HOMEWORK 4

1. Equation 9-17: $(\hat{n} \times \vec{F}) \times \hat{n} = \vec{F} - F_n \hat{n} = \vec{F_t}$

Equation 1-23: $\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B})$

Equation 1-30: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Equation 1-17: $\vec{A}^2 = \vec{A} \cdot \vec{A} = A^2$

Equation 9-14: $\vec{F} = \vec{F}_n + \vec{F}_t = F_n \hat{n} + \vec{F_t} \Rightarrow \vec{F} - F_n \hat{n} = \vec{F_t}$

$\Rightarrow (\hat{n} \times \vec{F}) \times \hat{n} = -\hat{n} \times (\hat{n} \times \vec{F})$ (1-23) $= -\hat{n}(\hat{n} \cdot \vec{F}) + \vec{F}(\underbrace{\hat{n} \cdot \hat{n}}_{|\hat{n}|^2 = 1 \text{ (1-17)}})$ (1-30)

$= \vec{F} - \hat{n}(\hat{n} \cdot \vec{F})$

$\Rightarrow \hat{n} \cdot \vec{F} = \hat{n} \cdot (F_n \hat{n} + \vec{F_t}) = F_n |\hat{n}|^2 + \underbrace{\cancel{\hat{n} \vec{F_t}}^{\,0}}_{\hat{n} \perp \vec{F_t}} = F_n$

$\Rightarrow \vec{F} - \hat{n}(\hat{n} \cdot \vec{F}) = \vec{F} - \hat{n} F_n = \vec{F_t}$.

2. Solution to 5-17: $\phi = \frac{\sigma}{2\varepsilon_0}\left[(a^2+z^2)^{1/2} - |z|\right]$, $\vec{E} = \hat{z}\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{|z|}\right)\left[1 - \frac{|z|}{(a^2+z^2)^{1/2}}\right]$

$\Rightarrow A = 2\pi r\,dr,\quad dq = \sigma 2\pi r\,dr$

$$\Rightarrow d\phi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{r^2+|z|^2}} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 2\pi r\,dr}{\sqrt{r^2+|z|^2}} = \frac{\sigma r\,dr}{2\varepsilon_0\sqrt{r^2+|z|^2}}$$

$$\Rightarrow \phi = \int_0^a \frac{\sigma r\,dr}{2\varepsilon_0\sqrt{r^2+|z|^2}} = \frac{\sigma}{2\varepsilon_0}\int_0^a \frac{r\,dr}{\sqrt{r^2+|z|^2}} = \frac{\sigma}{2\varepsilon_0}\sqrt{r^2+|z|^2}\,\Big|_0^a$$

$$\Rightarrow \phi = \frac{\sigma}{2\varepsilon_0}\left[(a^2+z^2)^{1/2} - |z|\right]$$

$$\Rightarrow \vec{E} = -\frac{dV}{da} = \frac{\sigma}{2\varepsilon_0}\left(\frac{z}{|z|}\right)\left[1 - \frac{|z|}{(a^2+z^2)^{1/2}}\right]\hat{z}$$ (No tangential field due to symmetry).

3 (Wagsness 9-1). $2x+y+z=1$, $\vec{E_1} = 4\hat{x}+\hat{y}-3\hat{z}$

$$\Rightarrow \vec{n} = 2\hat{x}+\hat{y}+\hat{z} \Rightarrow \hat{n}_1 = \frac{2\hat{x}+\hat{y}+\hat{z}}{\sqrt{4+1+1}} = \frac{2\hat{x}+\hat{y}+\hat{z}}{\sqrt{6}}$$

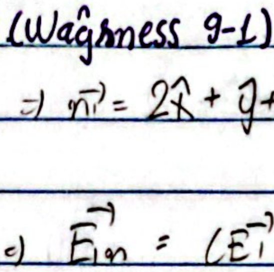
$$\Rightarrow \vec{E_{1n}} = (\vec{E_1}\cdot\hat{n}_1)\hat{n}_1 = \left(4\hat{x}+\hat{y}-3\hat{z} \cdot \frac{2\hat{x}+\hat{y}+\hat{z}}{\sqrt{6}}\right)\left(\frac{2\hat{x}+\hat{y}+\hat{z}}{\sqrt{6}}\right)$$
$$= \underbrace{(8+1-3)}_{\cancel{6}}\left(\frac{2\hat{x}+\hat{y}+\hat{z}}{\cancel{6}}\right) = 2\hat{x}+\hat{y}+\hat{z}$$

$$\Rightarrow \vec{E_{1t}} = \vec{E_1} - \vec{E_{1n}} = (4\hat{x}+\hat{y}-3\hat{z}) - (2\hat{x}+\hat{y}+\hat{z}) = 2\hat{x}-4\hat{z}$$

$$\Rightarrow \vec{E_{1n}}\cdot\vec{E_{1t}} = (2\hat{x}+\hat{y}+\hat{z})\cdot(2\hat{x}-4\hat{z}) = 4-4 = 0 \Rightarrow \vec{E_{1n}} \perp \vec{E_{1t}}$$

4 (Wangsness 9-3). $\vec{E_1} = \alpha\hat{x}+\beta\hat{y}+\gamma\hat{z}$, $\alpha=\beta=0$, $\gamma=E_0$, $\sigma=\sigma_0\cos\theta$

According to boundary con: $\vec{E_{2n}} - \vec{E_{1n}} = \frac{\sigma}{\varepsilon_0}\hat{n}$ & $\vec{E_{2t}} - \vec{E_{1t}} = 0$



$$\Rightarrow \vec{E_2} - \vec{E_1} = \frac{\sigma}{\varepsilon_0}\hat{n} = \frac{\sigma}{\varepsilon_0}\frac{\vec{a}}{|\vec{a}|} \Rightarrow \vec{E_2} = \vec{E_1} + \frac{\sigma}{\varepsilon_0}\frac{\vec{a}}{|\vec{a}|}$$

$$\Rightarrow \vec{a} = x\hat{x}+y\hat{y}+z\hat{z} \Rightarrow |\vec{a}| = \sqrt{x^2+y^2+z^2}$$

$$z = |\vec{a}|\cos\theta \Rightarrow \cos\theta = \frac{z}{|\vec{a}|}$$

$$\Rightarrow \vec{E_2} = (\alpha\hat{x}+\beta\hat{y}+\gamma\hat{z}) + \frac{\sigma_0\cos\theta}{\varepsilon_0}\times\frac{\vec{a}}{|\vec{a}|} = (\alpha\hat{x}+\beta\hat{y}+\gamma\hat{z}) + \frac{\sigma_0}{\varepsilon_0}\times\frac{z}{|\vec{a}|}\times\frac{\vec{a}}{|\vec{a}|}$$
$$= (\alpha\hat{x}+\beta\hat{y}+\gamma\hat{z}) + \frac{\sigma_0}{\varepsilon_0}\times\frac{z}{|\vec{a}|^2}\vec{a} = E_0\hat{z} + \frac{\sigma_0}{\varepsilon_0}\frac{z}{\underbrace{(x^2+y^2+z^2)}_{=a^2}}(x\hat{x}+y\hat{y}+z\hat{z})$$

$$\Rightarrow \vec{E_2} = \frac{\sigma_0 xz}{\varepsilon_0 a^2}\hat{x} + \frac{\sigma_0 yz}{\varepsilon_0 a^2}\hat{y} + \left(E_0 + \frac{\sigma z^2}{\varepsilon_0 a^2}\right)\hat{z}$$

Grading Space for 1-4:

5. $l = 0.001m$, $V = 500V$, $R = 0.529\mathring{A}$ $= 0.529 \cdot 10^{-10}m$, $\frac{\alpha}{4\pi\varepsilon_0} = 0.66 \cdot 10^{-30} m^3$

Magnitude of dipole moment of atom: $p = ed$

Magnitude of dipole momentum of the hydrogen atom: $p = \alpha E$ $\left.\right\} \Rightarrow ed = \alpha E$, $E = V/l$

$\Rightarrow$ Separation: $d = \frac{\alpha E}{e} = \frac{\alpha V}{el} = \frac{0.66 \cdot 10^{-30} \cdot 4\pi\varepsilon_0 \cdot 500}{1.6 \cdot 10^{-19} \cdot 0.001} \approx 229 \cdot 10^{-16} m$

$\Rightarrow \frac{d}{R} = \frac{229 \cdot 10^{-16}}{0.529 \cdot 10^{-10}} \approx 4.34 \cdot 10^{-6} m$

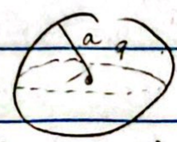
6. (Wangsness 10-12) $\vec{P} = P\hat{x}$

$\int \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0} = \frac{\sigma\pi r^2 h}{\varepsilon_0} = \frac{P\pi r^2 h}{\varepsilon_0}$ (h is height of Gaussian cylinder)

$\int \vec{E} \cdot d\vec{A} = \int \vec{E} 2\pi r h dz = \vec{E} 2\pi r h = \frac{\vec{P}\pi r^2 h}{\varepsilon_0} \Rightarrow \vec{E} = \frac{P\hat{x}}{2\varepsilon_0}$

$\sigma = P$ since the cylinder is uniformly polarized.

The total bound charge $= 0$.



7. (Wangsness 10-17).

Electric Displacement: $\oint \vec{D} \cdot da = D 4\pi r^2 = q \Rightarrow \vec{D} = \frac{q}{4\pi r^2}\hat{r}$

Electric Field: $\vec{D} = K_e \varepsilon_0 \vec{E} = (1 + \chi_e)\varepsilon_0 \vec{E}$, $\chi_e = 0$ when $r > a$

$\Rightarrow \vec{E}_{r>a} = \frac{q}{4\pi r^2 \varepsilon_0}\hat{r}$ $\Rightarrow \vec{E}_{r<a} = \frac{q}{4\pi r^2 \varepsilon_0 k_e}\hat{r}$ $(\chi_e > 0 \rightarrow (1 + \chi_e) = K_e)$

Polarization: $\vec{P} = \varepsilon_0 \chi_e \vec{E} = (K_e - 1)\varepsilon_0 \vec{E}$

$\chi_e = 0$ when $r > a \Rightarrow \vec{P}_{r>a} = 0$

$\Rightarrow \vec{P}_{r<a} = (K_e - 1) \cdot \frac{q}{4\pi r^2 \varepsilon_0 K_e}\hat{r}$

Grading Space for 5 & 6:

Bound charge on surface. $Q = \sigma_b A$

$\sigma_b = \vec{P}\cdot\hat{n} = (K_e - 1)\dfrac{q}{4\pi r^2 \varepsilon_0 K_e}\hat{r}\cdot\hat{r} = (K_e-1)\dfrac{q}{4\pi r^2 \varepsilon_0 K_e}$

$\Rightarrow Q = (K_e-1)\dfrac{q}{4\pi r^2 \varepsilon_0 K_e}\cdot 4\pi r^2 = \dfrac{q}{\varepsilon_0 K_e}(K_e-1)$

b) $\rho_b = -\nabla\cdot\vec{P} = -\nabla\cdot(K_e-1)\dfrac{q}{4\pi r^2 \varepsilon_0 K_e}\hat{r} = \dfrac{-q}{-K_e}\delta^3(\vec{r})(K_e-1)$

The total bound charge is 0.

a) $\vec{E}_n = (\vec{E}\cdot\hat{n})\hat{n} = \dfrac{q}{4\pi r^2 \varepsilon_0}\hat{r}$ (for $r > a$) & $= \dfrac{q}{4\pi r^2 \varepsilon_0 K_e}\hat{r}$ (for $r < a$)

$\vec{D}_n = (\vec{D}\cdot\hat{n})\hat{n} = \dfrac{q}{4\pi r^2}\hat{r}$ $\Rightarrow$ satisfy the boundary conditions.

8. (Wangeness 10-19).

$\vec{E}_{b>\rho>a} = \dfrac{\lambda}{2\pi\varepsilon_0\rho}\hat{\rho}$ & $\vec{E} = 0$ else where

$= \dfrac{\lambda f}{2\pi\varepsilon_0 \alpha\rho^n}\hat{\rho} \Rightarrow \vec{D} = K_e\varepsilon_0\vec{E} = \dfrac{\lambda f}{2\pi\rho^n}\hat{\rho}$

$\rho_b = -\nabla\cdot\vec{P}$, $\vec{P} = (K_e-1)\varepsilon_0\vec{E} = \dfrac{\lambda f}{2\pi\rho^n}\hat{\rho}$

$\Rightarrow \rho_b = -\nabla\cdot\dfrac{\lambda f}{2\pi\rho^n}\hat{\rho} = \dfrac{\lambda f}{2\pi\rho^{n+1}}$

When $n = 1$, $\vec{E}$ would be constant. $\vec{D} = \dfrac{\lambda f}{2\pi\rho}$ & $\rho_b = \dfrac{\lambda\rho}{2\pi\rho^2}$

$U_e = \dfrac{1}{2}\int_0^l\int_0^{2\pi}\Delta\phi = \dfrac{1}{2}\int_0^l\int_0^{2\pi}\dfrac{\lambda f}{\alpha n 2\pi\varepsilon_0}\left(\dfrac{1}{a^n} - \dfrac{1}{b^n}\right)\dfrac{\lambda f}{2\pi a}\,a\,dr\,d\theta$

$\Rightarrow U_e = \dfrac{l\lambda f^2}{4\pi\varepsilon_0\alpha n}\left(\dfrac{1}{a^n} - \dfrac{1}{b^n}\right)$

9. (Wangsness 10-25). $C = \dfrac{2\pi(\varepsilon_1+\varepsilon_2)ab}{b-a}$

$C = Q/V$ Let l be the distance from K_{e1} → plate & d be the distance between 2 plate

$\Rightarrow K_e = K_{e1} + (K_{e2} - K_{e1})\dfrac{l}{d}$ The E field @ distance l from plate. $E = \sigma/\varepsilon_0$

$\Rightarrow \varepsilon = \varepsilon_1 + (\varepsilon_2 - \varepsilon_1)\dfrac{l}{d}$ $\Rightarrow \sigma = \varepsilon\cdot E \Rightarrow \sigma = \dfrac{\varepsilon_1 + (\varepsilon_2-\varepsilon_1)l}{d}\cdot\dfrac{\sigma_1 + (\sigma_2-\sigma_1)l}{\varepsilon_0 d}$

$Q = \int A\sigma\,dl = \dfrac{Al(\varepsilon_1+\varepsilon_2)(\sigma_1+\sigma_2)}{2\varepsilon_0}$

$V = \int_{ab} E\,dl = \dfrac{(\sigma_1+\sigma_2)l}{2\varepsilon_0}\cdot\ln\left(\dfrac{b-a}{K_{e2}-K_{e1}}\right)$

$\Rightarrow C = Q/V = \dfrac{2\pi(\varepsilon_1+\varepsilon_2)ab}{b-a}$

Grading Space for 7, 8, 9:

10. Boundary Condition: $\frac{K_{e1}}{K_{e2}} = \frac{\tan\theta_1}{\tan\theta_2} \Rightarrow \tan\theta_2 = 1/4 \Rightarrow \theta = \tan^{-1}(1/4) = 14°$

E is at 45° to the surface $\Rightarrow E_0 \cos 45° = \frac{E_0}{\sqrt{2}} = E$

$\sigma = \pm \frac{K_e - 1}{K_e} \sigma_{free} = \pm \frac{K_e - 1}{K_e} \varepsilon_0 E = \pm \frac{K_e - 1}{K_e} \varepsilon_0 \frac{E_0}{\sqrt{2}}$

Grading space for 10: