1 (CO)HOMOLOGY WITH COEFFICIENTS

Let $\mathcal{C}$ be a chain complex of free $\mathbb{Z}$-modules (free abelian groups)

$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots,$$

then we can apply any functor $\mathcal{F} : \textsf{Ab} \to \textsf{Ab}$ (perhaps contravariant) to get another complex $\mathcal{F}\mathcal{C}$. In particular, we can use the following two functors, where G is some abelian group.

- $- \otimes G$ (covariant) maps $C \mapsto C \otimes G$ and $\phi \mapsto \phi \otimes \mathrm{id}$; since G is an abelian group, we're implicitly using $\otimes_{\mathbb{Z}}$;
- $\mathrm{Hom}(-, G)$ (contravariant) maps $C \mapsto \mathrm{Hom}(C, G)$ and $\phi \mapsto \phi^*$ (precomposition with ϕ).

Definition 1. For a chain complex $\mathcal{C}$, its **homology with G coefficients** is

$$H_*(\mathcal{C}; G) := H_*(\mathcal{C} \otimes G).$$

Its **cohomology with G coefficients** is

$$H^*(\mathcal{C}; G) := H_*(\mathrm{Hom}(\mathcal{C}, G)).$$

Note that $C \otimes_{\mathbb{Z}} \mathbb{Z} \cong C$ for any abelian group C, so $H_*(\mathcal{C}; \mathbb{Z}) \cong H_*(\mathcal{C})$. Also, when dealing with H^*, we throw "co-" on the front of all the vocab words, e.g. "cocyle" instead of "cycle".

go over why using TP and hom instead of hom and hom...

2 EXT AND TOR

Derived functors measure the extent to which a functor fails to preserve exactness. Ext and Tor are two examples of derived functors, which we will use in a bit to formulate the Universal Coefficient Theorem.

Definition 2. A covariant functor $\mathcal{F}$ is one of the below if it preserves exactness in the manner depicted.

$$\begin{array}{rrcl} \textbf{exact} & A \to B \to C & \rightsquigarrow & 0 \to \mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C \to 0 \\ \textbf{left exact} & 0 \to A \to B \to C & \rightsquigarrow & 0 \to \mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C \\ \textbf{right exact} & A \to B \to C \to 0 & \rightsquigarrow & \mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C \to 0 \end{array}$$

The following apply to a contravariant functor $\mathcal{G}$ instead.

$$\begin{array}{rrcl} \textbf{exact} & A \to B \to C & \rightsquigarrow & 0 \to \mathcal{G}C \to \mathcal{G}B \to \mathcal{G}A \to 0 \\ \textbf{left exact} & A \to B \to C \to 0 & \rightsquigarrow & 0 \to \mathcal{G}C \to \mathcal{G}B \to \mathcal{G}A \\ \textbf{right exact} & 0 \to A \to B \to C & \rightsquigarrow & \mathcal{G}C \to \mathcal{G}B \to \mathcal{G}A \to 0 \end{array}$$

In Ab, the above definitions are equivalent to those given by including 0's on the left and right of each LHS, but the forms above are a bit easier to work with since we won't always have things with 0's bookending them.

Definition 3. A **free** resolution of an abelian group A is an exact sequence of abelian groups

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0,$$

where each F_i is free.

Note 1. We're really only concerned with the derived functors Ext and Tor, which are both formulated in terms of projective resolutions. But that's okay, since a free module is projective. Thus we only need to concern ourselves with free resolutions.

Supose we have a right exact covariant functor $\mathcal{F}$ and a free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \twoheadrightarrow A \to 0,$$

then applying $\mathcal{F}$ gives

$$\cdots \to \mathcal{F}F_2 \to \mathcal{F}F_1 \to \mathcal{F}F_0 \twoheadrightarrow \mathcal{F}A \to 0.$$

Since $\mathcal{F}$ is right exact, the blue subsequence above is still exact. Removing $\mathcal{F}A$, we get a new sequence

$$\cdots \to \mathcal{F}F_2 \to \mathcal{F}F_1 \to \mathcal{F}F_0 \to 0,$$

Taking homology gives us the **derived functors** of $\mathcal{F}$. A similar story holds when $\mathcal{F}$ is a contravariant left exact functor instead. **check that still functor, i.e. a morphism $X \to Y$ induces morphism $L_i X \to L_i Y$.**

Theorem 1. Different free resolutions yield **isomorphic** derived functors.

Proof. **Do this.** ☐

Note 2. A nice thing about working with abelian groups is that you can find short free resolutions, which makes calculating derived functors much easier by Theorem 1.

Proposition 1. Every abelian group A has a free resolution

$$0 \to \operatorname{Ker}\varepsilon \hookrightarrow \langle A\rangle \overset{\varepsilon}{\twoheadrightarrow} A \to 0.$$

Proof. First note that all objects in the sequence are free abelian since the kernel of a free abelian group is itself free abelian. Construct ε by extenting id_A. Exactness is clear. ☐

Note 3. To be clear, $\langle A\rangle$ is not necessarily the same thing as A, since A might have extra relations. None of these relations are in $\langle A\rangle$. Thus $\operatorname{Ker}\varepsilon$ is generated by the relations of A.

Corollary 1. **calc derived functors for abelian group.**

how to turn $0 \to A \to B \to C \to 0$ SES into LES using derived functors?

With all this in place, we can finally define Ext and Tor as the derived functors of particular functors.

Definition 4. **Ext** is the derived functors of $\operatorname{Hom}(-, G)$, and **Tor** is the derived functors of $- \otimes G$.

Note that both of these use projetive resolutions, as $\operatorname{Hom}(-, G)$ is contravariant and left exact and $- \otimes G$ is covariant and right exact. **Go over earlier in more detail why contra/left and cov/right work with free resolutions.**

3 THE UNIVERSAL COEFFICIENT THEOREM

Homology with coefficients is useful for simplifying certain calculations, but as it turns out, it encodes the exact same information that the usual homology with $\mathbb{Z}$ coefficients does. The idea is that although $H_n(\mathcal{C} \otimes G) \ncong H_n(\mathcal{C}) \otimes G$ and $H^*(\mathrm{Hom}(\mathcal{C}, G)) \ncong \mathrm{Hom}(H_n\mathcal{C}, G)$ in general, we can use these as approximations and introduce some correction terms. These corrections are Ext and Tor.

Theorem 2 (The Universal Coefficient Theorem)**.** Let $\mathcal{C}$ be a chain complex of free abelian groups, and let G be any abelian group. Then there are short exact sequences

$$0 \longrightarrow H_n\mathcal{C} \otimes G \longrightarrow H_n(\mathcal{C}; G) \longrightarrow \mathrm{Tor}(H_{n-1}\mathcal{C}, G) \longrightarrow 0,$$

$$0 \longleftarrow \mathrm{Hom}(H_n\mathcal{C}, G) \longleftarrow H^n(\mathcal{C}; G) \longleftarrow \mathrm{Ext}(H_{n-1}\mathcal{C}, G) \longleftarrow 0$$

that are natural and split (although the splitting isn't natural). In other words,

$$\begin{aligned} H_n(\mathcal{C}; G) &\cong (H_n\mathcal{C} \otimes G) \oplus \mathrm{Tor}(H_{n-1}\mathcal{C}, G), \\ H^n(\mathcal{C}; G) &\cong \mathrm{Hom}(H_n\mathcal{C}, G) \oplus \mathrm{Ext}(H_{n-1}\mathcal{C}, G). \end{aligned}$$

We'll be mostly interested in the second statement since we care more about cohomology. If we're working with field coefficients, then the UCT gets simpler.

Theorem 3 (UCT for Fields)**.** If $\mathbf{k}$ is a field, then

$$H^n(\mathcal{C}, \mathbf{k}) \cong \mathrm{Hom}_{\mathbf{k}}\left(H_n(\mathcal{C}; \mathbf{k}), \mathbf{k}\right).$$

Note that it's $H_n(\mathcal{C}; \mathbf{k})$, *not* $H_n(\mathcal{C})$.