HOMEWORK 8

4.1. a) For $f(x,y,z)$, according to Clairaut's Theorem:

$$[x,\hat{y}]f = (\hat{x}\hat{y}-\hat{y}\hat{x})f = (xy-yx)f = 0 = [\hat{y},\hat{z}]f = [\hat{x},\hat{z}]f.$$

$$\rightarrow [\hat{p}_x,\hat{p}_y]f = (\hat{p}_x\hat{p}_y-\hat{p}_y\hat{p}_x)f = \left[\left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial y}\right)-\left(-i\hbar\frac{\partial}{\partial y}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\right]f$$
$$= \left(-\hbar^2\frac{\partial^2}{\partial x\partial y}+\hbar^2\frac{\partial^2}{\partial y\partial x}\right)f = -\hbar^2\frac{\partial^2 f}{\partial x\partial y}+\hbar^2\frac{\partial^2 f}{\partial y\partial x} = 0$$

$$\rightarrow [\hat{p}_y,\hat{p}_z] = [\hat{p}_x,\hat{p}_z] = 0$$

$$\rightarrow [\hat{x},\hat{p}_x]f = (\hat{x}\hat{p}_x-\hat{p}_x\hat{x})f = \left[x\left(-i\hbar\frac{\partial}{\partial x}\right)-\left(-i\hbar\frac{\partial}{\partial x}\right)x\right]f = -i\hbar x\frac{\partial}{\partial x}f+i\hbar\frac{\partial}{\partial x}(xf)$$
$$= -i\hbar x\frac{\partial f}{\partial x}+i\hbar x\frac{\partial f}{\partial x}+i\hbar f = i\hbar f$$

$$\rightarrow [\hat{y},\hat{p}_y]f = [\hat{z},\hat{p}_z]f = i\hbar f$$

$$\rightarrow [\hat{x},\hat{p}_y]f = (\hat{x}\hat{p}_y-\hat{p}_y\hat{x})f = \left[x\left(-i\hbar\frac{\partial}{\partial y}\right)-\left(-i\hbar\frac{\partial}{\partial y}\right)x\right]f = -i\hbar x\frac{\partial}{\partial y}f+i\hbar\frac{\partial}{\partial y}(xf)$$
$$= -i\hbar x\frac{\partial f}{\partial y}+i\hbar x\frac{\partial f}{\partial y} = 0$$

$$\rightarrow [\hat{x},\hat{p}_z]f = [\hat{y},\hat{p}_x]f = [\hat{y},\hat{p}_z]f = [\hat{z},\hat{p}_x]f = [\hat{z},\hat{p}_y]f = 0$$

$$\rightarrow [\hat{x},\hat{y}] = [\hat{y},\hat{x}] = [\hat{y},\hat{z}] = [\hat{z},\hat{y}] = [\hat{x},\hat{z}] = [\hat{z},\hat{x}] = 0$$

$$\rightarrow [\hat{p}_x,\hat{p}_y] = [\hat{p}_y,\hat{p}_x] = [\hat{p}_y,\hat{p}_z] = [\hat{p}_z,\hat{p}_y] = [\hat{p}_x,\hat{p}_z] = [\hat{p}_z,\hat{p}_x] = 0$$

$$\rightarrow [\hat{y},\hat{p}_x] = [\hat{p}_x,\hat{y}] = [\hat{x},\hat{p}_y] = [\hat{p}_y,\hat{x}] = [\hat{x},\hat{p}_z] = [\hat{p}_z,\hat{x}] = [\hat{y},\hat{p}_z] = [\hat{p}_z,\hat{y}]$$
$$= [\hat{z},\hat{p}_x] = [\hat{z},\hat{p}_y] = [\hat{p}_x,\hat{z}] = [\hat{p}_y,\hat{z}] = 0$$

$$\rightarrow [\hat{p}_x,\hat{x}] = [\hat{p}_y,\hat{y}] = [\hat{p}_z,\hat{z}] = -[\hat{x},\hat{p}_x] = -[\hat{y},\hat{p}_y] = -[\hat{z},\hat{p}_z] = -i\hbar.$$

b) $$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi+V(x,y,z,t)\Psi(x,y,z,t)$$

$$\Leftrightarrow \frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2\Psi-\frac{i}{\hbar}V\Psi \Leftrightarrow \frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\nabla^2\Psi^*+\frac{i}{\hbar}V\Psi^*$$

$$\rightarrow \frac{d}{dt}\langle x_j\rangle = \frac{d}{dt}\langle\Psi|\hat{x_j}|\Psi\rangle = \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*(x,y,z,t)(x_j)\Psi(x,y,z,t)\,dxdydz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\left[\left(\frac{-i\hbar}{2m}\nabla^2\Psi^*+\frac{i}{\hbar}V\Psi^*\right)\Psi+\Psi^*\left(\frac{i\hbar}{2m}\nabla^2\Psi-\frac{i}{\hbar}V\Psi\right)\right]dxdydz$$

$$= \frac{i\hbar}{2m}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\Psi^*\nabla^2\Psi\,dxdydz-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\Psi\nabla^2\Psi^*\,dxdydz\right)$$

$$= \frac{i\hbar}{2m}\left[-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nabla(x_j\Psi^*)\cdot\nabla\Psi\,dxdydz+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nabla(x_j\Psi)\cdot\nabla\Psi^*\,dxdydz\right]$$

(I don't wanna do this...)

$$= \frac{1}{m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*\left(-i\hbar\frac{\partial}{\partial x_j}\right)\Psi\,dxdydz = \frac{1}{m}\langle\Psi|\hat{p_j}|\Psi\rangle = \frac{1}{m}\langle p_j\rangle \rightarrow \frac{d}{dt}\langle\vec{r}\rangle = \frac{1}{m}\langle\vec{p}$$

$$\frac{d}{dt}\langle \vec{p}\rangle = \frac{d}{dt}\langle \psi|\hat{p}|\psi\rangle = \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x,y,z,t)(-i\hbar\nabla)\psi(x,y,z,t)\,dx\,dy\,dz$$

$$= -i\hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left(\frac{-i\hbar}{2m}\nabla^2\psi^* + \frac{i}{\hbar}V\psi^*\right)\nabla\psi + \psi^*\nabla\left(\frac{i\hbar}{2m}\nabla^2\psi - \frac{i}{\hbar}V\psi\right)\right]dx\,dy\,dz$$

$$= -i\hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{-\hbar^2}{2m}\nabla^2\psi^*\nabla\psi + \frac{\hbar^2}{2m}\psi^*\nabla^2(\nabla\psi) - \psi^*\nabla V\psi\right]dx\,dy\,dz$$

$$= \frac{\hbar^2}{2m}\cdot 0 - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi^*\nabla V\psi\,dx\,dy\,dz = \langle\psi|-\nabla V|\psi\rangle = \langle -\nabla V\rangle$$

c) $\sigma_x^2\sigma_p^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2$

$[\hat{x},\hat{y}] = [\hat{y},\hat{z}] = [\hat{x},\hat{z}] = [\hat{p_x},\hat{p_y}] = [\hat{p_y},\hat{p_z}] = [\hat{p_x},\hat{p_z}] = 0$

$\Rightarrow \sigma_x\sigma_y \geq 0,\ \sigma_y\sigma_z \geq 0,\ \sigma_x\sigma_z \geq 0,\ \sigma_{p_x}\sigma_{p_y} \geq 0,\ \sigma_{p_y}\sigma_{p_z} \geq 0,\ \sigma_{p_x}\sigma_{p_z} \geq 0$

$[\hat{x},\hat{p_y}] = [\hat{x},\hat{p_z}] = [\hat{y},\hat{p_x}] = [\hat{y},\hat{p_z}] = [\hat{z},\hat{p_x}] = [\hat{z},\hat{p_z}] = 0$

$\Rightarrow \sigma_x\sigma_{p_y} \geq 0,\ \sigma_x\sigma_{p_z} \geq 0,\ \sigma_y\sigma_{p_x} \geq 0,\ \sigma_y\sigma_{p_z} \geq 0,\ \sigma_z\sigma_{p_x} \geq 0,\ \sigma_z\sigma_{p_y} \geq 0$

$[\hat{x},\hat{p_x}] = [\hat{y},\hat{p_y}] = [\hat{z},\hat{p_z}] = i\hbar$

$\Rightarrow \sigma_x\sigma_{p_x} \geq \frac{\hbar}{2},\ \sigma_y\sigma_{p_y} \geq \frac{\hbar}{2},\ \sigma_z\sigma_{p_z} \geq \frac{\hbar}{2}$

4.2. $V(x,y,z) = \begin{cases} 0, & x,y,z \text{ all between } 0 \,\&\, a \\ \infty, & \text{otherwise} \end{cases}$

a) $i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V\psi, \quad -\infty < x,y,z < \infty, t > 0$

$\psi(x,y,z,0) = \psi_0(x,y,z)$

$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + \infty\psi,\ x,y,z \notin (0,a)$

outside cube

$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right),\ 0 < x,y,z < a, t > 0$

$\psi(0,y,z,t) = 0,\ \psi(x,0,z,t) = 0,\ \psi(x,y,0,t) = 0,\ \psi(x,y,z,0) = \psi_0(x,y,z)$

$\psi(a,y,z,t) = 0,\ \psi(x,a,z,t) = 0,\ \psi(x,y,a,t) = 0$

For $\psi(x,y,z,t) = X(x)\,Y(y)\,Z(z)\,T(t)$ and apply boundary conditions:

$\Rightarrow X(0) = X(a) = Y(0) = Y(a) = Z(0) = Z(a) = 0$

Apply Separation of Variables:

$$i\hbar\frac{T'(t)}{T(t)} = \frac{-\hbar^2}{2m}\left[\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)}\right] = E \text{ (const)}$$

$\Rightarrow \dfrac{i\hbar T'(t)}{T(t)} = E \text{ (const)}$

$\dfrac{X''(x)}{X(x)} = F \text{ (const)}$

$\dfrac{Y''(y)}{Y(y)} = G \text{ (const)}$

$-\dfrac{2mE}{\hbar^2} - F - \dfrac{Z''(z)}{Z(z)} = G$

$X(x) = \cancel{C_1 \cosh \mu x}^{\,0} + C_2 \sinh \mu x \quad (F = \mu^2) \Rightarrow X'' = 0$

$\Rightarrow X(x) = C_3 x + C_4^{\,=0} \quad (F = 0) \Rightarrow X'' = -\gamma^2 X$

$X(x) = C_5 \cos \gamma x^{\,=0} + C_6 \sin \gamma x \quad (F = -\gamma^2)$

$\hookrightarrow F = -\gamma^2 = -\dfrac{J^2\pi^2}{a^2},\ J = 1, 2, 3, \ldots \Rightarrow X(x) = C_6 \sin \dfrac{J\pi x}{a}$

$\Rightarrow$ For $G = -\dfrac{K^2\pi^2}{a^2}$ $(K = 1, 2, 3) \Rightarrow Y(y) = C_7 \sin \dfrac{K\pi y}{a}$

$\Rightarrow Z(z) = \cancel{C_8 \cos \sqrt{\dfrac{2mE}{\hbar^2} - \dfrac{J^2\pi^2}{a^2} - \dfrac{K^2\pi^2}{a^2}}}^{\,0} + C_9 \sin \sqrt{\dfrac{2mE}{\hbar^2} - \dfrac{J^2\pi^2}{a^2} - \dfrac{K^2\pi^2}{a^2}}\, z$

$\Rightarrow Z(z) = \dfrac{2mE}{\hbar^2} - \dfrac{J^2\pi^2}{a^2} - \dfrac{K^2\pi^2}{a^2} = \dfrac{\ell^2\pi^2}{a^2} \quad (\ell = 0, \pm 1, \pm 2 \ldots)$

$\Rightarrow E_{JK\ell} = \dfrac{\pi^2\hbar^2}{2ma^2}(J^2 + K^2 + \ell^2)$

$\Rightarrow$ Stationary states: $\Psi_{JK\ell}(x, y, z, t) = X_J(x)\, Y_K(y)\, Z_\ell(z)\, T_{JK\ell}(t)$

$= A \sin \dfrac{J\pi x}{a} \sin \dfrac{K\pi y}{a} \sin \dfrac{\ell\pi z}{a}\, e^{-i\pi^2\hbar/2ma^2 (J^2+K^2+\ell^2)t}$

Normallized to find $A: \pm\left(\dfrac{2}{a}\right)^{3/2}$

$\Rightarrow \Psi_{JK\ell}(x, y, z, t) = \left(\dfrac{2}{a}\right)^{3/2} \sin \dfrac{J\pi x}{a} \sin \dfrac{K\pi y}{a} \sin \dfrac{\ell\pi z}{a}\, e^{-i\pi^2\hbar/2ma^2 (J^2+K^2+\ell^2)t}$

According to Principle of Superposition:

$\Psi(x, y, z, t) = \sum_{J=1}^{\infty} \sum_{K=1}^{\infty} \sum_{\ell=1}^{\infty} c_{JK\ell}\, \Psi_{JK\ell}(x, y, z, t)$ (do math...)

$\Rightarrow c_{JK\ell} = \left(\dfrac{2}{a}\right)^{3/2} \int_0^a \int_0^a \int_0^a \Psi_0(x, y, z) \sin \dfrac{J\pi x}{a} \sin \dfrac{K\pi y}{a} \sin \dfrac{\ell\pi z}{a}\, dx\, dy\, dz$

	NRG	Degen	$J^2+K^2+\ell^2$	
b)	E_{111}	1	3	$= E_1$
	$E_{211/121/112}$	3	6	$= E_2$
	$E_{221/212/122}$	3	9	$= E_3$
	$E_{311/131/113}$	3	11	$= E_4$
	E_{222}	1	12	$= E_5$
	$E_{312/321/132/123/213/231}$	6	14	$= E_6$

c) For E_{14} : $E_{333/511/151/115}$ ⇒ degen = 4. We have the same set of number for E_{14} (333) similar to E_1 (111) & E_5 (222) but unlike these 2 where degen = 1, E_{14} has a degen = 4 by having other set of numbers (511, 151, 115).

4.3. a) $i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \Psi + V(r)\Psi(r,t) = -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) + V(r)\Psi(r,t)$

For $\Psi(r,t) = \psi(r)T(t)$

$\Rightarrow i\hbar \frac{\partial}{\partial t}[\psi(r)T(t)] = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}[\psi(r)T(t)]\right] + V(r)[\psi(r)T(t)]$

$\Leftrightarrow i\hbar\, \psi(r)T'(t) = -\frac{\hbar^2}{2m}\frac{T(t)}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) + V(r)[\psi(r)T(t)]$

Separating variables:

$\Rightarrow i\hbar \frac{T'(t)}{T(t)} = E$

$-\frac{\hbar^2}{2m}\frac{1}{r^2\psi(r)}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + V(r) = E$ } $\Rightarrow V(r) = E + \frac{\hbar^2}{2m r^2 \psi(r)}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right)$

If $\psi(r) = Ae^{-r/a} \Rightarrow V(r) = E + \frac{\hbar^2}{2mr^2\psi(r)}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)$

$= E + \frac{\hbar^2}{2mr^2(Ae^{-r/a})}\frac{d}{dr}\left[r^2 \frac{d}{dr}(Ae^{-r/a})\right] = E - \frac{\hbar^2}{2mar^2e^{-r/a}}\left(2re^{-r/a} - \frac{r^2}{a}e^{-r/a}\right)$

$= E - \frac{\hbar^2}{2ma^2}\left(\frac{2a}{r} - 1\right)$

$V(r) \to 0$ as $r \to \infty \Rightarrow \lim_{r\to\infty} V(r) = E - \frac{\hbar^2}{2ma^2}(-1) = 0 \Rightarrow \underline{E} = \frac{-\hbar^2}{2ma^2}$

$\Rightarrow V(r) = -\frac{\hbar^2}{mar}$

b) If $\psi(r) = Ae^{-r^2/a^2} \Rightarrow V(r) = E + \frac{\hbar^2}{2mr^2\psi(r)}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)$

$= E + \frac{\hbar^2}{2mr^2(Ae^{-r^2/a^2})}\frac{d}{dr}\left[r^2\frac{d}{dr}(Ae^{-r^2/a^2})\right] = E - \frac{\hbar^2}{ma^2r^2e^{-r^2/a^2}}\frac{d}{dr}\left(r^3e^{-r^2/a^2}\right)$

$= E - \frac{\hbar^2}{ma^2}\left(3 - 2\frac{r^2}{a^2}\right)$

$V(0) = 0 \Rightarrow E - \frac{\hbar^2}{ma^2}\cdot 3 = 0 \Rightarrow E = \frac{3\hbar^2}{ma^2} \Rightarrow V(r) = \frac{2\hbar^2}{ma^4}r^2$

4.7. Eqn 4.32: $Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}\, e^{im\phi} P_l^m \cos\theta$

$\Rightarrow Y_3^2(\theta,\phi) = \sqrt{\frac{(2\cdot 3+1)(3-2)!}{4\pi(3+2)!}}\, e^{2i\phi} P_3^2 \cos\theta$

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$$= \sqrt{\frac{7}{4\pi}\frac{1!}{5!}}\, e^{2i\phi}\left[\frac{(-1)^2}{2^3 3!}(1-x^2)^{2/2}\frac{d^{3+2}}{dx^{3+2}}(x^2-1)^3\right]\Big|_{x=\cos\theta}$$

$$= \sqrt{\frac{7}{480\pi}}\, e^{2i\phi}\left[\frac{1}{48}(1-x^2)\frac{d^5}{x^5}(x^6-3x^4+3x^2-1)\right]\Big|_{x=\cos\theta}$$

$$= \frac{1}{4}\sqrt{\frac{7}{30\pi}}\, e^{2i\phi}\left[\frac{1}{48}(1-x^2)(720x)\right]\Big|_{x=\cos\theta} = \frac{15}{4}\sqrt{\frac{7}{30\pi}}\, e^{2\phi}(1-\cos^2\theta)\cos\theta$$

$$\Rightarrow Y_3^2(\theta,\phi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\, e^{2i\phi}\cos\theta\sin^2\theta$$

Eqn 4.18: $\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)\sin^2\theta Y$, $l=3$

Satisfy check.

$$\Rightarrow \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_3^2}{\partial\theta}\right) + \frac{\partial^2 Y_3^2}{\partial\phi^2} = \sin\theta\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{4}\sqrt{\frac{105}{2\pi}}\, e^{2i\phi}\cos\theta\sin^2\theta\right)\right] + \frac{\partial^2}{\partial\phi^2}\left(\frac{1}{4}\sqrt{\frac{105}{2\pi}}\, e^{2i\phi}\cos\theta\sin^2\theta\right)$$

$$= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\left(4e^{2i\phi}\cos\theta\sin^2\theta - 4e^{2i\phi}\cos\theta\sin^2\theta - 12e^{2i\phi}\cos\theta\sin^4\theta\right)$$

$$= -(12\sin^2\theta)\left(\frac{1}{4}\sqrt{\frac{105}{2\pi}}\, e^{2i\phi}\cos\theta\sin^2\theta\right) = -3(3+1)\sin^2\theta\, Y_3^2(\theta,\phi)\ ✓$$

$$Y_l^l(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-l)!}{(l+l)!}}\, e^{il\phi}P_l^l(\cos\theta) = \sqrt{\frac{2l+1}{4\pi}\frac{0!}{(2l)!}}\, e^{il\phi}\left[\frac{(-1)^l}{2^l l!}(1-x^2)^{l/2}\frac{d^{l+l}}{dx^{l+l}}(x^2-1)^l\right]\Big|_{x=\cos\theta}$$

$$= \sqrt{\frac{2l+1}{4\pi(2l)!}}\, e^{il\phi}\left[\frac{(-1)^l}{2^l l!}(1-x^2)^{l/2}\frac{d^{2l}}{dx^{2l}}(x^2-1)^l\right]\Big|_{x=\cos\theta}$$

$l=0 \Rightarrow \frac{d^0}{x^0}(x^2-1)^0=1$, $l=1\Rightarrow\frac{d^2}{x^2}(x^2-1)^1=2$, $l=2\Rightarrow\frac{d^4}{x^4}(x^2-1)^2=24$, $l=3\Rightarrow\frac{d^6}{x^6}(x^2-1)^3=720$

$$\Rightarrow \frac{d^{2l}}{dx^{2l}}(x^2-1)^l = (2l)! \Rightarrow Y_l^l(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi(2l)!}}\, e^{il\phi}\left[\frac{(-1)^l}{2^l l!}(1-x^2)^{l/2}(2l)!\right]\Big|_{x=\cos\theta}$$

$$= \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\, e^{il\phi}\sin^l\theta$$

Satisfy check:

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_l^l}{\partial\theta}\right) + \frac{\partial^2 Y_l^l}{\partial\phi^2} = \sin\theta\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\left[\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\, e^{il\phi}\sin^l\theta\right]\right\}$$

$$+\frac{\partial^2}{\partial\phi^2}\left[\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\, e^{il\phi}\sin^l\theta\right] = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\, e^{il\phi}\sin\theta\frac{d}{d\theta}\left[\sin\theta\frac{d}{d\theta}(\sin^l\theta)\right]$$

$$+\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\sin^l\theta\frac{d^2}{d\phi^2}(e^{il\phi}) = -l(l+1)\sin^2\theta\left[\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\, e^{il\phi}\sin^l\theta\right]$$

$$= -l(l+1)\sin^2\theta\, Y_l^l(\theta,\phi)$$