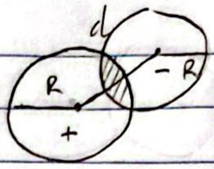
HW 2+1=3

1. (2.19).



Efield the sphere covers : $(r=R)$

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} \Rightarrow E\cdot 4\pi R^2 = \frac{\rho}{\varepsilon_0}\frac{4}{3}\pi R^3 \Rightarrow E = \frac{\rho}{3\varepsilon_0}R$$

$\Rightarrow$ For the positive sphere: $\vec{E}_+ = \frac{\rho}{3\varepsilon_0}R(+\hat{r})$, for the negative sphere: $\vec{E}_- = \frac{-\rho}{3\varepsilon_0}R(-\hat{r})$

$\Rightarrow$ $\vec{E}_{net}$ (overlap region): $\vec{E}_+ + \vec{E}_- = \frac{\rho}{3\varepsilon_0}R(+\hat{r}) - \frac{\rho}{3\varepsilon_0}R(-\hat{r}) = \frac{\rho}{3\varepsilon_0}\Big(\underbrace{R\hat{r} - R(-\hat{r})}_{\vec{d}}\Big)$

$\Rightarrow \vec{E}_{net} = \frac{\rho}{3\varepsilon_0}\vec{d} = \text{const}$

2. (2.21). a) $\vec{E} = k[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}]$

b) $\vec{E} = k[y^2\hat{x} + (2xy+z^2)\hat{y} + 2yz\hat{z}]$

To check the impossible field, we will check if $\vec{\nabla}\times\vec{E} = 0$

For (a): $\vec{\nabla}\times\vec{E} = \hat{x}\Big(\underbrace{\frac{\partial}{\partial y}3xz}_{0} - \underbrace{\frac{\partial}{\partial z}2yz}_{=-2y\hat{x}\neq 0}\Big) + \hat{y}\ldots \Rightarrow \vec{\nabla}\times\vec{E} \neq 0$

$\Rightarrow$ (a) is invalid

Check for (b): $\vec{\nabla}\times\vec{E} = \hat{x}\Big(\frac{\partial}{\partial y}2yz - \frac{\partial}{\partial z}(2xy+z^2)\Big) + \hat{y}\Big(\frac{\partial}{\partial x}2yz - \frac{\partial}{\partial z}y^2\Big)$

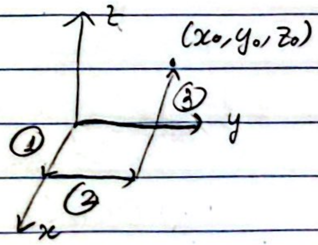
$+\hat{z}\Big(\frac{\partial}{\partial x}(2xy+z^2) - \frac{\partial}{\partial y}y^2\Big) = (2z-2z)\hat{x} + 0\hat{y} + (2y-2y)\hat{y} = 0$ ✓

$\Rightarrow$ (b) is valid

Check by computing ∇V: $V = -\int \vec{E}\cdot d\vec{l}$

Assuming $\vec{E}$ is the e-field @ random (x_0, y_0, z_0) :

Pick random path ①, ② & ③



$\Rightarrow V = V_1 + V_2 + V_3$

$$= -\int_{①}\vec{E}\cdot d\vec{l} - \int_{②}\vec{E}\cdot d\vec{l} - \int_{③}\vec{E}\cdot d\vec{l}$$

$$= -K\Big[\cancel{\int_0^{x_0} y^2 dx}_{\,y=0} + \int_0^{y_0}(2xy+\cancel{z^2})\,dy + \int_0^{z_0} 2yz\,dz\Big]$$

$x=x_0,\ y=y_0,\ z=0$

$$= -K\,[\ 0 + x_0y_0^2 + y_0z_0^2\ ] \text{ at } (x_0, y_0, z_0)$$

$\Rightarrow V = -K(xy^2 + yz^2)$ (valid potential)

3.(2.22). Uniformly charged sphere $\Rightarrow \phi = 0$ @ ∞

E-field: $r \leq R \Rightarrow \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E\cdot 4\pi r^2 = \frac{q}{\varepsilon_0}\frac{r^3}{R^3} \Rightarrow \vec{E} = \frac{q}{4\pi R^3 \varepsilon_0} r\hat{r}$

$r \geq R \Rightarrow \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E\cdot 4\pi r^2 = \frac{q}{\varepsilon_0} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$

Potential: $r \leq R$: potential goes from $\infty \to R \to r$

$$\Rightarrow V_1 = -\int_\infty^R \underbrace{\vec{E}\cdot d\vec{l}}_{\text{outside}} - \int_R^r \underbrace{\vec{E}\cdot d\vec{l}}_{\text{inside}} = -\int_\infty^R \frac{q}{4\pi\varepsilon_0 r^2}dr - \int_R^r \frac{q}{4\pi R^3\varepsilon_0} r\,dr$$

$$= \frac{-q}{4\pi\varepsilon_0}\left(\int_\infty^R r^{-2}dr + \frac{1}{R^3}\int_R^r r\,dr\right) = \frac{-q}{4\pi\varepsilon_0}\left[\left(\frac{-1}{r}\right)_\infty^R + \frac{1}{R^3}\left(\frac{r^2}{2}\right)_R^r\right]$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\left(\frac{-1}{R}+0\right) + \frac{1}{R^3}\left(\frac{r^2}{2} - \frac{R^2}{2}\right)\right] = \frac{-q}{4\pi\varepsilon_0}\left[\frac{-1}{R} + \frac{r^2}{2R^3} - \frac{1}{2R}\right]$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\frac{-3}{2R} + \frac{r^2}{2R^3}\right] = \frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

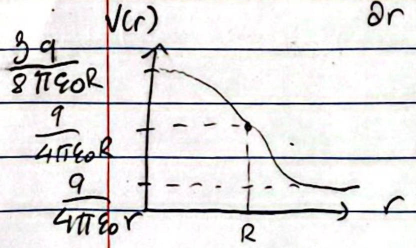
Potential: $r \geq R$: potential goes from $\infty \to r$

$$\Rightarrow V_2 = -\int_\infty^r \underbrace{\vec{E}\cdot d\vec{l}}_{\text{outside}} = -\int_\infty^r \frac{q}{4\pi\varepsilon_0 r^2}dr = -\frac{q}{4\pi\varepsilon_0}\left(\frac{-1}{r}\right)_\infty^r = \frac{q}{4\pi\varepsilon_0 r}$$

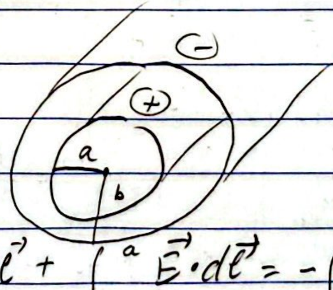
Check: $\vec{\nabla} V_1 = \frac{\partial}{\partial r} V_1 \hat{r} = \frac{\partial}{\partial r}\left[\frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)\right]\hat{r}$

$$= \frac{q}{8\pi\varepsilon_0 R}\frac{\partial}{\partial r}\left(3 - \frac{r^2}{R^2}\right)\hat{r} = \frac{-q}{4\pi R^3\varepsilon_0} r\hat{r} = -\vec{E}_{in} \checkmark$$

$$\vec{\nabla} V_2 = \frac{\partial}{\partial r}\frac{q}{4\pi\varepsilon_0 r}\hat{r} = \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial r} r^{-1} = \frac{-q}{4\pi\varepsilon_0 r^2}\hat{r} = -\vec{E}_{out} \checkmark$$



For Prob. 4 (2.25):



4.(2.25). Eqn 2.22: $V(\vec{b}) - V(\vec{a}) = -\int_0^b \vec{E}\cdot d\vec{l} + \int_0^a \vec{E}\cdot d\vec{l} = -\int_a^b \vec{E}\cdot d\vec{l}$

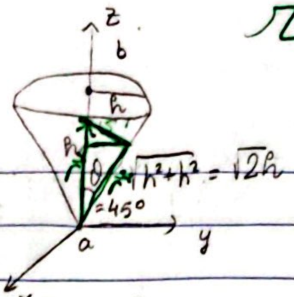
Based on problem 2.17 (HW2): $s < a \Rightarrow \vec{E}_{in} = \frac{\rho s}{2\varepsilon_0}\hat{s}$

$a < s < b \Rightarrow \vec{E}_{out} = \frac{\rho a^2}{2\varepsilon_0 s}\hat{s}$, No E field beyond b.

$$\Rightarrow V(\vec{b}) - V(\vec{a}) = -\int_a^b \vec{E}_{out}\cdot d\vec{l} = -\int_a^b \frac{\rho a^2}{2\varepsilon_0 s}ds = \frac{-a^2\rho}{2\varepsilon_0}\left(\ln s\right)_a^b$$

$$= \frac{-a^2\rho}{2\varepsilon_0}(\ln b - \ln a) = \frac{-a^2\rho}{2\varepsilon_0}(\ln b/\ln a)$$

$$\mathscr{R} = \vec{r} - \vec{r}\,'$$



(*) 5.(2.27). $\Delta V = V(a) - V(b) = -\int_a^b \vec{E} \cdot d\vec{l}$

Since finding efield everywhere on this entire cone scared me (because no Gauss' Law ;n;), I will just find V(a) & V(b) and subtract them.

$\Rightarrow V_a = V$ at $(0,0,0)$

$V_b = V$ at $(0,0,h)$

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, $\vec{r}\,' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$ $\quad q = \sigma$

$\Rightarrow R = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}$ (not dense.)

(?) $\Rightarrow V_a = \dfrac{\sigma}{4\pi\varepsilon_0} \displaystyle\int_S \frac{1}{R}\, da' = \frac{\sigma}{4\pi\varepsilon_0} \int_S \frac{1}{[(x-x')^2 + \dots (z-z')^2]^{1/2}}\, da'$

Spherical $\Rightarrow x = r\sin\theta\cos\phi$; $r\sin\theta\sin\phi = y$; $z = r\cos\theta$ } prime coord since integrate over da'

$x = r\sin 45\cos\phi$ $\quad r\sin 45\sin\phi = y$; $z = r\cos 45$

Va @ $(0,0,0) \Rightarrow x = y = z = 0$ $\qquad (x-x')^2 = (x'-x)^2$

$$\Rightarrow V_a = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^{\sqrt{2}h} \frac{r'\sin 45^\circ\, dr'\, d\phi'}{[(r'\sin 45^\circ\cos\phi - x)^2 + (r'\sin 45^\circ\sin\phi - y)^2 + (r'\cos 45 - z)^2]^{1/2}}$$

$$= \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^{\sqrt{2}h} \frac{r'\sin 45^\circ\, dr'\, d\phi'}{[(r'^2\sin^2 45^\circ\cos^2\phi - 2r'\sin 45^\circ\cos\phi x + x^2)] + [(r'^2\sin^2 45^\circ \dots + y^2)] + [(r'^2 \dots + z^2)]^{1/2}}$$

$$= \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^{\sqrt{2}h} \frac{r'\sin 45^\circ\, dr'\, d\phi'}{[r'^2(\sin^2 45^\circ\cos^2\phi + \sin^2 45^\circ\sin^2\phi + \cos^2 45) - 2r'\underbrace{(\sin 45\cos\phi x + \sin 45\sin\phi y + \cos 45 z)}_{0\ (x=y=z=0)} \underbrace{+ x^2 + y^2 + z^2}_{0}]^{1/2}}$$

$$= \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^{\sqrt{2}h} \frac{\cancel{r'}\sin 45^\circ\, dr'\, d\phi'}{[\cancel{r'^2}(\sin^2 45^\circ\cos^2\phi + \sin^2 45^\circ\sin^2\phi + \cos^2 45)]^{1/2}}$$

$$= \sqrt{2}h\frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} \frac{\cancel{\sin 45^\circ}\, d\phi'}{\cancel{\sin 45^\circ}(\underbrace{\cos^2\phi + \sin^2\phi}_{1} + \underbrace{\tan^2 45}_{1})^{1/2}} = \sqrt{2}h\frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} \frac{1}{\sqrt{2}}\, d\phi'$$

$= \dfrac{h\sigma}{4\pi\varepsilon_0} 2\pi = \dfrac{h\sigma}{2\varepsilon_0}$ (Gosh maybe I picked the wrong method after all). ;n;

Vb @ $(0,0,h) \Rightarrow x = y = 0,\ z = h$

$$\Rightarrow V_b = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^{\sqrt{2}h} \frac{r'\sin 45^\circ\, dr'\, d\phi'}{[2r'^2 \sin^2 45^\circ - 2r'(\cos 45^\circ h) + h^2]^{1/2}}$$

$\sin 45^\circ = 1/\sqrt{2}$, $\tan 45^\circ = 1$

$\Rightarrow \sin^2 45^\circ = 1/2$

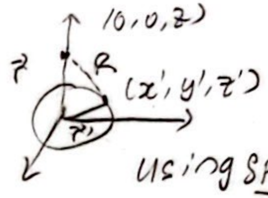
$$= \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^{\sqrt{2}h} \frac{r'\sin 45^\circ\, dr'\, d\phi'}{[r'^2 - 2/\sqrt{2}\, r'h + h^2]^{1/2}}$$

$$= \frac{\sigma}{4\sqrt{2}\pi\varepsilon_0} 2\pi \int_0^{\sqrt{2}h} \frac{r'\, dr'}{[r'^2 - 2/\sqrt{2}\, r'h + h^2]^{1/2}} = \frac{\sigma}{2\sqrt{2}\varepsilon_0} h\sqrt{2}\ln(\sqrt{2}+1)$$

$$= \frac{\sigma h}{2\varepsilon_0}\ln(\sqrt{2}+1) \Rightarrow V(a) - V(b) = \frac{h\sigma}{2\varepsilon_0} - \frac{\sigma h}{2\varepsilon_0}\ln(\sqrt{2}+1) = \frac{\sigma h}{2\varepsilon_0}\left[1 - \ln(\sqrt{2}+1)\right]$$

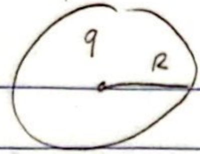
Hint from prof. Burant: Set Field point on z-axis $\Rightarrow$

$\vec{r} = 0\hat{x} + 0\hat{y} + z\hat{z}$
$\vec{r}' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$
$R = \sqrt{x'^2 + y'^2 + (z-z')^2}$



using spherical

6. (2.29)



Eqn 2.29 $V(r) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}')}{R}\,d\tau'$

Uniform $q \Rightarrow \rho(\vec{r}') = \rho = \frac{3q}{4\pi R^3}$

$\Rightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\frac{3q}{4\pi R^2}\int_V \frac{1}{R}\,d\tau' = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \frac{r'^2 \sin\theta' dr' d\phi' d\theta'}{[(r'\sin\theta'\cos\phi')^2 + (r'\sin\theta'\sin\phi')^2 + (z - r'\cos\theta')^2]^{1/2}}$

$= \frac{3q}{16\pi^2\varepsilon_0 R^2}\int_0^{\pi}\int_0^{2\pi}\int_0^{R} \frac{r'^2\sin\theta' dr' d\phi' d\theta'}{[r'^2\sin^2\theta'\cos^2\phi' + r'^2\sin^2\theta'\sin^2\phi' + z^2 - 2zr'\cos\theta' + r'^2\cos^2\theta']^{1/2}}$ $\quad 1 + \cot^2\theta = \csc^2\theta$

$= r'^2(\sin^2\theta'\cos^2\phi' + \sin^2\theta'\sin^2\phi' + \cos^2\theta') = r'^2\sin^2\theta'(\cos^2\phi' + \sin^2\phi' + \cot^2\theta')$

$\Rightarrow V(r) = \frac{3q}{16\pi^2\varepsilon_0 R^2}\,2\pi\int_0^{\pi}\int_0^{R}\frac{r'^2\sin\theta' dr' d\theta'}{[r'^2 - 2zr'\cos\theta' + z^2]^{1/2}}$ $\quad u = r'^2 - 2r'z\cos\theta' + z^2 \Rightarrow du = 2r'z\sin\theta' d\theta'$
$\Rightarrow du/2z = r'\sin\theta' d\theta'$

$\Rightarrow V(r) = \frac{3q}{8\pi\varepsilon_0 R^2 z}\int_0^{R}\int_{r'^2 - 2r'z + z^2}^{r'^2 + 2r'z + z^2} \frac{r'}{2\sqrt{u}}\,du\,dr' = \frac{3q}{8\pi\varepsilon_0 R^2 z}\int_0^{R} r'[\sqrt{(r'+z)^2} - \sqrt{(r'-z)^2}]\,dr'$
$\frac{}{2\sqrt{u}} = \sqrt{u} \Rightarrow$ $= r'[|r'+z| - |r'-z|]\,dr'$

$\Rightarrow V(z<R) = \frac{3q}{8\pi\varepsilon_0 R^2 z}\left(\int_0^{z} 2r'^2 dr' + 2z\int_z^{R} r' dr'\right) = \frac{3q}{8\pi\varepsilon_0 R^2 z}\left(zR^2 - \frac{z^3}{3}\right) = \frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{z^2}{R^2}\right)$

$\Rightarrow V(z>R) = \frac{3q}{8\pi\varepsilon_0 R^2 z}\int_0^{R} 2r'^2 dr' = \frac{3q}{8\pi\varepsilon_0 R^2 z}\frac{2R^3}{3} = \frac{q}{4\pi\varepsilon_0 z}$ ✓

$\Rightarrow$ similar to 2.22