Longwood Invitational(Fall 2011)

Problem 8: Permutation Multiplication (Contributed by Phillip Poplin)

A **permutation of a set** A is a function $\phi : A \to A$ that is both one-to-one and onto. If we use the set $A = \{1, 2, 3, 4, 5\}$, then one permutation would be $\phi(1) = 3$, $\phi(2) = 1$, $\phi(3) = 5$, $\phi(4) = 2$, $\phi(5) = 4$.

We could write this as: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$ or simply as $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$.

We define *permutation multiplication* as function composition. We write $\sigma\tau$ instead of $\sigma \circ \tau$. Many books define permutation multiplication to be right-to-left, but in this problem we will perform permutation multiplication left-to-right so $\sigma\tau(a) = \tau(\sigma(a))$.

For example, let $A = \{1, 2, 3, 4\}$, and let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$.

Then $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$,

and $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$.

This is a lot of writing. We can simplify our notation by using *cycle notation* to describe a permutation. To write a permutation in cycle notation, we omit any elements that map to themselves (i.e. they are "not moved" by the permutation). Then we find the smallest remaining element and write it at the beginning of a new list. We add the element it maps to to the end of the list. If that element maps to a number not already in the list, we append that new number to the list. We repeat the process until we encounter an element that maps to the first element we put into the list.

If there are any remaining elements, we then start a new list which begins with the smallest remaining integer. We continue this process until we exhaust all values in A. We distinguish each list by enclosing it in parentheses.

To illustrate this new notation, consider $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 6 & 5 & 9 & 7 & 8 \end{pmatrix}$.

We start with "1" and we note that $1 \to 3$, $3 \to 4$, $4 \to 2$, and $2 \to 1$. We write this as (1 3 4 2). This leaves the values 5, 6, 7, 8, and 9. Continuing the process, we get:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 6 & 5 & 9 & 7 & 8 \end{pmatrix} = (\ 1\ 3\ 4\ 2\ )\ (\ 5\ 6\ )\ (\ 7\ 9\ 8\ )$$

Note that we have written the permutation as a product of *disjoint* cycles – cycles that do not have elements in common.

$$\text{Consider also } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = (\ 1\ 3\ 4\ ).$$

Although the cycle (1 3 4) is the same as (3 4 1), the standard notation is to write the smallest number first. We can multiply cycles, just as we multiplied permutations. Although the cycles in the input product may not be given in standard notation, we require all output to be in standard notation. We will also use the convention that the identity permutation $\iota = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$, which has all elements fixed, is written as $\iota = (\ 1\ 2\ )\ (\ 1\ 2\ )$.

Problem
Write a program that will compute the product of two or more cycles.

Input
The first line will be the number of permutation products, $1 \leq n \leq 40$. The next n lines will be the products to calculate. Each product will consist of a sequence of permutations all on one line and separated by spaces. Each permutation will be written with a space between the parentheses and the elements and a space between between the elements.

You may assume that the set A which is both the domain and range of your permutation contains the numbers $1, 2, \ldots, N$, where N is the largest number in any cycle of the input and $2 \leq N \leq 100$.

Output
You output will be the product written as a product of disjoint cycles in the same format as the input.

Examples
For $A = \{1, 2, 3, 4, 5, 6\}$,
$(\ 4\ 6\ 5\ )\ (\ 1\ 3\ 2\ ) = (\ 1\ 3\ 2\ )\ (\ 4\ 6\ 5\ )$ [The two cycles have no common elements.]

$(\ 2\ 3\ )\ (\ 1\ 3\ 4\ ) = (\ 1\ 3\ 2\ 4\ )$

$(\ 3\ 6\ )(\ 4\ 5\ 2\ )\ (\ 2\ 3\ )\ (\ 1\ 3\ 4\ ) = (\ 1\ 3\ 6\ 2\ )\ (\ 4\ 5\ )$

Input:

```
3
( 4 6 5 ) ( 1 3 2 )
( 2 3 ) ( 1 3 4 )
( 3 6 ) ( 4 5 2 ) ( 2 3 ) ( 1 3 4 )
```

Output:

```
( 1 3 2 ) ( 4 6 5 )
( 1 3 2 4 )
( 1 3 6 2 ) ( 4 5 )
```

Input:

```
2
( 1 3 2 ) ( 1 3 )
( 5 6 ) ( 5 6 )
```

Output:

```
( 2 3 )
( 1 2 ) ( 1 2 )
```