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THE RATIO TEST

Rauf

let $\sum a_n$ be a series with positive terms and let

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$$

i) if $l < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.

ii) if $l > 1$, or $l = \infty$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

iii) if $l = 1$, the test is inconclusive.

eg. let $\sum a_n = \sum \frac{1}{n^2}$

$$\lim_{n\to\infty} \sum a_n \Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2}$$

$$= \underline{\underline{1}}$$

But $\sum \frac{1}{n^2}$ converges. {p-series, $p = 2 (>1)$}

Q.) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \Rightarrow \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(2(n+1)-1)}}{\frac{1}{2n-1}} = \lim_{n\to\infty} \frac{2n-1}{2n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(2-1/2)}{\cancel{n}(2+1/2)} = 1$$

The test is incoclusive

We can use Integral or limit comparison test to determine convergence or divergence.

let $v_n = \frac{1}{2n}$, $u_n = \frac{1}{2n-1}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2n}{2n-1} = \lim_{n\to\infty} \left(\frac{2}{2-1/n}\right) = 1 \rightarrow \text{non-ze}$$

$\sum v_n = \sum \frac{1}{2n}$ divergent a/c to harmonic series

so. according to limit comp.

$\sum u_n$ also diverges.

Q.) Use ratio test to determine whether the given series converges or diverges.

i) $\sum \frac{n}{e^n}$

$a_{n+1} = \frac{n+1}{e^{n+1}}$, $a_n = \frac{n}{e^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{e^{n+1}}\right)\left(\frac{e^n}{n}\right)$$

$$= \frac{1}{e} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)$$

$$= \frac{1}{e} < 1$$

since, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, series converges by ratio test.

ii) $\sum_{n=1}^{\infty} \frac{n^n}{e^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{e^{n+1}} \times \frac{e^n}{n^n}\right)$$

$\frac{n}{e}$

$$= \frac{1}{e} \cdot \left(\lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^2 (n+1)\right)\right)$$

$$= \frac{1}{e} \cdot \infty = \infty$$

By ratio test, $\sum_{n=1} \frac{a_{n+1}}{a_n} = \infty$, series diverges.

iii) $\sum \frac{1}{n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{1}{(n+1)!} \cdot n!\right] = \lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0 < 1.$$

so, by ratio test series converges. when $\sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n} = 0$

iv.) $\sum \frac{(2n)!}{(n!)^2}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{(2(n+1))!}{((n+1)!)^2} \times \frac{(n!)^2}{(2n)!}\right) = \lim_{n\to\infty}\left(\frac{(2n+2)(2n+1)}{(n+1)(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2+2/n)(2+1/n)}{(1+1/n)(1+1/n)}\right)$$

$$= 4 > 1 \qquad \sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n} > 1 \text{ series diverges.}$$

so, series diverges, a/c to ratio test

v.) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^n \cancel{(n+1)} \cdot \cancel{n!}}{n^n \cdot \cancel{(n+1)}\, \cancel{n!}}\right]$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e > 1 \qquad \sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n} > 1 \text{ series diverges.}$$

so, series diverges, A/c to ratio test

vi.) $\sum \frac{n^3}{(\ln 3)^n}$

$$l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\frac{(n+1)^3}{(\ln 3)^{n+1}} \times \frac{(\ln 3)^n}{n^3}\right]$$

$$= \lim_{n\to\infty} \frac{(1+1/n)^3}{\ln 3} = \frac{1}{\ln 3}$$

since $\ln 3 > \ln e = 1$

$\frac{1}{\ln 3} < 1$, series converges a/c to ratio test.

Q:) $\sum_{n=1}^{\infty} \frac{n!}{3.5.7 \ldots (2n+1)}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{(n+1)!}{3.5.7 \ldots (2n+1)} \cdot \frac{3.5.7 \ldots (2n+1)}{n!} \right]$$

$$= \lim_{n\to\infty} \frac{(n+1)\, \cancel{n!}}{\cancel{3.5.7 \ldots (2n+1)}\,(2n+3)} \times \frac{\cancel{3.5.7 \ldots (2n+1)}}{\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n+3}$$

$$= \lim_{n\to\infty} \frac{1+1/n}{2+3/n}$$

$= 1/2 < 1.$ series converges a/c to ratio test

$\sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n} < 1$ series converges.

THE ROOT TEST

let $\sum a_n$ be a positive term series & let $\lim_{n\to\infty} (a_n)^{1/n} = l$.

1:) if, $l < 1$, series $\sum a_n$ converges.

2:) if, $l > 1$ series $\sum a_n$ diverges.

3:) if $l = 1$, ~~series~~ test is inconclusive. series may converges or diverges.

Q:) Use root test to determine whether given series converges or diverges

:) $\sum_{n=1}^{\infty} \left(\frac{1}{(1+n)}\right)^n$

⇒ let, $l = \lim_{n\to\infty} (a_n)^{1/n}$

$$= \lim_{n\to\infty} \left[\left(\frac{1}{1+n}\right)^{n}\right]^{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{1+n} = 0 < 1$$

since, $l < 1$, series converges a/c to root test $\lim_{n\to\infty} (a_n)^{1/n} < 1$ series converges.

Q.) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$\Rightarrow \quad l = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{n}{2^n}\right)^{1/n}$

$= \frac{1}{2} \lim_{n\to\infty} n^{1/n}$

let $y = n^{1/n} \Rightarrow \log y = \log n^{1/n} = 1/n \log n$

$\lim_{n\to\infty} \log y = \lim_{n\to\infty} \left(\frac{\log n}{n}\right)$ $\{\frac{\infty}{\infty}$ form$\}$

$= \lim_{n\to\infty} \left(\frac{1/n}{1}\right)$ (By L'Hospital Rule)

$= \cancel{\infty}\ 0$

$\log\left(\lim_{n\to\infty} y\right) = 0$

$\lim_{n\to\infty} y = e^0 = 1$

$\therefore\ l = \lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{2} = \frac{1}{2} < 1 \Rightarrow$ converges by ~~ratio~~ root test

Q.) $\sum_{n=1}^{\infty} \left(\frac{n^n}{2^{3n-1}}\right)$

$\Rightarrow l = \lim_{n\to\infty} \left(\frac{n^n}{2^{3n-1}}\right)^{1/n} = \lim_{n\to\infty} \frac{n}{(2^{3n-1})^{1/n}} = \lim_{n\to\infty} \frac{n \cdot 2^{1/n}}{2^3} = \infty$

since, $l = \infty$, by root test the given series diverges.