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Connecting Quantum Contextuality and Genuine Multipartite Nonlocality with the Quantumness Witness *

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The Clauser–Horne–Shimony–Holt-type noncontextuality inequality and the Svetlichny inequality are derived from the Alicki–van Ryn quantumness witness. Thus connections between quantumness and quantum contextuality, and between quantumness and genuine multipartite nonlocality are established.

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Quantumness is often revealed by the negativity of a certain quantumness witness (QW).[1] Specifically, given a commutative C^*-algebra $\mathcal{A}$, for any pair $X, Y \in \mathcal{A}$ with $X \geq 0$ and $Y \geq 0$, the following anti-commutation relation always holds

$$\{X, Y\} := XY + YX \geq 0. \qquad (1)$$

In quantum mechanics, however there exist noncommutative positive-definite operators such that the above relation can be violated. In Alicki and van Ryn's series works, they have proven that a certain QW can always be found to detect quantumness for a two-dimensional system, except that the system is in the maximally mixed state $\mathbb{I}/2$.

Quantum contextuality, on the other hand, serves as a distinct correlation which shows an incompatibility of quantum mechanics with the noncontextual hidden variable theory.[2] Such a theory assumes that the measurement outcome of $\boldsymbol{A}$ is independent of whether $\boldsymbol{A}$ is measured together with $\boldsymbol{B}$ or with $\boldsymbol{C}$. In general, however, this is not the case in quantum mechanics. For a system consisting of a number of subsystems, in particular, there may exist the genuine multipartite nonlocality, a stronger form of quantum contextuality, which is usually detected by the Svetlichny inequality.[3,4]

The aforementioned three types of nonclassicality, i.e., quantumness, contextuality and nonlocality, have seemingly been investigated by using quite different means. Also, the relations between the last two types have been widely explored so far (see, e.g., Ref. [5] and references therein), relatively less is known about them with the first type of quantumness, and thus, this is what we would like to address in the study.

Here we shall show that with the QW Eq. (1) as the starting point, one is able to construct the Clauser–Horne–Shimony–Holt-type (CHSH-type) noncontextual inequality[6] and the Svetlichny inequality.[3,4] Therefore, the violation of the inequalities clearly implies the Alicki–van Ryn quantumness.[1,7,8] That is, both quantum contextuality and genuine multipartite nonlocality belong to the Alicki–van Ryn quantumness.

To see the connection between contextuality and quantumness, we start with an example of the two-qubit CHSH inequality, as shown in Ref. [1]. Two positive-definite operators can be written as

$$\begin{aligned} X &= 2 - (A_1 \otimes B_1 - A_2 \otimes B_2) \geq 0, \\ Y &= 2 - (A_1 \otimes B_2 + A_2 \otimes B_1) \geq 0, \end{aligned}$$

where $A_{1,2}, B_{1,2}$ are dichotomic observables, taking values ± 1 for each party. Then

$$\begin{aligned} XY &= 2E + [A_1, A_2] \otimes \mathbb{I} + \mathbb{I} \otimes [B_1, B_2], \\ YX &= 2E - [A_1, A_2] \otimes \mathbb{I} - \mathbb{I} \otimes [B_1, B_2], \end{aligned}$$

with $E \geq 0$ being the CHSH inequality, and

$$E = 2 - (A_1 \otimes A_2 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2).$$

Thus one obtains the witness as

$$Q = \{X, Y\} = 4E,$$

where $Q < 0$ if ρ is an entangled pure state, since any entangled pure state violates the CHSH inequality. This connection between Bell nonlocality and quantumness was constructed in Refs. [1,7,8]. Thus Bell nonlocality clearly implies quantumness.

Then a natural question is: does contextuality also imply quantumness? In other words, does there exist a quantumness witness Q_c to detect contextuality? The above example may give an affirmative answer, provided that the two-qubit state can effectively be seen as a four-level system.

To see this clearly, one can construct observables from $\{A_{1,2}, B_{1,2}\}$ as follows:

$$\begin{aligned} A_1 \otimes \mathbb{I} = B, \quad A_2 \otimes \mathbb{I} = D, \\ \mathbb{I} \otimes B_1 = C, \quad \mathbb{I} \otimes B_2 = A, \end{aligned}$$

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and then we write two positive-definite operators as

$$X = 2 - (BC - AD) \geq 0,$$
$$Y = 2 - (AB + CD) \geq 0,$$

thus

$$XY = 2E_{\rm c} + [B, D] + [C, A],$$
$$YX = 2E_{\rm c} - [B, D] - [C, A],$$

with $E_{\rm c} \geq 0$ being the CHSH-type noncontextuality inequality, and

$$E_{\rm c} = 2 - (AB + BC + CD - AD).$$

Eventually we obtain the witness as

$$Q_{\rm c} = \{X, Y\} = 4E_{\rm c}.$$

As shown in Ref. [5], the relation $E_{\rm c} \geq 0$ can be violated by some four-dimensional state, making negative. Thus $Q_{\rm c}$ is a quantumness witness for contextuality of an arbitrary four-dimensional pure state.

We now connect the QW with the N-qubit Svetlichny inequality, whose quantum violation directly indicates the genuine multipartite nonlocality.

It is worthwhile to present two preliminaries before we proceed. First, the sum of QW's is still a QW. That is, given a group of QW's

$$Q_\xi = \{X_\xi, Y_\xi\}, \quad \xi = 1, 2, \dots$$

the summation, that is,

$$Q^{\rm tot} = \sum_\xi Q_\xi, \tag{2}$$

is still a QW. Note that $Q^{\rm tot} \geq 0$ still holds for classical states.

Secondly, we have known from Ref. [9] that the N-qubit Svetlichny inequality can be constructed by summing up 2^{N-2} CHSH-type inequalities, i.e.,

$$\mathcal{I}_{\rm Svet} = \sum_\xi \mathcal{I}_\xi \leq 2^{N-1}, \tag{3}$$

where each $\mathcal{I}_\xi$ belongs to the CHSH-type. The idea is that each $\mathcal{I}_\xi$ consists of four correlations, for which one can group some parties effectively as a single party, $\tilde{A}$, and group the others as a second single party, $\tilde{B}$, thus $\mathcal{I}_\xi$ can be expressed formally as the CHSH inequality for two effective parties $\tilde{A}$ and $\tilde{B}$.

With Eqs. (2) and (3), we are ready now to connect the QW with the Svetlichny inequality. As a first step, let us consider the three-qubit case. The three-qubit Svetlichny inequality reads

$$\mathcal{I}(\rho) = Q_{000} + Q_{001} + Q_{010} - Q_{011} + Q_{100} - Q_{101} - Q_{110} - Q_{111} \leq 4.$$

Now we divide up the left-side of the inequality into two parts: $\mathcal{I}(\rho) = \mathcal{I}_1(\rho) + \mathcal{I}_2(\rho)$ with

$$\mathcal{I}_1(\rho) = Q_{000} + Q_{001} + Q_{010} - Q_{011},$$
$$\mathcal{I}_2(\rho) = Q_{100} - Q_{101} - Q_{110} - Q_{111}.$$

It is found that if we group A and B together as a single party $\tilde{A}$, and denote C as a $\tilde{B}$, then we consider an effective bipartite scenario that is similar to the CHSH inequality. Specifically,

$$\mathcal{I}_1(\rho) = \tilde{Q}_{00} + \tilde{Q}_{01} + \tilde{Q}_{10} - \tilde{Q}_{11},$$
$$\mathcal{I}_2(\rho) = \tilde{Q}_{00} - \tilde{Q}_{01} - \tilde{Q}_{10} - \tilde{Q}_{11},$$

where

$$\text{for } \mathcal{I}_1(\rho): \ \tilde{Q}_{0i} = Q_{00i}, \ \tilde{Q}_{1i} = Q_{01i},$$
$$\text{for } \mathcal{I}_2(\rho): \ \tilde{Q}_{0i} = Q_{10i}, \ \tilde{Q}_{1i} = Q_{11i}.$$

Here the correlation is defined as

$$\tilde{Q}_{ij} = P(\alpha_i + \beta_j \doteq 0) - P(\alpha_i + \beta_j \doteq 1),$$

where $\doteq$ indicates a modulo 2, i and j run over 0 and 1, and

$$\text{for } \mathcal{I}_1(\rho): \ \alpha_0 = a_0 + b_0, \ \alpha_1 = a_0 + b_1, \ \beta_j = c_j,$$
$$\text{for } \mathcal{I}_2(\rho): \ \alpha_0 = a_1 + b_0, \ \alpha_1 = a_1 + b_1, \ \beta_j = c_j,$$

with a_i, b_j, c_k being the local outcomes of A, B, C, respectively. Of course, one can alternatively group A and C (or B and C) as $\tilde{A}$, and leave the last party B (or A) as $\tilde{B}$, while this does not substantially affect our argument above and below. What we would like to stress is just that such a procedure of dividing up the Svetlichny inequality into the CHSH-type elements can always be made.

Similar to the previous section, the inequality can be derived from the quantumness witness as follows. We write

$$X_1 = 2 - (\tilde{Q}_{00} - \tilde{Q}_{11}),$$
$$Y_1 = 2 - (\tilde{Q}_{01} + \tilde{Q}_{10}),$$

for $\mathcal{I}_1(\rho)$, and

$$X_2 = 2 - (\tilde{Q}_{00} - \tilde{Q}_{11}),$$
$$Y_2 = 2 + (\tilde{Q}_{01} + \tilde{Q}_{10}),$$

for $\mathcal{I}_2(\rho)$, thus

$$\begin{aligned} Q_1 &= \{X_1, Y_1\} \\ &= 4(2 - (\tilde{Q}_{00} + \tilde{Q}_{01} + \tilde{Q}_{10} - \tilde{Q}_{11})) \\ &= 4(2 - (Q_{000} + Q_{001} + Q_{010} - Q_{011})), \end{aligned}$$

and

$$\begin{aligned} Q_2 &= \{X_2, Y_2\} \\ &= 4(2 - (\tilde{Q}_{00} + \tilde{Q}_{01} + \tilde{Q}_{10} - \tilde{Q}_{11})) \\ &= 4(2 - (Q_{100} + Q_{101} + Q_{110} - Q_{111})). \end{aligned}$$

Then a summation yields the total QW as

$$Q^{\rm tot} = Q_1 + Q_2 = 4(4 - \mathcal{I}(\rho)).$$

It is immediate to observe that $Q^{\rm tot} \geq 0$ for classical theories; however, the fact that $\mathcal{I}(\rho)$ could be larger than four for the genuine three-qubit entangled state makes the QW negative as well. Thus the genuine three-qubit nonlocality clearly implies the quantumness.

Now we are in a position to derive the N-qubit Svetlichny inequality $\mathcal{I}_{\rm Svet}$ from the QWs. For the sake of convenience, we focus on a particular $\mathcal{I}_\xi$ and expand it into probabilities, that is,

$$\mathcal{I}_\xi = \tilde{Q}_{00} + \tilde{Q}_{01} + \tilde{Q}_{10} - \tilde{Q}_{11}, \tag{4}$$

where

$$\tilde{Q}_{ij} = P(\alpha_i + \beta_j \doteq 0) - P(\alpha_i + \beta_j \doteq 1), \tag{5}$$

α_i is a sum of outcomes of the first m parties, and β_j is that of the remaining $(N-m)$ parties. For instance, we have $m = 2$ in the above three-qubit case. The specific index for each local outcome depends on ξ, i.e., which part of $\mathcal{I}_{\rm Svet}$ these four correlations are included in (see, e.g., the three-qubit case).

The notation we take here is different from that in Ref. [9], in which the correlation $\tilde{Q}$ is defined differently and we have $\mathcal{I}_\xi = -\tilde{Q}'_{00} - \tilde{Q}'_{01} - \tilde{Q}'_{10} - \tilde{Q}'_{11}$. In fact, this is equivalent to Eq. (4). Here $\tilde{Q}_{ij}$ is an N-qubit correlation. The reason why we only write two indices here is that the N qubits can always be divided into two parts, each of which acts as a single qubit and can be detected by the CHSH-type inequality $\mathcal{I}_\xi$ (such a division is always possible; see, e.g., the above three-qubit case, and also Ref. [9] for more details).

Now we write

$$\begin{aligned} X_\xi &= 2 - (\tilde{Q}_{00} - \tilde{Q}_{11}), \\ Y_\xi &= 2 - (\tilde{Q}_{01} + \tilde{Q}_{10}). \end{aligned}$$

For a different $\xi' \neq \xi$, $\mathcal{I}_{\xi'}$ could also be as follows: $\mathcal{I}_{\xi'} = \tilde{Q}_{00} - \tilde{Q}_{01} - \tilde{Q}_{10} - \tilde{Q}_{11}$. For this form, instead we must write

$$\begin{aligned} X_\xi &= 2 - (\tilde{Q}_{00} - \tilde{Q}_{11}), \\ Y_\xi &= 2 + (\tilde{Q}_{01} + \tilde{Q}_{10}). \end{aligned}$$

However, this difference in signs does not affect the analysis in the following, thus hereafter we do not point it out but a form that is different from Eq. (4) being considered. To continue, we have

$$\begin{aligned} X_\xi Y_\xi &= 2(2 - \mathcal{I}_\xi) + \tilde{Q}_{00}\tilde{Q}_{01} + \tilde{Q}_{00}\tilde{Q}_{10} \\ &\quad - \tilde{Q}_{11}\tilde{Q}_{01} - \tilde{Q}_{11}\tilde{Q}_{10}, \\ Y_\xi X_\xi &= 2(2 - \mathcal{I}_\xi) + \tilde{Q}_{01}\tilde{Q}_{00} + \tilde{Q}_{10}\tilde{Q}_{00} \\ &\quad - \tilde{Q}_{01}\tilde{Q}_{11} - \tilde{Q}_{10}\tilde{Q}_{11}. \end{aligned}$$

Due to orthogonal relations

$$\begin{aligned} P(\alpha_i = s)P(\alpha_i = t) &= \delta_{st} P(\alpha_i = s), \\ P(\beta_j = s)P(\beta_j = t) &= \delta_{st} P(\beta_j = s), \end{aligned}$$

along with the definition (4), we further have

$$\begin{aligned} \tilde{Q}_{00}\tilde{Q}_{01} &= P(\beta_1 = 0)P(\beta_2 = 0) \\ &\quad - P(\beta_1 = 0)P(\beta_2 = 1) \\ &\quad - P(\beta_1 = 1)P(\beta_2 = 0) \\ &\quad + P(\beta_1 = 1)P(\beta_2 = 1), \\ \tilde{Q}_{00}\tilde{Q}_{10} &= P(\alpha_1 = 0)P(\alpha_2 = 0) \\ &\quad - P(\alpha_1 = 0)P(\alpha_2 = 1) \\ &\quad - P(\alpha_1 = 1)P(\alpha_2 = 0) \\ &\quad + P(\alpha_1 = 1)P(\alpha_2 = 1), \\ \tilde{Q}_{11}\tilde{Q}_{01} &= P(\alpha_2 = 0)P(\alpha_1 = 0) \\ &\quad - P(\alpha_2 = 0)P(\alpha_1 = 1) \\ &\quad - P(\alpha_2 = 1)P(\alpha_1 = 0) \\ &\quad + P(\alpha_2 = 1)P(\alpha_1 = 1), \\ \tilde{Q}_{11}\tilde{Q}_{10} &= P(\beta_2 = 0)P(\beta_1 = 0) \\ &\quad - P(\beta_2 = 0)P(\beta_1 = 1) \\ &\quad - P(\beta_2 = 1)P(\beta_1 = 0) \\ &\quad + P(\beta_2 = 1)P(\beta_1 = 1), \\ \tilde{Q}_{01}\tilde{Q}_{00} &= P(\beta_2 = 0)P(\beta_1 = 0) \\ &\quad - P(\beta_2 = 0)P(\beta_1 = 1) \\ &\quad - P(\beta_2 = 1)P(\beta_1 = 0) \\ &\quad + P(\beta_2 = 1)P(\beta_1 = 1), \\ \tilde{Q}_{10}\tilde{Q}_{00} &= P(\alpha_2 = 0)P(\alpha_1 = 0) \\ &\quad - P(\alpha_2 = 0)P(\alpha_1 = 1) \\ &\quad - P(\alpha_2 = 1)P(\alpha_1 = 0) \\ &\quad + P(\alpha_2 = 1)P(\alpha_1 = 1), \\ \tilde{Q}_{01}\tilde{Q}_{11} &= P(\alpha_1 = 0)P(\alpha_2 = 0) \\ &\quad - P(\alpha_1 = 0)P(\alpha_2 = 1) \\ &\quad - P(\alpha_1 = 1)P(\alpha_2 = 0) \\ &\quad + P(\alpha_1 = 1)P(\alpha_2 = 1), \\ \tilde{Q}_{10}\tilde{Q}_{11} &= P(\beta_1 = 0)P(\beta_2 = 0) \\ &\quad - P(\beta_1 = 0)P(\beta_2 = 1) \\ &\quad - P(\beta_1 = 1)P(\beta_2 = 0) \\ &\quad + P(\beta_1 = 1)P(\beta_2 = 1). \end{aligned}$$

All these will be canceled with one another, hence we obtain the QW for ξ,

$$Q_\xi = \{X_\xi, Y_\xi\} = 4(2 - \mathcal{I}_\xi).$$

For other $\xi' \neq \xi$, one can similarly construct the corresponding QW $Q_{\xi'}$. In total, we have 2^{N-2} such Q_ξ's. Eventually we obtain the total QW by a summation

$$Q^{\rm tot} = \sum_\xi Q_\xi = \sum_{\xi=1}^{2^{N-2}} 4(2 - \mathcal{I}_\xi) = 4(2^{N-1} - \mathcal{I}_{\rm Svet}).$$

In the last step, we have used $\sum_\xi \mathcal{I}_\xi = \mathcal{I}_{\rm Svet}$. Thus we have connected the witness $Q^{\rm tot}$ with the N-qubit Svetlichny inequality and, similarly to the aforementioned three-qubit case, when the Svetlichny inequality is violated by the genuine multipartite entangled state, $Q^{\rm tot}$ becomes negative as well. Therefore, the genuine multipartite nonlocality clearly implies the quantumness.

In summary, we have demonstrated that the CHSH-type noncontextuality inequality and the Svetlichny inequality can be derived out from the Alicki–van Ryn quantumness witness. Connections between the quantumness and quantum contextuality and between the quantumness and genuine multipartite nonlocality have therefore been established. However, a few questions are still open. For instance, how can one construct the Mermin–Ardehali–Belinskii–Klyshko inequality[10] from the witness? We only consider the Svetlichny inequality of qubits, then how can the derivation be generalized to that of qudits? Can one derive state-independent noncontextuality inequalities (like, e.g., the one in Ref. [11]) from the witness as well? In the researchers' opinion, answering all these is very worthwhile and certainly deserves further investigations.

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