GATE IN-13 2022

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Q: A periodic function $f(x)$, with period 2, is defined as

$$f(x) = \begin{cases} -1 - x & -1 \le x < 0 \\ 1 - x & 0 < x \le 1 \end{cases} \tag{1}$$

The Fourier series of this function contains

A. Both $\cos(n\pi x)$ and $sin(n\pi x)$ where n=1,2,3...
B. Only $\sin(n\pi x)$ where n=1,2,3...
C. Only $\cos(n\pi x)$ where n=1,2,3...
D. Only $\cos(2n\pi x)$ where n=1,2,3...

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Solution:

Parameter	Description
$f(x)$	Polynomial function
$2L$	Period of the Polynomial function
$c(n)$	Complex Fourier Coefficients
$a(0)$, $a(n)$, $b(n)$	Trigonometric Fourier Coefficients

TABLE 4
INPUT PARAMETERS

The complex exponential Fourier Series of $f(x)$ is,

$$f(x) = \sum_{n=-\infty}^{\infty} c(n) e^{jn\omega x} \tag{2}$$

$$\implies c(n) = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-jn\omega x}\, dx \tag{3}$$

(4)

For $n \neq 0$;

$$\begin{aligned}
c(n) &= \frac{1}{2}\int_{-1}^{1} f(x)e^{-jn\omega x}\,dx && (5)\\
&= \frac{1}{2}\left(\int_{-1}^{0}(-1-x)\,e^{-jn\omega x}\,dx + \int_{0}^{1}(+1-x)\,e^{-jn\omega x}\,dx\right) && (6)\\
&= \frac{1}{2}\left(-\int_{-1}^{0} e^{-jn\omega x}\,dx - \int_{-1}^{1} xe^{-jn\omega x}\,dx + \int_{0}^{1} e^{-jn\omega x}\,dx\right) && (7)\\
&= \frac{1}{2}\left[\frac{-1}{jn\omega}\left[-\left(1-e^{+jn\omega}\right)+\left(e^{-jn\omega}-1\right)\right] - \int_{-1}^{1} xe^{-jn\omega x}\,dx\right] && (8)\\
&= \frac{1}{2}\left[\frac{-1}{jn\omega}\left[-2+e^{+jn\omega}+e^{-jn\omega}\right] + \left(\frac{e^{-jn\omega x}}{jn\omega}\left[x+\frac{1}{jn\omega}\right]\right)_{-1}^{1}\right] && (9)\\
&= \frac{-1}{jn\omega}\left[-1+\cos(n\omega)\right] + \frac{1}{2(jn\omega)^2}\left[\left(e^{-jn\omega}\right)(1+jn\omega) - \left(e^{jn\omega}\right)(-jn\omega+1)\right] && (10)\\
\implies c\,(n) &= \frac{-1}{(jn\omega)^2}\left[-jn\omega + j\sin(n\omega)\right] && (11)
\end{aligned}$$

For $n = 0$;

$$\begin{aligned}
c(0) &= \frac{1}{2}\int_{-1}^{1} f(x)\,dx && (12)\\
&= \frac{1}{2}\left[\int_{-1}^{0}(-1-x)\;dx + \int_{0}^{1}(1-x)\;dx\right] && (13)\\
&= \frac{1}{2}\left[\left(-x-\frac{x^2}{2}\right)_{-1}^{0} + \left(x-\frac{x^2}{2}\right)_{0}^{1}\right] && (14)\\
&= \frac{1}{2}\left[0-1+\frac{1}{2}+1-\frac{1}{2}-0\right] && (15)\\
\implies c(0) &= 0 && (16)
\end{aligned}$$

The trigonometric Fourier Series of $f\,(x)$ is,

$$f\,(x) = a(0) + \sum_{n=1}^{\infty}\left\{a(n)\cos\,(n\omega x) + b(n)\sin\,(n\omega x)\right\} \tag{17}$$

Finding the Fourier Coefficient a_0,

$$\begin{aligned}
a(0) &= c(0) && (18)\\
\implies a(0) &= 0 && (19)
\end{aligned}$$

Finding the Fourier Coefficients $a(n)$,

$$\begin{aligned}
a(n) &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\,(n\omega x)\;\,dx, n \geq 0 && (20)\\
&= \frac{1}{L}\int_{-L}^{L} f(x)\left(e^{-jn\omega x}+e^{jn\omega x}\right)\;dx && (21)\\
\implies a(n) &= c(n) + c(-n) && (22)\\
\implies a(n) &= 0 && (23)\\
& && (24)
\end{aligned}$$

Finding the Fourier Coefficients $b(n)$,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\omega x)\ dx, n \geq 0 \tag{25}$$

$$= \frac{1}{L}\int_{-L}^{L} f(x) j\left(e^{-jn\omega x} - e^{jn\omega x}\right)\ dx \tag{26}$$

$$\implies b(n) = j\left(c_n - c_{-n}\right) \tag{27}$$

$$\implies b(n) = \frac{-2}{(n\omega)^2}\left[-n\omega + \sin(n\omega)\right] \tag{28}$$

$\implies$ The trigonometric Fourier Series of $f(x)$ is,

$$f(x) = \sum_{n=1}^{\infty}\{0 + 0 + b(n)\sin(n\omega x)\} \tag{29}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{-2}{(n\omega)^2}\left[-n\omega + \sin(n\omega)\right]\sin(n\omega x)\right\} \tag{30}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{-2}{(n\pi)^2}\left[-n\pi + \sin(n\pi)\right]\sin(n\pi x)\right\} \tag{31}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}\sin(n\pi x)\right\} \tag{32}$$

$\therefore$ The Fourier series of this function contains only $\sin(n\pi x)$ where n=1,2,3...

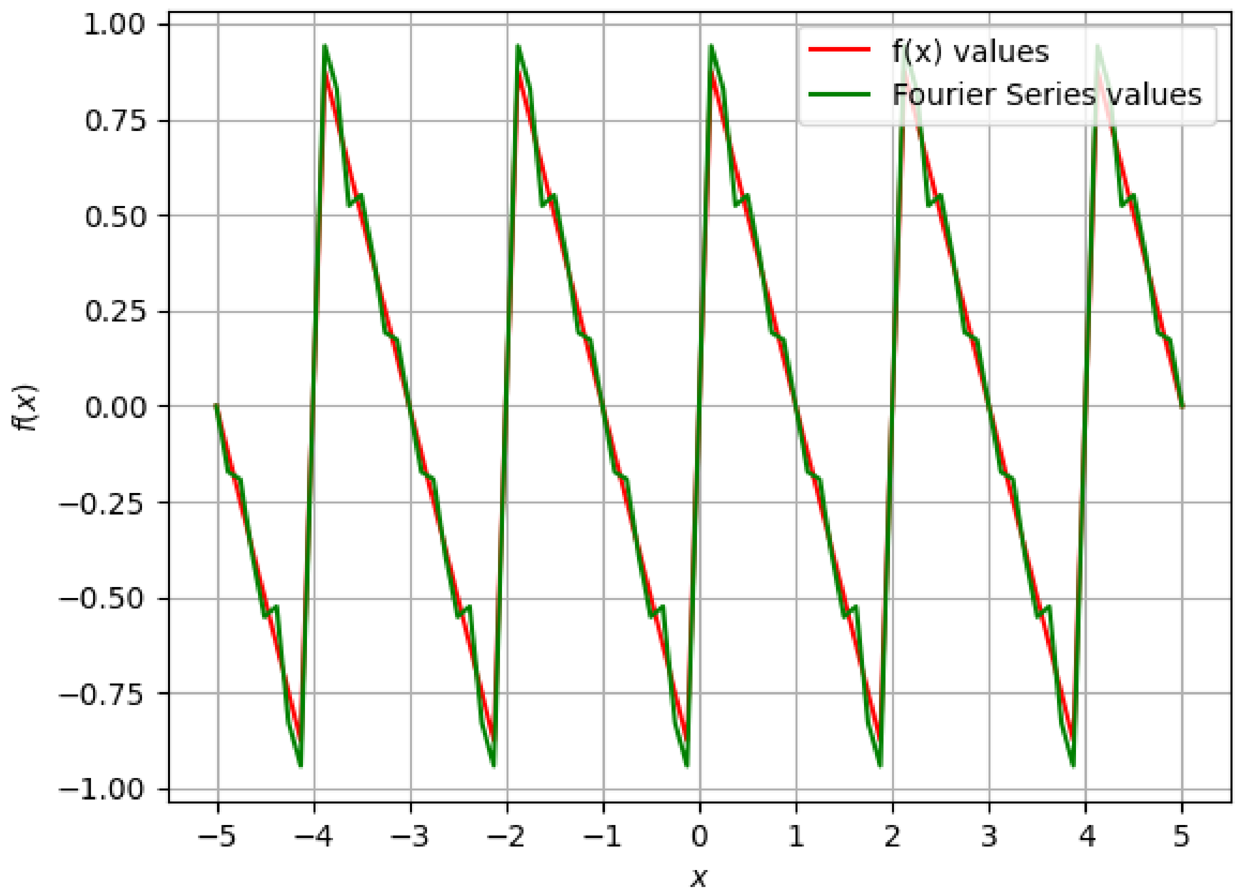
f(x) values
Fourier Series values
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
$f(x)$
−5
−4
−3
−2
−1
0
1
2
3
4
5
x

Fig. 4.