HOMEWORK 6

A.7. Prove: $\|(|\alpha\rangle+|\beta\rangle)\| \leq \|\alpha\| + \|\beta\|$

Set $|\gamma\rangle = |\alpha\rangle + |\beta\rangle$ & $\langle\gamma|\gamma\rangle = \langle\gamma|\alpha\rangle + \langle\gamma|\beta\rangle$

$\langle\gamma|\alpha\rangle^* = \langle\alpha|\gamma\rangle = \langle\alpha|\alpha\rangle + \langle\alpha|\beta\rangle$

$\langle\gamma|\beta\rangle^* = \langle\beta|\gamma\rangle = \langle\beta|\alpha\rangle + \langle\beta|\beta\rangle$

$\|(|\alpha\rangle+|\beta\rangle)\|^2 = \langle\gamma|\gamma\rangle = \langle\alpha|\alpha\rangle + \underbrace{\langle\alpha|\beta\rangle + \langle\beta|\alpha\rangle}_{} + \langle\beta|\beta\rangle$

Schwarz inequality: $= 2\,\mathrm{Re}(\langle\alpha|\beta\rangle) \leq 2|\langle\alpha|\beta\rangle| \leq 2\sqrt{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle}$

$\rightarrow \|(|\alpha\rangle+|\beta\rangle)\|^2 \leq \|\alpha\|^2 + \|\beta\|^2 + 2\|\alpha\|\|\beta\| = (\|\alpha\|+\|\beta\|)^2$

$\Rightarrow \|(|\alpha\rangle+|\beta\rangle)\| \leq \|\alpha\| + \|\beta\|$ ✓

A.8. $A = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix}$

a) $A+B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 3i & 3-2i & 4 \end{pmatrix}$ b) $AB = \begin{pmatrix} -3 & 1+3i & 3i \\ 4+3i & 9 & 6-2i \\ 6i & 6-2i & 6 \end{pmatrix}$

c) $[A,B] = AB - BA = \begin{pmatrix} -3 & 1+3i & 3i \\ 4+3i & 9 & 6-2i \\ 6i & 6-2i & 6 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 3 \\ 6+3i & -3i & 12 \end{pmatrix} = \begin{pmatrix} -3 & 1+3i & 3i \\ 2+3i & 9 & 3-2i \\ -6+3i & 6+i & -6 \end{pmatrix}$

d) $\tilde{A} = \begin{pmatrix} -1 & 2 & 2i \\ 1 & 0 & -2i \\ i & 3 & 2 \end{pmatrix}$ e) $A^* = \begin{pmatrix} -1 & 1 & -i \\ 2 & 0 & 3 \\ -2i & 2i & 2 \end{pmatrix}$ f) $A^\dagger = \begin{pmatrix} -1 & 2 & -2i \\ 1 & 0 & +2i \\ -i & 3 & 2 \end{pmatrix}$

g) $\det(B) = 2 \cdot 2 - 0 + (-i)(-i) = 4 - 1 = 3$

h) $B^{-1} = \left(\begin{array}{ccc|ccc} 2 & 0 & -i & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ i & 3 & 2 & 0 & 0 & 1 \end{array}\right) \overset{R_1 \to R_1/2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -i/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ i & 3 & 2 & 0 & 0 & 1 \end{array}\right) R_3 \to R_3 - iR_1$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -i/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 3/2 & -i/2 & 0 & 1 \end{array}\right) \overset{R_3 \to R_3 - 3R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -i/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3/2 & -i/2 & -3 & 1 \end{array}\right) R_3 \to 2/3 R_3$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -i/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -i/3 & -2 & 2/3 \end{array}\right) \overset{R_1 \to R_1 + i/2 R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -i & i/3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -i/3 & -2 & 2/3 \end{array}\right)$

$\rightarrow B^{-1} = \begin{pmatrix} 2/3 & -i & i/3 \\ 0 & 1 & 0 \\ -i/3 & -2 & 2/3 \end{pmatrix}$ $BB^{-1} = \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix}\begin{pmatrix} 2/3 & -i & i/3 \\ 0 & 1 & 0 \\ -i/3 & -2 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

$\det(A) = -1(+6i) - 1(4-6i) + i(-4i) = -6i - 4 + 6i + 4 = 0$

$\rightarrow$ A doesn't have an inverse.

A.19. $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$

$\Rightarrow (1-\lambda)x + y = 0 \Rightarrow 0x + y = 0 \Rightarrow y = 0 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

A.22. $M = \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}$ a) $MM^H = \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} 2 & 1-i \\ 1+i & 2 \end{pmatrix} \neq M^H M = \begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} = \begin{pmatrix} 2 & 1+i \\ 1-i & 2 \end{pmatrix}$

$\Rightarrow$ M is not normal

b) $M^T = \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} = M \Rightarrow$ M is diagonalizable

A.26. $A = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix}$

a) $A^T = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} = A$, $B^T = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} = B \Rightarrow$ They're diagonalizable

$AB = \begin{pmatrix} 0 & 9 & 0 \\ 9 & -9 & 9 \\ 0 & 9 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 9 & 0 \\ 9 & -9 & 9 \\ 0 & 9 & 0 \end{pmatrix} = AB \Rightarrow$ They commute.

b) $\begin{vmatrix} 2-\lambda & 2 & -1 \\ 2 & -1-\lambda & 2 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)[(-1-\lambda)(2-\lambda) - 4] - 2[2(2-\lambda) + 2] - 1[4 - (1+\lambda)]$
$= 0$

$\Leftrightarrow (2-\lambda)[-2 + \lambda - 2\lambda + \lambda^2 - 4] - 2[4 - 2\lambda + 2] - [4 - 1 - \lambda] = 0$

$\Leftrightarrow (2-\lambda)\cdot[\lambda^2 - \lambda - 6] + 4\lambda - 12 - 3 + \lambda = 0$

$\Leftrightarrow (2-\lambda)[\lambda^2 - \lambda - 6] + 4(\lambda - 3) + (\lambda - 3) = 0 \Leftrightarrow (2-\lambda)(\lambda^2 \overset{-3\lambda+2\lambda}{-\lambda} - 6) + 5(\lambda - 3) = 0$

$\Leftrightarrow (2-\lambda)[\lambda(\lambda-3) + 2(\lambda-3)] + 5(\lambda-3) = 0 \Leftrightarrow (2-\lambda)(\lambda-3)(\lambda+2) + 5(\lambda-3) = 0$

$\Leftrightarrow (\lambda-3)[5 + (\lambda+2)(2-\lambda)] = 0 \Leftrightarrow (\lambda-3)(9 + \lambda^2) = 0 \Rightarrow \lambda = 3, \lambda = 3$ & $\lambda = -3$

Since $\lambda = 3$ repeated twice $\Rightarrow$ the spectrum of A is degenerate.

(*) $\lambda = 3 \Rightarrow \begin{pmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0} \Rightarrow -x + 2y - z = 0 \Rightarrow 2y = z + x \Rightarrow x = 1, y = 1, z = 1$

$\Rightarrow \vec{v} = 1/\sqrt{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(*) $\lambda = -3 \Rightarrow \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0} \Rightarrow 5x + 2y - z = 0 \Rightarrow z = 2y + 5x$

$\Rightarrow x + y + z = 0 \Rightarrow x = -y - z$

$\Rightarrow -x + 2y + 5z = 0 \Rightarrow x = 2y + 5z = 2y + 5(5x + 2y)$

$\Rightarrow x = 25x + 12y \Rightarrow 24x + 12y = 0 \Rightarrow 2x + y = 0$

$\Rightarrow x = 1, y = -2, z = 1 \Rightarrow \vec{v} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3/\sqrt{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is also an eigenvector of B for $\lambda = 3$

$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 6 \\ -12 \\ 6 \end{pmatrix} = 6/\sqrt{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow 6/\sqrt{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is also an eigen vector of B for $\lambda = 6$