HOMEWORK 9

4.11. $V(r) = \begin{cases} -V_0; & r \le a; \\ 0, & r > a. \end{cases}$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2M} \nabla^2 \Psi + V\Psi$$

$$= -\frac{\hbar^2}{2M} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Psi}{\partial \phi^2} \right] + V(r)\Psi(r,\theta,\phi,t)$$

For $\Psi(r,\theta,\phi,t) = R(r)\,\Theta(\theta)\,\xi(\phi)\,T(t)$

$$\Rightarrow \left. \begin{aligned} & i\hbar \frac{T'(t)}{T(t)} = E \\ & \frac{1}{R(r)} \frac{d}{dr}\left(r^2 R'(r) \right) - \frac{2Mr^2}{\hbar^2} [V(r) - E] = F \\ & \frac{\sin\theta}{\Theta(\theta)} \frac{d}{d\theta}\left(\Theta'(\theta) \sin\theta \right) + F\sin^2\theta = G \\ & -\frac{\xi''(\phi)}{\xi(\phi)} = G \end{aligned} \right\}$$

$\Rightarrow \Theta(\theta) = C_2 P_\ell^m(\cos\theta)$

For 3rd Eigenvalue: $F = \ell(\ell+1)$; For 4th Eigenvalue: $G = m^2$ & $\Phi(\phi) = C_1 e^{im\phi}$

$\Rightarrow$ 2nd Eigenvalue: $\frac{1}{R(r)}\frac{d}{dr}\left(r^2 R'(r)\right) - \frac{2Mr^2}{\hbar^2}[V(r)-E] = \ell(\ell+1)$

$$\frac{d}{dr}\left(r^2 R'(r)\right) - \frac{2Mr^2}{\hbar^2}[V(r)-E]R(r) = \ell(\ell+1)R(r)$$

$$r^2 R''(r) + 2rR'(r) - \frac{2Mr^2}{\hbar^2}[V(r)-E]R(r) = \ell(\ell+1)R(r)$$

For $E<0$ & $\ell=0$ (ground)

$$\Rightarrow r^2R''(r) + 2rR'(r) - \frac{2Mr^2}{\hbar^2}[V(r)-E]R(r) = 0.$$

For $u(r) = rR(r) \Rightarrow r^2\left[\frac{u(r)}{r}\right]'' + 2r\left[\frac{u(r)}{r}\right]' - \frac{2Mr^2}{\hbar^2}[V(r)-E]\frac{u(r)}{r} = 0$

$$\Rightarrow u''(r) = \frac{2M}{\hbar^2}[V(r)-E]u(r)$$

$$\Rightarrow u''(r) = \begin{cases} \frac{2M}{\hbar^2}(-V_0-E)u(r) & \text{if } r\le a \\ \frac{2M}{\hbar^2}(0-E)u(r) & \text{if } r>a \end{cases} = \begin{cases} -\frac{2M(V_0+E)}{\hbar^2}u(r) & \text{if } r\le a \\ -\frac{2ME}{\hbar^2}u(r) & \text{if } r>a \end{cases}$$

$$\Rightarrow u(r) = \begin{cases} C_3\cos lr + C_4\sin lr & \text{if } r\le a \\ C_5 e^{-kr} + C_6 e^{kr} & \text{if } r>a \end{cases} \quad \text{where } l = \frac{\sqrt{2M(V_0+E)}}{\hbar} \ \& \ k = \frac{\sqrt{-2ME}}{\hbar}$$

Boundary Conditions: $\begin{cases} \lim_{r\to 0} u(r) = \lim_{r\to 0} rR(r) = 0 \\ \lim_{r\to\infty} u(r) = \lim_{r\to\infty} rR(r) = 0 \end{cases} \Rightarrow u(r) = \begin{cases} C_4\sin lr & \text{if } r\le a \\ C_5 e^{-kr} & \text{if } r>a \end{cases}$

spherical Bessel eqn with $\ell=0$: $R(r) = C_7 j_0(lr) + C_8 \overset{\to 0}{n_0}(lr)$ ($R(0)$ is finite)

$\Rightarrow R(r) = C_7 j_0(lr) = C_4\left(\frac{\sin lr}{r}\right)$. To determine C_4 & $C_5 \Rightarrow$ derivative continuous @ $r=a$

$$\Rightarrow \lim_{r\to a^-} u(r) = \lim_{r\to a^+} u(r): C_4\sin la = C_5 e^{-ka}$$

$$\Rightarrow \lim_{r\to a^-}\frac{du}{dr} = \lim_{r\to a^+}\frac{du}{dr}: C_4 l\cos la = -C_5 k e^{-ka}$$

Assume $C_4 \neq 0 \Rightarrow C_4 l\cos la = -k(C_4\sin la) \Rightarrow -la\cot la = ka$

$$\Rightarrow -\frac{\sqrt{2M(V_0+E)}}{\hbar}a\cot\left[\frac{\sqrt{2M(V_0+E)}}{\hbar}a\right] = \frac{\sqrt{-2ME}}{\hbar}a = \sqrt{\left(\frac{\sqrt{2MV_0}}{\hbar}a\right)^2 - \left[\frac{\sqrt{2M(V_0+E)}}{\hbar}a\right]^2}$$

Introducing $z_0 = \frac{\sqrt{2MV_0}}{\hbar}a$ & $z = \frac{\sqrt{2M(V_0+E)}}{\hbar}a$ to get eqn for the eigenvalues:

$$-z\cot z = \sqrt{z_0^2 - z^2} \Leftrightarrow -\cot z = \sqrt{(z_0/z)^2 - 1}$$

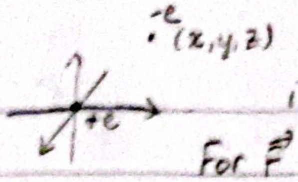
$$\Rightarrow \frac{\pi}{2} \le z < \pi \Leftrightarrow \frac{\pi}{2} \le la < \pi \Leftrightarrow \frac{\pi}{2} \le \frac{\sqrt{2M(V_0+E)}}{\hbar}a < \pi \Leftrightarrow \frac{\pi^2\hbar^2}{8Ma^2} \le V_0 + E < \frac{\pi^2\hbar^2}{2Ma^2}$$

$$\Rightarrow z_0 = \frac{\sqrt{2MV_0}}{\hbar}a < \frac{\pi}{2}$$

$\Rightarrow V_0 a^2 < \frac{\pi^2\hbar^2}{8M} \Rightarrow$ There is no ground state.

Scanned with CamScanner

4.12.



$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_e}\nabla^2\Psi + V(x,y,z)\Psi(x,y,z,t)$$

For $\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{(-e)e}{r^2}$ $\quad F = -\nabla V$

Since $\vec{F}$ is only dependent on spherical coord $\Rightarrow -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = -\frac{dV}{dr}$

$\Leftrightarrow \int_\infty^r \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_0^2}\,dr_0 = \int_\infty^r \frac{dV}{dr}(r_0)\,dr_0 \Leftrightarrow -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_0}\Big|_\infty^r = V(r) - \underbrace{V(\infty)}_{0} \Rightarrow V(r) = \frac{-e^2}{4\pi\varepsilon_0 r}$

$\Rightarrow i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_e}\nabla^2\Psi + V(r)\Psi(r,\phi,\theta,t)$, similar to 4.11 $\Rightarrow \begin{cases} i\hbar\frac{T'(t)}{T(t)} = E \\ \frac{1}{R(r)}\ldots = F \\ \frac{\sin\theta}{\Theta(\theta)}\ldots = G \\ -\frac{\Phi''(\phi)}{\Phi(\phi)} = G \end{cases}$

$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\,e^{im\phi}P_l^m(\cos\theta)\ \begin{cases} l = 0,1,2,\ldots \\ m = -l,-l+1,\ldots,l-1,l \end{cases}$

$\Rightarrow \frac{1}{R(r)}\frac{d}{dr}\left(r^2R'(r)\right) - \frac{2m_er^2}{\hbar^2}\left(-\frac{e^2}{4\pi\varepsilon_0 r} - E\right) = l(l+1)$

$\Rightarrow \frac{d}{dr}\left[r^2\frac{dR}{dr}(r)\right] + \left[2\left(\frac{m_ee^2}{4\pi\varepsilon_0\hbar^2}\right)r + \frac{2m_er^2}{\hbar^2}E\right]R(r) - l(l+1)R(r) = 0$. For $s = kr$ & $k = \frac{\sqrt{-8m_eE}}{\hbar}$

$\Rightarrow \frac{ds}{dr}\frac{d}{ds}\left[\left(\frac{s}{k}\right)^2\frac{ds}{dr}\frac{d}{ds}R\left(\frac{s}{k}\right)\right] + \left[\frac{2}{a_0}\left(\frac{s}{k}\right) + \frac{2m_es^2}{\hbar^2k^2}E\right]R\left(\frac{s}{k}\right) - l(l+1)R\left(\frac{s}{k}\right) = 0$

$w(s) = s^le^{-s/2}u(s) \Rightarrow 0 = \frac{d}{s}\left\{s^2\frac{d}{ds}\left[s^le^{-s/2}u(s)\right]\right\} + \left[\frac{2s}{a_0k} - \frac{s^2}{4} - l(l+1)\right]s^le^{-s/2}u(s)$

$\Rightarrow 0 = s^{l+1}e^{-s/2}\frac{d^2u}{ds^2} + (2l+2-s)s^{l+1}e^{-s/2}\frac{du}{ds} + \left(\frac{2}{a_0k} - l - 1\right)s^{l+1}e^{-s/2}u(s)$.

$\Leftrightarrow s\frac{d^2u}{ds^2} + [(2l+1)+1-s]\frac{du}{ds} + \left(\frac{2}{a_0k} - l - 1\right)u(s) = 0, \quad 0 < s < \infty$

$\Rightarrow \frac{2}{a_0k} - l - 1 = N$ (for N is a non negative integer) $\Rightarrow \frac{2}{a_0k} = N + l + 1 \Rightarrow \frac{2}{a_0k} = n$

$\Rightarrow n = 2\left(\frac{m_ee^2}{4\pi\varepsilon_0\hbar^2}\right)\left(\frac{\hbar}{\sqrt{-8m_eE}}\right) \Rightarrow \boxed{E_n = -\frac{m_ee^4}{32\pi^2\varepsilon_0^2\hbar^2n^2} = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2}\quad n = 1,2,\ldots}$

$\Rightarrow u_s = AL_{n-l-1}^{2l+1}(s)$

$\Rightarrow u_s = A(n+l)!\sum_{J=0}^{n-l-1}\frac{(-1)^J}{J!(n-l-J-1)!(2l+J+1)!}s^J \Rightarrow u(s) = As^le^{-s/2}L_{n-l-1}^{2l+1}(s)$

$\Rightarrow R(r) = A(kr)^le^{-(kr)/2}L_{n-l-1}^{2l+1}(kr) = A\left(\frac{2}{na_0}r\right)^le^{-r/na_0}L_{n-l-1}^{2l+1}\left(\frac{2}{na_0}r\right)$

Normalized to find $A = \sqrt{\left(\frac{2}{na_0}\right)^3\frac{(n-l-1)!}{2n(n+l)!}}$

$$\boxed{\Rightarrow R_{nl}(r) = \frac{2}{n^2}\sqrt{\frac{(n+l)!(n-l-1)!}{a_0^3}}\left[\sum_{J=0}^{n-l-1}\frac{(-1)^J}{J!(n-l-J-1)!(2l+J+1)!}\left(\frac{2}{na_0}r\right)^{J+l}\right]e^{-r/na_0}}$$

for $n = 1,2,3,\ \ l = 0,1,2$ & $n > l$

$\Rightarrow R_{30}(r) = \frac{2}{3^2}\sqrt{\frac{(3+0)!(3-0-1)!}{a_0^3}}\left[\sum_{J=0}^{3-0-1}\frac{(-1)^J}{J!(3-0-J-1)![2(0)+J+1]!}\left(\frac{2}{3a_0}r\right)^{J+0}\right]e^{-r/3a_0}$

$= \frac{2}{9}\sqrt{\frac{3}{a_0^3}}\left[1 - \frac{2}{3}\left(\frac{r}{a_0}\right) + \frac{2}{27}\left(\frac{r}{a_0}\right)^2\right]e^{-r/3a_0}$

$$\Rightarrow R_{31}(r) = \frac{2}{3^2}\sqrt{\frac{(3+1)!\,(3-1-1)!}{a_0^3}}\left[\sum_{J=0}^{3-1-1}\frac{(-1)^J}{J!\,(3-1-J-1)!\,[(2)(1)+J+1]!}\left(\frac{2}{3a_0}r\right)^{J+1}\right]e^{-r/3a_0}$$

$$= \frac{4}{81}\sqrt{\frac{6}{a_0^3}}\left(\frac{r}{a_0}\right)\left[1-\frac{1}{6}\left(\frac{r}{a_0}\right)\right]e^{-r/3a_0}$$

$$\Rightarrow R_{32}(r) = \frac{2}{3^2}\sqrt{\frac{(3+2)!\,(3-2-1)!}{a_0^3}}\left[\sum_{J=0}^{3-2-1}\frac{(-1)^J}{J!\,(3-2-J-1)!\,[2\cdot2+J+1]!}\left(\frac{2}{3a_0}r\right)^{J+2}\right]e^{-r/3a_0}$$

$$= \frac{1}{1215}\sqrt{\frac{120}{a_0^3}}\left(\frac{r}{a_0}\right)^2 e^{-r/3a_0}$$

4.15. $\Psi_{100}(r,\theta,\phi,t) = R_{10}(r)Y_0^0(\theta,\phi)\,T_1(t) = \left(\sqrt{\frac{4}{a_0^3}}\,e^{-r/a_0}\right)\left(\sqrt{\frac{1}{4\pi}}\right)e^{-iE_1t/\hbar}$

$$= \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}e^{-iE_1t/\hbar}$$

$$\Rightarrow \Psi_{211} = \left[\frac{1}{2\sqrt{6a_0^3}}\left(\frac{r}{a_0}\right)e^{-r/(2a_0)}\right]\left(-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\right)e^{-iE_2t/\hbar} = \frac{-1}{8\sqrt{\pi a_0^5}}\,r e^{-r/2a_0}\sin\theta e^{i\phi}e^{-iE_2t/\hbar}$$

a) $\langle r\rangle = \langle\Psi_{100}|r|\Psi_{100}\rangle = \iiint \Psi^*_{100}(r,\theta,\phi,t)\,r\,\Psi_{100}(r,\theta,\phi,t)\,dV$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\left(\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}e^{iE_1t/\hbar}\right)r\left(\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}e^{-iE_1t/\hbar}\right)(r^2\sin\theta\,dr\,d\phi\,d\theta)$$

$$= \frac{1}{\pi a_0^3}(2)(2\pi)\int_0^{\infty}\frac{\partial^3}{\partial u^3}(-e^{-ur})\Big|_{u=2/a_0}dr = \frac{-4}{a_0^3}\frac{d^3}{du^3}\left(\int_0^{\infty}e^{-ur}dr\right)\Big|_{u=2/a_0}$$

$$= \frac{-4}{a_0^3}\frac{d^3}{du^3}\left(\frac{1}{u}\right)\Big|_{u=2/a_0} = \frac{-4}{a_0^3}\left(-\frac{6a_0^4}{16}\right) = \frac{3}{2}a_0$$

$\langle r^2\rangle = \langle\Psi_{100}|r^2|\Psi_{100}\rangle = \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\left(\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}e^{iE_1t/\hbar}\right)r^2\left(\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}e^{-iE_1t/\hbar}\right)(r^2\sin\theta\,dr\,d\phi\,d\theta)$

$$= \frac{4}{a_0^3}\frac{d^4}{du^4}\left(\frac{1}{u}\right)\Big|_{u=2/a_0} = \frac{4}{a_0^3}\frac{d^3}{du^3}\left(\frac{-1}{u^2}\right)\Big|_{u=2/a_0} = \frac{4}{a_0^3}\left(\frac{24a_0^5}{32}\right) = 3a_0^2$$

b) $\langle x\rangle = \langle\Psi_{100}|x|\Psi_{100}\rangle = \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\left(\frac{1}{\sqrt{a_0^3}}\cdots\right)(r\sin\theta\cos\phi)\left(\frac{1}{\sqrt{\pi a_0^3}}\cdots\right)(r^2\sin\theta\,dr\,d\phi\,d\theta)$

$$= \frac{1}{\pi a_0^3}\left(\int_0^{\pi}\sin^2\theta\,d\theta\right)\left(\underbrace{\int_0^{2\pi}\cos\phi\,d\phi}_{0}\right)\left(\int_0^{\infty}r^3e^{-2r/a_0}dr\right) = 0$$

$\langle x^2\rangle = \langle\Psi_{100}|x^2|\Psi_{100}\rangle = \frac{1}{\pi a_0^3}\left(\int_0^{\pi}\sin^3\theta\,d\theta\right)\left(\int_0^{2\pi}\cos^2\phi\,d\phi\right)\left(\int_0^{\infty}r^4e^{-2r/a_0}dr\right)$

$$= \frac{1}{\pi a_0^3}\left[\int_0^{\pi}(1-\cos^2\theta)\sin\theta\,d\theta\right]\left[\int_0^{2\pi}\frac{1}{2}(1+2\cos\phi)\,d\phi\right]\left(\frac{24a_0^5}{32}\right)$$

$w = \cos\theta \Rightarrow dw = -\sin\theta\,d\theta$

$$\Rightarrow \langle x^2\rangle = \frac{1}{\pi a_0^3}\left[\int_{\cos 0}^{\cos\pi}(1-w^2)(-dw)\right]\left[\frac{1}{2}\int_0^{2\pi}(1+\cos 2\phi)\,d\phi\right]\left(\frac{24a_0^5}{32}\right)$$

$$= \frac{1}{a_0^3}\left[2\left(w - \frac{u^3}{2}\right)\Big|_0^1\right]\left(\frac{24a_0^5}{32}\right) = \frac{1}{a_0^3}\left[2\left(1-\frac{1^3}{3}\right)\right]\left(\frac{24a_0^5}{32}\right) = \frac{1}{a_0^3}\left(\frac{4}{3}\right)\left(\frac{3a_0^5}{4}\right) = a_0^2$$

c) $\langle x^2\rangle = \langle \psi_{211}|x^2|\psi_{211}\rangle$

$$= \int_0^\pi\int_0^{2\pi}\int_0^\infty \left[\frac{-1}{8\sqrt{\pi a_0^5}}\, r e^{-r/(2a_0)} \sin\theta e^{-i\phi} e^{iE_2t/\hbar}\right](r\sin\theta\cos\phi)^2\left[\frac{-1}{8\sqrt{\pi a_0^5}}\, r e^{-r/2a_0}\sin\theta e^{i\phi}e^{-iE_2t/\hbar}\right](r^2\sin\theta\, dr\, d\phi\, d\theta)$$

$$= \frac{1}{64\pi a_0^5}\left(\int_0^\pi \sin^3\theta d\theta\right)\left(\int_0^{2\pi}\cos^2\phi d\phi\right)\left(\int_0^\infty r^6 e^{-r/a_0}dr\right)$$

$$= \frac{1}{64a_0^5}\left[\int_{\cos 0}^{\cos\pi}(1-w^2)^2(-dw)\right]\frac{d^6}{du^6}\left(\int_0^\infty e^{-ur}dr\right)\Big|_{u=1/a_0} \quad (\text{sub } w=\cos\theta)$$

$$= \frac{1}{64a_0^5}\left[2\int_0^1(1-2w^2+w^4)dw\right]\frac{d^5}{du^5}\left(\frac{-1}{u^2}\right)\Big|_{u=1/a_0}$$

$$= \frac{1}{32a_0^5}\left(w-\frac{2}{3}w^3+\frac{1}{5}w^5\right)\Big|_0^1 \frac{d^4}{du^4}\left(\frac{2}{u^3}\right)\Big|_{u=1/a_0} = \frac{1}{32a_0^5}\left(1-\frac{2}{3}+\frac{1}{5}\right)\frac{d^3}{du^3}\left(\frac{-6}{u^4}\right)\Big|_{u=1/a_0}$$

$$= \frac{1}{32a_0^5}\left(\frac{8}{15}\right)\frac{d^2}{du^2}\left(\frac{24}{u^5}\right)\Big|_{u=1/a_0} = \frac{1}{60a_0^5}\left(\frac{720}{u^7}\right)\Big|_{u=1/a_0} = \frac{1}{60a_0^5}(720a_0^7) = 12a_0^2$$

4.18. $\Psi(\vec{r},0) = \frac{1}{\sqrt{2}}(\psi_{211}+\psi_{21-1})$

a) $\Psi(r,\theta,\phi,t) = \frac{1}{\sqrt{2}}\left(\psi_{211}e^{-iE_2t/\hbar} + \psi_{21-1}e^{-iE_2t/\hbar}\right)$

$$= \frac{1}{\sqrt{2}}\left[R_{21}(r)Y_1^1(\theta,\phi)e^{-iE_2t/\hbar} + R_{21}(r)Y_1^{-1}(\theta,\phi)e^{-iE_2t/\hbar}\right]$$

$$= \frac{1}{\sqrt{2}}R_{21}(r)\left[Y_1^1(\theta,\phi)+Y_1^{-1}(\theta,\phi)\right]e^{-iE_2t/\hbar}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{2\sqrt{6a_0^3}}\left(\frac{r}{a_0}\right)e^{-r/2a_0}\right]\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}+\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)e^{-\frac{iE_2t}{\hbar}}$$

$$= \frac{-i}{\sqrt{32\pi a_0^5}}(r\sin\theta\sin\phi)e^{-r/(2a_0)}e^{-iE_2t/\hbar}$$

$$\Rightarrow \Psi(x,y,z,t) = \frac{-i}{\sqrt{32\pi a_0^5}}\, y\, e^{(-1/2a_0\sqrt{x^2+y^2+z^2})}e^{-iE_2t/\hbar}$$

for $a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} \approx 0.529\cdot 10^{-10}$ m

$$E_n = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} \approx \frac{-2.18\cdot 10^{-18}}{n^2}$$

b) $\langle V\rangle = \langle\Psi|V|\Psi\rangle = \iiint_{\text{all space}} \Psi^*(r,\theta,\phi,t)V(r)\Psi(r,\theta,\phi,t)\,dV$

$$= \int_0^\pi\int_0^{2\pi}\int_0^\infty\left[\frac{i}{\sqrt{32\pi a_0^5}}(r\sin\theta\sin\phi)e^{-r/2a_0}e^{iE_2t/\hbar}\right]\left(\frac{-e^2}{4\pi\epsilon_0 r}\right)\left[\frac{-i}{\sqrt{32\pi a_0^5}}\cdots\right](r^2\sin\theta\, d\phi\, d\theta\, dr)$$

$$= \frac{-e^2}{4\pi\epsilon_0(32\pi a_0^5)}\left(\int_0^\pi \sin^3\theta d\theta\right)\left(\int_0^{2\pi}\sin^2\phi\, d\phi\right)\left(\int_0^\infty r^3 e^{-r/a_0}dr\right.$$

$$= \frac{-e^2}{128\pi^2\epsilon_0 a_0^5}\left[\int_0^\pi(1-\cos^2\theta)\sin\theta d\theta\right]\left[\int_0^{2\pi}\frac{1}{2}(1-\cos 2\phi)d\phi\right]\left[\int_0^\infty \frac{\partial^3}{\partial u^3}(-e^{-ur})\Big|_{u=1/a_0}dr\right]$$

sub $\omega = \cos\theta$

$$\Rightarrow \langle V\rangle = \frac{e^2}{128\pi^2\varepsilon_0 a_0^5}\left[\int_{\cos 0}^{\cos\pi}(1-\omega^2)(-d\omega)\right]\left[\frac{1}{2}\int_0^{2\pi}(1-\cos 2\phi)\,d\phi\right]\left[\int_0^\infty \frac{\partial^3}{\partial u^3}(-e^{-ur})\Big|_{u=1/a_0} dr\right]$$

$$= \frac{e^2}{128\pi^2\varepsilon_0 a_0^5}\left[2\int_0^1 (1-\omega^2)d\omega\right]\left[\frac{1}{2}(2\pi)\right]\left(-\frac{6}{u^4}\right)\Big|_{u=1/a_0}$$

$$= \frac{e^2}{128\pi^2\varepsilon_0 a_0^5}\left(\frac{4}{3}\right)(\pi)(-6a_0^4) = -\frac{e^2}{16\pi\varepsilon_0 a_0} \approx -\frac{(1.60\cdot 10^{-19}C)^2}{16(3.14)(8.85\cdot 10^{-12}\frac{C^2}{J\,m})(0.529\cdot 10^{-10}m)}$$

$$\approx -1.09\cdot 10^{-18}J \approx -6.80\,eV$$

4.22. a) $\cdot [L_z, x] = [xp_y - yp_x, x] = [xp_y, x] - [yp_x, x] = (-x[x,p_y] + 0p_y) - (-y[x,p_x] + 0\cdot p_x) = (-x\cdot 0) - (-y\cdot i\hbar) = i\hbar y$

$\cdot [L_z, y] = [xp_y - yp_x, y] = [xp_y, y] - [yp_x, y] = (-x[y,p_y] + 0\cdot p_y) - (-y[y,p_x] + 0\cdot p_x] = -i\hbar x$

$\cdot [L_z, p_x] = [xp_y - yp_x, p_x] = [xp_y, p_x] - [yp_x, p_x] = (x[p_x,p_x] + [xp_x]p_y) - (y[p_x,p_x] + [y,p_x]p_x) = (x\cdot 0 + i\hbar\cdot p_y) - (y\cdot 0 + 0\cdot p_x) = i\hbar p_y$

$\cdot [L_z, p_y] = [xp_y - yp_x, p_y] = [xp_y, p_y] - [yp_x, p_y] = (x\cdot 0 + 0\cdot p_y) - (y\cdot 0 + i\hbar p_x) = -i\hbar p_x$

$\cdot [L_z, p_z] = [xp_y - yp_x, p_z] = [xp_y, p_z] - [yp_x, p_z] = (x\cdot 0 + 0\cdot p_y) - (y\cdot 0 + 0\cdot p_x) = 0$

$\cdot [L_z, z] = [xp_y - yp_z, z] = [xp_y, z] - [yp_x, z] = (-x\cdot 0) - (-y\cdot 0) = 0$

b) $[L_z, L_x] = -[L_x, L_z] = -[yp_z - zp_y, L_z] = -([yp_z - zp_y, L_z] - [zp_y, L_z])$

$= -\{(y[p_z + L_z] + [y, L_z]p_z) - (z[p_y, L_z] + [z, L_z]p_y)\}$

$= -y[p_z, L_z] + [L_z, y]p_z - z[L_z, p_y] - [L_z, z]p_y$

$= y\cdot 0 + (-i\hbar x)p_z - z(-i\hbar p_x) - 0\cdot p_y = -i\hbar xp_z + i\hbar zp_x$

$= i\hbar(zp_x - xp_z) = i\hbar L_y$

c) $\cdot [L_z, r^2] = -[r^2, L_z] = -[x^2+y^2+z^2, L_z] = -([xx, L_z] + [yy, L_z] + [zz, L_z])$

$= -x[x, L_z] - [x, L_z]x - y[y, L_z] - [y, L_z]y - z[z, L_z] - [z, L_z]z$

$= x[L_z, x] + [L_z, x]x + y[L_z, y] + [L_z, y]y + z\cancel{[L_z, z]}^{0} + \cancel{[L_z, z]}^{0}z$

$= i\hbar(xy + yx - yx - xy) = i\hbar\cdot 0 = 0$

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$\cdot\ [L_z, p^2] = -[p^2, L_z] = -[p_x^2 + p_y^2 + p_z^2, L_z] = p_x[L_z, p_x] + [L_z, p_x]p_x + p_y[L_z, p_y]$
$+ [L_z, p_y]p_y + p_z[L_z, p_z] + [L_z, p_z]p_z = p_x(i\hbar p_y) + (i\hbar p_y)p_x + p_y(-i\hbar p_x)$
$+ (-i\hbar p_x)p_y + p_z(0) + 0 \cdot p_z = i\hbar(p_xp_y - p_yp_x + p_yp_x - p_xp_y) = i\hbar \cdot 0 = 0$

d) $[L_x, p^2] = [L_y, p^2] = 0$

Assume $H = (p^2/2m) + V$

$\cdot\ [H, L_x]f = [\frac{p^2}{2m} + V, L_x]f = (\frac{1}{2m}[p^2, L_x] + [V, L_x])f = (\frac{-1}{2m}[L_x, p^2] + [V, L_x])f$
$= (\frac{-1}{2m} \cdot 0 + [V, L_x])f = [V, L_x]f = (VL_x - L_xV)f = VL_xf - L_x(Vf)$
$= V(yp_z - zp_y)f - (yp_z - zp_y)(Vf)$
$= V[y(-i\hbar\frac{\partial}{\partial z}) - z(-i\hbar\frac{\partial}{\partial y})]f - [y(-i\hbar\frac{\partial}{\partial z}) - z(-i\hbar\frac{\partial}{\partial y})]Vf$
$= \cancel{-i\hbar yV\frac{\partial f}{\partial z}} + \cancel{i\hbar zV\frac{\partial f}{\partial y}} + i\hbar y\frac{\partial}{\partial z}(Vf) - i\hbar z\frac{\partial}{\partial y}(Vf) + \cancel{i\hbar yV\frac{\partial f}{\partial z}} - i\hbar z\frac{\partial V}{\partial y}f$
$= i\hbar(y\frac{\partial V}{\partial z} - z\frac{\partial V}{\partial y})f = i\hbar(y\frac{dV}{dr}\frac{\partial r}{\partial z} - z\frac{dV}{dr}\frac{\partial r}{\partial y})f$
$= i\hbar\frac{dV}{z}(y\frac{\partial r}{\partial z} - z\frac{\partial r}{\partial y})f$
$= i\hbar\frac{dV}{dr}[y \cdot \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2z - z \cdot \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2y]f = i\hbar\frac{dV}{dr}(0)f = 0$

Similarly: $\cdot\ [H, L_y]f = i\hbar\frac{dV}{dr}(z\frac{\partial r}{\partial x} - x\frac{\partial r}{\partial z})f = i\hbar\frac{dV}{dr}(0)f = 0$
$\cdot\ [H, L_z]f = i\hbar\frac{dV}{dr}(x\frac{\partial r}{\partial y} - y\frac{\partial r}{\partial x})f = i\hbar\frac{dV}{dr}(0)f = 0$

$\Rightarrow$ The Hamiltonian $H = (p^2/2m) + V$ commutes with all 3 components of L.