10/01/2022

Unit - 2 INFINITE SERIES OF REAL NUMBER

Sequence :-

A sequence of real number is a function whose domain in the set N of positive integers and whose range is contained in R of real numbers

Denoted by $\langle x_n \rangle = (\frac{1}{n})$

$= \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} - - , \frac{1}{n}, . .)$

$1, 1/2, 1/3, 1/4, - -$ } terms of sequences.

$\Rightarrow \langle x_n \rangle = \langle (-1)^n \rangle = -1, 1, -1, - - ,$

$\Rightarrow \langle x_n \rangle = \langle (-1)^n n \rangle_{n \geq 1}^{\infty} = -1, 2, -3, +4, -5, 6, - -$

$\Rightarrow \langle x_n \rangle$ where $x_n = 3 \; \forall n$

$\Rightarrow$ fibonacci ~~series~~ sequence

$f_1 = 1, \; f_2 = 1, \; f_n = f_{n-1} + f_{n-2}; \quad n \geq 3$

$1, 1, 2, 3, 5, 8, 13 . - -$

Convergence of sequences :-

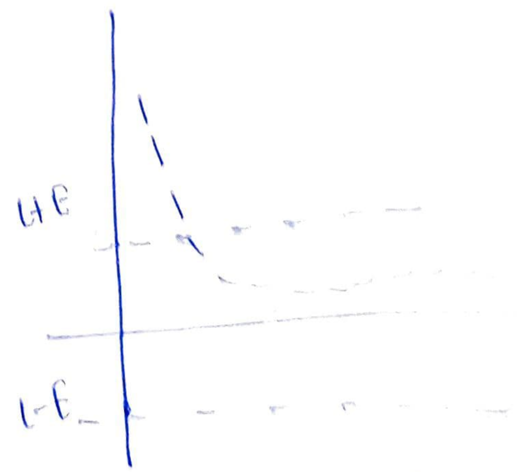
let $\langle x_n \rangle$ be a sequence of real numbers sequence $\langle x_n \rangle$ converges to the real number

$L (\lim_{n \to \infty} x_n = L)$

if for every $\varepsilon > 0$,

ε is a rational number, m (dependent on ε)

$\langle x_n \rangle = \frac{1}{n}$



sequences is converging to L.

Infinite series

let $\langle a_n \rangle$ be an infinite sequence of real numbers, then an expression of the form $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \cdots$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n!} = x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Sequences of Partial Sums :-

Suppose Σa_n is an infinite series. let S_n denote sum of first n terms of the series.

We define sequences $\langle S_n \rangle$ as

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

Here, S_n is called the n'th partial sum of the series & sequence $\langle S_n \rangle$ is called the sequence of partial sums. converges to a limit s, we say series converges as its sum is s.

Converges of an Infinite Series :-

A series Σa_n is said to be convergent if the sequences of partial sums $\langle S_n \rangle$ is convergent.

if $\lim_{n \to \infty} S_n = S$,

Then S is called the sum of series ΣS_n.

Note :- $\Rightarrow$ If $\langle S_n \rangle$ diverges, the series $\sum_{a=1}^{\infty} a_n$ is said to diverge.

series has no sum.

$\Rightarrow$ A divergent

Eg. s_{kv}

Eg. show that the series $\Sigma(-1)^{n-1}$ diverges.

Sol^n:- let $\{s_n\}$ be sequence of partial sums for given series

then, $s_1 = 1$

$s_2 = a_1 + a_2 = 1-1 = 0$

$s_3 = 1-1+1 = 1$

$s_4 = 1-1+1-1 = 0$

In general,

$$s_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus, sequence of partial sums is

$1, 0, 1, 0, \ldots$

since this is divergent sequence, the given series diverges.

Geometric Series

An infinite series in which each term after first term is obtained by multiplying proceeding term by a fixed constant

let first term $= a$, fixed constant $= r$.

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots + ar^n \quad (a \neq 0)$$

$r \rightarrow$ ratio of the series.

Eg. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$

$(a=1, r=1/3)$

$\rightarrow \displaystyle\sum_{n=1}^{\infty} 3^n = 3 + 3^2 + 3^3 + \ldots$

$(a=3, r=3)$

Theorem : Converges of a Geometric series.

A geometric series $\sum_{n=0}^{\infty} ar^n$ converges $(a \neq 0)$ if $|r| < 1$ & diverges if $|r| \geq 1$

$$\begin{cases} \text{converges} : |r| < 1 \ (a \neq 0) \\ \text{diverges} : |r| \geq 1 \ (a \neq 0) \end{cases}$$

Proof :-

Case 1 :

if $(r = 1)$

$$\sum_{n=1}^{\infty} ar^n = a + a + a + \cdots$$

The n^{th} partial sum is

$$S_n = \underbrace{a + a + \cdots + a}_{n \text{ terms}}$$

$$\boxed{S_n = na}$$

$$\lim_{n \to \infty} S_n = \pm \infty$$ depending on sign of a.

$\therefore$ If $r = 1$, the series diverges.

Case 2 : If $r = -1$,

then, series is $a - a + a - a + a - \cdots$

So, n^{th} partial sum is

$$S_n = \begin{cases} a, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

since sequence has no limit the series diverges.

Case 3 : $r \neq \pm 1$

The n^{th} partial sum of geometric series is

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad \text{--- (1)}$$

Multiply by r on both side

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad \text{--- (2)}$$

① − ②

$(1-r) S_n = a - ar^n$

$$S_n = \frac{a - ar^n}{1-r}$$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

i.) if $|r| < 1$, $r^n \to 0$ as $n \to \infty$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left[\frac{a}{1-r}(1-r^n)\right)$$

$$= \frac{a}{1-r} \lim_{n \to \infty}(1-r^n)$$

$$= \frac{a}{1-r}$$

series converges & its sum is $\frac{a}{1-r}$

ii.) if $|r| > 1$, either $r > 1$ or $r < -1$

$\Rightarrow$ if $r > 1$, r^n would increase without bounds as $n \to \infty$

$\Rightarrow$ if $r < -1$, r^n would oscillates b/w +ve & −ve values that ~~generates~~ grows.

$\therefore$ sequence of partial sum for both $r > 1$ & $r < -1$ diverges

$\Rightarrow \sum_{a=1}^{\infty} ar^n$ diverges.