Mathematical Induction

Exercise 1

Let $n \geq 1$. Prove that $\sum_{j=1}^{n}(-1)^{j-1}j^2 = (-1)^{n-1}\frac{n(n+1)}{2}$.

Base case:

For $n = 1$.

$$\sum_{j=1}^{1}(-1)^{1-1}1^2 = 1 = (-1)^{1-1}\frac{1(1+1)}{2} = 1\frac{2}{2} = 1$$

Inductive steps (Inductive Hypothesis) Assume that the result is true for $n = k$ where $k \geq 1$:

$$\sum_{j=1}^{k}(-1)^{j-1}1^2 = (-1)^{k-1}\frac{k(k+1)}{2}$$

Now let's assume that the result is also true for $n = k + 1$ where $k \geq 1$.

$$\sum_{j=1}^{k+1}(-1)^{j-1}j^2 = (-1)^{k}\frac{(k+1)(k+2)}{2}$$

Assume that $k \geq 1$ Then:

$$\sum_{j=1}^{k+1}(-1)^{j-1}j^2$$

$=$

$$\sum_{j=1}^{k}(-1)^{j-1}j^2 + (-1)^{k}(k+1)^2$$

$=$

$$(-1)^{k-1}\frac{k(k+1)}{2} + (-1)^{k}(k+1)^2$$

$=$

$$(-1)^k\left((k+1)^2+(-1)^{-1}\frac{k(k+1)}{2}\right)$$

$=$

$$(-1)^k\left(\frac{2(k+1)^2+(-k)(k+1)}{2}\right)$$

$=$

$$(-1)^k\frac{(k+1)(2(k+1)-k)}{2}$$

$=$

$$(-1)^k\frac{(k+1)(2k+2-k)}{2}$$

$=$

$$(-1)^k\frac{(k+1)(k+2)}{2}$$

By Principle of Mathematical induction this equation $\sum_{j=1}^{k+1}(-1)^{j-1}j^2=(-1)^k\frac{(k+1)(k+2)}{2}$ holds $\forall n\in\mathbb{N}$

Exercise 2

Let $n\geq 1$, $a\in\mathbb{R}$, $a^n=(a-1)(a^{n-1}+a^{n-2}+a^{n-3}+\cdots+a+1)$.

Base case for $n=1$:

$$a^1-1=(a-1)(a^{1-1})=(a-1)\times 1=a-1$$

Let's assume that the result is also true for $n=k$ where $k\geq 1$:

$$a^k-1=(a-1)(a^{k-1}+a^{k-2}+\cdots+a+1)$$

Now let's show that the result is also true $n=k+1$ where $k\geq 1$:

$$a^{k+1}-1=(a-1)(a^k+a^{k-1}+a^{k-2}\cdots+a+1)$$

Then we have:

$$a^{k+1}-1$$

$=$

$$a^k-1+(a-1)a^k$$

$=$

$$(a-1)(a^{k-1}+a^{k-2}+\cdots+a+1)+(a-1)a^k$$

$=$

$$(a-1)(a^k+a^{k-1}+a^{k-2}\cdots+a+1)$$

By the principle of mathematical induction this result holds $\forall n \in \mathbb{N}$.

Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

1. Show that $A^2 - 2A + I = 0$ with $I \in \mathbb{R}^2, 0 \in \mathbb{R}^{2\times 2}$
2. Deduce that A^{-1} exists and determine it
3. Show for all $n \in \mathbb{N}$ with $n \geq 1$ that $A^n = nA - (n-1)I$
4. $A^2 - 2A + I$

   $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} - 2\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
   $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} 4 & -2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
   $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
   $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
   So $A^2 - 2A + I = 0$
5. $A^2 - 2A = -I$
   $A(A - 2I) = -I$
   $A(-A + 2I) = I$
   Then it exists a matrix B such that $AB = I$ and here $B = -A + 2I$
   $-\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
   $= \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

   Then $A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$
6. Let $n \in \mathbb{N}$ with $n \geq 1$, Then we will prove that $A^n = nA - (n-1)I$

Base case for $n = 1$:
$A^1 = A$
$A - (1-1)I = A$
Then $A^n = nA - (n-1)I$ is true for $n = 1$

Inductive Step:

Let's assume that the result is also true for $n = k$ where $k \geq 1, k \in \mathbb{N}$:

Then we have $A^k = kA - (k-1)I$

Now let's show that the result is also true $n = k+1$ where $k \geq 1$:
Then we will show that $A^{k+1} = (k+1)A - kI$

$A^k = kA - (k-1)I$
$AA^n = A[kA - (k-1)I]$
$A^{n+1} = kA^2 - Ak + A$

But we know from above that $A^2 - 2A + I = 0$, Hence $A^2 = 2A - I$. We substitue in our expression:

$A^{k+1} = k(2A - I) - Ak + A$
$A^{k+1} = k2A - kI - Ak + A$
$A^{k+1} = Ak - kI + A$

We factorize:

$(n+1)A - kI$

Then, by the principle of mathematical induction the results hold true $\forall n \in \mathbb{N}$.