Elements of Analysis

SONAM
BHASKARACHARYA COLLEGE
UNIVERSITY OF DELHI

January 2022

Sets

- ***Sets***: A set is a well defined collection of distinct objects.

- The objects in a set are called the elements or members of the set.

Examples

1. $A = \{1, 2, 3, 4, 5\}$
2. N: Set of all natural numbers
3. Z: Set of all integers
4. Q: Set of all rational numbers
5. R: Set of all real numbers

Which of the following is a set?

1. $\{2,3,4,\ldots\}$

2. $\{1,2,3,\ldots.\}$

3. $\{\frac{p}{q}: q \neq 0, p, q \in Z\}$

4. $\{\frac{p}{q}: q \neq 0, p, q \in Z, (p,q)=1\}$

- ***Finite set***: A set which contains a finite number of elements is called a finite set.

  Example: $S = \{ x \mid x \in N \text{ and } 70 > x > 50 \}$

- ***Infinite set***: A set which contains infinite number of elements is called an infinite set.

  Example: $S = \{ x \mid x \in N \text{ and } x > 10 \}$

- ***Empty Set or Null Set***: An empty set contains no elements. It is denoted by φ.

  Example: $S = \{ x \mid x \in N \text{ and } 7 < x < 8 \} = \emptyset$

- ***Singleton Set or unit set***: A set which contains only one element. A singleton set is denoted by $\{ s \}$.

  Example: $S = \{ x \mid x \in N, 7 < x < 9 \} = \{ 8 \}$

- ***Equal Set***: If two sets contain the same elements they are said to be equal.

  Example : If $A = \{ 1, 2, 6 \}$ and $B = \{ 6, 1, 2 \}$, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

- ***Equivalent Set***: If the cardinalities of two sets are same, they are called equivalent sets.

  Example: If $A = \{ 1, 2, 6 \}$ and $B = \{ 16, 17, 22 \}$, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A| = |B| = 3$

- ***Subset***: A set X is a subset of set Y (Written as $X \subseteq Y$) if every element of X is an element of set Y.

  Example 1: Let X = { 1, 2, 3, 4, 5, 6 } and Y = { 1, 2 }. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

  Example 2: Let X = { 1, 2, 3 } and Y = { 1, 2, 3 }. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

- ***Superset***: A set X is the superset of Y, if all the elements of set Y are the elements of set X. The superset relationship is denoted as $X \supset Y$

  Example: Let set X = {1, 2, 3, 4} and set Y = {1, 3, 4},. Here, set X is the superset of Y.

- ***Proper Subset***: A Set X is a proper subset of set Y (Written as $X \subset Y$) if every element of X is an element of set Y and $|X| < |Y|$.

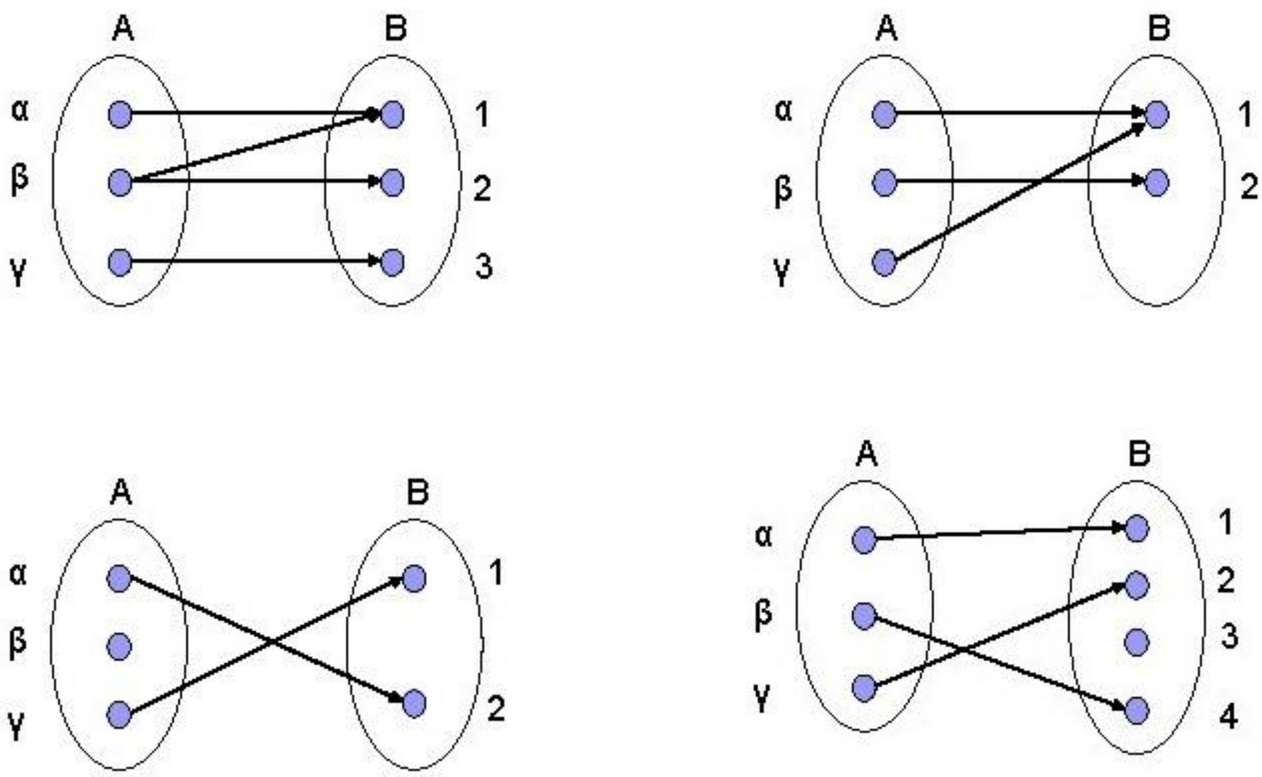
  Example: Let, $X = \{ 1, 2, 3, 4, 5, 6 \}$ and $Y = \{ 1, 2 \}$. Here set $Y \subset X$ since all elements in X are contained in X too and X has at least one element is more than set Y.

- ***Universal Set***: A universal set (usually denoted by U) is a set which has elements of all the related sets, without any repetition of elements.

  Example: A and B are two sets, such as $A = \{1,2,3\}$ and $B = \{1,a,b,c\}$, then the universal set associated with these two sets is given by $U = \{1,2,3,a,b,c\}$.

Function or Mapping:

- A function f : A ⟶ B is an assignment of exactly one element of a set B to each element of set A.
- Which of the following is a function?



- ***Injection / One-one function*:** A function f: A $\longrightarrow$ B is injective or one-to-one function if for every b ∈ B, there exists at most one a ∈ A such that f(s) = t.

  This means a function **f** is injective if $a_1 \neq a_2$ implies

  f(a1) ≠ f(a2).

  Example:

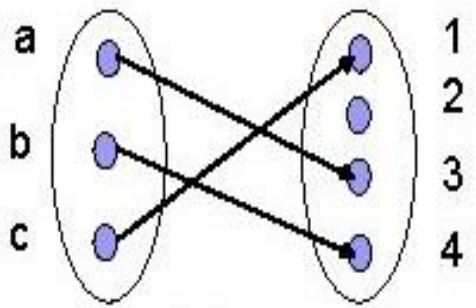
  1. f: N $\longrightarrow$ N, f(x) = 5x is injection.
  2. f: N $\longrightarrow$ N, f(x) = x^2 is injection.

- ***Surjection / Onto function*** : A function f: A $\longrightarrow$ B is surjection (onto) if the image of f equals its range. Equivalently, for every b ∈ B, there exists some a ∈ A such that f(a) = b.

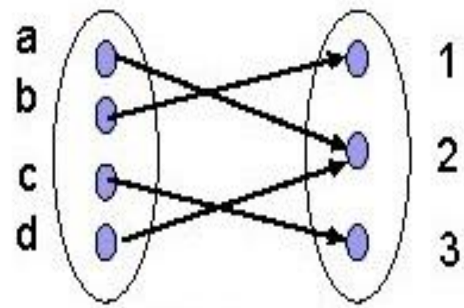
  Example: f : N $\longrightarrow$ N, f(x) = x + 2 is not surjection.

* ***Bijection***: A function f: A ⟶ B is bijection if and only if f is both injective and surjection.

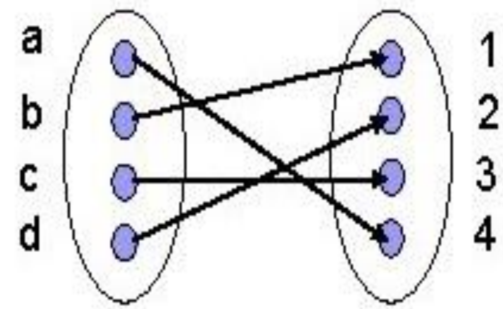
Examples



1-1 , not onto



not 1-1 , onto



1-1 and onto

Countable and Uncountable Sets

- A set S is said to be **denumerable** (or countably infinite) if there exists a bijection of N onto S.
- A set S is said to be **countable** if it is either finite or denumerable.
- A set S is said to be **uncountable** if it is not countable.

Examples:

- The set $E : \{2n : n \in N\}$ of even natural numbers is denumerable, since the mapping $f : N \to E$ defined by $f(n) = 2n$ for $n \in N$ is a bijection of N onto E.

Which is countable and uncountable set in the following?

1. The set of all real numbers in the interval $(0, 1)$.
2. The set of all rational numbers in the interval $(0, 1)$.
3. The set of all points in the plane with rational coordinates.
4. The set of all infinite sequences of integers.
5. The set of all functions $f : \{0, 1\} \rightarrow N$.
6. The set of all functions $f : N \rightarrow \{0, 1\}$.
7. The set of all "words" (defined as finite strings of letters in the alphabet).

Absolute Value and the Real Line

- ***Definition***: The **absolute value** of a real number x, denoted by | x | , is defined by

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

Example: | 3 | = 3 and | -8 | = 8

Properties

- $|x| = \max.\{x, -x\}$
- $|x|^2 = x^2$
- $|x+y| \leq |x| + |y|$
- $|x-y| \geq ||x| - |y||$

Bounded Sets

- ***Definition***: Let S be a nonempty subset of R.

a) The set S is said to be **bounded above** if there exists a number u ∈ R such that s ≤ u for all s ∈ S. Each such number u is called an upper bound of S.

b) The set S is said to be **bounded below** if there exists a number w ∈ R such that w ≤ s for all s ∈ S. Each such number w is called a lower bound of S.

c) A set is said to be **bounded** if it is both bounded above and bounded below. A set is said to be **unbounded** if it is not bounded.

Find the lower bound and upper bound, if they exist, of the following sets:

1. The set of natural numbers
2. $\{2,3,4\}$
3. $\{1,1/2,1/4,\ldots\}$
4. Open interval $(1,4)$
5. $\{-2,-1,0,1,\ldots\}$
6. The set of rational numbers
7. Interval $[3,\infty)$

Supremum and Infimum

- ***Definition:*** Let S be a nonempty subset of R.

A. If S is **bounded above**, then a number u is said to be a supremum (or a least upper bound) of S if it satisfies the conditions:

1. u is an upper bound of S, and
2. if v is any upper bound of S, then $u \leq v$.

B. If S is **bounded below**, then a number w is said to be an infimum (or a greatest lower bound) of S if it satisfies the conditions:

1. w is a lower bound of S, and
2. if t is any lower bound of S, then t w.

Thank you