Contents

Chapter 1

Topological Spaces and Continuous Functions

1.1 Topological Spaces

Definition 1: Topology

Let X be a set, then a **topology** on X is a collection $\mathcal{T}$ of subsets of X such that

1. $\varnothing, X \in \mathcal{T}$,
2. $\bigcup_{\alpha \in \mathcal{J}} U_\alpha \in \mathcal{T}$, and
3. $\bigcap_{i=1}^{N} U_i \in \mathcal{T}$.

Note that this definition matches the properties of open sets from real analysis. In fact, we call elements of a topology open sets.

Definition 2: Open Set

An **open set** is a member of a topology $\mathcal{T}$.

Example 1

In the standard topology on $\mathbb{R}$, consider the intersection

$$\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}\,.$$

This is not open, so the intersection of arbitrary intersections of open sets is not necessarily open.

Example 2: Topological Spaces

1. "Indiscrete" topology: $\mathcal{T}_i = \{\varnothing, X\}$
2. "Discrete" topology: $\mathcal{T}_d = \{\text{all subsets of } X\}$

Definition 3: Finer/Coarser

Let $\mathcal{T}, \mathcal{T}'$ be topologies on a set X, then $\mathcal{T}$ is **finer** than $\mathcal{T}'$ if $\mathcal{T}' \subset \mathcal{T}$. It is **strictly finer** if, in addition, $\mathcal{T}' \neq \mathcal{T}$.
$\mathcal{T}$ is **coarser** than $\mathcal{T}'$ if $\mathcal{T} \subset \mathcal{T}'$. It is **strictly coarser** if, in addition, $\mathcal{T}' \neq \mathcal{T}$.

From this we see that "fine" is a notion of a large topology, and "coarse" is a notion of a small topology.

Example 3: Finite Complement Toplogy

Let X be any set, then the **finite complement topology** is defined

$$\mathcal{T}_f = \{U \subset X \mid X - U \text{ is finite}\} \cup \{\varnothing\}\,,$$

where $X - U$ denotes the complement of U in X, i.e. $X \backslash U$.

Proposition 1. *$\mathcal{T}_f$ is a topology.*

Proof. We must show the three properties of a topology. The latter two are trivial if $U_\alpha = \varnothing$, so we assume $U_\alpha \neq \varnothing$.

1. $\varnothing$ and X are clearly in $\mathcal{T}_f$.
2. Let $U_\alpha \in \mathcal{T}_f$, then $X - U_\alpha$ is finite. By DeMorgan's laws, $X - \bigcup_\alpha U_\alpha = \bigcap_\alpha (X - U_\alpha)$. This is the intersection of finite sets, so it must be finite itself. Thus $\bigcup_\alpha U_\alpha \in \mathcal{T}_f$.

3. By DeMorgan's laws, $X - \bigcap_{i=1}^N U_i = \bigcup_{i=1}^N (X - U_i)$. Each $X - U_i$ is finite, so $\bigcup_{i=1}^N (X - U_i)$ is also finite. Thus $\bigcap_{i=1}^N U_i \in \mathcal{T}_f$.

$\square$

1.2 Basis for a Topology

Definition 4: Basis

A **basis** for a topology on a set X is a collection $\mathcal{B}$ of subsets of X such that

1. For each $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$, and
2. Given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Condition (1) is saying that $\mathcal{B}$ covers X. Since each $B \in \mathcal{B}$ is a subset of X, this implies that $\bigcup_{B \in \mathcal{B}} B = X$.

Additionally, note that since $\mathcal{B}$ exists independently from any topology, it doesn't make sense to describe its members as "open". After generating a topology using $\mathcal{B}$, however, it should be clear from the definition that every $B \in \mathcal{B}$ is a member of that topology, i.e. open in that topology.

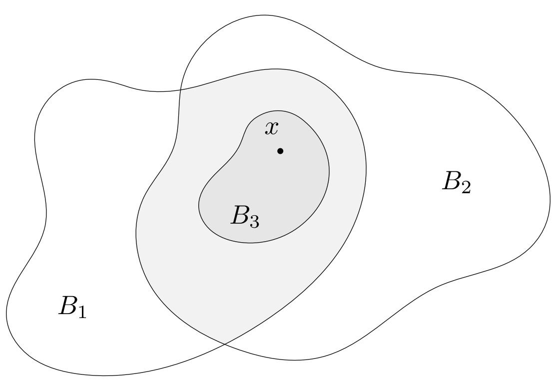

Figure 1.1: Condition (2) in the definition of a basis.

Definition 5: Generated Topology

A set U is an element of the topology $\mathcal{T}$ generated by the basis $\mathcal{B}$ if for every $x \in U$, there is a $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$.

As mentioned earlier, note that each $B \in \mathcal{B}$ clearly meets the requirements to be an element of $\mathcal{T}$.

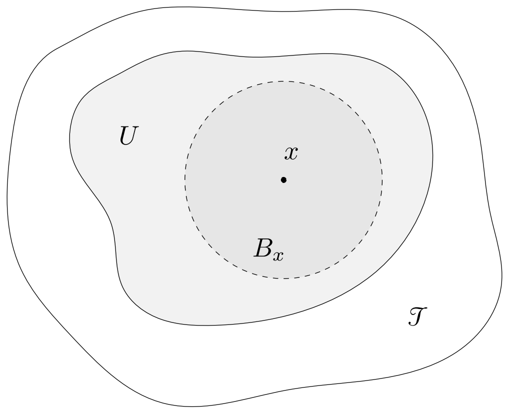

Figure 1.2: For any $U \in \mathcal{T}$, each $x \in U$ lies in some $B_x \in \mathcal{B}$ for $B_x \subset U$.

Proposition 2. *$\mathcal{T}$ is a topology.*

Proof. We again must show all three properties of a topology.

1. Clearly $\varnothing \in \mathcal{T}$. By condition (1) of the definition of a basis, $X \in \mathcal{T}$.

2. Let $x \in \bigcup_{\alpha \in \mathcal{J}} U_\alpha$, where $U_\alpha \in \mathcal{T}$ for each α, then $x \in U_\beta$ for some $B \in \mathcal{J}$. Then there is some $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_\beta$, so $x \in B_x \subset \bigcup_\alpha U_\alpha$. Thus $\bigcup_\alpha U_\alpha$ is in $\mathcal{T}$.

3. By induction, it suffices to prove that the intersection of only 2 elements. This is valid since the intersection is only over a finite number of sets.

   Let $U_1, U_2 \in \mathcal{T}$, and let $x \in U_1 \cap U_2$ (if this isn't possible, then the intersection is empty, which is in $\mathcal{T}$), then $x \in U_1$ and $x \in U_2$. Since U_1 and U_2 are in $\mathcal{T}$, we can find $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. Then $x \in B_1 \cap B_2$, so by condition (2) of the definition of a basis, there is a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$. Thus $U_1 \cap U_2$ is in $\mathcal{T}$.

   Then inductively, $\bigcap_{i=1}^N U_i \subset \mathcal{T}$ for any finite N.

□

If $\mathcal{T}$ is generated by $\mathcal{B}$, then we expect it to be comprised of unions of elements of B. Compare this to linear algebra, where a basis of a vector space spans that space.

Proposition 3. *Let $\mathcal{T}$ on X be generated by $\mathcal{B}$, then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.*

Proof. First we show $\bigcup_\alpha B_\alpha \subset \mathcal{T}$. Since $\mathcal{B} \subset \mathcal{T}$, arbitrary unions of $\mathcal{B}$ are in $\mathcal{T}$. Now we show $\mathcal{T} \subset \bigcup_\alpha$. Let $U \subset \mathcal{T}$, then for every $x \in U$, there is a $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. This implies $U = \bigcup_{x \in U} B_x$ (both directions of this equality should be clear). □

We can also get a notion of how relatively fine or coarse a topology is by using its basis.

Proposition 4. *Let $\mathcal{B}, \mathcal{B}'$ be bases for the topologies $\mathcal{T}, \mathcal{T}'$ on X, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for all $B \in \mathcal{B}$ and $x \in B$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subset \mathcal{B}$.*

Proof. First we show the backward implication. Let $U \in \mathcal{T}$, and let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is a $B \subset \mathcal{B}$ such that $x \in B \subset U$. By assumption, there is then a $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$. Thus $U \in \mathcal{T}'$, so $\mathcal{T}'$ is finer than $\mathcal{T}$.

Now we show the forward implication. Let $B \in \mathcal{B}$, and let $x \in B$, then $B \in \mathcal{T}$. By assmption, $\mathcal{T} \subset \mathcal{T}'$, so $B \in \mathcal{T}'$ as well. Then by thedefinition of a generated topology, there is a $B' \subset \mathcal{B}'$ such that $x \in B' \subset B$. □

Proposition 5. *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{C}$ be a collection of open subsets of X such that for each open set $U \in \mathcal{T}$ and each $x \in U$, there is a $C \in \mathcal{C}$ such that $x \in C \subset U$.*

Then $\mathcal{C}$ is a basis for some topology on X, and the topology it generates is $\mathcal{T}$.

Proof. First we show that $\mathcal{C}$ is a basis for some topology on X. The first condition of a basis is clear from the assumption (X is open in $\mathcal{T}$, so set $U = X$). For the second condition of a basis, consider $C_1, C_2 \in \mathcal{C}$. They are open, so $C_1 \cap C_2$ is also open. Then by assumption for any $x \in C_1 \subset C_2$, there is a C_3 such that $x \in C_3 \subset C_1 \cap C_2$.

Now we show that the topology $\mathcal{T}'$ generated by $\mathcal{C}$ is equal to $\mathcal{T}$. Since each $C \in \mathcal{C}$ is in $\mathcal{T}$, arbitrary unions of open sets are open, and $\mathcal{T}'$ is the collection of all unions of elements of $\mathcal{C}$, $\mathcal{T}' \subset \mathcal{T}$. By assumption, any element of $\mathcal{T}$ lies in a $C \in \mathcal{C}$. Then it is also in $\mathcal{T}'$. Thus $T \subset \mathcal{T}'$, so $\mathcal{T}' = \mathcal{T}$. □

Definition 6: Topologies on $\mathbb{R}$

If $\mathcal{B}$ is the collection of all open intervals of the real line, the topology generated by $\mathcal{B}$ is the **standard topology** on the real line.
If $\mathcal{B}$ is the collection of half-open intervals of the form $[a, b)$, it generates the **lower limit topology** on $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_l$.
Let K denote the set of all numbers of the form $1/n$ for $n \in \mathbb{Z}_+$, and let $\mathcal{B}$ be the collection of all open intervals (a, b) along with all sets of the form $(a, b) - K$. Then $\mathcal{B}$ generates the **K-topology** on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_K$.

Proposition 6. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with each other.*

Proof. Let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_l$, and $\mathbb{R}_K$, respectively. Given a basis element (a, b) of $\mathcal{T}$ and $x \in (a, b)$, the basis element $[x, b)$ of $\mathcal{T}'$ contains x and lies in (a, b). However, given the basis element $[x, d)$ of $\mathcal{T}'$, there is no open interval that contains x and lies in $[x, d)$. Thus $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

Similarly, given a basis element (a, b) of $\mathcal{T}$ and $x \in (a, b)$, this same interval is a basis element for $\mathcal{T}''$ that contains x. However, given the basis element $B = (-1, 1) - K$ for $\mathcal{T}''$ and the point $0 \in B$, there is no open interval that contains 0 and lies in B (since $\mathbb{Q}$ is dense in $\mathbb{R}$). Thus $\mathcal{T}''$ is strictly finer than $\mathcal{T}$.

Show $\mathcal{T}'$ and $\mathcal{T}''$ aren't comparable. □

Definition 7: Subbasis

A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Subbases are easier to construct than bases (1 condition vs. 2), but notice that the construction of a topology from a subbasis involves an extra step, namely the finite intersections. What we are doing is creating a basis $\mathcal{B}$ from $\mathcal{S}$ by taking finite intersections of the subbasis elements. Then we are taking $\mathcal{B}$ and constructing $\mathcal{T}$ by taking arbitrary unions, as is usual.

$$\mathcal{S} \xrightarrow{\bigcap_{i=1}^{N}} \mathcal{B} \xrightarrow{\bigcup_{\alpha \in \mathcal{J}}} \mathcal{T}$$

Figure 1.3: The process for constructing a topology using a subbasis $\mathcal{S}$.

Proposition 7. *$\mathcal{T}$ generated by $\mathcal{S}$ is a topology.*

Proof. It suffices to show that the collection of finite intersections of $\mathcal{S}$ is a valid basis $\mathcal{B}$, as we know that the arbitrary unions of a basis construct a valid topology. Given $x \in X$, it belongs to an element of $\mathcal{S}$ and hence to an element of $\mathcal{B}$, which is the first condition of a basis. For the second condition, let

$$B_1 = S_1 \cap \cdots \cap S_m \quad \text{and} \quad B_2 = S_1' \cap \cdots \cap S_n'$$

be two elements of $\mathcal{B}$. Their intersection

$$B_1 \cap B_2 = (S_1 \cap \cdots \cap S_m) \cap (S_1' \cap \cdots \cap S_n')$$

is also a finite intersection of elements of $\mathcal{S}$, so it belongs to $\mathcal{B}$. Then for $x \in B_1 \cap B_2$, we can always find $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$ (just set $B = B_1 \cap B_2$, since we now know this is a basis element). $\square$

1.3 The Order Topology

Definition 8: Total Order

A binary relation $\leq$ is a total order on a set X if

1. $a \leq b, b \leq a \implies a = b$,
2. $a \leq b, b \leq c \implies a \leq c$, and
3. $a \leq b$ or $b \leq a$.

A totally ordered set X can be divided into intervals of the form (a, b), $(a, b]$, $[a, b)$, and $[a, b]$. Note that even though this is reminiscent of the real number line, this is for any arbitrary ordered set.

Definition 9: The Order Topology

Let X be a totally ordered set with more than one element, and let $\mathscr{B}$ be a collection of

1. All open intervals (a, b) in X,
2. All half-open intervals $[a_0, b)$, where a_0 is the smallest element of X, and
3. All intervals of the form $(a, b_0]$, where b_0 is the largest element of X.

If X has no smallest element there are no sets of type (2), and if X has no largest elemtn, there are no sets of type (3). Then $\mathscr{B}$ is a basis for the **order topology** on X.

Example 4

The standard topology on $\mathbb{R}$, as defined in the previous section, is just the order topology derived from the usual order on $\mathbb{R}$.

Definition 10: Ray

Let a be an element of an ordered set X, then the **rays** of X determined by a are the subsets $(a, +\infty)$, $(-\infty, a)$, $[a, +\infty)$, and $(-\infty, a]$. The first two are **open rays** and the latter two are **closed rays**.

Based on the name, we expect open rays to be open sets in the order topology, and so they are. Consider (a, ∞). If X has a largest element b_0, then $(a, \infty) = (a, b_0]$, which is in the topology. If X has no largest element, then $(a, \infty) = \bigcup_{x>a}(a, x)$, so it is open. The argument is similar for $(-\infty, a)$.

In fact, the open rays form a subbasis for the order topology on X. Since the open rays are open in the order topology, the topology they generate is contained in the order topology. On the other hand, every basis element for the order topology is a finite intersection of open rays. The interval (a, b) is the intersection of $(-\infty, b)$ and (a, ∞). If $[a_0, b)$ or $(a, b_0]$ exist, they are themselves open rays. Thus the topology generated by the open rays contains the order topology.

1.4 The Product Topology

Definition 11: Product Topology

If X and Y are topological spaces, then the **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is a open subset of X and V is an open susbet of Y.

Since $X \times Y$ is itself a basis element, the first condition for a basis is trivial. Note that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Since each of these is a finite intersection of open sets, they are themselves open. Thus the intersection is itself a basis element, and the second condition for a basis is satisfied.

Theorem 1

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Proof. Here we use Proposition 5. Given an open set W of $X \times Y$ and $x \times y \in W$, by definition of the product topology there is a basis element $U \times V$ such that $x \times y \in U \times V \subset W$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, respectively, we can choose an element $B \in \mathcal{B}$ such that $x \in B \subset U$ and an element $C \in \mathcal{C}$ such that $y \in C \subset V$. Then $x \times y \in B \times C \subset W$. Since $B \times C \in \mathcal{D}$, by Proposition 5, $\mathcal{D}$ is a basis for $X \times Y$. $\square$

We can also express the product topology in terms of subbases, but we must first define projections.

Definition 12: Projection

Let $\pi_1 : X \times Y \to X$ be defined by

$$\pi_1(x, y) = x,$$

and let $\pi_2 : X \times Y \to Y$ be defined by

$$\pi_2(x, y) = y,$$

then the maps π_1 and π_2 are called the **projections** of $X \times Y$ onto its first and second factors, respectively.

Note that π_1 and π_2 are surjective (unless X or Y is empty, in which case $X \times Y$ is empty, but this is a meaningless case).

If U is an open subset of X, then $\pi_1^{-1}(U) = U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then $\pi_2^{-1}(V) = X \times V$, which is open in $X \times Y$ as well. The intersection of these two sets is $U \times V$, which leads to the following theorem.

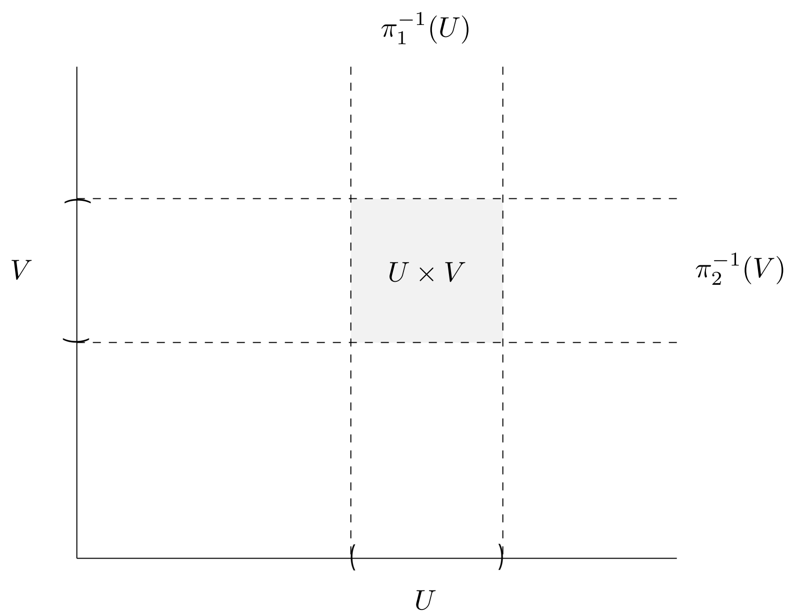

Figure 1.4: The open set in $X \times Y$ formed by the intersection of the preimages of the projections onto open sets $U \subset X$ and $V \subset Y$.

Theorem 2

The collection

$$\mathcal{S} = \left\{\pi_1^{-1}(U) \mid U \text{ open in } X\right\} \cup \left\{\pi_2^{-1}(V) \mid V \text{ open in } Y\right\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Every element of $\mathcal{S}$ is in $\mathcal{T}$, so arbitrary unions of finite intersections of elements of $\mathcal{S}$ are also in $\mathcal{T}$. Thus $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite inersectino of elements of $\mathcal{S}$, since

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

Then $U \times V \in \mathcal{T}'$, so $\mathcal{T} \subset \mathcal{T}'$. Thus $\mathcal{T} = \mathcal{T}'$. □

1.5 The Subspace Topology

Definition 13: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space. If $Y \subset X$, then

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is the **subspace topology** on Y. With this topology, Y is called a **subspace** of X.

Proposition 8. *$\mathcal{T}_Y$ is a topology on Y.*

Proof. $\varnothing \cap Y = \varnothing \in \mathcal{T}_Y$. Additionally, $Y \subset X$ implies $Y = Y \cap X$, so $Y \in \mathcal{T}_Y$. Arbitrary unions are open since

$$\bigcup_{\alpha \in \mathcal{J}} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in \mathcal{J}} U_\alpha\right) \cap Y \in \mathcal{T}_Y.$$

Finite intersections are also open since

$$\bigcap_{i=1}^{N} (U_i \cap Y) = \left(\bigcap_{i=1}^{N} U_i\right) \cap Y \in \mathcal{T}_Y.$$

□

Proposition 9. *Let $\mathcal{B}$ be a basis for the topology of X, then*

$$\mathcal{B}_Y \doteq \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let $U \in X$ and $y \in Y \cap U$. There exists $B \in \mathcal{B}$ such that $y \in B \subset U$, so $y \in Y \cap B \subset Y \cap U$. Then by Proposition 5, $\mathcal{B}_Y$ is a basis for the subspace topology on Y. □

Proposition 10. *Let Y be a subspace of X, and let U be open in Y and Y be open in X. Then U is open in X.*

Proof. U is open in Y, so $U = Y \cap V$ for some V open in X. Both sets Y and V are open in X, so their intersection U must be as well. □

Theorem 3

Let A be a subspace of X and B be a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof. $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y. Thus $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ is the general basis element for the subspace topology on $A \times B$.

Since $U \cap A$ and $V \cap B$ are the general open sets for the supspace topology on A and B, respectively, $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$.

This shows that the two topologies have the same bases, so the topologies themselves must be the same. □

This equivalence of topologies holds for the product topology, but not in general for the order topology, however.

Example 5

Let $Y = [0, 1] \cap \{2\}$ be a subset of $\mathbb{R}$. $\{2\}$ is open in the subspace topology since $(3/2, 5/2) \cap Y = \{2\}$, but $\{2\}$ is not open in the order topology since no sets of the form (a, b), $(a, b]$, or $[a, b)$ contain $\{2\}$ without also containing points not in Y.

It *is* possible to have equivalence with the ordered topology, though. We can define a condition in which this equivalence holds, but we'll need another definition first.

Definition 14: Convex

Let X be an ordered set, then $Y \subset X$ is **convex** in X if for all pairs a and b in Y such that $a < b$, the interval $(a, b) \subset X$ also lies in Y.

Note that intervals and rays in X are convex in X.

Theorem 4

Let X be an ordered set in the order topology, and let $Y \subset X$ be convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Proof. Consider $(a, \infty) \subset X$. If $a \in Y$, then $(a, \infty) \cap Y = \{x \mid x \in Y, x > a\}$. This is an open ray of Y. If $a \notin Y$, then a is either a lower bound or upper bound of Y. If a is a lower bound of Y, then $(a, \infty) \cap Y = Y$. If a is an upper bound of Y, then $(a, \infty) \cap Y = \varnothing$.

Similarly, $(-\infty, a) \cap Y$ is either $\varnothing$, Y, or an open ray of Y.

Since $(a, \infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y, and since each is open in the order topology, the order topology contains the subspace topology.

Any open ray of Y is the intersection of open rays of X with Y, so it is open in the subspace topology on Y. Since the open rays of Y are a subbasis for the order topology on Y, the order topology is contained in the subspace topology. Thus the order topology is the same as the subspace topology in this case. $\square$

From here on out, assume that if X is an ordered set in the order topology and $Y \subset X$, then Y has the subspace topology. If Y is convex in X, this is the same as the order topology on Y, otherwise it might not be.

1.6 Closed Sets and Limit Points

Definition 15: Closed Set

A set $A \subset (X, \mathcal{T})$ is closed if $X - A$ is open in X.

Theorem 5

Let $(X, \mathcal{T})$ be a topological space, and let F denote a closed set of X, then

1. $\varnothing$ and X are closed,
2. $\bigcap_{\alpha \in \mathcal{J}} F_\alpha$ is closed, and
3. $\bigcup_{i=1}^{N} F_i$ is closed.

Proof. The first condition is trivial and the second two can be established with DeMorgan's law.

1. $\varnothing = X - X$ and $X = X - \varnothing$.
2. By DeMorgan's law, $X - \bigcap_\alpha F_\alpha = \bigcup_\alpha (X - F_\alpha)$, which is open since each $X - F_\alpha$ is open.
3. Again by DeMorgan's law, $X - \bigcup_{i=1}^{N} F_i = \bigcap_{i=1}^{N} (X - F_i)$, which is similarly open.

$\square$

Proposition 11. *Let Y be a subspace of X. Then A is closed in Y if and only if it is equal to the intersection of a closed set of X with Y.*

Proof. First we show the forward implication. Assume A is closed in Y, then $Y - A$ is open in Y, so be definition $Y - A = U \cap Y$ for some U open in X. $X - U$ is closed in X, and $A = Y \cap (X - U)$, so A is the intersection of a closed set of X with Y.

Now we show the backward implication. Assume $A = C \cap Y$ for C closed in X. Then $X - C$ is open in X, so $(X - C) \cap Y$ is open in Y by the definition of the subspace topology. But $(X - C) \cap Y = Y - A$, so $Y - A$ is open in Y. Thus A is closed in Y. $\square$

Proposition 12. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. $A = F \cap Y$ for some F closed in X. A is then the intersection of closed sets of X, so it is itself closed in X. $\square$

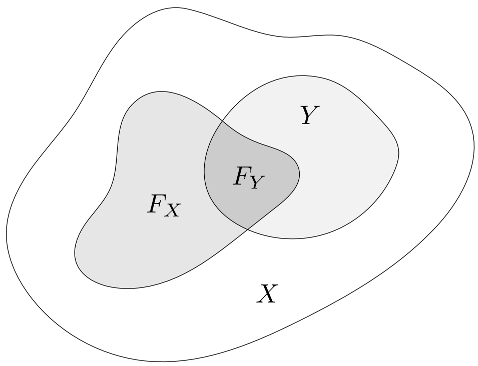

Figure 1.5: A set F_Y closed in Y, where $F_Y = Y \cap F_X$ for F_X closed in X.

Definition 16: Interior of a Set

The **interior** of a set A, denoted A^o, is the union of all open sets contained in A.

Definition 17: Closure of a Set

The **closure** of a set A, denoted $\overline{A}$ is the intersection of all closed sets containing A.

The closure of a set is clearly closed, and the interior of a set is clearly open. It is also clear that if A is open, then $A^o = A$, and if A is closed, then $\overline{A} = A$. We also have the obvious relation $A^o \subset A \subset \overline{A}$.

We have to be careful when describing closures. Given a subspace Y of X, the closure of A in X is generally not the same as the closure of A in Y. In this case, we use $\overline{A}$ to denote the closure of A in X (the overall space). We relate this to the closure of A in Y (the subspace) with the following proposition.

Proposition 13. *Let Y be a subspace of X, and let $A \subset Y$. Denote the closure of A in X by $\overline{A}$. Then the closure of A in Y is equal to $\overline{A} \cap Y$.*

Proof. Let B denote the closure of A in Y. $\overline{A}$ is closed in X, so by Proposition 11, $\overline{A} \cap Y$ is closed in Y. Since $\overline{A} \cap Y$ contains A, and since by definition B is the intersection of all closed subsets of Y containing A, we have $B \subset \overline{A} \cap Y$.

On the other hand, we know B is closed in Y. Again by Proposition 11, $B = C \cap Y$ for some C closed in X. Then C is a closed set of X containing A. Since $\overline{A}$ is the intersection of all such closed sets, we have $\overline{A} \subset C$, so $(\overline{A} \cap Y) \subset (C \cap Y) = B$. □

Definition 18: Intersects

We say A **intersects** B if $A \cap B$ is nonempty.

Definition 19: Neighborhood

A **neighborhood** of a point X is an open set containing x.

Theorem 6

Let A be a subset of a topological space X, then

1. $x \in \overline{A}$ if and only if every neighborhood of x intersects A, and
2. Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Proof.

1. We'll prove the contrapositive: $x \notin \overline{A}$ if and only if there exists an open neighborhood of x that does not intersect A. We will first show the forward implication. Assume $x \notin \overline{A}$, then $U = X - \overline{A}$ is open, contains x, and does not intersect A.

   Now we show the backward implication. Assume there exists an open neighborhood U of x that does not intersect A, then $X - U$ is a closed set containing A. By the definition of $\overline{A}$, $X - U$ must contain $\overline{A}$. Thus $x \notin \overline{A}$.

2. First we show the forward implication. By (1) we know if $x \in \overline{A}$, then every open neighborhood of x intersects A. Since basis elements are open, this implies that every basis element containing x intersects A.

   Now we show the backward implication. Assume every basis element containing x intersects A. Since every open neighborhood U of x contains some basis element, every such U also intersects A. Again by (1), this implies $x \in \overline{A}$.

□

Definition 20: Limit Point

Let $A \subset (X, \mathcal{T})$, then $x \in X$ is a **limit point**/cluster point/accumulation point of A if every open neighborhood of x intersects A at some point *other than* x .
Equivalently, x belongs to the closure of $A - \{x\}$. Note that x need not lie in A.

Theorem 7

Let $A \subset (X, \mathcal{T})$, and denote the set of limit points of A by A'. Then $\overline{A} = A \cup A'$.

Proof. First we show $A' \cup A \subset \overline{A}$. Let $x \in A'$, then every open neighborhood of x intersects A (at a point other than x). Thus by Theorem 6, $x \in \overline{A}$, so $A' \subset \overline{A}$. Since $A \subset \overline{A}$ by definition, we have $A \subset A' \subset \overline{A}$.

Now we show $\overline{A} \subset A' \cup A$. Let $x \in \overline{A}$. If $x \in A$, then this is trivial, so assume $x \notin A$. Since $x \in \overline{A}$, every neighborhood of x intersects A. Since $x \notin A$, this must be at a point other than x. Thus $x \in A' \subset A' \cup A$, so $\overline{A} \subset A' \cup A$. □

Corollary 1. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. Let $A \subset (X, \mathcal{T})$. Then A is closed if and only if $A = \overline{A} = A \cup A'$, and $A = A \cup A'$ if and only if $A' \subset A$. □

1.6.1 Hausdorff Spaces

To motivate Hausdorff Spaces, consider the set $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a, b\}, \{b\}, \{b, c\}\}$.

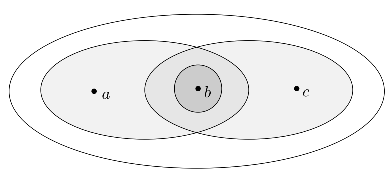

Figure 1.6: The open sets of our problematic topological space.

It is geometrically intuitive that a single point in a space should be closed; however, this is not the case for this example since $X - \{b\} = \{a, c\} \notin \mathcal{T}$. We would like the spaces we work with to not have this unintuitive behavior. We get other strange behavior when we consider convergence in this space.

Definition 21: Convergence

A sequence $\{x_n\}_{n=1}^{\infty}$ **converges** to $x \in X$ if for every open neighborhood U of x, there exists N such that $x_n \in U$ when $n > N$.

Consider the sequence $\{b, b, b, \ldots\}$. This sequence converges to a, b, and c. We dont want such behavior to be a possibility since it will lead to overly restricted results. Thus we often impose an additional condition on our topological spaces to avoid these issues.

Definition 22: Hausdorff Space

A topological space $(X, \mathcal{T})$ is a **Hausdorff space** if for every pair x_1, x_2 of distinct points of X, there exist open neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

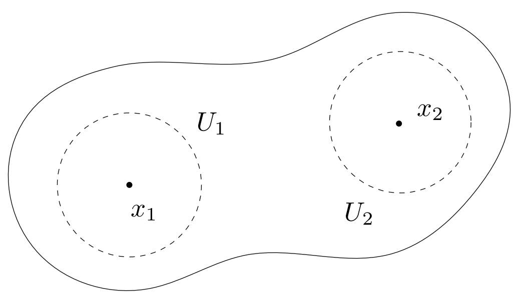

Figure 1.7: Two points in a Hausdorff space.

Proposition 14. *Every finite set in a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space. It suffices to show the arbitrary single point set $\{x_0\}$ is closed, as any finite set is the finite union of single points. Let $x \in X$ such that $x \neq x_0$, then x and x_0 have disjoint neighborhoods. Thus x's neighborhood does not intersect $\{x_0\}$, so x is not in the closure of $\{x_0\}$. Since x was arbitrary, the closure of $\{x_0\}$ is just $\{x_0\}$ itself, so it is closed. $\square$

Definition 23: T_1 Axiom

The T_1 axiom is the condition that finite point sets are closed.

Proposition 15. *Let X be a topological space satisfying the T_1 axiom, and let A be a subset of X. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. First we show the backward implication. Every neighborhood of x intersects A at infinitely many points, so it they surely all intersect A at a point other than x. Thus x is a limit point.

Now we show the forward implication. Let x be a limit point of A, and suppose some neighborhood U of x intersects A at only finitely many points.

Let $\{x_1, \dots, x_m\} = U \cap (A - \{x\})$, then $X - \{x_1, \dots, x_m\}$ is open in X since X satisfies the T_1 axiom. Then $U \cap (X - \{x_1, \dots, x_m\})$ is a neighborhood of x that doesn't intersect $A - \{x\}$ at all. This contradicts x being a limit point of A, so every neighborhood of x must intersect A at infinitely many points. □

Note that the Hausdorff condition is stronger than the T_1 axiom, so the previous proposition applies to Hausdorff spaces as well.

Proposition 16. *Let X be a Hausdorff space, then a sequence of points in X converges to at most one point in X.*

Proof. Suppose $\{x_n\} \subset X$ such that $x_n \to x \in X$. If $y \neq x$, then since X is Hausdorff we can find open neighborhoods U and $V$of x and y, respectively. The set U contains all but finitely many of the points in $\{x_n\}$, so V can only contain finitely many of the points in $\{x_n\}$. Thus x_n cannot converge to y. □

Proposition 17. *Every simply ordered set is Hausdorff in the order topology.*

Proof. Do this. □

Proposition 18. *The product of two Hausdorff spaces is a Hausdorff space.*

Proof. Do this. □

Proposition 19. *A subspace of a Hausdorff space is Hausdorff.*

Proof. Do this. □

1.7 Continuous Functions

Definition 24: Continuous

Let X, Y be topological spaces, then $f : X \to Y$ is **continuous** if for all U open in Y, $f^{-1}(U)$ is open in X.

Continuity obviously depends on f, but it also depends on the topologies of X and Y. We can emphasize this by saying that f is continuous *relative* to the topologies on X and Y.

Proposition 20. *Let Y be given by the basis $\mathcal{B}$. If $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$, then $f : X \to Y$ is continuous.*

Proof. Any U open in Y can be written $U = \bigcup_{\alpha \in \mathcal{J}} B_\alpha$ for $B_\alpha \in \mathcal{B}$, so the preimage of U is $f^{-1}(U) = \bigcup_{\alpha \in \mathcal{J}} f^{-1}(B_\alpha)$. Since each $f^{-1}(B_\alpha)$ is open, $f^{-1}(U)$ is also open. □

Proposition 21. *Let Y be given by the subbasis $\mathcal{S}$. If $f^{-1}(S)$ is open in X for all $S \in \mathcal{S}$, then $f : X \to Y$ is continuous.*

Proof. Any basis B of Y can be written as the finite intersection of subbasis elements, i.e. $B = S_1 \cap \cdots \cap S_n$. Then $f^{-1}(B) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$, so if each $f^{-1}(S_i)$ is open, then $f^{-1}(B)$ is open. Then by the previous proposition, f is continuous. □

Theorem 8

Let X and Y be topological spaces, and let $f : X \to Y$, then the following are equivalent:

1. f is continuous.
2. For all $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.
3. For all B closed in Y, $f^{-1}(B)$ is closed in X.
4. For all $x \in X$ and for each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Proof. First we will show (1) $\implies$ (2) $\implies$ (3) $\implies$ (1), then we will show (1) $\implies$ (4) $\implies$ (1).

1 implies 2: Assume f is continuous, and let $A \subset X$. We will show that if $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$, then by assumption $f^{-1}(V)$ is open in X and contains x. Since x is a limit point of A, $f^{-1}(V)$ must intersect A at some point y, so V intersects $f(A)$ at $f(y)$. This was for arbitrary V, so $f(x)$ is a limit point of $f(A)$. Thus $f(x) \in \overline{f(A)}$.

2 implies 3: Let B be closed in Y, and let $A = f^{-1}(B)$. We want to show that A is closed in X, which we accomplish by showing $A = \overline{A}$. Now $f(A) = f(f^{-1}(B)) \subset B$, so if $x \in \overline{A}$, then

$$f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B.$$

Thus $x \in f^{-1}(B) = A$, so $\overline{A} \subset A$. Clearly $A \subset \overline{A}$, so $A = \overline{A}$.

3 implies 1: Let V be open in Y, and let $B = Y - V$. Since B is closed in y, by assumption $f^{-1}(B)$ is closed in X. But

$$f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V),$$

so $X - f^{-1}(V)$ is closed in X, so $f^{-1}(V)$ is open in X.

1 implies 4: Let $x \in X$ and let V be a neighborhood of $f(x)$, then $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subset V$.

4 implies 1: Let V be open in Y, and let $x \in f^{-1}(V)$, then $f(x) \in V$. By assumption, there is a neighborhood U_x of x such that $f(U_x) \subset V$, so $U_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ is the union of the open sets $\{U_x\}$, so $f^{-1}(V)$ is open in X. $\square$

Definition 25: Continuous at a Point

If condition (4) above holds for the point $x \in X$, then we say that f is **continuous at** x.

Definition 26: Homeomorphism

Let X and Y be topological spaces, and let $f : X \leftrightarrow Y$ be bijective. If f and f^{-1} are continuous, then f is a **homeomorphism**.
Equivalently, a homeomorphism is a bijective function $f : X \leftrightarrow Y$ such that U is open in X if and only if $f(U)$ is open in Y.

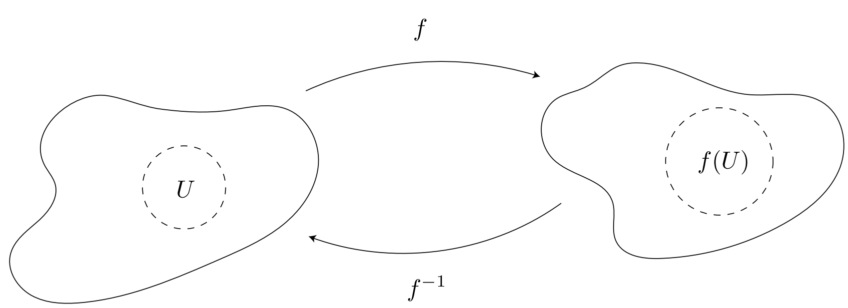

Figure 1.8: A homeomorphism f.

Definition 27: Topological Property

A **topological property** is a property of topological space X expressed entirely in terms of the topology on X (the open sets of X).

If $f : X \to Y$ is homeomorphic, then Y has the topological properties of X. From this viewpoint, a homeomorphism is a bijective correspondence that preserves topological structure. This can be thought of as the topological analogue to isomorphisms in algebra.

Definition 28: Topological Embedding

Suppose $f : X \hookrightarrow Y$ is injective and continuous with X, Y topological spaces. Let Z be $f(X)$ as a subspace of Y, then $f' : X \leftrightarrow Z$ (obtained by restricting the range of f) is bijective. If f' is a homeomorphism of X with Z, we say $f : X \hookrightarrow Y$ is a **(topological) embedding** of X in Y.