GATE NM-54 2022

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Q: A system with two degrees of freedom, as shown in the figure, has masses $m_1 = 200kg$ and $m_2 = 100kg$ and stiffness coefficients $k_1 = k_2 = 200N/m$. Then the lowest natural frequency of the system is ___________ rad/s (rounded off to one decimal place).

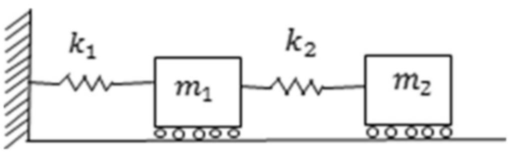

Fig. 0.

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Solution:

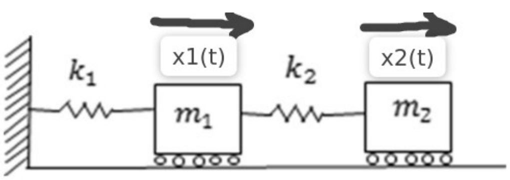

Fig. 0.

METHOD-1:

Variable	Description	Value
m_1	Mass of block 1	$200kg$
m_2	Mass of block 2	$100kg$
k_1	Stiffness coefficient of spring1	$200N/m$
k_2	Stiffness coefficient of spring2	$200N/m$
$x_i(t)$	Displacement of i^{th} block	$x_i(t) = \cos(\omega t + \phi_i), x_i \geq 0$

TABLE 0
Input Parameters

$$m_1\ddot{x}_1(t) - k_2\left(x_2(t) - x_1(t)\right) + k_1x_1(t) = 0 \tag{1}$$

$$m_2\ddot{x}_2(t) + k_2\left(x_2(t) - x_1(t)\right) = 0 \tag{2}$$

Writing (1) and (2) in matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \tag{4}$$

$$x^{''}(t) \xleftrightarrow{\mathcal{L}} s^2X(s) - sx(0) - x^{'}(0) \tag{5}$$

Applying Laplace transform for (1) and (2) assuming they are at their repective maximum displacement at t=0

$$m_1\left(s^2X_1(s) - sx_1(0) - 0\right) - k_2\left(X_2(s) - X_1(s)\right) + k_1X_1(s) = 0 \tag{6}$$

$$m_2\left(s^2X_2(s) - sx_2(0) - 0\right) + k_2\left(X_2(s) - X_1(s)\right) = 0 \tag{7}$$

Writing (6) and (7) in matrix form:

$$\begin{pmatrix} m_1s^2 + (k_1 + k_2) & -k_2 \\ -k_2 & m_2s^2 + k_2 \end{pmatrix}\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} m_1sx_1(0) \\ m_2sx_2(0) \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{1}{m_1m_2s^4 + (k_1 + k_2)m_2s^2 + k_2m_1s^2 + k_1k_2} \begin{pmatrix} m_2s^2 + k_2 & k_2 \\ k_2 & m_1s^2 + (k_1 + k_2) \end{pmatrix}\begin{pmatrix} m_1sx_1(0) \\ m_2sx_2(0) \end{pmatrix} \tag{9}$$

$$\implies X_1(s) = \frac{m_1m_2s^3x_1(0) + k_2m_1sx_1(0) + m_2k_2sx_2(0)}{m_1m_2s^4 + (k_1 + k_2)m_2s^2 + k_2m_1s^2 + k_1k_2} \tag{10}$$

$$\implies X_2(s) = \frac{m_1k_2sx_1(0) + m_1m_2s^3x_2(0) + (k_1 + k_2)m_2sx_2(0)}{m_1m_2s^4 + (k_1 + k_2)m_2s^2 + k_2m_1s^2 + k_1k_2} \tag{11}$$

Considering denominator of $X_i(s)$ for $i = 1, 2$(ie: characteristic equation of the system):

$$m_1 m_2 s^4 + (k_1 + k_2) m_2 s^2 + k_2 m_1 s^2 + k_1 k_2 = 0 \quad (12)$$

$$s^4 + 4s^2 + 2 = 0 \quad (13)$$

$$\implies s^2 = \frac{-4 \pm \sqrt{16 - 8}}{2} \quad (14)$$

$$= -2 \pm \sqrt{2} \quad (15)$$

$$\omega = \pm\sqrt{2 \mp \sqrt{2}} \quad (16)$$

$$\implies \omega_{least} = 0.765 rad/s \quad (17)$$

METHOD-2:CONVERTING MECHANICAL SYSTEM INTO ITS ANALOGOUS ELECTRICAL CIRCUIT BY FORCE VOLTAGE METHOD

Mechanical Quantity	**Electrical Quantity**
Force	Voltage
Velocity	Current
Mass	Inductance
Stiffness coefficient of spring	Inverse of Capacitance

TABLE 0
INPUT PARAMETERS

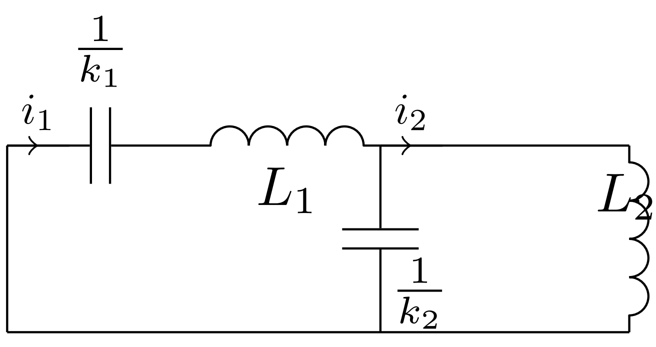

Fig. 0.

$$k_1 \int i_1 \, dt + L_1 \frac{di_1}{dt} + k_2 \int (i_1 - i_2) \, dt = 0 \quad (18)$$

$$L_2 \frac{di_2}{dt} - k_2 \int (i_1 - i_2) \, dt = 0 \quad (19)$$

Differentiating (18) and (19) wrt t;

$$k_1 i_1 + L_1 \frac{d^2 i_1}{dt^2} + k_2 (i_1 - i_2) = 0 \quad (20)$$

$$L_2 \frac{d^2 i_2}{dt^2} - k_2 (i_1 - i_2) = 0 \quad (21)$$

Writing (20) and (21) in matrix form:

$$\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} \ddot{i}_1(t) \\ \ddot{i}_2(t) \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} i_1(t) \\ i_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (22)$$

$$i(t) \overset{\mathcal{L}}{\longleftrightarrow} I(s) \quad (23)$$

$$i''(t) \overset{\mathcal{L}}{\longleftrightarrow} s^2 I(s) - s i(0) - i'(0) \quad (24)$$

Applying Laplace transform assuming current to be maximum;

$$L_1 \left(s^2 I_1(s) - i_1'(0) \right) - k_2 (I_2(s) - I_1(s)) + k_1 I_1(s) = 0 \quad (25)$$

$$L_2 \left(s^2 I_2(s) - i_2'(0) \right) + k_2 (I_2(s) - I_1(s)) = 0 \quad (26)$$

Writing (25) and (26) in matrix form:

$$\begin{pmatrix} L_1 s^2 + (k_1 + k_2) & -k_2 \\ -k_2 & L_2 s^2 + k_2 \end{pmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} L_1 i_1'(0) \\ L_2 i_2'(0) \end{pmatrix} \quad (27)$$

$$\begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \frac{1}{L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2} \begin{pmatrix} L_2 s^2 + k_2 & k_2 \\ k_2 & L_1 s^2 + (k_1 + k_2) \end{pmatrix} \begin{pmatrix} L_1 i_1'(0) \\ L_2 i_2'(0) \end{pmatrix} \quad (28)$$

$$\implies I_1(s) = \frac{\left(L_2 s^2 + k_2\right) L_1 i_1'(0) + k_2 L_2 i_2'(0)}{L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2} \quad (29)$$

$$\implies I_2(s) = \frac{k_2 L_1 i_1'(0) + \left(L_1 s^2 + (k_1 + k_2)\right) L_2 i_2'(0)}{L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2} \quad (30)$$

Considering denominator of $I_i(s)$ for $i = 1, 2$(ie: characteristic equation of the system):

$$L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2 = 0 \tag{31}$$

$$s^4 + 4s^2 + 2 = 0 \tag{32}$$

$$\implies s^2 = \frac{-4 \pm \sqrt{16 - 8}}{2} \tag{33}$$

$$= -2 \pm \sqrt{2} \tag{34}$$

$$\omega = \pm \sqrt{2 \mp \sqrt{2}} \tag{35}$$

$$\implies \omega_{least} = 0.765 rad/s \tag{36}$$