Problems completed: All.

Exercise 1. Munkres §25, pg. 162 #4.

Collaborators: None.

Let C be an open connected set in X. Then for all $x \in C$, since C is a neighborhood of x and X is locally path connected, there is some path component $P_x \subset C$. Then $C = \bigcup_{x \in X} P_x$.

By Munkres Theorem 25.4, each P_{x_i} is open in X. Since C is open in X, this means that each P_{x_i} is open in C as well. Thus if any two P_{x_i}, P_{x_j} are disjoint, we have contradicted the connectedness of C, so all P_{x_i} must intersect at least one other P_{x_j}. Thus C is path connected.

Exercise 2. Munkres §25, pg. 162 #5.

Collaborators: None.

a. **Path Connected:** Let $x, y \in T$, then x lies on a line L_x and y lies on a line L_y. To get from x to y, follow L_x to p, then follow L_y to y. Thus T is path connected.

   Locally Connected at p: Let U be any neighborhood of p, then we can find an open rectangle inside of U that contains p. But then this region is homeomorphic to the entire space (it's just a "scaled down" version), which we know now to be path connected, so it must also be connected. Thus T is locally connected at p.

   Not Locally Connected Elsewhere: Now consider any point $x \neq p$ in T on the line L_{q_1}, and let $B(x, \|x - p\|) \cap T$ be a neighborhood of x that does not contain p. We want to show that any open set U that lies in this open ball is disconnected. Any such U must contain at least one other distinct line, say L_{q_2}, and since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a line L_r for $r \in \mathbb{R}$ that lies between L_{q_1} and L_{q_2} that is not in T.

   Let U_1 denote the intersection of all of T to the left of L_r with U, and let U_2 be the intersection of all of T to the right of L_r with U. These are both open sets in U, as they are open sets of $[0,1] \times [0,1]$ intersected with U. And since U is the disjoint union of U_1 and U_2, it is disconnected.

b. Let T' be the union of all lines connecting the origin to the rational points of the interval $[-1, 0] \times \{1\}$, and consider the space given by the union of T with T'. The argument that this space is path connected everywhere and disconnected at all of the non-hub points is identical to the arguments before.

   At the hub points, the previous argument no longer holds, as any open set containing a hub point also contains lines from the opposing set of lines. Then the same argument for all the non-hub points can be used to show that our space is disconnected at the two hub points, so it is nowhere locally connected.

Exercise 3. Munkres §26, pg. 171 #7.

Collaborators: None.

Let $A \times B$ be closed in $X \times Y$, the we want to show that $\pi_1(A \times B) = A$ is closed in X. Let $a \notin A$, then $\pi_1^{-1}(a) = \{a\} \times Y$ is disjoint from $A \times B$. Since $A \times B$ is closed, its complement is open, so there is some open set U around $\{a\} \times Y$ that does not intersect $A \times B$.

Then since Y is compact, by the tube lemma we can find a neighborhood V of a such that $V \times Y \subset U$. Since U does not intersect $A \times B$, neither does $V \times Y$. But since Y clearly intersects B (B is a subset of Y), the only way this is possible is if V does not intersect A.

We have found a neighborhood of $a \notin A$ that does not intersect A, so $X - A$ is open, so A is closed. Thus π_1 is a closed map.

Exercise 4. Munkres §26, pg. 171 #8.

Collaborators: None.

Forward: Suppose f is continuous. If $x \times y \notin G_f$, then $y \neq f(x)$. Since Y is Hausdorff, we can find disjoint neighborhoods U, V of $y, f(x)$, respectively. Additionally, the continuity of f lets us find a neighborhood W of x such that $f(W) \subset V$. Then $W \times U$ is a neighborhood of $x \times y$ that does not intersect G_f, so $(X \times Y) - G_f$ is open, so G_f is closed.

Backward: Supose G_f is closed, and let V be a neighborhood of $f(x)$. Then $Y - V$ is closed, so $X \times (Y - V)$ is closed as it is the product of two closed sets. Then $G_f \cap (X \times (Y - V))$ is closed as it is the intersection of closed sets.

Since Y is compact, by the previous exercise, π_1 is a closed map. Thus

$$\pi_1 \left(G_f \cap (X \times (Y - V)) \right) = \{x \mid f(x) \in Y - V\}$$

is closed. Then its complement $U \cdot \{x \mid f(x) \in V\}$ is open. We have found a neighborhood U of x such that $f(U) \subset V$, so f is continuous.

Exercise 5. Munkres §26, pg. 171 #11.

Collaborators: Saloni Bulchandani.

> **Lemma 1.** *If C, D are disjoint compact subsets of a Hausdorff space X, then there exist disjoint open sets U, V of X containing C, D, respectively.*

Proof. Fix $d \in D$. Since X is Hausdorff, for all $c \in C$ we can find disjoint neighborhoods $U_{c,d}, V_{c,d}$ of c, d, respectively. Since $\{U_{c,d}\}_{c \in C}$ covers C and C is compact, there is a finite subcover $\{U_{c_i,d}\}_{i=1}^N$. Then $U_d \doteq \bigcup_{i=1}^N U_{c_i,d}$ contains C and does not intersect $V_d \doteq \bigcap_{i=1}^N V_{c_i,d}$, which is a neighborhood of d since the intersection is finite.

Now $\{V_d\}_{d \in D}$ is an open cover of D and D is compact, we can find a finite subcover $\left\{V_{d_j}\right\}_{j=1}^M$. Since U_d doesn't intersect V_d for all d, we can define two disjoint open sets

$$U \doteq \bigcap_{j=1}^M U_{d_j}, \quad V \doteq \bigcup_{j=1}^M V_{d_j}.$$

Since each U_{d_j} contains C, so does their intersection U, and V is a cover of D by definition. Thus we have found disjoint open sets containing C and D. $\square$

Suppose C, D separate Y. Since C and D are open and are each others' complements in Y, they are also both closed in Y. Since Y is the intersection of closed sets in X, Y is closed in X. Then C, D are closed in X, and since X is compact, C, D are also compact.

By the lemma, we can find disjoint open sets U, V of C, D. Now for any $A \in \mathcal{A}$, the set $A - (U \cup V)$ is nonempty. Otherwise, $U \cap A, V \cap A$ would separate A, contradicting the connectedness of A. Since any finite subset of $\mathcal{A}$ is nested, we know that $\mathcal{A}$ (and by extension Y) satisfies the finite intersection property.

Since $U \cup V$ is open and A is closed, $A - (U \cup V) = A \cap (X - (U \cup V))$ is the intersection of two closed sets and so must be closed itself. Then since X is compact, by Theorem 26.9, the intersection $\bigcap_{A \in \mathcal{A}} (A - (U \cup V))$ is nonempty. Then $\bigcap_{A \in \mathcal{A}} (A - (C \cup D))$ must also be nonempty, which contradicts the fact that C and D cover Y. Thus by contradiction, Y must be connected.