# CONTENTS

# 1 FOUNDATIONS

## 1.1 TOPOLOGY

**Definition 1.** Given a set $X$, a **topology** $\mathcal{T}$ on $X$ is a set of subsets of $X$ satisfying:

1. $\varnothing, X \in \mathcal{T}$;
2. it's closed under arbitrary unions; and
3. it's closed under finite intersections.

The pair $(X, \mathcal{T})$ is then a **topological space**.

If $\mathcal{T} \subseteq \mathcal{T}'$, then we say $\mathcal{T}$ is **coarser** than $\mathcal{T}'$, or $\mathcal{T}'$ is **finer** than $\mathcal{T}$.

**Definition 2.** A set $\mathcal{B}$ of subsets of $X$ is a **basis** for $\mathcal{T}$ on $X$ if

1. $\mathcal{B}$ covers $X$; and
2. if $x \in A, B \in \mathcal{B}$, then there's at least one $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

The topology generated by a basis $\mathcal{B}$ is defined to be the coarsest topology containing $\mathcal{B}$, which ends up being the set of all unions of elements of $\mathcal{B}$.

## 1.2 CATEGORY THEORY

**Definition 3.** A **category** $\mathbf{C}$ consists of a class of **objects** and, for all objects $X, Y \in \mathbf{C}$, a set of **morphisms** $\mathbf{C}(X, Y)$. Morphisms have an associative composition rule

$$f(gh) = (fg)h,$$

and each object has an identity morphism: for all $f : X \to Y$,

$$f = f \ \mathrm{id}_X = \mathrm{id}_Y \ f.$$

Identity morphisms are unique: Suppose $\text{id}_X$ and $\text{id}'_X$ are both identity morphisms $X \to X$, then $\text{id}_X = \text{id}_X \,\text{id}'_X = \text{id}'_X \,\text{id}_X = \text{id}'_X$.

Take any category and reverse the morphisms to get an **opposite/dual category $\mathbf{C}^{op}$**. More formally, $\mathbf{C}^{op}$ has the same objects as $\mathbf{C}$, but $\mathbf{C}^{op}(X, Y) = \mathbf{C}(Y, X)$.

**Definition 4.** Let $f : X \to Y$.

- $f$ is **left invertible** if there's a $g : Y \to X$ such that $gf = \text{id}_X$.
- $f$ is **right invertible** if there's an $h : Y \to X$ such that $fh = \text{id}_Y$.
- If both $g$ and $h$ exist, then $g = h$ is the **inverse** of $f$, and $f$ is an **isomorphism** between $X$ and $Y$.

If both left and right inverses exist, they're the same: $h = \text{id}_X \, h = gfh = g \,\text{id}_Y = g$. Inverses are unique: if $g$ and $g'$ are both inverses of $f$, then $g = \text{id}_X \, g = (g'f)g = g'(fg) = g' \,\text{id}_X = g'$.

Isos are reflexive, symmetric, and associative, so they're an equivalence relation. The equivalence classes induced by an iso in a category are **isomorphism classes** of that category.

### 1.2.1 FUNCTORS

**Functors**

**Proposition 1.** Functors map isos to isos.

Consequentially, functors encode invariants of iso classes, since if $FX \ncong FY$, then $X \ncong Y$.

**Example 1.** Algebraic topology considers functors $\mathbf{Top} \to \mathbf{C}$, where $\mathbf{C}$ is some algebraic category. For instance, homology is a functor $H : \mathbf{Top} \to R\mathbf{Mod}$, so if $HX \ncong HY$, then $X \ncong Y$.

### 1.2.2 HOM FUNCTORS

**Definition 5.** Let $f : X \to Y$. The **pushforward** $f_* : \mathbf{C}(\bullet, X) \to \mathbf{C}(\bullet, Y)$is given by postcomposition with $f$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \bullet \arrow[u, "g"] \arrow[ur, dashed, "f_*(g)=fg"'] & \end{tikzcd}$$

The **pullback** $f^* : \mathsf{C}(Y, \bullet) \to \mathsf{C}(X, \bullet)$ is given by precomposition with $f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "f^*(g)=gf"'] & Y \arrow[d, "g"] \\ & \bullet \end{tikzcd}$$

As a first step toward the Yoneda Lemma, the following attempts to formalize the idea that understanding an object is equivalent to understanding all maps into or out of it.

**Proposition 2.** TFAE:

1. $f : X \to Y$ is an iso.
2. $f_* : \mathsf{C}(\bullet, X) \to \mathsf{C}(\bullet, Y)$ is an iso of sets for all $\bullet$.
3. $f^* : \mathsf{C}(Y, \bullet) \to \mathsf{C}(X, \bullet)$ is an iso of sets for all $\bullet$.

**Hom functors**

### 1.2.3 NATURAL TRANSFORMATIONS

**natural transformations**

## 1.3 THE YONEDA LEMMA

**Motivation**

**Theorem 1** (Yoneda Lemma)**.** Fix a category $\mathsf{C}$.
**Contravariant:** For all $X \in \mathsf{C}$ and functors $F : C^{op} \to \mathsf{Set}$,

$$\mathrm{Nat}(\mathsf{C}(-, X), F) \cong FX.$$

**Covariant:** For all $X \in \mathsf{C}$ and functors $F : C \to \mathsf{Set}$,

$$\mathrm{Nat}(\mathsf{C}(X, -), F) \cong FX.$$

Note that these are isomorphisms of sets (bijections).

*Proof.* This is a proof of the contravariant version, as the covariant proof is analogous. Since $F$ is a contravariant functor, the following diagram commutes for any $f : \bullet \to X$. Note that $\eta f$ is completely determined by where $\eta$ sends $\mathrm{id}_X$, which I've denoted $\phi$; since $f$ is arbitrary, this means the choice of $\phi$

completely determines $\eta$ as a whole. There's a clear bijection between choices of $\phi$ and elements of $FX$.

$$\begin{tikzcd}
\mathbf{C}(\bullet, X) \arrow[ddd, "\eta_\bullet"] & & & \mathbf{C}(X, X) \arrow[lll, "f^*"] \arrow[ddd, "\eta_X"] \\
& \mathrm{id}_X\ f = f \arrow[d, mapsto] & \mathrm{id}_X \arrow[l, mapsto] \arrow[d, mapsto] & \\
& \eta f = (Ff)\phi & \phi := \eta\ \mathrm{id}_X \arrow[l, mapsto] & \\
F\bullet & & & FX \arrow[lll, "Ff"]
\end{tikzcd}$$

$\square$

**Corollary 1.** Let $F$ be the compatible hom functor to get the following.
Contravariant:

$$\mathrm{Nat}\left(\mathbf{C}(-, X), \mathbf{C}(-, Y)\right) \cong \mathbf{C}(X, Y).$$

Covariant:

$$\mathrm{Nat}\left(\mathbf{C}(X, -), \mathbf{C}(Y, -)\right) \cong \mathbf{C}(Y, X).$$

## 1.4 SET THEORY

# 2 CONSTRUCTING SPACES

## 2.1 THE SUBSPACE TOPOLOGY

**Proposition 3** (UP of the subspace topology)**.** Let $i : A \hookrightarrow B$ be an injective map. The subspace topology (induced by $i$) on $A$ is the unique topology on $A$ such that for all maps $f$ into $A$, the composition $if$ is continuous $\iff$ $f$ is continuous.

$$\begin{tikzcd}
\bullet \arrow[d, dashed, "f"'] \arrow[dr, dashed, "if"] & \\
A \arrow[r, hook, "i"] & B
\end{tikzcd}$$

This topology has the form

$$\left\{i^{-1}(U) \mid U \text{ open in } B\right\} = \{U \cap iA \mid U \text{ open in } B\}.$$

It's the coarsest topology on $A$ for which $i : A \hookrightarrow B$ is continuous. When $i$ is the natural inclusion, this all coincides with the usual definition of the subspace topology.

It's fine to do this even though $A$ might not be a literal subset of $B$. If $i : A \hookrightarrow B$ is an injective map, then $A$ is isomorphic as a set to its image $iA \subseteq B$. Using the UP, the space $A$ with the subspace topology induced by $i$ is homeomorphic to the space $iA \subseteq B$ with the subspace topology induced by the natural inclusion; so we can think of $A$ as being embedded in $B$.

**Definition 6.** Suppose $f : A \hookrightarrow B$ is a continuous injection (so $A$ and $B$ are already equipped with topologies). It's an **embedding** when the topology on $A$ is the same as the subspace topology induced by $f$.