Proposition 1. *For all $\varepsilon > 0$, whp there is a giant component after at most $(1+\varepsilon)n$ steps.*

Proof. We can assume $0 < \varepsilon \leq 2$. Then set $k = 2^\ell/\varepsilon^{\ell-1} \geq 1$, and $\Delta = 2\left\lceil \frac{2^\ell n}{\varepsilon^{\ell-1}k} \right\rceil$.

Let W denote the union of all components of size at least k at time $m = 0$, and take $\tilde{\varepsilon} \leq \varepsilon$ such that $W \geq \tilde{\varepsilon}n$. Let H denote the event that there are *not* $\ell - 1$ components together containing $|W| - \varepsilon n/2$ vertices. While H holds, the probability of choosing ℓ vertices in distinct components of W is at least $\frac{|W|}{n}\left(\frac{\tilde{\varepsilon}}{2}\right)^{\ell-1}$. In this case, the number of components of W must decrease by 1.

But as in the proof of Lemma 4, the probability of H holding is exponentially small in n/k, so whp we have $\ell - 1$ components of W together containing at least

$$|W| - \varepsilon n/2 \quad \geq \quad \tilde{\varepsilon}n - \varepsilon n/2 \quad \geq \quad \varepsilon n/2$$

vertices after $\Delta/2 = n < (1+\varepsilon)n$ steps. At least one of these components must have at least $\frac{\varepsilon}{2(\ell-1)}n$ vertices, so it is a giant component. □