

CONTENTS

1	Foundations	1
1.1	Topology	1
1.2	Category Theory	1
1.2.1	Functors	2
1.2.2	Hom Functors	2
1.2.3	Natural Transformations	3
1.3	The Yoneda Lemma	3
1.4	Set Theory	4
2	Constructing Spaces	5
2.1	The Subspace Topology	5

1 FOUNDATIONS

1.1 TOPOLOGY

Definition 1. Given a set X , a **topology** \mathcal{T} on X is a set of subsets of X satisfying:

1. $\emptyset, X \in \mathcal{T}$;
2. it's closed under arbitrary unions; and
3. it's closed under finite intersections.

The pair (X, \mathcal{T}) is then a **topological space**.

If $\mathcal{T} \subseteq \mathcal{T}'$, then we say \mathcal{T} is **coarser** than \mathcal{T}' , or \mathcal{T}' is **finer** than \mathcal{T} .

Definition 2. A set \mathcal{B} of subsets of X is a **basis** for \mathcal{T} on X if

1. \mathcal{B} covers X ; and
2. if $x \in A, B \in \mathcal{B}$, then there's at least one $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

The topology generated by a basis \mathcal{B} is defined to be the coarsest topology containing \mathcal{B} , which ends up being the set of all unions of elements of \mathcal{B} .

1.2 CATEGORY THEORY

Definition 3. A **category** \mathbf{C} consists of a class of **objects** and, for all objects $X, Y \in \mathbf{C}$, a set of **morphisms** $\mathbf{C}(X, Y)$. Morphisms have an associative composition rule

$$f(gh) = (fg)h,$$

and each object has an identity morphism: for all $f : X \rightarrow Y$,

$$f = f \operatorname{id}_X = \operatorname{id}_Y f.$$

Identity morphisms are unique: Suppose id_X and id'_X are both identity morphisms $X \rightarrow X$, then $\text{id}_X = \text{id}_X \text{id}'_X = \text{id}'_X \text{id}_X = \text{id}'_X$.

Take any category and reverse the morphisms to get an **opposite/dual category** \mathbf{C}^{op} . More formally, \mathbf{C}^{op} has the same objects as \mathbf{C} , but $\mathbf{C}^{op}(X, Y) = \mathbf{C}(Y, X)$.

Definition 4. Let $f : X \rightarrow Y$.

- f is **left invertible** if there's a $g : Y \rightarrow X$ such that $gf = \text{id}_X$.
- f is **right invertible** if there's an $h : Y \rightarrow X$ such that $fh = \text{id}_Y$.
- If both g and h exist, then $g = h$ is the **inverse** of f , and f is an **isomorphism** between X and Y .

If both left and right inverses exist, they're the same: $h = \text{id}_X h = gfh = g\text{id}_Y = g$. Inverses are unique: if g and g' are both inverses of f , then $g = \text{id}_X g = (g'f)g = g'(fg) = g'\text{id}_X = g'$.

Isos are reflexive, symmetric, and associative, so they're an equivalence relation. The equivalence classes induced by an iso in a category are **isomorphism classes** of that category.

1.2.1 FUNCTORS

Functors

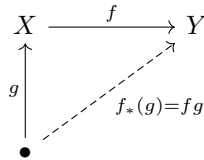
Proposition 1. Functors map isos to isos.

Consequentially, functors encode invariants of iso classes, since if $FX \not\cong FY$, then $X \not\cong Y$.

Example 1. Algebraic topology considers functors $\mathbf{Top} \rightarrow \mathbf{C}$, where \mathbf{C} is some algebraic category. For instance, homology is a functor $H : \mathbf{Top} \rightarrow R\mathbf{Mod}$, so if $HX \not\cong HY$, then $X \not\cong Y$.

1.2.2 HOM FUNCTORS

Definition 5. Let $f : X \rightarrow Y$. The **pushforward** $f_* : \mathbf{C}(\bullet, X) \rightarrow \mathbf{C}(\bullet, Y)$ is given by postcomposition with f .



The **pullback** $f^* : \mathbf{C}(Y, \bullet) \rightarrow \mathbf{C}(X, \bullet)$ is given by precomposition with f .

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow f^*(g)=gf & \downarrow g \\ & & \bullet \end{array}$$

As a first step toward the Yoneda Lemma, the following attempts to formalize the idea that understanding an object is equivalent to understanding all maps into or out of it.

Proposition 2. TFAE:

1. $f : X \rightarrow Y$ is an iso.
2. $f_* : \mathbf{C}(\bullet, X) \rightarrow \mathbf{C}(\bullet, Y)$ is an iso of sets for all \bullet .
3. $f^* : \mathbf{C}(Y, \bullet) \rightarrow \mathbf{C}(X, \bullet)$ is an iso of sets for all \bullet .

Hom functors

1.2.3 NATURAL TRANSFORMATIONS

natural transformations

1.3 THE YONEDA LEMMA

Motivation

Theorem 1 (Yoneda Lemma). Fix a category \mathbf{C} .

Contravariant: For all $X \in \mathbf{C}$ and functors $F : \mathbf{C}^{op} \rightarrow \mathbf{Set}$,

$$\text{Nat}(\mathbf{C}(-, X), F) \cong FX.$$

Covariant: For all $X \in \mathbf{C}$ and functors $F : \mathbf{C} \rightarrow \mathbf{Set}$,

$$\text{Nat}(\mathbf{C}(X, -), F) \cong FX.$$

Note that these are isomorphisms of sets (bijections).

Proof. This is a proof of the contravariant version, as the covariant proof is analogous. Since F is a contravariant functor, the following diagram commutes for any $f : \bullet \rightarrow X$. Note that ηf is completely determined by where η sends id_X , which I've denoted ϕ ; since f is arbitrary, this means the choice of ϕ

completely determines η as a whole. There's a clear bijection between choices of ϕ and elements of FX .

$$\begin{array}{ccc}
 \mathbf{C}(\bullet, X) & \xleftarrow{f^*} & \mathbf{C}(X, X) \\
 \downarrow \eta_\bullet & & \downarrow \eta_X \\
 & \begin{array}{ccc}
 \text{id}_X \circ f = f & \xleftarrow{\quad} & \text{id}_X \\
 \downarrow & & \downarrow \\
 \eta f = (Ff)\phi & \xleftarrow{\quad} & \phi := \eta \circ \text{id}_X
 \end{array} & \\
 F\bullet & \xleftarrow{Ff} & FX
 \end{array}$$

□

Corollary 1. Let F be the compatible hom functor to get the following.

Contravariant:

$$\text{Nat}(\mathbf{C}(-, X), \mathbf{C}(-, Y)) \cong \mathbf{C}(X, Y).$$

Covariant:

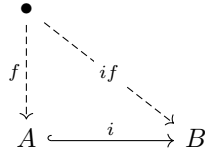
$$\text{Nat}(\mathbf{C}(X, -), \mathbf{C}(Y, -)) \cong \mathbf{C}(Y, X).$$

1.4 SET THEORY

2 CONSTRUCTING SPACES

2.1 THE SUBSPACE TOPOLOGY

Proposition 3 (UP of the subspace topology). Let $i : A \hookrightarrow B$ be an injective map. The subspace topology (induced by i) on A is the unique topology on A such that for all maps f into A , the composition if is continuous $\iff f$ is continuous.



This topology has the form

$$\{i^{-1}(U) \mid U \text{ open in } B\} = \{U \cap iA \mid U \text{ open in } B\}.$$

It's the coarsest topology on A for which $i : A \hookrightarrow B$ is continuous. When i is the natural inclusion, this all coincides with the usual definition of the subspace topology.

It's fine to do this even though A might not be a literal subset of B . If $i : A \hookrightarrow B$ is an injective map, then A is isomorphic as a set to its image $iA \subseteq B$. Using the UP, the space A with the subspace topology induced by i is homeomorphic to the space $iA \subseteq B$ with the subspace topology induced by the natural inclusion; so we can think of A as being embedded in B .

Definition 6. Suppose $f : A \hookrightarrow B$ is a continuous injection (so A and B are already equipped with topologies). It's an **embedding** when the topology on A is the same as the subspace topology induced by f .