Question 4:

$x = \{1, 2, 3, 6, 7, 8\}$

Component 1 | Component 2

$\mu_1 = 2$ | $\mu_2 = 7$

$\sigma_1^2 = 1$ | $\sigma_1^2 = 1$

$\pi_1 = 0.5$ | $\pi_2 = 0.5$

Uses

$$N(x_i | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$Y_{i1} = \frac{\pi_1 \cdot N(x_i | \mu_1, \sigma_1^2)}{\pi_1 \cdot N(x_i | \mu_1, \sigma_1^2) + \pi_2 \cdot N(x_i | \mu_2, \sigma_2^2)}$$

$$Y_{i2} = 1 - Y_{i1}$$

(E-Step)

Iteration 1

For $x = 1$

Component 1: $N(1 | 2, 1) = \frac{1}{\sqrt{2\pi(1)}} \exp\left(-\frac{(1-2)^2}{2(1)}\right)$

$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\right)$

$= 0.242$

Component 2: $N(1 | 7, 1) = \frac{1}{\sqrt{2\pi(1)}} \exp\left(-\frac{(1-7)^2}{2(1)}\right)$

$= \frac{1}{\sqrt{2\pi}} \exp(-18)$

$= 6.076 \times 10^{-9}$

$$Y_{11} = \frac{0.5 \times 0.242}{(0.5 \times 0.242) + (0.5 \times 6.076 \times 10^{-9})}$$

$$= \frac{0.121}{0.121 + 3.038 \times 10^{-9}}$$

$$= 0.99999974a \simeq 1$$

$$Y_{12} = 2.51 \times 10^{-8} \simeq 0$$

For $x=2$

Component 1: $N(2|2,1) = 0.399$

Component 2: $N(2|7,1) = 1.487 \times 10^{-6}$

$Y_{21} = 0.999962 \simeq 1$

$Y_{22} = 3.726639 \times 10^{-6} \simeq 0$

For $x = 3$

Component 1: $N(3|2,1) = 0.24197$

" 2: $N(3|7,1) = 0.000134$

$Y_{31} = 0.99945$

$Y_{32} = 0.000553$

For $x = 6$

Comp 1: $N(6|2,1) = 0.0001338$

∴ 2: $N(6|7,1) = 0.2419707$

$Y_{41} = 0.00055278$

$Y_{42} = 0.99944722$

For $x = 7$

Comp 1: $N(7|2,1) = 1.48767 \times 10^{-6}$

∴ 2: $N(7|7,1) = 0.39894$

$Y_{51} = 3.7266 \times 10^{-6} \simeq 0$

$Y_{52} = 0.999996 \simeq 1$

For $x = 8$

Comp. 1: $N(8|2,1) = 6.07588 \times 10^{-9}$

∴ 2: $N(8|7,1) = 0.24197$

$Y_{61} = 2.510999 \times 10^{-8} \simeq 0$

$Y_{62} = 0.99999 \simeq 1$

M-Step (Maxima) Updates.

① Means

$$\mu_1 = \frac{\sum_{i=1}^{n} y_{i1} x_i}{\sum_{i=1}^{n} y_{i1}}$$

$$= \frac{(1\times 1) + (2\times 1) + (3\times 0.99945) + (6\times 0.00055278) + (7\times 0) + (8\times 0)}{1+1+0.99945+0.00055278+0+0} = \frac{6.001666,8}{3.000072}$$

$$\mu_1 = 2.00055$$

$$\mu_2 = \frac{\sum_{i=1}^{n} y_{i2} x_i}{\sum_{i=1}^{n} y_{i2}}$$

$$= \frac{(1\times 0) + (2\times 0) + (3\times 0.00055) + (6\times 0.99944722) + (7\times 1) + (8\times 1)}{0+0+0.00055+0.99944722+1+1}$$

$$\mu_2 = 6.99944$$

② Variance:

$$\sigma_1^2 = \frac{\sum_{i=1}^{n} y_{i1}(x_i-\mu_1)^2}{\sum_{i=1}^{n} y_{i1}} \quad \& \quad \sigma_2^2 = \frac{\sum_{i=1}^{n} y_{i2}(x_i-\mu_2)^2}{\sum_{i=1}^{n} y_{i2}}$$

$$\sigma_1^2 = (0.2(1(1-2.00055)^2) + (1(2-2.00055)^2) + 0.99945(3-2.00055) + 0.00055278(6-2.00055)^2 + 0(7-2.00055) + 0(8-2.00055)$$

$$= \frac{2.0082930603}{3.0000278}$$

$$\sigma_1^2 = 0.66943$$

$$\sigma_2^2 = 0(1-6.99944)^2 + 0 + 0.00053(3-6.9944) + 0.99944(6-6.99944) + 1(7-6.99944) + 1(8-0.99944)$$

$$\sigma_2^2 = 0.66943$$

(3) Mixing proportions π

$$\pi_1 = \frac{1}{n}\sum_{i=1}^{n} y_{i1} \qquad \pi_2 = \frac{1}{n}\sum_{i=1}^{n} y_{i2}$$

$$\pi_1 = \frac{1+1+0.99945+0.00055+0+0}{6}$$

$$= 0.500004633$$

$$\pi_2 = \frac{0+0+0.00055+0.99945+1+1}{6}$$

$$= 0.500004633$$

Updated Parameters

$\mu_1 = 2.00055$

$\sigma_1^2 = 0.66943$

$\pi_1 = 0.5$

$\mu_2 = 6.99944$

$\sigma_2^2 = 0.66943$

$\pi_2 = 0.5$

Iteration 2:

$N(x_i | \mu, \sigma^2)$

for $x=1$

Comp 1 : $N(1 | 2.0005, 0.66943)$

$= \frac{1}{\sqrt{2\pi(0.66943)}} \exp\left(-\frac{(1-2.0005)}{2(0.66943)}\right)$

$= 0.2308$

Comp 2 : $N(1 | 6.99944, 0.66943)$

$= \frac{1}{\sqrt{2\pi 0.66943}} \exp\left(-\frac{(1-6.99944)}{2(0.66943)}\right)$

$= 1.0296 \times 10^{-12}$

$Y_{11} = 0.994 \simeq 1$

$Y_{12} = 4.4603 \times 10^{-12} \simeq 0$

for $x=2$

Comp 1 : $N(2 | 2.0005, 0.66943) = 0.4876$

" 2 : $N(2 | 6.99944, 0.6694) = 3.806 \times 10^{-9}$

$Y_{21} = 0.994 \simeq 1$

$Y_{22} = 7.8056 \times 10^{-9} \simeq 0$

for $x=3$

Comp 1 : $N(3 | 2.0005, 0.66943) = 0.2312$

" 2 : $N(3 | 6.99944, 0.604) = 2.1586 \times 10^{-6}$

$Y_{31} = 0.9999 \simeq 1$

$Y_{32} = 1.366 \times 10^{-5} \simeq 0$

for $x=6$

Comp 1 : $N(6 | 2.0005, 0.66943) = 3.158 \times 10^{-6}$

" 2 : $N(6 | 6.99944, 0.6694) = 0.2312$

$Y_{41} = 1.3659 \times 10^{-5} \simeq 0$

$Y_{42} = 0.999986 \simeq 1$

for $x = 7$

Comp 1 : $N(7 \mid 2.00055, 0.66443) = 3.8059 \times 10^{-9}$

.. 2 : $N(7 \mid 6.9944, 0.6643) = 0.4875$

$Y_{51} := 7.805 \times 10^{-9} \simeq 0$

$Y_{52} := 0.9999 = 1$

for $x = 8$

Comp 1 : $N(8 \mid 2.00055, 0.66443) = 1.0296 \times 10^{-12}$

.. 2 : $N(8 \mid 6.9999, 0.0043) = 0.23024$

$Y_{61} = 4.45998 \times 10^{-12} \simeq 0$

$Y_{62} = 0.9999\ldots \simeq 1$

Maximization

Means (μ)

$\mu_1 = 2.0000137$

$\mu_2 = 6.99999$

Variance σ^2

$\sigma_1^2 = 0.66674$

$\sigma_2^2 = 0.66674$

Mixing Proportion π

$\pi_1 = 0.5$

$\pi_2 = 0.5$

Iteration 3

for $x=1$:

Comp 1: $N(1 \mid 2.0000137, 0.66674)$

$= 0.2308$

$\therefore$ 2: $N(1 \mid 6.99999, 0.66674)$

$= 9.211 \times 10^{-13}$

$Y_{11} := 0.999\ldots \simeq 1$

$Y_{21} := 3.9909 \times 10^{-12} \simeq 0$

for $x=2$:

Comp 1: $N(2 \mid 2.0000137, 0.66674)$

$= 0.4886$

$\therefore$ 2: $N(2 \mid 6.99999 / 2.0000137, 0.66674)$

$= 3.522 \times 10^{-9}$

$Y_{21} := 0.999 \simeq 1$

$Y_{22} := 7.2045 \times 10^{-9} \simeq 0$

for $x=3$:

Comp 1: $N(3 \mid 2.0000137, 0.66674)$

$= 0.2308$

$\therefore$ 2: $N(3 \mid 6.99999, 0.66674)$

$= 3.066 \times 10^{-6}$

$Y_{31} := 0.9999 \simeq 1$

$Y_{32} := 1.3023 \times 10^{-5} \simeq 0$

for $X = 6$:

Comp 1: $N(6 \mid 2.0000137, 0.66674)$

$= 3.0006 \times 10^{-6}$

$\propto 2: N(6 \mid 6.99999, 0.66674)$

$= 0.2308$

$Y_{41} \propto 1.3024 \times 10^{-5} \simeq 0$

$Y_{42} = 0.9999 \simeq 1$

for $X = 7$

Comp 1: $N(7 \mid 2.0000137, 0.66674)$

$= 3.5225 \times 10^{-9}$

$\propto 2: N(7 \mid 6.9999, 0.66674)$

$= 0.4886$

$Y_{51} \propto 7.2097 \times 10^{-9} \simeq 0$

$Y_{52} \propto 0.999 \simeq 1$

for $X = 8$

Comp 1: $N(8 \mid 2.0000137, 0.66674)$

$= 9.2114 \times 10^{-13}$

$\propto 2 N(8 \mid 6.99999, 0.66674)$

$= 0.2308$

$Y_{61} = 3.941 \times 10^{-12} \simeq 0$

$Y_{62} = 0.9999 \simeq 1$

Maximization

Means (μ)

$$\mu_1 = 2.000013 \simeq 2$$

$$\mu_2 = 6.999999 \simeq 7$$

Variance (σ^2)

$$\sigma_1^2 = 0.6667$$

$$\sigma_2^2 = 0.6667$$

Mixing power (π)

$$\pi_1 = 0.5$$

$$\pi_2 = 0.5$$