0.1 The De Rham Complex on $\mathbb{R}^n$

Denote the space of k-forms on an n-dimensional manifold N by $\Omega^k(N)$, then the C^∞ differential forms on N form the vector space

$$\Omega^*(N) \doteq \bigoplus_{k=0}^{n} \Omega^k(N).$$

The exterior derivative is defined as usual: if f is a smooth function, then $df \doteq \sum \partial_i f \; dx_i$, and if $\omega = \sum f_I dx_I$ is a differential form, then $d\omega \doteq \sum df_I \wedge dx_I$.

Definition 1. $(\Omega^*(N), d)$ is the **de Rham complex** on N, which we represent by the cochain complex

$$0 \longrightarrow \Omega^0(N) \xrightarrow{d} \Omega^1(N) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(N) \longrightarrow 0.$$

The k-th **de Rham cohomology** of N is then the vector space

$$H^k(N) \doteq \frac{\ker d \, \cap \, \Omega^k(N)}{\operatorname{im} d \, \cap \, \Omega^k(N)}.$$

Since our complex is finite, the 0-th and n-th cohomologies will always be a bit simpler:

$$H^0(N) = \ker d \, \cap \, \Omega^0(N),$$
$$H^n(N) = \frac{\Omega^n(N)}{\operatorname{im} d \, \cap \, \Omega^n(N)}.$$

Any differential form in the kernel of d is **closed**, and any in the image of d is **exact**. Note that since $d^2 = 0$, an exact form must also be closed.

0.2 Functoriality of de Rham Cohomology

Suppose we have a smooth map of manifolds $f : M \to N$, then this induces a pullback

$$\begin{aligned} f^* : \Omega^*(N) &\to \Omega^*(M) \\ g &\mapsto g \circ f, \end{aligned}$$

which is easily seen from the following diagram.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "g\circ f"'] & N \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

Given smooth maps between manifolds A, B, C, we can show that the pullbacks satisfy a reversed composition law: $g^* \circ f^* = (f \circ g)^*$. It's straightforward to do this calculation, but the following picture makes it clear.

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C$$

$$\Omega^*(A) \xleftarrow{f^*} \Omega^*(B) \xleftarrow{g^*} \Omega^*(C)$$

All this shows that Ω^* is a contravariant functor from the category of smooth manifolds to the category of commutative differential graded algebras. The commutativity bit refers to the identity

$$\tau \wedge \omega = (-1)^{\deg \tau \deg \omega}\, \omega \wedge \tau.$$

We can check that f^* commutes with the exterior derivative: $f^*(d_N\omega) = d_M(f^*\omega)$ for any differential form ω on N. (Do this) This means f^* is a chain map $\Omega^*(N) \to \Omega^*(M)$, so it induces homomorphisms $H^k(N) \to H^k(M)$ for all k.

$$\begin{tikzcd} 0 \arrow[r] & \Omega^0(N) \arrow[r, "d_N"] \arrow[d, "f^*"] & \cdots \arrow[r, "d_N"] & \Omega^k(N) \arrow[r, "d_N"] \arrow[d, "f^*"] & \cdots \\ 0 \arrow[r] & \Omega^0(M) \arrow[r, "d_M"] & \cdots \arrow[r, "d_M"] & \Omega^k(M) \arrow[r, "d_M"] & \cdots \end{tikzcd}$$

Then since taking the induced homological structure is functorial (check), this means that H^* is also a contravariant functor (be specific about the category it's going to).