by ρA and noting that products of differentials are negligible we obtain:

$$-\frac{\mathrm{d}p}{\rho} - g\,\mathrm{d}z = V_s\,\mathrm{d}V_s = d\left(\frac{V_s^2}{2}\right)$$

or

$$\frac{\mathrm{d}p}{\rho} + d\left(\frac{V_s^2}{2}\right) + g\,\mathrm{d}z = 0.$$

As the flow is incompressible and $\rho =$ constant we can integrate this to obtain:

$$\frac{p}{\rho} + \frac{V_s^2}{2} + gz = \text{constant}.$$

This is one form of the Bernoulli equation and it is only valid under the following restrictions:

1. Steady flow
2. No friction
3. Flow along a streamline
4. Incompressible flow

3.4 Momentum equation for Control Volume with Rectilinear Acceleration

To develop the momentum equation for a linearly accelerating control volume, it is necessary to relate $\mathbf{P}_{XYZ}$ to $\mathbf{P}_{xyz}$. We begin by writing Newton's second law for a system, remembering the acceleration must be measured relative to an inertial reference frame that we have designated XYZ. We get

$$\mathbf{F} = \left.\frac{\mathrm{d}\mathbf{P}_{XYZ}}{\mathrm{d}t}\right)_{\text{system}} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{M(\text{system})} \mathbf{V}_{XYZ}\,\mathrm{d}m = \int_{M(\text{system})} \frac{\mathrm{d}\mathbf{V}_{XYZ}}{\mathrm{d}t}\,\mathrm{d}m.$$

The velocities with respect to the inertial (XYZ) and the control volume coordinates (xyz) are related by the relative-motion equation:

$$\mathbf{V}_{XYZ} = \mathbf{V}_{xyz} + \mathbf{V}_{rf}$$

where $\mathbf{V}_{rf}$ is the velocity of the control volume coordinates xyz with respect to the absolute XYZ.

We assume the motion of xyz is purely translational relative to the inertial reference frame XYZ, so:

$$\frac{\mathrm{d}\mathbf{V}_{XYZ}}{\mathrm{d}t} = \mathbf{a}_{XYZ} = \frac{\mathrm{d}\mathbf{V}_{xyz}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{V}_{rf}}{\mathrm{d}t} = \mathbf{a}_{xyz} + \mathbf{a}_{rf}.$$

Substituting this into Newton's second law from above we get:

$$\mathbf{F} = \int_{M(\text{system})} \mathbf{a}_{rf}\,\mathrm{d}m + \int_{M(\text{system})} \frac{\mathrm{d}\mathbf{V}_{xyz}}{\mathrm{d}t}\,\mathrm{d}m$$

or

$$\mathbf{F} - \int_{M(\text{system})} \mathbf{a}_{rf}\,\mathrm{d}m = \left.\frac{\mathrm{d}\mathbf{P}_{xyz}}{\mathrm{d}t}\right)_{\text{system}}$$

where the linear momentum of the system is

$$\left.\mathbf{P}_{xyz}\right)_{\text{system}} = \int_{M(\text{system})} \mathbf{V}_{xyz}\,\mathrm{d}m = \int_{V(\text{system})} \mathbf{V}_{xyz}\rho\,\mathrm{d}V$$

and the force $\mathbf{F}$ includes all surface and body forces that are acting on the system.

To derive the control volume formulation of Newton's second law, we set $N = \mathbf{P}_{xyz}$ and $\eta = \mathbf{V}_{xyz}$. Substituting this into Equation 9 we obtain:

$$\left.\frac{\mathrm{d}\mathbf{P}_{xyz}}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}_{xyz}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{V}_{xyz}\rho\mathbf{V}_{xyz}\cdot\mathrm{d}\mathbf{A}.$$

If we combine this with the linear momentum equation for the system we obtain:

$$\mathbf{F} - \int_{\text{CV}} \mathbf{a}_{rf}\rho\,\mathrm{d}V = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}_{xyz}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{V}_{xyz}\rho\mathbf{V}_{xyz}\cdot\mathrm{d}\mathbf{A}.$$

And since $\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B$ this becomes

$$\mathbf{F}_S + \mathbf{F}_B - \int_{\text{CV}} \mathbf{a}_{rf}\rho\,\mathrm{d}V = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}_{xyz}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{V}_{xyz}\rho\mathbf{V}_{xyz}\cdot\mathrm{d}\mathbf{A} \tag{12}$$

Comparing Equation 12 to that for a non-accelerating control volume we see that they only differ by the introduction of the term $-\int_{\text{CV}} \mathbf{a}_{rf}\rho\,\mathrm{d}V$, and for a non-accelerating reference frame $\mathbf{a}_{rf} = 0$ and it reduces to the equation for a non-accelerating reference frame.

This can also be written in components as:

$$
\begin{aligned}
F_{S_x} + F_{B_x} - \int_{\mathrm{CV}} a_{rf_x}\rho \,\mathrm{d}V &= \frac{\partial}{\partial t}\int_{\mathrm{CV}} u_{xyz}\rho\,\mathrm{d}V + \int_{\mathrm{CS}} u_{xyz}\rho\mathbf{V}_{xyz}\cdot \mathrm{d}\mathbf{A} \\
F_{S_y} + F_{B_y} - \int_{\mathrm{CV}} a_{rf_y}\rho \,\mathrm{d}V &= \frac{\partial}{\partial t}\int_{\mathrm{CV}} v_{xyz}\rho\,\mathrm{d}V + \int_{\mathrm{CS}} v_{xyz}\rho\mathbf{V}_{xyz}\cdot \mathrm{d}\mathbf{A} \\
F_{S_z} + F_{B_z} - \int_{\mathrm{CV}} a_{rf_z}\rho \,\mathrm{d}V &= \frac{\partial}{\partial t}\int_{\mathrm{CV}} w_{xyz}\rho\,\mathrm{d}V + \int_{\mathrm{CS}} w_{xyz}\rho\mathbf{V}_{xyz}\cdot \mathrm{d}\mathbf{A}.
\end{aligned}
$$

3.5 The Angular-Momentum Principle

3.5.1 Equation for Fixed Control Volume

The angular-momentum principle for a system in an inertial frame is

$$
\mathbf{T} = \left.\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}\right)_{\text{system}}
$$

where $\mathbf{T}$ is the total torque exerted on the system by its surroundings and $\mathbf{H}$ is the angular momentum of the system.

$$
\mathbf{H} = \int_{M(\text{system})} \mathbf{r}\times\mathbf{V}\,\mathrm{d}m = \int_{V(\text{system})} \mathbf{r}\times\mathbf{V}\rho\,\mathrm{d}V.
$$

If we let $\mathbf{r}$ locate each mass or volume element of the system with respect to the coordinate system, then the torque $\mathbf{T}$ applied to the system may be written:

$$
\mathbf{T} = \mathbf{r}\times\mathbf{F}_s + \int_{M(\text{system})} \mathbf{r}\times\mathbf{g}\,\mathrm{d}m + \mathbf{T}_{\text{shaft}}.
$$

The relation between the system and fixed control volume formulation is

$$
\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \eta\rho\,\mathrm{d}V + \int_{\mathrm{CS}} \eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.
$$

where $N_{\text{system}} = \int_{M(\text{system})} \eta\,\mathrm{d}m$. If we set $N = \mathbf{H}$ and $\eta = \mathbf{r}\times\mathbf{V}$ then:

$$
\left.\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \mathbf{r}\times\mathbf{V}\rho\,\mathrm{d}V + \int_{\mathrm{CS}} \mathbf{r}\times\mathbf{V}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.
$$

Combining these we obtain:

$$
\mathbf{r}\times\mathbf{F}_s + \int_{M(\text{system})} \mathbf{r}\times\mathbf{g}\,\mathrm{d}m + \mathbf{T}_{\text{shaft}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \mathbf{r}\times\mathbf{V}\rho\,\mathrm{d}V + \int_{\mathrm{CS}} \mathbf{r}\times\mathbf{V}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.
$$

Since the system and control volume coincide at t_0 we have that $\mathbf{T} = \mathbf{T}_{\mathrm{CV}}$ and therefore:

$$
\mathbf{r}\times\mathbf{F}_s + \int_{\mathrm{CV}} \mathbf{r}\times\mathbf{g}\rho\,\mathrm{d}V + \mathbf{T}_{\text{shaft}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \mathbf{r}\times\mathbf{V}\rho\,\mathrm{d}V + \int_{\mathrm{CS}} \mathbf{r}\times\mathbf{V}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.
$$

3.6 The First Law of Thermodynamics

We recall the system formulation of the first law of thermodynamics was

$$\dot{Q} - \dot{W} = \left.\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{system}}$$

where the total energy of the system is given by

$$E_{\text{system}} = \int_{M(\text{system})} e\,\mathrm{d}m = \int_{V(\text{system})} e\rho\,\mathrm{d}V$$

and

$$e = u + \frac{V^2}{2} + gz.$$

We set $N = E$ and $\eta = e$ in Equation 9 and obtain:

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\rho\,\mathrm{d}V + \int_{\text{CS}} e\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

Since the system and control volume coincide at t_0 we have that $\left[\dot{Q} - \dot{W}\right]_{\text{system}} = \left[\dot{Q} - \dot{W}\right]_{\text{control volume}}$. In light of this we get the control volume form of the first law of thermodynamics as:

$$\dot{Q} - \dot{W} = \frac{\partial}{\partial t}\int_{\text{CV}} e\rho\,\mathrm{d}V + \int_{\text{CS}} e\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A} \tag{13}$$

where

$$e = u + \frac{V^2}{2} + gz.$$

Note that for steady flow the right hand side of Equation 13 is zero.

3.6.1 Rate of Work Done by a Control Volume

The rate of work done by a control volume is subdivided into four classifications,

$$\dot{W} = \dot{W}_s + \dot{W}_{\text{normal}} + \dot{W}_{\text{shear}} + \dot{W}_{\text{other}}.$$

Shaft Work We will designate the shaft work W_s and hence the rate of work transferred out through the control system by shaft work is designated $\dot{W}_s$.

Work Done by Normal Stresses at the Control Surface Work requires a force to act through a distance. Thus, when a force $\mathbf{F}$ acts through an infinitesimal displacement d$\mathbf{s}$ the work done is given by

$$\delta W = \mathbf{F}\cdot\mathrm{d}\mathbf{s}.$$

If we divide this by the time increment Δt and take the limit as $\Delta t \to 0$ we obtain the rate of work done by the force $\mathbf{F}$ as:

$$\dot{W} = \lim_{\Delta t\to 0}\frac{\delta W}{\Delta T} = \lim_{\Delta t\to 0}\frac{\mathbf{F}\cdot\mathrm{d}\mathbf{s}}{\Delta t} \implies \dot{W} = \mathbf{F}\cdot\mathbf{V}.$$

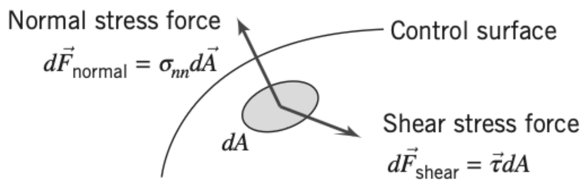

Figure 3.4: Normal and shear stress forces.

We can use this to compute the rate of work done by normal and shear stresses. We consider the segment og control surface shown on Figure 3.4. For an element of area $\mathrm{d}\mathbf{A}$ we can write an expression for the normal stress force $\mathrm{d}\mathbf{F}_{\text{normal}}$. This will be given by the normal stress σ_{nn} multiplied by the vector element $\mathrm{d}\mathbf{A}$. Hence:

$$\mathrm{d}\mathbf{F}_{\text{normal}} \cdot \mathbf{V} = \sigma_{nn}\,\mathrm{d}\mathbf{A} \cdot \mathbf{V}.$$

Since the work out from the control volume is the negative work done on the control volume we get:

$$\dot{W}_{\text{normal}} = -\int_{\text{CS}} \sigma_{nn}\,\mathrm{d}\mathbf{A} \cdot \mathbf{V} = -\int_{\text{CS}} \sigma_{nn} \mathbf{V} \cdot \mathrm{d}\mathbf{A}.$$

Work Done by Shear Stresses at the Control Surface As shown on Figure 3.4 the shear force acting on an element of the control surface is given by:

$$\mathrm{d}\mathbf{F}_{\text{shear}} = \boldsymbol{\tau}\,\mathrm{d}A$$

where the shear stress vector, $\boldsymbol{\tau}$, is the shear stress acting in some direction in the plane of $\mathrm{d}A$. The rate of work done on the entire control surface by shear stresses is thus:

$$\int_{\text{CS}} \boldsymbol{\tau}\,\mathrm{d}A \cdot \mathbf{V} = \int_{\text{CS}} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A.$$

Since the work out from the control volume is the negative of the work done on the control volume we get:

$$\dot{W}_{\text{shear}} = -\int_{\text{CS}} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A.$$

This is better expressed as three terms:

$$\dot{W}_{\text{shear}} = -\int_{\text{CS}} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A = -\int_{A(\text{shafts})} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A - \int_{A(\text{solid surface})} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A - \int_{A(\text{ports})} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A.$$

We have already accounted for the first term as $\dot{W}_s$ previously. At solid surfaces, $\mathbf{V} = 0$, so the second term is zero (for a fixed control volume). Thus

$$\dot{W}_{\text{shear}} = -\int_{A(\text{ports})} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A.$$

For a control surface perpendicular to $\mathbf{V}$ we get $\boldsymbol{\tau} \cdot \mathbf{V} = 0$ and therefore $\dot{W}_{\text{shear}} = 0$.

Other Work This includes electrical and electromagnetic energy that could be absorbed by the control volume. This is absent in most cases, but for the general formulation it must be included.

With all terms in $\dot{W}$ evaluated, we get:

$$\dot{W} = \dot{W}_s - \int_{\mathrm{CS}} \sigma_{nn} \mathbf{V} \cdot \mathrm{d}\mathbf{A} + \dot{W}_{\mathrm{shear}} + \dot{W}_{\mathrm{other}}.$$

Substituting this into the original expression we get:

$$\dot{Q} - \dot{W}_s + \int_{\mathrm{CS}} \sigma_{nn} \mathbf{V} \cdot \mathrm{d}\mathbf{A} - \dot{W}_{\mathrm{shear}} - \dot{W}_{other} = \frac{\partial}{\partial t} \int_{\mathrm{CV}} e\rho \, \mathrm{d}V + \int_{\mathrm{CS}} e\rho \mathbf{V} \cdot \mathrm{d}\mathbf{A}.$$

Rearranging this and remembering $\rho = \frac{1}{v}$, where v is specific volume we get:

$$\dot{Q} - \dot{W}_s - \dot{W}_{\mathrm{shear}} - \dot{W}_{\mathrm{other}} = \frac{\partial}{\partial t} \int_{\mathrm{CV}} e\rho \, \mathrm{d}V + \int_{\mathrm{CS}} (e + \rho v) \, \rho \mathbf{V} \cdot \mathrm{d}\mathbf{A}.$$

Substituting $e = u + \frac{V^2}{2} + gz$ into the last term we obtain the first law of thermodynamics for a control volume as:

$$\dot{Q} - \dot{W}_s - \dot{W}_{\mathrm{shear}} - \dot{W}_{\mathrm{other}} = \frac{\partial}{\partial t} \int_{\mathrm{CV}} e\rho \, \mathrm{d}V + \int_{\mathrm{CS}} \left(u + pv + \frac{V^2}{2} + gz \right) \rho \mathbf{V} \cdot \mathrm{d}\mathbf{A}.$$

3.6.2 The Second Law of Thermodynamics

The second law of thermodynamics applies to all fluid systems and is formulated as:

$$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right)_{\mathrm{system}} \geq \frac{1}{T} \dot{Q}$$

where the total entropy of the system is given by

$$S_{\mathrm{system}} = \int_{M(\mathrm{system})} s \, \mathrm{d}m = \int_{V(\mathrm{system})} s\rho \, \mathrm{d}V.$$

We set $N = S$ and $\eta = s$ in Equation 9 and obtain

$$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right)_{\mathrm{system}} = \frac{\partial}{\partial t} \int_{\mathrm{CV}} s\rho \, \mathrm{d}V + \int_{\mathrm{CS}} s\rho \mathbf{V} \cdot \mathrm{d}\mathbf{A}.$$

As the system and control volume coincide at t_0 we have that

$$\frac{1}{T} \dot{Q})_{\mathrm{system}} = \frac{1}{T} \dot{Q})_{\mathrm{CV}} = \int_{\mathrm{CS}} \frac{1}{T} \left(\frac{\dot{Q}}{A} \right) \mathrm{d}A.$$

Thus the control volume formulation of the second law of thermodynamics is:

$$\frac{\partial}{\partial t} \int_{\mathrm{CV}} s\rho \, \mathrm{d}V + \int_{\mathrm{CS}} s\rho \mathbf{V} \cdot \mathrm{d}\mathbf{A} \geq \int_{\mathrm{CS}} \frac{1}{T} \left(\frac{\dot{Q}}{A} \right) \mathrm{d}A.$$

Here the factor $\frac{\dot{Q}}{A}$ represents the heat flux per unit area into the control volume through the area element $\mathrm{d}A$. To evaluate the term

$$\int_{\mathrm{CS}} \frac{1}{T} \left(\frac{\dot{Q}}{A} \right) \mathrm{d}A.$$

both the local heat flux and local temperature T must be known for each area element of the control surface.