Predictive Assignment

Parameter Estimation

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Ques 1: Let $(X_1, X_2, \ldots)$ be a random sample of size n taken from a Normal Population with parameters: mean = θ_1 and variance = θ_2. Find the Maximum Likelihood Estimates of these two parameters.

Sol: → Probability Density Function (PDF) of Normal Distribution :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

ATQ :
$$\mu = \theta_1$$
$$\sigma^2 = \theta_2$$

$$\therefore \quad f(x) = \frac{1}{\sqrt{\theta_2}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\theta_1}{\sqrt{\theta_2}}\right)^2}$$

→ Given: $X_1, X_2, \ldots X_n$ are random sample values from the Normal Distribution hence, the likelihood function is:

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\left(\frac{x_i-\theta_1}{\sqrt{\theta_2}}\right)^2}$$

→ Taking Log on both sides (Natural Logarithm)

$$\ln(L) = \ln\left(\sqrt{2\pi\theta_2}^{\,-n} \prod_{i=1}^{n} e^{-\frac{1}{2}\left(\frac{x_i-\theta_1}{\sqrt{\theta_2}}\right)^2}\right)$$

(2)

$$\ln(L) = -\frac{n}{2}\ln(2\pi\theta_2) + \left(\frac{-1}{2\theta_2}\right)\sum_{i=1}^{n}(x_i-\theta_1)^2 \quad \text{—(1)}$$

→ Differentiating the above equation (1) with respect to θ_1:

$$\frac{1}{L}\frac{\partial L}{\partial \theta_1} = \frac{-1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1)(-1)$$

$$\frac{1}{L}\frac{\partial L}{\partial \theta_1} = \frac{1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1)$$

→ equating $\frac{\partial L}{\partial \theta_1} = 0$

$$L \cdot \frac{1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1) = 0$$

equating to 0, will give us the estimator that is obtained using Maximum Likelihood Estimation.

either $L = 0$ or $\frac{1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1) = 0$

$L = 0$ can't be possible

$$\therefore \frac{1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1) = 0$$

$$\sum_{i=1}^{n} 2x_i = 2\sum_{i=1}^{n}\theta_1$$

$$n\theta_1 = \sum_{i=1}^{n} x_i$$

$$\theta_1 = \left(\sum_{i=1}^{n} x_i\right)/n$$

$$\boxed{\theta_1 = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

θ_1 = Sample Mean

» Differentiating equation (1) wrt Θ_2:

$$\frac{1}{L}\frac{\partial L}{\partial \Theta_2} = \frac{-n}{2}\frac{2\pi}{2\pi\Theta_2} + \sum_{i=1}^{n}(x_i-\Theta_1)^2 \frac{1}{(2\Theta_2^2)}$$

» Equating to 0:

$\frac{1}{L}\frac{\partial L}{\partial \Theta_2}$ ⇒ L can't be 0

$$\therefore \frac{-n}{2}\frac{2\pi}{2\pi\Theta_2} + \sum_{i=1}^{n}(x_i-\Theta_1)^2 \frac{1}{(2\Theta_2^2)} = 0$$

$$\frac{-n}{2\Theta_2} + \sum_{i=1}^{n}(x_i-\Theta_1)^2 \frac{1}{2\Theta_2^2} = 0$$

$$\sum_{i=1}^{n}(x_i-\Theta_1)^2 = n\Theta_2$$

$$\boxed{\Theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\Theta_1)^2}$$

Θ_2 = Sample Variance

Ques2: Let $X_1, X_2, \ldots, X_n$ be a random sample from $B(m, \Theta)$ distribution, where $\Theta \in \Theta = (0,1)$ is unknown and 'm' is a known positive integer. Compute value of Θ using the M.L.E.

Sol: Probability Mass Function (PMF) of Binomial Distribution:

$$P(x=K) = {}^{m}C_K\, \Theta^K (1-\Theta)^{m-k}$$

$$\left[\, P(x=X) = {}^{n}C_x\, p^x\, \cancel{q}\, (1-p)^{n-x} \,\right]$$

» ATQ: Let $x_1, x_2 \ldots x_n$ be random sample from $B(m,\Theta)$ distribution for a x_i, it represents number of successes in i^{th} trial.

$$L(\theta) = \prod_{i=1}^{n} {}^{m}C_{x_i} \; \theta^{x_i} (1-\theta)^{m-x_i}$$

⇒ Taking Natural log on both sides:

$$\ln L = \ln \left(\prod_{i=1}^{n} {}^{m}C_{x_i} \theta^{x_i} (1-\theta)^{m-x_i} \right)$$

$$\ln L = \sum_{i=1}^{n} \left[\ln {}^{m}C_{x_i} + x_i \log\theta + (m-x_i) \ln(1-\theta) \right]$$

⇒ Differentiating wrt θ:

$$\frac{1}{L}\frac{dL}{d\theta} = \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\sum_{i=1}^{n}(m-x_i)$$

⇒ Equating to 0:

$$\frac{dL}{d\theta} = 0$$

$$L\left(\frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\sum_{i=1}^{n} m-x_i\right) = 0$$

L can't be zero,

$$\therefore \frac{1}{\theta}\sum_{i=1}^{n} x_i = \frac{1}{1-\theta}\sum_{i=1}^{n} m-x_i$$

$$(1-\theta)\sum_{i=1}^{n} x_i = \theta n m - \theta\sum_{i=1}^{n} x_i$$

$\theta = \frac{\Sigma x_i}{nm}$ where i goes from 1 to n

$$\theta = \frac{1}{m}\left(\frac{1}{n}\Sigma x_i\right)$$

θ = Sample mean $/m$