Lisa

Momentum Transport

First, let's examine the general transport equation:

$$\text{Rate of Transfer Process} = \frac{\text{driving force}}{\text{resistance}}$$

example: i (current) $= \dfrac{V \text{ (voltage)}}{R \text{ (resistance)}}$ Ohm's Law

This equation has three major applications in chemical engineering:

1) Newton's Law (of viscosity): $\tau_{zx} = -\nu \dfrac{d(v_x \rho)}{dz}$ for constant ρ

$\nu = \dfrac{\mu}{\rho} \equiv$ momentum diffusivity $[=] \dfrac{m^2}{s}$

$\tau_{zx} \equiv$ flux of x-directed momentum in z-direction $[=] \dfrac{kg \cdot m/s}{s \cdot m^2}$

2) Fourier's Law: $\dfrac{q_z}{A} = -\alpha \dfrac{d(\rho C_p T)}{dz}$ for constant ρ, C_p

$\alpha = \dfrac{k}{\rho C_p} \equiv$ thermal diffusivity $[=] \dfrac{m^2}{s}$

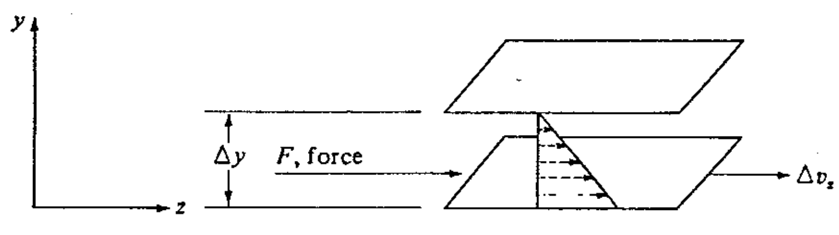
$q_z/A \equiv$ heat flux $[=] \dfrac{J}{s \cdot m^2}$

3) Fick's Law: $J^*_{Az} = -D_{AB} \dfrac{dc_A}{dz}$ for constant C

$D_{AB} \equiv$ molecular diffusivity $[=] \dfrac{m^2}{s}$

$J^*_{Az} \equiv$ mass flux $[=] \dfrac{kgmol\,A}{s \cdot m^2}$

Obviously, this discussion will only pertain to the first of these relations. We'll begin by deriving Newton's law and defining the viscosity of a fluid.



$$\frac{F}{A} = -\mu \frac{\Delta v_z}{\Delta y}$$

as $\Delta y \to 0$: $\tau_{yz} = -\mu \dfrac{dv_z}{dy}$

where the proportionality constant μ is the viscosity, defined as the property of a fluid that gives rise to forces that resist the relative movement of adjacent layers in the fluid (analogous forces arising in solids are called shear forces) $\mu [=] \dfrac{g}{cm \cdot s}$

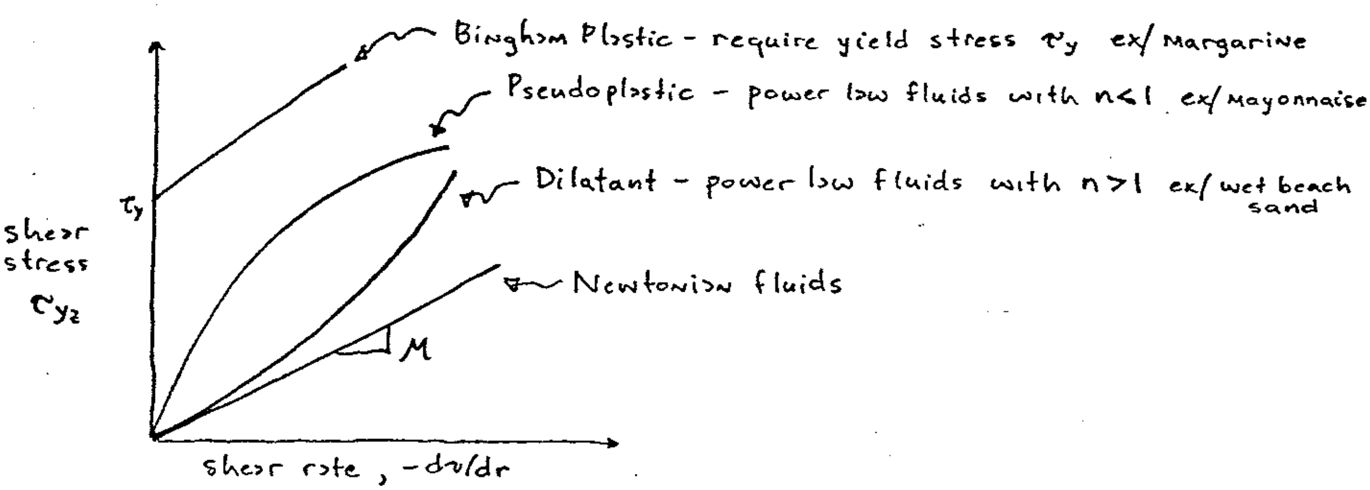
also Note: $\tau_{yz} = \frac{F}{A} [=] \frac{N}{m^2} = \frac{kg \cdot m/s}{s \cdot m^2} = \frac{momentum}{s \cdot m^2}$

This can be referred to as the momentum flux or shear stress.

Remember, Newton's law holds for Newtonian fluids in laminar flow. Turbulence, or the onset of lateral mixing, confuses the issue. Non-newtonian fluids have non-linear relationships between shear stress τ_{yz} and shear rate $\frac{dv_z}{dy}$. Therefore, the viscosity μ does not remain constant, but is a function of shear rate. While we are on the subject, let's examine some non-newtonian fluids. These can generally be divided into two categories: those whose shear stress is independent of time or duration of shear and those whose shear stress is dependent on time or duration of shear. The former class, called time-independent, are the most common and easiest to model. Generally these can be fit to Newton's law with the addition of one adjustable parameter, n, the flow behavior index. This eqn. is also called the power-law:

$$\tau = K\left(-\frac{dv}{dr}\right)^n$$

The three major classes of non-Newtonian time-independent fluids:



Equations governing laminar flow of time-independent power-law fluids have been developed using the additional parameter n.

Other non-newtonian fluids include two major classes of time-dependent fluids. Thixotropic fluids exhibit a reversible decrease in shear stress with time at a constant rate of shear. Rheopectic fluids exhibit a reversible increase in shear stress with time at constant shear rate. The latter of these are quite rare. The last class of non-Newtonian fluids we'll mention are viscoelastic. These exhibit elastic recovery from the deformations that occur during flow. This combination of viscous and elastic properties leads to their name. One example is flour dough.

Let's get back to momentum:

$$\vec{P} = M\vec{v} \equiv \text{momentum}$$

Newton's law of momentum: $\Sigma \vec{F} = \frac{d\vec{P}}{dt}$

These forces may come from gravity, pressure, friction, or solid surfaces. We can develop two statements of conservation of momentum, the integral (or overall) momentum balance and the differential (shell) momentum balance. Remember, momentum is generated by external forces on the system so it is not conserved unless these forces are all absent.

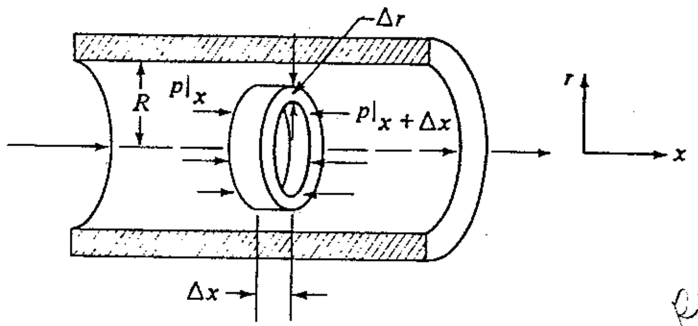
The integral momentum balance on a control volume:

$$\begin{pmatrix}\text{sum of forces}\\ \text{acting on the}\\ \text{control volume}\end{pmatrix} = \begin{pmatrix}\text{rate of } \vec{P}\\ \text{out of control}\\ \text{volume}\end{pmatrix} - \begin{pmatrix}\text{rate of } \vec{P}\\ \text{into control}\\ \text{volume}\end{pmatrix} + \begin{pmatrix}\text{rate of accumulation}\\ \text{of } \vec{P} \text{ in control volume}\end{pmatrix}$$

$$\Sigma \vec{F} = \iint_A \rho\vec{v}(\vec{v}\cdot\vec{n})\,dA + \frac{\partial}{\partial t}\iiint_V \rho\vec{v}\,dV$$

This equation is used to solve for overall (total) values of forces or changes in momentum in the control volume. To obtain details of what is occuring inside the control volume, we shrink the control volume to differential size. This is the shell momentum balance.

I'll do the simplest case of a shell balance on a horizontal pipe with fully-developed flow of a Newtonian fluid. Also assume the fluid is incompressible, laminar, and in steady-state:



Rate of Mom. in − Rate of Mom. out + $\Sigma \vec{F} = 0$

At steady-state: ΣF = rate $\vec{P}$ out − rate $\vec{P}$ in

gravity zero

$$\text{pressure forces} = pA|_x - pA|_{x+\Delta x} = p(2\pi r\Delta r)|_x - p(2\pi r \Delta r)_{x+\Delta x}$$

$$\text{net efflux of momentum} = \tau_{rx}A|_{r+\Delta r} - \tau_{rx}A|_r = \tau_{rx}(2\pi r\Delta x)|_{r+\Delta r} - \tau_{rx}(2\pi r \Delta x)|_r$$

Notice that the net convective momentum flux is zero because the flow is fully-developed (v_x at x = v_x at $x+\Delta x$).

Equating terms:

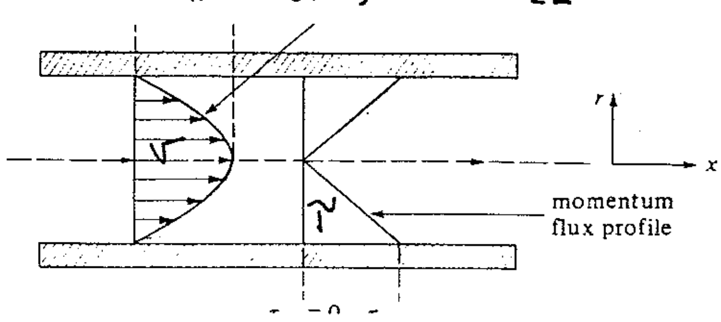
$$\frac{(r\tau_{rx})|_{r+\Delta r} - (r\tau_{rx})|_r}{\Delta r} = \frac{r(p|_x - p|_{x+\Delta x})}{\Delta x}$$

$$\frac{d(r\tau_{rx})}{dr} = \left(\frac{\Delta P}{L}\right) r$$

assuming constant pressure gradient = $\Delta P/L$ over length of pipe L.

Integrating and using the boundary condition that the flux is finite at the center ($r=0$) gives:

$$\tau_{rx} = \left(\frac{\Delta P}{2L}\right) r = \frac{(P_0 - P_L)}{2L} r$$



We have found that momentum flux varies linearly with the radius and the maximum value occurs at $r = R$ at the wall. A major application of shell momentum balances is in deriving velocity profiles. Using Newton's law of viscosity:

$$\tau_{rx} = -\mu \frac{dv_x}{dr}$$

$$\frac{dv_x}{dr} = -\frac{(p_0 - p_L)}{2\mu L} r$$

Integrating and using the boundary condition that $v_x = 0$ at $r = R$:

$$v_x = \frac{(p_0 - p_L)}{4\mu L} R^2 \left[1 - \left(\frac{r}{R}\right)^2\right] \quad \text{parabolic velocity profile}$$

We can find the average and maximum velocities from this equation. The average velocity is found by summing up all the velocities over the cross-section and dividing by the cross-sectional area:

HAGEN-POIS → V_{avg}

$$v_{x\,av} = \frac{1}{A} \iint_A v_x \, dA = \frac{1}{\pi R^2} \int_0^R v_x \, 2\pi r \, dr$$

Plugging in v_x and integrating gives:

$$v_{x\,av} = \frac{(p_0 - p_L) R^2}{8\mu L} = \frac{(p_0 - p_L) D^2}{32\mu L} \quad \text{Hagen-Poiseuille eqn.}$$

This equation relates the pressure drop and average velocity for laminar flow in a horizontal pipe.

The maximum velocity for a pipe is found at $r = 0$:

$$v_{x\,max} = \frac{(p_0 - p_L)}{4\mu L} R^2$$

giving the well-known result that:

$$v_{x\,av} = \frac{v_{x\,max}}{2} \quad \text{for laminar flow}$$

For turbulent flow no simple velocity profile can be derived, but the curve tends to be more flattened ($\sim$ plug flow) and $\frac{v_{av}}{v_{max}} \approx 0.8$.

The Hagen-Poiseuille equation is a good lead in to a brief discussion of the pressure drop due to friction in a pipe. Rewriting Hagen-Poiseuille:

$$\Delta p_f = (p_1 - p_2)_f = \frac{32 \mu v (L_2 - L_1)}{D^2}$$

This quantity Δp_f is the pressure loss due to skin friction in a pipe. The mechanical energy loss due to friction is written as:

$F_K = \frac{\Delta P}{\rho}$ $$F_f = \frac{(p_1 - p_2)_f}{\rho} \quad [=] \quad \frac{N \cdot m}{kg}$$

This is only the skin friction loss of energy; frictional losses also occur due to disruption of velocity caused by sudden expansions, contractions, and pipe fittings. These losses are related to the kinetic energy of the stream through empirical factors. The total mechanical energy loss due to friction is the sum of all of these losses.

A common parameter used to describe skin friction losses in pipes is the fanning friction factor, f. This dimensionless parameter is defined as the shear stress at the surface divided by the product of density times velocity head: 2 velocity head $= \frac{v^2}{2g}$ $\quad \frac{\rho v^2}{2} = \frac{KE}{volume}$

$$f = \frac{\tau_s}{\rho \cdot (v^2/2)} = \frac{\dfrac{\Delta p_f \cdot \cancel{\pi} R^{\cancel{2}}}{2 \cancel{\pi} \cancel{R} \Delta L}}{(\rho v^2/2)} \begin{array}{l} \leftarrow \text{drag force} \\ \leftarrow \text{wetted surface area} \end{array}$$

?

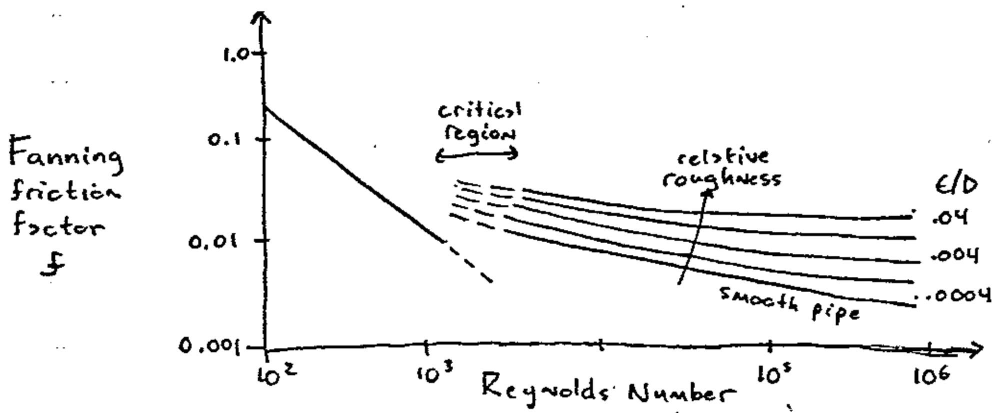
This gives us for laminar or turbulent flow:

$$F_f = \frac{\Delta p_f}{\rho} = 4 f \frac{\Delta L}{D} \frac{v^2}{2}$$

We can combine this with the Hagen-Poiseuille equation in the form given ~~about~~ above to find for laminar-flow only:

$$f = \frac{16}{D v \rho / \mu} = \frac{16}{N_{Re}}$$

For turbulent flow, f varies not only with N_{Re} but also with the surface roughness of the pipe. We can define a relative roughness as the equivalent roughness ϵ over pipe diameter D and correlate data for f vs. N_{Re} into one graph:



One more thing to note about f is that all of the above equations are given for isothermal flow, i.e. no heat transfer. A temperature gradient will cause the physical properties of the fluid to change, in particular the viscosity. Generally, when a liquid is being heated f will decrease.

A very similar situation to mechanical energy losses due to skin friction is mechanical energy losses due to flow past immersed objects. Immersed objects cause disruptions in the velocity profile leading to skin drag and form drag. We can define a dimensionless drag coefficient, C_D, analogous to the friction factor. This parameter describes the total drag force due to a given object:

$$C_D = \frac{F_D/A_p}{\rho(v_0^2/2)}$$

$$\text{or} \quad F_D = C_D \frac{v_0^2}{2} \rho A_p \quad \text{total form drag}$$

As was done with f, C_D can be correlated to N_{Re} for various geometries. First we must define a Reynold's number for a given solid immersed in a flowing liquid:

$$N_{Re} = \frac{D_p v_o \rho}{\mu} \quad \text{where } D_p \text{ is sphere diameter}$$

Using this definition, a linear relationship between C_D and N_{Re} is found to exist in the Stokes law region ($N_{Re} < 1$). In this region creeping flow occurs and inertial effects are negligible in the Navier-Stokes equation. From Navier-Stokes comes Stokes Law:

$$F_D = 3\pi\mu D_p v_o \quad \times 4 \text{ Needed!}$$

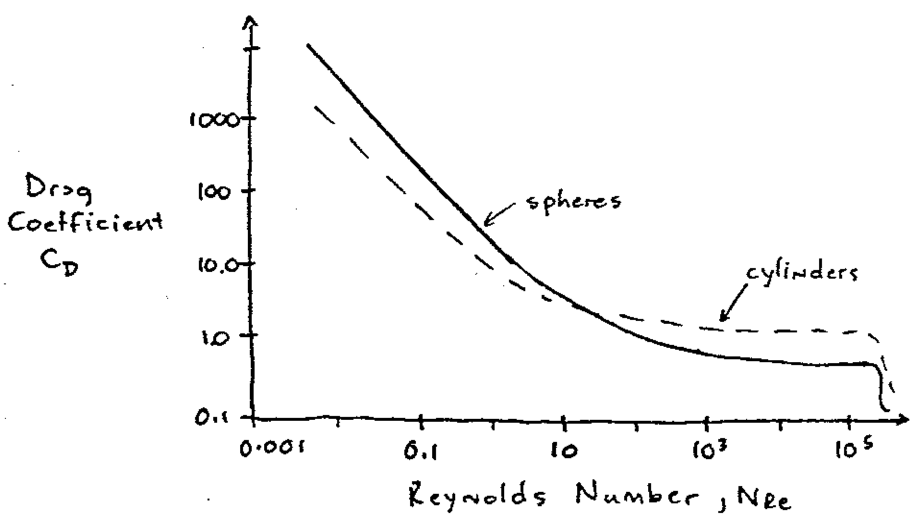
$$4\pi\mu r v$$

$$F_D = \text{skin drag} + \text{form drag}$$

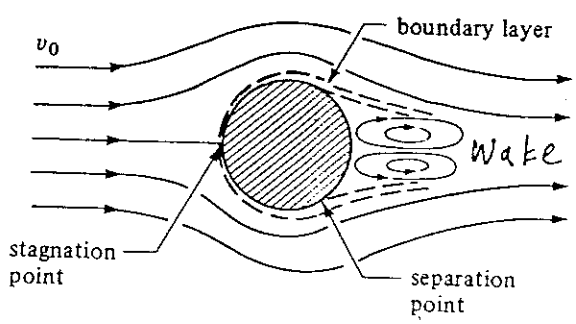
Combining the two equations we have for drag force gives:

$$C_D = \frac{24}{D_p v_o \rho/\mu} = \frac{24}{N_{Re}}$$

We can then graph C_D vs N_{Re} just as we did with the Fanning friction factor:



One last thing to include here is a diagram describing the physical reason mechanical energy losses occur in flow past objects, i.e. eddies and wake formation:



Earlier we discussed the differential momentum balance. We found that a balance of this kind allows us to investigate what goes on inside of our control volume, giving us velocity distributions and pressure drop. However, it is not necessary to formulate new balances for each flow problem. It is often easier to start with general differential equations for the conservation of mass and momentum and discard unneeded terms for each particular problem. These equations are referred to as the Equations of Change, since they describe the variations in the properties of a fluid with respect to position and time.

Before deriving these equations, we'll do a brief review of some calculus. It becomes necessary to use three types of time derivatives:

Partial time derivative: $\frac{\partial \rho}{\partial t}$

this is the change in ρ with time at a fixed point (x, y, z).

total time derivative: $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt}$

this is the change in ρ with time with respect to our motion in the x, y, and z directions, given by $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ respectively.

substantial time derivative: $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v_x\frac{\partial \rho}{\partial x} + v_y\frac{\partial \rho}{\partial y} + v_z\frac{\partial \rho}{\partial z}$

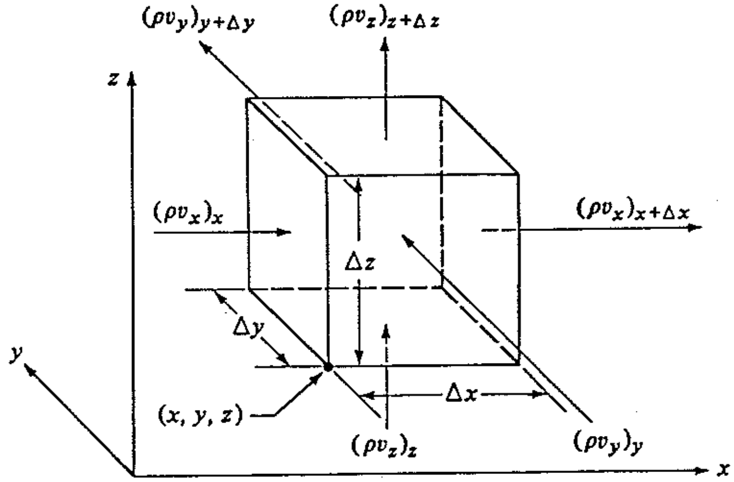
this is the change in ρ with time with respect the velocity $\vec{v}$ of the flowing stream.

Gradient ("grad") of a scalar: $\vec{\nabla}\rho = \vec{i}\frac{\partial \rho}{\partial x} + \vec{j}\frac{\partial \rho}{\partial y} + \vec{k}\frac{\partial \rho}{\partial z}$

Divergence ("div") of a vector: $(\vec{\nabla}\cdot\vec{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Laplacian of a scalar: $\vec{\nabla}^2\rho = \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}$

Now we can derive the differential equation of continuity (mass) and equation of motion (momentum). We'll use terms in the x-direction remembering the analogous terms in the y & z directions exist also.



Mass balance: $\begin{pmatrix}\text{rate of}\\ \text{mass in}\end{pmatrix} - \begin{pmatrix}\text{rate of}\\ \text{mass out}\end{pmatrix} = \begin{pmatrix}\text{rate of mass}\\ \text{accumulated}\end{pmatrix}$

$$\underbrace{(\rho v_x)_x \Delta y \Delta z - (\rho v_x)_{x+\Delta x} \Delta y \Delta z}_{\text{x-term only}} = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

Note that the term (ρv_x) is a mass flux in units of $kg/s{\cdot}m^2$ so multiplying by the area gives $\frac{kg}{s}$. Dividing by the volume and using the definition of the derivative gives:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right] = -(\vec{\nabla}\cdot\rho\vec{v})$$

Rearranging can give: $\frac{D\rho}{Dt} = -\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = -\rho(\vec{\nabla}\cdot\vec{v})$

Often we can say that a liquid is incompressible and its density is essentially constant. This means that as a fluid element follows the path of fluid motion ρ is constant, or $\frac{D\rho}{Dt} = 0$:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = (\vec{\nabla}\cdot\vec{v}) = 0$$

At steady state we can say $\partial\rho/\partial t = 0$.

Now, starting with the same diagram above:

Momentum balance: $\begin{pmatrix}\text{rate of}\\ \vec{P}\text{ in}\end{pmatrix} - \begin{pmatrix}\text{rate of}\\ \vec{P}\text{ out}\end{pmatrix} + \begin{pmatrix}\Sigma F\text{ acting}\\ \text{on system}\end{pmatrix} = \begin{pmatrix}\text{rate of }\vec{P}\\ \text{accumulation}\end{pmatrix}$

Momentum enters by convection $(\rho v)v$ and by molecular transfer τ. For the x-direction only:

$$\begin{matrix}\text{Net convective}\\ \text{momentum flow}\end{matrix} = [(\rho v_x v_x)_x - (\rho v_x v_x)_{x+\Delta x}]\Delta y \Delta z + [(\rho v_y v_x)_y - (\rho v_y v_x)_{y+\Delta y}]\Delta x \Delta z + [(\rho v_z v_x)_z - (\rho v_z v_x)_{z+\Delta z}]\Delta x \Delta y$$

$$\text{Net molecular transfer of momentum} = \left[(\tau_{xx})_x - (\tau_{xx})_{x+\Delta x}\right]\Delta y \Delta z + \left[(\tau_{yx})_y - (\tau_{yx})_{y+\Delta y}\right]\Delta x \Delta z + \left[(\tau_{zx})_z - (\tau_{zx})_{z+\Delta z}\right]\Delta x \Delta y$$

The forces acting on the system include pressure and gravity:

$$\text{Net fluid pressure force} = (p_x - p_{x+\Delta x})\Delta y \Delta z$$

$$\text{Net gravitational force} = \rho g_x \Delta x \Delta y \Delta z$$

The rate of accumulation of momentum in the x-direction:

$$\Delta x \Delta y \Delta z \frac{\partial(\rho v_x)}{\partial t}$$

Combining these terms, ~~diving~~ dividing by $\Delta x \Delta y \Delta z$ and taking the limit as Δx, Δy, and $\Delta z \to 0$ gives:

$$\frac{\partial(\rho v_x)}{\partial t} = -\left[\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right] - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x$$

Combining this with the equation of continuity gives:

$$\rho \frac{D v_x}{Dt} = -\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] - \frac{\partial p}{\partial x} + \rho g_x$$

And for all 3 directions:

$$\boxed{\rho \frac{D\vec{v}}{Dt} = -[\vec{\nabla} \cdot \tau] - \vec{\nabla} p + \rho \vec{g}}$$

To use this equation we must relate τ to velocity gradients and fluid properties. If we assume Newtonian fluids with constant density and viscosity we obtain the Navier-Stokes equation:

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p + \rho \vec{g} + \mu \vec{\nabla}^2 \vec{v}$$

Also, it is often useful to assume we have an "ideal" fluid, one which has constant density and zero viscosity. This gives

Euler's equations:

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x \quad \text{(x-direction only)}$$

One last simplification that is useful is at $N_{Re} < 1$, when creeping flow occurs. In this case we ignore inertial effects and obtain for an incompressible fluid:

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) \quad \text{(x-direction only)}$$

I've been waiting until now to discuss the subject of dimensionless numbers for a reason. Dimensionless parameters are important because their physical meanings allow us to correlate and predict transport phenomena without having to solve gigantic and unwieldy equations like Navier-Stokes. They also allow us to extend small-scale model results to the large-scale prototype.

First we must determine which dimensionless parameters are important in a given situation. As an example of this, let's look at the x-component of the Navier-Stokes equation at steady-state:

$$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

Each of these terms have units of L/time^2; also each term has a physical significance. Using a single characteristic velocity v and length L:

$$\underbrace{\left[\frac{v^2}{L}\right]}_{\text{inertial force}} = \underbrace{[g]}_{\text{gravity force}} - \underbrace{\left[\frac{p}{\rho L}\right]}_{\text{pressure force}} + \underbrace{\left[\frac{\mu v}{\rho L^2}\right]}_{\text{viscous force}} \quad \text{dimensional equality}$$

Now we can obtain the following dimensionless groups:

$$\frac{[v^2/L]}{[g]} = \frac{\text{inertia force}}{\text{gravity force}} = \frac{v^2}{gL} = N_{Fr}, \text{ Froude number}$$

$$\frac{[p/\rho L]}{[v^2/L]} = \frac{\text{pressure force}}{\text{inertia force}} = \frac{p}{\rho v^2} = N_{Eu}, \text{ Euler number}$$

$$\frac{[v^2/L]}{[\mu v/\rho L^2]} = \frac{\text{inertia force}}{\text{viscous force}} = \frac{Lv\rho}{\mu} = N_{Re}, \text{ Reynolds number}$$

Okay, a few last words. There are some things I didn't cover that it will come in handy to be familiar with. I didn't do any overall energy or mechanical energy balances. These are similar to the overall momentum balance and fairly simple. I also didn't cover stream functions ψ, which allow some complicated forms of Navier-Stokes to be solved. These are covered in BSL. I didn't cover packed beds, but that's pretty simple too. Lastly, I didn't cover the major subjects of boundary layer and enterance regions. You should examine these problems, which often involve solving Navier-Stokes and using a lot of the math techniques from 230. I've included a couple of examples from BSL for your enjoyment.