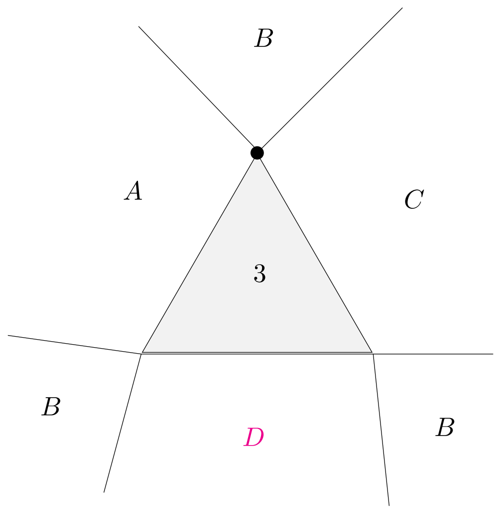
Exercise 1 (5.22)**.** 4 faces per vertex and 1 is triangle $\implies$ 2 of the others must be identical.

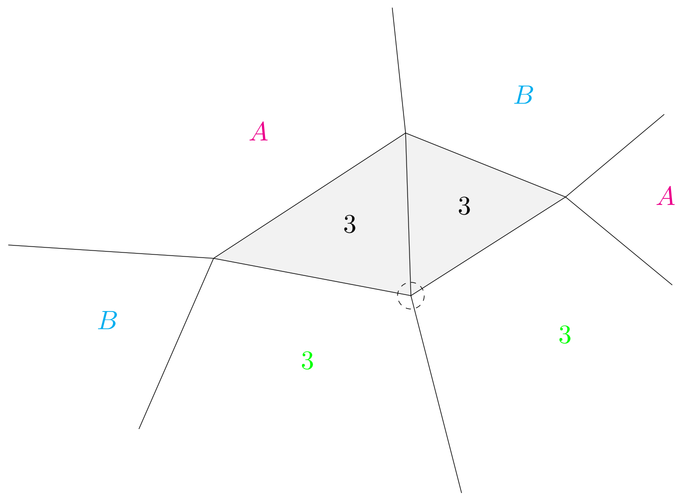
Suppose A, B, C are distinct integers, then we can depict one vertex as below.



But extending to multiple vertices, we arrive at a contradiction: neither A nor C can occupy the pink face while making all vertices identical.

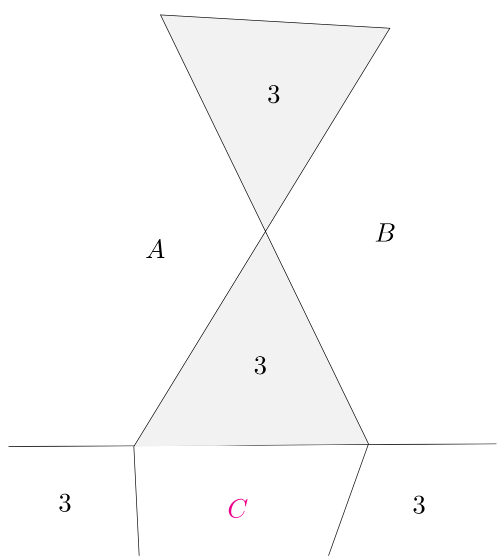
Exercise 2 (5.23)**.** 4 faces per vertex. If two triangles, they can't be adjacent; the other two faces must also be identical.

First part: Suppose the two triangles are adjacent, then we have the situation below.



Since each vertex must be identical, the blue (pink) faces are taken up by another face A (B). But then since each vertex has two adjacent triangles touching it, both green faces are also triangles. But then the highlighted vertex is surrounded by only triangles, a contradiction.

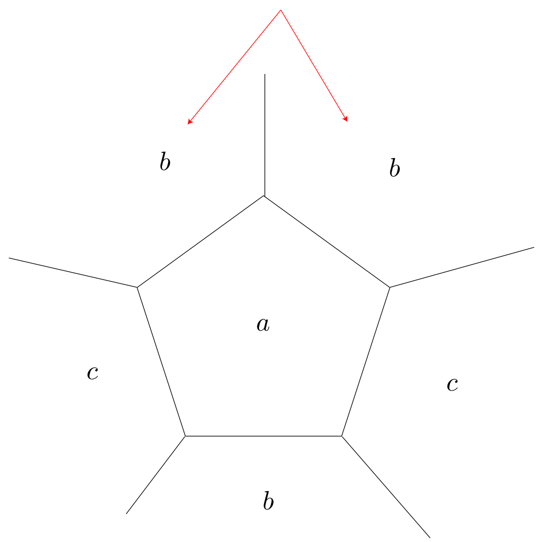
Second part: Suppose $A \neq B$, then we get the following situation.



But the pink face cannot be A or B, as then not all the vertices would be identical. Thus $A = B$.

Exercise 3 (5.25)**.** Classify the semiregular polyhedra with 3 faces per vertex.

All possible semiregular polyhedra with three faces per vertex will have representation (a, b, c); we have to find all valid a, b, and c. First note that if a is odd, then $b = c$. To see why, note that for all vertices to be identical, b and c must alternate around the face with a edges, but this is not possible if a is odd. This situation when $a = 5$ is pictured below.



Using this same logic, we have that if a is odd and b is odd, then $a = b = c$. The only time $b = c$ and neither equals a is when $b = c$ is even. Finally, we can eliminate all combinations of a and $b = c$ whose angles add up to $\geq 360°$. This gives us the list

$$(3,3,3),\ (3,4,4),\ (3,6,6),\ (3,8,8),\ (3,10,10),\ (3,12,12), (5,5,5),\ (5,6,6).$$

Now suppose a is even. We can apply the same reasoning about odds from before to conclude that b and c must also be even. Similar to above, we have

$$(4,4,4),\ (4,6,6),\ (4,8,8),\ (6,6,6).$$

We no longer have the restriction that $b = c$, though, so we can take all the triples with angle sums $\leq 360°$, giving

$$(4,4,n) \text{ for any } n,\ (4,6,8),\ (4,6,10), \ \text{ and } (4,6,12).$$

Exercise 4 (5.26)**.** What are all the Euclidean tilings with five or six faces?

Five faces: There are three possible tilings with five faces, since there are only three combinations of five polygons such that their angles add to exactly $360°$. They are

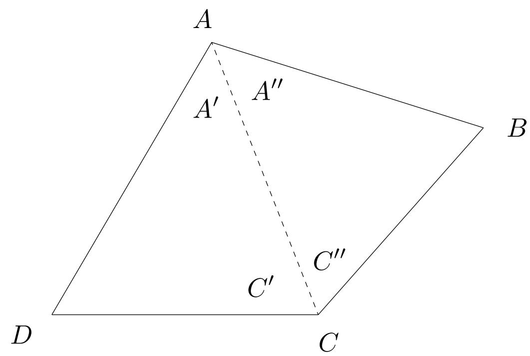
$$(3,3,3,3,6),(3,3,3,4,4),(3,3,4,3,4).$$

These are unique up to cyclic rotation. Any other combination of polygons give angles that don't sum to exactly $360°$.

Six faces: For six faces, the only tiling is $(3,3,3,3,3,3)$. Any other combination clearly gives an angle sum greater than $360°$, so any other combination of six polygons is neither a tiling nor a polyhedron.

Exercise 5 (6.5)**.** Show that any quadrilateral's angles sum to $\leq 360°$.

Take any quadrilateral $ABCD$ and decompose it into two triangles, as shown below.

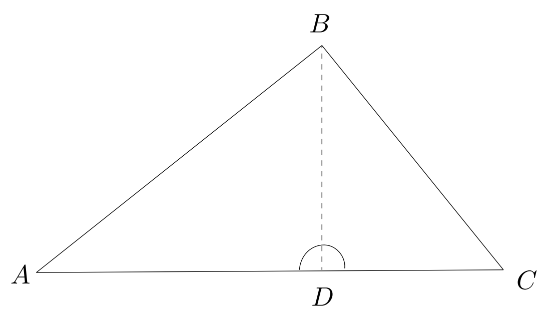


Since by Theorem 6.2 (which is true in *any* geometry) the angles in a triangle sum to $\leq 180°$, we have

$$A + B + C + D = (A + A' + C') + (B + C'' + A'') \leq 180° + 180° = 360°.$$

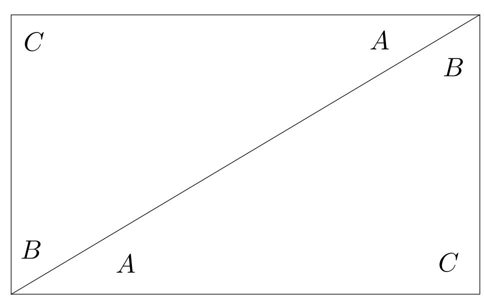
Exercise 6 (6.7)**.** If a triangle's angles sum to 180, then there is a right triangle whose angles sum to 180. This means we can contruct a rectangle.

Suppose ΔABC's angles add up to $180°$. At least one of its altitudes must intersect one of its component lines, as depicted below.



Note that we've added 180° worth of angles here since we added two right angles. Thus $A+B+C+D = 180° + 180° = 360°$. Now we know by Theorem 6.2 that the angles in a triangle are at most $180°$. But since the sums of the angles of the two triangles above give exactly $360°$, they must both sum to exactly $180°$.

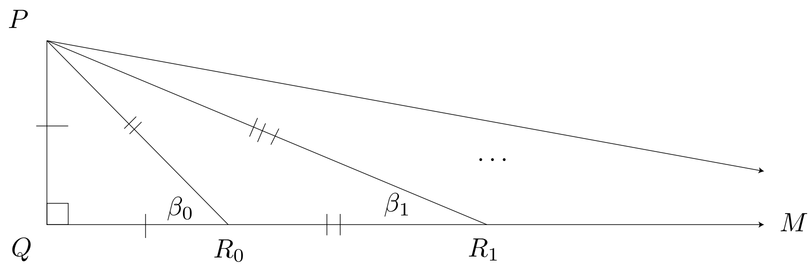
Now take some right triangle ΔABC with angles summing to 180° (which we just proved existed under our assumptions). Then assume, as shown below, that $C = 90°$ and $A + B = 90°$.



Then copying ΔABC and aligning the two triangles' hypotnuses with each other gives a rectangle since, as noted earlier, $C = B + C = 90°$.

Exercise 7 (6.16)**.** For any $\varepsilon > 0$, there's an R on QM such that $\angle PRQ < \varepsilon$.

Fix $\varepsilon > 0$. Now construct a sequence of triangles as follows: Start with right angle $\angle PQM$, then mark the point R_0 on QM that is distance $|PQ|$ from Q.



Now find the point on QM that is $|PR_0|$ to the right of R_0, and use this to form another triangle with bottom angle β_2. Now continue inductively to define a sequence of angles $\{\beta_n\}_{n\in\mathbb{N}_0}$.

Note since PQR_0 is isosceles, $\beta_0 = 45°$. Then its complement is $(180 - 45)°$, so PR_0R_1 being isosceles implies $\beta_1 = \frac{45}{2}^\circ$. In general, $\beta_n = \frac{45}{2^n}^\circ$. Then if $N > \log_2(45/\varepsilon)$, we have

$$\beta_N = \frac{45}{2^N}^\circ < \frac{45}{(45/\varepsilon)}^\circ \varepsilon.$$

Choose any such N, then R_N satisfies the problem statement.