

GATE NM-54 2022

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Q: A system with two degrees of freedom, as shown in the figure, has masses $m_1 = 200\text{kg}$ and $m_2 = 100\text{kg}$ and stiffness coefficients $k_1 = k_2 = 200\text{N/m}$. Then the lowest natural frequency of the system is _____ rad/s (rounded off to one decimal place).

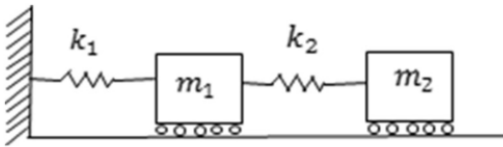


Fig. 0.

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Solution:

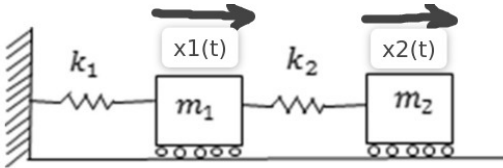


Fig. 0.

METHOD-1:

Variable	Description	Value
m_1	Mass of block 1	200kg
m_2	Mass of block 2	100kg
k_1	Stiffness coefficient of spring1	200N/m
k_2	Stiffness coefficient of spring2	200N/m

TABLE 0
INPUT PARAMETERS

$$m_1 \ddot{x}_1(t) - k_2 (x_2(t) - x_1(t)) + k_1 x_1(t) = 0 \quad (1)$$

$$m_2 \ddot{x}_2(t) + k_2 (x_2(t) - x_1(t)) = 0 \quad (2)$$

Writing (1) and (2) in matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3)$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad (4)$$

$$x''(t) \xleftrightarrow{\mathcal{L}} s^2 X(s) - sx(0) - x'(0) \quad (5)$$

Applying Laplace transform for (1) and (2) assuming they are at their respective maximum displacement at $t=0$

$$m_1 (s^2 X_1(s) - sx_1(0) - 0) - k_2 (X_2(s) - X_1(s)) + k_1 X_1(s) = 0 \quad (6)$$

$$m_2 (s^2 X_2(s) - sx_2(0) - 0) + k_2 (X_2(s) - X_1(s)) = 0 \quad (7)$$

Writing (6) and (7) in matrix form:

$$\begin{pmatrix} m_1 s^2 + (k_1 + k_2) & -k_2 \\ -k_2 & m_2 s^2 + k_2 \end{pmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} m_1 s x_1(0) \\ m_2 s x_2(0) \end{pmatrix} \quad (8)$$

$$\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{1}{m_1 m_2 s^4 + (k_1 + k_2) m_2 s^2 + k_2 m_1 s^2 + k_1 k_2} \begin{pmatrix} m_2 s^2 + k_2 & k_2 \\ k_2 & m_1 s^2 + (k_1 + k_2) \end{pmatrix} \begin{pmatrix} m_1 s x_1(0) \\ m_2 s x_2(0) \end{pmatrix} \quad (9)$$

$$\Rightarrow X_1(s) = \frac{m_1 m_2 s^3 x_1(0) + k_2 m_1 s x_1(0) + m_2 k_2 s x_2(0)}{m_1 m_2 s^4 + (k_1 + k_2) m_2 s^2 + k_2 m_1 s^2 + k_1 k_2} \quad (10)$$

$$\Rightarrow X_2(s) = \frac{m_1 k_2 s x_1(0) + m_1 m_2 s^3 x_2(0) + (k_1 + k_2) m_2 s x_2(0)}{m_1 m_2 s^4 + (k_1 + k_2) m_2 s^2 + k_2 m_1 s^2 + k_1 k_2} \quad (11)$$

Considering denominator of $X_i(s)$ for $i = 1, 2$ (ie: characteristic equation of the system):

$$m_1 m_2 s^4 + (k_1 + k_2) m_2 s^2 + k_2 m_1 s^2 + k_1 k_2 = 0 \quad (12)$$

$$s^4 + 4s^2 + 2 = 0 \quad (13)$$

$$\Rightarrow s^2 = \frac{-4 \pm \sqrt{16-8}}{2} \quad (14)$$

$$= -2 \pm \sqrt{2} \quad (15)$$

The roots of s are imaginary so it's Sinusoid.

$$\omega = \pm \sqrt{2 \mp \sqrt{2}} \quad (16)$$

$$\Rightarrow \omega_{least} = 0.765 \text{ rad/s} \quad (17)$$

METHOD-2: CONVERTING MECHANICAL SYSTEM INTO ITS ANALOGOUS ELECTRICAL CIRCUIT BY FORCE VOLTAGE METHOD

Mechanical Quantity	Electrical Quantity
Force	Voltage
Velocity	Current
Mass	Inductance
Stiffness coefficient of spring	Reciprocal of Capacitance

TABLE 0
INPUT PARAMETERS

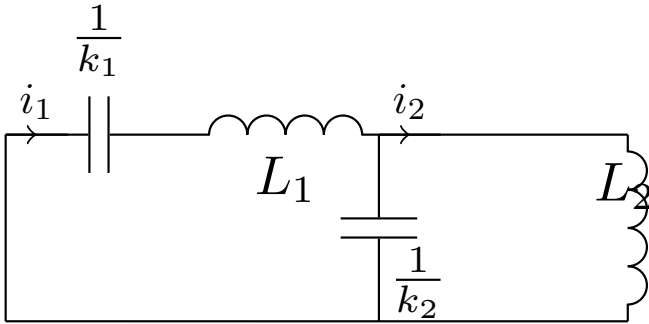


Fig. 0.

$$k_1 \int i_1 dt + L_1 \frac{di_1}{dt} + k_2 \int (i_1 - i_2) dt = 0 \quad (18)$$

$$L_2 \frac{di_2}{dt} - k_2 \int (i_1 - i_2) dt = 0 \quad (19)$$

but we know, $i = \frac{dq}{dt}$

$$\Rightarrow L_1 \ddot{q}_1 - k_2 (q_2 - q_1) + k_1 q_1 = 0 \quad (20)$$

$$\Rightarrow L_2 \ddot{q}_2 + k_2 (q_2 - q_1) = 0 \quad (21)$$

Writing (20) and (21) in matrix form:

$$\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (22)$$

$$q(t) \xleftrightarrow{\mathcal{L}} Q(s) \quad (23)$$

$$q''(t) \xleftrightarrow{\mathcal{L}} s^2 Q(s) - sq(0) - q'(0) \quad (24)$$

Assuming charge to be maximum at $t=0$;

$$L_1 (s^2 Q_1(s) - sq_1(0)) - k_2 (Q_2(s) - Q_1(s)) + k_1 Q_1(s) = 0 \quad (25)$$

$$L_2 (s^2 Q_2(s) - sq_2(0)) + k_2 (Q_2(s) - Q_1(s)) = 0 \quad (26)$$

Writing (25) and (26) in matrix form:

$$\begin{pmatrix} L_1 s^2 + (k_1 + k_2) & -k_2 \\ -k_2 & L_2 s^2 + k_2 \end{pmatrix} \begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix} = \begin{pmatrix} L_1 sq_1(0) \\ L_2 sq_2(0) \end{pmatrix} \quad (27)$$

$$\begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix} = \frac{1}{L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2} \begin{pmatrix} L_2 s^2 + k_2 & k_2 \\ k_2 & L_1 s^2 + (k_1 + k_2) \end{pmatrix} \begin{pmatrix} L_1 sq_1(0) \\ L_2 sq_2(0) \end{pmatrix} \quad (28)$$

$$\Rightarrow Q_1(s) = \frac{L_1 L_2 s^3 q_1(0) + k_2 L_1 sq_1(0) + k_2 L_2 sq_2(0)}{L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2} \quad (29)$$

$$\Rightarrow Q_2(s) = \frac{k_2 L_1 sq_1(0) + L_1 L_2 s^3 q_2(0) + (k_1 + k_2) L_2 sq_2(0)}{L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2} \quad (30)$$

Considering denominator of $Q_i(s)$ for $i = 1, 2$ (ie: characteristic equation of the system):

$$L_1 L_2 s^4 + (k_1 + k_2) L_2 s^2 + k_2 L_1 s^2 + k_1 k_2 = 0 \quad (31)$$

$$s^4 + 4s^2 + 2 = 0 \quad (32)$$

$$\Rightarrow s^2 = \frac{-4 \pm \sqrt{16-8}}{2} \quad (33)$$

$$= -2 \pm \sqrt{2} \quad (34)$$

The roots of s are imaginary so it's Sinusoid.

$$\omega = \pm \sqrt{2 \mp \sqrt{2}} \quad (35)$$

$$\Rightarrow \omega_{least} = 0.765 rad/s \quad (36)$$

DOUBT:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (37)$$

Directly substituting $x'' = -\omega^2 x$

$$\begin{pmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (38)$$

And then substituting $s = j\omega$;

$$\begin{pmatrix} m_1 s^2 + k_1 + k_2 & -k_2 \\ -k_2 & m_2 s^2 + k_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (39)$$

But when i apply laplace transform and then write in a matrix with our assumption ie:

$$\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{1}{m_1 m_2 s^4 + (k_1 + k_2) m_2 s^2 + k_2 m_1 s^2 + k_1 k_2} \begin{pmatrix} m_2 s^2 + k_2 & k_2 \\ k_2 & m_1 s^2 + (k_1 + k_2) \end{pmatrix} \begin{pmatrix} m_1 s x_1(0) \\ m_2 s x_2(0) \end{pmatrix} \quad (40)$$

These both are also not matching...

Is there any error in this..