NONLINEAR ANALYSIS OF TRUSSES

Part I

A Simple Example of Nonlinear Behaviour

Bence Balogh

Introduction

This document is the first of a sequence materials, intended to introduce the basic aspects of nonlinear behaviour and numerical analysis of structures. We will start with the simplest of solutions of a shallow truss system (Figure 1) accounting only for small strains and linear material behaviour, and then we are going to drop these limitations one-by-one in later parts of the series. Finally, we are going to extend our solution to be able to handle large systems by means of an efficient and systematic numerical approach implemented in Python.

Problem Statement

We have a simple truss supported with a spring, shown in Fig. 1. The members of the truss have length l, cross-sectional area A and Young's modulus E, the spring is characterized by the stiffness parameter k. Here, A and l are understood in the actual configuration. At the initial configuration, displacement-like quantities are eppended with a zero index, hence l_0 and A_0 are the initial length and area of the members. Here in this document, we will use the approximation $A = A_0$ and use an engineering measure of strain. Otherwise, the formulation allows for arbitrary large strains and displacements, which is enough to result in a nonlinear behaviour.

> The parameter set $\mathbf{P} = \{b = 10m, h = 0.5m, EA = 1000kN/m^2, k = 1kN/m, F = 1.5kN\}$ defines a well-behaving problem.

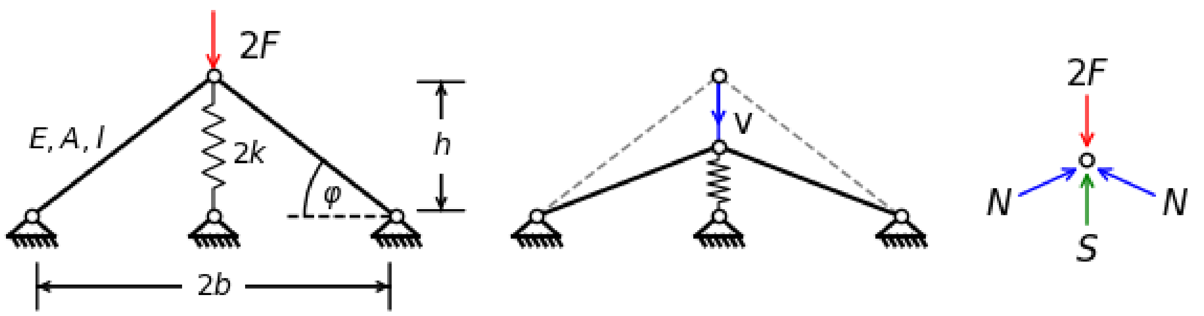

Figure 1: Geometry and parameters of a shallow truss system, with a free-body diagram (FBD) of the middle hinge.

QUESTIONS

A : What is the value of the unknown displacement v as a response of the known force F?

B : What is the value of the unknown force F, whose response is the known displacement v?

C : How does the behaviour of the structure depend on the parameters?

Equilibrium Approach

The first thing to note is that the structure has one degree of freedom. This means that the vertical displacement v of the middle hinge, the inclination angle φ and the length of the members l mutually define each other. If any of these three values are given, the other two can easily be calculated. This further suggests, that all unknowns in the system (and hence the governing equations) could be formulated as functions of one unknown parameter. For now, we select the parameter v as our control parameter, and we are going to express other quantities as functions of v, thus:

$$l(v) = \sqrt{(h-v)^2 + b^2}, \tag{1}$$

$$\varphi(v) = arcsin\left(\frac{h-v}{l}\right). \tag{2}$$

From elementary considerations we can say, that the equilibrium of the external and internal forces at the middle hinge requires that

$$2F - S(v) - 2N(v)sin(\varphi) = 0 \tag{3}$$

for any given value of the variables v and φ. To express the unknown internal forces S and N as functions of the control parameter v, we use the material equations. Since the material behaviour is assumed to be linear, we use a simple Hooke-model, wich for the spring gives

$$S(v) \approx 2kv. \tag{4}$$

To express the internal force N as a function of our control parameter v, we need to construct a material model for the trusses in the system, where analogously to the spring we assume, that the force in the truss is **linearly proportional** to its elongation. If the **factor of proportionality** is denoted with K, the expression takes the following form:

$$N(\Delta l) \approx K\Delta l, \tag{5}$$

where Δl is the elongation of a single member at a deformed configuration, that is

$$\Delta l(v) = l(v) - l_0. \tag{6}$$

Now we are in the position to rewrite the equilibrium equation (7) as

$$R(v) = F - kv - K\left(l(v) - l_0\right)\frac{h-v}{l(v)} = 0, \tag{7}$$

where the expression $R(v)$ is a nonlinear function of the parameter v. The actual form of eq. (7) is governed by the way the unknown forces are depicted on the FBD of Figure 1, which is arbitrary. However, in this simple case we know from intuition, that if $F > 0$, then $v > 0$ and from eq. (6) it follows that $\Delta l < 0$. Also from simple intuition, $F > 0$ and the applied sign conventions together imply that $N > 0$. It follows, that at an admissible solution N and Δl differ in sign, so for any real material, the parameter K in eq. (5) must be positive *for this particular set of sign conventions*. Furthermore, if $N > 0$, then according to the FBD of Fig. 1 the truss members are in compression, which contradicts the customary conventions of engineering practice, where a positive force in a truss member ought to mean tension. To rule out this awkwardness, it is more practical to use the modified residual

$$R(v) = F - kv + K\left(l(v) - l_0\right)\frac{h-v}{l(v)} = 0. \tag{8}$$

We can use this expression to obtain the vertical displacement v of the middle hinge as a response to the external load F. If we say that we have a candidate solution $\hat{v}$, the quantity $R(\hat{v})$ is called the **residual**, and has the meaning of an unbalanced external force. If the residual for a given candidate is other than zero, we say that it's not an **admissible solution** of the problem. In other words, the solution of the problem v^* is recognized by having a zero residual, that is $R(v^*) = 0$. So, we can answer Question A by using eq. (8). Two answer Question B, we can use the same expression, but this time, we consider F as the unknown, and v to be provided. From (8):

$$Q(F) = F - kv + K\left(l(v) - l_0\right)\frac{h - v}{l(v)} = 0. \tag{9}$$

Let's note a few things before moving on:

1. The residual is not a sign definite quantity, it might be positive, negative or zero.
2. The approximations of equations (4) and (5) don't necessarily mean a limitation. They simply mean, that we assume a linear relationsip of some quantities, which may not be true. In fact, there are a decent amount of materials, especially those related to civil engineeirng practice, that are in good agreement with these approximations, at least until failure. Apart from that, equations (8) and (9) do not contain other approximations, and both strains and displacements can be arbitrarily large.

§ Stress, Strain and Displacement

So far we didn't utilize the concepts of stress and strain, simply because we didn't need to, and the material parameters k and K in equations (4) and (5) can be readily determined from simple experiments. The reason why we could get away with this is that the kinematics of a one-dimensional truss model is so simple, that at this point there is no real difference between a truss member and a simple spring. The fact that truss members have cross sections and thus usually are much stiffer than springs, is insignificant from the point of the present discussion. The different stress and strain measures are going to be introduced in Part II of this series. Nonetheless, it is informative to reformulate our expressions by using simple concepts of stress and strain. Fow now, let use the well-known engineering stress and strain definitions

$$\sigma_{eng} = \frac{N}{A_0} \quad \text{and} \quad \varepsilon_{eng} = \frac{\Delta l}{l_0} \tag{10}$$

and a Hooke-model

$$\sigma_{eng} = E\varepsilon_{eng} \tag{11}$$

where E is the Young's modulus of the material. Then, combining equations (10) and (11) we get

$$N(\Delta l) = EA_0\frac{\Delta l}{l_0}, \tag{12}$$

and the modified residual expression (8) takes the form

$$R(v) = F - kv + EA_0\frac{l(v) - l_0}{l_0}\frac{h - v}{l(v)} = 0, \tag{13}$$

from which the relationship between the material parameters K and E can be observed as

$$K = E\frac{A_0}{l_0}. \tag{14}$$

§ Force- and Displacement-Control

To find the roots of eq. (8) or (9), we can use Newton's method, which, upon provided a function $f(x)$ and an initial guess x_0, determines the next solution by the condition

$$f(x_0 + \Delta x) \approx f(x_0) + \Delta x f^{'}(x_0) = 0. \tag{15}$$

From here, an initial guess x_0 and the formula

$$x_{i+1} = x_i - \frac{f(v_i)}{f'(v_i)} \tag{16}$$

defines a sequence, and it can be show that among certain circumstances, the sequence converges to zero. Examples of solutions for the parameter set $\mathbf{P}$ are shown in Fig. 2. On the left, we show the result of a so called **force-controlled** analysis, where we increase the load in small steps and determine the solution of (8). On the right, we show the result of a so called **displacement-controlled** analysis, where we increase the displacement in small steps and determine the solution of (9).

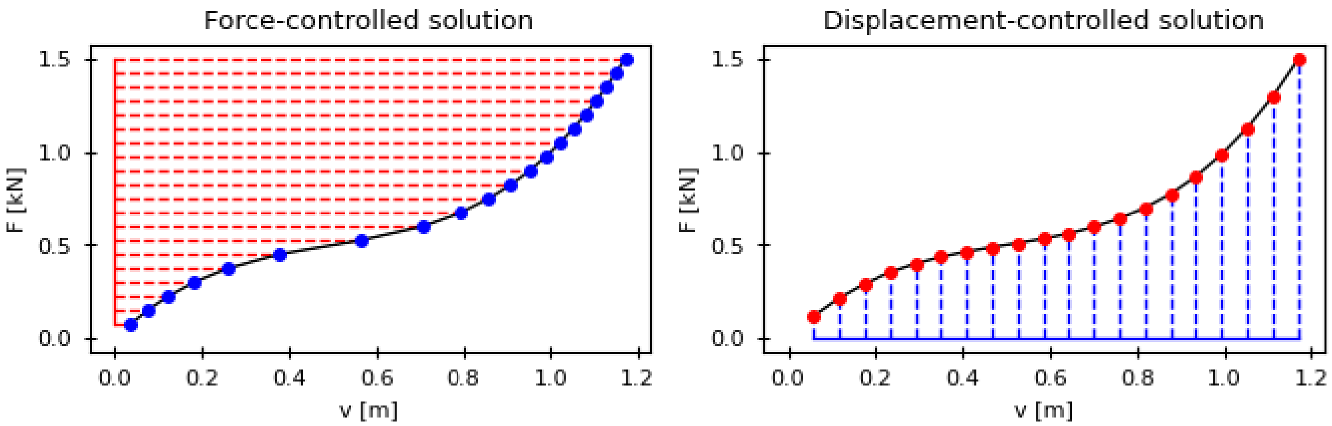

Figure 2: Load history diagrams created by finding the roots of the equilibrium equations.

§ The Error Measure

We already know about one way of how to answer our original question. Namely, we can use equation (7) or (8) and Newton's method to find the roots of it. The roots have the meaning of displacement, hence the presented methodology historically classifies as a **displacement method**, which is a family of methods where the primary of unknowns are *displacement-like* quantities. However, we know that at the solution v^*, the residual must be zero. Now let's define the function

$$G(v) = R(v)^2, \quad v \in \mathbb{R}. \tag{17}$$

We can observe, that $G(v) \in \mathbb{R}_{>0}$, and consequently, that from all the possible values $G(v)$ can take, it takes the smallest at v^*, if and only if $R(v^*) = 0$ holds. In other words, v^* is an admissible solution of the equilibrium equations (7) or (8) if and only if v^* **is a minimizer of** G. More formally:

$$G(v^*) \leq G(v), \; \forall v \in \mathbb{R} \quad \Longleftrightarrow \quad R(v^*) = 0.$$

Of course, we can also define a custom error measure for displacement-controlled analysis:

$$H(F) = Q(F)^2, \quad F \in \mathbb{R}. \tag{18}$$

and say

$$H(F^*) \leq H(F), \; \forall F \in \mathbb{R} \quad \Longleftrightarrow \quad Q(F^*) = 0.$$

The solution of the related optimization problems is discussed in the next section. In a later part of this series, we will discuss the generalization of the approach presented here. We will call it the **method of weighted residuals**, and the particular method presented here will be recognized as the **method of lest-squares**, a special case of the weighted-residual approach.

Energy Approach

We can also find the answer to our original questions by using one of the variational principles of mechanics. To use the **principle of minimum total potential energy**, we need to form an expression of the total potential energy of the system.

> The total potential energy $\mathbf{\Pi}$ is defined as the sum of the elastic strain energy, $\mathbf{U}$, stored in the deformed body and the potential energy, $\mathbf{V}$, associated to the applied forces:
> $$\mathbf{\Pi} = \mathbf{U} + \mathbf{V}$$

Without the details now, for our example this looks like

$$\mathbf{\Pi}(v) = \frac{1}{2}(2k)v^2 + \frac{1}{2}K\Delta l(v)^2 + \frac{1}{2}K\Delta l(v)^2 - 2Fv = kv^2 + K\Delta l(v)^2 - 2Fv. \tag{19}$$

According to the principle, at low temperatures a structure or body shall deform or displace to a position that (locally) minimizes the total potential energy, with the lost potential energy being converted into kinetic energy. We can apply this principle to the problem at hand, and say that

> $v^* \in \mathbb{R}$ is an admissible equilibrium solution if and only if $\mathbf{\Pi}(v)$ takes its minimum at v^*.

Parameter Study

To answer Question C, we usually try to explore a meaningful portion of the parameter space. Figure 3 shows a comparison of force- and displacement-controlled load histories for the parameter set $\mathbf{P}$, but with a sequence of spring stiffness values (k_1, k_2, k_3). On the right of the figure, we can see the internal force distributions of the springs and the members in correspondance with the three spring stiffnesses. Note, that the internal force history in the members is the same for all three configurations.

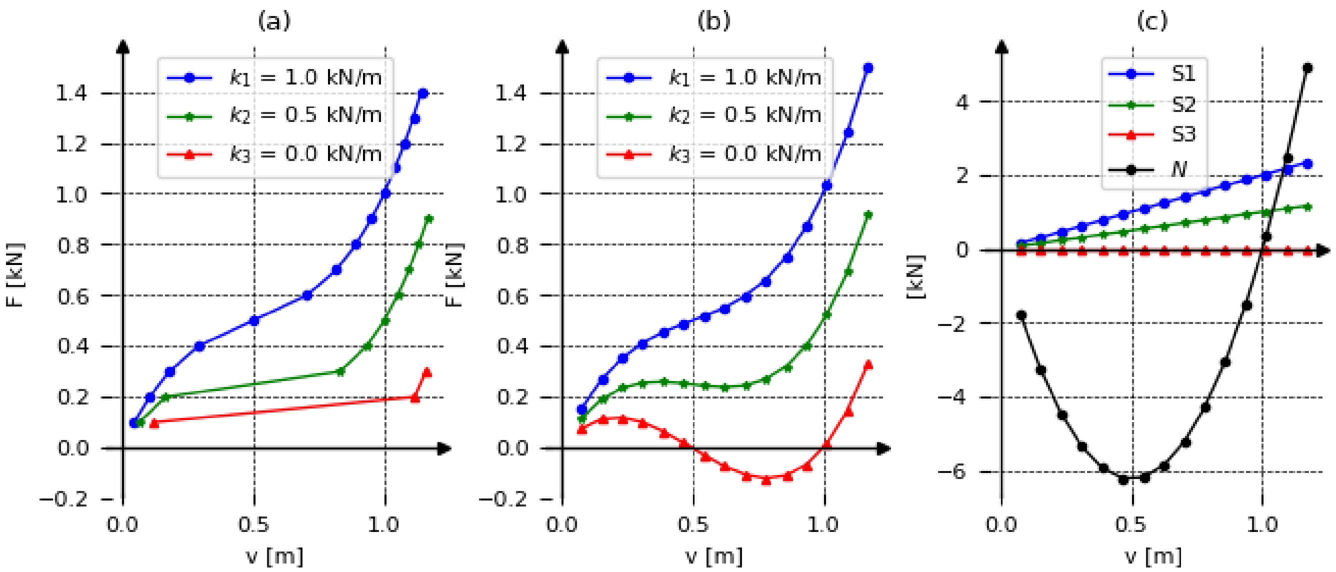

Figure 3: Load history for different configurations.
(a) force-ctonrol, (b) displacement-control, (c) internal-forces.

Related Content

There are two Jupyter Notebooks related to this document. The first one shows how to find roots and minimizers to our functions using *scipy*, the second shows how the figures of this document can be reproduced using *matplotlib*.