BC Calculus
Series

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Contents

Chapter 1

Sequences And Series

1.1 Geometric Series

Definition 1.1.1: What is a Geometric Series

A Geomtric Series is a series multipled by a constant

Example 1.1.1 (Example of a Geometric Series)
$\sum_{n=1}^{\infty} \frac{1}{n^2}$

Definition 1.1.2: Sum of a Geometric Series

The sum of a Geometric Series is $\frac{a}{1-r}$

1.2 Series convergence

Theorem 1.2.1 Limit test for convergence
If the limit of a series is > 1, then it diverges, if it is < 1, then it converges

Theorem 1.2.2 nth term test
If the limit of $\lim_{n\to\infty} a_n$ is equal to zero then the series converges.

Theorem 1.2.3 The Ratio Test
If the limit of $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ is greater than 1 the series will diverge, if it is less than 1 it will converge and if it is equal to 0 then another test is needed to be used.

Theorem 1.2.4 The integral test
Suppos f is continuous, positive, and decreasing on $[1, \text{inf})$, and let $a_n = f(n)$.

1. If $\int_n^{\infty} f(n)dn$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.
2. If $\int_n^{\infty} f(n)dn$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 1.2.5 p-series test for convergence

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$

Theorem 1.2.6 The comparison test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.
Then

1. If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.
2. If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Theorem 1.2.7 The alternating series test

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots (a_n > 0)$$

satisfies

(a) $a_{n+1} \leq a_n$ for all n, and

(b) $\lim_{n\to\infty} a_n = 0$,

then the series converges.

Theorem 1.2.8 Alternating series estimation

For any alternating series that converges, where S is the sum of that converging alternating series, then

$$|S - S_n| \leq a_{n+1}$$

where S_n is the sum of the first n terms in the series.

Definition 1.2.1: Interval of convergence

The interval of convergence of a series is the set of values of x for which the series converges.

Theorem 1.2.9 Finding the radius of convergence

The radius of convergence of a series is the largest value of x for which the series converges.
The radius of convergence is given by the formula: $|\frac{a_{n+1}}{a_n}| < 1$ where the value for x is the radius.
ex: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has a radius of convergence of $\frac{1}{2}$

1.3 Convergence conditionally/absolutely

Definition 1.3.1

A series $\sum_{n=1}^{\infty} a_n$ is called

- absolutely convergent if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty}|a_n|$ both converge.
- conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges.

Chapter 2

Approximating Function Using Series

2.1 Taylor Series

Definition 2.1.1: Taylor Polynomial

Taylor Polynomials are polynomials that approximate a function at a point a based on the function's derivatives at a.
ex: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
ex: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$
ex: $P_n(x) = T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Definition 2.1.2: Maclaurin polynomials

Maclaurin polynomials are Taylor polynomials that are centered at $a = 0$.
ex: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$
ex: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$
ex: $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$

Definition 2.1.3: Binomial Series Expansion

The binomial series expansion is a way to expand a binomial to a power.
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
ex: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$
ex: $(1-x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$

2.1.1 Taylor Series to memorize

- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
  - General term: $(-1)^n \frac{x^{2n+1}}{(2n+1)!}$
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
  - General term: $(-1)^n \frac{x^{2n}}{(2n)!}$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
  - General term: $\frac{x^n}{n!}$

- $\frac{1}{1+x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$
  - General term: $(-1)^{n-1}\frac{x^n}{n!}$

2.2 New series by substitution

Using known series you can find new series by substituting in a new variable.
ex: we already know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
thus we can find e^{x^2} by substituting x^2 for x in the series for $e^x \ldots$
$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots$

2.3 The error in Taylor Polynomial Approximation

Definition 2.3.1: The error in Taylor Polynomial Approximation

$E_n(x) = f(x) - T_n(x)$
$|E_n(x)| \leq |\frac{M(x-a)^{n+1}}{(n+1)!}|$
M is the maximum absolute value of the $n+1$ derivative of $f(x)$ on $[a, b]$

2.4 The convergence of Taylor Series

Definition 2.4.1: The convergence of Taylor Series

ex: cos(x) - $E_n(x) = \cos x - P_n(x) = \cos x - (1 - \frac{x^2}{2!} + \ldots (-1)^{\frac{n}{2}} \frac{x^n}{n!})$,
giving: $\cos x = 1 - \frac{x^2}{2!} + \ldots (-1)^{\frac{n}{2}} \frac{x^n}{n!} + E_n(x)$
Thus, for the Taylor series to converge to cox x, we must have $E_n(x) \to 0 as n \to \infty$.