CRACKS INTERACTION IN A PRE-STRESSED AND PRE-POLARIZED PIEZOELECTRIC MATERIAL
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Abstract

- Mathematical formulation and solution for antiplane cracks in pre-stressed, pre-polarized piezoelectric material with static initial fields.
- Assumption of local stability in initially deformed configuration of the body.
- Derivation of Riemann-Hilbert problems using boundary conditions of antiplane cracks.
- Obtaining nonhomogeneous linear complex differential equations for unknown complex potential.
- Determination of complex potentials, incremental displacement, and stress fields for third mode of classical fracture under constant incremental forces.
- Study of interaction between two collinear, unequal cracks in pre-stressed, pre-polarized piezoelectric material.

Keywords: piezoelectricity, pre-stressed, pre-polarized, cracks interaction, antiplane mode.

Introduction

- Piezoelectric materials are essential in engineering for sensing, actuation, and transduction.
- Understanding fracture in these materials is crucial for optimal design.
- Limited analytical studies exist on crack interactions, especially in pre-stressed and pre-polarized materials.
- Our focus is on the interaction of two unequal antiplane cracks.
- Objectives: Develop boundary conditions using complex potentials and analyze critical values for crack propagation.
- Findings suggest that in weak interaction scenarios, longer crack tips propagate first, while in strong interaction cases, inner cracks initiate propagation.

Notations and Basic Equations

- Analysis of antiplane states in pre-stressed and pre-polarized piezoelectric materials.
- Incorporate static initial applied stress (σ°) and electric fields (E°).
- Solution based on complex potentials, influenced by Soos's work (**soos1-soos5**).
- Incremental components: u_3 (displacement), φ (electric potential), θ_{13}, θ_{23} (stress fields), e_1, e_2 (electric fields), Δ_1, Δ_2 (electric displacement fields).

- Incremental components of displacement field u_3.
- Electric potential φ.
- Stress field θ_{13}, θ_{23}.
- Electric field e_1, e_2.
- Displacement field Δ_1, Δ_2.

$$
\begin{aligned}
u_3 &= 2\mathrm{Re}\{b_1\Phi_1(z_1) + b_2\Phi_2(z_2)\}, \\
\varphi &= 2\mathrm{Re}\{c_1\Phi_1(z_1) + c_2\Phi_2(z_2)\}, \\
\theta_{13} &= -2\mathrm{Re}\{d_1\Psi_1(z_1) + d_2\Psi_2(z_2)\}, \\
\theta_{23} &= 2\mathrm{Re}\{f_1\Psi_1(z_1) + f_2\Psi_2(z_2)\}, \\
\Delta_1 &= -2\mathrm{Re}\{\mu_1\Psi_1(z_1) + \mu_{12}\Psi_2(z_2)\}, \\
\Delta_2 &= 2\mathrm{Re}\{\Psi_1(z_1) + \Psi_2(z_2)\}.
\end{aligned}
\tag{1}
$$

cont...

Where:

$$
\begin{aligned}
d_\alpha &= \frac{1}{\epsilon}[2e\frac{\mu_\alpha}{1+\mu_\alpha^2} - (1+\epsilon)E^0 - \frac{\epsilon\sigma^0}{e}\frac{\mu_\alpha}{1-\mu_\alpha^2}] \\
f_\alpha &= \frac{1}{\epsilon}[2e\frac{\mu_\alpha}{1+\mu_\alpha^2} - (1+\epsilon)E^0 - \frac{\epsilon\sigma^0}{e}\frac{1}{1-\mu_\alpha^2}] \\
Z_\alpha &= x_1 + \mu_\alpha x_2
\end{aligned} \tag{2}
$$

- Characteristic equation:

$$(\epsilon C-\chi E^2)\mu^2-4E\mu^3+[2\epsilon C+\epsilon\sigma+4e^2-2\chi E^2]\mu^2-4eE\mu+(\epsilon C-\chi E^2+\epsilon\sigma)=0$$

- Assumption: Unequal roots, i.e.,

$$\mu_1 \neq \mu_2$$

.

Main Result

- Consideration of a pre-stressed and pre-polarized piezoelectric material with two cracks positioned as (-a,-k a) and (k a,a) on axis Ox_1 in the material's symmetry plane.
- Pre-stress σ^0 and initial electric field E^0 act in the plane of the cracks, first along the cracks line (Ox_1 axis) and second perpendicular to it (Ox_3 axis).
- The material is subjected to incremental shear stress $\tau = \tau(x_1) > 0$ and incremental surface charge density $w = w(x_1) > 0$, leading to an antiplane state in the body with respect to the Ox_1X_2 plane.
- Boundary conditions on the crack faces are:

$$\begin{aligned} \theta_{23}(t, 0^+) &= \theta_{23}(t, 0^-) = -\tau(t), \\ \Delta_2(t, 0^+) &= \Delta_2(t, 0^-) = -w(t), \quad t \in L, \end{aligned} \tag{3}$$

where $L = (-a, -ka) \cup (k_2 a, a)$, with $0 \leq k_1, k_2 < 1$.

cont...

- Assume unperturbed pre-stressed and pre-polarized equilibrium state at long distances from cracks.
- $U_2(X_1, X_2)$, $\theta_{23}(X_1, X_2)$, $\Delta_\alpha(X_1, X_2)$, and $\phi(x_1, x_2)$ vanish, where $\alpha = 1, 2$.
- From equations $(1)_6$ and (3), we obtain:

$$\begin{aligned} \Psi_1^+(t) + \Psi_2^+(t) + \overline{\Psi}_1^-(t) + \overline{\Psi}_2^-(t) &= -w(t) \\ \Psi_1^-(t) + \Psi_2^-(t) + \Psi_1^+(t) + \Psi_2^+(t) &= -w(t). \end{aligned} \tag{4}$$

- Adding and subtracting, we get:

$$\begin{aligned} (\Psi_1 + \Psi_2 + \overline{\Psi}_1 + \overline{\Psi}_2)^+(t) + (\Psi_1 + \Psi_2 + \overline{\Psi}_1 + \overline{\Psi}_2)^-(t) &= -2w(t) \\ (\Psi_1 + \Psi_2 - \overline{\Psi}_1 - \overline{\Psi}_2)^+(t) - (\Psi_1 + \Psi_2 - \overline{\Psi}_1 - \overline{\Psi}_2)^-(t) &= 0. \end{aligned} \tag{5}$$

cont...

- From Eq. $(5)_2$, we get that in the whole complex plane z, our complex potential satisfies:

$$\Psi_1(z) + \Psi_2(z) - \overline{\Psi}_1(z) - \overline{\Psi}_2(z) = 0, \quad z \in \mathbb{C}. \tag{6}$$

- Eq. $(5)_1$ represents a Riemann-Hilbert problem having solution

$$\Psi_1(z)+\Psi_2(z)+\overline{\Psi}_1(z)+\overline{\Psi}_2(z) = \frac{X(z)}{\pi i}\int_L \frac{w(t)dt}{X+(t)(t-z)} + P_1(z)X(z), \tag{7}$$

- with $P_1(z)$ an arbitrary first-degree polynomial and

$$X(z) = \frac{1}{\sqrt{(z^2-a^2)(z-k_1a)(z+k_2a)}} \tag{8}$$

- From Eqs. (6) and (7), we obtain

$$\Psi_1(z) + \Psi_2(z) = \frac{X(z)}{2\pi i}\int_L \frac{w(t)dt}{X^+(t)(t-z)} + \frac{1}{2}P(z)X(z). \tag{9}$$

- Using a similar procedure from boundary condition $(3)_1$, we get

$$f_1\Psi_1(z)+f_2\Psi_2(z)=\frac{X(z)}{2\pi i}\int_L\frac{\tau(t)dt}{X^+(t)(t-z)}+\frac{1}{2}P(z)X(z) \tag{10}$$

- With $P_2(z)$ as an arbitrary first-degree polynomial.
- Potentials $\Psi_j(z_j)$ can be obtained from equations (9) and (10).

$$\Psi_j(z_j)=\frac{(-1)^{j-1}X(z)}{2f\pi}\int_L\Omega_j(t)dt+P(z_j)X(z_j) \tag{11}$$

with

$$\begin{aligned}\Omega_1(t)&=\frac{\tau(t)-l_2w(t)}{X^+(t)(t-z_1)},\\ \Omega_2(t)&=\frac{\tau(t)-l_1w(t)}{X^+(t)(t-z_2)},\\ P(z)&=\frac{1}{2}(P_1(z)+P_2(z)),\quad f=f_1-f_2\end{aligned} \tag{12}$$

- Complex potentials $\Phi_j(z_j) = \int_L \Phi_j(z_j)dz_j$, $j = 1, 2$, can be multivalued functions even if $\Psi_j(z_j)$ are univalued.
- $X(z)$ is the branch of a multivalued function which satisfies specific conditions.

$$X^+(t) = \begin{cases} iA^{-1}(t), & -a < t < -k_1 a \\ -iA(t), & k_2 a < t < a \end{cases} \tag{13}$$

$$A(t) = \sqrt{(a^2 - t^2)(t - k_1 a)(t + k_2 a)}, \quad t \in L$$

and

$$[X(z)]^{-1} \to +\infty$$

- We consider crack faces subjected only to a constant positive incremental shear stress τ.
    - $w(t) = 0$
    - $\tau(t) = \tau > 0$, $t \in L$.

- Uniform incremental displacements lead to uniform potentials $\phi_j(z_j)$ around the cracks, yielding the following complex potentials:

$$\Psi_j(z_j) = (-1)^{j-1}\frac{\tau}{2f}\left[\frac{z_j^2 - aT_1 z_f - a^2 T_2}{\sqrt{(z_j - a^2)(z_j - k_1 a)(z_j + k_2 a)}}\right] \tag{14}$$

with

$$\begin{aligned} T_1 &= T_1(k, k_2) = \frac{I_2 J_0 - I_0 J_2}{I_1 J_0 - I_0 J_1}, \\ T_2 &= T_2(k, k_2) = \frac{I_2 J_1 - I_1 J_2}{I_1 J_1 - I_1 J_0}, \\ I_j &= \int_{k_1}^{1} \frac{t_j dt}{U(t)}, J_j = \int_{k_2}^{1} \frac{t^j dt}{V(t)}, j = 1, 2, 3 \\ U(t) &= \sqrt{(1 - t^2)(t + k_1)(t - k_2)}, \\ V(t) &= \sqrt{(1 - t^2)(t - k_1)(t + k_2)} \end{aligned} \tag{15}$$

Asymptotical behavior of the incremental fields

- In the vicinity of crack tip a of the right crack, we have:

$$\begin{aligned} x_1 = a + r\cos\phi, x_2 = r\sin\phi, z_j \approx a, \\ z_j - a = rx_j(\phi), x_j(\phi) = \cos\phi - \mu_j \sin\phi, \end{aligned} \tag{16}$$

$$A(z_1) \approx a\sqrt{2arx_j(\phi)}\sqrt{(1-k_1)(1+k_2)} \tag{17}$$

- r: Radial distance from crack tip a.
- ϕ: Oriented angle from crack line to current point direction.
- Using Eqs. $(1)_{1,4}$, (14), (16), we obtain representations in a small neighborhood of crack tip a.

$$\psi_j(z_j) = (-1)^{j-1}\frac{\tau\sqrt{a}}{2f}\frac{\sigma_1}{2r\chi_j(\phi)},$$

$$\Phi_j(z_j) = (-1)^{j-1}\frac{\tau\sqrt{a}}{2f}\sigma_j\sqrt{2r\chi_j(\phi)},$$

$$\theta_{23}(r,\phi) = \sigma_1\tau\sqrt{\frac{a}{2r}}R_e\left[\frac{1}{f}\left(\frac{f_1}{\sqrt{\chi_1(\phi)}} - \frac{f_2}{\sqrt{\chi_2(\phi)}}\right)\right],$$
$$u_3(r,\phi) = \sigma_1\tau\sqrt{2ar}R_e\left[\frac{1}{f}\left(\frac{b_1}{\sqrt{\chi_1(\phi)}} - \frac{b_2}{\sqrt{\chi_2(\phi)}}\right)\right], \tag{18}$$

- Let G(a) the energy release rate corresponding the tip a. Using Irwin's formula, (see [11]) we have:

$$G(a) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon}\theta_{23}(\epsilon - t, 0)u_3(t,\phi) \tag{19}$$

With $Eqs.(18)_{3,4}$ in (19) we get:

$$G(a) = \Gamma\pi a\tau^2/r_1^2 \tag{20}$$

Table 1: Numerical values of of dimensionless quantities $r_j, j = 1{,}4$ for different types of cracks

k_1	k_2	r_1	r_2	r_3	r_4
0.01	0.01	1.697541	0.8475714	0.8475714	1.697541
0.1	0.1	1.940640	1.679665	1.679665	1.940640
0.1	0.3	2.029097	1.929642	2.108677	2.240478
0.1	0.9	2.107195	2.106345	6.022558	6.042497
0.2	0.2	2.126183	2.010003	2.010003	2.126183
0.2	0.8	2.231839	2.228758	4.297526	4.319739
0.33	0.33	2.375949	2.369404	2.369404	2.375949
0.5	0.5	2.793542	2.779464	2.779464	2.793542
0.8	0.2	4.319739	4.297526	2.228758	2.231839
0.8	0.8	4.465573	4.464800	4.464800	4.465573
0.9	0.1	6.042497	6.022558	2.106345	2.107195
0.9	0.9	6.322420	6.322304	6.322304	6.322420
0.99	0.99	19.99994	19.99994	19.99994	19.99994

Where

$$r_1 = \frac{\sqrt{2(1-k_1)(1+k_2)}}{1-T_1-T_2}, and\Gamma \quad = Re\left(\frac{k_1-k_2}{f}\right) \tag{21}$$

cont...

- From Fig. 1 (left), denominators for dimensionless functions r_j, ($j = 1$ to 4), are strictly positive for $k \in (0, 1)$.
- Energy release rates for other crack tips $k_2 a$, $-k_1 a$, and $-a$ are similarly obtained.

$$
\begin{aligned}
G(k_2 a) &= \Gamma \pi r^2 / r_2^2, \\
G(k_1 a) &= \Gamma \pi r^2 / r_3^2, \\
G(-a) &= \Gamma \pi r^2 / r_4^2
\end{aligned}
\tag{22}
$$

cont...

With

$$
\begin{aligned}
r_2 = r_2(k_1, k_2) &= \frac{\sqrt{2(k_1 + k_2)(1 - k_1^2)}}{k_1^2 - k_1 T_1 - T_2}, \\
r_3 = r_3(k_1, k_2) &= \frac{\sqrt{(k_1 + k_2)(1 - k_2^2)}}{k_2^2 + k_2 T_1 - T_2}, \\
r_4 = r_4(k_1, k_2) &= \frac{\sqrt{2(1 + k_1)(1 - k_2)}}{1 - T_1 - T_2}
\end{aligned}
\tag{23}
$$

- Crack tip propagation determined by Griffith's criterion: crack initiates propagation when condition is met.

$$
G(a) = 2\gamma \tag{24}
$$

- From Eqs. (20) and (24), we derive the critical value of the applied stress $\tau(\alpha)$ inducing tip a propagation.

propagation is :

$$\tau(a) = Cr_1, C = \sqrt{\frac{2\gamma}{\Gamma \pi a}} \tag{25}$$

- Using Eqs. (23)-(25), critical values for tip propagation $k_1 a$, $k_2 a$, and $-a$ are obtained.

$$\begin{aligned} \tau(k_1 a) &= Cr_2, \\ \tau(-k_2 a) &= Cr_3, \\ \tau(-a) &= Cr_4 \end{aligned} \tag{26}$$

- The study of crack interaction involves finding the minimum values of four incremental stresses: $\tau(a)$, $\tau(k_1 a)$, $\tau(-ka)$, and $\tau(-k_2 a)$.
- These minimum values correspond to the minimum of dimensionless functions r_j, where $j = 1, 4$.
- Numerical calculus techniques are used to obtain these minimum values.

Case of equal Cracks

- Equal cracks considered: $k_1 = k_2 = k$.
- Critical values for crack tip propagation obtained using Eqs. (20)-(27).

$$\begin{aligned} \tau(a) = \tau(-a) &= Cr_a, \\ \tau(ka) &= Cr_{ka} \end{aligned} \tag{27}$$

With

$$\begin{aligned} C = \sqrt{\frac{2\gamma}{\Gamma\pi a}}, \quad r_a = \frac{\sqrt{1-k^2}}{1-T_2(k)}, \quad r_{ka} = \frac{\sqrt{k(1-k^2)}}{T_2(k)-k^2}, \\ T_2(k) = \frac{I_2(k)}{I_0(k)}, I_j = \int_k^1 \frac{t_j}{\sqrt{(1-t^2)(t^2-k^2)}} dt \end{aligned} \tag{28}$$

We introduce parameters

$$\Pi = \frac{\tau(a)}{\tau(ka)} = \frac{T_2(k)-k^2}{\sqrt{k}(1-T_2(k))}, k \in (0,1). \tag{29}$$

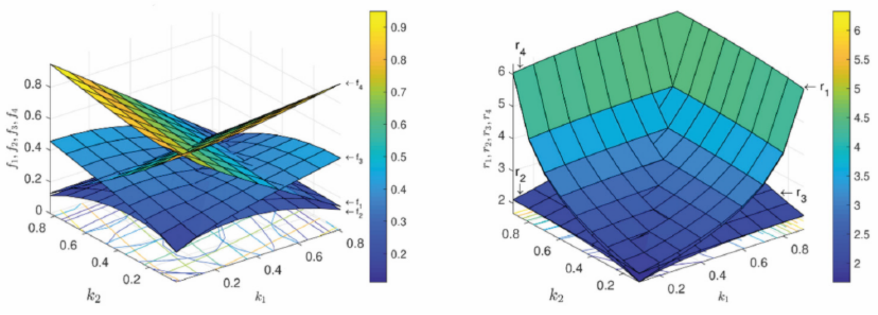

Fig. 1: Representation of denominators f_j and of dimensionless quantities $r_j, j = 1,4$, versus k_1 and k_2

- Interior tips $\pm ka$ propagate first when $\Pi > 1$.
- Exterior tips propagate first when $\Pi < 1$.

Discussion of Numerical Results

- Establish which tips propagate first.
- Consider fixed parameters $k_1, k_2 \in (0, 1)$.
- Determine minimum values of incremental stresses $\tau(\alpha), \tau(k_1 a), \tau(-k_2 a), \tau(-k_2 a)$.
- Tips corresponding to minimum dimensionless functions r_j, ($j = 1$ to 4) propagate first.
- Dimensionless functions $r_j, j = 1, 4$ were plotted against $k_1, k_2 \in (0, 1)$ (Figure 1, right).
- Various pairs of cracks were considered, as listed in Table 1.
- Table 1 presents computed values of r_j, ($j = 1$ to 4) for different crack types.

Conclusions

- **Equal Cracks:**
  - Cases 1, 2, 5, 7, 8, 10, 12, 13.
  - Small distance between cracks compared to their length (Cases 1, 2, 5) indicates strong interaction.
  - In Cases 12 and 131114, inner tips propagate first, tending to unify cracks.
  - Greater or approximately equal distance to length ratio (Cases 8, 10, 12, 13) indicates weak interaction, leading to simultaneous propagation of all tips.
- **Unequal Cracks:**
  - Cases 3, 4, 6, 9, 11.
  - Strong interaction when crack lengths exceed distance between them (Case 3), with the inner tip of the longer crack propagating first.
  - Weak interaction when crack lengths are smaller than distance between them (Cases 4, 6, 9, 11), resulting in both tips of the longer crack propagating first.

We note that the obtained results are plausible and in good agreement with observed cracks behavior and do not depend on the initial applied fields.

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Thank You

Thank You!