31/01/2022

ALTERNATING SERIES

⇒ series that alternates b/w both +ve and -ve terms.

eg. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} \cdots$

Leibnitz rule :-

An alternating series ~~$\sum (-1)^{n+1}$~~ $\sum (-1)^{n+1} a_n$ where $a_n > 0$

converges if the following two condition satisfies:-

i) $\lim_{n \to \infty} a_n = 0$

ii) $\langle a_n \rangle$ is a decreasing sequence i.e. $a_{n+1} \leq a_n \quad \forall n$

Q.) Determine the following series converges or diverges.

i) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln(n)}{n}$

⇒ Given series is an alternating series with $a_n = \frac{\ln n}{n}$

i) $\lim_{n \to \infty} \frac{\ln n}{n}$ $\quad \left(\frac{\infty}{\infty} \text{ form}\right)$

$= \lim_{n \to \infty} \frac{1/n}{1}$ (by L'Hospital Rule)

$= 0$

ii) let $f(n) = \frac{\ln n}{n}$

$f'(n) = \frac{1 - \ln n}{n^2}$

i.e. $f'(n) < 0$ for $n > 3$

so, $f'(n)$ is decreasing function for $n \geq 3$.

$\langle a_n \rangle$ is decreasing function for $n \geq 3$

∴ By alternating series test

$\sum (-1)^{n+1} \frac{\ln n}{n}$ converges.

ii) $\sum (-1)^{n+1} e^{-n}$

This is alternating series with $a_n = e^{-n}$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{e^n} = 0$

also, $f'(n) = \frac{d}{dx}(e^{-x}) = \frac{-1}{e^x} < 0 \quad \forall\ n > 1.$

$\forall\ x$

$\therefore$ $f'(n)$ is decreasing function

by libnitz form, given series converges.

iii) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+3}$

This series alternating series with $a_n = \frac{n^2-1}{n^2+3}$

$\lim_{n \to \infty} \frac{n^2-1}{n^2+3} = \lim_{n \to \infty} \left(\frac{1-1/n^2}{1+3/n^2}\right) = 1 \neq 0$

$\therefore$ by n^{th} term test for divergence the given series diverges.

Try it yourself

Q.)

i) $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{3n-1}$

ii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+2)}$

iii) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

Absolute and conditional convergence

→ consider a series with both +ve & -ve terms but not alternating series.

eg. $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$

Absolute convergence:-

A series $\sum a_n$ is said to be absolute convergence if the series of absolute values $\sum |a_n|$ is convergent

Conditionally convergent:-

A series $\sum a_n$ is convergent but not absolutely convergent

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

This is alternating series

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$f(n) = \frac{1}{n}$

$f'(n) = \frac{-1}{n^2} < 0 \quad \forall\ n > 0$

$f(n)$ is decreasing sequence.

∴ By Leibnitz test, the series is absolutely convergent but $\sum |a_n| = \sum 1/n$ is divergent harmonic series.

so given series convergent, but not completely convergent.

Absolute convergence implies convergence

Theorem:-

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

i.e. if $\sum |a_n|$ is convergent then $\sum a_n$ is convergent.

Proof

Let $\sum |a_n|$ be convergent.

then, by Cauchy Creterian of convergence of series for any $\epsilon > 0$, $\exists$ a +ve integer M such that

$$|s_n - s_m| < \epsilon \quad \forall \; n > m \geq M$$

where $s_n = |a_1| + |a_2| + \dots + |a_m| + |a_{m+1}| + \dots + |a_n|$

$n > m$

$$|s_n - s_M| = \big|\, |a_{M+1}| + |a_{M+2}| + \dots + |a_n| \,\big|$$

By Cauchy Critery of convergence

$$\big|\, |a_{M+1}| + \dots + |a_n| \,\big| < \epsilon \quad \forall \; n > m \geq M$$

$$|a_{M+1}| + |a_{M+2}| + \dots + |a_n| < \epsilon \quad \forall \; n > m \geq M \quad -(1)$$

since, $|a+b| \leq |a| + |b|$

To show $\sum a_n$ is convergence.

$$|a_{M+1} + a_{M+2} + \dots + a_n| \leq |a_{M+1}| + |a_{M+2}| + \dots + |a_n| < \epsilon \quad \forall \; n > m \geq M \quad [\text{using } (1)]$$

$$|a_{M+1} + a_{M+2} + \dots + a_n| \rightarrow < \epsilon \quad \forall \; n > m \geq M$$

$\Rightarrow \sum a_n$ is convergent by Cauchy criterial of convergence.

Converse is not true.

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent but not absolutely.

Theorem :-

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$n > m$

$$|s_n - s_M| = \big| |a_{M+1}| + |a_{M+2}| + - + |a_n| \big|$$

By Cauchy Criteria of convergence

$$\big| |a_{M+1}| + -- + |a_n| \big| < \epsilon \quad \forall \; n > m \geq M$$

$$|a_{M+1}| + |a_{M+2}| + - + |a_n| < \epsilon \quad \forall \; n > m \geq M \quad - (1)$$

since, $|a+b| \leq |a| + |b|$

To show $\sum a_n$ is convergence,

$$|a_{M+1} + a_{M+2} + .. + a_n| \leq |a_{M+1}| + |a_{M+2}| + -- + |a_n|$$

$$< \epsilon \quad \forall \; n > m \geq M \quad [\text{using } (1)]$$

$$|a_{M+1} + a_{M+2} + - + a_n| \rightarrow < \epsilon \quad \forall \; n > m \geq M$$

$\Rightarrow \sum a_n$ is convergent. by cauchy criterial of convergence.

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