GATE NM-54 2022

EE23BTECH11011- Batchu Ishitha*

Q: A system with two degrees of freedom, as shown in the figure, has masses $m_1 = 200kg$ and $m_2 = 100kg$ and stiffness coefficients $k_1 = k_2 = 200N/m$. Then the lowest natural frequency of the system is ___________ rad/s (rounded off to one decimal place).

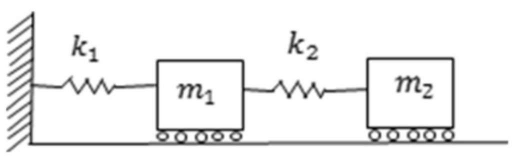

Fig. 0.

GATE NM 2022

Solution:

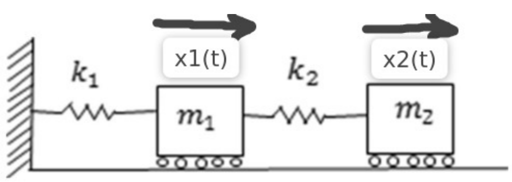

Fig. 0.

METHOD-1:

Variable	Description	Value
m_1	Mass of block 1	$200kg$
m_2	Mass of block 2	$100kg$
k_1	Stiffness coefficient of spring1	$200N/m$
k_2	Stiffness coefficient of spring2	$200N/m$

TABLE 0
INPUT PARAMETERS

$$m_1\ddot{x}_1(t) - k_2\left(x_2(t) - x_1(t)\right) + k_1x_1(t) = 0 \tag{1}$$

$$m_2\ddot{x}_2(t) + k_2\left(x_2(t) - x_1(t)\right) = 0 \tag{2}$$

Writing (1) and (2) in matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} + \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \tag{4}$$

$$x''(t) \overset{\mathcal{L}}{\longleftrightarrow} s^2X(s) - sx(0) - x'(0) \tag{5}$$

Applying Laplace transform for (1) and (2) assuming they are at their repective maximum displacement at t=0

$$m_1\left(s^2X_1(s) - sx_1(0) - 0\right) - k_2\left(X_2(s) - X_1(s)\right) + k_1X_1(s) = 0 \tag{6}$$

$$m_2\left(s^2X_2(s) - sx_2(0) - 0\right) + k_2\left(X_2(s) - X_1(s)\right) = 0 \tag{7}$$

Writing (6) and (7) in matrix form:

$$\begin{pmatrix} m_1s^2 + (k_1+k_2) & -k_2 \\ -k_2 & m_2s^2 + k_2 \end{pmatrix}\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} m_1sx_1(0) \\ m_2sx_2(0) \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{1}{m_1m_2s^4 + (k_1+k_2)m_2s^2 + k_2m_1s^2 + k_1k_2} \begin{pmatrix} m_2s^2 + k_2 & k_2 \\ k_2 & m_1s^2 + (k_1+k_2) \end{pmatrix}\begin{pmatrix} m_1sx_1(0) \\ m_2sx_2(0) \end{pmatrix} \tag{9}$$

$$\implies X_1(s) = \frac{m_1m_2s^3x_1(0) + k_2m_1sx_1(0) + m_2k_2sx_2(0)}{m_1m_2s^4 + (k_1+k_2)m_2s^2 + k_2m_1s^2 + k_1k_2} \tag{10}$$

$$\implies X_2(s) = \frac{m_1k_2sx_1(0) + m_1m_2s^3x_2(0) + (k_1+k_2)m_2sx_2(0)}{m_1m_2s^4 + (k_1+k_2)m_2s^2 + k_2m_1s^2 + k_1k_2} \tag{11}$$

Considering denominator of $X_i(s)$ for $i = 1, 2$(ie: characteristic equation of the system):

$$m_1 m_2 s^4 + (k_1 + k_2) m_2 s^2 + k_2 m_1 s^2 + k_1 k_2 = 0 \tag{12}$$

$$s^4 + 4s^2 + 2 = 0 \tag{13}$$

$$\implies s^2 = \frac{-4 \pm \sqrt{16 - 8}}{2} \tag{14}$$

$$= -2 \pm \sqrt{2} \tag{15}$$

The roots of s are imaginary so it's Sinusoid.

$$\omega = \pm \sqrt{2 \mp \sqrt{2}} \tag{16}$$

$$\implies \omega_{least} = 0.765 rad/s \tag{17}$$

METHOD-2:CONVERTING MECHANICAL SYSTEM INTO ITS ANALOGOUS ELECTRICAL CIRCUIT BY FORCE VOLTAGE METHOD

Mechanical Quantity	Electrical Quantity
Force	Voltage
Velocity	Current
Mass	Inductance
Stiffness coefficient of spring	Reciprocal of Capacitance

TABLE 0
INPUT PARAMETERS

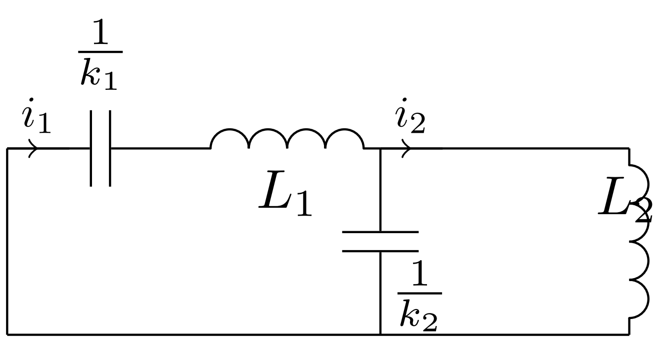

Fig. 0.

$$k_1 \int i_1 \, dt + L_1 \frac{di_1}{dt} + k_2 \int (i_1 - i_2) \, dt = 0 \tag{18}$$

$$L_2 \frac{di_2}{dt} - k_2 \int (i_1 - i_2) \, dt = 0 \tag{19}$$

but we know, $i = \frac{dq}{dt}$

$$\implies L_1 \ddot{q}_1 - k_2 (q_2 - q_1) + k_1 q_1 = 0 \tag{20}$$

$$\implies L_2 \ddot{q}_2 + k_2 (q_2 - q_1) \, dt = 0 \tag{21}$$

Writing (20) and (21) in matrix form:

$$\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{22}$$

$$q(t) \overset{\mathcal{L}}{\longleftrightarrow} Q(s) \tag{23}$$

$$q^{''}(t) \overset{\mathcal{L}}{\longleftrightarrow} s^2 Q(s) - sq(0) - q^{'}(0) \tag{24}$$

Assuming charge to be maximum at t=0;

$$L_1 \left(s^2 Q_1(s) - s q_1(0) \right) - k_2 \left(Q_2(s) - Q_1(s) \right) + k_1 Q_1(s) = 0 \tag{25}$$

$$L_2 \left(s^2 Q_2(s) - s q_2(0) \right) + k_2 \left(Q_2(s) - Q_1(s) \right) = 0 \tag{26}$$

Writing (25) and (26) in matrix form:

$$\begin{pmatrix} L_1 s^2 + (k_1 + k_2) & -k_2 \\ -k_2 & L_2 s^2 + k_2 \end{pmatrix} \begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix} = \begin{pmatrix} L_1 s q_1(0) \\ L_2 s q_2(0) \end{pmatrix} \tag{27}$$

$$\begin{pmatrix} Q_1(s) \\ Q2(s) \end{pmatrix} = \frac{1}{L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2} \begin{pmatrix} L_2 s^2 + k_2 & k_2 \\ k_2 & L_1 s^2 + (k_1 + k_2) \end{pmatrix} \begin{pmatrix} L_1 s q_1(0) \\ L_2 s q_2(0) \end{pmatrix} \tag{28}$$

$$\implies Q_1(s) = \frac{L_1 L_2 s^3 q_1(0) + k_2 L_1 s q_1(0) + k_2 L_2 s q_2(0)}{L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2} \tag{29}$$

$$\implies Q_2(s) = \frac{k_2 L_1 s q_1(0) + L_1 L_2 s^3 q_2(0) + (k_1 + k_2) L_2 s q_2(0)}{L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2} \tag{30}$$

Considering denominator of $Q_i(s)$ for $i = 1, 2$(ie: characteristic equation of the system):

$$L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2 = 0 \tag{31}$$

$$s^4 + 4s^2 + 2 = 0 \tag{32}$$

$$\implies s^2 = \frac{-4 \pm \sqrt{16 - 8}}{2} \tag{33}$$

$$= -2 \pm \sqrt{2} \tag{34}$$

The roots of s are imaginary so it's Sinusoid.

$$\omega = \pm\sqrt{2 \mp \sqrt{2}} \tag{35}$$

$$\implies \omega_{least} = 0.765 rad/s \tag{36}$$