Lagrange polynomial

$$W(x) = \sum_{i=0}^{n} w_i(x) f(x_i)$$

$$w_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

For instance for linear function:

$$W(x) = \sum_{i=0}^{1} w_i(x) f(x_i)$$

$$W(x) = \frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1)$$

For quadratic function:

$$W(x) = \sum_{i=0}^{2} w_i(x) f(x_i)$$

$$W(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Roots of Chebyshev polynomial – *n* roots:

$$z_m = \cos\left(\frac{2m+1}{2n}\pi\right), m = 0, 1, \ldots, n-1, z_m \in (-1,1)$$

For instance

$$n = 3 \Rightarrow m = 0, 1, 2$$

$$z_0 = \cos\left(\frac{2\cdot 0+1}{2\cdot 3}\pi\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},$$

$$z_1 = \cos\left(\frac{2\cdot 1+1}{2\cdot 3}\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0,$$

$$z_2 = \cos\left(\frac{2\cdot 2+1}{2\cdot 3}\pi\right) = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

Re-scaling to [*a*, *b*] interval:

$$x_m = \frac{b-a}{2} z_m + \frac{b+a}{2}, m = 0, 1, \ldots, n-1, x_m \in (a,b)$$

Cubic spline polynomial

$$y = Ay_j + By_{j+1} + Cy_j'' + Dy_{j+1}''$$

$$y \equiv W(x),\ y_j \equiv W(x_j) = f(x_j),\ y_j'' \equiv f''(x_j)$$

$$A = \frac{x - x_{j+1}}{x_j - x_{j+1}} = \frac{x_{j+1} - x}{x_{j+1} - x_j},$$

$$B = \frac{x - x_j}{x_{j+1} - x_j}$$

$$C = \frac{1}{6}(A^3 - A)(x_{j+1} - x_j)^2$$

$$D = \frac{1}{6}(B^3 - B)(x_{j+1} - x_j)^2$$

Second derivatives calculated in intrpolation nodes:

$$\frac{y_{j+1}-y_j}{x_{j+1}-x_j}-\frac{y_j-y_{j-1}}{x_j-x_{j-1}}=\frac{1}{6}(x_j-x_{j-1})y_{j-1}''+\frac{1}{3}(x_{j+1}-x_{j-1})y_j''+\frac{1}{6}(x_{j+1}-x_j)y_{j+1}''$$

$j=1,\ldots,n-1$

Assumption: $y_0'' = y_n'' = 0$

Example

	x_i	y_i
x_0	1	2
x_1	3	3.5
x_2	5	3.7

$$\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0} = \frac{1}{6}(x_1 - x_0)\overset{=0}{y_0''} + \frac{1}{3}(x_2 - x_0)y_1'' + \frac{1}{6}(x_2 - x_1)\overset{=0}{y_2''}$$

$$\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0} = \frac{1}{3}(x_2 - x_0)y_1''$$

$$y_1'' = \frac{3}{5-1}\left(\frac{3.7 - 3.5}{5-3} - \frac{3.5 - 2}{3-1}\right) = -0.4875$$

Polynomial

$$y = Ay_j + By_{j+1} + Cy_j'' + Dy_{j+1}''$$

$[x_0, x_1]$ interval, $y_0'' = 0$

	x_i	y_i
x_0	1	2
x_1	3	3.5
x_2	5	3.7

$$A = \frac{x - x_{j+1}}{x_j - x_{j+1}} = \frac{x-3}{1-3} = -\frac{1}{2}(x-3)$$

$$B = \frac{x - x_j}{x_{j+1} - x_j} = \frac{x-1}{3-1} = \frac{1}{2}(x-1)$$

$$D = \frac{1}{6}(B^3 - B)(x_{j+1} - x_j)^2 = \frac{1}{6}\left[\frac{1}{8}(x-1)^3 - \frac{1}{2}(x-1)\right](3-1)^2 =$$

$$= \frac{1}{12}\left(x^3 - 3x^2 - x + 3\right)$$

Polynomial:

$$\boxed{\begin{aligned} y &= -\frac{1}{2}(x-3)2 + \frac{1}{2}(x-1)3.5 + \frac{1}{12}(x^3 - 3x^2 - x + 3)(-0.4875) = \\ &= -0.040625x^3 + 0.121875x^2 + 0.790625x + 1.128125 \end{aligned}}$$

$[x_1, x_2]$ interval, $y_2'' = 0$

$$A = \frac{x - x_{j+1}}{x_j - x_{j+1}} = \frac{x-5}{3-5} = -\frac{1}{2}(x-5)$$

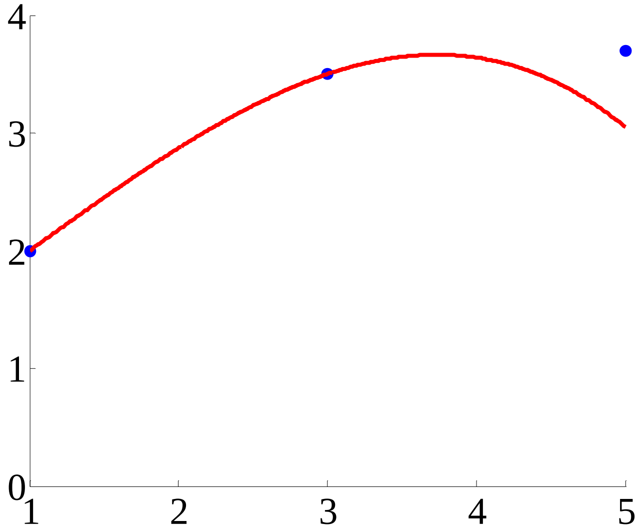
	x_i	y_i
x_0	1	2
x_1	3	3.5
x_2	5	3.7

$$B = \frac{x - x_j}{x_{j+1} - x_j} = \frac{x-3}{5-3} = \frac{1}{2}(x-3)$$

$$C = \frac{1}{6}(A^3 - A)(x_{j+1} - x_j)^2 = \frac{1}{6}\left[-\frac{1}{8}(x-5)^3 + \frac{1}{2}(x-5)\right](5-3)^2 =$$

$$= \frac{1}{12}\left(-x^3 + 15x^2 - 71x + 105\right)$$

$$\boxed{\begin{aligned} y &= -\frac{1}{2}(x-5)3.5 + \frac{1}{2}(x-3)3.7 + \frac{1}{12}(-x^3 + 15x^2 - 71x + 105)(-0.4875) = \\ &= -0.040625x^3 - 0.609375x^2 + 2.984375x - 1.065625 \end{aligned}}$$



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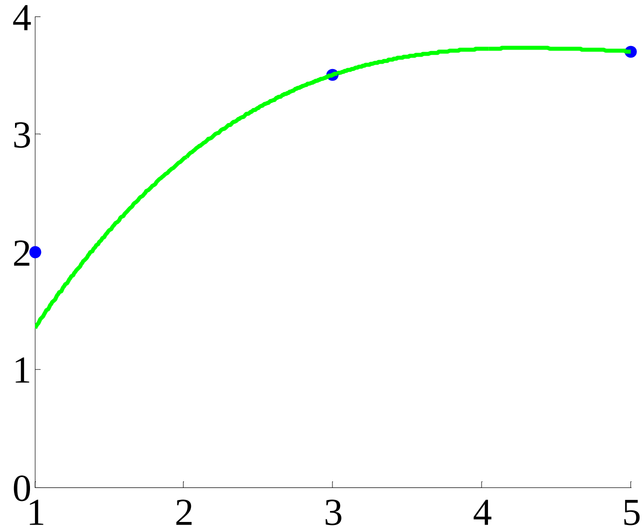

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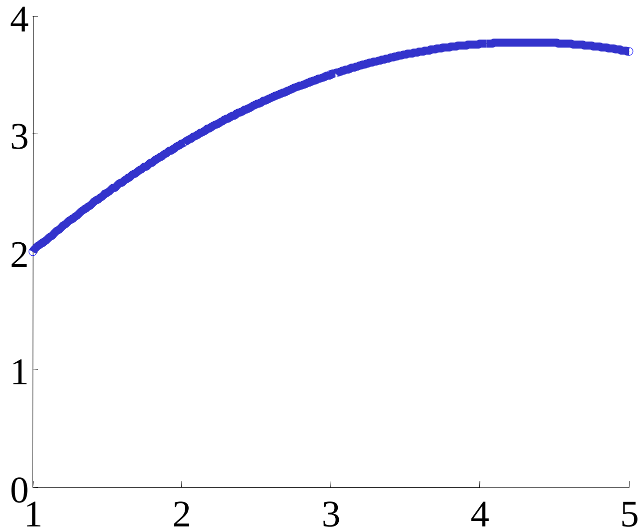

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