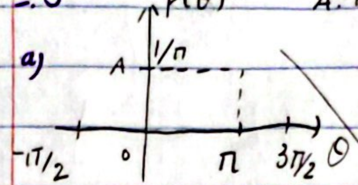
HOMEWORK 1

1.3 (wrong edition!!)

a) 

$A.\pi = 1 \Rightarrow A = \frac{1}{\pi} \Rightarrow p(\theta) = \frac{1}{\pi}$ for $0 \le \theta \le \pi$

b) $\langle\theta\rangle = \int_{-\infty}^{\infty} \theta p(\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi} \theta\, d\theta = \frac{1}{2\pi}\pi^2 = \frac{\pi}{2}$

$\langle\theta^2\rangle = \int_0^{\pi} \theta^2 p(\theta)\, d\theta = \frac{1}{3\pi}\pi^3 = \frac{\pi^2}{3}$

$\theta^2 = \langle\theta^2\rangle - \langle\theta\rangle^2 = \frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{\pi^2}{12} \Rightarrow \theta = \frac{\pi}{2\sqrt{3}}$

c) $\langle\sin\theta\rangle = \int_0^{\pi} \sin\theta\, p(\theta)\, d\theta = \frac{-1}{\pi}\cos\theta\Big|_0^{\pi} = \frac{-1}{\pi}(-2) = \frac{2}{\pi}$

$\langle\cos\theta\rangle = \int_0^{\pi} \cos\theta\, p(\theta)\, d\theta = \frac{1}{\pi}\sin\theta\Big|_0^{\pi} = 0$

$\cos(2\theta) = 2\cos^2\theta - 1$

$\langle\cos^2\theta\rangle = \int_0^{\pi} \cos^2\theta\, p(\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi} \boxed{\frac{\cos(2\theta)+1}{2}}\, d\theta = \frac{1}{2\pi}\left(\frac{1}{2}\sin 2\theta + \theta\right)\Big|_0^{\pi}$

$= \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$

1.3 (3^{rd} edition this time) $p(x) = Ae^{-\lambda(x-a)^2}$ $u = x-a \Rightarrow du = dx$

a) $\int_{-\infty}^{\infty} p(x)\,dx = 1 \Leftrightarrow \int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}dx = 1 \Leftrightarrow A\int_{-\infty}^{\infty} e^{-\lambda u^2}du = 1 \Leftrightarrow 2A\int_0^{\infty} e^{-u^2/(1/\sqrt{\lambda})^2}du = 1$

Equation from book: $\int_0^{\infty} x^{2n} e^{-x^2/a^2}dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$, using this:

$\Rightarrow 2A\sqrt{\pi}\frac{\sqrt{1/\lambda}}{2} = 1 \Rightarrow A = \sqrt{\frac{\lambda}{\pi}}$

b) $\langle x\rangle = \int_{-\infty}^{\infty} x p(x)\,dx = \int_{-\infty}^{\infty} x\sqrt{\frac{\lambda}{\pi}}e^{-\lambda(x-a)^2}dx$ (I don't wanna do this ˄'ö)

$u = x-a \Rightarrow du = dx$

$\Rightarrow \langle x\rangle = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty}(u+a)e^{-\lambda u^2}du = \sqrt{\frac{\lambda}{\pi}}\overbrace{\int_{-\infty}^{\infty} ue^{-\lambda u^2}du}^{0\text{ (odd function)}} + a\int_{-\infty}^{\infty} e^{-\lambda u^2}du$

$\Rightarrow \langle x\rangle = \sqrt{\frac{\lambda}{\pi}}\, 2a\int_0^{\infty} e^{-u^2/(1/\sqrt{\lambda})^2}du$ (same with part a $\Rightarrow 2a\sqrt{\frac{\lambda}{\pi}}\sqrt{\lambda}\frac{1}{2\sqrt{\lambda}} = a$

$u = x-a$

$\langle x^2\rangle = \sqrt{\frac{\lambda}{\pi}}\int^{\infty} x^2e^{-\lambda(x-a)^2}dx = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty}(u+a)^2e^{-\lambda u^2}du = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} u^2e^{-\lambda u^2}du + 2a$

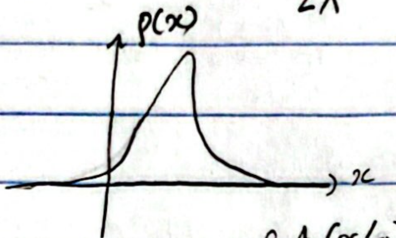
$\underbrace{\int_{-\infty}^{\infty} ue^{-\lambda u^2}du}_{0} + a^2\int_{-\infty}^{\infty} e^{-\lambda u^2}du$

$\Rightarrow \langle x^2\rangle = \sqrt{\frac{\lambda}{\pi}}\left(2\int_0^\infty u^2 e^{-u^2/(1/\sqrt{\lambda})^2}du + 2a^2\int_0^\infty e^{-u^2/(1/\sqrt{\lambda})^2}du\right)$

Use same equation from book $\Rightarrow \sqrt{\frac{\lambda}{\pi}}\left(2\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{1/\sqrt{\lambda}}{2}\right)^3 + 2a^2\sqrt{\lambda}\left(\frac{1/\sqrt{\lambda}}{2}\right)\right)$

$\Rightarrow \langle x^2\rangle = \sqrt{\frac{\lambda}{\pi}}\left(\frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}} + a^2\sqrt{\frac{\pi}{\lambda}}\right) = \frac{1}{2\lambda} + a^2$

$\Rightarrow \sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{2\lambda} + a^2 - a^2 = \frac{1}{2\lambda} \Rightarrow \sigma = \frac{1}{\sqrt{2\lambda}}$

c) 

$\rho(x) = \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2}$

1.4. $t=0 \Rightarrow \Psi(x,0) = \begin{cases} A(x/a) & 0\le x\le a \\ A(b-x)/(b-a) & a\le x\le b \\ 0 & \text{otherwise} \end{cases}$

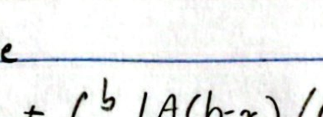
a) $\int_{-\infty}^{\infty}|\Psi(x,0)|^2dx = 1 \Leftrightarrow \int_0^a |A(x/a)|^2dx + \int_a^b |A(b-x)/(b-a)|^2dx = 1$

$\Leftrightarrow \frac{A^2}{a^2}\int_0^a x^2dx + \frac{A^2}{(b-a)^2}\int_a^b (b-x)^2dx = 1 \Leftrightarrow \frac{A^2}{a}\frac{a^3}{3} + \frac{A^2}{(b-a)^2}\left(b^2x - 2b\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_a^b$

$\Leftrightarrow \frac{A^2a^2}{3} + \frac{A^2}{(b-a)^2}\Big[\underbrace{b^2(b-a) - b(b^2-a^2) + \frac{b^3-a^3}{3}}_{=b^3-ab^2-b^3+a^2b+\frac{b^3-a^3}{3}}\Big] \Leftrightarrow \frac{A^2a^2}{3} + \frac{A^2}{(b-a)^2}\frac{(b-a)^3}{3} = 1$

$\Leftrightarrow \frac{A^2}{3}(a+b-a) = 1 \Rightarrow A = \sqrt{\frac{3}{b}}$

b) $\Psi(x,0) = \begin{cases} \sqrt{3/b}\,(x/a) & 0\le x\le a \\ \sqrt{3/b}\,(b-x)/(b-a) & a\le x\le b \\ 0 & \text{otherwise} \end{cases}$



c) At $t=0$, the particle is most likely be found at a.

d) Left of $a \Rightarrow P = \int_{-\infty}^a |\Psi(x,0)|^2dx = \int_0^a|\Psi(x,0)|^2dx = \int_0^a \frac{3}{b}\frac{x^2}{a}dx$

$\Rightarrow P = \frac{3}{a^2b}\int_0^a x^2dx = \frac{3}{a^2b}\frac{a^3}{3} = \frac{a}{b}$. $\Rightarrow a=b,\ P=1$

$\Rightarrow 2a=b,\ P=1/2$

e) $\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,0)|^2 dx = \int_0^a x \frac{3}{b}\frac{x^2}{a^2}\, dx + \int_a^b x \frac{3}{b}\left(\frac{b-x}{b-a}\right)^2 dx$

$\Rightarrow \langle x\rangle = \frac{3}{a^2 b}\int_0^a x^3 dx + \frac{3}{b(b-a)^2}\int_a^b x(b-x)^2 dx = \frac{3}{a^2 b}\frac{a^4}{4} + \frac{3}{b(b-a)^2}\cdot\frac{1}{12}(b-a)^3(3a+b)$

$\Rightarrow \langle x\rangle = \frac{3a^2 + (b-a)(3a+b)}{4b} = \frac{b^2+2ab}{4b} = \frac{b+2a}{4}$

1.5. $\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t} \Rightarrow \Psi^*(x,t) = Ae^{-\lambda|x|}e^{i\omega t}$

a) $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1 \Leftrightarrow \int_{-\infty}^{\infty}\Psi(x,t)\,\Psi^*(x,t)\,dx = 1 \Leftrightarrow A^2\int_{\infty}^{\infty} e^{-2\lambda|x|} dx = 1$

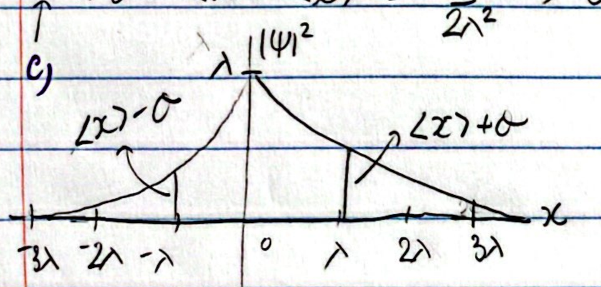
$\Leftrightarrow A^2\frac{1}{\lambda} = 1 \Rightarrow A = \sqrt{\lambda}$

b) $\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx = \lambda\int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx = 0$

$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2 dx = \lambda\int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx = 2\lambda\int_0^{\infty} x^2 e^{-2\lambda x} dx = 2\lambda\left[-\frac{(2\lambda)^2x^2+2\lambda x-}{4a^3}e^{-2\lambda x}\right]_0^{\infty} = 2\lambda\cdot\frac{1}{4\lambda^3} = \frac{1}{2\lambda^2}$

$\Rightarrow \sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{2\lambda^2} \Rightarrow \sigma = \frac{1}{\sqrt{2}\lambda}$

c)



$\Rightarrow$ Probability of finding the particle outside $(\langle x\rangle-\sigma, \langle x\rangle+\sigma)$:

$P = \int_{-\infty}^{\langle x\rangle-\sigma}|\Psi(x,t)|^2 dx + \int_{\langle x\rangle+\sigma}^{\infty}|\Psi(x,t)|^2 dx$

$= \int_{-\infty}^{-1/(\sqrt{2}\lambda)}\lambda e^{-2\lambda|x|} dx + \int_{1/(\sqrt{2}\lambda)}^{\infty}\lambda e^{-2\lambda|x|} dx$

$= \lambda\left(\frac{1}{2\lambda}e^{2\lambda x}\Big|_{-\infty}^{-1/\sqrt{2}\lambda} + \frac{1}{-2\lambda}e^{-2\lambda x}\Big|_{1/\sqrt{2}\lambda}^{\infty}\right) = \lambda\frac{1}{\lambda}e^{-\sqrt{2}} = e^{-\sqrt{2}}$