CSE 215

Homework 4

Ben Feuer

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Question 1: Problem 1

Prove that there exists a unique prime number of the form $n^2 + 2n - 3$, where n is a positive integer.
Proof:
Let $p = n^2+2n-3$. We can rewrite this as $p = (n+3)(n-1)$. Since p is prime, either $n+3 = 1$ or $n-1 = 1$. If $n + 3 = 1$, then $n = -2$, which is not a positive integer. Therefore, $n - 1 = 1$, so $n = 2$. Substituting this back into the original equation, we get $p = 2^2 + 2(2) - 3 = 4 + 4 - 3 = 5$. Since 5 is prime, it is the unique prime number of the form $n^2 + 2n - 3$.

Question 2: Problem 2

Prove that for all integers m and n, $m + n$ and $m - n$ are either both odd or both even.
Proof:
Let m and n be integers.
Case 1: m and n are both odd.
$m = 2a + 1$ and $n = 2b + 1$
$m + n = 2(a + b) + 2 = 2(a + b + 1) = 2k$
$m - n = 2(a - b) + 0 = 2k$
Case 2: m and n are both even.
$m = 2a$ and $n = 2b$
$m + n = 2(a + b) = 2k$
$m - n = 2(a - b) = 2k$
Case 3: m is odd and n is even.
$m = 2a + 1$ and $n = 2b$
$m + n = 2(a + b) + 1 = 2k + 1$
$m - n = 2(a - b) + 1 = 2k + 1$
Case 4: m is even and n is odd.
$m = 2a$ and $n = 2b + 1$
$m + n = 2(a + b) + 1 = 2k + 1$
$m - n = 2(a - b) - 1 = 2k - 1$
Therefore, $m + n$ and $m - n$ are either both odd or both even, if both are the same parity then they are both even, and if they are different parities then they are both odd.

Question 3: Problem 3

Prove that for all integers a, b, and c, if $a|b$ and $a|c$ then $a|(b - c)$.
Proof:
Let a, b, and c be integers such that $a|b$ and $a|c$.
By definition, $b = ak$ and $c = al$ for some integers k and l.
$b - c = ak - al = a(k - l)$.
Since k and l are integers, $k - l$ is also an integer.
Therefore, $a|(b - c)$.

Question 4: Problem 4

If $n = 4k + 3$, does 8 divide $n^2 - 1$?
Proof:
$n^2 - 1 = (4k + 3)^2 - 1 = 16k^2 + 24k + 9 - 1 = 16k^2 + 24k + 8 = 8(2k^2 + 3k + 1)$.
Since $n^2 - 1 = 8(a)$, 8 divides $n^2 - 1$.

Question 5: Problem 5

Prove that if r is any rational number, then $2r^2 - r + 1$ is rational.
Proof:
Let $r = \frac{a}{b}$ where a and b are integers and $b \neq 0$.

$2r^2 - r + 1 = 2(\frac{a}{b})^2 - \frac{a}{b} + 1 = 2(\frac{a^2}{b^2}) - \frac{a}{b} + 1 = \frac{2a^2}{b^2} - \frac{a}{b} + 1 = \frac{2a^2 - ab + b^2}{b^2}$.
Since a, b, and $2a^2 - ab + b^2$ are all integers and b and b^2 aren't equal to zero, $2r^2 - r + 1$ is rational.

Question 6: Problem 6

Prove or disprove: For all integers n and m, if $n - m$ is even, then $n^3 - m^3$ is even.
Proof
Case 1: n and m are both even
$n = 2a$ and $m = 2b$ for some integers a and b.
$n - m = 2a - 2b = 2(a - b) = 2k$
$n^3 - m^3 = (2a)^3 - (2b)^3 = 8a^3 - 8b^3 = 8(a^3 - b^3) = 2(4a^3 - 4b^3) = 2k$
Case 2: n and m are both odd
$n = 2a + 1$ and $m = 2b + 1$ for some integers a and b.
$n - m = 2a + 1 - 2b - 1 = 2(a - b) = 2k$
$n^3 - m^3 = (2a+1)^3 - (2b+1)^3 = 8a^3 + 12a^2 + 6a + 1 - 8b^3 - 12b^2 - 6b - 1 = 8(a^3 - b^3) + 12(a^2 - b^2) + 6(a - b) = 8(a^3 - b^3) + 12(a + b)(a - b) + 6(a - b) = 8(a^3 - b^3) + 12k + 6k = 2(4(a^3 - b^3) + 6k) = 2k$
Therefore, $n - m$ being even implies that $n^3 - m^3$ is even.

Question 7: Problem 7

Prove that the sum of any two odd integers is even.
Proof:
Let n and m be odd integers.
$n = 2a + 1$ and $m = 2b + 1$ for some integers a and b.
$n + m = 2a + 2b + 2 = 2(a + b + 1) = 2k$
Therefore, the sum of any two odd integers is even.

Question 8: Problem 8

Prove whether the following statement is valid: For all real numbers a and b, if $a < b$ then $a^2 < b^2$.
Proof (by counterexample):
Let $a = -2$ and $b = 1$.
$a < b$ since $-2 < 1$.
$a^2 = (-2)^2 = 4$ and $b^2 = 1^2 = 1$.
$a^2 > b^2$ since $4 > 1$.
Therefore, the statement is not valid.

Question 9: Problem 9

If r and s are both positive integers, is $r^2 + 2rs + s^2$ composite?
Proof:
$r^2 + 2rs + s^2 = (r + s)^2$.
Since r and s are both positive integers, $r + s$ is also a positive integer, that must be greater than 1.
Therefore, $r^2 + 2rs + s^2$ is composite.