Matrix World

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Matrix World : The Picture of All Matrices

I am happy to tell the history of Matrix World—the creation of Kenji Hiranabe in Japan. In April 2020 his friend Satomi Joba asked if I would send him a birthday message as a surprise. He was happy (and very surprised). Kenji combines mathematics with art and with computing : three talents in one. I was the one to be surprised when he sent Matrix World in its first form—without a name, without many of the entries and ideas that you see now, but with the central idea of displaying the wonderful variety of matrices.

Since that first form, Matrix World has steadily grown. It includes every property that would fit and every factorization that would display that property. Interesting that the SVD is in the outer circle and the identity matrix is at the center—it has all the good properties : the matrix $\boldsymbol{I}$ is diagonal, positive definite symmetric, orthogonal, projection, normal, invertible, and square.

Lek-Heng Lim has pointed out the usefulness of matrices $\boldsymbol{M}$ that are **symmetric and orthogona**l—kings and also queens. Their eigenvalues are 1 and −1. They have the form $\boldsymbol{M = I - 2P}$ ($\boldsymbol{P}$ = symmetric projection matrix). There is a neat match between all those matrices M and all subspaces of $\mathbf{R}^n$. You may see something interesting (or something missing) in Matrix World. We hope you will ! Thank you to Kenji.

Gilbert Strang

Walk through the "Matrix World"

There are many categories of matrices: **Symmetric**, **Orthogonal**, **Singular**, **Invertible**, **Square**, **Diagonalizable**, and more. This is a map of the categories using a venn diagram. From the top to the bottom, I'm walking you through this diagram. All matrices ($m \times n$) can be factorized to $\boldsymbol{A} = \boldsymbol{CR}$ and also be factorized into $\boldsymbol{A} = \boldsymbol{U\Sigma V}^{\mathrm{T}}$.

Square matrices ($n \times n$) are either **Invertible** or **Singular**. Invertibility, indicated by the dotted line through the center, can be checked by whether $\boldsymbol{A} = \boldsymbol{LU}$ has full pivots, $\det(\boldsymbol{A}) \neq \boldsymbol{0}$ or all eigenvalues are nonzero.

An **Invertible** matrix is factorized into $\boldsymbol{A} = \boldsymbol{QR}$ using **Orthogonal** matrix $\boldsymbol{Q}$ (Gram-Schmidt). A **Square** matrix is either **Diagonalizable** if it has n independent eigenvectors.

A **Diagonalizable** matrix is factorized into $\boldsymbol{A} = \boldsymbol{X\Lambda X}^{-\mathbf{1}}$ with a **Diagonal** eigenvalue matrix $\boldsymbol{\Lambda}$ and an **Invertible** eigenvector matrix $\boldsymbol{X}$. A Non-diagonalizable matrix has its **Jordan** normal form $\boldsymbol{J}$ instead of the diagonal matrix $\boldsymbol{\Lambda}$.

There is an important category **Normal** which includes **Symmetric**, **Orthogonal**, **Anti-Symmetric** matrices. A matrix is **Diagonalizable** as $\boldsymbol{Q\Lambda Q}^{-\mathbf{1}}$ by an **Orthogonal** matrix $\boldsymbol{Q}$ when and only when it is **Normal** ($\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{AA}^{\mathrm{T}}$).

Symmetric matrices ($\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}}$) and **Orthogonal** matrices ($\boldsymbol{Q}^{-\mathbf{1}} = \boldsymbol{Q}^{\mathrm{T}}$) are important **Normal** matrices. Some matrices are both **Symmetric** and **Orthogonal**. They include reflection matrices with eigenvalues 1 and -1.

All eigenvalues of an **Orthogonal** matrix have $|\lambda| = 1$. All eigenvalues of a Symmetric matrix are real. A Symmetric matrix is called **Positive Semidefinite** if all eigenvalues $\lambda \geq 0$, and **Positive Definite** if all eigenvalues $\lambda > 0$.

Projection matrices ($\boldsymbol{P}^{\mathbf{2}} = \boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}}$) are **Symmetric** and **Positive Semidefinite**, with all the eigenvalues are 1 or 0.

The **Identity** is the only **Invertible Projection** matrix.

$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ for any matrix $\boldsymbol{A}$ is Positive Semidefinite and is Positive Definite if and only if the columns of $\boldsymbol{A}$ are independent.

If a matrix is **Invertible**, its inverse can be expressed as $\boldsymbol{A}^{-\mathbf{1}} = \boldsymbol{V\Sigma U}^{\mathrm{T}}$. Any matrix even if it is not **Invertible** nor even **Square**, there exists a pseudoinverse $\boldsymbol{A}^{-\mathbf{1}} = \boldsymbol{V\Sigma U}^{\mathrm{T}}$.

Matrix World

in
Linear Algebra
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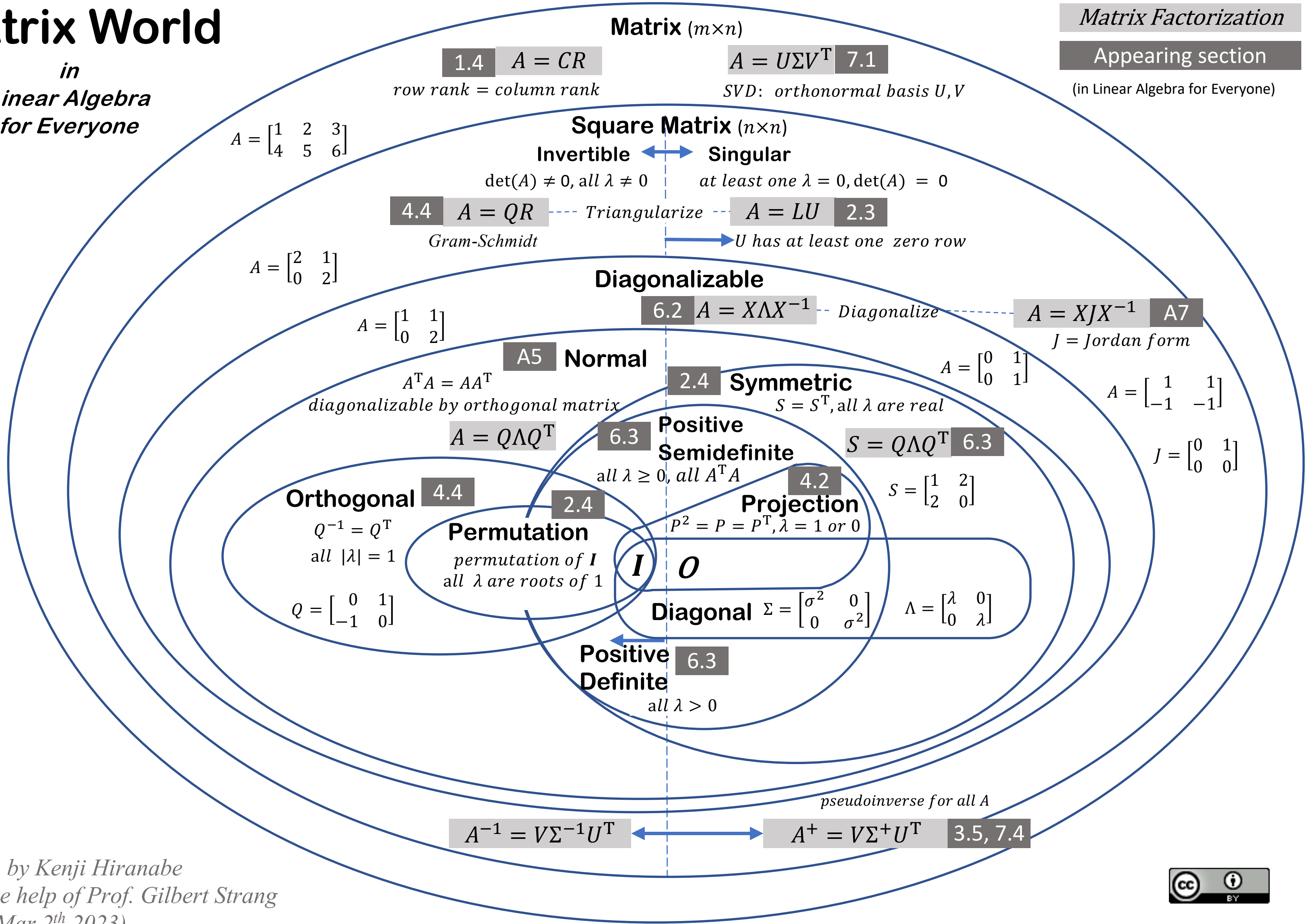

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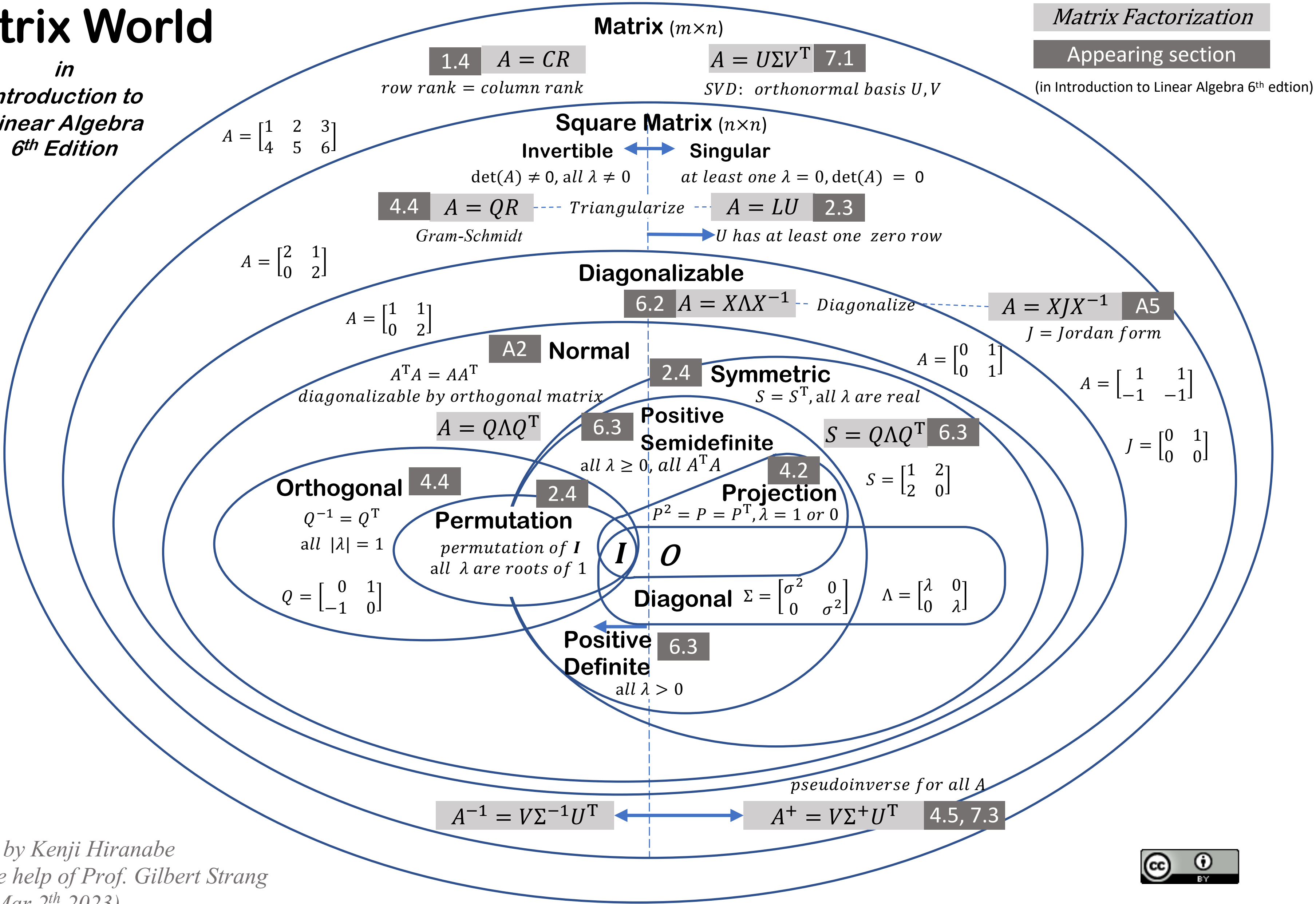

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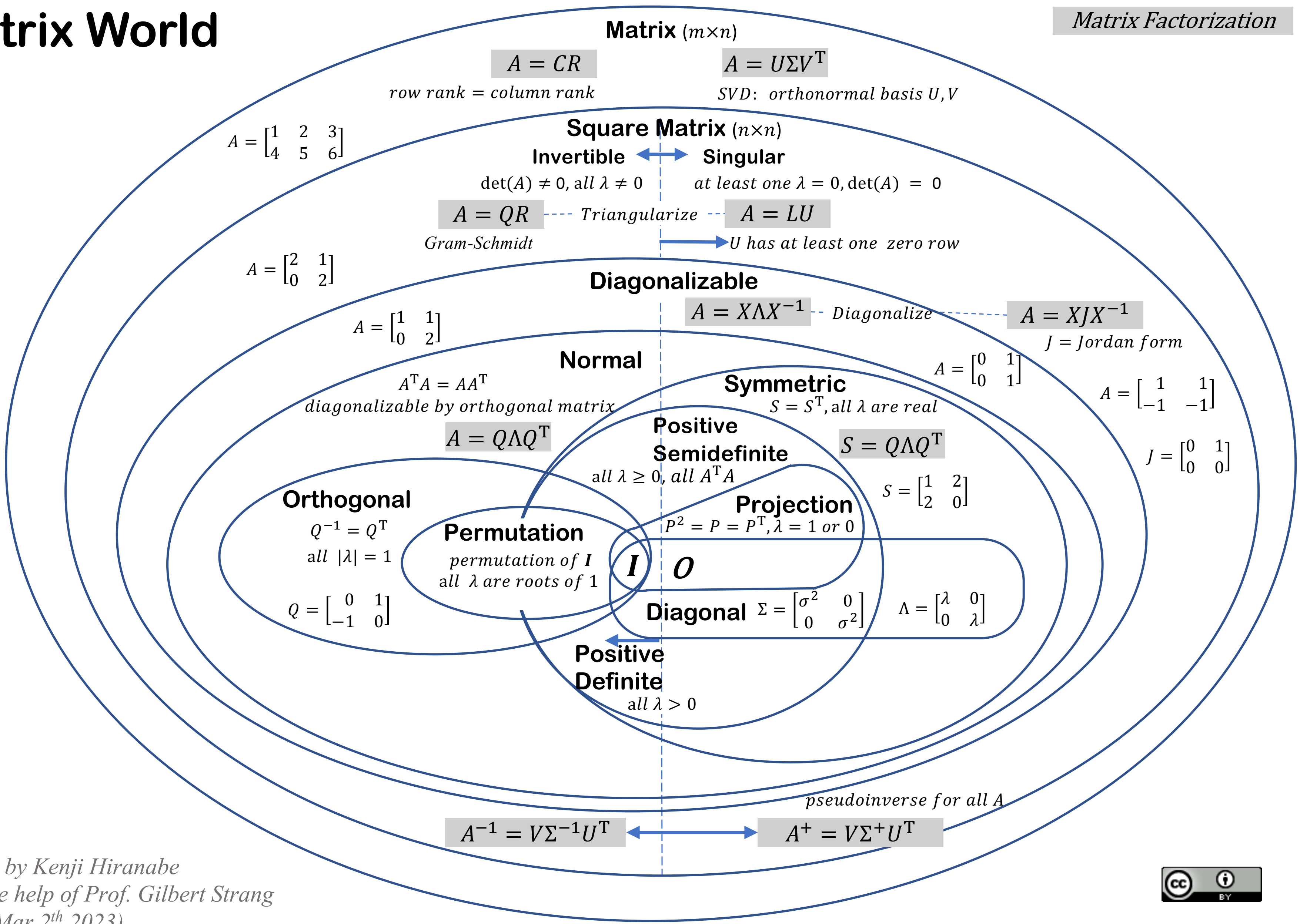

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Matrix World

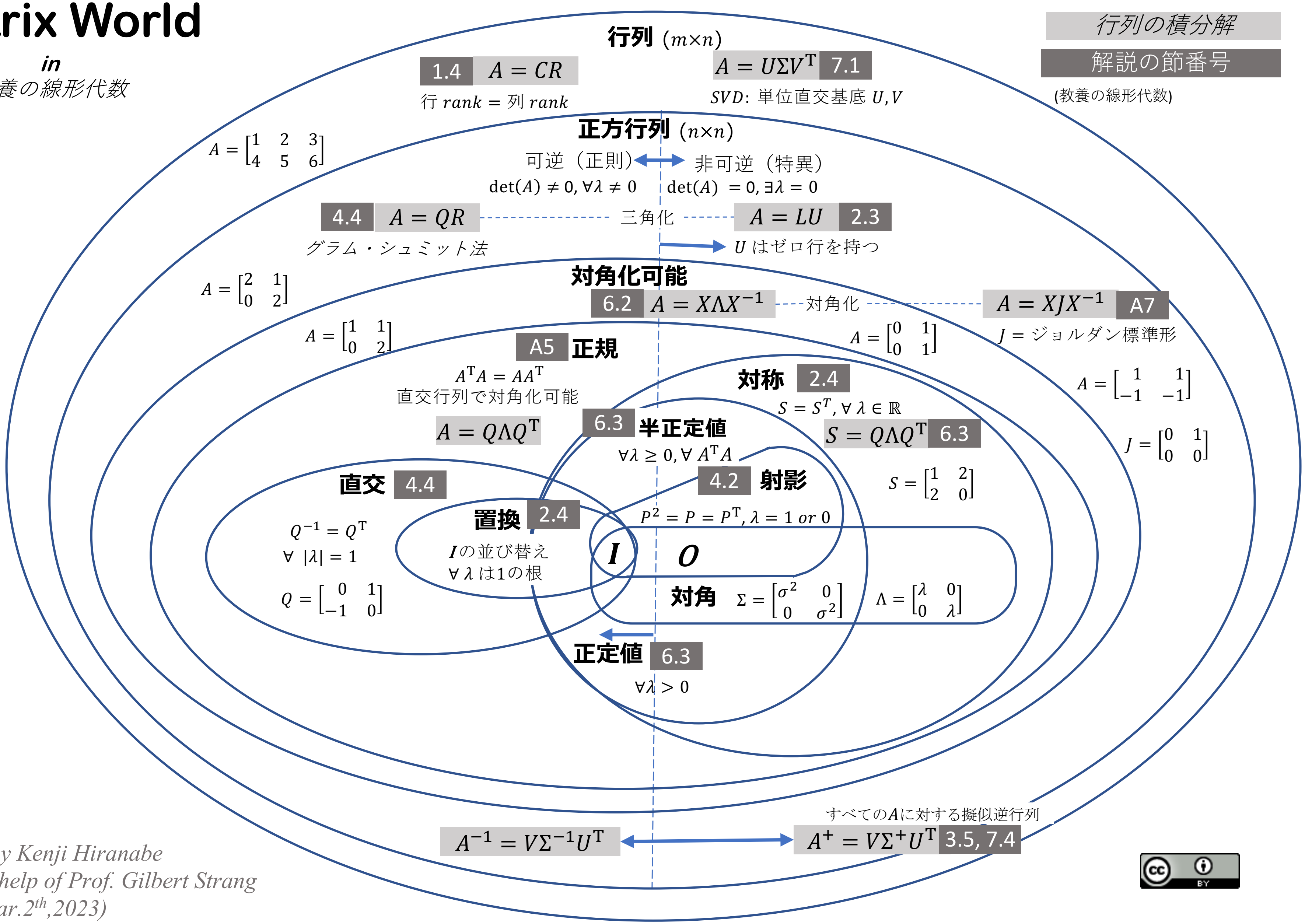
in

教養の線形代数

行列の積分解

解説の節番号

(教養の線形代数)



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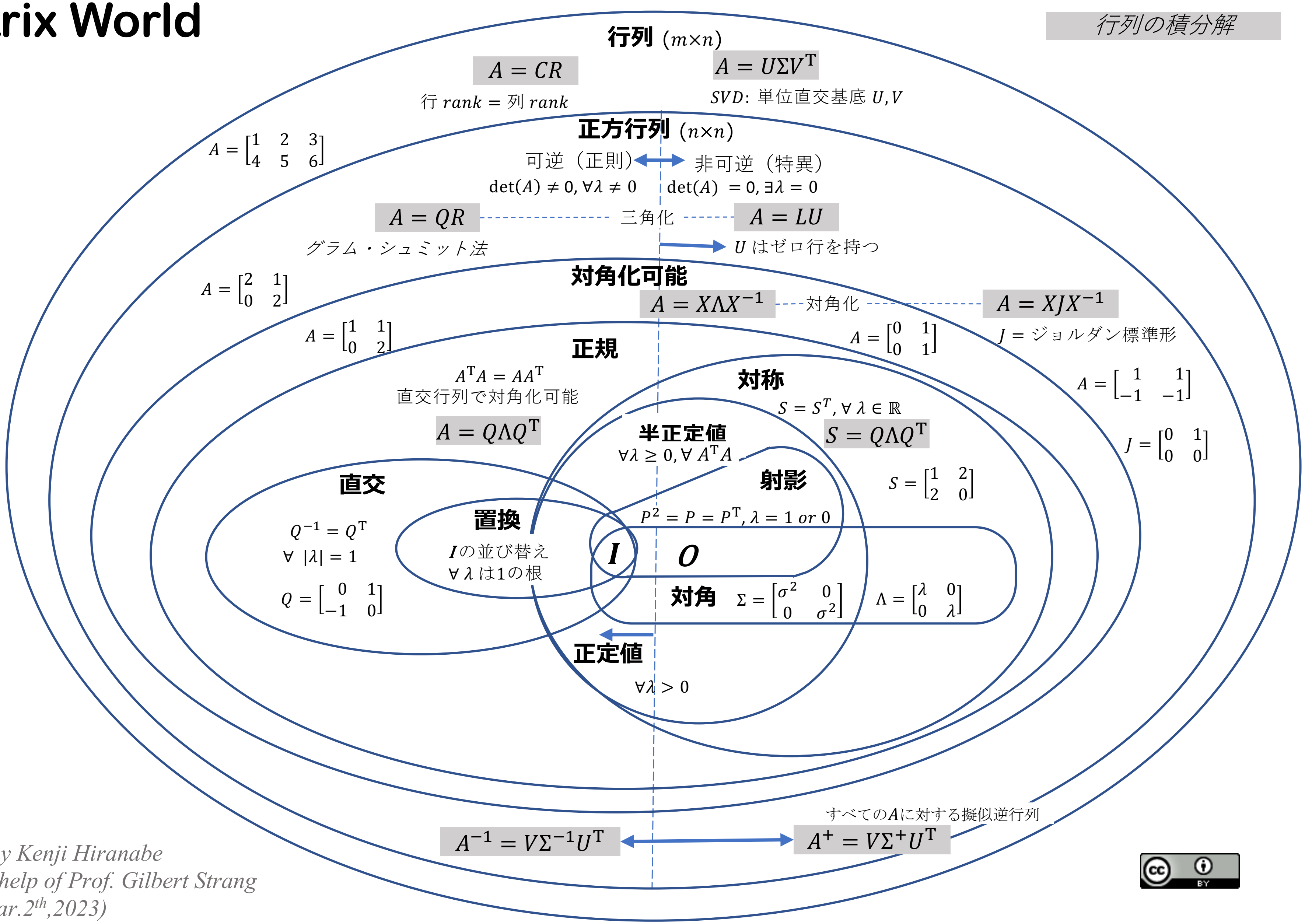

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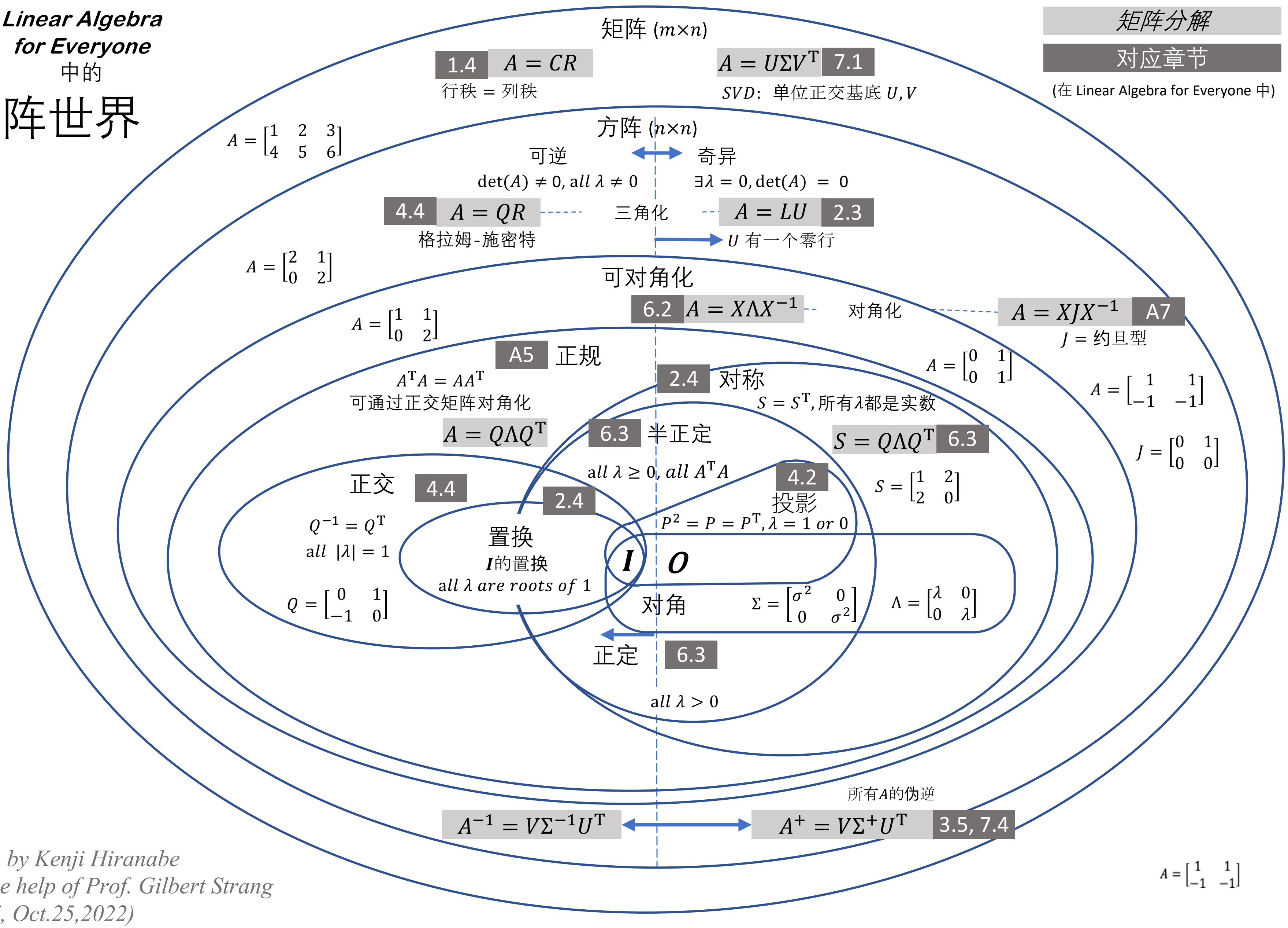

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