HOMEWORK 8

5.2 #4. $\begin{bmatrix} 5 & -5 \\ -2 & 3 \end{bmatrix} = A$

$\det[A-\lambda I] = 0 = (5-\lambda)(3-\lambda) - (-2)(-5) = 15 - 5\lambda - 3\lambda + \lambda^2 - 10$

$\Rightarrow \lambda^2 - 8\lambda + 5 = 0 \Rightarrow \lambda = \dfrac{8 \pm \sqrt{64 - 4(1)(5)}}{2} = \dfrac{8 \pm 2\sqrt{11}}{2} = 4 \pm \sqrt{11}$

$\Rightarrow$ The characteristic polynomial of A is $\lambda^2 - 8\lambda + 5$ & the e-vals are $4+\sqrt{11}$ & $4-\sqrt{11}$.

5.2 #12. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix} = B$

$\Rightarrow \det B = \det \begin{bmatrix} 1 & 0 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 6 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

$\Rightarrow \det[B-\lambda I] = 0 = (1-\lambda)(6-\lambda)(4-\lambda) = (6 - \lambda - 6\lambda + \lambda^2)(4-\lambda)$

$\Leftrightarrow 24 - 6\lambda - 4\lambda + \lambda^2 - 24\lambda + 24\lambda^2 + 4\lambda^2 - \lambda^3 = -\lambda^3 + 29\lambda^2 - 34\lambda + 24 = 0$

$\Rightarrow$ The characteristic polynomial of B is $-\lambda^3 + 29\lambda^2 - 34\lambda + 24 = 0$

5.2 #18. $A = \begin{bmatrix} 6 & 3 & 9 & -5 \\ 0 & 9 & h & 2 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 7 \end{bmatrix}, \lambda = 6$

$\Rightarrow A - \lambda I = \begin{bmatrix} 6 & 3 & 9 & -5 \\ 0 & 9 & h & 2 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 7 \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 9 & -5 \\ 0 & 3 & h & 2 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{R_2 \to R_2 - R_1}{\sim}$

$\begin{bmatrix} 0 & 3 & 9 & -5 \\ 0 & 0 & h-9 & 7 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Since $\lambda = 6$ is two-dimensional $\Rightarrow$ matrix also has 2 pivot columns $\Rightarrow h - 9 = 0$ for matrix to have 2 pivot entries $\Rightarrow h = 9$

5.3 #4 (Note: write as $A = PDP^{-1}$) ↪ (Am I doing the right problem?)

$$\underset{A}{\begin{bmatrix} 15 & -36 \\ 6 & -15 \end{bmatrix}} = \underset{P}{\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}}, \text{ find } A^k$$

$$\Rightarrow A^k = PD^kP^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}^k \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

$$\Rightarrow A^k = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-3)^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & (-1)^k \cdot 3^{k+1} \\ 3^k & -2 \cdot 3^k \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 3^k + 3^{k+1} & (-1)^k 2 \cdot 3^{k+1} - 2 \cdot 3^{k+1} \\ 3^k + 3^k & (-1)^k 3^{k+1} - 2 \cdot 3^k \end{bmatrix} = 3^k \begin{bmatrix} 2+3 & (-1)^k \cdot 2 \cdot 3 - 2 \cdot 3 \\ 1+1 & (-1)^k 3 - 2 \end{bmatrix}$$

$$= 3^k \begin{bmatrix} 5 & (-1)^k 6 - 6 \\ 2 & (-1)^k 3 - 2 \end{bmatrix} \Rightarrow A^k = \begin{cases} 3^k \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} & (\text{even } k) \\ 3^k \begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix} & (\text{odd } k) \end{cases}$$

$$\Rightarrow A^k = PD^kP^{-1} \Rightarrow \begin{cases} 3^k \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}^k \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} & (\text{even } k) \\ 3^k \begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}^k \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} & (\text{odd } k) \end{cases}$$

5.3 #16 $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$ $\lambda = 2, 2, 1$

$$\lambda = 2 \Rightarrow A - 2I = \begin{bmatrix} 0-2 & -4 & -6 \\ -1 & 0-2 & -3 \\ 1 & 2 & 5-2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix} = 0$$

$$\Rightarrow \text{Augmented matrix: } \begin{bmatrix} -2 & -4 & -6 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix} \underset{R_3 \to 2R_3 + R_1}{\overset{R_2 \to 2R_2 - R_1}{\sim}} \begin{bmatrix} -2 & -4 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1/-2 \to R_1}{\sim}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 + 2x_2 + 3x_3 = 0 \Rightarrow \left.\begin{matrix} x_1 = -2x_2 - 3x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{matrix}\right\} \Rightarrow x = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ The e-vects basis for the e-space when $\lambda = 2$: $\{v_1, v_2\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\lambda = 1 \Rightarrow A - I = \begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix} = 0 \Rightarrow \text{Augmented: } \begin{bmatrix} -1 & -4 & -6 & 0 \\ -1 & -1 & -3 & 0 \\ 1 & 2 & 4 & 0 \end{bmatrix} \underset{R_3 \to R_3 + R_1}{\overset{R_2 \to R_2 - R_1}{\sim}}$$

$$\begin{bmatrix} -1 & -4 & -6 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix} \underset{R_3 \to -R_3/2}{\overset{R_2 \to R_2/3}{\sim}} \begin{bmatrix} -1 & -4 & -6 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \overset{R_3 \to R_3 - R_2}{\sim} \begin{bmatrix} -1 & -4 & -6 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} x_1 = -4x_2 - 6x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x_1 = -2x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{matrix}\right\} \Rightarrow x = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ The e-vects basis for e-space when $\lambda = 1$: $v_3 = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$\Rightarrow P = [v_1 \; v_2 \; v_3] = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\lambda = 2, 2, 1)$$

$$P^{-1} = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 2 & 4 \\ -1 & -2 & -3 \end{bmatrix} \Rightarrow A = PDP^{-1} = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -3 \\ 1 & 2 & 4 \\ -1 & -2 & -3 \end{bmatrix}$$