CONTENTS

1 MODULES

1.1 MODULES AND ALGEBRAS

Modules are a generalization of vector spaces, replacing the field of scalars with a unital ring of scalars.

Definition 1. Let R be a unital ring. A **(left)** R**-module** is an additive abelian group M with a left action $R \times M \to M$ satisfying

1. $\lambda(x + y) = \lambda x + \lambda y$;
2. $(\lambda + \mu)x = \lambda x + \mu x$;
3. $\lambda(\mu x) = (\lambda\mu)x$; and
4. $1_R x = x$.

Right R**-modules** are defined similarly. An (R, S)**-bimodule** is both a left R-module and a right S-module satisfying $(rm)s = r(ms)$.

I denote left modules by $M : (R, -)$, right modules by $M : (-, R)$, and bimodules by $M : (R, S)$.

Example 1. $\mathbb{Z}$-modules and abelian groups are the same thing. Every right R-module is also a $(\mathbb{Z}, R)$-bimodule.

Proposition 1. Basic properties of modules:

1. $\lambda 0_M = 0_M$;
2. $0_R x = 0_M$;
3. $\lambda(-x) = -(\lambda x) = (-\lambda)x$.

If R is a division ring, then we also have

4. $\lambda x = 0_M \implies \lambda = 0_R$ or $x = 0_M$.

When R is commutative, any left R-module can be given the structure of a right R-module (and vice versa) by defining $x\lambda \doteq \lambda x$. Thus left and right R-modules are the same thing in this case. If F is a field, then an F-module is the same thing as an F-vector space.

The definition of modules gives us addition and scalar multiplication, but we still don't have a way of multiplying module elements together. Providing this is exactly the role of an algebra, which adds a ring structure to a module. **It seems like there isn't much of a difference between a ring and an algebra, so you should ask someone about this...**

Definition 2. Let R be a commutative unital ring. An R-**algebra** is an R-module M along with a "multiplication" map

$$\begin{aligned} M \times M &\to M \\ (x, y) &\mapsto xy. \end{aligned}$$

This map distributes over addition and satisfies

$$\lambda(xy) = (\lambda x)y = x(\lambda y).$$

Why is R commutative? We can form more specific types of algebras by putting restrictions on the multiplication map. **Associative and commutative algebras** have associative and commutative multiplication maps, respectively. A **unital** algebra has a multiplicative identity. A **division algebra** is a unital associative algebra in which every nonzero element has a multiplicative inverse.

1.2 SUBMODULES

A module is just an abelian group with a left action, so we can define a submodule to be just a subgroup that respects this action.

Definition 3. A **submodule** of an R-module M is a subgroup of M that is closed under the left action of R on M.

A module N is a submodule of M if and only if N is closed under subtraction and scalar multiplication (the subtraction emcompasses both addition and additive inverses). From this we infer the following simple characterization of a submodule.

Proposition 2. N is an submodule of M if and only if

$$\lambda x + \mu y \in N$$

for all $x, y \in N$ and $\lambda, \mu \in R$.

Thus given any set $S \subseteq M$, we can form a submodule of M by adding in all linear combinations of the elements of S (remember that linear combinations are by definition finite sums, so the induction works). This could be a good enough definition of $\langle S \rangle$, but we have to make sure that we aren't adding in any unnecessary terms. The following definition ensures this is the case, the next proposition shows that the definition makes sense, and the following theorem shows that our definition is equivalent to the linear combination approach.

Definition 4. Given a set $S \subseteq M$, let $\langle S \rangle$ denote the intersection of all submodules of M containing S.

Proposition 3. If $\{M_\alpha\}_\alpha$ is a family of submodules of M, then $\bigcap_\alpha M_\alpha$ is also a submodule of M.

Theorem 1. Let $S \subseteq M$, and let $LC(S)$ denote the set of all linear combinations of S. Then

$$\langle S \rangle = \begin{cases} \{0\} & \text{if } S = \varnothing, \\ LC(S) & \text{otherwise.} \end{cases}$$

Proof. The case $S = \varnothing$ is clear since all subgroups must contain 0, so assume S is nonempty. It's clear that $LC(S)$ is a submodule of M. Since $S \subseteq LC(S)$, this means $LC(S)$ is a submodule of M containing S, i.e. $\langle S \rangle \subseteq LC(S)$. But every linear combination of S must be in any submodule containing S, so $LC(S) \subseteq \langle S \rangle$ too. Thus $\langle S \rangle = LC(S)$. $\square$

If $\{M_\alpha\}_\alpha$ is a family of submodules of M, then $\bigcup_\alpha M_\alpha$ won't be a submodule in general (unlike $\bigcap_\alpha M_\alpha$), but it can certainly generate one. $\langle \cup_\alpha M_\alpha \rangle$ can be interpreted as the smallest submodule of M containing each of the M_α, and we can construct it by once again filling in all the missing linear combinations.

Proposition 4. Let $\mathcal{A}$ be some index set, and let $\mathbb{P}^\star(\mathcal{A})$ denote the set of all nonempty finite subsets of $\mathcal{A}$. Then $\langle \bigcup_\alpha M_\alpha \rangle$ is all finite sums of the form

$$\sum_{\beta \in \mathcal{B}} m_\beta,$$

where $\mathcal{B} \in \mathbb{P}^\star(\mathcal{A})$ and $m_\beta \in M_\beta$.

Proof. All linear combinations of the elements of $\bigcup_\alpha M_\alpha$ is this form, and $LC = \langle \bigcup_\alpha M_\alpha \rangle$ by Theorem 1 since $\bigcup_\alpha M_\alpha$ is nonempty (it must contain 0). $\square$

This motivates the notation

$$\sum_\alpha M_\alpha \doteq \langle \bigcup_\alpha M_\alpha \rangle$$

and the terminology "sum of the family $\{M_\alpha\}_\alpha$."

Theorem 2 (Modular Law)**.** Let M be an R-module, and let A, B, C be submodules of M with $C \subseteq A$. Then

$$A \cup (B + C) = (A \cup B) + C.$$

I have no idea why the book introduced this now.

1.3 MORPHISMS

As usual, an R-morphism respects the structure of R-modules.

Definition 5. An R-**morphism** is a map $f : M \to N$ between R-modules satisfying

1. $f(x + y) = f(x) + f(y)$;
2. $f(\lambda x) = \lambda f(x)$.

Note that if R is a field, then an R-morphism is just a linear map. Also note that if $f : M \to N$ is an R-morphism, then $\operatorname{Ker} f$ is a submodule of M and $\operatorname{Im} f$ is a submodule of N.

Proposition 5. Basic properties an R-morphism $f : M \to N$.

1. $f(0_M) = 0_N$.
2. $f(-x) = -f(x)$.

Because we like to be fancy, we'll use categorical language to describe specific types of R-morphisms:

$$\begin{aligned} R\text{-monomorphism} : \quad & M \rightarrowtail N, \\ R\text{-epimorphism} : \quad & M \twoheadrightarrow N. \end{aligned}$$

It's straightforward to show that the inverse of a bijective R-morphism is also an R-morphism, i.e. an R-isomorphism is just a bijective R-morphism. The usual properties of composed morphisms of course hold too:

- The composition of morphisms/monos/epis is a morphism/mono/epi.
- If gf is mono, then so is f.
- If gf is epi, then so is g.

As you might expect, a map between modules induces maps between their submodules.

Proposition 6. Suppose we have an R-morphism $f : M \to N$. Then for any submodule X of M, the image $f(X)$ is a submodule of N. Additionally, for any submodule Y of N, the preimage $f^{-1}(Y)$ is a submodule of M.

These maps induce maps between the entire submodule lattices $L(M)$ and $L(N)$:

$$\begin{tikzcd} L(M) \arrow[r, bend left, "f^{\rightarrow}"] & L(N) \arrow[l, bend left, "f^{\leftarrow}"] \end{tikzcd} \qquad \begin{aligned} f^{\rightarrow} &: X \mapsto f(X) \\ f^{\leftarrow} &: Y \mapsto f^{-1}(Y) \end{aligned}$$

Note that $f^{\rightarrow}$ and $f^{\leftarrow}$ are inclusion-preserving. We can also show how they interact with each other.

Proposition 7. Let f be an R-morphism $M \to N$. If $A \in L(M)$ and $B \in L(N)$, then

1. $f^{\to}(A \cap f^{\leftarrow}(B)) = f^{\to}(A) \cap B$;
2. $f^{\leftarrow}(B + f^{\to}(A)) = f^{\leftarrow}(B) + A$.

Prove this.

Corollary 1. If $A \in L(M)$ and $B \in L(N)$, then

1. $f^{\to}(f^{\leftarrow}(B)) = B \cap \operatorname{Im} f$;
2. $f^{\leftarrow}(f^{\to}(A)) = A + \operatorname{Ker} f$.

Is there a way to generalize this to something other than modules? If we have a morphism $f : X \to Y$, will $f(x)$ and $f^{-1}(y)$ have that proerty if x and y have the property, respectively?

Is the defn of R-morphism really just saying that it preserves module-ness by respecting linear combs?

f inj: There is a map $g : B \to A$ such that $fg = 1_A$.

f surj: There is a map $g : B \to A$ such that $gf = 1_B$.

1.4 LIFTS AND EXTENSIONS OF R-MORPHISMS

It's common to want to extend or lift an R-morphism. The following propositions give criteria for when this is possible.

Proposition 8. Suppose A, B, C are nonempty.

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, dashed, "\exists!\, h"] \arrow[r, "g"'] & A \end{tikzcd}$$

Suppose f is monic. Then there is a unique R-morphism h lifting g if and only if $\operatorname{Im} g \subseteq \operatorname{Im} f$. In this case, h is epic if and only if $\operatorname{Im} g = \operatorname{Im} f$.

Proof. The forward direction of the first statement is clear. To go backwards, note that any c, there is a b such that $g(c) = f(b)$ since $\operatorname{Im} g \subseteq \operatorname{Im} f$. Define h by $c \mapsto b$, then $f(h(c)) = f(b) = g(c)$, so h lifts g. This map is well-defined and unique since f is monic. To show it's an R-morphism, use the morphism properties of f and g to show $f(h(\lambda c)) = f(\lambda h(c))$ and $f(h(c_1 + c_2)) = f(h(c_1) + h(c_2))$, then use the fact that f is monic.

If h is epic, it's straightforward to show that $\operatorname{Im} f \subseteq \operatorname{Im} g$, which proves their equality. Conversely, fix b and suppose $\operatorname{Im} f = \operatorname{Im} g$. Then $f(b) = g(c) = f(h(c))$ for some c, which implies $b = h(c)$ since f is monic. □

Lemma 1. Suppose f and g are R-morphisms. If $\operatorname{Ker} f \subseteq \operatorname{Ker} g$, then

$$f(x) = f(y) \implies g(x) = g(y).$$

Proof. If $f(x) = f(y)$, then $f(x - y) = 0$, so $x - y \in \operatorname{Ker} f \subseteq \operatorname{Ker} g$. Thus $g(x - y) = 0$, so $g(x) = g(y)$. □

Proposition 9. Suppose A, B, C are nonempty.

$$\begin{tikzcd} & B \arrow[d, dashed, "\exists!\, h"] \\ A \arrow[ur, "f"] \arrow[r, "g"'] & C \end{tikzcd}$$

Suppose f is epic. Then there is a unique R-morphism h extending g if and only if $\operatorname{Ker} f \subseteq \operatorname{Ker} g$. In this case, h is monic if and only if $\operatorname{Ker} f = \operatorname{Ker} g$.

Proof. The forward direction of the first statement is clear. To go backwards, since f is epic, any b can be written $b = f(a)$ for some a. Then define $h : b \mapsto g(a)$. This clearly lifts g, and it is well-defined and unique by the preceding lemma. Showing it's an R-morphism is a standard check by writing $b = f(a)$ and using the morphism properties of f and g.

If h is monic, then for $a \in \operatorname{Ker} g$, we have $h(f(a)) = g(a) = 0$. But since f is monic, this implies $f(a) = 0$, so $a \in \operatorname{Ker} f$. Thus $\operatorname{Ker} g \subseteq \operatorname{Ker} f$, and we already know the opposite inclusion. Conversely, using the $b = f(a)$ fact, $h(b_1) = b(b_2) \implies g(a_1) = g(a_2)$, so $a_1 - a_2 \in \operatorname{Ker} g = \operatorname{Ker} f$, so $b_1 = f(a_1) = f(a_2) = b_2$. □

1.4.1 CONSEQUENCES OF EXACTNESS

We'll start out by noting some obvious characterizations of morphisms in terms of exact sequences. Quick reminder if if a sequence is exact, the composition of any two subsequent morphisms is 0.

Proposition 10. Monos, epis, and isos in terms of exact sequences:

1. f is monic $\iff 0 \to M \xrightarrow{f} N$ is exact.
2. f is epic $\iff M \xrightarrow{f} N \to 0$ is exact.
3. f is iso $\iff 0 \to M \xrightarrow{f} N \to 0$ is exact.

Now to prove that a bunch of diagrams commute if some exactness condition holds. I don't include any diagram chases, but luckily only the Four Lemma needs one. Also, everything below is implicitly assumed to be using R-modules and R-morphisms.

Proposition 11. The diagram commutes if the row is exact and $\theta g = 0$.

$$\begin{tikzcd}
 & & A \arrow[d, "g"] \arrow[dl, dashed, "\exists!\, h"'] & \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "\theta"] & Z
\end{tikzcd}$$

Proof. f must be monic and $\operatorname{Im} g \subseteq \operatorname{Im} f$, so a unique h exists by Proposition 8. □

Note that if $X = \operatorname{Ker} \theta$ and f is an inclusion map, then the row will always be exact.

Proposition 12. The diagram commutes if the row is exact and $g\theta = 0$.

$$\begin{tikzcd}
 & A & & \\
X \arrow[r, "\theta"] & Y \arrow[u, "g"] \arrow[r, "f"] & Z \arrow[ul, dashed, "\exists!\, h"'] \arrow[r] & 0
\end{tikzcd}$$

Proof. f must be epic and $\operatorname{Ker} f \subseteq \operatorname{Ker} g$, so a unique h exists by Proposition 9. □

Note that if $Z = X/\operatorname{Im} \theta$ and the f is a projection map, then the row will always be exact.

Theorem 3 (Four Lemma)**.** Suppose the following diagram commutes and has exact rows.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \end{tikzcd}$$

Then the following hold:

1. If α, γ are epic and δ is monic, then β is epic.
2. If α is epic and β, γ are monic, then γ is monic.

Theorem 4 (Five Lemma)**.** Suppose the following diagram commutes and has exact rows.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha_1"] & B \arrow[r] \arrow[d, "\alpha_2"] & C \arrow[r] \arrow[d, "\alpha_3"] & D \arrow[r] \arrow[d, "\alpha_4"] & E \arrow[d, "\alpha_5"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

If $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are iso, then so is α_3.

Proof. Apply the Four Lemma to the first three squares to show that α_3 is monic, and to the last three squares to show that α_3 is epic. Since it's an R-morphism, this is enough to show it's iso. □

Corollary 2 (Short Five Lemma)**.** Suppose the following diagram commutes and has exact rows.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

If α, γ are iso, then so is β.

Check D&F about the case "any two are iso".

This last corollary is just a special case of the Five Lemma when our two rows are short exact sequences. **If the D&F case applies here, that would be very useful for determining when a map of SESs is iso.**

Are all epis split in the category of modules?

2 CONSTRUCTING MODULES

2.1 QUOTIENT MODULES

Hello there.

2.2 PRODUCTS AND COPRODUCTS

We can make a **direct product** of R-modules $\prod_a M_\alpha$ into an R-module itself by defining

$$(x_\alpha)_\alpha + (y_\alpha)_\alpha \doteq (x_\alpha + y_\alpha)_\alpha,$$
$$\lambda(x_\alpha)_\alpha \doteq (\lambda x_\alpha)_\alpha.$$

If we add the restriction that only a finite number of the coordinates can be nonzero, then we get the (**external**) **direct sum** $\bigoplus_\alpha M_\alpha$. In this context, π_α denotes the canonical projection onto the α-th coordinate, and i_α denotes the α-th canonical injection

$$x \mapsto (\dots, 0, x, 0, \dots),$$

where the single nonzero coordinate is the α-th coordinate.

Instead of worrying about individual elements, we can use the universal properties of the product and coproduct to characterize direct products and sums.

Note 1. I still use the notation π_α and i_α in the general categorical setting, but unless I'm specifically using them with a direct product or direct sum, they're just ordinary morphisms instead of special projections or injections.

Definition 6. Fix a category C and objects $\{M_\alpha\}_\alpha$. A **product** of $\{M_\alpha\}_\alpha$ is an object P with morphisms $\pi_\alpha : P \to M_\alpha$ such that for any other object N and morphisms $f_\alpha : N \to M_\alpha$, there is a unique morphism $f : N \to P$ lifting each f_α.

$$\begin{tikzcd}
& P \arrow[d, "\pi_\alpha"] \\
N \arrow[ur, dashed, "f"] \arrow[r, "f_\alpha"'] & M_\alpha
\end{tikzcd}$$

Dually, a **coproduct** of $\{M_\alpha\}_\alpha$ is an object C with morphisms $i_\alpha : M_\alpha \to C$ such that for any other object object N and morphisms $f_\alpha : M_\alpha \to N$, there is a unique morphism $f : C \to N$ extending each f_α.

$$\begin{tikzcd}
& C \arrow[dl, dashed, "f"'] \\
N & M_\alpha \arrow[l, "f_\alpha"] \arrow[u, "i_\alpha"']
\end{tikzcd}$$

Proposition 13. If $(P, \{\pi_\alpha\})$ is a product, then each π_α is epic. If $(C, \{i_\alpha\})$ is a coproduct, then each i_α is monic.

Proof. Fix α, let $N = M_\alpha$, and let f_α be the identity. Then there are unique f_P, f_C such that $\pi_\alpha f_P = 1$ and $f_C i_\alpha = 1$, i.e. π_α is epic and i_α is monic. $\square$

Theorem 5 (Uniqueness)**.** If $(P, \{\pi_\alpha\})$ is a product, then $(Q, \{\phi_\alpha\})$ is too $\iff$ there is a unique isomorphism $P \cong Q$ such that the first diagram commutes for all α. Dually, if $(C, \{i_\alpha\})$ is a coproduct, then $(D, \{j_\alpha\})$ is too $\iff$ there is a unique isomorphism $C \cong D$ such that the second diagram commutes for all α.

$$\begin{tikzcd} P \arrow[d, "\pi_\alpha"'] & Q \arrow[l, "\sim"'] \arrow[dl, "\phi_\alpha"] \\ M_\alpha & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\sim"] & D \\ M_\alpha \arrow[u, "i_\alpha"] \arrow[ur, "j_\alpha"'] & \end{tikzcd}$$

Proof. We need only prove the case for products, since the coproduct case is dual. The forward direction is straightforward. For the backward direction, extend P's unique lift gotten with the unique isomorphism's inverse to get Q's unique lift. $\square$

Theorem 6 (Existence)**.** $\left(\prod_{\alpha \in \mathcal{A}} M_\alpha, \{\pi_\alpha\}\right)$ is a product of $\{M_\alpha\}$.

Proof. Given N and morphisms $f_\alpha : N \to M_\alpha$, we define f in the obvious way by

$$x \mapsto (f_\alpha(x))_\alpha.$$

It's an R-morphism, it satisfies the universal property, and it clearly must be unique. $\square$

Note 2. Thus up to (unique) isomorphism, every family of R-modules has a unique product and coproduct. We can then call the direct product (direct sum) *the* product (coproduct).

A consequence of the uniqueness of the product and coproduct is that both $\prod$ and $\bigoplus$ are commutative and associative (no matter what order we do things in, we end up with a product/coproduct, which must be isomorphic to the product/coproduct we got with the original ordering).

Do proof of associativity for practice.

Finish this section.

2.3 THE TENSOR PRODUCT

Note 3. Big idea: the tensor product of two modules is a space in which we can describe bilinear maps in terms of linear maps.

Definition 7. Suppose M is a right R-module and N is a left R-module. If G is a $\mathbb{Z}$-module, then $F : M \times N \to G$ is **balanced** if

1. $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$;
2. $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$;
3. $f(m\lambda, n) = f(m, \lambda n)$.

Definition 8. The **tensor product** of a right module M and left module N is a $\mathbb{Z}$-module $M \otimes N$ with a balanced **tensor map** $\otimes$ such that for all $\mathbb{Z}$-modules G and balanced maps $f : M \times N \to G$, there is a unique $\mathbb{Z}$-morphism lifting f through $\otimes$.

$$\begin{tikzcd} M \otimes N \arrow[dr, dashed, "\exists!\ \phi"] & \\ M \times N \arrow[u, "\otimes"] \arrow[r, "f"'] & G \end{tikzcd}$$

Note that if M and N are not both trivial, then $\otimes$ is *never* injective. Since $\otimes(m\lambda, n) = \otimes(m, \lambda n)$, set $\lambda = 0$ to get $\otimes(0, n) = \otimes(m, 0)$ for all m, n.

Proposition 14. $\langle \mathrm{Im} \otimes \rangle = M \otimes N$.

Thus every element in $M \otimes N$ can be written

$$\sum_{i=1}^{\ell} k_i(\tilde{m}_i \otimes \tilde{n}_i) = \sum_{i=1}^{\ell} m_i \otimes n_i.$$

In general, this representation is not unique, so we are *not* working with a basis.

Lemma 2. If a $\mathbb{Z}$-morphism has an addition-respecting property on a single $m \otimes n$, then it has that property on all of $M \otimes N$.

Proof. You can express any element of $M \otimes N$ as $\sum_i m_i \otimes n_i$, and $\mathbb{Z}$-morphisms respect addition. $\square$

Theorem 7 (Uniqueness)**.** The tensor product is unique up to (unique) isomorphism.

$$M\tilde{\otimes}N \text{ is also a tensor product} \iff \begin{tikzcd} M \otimes N \arrow[r, dashed, "\exists! \sim"] & M\tilde{\otimes}N \\ M \times N \arrow[u, "\otimes"] \arrow[ur, "\tilde{\otimes}"'] & \end{tikzcd}$$

Let F be the free module on $M \times N$, and let H be the subgroup of F generated by all elements of the form

$$\begin{aligned} i(m_1 + m_2, n) - i(m_1, n) - i(m_2, n), \\ i(m, n_1 + n_2) - i(m, n_1) - i(m, n_2), \\ i(m\lambda, n) - i(m, \lambda n). \end{aligned}$$

If $M \times N \xrightarrow{i} F \xrightarrow{\pi} F/H$, define

$$\begin{aligned} M \otimes_R N &\doteq F/H, \\ \otimes_R &\doteq \pi i. \end{aligned}$$

This gives us the canonical tensor product of $M \times N$.

Theorem 8 (Existence)**.** $M \otimes_R N$ is a tensor product of $M \times N$.

Proof. Recall that $M \otimes_R N = F/H$ and $\otimes_R = \pi i$.

$$\begin{tikzcd} & F \arrow[r, two heads, "\pi"] \arrow[dr, dotted, "\exists! h"] & F/H \arrow[d, dashed, "\exists! \phi"] \\ M \times N \arrow[ur, hook, "i"] \arrow[rr, "f"'] & & G \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} M \otimes_R N \arrow[dr, dashed, "\exists! \phi"] & \\ M \times N \arrow[u, "\otimes_R"] \arrow[r, "f"'] & G \end{tikzcd}$$

Since F is free, we get h extending f. Then since f is balanced, the definition of H gives $\operatorname{Ker} \pi = H \subseteq \operatorname{Ker} f$. Then since π is epic, Proposition 9 gives us ϕ extending h. Now ϕ is the only morphism extending h through π, but it is also the only morphism extending f through πi: if $\tilde{\phi}$ also extends f, then $\phi\pi i = \tilde{\phi}\pi i = f = hi$. But h is unique, so $\phi\pi = \tilde{\phi}\pi$, which implies $\phi = \tilde{\phi}$ since π is epic. □

Note 4. Thus up to (unique) isomorphism, there is a unique tensor product of $M \times N$. We'll then call $M \otimes_R N$ *the* tensor product of $M \times N$, and we'll also denote $m \otimes_R n \doteq \otimes_R(m, n)$.

The tensor product preserves module-ness in a manner similar to how dimensions work with matrix multiplication. The bimodules need to align in the middle, and the bimodules on the outside determine the bimodules of the tensor product.

Proposition 15.

$$M : (S, R), \quad N : (R, T) \quad \Longrightarrow \quad M \otimes_R N : (S, T)$$

Proof. The left S-action and right T-action are

$$s\left(\sum_i m_i \otimes_R n_i\right) \doteq \sum_i sm_i \otimes_R m_i,$$
$$\left(\sum_i m_i \otimes_R n_i\right) t \doteq \sum_i m_i \otimes_R m_i t.$$

□

Note 5. If $M : (-, R)$ and $N : (R, -)$, then $M \otimes_R N$ is a $\mathbb{Z}$-module since every right R-module is also a $(\mathbb{Z}, R)$-bimodule.

Corollary 3. If R is commutative and M, N are R-modules, then $M \otimes_R N$ is also an R-module.

Proposition 16. $\otimes$ is associative: if $M : (-, R), N : (R, S), P : (S, -)$, then there is a unique $\mathbb{Z}$-iso

$$(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P).$$

If $M : (T, R)$ too, then the iso is also a T-iso.

Corollary 4. If R is commutative and M, N, P are R-modules, then there is a unique R-iso

$$(M \otimes N) \otimes P \cong M \otimes (N \otimes P).$$

2.3.1 MULTILINEARITY TO LINEARITY

Definition 9. Suppose R is commutative and unital and $M_1, \ldots, M_n$ and N are R-modules. We say that

$$\phi : M_1 \times \cdots \times M_n \to N$$

is n-**multilinear** over R if it's an R-morphism (i.e. R-linear) in each factor. We call 2-multilinear maps **bilinear** and 3-multilinear maps **trilinear**.

Since $\otimes$ is associative, the following theorem is unambiguous.

Theorem 9. Suppose R is commutative and unital and $M_1, \ldots, M_n$ and N are R-modules. If $f : M_1 \times \cdots \times M_n \to N$ is n-multilinear, then it extends uniquely through the tensor product to an R-morphism (i.e. an R-linear map).

$$\begin{tikzcd}
M_1 \otimes \cdots \otimes M_n \arrow[dr, dashed, "\exists! \ \phi"] & \\
M_1 \times \cdots \times M_n \arrow[u] \arrow[r, "f"'] & N
\end{tikzcd}$$

The map $(m_1, \ldots, m_n) \mapsto m_1 \otimes \cdots \otimes m_n$ is also n-multilinear.

3 SPECIAL MODULES

3.1 CHAIN CONDITIONS AND TOWERS

Any modules can be broken down into some ascending or descending sequences of submodules. If we restrict our attention to only modules with finite such sequences, then we characterize them further.

Definition 10. An R-module M is **Noetherian** if for all ascending submodule chains

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots,$$

there is some $n \in \mathbb{N}$ such that $M_{n+k} = M_n$ for all $k \in \mathbb{N}$, i.e. the chain stabilizes at n. We say that M is **Artinian** if for all descending chains

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots,$$

there is again some n at which the chain stabilizes. We call these two qualities **chain conditions**.

We can also define similar concepts for unordered sets of submodules.

Definition 11. An R-module M has the **maximal (minimal) condition** if every nonempty collection of submodules of M has some maximal (minimal) submodule w.r.t. set inclusion.

Note that we're using maximal/minimal, *not* maximum/minimum. This is important.

Theorem 10. TFAE:

1. M is Noetherian.
2. M satisfies the maximal condition.
3. Every submodule of M is finitely generated.

Theorem 11. TFAE:

1. M is Artinian.
2. M satisfies the minimal condition.

Is there any similar thing about being finitely generated, or is that just a property of Noetherian modules?

A nice property of chain conditions is that they are passed onto submodules and quotient modules. The converse also holds.

Proposition 17. If M has some chain condition, then each of its submodules and quotient modules has it too. Conversely, if every submodule N of M and every quotient module M/N has the same chain condition, then so does M.

3.1.1 SIMPLE MODULES

A very extreme case of the above conditions is when a module's only proper submodule is the trivial submodule. These modules are called **simple**. As you might expect (since R-morphisms induce maps between submodules), modules going to or coming from a simple module are pretty restricted.

Proposition 18. If $f : M \to N$ is a nonzero R-morphism, then:

1. If M is simple, then f is monic.
2. If N is simple, then f is epic.

Proof. $\operatorname{Ker} f$ and $\operatorname{Im} f$ are submodules of M and $f \neq 0 \implies \operatorname{Ker} f = 0$ and $\operatorname{Im} f = N$. □

Corollary 5 (Schur)**.** If M is simple, then $\operatorname{End}_R(M)$ is a division ring.

Proof. Every nonzero endomorphism is necessarily iso. Since the natural multiplication on $\operatorname{End}_R(M)$ is composition, this means every nonzero element has a multiplicative inverse. □

3.1.2 SUBMODULE TOWERS

Stuff here.

Extra nice modules will be both Noetherian and Artinian, and its these modules that have a special "height" characterization based on their submodule towers.

3.2 FREE MODULES

Note 6. Big idea: free modules are modules with a basis.

Given a nonempty set S and a unital ring R, we can fill in all the missing linear combinations of S to get a module $\langle S \rangle$. This module is "free" of any unnecessary relations between its elements: it contains every possible linear combination of terms, with nothing simplified via some other relation.

Definition 12. Fix a category, then a **free object** on a set S is an object F with a map $i : S \to F$ such that for all other objects M, every map $f : S \to M$ extends uniquely through i to a morphism $F \to M$.

$$\begin{tikzcd} F \arrow[dr, dashed, "\exists!\, h"] & \\ S \arrow[u, "i"] \arrow[r, "f"'] & M \end{tikzcd}$$

We denote this by (F, i) and say that F is free on S.

Proposition 19. If (F, i) is a free module, then f is injective and $\langle \operatorname{Im} i \rangle = F$.

Theorem 12 (Uniqueness)**.** Suppose (F, i) is free on S. Then so is (G, j) $\iff$ there is a unique isomorphism $F \cong G$ making the following diagram commute.

$$\begin{tikzcd} F \arrow[r, dashed, "\exists!\, \sim"] & G \\ S \arrow[u, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

Proof. To go forwards, plug F into G's universal property, then plug G into F's. The resulting two unique morphisms are isomorphisms that make the diagram commute. To go backwards, lift i's unique extesion by the unique isomorphism $G \to F$ (the inverse of the one in the diagram) to get j's unique extension. $\square$

Theorem 13 (Existence)**.** For every nonempty set S and unital ring R, there is a free R-module on S.

Proof. Let $F = \bigoplus_{s \in S} Rs$ denote the set of all formal linear combinations of S, which has elements of the form $\sum_s r_s s$, where only finitely many of the r_s are nonzero. There's a natural inclusion $i : S \hookrightarrow F$. Given M and g, define h on $i(S)$ by $h(s) = g(s)$, then extend by linearity to all of F. It's necessarily a unique R-morphism that satisfies the universal property. $\square$

Note 7. Thus up to (unique) isomorphism, every nonempty set S has a unique free R-module. We can then call $\bigoplus_{s \in S} Rs$ *the* free R-module on S.

Note that the map $s \mapsto \mathbf{e}_s$ shows $\bigoplus_s Rs \cong \bigoplus_s R$, so we can also describe the free R-module on S as a direct sum of copies of R, indexed by S.

So that we don't have to deal with the map i when describing free modules, we say that a module M is free if there is some free module (F, i) and an isomorphism $M \cong F$. Then the "inclusion" map for M is i extended by the isomorphism.

3.2.1 BASES

Definition 13. A **basis** of an R-module M is a linearly independent subset of M that generates M.

Theorem 14. A nonempty subset $S \subseteq M$ is a basis of $M \iff$ each element of M can be uniquely expressed as a linear combination of elements of S.

Proposition 20. If (F, i) is a free module, then $\operatorname{Im} i$ is a basis of F.

Proof. Suppose (F, i) is free over some nonempty S, then we know $F \cong \bigoplus_s Rs$, and it's clear that S is a basis of $\bigoplus_s Rs$. We can then translate this basis for $\bigoplus_s Rs$ into a basis for F since the isomorphism necessarily commutes with both modules' inclusion maps by Theorem 12. □

Theorem 15. A module is free $\iff$ it has a basis.

Proof. If F is free, then isomorphic to $\bigoplus_s Rs$, so its basis is the basis S of $\bigoplus_s Rs$ mapped through the isomorphism. Conversely, if S is a basis of F, then there is a natural inclusion $i : S \hookrightarrow F$. Fix another module M and a map $f : S \to M$, then the only way to get an R-morphism $h : F \to M$ is to define $h(s) \doteq g(s)$ and then extend by linearity, which is unique. Thus F is free. □

Fill in other notes here.

Theorem 16. The coproduct of free objects is itself free. Explicitly, if F_α is free over S_α, then $\coprod_\alpha F_\alpha$ is free over $\bigsqcup_\alpha S_\alpha$.

Proof. Suppose we have a family of free objects F_α over S_α. Fix α, and let M and $g : \bigsqcup_\alpha S_\alpha \to M$

be arbitrary.

$$
\begin{tikzcd}
F_\alpha \arrow[r, hook, "j_\alpha"] & \coprod_\alpha F_\alpha \arrow[ddr, dashed, bend left, "\exists!\ \phi"] & \\
S_\alpha \arrow[u, hook, "i_\alpha"] \arrow[r, hook, shift left] \arrow[ddrr, bend right, "f_\alpha"'] & \bigsqcup_\alpha S_\alpha \arrow[u, hook, "i"'] \arrow[l, two heads, shift left, "\pi_\alpha"] \arrow[dr, "g"] & \\
& & M
\end{tikzcd}
\qquad
\begin{tikzcd}
F_\alpha \arrow[dr, dashed, "\exists!\ f"] & \\
& M
\end{tikzcd}
$$

The diagram's got a lot going on, but it's straightforward. All four inclusions and the one projection are the natural ones, so the square commutes. The map g induces f_α by $f_\alpha \pi_\alpha = g|_{S_\alpha}$. Then f comes from the universal property of free modules, so $f i_\alpha = f_\alpha$. Then ϕ comes from the universal property of the coproduct, so $\phi j_\alpha = f$.

To show that $\coprod_\alpha F_\alpha$ is free, we have to show that ϕ extends g through i. But for any $s \in \bigsqcup_\alpha S_\alpha$ coming from S_α,

$$(\phi i)(s) = (\phi j_\alpha i_\alpha \pi_\alpha)(s) = (f_\alpha \pi_\alpha)(s) = g(s),$$

so $\phi i = g$. Thus $\coprod_\alpha F_\alpha$ is free on $\bigsqcup_\alpha S_\alpha$. □

Corollary 6. The direct sum of free R-modules is itself free.

Finish this section

3.3 HOM SETS

Given R-modules M, N, the set $\mathrm{Hom}(M, N)$ is an abelian group under function addition, but the left action $(\lambda, f) \mapsto \lambda f$ doesn't necessarily make $\mathrm{Hom}(M, N)$ into an R-module (λf might not be a morphism). This is only true if R is commutative.

> **Note 8.** An abelian group iso is the same thing as a $\mathbb{Z}$-iso.

Theorem 17. The following are $\mathbb{Z}$-isos.

1. $\mathrm{Hom}\left(\coprod_\alpha M_\alpha, N\right) \cong \prod_\alpha \mathrm{Hom}\left(M_\alpha, N\right)$.
2. $\mathrm{Hom}\left(N, \prod_\alpha M_\alpha\right) \cong \prod_\alpha \mathrm{Hom}(N, M_\alpha)$.

Corollary 7. If R is commutative, then the above $\mathbb{Z}$-isos are also R-isos.

Corollary 8. If we're dealing with a finite set $M_1, \ldots, M_n$, then we have $\mathbb{Z}$-isos

1. $\mathrm{Hom}(\bigoplus_{i=1}^n M_i, N) \cong \bigoplus_{i=1}^n \mathrm{Hom}(M_i, N)$;
2. $\mathrm{Hom}(N, \bigoplus_{i=1}^n M_i) \cong \bigoplus_{i=1}^n \mathrm{Hom}(N, M_i)$.

3.3.1 HOM FUNCTORS

Fix a module M, then for any other module A, there are associated abelian groups $\mathrm{Hom}(A, M)$ and $\mathrm{Hom}(M, A)$. A morphism $f : A \to B$ also induces maps on the hom sets via pre/post composition.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[dl, dashed] \\ M \arrow[u, dashed] & \end{tikzcd}$$

$$\begin{aligned} f_* : \mathrm{Hom}(M, A) &\to \mathrm{Hom}(M, B) \\ g &\mapsto fg \end{aligned}$$

$$\begin{aligned} f^* : \mathrm{Hom}(A, M) &\leftarrow \mathrm{Hom}(B, M) \\ gf &\mapsfrom g \end{aligned}$$

Proposition 21. Both induced maps respect addition. Also, $(gf)_* = g_* f_*$ and $(gf)^* = f^* g^*$.

> **Note 9.** This says that $\mathrm{Hom}(M, -)$ is a covariant functor, while $\mathrm{Hom}(-, M)$ is contravariant.

Definition 14. The following definitions apply for a covariant functor $\mathcal{F}$ if, given any short exact $0 \to A \to B \to C \to 0$, the given induced sequences are also exact.

$$\begin{array}{rl} \textbf{exact} & 0 \to \mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C \to 0 \\ \textbf{left exact} & 0 \to \mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C \\ \textbf{right exact} & \quad\;\; \mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C \to 0 \end{array}$$

There are similar definitions for a contravariant functor $\mathcal{G}$.

$$\begin{array}{rl} \textbf{exact} & 0 \to \mathcal{G}C \to \mathcal{G}B \to \mathcal{G}A \to 0 \\ \textbf{left exact} & 0 \to \mathcal{G}C \to \mathcal{G}B \to \mathcal{G}A \\ \textbf{right exact} & \quad\;\; \mathcal{G}C \to \mathcal{G}B \to \mathcal{G}A \to 0 \end{array}$$

Theorem 18. $\text{Hom}(M, -)$ and $\text{Hom}(-, M)$ are left exact. Explicitly, for all short exact

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

the following induced sequences are exact.

$$0 \longrightarrow \text{Hom}(M, A) \xrightarrow{f_*} \text{Hom}(M, B) \xrightarrow{g_*} \text{Hom}(M, C)$$

$$\text{Hom}(A, M) \xleftarrow{f^*} \text{Hom}(B, M) \xleftarrow{g^*} \text{Hom}(C, M) \longleftarrow 0$$

Corollary 9. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact, then the following induced sequences are also split exact.

$$0 \longrightarrow \text{Hom}(M, A) \xrightarrow{f_*} \text{Hom}(M, B) \xrightarrow{g_*} \text{Hom}(M, C) \longrightarrow 0$$

$$0 \longleftarrow \text{Hom}(A, M) \xleftarrow{f^*} \text{Hom}(B, M) \xleftarrow{g^*} \text{Hom}(C, M) \longleftarrow 0$$

Proof. We only do this for the first induced sequence, as the second one is dual. Since the original SES splits, g has a right inverse $\tilde{g}$. Then $g_*\tilde{g}_* = (g\tilde{g})_* = (1_C)_*$, which is the identity on $\text{Hom}(M, C)$. Thus g_* is epic and the sequence splits. By the previous theorem, the rest of the sequence is exact. $\square$

In general, though, we can't guarantee that g_* or f^* is surjective. This motivates the definition of projective and injective modules.

3.4 PROJECTIVE MODULES

Note 10. Big idea: a projective module P makes any SES $0 \to A \to B \to P \to 0$ split.

Definition 15. An R-module M is **projective** if for all short exact sequences

$$0 \longrightarrow A \longrightarrow B \xrightarrow{g} C \longrightarrow 0,$$

the following sequence is exact.

$$\mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \longrightarrow 0$$

Equivalently, if g is epic, then so is g_*.

Proposition 22. The functor $\mathrm{Hom}(M, -)$ is exact $\iff$ M is projective.

Proposition 23. P is projective $\iff$ any morphism $P \to C$ can be lifted (not necessarily uniquely) through epis, i.e. whenever $B \to C \to 0$ is exact.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\,\phi"'] \arrow[d] & \\ B \arrow[r, two heads] & C \arrow[r] & 0 \end{tikzcd}$$

Theorem 19. TFAE:

1. P is a projective module.
2. Every SES $0 \to L \to M \to P \to 0$ splits.
3. P is a direct summand of a free module, i.e. there is some Q such that $P \oplus Q$ is free.

Corollary 10. Free modules are projective (the converse isn't true in general).

Corollary 11. Every module is the quotient of a projective module.

Proof. Every module is the quotient of a free module, and free modules are projective. □

3.4.1 RESULTS FOR VECTOR SPACES

If we're working with vector spaces, then these results can tell us a lot about our spaces' dimensions.

Proposition 24. Every SES of vector spaces splits.

Proof. Every vector space has a basis, so it's free, so it's projective. $\square$

Corollary 12. If W is a subspace of a vector space V, then $V \cong W \oplus V/W$.

Proof. The sequence $0 \to W \xrightarrow{i} V \xrightarrow{\pi} V/W \to 0$ is exact, so it splits, so $V \cong W \oplus V/W$. $\square$

Corollary 13. If W is a subspace of V, then $\dim V = \dim W + \dim(V/W)$.

Corollary 14. If W is a subspace of finite-dimensional vector space V, then $\dim V = \dim W \iff V = W$.

Proof. If $\dim V = \dim W$, then by Corollary 13, $\dim(V/W) = 0$. Thus $V/W = 0$, so $V = W$. The other direction is clear. $\square$

This isn't true for free modules in general. For example, if $n \neq 0, 1$, then $n\mathbb{Z}$ is a strict submodule of $\mathbb{Z}$, yet both have dimension 1 since they each have a 1-element basis.

Theorem 20 (Rank-Nullity)**.** If $\phi : V \to W$ is a linear map, then

$$\dim V = \dim(\operatorname{Im}\phi) + \dim(\operatorname{Ker}\phi).$$

Proof. $\operatorname{Ker}\phi$ is a subspace of V, so $V \cong \operatorname{Ker} f \oplus V/\operatorname{Ker} f \cong \operatorname{Ker} f \oplus \operatorname{Im} f$ (by 1st iso theorem). $\square$

Corollary 15. If V, W are finite-dimensional vector spaces of equal dimension, and if $\phi : V \to W$ is linear, then TFAE:

1. ϕ is injective;
2. ϕ is surjective;
3. ϕ is bijective.

Proof. By rank-nullity, f is injective $\iff$ $\operatorname{Ker}\phi = 0$ $\iff$ $\dim(\operatorname{Ker}\phi) = 0$ $\iff$ $\dim V = \dim(\operatorname{Im}\phi)$ $\iff$ $V = \operatorname{Im}\phi$ $\iff$ ϕ is surjective. $\square$