HOMEWORK 12

4.35. a) $P\left(\frac{\pi}{2}\right) = |c_+^{(x)}|^2 = |\langle \chi_+^{(x)} | \chi \rangle|^2 = |\chi_+^{(x)\dagger} \chi|^2$ where $\chi_+^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow P\left(\frac{\pi}{2}\right) = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\dagger \begin{pmatrix} \cos(\alpha/2)\, e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)\, e^{-i\gamma B_0 t/2} \end{pmatrix} \right|^2 = \frac{1}{2} \left| \cos(\alpha/2) e^{i\cdots} + \sin(\alpha/2) e^{-i\cdots} \right|^2$

$= \frac{1}{2} \left[\cos(\alpha/2) e^{i\cdots} + \sin(\alpha/2) e^{-i\cdots} \right] \left[\cos(\alpha/2) e^{-i\cdots} + \sin(\alpha/2) e^{i\cdots} \right]$

$= \frac{1}{2} \left[1 + \sin(\alpha/2)\cos(\alpha/2)\left(e^{i\cdots} + e^{-i\cdots}\right) \right] = \frac{1}{2}\left(1 + \frac{1}{2} \sin\alpha \cdot 2\cos\gamma B_0 t\right)$

$= \frac{1}{2}\left(1 + \sin\alpha \cos\gamma B_0 t\right)$

b) $P\left(\frac{\pi}{2}\right) = |C_+^{(y)}|^2 = |\chi_+^{(y)\dagger} \chi|^2$ where $\chi_+^{(y)} = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\Rightarrow P\left(\frac{\pi}{2}\right) = \frac{1}{2} \left| (1 \;\; -i) \begin{pmatrix} \cos(\alpha/2) e^{i\cdots} \\ \sin(\alpha/2) e^{-i\cdots} \end{pmatrix} \right|^2 = \frac{1}{2} \left| \cos(\alpha/2) e^{i\cdots} - i \sin(\alpha/2) e^{-i\cdots} \right|^2$

$= \frac{1}{2} \left[1 + i \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) \left(e^{i\cdots} - e^{-i\cdots}\right) \right] = \frac{1}{2} \left[1 + i\left(\frac{1}{2}\sin\alpha\right)(2i \sin\gamma B_0 t) \right]$

$= \frac{1}{2}\left(1 - \sin\alpha \sin\gamma B_0 t\right)$

c) $P(\frac{\hbar}{2}) = |\chi_+^\dagger \chi|^2$ where $\chi_+ = \binom{1}{0} \to P(\frac{\hbar}{2}) = \left|(1\ 0)\begin{pmatrix}\cos(\alpha/2)e^{i\ldots}\\ \sin(\alpha/2)e^{-i\ldots}\end{pmatrix}\right|^2 = \cos^2(\frac{\alpha}{2})$

4.87. a) $S_-|1 0\rangle = \sqrt{2}\frac{\hbar}{\sqrt{2}}|1,-1\rangle$ (equation 4.175)

If we start with the state $|1 0\rangle$ where $n=1, m=0$ and applying the angular momentum lowering operator $S_- \Rightarrow S_-|1 0\rangle = \sqrt{2}\frac{\hbar}{\sqrt{2}}|1,-1\rangle$ = equation 4.175

Or $S_-|1 0\rangle = \frac{\hbar}{\sqrt{2}}\sqrt{(1+0)\cdot(1-0+1)}|1, 0-1\rangle = \frac{\hbar}{\sqrt{2}}\sqrt{2}|1,-1\rangle$

b) $S_+|0 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ (equation 4.176)

Start with $n=0, m=0$ state then applying angular momentum raising S_+:

$\to S_+|0 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \Leftarrow S_+|0 0\rangle = \frac{\hbar}{\sqrt{2}}\sqrt{(0-0)(0+0+1)}|0, 0+1\rangle$

c) Expression for S^2: $S^2|s,m\rangle = s(s+1)\frac{\hbar^2}{(2\pi)^2}|s,m\rangle \Leftarrow S^2|1\ 1\rangle = 1(1+1)\frac{\hbar^2}{(2\pi)^2}|1,1\rangle$

$\to S^2|1\ 1\rangle = \frac{2\hbar^2}{(2\pi)^2}|1,1\rangle \Rightarrow S^2|1\ -1\rangle = \frac{2\hbar^2}{(2\pi^2)}|1,-1\rangle$

$\Rightarrow$ The state $|1,\pm1\rangle$ are estate of S^2.

4.58. Resultant spin of S_1 & S_2: $S = |S_1+S_2| \to |S_1-S_2|$ with interval 1

Resultant spin of 2 quarks: $S = |1/2+1/2| \to |1/2-1/2| = 1, 0$

$\Rightarrow$ " 3 quarks: $S = |1+1/2| \to |1-1/2| = 3/2, 1/2$ (2 quarks + 1 quark)

$\Rightarrow$ " " of $S_1 = 0$ & $S_2 = 1/2$: $S = |0+1/2| \to |0-1/2| = 1/2$

a) Possible spins for Baryons (3 quarks): $S = 3/2$, $S = 1/2$

b) Possible spins for Mesons (2 quarks): $S = 1$, $S = 0$

4.64. $R_{21}(\sqrt{1/3}Y_1^0\chi_+ + \sqrt{2/3}Y_1^1\chi_-)$ Spherical harmonic with $l=1$

a) $L^2 = l(l+1)\hbar^2 = 2\hbar^2$ $(l=1) \to P=1$ c) Both are spins 1/2 so $S^2 = \frac{3\hbar^2}{4} \Rightarrow P=1$

b) $L_z = 0 \Rightarrow P = 1/3$ & $L_z = \hbar \Rightarrow P = 2/3$ d) $S_z = \frac{1}{2} \Rightarrow P = \frac{1}{3}$ & $S_z = -\frac{1}{2} \Rightarrow P = \frac{2}{3}$

e) $\vec{J} = \vec{L} + \vec{S} \to \frac{1}{\sqrt{3}}|\frac{1}{2}\ \frac{1}{2}\rangle|1\ 0\rangle + \sqrt{\frac{2}{3}}|\frac{1}{2}\ \frac{-1}{2}\rangle|1\ 1\rangle$

$= \frac{1}{\sqrt{3}}\left[\sqrt{\frac{2}{3}}|\frac{3}{2}\ \frac{1}{2}\rangle - \frac{1}{\sqrt{3}}|\frac{1}{2}\ \frac{1}{2}\rangle\right] + \sqrt{\frac{2}{3}}\left[\frac{1}{\sqrt{3}}|\frac{3}{2}\ \frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|\frac{1}{2}\ \frac{1}{2}\rangle\right]$

$= (\frac{2\sqrt{2}}{3})|\frac{3}{2}\ \frac{1}{2}\rangle + (\frac{1}{3})|\frac{1}{2}\ \frac{1}{2}\rangle$. So $S = \frac{3}{2}$ or $\frac{1}{2} \Rightarrow \frac{15}{4}\hbar^2, P = \frac{8}{9}$ or $\frac{3}{4}\hbar^2, P = \frac{1}{9}$.

f) $\frac{1}{2}\hbar$, $P = 1$.

g) $|\Psi|^2 = |R_{21}|^2 \Big\{ \frac{1}{3} |Y_1^0|^2 \underbrace{(\chi_+^\dagger \chi_+)}_{1} + \frac{\sqrt{2}}{3} \Big[Y_1^{0*} Y_1^1 \underbrace{(\chi_+^\dagger \chi_-)}_{0} + Y_1^{1*} Y_1^0 \underbrace{(\chi_-^\dagger \chi_+)}_{0} \Big]$

$+ \frac{2}{3} |Y_1^1|^2 \underbrace{(\chi_-^\dagger \chi_-)}_{1} \Big\} = \frac{1}{3} |R_{21}|^2 (|Y_1^0|^2 + 2|Y_1^1|^2) = \frac{1}{3} \cdot \frac{1}{24} \cdot \frac{1}{a^3} \cdot \frac{r^2}{a^2} e^{-r/a}$

$\cdot \Big[\frac{3}{4\pi} \cos^2\theta + 2 \frac{3}{8\pi} \sin^2\theta \Big] = \frac{1}{96\pi a^5} r^2 e^{-r/a}$

h) $\frac{1}{3} |R_{21}|^2 \int |Y_1^0|^2 \sin^2\theta \, d\theta \, d\phi = \frac{1}{3} |R_{21}|^2 = \frac{1}{3} \cdot \frac{1}{24 a^3} r^2 e^{-r/a} = \frac{1}{72 a^5} r^2 e^{-r/a}$

4.66. $\chi = \binom{a}{b}$ for $|a|^2 + |b|^2 = 1$

$\langle S_z \rangle = \frac{\hbar}{2} (|a|^2 - |b|^2)$, $\langle S_x \rangle = \hbar \,\mathrm{Re}(ab^*)$, $\langle S_y \rangle = -\hbar \,\mathrm{Im}(ab^*)$

$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \frac{\hbar^2}{4}$

For $a = |a| e^{i\phi_a}$, $b = |b| e^{i\phi_b} \to ab^* = |a||b| e^{i(\phi_a - \phi_b)} = |a||b| e^{i\theta}$ $(\theta = \phi_a - \phi_b)$

$\to \langle S_x \rangle = \hbar \,\mathrm{Re}(|a||b| e^{i\theta}) = \hbar |a||b| \cos\theta$

$\langle S_y \rangle = -\hbar \,\mathrm{Im}(|a||b| e^{i\theta}) = -\hbar |a||b| \sin\theta$

$\Rightarrow \sigma_{S_x}^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - \hbar^2 |a|^2 |b|^2 \cos^2\theta$

$\sigma_{S_y}^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - \hbar^2 |a|^2 |b|^2 \sin^2\theta$

$\sigma_{S_x}^2 \sigma_{S_y}^2 = \frac{\hbar^2}{4} \langle S_z \rangle^2$

$\Rightarrow \frac{\hbar^2}{4} (1 - 4|a|^2|b|^2 \cos^2\theta) \frac{\hbar^2}{4} (1 - 4|a|^2|b|^2 \sin^2\theta) = \left(\frac{\hbar^2}{4}\right)^2 (|a|^2 - |b|^2)^2$

$\Leftrightarrow 1 - 4|a|^2|b|^2 (\cos^2\theta + \sin^2\theta) + 16 |a|^4 |b|^4 \sin^2\theta \cos^2\theta = |a|^4 - 2|a|^2|b|^2 + |b|^4$

$\Leftrightarrow 1 + 16 |a|^4 |b|^4 \sin^2\theta \cos^2\theta = |a|^4 + 2|a|^2|b|^2 + |b|^4 = (|a|^2 + |b|^2)^2 = 1$

$\Rightarrow |a|^2 |b|^2 \sin\theta \cos\theta = 0.$

$\Rightarrow \theta = 0$ or $\theta = \pi$, then a & b are real.

$\Rightarrow \theta = \pm \pi/2$, then a & b are imaginary.