Differential Equations 2024-25
Review HW

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Differentiation

Question 1: Find the derivative of $y = x^3 \sin x$

$$y' = 3x^2 \sin x + x^3 \cos x$$

Question 2: Find the derivative of $y = \frac{\ln x}{\cos x}$

$$y' = \frac{\frac{1}{x}\cos x + \ln x \sin x}{\cos^2 x}$$

Question 3: Find the derivative of $y = \ln(\sin e^{2x})$

$$y' = \frac{2e^{2x}\cos e^{2x}}{\sin e^{2x}}$$

Integration

Question 4: Evaluate $\int x\sqrt{x^2+1}dx$

$$\int x\sqrt{x^2+1}dx$$

Let $u = x^2 + 1$, then $du = 2xdx$, so $\frac{1}{2}du = xdx$.

$$= \frac{1}{2}\int \sqrt{u}du$$

$$= \frac{1}{2}\frac{2}{3}u^{3/2} + C$$

$$= \frac{1}{3}(x^2+1)^{3/2} + C$$

Question 5: Evaluate $\int \frac{\sin x}{\cos x}dx$

$$\int \frac{\sin x}{\cos x}dx$$

Let $u = \cos x$ so $du = -\sin x dx$.

$$= -\int \frac{1}{u}du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

Question 6: Evaluate $\int xe^{3x}dx$

$$\int xe^{3x}dx$$

$$= \left(x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}dx\right) + C$$

$$= \left(x\frac{e^{3x}}{3} - \frac{1}{9}e^{3x}\right) + C$$

$$= \left(\frac{3x-1}{9}e^{3x}\right) + C$$

Question 7: Given $\int u^n \ln u du = \frac{u^{n+1}\ln u}{n+1} - \frac{u^{n+1}}{(n+1)^2} + C$, evaluate $\int x^2 \ln(2x)dx$

$$\int x^2 \ln(2x)dx$$

Let $u = 2x$, so $du = 2dx$, and $\frac{1}{2}du = dx$.

$$= \frac{1}{2}\int \frac{1}{4}u^2 \ln u du$$

$$= \frac{1}{8}\int u^2 \ln u du$$

$$= \frac{1}{8}\left(\frac{u^3 \ln u}{3} - \frac{u^3}{9}\right) + C$$

$$= \frac{1}{72}\left(3(2x)^3 \ln(2x) - (2x)^3\right) + C$$

Basic Integration Formulas

Question 8: $\int \sin(ax)dx$

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C$$

a. $\int \sin 16x dx$
$= -\frac{1}{16}\cos 16x + C$
b. $\int \sin \frac{1}{2}x dx$
$= -2\cos\frac{1}{2}x + C$

Question 9: $\int \tan(ax)dx$

$$\int \tan(ax)dx = -\frac{1}{a}\ln|\cos(ax)| + C$$

a. $\int \tan 3x dx$
$-\frac{1}{3}\ln\cos 3x + C$
b. $\int \tan\frac{1}{3}x dx$
$-3\ln|\cos\frac{1}{3}x| + C$

U-substitution

Question 10: $\int \frac{1}{\theta^2} \cos \frac{1}{\theta} d\theta$

Let $u = \frac{1}{\theta}$, so $du = -\frac{1}{\theta^2} d\theta$.

$$= -\int \cos u du$$
$$= -\sin u + C$$

Question 11: $\int \frac{\sin x}{\cos^2 x} dx$

Let $u = \cos x$ so $du = -\sin x dx$.

$$-\int \frac{1}{u^2} du$$
$$= \frac{1}{u} + C$$
$$= \frac{1}{\cos x} + C$$

Question 12: $\int \tan^4 x \sec^2 x dx$

Let $u = \tan x$ so $du = \sec^2 x dx$.

$$= \int u^4 du$$
$$= \frac{u^5}{5} + C$$
$$= \frac{\tan^5 x}{5} + C$$

Integration by Parts

Question 13: $\int t \ln(t+1) dt$

Let $u = \ln(t+1)$ so $du = \frac{1}{t+1} dt$. Let $dv = t dt$ so $v = \frac{1}{2} t^2$.

$$= \frac{t^2}{2} \ln(t+1) - \int \frac{1}{t+1} \frac{1}{2} t^2 dt$$
$$= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \int \frac{t^2}{t+1} dt$$
$$\frac{t^2}{t+1} = (t-1) - \frac{1}{t+1}$$
$$= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \int (t-1) dt - \frac{1}{2} \int \frac{1}{t+1} dt$$
$$= \frac{t^2}{2} \ln(t+1) - \frac{t^2}{4} + \frac{t}{2} - \frac{1}{2} \ln|t+1| + C$$

Question 14: $\int \cos^{-1} x dx$

$$u = \cos^{-1} x$$
$$du = -\frac{1}{\sqrt{1-x^2}}dx$$
$$dv = dx$$
$$v = x$$
$$= x\cos^{-1} x - \int x(-\frac{1}{\sqrt{1-x^2}})dx$$

Let $w = 1 - x^2$, so $dw = -2xdx$, and $-\frac{1}{2}dw = xdx$.

$$= x\cos^{-1} x - \frac{1}{2}\int \frac{1}{\sqrt{w}}dw$$
$$= x\cos^{-1} x - \sqrt{w} + C$$
$$= x\cos^{-1} x - \sqrt{1-x^2} + C$$

Question 15: $\int e^{2x} \sin x dx$

$$u = e^{2x}$$
$$du = 2e^{2x}dx$$
$$dv = \sin x dx$$
$$v = -\cos x$$
$$= -e^{2x}\cos x - \int -\cos x 2e^{2x}dx$$
$$= -e^{2x}\cos x + \int 2e^{2x}\cos x dx$$
$$u = 2e^{2x}$$
$$du = 4e^{2x}dx$$
$$dv = \cos x dx$$
$$v = \sin x$$
$$= -e^{2x}\cos x + 2e^{2x}\sin x - \int 4e^{2x}\sin x dx$$
$$5\int e^{2x}\sin x dx = -e^{2x}\cos x + 2e^{2x}\sin x$$
$$\int e^{2x}\sin x dx = \frac{-e^{2x}\cos x + 2e^{2x}\sin x}{5} + C$$

Partial Fractions

Question 16: Find the partial fraction decomposition of $\frac{x^3-x-3\sqrt{2}}{x^2-2}$

$$\frac{x^3 - x - 3\sqrt{2}}{x^2 - 2} = x + \frac{x - 3\sqrt{2}}{x^2 - 2}$$

Question 17: What is wrong?

What happens when you try to solve for A, B, and C for the partial fraction decom- position:

$$\frac{1}{(x^2 - 1)(x - 1)} = \frac{A}{x - 1} + fracBx + Cx^2 - 1$$

Bx + C should be B/(x+1) + C/(x-1)