HW2

1. (1.44)

a) $\int_2^6 (3x^2 - 2x - 1)\,\delta(x-3)\,dx$ $\left(\int_{x_1}^{x_2} f(x)\delta(x-a)\,dx = \begin{cases} f(a) \text{ if } x_1 < a < x_2 \\ 0 \text{ otherwise} \end{cases} \right)$

$= 3\cdot 3^2 - 2\cdot 3 - 1 = 27 - 6 - 1 = 20$

b) $\int_0^5 \cos x\,\delta(x-\pi)\,dx = \cos\pi = -1$

c) $\int_0^3 x^3\,\delta(x+1)\,dx = 0$

d) $\int_{-\infty}^{\infty} \ln(x+3)\,\delta(x+2)\,dx = \ln(-2+3) = \ln(1) = 0$

2. (1.63). a) $\vec{v} = \frac{\hat{r}}{r}$

$\Rightarrow \nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r} r = \frac{1}{r^2}$

Divergence Theorem: $\int_V (\nabla\cdot\vec{v})\,dV = \oint_S \vec{v}\cdot d\vec{a}$

$\Leftrightarrow \int_0^\pi\int_0^{2\pi}\int_0^R \frac{1}{r^2} r^2\sin\theta\,dr\,d\phi\,d\theta = \int_0^\pi\int_0^{2\pi} \left.\frac{\hat{r}}{r}\right|_{r=R} \cdot R^2\sin\theta\,\hat{r}\,d\phi\,d\theta$

$\Leftrightarrow 2\pi R(1-(-1)) \qquad = 2\pi(1-(-1))\cdot R$

$\Rightarrow 4\pi R = 4\pi R$ ($\checkmark$) $\Rightarrow$ No delta func @ origin.

$\nabla\cdot(r^n\hat{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot r^n) = \frac{1}{r^2}\frac{\partial}{\partial r} r^{2+n} = \frac{1}{r^2}(2+n)r^{1+n} = (2+n)r^{n-1}$

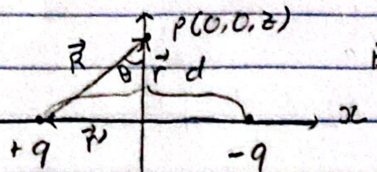
b) $\nabla\times(r^n\hat{r}) = \frac{\hat{r}}{r\sin\theta}\left(\frac{\partial}{\partial\theta}0 - \frac{\partial}{\partial\theta}0\right) - \frac{\hat{\theta}}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}r^n - \frac{\partial}{\partial r}0\right) + \frac{\hat{\phi}}{r}\left(\frac{\partial}{\partial r}0 - \frac{\partial}{\partial\theta}r^n\right)$

$= 0$

Prob 1.61: $\int_V (\nabla\times r^n\hat{r})\,dV = -\oint_A (r^n\hat{r})\times d\vec{a}$

$\Leftrightarrow 0 \qquad = -\int_0^\pi\int_0^{2\pi} R^n\hat{r}\times R^2\sin\theta\,\hat{r}\,d\phi\,d\theta = 0$ ($\checkmark$)

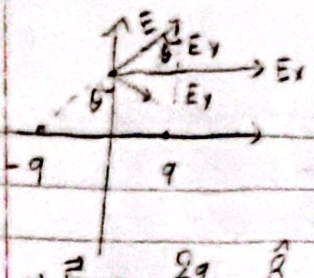
3. (2.2).



E field @ P

$\vec{r}' = \pm d/2\,\hat{x},\ \vec{r} = z\hat{z} \Rightarrow \vec{R} = \vec{r} - \vec{r}' = (z \pm d/2) \Rightarrow R = \sqrt{z^2 + d^2/4}$

$\hat{R}$



$\Rightarrow$ y-comps cancelled out $\Rightarrow$ only x-left

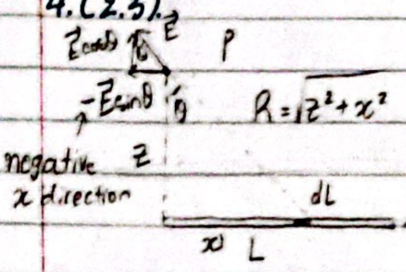
$\Rightarrow E = E_{x,q} + E_{x,-q} = 2E_x = 2E\sin\theta = 2E\dfrac{d}{2R}$

$$\Rightarrow \vec{E} = \frac{2q}{4\pi\varepsilon_0}\frac{\hat{R}}{R^2}\cdot\frac{d}{2R} = \frac{q}{4\pi\varepsilon_0}\frac{d}{(z^2+d^2/4)^{3/2}}\hat{x}$$

$z \gg d \Rightarrow \vec{E} = \dfrac{qd}{4\pi\varepsilon_0 z^3}$ (point charge!)

↳ No, dipole

4.(2.3)



$\lambda = \frac{q}{L} \Rightarrow dq' = \lambda dL'$

$$\Rightarrow \vec{E}_z = \vec{E}\cos\theta = \vec{E}\frac{z}{R} = \vec{E}\frac{z}{(z^2+x'^2)^{1/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int_0^L \frac{1}{R^2}\hat{R}\frac{z}{(z^2+x'^2)^{1/2}}$$

$$= \frac{\lambda z}{4\pi\varepsilon_0}\int_0^L \frac{1}{(z^2+x'^2)^{3/2}}dx'$$

$r = z,\ r' = x' \Rightarrow R = \sqrt{z^2+x'^2}$

$\Rightarrow u' = \arctan x/z\ ;\ \sin u' = x/(x^2+z^2)^{1/2}$

$$\int\frac{1}{(z^2+x^2)^{3/2}}dx \quad x' = z\tan u' \Rightarrow dx' = z\sec^2 u'\,du' \quad = \int\frac{z\sec^2 u'}{(z^2\tan^2 u' + z^2)^{3/2}}du = \frac{1}{z^2}\int\frac{1}{\sec u}du$$

$$= \frac{1}{z^2}\int\cos u'\,du' = \frac{\sin u'}{z^2} = \frac{x'}{z^2\sqrt{x'^2+z^2}} \Rightarrow \vec{E}_z = \frac{\lambda z}{4\pi\varepsilon_0}\left(\frac{x}{z^2(x^2+z^2)^{1/2}}\right)_0^L$$

$$\Rightarrow \vec{E}_z = \frac{\lambda L}{4\pi z\varepsilon_0}\frac{1}{\sqrt{z^2+L^2}}\hat{z}$$

$$\Rightarrow \vec{E}_x = -\vec{E}\sin\theta = -\vec{E}\frac{x}{R} = \frac{-x'}{(z^2+x'^2)^{1/2}}\vec{E} = \frac{-\lambda}{4\pi\varepsilon_0}\int_0^L\frac{1}{(x'^2+z^2)^{3/2}}x\,dx \qquad u = x^2+z^2 \Rightarrow du' = 2x\,dx$$

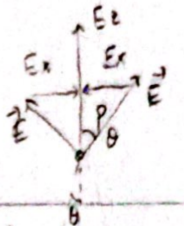
$$\Rightarrow \vec{E}_x = -\frac{\lambda}{4\pi\varepsilon_0}\int\frac{1}{2u^{3/2}}du' = \frac{\lambda}{4\pi\varepsilon_0}\int\frac{1}{u^{1/2}}du = \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{1}{(z^2+x^2)^{1/2}}\right)_0^L$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{L^2+z^2}} - \frac{1}{z}\right)\hat{x}$$

$$\Rightarrow \vec{E} = \vec{E}_x + \vec{E}_z = \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{L^2+z^2}} - \frac{1}{z}\right)\hat{x} + \frac{\lambda L}{4\pi z\varepsilon_0}\frac{1}{\sqrt{z^2+L^2}}\hat{z}$$

$$z \gg L \Rightarrow \vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{1}{z} - \frac{1}{z}\right)\hat{x} + \frac{\lambda L}{4\pi\varepsilon_0 z}\frac{1}{z}\hat{z}$$

$$= 0\,\hat{x} + \frac{\lambda L}{4\pi\varepsilon_0 z^2}z$$

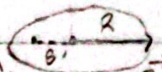


$Q = \sigma A = \sigma 2\pi r$

$\Rightarrow dQ = \sigma 2\pi r dr$ or $s'ds'$ in this case

5.(2.6).

Assuming total charge Q



Symmetry $\Rightarrow$ horizontal comp (x) cancelled

$\vec{r} = z\hat{z}$, $\vec{r}' = s'\hat{x}'$, $\vec{R} = (z-s')\hat{R}$

$\Rightarrow R = \sqrt{z^2 + s'^2}$

$$\Rightarrow \vec{E} = \vec{E}_z = \vec{E}\cos\theta = \vec{E}\frac{z}{\sqrt{s'^2+z^2}}$$

$$= \frac{Q}{4\pi\varepsilon_0}\int \frac{\hat{R}}{R^2}\cdot\frac{z}{\sqrt{s'^2+z^2}}\, d\vec{A}$$

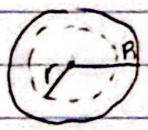
$$= \frac{\sigma 2\pi}{4\pi\varepsilon_0}\int_0^R \frac{z}{(s'^2+z^2)^{3/2}}\, s'ds'$$

$$\Rightarrow \vec{E} = \frac{\sigma z}{2\varepsilon_0}\left(\frac{-1}{\sqrt{s'^2+z^2}}\right)\Bigg|_0^R = \frac{\sigma z}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right) = \left(\frac{\sigma}{2\varepsilon_0} - \frac{\sigma z}{2\varepsilon_0\sqrt{z^2+R^2}}\right)\hat{z}$$

$\therefore R \to \infty \Rightarrow \lim_{R\to\infty} \vec{E} = \frac{\sigma}{2\varepsilon_0} - \frac{1}{\infty} = \frac{\sigma}{2\varepsilon_0}\hat{z}$ (infinite sheet)

$\therefore z \gg R \Rightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0} - \frac{\sigma z}{2\varepsilon_0 z} = 0$ (this seems wrong. shouldn't it equal to a point charge if $z \to \infty$?)

(2.15).6.



$\rho = kr$ $\quad Q = \rho V \Rightarrow dQ = \rho dV$

E field points radially outwards. ($\hat{r}$)

*Inside: $\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} \Leftrightarrow E\cdot 4\pi r^2 = \frac{Q_{encl}}{\varepsilon_0}$

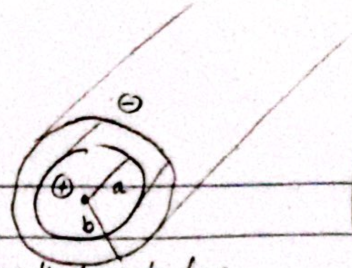
Since ρ is not constant (r dependent) $\Rightarrow$ We can know Q_{encl} if we integrate over the Gaussian's sphere volume

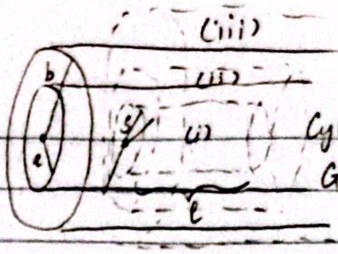
$$Q_{encl} = \int \rho dV = k\int_V r' dV' = k\int_0^{2\pi}\int_0^{\pi}\int_0^{r} r'\cdot r'^2 \sin\theta\, dr' d\theta' d\phi'$$

$$= 4\pi k\int_0^r r'^3 dr' = 4\pi k\frac{r^4}{4} = \pi k r^4$$

$$\Rightarrow E\cdot 4\pi r^2 = \frac{\pi k r^4}{\varepsilon_0} \Rightarrow \vec{E} = \frac{kr^2}{4\varepsilon_0}\hat{r}$$

HW (cont): 7. (2.17).





Cylindrical Gaussian surface length l

Horizontal components cancelled out due to symmetry $\Rightarrow \vec{E}$ field points radially outward in all 3 cases

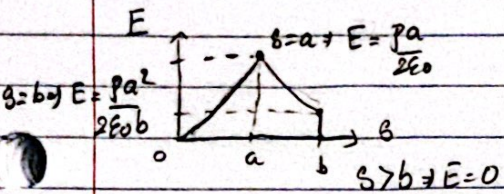
(i) $s < a$

$$\Rightarrow \oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} \Leftrightarrow E \cdot 2\pi s l = \frac{Q_{encl}}{\varepsilon_0} = \frac{\rho V_{encl}}{\varepsilon_0} = \frac{\rho \cdot \pi s^2 l}{\varepsilon_0}$$

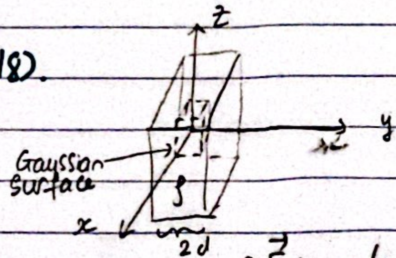
$$\Rightarrow \vec{E} = \frac{\rho s}{2\varepsilon_0} \hat{s}$$

(ii) $a < s < b \Rightarrow E \cdot 2\pi s l = \frac{Q_{encl}}{\varepsilon_0} = \rho \cdot \frac{\pi a^2 l}{\varepsilon_0} \Rightarrow \vec{E} = \frac{\rho \cdot a^2}{2\varepsilon_0 \cdot s} \hat{s}$

(iii) $s > b \Rightarrow E \cdot 2\pi s l = \frac{Q_{encl}}{\varepsilon_0}$ (neutral cable $\Rightarrow Q_{encl} = 0$) $\Rightarrow \vec{E} = 0$



8. (2.18).



y-dependent $\Rightarrow \vec{E} = \vec{E}\hat{y}$; Gaussian surface: rectangular box with $x = l$, $z = h$, $y = y$.

(i) $y = 0 \Rightarrow \vec{E} = 0$ (x & z components cancelled)

$\vec{E}$ is y dependent $\Rightarrow$ integrate on x & z surface only

(ii) $0 < y < d \Rightarrow \oint_S \vec{E} \cdot d\vec{a} = \int_0^l \int_0^h (E \cdot dx\,dz)\hat{y} = Ehl = \frac{Q_{encl}}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} yhl \Rightarrow \vec{E} = \frac{\rho y}{\varepsilon_0} \hat{y}$

(iii) $y > d \Rightarrow \oint_S \vec{E} \cdot d\vec{a} = Ehl = \frac{Q_{encl}}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} dhl \Rightarrow \vec{E} = \frac{\rho d}{\varepsilon_0} \hat{y}$

$\Rightarrow$ Same for $-y$:
$$\begin{cases} \vec{E} = -\frac{\rho(-y)}{\varepsilon_0} \hat{y} = \frac{\rho y}{\varepsilon_0} \hat{y} \text{ for } -d < y < 0 \\ \vec{E} = -\frac{\rho d}{\varepsilon_0} \hat{y} \text{ for } y < -d \end{cases}$$