HOMEWORK 9

1. Eqn 18-21: $U_m = \int_{allspace} \frac{\vec{B}^2}{2\mu_0} d\tau$

⊛ For $\rho < a \Rightarrow \oint B \cdot dl = \mu_0 I_{encl} = \mu_0 \frac{I \pi \rho^2}{\pi a^2} = \mu_0 \frac{I\rho^2}{a^2}$

$\Rightarrow B_{\rho<a} = \frac{I\rho\mu_0}{2\pi a^2}$

⊛ For $\rho > a \Rightarrow \oint B \cdot dl = \mu_0 I_{encl} \Rightarrow \frac{\mu_0 I}{2\pi\rho} = B_{\rho>a} (I_{encl} = I)$

$\Rightarrow U_m = \frac{1}{2\mu_0}\left(\underbrace{\int_{allspace} B_{\rho<a}^2 \, d\tau}_{U_{in}} + \underbrace{\int_{allspace} B_{\rho>a}^2 \, d\tau}_{U_{out}}\right)$

$= \frac{1}{2\mu_0}\left(\frac{\mu_0^2 I^2}{4\pi^2 a^4}\int_0^{2\pi}\int_0^{\ell}\int_0^{a} \rho^2 \cdot \rho \, d\rho \, dz \, d\theta + \frac{\mu_0^2 I^2}{4\pi^2}\int_0^{2\pi}\int_0^{\ell}\int_a^{R} \frac{1}{\rho^2} \cdot \rho \, d\rho \, dz \, d\theta\right)$ $= \frac{\mu_0 I^2 \ell}{16\pi}\left(1 + 4\ln\frac{R}{a}\right)$

$= \frac{1}{2\mu_0} \cdot \frac{\mu_0^2 I^2 \ell}{4\pi^2} \cdot 2\pi \left(\frac{1}{a^4}\frac{a^4}{4} + \ln\frac{R}{a}\right) = \frac{\mu_0 I^2 \ell}{4\pi}\left(\frac{1}{4} + \ln\frac{R}{a}\right)$ ✓

2. Eqn 18-12: $U_m = \frac{1}{2}\int_{allspace} \vec{J}_f(\vec{r}) \cdot \vec{A}(\vec{r}) \, d\tau$

$\vec{J}_f(\vec{r}) = \frac{1}{\mu_0} |\vec{\nabla} \times \vec{B}| \Rightarrow U_m = \frac{1}{2\mu_0}\int_{allspace} [(\vec{\nabla} \times \vec{B}) \cdot A] \, d\tau = \frac{1}{2\mu_0}\int_{allspace} B^2 - \underbrace{[\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})]}_{=0 \text{ since curl}=0} d\tau$

$B = \mu_0 n I \Rightarrow U_m = \frac{1}{2\mu_0}\int_{allspace} (\mu_0 n I)^2 d\tau = \frac{1}{2}\mu_0 n^2 I^2 A \ell$ ✓

Grading space for 1&2: All correct. I did combine both U_{in} and U_{out} right at the get go instead of doing it at the end but similar answer so I think I'm good.

3. Wangsness 17-24:

$\Phi = \int \vec{B}\cdot d\vec{a} = \int_0^l \int_a^b \frac{\mu_0 I_{enc}}{2\pi r}\hat{\phi}\cdot l\,dr\,\hat{\phi} = \frac{\mu_0 I_{enc}}{2\pi} l \ln\left(\frac{b}{a}\right)$

$\Rightarrow L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right)$

Using Magnetic NRG: $U_m = \frac{1}{2\mu_0}\int B^2 d\tau' = \frac{1}{2\mu_0}\int_0^{2\pi}\int_0^l\int_b^a \frac{\mu_0^2 I_{enc}^2}{4\pi^2 r^2}\, r\,dr\,dz\,d\theta$

$\Rightarrow U = \frac{\mu_0 I_{enc}^2}{4\pi} l \ln\left(\frac{b}{a}\right) \Rightarrow L = \frac{2U}{I^2} = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right)$ ✓

$\Rightarrow f_m = \frac{B^2}{2\mu_0} = \frac{\mu_0^2 I^2}{4\pi^2 a^2 \cdot 2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 a^2} \Rightarrow I = \sqrt{\frac{8\pi^2 a^2}{\mu_0} f_m} = \frac{4\pi \cdot 1\cdot 10^{-2} m \cdot 1\,atm}{\sqrt{\mu_0}}$

$\Rightarrow I \approx 25000\,A.$

4. Wangsness 18-1. $\vec{m} = \frac{1}{2}\int_V \vec{r}'\times\vec{J}\,d\tau' = \frac{1}{2}\int_S \vec{r}'\times I\,d\vec{s}'$

$= \frac{1}{2}\int_S (a\hat{\rho} + b\sin n\phi\,\hat{z})\times I\,d\vec{s}'$

$d\vec{s}' = d\vec{r}' = \underbrace{a\,d\hat{\rho} + bn\cos\phi\,d\phi\,\hat{z}}$

$= \frac{1}{2}\int_0^{2\pi}(a\hat{\rho} + b\sin n\phi\,\hat{z})\times I(a\,d\phi\,\hat{\phi} + bn\cos n\phi\,d\phi\,\hat{z})$

n has to be ≥ 2 because if $n=1$

$\Rightarrow \vec{m} = \frac{1}{2}\int_0^{2\pi}(a\hat{\rho} + b\sin\phi\,\hat{z})\times I(a\,d\phi\,\hat{\phi} + b\cos\phi\,d\phi\,\hat{z})$

$= \frac{1}{2} I\pi a^2\hat{z}$ ✓ $\int_0^{2\pi}\cos n\phi(\sin\phi\,\hat{x} - \cos\phi\,\hat{y})\,d\phi$

$\Rightarrow = \frac{1}{2} I \int_0^{2\pi}(\overbrace{a^2\hat{z}\,d\phi}^{2\pi a^2\hat{z}} + abn\cos\phi\,d\phi(-\hat{\phi}) + ab\sin n\phi\,d\phi(-\hat{\rho}))$

5. Wangsness 19-3. $\vec{m} = m\hat{z}$ @ $z = z_0 \gg \frac{l}{2}$, $\vec{J} = \frac{Q}{\pi a^2 l}\rho'\omega\hat{\phi}$

$B_r = \frac{\mu_0 m}{4\pi}\frac{2\cos\theta}{r^3}$, $B_\theta = \frac{\mu_0 m}{4\pi}\frac{\sin\theta}{r^3}$ (eq'n 19-24) $\Rightarrow \vec{B} = \vec{B_r} + \vec{B_\theta}$

$\Rightarrow \vec{m} = \frac{1}{2}\int_0^{2\pi}\int_0^l\int_0^a (\rho'\hat{\rho} + \underbrace{z'\hat{z}}_{\text{cancelled}}) \times \frac{Q}{\pi a^2 l}\rho'\omega\hat{\phi}\,\rho' d\rho' dz' d\phi' = \frac{Q\omega a^2}{4}\hat{z}$ ✓

$\Rightarrow \vec{B} = \frac{\mu_0 Q\omega a^2}{16\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$

$\Rightarrow \vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = \vec{\nabla}\left(\frac{\mu_0 m Q\omega a^2}{8\pi r^3}\right) = -\frac{3\mu_0}{8}\frac{mQ\omega a^2}{\pi r^4}\hat{z}$ ✓ $\to z_0^4$

Grading Space for 3,4,5: All correct for 3 & 5. I got it correct for 4 (especially at n=1) but I didn't expand on the results so I don't know if it's necessary.

6. Wangsness 19-7. $\vec{B}_{net} = B_r + B_\theta = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$; $m = I a^2 \pi$

$$\Phi = \oint \vec{B}\cdot d\vec{a} = \int_0^{\theta'}\int_0^{2\pi} \frac{\mu_0 I \pi a^2}{4\pi r^3}(2\cos\theta)\, r^2 \sin\theta\, d\theta\, d\phi$$

$$\boxed{r = \sqrt{b^2+z^2} = c}$$

$$= \frac{\mu_0 I \pi a^2}{2}\int_0^{\theta'} \cos\theta \sin\theta\, d\theta = \frac{\mu_0 I \pi a^2}{2r}\left(\frac{b^2}{b^2+z^2}\right) = \frac{\mu_0 \pi a^2 b^2 I}{2(b^2+z^2)^{3/2}}$$

$$\Phi = MI \Rightarrow M = \frac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}} = \frac{\mu_0 \pi a^2 b^2}{2c^3} \checkmark$$

7. Wangsness 19-11. $U_{DD} = \frac{\mu_0}{4\pi R^3}[(\vec{m}_1\cdot\vec{m}_2) - 3(\vec{m}_1\cdot\hat{R})(\vec{m}_2\cdot\hat{R})]$ (eqn 19-56)

$$\Rightarrow U_{DD} = \frac{\mu_0}{4\pi R^3}\{[m_1 m_2 \cos(\alpha_2-\alpha_1)] - 3m_1\cos\alpha_1\, m_2\cos\alpha_2\}$$

$$= \frac{\mu_0}{4\pi R^3}\{m_1 m_2 [\cos(\alpha_2-\alpha_1) - 3\cos\alpha_1\cos\alpha_2]\} \qquad \boxed{\cos(A-B) = \cos A\cos B + \sin A \sin B}$$

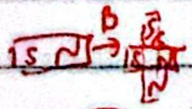
$$= \frac{\mu_0 m_1 m_2}{4\pi R^3}(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2 - 3\cos\alpha_1\cos\alpha_2)$$

$$= \frac{\mu_0 m_1 m_2}{4\pi R^3}(-2\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2)$$

m_1 fixed & m_2 rotates $\Rightarrow \tau_2 = 0 \Rightarrow \frac{\partial U}{\partial\alpha_2} = 0$ (torque at $m_2 = 0$)

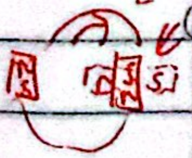
$$\Rightarrow -2\cos\alpha_1(-\sin\alpha_2) + \sin\alpha_1\cos\alpha_2 = 0 \Rightarrow \sin\alpha_1\cos\alpha_2 = -2\cos\alpha_2\sin\alpha_2$$

$$\Rightarrow \frac{\sin\alpha_1}{\cos\alpha_1} = -2\frac{\sin\alpha_2}{\cos\alpha_2} \Rightarrow \tan\alpha_1 = -2\tan\alpha_2. \checkmark \left(\tan\alpha_2 = -\frac{1}{2}\tan\alpha_1\right)$$



(*) $\alpha = 0° \Rightarrow 0 = -2\tan\alpha_2 \Rightarrow \alpha_2 = 0$ or π $\checkmark$ $\Rightarrow U_{DD} = \frac{\mu_0}{4\pi R^3}\cdot 2m_1 m_2 < 0$ (stable) $\checkmark$

(*) $\alpha = \frac{\pi}{2} \Rightarrow \tan\frac{\pi}{2} = -2\tan\alpha_2 \Rightarrow \infty = -2\tan\alpha_2 \Rightarrow \alpha_2 = \frac{-\pi}{2}$ $\checkmark$ $\Rightarrow U_{DD} = \frac{\mu_0}{4\pi R^3} > 0$ (unstable) ?



8. Wangsness 19-16.

$$\vec{A} = \frac{\mu_0}{4\pi}\left(\frac{\vec{m}\times\vec{r}_1}{|\vec{r}_1|^3} + \frac{-\vec{m}\times\vec{r}_2}{|\vec{r}_2|^3}\right) = \frac{\mu_0 m}{4\pi}\left(\frac{\hat{z}\times(r\hat{r} - a\hat{z})}{(r^2+a^2-2ar\cos\theta)^{3/2}}\right)$$

$$= \frac{\mu_0 m r\sin\theta\,\hat{\phi}}{4\pi r^3}\left[\frac{1}{(1+\frac{a^2}{r^2}-\frac{2a\cos\theta}{r})^{3/2}} - \frac{1}{(1+a^2/r^2+\frac{2a\cos\theta}{r})^{3/2}}\right]$$

$$= \frac{3\mu_0 m a\sin\theta\cos\theta}{2\pi r^3} \checkmark \Rightarrow B = \vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta A_\phi) - \frac{1}{r}\frac{d}{dr}(rA_\phi)$$

$$\Rightarrow \vec{B} = \frac{3\mu_0 m a}{2\pi}\left[\frac{1}{r^4}(3\cos^2\theta - 1)\hat{r} + \frac{2\sin\theta\cos\theta}{r^4}\hat{\theta}\right] + 0\hat{\phi}$$

$$C'' = \frac{8Q}{16\pi\varepsilon_0}$$

$\Rightarrow$ Analogue to $E_r = \left(\frac{3Qa}{8\pi\varepsilon_0}\right)\frac{3\cos^2\theta - 1}{2r^4}$ $\quad E_\phi = 0$

$$E_\theta = \left(\frac{3Qa}{8\pi\varepsilon_0}\right)\frac{\cos\theta\sin\theta}{r^4} \checkmark$$

Grading Space for 6,8,7: All Correct for 6 & similar answers for 8. As for 7, I did not explain conceptually why the answer makes sense.

Overall: Math wise, this is an easy homework. However, I am still stuck at conceptual questions like 7 & 4 as I am not fully understand the concepts behind the math, which is something I need to work on more.