24/01/2022

Comparison Test

1.) Direct comparison test :-

If $\sum u_n$ and $\sum v_n$ are two positive term series and $k \neq 0$, a fixed positive number (independent on n) & $\exists$ a positive integer m such that

$$0 < u_n \leq k v_n \quad \forall \; n \geq m.$$

Then.

i.) if $\sum v_n$, converges, then $\sum u_n$ also converges.

ii.) if $\sum v_n$ diverges, then $\sum u_n$ also diverges.

Q.) Use Direct comparison test to determine convergence or divergence of the following series

i.) $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

for $n \geq 1$, $n^2+4 > n^2$

$$\frac{1}{n^2+4} < \frac{1}{n^2}$$

$$0 < \frac{1}{n^2+4} < \frac{1}{n^2}$$

let $u_n = \frac{1}{n^2+4}$, $v_n = \frac{1}{n^2}$

then, $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are positive series, such that

$$0 < u_n < v_n$$

since, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is p-series. & converges as $p > 1$

$\therefore$ By direct comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ also converges.

$n=1$

let $u_n = \frac{1}{3^n+1}$, $v_n = \frac{1}{3^n}$

then, $0 < u_n \leq v_n$, for $n > 1$ $\quad \Sigma \frac{1}{3^n}$ is geometric series with $r = 1/3 < 1$

& thus is converges.

Hence, by direct comparison test,

$\Sigma \frac{1}{3^n+1}$ also converges.

iii) $\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$

since, $\ln n < n \quad \forall \; n \geq 1$

$\frac{\ln n}{n^4} < \frac{n}{n^4} = \frac{1}{n^3}$

let $u_n = \frac{\ln n}{4}$, $v_n = \frac{1}{n^3}$

then, $0 < u_n \leq v_n \quad \forall \; n \geq 1$

since, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3 \; (>1)$

thus, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is converges.

Hence, by direct comparision test,

$\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$ also converges.

iv) $\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots < 1 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \cdots$

$\Rightarrow 0 < \frac{1}{n!} < \frac{1}{2^{n-1}}$

let $u_n = \frac{1}{n!}$, $v_n = \frac{1}{2^{n-1}}$

since, $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is convergent geometric series $r = 1/2 > 1$, so

by direct comparision test $\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges.

v) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$

for $n \geq 1$, $0 < e^{1/n} < e$

$\Rightarrow \frac{e^{1/n}}{n^3} \leq \frac{e}{n^3}$

let $u_n = \frac{e^{1/n}}{n^3}$, $v_n = \frac{e}{n^3}$

$\boxed{k = e^{1/n}}$

$0 < u_n \leq k v_n \quad \forall\ n \geq 1$

since, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3 (>1)$.

$\sum \frac{1}{n^3}$ converges.

Hence, by direct comparison test, $\sum \frac{e^{1/n}}{n^3}$ also converges.

vi) $\sum_{n=2}^{\infty} \frac{n^2}{n^2-3}$

$n^2 - 3 < n^3 \quad \forall\ n \geq 2$

$\frac{1}{n^2-3} > \frac{1}{n^3}$

$\frac{n^2}{n^3-3} > \frac{n^2}{n^3} = \frac{1}{n}$

$\frac{1}{n} < \frac{n^2}{n^3-3}$

let, $u_n = 1/n$, $v_n = \frac{n^2}{n^3-3}$.

then, $0 < u_n \leq v_n \quad \forall\ n \geq 2$

Now, $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent harmonic series.

Hence, by direct comparison test -

$\sum_{n=2}^{\infty} \frac{n^2}{n^3-3}$ is also divergent.

ry It Yourself)

Q.) $\sum n^3/_{2n^4-1}$ ~~(ii.) $\sum \frac{1}{n^3+8n}$~~ (ii.) $\sum \frac{1}{n^3+8n}$

$\sum \frac{n^3}{2n^4-1}$

$2n^4-1 < 2n^4$

$\frac{1}{2n^4-1} > \frac{1}{2n^4}$

so, $\frac{n^3}{2n^4-1} > \frac{n^3}{2n^4} = \frac{1}{2n} \Rightarrow \frac{1}{2n} < \frac{n^3}{2n^4-1}$

let $u_n = \frac{n^3}{2n^4-1}$, $v_n = \frac{1}{2n}$, then, $0 < u_n \leq v_n$

Now $\sum \frac{1}{2n}$ is divergence harmonic series

so, by direct comparison test, $\sum \frac{n^3}{2n^4-1}$ also diverges.

(ii.) $\sum \frac{1}{n^3+8n}$

$n^3+8n > n^3$

$\frac{1}{n^3+8n} < \frac{1}{n^3}$

$u_n = \frac{1}{n^3+8n}$, $v_n = \frac{1}{n^3}$ $\quad 0 < u_n \leq v_n \quad \forall\ n \geq 1$

$\sum \frac{1}{n^3}$ is p-series where $p = 3 (>1)$ converges.

by direct comparison test, $\sum \frac{1}{n^3+8n}$ it also converges.

Q.) Prove that if Σa_n is a convergent series of positive series then the series Σa_n^2 also converges.

Proof :-

since, Σa_n is a convergent series,

$$\lim_{n\to\infty} a_n = 0$$

i.e. for $\epsilon = 1$ $\exists$ $m \in \mathbb{N}$ such that

$|a_n - 0| < 1$ $\forall$ $n \geq m$

$\because$ since $a_n > 0$ $\forall n$

$\Rightarrow$ $a_n < \frac{1}{2}$, $\forall$ $n \geq m$

$\Rightarrow$ $0 < a_n^2 < a_n$ $\forall$ $n \geq m$

since. Σa_n is convergent series.

By Direct comparison test

Σa_n^2 is convergent.

Note :-

Converse may not be true.

eg. let $\Sigma a_n = 1/n$

$\Sigma a_n^2 = \frac{1}{n^2}$

$\Sigma a_n^2 = \Sigma \frac{1}{n^2}$ is convergent p-series with $p=2$ (>1)

However, Σa_n is not convergent, it is divergent

LIMIT COMPARISON TEST

If $\sum u_n$ and $\sum v_n$ are positive term series such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, where l is non-zero finite series, then the two series $\sum u_n$ and $\sum v_n$ converges or diverges together.

Proof:-

Let $\sum v_n$ converges.

$\lim_{n\to\infty} \frac{u_n}{v_n} = l$ $\quad \exists\, N \in \mathbb{N}$

such that, $\left|\frac{u_n}{v_n} - l\right| < \varepsilon\ (=1)$ $\quad \forall\ n \geq N$

$0 < \frac{u_n}{v_n} < l+1$ $\quad \forall\ n \geq N$

$u_n < (l+1) < v_n$ $\quad \forall\ n \geq N$

since, $\sum v_n$ converges,

$\sum (l+1) v_n$ converges.

By direct comparison test $\sum u_n$ also converges.

Note:- If $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ or ∞, the test may not hold.

Example:- let $\sum u_n = \frac{1}{n^2}$, $\sum v_n = 1/n$

$$\frac{u_n}{v_n} = \frac{1/n^2}{1/n} = 1/n$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\sum 1/n$ diverges whereas $\sum 1/n^2$ converges.

or, let $\sum u_n = 1/n$, $\sum v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{1/n}{1/n^2} = n$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} (n) = \infty$$

$\sum u_n$ diverges, $\sum v_n$ converges.

Thus, limit test may not hold if $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ or ∞

Q) Use limit comparison test to determine convergence or divergence of the given series :-

i) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

let $u_n = \frac{1}{2^n - 1}$, $v_n = \frac{1}{2^n}$

u_n and v_n are positive for $n \geq 1$

then, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{2^n}{2^n - 1}\right)$

$= \lim_{n\to\infty} \left(\frac{1}{1 - 2^{-n}}\right)$

$= 1$, a finite positive number.

Now, $\sum \frac{1}{2^n}$ is convergent geometric series with $(r = \frac{1}{2} < 1)$

$\therefore$ By limit comparison test,

$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges.

ii) $\sum_{n=1}^{\infty} \frac{3n - 1}{2n^3 - 4n + 5}$

let $u_n = \frac{3n - 1}{2n^3 - 4n + 5}$, $\boxed{v_n = \frac{1}{n^2}}$

$= \frac{n(3 - 1/n)}{n^3(2 - 4/n^2 + 5/n^3)}$

$\boxed{u_n = \frac{1}{n^2}\left(\frac{3 - 1/n}{2 - 4/n^2 + 5/n^3}\right)}$

u_n & v_n both are positive for $n \geq 1$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{3 - 1/n}{2 - 4/n^2 + 5/n^3}\right) = \frac{3}{2}$, a finite positive number.

$\therefore$ $\sum u_n$ and $\sum v_n$ behave alike (by limit comparison test)

since, $\sum v_n = \sum \frac{1}{n^2}$ converges by p-series $p = 2 > 1$, so,

$\sum u_n = \sum \frac{3n-1}{2n^3 - 4n + 5}$ also converges.

Q:) $\frac{1}{\sqrt{1\cdot 2}} + \frac{1}{\sqrt{2\cdot 3}} + \frac{1}{\sqrt{3\cdot 4}} + \cdots$

let $u_n = \frac{1}{\sqrt{n(n+1)}}$, $v_n = \frac{1}{n}$

$u_n, v_n > 0 \quad \forall\ n \geq 1$

$$\frac{u_n}{v_n} = \frac{n}{\sqrt{n^2+n}} = \frac{n}{n\sqrt{1+1/n}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{\sqrt{1+1/n}}\right)$$

$= 1$, finite positive numbers.

$\therefore$ By limit comparison test, Σu_n & Σv_n converges or diverges together.

Now, $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent harmonic series.

$\therefore \Sigma u_n = \Sigma \frac{1}{\sqrt{n(n+1)}}$ diverges.

Q:) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}}$

$$u_n = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{(\cancel{n}+1) - (\cancel{n})}{\sqrt{n}(\sqrt{n+1}+\sqrt{n})}$$

$$= \frac{1}{\sqrt{n}\cdot\sqrt{n}(\sqrt{1+\frac{1}{n}}+1)}$$

$$= \frac{1}{n(\sqrt{1+1/n}+1)}$$

$v_n = \frac{1}{n}$.

u_n & v_n both are positive.

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{\sqrt{1+1/n}+1}\right) = 1$, a positive finite numb

$\longrightarrow$ same

Q) $\frac{1}{3.7}+\frac{1}{4.9}+\frac{1}{5.11}+\frac{1}{6.13}+ - - -$

n^{th} term of given series.

$$a_n=\frac{1}{(n+2)(2n+5)}=\frac{1}{n^2(1+2/n)(2+5/n)}$$

$$u_n=\frac{1}{n^2(1+2/n)(2+5/n)}$$

$$v_n=\frac{1}{n^2}$$

u_n & v_n both positive.

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left[\frac{1}{(1+2/n)(2+5/n)}\right]=\frac{1}{2}, \text{ finite number (+ve)}$$

so, by limit comparison test, given series converge or diverge together

since, $\Sigma v_n=\Sigma\frac{1}{n^2}$ is converges p-series $p=2\ (>1)$

also Σu_n converges.

Q) $\sum_{n=1}^{\infty}\frac{1}{3^n+5}$

$u_n=\frac{1}{3^n+5}$, $v_n=\frac{1}{3^n}$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(\frac{1}{3^n+5}\right)\cdot 3^n$$

$$=\lim_{n\to\infty}\left(\frac{1}{3^n(1+5/3^n)}\right)\cdot 3^n$$

$$=\lim_{n\to\infty}\left(\frac{1}{1+5/3^n}\right)$$

$= 1$, non-zero finite number.

So, a/c to limit comp. test, series may converge or diverge together.

$\Sigma v_n=\Sigma\frac{1}{3^n}$ g.s. $r=1/3<1$,

~~diverges~~. converges.

(i) $\sum_{n=1}^{\infty}\frac{n+1}{n^p}$

$$a_n=\frac{n+1}{n^p}=\frac{1+1/n}{n^{p-1}}$$

$$v_n=\frac{1}{n^{p-1}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$=1$, non-zero finite no.

$\Sigma\frac{1}{n^{p-1}}$ converges for $p-1>1$ i.e $p>2$.

diverges for $p\leq 2$.

Q.) $\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$

$u_n = \sin \frac{1}{n^2}$, $v_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{\sin 1/n^2}{1/n^2}\right) = 1$, finite positive no.

$\sum \frac{1}{n^2}$ is convergent p-series $p=2 (>1)$.

so, $\sum u_n$ also converges.

Q.) $\sum_{n=1}^{\infty} \sin 1/n$

$u_n = \sin \frac{1}{n}$, $v_n = \frac{1}{n}$ $\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{\sin 1/n}{1/n}\right) = 1 \neq 0$

$\sum v_n = \sum 1/n$ diverges.

$\sum u_n$ also diverges.

Q.) Use direct comparision test to show that,

$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$

$\sum u_n = \sum \frac{1}{n^n}$