CONTENTS

0.1 CW COMPLEXES

For $n \geq 1$, an n-**cell** (or a cell of dimension n) is a space homeomorphic to the open ball B^n. A 0-cell is said to just be a point.

Figure for 0, 1, 2-cells.

Figure for simple CW complex.

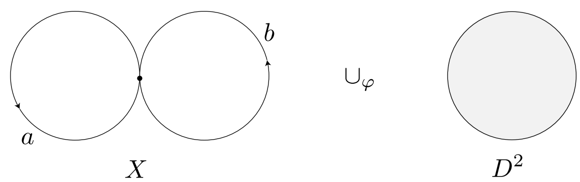
A CW complex X is then built inductively by gluing cells of different dimension together. We can represent this gluing via continuous maps $D^n \to X$:

- $\partial D^n \to X$ says how the cell is glued to existing lower-dimensional cells in X;
- $B^n \to X$ says where the new stuff goes.

Thus we only really need to specify $\partial D^n \to X$. Since $\partial D^n \cong S^{n-1}$, this is the same as specifying a map $S^{n-1} \to X$. Formally, gluing D^n to X via a map $\varphi : D^n \to X$ is represented with the space

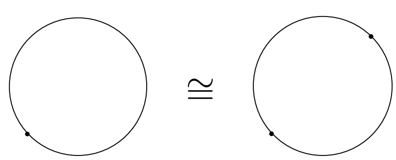
$$\frac{X \sqcup D^n}{x \sim \varphi(x)}.$$

But this notation is messy, so we'll use the notation $X \cup_\varphi D^n$ instead.



The map φ is important, as two different gluing maps can yield spaces that aren't even homotopy equivalent. For example, if we glue D^2 to X above via the constant map $x \mapsto \bullet$ (the intersection point of X), then we get $S^1 \vee S^1 \vee S^2$, which has $\pi_1 = \mathbb{Z} * \mathbb{Z}$. If we instead map ∂D^2 to $aba^{-1}b^{-1}$, then we get the torus, which has fundamental group $\mathbb{Z}^2 \ncong \mathbb{Z} * \mathbb{Z}$.

In general, cell structures on a topological space are not unique. For example, the circle can be represented with n 0-cells and n 1-cells for any $n \geq 1$.



Definition 1. A **CW complex** X is a collection of disjoint cells $e_\alpha \subset X$ such that

1. X is Hausdorff;
2. $\bigcup_\alpha e_\alpha = X$;

3. Closure finiteness: for each e_α with dimension ≥ 1, there is a continuous map $\varphi_\alpha : D^n \to X$ such that
    - $B^n \cong e_\alpha$ via φ_α;
    - φ_α maps ∂D^n to a finite union of cells, each of dimension $\leq n-1$.
4. Weak topology: $A \subseteq X$ is closed $\iff$ $A \cap \overline{e}_\alpha$ is closed in $\overline{e}_\alpha$ for all α. Here, $\overline{e}_\alpha = \varphi_\alpha(D^n)$. **is this closure of e_α?**

Note 1. Each φ is a map $D^n \to X$, but the cell being added is only the image of B^n. The image of ∂D^n is the union of lower-dimensional cells that the new cell is being glued to.

A CW complex is **finite** if it has finitely many e_α, and it is n**-dimensional** if the max dimension of all its cells is n.

We can build CW complexes inductively, starting with 0-cells and adding in higher dimensional cells to fill in the gaps.

Definition 2. The m**-skeleton** of a CW complex X is

$$X_m \doteq \bigcup \{e_\alpha \mid \dim e_\alpha \leq m\}.$$

Thus when X is n-dimensional, $X_n = X$. To build X, we start with its 0-skeleton and then glue on its higher dimensional cells to get its higher dimensional skeletons, eventually recovering X itself. If e_α^n denotes e_α being an n-cell, then

$$\begin{aligned} X_n &= X_{n-1} \cup (\cup_\alpha e_\alpha^n) \\ &= X_{n-1} \cup_{\varphi_\alpha} (\cup_\alpha D^n) \end{aligned}$$

since $\varphi_\alpha(\partial D^n)$ is contained in X_{n-1}.

Example 1. The torus is 2-dimensional since its 2-skeleton is just itself.

- 0-skeleton: $\bullet$.
- 1-skeleton: $S^1 \vee S^1$.
- 2-skeleton: T^2.

1 THE FUNDAMENTAL GROUP

1.1 THE FUNDAMENTAL GROUP

Add in notes from last year?

Proposition 1. All paths (regardless of endpoints) are homotopic in a path-connected space.

Proof. All paths are clearly homotopic to both of their endpoints. Thus we can shrink down one path to an endpoint, follow a path to one of the other path's endpoints, then expand out. $\square$

Note 2. If two paths with the same endpoints are homotopic, assume they're homotopic rel the endpoints, i.e. path homotopic.

Proposition 2. Homotopy rel A is an equivalence relation.

Path multiplication is just concatenation (with normalization to ensure that everything happens on I still). Although paths are maps, we write path multiplication left to right in the algebraic style.

This is *not* a group operation on paths because of the time normalization; however, $[fg] \doteq [f][g]$ *is* a group operation on homotopy classes. It's also well defined in the first place since if $f \simeq f'$ and $g \simeq g'$ as paths, then $fg \simeq f'g'$.

Definition 3. The **fundamental group** of X at x_0 is the homotopy classes of loops at x_0 with the group operation $[fg] \doteq [f][g]$.

Proposition 3. In the same path component, all fundamental groups are iso via the **change of basepoint iso**: if h is a path from x to y, then

$$\pi_1(X, x_1) \xrightarrow{\sim} \pi_1(X, x_0)$$
$$[f] \mapsto [hf\overline{h}].$$

Proof. The change of basepoint map is a homomorphism with inverse $[f] \mapsto [\overline{h}fh]$. $\square$

π_1 is a covariant functor $\textsf{\textbf{Top}}_* \to \textsf{\textbf{Grp}}$.

- A map $\phi : X \to Y$ induces a homomorphism $\phi_* : [f] \mapsto [\phi f]$.
- The induced maps are covariant: $(\phi\psi)_* = \phi_*\psi_*$.
- Identities are preserved: $(1_X)_* = 1_{\pi_1(X)}$.

Proposition 4. If $f, g : X \to Y$ are homotopic, then $f_* = g_*$.

Proposition 5. If $X \cong Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof. Functors preserve isos. □

1.2 HOMOTOPY EQUIVALENCE

Definition 4. We say X and Y are **homotopy equivalent**, written $X \simeq Y$, if there exist

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

such that $gf \simeq 1_X$ and $fg \simeq 1_Y$.

As it turns out, this is all we need to have isomorphic fundamental groups.

Theorem 1. If $f, g : X \to Y$ are homotopic, then

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(Y, f(x_0)) \arrow[d, dashed, "\sim"] \\ & \pi_1(Y, g(x_0)) \end{tikzcd}$$

Proof. Suppose $f \simeq g$ via F, then $h(t) \doteq F(x_0, t)$ is a path from $f(x_0)$ to $g(x_0)$. We then use this h in the change of basis iso. **Show it commutes.** □

Corollary 1. If $X \simeq Y$ via $f : X \to Y$, then $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$ via f_*.

Definition 5. A space is **contractible** if it's homotopy equivalent to a point. This is the same thing as requiring the identity map to be **nullhomotopic**, i.e. homotopic to a point.

1.3 DEFORMATION RETRACTS

Definition 6. Suppose $A \subseteq X$, then $r : X \to A$ is a **retraction** onto A if $ri = 1_A$, i.e. it fixes A.

Definition 7. A **deformation retract** of X onto A is a homotopy $F : X \times I \to X$ such that

$$\begin{aligned} F_0 &= 1_X, \\ F_1(x) &\in A, \\ F_t|_A &= 1_A. \end{aligned}$$

A **weak deformation retract** is the same, exact F_t need only map A into itself.

Note that a deformation retract is a homotopy between the identity on X and a retraction F_1 of X onto A.

Proposition 6. If A is a deformation retract of X, then $A \simeq X$ via the inclusion $A \hookrightarrow X$.

Proof. By definition, $F_1 \circ i = 1_A$. Also, $i \circ F_1 \simeq 1_X$ via F. □

Corollary 2. If A is a deformation retract of X, then $\pi_1(A, a_0) \cong \pi_1(X, a_0)$.

Proof. Apply Corollary 1 with the homotopy equivalence $A \hookrightarrow X$. □

still homo equiv if weak DR. See HW.
How much will we use weak DR's?