HOMEWORK 10.

4.23. $\frac{d}{dt}\langle \vec{L} \rangle = \langle N \rangle$ where $N = r \times (-\nabla V)$

a) $\frac{d}{dt}\langle \vec{L} \rangle = \frac{d}{dt}\langle \Psi | \vec{L} | \Psi \rangle = \frac{d}{dt}\iiint \Psi^* \vec{L} \Psi \, dV = \frac{d}{dt}\iiint \Psi^* \left(\sum_{J=1}^{3} \delta_J L_J \right) \Psi \, dV$

$$= \sum_{J=1}^{3} \delta_J \frac{d}{dt} \iiint \Psi^* L_J \Psi \, dV = \sum_{J=1}^{3} \delta_J \iiint \frac{\partial}{\partial t} (\Psi^* L_J \Psi) \, dV$$

$$= \sum_{J=1}^{3} \delta_J \iiint \left(\frac{\partial \Psi^*}{\partial t} L_J \Psi + \Psi^* \frac{\partial L_J}{\partial t} \Psi + \Psi^* L_J \frac{\partial \Psi}{\partial t} \right) dV$$

$$= \sum_{J=1}^{3} \delta_J \iiint \left\{ \left(\frac{1}{i\hbar} \hat{H} \Psi \right)^* L_J \Psi + \Psi^* \left[\frac{\partial}{\partial t} (r \times p)_J \right] \Psi + \Psi^* L_J \left(\frac{1}{i\hbar} \hat{H} \Psi \right) \right\} dV$$

$$= \sum_{J=1}^{3} \delta_J \iiint \left\{ \left(\frac{-1}{i\hbar}\psi^* \hat{H}^+\right) L_J \psi + \psi^* \left[\frac{\partial}{\partial t}\left(\cancelto{0}{\sum_{K=1}^{3} \sum_{l=1}^{3} \varepsilon_{JKl}\, x_K\, p_l}\right)\right]\psi + \psi^* L_J \left(\frac{1}{i\hbar}\hat{H}\psi\right) \right\} dV$$

$$= -\frac{1}{i\hbar}\sum_{J=1}^{3} \delta_J \iiint \left[\psi^* \hat{H}(L_J\psi) - \psi^* L_J(\hat{H}\psi)\right] dV \qquad \boxed{\vec{L} = \vec{r}\times\vec{p}}$$

$$= -\frac{1}{i\hbar}\sum_{J=1}^{3} \delta_J \iiint \psi^* \left\{\hat{H}\left[\vec{r}\times\vec{p}\right]_J\psi\right] - (\vec{r}\times\vec{p})_J(\hat{H}\psi)\right\} dV$$

$$= \frac{-1}{i\hbar}\sum_{J,K,l=1}^{3} \varepsilon_{jkl}\,\delta_J \iiint \psi^* \left[\hat{H}(x_k p_l \psi) - x_k p_l (\hat{H}\psi)\right] dV$$

$$= \text{"} \quad \iiint \psi^* \left[\hat{H}\left(x_k \frac{\partial\psi}{\partial x_l}\right) - x_k \frac{\partial}{\partial x_l}(\hat{H}\psi)\right] dV$$

$$= \text{"} \quad \iiint \psi^* \left[-\frac{\hbar^2}{2m}\nabla^2\left(x_k\frac{\partial\psi}{\partial x_l}\right) + \cancelto{0}{V x_k \frac{\partial\psi}{\partial x_l}} + \frac{\hbar^2}{2m} x_k \frac{\partial}{\partial x_l}(\nabla^2\psi) - \cancelto{0}{x_k\frac{\partial}{\partial x_l}(V\psi)}\right] dV$$

$$= \text{"} \quad \iiint \psi^* \left\{-\frac{\hbar^2}{2m}\sum_{n=1}^{3}\frac{\partial^2}{\partial x_n^2}\left(x_k\frac{\partial\psi}{\partial x_l}\right) + \frac{\hbar^2}{2m} x_k \frac{\partial}{\partial x_l}(\nabla^2\psi) - x_k\frac{\partial V}{\partial x_l}\psi\right\} dV$$

$$= \text{"} \quad \iiint \psi^* \left\{-\frac{\hbar^2}{2m}\left[\sum_{n=1}^{3}\frac{\partial}{\partial x_n}\left(\frac{\partial x_k}{\partial x_n}\frac{\partial\psi}{\partial x_l}\right) + \sum_{n=1}^{3}\frac{\partial}{\partial x_n}\left(x_k\frac{\partial^2\psi}{\partial x_n\partial x_l}\right)\right] + \frac{\hbar^2}{2m} x_k \frac{\partial}{\partial x_l}(\nabla^2\psi) - x_k\frac{\partial V}{\partial x_l}\psi\right\} dV$$

$$= \text{"} \quad \iiint \psi^* \left\{-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial x_k\partial x_l} + \frac{\partial^2\psi}{\partial x_k\partial x_l} + \cancelto{0}{x_k\frac{\partial}{\partial x_l}(\nabla^2\psi)}\right] + \cancelto{0}{\frac{\hbar^2}{2m} x_k\frac{\partial}{\partial x_l}(\nabla^2\psi)} - x_k\frac{\partial V}{\partial x_l}\psi\right\} dV$$

$$\nabla\times\nabla f = 0 \rightarrow \quad \text{"} \quad \iiint \psi^* \left(-\frac{\hbar^2}{m}\frac{\partial^2\psi}{\partial x_k\partial x_l} - x_k\frac{\partial V}{\partial x_l}\psi\right) dV$$

$$= \iiint \psi^* \left[-\frac{\hbar^2}{m}\sum_{J,K,l=1}^{3}\varepsilon_{jkl}\,\delta_J\frac{\partial}{\partial x_k}\left(\frac{\partial\psi}{\partial x_l}\right) - \left(\sum_{J,K,l=1}^{3}\varepsilon_{Jkl}\,\delta_J\, x_k\frac{\partial V}{\partial x_l}\right)\psi\right] dV$$

$$= \iiint \psi^* \left[\cancelto{0}{-\frac{\hbar^2}{m}(\nabla\times\nabla\psi)} - (r\times\nabla V)\psi\right) dV = \iiint \psi^* \left[r\times(-\nabla V)\right]\psi\, dV$$

$$= \langle\psi|r\times(-\nabla V)|\psi\rangle = \langle r\times(-\nabla V)\rangle = N = \frac{d}{dt}\langle\vec{L}\rangle$$

b) $V = V(r)$, $\frac{d}{dt}\langle \vec{L} \rangle = \langle \vec{N} \rangle = \langle \vec{r} \times (-\nabla V) \rangle = \langle (r\hat{r}) \times \left(-\frac{\partial V}{\partial r}\hat{r}\right) - \cancel{\frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}}^{\,0} - \cancel{\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}}^{\,0} \rangle$

$= \langle -r\frac{dV}{dr}\underbrace{(\hat{r}\times\hat{r})}_{0} \rangle = \langle -r\frac{dV}{dr}(0) \rangle = \langle 0 \rangle = 0$

4.26. $Y_2^1(\theta,\phi) = -\sqrt{15/8\pi}\,\sin\theta\cos\theta e^{i\phi}$

• $L_+ f_l^m = A_l^m f_l^{m+1} = \hbar\sqrt{l(l+1)-m(m+1)}\,f_l^{m+1} \Rightarrow L_+ Y_2^1(\theta,\phi) = \hbar\sqrt{2(3)-1(2)}\,Y_2^2(\theta,\phi) = 2\hbar Y_2^2$

$\Rightarrow Y_2^2(\theta,\phi) = \frac{1}{2\hbar}L_+Y_2^1(\theta,\phi) = \frac{1}{2\hbar}\left[\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\right]\left(-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}\right)$

$= \frac{-e^{i\phi}}{2}\sqrt{\frac{15}{8\pi}}\left[\frac{\partial}{\partial\theta}(\sin\theta\cos\theta e^{i\phi}) + i\cot\theta\frac{\partial}{\partial\phi}(\sin\theta\cos\theta e^{i\phi})\right]$

$= -\frac{e^{i\phi}}{2}\sqrt{\frac{15}{8\pi}}\left[e^{i\phi}\underbrace{\frac{d}{d\theta}(\sin\theta\cos\theta)}_{\cos^2\theta-\sin^2\theta} + i\left(\frac{\cos\theta}{\sin\theta}\right)\underbrace{\sin\theta\cos\theta\frac{d}{d\phi}(e^{i\phi})}_{\cos^2\theta\,(ie^{i\phi})}\right]$

$= -\frac{e^{i\phi}}{2}\sqrt{\frac{15}{8\pi}}\left(-e^{i\phi}\sin^2\theta\right) = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi}$

4.27. a) $\chi^\dagger\chi = |A^2|(9+16) = 25|A^2| = 1 \Rightarrow A = 1/5$

b) $\langle S_x \rangle = \chi^\dagger S_x \chi = \frac{1}{25}\frac{\hbar}{2}(-3i\ 4)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix} = \frac{\hbar}{50}(-3i\ 4)\begin{pmatrix}4\\3i\end{pmatrix} = \frac{\hbar}{50}(12i+12i) = 0$

$\langle S_y \rangle = \chi^\dagger S_y \chi = \frac{1}{25}\frac{\hbar}{2}(-3i\ 4)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix} = \frac{\hbar}{50}(-3i\ 4)\begin{pmatrix}-4i\\-3\end{pmatrix} = \frac{\hbar}{50}(-12-12) = \frac{-12}{25}\hbar$

$\langle S_z \rangle = \chi^\dagger S_z \chi = \frac{1}{25}\frac{\hbar}{2}(-3i\ 4)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix} = \frac{\hbar}{50}(-3i\ 4)\begin{pmatrix}3i\\-4\end{pmatrix} = \frac{\hbar}{50}(9-16) = \frac{-7}{50}\hbar$

c) $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4}$ (for spin 1/2) $\Rightarrow \sigma^2_{S_x} = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - 0 \Rightarrow \sigma_{S_x} = \frac{\hbar}{2}$

$\sigma^2_{S_y} = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - \left(\frac{12}{25}\right)^2\hbar^2 = \frac{\hbar^2}{2500}(625-576) = \frac{49}{2500}\hbar^2 \Rightarrow \sigma_{S_y} = \frac{7}{50}\hbar$

$\sigma^2_{S_z} = \langle S_z^2 \rangle - \langle S_z \rangle^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2\hbar^2 = \frac{\hbar^2}{2500}(625-49) = \frac{576}{2500}\hbar^2 \Rightarrow \sigma_{S_z} = \frac{12}{25}\hbar$

d) $\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\cdot\frac{7}{50}\hbar \geq \frac{\hbar}{2}|\langle S_z \rangle| = \frac{\hbar}{2}\cdot\frac{7}{50}\hbar$ (@ the uncertainty limit)

$\sigma_{S_y}\sigma_{S_z} = \frac{7}{50}\hbar\,\frac{12}{25}\hbar \geq \frac{\hbar}{2}|\langle S_z \rangle| = 0$ (trivial)

$\sigma_{S_z}\sigma_{S_x} = \frac{12}{25}\hbar\cdot\hbar \geq \frac{\hbar}{2}|\langle S_z \rangle| = \frac{\hbar}{2}\cdot\frac{12}{25}\hbar$ (@ the uncertainty limit)

4.29. a) Spin matrix eqn: $S_x = \frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $S_y = \frac{\hbar}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix}$ $S_z = \frac{\hbar}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

Commutation relation: $[S_x, S_y] = i\hbar S_x$ $[S_y, S_z] = i\hbar S_x$ $[S_z, S_x] = i\hbar S_y$

* $[S_x, S_y] = S_xS_y - S_yS_x = \frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix} - \frac{\hbar}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}$

$= \frac{\hbar^2}{4}\begin{bmatrix}i&0\\0&-i\end{bmatrix} - \frac{\hbar^2}{4}\begin{bmatrix}-i&0\\0&i\end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix}2i&0\\0&-2i\end{bmatrix} = i\hbar\left(\frac{\hbar}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right) = \underline{i\hbar S_x}$ ✓

* $[S_y, S_z] = S_yS_z - S_zS_y = \frac{\hbar}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix} - \frac{\hbar}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&-i\\i&0\end{bmatrix}$

$= \frac{\hbar^2}{4}\begin{bmatrix}0&i\\i&0\end{bmatrix} - \frac{\hbar^2}{4}\begin{bmatrix}0&-i\\-i&0\end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix}0&2i\\2i&0\end{bmatrix} = i\hbar\left(\frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}\right) = \underline{i\hbar S_x}$ ✓

* $[S_z, S_x] = S_zS_x - S_xS_z = \frac{\hbar}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix} - \frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

$= \frac{\hbar^2}{4}\begin{bmatrix}0&1\\-1&0\end{bmatrix} - \frac{\hbar^2}{4}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix}0&2\\-2&0\end{bmatrix} = i\hbar\left(\frac{\hbar}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix}\right) = \underline{i\hbar S_y}$ ✓

b) $\sigma_x = \begin{bmatrix}0&1\\1&0\end{bmatrix}$, $\sigma_y = \begin{bmatrix}0&-i\\i&0\end{bmatrix}$, $\sigma_z = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$

General formula for Pauli matrices: $\sigma_J = \begin{bmatrix}\delta_{3J} & \delta_{1J} - i\delta_{2J}\\ \delta_{1J} + i\delta_{2J} & -\delta_{3J}\end{bmatrix}$

$\Rightarrow \sigma_J\sigma_K = \begin{bmatrix}\delta_{3J} & \delta_{1J} - i\delta_{2J}\\ \delta_{1J}+i\delta_{2J} & -\delta_{3J}\end{bmatrix}\begin{bmatrix}\delta_{3K} & \delta_{1K} - i\delta_{2K}\\ \delta_{1K}+i\delta_{2K} & -\delta_{3K}\end{bmatrix}$

$= \delta_{JK}\begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix} i\sum_{l=1}^{3}\varepsilon_{12l}\varepsilon_{JKl} & -\sum_{l=1}^{3}\varepsilon_{13l}\varepsilon_{JKl} + i\sum_{l=1}^{3}\varepsilon_{23l}\varepsilon_{JKl} \\ \sum_{l=1}^{3}\varepsilon_{13l}\varepsilon_{JKl} + i\sum_{l=1}^{3}\varepsilon_{23l}\varepsilon_{JKl} & -i\sum_{l=1}^{3}\varepsilon_{12l}\varepsilon_{JKl}\end{bmatrix}$

$= \delta_{JK} + i\begin{bmatrix}\varepsilon_{JK3} & -i\varepsilon_{JK2} + \varepsilon_{JK1}\\ i\varepsilon_{JK2} + \varepsilon_{JK1} & -\varepsilon_{JK3}\end{bmatrix}$

$= \delta_{JK}I + i\left(\varepsilon_{JK1}\begin{bmatrix}0&1\\1&0\end{bmatrix} + \varepsilon_{JK2}\begin{bmatrix}0&-i\\i&0\end{bmatrix} + \varepsilon_{JK3}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right)$

$= \delta_{JK}I + i(\varepsilon_{JK1}\sigma_x + \varepsilon_{JK2}\sigma_y + \varepsilon_{JK3}\sigma_z) = \delta_{JK}I + i\sum_{l=1}^{3}\varepsilon_{JKl}\sigma_l$

4.30. $\chi = A\begin{pmatrix}3i\\4\end{pmatrix}$

a) $\chi = \begin{vmatrix}3iA\\4A\end{vmatrix} \Rightarrow |3iA|^2 + |4A|^2 = 1 = 9A^2 + 16A^2 = 1 \Rightarrow A = \pm\frac{1}{5} \Rightarrow \chi = \begin{bmatrix}3i/5\\4/5\end{bmatrix}$

b) $\langle S_x\rangle = \frac{\langle\chi|S_x|\chi\rangle}{\langle\chi|\chi\rangle} = \frac{\chi^+ S_x\chi}{\chi^+\chi} = \frac{\begin{bmatrix}3i/5\\4/5\end{bmatrix}^+\frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}3i/5\\4/5\end{bmatrix}}{\begin{bmatrix}3i/5\\4/5\end{bmatrix}^+\begin{bmatrix}3i/5\\4/5\end{bmatrix}}$

$= \frac{\hbar}{2}\frac{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}4/5\\3i/5\end{bmatrix}}{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}3i/5\\4/5\end{bmatrix}} = \frac{\hbar}{2}\frac{\cancel{-\frac{12i}{25} + \frac{12i}{25}}^{\,0}}{\frac{9}{25}+\frac{16}{25}} = 0$

$\langle S_x^2\rangle = \frac{\langle\chi|S_x^2|\chi\rangle}{\langle\chi|\chi\rangle} = \frac{\chi^+S_x^2\chi}{1} = \frac{\hbar^2}{4}\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}3i/5\\4/5\end{bmatrix}$

$= \frac{\hbar^2}{4}\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}3i/5\\4/5\end{bmatrix} = \frac{\hbar^2}{4}\left(\frac{9}{25}+\frac{16}{25}\right) = \frac{\hbar^2}{4}$

$\langle S_z\rangle = \frac{\langle\chi|S_z|\chi\rangle}{\langle\chi|\chi\rangle} = \frac{\hbar}{2}\frac{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}-3i/5\\-4/5\end{bmatrix}}{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}3i/5\\4/5\end{bmatrix}} = \frac{\hbar}{2}\frac{9/25 - 16/25}{9/25 + 16/25} = -\frac{7\hbar}{50}$

$$\langle S_z^2\rangle = \frac{\langle \chi|S_z^2|\chi\rangle}{\langle\chi|\chi\rangle} = \frac{\hbar^2}{4}\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\cancel{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}}\begin{bmatrix}3i/5\\ 4/5\end{bmatrix} = \frac{\hbar^2}{4}\left(\frac{9}{25}+\frac{16}{25}\right) = \frac{\hbar^2}{4}$$

$$\langle S_y\rangle = \frac{\langle\chi|S_y|\chi\rangle}{\langle\chi|\chi\rangle} = \frac{\hbar}{2}\frac{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}3i/5\\ 4/5\end{bmatrix}}{\begin{bmatrix}-3i/5 & 4/5\end{bmatrix}\begin{bmatrix}3i/5\\ 4/5\end{bmatrix}} = \frac{\hbar}{2}\frac{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}-4i/5\\ -3/5\end{bmatrix}}{\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}3i/5\\ 4/5\end{bmatrix}}$$

$$= \frac{\hbar}{2}\,\frac{-\frac{12}{25}-\frac{12}{25}}{\frac{9}{25}+\frac{16}{25}} = \frac{-12}{25}\hbar$$

$$\langle S_y^2\rangle = \frac{\langle\chi|S_y^2|\chi\rangle}{\langle\chi|\chi\rangle} = \frac{\hbar^2}{4}\begin{bmatrix}-\frac{3i}{5} & \frac{4}{5}\end{bmatrix}\cancel{\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}}\begin{bmatrix}3i/5\\ 4/5\end{bmatrix}$$

$$= \frac{\hbar^2}{4}\left(\frac{9}{25}+\frac{16}{25}\right) = \frac{\hbar^2}{4}$$

c) $\sigma_{S_x} = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \left(\frac{\hbar^2}{4} - 0^2\right)^{1/2} = \frac{\hbar}{2}$

$\sigma_{S_y} = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2} = \left(\frac{\hbar^2}{4} - \left(-\frac{12}{25}\hbar\right)^2\right)^{1/2} = \frac{7\hbar}{50}$

$\sigma_{S_z} = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \left(\frac{\hbar^2}{4} - \left(-\frac{7\hbar}{50}\right)\right)^{1/2} = \frac{12\hbar}{25}$

d) $\left(\frac{\hbar}{2}\right)\left(\frac{7\hbar}{50}\right) = \frac{7\hbar^2}{100} \geq \frac{7\hbar^2}{100} = \frac{\hbar}{2}\left|-\frac{7\hbar}{50}\right|$ $(\sigma_{S_x}\sigma_{S_y}) \geq \frac{\hbar}{2}|\langle S_z\rangle|$

$\left(\frac{7\hbar}{50}\right)\left(\frac{12\hbar}{25}\right) = \frac{42\hbar^2}{625} \geq 0 = \frac{\hbar}{2}|0|$ $(\sigma_{S_y}\sigma_{S_z} \geq \frac{\hbar}{2}|\langle S_x\rangle|$

$\left(\frac{12\hbar}{25}\right)\left(\frac{\hbar}{2}\right) = \frac{6\hbar^2}{25} \geq \frac{6\hbar^2}{25} = \frac{\hbar}{2}\left|-\frac{12\hbar}{25}\right|$ $(\sigma_{S_z}\sigma_{S_x} \geq \frac{\hbar}{2}|\langle S_y\rangle|$