CONTENTS

1 SIMPLICIAL HOMOLOGY

1.1 GRAPHS

Definition 1. A (**simple undirected**) **graph** G is a set of vertices V and undirected edges E, where E has no self-loops or duplicate edges.

Definition 2. A **path** between vertices x and y is a sequences of vertices

$$x = u_0, \quad u_1, \quad \dots, \quad u_m = y$$

such that $[u_i, u_{i+1}]$ is an edge for all i.

Definition 3. A graph is **connected** if there is a path between every pair of vertices.

A **separation** of G is two nonempty subsets of G with no edges going between them. We can then equivalently define a graph to be connected if it has no separation.

Proposition 1. Let $x \sim_p y$ if there is a path from x to y. Then $\sim_p$ is an equivalence relation.

We call the equivalence classes of $\sim_p$ **connected components**. Since equivalence relations naturally form partitions, the connected components of a graph union to the entire graph.

Example 1. Let V be a vector space with subspace N, then $x \sim y \iff x - y \in N$ is an equivalence relation (since N has 0 and is closed under addition and additive inverse). The quotient V/N can then be defined as the equivalence classes of $\sim$, which is also a vector space with the operations $\alpha[x] = [\alpha x]$ and $[x] + [y] = [x + y]$.

1.2 SIMPLICIAL COMPLEXES

Definition 4. An **abstract simplicial complex** is a collection K of nonempty subsets of a finite set V such that

1. If $\{v\} \in K$, then $v \in V$;
2. If $\sigma \in K$ and $\tau \subset \sigma$ is nonempty, then $\tau \in K$.

In this setting, each singleton $\{v\}$ represents a vertex, and V represents all the vertices in the complex. Then for a simplex σ, any subset $\tau \subset \sigma$ is one of its **faces**. Note that a face need not be 1 dimension lower (e.g. a vertex is a face of a tetrahedron).

We say that the **dimension** of a complex $\sigma \in K$ is $\dim \sigma = |\sigma| - 1$. The dimension of the whole simplex K is then $\max_{\sigma \in K} \{\dim \sigma\}$. We need the -1 in the definition to make edges 1-dimensional, triangles 2-dimensional, etc.

1.3 SIMPLICIAL HOMOLOGY

Definition 5. Suppose X is a simplicial complex, then let $C_n(X)$ be the vector space over $\mathbb{Z}_2$ with basis the n-simplices in X. Elements of $C_n(X)$ are called n**-chains**.

- C_0: vertices
- C_1: edges
- C_2: triangles

Note 1. The choice to do this over $\mathbb{Z}_2$ was pretty arbitrary. It makes lots of things nicer from a computation viewpoint, and it simplifies a few things later on since $-1 = 1$, but we can also get more information by taking homology with coefficients in more complicated rings. In general, C_n will be some module. Then if they're over $\mathbb{Z}$, they're abelian groups.

Definition 6. The **boundary map** ∂_n is the linear map

$$\begin{aligned} C_n(X) &\to C_{n-1}(X) \\ [v_0, \ldots, v_n] &\mapsto \sum_i [v_0, \ldots, \hat{v}_i, \ldots, v_n], \end{aligned}$$

where $\hat{v}_i$ indicates that v_i has been removed from the simplex.

Proposition 2. $\partial^2 = 0$.

Proof. Applying the defintion of ∂ gives

$$\begin{aligned} \partial^2([v_0, \ldots, v_n]) &= \sum_i \partial([v_0, \ldots, \hat{v}_i, \ldots, v_n]) \\ &= \sum_{j<i} [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n] + \sum_{i<j} [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]. \end{aligned}$$

Now we can swap the roles of i and j in the second sum to get a sum identical to the first. This gives

$$\begin{aligned} &= 2 \sum_{j<i} [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n] \\ &= 0 \end{aligned}$$

since we're working over $\mathbb{Z}_2$. $\square$

This result shows that

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

is a chain complex. Thus we call $Z_k(X) \doteq \operatorname{Ker} \partial_k$ the space of k**-cycles** and $B_k(X) \doteq \operatorname{Im} \partial_{k+1}$ the space of k**-boundaries**.

Definition 7. The k**-th homology** of X is $H_k(X) \doteq Z_k(X)/B_k(X)$, and its dimension β_k is the k**-th Betti number**.

Proposition 3. β_0 is the number of connected components of X. **Infinite case?**

Proof. Suppose X has connected components $X_1, \ldots, X_\ell$. Then since the homology functor commutes with direct sums,

$$H_0(X) = H_0\left(\bigoplus_{i=1}^{\ell} X_i\right) = \bigoplus_{i=1}^{\ell} H_0(X_i).$$

Show that $\beta_0 = 1$ when X is connected. Then since β_0 of a connected complex is 1,

$$\beta_0 = \dim\left(\bigoplus_{i=1}^{\ell} H_0(X_i)\right) = \sum_{i=1}^{\ell} \dim H_0(X_i) = \sum_{i=1}^{\ell} 1 = \ell.$$

□

1.4 FUNCTORIALITY OF SIMPLICIAL HOMOLOGY

I think I can build in the chain map-ness of all the functors better.

Suppose K and L are simplicial complexes, then we can turn morphisms between them into morphisms between homologies. This boils down to defining two covariant functors, then composing them.

Definition 8. Let K, L be simplicial complexes. We say $f : K \to L$ is a **simplicial map** if

1. $f : V(K) \to V(L)$;
2. it's determined by where it sends vertices: $f([x_0, \dots, x_p]) = [f(x_0), \dots, f(x_p)]$;
3. $f(\sigma) \in L$ when $\sigma \in K$.

Theorem 1. Fix $p \in \mathbb{N}_0$, then taking p-th chain groups is a functor. It sends simplicial maps $f : K \to L$ to the morphism $f_\# : C_p(K) \to C_p(L)$ uniquely determined by

$$\sigma \mapsto \begin{cases} f(\sigma) & \text{if } |f(\sigma)| = |\sigma|, \\ 0 & \text{else.} \end{cases}$$

Proof. **Do this.** □

Proposition 4. $f_\#$ respects ∂, i.e. the following diagram commutes.

$$\begin{tikzcd} C_p(K) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{p-1}(K) \arrow[d, "f_\#"] \\ C_p(L) \arrow[r, "\partial"] & C_{p-1}(L) \end{tikzcd}$$

Proof. **Do this.** □

Theorem 2. Taking p-th homology is a functor. It sends $f_\# : C_p(K) \to C_p(L)$ to the morphism $f_* : H_p(K) \to H_p(L)$ uniquely determined by

$$[x] \mapsto [f_\#(x)].$$

Want a similar commutative diagram prop.

2 PERSISTENT HOMOLOGY

2.1 PERSISTENCE MODULES

The following definition of a persistence module is restricted, but it's all we'll need for our purposes (in general, you can define it to be a functor from a specific poset to the category of R-modules). **You should read more about this.**

Definition 9. A **persistence module** V is a collection of vector spaces $\{V_r\}_{r\in\mathbb{R}}$ over $\mathbb{Z}_2$ with functorial maps $v_i^j : V_i \to V_j$ whenever $i \leq j$. **Covariant functor from $\mathbb{R}$ to Vector spaces over $\mathbb{Z}_2$...**

The functoriality condition means

1. $v_i^i = \text{id}$;
2. $v_i^k = v_j^k v_i^j$ whenver $i \leq j \leq k$.

Suppose we're indexing our vectors spaces via $\mathbb{Z}$ instead of $\mathbb{R}$, then the situation looks as follows.

$$\begin{tikzcd} \cdots \arrow[r] & V_{i-1} \arrow[r, "v_{i-1}^{i}"] \arrow[rr, bend right, "v_{i-1}^{i+1}"'] & V_i \arrow[r, "v_i^{i+1}"] & V_{i+1} \arrow[r] & \cdots \end{tikzcd}$$

2.2 INTERVAL DECOMPOSITION

2.3 IDK WHERE TO PUT THIS STUFF

Given a function $f : G \to \mathbb{R}$, we can think of $f(x)$ as the time at which x appears.

Definition 10. $F : G \to \mathbb{R}$ is **monotonic** if $f(v) \leq f(e)$ whenever e is an edge containing vertex v. **gen to complexes...**

Thus for monotonic functions, no edge will appear until both its vertices have also appeared.
0 dim Persistent Homology stuff...
Note that every birth-death pair is an element of

$$\overline{\mathbb{R}}^2_< \doteq \{(x, y) \mid x \in \mathbb{R},\ y \in \mathbb{R} \cup \{\infty\}\}.$$

Figure.

Definition 11. A **partial mapping** between multisets $P, Q \subset \overline{\mathbb{R}}^2_<$ is a bijection $\eta : P_0 \to Q_0$, where $P_0 \subset P$ and $Q_0 \subset Q$. We denote it

$$\eta : P \leftrightarrow Q.$$

We define the cost of a partial matching $\eta : P \leftrightarrow Q$ **finish...**