Ordinary Differential Equations, Linear Algebra and Vector Calculus – Exam

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Exercise 1

Let a be a real number and
$$A = \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix}.$$
Calculate all values of a such that the inverse A^{-1} doesn't exist.

For the inverse to exist, we must have $\det(A) \neq 0$. The determinant of A can simply be found as:
$$\det(A) = \begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix} = 1 - a \neq 0.$$
This equality does not hold for $a = 1$, hence no inverse A^{-1} exists for $a = 1$. For $a \neq 1$ we have $\det(A) \neq 0$ and therefore an inverse exists.

Exercise 2

Let a be a real number and
$$A = \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix}.$$
Calculate all values of a such that A has the eigenvalue $\lambda = 1$.

We know that λ is an eigenvalue to a matrix, if and only if:
$$\det(A - \lambda I) = 0.$$
Here we are looking for solutions with $\lambda = 1$ and therefore we get:
$$\det(A - I) = \begin{vmatrix} 0 & a \\ 1 & 0 \end{vmatrix} = 0 \implies 0 - a = 0 \implies a = 0.$$
Therefore $a = 0$ is the solution we are looking for.

Exercise 3

Consider the linear system of m equations with n unknowns
$$A\vec{x} = \vec{0}.$$
Suppose all solutions $\vec{x}$ are of the form
$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$
Find n and $\operatorname{rank}(A)$.

As our solution is a linear combination of two vectors with length 4 we have exactly 4 unknowns as well. We can write our solution as

$$\vec{x} = \begin{pmatrix} c_1 \\ c_1 + c_2 \\ c_1 + c_2 \\ c_2 \end{pmatrix}.$$

And here we can quickly see that $\operatorname{rank}(\vec{x}) = 2$ as we have two linearly independent vectors (entry 1 and entry 4) and the rest (entry 2 and 3) are simply the sum of these.

Exercise 4

Consider the nonlinear system of first order ODEs

$$\begin{aligned} y_1'(t) &= e^{y_1(t)} + y_2(t) \\ y_2'(t) &= e^{y_1(t)} - 2. \end{aligned}$$

Calculate the location of all critical points.

We define the system as

$$\begin{aligned} y_1'(t) &= e^{y_1(t)} + y_2(t) = f_1(y_1(t), y_2(t)) \\ y_2'(t) &= e^{y_1(t)} - 2 = f_2(y_1(t), y_2(t)). \end{aligned}$$

For a critical point we have that:

$$\frac{\mathrm{d}y_2}{\mathrm{d}y_1} = \frac{y_2'(t)\,\mathrm{d}t}{y_1'(t)\,\mathrm{d}t} = \frac{y_2'(t)}{y_1'(t)} = \frac{f_2(y_1(t), y_2(t))}{f_1(y_1(t), y_2(t))} = \frac{0}{0}.$$

For $f_2 = 0$ we have:

$$\begin{aligned} e^{y_1(t)} - 2 &= 0 \\ e^{y_1(t)} &= 2 \\ y_1(t) &= \ln 2. \end{aligned}$$

And for $f_1 = 0$ we have:

$$\begin{aligned} e^{\ln 2} + y_2(t) &= 0 \\ 2 + y_2(t) &= 0 \\ y_2(t) &= -2. \end{aligned}$$

Therefore for the condition

$$\frac{\mathrm{d}y_2}{\mathrm{d}y_1} = \frac{0}{0}$$

to hold we must have $y_1(t) = \ln 2$ and $y_2(t) = -2$.

Exercise 5

Calculate the general solution of

$$y''(x) - 4y(x) = x + e^x.$$

We compare this to the standard form for a nonhomogeneous second order linear ODE:

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x).$$

Here we see that $p(x) = 0$, $q(x) = -4$ and $r(x) = x + e^x$. To find the solution of this we start by finding the solution to the corresponding homogeneous ODE:

$$y_h''(x) - 4y_h(x) = 0.$$

This is a homogeneous linear second order ODE with constant coefficients. The characteristic equation is:

$$\lambda^2 - 4 = 0 \implies \lambda = \{-2, 2\}.$$

As these are both real and different we have the two solutions:

$$y_{h_1}(x) = c_1 e^{2x} \quad \text{and} \quad y_{h_2} = c_2 e^{-2x}.$$

These are linearly independent and constitute a basis. The general solution to the homogeneous ODE is therefore:

$$y_h(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

To find a general solution for the nonhomogeneous ODE we employ the method of undetermined coefficients. Using the basic and sum rules we get a solution of the form:

$$y_r(x) = Ce^x + K_1 x + K_0.$$

This has the derivatives:

$$\begin{aligned} y_r'(x) &= Ce^x + K_1 \\ y_r''(x) &= Ce^x. \end{aligned}$$

We can now insert this into the nonhomogeneous ODE as:

$$\begin{aligned} y_r''(x) - 4y_r(x) &= x + e^x \\ Ce^x - 4Ce^x - 4K_1 x + 4K_0 &= x + e^x \\ -3Ce^x - 4K_1 x + 4K_0 &= x + e^x. \end{aligned}$$

$$\begin{aligned} -3Ce^x &= e^x \\ -3C &= 1 \\ C &= -\frac{1}{3} \end{aligned} \qquad\qquad \begin{aligned} -4K_1 x &= x \\ -4K_1 &= 1 \\ K_1 &= -\frac{1}{4}. \end{aligned}$$

Therefore

$$y_r(x) = -\frac{1}{3}e^x - \frac{1}{4}x.$$

And the general solution therefore is:

$$y(x) = y_h(x) + y_r(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3}e^x - \frac{1}{4}x = c_1 e^{2x} + c_2 e^{-2x} - \frac{4e^x + 3x}{12}.$$

Exercise 6

Solve the initial value problem for $x > 0$

$$2xy(x)^2 + 2x^2 y(x)y'(x) = 0, y(1) = 1.$$

using the theory of exact ODEs.

We start by comparing the given ODE to the standard form for an exact ODE:

$$M(x,y) + N(x,y)y' = 0.$$

Here we see that:

$$M(x,y) = 2xy^2 \quad \text{and} \quad N(x,y) = 2x^2y.$$

We can check if it actually is exact as:

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x}.$$

As this holds the ODE is exact. We then get:

$$u(x,y) = \int M(x,y)\,\mathrm{d}x + k(y) = \int 2 \cdot x \cdot y^2\,\mathrm{d}x + k(y) = x^2y^2 + k(y).$$

We also have that:

$$N(x,y) = \frac{\partial u(x,y)}{\partial y} \implies 2x^2y(x) = 2x^2y + k(y) \implies k(y) = 0.$$

Our function $u(x,y)$ is therefore:

$$u(x,y) = x^2y^2.$$

We now choose a constant c

$$\begin{aligned} u(x,y) &= c \\ x^2y^2 &= c \\ y^2 &= \frac{c}{x^2} \\ y &= \sqrt{\frac{c}{x}} \\ y &= \frac{k}{x}, k = \sqrt{c}. \end{aligned}$$

And therefore our general solution is:

$$y(x) = \frac{k}{x}.$$

We can now insert our initial value as:

$$1 = \frac{k}{1} \implies k = 1.$$

And the particular solution is therefore:

$$y(x) = \frac{1}{x}.$$

Exercise 7

Solve the initial value problem for $x > 0$

$$xy'(x) + y(x) = 0, y(1) = 1.$$

Using the theory of separable ODEs.

We start by rewriting the ODE as:

$$y'(x) = \frac{y(x)}{x}.$$

This can be compared to the standard form for a separable ODE:

$$y'(x) = \frac{h(x)}{g(y(x))}.$$

We see that $h(x) = \frac{1}{x}$ and $g(y(x)) = -\frac{1}{y(x)}$. We can now integrate as:

$$\begin{aligned}
\int_{y(x_0)}^{y(x)} g(y)\,\mathrm{d}y &= \int_{x_0}^{x} h(\hat{x})\,\mathrm{d}\hat{x} \\
[-\ln(y)]_{y(x_0)}^{y(x)} &= [\ln x]_{x_0}^{x} \\
-\ln y(x) + \ln 1 &= \ln x - \ln 1 \\
\ln y(x) &= -\ln x - \ln 2 \\
y(x) &= e^{-\frac{x}{2}}.
\end{aligned}$$

It seems like something has gone wrong here as this does not fulfill our initial condition ($y(1) = \frac{1}{\sqrt{e}} \neq 1$). The actual solution to the ODE is expected to have the form:

$$y(x) = \frac{c_1}{x}.$$

In which case the initial condition gives the solution:

$$y(x) = \frac{1}{x}.$$

Exercise 8

Consider the curve C given by all points $P(x, y, z)$ such that

$$x - y^2 = 0, z = x, 0 \leq x \leq 1, y \leq 0.$$

Find a parametric representation.

We start by rewriting the surface as:

$$y^2 = x, z = x, 0 \leq x \leq 1, y \leq 0.$$

We can now quickly see that we can parameterize it as: $(t, \pm\sqrt{t}, t)$. As we are only looking for the solutions with $y \leq 0$ we can reduce it to $(t, -\sqrt{t}, t)$. And finally we can add the boundaries for x:

$$\vec{r}(t)(t, -\sqrt{t}, t), 0 \leq t \leq 1.$$

Exercise 9

Consider the surface S given by all points $P(x, y, z)$ such that

$$x - y^2 = 0, z \leq x, 0 \leq x \leq 1, y \geq 0.$$

Find a parametric representation.

This case has a lot of similarities with the above. However, this time the z-value is free (as we are working with a surface and not a curve anymore). We therefore get this again:

$$\vec{r}(u,v) = (u, \pm\sqrt{u}, v).$$

This time we are only looking for the positive values of y, hence:

$$\vec{r}(u,v) = (u, \sqrt{u}, v).$$

And now the boundaries can be added as:

$$\vec{r}(u,v) = (u, \sqrt{u}, v), 0 \leq u \leq 1, v \leq u.$$

Exercise 10

Consider the vector field $\vec{F}(\vec{r}) = (ye^{xy} + ze^x, xe^{xy} + 1, e^x)$. Calculate a scalar field $f(\vec{r})$ such that $\nabla f(\vec{r}) = \vec{F}(\vec{r})$

For the condition to hold, we have that:

$$\begin{aligned}
F_1(\vec{r}) &= \frac{\partial f(\vec{r})}{\partial x} \\
\implies ye^{xy} + ze^x &= \frac{\partial f(\vec{r})}{\partial x} \\
\implies f(\vec{r}) &= e^{xy} + e^x z + g(y,z) \\
F_2(\vec{r}) &= \frac{\partial f(\vec{r})}{\partial y} \\
\implies xe^{xy} + 1 &= xe^{xy} + g(z) \\
\implies g(z) &= 1.
\end{aligned}$$

Therefore the surface becomes:

$$f(\vec{r}) = e^{xy} + e^x z + 1.$$