GATE NM-54 2022

EE23BTECH11011- Batchu Ishitha*

Q: A system with two degrees of freedom, as shown in the figure, has masses $m_1 = 200kg$ and $m_2 = 100kg$ and stiffness coefficients $k_1 = k_2 = 200N/m$. Then the lowest natural frequency of the system is ___________ rad/s (rounded off to one decimal place).

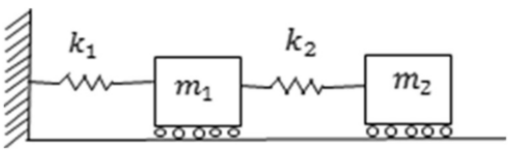

Fig. 0.

GATE NM 2022

Solution:

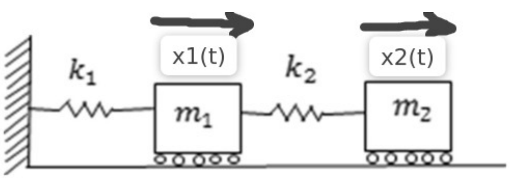

Fig. 0.

Variable	Description	Value
m_1	Mass of block 1	$200kg$
m_2	Mass of block 2	$100kg$
k_1	Stiffness coefficient of spring1	$200N/m$
k_2	Stiffness coefficient of spring2	$200N/m$
$x_i(t)$	Displacement of i^{th} block	$x_i(t) = \cos(\omega t + \phi_i), x_i \geq 0$

TABLE 0
INPUT PARAMETERS

$$m_1\ddot{x}_1(t) - k_2\left(x_2(t) - x_1(t)\right) + k_1x_1(t) = 0 \tag{1}$$

$$m_2\ddot{x}_2(t) + k_2\left(x_2(t) - x_1(t)\right) = 0 \tag{2}$$

Writing (1) and (2) in matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \tag{3}$$

$$x''(t) \overset{\mathcal{L}}{\longleftrightarrow} s^2X(s) - sx(0) - x'(0) \tag{4}$$

Applying Laplace transform for (1) and (2) assuming they are at their repective maximum displacement at t=0

$$m_1\left(s^2X_1(s) - sx_1(0) - 0\right) - k_2\left(X_2(s) - X_1(s)\right) + k_1X_1(s) = 0 \tag{5}$$

$$m_2\left(s^2X_2(s) - sx_2(0) - 0\right) + k_2\left(X_2(s) - X_1(s)\right) = 0 \tag{6}$$

Writing (5) and (6) in matrix form:

$$\begin{bmatrix} m_1s^2 + (k_1 + k_2) & -k_2 \\ -k_2 & m_2s^2 + k_2 \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} m_1sx_1(0) \\ m_2sx_2(0) \end{bmatrix}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} m_1sx_1(0) \\ m_2sx_2(0) \end{bmatrix}\begin{bmatrix} m_2s^2 + k_2 & k_2 \\ k_2 & m_1s^2 + (k_1 + k_2) \end{bmatrix}$$

Finding inverse Laplace transform for this on both sides;

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} m_1x_1(0)u(t) \\ m_2x_2(0)u(t) \end{bmatrix}\begin{bmatrix} m_2tu(t) + k_2\delta(t) & k_2\delta(t) \\ k_2\delta(t) & m_1tu(t) + (k_1 + k_2)\delta(t) \end{bmatrix}$$

For frequency put $s = j\omega$,

$$\begin{bmatrix} -m_1\omega^2 + (k_1 + k_2) & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{bmatrix}\begin{bmatrix} X_1(j\omega) \\ X_2(j\omega) \end{bmatrix} = \begin{bmatrix} m_1j\omega x_1(0) \\ m_2j\omega x_2(0) \end{bmatrix}$$

For free vibration

$$\det\begin{pmatrix} -m_1\omega^2 + (k_1 + k_2) & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{pmatrix} = 0$$

$$\left(-m_1\omega^2 + (k_1 + k_2)\right)\left(-m_2\omega^2 + k_2\right) - (-k_2)(-k_2) = 0 \tag{7}$$

$$m_1 m_2 \omega^4 - \left[(k_1 + k_2) m_2 + k_2 m_1\right] \omega^2 + k_1 k_2 = 0 \tag{8}$$

$$\implies \omega^4 - \frac{\left[(k_1 + k_2) m_2 + k_2 m_1\right]}{m_1 m_2}\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \tag{9}$$

Substituting the values;

$$\implies \omega^4 - 4\omega^2 + 2 = 0 \tag{10}$$

$$\omega^2 = \frac{4 \pm \sqrt{16 - 8}}{2} \tag{11}$$

$$= 2 \pm \sqrt{2} \tag{12}$$

$$\omega = \pm\sqrt{2 \pm \sqrt{2}} \tag{13}$$

$$\implies \omega_{least} = 0.765 rad/s \tag{14}$$