HOMEWORK 6

4.4 #12. $B = \left\{ \underbrace{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}_{\vec{b_1}}, \underbrace{\begin{bmatrix} 6 \\ 7 \end{bmatrix}}_{\vec{b_2}} \right\}, \quad \vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\Rightarrow P_B = [\vec{b_1} \ \vec{b_2}]$

$= \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix} \Rightarrow P_B^{-1} = \dfrac{1}{4 \cdot 7 - 5 \cdot 6} \begin{bmatrix} 7 & -6 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} -7/2 & 3 \\ 5/2 & -2 \end{bmatrix}$

$\Rightarrow [\vec{x}]_B = P_B^{-1} \vec{x} = \begin{bmatrix} -7/2 & 3 \\ 5/2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

4.4. #14. $B = \{1-t^2, t-t^2, 2-2t+t^2\}$; $\vec{p}(t) = 3+t-6t^2$

$a(1-t^2) + b(t-t^2) + c(2-2t+t^2) = 3+t-6t^2$

$\Leftrightarrow a - at^2 + bt - bt^2 + 2c - 2ct + ct^2 = 3 + t - 6t^2$

$\Leftrightarrow (a+2c) + (b-2c)t + (-a-b+c)t^2 = 3 + t - 6t^2$

$$\Rightarrow \left.\begin{aligned} a + 2c &= 3 \\ b - 2c &= 1 \\ -a - b + c &= -6 \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}}_{x}$$

$$\Rightarrow A^{-1}: \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{R_3 \to R_3 + R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 1 \end{array}\right] \underset{R_3 \to R_3 + R_2}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \underset{\substack{R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 + 2R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -2 \\ 0 & 1 & 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$\Rightarrow A^{-1} = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow [\vec{x}]_B = A^{-1}\vec{x}$

$\Rightarrow \vec{x}_B = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix} = \begin{bmatrix} +7 \\ -3 \\ -2 \end{bmatrix}$

4.4. #32. $\{1-2t^2-t^3,\ t+2t^3,\ 1+t-2t^2\}$

$\begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$

$\Rightarrow$ Augmented matrix $= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -2 & 0 & -2 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \underset{R_3 \to R_3 + 2R_1}{\overset{R_4 \sim R_4 + R_1}{\sim}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$

$\overset{R_3 \leftrightarrow R_4}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_3 \to R_3 - R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_2 \to R_2 - R_3}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\overset{R_1 \to R_1 - R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ There's no free variables $\Rightarrow$ The polynomial's linear indep.

4.5 #10. $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$ ($\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, $\vec{v_4}$)

$\Rightarrow A = [\vec{v_1}\ \vec{v_2}\ \vec{v_3}\ \vec{v_4}]$

$= \begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \overset{R_2 \to R_2 + 2R_1}{\sim} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 1 & 5 & 7 \end{bmatrix}$

$\overset{R_2 \to R_2/2}{\sim} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 5 & 7 \end{bmatrix} \overset{R_3 \to R_3 - R_2}{\sim} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$\overset{R_3 \to R_3/4}{\sim} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{R_2 \to R_2 - 3R_3}{\sim} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\overset{R_1 \to R_1 + 3R_2}{\sim} \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{R_1 \to R_1 + 3R_3}{\sim} \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\rightarrow$ # of pivot columns $= 3 \Rightarrow$ Basis of the subspace $= \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$

$\Rightarrow$ Dimension of subspace $=$ # elements in the subspace $= \underline{3}$

3.1. #6.

$$A = \begin{vmatrix} 5 & -2 & 2 \\ 0 & 3 & -3 \\ 2 & -4 & 7 \end{vmatrix}$$

Down the 2^{nd} column

$$\Rightarrow \det A = -a_{12} \det A_{12} + a_{22} \det A_{22} - a_{32} \det A_{32}$$

$$\Rightarrow \det A = +2 \begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix} + 3 \begin{vmatrix} 5 & 2 \\ 2 & 7 \end{vmatrix} + 4 \begin{vmatrix} 5 & 2 \\ 0 & -3 \end{vmatrix}$$

$$= 2 \cdot 6 + 3 \cdot 31 + 4 \cdot (-15) = 12 + 93 - 60 = 45$$

3.1. #10.

$$A = \begin{vmatrix} 1 & -2 & 4 & 2 \\ \boxed{0 \quad 0 \quad 3 \quad 0} \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix} \leftarrow \text{three } 0!$$

$$\Rightarrow \det A = -0 + 0 - 3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix} + 0 = -3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix} \leftarrow \text{one } 0$$

$$\Rightarrow \det A = -3 \left(2 \underbrace{\begin{vmatrix} -2 & 2 \\ -4 & 5 \end{vmatrix}}_{=\,-2} - 0 + 5 \underbrace{\begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix}}_{=\,0} \right)$$

$$2(-2) + 5(0)$$

$$= -6(-2) = 12$$