Linear Algebra

1. Demonstration

Let x and y two vectors such as:

$$x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix} \text{ and } y = \begin{pmatrix} \Psi_0 \\ \vdots \\ \Psi_{n-1} \end{pmatrix} \text{ of size } \mathbb{R}^n.$$

Prove that:

$$\|x\| \times \|y\| \times \cos\theta = \sum_{n=0}^{n-1} \chi_i \Psi_i$$

Demonstration:

$$\|x\| \times \|y\| \times \cos\theta$$

$$\Leftrightarrow \sqrt{\sum_{i=0}^{n-1} \chi_i^2} \times \sqrt{\sum_{i=0}^{n-1} \Psi_i^2} \times \cos\theta$$

$$\Leftrightarrow \sqrt{\sum_{i=0}^{n-1} \chi_i^2} \times \sqrt{\sum_{i=0}^{n-1} \Psi_i^2} \times \frac{\sum_{n=0}^{n-1} \chi_i \Psi_i}{\sqrt{\sum_{i=0}^{n-1} \chi_i^2} \times \sqrt{\sum_{i=0}^{n-1} \Psi_i^2}}$$

$$\Leftrightarrow \sum_{n=0}^{n-1} \chi_i \Psi_i$$

So $\|x\| \times \|y\| \times \cos\theta = \sum_{n=0}^{n-1} \chi_i \Psi_i$

This demonstration was made to show why when we have coordinates you can simply compute the dot product like this $\sum_{n=0}^{n-1} \chi_i \Psi_i$ and that you are not obligated to use this formula $\|x\| \times \|y\| \times \cos\theta$.

2. Demonstration

For x, $y \in \mathbb{R}^n$, $x^T y = y^T x$ (We want to find out if the dot product is commutative):

$x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$ and $y = \begin{pmatrix} \Psi_0 \\ \vdots \\ \Psi_{n-1} \end{pmatrix}$ of size $\mathbb{R}^n$ with $y = \sum_{n=0}^{n-1} \Psi_i \chi_i$

$$x^T y$$

$$\Leftrightarrow \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix}^T \begin{pmatrix} \Psi_0 \\ \vdots \\ \Psi_{n-1} \end{pmatrix}$$

$$\Leftrightarrow (\chi_0 \ldots \chi_{n-1}) \begin{pmatrix} \Psi_0 \\ \vdots \\ \Psi_{n-1} \end{pmatrix}$$

$$\Leftrightarrow \sum_{n=0}^{n-1} \chi_i \Psi_i$$

$$\Leftrightarrow \sum_{n=0}^{n-1} \Psi_i \chi_i$$

$$\Leftrightarrow y^T x$$

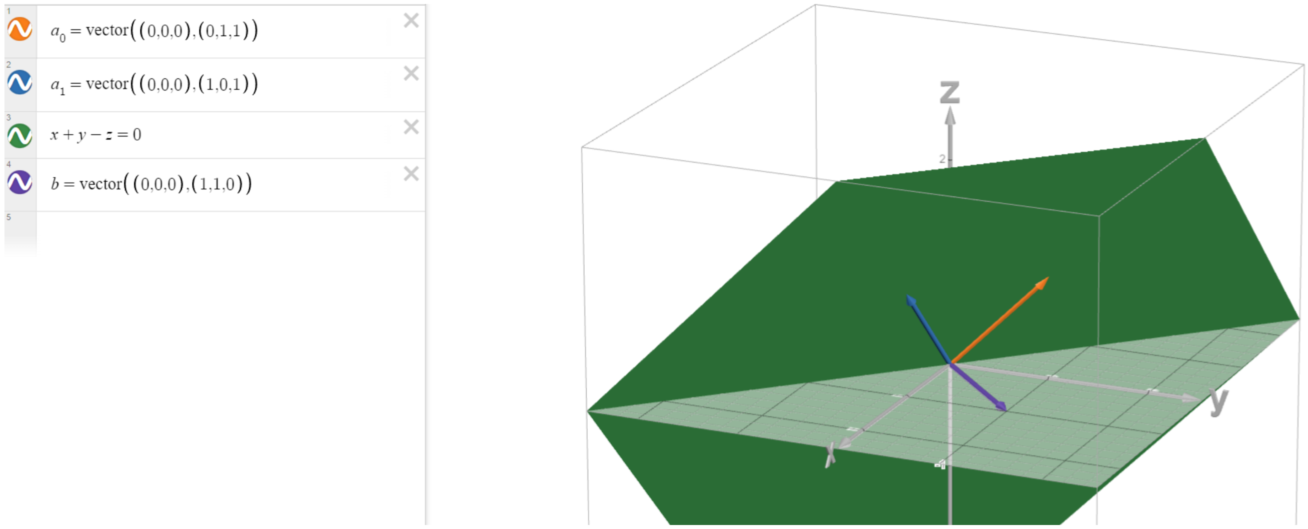
So $x^T y = y^T x$ because we know that components are real numbers so multiplication is commutative therefore dot product is commutative.

Theorem:

Let $A \in \mathbb{R}^{m \times n}$ Every $x \in \mathbb{R}^n$ can be written as
$x = x_r + x_n$
where $x_r \in \mathcal{R}(A)$ and $x_n \in \mathcal{N}(A)$

Finding the best solution

Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$



We can see that $b \notin \mathcal{C}(A)$. Hence the system of equation doesn't have any solution so we gonna try to find the best approximate solution.

We know that we can write $b = z + w$ where $w^T z = 0$. Hence, if w is orthogonal to z where $z \in \mathcal{C}(A)$,therefore it is also orthogonal to the column space of the matrix A which means that $w \in \mathcal{C}(A)^{\perp}$. Consequently $w \in \mathcal{N}(A^T)$.

Let $\hat{x}$ be the solution of the equation $A\hat{x} = z$
Since $w \in \mathcal{N}(A^T)$ we have:

$$A^T w = 0$$
$$A^T(b - z) = 0$$
$$A^T b - A^T z = 0$$
$$A^T b = A^T A\hat{x}$$

We know that A has linearly independents columns since $\mathcal{N}(A) = \{0\}$ the zero vector. Consequently, $A^T A$ is nonsingular.

Proof by contrapositive:

- Let $\neg P$ represents "$A \in \mathbb{R}^{m \times n}$ has linearly dependents columns"
- Let $\neg Q$ represents "$A^T A$" is singular
- Let $x \in \mathbb{R}^n$

We will prove $\neg P \Rightarrow \neg Q$:

If $A \in \mathbb{R}^{m \times n}$ has linearly dependents columns, it means that $\mathcal{N}(A) \neq \{0\}$ since it exists a vector $x \neq 0$ such as $Ax = 0$.

But then x is also a solution to the equation $A^T Ax = 0$ since $Ax = 0$ we have $A^T 0 = 0$. Thereby $x \in \mathcal{N}(A), \mathcal{N}(A^T A)$.

Therefore $A^T A$ is singular which means $A^T A$ has an inverse.

Then since we know that $\hat{x}$ represent the best approximate solution

$(A^T A)^{-1} A^T b = A^T A (A^T A)^{-1} \hat{x}$
$\hat{x} = A^\dagger b$

The vector z is given by:

$z = AA^\dagger b$

Using the Cholesky Factorization:

We know that $A^T A$ is a symmetric matrix, let's prove that $A^T A$ is always positive because if it's not we will have to do an LDL^T factorization to avoid taking the square root of a negative number.

In this example it its trivial that $A^T A$ is positive but we check anyways:
$\det(A^T A) = 4 - 1 = 3 > 0$ so
$x^T A^T A x > 0 \; \forall x \neq 0 \in \mathbb{R}^n$

Let $B = A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and let L be a lower triangular matrix such that $B = LL^T$. with
$L = \left(\begin{array}{c|c} \lambda_{00} & 0 \\ \hline l_{01} & L_{11} \end{array}\right)$
Hence we have

$$B = \left(\begin{array}{c|c} \lambda_{00}^2 & \star \\ \hline l_{01}\lambda_{00} & l_{01}l_{01}^T + L_{11}L_{11}^T \end{array}\right)$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \left(\begin{array}{c|c} \lambda_{00}^2 & \star \\ \hline l_{01}\lambda_{00} & l_{01}l_{01}^T + L_{11}L_{11}^T \end{array}\right)$$

Since B is a 2×2 matrix we have:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \left(\begin{array}{c|c} \lambda_{00}^2 & \star \\ \hline \lambda_{01}\lambda_{00} & \lambda_{01}^2 + \lambda_{11}^2 \end{array}\right)$$

$$B = \left(\begin{array}{c|c} \lambda_{00} = \sqrt{2} & \star \\ \hline \lambda_{01} = \frac{1}{\sqrt{2}} & \lambda_{11} = \sqrt{\frac{3}{2}} \end{array}\right)$$

Therefore $L = \begin{pmatrix} \sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} \end{pmatrix}$

Since $B = LL^T$ we have:

$$B = \begin{pmatrix} \sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} \end{pmatrix}^T$$

We can then resolve:

$A^T b = B\hat{x}$ with $\hat{x} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}$, we solve $A^T b = LL^T\hat{x}$

$$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}\gamma_0\\\gamma_1\end{pmatrix}$$
$$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}\sqrt{2} & 0\\\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}}\end{pmatrix}^T\begin{pmatrix}\gamma_0\\\gamma_1\end{pmatrix}$$

We first resolve $L\tilde{x} = A^T b$ with $\tilde{x} = L^T\hat{x}$

$$\begin{pmatrix}\sqrt{2} & 0\\\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}}\end{pmatrix}\begin{pmatrix}\Gamma_0\\\Gamma_1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

By Forward substitution we get $\Gamma_0 = \frac{1}{\sqrt{2}}$ hence

$\Gamma_1 = \frac{1-\frac{1}{2}}{\sqrt{\frac{3}{2}}} = \frac{\sqrt{2}}{2\sqrt{3}}$ so get $\tilde{x} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{\sqrt{2}}{2\sqrt{3}}\end{pmatrix}$

But we know that $\tilde{x} = L^T\hat{x}$ so we just need to resolve:

$$\begin{pmatrix}\sqrt{2} & \frac{1}{\sqrt{2}}\\0 & \sqrt{\frac{3}{2}}\end{pmatrix}\begin{pmatrix}\gamma_0\\\gamma_1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{\sqrt{2}}{2\sqrt{3}}\end{pmatrix}$$

$\gamma_1 = \frac{\frac{\sqrt{2}}{2\sqrt{3}}}{\sqrt{\frac{3}{2}}} = \frac{2}{2\times 3} = \frac{1}{3}$ and $\gamma_0 = \frac{\frac{1}{\sqrt{2}} - \frac{1}{3}\times\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{2}{6} = \frac{1}{3}$

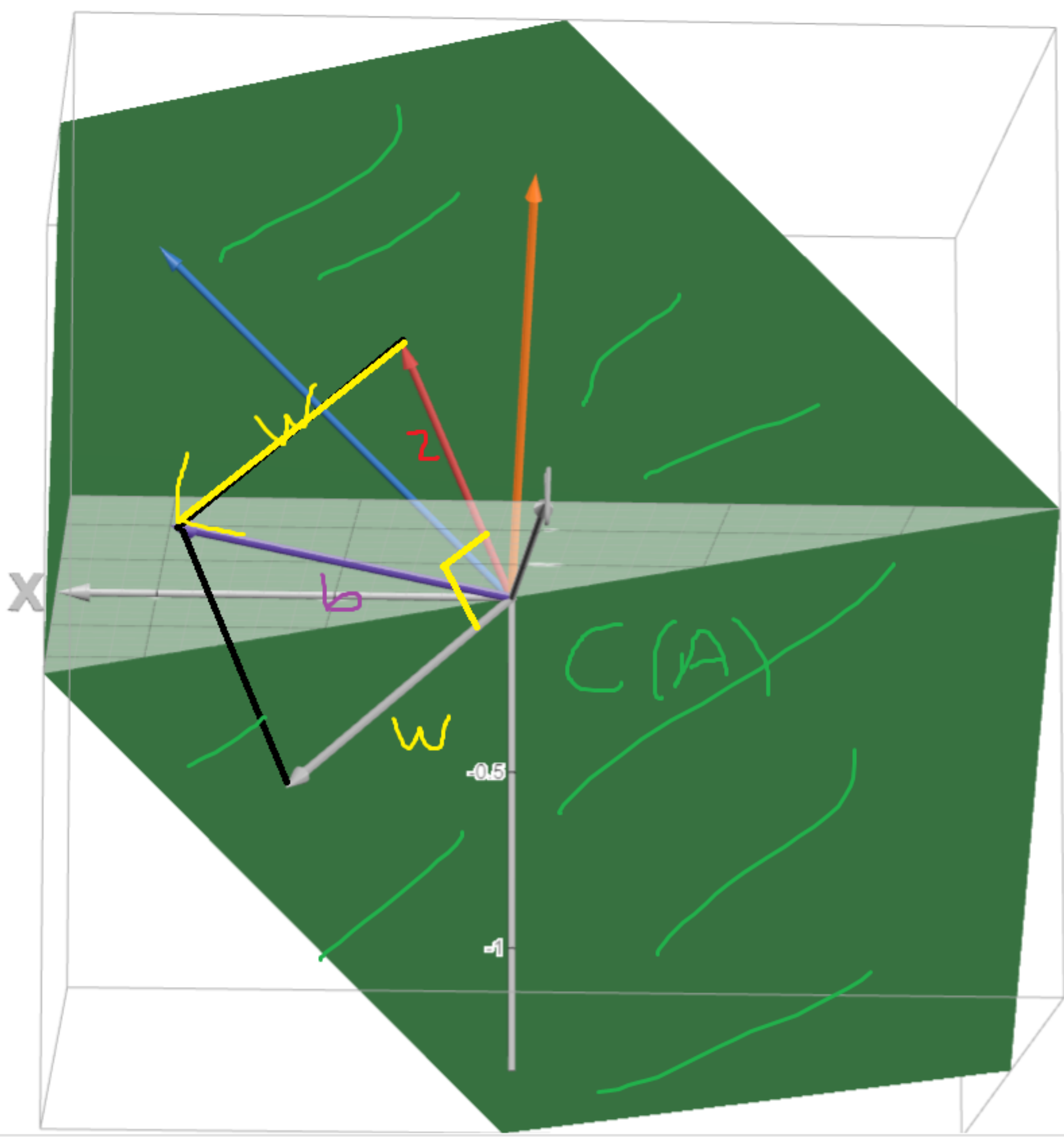
Consequently $\hat{x} = \begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\end{pmatrix}$

Then our vector $z = A\hat{x}$:

$$z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

We can retrieve our vector w since we know that $w = b - z$:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$



Find the line, $y = \gamma_0 + \gamma_1 x$, that best fits the following data:

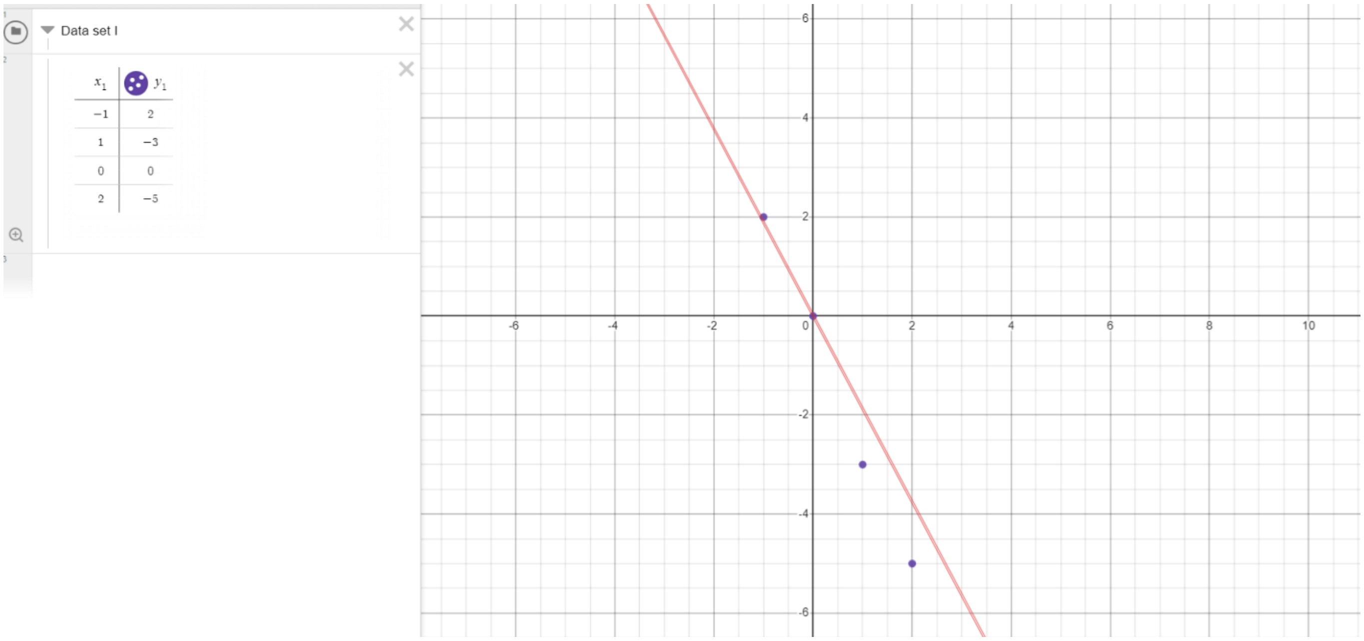
x	y
-1	2
1	-3
0	0
2	-5

We can represent this problem by:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \\ -5 \end{pmatrix}$$

Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}$, $x = \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ -3 \\ 0 \\ -5 \end{pmatrix}$

We will notice that $b \notin \mathcal{C}(A)$ so the equation $Ax = b$ doesn't have any solution. If we plug the points in a x-y plane we get this.



We clearly that the line can't get through all the points since they aren't aligned.

We will try then to find an approximate solution.

Our approximate solution will be a vector z that is the projection of the vector b into the column space of A. It means that we can write our vector $b$as follow:

$b = z + w$ but since z is the projection of b into the column space of A we know that w is orthogonal to z. Hence $w^T z = 0$

We know then that $w \in \mathcal{C}(A)^\perp$, so $w \in \mathcal{N}(A^T)$. We know then that we have:
$A^T w = 0$
$A^T b - A^T z = 0$

Let $z = A\hat{x}$ with $\hat{x}$ the approximate solution. We have then:

$$A^T b = A^T A \hat{x}$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} -6 \\ -15 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$$

$\mathcal{N}(A^T A) = \{0\}$ hence $A^T A$ is nonsingular so we can write $\hat{x}$ as follow:

$$\hat{x} = A^\dagger b$$

$$(A^T A)^{-1} = \begin{pmatrix} \frac{3}{10} & \frac{-1}{10} \\ \frac{-1}{10} & \frac{2}{10} \end{pmatrix}$$

Hence $A^\dagger b$:

$$\begin{pmatrix} \frac{3}{10} & \frac{-1}{10} \\ \frac{-1}{10} & \frac{2}{10} \end{pmatrix} \begin{pmatrix} -6 \\ -15 \end{pmatrix} = \begin{pmatrix} \frac{-3}{10} \\ \frac{-12}{5} \end{pmatrix}$$

So $\hat{x} = \begin{pmatrix} \frac{-3}{10} \\ \frac{-12}{5} \end{pmatrix}$

We can then write our line as $y = -\frac{3}{10} - \frac{12}{5}x$