HOMEWORK 5

2.20. $\Psi(x,0) = Ae^{-a|x|}$

a) $1 = \int_{-\infty}^{\infty} A^2 e^{-2a|x|} dx = 2A^2 \int_0^{\infty} e^{-2a|x|} dx = 2A^2 \left(\frac{e^{-2ax}}{-2a}\right)_0^{\infty} = 2A^2 \frac{1}{2a} = \frac{A2}{a}$

$\Rightarrow A^2 = a \Rightarrow A = \sqrt{a} \Rightarrow \Psi(x,0) = \sqrt{a}\, e^{-a|x|}$

b) $\phi(k)$ = Fourier transform of $\Psi_0(x)$

$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-ikx} \Psi_0(x)\, dx = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-ikx - a|x|} dx$

$= \sqrt{\frac{a}{2\pi}} \left(\int_{\infty}^{0} e^{-ikx + ax} dx + \int_0^{\infty} e^{-ikx - ax} dx \right)$

$= \sqrt{\frac{a}{2\pi}} \left(\frac{1}{-ik+a} + \frac{1}{ik+a} \right) = \sqrt{\frac{a}{2\pi}} \left(\frac{2a}{k^2+a^2} \right)$

c) $\Psi(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-ikx - \frac{i\hbar}{2m}k^2 t} \tilde{\Psi_0}(k)\, dk$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx - \frac{i\hbar}{2m}k^2 t} \sqrt{\frac{a}{\cancel{2}\pi}} \left(\frac{\cancel{2}a}{k^2+a^2} \right) dk$

$= \frac{a\sqrt{a}}{\pi} \int_{-\infty}^{\infty} e^{ikx - \frac{i\hbar}{2m}k^2 t} \frac{1}{k^2+a^2} dk$

d) * a very small $\rightarrow a \rightarrow 0 \Rightarrow \lim_{a \to 0} \Psi(x,t) = 0 \int_{-\infty}^{\infty} e^{ikx - \dots} \frac{1}{k^2} dk = 0$

* a very large $\Rightarrow a \rightarrow \infty$

$\Rightarrow \lim_{a \to \infty} \Psi(x,t) = \frac{\infty}{\pi} \int_{-\infty}^{\infty} e^{ikx - \dots} \cdot \underbrace{\frac{1}{\infty}}_{\to 0} dk = \frac{\infty}{\pi} 0 = 0.$

2.22. a) $\int_{-3}^{1} (x^3-3x^2+2x-1)\,\delta(x+2)\,dx$

$x+2=0 \to x=-2 \Rightarrow -8-12-4-1=-25$

b) $\int_0^{\infty} |\cos(3x)+2|\,\delta(x-\pi)\,dx$

$x=\pi \Rightarrow \cos(3\pi)+2=-1+2=1$

c) $\int_{-1}^{1} e^{|x|+3}\,\delta(x-2)\,dx$

$x=2 \Rightarrow \int_{-1}^{1} e^{|x|+3}\delta(x-2)\,dx=0$ (x lies outside interval -1 & 1)

2.26. Fourier $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,\underbrace{f(x)}_{\delta(x)}\,dx$

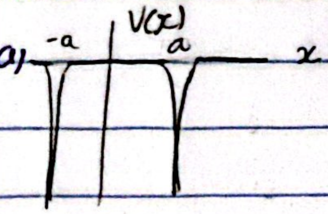
$F(\delta(x))=\frac{1}{\sqrt{2\pi}}e^0=\frac{1}{\sqrt{2\pi}}$

$\Rightarrow F^{-1}(F(\delta(x)))=F^{-1}\left(\frac{1}{\sqrt{2\pi}}\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\frac{1}{\sqrt{2\pi}}\,dk=\delta(x)$

$\Rightarrow \delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dx.$ ✓

2.27. $V(x)=-\alpha[\delta(x+a)+\delta(x-a)]$

a)



b) S.E: $i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}-\alpha[\delta(x+a)+\delta(x-a)]\Psi(x,t)$

$\Rightarrow i\hbar\frac{\phi'(t)}{\phi(t)}=-\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)}-\alpha[\delta(x+a)+\delta(x-a)]=E$

$\Rightarrow -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)}-\alpha[\delta(x+a)+\delta(x-a)]=E \Leftrightarrow \frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}\{\alpha[\delta(x+a)+\delta(x-a)]+E\}\psi(x)=$

$\Rightarrow \frac{d^2\psi}{dx^2}+\frac{2mE}{\hbar^2}\psi=0$ for $x\neq a$ & $-a$. Boundstate $\to E<0$

$\Rightarrow \psi(x)=\begin{cases} C_1e^{\frac{\sqrt{-2mE}}{\hbar}x}+\cancel{C_2e^{-\frac{\sqrt{-2mE}}{\hbar}x}}\to 0 & \text{for } x<-a\\ C_3e^{\frac{\sqrt{\cdots}}{\hbar}x}+C_4e^{-\frac{\sqrt{\cdots}}{\hbar}x} & \text{for } -a<x<a\\ C_5e^{\frac{\sqrt{\cdots}}{\hbar}x}+\cancel{C_6e^{-\frac{\sqrt{\cdots}}{\hbar}x}}\to 0 & \text{for } x>a\end{cases}$

for $k=\frac{\sqrt{-2mE}}{\hbar}$ and since the wave function has to be continuous at $x=-a$ & $x=a$

$$\Rightarrow \lim_{x\to a^-} \psi(x) = \lim_{x\to a^+} \psi(x)$$

$$\Rightarrow C_1 e^{-ka} = C_3 e^{-ka} + C_4 e^{ka} \; \& \; C_6 e^{-ka} = C_3 e^{ka} + C_4 e^{-ka} \text{ (condition 1\&2)}$$

- Condition 3:

$$\int_{-a-\varepsilon}^{-a+\varepsilon} \left\{ \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \{ \alpha[\delta(x+a)+\delta(x-a)] + E \} \psi(x) \right\} dx = \int_{-a-\varepsilon}^{-a+\varepsilon} 0\, dx$$

where $\varepsilon \to 0$

$$\Rightarrow \frac{2m\alpha}{\hbar^2} \psi(-a) = \lim_{x\to -a^-} \frac{d\psi}{dx} - \lim_{x\to -a^+} \frac{d\psi}{dx}$$

$$\Rightarrow \frac{2m\alpha}{\hbar^2} C_1 e^{-ka} = kC_1 e^{-k\alpha} - (kC_3 e^{-k\alpha} - kC_4 e^{k\alpha})$$

- Condition 4:

$$\int_{a-\varepsilon}^{a+\varepsilon} \left\{ \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \{ \alpha[\delta(x+a)+\delta(x-a)] + E \} \psi(x) \right\} dx = \int_{a-\varepsilon}^{a+\varepsilon} 0\, dx$$

where $\varepsilon \to 0$

$$\Rightarrow \frac{2m\alpha}{\hbar^2} \psi(a) = \lim_{x\to a^-} \frac{d\psi}{dx} - \lim_{x\to a^+} \frac{d\psi}{dx}$$

$$\Rightarrow \frac{2m\alpha}{\hbar^2} C_6 e^{-ka} = kC_3 e^{ka} - kC_4 e^{-ka} + kC_6 e^{-ka}$$

$$\Rightarrow \begin{cases} C_1 e^{-ka} = C_3 e^{-ka} + C_4 e^{ka} & (1) \\ C_6 e^{-ka} = C_3 e^{ka} + C_4 e^{-ka} & (2) \\ \frac{2m\alpha}{\hbar^2} C_1 e^{-ka} = kC_1 e^{-k\alpha} - kC_3 e^{-k\alpha} + kC_4 e^{k\alpha} & (3) \\ \frac{2m\alpha}{\hbar^2} C_6 e^{-ka} = kC_6 e^{-ka} + kC_3 e^{ka} - kC_4 e^{-ka} & (4) \end{cases}$$

From (3) & (4) $\Rightarrow \begin{cases} C_3 = \left(\frac{\hbar^2 k}{m\alpha} - 1\right) C_4 e^{2ka} \\ C_4 = \left(\frac{\hbar^2 k}{m\alpha} - 1\right) C_3 e^{2ka} \end{cases} \Rightarrow C_4 = \left(\frac{\hbar^2 k}{m\alpha} - 1\right)^2 C_4 e^{4ka}$

$$\Rightarrow 1 = \left(\frac{\hbar^2 k}{m\alpha} - 1\right)^2 e^{4ka}$$

(*) $\alpha = \hbar^2/(ma) \Rightarrow 1 = (k\alpha - 1)^2 e^{4ka} \Rightarrow \begin{cases} ka \approx 0.797 \to E \approx \frac{0.317\hbar^2}{ma^2} \\ ka = 0 \to E = 0 \\ ka \approx 1.11 \to E \approx \frac{-0.615\hbar^2}{ma^2} \end{cases}$

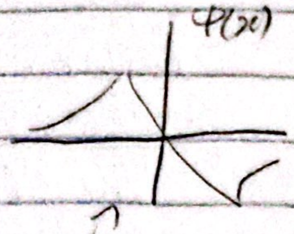
2 negative energies $\Rightarrow$ 2 boundstates. For $E \approx \frac{0.317\hbar^2}{ma^2}$

Substitute C_3 & $C_4 \Rightarrow \psi(x) = \begin{cases} 3.92 C_4 e^{\frac{0.797}{a}x} & \text{for } x < a \\ C_4 \left(-e^{+\frac{0.797}{a}x} + e^{-\frac{0.797}{a}x}\right) & \text{for } -a < x < a \\ -3.92 C_4 e^{-\frac{0.797}{a}x} & \text{for } x > a \end{cases}$

$$1=\int_{-\infty}^{\infty}|\psi(x)|^2dx=\overbrace{\int_{-\infty}^{-a}|\psi(x)|^2dx}^{(3.92C_4e^{\frac{0.797}{a}x})^2}+\overbrace{\int_{-a}^{a}|\psi(x)|^2dx}^{[C_4(-e^{\frac{0.797}{a}x})]}+\overbrace{\int_{a}^{\infty}|\psi(x)|^2dx}^{(-3.92C_4e^{-\frac{0.797}{a}x})^2}$$

$\rightarrow C_4\approx\frac{0.414}{\sqrt{a}}$ → Bound state with $E\approx-0.317\hbar^2/ma^2$:

$$\psi(x)=\begin{cases}\frac{1.62}{\sqrt{a}}e^{\frac{0.797}{a}x} & \text{for } x<-a\\ \frac{0.414}{\sqrt{a}}(e^{\frac{0.797}{a}x}+e^{-\frac{0.797}{a}}) & \text{for } -a<x<a\\ -\frac{1.62}{\sqrt{a}}e^{-\frac{0.797}{a}x} & \text{for } x>a\end{cases}$$



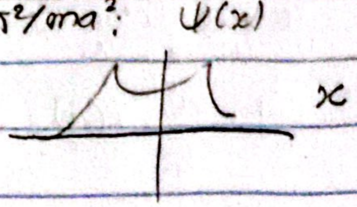
For $E\approx-0.615\hbar^2/ma^2$

$$\rightarrow\psi(x)=\begin{cases}10.2C_4e^{\frac{1.11}{a}x} & \text{for } x<-a\\ C_4(e^{\frac{1.11}{a}x}+e^{-\frac{1.11}{a}x}) & \text{for } -a<x<a\\ 10.2C_4e^{-\frac{1.11}{a}x} & \text{for } x>a\end{cases}$$

$a=3$

Normalized for C_4 → $C_4\approx\frac{0.211}{\sqrt{a}}$ → Bound state with $E\approx-0.615\hbar^2/ma^2$:

$$\psi(x)=\begin{cases}\frac{2.15}{\sqrt{a}}e^{\frac{1.11}{a}x} & \text{for } x<-a\\ \frac{0.211}{\sqrt{a}}(e^{\frac{1.11}{a}x}+e^{-\frac{1.11}{a}x}) & \text{for } -a<x<a\\ \frac{2.15}{\sqrt{a}}e^{-\frac{1.11}{a}x} & \text{for } x>a\end{cases}$$

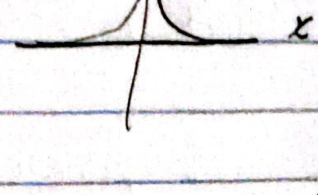


⊛ $\alpha=\hbar^2/4ma\rightarrow 1=(4Ka-1)^2\rightarrow\begin{cases}Ka\approx-0.628\ \text{(no value).}\\ Ka=0\rightarrow E=0\\ Ka\approx0.369\rightarrow E\approx\frac{-0.0682\hbar^2}{ma^2}\end{cases}$

$$\rightarrow\psi(x)=\begin{cases}3.09e^{\frac{0.369}{a}x} & \text{for } x<-a\\ C_4(e^{\frac{0.369}{a}x}+e^{-\frac{0.369}{a}x}) & \text{for } -a<x<a\\ 3.09C_4e^{-\frac{0.369}{a}x} & \text{for } x>a\end{cases}$$

Normalized for C_4 → $C_4\approx\frac{0.220}{\sqrt{a}}$ → Bound state with $E\approx-0.0682\hbar^2/ma^2$:

$$\psi(x)=\begin{cases}\frac{0.679}{\sqrt{a}}e^{\frac{0.369}{a}x} & \text{for } x<-a\\ \frac{0.220}{\sqrt{a}}(e^{\frac{0.369}{a}x}+e^{-\frac{0.369}{a}x}) & \text{for } -a<x<a\\ \frac{0.679}{\sqrt{a}}e^{-\frac{0.369}{a}x} & \text{for } x>a\end{cases}$$



c) ⊛ $a \to 0 \Rightarrow \begin{cases} C_1 = C_3 + C_4 \\ C_6 = C_3 + C_4 \\ \left(\frac{2m\alpha}{\hbar^2} - K\right) C_1 = -(KC_3 - KC_4) \\ \left(\frac{2m\alpha}{\hbar^2} - K\right) C_6 = (KC_3 - KC_4) \end{cases} \Rightarrow C_1 = C_6$

$\Rightarrow \left(\frac{2m\alpha}{\hbar^2} - K\right) C_1 = -(KC_3 - KC_4) = KC_3 - KC_4$, Assuming that $C_1 \neq 0$

$\Rightarrow K = \frac{2m\alpha}{\hbar^2} \Rightarrow E = -\frac{2m\alpha^2}{\hbar^2}$

⊛ $a \to \infty \Rightarrow \begin{cases} \left(\frac{2m\alpha}{\hbar^2} - K\right) C_4 e^{ka} = +KC_4 e^{ka} \\ \left(\frac{2m\alpha}{\hbar^2} - K\right) C_3 e^{ka} = KC_3 e^{ka} \end{cases}$ (Substituting (1) & (2) into (3) & (4))

$\Rightarrow \frac{2m\alpha}{\hbar^2} - K = K \Rightarrow E = -\frac{m\alpha^2}{2\hbar^2}$

$\Rightarrow$ Reasonable equation because single-well potential $V(x) = -\alpha\delta(x)$ has bound NRG $E = -\frac{m\alpha^2}{2\hbar^2}$, while the double well potential has $E = -\frac{2m\alpha^2}{\hbar^2}$.

2.29. $\psi(x) = \begin{cases} C_1 e^{Kx} \text{ for } x < -a \\ C_3 \cos\left[\frac{\sqrt{2m(V_0+E)}}{\hbar} x\right] + C_4 \sin\left[\frac{\sqrt{2m(V_0+E)}}{\hbar} x\right] \text{ for } -a \le x \le a \\ C_6 e^{-Kx} \text{ for } x > a \end{cases}$ For $K = \frac{\sqrt{-2mE}}{\hbar}$

Odd bound state $\Rightarrow C_3 = 0$ & $C_1 = -C_6$

$\Rightarrow \psi(x) = \begin{cases} -C_6 e^{Kx} \text{ for } x < -a \\ C_4 \sin\left[\frac{\sqrt{2m(V_0+E)}}{\hbar} x\right] \text{ for } -a \le x \le a \\ C_6 e^{-Kx} \text{ for } x > a \end{cases}$

Continuous at $x = a$ & $x = -a$

$\Rightarrow C_4 \sin\left[\frac{\sqrt{2m(V_0+E)}}{\hbar} a\right] = C_6 e^{-Kx} \Rightarrow C_6 = C_4 \sin[\dots] e^{-kx}$

$\Rightarrow \psi(x) = \begin{cases} -C_4 \sin\left[\frac{\sqrt{2m(V_0+E)}}{\hbar} a\right] e^{\frac{\sqrt{-2mE}}{\hbar}(x+a)} \text{ for } x < -a \\ C_4 \sin\left[\frac{\sqrt{2m(V_0+E)}}{\hbar} a\right] \text{ for } -a \le x \le a \\ C_4 \sin\left[\frac{\sqrt{2m(V_0+E)}}{\hbar} a\right] e^{-\frac{\sqrt{2mE}}{\hbar}(x-a)} \text{ for } x > a \end{cases}$

Normalizing and solve for C_4:

$$C_4 = \frac{1}{\frac{\hbar}{\sqrt{-2mE}}\sin^2\left[\frac{\sqrt{2m(V_0+E)}}{\hbar}a\right] + a - \frac{\hbar}{2\sqrt{2m(V_0+E)}}\sin\left[2\frac{\sqrt{2m(V_0+E)}}{\hbar}a\right]}$$

I won't plug this abysmal thing in the wave function but you know the drill.

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