GATE IN-13 2022

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Q: A periodic function $f(x)$, with period 2, is defined as

$$f(x) = \begin{cases} -1-x & -1 \leq x < 0 \\ 1-x & 0 < x \leq 1 \end{cases} \tag{1}$$

The Fourier series of this function contains

A. Both $\cos(n\pi x)$ and $sin(n\pi x)$ where n=1,2,3...
B. Only $\sin(n\pi x)$ where n=1,2,3...
C. Only $\cos(n\pi x)$ where n=1,2,3...
D. Only $\cos(2n\pi x)$ where n=1,2,3...

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Solution:

Parameter	Description
$f(x)$	Polynomial function
$2T$	Period of the Polynomial function
c_n	Complex Fourier Coefficients
a_0, a_n, b_n	Trigonometric Fourier Coefficients

TABLE 4
INPUT PARAMETERS

The complex exponential Fourier Series of $f(x)$ is,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 x} \tag{2}$$

$$\implies c_n = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{-T}{2}} f(x)\, e^{-jn\omega_0 x}\, dx \tag{3}$$

$$\tag{4}$$

For $n \neq 0$;

$$\begin{aligned}
c_n &= \frac{1}{2}\int_{-1}^{1} f(x)e^{-jn\omega_0 x}\,dx && (5)\\
&= \frac{1}{2}\left(\int_{-1}^{0}(-1-x)\,e^{-jn\omega_0 x}\,dx + \int_{0}^{1}(+1-x)\,e^{-jn\omega_0 x}\,dx\right) && (6)\\
&= \frac{1}{2}\left(-\int_{-1}^{0} e^{-jn\omega_0 x}\,dx - \int_{-1}^{1} xe^{-jn\omega_0 x}\,dx + \int_{0}^{1} e^{-jn\omega_0 x}\,dx\right) && (7)\\
&= \frac{1}{2}\left[\frac{-1}{jn\omega_0}\left[-\left(1-e^{+jn\omega_0}\right)+\left(e^{-jn\omega_0}-1\right)\right] - \int_{-1}^{1} xe^{-jn\omega_0 x}\,dx\right] && (8)\\
&= \frac{1}{2}\left[\frac{-1}{jn\omega_0}\left[-2+e^{+jn\omega_0}+e^{-jn\omega_0}\right] + \left(\frac{e^{-jn\omega_0 x}}{jn\omega_0}\left[x+\frac{1}{jn\omega_0}\right]\right)_{-1}^{1}\right] && (9)\\
&= \frac{-1}{jn\omega_0}\left[-1+\cos(n\omega_0)\right] + \frac{1}{2(jn\omega_0)^2}\left[\left(e^{-jn\omega_0}\right)(1+jn\omega_0) - \left(e^{jn\omega_0}\right)(-jn\omega_0+1)\right] && (10)\\
\implies c_n &= \frac{-1}{(jn\omega_0)^2}\left[-jn\omega_0 + j\sin(n\omega_0)\right] && (11)
\end{aligned}$$

For $n = 0$;

$$\begin{aligned}
c_0 &= \frac{1}{2}\int_{-1}^{1} f(x)\,dx && (12)\\
&= \frac{1}{2}\left[\int_{-1}^{0}(-1-x)\ dx + \int_{0}^{1}(1-x)\ dx\right] && (13)\\
&= \frac{1}{2}\left[\left(-x-\frac{x^2}{2}\right)_{-1}^{0} + \left(x-\frac{x^2}{2}\right)_{0}^{1}\right] && (14)\\
&= \frac{1}{2}\left[0-1+\frac{1}{2}+1-\frac{1}{2}-0\right] && (15)\\
&= 0 && (16)
\end{aligned}$$

The trigonometric Fourier Series of $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left\{a_n\cos(n\omega_0 x) + b_n\sin(n\omega_0 x)\right\} \tag{17}$$

Finding the Fourier Coefficient a_0,

$$\begin{aligned}
a_0 &= c_0 && (18)\\
\implies a_0 &= 0 && (19)
\end{aligned}$$

Finding the Fourier Coefficients a_n,

$$\begin{aligned}
a_n &= \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(x)\cos(n\omega_0 x)\ dx, n \geq 0 && (20)\\
&= \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(x)\left(e^{-jn\omega_0 x}+e^{jn\omega_0 x}\right)\ dx && (21)\\
\implies a_n &= c_n + c_{-n} && (22)\\
\implies a_n &= 0 && (23)\\
& && (24)
\end{aligned}$$

Finding the Fourier Coefficients b_n,

$$b_n = \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(x)\sin\left(n\omega_0 x\right)\, dx, n \geq 0 \tag{25}$$

$$= \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(x) j\left(e^{-jn\omega_0 x} - e^{jn\omega_0 x}\right)\, dx \tag{26}$$

$$\implies b_n = j\left(c_n - c_{-n}\right) \tag{27}$$

$$\implies b_n = \frac{-2}{(n\omega_0)^2}\left[-n\omega_0 + \sin(n\omega_0)\right] \tag{28}$$

$\therefore$ The Fourier series of this function contains only $\sin(n\pi x)$ where n=1,2,3...

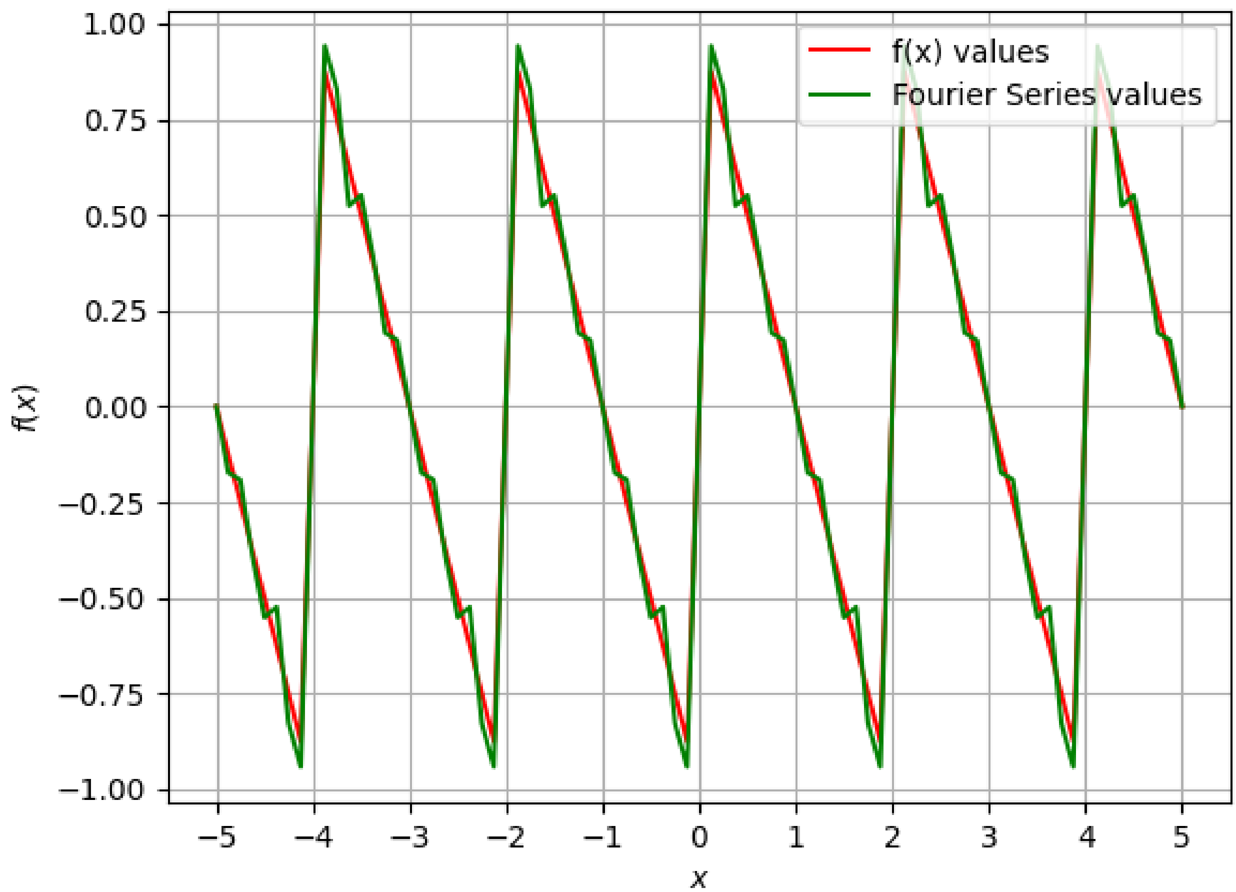

Fig. 4.