HOMEWORK 7

3.3. If $\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$ & $h = f+g$ then:

$$\langle h|\hat{Q}h\rangle = \langle h|\hat{Q}|h\rangle = \int_a^b h^*(x)\hat{Q}h(x)dx = \int_a^b [f(x)+g(x)]^*\{\hat{Q}[f(x)+g(x)]\}dx$$
$$= \int_a^b [f^*(x)\hat{Q}f(x) + f^*(x)\hat{Q}g(x) + g^*(x)\hat{Q}f(x) + g^*(x)\hat{Q}g(x)]dx$$
$$= \langle f|\hat{Q}|f\rangle + \langle f|\hat{Q}|g\rangle + \langle g|\hat{Q}|f\rangle + \langle g|\hat{Q}|g\rangle$$
$$= \langle f|\hat{Q}f\rangle + \langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle$$

$$\langle \hat{Q}h|h\rangle = \langle h|\hat{Q}^\dagger|h\rangle = \int_a^b h^*(x)\hat{Q}^\dagger h(x)dx = \int_a^b h^*(x)[\hat{Q}^\dagger h(x)]dx$$
$$= \int_a^b [f^*(x)+g^*(x)][\hat{Q}^\dagger f(x) + \hat{Q}^\dagger g(x)]dx$$
$$= \int_a^b [f^*(x)\hat{Q}^\dagger f(x) + f^*(x)\hat{Q}^\dagger g(x) + g^*(x)\hat{Q}^\dagger f(x) + g^*(x)\hat{Q}^\dagger g(x)]dx$$
$$= \langle \hat{Q}f|f\rangle + \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle + \langle \hat{Q}g|g\rangle$$

Since $\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$ and $\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$ & $\langle g|\hat{Q}g\rangle = \langle \hat{Q}g|g\rangle$

$$\Rightarrow \langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle + \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle \quad (1)$$

If $h = f+ig$ then: $\langle h|\hat{Q}h\rangle = \langle h|\hat{Q}|h\rangle = \int_a^b h^*(x)\hat{Q}h(x)dx = \int_a^b h^*(x)[\hat{Q}h(x)]dx$

$$= \int_a^b [f(x)+ig(x)]^*\{\hat{Q}[f(x)+ig(x)]\}dx$$
$$= \langle f|\hat{Q}f\rangle + i\langle f|\hat{Q}g\rangle - i\langle g|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle$$

$$\langle \hat{Q}h|h\rangle = \langle h|\hat{Q}^\dagger|h\rangle = \int_a^b h^*(x)\hat{Q}^\dagger h(x)dx = \int_a^b [f(x)+ig(x)]^*\{\hat{Q}^\dagger[f(x)+ig(x)]\}dx$$
$$= \langle \hat{Q}f|f\rangle + i\langle \hat{Q}f|g\rangle - i\langle \hat{Q}g|f\rangle + \langle \hat{Q}g|g\rangle$$

Since $\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$ and $\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$ and $\langle g|\hat{Q}g\rangle = \langle \hat{Q}g|g\rangle$

$$\Rightarrow \langle f|\hat{Q}g\rangle - \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle - \langle \hat{Q}g|f\rangle \quad (2)$$

$(1)+(2) \Rightarrow 2\langle f|\hat{Q}g\rangle = 2\langle \hat{Q}f|g\rangle \Leftrightarrow \langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$ ✓

3.10. Ground state of ∞ square well: $\psi_1(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\Rightarrow \hat{p}\psi_1(x) = -i\hbar\frac{\partial}{\partial x}\left(\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\right) = -i\hbar\left(\frac{\pi}{a}\right)\left(\sqrt{\frac{2}{a}}\cos\frac{\pi x}{a}\right) \neq p\psi_1(x)$$

$\Rightarrow$ ground state of ∞ square well is not an eigenfunction of momentum operator.