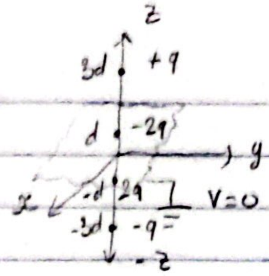
HW5

1. (3.7).



Force on +q

$$F_{-2q\,to\,q} = \frac{1}{4\pi\varepsilon_0}\frac{q\cdot(-2q)}{(2d)^2} = \frac{-2q^2}{4\pi\varepsilon_0 4d^2} \quad (1)$$

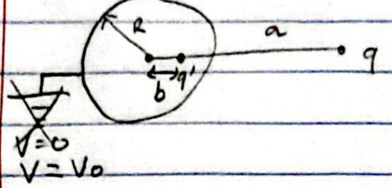
$$F_{-d\,to\,q} = \frac{1}{4\pi\varepsilon_0}\frac{q\,2q}{(4d)^2} = \frac{2q^2}{4\pi\varepsilon_0 16d^2} \quad (2)$$

$$F_{-q\,to\,q} = \frac{1}{4\pi\varepsilon_0}\frac{-q\cdot q}{(6d)^2} = \frac{-q^2}{4\pi\varepsilon_0 36d^2} \quad (3)$$

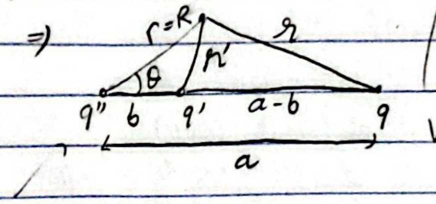
$$\Rightarrow F_{net} = (1)+(2)+(3) = \frac{q^2}{4\pi\varepsilon_0 d^2}\left(\frac{-2}{4}+\frac{2}{16}-\frac{1}{36}\right)$$

$$= \frac{q^2}{4\pi\varepsilon_0 d^2}\left(\frac{-72+18-4}{144}\right) \Rightarrow \vec{F}_{net} = \frac{-q^2}{4\pi\varepsilon_0 d^2}\,\frac{29}{72}\,\hat{z}$$

2. (3.10). (Wait! Kinda similar to discussion) $\Rightarrow$ Image charge distance b away from center such that $b < R$



$\Rightarrow$



$$\boxed{b = \frac{R^2}{a} \ \&\ q' = -\frac{R}{a}q}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}+\frac{q'}{r'}\right) = 0 \quad \text{(for } V=0\text{)}$$

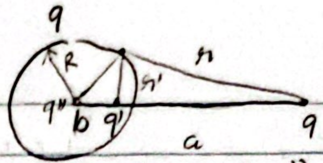
But since $V=V_0$ now $\Rightarrow$ A 3rd charge needs to be added. But since this conductor sphere needs to be at equipotential on the surface $\Rightarrow$ The only place for the 3rd charge to go is at the center of the sphere since then the distance r'' to charge q'' is $= R =$ const

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{q''}{R} = \text{const} = V_0 \Rightarrow q'' = 4\pi\varepsilon_0 R V_0 = -q' \text{ (for net charge inside the sphere} = 0)$$

$$\Rightarrow F = \frac{1}{4\pi\varepsilon_0}\left(\frac{qq'}{(a-b)^2}+\frac{qq''}{a^2}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{qq'}{(a-b)^2}-\frac{qq'}{a^2}\right) \quad \underline{q''=-q'}$$

$$= \frac{qq'}{4\pi\varepsilon_0}\left(\frac{a^2-a^2+2ab-b^2}{a^2(a-b)^2}\right) = \frac{qq'}{4\pi\varepsilon_0}\left[\frac{b(2a-b)}{a^2(a-b)^2}\right]$$

$$= \frac{-q^2R}{4\pi\varepsilon_0 a}\left[\frac{R^2/a\,(2a-R^2/a)}{a^2(a-R^2/a)^2}\right] = \frac{-q^2}{4\pi\varepsilon_0}\left(\frac{R}{a}\right)^3\frac{2a^2-R^2}{(a^2-R^2)^2}$$

3(3.11.) 

$b = \frac{R^2}{a}$; $q' = -\frac{R}{a}q$; $q'' = -q' + q$ for total charge of sphere $= q$

$$\Rightarrow F = \frac{1}{4\pi\varepsilon_0}\left[\frac{qq'}{(a-b)^2} + \frac{(-qq') + q^2}{a^2}\right] = \frac{q}{4\pi\varepsilon_0}\left[\frac{a^2q' + (-q'+q)(a-b)^2}{a^2(a-b)^2}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{a^2q + 2abq' - 2abq - b^2q' + b^2q}{a^2(a-b)^2}\right] = \frac{q}{4\pi\varepsilon_0}\left[\frac{q(a^2-2ab+b^2) + q'(2ab-b^2)}{a^2(a-b)^2}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{q(a^2-2R^2+R^4/a^2) - R/a\, q\,(2R^2 - R^4/a^2)}{a^2(a-R^2/a)^2}\right] = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{(a-R^2/a)^2 - 2R^3/a + R^5/a^3}{a^2(a-R^2/a)^2}\right]$$

$$= \frac{-q^2}{4\pi\varepsilon_0}\left[\frac{2R^3/a - R^5/a^3 - a^2 + 2R^2 - R^4/a^2}{a^2(a-R^2/a)^2}\right] = \frac{-q^2}{4\pi\varepsilon_0}\left[\frac{2R^3a^2 - R^5 - a^5 + 2R^2a^3 - R^4a}{a^5(a-R^2/a)^2}\right]$$

$\Rightarrow$ Force is negative $\rightarrow$ attractive force $\rightarrow a_c = a$ when $F = 0$

$\Rightarrow 0 = 2R^3a^2 - R^5 - a^5 + 2R^2a^3 - R^4a \rightarrow \frac{a}{R} \approx 1.618 \rightarrow a = a_c = \frac{1+\sqrt{5}}{2}$

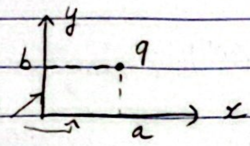
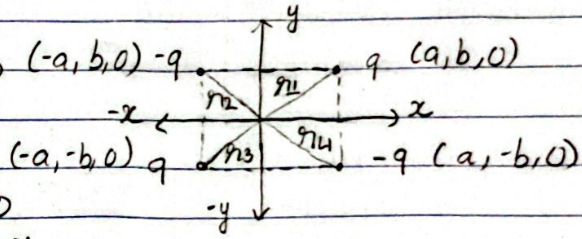
4.(3.13). 

image → 

$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$; $z' = 0$

$$\Rightarrow V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}} + \frac{1}{\sqrt{(x+a^2)+(y+b)^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+(y+b^2)+z^2}}\right]$$

$\Rightarrow$ Force on q due to -q charges: $\frac{-1}{4\pi\varepsilon_0}\frac{q^2}{4a^2}\hat{i} - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4b^2}\hat{j}$

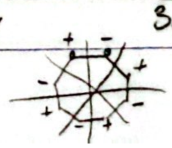
$\Rightarrow$ Force on q due to q charge: $\frac{1}{4\pi\varepsilon_0}\frac{q}{4a^2+4b^2}\frac{a\hat{i}+b\hat{j}}{(a^2+b^2)^{1/2}} = \frac{q}{16\pi\varepsilon_0}\frac{a\hat{i}+b\hat{j}}{(a^2+b^2)^{3/2}}$

$$\Rightarrow \vec{F}_{net} = \frac{-q^2}{16\pi\varepsilon_0}\left\{\left[\frac{1}{a^2} - \frac{a}{(a^2+b^2)^{3/2}}\right]\hat{i} - \left[\frac{1}{b^2} - \frac{b}{(a^2+b^2)^{3/2}}\right]\hat{j}\right\}$$

$$\rightarrow W_{net} = \frac{q^2}{4\pi\varepsilon_0}\left(-\frac{1}{2a} + \frac{1}{\sqrt{4a^2+4b^2}} - \frac{1}{2b}\right) = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{a} - \frac{1}{b}\right)$$

→ fake

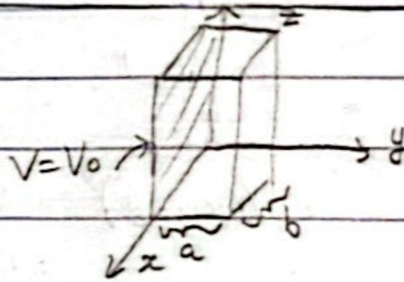
But since the other 3 charges are not real $\rightarrow W_{real} = 1/4\, W_{net} = \frac{q^2}{32\pi\varepsilon_0}\left(\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{a} - \frac{1}{b}\right)$

I guess by symmetry, if it works for 90°, it could work for 45° too if you split every thing in half  ← like this?

HW5 Cont

5.(3.17). Boundary Conditions: $V=0$ at $y=0,\ y=a\ \&\ x=0$

a) $V=V_0(y)$ at $x=b$



$\Rightarrow \phi(x,y)=X(x)Y(y)=V(x,y)$; $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}=0$ (Laplace)

$\Rightarrow \dfrac{1}{X}\dfrac{\partial^2 X}{\partial x^2}+\dfrac{1}{Y}\dfrac{\partial^2 Y}{\partial y^2}=0 \Rightarrow \dfrac{1}{X}\dfrac{\partial^2 X}{\partial x^2}=-\dfrac{1}{Y}\dfrac{\partial^2 Y}{\partial y^2}=+\alpha^2$

$\Rightarrow \dfrac{1}{X}\dfrac{\partial^2 X}{\partial x^2}-\alpha^2=0$

$\Leftrightarrow \dfrac{\partial^2 X}{\partial x^2}-\alpha^2 X=0$

$\Leftrightarrow X=Ae^{-\alpha x}+Be^{\alpha x}$

$\Rightarrow \dfrac{1}{Y}\dfrac{\partial^2 Y}{\partial y^2}+\alpha^2=0$

$\Leftrightarrow \dfrac{\partial^2}{\partial y^2}+\alpha^2 Y=0$

$\Leftrightarrow Y=C\cos\alpha y+D\sin\alpha y$

Bound Cond: $X(0)=0=A+B \Rightarrow A=-B$

$\Rightarrow Y(0)=0=C\cdot 1+D\cdot 0=C \Rightarrow C=0$

$\Rightarrow Y(a)=0=D\sin\alpha a \Rightarrow \alpha=\dfrac{n\pi}{a}\ (n\neq 0)$

$\Rightarrow V(x,y)=A(e^{\alpha x}-e^{-\alpha x})D\sin\alpha y$

$=2AD\sinh(\alpha x)\sin(\alpha y)$

$\Rightarrow V(b,y)=V_0(y)=2AD\sinh(\alpha b)\sin(\alpha y)$

5.(3.17) Cont: $AD = \dfrac{\dfrac{1}{2\sinh(\alpha b)}\int_0^a V_0(y)\sin(\alpha y)dy}{\int_0^a \sin^2(\alpha y)dy = a/2}$

$\Rightarrow AD = \dfrac{1}{a\sinh(\alpha b)}\int_0^a V_0(y)\sin(\alpha y)dy$, $V(x,y) = 2AD\sinh(\alpha x)\sin(\alpha y)$

y-dependent

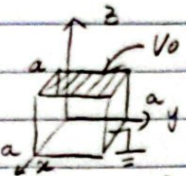
b) Solve for $V_0(y) = V_0$ (I'm doing it right now)

$\Rightarrow AD = \dfrac{V_0}{a\sinh(\alpha b)}\int_0^a \sin(\alpha y)dy = \dfrac{V_0}{2\sinh(\alpha b)}\left(\dfrac{-\cos\alpha y}{\alpha}\right)_0^a$

$= \dfrac{-V_0}{\sinh\alpha b}\left(\dfrac{\cos n\pi - 1}{n\pi}\right) \Big| \alpha = \dfrac{n\pi}{a} \Big| = \dfrac{2V_0}{n\pi\sinh(\alpha b)}$

$\Rightarrow \boxed{V(x,y) = \dfrac{4V_0}{n\pi\sinh(\alpha b)}\sinh(\alpha x)\sin(\alpha y)}$

6.(3.18).



$V = X(x)\,Y(y)\,Z(z)$

$\Rightarrow \dfrac{1}{X}\dfrac{\partial^2 X}{\partial x^2} + \dfrac{1}{Y}\dfrac{\partial^2 Y}{\partial y^2} + \dfrac{1}{Z}\dfrac{\partial^2 Z}{\partial z^2} = 0$

We did this once on the lecture, ain't no way I'll suffer again...

$\Rightarrow X = A\cos\alpha x + B\sin\alpha x$; $Y = C\cos(\beta y) + D\sin(\beta y)$; $Z = Ee^{\sqrt{\alpha^2+\beta^2}z} + Fe^{-\sqrt{\alpha^2+\beta^2}z}$

Boundary conditions:

$X(0) = 0 \Rightarrow A = 0$; $X(a) = B\sin\alpha a = 0 \Rightarrow \alpha = \dfrac{n\pi}{a}$

$Y(0) = 0 \Rightarrow C = 0$; $Y(a) = D\sin(\beta y) = 0 \Rightarrow \beta = \dfrac{m\pi}{a}$

$Z(0) = 0 \Rightarrow E + F = 0 \Rightarrow E = -F$: sub in the value for α & β

$\Rightarrow Z(z) = 2E\sinh\left(\dfrac{\pi z\sqrt{m^2+n^2}}{a}\right)$

$\Rightarrow V(x,y,z) = \dfrac{16}{\pi^2}V_0\sum_{m,n\ odd}\dfrac{1}{mn}\left[\sin\left(\dfrac{m\pi x}{a}\right)\right]\left[\sin\left(\dfrac{n\pi y}{a}\right)\right]\dfrac{\sinh(\sqrt{m^2+n^2}\pi z/a)}{\sinh(\sqrt{m^2+n^2}\pi)}$

7.(3.20). Ex 3.6: $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ (inside) $= V_0(\theta)$ at $r = R$

Ex 3.7: $V(r,\theta) = \sum_{l=0}^{\infty}\dfrac{B_l}{r^{l+1}}P_l(\cos\theta)$ (outside)

a) From 3.6: $A_l = \dfrac{2l+1}{2R^l}\int_0^\pi \underbrace{V_0(\theta)}_{V_0} P_l(\cos\theta)\sin\theta\, d\theta$ (inside)

⊛ For Legendre polynomials: $l = 0 \Rightarrow \int_0^\pi \underbrace{P_0(\cos\theta)}_{=1} P_{l'}(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0 & (l' \neq 0) \\ 2 & (l' = 0)\end{cases}$

$l \neq 0 \Rightarrow \int_0^\pi P_l\cos(\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0 & (l' \neq l) \\ \dfrac{2}{2l+1} & (l' = l)\end{cases}$

$\Rightarrow A_\ell = \frac{(2\ell+1)}{2R^\ell} V_0 \int_0^\pi \underbrace{P_0(\cos\theta)}_{1} P_\ell(\cos\theta)\sin\theta\, d\theta \Rightarrow A_\ell = \begin{cases} 0 & (\ell \neq 0) \\ V_0 & (\ell = 0) \end{cases}$

$\Rightarrow V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = A_0 r^0 \overbrace{P_0 \cos\theta}^{1} = A_0 = V_0$

From 3.7 $B_\ell = \left(\frac{2\ell+1}{2}\right) R^{\ell+1} \int_0^\pi V_0(\theta) P_\ell \cos\theta \sin\theta\, d\theta$ (<u>outside</u>)

(*) Using Legendre Polynomial: $\Rightarrow B_\ell = \left(\frac{2\ell+1}{2}\right) R^{\ell+1} V_0 \int_0^\pi P_0(\cos\theta) P_\ell(\cos\theta) \sin\theta\, d\theta$

$\Rightarrow B_\ell = \begin{cases} 0 & (\ell \neq 0) \\ \left(\frac{2\ell+1}{2}\right) \frac{2}{2\ell+1} R^{0+1} V_0 = R V_0 & (\ell = 0) \end{cases}$

$\Rightarrow V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) = \frac{B_0}{r} P_0 \cos\theta = \frac{R V_0}{r}$

b) Ex 3.9: Potential is continuous as $r = R \Rightarrow V_{3.6} = V_{3.7} \Rightarrow B_\ell = A_\ell R^{2\ell+1}$

$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\Big|_{r=R} = \frac{-1}{\varepsilon_0} \sigma_0(\theta) \Leftrightarrow -\sum_{\ell=0}^{\infty} (\ell+1) \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) - \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta)$
$= -1/\varepsilon_0\, \sigma_0(\theta)$

Plug in $B_\ell = A_\ell R^{2\ell+1}$: $\sum_{\ell=0}^{\infty} (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta) = \frac{1}{\varepsilon_0} \sigma_0(\theta)$

$\Rightarrow A_\ell = \frac{1}{2\varepsilon_0 R^{\ell-1}} \int_0^\pi \underbrace{\sigma_0(\theta)}_{\sigma_0} P_\ell(\cos\theta) \sin\theta\, d\theta \Rightarrow A_0 = \frac{1}{2\varepsilon_0 R^{0-1}} \sigma_0 \int_0^\pi \sin\theta\, d\theta = \frac{\sigma_0}{2\varepsilon_0 R^{-1}} (-\cos\theta)\Big|_0^\pi$

$\Rightarrow A_0 = \frac{R\sigma_0}{2\varepsilon_0} \cancel{2} = \frac{R\sigma_0}{\varepsilon_0} \Rightarrow A_\ell = \begin{cases} \frac{R\sigma_0}{\varepsilon_0} & (\ell = 0) \\ 0 & (\ell \neq 0) \end{cases}$

Notice from part a: $V_{in} = A_0 = V_0 = V_{out} \frac{r}{R} \Rightarrow V(r,\theta) = \begin{cases} A_0 = \frac{R\sigma_0}{\varepsilon_0} & \text{(inside)} \\ \frac{R^2 \sigma_0}{\varepsilon_0 r} & \text{(outside)} \end{cases}$

8. (3.23) From problem 7 (3.21): $V(r,\theta) = \sum_{\ell=0}^{\infty} \left(\ell A_\ell R^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

$\Rightarrow B_\ell = -A_\ell r^{2\ell+1}$ We have $\vec{E} = E_0 \hat{z}$ (whatever direction)

when $r = R \Rightarrow V = -E_0 r\cos\theta$ (only tangential component)

$\Rightarrow$ At boundary condition: $V(r,\theta) = V = -E_0 r\cos\theta = \sum_{\ell=0}^{\infty} A_\ell \underbrace{\left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}}\right)}_{①} P_\ell(\cos\theta)$ substitute B_ℓ

$\Rightarrow \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = -E_0 r\cos\theta \Rightarrow \begin{cases} A_1 = -E_0 \\ A_\ell = 0 \ (\ell \neq 1) \end{cases}$

$\Rightarrow V(r,\theta) = \underbrace{-E_0\left(r - \frac{R^3}{r^2}\right)}_{①} + V_{sphere} = -E_0\left(r - \frac{R^3}{r^2}\right) + \underbrace{\frac{Q}{4\pi\varepsilon_0 r}}_{\text{contribution from the sphere}}$