CONTENTS

1 Polynomial dynamics **1**

1.1 Introduction 1

1 POLYNOMIAL DYNAMICS

1.1 INTRODUCTION

Take some polynomial $p : \mathbb{C} \to \mathbb{C}$, then we can consider the orbits $\{p^n(z)\}_{n=1}^{\infty}$ for any $z \in \mathbb{C}$, where p^n denotes repeatedly applying p a total of n times.

Definition 1. Given a polynomial $p : \mathbb{C} \to \mathbb{C}$,

- $I_p := \{z \in \mathbb{C} \mid \lim_{n\to\infty} p^n(z)\}$ is the **set of escaping points** of p;
- $K_p := \mathbb{C} - I_p$ is the **filled Julia set** of p;
- $J_p := \partial K_p$ is the **Julia set** of p.

I_p is nonempty if $\deg(p) \geq 2$. This is because as $|z|$ gets large enough, $|p^n(z)| \approx |z|^{(k^n)}$, which clearly diverges as $n \to \infty$.

One way of thinking of I_p is as the set of points converging to the point at infinity inside the Riemann sphere $\hat{\mathbb{C}}$. This is helpful because, as we'll see later on, the Julia set contains the "chaotic" points that neither converge nor get stuck in a cycle during the process of iterating p. The rest of the points in the plane (so I_p and the interior of K_p) either converge (perhaps to the point at infinity) or get stuck in a cycle.

Example 1. Let $p(z) = z^2$, then

- $I_p = \{z \in \mathbb{C} \mid |z| > 1\}$;
- $K_p = \{z \in \mathbb{C} \mid |z| \leq 1\}$;
- $J_p = \{z \in \mathbb{C} \mid |z| = 1\}$.

For many simple polynomials, the Julia set is a fractal. The below is a first attempt at giving this a concrete definition, although we'll move to better definitions later on.

Definition 2. A closed set $E \subseteq \mathbb{C}$ is a **fractal set** if it's *not* a countable union of rectifiable (finite length) curves. A domain (nonempty, connected, open set) D in $\mathbb{C}$ is a **fractal domain** if ∂D is a fractal set.

Fill in gap here...

Theorem 1. Let $\deg(p) \geq 2$, then

1. K_p is a nonempty compact set;
2. I_p is a domain (nonempty, connected, open).