HOMEWORK 4

2.10.

a) $\Psi_g = A_0 e^{-m\omega x^2/2\hbar}$ where $A_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$ $\quad a_+ = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x - ip)$

$\Psi_n = A_n (a_+)^n \Psi_g$ $\quad a_- = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip)$

$\Rightarrow \Psi_2 = A_2 (a_+)^2 \Psi_g$

$= A_2 \frac{1}{2m\hbar\omega}(m\omega x - i\hat{p})^2 \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$

$= A_2 \left[\frac{2m\omega}{\hbar}(x^2 - 1)\right] \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$

$\Rightarrow 1 = \int_{-\infty}^{\infty} |\Psi_2(x)|^2 dx = \int_{-\infty}^{\infty} A_2^2 \left[\frac{2m\omega}{\hbar}(x^2 - 1)\right]^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega x^2/\hbar} dx$

$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$

$= 2A_2^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_0^{\infty} \left[\frac{2m\omega}{\hbar}(x^2 - 1)\right]^2 e^{-m\omega x^2/\hbar} dx \quad \Rightarrow d\xi = \sqrt{\frac{m\omega}{\hbar}}\, dx$

$= 2A_2^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_0^{\infty} (2\xi^2 - 1)^2 e^{-\xi^2} \left(\sqrt{\frac{\hbar}{m\omega}}\, d\xi\right)$

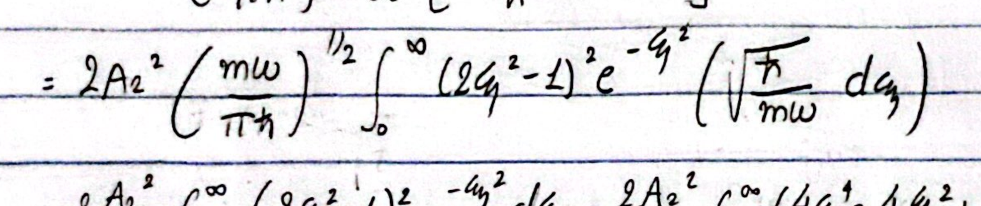
$= \frac{2A_2^2}{\sqrt{\pi}} \int_0^{\infty} (2\xi^2 - 1)^2 e^{-\xi^2} d\xi = \frac{2A_2^2}{\sqrt{\pi}} \int_0^{\infty} (4\xi^4 - 4\xi^2 + 1) e^{-\xi^2} d\xi$

$= \frac{2A_2^2}{\sqrt{\pi}} \left(4\int_0^{\infty} (\xi^4 e^{-\xi^2} d\xi) - 4\int_0^{\infty} (\xi^2 e^{-\xi^2} d\xi) + \int_0^{\infty} e^{-\xi^2} d\xi\right)$

$= \frac{2A_2^2}{\sqrt{\pi}} \left[4\sqrt{\pi}\frac{4!}{2!}\left(\frac{1}{2}\right)^5 - 4\sqrt{\pi}\frac{2!}{1!}\left(\frac{1}{2}\right)^3 + \sqrt{\pi}\frac{1}{2}\right]$

$= 2A_2^2 \Rightarrow A_2 = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \Psi_2 = \frac{\sqrt{2}\, m\omega}{\hbar}(x^2 - 1)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$

b)



$\Psi_1 = A_1 (a_+)\Psi_g = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x e^{-m\omega x^2/2\hbar}$, for $A_1 = 1$.

c) $\int_{-\infty}^{\infty}\psi_g^*\psi_1\,dx = \int_{-\infty}^{\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x e^{-m\omega x^2/2\hbar}\,dx$

$= \sqrt{\frac{2}{\pi}}\frac{m\omega}{\hbar}\int_{-\infty}^{\infty}\underbrace{x e^{-m\omega x^2/2\hbar}}_{odd.}\,dx = 0 \Rightarrow \psi_g$ & ψ_1 are orthogonal.

$\int_{-\infty}^{\infty}\psi_1^*\psi_2\,dx = \int_{-\infty}^{\infty}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}\underbrace{x e^{-m\omega x^2/2\hbar}}_{odd}\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}(x^2-1)\right)e^{-m\omega x^2/2\hbar}\,dx$

$= \frac{m\omega}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} x\left(\frac{2m\omega}{\hbar}(x^2-1)\right)e^{-m\omega x^2/\hbar}\,dx = 0$

$\Rightarrow \psi_1$ & ψ_2 are orthogonal

$\int_{-\infty}^{\infty}\psi_g^*\psi_2\,dx = \int_{-\infty}^{\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}(x^2-1)\right)e^{-m\omega x^2/2\hbar}\,dx$

$= 2\sqrt{\frac{m\omega}{2\pi\hbar}}\int_{0}^{\infty}\left(\frac{2m\omega x^2-1}{\hbar}\right)e^{-m\omega x^2/\hbar}\,dx \quad , \xi = \sqrt{\frac{m\omega}{\hbar}}x$

$= 2\sqrt{\frac{m\omega}{2\pi\hbar}}\int_0^{\infty}(2\xi^2-1)e^{-\xi^2}\left(\sqrt{\frac{\hbar}{m\omega}}\,d\xi\right) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}(2\xi^2-1)e^{-\xi^2}d\xi$

$= \sqrt{\frac{2}{\pi}}\left(2\int_0^{\infty}\xi^2 e^{-\xi^2}d\xi - \int_0^{\infty}e^{-\xi^2}d\xi\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}\right) = 0$

$\Rightarrow \psi_g$ & ψ_2 are orthogonal.

2.13. $\Psi(x,0) = A[3\psi_0(x) + 4\psi_1(x)]$

a) $1 = A^2\int_{-\infty}^{\infty}[9\psi_0(x)\psi_0^*(x) + 12\psi_0(x)\psi_1^*(x) + 12\psi_1(x)\psi_0^*(x) + 16\psi_1(x)\psi_1^*(x)]\,dx$

$= A^2\Big[9\int_{-\infty}^{\infty}\underbrace{\psi_0(x)\psi_0^*(x)}_{1}dx + 12\cancel{\int_{-\infty}^{\infty}\psi_0(x)\psi_1^*(x)dx}^{0} + 12\cancel{\int_{-\infty}^{\infty}\psi_1(x)\psi_0^*(x)dx}^{0} + 16\int_{-\infty}^{\infty}\underbrace{\psi_1(x)\psi_1^*(x)}_{1}dx$

$= 9A^2 + 16A^2 = 25A^2 \Rightarrow A = \frac{1}{5}$

b) $\Rightarrow \Psi(x,0) = \underset{c_0\psi_0+}{\frac{3}{5}\psi_0(x)} + \frac{4}{5}\psi_1(x) = \frac{3}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar} + \frac{4}{5}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x e^{-\frac{m\omega x^2}{2\hbar}}$

$\Rightarrow \Psi(x,0) = c_1\psi_1 + c_2\psi_2 + c_3\psi_3 + \ldots + c_n\psi_n \Rightarrow c_0 = 3/5,\ c_1 = 4/5,\ c_n = 0,\ n \geq 2$

$\Rightarrow \Psi(x,t) = c_0\psi_0(x)\phi_0(t) + c_1\psi_1(x)\phi_1(t) + \ldots = c_0\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}e^{-i\omega t/2}$

$+ c_1\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x\,e^{-m\omega x^2/2\hbar}e^{-3i\omega t/2}$

$$\to \Psi(x,t) = \frac{3}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} e^{-i\omega t/2} + \frac{4}{5}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x\, e^{-\frac{m\omega x^2}{2\hbar}} e^{-3i\omega t/2}$$

$$= \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(3e^{-i\omega t/2} + 4\sqrt{\frac{2m\omega}{\hbar}}\, x e^{-3i\omega t/2}\right) e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\to |\Psi(x,t)|^2 = \Psi(x,t)\Psi^*(x,t)$$

$$= \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(3e^{-i\omega t/2} + 4\sqrt{\frac{2m\omega}{\hbar}}\, x e^{-3i\omega t/2}\right) e^{-\frac{m\omega x^2}{2\hbar}} \cdot$$

$$\frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(3e^{i\omega t/2} + 4\sqrt{\frac{2m\omega}{\hbar}}\, x e^{3i\omega t/2}\right) e^{-\frac{m\omega x^2}{2\hbar}}$$

$$= \frac{1}{25}\sqrt{\frac{m\omega}{\pi\hbar}}\left[9 + 12\left(\underbrace{\sqrt{\frac{2m\omega}{\hbar}}\, x e^{i\omega t} + \sqrt{\frac{2m\omega}{\hbar}}\, x e^{-i\omega t}}_{2\cos\omega t}\right) + 16\,\frac{2m\omega x^2}{\hbar}\right] e^{-\frac{m\omega x^2}{\hbar}}$$

$$\to |\Psi(x,t)|^2 = \frac{1}{25}\sqrt{\frac{m\omega}{\pi\hbar}}\left(9 + \frac{32m\omega}{\hbar}x^2 + 24\sqrt{\frac{2m\omega}{\hbar}}\, x\cos\omega t\right) e^{-\frac{m\omega x^2}{\hbar}}$$

c) $$\langle x\rangle = \frac{1}{25}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\left(9 + \frac{32m\omega}{\hbar}x^2 + 24\sqrt{\frac{2m\omega}{\hbar}}\, x\cos\omega t\right) e^{-\frac{m\omega x^2}{\hbar}}\, dx$$

$$= \frac{1}{25}\sqrt{\frac{m\omega}{\pi\hbar}}\left[\cancel{9\int_{-\infty}^{\infty} x e^{-\frac{m\omega x^2}{\hbar}}dx}^{\,0} + \cancel{\frac{32m\omega}{\hbar}\int_{-\infty}^{\infty} x^3 e^{-\frac{m\omega x^2}{\hbar}}dx}^{\,0}\right.$$

$$\left. + 24\sqrt{\frac{2m\omega}{\hbar}}\cos\omega t\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega x^2}{\hbar}}dx\right]$$

$$= \frac{24}{25}\sqrt{\frac{2}{\pi}}\cos\omega t\int_{-\infty}^{\infty}\frac{m\omega x^2}{\hbar} e^{-\frac{m\omega x^2}{\hbar}}dx \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}\,x \Rightarrow d\xi = \sqrt{\frac{m\omega}{\hbar}}\,dx$$

$$= \frac{24}{25}\sqrt{\frac{2}{\pi}}\cos\omega t\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2}\left(\sqrt{\frac{\hbar}{m\omega}}\,d\xi\right)$$

$$= \frac{48}{25}\sqrt{\frac{2\hbar}{\pi m\omega}}\cos\omega t\int_0^{\infty}\xi^2 e^{-\xi^2}d\xi = \frac{48}{25}\sqrt{\frac{2\hbar}{\pi m\omega}}\cos\omega t\,\sqrt{\pi}\left(\frac{2!}{1!}\right)\left(\frac{1}{2}\right)^3$$

$$= \frac{12}{25}\sqrt{\frac{2\hbar}{m\omega}}\cos\omega t$$

$$\to \langle p\rangle = m\langle v\rangle = m\frac{d\langle x\rangle}{dt} = m\frac{d}{dt}\left(\frac{12}{25}\sqrt{\frac{2\hbar}{m\omega}}\cos\omega t\right) = -\frac{12}{25}\sqrt{2\hbar m\omega}\,\sin\omega t$$

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x,t)\,dx = \frac{-12}{25}\sqrt{2\hbar m\omega}\,\sin\omega t$$

(sorry, I don't want to do this integral again :'D)

$$\frac{d\langle p\rangle}{dt} = \frac{d}{dt}\left(\frac{-12}{25}\sqrt{2\hbar m\omega}\,\sin\omega t\right) = -\frac{12}{25}\sqrt{2\hbar m\omega^2}\,\cos\omega t = -m\omega^2\langle x\rangle$$

$$= \left\langle -\frac{\partial}{\partial x}\left(\frac{1}{2}m\omega^2 x^2\right)\right\rangle$$

$$= \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

$\Rightarrow$ Confirmed the Ehrenfest's theorem.

d) $\Psi(x,t) = \frac{3}{5}\psi_0(x)e^{-iE_0t/\hbar} + \frac{4}{5}\psi_1(x)e^{-iE_1t/\hbar}$ $\quad E_0 = \frac{\hbar\omega}{2},\ E_1 = \frac{3\hbar\omega}{2}$

$$\Rightarrow \begin{cases} P(E_0) = \left|\frac{3}{5}\right|^2 = \frac{9}{25} \\ P(E_1) = \left|\frac{4}{5}\right|^2 = \frac{16}{25} \end{cases}$$

2.14. Classically allowed region: $-\sqrt{2E/m\omega^2} \rightarrow \sqrt{2E/m\omega^2}$

$\Rightarrow$ Probability of the particle not in the region:

$$1-\int_{-a}^{a}\Psi_0(x,t)\Psi_0^*(x,t)\,dx = 1-\int_{-a}^{a}[\psi_0(x)]^2\,dx = 1-\int_{-a}^{a}\sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-\frac{m\omega x^2}{\hbar}}\,dx$$

$$= 1-\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-a}^{a} e^{-\frac{m\omega x^2}{\hbar}}\,dx \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}\,x \Rightarrow d\xi = \sqrt{\frac{m\omega}{\hbar}}\,dx$$

$$\sqrt{\frac{m\omega}{\pi\hbar}}$$

$$\Rightarrow 1-\int_{-\xi/x\,a}^{\xi/x\,a} e^{-\xi^2}\left(\sqrt{\frac{\hbar}{m\omega}}\,d\xi\right) = 1-\frac{1}{\sqrt{\pi}}\int_{-\xi/x\,a}^{\xi a/x} e^{-\xi^2}\,d\xi$$

$$= 1-\underbrace{\frac{2}{\sqrt{\pi}}\int_0^{\xi a/x} e^{-\xi^2}\,d\xi}_{\text{error function of }\left(\sqrt{\frac{m\omega}{\hbar}}\,a\right) = e_f} \qquad \frac{\hbar\omega}{2} = \frac{1}{2}m\omega^2a^2$$

$$\Rightarrow 1-\int_{-a}^{a}\Psi_0(x,t)\Psi_0^*(x,t)\,dx = 1-e_f\left(\frac{m\omega}{\hbar}a\right) \approx 0.157$$

2.15. $a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)}\,a_j$ (Equation 2.85)

$h(\xi) = \sum_{j=0}^{\infty} a_j\,\xi^j \leftarrow$ coefficients to the power series

$J=0 \Rightarrow a_{J+2}=0$
$\Rightarrow a_2, a_4 \dots = 0$, set $a_1 = 0$

* $n=0 \Rightarrow a_{J+2} = \dfrac{+2J}{(J+1)(J+2)} a_J \Rightarrow a_0 = 2^0 = 1 \Rightarrow h(\xi) = 1$

$J=1 \Rightarrow a_{J+2} = 0$

* $n=1 \Rightarrow a_{J+2} = \dfrac{-2(1-J)}{(J+1)(J+2)} a_J \Rightarrow a_3, a_5 \dots = 0$, Set $a_0 = 0$

$\Rightarrow a_1 = 2^1 = 2 \Rightarrow h(\xi) = 2\xi$

$J=2 \Rightarrow a_{J+2} = 0$

* $n=2 \Rightarrow a_{J+2} = \dfrac{-2(2-J)}{(J+1)(J+2)} a_J \Rightarrow a_4, a_6 \dots = 0$, Set $a_1 = 0$ $\begin{cases} a_0 = -2 \\ a_2 = 4 \end{cases}$

$\Rightarrow a_2 = \dfrac{-2.2}{1.2} a_0 = 2^2 \Rightarrow$

$\Rightarrow h(\xi) = 4\xi^2 - 2$

$J=3 \Rightarrow a_{J+2} = 0$

* $n=3 \Rightarrow a_{J+2} = \dfrac{-2(3-J)}{(J+1)(J+2)} a_J \Rightarrow a_5, a_7 \dots = 0$, Set $a_0 = 0$ $\begin{cases} a_1 = -12 \\ a_3 = 8 \end{cases}$

$\Rightarrow a_3 = \dfrac{-2 \cdot 3}{2 \cdot 3} a_1 = 2^3 \Rightarrow$

$\Rightarrow h(\xi) = 8\xi^3 - 12\xi$

$J=4 \Rightarrow a_{J+2} = 0$

* $n=4 \Rightarrow a_{J+2} = \dfrac{-2(4-J)}{(J+1)(J+2)} a_J \Rightarrow a_6, a_8 \dots = 0$, Set $a_1 = 0$

$\Rightarrow a_4 = \dfrac{-2.2}{3.4} a_2 = \dfrac{-2.2}{3.4} \dfrac{-2.4}{1.2} a_0 = 2^4 \Rightarrow \begin{cases} a_0 = 12 \\ a_2 = -48 \\ a_4 = 16 \end{cases}$

$\Rightarrow h(\xi) = 16\xi^4 - 48\xi^2 + 12$

$J=5 \Rightarrow a_{J+2} = 0$

* $n=5 \Rightarrow a_{J+2} = \dfrac{-2(5-J)}{(J+1)(J+2)} a_J \Rightarrow a_7, a_9 \dots = 0$, Set $a_0 = 0$

$\Rightarrow a_5 = \dfrac{-2.2}{4.5} a_3 = \dfrac{-2.2}{4.5} \dfrac{-2.4}{2.3} a_1 = 2^5 \Rightarrow \begin{cases} a_1 = 60 \\ a_3 = -160 \\ a_5 = 32 \end{cases}$

$\Rightarrow h(\xi) = 32\xi^5 - 160\xi^3 + 60\xi$

$J=6 \Rightarrow a_{J+2} = 0$

* $n=6 \Rightarrow a_{J+2} = \dfrac{-2(6-J)}{(J+1)(J+2)} a_J \Rightarrow a_8, a_{10} \dots = 0$, Set $a_1 = 0$

$\Rightarrow a_6 = \dfrac{-2.2}{5.6} a_4 = \dfrac{-2.2}{5.6} \dfrac{-2.4}{3.4} a_2$

$= \dfrac{-2.2}{5.6} \cdot \dfrac{-2.4}{3.4} \cdot \dfrac{-2.6}{1.2} a_0 = 2^6 = a_6$

$\Rightarrow \begin{cases} a_0 = -120 \\ a_2 = 720 \\ a_4 = -480 \end{cases} \Rightarrow h(\xi) = 64\xi^6 - 480\xi^4 + 720\xi^2 - 120$