11/01/2022

Q) Show that the series

$1+r+r^2+\cdots+r^n+\cdots \quad (r>0)$

converges if $r<1$ & diverges if $(r\geq 1)$.

Sol^n:- The n^{th} partial sum of given series.

$$S_n = 1+r+r^2+\cdots+r^{n-1}$$

$$\boxed{S_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}}$$

Case 1: $r<1$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{1-r} - \frac{r^n}{1-r}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{1-r}\right) - \lim_{n\to\infty}\left(\frac{r^n}{1-r}\right)$$

$$= \frac{1}{1-r} - \frac{1}{1-r}\lim_{n\to 0} r^n$$

$$= \frac{1}{1-r}$$

$\therefore \langle S_n\rangle$ is a convergent sequence

$\therefore$ given series is convergent.

Case 2: $r=1$

$$S_n = 1+1+1+\cdots+1$$

$$\boxed{S_n = n}$$

$\therefore \langle S_n\rangle = \langle 1, 2, 3, 4, \ldots, n, \ldots\rangle$ diverges.

$\therefore$ given series diverges.

Case 3: $r>1$

$$S_n = 1+r+\cdots+r^{n-1}$$

$$S_n > n$$

$\therefore \lim_{n\to\infty} S_n = +\infty$ & this given series diverges.

eg. 1) $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{3^2} + - - + \frac{1}{3^n} + - - -$

Sol: This is geometric series with $a = \frac{1}{3}$, $r = \frac{1}{3} < 1$

since, $|r| = \frac{1}{3} < 1$.

Then given series converges.

sum of series $= \frac{a}{1-r} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = 1/2$

2) $\sum_{n=0}^{\infty} \frac{(-2)^n}{5^{2n+1}} = \sum_{n=0}^{\infty} \frac{1}{5}\left(\frac{-2}{25}\right)^n$

$= \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{-2}{25}\right)^n$

∵ This is geometric series with $a = 1/5$, $|r| = |-2/25| < 1$

∴ Series converges & sum of series $= \frac{1/5}{1-(-2/25)}$

$= \frac{1/5}{27/25} = \frac{1}{27}$

Q.) Determine the values of x for which the series converges and find the sum of series for the value of x.

a) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

$= 1 - x^2 + x^4 - x^6 + - -$

∵ This is geometric series with $a = 1$, $r = -x^2$

∴ series converges if $|r| < 1$

i.e. $|-x^2| < 1$

$\Rightarrow -1 < x^2 < 1$

$\Rightarrow -1 < x < 1$

and for $-1 < x < 1$, sum of series $= \frac{a}{1-r}$

$= \frac{a}{1-(-x^2)} = \frac{1}{1+x^2}$

b.) $\sum_{n=0}^{\infty} \frac{3(-1)^n}{2^n} x^n$

$= 3 - \frac{3x}{2} + \frac{3}{4}x^2 - \frac{3}{8}x^3 + \cdots$

Here, $a = 3$, $r = -x/2$

series converges if $|r| < 1$

i.e. $|-x/2| < 1$

$\Rightarrow |x| < 2$

& sum of series $= \frac{a}{1-r} = \frac{3}{1+x/2} = \frac{6}{2+x}$

Note :- $\boxed{\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}}$

$\left\{ \frac{1}{k} - \frac{1}{k+1} = \frac{\cancel{k}+1-\cancel{k}}{k(k+1)} = \frac{1}{k(k+1)} \right\}$

Telescopic series

Series of the form

$(b_1 - b_2) + (b_2 - b_3) + \cdots$

Here, n^{th} partial sum of series

$\boxed{S_n = b_1 - b_{n+1}}$

Q.) for the following telescoping series, show that series is converge.

a.) $\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)}$

$= 8\left(\frac{1}{2} - \frac{1}{3}\right) + 8\left(\frac{1}{3} - \frac{1}{4}\right) + 8\left(\frac{1}{4} - \frac{1}{5}\right) \cdots + 8\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$= 8\left(\frac{1}{2}\right) - 8\left(\frac{1}{n+2}\right)$

$= 4 - \frac{8}{n+2}$

limit of sequence of partial sum

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(4 - \frac{8}{n+2}\right) = 4$

∴ series converges & its sum is 4.

Q:) Show that series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$ is convergent $\left\{\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\right.$

$\Rightarrow$ sequence of partial sums of given series is

$$S_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)$$

$$=1-\lim_{n\to\infty}\left(\frac{1}{n+1}\right)$$

$$=1$$

Hence, series converges and its sum is 1.

Try it Yourself $\boxed{\text{H.W}}$ $\underline{\text{WS}}$!!!

Q:) For the given telescoping series, find a formula for the n terms S_n of the sequence of partial sums $\langle S_n \rangle$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the sum of series.

a:) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$ b.) $\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$

1.) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

$\frac{1}{2n-1} - \frac{1}{2n+1}$

$\cancel{2}n+1 - \cancel{2}n+1$

2

$\sum_{n=1}^{\infty} \frac{1}{2} \times \frac{2}{(2n-1)(2n+1)}$

sum of n^{th} partial sum,

$Sn = \sum_{n=1}^{\infty} \frac{1}{2} \times \frac{2}{(2n-1)(2n+1)}$

$= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$

$= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

$Sn = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{\cancel{2}n+\cancel{1}-1}{\cancel{2}(2n+1)} = \frac{n}{2n+1}$

~~$\lim_{n\to\infty} Sn = \frac{1}{2} \lim_{n\to\infty} \left(1 - \frac{1}{2n+2}\right)$~~

$\lim_{n\to\infty} Sn = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \left(\frac{1}{2+1/n}\right) = 1/2$

$\boxed{\text{sum of } = 1/2}$

2) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

n^{th} partial sum,

$Sn = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$Sn = \frac{1}{2} - \frac{1}{n+2} = \frac{n+2-2}{2(n+2)} = \frac{n}{2(n+2)}$

$\lim_{n\to\infty} Sn = \lim_{n\to\infty} \frac{n}{2(n+2)} = \frac{1}{2} \lim_{n\to\infty} \frac{1}{1+2/n} = 1/2$

$\therefore$ $\boxed{\text{sum of series} = 1/2}$

Theorem

A necessary condition for convergence.

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\lim_{n\to\infty} a_n = 0$$

Proof :- let the series $\sum a_n$ be convergent, then sequence of partial sums $\langle S_n \rangle$ is convergent where $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ denotes the n^{th} partial sum of the series.

let $\lim_{n\to\infty} S_n = S$.

$\lim_{n\to\infty} S_{n-1} = S$.

Now, $a_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0$$

Note :- Converse may not be true.

eg. let $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n}$

Here, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

n^{th} term test for Divergence of series

If $\lim_{n\to\infty} a \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

so, a/c to $\boxed{n^{th}\text{ term test,}}$

if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: n^{th} term test can't be used to prove convergence of series.

Q.> Using n^{th} term test for divergence, determine whether following series diverges.

1.> $\sum_{n=1}^{\infty} \frac{n}{3n+5}$

$\Rightarrow \lim_{n\to\infty} \frac{n}{3n+5} = \lim_{n\to\infty} \frac{1}{3+5/n} = \frac{1}{3} \neq 0$

By n^{th} term test, since $\lim_{n\to\infty} a_n \neq 0$,

the given series diverges.

2.> $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty}\left(\frac{1/n}{1+1/n^2}\right) = 0$

~~By~~, since, $\lim_{n\to\infty} a_n = 0$, by n^{th} term test for diverge is inconclusive.

Try It Yourself Again !!??!

a.> $\sum_{n=1}^{\infty} \frac{e^n}{n}$ b.> $\sum_{n=1}^{\infty} \sqrt[n]{3}$

a.> $\sum_{n=1}^{\infty} \frac{e^n}{n}$ $\{\because \frac{\infty}{\infty}\text{ form}\}$

$\lim_{n\to\infty}\left(\frac{e^n}{n}\right) = \lim_{n\to\infty}\left(\frac{e^n}{1}\right)$ {by L Hospital}

$= \infty$

result is inconclusive

b.> $\sum_{n=1}^{\infty} \sqrt[n]{3}$

$\lim_{n\to\infty} \sqrt[n]{3} = \lim_{n\to\infty} (3)^{1/n}$

$= (3)^0 = 1$

$\lim_{n\to\infty} \neq 0$, it diverges.