1 **THE DE RHAM COMPLEX**

Denote the space of k-forms on an n-dimensional manifold M by $\Omega^k(M)$, then the C^∞ differential forms on M form the vector space

$$\Omega^*(M) \doteq \bigoplus_{k=0}^{n} \Omega^k(M).$$

The exterior derivative is defined as usual: if f is a smooth function, then $df \doteq \sum \partial_i f \ dx_i$, and if $\omega = \sum f_I dx_I$ is a differential form, then $d\omega \doteq \sum df_I \wedge dx_I$.

Definition 1. $(\Omega^*(M), d)$ is the **de Rham complex** on M, which we represent by the cochain complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\ d\ } \Omega^1(M) \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega^n(M) \longrightarrow 0.$$

The k-th **de Rham cohomology** of M is then the vector space

$$H^k(M) \doteq \frac{\ker d \ \cap \ \Omega^k(M)}{\operatorname{im} d \ \cap \ \Omega^k(M)}.$$

Since our complex is finite, the 0-th and n-th cohomologies will always be a bit simpler:

$$H^0(M) = \ker d \ \cap \ \Omega^0(M),$$

$$H^n(M) = \frac{\Omega^n(M)}{\operatorname{im} d \ \cap \ \Omega^n(M)}.$$

Any differential form in the kernel of d is **closed**, and any in the image of d is **exact**. Note that since $d^2 = 0$, an exact form must also be closed.

2   FUNCTORIALITY OF DE RHAM COHOMOLOGY

Suppose we have a smooth map of manifolds $f : M \to N$, then this induces a pullback

$$f^* : \Omega^*(N) \to \Omega^*(M)$$
$$g \mapsto g \circ f,$$

which is easily seen from the following diagram.

$$M \xrightarrow{\ f\ } N$$
$$\underset{g \circ f}{\searrow} \quad \downarrow{\scriptstyle g}$$
$$\mathbb{R}$$

Given smooth maps between manifolds A, B, C, we can show that the pullbacks satisfy a reversed composition law: $g^* \circ f^* = (f \circ g)^*$. It's straightforward to do this calculation, but the following picture makes it clear.

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

$$\Omega^*(A) \xleftarrow{\ f^*\ } \Omega^*(B) \xleftarrow{\ g^*\ } \Omega^*(C)$$

All this shows that Ω^* is a contravariant functor from the category of smooth manifolds to the category of commutative differential graded algebras. The commutativity bit refers to the identity

$$\tau \wedge \omega = (-1)^{\deg \tau \deg \omega}\ \omega \wedge \tau.$$

We can check that f^* commutes with the exterior derivative: $f^*(d_N \omega) = d_M(f^*\omega)$ for any differential form ω on N. (Do this) This means f^* is a chain map $\Omega^*(N) \to \Omega^*(M)$, so it induces homomorphisms $H^k(N) \to H^k(M)$ for all k.

$$0 \longrightarrow \Omega^0(N) \xrightarrow{\ d_N\ } \cdots \xrightarrow{\ d_N\ } \Omega^k(N) \xrightarrow{\ d_N\ } \cdots$$
$$\downarrow{\scriptstyle f^*} \qquad\qquad\qquad \downarrow{\scriptstyle f^*}$$
$$0 \longrightarrow \Omega^0(M) \xrightarrow{\ d_M\ } \cdots \xrightarrow{\ d_M\ } \Omega^k(M) \xrightarrow{\ d_M\ } \cdots$$

Then since taking the induced homological structure is functorial (check), this means that H^* is also a contravariant functor (be specific about the category it's going to).

3   THE MAYER-VIETORIS SEQUENCE

Suppose $M = U \cup V$, where U and V are both open (why do they have to be open?). There's a natural sequence of inclusions

$$M \longleftarrow U \coprod V \overset{i_V}{\underset{i_U}{\rightleftarrows}} U \cap V,$$

(go over use of coproduct) where i_U and i_V are the inclusions into U and V, respectively. Applying the Ω^* functor then gives

$$\Omega^*(M) \longrightarrow \Omega^*(U) \bigoplus \Omega^*(V) \overset{i_V^*}{\underset{i_U^*}{\rightleftarrows}} U \cap V.$$

We can take the difference of i_V^* and i_U^* to get a new sequence.

Definition 2. The sequence

$$\Omega^*(M) \longrightarrow \Omega^*(U) \bigoplus \Omega^*(V) \longrightarrow U \cap V$$

$$(\omega, \tau) \longmapsto \tau - \omega$$

is the **Mayer-Vietoris sequence**.

You should go through this and make some of the maps explicit to make sure you understand what they each represent.

Theorem 1. The Mayer-Vietoris sequence is exact.