1. **Mathematical logic** is the study of formal logic within mathematics. Major subareas include model theory, proof theory, set theory, and recursion theory (also known as computability theory). Research in mathematical logic commonly addresses the mathematical properties of formal systems of logic such as their expressive or deductive power. However, it can also include uses of logic to characterize correct mathematical reasoning or to establish foundations of mathematics.

2. Since its inception, mathematical logic has both contributed to and been motivated by the study of foundations of mathematics. This study began in the late 19th century with the development of axiomatic frameworks for geometry, arithmetic, and analysis. In the early 20th century it was shaped by David Hilbert's program to prove the consistency of foundational theories. Results of Kurt Gödel, Gerhard Gentzen, and others provided partial resolution to the program, and clarified the issues involved in proving consistency. Work in set theory showed that almost all ordinary mathematics can be formalized in terms of sets, although there are some theorems that cannot be proven in common axiom systems for set theory. Contemporary work in the foundations of mathematics often focuses on establishing which parts of mathematics can be formalized in particular formal systems (as in reverse mathematics) rather than trying to find theories in which all of mathematics can be developed.

Subfields and scope

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The *Handbook of Mathematical Logic*[1] in 1977 makes a rough division of contemporary mathematical logic into four areas:

1. set theory
2. model theory
3. recursion theory, and
4. proof theory and constructive mathematics (considered as parts of a single area).

Additionally, sometimes the field of computational complexity theory is also included as part of mathematical logic.[2] Each area has a distinct focus, although many techniques and results are shared among multiple areas. The borderlines amongst these fields, and the lines separating mathematical logic and other fields of mathematics, are not always sharp. Gödel's incompleteness theorem marks not only a milestone in recursion theory and proof theory, but has also led to Löb's theorem in modal logic. The method of forcing is employed in set theory, model theory, and recursion theory, as well as in the study of intuitionistic mathematics.

The mathematical field of category theory uses many formal axiomatic methods, and includes the study of categorical logic, but category theory is not ordinarily considered a subfield of mathematical logic. Because of its applicability in diverse fields of mathematics, mathematicians including Saunders Mac Lane have proposed category theory as a foundational system for mathematics, independent of set theory. These foundations use toposes, which resemble generalized models of set theory that may employ classical or nonclassical logic.

History

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Mathematical logic emerged in the mid-19th century as a subfield of mathematics, reflecting the confluence of two traditions: formal philosophical logic and mathematics.[3] Mathematical logic, also called 'logistic', 'symbolic logic', the 'algebra of logic', and, more recently, simply 'formal logic', is the set of logical theories elaborated in the course of the nineteenth century with the aid of an artificial notation and a rigorously deductive method.[4] Before this emergence, logic was studied with rhetoric, with *calculationes*,[5] through the syllogism, and with philosophy. The first half of the 20th century saw an explosion of fundamental results, accompanied by vigorous debate over the foundations of mathematics.

Early history

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Further information: History of logic

Theories of logic were developed in many cultures in history, including China, India, Greece and the Islamic world. Greek methods, particularly Aristotelian logic (or term logic) as found in the *Organon*, found wide application and acceptance in Western science and mathematics for millennia.[6] The Stoics, especially Chrysippus, began the development of predicate logic. In 18th-century Europe, attempts to treat the operations of formal logic in a symbolic or algebraic way had been made by philosophical mathematicians including Leibniz and Lambert, but their labors remained isolated and little known.

19th century

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In the middle of the nineteenth century, George Boole and then Augustus De Morgan presented systematic mathematical treatments of logic. Their work, building on work by algebraists such as George Peacock, extended the traditional Aristotelian doctrine of logic into a sufficient framework for the study of foundations of mathematics.[7] In 1847, Vatroslav Bertić made substantial work on algebraization of logic, independently from Boole.[8] Charles Sanders Peirce later built upon the work of Boole to develop a logical system for relations and quantifiers, which he published in several papers from 1870 to 1885.

Gottlob Frege presented an independent development of logic with quantifiers in his *Begriffsschrift*, published in 1879, a work generally considered as marking a turning point in the history of logic. Frege's work remained obscure, however, until Bertrand Russell began to promote it near the turn of the century. The two-dimensional notation Frege developed was never widely adopted and is unused in contemporary texts.

From 1890 to 1905, Ernst Schröder published *Vorlesungen über die Algebra der Logik* in three volumes. This work summarized and extended the work of Boole, De Morgan, and Peirce, and was a comprehensive reference to symbolic logic as it was understood at the end of the 19th century.

Foundational theories

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Concerns that mathematics had not been built on a proper foundation led to the development of axiomatic systems for fundamental areas of mathematics such as arithmetic, analysis, and geometry.

In logic, the term *arithmetic* refers to the theory of the natural numbers. Giuseppe Peano[9] published a set of axioms for arithmetic that came to bear his name (Peano axioms), using a variation of the logical system of Boole and Schröder but adding quantifiers. Peano was unaware of Frege's work at the time. Around the same time Richard Dedekind showed that the natural numbers are uniquely characterized by their induction properties. Dedekind proposed a different characterization, which lacked the formal logical character of Peano's axioms.[10] Dedekind's work, however, proved theorems inaccessible in Peano's system, including the uniqueness of the set of natural numbers (up to isomorphism) and the recursive definitions of addition and multiplication from the successor function and mathematical induction.

The **Semantic Table Method** in a **LogicMath** course is a systematic technique for determining the validity, satisfiability, or contradiction of logical statements. It involves creating a table of all possible truth assignments for atomic propositions and evaluating the truth values of complex formulas step by step. This method helps identify whether a formula is:

1. **Valid**: True in all cases.
2. **Satisfiable**: True in at least one case.
3. **Contradictory**: False in all cases.

Example: For (A and (B or not C)), truth values are calculated row by row, revealing whether the formula holds under specific conditions.