GATE IN-13 2022

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Q: A periodic function $f(x)$, with period 2, is defined as

$$f(x) = \begin{cases} -1 - x & -1 \leq x < 0 \\ 1 - x & 0 < x \leq 1 \end{cases} \tag{1}$$

The Fourier series of this function contains

A. Both $\cos(n\pi x)$ and $sin(n\pi x)$ where n=1,2,3...
B. Only $\sin(n\pi x)$ where n=1,2,3...
C. Only $\cos(n\pi x)$ where n=1,2,3...
D. Only $\cos(2n\pi x)$ where n=1,2,3... GATE IN 2022

Solution:

Parameter	Description
$f(x)$	Polynomial function
$2L$	Period of the Polynomial function
$c(n)$	Complex Fourier Coefficients

TABLE 4
INPUT PARAMETERS

The complex exponential Fourier Series of $f(x)$ is,

$$f(x) = \sum_{n=-\infty}^{\infty} c(n) e^{jn\omega x} \tag{2}$$

$$\implies c(n) = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-jn\omega x}\, dx \tag{3}$$

$$\tag{4}$$

For $n \neq 0$;

$$c(n) = \frac{1}{2} \int_{-1}^{1} f(x) e^{-jn\omega x}\, dx \tag{5}$$

$$= \frac{1}{2} \left(\int_{-1}^{0} (-1 - x)\, e^{-jn\omega x}\, dx + \int_{0}^{1} (+1 - x)\, e^{-jn\omega x}\, dx \right) \tag{6}$$

$$= \frac{1}{2} \left(- \int_{-1}^{0} e^{-jn\omega x}\, dx - \int_{-1}^{1} x e^{-jn\omega x}\, dx + \int_{0}^{1} e^{-jn\omega x}\, dx \right) \tag{7}$$

$$= \frac{1}{2} \left[\frac{-1}{jn\omega} \left[-\left(1 - e^{+jn\omega}\right) + \left(e^{-jn\omega} - 1\right) \right] - \int_{-1}^{1} x e^{-jn\omega x}\, dx \right] \tag{8}$$

$$= \frac{1}{2} \left[\frac{-1}{jn\omega} \left[-2 + e^{+jn\omega} + e^{-jn\omega} \right] + \left(\frac{e^{-jn\omega x}}{jn\omega} \left[x + \frac{1}{jn\omega} \right] \right)_{-1}^{1} \right] \tag{9}$$

$$= \frac{-1}{jn\omega} \left[-1 + \cos(n\omega) \right] + \frac{1}{2(jn\omega)^2} \left[\left(e^{-jn\omega}\right)(1 + jn\omega) - \left(e^{jn\omega}\right)(-jn\omega + 1) \right] \tag{10}$$

$$\implies c(n) = \frac{-1}{(jn\omega)^2} \left[-jn\omega + j \sin(n\omega) \right] \tag{11}$$

For $n = 0$;

$$c(0) = \frac{1}{2}\int_{-1}^{1} f(x)\,dx \tag{12}$$

$$= \frac{1}{2}\left[\int_{-1}^{0}(-1-x)\ dx + \int_{0}^{1}(1-x)\ dx\right] \tag{13}$$

$$= \frac{1}{2}\left[\left(-x-\frac{x^2}{2}\right)_{-1}^{0} + \left(x-\frac{x^2}{2}\right)_{0}^{1}\right] \tag{14}$$

$$= \frac{1}{2}\left[0-1+\frac{1}{2}+1-\frac{1}{2}-0\right] \tag{15}$$

$$\implies c(0) = 0 \tag{16}$$

The trigonometric Fourier Series of $f(x)$ is,

$$f(x) = a(0) + \sum_{n=1}^{\infty}\{a(n)\cos(n\omega x) + b(n)\sin(n\omega x)\} \tag{17}$$

Finding the Fourier Coefficient a_0,

$$a(0) = c(0) \tag{18}$$

$$\implies a(0) = 0 \tag{19}$$

Finding the Fourier Coefficients $a(n)$,

$$a(n) = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\omega x)\ dx, n \geq 0 \tag{20}$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\left(e^{-jn\omega x} + e^{jn\omega x}\right)\ dx \tag{21}$$

$$\implies a(n) = c(n) + c(-n) \tag{22}$$

$$\implies a(n) = 0 \tag{23}$$

$$\tag{24}$$

Finding the Fourier Coefficients $b(n)$,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\omega x)\ dx, n \geq 0 \tag{25}$$

$$= \frac{1}{L}\int_{-L}^{L} f(x) j\left(e^{-jn\omega x} - e^{jn\omega x}\right)\ dx \tag{26}$$

$$\implies b(n) = j\left(c_n - c_{-n}\right) \tag{27}$$

$$\implies b(n) = \frac{-2}{(n\omega)^2}\left[-n\omega + \sin(n\omega)\right] \tag{28}$$

$\implies$ The trigonometric Fourier Series of $f(x)$ is,

$$f(x) = \sum_{n=1}^{\infty} \{0 + 0 + b(n) \sin(n\omega x)\} \tag{29}$$

$$f(x) = \sum_{n=1}^{\infty} \left\{ \frac{-2}{(n\omega)^2} \left[-n\omega + \sin(n\omega)\right] \sin(n\omega x) \right\} \tag{30}$$

$$f(x) = \sum_{n=1}^{\infty} \left\{ \frac{-2}{(n\pi)^2} \left[-n\pi + \sin(n\pi)\right] \sin(n\pi x) \right\} \tag{31}$$

$$f(x) = \sum_{n=1}^{\infty} \left\{ \frac{2}{n\pi} \sin(n\pi x) \right\} \tag{32}$$

$\therefore$ The Fourier series of this function contains only $\sin(n\pi x)$ where n=1,2,3...

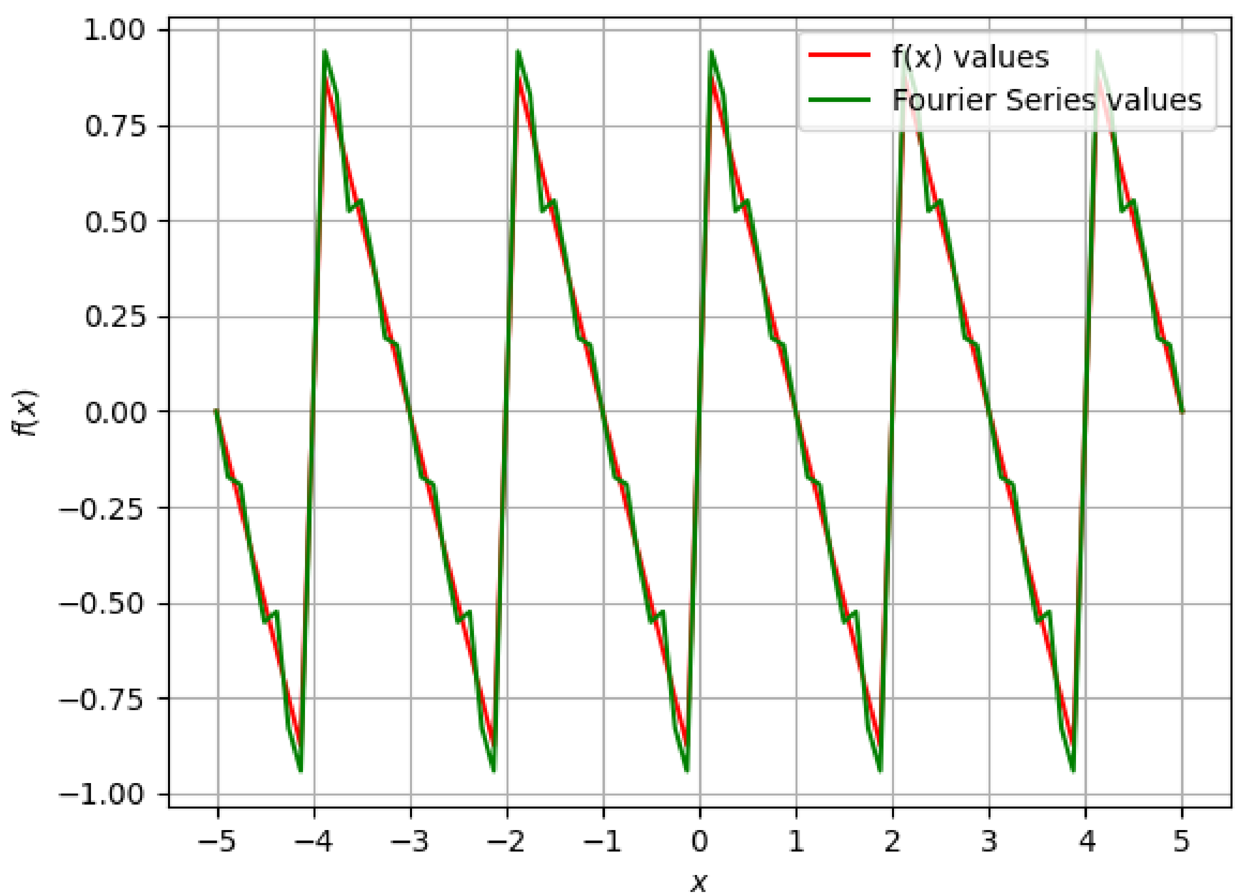

Fig. 4.