HOMEWORK 9

1. Eqn 18-21: $U_m = \int_{allspace} \frac{\vec{B}^2}{2\mu_0} d\tau$

* for $\rho < a \Rightarrow \oint B \cdot dl = \mu_0 I_{encl} = \mu_0 \frac{I \pi \rho^2}{\pi a^2} = \mu_0 \frac{I \rho^2}{a^2}$

$\Rightarrow B_{\rho<a} = \frac{I \rho \mu_0}{2\pi a^2}$

* for $\rho > a \Rightarrow \oint B \cdot dl = \mu_0 I_{encl} \Rightarrow \frac{\mu_0 I}{2\pi \rho} = B_{\rho>a} (I_{encl} = I)$

$\Rightarrow U_m = \frac{1}{2\mu_0} \left(\int_{allspace} B_{\rho<a}^2 d\tau + \int_{allspace} B_{\rho>a}^2 d\tau \right)$

$= \frac{1}{2\mu_0} \left(\frac{\mu_0^2 I^2}{4\pi^2 a^4} \int_0^{2\pi} \int_0^{l} \int_0^{a} \rho^2 \cdot \rho d\rho dz d\theta + \frac{\mu_0^2 I^2}{4\pi^2} \int_0^{2\pi} \int_0^{l} \int_a^{R} \frac{1}{\rho^2} \cdot \rho d\rho dz d\theta \right)$

$= \frac{1}{2\mu_0} \cdot \frac{\mu_0^2 I^2 l}{4\pi^2} 2\pi \left(\frac{1}{a^4} \frac{a^4}{4} + \ln \frac{R}{a} \right) = \frac{\mu_0 I^2 l}{4\pi} \left(\frac{1}{4} + \ln \frac{R}{a} \right)$

2. Eqn 18-12: $U_m = \frac{1}{2} \int_{allspace} \vec{J}_f(\vec{r}) \cdot \vec{A}(\vec{r}) d\tau$

$\vec{J}_f(\vec{r}) = \frac{1}{\mu_0} |\vec{\nabla} \times \vec{B}| \Rightarrow U_m = \frac{1}{2\mu_0} \int_{allspace} [(\vec{\nabla} \times \vec{B}) \cdot A] d\tau = \frac{1}{2\mu_0} \int_{allspace} B^2 - \overbrace{[\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})}^{=0 \text{ since cont}} d\tau$

$B = \mu_0 n I \Rightarrow U_m = \frac{1}{2\mu_0} \int_{allspace} (\mu_0 n I)^2 d\tau = \frac{1}{2} \mu_0 n^2 I^2 A l$

Grading space for 1&2:

3. Wangsness 17-24:

$\Phi = \int \vec{B} \cdot d\vec{A} = \int_0^l \int_a^b \frac{\mu_0 I_{encl}}{2\pi r} \hat{\phi} \cdot l\,dr\,\hat{\phi} = \frac{\mu_0 I_{encl}}{2\pi} l \ln\left(\frac{b}{a}\right)$

$\Rightarrow L = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)$

Using Magnetic NRG: $U_m = \frac{1}{2\mu_0} \int B^2 d\tau' = \frac{1}{2\mu_0} \int_0^{2\pi} \int_0^{l} \int_b^{a} \frac{\mu_0^2 I_{encl}^2}{4\pi^2 r^2} r\,dr\,dz\,d\theta$

$\Rightarrow U = \frac{\mu_0 I_{encl}^2}{4\pi} l \ln\left(\frac{b}{a}\right) \Rightarrow L = \frac{2U}{I^2} = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)$ ✓

$\Rightarrow f_m = \frac{B^2}{2\mu_0} = \frac{\mu_0^2 I^2}{4\pi^2 a^2 \cdot 2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 a^2} \Rightarrow I = \sqrt{\frac{8\pi^2 a^2}{\mu_0} f_m} = \frac{4\pi \cdot 1 \cdot 10^{-2} m \cdot 1 atm}{\sqrt{\mu_0}}$

$\Rightarrow I \approx 25000 A.$

4. Wangsness 18-1. $\vec{m} = \frac{1}{2} \int_{V'} \vec{r}' \times \vec{J} d\tau' = \frac{1}{2} \int_S \vec{r}' \times I d\vec{s}'$

$= \frac{1}{2} \int_S (a\hat{\rho} + b \sin n\phi\, \hat{z}) \times I \underset{\sim}{d\vec{s}'}$

$d\vec{s}' = d\vec{r}' = \underbrace{a\,d\hat{\rho} + bn\cos\phi\, d\phi\, \hat{z}}$

$= \frac{1}{2} \int_0^{2\pi} (a\hat{\rho} + b \sin n\phi\, \hat{z}) \times I (a\,d\phi\,\hat{\phi} + bn\cos\phi\, d\phi\, \hat{z})$

n has to be ≥ 2 because if $n = 1$

$\Rightarrow \vec{m} = \frac{1}{2} \int_0^{2\pi} (a\hat{\rho} + b\sin\phi\, \hat{z}) \times I (a\,d\phi\,\hat{\phi} + b\cos\phi\, d\phi\, \hat{z})$

$= \frac{1}{2} I \pi a^2 \hat{z}$

5. Wangsness 19-3. $\vec{m} = m\hat{z}$ @ $z = z_0 \gg \frac{1}{2}$, $\vec{J} = \frac{Q}{\pi a^2 l} \rho' \omega \hat{\phi}$

$B_r = \frac{\mu_0 m}{4\pi} \frac{2\cos\theta}{r^3}$, $B_\theta = \frac{\mu_0 m}{4\pi} \frac{\sin\theta}{r^3}$ (eqn 19-24) $\Rightarrow \vec{B} = \vec{B_r} + \vec{B_\theta}$

$\Rightarrow \vec{m} = \frac{1}{2} \int_0^{2\pi} \int_0^{l} \int_0^{a} (\rho' \underbrace{\hat{\rho} + z' \hat{z}}_{\text{cancelled}}) \times \frac{Q}{\pi a^2 l} \rho' \omega \hat{\phi}\, \rho' d\rho' dz' d\phi' = \frac{Q\omega a^2}{4} \hat{z}$

$\Rightarrow \vec{B} = \frac{\mu_0 Q \omega a^2}{16\pi r^3} (2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta})$

$\Rightarrow \vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B}) = \vec{\nabla}\left(\frac{\mu_0 m Q \omega a^2}{8\pi r^3}\right) = -\frac{3\mu_0}{8} \frac{m Q \omega a^2}{\pi r^4} \hat{z}$

Grading Space for 3, 4, 5:

6. Wangsness 19-7. $\vec{B}_{net} = B_r + B_\theta = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$; $m = Ia^2\pi$

$\Phi = \oint \vec{B}\cdot d\vec{a} = \int_0^{\theta'}\int_0^{2\pi} \frac{\mu_0 I\pi a^2}{4\pi r^3}(2\cos\theta)\, r^2\sin\theta\, d\theta\, d\phi$

$\left(r = \sqrt{b^2+z^2} = c \right)$

$= \frac{\mu_0 I\pi a^2}{2}\int_0^{\theta'} \cos\theta\sin\theta\, d\theta = \frac{\mu_0 I\pi a^2}{2r}\left(\frac{b^2}{b^2+z^2}\right) = \frac{\mu_0 \pi a^2 b^2 I}{2(b^2+z^2)^{3/2}}$

$\Phi = MI \Rightarrow M = \frac{\mu_0\pi a^2 b^2}{2(b^2+z^2)^{3/2}} = \frac{\mu_0\pi a^2 b^2}{2c^3}$

7. Wangsness 19-11. $U_{DD} = \frac{\mu_0}{4\pi R^3}\left[(\vec{m}_1\cdot\vec{m}_2) - 3(\vec{m}_1\cdot\hat{R})(\vec{m}_2\cdot\hat{R})\right]$ (Eqn 19-56)

$\Rightarrow U_{DD} = \frac{\mu_0}{4\pi R^3}\left\{\left[m_1 m_2\cos(\alpha_2-\alpha_1)\right] - 3m_1\cos\alpha_1\, m_2\cos\alpha_2\right\}$

$= \frac{\mu_0}{4\pi R^3}\left\{m_1 m_2\left[\cos(\alpha_2-\alpha_1) - 3\cos\alpha_1\cos\alpha_2\right]\right\}$ $\boxed{\begin{array}{l}\cos(A-B) = \cos A\cos B\\ +\sin A\sin B\end{array}}$

$= \frac{\mu_0 m_1 m_2}{4\pi R^3}(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2 - 3\cos\alpha_1\cos\alpha_2)$

$= \frac{\mu_0 m_1 m_2}{4\pi R^3}(-2\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2)$

m_1 fixed & m_2 rotates $\Rightarrow \tau_2 = 0 \Rightarrow \frac{\partial u}{\partial\alpha_2} = 0$ (torque at $m_2 = 0$)

$\Rightarrow -2\cos\alpha_1(-\sin\alpha_2) + \sin\alpha_1\cos\alpha_2 = 0 \Rightarrow \sin\alpha_1\cos\alpha_2 = -2\cos\alpha_1\sin\alpha_2$

$\Rightarrow \frac{\sin\alpha_1}{\cos\alpha_1} = -2\frac{\sin\alpha_2}{\cos\alpha_2} \Rightarrow \tan\alpha_1 = -2\tan\alpha_2$.

(i) $\alpha = 0^\circ \Rightarrow 0 = -2\tan\alpha_2 \Rightarrow \alpha_2 = 0 \Rightarrow U_{DD} = \frac{\mu_0}{4\pi R^3}\cdot 2m_1 m_2 < 0$ (stable)

(ii) $\alpha = \frac{\pi}{2} \Rightarrow \tan\frac{\pi}{2} = -2\tan\alpha_2 \Rightarrow \infty = -2\tan\alpha_2 \Rightarrow \alpha_2 = \frac{\pi}{2} \Rightarrow U_{DD} = \frac{\mu_0}{4\pi R^3} > 0$ (unstable)

8. Wangsness 19-16.

$\vec{A} = \frac{\mu_0}{4\pi}\left(\frac{\vec{m}\times\vec{r}_1}{|\vec{r}_1|^3} + \frac{-\vec{m}\times\vec{r}_2}{|\vec{r}_2|^3}\right) = \frac{\mu_0 m}{4\pi}\left(\frac{\hat{z}\times(r\hat{r} - a\hat{z})}{(r^2+a^2-2ar\cos\theta)^{3/2}}\right)$

$= \frac{\mu_0 m r\sin\theta\,\hat{\phi}}{4\pi r^3}\left[\frac{1}{\left(1+\frac{a^2}{r^2} - \frac{2a\cos\theta}{r}\right)^{3/2}} - \frac{1}{\left(1+a^2/r^2 + \frac{2a\cos\theta}{r}\right)^{3/2}}\right]$

$= \frac{3\mu_0 m a\sin\theta\cos\theta}{2\pi r^2} \Rightarrow B = \vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta A_\phi) - \frac{1}{r}\frac{d}{dr}(rA_\phi)$

$\Rightarrow \vec{B} = \frac{3\mu_0 m a}{2\pi}\left[\frac{1}{r^4}(3\cos^2\theta - 1)\,\hat{r} + \frac{2\sin\theta\cos\theta}{r^4}\,\hat{\theta}\right] + 0\,\hat{\phi}$

$\Rightarrow$ Analogue to $E_r = \left(\frac{3Q\,a}{8\pi\varepsilon_0}\right)\frac{3\cos^2\theta - 1}{2r^4}$ $E_\phi = 0$

$E_\theta = \left(\frac{3Qa}{8\pi\varepsilon_0}\right)\frac{\cos\theta\sin\theta}{r^4}$