GATE NM-54 2022

EE23BTECH11011- Batchu Ishitha*

Q: A system with two degrees of freedom, as shown in the figure, has masses $m_1 = 200kg$ and $m_2 = 100kg$ and stiffness coefficients $k_1 = k_2 = 200N/m$. Then the lowest natural frequency of the system is _____ rad/s (rounded off to one decimal place).
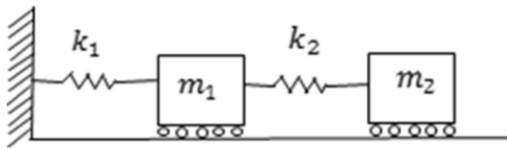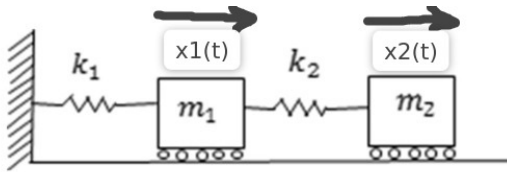

Fig. 0.

GATE NM 2022

Solution:

Fig. 0.

METHOD-1:

Variable	Description	Value
m_1	Mass of block 1	$200kg$
m_2	Mass of block 2	$100kg$
k_1	Stiffness coefficient of spring1	$200N/m$
k_2	Stiffness coefficient of spring2	$200N/m$
$x_i(t)$	Displacement of ith block	$x_i(t) = \cos(\omega t + \phi_i), x_i \geq 0$

TABLE 0

Input Parameters

$$m_1\ddot{x}_1(t) - k_2(x_2(t) - x_1(t)) + k_1 x_1(t) = 0 \quad (1)$$
$$m_2\ddot{x}_2(t) + k_2(x_2(t) - x_1(t)) = 0 \quad (2)$$

Writing (1) and (2) in matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3)$$

$$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s) \quad (4)$$

$$x''(t) \overset{\mathcal{L}}{\leftrightarrow} s^2 X(s) - sx(0) - x'(0) \quad (5)$$

Applying Laplace transform for (1) and (2) assuming they are at their repective maximum displacement at t=0

$$m_1\left(s^2 X_1(s) - sx_1(0) - 0\right) - k_2\left(X_2(s) - X_1(s)\right) + k_1 X_1(s) = 0 \quad (6)$$

$$m_2\left(s^2 X_2(s) - sx_2(0) - 0\right) + k_2\left(X_2(s) - X_1(s)\right) = 0 \quad (7)$$

Writing (6) and (7) in matrix form:

$$\begin{pmatrix} m_1 s^2 + (k_1 + k_2) & -k_2 \\ -k_2 & m_2 s^2 + k_2 \end{pmatrix}\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} m_1 sx_1(0) \\ m_2 sx_2(0) \end{pmatrix} \quad (8)$$

$$\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{1}{m_1 m_2 s^4 + (k_1 + k_2)m_2 s^2 + k_2 m_1 s^2 + k_1 k_2} \begin{pmatrix} m_2 s^2 + k_2 & k_2 \\ k_2 & m_1 s^2 + (k_1 + k_2) \end{pmatrix}\begin{pmatrix} m_1 sx_1(0) \\ m_2 sx_2(0) \end{pmatrix} \quad (9)$$

$$\implies X_1(s) = \frac{m_1 m_2 s^3 x_1(0) + k_2 m_1 sx_1(0) + m_2 k_2 sx_2(0)}{m_1 m_2 s^4 + (k_1 + k_2)m_2 s^2 + k_2 m_1 s^2 + k_1 k_2} \quad (10)$$

$$\implies X_2(s) = \frac{m_1 k_2 sx_1(0) + m_1 m_2 s^3 x_2(0) + (k_1 + k_2)m_2 sx_2(0)}{m_1 m_2 s^4 + (k_1 + k_2)m_2 s^2 + k_2 m_1 s^2 + k_1 k_2} \quad (11)$$

Considering denominator of $X_i(s)$ for $i = 1, 2$(ie: characteristic equation of the system):

$$m_1 m_2 s^4 + (k_1 + k_2)m_2 s^2 + k_2 m_1 s^2 + k_1 k_2 = 0 \quad (12)$$

$$s^4 + 4s^2 + 2 = 0 \quad (13)$$

$$\implies s^2 = \frac{-4 \pm \sqrt{16 - 8}}{2} \quad (14)$$

$$= -2 \pm \sqrt{2} \quad (15)$$

$$\omega = \pm \sqrt{2 \mp \sqrt{2}} \quad (16)$$

$$\implies \omega_{least} = 0.765 rad/s \quad (17)$$

METHOD-2:CONVERTING MECHANICAL SYSTEM INTO ITS ANALOGOUS ELECTRICAL CIRCUIT BY FORCE VOLTAGE METHOD

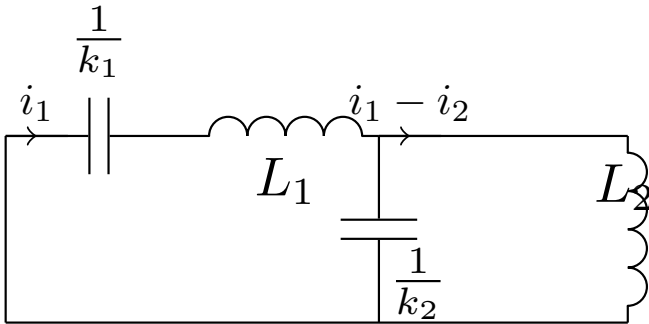| Mechanical Quantity	Electrical Quantity
Force	Voltage
Velocity	Current
Mass	Inductance
Stiffness coefficient of spring	Inverse of Capacitance

TABLE 0

INPUT PARAMETERS

Fig. 0.

$$k_1 \int i_1 \, dt + L_1 \frac{di_1}{dt} + L_2 \left(\frac{di_1}{dt} - \frac{di_2}{dt} \right) = 0 \quad (18)$$

$$k_2 \int i_2 \, dt + L_2 \left(\frac{di_2}{dt} - \frac{di_1}{dt} \right) = 0 \quad (19)$$

Differentiating (18) and (19) wrt t;

$$k_1 i_1 + L_1 \frac{d^2 i_1}{dt^2} + L_2 \left(\frac{d^2 i_1}{dt^2} - \frac{d^2 i_2}{dt^2} \right) = 0 \quad (20)$$

$$k_2 i_2 + L_2 \left(\frac{d^2 i_2}{dt^2} - \frac{d^2 i_1}{dt^2} \right) = 0 \quad (21)$$

From (21)

$$\implies \frac{d^2 i_1}{dt^2} = \frac{k_2 i_2 + L_2 \frac{d^2 i_2}{dt^2}}{L_2} \quad (22)$$

Differentiating (22) wrt t twice;

$$\implies \frac{d^4 i_1}{dt^4} = \frac{k_2 \frac{d^2 i_2}{dt^2} + L_2 \frac{d^4 i_2}{dt^4}}{L_2} \quad (23)$$

Differentiating (20) wrt t twice;

$$k_1 \frac{d^2 i_1}{dt^2} + L_1 \frac{d^4 i_1}{dt^4} + L_2 \left(\frac{d^4 i_1}{dt^4} - \frac{d^4 i_2}{dt^4} \right) = 0 \quad (24)$$

Substituting (22) and (23) in (24)

$$k_1 \left(\frac{k_2 i_2 + L_2 \frac{d^2 i_2}{dt^2}}{L_2} \right) + (L_1 + L_2) \left(\frac{k_2 \frac{d^2 i_2}{dt^2} + L_2 \frac{d^4 i_2}{dt^4}}{L_2} \right) - L_2 \frac{d^4 i_2}{dt^4} = 0 \quad (25)$$

$$\frac{d^4 i_2}{dt^4} (L_1 + L_2 - L_2) + \frac{d^2 i_2}{dt^2} \left(k_1 + \frac{k_2 (L_1 + L_2)}{L_2} \right) + \frac{k_1 k_2}{L_2} i_2 = 0 \quad (26)$$

$$L_1 L_2 \frac{d^4 i_2}{dt^4} + [k_1 L_2 + k_2 (L_1 + L_2)] \frac{d^2 i_2}{dt^2} + k_1 k_2 i_2 = 0 \quad (27)$$

Applying Laplace transform for (27)

$$L_1 L_2 s^4 I_2(s) + [k_1 L_2 + k_2 (L_1 + L_2)] s^2 I_2(s) + k_1 k_2 I_2(s) = 0 \quad (28)$$

Substituting the values in (28)

$$(200)(100)s^4 + [(200)(100) + (200)(200 + 100)] s^2 + (200)(200) = 0 \quad (29)$$

$$s^4 + 4s^2 + 2 = 0 \quad (30)$$

$$\implies s^2 = \frac{-4 \pm \sqrt{16 - 8}}{2} \quad (31)$$

$$= -2 \pm \sqrt{2} \quad (32)$$

$$\omega = \pm \sqrt{2 \mp \sqrt{2}} \quad (33)$$

$$\implies \omega_{least} = 0.765 rad/s \quad (34)$$