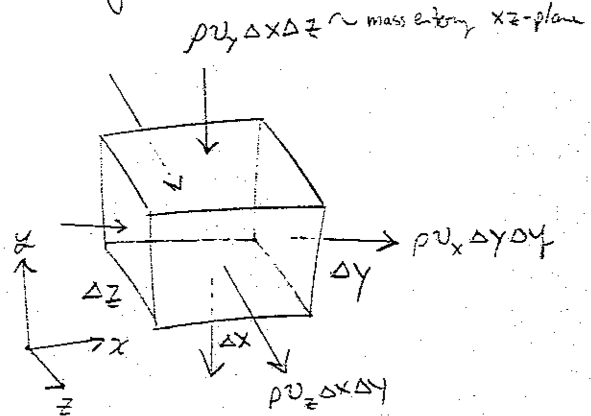
Fluid Mechanics

Continuity Eqn.



Mass Balance:

$$\text{In} - \text{Out} + \cancel{\text{gen.}}^{\,0} = \text{acc.}$$

(this balance can be made because $v_x, v_y,$ and v_z are orthagonal)

$$\left(\rho v_x|_x - \rho v_x|_{x+\Delta x}\right)\Delta y \Delta z + \left(\rho v_y|_y - \rho v_y|_{y+\Delta y}\right)\Delta x \Delta z$$

$$+ \left(\rho v_z|_z - \rho v_z|_{z+\Delta z}\right)\Delta y \Delta x = \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z$$

divide by $\Delta x \Delta y \Delta z$

$$\frac{\left(\rho v_x|_x - \rho v_x|_{x+\Delta x}\right)}{\Delta x} + \frac{\left(\rho v_y|_y - \rho v_y|_{y+\Delta y}\right)}{\Delta y} + \frac{\left(\rho v_z|_z - \rho v_z|_{z+\Delta z}\right)}{\Delta z} = \frac{\partial \rho}{\partial t}$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0}$$

$$-\frac{\partial \rho v_x}{\partial x} - \frac{\partial \rho v_y}{\partial y} - \frac{\partial \rho v_z}{\partial z} = \frac{\partial \rho}{\partial t}$$

Remember from the Calculus

$$\lim_{x \to 0} \frac{n|_x - n|_{x+\Delta x}}{\Delta x} = -\frac{dn}{dx}$$

The continuity eqn. is

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0}$$

or

$$\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial v_x}{\partial x} + v_y \frac{\partial \rho}{\partial y} + \rho \frac{\partial v_y}{\partial y} + v_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial v_z}{\partial z} = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$$

where $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho$

For an incompressible fluid $\rho = \text{const.}$

$$\frac{\partial \rho}{\partial t} = 0$$

$$\cancel{\frac{\partial \rho}{\partial t}} + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

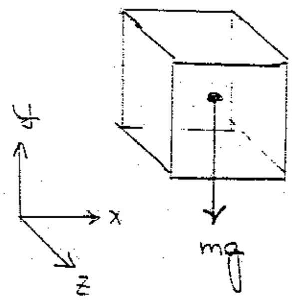
$\nabla \cdot \vec{v} = 0$ for an incompressible fluid

Equation of motion from a force shell balace:

two types of forces acting on a fluid

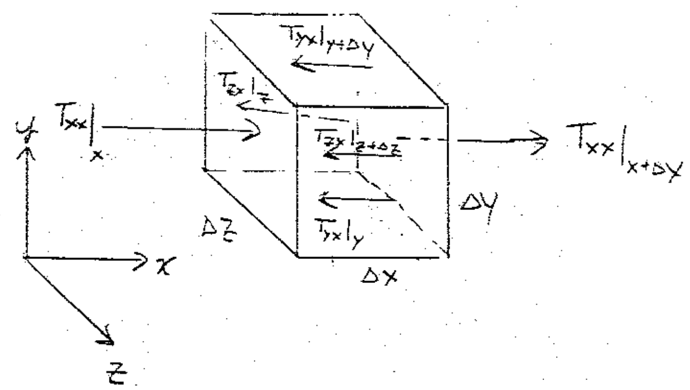
I. body forces

II. surface forces



Body force: $f_x = 0$
$f_y = -mg$
$f_z = 0$

Surface forces: $\vec{F} = m\vec{a}$



x-comp. $\sum F_x = ma_x$

y-comp. $\sum F_y = ma_y$

z-comp. $\sum F_z = ma_z$

$$(1)\quad a_x = \frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = \frac{\partial v_x}{\partial t} + \vec{v}\cdot\nabla v_x$$

$$(2)\quad a_y = \frac{Dv_y}{Dt} = \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}$$

$$(3)\quad a_z = \frac{Dv_z}{Dt} = \frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_x\frac{\partial v_z}{\partial z}$$

Including shear stress and body forces and using BS$L convention of Tension as negative

also

$$m = \rho \Delta x \Delta y \Delta z$$

$$\rho(\Delta x \Delta y \Delta z)\frac{DU_x}{\partial t} = -\Big\{\left(T_{xx}\big|_{x+\Delta x} - T_{xx}\big|_x\right)\Delta y \Delta z$$

$$+ \left(T_{yx}\big|_{y+\Delta y} - T_{yx}\big|_y\right)\Delta x \Delta z + \left(T_{zx}\big|_z - T_{zx}\big|_{z+\Delta z}\right)\Delta x \Delta y\Big\} + f_x \rho(\Delta x \Delta y \Delta z)$$

Divide by $\Delta x \Delta y \Delta z$

and then take the limit

$$\lim_{\Delta x, \Delta y, \Delta z \to 0}$$

$$\rho\frac{Dv_x}{Dt} = -\left\{\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z}\right\} + \rho f_x$$

$$\rho\frac{Dv_y}{Dt} = -\left\{\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z}\right\} + \rho f_y$$

$$\rho\frac{Dv_z}{Dt} = -\left\{\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z}\right\} + \rho f_z$$

General Sign Convention:

T_{ij} — i ~ surface, j ~ direction

- Positive Stress
  1. on a positive surface stress acts in positive direction
  2. on a negative surface stress acts in a negative direction
- Negative Stress ~ opposite the conditions above

Eqn. of motion in tensor notation

$$\underbrace{\begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}}_{\text{Stress Tensor}} = -p \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{pressure tensor}} + \underbrace{\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}}_{\text{Viscous tensor}}$$

$$\overset{\Rrightarrow}{T} = -p \overset{\Rrightarrow}{I} + \overset{\Rightarrow}{\tau}$$

from matrix addition

$$T_{xx} = -p + \tau_{xx}$$
$$T_{yy} = -p + \tau_{yy}$$
$$T_{zz} = -p + \tau_{zz}$$

$$\left.\begin{aligned} T_{xy} &= \tau_{xy} = T_{yx} = \tau_{yx} \\ T_{xz} &= \tau_{xz} = T_{zx} = \tau_{zx} \\ T_{yz} &= \tau_{yz} = T_{zy} = \tau_{zy} \end{aligned}\right\} \text{from symmetry}$$

On static fluid the only forces acting are pressure

$$\begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$ when $\Delta x \Delta y \Delta z = 0$ all forces are equal

Stokes Relations:

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x} + \frac{3}{2}\mu(\nabla \cdot \vec{v})$$
$$\tau_{yy} = 2\mu \frac{\partial v_y}{\partial y} + \frac{3}{2}\mu(\nabla \cdot \vec{v})$$
$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} + \frac{3}{2}\mu(\nabla \cdot \vec{v})$$
$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$
$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$
$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)$$

To complete the momentum eqn.

$$\rho \frac{D v_x}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial v_x}{\partial x} - \frac{2}{3}\mu(\nabla \cdot v)\right]$$
$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right] + \rho g_x$$

for const. ρ and μ the above eqn. can be educed to the navier-stokes eqns. with the aid of the continuity eqn. $\nabla \cdot \vec{v} = 0$

$$\rho \frac{D v_x}{Dt} = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] - \frac{\partial}{\partial x}\left[\frac{2}{3}\mu(\underbrace{\cancel{\nabla \cdot \vec{v}}}_{0})\right]$$
$$+ \mu \frac{\partial}{\partial x}\left[\underbrace{\cancel{\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}}}_{\nabla \cdot \vec{v} = 0}\right] + \rho g_x$$

$$\rho \frac{D v_x}{Dt} = -\frac{\partial P}{\partial x} + \mu \nabla^2 v_x + \rho g_x$$

In general the equation of motion has the following form:

$$\frac{\partial}{\partial t}\rho v = -[\nabla \cdot \rho\vec{v}\vec{v}] - \nabla P - [\nabla \cdot \tau] + \rho g$$

rate of increase of momentum per unit volume	rate of momentum in by convection per unit volume	pressure force on element per unit volume	rate of momentum gain by viscous transfer per unit volume	gravitational force on element per unit volume

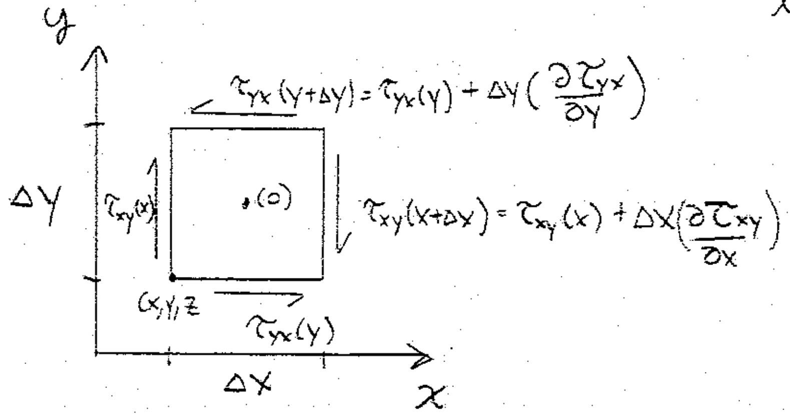
Limits of the Navier-Stokes Eqn.

$$Re = \frac{\text{inertial forces}}{\text{viscous forces}}$$

$Re \gg 1$	$Re \approx 1$	$Re \ll 1$
(Potential flow or inviscid flow) aerodynamics	Boundary Layer theory	Creeping flow

Why is the stress tensor symmetric?

let $\Delta z = 1$



Show that $\tau_{xy} = \tau_{yx}$

Consider the angular momentum M

$$\circlearrowright \sum M_0 = \underbrace{\frac{\Delta x}{2}\tau_{xy}(x)\Delta y}_{\substack{\text{Force} \times \text{Length} \\ \tau_{xy}(x)\Delta y \times \Delta x}} + \frac{\Delta x}{2}\tau_{xy}(x)\Delta y + \frac{\Delta x^2}{2}\cancelto{0}{\left(\frac{\partial \tau_{xy}}{\partial x}\right)}\Delta y$$

$$- \frac{\Delta x}{2}\tau_{yx}(y)\Delta x - \frac{\Delta y}{2}\tau_{yx}(y)\Delta x - \frac{\Delta y^2}{2}\cancelto{0}{\left(\frac{\partial \tau_{yx}}{\partial y}\right)}\Delta x$$

$\frac{\partial \tau_{xy}}{\partial x} = \frac{\partial \tau_{yx}}{\partial y} = 0$ $\because$ coupled forces

and because $\tau_{xy}(x) = \tau_{xy}(x+\Delta x) \leftarrow$ problem statement

Angular Acceleration:

$$\circlearrowright \sum M_0 = \bar{I}\alpha \qquad \bar{I} = \frac{1}{12}m(\Delta x^2 + \Delta y^2)$$

$$\Delta y \cancel{\Delta x}\, \tau_{xy}(x) - \Delta x \cancel{\Delta y}\, \tau_{yx}(y) = \frac{1}{12}(\Delta x \cancel{\Delta y})(\Delta x^2 + \Delta y^2)\alpha$$

$$\tau_{xy}(x) - \tau_{yx}(y) = \frac{1}{12}\alpha\left(\Delta x^2 + \Delta y^2\right)$$

$$\lim_{\Delta x, \Delta y \to 0} \tau_{xy}(x) - \tau_{yx}(y) = 0$$

$$\boxed{\tau_{xy}(x) = \tau_{yx}(y)}$$

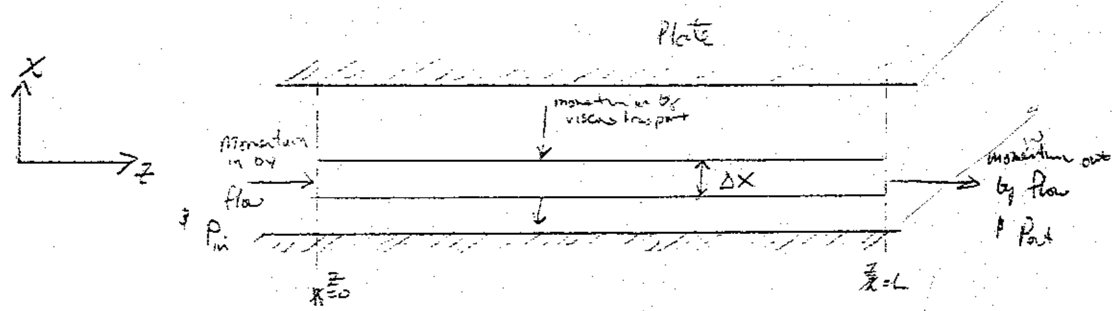
the angular acceleration would be infinite if the stress tensor was not symmetric

$\Rightarrow$ Conservation of angular momentum

General Shell Balance in Fluid Mechanics:

Cartesian coordinates:

consider 1-dimensional flow only



$$\text{In} - \text{Out} + \cancel{\text{Gen.}}^{\,0} = \cancel{\text{acc.}}^{\,S.S.}$$

momentum in

$$A\rho v_z\big|_{z=0} = W\Delta x\,\rho v_z\big|_{x=0}$$

momentum out

$$A\rho v_z\big|_{x=L} = W\Delta x\,\rho v_z\big|_{x=L}$$

from continuity $v_z|_{x=0} = v_z|_{x=L}$

Pressure in

$$AP_{in} = W\Delta x P_0$$

Pressure out

$$AP_{out} = W\Delta x P_L$$

viscous transport in

$$WL\,\tau_{xz}\big|_x$$

viscous transport out

$$WL\,\tau_{xz}\big|_{x+\Delta x}$$

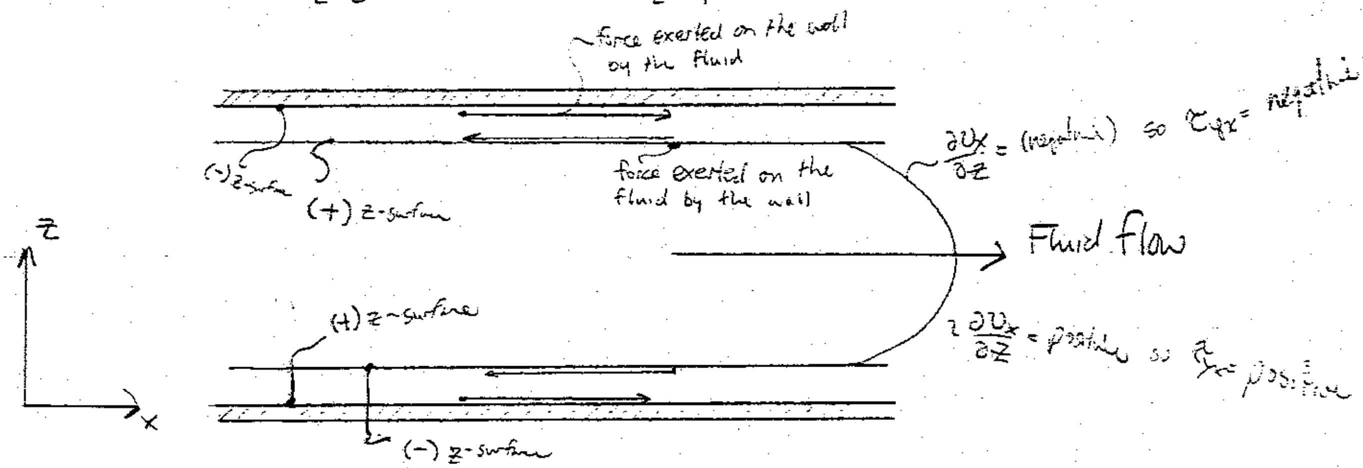
$$W\Delta x\rho \cancel{v_z}\big|_{x=0} - W\Delta x\rho \cancel{v_z}\big|_{x=L} + W\Delta x P_0 - W\Delta x P_L + WL\,\tau_{xz}\big|_x - WL\,\tau_{xz}\big|_{x+\Delta x} = 0$$

$$\frac{P_0 - P_L}{L} + \frac{\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x}}{\Delta x} = 0$$

Forces In fluid Systems

must calculate the Stress

$\tau_{zx}|_{z=0}$ $\qquad$ $\tau_{zx}|_{z=h}$

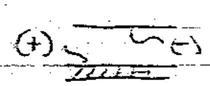


Positive stress

surface	+	—
direction stress acts	+	—

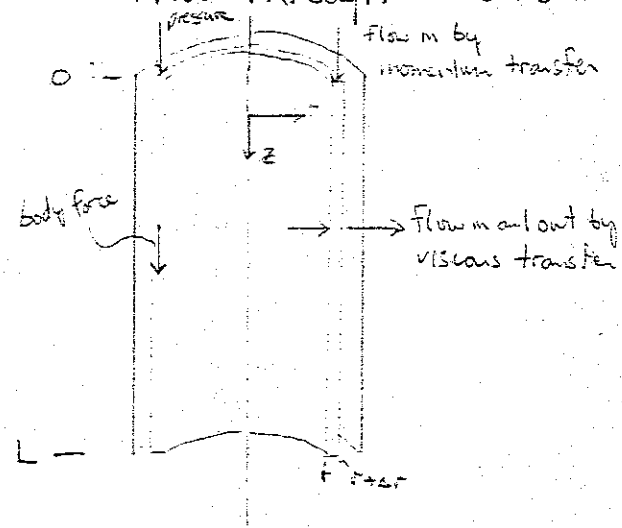
Negative Stress

surface	+	—
direction stress acts	—	+



Hagen-Poiseuille Flow

flow through a circular tube (Shell balance)



rate of momentum in $= \rho v_z|_z 2\pi r \Delta r$

rate of momentum out $= \rho v_z|_{z+\Delta z} 2\pi r \Delta r$

Pressure in $= P|_z \, 2\pi r \Delta r$

Pressure out $= P|_{z+\Delta z} \, 2\pi r \Delta r$

viscous transfer in $= (2\pi r L \tau_{rz})|_r$

viscous transfer out $= (2\pi r L \tau_{rz})|_{r+\Delta r}$

Body force $= +\rho g \, 2\pi r \Delta r L$

$V = \pi r^2 z \qquad A = \pi r^2$

$\Delta V = 2\pi r \Delta r \Delta z$

$\Delta A = 2\pi r \Delta r$

$$\text{In} - \text{out} + \cancel{\text{gen}}^{\,0} = \cancel{\text{acc.}}^{\,0}$$

$$(\rho v_z 2\pi r \Delta r)_z - (\rho v_z 2\pi r \Delta r)_{z+\Delta z} + (2\pi r \Delta r P_0) - (2\pi r \Delta r P_L)$$

$$+ (2\pi r L \tau_{rz})|_r - (2\pi r L \tau_{rz})|_{r+\Delta r} + (\rho g \, 2\pi r \Delta r L) = 0$$

If mass is conserved and the fluid is incompressible $v_z|_z = v_z|_{z+\Delta z}$

$$\cancel{2\pi r \Delta r} P_0 - \cancel{2\pi r \Delta r} P_L + 2\pi\left[(r\tau_{rz})|_r - (r\tau_{rz})|_{r+\Delta r}\right] + \rho g \cancel{2\pi r} \Delta r L = 0$$

$$r\frac{(P_0 - P_L)}{L} + \frac{\left[(r\tau_{rz})|_r - (r\tau_{rz})|_{r+\Delta r}\right]}{\Delta r} + \rho g r = 0$$

$$\lim_{\Delta r \to 0}$$

$$\frac{d(r\tau_{rz})}{dr} = \left[\frac{(P_0 - P_L)}{L} - \rho g\right] r$$

define $\mathcal{P} = p - \rho g z$ $\quad p_0 = \mathcal{P}_0 + \cancel{\rho g (0)}^{0}$ $\quad p_L = \mathcal{P}_L + \rho g L$

$$\frac{d(r\tau_{rz})}{dr} = \left[\frac{\mathcal{P}_0}{L} - \frac{\mathcal{P}_L}{L} - \rho g + \rho g\right] r$$

$$\frac{d(r\tau_{rz})}{dr} = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right) r$$

Boundary Conditions:

at $r = 0$ $\quad v_z = \text{finite}$

at $r = R$ $\quad v_z = 0$

$$\tau_{rz} = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{2L}\right) r + \frac{C_1}{r}$$

$$\tau_{rz} = -\mu \frac{dv_z}{dr} = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{2L}\right) r + \frac{C_1}{r}$$

$$v_z = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{\mu 4L} r^2 + \frac{\ln r}{\mu} C_1 + C_2$$

at $r = 0$ $\quad v_z = \text{finite}$ $\quad C_1 = 0$

at $r = R$ $\quad v_z = 0$

$$0 = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{4L} R^2 + C_2 \qquad C_2 = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{4\mu L} R^2$$

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{4\mu L}\left[R^2 - r^2\right]$$

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L) R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Using the Navier-Stokes:

Cylindrical coordinates: $v_r = 0 \quad v_\theta = 0 \quad v_z \neq 0$

Continuity eqn.:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Incompressible fluid:

$$\frac{d v_z}{dz} = 0$$

Navier-Stokes

z-direction:

$$\rho\left[\cancelto{S.S}{\frac{\partial v_z}{\partial t}} + \cancelto{v_r=0}{v_r}\frac{\partial v_z}{\partial r} + \cancelto{v_\theta=0}{\frac{v_\theta}{r}}\frac{\partial v_z}{\partial \theta} + v_z \cancelto{\text{continuity}}{\frac{\partial v_z}{\partial z}}\right] =$$

$$-\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right] + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial z^2}}\right] - \rho g$$

Symmetry

note direction of body force

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right] = \frac{\partial P}{\partial z} + \rho g$$

How does the pressure vary? Is $P = P(r, \theta, z)$?

1) Navier-Stoke:

r-component: $0 = -\frac{\partial P}{\partial r} + \cancelto{0}{\rho g_r}$

θ-component: $0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \cancelto{0}{\rho g_\theta}$

z-component: $0 = -\frac{\partial P}{\partial z} - \rho g + \frac{\mu}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right]$

Prove that $P = P(z)$

$$\frac{\partial}{\partial z}\left(-\frac{\partial P}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial z}\left(-\frac{\partial P}{\partial \theta}\right) = 0$$

$$\frac{\partial}{\partial z}\left(-\frac{\partial P}{\partial z} - \rho g + \frac{\mu}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right]\right) = 0$$

Can switch the order of differentiation:

$$\frac{\partial}{\partial x}\left(\frac{\partial X}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial X}{\partial x}\right)$$

$$\left.\begin{aligned} &\frac{\partial}{\partial r}\left(\frac{\partial P}{\partial z}\right) = 0 \\ &\frac{\partial}{\partial \theta}\left(\frac{\partial P}{\partial z}\right) = 0 \\ &-\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial z}\right) - \underbrace{\cancel{\frac{\partial}{\partial z}\rho g_z}}_{0} + \frac{\partial}{\partial z}\left[\frac{\mu}{r}\underbrace{\cancel{\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right]}}_{0}\right] = 0 \end{aligned}\right\} \frac{\partial P}{\partial z} = \text{const.}$$

$v_z \neq v_z(z)$

2) Easier way

assume: $p = p(z)$

$$-\frac{dP}{dz} = -\frac{dP}{dz}$$

$$\underbrace{\rho g}_{\text{not a function of position}} + \underbrace{\left(\frac{dP}{dz}\right)}_{\text{at most a function of } z} = \underbrace{\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dV_z}{dr}\right)}_{\text{only a function of } r} = \mu\ (\text{const})$$

same as separation of variables

now

$$\frac{\mu}{r}\frac{d}{dr}\left[r\frac{dv_z}{dr}\right] = \overbrace{\left(\frac{dP}{dz}\right)}^{\text{const.}} + \rho g$$

Boundary conditions:

at $r=0$ v_z = finite
at $r=R$ $v_z = 0$

$$v_z = \frac{\left(-\frac{\partial P}{\partial z}\right)R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2\right]$$

then $P = P(z)$ can be found with appropriate boundary conditions:

by inspection or using $\frac{dv_z}{\partial r} = 0$

$$v_{z,max} = \frac{\left(-\frac{\partial P}{\partial z}\right)R^2}{4\mu}$$

$$\langle v_z \rangle = \frac{\iint v_z(r)\,dA}{\iint dA} = \frac{\int_0^R v_z\, 2\pi r\, dr}{\int_0^R 2\pi r\, dr}$$

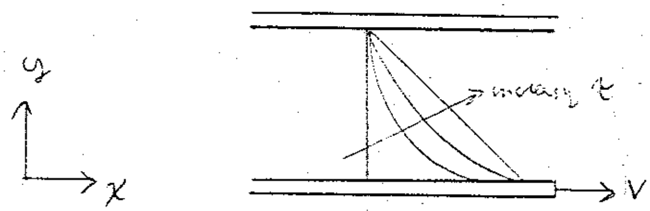
$$\langle v_z \rangle = \left(-\frac{\partial P}{\partial z}\right)\frac{R^2}{8\mu}$$

$$v_{z,max} = 2\langle v_z \rangle$$

$$Q = A\langle v_z \rangle = \iint v_z(r)\,dA = \pi\left(-\frac{\partial P}{\partial z}\right)\frac{R^4}{8\mu} = \frac{\pi(P_0 - P_L)R^4}{8\mu L}$$

Hagen-Poiseuille Law

Unsteady State Velocity Distributions



navier - stokes

$$\rho\left(\frac{\partial v_x}{\partial t} + \underset{0}{\cancel{v_x\frac{\partial v_x}{\partial x}}} + \underset{0}{\cancel{v_y\frac{\partial v_x}{\partial y}}} + \underset{0}{\cancel{v_z\frac{\partial v_x}{\partial z}}}\right) = \underset{\text{no } \Delta P}{\cancel{-\frac{\partial P}{\partial z}}} + \mu\left(\underset{0}{\cancel{\frac{\partial^2 v_x}{\partial x^2}}} + \frac{\partial^2 v_x}{\partial y^2} + \underset{\text{very thin}}{\cancel{\frac{\partial^2 v_x}{\partial z^2}}}\right) + \underset{\text{horizontal}}{\cancel{\rho g}}$$

continuity:

$$\underset{0 \text{ incomp}}{\cancel{\frac{\partial \rho}{\partial t}}} + \rho\frac{\partial v_x}{\partial x} + \underset{0}{\cancel{\rho\frac{\partial v_y}{\partial x}}} + \underset{0}{\cancel{\rho\frac{\partial v_z}{\partial z}}} = 0 \qquad \frac{\partial v_x}{\partial x} = 0$$

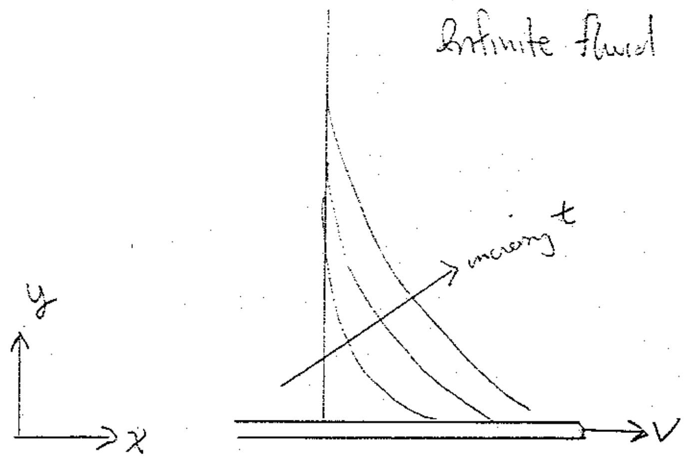
$$\rho\frac{\partial v_x}{\partial t} = \mu\frac{\partial^2 v_x}{\partial y^2} \Leftarrow$$ solve using Separation of variables

be clever in choice of non-dimensional form of $v_x \approx \theta = 1 - \frac{v_x}{V}$

B.C. #1 at $y=0$ $v_x = V$

$y = L$ $v_x = 0$

I.C. at $t=0$ $v_x = 0$



$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

I.C. at $t=0$ $v_x = 0$ $\forall y$

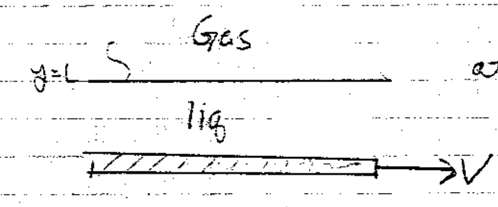
B.C. #1 at $y=0$ $v_x = V$ $t>0$

#2 at $y \to \infty$ $v_x = 0$ $t>0$

solve using Similarity transforms

$$\eta = \frac{y}{\sqrt{4\nu t}}$$ $\Leftarrow$ use chain rule to get dimensionless velocity profile out in terms of η

two Boundary conditions should collapse.



at $t=0$ $v_x = 0$ $\forall y$

the interesting question is what happens to the velocity profile at long time:

Consider G-L interface $\tau_{yx}^{g} = \tau_{yx}^{L}$ $\quad \frac{dv_x^l}{dy} = \frac{\mu_g}{\mu_l}\frac{dv_x^g}{dy} \approx 0$ $\quad \mu_g \ll \mu_l$

at steady state $\frac{d^2 v_x}{dy^2} = 0$ or $\frac{dv_x}{dy} = \text{Const.}$ $\quad$ if $\left.\frac{dv_x}{dy}\right|_{y=L} = 0$ then this must be true everywhere for $\frac{dv_x}{dy} = \text{const.}$

$v_x = C_1 y + C_2$

at $y=0$ $v_x = V$

$v_x = C_1 y + V$ $\quad C_1 = 0$

at long time the velocity of the fluid is constant i.e. not a function of y at all & $v_x = V$

Dimensionless Numbers

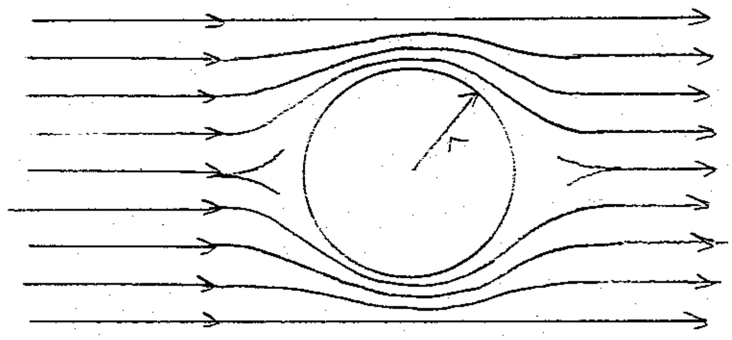
Fluid Mechanics:

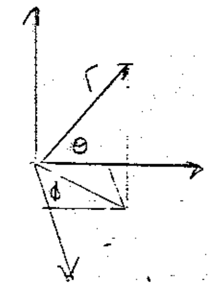
$$Re = \frac{\rho U D}{\mu} = \frac{\text{inertial forces}}{\text{viscous forces}}$$

characteristic length

$$Fr = \frac{v^2}{gD} = \frac{\text{kinetic energy}}{\text{potential energy}}$$

Creeping Flow Past a Sphere





Continuity:

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r^2}(\rho r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v_\theta \sin\theta) + \frac{1}{r\sin\theta}\cancel{\frac{\partial}{\partial\phi}}(\rho v_\phi) = 0$$

incompressible

Navier-Stokes:

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi}} - \cancel{\frac{v_\theta^2 + v_\phi^2}{r}}\right) =$$

S.S.

$$-\cancel{\frac{\partial P}{\partial r}} - \mu\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cancel{\frac{\partial^2 v_r}{\partial\phi^2}}\right)$$

not pressure driven flow

Symmetric

Creeping Flow: $Re \ll 1$

potential terms are negligible

$$0 = \left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right]$$

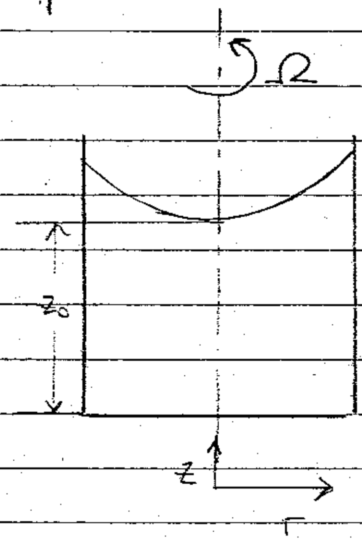
Using the streaming Potential, ψ
the boundary conditions become:

$$v_r = -\frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} = 0 \quad \text{at } r = R$$

$$v_\theta = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} = 0 \quad \text{at } r = R$$

$$\psi \rightarrow -\frac{1}{2} v_\infty r^2 \sin^2\theta \quad \text{as } r \rightarrow \infty$$

The shape of the surface of a rotating fluid:



v_θ
$v_z = 0$
$v_r = 0$

Eqn. of Continuity (incompressible fluid)

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{\rho r v_r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}\left(\cancel{\rho v_z}\right) = 0$$

$$\frac{\partial(\rho v_\theta)}{\partial \theta} = 0$$

Navier-Stokes Eqns.

r-component: $$-\rho\frac{v_\theta^2}{r} = -\frac{\partial P}{\partial r}$$

θ-component: $$0 = -\frac{1}{r}\cancel{\frac{\partial P}{\partial \theta}} + \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$$

z-component: $$0 = -\frac{\partial P}{\partial z} + \rho g$$

Boundary Conditions:

at $r=0$ $\frac{dv_\theta}{dr}=0$ = finite

at $r=R$ $v_\theta = \Omega R$

at $z=z_0$ $P=P_0$

at $r=0$ $P=P_0$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = C_1 \qquad v_\theta = \frac{C_1}{2} r + \frac{C_2}{r}$$

$C_2 = 0$ $\qquad$ $C_1 = 2\Omega$

$$v_\theta = \Omega r$$

$$\frac{\partial P}{\partial r} = \frac{\rho v_\theta^2}{r} = \rho \Omega^2 r$$

$$\frac{\partial P}{\partial z} = \rho g$$

$$dp = \frac{\partial P}{\partial r} dr + \frac{\partial P}{\partial z} dz$$

$$\int_P^{P_0} dP = \int_r^0 \rho \Omega^2 r \, dr + \int_z^{z_0} \rho g \, dz$$

$$P_0 - P = -\rho \frac{\Omega^2}{2} r^2 + \rho g (z_0 - z)$$

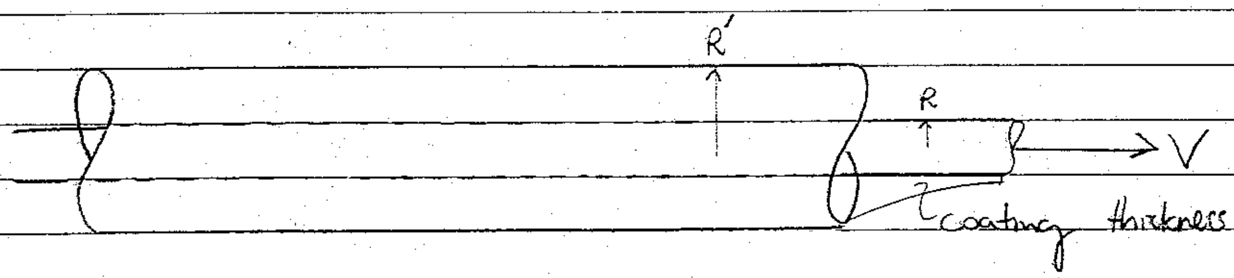
$$P - P_0 = -\rho g(z - z_0) + \frac{\rho \Omega^2 r^2}{2}$$

the locus of the free surface consists of all points consistent with $P = P_0$

$$z - z_0 = \left(\frac{\Omega^2}{2g}\right) r^2$$

The equation of pressure drop is not consistent with Bernoulli's eqn. because it describes constant angular velocity and NOT momentum

Thickness of Coating on a wire



$v_r = 0 \quad v_\theta = 0$ (not spinning)

Eqn. of Continuity: (Const. ρ)

$$\frac{1}{r}\frac{\partial(\cancel{r v_r})}{\partial r} + \frac{1}{r}\frac{\partial \cancel{v_\theta}}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Navier-Stokes:

z-component:

$$0 = \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

B.C. at $r = R \quad v_z = V$
at $r = R' \quad v_z = 0$

$$\frac{dv_z}{dr} = \frac{C_1}{r}$$

$$v_z = C_1 \ln r + C_2$$

$$0 = C_1 \ln R' + C_2 \qquad C_2 = -C_1 \ln R'$$

$$V = C_1 \ln R - C_1 \ln R'$$

$$C_1 = \frac{V}{\ln\left(\frac{R}{R'}\right)}$$

$$v_z = \frac{V}{\ln\left(\frac{R}{R'}\right)} \left[\ln r - \ln R' \right]$$

$$v_z = \frac{V}{\ln\left(\frac{R}{R'}\right)} \ln\left(\frac{r}{R'}\right)$$

Outside of Tube:

Navier-Stokes eqn.

$$\frac{dv_z}{dr} = \frac{C_1}{r} \qquad v_z = C_1 \ln r + C_2$$

Boundary Conditions:

$$\text{at } r = r' \quad \frac{dv_z}{dr} = 0 \qquad \text{at } r = R \quad v_z = V$$

$C_1 = 0$ $\quad v_z = \text{const.} = V$ (Plug flow)

Mass Balance:

$$\cancel{\rho} A_1 \langle v_z \rangle_1 = \cancel{\rho} A_2 \langle v_z \rangle_2$$

$$A_1 = \pi(R'^2 - R^2) \qquad A_2 = \pi(r'^2 - R^2)$$

Need $\langle v_z \rangle_1$

$$\langle v_z \rangle_1 = \frac{\iint_A v_z\, dA}{\iint_A dA} = \frac{2\pi \int_R^{R'} v_z\, r\, dr}{2\pi(R'^2 - R^2)}$$

$$\frac{V R'^2}{\ln\left(\frac{R}{R'}\right)} \int_R^{R'} \ln\left(\frac{r}{R'}\right)\left(\frac{r}{R'}\right) d\left(\frac{r}{R'}\right) = \frac{V R'^2}{\ln\left(\frac{R}{R'}\right)} \left[\frac{\left(\frac{r}{R'}\right)^2}{2} \ln\left(\frac{r}{R'}\right) - \frac{\left(\frac{r}{R'}\right)^2}{4} \right]_R^{R'}$$

$$= \frac{V R'^2}{\ln\left(\frac{R}{R'}\right)} \left[\frac{\left(\frac{R'}{R'}\right)^2}{2} \cancel{\ln\left(\frac{R'}{R'}\right)} - \frac{1}{4}\left(\frac{R'}{R'}\right)^2 - \frac{\left(\frac{R}{R'}\right)^2}{2} \ln\left(\frac{R}{R'}\right) + \frac{1}{4}\left(\frac{R}{R'}\right)^2 \right]$$

$$= \frac{V}{\ln\left(\frac{R}{R'}\right)} \left[\frac{R^2}{4} - \frac{R'^2}{4} - \frac{R^2}{2} \ln\left(\frac{R}{R'}\right) \right]$$

$$\langle v_z \rangle_1 = \frac{V\,\cancel{(R^2 - R'^2)}}{-4\ln\left(\frac{R}{R'}\right)\cancel{(R'^2 - R^2)}} - \frac{V R^2}{2(R'^2 - R^2)}$$

$$= \frac{V}{2}\left(\frac{1}{2\ln\frac{R'}{R}} - \frac{R^2}{(R'^2 - R^2)} \right)$$

$$\langle v_z \rangle_2 A_2 = \frac{\pi}{2}(R'^2 - R^2) V \left(\frac{1}{2\ln\left(\frac{R'}{R}\right)} - \frac{R^2}{(R'^2 - R^2)} \right)$$

$$= \frac{\pi V}{2} \left(\frac{R'^2 - R^2}{2\ln\left(\frac{R'}{R}\right)} - R^2 \right)$$

$\langle v_z \rangle_2 = V$

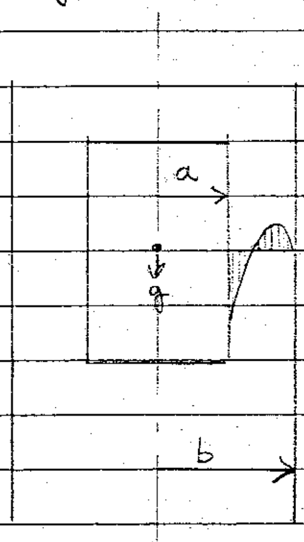
$$(r'^2 - R^2)V = \frac{V}{2}\left(\frac{R'^2 - R^2}{2\ln\left(\frac{R'}{R}\right)} - R^2\right)$$

$$r'^2 = R^2\left[\frac{\left(\frac{R'}{R}\right)^2 - 1}{4\ln\left(\frac{R'}{R}\right)}\right]$$

$$r' = R\sqrt{\frac{(R'/R)^2 - 1}{4\ln(R'/R)}}$$

$$\Delta r = r' - R = R\left[\sqrt{\frac{(R'/R)^2 - 1}{4\ln(R'/R)}} - 1\right]$$

Falling rod



$v_r = v_\theta = 0$

continuity: ($\rho = \text{const.}$)

$$\frac{\partial v_z}{\partial z} = 0$$

Navier-stokes:

z-direction:

$$\underbrace{\frac{\partial P}{\partial z} - \rho g}_{\text{function of } z \text{ only}} = \underbrace{\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{\text{function of } r \text{ only}} = \text{const.}$$

$$\frac{dP}{dz} - \rho g = C_1$$

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_1$$

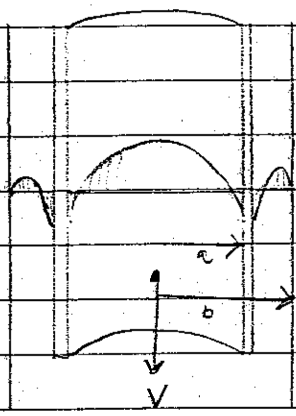
Boundary conditions:

at $r = a$ $\quad -\mu\frac{dv_z}{dr} = \frac{\text{Force}}{\text{Area}} = \frac{\pi a^2 \cancel{z}(\rho_s - \rho_f)g}{2\pi a \cancel{z}} = \frac{a(\rho_s - \rho_f)g}{2}$

at $r = b$ $\quad v_z = 0$

Conservation of mass $\quad \underbrace{2\pi\int_a^b v_z\, r\, dr}_{\text{[illegible]}} = \underbrace{v_z|_{r=a}\, \pi a^2}_{\text{volume displaced by rod}}$

Falling Hollow Cylinder



Split into two problems:

1) Hagen-Poissuelle flow

2) Falling Cylinder

$v_r = v_\theta = 0$ continuity: $\frac{\partial v_z}{\partial z} = 0$

① Navier-Stokes:

$$\left(\frac{\partial P}{\partial z} - \rho g\right) = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \text{Const.}$$

Boundary conditions:

at $r=0$ $\frac{d v_z}{\partial r} = 0$ at $r=a$ $v_z^1 = v_z^2$ $-\mu\frac{\partial v_z}{\partial r} = \frac{a}{2}(\rho_s - \rho_f)g$

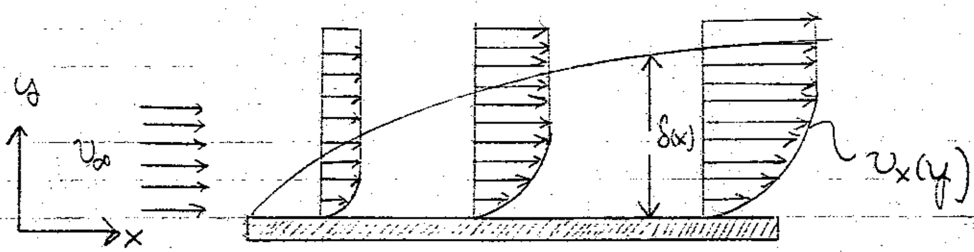
② $$\left(\frac{\partial P}{\partial z} - \rho g\right) = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \text{Const.}$$

Boundary Conditions:

at $r=b$ $v_z = 0$ at $r=a$ $-\mu\frac{\partial v_z}{\partial r} = \frac{F}{A} = \frac{a}{2}(\rho_s - \rho_f)g$

$$2\pi\int_0^b v_z r\,dr + 2\pi\int_b^a v_z r\,dr = \pi(b^2 - a^2)v_z\Big|_{z=a}$$

Boundary layer Theory



Prandtl: 1. δ is very small

2. Cannot neglect viscous forces in the boundary layer Region (forces are of the same order of magnitude as inertial forces)

In general the thickness of the boundary layer increases with with viscosity, or, more generally that it decreases as the Reynolds number increases:

$$\delta \sim \sqrt{\nu} \sim \frac{1}{\sqrt{Re}}$$

$$\delta \ll L$$

Consider: Continuity eqn. & Navier-Stokes eqns. (x & y-component) (i.e $v_z = 0$)

Continuity eqn.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

first consider order of magnitude analysis (scaling theory)

Non-dimensionalize

$$u = \frac{v_x}{V_\infty} \qquad v = \frac{v_y}{V_\infty} \qquad \bar{x} = \frac{x}{L} \qquad \bar{y} = \frac{y}{L}$$

order of magnitudes

$$v_x \; O1 \qquad \bar{x} \; O1 \qquad \bar{y} = \frac{\delta}{L} \; O\delta$$

what is the order of v_y?

$$\int_0^\delta \frac{\partial u}{\partial \bar{x}}\, d\bar{y} + \int_0^\delta \frac{\partial v}{\partial \bar{y}}\, d\bar{y} = 0$$

$$\int_0^\delta \underbrace{\frac{\partial u}{\partial \bar{x}}}_{O1} d\bar{y} + v\Big|_0^\delta = 0$$

$$v\Big|_0^\delta = -1 \int_0^\delta d\bar{y} = \delta$$

$$V_y \; O1$$

Continuity Eqn.

$$\underbrace{\frac{\partial v_x}{\partial x}}_{O1}^{\frac{1}{1} = O1} + \underbrace{\frac{\partial v_y}{\partial y}}_{O\delta}^{\frac{\delta}{\delta} = O1} = 0 \qquad \text{both } v_x \text{ \& } v_y \text{ are important}$$

Navier-Stokes Equations:

x-direction

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \bar{x}} + v\frac{\partial u}{\partial \bar{y}} = -\frac{\partial P}{\partial \bar{x}} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial \bar{x}^2} + \frac{\partial^2 u}{\partial \bar{y}^2}\right)$$

order $\quad 1 \quad 1\ 1 \quad \delta\ \frac{1}{\delta} \quad \delta^2 \quad 1 \quad \frac{1}{\delta^2}$

y-direction

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial \bar{x}} + v\frac{\partial v}{\partial \bar{y}} = -\frac{\partial P}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial \bar{x}^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\delta \quad 1 \quad \delta \quad \delta \quad 1 \qquad\qquad \delta^2 \quad \delta \qquad \frac{1}{\delta}$$

⇐ of order δ, which is small, so it is discarded

$\frac{dP}{dy} = 0$ inside boundary layer

Consider only x-direction

$$\frac{\partial u}{\partial \bar{x}}\ O\ 1 \qquad \frac{\partial^2 u}{\partial \bar{x}^2}\ O\ 1 \qquad \frac{\partial u}{\partial \bar{y}}\ O\ \frac{1}{\delta} \qquad \frac{\partial^2 u}{\partial \bar{y}^2}\ O\ \frac{1}{\delta^2}$$

$$\frac{\partial v}{\partial \bar{y}}\ O\ 1 \qquad \frac{\partial^2 v}{\partial \bar{y}^2}\ O\ \frac{1}{\delta} \qquad \frac{\partial v}{\partial \bar{x}}\ O\ \delta \qquad \frac{\partial^2 v}{\partial \bar{x}^2}\ O\ \delta$$

$$\frac{\partial u}{\partial \bar{x}} << \frac{\partial u}{\partial \bar{y}} \qquad \frac{\partial^2 u}{\partial \bar{y}^2} >> \frac{\partial^2 u}{\partial \bar{x}^2}$$

$$1 \qquad \frac{1}{\delta} \qquad \frac{1}{\delta^2} \qquad 1$$

Simplifying Navier-Stokes in x-direction

obtained from potential flow outside BL

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Boundary Conditions:

at $t=0$ $\quad v_x=0$

at $y=0$ $\quad v_x=0$; $v_y=0$

at $x=0$ $\quad v_x=V_\infty$

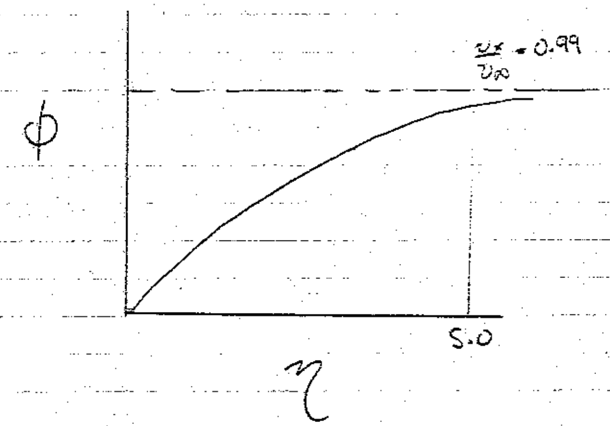
at $y=\infty$ $\quad v_x=V_\infty$

Boundary Layer Eqns.

Blasius Problem

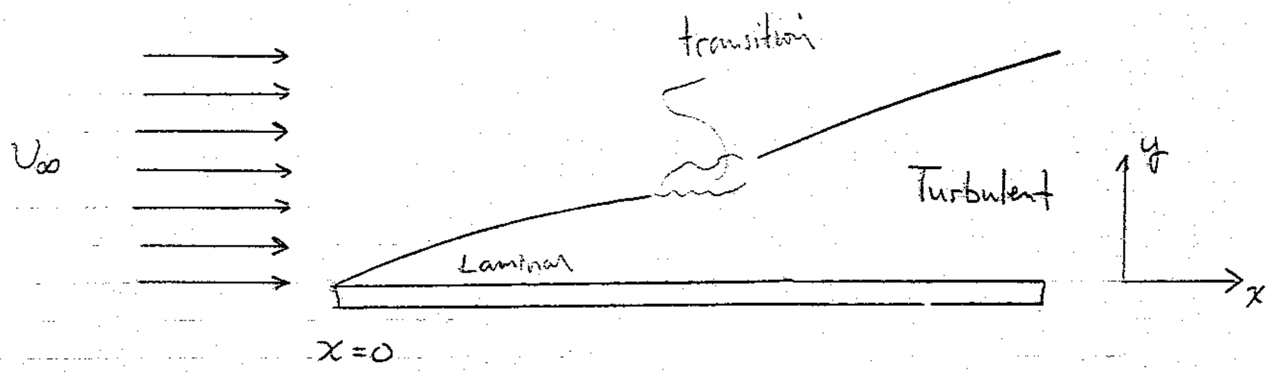
~ full solution using perturbation techniques

$$\phi(\eta) = \frac{v_x}{v_\infty} \qquad \eta = \frac{y}{\phi(x)}$$



$$\delta\sqrt{\frac{v_\infty}{\nu x}} = 5 \qquad \delta \approx 0.1 \text{ inch}$$

Turbulent Boundary layer



Prandtl's Solution

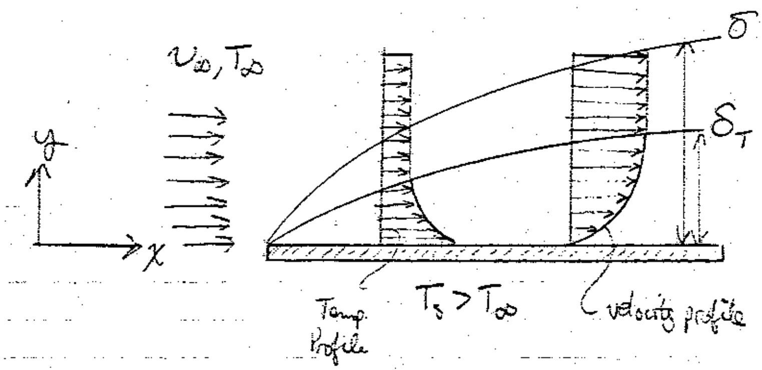
$$\delta = 0.37\,x\left(\frac{\nu}{V_\infty x}\right)^{1/5} \qquad \delta\downarrow \text{ as } Re\uparrow \qquad \delta \propto \frac{1}{Re^{1/5}}$$

the Drag coefficient for the turbulent boundary Layer

$$C_f = \frac{0.072}{Re^{1/5}} \qquad C_f\downarrow \quad Re\uparrow$$

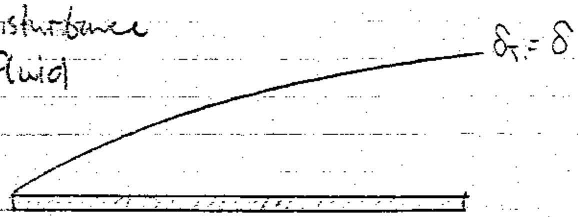
These eqns apply to a smooth plate, and give satisfactory representation in comparison to experimental data.

Heat Transfer Boundary Layer at a Flat Plate

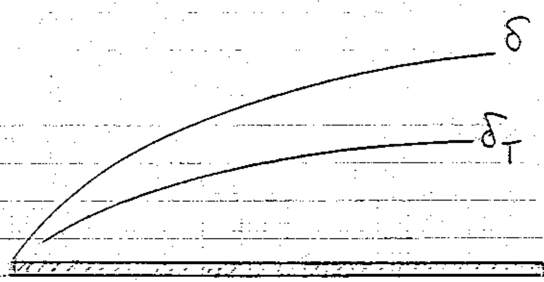


$P_r < 1$ momentum transfer is faster than heat transfer

δ - represents how far the heat or momentum disturbance penetrates into the fluid



$P_r = 1$

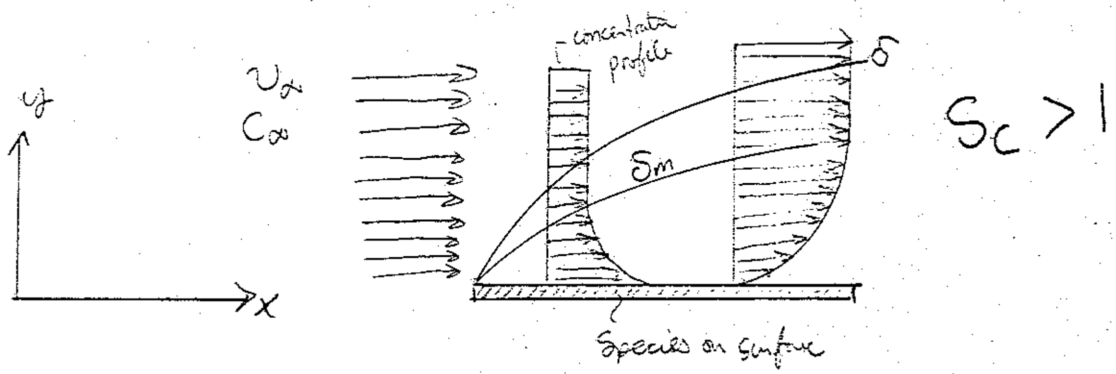


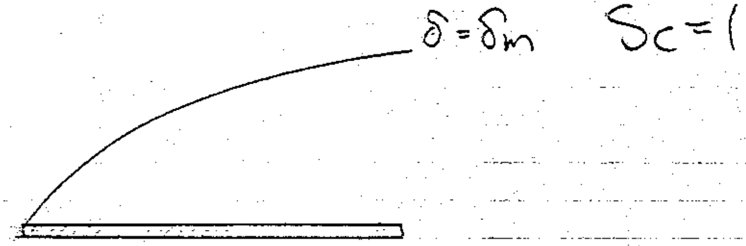
$P_r > 1$

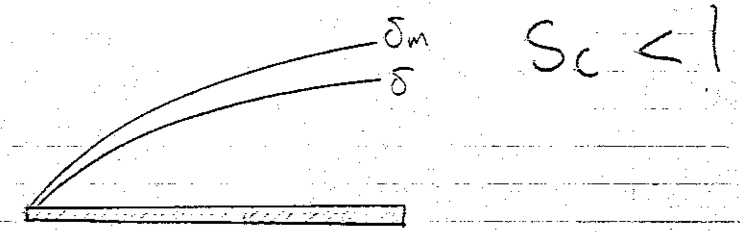
$$P_r = \frac{\nu}{\alpha} = \frac{\text{viscous momentum transport}}{\text{heat transport by conduction}}$$

$$\frac{\delta}{\delta_T} = P_r^{1/3}$$ ← leads to analogy

Mass Transfer at a Flat Plate

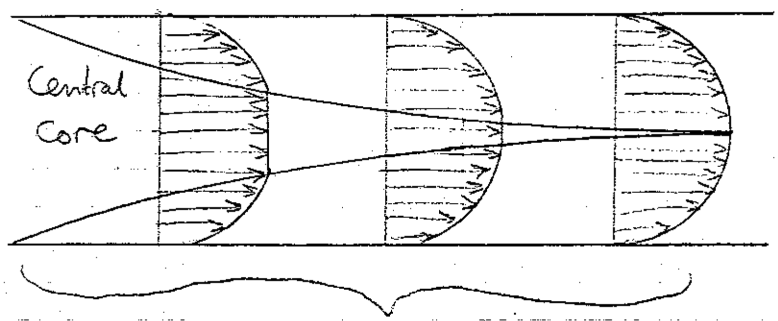






$$Sc = \frac{\nu}{\mathcal{D}_{AB}} = \frac{\text{Momentum transport}}{\text{ability of fluid to transport mass}}$$

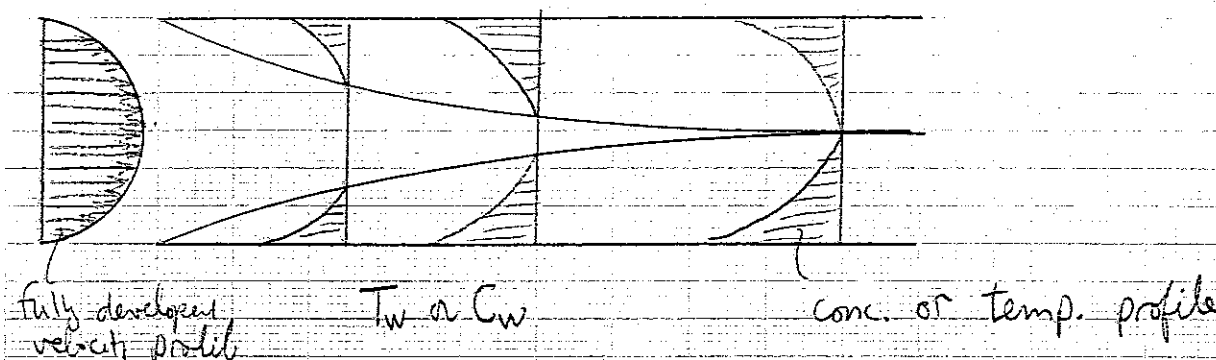
Boundary Layers in a Pipe



Entrance Region

For Laminar flow you need approx. 70 diameters to Reach fully developed flow.

Mass & Heat transfer



Solving Short z problem gives $Sc^{1/3}$ or $Pr^{1/3}$

keeping linear term only

$$v_z \frac{\partial C_A}{\partial z} = \mathcal{D} \frac{\partial^2 C_A}{\partial y^2} \qquad v_z \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial y^2} \qquad v_z = \frac{4\langle v_z \rangle}{R} y$$

$C_A = C_{Aeq}$ at $y = 0$ $\qquad$ $T = T_W$ at $y = 0$

$C_A = C_{A0}$ at $y = \infty$ $\qquad$ $T = T_\infty$ at $y \to \infty$

$C_A = C_{A0}$ at $z = 0$ $\qquad$ $T = T_\infty$ at $z = 0$

$$\frac{\pi r^2}{2\pi r} = \frac{r}{2}$$

Bernoulli's Eqn.

Mechanical Energy Balance:

$$\frac{\Delta P}{\rho} + g\Delta z + \frac{\Delta}{2}\left[\alpha\langle v\rangle^2\right] = -\mathcal{F} - \frac{\dot{W}}{\dot{m}}$$

if there is work done on the fluid then it is negative

$$\alpha = \frac{\langle v^3\rangle}{\langle v\rangle^3} = \frac{\int v^3\,dA / A}{\langle v\rangle^3}$$

this term arises because in the rigorous derivation the kinetic term is

$$\frac{1}{2}\frac{\langle v^3\rangle}{\langle v\rangle} \text{ so } \frac{1}{2}\frac{\overbrace{\langle v^3\rangle}^{\alpha}}{\langle v\rangle^3}\langle v\rangle^2$$

which simplifies analysis.

$\alpha = 2.0$ for laminar flow

$\alpha = 1.0$ for highly turbulent flow

The friction term represents all friction generated per unit mass (and therefore all the conversion of mechanical energy into heat)

$$-\mathcal{F} = \sum_i\left(\frac{1}{2}\langle v\rangle^2\frac{L}{R_h}f\right)_i + \sum_i\left(\frac{1}{2}\langle v\rangle^2 e_v\right)_i = 0$$

Sum on all sections of straight pipe

Sum on all fittings valves, meters, etc.

R_h = hydraulic radius = $\left(\dfrac{\text{cross section available for flow}}{\text{wetted perimeter}}\right)$

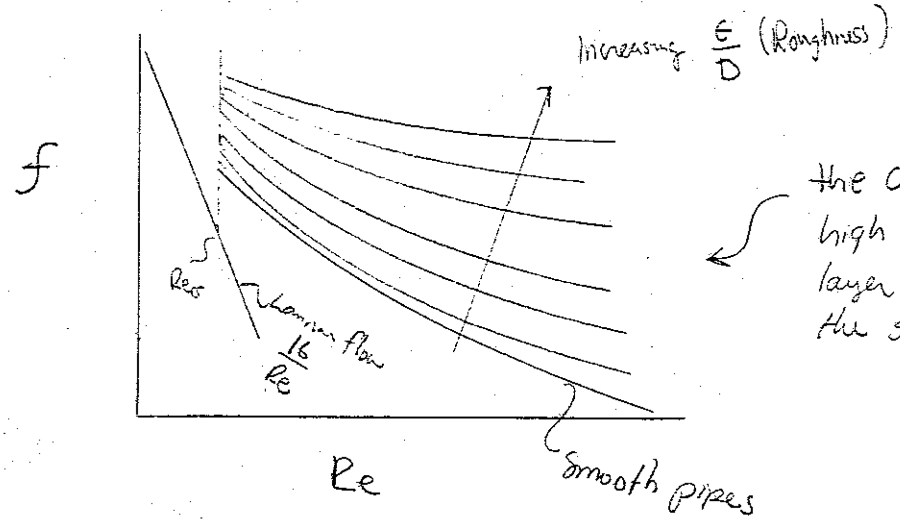
e_v ranges from

0.2 (Gate valve) – 1.5 (square 90° elbow)

f = Fanning friction factor

fanning friction factor

$$f_{\text{chemical, mechanical}} = \frac{f_{\text{civil}}}{4} = \frac{\mathcal{F}}{\left(\frac{\Delta X}{D}\right)\left(\frac{V^2}{2}\right)}$$

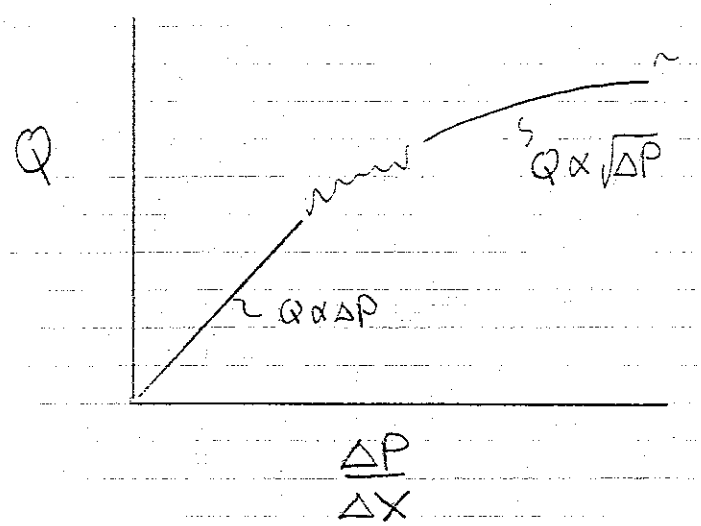


the curves becom asymtotic at high Re because the Boundary layer becomes of the order of the surface roughness

$\delta \propto \frac{1}{\sqrt{Re}}$ $Re \uparrow$ $\delta \downarrow$

Moody diagram

Flow rate vs. ΔP



turbulence uses up the energy supplied through the formation of eddies

Bernoulli Eqn. Friction

$$-\mathcal{F} = -\sum_{\substack{\text{all losses} \\ \text{in straight pipe} \\ \text{\& fittings}}} \frac{1}{2}\langle v\rangle^2 e$$

Straight pipe: $e = 4f\frac{L}{D}$ ~ length of pipe

Non-circular: $e = \frac{fL}{r_H}$

r_H = hydraulic radius = $\frac{\text{Area}}{\text{wetted perimeter}}$

for a pipe $r_H = \frac{\pi r^2}{2\pi r} = \frac{D}{4}$

The Reynolds Number

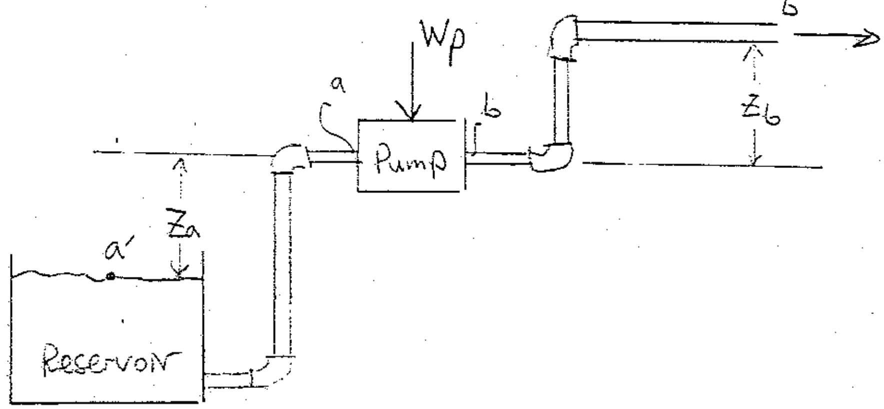
$$Re = \frac{(4r_H)\langle v\rangle\rho}{\mu}$$

Enlargements and Contractions

$$\mathcal{F} = K\frac{\langle v\rangle^2}{2} \qquad e = K$$

$$K = \left[1 - \frac{D_1^2}{D_2^2}\right]^2$$

Pumps and Metering



$$\eta W_p = \underbrace{\left(\frac{P_b}{\rho} + g z_b + \frac{\alpha_b}{2}\langle V_b\rangle^2\right) - \left(\frac{P_a}{\rho} + g z_a + \frac{\alpha_a}{2}\langle V_a\rangle^2\right)}_{\text{total Heads}}$$

$$H = \frac{P}{\rho} + gz + \frac{\alpha}{2}\langle v\rangle^2$$

Total discharge head: $W_p = \frac{H_b - H_a}{\eta} = \frac{\Delta H}{\eta}$

Developed Head: $\Delta H = H_b - H_a \ \left(\frac{m^2}{s^2}\right)\left(\frac{J}{kg}\right)$

Hydraulic Head: $\frac{1}{g}\left\{ \frac{\Delta\langle v\rangle^2}{2} + g\Delta h + \frac{\Delta P}{\rho} \right\}$

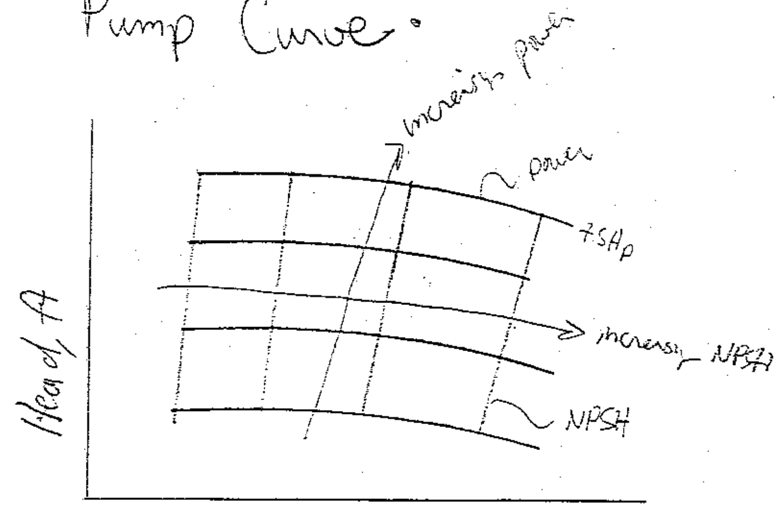
Total Head: $\Delta H_{fluid\ syst.} = \frac{1}{g}\left(\frac{\Delta P}{\rho} + g\Delta h + \Delta\left[\frac{\alpha\langle v\rangle^2}{2}\right] + \mathcal{F} \right)$

$$NPSH = \frac{1}{g}\left(\frac{P - P^{sat}}{\rho} - \mathcal{F}_{SH}\right) - z_a$$ ~ don't want cavitation

Power Supplied: $P_B = \dot{m} W_p = \frac{\dot{m}\Delta H}{\eta}$

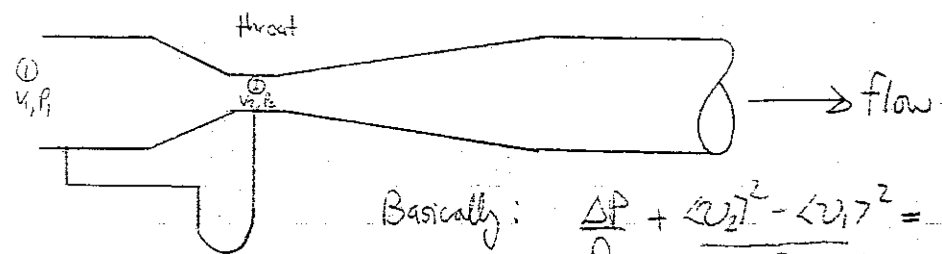
Power delivered to fluid: $P_f = \dot{m}\Delta H$

Pump Curve:



Metering of fluids:

Venturi meter:

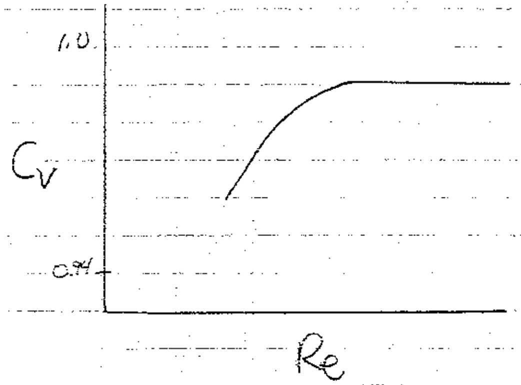


Basically: $\frac{\Delta P}{\rho} + \frac{\langle v_2 \rangle^2 - \langle v_1 \rangle^2}{2} = 0 \Leftarrow$ add correction

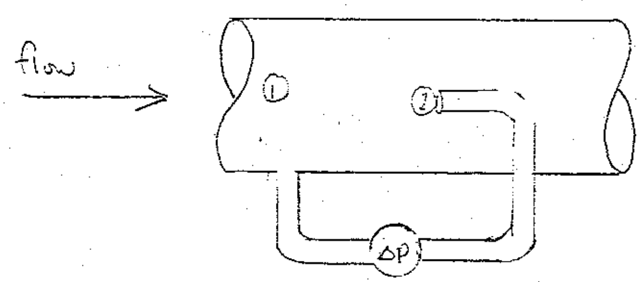
$$\langle v \rangle_2 = C_v \sqrt{\frac{2(P_1 - P_2)/\rho}{\alpha_1 - \alpha_2\left(\frac{D_2^4}{D_1^4}\right)}}$$

if C_v is unknown

1. Let $C_v = 1$ calc. $\langle v \rangle_2$ ← no
2. calc. Re from $\langle v \rangle_2$
3. find C_v from chart
4. is $C_v' = C_v$? —yes→ stop



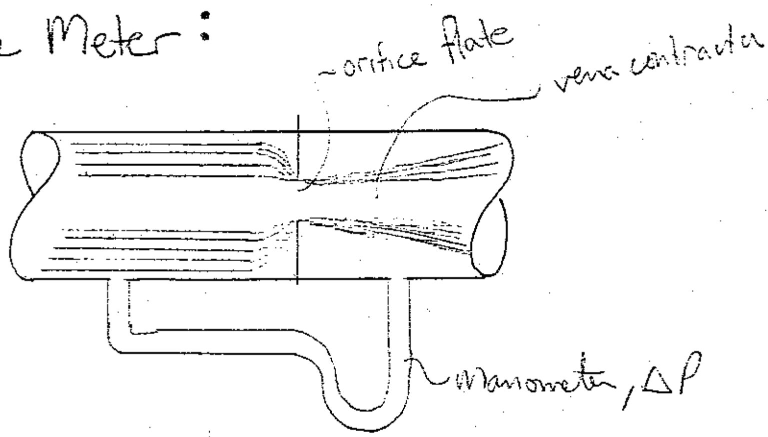
Pitot tube:



assuming no friction

$$V_1 = \left(\frac{2\Delta P}{\rho}\right)^{1/2}$$

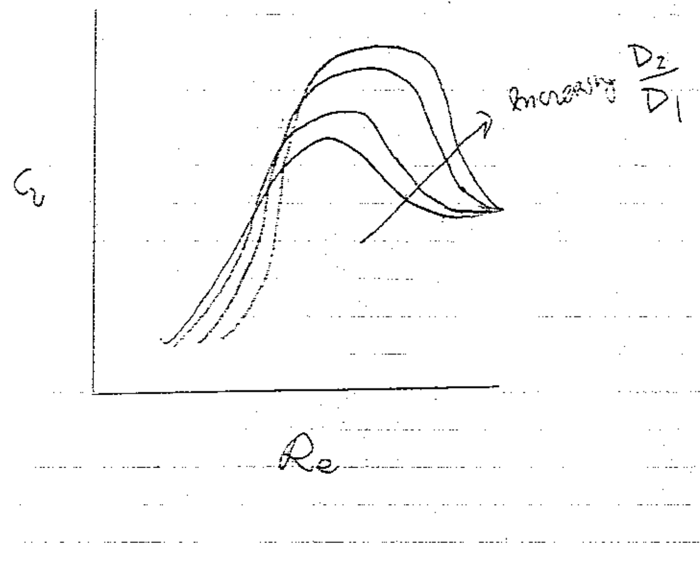
Orifice Meter:



$$\Delta P = \frac{\rho V_2^2}{2C_v^2}\left(1-\left(\frac{D_2}{D_1}\right)^4\right)$$

Mass balance:

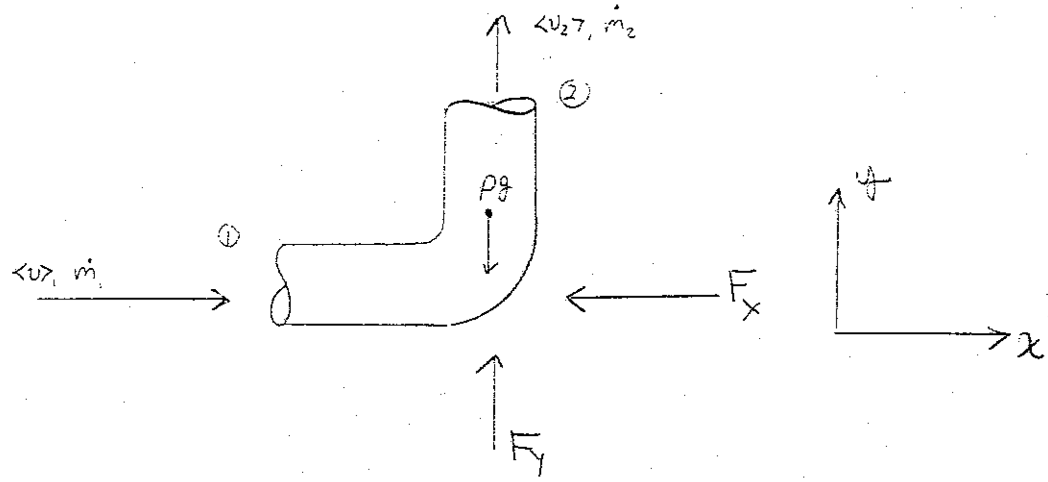
$$V_1 = V_2 \frac{A_2}{A_1}$$



for $Re > 30{,}000$

$C_v = 0.61$

Momentum



Force Balance:

$$\frac{d(mV)_{sys}}{dt} = \langle v_1 \rangle \dot{m}_1 - \langle v_2 \rangle \dot{m}_2 + \sum F$$

Mass Balance:

$$\dot{m}_1 = \dot{m}_2$$

y-direction:

$$\cancelto{0}{\frac{d(mV)_{sys}}{dt}} = \cancelto{0}{\langle v_{in} \rangle \dot{m}_{in}} - \langle v_{out} \rangle \dot{m}_{out} + \sum F_y$$

$$\sum F_y = \cancelto{0}{P_{in} A} - P_{out} A - \rho V g + F_y$$

$$0 = -\langle v_2 \rangle \dot{m}_2 - P_2 A_2 - \rho V g + F_y$$

$F_y = \langle v_2 \rangle \dot{m}_2 + P_2 A_2 + \rho V g$ $\Leftarrow$ the force is in the positive y-direction

X-direction:

$$\cancelto{0}{\frac{d(mV)_{sys}}{dt}} = \langle v_{in} \rangle \dot{m}_{in} - \cancelto{0}{\langle v_{out} \rangle \dot{m}_{out}} + \sum F_x$$

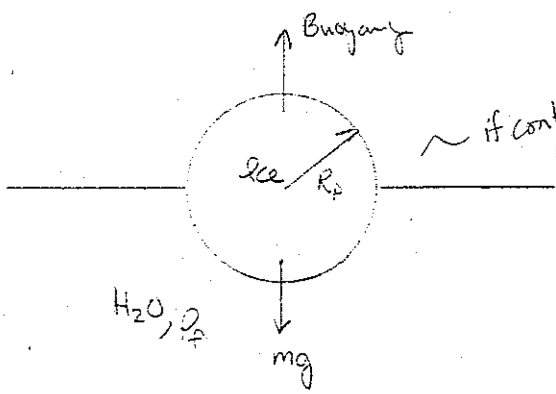
$$\sum F_x = P_{in} A - \cancelto{0}{P_{out} A} + F_x$$

$$0 = \langle v_1 \rangle \dot{m}_1 + P_1 A_1 + F_x$$

$F_x = -(\langle v_1 \rangle \dot{m}_1 + P_1 A_1)$ $\Leftarrow$ the force needed is in the negative x-direction

Buoyancy: (weight of liquid Displaced by solid)

$\rho_f V_p g$



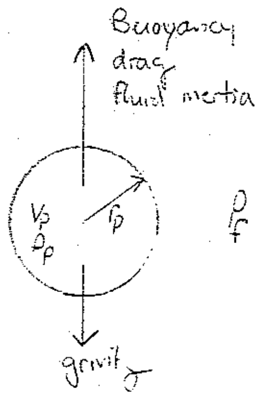
~ if contact angle is not equal to zero then surface tension must be included

$\sum F = 0$

$ma = \rho_p V_p \frac{dv_p}{dt} = \rho_f V_p g - \rho_p V_p g$

Flow Past a sphere

$\rightarrow$ determine μ_f ; D_p



General Eqn.

$$F = \underset{\text{acceleration of particle}}{\overset{\text{mass of particle}}{m a}} = \sum \text{Forces acting on sphere}$$

$$-\rho_p V_p \frac{dv_p}{dt} = -\rho_p V_p g + \rho_f V_p g + A_p \frac{v_p^2}{2} \rho_f C_D + \rho_f \frac{V_p}{2} \frac{dv_p}{dt}$$

(Gravity) (Buoyancy) (Drag) (Fluid inertia).

Fluid inertia = Force required to move fluid out of the way of the particle as it moves through the fluid

$$V_p = \frac{4}{3}\pi r_p^3 = \frac{\pi}{6} D_p^2 \qquad A_p = \frac{\pi}{4} D_p^2$$

$$Re_p = \frac{\rho_f v_p D_p}{\mu_f}$$

Drag Coefficients:

component of normal force (shape dependent)

$$F = F_{form} + F_{friction}$$ tangential force ($\tau_{total, R}$ integrate over the surface)

$$F = \frac{A_p}{2} v_p^2 \rho_f f$$

$f = f_{form} + f_{friction}$ ~ Integration of the normal force

$$f_{form} = \frac{2}{\pi}\int_0^{2\pi}\int_0^{\pi}\left[-P\cos\theta\right]_{r=R} \sin\theta \, d\theta \, d\phi$$

$$f_{friction} = \frac{4}{\pi}\frac{1}{Re}\int_0^{2\pi}\int_0^{\pi}\left\{-\left[\frac{r}{R}\frac{\partial}{\partial r^*}\left(\frac{v_\theta^*}{r^*}\right)+\frac{1}{r^*}\frac{\partial v_r^*}{\partial\theta}\right]\right\}\Bigg|_{r^*=1} \sin^2\theta \, d\theta \, d\phi$$

from full navier-stokes (stokes law Regime ~ creeping flow)

$$C_D = f = \frac{24}{Re_p} \qquad Re_p < 0.1$$

$$C_D = f = \frac{18}{Re_p^{0.6}} \qquad 1 \le Re_p \le 1000$$

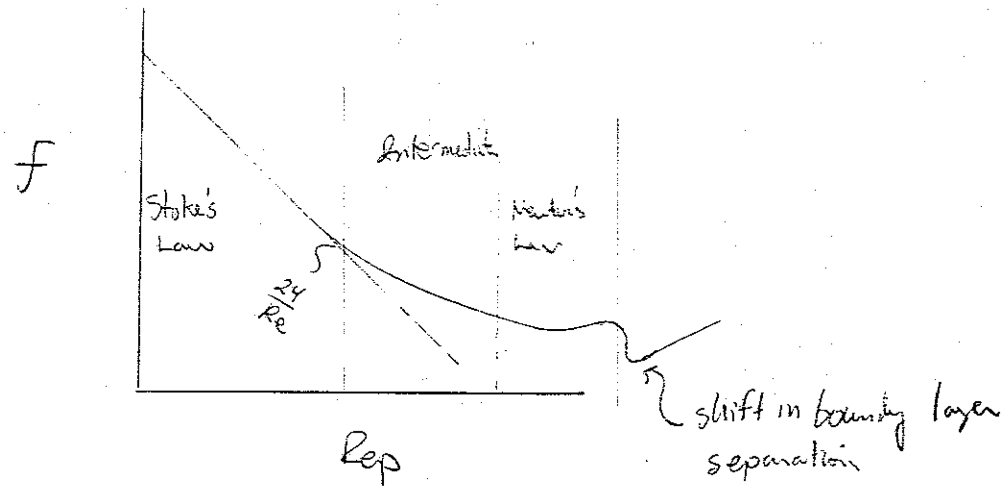
$$C_D = f = 0.44 \qquad 1000 \le Re_p \le 2\times10^5$$

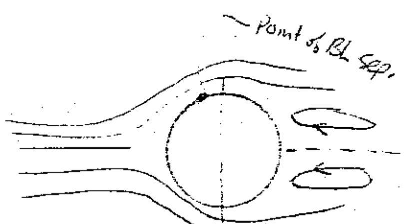
from the general eqn. and after substituting in for A_p & V_p the terminal velocity, $\frac{dv_p}{dt} = 0$, and transient settling can be determined.

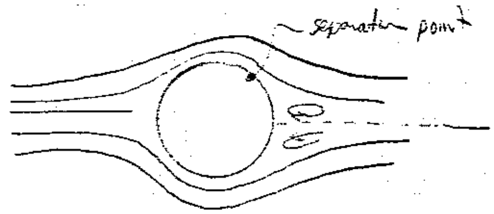
$$v_{p,f} = \frac{D_p^2 g(\rho_p - \rho_f)}{18\mu_f} \qquad Re_p < 0.1$$

terminal velocity

Drag around a sphere







$Re_p > 10$ BL separation

occurs just before the equator & a wake is formed covering the entire back of the sphere
~ characterized by large frictional losses & P at back of sphere < P_{front}
$\Delta P \Rightarrow$ direction of flow

as Re is increased both friction & drag decrease due to movement of the BL to the back of the sphere.

Stokes Law:

$$F_k = 6\pi\mu R v_\infty$$

Packed Bed

Bernoulli's Eqn.

$$\frac{\alpha_2\langle v_2\rangle^2-\alpha_1\langle v_1\rangle^2}{2}+g(h_2-h_1)+\frac{P_2-P_1}{\rho_f}=-\mathcal{F}$$

length of packed bed

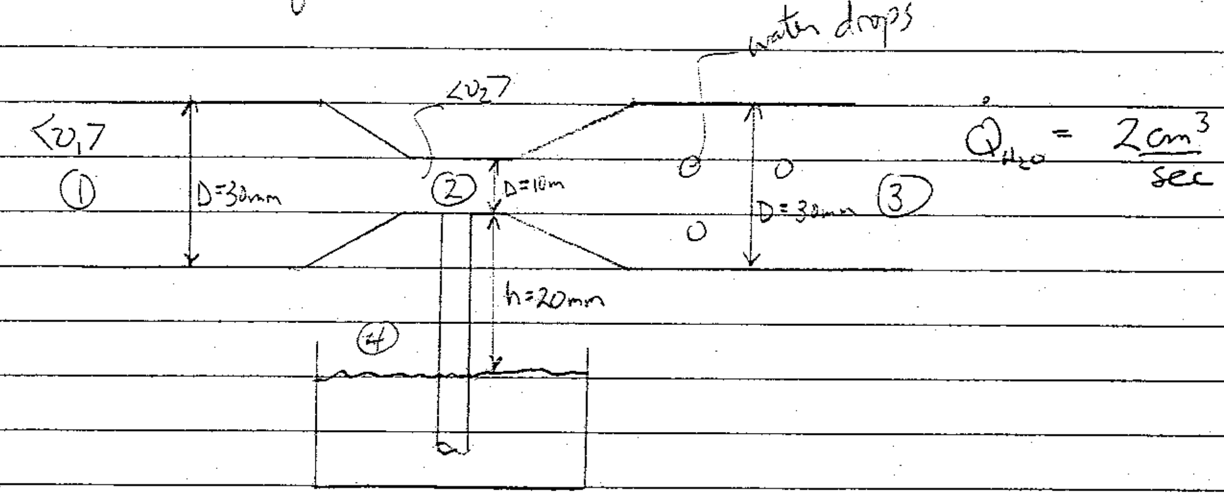
$$\mathcal{F}=\frac{v_\infty^2(1-\varepsilon)L_{pb}}{\varepsilon^2 D_p}f_{pb}$$

$$f_{pb}=\frac{150}{Re_{pb}}+1.75 \qquad Re_{pb}=\frac{D_p v_\infty \rho_f}{\mu_f(1-\varepsilon)}$$

$$\dot{m}=A_{pb}v_\infty\rho_f=\frac{\pi D_{pb}^2}{4}v_\infty\rho_f$$

$\frac{M}{S}$ m^2

Given the following system, determine the volumetric flow rate of air.



Bernoulli between 2 & 4

$$\frac{\Delta P}{\rho} + \frac{\Delta(\alpha \langle v \rangle^2)}{2} + \rho \Delta z = -\hat{F}$$

$$\frac{P_2 - P_4}{\rho} + \frac{\alpha \langle v \rangle_2^2}{2} + \rho h = -\cancel{\hat{F}}^{\,0} \text{ negligible}$$

Mass Balance Between 2 & 3

$$\dot{Q}_{H_2O}^{(2)} = \dot{Q}_{H_2O}^{(3)} \qquad \dot{Q}_{H_2O}^{(2)} = A_2 \langle v_2^{H_2O} \rangle = \dot{Q}_{H_2O}^{(3)} \qquad \langle v_2^{H_2O} \rangle = \frac{\dot{Q}_{H_2O}^{(3)}}{A_2}$$

$$\frac{P_2 - P_4}{\rho} + \frac{\alpha \dot{Q}_{H_2O}^{(3)\,2}}{2 A_2^2} + \rho h = 0$$

Bernoulli between 1 and 2

$$\frac{\Delta P}{\rho} + \frac{\Delta(\alpha \langle v \rangle^2)}{2} + \rho \cancel{\Delta z}^{\,0} = -\hat{F}$$

$$\frac{P_2-P_1}{\rho} + \frac{\alpha\langle v_2^a\rangle^2}{2} - \frac{\alpha\langle v_1^a\rangle^2}{2} = 0$$ ~ for simplicity

Mass balance: $m_{in} = m_{out}$ $\rho = \text{const.}$

$$\dot{Q}_{in}^a = \dot{Q}_{out}^a \qquad A_1\langle v_1\rangle^a = A_2\langle v_2\rangle^a$$

$$\langle v_1^a\rangle = \frac{A_2}{A_1}\langle v_2^a\rangle$$

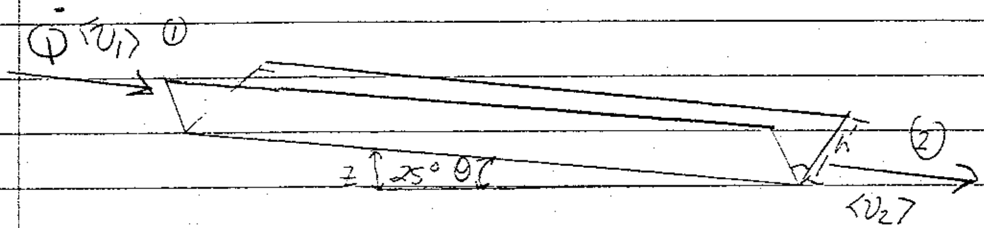
$$\frac{P_2-P_1}{\rho} + \frac{\alpha}{2}\langle v_2^a\rangle^2 - \frac{\alpha}{2}\langle v_2^a\rangle^2\frac{A_2^2}{A_1^2} = 0$$

$$\frac{P_2-P_1}{\rho} + \frac{\alpha}{2}\langle v_2^a\rangle\left[1-\frac{A_2^2}{A_1^2}\right] = 0$$

$$\langle v_2^a\rangle = \frac{(P_1-P_2)}{\rho}\frac{2}{\alpha}\frac{1}{\left[1-\frac{A_2^2}{A_1^2}\right]}$$

$\dot{Q}_1 = \langle v_2^a\rangle A_2$ $\;$ & P_2 is found from previous relationship

Open wooden flume:



$\epsilon = 0.036$ inches

Apply Bernoulli's eqn.

$$\frac{\cancel{\Delta p}}{\rho} + \frac{\Delta(\alpha\langle v_1\rangle^2)}{2} + \Delta z\, g = -\hat{F} = -\sum_i \frac{1}{2}\langle v\rangle^2 e_i$$

Mass Balance:

$$\dot{m}_1 A_1 \langle v_1\rangle = \dot{m}_2 A_2 \langle v_2\rangle \qquad -\Delta z = L\sin\theta$$

$$e_i = \frac{L}{r_H} f \qquad Re = \frac{4 r_H \langle v\rangle \rho}{\mu}$$

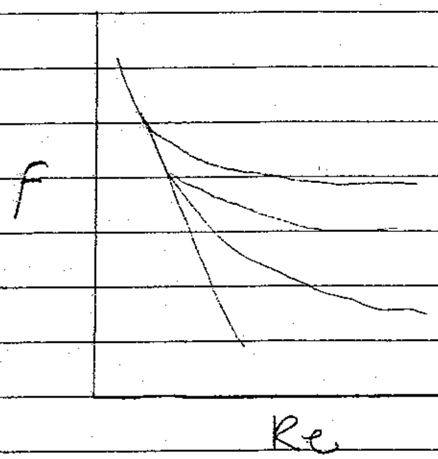
$$r_H = \frac{\frac{1}{2}h^2}{2h} = \frac{h}{4} \qquad Re = \frac{h\langle v\rangle\rho}{\mu}$$

$$e_i = 4\frac{L}{h} f$$

$$\Delta z\, g = \cancel{L}\sin\theta\, g = \frac{1}{2}\langle v\rangle^2\, 4\frac{\cancel{L}}{h} f$$

$$\sqrt{\langle v\rangle^2} = \sqrt{\frac{(\sin\theta\, g)h}{2f}}$$

determine f from chart

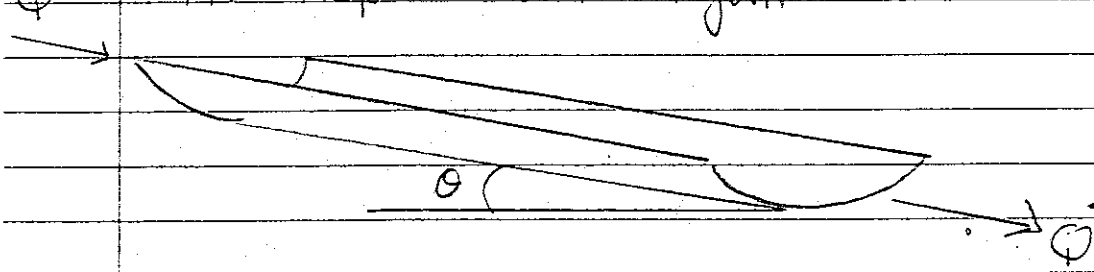


$\frac{\epsilon}{D}$ $\qquad r_H = \frac{D}{4} \quad D = 4r_H$

$$\frac{\epsilon}{4r_H} = \frac{\epsilon}{h}$$

$$\dot{Q} = \langle v \rangle A = \frac{1}{2}h^2\sqrt{\frac{\sin\theta\, g h}{2f}} \Leftarrow$$

$\dot{Q}$ Hemispherical rain gutter:

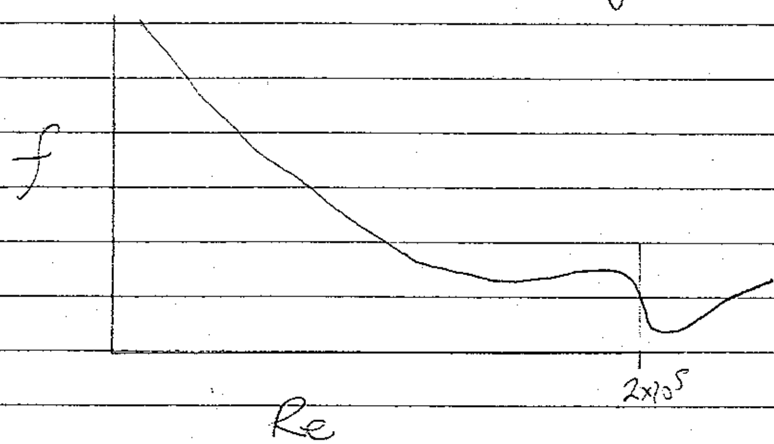


Exact same analysis as before but

$$r_H = \frac{\frac{1}{2}\pi r^2}{\pi r} = \frac{r}{2} = \frac{D}{4}$$

Baseball:

How fast should a baseball be pitched so that a rapid increase in drag takes place?



at a $Re \approx 2\times10^5$ there is a rapid increase in drag.

$$Re = \frac{\rho \langle v \rangle D_p}{\mu} = 2\times10^5$$

$$\langle v \rangle = 77 \text{ mph}$$