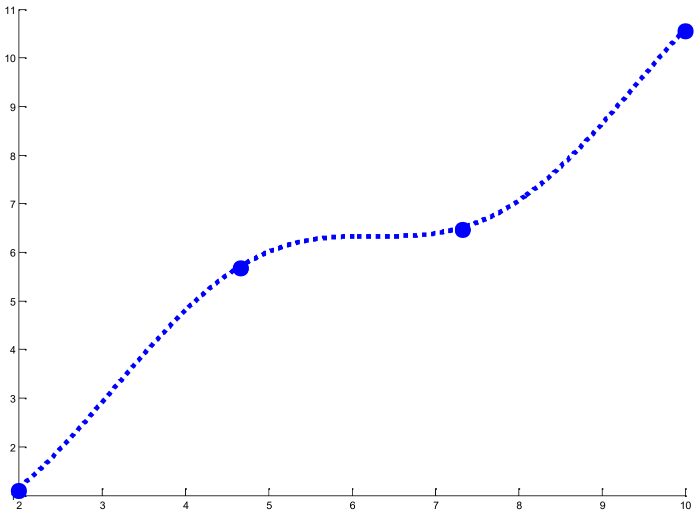
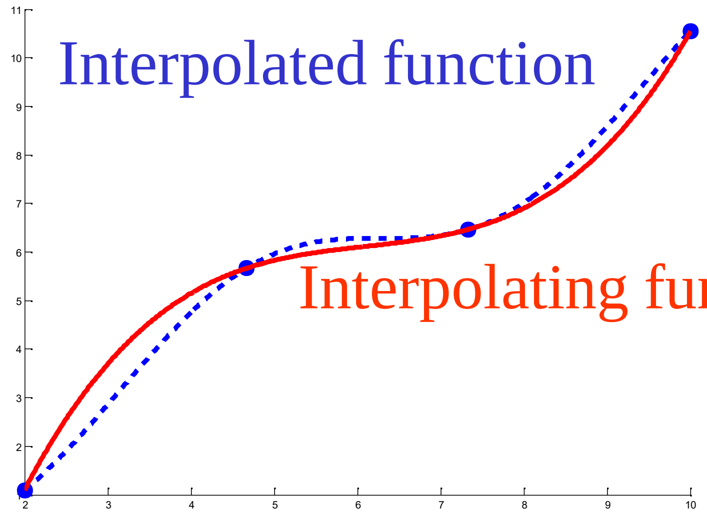
Interpolation

We have values of $f(x)$ in points: x_i i=0, 1, ..., n. We determine $W(x)$, for which:

$$W(x_i) = f(x_i)\quad i = 0,1,\ldots,n$$

Interpolation nodes

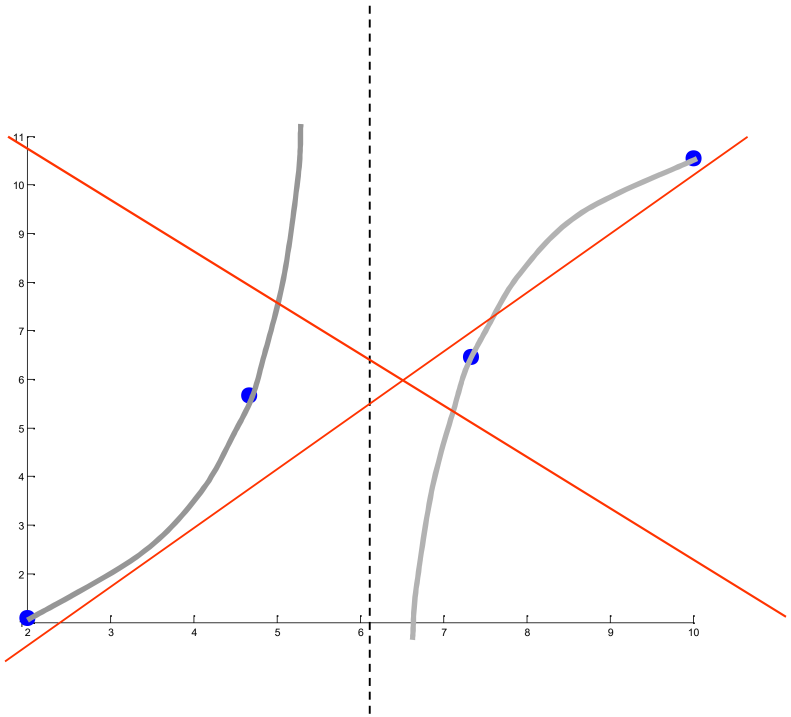


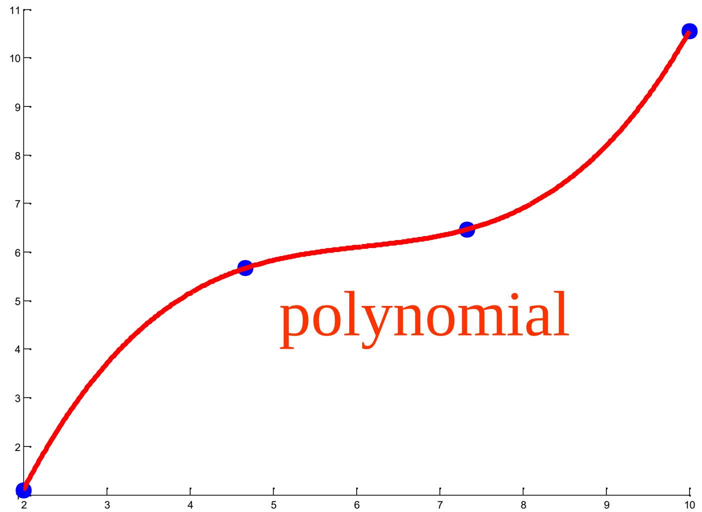


Necessary assumptions:

1) Interpolation on function domain

2) Chosen form of the function: polynomial





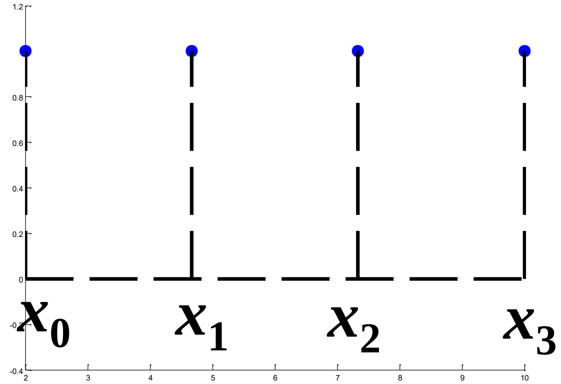
How to find the polynomial?

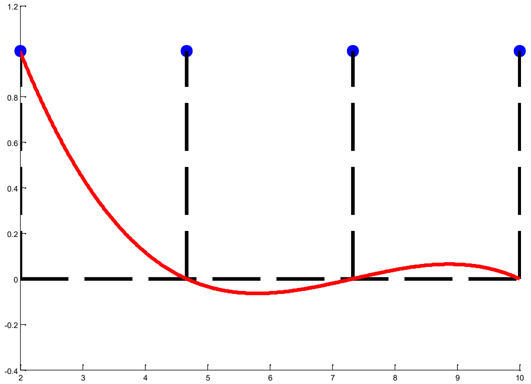
$$a_0 + a_1 x_0 + a_2 x_0^2 \ldots + a_n x_0^n = f(x_0)$$
$$a_0 + a_1 x_1 + a_2 x_1^2 \ldots + a_n x_1^n = f(x_1)$$
$$\ldots$$
$$a_0 + a_1 x_n + a_2 x_n^2 \cdots + a_n x_n^n = f(x_n)$$

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_0^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix}, \quad \det(V) \neq 0 \text{ iff } x_i \neq x_j \text{ for } i \neq j.$$

Inconvenient!

It’s better to use auxillary polynomials.





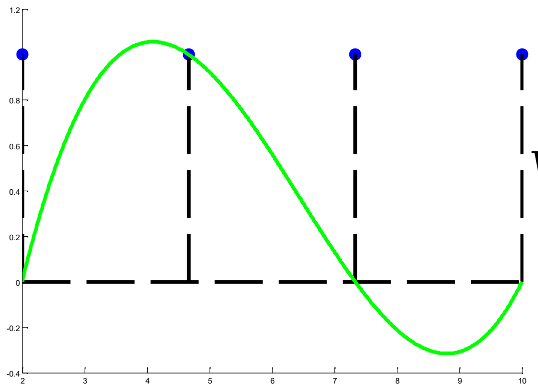
$$w_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$w_0(x_0) = \frac{(x_0-x_1)(x_0-x_2)(x_0-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = 1$$

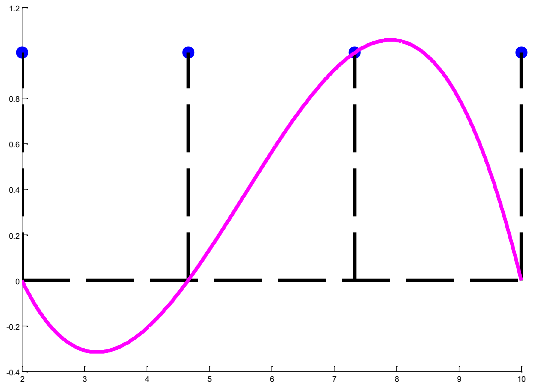
$$w_0(x_1) = \frac{\overset{=0}{(x_1-x_1)}(x_1-x_2)(x_1-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = 0$$

$$w_0(x_2) = \frac{(x_2-x_1)\overset{=0}{(x_2-x_2)}(x_2-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = 0$$

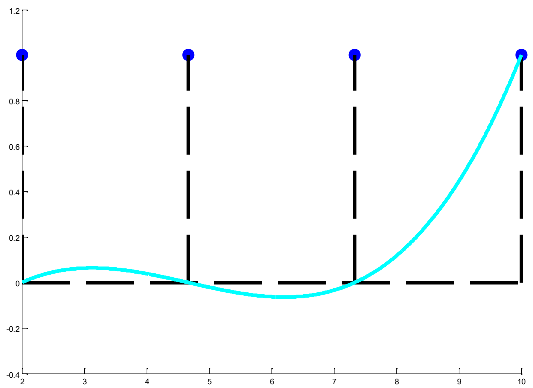
$$w_0(x_3) = \frac{(x_3-x_1)(x_3-x_2)\overset{=0}{(x_3-x_3)}}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = 0$$



$$w_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$



$$w_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$



$$w_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

Lagrange's formula

$$W(x) = \sum_{i=0}^{n} w_i(x) f(x_i)$$

$$w_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

For the linear interpolation:

$$W(x) = \sum_{i=0}^{1} w_i(x) f(x_i)$$

$$W(x) = \frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1)$$

For the square interpolation:

$$W(x) = \sum_{i=0}^{2} w_i(x) f(x_i)$$

$$W(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Example

$$W(x)=\sum_{i=0}^{3} w_i(x)\, f(x_i)$$

	x_i	$f(x_i)$
$i=0$	2	-0,750
$i=1$	3	8,000
$i=2$	4	1,875
$i=3$	6	1,250

$$w_0(x)=\frac{(x-3)(x-4)(x-6)}{(2-3)(2-4)(2-6)}=\frac{x^3-13x^2+54x-72}{-8}=$$

$$=-0.125x^3+1.625x^2-6.75x+9$$

$$w_1(x)=\frac{(x-2)(x-4)(x-6)}{(3-2)(3-4)(3-6)}=\frac{x^3-12x^2+44x-48}{3}=$$

$$=0.333x^3-4x^2+14.667x-16$$

$$w_2(x)=\frac{(x-2)(x-3)(x-6)}{(4-2)(4-3)(4-6)}=\frac{x^3-11x^2+36x-36}{-4}=$$

$$=-0.25x^3+2.75x^2-9x+9$$

$$w_3(x)=\frac{(x-2)(x-3)(x-4)}{(6-2)(6-3)(6-4)}=\frac{x^3-9x^2+26x-24}{24}=$$

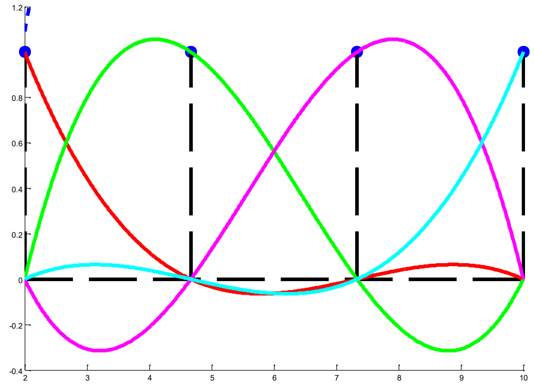
$$=0.0417x^3-0.375x^2+1.0833x-1$$

$$
\begin{aligned}
W(x) = {} & \left(-0.125x^3 + 1.625x^2 - 6.75x + 9\right)(-0.750) + \\
& + \left(0.333x^3 - 4x^2 + 14.667x - 16\right) \cdot 8.000 + \\
& + \left(-0.25x^3 + 2.75x^2 - 9x + 9\right) \cdot 1.875 + \\
& + \left(0.0417x^3 - 0.375x^2 + 1.08333x - 1\right) \cdot 1.250 = \\
& 0.09375x^3 - 1.21875x^2 + 5.0625x - 6.75 + \\
& + 2.66667x^3 - 32x^2 \quad + 117.33334x - 128 - \\
& - 0.46875x^3 + 5.15625x^2 - 16.875x + 16.875 + \\
& \quad 0.052083x^3 - 0.46875x^2 + 1.354167x - 1.25 = \\
= {} & 2.344x^3 - 28.531x^2 + 106.875x - 119.125
\end{aligned}
$$

Algorithm of calculations:

$$\begin{bmatrix} x-x_0 & x_0-x_1 & x_0-x_2 & \dots & x_0-x_n \\ x_1-x_0 & x-x_1 & x_1-x_2 & \dots & x_1-x_n \\ \dots & \dots & \dots & \dots & \dots \\ x_n-x_0 & x_n-x_1 & x_n-x_2 & \dots & x-x_n \end{bmatrix}. \qquad \frac{}{}$$

Sum of auxillary polynomials.

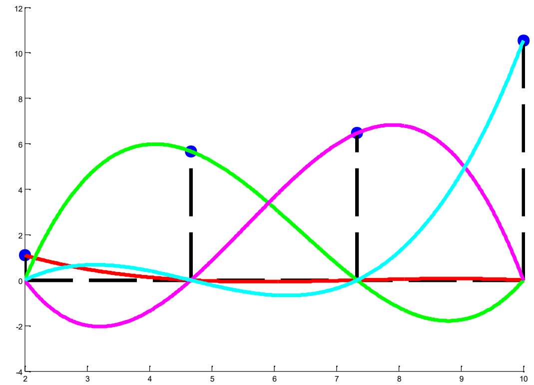


$w_0(x)$

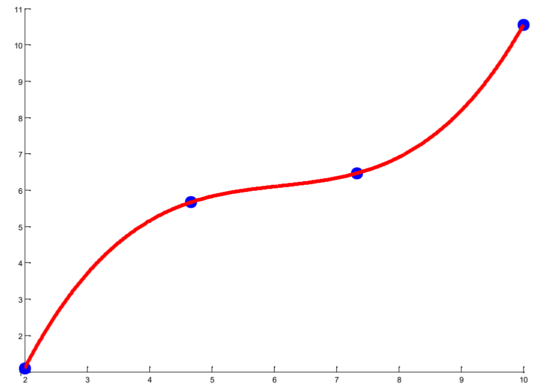
$w_1(x)$

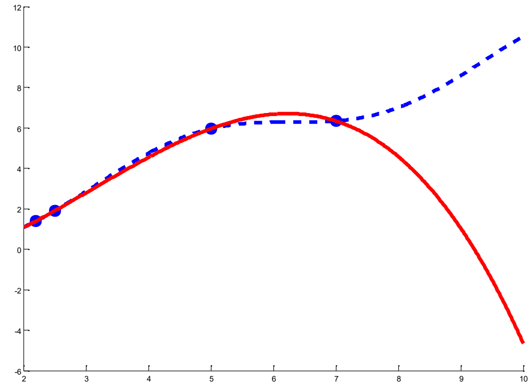
$w_2(x)$

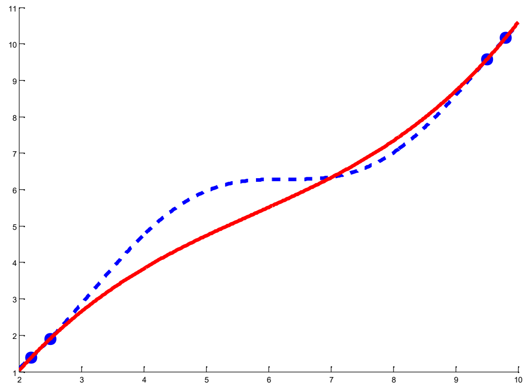
$w_3(x)$



$$w_0(x)f(x_0) + w_1(x)f(x_1) + w_2(x)f(x_2) + w_3(x)f(x_3) = W(x)$$







Interpolation nodes can be determined in such a way that the error ESTIMATION is minimal. From the definition of interpolation, we have:

$$R(x) = f(x) - W(x)$$

The formula for the error estimation uses a auxillary function $u(x)$:

$$u(x) = f(x) - W(x) - k\omega_{n+1}(x)$$

$$\omega_{n+1}(x) = (x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_i)(x-x_{i+1})\ldots(x-x_n)$$

function $u(x)$ is chosen in such a way that it has roots:

$x_0, \ldots, x_n$, $\boldsymbol{x^*}$, $x^* \in [a,b]$

For x_i, $i=1,\ldots, n$

$$(x-x_0)(x-x_1)\ldots(x-x_{i-1})\overset{=0}{(x-x_i)}(x-x_{i+1})\ldots(x-x_n)$$

$$u(x_i) = \underbrace{f(x_i) - W(x_i)}_{=0} - k\omega_{n+1}(x_i) = 0$$

k is set in such a way that $\boldsymbol{x^*}$ is a root. So, $u(x)$ has $n+2$ roots.

Rolle's theorem

If a real-valued function f is continuous on a closed interval $[a,b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that:

$$f'(c) = 0.$$

Since $u(x_i) = u(x_{i+1})$ (they equal 0), then in each of the intervals $u'(x)$ has one root, i.e.:

$u(x)$ has n+2 roots $\quad u(x) = f(x) - W(x) - k\omega_{n+1}(x)$

$u'(x)$ has n+1 roots

$u''(x)$ has n roots

...

$u^{(n+1)}(x)$ has 1 root

$W^{(n+1)}(x) = 0$ as W is n-ordered

$$\exists_{\xi\in[a,b]} u^{(n+1)}(\xi) = f^{(n+1)}(\xi) - 0 - k(n+1)! = 0$$

$\omega_{n+1}^{(n+1)}(x) = (n+1)!$

Eg.

$$\omega_3(x) = (x-x_0)(x-x_1)(x-x_2)$$

$$\omega_3'(x) = 1\cdot(x-x_1)(x-x_2) + (x-x_0)[(x-x_1)(x-x_2)]' =$$

$$= (x-x_1)(x-x_2) + (x-x_0)[1\cdot(x-x_2) + (x-x_1)\cdot 1] =$$

$$= (x-x_1)(x-x_2) + (x-x_0)(x-x_2) + (x-x_0)(x-x_1)$$

$$\omega_3''(x) = 1\cdot(x-x_2) + (x-x_1)\cdot 1 + 1\cdot(x-x_2) + (x-x_0)\cdot 1 + 1\cdot(x-x_1) + (x-x_0)\cdot 1$$

$$\omega_3^{(3)}(x) = 1+1+1+1+1+1 = 6 = 3!$$

$$f^{(n+1)}(\xi) - k(n+1)! = 0 \Rightarrow k = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

Since x^* is a root of $u(x)$, then:

$$u(x^*) = f(x^*) - W(x^*) - k\omega_{n+1}(x^*) = 0 \Rightarrow k = \frac{f(x^*) - W(x^*)}{\omega_{n+1}(x^*)}$$

x^* is any point in $[a, b]$ interval (except form nodes, for which the error = 0), then:

$$\frac{f^{(n+1)}(\xi)}{(n+1)!} = \frac{f(x) - W(x)}{\omega_{n+1}(x)} \Rightarrow \boxed{f(x) - W(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)}$$

$$\boxed{\left|f(x) - W(x)\right| = \frac{M_{n+1}}{(n+1)!}\left|\omega_{n+1}(x)\right|, \quad M_{n+1} = \sup_{x\in[a,b]}\left|f^{(n+1)}(x)\right|}$$

The smallest error estimation is obtained for nodes determined in the points responding Czebyszew polynomial roots.

This roots are:

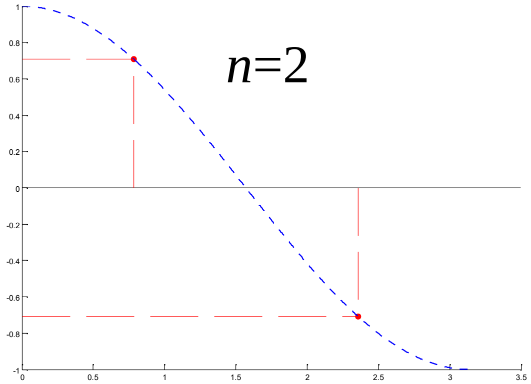
$$z_m = \cos\left(\frac{2m+1}{2n}\pi\right), m = 0, 1, \ldots, n-1, z_m \in (-1,1)$$

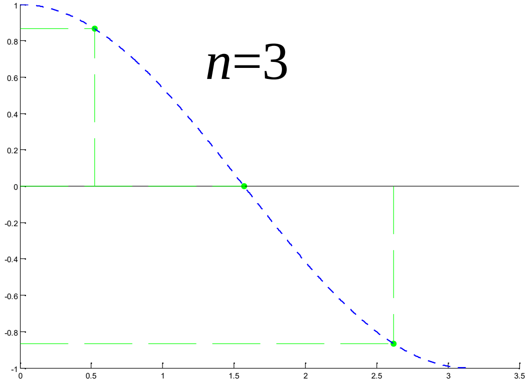
Eg. $n = 3 \Rightarrow m = 0, 1, 2$

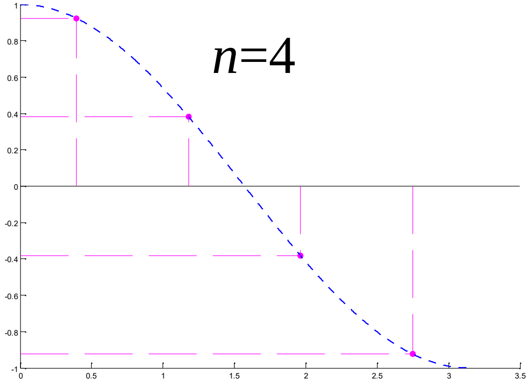
$$z_0 = \cos\left(\frac{2\cdot 0+1}{2\cdot 3}\pi\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},$$

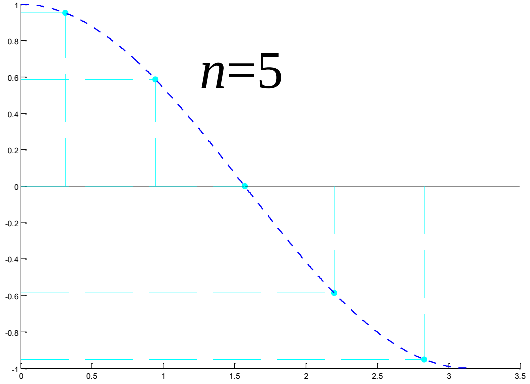
$$z_1 = \cos\left(\frac{2\cdot 1+1}{2\cdot 3}\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0,$$

$$z_2 = \cos\left(\frac{2\cdot 2+1}{2\cdot 3}\pi\right) = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$









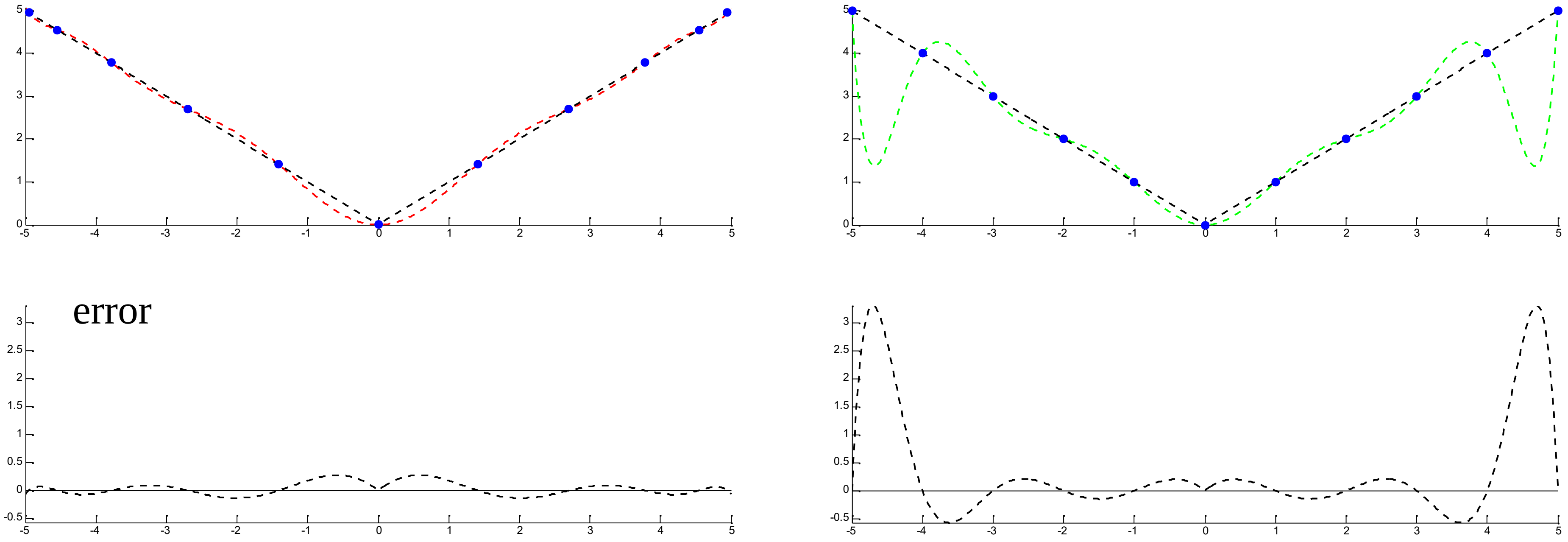
Scaling to [a, b]:

$$x_m = \frac{b-a}{2} z_m + \frac{b+a}{2}, m = 0, 1, \ldots, n-1, x_m \in (a, b)$$

$f(x)=|x|$, n=10

Function and polynomial

error



Example

$$f(x)=\sqrt{x},\quad x\in[1,4],\quad n=3$$

$$f(x)=\sqrt{x},\ f'(x)=\frac{1}{2}x^{-\frac{1}{2}};\ f''(x)=-\frac{1}{4}x^{-\frac{3}{2}};\ f^{(3)}(x)=\frac{3}{8}x^{-\frac{5}{2}};\ f^{(4)}(x)=-\frac{15}{16}x^{-\frac{7}{2}};$$

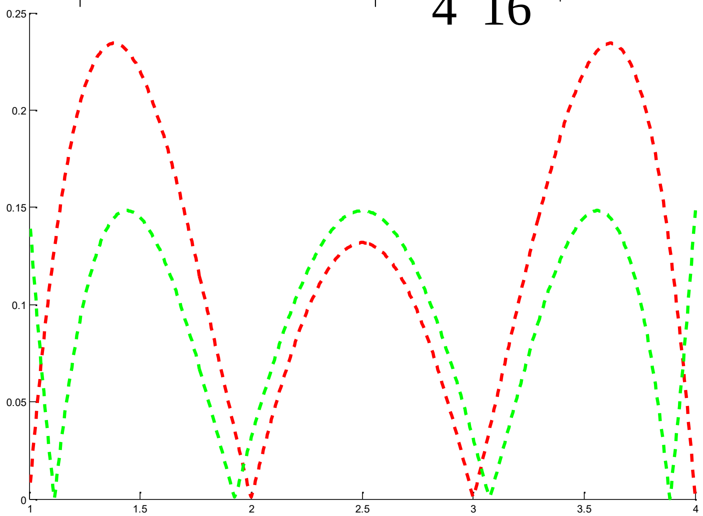
$$M_4=\sup_{x\in[1,4]}\left|-\frac{15}{16}\frac{1}{\sqrt{x^7}}\right|=\frac{15}{16}\frac{1}{\sqrt{1^7}}=\frac{15}{16}$$

			x	
	1,25	1,69	2,25	3,24
oszac. bł. rów.	0,2115	0,1517	0,0961	0,1187
oszac. bł. Czeb.	0,1035	0,0968	0,1163	0,0702
bł. rów.	0,0029	0,0017	0,0009	0,0008
bł. Czeb.	0,0014	0,0011	0,001	0,0005

For nodes

x	f(x)
1	1
2	1,41
3	1,73
4	2

estimation $$\left|\sqrt{x}-W_{równe}(x)\right|\le\frac{1}{4}\cdot\frac{15}{16}\cdot\left|(x-1)(x-2)(x-3)(x-4)\right|$$



x	f(x)
1,11	1,06
1,93	1,39
3,07	1,75
3,89	1,97

estimation $$\left|\sqrt{x}-W_{Czeb}(x)\right|\le\frac{1}{4}\cdot\frac{15}{16}\cdot\left|(x-1.11)(x-1.93)(x-3.07)(x-3.89)\right|$$

If distances among interpolation nodes are equal, the Newton's interpolation formula is the most convenient. To this end forward, backward and central differences.

Forward differences:

$$\Delta^0 f(x) = f(x)$$

$$\Delta^k f(x) = \Delta^{k-1} f(x+h) - \Delta^{k-1} f(x),\ \ k = 1,2,\ldots$$

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
1.5	2				
		2			
2.5	4		2		
		4		-7	
3.5	8		-5		3
		-1		-4	
4.5	7		-1		
		-2			
5.5	5				

Backward differences:

$$\nabla^0 f(x) = f(x)$$

$$\nabla^k f(x) = \nabla^{k-1} f(x) - \nabla^{k-1} f(x-h),\ \ k = 1,2,\ldots$$

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
1.5	2				
		2			
2.5	4		2		
		4		-7	
3.5	8		-5		3
		-1		-4	
4.5	7		-1		
		-2			
5.5	5				

Central differences:

$$\delta^0 f(x) = f(x)$$

$$\delta^k f(x) = \delta^{k-1} f(x + \frac{1}{2}h) - \delta^{k-1} f(x - \frac{1}{2}h), \ k = 1,2,\ldots$$

x_0 1.5 h $f($ 2 $-2h)$

δf 2 $-\frac{3}{2}h)$

x 2.5 $f($ 4 $-h)$ $\delta^2 f(x$ 2 $-h)$

δf 4 $-\frac{1}{2}h)$ $\delta^3 f($ -7 $-\frac{1}{2}h)$

3.5 8 $x_0)$ $\delta^2 f$ -5 $)$ δ^4 3 (x_0)

δf -1 $+\frac{1}{2}h)$ $\delta^3 f($ -4 $+\frac{1}{2}h)$

x 4.5 $f($ 7 $+h)$ $\delta^2 f(x$ -1 $h)$

δ -2 $x_0)$

x_0 5.5 h $f(x$ 5 $+\frac{3}{2}h)$

1. 1. 1.5	2 2 2				
		2 2 2			
2. 2. 2.5	4 4 4		2 2 2		
		4 4 4		-7 -7 -7	
3. 3. 3.5	8 8 8		-5 -5 -5		3 3 3
		-1 -1 -1		-4 -4 -4	
4. 4. 4.5	7 7 7		-1 -1 -1		
		-2 -2 -2			
5. 5. 5.5	5 5 5				

$x_0 =$ $\quad f(x_0) =$

$x_0' - 4h =$ $\quad f(x_0' - 4h) =$

$x_0'' - 2h$ $\quad f(x_0'' - 2h)$

$\Delta f(x_0) =$

$\nabla f(x_0' - 3h) =$

$\delta f(x_0'' - \frac{3}{2}h)$

$x_0 + h =$ $\quad f(x_0 + h) =$

$x_0' - 3h =$ $\quad f(x_0' - 3h) =$

$x_0'' - h$ $\quad f(x_0'' - h)$

Newton-Cotes formulas:

$$x_i = x_0 + ih \quad u = \frac{x - x_0}{h}$$

$$W_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

$$(x - x_0)^{[r]} = (x - x_0)(x - x_1)\ldots(x - x_{r-1})$$

$$W_n(x) = a_0 + a_1(x - x_0)^{[1]} + a_2(x - x_0)^{[2]} + \ldots + a_n(x - x_0)^{[n]}$$

If a_0, ... a_n are found then interpolation is done.
We know that: $W_n(x_i)= y_i$.

$$x = x_0 \Rightarrow W_n(x_0) = a_0 = y_0$$

Forward differences are calculated.

$$\Delta W_n(x) = W_n(x+h) - W_n(x) =$$

$$a_0 + a_1(x+h-x_0)^{[1]} + a_2(x+h-x_0)^{[2]} + \ldots + a_n(x+h-x_0)^{[n]}$$

$$-a_0 - a_1(x-x_0)^{[1]} \qquad - a_2(x-x_0)^{[2]} - \ldots \qquad - a_n(x-x_0)^{[n]} =$$

$$= a_1(\cancel{x} + h - \cancel{x_0} - \cancel{x} + \cancel{x_0}) + a_2\{(x+h-x_0)(x+\cancel{h}-x_0-\cancel{h}) - (x-x_0)(x-x_0-h)\} + \ldots$$

$$+ a_n\{(x+h-x_0)^{[n]} - (x-x_0)^{[n]}\} =$$

$$= a_1 h + a_2\{(x+h-x_0)(x-x_0) - (x-x_0)(x-x_0-h)\}$$

$$+ a_n\{(x+h-x_0)^{[n]} - (x-x_0)^{[n]}\} =$$

$$= a_1 h + a_2\{(x-x_0)(\cancel{x} + h - \cancel{x_0} - \cancel{x} + \cancel{x_0} + h)) - (x-x_0)\}$$

$$= a_1 h + 2a_2(x-x_0)^{[1]}h + 3a_3(x-x_0)^{[2]}h + \ldots + na_n(x-x_0)^{[n-1]}h$$

$$x = x_0 \Rightarrow \Delta W_n(x_0) = a_1 h,\ \ \Delta W_n(x_0) = \Delta y_0 \Rightarrow a_1 = \frac{\Delta y_0}{h} = \frac{\Delta y_0}{1!\,h}$$

$$
\begin{aligned}
&\Delta^2 W_n(x) = \Delta W_n(x+h) - \Delta W_n(x) = \\
&= \cancel{a_1 h} + 2a_2(x+h-x_0)^{[1]}h + 3a_3(x+h-x_0)^{[2]}h + \ldots + na_n(x+h-x_0)^{[n-1]}h \\
&- \cancel{a_1 h} - 2a_2(x-x_0)^{[1]}h - 3a_3(x-x_0)^{[2]}h - \ldots - na_n(x-x_0)^{[n-1]}h \\
&= 2a_2(\cancel{x}+h-\cancel{x_0}-\cancel{x}+\cancel{x_0})h + 3a_3\{(x+h-x_0)(x+\cancel{h}-x_0-\cancel{h}) - (x-x_0)(x-x_0-h)\}h \\
&+ na_n\{(x+h-x_0)^{[n-1]} - (x-x_0)^{[n-1]}\}h \\
&= 2a_2h^2 + 3a_3\{(x-x_0)(\cancel{x}+h-\cancel{x_0}-\cancel{x}+\cancel{x_0}+h)\}h \\
&+ na_n\{(x+h-x_0)^{[n-1]} - (x-x_0)^{[n-1]}\}h \\
&= 2a_2h^2 + 2\cdot 3\cdot h^2 a_3(x-x_0)^{[1]} + \ldots + (n-1)nh^2 a_n(x-x_0)^{[n-2]}
\end{aligned}
$$

$$
x = x_0 \Rightarrow \Delta^2 W_n(x_0) = 2h^2 a_2 = \Delta^2 y_0 \Rightarrow a_2 = \frac{\Delta^2 y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2}
$$

Thus:

$$
a_i = \frac{\Delta^i y_0}{i!h^i}, \quad i = 0,1,\ldots,n
$$

$$W_n(x) = a_0 + a_1(x-x_0)^{[1]} + a_2(x-x_0)^{[2]} + \ldots + a_n(x-x_0)^{[n]}$$

$$W_n(x) = y_0 + \frac{\Delta y_0}{1!h}(x-x_0)^{[1]} + \frac{\Delta^2 y_0}{2!h^2}(x-x_0)^{[2]} + \ldots + \frac{\Delta^n y_0}{n!h^n}(x-x_0)^{[n]}$$

$$u = \frac{x-x_0}{h} \Rightarrow x = x_0 + uh$$

$$(x-x_0)^{[1]} = x - x_0 = uh$$

$$(x-x_0)^{[2]} = (x-x_0)(x-x_1) = (x-x_0)(\overset{uh}{x-x_0} - h) = uh \cdot h(u-1) = h^2 u(u-1)$$

$$(x-x_0)^{[n]} = (x-x_0)(x-x_1)\ldots(x-x_{n-1}) = uh \cdot (u-1)h \ldots (u-n+1)h =$$

$$= h^n u(u-1)\ldots(u-n+1)$$

$$W_n(x) = y_0 + \frac{\Delta y_0}{1!h}uh + \frac{\Delta^2 y_0}{2!h^2}u(u-1)h^2 + \ldots + \frac{\Delta^n y_0}{n!h^n}u(u-1)\ldots(u-n+1)h^n$$

$$\boxed{W_n(x) = y_0 + \frac{\Delta y_0}{1!}u + \frac{\Delta^2 y_0}{2!}u(u-1) + \ldots + \frac{\Delta^n y_0}{n!}u(u-1)\ldots(u-n+1)}$$

$$\boxed{W_n(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_0 + \ldots + \binom{u}{n}\Delta^n y_0}$$

By analogy for the backward differences:

$$\boxed{W_n(x) = y_0 + \frac{\nabla y_0}{1!}u + \frac{\nabla^2 y_0}{2!}u(u+1) + \ldots + \frac{\nabla^n y_0}{n!}u(u+1)\ldots(u+n-1)}$$

Error for Newton-Cotes formulas:

$$\boxed{\left|R_n(x)\right| = \frac{\left|f^{(n+1)}(\xi)\right|}{(n+1)!}h^{n+1}\left|u(u-1)\ldots(u-n)\right|}$$

$$\boxed{\left|R_n(x)\right| = \frac{\left|f^{(n+1)}(\xi)\right|}{(n+1)!}h^{n+1}\left|u(u+1)\ldots(u+n)\right|}$$

Example

$x=1$

x	f(x)
1	2
2	6
3	8
4	4

$$\begin{array}{cccccc}
x & y & & & \\
1 & 2 & & & \\
 & & 4 & & \\
2 & 6 & & -2 & \\
 & & 2 & & -4 \\
3 & 8 & & -6 & \\
 & & -4 & & \\
4 & 4 & & &
\end{array}$$

$$x_0 = 1 \Rightarrow u = \frac{x-1}{1} = x-1$$

$$W_3(x) = y_0 + \frac{\Delta y_0}{1!}u + \frac{\Delta^2 y_0}{2!}u(u-1) + \frac{\Delta^3 y_0}{3!}u(u-1)(u-2) =$$

$$= 2 + \frac{4}{1}u + \frac{-2}{2}u(u-1) + \frac{-4}{6}u(u-1)(u-2) =$$

$$= 2 + 4u - u(u-1) - \frac{2}{3}u(u-1)(u-2) =$$

$$= 2 + 4(x-1) - (x-1)(x-2) - \frac{2}{3}(x-1)(x-2)(x-3) =$$

$$= -\frac{2}{3}x^3 + 3x^2 - \frac{1}{3}x$$

Example

x=4

x	f(x)
1	2
2	6
3	8
4	4

$$
\begin{array}{ccccc}
x & y & & & \\
1 & 2 & & & \\
 & & 4 & & \\
2 & 6 & & -2 & \\
 & & 2 & & -4 \\
3 & 8 & & -6 & \\
 & & -4 & & \\
4 & 4 & & &
\end{array}
$$

$$x_0 = 4 \Rightarrow u = \frac{x-4}{1} = x-4$$

$$W_3(x) = y_0 + \frac{\nabla y_0}{1!}u + \frac{\nabla^2 y_0}{2!}u(u+1) + \frac{\nabla^3 y_0}{3!}u(u+1)(u+2) =$$

$$= 4 + \frac{-4}{1}u + \frac{-6}{2}u(u+1) + \frac{-4}{6}u(u+1)(u+2) =$$

$$= 4 - 4u - 3u(u+1) - \frac{2}{3}u(u+1)(u+2) =$$

$$= 4 - 4(x-4) - 3(x-4)(x-3) - \frac{2}{3}(x-4)(x-3)(x-2) =$$

$$= -\frac{2}{3}x^3 + 3x^2 - \frac{1}{3}x$$

The same polynomial is obtained regardless the method. Then – what for we use different formulas?

Data complement

$x=1$

x	f(x)
1	2
2	6
3	8
4	4
5	2

x	y				
1	2				
		4			
2	6		-2		
		2		-4	
3	8		-6		12
		-4		8	
4	4		2		
		-2			
5	2				

$$x_0 = 1 \Rightarrow u = \frac{x-1}{1} = x-1$$

$$W_4(x) = y_0 + \frac{\Delta y_0}{1!}u + \frac{\Delta^2 y_0}{2!}u(u-1) + \frac{\Delta^3 y_0}{3!}u(u-1)(u-2) + \frac{\Delta^4 y_0}{4!}u(u-1)(u-2)(u-3) =$$

$$= -\frac{2}{3}x^3 + 3x^2 - \frac{1}{3}x + \frac{12}{4!}(u-1)(u-2)(u-3) =$$

$$= -\frac{2}{3}x^3 + 3x^2 - \frac{1}{3}x + \frac{12}{24}(x-1)(x-2)(x-3)(x-4) = \frac{1}{2}x^4 - \frac{17}{3}x^3 + \frac{41}{2}x^2 - \frac{76}{3}x + 12$$

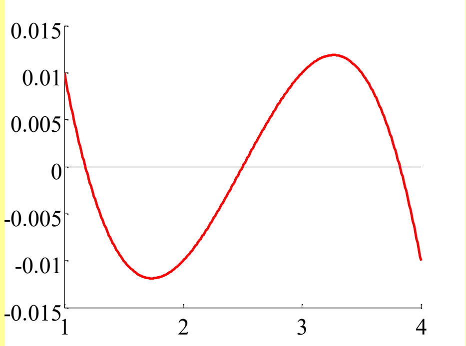
$$x_0 = 4 \Rightarrow u = \frac{x-4}{1} = x-4$$

Error accumulation

x	f(x)
1	1,99
2	6,01
3	7,99
4	4,01

$$\begin{array}{llllll}
x & y & & & & \\
1 & 1.99 & & & & \\
 & & 4.02 & & & \\
2 & 6.01 & & -2.04 & & \\
 & & 1.98 & & -3.92 & \\
3 & 7.99 & & -5.96 & & \\
 & & -3.98 & & & \\
4 & 4.01 & & & &
\end{array}$$

Error W_3- V_3



$$V_3(x) = y_0 + \frac{\nabla y_0}{1!}u + \frac{\nabla^2 y_0}{2!}u(u+1) + \frac{\nabla^3 y_0}{3!}u(u+1)(u+2) =$$

$$= 4.01 + \frac{-3.98}{1}u + \frac{-5.96}{2}u(u+1) + \frac{-3.92}{6}u(u+1)(u+2) =$$

$$= 4.01 - 3.98u - 2.98u(u+1) - \frac{1.96}{3}u(u+1)(u+2) =$$

$$= 4 - 3.98(x-4) - 2.98(x-4)(x-3) - \frac{1.96}{3}(x-4)(x-3)(x-2)$$

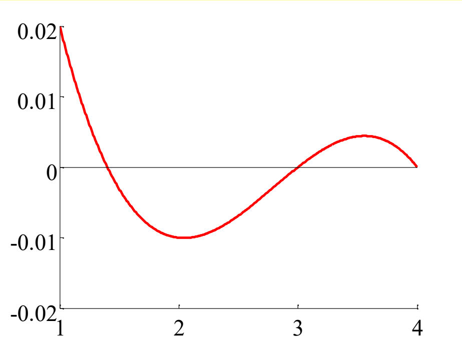
$$W_3 = 4 - 4(x-4) - 3(x-4)(x-3) - \frac{2}{3}(x-4)(x-3)(x-2)$$

$$x_0 = 4 \Rightarrow u = \frac{x-4}{1} = x-4$$

x	f(x)
1	1,99
2	6,01
3	8
4	4

$$\begin{array}{cccccc}
x & y & & & \\
1 & 1.99 & & & \\
 & & 4.02 & & \\
2 & 6.01 & & -2.04 & \\
 & & 1.99 & & -3.95 \\
3 & 8 & & -5.99 & \\
 & & -4 & & \\
4 & 4 & & &
\end{array}$$

error W_3- V_3



$$V_3(x) = y_0 + \frac{\nabla y_0}{1!}u + \frac{\nabla^2 y_0}{2!}u(u+1) + \frac{\nabla^3 y_0}{3!}u(u+1)(u+2) =$$

$$= 4 + \frac{-4}{1}u + \frac{-5.99}{2}u(u+1) + \frac{-3.95}{6}u(u+1)(u+2) =$$

$$= 4 - 4u - 2.995u(u+1) - \frac{1.975}{3}u(u+1)(u+2) =$$

$$= 4 - 4(x-4) - 2.995(x-4)(x-3) - \frac{1.975}{3}(x-4)(x-3)(x-2)$$

$$W_3 = 4 - 4(x-4) - 3(x-4)(x-3) - \frac{2}{3}(x-4)(x-3)(x-2)$$

Polynomial is AT MOST OF the *n*-th order

x	f(x)
1	1
2	4
3	9
4	16
5	25
6	36

x	*y*					
1	1					
		3				
2	4		2			
		5		0		
3	9		2		0	
		7		0		0
4	16		2		0	
		9		0		
5	25		2			
		11				
6	36					

Formulas with central differences

Stirling's formula for $u \le 0.25$

$$f(x) \approx f(x_0) + \sum_{k=0}^{m-1} \binom{u+k}{2k+1} \frac{\delta^{2k+1} f\left(x_0 - \frac{1}{2}h\right) + \delta^{2k+1} f\left(x_0 + \frac{1}{2}h\right)}{2} + \sum_{k=1}^{m} \frac{u}{2k} \binom{u+k-1}{2k-1} \delta^{2k} f(x_0)$$

Reszta

$$R(\xi) = \binom{u+m}{2m+1} f^{(2m+1)}(\xi) h^{2m+1}$$

Bessel's formula for $0.25 \le u \le 0.75$

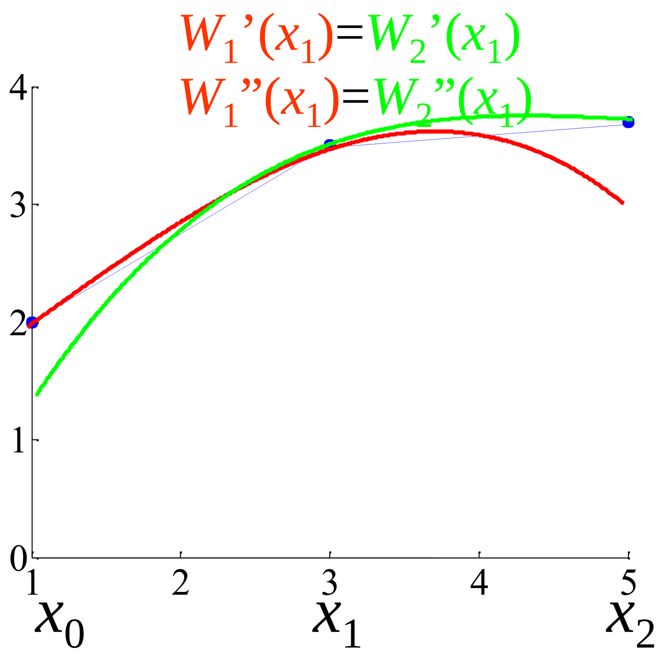
$$f(x) \approx \frac{f(x_0) + f(x_0+h)}{2} + \sum_{k=0}^{m-1} \frac{u - \frac{1}{2}}{2k+1} \binom{u+k-1}{2k} \delta^{2k+1} f\left(x_0 + \frac{1}{2}h\right) +$$
$$+ \sum_{k=0}^{m-1} \binom{u+k-1}{2k} \frac{\delta^{2k} f(x_0) + \delta^{2k} f(x_0+h)}{2}$$

Reszta

$$R(\xi) = \binom{u+m-1}{2m} f^{(2m)}(\xi) h^{2m}$$

Spline interpolation

Sampled signal need to be smoothly reproduced. This cannot be done by means of Lagrange' interpolation (too many nodes or rough junctions). This is usually done by means of the third-order polynomials.



$$y = y_j \qquad y = Ay_j + By_{j+1} + Cy_j'' + Dy_{j+1}''$$

$$y = y_{j+1} \qquad y \equiv W(x),\ y_j \equiv W(x_j) = f(x_j),\ y_j'' \equiv f''(x_j)$$

$$x = x_j \Rightarrow \qquad A = \frac{x - x_{j+1}}{x_j - x_{j+1}} = \frac{x_{j+1} - x_{j+1}}{x_{j+1} - x_j}, \qquad A = 1$$

$$x = x_{j+1} \Rightarrow \qquad A = 0$$

$$B = \frac{x_{j+1} - x_j}{x_{j+1} - x_j} \qquad B = 0 \qquad B = 1$$

$$C = \frac{1}{6}(A^3 - A)(x_{j+1} - x_j)^2 \qquad C = 0 \qquad C = 0$$

$$D = \frac{1}{6}(B^3 - B)(x_{j+1} - x_j)^2 \qquad D = 0 \qquad D = 0$$

$$\frac{dy}{dx} = \frac{dA}{dx} y_j + \frac{dB}{dx} y_{j+1} + \frac{dC}{dx} y_j'' + \frac{dD}{dx} y_{j+1}''$$

$$A = \frac{x_{j+1} - x}{x_{j+1} - x_j} \Rightarrow \frac{dA}{dx} = -\frac{1}{x_{j+1} - x_j}$$

$$B = \frac{x - x_j}{x_{j+1} - x_j} \Rightarrow \frac{dB}{dx} = \frac{1}{x_{j+1} - x_j}$$

$$C = \frac{1}{6}(A^3 - A)(x_{j+1} - x_j)^2 \Rightarrow \frac{dC}{dx} = \frac{dC}{dA} \cdot \frac{dA}{dx} =$$

$$= \frac{1}{6}(3A^2 - 1)(x_{j+1} - x_j)^2 \frac{dA}{dx} = -\frac{1}{6}(3A^2 - 1)(x_{j+1} - x_j)$$

$$D = \frac{1}{6}(B^3 - B)(x_{j+1} - x_j)^2 \Rightarrow \frac{dD}{dx} = \frac{dD}{dB} \cdot \frac{dB}{dx} =$$

$$= \frac{1}{6}(3B^2 - 1)(x_{j+1} - x_j)^2 \frac{dB}{dx} = \frac{1}{6}(3B^2 - 1)(x_{j+1} - x_j)$$

$$y = Ay_j + By_{j+1} + Cy_j'' + Dy_{j+1}''$$

$$\frac{dy}{dx} = -\frac{y_j}{x_{j+1} - x_j} + \frac{y_{j+1}}{x_{j+1} - x_j} - \frac{1}{6}(3A^2 - 1)(x_{j+1} - x_j)y_j'' + \frac{1}{6}(3B^2 - 1)(x_{j+1} - x_j)y_{j+1}''$$

$$\frac{d^2y}{dx^2} = 0 + 0 - \frac{1}{6}6A(x_{j+1} - x_j)\underbrace{\frac{dA}{dx}}_{= -\frac{1}{x_{j+1} - x_j}} y_j'' + \frac{1}{6}6B(x_{j+1} - x_j)\underbrace{\frac{dB}{dx}}_{= \frac{1}{x_{j+1} - x_j}} y_{j+1}'' =$$

$$= Ay_j'' + By_{j+1}''$$

So, the second derivative is as good interpolated, as the original function. However, how to find y_j'', y_{j+1}'' necessary for the polynomial formula?

The value of the first derivative in the point x_j (e.i. y_j) should be the same either it is determined for the (x_{j-1}, x_j) interval or (x_j, x_{j+1}).

For $(x_j,\ x_{j+1})$, in point $x_j \Rightarrow A=1,\ B=0$:

$$\frac{dy}{dx} = -\frac{y_j}{x_{j+1}-x_j} + \frac{y_{j+1}}{x_{j+1}-x_j} - \frac{1}{6}\overset{(3-1)}{(3A^2-1)}(x_{j+1}-x_j)y_j'' +$$

$$+\frac{1}{6}\overset{(0-1)}{(3B^2-1)}(x_{j+1}-x_j)y_{j+1}'' =$$

$$= \frac{y_{j+1}-y_j}{x_{j+1}-x_j} - \frac{1}{3}(x_{j+1}-x_j)y_j'' - \frac{1}{6}(x_{j+1}-x_j)y_{j+1}''$$

For $(x_{j-1},\ x_j)$, in point $x_j \Rightarrow A=0,\ B=1$:

$$\frac{dy}{dx} = -\frac{y_{j-1}}{x_j-x_{j-1}} + \frac{y_j}{x_j-x_{j-1}} - \frac{1}{6}\overset{(0-1)}{(3A^2-1)}(x_j-x_{j-1})y_{j-1}'' + \qquad =$$

$$+\frac{1}{6}\overset{(3-1)}{(3B^2-1)}(x_j-x_{j-1})y_j'' =$$

$$= \frac{y_j-y_{j-1}}{x_j-x_{j-1}} + \frac{1}{6}(x_j-x_{j-1})y_{j-1}'' + \frac{1}{3}(x_j-x_{j-1})y_j''$$

$$\frac{y_{j+1}-y_j}{x_{j+1}-x_j}-\frac{1}{3}(x_{j+1}-x_j)y_j''-\frac{1}{6}(x_{j+1}-x_j)y_{j+1}''=$$

$$=\frac{y_j-y_{j-1}}{x_j-x_{j-1}}+\frac{1}{6}(x_j-x_{j-1})y_{j-1}''+\frac{1}{3}(x_j-x_{j-1})y_j''$$

$$\frac{y_{j+1}-y_j}{x_{j+1}-x_j}-\frac{y_j-y_{j-1}}{x_j-x_{j-1}}=$$

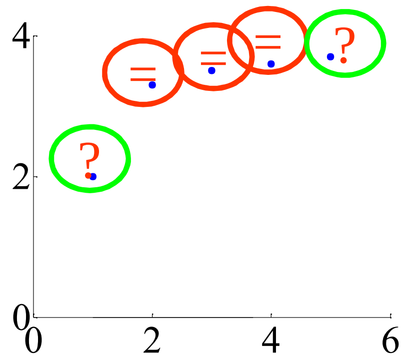
$$=\frac{1}{6}(x_j-x_{j-1})y_{j-1}''+\frac{1}{3}(\cancel{x_j}-x_{j-1}+x_{j+1}-\cancel{x_j})y_j''+\frac{1}{6}(x_{j+1}-x_j)y_{j+1}''$$

$$\frac{y_{j+1}-y_j}{x_{j+1}-x_j}-\frac{y_j-y_{j-1}}{x_j-x_{j-1}}=\frac{1}{6}(x_j-x_{j-1})y_{j-1}''+\frac{1}{3}(x_{j+1}-x_{j-1})y_j''+\frac{1}{6}(x_{j+1}-x_j)y_{j+1}''$$

For $j=1,2,\ldots,n-1$

Three equations, from which: y_1'', y_2'', y_3''.

What about y_0'' i y_4''? Assume: $y_0''=0$ i $y_4''=0$



It can be also assumed that we know y_0' i y_4', and the second derivatives can be calculated from formulas for the first derivatives.

For $(x_j, \ x_{j+1})$, in point $x_j \Rightarrow A=1, B=0$:
For $(x_0, \ x_1)$, in point $x_1 \Rightarrow A=1, B=0$:

assumed

$$y_0' = \frac{y_1 - y_0}{x_1 - x_0} - \frac{1}{3}(x_1 - x_0)y_0'' - \frac{1}{6}(x_1 - x_0)y_1''$$

cal. From next eqs.

$$\frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}} = \frac{1}{6}(x_j - x_{j-1})y_{j-1}'' + \frac{1}{3}(x_{j+1} - x_{j-1})y_j'' + \frac{1}{6}(x_{j+1} - x_j)y_{j+1}''$$

For $(x_{j-1}, \ x_j)$, in point $x_j \Rightarrow A=0, B=1$:
For $(x_{n-1}, \ x_n)$, in point $x_n \Rightarrow A=0, B=1$:

$$y_n' = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} + \frac{1}{6}(x_n - x_{n-1})y_{n-1}'' + \frac{1}{3}(x_n - x_{n-1})y_n''$$

assumed

From previous eqs.

cal.

Thus, we have $(n-2)+2$ equations and n variables.

However, most often: y_0”=0 i y_n”=0.

Example

Spline interpolation for the following nodes and y_0”=y_2”=0.

	x_i	y_i
x_0	1	2
x_1	3	3.5
x_2	5	3.7

$$\frac{y_{j+1}-y_j}{x_{j+1}-x_j}-\frac{y_j-y_{j-1}}{x_j-x_{j-1}}=\frac{1}{6}(x_j-x_{j-1})y_{j-1}''+\frac{1}{3}(x_{j+1}-x_{j-1})y_j''+$$
$$+\frac{1}{6}(x_{j+1}-x_j)y_{j+1}''$$

$$\frac{y_2-y_1}{x_2-x_1}-\frac{y_1-y_0}{x_1-x_0}=\frac{1}{6}(x_1-x_0)\overset{=0}{y_0''}+\frac{1}{3}(x_2-x_0)y_1''+\frac{1}{6}(x_2-x_1)\overset{=0}{y_2''}$$

$$\frac{y_2-y_1}{x_2-x_1}-\frac{y_1-y_0}{x_1-x_0}=\frac{1}{3}(x_2-x_0)y_1''$$

$$y_1''=\frac{3}{5-1}\left(\frac{3.7-3.5}{5-3}-\frac{3.5-2}{3-1}\right)=-0.4875$$

Polynomial: $y = Ay_j + By_{j+1} + Cy_j'' + Dy_{j+1}''$

For $(x_0, x_1) \equiv (1, 3)$:

$$y = Ay_0 + By_1 + C\underbrace{y_0''}_{=0} + Dy_1''$$

	x_i	y_i
x_0	1	2
x_1	3	3.5
x_2	5	3.7

$$A = \frac{x - x_{j+1}}{x_j - x_{j+1}} = \frac{x-3}{1-3} = -\frac{1}{2}(x-3)$$

$$B = \frac{x - x_j}{x_{j+1} - x_j} = \frac{x-1}{3-1} = \frac{1}{2}(x-1)$$

$$D = \frac{1}{6}(B^3 - B)(x_{j+1} - x_j)^2 = \frac{1}{6}\left[\frac{1}{8}(x-1)^3 - \frac{1}{2}(x-1)\right](3-1)^2 =$$

$$= \frac{1}{12}\left(x^3 - 3x^2 - x + 3\right)$$

So, for the (x_0, x_1) interval the polynomial is:

$$\boxed{\begin{aligned} y &= -\frac{1}{2}(x-3)2 + \frac{1}{2}(x-1)3.5 + \frac{1}{12}(x^3 - 3x^2 - x + 3)(-0.4875) = \\ &= -0.040625x^3 + 0.121875x^2 + 0.790625x + 1.128125 \end{aligned}}$$

For $(x_1, x_2)\equiv(3, 5)$:

$$y = Ay_1 + By_2 + Cy_1'' + D\underbrace{y_2''}_{=0}$$

	x_i	y_i
x_0	1	2
x_1	3	3.5
x_2	5	3.7

$$A = \frac{x - x_{j+1}}{x_j - x_{j+1}} = \frac{x-5}{3-5} = -\frac{1}{2}(x-5)$$

$$B = \frac{x - x_j}{x_{j+1} - x_j} = \frac{x-3}{5-3} = \frac{1}{2}(x-3)$$

$$C = \frac{1}{6}(A^3 - A)(x_{j+1} - x_j)^2 = \frac{1}{6}\left[-\frac{1}{8}(x-5)^3 + \frac{1}{2}(x-5)\right](5-3)^2 =$$

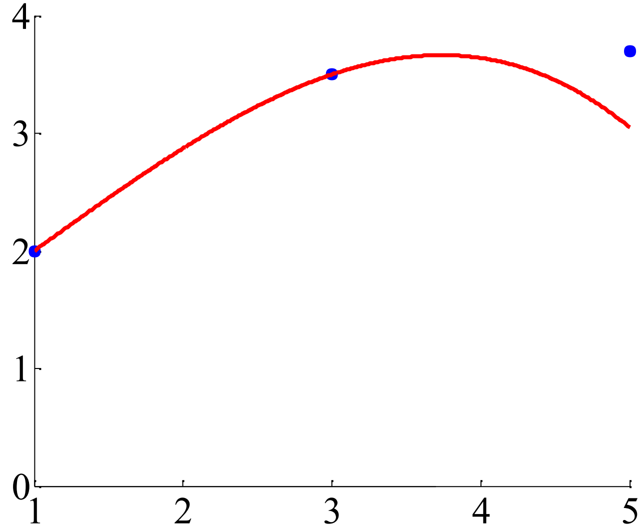
$$= \frac{1}{12}\left(-x^3 + 15x^2 - 71x + 105\right)$$

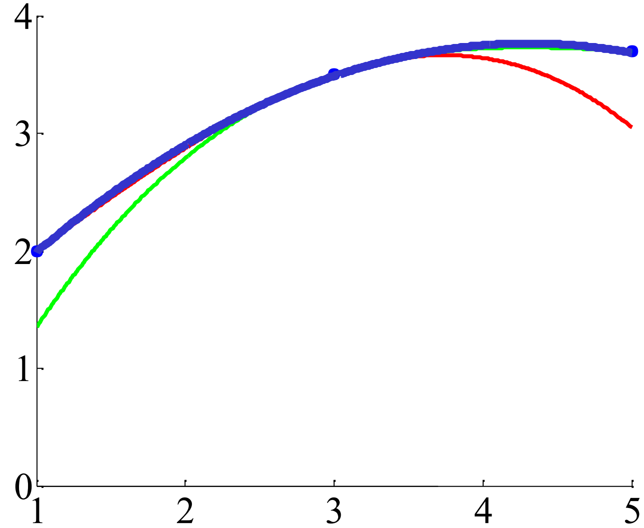
For the (x_1, x_2) interval the polynomial is:

$$y = -\frac{1}{2}(x-5)3.5 + \frac{1}{2}(x-3)3.7 + \frac{1}{12}(-x^3 + 15x^2 - 71x + 105)(-0.4875) =$$

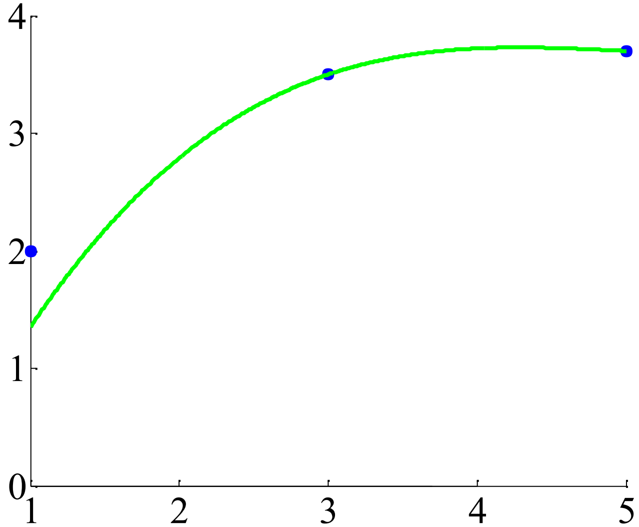
$$= -0.040625x^3 - 0.609375x^2 + 2.984375x - 1.065625$$

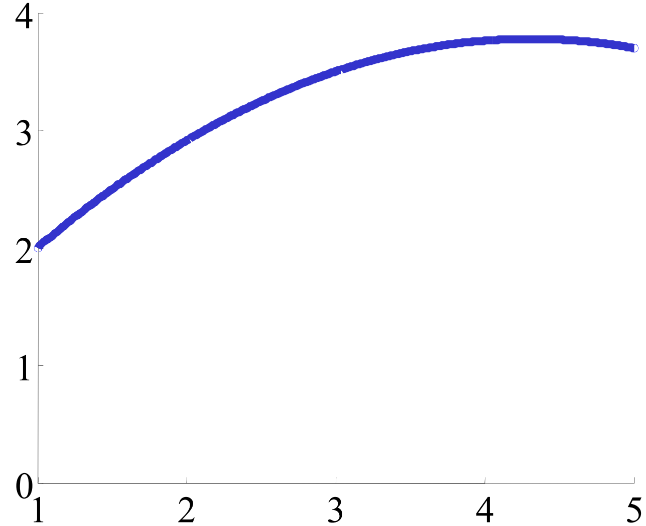
$$y=-\frac{1}{2}(x-3)2+\frac{1}{2}(x-1)3.5+\frac{1}{12}(x^3-3x^2-x+3)(-0.4875)=$$
$$=-0.040625x^3+0.121875x^2+0.790625x+1.128125$$





$$y=-\frac{1}{2}(x-5)3.5+\frac{1}{2}(x-3)3.7+\frac{1}{12}(-x^3+15x^2-71x+105)(-0.4$$
$$=-0.040625x^3-0.609375x^2+2.984375x-1.065625$$





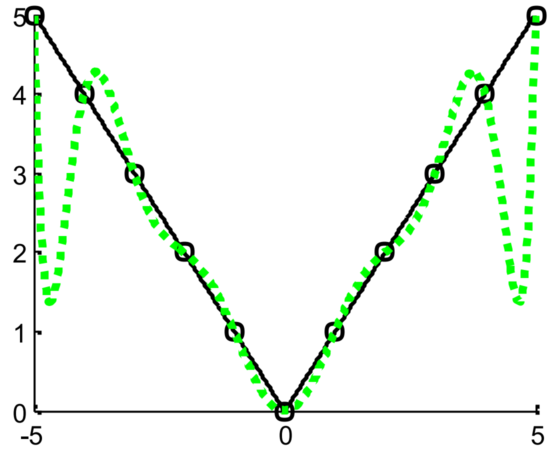
Interpolation

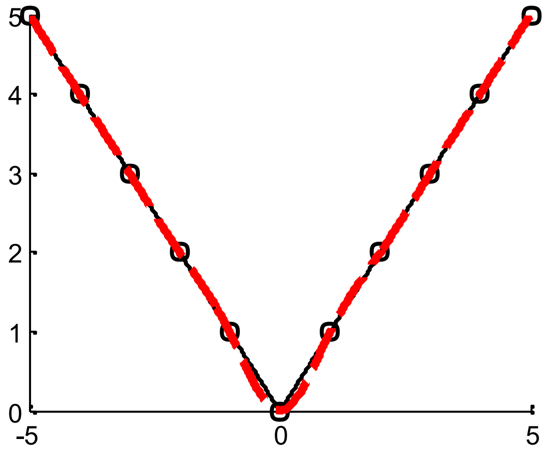
Lagrange’

Spline

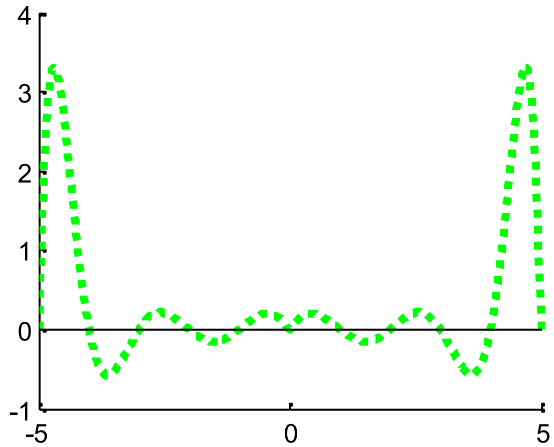
$y=|x|$

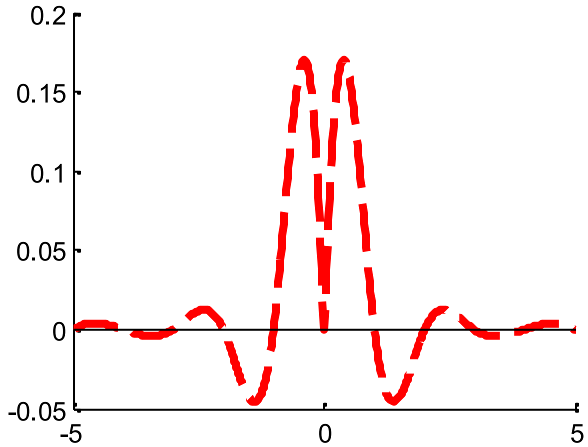
Runge’s effect





Error

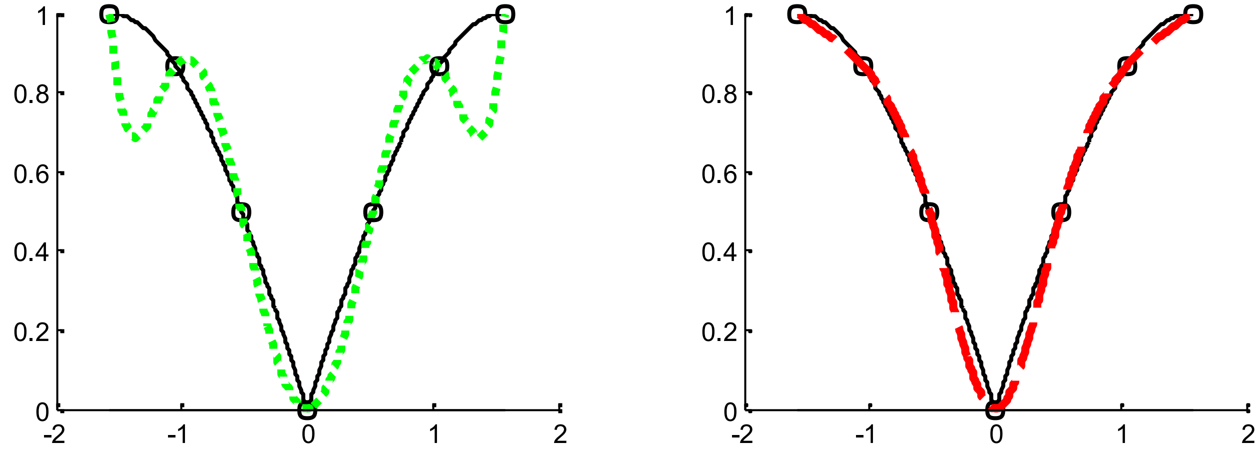




Interpolation

Lagrange' **Spline**

$y=\sin(|x|)$



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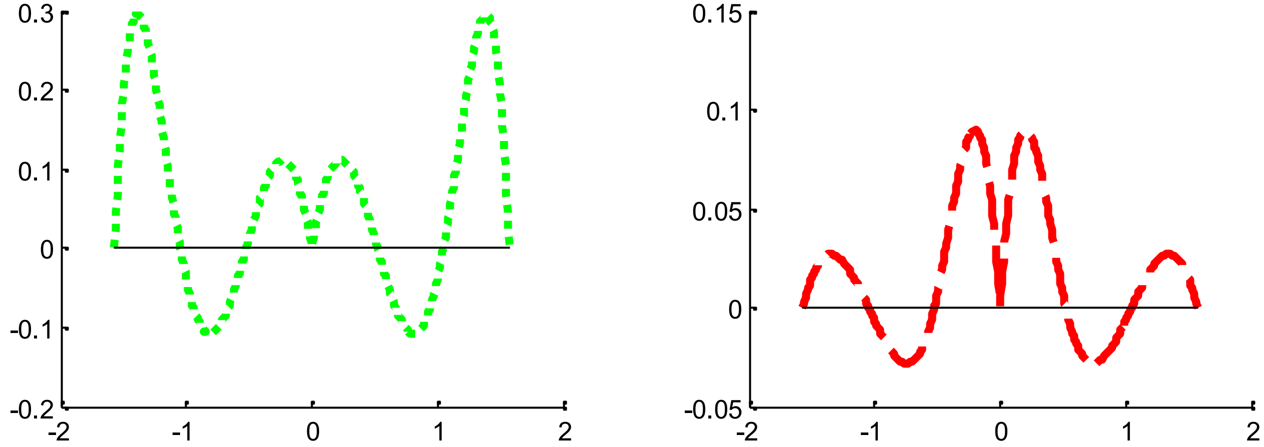
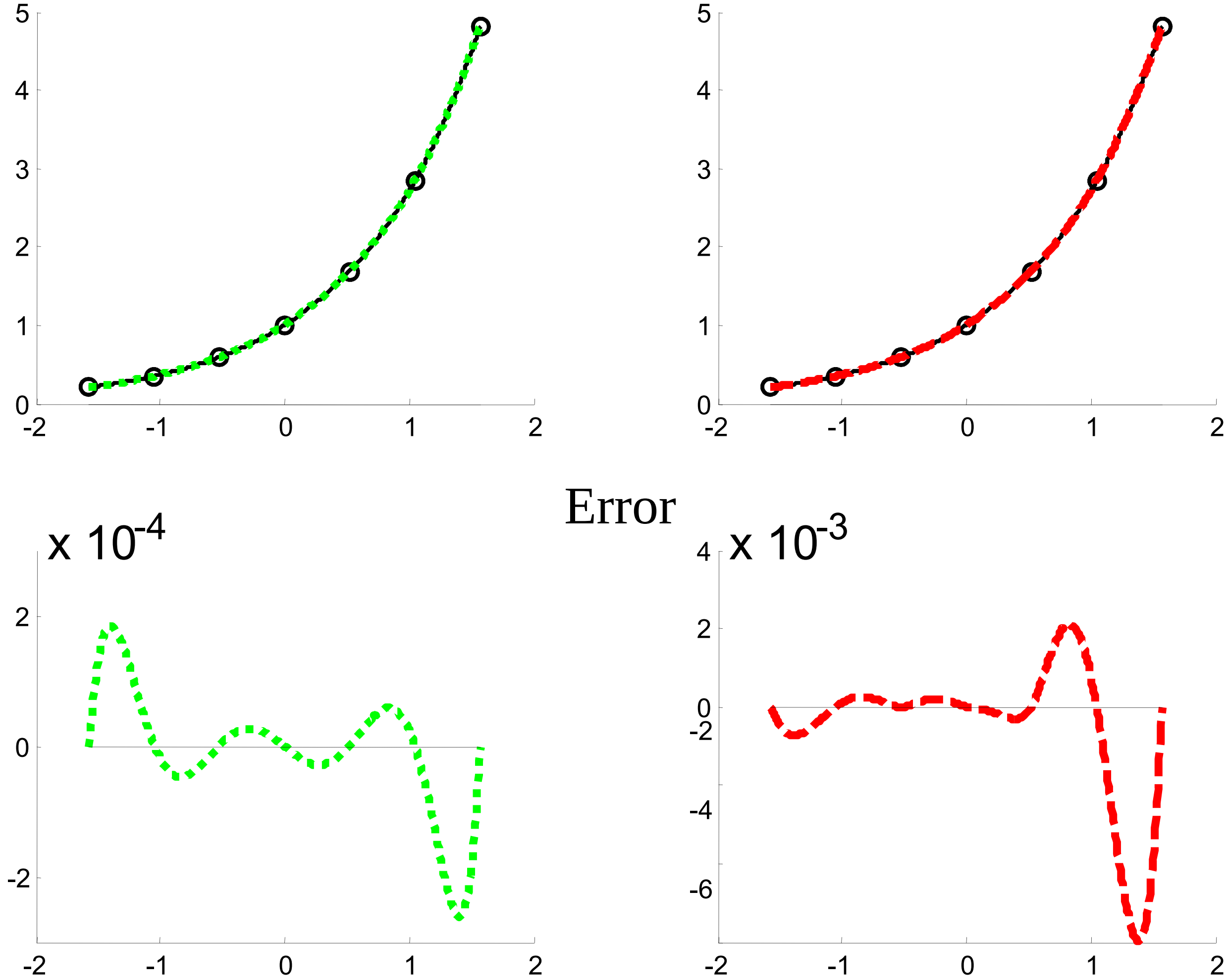


Interpolation

Lagrange'

Spline

y=exp(x)



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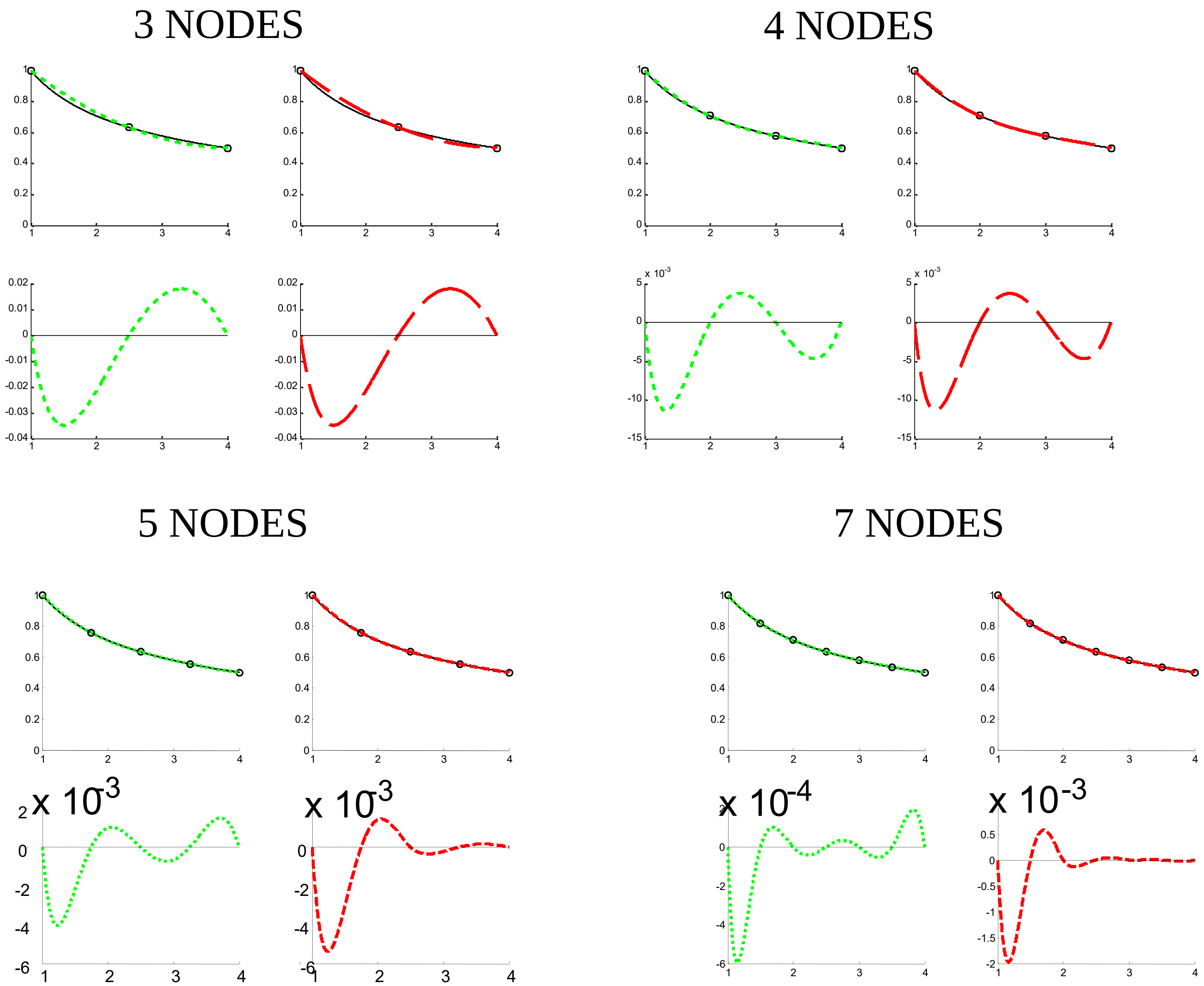
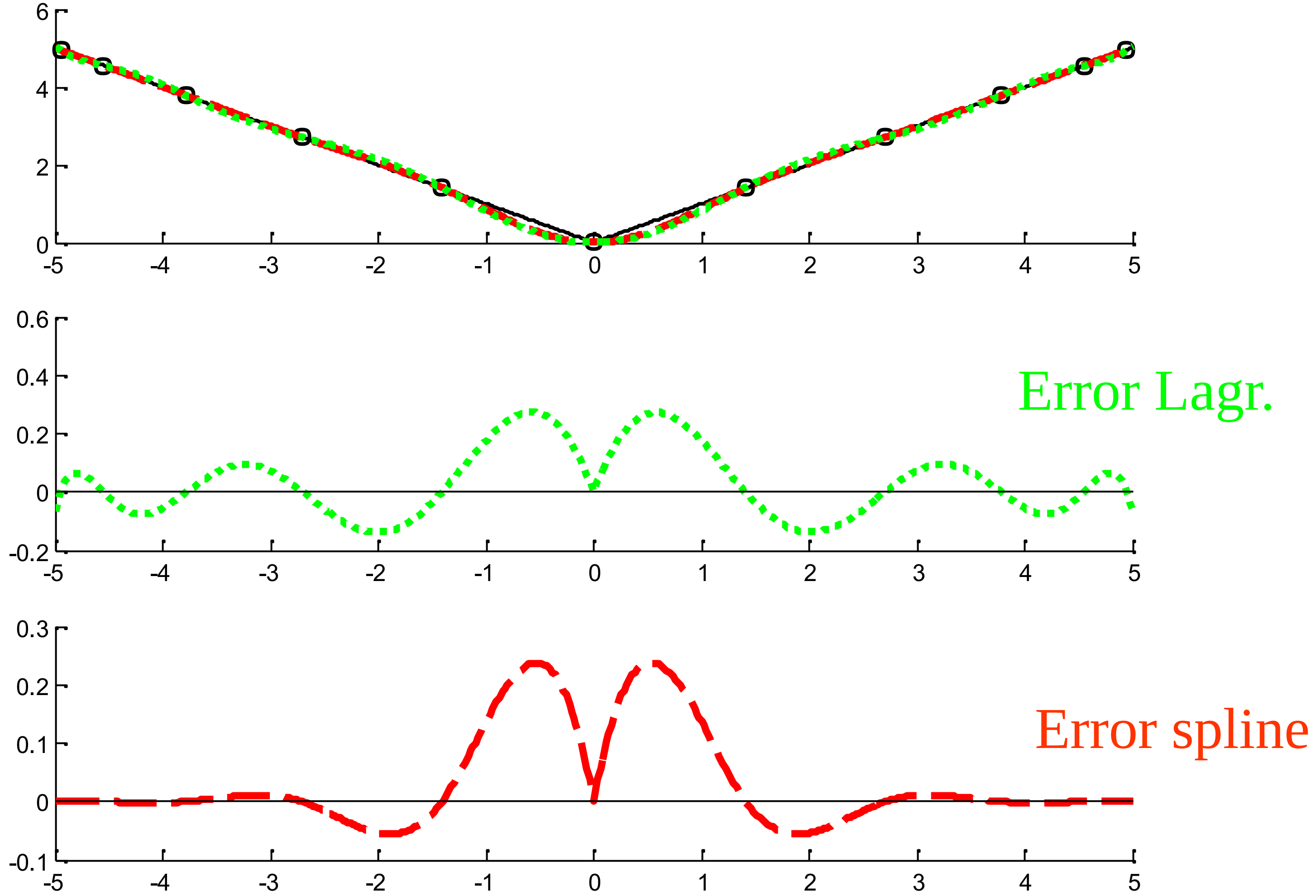


3 NODES
4 NODES
x 10-3
x 10-3
5 NODES
7 NODES
x 10-3
x 10-3
x 10-4
x 10-3



Error Lagr.
Error spline

If the first derivative value is forced
(slope at the first and the last node are given)

First derivative assumed by Matlab

ya(i)=1/sqrt(xa(i));