Copula methods in finance
——a way to capture the tail dependence

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1 Introduction

Over the past few decades, the dependency structure of assets and risk factors has been playing a critical rule in financial market. In trading strategy, a popular method named pair trading is basically built upon the estimated correlation among different financial products. In risk management, the new term correlation risk [1] is meant to identify the shift of risk exposures triggered by changes in correlation of risky assets. In portfolio management, a reliable estimate of the dependency of underlying assets helps investors do a risk arbitrage efficiently. To better investigate the dependency structure, the dependency functions or copula functions come into play.

Copula methods in finance can be briefly summarized as a bottom-up methodology which constructs the joint distribution of underlying assets by the marginal distributions of each asset which are easily to observe. Because of the flexibility of copula functions, this project will mainly focus on the application of copula in multivariate process which represents a collection of variables representing the value of each asset or risk factors.([1]) One of the advantages of using copula to model dependency rather than the vector autoregression(VAR) is that the we can extent copula model to other non-linear of non-Gaussian cases which are not available in the VAR. What's more, the dependency structure captured by copula method is more complete than that from the sample correlation coefficient which only reveals the strength of the linear dependence between the underlying random variables

The structure of this project as follows:

- Introduce the basic idea of copula and techniques used in parametric and nonparametric estimation.
- Estimate the tail dependency structure of S&P500 and DAX30 before and after the financial crisis in 2008 with nonparmatric method.
- Estimate the dynamic tail dependency structure of S&P500 and DAX30 in the period around the financial crisis in 2008 with different copulas such as Gaussian copula,Student t copula and Joe-Clayton copula.
- Compare the results

2 Methodology

2.1 Copula

Definition 2.1. *A n-dimensional copula is a function $C : [0,1]^n \to [0,1]$ satisfying the following properties:*

- *Grounded: for every* $\mathbf{u}$ *in* $[0;1]^n$, $C(u) = 0$ *if at least one coordinate* $u_j = 0, j = 1, ..., n$
- *Uniform marginals: If all coordinates of u are 1 except for some* $u_j, j = 1, ..., n$*, then* $C(\mathbf{u}) = u_j$.
- *n-Increasing: i.e. for every a and b in* $[0;1]^n$ *such that* $\mathbf{a} \leq \mathbf{b}$ *the C-volume* $V_C([\mathbf{a}, \mathbf{b}])$ *of the box* $[\mathbf{a}, \mathbf{b}]$ *is positive*

The usefulness of the copula benefits from the following theorem by Sklar(1959) [11] which reveals the link between the joint distribution and its margins.

Theorem 2.1 (Sklar's Theorem)**.** *Let F be an n-dimensional distribution function with margins* $F_1, ..., F_n$*. Then there exists an n-copula C such that for all* $\mathbf{y}$ *in* R^n*:*

$$\mathbf{F}(\mathbf{y}) = C(F_1(y_1), ..., F_n(y_n)) \tag{1}$$

If $F_1, ..., F_n$ *are all continuous, then C is uniquely determined. Otherwise, C is uniquely determined on* $rangeF1 \times ... \times rangeFn$*. Conversely, if C is an n-copula and* $F_1, ..., F_n$ *are distribution functions, then the function F determined by (1) is an n-dimensional distribution function with margins* $F, ..., Fn$*.*

An immediate corollary of Sklar's Theorem is:

$$C(u_1, ..., u_n) = \mathbf{F}(F_1^{-1}(u_1), ..., F_n^{-1}(u_n)) \tag{2}$$

where $F_1^{-1}, ..., F_n^{-1}$ are quasi inverses of $F_1, ..., F_n$, namely

$$F_i^{-1}(u_j) = inf\{y | F_j(u) \geq u_j\}$$

Note that when F_j is strictly increase, the quasi-inverse becomes the ordinary inverse.

The theorem implies that copulas are thus multivariate uniform distributions capturing the dependence structure of a set of random variables. The expression (2) provides us a way to rebuild the dependence structure by the knowledge of the joint distribution F and its margins F_j.

In the view of financial market, we can only observe marginal information of the market, for example, the stocking prices of each company. People prone to ignore some key factors when they model the financial assets by the top-down method. However, utilizing the power of copula model, we can construct the model by what we already known instead of making so many assumptions to apply a fancy model. That's one of the advantages of the copula model.

Another interesting feature of copula is the possibility to describe association between extreme events which are critical especially in financial market. As we we all know that the financial crisis in 2008 resulted from a series of extreme events which were not anticipated by almost all financial models. One of the main reasons is that most models are based on assumption of the normal distribution of the market which doesn't make sense when the debt is soaring.

2.2 Tail dependence

Utilizing the flexibility of copula model, we can avoid such mistake by monitoring the tail dependence of underlying assets. One way to measure the tail dependence is to calculate upper and lower tail indices which are defined as following. For simplicity, I will only consider the two-dimensional copula in most part of the project since the tail dependence mainly focus on the relationship between two random variables.

We first consider the conditional probability of observing an extreme price movement of an underlying asset B (assume the the low probability is p), given that there has been an extreme price movement in other asset A which has the same low probability p. By Bayes' theorem, we can express this conditional probablity as:

$$P(X_2 \leq F_2^{-1}(p)|X_1 \leq F_1^{-1}(p)) \tag{3}$$

where (X_1, X_2) is the random vector with marginal dfs F_1, F_2 corresponding to the asset A and B.

According to [1], we will take the limit of (3) with p approaching to 0. Then the lower tail index is defined as:

$$\lambda_L = \lim_{p \to 0^+} P(X_2 \leq F_2^{-1}(p)|X_1 \leq F_1^{-1}(p)) \tag{4}$$

which represents the association of simultaneous crashes in the markets.

Similarly, we can define the upper tail index as:

$$\lambda_U = \lim_{p \to 1^-} P(X_2 > F_2^{-1}(p)|X_1 > F_1^{-1}(p)) \tag{5}$$

To estimate the upper and lower tails indices, I will apply two methods and compare the results. One is non-parametric estimation of tail dependence.(Schmidt 2006 [10]). The other one is the parametric estimation of dynamic tail dependence.(Cortese 2019 [2],Patton 2006 [9]). Next,I will introduce the main idea in each method.

In literature, people may denote upper and lower tails indices as tail dependence coefficients(TDC). In the following context, I may use those two notations interchangeably.

2.3 Non-Parametric Estimation of Tail Dependence

For convenience, I will use the notation in Schimidt(2006) [10]. Suppose we have m two-dimensional random vectors indexed as $(X, Y)', (X^{(1)}, Y^{(1)})', ...(X^{(m)}, Y^{(m)})'$, which are iid with multivariate distribution function **F**. The corresponding marginal distribution functions are G and H which are assumed to be strictly increasing and continuous. By Sklar's theorem, we know that the corresponding copula C is unique.

Let C_m denote the empirical copula defined by

$$C_m(u, v) = \mathbf{F}_m(G_m^{-1}(u), H_m^{-1}(v)), (u, v)' \in [0, 1]^2 \tag{6}$$

where F_m, G_m, H_m are the empirical distribution functions corresponding to F,G,H.

Analogously, Schimidt defined the "empirical survival copula" by $\bar{C}_m(u, v) = \bar{\mathbf{F}}_m(\bar{G}_m^{-1}(u), \bar{H}_m^{-1}(v)), (u, v)' \in [0, 1]^2$ with

$$\bar{\mathbf{F}}_m(x, y) = \frac{1}{m} \sum_{j=1}^{m} \mathbf{1}_{\{X^{(j)} > x, Y^{(j)} > y\}}$$

and $\bar{G}_m = 1 - G_m$, $\bar{H}_m = 1 - H_m$.

Next, let $R_{m1}^{(j)}$ and R_{m2}^{j} denote the rank(ascending order) of $X^{(j)}$ and $Y^{(j)}$, $j = 1, ..., m$ respectively. Then the non-parametric estimator of lower and upper tail indices are defined as :

$$\hat{\Lambda}_{L,m}(x,y) = \frac{m}{k} C_m(\frac{kx}{m}, \frac{ky}{m}) \approx \frac{1}{k} \sum_{j=1}^{m} \mathbf{1}_{\{R_{m1}^{(j)} \leq kx \quad and \quad R_{m2}^{(j)} \leq ky\}} \tag{7}$$

and

$$\hat{\Lambda}_{U,m}(x,y) = \frac{m}{k} \bar{C}_m(\frac{kx}{m}, \frac{ky}{m}) \approx \frac{1}{k} \sum_{j=1}^{m} \mathbf{1}_{\{R_{m1}^{(j)} > m-kx \quad and \quad R_{m2}^{(j)} > m-ky\}} \tag{8}$$

parameter $k \in \{1, 2, ..., m\}$ is chosen by the statistician or optimized in some sense.

Schimidt assumed that $k = k(m) \to \infty$ and $\frac{k}{m} \to 0$ as $m \to \infty$. $\hat{\Lambda}_{L,m}(x,y)$ and $\hat{\Lambda}_{U,m}(x,y)$ are referred to as "empirical tail copulae".

The optimal choice of parameter k is the most difficulty in practical problem, which is related to the usual variance-bias problem.

He then proposed the

$$\hat{\lambda}_{U,m} = \hat{\Lambda}_{U,m}(1,1), \hat{\lambda}_{L,m} = \hat{\Lambda}_{L,m}(1,1)$$

as non-parametric estimators for the upper tail index and lower tail index.

The consistency of empirical tail copulae(ETC) and tail dependence coefficients(TDC)($\hat{\lambda}_{U,m}, \hat{\lambda}_{L,m}$) has been established by Schimidt(2006) [10] as well.

Note that this method is static which means that it doesn't take the time variation into account. Even though there hasn't be any literature of the non-parametric estimation of TDCs in dynamic copula model, I will split the time period as two parts. The first part of data is before the financial crisis and the other part is the data after the financial crisis. I will do the nonparametric estimation for these two periods to get some insights.

2.4 Parametric Estimation of Tail Dependence

Unlike the non-parametric estimation of dynamic tail dependence, the parametric estimation has received lots of attention these years and plenty of interesting results have emerged.

Pattern(2006) [9] extended Sklar's theorem, allowing the dependence parameters to be conditional and time varying. Furthermore, Oh and Patton (2018) [8] improved the time-varying equation for copulas via the so-called generalized autoregressive score (GAS) model.Cortese(2019) [2] introduced an approach to modeling dependence between financial returns allowing for two time-varying structures. Frahm(2006) [4] introduced three ways in parametric estimation of TDCs. To give a brief summary of his introduction, TDCs λ will be written without the subscript L or U whenever we know that $\lambda_L = \lambda_U$. What's more, the subscript is dropped whenever we neither specifically refer to the upper nor to the lower TDC. Those notations are in Frahm(2006) [4].

2.4.1 Estimation using a specific distribution

Suppose that the distribution of underlying financial asset $F(\cdot;\theta)$ is known (where θ is the parameter of the distribution) and λ can be represented via a known function of λ, i.e., $\lambda = \lambda(\theta)$. Further assume that F allows for tail dependence. Then the parameter θ can be estimated by maximum-likelihood (ML), which gives the estimator $\hat{\lambda} = \lambda(\hat{\theta})$. Under the usual regularity conditions of

ML-theory, it follows that $\hat{\lambda}$ possesses the well-known consistency and asymptotic normality properties.

2.4.2 Estimation using a specific copula

Now we assume that the copula $C(\cdot;\theta)$ is known, then we need to estimate θ first. This can be performed in two steps. First, we transform the observations of random vectors $(X,Y)'$ via estimates of the marginal distribution functions G and H and fit the copula from the transformed data in a second step.

In most cases, G and H will be estimated by the empirical distribution function $\hat{G}_n$, $\hat{H}_n$ so that an incorrect specification of the margins could be avoid.

Of course, if $G(\cdot;\theta_G)$ and $H(\cdot;\theta_H)$ are assumed to be specific distributions and θ_G and θ_H do not analytically depend on the copula parameter θ, then they can be estimated via ML methods. The corresponding estimation procedure is called IFM (method of inference functions for margins) which consists of estimating θ_G and θ_H through ML method and, in a second step, estimating the parameter θ of the copula $C(\cdot;\theta)$ via ML as well.

2.4.3 Estimation within a class of copulas

Here I will introduce an important class of copulas called Archimedean copulas. According to Juri and W¨uthrich (2002) [5], the bivariate excess distribution of Archimedean copulas has an explicit limiting result.

Define

$$F_t(x) = \frac{C\{min(x,t),t\})}{C(t,t)}, \quad 0 \leq x \leq 1$$

, where $0 < t < 1$ is a low threshold.

And define the "copula of small values" by

$$C_t(u,v) = \frac{C\{F_t^{-1}(t), F_t^{-1}(v)\}}{C(t,t)} \tag{9}$$

,where F_t^{-1} is the generalized inverse of F_t.

Under certain regular conditions, it can be shown that

$$\lim_{t\to 0^+} C_t(u,v) = C_{Cl}(u,v;\alpha)$$

for every $0 \leq u,v \leq 1$,where

$$C_{Cl}(u,v;\alpha) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$$

is the Clayton copula with parameter α.

We can verify that $\lambda_L = \lambda_U = 2^{-1/\alpha}$. Therefore the lower TDC can be estimated by fitting the Clayton copula and set $\hat{\lambda}_L = 2^{-1/\hat{\alpha}}$

3 Examples of copula [6]

For simplicity and comparison, I will introduce four different distribution which are denoted by G,T, C and GH. G represents the bivariate symmetric gaussian distribution with $\mu = (1,1), \rho = 0.5$. T refers to the bivariate t-distribution with $\nu = 1.5$ degrees of freedom and $\rho = 0.5$.

C is the bivariate Clayton copular :

$$C_{Cl}(u, v; \delta) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$$

whose generator function is

$$\phi_C(t) = (1+t)^{-1/\delta}$$

Finally, the GH represents the distribution Gumbel-Hougaard copula:

$$F_{GH}(x, y) = C_{AG}\{\phi_G(x), \phi_G(y); \delta\},$$

where

$$C_{GH}(u, v; \delta) = exp\bigg(- \Big[\{-ln(u)\}^{\delta} + \{-ln(v)\}^{\delta} \Big]^{1/\delta} \bigg)$$

and the generator function is

$$\phi_G(t) = exp(-t^{1/\delta})$$

For Archimedean copulas with differentiable generator function ϕ, the coefficients of tail dependence can be expressed in terms of the generator function $\phi(t)$ via

$$\lambda_L = \lim_{t\to\infty} \frac{\phi(2t)}{\phi(t)} = 2 \lim_{t\to\infty} \frac{\phi'(2t)}{\phi'(t)}, \lambda_U = 2 - \lim_{t\to 0^+} \frac{1-\phi(2t)}{1-\phi(t)} = 2 - 2 \lim_{t\to 0^+} \frac{\phi'(2t)}{\phi'(t)}$$

By this way, we can now verify that C_{Cl} has zero upper tail dependence but nonzero lower dependence, i.e. $\lambda_U^A = 0, \lambda_L^A = 2^{1/\delta}$. The TDCs for Gumbel-Hougaard copula would be $\lambda_L^{GH} = 0, \lambda_U^{GH} = 2 - 2^{1/\delta}$. As to the TDCs for t-distribution, it is kind of complicated but has closed form as following:

$$\lambda_L^T = \lambda_L^T = 2t_{\nu+1}\bigg(- \sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}} \bigg) \tag{10}$$

The following graphs display the TDC of t-copula as a function of ρ(left) and ν(right):

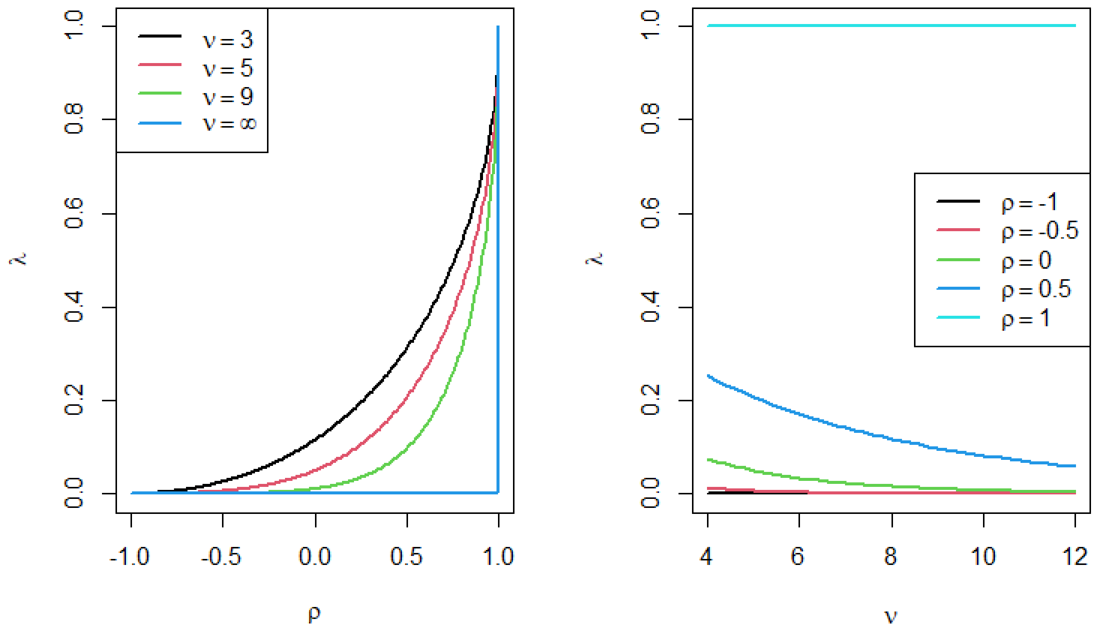

Figure 1: Coefficient of tail dependence w.r.t t-copula case

As we can see in the graphs above, for fixed ν, the tail dependence increases as ρ increases. For fixed ρ,the tail dependence decreases as ν increases. Such performance of the tail dependence is very different than that of the gaussian copula which have independent tail dependency, which is usually violated by underlying assets in financial market.

For comparison, the tail dependence coefficients of gaussian copula are actually all zero, which implies that gaussian copula is tail independent. We can derive this result as following:

Since the gaussian copula is radial symmetric, we only need to calculate the lower tail dependence λ_L^G.

According to the definition of lower tail dependence, for the random vector $(U_1, U_2)$with continuous marginal dfs and copula C,we have

$$\begin{aligned}
\lambda_L^G = \lambda_L^G(U_1, U_2) = \lim_{p \to 0^+} P(U_2 \leq F_G^{-1}(p) | U_1 \leq F_G^{-1}(p)) \\
= \lim_{p \to 0^+} \frac{P(U_1 \leq F_G^{-1}(p), U_2 \leq F_G^{-1}(p))}{P(U_1 \leq F_G^{-1}(p))} \\
= \lim_{p \to 0^+} \frac{P(F_G(U_1) \leq p, F_G(U_2) \leq p)}{P(F_G(U_1) \leq p)} \\
= \lim_{p \to 0^+} \frac{C_G(p, p)}{p}
\end{aligned} \tag{11}$$

where $C_G(u_1, u_2; \rho) = \Phi_\rho(\Phi^{-1}(u_1), \Phi^{-1}(u_2))$ and Φ_ρ is a joint distribution of a multi-dimensional standard normal distribution, with linear correlation coefficient ρ, and Φ being the standard normal distribution function. $F_G(\cdot)$ is the cumulative distribution function of the standard normal distribution.

By l'Hopital's rule, (10) can be further derived as:

$$\begin{aligned}
\lambda_L^G &= \lim_{p \to 0^+} \frac{C_G(p,p)}{p} \\
&= \lim_{p \to 0^+} \frac{d}{dp} C_G(p,p) = \lim_{p \to 0^+} \frac{d}{dp} \left(\int_{-\infty}^{p} \int_{-\infty}^{p} f(u_1,u_2) du_1 d_2 \right) \\
&= \lim_{p \to 0^+} \frac{d}{dp} \left(\int_{-\infty}^{p} du_1 \left[\int_{-\infty}^{p} f(u_1,u_2) du_2 \right] \right) \\
&= \lim_{p \to 0^+} \left(\int_{-\infty}^{p} f(p,u2) du_2 + \int_{-\infty}^{p} du_1 \frac{\partial}{\partial p} \left[\int_{-\infty}^{p} f(u_1,u_2) du_2 \right] \right) \\
&= \lim_{p \to 0^+} \left(\int_{-\infty}^{p} f(p,u2) du_2 + \int_{-\infty}^{p} du_1 \left[f(u_1,p) + \int_{-\infty}^{p} \frac{\partial}{\partial p} f(u_1,u_2) du_2 \right] \right) \\
&= \lim_{p \to 0^+} \left(\int_{-\infty}^{p} f(p,u2) du_2 + \int_{-\infty}^{p} f(u_1,p) du_1 \right) \\
&= 2 \lim_{p \to 0^+} \int_{-\infty}^{p} f(p,u2) du_2 = 2 \lim_{p \to 0^+} P(X_2 \le p | X_1 = p)
\end{aligned} \tag{12}$$

where $f(\cdot,\cdot)$ is the density of bivariate normal distribution of (X_1, X_2), which is symmetry w.r.t u_1, u_2 and leads to the last equality in (11). The main technique used is the Leibniz integral.

For the correlation coefficient being ρ, we know that $X_2|X_1 \sim N(\rho X_1, 1-\rho^2)$. It follows that

$$\begin{aligned}
\lambda_L^G &= 2 \lim_{p \to 0^+} P(X_2 \le p | X_1 = p) \\
&= 2 \lim_{p \to 0^+} P\left(\frac{X_2 - p\rho}{\sqrt{1-\rho^2}} \le \frac{p - p\rho}{\sqrt{1-\rho^2}} | X_1 = p\right) \\
&= 2 \lim_{p \to 0^+} \Phi\left(p \sqrt{\frac{1-\rho}{1+\rho}} \right) = 0
\end{aligned} \tag{13}$$

Therefore we can conclude that gaussian copula is tail independent which is very different with t-copula.

4 Empirical study ——Time varying tail dependence in financial market

Having observed such a big difference between t-copula and gaussian copula, it's natural for us to compare the time varying tail dependence in real data under such two copulas, which might give us some insights of the financial crisis in 2008. Criticism has been made w.r.t the model based on normal assumption after that crisis. If we can recognize certain signals from the time evolution patterns of tail dependence in financial market with t-copula(or other special one), that will be a good leading predictor for us to monitor the abnormality in the market.

4.1 Dataset

The choice of underlying assets and macroeconomics indicators is critical in financial analysis. This requires some professional sense. Furthermore, in order to make sure the size of dataset is appropriate for calculating the time varying TDCs, I will choose some high frequency variable. The time period of dataset would be 2006/1/1-2010/12/31 so that our study covers the most part of the life-cycle of 2008-2009 financial crisis.

Since the financial crisis in 2008 is a global crisis, I would like to select the exchange index in two different countries. One is the DAX30 which is a stock market index consisting of the 40 major German blue chip companies trading on the Frankfurt Stock Exchange. The other is the S&P 500 index.

4.2 Methodology——Nonparametric estimator of the tail dependence coefficient

The main idea has been explained in section 2.3. To get a more stable upper tail dependence, I will use the estimator proposed in [7]:

$$\hat{\lambda}_U(\frac{k}{n}) = \frac{1 - 2\frac{k}{n} + \hat{C}(\frac{k}{n}, \frac{k}{n})}{1 - \frac{k}{n}}$$

where k is a threshold depends on an appropriate selection of $k \in (1, n)$, n is the sample size and $\hat{C}$ is the empirical copula.

The estimator of lower tail dependence is the same with (7) in section 2.3

4.3 Methodology—-Dynamic copula model

Since most time series data in financial market are dependent, we cannot estimate the TDCs by the simply moving windows of data. One possible way to overcome this issue is by the theorem of conditional copula proposed by Patton(2006):

Theorem 4.1 (Conditional copula). *Let $F_{X|W}(\cdot|w)$ be the conditional distribution of $X|W = w$, $F_{Y|W}(\cdot|w)$ be the conditional distribution of $Y|W = w$, $F_{XY|W}(\cdot|w)$ be the joint conditional distribution of $(X,Y)|W = w$, and $\mathbf{W}$ be the support of W. Assume that $F_{X|W}(\cdot|w)$ and $F_{Y|W}(\cdot|w)$ are continuous in x and y for all $w \in \mathbf{W}$. Then there exists a unique conditional copula $C(\cdot|w)$ such that :*

$$F_{XY|W}(x,y|w) = C(F_{X|w}(x|w), F_{Y|W}(y|w)|w), \forall (x,y) \in \tilde{R} \times \tilde{R} \quad and \quad each \quad w \in \mathbf{W} \tag{14}$$

Conversely, if we let $F_{X|W}(\cdot|w)$ be the conditional distribution of $X|W = w$, $F_{Y|W}(\cdot|w)$ be the conditional distribution of $Y|W = w$, and $\{C(\cdot|w)\}$ be a family of conditional copula that is measurable in w, then the function $F_{XY|W}(\cdot|w)$ defined by equation (13) is a conditional bivariate distribution function with conditional marginal distribution $F_{X|W}(\cdot|w)$ and $F_{Y|W}(\cdot|w)$

Theorem 4.1 tells us that conditioning on a common variable W, there exists a unique copula connecting the conditional joint distributions and marginal distributions.

So we can model the conditional copula and calculate the TDCs by the parameters in the conditional copula. In time series data, we can typically condition on the past observation of underlying assets and this is where the word "dynamic" comes from. My strategy of dynamic copula model mainly follows the spirit of Patton(2006) [9] and Fantazzini(2006) [3]:

4.3.1 Estimation: The Inference for Margins method (IFM)

Under the time series case, we typically care about the conditional density/distribution function based on past observations $\mathcal{F}_{t-1}$.

Let the conditional joint distribution of two time series X_t and Y_t be $H(x, y|\mathcal{F}_{t-1}; \theta_h)$ where θ_h is the parameter of the joint distribution. $F_t(x|\mathcal{F}_{t-1}; \theta_f)$ is the conditional margin distribution of X_t and $G_t(y|\mathcal{F}_{t-1}; \theta_g)$ is the conditional margin distribution of Y_t. θ_f, θ_g are parameters of the conditional margin distribution of X_t, Y_t, respectively.

By (14), we can derive the joint density function $h(x, y|\mathcal{F}_{t-1}; \theta_h)$ as following :

$$\begin{aligned} h(x, y|\mathcal{F}_{t-1}; \theta_h) &= \frac{\partial^2 H(x, y|\mathcal{F}_{t-1}; \theta_h)}{\partial x \partial y} = \frac{\partial^2 C_t(F_t(x|\mathcal{F}_{t-1}), G_t(x|\mathcal{F}_{t-1})|\mathcal{F}_{t-1})}{\partial x \partial y} \\ &= \frac{\partial^2 C_t(F_t(x|\mathcal{F}_{t-1}), G_t(y|\mathcal{F}_{t-1})|\mathcal{F}_{t-1}}{\partial F_t(x|\mathcal{F}_{t-1}) \partial G_t(y|\mathcal{F}_{t-1}}) \frac{\partial F_t(x|\mathcal{F}_{t-1})}{\partial x} \frac{\partial G_t(x|\mathcal{F}_{t-1})}{\partial y} \\ &= c_t(u, v|\mathcal{F}_{t-1}) f_t(x|\mathcal{F}_{t-1}) g_t(y|\mathcal{F}_{t-1}) \end{aligned} \tag{15}$$

where f_t, g_t are the marginal density function of variable X, Y, c_t is the second order partial derivative of original copula C w.r.t u, v and $u = F_t(x|\mathcal{F}_{t-1}), v = G_t(x|\mathcal{F}_{t-1})$. Assume that the parameter of joint density function is separable w.r.t each component above, (15) can be written as

$$h(x, y|\mathcal{F}_{t-1}; \theta_h) = c_t(u, v|\mathcal{F}_{t-1}; \theta_c) f_t(x|\mathcal{F}_{t-1}; \theta_f) g_t(y|\mathcal{F}_{t-1}; \theta_g) \tag{16}$$

where θ_c is the parameter of the copula.

By IFM method mentioned in 2.4.2, the parameters can be estimated by two steps:

1.Estimating θ_f and θ_g of the marginal distributions F_t and G_t by maximizing the loglikelihood below:

$$\hat{\theta}_f = argmaxL(\theta_f) = argmax \sum_{t=1}^{T} log f_t(x_t; \theta_f)$$

$$\hat{\theta}_g = argmaxL(\theta_g) = argmax \sum_{t=1}^{T} log g_t(x_t; \theta_g)$$

since we assume that parameters of two marginal distributions are separable.

2.Given $\hat{\theta}_f, \hat{\theta}_g$ obtained in step1, we estimate θ_c by:

$$\hat{\theta}_c = argmaxL(\theta_c) = argmax \sum_{t=1}^{T} log[c_t(F_t(x_t; \hat{\theta}_f), G_t(y_t; \hat{\theta}_g); \theta_c)]$$

These two steps are illustrated in Fantazzini(2006) [3].

In dynamic sense, we can give an evolution equation for θ_c such as $\theta_{c,t} = \Lambda(\xi + \delta\epsilon_{t-1})$. I will give a more detailed example in next section.

4.3.2 Models for Marginal distributions

Based on the descriptive statistics above, there is a sign of excess of kurtosis in log returns of S&P500 and DAX30. For this reason, I will assume that their errors have a Student-t distribution. The choice of order depends on the observation of ACF and PACF plots.(Shown in slides. I put them into the Rmarkdown file)

Descriptive statistics of log returns		
	SP500	DAX30
# of obs	1740	1740
Mean	0	0
Median	0	0
Maximum	0.11	0.11
Minimum	-0.09	-0.08
Std. Dev	0.01	0.01
Skewness	-0.24	0.12
Kurtosis	10.92	8.47

Table 1: Descriptive statistics of log returns

Denote the log return of S&P500 as X_t and log return of DAX30 Y_t, then the marginal distribution of X_t is characterized by the ARMA(1,1),t-GARCH(1,1) structure :

$$X_t = \mu_x + \phi_{1x} X_{t-1} + \epsilon_t \tag{17}$$

$$\epsilon_t = \eta_{t,x} h_{t,x}, \quad \eta_{t,x} \sim iidF_t \tag{18}$$

$$h_{t,x}^2 = \omega_x + \alpha_x \epsilon_{t-1}^2 + \beta_x h_{t-1,x}^2 \tag{19}$$

The choice of order depends on the observation of ACF and PACF plots(Shown in slides. I will put them into the Rmarkdown file).

Similarly, the marginal distribution of log return of DAX30 is characterized by the tGARCH(1,1) structure:

$$Y_t = \mu_y + \delta_t \tag{20}$$

$$\delta_t = \eta_{t,y} h_{t,y}, \quad \eta_{t,y} \sim iidG_t \tag{21}$$

$$h_{t,y}^2 = \omega_y + \alpha_y \delta_{t-1}^2 + \beta_y h_{t-1,y}^2 \tag{22}$$

4.3.3 Copula models

After we obtain the estimate of parameters in stage 1(marginal distribution), we proceed to the stage 2 where we estimate the copula parameters. I will fit the Normal and T-copula to this asset pair.

$$\begin{aligned} \left(\frac{X_t - \mu_t^x}{\sqrt{h_t^x}}, \frac{Y_t - \mu_t^y}{\sqrt{h_t^y}} \right) &\sim C_{Gaussian-copula}(F_t(\eta_t^x), G_t(\eta_t^y); \rho_t | \mathbf{F}_{t-1}) \\ \left(\frac{X_t - \mu_t^x}{\sqrt{h_t^x}}, \frac{Y_t - \mu_t^y}{\sqrt{h_t^y}} \right) &\sim C_{T-copula}(F_t(\eta_t^x), G_t(\eta_t^y); \rho_t, \nu_t | \mathbf{F}_{t-1}) \end{aligned} \tag{23}$$

where (ρ_t, ν_t) are the correlation and conditional degrees of freedom conditioning on the previous information $\mathbf{F}_{t-1}$, respectively. $\mu_t^x, h_t^x, \mu_t^y, h_t^y$ are the conditional means and variances for time series X_t and Y_t. $\{F_t, G_t\}$ can be Normal / Student's T cumulative distribution function. (Fantazzini(2006) [3]) After we obtain the standard residual, we should transform the margin of residuals into the uniform distribution. In R, we can do this by "pobs" function in the "copula" package.

According to Fantazzini(2006), the normal-copula(copula of bivariate Normal distribution) has the joint density function as following:

$$
\begin{aligned}
c(u_t, v_t; \rho_t | \mathcal{F}_{t-1}) = & \frac{1}{\sqrt{1-\rho_t^2}} \\
& \cdot exp\Big(-\frac{1}{2} \frac{[(\Phi^{-1}(u_t))^2 + (\Phi^{-1}(v_t))^2 - 2\rho_t \Phi^{-1}(u_t)\Phi^{-1}(v_t)]}{1-\rho_t^2} \Big) \\
& \cdot exp\Big(\frac{1}{2}[(\Phi^{-1}(u_t))^2 + (\Phi^{-1}(v_t))^2] \Big)
\end{aligned} \tag{24}
$$

where Φ^{-1} is the inverse function of standard normal cdf.

The T-copula (copula of bivariate Student's t-distribution) has the joint density function as following:

$$
\begin{aligned}
c(u_t, v_t; \rho_t, \nu_t | \mathcal{F}_{t-1}) = & \frac{\Gamma(\frac{\nu_t+2}{2})\Gamma(\frac{\nu_t}{2})}{\Gamma^2(\frac{\nu_t+1}{2})\sqrt{1-\rho_t^2}} \\
& \cdot \Big(1 + \Big[\frac{(t_{\nu_t}^{-1}(u_t))^2 + (t_{\nu_t}^{-1}(v_t))^2 - 2\rho_t t_{\nu_t}^{-1}(u_t) t_{\nu_t}^{-1}(v_t)}{(1+\rho_t^2)\nu_t} \Big] \Big)^{-\frac{\nu_t+2}{2}} \\
& \cdot \Big[\Big(1 + \frac{t_{\nu_t}^{-1}(u_t))^2}{\nu_t} \Big) \Big(1 + \frac{t_{\nu_t}^{-1}(v_t))^2}{\nu_t} \Big) \Big]^{\frac{\nu_t+1}{2}}
\end{aligned} \tag{25}
$$

where $t_{\nu_t}^{-1}$ is the inverse of the Student's t cdf. To capture the time variation in conditional copula, I will use the follow evolution equations to specify the dynamic copula

$$
\rho_t = \Lambda(\xi + \alpha\rho_{t-1} + \delta \cdot \frac{\sum_{j=1}^{15} |u_{t-j} - v_{t-j}|}{15}) \tag{26}
$$

$$
\nu_t = \Gamma(\theta + \beta\nu_{t-1} + \phi \cdot \frac{\sum_{j=1}^{15} |u_{t-j} - v_{t-j}|}{15}) \tag{27}
$$

where $\Lambda(x) = (1-e^{-x})/(1+e^{-x})$ is the modified logistic transformation which keeps the conditional correlation in $(-1, 1)$ all the time. And $\Gamma(x) = \frac{8}{1+e^{-x}} + 2$ is designed to keep the conditional degree of freedom in $(2, 10)$ all the time.

For (24),(25), we can of course use more complicated evolution equation (e.g including more autoregressive terms). For the project, I will just use the relative simple one. The mean absolute value term on right hand sides of (24,25) is a forcing variable proposed by Patton(2006) [9]. In this project, I propose using the MAD between u_t, v_t over the previous 10 or 15 observations. The initial value of ρ_t, ν_t are estimated by first 100 observations of series. Once we obtained the estimate of copula parameters, we can get the time varying tail dependence under t-copula by plugging the estimate into formula (10) directly.

For comparison, I will plot the estimated ρ_t when gaussian-copula is used (no TDCs will be given since gaussain copula has zero tail dependence) and the estimated TDCs when T-copula is used. To get the estimate of TDCs of T-copula, we can plug in the estimate of ρ_t and ν_t directly into the formula (1). Typically we can use maximum likelihood method to do the estimation.

Repeat the procedure above for other pairs of underlying assets, we can obtain several estimated TDCs and correlations. The results of parameter estimation are given in the following section.

4.4 Empirical results

Firstly, I will display the nonparametric estimation of the tail dependency coefficients by the method described in (2.3) for the log-return of S&P500 and DAX 30 . The estimation is based on the date before 2008/09/15 and after 2008/09/15 (when the global crisis began) and the 95% confidence intervals are given by the 1000 times of nonparametric bootstrap. The optimal threshold k is selected by the Plateau-finding algorithm(Garcin&Nicolas(2021) [7],Frahm(2006) [4]).

Nonparametric estimation of tail dependency before 2008/9/15			
	Estimate(std)	95% lower bound	95% upper bound
S&P500 vs.DAX30 (Before)	$\hat{\lambda}_L = 0.4250(0.0352)$	0.3548	0.4908
	$\hat{\lambda}_U = 0.2973(0.0536)$	0.2032	0.4107

Table 2: Nonparametric estimation of tail dependency before 2008/9/15

Nonparametric estimation of tail dependency after 2008/9/15			
	Estimate(std)	95% lower bound	95% upper bound
S&P500 vs.DAX30 (After)	$\hat{\lambda}_L = 0.6772(0.0419)$	0.5770	0.7452
	$\hat{\lambda}_U = 0.5455(0.0559)$	0.4338	0.6485

Table 3: Nonparametric estimation of tail dependency after 2008/9/15

4.4.1 Plot of S&P500 and DAX30

Figure 2 plots all five indicators on the same graph

We can see that there is a common peak of those two index during the 2008-2009 financial crisis. Because of the different magnitudes of those indicators, the plot of moving averaged S&P 500 index doesn't seem to have an obvious trend in figure 2. So we will plot this time series separately to get more insights.

Now it is clear that both price indices experienced a peak during the crisis. From those plots, we can expect that there will be a high tail dependency coefficients among those indicators.

4.4.2 Plot of time varying ρ when gaussian copula is used

Since the gaussian copula is tail independent, we can only plot the time varing correlation coefficients ρ_t. The evoluation equation is:

$$\rho_t = \Lambda(\xi + \alpha\rho_{t-1} + \delta \cdot \frac{\sum_{j=1}^{15} |u_{t-j} - v_{t-j}|}{15})$$

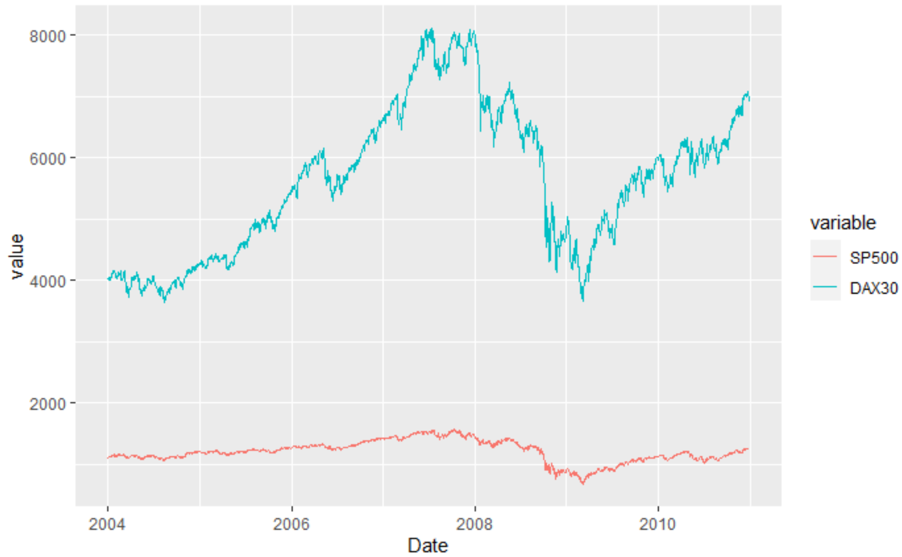

Figure 2: S&P500 and DAX30 series

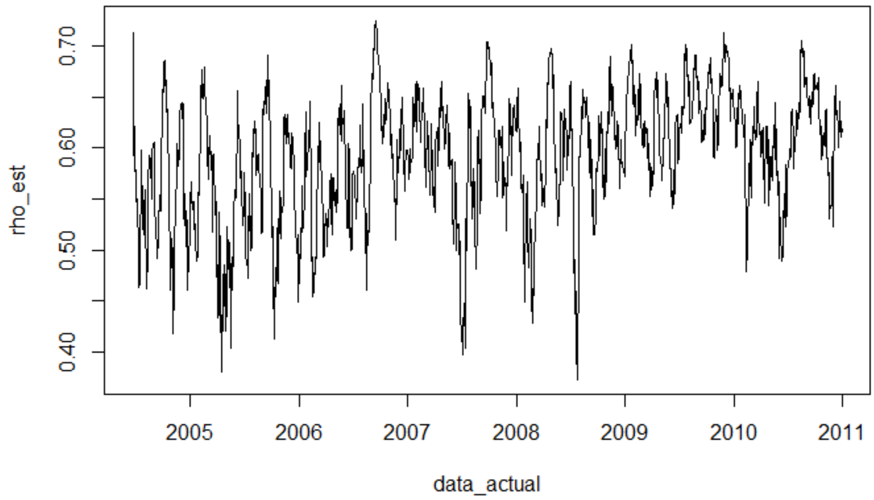

Figure 3: Plot of correlation coefficient between S&P500 and DAX300

4.4.3 Plot of time varying TDCs when different copulas are used

We have two parameters ρ_t, ν_tin t-copula and there is a closed form for upper and lower tail dependency (Formula (10)). Evolution equations are:

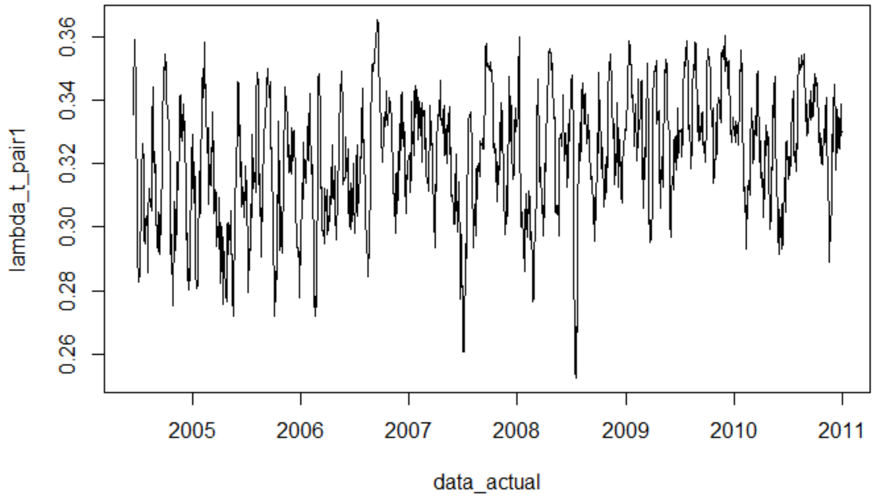

Figure 4: Plot of time varying TDCs when t-copula is used

$$\rho_t = \Lambda(\xi + \delta \cdot \frac{\sum_{j=1}^{10} |u_{t-j} - v_{t-j}|}{10})$$

$$\nu_t = \Gamma(\theta + \phi \cdot \frac{\sum_{j=1}^{10} |u_{t-j} - v_{t-j}|}{10})$$

Note that I don't include the autoregrresive term here since the estimation of standard deviations will be unstable when I put them into the optimization procedure. The forcing variable only depends on the previous 10 observations.

Since the upper than lower tail dependence are the same for t-copula which might not be true in general in financial market, I will try another copula called Joe-Clayton copula who has asymmetric tail dependence: (copula density can be found by the symbolic derivative in R or Maple)

In figure 5, we can see that S&P500 and DAX30 have relatively larger lower tail dependence at many points during the whole period. Before the crisis began, there are few dates whose lower tail dependence is even around 0.6, which might be a sign of the global crisis. Since the higher lower tail dependence implies that when S&P500 index drops a lot, it's highly likely that DAX30 also drops a lot. And we know it it true since when the financial market in US collapsed, the fear spread to the Europe market quickly. The upper TDCs are all around 0.4 which implies that the market does not have enough recovery emotion. So the market is more cautious to the positive signal than to the negative signal. In other words, when the upper TDC goes up, the lower TDC goes up even further, leading to a high risk. When the upper TDC goes down, the lower TDC won't go down too much. These results are also consistent with the nonparmatric estimates in table 2 and table 3. The Joe-Clayton copula is defined as:

$$C_{JC}(u, v; k, \gamma) = 1 - (1 - ((1 - (1 - u)^k)^{-\gamma} + (1 - (1 - v)^k)^{-\gamma} - 1)^{-1/\gamma})^{1/k}$$

where $k = 1/log_2(2 - \lambda^U)$ and $\gamma = -1/log_2(\lambda^L)$. Evolution equations are:

Figure 5: Plot of time varying TDCs when Joe-Clayton-copula is used

$$\lambda_t^U = \Lambda(\xi + \alpha\lambda_t^U + \delta \cdot \frac{\sum_{j=1}^{10} |u_{t-j} - v_{t-j}|}{10})$$

$$\lambda_t^L = \Lambda(\theta + \beta\lambda_t^L + \phi \cdot \frac{\sum_{j=1}^{10} |u_{t-j} - v_{t-j}|}{10})$$

Function $\Lambda(x)$ ensures that the time varying TDCs are in $(0, 1)$ all the time.

4.4.4 Fitted results

Fitted results and standard errors			
	Gaussian copula	T copula	Joe-Clayton copula
ξ (Intercept 1)	2.4486 (0.7678)	1.0568 (0.1413)	1.7140 (1.0345)
α (Autoregressive term 1)	-0.4793 (0.8984)	-	-2.0783 (1.5868)
δ (Forcing term)	-3.8637 (1.3874)	-1.4191 (0.6381)	-5.3341 (2.4004)
θ (Intercept)	-	-16.0594 (30.8271)	3.2397 (0.5201)
β (Autoregressive term 2)	-	-	-4.1304 (0.1042)
ϕ (Forcing term)	-	-3.7219 (151.7411)	-10.3035(2.5245)

Table 4: Fitted results and standard errors

From the table above, we can find that most estimates for lower tail dependency in Joe-Clayton copula are significant. The forcing term in upper tail dependency is significant as well. In T copula and gaussian copula, the estimates for ρ_t are mostly significant.

5 Conclusion

During this project, I found that it is relatively easy to do the nonparametric estimation of the tail dependence coefficients since doesn't include optimization process which is time consuming. The only turning parameter is the threshold k which can be quickly found by the plateau finding algorithm. Furthermore, we don't have to assume any copula of the two series which avoids the model misspecification problem. As what we did in the parametric estimation, we need to specify a particular copula connecting two target series. Somehow we need a careful check of the dataset in order to have a reasonable model setting.

However, one big drawback of the nonparametric TDCs is the requirement of independent pair of random vector. In time series case, especially in financial market, this is not the case in general. That's why the dynamic copula model comes into play. Benifiting from the conditional copula theorem given by Patton(2006), we can introduce evolution equations for the copula parameters so that we can capture the time varying pattern of the TDCs.

Looking from the nonparametric estimation in two time periods, there is a huge difference before and after the financial crisis. Before the crisis, although both lower and upper tail dependence are relatively low , there is a sign of high risk of extreme event happening together ($\hat{\lambda}_L = 0.4250$) for DAX30 and S&P500. High lower TDC implies that there is a tendency of DAX30 dropping heavily when the price index of S&P500 decreases. After the crisis, all TDCs are higher than previous period which makes sense since when the crisis begin, the global markets are affected by the fear of losing money and default. When the fear spread, the market collapsed quickly. The increasing TDCs imply the higher uncertainty of the global financial market after the crisis.

The plots of time varying TDCs given by dynamic copula give us more information. But again, dynamic t-copula model will generate equal upper and lower tail dependence. That may not be true during the financial crisis since we can observe that the upper TDCs are all higher than the lower TDCs in nonparametric estimation. So I tried Joe-Clayton copula to avoid that issue. One advantage of the t-copula/Joe-Clayton-copula over the gaussian copula is that it at least gives us the information of tail dependency. This probably explains why most models failed to predict the crisis since they all used the normal distribution. In future, I may think of a better evolution equation to find the time varying pattern of the copula parameter. Dynamic nonparametric TDC is a potential direction as well.

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