Pariial Differential Equations – Exercises

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September 1, 2025

Contents

Lecture 1: Fourier Theory

29. August 2025

Exercise 1.

Consider the following periodic functions. Find possible values for the period, using the definition of periodic functions.

(a) $f(x) = e^{\cos x}$

(b) $f(x) = \cos\left(x + \frac{\pi}{3}\right)$

(c) $f(x) = \cos\frac{\pi x}{3}$

(d) $f(x) = \cos x + \cos 3x$

(e) $f(x) = \cos x \cos 3x$

(f) $f(x) = \cos x + \cos(0{,}6x)$

For (a) the exponent to e, i.e. $\cos x$ only has a range of $(-1; 1)$. This means that the exponent will change between -1 and 1 with a period of 2π, therefore the period $p_a = 2\pi$ will also be the case for the function described in (a).

For (b) we are once again working with a cos function. This time it is however shifted by $\frac{\pi}{3}$ along the x-axis. A translation along the x-axis will, however, not affect the period of the function and therefore the period here is once again $p_b = 2\pi$.

For (c) the x is multiplied by $\frac{\pi}{3} \approx 1{,}047$. This means that the argument to the cosine will reach 2π at a speed that is $\frac{\pi}{3} \approx 1{,}047$ times faster than a normal cosine. Therefore the period of this is $p_c = \frac{3\cdot 2\pi}{\pi} = 6$

For (d) we use that

$$f(x) = f(x + p)$$

for a periodic function to get

$$\cos x + \cos 3x = \cos(x+p) + \cos(3(x+p))$$
$$\cos x + \cos 3x = \cos(x+p) + \cos(3x+3p).$$

We see that this is true when both p and $3p$ are integer multiples of 2π so $p_d = 2\pi$

For (e) we use the same definition to get

$$\cos x \cos 3x = \cos(x+p)\cos(3x+3p).$$

We see that this is also true only when both p and $3p$ are integer multiples of 2π so $p_e = 2\pi$

For (f) we can once again use the same definition to get

$$\cos x + \cos(0{,}6x) = \cos(x+p) + \cos(0{,}6x + 0{,}6p).$$

This is true when p and $0{,}6p$ are integer multiples of 2π, so $p_f = 10\pi$.

Exercise 2.

Let

$$f(x) = x, \quad -1 \leq x \leq 1, \quad f(x) = f(x+p), \quad p = 2.$$

- Sketch f.
- Find a Fourier series representation of f.

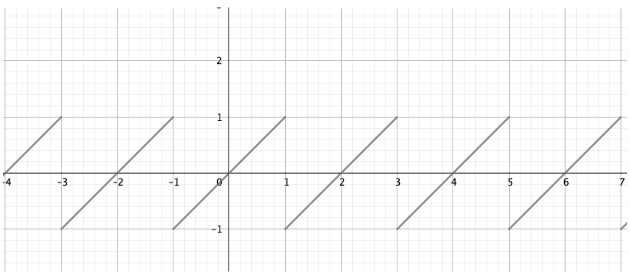

Figure 0.1:

f has been sketched on **Figure 0.1**. The Fourier series of this can, as it is a "nice" function be found by the formula:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

Where $L = \frac{p}{2} = 1$ with $p = 2$ being the period of the function. The Fourier coefficients can be found by the Euler formulas:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,\mathrm{d}x$$
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,\mathrm{d}x, \quad n = 1, 2, 3, \ldots$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,\mathrm{d}x, \quad n = 1, 2, 3, \ldots.$$

We start by solving for a_0 as:

$$a_0 = \int_{-1}^{1} x \,\mathrm{d}x = \left[\frac{1}{2}x^2\right]_{-1}^{1} = \cdot 0 = 0.$$

We now solve for a_n using integration by parts as:

$$\begin{aligned} a_n &= \int_{-1}^{1} x \cos n\pi x \,\mathrm{d}x \\ &= \left[x\frac{\sin n\pi x}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1} \frac{\sin n\pi x}{n\pi} \,\mathrm{d}x \\ &= \left(-\frac{1}{n\pi}\right) \int_{-1}^{1} \sin n\pi x \,\mathrm{d}x \\ &= \left[-\frac{1}{n\pi}\right] \left[-\frac{\cos n\pi x}{n\pi}\right]_{-1}^{1} \\ &= 0. \end{aligned}$$

We similarly solve for b_n as:

$$\begin{aligned} b_n &= \int_{-1}^{1} x \sin n\pi x \,\mathrm{d}x \\ &= \left[x\frac{-\cos n\pi x}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1} \frac{-\cos n\pi x}{n\pi} \,\mathrm{d}x \\ &= \left(-\frac{2}{n\pi}\right) \cos n\pi + \frac{1}{n\pi}\left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{1} \\ &= \left(-\frac{2}{n\pi}\right) \cos n\pi. \end{aligned}$$

The full Fourier series therefore becomes:

$$f(x) = \sum_{n=1}^{\infty} \left(-\frac{2}{n\pi}\right) \cos n\pi \cdot \sin n\pi x.$$

Exercise 3.

Let $a \neq 0$ and $b \neq 0$. Let $f(x) = f(x + p_0)$ be a periodic function with period p_0. Is $f(ax + b)$ periodic? If yes, find a period. Give arguments for your answer.

The b only serves to translate the function $f(x)$ along the x-axis, this does not change the period. The value of a corresponds to how "quickly" some value is reached, e.g. if ax is an argument to a function a doubling of the value of a will mean that the function will reach its maximum for a value of x only half as big. I.e. there is an inverse relation between the size of a and the period as:

$$p_2 = \frac{p_1}{a}.$$

Exercise 4.

Calculate the left hand limit and the right hand limit of

$$f(x) = \frac{|x|}{x}$$

at $x = 0$.

We start with the right-hand limit as:

$$f(0+) = \lim_{h \to 0} (0 + h) = \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

And now we can similarly do the left-hand limit as:

$$f(0-) = \lim_{h \to 0} (0 - h) = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1.$$

Exercise 5.

Calculate the left-hand derivative and the right-hand derivative of

$$f(x) = x\,|x|$$

at $x = 0$.

We start with the right-hand derivative as:

$$f'(0+) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h|h|}{h} = \lim_{h \to 0} h = 0.$$

And the left-hand derivative can likewise be computed as:

$$f'(0-) = \lim_{h \to 0} \frac{f(0) - f(0-h)}{h} = \lim_{h \to 0} \frac{-(-h)| - h|}{h} = \lim_{h \to 0} h = 0.$$