Exercise 1. §5.6 #2.

a. Principal, asymptotic.

b. Principal, geodesic.

c. Asymptotic, geodesic.

Exercise 2. §6.2 #2.

a. We know $\omega_{12} = f_1\theta_1 + f_2\theta_2$, where $f_1 = \omega_{12}(E_1), f_2 = \omega_{12}(E_2)$, so by §6.2 Corollary 2.3,

$$\begin{aligned} -K\theta_1 \wedge \theta_2 &= d\omega_{12} \\ &= df_1 \wedge \theta_1 + f_1 d\theta_1 + df_2 \wedge \theta_2 + f_2 d\theta_2 \\ &= df_1 \wedge \theta_1 + f_1\omega_{12} \wedge \theta_2 + df_2 \wedge \theta_2 - f_2\omega_{12} \wedge \theta_1 \\ &= (df_1 - f_2\omega_{12}) \wedge \theta_1 + (df_2 + f_1\omega_{12}) \wedge \theta_2. \end{aligned}$$

Applying this at (E_1, E_2) and using the fact that $\theta_i(E_j) = \delta_{ij}$ then gives

$$-K = (-K\theta_1 \wedge \theta_2)(E_1, E_2) = -E_2[f_1] + E_1[f_2] + f_1^2 + f_2^2.$$

Negating both sides then gives

$$K = E_2[f_1] - E_1[f_2] - f_1^2 - f_2^2,$$

as desired.

b. Since

$$\begin{aligned} \omega_{12} &= f_1\theta_1 + f_2\theta_2 \\ \sin\phi\; d\theta &= f_1 r\cos\phi\; d\theta + f_2 r\; d\phi, \end{aligned}$$

we have $f_1 = \frac{\tan\phi}{r}$ and $f_2 = 0$. The formula for K that we derived in part (a) then gives

$$\begin{aligned} K &= E_2[f_1] - E_1[f_2] - f_1^2 - f_2^2 \\ &= E_2\left[\frac{\tan\phi}{r}\right] - \frac{\tan^2\phi}{r^2} \\ &= \frac{\sec^2\phi}{r^2} - \frac{\tan^2\phi}{r^2} \\ &= \frac{1}{r^2}. \end{aligned}$$

We already know the Gaussian curvature of a sphere is $1/r^2$ everywhere, so our formula from part (a) was correct.

Exercise 3. §6.4 #1.

a implies c: Suppose F_* preserves inner products, and let $\mathbf{e}_1, \mathbf{e}_2$ be a tangent frame at $\mathbf{p}$. Then $F_*(\mathbf{e}_i) \cdot F_*(\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, so $F_*(\mathbf{e}_1), F_*(\mathbf{e}_2)$ is a tangent frame at $F(\mathbf{p})$.

c implies d: Suppose F_* preserves frames, then $\|F_*(\mathbf{e}_i)\| = 1 = \|\mathbf{e}_i\|$ and $F_*(\mathbf{e}_1) \cdot F_*(\mathbf{e}_2) = 0 = \mathbf{e}_1 \cdot \mathbf{e}_2$. Note that $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent because they are a frame, so they are the $\mathbf{u}, \mathbf{v}$ we are looking for.

d implies b: Let $\mathbf{z} \in T_{\mathbf{p}}(M)$, then because $\mathbf{v}, \mathbf{w}$ are linearly independent, they span $T_{\mathbf{p}}(M)$, i.e. $\mathbf{z} = a\mathbf{v} + b\mathbf{w}$ for some scalars a, b. Then by the linearity of F_*,

$$\begin{aligned}
\|F_*(\mathbf{z})\|^2 &= \|F_*(a\mathbf{v} + b\mathbf{w})\|^2 \\
&= \|aF_*(\mathbf{v}) + bF_*(\mathbf{w})\|^2 \\
&= a^2\|F_*(\mathbf{v})\|^2 + 2abF_*(\mathbf{v}) \cdot F_*(\mathbf{w}) + b^2\|F_*(\mathbf{w})\|^2 \\
&= a^2\|\mathbf{v}\|^2 + 2ab\mathbf{v} \cdot \mathbf{w} + b^2\|\mathbf{w}\|^2 \\
&= \|\mathbf{z}\|^2,
\end{aligned}$$

so $\|F_*(\mathbf{z})\| = \|\mathbf{z}\|$.

b implies a: Suppose F_* preserves norms. We can expand the dot product between arbitrary $\mathbf{x}$ and $\mathbf{y}$ as

$$x \cdot y = \frac{1}{4}\left(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2\right).$$

Then using the linearity of F_*, the dot product between $F_*(\mathbf{v})$ and $F_*(\mathbf{w})$ is

$$\begin{aligned}
F_*(\mathbf{v}) \cdot F_*(\mathbf{w}) &= \frac{1}{4}\left(\|F_*(\mathbf{v}) + F_*(\mathbf{w})\|^2 - \|F_*(\mathbf{v}) - F_*(\mathbf{w})\|^2\right) \\
&= \frac{1}{4}\left(\|F_*(\mathbf{v} + \mathbf{w})\|^2 - \|F_*(\mathbf{v} - \mathbf{w})\|^2\right) \\
&= \frac{1}{4}\left(\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2\right) \\
&= \mathbf{v} \cdot \mathbf{w}.
\end{aligned}$$

Exercise 4. §6.4 #8.

a.
- F_* preserves inner products up to a scalar multiple $\lambda(\mathbf{p})^2$.
- $\|F_*(\mathbf{v})\| = \lambda_{\mathbf{p}}\|\mathbf{v}\|$ for all $\mathbf{v}$ at $\mathbf{p}$.
- If $\mathbf{e}_1, \mathbf{e}_2$ is a frame of $\mathbf{p}$, then

$$\frac{F_*(\mathbf{e}_1) \cdot F_*(\mathbf{e}_2)}{\lambda(\mathbf{p})^2}$$

  is a frame of $F(\mathbf{p})$.

- For one pair of linearly independent $\mathbf{v}, \mathbf{w}$, we have

$$\begin{aligned} \|F_*(\mathbf{v})\| &= \lambda(\mathbf{p})\|\mathbf{v}\| \\ \|F_*(\mathbf{w})\| &= \lambda(\mathbf{p})\|\mathbf{w}\| \\ F_*(\mathbf{v}) \cdot F_*(\mathbf{w}) &= \lambda(\mathbf{p})^2 \mathbf{v} \cdot \mathbf{w}. \end{aligned}$$

b. **Forward:** Suppose $\mathbf{x}$ is conformal. It suffices to evaluate $\mathbf{x}_*$ at $\mathbf{e}_1, \mathbf{e}_2$, which gives $\mathbf{x}_*(\mathbf{e}_1) = \mathbf{x}_u$, $\mathbf{x}_*(\mathbf{e}_2) = \mathbf{x}_v$. Then

$$F = \mathbf{x}_u \cdot \mathbf{x}_v = \lambda(\mathbf{p})^2 \mathbf{e}_1 \cdot \mathbf{e}_2 = 0$$

and

$$E = \|x_u\| = \lambda(\mathbf{p})\|\mathbf{e1}\| = \lambda(\mathbf{p})\|\mathbf{e}_2\| = \|\mathbf{x}_v\| = G.$$

Backward: Suppose $E = G$ and $F = 0$, then

$$\|\mathbf{x}_u\| = \mathbf{x}_u \cdot \mathbf{x}_u = E = G = \mathbf{x}_v \cdot \mathbf{x}_v = \|\mathbf{x}_v\|.$$

Any tangent vector $\mathbf{v}$ can be written $a\mathbf{x}_u + b\mathbf{x}_v$, so

$$\|a\mathbf{x}_u + b\mathbf{x}_v\| = a^2\|\mathbf{x}_v\|^2 + 2ab\mathbf{x}_u \cdot \mathbf{x}_v + b^2\|\mathbf{x}_v\|^2.$$

Then since since $\|\mathbf{x}_u\| = \|\mathbf{x}_v\| = E$ and $\mathbf{x}_u \cdot \mathbf{x}_v = F = 0$, this becomes

$$\begin{aligned} &= E^2(a^2 + b^2) \\ &= E^2\|a\mathbf{e}_1 + b\mathbf{e}_2\|. \end{aligned}$$

Thus $\mathbf{x}$ is conformal.

c. Since F_* preserves inner products up to a scalar multiple $\lambda(-\mathbf{p})^2$,

$$\begin{array}{ccc} F_*(\mathbf{v}) \cdot F_*(\mathbf{w}) & = & \|F_*(\mathbf{v})\|\|F_*(\mathbf{w})\| \cos\theta \\ \shortparallel & & \shortparallel \\ \lambda(\mathbf{p})^2 \mathbf{v} \cdot \mathbf{w} & = & \lambda^2(\mathbf{p})\|\mathbf{v}\|\|\mathbf{w}\| \cos\tilde{\theta}. \end{array}$$

Now since $\|F_*(\mathbf{x})\| = \lambda(\mathbf{p})\|\mathbf{x}\|$ for all $\mathbf{x}$ at $\mathbf{p}$, this implies that $\cos\theta = \cos\tilde{\theta}$. Thus $\theta = \tilde{\theta}$, i.e. F_* preserves angles.