# MIN-PLUS ALGEBRA AND GRAPH DOMINATION

by

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Min-Plus Algebra and Graph Domination

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ABSTRACT


This thesis describes the Min-Plus algebra and uses Min-Plus algebra to implement dynamic programming algorithms. Min-plus algebra is defined by replacing the standard algebra operators of addition and multiplication by minimization and addition respectively. There are several applications which can be solved using these dynamic programming algorithms in Min-Plus algebra. We will illustrate these algorithms by using them to solve the following problems involving domination.

1 A domination $D$ of a graph $G$ is a subset of vertices so that each vertex of $G$ is either in $D$ or adjacent to a vertex in $D$. Let $\gamma(G)$ (the domination number of $G$) be the minimum size of a domination of $G$. We find the domination number of a large class of graphs including the cartesian product graphs and circulant graphs.

3 From chess, a knight move is two squares in one direction and one square in the perpendicular direction. A Knight domination of a $k \times r$ board

is a placement of Knights so each square either has a Knight on it or attacking it. The $k \times r$ **Knight domination number** is the minimum number of Knights in a Knight domination of a $k \times r$ board. We find the the Knight domination number of a $k \times r$ chessboard for $k \leq 7$ and for all $r$.

The defining feature of these dynamic programming algorithms is that they find infinitely many solutions in a finite amount of time. This is possible because the algorithms are periodic after a finite number of iterations. We show that periodicity is dependent on the size and the entries in the state transition matrix, and we determine the necessary conditions on the matrix for periodicity to occur. We also look at random matrices with entries independently chosen from discrete and continuous distributions and find the expected number of iterations needed for periodicity in each case. The results of this analysis shows why the algorithms are successful in solving the graph domination problems and what conditions are necessary to use similar dynamic programming algorithms in other problems.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ______________________

David C. Fisher

# DEDICATION

I would like to dedicate this thesis to my parents and my family who have always stood behind me supporting me and driving me forward; to Ray who has always stood in front of me and shown me the way; and to Steve, who always encouraged me to stretch my limits... to ride faster, climb higher and play harder. Thank you.

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I would like to thank my advisor, David C. Fisher, for his encouragement in preparation of this thesis, his support, guidance, friendship, and time spent in discussions throughout my studies at the University of Colorado at Denver.

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# 1. Introduction

This thesis describes the Min-Plus algebra and uses Min-Plus algebra to implement dynamic programming algorithms. Min-plus algebra is defined by two operators $\boxtimes$ and $\boxplus$ on the real numbers. The symbol $\boxtimes$ represents standard addition and the symbol $\boxplus$ represents minimization. In Chapter 2 we will look at several properties of Min-Plus algebra. Using the standard algebra properties, we develop analogues to linear algebra including eigenvalues and eigenvectors. Eigenvalues are used in the implementation of dynamic programming algorithms in Min-Plus algebra. We will discuss two methods of finding eigenvalues and we will show how their existence ensures finite time execution of the dynamic programming algorithms.

There are several applications which can be solved using dynamic programming algorithms in Min-Plus algebra. We will illustrate these algorithms by using them to solve problems in graph domination.

**Definition 1.1** *A* domination *$D$ of a graph $G$ is a subset of vertices so that each vertex of $G$ is either in $D$ or adjacent to a vertex in $D$. Let $\gamma(G)$ (the* domination number *of $G$) be the minimum size of a domination of $G$.*

The domination problem is NP-hard. Most algorithms for finding the domination number of a graph involve using Branch and Bound techniques to reduce the graph to other graphs for which the domination number is known. We develop a dynamic programming algorithm to find the domination number of large classes of graphs. The first set of graphs we consider are the cartesian product graphs.

**Definition 1.2** *Let $G = (V, E)$ be a graph with vertices $\{v_1, v_2, \ldots v_n\}$ and let $H = (V, E)$ be a graph with vertices $\{w_1, w_2 \ldots w_\ell\}$. Then the* cartesian product *$G \times H$ has vertices of the form $(v_i, w_j)$, with $1 \leq i \leq n$ and $1 \leq j \leq \ell$. The edges of $G \times H$ have the form $((v_i, w_j), (v_p, w_q))$ if $i = p$ and $(w_j, w_q)$ is and edge of $H$, or if $j = q$ and $(v_i, v_p)$ is an edge of $G$. The $k^{th}$ copy of $G$ in $G \times H$ is $G_k$ and its vertices are $(v_1, w_k), (v_2, w_k), \ldots (v_n, w_k)$.*

In Chapter 3, we consider cartesian product graphs with $H = P_r$ where $P_r$ is the path on $r$ vertices with vertix $w_i$ adjacent to vertex $w_k$ if and only if $|i - k| = 1$.

We begin by looking at a special cartesian product graph called the complete grid graph. The complete grid graph on $n, r$ vertices, $P_n \times P_r$ has $nr$ vertices with vertex $(v_i, w_j)$ adjacent to vertex $(v_k, w_l)$ if and only if $|i-k|+|j-l| = 1$. The domination number of complete grid graphs $\gamma(P_n \times P_r)$ has been analyzed for small values of $n$. Jacobson and Kinch [13] found results for $n \leq 4$, and Chang and Clark [2] found results for $n = 5, 6$. In [11] and [15] Branch and

Bound algorithms are used to find domination numbers for larger values of $n$ and some values of $r$. These results were then expanded by the use of dynamic programming algorithms [11]. We improve these algorithms by incorporating the periodic property of Min-Plus algebra. We develop two algorithms, the Exact Algorithm and the At Least Algorithm and are able to find the domination number for all grid graphs $(P_n \times P_r)$ with $n \leq 19$ and for all values of $r$. Livingston and Stout [14] use an algorithm which we will call the Exact Algorithm to find the domination number of general cartesian product graphs $(G \times P_r)$. We use the faster At Least Algorithm to find the domination number of graphs $(G \times P_r)$ for any graph $G$ on $n \leq 6$ vertices and for all $r$.

In Chapter 4, we modify the Exact Algorithm to find the domination number of circulant graphs.

**Definition 1.3** *Given an integer $r > 0$ and subset $S$ of the positive integers, the* circulant graph *$C_r[S]$ is a graph on vertices 0, 1, $\ldots$, $r-1$ with vertex $i$ adjacent to vertex $j$ if $i - j \in S$ or $j - i \in S$ (mod $r$).*

We find the domination numbers of all circulant graphs $C_r[S]$ for all $r$ and for any subset $S$ of the set $\{1, 2, 3, \ldots, 9\}$. The algorithms developed in Chapters 3 and 4 can be modified to solve many types of domination problems. To illustrate this, we solve the knight domination problem for rectangular chessboards.

**Definition 1.4** *A* Knight domination *of a $k \times r$ board is a placement of Knights* *so each square either has a Knight on it or attacking it. The $k \times r$* Knight domination number *is the minimum number of Knights in a Knight domination of a $k \times r$ board.*

The knight domination problem is a variation of the domination problem and wide interest in this problem was first generated by Gardner [9] for square chessboards. Hare and Hedetniemi [11] found the domination numbers for $k \times r$ chessboards with $k \leq 6$ and some values of $r$. We are able to verify these results and find the knight domination number for all $k \times r$ chessboards with $k \leq 7$ and for all $r$.

The defining feature of the applications in Chapters 3-5 is that the dynamic programming algorithms are able to find infinitely many solutions in a finite amount of time. This is possible because the algorithms are periodic after a finite number of iterations. The periodicity of the algorithms can be described using the eigenvalues of a state transition matrix. In Chapter 6, we show that periodicity is dependent on the size and the entries in the state transition matrix. First, we determine the necessary conditions on the matrix for periodicity to occur. Then for special cases, we find exact bounds on the number of iterations of the algorithm that are needed before periodicity occurs. Next we look at random matrices with entries independently chosen from discrete and continuous distributions and find the expected number of iterations needed for periodicity in each case. The results of this analysis shows why the algorithms are successful in solving the graph domination problems in Chapters 3-5 and what conditions are necessary to solve other types of problems with similar algorithms.

## 2. Properties of Min-Plus

In this chapter we will define the Min-Plus algebra and look at several of its properties. The majority of this material is derived from Baccelli *et al.* [1] Min-plus algebra is defined by two operators $\boxtimes$ and $\boxplus$ on the real numbers. The symbol $\boxtimes$ represents standard addition and the symbol $\boxplus$ represents minimization. An identity $e$ for $\boxplus$ satisfies $\min(a, e) = a \boxplus e = a$ for all $a$. Since $\min(a, e) = a$ only if $e \geq a$, we must have $e \geq a$ for all $a$. So to have an identity for $\boxplus$ in an algebra which includes all real numbers, we must append $\infty$ to the real numbers. Let $\Re_+ = \Re \cup \{\infty\}$. We must then define the operations for $\infty$: let $\infty \boxplus a = a \boxplus \infty = a$ and $\infty \boxtimes a = a \boxtimes \infty = \infty$ for all $a \in \Re_+$. Then these properties hold for all $a, b, c \in \Re_+$.

- (Commutative) $a \boxplus b = b \boxplus a$ (*i.e.*, $\min(a, b) = \min(b, a)$),

  and $a \boxtimes b = b \boxtimes a$ (*i.e.*, $a + b = b + a$);
- (Associative) $a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c$

  (*i.e.*,$\min(a, \min(b, c)) = \min(\min(a, b), c)$),

  and $a \boxtimes (b \boxtimes c) = (a \boxtimes b) \boxtimes c$, (*i.e.*, $a + (b + c) = (a + b) + c$);
- (Distributive) $a \boxtimes (b \boxplus c) = (a \boxtimes b) \boxplus (a \boxtimes c)$

  (*i.e.*, $a + \min(b, c) = \min(a + b, a + c)$);

- (Identity) $\infty \boxplus a = a$ (*i.e.*, $\min(\infty, a) = a$),

  and $0 \boxtimes a = a$ (*i.e.*, $0 + a = a$).

- (Inverses under $\boxtimes$) $(-a) \boxtimes a = 0$ (*i.e.*, $-a + a = 0$).

This would be a field if there were inverses under $\boxplus$. But $a \boxplus x = \infty$ (*i.e.*, $\min(a, x) = \infty$) has no solution for $x$ unless $a = \infty$.

## 2.1 Min-Plus Linear Algebra

Several of the applications presented in Chapters 3-5 use linear algebra operations within the Min-Plus algebra. Using the properties on real numbers defined above we can define analogues to standard matrix algebra concepts.

**Matrix addition** is defined as componentwise minimization. For example:

$$\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \boxplus \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 \boxplus 1 & 2 \boxplus 5 \\ 2 \boxplus 3 & 4 \boxplus 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}.$$

**Matrix multiplication** is denoted by

$$(A \boxtimes B)_{ij} = \boxplus_{k=1}^{n} A_{ik} \boxtimes B_{kj};$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \boxtimes \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} (3 \boxtimes 1) \boxplus (2 \boxtimes 3) & (3 \boxtimes 5) \boxplus (2 \boxtimes 2) \\ (2 \boxtimes 1) \boxplus (4 \boxtimes 3) & (2 \boxtimes 5) \boxplus (4 \boxtimes 2) \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 3 & 6 \end{bmatrix}.$$

**The "zero" matrix** $(A \boxplus \infty = \infty \boxplus A = A)$ is:

$$Z = \infty = \begin{bmatrix} \infty & \infty & \infty & \dots & \infty \\ \infty & \infty & \infty & \dots & \infty \\ \infty & \infty & \infty & \dots & \infty \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \infty & \dots & \infty \end{bmatrix}.$$

**The "identity" matrix** $(A \boxtimes I = I \boxtimes A = A)$ is:

$$I = \begin{bmatrix} 0 & \infty & \infty & \dots & \infty \\ \infty & 0 & \infty & \dots & \infty \\ \infty & \infty & 0 & \dots & \infty \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \infty & \dots & 0 \end{bmatrix}.$$

**Matrix Exponents.** Let $A$ be a square matrix. Let $A^1 = A$, and recursively define $A^r = A \boxtimes A^{r-1}$ for all $k > 1$. Associativity ensures that $A^r = A^{r-i} \boxtimes A^i$ for all $i = 1, 2, \dots, r-1$.

**Scalar multiplication.** The $i, j$ entry of $k \boxtimes A$ is $(k \boxtimes A)_{i,j} = k \boxtimes A_{i,j}$.

**Trace.** For a square matrix $A$, let trace$(A)$ be the minimum of the diagonal entries of $A$.

The properties above show that matrix multiplication is associative. Since there is no inverse for $\boxplus$, matrix inverses are not defined except in special cases. There are also analogues to solving linear systems, eigenvectors, eigenvalues, characteristic equation, and even a Cayley-Hamilton theorem. We describe only those properties that are used in the applications presented.

## 2.2 Eigenvalues and Eigenvectors: The Power Method

Several applications of Min-Plus algebra use dynamic programming algorithms. The defining property of these algorithms is that they use periodicity to find infinitely many solutions in finite time. Periodicity of matrices in Min-Plus algebra is defined in terms of eigenvalues and eigenvectors. In this section we will look at the definition of eigenvalues and eigenvectors, and the Power Method of finding them. To illustrate this method, consider the following example.

> A country has 4 cities with overnight trains between them. The overnight rates can be represented as a matrix $A$ where $A_{ij}$ is the cost to start in city $i$ and take the overnight train to city $j$. Below is the matrix $A$ with the costs to stay in each city and for overnight train rides between them. *Find the cheapest way to stay $n$ nights starting at city $i$ and ending at city $j$.*
>
> $$A = \begin{bmatrix} 8 & 0 & 2 & 1 \\ 7 & 3 & 3 & 6 \\ 4 & 5 & 5 & 6 \\ 3 & 3 & 7 & 9 \end{bmatrix}.$$

For the cost matrix $A$, $(A^2)_{ij}$ represents the minimum cost to start at city $i$ , travel for 2 nights, and end in city $j$. Likewise, $(A^r)_{ij}$ represents the minimum cost for a trip of $r$ nights. Next, look at various matrix powers of the cost matrix $A$.

$$A = \begin{bmatrix} 8 & 0 & 2 & 1 \\ 7 & 3 & 3 & 6 \\ 4 & 5 & 5 & 6 \\ 3 & 3 & 7 & 9 \end{bmatrix}, \; A^2 = \begin{bmatrix} 4 & 3 & 3 & 6 \\ 7 & 6 & 6 & 8 \\ 9 & 4 & 6 & 5 \\ 10 & 3 & 5 & 4 \end{bmatrix}, \; A^3 = \begin{bmatrix} 7 & 4 & 6 & 5 \\ 10 & 7 & 9 & 8 \\ 8 & 7 & 7 & 10 \\ 7 & 6 & 6 & 9 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 8 & 7 & 7 & 8 \\ 11 & 10 & 10 & 11 \\ 11 & 8 & 10 & 9 \\ 10 & 7 & 9 & 8 \end{bmatrix}, \; A^5 = \begin{bmatrix} 11 & 8 & 10 & 9 \\ 14 & 11 & 13 & 12 \\ 12 & 11 & 11 & 12 \\ 11 & 10 & 10 & 11 \end{bmatrix}, \; A^6 = \begin{bmatrix} 12 & 11 & 11 & 12 \\ 15 & 14 & 14 & 15 \\ 15 & 12 & 14 & 13 \\ 14 & 11 & 13 & 12 \end{bmatrix}.$$

Note that $A^6 = 4 \boxtimes A^4$ (under "scalar addition"). Then

$$A^7 = A^6 \boxtimes A = (4 \boxtimes A^4) \boxtimes A = 4 \boxtimes (A^4 \boxtimes A) = 4 \boxtimes A^5,$$

$$A^8 = A^7 \boxtimes A = (4 \boxtimes A^5) \boxtimes A = 4 \boxtimes (A^5 \boxtimes A) = 4 \boxtimes A^6,$$

$$A^9 = A^8 \boxtimes A = (4 \boxtimes A^6) \boxtimes A = 4 \boxtimes (A^6 \boxtimes A) = 4 \boxtimes A^7.$$

By induction, we can then show that for all $r \geq 6$, we have $A^r = 4 \boxtimes A^{r-2}$ and hence

$$A^r = \begin{cases} 2(r-4) \boxtimes A^4 & \text{if } r \text{ is even} \\ 2(r-5) \boxtimes A^5 & \text{if } r \text{ is odd.} \end{cases}$$

So for example, the minimum cost of start in city 3 and ending in city 2 while staying 13 nights is $2(13-5)+11 = 27$. We will now show how this relates to the eigenvalues and eigenvectors of $A$.

**Definition 2.1** *Given an $m \times m$ matrix $A$ in $\Re_+$, let $\lambda$ be an eigenvalue and $\mathbf{x} \neq \infty$ be the associated eigenvector if $A \boxtimes \mathbf{x} = \lambda \boxtimes \mathbf{x}$. For example, let*

$$A = \begin{bmatrix} 4 & 3 & 4 \\ 2 & 4 & 3 \\ 1 & 3 & 3 \end{bmatrix}.$$

*Then $\lambda = \frac{7}{3}$ is an eigenvalue with associated eigenvector $\mathbf{x} = (\frac{4}{3}, \frac{2}{3}, 0)^T$, because*

$$A \boxtimes \mathbf{x} = \begin{bmatrix} 4 & 3 & 4 \\ 2 & 4 & 3 \\ 1 & 3 & 3 \end{bmatrix} \boxtimes \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{11}{3} \\ 3 \\ \frac{7}{3} \end{pmatrix} = \frac{7}{3} \boxtimes \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} = \lambda \boxtimes \mathbf{x}$$

This leads us to the Power Method for finding the eigenvalues of a matrix. In our example we found that for large enough $r$, we have $A^r = 2 \boxtimes A^{r-1}$. Now, let $\mathbf{x}_0$ be a vector and define $\mathbf{x}_r = A \boxtimes \mathbf{x}_{r-1}$ for $r > 0$, then

$$x_r = A^r \boxtimes x_0 = 2 \boxtimes A^{r-1} \boxtimes \mathbf{x}_0 = 2 \boxtimes x_{r-1}.$$

In general, suppose that there is a $k$, $p > 0$, and $q$ where

$$\mathbf{x}_k = q \boxtimes \mathbf{x}_{k-p}.$$

Let $\lambda = q/p$ and let

$$\mathbf{x}^* = \mathbf{x}_{k-1} \boxplus (\lambda \boxtimes \mathbf{x}_{k-2}) \boxplus (\lambda^2 \boxtimes \mathbf{x}_{k-3}) \boxplus \cdots \boxplus (\lambda^{p-1} \boxtimes \mathbf{x}_{k-p}).$$

Then

$$A \boxtimes \mathbf{x}^* = A \boxtimes (\mathbf{x}_{k-1} \boxplus (\lambda \boxtimes \mathbf{x}_{k-2}) \boxplus (\lambda^2 \boxtimes \mathbf{x}_{k-3}) \boxplus \cdots \boxplus (\lambda^{p-1} \boxtimes \mathbf{x}_{k-p}))$$

$$= \mathbf{x}_k \boxplus (\lambda \boxtimes \mathbf{x}_{k-1}) \boxplus (\lambda^2 \boxtimes \mathbf{x}_{k-2}) \boxplus \cdots \boxplus (\lambda^{p-1} \boxtimes \mathbf{x}_{k-p+1}))$$

$$= (\lambda^p \boxtimes \mathbf{x}_{k-p} \boxplus (\lambda \boxtimes \mathbf{x}_{k-1}) \boxplus (\lambda^2 \boxtimes \mathbf{x}_{k-2}) \boxplus \cdots \boxplus (\lambda^{p-1} \boxtimes \mathbf{x}_{k-p+1}))$$

$$= \lambda \boxtimes (\mathbf{x}_{k-1} \boxplus (\lambda \boxtimes \mathbf{x}_{k-2}) \boxplus \cdots \boxplus ((p-2)\lambda \boxtimes \mathbf{x}_{k-p+1}) \boxplus (\lambda^{p-1} \boxtimes \mathbf{x}_{k-p}))$$

$$= \lambda \boxtimes \mathbf{x}^*.$$

So $\lambda$ is an eigenvalue with associated eigenvector $\mathbf{x}^*$. In our example, we had

$$A^4 = \begin{bmatrix} 8 & 7 & 7 & 8 \\ 11 & 10 & 10 & 11 \\ 11 & 8 & 10 & 9 \\ 10 & 7 & 9 & 8 \end{bmatrix}, A^5 = \begin{bmatrix} 11 & 8 & 10 & 9 \\ 14 & 11 & 13 & 12 \\ 12 & 11 & 11 & 12 \\ 11 & 10 & 10 & 11 \end{bmatrix},$$

and $A^r = 4 \boxtimes A^{r-2}$ for all $r \geq 6$.

Thus the eigenvalue is $\lambda = 4/2 = 2$ with associated eigenvector

$$\mathbf{x} = \left(2 \otimes (A^4)_1\right) \oplus (A^5)_1 = \begin{pmatrix} 2+8 \\ 2+11 \\ 2+11 \\ 2+10 \end{pmatrix} \oplus \begin{pmatrix} 11 \\ 14 \\ 12 \\ 11 \end{pmatrix} = \begin{pmatrix} 10 \\ 13 \\ 12 \\ 11 \end{pmatrix}.$$

Then

$$A \boxtimes \mathbf{x} = \begin{bmatrix} 8 & 0 & 2 & 1 \\ 7 & 3 & 3 & 6 \\ 4 & 5 & 5 & 6 \\ 3 & 3 & 7 & 9 \end{bmatrix} \boxtimes \begin{pmatrix} 10 \\ 13 \\ 12 \\ 11 \end{pmatrix} = \begin{pmatrix} 12 \\ 15 \\ 14 \\ 13 \end{pmatrix} = 2 \times \begin{pmatrix} 10 \\ 13 \\ 12 \\ 11 \end{pmatrix}.$$

In the next chapters, we will look at several application of Min-Plus algebra. In each application, the solution relies on periodicity of powers of a matrix $A$. In our example $A^r = 2 \boxtimes A^{r-2}$ for all $r \geq 6$, in general we have $A^r = q \boxtimes A^{r-p}$ for all $r \geq R_0$. The existence of $R_0$ depends on properties of the matrix $A$.

**Definition 2.2** *A matrix is* irreducible *if there is some $K$ so that for all $k \geq K$ the matrix $A^k$ has no infinity entries.*

This is analogous to the definition of irreducible matrices in regular algebra, Perkins [17] proves that for an irreducible matrix $K \leq m^2 - m + 1$ where $m$ is the size of the matrix. This bound holds in the Min-Plus definition of irreducible. The following theorem will be refered to throughout the thesis and we will prove this theorem in Chapter 6.

**Theorem 2.3** *If $A$ is an irreducible matrix on $m$ vertices, then there exists a $p, q$ and $R_0$ so that*

$$A^r = q \boxtimes A^{r-p} \qquad \textit{for all} \quad r \geq R_0$$

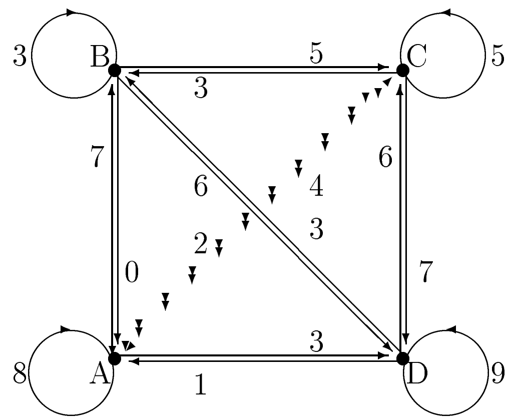
This shows that the power method for finding when $A$ repeats, which gives the eigenvalue of the matrix, will work if $A$ is irreducible. In our example, the cost matrix $A$ is irreducible and thus we know that if we compute the sequence $A, A^2, A^3, \ldots A^K$ there will be some $K$ so that two members of this sequence differ by a constant.

## 2.3 Minimum Cycle Means and Karp's Theorem

In the previous section, we showed that if $A$ is an irreducible matrix, then there is some $p$, $q$ so that $A^r = q \boxtimes A^{r-p}$. We will now look use the properties of the entries in $A$ to find the actual values of $p$ and $q$.

**Definition 2.4** *The* precedence digraph *of $A$ denoted $D_A$, is the weighted digraph with nodes $v_1, v_2, \ldots, v_m$ with an arc from $v_j$ to $v_i$ if $A_{ij} \neq \infty$, and the weight of this arc is $A_{ij}$. The precedence digraph is irreducible if the matrix $A$ is irreducible which means that for some $K$, there is a path of any length $k \geq K$ between every pair of vertices in the precedence digraph.*

The precedence digraph for our example is shown below.




**Definition 2.5** *The* length *of a path between two vertices is the number of arcs in the path, and is denoted $|P|$. A* cycle *is a path that begins and ends with the same vertex but has no other repeated vertices. The* weight *of a path is the sum of the weights of its arcs and is denoted $w(P)$.*

**Definition 2.6** *The precedence digraph of an adjacency matrix $A$ is* strongly connected *if there is a path from every vertex $v_j$ to every other vertex $v_i$. The adjacency matrix $A$ is strongly connected if its precedence digraph is. Any irreducible matrix is also strongly connected.*

The periodic property of a matrix is determined by paths in the precedence digraph. We will now look at the minimum weight paths in the precedence digraph.

**Theorem 2.7** *Let $A$ be an $m \times m$ matrix with precedence digraph $D_A$. Then the minimum weight of a path from vertex $j$ to vertex $i$ in $D_A$ of length $k > 0$ is $(A^k)_{i,j}$.*

*Proof:* Since $A_{i,j}$ is the weight of an arc from $j$ to $i$ in $D_A$ if it exists and $\infty$ otherwise, the result holds for $k = 1$. For $k > 1$, assume for induction purposes that $(A^{k-1})_{i,j}$ is minimum weight path from $j$ to $i$ of length $k - 1$. Then

$$(A^k)_{i,j} = \min\left((A^{k-1})_{i,1} + A_{1,j}, (A^{k-1})_{i,2} + A_{2,j}, \ldots, (A^{k-1})_{i,m} + A_{m,j}\right).$$

This is the minimum of all paths of length $k-1$ from $h$ to $i$ plus the minimum weight path of length 1 from $j$ to $h$. This gives the minimum weight of a path of length $k$ from $j$ to $i$. $\square$

Let

$$A^+ = A \boxtimes A^2 \boxtimes A^3 \ldots \boxtimes A^{n-1} \boxtimes A^n \boxtimes A^{n+1} \ldots,$$

and

$$A^* = I \boxtimes A \boxtimes A^2 \boxtimes A^3 \ldots \boxtimes A^{n-1} \boxtimes A^n \boxtimes A^{n+1} \ldots.$$

Then $(A^+)_{ij}$ is the minimum weight path of any length from $j$ to $i$, and $(A^*)_{ij}$ is the minimum weight path of any length from $j$ to $i$ allowing for the possibility of staying on a vertex. If $A$ is irreducible, then $A^*$ and $A^+$ have no infinity entries.

In our example, the weight of a path of length $r$ represents the cost of starting in city $j$, traveling for $r$ nights and ending in city $i$. The eigenvalue shows us that the cost to stay $r$ nights is \$2 more than the cost to stay $r - 2$ nights, but this does not show us how to travel at this cost. If we look at the precedence digraph, we see that there is a cycle from city A to city D and back to city A which has weight 4 and length 2. Thus we must use this cycle to stay 2 more nights and only pay \$4. This leads us to the next method for finding the eigenvalues of a matrix by looking at the cycles of the matrix.

**Definition 2.8** *The* cycle mean *of a cycle is the sum of the weights of the arcs divided by the number of arcs in the cycle. For a cycle c the cycle mean is denoted* $\rho(c) = \frac{w(c)}{|c|}$. *The* minimum cycle mean *of a graph is the smallest value for* $\rho(c)$ *over all cycles in the graph. All cycles with cycle mean equal to the minimum cycle mean are called* critical cycles.

**Theorem 2.9** *If* $D_A$ *is strongly connected, there exists exactly one eigenvalue* $\lambda$ *which solves the equation* $A \boxtimes \mathbf{x} = \lambda \boxtimes \mathbf{x}$ *for some* $\mathbf{x} \neq \infty$. *This eigenvalue is equal to the minimum cycle mean of the graph.*

$$\lambda \;=\; \min_c \rho(c) \;=\; \min_c \frac{w(c)}{|c|}$$

*where c ranges over all cycles in the graph.*

*Proof:* **Existence of x and** $\boldsymbol{\lambda}$**.** Consider matrix $B = -\lambda A$, where $\lambda = \min_c \rho(c) = \min_c \frac{w(c)}{|c|}$ and let $D_B$ be the precedence digraph of $B$. The minimum cycle weight of $D_B$ is 0, thus the matrices $B^*$ and $B^+ = BB^*$ exist and $B^+$ has some columns with diagonal entries equal to 0. Suppose a vertex $k$ is in some minimum cycle of $B$, then the minimum weight of paths from $k$ to $k$ is 0. Therefore, we have $0 = (B^+)_{kk}$. Let $B_k$ denote the $k^{\underline{\text{th}}}$ column of $B$. Then, since $B = -\lambda A$, $B^+ = BB^*$, and $B^* = \mathbf{I} \boxplus B^+$, for a given $k$ we have,

$$(B^+)_k = (B^*)_k$$

$$\Rightarrow (BB^*)_k = (B^+)_k = (B^*)_k$$

$$\Rightarrow -\lambda A(B^*)_k = (B^*)_k$$

$$\Rightarrow A(B^*)_k = \lambda(B^*)_k.$$

Hence $\mathbf{x} = (B^*)_k$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$.

**Uniqueness of $\lambda$.** Let $A$ be an strongly connected matrix with eigenvalues $\lambda_1 \leq \lambda_2$ with associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Let $\mathbf{1}$ be the vector of all ones and pick $t$ large enough so that

$$t\mathbf{1} + \mathbf{v_1} \geq \mathbf{v_2} \Longrightarrow \mathbf{t} \boxtimes \mathbf{v_1} \geq \mathbf{v_2}.$$

Then

$$(t \boxtimes \mathbf{v}_1) \boxplus \mathbf{v}_2 = t \boxtimes \mathbf{v}_1 \Longrightarrow A^r((t \boxtimes \mathbf{v}_1) \boxplus \mathbf{v}_2) = A^r(t \boxtimes \mathbf{v}_1)$$

$$\Longrightarrow tA^r\mathbf{v}_1 \boxplus A^r\mathbf{v}_2 = tA^r\mathbf{v}_1$$

$$\Longrightarrow (t \boxtimes \lambda_1^r)\mathbf{v}_1 \boxplus \lambda_2^r\mathbf{v}_2 = (t \boxtimes \lambda_1^r)\mathbf{v}_1.$$

$$\Longrightarrow \min((t + r\lambda_1)\mathbf{1} + \mathbf{v_1}, \mathbf{r}\boldsymbol{\lambda_2}\mathbf{1} + \mathbf{v_2}) = (\mathbf{t} + \mathbf{r}\boldsymbol{\lambda_1})\mathbf{1} + \mathbf{v_1}.$$

If $\lambda_1 < \lambda_2$, then $r\lambda_2\mathbf{1} + \mathbf{v_2} \not\leq (\mathbf{t} + \mathbf{r}\boldsymbol{\lambda_1})\mathbf{1} + \mathbf{v_1}$ for some $r$. Thus $\lambda_1 = \lambda_2$. $\square$

Since any irreducible matrix is strongly connected, we also have that the minimum cycle mean of a irreducible matrix is its eigenvalue.

Next we look at one method of finding the minimum cycle mean using the following theorem which is analogous to a theorem by Karp which finds the maximum cycle mean but not the critical cycles.

**Theorem 2.10** *Let $A$ be an $m \times m$ matrix with corresponding precedence digraph $D_A$. Then, for any $j$ the minimum cycle mean $\rho(A)$ is*

$$\rho(A) = \min_{i=1\ldots m} \max_{k=1\ldots m} \frac{(A^m)_{ij} - (A^k)_{ij}}{m-k},$$

*where $A^m$, $A^k$ are computed in Min-Plus algebra and the other computations are conventional.*

*Proof:* Without loss of generality, assume that $D_A$ is strongly connected. If $D_A$ is not strongly connected, then the theorem holds componentwise. Also, the index $j$ is arbitrary and the computation of $\rho(A)$ is independent of $j$.

First assume that the minimum cycle mean is 0. Then we must show

$$\min_{i=1\ldots m} \max_{k=0\ldots m-1} \frac{(A^m)_{ij} - (A^k)_{ij}}{m-k} = 0.$$

Since $\rho(A) = 0$, there exists a cycle of weight 0 and there are no cycles with negative weight. Because there are no cycles with negative weight, there exists a minimum weight path of length $k$ from vertex $i$ to vertex $j$. When $k \geq m$ the path would contain a cycle of non-negative weight. Thus a minimum weight path can be found by restricting $k < m$. This path is defined as

$$(A^+)_{ij} = \min_{k=0\ldots m-1} (A^k)_{ij}.$$

Also, $(A^m)_{ij} \geq (A^+)_{ij}$, and hence

$$(A^m)_{ij} - (A^+)_{ij} = \max_{k=0\ldots m-1} \left[(A^m)_{ij} - (A^k)_{ij}\right] \geq 0.$$

Equivalently,

$$\max_{k=0\ldots m-1} \frac{(A^m)_{ij} - (A^k)_{ij}}{m-k} \geq 0.$$

Equality holds only if $(A^m)_{ij} = (A^+)_{ij}$.

Now we will show that there is an index for $i$ where this is true. Let $C$ be a cycle of weight 0 and let $l$ be a vertex of $C$. Let $P_{lj}$ be a path from $l$ to $j$ with minimum weight $w(P_{lj}) = (A^+)_{lj}$. Now this path is extended by appending to it a number of repetitions of $C$ such that the total length of this extended path , denoted $P_e$, is greater than or equal to $m$. This is again a path of minimum weight from $j$ to $l$. Now consider the path consisting of the first $m$ arcs of $P_e$. Its initial vertex is $j$ and denote its final vertex $i$. Note that $i$ is in $C$. Since any subpath of a minimum weight path is also of minimum weight, the path from $j$ to $i$ is of minimum weight. Therefore $(A^m)_{ji} = (A^+)_{ji}$ and

$$\min_{i=1\ldots m} \max_{k=0\ldots m-1} \frac{(A^m)_{ij} - (A^k)_{ij}}{m-k} = 0.$$

This completes the case where $\rho(A) = 0$.

Now consider an arbitrary finite $\rho(A)$. A constant $c$ is now subtracted from each weight in $A_{ij}$. Then clearly $\rho(A)$ will be changed $c$ units and each entry in $A^k$ will be changed by $kc$ units. Thus

$$\min_{i=1\ldots n}\max_{k=0\ldots m-1}\frac{(A^m)_{ij}-(A^k)_{ij}}{m-k}.$$

is changed by $c$ and both sides of the equation are affected equally when all weights $A_{ij}$ are changed by the same amount. Now choose $c$ so that $\rho(A)-c=0$ and we are back to the original case. $\square$

Like the power method, Karp's theorem gives the minimum cycle mean but not the critical cycles. To find the critical cycles, backtracking must be used in both of these methods. There are many applications of Min-Plus algebra in which the solution is found using eigenvalues. In practice, the power method is a faster way to find the eigenvalues of an irreducible matrix, although this is not true for all matrices as we will see in Chapter 6. In the next three Chapters we present applications of Min-Plus algebra which involve finding the domination numbers of certain graphs.

# 3. Domination of Cartesian Product Graphs

In this chapter, we will use two dynamic programming algorithms in Min-Plus algebra to find the domination numbers of a large class of graphs.

**Definition 3.1** *Let $G = (V, E)$ be a graph with vertices $\{v_1, v_2, \ldots v_n\}$ and let $H = (V, E)$ be a graph with vertices $\{w_1, w_2 \ldots w_\ell\}$. Then the* cartesian product *$G \times H$ has vertices of the form $(v_i, w_j)$, with $1 \le i \le n$ and $1 \le j \le \ell$. The edges of $G \times H$ have the form $((v_i, w_j), (v_p, w_q))$ if $i = p$ and $(w_j, w_q)$ is and edge of $H$, or if $j = q$ and $(v_i, v_p)$ is an edge of $G$. The $k^{th}$ copy of $G$ in $G \times H$ is $G_k$ and its vertices are $(v_1, w_k), (v_2, w_k), \ldots (v_n, w_k)$.*

To introduce the algorithms, we consider complete grid graphs which are cartesian product graphs with $H = P_r$, the path on $r$ vertices, and $G = P_n$, the path on $n$ vertices. The domination number of $P_n \times P_r$ is denoted $\gamma(P_n \times P_r)$.

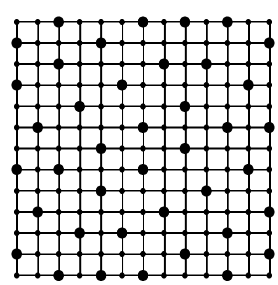


**Figure 3.1** Minimum Dominations of $P_{13} \times P_{13}$. Above is a domination of $P_{13} \times P_{13}$ using 40 vertices. Since Theorem 3.12 shows that $\gamma_{13,13} = 40$, no domination has fewer vertices.

Considerable work has been directed toward finding $\gamma(P_n \times P_r)$. Jacobson and Kinch [13] showed:

$$\gamma_{1,r} = \left\lceil \frac{r}{3} \right\rceil,\ \gamma_{2,r} = \left\lceil \frac{r+1}{2} \right\rceil,\ \gamma_{3,r} = \left\lceil \frac{3r+1}{4} \right\rceil,$$

$$\text{and } \gamma_{4,r} = \begin{cases} r+1 & \text{if } r = 1,2,3,5,6,9 \\ r & \text{otherwise.} \end{cases} \tag{1}$$

Chang and Clark [3] showed that

$$\gamma_{5,r} = \begin{cases} \left\lceil \frac{6r+4}{5} \right\rceil - 1 & \text{if } r = 2,3,7 \\ \left\lceil \frac{6r+4}{5} \right\rceil & \text{otherwise} \end{cases}$$

$$\text{and } \gamma_{6,r} = \begin{cases} \left\lceil \frac{10r+4}{7} \right\rceil + 1 & \text{if } r \bmod 7 = 3 \text{ and } r \neq 3 \\ \left\lceil \frac{10r+4}{7} \right\rceil & \text{otherwise.} \end{cases} \tag{2}$$

Clark, Colbourn and Johnson [6] showed that finding the domination number of a grid graph (a subgraph of $P_n \times P_r$) is NP-hard. All known algorithms for finding $\gamma(P_n \times P_r)$ could be adapted to find the domination number of a grid graph. So it is not surprising that the time needed for these algorithms is not polynomial in $nr$. Nevertheless, profound differences exist in the speed of these algorithms.

Perhaps the most natural non-trivial algorithms for finding $\gamma(P_n \times P_r)$ are "Branch and Bound" algorithms. Before trying a different approach, Hare, Hedetniemi, and Hare [11] used a branch and bound algorithm that took 20 hours to find $\gamma(P_7 \times P_7) = 12$. Ma and Lam [15] gave a more complicated branch and bound algorithm which they used to show that $\gamma(P_8 \times P_8) = 16$,

$\gamma(P_8 \times P_9) \geq 17$, and $\gamma(P_9 \times P_9) \geq 19$. However since the time needed for these algorithms seem to be exponential in both $n$ and $r$, they are limited to modest values of $n$ and $r$.

Much superior are dynamic programming algorithms for finding $\gamma(P_n \times P_r)$ introduced by Hare, Hedetniemi and Hare [11]. While these algorithms are exponential in $n$, they are linear in $r$. In section 3.1, we present this algorithm of Hare, Hedetniemi and Hare in which they found $\gamma(P_n \times P_r)$ for $n \leq 8$ and $r \leq 500$, and then used these results to make conjectures for $\gamma(P_n \times P_r)$ for all values of $r$. Fisher (private communications) expanded on this algorithm by stating it in terms of Min-Plus algebra and by looking for periodicity in the algorithm. When periodicity is detected, induction gives $\gamma(P_n \times P_r)$ for a given $n$ and for all $r$. Thus $\gamma(P_n \times P_r)$ is calculated for an infinite number of values in a finite amount of time. Using this expanded algorithm, which we will call the Exact Algorithm, Fisher was able to verify the conjectures of Hare, Hedetniemi and Hare for all values of $r$. While it does not seem possible to find an algorithm that is not exponential in $n$, the Exact Algorithm can be modified to reduce the base of the exponent. Section 3.2 describes this modified algorithm which will be called the *At Least Algorithm*. Using the At Least Algorithm and periodicity, we find formulas for $\gamma(P_n \times P_r)$ for all $n \leq 19$.

## 3.1 The Exact Algorithm

To illustrate this algorithm, we will look at the domination of $P_3 \times P_r$. Let $S$ be a set of vertices of $P_3 \times P_r$. Any vertex in this graph is either in $S$, adjacent to a vertex in $S$, or not adjacent to any vertices in $S$. These vertices will be labeled 0, 1, and 2 respectively. Now, any column of $P_3 \times P_r$ is a copy of $P_3$, and its vertices can be labeled either 0,1,or 2. A labeling of $P_3$ which can occur in a dominating set $S$ of $P_3 \times P_r$ will be called a state. Since no state can contain vertices labeled 0 adjacent to vertices labeled 2, there are 17 possible states for $P_3$. These states are shown below.

$$\begin{array}{lll}
s_1 = 000 & s_7 = 110 & s_{13} = 211 \\
s_2 = 001 & s_8 = 012 & s_{14} = 122 \\
s_3 = 010 & s_9 = 111 & s_{15} = 212 \\
s_4 = 100 & s_{10} = 210 & s_{16} = 221 \\
s_5 = 011 & s_{11} = 112 & s_{17} = 222 \\
s_6 = 101 & s_{12} = 121 &
\end{array}$$

To find the domination number of $P_3 \times P_{r+1}$ we will add a column onto dominating sets of $P_3 \times P_r$. If column $r$ is in state $s_j$, then it is possible for column $r+1$ to be in state $s_i$ if the following conditions hold for all vertices p where $s_j(p)$=the $p^{\underline{\text{th}}}$ vertex of $s_j$.

If $s_j(p) = 0$, then $s_i(p) \neq 2$.

If $s_j(p) = 2$, then $s_i(p) = 0$.

If $s_i(p) = 1$, then $s_j(p) = 2$ if some vertex adjacent to $p$ in $s_j$ has value 0, and $s_j(p) = 0$ otherwise.

Using these rules, we form the **state transition matrix** A where $A_{ij}$ = the number of vertices with value 0 in state $s_i$ if $s_i$ can follow $s_j$ and infinity (denoted by $-$) otherwise. This matrix is shown below.

$$\begin{bmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & - & 2 & 2 & - & 2 & 2 & - & - & 2 & - \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & - & 2 & - & 2 & - & - \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & - & 2 & 2 & - & 2 & - & - & - \\
1 & - & 1 & 1 & - & - & 1 & - & - & 1 & - & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & - & - & - \\
1 & 1 & 1 & - & 1 & - & - & 1 & - & - & - & - & - & - & - & - & - \\
- & 1 & - & - & 1 & 1 & - & - & 1 & - & - & - & 1 & - & - & - & - \\
0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & - & - & - \\
- & 0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & 0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & 0 & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & 0 & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & 0 & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & 0 & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & 0 & - & - & - & - & - & - & - & -
\end{bmatrix}$$

Next, we form a **states vector** $\mathbf{x}_r$ where $\mathbf{x}_r(s_i)$ is the minimum number of vertices needed to dominate $P_3 \times P_r$ and have the $r^{\underline{\text{th}}}$ column be in state $s_i$. If it is not possible to dominate $P_3 \times P_r$ and have the $r^{\underline{\text{th}}}$ column in state $s_i$ then $\mathbf{x}_r(s_i) = \infty$. Thus

$$\mathbf{x}_1^T = \begin{bmatrix} 3 & 2 & 2 & 2 & - & 1 & - & 1 & - & 1 & - & - & - & - & - & - & 0 \end{bmatrix}$$

The infinity ($-$) entries in $\mathbf{x}_1$ correspond to states in which $s_i(p) = 1$ and which must be preceded by a state with $s_j(p) = 0$ in column 0, which does not exist. To find $\mathbf{x}_2(s_i)$ we consider all labelings of the first column of $P_3 \times P_2$ when the second column is in state $s_i$. These labelings correspond to the states $s_j$ for which $A_{ij} \neq \infty$. From all these possible labelings of the first column, the one with the minimum number of vertices with value 0 will give a minimal domination of $P_3 \times P_2$ with column 2 in state $s_i$. Thus $\mathbf{x}_2(s_i) = \min(\sum_j A_{ij} + \mathbf{x}_1(s_j))$. Stating this in terms of Min-Plus Algebra, $\mathbf{x}_2 = A \boxtimes \mathbf{x}_1$ and in general $\mathbf{x}_{r+1} = A \boxtimes \mathbf{x}_r$. Repeating this process we have a recursive algorithm to find $\mathbf{x}_r$ where $\mathbf{x}_r(s_i)$ is the number of vertices needed to dominate $P_3 \times P_r$ and leave the $r^{\underline{\text{th}}}$ column in state $s_i$. To find the domination number of $P_3 \times P_r$ we find the minimum entry in $\mathbf{x}_r$ for which the state leaves the $r^{\underline{\text{th}}}$ column dominated. This is done by forming an ending vector $\mathbf{y}$ where $\mathbf{y}(i) = 0$ if state $s_i$ leaves the $r^{\underline{\text{th}}}$ column dominated (these are all the states with no vertex labeled 2) and $\mathbf{y}(i) = \infty$ otherwise. The domination number for $P_3 \times P_r$ is $\mathbf{x}_r \boxtimes \mathbf{y}^T$. The ending vector $\mathbf{y}$ and the states vectors $\mathbf{x}_1$ through $\mathbf{x}_{11}$ for $P_3 \times P_r$ are shown below.

$$
\begin{array}{lll}
y & = & [\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -\ 0\ -\ -\ -\ -\ -\ -\ -\ -\ ] \\
\mathbf{x}_1 & = 0 \boxtimes & [\ 3\ 2\ 2\ 2\ -\ 1\ -\ 1\ -\ 1\ -\ -\ -\ -\ -\ -\ 0\ ] \\
\mathbf{x}_2 & = 0 \boxtimes & [\ 3\ 3\ 3\ 3\ 2\ 2\ 2\ 2\ 3\ 2\ 2\ 2\ 2\ -\ 1\ -\ -\ ] \\
\mathbf{x}_3 & = 2 \boxtimes & [\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ ] \\
\mathbf{x}_4 & = 3 \boxtimes & [\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ ] \\
\mathbf{x}_5 & = 4 \boxtimes & [\ 2\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ ] \\
\mathbf{x}_6 & = 4 \boxtimes & [\ 3\ 2\ 2\ 2\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ ] \\
\mathbf{x}_7 & = 5 \boxtimes & [\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ ] \\
\mathbf{x}_8 & = 6 \boxtimes & [\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ ] \\
\mathbf{x}_9 & = 7 \boxtimes & [\ 2\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ ] \\
\mathbf{x}_{10} & = 7 \boxtimes & [\ 3\ 2\ 2\ 2\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ ] \\
\mathbf{x}_{11} & = 8 \boxtimes & [\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ ]
\end{array}
$$

Notice that $\mathbf{x}_{10} = 3 \boxtimes \mathbf{x}_6$ and $\mathbf{x}_{11} = 3 \boxtimes \mathbf{x}_7$.

**Theorem 3.2** $\mathbf{x}_{r+4} = 3 \boxtimes \mathbf{x}_r$ *for all* $r \geq 6$.

*Proof:* By Induction. The base case is $\mathbf{x}_{10} = 3 \boxtimes \mathbf{x}_6$ or $A\mathbf{x}_9 = 3 \boxtimes \mathbf{x}_6$. Then for any $r > 6$ assume $A \boxtimes \mathbf{x}_{r+2} = 3 \boxtimes \mathbf{x}_{r-1}$ Then $A\mathbf{x}_{r+2}\mathbf{x} = \mathbf{x}_{r-1}\mathbf{x}$ and $\mathbf{x}_{r+4} = \mathbf{x}_r$ □

**Theorem 3.3** $\gamma(P_3 \times P_{r+4}) = 3 + \gamma(P_3 \times P_r)$ *for all* $r \geq 6$

*Proof:*

$$
\begin{aligned}
\gamma(P_3 \times P_{r+4}) &= y^T \boxtimes \mathbf{x}_{r+4} \\
&= y^T \boxtimes (3 \boxtimes \mathbf{x}_r) \\
&= 3 \boxtimes (y^T \boxtimes \mathbf{x}_r) \\
&= 3 + \gamma(P_3 \times P_r)
\end{aligned}
$$

□

Using the recursive definition above, we can find a closed form for $\gamma(P_3 \times P_r)$

for all $r \geq 6$.

$$\gamma(P_3 \times P_r) = \left\lceil \frac{3r+1}{4} \right\rceil.$$

Finding the domination number of $P_3 \times P_r$ for all $r$ relies on the periodicity of the dynamic programming algorithm. According to Theorem 2.3, periodicity will occur if the precedence digraph of the state transition matrix is irreducible. The possible states for $P_k \times P_r$ can be partitioned into three groups. Let $A$ be the set containing the state with all vertices labeled $0 = 00\ldots0$, $B$ be the set containing all states with vertices labeled 0 or 1 but not 2, and let $C$ be all remaining states which have at least one vertex labeled 2.

**Theorem 3.4** *The precedence digraph of the state transition matrix for finding the domination number of $P_k \times P_r$ is irreducible.*

*Proof:* Using the rules of state transition , all states can be followed by the states of all zeros. All states in set $B$ can follow the zero state and for every state in set $C$ there is at least one state in set $B$ that it can follow. Thus there is a path of length 3 or less from any state to any other state in the precedence digraph by going through the zero state. This path can then be extended by repeating the zero state. Thus there is a path of any length $k \geq 3$ between any pair of vertices and the precedence digraph is irreducible. □

For any $n$ , the exact algorithm can be used to find the domination number of $P_n \times P_r$. The algorithm will reach periodicity after some number of iterations allowing us to find a closed form for the domination number of $P_n \times P_r$ for all $r$. The limiting feature of all dynamic programming algorithms is computation time. For the exact algorithm on $P_n \times P_r$, there are $O((1+\sqrt{2})^n)$ states each of which matches to at most $2^n$ other states and each match takes $O(n)$ computations to find. Thus the computational complexity for finding the domination number of $P_n \times P_r$ is $O(nr(2+2\sqrt{2})^n)$, and the maximum size of $n$ is greatly limited by computational complexity. In the next section, we will look at another dynamic programming algorithm which greatly reduces this complexity. Although this algorithm is similar to the exact algorithm, it is not a direct descendant.

## 3.2 The At Least Algorithm

In this algorithm, we will reduce the computational complexity by redefining what it is that we are computing. Fisher (private communications) used this new algorithm to find $\gamma(P_n \times P_r)$ for $n \leq 19$ and for all $r$. These results are presented below. In section 3.4 we will expand this algorithm to find $\gamma(G \times P_r)$ for any graph $G$ on 6 or less vertices.

**Definition 3.5** *A column of $P_n \times P_r$ is in* at least *state $s_i$ if it is in state $s_i$ or any state $s_j$ which has $s_i(p) \geq s_j(p)$ for all $p$.*

This means that $s_j$ dominates at least the same set of vertices that $s_i$ does and possibly more.

**Definition 3.6** *A* t-domination *of $P_n \times P_r$ is a set of vertices that dominates the first $r-1$ columns of $P_n \times P_r$ and leaves the last column in at least state $s_i$.*

To illustrate the algorithm, we will find the least-domination number of $P_3 \times P_r$. As in the exact algorithm, there are seventeen states. These states form a partial ordering which is illustrated in Figure 3.2.

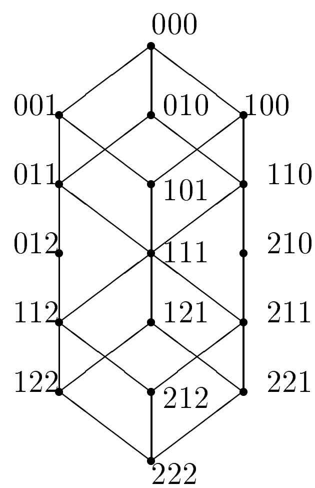



**Figure 3.2** The partial ordering of the 17 states for $P_3 \times P_r$

State $s_i$ is at least state $s_j$ if state $s_j$ is an ancestor of $s_i$ in the partial ordering. Next, we form the states vector $\mathbf{t}_r$ where $\mathbf{t}_r(s_i)$ is the number of vertices in a dominating set for $P_3 \times P_r$ with column $r$ in at least state $s_i$. Since a state is in at least $s_i$ if it is in exactly $s_i$ or exactly $s_j$ for all $s_j > s_i$, the entries in $\mathbf{t}_r$ can be defined using the states vector $\mathbf{x}_r$ with $\mathbf{x}_r(s_i)$ representing the number of vertices in a domination of $P_3 \times P_r$ with column $r$ in exactly $s_i$ as described in the Exact Algorithm. For example

$$\begin{aligned} \mathbf{t}_r(s_7) &= \min(\mathbf{x}_r(s_7), \mathbf{x}_r(s_4), \mathbf{x}_r(s_3), \mathbf{x}_r(s_1)) \\ \mathbf{t}_r(s_{10}) &= \min(\mathbf{x}_r(s_{10}), \mathbf{x}_r(s_7), \mathbf{x}_r(s_4), \mathbf{x}_r(s_3), \mathbf{x}_r(s_1)) \end{aligned}$$

Now the last 4 entries for $\mathbf{t}_r(s_{10})$ are the same as $\mathbf{t}_r(s_7)$ so we have

$$\mathbf{t}_r(s_{10}) = \min(\mathbf{x}_r(s_{10}), \mathbf{t}_r(s_7))$$

Next, if the $r^{\underline{\text{th}}}$ column is in exactly state $s_i$ then the entry in $\mathbf{x}_r(s_i)$ represents the number of vertices needed to dominate $P_3 \times P_{r-1}$ and leave the $(r-1)^{\text{st}}$ column in the minimum previous state that can precede state $s_i$ plus the number of vertices that are labeled 0 in $s_i$.

**Definition 3.7** *Given a state $s_i$, the* minimum previous state $M(s_i)$ *is defined as*

$$M(s_i(p)) = \begin{cases} 2 & \text{if } s_i(p) = 0 \\ 0 & \text{if } s_i(p) = 1 \text{ and no neighbors of } p \text{ in } s_i = 0 \\ 1 & \text{otherwise} \end{cases}$$

The minimum previous state for $s_{10} = 210$ is $s_{11} = 112$ and $s_{10}$ has one vertex in the dominating set, so $\mathbf{x}_r(s_{10}) = 1 + \mathbf{t}_{r-1}(s_{11})$ and,

$$\mathbf{t}_r(s_{10}) = \min(1 + \mathbf{t}_{r-1}(s_{11}), \mathbf{t}_r(s_7)).$$

In general,

$$\mathbf{t}_r(i) = \min(|s_i| + \mathbf{t}_{r-1}(M(s_i)), \min_{s_j \geq s_i} \mathbf{t}_r(s_j))$$

Completing a similar computation for each state, we form two matrices. The **at least matrix A** has $A_{ij} = 0$ if state $s_i$ is at least state $s_j$ and there is no state $s_k$ where $s_i$ is at least $s_k$ and $s_k$ is at least $s_j$, and $A_{ij} = \infty$ otherwise. This matrix is lower triangular because of the partial ordering of the states. The second matrix is the **minimum previous matrix B** with $B_{ij} =$ the number of vertices in the dominating set for state $s_i$ if state $s_j$ is the minimum previous state of $s_i$ and $\infty$ otherwise. The matrices $A$ and $B$ for the at least domination of $P_3 \times P_r$ are shown below.

$$A = \begin{bmatrix}
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & 0 & 0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & 0 & - & 0 & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & 0 & 0 & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & 0 & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & 0 & 0 & 0 & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & 0 & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & 0 & 0 & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & 0 & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & 0 & 0 & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & 0 & 0 & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & 0 & - & 0 & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & 0 & 0 & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & 0 & 0 & 0 & -
\end{bmatrix}$$

$$B = \begin{bmatrix}
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & 3 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & 2 & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & 2 & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & 2 & - & - & - \\
- & - & - & - & - & - & - & - & - & 1 & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & 1 & - & - & - & - & - \\
- & - & - & - & - & - & - & 1 & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & 1 & - & - & - & - \\
0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & 1 & - & - & - & - & - & - \\
- & 0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & 0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & 0 & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & 0 & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & 0 & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & 0 & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & 0 & - & - & - & - & - & - & - & -
\end{bmatrix}$$

Now, in terms of min plus algebra,

$$\mathbf{t}_r = A\mathbf{t}_r + B\mathbf{t}_{r-1}.$$

To find $\mathbf{t}_r$ we iteratively multiply by $B$ and $A$ one row at a time, updating $\mathbf{t}_r$ after each new entry is found. This is possible since $A$ is lower triangular. To find the domination number of $P_3 \times P_r$ we need the minimum of the entries in $\mathbf{t}_r$ which correspond to states that dominate the last column. All states containing no vertices labeled 2 dominate the last column, and the state of all 1's denoted $\mathbf{1}$ is at least all of these states so the ending vector $\mathbf{y}$ has a 0 entry for the $\mathbf{1}$ state and infinity entries for all other states. For $P_3 \times P_r$ $\mathbf{1}$ is state $s_9$, so $\gamma(P_3 \times P_r) = \mathbf{t}_r(s_9) \boxtimes \mathbf{y}^T$ and in general $\gamma(P_k \times P_r) = \mathbf{t}_r(\mathbf{1} \boxtimes \mathbf{y}^T)$ Now computing $\mathbf{t}_r$ iteratively, we find the states vectors for the t-domination of $P_3 \times P_r$, shown below.

$$
\begin{array}{lll}
\mathbf{y} = & & [\ \texttt{- - - - - - - - 0 - - - - - - - -}\ ] \\
\mathbf{t}_1 = & 0 \boxtimes & [\ 3\ 2\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ ] \\
\mathbf{t}_2 = & 0 \boxtimes & [\ 3\ 3\ 3\ 3\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 2\ 1\ ] \\
\mathbf{t}_3 = & 2 \boxtimes & [\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ ] \\
\mathbf{t}_4 = & 3 \boxtimes & [\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ ] \\
\mathbf{t}_5 = & 4 \boxtimes & [\ 2\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ ] \\
\mathbf{t}_6 = & 4 \boxtimes & [\ 3\ 2\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ ] \\
\mathbf{t}_7 = & 5 \boxtimes & [\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ ] \\
\mathbf{t}_8 = & 6 \boxtimes & [\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ ] \\
\mathbf{t}_9 = & 7 \boxtimes & [\ 2\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ ] \\
\mathbf{t}_{10} = & 7 \boxtimes & [\ 3\ 2\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ ] \\
\mathbf{t}_{11} = & 8 \boxtimes & [\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ ]
\end{array}
$$

Notice that $\mathbf{t}_9 = 3 \boxtimes \mathbf{t}_5$ and $\mathbf{t}_{10} = 3 \boxtimes \mathbf{t}_6$ and thus by induction $\mathbf{t}_r = 3 \boxtimes \mathbf{t}_{r-4}$ for all $r \geq 5$ and we have the domination number for infinite $r$ after 9 iterations of the algorithm. We now need to show that periodicity in the At Least Algorithm will always occur.

**Lemma 3.8** *If $s_i$ is a direct ancestor of $s_j$ in the partial ordering (so $s_i \geq s_j$ and there is no $s_k$ with $s_i \geq s_k \geq s_j$) then $\mathbf{t}_r(s_i) - \mathbf{t}_r(s_j)$ is either 0 or 1.*

*Proof:* $s_i$ and $s_j$ have at most one vertex with a different label. If $s_i(p) = 0$ and $s_j(p) \neq 0$, then any domination ending in state $s_j$ can be changed into a domination ending in $s_i$ by changing $s_j(p)$ to 0 and this increases $\mathbf{t}_r(s_j)$ by 1. Otherwise $\mathbf{t}_r(s_i) = \mathbf{t}_r(s_j)$. $\square$

**Definition 3.9** *A state $s_j$ is* maximal *for a given $r$ if $\mathbf{t}_r(s_j) < \mathbf{t}_r(s_i)$ whenever $s_i$ is a direct ancestor of $s_j$ in the partial ordering. Let $R_{n,r}$ be the set of all* *maximal states for $P_n \times P_r$*

Using the states vectors $\mathbf{t}_1 \ldots \mathbf{t}_{11}$ shown above, we have

$$\begin{aligned}
R_{3,1} = &\quad \{s_1, s_2, s_3, s_4, s_6, s_8, s_9, s_{10}, s_{17}\} \\
R_{3,2} = &\quad \{s_1, s_5, s_6, s_{14}\} \\
R_{3,3} = &\quad \{s_1, s_3, s_6, s_{14}, s_{15}, s_{16}\} \\
R_{3,4} = &\quad \{s_1, s_2, s_3, s_4, s_{12}, s_{15}\} \\
R_{3,5} = &\quad \{s_1, s_2, s_3, s_4, s_6\} \\
\vdots \quad &\quad \vdots \\
R_{3,9} = &\quad \{s_1, s_2, s_3, s_4, s_6\}
\end{aligned}$$

**Lemma 3.10** *There exists positive integers $j, k, n$ so that $\mathbf{t}_k(s_i) - \mathbf{t}_j(s_i) = q$ for all states $s_i$ if and only if $R_{n,j} = R_{n,k}$*

*Proof:* If $\mathbf{t}_k(s_i) - \mathbf{t}_j(s_i) = q$ then certainly the maximal states for $j$ and $k$ must be the same and $R_{n,j} = R_{n,k}$. Next, for all $s_i, s_j \in R_{n,k}$ Lemma 3.8 states that $\mathbf{t}_k(s_i) = \mathbf{t}_k(s_j) - 1$ for all $s_j > s_i$. Let $\mathbf{2}$ be the state with all vertices labeled 2, then for any $s_i \neq \mathbf{2}$ there is a sequence of states $\mathbf{2} = s_1 < s_2 < \ldots s_h < s_i$. Let $N(s_i)$ be the states in this sequence, $N(s_i) = \{s_1, s_2, \ldots s_h\}$. Now the value of $\mathbf{t}_k$ increases by one each time a maximal state occurs in the sequence and the number of maximal states is $N(s_i) \bigcap R_{n,k}$. Thus

$$\mathbf{t}_k(s_i) = \mathbf{t}_k(\mathbf{2}) + |N(s_i) \bigcap R_{n,k}|,$$

and the domination number of any state can be constructed using this sequence.

Now if $R_{n,j} = R_{n,k}$ then let $q = \mathbf{t}_k(\mathbf{2}) - \mathbf{t}_j(\mathbf{2})$ and for all states $s_i$ we have

$$\begin{aligned} \mathbf{t}_k(s_i) &= \mathbf{t}_k(\mathbf{2}) + |N(s_i) \bigcap R_{n,k}| \\ &= \mathbf{t}_k(\mathbf{2}) + |N(s_i) \bigcap R_{n,j}| \\ &= \mathbf{t}_k(\mathbf{2}) + \mathbf{t}_j(s_i) - \mathbf{t}_j(\mathbf{2}) \\ &= q + \mathbf{t}_j(s_i) \end{aligned}$$

□

**Lemma 3.11** *Let $C_m$ be the number of states, then there exists $0 < j < k < 2^{C_m} + 1$ where $R_{n,j} = R_{n,k}$.*

*Proof:* The entries in $R_{n,k}$ are a subset of the states so there are at most $2^{C_m}$ possibilities for $R_{n,k}$. The pigeonhole principle ensures that $R_{n,j} = R_{n,k}$ for some $0 < j < k < 2^{C_m} + 1$. □

Now by Lemma 3.10 and Lemma 3.11 periodicity is guaranteed for the At Least Algorithm. Thus, the At Least Algorithm allows us to compute the domination number for $P_n \times P_r$ for all $r$ using the periodicity of the algorithm and we have greatly reduced the number of computations. For each state there are at most $n$ entries in the at least matrix and one entry in the minimum previous matrix, reducing the computational complexity to $O(nr(1 + \sqrt{2})^n)$.

## 3.3 Formulas for the Domination Number

A program implementing the At Least algorithm verified (1) and (2) in about a second. It took about a week on a Vax 8800 to find formulas for $\gamma(P_n \times P_r)$ for all $n \leq 19$. Since $\gamma(P_n \times P_r) = \gamma(P_r \times P_n)$, this gives $\gamma(P_n \times P_r)$

for all $n$ and $r$ with $\min(n, r) \leq 19$.

**Theorem 3.12** *For all $n \leq r$ with $n \leq 19$, we have*

$$\gamma(P_n \times P_r) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil & \textit{if } n = 1 \\ \left\lceil \frac{r+1}{2} \right\rceil & \textit{if } n = 2 \\ \left\lceil \frac{3r+1}{4} \right\rceil & \textit{if } n = 3 \\ r & \textit{if } n = 4 \textit{ and } r \neq 5, 6, 9 \\ r + 1 & \textit{if } n = 4 \textit{ and } r = 5, 6, 9 \\ \left\lceil \frac{6r+4}{5} \right\rceil & \textit{if } n = 5 \textit{ and } r \neq 7 \\ \left\lceil \frac{6r+4}{5} \right\rceil - 1 & \textit{if } n = 5 \textit{ and } r = 7 \\ \left\lceil \frac{10r+4}{7} \right\rceil & \textit{if } n = 6 \\ \left\lceil \frac{5r+1}{3} \right\rceil & \textit{if } n = 7 \\ \left\lceil \frac{15r+7}{8} \right\rceil & \textit{if } n = 8 \\ \left\lceil \frac{23r+10}{11} \right\rceil & \textit{if } n = 9 \\ \left\lceil \frac{30r+15}{13} \right\rceil & \textit{if } n = 10 \textit{ and } r \textit{ mod } 13 \neq 10 \textit{ and } r \neq 13, 16 \\ \left\lceil \frac{30r+15}{13} \right\rceil - 1 & \textit{if } n = 10 \textit{ and } r \textit{ mod } 13 = 10 \textit{ or } r = 13, 16 \\ \left\lceil \frac{38r+22}{15} \right\rceil & \textit{if } n = 11 \textit{ and } r \neq 11, 18, 20, 22, 33 \\ \left\lceil \frac{38r+22}{15} \right\rceil - 1 & \textit{if } n = 11 \textit{ and } r = 11, 18, 20, 22, 33 \\ \left\lceil \frac{80r+38}{29} \right\rceil & \textit{if } n = 12 \\ \left\lceil \frac{98r+54}{33} \right\rceil & \textit{if } n = 13 \textit{ and } r \textit{ mod } 33 \neq 13, 16, 18, 19 \\ \left\lceil \frac{98r+54}{33} \right\rceil - 1 & \textit{if } n = 13 \textit{ and } r \textit{ mod } 33 = 13, 16, 18, 19 \\ \left\lceil \frac{35r+20}{11} \right\rceil & \textit{if } n = 14 \textit{ and } r \textit{ mod } 22 \neq 7 \\ \left\lceil \frac{35r+20}{11} \right\rceil - 1 & \textit{if } n = 14 \textit{ and } r \textit{ mod } 22 = 7 \\ \left\lceil \frac{44r+28}{13} \right\rceil & \textit{if } n = 15 \textit{ and } r \textit{ mod } 26 \neq 5 \\ \left\lceil \frac{44r+28}{13} \right\rceil - 1 & \textit{if } n = 15 \textit{ and } r \textit{ mod } 26 = 5 \\ \left\lfloor \frac{(n+2)(r+2)}{5} \right\rfloor - 4 & \textit{if } n = 16, 17, 18, 19. \end{cases}$$

Even with Theorem 3.12, it is difficult to find the pattern that dominates the $n \times r$ grid for all $r$ using the minimum number of vertices. Jacobson and Kinch [13] found minimum domination patterns of $1 \times r$, $2 \times r$, $3 \times r$ and $4 \times r$

grids. Chang, Clark and Hare [4] use the algorithm of Hare, Hedetniemi and Hare to generate minimum domination patterns of $5 \times r$ and $6 \times r$ grids, and what they conjectured to be minimum dominations of $7 \times r$, $8 \times r$, $9 \times r$ and $10 \times r$ grids for all $r$. Theorem 3.12 shows these dominations are minimum. They also found formulas for $\gamma_{11,r}$ when $r \leq 122$ and for $\gamma_{12,r}$ when $r \leq 33$ (the formula for $\gamma_{11,r}$ holds for all $r$, while the formula for $\gamma_{12,r}$ does not).

The program in this paper was modified to store backtracking information to give minimum dominations of $11 \times r$, $12 \times r$, $13 \times r$, $14 \times r$, and $15 \times r$ grids for all $r$ up to a certain point. These were used to find minimum dominations for $n \times r$ grids for a given $n$ and for all $r$ (this is actually quite difficult because only one of what is usually many minimum dominations is generated). Chang, Fisher and Hare [5] report these minimum dominations.

For $n, r \geq 8$, Chang [2] gives a construction which dominates an $n \times r$ grid with $\lfloor (n+2)(r+2)/5 \rfloor - 4$ vertices. This is a generalization of a construction of Cockayne, Hare, Hedetniemi and Wimer [7] for $r \times r$ grids. Theorem 3.12 shows this construction is minimum for $16 \times r$, $17 \times r$, $18 \times r$, and $19 \times r$ grids when $r \geq 16$. This causes one to suspect the following conjecture (originally due to Chang [2]) might be true.

**Conjecture 3.13** *For all $n, r \geq 16$,*

$$\gamma(P_n \times P_r) = \left\lfloor \frac{(n+2)(r+2)}{5} \right\rfloor - 4.$$

If Conjecture 1 is true, dominating an $n \times r$ grid would be analogous to 2-packing (Fisher [8]) in that a simple formula holds for $n$ and $r$ large enough. Further, it would answer a question of Hedetniemi and Laskar [12]. In commenting on [6], they asked *is there a polynomial algorithm for finding* $\gamma(P_n \times P_r)$*?* Conjecture 3.13 together with Theorem 3.12 would give an affirmative answer to the question.

## 3.4 Domination of $G \times P_r$: The At Least Algorithm

We will now use the At Least Algorithm to find the domination number of $G \times P_r$ where $G$ is any graph on $n \leq 6$ vertices.

Livingstone and Stout [14] develop a dynamic programming algorithm for finding $\gamma(G \times P_r)$ using an algorithm that is similar to the Exact Algorithm. We are able to significantly reduce the complexity of this algorithm by stating in terms of the At Least Algorithm.

To illustrate the At Least Algorithm for $\gamma(G \times P_r)$ we will let $G$ be the house graph on five vertices as shown in Figure 3.3.

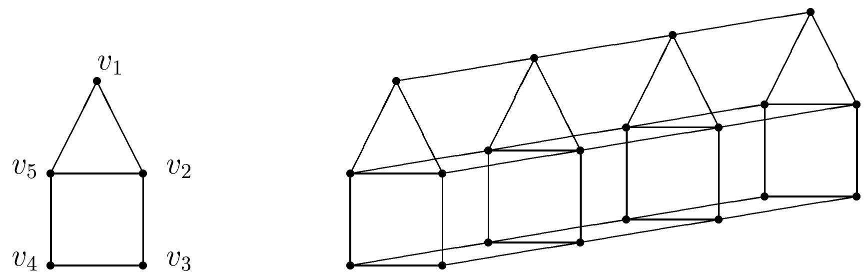



**Figure 3.3** The house graph on five vertices $G$ and $G \times P_4$.

There are 77 possible states for the domination of the house graph on five vertices. These states are shown below and the vertices are ordered $v_1 \ldots v_5$ as labeled in Figure 3.3.

| | | | |
|---|---|---|---|
| $s_1 = 00000$ | $s_{21} = 01221$ | $s_{41} = 11121$ | $s_{61} = 21111$ |
| $s_2 = 00001$ | $s_{22} = 10000$ | $s_{42} = 11122$ | $s_{62} = 21112$ |
| $s_3 = 00010$ | $s_{23} = 10001$ | $s_{43} = 11210$ | $s_{63} = 21121$ |
| $s_4 = 00011$ | $s_{24} = 10010$ | $s_{44} = 11211$ | $s_{64} = 21122$ |
| $s_5 = 00100$ | $s_{25} = 10011$ | $s_{45} = 11212$ | $s_{65} = 21211$ |
| $s_6 = 00101$ | $s_{26} = 10101$ | $s_{46} = 11221$ | $s_{66} = 21212$ |
| $s_7 = 00110$ | $s_{27} = 10101$ | $s_{47} = 11222$ | $s_{67} = 21221$ |
| $s_8 = 00111$ | $s_{28} = 10110$ | $s_{48} = 12101$ | $s_{68} = 21222$ |
| $s_9 = 00121$ | $s_{29} = 10111$ | $s_{49} = 12111$ | $s_{69} = 22101$ |
| $s_{10} = 01000$ | $s_{30} = 10121$ | $s_{50} = 12112$ | $s_{70} = 22111$ |
| $s_{11} = 01001$ | $s_{31} = 11000$ | $s_{51} = 12121$ | $s_{71} = 22112$ |
| $s_{12} = 01010$ | $s_{32} = 11001$ | $s_{52} = 12122$ | $s_{72} = 22121$ |
| $s_{13} = 01011$ | $s_{33} = 11010$ | $s_{53} = 12211$ | $s_{73} = 22122$ |
| $s_{14} = 01100$ | $s_{34} = 11011$ | $s_{54} = 12212$ | $s_{74} = 22211$ |
| $s_{15} = 01101$ | $s_{35} = 11012$ | $s_{55} = 12221$ | $s_{75} = 22212$ |
| $s_{16} = 01110$ | $s_{36} = 11100$ | $s_{56} = 12222$ | $s_{76} = 22221$ |
| $s_{17} = 01111$ | $s_{37} = 11101$ | $s_{57} = 21001$ | $s_{77} = 22222$ |
| $s_{18} = 01121$ | $s_{38} = 11110$ | $s_{58} = 21011$ | |
| $s_{19} = 01210$ | $s_{39} = 11111$ | $s_{59} = 21012$ | |
| $s_{20} = 01211$ | $s_{40} = 11112$ | $s_{60} = 21101$ | |

These states form a partial ordering were $s_i$ follows $s_j$ in the ordering if $s_i$ is at least $s_j$ which means the labelings $s_j(p) < s_i(p)$ for the corresponding vertices $p$ in each state. The partial ordering is used to form the At Least Matrix $A$ and the Minimum Previous Matrix $B$ as described for $P_n \times P_r$.

Next, we form the states vector $\mathbf{t}_r$ where $\mathbf{t}_r(s_i)$ is the number of vertices needed to dominate $G \times P_r$ and leave the last copy of $G$ in at least state $s_i$ and the ending vector $\mathbf{y}$ which has a 0 for the state $\mathbf{1}$ and infinity otherwise. For this example, $\mathbf{1}$ is state $s_{39}$. The vectors for the house graph are shown below.

| | | |
|---|---|---|
| $\mathbf{y}$ | $=$ | [--------------------------------------0--------------------------------------] |
| $\mathbf{t}_1$ | $= 0 \boxtimes$ | [00000000-00000000----00000000-0000-0000---------------------------------------] |
| $\mathbf{t}_2$ | $= 2 \boxtimes$ | [21111110111111110010011111100011111110000000011100010112100001010112011121223] |
| $\mathbf{t}_3$ | $= 3 \boxtimes$ | [11111111111111111111010101000000000000000000000001001010112000000010112001121222] |
| $\vdots$ | $\vdots$ | $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ |
| $\mathbf{t}_{18}$ | $= 17 \boxtimes$ | [00000000000000000001000000000000000000001101112001121223001011121222112232323] |
| $\mathbf{t}_{19}$ | $= 18 \boxtimes$ | [11000000010000000000100000000000000000000110111200112122300101112122311223 2334] |
| $\mathbf{t}_{20}$ | $= 19 \boxtimes$ | [110000000100000000001000000000000000000001101112001121223001011121222112232323] |
| $\mathbf{t}_{21}$ | $= 20 \boxtimes$ | [110000000100000000001000000000000000000001101112001121223001011121222112232323] |
| $\mathbf{t}_{22}$ | $= 21 \boxtimes$ | [110000000100000000001000000000000000000001101112001121223001011121222112232323] |
| $\mathbf{t}_{23}$ | $= 22 \boxtimes$ | [110000000100000000001000000000000000000001101112001121223001011121222112232323] |

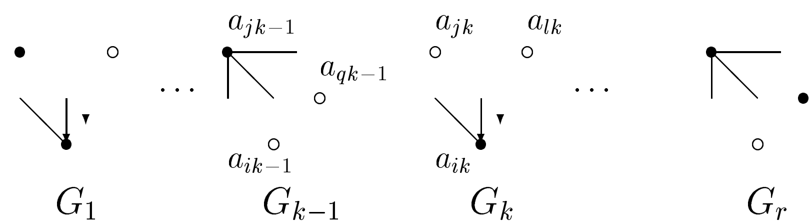
Notice that $\mathbf{t}_{20} = 1 \boxtimes \mathbf{t}_{19}$ and $\mathbf{t}_{21} = 1 \boxtimes \mathbf{t}_{20}$. Thus after 20 iterations, periodicity occurs and we have $\mathbf{t}_r = 1 \boxtimes \mathbf{t}_{r-1}$ for all $r \geq 20$. This allows us to find a closed form for the domination number of the house graph $G$. For all $r \geq 1$, we have

$$\gamma(G \times P_r) = r + \begin{cases} 1 & \text{if } r = 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 11,\ 12,\ 16 \\ 0 & \text{otherwise.} \end{cases}$$

Now to find the domination number of $G \times P_r$, first note that if $G$ is disconnected then so is $G \times P_r$. Thus, if $G_1, G_2, \ldots$ are the components of $G$, then the domination number of $G$ is just the sum of the domination numbers of its components. Thus we only need to find the domination number of connected graphs. To further reduce the number of graphs we need to consider, the following theorem gives the domination number for certain graphs with high degree.

**Theorem 3.14** *If a graph on $n$ vertices $G$ has a set of vertices $a_i \in A$ with $deg(a_i) \geq n-3$ and for each $a_i \in A$, all vertices not in the closed neighborhood of $a_i$ are also in $A$, then $\gamma(G \times P_r) = r$ or $r+1$ or $r+2$.*

*Proof:* Let $a_1, a_2, \ldots$ be the vertices in set $A$ as described above. To dominate $(G \times P_r)$ start with the $k^{\underline{\text{th}}}$ copy of $G = G_k$ where $k \neq 1, r$. Place $a_{ik}$ in the dominating set. Now there are at most two undominated vertices in $G$ both of which are also in $A$, call these vertices $a_{jk}$ and $a_{lk}$. Place $a_{jk-1}$ and $a_{lk+1}$ in the dominating set. Now $a_{hk\pm1}$ is adjacent to $a_{hk}$ by definition of $G \times P_r$, so all the vertices in $G_k$ are now dominated. Next we have that in $G_{k-1}$, $a_{ik-1}$ and the closed neighborhood of $a_{jk-1}$ are dominated and only one vertex $a_{qk-1}$ remains undominated. Place $a_{qk-2}$ in the dominating set and then $G_{k-1}$ is dominated and $G_{k-2}$ has only one undominated vertex. The same process is complete for $G_{k+2}, G_{k\pm3} \ldots$ until $G_1$ and $G_r$ are reached. At each step only one vertex is added to the dominating set and both $G_1$ and $G_r$ have one undominated vertex so adding these vertices to the set gives a domination of $G \times P_r$ with $r+2$ vertices. In the diagram below, dark circles represent vertices in the dominating set, all other vertices are dominated by the corresponding vertex in an adjacent copy of $G$.




If one vertex $a_p$ in $A$ has degree $n-2$ then it may be possible to rearrange the order in which the vertices are place in the dominating set so that $a_{p1}$ is is used when domination $G_2$. If this is possible, then no additional vertex will be necessary to dominate $G_1$ and $\gamma(G \times P_r) = r + 1$. Likewise it may be possible to order the vertices in the dominating set so that both $G_1$ and $G_r$ use vertices of degree $n-2$ and then $\gamma(G \times P_r) = r$. If two vertices $a_p$ and $a_q$ of $A$ have degree $n-2$ then place $a_{pk}$ and $a_{qk+1}$ in the dominating set for $k$ odd. This gives a domination number of $r$. Finally if any vertex of $G$ has degree $n-1$ place one copy of this vertex in the dominating set for each copy of $G$ and this gives $\gamma(G \times P_r) = r$. $\square$

Next, for the connected graphs on $n \leq 6$ vertices that are not covered by Theorem 3.14 the domination number is computed using the At Least Algorithm.

## 3.5 Values of $\gamma(G \times P_r)$ for graphs on $\leq 6$ vertices.

**Theorem 3.15** *For all connected graphs $G$ on $n = 1$ vertices, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{r}{3} \right\rceil$$

**Theorem 3.16** *For all connected graphs $G$ on $n = 2$ vertices, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{r}{2} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0 \pmod{2} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 3.17** *For all connected graphs $G$ on $n = 3$ vertices, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{3r+1}{4} \right\rceil.$$

**Theorem 3.18** *For all connected graphs $G$ on $n = 4$ vertices, we have*

$$\gamma(G \times P_r) = r + \begin{cases} 1 & \text{if } r = 5, 6, 9 \text{ and } G \text{ is } P_4 \text{ or } r = 1 \text{ and } G = C_4 \\ 0 & \text{otherwise.} \end{cases}$$

On five vertices, there are 34 non-isomorphic graphs 13 of which are disconnected and their domination number is found by taking the sum of the domination numbers of the components. There are 21 connected non-isomorphic graphs on 5 vertices, 20 of which are covered by Theorem 3.14.

**Theorem 3.19** *For all connected graphs $G$ on $n = 5$ vertices, we have*

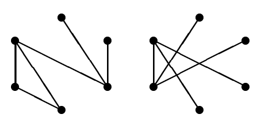
$$\gamma(G \times P_r) = \left\lceil \frac{6r}{5} \right\rceil + \begin{cases} 1 & \text{if } r = 7 \\ 0 & \text{otherwise.} \end{cases}$$

*If $G$ is the path on 5 vertices, and*

$$\gamma(G \times P_r) = r, r+1, \text{ or } r+2 \text{ otherwise}$$

There are 156 graphs on 6 vertices, 44 of these graphs are disconnected and their domination number is found by taking the sum of the domination number of there components which are graphs previously discussed. There are 112 non-isomorphic connected graphs on 6 vertices, 47 of them are covered by Theorem 3.14. The remaining 65 connected graphs fall into 28 different groups and the domination numbers of these graphs are given below.

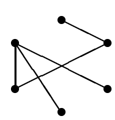
**Theorem 3.20** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{3r}{2} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 2 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

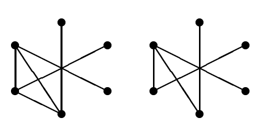
**Theorem 3.21** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{3r}{2} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0 \pmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

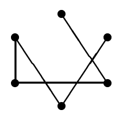
**Theorem 3.22** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{10r}{7} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 2 \pmod 7\text{, or } r = 1,\ 6 \\ 0 & \text{otherwise.} \end{cases}$$

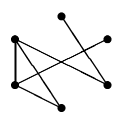
**Theorem 3.23** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{10r}{7} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 2,\ 3,\ 4,\ 6 \pmod 7 \text{ except } r = 3 \\ 0 & \text{otherwise.} \end{cases}$$

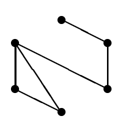
**Theorem 3.24** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{7r}{5} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 2,\ 4 \pmod 5 \text{ except } r = 4,\ 7,\ 9, \\ & 14 \\ 0 & \text{otherwise.} \end{cases}$$

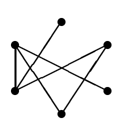
**Theorem 3.25** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{7r}{5} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 2,\ 4,\ 5,\ 6,\ 7,\ 9 \pmod{10} \\ 0 & \text{otherwise.} \end{cases}$$

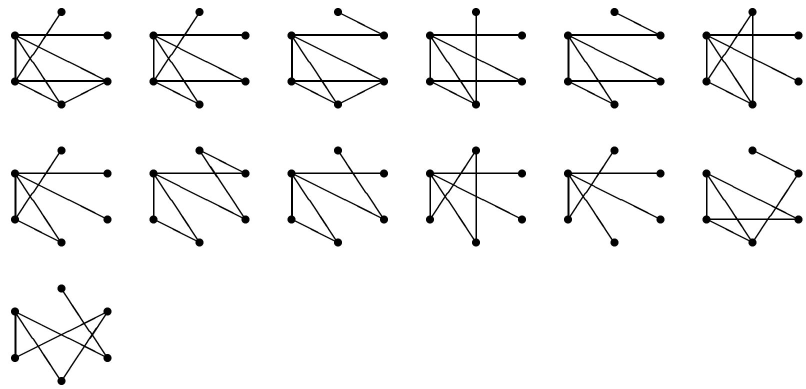
**Theorem 3.26** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{11r}{8} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 2,\ 4,\ 5,\ 7 \pmod 8 \text{ except } r = 4,\ 7, \\ & \text{or } r = 6,\ 9 \\ 0 & \text{otherwise.} \end{cases}$$

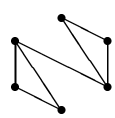
**Theorem 3.27** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{4r}{3} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 2 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

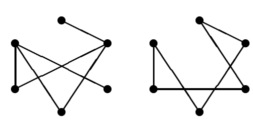
**Theorem 3.28** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{4r}{3} \right\rceil + \begin{cases} 2 & \text{if } r \equiv 0 \pmod 3 \text{ except } r = 3 \\ 1 & \text{if } r \equiv 1,\ 2 \pmod 3 \text{ except } r = 1,\ 4, \text{ or } r = 3 \\ 0 & \text{otherwise.} \end{cases}$$

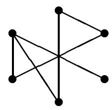
**Theorem 3.29** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{4r}{3} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 2 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

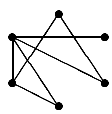
**Theorem 3.30** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{13r}{10} \right\rceil + \begin{cases} 2 & \textit{if } r \equiv 0 \textit{ (mod 10) except } r = 10 \\ 1 & \textit{if } r \equiv 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9 \textit{ (mod 10)} \\ & \textit{except } r = 1,\ 4,\ 7, \textit{ or } r = 10 \\ 0 & \textit{otherwise.} \end{cases}$$

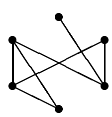
**Theorem 3.31** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{9r}{7} \right\rceil + \begin{cases} 1 & \textit{if } r \equiv 7 \textit{ (mod 14)} \\ 0 & \textit{otherwise.} \end{cases}$$

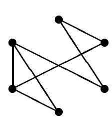
**Theorem 3.32** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{14r}{11} \right\rceil + \begin{cases} 1 & \textit{if } r = 2,\ 3,\ 7 \\ 0 & \textit{otherwise.} \end{cases}$$

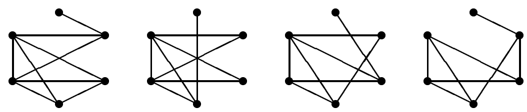
**Theorem 3.33** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{14r}{11} \right\rceil + \begin{cases} 1 & \textit{if } r \equiv 0,\ 2,\ 3,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10 \textit{ (mod 11)} \\ & \textit{except } n = 5,\ 8,\ 9 \\ 0 & \textit{otherwise.} \end{cases}$$

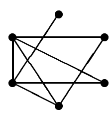
**Theorem 3.34** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

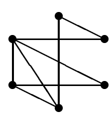
$$\gamma(G \times P_r) = \left\lceil \frac{5r}{4} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 3.35** *Let $G$ be one of the following graphs:*



*Then $\gamma(G \times P_r) = \left\lceil \frac{5r}{4} \right\rceil$ for all $r \geq 1$.*

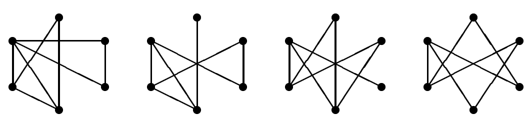
**Theorem 3.36** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{5r}{4} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 4 \pmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

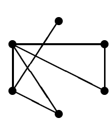
**Theorem 3.37** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{5r}{4} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 2,\ 3 \pmod 4 \text{ except } r = 2,\ 3,\ 6 \\ 0 & \text{otherwise.} \end{cases}$$

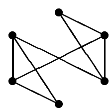
**Theorem 3.38** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{5r}{4} \right\rceil + \begin{cases} 2 & \text{if } r \equiv 0 \pmod 4 \text{ except } r = 4,\ 8,\ 12 \\ 1 & \text{if } r \equiv 1,\ 2,\ 3 \pmod 4 \text{ except } r = 1,\ 2,\ 3,\ 5, \\ & \quad 6,\ 9, \text{ or } r = 4,\ 8,\ 12 \\ 0 & \text{otherwise.} \end{cases}$$

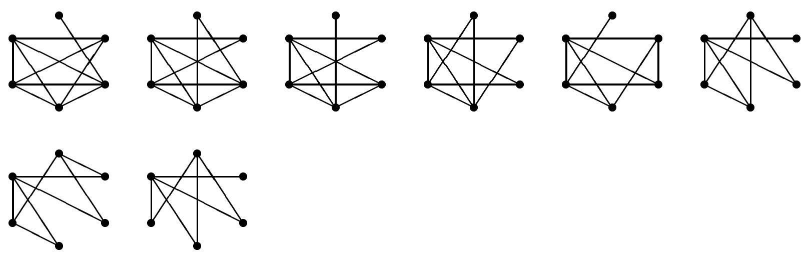
**Theorem 3.39** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

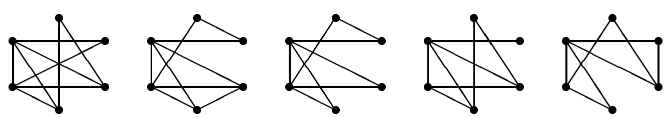
$$\gamma(G \times P_r) = \left\lceil \frac{5r}{4} \right\rceil + \begin{cases} 1 & \text{if } r = 2,\ 3,\ 4,\ 7 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 3.40** *Let $G$ be one of the following graphs:*



*Then $\gamma(G \times P_r) = \left\lceil \frac{6r}{5} \right\rceil$ for all $r \geq 1$.*

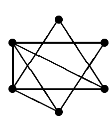
**Theorem 3.41** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{6r}{5} \right\rceil + \begin{cases} 1 & \text{if } r = 4 \\ 0 & \text{otherwise.} \end{cases}$$

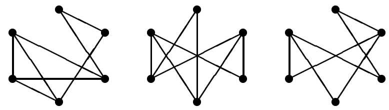
**Theorem 3.42** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{6r}{5} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 4 \pmod 5 \text{ except } r = 4 \\ 0 & \text{otherwise.} \end{cases}$$

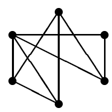
**Theorem 3.43** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{6r}{5} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 2,\ 3,\ 4 \pmod 5 \text{ except } r = 2,\ 3 \\ 0 & \text{otherwise.} \end{cases}$$

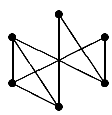
**Theorem 3.44** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{7r}{6} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 4,\ 5 \pmod 6 \text{ except } r = 4,\ 5,\ 10 \\ 0 & \text{otherwise.} \end{cases}$$

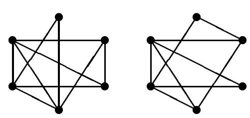
**Theorem 3.45** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{7r}{6} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 2,\ 3,\ 4,\ 5 \pmod 6 \text{ except } r = 2,\ 3 \\ 0 & \text{otherwise.} \end{cases}$$

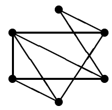
**Theorem 3.46** *Let $G$ be one of the following graphs:*



*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{8r}{7} \right\rceil + \begin{cases} 1 & \text{if } r = 6 \\ 0 & \text{otherwise.} \end{cases}$$

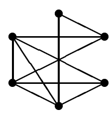
**Theorem 3.47** *Let $G$ be one of the following graphs:*

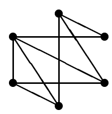


*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{8r}{7} \right\rceil + \begin{cases} 1 & \text{if } r \equiv 0,\ 2,\ 3,\ 4,\ 5,\ 6 \pmod 7 \text{ except } r = 2, \\ & 3,\ 5,\ 9,\ 10 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 3.48** *Let $G$ be one of the following graphs:*





*Then for all $r \geq 1$, we have*

$$\gamma(G \times P_r) = \left\lceil \frac{10r}{9} \right\rceil + \begin{cases} 1 & \text{if } r = 8 \\ 0 & \text{otherwise.} \end{cases}$$

# 4. Domination of Circulant Graphs

In this chapter we will find the domination number of circulant graphs.

**Definition 4.1** *Given an integer $r > 0$ and subset of the positive integer $S$, the* circulant graph *$C_r[S]$ is a graph on vertices 0, 1, ..., $r-1$ with vertex $i$ adjacent to vertex $j$ if $i - j \in S$ or $j - i \in S$ (mod $r$).*

Let $C_r[1,3]$ be shorthand for $C_r[\{1,3\}]$, where $C_r[1,3]$ is a graph on vertices 0, 1, ..., $r-1$ with vertex $i$ adjacent to $i \pm 1$ and $i \pm 3$ (mod $r$). To illustrate the algorithm we will find $\gamma(C_r[1,3])$ which has domination number 4 (see Figure 4.1). To find the domination number for all $r$, we formulate the problem as a dynamic programming algorithm similar to the algorithm used in Hare and Hedetniemi [10] and Fisher [8] and Chapter 3 to find the domination number of complete grid graphs. Then, like is done in [8], we iterate the algorithm until a periodicity is detected. When this happens, we declare that we know the answer for all $r$, and stop iterating. This allows us to find $C_r[1,3]$ for all $r$ in a finite amount of time.

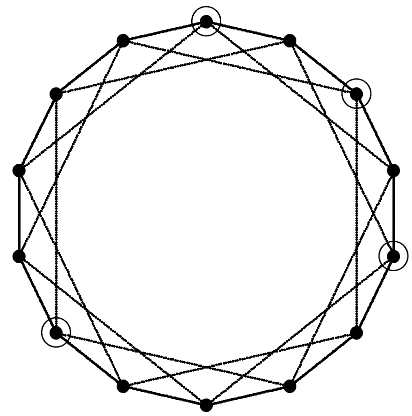


**Figure 4.1** The circulant graph $C_{14}[1,3]$. The circled vertices are a minimum domination.

## 4.1 The States and Transition Matrix

We can think of $C_r[1,3]$ as a sequences of overlapping paths on three vertices ($P_3$) where adjacent $P_3$'s share two vertices and there is an edge between the nonshared vertices. Connecting together $r$ of these $P_3$'s so the $r^{\underline{\text{th}}}$ $P_3$ overlaps the first builds $C_r[1,3]$.

We will build a domination by appending $P_3$'s clockwise from the existing sequence. The vertices of $P_3$ are either in the domination, dominated but not in the domination, or not yet dominated (and thus need to be dominated by a vertex in the clockwise direction). The states will be all valid combinations of these three attributes.

Figure 4.2 shows the 15 valid states. This is less than the 27 ways in which 3 vertices can be assigned 3 attributes. This is because 10 potential states have a dominated vertex next to an undominated vertex, while two others have the first vertex undominated and the last vertex is dominated and these are not valid since the last vertex cannot be dominated without dominating the first vertex.

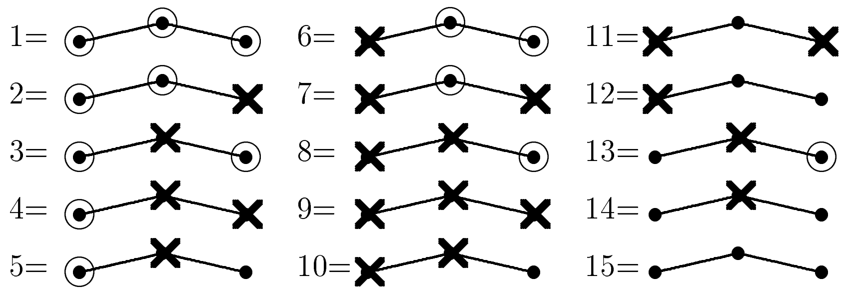



**Figure 4.2** The fifteen states of $P_3$. Circled vertices are in the domination. Dominated vertices have a $\times$ on them.

Going clockwise, we will refer to the vertices of a state as being vertex $a$, $b$, and $c$. *When can two states overlap?* Let vertices $k$, $k+1$, and $k+2$ be in state $j$ and $k+1$, $k+2$, and $k+3$ be in state $i$. To be consistent, we must have the following.

(1) Vertex $b$ of state $j$ must be the same as vertex $a$ of state $i$.

(2) Vertex $c$ of state $j$ is the same as vertex $b$ of state $i$ unless vertex $c$ of state $i$ is in the domination. In this case, if vertex $c$ of state $j$ is undominated, then vertex $b$ of state $i$ is “dominated but not in the domination”.

(3) If vertex $a$ of state $j$ is in the domination, then vertex $c$ of state $i$ is either be in the domination, or “dominated but not in the domination”. If vertex $a$ of state $j$ is “dominated but not in the domination”, then

vertex $c$ of state $i$ can either be in the domination, or not dominated. Finally, if vertex $a$ of state $j$ is not dominated, then vertex $c$ of state $i$ must be in the domination.

The last condition forces vertex $a$ of state $j$ to be dominated by vertex $c$ in state $i$ if it has not already been dominated. Thus states that have vertex $a$ dominated match to two states, while states in which vertex $a$ is not dominated match to only one state.

We will represent which states match to other states by a transition matrix $A$. The $i, j$ entry of $A$ is:

$$A_{i,j} = \begin{cases} 0 & \text{if state } j \text{ matches to state } i \\ & \text{and vertex } c \text{ of state } i \text{ is not in the domination} \\ 1 & \text{if state } j \text{ matches to state } i \\ & \text{and vertex } c \text{ of state } j \text{ is in the domination} \\ \infty & \text{if state } j \text{ does not match to state } i. \end{cases}$$

Then the transition matrix is:

$$A = \begin{bmatrix}
1 & \infty & \infty & \infty & \infty & 1 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
0 & \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & 1 & \infty & \infty & \infty & \infty & 1 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & 1 & \infty & \infty & \infty & \infty & 1 & \infty & \infty & \infty & \infty & 1 & \infty & \infty \\
\infty & \infty & 0 & \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & 1 & 1 & \infty & \infty & \infty & 1 & 1 & \infty & \infty & \infty & 1 & \infty \\
\infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 1 & \infty & \infty & 1 \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty
\end{bmatrix}.$$

The information in the matrix can also be represented in the precedence digraph $D_A$ which is shown in Figure 4.3.

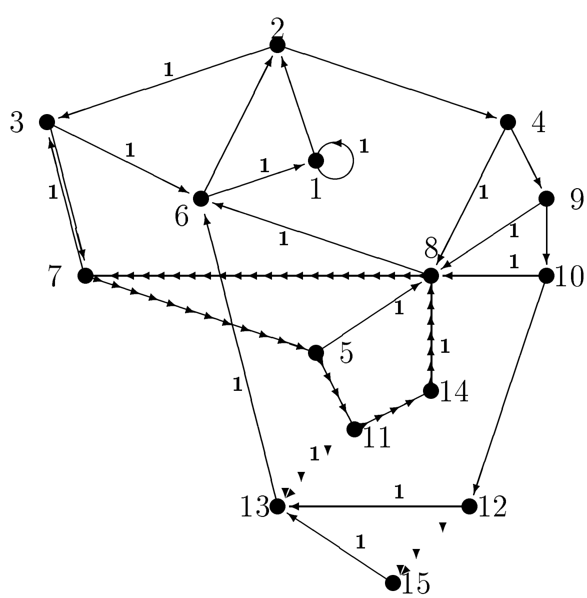



**Figure 4.3** Precedence Digraph for Circulant graphs. The unlabeled arcs have weight zero. The arrowed arcs show a cycle whose cycle mean is minimum.

Since the states in a domination must match, a domination of $C_r[1,3]$ corresponds to a path (not necessarily simple) in $D_A$. Further a domination must be a cycle because the first state and the $r^{\underline{\text{th}}}$ state must match to build $C_r[1,3]$. So a minimum domination of $C_r[1,3]$ corresponds to a cycle in $D_A$ of length $r$ with minimum weight.

**Theorem 4.2** *Let $A$ be an $15 \times 15$ matrix shown above. Then for all $r > 0$, we have that*

$$\gamma(C_r[1,3]) = trace(A^r)$$

*where matrix exponentiation and trace are in Min-Plus algebra.*

*Proof:* By Theorem 2.7, $(A^r)_{i,i}$ is the minimum weight cycle of length $r$ from $i$ to $i$ in $D_A$. By the definition of $D_A$, this is the number of vertices in a domination starting and ending in state $i$. Thus $\text{trace}(A^r) = \min_i (A^r)_{i,i}$ is the minimum size of a domination of $C_r[1,3]$. $\Box$

## 4.2 Periodicity of the Transition Matrix

Theorem 4.2 gives us a method for finding the domination number of $C_r[1,3]$ for any $r$. However, we cannot use it to find $\gamma(C_r[1,3])$ for all $r$. This depends on the periodic property of Min Plus algebra. In this example, this property can be shown by looking at $A^{19}$ and $A^{24}$.

$$A^{19} = \begin{bmatrix} 6&6&6&6&6&6&6&6&6&7&6&7&7&7&7 \\ 5&5&5&5&5&5&5&5&5&6&5&6&6&6&6 \\ 5&5&5&5&6&5&5&5&5&6&5&6&6&6&6 \\ 5&5&5&5&5&5&5&5&5&5&5&6&5&5&6 \\ 4&4&4&4&5&4&4&4&4&5&4&5&5&5&5 \\ 5&5&5&5&5&5&5&5&6&6&6&6&6&6&7 \\ 4&4&4&4&4&4&5&4&5&5&5&5&5&5&6 \\ 4&4&5&5&5&4&4&5&5&5&5&6&5&5&6 \\ 4&5&4&5&5&4&5&4&5&5&5&6&5&5&6 \\ 4&4&4&5&5&4&4&4&5&5&5&6&5&5&6 \\ 4&4&4&4&4&4&4&4&4&4&5&5&4&4&5 \\ 4&4&4&4&4&4&4&4&5&5&5&5&5&5&6 \\ 4&5&4&5&5&4&5&4&5&5&5&6&5&5&6 \\ 3&4&3&4&4&3&4&3&4&5&4&5&4&5&5 \\ 4&4&4&4&4&4&4&4&4&5&4&5&5&5&5 \end{bmatrix}$$

$$A^{24} = \begin{bmatrix} 7&7&7&7&7&7&7&7&7&8&7&8&8&8&8 \\ 6&6&6&6&6&6&6&6&6&7&6&7&7&7&7 \\ 6&6&6&6&7&6&6&6&6&7&6&7&7&7&7 \\ 6&6&6&6&6&6&6&6&6&6&6&7&6&6&7 \\ 5&5&5&5&6&5&5&5&5&6&5&6&6&6&6 \\ 6&6&6&6&6&6&6&6&7&7&7&7&7&7&8 \\ 5&5&5&5&5&5&6&5&6&6&6&6&6&6&7 \\ 5&5&6&6&6&5&5&6&6&6&6&7&6&6&7 \\ 5&6&5&6&6&5&6&5&6&6&6&7&6&6&7 \\ 5&5&5&6&6&5&5&5&6&6&6&7&6&6&7 \\ 5&5&5&5&5&5&5&5&5&5&6&6&5&5&6 \\ 5&5&5&5&5&5&5&5&6&6&6&6&6&6&7 \\ 5&6&5&6&6&5&6&5&6&6&6&7&6&6&7 \\ 4&5&4&5&5&4&5&4&5&6&5&6&5&6&6 \\ 5&5&5&5&5&5&5&5&5&6&5&6&6&6&6 \end{bmatrix}$$

Note every entry in $A^{24}$ is one more than the corresponding entry in $A^{19}$. This can be notated as $A^{24} = 1 \boxtimes A^{19}$. Then

$$A^{25} = A \boxtimes A^{24} = A \boxtimes (1 \boxtimes A^{19}) = 1 \boxtimes (A \boxtimes A^{19}) = 1 \boxtimes A^{20}.$$

Continuing in this manner, we get the following theorem.

**Theorem 4.3** *Let $A$ be the $15 \times 15$ matrix given above. Then for all $r \geq 24$, we have*

$$A^r = 1 \boxtimes A^{r-5}.$$

*Proof:* Since $A^{24} = 1 \boxtimes A^{19}$, the result holds for $r = 24$. For $n > 24$, assume for induction purposes that $A^{r-1} = 1 \boxtimes A^{r-6}$. Then

$$A^r = A \boxtimes A^{r-1} = A \boxtimes (1 \boxtimes A^{r-6}) = 1 \boxtimes (A \boxtimes A^{r-6}) = 1 \boxtimes A^{r-5}.$$

□

Thus, we can use this repetition to find the domination number of $C_r[1,3]$ for all $r$. By observing the domination numbers when $r < 24$, we find an explicit formula for $C_r[1,3]$.

**Theorem 4.4** *For all $r \geq 1$, we have*

$$\gamma(C_r[1,3]) = \begin{cases} \lceil \frac{n}{5} \rceil + 1 & \textit{if } n = 4 \textit{ (mod 5)} \\ \lceil \frac{n}{5} \rceil & \textit{otherwise.} \end{cases}$$

Using this method, we let $S$ be any set from the integers $\{1, 2, \ldots 9\}$ and we find the domination number of $C_r[S]$ for all $r$. There are 512 different circulant graphs on $S$ and the domination numbers of these graphs is given in Appendix A.

# 5. Knights Domination of $k \times r$ Chessboards

In this chapter we use Min-Plus Algebra to develop a dynamic programming algorithm which solves the Knight domination problem for $k \times r$ chessboards. The game of Chess is played on an $8 \times 8$ board on which a number of pieces are placed. One piece, a Knight, can move (or attack) "from corner to diagonally-opposite corner of a rectangle three squares by two" (Dr. Lasker as quoted in [12]). See Figure 5.1.

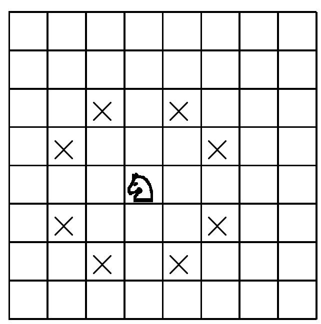


**Figure 5.1** Knight Moves: A Knight (♘) attacks the squares marked with $\times$.

**Definition 5.1** *A* Knight domination *of a $k \times r$ board is a placement of Knights so each square either has a Knight on it or attacking it. The $k \times r$* Knight domination number *is the minimum number of Knights in a Knight domination of a $k \times r$ board.*

Figure 5.2 shows a domination of a $3 \times 14$ board with 11 Knights. Since 11 is the minimum number, the $3 \times 14$ Knight domination number is 11.

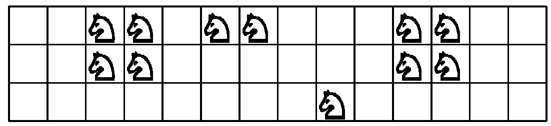


**Figure 5.2** A Minimal $3 \times 14$ Knight Domination.

The $k \times r$ Knight Graph, denoted $N_{k,r}$ ($N$ is the standard initial for "Knight", as $K$ is used for "King"), has vertices $\{1, 2, \ldots, k\} \times \{1, 2, \ldots, r\}$ with an edge between vertices $(g, h)$ and $(i, j)$ if $|i - g||j - h| = 2$ (see Figure 5.3). Then the $k \times r$ Knight domination number is $\gamma(N_{k,r})$.

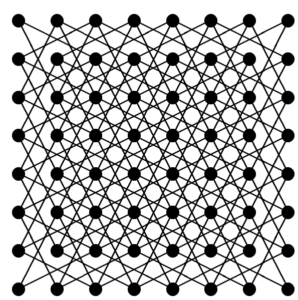
$N_{8,8} =$ 

**Figure 5.3** The $8 \times 8$ Knight Graph. Vertices represent squares of an $8 \times 8$ board with edges showing moves that a Knight can make. Its domination number is the Knight domination number of an $8 \times 8$ board.

Gardner [9] generated considerable interest in Knight domination of square boards (when $k = r$). Gardner and his readers found small Knight dominations of a $k \times k$ board for $k \leq 15$. These are known to be optimum when $k \leq 10$.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\gamma(N_{k,k})$ | 1 | 4 | 4 | 4 | 5 | 6 | 10 | 12 | 14 | 16 | $\leq 21$ | $\leq 24$ | $\leq 28$ | $\leq 32$ | $\leq 37$ |

Hare and Hedetniemi [10] developed a dynamic programming algorithm for the $k \times r$ Knight domination number which is exponential in $k$, but linear in $r$. This allowed them to conjecture formulas for $\gamma(N_{k,r})$ for $k = 3$, 4, and 6 ($k = 1$

and 2 are trivial; $k = 5$ was "not fully analyzed"). They also found values for $k \leq 10$ for some values of $r$. In this chapter, we restate the algorithm in Min-Plus algebra and looks for periodic solutions to the dynamic programming algorithm. By finding periodicity, we prove the conjectures in [10], and go on to find the $k \times r$ Knight domination number for all $k \leq 8$ and for all $r$. To illustrate the dynamic programming algorithm, we will use it to find the $2 \times r$ Knight domination number.

## 5.1 The States

A state will be a configuration of two columns of a $k \times r$ board. A square either has a Knight on it (shown by ♘), is dominated by a Knight from the left or within the two columns (shown by $\times$), or to be dominated from the right of the two columns (shown by a blank). Since a blank cannot be a Knight move from a Knight, states correspond to labellings of $N_{k,2}$ with 0, 1 and 2 (for squares with a ♘, squares with a $\times$, and blank squares, resp.) where 0 and 2 cannot be adjacent.

*How many such labellings are possible?* Let $a_r$ be the number of such labellings on a path on $r$ vertices. It is easy to show that $a_1 = 3$, $a_2 = 7$, and $a_r = 2a_{r-1} + a_{r-2}$ for all $r > 2$. So $a_3 = 17$, $a_4 = 41$, $a_5 = 99$, etc. We can then solve this recursion to show that

$$a_r = \frac{\left(1+\sqrt{2}\right)^{r+1} + \left(1-\sqrt{2}\right)^{r+1}}{2}.$$

Since $N_{k,2}$ consists of two paths on $\lfloor k/2 \rfloor$ vertices and two paths on $\lceil k/2 \rceil$ vertices, the number of states needed to find the $k \times r$ Knight domination number for all $r$ is (Figure 5.4 shows the 81 states when $k = 2$):

$$\text{Number of States} = a^2_{\lfloor k/2 \rfloor} a^2_{\lceil k/2 \rceil} \approx \frac{\left(1+\sqrt{2}\right)^{2k+4}}{16}. \tag{5.1}$$

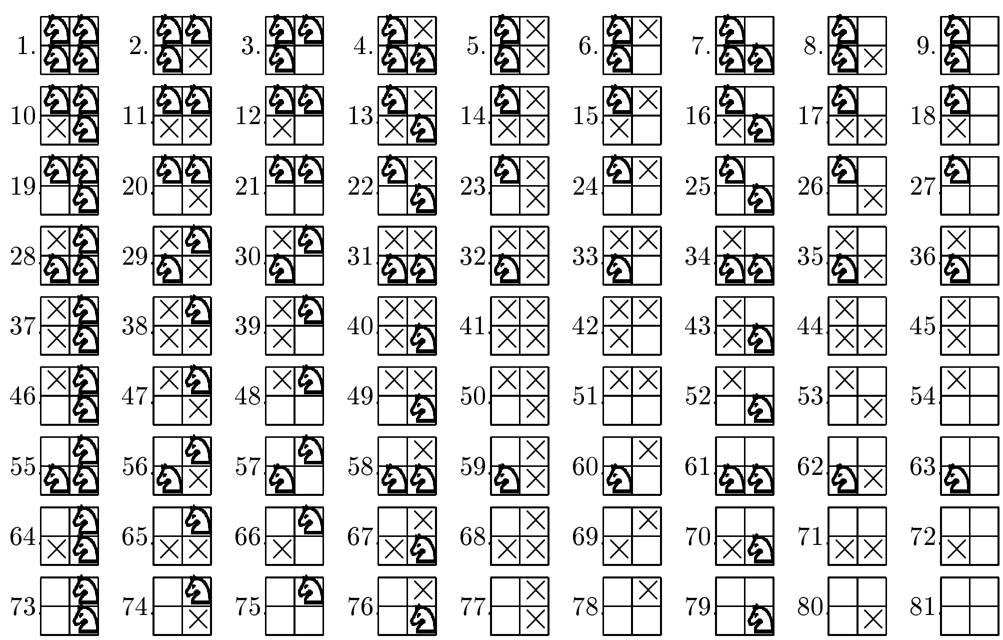


**Figure 5.4** The 81 Knights configurations of a $2 \times 2$ board.

## 5.2 The Transition Matrix

Of the $3^6$ possible $2 \times 3$ boards, only 81 need to be considered for domination patterns. This is because each square in the third column of this board is a Knight move from a square in the first column (square (1,1) and (2,3) are a Knight move apart as are square (2,1) and (1,3)). The possible $2 \times 3$ boards are created by matching two of the $2 \times 2$ boards shown in Figure 5.4 together so that the middle columns overlap and the following conditions are satisfied.

(1) Any Knights in the overlapping columns must be in the same square in each pattern.

(2) All dominated ($\times$) squares in the last column of the second $k \times 2$ board must be dominated by Knights within the resulting $k \times 3$ board.

(3) All undominated (blank) squares in the first column of the first $k \times 2$ board must be dominated by Knights in the resulting $k \times 3$ board.

(4) When matching a $k \times r$ board to a $k \times 2$ board, condition 2 guarantees that all dominated squares in the last column of the $k \times r$ board are dominated by Knights within previous columns. Thus, an undominated square in the first column of the second pattern can match with a dominated square in the last column of the first pattern since the undominated square will become dominated in the match. (This matching does not apply to $2 \times 2$ boards since the pattern is not possible).

All possible domination patterns of a $k \times 3$ board are found similarly by matching together two of the domination patterns of a $k \times 2$ board following the rules listed above. In Figure 5.5, possible matchings of $k \times 2$ boards are shown to illustrate the matching requirements.

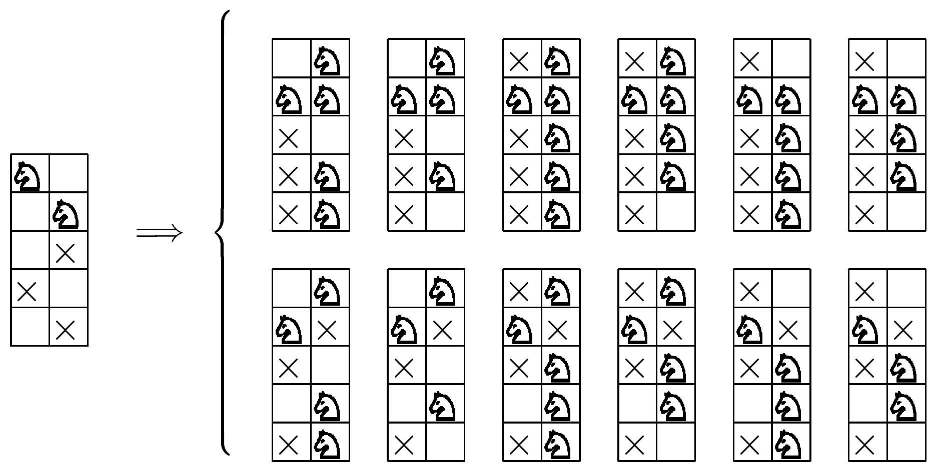


**Figure 5.5** Possible Matchings of $k \times 2$ Boards

To find the domination number of a $2 \times r$ board, a column is added on to all of the possible patterns on $2 \times (r-1)$ boards and then each of these new patterns is checked to see if it is a domination. The minimum number of Knights in the patterns which are dominations is then the domination number of a $2 \times r$ board. The $r^{\underline{\text{th}}}$ column that is being added onto the $2 \times (r-1)$ board is dependent on the last two columns of the $2 \times (r-1)$ board. Thus, the column is added by matching the last column of the $2 \times (r-1)$ pattern to the first column of one of the $2 \times 2$ boards following the matching constraints listed above. To keep track of which patterns can be added onto other patterns by using this matching, a transition matrix $A$ is formed with rows and columns being represented by the initial $2 \times 2$ boards. If the $i^{\underline{\text{th}}}$ $k \times 2$ board matches the $j^{\underline{\text{th}}}$ $2 \times 2$ board then the $(i, j)$ entry in the transition matrix is the number of Knights in the last column of the $j^{\underline{\text{th}}}$ $2 \times 2$ board. If the $i^{\underline{\text{th}}}$ $2 \times 2$ board does not match the $j^{\underline{\text{th}}}$ board then the $(i, j)$ entry in the transition matrix is infinity or (-) (see below).

```
2--------2--------2--------2--------2--------2--------2--------2--------2--------
1--------1--------1--------------------------------------------------------------
---------------------------1--------1--------1-----------------------------------
1--------------------------1--------------------------1--------------------------
0--------------------------------------------------------------------------------
---------------------------0-----------------------------------------------------
---------1--------------------------1--------------------------1-----------------
---------0-----------------------------------------------------------------------
------------------------------------0--------------------------------------------
-2--------2--------2--------2--------2--------2--------2--------2--------2-------
-1--------1--------1-------------------------------------------------------------
----------------------------1--------1--------1----------------------------------
-1--------------------------1--------------------------1-------------------------
-0-------------------------------------------------------------------------------
----------------------------0----------------------------------------------------
----------1--------------------------1--------------------------1----------------
----------0----------------------------------------------------------------------
-------------------------------------0-------------------------------------------
--2--------2--------2--------2--------2--------2--------2--------2--------2------
--1--------1--------1------------------------------------------------------------
-----------------------------1--------1--------1---------------------------------
--1--------------------------1--------------------------1------------------------
--0------------------------------------------------------------------------------
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---2--------2--------2--------2--------2--------2--------2--------2--------2-----
---1--------1--------1-----------------------------------------------------------
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---0-----------------------------------------------------------------------------
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----1--------1--------1----------------------------------------------------------
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## 5.3 The States Vector

Next, a **states vector** $\mathbf{x}_r$ is formed to keep track of the 81 possible minimal states of a $2 \times r$ board. The $i^{\underline{\text{th}}}$ entry in this vector is the number of Knights in a $2 \times r$ pattern whose last 2 columns are the same as the $i^{\underline{\text{th}}}$ $2 \times 2$ board in Figure 5.4. If one of these $2 \times 2$ boards does not form the last 2 columns of any of the possible $2 \times r$ patterns then the entry in $\mathbf{x}_r$ is infinity or (-). Now, to find the possible patterns of a $2 \times (r+1)$ board, the transition matrix is multiplied by $\mathbf{x}_r$ using Min-Plus matrix multiplication to form a new vector $\mathbf{x}_{r+1}$. The $i^{\underline{\text{th}}}$ entry in $\mathbf{x}_{r+1}$ represents the number of Knights in a $2 \times (r+1)$ pattern whose last 2 columns are the same as the $i^{\underline{\text{th}}}$ $2 \times 2$ board. For each of these $2 \times (r+1)$ patterns, the first $r-1$ columns are dominated. If the $i^{\underline{\text{th}}}$ $2 \times 2$ board, which forms the $r$, and $r+1$ columns of the pattern, is also dominated, then the entire $2 \times (r+1)$ pattern will be dominated. Out of these dominated states, the one with the least number of Knights in it (the lowest entry in $\mathbf{x}_{r+1}$) forms a minimal domination of a $2 \times (r+1)$ board and the domination number of a $2 \times (r+1)$ board is the number of Knights in this pattern.

This iterative process is started with an initial vector $\mathbf{x}_1$ which is used to find the domination number of a $2 \times 1$ board. To do this, the first column of the 81 $2 \times 2$ boards is considered column 0 of the $2 \times 1$ board. Only the $2 \times 2$ boards in which the first column is dominated and contains no Knights are possible patterns for a $2 \times 1$ board. Thus, the $i^{\underline{\text{th}}}$ entry in the initial vector $\mathbf{x}_1$ is either the number of Knights in the last column of the $i^{\underline{\text{th}}}$ $2 \times 2$ board if the first column of this board is dominated and contains no Knights, or infinity (-) if the first column is not dominated or has Knights in it. Using this initial vector, the domination number of a $2 \times r$ board is found by iterative use of Min-Plus matrix multiplication of the states vectors and the transition matrix $A$ with

$$\mathbf{x}_r = A\mathbf{x}_{r-1}.$$

The states vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{18}$ are shown below.

$\mathbf{x}_1 = 0$ ⊠ [------------------------------------2-1---1-0------------------------------------]

$\mathbf{x}_2 = 0$ ⊠ [4-3---3-2---------3-2---2-1---------------------------3-2---2-1---------2-1---1-0]

$\mathbf{x}_3 = 2$ ⊠ [22-22-------------11-11-------------------------------11-11-------------00-00----]

$\mathbf{x}_4 = 4$ ⊠ [00-00----00-00-------------00-00----00-00----------------------------------------]

$\mathbf{x}_5 = 4$ ⊠ [211100100211100100---------211100100211100100------------------------------------]

$\mathbf{x}_6 = 4$ ⊠ [433322322322211211322211211322211211211100100211100100322211211211100100211100100]

$\mathbf{x}_7 = 4$ ⊠ [443443332332332221332332221332332221221221110221221110332332221221221110221221110]

$\mathbf{x}_8 = 4$ ⊠ [443443332443443332332332221443443332443443332332332221332332221332332221221221110]

$\mathbf{x}_9 = 6$ ⊠ [222222222222222222111111111222222222222222222111111111111111111111111111000000000]

$\mathbf{x}_{10} = 8$ ⊠ [000000000000000000000000000000000000000000000000000000000000000000000000000000000]

$\mathbf{x}_{11} = 8$ ⊠ [211100100211100100211100100211100100211100100211100100211100100211100100211100100]

$\mathbf{x}_{12} = 8$ ⊠ [433322322322211211322211211322211211211100100211100100322211211211100100211100100]

## 5.4 Detecting Periodicity

To find the domination number of a $2 \times r$ board and the pattern that gives this domination number, repetition can be used. Looking at the 18 states vectors, note that $\mathbf{x}_{12} = \mathbf{x}_6 + 4$. Since the values in $x_r$ only depend on the values in $x_{r-1}$ and the transition matrix, we have $\mathbf{x}_{r+6} = \mathbf{x}_r + 4$ for all $r \geq 6$. So the domination pattern for a $2 \times (r+6)$ board is found by finding the domination pattern for a $2 \times r$ board and adding onto this a dominated pattern with 6 columns and 4 Knights. The domination pattern for a $2 \times r$ board for all $r$ is shown in Figure 5.6.

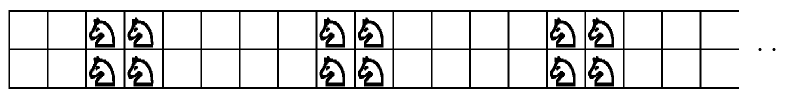

**Figure 5.6** Domination Pattern for a $2 \times r$ board.

Periodicity is detected by storing each $\mathbf{x}_r$ and comparing its entries with the entries in all previous state vectors.

The algorithm for finding the domination number of a $k \times r$ board is essentially the same as for a $2 \times r$. In each case, all possible $k \times 2$ boards are formed. The number of possible $k \times 2$ boards is equal to $3^{2k}$ minus the number of impossible patterns which is approximated by the formula 5.1. A transfer matrix $A$ and and initial vector $\mathbf{x}_1$ are then formed from these $k \times 2$ boards. Min-Plus matrix multiplication is then used to find $\mathbf{x}_r$ and the domination number of a $k \times r$ board. As with $2 \times r$ the domination number is the smallest entry in $\mathbf{x}_r$ which corresponds to a dominated $k \times r$ board. Successive $\mathbf{x}_r$ vectors are computed until a repetition is found and this is used to determine the pattern corresponding to the domination number for all $r$. The results for

$k \leq 7$ are given in the following sections.

## 5.5 Domination Number for a $3 \times r$ board

For a $3 \times r$ board, the periodicity is given by

$$\mathbf{x}_{r+6} = \mathbf{x}_r + 4.$$

This means that once an initial pattern is established, the pattern can be extended by adding on a pattern of 6 columns which is dominated by 4 Knights. The domination patterns are shown in Figures 5.7 5.8 5.9.

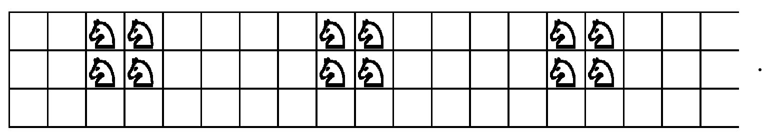

**Figure 5.7** Domination Pattern of $3 \times r$ for $r = 0$, 3, 4, and 5. For $r = 0$, 3, 4, and 5 (mod 6) $r \geq 4$. For $r = 3$ use columns 3, 4, and 5 of this pattern. For $r = 8$, remove columns 4 and 5 of this pattern.

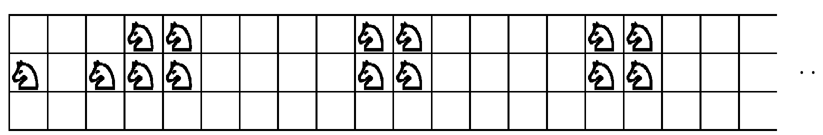

**Figure 5.8** Domination Pattern of $3 \times r$ for $r = 1 \pmod 6$ $r \geq 4$

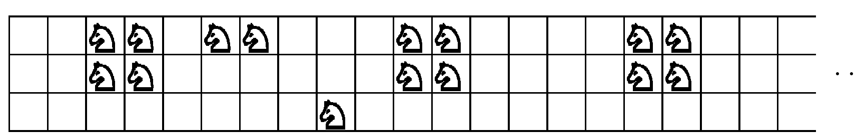

**Figure 5.9** Domination Pattern of $3 \times r$ for $r = 2 \pmod 6$ $r \geq 14$

## 5.6 Domination Number for a $4 \times r$ board

For a $4 \times r$ board, the periodicity is given by

$$\mathbf{x}_{r+6} = \mathbf{x}_r + 4.$$

This means that once an initial pattern is established, the pattern can be extended by adding on a pattern of 6 columns which is dominated by 4 Knights. The algorithm detects periodicity at $r = 27$. The domination patterns are shown in Figures 5.10 5.11 5.12

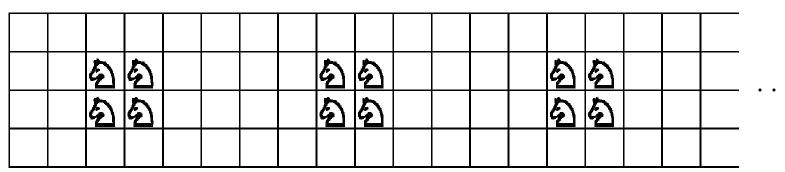

**Figure 5.10** Domination Pattern of $4 \times r$ Board, for $r = 0$, 3, 4, and 5 (mod 6) when $r \geq 4$. For $r = 8$, remove columns 4 and 5 of this pattern.

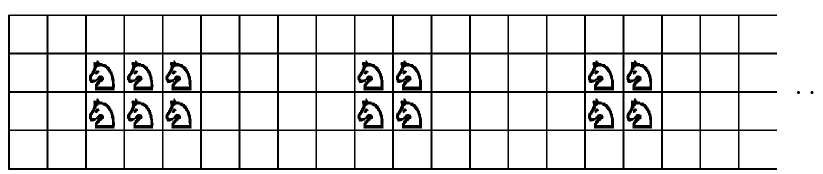

**Figure 5.11** Domination Pattern of $4 \times r$ Board, for $r = 1$ (mod 6) $r \geq 4$.

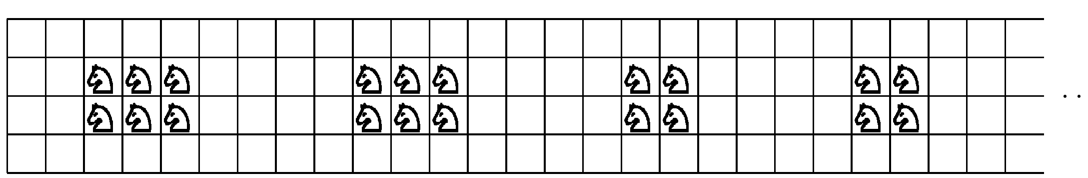

**Figure 5.12** Domination Pattern of $4 \times r$ Board for $r = 2$ (mod 6) $r \geq 14$.

## 5.7 Domination Number for a $5 \times r$ board

For a $5 \times r$ board, the periodicity is given by

$$\mathbf{x}_{r+18} = \mathbf{x}_r + 14.$$

This means that once an initial pattern is established, the pattern can be extended by adding on a pattern of 18 columns which is dominated by 14 Knights. The algorithm detects periodicity at $r = 39$. The domination patterns can be described in terms of the separate components shown in Figure 5.13.

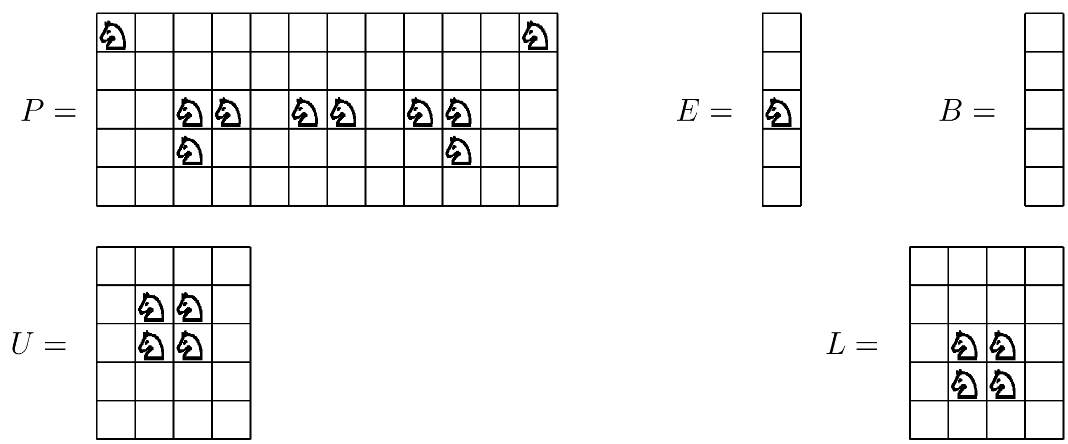


**Figure 5.13** Components in Domination Patterns of $5 \times r$ Boards.

These components are combined together to form the patterns for a $5 \times r$ board. The table below shows the patterns with the repeating 18 columns of each pattern in parentheses.

| $n$ | $\gamma(N_{5,r})$ | Pattern for $r=0$ | Pattern for $r>0$ |
|---|---|---|---|
| $18r+0$ | $14r+1$ | Meaningless | $(PBE^4B)^{r-1}PBE^5$ |
| $18r+1$ | $14r+2$ | See $1\times 5$ result | $(PBE^4B)^{r-1}PBE^6$ |
| $18r+2$ | $14r+3$ | See $2\times 5$ result | $(PBE^4B)^{r-1}PBE^7$ |
| $18r+3$ | $14r+4$ | See $3\times 5$ result | $(PBE^4B)^{r-1}PBE^8$ |
| $18r+4$ | $14r+4$ | $(PBE^4B)^rU$ | |
| $18r+5$ | $14r+5$ | $E^5$ | $(PBE^4B)^{r-1}PBE^5BU$ |
| $18r+6$ | $14r+6$ | $E^6$ | $(PBE^4B)^{r-1}PBE^6BU$ |
| $18r+7$ | $14r+7$ | $E^7$ | $(PBE^4B)^{r-1}PBE^7BU$ |
| $18r+8$ | $14r+8$ | $E^8$ | $(PBE^4B)^{r-1}PBE^8BU$ |
| $18r+9$ | $14r+8$ | $(PBE^4B)^rUBE^4$ | |
| $18r+10$ | $14r+9$ | $(PBE^4B)^rUBE^5$ | |
| $18r+11$ | $14r+10$ | $(PBE^4B)^rUBE^6$ | |
| $18r+12$ | $14r+10$ | $(PBE^4B)^rP$ | |
| $18r+13$ | $14r+11$ | $UBE^8$ | $(PBE^4B)^{r-1}PBE^4EBP$ |
| $18r+14$ | $14r+12$ | $(PBE^4B)^rUBE^4BU$ | |
| $18r+15$ | $14r+13$ | $(PBE^4B)^rUBE^5BU$ | |
| $18r+16$ | $14r+14$ | $(PBE^4B)^rUBE^6BU$ | |
| $18r+17$ | $14r+14$ | $(PBE^4B)^rPBE^4$ | |

## 5.8 Domination Number for a $6\times r$ board

For a $6\times r$ board, the periodicity is given by

$$\mathbf{x}_{r+4} = \mathbf{x}_r + 4.$$

This means that once an initial pattern is established, the pattern can be extended by adding on a pattern of 4 columns which is dominated by 4 Knights. The algorithm detects periodicity at $r = 15$. The domination patterns are shown in Figures 5.14 5.15 5.16

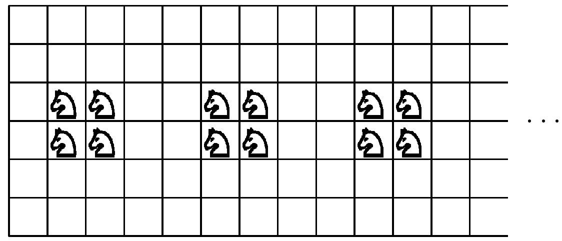

**Figure 5.14** Domination Pattern of $6 \times r$ Board, for $r = 0$, and 3 (mod 4) $r \geq 4$.

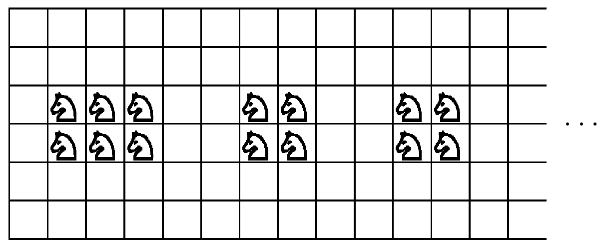

**Figure 5.15** Domination Pattern of $6 \times r$ Board, for $r = 1$ (mod 4) $r \geq 4$

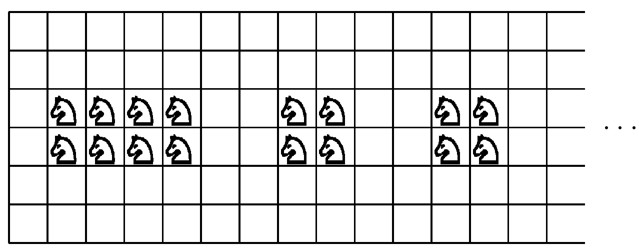

**Figure 5.16** Domination Pattern of $6 \times r$ Board, for $r = 2$ (mod 4) $r \geq 4$

## 5.9 Domination Number for a $7 \times r$ board

For a $7 \times r$ board, the periodicity is given by

$$\mathbf{x}_{r+5} = \mathbf{x}_r + 6.$$

This means that once an initial pattern is established, the pattern can be extended by adding on a pattern of 4 columns which is dominated by 4 Knights.

The algorithm detects periodicity at $r = 32$. The domination patterns can be described in terms of the components shown in Figure 5.17.

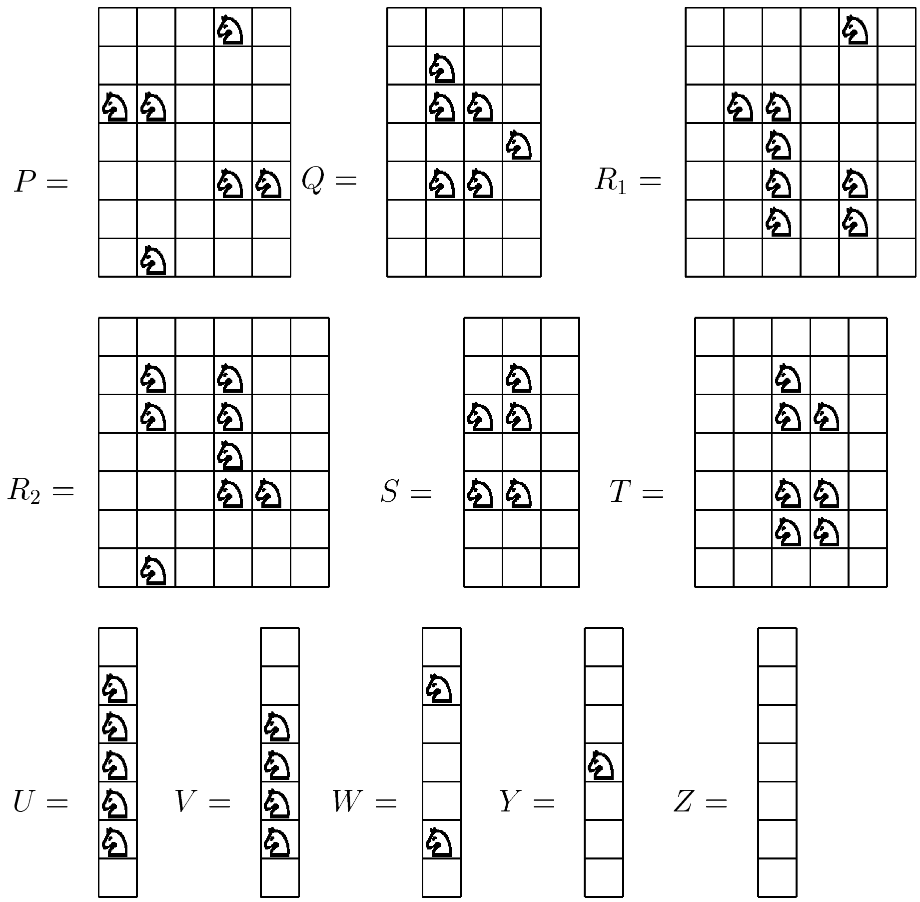


**Figure 5.17** Components in Domination Patterns of $7 \times r$ Boards.

These components are combined together to form the patterns for a $7 \times r$ board. The table below shows the patterns with the repeating 6 indicated by raising the pattern to the exponent $r$.

| $n$ | $\gamma(N_{7,r})$ | Pattern for $r \geq 0$ |
|---|---|---|
| $1-6$ | | See Previous results |
| 7 | 10 | $ZVZWZVZ$ |
| 8 | 11 | $ZVZWZUZZ$ |
| 9 | 12 | $ZZUZWZUZZ$ |
| 10 | 14 | $ZVZWYYWZVZ$ |
| 11 | 15 | $ZVZWYYWZUZZ$ |
| 12 | 16 | $ZZUZWYYWZUZZ$ |
| $5r+13$ | $6r+18$ | $T(P)^rS$ |
| $5r+14$ | $6r+19$ | $R_1(P)^rS$ |
| $5r+15$ | $6r+20$ | $R_1(P)^rQ$ |
| $5r+16$ | $6r+21$ | $T(P)^rR_2$ |
| $5r+17$ | $6r+22$ | $R_1(P)^rR_2$ |

## 5.10 Conclusions

The algorithm developed in this paper can be used to find the domination number of a $k \times r$ board for all $r$. The algorithm is not linear in $k$ so the size of $k$ is limited by computer time.

**Theorem 5.2** *The domination number of a $k \times r$ board $\gamma(N_{k,r})$ for $k \leq 6$ and for all $r$ is:*

$$\gamma(N_{k,r}) = \begin{cases} n & \text{if } k=1 \\ 4\left\lceil \frac{r}{6} \right\rceil & \text{if } k=2 \\ \frac{2r+4}{3} & \text{if } k=2,\ r=1 \text{ (mod 6) and } r \geq 6 \\ \frac{2r+4}{3} & \text{if } k=3,\ r=1 \text{ (mod 6), and } r \geq 6 \\ \frac{2r+5}{3} & \text{if } k=3,\ r=2 \text{ (mod 6), and } r \geq 6 \\ 4\left\lceil \frac{r}{6} \right\rceil & \text{if } k=3,\ r \neq 1,\ 2 \text{ (mod 6) and } r \geq 6 \\ \frac{2r+4}{3} & \text{if } k=3,\ r=1 \text{ (mod 6), and } r \geq 7 \\ 4\left\lceil \frac{r}{6} \right\rceil & \text{if } k=4,\ r \neq 1 \text{ (mod 6) and } r \geq 7 \\ \left\lceil \frac{7r}{9} \right\rceil & \text{if } k=5,\ r \neq 4,12,13, \text{ or } 17 \text{ (mod 18), and } r \geq 13 \\ \left\lceil \frac{7r}{9} - 1 \right\rceil & \text{if } k=5,\ r=4,12,13, \text{ or } 17 \text{ (mod 18), and } r \geq 13 \\ r+1 & \text{if } k=6,\ r=1 \text{ (mod 4), and } r \geq 4 \\ 4\left\lceil \frac{r}{4} \right\rceil & \text{if } k=6,\ r \neq 1 \text{ (mod 4), and } r \geq 4 \\ \left\lfloor \frac{6r+12}{5} \right\rfloor & \text{if } k=7, \text{ and } r \geq 13 \end{cases}$$

# 6. Time Complexity of The Power Method

All of the applications presented in the previous chapters rely on the periodic property of dynamic programming algorithms in Min-Plus algebra. The algorithms terminate when periodicity occurs, and this allows us to find infinitely many solutions in a finite number of iterations of the algorithm.

**Definition 6.1** *The* pre-periodic interval *of a matrix $A$ denoted $R_0(A)$, is the smallest $R_0$ such that $A^r = q \boxtimes A^{r-p}$ for all $r \geq R_0$.*

In Theorem 2.3 we stated that if $A$ is the irreducible transfer matrix for the dynamic programming algorithm, then periodicity implies that for some $R_0$, $p$ and $q$, we have that $A^r = q \boxtimes A^{r-p}$ for all$r \geq R_0$.

Although some matrices that are not irreducible are periodic, this is not the case in general. Consider a matrix whose precedence digraph has two disjoint cycles of length one with different weights as shown below.

$$A = \begin{bmatrix} t & \infty \\ \infty & s \end{bmatrix} \quad A^{100} = \begin{bmatrix} 100t & \infty \\ \infty & 100s \end{bmatrix}$$

In this case $A^r_{11} = rt$ and $A^r_{22} = rs$ and thus if $t \neq s$ , there is no $r$ where $A^r = q \boxtimes A^{r-p}$ and $R_0(A)$ is undefined. Thus when using the periodicity of Min-Plus algebra to solve problems, we must make sure that the matrix is irreducible.

## 6.1 Finite Termination of the Power Method

In Chapter 2, we stated Theorem 2.3 without proof. We have referred to this theorem throughout all of the applications presented in this thesis because it guarantees that at some point the dynamic programming algorithms will be periodic and the power method can terminate. We will now restate the theorem and give a proof based on minimum cycle means and the diameter of $A$.

**Definition 6.2** *The* diameter of $A$ *denoted* $diam(A)$ *is the maximum weight of all minimum weight paths in* $A$.

**Theorem 6.3** *If* $A$ *is an irreducible matrix on* $m$ *vertices, then there exists a* $p, q$ *and* $R_0$ *so that*

$$A^r = q \boxtimes A^{r-p} \qquad \textit{for all} \quad r \geq R_0$$

*Proof:* Let $\rho(A)$ be the minimum cycle mean of $A$. Since $A$ is irreducible, we know that the precedence digraph of $A^{m^2-m+1}$ has paths between all pairs of vertices. Let $k$ be the smallest common divisor of the lengths of the critical cycles of $A$, with the property that $k \geq m^2 - m + 1$. Now, consider $A^k$ and let $S$ be the subgraph of the precedence digraph of $A^k$, $(D_{A^k})$ whose vertices are contained in at least one of the critical cycles in $A$, and let $T$ be the subgraph of $D_{A^k}$ on the remaining vertices. In $D_A$, every vertex of $S$ has a minimum weight path of length $k$ to itself using the critical cycle that contains it. The weight of this path is $k\rho(A)$, so the minimum cycle mean of $A^k$ is $k\rho(A)$ and the critical cycles are 1-loops on every vertex in $S$.

We will now look at possible paths in the precedence digraph of $A^k$. Let $P$ be any minimum weight path of length $r$ between $v_i$ and $v_j$ that uses no vertices of $S$. This path consists of cycles and simple paths in $T$. Let $b(A^k)$ be the minimum entry in $A^k$, then the weight of each arc in the simple paths is at least $b(A^k)$. If there are $h$ such arcs, the weight of the simple path portion of $P$ is at least $hb(A^k)$. Next if $\rho(T)$ is the minimum cycle mean of $T$, then the arcs in every cycle in $T$ have weight of at least $\rho(T)$. $P$ contains $r-h$ arcs that are part of some cycle, so the weight of these cycles is at least $(r-h)\rho(T)$. Thus

$$w(P) \geq (r-h)\rho(T) + hb(A^k)$$

This is a non-negative value since $\rho(T) \geq b(A^k)$, and since $h \leq m-1$ we have

$$w(P) \geq r\rho(T) + (m-1)(b(A^k) - \rho(T))$$

Next consider all minimum weight paths $Q$ from $v_i$ to $v_j$ of length $r$ that use vertices of $S$. These paths consist of a path from $v_i$ to $S$, loops on the vertices of $S$, and a path from $S$ to $v_j$. The paths to and from $S$ have weight at most $\mathrm{diam}(A^k)$ and the loops in $S$ each have weight $k\rho(A) = \rho(A^k)$. Since $Q$ contains at most $r$ loops in $S$, we have

$$w(Q) \leq 2\mathrm{diam}(A^k) + r\rho(A^k).$$

Now we want to show that all minimal paths of length $r \geq R_0$ must use vertices of $S$. This is true if $w(P) \geq w(Q)$ for all paths of type $P$ and $Q$ of length $r$. Since $\rho(T) \geq \rho(A^k)$ we have $w(P) \geq w(Q)$ for all $r \geq R_0$ when

$$R_0 = \frac{2\text{diam}(A^k) - (m-1)(b(A^k) - \rho(T))}{\rho(T) - \rho(A^k)} \tag{6.1}$$

Now for any minimal path of length $r \geq R_0$ a minimal path of length $r+1$ is obtained by looping on some vertex of $S$ that was in the minimal path of length $r$, and the weight of these paths differs by $\rho(A^k)$ in every case. Thus

$$(A^k)^r = \rho(A^k)\ (A^k)^{r+1} \text{ for all} r \geq R_0$$

This proves that $A^k$ is periodic and hence $A$ is periodic with $R_0(A) = kR_0(A^k)$. The construction of the proof also gives a bound for $R_0$ defined in terms of the entries of $A^k$. $\square$

We will now illustrate this method by finding the bound on $R_0(A)$ where $A$ is the $15 \times 15$ transition matrix used in Chapter 4 to find the domination number of $C_r[1,3]$. The precedence digraph in Figure 4.3 shows that there is only one critical cycle $C$ for this matrix and that $\rho(A) = \frac{w(C)}{|C|} = \frac{1}{5}$. To find the bound for $R_0$ using the method described above, we need to find the smallest $k \geq m^2 - m + 1$ with $k$ being a common divisor of the lengths of the critical cycles in $A$. Thus since $m = 15$, and there is only one critical cycle of length 5, we have $k = 215$. This bound is large because we are required to have

$k \geq m^2 - m + 1$ to guarantee that the matrix has paths of any length greater than $k$. (In this example, we have that $A^8$ has no infinity enties so $k$ could be reduced to 10). We will illustrate the bound using $k = 215$ and considering possible paths in $A^{215}$ which is shown below.

$$A^{215} = 39 \boxtimes A^{20} = 39 \boxtimes \begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 8 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 7 \\ 5 & 5 & 5 & 5 & 5 & 5 & 6 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 7 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 5 & 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 & 4 & 4 & 5 & 4 & 5 & 5 & 5 & 5 & 5 & 5 & 6 \\ 5 & 5 & 5 & 6 & 6 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 6 & 6 & 7 \\ 4 & 4 & 5 & 5 & 5 & 4 & 4 & 5 & 5 & 5 & 5 & 6 & 5 & 5 & 6 \\ 4 & 5 & 4 & 5 & 5 & 4 & 5 & 4 & 5 & 6 & 5 & 6 & 5 & 6 & 6 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 5 & 5 & 6 \\ 4 & 5 & 4 & 5 & 5 & 4 & 5 & 4 & 5 & 5 & 5 & 6 & 5 & 5 & 6 \\ 4 & 4 & 4 & 4 & 5 & 4 & 4 & 4 & 4 & 5 & 4 & 5 & 5 & 5 & 5 \\ 4 & 4 & 4 & 5 & 5 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 5 & 5 & 6 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 5 & 5 & 6 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 5 & 5 & 4 & 4 & 5 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 & 5 & 6 \end{bmatrix}$$

Using this matrix and the fact that $S = \{v_5, v_7, v_8, v_{11}, v_{14}\}$, we have that

$$b(A^k) = 43, \quad \rho(T) = \frac{87}{2} \text{ and } \rho(A^k) = k\rho(A) = \frac{1}{5}(215) = 43.$$

Next, for any matrix, it's diameter is less than the number of vertices times the maximum entry. Using this bound we have

$$\mathrm{diam}(A^k) \leq 690.$$

This is considerably larger than the actual diameter of $A$ which is 269 and thus the bound on $R_0$ will be significantly effected by using the bound instead of the actual diameter. Putting all of this together in equation 6.1 we have

$$R_0 = \frac{2 \times 705 - (14) \times (43 - 43.5)}{43.5 - 43} = 2806$$

Thus, the bound is $R_0(A^k) = 2806$ and the bound for $A$ is $R_0(A) = (215)(2806) = 603290$, whereas the actual value of $R_0(A)$ was 24. In computing this bound, we used $k = 215$ as the smallest value where $A^k$ had no infinity entries instead of the actual value of $k = 20$. If we compute the bound with $k = 10$ and $diam(A) = 269$ it gives $R_0(A) = (10)(1062) = 10620$ which is still quite large. Thus although the bound shows that $R_0$ is finite, it is not a very good approximation of $R_0(A)$ in most cases.

The large value of $k$ and the large approximation of $diam(A)$ both contribute to the difference between the bound and the actual value of $R_0(A)$. In addition, if the growth of the entries in successive powers of $A$ is large, then the fact that the bound requires us to consider paths in $A^k$ where $k$ is large may have a much more significant effect. In the next section we will restrict $A$ and find a bound for $R_0$ that is strong.

## 6.2 Exact Bound for $R_0$ when $A$ is MCM-Hamiltonian

In this section, we will determine the bound for $R_0$ when $A$ is restricted to a special case.

**Definition 6.4** *A* minimum cycle mean (MCM) Hamiltonian digraph *is a digraph where the minimum cycle mean occurs on a Hamiltonian cycle.*

**Definition 6.5** *A matrix is* MCM Hamiltonian *if its precedence digraph is.*

Note that if $A$ is MCM Hamiltonian then the subgraph $T$ as described in the previous section does not exist. Thus, we cannot use the bound from that section and we must look at other ways to bound $R_0$ in this case.

**Definition 6.6** *If $B$ is an MCM Hamiltonian matrix, define the following.*

*(1) Let $\lambda = \frac{q}{p}$ be the unique eigenvalue of $B$.*

*(2) Let $P$ be the permutation matrix that permutes the rows and columns of $B$ so that $F = P^T \boxtimes (B \boxtimes (-\lambda)) \boxtimes P$ is a matrix with minimum cycle mean 0 and the minimum cycle $C$ oriented on the vertices 1, 2, ..., $m$ where vertex 1 is represented by row 1 of $F$. Thus the arcs of $C$ are $(i, i+1)$ for all $i < m$, and the arc $(m, 1)$.*

*(3) Let $D$ be a diagonal matrix with $D_{jj} = \sum_{i=j}^{m-1}(F_{i,i+1}) + F_{m,1}$*

**Definition 6.7** *If $B$ is an MCM Hamiltonian matrix on $m$ vertices, then the* normalized form *of $B$ is $A = (-D) \boxtimes P^T \boxtimes (B \boxtimes (-\lambda)) \boxtimes P \boxtimes D$. A* normalized MCM Hamiltonian matrix *is a matrix that is equal to its normalized form.*

**Theorem 6.8** *Let $B$ be an MCM Hamiltonian matrix with minimum cycle mean $\lambda$, and normal form $A = (-D) \boxtimes P^T \boxtimes (B \boxtimes (-\lambda)) \boxtimes P \boxtimes D$. Then*

*(1) $A_{1,2} = A_{2,3} = \ldots = A_{m-1,m} = A_{m,1} = 0$*

*(2) The minimum cycle mean of $A$ is 0.*

*(3) $A$ is non-negative.*

*(4) $B^r = \lambda A^r \boxtimes (-D) \boxtimes P^T \boxtimes A^r \boxtimes P \boxtimes D$ and*

*(5) $B^r = q \boxtimes B^{r-p}$ then $A^r = A^{r-p}$.*

*Proof:* Let $C_B$, $C_R$, $C_A$ be the cycles with minimum cycle means in $F$, $R$, and $A$. Then $F = P^T \boxtimes B \boxtimes (-\lambda) \boxtimes P$ so the arcs of $C_B$ have been permuted to the arcs of $C_F + \{F_{1,2}, F_{2,3}, \ldots F_{m,1}\}$ and there weights have been reduced by $\lambda$. Also for any arc in $A$,

$$A_{i,j} = F_{i,j} - D_{i,i} + D_{j,j}$$

$$= F_{i,j} - [F_{i,i+1} + \ldots + F_{m-1,m} + F_{m,1}] + [F_{j,j+1} + \ldots + F_{m-1,m} + F_{m,1}].$$

which gives a direct correspondence between the cycles in $A$ and the cycles in $B$.

1) By construction,

$$A_{i,i+1} = F_{i,i+1} - [F_{i,i+1} + \ldots + F_{m-1,m} + F_{m,1}] + [F_{i+1,i+2} + \ldots + F_{m-1,m} + F_{m,1}] = 0$$

for all $i < m$, and

$$A_{m,1} = F_{m,1} - [F_{m,1}] + [F_{1,2} + F_{2,3} + \ldots + F_{m-1,m} + F_{m,1}] = 0.$$

Thus all the arcs in $C = \{A_{1,2} \ldots A_{m-1,m}, A_{m,1}\}$ have weight 0 and form a hamiltonian cycle of weight 0.

2) For any cycle $C_{A_i}$ of length $i$ in $A$ which has weight $w(C_{A_i})$, there is a corresponding cycle $C_{B_i}$ of length $i$ in $B$ which has weight $w(C_{B_i}) = w(C_{A_i}) + \lambda i$. By part 1 , $C$ is a hamiltonian cycle with cycle mean 0 so $w(C_A) \leq 0$. If $w(C_A) < 0$ then $w(C_B) + w(C_A) + \lambda m$ has cycle mean less than $\lambda$ which is a contradiction. Thus the minimum cycle mean of $A$ is 0.

3) Suppose $A$ has some entry $A_{i,j} < 0$ . Consider the cycle $\{A_{i,j}, A_{j,j+1}, \ldots, A_{m-1,m}, A_{m,1}, \ldots, A_{i-1,i}\}$. Since all the arcs of this cycle except $A_{i,j}$ are contained in the cycle with minimum cycle mean, they have weight

0 and the weight of the cycle is $A_{i,j} < 0$ which is a contradiction since the minimum cycle mean of $A$ is 0.

4 & 5) The results are found by using the commutative and associative properties in Min-Plus algebra.

□

To illustrate the normalization process consider the following matrix which has the arcs in the cycle with minimum cycle mean underlined.

$$B = \begin{bmatrix} 7 & 9 & \underline{2} & 8 \\ \underline{1} & 8 & 6 & 6 \\ 9 & 8 & 6 & \underline{5} \\ 3 & \underline{4} & 7 & 8 \end{bmatrix} \text{which has } \lambda = 3$$

Then using the definitions above,

$$P = \begin{bmatrix} \infty & 0 & \infty & \infty \\ 0 & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & 0 \end{bmatrix} \quad F = \begin{bmatrix} 5 & \underline{-2} & 3 & 3 \\ 6 & 4 & \underline{-1} & 5 \\ 5 & 6 & 3 & \underline{2} \\ \underline{1} & 0 & 4 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 0 & \infty & \infty & \infty \\ \infty & 2 & \infty & \infty \\ \infty & \infty & 3 & \infty \\ \infty & \infty & \infty & 1 \end{bmatrix}$$

and finally

$$A = \begin{bmatrix} 5 & \underline{0} & 6 & 4 \\ 4 & 4 & \underline{0} & 4 \\ 2 & 5 & 3 & \underline{0} \\ \underline{0} & 1 & 6 & 5 \end{bmatrix}$$

Thus any min-plus problem with an MCM Hamiltonian transfer matrix $B$ can be solved using the normalized MCM Hamiltonian matrix $A$ and then converting the solution using parts 4 and 5 of Theorem 6.8

To find a bound on the number of iterations needed for the transfer matrix to repeat, we will consider normalized MCM Hamiltonian matrices. The precedence digraph for a normalized MCM Hamiltonian matrix has a Hamiltonian cycle with all arcs having weight zero and other arcs of non-negative weight. The bounds on repetition will be formed by considering paths in the precedence digraph since if $A^r = A^{r-m}$ where $m$ is the size of the matrix, then in the precedence graph the minimum weight of a path of length $r$ from $i$ to $j$ is the same as the minimum weight of a path from $i$ to $j$ of length $r-m$ for all $i$ and $j$.

**Definition 6.9** *Let $A$ be a normalized MCM Hamiltonian matrix and let $b_i$ be the $i^{\underline{\text{th}}}$ arc in the minimum cycle of $A$ which by definition has weight 0.*

**Definition 6.10** *Let $a_k$ be the number of arcs from $i$ to $j$ used in the path where $i - j + 1 = k$ (mod $m$).*

**Theorem 6.11** *A path $P$ of length $r$ from vertex $i$ to vertex $j$ in a MCM Hamiltonian digraph on $m$ vertices satisfies the following:*

$$a_1 + 2a_2 + \cdots + (m-1)a_{m-1} - r = (i-j) \ mod \ m \tag{6.2}$$

*Proof:* First we will prove this for $r = 1$. There are two cases for a path of length 1. Case 1: $j = i + 1$. In this case, the arc from $i$ to $j$ is one of the $b_i$ arcs from the minimal cycle. Thus no $a_i$ arcs are used and equation 6.2 becomes $-1 = (i - (i+1))$ (mod $m$) which holds. Case 2: $j \neq i + 1$ In this case, if $j = i + k$ the arc used is $a_l$ where $l = i - j + 1 = -k + 1$ and equation 6.2 becomes $l - 1 = (i - j)$ (mod $m$) or $(i - (i + k) + 1) - 1 = (i - (i + k))$ (mod $m$) which also holds.

For induction purposes, we will assume that a path of length $r-1$ satisfies ( 6.2) Now any path of length $r$ contains a path of length $r-1$ plus one arc. Let j be the vertex at the end of the path of length $r-1$ and k be last vertex in the path. Thus the last arc in this path of length $r$ goes from vertex j to vertex k. Let $a'_1, a'_2, \ldots a'_{m-1}$ be the coefficients that satisfy ( 6.2) for the path of length $r-1$. Then ( 6.2) is satisfied for the path of length $r$ if $a'_1 + 2a'_2 + \cdots + (m-1)a'_m +$ arc from $j$ to $k - r = (i-k) \pmod m$. Using the induction hypothesis, this will hold if $i - j \pmod m +$ arc from $j$ to $k - 1 = i - k \pmod m$.

Case 1: The additional arc is one of the $b_i$ arcs on the minimal cycle. This gives $k = j + 1 \pmod m$ by the labeling of the minimal cycle, and ( 6.2) becomes $i - j - 1 = i - (j+1) \pmod m$ since there is no additional $a_i$ arc added to the equation.

Case 2: The additional arc is $a_l$ where $l = j - k + 1$. Then ( 6.2) becomes $i - j + (j - k + 1) - 1 = i - k \pmod m$ which holds. □

**Definition 6.12** *Given a path P the arcs can be divided into three exclusive classes:*

*Type 1 Arcs used in a minimal path from the initial vertex to the final vertex in the path. There are $\leq m$ of these.*

*Type 2 Arc used in any minimal cycle containing an $a_i$. There are $\sum_{i=1}^{m} ia_i$ of these.*

*Type 3 Arcs that are in the minimal cycle.*

*Let $N_i$ be the number of arcs of type i used in a given path.*

Each type is exclusive so once an arc has been designated as being in type one it cannot be in type two or three. Also the determination of the arc type must be done in the order they are described. For example using the graph in Figure 6.1, consider the following path of length 13 from vertex 1 to vertex 3.

$$P = \mathbf{b_1 b_2}\, a_3\, b_1\, b_2\, \mathtt{b_3 b_4 b_5}\, a_4\, b_4\, b_5\, \mathtt{b_1 b_2}$$

In this path, the bold arcs are designated type one, the italic arcs are designated type two and the remaining arcs are type three.

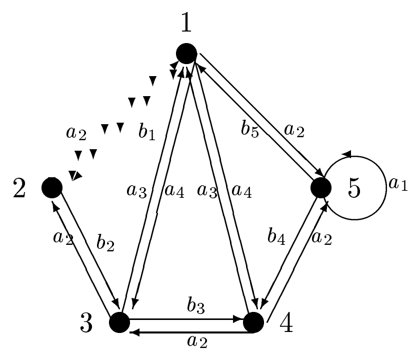


**Figure 6.1** Precedence digraph of a normalized MCM Hamiltonian matrix. The $b_i$ arcs represent the arcs in the minimal cycle and have weight 0. The weight of the $a_i$ arcs is non-negative.

Now we need to show that for some $r$, any minimum weight path in the precedence digraph of length $r$ has the same weight as a minimum weight path of length $r - m$. This will be possible if there are always at least $m$ type 3 arcs in a minimum weight path of length $r$. Since $m$ of these arcs form a Hamiltonian cycle and we assumed the type 3 arcs have weight zero by theorem 3.2, their removal will not change the weight of the path and will give a path of equal weight and length $r - m$.

To find the value of $r$ for which we are guaranteed to have $N_3 \geq m$, we must first show that we need only consider paths which use less than $m$ of the $a_i$ arcs. To do this we will show that for any path of length $r$ from $i$ to $j$ that uses more than $m$ of the $a_i$ arcs there is a path of length $r$ from $i$ to $j$ that uses less than $m$ of these arcs. This is done by showing that such a path satisfies 6.2 in the same way that the original path does.

**Definition 6.13** *Let* $\mathbf{a} = (a_1, \ldots, a_{m-1})$ *and* $\mathbf{c} = (c_1, \ldots, c_{m-1})$ *and let* $f(\mathbf{a}) = a_1 + 2a_2 + \cdots + (m-1)a_{m-1}$. *Next define* $\mathbf{a} < \mathbf{c}$ *if* $a_i \leq c_i$ *for all* $1 \leq i \leq m$.

Now $\mathbf{a}$ and $\mathbf{c}$ are the coefficients of two paths described by 6.2. We want to show that if $\mathbf{a}$ satisfies 6.2 then there is some $\mathbf{c}$ that also satisfies 6.2 and uses less than $m$ of the $a_i$ arcs.

**Theorem 6.14** *If* $\mathbf{a}$ *satisfies* $\sum_{i=1}^{m-1} a_i = m$, *then there is a* $\mathbf{c}$ *with* $\mathbf{0} \leq \mathbf{c} < \mathbf{a}$ *and* $f(\mathbf{c}) \equiv f(\mathbf{a}) \pmod{m}$.

*Proof:* Construct a chain of $\mathbf{c}_j$, $j = 1, 2, \ldots, m$ with $\mathbf{a} = \mathbf{c_0} > \mathbf{c_1} > \mathbf{c_2} > \ldots > \mathbf{c_m}$ by recursively defining $\mathbf{c_{j+1}}$ by decreasing one non-zero entry of $\mathbf{c_j}$ by 1. After $a_1 + a_2 + \ldots + a_{m-1} = m$ we reach $\mathbf{0}$ so $\mathbf{c_m} = \mathbf{0}$

Now consider the values of $f(\mathbf{c_1}), \ldots, f(\mathbf{c_m}) \pmod{m}$. If they are all distinct, then $f(\mathbf{a})$ is congruent mod $m$ to a unique $f(\mathbf{c_j})$, satisfying the conclusion of the theorem. Otherwise, for some $j, k$ with $1 \leq j < k \leq m$, we have $f(\mathbf{c_j}) \equiv f(\mathbf{c_k}) \pmod{m}$, and $\mathbf{c_j} > \mathbf{c_k}$. Then $\mathbf{a} > \mathbf{c_j} - \mathbf{c_k} > \mathbf{0}$ and $\mathbf{a} > \mathbf{a} - (\mathbf{c_j} - \mathbf{c_k})$ with

$f(\mathbf{a} - (\mathbf{c_j} - \mathbf{c_k})) = f(\mathbf{a}) - f(\mathbf{c_j}) + f(\mathbf{c_k}) \equiv f(\mathbf{a}) \pmod{m}$ [16]. □

Thus the maximum number of type two arcs needed in a minimal path of length $r$ has $\sum a_i \leq m$. Using this, we are now ready to prove the final theorem of this paper which gives a bound on repetition in the min-plus algebra in the special case where the matrix is MCM Hamiltonian.

**Theorem 6.15** *If $A$ is a normalized MCM Hamiltonian matrix on $m$ vertices, then*

$$A^{(m-1)^2+1} = A^{(m-1)^2+1+m}.$$

*Proof:* Using Theorem 6.11, all paths of length $r = (m-1)^2 + 1 + m$ in $D_A$ satisfy (6.2). Since $(m-1)^2+1+m = m^2-m+2 = 2 \text{ mod} m$, equation 6.2 becomes

$$a_1 + 2a_2 + \ldots (m-1)a_{m-1} - 2 = (i-j)\text{mod} m.$$

Let $N_i$ be defined as in Definition 6.12, then the number of arcs in such a path can be divided into $N_1 + N_2 + N_3$, so that

$$N_1 + N_3 + N_3 = (m-1)^2 + 1 + m$$

$$\text{and,} \qquad N_1 + N_2 + N_3 \leq m + \sum ia_i + N_3.$$

Next, combining these gives

$$N_3 \geq ((m-1)^2 + 1 + m) - m - \sum ia_i. \tag{6.3}$$

Now given that $a_{m-1} = m - k$, Theorem 6.14 says that a minimal path has $\sum_{i=1}^{m-2} a_i < k$. Thus the number of type one arcs is maximized by letting $a_{m-2} = k-1$.

Next, using this and Theorem 3.4, we find that a path in $D_A$ satisfies

$$(m-2)(k-1)+(m-1)(m-k)-2=-k \bmod m$$

so $(i-j)=k$ and the number of type one arcs is $k$. Thus 6.3 becomes

$$N_3 \quad = [(m-1)^2+1+m]-k-[(m-2)(k-1)+(m-1)(m-k)] \quad = m.$$

Now since we maximized the number of type one arcs in this calculation, we have the number of $N_3 \geq m$ for any path in $D_A$. This means that $m$ of these type three arcs which form one cycle of the minimum cycle in which all arcs have weight 0 can be removed from any minimal path of length $(m-1)^2+1+m$ to obtain a path of length $(m-1)^2+1$ with the same weight.

Finally, we need to show that the removal of the $m$ type three arcs does not disconnect the remaining path. Let $D$ be the path that remains after removing the $m$ type three arcs. Suppose $D$ is disconnected and let $H_1,\ H_2,\ \ldots H_k$ represent the connected components of D. In each component $H_i$, the minimum weight path between any pair of vertices has length at most $|H_i|-1$. The remaining arcs of the path in each component must be part of cycles within this component. Since $D$ has $m^2-2m+2$ arcs and at most $m-2$ of these are neeeded for simple paths in each component, there are at least $m^2-3m$ arcs which are part of cycles in these components. Now, since $m^2-3m \geq m$ for all $m \geq 4$, consider a new path $D'$ in which $m$ of these arcs from cycles in the components are removed, and replace by a single cycle of the MCM cycle. This path would then be connected, and since the weight of any cycle in a $H_i$ is at least the weight of the minimum cycle mean we have that $w(D) \geq w(D')$. Thus we can always find a minimum weight path in which the removal of the $m$ type three arcs does not disconnect the remaining path. $\square$

This bound is strong which can be seen by considering a graph with only one $a_{m-1}$ arc from vertex $m$ to vertex 2, a Hamiltonian cycle with all arcs having weight zero and no other arcs. Now, in this graph, there is no path of length $(m-1)^2$ from vertex 1 to itself but there is a path of length $(m-1)^2 + m$ and weight $m-1$ from vertex 1 to itself. To see this, consider the normalized MCM Hamiltonian transfer matrix A which has this graph as its precedence graph.

$$A = \begin{bmatrix} \infty & 0 & \infty & \dots & \dots & \infty \\ \infty & \infty & 0 & \infty & \dots & \infty \\ \vdots & & & \ddots & & \infty \\ \infty & \dots & \dots & \infty & 0 & \infty \\ 0 & 1 & \infty & \dots & \dots & \infty \end{bmatrix}$$

The powers of the matrix are

$$A^{(m-1)^2} = \begin{bmatrix} \infty & 0 & 1 & 2 & \dots & m-2 \\ m-2 & m-1 & 0 & 1 & \dots & m-3 \\ m-3 & m-2 & \ddots & 0 & \dots & m-4 \\ m-4 & m-3 & \ddots & \ddots & \dots & m-5 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 2 & \dots & \dots & m-1 \end{bmatrix}$$

$$A^{(m-1)^2+1} = \begin{bmatrix} m-2 & m-1 & 0 & 1 & \dots & m-3 \\ m-3 & m-2 & m-1 & 0 & \dots & m-4 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 2 & \dots & \ddots & \dots & 0 \\ 0 & 1 & \ddots & \dots & \dots & m-1 \\ m-1 & 0 & 1 & 2 & \dots & m-2 \end{bmatrix}$$

.........

$$A^{(m-1)^2+m} = \begin{bmatrix} m-1 & 0 & 1 & 2 & \dots & m-2 \\ m-2 & m-1 & 0 & 1 & \dots & m-3 \\ m-3 & m-2 & \ddots & 0 & \dots & m-4 \\ m-4 & m-3 & \ddots & \ddots & \dots & m-5 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 2 & \dots & \dots & m-1 \end{bmatrix}$$

$$A^{(m-1)^2+1+m} = \begin{bmatrix} m-2 & m-1 & 0 & 1 & \dots & m-3 \\ m-3 & m-2 & m-1 & 0 & \dots & m-4 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 2 & \dots & \ddots & \dots & 0 \\ 0 & 1 & \ddots & \dots & \dots & m-1 \\ m-1 & 0 & 1 & 2 & \dots & m-2 \end{bmatrix}$$

Thus $(A^{(m-1)^2})_{11} = \infty$ while $(A^{(m-1)^2+m})_{11} = m-1$ so the bound is strong. Also, note that $A^{(m-1)^2+1} = A^{(m-1)^2+1+m}$ so the bound holds and thus for any $r \geq (m-1)^2+1+m$, we have that $A^r = A^{r-m}$.

## 6.3 Periodicity in Matrices with Integer entries.

In the previous two sections, we show that $R_0(A)$ is finite if $A$ is irreducible. However, if $A$ is not MCM Hamiltonian then the size of $R_0$ may be made arbitrarily large by increasing the weight of the arcs from the critical cycles to the rest of the graph. Consider the matrix $A$ and its powers shown below.

$$A = \begin{bmatrix} 1 & 0 \\ t & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 2 & 0 \\ t & 0 \end{bmatrix} \quad A^{t-1} = \begin{bmatrix} t-1 & 0 \\ t & 0 \end{bmatrix} \quad A^t = \begin{bmatrix} t & 0 \\ t & 0 \end{bmatrix}$$

This matrix has minimum cycle mean 0 on a cycle of length 1 so $A^r = A^{r+1}$ for all $r \geq t$ and thus $R_0(A)$ is dependent on the entries in $A$ and as we increase $t$, we increase $R_0(A)$.

In the applications in the previous chapters, the transition matrix was not MCM

Hamiltonian, yet the size of $R_0$ was much smaller than the bounds given in either of the previous two sections. Thus we will now look at how the size of $R_0$ might be affected by the entries in $A$.

**Definition 6.16** *$A$ random matrix has entries which are independently chosen from some distribution of real numbers. These entries will be represented by a random variable $X$.*

To analyze the dependency of $A$ on the size of $A$ and the size of the entries in $A$, we compared six sets of random matrices with different size limits on the entries between sets and different sizes of matrices within sets. For each set, the entries in the matrices were integers ranging from 0 to 2, 0 to 4, 0 to 8, 0 to 16, 0 to 32 and, 0 to 64 respectively. In each set we look at matrices of size $1 \times 1$, $2 \times 2 \ldots 100 \times 100$. For each size matrix in each set, we create 1000 random matrices with the appropriate entries and then compute the pre-periodic interval for each and take the average. The results of these experiments are shown in Figure 6.2.

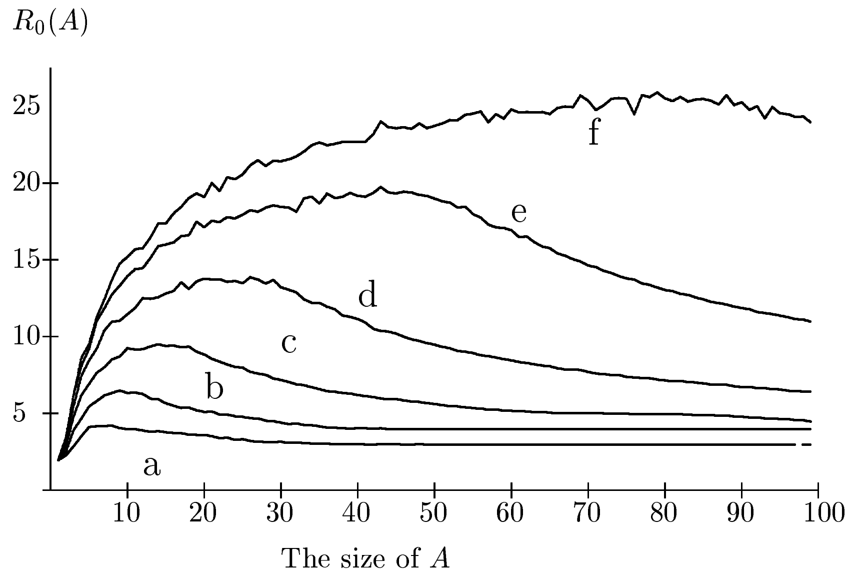


**Figure 6.2** The Pre-Periodic Interval for Random Matrices With Integer Entries. Labels a ... f correspond to matrices with entries from 0 to 2, 0 to 4, 0 to 8, 0 to 16, 0 to 32 , and 0 to 64 respectively.

This figure shows that in the discrete case $R_0(A)$ increases as the size of the entries in $A$ increase. This effect is greatest for smaller size matrices. We also notice that $R_0(A)$ decreases to a limit as the size of $A$ increases.

**Theorem 6.17** *Let $A$ be an $m \times m$ matrix with entries independently chosen from a distribution that has a minimum value $v$. Assume that the random variable $X$ is equal to $v$ with positive probability. Then the probability that $R_0(A) = 3$ approaches 1 as the size of $A$ goes to infinity.*

*Proof:* Let $D_A$ be the precedence digraph of $A$ and let $p$ be the probability that an arc in $D_A$ has weight $v$. Then the probability of a path of weight $2v$ from vertex $a$ to vertex $b$ through vertex $c$ is $p^2$. The probability that no such path through $c$

exists is $1-p^2$, and the probability that there is no such a path trough any vertex in $A$ is $(1-p^2)^m$ where $m$ is the number of vertices in $D_A$. Finally the probability that there is such a path through some vertex in $A$ is $1-(1-p^2)^m$ and thus the probability that there is such a path through all vertices is $(1-(1-p^2)^m)^{m^2}$. Next we have that $1-x \leq e^{-x}$ so

$$\lim_{m\to\infty}(1-(1-p^2)^m)^{m^2} \leq \lim_{m\to\infty}(e^{-m^2(1-p^2)^m}) = e^0 = 1.$$

Thus the probability that there is a path of length 2v between every pair of vertices in $A$ approaches 1 as $m$ approaches infinity. Now, consider the probability that there is a path of length $3v$, between every pair of vertices. This is a path from vertex $a$ to vertex $d$ which travels through two other vertices $c$ and $d$. If an arc of weight $v$ occurs with probability $p$, then by the argument above, the probability that there is no path of weight $3v$ between a given pair of vertices is $1-p^3$. Next the probability that no such path exist through $c$ and $d$ is $(1-p^3)^{m^2}$. Then by the argument for paths of length 2v, the probability that there is a path of weight $3v$ or less between every pair of vertices is $(1-(1-p^3)^{m^2})^{m^2}$. Next by the argument above,

$$\lim_{m\to\infty}(1-(1-p^3)^{m^2})^{m^2} \leq \lim_{m\to\infty} e^{-m^2(1-p^3)^{m^2}} = e^0 = 1.$$

Thus the probability that $A^2 = 2v \boxtimes \mathbf{0}$ and $A^3 = 3v \boxtimes \mathbf{0}$ where $\mathbf{0}$ is the matrix of all zeros approaches 1 as $m$ approaches infinity. When this happens, we have

$$A^3 = v \boxtimes A^2 \text{and} A^r = v \boxtimes A^{r-1} \text{ for all } r \geq 3$$

and thus $R_0(A) = 3$. $\square$

Note that this theorem does not state that $R_0(A) = 3$ for any size of $A$ mearly that for large $A$ there is a positive probability of this. Indeed, in the applications from Chapters 3-5 the value for $R_0$ was larger than 3 for all sizes of the transition matrix. This is not unexpected since the size of these matrices was relatively small and Figure 6.3 indicates that the average value for $R_0$ does not approach 3 until the size of matrix gets larger.

## 6.4 Periodicity in Random Matrices With Real Entries

The next question is what happens to the pre-periodic interval when the entries in $A$ are independently and identically distributed random variables chosen from a distribution with a continuous density function. To analyze this we will look at $2 \times 2$ matrices with random entries from different continuous distributions. These entries will be represented by a random variable $X$.

**Definition 6.18** *A random variable $X$ is* continuous *if there exists a non-negative function $f(x)$, defined for all real $x \in (-\infty, \infty)$, having the property that for any set $B$ of real numbers, the probability that the $X$ is in this set is defined as*

$$P\{X \in B\} = \int_B f(x)dx.$$

*The function $f(x)$ is called the* probability density function *of the random variable $X$.*

Thus, if $B = [a, b]$ then

$$P\{a \leq X \leq b\} = \int_b^a f(x)dx \quad \text{and} \quad P\{X = a\} = \int_a^a f(x)dx = 0.$$

This shows that the probability that the random variable will assume any particular value is zero. To analyze the pre-periodic interval of our random matrix $A$, we will first simplify the problem by limiting the entries in $A$ to real numbers from a uniform distribution in the interval (0,1).

**Definition 6.19** *A random variable is* uniformly distributed *over the interval (0,1) if its probability function is given by*

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

*In general , $X$ is a uniform random variable on $(\alpha, \beta)$ if its probability density function is given by*

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

Now, to find the pre-periodic interval of a random matrix $A$ with entries from a uniform distribution, we must first determine how the value of $R_0$ is dependent on the entries in $A$. We will begin by looking at the values of $R_0(A)$ for a $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Since $R_0$ is determined by comparing powers of $A$, we need to know what these powers look like in the general case. This is shown in the next two lemmas.

**Lemma 6.20** *For $A$ as stated above, assume $b+c < 2a$ and $a \leq d$, then the following powers of $A$ are found by Min-Plus matrix exponentiation.*

$$\begin{aligned}
A^2 &= \begin{bmatrix} b+c & a+b \\ a+c & b+c \end{bmatrix}, \\
A^3 &= \begin{bmatrix} a+b+c & 2b+c \\ b+2c & a+b+c \end{bmatrix}, \\
A^4 &= \begin{bmatrix} 2b+2c & a+2b+c \\ a+b+2c & 2b+2c \end{bmatrix} \\
A^r &= (r-2)(\tfrac{b+c}{2}) \boxplus A^2 \textit{if } r \textit{ even} \\
\textit{and } A^r &= (r-3)(\tfrac{b+c}{2}) \boxplus A^3 \textit{if } r \textit{ odd.}
\end{aligned}$$

**Lemma 6.21** *Now assume $2a \leq b+c$ and $a \leq d$. Then matrix exponentiation gives*

$$\begin{aligned}
A^2 &= \begin{bmatrix} \min(2a, b+c) & \min(a+b, b+d) \\ \min(a+c, c+d) & \min(b+c, 2d) \end{bmatrix} = \begin{bmatrix} 2a & a+b \\ c+b & \min(b+c, 2d) \end{bmatrix}. \\
A^r &= \begin{bmatrix} ra & (r-1)a+b \\ (r-1)a+c & \min((r-2)a+b+a, rd) \end{bmatrix} \quad \textit{for } r > 2.
\end{aligned}$$

Thus for any value of the entries in $A$, the powers of $A$ are defined by the previous lemmas. Now we need to compare these powers and find the values of $R_0$ for all possible combinations of the entries in $A$.

**Lemma 6.22** *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $A^1 = A$ and recursively define $A^r = A \boxtimes A^{r-1}$ where the matrix multiplication is in min-plus algebra. Let $R_0(A)$ be the minimum $R_0$ where $A^r = q \boxtimes A^{r-p}$ $\forall r \geq R_0$ for some positive values $p$ and $q$. If $a \leq d$, then*

$$R_0(A) = \begin{cases} 2 & \textit{if } 2a = 2d \le b+c \\ 3 & \textit{if } b+c < 2a = 2d, \textit{ or } 2a \le b+c \le 2d \textit{ and } a < d \\ 4 & \textit{if } b+c < 2a \textit{ or } \frac{a+b+c}{3} \le d < \frac{b+c}{2}, \textit{ and } a < d \\ r > 4 & \textit{if } \frac{(r-3)a+b+c}{r-1} \le d < \frac{(r-4)a+b+c}{r-2} \textit{ and } a < d. \end{cases}$$

*Proof:* From Lemma 1 we have $A^4 = (b+c) \boxtimes A^2$, and $R_0(A) \le 4$. Since $b+c < 2a$, we do not have $A^2 = q \boxtimes A$ nor $A^3 = q \boxtimes A^2$ for any $q$. If $R_0(A) = 3$, then $A^3 = (b+c) + A$ giving $a = d$. Then from Lemma 2, $R_0(A) = 2$ only if $A^2 = a \boxtimes A$. This occurs only if $a = d$. For $r > 2$, we have $A^r = a \boxtimes A^{r-1}$ only if $(r-3)a+b+c \le (r-1)d$. Thus $R_0(A) = 3$ only if $a < d$ and $b+c \le 2d$. And for $r > 3$, we have $R_0(A) = r$ only if $(r-3)c + b + c \le (r-1)d$ and $(r-4)a + b + c > (r-2)d$. Compiling these observations gives the result and a similar result is obtained if $a > d$ by symmetry.□

**Theorem 6.23** *Let $A$ be a $2 \times 2$ matrix whose elements are independently chosen from the uniform distribution on $[0,1]$. Let $R_0(A)$ be as above. Then*

$$P(R_0(A) = r) = \begin{cases} 0 & \textit{if } r \le 2 \\ \frac{5}{12} & \textit{if } r = 3 \\ \frac{7}{18} & \textit{if } r = 4 \\ \frac{7}{12(r-2)(r-1)} & \textit{if } r > 4. \end{cases}$$

*Proof:* For any continuous distribution, the probability that $a = d$ is zero. Thus $P(R_0(A) = 2)$ is zero. In the uniform distribution, the probability density function $f(x)$ is exactly 1 on the interval (0,1), and

$$
\begin{aligned}
P(R_0(A)=3) &= 2\times P(2a\le b+c\le 2d) \\
&=2\int_0^1\int_0^1\int_0^{(b+c)/2}\int_{(b+c)/2}^1 dd\,da\,db\,dc \\
&=2\int_0^1\int_0^1 \frac{b+c}{2}\left(1-\frac{b+c}{2}\right)^2 db\,dc \\
&=2\left(\int_0^1\int_0^1 \frac{b+c}{2}\,db\,dc-\int_0^1\int_0^1\left(\frac{b+c}{2}\right)^2 db\,dc\right) \\
&=2\left(\frac{1}{2}-\frac{7}{24}\right)=\frac{5}{12}.
\end{aligned}
$$

For $r=4$, we have

$$
\begin{aligned}
&P(b+c\le 2\min(a,d)) \\
&=\int_0^1\int_0^1\int_{(b+c)/2}^1\int_{(b+c)/2}^1 dd\,da\,db\,dc \\
&=\int_0^1\int_0^1\left(1-\frac{b+c}{2}\right)^2 db\,dc \\
&=\int_0^1\int_0^1 db\,dc-2\int_0^1\int_0^1\frac{b+c}{2}\,db\,dc+\int_0^1\int_0^1\left(\frac{b+c}{2}\right)^2 db\,dc \\
&=1-2\cdot\frac{1}{2}+\frac{7}{24}=\frac{7}{24}.
\end{aligned}
$$

For $r \geq 4$, we also have

$$P\left(\frac{(r-3)a+b+c}{3} \leq (r-2)d < \frac{(r-4)a+b+c}{2}\right)$$

$$= \int_0^1 \int_0^1 \int_0^{(b+c)/2} \int_{((r-3)a+b+c)/(r-1)}^{((r-4)a+b+c)/(r-2)} dd\, da\, db\, dc$$

$$= \int_0^1 \int_0^1 \int_0^{(b+c)/2} \left(\frac{(r-4)a+b+c}{r-2} - \frac{(r-3)a+b+c}{r-1}\right) da\, db\, dc$$

$$= \int_0^1 \int_0^1 \int_0^{(b+c)/2} \frac{b+c-2a}{(r-2)(r-1)}\, da\, db\, dc$$

$$= \frac{1}{(r-2)(r-1)} \int_0^1 \int_0^1 \left((b+c)\frac{a+b}{2} - \left(\frac{b+c}{2}\right)^2\right) da\, db\, dc$$

$$= \frac{1}{(r-2)(r-1)} \int_0^1 \int_0^1 \left(\frac{b+c}{2}\right)^2 da\, db\, dc$$

$$= \frac{7}{24(r-2)(r-1)}.$$

Thus

$$P(R_0(A) = 4) = \frac{7}{24} + 2\frac{7}{24 \cdot (4-2) \cdot (4-1)} = \frac{7}{18}$$

and for $r > 4$, we have

$$P(R_0(A) = r) = 2\frac{7}{24 \cdot (r-2)(r-1)} = \frac{7}{12(r-2)(r-1)}.$$

Then putting these together gives the result.

Note that

$$\sum_{r=3}^{\infty} P(R_0(A) = r) = \frac{5}{12} + \frac{7}{18} + \frac{7}{12}\left(\frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots\right)$$
$$= \frac{5}{12} + \frac{7}{18} + \frac{7}{12} \cdot \frac{1}{3} = 1,$$

as it should. □

**Corollary 6.24** *Under the conditions above, $E(R_0(A)) = \infty$.*

*Proof:* We have

$$E(R_0(A)) = \sum_{r=3}^{\infty} rP(R_0(A) = r)$$

$$= \frac{3 \cdot 5}{12} + \frac{4 \cdot 7}{18} + \frac{7}{12}\left(\frac{5}{3 \cdot 4} + \frac{6}{4 \cdot 5} + \frac{7}{5 \cdot 6} + \cdots\right) = \infty.$$

□

Next, we will extend this result to show that $E(R_0(A)) = \infty$ for all irreducible matrices $A$. First, we need to show that if the entries in $A_{2\times 2}$ come from any continuous distribution then $E(R_0(A)) = \infty$.

**Theorem 6.25** *Let $A$ be a $2 \times 2$ matrix whose elements are independently chosen from the continuous distribution on $\Re$, then $E(R_0(A)) = \infty$.*

*Proof:* Let $f(x)$ be the probability density function for the random variable $X$ for each of the entries in $A$. Since $f(x) \geq 0$ and continuous, there exist a constant $x_0$ so that $f(x_0) > 0$. Let $\epsilon = \frac{f(x_0)}{2}$. Then there is a $\delta > 0$ so that $|f(x) - f(x_0)| < \epsilon$ whenever $x \in (x_0 - \delta, x_0 + \delta)$. Now, each of the probability functions for the uniform distribution are the same for the continuous distribution except that probability distribution functions are $f(x)$ instead of 1, and the functions are integrated over all of $\Re$ instead of just the interval (0,1). For example, in the continuous distribution

$$P(R_0(A) = 3) = 2 \times P(2a \leq b + c \leq 2d)$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{(b+c)/2}\int_{(b+c)/2}^{\infty} f(d)f(a)f(b)f(c)\, dd\, da\, db\, dc$$

By continuity the probability distribution function $f(x)$ for the continuous distribution is greater than the constant function $\frac{f(x_0)}{2}$ on the interval defined above. Thus if we define

$$g(x) = \begin{cases} \frac{f(x_0)}{2} & \text{if } x \in (x_0 - \delta, x_0 + \delta) \\ 0 & \text{otherwise} \end{cases}$$

Then $g(x)$ is the probability density function of a uniform distribution. Now since $f(x) \geq g(x)$, we have

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{(b+c)/2}\int_{(b+c)/2}^{\infty} f(d)f(a)f(b)f(c)\,dd\,da\,db\,dc$$

$$\geq\ 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{(b+c)/2}\int_{(b+c)/2}^{\infty} g(d)g(a)g(b)g(c)\,dd\,da\,db\,dc$$

$$= 2\int_{x_0-\delta}^{x_0+\delta}\int_{x_0-\delta}^{x_0+\delta}\int_{x_0-\delta}^{(b+c)/2}\int_{(b+c)/2}^{x_0+\delta} \frac{f(x_0)}{2}\frac{f(x_0)}{2}\frac{f(x_0)}{2}\frac{f(x_0)}{2}\,dd\,da\,db\,dc$$

$$= 2\left(\frac{f(x_0)}{2}\right)^4 \int_{x_0-\delta}^{x_0+\delta}\int_{x_0-\delta}^{x_0+\delta}\int_{x_0-\delta}^{(b+c)/2}\int_{(b+c)/2}^{x_0+\delta} dd\,da\,db\,dc \tag{6.4}$$

Now to map the interval $(x-\delta, x+\delta)$ to the interval (0,1), define a linear change of variable with $a = x_0 - \delta + 2\delta a_0$, $b = x_0 - \delta + 2\delta b_0$, $c = x_0 - \delta + 2\delta c_0$, $d = x_0 - \delta + 2\delta d_0$. Then **[6.4]** becomes

$$2\left(\frac{f(x_0)}{2}\right)^4 \int_0^1\int_0^1\int_0^{(b_0+c_0)/2}\int_{(b_0+c_0)/2}^{\infty} f(d_0)f(a_0)f(b_0)f(c_0)\,dd_0\,da_0\,db_0\,dc_0$$

This is the same equation obtained in the uniform distribution so $P(R_0 = 3)$ for the continuous distribution is bounded below by a constant multiple of $P(R_0(A))$ for the uniform distribution. Similar computations also bounds the values for $P(R_0(A) = 4)$

and $P(R_0(A)=r)$. Now we use these bounds and the computation of the expected value for the uniform distribution denoted $E_U(R_0(A))$ to find the expected value of for the continuous distribution denoted $E_C(R_0(A))$.

$$
\begin{aligned}
E_C(R_0(A)) &= \sum_{r=3}^{\infty} rP(R_0(A)=r) \\
&\geq \left(\frac{f(x_0)}{2}\right)^4 E_U(R_0(A)) \\
&= \left(\frac{f(x_0)}{2}\right)^4 \left(\frac{3\cdot 5}{12}+\frac{4\cdot 7}{18}+\frac{7}{12}\left(\frac{5}{3\cdot 4}+\frac{6}{4\cdot 5}+\frac{7}{5\cdot 6}+\cdots\right)\right) \\
&= \left(\frac{f(x_0)}{2}\right)^4 \infty = \infty.
\end{aligned}
$$

□

Next, we need to show that $E(R_0(A))=\infty$ for any $m\times m$ matrix $A$. We will show that the behavior of $A$ is dictated by the behavior of a $2\times 2$ block of $A$ and then we can apply the previous theorems to the $2\times 2$ block and hence all of $A$.

**Definition 6.26** *$A$ critical block of a matrix $A$ is a principle submatrix of $A$ where the entries within this submatrix are smaller than all the other entries in $A$.*

**Lemma 6.27** *If $A$ is a $m\times m$ matrix with a critical $2\times 2$ block $U$, then $R_0(U)\leq R_0(A)$.*

*Proof:* Without loss of generality, we can do a symmetric permutation of $A$ to obtain

$$A=\begin{bmatrix} U & \vdots \\ \cdots & \cdots \end{bmatrix}.$$

Let $u_i$ and $u_j$ be the vertices of $U$. Consider all paths of length $r$ from $u_i$ to $u_j$, if this path uses arcs between $U$ and the rest of $A$, then each of these arcs could be replaced by an arc of lesser weight in $U$. Thus any minimal path between $u_i$ and $u_j$ uses only arcs of $U$. This gives.

$$A^r = \begin{bmatrix} U^r & \vdots \\ \cdots & \cdots \end{bmatrix}.$$

Let $k \leq R_0(A)$ be the smallest value so

$$A^k = \begin{bmatrix} U^k & \vdots \\ \cdots & \cdots \end{bmatrix} \text{ and } A^{k-p} = \begin{bmatrix} q \boxtimes U^{k-p} & \vdots \\ \cdots & \cdots \end{bmatrix}.$$

The inequality holds because the submatrix $U$ may be periodic before the entire matrix $A$ is. Thus $R_0(U) \leq R_0(A)$. □

**Theorem 6.28** *Let $A$ be a $m \times m$ matrix whose entries are independently chosen from the continuous distribution on $\Re$. Then $E(R_0(A)) = \infty$.*

*Proof:* Let

$$R_0^*(A) = \begin{cases} R_0(A) & \text{if } A \text{ has a } 2 \times 2 \text{ critical block } U \\ 0 & \text{otherwise} \end{cases}$$

The probability that $A_{m\times m}$ contains a critical block $U_{2\times 2}$ is given by

$$P(U \in A) = \frac{1}{\begin{pmatrix} m^2 \\ 4 \end{pmatrix}}.$$

Then

$$\begin{aligned}
E(R_0^*(A)) &= E(R_0(A)) + 0 \\
&= \sum_{r=3}^{\infty} r \ P(R_0(A) = r) \\
&\geq \sum_{r=3}^{\infty} r \ P(R_0(U) = r) \ P(U \in A) \\
&= \frac{1}{\binom{m^2}{4}} E(R_0(U)) \\
&= \frac{1}{\binom{m^2}{4}} \infty = \infty
\end{aligned}$$

□

## 6.5 Conclusions

In section 6.1, we have a bound for $R_0(A)$ in terms of its entries. We can compare this bound with the actual values of $R_0$ in the applications. In computing this bound, we needed to consider entries in $A^k$ where $k \geq m^2 - m + 1$ guarantees that there is a path of any length greater than $k$. In most cases, the actual value of $k$ is much smaller than $m^2 - m + 1$. If the entries in the successive powers of the matrix grow very slowly this will not have a large effect on the accuracy of the bound. However if the entries grow significantly from one power of the matrix to the next, then this bound may be far less accurate.

We cannot illustrate the bound developed in Section 6.2 in any of our applications because none of the transitions matrices were MCM Hamiltonian. However, in every application, we had $R_0 \leq m^2 - m + 1$. For the bound in Section 6.1 to be larger than the bound in Section 6.2, the diameter of the matrix would have to be large. In our examples, the diameter of the matrix was less than $2m$ where $m$ is the size of the matrix, so the bound from Section 6.1 is closer to the actual value of $R_0$ for these matrices.

Finally, looking at the results from sections 6.3 and 6.4 we see that for any irreducible matrix $A$, the pre-periodic interval $R_0(A)$ is finite, but the expected value of $R_0(A)$ for all such $A$ is infinite. Given this result, it may seem somewhat surprising that in all of our examples $R_0(A)$ was relatively small. However, when we also consider the dependency of $R_0(A)$ on the size of the entries in $A$ which were restricted to $\{0, 1, 2\}$ in all of our examples, we might expect reasonably small values for $R_0(A)$. In fact, looking at Figure 6.2 we see that in our experiment, the average value for $R_0(A)$ with the entries in $A$ chosen from $\{0, 1, 2\}$ is less than 5 for all sizes of $A$. This average value is less than the values found for the transition matrices in our applications.

The applications presented in Chapters 3-5 are just a sample of the problems that can be solved using Min-Plus algebra. Other finite state problems that have been solved using similar methods include independence number of graphs, coloring of graphs, shipping problems, and even an optimal currency trading problem. The algorithms for solving all of these problems are similar to those discussed in this thesis, and the defining feature of these algorithms is periodicity which allows us to find infinitely many solutions in finite time. The main limitation of these algorithms is the number of iterations needed before periodicity occurs. The bounds developed in this Chapter can be used to determine the size of the pre-periodic interval and thus the feasibility of solving problems with Min-Plus algebra.

# A. APPENDIX Domination Numbers of Circulant Graphs

**Theorem 1.** *Let $S = \{1\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{3} \rceil$ for all $n \geq 2$.*

**Theorem 2.** *Let $S = \{2\}$. Then for all $n \geq 4$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{3} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2 \pmod 6 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 3.** *Let $S = \{1, 2\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{5} \rceil$ for all $n \geq 4$.*

**Theorem 4.** *Let $S = \{3\}$. Then for all $n \geq 6$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{3} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 3 \pmod 9 \\ 1 & \text{if } n \equiv 6 \pmod 9 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 5.** *Let $S = \{1, 3\}$. Then for all $n \geq 6$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4 \pmod 5 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 6.** *Let $S = \{2, 3\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{4} \rceil$ for all $n \geq 6$.*

**Theorem 7.** *Let $S = \{1, 2, 3\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ for all $n \geq 6$.*

**Theorem 8.** *Let $S = \{4\}$. Then for all $n \geq 8$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{3} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 4 \pmod{12} \\ 1 & \text{if } n \equiv 2,\ 8 \pmod{12} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 9.** *Let $S = \{1, 4\}$. Then for all $n \geq 8$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 8 \pmod 9 \text{ except } n = 13,\ 26,\ 39 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 10.** *Let $S = \{2, 4\}$. Then for all $n \geq 8$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4 \pmod{10} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 11.** *Let $S = \{1, 2, 4\}$. Then for all $n \geq 8$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6 \pmod{7} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 12.** *Let $S = \{3, 4\}$. Then for all $n \geq 8$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4 \pmod{5} \text{ except } n = 12,\ 13 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 13.** *Let $S = \{1, 3, 4\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{6} \rceil$ for all $n \geq 8$.*

**Theorem 14.** *Let $S = \{2, 3, 4\}$. Then for all $n \geq 8$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6 \pmod{12} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 15.** *Let $S = \{1, 2, 3, 4\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{9} \rceil$ for all $n \geq 8$.*

**Theorem 16.** *Let $S = \{5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{3} \right\rceil + \begin{cases} 3 & \text{if } n \equiv 5 \pmod{15} \\ 1 & \text{if } n \equiv 10 \pmod{15} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 17.** *Let $S = \{1, 5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 1,\ 3,\ 4,\ 6,\ 8 \pmod{9} \text{ except } n = 10,\ 15,\ 19,\ 24, \\ & 28,\ 37,\ 42 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 18.** *Let $S = \{2, 5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4 \pmod{9} \text{ except } n = 13 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 19.** *Let $S = \{1, 2, 5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 11,\ 16 \pmod{17} \text{ except } n = 11,\ 33 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 20.** *Let $S = \{3,5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 8,\ 9,\ 10,\ 12,\ 13,\ 15,\ 17 \pmod{18} \text{ except} \\ & n = 10,\ 12,\ 13,\ 15,\ 26,\ 28,\ 33,\ 39,\ 46,\ 51,\ 64 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 21.** *Let $S = \{1,3,5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6 \pmod{7} \text{ except } n = 11 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 22.** *Let $S = \{2,3,5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 12 \pmod{13} \text{ except } n = 12,\ 18 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 23.** *Let $S = \{1,2,3,5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8 \pmod{9} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 24.** *Let $S = \{4,5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{14} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 7,\ 8,\ 9 \pmod{14}, \text{ or } n = 12 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 25.** *Let $S = \{1,4,5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6 \pmod{7} \text{ except } n = 11,\ 12,\ 13,\ 25,\ 26, \\ & 39 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 26.** *Let $S = \{2,4,5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6 \pmod{12} \text{ except } n = 15 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 27.** *Let $S = \{1,2,4,5\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{8} \rceil$ for all $n \geq 10$.*

**Theorem 28.** *Let $S = \{3,4,5\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{6} \rceil$ for all $n \geq 10$.*

**Theorem 29.** *Let $S = \{1,3,4,5\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ for all $n \geq 10$.*

**Theorem 30.** *Let $S = \{2,3,4,5\}$. Then for all $n \geq 10$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7 \text{ (mod 15)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 31.** *Let $S = \{1,2,3,4,5\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ for all $n \geq 10$.*

**Theorem 32.** *Let $S = \{6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{3} \right\rceil + \begin{cases} 4 & \text{if } n \equiv 6 \text{ (mod 18)} \\ 2 & \text{if } n \equiv 3,\ 12 \text{ (mod 18)} \\ 1 & \text{if } n \equiv 2,\ 8,\ 14,\ 15 \text{ (mod 18)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 33.** *Let $S = \{1,6\}$. Then $\gamma(C_n[S]) = \left\lceil \frac{3n}{13} \right\rceil$ for all $n \geq 12$.*

**Theorem 34.** *Let $S = \{2,6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 8 \text{ (mod 10)} \\ 1 & \text{if } n \equiv 2,\ 4,\ 9 \text{ (mod 10)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 35.** *Let $S = \{1,2,6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 8,\ 10 \text{ (mod 11) except } n = 14,\ 15,\ 19, \\ & 25,\ 30 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 36.** *Let $S = \{3,6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 3 \text{ (mod 15)} \\ 1 & \text{if } n \equiv 6,\ 9 \text{ (mod 15)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 37.** *Let $S = \{1,3,6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 9,\ 11,\ 12,\ 14,\ 16,\ 17 \text{ (mod 18) except} \\ & n = 14,\ 22,\ 29,\ 32,\ 40,\ 50,\ 58,\ 68 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 38.** *Let $S = \{2,3,6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 5,\ 6 \text{ (mod 7)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 39.** *Let* $S = \{1,2,3,6\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ *for all* $n \geq 12$.

**Theorem 40.** *Let* $S = \{4,6\}$. *Then for all* $n \geq 12$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{4} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4,\ 10,\ 12 \pmod{16} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 41.** *Let* $S = \{1,4,6\}$. *Then for all* $n \geq 12$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{17} \right\rceil + \begin{cases} 1 & \text{if } n = 16,\ 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 42.** *Let* $S = \{2,4,6\}$. *Then for all* $n \geq 12$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4,\ 6 \pmod{14} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 43.** *Let* $S = \{1,2,4,6\}$. *Then for all* $n \geq 12$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8 \pmod{9} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 44.** *Let* $S = \{3,4,6\}$. *Then for all* $n \geq 12$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 12 \pmod{13} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 45.** *Let* $S = \{1,3,4,6\}$. *Then for all* $n \geq 12$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7 \pmod{8} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 46.** *Let* $S = \{2,3,4,6\}$. *Then for all* $n \geq 12$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 8,\ 15 \pmod{16} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 47.** *Let* $S = \{1,2,3,4,6\}$. *Then for all* $n \geq 12$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 10 \pmod{11} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 48.** *Let $S = \{5, 6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{5n}{23} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 18 \text{ (mod 23) except } n = 18,\ 41,\ 64 \\ 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 8,\ 9,\ 11,\ 12,\ 13,\ 17,\ 19,\ 20,\ 21,\ 22 \\ & \text{(mod 23) except } n = 12,\ 13,\ 17,\ 19,\ 20,\ 21,\ 25,\ 26, \\ & 28,\ 34,\ 42,\ 43,\ 51,\ 65, \text{ or } n = 18,\ 41,\ 64 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 49.** *Let $S = \{1, 5, 6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6 \text{ (mod 13) except } n = 17,\ 18 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 50.** *Let $S = \{2, 5, 6\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{5} \rceil$ for all $n \geq 12$.*

**Theorem 51.** *Let $S = \{1, 2, 5, 6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8 \text{ (mod 9) except } n = 15 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 52.** *Let $S = \{3, 5, 6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6 \text{ (mod 7) except } n = 16,\ 17,\ 18 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 53.** *Let $S = \{1, 3, 5, 6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6 \text{ (mod 14)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 54.** *Let $S = \{2, 3, 5, 6\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ for all $n \geq 12$.*

**Theorem 55.** *Let $S = \{1, 2, 3, 5, 6\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{10} \rceil$ for all $n \geq 12$.*

**Theorem 56.** *Let $S = \{4, 5, 6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6 \text{ (mod 7) except } n = 16,\ 17,\ 18,\ 19, \\ & 23,\ 37,\ 38 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 57.** *Let $S = \{1, 4, 5, 6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n = 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 58.** *Let $S = \{2,4,5,6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8 \pmod{16} \text{ except } n = 21 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 59.** *Let $S = \{1,2,4,5,6\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{9} \rceil$ for all $n \geq 12$.*

**Theorem 60.** *Let $S = \{3,4,5,6\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ for all $n \geq 12$.*

**Theorem 61.** *Let $S = \{1,3,4,5,6\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{8} \rceil$ for all $n \geq 12$.*

**Theorem 62.** *Let $S = \{2,3,4,5,6\}$. Then for all $n \geq 12$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9 \pmod{18} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 63.** *Let $S = \{1,2,3,4,5,6\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{13} \rceil$ for all $n \geq 12$.*

**Theorem 64.** *Let $S = \{7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{3} \right\rceil + \begin{cases} 4 & \text{if } n \equiv 7 \pmod{21} \\ 2 & \text{if } n \equiv 14 \pmod{21} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 65.** *Let $S = \{1,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 1 \pmod{5} \text{ except } n = 16,\ 21,\ 26,\ 31,\ 36,\ 41,\ 46, \\ & \quad 51,\ 56,\ 61,\ 66,\ 71,\ 76,\ 81 \\ 1 & \text{if } n \equiv 2,\ 3,\ 4 \pmod{5} \text{ except } n = 18,\ 22,\ 27, \text{ or } n = 21, \\ & \quad 41,\ 46,\ 51,\ 56,\ 61,\ 66,\ 71,\ 76,\ 81 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 66.** *Let $S = \{2,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{5n}{23} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 9 \pmod{23} \text{ except } n = 32,\ 55,\ 78,\ 124,\ 147,\ 216 \\ 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 7,\ 8,\ 10,\ 13,\ 14,\ 16,\ 17,\ 18,\ 20,\ 21,\ 22 \\ & \pmod{23} \text{ except } n = 14,\ 16,\ 20,\ 26,\ 27,\ 37,\ 39,\ 43, \\ & 54,\ 60,\ 66,\ 79,\ 108,\ 135, \text{ or } n = 32,\ 55,\ 78,\ 124,\ 147, \\ & 216 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 67.** *Let $S = \{1,2,7\}$. Then $\gamma(C_n[S]) = \left\lceil \frac{2n}{11} \right\rceil$ for all $n \geq 14$.*

**Theorem 68.** *Let $S = \{3, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13 \pmod{18} \\ & \text{except } n = 23,\ 24,\ 25,\ 26,\ 28,\ 29,\ 30,\ 38,\ 39,\ 41,\ 42, \\ & 43,\ 46,\ 47,\ 56,\ 59,\ 60,\ 64,\ 65,\ 77,\ 78,\ 82,\ 95 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 69.** *Let $S = \{1, 3, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6 \pmod{12} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 70.** *Let $S = \{2, 3, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 5,\ 6,\ 9,\ 10,\ 12,\ 16,\ 17,\ 18 \pmod{19} \text{ except} \\ & n = 18,\ 21,\ 24,\ 36,\ 48,\ 54,\ 66,\ 78 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 71.** *Let $S = \{1, 2, 3, 7\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ for all $n \geq 14$.*

**Theorem 72.** *Let $S = \{4, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 1,\ 4 \pmod{5} \text{ except } n = 14,\ 16,\ 19,\ 21,\ 24,\ 26, \\ & 31,\ 34,\ 36,\ 41,\ 46,\ 51,\ 54,\ 71,\ 81 \\ 1 & \text{if } n \equiv 2,\ 3 \pmod{5} \text{ except } n = 17,\ 18,\ 27, \text{ or } n = 14, \\ & 19,\ 21,\ 24,\ 26,\ 31,\ 34,\ 41,\ 46,\ 51,\ 54,\ 71,\ 81 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 73.** *Let $S = \{1, 4, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 8,\ 11,\ 12,\ 15,\ 17,\ 18 \pmod{19} \text{ except} \\ & n = 15,\ 17,\ 24,\ 27,\ 34,\ 36,\ 46,\ 72,\ 84 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 74.** *Let $S = \{2, 4, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6 \pmod{13} \text{ except } n = 17 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 75.** *Let $S = \{1, 2, 4, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{22} \right\rceil + \begin{cases} 1 & \text{if } n = 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 76.** *Let* $S = \{3,4,7\}$*. Then for all* $n \geq 14$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 6,\ 11,\ 12 \pmod{13} \text{ except } n = 16,\ 17,\ 24, \\ & 29,\ 30,\ 38,\ 42,\ 55,\ 76 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 77.** *Let* $S = \{1,3,4,7\}$*. Then for all* $n \geq 14$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 7,\ 8 \pmod{9} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 78.** *Let* $S = \{2,3,4,7\}$*. Then for all* $n \geq 14$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7 \pmod{8} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 79.** *Let* $S = \{1,2,3,4,7\}$*. Then* $\gamma(C_n[S]) = \lceil \frac{n}{9} \rceil$ *for all* $n \geq 14$*.*

**Theorem 80.** *Let* $S = \{5,7\}$*. Then for all* $n \geq 14$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{19} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 8 \pmod{19} \text{ except } n = 27,\ 46,\ 65 \\ 1 & \text{if } n \equiv 2,\ 4,\ 5,\ 7,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 18 \pmod{19} \\ & \text{except } n = 18,\ 24,\ 26,\ 43,\ 45,\ 62,\ 81, \text{ or } n = 46,\ 65 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 81.** *Let* $S = \{1,5,7\}$*. Then for all* $n \geq 14$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4,\ 5,\ 6,\ 12 \pmod{13} \text{ except } n = 15 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 82.** *Let* $S = \{2,5,7\}$*. Then for all* $n \geq 14$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 12,\ 18 \pmod{19} \text{ except } n = 37 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 83.** *Let* $S = \{1,2,5,7\}$*. Then for all* $n \geq 14$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 5,\ 6,\ 7,\ 12,\ 14 \pmod{15} \text{ except } n = 21,\ 42, \\ & 63 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 84.** *Let $S = \{3,5,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 12 \text{ (mod 24) except } n = 36,\ 60,\ 84 \\ 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 10,\ 11,\ 14,\ 16,\ 17,\ 18,\ 19,\ 21,\ 23 \\ & \text{(mod 24) except } n = 14,\ 16,\ 17,\ 19,\ 21,\ 34,\ 38,\ 43, \\ & 51,\ 67, \text{ or } n = 36,\ 60,\ 84 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 85.** *Let $S = \{1,3,5,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 8 \text{ (mod 9) except } n = 15,\ 31 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 86.** *Let $S = \{2,3,5,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 5,\ 7,\ 8,\ 14,\ 16 \text{ (mod 17) except } n = 14,\ 16, \\ & 31,\ 48 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 87.** *Let $S = \{1,2,3,5,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 10 \text{ (mod 11)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 88.** *Let $S = \{4,5,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{16} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 8,\ 9,\ 10,\ 15 \text{ (mod 16) except } n = 15,\ 20, \\ & 21,\ 24,\ 25,\ 31,\ 36,\ 40,\ 41,\ 42,\ 52,\ 56,\ 57,\ 63,\ 72,\ 73, \\ & 84,\ 88,\ 104,\ 105,\ 120,\ 136,\ 168 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 89.** *Let $S = \{1,4,5,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 7,\ 14 \text{ (mod 15) except } n = 14,\ 19,\ 21, \\ & 33,\ 35,\ 49,\ 63 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 90.** *Let $S = \{2,4,5,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7 \text{ (mod 16)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 91.** *Let $S = \{1, 2, 4, 5, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 9 \text{ (mod 10)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 92.** *Let $S = \{3, 4, 5, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 7,\ 14 \text{ (mod 15)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 93.** *Let $S = \{1, 3, 4, 5, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 17 \text{ (mod 18)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 94.** *Let $S = \{2, 3, 4, 5, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 7,\ 8,\ 9,\ 18 \text{ (mod 19) except } n = 24 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 95.** *Let $S = \{1, 2, 3, 4, 5, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 12 \text{ (mod 13)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 96.** *Let $S = \{6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 1,\ 2,\ 3 \text{ (mod 5) except } n = 16,\ 17,\ 18,\ 21,\ 22, \\ & 23,\ 26,\ 27,\ 28,\ 31,\ 32,\ 36,\ 37,\ 38,\ 41,\ 42,\ 43,\ 46,\ 47, \\ & 51,\ 57,\ 58,\ 61,\ 62,\ 66,\ 68,\ 73,\ 76,\ 77,\ 81,\ 87,\ 92,\ 96, \\ & 103,\ 106,\ 111,\ 122,\ 133,\ 141,\ 152,\ 171 \\ 1 & \text{if } n \equiv 4 \text{ (mod 5) except } n = 19, \text{ or } n = 17,\ 18,\ 21,\ 22, \\ & 23,\ 26,\ 28,\ 31,\ 32,\ 36,\ 37,\ 41,\ 42,\ 43,\ 47,\ 51,\ 58,\ 61, \\ & 62,\ 66,\ 68,\ 73,\ 77,\ 81,\ 87,\ 92,\ 96,\ 103,\ 106,\ 111,\ 122, \\ & 133,\ 141,\ 152,\ 171 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 97.** *Let $S = \{1, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{20} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 12,\ 13,\ 16,\ 17,\ 18,\ 19 \text{ (mod 20) except} \\ & n = 16,\ 17,\ 18,\ 19,\ 24,\ 36,\ 37,\ 38,\ 56,\ 57,\ 76 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 98.** *Let $S = \{2, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{4, 5, 6, 10, 11, 12 (mod 18) except } n = \textit{22, 28,} \\ & \textit{or } n = \textit{16, 26} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 99.** *Let $S = \{1, 2, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{5, 6, 7 (mod 15) except } n = \textit{20, 21, 22, 65, 66,} \\ & \textit{110} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 100.** *Let $S = \{3, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{4, 5, 6, 7, 8, 9, 10, 11, 12 (mod 19) except} \\ & n = \textit{23, 26, 27, 46, 64} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 101.** *Let $S = \{1, 3, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{2, 6, 7, 8, 13, 14, 15 (mod 16) except } n = \textit{18} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 102.** *Let $S = \{2, 3, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{6, 13, 20, 27, 34 (mod 42)} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 103.** *Let $S = \{1, 2, 3, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{8, 9, 10 (mod 11)} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 104.** *Let $S = \{4, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{23} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{4, 5, 10, 11, 15, 16, 17, 22 (mod 23) except} \\ & n = \textit{15, 17, 34, 51, 68, 85, 102, 119, 153, or } n = \textit{36} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 105.** *Let $S = \{1, 4, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{4, 5, 6, 7, 8 (mod 9)} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 106.** *Let $S = \{2,4,6,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 7,\ 8 \pmod{16} \text{ except } n = 19,\ 21,\ 35 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 107.** *Let $S = \{1,2,4,6,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8 \pmod{9} \text{ except } n = 15 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 108.** *Let $S = \{3,4,6,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8 \pmod{17} \text{ except } n = 24,\ 40, \text{ or } n = 15, \\ & 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 109.** *Let $S = \{1,3,4,6,7\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{9} \rceil$ for all $n \geq 14$.*

**Theorem 110.** *Let $S = \{2,3,4,6,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8 \pmod{17} \text{ except } n = 23,\ 24,\ 40 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 111.** *Let $S = \{1,2,3,4,6,7\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{12} \rceil$ for all $n \geq 14$.*

**Theorem 112.** *Let $S = \{5,6,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 9,\ 10,\ 11,\ 12 \pmod{19} \text{ except } n = 24,\ 47, \\ & 48, \text{ or } n = 16 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 113.** *Let $S = \{1,5,6,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8 \pmod{18} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 114.** *Let $S = \{2,5,6,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14 \pmod{21} \\ & \text{except } n = 25,\ 26,\ 29,\ 30,\ 50,\ 51,\ 52 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 115.** *Let $S = \{1, 2, 5, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} \mathit{1} & \text{if } n = 26 \\ \mathit{0} & \text{otherwise.} \end{cases}$$

**Theorem 116.** *Let $S = \{3, 5, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} \mathit{1} & \text{if } n \equiv 4,\ 5,\ 6 \pmod{7} \text{ except } n = 18,\ 19 \\ \mathit{0} & \text{otherwise.} \end{cases}$$

**Theorem 117.** *Let $S = \{1, 3, 5, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} \mathit{1} & \text{if } n \equiv 4,\ 6,\ 8 \pmod{18} \\ \mathit{0} & \text{otherwise.} \end{cases}$$

**Theorem 118.** *Let $S = \{2, 3, 5, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} \mathit{1} & \text{if } n \equiv 7,\ 8 \pmod{17} \\ \mathit{0} & \text{otherwise.} \end{cases}$$

**Theorem 119.** *Let $S = \{1, 2, 3, 5, 6, 7\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ for all $n \geq 14$.*

**Theorem 120.** *Let $S = \{4, 5, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} \mathit{1} & \text{if } n = 34 \\ \mathit{0} & \text{otherwise.} \end{cases}$$

**Theorem 121.** *Let $S = \{1, 4, 5, 6, 7\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{9} \rceil$ for all $n \geq 14$.*

**Theorem 122.** *Let $S = \{2, 4, 5, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} \mathit{1} & \text{if } n \equiv 6,\ 7,\ 8,\ 9 \pmod{18} \text{ except } n = 24 \\ \mathit{0} & \text{otherwise.} \end{cases}$$

**Theorem 123.** *Let $S = \{1, 2, 4, 5, 6, 7\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{10} \rceil$ for all $n \geq 14$.*

**Theorem 124.** *Let $S = \{3, 4, 5, 6, 7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} \mathit{1} & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10 \pmod{20} \\ \mathit{0} & \text{otherwise.} \end{cases}$$

**Theorem 125.** *Let $S = \{1,3,4,5,6,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10 \pmod{20} \text{ except } n = 26,\ 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 126.** *Let $S = \{2,3,4,5,6,7\}$. Then for all $n \geq 14$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{21} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9,\ 10 \pmod{21} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 127.** *Let $S = \{1,2,3,4,5,6,7\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{15} \rceil$ for all $n \geq 14$.*

**Theorem 128.** *Let $S = \{8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{3} \right\rceil + \begin{cases} 5 & \text{if } n \equiv 8 \pmod{24} \\ 2 & \text{if } n \equiv 4,\ 16 \pmod{24} \\ 1 & \text{if } n \equiv 2,\ 14,\ 20 \pmod{24} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 129.** *Let $S = \{1,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 3 & \text{if } n \equiv 4 \pmod{5} \text{ except } n = 19,\ 24,\ 29,\ 34,\ 39,\ 44,\ 49, \\ & \quad 54,\ 59,\ 64,\ 69,\ 74,\ 79,\ 109 \\ 2 & \text{if } n = 24,\ 39,\ 49,\ 59,\ 64,\ 74,\ 79,\ 109 \\ 1 & \text{if } n \equiv 1,\ 2,\ 3 \pmod{5} \text{ except } n = 17,\ 21,\ 23,\ 27,\ 46, \\ & \quad \text{or } n = 19,\ 29,\ 34,\ 44,\ 54,\ 69 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 130.** *Let $S = \{2,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{9} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 6,\ 8,\ 16 \pmod{18} \text{ except } n = 26,\ 52,\ 78 \\ 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 10,\ 12,\ 13,\ 17 \pmod{18} \text{ except } n = 39 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 131.** *Let $S = \{1,2,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6 \pmod{12} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 132.** *Let $S = \{3, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{19} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 3,\ 8,\ 9,\ 18 \pmod{19} \text{ except } n = 18,\ 22,\ 27,\ 28, \\ & 37,\ 41,\ 46,\ 47,\ 56,\ 60,\ 65,\ 66,\ 75,\ 79,\ 84,\ 85,\ 94, \\ & 98,\ 103,\ 113,\ 117,\ 122,\ 123,\ 136,\ 141,\ 151,\ 155,\ 179, \\ & 180,\ 193,\ 208,\ 236,\ 237,\ 250,\ 265,\ 293,\ 294,\ 307, \\ & 322,\ 350,\ 364 \\ 1 & \text{if } n \equiv 1,\ 4,\ 5,\ 6,\ 7,\ 10,\ 11,\ 12,\ 13,\ 14,\ 16 \pmod{19} \\ & \text{except } n = 20,\ 24,\ 26,\ 29,\ 39,\ 42,\ 43,\ 54,\ 67,\ 68, \\ & 69,\ 70,\ 81,\ 82,\ 83,\ 96,\ 111,\ 124,\ 125,\ 126,\ 138,\ 139, \\ & 140,\ 153,\ 168,\ 181,\ 182,\ 195,\ 196,\ 210,\ 238,\ 252, \text{ or} \\ & n = 18,\ 22,\ 37,\ 46,\ 47,\ 60,\ 65,\ 66,\ 75,\ 79,\ 85,\ 94,\ 103, \\ & 113,\ 117,\ 122,\ 123,\ 136,\ 141,\ 151,\ 155,\ 179,\ 180,\ 193, \\ & 208,\ 236,\ 237,\ 250,\ 265,\ 293,\ 294,\ 307,\ 322,\ 350,\ 364 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 133.** *Let $S = \{1, 3, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5 \pmod{6} \text{ except } n = 20,\ 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 134.** *Let $S = \{2, 3, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 5 \pmod{7} \text{ except } n = 19,\ 26,\ 33,\ 47,\ 61,\ 82,\ 117 \\ 1 & \text{if } n \equiv 3,\ 4,\ 6 \pmod{7} \text{ except } n = 17,\ 24,\ 39,\ 52, \text{ or} \\ & n = 19,\ 33,\ 47,\ 61,\ 82,\ 117 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 135.** *Let $S = \{1, 2, 3, 8\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ for all $n \geq 16$.*

**Theorem 136.** *Let $S = \{4, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 3 & \text{if } n \equiv 4 \pmod{20} \\ 2 & \text{if } n \equiv 8 \pmod{20} \\ 1 & \text{if } n \equiv 2,\ 12,\ 14 \pmod{20} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 137.** *Let $S = \{1, 4, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 12 \pmod{13} \text{ except } n = 25,\ 38,\ 51 \\ 1 & \text{if } n \equiv 1,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10 \pmod{13} \text{ except } n = 17,\ 21, \\ & 27,\ 34, \text{ or } n = 25,\ 38,\ 51 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 138.** *Let $S = \{2,4,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 12 \text{ (mod 14)} \\ 1 & \text{if } n \equiv 2,\ 4,\ 6,\ 13 \text{ (mod 14)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 139.** *Let $S = \{1,2,4,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6 \text{ (mod 14)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 140.** *Let $S = \{3,4,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 11,\ 12,\ 16,\ 17,\ 18 \text{ (mod 19)} \\ & \text{except } n = 17,\ 22,\ 26,\ 35,\ 37,\ 41,\ 46,\ 54,\ 59,\ 61,\ 65, \\ & 74,\ 78,\ 83,\ 98,\ 102,\ 111,\ 122,\ 135,\ 159 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 141.** *Let $S = \{1,3,4,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 14,\ 15,\ 22 \text{ (mod 23) except } n = 30,\ 45,\ 60 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 142.** *Let $S = \{2,3,4,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 6,\ 7,\ 8 \text{ (mod 9)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 143.** *Let $S = \{1,2,3,4,8\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{9} \rceil$ for all $n \geq 16$.*

**Theorem 144.** *Let $S = \{5,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{5n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 15,\ 16,\ 17,\ 18, \\ & 22 \text{ (mod 23) except } n = 18,\ 27,\ 29,\ 30,\ 36,\ 38,\ 39,\ 45, \\ & 54,\ 56,\ 57,\ 61,\ 63,\ 72,\ 75,\ 79,\ 81,\ 84,\ 99,\ 102,\ 108, \\ & 126,\ 144,\ 153,\ 171, \text{ or } n = 24 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 145.** *Let $S = \{1,5,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 1,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 11 \text{ (mod 12) except } n = 20, \\ & 22,\ 25,\ 37,\ 44 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 146.** *Let $S = \{2, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12 \pmod{19} \text{ except} \\ & n = 21,\ 22,\ 23,\ 24,\ 26,\ 27,\ 28,\ 29,\ 40,\ 41,\ 45,\ 46,\ 47, \\ & 48,\ 59,\ 60,\ 64,\ 65,\ 83,\ 84 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 147.** *Let $S = \{1, 2, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{31} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 13,\ 15,\ 21,\ 22,\ 23,\ 29,\ 30 \pmod{31} \text{ except} \\ & n = 23,\ 30,\ 46,\ 60,\ 69,\ 75,\ 92,\ 115,\ 161,\ 184,\ 207, \\ & 230 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 148.** *Let $S = \{3, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 5 \pmod{7} \text{ except } n = 19,\ 26,\ 33,\ 40,\ 47,\ 54,\ 61, \\ & 75 \\ 1 & \text{if } n \equiv 1,\ 3,\ 4,\ 6 \pmod{7} \text{ except } n = 17,\ 18,\ 22,\ 24,\ 29, \\ & 36,\ 38,\ 57, \text{ or } n = 26,\ 33,\ 40,\ 47,\ 54,\ 61,\ 75 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 149.** *Let $S = \{1, 3, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7 \pmod{15} \text{ except } n = 21,\ 35 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 150.** *Let $S = \{2, 3, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 5,\ 7,\ 8 \pmod{9} \text{ except } n = 29,\ 47 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 151.** *Let $S = \{1, 2, 3, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n = 17 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 152.** *Let $S = \{4, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 5 \pmod{7} \text{ except } n = 19,\ 26,\ 33 \\ 1 & \text{if } n \equiv 1,\ 2,\ 4,\ 6 \pmod{7} \text{ except } n = 22, \text{ or } n = 19,\ 26, \\ & 33 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 153.** *Let $S = \{1, 4, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{3, 4, 5, 6, 7, 8, 16 (mod 17) except } n = \textit{21} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 154.** *Let $S = \{2, 4, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{5, 6, 7 (mod 15) except } n = \textit{21, 35} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 155.** *Let $S = \{1, 2, 4, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{6, 9, 10 (mod 11) except } n = \textit{17} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 156.** *Let $S = \{3, 4, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{4, 5, 7, 13, 14 (mod 15) except } n = \textit{19, 28, 35,} \\ & \textit{43, 49} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 157.** *Let $S = \{1, 3, 4, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{8, 9 (mod 10)} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 158.** *Let $S = \{2, 3, 4, 5, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{7, 8 (mod 9)} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 159.** *Let $S = \{1, 2, 3, 4, 5, 8\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ for all $n \geq 16$.*

**Theorem 160.** *Let $S = \{6, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} \textit{3} & \textit{if } n \equiv \textit{4 (mod 10) except } n = \textit{24} \\ \textit{2} & \textit{if } n \equiv \textit{6, 8 (mod 10) except } n = \textit{26} \\ \textit{1} & \textit{if } n \equiv \textit{2, 3, 7, 9 (mod 10), or } n = \textit{24} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 161.** *Let* $S = \{1, 6, 8\}$*. Then for all* $n \geq 16$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 9,\ 11,\ 12 \pmod{13} \text{ except } n = 17,\ 18, \\ & 19,\ 22,\ 38,\ 57,\ 61,\ 74,\ 76,\ 95,\ 152 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 162.** *Let* $S = \{2, 6, 8\}$*. Then for all* $n \geq 16$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4,\ 6 \pmod{12} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 163.** *Let* $S = \{1, 2, 6, 8\}$*. Then for all* $n \geq 16$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 14 \pmod{15} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 164.** *Let* $S = \{3, 6, 8\}$*. Then for all* $n \geq 16$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12 \pmod{19} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 165.** *Let* $S = \{1, 3, 6, 8\}$*. Then for all* $n \geq 16$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 12,\ 14,\ 15,\ 20,\ 22 \pmod{23} \text{ except} \\ & n = 27,\ 35,\ 58 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 166.** *Let* $S = \{2, 3, 6, 8\}$*. Then for all* $n \geq 16$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 8,\ 15,\ 16,\ 22,\ 23,\ 24,\ 29,\ 31 \pmod{32} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 167.** *Let* $S = \{1, 2, 3, 6, 8\}$*. Then for all* $n \geq 16$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 14,\ 16 \pmod{17} \text{ except } n = 23,\ 25, \\ & 31,\ 39,\ 48,\ 50,\ 56,\ 57,\ 73,\ 75,\ 82,\ 107,\ 125,\ 150,\ 175 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 168.** *Let* $S = \{4, 6, 8\}$*. Then for all* $n \geq 16$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 8,\ 10,\ 12 \pmod{24} \\ 1 & \text{if } n \equiv 2,\ 4,\ 5,\ 6,\ 14,\ 16,\ 17,\ 18 \pmod{24} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 169.** *Let $S = \{1,4,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 12,\ 14,\ 20,\ 25,\ 27,\ 33,\ 34,\ 35,\ 46,\ 48, \\ & 49,\ 55,\ 62,\ 68,\ 70,\ 76,\ 77,\ 83 \pmod{84} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 170.** *Let $S = \{2,4,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4,\ 6,\ 8 \pmod{18} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 171.** *Let $S = \{1,2,4,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 10 \pmod{11} \text{ except } n = 17 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 172.** *Let $S = \{3,4,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{20} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 13,\ 20,\ 32,\ 33 \pmod{40}, \text{ or } n = 45,\ 51,\ 71 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 173.** *Let $S = \{1,3,4,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 7,\ 8,\ 9,\ 16,\ 18 \pmod{19} \text{ except } n = 16,\ 18, \\ & 24,\ 35,\ 54 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 174.** *Let $S = \{2,3,4,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 8,\ 9,\ 10,\ 17,\ 19 \pmod{20} \text{ except } n = 17 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 175.** *Let $S = \{1,2,3,4,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 10,\ 12 \pmod{13} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 176.** *Let $S = \{5,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 10,\ 11,\ 16 \pmod{22} \text{ except } n = 16,\ 32,\ 48, \\ & \text{or } n = 21,\ 36,\ 37,\ 47,\ 58,\ 69 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 177.** *Let $S = \{1, 5, 6, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4,\ 5,\ 6,\ 7,\ 14 \pmod{15} \text{ except } n = 17,\ 19 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 178.** *Let $S = \{2, 5, 6, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4,\ 5,\ 6,\ 7,\ 8 \pmod{9} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 179.** *Let $S = \{1, 2, 5, 6, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8 \pmod{9} \text{ except } n = 17,\ 34,\ 51 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 180.** *Let $S = \{3, 5, 6, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 5,\ 6,\ 8,\ 13,\ 15,\ 16 \pmod{17} \text{ except } n = 20, \\ & 23 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 181.** *Let $S = \{1, 3, 5, 6, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 9 \pmod{10} \text{ except } n = 16,\ 18,\ 36 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 182.** *Let $S = \{2, 3, 5, 6, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 18 \pmod{19} \text{ except } n = 18,\ 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 183.** *Let $S = \{1, 2, 3, 5, 6, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 11 \pmod{12} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 184.** *Let $S = \{4, 5, 6, 8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n = 20,\ 27,\ 33,\ 34,\ 41,\ 48 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 185.** *Let $S = \{1,4,5,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9 \pmod{18} \text{ except } n = 22,\ 23,\ 24, \\ & 40 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 186.** *Let $S = \{2,4,5,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 8,\ 9,\ 10,\ 19 \pmod{20} \text{ except } n = 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 187.** *Let $S = \{1,2,4,5,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 10 \pmod{11} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 188.** *Let $S = \{3,4,5,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 16 \pmod{17} \text{ except } n = 23 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 189.** *Let $S = \{1,3,4,5,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{21} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9,\ 10,\ 20 \pmod{21} \text{ except } n = 20,\ 28,\ 30, \\ & 50,\ 70 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 190.** *Let $S = \{2,3,4,5,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 9,\ 10,\ 11,\ 21 \pmod{22} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 191.** *Let $S = \{1,2,3,4,5,6,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 14 \pmod{15} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 192.** *Let $S = \{7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{5n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 7,\ 8,\ 9,\ 10,\ 12,\ 13,\ 14,\ 15,\ 16,\ 18 \pmod{23} \\ & \text{except } n = 18,\ 27,\ 31,\ 35,\ 36,\ 39,\ 53,\ 54,\ 58,\ 61,\ 62, \\ & 76,\ 79,\ 81,\ 83,\ 84,\ 85,\ 99,\ 102,\ 106,\ 107,\ 108,\ 125, \\ & 129,\ 130,\ 148,\ 152,\ 153,\ 171,\ 175,\ 198, \text{ or } n = 21,\ 24 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 193.** *Let $S = \{1,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 12 \pmod{13} \text{ except } n = 17,\ 56,\ 57,\ 58, \\ & 116,\ 121,\ 173,\ 174,\ 290 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 194.** *Let $S = \{2,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 9 \pmod{12} \text{ except } n = 21,\ 33,\ 45,\ 57,\ 69,\ 81, \\ & 93,\ 105,\ 117,\ 153 \\ 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 11 \pmod{12} \text{ except } n = 16, \\ & 17,\ 20,\ 22,\ 32,\ 34,\ 44,\ 50,\ 51,\ 68, \text{ or } n = 45,\ 57,\ 69, \\ & 81,\ 93,\ 105,\ 117,\ 153 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 195.** *Let $S = \{1,2,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 5,\ 6,\ 7,\ 8,\ 14,\ 15,\ 16 \pmod{17} \text{ except} \\ & n = 19,\ 20,\ 24,\ 36,\ 37,\ 48,\ 53 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 196.** *Let $S = \{3,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 8,\ 10,\ 11,\ 15,\ 16,\ 17,\ 20,\ 23,\ 26,\ 27, \\ & 28,\ 29 \pmod{30} \text{ except } n = 20,\ 23,\ 26,\ 45,\ 46,\ 68,\ 92 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 197.** *Let $S = \{1,3,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 6,\ 7,\ 14,\ 15,\ 19,\ 20,\ 21,\ 22,\ 27,\ 28,\ 29 \\ & \pmod{30} \text{ except } n = 19,\ 20,\ 21,\ 33,\ 34,\ 63 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 198.** *Let $S = \{2,3,7,8\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ for all $n \geq 16$.*

**Theorem 199.** *Let $S = \{1,2,3,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{26} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 15,\ 16,\ 17 \pmod{26} \text{ except } n = 16,\ 17,\ 34, \\ & 67,\ 68,\ 119, \text{ or } n = 22,\ 23 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 200.** *Let $S = \{4,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 11,\ 12 \pmod{19}, \text{ or } n = 16,\ 17,\ 35,\ 40 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 201.** *Let $S = \{1,4,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{4, 6, 7, 14, 15 (mod 23), or } n = \textit{43} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 202.** *Let $S = \{2,4,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{6 (mod 14)} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 203.** *Let $S = \{1,2,4,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{6, 7, 8, 9 (mod 18) except } n = \textit{24, or } n = \textit{16, 33} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 204.** *Let $S = \{3,4,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{2, 3, 4, 6, 7, 8 (mod 9) except } n = \textit{20, 21} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 205.** *Let $S = \{1,3,4,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{7, 8, 9 (mod 19) except } n = \textit{27, 45} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 206.** *Let $S = \{2,3,4,7,8\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{9} \rceil$ for all $n \geq 16$.*

**Theorem 207.** *Let $S = \{1,2,3,4,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{10, 11, 12 (mod 13)} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 208.** *Let $S = \{5,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} \textit{2} & \textit{if } n = \textit{36} \\ \textit{1} & \textit{if } n \equiv \textit{4, 5, 6, 10, 11, 12, 14, 15, 16, 17, 18 (mod 24)} \\ & \textit{except } n = \textit{17, 28, 29, 34, 36, 38, 39, 58, 62, 87, or} \\ & n = \textit{26, 32} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 209.** *Let* $S = \{1,5,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n = 18,\ 33,\ 34 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 210.** *Let* $S = \{2,5,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{22} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 10,\ 11,\ 12,\ 13,\ 14 \pmod{22} \text{ except} \\ & n = 27,\ 28,\ 54,\ 55,\ 56, \text{ or } n = 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 211.** *Let* $S = \{1,2,5,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10 \pmod{11} \text{ except } n = 17,\ 18 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 212.** *Let* $S = \{3,5,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \pmod{9} \text{ except } n = 21 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 213.** *Let* $S = \{1,3,5,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 8 \pmod{18} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 214.** *Let* $S = \{2,3,5,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8 \pmod{18} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 215.** *Let* $S = \{1,2,3,5,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 10 \pmod{11} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 216.** *Let* $S = \{4,5,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 12,\ 13,\ 18,\ 19,\ 20 \pmod{28} \text{ except } n = 18, \\ & 19 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 217.** *Let $S = \{1,4,5,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 8,\ 9 \pmod{10} \text{ except } n = 16,\ 18,\ 19,\ 35, \\ & 36,\ 38,\ 55,\ 56,\ 75,\ 76,\ 95 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 218.** *Let $S = \{2,4,5,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9 \pmod{18} \text{ except } n = 21,\ 22, \\ & 24,\ 40 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 219.** *Let $S = \{1,2,4,5,7,8\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ for all $n \geq 16$.*

**Theorem 220.** *Let $S = \{3,4,5,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9,\ 10 \pmod{20} \text{ except } n = 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 221.** *Let $S = \{1,3,4,5,7,8\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{10} \rceil$ for all $n \geq 16$.*

**Theorem 222.** *Let $S = \{2,3,4,5,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9,\ 10 \pmod{20} \text{ except } n = 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 223.** *Let $S = \{1,2,3,4,5,7,8\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{14} \rceil$ for all $n \geq 16$.*

**Theorem 224.** *Let $S = \{6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6,\ 10,\ 11,\ 12,\ 14,\ 15,\ 16,\ 17,\ 18,\ 22, \\ & 23,\ 24 \pmod{30} \text{ except } n = 16,\ 17,\ 22,\ 32,\ 33,\ 34,\ 44, \\ & 45,\ 62 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 225.** *Let $S = \{1,6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 14,\ 15 \pmod{23} \text{ except } n = 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 226.** *Let $S = \{2,6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15 \pmod{23} \\ & \text{except } n = 28,\ 29,\ 32,\ 33,\ 56,\ 57,\ 58,\ 78 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 227.** *Let $S = \{1,2,6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10 \pmod{11} \text{ except } n = 17,\ 18,\ 19, \\ & 20,\ 39,\ 40 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 228.** *Let $S = \{3,6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14 \pmod{21} \text{ except} \\ & n = 30 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 229.** *Let $S = \{1,3,6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8 \pmod{17} \text{ except } n = 23, \text{ or } n = 31 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 230.** *Let $S = \{2,3,6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9 \pmod{19}, \text{ or } n = 16,\ 17,\ 24,\ 34,\ 35,\ 41, \\ & 42,\ 53,\ 60 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 231.** *Let $S = \{1,2,3,6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} 1 & \text{if } n = 21,\ 32 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 232.** *Let $S = \{4,6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \pmod{9} \text{ except } n = 21,\ 22,\ 23 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 233.** *Let $S = \{1,4,6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8 \pmod{9} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 234.** *Let $S = \{2,4,6,7,8\}$. Then for all $n \geq 16$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10 \pmod{20} \text{ except } n = 25,\ 27, \\ & 45 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 235.** *Let* $S = \{1, 2, 4, 6, 7, 8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 10 \pmod{11} \text{ except } n = 17,\ 19,\ 39 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 236.** *Let* $S = \{3, 4, 6, 7, 8\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{9} \rceil$ *for all* $n \geq 16$.

**Theorem 237.** *Let* $S = \{1, 3, 4, 6, 7, 8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9 \pmod{19} \text{ except } n = 26,\ 27,\ 45 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 238.** *Let* $S = \{2, 3, 4, 6, 7, 8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10 \pmod{20} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 239.** *Let* $S = \{1, 2, 3, 4, 6, 7, 8\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{13} \rceil$ *for all* $n \geq 16$.

**Theorem 240.** *Let* $S = \{5, 6, 7, 8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \pmod{9} \text{ except } n = 20,\ 21, \\ & 22,\ 23,\ 24,\ 25,\ 29,\ 30,\ 47,\ 48,\ 49,\ 50,\ 74,\ 75 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 241.** *Let* $S = \{1, 5, 6, 7, 8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n = 26 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 242.** *Let* $S = \{2, 5, 6, 7, 8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8 \pmod{9} \text{ except } n = 22,\ 23,\ 24,\ 25, \\ & 49,\ 50 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 243.** *Let* $S = \{1, 2, 5, 6, 7, 8\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ *for all* $n \geq 16$.

**Theorem 244.** *Let* $S = \{3, 5, 6, 7, 8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8 \pmod{9} \text{ except } n = 17 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 245.** *Let* $S = \{1,3,5,6,7,8\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{10} \rceil$ *for all* $n \geq 16$.

**Theorem 246.** *Let* $S = \{2,3,5,6,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{21} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 9,\ 10 \pmod{21} \text{ except } n = 26,\ 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 247.** *Let* $S = \{1,2,3,5,6,7,8\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{12} \rceil$ *for all* $n \geq 16$.

**Theorem 248.** *Let* $S = \{4,5,6,7,8\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{9} \rceil$ *for all* $n \geq 16$.

**Theorem 249.** *Let* $S = \{1,4,5,6,7,8\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{10} \rceil$ *for all* $n \geq 16$.

**Theorem 250.** *Let* $S = \{2,4,5,6,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10,\ 11 \pmod{22} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 251.** *Let* $S = \{1,2,4,5,6,7,8\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ *for all* $n \geq 16$.

**Theorem 252.** *Let* $S = \{3,4,5,6,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9,\ 10,\ 11 \pmod{23} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 253.** *Let* $S = \{1,3,4,5,6,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9,\ 10,\ 11 \pmod{23} \text{ except } n = 30 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 254.** *Let* $S = \{2,3,4,5,6,7,8\}$. *Then for all* $n \geq 16$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10,\ 11,\ 12 \pmod{24} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 255.** *Let* $S = \{1,2,3,4,5,6,7,8\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{17} \rceil$ *for all* $n \geq 16$.

**Theorem 256.** *Let* $S = \{9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{3} \right\rceil + \begin{cases} 6 & \text{if } n \equiv 9 \pmod{27} \\ 3 & \text{if } n \equiv 18 \pmod{27} \\ 2 & \text{if } n \equiv 3,\ 12,\ 21 \pmod{27} \\ 1 & \text{if } n \equiv 6,\ 15,\ 24 \pmod{27} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 257.** *Let* $S = \{1, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{19} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 3,\ 6,\ 8,\ 9,\ 13,\ 18 \pmod{19} \text{ except } n = 18,\ 22, \\ & 25,\ 27,\ 28,\ 32,\ 37,\ 41,\ 44,\ 47,\ 51,\ 56,\ 60,\ 63,\ 65,\ 70, \\ & 75,\ 79,\ 82,\ 84,\ 98,\ 101,\ 108,\ 120,\ 123,\ 136,\ 146,\ 151, \\ & 174,\ 177,\ 179,\ 196,\ 241,\ 269,\ 291,\ 336,\ 364 \\ 1 & \text{if } n \equiv 1,\ 2,\ 4,\ 10,\ 11,\ 14,\ 15,\ 16,\ 17 \pmod{19} \text{ except} \\ & n = 30,\ 34,\ 39,\ 58,\ 67,\ 73,\ 112,\ 129,\ 168,\ 224, \text{ or} \\ & n = 18,\ 22,\ 25,\ 27,\ 32,\ 37,\ 44,\ 47,\ 51,\ 60,\ 63,\ 65,\ 70, \\ & 75,\ 79,\ 98,\ 108,\ 120,\ 123,\ 136,\ 146,\ 151,\ 174,\ 177, \\ & 179,\ 241,\ 269,\ 291,\ 336,\ 364 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 258.** *Let* $S = \{2, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 3 & \text{if } n \equiv 2 \pmod{5} \text{ except } n = 22,\ 27,\ 32,\ 37,\ 42,\ 47,\ 52, \\ & 57,\ 62,\ 67,\ 72,\ 82,\ 97,\ 107 \\ 2 & \text{if } n \equiv 4 \pmod{5} \text{ except } n = 19,\ 24,\ 29,\ 34,\ 39, \text{ or} \\ & n = 57,\ 62,\ 67,\ 72,\ 82,\ 97,\ 107 \\ 1 & \text{if } n \equiv 1,\ 3 \pmod{5} \text{ except } n = 21,\ 26, \text{ or } n = 19,\ 22, \\ & 24,\ 27,\ 29,\ 32,\ 34,\ 37,\ 39,\ 42,\ 47,\ 52 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 259.** *Let* $S = \{1, 2, 9\}$*. Then* $\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil$ *for all* $n \geq 18$*.*

**Theorem 260.** *Let* $S = \{3, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 3 & \text{if } n \equiv 12 \pmod{15} \\ 2 & \text{if } n \equiv 3 \pmod{15} \\ 1 & \text{if } n \equiv 4,\ 6,\ 9,\ 14 \pmod{15} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 261.** *Let* $S = \{1, 3, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 3 \pmod{7} \text{ except } n = 24,\ 31,\ 38 \\ 1 & \text{if } n \equiv 2,\ 4,\ 5,\ 6 \pmod{7}, \text{ or } n = 24,\ 31,\ 38 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 262.** *Let* $S = \{2, 3, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{5n}{29} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 9,\ 10,\ 11,\ 14,\ 16,\ 17,\ 22,\ 23,\ 26,\ 27,\ 28 \\ & \pmod{29} \text{ except } n = 23,\ 27,\ 32,\ 34,\ 38,\ 46,\ 51,\ 55, \\ & 56,\ 57,\ 61,\ 67,\ 68,\ 69,\ 72,\ 84,\ 85,\ 92,\ 101,\ 113,\ 114, \\ & 115,\ 119,\ 130,\ 138,\ 161,\ 171,\ 184,\ 207,\ 230,\ 299, \\ & 322,\ 345,\ 391 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 263.** *Let $S = \{1,2,3,9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ for all $n \geq 18$.*

**Theorem 264.** *Let $S = \{4,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{14} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 2,\ 4,\ 7,\ 10,\ 13 \pmod{14} \text{ except } n = 18,\ 21,\ 24, \\ & 27,\ 30,\ 32,\ 35,\ 38,\ 41,\ 44,\ 46,\ 49,\ 52,\ 55,\ 58,\ 60,\ 63, \\ & 66,\ 69,\ 72,\ 77,\ 80,\ 83,\ 86,\ 88,\ 91,\ 94,\ 100,\ 105,\ 108, \\ & 111,\ 114,\ 116,\ 122,\ 133,\ 136,\ 139,\ 150,\ 156,\ 158,\ 161, \\ & 178,\ 181,\ 184,\ 203,\ 206,\ 226,\ 228,\ 231,\ 248,\ 251,\ 254, \\ & 273,\ 276,\ 296,\ 298,\ 318,\ 321,\ 343,\ 346,\ 366,\ 368,\ 388, \\ & 391,\ 413,\ 436,\ 458,\ 483,\ 506,\ 528,\ 598 \\ 1 & \text{if } n \equiv 1,\ 5,\ 8,\ 9,\ 12 \pmod{14} \text{ except } n = 19,\ 22,\ 23, \\ & 29,\ 33,\ 43,\ 47,\ 61,\ 68,\ 89,\ 92,\ 113,\ 138,\ 159,\ 183, \\ & 229,\ 253,\ 299, \text{ or } n = 18,\ 27,\ 30,\ 32,\ 35,\ 41,\ 49,\ 52, \\ & 55,\ 58,\ 60,\ 63,\ 72,\ 77,\ 80,\ 83,\ 86,\ 88,\ 94,\ 100,\ 105, \\ & 108,\ 111,\ 116,\ 122,\ 133,\ 139,\ 150,\ 156,\ 158,\ 178, \\ & 181,\ 203,\ 226,\ 228,\ 231,\ 248,\ 251,\ 254,\ 273,\ 296, \\ & 298,\ 318,\ 321,\ 343,\ 346,\ 366,\ 368,\ 388,\ 391,\ 413, \\ & 436,\ 458,\ 483,\ 506,\ 528,\ 598 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 265.** *Let $S = \{1,4,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 3 \pmod{7} \text{ except } n = 24,\ 31,\ 38,\ 45,\ 52,\ 59,\ 66 \\ 1 & \text{if } n \equiv 1,\ 4,\ 5,\ 6 \pmod{7} \text{ except } n = 18,\ 22,\ 29,\ 36, \text{ or} \\ & n = 31,\ 38,\ 45,\ 52,\ 59,\ 66 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 266.** *Let $S = \{2,4,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{5n}{31} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 12 \pmod{31} \text{ except } n = 43,\ 74,\ 105 \\ 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6,\ 11,\ 13,\ 14,\ 17,\ 18,\ 21,\ 22,\ 23,\ 24, \\ & 27,\ 28,\ 29,\ 30 \pmod{31} \text{ except } n = 22,\ 24,\ 27,\ 29,\ 34, \\ & 35,\ 36,\ 44,\ 48,\ 53,\ 58,\ 60,\ 65,\ 75,\ 84,\ 89,\ 96,\ 106, \\ & 120,\ 137,\ 168, \text{ or } n = 43,\ 74,\ 105 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 267.** *Let $S = \{1,2,4,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 13 \pmod{21} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 268.** *Let $S = \{3, 4, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 18 \text{ (mod 19) except } n = 18,\ 37,\ 56 \\ 1 & \text{if } n \equiv 1,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 16 \text{ (mod 19)} \\ & \text{except } n = 20,\ 22,\ 27,\ 46, \text{ or } n = 18,\ 37,\ 56 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 269.** *Let $S = \{1, 3, 4, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 10,\ 12,\ 13 \text{ (mod 14) except } n = 18,\ 19, \\ & 20,\ 24,\ 26,\ 32,\ 38,\ 40,\ 46,\ 52,\ 60,\ 66,\ 80 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 270.** *Let $S = \{2, 3, 4, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 12,\ 13,\ 15,\ 20,\ 21,\ 22,\ 23 \text{ (mod 24)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 271.** *Let $S = \{1, 2, 3, 4, 9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{9} \rceil$ for all $n \geq 18$.*

**Theorem 272.** *Let $S = \{5, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{19} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 9,\ 18 \text{ (mod 19) except } n = 18,\ 47,\ 66,\ 75,\ 104, \\ & 113,\ 161, \text{ or } n = 33 \\ 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 8,\ 10,\ 11,\ 12,\ 14,\ 15,\ 16,\ 17 \text{ (mod} \\ & \text{19) except } n = 21,\ 22,\ 23,\ 24,\ 29,\ 33,\ 34,\ 35,\ 43,\ 46, \\ & 67,\ 68,\ 69,\ 81,\ 91,\ 92,\ 138, \text{ or } n = 18,\ 47,\ 66,\ 75, \\ & 104,\ 113,\ 161 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 273.** *Let $S = \{1, 5, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 6,\ 10,\ 12 \text{ (mod 13) except } n = 19,\ 23,\ 25,\ 32, \\ & 36,\ 38,\ 49,\ 51,\ 58,\ 62,\ 64,\ 75,\ 77,\ 84,\ 88,\ 101,\ 103, \\ & 110,\ 114,\ 127,\ 129,\ 136,\ 149,\ 153,\ 155,\ 168,\ 179, \\ & 205,\ 231,\ 244,\ 266,\ 285,\ 361 \\ 1 & \text{if } n \equiv 1,\ 3,\ 4,\ 5,\ 8 \text{ (mod 13) except } n = 21,\ 27,\ 34,\ 40, \\ & 53,\ 57,\ 60,\ 73,\ 79,\ 92,\ 95,\ 133,\ 190,\ 209, \text{ or } n = 23, \\ & 25,\ 32,\ 49,\ 51,\ 58,\ 62,\ 64,\ 75,\ 77,\ 84,\ 88,\ 101,\ 103, \\ & 110,\ 127,\ 129,\ 136,\ 149,\ 153,\ 155,\ 168,\ 179,\ 205, \\ & 231,\ 244,\ 266,\ 285,\ 361 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 274.** *Let $S = \{2,5,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{6n}{37} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 9,\ 10,\ 11,\ 12,\ 18,\ 23,\ 24,\ 28,\ 29,\ 30, \\ & 35,\ 36 \pmod{37} \text{ except } n = 18,\ 23,\ 24,\ 28,\ 29,\ 30,\ 36, \\ & 40,\ 41,\ 42,\ 46,\ 48,\ 60,\ 65,\ 66,\ 72,\ 77,\ 78,\ 84,\ 102, \\ & 114,\ 120 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 275.** *Let $S = \{1,2,5,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 1,\ 3,\ 4,\ 5,\ 6,\ 7,\ 10,\ 12,\ 14 \pmod{15} \text{ except} \\ & n = 18,\ 19,\ 21,\ 25,\ 27,\ 31,\ 33,\ 34,\ 40,\ 42,\ 46,\ 48,\ 55, \\ & 61,\ 63,\ 76 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 276.** *Let $S = \{3,5,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{23} \right\rceil + \begin{cases} 1 & \text{if } n = 28,\ 45 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 277.** *Let $S = \{1,3,5,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 15 \pmod{16} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 278.** *Let $S = \{2,3,5,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4,\ 5,\ 6,\ 7,\ 8,\ 13,\ 14,\ 15 \pmod{16} \text{ except } n = \\ & 18 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 279.** *Let $S = \{1,2,3,5,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8 \pmod{9} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 280.** *Let $S = \{4,5,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 12 \pmod{19} \text{ except } n = 24,\ 25,\ 50,\ 62, \\ & 100, \text{ or } n = 28 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 281.** *Let $S = \{1, 4, 5, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 8,\ 16,\ 20,\ 23,\ 24,\ 25,\ 31,\ 33 \pmod{34} \\ & \text{except } n = 23 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 282.** *Let $S = \{2, 4, 5, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 13,\ 14,\ 15 \pmod{16} \text{ except } n = 22,\ 30, \\ & 45 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 283.** *Let $S = \{1, 2, 4, 5, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 9,\ 17,\ 18,\ 25,\ 26 \pmod{27} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 284.** *Let $S = \{3, 4, 5, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 13,\ 14,\ 20,\ 28,\ 34,\ 35,\ 41 \pmod{42} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 285.** *Let $S = \{1, 3, 4, 5, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 8,\ 9,\ 10 \pmod{11} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 286.** *Let $S = \{2, 3, 4, 5, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9 \pmod{10} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 287.** *Let $S = \{1, 2, 3, 4, 5, 9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ for all $n \geq 18$.*

**Theorem 288.** *Let $S = \{6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{4} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 3,\ 15 \pmod{24} \\ 1 & \text{if } n \equiv 6,\ 18 \pmod{24} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 289.** *Let $S = \{1, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 7,\ 9,\ 10,\ 11,\ 13,\ 14,\ 15,\ 16,\ 17,\ 22 \text{ (mod} \\ & \text{23) except } n = 22,\ 27,\ 30,\ 32,\ 33,\ 34,\ 36,\ 37,\ 38,\ 51, \\ & 53,\ 55,\ 59,\ 60,\ 68,\ 72,\ 76,\ 82,\ 85,\ 99,\ 102,\ 105,\ 106, \\ & 119,\ 122,\ 153,\ 170,\ 174,\ 187,\ 191,\ 221,\ 289,\ 306 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 290.** *Let $S = \{2,6,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 9,\ 11,\ 12 \pmod{13} \text{ except} \\ & n = 19,\ 21,\ 22,\ 24,\ 29,\ 34,\ 38,\ 41,\ 43,\ 48,\ 55,\ 57,\ 60, \\ & 67,\ 74,\ 76,\ 81,\ 86,\ 93,\ 95,\ 100,\ 112,\ 119,\ 133,\ 138, \\ & 152,\ 171,\ 190 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 291.** *Let $S = \{1,2,6,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 13 \pmod{14} \text{ except } n = 18 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 292.** *Let $S = \{3,6,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 3,\ 6 \pmod{21} \\ 1 & \text{if } n \equiv 9,\ 12 \pmod{21} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 293.** *Let $S = \{1,3,6,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 8,\ 9,\ 12,\ 14,\ 15,\ 17,\ 19,\ 20,\ 22,\ 23,\ 28, \\ & 30,\ 31,\ 33,\ 36,\ 38,\ 39,\ 40,\ 41,\ 44,\ 46,\ 47 \pmod{48} \\ & \text{except } n = 19,\ 20,\ 28,\ 38,\ 41,\ 52,\ 62,\ 65,\ 76,\ 89,\ 100, \\ & 113,\ 124,\ 137 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 294.** *Let $S = \{2,3,6,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 16 \pmod{24} \text{ except } n = 40,\ 64,\ 88,\ 112,\ 136, \\ & 160,\ 184 \\ 1 & \text{if } n \equiv 2,\ 5,\ 6,\ 8,\ 9,\ 12,\ 13,\ 15,\ 19,\ 20,\ 22,\ 23 \pmod{24} \text{ except } n = 19,\ 20,\ 26,\ 37,\ 50,\ 61,\ 74,\ 85,\ 98, \\ & 122,\ 146,\ 170, \text{ or } n = 40,\ 64,\ 88,\ 112,\ 136,\ 160,\ 184 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 295.** *Let $S = \{1,2,3,6,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 15,\ 16,\ 17,\ 18,\ 22,\ 23,\ 25,\ 26 \pmod{27} \\ & \text{except } n = 35,\ 49,\ 70 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 296.** *Let $S = \{4, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 2 & \textit{if } n \equiv 3,\ 6 \pmod{7} \textit{ except } n = 20,\ 24,\ 27,\ 31,\ 45,\ 48 \\ 1 & \textit{if } n \equiv 1,\ 2,\ 4,\ 5 \pmod{7} \textit{ except } n = 22,\ 23, \textit{ or } n = 20, \\ & 27,\ 31,\ 45,\ 48 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 297.** *Let $S = \{1, 4, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 4,\ 7,\ 13,\ 15,\ 21,\ 22 \pmod{23} \textit{ except } n = 21,\ 27, \\ & \textit{or } n = 33 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 298.** *Let $S = \{2, 4, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8 \pmod{17} \textit{ except } n = 21 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 299.** *Let $S = \{1, 2, 4, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 6,\ 8 \pmod{9} \textit{ except } n = 33,\ 51 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 300.** *Let $S = \{3, 4, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 1,\ 4,\ 6,\ 7 \pmod{8} \textit{ except } n = 20,\ 25,\ 28,\ 41,\ 49, \\ & 65 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 301.** *Let $S = \{1, 3, 4, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 4,\ 7,\ 9,\ 10 \pmod{11} \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 302.** *Let $S = \{2, 3, 4, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 8 \pmod{17}, \textit{ or } n = 22,\ 23,\ 39 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 303.** *Let $S = \{1, 2, 3, 4, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 10 \pmod{11} \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 304.** *Let $S = \{5, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{5n}{29} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 9,\ 10,\ 11,\ 14,\ 15,\ 16,\ 17,\ 19, \\ & 22,\ 23,\ 24,\ 27,\ 28 \ (\textit{mod } 29) \textit{ except } n = 19,\ 24,\ 27,\ 31, \\ & 32,\ 35,\ 36,\ 48,\ 53,\ 64,\ 65,\ 77 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 305.** *Let $S = \{1, 5, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 2,\ 4,\ 5,\ 6,\ 7,\ 8,\ 14,\ 15 \ (\textit{mod } 16) \textit{ except } n = 18, \\ & 21 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 306.** *Let $S = \{2, 5, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{22} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 7,\ 11,\ 14,\ 20,\ 21 \ (\textit{mod } 22) \textit{ except } n = 20,\ 21, \\ & 42,\ 55 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 307.** *Let $S = \{1, 2, 5, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 18 \ (\textit{mod } 19) \textit{ except } n = 18,\ 23, \\ & 25,\ 27,\ 43,\ 45,\ 61,\ 63,\ 81,\ 99 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 308.** *Let $S = \{3, 5, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 2,\ 5,\ 6,\ 8,\ 12,\ 15,\ 16 \ (\textit{mod } 17) \textit{ except } n = 19 \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 309.** *Let $S = \{1, 3, 5, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 2,\ 5,\ 6,\ 8,\ 9,\ 10,\ 15,\ 18,\ 19 \ (\textit{mod } 20) \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 310.** *Let $S = \{2, 3, 5, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 5,\ 8,\ 9 \ (\textit{mod } 10) \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 311.** *Let $S = \{1, 2, 3, 5, 6, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 8,\ 11,\ 12 \ (\textit{mod } 13) \\ 0 & \textit{otherwise.} \end{cases}$$

**Theorem 312.** *Let* $S = \{4,5,6,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 1,\ 2,\ 4,\ 5,\ 6,\ 7 \pmod 8 \text{ except } n = 18,\ 21,\ 25, \\ & 26,\ 28,\ 33,\ 42,\ 49 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 313.** *Let* $S = \{1,4,5,6,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 8,\ 9,\ 18 \pmod{19} \text{ except } n = 24 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 314.** *Let* $S = \{2,4,5,6,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 18,\ 21,\ 23,\ 24,\ 25,\ 26,\ 27,\ 35 \\ & \pmod{36} \text{ except } n = 21,\ 24,\ 40 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 315.** *Let* $S = \{1,2,4,5,6,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 10,\ 11 \pmod{12} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 316.** *Let* $S = \{3,4,5,6,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 8,\ 15,\ 16 \pmod{17} \text{ except } n = 32,\ 40,\ 56 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 317.** *Let* $S = \{1,3,4,5,6,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 9,\ 10 \pmod{11} \text{ except } n = 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 318.** *Let* $S = \{2,3,4,5,6,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9 \pmod{10} \text{ except } n = 18,\ 38 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 319.** *Let* $S = \{1,2,3,4,5,6,9\}$*. Then* $\gamma(C_n[S]) = \lceil \frac{n}{13} \rceil$ *for all* $n \geq 18$*.*

**Theorem 320.** *Let $S = \{7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} 3 & \text{if } n \equiv 2,\ 4 \pmod 5 \text{ except } n = 19,\ 22,\ 24,\ 27,\ 29,\ 32, \\ & 34,\ 37,\ 39,\ 42,\ 44,\ 47,\ 49,\ 52,\ 54,\ 57,\ 59,\ 62,\ 67,\ 69, \\ & 72,\ 74,\ 77,\ 79,\ 82,\ 92,\ 94,\ 97,\ 104,\ 109,\ 117,\ 127, \\ & 132 \\ 2 & \text{if } n \equiv 1 \pmod 5 \text{ except } n = 21,\ 26,\ 31,\ 36,\ 46,\ 51,\ 56, \\ & 71,\ 81,\ 86, \text{ or } n = 39,\ 44,\ 49,\ 54,\ 59,\ 62,\ 67,\ 72,\ 74, \\ & 77,\ 79,\ 82,\ 94,\ 97,\ 104,\ 109,\ 117,\ 127,\ 132 \\ 1 & \text{if } n \equiv 3 \pmod 5 \text{ except } n = 23, \text{ or } n = 19,\ 21,\ 22,\ 24, \\ & 26,\ 27,\ 29,\ 31,\ 32,\ 34,\ 36,\ 37,\ 42,\ 47,\ 51,\ 52,\ 56,\ 57, \\ & 69,\ 71,\ 81,\ 86,\ 92 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 321.** *Let $S = \{1, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 10,\ 12,\ 19,\ 25 \pmod{26} \text{ except } n = 19,\ 25,\ 36, \\ & 38,\ 45,\ 51,\ 62,\ 71,\ 77 \\ 1 & \text{if } n \equiv 1,\ 3,\ 4,\ 5,\ 6,\ 8,\ 13,\ 14,\ 16,\ 17,\ 18,\ 21,\ 23 \pmod{26} \text{ except } n = 21,\ 23,\ 27,\ 29,\ 34,\ 47,\ 53,\ 60, \text{ or} \\ & n = 19,\ 25,\ 38,\ 45,\ 51,\ 62,\ 71,\ 77 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 322.** *Let $S = \{2, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{20} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 19 \pmod{20} \\ & \text{except } n = 22,\ 24,\ 25,\ 26,\ 27,\ 28,\ 29,\ 30,\ 39,\ 42, \\ & 47,\ 48,\ 49,\ 50,\ 51,\ 52,\ 62,\ 64,\ 65,\ 87,\ 88,\ 89,\ 90, \\ & 91,\ 102,\ 104,\ 127,\ 128,\ 129,\ 130,\ 142,\ 167,\ 168,\ 169, \\ & 182,\ 207,\ 208,\ 247 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 323.** *Let $S = \{1, 2, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 1,\ 3,\ 5,\ 6,\ 7,\ 8,\ 12,\ 14,\ 16 \pmod{17} \text{ except} \\ & n = 18,\ 20,\ 35,\ 37,\ 52,\ 69 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 324.** *Let $S = \{3, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 12 \pmod{13} \text{ except } n = 25 \\ 1 & \text{if } n \equiv 1,\ 3,\ 4,\ 5,\ 6,\ 10,\ 11 \pmod{13} \text{ except } n = 27, \text{ or} \\ & n = 25 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 325.** *Let $S = \{1,3,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{29} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 20,\ 21,\ 29,\ 35,\ 36,\ 43,\ 49,\ 50,\ 57 \text{ (mod} \\ & \text{58) except } n = 21,\ 35,\ 49 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 326.** *Let $S = \{2,3,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n = 35 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 327.** *Let $S = \{1,2,3,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 16 \text{ (mod 17)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 328.** *Let $S = \{4,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 1,\ 4,\ 5 \text{ (mod 6) except } n = 19,\ 22,\ 25,\ 43,\ 49, \\ & 67 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 329.** *Let $S = \{1,4,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 11,\ 13 \text{ (mod 21)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 330.** *Let $S = \{2,4,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{6n}{47} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 14,\ 19,\ 21,\ 22,\ 23,\ 29,\ 31,\ 36,\ 37, \\ & 38,\ 39,\ 44,\ 46 \text{ (mod 47) except } n = 19,\ 21,\ 29,\ 44 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 331.** *Let $S = \{1,2,4,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 16,\ 18 \text{ (mod 19) except } n = 25,\ 27, \\ & 54 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 332.** *Let $S = \{3,4,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 7,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14 \text{ (mod 21)} \\ & \text{except } n = 24,\ 25,\ 26,\ 27,\ 30,\ 31,\ 45,\ 46,\ 51,\ 52,\ 53, \\ & 54,\ 66,\ 67,\ 72,\ 73,\ 93,\ 94,\ 108,\ 135 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 333.** *Let* $S = \{1, 3, 4, 7, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 7,\ 8 \pmod 9 \text{ except } n = 31,\ 34,\ 49,\ 85 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 334.** *Let* $S = \{2, 3, 4, 7, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{37} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 9,\ 18,\ 26,\ 27,\ 36 \pmod{37} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 335.** *Let* $S = \{1, 2, 3, 4, 7, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{29} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 19,\ 28 \pmod{29} \text{ except } n = 19,\ 57 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 336.** *Let* $S = \{5, 7, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18, \\ & 19,\ 25 \pmod{26} \text{ except } n = 19,\ 30,\ 33,\ 38,\ 40,\ 41,\ 57, \\ & 59,\ 68,\ 87,\ 95,\ 114,\ 144,\ 163,\ 171, \text{ or } n = 28 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 337.** *Let* $S = \{1, 5, 7, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 11,\ 12,\ 13,\ 14,\ 15,\ 22 \pmod{23} \\ & \text{except } n = 27,\ 28,\ 30,\ 34,\ 45,\ 57,\ 58,\ 60,\ 73,\ 75, \\ & 103,\ 105,\ 120,\ 150,\ 165,\ 195 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 338.** *Let* $S = \{2, 5, 7, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{25} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 7 \pmod{25} \text{ except } n = 32,\ 57,\ 82 \\ 1 & \text{if } n \equiv 1,\ 3,\ 5,\ 8,\ 9,\ 11,\ 13,\ 14,\ 15,\ 16,\ 20,\ 22,\ 24 \pmod{} \\ & 25) \text{ except } n = 20,\ 26,\ 34,\ 36,\ 45,\ 59, \text{ or } n = 32,\ 57, \\ & 82 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 339.** *Let* $S = \{1, 2, 5, 7, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n = 22,\ 24,\ 33,\ 39,\ 41,\ 58 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 340.** *Let $S = \{3,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 14 \text{ (mod 30) except } n = 44,\ 74,\ 104 \\ 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 7,\ 12,\ 13,\ 15,\ 16,\ 18,\ 20,\ 21,\ 22,\ 23, \\ & 25,\ 27,\ 29 \text{ (mod 30) except } n = 18,\ 20,\ 21,\ 23,\ 25,\ 27, \\ & 42,\ 46,\ 48,\ 53,\ 55,\ 63,\ 76,\ 83, \text{ or } n = 44,\ 74,\ 104 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 341.** *Let $S = \{1,3,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 8,\ 10 \text{ (mod 11) except } n = 19,\ 37,\ 39,\ 59 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 342.** *Let $S = \{2,3,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10 \text{ (mod 20) except } n = 23, \\ & 25,\ 44,\ 63 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 343.** *Let $S = \{1,2,3,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 10,\ 12 \text{ (mod 13)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 344.** *Let $S = \{4,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 7,\ 13,\ 14,\ 15,\ 22 \text{ (mod 23) except } n = 28,\ 30, \\ & 45,\ 60,\ 105,\ 120 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 345.** *Let $S = \{1,4,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 16 \text{ (mod 17) except } n = 23 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 346.** *Let $S = \{2,4,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 9 \text{ (mod 20) except } n = 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 347.** *Let $S = \{1,2,4,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 9 \text{ (mod 20)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 348.** *Let $S = \{3,4,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 5,\ 7,\ 8,\ 14,\ 16 \pmod{17} \text{ except } n = 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 349.** *Let $S = \{1,3,4,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 10,\ 19,\ 21 \pmod{22} \text{ except } n = 19 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 350.** *Let $S = \{2,3,4,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 9,\ 10,\ 11,\ 20,\ 22 \pmod{23} \text{ except } n = 20,\ 30 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 351.** *Let $S = \{1,2,3,4,5,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 12,\ 14 \pmod{15} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 352.** *Let $S = \{6,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{25} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 11 \pmod{25} \text{ except } n = 36,\ 61 \\ 1 & \text{if } n \equiv 3,\ 5,\ 6,\ 7,\ 9,\ 10,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18,\ 24 \\ & \pmod{25} \text{ except } n = 32,\ 34,\ 39, \text{ or } n = 36,\ 61 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 353.** *Let $S = \{1,6,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{6n}{43} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 14,\ 35 \pmod{43}, \text{ or } n = 27,\ 34,\ 48 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 354.** *Let $S = \{2,6,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{22} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 11,\ 12,\ 13,\ 14,\ 20,\ 21 \pmod{22} \text{ except} \\ & n = 20,\ 21,\ 26,\ 33,\ 34,\ 42,\ 55 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 355.** *Let $S = \{1,2,6,7,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 16 \pmod{17} \text{ except } n = 23 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 356.** *Let $S = \{3, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 8,\ 12,\ 14,\ 15,\ 16,\ 23,\ 31,\ 36,\ 38,\ 39, \\ & 40,\ 47,\ 52,\ 54,\ 55,\ 56,\ 63,\ 71,\ 76,\ 78,\ 79,\ 80,\ 84,\ 86, \\ & 87,\ 88,\ 95 \pmod{96}, \text{ or } n = 33,\ 57 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 357.** *Let $S = \{1, 3, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 8,\ 9,\ 10 \pmod{11} \text{ except } n = 19 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 358.** *Let $S = \{2, 3, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 15,\ 23,\ 39,\ 40,\ 47 \pmod{48}, \text{ or } n = 36,\ 54 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 359.** *Let $S = \{1, 2, 3, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10 \pmod{11} \text{ except } n = 19 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 360.** *Let $S = \{4, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{29} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 7,\ 13,\ 14 \pmod{29} \text{ except } n = 36,\ 42,\ 43,\ 71, \\ & 72,\ 94,\ 100,\ 101,\ 129,\ 152,\ 159,\ 187,\ 210,\ 217,\ 245 \\ 1 & \text{if } n \equiv 3,\ 5,\ 6,\ 9,\ 10,\ 11,\ 12,\ 15,\ 16,\ 17,\ 18,\ 19,\ 20, \\ & 21,\ 28 \pmod{29} \text{ except } n = 18,\ 19,\ 20,\ 21,\ 28,\ 32,\ 35, \\ & 38,\ 40,\ 44,\ 45,\ 46,\ 47,\ 49,\ 61,\ 63,\ 68,\ 70,\ 73,\ 75,\ 77, \\ & 96,\ 98,\ 103,\ 105,\ 119,\ 126,\ 131,\ 133,\ 154,\ 161,\ 189, \\ & \text{or } n = 36,\ 43,\ 71,\ 72,\ 94,\ 100,\ 101,\ 129,\ 152,\ 159, \\ & 187,\ 210,\ 217,\ 245 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 361.** *Let $S = \{1, 4, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 8 \pmod{9} \text{ except } n = 22,\ 25,\ 49,\ 51,\ 76 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 362.** *Let $S = \{2, 4, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10 \pmod{20} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 363.** *Let $S = \{1, 2, 4, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 10,\ 11 \pmod{12} \text{ except } n = 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 364.** *Let $S = \{3, 4, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 9,\ 18 \pmod{19} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 365.** *Let $S = \{1, 3, 4, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 10 \pmod{11} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 366.** *Let $S = \{2, 3, 4, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 9,\ 10,\ 11,\ 21 \pmod{22}, \text{ or } n = 26 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 367.** *Let $S = \{1, 2, 3, 4, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{14} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 13 \pmod{14} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 368.** *Let $S = \{5, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 12,\ 13,\ 14,\ 20,\ 32,\ 34,\ 35,\ 41,\ 48,\ 49,\ 62, \\ & \quad 68,\ 69,\ 70,\ 76,\ 77 \pmod{84} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 369.** *Let $S = \{1, 5, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 19 \pmod{20} \text{ except } n = 27, \\ & \quad 45 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 370.** *Let $S = \{2, 5, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9 \pmod{10} \text{ except } n = 25 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 371.** *Let* $S = \{1, 2, 5, 6, 7, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n = 26 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 372.** *Let* $S = \{3, 5, 6, 7, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 9 \pmod{27} \\ 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 8,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18,\ 26 \\ & \pmod{27} \text{ except } n = 31,\ 34,\ 38,\ 41,\ 58,\ 65,\ 68,\ 85, \\ & 92,\ 119 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 373.** *Let* $S = \{1, 3, 5, 6, 7, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 8,\ 10,\ 21 \pmod{22} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 374.** *Let* $S = \{2, 3, 5, 6, 7, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9 \pmod{20} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 375.** *Let* $S = \{1, 2, 3, 5, 6, 7, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 12 \pmod{13} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 376.** *Let* $S = \{4, 5, 6, 7, 9\}$. *Then* $\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil$ *for all* $n \geq 18$.

**Theorem 377.** *Let* $S = \{1, 4, 5, 6, 7, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10 \pmod{20} \text{ except } n = 26,\ 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 378.** *Let* $S = \{2, 4, 5, 6, 7, 9\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{10} \rceil$ *for all* $n \geq 18$.

**Theorem 379.** *Let* $S = \{1, 2, 4, 5, 6, 7, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 11 \pmod{12} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 380.** *Let $S = \{3, 4, 5, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{6, 7, 8, 9, 10, 11, 21 (mod 22)} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 381.** *Let $S = \{1, 3, 4, 5, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{6, 8, 9, 10, 11, 12, 23 (mod 24) except } n = \textit{30,} \\ & \textit{33} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 382.** *Let $S = \{2, 3, 4, 5, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{25} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{7, 9, 10, 11, 12, 24 (mod 25)} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 383.** *Let $S = \{1, 2, 3, 4, 5, 6, 7, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{17} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{16 (mod 17)} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 384.** *Let $S = \{8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{5} \right\rceil + \begin{cases} \textit{3} & \textit{if } n \equiv \textit{2, 3, 4 (mod 5) except } n = \textit{18, 19, 22, 23, 24,} \\ & \textit{27, 28, 29, 32, 33, 34, 37, 38, 39, 42, 43, 44, 47, 48,} \\ & \textit{49, 52, 53, 54, 57, 58, 59, 62, 63, 64, 67, 72, 73, 74,} \\ & \textit{77, 78, 79, 82, 84, 87, 88, 92, 93, 97, 98, 99, 107,} \\ & \textit{112, 113, 117, 118, 132, 139, 158, 172, 198} \\ \textit{2} & \textit{if } n = \textit{34, 42, 43, 44, 48, 49, 53, 54, 57, 58, 62, 63,} \\ & \textit{64, 67, 72, 74, 77, 78, 79, 82, 84, 87, 88, 93, 97, 98,} \\ & \textit{107, 112, 113, 117, 118, 139, 158, 172, 198} \\ \textit{1} & \textit{if } n \equiv \textit{1 (mod 5) except } n = \textit{26, 66, or } n = \textit{18, 19, 22,} \\ & \textit{23, 24, 27, 28, 29, 32, 37, 38, 39, 47, 52, 59, 73, 92,} \\ & \textit{99, 132} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 385.** *Let $S = \{1, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{13} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{4, 5, 6, 8, 9, 12, 17, 18, 19, 23, 24, 25 (mod} \\ & \textit{26) except } n = \textit{18, 19, 23, 24, 25, 30, 31, 34, 35, 38,} \\ & \textit{43, 44, 49, 50, 56, 57, 60, 61, 69, 75, 76, 82, 86, 87,} \\ & \textit{95, 101, 112, 113, 138, 139, 164, 190} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 386.** *Let $S = \{2, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{2, 4, 5, 6, 8, 9, 10, 11 (mod 12) except } n = \textit{20,} \\ & \textit{21, 28, 32, 56} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 387.** *Let $S = \{1, 2, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{31} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{5, 6, 7, 20, 21, 22, 23 (mod 31), or } n = \textit{29, 44} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 388.** *Let $S = \{3, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} \textit{2} & \textit{if } n \equiv \textit{3, 4, 5 (mod 7) except } n = \textit{18, 19, 24, 25, 26,} \\ & \textit{33, 38, 39, 45, 46, 47, 52, 73, 74, 75} \\ \textit{1} & \textit{if } n \equiv \textit{2, 6 (mod 7) except } n = \textit{23, or } n = \textit{18, 19, 26,} \\ & \textit{33, 38, 39, 45, 46, 47, 52, 73, 74, 75} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 389.** *Let $S = \{1, 3, 8, 9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ for all $n \geq 18$.*

**Theorem 390.** *Let $S = \{2, 3, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{5, 6, 7, 10, 11, 12, 13, 14, 15 (mod 23) except} \\ & n = \textit{28, 35, 36, 56, 57, 58, 79, 80, 102} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 391.** *Let $S = \{1, 2, 3, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} \textit{1} & \textit{if } n \equiv \textit{5, 6, 7, 8, 9, 16, 17, 18 (mod 19) except } n = \textit{28,} \\ & \textit{35, 43, 56, 62, 63, 84, 112, 119, 140, 168, 196, 252} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 392.** *Let $S = \{4, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} \textit{2} & \textit{if } n \equiv \textit{3, 4, 5 (mod 7) except } n = \textit{18, 19, 24, 25, 26,} \\ & \textit{31, 32, 33, 38, 45, 46, 47, 52, 53, 59, 66, 73, 74, 75} \\ \textit{1} & \textit{if } n \equiv \textit{1, 2, 6 (mod 7) except } n = \textit{22, 23, 50, or } n = \textit{18,} \\ & \textit{19, 26, 31, 32, 33, 38, 45, 46, 47, 52, 53, 59, 66, 73,} \\ & \textit{74, 75} \\ \textit{0} & \textit{otherwise.} \end{cases}$$

**Theorem 393.** *Let $S = \{1,4,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{25} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 13,\ 15,\ 16,\ 19,\ 20,\ 21,\ 22,\ 23, \\ & 24 \pmod{25} \text{ except } n = 19,\ 20,\ 21,\ 22,\ 44 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 394.** *Let $S = \{2,4,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7 \pmod{15} \text{ except } n = 22, \text{ or } n = 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 395.** *Let $S = \{1,2,4,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 8,\ 16,\ 17,\ 23,\ 24,\ 25,\ 31,\ 32,\ 33 \pmod{34} \\ & \text{except } n = 25,\ 31,\ 32,\ 50,\ 57,\ 65,\ 72,\ 100,\ 106,\ 125, \\ & 140 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 396.** *Let $S = \{3,4,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 7 \pmod{23} \text{ except } n = 30,\ 53 \\ 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15 \pmod{23} \\ & \text{except } n = 26,\ 28,\ 31,\ 32,\ 33,\ 34,\ 35,\ 49,\ 54,\ 56,\ 77, \\ & \text{or } n = 30,\ 53 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 397.** *Let $S = \{1,3,4,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9,\ 17,\ 18,\ 23,\ 24,\ 25,\ 26,\ 27 \pmod{36} \\ & \text{except } n = 23,\ 24,\ 25, \text{ or } n = 33 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 398.** *Let $S = \{2,3,4,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 9 \pmod{19} \text{ except } n = 24,\ 27,\ 45,\ 63, \\ & 81 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 399.** *Let $S = \{1,2,3,4,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 10,\ 19,\ 20 \pmod{30} \text{ except } n = 19 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 400.** *Let $S = \{5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{6} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 5 \pmod 6 \text{ except } n = 23,\ 29,\ 35,\ 41,\ 53,\ 59,\ 65, \\ & 77,\ 83,\ 95 \\ 1 & \text{if } n \equiv 2,\ 4 \pmod 6 \text{ except } n = 20,\ 22,\ 26,\ 44,\ 56, \text{ or} \\ & n = 23,\ 29,\ 35,\ 41,\ 53,\ 59,\ 65,\ 77,\ 83,\ 95 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 401.** *Let $S = \{1, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 20,\ 21,\ 22,\ 36,\ 37,\ 38 \pmod{46}, \text{ or} \\ & n = 28,\ 29,\ 32,\ 43,\ 44,\ 59,\ 74,\ 89 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 402.** *Let $S = \{2, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 7,\ 10,\ 11,\ 13,\ 14 \pmod{21} \text{ except } n = 31, \\ & 32,\ 52 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 403.** *Let $S = \{1, 2, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 10,\ 17,\ 18,\ 19 \pmod{20} \\ & \text{except } n = 22,\ 23,\ 26,\ 27,\ 42 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 404.** *Let $S = \{3, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 13 \pmod{21}, \text{ or } n = 31 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 405.** *Let $S = \{1, 3, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 17,\ 24,\ 25,\ 33 \pmod{34} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 406.** *Let $S = \{2, 3, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 9,\ 17,\ 18 \pmod{19} \text{ except } n = 18,\ 24, \\ & 27,\ 36,\ 45,\ 63,\ 81 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 407.** *Let $S = \{1, 2, 3, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10 \pmod{20} \text{ except } n = 26,\ 27,\ 28, \\ & 46 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 408.** *Let $S = \{4, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n = 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 409.** *Let $S = \{1, 4, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 8,\ 9,\ 10 \pmod{11} \text{ except } n = 19,\ 20, \\ & 21,\ 27,\ 37,\ 38,\ 39,\ 41,\ 42,\ 59,\ 60,\ 61,\ 63,\ 81,\ 82,\ 83, \\ & 103,\ 104,\ 105,\ 125,\ 126,\ 147 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 410.** *Let $S = \{2, 4, 5, 8, 9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{8} \rceil$ for all $n \geq 18$.*

**Theorem 411.** *Let $S = \{1, 2, 4, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 9,\ 10 \pmod{11} \text{ except } n = 20,\ 21,\ 39,\ 42,\ 61, \\ & 83,\ 105 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 412.** *Let $S = \{3, 4, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 7,\ 8,\ 9 \pmod{10} \text{ except } n = 23,\ 24 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 413.** *Let $S = \{1, 3, 4, 5, 8, 9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ for all $n \geq 18$.*

**Theorem 414.** *Let $S = \{2, 3, 4, 5, 8, 9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{10} \rceil$ for all $n \geq 18$.*

**Theorem 415.** *Let $S = \{1, 2, 3, 4, 5, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 12,\ 13,\ 14 \pmod{15} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 416.** *Let $S = \{6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 14,\ 15,\ 16,\ 17 \pmod{23} \text{ except } n = 27,\ 37 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 417.** *Let $S = \{1, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 7 \pmod{24} \text{ except } n = 31 \\ 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 14,\ 15,\ 16,\ 17,\ 18,\ 19,\ 20, \\ & 21,\ 22,\ 23 \pmod{24} \text{ except } n = 18,\ 19,\ 20, \text{ or } n = 31 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 418.** *Let $S = \{2, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 11,\ 13,\ 14,\ 15,\ 20,\ 22 \pmod{23} \text{ except} \\ & n = 57 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 419.** *Let $S = \{1, 2, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 15,\ 17 \pmod{18} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 420.** *Let $S = \{3, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 12,\ 13,\ 14,\ 15,\ 16 \pmod{24} \text{ except} \\ & n = 31,\ 55,\ 62,\ 86,\ 110, \text{ or } n = 57 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 421.** *Let $S = \{1, 3, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 15,\ 16,\ 17,\ 25,\ 32,\ 33,\ 34,\ 39,\ 40, \\ & 41,\ 42 \pmod{51} \text{ except } n = 32 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 422.** *Let $S = \{2, 3, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 15,\ 16,\ 17,\ 18,\ 19 \pmod{20} \\ & \text{except } n = 24 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 423.** *Let $S = \{1, 2, 3, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10,\ 11,\ 12 \pmod{13} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 424.** *Let $S = \{4, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{29} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 12,\ 13,\ 14,\ 19,\ 20,\ 21,\ 26,\ 28 \pmod{29} \text{ except } n = 19,\ 21,\ 26,\ 28,\ 35,\ 42,\ 49,\ 63,\ 70,\ 77,\ 84,\ 91 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 425.** *Let $S = \{1, 4, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10 \pmod{11} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 426.** *Let $S = \{2, 4, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10 \pmod{20} \text{ except } n = 23,\ 25,\ 27,\ 43,\ 45,\ 63 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 427.** *Let $S = \{1, 2, 4, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 10 \pmod{11} \text{ except } n = 19,\ 39 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 428.** *Let $S = \{3, 4, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 15,\ 39 \pmod{48}, \text{ or } n = 30,\ 36,\ 38,\ 54 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 429.** *Let $S = \{1, 3, 4, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 7,\ 9,\ 10 \pmod{11} \text{ except } n = 18,\ 20,\ 37,\ 40,\ 59 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 430.** *Let $S = \{2, 3, 4, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 8,\ 9,\ 10 \pmod{20} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 431.** *Let $S = \{1, 2, 3, 4, 6, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 10,\ 12 \pmod{13} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 432.** *Let $S = \{5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{4n}{31} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 14,\ 15 \pmod{31} \\ 1 & \text{if } n \equiv 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 10,\ 11,\ 13,\ 17,\ 18,\ 20,\ 21, \\ & 22,\ 23,\ 26,\ 27,\ 29,\ 30 \pmod{31} \text{ except } n = 18,\ 26,\ 27, \\ & 32,\ 33 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 433.** *Let $S = \{1,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 38,\ 39,\ 40 \pmod{48} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 434.** *Let $S = \{2,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 2,\ 4,\ 5,\ 6,\ 7,\ 8 \pmod{9} \text{ except } n = 20,\ 22,\ 23, \\ & 25,\ 26,\ 29,\ 31,\ 47,\ 49,\ 50,\ 51,\ 52,\ 56,\ 74,\ 76,\ 77,\ 78, \\ & 83,\ 101,\ 103,\ 104,\ 128,\ 130,\ 155,\ 182 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 435.** *Let $S = \{1,2,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 10,\ 11 \pmod{12} \text{ except } n = 19,\ 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 436.** *Let $S = \{3,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9 \pmod{20}, \text{ or } n = 18,\ 25 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 437.** *Let $S = \{1,3,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9 \pmod{20} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 438.** *Let $S = \{2,3,5,6,8,9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{10} \rceil$ for all $n \geq 18$.*

**Theorem 439.** *Let $S = \{1,2,3,5,6,8,9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{13} \rceil$ for all $n \geq 18$.*

**Theorem 440.** *Let $S = \{4,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9 \pmod{19} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 441.** *Let $S = \{1,4,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 9,\ 10 \pmod{11} \text{ except } n = 20,\ 21,\ 28,\ 39,\ 42, \\ & 61,\ 83,\ 105 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 442.** *Let $S = \{2,4,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{21} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10 \pmod{21} \text{ except } n = 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 443.** *Let $S = \{1,2,4,5,6,8,9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{12} \rceil$ for all $n \geq 18$.*

**Theorem 444.** *Let $S = \{3,4,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10,\ 11 \pmod{23}, \text{ or } n = 21,\ 28 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 445.** *Let $S = \{1,3,4,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10,\ 11 \pmod{23} \text{ except } n = 31,\ 32,\ 33,\ 54, \\ & 55,\ 77 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 446.** *Let $S = \{2,3,4,5,6,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10,\ 11 \pmod{23} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 447.** *Let $S = \{1,2,3,4,5,6,8,9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{16} \rceil$ for all $n \geq 18$.*

**Theorem 448.** *Let $S = \{7,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{5n}{33} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 6,\ 25,\ 26 \pmod{33} \text{ except } n = 25,\ 26,\ 39,\ 58, \\ & 59,\ 72,\ 91,\ 92,\ 105,\ 124,\ 125,\ 138,\ 157,\ 158,\ 171,\ 190 \\ 1 & \text{if } n \equiv 2,\ 3,\ 4,\ 5,\ 7,\ 8,\ 11,\ 12,\ 13,\ 15,\ 16,\ 17,\ 18,\ 19, \\ & 24,\ 27,\ 28,\ 29,\ 30,\ 31,\ 32 \pmod{33} \text{ except } n = 18,\ 19, \\ & 24,\ 27,\ 28,\ 29,\ 30,\ 31,\ 35,\ 36,\ 37,\ 38,\ 44,\ 48,\ 49,\ 50, \\ & 57,\ 60,\ 61,\ 62,\ 63,\ 68,\ 69,\ 73,\ 74,\ 81,\ 82,\ 93,\ 94,\ 95, \\ & 101,\ 106,\ 107,\ 114,\ 126,\ 139, \text{ or } n = 26,\ 39,\ 58,\ 59, \\ & 72,\ 91,\ 92,\ 105,\ 124,\ 125,\ 138,\ 157,\ 158,\ 171,\ 190 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 449.** *Let* $S = \{1,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{25} \right\rceil + \begin{cases} 2 & \text{if } n = 32 \\ 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 15,\ 16 \pmod{25} \text{ except } n = 29, \\ & 30,\ 32,\ 54,\ 79 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 450.** *Let* $S = \{2,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{25} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 7,\ 8 \pmod{25} \text{ except } n = 32,\ 33,\ 57,\ 58,\ 82 \\ 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16 \pmod{25} \\ & \text{except } n = 34,\ 35,\ 36,\ 59, \text{ or } n = 32,\ 33,\ 57,\ 58,\ 82 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 451.** *Let* $S = \{1,2,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9 \pmod{19} \text{ except } n = 25,\ 26,\ 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 452.** *Let* $S = \{3,7,8,9\}$*. Then* $\gamma(C_n[S]) = \lceil \frac{n}{7} \rceil$ *for all* $n \geq 18$*.*

**Theorem 453.** *Let* $S = \{1,3,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9 \pmod{19} \text{ except } n = 25, \text{ or } n = 34, \\ & 35,\ 40,\ 53 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 454.** *Let* $S = \{2,3,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{17} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 15,\ 16,\ 17,\ 23,\ 24,\ 25 \pmod{34} \\ & \text{except } n = 23,\ 24,\ 25,\ 49,\ 50,\ 72,\ 73,\ 74,\ 75,\ 125, \\ & 174,\ 175, \text{ or } n = 31 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 455.** *Let* $S = \{1,2,3,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10,\ 11,\ 12 \pmod{13} \text{ except } n = 21,\ 22 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 456.** *Let* $S = \{4,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{7} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 13,\ 20 \pmod{28} \text{ except } n = 20,\ 34 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 457.** *Let* $S = \{1, 4, 7, 8, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{26} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 13,\ 14,\ 15,\ 16,\ 17 \pmod{26} \text{ except } n = 39 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 458.** *Let* $S = \{2, 4, 7, 8, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{25} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16 \pmod{25} \\ & \text{except } n = 29,\ 30,\ 36,\ 37,\ 38,\ 61,\ 62 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 459.** *Let* $S = \{1, 2, 4, 7, 8, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9 \pmod{19} \text{ except } n = 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 460.** *Let* $S = \{3, 4, 7, 8, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 8 \pmod{9} \text{ except } n = 22,\ 24,\ 31,\ 49,\ 51 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 461.** *Let* $S = \{1, 3, 4, 7, 8, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{21} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 9,\ 10 \pmod{21} \text{ except } n = 26,\ 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 462.** *Let* $S = \{2, 3, 4, 7, 8, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 9,\ 10,\ 11 \pmod{22}, \text{ or } n = 28,\ 49 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 463.** *Let* $S = \{1, 2, 3, 4, 7, 8, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{14} \right\rceil + \begin{cases} 1 & \text{if } n = 24,\ 25,\ 38 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 464.** *Let* $S = \{5, 7, 8, 9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{15} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 7,\ 12,\ 14,\ 15,\ 16,\ 18,\ 20,\ 21,\ 22,\ 27, \\ & 29 \pmod{30} \text{ except } n = 18,\ 21,\ 27,\ 33,\ 42,\ 63 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 465.** *Let* $S = \{1, 5, 7, 8, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10 \pmod{11} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 466.** *Let* $S = \{2, 5, 7, 8, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{25} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 13,\ 14,\ 15,\ 16 \pmod{25} \text{ except } n = 32,\ 63, \\ & 64 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 467.** *Let* $S = \{1, 2, 5, 7, 8, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10 \pmod{11} \text{ except } n = 18,\ 19,\ 20, \\ & 39,\ 40 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 468.** *Let* $S = \{3, 5, 7, 8, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8 \pmod{9} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 469.** *Let* $S = \{1, 3, 5, 7, 8, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 8,\ 10 \pmod{22} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 470.** *Let* $S = \{2, 3, 5, 7, 8, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10 \pmod{20} \text{ except } n = 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 471.** *Let* $S = \{1, 2, 3, 5, 7, 8, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 10,\ 12 \pmod{13} \text{ except } n = 21 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 472.** *Let* $S = \{4, 5, 7, 8, 9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9 \pmod{10} \text{ except } n = 23,\ 24, \\ & 25,\ 26,\ 27,\ 33,\ 53,\ 54 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 473.** *Let* $S = \{1, 4, 5, 7, 8, 9\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ *for all* $n \geq 18$.

**Theorem 474.** *Let* $S = \{2,4,5,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 9 \pmod{20} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 475.** *Let* $S = \{1,2,4,5,7,8,9\}$*. Then* $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ *for all* $n \geq 18$*.*

**Theorem 476.** *Let* $S = \{3,4,5,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{21} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9,\ 10 \pmod{21} \text{ except } n = 28,\ 29,\ 30,\ 49, \\ & 50,\ 70 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 477.** *Let* $S = \{1,3,4,5,7,8,9\}$*. Then* $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ *for all* $n \geq 18$*.*

**Theorem 478.** *Let* $S = \{2,3,4,5,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 9,\ 10,\ 11 \pmod{23} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 479.** *Let* $S = \{1,2,3,4,5,7,8,9\}$*. Then* $\gamma(C_n[S]) = \lceil \frac{n}{15} \rceil$ *for all* $n \geq 18$*.*

**Theorem 480.** *Let* $S = \{6,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{8} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 12,\ 13,\ 14,\ 15,\ 16 \pmod{24} \text{ except} \\ & n = 30,\ 31,\ 60,\ 61,\ 62, \text{ or } n = 20 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 481.** *Let* $S = \{1,6,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10 \pmod{11} \text{ except } n = 26,\ 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 482.** *Let* $S = \{2,6,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{25} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 13,\ 14,\ 15,\ 16 \pmod{25} \text{ except } n = 32,\ 63, \\ & 64, \text{ or } n = 37 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 483.** *Let* $S = \{1,2,6,7,8,9\}$*. Then for all* $n \geq 18$*, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} 1 & \text{if } n = 30,\ 31,\ 32,\ 54 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 484.** *Let $S = \{3, 6, 7, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 2 & \text{if } n \equiv 9 \pmod{27} \\ 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18 \\ & \pmod{27} \text{ except } n = 32,\ 37,\ 64 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 485.** *Let $S = \{1, 3, 6, 7, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 7,\ 8,\ 9,\ 10 \pmod{11} \text{ except } n = 18,\ 19,\ 26, \\ & 37 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 486.** *Let $S = \{2, 3, 6, 7, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10 \pmod{20}, \text{ or } n = 35 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 487.** *Let $S = \{1, 2, 3, 6, 7, 8, 9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{13} \rceil$ for all $n \geq 18$.*

**Theorem 488.** *Let $S = \{4, 6, 7, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{9} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 5,\ 6,\ 7,\ 8 \pmod{9} \text{ except } n = 22,\ 23,\ 24,\ 31 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 489.** *Let $S = \{1, 4, 6, 7, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 4,\ 6,\ 8,\ 10 \pmod{11} \text{ except } n = 26 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 490.** *Let $S = \{2, 4, 6, 7, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11 \pmod{22} \text{ except } n = 27 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 491.** *Let $S = \{1, 2, 4, 6, 7, 8, 9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{12} \rceil$ for all $n \geq 18$.*

**Theorem 492.** *Let $S = \{3, 4, 6, 7, 8, 9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{19} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 9 \pmod{19} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 493.** *Let $S = \{1,3,4,6,7,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 10 \text{ (mod 11) except } n = 21 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 494.** *Let $S = \{2,3,4,6,7,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{11} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 9,\ 10,\ 11 \text{ (mod 22)} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 495.** *Let $S = \{1,2,3,4,6,7,8,9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{14} \rceil$ for all $n \geq 18$.*

**Theorem 496.** *Let $S = \{5,6,7,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n = 35,\ 42,\ 43,\ 44 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 497.** *Let $S = \{1,5,6,7,8,9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ for all $n \geq 18$.*

**Theorem 498.** *Let $S = \{2,5,6,7,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{10} \right\rceil + \begin{cases} 1 & \text{if } n = 35,\ 44 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 499.** *Let $S = \{1,2,5,6,7,8,9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{12} \rceil$ for all $n \geq 18$.*

**Theorem 500.** *Let $S = \{3,5,6,7,8,9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{10} \rceil$ for all $n \geq 18$.*

**Theorem 501.** *Let $S = \{1,3,5,6,7,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12 \text{ (mod 24) except } n = 30, \\ & 31,\ 32,\ 33,\ 54,\ 55 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 502.** *Let $S = \{2,3,5,6,7,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{23} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9,\ 10,\ 11 \text{ (mod 23) except } n = 30 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 503.** *Let $S = \{1,2,3,5,6,7,8,9\}$. Then $\gamma(C_n[S]) = \lceil \frac{n}{13} \rceil$ for all $n \geq 18$.*

**Theorem 504.** *Let $S = \{4,5,6,7,8,9\}$. Then for all $n \geq 18$, we have*

$$\gamma(C_n[S]) = \left\lceil \frac{3n}{31} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 7,\ 8,\ 9,\ 10,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18,\ 19,\ 20 \text{ (mod} \\ & \text{31) except } n = 18,\ 19,\ 20,\ 38,\ 39,\ 40,\ 44,\ 45,\ 46,\ 47, \\ & 48,\ 49,\ 50,\ 69,\ 70,\ 75,\ 76,\ 77,\ 78,\ 79,\ 80,\ 100,\ 106, \\ & 107,\ 108,\ 109,\ 110,\ 137,\ 138,\ 139,\ 140,\ 168,\ 169, \\ & 170,\ 199,\ 200,\ 230 \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 505.** *Let* $S = \{1,4,5,6,7,8,9\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{11} \rceil$ *for all* $n \geq 18$.

**Theorem 506.** *Let* $S = \{2,4,5,6,7,8,9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{12} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10,\ 11,\ 12 \pmod{24} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 507.** *Let* $S = \{1,2,4,5,6,7,8,9\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{12} \rceil$ *for all* $n \geq 18$.

**Theorem 508.** *Let* $S = \{3,4,5,6,7,8,9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10,\ 11,\ 12,\ 13 \pmod{26} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 509.** *Let* $S = \{1,3,4,5,6,7,8,9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{n}{13} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 8,\ 9,\ 10,\ 11,\ 12,\ 13 \pmod{26} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 510.** *Let* $S = \{2,3,4,5,6,7,8,9\}$. *Then for all* $n \geq 18$, *we have*

$$\gamma(C_n[S]) = \left\lceil \frac{2n}{27} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 9,\ 10,\ 11,\ 12,\ 13 \pmod{27} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 511.** *Let* $S = \{1,2,3,4,5,6,7,8,9\}$. *Then* $\gamma(C_n[S]) = \lceil \frac{n}{19} \rceil$ *for all* $n \geq 18$.

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