# HOMEWORK 7

1. (4.3). Eqn 4.1: $\vec{p} = \alpha \vec{E}$ ; $\rho(r) \sim A r$

Gauss' Law: $|\vec{E}| \cdot 4\pi r^2 = \frac{Q_{enc}}{\varepsilon_0} = \frac{\int \rho(r) dV}{\varepsilon_0} = \frac{\int_0^r A r 4\pi r^2 dr}{\varepsilon_0} = \frac{4\pi A r^4}{4\varepsilon_0}$

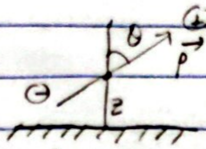
$\Rightarrow E = \frac{A r^2}{4\varepsilon_0} \Rightarrow r = \sqrt{\frac{4\varepsilon_0 E}{A}} = d$

$|\vec{p}| = q d = q\sqrt{\frac{4\varepsilon_0 E}{A}} = 2q\sqrt{\frac{4\varepsilon_0 E}{A}} \Rightarrow p \propto E^{1/2}$

$\underline{P}$: polarization, p: dipole

2.(4.4). $\vec{p} = \alpha\vec{E} = \frac{\alpha}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \Rightarrow \vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}_{ext} = \frac{\alpha q}{4\pi\varepsilon_0 r^2}(\hat{r}\cdot\nabla)\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

$\Rightarrow \vec{F} = \frac{\alpha q^2}{(4\pi\varepsilon_0)^2}\frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{r^2}\hat{r} = \frac{-2\alpha q^2}{16\pi^2\varepsilon_0^2}\frac{1}{r^5}\hat{r} = \frac{-\alpha q^2}{4\pi^2 r^5\varepsilon_0^2}\hat{r}$

3.(4.6). 


$\vec{N} = \vec{p}\times\vec{E}$ where $\vec{p} = p\cos\theta\hat{r} + p\sin\theta\hat{\theta}$

$\vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$

→ The distance between 2 dipoles is: $r = 2z$

$\Rightarrow \vec{E} = \frac{p}{32\pi\varepsilon_0 z^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \Rightarrow \tau = (p\cos\theta\hat{r} + p\sin\theta\hat{\theta})\times\frac{p}{32\pi\varepsilon_0 z^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$

$\Rightarrow \vec{N} = \frac{p^2}{32\pi\varepsilon_0 z^3}[(\cos\theta\hat{r} + \sin\theta\hat{\theta})\times(2\cos\theta\hat{r} + \sin\theta\hat{\theta})]$

$$\begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \cos\theta & \sin\theta & 0 \\ 2\cos\theta & \sin\theta & 0 \end{vmatrix}$$

$= \frac{p^2}{32\pi\varepsilon_0 z^3}[\cos\theta\sin\theta\hat{\phi} - 2\cos\theta\sin\theta\hat{\phi}]$

$= \frac{p^2}{32\pi\varepsilon_0 z^3}(\underbrace{-\cos\theta\sin\theta}_{\frac{\sin 2\theta}{2}})\hat{\phi} = \frac{-p^2\sin 2\theta}{64\pi\varepsilon_0 z^3}\hat{\phi}$

4.(4.10). $\vec{P}(\vec{r}) = k\vec{r} = kR\hat{r}$ (for $\underbrace{\text{surface bound charge}}_{\text{center→outside sphere}}$) $= kr\hat{r}$ (for $\underbrace{\text{volume bound charge}}_{\text{center→inside sphere}}$)

a) $\sigma_b = \vec{P}\cdot\hat{n} = kR\hat{r}\cdot\hat{r} = kR$

$\rho_b = -\nabla\cdot P = -\frac{1}{r^2}\frac{\partial}{\partial r}r^2 P = -\frac{k}{r^2}\frac{\partial}{\partial r}r^3 = -\frac{k}{r^2}3r^2 = -3k$

b) • Inside ($r \le R$): $Q_{in} = \rho_b V = -3k\frac{4}{3}\pi r^3 = -4k\pi r^3$

$\Rightarrow \vec{E}\cdot 4\pi r^2 = \frac{Q_{in}}{\varepsilon_0} = \frac{-4k\pi r^3}{\varepsilon_0} = -\frac{kr}{\varepsilon_0}\hat{r}$

• Outside ($r > R$): $Q_{net} = Q_{in} + Q_{out} = Q_{volume} + Q_{surface}$

$= -4k\pi r^3 + \sigma_b(4\pi r^2) = -4k\pi r^3 + 4k\pi r^3 = 0$

$\Rightarrow \vec{E} = 0$ ($Q_{encl}$ for the whole sphere is 0)

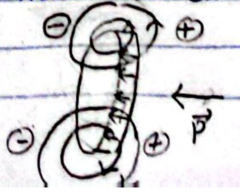
5.(4.11). Uniform polarization → $\nabla\cdot\vec{P} = 0 = \rho_b \Rightarrow \sigma_b = \vec{P}\cdot\hat{n} = \pm P$ (for parallel comp)

→ Plus sign @ one end → Minus sign @ another end.

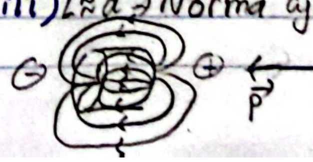
i) $L \gg a$ → a looks like a point charge

→ Dipole slab length L


ii) $L \ll a$ ⇒ circular parallel charge capacitor



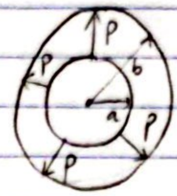

iii) $L \approx a$ → Normal cylinder



$\vec{n}$ points from medium 1 → medium 2, works similar everywhere

# HOMEWORK 7 (Cont).

**6. (4.15).** $\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$ — Const P → no free charge




a) • $r<a \rightarrow$ No bound charge $\Rightarrow Q_{encl}=0$

$\Rightarrow \vec{E}_{r<a} = 0$

• $a<r<b \rightarrow \rho_b = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = \frac{-k}{r^2}$

$\sigma_b = \vec{P}\cdot\hat{n}$

$= \begin{cases} -\frac{k}{a} & \text{(on the inner surface)} \\ \frac{k}{b} & \text{(on the outter surface)} \end{cases}$

$\Rightarrow Q_{encl} = 0 - \frac{k}{a}4\pi a^2 + \int_a^r \rho_b dV$

$= -4\pi a k - k\int_0^{2\pi}\int_0^{\pi}\int_a^r \frac{1}{r^2} r^2 \sin\theta \, dr d\theta d\phi$

$= -4\pi a k - 4\pi k(r-a) = -4\pi k r$

$\Rightarrow \vec{E}_{a<r<b} \cdot 4\pi r^2 = \frac{-4\pi k r}{\varepsilon_0} = \frac{-k}{\varepsilon_0 r}\hat{r}$

? • $r>b \rightarrow Q_{encl} = -4\pi k r + 4\pi b^2 \frac{k}{b} - k\int_0^{2\pi}\int_0^{\pi}\int_r^b \frac{1}{r^2} r^2 \sin\theta \, dr d\theta d\phi$

$= -4\pi k r + 4\pi k b - 4\pi k(b-r) = 0$

$\Rightarrow \vec{E}_{r>b} = 0$

b) Eqn 4.23: $\oint \vec{D}\cdot d\vec{a} = Q_{f\,enc}$ * No free charge in the problem

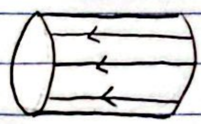
Eqn 4.21: $\vec{D} \equiv \varepsilon_0\vec{E} + \vec{P}$ $\Rightarrow \vec{D} = 0$ everywhere

• $r<a$ : $\vec{P} = 0 \Rightarrow \vec{E} = 0$, same with $r>b$

• $a<r<b$ : $\vec{P}(\vec{r}) = \frac{k}{r}\hat{r} \Rightarrow \vec{E} = -\frac{\vec{P}}{\varepsilon_0} = \frac{-k}{r\varepsilon_0}\hat{r}$

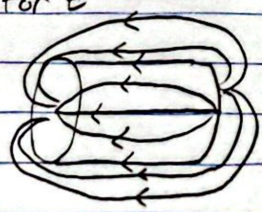
**7. (4.17).**

• For $\vec{P}$



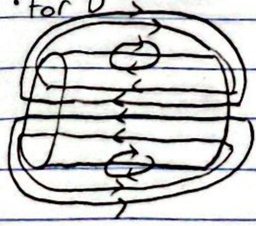
Uniform field lines

• For $\vec{E}$



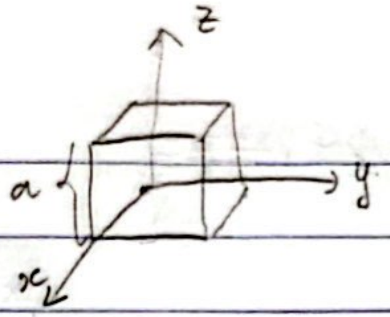
Similar to two circular plates

• For $\vec{D}$



continuous because $\nabla\cdot\vec{D} = 0$

8. (4.34).




$\vec{P} = K\vec{r} = K(x\hat{x} + y\hat{y} + z\hat{z})$

$\rho_b = -\vec{\nabla}\cdot\vec{P} = -K\left(\hat{x}\frac{\partial x}{\partial x} + \hat{y}\frac{\partial y}{\partial y} + \hat{z}\frac{\partial z}{\partial z}\right) = -3K$

$\Rightarrow Q_v = \rho_b V = -3Ka^3$

$\sigma_b = \vec{P}\cdot\hat{n}$ , at the surface of $x, -x, y, -y, z, -z$, $\hat{n} = \hat{x}, -\hat{x}, \hat{y}, -\hat{y}, \hat{z}, -\hat{z}$ respectively. The normal component is always at the distance $a/2$ on the positive surface & $-a/2$ on the negative surface.

$\Rightarrow \sigma_{b(+)} = K(x\hat{x} + y\hat{y} + z\hat{z})\cdot\hat{x}/\hat{y}/\hat{z} = \frac{Ka}{2}$ $(x = y = z = a/2)$

$\sigma_{b(-)} = K(x\hat{x} + y\hat{y} + z\hat{z})\cdot -\hat{x}/-\hat{y}/-\hat{z} = \frac{Ka}{2}$ $(x = y = z = -a/2)$

$\Rightarrow \sigma_b = \frac{Ka}{2}$ everywhere $\Rightarrow Q_s = a^2 \cdot \underset{\text{(6 surfaces)}}{6}\frac{Ka}{2} = 3Ka^3$

$\Rightarrow Q_{total} = Q_v + Q_s = -3Ka^3 + 3Ka^3 = 0$