DIFFERENTIAL GEOMETRY

BRADEN HOAGLAND

via Visual Differential Geometry and Forms

1 GEOMETRIES

1.1 CURVATURE OF A GEOMETRY

Euclid's *Elements* is axiomatic, and the changing of his fifth axiom is enough to change the geometry from that of the plane to either that of a sphere or of hyperbolic space. Suppose we have a straight line/geodesic L and a point p that doesn't lie on L, then

- **Euclidean:** there is exactly 1 line parallel to L and going through p;
- **Hyperbolic:** there are 2+ lines parallel to L and going through p;
- **Spherical:** there are no lines pallel to L and going through p.

In Euclidean space, the sum of the angles of a triangle Δ sum to π, but in the other geometries, this isn't necessarily true.

Definition 1. The **angular excess** $\mathcal{E}(\Delta)$ of a triangle Δ is

$$(\text{angle sum of } \Delta) - \pi.$$

Theorem 1. For any geometry, the **curvature** $\mathcal{K} = \frac{\mathcal{E}(\Delta)}{\mathcal{A}(\Delta)}$ is a constant for any triangle in the geometry:

- **Euclidean:** $\mathcal{K} = 0$;
- **Hyperbolic:** $\mathcal{K} < 0$;
- **Spherical:** $\mathcal{K} > 0$.

There are some easy corollaries:

- There are infinitely many spherical and hyperbolic geometries: just change $\mathcal{K}$;
- The sum of angles in a triangle can't be negative $\implies \mathcal{E} \geq -\pi \implies \mathcal{A} \leq |\pi/\mathcal{K}|$ in hyperbolic geometry;
- In the same geometry (i.e. same $\mathcal{K}$), triangles of different areas must have different $\mathcal{E}$, i.e. there are no similar triangles when $\mathcal{K} \neq 0$;

Proposition 1. On a sphere of radius R, the area of a triangle is $\mathcal{A} = \mathcal{E}R^2$. Thus $\mathcal{K} = 1/R^2$ on the sphere.

Proof. Take a triangle and extend each of its lines, dividing the sphere into 2 triangles and 6 other strips. Using certain pairs of strips and triangles, we can make shapes bounded by two meridians. Suppose the angle at the endpoints of such a shape is α, then its area is $4\pi R^2 \cdot (\alpha/2\pi) = 2\alpha R^2$. Using some other basic relationships and a bit of algebra, we get the result. □

Note 1. The big idea here is that $\mathcal{E} = \mathcal{K}\mathcal{A}$, i.e. the angular excess of a triangle is the "total amount" of curvature inside the triangle. This generalizes in the next section to the local Gauss-Bonnet theorem.

1.2 LOCAL GAUSS-BONNET

In spherical geometry, $\mathcal{K} = 1/R^2$, and in hyperbolic geometry, $\mathcal{K} = -1/R^2$. But we're interested in what happens on more general surfaces, not just ones that are entirely spherical or entirely flat or entirely hyperbolic. Thus we need a notion of curvature that varies point by point.

Definition 2. Fix a point p on a surface, and let $\Delta_p \to p$ denote a sequence of geodesic triangles containing p and shrinking towards p. The **Gaussian curvature** at a point p is then

$$\mathcal{K}(p) = \lim_{\Delta_p \to p} \frac{\mathcal{E}(\Delta_p)}{\mathcal{A}(\Delta_p)}.$$

Note that this agrees with Theorem 1.

This allows us to generalize Proposition 1 to general curved surfaces.

Theorem 2 (Local Gauss-Bonnet)**.** If Δ is a geodesic triangle, then

$$\mathcal{E}(\Delta) = \iint_\Delta \mathcal{K} \, d\mathcal{A}.$$

It's easy to check that $\mathcal{E}$ is additive in the sense that if the geodesic triangle Δ is subdivided into geodesic triangles Δ_1 and Δ_2, then

$$\mathcal{E}(\Delta) = \mathcal{E}(\Delta_1) + \mathcal{E}(\Delta_2).$$

Then we can take any geodesic triangle Δ and continually subdivide it into finer parts. As the parts get finer, the Gaussian curvature over each individual part approaches a constant. By Theorem 1, we know that $\mathcal{E} = \mathcal{K}\mathcal{A}$ when $\mathcal{K}$ is constant. This is exactly what the integral $\iint_\Delta \mathcal{K} \, d\mathcal{A}$ captures.

2 METRICS

2.1 MAPPINGS AND METRICS