GATE IN-13 2022

EE23BTECH11011- Batchu Ishitha*

Q: A periodic function $f(x)$, with period 2, is defined as

$$f(x) = \begin{cases} -1 - x & -1 \leq x < 0 \\ 1 - x & 0 < x \leq 1 \end{cases} \tag{1}$$

The Fourier series of this function contains

A. Both $cos(n\pi x)$ and $sin(n\pi x)$ where n=1,2,3...
B. Only $sin(n\pi x)$ where n=1,2,3...
C. Only $cos(n\pi x)$ where n=1,2,3...
D. Only $cos(2n\pi x)$ where n=1,2,3... GATE IN 2022

Solution:

The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n cos(n\pi x) + b_n sin(n\pi x)\right] \tag{2}$$

$$a_0 = \int_{-1}^{1} f(x)\, dx \tag{3}$$

$$= \int_{-1}^{0} (-1 - x)\ dx + \int_{0}^{1} (+1 - x)\ dx \tag{4}$$

$$= \left(-x - \frac{x^2}{2}\right)_{-1}^{0} + \left(x - \frac{x^2}{2}\right)_{0}^{1} \tag{5}$$

$$= (0) - \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) - (0) \tag{6}$$

$$\implies a_0 = 0 \tag{7}$$

$$a_n = \int_{-1}^{1} f(x) cos(n\pi x)\, dx \tag{8}$$

$$= \int_{-1}^{0} (-1 - x)\, cos(n\pi x)\, dx + \int_{0}^{1} (+1 - x)\, cos(n\pi x)\, dx \tag{9}$$

$$= -\left(\frac{sin(n\pi x)}{n\pi}\right)_{-1}^{0} - \left(x\frac{sin(n\pi x)}{n\pi} - \int_{-1}^{0} \frac{sin(n\pi x)}{n\pi}\, dx\right) + \left(\frac{sin(n\pi x)}{n\pi}\right)_{0}^{1} - \left(x\frac{sin(n\pi x)}{n\pi} - \int_{0}^{1} \frac{sin(n\pi x)}{n\pi}\, dx\right) \tag{10}$$

$$= 0 - \left(x\frac{sin(n\pi x)}{n\pi} + \frac{cos(n\pi x)}{(n\pi)^2}\right)_{-1}^{0} + 0 - \left(x\frac{sin(n\pi x)}{n\pi} + \frac{cos(n\pi x)}{(n\pi)^2}\right)_{0}^{1} \tag{11}$$

$$= -\left(\frac{1}{(n\pi)^2} - \frac{(-1)^n}{(n\pi)^2}\right) - \left(\frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right) \tag{12}$$

$$\implies a_n = 0 \tag{13}$$

$$
\begin{aligned}
b_n &= \int_{-1}^{1} f(x) sin(n\pi x)\, dx && (14)\\
&= \int_{-1}^{0} (-1-x)\, sin(n\pi x)\, dx + \int_{0}^{1} (+1-x)\, sin(n\pi x)\, dx && (15)\\
&= -\left(\frac{-cos(n\pi x)}{n\pi}\right)_{-1}^{0} - \left(x\frac{-cos(n\pi x)}{n\pi} - \int_{-1}^{0} \frac{-cos(n\pi x)}{n\pi}\, dx\right) + \left(\frac{-cos(n\pi x)}{n\pi}\right)_{0}^{1} - \left(x\frac{-cos(n\pi x)}{n\pi} - \int_{0}^{1} \frac{-cos(n\pi x)}{n\pi}\, d. && (16)\\
&= -\left(\frac{1}{n\pi} - \frac{(-1)^n}{n\pi}\right) - \left(x\frac{-cos(n\pi x)}{n\pi} + \frac{sin(n\pi x)}{(n\pi)^2}\right)_{-1}^{0} + \left(\frac{1}{n\pi} - \frac{(-1)^n}{n\pi}\right) - \left(x\frac{-cos(n\pi x)}{n\pi} + \frac{sin(n\pi x)}{(n\pi)^2}\right)_{0}^{1} && (17)\\
&= 0 - \left(0 - \frac{(-1)^n}{n\pi}\right) - \left(-\frac{(-1)^n}{n\pi} - 0\right) && (18)\\
\implies b_n &= 2\frac{(-1)^n}{n\pi} && (19)
\end{aligned}
$$

$\implies$ Fourier series expansion is $f(x) = \sum_{n=1}^{\infty} 2\frac{(-1)^n}{n\pi} sin(n\pi x)$.
$\therefore$ The Fourier series of this function contains only $sin(n\pi x)$ where n=1,2,3...