SPLINE INTERPOLATION

Remark: the exercise requires knowledge of Lagrange polynomial interpolation.

1. Third order polynomials as splines.

In majority of cases an increase of numbers of interpolation knots causes an increase of the interpolation error. Thus, polynomials that interpolate functions given by their points (samples) should be found in sub-intervals of the main interpolation interval. However, if we use first order polynomials joining neighboring knots, the resulting function is not "smooth" – it is impossible to determine the value of its derivative, since it is different for the polynomials that "meet" in the knot (see fig.1). Therefore, usually the third order polynomials are used, which apart from the usual interpolation assumption: $W_1(x_i) = W_2(x_i) = f(x_i)$, fit also: $W_1'(x_i) = W_2'(x_i)$, $W_1''(x_i) = W_2''(x_i)$.

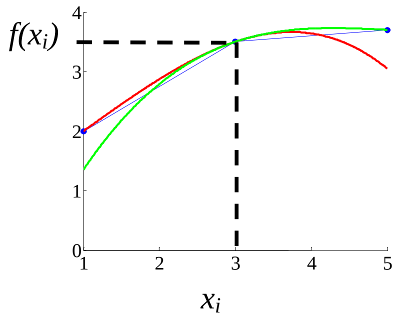


Fig.1. Interpolation by first and third order polynomials.

The polynomial for the $[x_j, x_{j+1}]$ interval is of the following form:

$$y = Ay_j + By_{j+1} + Cy_j'' + Dy_{j+1}'', \tag{1}$$

where:

$$y \equiv W(x),\ y_j \equiv W(x_j) = f(x_j),\ y_j'' \equiv f''(x_j).$$

Its coefficients are calculated as:

$$A = \frac{x - x_{j+1}}{x_j - x_{j+1}} = \frac{x_{j+1} - x}{x_{j+1} - x_j}, \tag{2}$$

$$B = \frac{x - x_j}{x_{j+1} - x_j}, \tag{3}$$

$$C = \frac{1}{6}(A^3 - A)(x_{j+1} - x_j)^2, \tag{4}$$

$$D = \frac{1}{6}(B^3 - B)(x_{j+1} - x_j)^2. \tag{5}$$

In each interpolation knot one coefficient is equal to 1 and the other to 0:

$$x = x_j \Rightarrow A = \frac{x_{j+1} - x_j}{x_{j+1} - x_j} = 1,\ x = x_{j+1} \Rightarrow A = \frac{x_{j+1} - x_{j+1}}{x_{j+1} - x_j} = 0, \tag{6}$$

$$x = x_j \Rightarrow B = \frac{x_j - x_j}{x_{j+1} - x_j} = 0, x = x_{j+1} \Rightarrow B = \frac{x_{j+1} - x_j}{x_{j+1} - x_j} = 1. \tag{7}$$

For $x=x_j$ or $x=x_{j+1}$ the cofficients $C=0$ i $D=0$, which can be shown using (6) and (7) in formulas (4) and (5). Hence, in interpolation knots the polynomial (1) fits the interpolation assumtion, i.e. $x=x_j \Rightarrow y=y_j$, $x=x_{j+1} \Rightarrow y=y_{j+1}$.

The (1) polynomial cannot be determined if y_j'', y_{j+1}'' are unknown. The latter can be found if we assume that first and the second derivatives have the same values for both polynomials "meeting" in one knot. Let us calculate the first derivatives.

$$\frac{dy}{dx}=\frac{dA}{dx}y_j+\frac{dB}{dx}y_{j+1}+\frac{dC}{dx}y_j''+\frac{dD}{dx}y_{j+1}'', \tag{8}$$

$$A=\frac{x_{j+1}-x}{x_{j+1}-x_j}\Rightarrow\frac{dA}{dx}=-\frac{1}{x_{j+1}-x_j}, \tag{9}$$

$$B=\frac{x-x_j}{x_{j+1}-x_j}\Rightarrow\frac{dB}{dx}=\frac{1}{x_{j+1}-x_j}, \tag{10}$$

$$\begin{aligned}C=\frac{1}{6}(A^3-A)(x_{j+1}-x_j)^2\Rightarrow\frac{dC}{dx}&=\frac{1}{6}(3A^2-1)(x_{j+1}-x_j)^2\frac{dA}{dx}=\\&=-\frac{1}{6}(3A^2-1)(x_{j+1}-x_j)\end{aligned}, \tag{11}$$

$$\begin{aligned}D=\frac{1}{6}(B^3-B)(x_{j+1}-x_j)^2\Rightarrow\frac{dD}{dx}&=\frac{1}{6}(3B^2-1)(x_{j+1}-x_j)^2\frac{dB}{dx}=\\&=\frac{1}{6}(3B^2-1)(x_{j+1}-x_j)\end{aligned}. \tag{12}$$

Using (9)-(12) in (8) we have:

$$\begin{aligned}\frac{dy}{dx}=-\frac{y_j}{x_{j+1}-x_j}+\frac{y_{j+1}}{x_{j+1}-x_j}&-\frac{1}{6}(3A^2-1)(x_{j+1}-x_j)y_j''+\\&+\frac{1}{6}(3B^2-1)(x_{j+1}-x_j)y_{j+1}''\end{aligned}. \tag{13}$$

From (13) the second derivative can be calculated as:

$$\begin{aligned}\frac{d^2y}{dx^2}&=0+0-\frac{1}{6}6A(x_{j+1}-x_j)\frac{dA}{dx}y_j''+\frac{1}{6}6B(x_{j+1}-x_j)\frac{dB}{dx}y_{j+1}''=\\&=-A(x_{j+1}-x_j)\left(-\frac{1}{x_{j+1}-x_j}\right)y_j''+B(x_{j+1}-x_j)\left(\frac{1}{x_{j+1}-x_j}\right)y_{j+1}''=Ay_j''+By_{j+1}''\end{aligned}. \tag{14}$$

Since A and B in interpolation knots are equal either 0 or 1, then from (14) we get:

$$x=x_j\Rightarrow\frac{d^2y}{dx^2}=y_j'',\ x=x_{j+1}\Rightarrow\frac{d^2y}{dx^2}=y_{j+1}''. \tag{15}$$

It means that the interpolated function and the interpolating polynomial in interpolation knots have equal not only values, but also values of second derivatives. Yet, (14) cannot be used unless we do not know values of the second derivative for the interpolated function. However, we can benefit from the assumption that values of the first and the second derivatives are the same for polynomials "meeting" in the knot. Then, from (13) for the (x_j, x_{j+1}) interval in the x_j knot, i.e. for A=1, B=0:

$$\begin{aligned}\frac{dy}{dx}&=-\frac{y_j}{x_{j+1}-x_j}+\frac{y_{j+1}}{x_{j+1}-x_j}-\frac{1}{6}(3A^2-1)(x_{j+1}-x_j)y_j''+\frac{1}{6}(3B^2-1)(x_{j+1}-x_j)y_{j+1}''=\\&=\frac{y_{j+1}-y_j}{x_{j+1}-x_j}-\frac{1}{3}(x_{j+1}-x_j)y_j''-\frac{1}{6}(x_{j+1}-x_j)y_{j+1}''.\end{aligned} \tag{16}$$

For the (x_{j-1}, x_j) interval, and the same knot x_j : A=0, B=1: (17)

$$\frac{dy}{dx}=-\frac{y_{j-1}}{x_j-x_{j-1}}+\frac{y_j}{x_j-x_{j-1}}-\frac{1}{6}(3A^2-1)(x_j-x_{j-1})y_{j-1}''+\frac{1}{6}(3B^2-1)(x_j-x_{j-1})y_j''=$$

$$=\frac{y_j-y_{j-1}}{x_j-x_{j-1}}+\frac{1}{6}(x_j-x_{j-1})y_{j-1}''+\frac{1}{3}(x_j-x_{j-1})y_j''.$$

Comparing (16) and (17) and ordering the formula, we obtain:

$$\frac{y_{j+1}-y_j}{x_{j+1}-x_j}-\frac{y_j-y_{j-1}}{x_j-x_{j-1}}=\frac{1}{6}(x_j-x_{j-1})y_{j-1}''+\frac{1}{3}(x_{j+1}-x_{j-1})y_j''+\frac{1}{6}(x_{j+1}-x_j)y_{j+1}''. \quad (18)$$

The (18) formula makes it possible to determine the second derivatives in all x_j knots besides x_0 and x_n. In this points the values of the second derivatives must be assumed. Most often y_0”= y_n”=0. It is also possible to assume the y_0’ and y_n’ values, and calculate the second derivatives from (17). The first assumption is used in the following example.

Example 1.

Determine spline polynomials of the third order (cubic polynomials), assuming that y_0”=y_2”=0.

	x_i	y_i
x_0	1	2
x_1	3	3.5
x_2	5	3.7

We find y_1” from (18).

$$\frac{y_2-y_1}{x_2-x_1}-\frac{y_1-y_0}{x_1-x_0}=\frac{1}{6}(x_1-x_0)y_0''+\frac{1}{3}(x_2-x_0)y_1''+\frac{1}{6}(x_2-x_1)y_2''=$$

$$\frac{1}{6}(x_1-x_0)0+\frac{1}{3}(x_2-x_0)y_1''+\frac{1}{6}(x_2-x_1)0=\frac{1}{3}(x_2-x_0)y_1'',$$

stąd: $y_1''=\dfrac{3}{5-1}\left(\dfrac{3.7-3.5}{5-3}-\dfrac{3.5-2}{3-1}\right)=-0.4875$.

Polynomial for the [x_0, x_1] interval:

$$y=Ay_0+By_1+Cy_0''+Dy_1''=Ay_0+By_1+0y_0''+Dy_1''=Ay_0+By_1+Dy_1'',$$

$$A=\frac{x-x_{j+1}}{x_j-x_{j+1}}=\frac{x-3}{1-3}=-\frac{1}{2}(x-3),$$

$$B=\frac{x-x_j}{x_{j+1}-x_j}=\frac{x-1}{3-1}=\frac{1}{2}(x-1),$$

$$D=\frac{1}{6}(B^3-B)(x_{j+1}-x_j)^2=\frac{1}{6}\left[\frac{1}{8}(x-1)^3-\frac{1}{2}(x-1)\right](3-1)^2=\frac{1}{12}\left(x^3-3x^2-x+3\right).$$

Hence:

$$y=-\frac{1}{2}(x-3)2+\frac{1}{2}(x-1)3.5+\frac{1}{12}(x^3-3x^2-x+3)(-0.4875)=$$

$$=-0.040625x^3+0.121875x^2+0.790625x+1.128125.$$

In analogy, for the [x_1, x_2] interval, the polynomial is found as:

$$y = Ay_1 + By_2 + Cy_1'' + Dy_2'' = Ay_1 + By_2 + Cy_1'' + 0y_2'' = Ay_1 + By_2 + Cy_1'',$$

$$A = \frac{x - x_{j+1}}{x_j - x_{j+1}} = \frac{x-5}{3-5} = -\frac{1}{2}(x-5),$$

$$B = \frac{x - x_j}{x_{j+1} - x_j} = \frac{x-3}{5-3} = \frac{1}{2}(x-3),$$

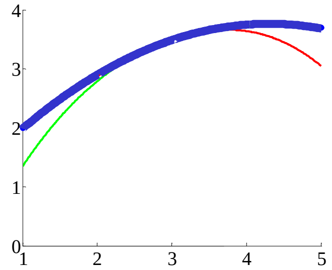
$$C = \frac{1}{6}(A^3 - A)(x_{j+1} - x_j)^2 = \frac{1}{6}\left[-\frac{1}{8}(x-5)^3 + \frac{1}{2}(x-5)\right](5-3)^2 = \frac{1}{12}\left(-x^3 + 15x^2 - 71x + 105\right),$$

Hence:

$$y = -\frac{1}{2}(x-5)3.5 + \frac{1}{2}(x-3)3.7 + \frac{1}{12}(-x^3 + 15x^2 - 71x + 105)(-0.4875) =$$

$$= -0.040625x^3 - 0.609375x^2 + 2.984375x - 1.065625.$$

The polynomials (thin lines) as well as the spline function (thick line) are illustrated in Fig.2.



Rys.2. Polynomials and the spline function..

TAKS

1) Calculate cubic polynomials that make the spline function of given three points.
2) In Matlab environment determine linear and cubic spline functions that interpolate a function named by your teacher for four points (knots). Make plots of the functions and calculate the maximal absolute value of the interpolation error. Decide which interpolation was better in your case.
3) perform The same tasks as in point 2) for the Lagrange and spline interpolations and different number of equidistant knots. Draw conclusions.