Matrix properties and algebra

1. **Definitions and properties.** Know what is means and implies if a matrix is: (a) square, (b) triangular, (c) diagonal, (d) symmetric, (e) stochastic, (f) singular, (g) invertible, (h) defective, (i) diagonalizable, (j) similar to another matrix.

2. **Methods.** Given a matrix, know how to find (a) the determinant, (b) the trace, (c) the transpose, (d) the matrix of cofactors, (e) the inverse, (f) the eigenvalues, (g) the eigenvectors, (h) the diagonalization.

Transformations

1. **Definitions and properties.** Know what it means and implies if a transformation is (a) a linear transformation, (b) a matrix transformation.

2. **Methods.** Know how to find (a) images and preimages of vectors under a given transformation, (b) the domain and codomain of a transformation, (c) the standard matrix of a matrix transformation.

3. **Transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.** Know how to build a matrix transformation for a (a) dilation/contraction, (b) reflection, (c) rotation, (d) composition.

Applications

1. **Markov Chains.** Know how to build a Markov chain model as a system of recurrence relations in matrix-vector form. Know how to find the long-term steady state solution.

2. **Leslie models.** Know how to build a Leslie model as a system of recurrence relations in matrix-vector form. Know how to find the long-term outcome of the system.

3. **Digraphs.** Know how to find the adjacency matrix and distance matrix of a digraph.

Exercises

1. Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ 2x+3y \\ 1 \end{bmatrix}$.

   (a) Find the domain and codomain of T.

   (b) Find the image of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under T.

   (c) Show that T is not a linear transformation.

2. Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 2x-y \\ x+y \end{bmatrix}$.

   (a) Find the domain and codomain of T.

   (b) Find the standard matrix of T.

   (c) Find the image of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under T.

   (d) Find a preimage of $\begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$ under T.

3. Find the standard matrix of a transformation that maps a 2D vector to its image by doing a reflection across the x-axis, followed by a counterclockwise rotation by 30 degrees, followed by a contraction by a factor of 0.5.

    - Find the image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ using the standard matrix.

4. Find the equation of the curve obtained by rotating the parabola $y = x^2 - 1$ around the origin counterclockwise by $60°$.

5. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$. Find $\det A$.

6. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 1 & 0 & -2 \end{bmatrix}$. Find A^{-1}.

7. Let $A = \begin{bmatrix} -4 & 0 & 6 \\ -15 & 2 & 15 \\ -3 & 0 & 5 \end{bmatrix}$.

    (a) Find the eigenspace for $\lambda = 2$.

    (b) Find a basis for the eigenspace for $\lambda = 2$.

    (c) Find the dimension of the eigenspace for $\lambda = 2$.

    (d) Find the eigenspace for $\lambda = -1$.

    (e) Find a basis for the eigenspace for $\lambda = -1$.

    (f) Find the dimension of the eigenspace for $\lambda = -1$.

    (g) Diagonalize A (if possible).