13/01/2022

Q:) Use the nth term test for divergence to determine whether the following series diverges.

a:) $\sum \frac{n^3}{n^3+1}$, b:) $\frac{1}{\sqrt{n}}$ c:) $\sqrt{\frac{n}{n+1}}$

Properties of convergent series:-

If $\sum a_n$ and $\sum b_n$ are convergent series and c is a real number then

i:) series $\sum c a_n$ is convergent and

$$\sum c a_n = c \sum a_n$$

ii:) series $\sum (a_n + b_n)$ is convergent and

$$\sum (a_n + b_n) = \sum a_n + \sum b_n$$

iii:) series $\sum (a_n - b_n)$ is convergent and

$$\sum (a_n - b_n) = \sum a_n - \sum b_n$$

Q:) Using properties of infinite series, evaluate the following:-

i:) $\sum_{n=0}^{\infty} \frac{2+3^n}{5^n}$

Let $a_n = \frac{2+3^n}{5^n} = \frac{2}{5^n} + \left(\frac{3}{5}\right)^n = 2\left(\frac{1}{5}\right)^n + \left(\frac{3}{5}\right)^n$

Now, $\sum 2\left(\frac{1}{5}\right)^n$ is geometric series with $a = \cancel{1}\,2$, $r = \frac{1}{5} < 1$

so, this series converges.

series $\sum (3/5)^n$ is geometric series with $a=1$, $r = 3/5 < 1$

so, it converges.

then $\sum \frac{2+3^n}{5^n}$ converges and

$$\sum_{n=0}^{\infty} \frac{2+3^n}{5^n} = 2\sum_{n=0}^{\infty} \frac{1}{5^n} + \sum_{n=0}^{\infty} (3/5)^n$$

$$= 2\left(\frac{1}{1-1/5}\right) + \frac{1}{1-3/5}$$

$$= \frac{10}{4} + \frac{5}{2} = 5$$

$\frac{1}{5^n} = \cancel{1} + \frac{1}{5} + \frac{1}{25} + \cancel{\frac{1}{5^n}}$

$S_n = \frac{a}{1-r}$ $a = 1$, $r = 1/5 < 1$

(ii) $\sum_{n=1}^{\infty} \left[\frac{1}{n(n+1)} + \frac{1}{2^n}\right]$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{n(n+1)}\right] + \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)$$

Consider, $\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)}\right)$

n^{th} partial sum is

$$S = a_1 + a_2 + a_3 + \ldots + a_3$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right) = 1$$

$\therefore$ series converges & its series sum is 1.

Also, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = 1/2$, $r = 1/2 < 1$

The series converges with sum $= \frac{a}{1-r} = \frac{1/2}{1-1/2} = 1$.

$\therefore$ Given series converges with sum $= 1 + 1 = 2$.

Q.) Use prop. of infinite series to evaluate the following series.

i.) $\sum_{n=1}^{\infty}\left[5\left(\frac{2}{3}\right)^n - \frac{2^{n-1}}{7^n}\right]$ ii.)

Try It yourself $\boxed{Ans = 47/5}$

$\Rightarrow \sum_{n=1}^{\infty}\left[5\left(\frac{2}{3}\right)^n - \left(\frac{2}{7}\right)^n \cdot \frac{1}{2}\right]$

$= \sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^n - \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{2}{7}\right)^n$

consider $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^n$ $\frac{10}{3}, \frac{20/9}{10/3}$

it's a geometric series with $a = 10/3$, $r = 2/3 < 1$.

$sum = \frac{a}{1-r} = \frac{10/3}{1-2/3} = \frac{10}{1}$

$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{2}{7}\right)^n$ $\frac{1}{2} \cdot \frac{2}{7}$ $\frac{1}{7} \cdot \frac{\cancel{4}2}{\cancel{49}7} \over 1/7$

is a geometric series with $a = 1/7$, $r = 2/7 < 1$

$S = \frac{1/7}{1-2/7} = \frac{1}{5}$

Given series converges & its sum $= 10 - 1/5 = 49/5$.

Try It Yourself

Q.) Determine if the geometric series converges or diverges if the series converges find sum?

a.) $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^{-n}$ b.) $\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$, c.) $\sum_{n=1}^{\infty} 3^{2n} \cdot 5^{1-n}$

d.) $\sum_{n=0}^{\infty} (1.1)^n$ e.) $\sum_{n=0}^{\infty} 5\left(2/3\right)^n$

a.) $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^{-n} \Rightarrow \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ is a geometric series with $a = 1, r = 3/4 < 1$

$S = \frac{1}{1-3/4} = 4$

b.) $\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} \Rightarrow \sum 2\left(\frac{2}{5}\right)^n$ is a geometric series with $a = 2, r = 2/5 < 1$

$S = \frac{2}{1-2/5} = \frac{10}{3}$

c.) $\sum_{n=1}^{\infty} 3^{2n} \cdot 5^{1-n} \Rightarrow \sum_{n=1}^{\infty} 3 \cdot 3^n \cdot \frac{5}{5^n} \Rightarrow \sum_{n=1}^{\infty} 15\left(\frac{3}{5}\right)^n$ is geometric series

with $a = 6, r = 3/5 < 1$

$S = \frac{6}{1-3/5} = \frac{30}{2} = 15$

d.) $\sum_{n=0}^{\infty} (1.1)^n$ $a = 1$, $\underline{\underline{r = 1 > 1}}$ it's diverges.

This is divergent series, so it diverges.

e.) $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^n$ is converging geometric series with $a = 10/3$
$r = 2/3 < 1$

$S = \frac{10/3}{1-2/3} = 10$

Cauchy criterion of convergence of series

$\exists \Rightarrow$ there exist

A series $\sum a_n$ converges iff for every $\varepsilon > 0$, $\exists$ $m \in \mathbb{N}$ such that

$$|S_n - S_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|$$

$$(a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_m)$$

$$< \varepsilon \quad \forall\ n > m \geq M$$

Proof:- let series $\sum a_n$ converges.

$\Leftrightarrow$ the sequence of partial sums $\langle S_n \rangle$ converges.

$\Leftrightarrow$ the sequence of partial sums is couchy sequence.

$\Leftrightarrow$ for each $\varepsilon > 0$ $\exists$ $m \in N$ such that

$$|S_n - S_m| < \varepsilon \quad \forall\ n > m \geq M$$

~~i.e. $(a_1 + a_2 + \cdots + a_n) - ($~~

i.e. $|a_{M+1} + a_{M+2} + \cdots + a_n| < \varepsilon \quad \forall\ n > m \geq M$

Q:- show that the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ doesn't converge?

sol:- let on the contrary $\sum_{n=0}^{\infty} 1/n$ converges, Then by Couchy criteria for convergence, for $\varepsilon = 1/2$, $\exists$ a positive integer M such that

$$|S_n - S_M| = \left|\frac{1}{M+1} + \frac{1}{M+2} + \cdots + \frac{1}{n}\right|$$

$$= \frac{1}{M+1} + \frac{1}{M+2} + \cdots + \frac{1}{n} < \varepsilon \left(= \frac{1}{2}\right)^{*} \quad \forall\ n > m \geq M$$

In particular for $n = 2m$

$$\frac{1}{M+1} + \frac{1}{M+2} + \cdots + \frac{1}{2m} < 1/3$$

But,
$$\frac{1}{M+1} + \frac{1}{M+2} + \cdots + \frac{1}{2m} > \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m}$$

$$= m\left(\frac{1}{2m}\right) = \frac{1}{2}$$

$$\frac{1}{M+1} + \frac{1}{M+2} + \cdots + \frac{1}{2m} > 1/2$$

which contradicts (*).

Hence, the given series doesn't converges.