Pariial Differential Equations – Exercises

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Contents

Lecture 1: Fourier Theory

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Exercise 1.

Consider the following periodic functions. Find possible values for the period, using the definition of periodic functions.

(a) $f(x) = e^{\cos x}$

(b) $f(x) = \cos\left(x + \frac{\pi}{3}\right)$

(c) $f(x) = \cos \frac{\pi x}{3}$

(d) $f(x) = \cos x + \cos 3x$

(e) $f(x) = \cos x \cos 3x$

(f) $f(x) = \cos x + \cos(0{,}6x)$

For (a) the exponent to e, i.e. $\cos x$ only has a range of $(-1; 1)$. This means that the exponent will change between -1 and 1 with a period of 2π, therefore the period $p_a = 2\pi$ will also be the case for the function described in (a).

For (b) we are once again working with a cos function. This time it is however shifted by $\frac{\pi}{3}$ along the x-axis. A translation along the x-axis will, however, not affect the period of the function and therefore the period here is once again $p_b = 2\pi$.

For (c) the x is multiplied by $\frac{\pi}{3} \approx 1{,}047$. This means that the argument to the cosine will reach 2π at a speed that is $\frac{\pi}{3} \approx 1{,}047$ times faster than a normal cosine. Therefore the period of this is $p_c = \frac{3 \cdot 2\pi}{\pi} = 6$

For (d) we use that

$$f(x) = f(x + p)$$

for a periodic function to get

$$\cos x + \cos 3x = \cos(x+p) + \cos(3(x+p))$$
$$\cos x + \cos 3x = \cos(x+p) + \cos(3x+3p).$$

We see that this is true when both p and $3p$ are integer multiples of 2π so $p_d = 2\pi$

For (e) we use the same definition to get

$$\cos x \cos 3x = \cos(x+p)\cos(3x+3p).$$

We see that this is also true only when both p and $3p$ are integer multiples of 2π so $p_e = 2\pi$

For (f) we can once again use the same definition to get

$$\cos x + \cos(0{,}6x) = \cos(x+p) + \cos(0{,}6x + 0{,}6p).$$

This is true when p and $0{,}6p$ are integer multiples of 2π, so $p_f = 10\pi$.

Exercise 2.

Let

$$f(x) = x, \quad -1 \le x \le 1, \quad f(x) = f(x+p), \quad p = 2.$$

- Sketch f.
- Find a Fourier series representation of f.

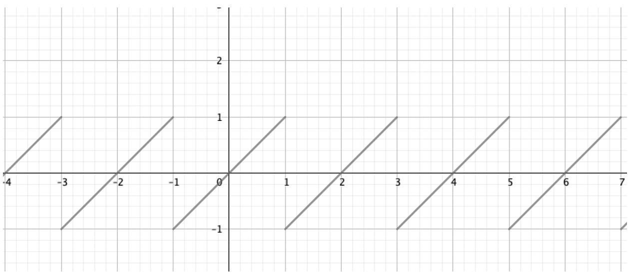

Figure 0.1:

f has been sketched on **Figure 0.1**. The Fourier series of this can, as it is a "nice" function be found by the formula:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

Where $L = \frac{p}{2} = 1$ with $p = 2$ being the period of the function. The Fourier coefficients can be found by the Euler formulas:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x, \quad n = 1,2,3,\ldots$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x, \quad n = 1,2,3,\ldots.$$

We start by solving for a_0 as:

$$a_0 = \int_{-1}^{1} x \,\mathrm{d}x = \left[\frac{1}{2}x^2\right]_{-1}^{1} = \cdot 0 = 0.$$

We now solve for a_n using integration by parts as:

$$\begin{aligned}
a_n &= \int_{-1}^{1} x \cos n\pi x \,\mathrm{d}x \\
&= \left[x \frac{\sin n\pi x}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1} \frac{\sin n\pi x}{n\pi} \,\mathrm{d}x \\
&= \left(-\frac{1}{n\pi}\right) \int_{-1}^{1} \sin n\pi x \,\mathrm{d}x \\
&= \left[-\frac{1}{n\pi}\right] \left[-\frac{\cos n\pi x}{n\pi}\right]_{-1}^{1} \\
&= 0.
\end{aligned}$$

We similarly solve for b_n as:

$$\begin{aligned}
b_n &= \int_{-1}^{1} x \sin n\pi x \,\mathrm{d}x \\
&= \left[x \frac{-\cos n\pi x}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1} \frac{-\cos n\pi x}{n\pi} \,\mathrm{d}x \\
&= \left(-\frac{2}{n\pi}\right) \cos n\pi + \frac{1}{n\pi} \left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{1} \\
&= \left(-\frac{2}{n\pi}\right) \cos n\pi.
\end{aligned}$$

The full Fourier series therefore becomes:

$$f(x) = \sum_{n=1}^{\infty} \left(-\frac{2}{n\pi}\right) \cos n\pi \cdot \sin n\pi x.$$

Exercise 3.

Let $a \neq 0$ and $b \neq 0$. Let $f(x) = f(x + p_0)$ be a periodic function with period p_0. Is $f(ax + b)$ periodic? If yes, find a period. Give arguments for your answer.

The b only serves to translate the function $f(x)$ along the x-axis, this does not change the period. The value of a corresponds to how "quickly" some value is reached, e.g. if ax is an argument to a function a doubling of the value of a will mean that the function will reach its maximum for a value of x only half as big. I.e. there is an inverse relation between the size of a and the period as:

$$p_2 = \frac{p_1}{a}.$$

Exercise 4.

Calculate the left hand limit and the right hand limit of

$$f(x) = \frac{|x|}{x}$$

at $x = 0$.

We start with the right-hand limit as:

$$f(0+) = \lim_{h \to 0} (0 + h) = \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

And now we can similarly do the left-hand limit as:

$$f(0-) = \lim_{h \to 0} (0 - h) = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1.$$

Exercise 5.

Calculate the left-hand derivative and the right-hand derivative of

$$f(x) = x\,|x|$$

at $x = 0$.

We start with the right-hand derivative as:

$$f'(0+) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h|h|}{h} = \lim_{h \to 0} h = 0.$$

And the left-hand derivative can likewise be computed as:

$$f'(0-) = \lim_{h \to 0} \frac{f(0) - f(0-h)}{h} = \lim_{h \to 0} \frac{-(-h)| - h|}{h} = \lim_{h \to 0} h = 0.$$

Lecture 2: Even and odd functions

5. September 2025

Exercise 1.

Find a period and the corresponding Fourier coefficients of

a) $f(x) = \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{2x}{4}\right) + \cos\left(\frac{\pi}{5}\right)\sin\left(\frac{3x}{7}\right)$

We rewrite the given function as:

$$\begin{aligned} f(x) &= \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{2x}{4}\right) + \cos\left(\frac{\pi}{5}\right)\sin\left(\frac{3x}{7}\right) \\ &= \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{7\pi x}{14\pi}\right) + \cos\left(\frac{\pi}{5}\right)\sin\left(\frac{6\pi x}{14\pi}\right). \end{aligned}$$

We will now compare this to the general formulation of the Fourier series as:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

We can quickly see that $L = 14\pi \implies p = 28\pi$, $b_6 = \cos\left(\frac{\pi}{5}\right)$, $b_7 = \sin\left(\frac{\pi}{4}\right)$. All other unmentioned Fourier coefficients are zero.

b) $f(x) = \sin\left(\sqrt{2}x\right) + \cos\left(\frac{x}{\sqrt{2}}\right)$

We once again start by rewriting as:

$$\begin{aligned} f(x) &= \sin\left(\sqrt{2}x\right) + \cos\left(\frac{x}{\sqrt{2}}\right) \\ &= \sin\left(\frac{2\pi x}{\sqrt{2}\pi}\right) + \cos\left(\frac{\pi x}{\sqrt{2}\pi}\right). \end{aligned}$$

Comparing this with the general formulation of the Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

We see that $L = \sqrt{2}\pi \implies p = 2\sqrt{2}\pi$, $a_1 = 1$, $b_2 = 1$. All other Fourier coefficients are zero.

c) $f(x) = \sin\left(\frac{\pi}{3} + \frac{3\pi x}{4}\right)$

We use that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

to rewrite the equation as:

$$\begin{aligned} f(x) &= \sin\left(\frac{\pi}{3} + \frac{3\pi x}{4}\right) \\ &= \sin\frac{\pi}{3} \cdot \cos\frac{3\pi x}{4} + \sin\frac{3\pi x}{4} \cdot \cos\frac{\pi}{3}. \end{aligned}$$

Comparing this to the general formulation of the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

we see that $L = 4 \implies p = 8$, $a_3 = \sin\frac{\pi}{3}$, $b_3 = \cos\frac{\pi}{3}$. All other Fourier coefficients are zero

Exercise 2.

Are the following functions even or odd? Give arguments.

a) $f(x) = \left(\cos x^2 + \cos x\right)\sin x$

We check if the function is even or odd by checking what happens for an input of $-x$ as:

$$\begin{aligned} f(-x) &= \left(\cos(-x)^2 + \cos(-x)\right)\sin(-x) \\ &= \left(\cos x^2 + \cos(-x)\right)(-\sin x) \\ &= -f(x). \end{aligned}$$

Hence the function is odd.

b) $f(x) = \frac{1+\cos x}{2+\cos x^2}$

We follow the same procedure:

$$\begin{aligned} f(-x) &= \frac{1+\cos(-x)}{2+\cos(-x)^2} \\ &= \frac{1+\cos x}{2+\cos x} \\ &= f(x). \end{aligned}$$

Hence the function is even.

c) $f(x) = \frac{1+\sin x}{2+\sin x^2}$

The same procedure is applied:

$$\begin{aligned} f(-x) &= \frac{1+\sin(-x)}{2+\sin(-x)^2} \\ &= \frac{1-\sin x}{2+\sin x} \\ &\neq \begin{cases} f(x) \\ -f(x) \end{cases} \end{aligned}$$

Hence the function is neither even nor odd.

d) $f(x) = x|x|$

The same procedure is applied:

$$\begin{aligned} f(-x) &= (-x)\,|-x| \\ &= -x\,|x| \\ &= -f(x). \end{aligned}$$

Hence the function is odd.

Exercise 3.

Consider the periodic function

$$f(x) = \begin{cases} x, & -2 < x < -1 \\ x + k, & -1 < x < 1 \\ x, & 1 < x < 2 \end{cases}$$
$$f(x) = f(x+p), p = 4$$

where k is a constant. Decompose $f(x)$ into its even and odd part.

We apply the definition of decomposition into odd and even parts for the first interval, $-2 < x < -1$ as:

$$f_1(x) = \frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}(x + (-x)) = 0$$
$$f_2(x) = \frac{1}{2}(f(x) - f(-x) = \frac{1}{2}(x - (-x)) = x.$$

For the second interval, $-1 < x < 1$ we get

$$f_1(x) = \frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}((x + k) + (-x + k)) = k$$
$$f_2(x) = \frac{1}{2}(f(x) - f(-x) = \frac{1}{2}((x + k) - (-x + k)) = x.$$

And for the last interval, $1 < x < 2$ we get:

$$f_1(x) = \frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}(x + (-x)) = 0$$
$$f_2(x) = \frac{1}{2}(f(x) - f(-x) = \frac{1}{2}(x - (-x)) = x.$$

We thus have:

$$f(x) = f_1(x) + f_2(x)$$
$$f_1(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2. \end{cases}$$
$$f_1(x) = f_1(x + p), \quad p = 4$$
$$f_2(x) = x, \qquad -2 < x < 2$$
$$f_2(x) = f_2(x + p), \quad p = 4.$$

Exercise 4.

Consider the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 2x, & 0 < x < \pi \end{cases}$$
$$f(x) = f(x + p), p = 2\pi.$$

Decompose $f(x)$ into its even and odd part and find its Fourier series.

Hint: You may want to use examples from the lecture.

For the interval $-\pi < x < 0$ we have

$$f_1(x) = \frac{1}{2}(0 + (-2x)) = -x$$
$$f_2(x) = \frac{1}{2}(0 - (-2x)) = x.$$

And for $0 < x < \pi$ we have

$$f_1(x) = \frac{1}{2}(2x + 0) = x$$
$$f_2(x) = \frac{1}{2}(2x - 0) = x.$$

We thus have

$$\begin{aligned} f(x) &= f_1(x) + f_2(x) \\ f_1(x) &= |x|, -\pi < x < \pi \\ f_1(x) &= f_1(x+p), \quad p = 2\pi \\ f_2(x) &= x, \quad -\pi < x < \pi \\ f_2(x) &= f_2(x+p) \quad p = 2\pi. \end{aligned}$$

From the lecture we know that f_1 has the Fourier cosine series

$$f_1(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} \left(\cos n\pi - 1\right) \cos nx$$

and that f_2 has the Fourier sine series:

$$f_2(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx.$$

Combining these we get the Fourier series for $f(x)$ as:

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n^2}(\cos n\pi - 1) \cos nx + (-1)^{n+1} \frac{2}{n} \sin nx \right).$$

Exercise 5.

Decompose

$$f(x) = e^{ix}$$

into its even and odd part.

Hint; you may want to use the representation of $f(x)$ in terms of $\cos x$ and $\sin x$.

We apply the same procedure as in the last couple of exercises whilst remembering that:

$$e^{ix} = \cos x + i \sin x.$$

We therefore get:

$$\begin{aligned} f_1(x) &= \frac{1}{2}(f(x) + f(-x)) \\ &= \frac{1}{2}\left(\cos x + i \sin x + \cos x - i \sin x\right) \\ &= \cos x \\ f_2(x) &= \frac{1}{2}\left(f(x) - f(-x)\right) \\ &= \frac{1}{2}\left(\cos x + i \sin x - (\cos x - i \sin x)\right) \\ &= i \sin x. \end{aligned}$$

This also is what would be expected when looking at the representation of e^{ix} introduced at the start of this answer.