HOMEWORK 10

1. $\vec{u_1} = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}$, and $\vec{u_3} = \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}$

$\vec{u_1} \cdot \vec{u_2} = -20 - 4 + 0 + 24 = 0$, $\vec{u_2} \cdot \vec{u_3} = -12 + 3 - 15 - 8 = -32$

$\Rightarrow \vec{u_1} \perp \vec{u_2}$ $\Rightarrow \vec{u_2}$ is not orthogonal with $\vec{u_3}$

$\vec{u_1} \cdot \vec{u_3} = 15 - 12 + 0 - 3 = 0 \Rightarrow \vec{u_1} \perp \vec{u_3}$

2. $\vec{y} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \hat{y} + \vec{z}$ where $\hat{y} \in W = \text{Span}\left\{\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right\}$ and $z \in W^{\perp}$

Let $\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \vec{u} \Rightarrow \hat{y} = \dfrac{\vec{y} \cdot \vec{u}}{\|u\|^2} \vec{u} = \dfrac{12 - 2}{9 + 4} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 30/13 \\ -20/13 \end{bmatrix} = \dfrac{10}{13} \vec{u}$

$\Rightarrow \vec{z} = \vec{y} - \dfrac{10}{13} \vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 30/13 \\ -20/13 \end{bmatrix} = \begin{bmatrix} 22/13 \\ 33/13 \end{bmatrix}$

So $\vec{y} = \begin{bmatrix} 30/13 \\ -20/13 \end{bmatrix} + \begin{bmatrix} 22/13 \\ 33/13 \end{bmatrix}$ Check: $\hat{y} \cdot \vec{z} = 1/13\,(660 - 660) = 0$

3. Let $S = \left\{ \underbrace{\begin{bmatrix} -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ -8 \\ 6 \\ 1 \end{bmatrix}}_{u_2}, \underbrace{\begin{bmatrix} -4 \\ 0 \\ -1 \\ 6 \end{bmatrix}}_{u_3} \right\}$

a) $\left.\begin{aligned} \vec{u_1} \cdot \vec{u_2} &= 0 - 24 + 24 + 0 = 0 \\ \vec{u_2} \cdot \vec{u_3} &= 0 + 0 - 6 + 6 = 0 \\ \vec{u_3} \cdot \vec{u_1} &= 4 + 0 - 4 + 0 = 0 \end{aligned}\right\} \Rightarrow$ S is an orthogonal set

b) Using Gram-schmidt method:

$\vec{v_1} = \vec{u_1}$, $\vec{v_2} = \vec{u_2} - \dfrac{\vec{u_2} \cdot \vec{v_1}}{\|v_1\|^2} \vec{v_1} = \vec{u_2} = \begin{bmatrix} 0 \\ -8 \\ 6 \\ 1 \end{bmatrix}$

$\vec{v_3} = \vec{u_3} - \dfrac{\vec{u_3} \cdot \vec{v_1}}{\|v_1\|^2} \vec{v_1} - \dfrac{\vec{u_3} \cdot \vec{v_2}}{\|v_2\|^2} \vec{v_2} = \vec{u_3} = \begin{bmatrix} -4 \\ 0 \\ -1 \\ 6 \end{bmatrix}$

$\Rightarrow$ The orthogonal basis for $\mathbb{R}^4$ contains S is $\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -8 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -1 \\ 6 \end{bmatrix} \right\}$

c) $\vec{v_1} = \vec{u_1} = \dfrac{\vec{u_1}}{\|\vec{u_1}\|} = \dfrac{1}{\sqrt{26}} \begin{bmatrix} -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}$ $\vec{v_3} = \vec{u_3} = \dfrac{1}{\sqrt{53}} \begin{bmatrix} -4 \\ 0 \\ -1 \\ 6 \end{bmatrix}$

$\vec{v_2} = \vec{u_2} = \dfrac{\vec{u_2}}{\|\vec{u_2}\|} = \dfrac{1}{\sqrt{101}} \begin{bmatrix} 0 \\ -8 \\ 6 \\ 1 \end{bmatrix}$

$\Rightarrow$ The orthonormal basis of $W = \{v_1, v_2, v_3\} = \left\{ \dfrac{1}{\sqrt{26}} \begin{bmatrix} -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \dfrac{1}{\sqrt{101}} \begin{bmatrix} 0 \\ -8 \\ 6 \\ 1 \end{bmatrix}, \dfrac{1}{\sqrt{53}} \begin{bmatrix} -4 \\ 0 \\ -1 \\ 6 \end{bmatrix} \right\}$

$$d)\ W = \text{Span}(S) = \text{Span}\left\{ \underbrace{\begin{bmatrix}-1\\3\\4\\0\end{bmatrix}}_{\vec{u_1}}, \underbrace{\begin{bmatrix}0\\-8\\6\\1\end{bmatrix}}_{\vec{u_2}}, \underbrace{\begin{bmatrix}-4\\0\\-1\\6\end{bmatrix}}_{\vec{u_3}} \right\}, \text{ find distance to}$$

$$y = \begin{bmatrix}5\\-1\\3\\2\end{bmatrix}$$

$$\hat{y} = \frac{\vec{y}\cdot\vec{u_1}}{\|\vec{u_1}\|^2}\vec{u_1} + \frac{\vec{y}\cdot\vec{u_2}}{\|\vec{u_2}\|^2}\vec{u_2} + \frac{\vec{y}\cdot\vec{u_3}}{\|\vec{u_3}\|^2}\vec{u_3} = \frac{-5-3+12}{1+9+16}\vec{u_1} + \frac{8+18+2}{64+36+1}\vec{u_2} + \frac{-20-3+12}{24+1+36}\vec{u_3}$$

$$= \frac{4}{26}\begin{bmatrix}-1\\3\\4\\0\end{bmatrix} + \frac{28}{101}\begin{bmatrix}0\\-8\\6\\1\end{bmatrix} - \frac{11}{53}\begin{bmatrix}-4\\0\\-1\\6\end{bmatrix}$$

$$= \begin{bmatrix}-4/26 + 0 + 44/53\\ 12/26 - 224/101 - 0\\ 16/26 + 168/101 + 11/53\\ 0 + 28/101 - 66/53\end{bmatrix} = \begin{bmatrix}466/689\\ -2306/1313\\ 173019/69589\\ -5182/5353\end{bmatrix}$$

$\Rightarrow$ The distance from W to y:

$$\|y-\hat{y}\| = \sqrt{\left\| \begin{bmatrix}5-466/689\\ -1+2306/1313\\ 3-173019/69589\\ 2+5182/5353\end{bmatrix} \right\|^2} = \sqrt{\left(\frac{2979}{689}\right)^2 + \left(\frac{993}{1313}\right)^2 + \left(\frac{35748}{69589}\right)^2 + \left(\frac{15888}{5353}\right)^2}$$

$\approx \underline{\underline{5.3}}$

Scanned with CamScanner

4. Have submitted Tutorial 2