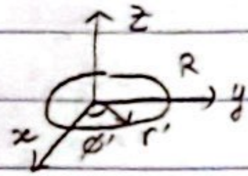
HW6

1. (3.30).



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2r^5}\sum_\alpha\sum_\beta \alpha\beta Q_{\alpha\beta}\right\}$$

→ Monopole term ($n=0$): $\frac{Q}{r} = \frac{\lambda \cdot 2\pi R}{R} = 2\pi\lambda$

⇒ Dipole term ($n=1$): $\vec{p}=0$ due to symmetry of the charge distribution (for every small charge element, there'll be = & opposite charge to cancel out)

⇒ Quadrupole term ($n=2$): Special case of rotational symmetry → $Q_{\alpha\beta}=0$ for $\alpha\neq\beta$

→ $Q_{xx}=Q_{yy}=-\frac{1}{2}Q_{zz}$; $Q_{xx}=\int(3x'^2-r'^2)\rho(\vec{r}')dV'$

$r'=\sqrt{x'^2+y'^2+z'^2}=R$; $x'=R\cos\phi$; $y'=R\sin\phi$; $z'=0$ $\qquad \rho = \text{const} = \lambda = \frac{Q}{2\pi R}$

$dV' \sim dl' = Rd\phi$

$$\Rightarrow Q_{xx} = \int_0^{2\pi}(3x'^2-r'^2)\lambda R d\phi = \int_0^{2\pi}(3R^2\cos^2\phi - R^2)\frac{Q}{2\pi R}Rd\phi$$

$$= \frac{QR^2}{2\pi}\int_0^{2\pi}(3\cos^2\phi - 1)d\phi = \frac{QR^2}{2\pi}(3\pi-2\pi) = \frac{QR^2}{2} = Q_{yy}$$

$\int_0^{2\pi}\cos^2\phi\, d\phi = \pi$; 2π

⇒ $Q_{zz} = -QR^2$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left\{2\pi\lambda + 0 + \frac{QR^2}{2r^5}\left(\frac{x^2+y^2}{2} - z^2\right)\right\}$$

$x = \sin\theta\cos\phi\, r$

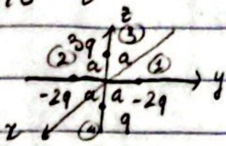
$y = \sin\theta\sin\phi\, r$ $\qquad r=R$

$z = \cos\theta\, r$

$\rightarrow V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left\{ 2\pi\lambda + \frac{2\pi R^3}{2R^5} R^2 \left(\frac{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi}{2} - \cos^2\theta \right) \right\}$

$= \frac{1}{2\varepsilon_0}\left\{ \lambda + \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi - 2\cos^2\theta \right\}$

2. (3.31).



• Monopole term: $Q_{total} = -2q + (-2q) + 3q + q = 0$

• Dipole term: $\vec{p} = \sum_i q_i \vec{r_i}$

$\rightarrow \vec{p} = q_1\vec{r_1} + q_2\vec{r_2} + q_3\vec{r_3} + q_4\vec{r_4}$

$= -2qa\hat{y} + (-2q)(-a)\hat{y} + 3qa\hat{z} + q(-a)\hat{z} = 2qa\hat{z} + 0\hat{y}$

$\Rightarrow \frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{2qa\hat{z}\cdot(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z})}{r^2} = \frac{2qa\cos\theta}{r^2}$

• Quadrupole term: No need to be calculated

$\rightarrow V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{2qa\cos\theta}{r^2}$

3. (3.32). From Ex 3.9: $V(r,\theta) = \frac{k}{3\varepsilon_0} r\cos\theta \quad (r \le R)$ (ignore this, only use the shell) for $\sigma = k\cos\theta$

$= \frac{kR^3}{3\varepsilon_0}\frac{1}{r^2}\cos\theta \quad (r \ge R)$

a) Dipole moment: $\vec{p} = \sum_i q_i\vec{r_i} = \int_{V'} \rho(\vec{r'})\vec{r'}\,dV' = \int_{S'} \sigma(\theta)\underbrace{z'}_{\cos\theta r' = \cos\theta R}\,da = \int_0^{2\pi}\int_0^{\pi} (k\cos\theta)(\cos\theta R)\,R^2\sin\theta\,d\theta\,d\phi$

$\Rightarrow \vec{p} = 2\pi kR^3 \int_0^{\pi} \underbrace{\cos^2\theta\sin\theta\,d\theta}_{= -u^2} \quad u = \cos\theta \Rightarrow du = -\sin\theta\,d\theta$

$\rightarrow \vec{p} = 2\pi kR^3 \left(\frac{\cos^3\theta}{3}\right)_0^{\pi} = 2\pi kR^3\left(\frac{1+1}{3}\right) = \frac{4k\pi R^3}{3}\hat{z}$

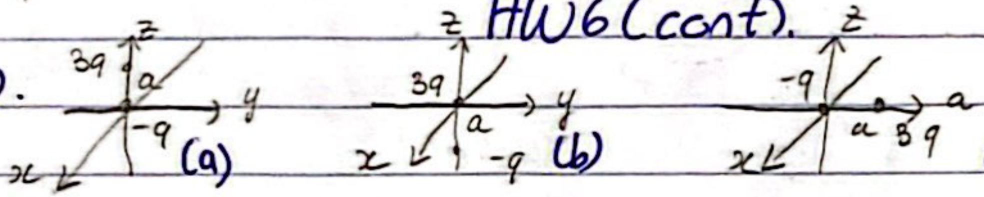
b) $\rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{4\pi R^3 k\hat{z}\cdot(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z})}{3r^2} = \frac{4\pi kR^3\cos\theta}{4\pi\varepsilon_0 r^2 3}$

$\Rightarrow V = \frac{kR^3}{3\varepsilon_0}\frac{1}{r^2}\cos\theta = V(r,\theta)$ from example 3.9!

$\Rightarrow$ Since $V(r,\theta) = \frac{kR^3}{3\varepsilon_0}\frac{1}{r^2}\cos\theta$ is for $r \ge R$ going onward $\rightarrow$ higher multipole terms $= 0$. (max at dipole)

HW6 (cont).

4. (3.34).



(i) For monopole: (a) = (b) = (c) = $3q - q = 2q$

(ii) For dipole: (a) $\vec{p} = \sum_i q_i \vec{r_i} = 3qa\hat{z} + (-q)0 = 3qa\hat{z}$

(b) $\vec{p} = \sum_i q_i \vec{r_i} = 3q0 + (-q)(-a)\hat{z} = qa\hat{z}$

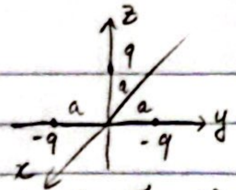
(c) $\vec{p} = \sum_i q_i \vec{r_i} = 3qa\hat{y} + (-q)0 = 3qa\hat{y}$

(iii) For potential: (a) $V = \frac{Q}{4\pi\varepsilon_0} + \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{q}{2\pi\varepsilon_0} + \frac{3qa\cos\theta}{4\pi\varepsilon_0 r^2}$

(b): $V = \frac{q}{2\pi\varepsilon_0} + \frac{qa\cos\theta}{4\pi\varepsilon_0 r^2}$

(c): $V = \frac{q}{2\pi\varepsilon_0} + \frac{3qa\sin\theta\sin\phi}{4\pi\varepsilon_0 r^2}$

5.(3.36).



$\rightarrow$ Monopole moment $= -2q + q = -q$

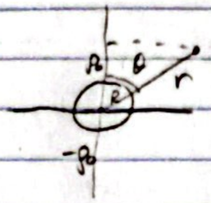
$\rightarrow$ Dipole moment: $\vec{p} = \sum q_i \vec{r_i} = qa\hat{z} - qa\hat{y} + qa\hat{y} = qa\hat{z}$

$\rightarrow V(r,\theta) = \dfrac{-q}{4\pi\varepsilon_0 r} + \dfrac{qa\cos\theta}{4\pi\varepsilon_0 r^2}$ (sorry I skipped steps here since $\hat{z}\cdot\hat{r}$ has been done multiple times already)

$$\rightarrow E(r,\theta) = -\nabla V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}\left(\frac{1}{r} - \frac{\cos\theta a}{1\cdot r^2}\right)\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{+1}{r} - \frac{\cos\theta a}{r^2}\right)\hat{\theta}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\left(\frac{-1}{r^2} + \frac{2\cos\theta a}{r^3}\right)\hat{r} + \frac{\sin\theta a}{r^3}\hat{\theta}\right]$$

6.(3.37).



$\rightarrow$ Monopole moment: $Q_{total} = \rho_0 V - \rho_0 V = V(\rho_0 - \rho_0) = 0$

$\rightarrow$ Dipole moment: $\vec{p} = \int_{V'} \rho(\vec{r}')dV' = 2\int_0^{\pi/2}\int_0^{\pi}\int_0^{R} \rho_0\, r^2 \sin\theta\, dr\, d\phi\, d\theta$ (Double the dipole of only the top half to get the whole sphere).

$= 2\dfrac{R^3}{3}\pi\rho_0\hat{z} \rightarrow V(r,\theta) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\vec{p}\cdot\hat{r}}{r^2} = \dfrac{1}{6\varepsilon_0}\dfrac{R^3\rho_0\cos\theta}{r}$

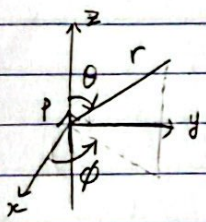
$$\rightarrow E(r,\theta) = \frac{R^3\rho_0}{6\varepsilon_0}\left[\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r}\right)\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r}\right)\right]$$

not $2\cos\theta\hat{r} + \sin\theta\hat{\theta}$?

$$= \frac{R^3\rho_0}{6\varepsilon_0}\left[\frac{-\cos\theta}{r^2}\hat{r} - \frac{\sin\theta}{r^2}\hat{\theta}\right] = \frac{-R^3\rho_0}{6\varepsilon_0 r^2}\left(\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

this is definitely wrong

7.(3.61).

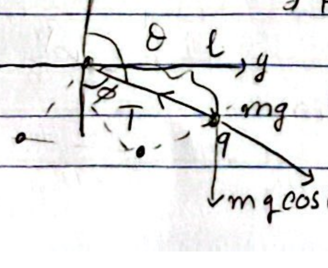


Field lines for a pure ("ideal") dipole:

$$\vec{E}_D = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

$$\rightarrow \vec{F}_D = \frac{qp}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) \quad (\vec{F} = q\vec{E})$$

If the electric charge swing like a pendulum:



T: Tension, mg: gravitational force

$\rightarrow \vec{F}_D = \vec{F}_{net} = -mg\hat{z} - T\hat{r}$

$\rightarrow$ Centripetal force for pendulum: $F_c = T - mg\cos\phi = \dfrac{mv^2}{\ell}$

I need to study 321 again !!

At $\phi = 90°$ (the starting point aka initial condition) $\Rightarrow U_i = mgl$ & $K_i = 0$

$\Rightarrow \cancel{m}gl = \frac{1}{2}\cancel{m}v^2 + \cancel{m}gl(1-\cos\phi)$ ($F_i = F_f$) $\Rightarrow U_f = mgl(1-\cos\phi)$ & $K_f = \frac{1}{2}mv^2$

$\Rightarrow v^2 = 2gl\cos\phi \Rightarrow T - \frac{m(2gl\cos\phi)}{l} = mg\cos\phi$

$\Rightarrow T = 2mg\cos\phi + mg\cos\phi = 3mg\cos\phi$

$\Rightarrow \vec{F}_{net} = -mg(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) + 3mg\cos\theta\,\hat{r}$ ($\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$)

$= mg(-\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta} + 3\cos\theta\,\hat{r})$

$= mg(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) = \frac{pq}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) = \vec{F}_D$

$\Rightarrow mg = \frac{pq}{4\pi\varepsilon_0 r^3} = F_q \Rightarrow$ q moves like a pendulum (I really need to relearn 321, don't even know why I passed this clas