Course Name

January, 2024

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# About this Course

## 0.1 Available course formats

This course is available in multiple formats which allows you to take it in the way that best suites your needs. You can take it for certificate which can be for free or fee.

* The material for this course can be viewed without login requirement on this [Bookdown website](LINK%20HERE). This format might be most appropriate for you if you rely on screen-reader technology.
* This course can be taken for [free certification through Leanpub](LINK%20HERE).
* This course can be taken on [Coursera for certification here](LINK%20HERE) (but it is not available for free on Coursera).
* Our courses are open source, you can find the [source material for this course on GitHub](LINK%20HERE).

## Installing package into '/usr/local/lib/R/site-library'  
## (as 'lib' is unspecified)  
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## Installing package into '/usr/local/lib/R/site-library'  
## (as 'lib' is unspecified)

## Loading required package: MASS

## Loading required package: HistData

## Loading required package: Hmisc

## Loading required package: lattice

## Loading required package: survival

## Loading required package: Formula

## Loading required package: ggplot2

##   
## Attaching package: 'Hmisc'

## The following objects are masked from 'package:base':  
##   
## format.pval, units

##   
## Attaching package: 'UsingR'

## The following object is masked from 'package:survival':  
##   
## cancer

## Warning: package 'reshape' was built under R version 4.0.3

# 1 Week 01

## 1.1 Introduction

### 1.1.1 Welcome to Regression Models

I am happy that you’ve chosen to take Regression Models, part of the Johns Hopkins Data Science Specialization on Coursera! This course presents the fundamentals of regression modeling that you will need for the rest of the specialization and ultimately for your work in the field of data science.

We believe that the key word in Data Science is “science”. Our course track is focused on providing you with three things: (1) an introduction to the key ideas behind working with data in a scientific way that will produce new and reproducible insight, (2) an introduction to the tools that will allow you to execute on a data analytic strategy, from raw data in a database to a completed report with interactive graphics, and (3) on giving you plenty of hands on practice so you can learn the techniques for yourself.

Regression Models represents a both fundamental and foundational component of the series, and it presents the single most practical data analysis toolset. Using only a bare minimum of mathematics, we will attempt to provide you with the fundamentals for the application and practice of regression.

We are excited about the opportunity to attempt to scale Data Science education. We intend for the courses to be self-contained, fast-paced, and interactive, and we intend to run them frequently to give people with busy schedules the opportunity to work on material at their own pace.

### 1.1.2 Some Basics

A couple of first week housekeeping items. First, make sure that you’ve had [R Programming](https://www.coursera.org/learn/r-programming) , the [Data Scientist’s Toolbox](https://www.coursera.org/learn/data-scientists-tools), [Reproducible Research](https://www.coursera.org/learn/regression-models/supplement/uCPA0/welcome-to-regression-models) and [Statistical Inference](https://www.coursera.org/learn/statistical-inference) before taking this class. At a minimum you must know: very basic git, basic R and most of the Statistical Inference Coursera class. The small amount of knitr that you need for the project you can pick up quickly.

An important aspect of this class is to peruse the materials in the github repository. All of the most up to date material can be found [here](https://github.com/bcaffo/courses/tree/master/07_RegressionModels). You should clone this repository as your first step in this class and make sure to fetch updates periodically. (Please issue pull requests so that we may improve the materials!) It is one of the most essential components of the Specialization that you start to use Git frequently. We’re practicing what we preach as well by using the tools in the series to create the series, especially git. Note my [GitHub repo](https://github.com/bcaffo/courses) will generally be more up to date than the Data Science Specialization Repo.

The lectures are in the index.Rmd lecture files. In [Developing Data Products](https://www.coursera.org/learn/data-products), we cover how to create these sorts of slides. However, for the time being, you should be able to open them in R Studio and look at their contents. You will see all of the R code to recreate the lectures. Going through the R code is the best way to familiarize yourself with the lecture materials.

#### 1.1.2.1 YouTube

If you’d prefer to watch the videos on YouTube, you can find them [here](https://www.youtube.com/playlist?list=PLpl-gQkQivXjqHAJd2t-J_One_fYE55tC) and [here](https://www.youtube.com/playlist?list=PLpl-gQkQivXhdgUCdaUQcdb31CRe8Mm2y). If you’d like to keep up with the instructors I’m ([**bcaffo?**](#ref-bcaffo)) on twitter, Roger is ([**rdpeng?**](#ref-rdpeng)) and Jeff is ([**jtleek?**](#ref-jtleek)). The Department of Biostat here is ([**jhubiostat?**](#ref-jhubiostat)).

### 1.1.3 Syllabus (xxx)

Course Title: Regression Models

Course Instructor(s):The primary instructor of this class is [Brian Caffo](https://sites.google.com/view/bcaffo/home/). Brian is a professor at Johns Hopkins Biostatistics and co-directs the [SMART working group](https://www.smart-stats.org).

This class is co-taught by Roger Peng and Jeff Leek. In addition, Sean Kross and Nick Carchedi have been helping greatly.

#### 1.1.3.1 Course Description:

Linear models, as their name implies, relates an outcome to a set of predictors of interest using linear assumptions. Regression models, a subset of linear models, are the most important statistical analysis tool in a data scientist’s toolkit. This course covers regression analysis, least squares and inference using regression models. Special cases of the regression model, ANOVA and ANCOVA will be covered as well. Analysis of residuals and variability will be investigated. The course will cover modern thinking on model selection and novel uses of regression models including scatterplot smoothing.

#### 1.1.3.2 Course Content

This class has three main components:

* Least squares and linear regression
* Multivariable regression
* Generalized linear models

The full list of topics are as follows:

* Module 1, least squares and linear regression
  + 01\_01 Introduction
  + 01\_02 Notation
  + 01\_03 Ordinary least squares
  + 01\_04 Regression to the mean
  + 01\_05 Linear regression
  + 01\_06 Residuals
  + 01\_07 Regression inference
* Module 2, Multivariable regression
  + 02\_01 Multivariate regression
  + 02\_02 Multivariate examples
  + 02\_03 Adjustment
  + 02\_04 Residual variation and diagnostics
  + 02\_05 Multiple variables
* Module 3, Generalized linear models
  + 03\_01 GLMs
  + 03\_02 Binary outcomes
  + 03\_03 Count outcomes
  + 03\_04 Olio
* Module 4, Logistic Regression and Poisson Regression
  + 04\_01 Logistic Regression
  + 04\_02Poisson Regression
  + 04\_03 Hodgepodge

#### 1.1.3.3 Book: Regression Models for Data Science in R.

A companion book is available [here](https://leanpub.com/regmods). The book is published via leanpub, and the suggested price is $14.99. You can get it for free or pay what you feel it is worth.

#### 1.1.3.4 Quizzes

There are four weekly quizzes. You must earn a grade of at least 80% to pass a quiz. You may attempt each quiz up to 3 times in 8 hours. The score from your most successful attempt will count toward your final grade.

#### 1.1.3.5 Course Project

The Course Project is an opportunity to demonstrate the skills you have learned during the course. It is graded through peer assessment. You must earn a grade of at least 80% to pass the peer assessment.

#### 1.1.3.6 Grading Policy

You must score at least 80% on all assignments (Quizzes & Project) to pass the course.

Your final grade will be calculated as follows:

Quiz 1 = 15% Quiz 2 = 15% Quiz 3 = 15% Quiz 4 = 15% Course Project = 40%

##### 1.1.3.6.1 swirl Programming Assignment (optional)

In this course, you have the option to use the [swirl](https://swirlstats.com) R package to practice some of the concepts we cover in lectures.

While these lessons will give you valuable practice and you are encouraged to complete as many as possible, please note that they are completely optional and you can get full marks in the class without completing them.

#### 1.1.3.7 Differences of opinion

Keep in mind that currently data analysis is as much art as it is science - so we may have a difference of opinion - and that is ok! Please refrain from angry, sarcastic, or abusive comments on the message boards. Our goal is to create a supportive community that helps the learning of all students, from the most advanced to those who are just seeing this material for the first time.

### 1.1.4 Data Science Specialization Community Site

Since the beginning of the Data Science Specialization, we’ve noticed the unbelievable passion students have about our courses and the generosity they show toward each other on the course forums. A couple students have created quality content around the subjects we discuss, and many of these materials are so good we feel that they should be shared with all of our students.

We’re excited to announce that we’ve created a site using [GitHub Pages](http://datasciencespecialization.github.io/) to serve as a directory for content that the community has created. If you’ve created materials relating to any of the courses in the Data Science Specialization, please send us a pull request so we can add a link to your content on our site. You can find out more about contributing [here.](https://github.com/DataScienceSpecialization/DataScienceSpecialization.github.io#contributing)

We can’t wait to see what you’ve created and where the community can take this site!

### 1.1.5 Where to get more advanced material

If you want more advanced material, I’ve been working on another version of this class. Eventually I hope to have a second Coursera class as well. Currently, you can get the E-Book in progress [here](https://leanpub.com/lm) (it’s variable pricing including free!)

In addition, you can watch the videos as they’re being developed [here.](https://www.youtube.com/playlist?list=PLpl-gQkQivXhdgUCdaUQcdb31CRe8Mm2y)

## 1.2 Introduction to regression and least squares

Regression models are the workhorse of data science. They are the most well described, practical and theoretically understood models in statistics. A data scientist well versed in regression models will be able to solve an incredible array of problems.

Perhaps the key insight for regression models is that they produce highly interpretable model fits. This is unlike machine learning algorithms, which often sacrifice interpretability for improved prediction performance or automation. These are, of course, valuable attributes in their own rights. However, the benefit of simplicity, parsimony and intrepretability offered by regression models (and their close generalizations) should make them a first tool of choice for any practical problem.

### 1.2.1 Introduction to Regression

Hello, I’m Brian Caffo, and I’d like to welcome you to the introduction to regression lecture in the regression Coursera class, part of our data science specialization. Co-taught by my colleagues Jeff Leek and Roger Peng, we all belong to the Department of Biostatistics at the Johns Hopkins Bloomberg School of Public Health.

Regression is a cornerstone for data scientists. Before delving into complex machine learning, linear regression or its generalization, linear models, are often the go-to procedures. The roots of regression trace back to Francis Galton, who coined the term and concept, along with correlation, closely tied to linear regression.

Galton’s prediction of a child’s height from a parent’s height remains historically significant. Jeff Leek [highlights](https://www.nature.com/articles/ejhg20095) its continued relevance in modern genetic analysis, comparing it to Victorian Era measurements. Moving to a more contemporary example, a blog post by Rafael Irazarry on Simply Statistics explores the relationship between Kobe Bryant’s ball-hogging and the Lakers’ performance, utilizing linear regression.

In a modern example, [Simply Statistics](https://simplystatistics.org) blog talks about “[the Lakers wins](https://simplystatistics.org/posts/2013-01-28-data-supports-claim-that-if-kobe-stops-ball-hogging-the-lakers-will-win-more/)” that Data supports claim that if Kobe stops ball hogging the Lakers will win more.The heart of our class is understanding how to formulate and interpret statements like for example in the Simply Statistics blog post “Linear regression suggests an increase of 1% in the percent of shots taken by Kobe results in a drop of 1.16 points.” We’ll delve into good statistical practices, including providing standard errors.

We might want to find a parsimonious and easily described mean relationships between the parent’s and child’s height. So we don’t want anything complicated. We want the simplest possible relationship, and that is what regression is best at. While machine learning and other techniques generate highly elaborate, in many cases, accurate prediction models, they tend to not be parsimonious. They tend not to explain the data, and they tend not to generate new parsimonious knowledge, whereas this is what regression is good at. This is what regression is in fact best at. We can talk about variation that’s unexplained by the regression model. The so called residual variation.

We’re going to connect the results back to the subject of inference. How do we take our data, which is just a sample, it only talks about that data set, and try to figure out what assumptions are needed to extrapolate it to a larger population. This is a deep subject called statistical inference. We have a whole another course of Statistical Inference as part of data science specialization. But we’re going to apply the tools of inference, which we are hoping most of you will have had as a prerequisite. We’re going to apply the tools of inference to this new subject of regression.

Let’s look at Francis Galton’s data, he first used this data in 1885. He’s really an interesting character in history, in general and definitely in the history of statistics. You need to run install.packages("UsingR"). Here UsingR is the package for the book, [Using R for Introductory Statistics](https://cran.r-project.org/doc/contrib/Verzani-SimpleR.pdf). It is a great book, and they’ve very kindly packaged all these data sets together in a single R package. So you need to use UsingR then the library UsingR to get a lot of the data sets that we are going to talk about. So let’s first look at the marginal distribution of the parents. In other words, distribution of the parents disregarding children. And the marginal distribution of the children, disregarding parents.

install.packages("UsingR")

Parent distribution is all heterosexual couples, correcting for sex by multiplying the female heights by 1.08.

library(UsingR); data(galton); library(reshape); long<-melt(galton);

## Using as id variables

g<- ggplot(long, aes(x=value, fill=variable))   
g<- g+ geom\_histogram(color='black', binwidth=1)  
g<- g+ facet\_grid(.~variable)  
  
g



On the left, we have the children’s heights. The X-axis is in inches, the scale goes from 60 inches to 75. The Y-axis is the count, the number of children that fall in each bin of heights. On the right in the more bluish teal color, we have the parents heights. We’ve broken the association by the children and the parents by not doing a scatter plot, and only looking at the marginal distribution of the children, and the marginal distribution of the parents by themselves. We would like to use these distributions to introduce least squares, and then we’ll build on the bivaried association after that. So consider only the child’s height,forget for the moment about using the parent’s height to predict the child’s heights. We just want to find maybe the best prediction of the child’s heights without any other information. Well, probably the best predictor would be the middle and how could one define the middle?

One definition, let , be the height for child , where in this dataset . So the middle is the value of that minimizes

That’s how we define the middle. It’s also related to physics in this so called physical center of mass of the histogram that we showed on the previously. Imagine of those bars as being physical entities, having weight and you are trying to figure out where you would put your finger to balance it out. That would be the physical center of mass. You might have guessed that the center of the data has to be the mean.

Let’s use our studio’s manipulate function to experiment with trying to find that center of mass.

library(manipulate)   
myHist<-function(mu){  
 mse<-mean((galton$child - mu)^2)  
 g <- ggplot(galton, aes(x = child)) + geom\_histogram(fill = "salmon", colour = "black", binwidth=1)  
 g <- g + geom\_vline(xintercept = mu, size = 3)  
 g <- g + ggtitle(paste("mu = ", mu, ", MSE = ", round(mse, 2), sep = ""))  
 g  
}  
manipulate(myHist(mu),mu=slider(62,74,step=0.5))

fig xxx

Because we’re using manipulate we can move the slider around and monitor the value of and the mean squared error, that is the sum of the squared distances between the observed data points and that particular value of . If you move the slider around, you would notice notice as we get toward the center of the histogram, the mean squared error is going down and if you keep moving the slider way up, it get’s up large again. You can see is the point that balanced out this histogram.

**Notice** For those that are interested, we cover some simple proofs of some of the statements made. If this isn’t your thing, just skip these sections. However, if you’re interested, get a pencil and paper to work along!

The equations above show for any value of , the function is larger than or equal to the specific case when we plug in . Therefore, has to be the unique minimizer of that equation.

At this stage, we haven’t utilized the parent’s heights in our analysis. The initial step in examining this type of data is to construct a scatter plot of child heights against parent heights. Here we employ ggplot, but the plot has several shortcomings.

ggplot(galton, aes(x = parent, y = child)) + geom\_point()



Notably, there’s over-plotting due to numerous parent-child pairs sharing the same x, y values. To address this, we provide an improved plot where the point size reflects the number of parent-child combinations at a specific x, y location. Additionally, color indicates frequency, with lighter colors representing higher frequencies.

library(dplyr)

##   
## Attaching package: 'dplyr'

## The following object is masked from 'package:reshape':  
##   
## rename

## The following objects are masked from 'package:Hmisc':  
##   
## src, summarize

## The following object is masked from 'package:MASS':  
##   
## select

## The following objects are masked from 'package:stats':  
##   
## filter, lag

## The following objects are masked from 'package:base':  
##   
## intersect, setdiff, setequal, union

freqData <- as.data.frame(table(galton$child, galton$parent))  
names(freqData) <- c("child", "parent", "freq")  
freqData$child <- as.numeric(as.character(freqData$child))  
freqData$parent <- as.numeric(as.character(freqData$parent))  
g <- ggplot(filter(freqData, freq > 0), aes(x = parent, y = child))  
g <- g + scale\_size(range = c(2, 20), guide = "none" )  
g <- g + geom\_point(colour="grey50", aes(size = freq+20, show\_guide = FALSE))

## Warning: Ignoring unknown aesthetics: show\_guide

g <- g + geom\_point(aes(colour=freq, size = freq))  
g <- g + scale\_colour\_gradient(low = "lightblue", high="white")   
g



In order to find the best line, all we have to find is the slope. Well, here’s how we could potentially do that. We would want to find the slope beta that minimizes the sum of the squared distances between the observed data points the and the fitted data points on the line, . We’ll square that distance and add them up and this is directly analogous to finding the least squares mean. This is sort of using the origin as a pivot point and picking the line that minimizes the sum of the squared vertical distances between the points and the line. Notice that there is a point in regression to the origin is useful for explaining things, because we only have one parameter, the slope and we don’t have two parameters, the slope and the intercept. But it’s generally bad practice to force regression lines through the point (0, 0). So, an easy way around this is to subtract the mean from the parent’s heights and the mean from the child’s heights, so that the zero, zero point is right in the middle of the data and that will make this solution a little bit more palatable.

y <- galton$child - mean(galton$child)  
x <- galton$parent - mean(galton$parent)  
freqData <- as.data.frame(table(x, y))  
names(freqData) <- c("child", "parent", "freq")  
freqData$child <- as.numeric(as.character(freqData$child))  
freqData$parent <- as.numeric(as.character(freqData$parent))  
myPlot <- function(beta){  
 g <- ggplot(filter(freqData, freq > 0), aes(x = parent, y = child))  
 g <- g + scale\_size(range = c(2, 20), guide = "none" )  
 g <- g + geom\_point(colour="grey50", aes(size = freq+20, show\_guide = FALSE))  
 g <- g + geom\_point(aes(colour=freq, size = freq))  
 g <- g + scale\_colour\_gradient(low = "lightblue", high="white")   
 g <- g + geom\_abline(intercept = 0, slope = beta, size = 3)  
 mse <- mean( (y - beta \* x) ^2 )  
 g <- g + ggtitle(paste("beta = ", beta, "mse = ", round(mse, 3)))  
 g  
}  
  
myPlot(0.5)

## Warning: Ignoring unknown aesthetics: show\_guide



We can find the slope of the line very quickly in R using the lm function. The lm function stands for linear model. We’re going to regress the child’s height on the parent’s height. We’re going to subtract the mean from the child’s height and the mean from the parent’s height, to make sure line is going through the origin. Doing so will give us a line that has slope of 0.646.

lm(I(child - mean(child))~ I(parent - mean(parent)) - 1, data = galton)

##   
## Call:  
## lm(formula = I(child - mean(child)) ~ I(parent - mean(parent)) -   
## 1, data = galton)  
##   
## Coefficients:  
## I(parent - mean(parent))   
## 0.6463

Now what we’re going to do in subsequent sections is to talk about how we get these values? What is the motivation behind it and all the things we can do with this fitted line, we’re going to spend maybe the next several sections talking about this. You have actually learned a lot of material in this very first part, well done!

## 1.3 Linear least squares

### 1.3.1 Notations and background

Ordinary least squares (OLS) is the workhorse of statistics. It gives a way of taking complicated outcomes and explaining behavior (such as trends) using linearity. The simplest application of OLS is fitting a line through some data. In the next few sections, we cover the basics of linear least squares. We start with defining our notation. These are things you probably already saw in the prerequisite for this course in a Statistical Inference course in Data Science Specialization. However, because they’re so fundamental to regression, we’re going to cover them again, so they’re fresh in our minds. We will try to minimize the amount of mathematics that’s required for this class. Throughout the course we will neither require calculus nor linear algebra. And when it does get a little bit more mathematical, we will let you know when you can skip over those sections.

We might write to describe data points. As an example, consider the data set , where , , and in this case is 3. There’s nothing in particular about the letter . We could have just as easily described to . The last bit of notation that’s important, is we’re typically going to use Greek letters for things we don’t know, such as for a population mean and we’ll use non Greek letters or regular letters to denote things that we can observe. So, is something we can observe. is something we can’t observe and would like to estimate. We can define the empirical mean as

Notice if we subtract the mean from data points, we get data that has mean 0. That is, if we define The mean of the is 0.

* This process is called “centering” the random variables. Recall from the previous section that the mean is the least squares solution for minimizing .

Since we talked about means, let’s talk about variances. The variances is usually denoted by . It’s defined as

This is nothing other than basically the average squared deviation of the observations around the mean. The empirical standard deviation is defined as . Notice that the standard deviation has the same units as the data. It’s nice to work with standard deviations because the variance is expressed in whatever units has squared, whereas the standard deviation is just expressed in the normal units of . Another interesting fact related to standard deviation is scaling, so if we subtract a mean off from every observation, we get a resulting data set that has mean 0. If we divide every observation by the standard deviation, the resulting data set will have standard deviation 1. This is called **scaling** the data. If we take our original data now and subtract off , then take the resulting centered data and scale it by . We get a new data set, let’s call them .

This process of centering and then scaling is called *normalizing* the data. As an example, if something has a value 2 from normalized data, that means that the data point was 2 standard deviations larger than the mean. As its name would suggest, normalization is an attempt to make non-comparable data sets comparable.

The empirical covariance is the most central quantity in regression. Imagine we have two vectors, and , and they’re lined up. So might be the BMI and might be the blood pressure for subject . You could meaningfully do a scatter plot. Then we just define the covariance between X and Y as:

The correlation is defined as:

where and are the estimates of standard deviations for the observations and observations, respectively. In other words, the correlation is simply the covariance then standardized into a unitless quantity. So, the correlation is the covariance of and , which has units, basically units of X times units of Y.

Some facts about correlation: \* \* \* and only when the or observations fall perfectly on a positive or negative sloped line, respectively. \* measures the strength of the linear relationship between the and data, with stronger relationships as heads towards -1 or 1. \* implies no linear relationship.

### 1.3.2 Linear Least Squares

Consider again, when we’re looking at the scatter plot of the parent’s heights by the child’s heights from the Galton data, the size of the circle represents the frequency of that particular x, y combination.

## Warning: Ignoring unknown aesthetics: show\_guide



We’d like to use the parent’s heights to explain the child’s heights and we’re going to do it using linear regression. We’re going to use our notation that we developed in our last section. So let’s let be the child’s height and be the parents’ height. Now we want to find the best line, where we want the line to look like child’s height is an intercept. Child’s Height = + Parent’s Height , and are parameters we would like to know that we don’t know. Well, we need a criteria for the term best. We need to figure out what we mean by the best line that fits the data. Well, one criteria is the famous least squares criteria. And the basic gist of the equation is we want to minimize the sum of the squared vertical distances between the data points, the height of the data points, the child’s heights and the points on the line, on the fitted line. And we can write this as

This is the sum of the squared vertical distances between the data points and the fitted line. We want to minimize this quantity. We want to find the and that minimize this quantity. This is called the least squares criteria. We put little hats over and to indicate the estimated values. The least squares model fit to the line through the data pairs with as the outcome obtains the line where

The solution works out to be is the correlation between and times the standard deviation of divided by the standard deviation of . The estimated intercept . So let’s go through a couple of consequences of this being the result.

* has the units of , has the units of . We can see this because the correlation is a unitless quantity.
* The line passes through the point )
* The slope of the regression line with as the outcome and as the predictor is .
* The slope is the same one you would get if you centered the data, , and did regression through the origin.
* If you normalized the data, , the slope is .

### 1.3.3 Linear Least Squares Coding Example

Here we will go through a coding example to show how to calculate the least squares estimates. We plot the Galton parents’ height and childrens’ height data that we are going to look at.

library(dplyr)  
freqData <- as.data.frame(table(galton$child, galton$parent))  
names(freqData) <- c("child", "parent", "freq")  
freqData$child <- as.numeric(as.character(freqData$child))  
freqData$parent <- as.numeric(as.character(freqData$parent))  
g <- ggplot(filter(freqData, freq > 0), aes(x = parent, y = child))  
g <- g + scale\_size(range = c(2, 20), guide = "none" )  
g <- g + geom\_point(colour="grey50", aes(size = freq+20, show\_guide = FALSE))

## Warning: Ignoring unknown aesthetics: show\_guide

g <- g + geom\_point(aes(colour=freq, size = freq))  
g <- g + scale\_colour\_gradient(low = "lightblue", high="white")   
g

 Now we indicate that the solution that we specified is the same solution that R will give you with its built in regression function. The function lm in R stands for linear model. Regression is a component of linear models, and so, this function is the general function whether you want regression or you want some of the more elaborate versions of regression that we’re going to cover later on. So we want lm, the outcome , the predictor . coef takes the output of the linear model and just grabs the coefficients.

beta1 <- cor(y, x) \* sd(x) / sd(y)  
beta0 <- mean(x) - beta1 \* mean(y)  
rbind(c(beta0, beta1), coef(lm(x ~ y)))

## (Intercept) y  
## [1,] 8.207028e-16 0.3256475  
## [2,] 1.258492e-15 0.3256475

As we expected you see we get the same numbers, 23.94 and 0.64, 0.65. Very briefly now, we just want to mention that if we reverse the and relationship the formula, of course holds but now with standard deviation of in the numerator and standard deviation of in the denominator.

beta1 <- cor(y, x) \* sd(x) / sd(y)  
beta0 <- mean(x) - beta1 \* mean(y)  
rbind(c(beta0, beta1), coef(lm(x ~ y)))

## (Intercept) y  
## [1,] 8.207028e-16 0.3256475  
## [2,] 1.258492e-15 0.3256475

If we concatenate these slope and intercept estimates with those that you get with lm where is on the left hand side of the ~ and is on the right hand side of ~, reversed from what it was previously.

So our formula is correct and we know how to use it and we know what happens when we reverse the relationship. Another point that was made thus far in the course was that regression through the origin yielded the same slope as linear regression with a not necessarily zero intercept. If you mean centered the ’s and mean centered the ’s first. So let’s just check that computationally. Recall that the regression to the origin equation for the slope was just the sum of the variable times the variable, divided by the sum of the variable squared. So, let’s run that and get our coefficient that is estimated through a regression to the origin.

yc <- y - mean(y)  
xc <- x - mean(x)  
beta1 <- sum(yc \* xc) / sum(xc ^ 2)  
c(beta1, coef(lm(y ~ x))[2])

## x   
## 0.6462906 0.6462906

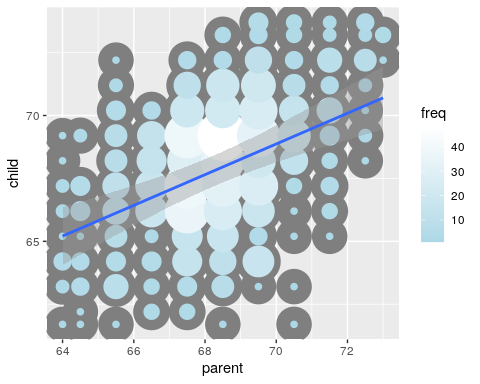
We want to very briefly also just show you how you can actually do regression to the origin. In this case I’ll get the same number if I take the centered and use the centered as a predictor, to subtract out the intercept, you put a minus one to get rid of the intercept.

Another point that was made before, was that if we were to normalize the or the so that they have standard deviation one, the slope would be the correlation. So let’s just double check that quickly. Here, We normalize the child’s heights by subtracting off the mean and dividing by the standard deviation. We do the same thing for variables. We have gotten rid of the, the original units, the inches.

yn <- (y - mean(y))/sd(y)  
xn <- (x - mean(x))/sd(x)  
c(cor(y, x), cor(yn, xn), coef(lm(yn ~ xn))[2])

## xn   
## 0.4587624 0.4587624 0.4587624

## Warning: Ignoring unknown aesthetics: show\_guide



Here we are showing the somewhat fancy plot for this data. We would also note that ggplot2 does a very good thing for us on our behalf. It automatically gives us a confidence interval around the line. We’ll talk about how to generate this confidence interval later on in the lecture. But it’s very nice that they’re thinking of statistical uncertainty automatically.

### 1.3.4 Mathematical Details (Optional) XXX

## 1.4 Regression to the Mean

Regression to the mean was an important milestone in the discovery of regression. So we’re going to talk about it. It was discovered by Francis Galton. Regression to mean asks questions like this.

* Why is it that the children of tall parents tend to be tall, but not as tall as their parents?
* Why do children of short parents tend to be short, but not as short as their parents?
* Why do parents of very short children, tend to be short, but not a short as their child? And the same with parents of very tall children?

We can try this with anything that is measured with error. Why do the best performing athletes this year tend to do a little worse the following? Why do the best performers on hard exams always do a little worse on the next hard exam?

These phenomena are all examples of so-called regression to the mean. Regression to the mean, was invented by Francis Galton in the paper “Regression towards mediocrity in hereditary stature” The Journal of the Anthropological Institute of Great Britain and Ireland , Vol. 15, (1886). The idea served as a foundation for the discovery of linear regression.

Regression to the mean often comes up in sports. If you have a player who has a phenomenal year, the next year they tend to do a little bit worse. If you have a player who has a terrible year, the next year they tend to do a little bit better. Another example would be often people talk about stocks in the same way. Some of the best performing stocks tend to go down. These phenomena could all be examples of so called regression to the mean. We will talk about why these happen and whether or not something is intrinsic or whether it is a regression to the mean effect. Regression to the mean was invented by Francis Galton. We like to think of regression to the mean by thinking of the case where it’s a 100% regression to the mean. So imagine if we were to simulate pairs of standard normals, i.e. they have nothing to do with one another, they’re independent standard normals. If we were to take the largest one, the chance that its pair in the second vector is smaller will be high. And this is simply saying that the probability that is less than , given is going to get bigger as heads to very large values. The same thing in other words, is that probability is greater than . Given that equals is going to get bigger as heads to smaller values. This extreme version of regression in the mean where there’s 100% regression to the mean is what we like to think about.

* gets bigger as heads into the very large values.
* gets bigger as heads to very small values.

However, in most cases there’s some blend of some, some intrinsic component, and a noise. For example, consider a scenario where every student in this class takes two very challenging quizzes. While those at the top likely have a better understanding of the material, quizzes are imperfect instruments, introducing inherent error or noise. This means that even the top performers might benefit from some luck or randomness. Consequently, a top performer, who probably knows the material a bit better than others, may experience a slight dip in performance on the second quiz due to this inherent variability. Conversely, even the worst performers might fare a bit better on one quiz due to chance. This concept extends beyond academics. It’s intriguing to reflect on how much of the discussion about sports revolves around the idea of regression to the mean. For instance, a baseball player with a phenomenal batting average one year might experience a slightly lower average the next year, illustrating the natural tendency for extreme performances to move closer to the average over time. The question is are these examples of just regression to the mean? If so, it would be nice to figure out how to quantify it. This is what Francis Galton did with regression in the first treatment of regression to the mean.

Let’s delve into how Francis Galton employed the concept of regression, particularly using correlation, which is intimately related to linear regression. The goal is to quantify regression to the mean, and I’ll illustrate this with a visual representation. Before delving into the R code, let me outline the setup.

In this case, I’m assigning to be the child’s height and to be the parent’s height. I’m using a dataset where the parent is a single parent, specifically the father. Both the and values have been normalized, meaning they have a mean of 0 and a variance of 1. Assuming you’re familiar with this normalization process, the regression line will pass through the point (0, 0). Notably, regardless of whether the child’s height is the outcome or the parent’s height is the outcome, the slope of the regression line is simply the correlation.

Now, a quirk worth mentioning when creating the plot is that if is the outcome and you happen to plot it on the horizontal axis, the slope of the line needs to be 1 over the correlation. This is due to the specific orientation of the axes. Keep this in mind as we proceed with the *R* code. In the code below we are using the dataset from the usingR library, specifically the father.son data. Here’s how we define the variables:

* Y: Son’s heights, normalized by subtracting the mean and dividing by the standard deviation.
* X: Father’s heights, similarly normalized.

Now, both X and Y should have a mean of 0 and a variance of 1.

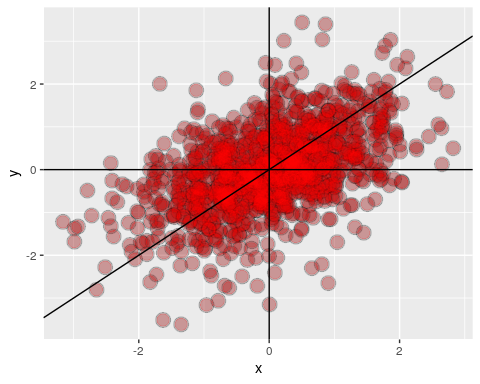
We use the Greek letter (rho) to represent the correlation between and . If you would check the value of rho, turns out to be about 0.5. This indicates a correlation of 0.5 between the father’s height and the son’s height.

Now, let’s create the plot. After loading the ggplot2 library, we assign the ggplot to the variable g and adding points with a black background and salmon-colored foreground. The use of alpha blending makes the points somewhat transparent. We set the x-axis and y-axis limits to be -4 to +4 on both axes. This range is chosen as it should cover most of the data, considering the extremely low probability of standardized random variables being below -4 or above +4. Chebyshev’s theorem supports this choice, especially if you’ve covered it in the Statistical Inference course. Next, we add a layer for the identity line. Afterward, we’ll add the horizontal and vertical axes.

library(UsingR)  
data(father.son)  
y <- (father.son$sheight - mean(father.son$sheight)) / sd(father.son$sheight)  
x <- (father.son$fheight - mean(father.son$fheight)) / sd(father.son$fheight)  
rho <- cor(x, y)  
library(ggplot2)  
g = ggplot(data.frame(x = x, y = y), aes(x = x, y = y))  
g = g + geom\_point(size = 6, colour = "black", alpha = 0.2)  
g = g + geom\_point(size = 4, colour = "salmon", alpha = 0.2)  
g = g + xlim(-4, 4) + ylim(-4, 4)  
g = g + geom\_abline(intercept = 0, slope = 1)  
g = g + geom\_vline(xintercept = 0)  
g = g + geom\_hline(yintercept = 0)  
g = g + geom\_abline(intercept = 0, slope = rho, size = 2)  
g = g + geom\_abline(intercept = 0, slope = 1 / rho, size = 2)  
g = ggplot(data.frame(x, y), aes(x = x, y = y))  
g = g + geom\_point(size = 5, alpha = .2, colour = "black")  
g = g + geom\_point(size = 4, alpha = .2, colour = "red")  
g = g + geom\_vline(xintercept = 0)  
g = g + geom\_hline(yintercept = 0)  
g = g + geom\_abline(position = "identity")

## Warning: Ignoring unknown parameters: position

g

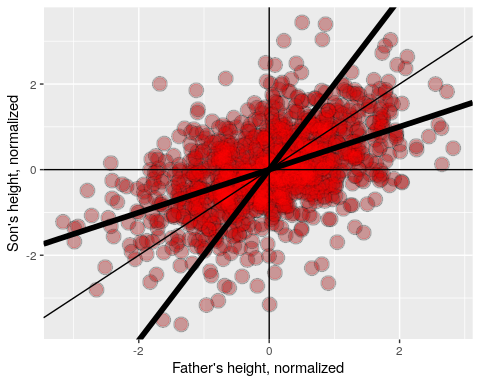


Now, let’s create two lines. First, we’ll treat the son’s height as the outcome and the father’s height as the predictor. Then, we’ll add the line treating the son’s height as the predictor and the father’s height as the outcome. Since the axes are rotated, the slope needs to be 1 over rho.

library(UsingR)  
data(father.son)  
y <- (father.son$sheight - mean(father.son$sheight)) / sd(father.son$sheight)  
x <- (father.son$fheight - mean(father.son$fheight)) / sd(father.son$fheight)  
rho <- cor(x, y)  
library(ggplot2)  
g = ggplot(data.frame(x = x, y = y), aes(x = x, y = y))  
g = g + geom\_point(size = 6, colour = "black", alpha = 0.2)  
g = g + geom\_point(size = 4, colour = "salmon", alpha = 0.2)  
g = g + xlim(-4, 4) + ylim(-4, 4)  
g = g + geom\_abline(intercept = 0, slope = 1)  
g = g + geom\_vline(xintercept = 0)  
g = g + geom\_hline(yintercept = 0)  
g = g + geom\_abline(intercept = 0, slope = rho, size = 2)  
g = g + geom\_abline(intercept = 0, slope = 1 / rho, size = 2)  
g = ggplot(data.frame(x, y), aes(x = x, y = y))  
g = g + geom\_point(size = 5, alpha = .2, colour = "black")  
g = g + geom\_point(size = 4, alpha = .2, colour = "red")  
g = g + geom\_vline(xintercept = 0)  
g = g + geom\_hline(yintercept = 0)  
g = g + geom\_abline(position = "identity")

## Warning: Ignoring unknown parameters: position

g = g + geom\_abline(intercept = 0, slope = rho, size = 2)  
g = g + geom\_abline(intercept = 0, slope = 1 / rho, size = 2)  
g = g + xlab("Father's height, normalized")  
g = g + ylab("Son's height, normalized")  
g



Now, let’s discuss regression to the mean in relation to this plot. If the observations perfectly aligned on a line, it would be the identity line, given that both and have been normalized. The father’s height is plotted as the variable, and the son’s height is plotted as the variable. For instance, if we had a father’s height of 2 with no noise, the prediction for the son’s height would also be 2, representing 2 standard deviations above the mean for both fathers and sons.

However, in the presence of noise, the prediction deviates from 2 but falls on the regression line. This prediction is obtained by multiplying the father’s height (=2) by the slope (=correlation). The result is a prediction between 2 and 0, precisely 2 multiplied by the correlation. This phenomenon is known as regression to the mean. The extent to which this correlation is shrunk towards the horizontal line indicates the degree of regression to the mean.

Consider the extreme cases for better understanding. In a scenario with no noise, the line would fall perfectly on the identity line. Conversely, if there was only noise, indicating no informative relationship between father’s and son’s heights (correlation = 0), the line would lie on the horizontal axis, predicting a constant height of 0 for sons based on fathers. This concept holds when considering the son’s height as the predictor and the father’s height as the outcome. The regression to the mean is observed in how much the line is shrunk towards the vertical axis. This notion, introduced by Francis Galton, played a pivotal role in the development of modern regression. Although it remains a fundamental idea, regression to the mean continues to have significance in statistical analyses, particularly in the study of longitudinal data where it’s crucial to consider this phenomenon.

In summary: \* If you had to predict a son’s normalized height, it would be \* If you had to predict a father’s normalized height, it would be \* Multiplication by this correlation shrinks toward 0 (regression toward the mean) \* If the correlation is 1 there is no regression to the mean (if father’s height perfectly determine’s child’s height and vice versa) \* Note, regression to the mean has been thought about quite a bit and generalized

## 1.5 Practical R Exercises in swirl

During this course we’ll be using the [swirl](https://swirlstats.com) software package for R in order to illustrate some key concepts. The swirl package turns the R console into an interactive learning environment. Using swirl will also give you the opportunity to construct and explore your own regression models.

1. Install R

swirl requires R 3.0.2 or later. If you have an older version of R, please update before going any further. If you’re not sure what version of R you have, type R.version.string at the R prompt. You can download the latest version of R from <https://www.r-project.org/>.

Optional but highly recommended: Install RStudio. You can download the latest version of RStudio at <https://www.rstudio.com/products/rstudio/>.

1. Install swirl

Since swirl is an R package, you can easily install it by entering a single command from the R console:

* + If you are on a Linux operating system, please visit our Installing swirl on Linux page for special instructions: install.packages("swirl")
  + If you’ve installed swirl in the past make sure you have version 2.2.21 or later. You can check this with: packageVersion("swirl")

1. Load swirl

Every time you want to use swirl, you need to first load the package. From the R console: library(swirl).

1. Install the Regression Models course

swirl offers a variety of interactive courses, but for our purposes, you want the one called Regression Models. If this is your first time using swirl, it will prompt you to install the Regression Models course automatically. If you’ve used swirl in the past, you will need to type the following from the R prompt: install\_course("Regression Models").

1. Start swirl and complete the lessons

Type the following from the R console to start swirl:

For the first part of this course you should complete the following lessons: - Introduction - Residuals - Least Squares Estimation

Good luck and have fun!

## 1.6 Week 1 Quiz

1. Consider the data set given by the R code x <- c(0.18, -1.54, 0.42, 0.95)and weights given by w <- c(2, 1, 3, 1) give the value of that minimizes the least squares equation .
2. Consider the following data set fit the regression through the origin and get the slope treating yas the outcome and x as the regressor. (Hint, do not center the data since we want regression through the origin, not through the means of the data.)

x <- c(0.8, 0.47, 0.51, 0.73, 0.36, 0.58, 0.57, 0.85, 0.44, 0.42)  
y <- c(1.39, 0.72, 1.55, 0.48, 1.19, -1.59, 1.23, -0.65, 1.49, 0.05)

1. Do data(mtcars) from the datasets package and fit the regression model with mpg as the outcome and weight as the predictor. What is the slope coefficient?
2. Consider data with an outcome () and a predictor (). The standard deviation of the predictor is one half that of the outcome. The correlation between the two variables is .5. What value would the slope coefficient for the regression model with as the outcome and as the predictor?
3. Students were given two hard tests and scores were normalized to have empirical mean 0 and variance 1. The correlation between the scores on the two tests was 0.4. What would be the expected score on Quiz 2 for a student who had a normalized score of 1.5 on Quiz 1?
4. Consider the data given by x <- c(8.58, 10.46, 9.01, 9.64, 8.86). What is the value of the first measurement if x were normalized (to have mean 0 and variance 1)?
5. Consider the following data set (used above as well). What is the intercept for fitting the model with x as the predictor and y as the outcome?

x <- c(0.8, 0.47, 0.51, 0.73, 0.36, 0.58, 0.57, 0.85, 0.44, 0.42)  
y <- c(1.39, 0.72, 1.55, 0.48, 1.19, -1.59, 1.23, -0.65, 1.49, 0.05)

1. You know that both the predictor and response have mean 0. What can be said about the intercept when you fit a linear regression?
2. It must be identically 0.
3. It is undefined as you have to divide by zero.
4. It must be exactly one.
5. Nothing about the intercept can be said from the information given.
6. Consider the data given by x <- c(0.8, 0.47, 0.51, 0.73, 0.36, 0.58, 0.57, 0.85, 0.44, 0.42). What value minimizes the sum of the squared distances between these points and itself?
7. Let the slope having fit as the outcome and as the predictor be denoted as . Let the slope from fitting as the outcome and as the predictor be denoted as . Suppose that you divide by ; in other words consider . What is this ratio always equal to?

# 2 Week 2

## 2.1 Statistical linear regression models

Up to this point, we’ve only considered estimation. Estimation is useful, but we also need to know how to extend our estimates to a population. This is the process of statistical inference. Our approach to statistical inference will be through a statistical model. At the bare minimum, we need a few distributional assumptions on the errors. However, we’ll focus on full model assumptions under Gaussianity.

### 2.1.1 Statistical Linear Regression Models

Finding a good regression line using least squares is a mathematical procedure. However, we’d like to do statistics. We’d like to draw emphasis based on our data. In other words we’d like to generalize from our data to a population using statistical models. Consider the probabilistic model for linear regression

The values of are the population parameters that we would like to estimate. is a collection of explanatory variables that we do know, and is iid Gaussian errors.

* Here the are assumed iid .

Understanding independent errors in regression can be approached in various ways. One relatively straightforward interpretation is to consider them as the cumulative effect of unmodeled variables that might collectively influence the response. These unmodeled variables act on the response in a manner that can be statistically modeled as independent and identically distributed Gaussian errors.

Setting aside the complexities of interpretation, let’s focus on the mechanics of working with statistical inference for regression. It’s important to note that the expected value of the response given a specific value of the regressor is simply the line at that regressor, represented as . Additionally, the variance of the response at any given value of the regressor is denoted as σ². It’s crucial to clarify that this variance pertains to the variation around the regression line and not the overall response variance. Conditioning on X reduces the variation, making it lower than the unconditional response variance.

* Note,
* Note, .

Both the expected value and variance mentioned here are population quantities. Although there are sample analogs that estimate these values, it’s essential to recognize that, at this point, we are referring to population quantities—these are the estimands that we ideally want to know.

Now that we have a formal statistical framework, we can interpret our regression coefficients with respect to that framework. Take for example, the intercept. It is the expected value given that the regressor is 0.

Note that the regressor being equal to zero is often not of interest in the study. For example, if the regression variable is blood pressure, probably you’re not interested in the response for among people with blood pressure of zero. However, there is an easy fix for this. Consider just shifting our regression variable by a constant .

We see a new regression line with a new intercept and the same slope. So, shifting your values by value changes the intercept, but not the slope.

* Often is set to so that the intercept is interpretted as the expected response at the average value.

For slope, we can interpret it as the expected change in response for a 1 unit change in the predictor.

\* Consider the impact of changing the units of .

We see a new regression line with a new slope and the same intercept. So, multiplying your values by value changes the slope, but not the intercept. For example, is height in and is weight in . Then is . Converting to implies multiplying by . To get in the right units, we have to divide by to get it to have the right units.

If we would like to guess the outcome at a particular value of the predictor, say , the regression model guesses

This doesn’t mean that we can only predict at the fitted values. We can predict at any value of by plugging in the value of into the equation. However, we’re going to have more reasonable predictions if the value of that we plug in is in the cloud of data that we used to build the model. Later on, we’ll also talk about how to account for that kind of uncertainty with prediction intervals. But for the time being, let’s just talk about how we get a prediction. Let’s go through an example to interpret the regression coefficients and show running of the regression coefficient. The dataset is the diamond dataset from the UsingR package. The data is diamond prices in Singapore dollars and diamond weight in carats, which is a standard measure of diamond mass.

library(UsingR)

## Loading required package: MASS

## Loading required package: HistData

## Loading required package: Hmisc

## Loading required package: lattice

## Loading required package: survival

## Loading required package: Formula

## Loading required package: ggplot2

##   
## Attaching package: 'Hmisc'

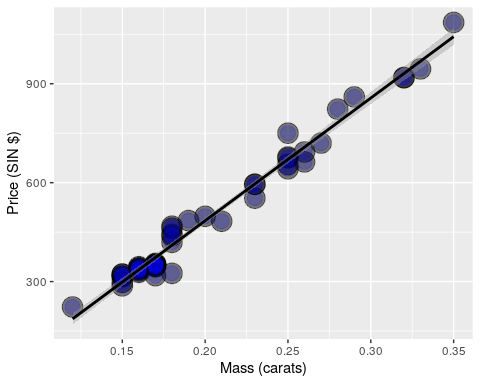
## The following objects are masked from 'package:base':  
##   
## format.pval, units

##   
## Attaching package: 'UsingR'

## The following object is masked from 'package:survival':  
##   
## cancer

data(diamond)  
library(ggplot2)  
g = ggplot(diamond, aes(x = carat, y = price))  
g = g + xlab("Mass (carats)")  
g = g + ylab("Price (SIN $)")  
g = g + geom\_point(size = 7, colour = "black", alpha=0.5)  
g = g + geom\_point(size = 5, colour = "blue", alpha=0.2)  
g = g + geom\_smooth(method = "lm", colour = "black")  
g

## `geom\_smooth()` using formula 'y ~ x'



In this code we assign variable g to the ggplot, the dataset is diamond, the aesthetic has the horizontal axis variable as carat and the y-axis variable as price, we add a layer where the xlab is Mass in carats and the y label price in Singapore dollars. We also add the points of the black background and then a light alpha blending color on top. Afterwards we add a layer that is geom\_smooth where method = "lm" will add the regression line. If you omit any arguments, it’s just going to assume the regression line with as the outcome and as the predictor. Finally, we indicate the color of the regression line as black and call the plot.

Notice what we are plotting is the fitted line, the line that minimizes the sum of the squared vertical distances between the points and the lines. By default, lm includes an intercept, if you don’t want an intercept, you have to explicitly force it in the model. We also want the dataset to be the diamond dataset in other words, we have to give it the data frame. Otherwise, lm looks in the regular R environment for variables in the model. After running the code it basically just prints out the coefficients , which are the intercept and labels it as Intercept and the regression variable for the carat, the slope for the carat regression variable.

fit <- lm(price ~ carat, data = diamond)  
coef(fit)

## (Intercept) carat   
## -259.6259 3721.0249

Let’s look at this variable and try to interpret it. It’s saying that we have an expected Singapore dollar increase in price for every carat increase in mass of the diamond. The intercept, is the expected price of a carat diamond not very interesting, because we’re not interested in zero carat diamonds.

**A side note**, if you want a much more detailed printout by doing summary(fit) which is the summary of the outputted variable from lm and you get this more elaborate printout.

##   
## Call:  
## lm(formula = price ~ carat, data = diamond)  
##   
## Residuals:  
## Min 1Q Median 3Q Max   
## -85.159 -21.448 -0.869 18.972 79.370   
##   
## Coefficients:  
## Estimate Std. Error t value Pr(>|t|)   
## (Intercept) -259.63 17.32 -14.99 <2e-16 \*\*\*  
## carat 3721.02 81.79 45.50 <2e-16 \*\*\*  
## ---  
## Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1  
##   
## Residual standard error: 31.84 on 46 degrees of freedom  
## Multiple R-squared: 0.9783, Adjusted R-squared: 0.9778   
## F-statistic: 2070 on 1 and 46 DF, p-value: < 2.2e-16

If we mean center our variable, so that the intercept is on a more interpretable scale. Here we assign the output to a different variable, fit2 instead of fit, because we don’t want to overwrite the original fit.

fit2 <- lm(price ~ I(carat - mean(carat)), data = diamond)  
coef(fit2)

## (Intercept) I(carat - mean(carat))   
## 500.0833 3721.0249

As you notice in code: lm is again the linear model procedure, the outcome stays the same and we use carat - mean(carat), and the I is to indicate that we want to do arithmetic on the variable. So, we want to subtract the mean of the carat variable from the carat variable. This is a way to mean center the variable. As we expected the slope stays the same, 3,721, but the intercept has changed to , meaning , Singapore dollars is the expected price of the average sized diamond. In this case, the average diamond is about 0.2 carats. A one carat increase is actually kind of big. What about changing the units to one-tenth of a carat? We can do this just by dividing the coefficient by ten. So we know that we would expect to see a increase in price for every of a carat increase in the mass of a diamond.

fit3 <- lm(price ~ I(carat/10), data = diamond)  
coef(fit3)

## (Intercept) I(carat/10)   
## -259.6259 37210.2485

In the linear model fit instead of putting in carat, we put in , the units of this new variable is one-tenth of a carat. The data is of course, still the diamond dataset.

Imagine if someone came to you with three new diamonds that they had 0.16 carats, 0.27 carats and 0.35 carats, and they wanted to know what you would estimate the price would be. Well, you could do it manually by grabbing the two coefficients in multiplying the intercept or adding the intercept plus the slope times these new values. Let’s do that:

newx <- c(0.16, 0.27, 0.34)  
coef(fit)[1] + coef(fit)[2] \* newx

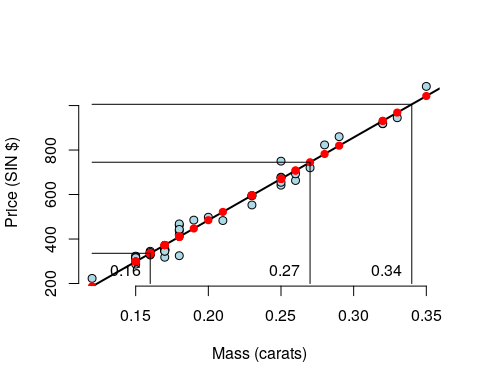
## [1] 335.7381 745.0508 1005.5225

predict(fit, newdata = data.frame(carat = newx))

## 1 2 3   
## 335.7381 745.0508 1005.5225

Often, you don’t want to do even that much coding, you want to more general method, especially when you get lots of regression variables. So there’s this general method called predict that will take the output from several different kinds of model fits. Linear models are one example, but predict is a generic function, and it applies to several different prediction models. The new data is a data.frame(catar=newx) that has the new values of for the carat variable. Then when we do that, what you’ll see is the same answer. The difference is that it scales up when we have lots of regressors in much more complicated settings. In general, we want to predict using the predict function. If you omit this new data statement if you just do predict fit, it predicts at the observed values, so it gives you the values. If you want it at new values, you have to give it this new data argument.

data(diamond)  
plot(diamond$carat, diamond$price,   
 xlab = "Mass (carats)",   
 ylab = "Price (SIN $)",   
 bg = "lightblue",   
 col = "black", cex = 1.1, pch = 21,frame = FALSE)  
abline(fit, lwd = 2)  
points(diamond$carat, predict(fit), pch = 19, col = "red")  
lines(c(0.16, 0.16, 0.12),   
 c(200, coef(fit)[1] + coef(fit)[2] \* 0.16,  
 coef(fit)[1] + coef(fit)[2] \* 0.16))  
lines(c(0.27, 0.27, 0.12),   
 c(200, coef(fit)[1] + coef(fit)[2] \* 0.27,  
 coef(fit)[1] + coef(fit)[2] \* 0.27))  
lines(c(0.34, 0.34, 0.12),   
 c(200, coef(fit)[1] + coef(fit)[2] \* 0.34,  
 coef(fit)[1] + coef(fit)[2] \* 0.34))  
text(newx, rep(250, 3), labels = newx, pos = 2)



To illustrate, here’s our observe data points in blue. The fitted values when we do the predict command, the fitted values in red all of the observed values and their associated fitted points on the line. These are if we were to draw vertical lines from the observed data points on to the fitted line, they would occur on these red points. When we predicted a new value of , we’re finding a point along this horizontal axis. In this example we want, 0.16, 0.27 and 0.34. We’re drawing a line up to the fitted regression line and then over to dollars and those are our predicted dollar amounts.

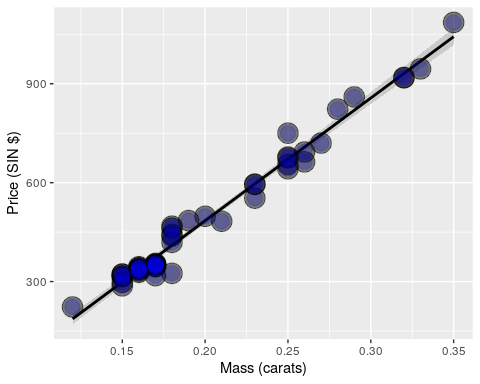
## 2.2 Residuals

Residuals represent variation left unexplained by our model. We emphasize the difference between residuals and errors. The errors unobservable true errors from the known coefficients, while residuals are the observable errors from the estimated coefficients. In a sense, the residuals are estimates of the errors.

To begin, let’s delve into our illustrative example featuring the diamond dataset. It’s important to recall that in this dataset, the diamonds are priced in Singapore dollars. The key variable under consideration is the weight of the diamonds, expressed in carats. Our objective is to explore the correlation between the weight of diamonds and their corresponding prices, seeking to understand how variations in diamond prices can be elucidated by their mass.

library(UsingR)  
data(diamond)  
library(ggplot2)  
g = ggplot(diamond, aes(x = carat, y = price))  
g = g + xlab("Mass (carats)")  
g = g + ylab("Price (SIN $)")  
g = g + geom\_smooth(method = "lm", colour = "black")  
g = g + geom\_point(size = 7, colour = "black", alpha=0.5)  
g = g + geom\_point(size = 5, colour = "blue", alpha=0.2)  
g

## `geom\_smooth()` using formula 'y ~ x'



Now, our focus is on elucidating the price (on the vertical axis) through the mass (on the horizontal axis). Without taking mass into account, we’d have a scatter of points projecting onto the vertical axis, displaying considerable variation. Disregarding mass would result in a notable amount of unexplained variation. However, when we factor in mass, the variation diminishes, as we’re now examining the variation around the regression line.

This remaining variation around the regression line is termed residual variation. It represents the portion of variation that persists even after accounting for mass. Initially, there is substantial variation, a significant portion of which is clarified by the linear relationship with mass. Nonetheless, there remains some residual variation. These residual distances are referred to as residuals, and they constitute the focal point of today’s lecture. Residuals prove to be valuable for various diagnostic purposes, including assessing model fit. Let’s refresh our memory regarding the model under consideration.

The outcome in our example,price, is , which we’re assuming is a line. Observed outcome is at predictor value , predicted outcome is at predictor value is . Residual, the between the observed and predicted outcome , which is the vertical distance between the observed data point and the regression line where least squares minimizes . In essence, it was minimizing the sum of the squared residual, summation squared. One way to think about the residuals are as an estimate of , though, you have to be careful with that, because as we will see later on, we can decrease the residuals just by adding irrelevant regressors into the equation. Let’s talk about some aspects of residuals that will help us interpret them.

* . (Their population’s expected value is zero.)
* If an intercept is included, (Their empirical sum, hence the empirical mean also, is zero if you include an intercept.If you don’t include an intercept, this property doesn’t have to hold.)
* If a regressor variable, , is included in the model . (The generalization of this property is, if you include any regression term in linear regression, the sum of the residuals times that regression variable has to be zero.)
* Residuals are useful for investigating poor model fit. (We can create plots that highlight the aspects of poor model fit.)
* Positive residuals are above the line, negative residuals are below.
* Residuals can be thought of as the outcome () with the linear association of the predictor () removed. (A common use of residuals is to think of them as the outcome with the linear influence of the predictor having been removed. For example, if we wanted to in some subsequent model or some subsequent analysis diamond prices, but in a way that has already been adjusted for their weight, calibrating all the diamond prices to be on the same scale regardless of their weight, we would take those residuals from the model fit that has diamond prices as the outcome, and weight as the predictor.)
* One differentiates residual variation (variation after removing the predictor) from systematic variation (variation explained by the regression model). (It’s very common to take residuals and carry them forward in a later analysis where you want to think of them as the, the new outcome, having removed the predictor at that point. But, remember with linear regression, you’re only removing the linear component of the predictor. One should differentiate between residual variation, which is variation that is left over after the explanatory variable has been accounted for in a linear fashion, from systematic variation, which is variation explained by the regression model. Again, residual plots can highlight poor model fit. And, we are going to go through some residual plots.)
* Residual plots highlight poor model fit.

Let’s walk through calculating residuals in this example we’re going to use the diamond dataset.

data(diamond)  
y <- diamond$price; x <- diamond$carat; n <- length(y)  
fit <- lm(y ~ x)  
e <- resid(fit)  
yhat <- predict(fit)  
max(abs(e -(y - yhat)))

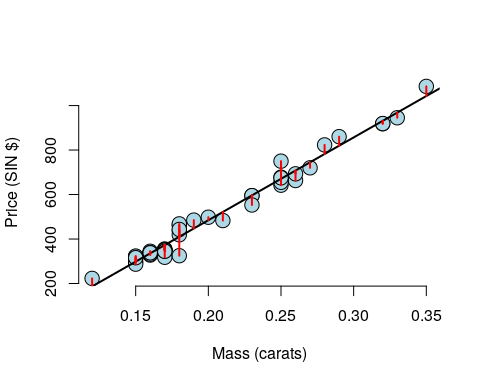
## [1] 8.242296e-13

max(abs(e - (y - coef(fit)[1] - coef(fit)[2] \* x)))

## [1] 8.242296e-13

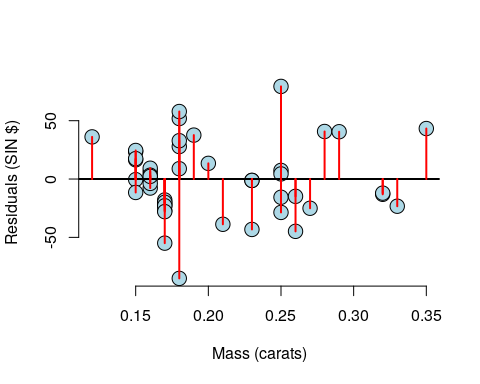
We redefine price as y and x as carat, n as the length of the number of pairs. We assign the linear regression object from lm to variable fit. To get the residuals we resid(fit), and we assign that to e. We also get the fitted values by predict(fit) and assign that to yhat. We can check that the residuals are the difference between the observed outcome and the predicted outcome. We can also check that the residuals are the difference between the observed outcome and the intercept plus the slope times the predictor. To show you that residual’s calculated via resid() functions are the same as the residuals that we calculate manually we take the absolute difference between y - yhat and e and find the one is on the scale of i.e, up to numerical precision, it’s the same thing. Then lastly, we want to show that the residuals are the difference between the observed outcome and the intercept plus the slope times the predictor, again up to numeric precision, exactly the same. To obtain the residuals, the preferred method is to use resid(). However, by demonstrating an alternative code, we aim to shed light on the underlying process of “res” and the specific computation performed by resid(). Ultimately, we would like to demonstrate that the total sum of the residuals equals zero. Technically, it’s , which is sufficiently close to zero. Additionally, the sum of the residuals multiplied by the price variable `x`` must also be zero—albeit at . Therefore, in numerical terms, both cases effectively amount to zero. These residuals represent the magnitudes of the deviations depicted by the red line in the accompanying plot.

plot(diamond$carat, diamond$price,   
 xlab = "Mass (carats)",   
 ylab = "Price (SIN $)",   
 bg = "lightblue",   
 col = "black", cex = 2, pch = 21,frame = FALSE)  
abline(fit, lwd = 2)  
for (i in 1 : n)   
 lines(c(x[i], x[i]), c(y[i], yhat[i]), col = "red" , lwd = 2)



Notice all of the blank space in the graph, making the plot kind of useless for that purpose, why don’t we plot the residuals on the vertical axis versus mass on the horizontal axis?

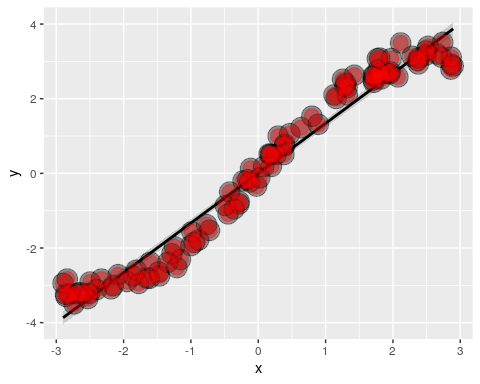
plot(x, e,   
 xlab = "Mass (carats)",   
 ylab = "Residuals (SIN $)",   
 bg = "lightblue",   
 col = "black", cex = 2, pch = 21,frame = FALSE)  
abline(h = 0, lwd = 2)  
for (i in 1 : n)   
 lines(c(x[i], x[i]), c(e[i], 0), col = "red" , lwd = 2)



Now we can see the residual variation much more clearly. One important point is: the residuals should be mostly patternless. Also, remember that if you include an intercept, residuals have to sum to zero. We can see some interesting patterns by honing in on the residual plot here. For example, we can see that there were lots of diamonds of exactly the same mass which gets lost in the scatter plot. Next, we want to go through some pathological residual plots, just to highlight what residual plots can do for us.

x = runif(100, -3, 3); y = x + sin(x) + rnorm(100, sd = .2);   
library(ggplot2)  
g = ggplot(data.frame(x = x, y = y), aes(x = x, y = y))  
g = g + geom\_smooth(method = "lm", colour = "black")  
g = g + geom\_point(size = 7, colour = "black", alpha = 0.4)  
g = g + geom\_point(size = 5, colour = "red", alpha = 0.4)  
g

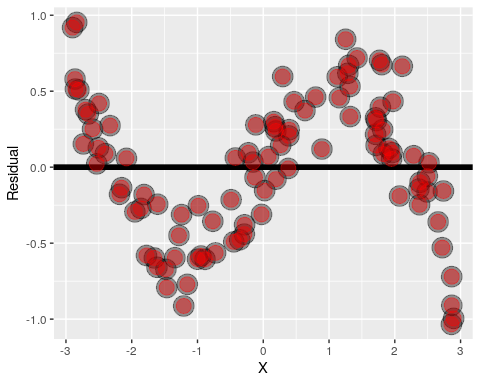
## `geom\_smooth()` using formula 'y ~ x'



Here X is just going to be uniform , y is equal to x, so it’s an identity line, but then we add another term that’s sin(x). This looks like an identity line, but kind of oscillating around it a little bit with some normal noise on top of it. Before we move on to the residual plot, let us make a comment. This model is actually not the correct model for this data and this might happen in practice. This doesn’t mean that this model is unimportant, right? There is a linear trend and the model is accounting for it, it’s just not accounting for the secondary variation in the sin term. To emphasize just because you aren’t fitting the actually correct model, that doesn’t mean the model is itself useless, in regression, having the exact right model is not always the primary goal. You can get meaningful information about trends from incorrect models.

Let’s me plot the residuals’ versus the x variable.

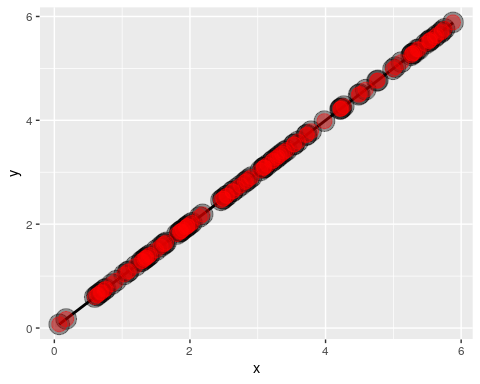
g = ggplot(data.frame(x = x, y = resid(lm(y ~ x))),   
 aes(x = x, y = y))  
g = g + geom\_hline(yintercept = 0, size = 2);   
g = g + geom\_point(size = 7, colour = "black", alpha = 0.4)  
g = g + geom\_point(size = 5, colour = "red", alpha = 0.4)  
g = g + xlab("X") + ylab("Residual")  
g



You can see that the sin term is now extremely apparent. This is what the residual plot has done highlighting the model inadequacy. Another example is the following plot, where by appearances, the plot falls perfectly on a line.

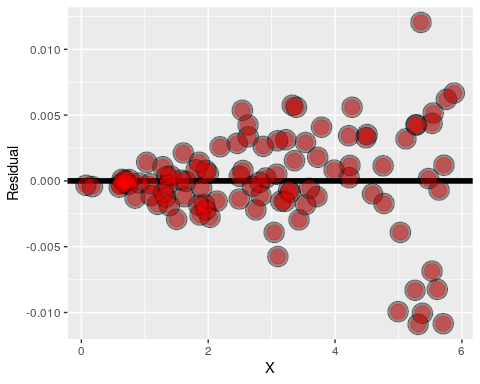
x <- runif(100, 0, 6); y <- x + rnorm(100, mean = 0, sd = .001 \* x);   
g = ggplot(data.frame(x = x, y = y), aes(x = x, y = y))  
g = g + geom\_smooth(method = "lm", colour = "black")  
g = g + geom\_point(size = 7, colour = "black", alpha = 0.4)  
g = g + geom\_point(size = 5, colour = "red", alpha = 0.4)  
g

## `geom\_smooth()` using formula 'y ~ x'



But when you highlight the residuals, it looks quite different.

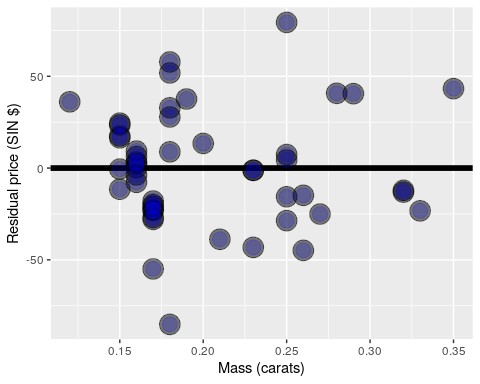
g = ggplot(data.frame(x = x, y = resid(lm(y ~ x))),   
 aes(x = x, y = y))  
g = g + geom\_hline(yintercept = 0, size = 2);   
g = g + geom\_point(size = 7, colour = "black", alpha = 0.4)  
g = g + geom\_point(size = 5, colour = "red", alpha = 0.4)  
g = g + xlab("X") + ylab("Residual")  
g



Plotting the residuals shows the trend toward greater variability as you head along the x variable. That property, where the variability increases with the x variables called heteroscedasticity. Heteroscedasticity is one of those things that residual plots are quite good at diagnosing and you couldn’t see it.

Let’s run the residual plot for the diamond data.

diamond$e <- resid(lm(price ~ carat, data = diamond))  
g = ggplot(diamond, aes(x = carat, y = e))  
g = g + xlab("Mass (carats)")  
g = g + ylab("Residual price (SIN $)")  
g = g + geom\_hline(yintercept = 0, size = 2)  
g = g + geom\_point(size = 7, colour = "black", alpha=0.5)  
g = g + geom\_point(size = 5, colour = "blue", alpha=0.2)  
g



The x-label is Mass in carats, the y-label is Residual price and just to emphasize the residuals have the same units as the ys. There doesn’t appear to be a lot of pattern in the plot, meaning it’s a pretty good fit.

Let us illustrate something about variability in a diamond dataset that will help us set the stage for defining some new properties about our regression model fit. So we create two residual vectors. The first residual vector is the one where we just fit an intercept, so the residuals are just the deviations around the average price. The second is the variation around the regression line with carats as the explanatory variable and price as the outcome. Then we create a factor variable that labels the set of residuals. The first one is labeled as a bunch of intercept only model residuals and the second set is labeled as a bunch of intercept and slope residuals.

e = c(resid(lm(price ~ 1, data = diamond)),  
 resid(lm(price ~ carat, data = diamond)))  
fit = factor(c(rep("Itc", nrow(diamond)),  
 rep("Itc, slope", nrow(diamond))))  
g = ggplot(data.frame(e = e, fit = fit), aes(y = e, x = fit, fill = fit))  
g = g + geom\_dotplot(binaxis = "y", size = 2, stackdir = "center", binwidth = 20)

## Warning: Ignoring unknown parameters: size

g = g + xlab("Fitting approach")  
g = g + ylab("Residual price")  
g



What we see on the left-hand plot with just the intercept is the variation in diamond prices around the average diamond price. What we’re seeing in the rightmost plot is displaying the variation around the regression line. So we have explained a lot of the variation with the relationship with mass. We’re going to talk about , which basically says, we can decompose the total variation, the variation explained by the regression model and the variation that’s left over after accounting for the regression model.

Residual variation is the variation around the regression line ( where ). The residuals are the vertical distances between the outcomes and the fitted regression line. If we include an intercept, the residuals have to sum to zero, which means their mean is zero. The variance of the residuals, is the average squared residual ( is ). Most people use , they instead of so that . The way to think about that is, we include the intercept the residuals have to sum to zero, that puts a constraint. If you know n minus one of them, then, you know the if you have a line term in there, if you have a co-variant in there, then, that puts a second constrain on the residuals. So, you lose two degrees of freedom. If you put another regression variable in there, you have another constraint, you lose three degrees of freedom. So in that sense you really don’t have n residuals, you have of them, because if you knew of them you could figure out the last two. And that’s why it’s one over . You can grab the residual variation out of the lm fit and assign it to a variable.

y <- diamond$price; x <- diamond$carat; n <- length(y)  
fit <- lm(y ~ x)  
summary(fit)$sigma

## [1] 31.84052

sqrt(sum(resid(fit)^2) / (n - 2))

## [1] 31.84052

If you want to grab it as an object that you can assign to something, just put dollar sign sigma. Then you can assign sigma to any other variable. The line sqrt(sum(resid(fit)^2) / (n - 2)) will result in the value and is showing what the lm function is doing behind the scenes.

Now let’s go back to the following plot where we look at the total variability in diamond prices, and compare what happens to the variability when we explain some of that variability with a regression line.

e = c(resid(lm(price ~ 1, data = diamond)),  
 resid(lm(price ~ carat, data = diamond)))  
fit = factor(c(rep("Itc", nrow(diamond)),  
 rep("Itc, slope", nrow(diamond))))  
g = ggplot(data.frame(e = e, fit = fit), aes(y = e, x = fit, fill = fit))  
g = g + geom\_dotplot(binaxis = "y", size = 2, stackdir = "center", binwidth = 20)

## Warning: Ignoring unknown parameters: size

g = g + xlab("Fitting approach")  
g = g + ylab("Residual price")  
g



The total variability is just the deviations of the data, the average squared deviation of the data around its mean. To make things easy, let’s forget about the denominator and just talk about the sum of the squared deviations. We might call the regression variability as the component of that variability that then gets explained away by the regression line. We would take the points on the regression line, the heights, which is the variability in the response and explained by the regression line, . The error variability is what’s leftover around the regression line . The interesting identity is that the total variability disregarding everything except for where they’re centered at is equal to the regression variability, that is the variability explained by the model plus the residual variability, the variability left over and not explained by the model.

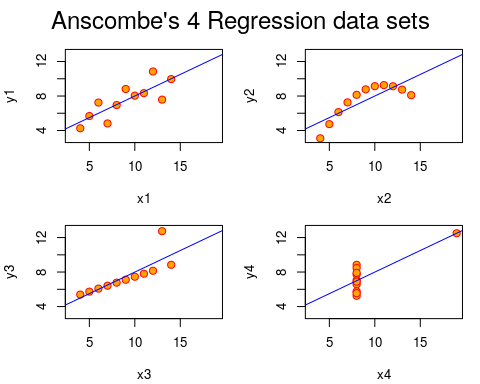
Because the residual variation and the regression model variation add up to the total variation we can define a quantity that represents the percentage of the total variation that’s represented by the model. This is called the coefficient of determination, . R squared is the percentage of the total variability that is explained by the linear relationship with the predictor

So R squared for our diamond example, is the percentage of the variation in diamond price, that is explained by the regression relationship with mass.

**Some facts about :**

* is the percentage of variation explained by the regression model.
* (because the regression variability and the error variability and the sums of the squares add up to the total sums of squares, and they are all positive)
* is the sample correlation squared. (If we define R as the sample correlation between the predictor and the outcome, then R squared is literally that sample correlation R, squared.)
* can be a misleading summary of model fit. (For example, if you have somewhat noisy data and delete a lot of the points in the middle you can get a much higher R squared. Or if you just add arbitrary regression variables into a linear model fit, you increase R squared and decrease mean squared error)
  + Deleting data can inflate .
  + (For later.) Adding terms to a regression model always increases .

Anscombe created a particularly stark example of a bunch of data sets with an equivalent R squared, equivalent mean, and variances in the x’s and the y’s, and identical regression relationships, but when you look at the scatter plots, you can see that the fit has very different meanings in each of the cases.

 The first is a nice regression line, exactly sort of along the lines of what we think of, when we think of just a slightly noisy x,y relationship. In the second one clearly there’s a missing term in order to address some of the curvature in the data. In the third one, there’s an outlier. Finally, in the fourth one, all the data stacked up at one particular location and there’s one point way out at the end. So you could imagine getting this if you had the first example and you deleted a lot of the points in the middle. In all these cases you have an equivalent R squared. But the summary to the single number certainly has thrown out a lot of the important information that you get from a simple scatter plot.

## 2.3 Inference in regression

Inference is the process of drawing conclusions about a population using a sample. In statistical inference, we must account for the uncertainty in our estimates in a principled way. Hypothesis tests and confidence intervals are among the most common forms of statistical inference.

These statements apply generally, and, of course, to the regression setting that we’ve been studying. In the next few lectures, we’ll cover inference in regression where we make some Gaussian assumptions about the errors.

## 2.4 For the project

## 2.5 Practical R Exercises in swirl

## 2.6 Week 2 Quiz

# 3 Week 3

## 3.1 Multivariable regression

## 3.2 Multivariable regression tips and tricks

## 3.3 Adjustment

## 3.4 Residuals again

## 3.5 Model selection

## 3.6 Practical R Exercises in swirl

## 3.7 Week 3 Quiz

## 3.8 (OPTIONAL) Practice exercise in regression modeling

# 4 Week 4

## 4.1 GLM

## 4.2 Logistic Regression

## 4.3 Poisson Regression

## 4.4 Hodgepodge

## 4.5 Practical R Exercises in swirl

## 4.6 Week 4 Quiz

## 4.7 Course Project

# About the Authors

These credits are based on our [course contributors table guidelines](https://www.ottrproject.org/more_features.html#giving-credits-to-contributors).

| Credits | Names |
| --- | --- |
| **Pedagogy** |  |
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| Content Editor(s)/Reviewer(s) | Checked your content |
| Content Director(s) | Helped guide the content direction |
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| Acknowledgments | Gave small assistance to content but not to the level of consulting |
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| Content Publisher(s) | Helped with publishing platform |
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| Course Publishing Engineer(s) | Helped with the code for the technical aspects related to the specific course generation |
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| **Art and Design** |  |
| Illustrator(s) | Created graphics for the course |
| Figure Artist(s) | Created figures/plots for course |
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| Videography Editor(s) | Edited film |
| Audiographer(s) | Recorded audio |
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## ─ Session info ───────────────────────────────────────────────────────────────  
## setting value   
## version R version 4.0.2 (2020-06-22)  
## os Ubuntu 20.04.5 LTS   
## system x86\_64, linux-gnu   
## ui X11   
## language (EN)   
## collate en\_US.UTF-8   
## ctype en\_US.UTF-8   
## tz Etc/UTC   
## date 2024-01-21   
##   
## ─ Packages ───────────────────────────────────────────────────────────────────  
## package \* version date lib source   
## assertthat 0.2.1 2019-03-21 [1] RSPM (R 4.0.5)   
## bookdown 0.24 2023-03-28 [1] Github (rstudio/bookdown@88bc4ea)   
## cachem 1.0.7 2023-02-24 [1] CRAN (R 4.0.2)   
## callr 3.5.0 2020-10-08 [1] RSPM (R 4.0.2)   
## cli 3.6.1 2023-03-23 [1] CRAN (R 4.0.2)   
## crayon 1.3.4 2017-09-16 [1] RSPM (R 4.0.0)   
## desc 1.2.0 2018-05-01 [1] RSPM (R 4.0.3)   
## devtools 2.3.2 2020-09-18 [1] RSPM (R 4.0.3)   
## digest 0.6.25 2020-02-23 [1] RSPM (R 4.0.0)   
## ellipsis 0.3.1 2020-05-15 [1] RSPM (R 4.0.3)   
## evaluate 0.20 2023-01-17 [1] CRAN (R 4.0.2)   
## fastmap 1.1.1 2023-02-24 [1] CRAN (R 4.0.2)   
## fs 1.5.0 2020-07-31 [1] RSPM (R 4.0.3)   
## glue 1.4.2 2020-08-27 [1] RSPM (R 4.0.5)   
## htmltools 0.5.5 2023-03-23 [1] CRAN (R 4.0.2)   
## knitr 1.33 2023-03-28 [1] Github (yihui/knitr@a1052d1)   
## magrittr 2.0.3 2022-03-30 [1] CRAN (R 4.0.2)   
## memoise 2.0.1 2021-11-26 [1] CRAN (R 4.0.2)   
## pkgbuild 1.1.0 2020-07-13 [1] RSPM (R 4.0.2)   
## pkgload 1.1.0 2020-05-29 [1] RSPM (R 4.0.3)   
## prettyunits 1.1.1 2020-01-24 [1] RSPM (R 4.0.3)   
## processx 3.4.4 2020-09-03 [1] RSPM (R 4.0.2)   
## ps 1.4.0 2020-10-07 [1] RSPM (R 4.0.2)   
## R6 2.4.1 2019-11-12 [1] RSPM (R 4.0.0)   
## remotes 2.2.0 2020-07-21 [1] RSPM (R 4.0.3)   
## rlang 1.1.0 2023-03-14 [1] CRAN (R 4.0.2)   
## rmarkdown 2.10 2023-03-28 [1] Github (rstudio/rmarkdown@02d3c25)  
## rprojroot 2.0.3 2022-04-02 [1] CRAN (R 4.0.2)   
## sessioninfo 1.1.1 2018-11-05 [1] RSPM (R 4.0.3)   
## stringi 1.5.3 2020-09-09 [1] RSPM (R 4.0.3)   
## stringr 1.4.0 2019-02-10 [1] RSPM (R 4.0.3)   
## testthat 3.0.1 2023-03-28 [1] Github (R-lib/testthat@e99155a)   
## usethis 1.6.3 2020-09-17 [1] RSPM (R 4.0.2)   
## withr 2.3.0 2020-09-22 [1] RSPM (R 4.0.2)   
## xfun 0.26 2023-03-28 [1] Github (yihui/xfun@74c2a66)   
## yaml 2.2.1 2020-02-01 [1] RSPM (R 4.0.3)   
##   
## [1] /usr/local/lib/R/site-library  
## [2] /usr/local/lib/R/library

# 5 References