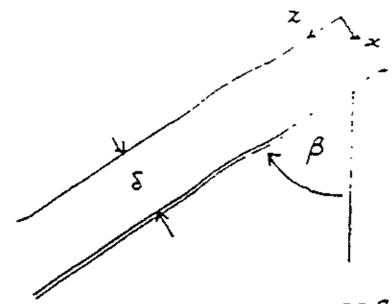
CHAPTER TWO: VELOCITY DISTRIBUTIONS IN LAMINAR FLOW

2.1 FALLING FILM



- differential mom. balance: $\frac{d}{dx}\tau_{xz} = \rho g \cos\beta$

  $\tau_{xz} = \rho g x \cos\beta$

  $-\tau_{xz} = -\mu \frac{dv_z}{dx}$ → $\boxed{v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right]}$

  $v_{max} = \frac{\rho g \delta^2 c\cdot}{2\mu}$

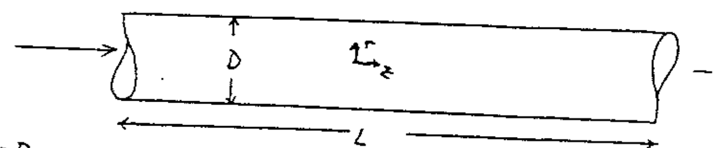
  $Re = \frac{4\delta \langle v_z \rangle}{\mu}$

  - laminar flow w/o rippling: $Re < 4$ to 25

    laminar flow w/ rippling: 4 to $25 < Re < 1000$ to 2000

    turbulent flow: $Re > 1000$ to 2000

2.2 SMOOTH PIPE FLOW



- diff. mom. balance:

  $\frac{d}{dr}(r\tau_{rz}) = \frac{P_0 - P_L}{L} r$

  $\tau_{rz} = \frac{P_0 - P_L}{2L} r$

  $-\tau_{rz} = -\mu \frac{dv_z}{dr}$ → $\boxed{v_z = \frac{(P_0 - P_L)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]}$

  $v_{max} = \frac{(P_0 - P_L)}{4\mu L}$

  $Re = \frac{D\langle v_z \rangle}{\mu}$

  - assumptions: laminar flow ($Re < 2100$) ← turbulent >

    Incompressible, newtonian flow (ρ, μ const)

    steady-state, no slip, no end-effects

- friction factor

  - laminar $f \equiv \frac{|\Delta P|}{2\rho v^2}\frac{D}{L} = \frac{16}{Re}$ → $\boxed{Q = \frac{\pi}{128}\frac{|\Delta P| D^4}{L\mu}}$ Hagen-Poiseuille Law

  - turbulent $f = 0.079\, Re^{-1/4}$ or $\frac{1}{\sqrt{f}} = 4.0 \log(Re\sqrt{f}) - 0.4$

- using continuity

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2\right) = -\left(\bar{\nabla}\cdot \tfrac{1}{2}\rho v^2 \bar{v}\right) - \left(\bar{\nabla}\cdot p\bar{v}\right) - p\left(-\bar{\nabla}\cdot\bar{v}\right)$$

rate of inc in KE per unit volume	net rate input of KE by bulk flow	rate of work done by pressure of surroundings	rate of reversible conversion to internal energy

$$-\left(\bar{\nabla}\cdot[\bar{\tau}\cdot\bar{v}]\right) - \left(-\bar{\tau}:\bar{\nabla}\bar{v}\right) + \rho\left(\bar{v}\cdot\bar{g}\right)$$

rate of work done by viscous forces	rate of irreversible conversion to internal energy	rate of work done by gravity force

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3.4 GOVERNING EQUATIONS IN RECTANGULAR COORDINATES (x,y,z)

- continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

- motion (Newtonian const ρ, μ)

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho$$

- energy (Newtonian const ρ & k)

$$\rho C_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

$$+ 2\mu\left\{\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right\} + \mu\left\{\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2\right.$$

$$\left. + \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)^2\right\}$$

CHAPTER 3 EQUATIONS OF CHANGE FOR ISOTHERMAL SYSTEMS

3.1 CONTINUITY EQUATION

- $$\frac{\partial \rho}{\partial t} = -(\bar{\nabla}\cdot\rho\bar{v}) = -(\rho\bar{\nabla}\cdot\bar{v} + \bar{v}\cdot\bar{\nabla}\rho) \Rightarrow \boxed{\frac{D\rho}{Dt} = -\rho(\bar{\nabla}\cdot\bar{v})}$$

- for incompressible fluid: $\frac{D\rho}{Dt} = 0 \longrightarrow \boxed{\bar{\nabla}\cdot\bar{v} = 0}$

3.2 EQUATIONS OF MOTION

- $$\frac{\partial}{\partial t}(\rho\bar{v}) = -(\bar{\nabla}\cdot\rho\bar{v}\bar{v}) - \bar{\nabla}p - \bar{\nabla}\cdot\bar{\tau} + \rho\bar{g}$$

rate of inc. of momentum per unit vol	rate of mom. gained by convection	pressure force per unit vol	rate of mom gain by viscous transfer per unit vol	gravitational force per unit volume

- using continuity

$$\boxed{\rho\frac{D\bar{v}}{Dt} = -\bar{\nabla}p - \bar{\nabla}\cdot\bar{\tau} + \rho\bar{g}}$$

local acceleration

$$\frac{D\bar{v}}{Dt} = \frac{\partial\bar{v}}{\partial t} + \bar{v}\cdot\nabla\bar{v}$$

convective acceleratio

- simplifications
  - constant ρ, μ, $\bar{\nabla}\cdot\bar{v} = 0$ (Navier-Stokes Equation)

$$\boxed{\rho\frac{D\bar{v}}{Dt} = -\bar{\nabla}p + \mu\bar{\nabla}^2\bar{v} + \rho\bar{g}}$$

  - $\bar{\nabla}\cdot\bar{\tau} = 0$ (Euler Equation)

$$\boxed{\rho\frac{D\bar{v}}{Dt} = -\bar{\nabla}p + \rho\bar{g}}$$

3.3 EQUATIONS OF ENERGY

- $$\boxed{\rho\frac{D}{Dt}(\tfrac{1}{2}v^2) = -(\bar{v}\cdot\nabla p) - (\bar{v}\cdot[\bar{\nabla}\cdot\bar{\tau}]) + \rho(\bar{v}\cdot\bar{g})}$$

CHAPTER 4. VELOCITY DISTRIBUTIONS IN MORE THAN ONE VARIABLE

4.1 FLOW NEAR WALL SUDDENLY SET IN MOTION

I.C. $t=0 \quad v_x = 0$

DEQ $$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

B.C. $y=0 \quad v_x = V$

$y=\infty \quad v_x = 0$

— substitute $$\frac{v_x}{V} = \phi(\eta)$$

$$\eta = \frac{y}{\sqrt{4\nu t}} \quad \longrightarrow \quad \phi'' + 2\eta\phi' = 0$$

$\eta = 0 \quad \phi = 1$

$\eta = \infty \quad \phi = 0$

$$\phi = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$

$$\frac{v_x}{V} = 1 - \operatorname{erf} \frac{y}{\sqrt{4\nu t}}$$

4.2 FLOW IN CIRCULAR TUBE

I.C. $t=0 \quad v_z = 0$

DEQ $$\rho \frac{\partial v_z}{\partial t} = \frac{p_0 - p_L}{L} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right)$$

$r=0 \quad v_z$ finite

$r=R \quad v_z = 0$

— substitute $$\phi = \frac{v_z}{(p_0 - p_L)R^2/4\mu L}$$

$\tau = 0 \quad \phi$

$$\xi = \frac{r}{R} \quad \longrightarrow \quad \frac{\partial \phi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right)$$

$\xi = 1 \quad \phi$.

$\xi = 0 \quad \phi$

$$\tau = \frac{\mu t}{\rho R^2}$$

let $\phi = \phi_{ss} - \phi_t$

$$\phi_{ss} = 1 - \xi^2 \quad \longrightarrow \quad \frac{\partial \phi_t}{\partial \tau} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi_t}{\partial \xi}\right)$$

$\tau = 0 \quad \phi_t = \phi_{ss}$

$\xi = 0 \quad \phi_t =$ fin

$\xi = 1 \quad \phi_t = 0$

$\phi_t = f(\xi) g(\tau)$ solve by SOV

$$\phi_t = \sum_{n=1}^{\infty} B_n \, e^{-\lambda_n^2 \tau} J_0(\lambda_n \xi)$$

$$\phi = (1 - \xi^2) - 8 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \xi)}{\lambda_n^3 J_1(\lambda_n)} e^{-\lambda_n^2 \tau}$$

35 DIMENSIONAL ANALYSIS FOR EQUATIONS OF CHANGE

$$\bar{v}^* = \frac{\bar{v}}{V} \qquad p^* = \frac{p - p_0}{\rho V^2} \qquad t^* = \frac{tV}{D} \qquad v^{**} = \frac{vD\rho}{\mu}$$

$$x^* = \frac{x}{D} \qquad y^* = \frac{y}{D} \qquad z^* = \frac{z}{D} \qquad t^{**} = \frac{t\mu}{\rho D^2}$$

(forced convection)

$$T^* = \frac{T - T_0}{T_1 - T_0}$$

continuity : $\bar{\nabla}^* \cdot \bar{v}^* = 0$

N. S. : $$\frac{D\bar{v}^*}{Dt^*} = -\nabla^* p^* + \frac{\mu}{DV\rho}\bar{\nabla}^{*2}\bar{v}^* + \frac{gD}{V^2}\frac{\bar{g}}{g}$$

$\Rightarrow$ $Re = \dfrac{DV\rho}{\mu} = \dfrac{DV}{\nu}$ Reynolds #

$Fr = \dfrac{V^2}{gD}$ Froude #

energy : $$\frac{DT^*}{Dt^*} = \frac{\mu}{DV\rho}\frac{k}{\hat{C}_p\mu}\left(\bar{\nabla}^{*2}T + \frac{\mu V^2}{k(T_1 - T_0)}\Phi_v^*\right)$$

$\Rightarrow$

$Br = \dfrac{\mu V^2}{k(T_1 - T_0)}$ Brinkman #

$Pr = \dfrac{\hat{C}_p\mu}{k}$ Prandtl #

(free convection)

continuity : $\bar{\nabla}^* \bar{v}^{**} = 0$

N. S. : $$\frac{Dv^{**}}{Dt^{**}} = \bar{\nabla}^{*2} v^{**} - T^* Gr \frac{\bar{g}}{g}$$

$\Rightarrow$ $Gr = \dfrac{g\rho^2\beta(T_1 - T_0)D^3}{\mu^2}$ Grashof #

energy : $$\frac{DT^*}{Dt^{**}} = \frac{1}{Pr}\bar{\nabla}^{*2}T^*$$

• conservation of momentum.

$$\frac{d}{dt}\int_{z_1}^{z_2} \rho\langle\bar{v}\rangle A\, dz = \rho_1 \langle\bar{v}V\rangle_1 A_1 - \rho_2\langle\bar{v}V\rangle_2 A$$

$$+ p_1\bar{A}_1 - p_1\bar{A}_2 - \vec{F} + \left(\int_{z_1}^{z_2} \rho A\, dz\right)\vec{g}$$

S.S. $\vec{v}$ normal to cross surface, $\beta \equiv \frac{\langle V^2\rangle}{\langle V\rangle^2}$

$\beta = 1$ for turbule.

$\beta = 4/3$ for laminar (pipe)

$$w\left(\beta_1\langle\bar{v}\rangle_1 - \beta_2\langle\bar{v}\rangle_2\right) + p_1A_1 - p_2A_2 - F + \int_{z_1}^{z} \rho A dz \cdot \vec{g}$$

CHAPTER 7 MACROSCOPIC BALANCES

- conservation of mass

$$\frac{d}{dt}\int_{z_1}^{z_2} \langle \rho \rangle A\,dz = \langle \rho V \rangle_1 A_1 - \langle \rho V \rangle_2 A_2$$

s.s, single fluid: $\langle V \rangle_1 A_1 = \langle V \rangle_2 A_2$

- conservation of energy

$$\frac{d}{dt}\int_{z_1}^{z_2} \langle \rho (e + \tfrac{1}{2}v^2 + gh) \rangle A\,dz =$$

$$\langle \rho (e+\tfrac{1}{2}v^2+gh) V \rangle_1 A_1 - \langle \rho (e+\tfrac{1}{2}v^2+gh) V \rangle_2 A_2 +$$
$$\langle p V \rangle_1 A_1 - \langle p V \rangle_2 A_2 + \dot{Q}_H + \dot{W}_S + l_v$$

s.s, single fluid, phase, uniform properties, uniform equivalent pressure

$$\langle \rho v^2 V \rangle = \rho \langle v^2 V \rangle$$
$$\langle \rho e V \rangle = \rho e \langle V \rangle$$
$$\langle (P + \rho g h) V \rangle = (P + \rho g h) \langle V \rangle$$
$$\alpha = \frac{\langle v^2 V \rangle}{\langle V \rangle}$$

$$\Delta \left(e + \frac{\alpha}{2}\langle v \rangle^2 + gh + \frac{P}{\rho} \right) = \delta Q_H + \delta W_S + l_v$$

CHAPTER 9: TEMPERATURE DISTRIBUTIONS IN LAMINAR FLOW

9.1 HEAT CONDUCTION WITH ELECTRICAL HEAT SOURCE

• diff energy balance: $\frac{d}{dr}(r q_r) = S_e r$ $S_e = \frac{I^2}{k_e} \equiv$ heat producti[on] per unit volu[me]

$$q_r = \frac{S_e r}{2}$$

$$-q_r = -k\frac{dT}{dr} \longrightarrow \boxed{T - T_0 = \frac{S_e R^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right]}$$

9.2 HEAT CONDUCTION W/ CHEMICAL RXN

packed-bed reactor

• diff energy balance:

$$\frac{dq_z}{dz} + \rho_1 v_1 C_p \frac{dT}{dz} = S_c$$

9.3 COMPOSITE CYLINDRICAL WALLS

• diff energy bal: $\frac{d}{dr}(r q_r) = 0 \longrightarrow T_0 - T_1 = r_0 q_0 \left(\frac{\ln r_1/r_0}{k^{01}}\right)$

Addition of resistances:

$$Q_0 = 2\pi L r_0 q_0 = \frac{2\pi L (T_a - T_b)}{\left(\frac{1}{r_0 h_0} + \frac{\ln r_1/r_0}{k^{01}} + \frac{\ln r_2/r_1}{k^{12}} + \dots + \frac{1}{r_1 \dots}\right.}$$

CHAPTER 10 THE EQUATIONS OF CHANGE FOR NON-ISOTHERMAL SYSTEMS

10.1 EQUATION OF ENERGY

$$\frac{\partial}{\partial t}\rho\left(\hat{U}+\tfrac{1}{2}v^2\right) = -\left(\bar{\nabla}\cdot\rho\bar{v}\left(\hat{U}+\tfrac{1}{2}v^2\right)\right) - \bar{\nabla}\cdot\bar{q} + \rho(\bar{v}\cdot\bar{g})$$

rate of gain of energy per unit volume — rate of energy input per unit volume by convection — rate of energy inp by conduction — rate of work done on fluid by gravity

$$-(\bar{\nabla}\cdot p\bar{v}) - (\bar{\nabla}\cdot[\bar{\tau}\cdot\bar{v}])$$

rate of work done on fluid by pressure — rate of work done on fluid by viscous forces

→

$$\boxed{\begin{aligned} \rho\frac{D}{Dt}\left(\tfrac{1}{2}v^2\right) &= -(\bar{v}\cdot\nabla p) - (\bar{v}\cdot[\bar{\nabla}\cdot\bar{\tau}]) + \rho(\bar{v}\cdot\bar{g}) && KE \\ \rho\frac{D}{Dt}(U) &= -(\bar{\nabla}\cdot\bar{q}) - p(\bar{\nabla}\cdot\bar{v}) - (\bar{\tau}:\bar{\nabla}\bar{v}) && IE \end{aligned}}$$

- Simplifications

$$\rho\hat{C}_v\frac{DT}{Dt} = -\bar{\nabla}\cdot\bar{q} - T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}}(\bar{\nabla}\cdot\bar{v}) - (\bar{\tau}:\bar{\nabla}\bar{v})$$

- ideal gas : const. k $\quad\boxed{\rho\hat{C}_V\frac{DT}{Dt} = k\bar{\nabla}^2T - p(\bar{\nabla}\cdot\bar{v})}$

- fluid at const p or with $\rho\neq f(T)$ and solids $\quad\boxed{\rho C_p\frac{DT}{Dt} = k\bar{\nabla}^2T}$