Linear Algebra Notes - Math 61 & 62

Ben Feuer

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Contents

Chapter 1

Introduction to Vectors and Matrices

1.1 Vectors

1.1.1 Linear Combinations of Vectors

Definition 1.1.1: Vector Addition, Scalar Multiplication, and Linear Combinations

- Vector Addition: $\vec{v} + \vec{w} = \vec{v} + \vec{w}$
- Scalar Multiplication: $c\vec{v} = c\vec{v}$
- Linear Combination: $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$

Note:-

All combinations
cu fills a line through the origin
$cu + dv$ fills a plane through the origin
$cu + dv + ew$ fills all of three-dimensional space

1.1.2 The Dot Product $\vec{v} \cdot \vec{w}$ and its Properties

Definition 1.1.2: The Dot Product

The multiplication of two vectors:

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} = 3 \cdot 4 + 1 \cdot 5 + 7 \cdot 2 = 31$$

Definition 1.1.3: The length of a vector

The length of a vector is the square root of the dot product of the vector with itself:

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$$

Definition 1.1.4: The unit vector

The unit vector is the vector divided by its length:

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}$$

The unit vector is a vector of length 1 in the same direction as the original vector.

Note:-

Perpendicular vectors
Two vectors are perpendicular if their dot product is zero.
$\vec{v} \cdot \vec{w} = 0$
ex: $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] = 0$
This has later implications for the nullspace of a matrix.

Definition 1.1.5: The Dot Product Cosine Formula

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}||||\vec{w}||}$$

Definition 1.1.6: Schwarz Inequality

$$|\vec{v} \cdot \vec{w}| \leqslant ||\vec{v}||||\vec{w}||$$

Definition 1.1.7: Triangle Inequality

$$||\vec{v} + \vec{w}|| \leqslant ||\vec{v}|| + ||\vec{w}||$$

1.2 Matrices

Definition 1.2.1: Matrix

A matrix is a rectangular array of numbers.

- $m \times n$ matrix has m rows and n columns
- A_{ij} is the entry in the ith row and jth column
- $A = \left[\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{smallmatrix}\right]$

1.2.1 Types of Matrices

Definition 1.2.2: List of Matrices

- Identity Matrix: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Zero Matrix: $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- Diagonal Matrix: $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
- Upper Triangular Matrix: $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$
- Lower Triangular Matrix: $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$
- Symmetric Matrix: $S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

1.2.2 Matrix Multiplication

Definition 1.2.3: Matrix times Vector

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 14 \\ 32 \end{bmatrix}$$

Definition 1.2.4: Matrix times Matrix

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}$$

1.2.3 The Column Space of a Matrix

Definition 1.2.5: Column Space

The column space of a matrix is the set of all possible linear combinations of the columns of the matrix.

$$\mathbf{A}x = \text{All possible linear combinations of the columns of A}$$

$$\text{Col } A = \text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$$

1.2.4 Independent & Dependent Columns

Definition 1.2.6: Linear Independence

A set of vectors is linearly independent if no vector in the set is a linear combination of the other vectors in the set.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0$$

$$c_1 = c_2 = \cdots = c_n = 0$$

Definition 1.2.7: Linear Dependence

A set of vectors is linearly dependent if at least one vector in the set is a linear combination of the other vectors in the set.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0$$

$$c_1 = c_2 = \cdots = c_n = 0$$

Definition 1.2.8: Rank of a Matrix

The rank of a matrix is the number of linearly independent columns in the matrix.

$$\text{rank } A = \text{number of linearly independent columns in A}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \mathbf{R}^3$$

1.2.5 $\mathbf{A} = \mathbb{C}\mathbf{R}$

Definition 1.2.9: Matrix Factorization

$\mathbb{C}$ is a matrix with the columns of $\mathbf{A}$ that are linearly independent.
$\mathbf{R}$ is essentially the **reduced row echelon form** of $\mathbf{A}$, which will have later significance.

$$\mathbf{A} = \mathbb{C}\mathbf{R}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Example 1.2.1 ($\mathbf{A} = \mathbb{C}\mathbf{R}$)

$\mathbf{A} = \begin{bmatrix} 2 & 6 & 4 \\ 4 & 12 & 8 \\ 1 & 3 & 5 \end{bmatrix}$

$\mathbb{C} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \\ 1 & 5 \end{bmatrix}$ because column 2 is 3 times column 1

$\mathbf{R} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathbf{A} = \mathbb{C}\mathbf{R} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Chapter 2

Elimination, A = ŁU, and Inverses

2.1 Elimination

Definition 2.1.1: Elimination

Elimination is the process of transforming a matrix $\mathbf{A}$ into a matrix $\mathbf{U}$ by adding multiples of one row to another row.
We do this by multiplying $\mathbf{A}$ by an elimination matrix $\mathbf{E}$, which is the identity matrix with the row we want to add to another row replaced with the row we want to add.

$$\mathbf{EA} = \mathbf{U}$$

We use elimination to solve systems of equations. ($\mathbf{A}x = b$) With elimination, we can transform $\mathbf{A}x = b$ into $\mathbf{U}x = c$, where c is a new vector, and we can determine if $\mathbf{A}$ is invertible, which is the case if $\mathbf{U}$ has no zeros on its main diagonal.

2.1.1 The process of elimination

Example 2.1.1 (Elimination)
(May be wrong)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\mathbf{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplies row 1 by 4 and subtracts that from row 2 to eliminate the nonzero value in $\mathbf{A}_{21}$.

$$\mathbf{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$$

Multiplies row 1 by 7 and subtracts that from row 3 to eliminate the nonzero value in $\mathbf{A}_{31}$.

$$\mathbf{E}_{31}\mathbf{E}_{21}\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

$$\mathbf{E}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Multiplies row 2 by 2 and subtracts that from row 3 to eliminate the nonzero value in $\mathbf{A}_{32}$.

$$\mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21}\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{U} = \mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21}\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{E} = \mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Because $\mathbf{U}$ has a zero on its main diagonal(third pivot), $\mathbf{A}$ is not invertible, not full rank.

Note:-

System of equations
$\mathbf{A}x = b$
Exactly one solution if $\mathbf{A}$ is invertible/has independent columns.
ex: $(x, y) = (1, 1)$; independent columns: $(2, 4)$ and $(3, 2)$

$$2x + 3y = 5$$
$$4x + 2y = 6$$

No solution if $\mathbf{A}$ is not invertible/has dependent columns.
ex: dependent columns: $(2, 4)$ and $(3, 6)$

$$2x + 3y = 5$$
$$4x + 6y = 15$$

Infinite solutions if $\mathbf{A}$ is not invertible/has dependent columns.
There will be infinitely many solutions to $\mathbf{A}X = 0$ when columns of $\mathbf{A}$ are dependent, because for instance $2a - 2a = 4a - 4a$. So if there is one solution to $\mathbf{A}x = b$, we have many solutions as shown below:

$$A(x + cX) = Ax + cAX = b + 0 = b$$

2.1.2 Augmented Matrices

Definition 2.1.2: Augmented Matrices

An augmented matrix is a matrix that contains the coefficients of a system of linear equations, as well as an additional column containing the constants.

$$\begin{bmatrix} A & | b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 4 & 5 & 6 & | & 7 \\ 7 & 8 & 9 & | & 10 \end{bmatrix}$$

From the augmented matrix, we can determine if the system of equations has a solution, and if so, how many solutions it has by performing elimination on the matrix as a whole.

2.1.3 Inverse Matrices

Definition 2.1.3: Inverse Matrices

The matrix $\mathbf{A}$ is invertible if there existes a matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

$$\mathbf{A}^{-1}\mathbf{A}x = \mathbf{A}^{-1}b$$

The inverse exists if and only if:

1. elimination produces n pivots (row exchanges are allowed). Elimination solves Ax = b.
2. The matrix $\mathbf{A}$ only has one inverse.
3. The one and only solution to $\mathbf{A}x = b$ is $x = \mathbf{A}^{-1}b$.
4. If $\mathbf{A}$ is invertible, then $\mathbf{A}x = 0$ has only the trivial solution $x = 0$.
5. A square matrix is invertible if and only if its columns are independent.
6. A 2 by 2 matrix is invertible if and only if $ad - bc \neq 0$.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note:-

Ł is the inverse of E.

$$\mathrm{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\ell_{32} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell 31 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ (\ell_{32}\ell_{21}-\ell_{31}) & -\ell_{32} & 1 \end{bmatrix}$$

$$\mathrm{E}^{-1} = L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Later we will see how $\mathbf{A} = \mathbf{ŁU}$

Definition 2.1.4: Gauss-Jordan Elimination

$$\begin{bmatrix}\mathbf{A} & \mathbf{I}\end{bmatrix} \to \begin{bmatrix}\mathbf{I} & \mathbf{A}^{-1}\end{bmatrix}$$

Definition 2.1.5: Cost of elimination

Will be added after exam!

2.1.4 $\mathbf{A} = \mathbf{ŁU}$

Definition 2.1.6: Finding $\mathbf{A} = \mathbf{ŁU}$

$$\mathbf{A} = \ell_i u_1 + \ell_2 u_2 + \ell_3 u_3$$

$$\mathbf{A} = \begin{bmatrix} \ell_1 & \dots & \ell_n \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_3 \end{bmatrix} = \mathbf{ŁU}$$

Definition 2.1.7: Finding $\mathbf{A} = \mathbf{Ł}D\mathbf{U}$

Same thing as above but with D in the middle, which is a diagonal matrix and is the same as $\mathbf{U}$ but with the pivots on the diagonal, meaning the pivots of $\mathbf{U}$ all equal 1.

2.1.5 Permutation Matrices

Definition 2.1.8: Permutation Matrices

A permutation matrix is a square binary matrix that has exactly one entry 1 in each row and each column and 0s elsewhere. Each permutation matrix represents a specific permutation of rows of the identity matrix.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

A permutation matrix has the same rows as the identity matrix, but in a different order($n!$ different orders).

$$P^{-1} = P^T$$

Definition 2.1.9: $P\mathbf{A} = \mathbf{ŁU}$

$P\mathbf{A} = \mathbf{ŁU}$ is the same as $\mathbf{A} = \mathbf{ŁU}$ just with A's rows properly sorted.

2.1.6 Transposes

Definition 2.1.10: Transposes

The transpose of a matrix $\mathbf{A}$ is denoted by $\mathbf{A}^T$ and is the matrix formed by turning all the rows of $\mathbf{A}$ into columns.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$(AB)^T = B^T A^T$
$(A+B)^T = A^T + B^T$
$(A^T)^T = A$
$(A^{-1})^T = (A^T)^{-1}$
$(A^T)^{-1} = (A^{-1})^T$

2.1.7 Symmetric Matrices

Definition 2.1.11: Symmetric Matrices

A symmetric matrix is a square matrix that is equal to its transpose.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$\mathbf{A}^T = \mathbf{A}$
$S = \mathbf{A}^T\mathbf{A}$ is symmetric.
S^{-1} is symmetric.

Chapter 3

$\mathbf{A}x = b$, $\mathbf{A}x = 0$, Subspaces, Independence, Basis, and Dimension

3.1 $\mathbf{A}x = b$ and $\mathbf{A}x = 0$

3.2 The Four Fundamental Subspaces

Definition 3.2.1: Subspace

A subspace of $\mathbf{R}^n$ is a set of vectors that satisfies two conditions:
1. The zero vector is in S.
2. If u and v are in S, then the sum of u and v is in S.
3. If u is in S and c is any scalar, then the scalar multiple of u is in S.

Example 3.2.1 (Possible Subspaces of $\mathbf{R}^3$)

$\mathbf{R}^3$

The plane through the origin

The line through the origin

The origin/zero vector

Definition 3.2.2: Span

The span of a set of vectors is the set of all linear combinations of the vectors.

$$\text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\} = \text{All vectors in } \mathbf{R}^3 \text{ that can be written as } c\begin{bmatrix}1\\2\\3\end{bmatrix} \text{ for some scalar } c$$

$$\text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\} = \text{All vectors in } \mathbf{R}^3 \text{ that can be written as } c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}4\\5\\6\end{bmatrix} \text{ for some scalars } c_1 \text{ and } c_2$$

Question 1: When do 10 vectors span R^5 ? This is very possible

Definition 3.2.3: Column Space

The column space of **A** is the set of all linear combinations of the columns of **A**.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\text{Col } \mathbf{A} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$$

$$\text{Col } \mathbf{A} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$$

$$\text{Col } \mathbf{A} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \right\}$$

$$\text{Col } \mathbf{A} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\text{Col } \mathbf{A} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Definition 3.2.4: Row Space

The row space of $\mathbf{A}$ is the set of all linear combinations of the rows of $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\text{Row } \mathbf{A} = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\right\}$$

$$\text{Row } \mathbf{A} = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right\}$$

$$\text{Row } \mathbf{A} = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$$

$$\text{Row } \mathbf{A} = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$\text{Row } \mathbf{A} = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Chapter 4

Orthoganality and Projections

4.1 Orthoganality of the Four Subspaces

Definition 4.1.1: Orthogonal

Two vectors are orthogonal if their dot product is zero.

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0$$

Note:-

The column space and the nullspace are orthogonal complements.

$$\text{Col } \mathbf{A} \perp \text{Null } \mathbf{A}$$

The row space and the left nullspace are orthogonal complements.

$$\text{Row } \mathbf{A} \perp \text{Null } \mathbf{A}^T$$

4.2 Projection onto Subspaces

4.2.1 Orthonormal

Definition 4.2.1: Orthonormal Vectors and Bases

A set of vectors is orthonormal if they are all orthogonal to each other and they all have length 1.

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}$$

4.3 Gram-Schmidt

Definition 4.3.1: Gram-Schmidt Process

The Gram-Schmidt process is a method for orthonormalizing a set of vectors.

$$v_1 = a_1$$
$$v_2 = a_2 - \text{proj}_{v_1} a_2$$
$$v_3 = a_3 - \text{proj}_{v_1} a_3 - \text{proj}_{v_2} a_3$$
$$\text{proj}_{v_1} a_2 = \frac{a_2 \cdot v_1}{v_1 \cdot v_1} v_1$$
$$\text{proj}_{v_2} a_3 = \frac{a_3 \cdot v_2}{v_2 \cdot v_2} v_2$$
$$v_2 = a_2 - \frac{a_2 \cdot v_1}{v_1 \cdot v_1} v_1$$
$$v_3 = a_3 - \frac{a_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{a_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

The Gram-Schmidt process is used to find an orthonormal basis for the column space of a matrix.

$$q_1 = \frac{v_1}{\|v_1\|}$$
$$q_2 = \frac{v_2}{\|v_2\|}$$
$$q_3 = \frac{v_3}{\|v_3\|}$$

Next, we find $A = QR$, where Q is the matrix with the orthonormal vectors as its columns and R is an upper triangular matrix.

$$Q^T Q = I \text{ because the columns of Q are orthonormal}$$
$$Q^T A = R$$

Chapter 5

Determinants and Linear Transformations

5.1 Determinants

Definition 5.1.1: Determinant

The determinant of a matrix is a scalar value that can be computed from the elements of a square matrix. It is denoted by $\det A$.
Geometric interpretation: the determinant of a matrix is the factor by which the matrix changes the area of a unit square.

Note:-

The determinant of a singular matrix is zero.
The determinant of a diagonal matrix is the product of the diagonal entries.
The determinant of a triangular matrix is the product of the diagonal entries.
$\det A = \det A^T$
$\det AB = \det A \det B$
From a geometric perspective, if B chagnes the area by a factor of $\det B$, then A will change the updated area by a factor of $\det A$, resulting in the original area being changed by a factor of the product of the determinants.
Orthogonal matrices have products/determinants equal to 1 or -1.
Invertible matrices have determinants equal to + or - the product of their pivots.
Consider $A = LU$, L has 1s on its diagonal. U has pivots on its diagonal.
If $PA = LU$, then depending on the number of row exchanges $\det P = 1$ or -1.
Linearity of determinant of a matrix:

$$\det \begin{bmatrix} a & b \\ c+k & d+l \end{bmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det \begin{bmatrix} a & b \\ k & l \end{bmatrix}$$

Definition 5.1.2: Determinant Rules & Shortcuts

- The determinant is unaltered by reflection.
- The all-zero propety. If all elements are zero, $\det A = 0$
- Repition/Proportionality property: if all rows are the same, $\det A = 0$
- Switching property: if two rows are switched, $\det A = -\det A$
- Scalar multiple property: if a row(or column) is multiplied by a scalar, $\det A = k \det A$
- Sum property: $\det \begin{bmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{bmatrix} = \det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix} + \det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}$
- Triangle property: $\det \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = a \cdot d \cdot f$

Definition 5.1.3: Cofactor

The cofactor of a matrix is the determinant of the matrix formed by removing the row and column of a specific entry and multiplying the result by $(-1)^{i+j}$, where i and j are the row and column of the entry.

$$\text{Cofactor } A_{ij} = (-1)^{i+j} \det A_{ij}$$

Each Cofactor is a 2 by 2 determinant for a 3 by 3 determinant

5.2 Cramer's Rule to solve $\mathbf{A}x = b$

Definition 5.2.1: Cramer's Rule

Key idea: we can rewite $\mathbf{A}x = b$ as follows:

$$\begin{bmatrix} \mathbf{A} \end{bmatrix} \begin{bmatrix} x_1 & & \\ x_2 & 1 & \\ x_3 & & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & b_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

Apply the determinant product rule to the above expression:

$$\det \mathbf{A}x_1 = \det \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & b_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = \det B_1$$

$$\text{so } x_1 = \frac{\det B_1}{\det \mathbf{A}}$$

Example 5.2.1 (Cramer's Rule)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$$

$$\det \mathbf{A} = -3$$

5.3 Linear Transformations

Definition 5.3.1: Linear Transformations

A transformation T assigns an output T(v) to each input vector v in V.
The transformation is linear if it satisfies the following two properties:
1. Linearity: $T(u+v) = T(u) + T(v)$
2. $T(cu) = cT(u)$ for all c

Definition 5.3.2: The derivitive is a linear transformation

$$u(x) = 6 - 4x + 3x^2$$

$$\frac{du}{dx} = -4 + 6x$$

Nullspace of $T(u) = \frac{du}{dx}$ For the nullspace we solve $T(u) = 0$.
Column space of $T(u) = \frac{du}{dx}$ is the set of all possible derivatives.