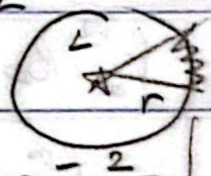
HW 16, 17

16. L_ν ; $Q_{\nu,abs} \sim \kappa_{\nu,abs} \sim \nu^{\beta} \sim \lambda^{-\beta}$; $\beta \sim 2$

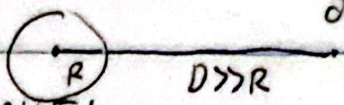


$\frac{L}{4\pi r^2}$

$u = \frac{1}{c} \oint I \, d\Omega \sim F \sim u_{ISRF} \sim \frac{L_\nu}{r^2}$

$T_d \sim u_{ISRF}^{1/(4+\beta)} = \left(\frac{L_\nu}{r^2}\right)^{1/(4+\beta)} = \boxed{L_\nu^{\frac{1}{4+\beta}} \, r^{-\frac{2}{4+\beta}}} = L_\nu^{\frac{1}{6}} \, r^{-\frac{1}{3}} \; (\beta \sim 2)$

17. $F_\nu \sim \nu^\alpha$; $\alpha = \frac{\partial \ln(F_\nu)}{\partial \ln(\nu)} = \frac{\nu}{F_\nu}\frac{\partial F_\nu}{\partial \nu} \neq \alpha_{abs}$



$\rho_d; T_d$

a) $F_\nu = \overbrace{\frac{4\pi}{3D^2} R^3}^{\text{ignore}} \rho_d K_\nu B_\nu(T_d)$ (HW15 a))

$K_{\nu,abs}(\nu) = K_{\nu,abs,0}\left(\frac{\nu}{\nu_0}\right)^\beta \sim \nu^\beta$

$$F_\nu \sim \nu^\alpha \Rightarrow \alpha = \frac{\nu}{F_\nu}\frac{\partial F_\nu}{\partial \nu} = \frac{\nu}{\nu^\beta B_\nu(T_d)}\frac{\partial}{\partial \nu}\Big(\underbrace{\nu^{\beta+3}\frac{2h}{c^2}\cdot\frac{1}{e^{h\nu/k_BT}-1}}_{C}\Big)$$

$$\frac{\partial}{\partial \nu}C = (\beta+3)\nu^{\beta+2}\frac{2h}{c^2}\frac{1}{e^{h\nu/k_BT}-1} + \nu^{\beta+3}\frac{2h}{c}(-1)(e^{h\nu/k_BT_d}-1)^{-2}\frac{h}{k_BT_d}\cdot e^{h\nu/k_BT_d}$$

$$\Rightarrow \alpha = \frac{\nu}{\nu^{\beta+3}}(\beta+3)\nu^{\beta+2} - \frac{\nu^{\beta+4}\cdot 2h^2\cdot e^{h\nu/k_BT_d}}{\nu^\beta B_\nu(T_d)\cdot(e^{h\nu/k_BT_d}-1)^2\cdot k_BT_d}$$

$$= \frac{\nu}{\nu^{\beta+3}}(\beta+3)\nu^{\beta+2} - \frac{\nu^4\cdot 2h^2\cdot e^{h\nu/k_BT_d}c^2(e^{h\nu/k_BT_d}-1)}{2h\nu^3\cdot(e^{h\nu/k_BT_d}-1)^2\cdot k_BT_d}$$

$$= \frac{\nu}{\nu^{\beta+3}}(\beta+3)\nu^{\beta+2} - \frac{\nu e^{h\nu/k_BT_d}c^2h}{(e^{h\nu/k_BT_d}-1)k_BT_d}$$ Let's call $\frac{h\nu}{k_BT_d} = x$

$$= (\beta+3) - \frac{\nu e^x c^2 h}{(e^x-1)k_BT_d} \Rightarrow \alpha \sim (\beta+3) - \frac{e^x}{(e^x-1)}x$$

b) Rayleigh - Jeans limit: $\frac{h\nu}{k_BT} \ll 1 \Rightarrow e^{h\nu/k_BT} \approx 1 + \frac{h\nu}{k_BT}$

$$\Rightarrow \alpha = (\beta+3) - \frac{x+1}{x}x = (\beta+3) - (x+1) \quad (x \ll 1 \Rightarrow \text{negligible})$$
$$= \beta+3-1 = \beta+2$$

$\beta > 0 \Rightarrow$ Compare to BB emission: steeper because $\alpha > 2$

$\Rightarrow F_\nu \sim \nu^{\alpha>2} \Rightarrow$ grows steeper (faster)

$\Rightarrow$ Dust is even more efficient in emitting specific wavelengths than even black body.