수치해석 | 과제 #4

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본과제에 작성된 내용은 (아래에 작성한 바와 같이 일부 참고문헌의 도움을 받거나 동료들과 상의는 하였지만) 기본적으로는 본인 스스로 해결하고 작성한 것임을 서약합니다. 서약이 거짓임이 밝혀지면 F학점을 받는 것에 동의합니다.

2022/5/9 신상원 신상원 참고문헌: 교재

4.3 - #25.

degree of precision n :

$E(x^k) = 0$ for each $k = 0, 1, \cdots, n$ & $E(x^{k+1}) \neq 0$

$\Longleftrightarrow$ $E(P(x)) = 0$ for all polynomials $P(x)$ of degree $k = 0, 1, \cdots, n$ but $E(P(x)) \neq 0$ for some polynomial $P(x)$ of degree $n+1$.

($\Rightarrow$) Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

최대 n차 다항식.

$$E(P(x)) = E(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)$$
$$= a_n E(x^n) + a_{n-1} E(x^{n-1}) + \cdots + E(a_0)$$
$$= 0 + 0 + 0 + \cdots + 0$$
$$= 0 \quad (\because \text{degree of precision } n)$$

Let $P_{n+1}(x) = a_{n+1} x^{n+1} + P(x)$, $a_{n+1} \neq 0$.

$n+1$차 다항식

$$E(P_{n+1}(x)) = E(a_{n+1} x^{n+1}) + 0$$
$$= a_{n+1} E(x^{n+1}) \neq 0$$

$\therefore$ n차 이하 모든 다항식 $P(x)$에서 $E(P(x)) = 0$ 이고

어떤 $n+1$차 다항식 $P(x)$에서 $E(P(x)) \neq 0$ 이다.

($\Leftarrow$) n차 이하 다항식에서 $E(P(x)) = 0$ 이므로

$\therefore$ $E(x^k) = 0$ for each $k = 0, 1, \cdots, n$

Let $P_{n+1}(x) = a_{n+1} x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$

$a_{n+1} \neq 0$ 인 $n+1$차 다항식 생각.

가정에 의해 $E(P_{n+1}(x)) \neq 0$

이제 $E(x^{n+1})$을 구해보면

$$x^{n+1} = \frac{P_{n+1}(x)}{a_{n+1}} - \frac{a_n x^n}{a_{n+1}} - \frac{a_{n-1} x^{n-1}}{a_{n+1}} - \cdots - \frac{a_0}{a_{n+1}}$$

$$E(x^{n+1}) = \frac{E(P_{n+1}(x))}{a_{n+1}} - \frac{a_n}{a_{n+1}} E(x^n) - \cdots - E\left(\frac{a_0}{a_{n+1}}\right)$$
$$= \frac{E(P_{n+1}(x))}{a_{n+1}} - 0 - 0 = \frac{E(P_{n+1}(x))}{a_{n+1}} \neq 0$$

$\therefore$ $E(x^{n+1}) \neq 0$.

$\therefore$ degree of Precision 은 n이다.

#26.

error term 이 $K f^{(4)}(\xi)$ 이므로 $f(x)$가 3차 다항식 일때 까지는 exact value를 가진다.

$$\int_{x_0}^{x_2} x \, dx = a_0 x_0 + a_1 x_1 + a_2 x_2$$

$$\int_{x_0}^{x_2} x^2 \, dx = a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2$$

$$\int_{x_0}^{x_2} x^3 \, dx = a_0 x_0^3 + a_1 x_1^3 + a_2 x_2^3$$

$x_1 = x_0 + h$, $x_2 = x_0 + 2h$.

$$\frac{x_2^2 - x_0^2}{2} = a_0 x_0 + a_1 x_1 + a_2 x_2$$

$$\frac{x_0^2 + 4x_0 h + 4h^2 - x_0^2}{2} = 2x_0 h + 2h^2$$

$$= a_0 x_0 + a_1 x_0 + a_1 h + a_2 x_0 + 2a_2 h$$

$$2h x_0 + 2h^2 = (a_0 + a_1 + a_2) x_0 + (a_1 + 2a_2) h$$

$$\therefore a_0 + a_1 + a_2 = 2h, \quad a_1 + 2a_2 = 2h$$

$$\frac{x_0^3 + 8h^3 + 12x_0 h^2 + 6x_0^2 h}{3} = a_0 x_0^2 + a_1 (x_0 + h)^2 + a_2 (x_0 + 2h)^2$$

$$\frac{8}{3}h^3 + 4h^2 x_0 + 2h x_0^2 = (a_0 + a_1 + a_2) x_0^2 + (2h a_1 + 4h a_2) x_0 + a_1 h^2 + a_2 4h^2$$

$$\therefore a_1 + 4a_2 = \frac{8}{3}h$$

$$\rightarrow a_0 = \frac{h}{3}, \quad a_1 = \frac{4}{3}h, \quad a_2 = \frac{h}{3}$$

$f(x) = x^4$ 일때 $f^{(4)}(\xi)$가 상수 이므로 K값 결정 가능

$$\int_{x_0}^{x_2} x^4 dx = \frac{x_2^5 - x_0^5}{5} = \frac{(x_0+2h)^5 - x_0^5}{5} = a_0 x_0^4 + a_1 (x_0 + h)^4 + a_2 (x_0 + 2h)^4 + 24K$$

상수 비교 : $\frac{(2h)^5}{5} = \frac{4}{3}h \cdot h^4 + \frac{h}{3}(2h)^4 + 24K$

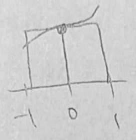
$$\frac{32}{5}h^5 = \frac{4}{3}h^5 + \frac{16}{3}h^5 + 24K$$

$$\underline{K = -\frac{1}{90}h^5}$$

$$\therefore \int_{x_0}^{x_2} f(x)\, dx = \frac{h}{3}f(x_0) + \frac{4}{3}h f(x_1) + \frac{h}{3}f(x_2) - \frac{h^5}{90}f^{(4)}(\xi)$$

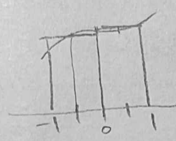
4.4 #10

by midpoint rule



$f(0) \times 2 = 12, \quad f(0) = 6$

by composite midpoint rule $\underline{n=2}$



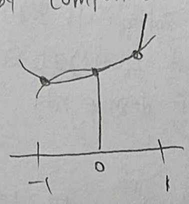
$f(-0.5) + f(0.5) = 5$

$f(-0.5) = f(0.5) - 1$

$f(-0.5) = 2$

$f(0.5) = 3$

by composite simpson's rule



$\frac{1}{3}(f(-1) + 4f(0) + f(1)) = 6$

$f(-1) + f(1) = -6$

$f(-1) = f(1)$

$f(-1) = -3 = f(1)$

$\therefore f(-1) = -3$

$f(-0.5) = 2$

$f(0) = 6$

$f(0.5) = 3$

$f(1) = -3$

4.5 - #11

$R_{11} = 8$, $R_{22} = \frac{16}{3}$, $R_{33} = \frac{208}{45}$

R_{11}

$\underbrace{R_{21}}$ R_{22}

$\underbrace{R_{31}}_{h^2}$ $\underbrace{R_{32}}_{h^4}$ $\underbrace{R_{33}}_{h^6}$

$R_{22} = R_{21} + \frac{1}{3}(R_{21} - R_{11})$

$\frac{16}{3} = \frac{4}{3}R_{21} - \frac{8}{3}$, $R_{21} = 6$

$R_{33} = R_{32} + \frac{1}{15}(R_{32} - R_{22})$

$\frac{208}{45} = \frac{16}{15}R_{32} - \frac{16}{45}$, $R_{32} = \frac{14}{3}$

$R_{32} = R_{31} + \frac{1}{3}(R_{31} - R_{21})$

$\frac{14}{3} = \frac{4}{3}R_{31} - \frac{6}{3}$, $\boxed{R_{31} = 5}$

4.6 - #3.

(a) $\left| \int_1^{1.5} x^2 \ln x \, dx - \text{근사} \right| < 10^{-3}$

$S(1, 1.5) = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$

$h = 0.25$ $= 0.1922453074$

$S(1, 1.25) = 0.03937243404$

$S(1.25, 1.5) = 0.1528860264$

$|S(1, 1.5) - S(1, 1.25) - S(1.25, 1.5)|$

$= +1.315304 \times 10^{-5}$

$< 15\varepsilon$ 이므로

$8.768669 \times 10^{-7} \varepsilon$ 이고 이는

이미 충분히 잘 근사됨을 의미한다.

$\therefore \int_1^{1.5} x^2 \ln x \, dx \approx S(1, 1.25) + S(1.25, 1.5)$

$= 0.1922584604$

(b) $\int_0^1 x^2 e^{-x} \, dx$

$S(0, 1) = 0.1624016835$

$S(0, \frac{1}{2}) = 0.0288610742$

$S(\frac{1}{2}, 1) = 0.1318614041$

$|S(0,1) - S(0, \frac{1}{2}) - S(\frac{1}{2}, 1)| = 1.679207768 \times 10^{-3}$

$\to \left| \int_0^1 x^2 e^{-x} dx - S(0, \frac{1}{2}) - S(\frac{1}{2}, 1) \right| < 1.11947 \times 10^{-4}$

$\therefore \int_0^1 x^2 e^{-x} dx \approx S(0, \frac{1}{2}) + S(\frac{1}{2}, 1)$

$= 0.1607224758$

(c) $\int_0^{0.35} \frac{2}{x^2 - 4} \, dx$

$S(0, 0.35) = -0.1768215692$

$S(0, 0.175) = -0.0877243828 6$

$S(0.175, 0.35) = -0.0890957362 7$

$|S(0, 0.35) - S(0, 0.175) - S(0.175, 0.35)| = 1.45007 \times 10^{-6}$

$\to \left| \int_0^{0.35} \frac{2}{x^2-4} dx - S(0, 0.175) - S(0.175, 0.35) \right| < 9.667 \times 10^{-}$

$\therefore \int_0^{0.35} \frac{2}{x^2-4} dx \approx S(0, 0.175) + S(0.175, 0.35)$

$= -0.1768201191$

(d) $\int_0^{\frac{\pi}{4}} x^2 \sin x \, dx$

$S(0, \frac{\pi}{4}) = 0.08799566847$

$S(0, \frac{\pi}{8}) = 0.005831519716$

$S(\frac{\pi}{8}, \frac{\pi}{4}) = 0.08287762423$

$|S(0, \frac{\pi}{4}) - S(0, \frac{\pi}{8}) - S(\frac{\pi}{8}, \frac{\pi}{4})| = -7.1353 \times 10^{-}$

$\to \left| \int_0^{\frac{\pi}{4}} x^2 \sin x - S(0, \frac{\pi}{8}) - S(\frac{\pi}{8}, \frac{\pi}{4}) \right| < -4.756 \times 10^{-5}$

$\therefore \int_0^{\frac{\pi}{4}} x^2 \sin x \, dx \approx S(0, \frac{\pi}{8}) + S(\frac{\pi}{8}, \frac{\pi}{4})$

$= 0.0887092039\overline{5}$

4.7 - #11

$$\int_{-1}^{1} f(x)\,dx = a f(-1) + b f(1) + c f'(-1) + d f'(1)$$

$f(x)$가 3차 함수까지 exact value

$\int_{-1}^{1} x\,dx = {}_{-1}^{1}\left[\frac{x^2}{2}\right] = 0 = -a+b+c+d$

$a = b+c+d$ --①

$\int_{-1}^{1} x^2\,dx = {}_{-1}^{1}\left[\frac{x^3}{3}\right] = \frac{2}{3} = a+b-2c+2d$

$a+b-2c+2d = \frac{2}{3}$ --②

$\int_{-1}^{1} x^3\,dx = 0 = -a+b+3c+3d$

$a = b+3c+3d$ --③

$\int_{-1}^{1} 1\,dx = 2 = a+b$

$a+b=2$ --④

$c+d=0$ (① & ③)

$\frac{4}{3} = 2c-2d, \ \frac{2}{3} = c-d$ (② & ④)

$c = \frac{1}{3}, \ d = -\frac{1}{3}$

$a=1, \ b=1$

$$\therefore \int_{-1}^{1} f(x)\,dx = f(-1) + f(1) + \frac{1}{3}f'(-1) - \frac{1}{3}f'(1)$$