HOMEWORK 5

1. 

$$\vec{F} = \frac{kq^2}{4z^2} \Rightarrow \vec{a} = \frac{kq^2}{4mz^2} = -v\frac{dv}{dx} \Rightarrow v^2 = 2\cdot\frac{kq^2}{4m}\left(\frac{1}{z} - \frac{1}{d}\right)$$

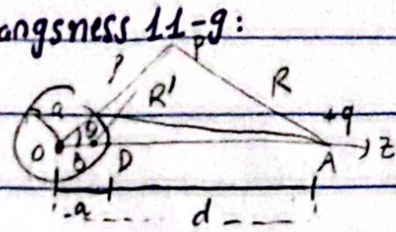
$$\Rightarrow v = \sqrt{\frac{kq^2}{2m}\left(\frac{d-z}{zd}\right)} = \frac{dz}{dt} \Rightarrow \sqrt{\frac{kq^2}{2m}}\int_0^t dt = \int_0^d \sqrt{\frac{z}{d-z}}\,dz$$

$z = d\cos^2\theta$, $dz = 2d\sin\theta\cos\theta\, d\theta$

$$\Rightarrow \sqrt{\frac{kq^2}{2m}}\,t = \int_0^d \sqrt{\frac{d\sin\theta}{d\cos^2\theta}}\,2d\sin\theta\cos\theta\, d\theta = 2d\sqrt{d}\int_{\pi/2}^{0}\sin^2\theta$$

$$\Rightarrow t = \frac{2d\sqrt{d}\sqrt{2m}}{\sqrt{kq^2}}\frac{\pi}{4} = \frac{\pi}{2}\sqrt{\frac{2md^3}{kq^2}}$$

3. Wangsness 11-9:



Potential at O due to charge $q\frac{a}{d}$:

$$\frac{q(a/d)}{4\pi\varepsilon_0 a} = \frac{q}{4\pi\varepsilon_0 d} = \text{potential on the sphere's surface.}$$

$\Rightarrow$ Potential at any point outside the sphere:

$$\Phi = \frac{q}{4\pi\varepsilon_0}\Bigg[\underbrace{\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}}}_{\text{from } +q \text{ @ } A} - \underbrace{\frac{a}{\sqrt{r^2d^2+a^4-2rda^2\cos\theta}}}_{\text{from } -qa/d \text{ @ } b} + \underbrace{\frac{a}{dr}}_{\text{from } qa/d \text{ @ } O}\Bigg] + \underbrace{\frac{Q}{4\pi r\varepsilon_0}}_{\text{total charge on sphere.}}$$

$$\Rightarrow E = -\frac{\partial\Phi}{\partial\rho} = \frac{q}{4\pi\varepsilon_0}\left[\frac{r-d\cos\theta}{(r^2+d^2-2rd\cos\theta)^{3/2}} - \frac{ad(rd-a^2\cos\theta)}{(r^2d^2+a^4-2rda^2\cos\theta)^{3/2}} + \frac{a}{dr^2}\right] + \frac{Q}{4\pi\varepsilon_0}$$

$\Rightarrow$ Force on q @ A:

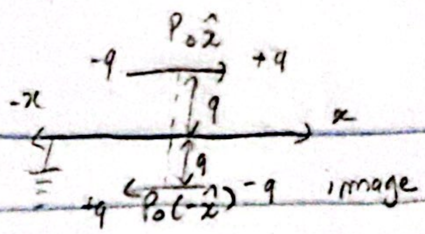
$$\vec{F} = \frac{-q^2}{4\pi\varepsilon_0}\left(\frac{a}{d}\right)^3\frac{2d^2-a^2}{(d^2-a^2)^2}\hat{z} + \frac{Q}{4\pi\varepsilon_0 d^2}\hat{z} = \frac{q}{4\pi\varepsilon_0}\left[\frac{Q}{d^2} - \frac{qa^3}{d^3}\frac{2d^2-a^2}{(d^2-a^2)^2}\right]\hat{z}$$

$\Rightarrow$ Potential on sphere:

$$\Phi_{sphere} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+d^2}} - \frac{a}{\sqrt{a^2d^2+a^4}} + \frac{a}{da}\right] + \frac{Q}{4\pi a\varepsilon_0} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{a^2+d^2}} - \frac{1}{\sqrt{a^2+d^2}} + \frac{1}{d}\right]$$

$$+ \frac{Q}{4\pi\varepsilon_0 a} = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{a} + \frac{q}{d}\right]$$

Grading Space 1 & 3.



4. There will be a force on the dipole b/c

$\Rightarrow$ Image of the dipole's also a dipole with opposite polarity $\Rightarrow$ there will be a force between their interaction.

$$U = -p_0\hat{x} \cdot \frac{(-p_0\hat{x})}{4\pi\varepsilon_0(2d)^3} = \frac{p_0^2}{16\pi\varepsilon_0 d^3} \Rightarrow F = -\frac{dU}{dr} = \frac{-P_0^2}{16\pi\varepsilon_0}\frac{d}{dr}\frac{1}{r^3} \quad (r=d)$$

$$\Rightarrow F = \frac{3P_0^2}{16\pi\varepsilon_0 d^4}$$ is the force on the dipole.

6. $$\nabla^2\Phi = 0 = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} \quad \text{for } \Phi(x,y) = X(x)\,Y(y)$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = K^2 \Rightarrow X = Ae^{Kx} + Be^{-Kx} \quad (k \text{ is a constant})$$

$$\Rightarrow \frac{1}{Y}\frac{d^2Y}{dy^2} = -K^2 \Rightarrow Y = C\cos(Ky) + D\sin(Ky)$$

$$\Rightarrow \Phi = (Ae^{kx} + Be^{-kx})[C\cos(ky) + D\sin(ky)]$$

$$\Rightarrow \Phi(x, y=0) = 0 \Rightarrow C = 0$$

$$\Phi(x, y=a) = 0 \Rightarrow Ka = m\pi \Rightarrow k = \frac{m\pi}{a}$$

$$\Rightarrow \Phi = (Ae^{kx} + Be^{-kx})\,D\sin\left(\frac{m\pi}{a}y\right)$$

$$\Phi'(x=0, y) = 0 = (A+B)\,D\sin\left(\frac{m\pi}{a}y\right) \Rightarrow A = -B$$

$$\Rightarrow \Phi' = \sum_m (Ae^{kx} + Be^{-kx})\,D\sin\left(\frac{m\pi}{a}y\right)$$

$$= \sum_m \sinh\left(\frac{m\pi}{a}x\right) D\sin\left(\frac{m\pi}{a}y\right)$$

$$\Phi'(x=a, y) = \Phi_1 = \sum_m \sinh(m\pi)\,D\sin\left(\frac{m\pi}{a}y\right)$$

Multiply both sides by $\sin\left(\frac{m\pi}{a}y\right)$ & integrating:

$$\int_a^b \sin\left(\frac{m\pi}{a}\right)\Phi_0\,dy = \sum_m \sinh\left(\frac{m\pi}{a}y\right) D\sin\left(\frac{m\pi}{a}y\right)dy$$

$$\Rightarrow \Phi' = \sum_m \frac{2\Phi_0}{n\pi}(1-\cos n\pi)\sin\left(\frac{m\pi}{a}y\right)\frac{\sinh\left(\frac{m\pi}{a}x\right)}{\sinh m\pi}$$

Since $\cos m\pi = (-1)^m$

$$\Rightarrow \Phi' = \sum_{m \text{ odd}} \frac{4\phi_0}{m\pi}\sin\left(\frac{m\pi}{a}y\right)\frac{\sinh\left(\frac{m\pi}{a}x\right)}{\sinh m\pi}$$

Grading Space for 4&6:

8. $\Phi = A + B\ln\rho + \sum_{m=1}^{\infty}\left(A_m\rho^m + \frac{B_m}{\rho^m}\right) \times (C_m\cos m\phi + D_m\sin m\phi)$

$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}$ for polar. Apply this on function $f(\rho,\phi)$:

For $f(\rho,\phi) = P(\rho)\Phi(\phi)$

$\Rightarrow \nabla^2 f = 0 \Rightarrow \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} = 0$

$\Rightarrow P''(\rho)\Phi(\phi) + \frac{1}{\rho}P'(\rho)\Phi(\phi) + \frac{1}{\rho^2}P(\rho)\Phi''(\phi) = 0 \Leftrightarrow \frac{\rho^2 P''(\rho)}{P(\rho)} + \frac{\rho P'(\rho)}{P(\rho)} + \frac{\Phi''(\phi)}{\Phi(\phi)} = 0$

$\Rightarrow \frac{\rho^2 P''(\rho)}{P(\rho)} + \frac{\rho P'(\rho)}{P(\rho)} = -\frac{\Phi''(\phi)}{\Phi(\phi)} = K$ (K is a const)

$\Rightarrow \begin{cases}\rho^2 P''(\rho) + \rho P'(\rho) - KP(\rho) = 0 \\ \Phi''(\phi) + K\Phi(\phi) = 0\end{cases} \Rightarrow \begin{cases}\Phi(\phi) = C\cos(\sqrt{K}\phi) + D\sin(\sqrt{K}\phi) \text{ for } K>0 \\ \Phi(\phi) = C + D\phi \text{ for } K=0\end{cases}$

$\Rightarrow K$ must be an integer & $\sqrt{K} \geq 0 \Rightarrow K = m^2$ for $m = 0, 1, 2, 3 \ldots$

$\Rightarrow \begin{cases}\Phi(\phi) = C_m\cos(m\phi) + D_m\sin(m\phi) \quad (m>0) \\ \Phi(\phi) = C_0 \quad (m=0)\end{cases}$

$\Rightarrow \begin{cases}P(\rho) = A_m\rho^m + B_m\rho^{-m} \quad (m>0) \\ P(\rho) = A_0 + B_0\ln\rho \quad (m=0)\end{cases}$ Are the solutions for $\rho^2 P''(\rho) + \rho P'(\rho) - m^2 P(\rho) = 0$

$\Rightarrow f(\rho,\phi) = A + B\ln\rho + \sum_{m=1}^{\infty}\left(A_m\rho^m + \frac{B_m}{\rho^m}\right)\cdot(C_m\cos m\phi + D_m\sin m\phi)$ (1)

9. From 8: $\Phi(\rho,\phi) = A_0 + B_0\ln\rho + \sum_{m=1}^{\infty}(A_m\cos(m\phi) + B_m\sin(m\phi))\frac{1}{\rho^m} + \sum_{m=1}^{\infty}$ $[C_m\cos(n\phi) + D_m\sin(n\phi)]\,\rho^m$

$A_0 = B_0 = 0$ for grounded neutral cylinder.

$r\to\infty$; $\Phi(\rho,\phi) = -E_0\rho\cos\phi \Rightarrow C_m = -E_0\ (m=1)$ & $C_m = 0\ (m\geq 2)$

$D_m = 0$ (all m)

$\Phi(\rho, -\Phi) = \Phi(\rho,\phi) \Rightarrow D_m = 0$

$A_m \neq 0\ (n=1)$ & $A_m = 0\ (n\geq 2)$

$\Rightarrow \Phi(\rho,\phi) = -\frac{A_1 \cos\phi}{\rho} - E_0\rho\cos\phi$ (only $m=1$ makes sense)

At $\rho = a$ $\Phi(\rho, \phi) = 0$

$\Rightarrow \Phi(a,\phi) = 0 \Rightarrow A_1 = -E_0 a^2 \Rightarrow \Phi(\rho,\phi) = \frac{E_0 a^2 \cos\phi}{\rho} - E_0 \rho\cos\phi$

$\Phi(\rho,\phi) = -E_0\rho\cos\phi\left(1 - \frac{a^2}{\rho^2}\right)$

From boundary conditions: $\sigma = \left(-\varepsilon_0 \frac{\partial V}{\partial r}\right)$

$\Rightarrow \sigma = \varepsilon_0\left[\frac{d}{dr}\left(\frac{E_0 a^2\cos\phi}{\rho} - E_0\rho\cos\phi\right)\right] = \frac{E_0 a^2 \cos\phi}{a^2} + E_0 a\cos\phi$

$\Rightarrow \sigma = 2E_0 a\cos\phi$

5. $\nabla^2\Phi = -\frac{\rho_0}{\varepsilon_0}$ (z-dependent only) $\Rightarrow \frac{\partial^2\Phi}{\partial z^2} = -\frac{\rho_0}{\varepsilon_0}$

$\Rightarrow \Phi = -\frac{\rho_0}{2\varepsilon_0} z^2 + Az + B$ Boundary Con: $\Phi(x,y,0) = -\frac{\rho_0}{2\varepsilon_0} 0^2 + A\cdot 0 + B$

$\Rightarrow B = 0$

$\Phi(x,y,d) = \Phi' = -\frac{\rho_0}{2\varepsilon_0} d^2 + Ad \Rightarrow A = \frac{\Phi'}{d} + \frac{\rho_0 d}{2\varepsilon_0}$

$\Rightarrow \Phi = -\frac{\rho_0}{2\varepsilon_0} z^2 + \left(\frac{\Phi'}{d} + \frac{\rho_0 d}{2\varepsilon_0}\right) z$

$\sigma_0 = -\varepsilon_0 \left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = -\frac{\varepsilon_0\Phi'}{d} - \frac{\rho_0 d}{2}$

$\sigma_d = \varepsilon_0 \left.\frac{\partial\Phi}{\partial z}\right|_{z=d} = -\rho_0 d + \frac{\varepsilon_0\Phi'}{d} + \frac{\rho_0 d}{2}$

$\Rightarrow 0 = \frac{-\rho_0}{\varepsilon_0} z + \left(\frac{\Phi'}{d} + \frac{\rho_0 d}{2\varepsilon_0}\right) \Leftrightarrow z = \frac{\Phi'\varepsilon_0}{d\rho_0} + \frac{d}{2}$

Grading Space for 8, 9, 5:

2. a) The images' charges are q @ $(-a,-b)$ & $-q$ @ $(-a,b)$ & $(a,-b)$
This satisfy the boundary condition for $\Phi=0$ since positive charge images and the negative charge images cancelled out along the planes.
In the region where $x>0$ & $y>0$, $\Phi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right)$

b) $\sigma = \varepsilon_0 E_n = -\varepsilon_0 \frac{\partial \Phi}{\partial y}\Big|_{y=0} = -\varepsilon_0 \frac{\partial}{\partial y}\left[\frac{q}{4\pi\varepsilon_0}\left(\cdots\right)\right]_{y=0}$

$\Rightarrow \sigma = \frac{-q}{4\pi}\left(\frac{b-y}{[(x-a)^2+(y-b)^2+z^2]^{3/2}} + \frac{y-b}{[(x+a)^2+(y-b)^2+z^2]^{3/2}} + \frac{b+y}{[(x-a)^2+(y+b)^2+z^2]^{3/2}} - \frac{b-y}{[(x+a)^2+(y+b)^2+z^2]^{3/2}}\right) = \frac{-qb}{2\pi}\left[\frac{1}{((x-a)^2+b^2+z^2)^{3/2}} - \frac{1}{((x+a)^2+b^2+z^2)^{3/2}}\right]$

c) $\vec{F} = \sum \vec{F}_{image} = \frac{-q^2}{16\pi\varepsilon_0 a^2}\hat{x} - \frac{q^2}{16\pi\varepsilon_0 b^2}\hat{y} + \frac{q^2}{16\pi\varepsilon_0(a^2+b^2)}\frac{a\hat{x}+b\hat{y}}{\sqrt{a^2+b^2}}$

$\Rightarrow \vec{F} = \frac{q^2}{16\pi\varepsilon_0}\left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\hat{x} + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\hat{y}\right]$

7. a) $\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell[\cos(\theta)]$

$\Phi(\infty,\theta) = 0 \Rightarrow A_\ell = 0$ (out) & $\Phi(0,\theta) = 0 \Rightarrow B_\ell = 0$ (in)

$\Phi = \Phi_{top} + \Phi_{bot} \Rightarrow \Phi_{bot}(r,\theta) = -\Phi_{top}(-r,\theta)$

$\Rightarrow A_\ell = \frac{1}{2\varepsilon_0 R^{\ell-1}}\int_0^\pi \sigma(\theta) P_\ell \cos\theta \sin\theta \, d\theta$

$= \frac{1}{2\varepsilon_0 R^{\ell-1}}\left(\int_0^{\pi/2} \sigma_0 P_\ell \cos\theta \sin\theta \, d\theta - \int_{\pi/2}^{\pi} \sigma_0 P_\ell \cos\theta \sin\theta \, d\theta\right)$

$= \frac{\sigma_0}{\varepsilon_0 R^{\ell-1}}\int_0^{\pi/2} P_\ell \cos\theta \sin\theta \, d\theta$

$\Rightarrow A_1 = \frac{\sigma_0}{\varepsilon_0 R^0}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta = \frac{\sigma_0}{2\varepsilon_0}$, $A_3 = \frac{\sigma_0}{\varepsilon_0 R^2}\int_0^{\pi/2}\frac{1}{2}(5\cos^3\theta - 3\cos\theta)\sin\theta\,d\theta = \frac{-\sigma}{8\varepsilon_0 R^2}$

$\Rightarrow B_\ell = A_\ell R^{2\ell+1} \Rightarrow B_1 = \frac{\sigma_0 R^3}{2\varepsilon_0} \Rightarrow B_3 = \frac{-\sigma R^5}{8\varepsilon_0}$

$$\Phi_{in} = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell \cos\theta \text{ (odd } \ell \text{ only)} = \frac{\sigma_0 r}{2\varepsilon_0} P_1 \cos\theta - \frac{\sigma r^3}{8\varepsilon_0 R^2} P_3 \cos\theta \ldots$$

$$\Phi_{out} = \sum_{\ell}^{\infty} \frac{b_\ell}{r^{\ell+1}} P_\ell \cos\theta \text{ (odd } \ell \text{ only)} = \frac{\sigma_0 R^3}{2\varepsilon_0 r^2} P_1 \cos\theta - \frac{\sigma R^5}{8\varepsilon_0 r^4} P_3 \cos\theta \ldots$$

b) $\vec{p} = \int_S \rho(\vec{r}')\, \vec{r}'\, d\tau' = 2\int_0^{\pi/2}\int_0^{2\pi} \sigma_0 R\cos\theta\, \hat{z}\, R^2 \sin\theta\, d\phi\, d\theta$

$\vec{p} = 2\pi\sigma_0 R^3 \hat{z} \Rightarrow \Phi = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2} = \frac{\sigma_0 R^3 \cos\theta}{2\varepsilon_0 r^2} = \Phi_{out}$ 1st term.

Grading Space for 2 & 7 & Conclusion: