- Eigenvalues and eigenvectors
- Diagonalization
- Symmetric Positive Definite matrices
- Systems of Differential Equations

Eigenvalues and Eigenvectors

Definition 0.0.1: Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there exists a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. The vector $\vec{x}$ is called an eigenvector of A corresponding to λ.

$$(A - \lambda I)x = 0$$

Definition 0.0.2: Eigenvector properties

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Then the following properties hold:

- The eigenvectors are linearly independent.
- The matrix A can be diagonalized as $A = X\Lambda X^{-1}$, where X is the matrix whose columns are the eigenvectors of A and Λ is the diagonal matrix with the eigenvalues of A on the diagonal.
- The eigenvectors of similar matrices are the same. EX: the eigenvectors of A are the same as the eigenvectors of $A^9 + cI$.

Definition 0.0.3: Eigenvalue properties

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Then the following properties hold:

- The sum of the eigenvalues is equal to the trace of the matrix: $\sum \lambda_i = \text{Trace}$.
- The product of the eigenvalues is equal to the determinant of the matrix: $\prod_{i=1}^{n} \lambda_i = \det(A)$.
- For a markov matrix, the largest eigenvalue is 1. A markov matrix is a matrix whose columns add to 1.
- For a singular matrix, the determinant is 0 and at least one eigenvalue is 0. A singular matrix is a matrix whose columns are linearly dependent.
- For a symmetric matrix, the eigenvalues are real and the eigenvectors are orthogonal.

Note:-

Imaginary Eigenvalues: If a matrix has imaginary eigenvalues, then the matrix is not diagonalizable.
Complex Eigenvectors: If a matrix has complex eigenvectors, then the matrix is not diagonalizable.
Eigenvalues of AB and A+B: Eigenvalues are not the same for AB and A+B. A and B share n independent Eigenvectors if AB=BA.

Diagonalization

Definition 0.0.4: Diagonalization

A matrix A is diagonalizable if it can be written as $A = X\Lambda X^{-1}$, where X is the matrix whose columns are the eigenvectors of A and Λ is the diagonal matrix with the eigenvalues of A on the diagonal.

Symmetric Positive Definite Matrices

Definition 0.0.5: Symmetric Positive Definite Matrices

A symmetric matrix has n real eigenvalues λ_i and n orthogonal eigenvectors q_i.
S is diagonalized by an orthogonal matrix Q: $S = Q\Lambda Q^T = Q\Lambda Q^{-1}$.
S is positive definite if all eigenvalues are positive.
Positive Semi-Definite: All eigenvalues are non-negative (allows $\lambda = 0$).

Definition 0.0.6: Positive Definite Matrices Tests

Energy Test: A matrix A is positive definite if for all nonzero vectors $\vec{x}$, $\vec{x}^T A\vec{x} > 0$.
Eigenvalue Test: A matrix A is positive definite if all of its eigenvalues are positive.
Cholesky Decomposition: A matrix A is positive definite if it can be written as $A = LL^T$, where L is a lower triangular matrix with positive diagonal entries. Works when L has independent columns.
Pivot Test: A matrix A is positive definite if all of its pivots are positive.
Upper Left Determinants: A matrix A is positive definite if all of its upper left determinants are positive.

Note:-

$S = CAC^T$ is pos definite if C is invertible

If S1 and S2 are pos def, then S1 + S2 is pos def

Systems of Differential Equations

Definition 0.0.7: Systems of Differential Equations

If $Ax = \lambda x$, then $u(t) = e^{\lambda t}x$ will solve $\frac{du}{dt} = Au$. Each λ and x is a solution $e^{\lambda t}x$.
If $A = X\Lambda X^{-1}$, then $u(t) = e^{At}u(0) = Xe^{\Lambda t}X^{-1}u(0)$ will solve $\frac{du}{dt} = Au$.
Matrix Exponential: $e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots$.
Taylor Series:

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$.
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$.

A is stable and u(t) approaches 0 as t approaches infinity if all eigenvalues of A have negative real parts ¡ 0.
Second order eqn: $\frac{d^2u}{dt^2} + 2\zeta\omega_n\frac{du}{dt} + \omega_n^2 u = 0\ldots$ First order System: $\frac{du}{dt} = Au$, where $A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$.

$$u'' + Bu' + Cu = 0 \to \begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}\begin{bmatrix} u \\ u' \end{bmatrix}$$