

Homework 4

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1 Distributed SGD

SGD is an optimization method for training ml models by taking steps in the direction of the negative gradient using samples of the data. In distributed SGD, instead of doing gradient computations on one machine:

- We have **m worker nodes** that each of them compute gradient based on the local data.
- The gradients from all worker nodes are aggregated to update the model parameters.
- This process is repeated iteratively until convergence.

There are two major types of SGD strategies:

1. **Synchronous SGD:** In this strategy, all worker nodes compute their gradients and send them to a central parameter server. The server aggregates the gradients and updates the model parameters. The updated parameters are then sent back to all worker nodes. This process ensures consistency but may suffer from delays caused by slower workers (stragglers).

Time per iteration = max of all X_i

2. **Asynchronous SGD:** In this strategy, worker nodes compute gradients and send them to the parameter server independently. The server updates the model parameters as soon as it receives gradients from any worker. This approach is faster but may lead to stale updates, as some workers might use outdated model parameters.

Time per iteration = min of all X_i

Consider that we have a system of m worker nodes and a parameter server performing distributed SGD (stochastic gradient descent). In each iteration, every worker node receives the model from the parameter server, computes one gradient step of the objective function locally using its local data, and sends the gradient to the parameter server. The parameter server does the aggregation of gradients using either synchronous SGD or asynchronous SGD.

The gradient calculation time X_i taken by each node i follows the exponential distribution with rate $\lambda = 2$, which has the following probability density function (PDF):

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{else} \end{cases}$$

- (a) **Cumulative Distribution Function (CDF):** What is the cumulative distribution function (CDF) of $f_X(x)$, i.e., $F_X(x)$?

Solution

We are given that $X \sim \text{Exponential}(\lambda = 2)$ and the pdf is:

$$f_X(x) = \begin{cases} 2e^{-2x}, & \text{if } x \geq 0 \\ 0, & \text{else} \end{cases}$$

The cumulative distribution function (CDF) is the integral of the PDF. For $x \geq 0$, we compute:

$$F_X(x) = \int_0^x f_X(t) dt = \int_0^x 2e^{-2t} dt$$

Integration:

$$F_X(x) = [-e^{-2t}]_0^x = -e^{-2x} + e^0 = 1 - e^{-2x}, \quad \text{for } x \geq 0$$

For $x < 0$, $F_X(x) = 0$ since the PDF is 0 for $x < 0$.

Therefore, the CDF is:

$$F_X(x) = \begin{cases} 1 - e^{-2x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

- (b) **Maximum of m Instances:** Define $X_{m:m}$ as the maximum of m i.i.d. (independently and identically distributed) instances X_1, \dots, X_m following the distribution X . What is the CDF of $X_{m:m}$, and what is the expected value $\mathbb{E}[X_{m:m}]$?

Solution

CDF of the Maximum $X_{m:m}$

The CDF of the maximum $X_{m:m}$ can be derived as follows:

Let $F_X(x)$ be the CDF of a single instance X . The CDF of the maximum $X_{m:m}$ is given by:

$$F_{X_{m:m}}(x) = P(X_{m:m} \leq x) = P(X_1 \leq x, X_2 \leq x, \dots, X_m \leq x)$$

Since X_1, X_2, \dots, X_m are i.i.d., we can write:

$$F_{X_{m:m}}(x) = \prod_{i=1}^m P(X_i \leq x) = [F_X(x)]^m$$

Substituting the CDF $F_X(x)$ from part (a):

$$F_{X_{m:m}}(x) = \begin{cases} (1 - e^{-2x})^m, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Expected Value of $X_{m:m}$

The expected value $\mathbb{E}[X_{m:m}]$ of m i.i.d with rate λ is:

$$\mathbb{E}[X_{m:m}] = \frac{1}{\lambda} \sum_{i=1}^m \frac{1}{i}$$

Since $\lambda = 2$, therefore:

$$\mathbb{E}[X_{m:m}] = \frac{1}{2} \sum_{i=1}^m \frac{1}{i}$$

- (c) **Minimum of m Instances:** Define $X_{1:m}$ as the minimum of m i.i.d. instances X_1, \dots, X_m following the distribution X . What is the CDF of $X_{1:m}$, and what is the expected value $\mathbb{E}[X_{1:m}]$?

Solution

CDF of minimum $X_{1:m}$

The CDF of the minimum $X_{1:m}$ is derived by:

If $F_X(x)$ is the CDF of a single instance X . The CDF of the minimum $X_{1:m}$ is:

$$F_{X_{1:m}}(x) = P(X_{1:m} \leq x) = 1 - P(X_{1:m} > x) = 1 - P(X_1 > x, X_2 > x, \dots, X_m > x)$$

Since X_1, X_2, \dots, X_m are i.i.d., we write:

$$F_{X_{1:m}}(x) = 1 - \prod_{i=1}^m P(X_i > x) = 1 - [1 - F_X(x)]^m$$

Substituting the CDF $F_X(x)$ from part (a):

$$F_{X_{1:m}}(x) = 1 - e^{-2mx}$$

$$F_{X_{1:m}}(x) = \begin{cases} 1 - e^{-2mx}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Expected Value of $X_{1:m}$

The expected value $\mathbb{E}[X_{1:m}]$ of the minimum of m i.i.d. exponential random variables with rate λ is:

$$\mathbb{E}[X_{1:m}] = \frac{1}{m\lambda}$$

Since $\lambda = 2$, we have:

$$\mathbb{E}[X_{1:m}] = \frac{1}{2m}$$

- (d) **Simulation of Expected Runtime:** Simulate and compare the expected runtime per iteration of synchronous SGD and asynchronous SGD for different values of m . The time for each worker node to finish one gradient computation is exponentially distributed as given in part (a) with $\lambda = 2$, and it is i.i.d. across workers and iterations. Assume there is no communication delay.

Simulate 5000 iterations of training using Python for different values of m ranging from 1 to 20, and obtain the average runtime per iteration. Make a comparative plot of the average runtimes per iteration of synchronous and asynchronous SGD versus m . Explain the trends observed in the plot in 1-2 sentences. You may use packages inside `numpy.random` to draw random samples from the exponential distribution. Attach your plot and code in PDF format to the end of your homework.

Solution

```
import numpy as np
import matplotlib.pyplot as plt

np.random.seed(42)

lambda_rate = 2
num_iter = 5000
m_values = range(1, 21)

sync_runtime = []
async_runtime = []

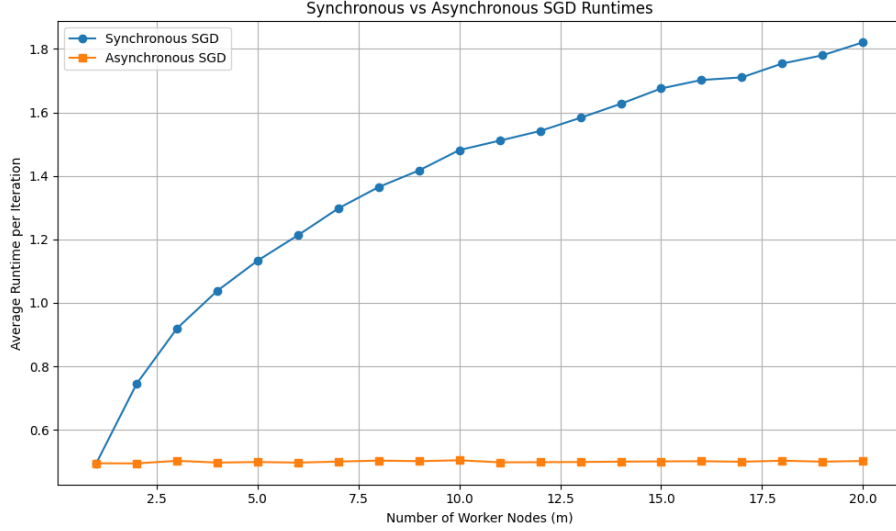
for m in m_values:
    sample = np.random.exponential(scale=1/lambda_rate, size=(num_iter, m))

    # Synchronous: max of each row
    sync_average = np.mean(np.max(sample, axis=1))
    sync_runtime.append(sync_average)

    # Asynchronous: mean of each row
    async_average = np.mean(np.mean(sample, axis=1))
    async_runtime.append(async_average)

# Plotting
plt.figure(figsize=(10, 6))
plt.plot(m_values, sync_runtime, label='Synchronous SGD', marker='o')
plt.plot(m_values, async_runtime, label='Asynchronous SGD', marker='s')
plt.xlabel('Number of Worker Nodes (m)')
plt.ylabel('Average Runtime per Iteration')
plt.title('Synchronous vs Asynchronous SGD Runtimes')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.savefig("sgd_runtimes_plot.pdf") # Save for submission
plt.show()
```

Output



Trend Explained

- The Synchronous SGD takes longer as m the number of workers increases because it waits for the slowest worker this leads to the max computation time grow with number of workers,
- The Assynchoronus SGD stabilizes with more workers this is beacuse the average of exponentially distributed varaibles converges quickly it becomes more efficeint as the number of workers increases.

- (e) **Theoretical Expressions for Expected Runtimes:** Write down the theoretical expressions for the expected runtimes per iteration of synchronous and asynchronous SGD in terms of m and λ . (Hint: You can use the expressions derived in parts (b) and (c)). On the figure generated in part (d), also plot the theoretical expected runtimes versus m . Check whether the theoretical and simulated values align.

Solution

Theoretical Expression for Synchronous

From part (b)

$$\mathbb{E}[X_{m:m}] = \frac{1}{\lambda} \sum_{i=1}^m \frac{1}{i}$$

Theoretical Expression for Assynchoronus

Sincew the average of i.i.d exponential random variable with rate λ has

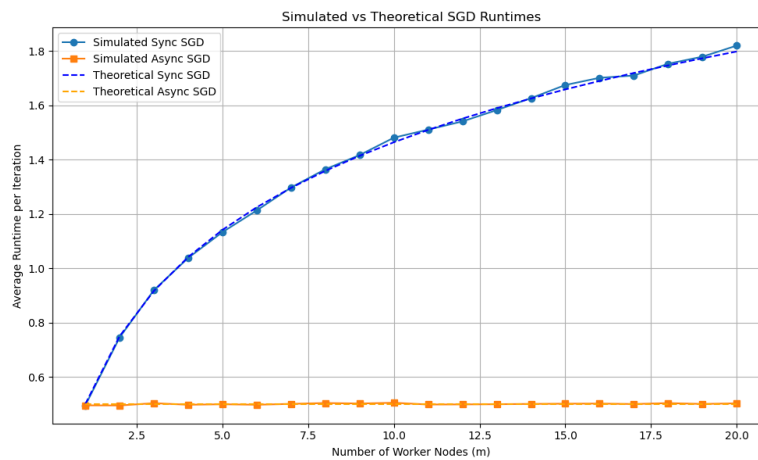
the same mean therefore:

$$\mathbb{E}[X_{1:m}] = \frac{1}{m\lambda}$$

```
# Harmonic numbers for synchronous SGD
harmonic_numbers = np.array([np.sum(1 / np.arange(1, m + 1)) for m in m_values])
expected_sync = (1 / lambda_rate) * harmonic_numbers

# Expected runtime for asynchronous SGD is constant
expected_async = np.full_like(m_values, 1 / lambda_rate, dtype=np.float64)

# Plot: Simulated vs Theoretical runtimes
plt.figure(figsize=(10, 6))
plt.plot(m_values, sync_runtime, label='Simulated Sync SGD', marker='o')
plt.plot(m_values, async_runtime, label='Simulated Async SGD', marker='s')
plt.plot(m_values, expected_sync, label='Theoretical Sync SGD', linestyle='--', color='blue')
plt.plot(m_values, expected_async, label='Theoretical Async SGD', linestyle='--', color='orange')
plt.xlabel('Number of Worker Nodes (m)')
plt.ylabel('Average Runtime per Iteration')
plt.title('Simulated vs Theoretical SGD Runtimes')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```



Outcome Explained

The theoretical curves follows the simulated curves.

2 K-means

K-means clustering is an unsupervised learning algorithm that partitions a set of N datapoints to K clusters with an aim of minimizing the within-cluster variance or distortion by assigning each data point to a cluster so that the sum of the squared distance between data points and their corresponding clusters is minimized.

Given a set of data points $\{x_n\}_{n=1}^N$, k-means clustering minimizes the following distortion measure (also called the “objective” or “clustering cost”):

$$D = \sum_{n=1}^N \sum_{k=1}^K r_{nk} \|x_n - \mu_k\|_2^2$$

where μ_k is the prototype of the k -th cluster and r_{nk} is a binary indicator variable. If x_n is assigned to the cluster k , r_{nk} is 1, and otherwise r_{nk} is 0. For each cluster, μ_k is the prototype representative for all the data points assigned to that cluster.

- (a) [10 points] In lecture, we stated but did not prove that μ_k is the mean of all points associated with the k -th cluster, thus motivating the name of the algorithm. You will now prove this statement. Assuming all r_{nk} are known (i.e., assuming you know the cluster assignments of all N data points), show that the objective D is minimized when each μ_k is chosen as the mean of all data points assigned to cluster k , for any k . This justifies the iterative procedure of k-means.

Solution

To minimize the objective D , we differentiate it with respect to μ_k and set the derivative to zero. The objective D is given by:

$$D = \sum_{n=1}^N \sum_{k=1}^K r_{nk} \|x_n - \mu_k\|_2^2$$

Expanding the squared norm:

$$\|x_n - \mu_k\|_2^2 = (x_n - \mu_k)^\top (x_n - \mu_k) = x_n^\top x_n - 2x_n^\top \mu_k + \mu_k^\top \mu_k$$

Substituting this into D :

$$D = \sum_{n=1}^N \sum_{k=1}^K r_{nk} (x_n^\top x_n - 2x_n^\top \mu_k + \mu_k^\top \mu_k)$$

Rearranging terms:

$$D = \sum_{n=1}^N \sum_{k=1}^K r_{nk} x_n^\top x_n - 2 \sum_{n=1}^N \sum_{k=1}^K r_{nk} x_n^\top \mu_k + \sum_{n=1}^N \sum_{k=1}^K r_{nk} \mu_k^\top \mu_k$$

The first term does not depend on μ_k , so we can ignore it when minimizing D . The remaining terms are:

$$D = -2 \sum_{n=1}^N \sum_{k=1}^K r_{nk} x_n^\top \mu_k + \sum_{n=1}^N \sum_{k=1}^K r_{nk} \mu_k^\top \mu_k$$

Now, differentiate D with respect to μ_k :

$$\frac{\partial D}{\partial \mu_k} = -2 \sum_{n=1}^N r_{nk} x_n + 2 \mu_k \sum_{n=1}^N r_{nk}$$

Set $\frac{\partial D}{\partial \mu_k} = 0$ to find the optimal μ_k :

$$-2 \sum_{n=1}^N r_{nk} x_n + 2 \mu_k \sum_{n=1}^N r_{nk} = 0$$

Simplify:

$$\mu_k \sum_{n=1}^N r_{nk} = \sum_{n=1}^N r_{nk} x_n$$

Solve for μ_k :

$$\mu_k = \frac{\sum_{n=1}^N r_{nk} x_n}{\sum_{n=1}^N r_{nk}}$$

This shows that μ_k is the mean of all points assigned to cluster k , as required.

- (b) [10 points] As discussed in lecture, sometimes we wish to scale each feature in order to ensure that “larger” features do not dominate the clustering. Suppose that each data point x_n is a d -dimensional feature vector and that we scale the j -th feature by a factor $w_j > 0$. Letting W denote a $d \times d$ diagonal matrix with the j -th diagonal entry being w_j , $j = 1, 2, \dots, d$, we can write our transformed features as $x' = Wx$.

Suppose we fix the r_{nk} , i.e., we take the assignment of data points x_n to clusters k as given. Our goal is then to find the cluster centers μ_k that minimize the distortion measure:

$$D = \sum_{n=1}^N \sum_{k=1}^K r_{nk} \|Wx_n - \mu_k\|_2^2.$$

Show that the cluster centers $\{\mu_k\}$ that do so are given by:

$$\mu_k = \left(\sum_{n=1}^N r_{nk} \right)^{-1} W \sum_{n=1}^N r_{nk} x_n.$$

Solution

To minimize the distortion D , we differentiate it with respect to μ_k and set the derivative to zero. The distortion D is given by:

$$D = \sum_{n=1}^N \sum_{k=1}^K r_{nk} \|Wx_n - \mu_k\|_2^2$$

Expanding the squared norm:

$$\|Wx_n - \mu_k\|_2^2 = (Wx_n - \mu_k)^\top (Wx_n - \mu_k) = x_n^\top W^\top Wx_n - 2x_n^\top W^\top \mu_k + \mu_k^\top \mu_k$$

Substituting this into D :

$$D = \sum_{n=1}^N \sum_{k=1}^K r_{nk} (x_n^\top W^\top Wx_n - 2x_n^\top W^\top \mu_k + \mu_k^\top \mu_k)$$

Rearranging terms:

$$D = \sum_{n=1}^N \sum_{k=1}^K r_{nk} x_n^\top W^\top Wx_n - 2 \sum_{n=1}^N \sum_{k=1}^K r_{nk} x_n^\top W^\top \mu_k + \sum_{n=1}^N \sum_{k=1}^K r_{nk} \mu_k^\top \mu_k$$

The first term does not depend on μ_k , so we can ignore it when minimizing D . The remaining terms are:

$$D = -2 \sum_{n=1}^N \sum_{k=1}^K r_{nk} x_n^\top W^\top \mu_k + \sum_{n=1}^N \sum_{k=1}^K r_{nk} \mu_k^\top \mu_k$$

Now, differentiate D with respect to μ_k :

$$\frac{\partial D}{\partial \mu_k} = -2 \sum_{n=1}^N r_{nk} W x_n + 2\mu_k \sum_{n=1}^N r_{nk}$$

Set $\frac{\partial D}{\partial \mu_k} = 0$ to find the optimal μ_k :

$$-2 \sum_{n=1}^N r_{nk} W x_n + 2\mu_k \sum_{n=1}^N r_{nk} = 0$$

Simplify:

$$\mu_k \sum_{n=1}^N r_{nk} = \sum_{n=1}^N r_{nk} W x_n$$

Solve for μ_k :

$$\mu_k = \left(\sum_{n=1}^N r_{nk} \right)^{-1} \sum_{n=1}^N r_{nk} W x_n$$

Thus, the cluster centers μ_k are given by:

$$\mu_k = \frac{\sum_{n=1}^N r_{nk} W x_n}{\sum_{n=1}^N r_{nk}}$$

3 3-Dimensional Principal Component Analysis

In this problem, we will perform PCA on 3-dimensional data step by step. We are given three data points:

$$x_1 = [0, -1, -2], \quad x_2 = [1, 1, 1], \quad x_3 = [2, 0, 1],$$

and we want to find 2 principal components of the given data.

(a) [8 points] First, find the covariance matrix $C_X = X^T X$ where:

$$X = \begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ x_3 - \bar{x} \end{bmatrix},$$

and $\bar{x} = \frac{1}{3}(x_1 + x_2 + x_3)$ is the mean of the data samples. Then, find the eigenvalues and the corresponding eigenvectors of C_X . Feel free to use any numerical analysis program such as `numpy`, e.g., `numpy.linalg.eig` can be useful. However, you should explain what you inputted into this program.

Solution

First, calculate the mean of the data samples:

$$\bar{x} = \frac{1}{3}(x_1 + x_2 + x_3) = \frac{1}{3}([0, -1, -2] + [1, 1, 1] + [2, 0, 1]) = [1, 0, 0].$$

Next, subtract the mean from each data point to form the matrix X :

$$X = \begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ x_3 - \bar{x} \end{bmatrix} = \begin{bmatrix} [0, -1, -2] - [1, 0, 0] \\ [1, 1, 1] - [1, 0, 0] \\ [2, 0, 1] - [1, 0, 0] \end{bmatrix} = \begin{bmatrix} -1 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The covariance matrix C_X is given by:

$$C_X = X^T X.$$

Compute X^T :

$$X^T = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}.$$

Now, calculate C_X :

$$C_X = X^\top X = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Matrix multiplication:

$$C_X = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}.$$

Found the Eigenvalues and eigenvectors:

```
import numpy as np

Cx = np.array([
    [2, 1, 3],
    [1, 2, 3],
    [3, 3, 6]
])

eigvals, eigvecs = np.linalg.eig(Cx)
print("EigenValues:", eigvals)
print("Eigen Vectors", eigvecs)
```

Eigenvalues:

$$\lambda_1 = 9.000, \quad \lambda_2 = 1.000, \quad \lambda_3 = 1.054 \times 10^{-16}.$$

Eigenvectors (corresponding to $\lambda_1, \lambda_2, \lambda_3$):

$$u_1 = \begin{bmatrix} -0.408 \\ -0.408 \\ -0.816 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -0.707 \\ 0.707 \\ 0.000 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -0.577 \\ -0.577 \\ 0.577 \end{bmatrix}.$$

- (b) [4 points] Using the result above, find the first two principal components of the given data.

Solution

From part a above, to find the first top 2 principal component we select the top 2 eigenvectors corresponding to the top eigenvalues.

$$\lambda_1 = 9.000, \quad \lambda_2 = 1.000,$$

The eigenvectors corresponding to λ_1 and λ_2 are:

$$u_1 = \begin{bmatrix} -0.408 \\ -0.408 \\ -0.816 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -0.707 \\ 0.707 \\ 0.000 \end{bmatrix}.$$

Therefore, the first two principal components are:

$$u_1 = \begin{bmatrix} -0.408 \\ -0.408 \\ -0.816 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -0.707 \\ 0.707 \\ 0.000 \end{bmatrix}.$$

- (c) [8 points] Now we want to represent the data x_1, \dots, x_3 using a 2-dimensional subspace instead of a 3-dimensional one. PCA gives us the 2-D plane which minimizes the difference between the original data and the data projected to the 2-dimensional plane. In other words, x_i can be approximated as:

$$\hat{x}_i = a_{i1}u_1 + a_{i2}u_2 + \bar{x},$$

where u_1 and u_2 are the principal components we found in 3.(b). Figure 1 gives an example of what this might look like.

Find a_{i1}, a_{i2} for $i = 1, 2, 3$. Then, find the \hat{x}_i 's and the difference between \hat{x}_i and x_i , i.e., $\|\hat{x}_i - x_i\|_2$ for $i = 1, 2, 3$. (Again, feel free to use any numerical analysis program to get the final answer, but show your calculation process.)

Solution

Have defined the calculation steps in the comments.

```
import numpy as np

# Original data
data = np.array([
    [0, -1, -2],
    [1, 1, 1],
    [2, 0, 1]
])

# Compute the mean
mean_vector = np.mean(data, axis=0)

# Define the Center the data
X_centered = data - mean_vector

# Given covariance matrix
```

```

Cx = np.array([
    [2, 1, 3],
    [1, 2, 3],
    [3, 3, 6]
])

# Eigen decomposition
eigvals, eigvecs = np.linalg.eig(Cx)
print("Eigenvalues:", eigvals)
print("Eigenvectors:\n", eigvecs)

# First 2 principal components (eigenvectors)
U = eigvecs[:, :2] # Take first two columns (PC1 and PC2)

# Project centered data onto 2D principal subspace
A = X_centered @ U

# Reconstructing the data using the top 2 PCs
X_reconstructed = A @ U.T + mean_vector

# Computing reconstruction errors
errors = np.sum((data - X_reconstructed)**2, axis=1)

# Display results
print("\nProjection coefficients (a_i):\n", A)
print("\nReconstructed data (x_i_hat):\n", X_reconstructed)
print("\nReconstruction errors ||x_i - x_i_hat||^2:\n", errors)

```

Output: Projection coefficients (a_i):

$$\begin{bmatrix} 2.44948974 \times 10^0 & 2.62480955 \times 10^{-17} \\ -1.22474487 \times 10^0 & 7.07106781 \times 10^{-1} \\ -1.22474487 \times 10^0 & -7.07106781 \times 10^{-1} \end{bmatrix}$$

Reconstructed data (\hat{x}_i):

$$\begin{bmatrix} 0.00000000 \times 10^0 & -1.00000000 \times 10^0 & -2.00000000 \times 10^0 \\ 1.00000000 \times 10^0 & 1.00000000 \times 10^0 & 1.00000000 \times 10^0 \\ 2.00000000 \times 10^0 & 1.66533454 \times 10^{-16} & 1.00000000 \times 10^0 \end{bmatrix}$$

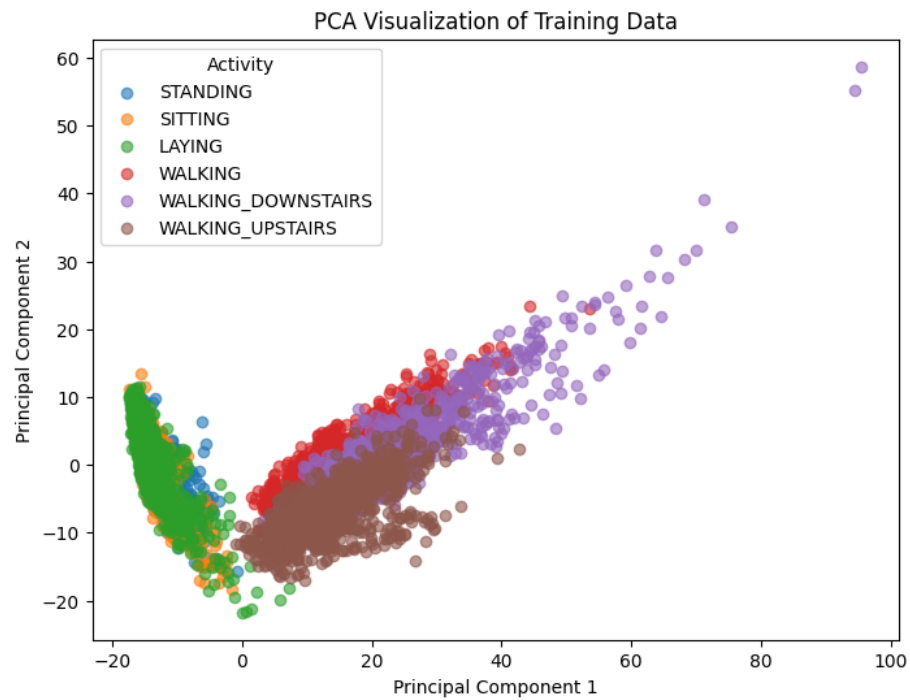
Reconstruction errors $\|\hat{x}_i - x_i\|_2^2$:

$$\begin{bmatrix} 1.97215226 \times 10^{-31} \\ 2.58844985 \times 10^{-31} \\ 2.77333912 \times 10^{-32} \end{bmatrix}$$

4 Clustering Human Activity using Inertial Sensors Data

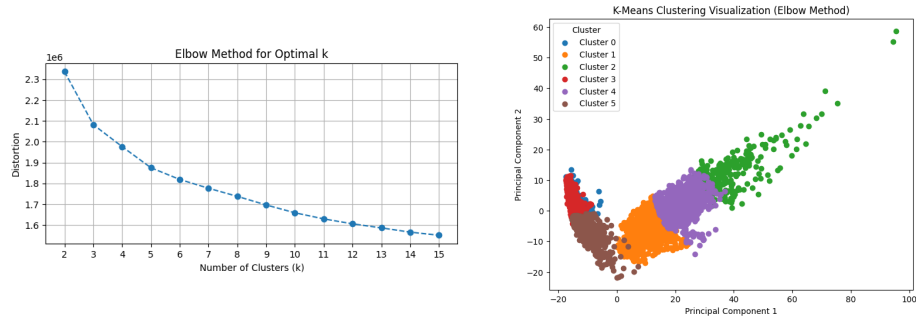
4.1 Import Data and Plotting

Visual of PCA wiht different colors with respective activity labels



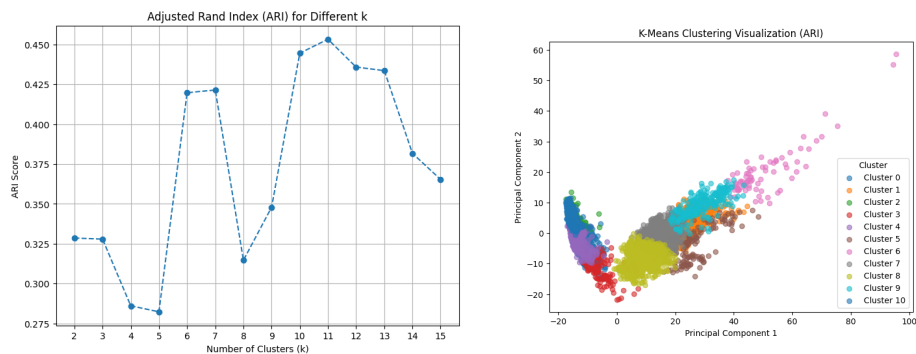
4.2 Choosing the optimal Number of Clusters

(a) Elbow Method



- The optimal number of cluster is 6.
- Distortion measures the sum of the squared distance between the data points and the assigned clusters this means an increase in the clusters leads to the decrease of the the distortion. This is mainly because increasing clusters allows the centroing to better fit the data which reduces the distance between points and their nearest centroid. However, as K becomes large the distortion the decreasing rate also diminishes which leads to the elbow point where adding more clusters provides the diminishing returns in reducing distortion.

(b) Adjusted Rand Index



- The optimal was 11.
- ARI measures the similarity between clustering results and the ground truth labels. As the number of clusters increase the following trend is observed.

- Plateau: Beyond the optimal K, the Ari score declines as the clusters become too granular, this leads reduction of alignment with the ground truth.
- Fluctuations: As K increases, the ARI fluctuates due to creation of more clusters that do not correspond well to the true data distributions or overfitting.

4.3 Prototype Selection using K-means clustering

- (a) **Random Selection** : Average Accuracy with Random Selection over 10 repetitions: 0.9223.
- (b) **Using K-means clustering by Class** : Accuracy with K-means Selection: 0.9057

Comparizon of the Random selection and the K-meands clusering by class model Accuracy

The Random prototype selection had a higher Accuracy of 92.23% compared to the K-means clustering that had 90.57%, this could be because Random selection relies on randomly selected prototypes that may capture diverse sample from each class while K-means clustering selects the prototype based on clusters that ensure that the selected samples are representative of the cluster centroids however, this may miss some outlier or diverse samples. And since the later focuess more on centroids rather than capturing the full variability of the data this may be the reason of the less performance of the models compared to the random selection model.