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Monotonic sequence

A sequence $\langle x_n \rangle = \langle x_1, x_2, x_3, \dots x_n \rangle$ is said to be

i) monotonically increasing if

$$x_{n+1} \geq x_n \quad \forall\, x \in \mathbb{N}$$

ii) monotonically decreasing if

$$x_{n+1} \leq x_n \quad \forall\, x \in \mathbb{N}$$

iii) monotonic, if it is either monotonically increasing or montonically decreasing.

eg. $\langle (-1)^n \rangle = \langle -1, 1, -1, 1, \dots \rangle$ isn't a monotonic sequence.

Monotonic Convergence theorem

A monotonic sequences of real numbers is convergent iff it's bounded.

Series with positive terms.

An infinite series $\sum a_n$ where $a_n > 0 \ \forall\, n$. sequence of partial sums of positive term series is monotonically increasing.

Let $\sum a_n$ be an infinite series of positive terms & let $\langle S_n \rangle$ be a sequence of its partial sums.

Now, $S_n = a_1 + a_2 + \dots + a_n$ denotes its n^{th} partial sum.

since, $a_i > 0 \ \forall\, i = 1, 2, \dots x$

$S_n > 0$

Also, $S_n - S_{n-1} = (a_1 + a_2 + \dots + a_n) - (a_1 + a_2 + \dots + a_{n-1})$

$= a_n > 0$

$\Rightarrow S_n > S_{n-1}$

sequence of partial sum is increasing.

Now by monotonically convergence theoram, sequence of partial sum converges iff it's bounded above.

If it's not bounded, then it diverges.

$\boxed{\text{Partial sums Theoram for series with positive terms}}$

A series $\sum a_n$ with positive terms converges iff sequence of ~~particular~~ partial sums $< S_n >$ is bounded above.

Result: Positive term series $\sum_{n=0}^{\infty} r^n$ $\begin{cases} \text{converges for } r<1 \text{ \&} \\ \text{diverges to } +\infty \text{ for } r>=1 \end{cases}$

Proof:– case 1, $0 \leq r < 1$

let $< S_n >$ be sequence of partial sum.

$$S_n = 1 + r + r^2 + \cdots + r^n$$

$$= \frac{1 - r^n}{1 - r}$$

$$= \frac{1}{1-r} - \frac{r^n}{1-r}$$

$$\leq \frac{1}{1-r}$$

$\Rightarrow < S_n >$ is bounded above.

$\therefore$ Sequence of partial sum is increasing and bounded above.

Then the series $\sum_{n=0}^{\infty} r^n$ converges for $0 \leq r < 1$.

Case 2 $r=1$, $\sum r^n = \sum 1^n$

then, $S_n = 1$.

$\Rightarrow \langle S_n \rangle$ is not bounded above.

$\therefore$ series diverges to $+\infty$

Case 3 :-

$r > 1$

every term of $\langle S_n \rangle$ after 1^{st} term is greater than 1

$$\sum_{n=0}^{\infty} r^n = \underbrace{\underbrace{\underbrace{1+r}_{S_1}+r^2}_{S_2}+ \cdots}_{S_3}$$

$\Rightarrow S_n > n$ $\quad r^n$

$\Rightarrow \langle S_n \rangle$ is not bounded

$\therefore$ Given series diverges.

Thus, series $\sum_{n=0}^{\infty} r^n$ converges for $r < 1$ & diverges for $r \geq 1$.

THE INTEGRAL TEST

Let $\sum a_n$ be a series with positive terms. Let f be a function that is positive continuous & decreasing on the interval $[1, \infty)$ such that $a_n = f(n)$.

Then, this series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ either both converges or both diverges.

Note :- The integral test also applies ~~for~~ if the integral test is satisfied $\forall\, n \geq N$ for some finite $N \geq 1$

(as convergence or divergence of infinite series is not affected by deleting finite numbers of terms.

So, we use integral.

$\int_N^\infty f(x)\,dx$ to test for convergence or divergence.

p-series :-

$$\sum_{n=1}^\infty \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

where p is a real constant.

for $p = 1$, $\sum 1/n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is called harmonic series.

Theorem :-

p-series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ & diverges if $p \leq 1$.

$\Rightarrow$ Case 1: $p > 0$, $p \neq 1$

Then, $f(x) = \frac{1}{x^p}$ is continuous, positive & decreasing on the interval $[1, \infty)$

Now, $\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx$

$$= \lim_{b\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^b$$

$$= \frac{1}{1-p}\left(\lim_{b\to\infty} \left[b^{1-p} - 1\right]\right)$$

Now, $\lim_{b\to\infty} b^{1-p} = \begin{cases} 0, & \text{if } p > 1 \\ \infty, & \text{if } p < 1 \end{cases}$

$$\therefore \int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \infty, & \text{if } 0 < p < 1 \end{cases}$$

$\boxed{\text{Thus, } \int_1^{\infty} \frac{1}{x^p}\,dx \text{ converges if } p>1 \text{ \& diverges if } 0<p<1.}$

Hence, by integral test,

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ & diverges if $0<p<1$.

<u>case 2: $p=1$</u>

then, $\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n}$

let $f(x) = \frac{1}{x}$

the function is positive, continuous and decreasing in $(1,\infty)$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} (\ln x)_1^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 1) = \infty$$

$\therefore \int_1^{\infty} \frac{1}{x}\,dx$ diverges.

using, integral test $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

<u>case 3: if $p \le 0$</u>

if $p=0$, $\sum \frac{1}{n^p} = \sum 1$

$\lim_{n\to\infty} \frac{1}{n^p} = 1$

If $p<0$, $\lim_{n\to\infty} \frac{1}{n^p} = \infty$

Thus, for $p \le 0$, $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$

By, n^{th} term test for divergence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \le 0$

Q.) For the convergence of given series.

i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\Rightarrow \frac{1}{\sqrt{n}}$ is a p-series with $\frac{1}{2} < 1$

$\therefore$ The given series diverges (by p-series test)

(ii) $\sum_{n=5}^{\infty} \frac{1}{(n-1)^2}$

$= \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + - -$

$= \sum_{n=4}^{\infty} \frac{1}{n^2}$

This is a p-series without first three terms with $p > 1$ $(p=2)$ (by p-series test).

$\therefore$ the given series converges (by p-series test).

Q.) Use integral test to determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge.

let $f(x) = \frac{1}{x^2+1}$

$f(x)$ is continuous positive integer $\forall x$, also $\forall x > 0$

$f'(x) = \frac{-2x}{x^2+1} < 0$. $\forall x > 0$

$\therefore$ $f(x)$ is decreasing

so, by integral test, $\int_1^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverge together.

Now, $\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b \to \infty} (\tan^{-1} x)_1^b = \lim_{b \to \infty} (\tan^{-1} b - \tan^{-1} 1)$

$= \pi/2 - \pi/4 = \underline{\pi/4}$

$\Rightarrow \int_1^{\infty} f(x)\,dx$ converges.

By integral test, series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Try it yourself ; why!!??

Q> Use integral test to determine convergence or divergence of series.

i> $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$, ii> $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

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i> $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

let $f(x) = \frac{x}{(x^2+1)^2}$

$f(x)$ is continuous positive integer $\forall\ x > 0$, also

$$f'(x) = \frac{(x^2+1)^2 - 4x^2(x^2+1)}{(x^2+1)^4}$$

$$= \frac{(x^2+1) - 4x^2}{(x^2+1)^3} = \frac{-3x^2+1}{(x^2+1)^3} < 0 \quad \forall\ x > 0$$

$\therefore$ $f(x)$ is decreasing $\forall\ x > 0$.

so, integral test $\int_1^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges or diverges

Now, $\int_1^{\infty} \frac{x}{(x^2+1)^2}\,dx$

let $x^2+1 = u \Rightarrow \frac{du}{dx} = 2x \Rightarrow x \cdot dx = \frac{du}{2}$

$$\Rightarrow \int_2^{\infty} \frac{1}{2} \cdot \frac{du}{u^2} = \lim_{b\to\infty} \left[\frac{-1}{2u} \right]_2^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{2b} + \frac{1}{4} \right] = \frac{1}{4} < 0$$

$\therefore$ $f(x)$ is converges

By integral test, series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges.

ii) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

let $f(x) \frac{1}{x(\ln x)^2}$

that is continuous positive integer $\forall\ x$ also,

decreasing $\forall\ x > 0$.

so, integral test $\int_{1}^{\infty} f(x)\,dx$ and $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.

Now, $\int_{2}^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x(\ln x)^2}\,dx$

$= \lim_{b \to \infty} \left[\frac{-1}{\ln x} \right]_{2}^{b}$

$= \lim_{b \to \infty} \left[\frac{-1}{\ln b} + \frac{1}{\ln 2} \right]$

$= \frac{1}{\ln 2}$

since, the integral $\int_{2}^{\infty} \frac{1}{x(\ln x)^2}\,dx$ converges, we conclude from the integral test that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.