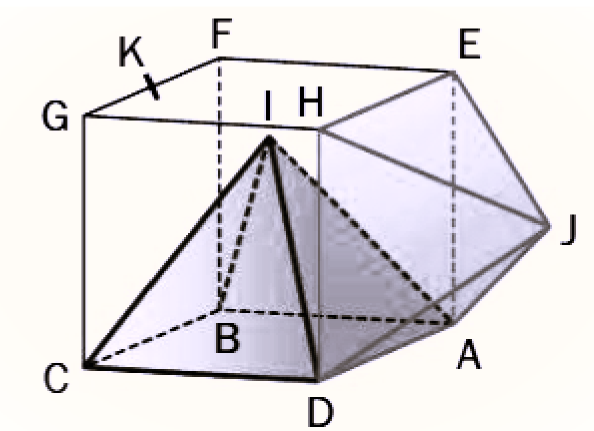
Pure Vector Geometry in Space

$ABCDEFGH$ is a cube with a edge length of a. We build inside this cube a pyramid $IABCD$ and outside a pyramid $JADHE$, identical from the first one where the triangular faces are isosceles triangles. We place the point K the midpoint of the segment $[GF]$

Determine the height of the pyramid that makes the points K, I and J aligned.



Solution:

To be able to demonstrate that the points K, I and J are aligned we need to show that it exits a real α such that $\vec{KI} = \alpha \vec{KJ}$

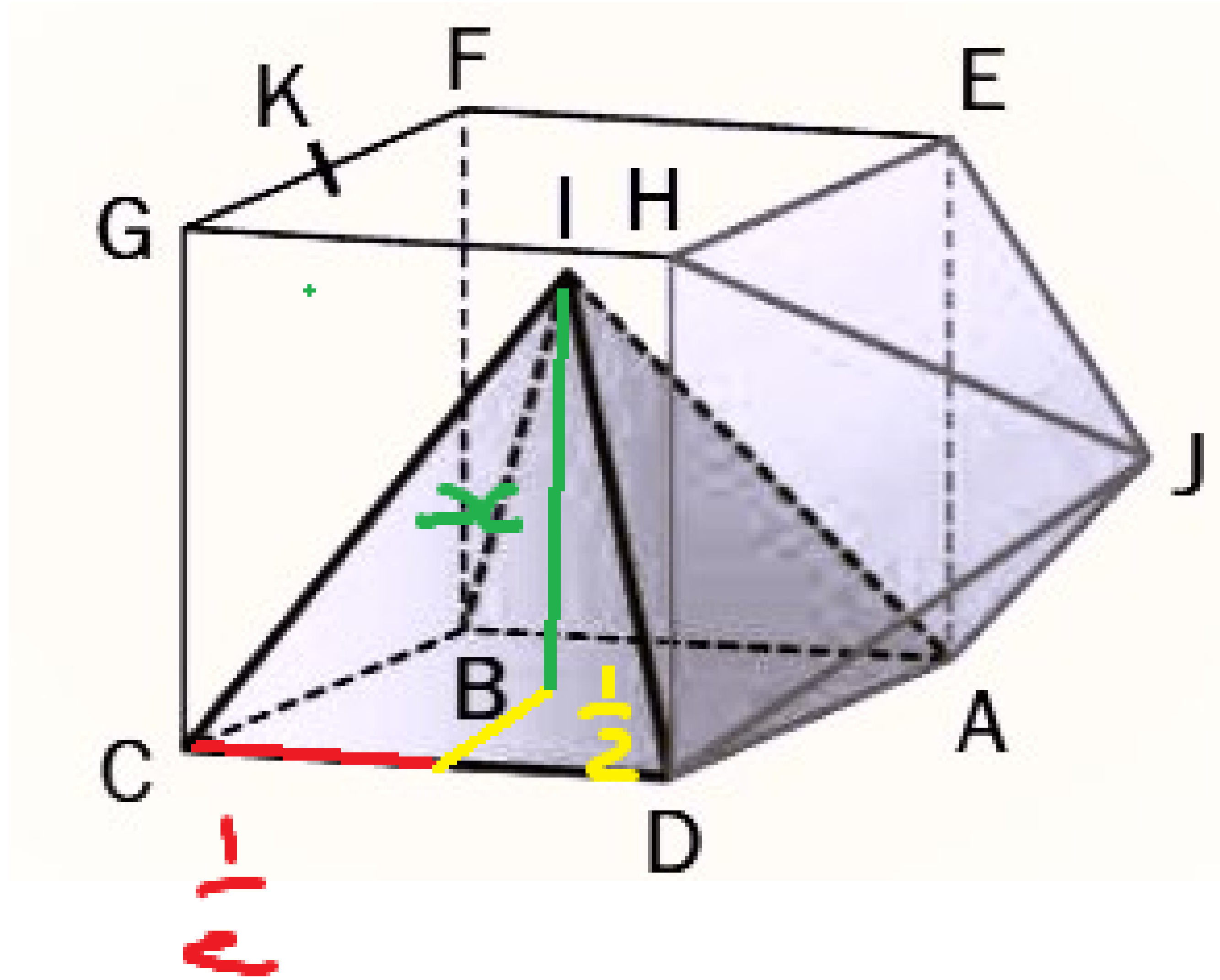
To do this in the simplest way we will do that with coordinates to then do a system to find the real α.

We first can determine a basis to use coordinates: $(A; \vec{CD}, \vec{CG}, \vec{CB})$

We know that K is the midpoint of $[GF]$ and $[GF] = [CB]$ hence using our base we have:

$$K \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

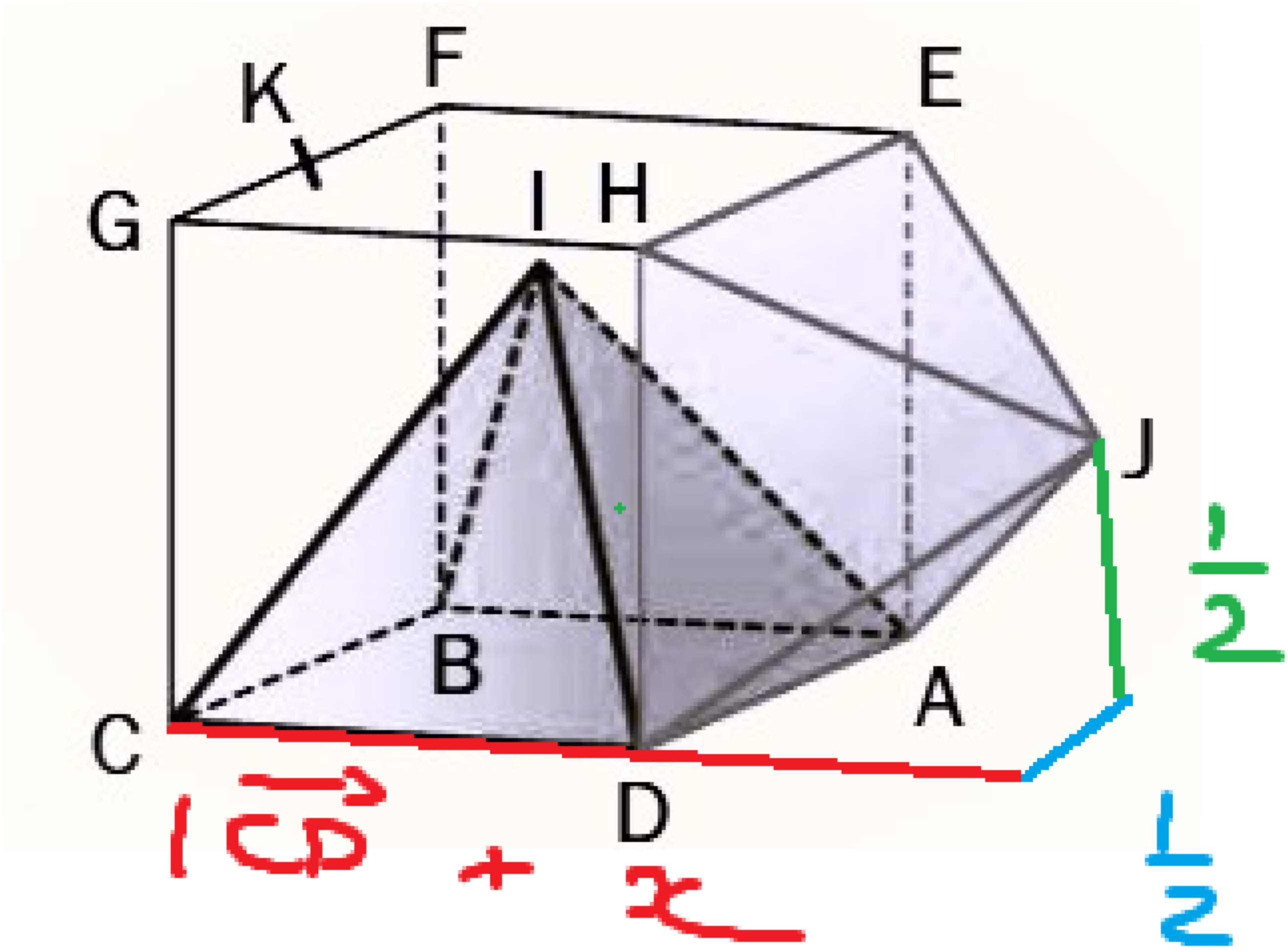
We know that the pyramid $IABCD$ of base $ABCD$ has isosceles triangular faces so we can deduce that the height is place in the intersect between the 2 diagonals of $ABCD$:



Hence using our bases we can determine the coordinates of I. $x \in \mathbb{R}$ and it's the height of the pyramid.

$$I\begin{pmatrix}\frac{1}{2}\\ x\\ \frac{1}{2}\end{pmatrix}$$

We can do the same method to determine the coordinates of J:



So:

$$J\begin{pmatrix}1+x\\ \frac{1}{2}\\ \frac{1}{2}\end{pmatrix}$$

Now that we have the coordinates of our points we can determine $\vec{KI}$ and $\vec{KJ}$:

$$\vec{KI}\begin{pmatrix}\frac{1}{2}\\ x-1\\ 0\end{pmatrix},\ \vec{KJ}\begin{pmatrix}1+x\\ -\frac{1}{2}\\ 0\end{pmatrix}$$

We create our system:

$$\begin{cases} \alpha & + & \alpha x & = & \frac{1}{2} & (L_1) \\ & - & \frac{1}{2}\alpha & = & x-1 & (L_2) \\ & & 0 & = & 0 & (L_3) \end{cases}$$

(L_3) is consistent so we can reduce our system as:

$$\begin{cases} \alpha & + & \alpha x & = & \frac{1}{2} & (L_1) \\ & - & \frac{1}{2}\alpha & = & x-1 & (L_2) \end{cases}$$

We can directly do it by substitution:

Then we have $\alpha = \frac{x-1}{-\frac{1}{2}} = 2(1-x)$

We substitue in (L_1)

$$2 - 2x + 2(1-x)x = \frac{1}{2}$$
$$-2x + 2(x - x^2) = -\frac{3}{2}$$
$$x^2 = \frac{3}{4}$$
$$x \pm \frac{\sqrt{3}}{2}$$

Thereby, The height of the pyramid is $\frac{\sqrt{3}}{2}$