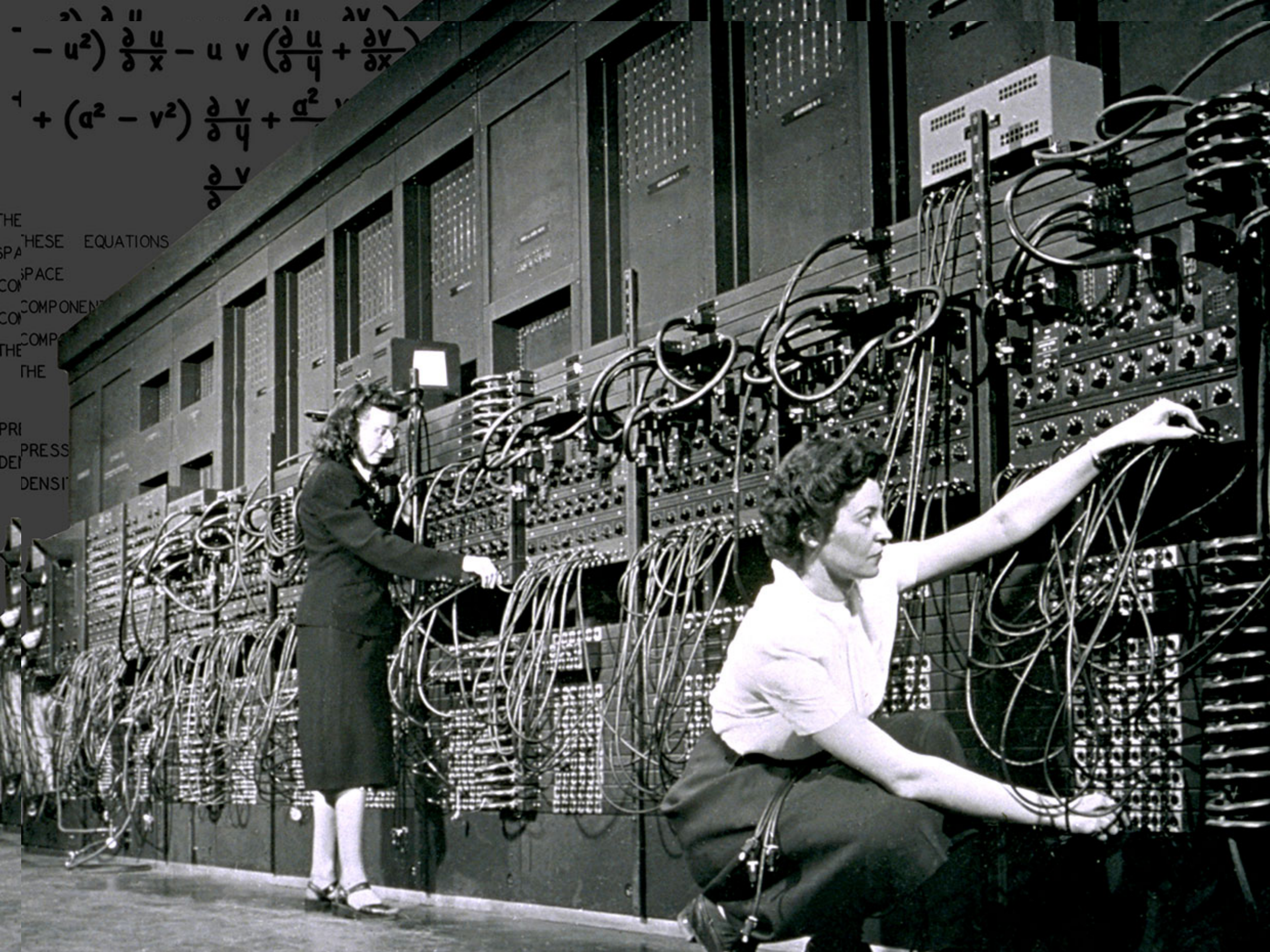
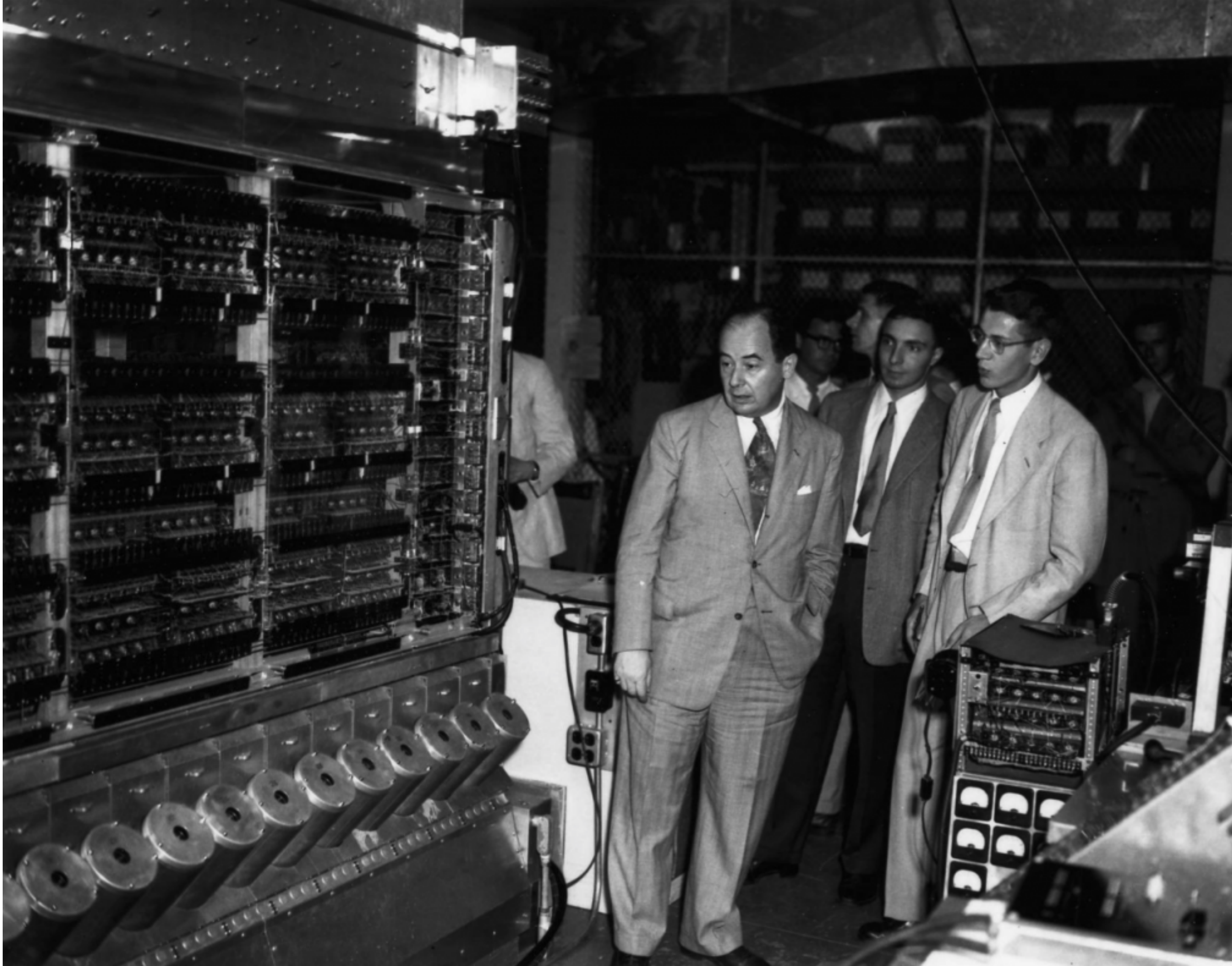
CSCI 1102 Computer Science 2

Meeting 15: Tuesday 3/23/2021
Order & Equality


John von Neumann


THESE EQUATIONS
SPACE
COMPONENT
THE
PRESS
DENSIT




Von Neumann w Richard Feynman & Stanislaw Ulam in Los Alamos

Order & Equality

The *Natural Order*

equals

compareTo

==

hashCode

compare

<=

null

mutation

new Comparator()

Sets

$$\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$$

Sets

$$A = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$$

$$B = \{\otimes, \oplus\}$$

Statements about Containment

$$\clubsuit \in \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$$

$$\otimes \notin \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$$

General Statements
Variables and their Quantifiers

x, y, z for variables (which *vary* over sets)

$$\forall x \in A \,.\, \textit{statement}$$

$$\exists x \in A \,.\, \textit{statement}$$

$$\forall x \in \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.\ x \textbf{ is a card suit.}$$

$\clubsuit$ is a card suit *and*

$\diamondsuit$ is a card suit *and*

$\heartsuit$ is a card suit *and*

$\spadesuit$ is a card suit.

$$\forall x \in \{\otimes, \oplus\}.\ x \textbf{ is a card suit.}$$

$\otimes$ is a card suit *and*

$\oplus$ is a card suit.

Statements can be true or false.

$$\exists x \in \{\otimes, \oplus\}.\ x \text{ is a card suit.}$$

$\otimes$ is a card suit *or*

$\oplus$ is a card suit.

Statements can be true or false.

Set Comprehensions

- { x | *statement* } means set of all x such that *statement* is true;

  Example

  $\mathbb{N} = \{\ k\ \mid k = 0 \text{ or } \exists k' \in \mathbb{N} \text{ such that } k = k' + 1\}$

  Evens = $\{x \mid x \in \mathbb{N} \text{ and } \exists y \in \mathbb{N} \text{ such that } x = 2y\}$

Tuples

$(x_1, \ldots, x_k)$ k-tuples.

Subset & Set Equality

$A \subseteq B$ means $\forall x \in A.\, x \in B$;

$A = B$ means $A \subseteq B$ and $B \subseteq A$.

Building Sets

- *Union* : $A_1 \cup \ldots \cup A_n = \{a \mid a \in A_i \text{ for some } i \in \{1, \ldots, n\}\}$;
- *Intersection* : $A_1 \cap \ldots \cap A_n = \{a \mid a \in A_i \text{ for every } i \in \{1, \ldots, n\}\}$;
- *Disjoint Union* : $A_1 + \ldots + A_n = \{(i, a) \mid a \in A_i \text{ for some } i \in \{1, \ldots, n\}\}$;
- *Product* : $A_1 \times \ldots \times A_n = \{(a_1, \ldots, a_n) \mid a_i \in A_i\}$;
- *Sequences* : Let A be a set of symbols and let ϵ denote the empty sequence.

  $A^* = \{w \mid w = \epsilon \text{ or } w = aw' \text{ with } a \in A \text{ and } w' \in A^*\}$

  Example: $\{a, b\}^* = \{\epsilon, a\epsilon, b\epsilon, aa\epsilon, ab\epsilon, \ldots\}$

Relations

- $R \subseteq A_1 \times \ldots \times A_n$ is called an n-ary *relation* on sets $A_1, \ldots, A_n$.
- When R is an n-ary relation on sets $A_1, \ldots, A_n$ and $A_1 = \ldots = A_n$ we say that R is an n-ary relation on A_1;
- We're especially interested in binary relations where $n = 2$.
- DomDef(R) = $\{a \in A \mid \text{for some } b \in B, (a, b) \in R\}$

$$A = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\} \qquad B = \{\otimes, \oplus\}$$

$$\begin{aligned} R_1 = A \times B = \{ & \\ & (\clubsuit, \otimes), (\diamondsuit, \otimes), (\heartsuit, \otimes), (\spadesuit, \otimes), \\ & (\clubsuit, \oplus), (\diamondsuit, \oplus), (\diamondsuit, \oplus), (\spadesuit, \oplus) \\ & \} \end{aligned}$$

$$R_2 = \{\}.$$

$$R_3 = \{(\clubsuit, \otimes), (\spadesuit, \oplus)\}$$

$$R_4 = \{(\clubsuit, \otimes), (\diamondsuit, \otimes), (\heartsuit, \otimes), (\spadesuit, \otimes)\}$$

Partial Maps (aka Partial Functions)

Let R be a binary relation on A, B.

R is a *partial map* from A to B if and only iff

$$\forall a \in A.\, b, b' \in B.\, \text{if } (a, b) \in R \text{ and } (a, b') \in R \text{ then } b = b'.$$

$$
\begin{aligned}
R_1 = A \times B = \{ \\
&(\clubsuit, \otimes), (\diamondsuit, \otimes), (\heartsuit, \otimes), (\spadesuit, \otimes), \\
&(\clubsuit, \oplus), (\diamondsuit, \oplus), (\diamondsuit, \oplus), (\spadesuit, \oplus) \\
&\}
\end{aligned}
$$

$$R_2 = \{\}.$$

Partial maps from A to B?

$$R_3 = \{(\clubsuit, \otimes), (\spadesuit, \oplus)\}$$

$$R_4 = \{(\clubsuit, \otimes), (\diamondsuit, \otimes), (\heartsuit, \otimes), (\spadesuit, \otimes)\}$$

$$
\begin{aligned}
R_1 = A \times B = \{ & \\
& (\clubsuit, \otimes), (\diamondsuit, \otimes), (\heartsuit, \otimes), (\spadesuit, \otimes), \\
& (\clubsuit, \oplus), (\diamondsuit, \oplus), (\diamondsuit, \oplus), (\spadesuit, \oplus) \\
& \}
\end{aligned}
$$

$$R_2 = \{\}.$$

Only R_2, R_3 and R_4 are partial maps from A to B.

$$R_3 = \{(\clubsuit, \otimes), (\spadesuit, \oplus)\}$$

$$R_4 = \{(\clubsuit, \otimes), (\diamondsuit, \otimes), (\heartsuit, \otimes), (\spadesuit, \otimes)\}$$

Notation:

- We usually use f, g, h, ... for partial maps;
- We use Euler's notation $f(a)$ to denote the unique $b \in B$ such that $(a, b) \in f$ or the special "undefined" symbol $\perp$ if there is no such b.

Total Map

Let f be a partial map from A to B. Then f is a *total map* from A to B iff DomDef(f) = A. In the examples above, only R4 is a total map from A to B.

Preorders

Let R be a binary relation on A. I.e., $R \subseteq A \times A$.

- R is *reflexive* iff $\forall x \in A. \, (x, x) \in R$;
- R is *transitive* iff
  $\forall x, y, z \in A. \text{ If } (x, y) \in R \text{ and } (y, z) \in R \text{ then } (x, z) \in R.$

A binary relation that is both reflexive and transitive is called a *preorder*.

Preorders

$\{(\clubsuit, \clubsuit), (\diamondsuit, \diamondsuit), (\heartsuit, \heartsuit), (\spadesuit, \spadesuit)\}$

$\{(\clubsuit, \clubsuit), (\diamondsuit, \diamondsuit), (\heartsuit, \heartsuit), (\spadesuit, \spadesuit), (\clubsuit, \spadesuit)\}$

$\{(\clubsuit, \clubsuit), (\diamondsuit, \diamondsuit), (\heartsuit, \heartsuit), (\spadesuit, \spadesuit), (\clubsuit, \spadesuit), (\spadesuit, \heartsuit)\}$

Preorders

$\{(\clubsuit, \clubsuit), (\diamondsuit, \diamondsuit), (\heartsuit, \heartsuit), (\spadesuit, \spadesuit)\}$ preorder

$\{(\clubsuit, \clubsuit), (\diamondsuit, \diamondsuit), (\heartsuit, \heartsuit), (\spadesuit, \spadesuit), (\clubsuit, \spadesuit)\}$ preorder

$\{(\clubsuit, \clubsuit), (\diamondsuit, \diamondsuit), (\heartsuit, \heartsuit), (\spadesuit, \spadesuit), (\clubsuit, \spadesuit), (\spadesuit, \heartsuit)\}$ not a preorder

Equivalence Relations & Partial Orders

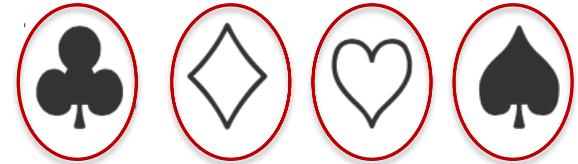
Let R be a binary relation on A.

- R is *symmetric* iff $\forall x, y \in A.\ \text{if } (x, y) \in R \text{ then } (y, x) \in R$;
- R is *antisymmetric* iff $\forall x, y \in A.\ \text{if } (x, y) \in R \text{ and } (y, x) \in R \text{ then } x = y$.

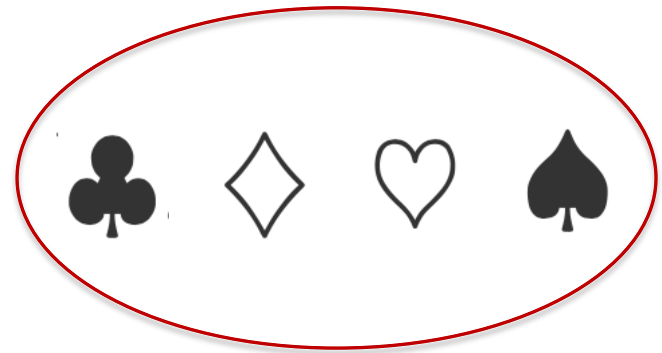
A symmetric preorder is called an *equivalence relation*.

An antisymmetric preorder is called a *partial order*.

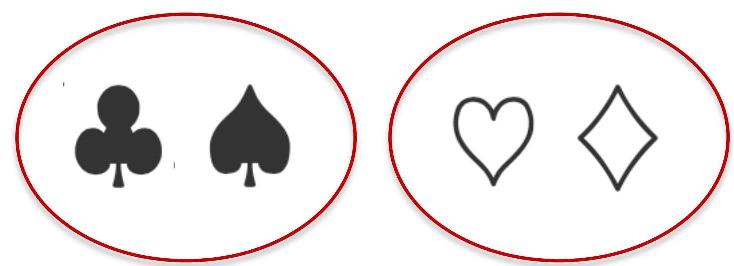
$$\{(\clubsuit,\clubsuit),(\diamondsuit,\diamondsuit),(\heartsuit,\heartsuit),(\spadesuit,\spadesuit)\}$$



$$\{$$
$$(\clubsuit,\clubsuit),(\clubsuit,\diamondsuit),(\clubsuit,\heartsuit),(\clubsuit,\spadesuit),$$
$$(\diamondsuit,\clubsuit),(\diamondsuit,\diamondsuit),(\diamondsuit,\heartsuit),(\diamondsuit,\spadesuit),$$
$$(\heartsuit,\clubsuit),(\heartsuit,\diamondsuit),(\heartsuit,\heartsuit),(\heartsuit,\spadesuit),$$
$$(\spadesuit,\clubsuit),(\spadesuit,\diamondsuit),(\spadesuit,\heartsuit),(\spadesuit,\spadesuit)$$
$$\}$$



$$\{$$
$$(\clubsuit,\clubsuit),(\clubsuit,\spadesuit),(\spadesuit,\clubsuit),(\spadesuit,\spadesuit),$$
$$(\diamondsuit,\diamondsuit),(\diamondsuit,\heartsuit),(\heartsuit,\diamondsuit),(\heartsuit,\heartsuit)$$
$$\}$$



$\{(\clubsuit, \clubsuit), (\diamondsuit, \diamondsuit), (\heartsuit, \heartsuit), (\spadesuit, \spadesuit)\}$

$\clubsuit \quad \diamondsuit \quad \heartsuit \quad \spadesuit$

$\{$

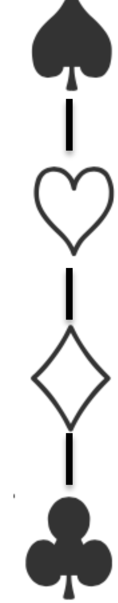
$(\clubsuit, \clubsuit), (\clubsuit, \diamondsuit), (\clubsuit, \heartsuit), (\clubsuit, \spadesuit),$

$(\diamondsuit, \diamondsuit), (\diamondsuit, \heartsuit), (\diamondsuit, \spadesuit),$

$(\heartsuit, \heartsuit), (\heartsuit, \spadesuit),$

$(\spadesuit, \spadesuit)$

$\}$



$\{$

$(\clubsuit, \clubsuit), (\clubsuit, \spadesuit), (\spadesuit, \clubsuit), (\spadesuit, \spadesuit),$

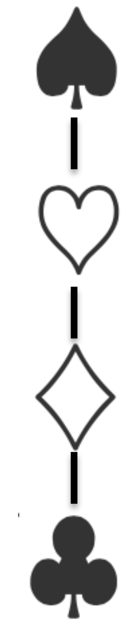
$(\diamondsuit, \diamondsuit), (\diamondsuit, \heartsuit), (\heartsuit, \diamondsuit), (\heartsuit, \heartsuit)$

$\}$

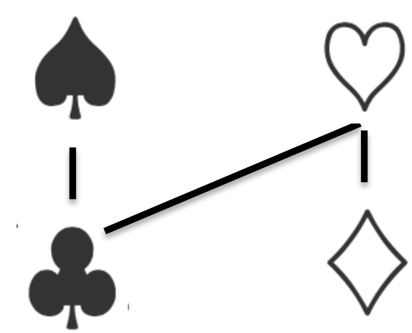
Not a partial order.

$\{(\clubsuit,\clubsuit),(\diamondsuit,\diamondsuit),(\heartsuit,\heartsuit),(\spadesuit,\spadesuit)\}$

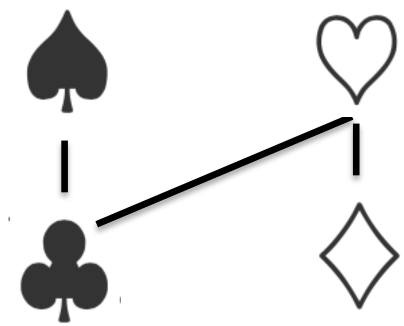
$$\{$$
$$(\clubsuit,\clubsuit),(\clubsuit,\diamondsuit),(\clubsuit,\heartsuit),(\clubsuit,\spadesuit),$$
$$(\diamondsuit,\diamondsuit),(\diamondsuit,\heartsuit),(\diamondsuit,\spadesuit),$$
$$(\heartsuit,\heartsuit),(\heartsuit,\spadesuit),$$
$$(\spadesuit,\spadesuit)$$
$$\}$$



$$\{$$
$$(\clubsuit,\clubsuit),(\clubsuit,\spadesuit),(\clubsuit,\heartsuit),(\spadesuit,\spadesuit),$$
$$(\diamondsuit,\diamondsuit),(\diamondsuit,\heartsuit),(\heartsuit,\heartsuit)$$
$$\}$$



Total Order



Diamond isn't comparable to either spade or club.

Good to go.

Java

- equals – should define an equivalence relation, for items x and y of the same type.
  - If x isn't null, y can't be null;
  - x.equals(x)
  - If x.equals(y) then y.equals(x)
  - If x.equals(y) and y.equals(z) then x.equals(z).

Java

- equals must be in synch with hashCode: any two items x and y such that x.equals(y) is true, it must be the case that

  x.hashCode() == y.hashCode()

- NB: it isn't the case that if x.equals(y) is false, then

  x.hashCode() =/= y.hashCode()

Java

- compareTo – should define a total order;

- compareTo should be *consistent* with equals: for items x and y of the same type, it should be the case that:

x.compareTo(y) == 0 if and only if x.equals(y)