CSE 215
Class Notes

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Contents

Chapter 1

Logic

Note:-

Why formalize logic?

- to remove ambiguity
- to represent facts on a computer and use it for proving, proof-checking, etc
- to detect unsound reasoning in arguments

Mathematical logic is a tool for dealing with formal reasoning.
Logic does: assess if an argument is valid/invalid
Logic does not directly: assess the truth of atomic statements.

1.1 Propositional logic

Definition 1.1.1: Propositional logic

Propositional logic is the study of:

- the structure (syntax) and
- the meaning (semantics) of (simple and complex) propositions.

A proposition is a sentence that is either true or false, but not both.

Example 1.1.1 (Simple propositions)

- P: It is raining.
- Q: The sun is shining.
- R: $2 + 2 = 4$.

Example 1.1.2 (Complex propositions)

- $P \wedge Q$: It is raining and the sun is shining.
- $P \vee Q$: It is raining or the sun is shining.
- $\neg P$: It is not raining.

- $P \implies Q$: If it is raining, then the sun is shining.
- $P \iff Q$: It is raining if and only if the sun is shining.

Definition 1.1.2: Operators and operands

- **Operators** are symbols that combine propositions to form complex propositions.
- **Operands** are the propositions that are combined by operators.

ex: $P \wedge Q$ where $\wedge$ is the operator and P and Q are the operands.

1.2 Quiz #1

Question 1: Table for implication

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Vacuously true when P is false.

Question 2: Complete rule for communativity as it relates to AND

$P \wedge Q \equiv Q \wedge P$

Question 3: Complete rule for absorbtion (both cases) as it relates to AND and OR

$P \wedge (P \vee Q) \equiv P$
$P \vee (P \wedge Q) \equiv P$

Got this wrong. Basically, these expressions above show Q become irrelevant when combined with P.

Question 4: What is the inverse of p *implies* q? Is it logically equivalent?

$\neg P \implies \neg Q$
No, it is not logically equivalent.

Question 5: What is the converse of p *implies* q? Is it logically equivalent?

$Q \implies P$
No, it is not logically equivalent.

Question 6: What is the contrapositive of p *implies* q? Is it logically equivalent?

$\neg Q \implies \neg P$
Yes, it is logically equivalent.

Question 7: What are De Morgan's Law?

$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

1.3 Logical Operators

Definition 1.3.1: Conditional

The **conditional** operator is denoted by $\implies$ and is read as "implies" or "if-then".
The proposition $P \implies Q$ is true when P is false or when Q is true.

Definition 1.3.2: Biconditional

The **biconditional** operator is denoted by $\iff$ and is read as "if and only if".
The proposition $P \iff Q$ is true when P and Q have the same truth value.

Definition 1.3.3: Conjunction

The **conjunction** operator is denoted by $\land$ and is read as "and".
The proposition $P \land Q$ is true when both P and Q are true.

Definition 1.3.4: Disjunction

The **disjunction** operator is denoted by $\lor$ and is read as "or".
The proposition $P \lor Q$ is true when at least one of P and Q is true.

Definition 1.3.5: Negation

The **negation** operator is denoted by $\neg$ and is read as "not".
The proposition $\neg P$ is true when P is false.

1.4 Truth Tables

Definition 1.4.1: Truth table

A **truth table** is a table that shows the truth value of a complex proposition for all possible truth values of its operands.

Example 1.4.1 (Truth table for conjunction)

P	Q	$P \land Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.4.2 (Truth table for disjunction)

P	Q	$P \lor Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.4.3 (Truth table for biconditional)

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

1.5 Logical Equivalence

Definition 1.5.1: Logical equivalence

Two propositions are **logically equivalent** if they have the same truth value for all possible truth values of their operands.

1.6 Logical Rules

Definition 1.6.1: Communativity

$P \land Q \equiv Q \land P$
$P \lor Q \equiv Q \lor P$

Definition 1.6.2: Associativity

$(P \land Q) \land R \equiv P \land (Q \land R)$
$(P \lor Q) \lor R \equiv P \lor (Q \lor R)$

Definition 1.6.3: Distributivity

$P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$
$P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$

Definition 1.6.4: Identity

$P \land T \equiv P$
$P \lor F \equiv P$

Definition 1.6.5: Domination

$P \land F \equiv F$
$P \lor T \equiv T$

Definition 1.6.6: Double negation

$\neg\neg P \equiv P$

Definition 1.6.7: Idempotent

$P \land P \equiv P$
$P \lor P \equiv P$

Definition 1.6.8: Absorption

$P \land (P \lor Q) \equiv P$
$P \lor (P \land Q) \equiv P$

Definition 1.6.9: Negation

$P \land \neg P \equiv F$
$P \lor \neg P \equiv T$

Definition 1.6.10: De Morgan's

$\neg(P \land Q) \equiv \neg P \lor \neg Q$
$\neg(P \lor Q) \equiv \neg P \land \neg Q$

1.7 Necessary and Sufficient Conditions

Definition 1.7.1: Necessary condition

A proposition P is a **necessary condition** for a proposition Q if Q is true only when P is true.

Definition 1.7.2: Sufficient condition

A proposition P is a **sufficient condition** for a proposition Q if P being true guarantees that Q is true.

1.8 Tautologies

Definition 1.8.1: Tautology

A proposition that is always true, regardless of the truth values of its operands, is called a **tautology**.

Example 1.8.1 (Tautologies)
A propositional formula p is logically equivalent to q if and only if p $\iff$ q is a tautology.

Chapter 2

Logical Arguments

Definition 2.0.1: Argument

An arguement (form) is a (finite) sequence of statements (forms), usually written as follows:

$$\begin{aligned} &P_1 \\ &\vdots \\ &P_n \\ \because\ &q \end{aligned}$$

We call $P_1, \ldots, P_n$ the **premises** and q the **conclusion**.
"p_1, p_2, p_n therefore q" OR "From premises p_1, p_2, p_n, we can conclude q."

Note:-

Arguement forms are also called inference rules.
An inference rule is said to be valid, or (logically sound), if it is the case that, for each truth valuation, if all the premises are true, tehn the conclusion is also true.
Theorem: An argument is valid if and only if the corresponding argument form is a tautology.

Note:-

If not all the premises are true, then the argument is vacuously true; it is neither valid nor invalid. Validity can only be assessed when all premises are true and determined whether the conclusion is true or false.

Definition 2.0.2: Syllogism

A syllogism is a form of reasoning in which a conclusion is drawn from two given or assumed propositions (premises).

2.0.1 Determining the validity or invalidity of an argument

Theorem 2.0.1 Formula to determine validity

1. Identify the premises and conclusion of the argument form.
2. Construct the truth table showing the truth values for all the premises and the conclusion.
3. A row of the table in which **all the premises are true is called a critical row**. *If there is a critical row in which the conclusion is false, then the arguement is invalid.* If the conclusion in every critical row is true, then the argument form is valid.

4. If there are no critical rows, then the argument is **vacuously true**.

2.1 Modus Ponens and Modus Tollens

Definition 2.1.1: Modus Ponens

If $P \implies Q$ and P are true, then Q is true. It is a method of affirming.

$$\begin{aligned} P &\implies Q \\ &P \\ &\therefore Q \end{aligned}$$

P	Q	$P \implies Q$	P	Q
T	T	T	T	T
T	F	F	T	
F	T	T	F	
F	F	T	F	

Example 2.1.1 (Modus Ponens)

- If the sum of the digits of 371487 is divisible by 3, then 371487 is divisible by 3.
- The sum of the digits of 371487 is divisible by 3.
- $\therefore$ 371487 is divisible by 3.

Definition 2.1.2: Modus Tollens

If $P \implies Q$ and $\neg Q$ are true, then $\neg P$ is true. It is a method of denying.

$$\begin{aligned} P &\implies Q \\ &\neg Q \\ &\therefore \neg P \end{aligned}$$

P	Q	$P \implies Q$	$\neg Q$	$\neg P$
T	T	T	F	T
T	F	F	T	F
F	T	T	F	
F	F	T	T	

Note:-

Modus Tollens is valid because:

- Modus ponens is valid and the fact that a conditional statement is logically equivalent to its contrapositive. Or
- It can be established formally using a truth table.

Example 2.1.2 (Modus Tollens)

- If Zeus is human, then Zeus is mortal.
- Zeus is not mortal.
- $\therefore$ Zeus is not human.

2.1.1 Generalization

Definition 2.1.3: Generalization

$$\begin{aligned} P \\ \therefore P \vee Q \end{aligned}$$

And

$$\begin{aligned} Q \\ \therefore P \vee Q \end{aligned}$$

EX: Anton is a junior. Therefore, Anton is a junior or Anton is a senior.

2.1.2 Specialization

Definition 2.1.4: Specialization

$$\begin{aligned} P \wedge Q \\ \therefore P \end{aligned}$$

And

$$\begin{aligned} P \wedge Q \\ \therefore Q \end{aligned}$$

EX: If it is raining and the sun is shining, then it is raining.

2.1.3 Elimination

Definition 2.1.5: Elimination

$$\begin{aligned} P \vee Q \\ \neg P \\ \therefore Q \end{aligned}$$

And

$$\begin{aligned} P \vee Q \\ \neg Q \\ \therefore P \end{aligned}$$

EX: If it is raining or the sun is shining, and it is not raining, then the sun is shining

2.1.4 Transitivity

Definition 2.1.6: Transitivity

$$\begin{aligned} P &\implies Q \\ Q &\implies R \\ \therefore P &\implies R \end{aligned}$$

EX: If it is raining, then the sun is shining. If the sun is shining, then the grass is wet. Therefore, if it is raining, then the grass is wet.

Chapter 3

Proof Techniques

3.1 Proof by Contradiction

Definition 3.1.1: Proof by contradiction

To prove a proposition P, assume that $\neg P$ is true and show that this assumption leads to a contradiction.