HOMEWORK 11 (Yeah! At Long Last!)
I survived!

1. $\text{Col}\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -4 & -3 \end{bmatrix}$ with $\underset{u_1}{\shortparallel}$ $\underset{u_2}{\shortparallel}$ $\underset{u_3}{\shortparallel}$ under the columns

Let $v_1 = u_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

- $$V_2 = u_2 - \frac{u_2 \circ V_2}{\|v_1\|^2} V_1 = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix} - \frac{-6-24-4}{1^2+3^2+1^2}\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix} - \frac{-34}{11}\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 - 34/11 \\ -8 + 102/11 \\ -4 + 34/11 \end{bmatrix}$$

$$\Rightarrow V_2 = \begin{bmatrix} 32/11 \\ 14/11 \\ -10/11 \end{bmatrix} \Rightarrow \|V_2\|^2 = \left(\frac{32}{11}\right)^2 + \left(\frac{14}{11}\right)^2 + \left(\frac{-16}{11}\right)^2 = \frac{120}{11}$$

- $$V_3 = u_3 - \frac{u_3 \circ V_1}{\|v_1\|^2} V_1 - \frac{u_3 \circ V_2}{\|V_2\|^2} V_2 = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} - \cancelto{0}{\frac{-6+9-3}{11}\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}} - \frac{\frac{192+42+30}{11}}{120/11}$$

$$\begin{bmatrix} 32/11 \\ 14/11 \\ -10/11 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} - \frac{11}{5}\begin{bmatrix} 32/11 \\ 14/11 \\ -10/11 \end{bmatrix} = \begin{bmatrix} 6 - 32/5 \\ 3 - 14/5 \\ -3 + 2 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 1/5 \\ -1 \end{bmatrix}$$

$\Rightarrow$ The orthogonal basis for Col $\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -4 & -3 \end{bmatrix}$ is $\{v_1, v_2, v_3\}$

$$= \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 32/11 \\ 14/11 \\ -10/11 \end{bmatrix}, \begin{bmatrix} -2/5 \\ 1/5 \\ -1 \end{bmatrix} \right\}$$

2. $\langle f, g \rangle = \int_{-1}^{1} fg \, dx$; standard basis for $\mathbb{P}_2 = \{1, t, t^2\}$

$\Rightarrow v_1 = 1$

$$\Rightarrow v_2 = t - \frac{\langle 1, t \rangle}{\langle t, t \rangle} t = t - \frac{\int_{-1}^{1} 1 \cdot t \, dt}{\int_{-1}^{1} t^2 dt} = t - \frac{\cancelto{0}{\frac{1}{2} - \frac{1}{2}}}{\frac{1}{3} + \frac{1}{3}} = t$$

$$\Rightarrow v_3 = t^2 - \frac{\langle t^2, t \rangle}{\langle t, t \rangle} t - \frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle} 1$$

$$= t^2 - \frac{\cancelto{0}{\int_{-1}^{1} t^3 dt}}{\langle t, t \rangle} - \frac{1/3 + 1/3}{\int_{-1}^{1} 1 dt} = t^2 - \frac{2}{3 \cdot 2} = t^2 - \frac{1}{3}$$

$\Rightarrow$ Orthogonal basis for $P_2 = \{v_1, v_2, v_3\} = \{1, t, t^2 - 1/3\}$

3. $\underbrace{\begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}}_{b} \Rightarrow (A^T A) x = A^T b$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} x = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 11 & 14 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 11 & 14 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 8 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \hat{x_1} = 1, \ \hat{x_2} = 1$$

$\Rightarrow$ Least square error: $\|b - \hat{b}\| = \|b - A\hat{x}\|$

$$A\hat{x} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \|b - \hat{b}\| = \left\| \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$W = \text{Span}\left\{ \begin{bmatrix} k \\ l \end{bmatrix} \right\}$

4. $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\lambda = \|z_1\|^2 + \|z_2\|^2$ where $z_i = b_i - \text{proj}_W b_i$

a) $\text{proj}_W b_i = \frac{b_i \cdot w}{\|w\|^2} w$ $\quad \|w\|^2 = k^2 + l^2$

$$\Rightarrow z_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{k}{k^2+l^2}\begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 1 - k^2/(k^2+l^2) \\ -kl/(k^2+l^2) \end{bmatrix} = \begin{bmatrix} l^2/(k^2+l^2) \\ -kl/(k^2+l^2) \end{bmatrix}$$

$$\Rightarrow z_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{k+l}{k^2+l^2}\begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 1 - k^2+lk/k^2+l^2 \\ 1 - kl+l^2/k^2+l^2 \end{bmatrix} = \begin{bmatrix} (l^2-lk)/(k^2+l^2) \\ (k^2-kl)/(k^2+l^2) \end{bmatrix}$$

$$\Rightarrow \lambda = \|z_1\|^2 + \|z_2\|^2 = \left(\frac{l^2}{k^2+l^2}\right)^2 + \left(\frac{kl}{k^2+l^2}\right)^2 + \left(\frac{l^2-lk}{k^2+l^2}\right)^2 + \left(\frac{k^2-kl}{k^2+l^2}\right)^2$$

$$= \frac{l^2l^2 + k^2l^2}{(k^2+l^2)^2} + \frac{l^2(l-k)^2 + k^2(k-l)^2}{(k^2+l^2)^2} = \frac{l^2(k^2+l^2)}{(k^2+l^2)^2} + \frac{(k^2+l^2)(k-l)^2}{(k^2+l^2)^2}$$

$$= \frac{l^2 + (k-l)^2}{k^2+l^2}$$

b) $\lambda = \frac{l^2+(k-l)^2}{k^2+l^2} = \frac{\left[\frac{l^2+(k-l)^2}{l^2}\right]}{\left[\frac{k^2+l^2}{l^2}\right]} = \frac{1 + (\frac{k}{l}-1)^2}{(\frac{k}{l})^2+1}$ Let $x = \frac{k}{l}$

$\Rightarrow \lambda = \frac{1+(x-1)^2}{x^2+1}$. λ min/max when $\frac{d\lambda}{dx} = 0$

$$\Rightarrow \frac{d\lambda}{dx} = \frac{d}{dx}\left[\frac{1+(x-1)^2}{x^2+1}\right] = \frac{2(x-1)(x^2+1) - [1+(x-1)^2]\cdot 2x}{(x^2+1)^2} = 0$$

$\Rightarrow (x-1)(x^2+1) - x[1+(x-1)^2] = 0$

$\Leftrightarrow x^3 + \cancel{x} - x^2 - 1 - \cancel{x} - x(x-1)^2 = \cancel{x^3} - x^2 - 1 - \cancel{x^3} + 2x^2 - x = 0$

$\Leftrightarrow x^2 - x - 1 = x^2 - 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{5}{4} = (x - 1/2)^2 - \frac{5}{4} = 0$

$$\Leftrightarrow (x-1/2)^2 - \left(\sqrt{\frac{5}{4}}\right)^2 = \left(\overset{x_1}{x - \frac{1}{2} - \sqrt{\frac{5}{4}}}\right)\left(\overset{x_2}{x - \frac{1}{2} + \sqrt{\frac{5}{4}}}\right) = 0$$

$\Rightarrow x_1 = \frac{1+\sqrt{5}}{2}$ & $x_2 = \frac{1-\sqrt{5}}{2}$

$(-\infty, x_1)$	(x_1, x_2)	(x_2, ∞)
$(+)$	$(-)$	$(+)$

$d\lambda/dx$ signs

$\Rightarrow \lambda \uparrow$ at $(-\infty, x_1)$, $\lambda \downarrow$ at (x_1, x_2), $\lambda \uparrow$ at $(x_2, \infty) \Rightarrow x_2$ is local min

$\Rightarrow x = \frac{1-\sqrt{5}}{2}$ for minimizing $\lambda \Rightarrow \frac{K}{\ell} = \frac{1-\sqrt{5}}{2} \Rightarrow K = \frac{1-\sqrt{5}}{2}\ell$