AP Calculus BC
ALL NOTES

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Contents

- 3 hours and 15 minutes long(1.5x for students with accommodations = 4 hours and 52 minutes)
- Section 1 has 2 parts: Part A and Part B
- Part A has 30 multiple choice questions, 60 minutes long, no calculator
- Part B has 15 multiple choice questions, 45 minutes long, calculator allowed
- Section 2 has 2 parts: Part A and Part B
- Part A has 2 problems, 30 minutes long, calculator allowed
- Part B has 4 problems, 60 minutes long, calculator allowed

Chapter 1

Topic 1 - Functions

1.1 Functions

Definition 1.1.1: Function

A function is a rule that assigns to each input exactly one output.
The input is called the independent variable, and the output is called the dependent variable.

1.1.1 Linear Functions

Definition 1.1.2: Linear Function

A linear function has the form

$$y = f(x) = b + mx$$

Its graph is a line such that

- b is the y-intercept, or the value of y when $x = 0$
- m is the slope of the line, or the rate of change of y with respect to x.

1.1.2 Exponential Functions

Definition 1.1.3: Exponential Function

An exponential function has the form

$$y = f(x) = a^x$$

Its graph is a curve that opens upward and to the right.
The constant a is called the base of the function.
The general exponential function is: $P = Pe^{rt}$

Note:-

The half-life of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.
The doubling time of an exponeentially increasing quantity is the time required for the quantity to double.

1.1.3 Logarithmic Functions

Definition 1.1.4: Logorithmic functions

The definition of a logorithm say, $\log_a x$, is the inverse of the exponential function, a^x.

Note:-

The natural logarithm is the logarithm with base e.
The natural logarithm is denoted by $\ln x$.

Definition 1.1.5: Properties of Logarithms

- $\log(AB) = \log A + \log B$
- $\log(\frac{A}{B}) = \log A - \log B$
- $\log(A^B) = B \log A$
- $\log(10^x) = x$
- $\ln(AB) = \ln A + \ln B$
- $\ln(\frac{A}{B}) = \ln A - \ln B$
- $\ln(e^x) = x$
- $\log_b x = \frac{\log_a x}{\log_a b}$

1.1.4 Composite funcitons

Definition 1.1.6: Composite Function

A composite function is a function that is the result of applying one function to the output of another function.
For example, if $f(x) = x^2$ and $g(x) = x + 1$, then $h(x) = f(g(x)) = (x + 1)^2$.

1.1.5 Inverse Functions

Definition 1.1.7: Inverse Function

An inverse function is a function that undoes the effect of another function.

$$f^{-1}(y) = x \rightarrow f(x) = y$$

Example 1.1.1

If $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$.

1.1.6 Parametric Functions

Definition 1.1.8: Parametric functions

Parametric functions are given as a set of parametric equations for x and y in terms of a third variable aka the parameter, t.
For example, $x = 2t$ and $y = t^2$.
The graph of a parametric function is a curve in the xy-plane.

1.1.7 Polar Functions

Definition 1.1.9: Polar functions

Polar coordinates are in the form (r, θ), where r is the distance from the origin and θ is the angle from the positive x-axis. A polar function defines a curve with the equation $f(\theta) = r$.

Note:-

Some common polar functions are:

- Spiral - $r = \theta$
- Circle - $r = a$
- Cardioid - $r = a(1 + \cos\theta) or a(1 + \sin\theta$ where sin and cos determine the orientation.
- Rose - $r = a\cos(n\theta)$
- Limacon - $r = a + b\cos\theta$

1.2 Properties of Functions

1.2.1 Domain and Range

Definition 1.2.1: Domain

The domain of a function is the set of all possible values of the independent variable.

Definition 1.2.2: Range

The range of a function is the set of all possible values of the dependent variable.

1.2.2 Increasing vs Decreasing

Definition 1.2.3: An increasing function

A function is increasing if $f(x) < f(x + \Delta x)$.

Definition 1.2.4: A decreasing function

A function is decreasing if $f(x) > f(x + \Delta x)$.

Definition 1.2.5: A monotonic function

A function is monotonic if it is either increasing or decreasing for all x.

1.2.3 Proportionality

Definition 1.2.6: Proportionality

A function is proportional if it has the form $f(x) = kx$ for some constant k.
The constant k is called the proportionality constant.

1.2.4 Concavity

Definition 1.2.7: Concave up

A fucntion is concave up if it bends upwards as we move left of right.

Definition 1.2.8: Concave down

A function is concvae down if it bends downwards as we move left to right.

Note:-

Concavity is also later defined using the second derivative of a function.

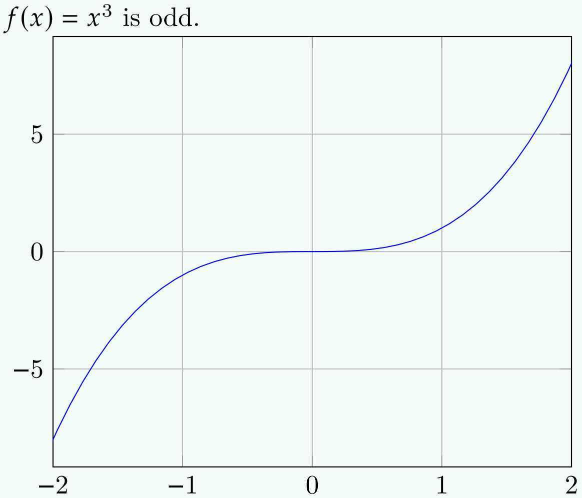
1.2.5 Symmetry

Definition 1.2.9: Odd functions

A function is odd if $f(-x) = -f(x)$.

Example 1.2.1

$f(x) = x^3$ is odd.

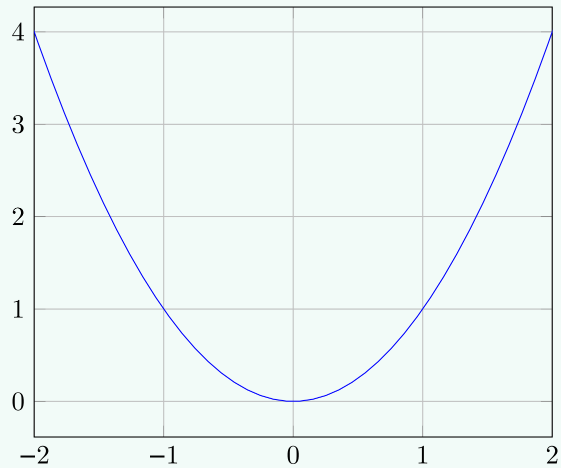


Definition 1.2.10: Even functions

A function is even if $f(-x) = f(x)$.

Example 1.2.2

$f(x) = x^2$ is even.



4
3
2
1
0
−2
−1
0
1
2

Chapter 2

Topic 2 - Limits and Continuity

2.1 Limits

Definition 2.1.1: Limit

The number L is the *limit of the function* $f(x)$ as x approaches c if, the values of x get arbitrarily close(but not equal) to c, the values of f(x) approach (or equal) L. We write

$$\lim_{x \to c} f(x) = L$$

Moreover, for $\lim_{x \to c} f(x) = L$ to exist, the left and right hand limits must be equal. Thus,

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

Note:-

If a piecewise function isn't continuous at a point, then the limit at that point does not exist. This is because the right hand and left hand limits aren't equal.

2.1.1 Theorems on Limits

IF k is a constant and the limits of $f(x)$ and $g(x)$ exist as $x \longrightarrow c$, and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then:

Theorem 2.1.1 The Constant Rule

$$lim_{x \to c} k = k$$

Theorem 2.1.2 The Constant Multiple Rule

$$lim_{x \to c} k f(x) = k \lim_{x \to c} f(x) = k \cdot L$$

Theorem 2.1.3 The Sum Rule

$$lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$$

Theorem 2.1.4 The Difference Rule

$$lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M$$

Theorem 2.1.5 The Product Rule

$$lim_{x\to c}(f(x)\cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = L \cdot M$$

Theorem 2.1.6 The Quotient Rule

$$lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$$

Provided that $M \neq 0$. Otherwise the limit does not exist.

Theorem 2.1.7 The Composition Rule

If the limit of $g(x)$ as $x \longrightarrow c$ is L and $f(x)$ is continuous at $x = L$, then

$$lim_{x\to c}f(g(x)) = f(\lim_{x\to c} g(x)) = f(L)$$

Theorem 2.1.8 * THE SQUEEZE THEOREM

If $f(x) \leq g(x) \leq h(x)$ and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

Note:-

The limit definition of euler's number(e) is:

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n$$

2.2 Continuity

Definition 2.2.1: Continuity

A function is continuous over an interval if we can draw its graph without tifting our pencil over that interval. The graph has no holes, breaks, or jumps on a continuous interval. The function y = f(x) is continuous at x = c if:

- f(c) exists (that is, c is a domain of f(x))
- $\lim_{x\to c} f(x)$ exists
- $\lim_{x\to c} f(x) = f(c)$

A function is continuous over the closed interval [a,b] if it is continuous at each x such that $a \leq x \leq b$
A function that is not continuous at x = c is said to be discontinuous at x = c. We then call x = c a *point of discontinuity*.

2.2.1 Theorems on Continuous Functions

Theorem 2.2.1 The Extreme Value Function

If f(x) is continuous over a closed interval [a,b], then f(x) attains a minimum and maximum value somewhere on the interval.

Note:-

This is important to keep in mind when finding the absolute maximum and minimum of a function. Because there will always be global minima and maxima

Theorem 2.2.2 The Intermediate Value Theorem

If f(x) is continuous over a closed interval [a,b], and if k is any number between f(a) and f(b), then there is at least one number c in [a,b] such that f(c) = k.

Theorem 2.2.3 The Continuous Functions Theorem

If you have two continuous functions, f(x) and g(x), then:

- $k \cdot f(x)$ is continuous
- $f(x) + g(x)$ is continuous
- $f(x) - g(x)$ is continuous
- $f(x) \cdot g(x)$ is continuous
- $\frac{f(x)}{g(x)}$ is continuous, provided that $g(x) \neq 0$

Theorem 2.2.4 The Composition of Continuous Functions Theorem

If f(x) is continuous at x = c and g(x) is continuous at x = f(c), then $f(g(x))$ is continuous at x = c.

Chapter 3

Topic 3 - Differentiation

Definition 3.0.1: The definition of the derivitive

At any x in the domain of the function y = f(x), the *derivative* is defined as:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{or} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

A function is said to be *differentiable* at every x for which the limit above exists, and its derivative may be denoted by $f'(x), y', \frac{dx}{dy}, \text{or} D_x y$.
The derivative of $y = f(x)$ at x = , denoted by $f'(a)$, is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

3.1 List of Derivatives

$$\frac{d}{dx} k = 0 \tag{3.1}$$

$$\frac{d}{dx} x = 1 \tag{3.2}$$

$$\frac{d}{dx} x^n = n x^{n-1} \text{(The Power Rule)} \tag{3.3}$$

$$\frac{d}{dx}(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx} \text{(The Product Rule)} \tag{3.4}$$

$$\frac{d}{dx} \frac{u}{v} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \text{(The Quotient Rule)} \tag{3.5}$$

$$\frac{d}{dx} e^x = e^x \tag{3.6}$$

$$\frac{d}{dx} a^x = a^x \ln a \tag{3.7}$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \tag{3.8}$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a} \tag{3.9}$$

$$\frac{d}{dx}\sin x = \cos x \tag{3.10}$$

$$\frac{d}{dx}\cos x = -\sin x \tag{3.11}$$

$$\frac{d}{dx}\tan x = \sec^2 x \tag{3.12}$$

$$\frac{d}{dx}\cot x = -\csc^2 x \tag{3.13}$$

$$\frac{d}{dx}\sec x = \sec x \tan x \tag{3.14}$$

$$\frac{d}{dx}\csc x = -\csc x \cot x \tag{3.15}$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \tag{3.16}$$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \tag{3.17}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \tag{3.18}$$

$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2} \tag{3.19}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \tag{3.20}$$

$$\frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \tag{3.21}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \tag{3.22}$$

3.2 The Chain Rule

Definition 3.2.1: The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 3.2.1 (

$$\frac{d}{dx}(x^2+1)^3 = 3u^2 \cdot \frac{du}{dx} = 3(x^2+1)^2 \cdot 2x = 6x(x^2+1)^2$$

)

3.3 Differentiablity and Continuity

Definition 3.3.1: The relationship between continuity and differentiability

If f is differentiable at $x = a$, then f is continuous at $x = a$.

3.4 Estimating Derivatives

Question 1: Given a table of values, estimate the derivative at a given point.

x	0	1	2	3	4	5	6	7	8
$f(x)$	98	94.95	93.06	91.90	91.17	90.73	90.45	90.28	90.17

Solution:

$$f'(0) \approx \frac{f(1) - f(0)}{1} = \frac{94.95 - 98}{1} = -3.05$$

$$f'(1) \approx \frac{f(2) - f(1)}{2 - 1} = \frac{93.06 - 94.95}{1} = -1.89$$

$$f'(2) \approx \frac{f(3) - f(2)}{3 - 2} = \frac{91.90 - 93.06}{1} = -1.16$$

$$f'(3) \approx \frac{f(4) - f(3)}{4 - 3} = \frac{91.17 - 91.90}{1} = -0.73$$

$$f'(4) \approx \frac{f(5) - f(4)}{5 - 4} = \frac{90.73 - 91.17}{1} = -0.44$$

And so on...

3.5 Derivatives of Parametrcally defined functions

Definition 3.5.1: Derivatives of Parametrically defined functions

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$$

3.6 Implicit Differentiation

Definition 3.6.1: Implicit Differentiation

Implicit Differentiation is thetechnique we use to find a derivative when y is not defined explicitly in terms of x but is differentiable.

Example 3.6.1

If $x^2 + y^2 - 9 = 0$, then

$$2x + 2y\frac{dy}{dx} = 0$$

And so,

$$\frac{dy}{dx} = -\frac{x}{y}$$

3.7 The mean value theorem

Theorem 3.7.1 The Mean Value Theorem

The Mean Value Theorem(MVT) states that If the function f(x) is continuous on the closed interval $a \leq x \leq b$ and has a derivative at each point on the open interval $a < x < b$, then there is at least one number c, $a < c < b$, such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Note:-

Rolle's Theorem is a special case of the MVT. Where if there is a function f(x) where f(a) = f(b) = k , then there is at least one number c, $a < c < b$, such that $f'(c) = 0$.

Question 2

Demonstrate Rolle's Theorem using $f(x) = x \sin x$ on the interval $[0, \pi]$.

Solution: First we check that the conditions of Rolle's Theorem are satisfied.

1. $f(x) = x \sin x$ is continuous on the *open interval* $(0, \pi)$ and exists for all x in the the closed interval $[0, \pi]$.
2. $f'(x) = \sin x + x \cos x$ exists for all x in the open interval $(0, \pi)$
3. $f(0) = 0 \sin 0 = 0$ and $f(\pi) = \pi \sin \pi = 0$

Since the conditions of Rolle's Theorem are satisfied, there must be at least one number c, $0 < c < \pi$, such that $f'(c) = 0$ and using a calculator we know that $c = 2.029$ (to three decimal places).

3.8 Indeterminate Forms and L'Hopital's Rule

Definition 3.8.1: Indeterminate Forms

An *Indeterminate Form* is an expression that can not be evaluated by substituting the limit. For example, $\frac{0}{0}$ is an indeterminate form.
The following are all indeterminate forms:

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $0 \cdot \infty$
- $\infty - \infty$
- 0^0
- 1^∞
- ∞^0

Theorem 3.8.1 L'Hospital's Rule

To find the limit of an indeterminate form, we can use L'Hospital's Rule.
The rule has several parts:

1. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ otherwise the rule can't be applied.
2. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ otherwise the rule can't be applied.

3. If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ and if $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ otherwise the rule can't be applied.
4. The above rules work for one-sided limits as well. (i.e. $\lim_{x\to a^-}$, $\lim_{x\to a^+}$, $\lim_{x\to +\infty}$, and $\lim_{x\to -\infty}$)
5. You can apply L'Hospital's Rule repeatedly if necessary.

Chapter 4

Topic 4 - Applications of Differentiation

4.1 Slope and Tangent Lines

Definition 4.1.1: Slope

The *slope* of a line at a given point is equal to the derivative of the function at that point.

Definition 4.1.2: Tangent Line

A *tangent line* is a line that touches a curve at a single point and has the same slope as the curve at that point.
The equation of a tangent line is $y - y_1 = m(x - x_1)$ where (x_1, y_1) is the point of tangency and m is the slope of the tangent line.

4.2 Increasing and Decreasing Functions

Definition 4.2.1: The relationship between the derivative the change in a function

Analyzing the signs of the first derivative of a function tells us the intervals over which f(x) is increasing or decreasing.
For intervals where f(x) is increasing, $f'(x) \geq 0$ and for intervals where f(x) is decreasing, $f'(x) \leq 0$.

Note:-

Endpoints of intervals where f(x) is increasing or decreasing are included in the intervals; however, points of discontinuity aren't such.

4.3 Maximum, Minimum and Inflection Points

Definition 4.3.1: Critical Points

A *critical point* of a function f(x) is a point in the domain of f(x) where either $f'(x) = 0$ or $f'(x)$ does not exist.

Definition 4.3.2: Local Maximum

A function f(x) has a *local maximum* at $x = c$ if $f'(c) = 0$ and $f''(x) < 0$

Definition 4.3.3: Local Minimum

A function f(x) has a *local minimum* at $x = c$ if $f'(c) = 0$ and $f''(x) > 0$

Definition 4.3.4: Inflection Point

A function f(x) has an *inflection point* at $x = c$ if $f''(c) = 0$ and $f''(x)$ switches signs at x = c.

Note:-

A function is concave up when $f'' \geq 0$ and concave down when $f'' \leq 0$.

Definition 4.3.5: Global Maximum and Minimum

Using the Extreme Value Theorem, we know that f(x) attains a global maximum and minimum or an interval, thus to solve for the global maximum and minimum we must use the closed interval or candidates test, comparing critical points with endpoints of the interval.

4.4 Motion on a line

Definition 4.4.1: Position, Velocity, and Acceleration

The *position* of an object at time t is given by the function $s(t)$.
The *velocity* of an object at time t is given by the function $v(t) = s'(t)$.
The *acceleration* of an object at time t is given by the function $a(t) = v'(t) = s''(t)$.

4.5 Motion along a curve

Definition 4.5.1: Position, Velocity, Acceleration Vectors

For parametrically defined functions the following definitions apply:
The *position vector* of an object at time t is given by the function $\vec{r}(t) = \langle x(t), y(t) \rangle$.
The *velocity vector* of an object at time t is given by the function $\vec{v}(t) = \vec{r}'(t) = \langle x'(t), y'(t) \rangle$.
The *acceleration vector* of an object at time t is given by the function $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \langle x''(t), y''(t) \rangle$.

Definition 4.5.2: Vector Velocity

The *speed* of an object at time t is given by the function $|\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.

Definition 4.5.3: Vector Magnitude

The *magnitude* of a vector's velocity $\vec{v} = \langle x', y' \rangle$ is given by $|\vec{v}| = \sqrt{x'^2 + y'^2}$.

4.6 Tangent-Line Approximation

Definition 4.6.1: Tangent-Line Approximation

If f'(a) exists, then the *local linear approximation* of f(x) at x = a is

$$f(x) = f'(a)(x - a)$$

Since the equation of the tangent line to y = f(x) at x = a is

$$y - f(a) = f'(a)(x - a)$$

Therefore, the tangent-line approximation of f(x) at x = a is

$$f(x) \approx f(a) + f'(a)(x - a)$$

4.7 Related Rates

If several variables are functions of time or a differing parameter related by an equation, we can obtain a relation involving their (time) rates of changeby differentiating with respect to t.

4.8 Slope of a Polar Curve

Definition 4.8.1: Slope of a Polar Curve

The slope of a polar curve at a point $P(r, \theta)$ is given by $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \sin\theta + r' \cos\theta}{r \cos\theta - r' \sin\theta}$. This is because the x component of a polar curve is $x = r\cos\theta$ and the y component is $y = r\sin\theta$.

Chapter 5

Topic 5 - Antidifferentiation

Definition 5.0.1: Antiderivative

The *antiderivative* or indefinite integral of a function f(x) is a function F(x) such that $F'(x) = f(x)$.
Since the derivative of a constant is zero, the antiderivative of a function f(x) is a family of functions that differ by a constant.
The antiderivative of f(x) is denoted by $\int f(x)dx$.
The process of finding the antiderivative of a function is called *antidifferentiation*.

$$\int f(x)dx = F(x) + C(\text{where C is a constant})$$

Example 5.0.1 (Basic Integrals)

- $\int kf(x)dx = k\int f(x)dx$
- $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (\ n \neq -1 \ )$
- $\int \frac{1}{x}dx = \ln|x| + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\ln a} + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \sec^2 x dx = \tan x + C$
- $\int \csc^2 x dx = -\cot x + C$
- $\int \sec x \tan x dx = \sec x + C$
- $\int \csc x \cot x dx = -\csc x + C$
- $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + C$
- $\int \frac{1}{\sqrt{x^2-a^2}} dx = \cos^{-1}\frac{x}{a} + C$
- $\int f^{-1}(x) = xf^{-1}(x) - F(f^{-1}(x)) + Cdx$

5.1 Methods of Integration

Theorem 5.1.1 Integration by Substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Theorem 5.1.2 Integration by Partial Fractions

The method of partial fractions makes it possible to express a rational function $\frac{f(x)}{g(x)}$ as a sum of simpler fractions.
If the degree of f(x) is less than the degree of g(x), then the rational function $\frac{f(x)}{g(x)}$ can be expressed as a sum of partial fractions.
ex: $\frac{2x^2+5x+1}{x^3+2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$
To find the constants A, B, and C, we can multiply both sides by the denominator and then substitute values for x that make the other terms zero.
ex: $2x^2 + 5x + 1 = A(x^2)(x + 2) + B(x + 2) + C(x^2)$
$x = 0 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$
$x = -2 \Rightarrow 4 = 4A \Rightarrow A = 1$
$x = -1 \Rightarrow -2 = -2B \Rightarrow B = 1$
Therefore, $\frac{2x^2+5x+1}{x^3+2x^2} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{2(x+2)}$
Thus,

$$\int \frac{2x^2 + 5x + 1}{x^3 + 2x^2} dx = \int \frac{1}{x} dx + \int \frac{1}{x^2} dx + \int \frac{1}{2(x + 2)} dx$$

Theorem 5.1.3 Integration by Parts

If u and v are differentiable functions, then

$$\int udv = uv - \int vdu$$

The acronym LIPET is used to remember the order of integration by parts, choose u in the followin g order.
L - Logarithmic
I - Inverse Trigonometric
P - Polynomial
E - Exponential
T - Trigonometric

Question 3: Find $\int x \cos x dx$

Using integrations by parts

Solution: We let u = x and dv = $\cos x dx$. Then du = dx and v = sinx

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

Theorem 5.1.4 The Tic-Tac-Toe Method

This method of integration is extremely useful when repeated integration of parts is necessary. To integrate $\int u(x)v(x)dx$, we construct a table where u' is the derivative of u(x) and $v_1(x)$ is the antiderivative of v(x) as follows:

u(x)	+	$v(x)$
u'	-	$v_1(x)$
u''	+	$v_2(x)$
u'''	-	$v_3(x)$
$u^{(4)}$		

Such that:

$$\int u(x)v(x)dx = u(x)v_1(x) - u'(x)v_2(x) + u''(x)v_3(x) - u'''(x)v_4(x) + \cdots$$

* this method is also called the tabular method!

Chapter 6

Topic 6 - Definite Integrals

Definition 6.0.1: The Fundamental Theorem of Calculus

If f is continuous on the closed interval [a,b] and F' = f, then according to the FTC,

$$\int_a^b f(x)dx = F(b) - F(a)$$

6.1 Reimann Sums

Definition 6.1.1: Reimann Sum

Let f be defined on the interval $[a, b]$ and let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$, where $a = x_0 < x_1 < \cdots < x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$ be the length of the ith subinterval. Let c_i be any point in the ith subinterval $[x_{i-1}, x_i]$. Then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

is called a Reimann Sum for f on [a,b] associated with the partition P and the choice of c_i in the ith subinterval.

Theorem 6.1.1 Approximations of the definite integral using Reimann Sums

Left Sum: $f(x_0)\Delta x_1 + \ldots$
Right Sum: $f(x_1)\Delta x_1 + \ldots$
Midpoint Sum: $f(\frac{x_0+x_1}{2})\Delta x_1 + \ldots$
Trapezoidal Sum: $\frac{1}{2}(f(x_0) + f(x_1))\Delta x_1 + \ldots$

Chapter 7

Topic 7 - Applications of Integration to Geometry

7.1 Finding Area using integrals

Theorem 7.1.1 Finding Area with integrals

$$A = \int_a^b f(x)dx$$

The area between two curves is:

$$\int_a^b [g(x) - f(x)]dx \text{(if there are no points of intersection, otherwise)}$$

$$\int_a^b [g(x) - f(x)]dx - \int_b^c [f(x) - g(x)]dx$$

Theorem 7.1.2 The Region bounded by a Polar Curve

$$A = \frac{1}{2}\int r^2 d\theta$$

7.2 Finding Volume using integrals

Theorem 7.2.1 Finding Volume of Solids with known cross sections

$$V = \int A(x)dx$$

Theorem 7.2.2 Finding the Volume of Disks, Washers, and Shells

Disk: $\Delta V = \pi r^2 \Delta x$ and $V = \pi \int_a^b r^2 dx$.

Washer: $\Delta V = \pi R^2 \Delta x - \pi r^2 \Delta x$ and $V = \pi \int_a^b (R^2 - r^2)dx$.

Shells: $\Delta V = 2\pi r h \Delta x$ and $V = 2\pi \int_a^b rhdx$

7.3 Arc Length

Theorem 7.3.1 The Formula for Arc Length

$$s = \int_a^b \sqrt{1 + \frac{dy}{dx}^2}\, dx$$

For parametric functions:

$$s = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$$

7.4 Improper Integrals

Definition 7.4.1: Improper Integrals

There are two kinds of improper integrals:

- Those in which at least one fo the limits of integration is infinite (the interval is not bounded)
- Those of the type $\int_a^b f(x)dx$, where f(x) has a point of discontinuity (becoming infinite) at x = c, $a \leq c \leq b$ (the function is not bounded).

Theorem 7.4.1 The comparison test for improper integral

If $f(x) \leq g(x)$ and both $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are improper integrals with g(x) converging, then $\int_a^b f(x)dx$ converges.
If $f(x) \geq g(x)$ and both $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are improper integrals with g(x) diverging, then $\int_a^b f(x)dx$ diverges.

Chapter 8

Topic 8 - More Applications of Integration

8.1 Average Value of a Function

Definition 8.1.1: Average Value of a Function

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

8.2 Distance a particle moves on a curve

Theorem 8.2.1 Distance a particle moves on a parametric curve

$$s = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$$

Chapter 9

Topic 9 - Differential Equations

Definition 9.0.1: Differential Equation

An equation that involves a derivative is called a differential equation.

9.1 Rate of Change + Slope Fields

Definition 9.1.1: Rate of Change

The rate of change of a quantity y with respect to another quantity x is defined as the derivative of y with respect to x.

Definition 9.1.2: Slope Field

A slope field is a graph of the slopes of a differential equation. This is a basically a graph of every derivative at many points on a graph.

9.2 Derivatives of Implicitly defined functions

Theorem 9.2.1 Derivatives of Implicitly defined functions

ex: $x^2 + y^2 = 25$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Now for implementing implicit functions in slope fields we can just plug in the x and y values into the derivative.

9.3 Euler's Method

Theorem 9.3.1 Euler's Method

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x$$

Where $\Delta x = \frac{b-a}{n}$ and $x_n = a + n\Delta x$