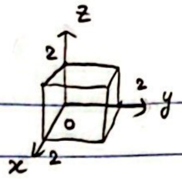
8.(1.33). $\vec{V} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$



Divergence theorem:

$$\iiint_V \underbrace{(\nabla\cdot\vec{A})}_{①}\,dV = \oint_S \underbrace{\vec{A}\cdot d\vec{a}}_{②}$$

① $\int_V \nabla\cdot d\vec{v}\,dV = \int_0^2\int_0^2\int_0^2 \left(\frac{\partial}{\partial x}xy + \frac{\partial}{\partial y}2yz + \frac{\partial}{\partial z}3xz\right)dx\,dy\,dz$

$= \int_0^2\int_0^2\int_0^2 (y+2z+3x)\,dx\,dy\,dz = \int_0^2\int_0^2 (2y+6x+4)\,dx\,dy$

$= \int_0^2 (12x+12)\,dx\,dy = 6\cdot4+24 = 48$

② $\oint_S \vec{V}\cdot d\vec{a} = \int_{S\,x=2} \vec{V}\cdot d\vec{a} + \int_{S\,x=0} \vec{V}\cdot d\vec{a} + \int_{S\,y=2} \vec{V}\cdot d\vec{a} + \int_{S\,y=0} \vec{V}\cdot d\vec{a} + \int_{S\,z=2} \vec{V}\cdot d\vec{a} + \int_{S\,z=0} \vec{V}\cdot d\vec{a}$

$= \int_0^2\int_0^2 xy\,dy\,dz\Big|_{x=2} - \cancel{\int_0^2\int_0^2 xy\,dy\,dz\Big|_{x=0}} + \int_0^2\int_0^2 2yz\,dx\,dz\Big|_{y=2} - \cancel{\int_0^2\int_0^2 2yz\,dx\,dz\Big|_{y=0}} + \int_0^2\int_0^2 3zx\,dy\,dx\Big|_{z=2}$

$+ \cancel{\int_0^2\int_0^2 3zx\,dx\,dy\Big|_{z=0}} = 2\int_0^2\int_0^2 y\,dy\,dz + 4\int_0^2\int_0^2 z\,d\,dx\,dz + 6\int_0^2\int_0^2 x\,dx\,dy$

$= 2\int_0^2 2y\,dy + 4\int_0^2 2z\,dz + 6\int_0^2 2x\,dx = 2\cdot4 + 4\cdot4 + 6\cdot4 = 48$

①=②= 48 $\Rightarrow$ Divergence theorem is proven.

$\bullet\ \nabla \times \vec{V}_c = \left(\frac{\partial}{\partial y}2yz - \frac{\partial}{\partial z}(2xy+z^2)\right)\hat{x} + \left(\frac{\partial}{\partial z}y^2 - \frac{\partial}{\partial x}2yz\right)\hat{y} + \left(\frac{\partial}{\partial x}(2xy+z^2) - \frac{\partial}{\partial y}(y^2)\right)$

$= \hat{x}(2z-2z) + \hat{y}(0) + \hat{z}(2y-2y) = 0\hat{x} + 0\hat{y} + 0\hat{z}$

(1.20) 5. Set vector $\vec{V} = a\hat{x} + b\hat{y} + c\hat{z}$ and div $\vec{V}$ & curl $\vec{V} = 0$

div $\vec{V} = 0 \Rightarrow \frac{\partial}{\partial x}a + \frac{\partial}{\partial y}b + \frac{\partial}{\partial z}c = 0 \Rightarrow \frac{\partial}{\partial x}a = \frac{\partial}{\partial y}b = \frac{\partial}{\partial z}c = 0$

curl $\vec{V} = 0 \Rightarrow \left(\frac{\partial}{\partial y}c - \frac{\partial}{\partial z}b\right)\hat{x} + \left(\frac{\partial}{\partial z}a - \frac{\partial}{\partial x}c\right)\hat{y} + \left(\frac{\partial}{\partial x}b - \frac{\partial}{\partial y}a\right)\hat{z} = 0\hat{x} + 0\hat{y} + 0\hat{z}$

$\Rightarrow a = yz,\ b = xz,\ c = xy$ (a indep from x, b indep from y, c indep from z)

$\Rightarrow$ div $\vec{V} = 0$

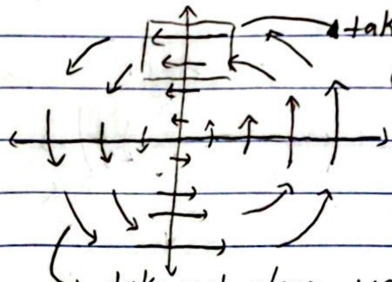
$\Rightarrow$ curl $\vec{V} = \left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\hat{x} + \left(\frac{\partial}{\partial z}yz - \frac{\partial}{\partial x}xy\right)\hat{y} + \left(\frac{\partial}{\partial x}xz - \frac{\partial}{\partial y}yz\right)\hat{z}$

$= (x-x)\hat{x} + (y-y)\hat{y} + (z-z)\hat{z} = 0\hat{x} + 0\hat{y} + 0\hat{z}$

$\Rightarrow \vec{V} = (yz)\hat{x} + (xz)\hat{y} + (xy)\hat{z}$

Sketch: I'm not really sure how to sketch this in 3D but in 2D it would be something like this:

Sorry for my poor artistic interpretation of this but the arrow in the same loop should be in the same size and each loop is evenly spaced from each other.



take this section for example: zoom in & it would be like this: no outward flow $\Rightarrow$ Div $= 0$

take out a loop: uniform flow $\Rightarrow$ no circulation $\Rightarrow$ Curl $= 0$

6. (1.26). a) $T_a = x^2 + 2xy + 3z + 4 \Rightarrow \nabla^2 T_a = \frac{\partial^2 T_a}{\partial x^2} + \frac{\partial^2 T_a}{\partial y^2} + \frac{\partial^2 T_a}{\partial z^2}$

$\Rightarrow \nabla^2 T_a = \frac{\partial}{\partial x}2x + 2y + \frac{\partial}{\partial y}2x + \frac{\partial}{\partial z}3 = \boxed{2}$

b) $T_b = \sin x \sin y \sin z \Rightarrow \nabla^2 T_b = \frac{\partial}{\partial x}\cos x \sin y \sin z + \frac{\partial}{\partial y}\sin x \cos y \sin z + \frac{\partial}{\partial z}\sin x \sin y \cos z$

$= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z = \boxed{-3(\sin x \sin y \sin z)}$

c) $T_c = e^{-5x}\sin 4y \cos 3z \Rightarrow \nabla^2 T_b = \frac{\partial}{\partial x} -5e^{-5x}\sin 4y \cos 3z + \frac{\partial}{\partial y}4e^{-5x}\cos 4y \cos 3z + \frac{\partial}{\partial z}(-3)e^{-5x}\sin 4y \sin 3z$

$\Rightarrow \nabla^2 T_b = 25e^{-5x}\sin 4y \cos 3z - 16e^{-5x}\sin 4y \cos 3z - 9e^{-5x}\sin 4y \cos 3z = \boxed{0}$

d) $\vec{V} = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z} \Rightarrow \nabla^2\vec{V} = \nabla^2\vec{V}_x + \nabla^2\vec{V}_y + \nabla^2\vec{V}_z$

$= \frac{\partial^2}{\partial x^2}x^2\hat{x} + \left(\frac{\partial^2}{\partial x^2}3xz^2 + \frac{\partial^2}{\partial z^2}3xz^2\right)\hat{y} + \left(\frac{\partial^2}{\partial x^2}(-2xz) + \frac{\partial^2}{\partial z^2}(-2xz)\right)\hat{z}$

$= 2\hat{x} + (0 + 6x)\hat{y} + 0\hat{z} = \boxed{2\hat{x} + 6x\hat{y} + 0\hat{z}}$

HW1

HW1: 1.(1.11)

a) $\nabla f(x,y,z) = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} = \hat{x}(2x) + \hat{y}(3y^2) + \hat{z}(4z^3)$

b) $\nabla f(x,y,z) = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} = \hat{x}(2xy^3z^4) + \hat{y}(3x^2y^2z^4) + \hat{z}(4x^2y^3z^3)$

c) $\nabla f(x,y,z) = \nabla(e^x \sin y \ln z) = \hat{x}(e^x \sin y \ln z) + \hat{y}(e^x \cos y \ln z) + \hat{z}(e^x \sin y \, 1/z)$

2.(1.12). $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$

a) Top of the hill $\Rightarrow$ no steepness $\Rightarrow \nabla h(x,y) = 0\hat{x} + 0\hat{y}$

$\Rightarrow 10\nabla h(x,y) \cdot [\hat{x}(2y - 6x - 18) + \hat{y}(2x - 8y + 28)] \cdot 10 = 0\hat{x} + 0\hat{y}$

$\Rightarrow \begin{cases} 2y - 6x - 18 = 0 \\ 2x - 8y + 28 = 0 \end{cases} \Rightarrow \begin{cases} y - 3x - 9 = 0 \\ x - 4y + 14 = 0 \end{cases} \Rightarrow \begin{cases} 4y - 12x - 36 = 0 \\ -4y + x + 14 = 0 \end{cases} \Rightarrow -11x = 22 \Rightarrow x = -2$

$\Rightarrow 2y = +6(-2) + 18 = -12 + 18 = 6 \Rightarrow y = 3$

$\Rightarrow$ The location of the top of the hill is $(-2,3)$, or 3 miles North & 2 miles West

b) $h(-2,3) = 10(2(-2)\cdot 3 - 3(-2)^2 - 4(3)^2 - 18(-2) + 28(3) + 12)$

$= 10(-12 - 12 - 36 + 36 + 84 + 12) = 10 \cdot 72 = 720$ ft

c) $\nabla h(1,1) = 10[\hat{x}(2 - 6 - 18) + \hat{y}(2 - 8 + 28)] = 10[\hat{x}(-22) + \hat{y}(22)]$

$= 220(\hat{y} - \hat{x}) = 220\sqrt{1^2 + 1^2} = 220\sqrt{2} \approx 311.13$ ft/mi

3.(1.15).

a) $\vec{v}_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z} \Rightarrow \nabla \cdot \vec{v}_a = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}3xz^2 - \frac{\partial}{\partial z}2xz$

$\Rightarrow \nabla \cdot \vec{v}_a = 2x - 2x = 0$

b) $\nabla \cdot \vec{v}_b = \frac{\partial}{\partial x}xy + \frac{\partial}{\partial y}2yz + \frac{\partial}{\partial z}3zx = y + 2z + 3x$

c) $\nabla \cdot \vec{v}_c = \frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}2yz = 2x + 2y$

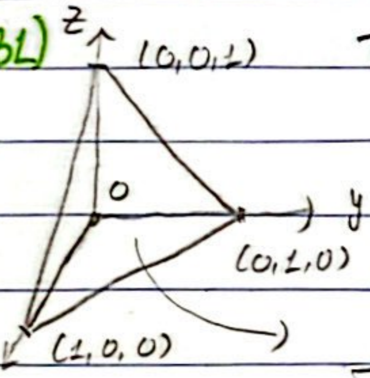
4.(1.18). $\cdot \nabla \times \vec{v}_a = \hat{x}\left(\frac{\partial}{\partial y}(-2xz) - \frac{\partial}{\partial z}3xz^2\right) + \hat{y}\left(\frac{\partial}{\partial z}x^2 - \frac{\partial}{\partial x}(-2xz)\right) + \hat{z}\left(\frac{\partial}{\partial x}3xz^2 - \frac{\partial}{\partial y}x^2\right)$

$= \hat{x}(-6xz) + \hat{y}(2z) + \hat{z}(3z^2)$

$\cdot \nabla \times \vec{v}_b = \left(\frac{\partial}{\partial y}3zx - \frac{\partial}{\partial z}2yz\right)\hat{x} + \left(\frac{\partial}{\partial z}xy - \frac{\partial}{\partial x}3zx\right)\hat{y} + \left(\frac{\partial}{\partial x}2yz - \frac{\partial}{\partial y}xy\right)$

$= \hat{x}(-2y) + \hat{y}(-3z) + \hat{z}(-x)$

HW1 (cont): 7(1.31)



$T = z^2$

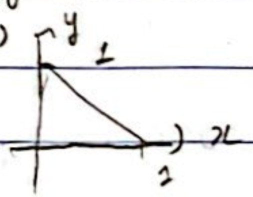
$z = 1,\ x = 0,\ y = 0$

$\Rightarrow z = 1 - x - y$

$\Rightarrow x = 1 - y - z$

$\Rightarrow y = 1 - x - z$

$$\Rightarrow \int_V T\,dV = \int_0^{1-x-y}\int_0^{1-x}\int_0^{1} z^2\,dx\,dy\,dz$$



$y = 1 \Rightarrow x = 0 \Rightarrow y = 1 - x$

$$\Rightarrow \int_V T\,dV = \int_0^{1-x}\int_0^{1} \frac{(1-x-y)^3}{3}\,dx\,dy$$

$u = 1 - x - y \Rightarrow du = -dy$

$v = 1 - x \Rightarrow dv = -dx$

$$= \frac{-1}{3}\int_0^1 \frac{u^4}{4}\Big|_0^{1-x} = \frac{-1}{12}\int_0^1 (1-x)^4\,dx = \frac{1}{12}\int_0^1 v^4\,dv$$

$$= \frac{1}{60}(1-x)^5\Big|_0^1 = \frac{1}{60}$$