HOMEWORK 11

4.32.

a) $S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0\end{bmatrix} \Rightarrow$ Evals: $\begin{vmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{\hbar^2}{4} = 0 \Rightarrow \lambda_- = -\frac{\hbar}{2}$ & $\lambda_+ = \frac{\hbar}{2}$

$\Rightarrow$ Evects: $(S_y - \lambda_- I)\chi_- = 0$ $\qquad$ $(S_y - \lambda_+ I)\chi_+ = 0$

$\begin{bmatrix} \hbar/2 & -i\hbar/2 \\ i\hbar/2 & \hbar/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} -\hbar/2 & -i\hbar/2 \\ i\hbar/2 & -\hbar/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} \frac{\hbar}{2}x_1 - \frac{i\hbar}{2}x_2 = 0 \\ \frac{i\hbar}{2}x_1 + \frac{\hbar}{2}x_2 = 0 \end{array}\right\} \Rightarrow x_2 = -ix_1 \Rightarrow \chi_- = x_1\begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\qquad$ $\Rightarrow \left.\begin{array}{l} -\frac{\hbar}{2}x_1 - \frac{i\hbar}{2}x_2 = 0 \\ \frac{i\hbar}{2}x_1 - \frac{\hbar}{2}x_2 = 0 \end{array}\right\} \Rightarrow x_2 = ix_1 \Rightarrow \chi_+ = x_1\begin{bmatrix} 1 \\ i \end{bmatrix}$

$\Rightarrow |x_1|^2 + |ix_1|^2 = 1$ $\qquad$ $\Rightarrow |x_1|^2 + |ix_1|^2 = 1$

$\Rightarrow x_1^2 + x_1^2 = 1 \Rightarrow 2x_1^2 = 1 \Rightarrow x_1 = \frac{1}{\sqrt{2}}$ $\qquad$ $\Rightarrow x_1^2 + x_1^2 = 1 \Rightarrow 2x_1^2 = 1 \Rightarrow x_1 = \frac{1}{\sqrt{2}}$

$\Rightarrow$ Evals & normalized espinors: $\begin{cases} \lambda_- = -\frac{\hbar}{2} \\ \chi_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \end{cases}$ and $\begin{cases} \lambda_+ = \frac{\hbar}{2} \\ \chi_+ = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \end{cases}$

b) If you measure S_y on a particle with spin $\frac{1}{2}$: $\chi = \begin{bmatrix} a \\ b \end{bmatrix}$ where $a^2+b^2=1 \Rightarrow$ Results: $-\frac{\hbar}{2}$ & $\frac{\hbar}{2}$

Prob of measuring $-\frac{\hbar}{2}$: $P(-\frac{\hbar}{2}) = |\langle \chi_- | \chi \rangle|^2 = |\chi_-^{\dagger}\chi|^2 = \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}^{\dagger}\begin{bmatrix} a \\ b \end{bmatrix}\right|^2$

$= \frac{1}{2}\left|\begin{bmatrix} 1 & i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}\right|^2 = \frac{1}{2}|a+ib|^2 = \frac{1}{2}(a+ib)(a+ib)^* = \frac{1}{2}(a+ib)(a^* - ib^*)$

$= \frac{1}{2}[aa^* + bb^* - i(ab^* - ba^*)] = \frac{1}{2}(|a|^2+|b|^2) + \frac{ab^* - (ab^*)^*}{2i} = \frac{1}{2}(|a|^2+|b|^2) + Im(ab^*)$

Prob of measuring $\frac{\hbar}{2}$: $P(\frac{\hbar}{2}) = \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}^{\dagger}\begin{bmatrix} a \\ b \end{bmatrix}\right|^2 = \frac{1}{2}\left|\begin{bmatrix} 1 & -i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}\right|^2 = \frac{1}{2}|a-ib|^2$

$= \frac{1}{2}(a-ib)(a^*+ib^*) = \frac{1}{2}(|a|^2+|b|^2) - \frac{ab^* - (ab^*)^*}{2i} = \frac{1}{2}(|a|^2+|b|^2) - Im(ab^*)$

$\Rightarrow P(-\frac{\hbar}{2}) + P(+\frac{\hbar}{2}) = [\frac{1}{2}(|a|^2+|b|^2) + Im(ab^*)] + [\frac{1}{2}(|a|^2+|b|^2) - Im(ab^*)]$

$= |a|^2+|b|^2 = 1$

c) $S_y^2 = S_yS_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{bmatrix}$

$\Rightarrow \begin{vmatrix} \frac{\hbar^2}{4} - \lambda & 0 \\ 0 & \frac{\hbar^2}{4} - 4 \end{vmatrix} = 0 \Rightarrow (\frac{\hbar^2}{4} - \lambda)^2 = 0 \Rightarrow \lambda = \frac{\hbar^2}{4} \Rightarrow P(\frac{\hbar^2}{4}) = 1$

4.36. a) $H = -\gamma\vec{B}\cdot\vec{S} = -\gamma(B_0\cos\omega t\,\hat{z})\cdot\vec{S} = -\gamma B_0\cos\omega t\,\hat{S}_z = -\gamma B_0\cos\omega t\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} -\gamma B_0\hbar/2\cos\omega t & 0 \\ 0 & \gamma B_0\hbar/2\cos\omega t \end{bmatrix}$

b) $i\hbar\frac{\partial\chi}{\partial t} = H\chi$, $t > 0$ + $\chi(0) = \chi_+ = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\Rightarrow i\hbar\frac{\partial\chi}{\partial t} = \frac{1}{i\hbar}\begin{bmatrix} -\frac{\gamma B_0\hbar}{2}\cos\omega t & 0 \\ 0 & \frac{\gamma B_0\hbar}{2}\cos\omega t \end{bmatrix}\chi \Rightarrow \begin{bmatrix} \chi_1'(t) \\ \chi_2'(t) \end{bmatrix} = \begin{bmatrix} \frac{i\gamma B_0}{2}\cos\omega t\,\chi_1(t) \\ -\frac{i\gamma B_0}{2}\cos\omega t\,\chi_2(t) \end{bmatrix}$

$\Rightarrow \begin{cases} \chi_1'(t) = \frac{i\gamma B_0}{2}\cos\omega t\,\chi_1(t) \\ \chi_2'(t) = -\frac{i\gamma B_0}{2}\cos\omega t\,\chi_2(t) \end{cases} \Rightarrow \begin{cases} \frac{\chi_1'}{\chi_1} = \frac{i\gamma B_0}{2}\cos\omega t \\ \frac{\chi_2'}{\chi_2} = -\frac{i\gamma B_0}{2}\cos\omega t \end{cases}$

$\Rightarrow \begin{cases} d/dt\, \ln \chi_1 = \frac{i\gamma B_0}{2}\cos\omega t \\ d/dt\, \ln\chi_2 = -\frac{i\gamma B_0}{2}\cos\omega t \end{cases} \Rightarrow \begin{cases} \ln\chi_1 = \frac{i\gamma B_0}{2}\left(\frac{\sin\omega t}{\omega}\right) + C_1 \\ \ln\chi_2 = -\frac{i\gamma B_0}{2}\left(\frac{\sin\omega t}{\omega}\right) + C_2 \end{cases} \Rightarrow \begin{cases} \chi_1 = A_1 e^{\frac{i\gamma B_0}{2\omega}\sin\omega t} \\ \chi_2 = A_2 e^{-\frac{i\gamma B_0}{2\omega}\sin\omega t} \end{cases}$

$\Rightarrow \chi(0) = \begin{bmatrix} A_1 e^0 \\ A_2 e^0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow \begin{cases} A_1 = 1/\sqrt{2} \\ A_2 = 1/\sqrt{2} \end{cases}$

$\Rightarrow \chi(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} e^{\frac{i\gamma B_0}{2\omega}\sin\omega t} \\ \frac{1}{\sqrt{3}} e^{-\frac{i\gamma B_0}{2\omega}\sin\omega t} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\cdots} \\ e^{-i\cdots} \end{bmatrix}$

c) $P\left(-\frac{\hbar}{2}\right) = |\langle\chi_- | \chi\rangle|^2 = |\chi_-^{+}\chi|^2 = \left| \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\cdots} \\ e^{-i\cdots} \end{bmatrix} \right|^2$

$= \frac{1}{4}[e^{i\cdots} - e^{-i\cdots}][e^{i\cdots} - e^{i\cdots}]^* = \frac{1}{4}[e^{i\cdots} - e^{-i\cdots}][e^{-i\cdots} - e^{i\cdots}]$

$= \frac{1}{4}[1 - e^{i\cdots} - e^{-i\cdots} + 1] = \frac{1}{2}\left[1 - \frac{e^{i\cdots} + e^{-i\cdots}}{2}\right] = \frac{1}{2}\left[1 - \cos\left(\frac{\gamma B_0}{\omega}\sin\omega t\right)\right]$

$= \frac{1}{2}\left\{1 - \cos\left[2\left(\frac{\gamma B_0}{\omega}\sin\omega t\right)\right]\right\} = \frac{1}{2}\left[2\sin^2\left(\frac{\gamma B_0}{2\omega}\sin\omega t\right)\right] = \sin^2\left(\frac{\gamma B_0}{2\omega}\sin\omega t\right)$

d) $P\left(-\frac{\hbar}{2}\right) = 1 \Rightarrow \sin^2\left(\frac{\gamma B_0}{2\omega}\sin\omega t\right) = 1 \Rightarrow \frac{\gamma B_0}{2\omega}\sin\omega t = \frac{\pi}{2} \Rightarrow B_0 = \frac{\pi\omega}{\gamma\sin\omega t}$

$\Rightarrow B_0 = \frac{\pi\omega}{\gamma}\ (\omega t = 1)$

4.54. a) $P = \iiint_{\text{nucleus}} |\Psi_{100}|^2 dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{b} |\Psi_{100}(r,\theta,\phi)e^{-iE_nt/\hbar}|^2 (r^2\sin\theta\, dr\, d\phi\, d\theta)$

$= \int_0^{\pi}\int_0^{2\pi}\int_0^{b}\left[\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} e^{-iE_nt/\hbar}\right]^* \left[\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} e^{-iE_nt/\hbar}\right](r^2\sin\theta\, dr\, d\phi\, d\theta)$

$= \frac{1}{\pi a_0^3} 2\cdot 2\pi \int_0^b r^2 e^{-2r/a_0} dr = \frac{4}{a_0^3}\int_0^b \frac{\partial^2}{\partial k^2}(a_0^2 e^{kr/a_0})\Big|_{k=-2} dr = \frac{4}{a_0}\frac{d^2}{dk^2}\left(\int_0^b e^{kr/a_0}dr\right)\Big|_{k=-2}$

$= \frac{4}{a_0}\frac{d^2}{dk^2}\left[\left(\frac{a_0}{k}e^{kr/a_0}\right)\Big|_0^b\right]\Big|_{k=-2} = 4\frac{d^2}{dk^2}\left(\frac{e^{kb/a_0} - 1}{k}\right)\Big|_{k=-2} = 4\frac{d}{dk}\left[\frac{e^{kb/a_0}(kb - a_0) + a_0}{a_0 k^2}\right]\Big|_{k=-2}$

$= 4\left[\frac{e^{-2b/a_0}(4b^2 + 4a_0 b + 2a_0^2) - 2a_0^2}{-8a_0^2}\right] = 1 - \left(\frac{2b^2 + 2a_0 b + a_0^2}{a_0^2}\right)e^{-2b/a_0}$

b) $P = 1 - \left(\frac{1}{2}\epsilon^2 + \epsilon + 1\right)e^{-\epsilon}\ (\epsilon = 2b/a_0) \Rightarrow$ Taylor expand: $P = 1 - \left(\frac{1}{2}\epsilon^2 + \epsilon + 1\right)\left(1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{6} + \cdots\right)$

$P = 1 - \left[1\left(\frac{1}{2}\epsilon^2 + \epsilon + 1\right) - \epsilon\left(\frac{1}{2}\epsilon^2 + \epsilon + 1\right) + \frac{\epsilon^2}{2}\left(\frac{1}{2}\epsilon^2 + \epsilon + 1\right) - \frac{\epsilon^3}{6}\left(\frac{1}{2}\epsilon^2 + \epsilon + 1\right) + \cdots\right]$

$\approx \frac{\epsilon^3}{6} \approx \frac{1}{6}\left(\frac{2b}{a_0}\right)^3 \approx \frac{4}{3}\left(\frac{b}{a_0}\right)^3$

c) $P = \iiint |\Psi_{100}|^2 dV \approx \int_0^{\pi}\int_0^{2\pi}\int_0^{b}\left[\frac{1}{\sqrt{\pi a_0^3}} e^0 e^{-iE_nt/\hbar}\right]^*\left[\frac{1}{\sqrt{\pi a_0^3}} e^0 e^{-iE_nt/\hbar}\right](r^2\sin\theta\, dr\, d\phi\, d\theta)$

$\approx \frac{1}{\pi a_0^3} 2\pi\cdot 2\pi \frac{b^3}{3} = \frac{4}{3}\left(\frac{b}{a_0}\right)^3$

d) $b \approx 10^{-15}$ & $a_0 \approx 0.5 \cdot 10^{-10} \Rightarrow P = 1 - \left(\frac{2b^2 + 2a_0 b + a_0^2}{a_0^2}\right) e^{-2b/a_0} \approx 1.0769 \cdot 10^{-14}$

$\Rightarrow P \approx 1.067 \cdot 10^{-14}$

$\Rightarrow$ Percent diffeence $= \frac{\text{Exact - Approx}}{\text{Approx}} \cdot 100 = \frac{1.0769 \cdot 10^{-14} - 1.067 \cdot 10^{-14}}{1.067 \cdot 10^{-14}} \approx 0.96\%$

4.57. a) $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle\right)^2 = \left(\frac{1}{2i} \langle [x^2, L_z] \rangle\right)^2 = \left(\frac{1}{2i} \langle -x[L_z, x] - [L_z, x]x \rangle\right)^2$

$= \left[\frac{1}{2i} \langle -x(i\hbar y) - (i\hbar y)x \rangle\right]^2 = \left(\frac{1}{2i} \langle -2i\hbar xy \rangle\right)^2 = \left[\frac{1}{2i}(-2i\hbar)\langle xy \rangle\right]^2 = (\hbar \langle xy \rangle)^2$

$\Rightarrow \sigma_A \sigma_B \geq \hbar |\langle xy \rangle|$

b) $\sigma_B = \sqrt{\langle B^2 \rangle - \langle B \rangle} = \sqrt{\langle L_z^2 \rangle - \langle L_z \rangle^2} = (\langle \psi_{nlm} | L_z^2 | \psi_{nlm} \rangle - \langle \psi_{nlm} | L_z | \psi_{nlm} \rangle^2)^{1/2}$

$= \left[\iiint_{\text{all space}} \psi^*_{nlm} L_z^2 \psi_{nlm} dV - \left(\iiint_{\text{all space}} \psi^* L_z \psi_{nlm} dV\right)^2\right]^{1/2}$

$= \left[\hbar m_l \iiint \psi^*_{nml} (\hbar m_l \psi_{nml}) dV - \hbar^2 m_l^2 \left(\iiint \psi^*_{nml} \psi_{nml} dV\right)^2\right]^{1/2}$

$= \left[\hbar^2 m_l^2 \left(\iiint \psi^*_{nml} \psi_{nml} dV\right) - \hbar^2 m_l^2 1^2\right]^{1/2} = \sqrt{\hbar^2 m_l^2 \cdot 1 - \hbar^2 m_l^2} = 0$

c) $\Rightarrow \sigma_A (0) \geq \hbar |\langle xy \rangle| \Rightarrow 0 \geq |\langle xy \rangle| \Rightarrow \langle xy \rangle = 0$

4.58. a) $\chi = \begin{bmatrix} A(1-2i) \\ 2A \end{bmatrix} \Rightarrow [A(1-2i)]^2 + [2A]^2 = 1 \Rightarrow A^2|1-2i|^2 + 4A^2 = 1$

$\Rightarrow A^2(1-2i)(1-2i)^* + 4A^2 = 1 \Rightarrow A^2(1-2i)(1+2i) + 4A^2 = 1 \Rightarrow 9A^2 = 1 \Rightarrow A = \pm\frac{1}{3}$

$\Rightarrow$ Spin state: $\chi = \frac{1}{3} \begin{bmatrix} 1-2i \\ 2 \end{bmatrix}$

b) $S_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix} \Rightarrow (\lambda - \frac{\hbar}{2})(\lambda + \frac{\hbar}{2}) = 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}$

$\Rightarrow (S_z - \lambda_- I)\chi_- = 0$

$\begin{bmatrix} 2\hbar/2 & 0 \\ 0 & -\hbar/2 + \hbar/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} \hbar & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \chi_- = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow (S_z - \lambda_+ I)\chi_+ = 0$

$\begin{bmatrix} 0 & 0 \\ 0 & -\hbar \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \chi_+ = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

$\Rightarrow \chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$P(-\frac{\hbar}{2}) = \left|\begin{bmatrix} 0 \\ 1 \end{bmatrix}^+ \frac{1}{3} \begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$

$= \frac{1}{9} \left|[0 \ \ 1] \begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$

$= \frac{1}{9} |0(1-2i) + 1(2)|^2 = \frac{4}{9}$

$P(+\frac{\hbar}{2}) = \left|\begin{bmatrix} 1 \\ 0 \end{bmatrix}^+ \frac{1}{3} \begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$

$= \frac{1}{9} \left|[1 \ \ 0] \begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$

$= \frac{1}{9} |1^2 + (-2)^2| = \frac{5}{9}$

$\Rightarrow \langle S_z \rangle = P(\lambda_-)(\lambda_-) + P(\lambda_+)(\lambda_+) = \frac{4}{9}(-\frac{\hbar}{2}) + \frac{5}{9}(\frac{\hbar}{2}) = \frac{\hbar}{18}$

c) $S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix} \Rightarrow \det(S_x - \lambda I) = \lambda^2 - \frac{\hbar^2}{4} = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2}$

$\Rightarrow (S_x - \lambda_- I)\chi_- = 0$ $\qquad (S_x - \lambda_+ I)\chi_+ = 0$

$\begin{bmatrix} \hbar/2 & \hbar/2 \\ \hbar/2 & \hbar/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad \begin{bmatrix} -\hbar/2 & \hbar/2 \\ \hbar/2 & -\hbar/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{aligned} \frac{\hbar}{2}x_1 + \frac{\hbar}{2}x_2 = 0 \\ \frac{\hbar}{2}x_1 + \frac{\hbar}{2}x_2 = 0 \end{aligned}\right\} \Rightarrow x_2 = -x_1 \Rightarrow \chi_- = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ $\qquad \left.\begin{aligned} -\frac{\hbar}{2}x_1 + \frac{\hbar}{2}x_2 = 0 \\ \frac{\hbar}{2}x_1 - \frac{\hbar}{2}x_2 = 0 \end{aligned}\right\} \Rightarrow x_2 = x_1 \Rightarrow \chi_+ = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$

$\Rightarrow \chi_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad \chi_+ = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P(-\frac{\hbar}{2}) = \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}^{+}\frac{1}{3}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$ $\qquad P(\frac{\hbar}{2}) = \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^{+}\frac{1}{3}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$

$= \frac{1}{18}\left|\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2 = \frac{1}{18}|1(1-2i)+(-1)2|^2$ $\qquad = \frac{1}{18}\left|\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$

$= \frac{1}{18}|-1-2i|^2 = \frac{1}{18}(1^2+2^2) = \frac{5}{18}$ $\qquad = \frac{1}{18}|3-2i|^2 = \frac{1}{18}(3^2+2^2) = \frac{13}{18}$

$\Rightarrow \langle S_x \rangle = P(\lambda_-)(\lambda_-) + P(\lambda_+)(\lambda_+) = \frac{5}{18}\left(-\frac{\hbar}{2}\right) + \frac{13}{18}\left(\frac{\hbar}{2}\right) = \frac{2\hbar}{9}$

d) $S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{bmatrix} \Rightarrow \det(S_y - \lambda I) = \lambda^2 - \frac{\hbar^2}{4} = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2}$

$\Rightarrow (S_y - \lambda_- I)\chi_- = 0$ $\qquad (S_y - \lambda_+ I)\chi_+ = 0$

$\begin{bmatrix} \hbar/2 & -i\hbar/2 \\ i\hbar/2 & \hbar/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad \begin{bmatrix} -\hbar/2 & -i\hbar/2 \\ i\hbar/2 & -\hbar/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{aligned} \hbar/2\, x_1 - i\hbar/2\, x_2 = 0 \\ i\hbar/2\, x_1 + \hbar/2\, x_2 = 0 \end{aligned}\right\} \Rightarrow x_2 = -ix_1 \Rightarrow \chi_- = \begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix}$ $\qquad \left.\begin{aligned} -\hbar/2\, x_1 - i\hbar/2\, x_2 = 0 \\ i\hbar/2\, x_1 - \hbar/2\, x_2 = 0 \end{aligned}\right\} \Rightarrow x_2 = ix_1 \Rightarrow \chi_+ = \begin{bmatrix} x_1 \\ ix_1 \end{bmatrix}$

$\Rightarrow \chi_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\qquad \chi_+ = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

$P(-\frac{\hbar}{2}) = \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}^{+}\frac{1}{3}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$ $\qquad P(\frac{\hbar}{2}) = \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}^{+}\frac{1}{3}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$

$= \frac{1}{18}\left|\begin{bmatrix} 1 & i \end{bmatrix}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$ $\qquad = \frac{1}{18}\left|\begin{bmatrix} 1 & -i \end{bmatrix}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2$

$= \frac{1}{18}|1(1-2i)+i(2)|^2 = \frac{1}{18}$ $\qquad = \frac{1}{18}|1(1-2i)+(-i)(2)|^2 = \frac{1}{18}|1-4i|^2 = \frac{17}{18}$

$\Rightarrow \langle S_y \rangle = P(\lambda_-)(\lambda_-) + P(\lambda_+)(\lambda_+) = \frac{1}{18}\left(-\frac{\hbar}{2}\right) + \frac{17}{18}\left(\frac{\hbar}{2}\right) = \frac{4\hbar}{9}$