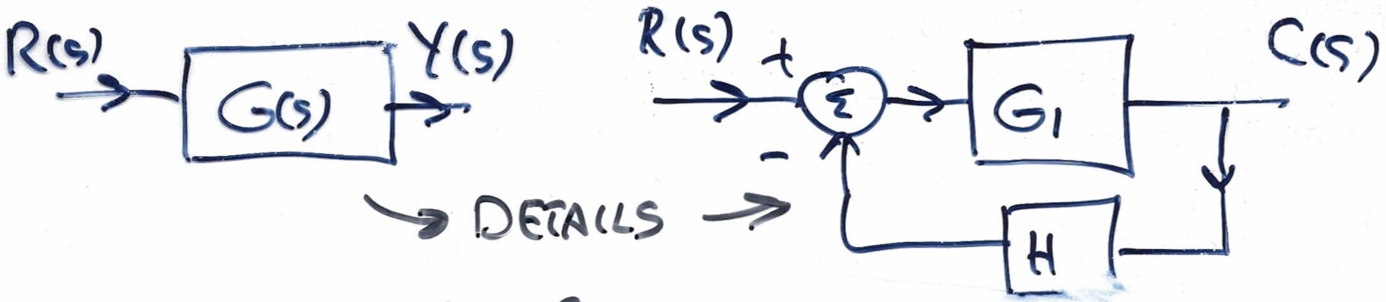
revised 14 July 2016

TIME RESPONSE OF A SECOND ORDER SYSTEM



CLOSED LOOP XFER FUNCTION

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ZETA

LOWER CASE OMEGA

- "prototype" SECOND ORDER SYSTEM
- CONSIDER STABLE SYSTEMS IN THIS DISCUSSION

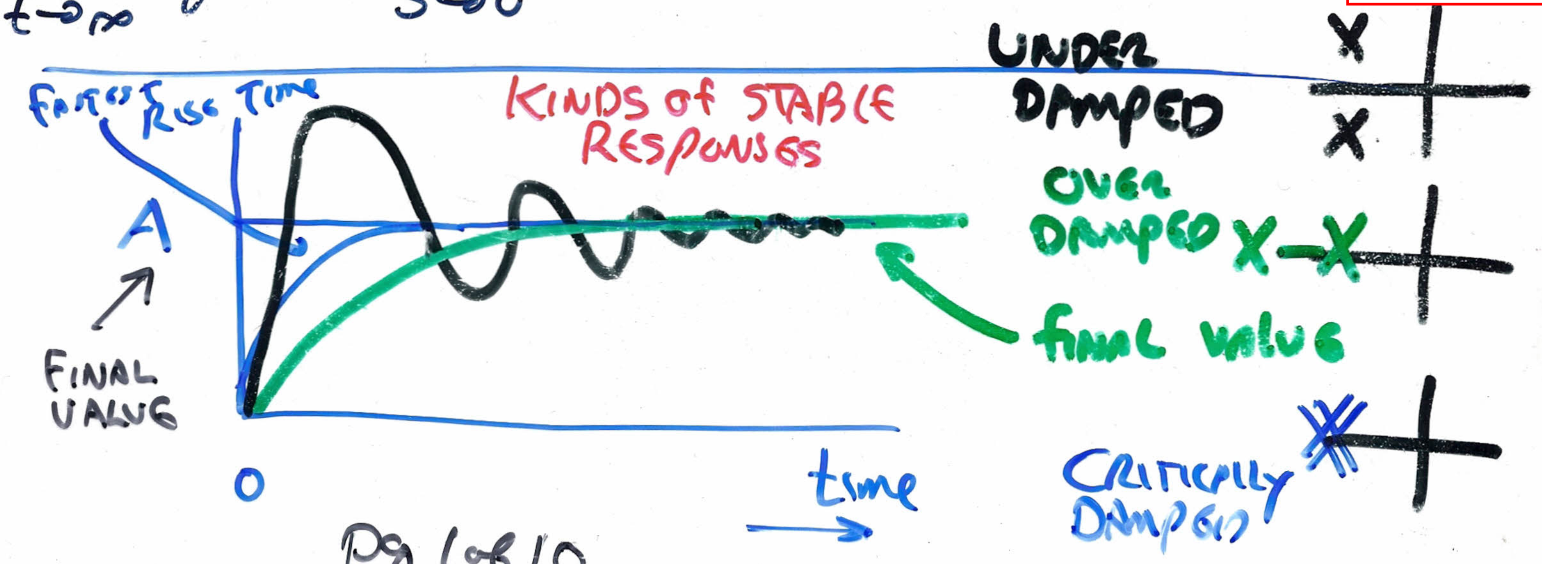
IF $R(s) = \frac{A}{s}$ STEP FUNCTION

what is $y(\infty)$ = STEADY STATE VALUE

$$y(\infty) = \lim_{t\to\infty} y(t)$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,\frac{A}{s}\,\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = A$$

if numerator is K (wn)^2 then steady state value is K*A
K can be any value



Given this background we would like to find / have relationships of 2nd order parameters $\{\zeta, \omega_n\}$ to time response performance.

Stated another way:
if we measure the overshoot, time of overshoot and settling time of some system we can find the parameters {zeta and omega_n} of the system. Note system could be 2nd order or larger with dominant poles that exhibit 2nd order response
The time domain response of a 2nd order system to a step has been studied extensively - see text for derivations

$$C(s) = \left[\frac{1}{s}\right] \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\Rightarrow \boxed{c(t) = 1 - e^{-\frac{\zeta\omega_n t}{\sqrt{1-\zeta^2}}} \sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t + \cos^{-1}\zeta\right)}$$

For underdamped case we will relate $\{\zeta, \omega_n\}$ to

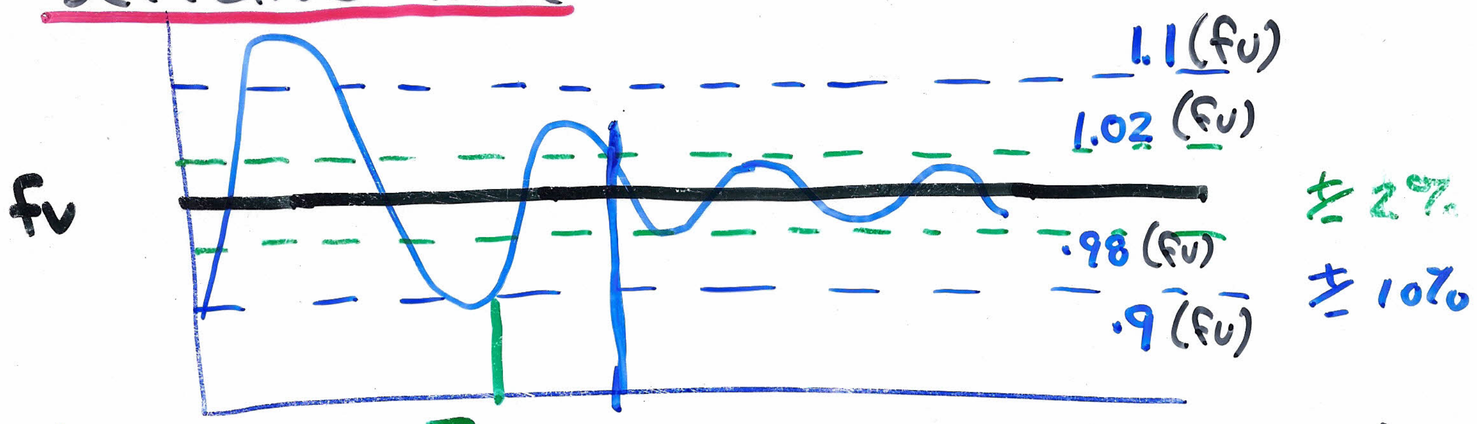
- maximum excursion (overshoot) c_p PEAK VALUE
- Settling time T_s
- time of peak value t_p
- 10 % to 90 % rise time t_r
- frequency of oscillation ω_d damped frequency of oscillation

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} \exp(-\zeta\omega_n t) \sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t + \cos^{-1}\zeta\right)$$

Compare to eq 4.28 of text slightly different form sin() used here vs cos(t) pg 173 - just a phase reference

SETTLING TIME

showing +/-2% and +/-10% band



o ↓

Ts — For 10% settling criterion

Ts — For 2% settling time criterion

Choose a performance level (settling band about final value)

$$\boxed{T_s = \frac{\#\text{ of TC}}{\zeta \omega_n}}$$

Settling time ::= Ts
Time constant of exponential ::= T.C.

This table comes from D'Azzo and Houpis and allows for different settling bands

# TC	% Error
1	36.8
2	13.5
3	5
NISE → 4	1.8
5	0.7

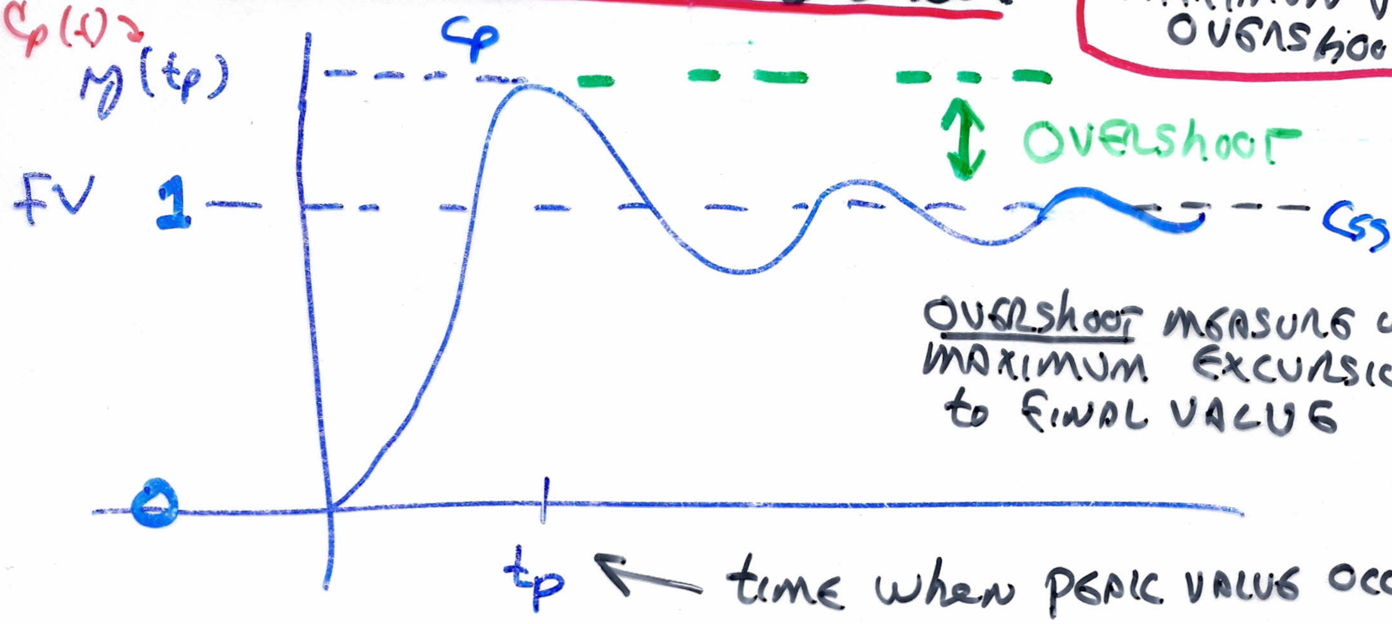
Criterion is to define Ts as the point at which the response falls within the settling band and STAYS within the band

TC is when argument is -1 $TC = \frac{1}{\zeta \omega_n}$

$$\boxed{3.59} \quad c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1} \zeta\right)$$

PEAK VALUE / OVERSHOOT

MAXIMUM VALUE
OVERSHOOT



<u>OVERSHOOT</u> MEASURE of MAXIMUM EXCURSION (PEAK) to FINAL VALUE

$t_p \leftarrow$ time when PEAK VALUE OCCURS

STEP RESPONSE of 2nd order SYSTEM $G(s) = \frac{Y(s)}{R(s)}$
UNDERDAMPED CASE

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$ time of 1st PEAK

$$y(t)\big|_{t=t_p} = \text{PEAK VALUE} = 1 + e^{-\zeta \omega_n t_p}$$

$$\text{ACTUAL OVERSHOOT} = e^{-\zeta \omega_n t_p} = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$ ONLY ζ

$$\%OS = 100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

$$\%OS = \left[\frac{y(t_p) - y_{ss}}{y_{ss}}\right] \times 100 = \left[\frac{c_p - c_{ss}}{c_{ss}}\right] \times 100$$

TREND as $\zeta \downarrow$ O.S. $\uparrow$

$\zeta \uparrow$ O.S. $\downarrow$

$\zeta = 1$ C.D $\leftrightarrow$ CRITICALLY DAMPED

$\underline{0} < \zeta \leq 1$

GRAPHS OF ξ, AND O.S.

the time response with damping ratio
and peak overshoot vs. damping ratio

this validates trend $\xi \downarrow$ % OS $\uparrow$
$\xi \uparrow$ % OS $\downarrow$

NOTE 40% OS $\Rightarrow \xi \approx 0.3$ LOOK AT CURVE

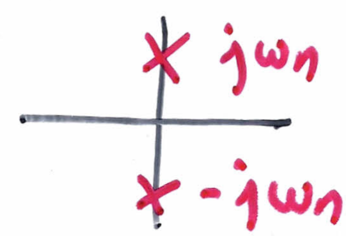
OSCILLATION FREQUENCY

ω_n defined as the undamped natural frequency

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$ CHARACTERISTIC EQ

If $\xi = 0$ there is no damping $\Rightarrow s^2 + \omega_n^2 = 0$

$$\Rightarrow s = \pm j\omega_n$$



From the original response equation

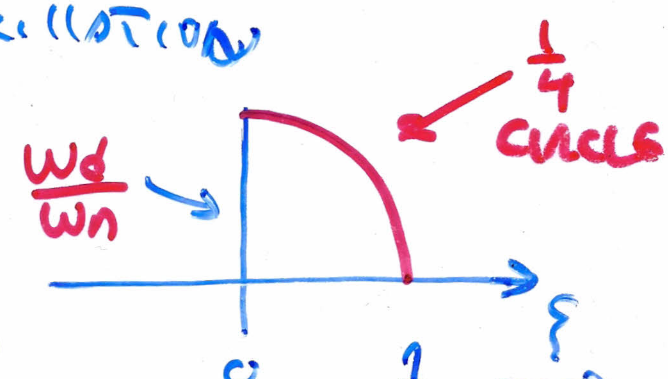
$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\left(\omega_n\sqrt{1-\xi^2}\right)t + \cos^{-1}\xi\right)$$

the frequency of oscillation $\omega_n\sqrt{1-\xi^2} = \omega_d$

ω_d = damped freq of oscillation

$$\boxed{\frac{\omega_d}{\omega_n} = \sqrt{1-\xi^2}}$$

RATIO



see graphical relationship Nise ed7 fig 4.16

RISE TIME, t_r

time from 10% of steady state value to 90% of steady state value

SEE BELOW

$0.1 = c(t_{10})$ $\quad$ $0.9 = c(t_{90})$

solve for t_{10}, t_{90} $\quad$ $t_r = t_{90} - t_{10}$

of course this assumes you have ζ, ω_n to plug into Eq $c(t)$ and $c(t)$'s final value $= 1$

YOU MAY WANT TO DO THIS NUMERICALLY

NOTE IF NECESSARY YOU CAN ADJUST $c(t)$ to compensate for a final value $\neq 1$

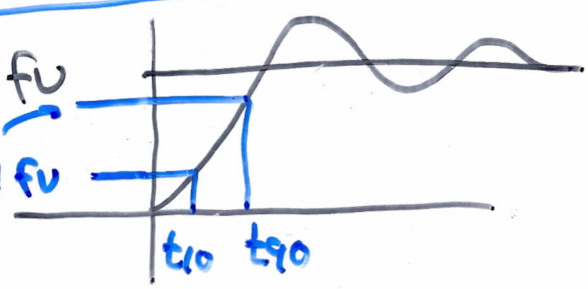
$$c(t) = \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \cos^{-1}\zeta)\right] A$$

A = FINAL VALUE S.S.

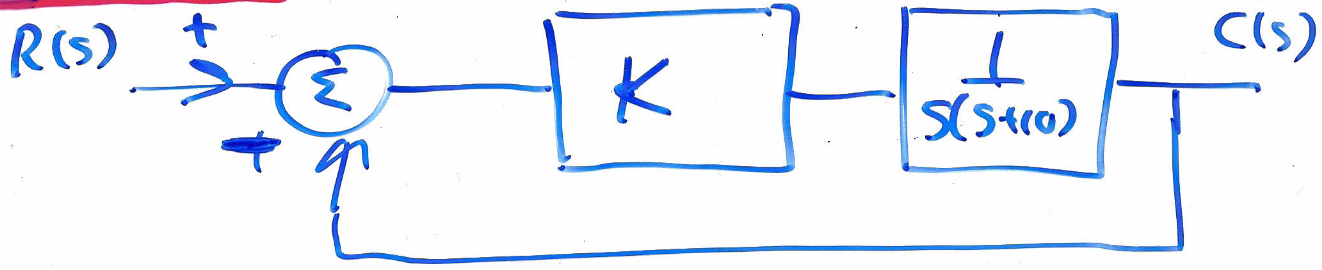
GRAPHICALLY FIND T_r

Now let's apply these concepts to some examples!

10-90% DEFINITION



Problem/Example 1



(1)

Find K to yield a step response with
$\zeta = 0.5$ OR A 16.5% OS

PROTOTYPE

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1 + \frac{K}{s(s+10)} \cdot 1} = \frac{K}{s^2 + 10s + K} \quad \boxed{\text{EQUIV}} \quad \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

HERE WE HAVE A 1:1 RELATIONSHIP

$$\boxed{\begin{aligned} 2\zeta\omega_n &= 10 \\ K &= \omega_n^2 \end{aligned}}$$

FROM EQS

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

.363 Sec

$2(.5)\,\omega_n = 10$ PLUG IN $\zeta = 0.5$

$2(.5)\sqrt{K} = 10$

$\sqrt{K} = 10 \rightarrow K = 100$

What ARE TIME SPECIFICATIONS from SYSTEM

defined by $\zeta = .5$

$$?\ \%\ OS = 100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) = 100(.163)$$

16.3 % OS

USE 4 TC FOR < 2% SETTLING (1.8%)

$$T_s = \frac{\#\,TC}{\zeta\omega_n} \qquad \frac{4}{(0.5)10} = \frac{4}{5}\ \text{SEC} = 800\ \text{ms}$$

< 2% SETTLING AT 0.8 SEC

EXAMPLE 2 SAME SYSTEM NEW SPECS

FIND K SO THAT

- T_S MEETS < 5% SETTLING CRITERIA IN LESS THAN 3 SEC
- % OS < 5%

$2\zeta\omega_n = 10 \quad \omega_n^2 = K$ FROM SYSTEM

$$\boxed{\begin{array}{ll} T_S = \dfrac{3}{\zeta\omega_n} \leftarrow \text{for } < 5\% & T_S < 3 \text{ SEC} \\ 5\% < 100 \exp\left(-\dfrac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)\% & \end{array}}$$

RELATIONSHIPS WITH ζ'S

LET $T_S = 2$ SEC $\therefore 2 = \dfrac{3}{\zeta\omega_N} \Rightarrow \underline{\zeta\omega_N = 1.5}$
GOOD CHOICE

CONFLICT

[but from $2\zeta\omega_N = 10$ we know $\underline{\zeta\omega_N = 5}$
IN OUR SYSTEM

USING $\zeta\omega_N = 5$ T_S would be $\dfrac{3}{5} = 0.6$
which <u>MEETS OUR SPEC</u> SO
WE WILL LET $\underline{\zeta\omega_N = 5}$

AND SAY $T_S = 0.6$ SEC within 5% SETTLING

Now for the <u>OVERSHOOT SPECIFICATION</u>

LET US PICK %OS = 4 (4C5)

$$0.04 = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

FUNCTION ONLY of ζ

$$-3.2188 = \frac{-\zeta\pi}{\sqrt{1-\zeta^2}}$$

$$\left(\sqrt{1-\zeta^2}\right)(3.2188) = \zeta\pi$$

$$(1-\zeta^2)\ 1.0498 = \zeta^2 \Rightarrow 1.0498 = 2.0498\,\zeta^2$$

$$\zeta^2 = 0.512$$

$$\underline{\zeta = 0.7156}$$

NOW SINCE $\zeta\omega_n = 5$

$$.7156\ \omega_n = 5$$

$$\omega_n = 6.986$$

$$\omega_n^2 = K \quad \text{SO} \quad K = 48.8$$

FOR TIME RESPONSE

$$e^{-\frac{\zeta\omega_n t}{\sqrt{1-\zeta^2}}} = e^{\frac{-5t}{.6985}} = e^{-7.1582t}$$

$$\omega_n\left(\sqrt{1-\zeta^2}\right) = 6.986(.6985) = \omega_D$$

FINAL SPECS OF DESIGN

$T_S = 0.6$ SEC WITHIN 5% SETTLING

% OS = 4

MEETS SYSTEM SPEC

$$c(t) = 1 - e^{-7.1582t}\sin(4.88t + 44.3^\circ)$$

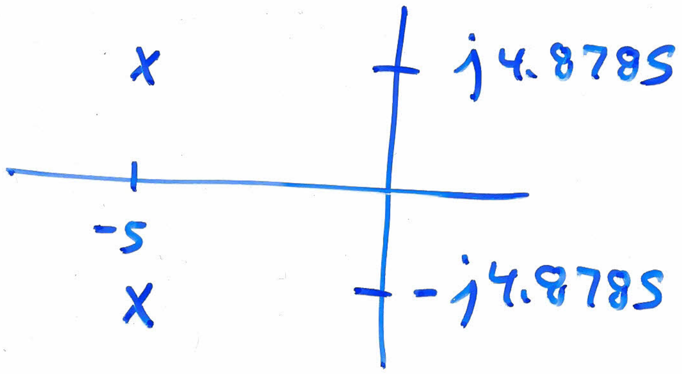
NOTE $\omega_D = 4.88$

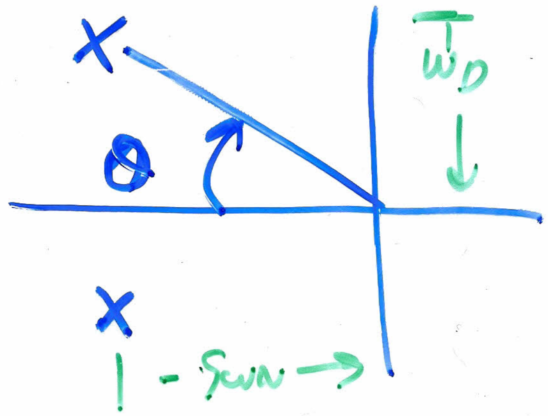
$\omega_n = 6.986$

$$\frac{C(s)}{R(s)} = \frac{48.8}{s^2 + 10s + 48.8}$$

FINAL SYSTEM

Roots $\dfrac{-10 \pm \sqrt{100 - 195.2}}{2}$ $\quad -5 \pm j4.8785$

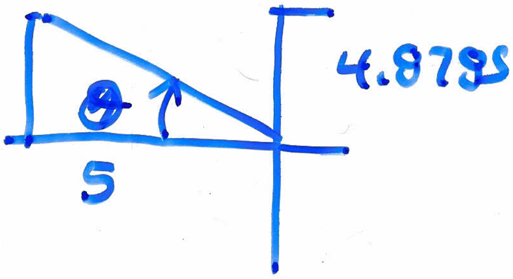




$\zeta = \cos\theta$

see figures in Nise

CHECK RELATIONSHIP



$$\theta = \tan^{-1}\frac{4.8785}{5}$$

$$\theta = 44.295^\circ$$

$$\cos\theta = 0.7157$$

this RELATIONSHIP is IMPORTANT for ROOT LOCUS Problems