

CONTENTS

0.1	CW Complexes	2
1	The Fundamental Group	4
1.1	The Fundamental Group	4
1.2	Homotopy Equivalence	6
1.3	Deformation Retracts	7

0.1 CW COMPLEXES

For $n \geq 1$, an n -**cell** (or a cell of dimension n) is a space homeomorphic to the open ball B^n . A 0-cell is said to just be a point.

Figure for 0, 1, 2-cells.

Figure for simple CW complex.

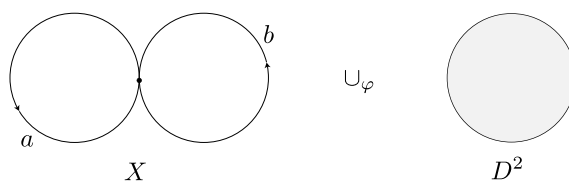
A CW complex X is then built inductively by gluing cells of different dimension together. We can represent this gluing via continuous maps $D^n \rightarrow X$:

- $\partial D^n \rightarrow X$ says how the cell is glued to existing lower-dimensional cells in X ;
- $B^n \rightarrow X$ says where the new stuff goes.

Thus we only really need to specify $\partial D^n \rightarrow X$. Since $\partial D^n \cong S^{n-1}$, this is the same as specifying a map $S^{n-1} \rightarrow X$. Formally, gluing D^n to X via a map $\varphi : D^n \rightarrow X$ is represented with the space

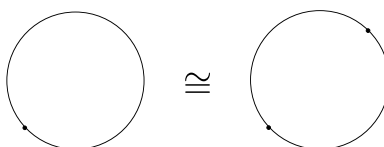
$$\frac{X \sqcup D^n}{x \sim \varphi(x)}.$$

But this notation is messy, so we'll use the notation $X \cup_{\varphi} D^n$ instead.



The map φ is important, as two different gluing maps can yield spaces that aren't even homotopy equivalent. For example, if we glue D^2 to X above via the constant map $x \mapsto \bullet$ (the intersection point of X), then we get $S^1 \vee S^1 \vee S^2$, which has $\pi_1 = \mathbb{Z} * \mathbb{Z}$. If we instead map ∂D^2 to $aba^{-1}b^{-1}$, then we get the torus, which has fundamental group $\mathbb{Z}^2 \not\cong \mathbb{Z} * \mathbb{Z}$.

In general, cell structures on a topological space are not unique. For example, the circle can be represented with n 0-cells and n 1-cells for any $n \geq 1$.



Definition 1. A **CW complex** X is a collection of disjoint cells $e_{\alpha} \subset X$ such that

1. X is Hausdorff;
2. $\bigcup_{\alpha} e_{\alpha} = X$;
3. Closure finiteness: for each e_{α} with dimension ≥ 1 , there is a continuous map $\varphi_{\alpha} : D^n \rightarrow X$ such that
 - $B^n \cong e_{\alpha}$ via φ_{α} ;

- φ_α maps ∂D^n to a finite union of cells, each of dimension $\leq n - 1$.

4. Weak topology: $A \subseteq X$ is closed $\iff A \cap \bar{e}_\alpha$ is closed in \bar{e}_α for all α . Here, $\bar{e}_\alpha = \varphi_\alpha(D^n)$. **is this closure of e_α ?**

Note 1. Each φ is a map $D^n \rightarrow X$, but the cell being added is only the image of B^n . The image of ∂D^n is the union of lower-dimensional cells that the new cell is being glued to.

A CW complex is **finite** if it has finitely many e_α , and it is **n -dimensional** if the max dimension of all its cells is n .

We can build CW complexes inductively, starting with 0-cells and adding in higher dimensional cells to fill in the gaps.

Definition 2. The m -skeleton of a CW complex X is

$$X_m \doteq \bigcup \{e_\alpha \mid \dim e_\alpha \leq m\}.$$

Thus when X is n -dimensional, $X_n = X$. To build X , we start with its 0-skeleton and then glue on its higher dimensional cells to get its higher dimensional skeletons, eventually recovering X itself. If e_α^n denotes e_α being an n -cell, then

$$\begin{aligned} X_n &= X_{n-1} \cup (\cup_\alpha e_\alpha^n) \\ &= X_{n-1} \cup_{\varphi_\alpha} (\cup_\alpha D^n) \end{aligned}$$

since $\varphi_\alpha(\partial D^n)$ is contained in X_{n-1} .

Example 1. The torus is 2-dimensional since its 2-skeleton is just itself.

- 0-skeleton: \bullet .
- 1-skeleton: $S^1 \vee S^1$.
- 2-skeleton: T^2 .

1 THE FUNDAMENTAL GROUP

1.1 THE FUNDAMENTAL GROUP

Add in notes from last year?

Proposition 1. All paths (regardless of endpoints) are homotopic in a path-connected space.

Proof. All paths are clearly homotopic to both of their endpoints. Thus we can shrink down one path to an endpoint, follow a path to one of the other path's endpoints, then expand out. \square

Note 2. If two paths with the same endpoints are homotopic, assume they're homotopic rel the endpoints, i.e. path homotopic.

Proposition 2. Homotopy rel A is an equivalence relation.

Path multiplication is just concatenation (with normalization to ensure that everything happens on I still). Although paths are maps, we write path multiplication left to right in the algebraic style.

This is *not* a group operation on paths because of the time normalization; however, $[fg] \doteq [f][g]$ is a group operation on homotopy classes. It's also well defined in the first place since if $f \simeq f'$ and $g \simeq g'$ as paths, then $fg \simeq f'g'$.

Definition 3. The **fundamental group** of X at x_0 is the homotopy classes of loops at x_0 with the group operation $[fg] \doteq [f][g]$.

Proposition 3. In the same path component, all fundamental groups are iso via the **change of base-point iso**: if h is a path from x to y , then

$$\begin{aligned}\pi_1(X, x_1) &\xrightarrow{\sim} \pi_1(X, x_0) \\ [f] &\mapsto [hf\bar{h}].\end{aligned}$$

Proof. The change of basepoint map is a homomorphism with inverse $[f] \mapsto [\bar{h}fh]$. \square

π_1 is a covariant functor $\mathbf{Top}_* \rightarrow \mathbf{Grp}$.

- A map $\phi : X \rightarrow Y$ induces a homomorphism $\phi_* : [f] \mapsto [\phi f]$.
- The induced maps are covariant: $(\phi\psi)_* = \phi_*\psi_*$.
- Identities are preserved: $(1_X)_* = 1_{\pi_1(X)}$.

Proposition 4. If $f, g : X \rightarrow Y$ are homotopic, then $f_* = g_*$.

Proposition 5. If $X \cong Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof. Functors preserve isos.

□

1.2 HOMOTOPY EQUIVALENCE

Definition 4. We say X and Y are **homotopy equivalent**, written $X \simeq Y$, if there exist

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \xleftarrow{g} & \end{array}$$

such that $gf \simeq 1_X$ and $fg \simeq 1_Y$.

As it turns out, this is all we need to have isomorphic fundamental groups.

Theorem 1. If $f, g : X \rightarrow Y$ are homotopic, then

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, f(x_0)) \\ & \searrow g_* & \downarrow \sim \\ & & \pi_1(Y, g(x_0)) \end{array}$$

Proof. Suppose $f \simeq g$ via F , then $h(t) \doteq F(x_0, t)$ is a path from $f(x_0)$ to $g(x_0)$. We then use this h in the change of basis iso. **Show it commutes.** \square

Corollary 1. If $X \simeq Y$ via $f : X \rightarrow Y$, then $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$ via f_* .

Definition 5. A space is **contractible** if it's homotopy equivalent to a point. This is the same thing as requiring the identity map to be **nullhomotopic**, i.e. homotopic to a point.

1.3 DEFORMATION RETRACTS

Definition 6. Suppose $A \subseteq X$, then $r : X \rightarrow A$ is a **retraction** onto A if $ri = 1_A$, i.e. it fixes A .

Definition 7. A **deformation retract** of X onto A is a homotopy $F : X \times I \rightarrow X$ such that

$$\begin{aligned} F_0 &= 1_X, \\ F_1(x) &\in A, \\ F_t|_A &= 1_A. \end{aligned}$$

A **weak deformation retract** is the same, exact F_t need only map A into itself.

Note that a deformation retract is a homotopy between the identity on X and a retraction F_1 of X onto A .

Proposition 6. If A is a deformation retract of X , then $A \simeq X$ via the inclusion $A \hookrightarrow X$.

Proof. By definition, $F_1 \circ i = 1_A$. Also, $i \circ F_1 \simeq 1_X$ via F . □

Corollary 2. If A is a deformation retract of X , then $\pi_1(A, a_0) \cong \pi_1(X, a_0)$.

Proof. Apply Corollary 1 with the homotopy equivalence $A \hookrightarrow X$. □

still homo equiv if weak DR. See HW.
How much will we use weak DR's?