Interior Point Methods

V. Leclère (ENPC)

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Contents

Convex differentiable optimization problem

We consider the following convex optimization problem

$$
\begin{aligned}
(\mathcal{P}) \quad \min_{x\in\mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & Ax = b \\
& g_i(x) \leq 0 \qquad\qquad \forall i \in [\![1, n_I]\!]
\end{aligned}
$$

where A is a $n_E \times n$ matrix, and all functions f and g_i are assumed convex, real valued and twice differentiable.

Introducing the Lagrangian II

$$
\begin{aligned}
(\mathcal{P}) \quad \min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & Ax = b \\
& g_i(x) \leq 0 \qquad\qquad \forall i \in [\![1, n_I]\!]
\end{aligned}
$$

is equivalent to

$$
\min_{x \in \mathbb{R}^n} \quad f(x) + \mathbb{I}_{\{0\}}(Ax - b) + \sum_{i=1}^{n_I} \mathbb{I}_{\mathbb{R}^-}(h_i(x))
$$

which we rewrite

$$
\min_{x \in \mathbb{R}^n} \quad f(x) + \sup_{\lambda \in \mathbb{R}^{n_E}} \lambda^\top (Ax - b) + \sum_{i=1}^{n_I} \sup_{\mu_i \geq 0} \mu_i h_i(x)
$$

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$$

which we rewrite

$$
\min_{x\in\mathbb{R}^n} \sup_{\lambda\in\mathbb{R}^{n_E},\mu\in\mathbb{R}^{n_I}_+} \quad f(x) + \lambda^\top(Ax-b) + \sum_{i=1}^{n_I} \mu_i h_i(x)
$$

Introducing the Lagrangian II

$$(\mathcal{P}) \quad \min_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^{n_E}, \mu \in \mathbb{R}_+^{n_I}} \underbrace{f(x) + \lambda^\top (Ax - b) + \sum_{i=1}^{n_I} \mu_i h_i(x)}_{:=\mathcal{L}(x;\lambda,\mu)}$$

$$(\mathcal{D}) \quad \sup_{\lambda \in \mathbb{R}^{n_E}, \mu \in \mathbb{R}_+^{n_I}} \min_{x \in \mathbb{R}^n} f(x) + \lambda^\top (Ax - b) + \sum_{i=1}^{n_I} \mu_i h_i(x)$$

As for any function ϕ we always have

$$\sup_y \inf_x \phi(x, y) \leq \inf_x \sup_y \phi(x, y)$$

we have that (weak duality)

$$val(\mathcal{D}) \leq val(\mathcal{P}).$$

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we have that (weak duality)

$$val(\mathcal{D}) \leq val(\mathcal{P}).$$

Lower bounds from duality

Define the dual function

$$d(\lambda, \mu) := \inf_x \mathcal{L}(x; \lambda, \mu)$$

Then we have $val(\mathcal{D}) = \sup_{\lambda \in \mathbb{R}^{n_E}, \mu \in \mathbb{R}^{n_I}_+} d(\lambda, \mu)$.

Thus, we can compute a lower bound to $val(\mathcal{D}) \leq val(\mathcal{P})$ by choosing an any admissible dual points $\lambda \in \mathbb{R}^{n_E}, \mu \in \mathbb{R}^{n_I}_+$ and solving the unconstrained problem

$$d(\lambda, \mu) = \inf_{x \in \mathbb{R}^n} f(x) + \lambda^\top (Ax - b) + \sum_{i=1}^{n_I} \mu_i h_i(x)$$

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$$d(\lambda,\mu) = \inf_{x\in\mathbb{R}^n} f(x) + \lambda^\top(Ax-b) + \sum_{i=1}^{n_I} \mu_i h_i(x)$$

Constraint qualification

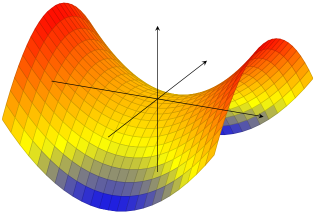
Recall that, for a convex differentiable optimization problem, the constraints are qualified if *Slater's condition* is satisfied : there exists a strictly admissible feasable point

$$\exists x_0 \in \mathbb{R}^n, \qquad Ax_0 = b, \quad \forall i \in [\![1, n_I]\!], \quad g_i(x_0) < 0$$

Saddle point

If $(\mathcal{P})$ is a convex optimization problem with qualified constraints, then

- $val(\mathcal{D}) = val(\mathcal{P})$
- any optimal solution $x^\sharp$ of $(\mathcal{P})$ is part of a saddle point $(x^\sharp; \lambda^\sharp, \mu^\sharp)$ of $\mathcal{L}$
- $(\lambda^\sharp, \mu^\sharp)$ is an optimal solution of $(\mathcal{D})$



Karush Kuhn Tucker conditions

If Slater's condition is satisfied, then $x^\sharp$ is an optimal solution to (P) if and only if there exists optimal multipliers $\lambda^\sharp \in \mathbb{R}^{n_E}$ and $\mu^\sharp \in \mathbb{R}^{n_I}$ satisfying

$$
\begin{cases}
\nabla f(x^\sharp) + A^\top \lambda^\sharp + \sum_{i=1}^{n_I} \mu_i^\sharp \nabla g_i(x^\sharp) = 0 & \text{first order condition} \\
Ax^\sharp = b & \text{primal admissibility} \\
g(x^\sharp) \leq 0 & \\
\mu \geq 0 & \text{dual admissibility} \\
\mu_i g_i(x^\sharp) = 0, \quad \forall i \in [\![1, n_I]\!] & \text{complementarity}
\end{cases}
$$

The three last conditions are sometimes compactly written

$$0 \geq g(x^\sharp) \perp \mu \geq 0$$

Karush Kuhn Tucker conditions

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$$\begin{cases} \nabla f(x^\sharp) + A^\top \lambda^\sharp + \sum_{i=1}^{n_I} \mu_i^\sharp \nabla g_i(x^\sharp) = 0 & \text{first order condition} \\ Ax^\sharp = b & \text{primal admissibility} \\ g(x^\sharp) \leq 0 & \\ \mu \geq 0 & \text{dual admissibility} \\ \mu_i g_i(x^\sharp) = 0, \quad \forall i \in [\![1, n_I]\!] & \text{complementarity} \end{cases}$$

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Contents

Intuition for Newton's method : unconstrained case

Newton's method is an iterative optimization method that minimizes a quadratic approximation of the objective function at the current point x_k.

Consider the following unconstrained optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x)$$

At x_k we have

$$f(x_k + d) = f(x_k) + \nabla f(x_k)^\top d + \frac{1}{2} d^\top \nabla^2 f(x_k) d + o(\|d\|^2)$$

And the direction d^k minimizing the quadratic approximation is given by

$$\nabla f(x_k) + \nabla^2 f(x_k) d_k = 0.$$

Intuition for Newton's method : linearly constrained case

Approximate the linearly constrained optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

by

$$\begin{aligned} \min_{d\in\mathbb{R}^n} \quad & f(x_k) + \nabla f(x_k)^\top d + \frac{1}{2} d^\top \nabla^2 f(x_k) d \\ \text{s.t.} \quad & A(x_k + d) = b \end{aligned}$$

Which is equivalent to solving (for admissible x_k)

$$\begin{aligned} \min_{d\in\mathbb{R}^n} \quad & \nabla f(x_k)^\top d + \frac{1}{2} d^\top \nabla^2 f(x_k) d \\ \text{s.t.} \quad & Ad = 0 \end{aligned}$$

Finding Newton's direction

$$\begin{aligned} \min_{d \in \mathbb{R}^n} \quad & \nabla f(x_k)^\top d + \frac{1}{2} d^\top \nabla^2 f(x_k) d \\ \text{s.t.} \quad & Ad = 0 \end{aligned}$$

By KKT the optimal d_k is given by

$$\begin{cases} \nabla f(x_k) + \nabla^2 f(x_k) d_k + A^\top \lambda = 0 \\ A d_k = 0 \end{cases}$$

Or in a matricial form

$$\begin{pmatrix} \nabla^2 f(x_k) & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} d_k \\ \lambda \end{pmatrix} = \begin{pmatrix} -\nabla f(x_k) \\ 0 \end{pmatrix}$$

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Newton's algorithm: linearly constrained case

Data: Initial admissible point x_0
Result: quasi-optimal point
$k = 0$;
while $|\nabla f(x_k)| \geq \varepsilon$ **do**

Solve for d

$$\begin{pmatrix} \nabla^2 f(x_k) & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ \lambda \end{pmatrix} = \begin{pmatrix} -\nabla f(x_k) \\ 0 \end{pmatrix}$$

Line-search for $\alpha \in [0,1]$ on $f(x_k + \alpha d)$
$x_{k+1} = x_k + \alpha d$
$k = k+1$

Algorithm 1: Newton's algorithm

Contents

Video explanation

A short video introduction to the content of this and the next section. https://www.youtube.com/watch?v=MsgpS15JRbI

Constrained optimization problem

We now want to consider a convex differentiable optimization problem with equality and inequality constraints.

$$(\mathcal{P}_\infty) \qquad \min_{x\in\mathbb{R}^n} \quad f(x)$$
$$\text{s.t.} \quad Ax = b$$
$$g_i(x) \leq 0 \qquad \forall i \in [\![1, n_I]\!]$$

where all functions f and g_i are assumed convex, finite valued and twice differentiable.

Which we rewrite

$$\min_{x\in\mathbb{R}^n} \quad f(x) + \sum_{i=1}^{n_I} \mathbb{I}_{\mathbb{R}^-}(g_i(x))$$
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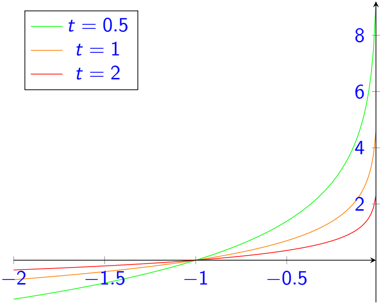
\end{aligned}
$$

The negative log function

- The idea of barrier method is to replace the indicator function $\mathbb{I}_{\mathbb{R}^-}$ by a smooth function.
- We choose the function $z \mapsto -1/t \log(-z)$
- Note that they also take value $+\infty$ on $\mathbb{R}^+$

Illustration of barrier functions



Calculus

- We define

$$\phi : x \mapsto -\sum_{i=1}^{n_I} \ln(-g_i(x))$$

- Thus we have $t\phi(x) \to \mathbb{I}_{\{g_i(x)<0,\ \forall i \in [n_I]\}}$
- We have

$$\nabla \phi(x) =$$

$$\nabla^2 \phi(x) =$$

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- We have

$$\nabla \phi(x) = \sum_{i=1}^{n_I} -\frac{1}{g_i(x)} \nabla g_i(x)$$

$$\nabla^2 \phi(x) =$$

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- We define
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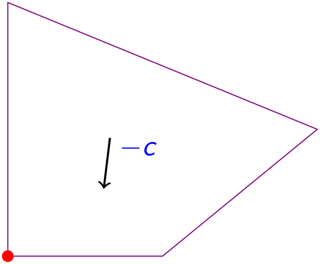
- Thus we have $t\phi(x) \to \mathbb{I}_{\{g_i(x)<0,\ \forall i\in[n_I]\}}$
- We have
$$\nabla\phi(x) = \sum_{i=1}^{n_I} -\frac{1}{g_i(x)}\nabla g_i(x)$$
$$\nabla^2\phi(x) = \sum_{i=1}^{n_I} \Big[\frac{1}{g_i^2(x)}\nabla g_i(x)\nabla g_i(x)^\top - \frac{1}{g_i(x)}\nabla^2 g_i(x)\Big]$$

Penalized problem

We consider

$$(\mathcal{P}_\infty) \quad \min_{x\in\mathbb{R}^n} \; f(x)$$
$$\text{s.t.} \quad Ax = b$$

with optimal solution x_t.

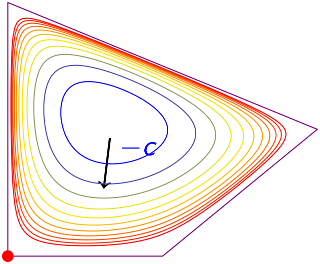


Penalized problem

We consider

$$(\mathcal{P}_t) \quad \min_{x \in \mathbb{R}^n} \; f(x) + t\phi(x) \quad \text{s.t.} \quad Ax = b$$

with optimal solution x_t.

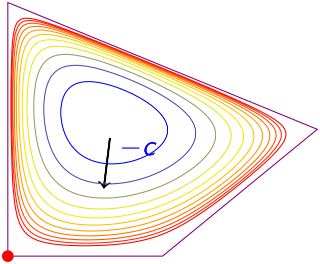


Penalized problem

We consider

$$(\mathcal{P}_t) \quad \min_{x\in\mathbb{R}^n} tf(x) + \phi(x)$$
$$\text{s.t.} \quad Ax = b$$

with optimal solution x_t.
Letting t goes to $+\infty$ get to solution of $(\mathcal{P})$ along the central path.



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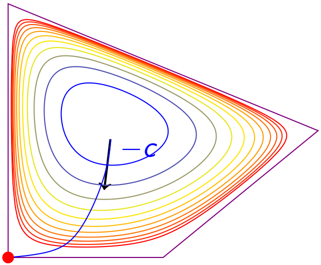


Characterizing central path I

x_t is solution of

$$(\mathcal{P}_t) \quad \min_{x\in\mathbb{R}^n} tf(x) + \phi(x)$$
$$\text{s.t.} \quad Ax = b$$

if and only if, there exists $\lambda_t \in \mathbb{R}^{n_E}$, such that

Characterizing central path I

x_t is solution of

$$(\mathcal{P}_t) \quad \min_{x \in \mathbb{R}^n} tf(x) + \phi(x)$$
$$\text{s.t.} \quad Ax = b$$

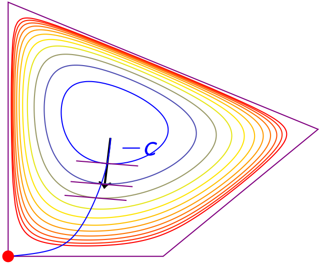
if and only if, there exists $\lambda_t \in \mathbb{R}^{n_E}$, such that

$$\begin{cases} Ax_t = b & \\ g_i(x_t) < 0 & \forall i \in [n_I] \\ t\nabla f(x_t) + \nabla \phi(x_t) + A^\top \lambda = 0 & \end{cases}$$

Characterizing central path II

$$\begin{cases} Ax_t = b \\ g(x_t) < 0 \\ t\nabla f(x_t) + \nabla\phi(x_t) + A^\top \lambda = 0 \end{cases}$$

If $A = 0$ it means that $\nabla f(x_t)$ is orthogonal to the level lines of ϕ



Duality

Recall the original optimization problem

$$(\mathcal{P}_\infty) \qquad \min_{x\in\mathbb{R}^n} \quad f(x)$$
$$\text{s.t.} \quad Ax = b$$
$$g_i(x) \leq 0 \qquad \forall i \in [\![1, n_I]\!]$$

with Lagrangian

$$\mathcal{L}(x;\lambda,\mu) := f(x) + \lambda^\top(Ax-b) + \sum_{i=1}^{n_I}\mu_i g_i(x)$$

and dual function

$$d(\lambda,\mu) := \inf_{x\in\mathbb{R}^n} \mathcal{L}(x;\lambda,\mu).$$

For any admissible dual point $(\lambda,\mu) \in \mathbb{R}^{n_E}\times\mathbb{R}^{n_I}_+$, we have

$$d(\lambda,\mu) \leq val(\mathcal{P}_\infty)$$

Duality

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$$\mathcal{L}(x;\lambda,\mu) := f(x) + \lambda^\top(Ax - b) + \sum_{i=1}^{n_I} \mu_i g_i(x)$$

and dual function

$$d(\lambda,\mu) := \inf_{x\in\mathbb{R}^n} \mathcal{L}(x;\lambda,\mu).$$

For any admissible dual point $(\lambda,\mu) \in \mathbb{R}^{n_E} \times \mathbb{R}_+^{n_I}$, we have

$$d(\lambda,\mu) \leq val(\mathcal{P}_\infty)$$

Getting a lower bound

For given admissible dual point $(\lambda, \mu) \in \mathbb{R}^{n_E} \times \mathbb{R}_+^{n_I}$, a point $x^\sharp(\lambda, \mu)$ minimizing $\mathcal{L}(\cdot, \lambda, \mu)$, is characterized by first order conditions

$$\nabla f(x^\sharp(\lambda, \mu)) + A^\top \lambda + \sum_{i=1}^{n_I} \mu_I \nabla g_i(x^\sharp(\lambda, \mu)) = 0$$

which gives

$$d(\lambda, \mu) = \mathcal{L}(x^\sharp(\lambda, \mu); \lambda, \mu) \leq val(\mathcal{P}_\infty)$$

Dual point on the central path

Now recall that x_t, solution of $(\mathcal{P}_t)$, is characterized by

$$\begin{cases} Ax_t = b, g(x_t) < 0 \\ t\nabla f(x_t) + \nabla \phi(x_t) + A^\top \lambda = 0 \end{cases}$$

And we have seen that

$$\nabla \phi(x) = \sum_{i=1}^{n_I} \frac{1}{-g_i(x)} g_i(x)$$

Thus,

$$\nabla f(x_t) + A^\top \lambda / t + \sum_{i=1}^{n_I} \underbrace{\frac{1}{-tg_i(x)}}_{(\mu_t)_i} \nabla g_i(x) = 0$$

which means that $x_t = x^\sharp(\lambda/t, \mu_t)$.

Bounding the error

Let x_t be a primal point on the central path satisfying

$$t\nabla f(x_t) + \nabla \phi(x_t) + A^\top \lambda_t = 0$$

We define a dual point $(\mu_t)_i = \frac{1}{-tg_i(x_t)} > 0$. We have

$$\begin{aligned} d(\mu_t, \lambda_t/t) &= \mathcal{L}(x_t, \mu_t, \lambda_t/t) \\ &= f(x_t) + \frac{1}{t}\lambda_t^\top \underbrace{(Ax_t - b)}_{=0} + \sum_{i=1}^{n_I} \frac{1}{-tg_i(x_t)} g_i(x_t) \\ &= f(x_t) - \frac{n_I}{t} \leq val(\mathcal{P}_\infty) \end{aligned}$$

And in particular x_t is an n_I/t-optimal solution of $(\mathcal{P}_\infty)$.

Bounding the error

Let x_t be a primal point on the central path satisfying

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And in particular x_t is an n_I/t-optimal solution of $(\mathcal{P}_\infty)$.

Interpretation through KKT condition

A point x_t is on the central path iff it is admissible and there exists λ such that

$$\nabla f(x_t) + A^\top \lambda + \sum_{i=1}^{n_I} \underbrace{\frac{1}{-tg_i(x)}}_{(\mu_t)_i} \nabla g_i(x) = 0$$

which can be rewritten

$$\begin{cases} \nabla f(x) + A^\top \lambda + \sum_{i=1}^{n_i} \mu_i \nabla g_i(x) = 0 \\ Ax = b, g_i(x) \leq 0 \\ \mu \geq 0 \\ -\mu_i g_i(x) = \frac{1}{t} \end{cases}$$

Contents

Taking a step back

- We saw that we can extend Newton's method to solve linearly constrained optimization problem.
- We saw that we can approximate inequality constraints through the use of logarithmic barrier $-1/t \sum_i \ln(-g_i(x))$.
- We proved that x_t is an n_I/t-optimal solution.
- The trade-off with t is : larger t means x_t closer to optimal solution x_∞ but the approximate problem $(\mathcal{P}_t)$ have worse conditionning.

Barrier method

Data: increase $\rho > 1$, error $\varepsilon > 0$, initial t

Result: ε-optimal point

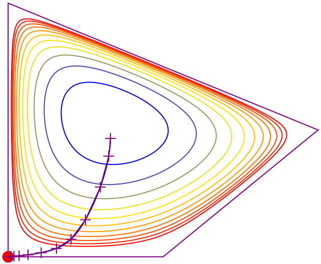
solve $(\mathcal{P}_t)$ and set $x = x_t$;

while $n_I/t \geq \varepsilon$ **do**

increase t*:* $t = \rho t$

centering step: solve $(\mathcal{P}_t)$ starting at x ;

update : $x = x_t$



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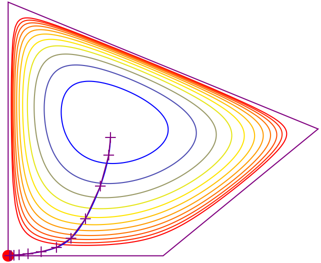
solve $(\mathcal{P}_t)$ and set $x = x_t$;
while $n_I/t \geq \varepsilon$ **do**
- *increase* t*:* $t = \rho t$
- *centering step:* solve $(\mathcal{P}_t)$ starting at x ;
- *update* : $x = x_t$



Question : why solve $(\mathcal{P}_t)$ to optimality ?

Solving $(\mathcal{P}_t)$ with Newton's method

$$(\mathcal{P}_t) \quad \min_{x\in\mathbb{R}^n} \quad tf(x) + \phi(x)$$
$$\text{s.t.} \quad Ax = b$$

is a linearly constrained optimization problem that can be solved by Newton's method.
More precisely we have $x_{k+1} = x_k + d_k$ with d_k a solution of

$$\begin{pmatrix} t\nabla^2 f(x_k) + \nabla^2\phi(x_k) & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} d_k \\ \lambda \end{pmatrix} = \begin{pmatrix} -t\nabla f(x_k) - \nabla\phi(x_k) \\ 0 \end{pmatrix}$$

Path following interior point method

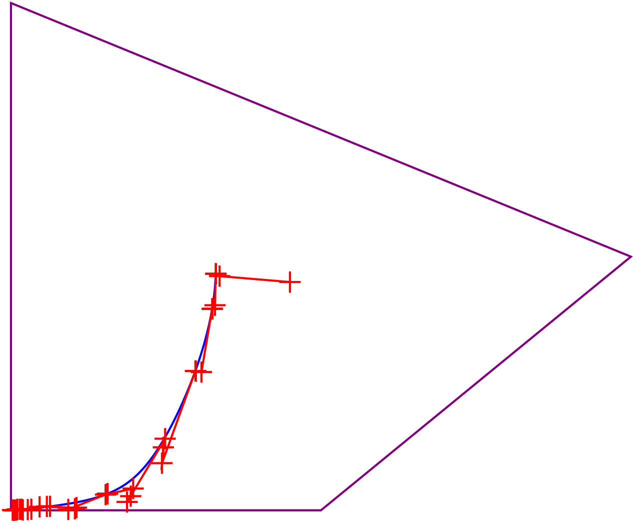
Data: increase $\rho > 1$, error $\varepsilon > 0$, initial t_0
initial strictly feasible point x_0
$k = 0$
for $k \in \mathbb{N}$ **do** `// Outer step`
  $y_0 = x_k$
  for $\kappa \in [K]$ **do** `// Inner step`
    solve for d ; `// Newton step for` $(\mathcal{P}_t)$

$$\begin{pmatrix} t_k \nabla^2 f(y_\kappa) + \nabla^2 \phi(y_\kappa) & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ \lambda \end{pmatrix} = \begin{pmatrix} -t_k \nabla f(y_\kappa) - \nabla \phi(x_\kappa) \\ 0 \end{pmatrix}$$

    $y_{\kappa+1} = y_\kappa + d$
  $t_{k+1} = \rho t_k$

Algorithm 2: Path following algorithm

Path following algorithm



Video explanation

A longer presentation to watch at a later time
https://www.youtube.com/watch?v=zm4mfr-QT1E

Contents

A linear problem - inequality form

We consider the following LP

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \leq b_i \qquad\qquad \forall i \in [n_I] \end{aligned}$$

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Newton step

Starting from x, the Newton direction for $(\mathcal{P}_t)$ is

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which, in algebraic form, yields

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with $d_i = 1/(b_i - a_i^\top x)$.

Newton step

Starting from x, the Newton direction for $(\mathcal{P}_t)$ is

$$dir_t(x) = -\left(\nabla^2\phi(x)\right)^{-1}\left(tc + \nabla\phi(x)\right)$$

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Theory tell us to use a step-size of 1 for Newton's method.
Practice teach us to use a smaller step-size (or linear-search).

Interior Point Method for LP pseudo code

Data: Initial admissible point x_0, initial penalization $t_0 > 0$;
parameter: $\rho > 1$, $N_{in} \geq 1$, $N_{out} \geq 1$;
Result: quasi-optimal point
$x = x0$, $t = t_0$;
for $k = 1..N_{out}$ **do**
  for $\kappa = 1..N_{in}$ **do**
    Compute d, with $d_i = 1/(b_i - a_i^T x)$;
    Solve for dir

$$A^\top \mathrm{diag}(d)^2 A \mathrm{dir} = -(tc + A^\top d)$$

    update $x \leftarrow x + \mathrm{dir}$;
  update $t \leftarrow \rho t$

Algorithm 3: Interior Point Method for LP