HOMEWORK 2

1.8. Schrödinger Equation: $\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Psi(x,t)$ for potential energy $V(x,t)$

Now, if we add V_0 $\Rightarrow \frac{\partial \Psi'}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi'}{\partial x^2} - \frac{i}{\hbar}[V(x,t)+V_0]\Psi'(x,t)$ for $\Psi'(x,t)$ is the wave function for $V(x,t)+V_0$.

$$\Rightarrow \frac{\partial \Psi'}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi'}{\partial x^2} - \frac{i}{\hbar}V(x,t) - \frac{i}{\hbar}V_0\Psi'(x,t)$$

$$\Rightarrow \frac{\partial \Psi'}{\partial t} + \frac{i}{\hbar}V_0\Psi'(x,t) = \frac{i\hbar}{2m}\frac{\partial^2 \Psi'}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Psi'(x,t)$$

Multiplying both sides with $e^{iV_0t/\hbar}$ $\Rightarrow e^{iV_0t/\hbar}\frac{\partial \Psi'}{\partial t} + \frac{i}{\hbar}V_0 e^{iV_0t/\hbar}\Psi'(x,t) = \frac{i\hbar}{2m}e^{iV_0t/\hbar}\frac{\partial^2\Psi'}{\partial x^2} - \frac{i}{\hbar}e^{iV_0t/\hbar}V(x,t)\Psi'(x,t)$

$$\Rightarrow \frac{\partial}{\partial t}(e^{iV_0t/\hbar}\Psi') = \frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}(e^{iV_0t/\hbar}\Psi') - \frac{i}{\hbar}V(x,t)(e^{iV_0t/\hbar}\Psi')$$

$$\Longrightarrow \Psi(x,t) = e^{iV_0t/\hbar}\Psi'(x,t) \Rightarrow \Psi'(x,t) = \Psi(x,t)e^{-iV_0t/\hbar} \quad \square$$

1.9. $\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]}$

a) $\int_{-\infty}^{\infty}|\Psi(x,t)|^2dx = 1 \Leftrightarrow \int_{-\infty}^{\infty} Ae^{-a[(mx^2/\hbar+it)]} \cdot Ae^{-a[(mx^2/\hbar-it)]}\,dx = 1$

$$\Leftrightarrow A^2\int_{-\infty}^{\infty}e^{-2amx^2/\hbar}dx = 1 \Leftrightarrow 2A^2\int_0^{\infty}e^{-x^2/[\sqrt{\hbar/(2am)}]^2}dx = 1 \Leftrightarrow 2A^2\sqrt{\pi}\left(\frac{\sqrt{\frac{\hbar}{2am}}}{2}\right) = 1$$

$$\Leftrightarrow A^2\sqrt{\frac{\hbar\pi}{2am}} = 1 \Rightarrow A = \sqrt[4]{\frac{2am}{\hbar\pi}}$$

b) $V(x,t) = \frac{\hbar}{i}\frac{1}{\Psi(x,t)}\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial \Psi}{\partial t}\right) = \frac{\hbar}{i}\sqrt[4]{\frac{\pi\hbar}{2am}}e^{a[(mx^2/\hbar+it)]}\left[\frac{i\hbar}{2m}\sqrt[4]{\frac{2am}{\pi\hbar}}(e^{-a[\dots]}\right.$

$$\left. - \sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[\dots]}(-ia)\right]$$

$$\Rightarrow V(x,t) = \hbar e^{a[(mx^2/\hbar)+it]} \left[\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} (e^{-a[\dots]} + a e^{-a[\dots]} \right]$$

$$= \hbar e^{a[\dots]} \left\{ \frac{\hbar}{2m} \left[-\frac{2am}{\hbar} \left(1 - \frac{2am}{\hbar} x^2\right) e^{-a[\dots]} \right] + a e^{-a[\dots]} \right\}$$

$$= \hbar a \frac{2am}{\hbar} x^2 - \hbar a + \hbar a = 2ma^2 x^2$$

c) $\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x,t)|^2 dx = \sqrt{\frac{2am}{\pi\hbar}} \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar} dx = 0$ (odd function over symmetrical interval)

$$\langle x^2 \rangle = \sqrt{\frac{2am}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 e^{-2amx^2/\hbar} dx = 2\sqrt{\frac{2am}{\pi\hbar}} \int_0^{\infty} x^2 e^{-x^2/[\sqrt{\hbar/(2am)}]^2} dx$$

$$= 2\sqrt{\frac{2am}{\pi\hbar}} . 2\sqrt{\pi} \left(\frac{\sqrt{\frac{\hbar}{2am}}}{2} \right)^3 = \frac{\hbar}{4am}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t) \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi(x,t) dx = -i\hbar \int_{-\infty}^{\infty} \left[\sqrt[4]{\frac{am}{\pi\hbar}} e^{-a[\dots-it]} \right] \frac{\partial}{\partial x} \left[\sqrt[4]{\frac{am}{\pi\hbar}} e^{-a[\dots+it]} \right] dx$$

$= i\sqrt{\frac{2am}{\pi\hbar}}\, 2am \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar} dx = 0$ (odd function over symmetrical interval).

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\, i^2\hbar^2 \frac{\partial^2}{\partial x^2} \Psi(x,t) dx = -\hbar^2 \int_{-\infty}^{\infty} \left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[\dots-it]} \right] \frac{\partial^2}{\partial x^2} \left[\sqrt[4]{\frac{2am}{\pi\hbar}} e^{-a[\dots+it]} \right] dx$$

$$= \hbar^2 \sqrt{\frac{2am}{\pi\hbar}} \frac{2am}{\hbar} \int_{-\infty}^{\infty} \left(1 - \frac{2am}{\hbar} x^2\right) e^{-2amx^2/\hbar} dx$$

$$= \hbar^2 \sqrt{\frac{2am}{\pi\hbar}} \frac{4am}{\hbar} \int_0^{\infty} e^{-x^2/[\sqrt{\hbar/(2am)}]^2} dx - \frac{2am}{\hbar} \int_0^{\infty} x^2 e^{-x^2/[\sqrt{\hbar/(2am)}]^2} dx$$

$$= \hbar^2 \frac{4am}{\hbar} \left(\frac{1}{2} - \frac{1}{4} \right) = \hbar am$$

d) $\sigma_x = \sqrt{\langle x^2 \rangle} = \frac{1}{2}\sqrt{\frac{\hbar}{am}}$, $\sigma_p = \sqrt{\langle p^2 \rangle} = \sqrt{\hbar am}$

$\Rightarrow \sigma_x \sigma_p = \frac{1}{2}\sqrt{\frac{\hbar}{am}} \sqrt{\hbar am} = \frac{\hbar}{2} \geq \frac{\hbar}{2}$ (consistent with Heisenberg's uncertainty principle.)

1.16. $\Psi(x,0) = \begin{cases} A(a^2-x^2) & -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$

a) $\int_{-a}^{a} |\Psi(x,0)|^2 dx = 1 \Leftrightarrow A^2 \int_{-a}^{a} (a^4 - 2a^2x^2 + x^4) dx = 1 \Leftrightarrow 2A^2 \left(a^4x - \frac{2}{3}a^2x^3 + \frac{x^5}{5}\right)\Big|_0^a = 1$

$\Leftrightarrow 2A^2\, a^5 - \frac{2}{3}a^5 + \frac{a^5}{5} = 1 \Leftrightarrow \frac{16a^5}{15} A^2 = 1 \Rightarrow A = \frac{1}{4}\sqrt{\frac{15}{a^5}}$

b, d) $\langle x \rangle = \frac{15}{16a^5} \int_{-a}^{a} x(a^2-x^2)^2 dx = 0$ (odd)

$\langle x^2 \rangle = \frac{15}{16a^5} \int_{-a}^{a} x^2(x^2-x^2)^2 dx = \frac{15}{8a^5}\left(a^4 \frac{a^3}{3} - 2a^2 \frac{a^5}{5} + \frac{a^7}{7}\right)\Big|_0^a = \frac{a^2}{7}$

c, e) $\langle p \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t) \, -i\hbar \frac{\partial}{\partial x} \Psi(x,t) dx$

$= -i\hbar \int_{-a}^{a} \left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(a^2-x^2) \frac{\partial}{\partial x} \frac{1}{4}\sqrt{\frac{15}{a^5}}(a^2-x^2)\right] dx$

$= \frac{15i\hbar}{8a^5} \int_{-a}^{a} x(a^2-x^2) dx = 0$ (odd)

$\langle p^2 \rangle = \frac{15\hbar^2}{8a^5} \int_{-a}^{a} (a^2-x^2) dx = \frac{15\hbar^2}{4a^5} \int_{0}^{a} (a^2-x^2) dx = \frac{15\hbar^2}{4a^5}\left(a^3 - \frac{a^3}{3}\right) = \frac{5\hbar^2}{2a^2}$

f, g) $\sigma_x = \sqrt{\langle x^2 \rangle} = \frac{a}{\sqrt{7}}$ $\quad \sigma_p = \sqrt{\langle p^2 \rangle} = \frac{\hbar}{a}\sqrt{\frac{5}{2}}$

h) $\sigma_x \sigma_p = \frac{a}{\sqrt{7}} \frac{\hbar}{a} \sqrt{\frac{5}{2}} = \sqrt{\frac{5}{14}}\, \hbar \approx 0.6\hbar \geq \frac{\hbar}{2}$ (consistent with Heisenberg's uncertainty principle)