305 HW6

Problem 2:

$$\left.\begin{aligned}
-x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 &= -1 \\
x_1 - 3x_2 + x_3 + 0x_4 + 0x_5 &= 0 \\
0x_1 + x_2 - 3x_3 + x_4 + 0x_5 &= 0 \\
0x_1 + 0x_2 + x_3 - 3x_4 + x_5 &= 0 \\
0x_1 + 0x_2 + 0x_3 - x_4 + x_5 &= 1
\end{aligned}\right\} \rightarrow \left[\begin{array}{ccccc|c}
-1 & 1 & 0 & 0 & 0 & -1 \\
1 & -3 & 1 & 0 & 0 & 0 \\
0 & 1 & -3 & 1 & 0 & 0 \\
0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 1
\end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c}
1 & -1 & 0 & 0 & 0 & 1 \\
0 & -2 & 1 & 0 & 0 & -1 \\
0 & 1 & -3 & 1 & 0 & 0 \\
0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 1
\end{array}\right] \sim \left[\begin{array}{ccccc|c}
1 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & -3 & 1 & 0 & 0 \\
0 & -2 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 1
\end{array}\right] \sim \left[\begin{array}{ccccc|c}
1 & 0 & -3 & 1 & 0 & 1 \\
0 & 1 & -3 & 1 & 0 & 0 \\
0 & 0 & -5 & 2 & 0 & -1 \\
0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 1
\end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c}
1 & 0 & -3 & 1 & 0 & 1 \\
0 & 1 & -3 & 1 & 0 & 0 \\
0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & -5 & 2 & 0 & -1 \\
0 & 0 & 0 & -1 & 1 & 1
\end{array}\right] \sim \left[\begin{array}{ccccc|c}
1 & 0 & 0 & -8 & 3 & 1 \\
0 & 1 & 0 & -8 & 3 & 0 \\
0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & -13 & 5 & -1 \\
0 & 0 & 0 & -1 & 1 & 0
\end{array}\right] \sim \left[\begin{array}{ccccc|c}
1 & 0 & 0 & -8 & 3 & 1 \\
0 & 1 & 0 & -8 & 3 & 0 \\
0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & -13 & 5 & -1
\end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & -5 & -7 \\
0 & 1 & 0 & 0 & -5 & -8 \\
0 & 0 & 1 & 0 & -2 & -3 \\
0 & 0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & -8 & -14
\end{array}\right] \sim \left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & 7/4 \\
0 & 1 & 0 & 0 & 0 & 3/4 \\
0 & 0 & 1 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 1 & 0 & 3/4 \\
0 & 0 & 0 & 0 & 1 & 7/4
\end{array}\right] \begin{array}{l} = x_1 \\ = x_2 \\ = x_3 \\ = x_4 \\ = x_5 \end{array}$$

Problem 3: $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u(1-u)$; $0<x<L$; B.C. $u(0,t)=1$ & $u(L,t)=0$

a) Backward Euler for $\frac{\partial u}{\partial t} \approx \frac{u_i^{j+1} - u_i^j}{\Delta t}$; Central for $\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta x)^2}$ (Lecture 15)

For semi-explicit: time deriv is evaluated at $j+1$ timestep

$\Rightarrow$ Plug in equation: $\frac{u_i^{n+1} - u_i^n}{\Delta t} = D\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} + u_i^{n+1}(1-u_i^{n+1})$

Unknowns on the leftside: (u_i^{j+1})

$$\frac{-D}{(\Delta x)^2} u_{i-1}^{j+1} + \left(1 + 2\frac{D}{(\Delta x)^2} + \Delta t\right) u_i^{j+1} - \frac{D}{(\Delta x)^2} u_{i+1}^{j+1} = u_i^j + \Delta t\, u_i^{j+1}(1-u_i^{j+1})$$

b) 5 notes $\Rightarrow i=0 \rightarrow i=4$. ($u_0^j = 1$ & $u_4^j = 0$ for B.C). $\Rightarrow$ Only solve for note $2 \rightarrow 4$

(next page)

• At $i=1$: $\underbrace{\frac{-D}{(\Delta x)^2}}_{A} \cdot \cancel{u_0^{j+1}} + \underbrace{\left(1+2\frac{D}{(\Delta x)^2} + \Delta t\right)}_{B} u_1^{j+1} - \frac{D}{(\Delta x)^2} u_2^{j+1} = u_1^j + \Delta t\, u_1^{j+1}(1-u_1^{j+1})$

Let's call $D/(\Delta x)^2 = A$ and $(1+2D/(\Delta x)^2+\Delta t) = B$ because I'm lazy.

• At $i=2$: $-A \cdot u_1^{j+1} + B u_2^{j+1} - A u_3^{j+1} = u_2^j + \Delta t\, u_2^{j+1}(1-u_2^{j+1})$

• At $i=3$: $-A \cdot u_2^{j+1} + B u_3^{j+1} - \cancel{A u_4}^{\,0} = u_3^j + \Delta t\, u_3^{j+1}(1-u_3^{j+1})$

So, the matrix would be:

c) $$\begin{bmatrix} B & -A & 0 \\ -A & B & -A \\ 0 & -A & B \end{bmatrix} \begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ u_3^{j+1} \end{bmatrix} = \begin{bmatrix} u_1^j + \Delta t \cdot u_1^{j+1}(1-u_1^{j+1}) \\ u_2^j + \Delta t \cdot u_2^{j+1}(1-u_2^{j+1}) \\ u_3^j + \Delta t\, u_3^{j+1}(1-u_3^{j+1}) \end{bmatrix}$$