**Exercise 1** (2.1: 14)**.**

a. Determine whether there exists a SES $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$.

b. Determine which abelian groups $A$ fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with $p$ prime.

c. What about $0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$?

a. By the first isomorphism theorem, exactness of the SES given is equivalent to finding injective $i : \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2$ and surjective $j : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$ such that

$$\frac{\mathbb{Z}_8 \oplus \mathbb{Z}_2}{i(\mathbb{Z}_4)} = \frac{\mathbb{Z}_8 \oplus \mathbb{Z}_2}{\operatorname{Ker} j} \cong \mathbb{Z}_4.$$

There aren't actually many candidates for $i$, so we can start our search with those. Note that in order for $i$ to be a homomorphism, we need $4 \cdot i(1) = i(4) = i(0) = 0$. Since $i$ must also be injective, this implies that $i(1)$ must be of order 4 exactly in $\mathbb{Z}_8 \oplus \mathbb{Z}_2$. There are two such elements: $(2, 0)$ and $(2, 1)$.

Mapping $1 \to (2, 0)$ doesn't work, as $\frac{\mathbb{Z}_8 \oplus \mathbb{Z}_2}{i(\mathbb{Z}_4)}$ in that case isn't cyclic and thus can't be isomorphic to $\mathbb{Z}_4$. So we instead define $i$ by mapping $1 \mapsto (2, 1)$. The image of $i$ is then

$$i(\mathbb{Z}_4) = \{(0, 0), (2, 1), (4, 0), (6, 1)\},$$

and we can use this to show that the cosets of $\frac{\mathbb{Z}_8 \oplus \mathbb{Z}_2}{i(\mathbb{Z}_4)}$ are

$$(0, 0) + i(\mathbb{Z}_4), \quad (1, 0) + i(\mathbb{Z}_4), \quad (0, 1) + i(\mathbb{Z}_4), \quad (1, 1) + i(\mathbb{Z}_4).$$

Note that this quotient group is generated by $[(1, 0)]$ since

$$\begin{aligned} 2 \cdot [(1, 0)] &= [(2, 0)] = [(0, 1)], \\ 3 \cdot [(1, 0)] &= [(3, 0)] = [(1, 1)], \\ 4 \cdot [(1, 0)] &= [(4, 0)] = [(0, 0)]. \end{aligned}$$

Thus the map determined by $[(1, 0)] \mapsto 1$ is an isomorphism $\frac{\mathbb{Z}_8 \oplus \mathbb{Z}_2}{i(\mathbb{Z}_4)}$. This means the sequence

$$0 \to \mathbb{Z}_4 \xrightarrow{i} \mathbb{Z}_8 \oplus \mathbb{Z}_2 \xrightarrow{j} \mathbb{Z}_4 \to 0$$

is exact, where $j$ is the composition of the canonical projection $\mathbb{Z}_8 \oplus \mathbb{Z}_2 \twoheadrightarrow \frac{\mathbb{Z}_8 \oplus \mathbb{Z}_2}{i(\mathbb{Z}_4)}$ and the above isomorphism.

b. In order to be exact, we need

$$\frac{A}{\mathbb{Z}_{p^m}} \cong \mathbb{Z}_{p^n},$$

which forces the order of $A$ to be $p^n p^m = p^{n+m}$. We know $A$ is abelian, and now we know it's finite, so it must then be finitely generated. Any finitely generated, finite group $A$ of order $p^{m+n}$ can be written

$$A \cong \bigoplus_{i=1}^{\ell} \mathbb{Z}_{p^{k_i}}$$

for some $\ell$ and natural numbers $k_i$. Similar to part (a), $i(1)$ must be order $p^m$ in order for $i$ to be an injective homomorphism, so $\max_i k_i \geq m$. We'll now show that $\ell = 2$ since $A$ is generated by 2 elements.

We claim $A = \langle i(1), \tilde{a} \rangle$, where $j(\tilde{a}) = 1$ (we know such an $\tilde{a}$ exists since $j$ is surjective). Suppose $a \in \operatorname{Im} i$, then since $i(\mathbb{Z}_{p^m})$ is cyclic, $a$ is generated by $i(1)$. Now suppose $a \notin \operatorname{Im} i = \operatorname{Ker} j$, then $j(a) \neq 0$, so $j(a) = r \cdot j(\tilde{a}) = j(r\tilde{a})$ for some $r \in \mathbb{N}$. Then $j(a - r\tilde{a}) = 0$, so $a - r\tilde{a} \in \operatorname{Ker} j = \operatorname{Im} i$. Since this element is then generated by $i(1)$, we have $a - r\tilde{a} = s \cdot i(1))$ for some $s \in \mathbb{N}$. Rearranging gives $a = s \cdot i(1) + r\tilde{a}$, so $a$ is generated by $i(1)$ and $\tilde{a}$.

By this argument, we know $A$ is the direct sum of exactly two groups $\mathbb{Z}_{p^{k_1}}$ and $\mathbb{Z}_{p^{k_2}}$. Since $\max_i k_i \geq m$, this leaves us with the following family of possible $A$:

$$\mathbb{Z}_{p^{m+n-k}} \oplus \mathbb{Z}_{p^k}$$

for $0 \leq k \leq \min\{n, m\}$. As it turns out, all of these work.

To construct $i$, we'll use the same observation from part (a) that $i(1)$ should have order $p^m$ and define $i$ by mapping $i : 1 \mapsto (p^{n-k}, 1)$. We now claim that the cosets of $\frac{\mathbb{Z}_{p^{m+n-k}} \oplus \mathbb{Z}_{p^k}}{i(\mathbb{Z}_{p^m})}$ are generated by $[(1,0)]$. Consider any coset $[(x,y)] = (x,y) + \operatorname{Im} i$, then

$$[(x,y)] = [(x,y) - y(p^{n-k}, 1)] = [(x - yp^{n-k}, 0)] = (x - yp^{n-k})[(1,0)].$$

Thus this quotient group is cyclic. To find its order, note that

$$p^n[(1,0)] = [(p^n, 0)] = p^k[(p^{n-k}, 0)] = p^k[(0,0)] = 0.$$

This is the smallest integer multiple that yields 0, so the order of the quotient is $p^n$, i.e. it's isomorphic to $\mathbb{Z}_{p^n}$. Then the sequence

$$0 \to \mathbb{Z}_{p^m} \overset{i}{\rightarrowtail} \mathbb{Z}_{p^{n+m-k}} \oplus \mathbb{Z}_{p^k} \overset{j}{\twoheadrightarrow} \mathbb{Z}_{p^n} \to 0$$

is exact, where $j$ is the composition of the canonical projection $\frac{\mathbb{Z}_{p^{n+m-k}} \oplus \mathbb{Z}_{p^k}}{i(\mathbb{Z}_{p^m})} \twoheadrightarrow i(\mathbb{Z}_{p^m})$ and the isomorphism of this quotient with $\mathbb{Z}_{p^n}$.

c. By the same argument as in part (b), $A$ is the direct sum of two cyclic groups. Since $\mathbb{Z} \rightarrowtail A$ is injective when the sequence is exact, we know that 1 of them must be $\mathbb{Z}$. Since $A \twoheadrightarrow \mathbb{Z}_n$ is surjective when the sequence is exact, we know the other must be $\mathbb{Z}_m$ for some $m$ dividing $n$.

As it turns out, any such direct sum works. Define $i$ by mapping $i : 1 \mapsto (1, n/d)$, and define $j : (x,y) \mapsto y - xn/d$. Since for all $x$, we have

$$(ji)(x) = j(x, xn/d) = xn/d - xn/d = 0,$$

we know $\operatorname{Im} i \subset \operatorname{Ker} j$. Conversely, suppose $j(x,y) = y - xn/d = 0 \mod n$, then $y - xn/d = nk$ for some $k \in \mathbb{N}_0$. Rearranging gives $y = nk + xn/d$, so we rewrite $(x,y)$ as

$$(x,y) = (x, nk + xn/d) = (x, xn/d) = i(x),$$

where the second equality follows from the second coordinate being mod $n$. Thus $\operatorname{Ker} j \subset \operatorname{Im} i$, so the two are actually equal. Thus the sequence

$$0 \to \mathbb{Z} \overset{i}{\rightarrowtail} \mathbb{Z} \oplus \mathbb{Z}_d \overset{j}{\twoheadrightarrow} \mathbb{Z}_n \to 0$$

is exact.

**Exercise 2** (2.1: 15)**.** For $A \to B \to C \to D \to E$ exact, show that $C = 0 \iff$ the map $A \to B$ is surjective and $D \to E$ is injective. Then show that $A \hookrightarrow X$ induces isomorphisms on all homology groups $\iff H_n(X, A) = 0$ for all $n$.

**First part:** Suppose $H_n(X, A) = 0$, then we have the exact sequence

$$A \xrightarrow{a} B \xrightarrow{b} 0 \xrightarrow{c} D \xrightarrow{d} E.$$

By exactness, $\operatorname{Im} a = \operatorname{Ker} b = B$ (i.e. $a$ is surjective) and $\operatorname{Ker} d = \operatorname{Im} c = 0$ (i.e. $d$ is injective).

Conversely, suppose we have the exact sequence

$$A \overset{a}{\twoheadrightarrow} B \xrightarrow{b} C \xrightarrow{c} D \overset{d}{\rightarrowtail} E,$$

where $a$ is surjective and $d$ is injective. By exactnes and the surjectivity of $a$, we have $\operatorname{Ker} b = \operatorname{Im} a = B$, so $b$ is the zero map. Then again by exactness, $\operatorname{Ker} c = \operatorname{Im} b = 0$. Finally, by exactness and the injectivity of $d$, we have $\operatorname{Im} c = \operatorname{Ker} d = 0$, so $c$ is also the zero map. In order for the zero map to have trivial kernel, $C = 0$.

**Second part:** Fix $A \subset X$. We will make frequent use of the long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

For the forward direction, suppose $H_n(A) \cong H_n(X)$ via $i_*$ for all $n$. Since $i_*$ is the map $H_n(A) \to H_n(X)$ in the long exact sequence above, we necessarily have exact sequences of the form

$$H_n(A) \twoheadrightarrow H_n(X) \to H_n(X, A) \to H_{n-1}(A) \rightarrowtail H_{n-1}(X)$$

Conversely, suppose $H_n(X, A) = 0$ for all $n$. We then have exact sequences of the form

$$H_n(A) \to H_n(X) \to 0 \to H_{n-1}(A) \to H_{n-1}(X)$$

for all $n$. By the first part of the problem, this implies $H_n(A) \twoheadrightarrow H_n(X)$ is surjective. Applying the first part of the problem to the exact sequence

$$H_{n+1}(A) \to H_{n+1}(X) \to 0 \to H_n(A) \to H_n(X)$$

shows that $H_n(A) \rightarrowtail H_n(X)$ is also injective, so it's an isomorphism. Thus $H_n(A) \cong H_n(X)$ for all $n$.

**Exercise 3** (2.1: 16). a. $H_0(X, A) = 0 \iff A$ meets each path component of $X$.

b. $H_1(X, A) = 0 \iff H_1(A) \to H_1(X)$ is surjective and each path component of $X$ contains at most one path component of $A$.

Suppose $X$ has path components $\{X_\alpha\}_\alpha$, then we know

(i) $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$; and

(ii) $H_0(X) \cong \bigoplus_\alpha \mathbb{Z}$.

**Lemma 1.** $H_0(X, A) = 0 \iff$ there is a surjective map $H_0(A) \cong H_0(X)$.

*Proof.* This is essentially just a restriction of the previous problem. Consider the following exact sequence taken from the long exact sequence of the pair $(X, A)$.

$$H_0(A) \to H_0(X) \to H_0(X, A) \to 0 \rightarrowtail 0.$$

The rightmost map $0 \rightarrowtail 0$ is necessarily injective, so by part (a) of the previous problem, $H_0(X, A) = 0 \iff H_0(A) \twoheadrightarrow H_0(X)$ is surjective. $\square$

a. $A$ intersects all $X_\alpha \iff$ the generators of $H_0(A \cap X_\alpha)$ also generate $H_0(X_\alpha)$ for all $\alpha$. This occurs iff $H_0(A \cap X_\alpha) \overset{i_*}{\twoheadrightarrow} H_0(X_\alpha)$ is surjective for all $\alpha$.

By (i), $H_0(A) \cong \bigoplus_\alpha H_0(A \cap X_\alpha)$ and $H_0(X) \cong \bigoplus_\alpha X_\alpha$. Then direct summing each $H_0(A \cap X_\alpha) \twoheadrightarrow H_0(X_\alpha)$ gives a surjective map $H_0(A) \twoheadrightarrow H_0(X)$. Then by the lemma, this happens $\iff H_0(X, A) = 0$.

b. We have the following exact sequence from the exact sequence of the pair $(X, A)$.

$$H_1(A) \to H_1(X) \to H_1(X, A) \to H_0(A) \to H_0(X)$$

By part (a) of the previous problem $H_1(X, A) = 0 \iff H_1(A) \twoheadrightarrow H_1(X)$ is surjective and $H_0(A) \rightarrowtail H_0(X)$ is injective. But $H_0(A) \rightarrowtail H_0(X)$ is injective if and only if any path component of $X$ contains at most 1 path component of $A$:

**Forward:** Suppose not, then by (ii), $H_0(A \cap X_\alpha)$ for some $\alpha$ is isomorphic to the direct sum of 2 or more copies of $\mathbb{Z}$. By (i), we know $H_0(X_\alpha) \cong \mathbb{Z}$ for that same $\alpha$, but there is no injective map $\bigoplus_{i \in \mathcal{J}} \mathbb{Z} \to \mathbb{Z}$ when $|\mathcal{J}| > 1$. So taking direct sums, there is no injective map $H_0(A) = \bigoplus_\alpha H_0(A \cap X_\alpha) \to \bigoplus_\alpha H_0(X_\alpha) = H_0(X)$, a contradiction.

**Backward:** Suppose there's at most 1 path component of $A$ in each path component of $X$. Then by (ii), $H_0(A) \cong \bigoplus_{\beta \in \mathcal{B}}$, where $\mathcal{B} \subset \mathcal{A}$ and $H_0(X) \cong \bigoplus_{\alpha \in \mathcal{A}}$. We can clearly include $H_0(A)$ inside $H_0(X)$.

**Exercise 4** (2.1: 17). a. Compute $H_n(X,A)$ when $X$ is $S^2$ or $S^1 \times S^1$ and $A$ is a finite set of points in $X$.

b. Compute $H_n(X,A)$ and $H_n(X,B)$ for $X$ a closed orientable surface of genus two with $A$ and $B$ the circles shown.

a. **2-Sphere:** We know $\tilde{H}_n(S^2) = \mathbb{Z}$ when $n = 2$ and 0 otherwise. We also know that $H_n(\text{pt}) = \mathbb{Z}$ when $n = 0$ and 0 otherwise. Then since the $n$-th homology of a space is the direct sum of the $n$-th homologies of its path components, we have $H_n(A) = \mathbb{Z}^m$ when $n = 0$ and 0 otherwise, where $m$ is the number of points in $A$. Its reduced homology is then $\tilde{H}_n(A) = \mathbb{Z}^{m-1}$ when $n = 0$ and 0 otherwise.

We can now form the long exact of the pair $(X,A)$ in reduced homology.

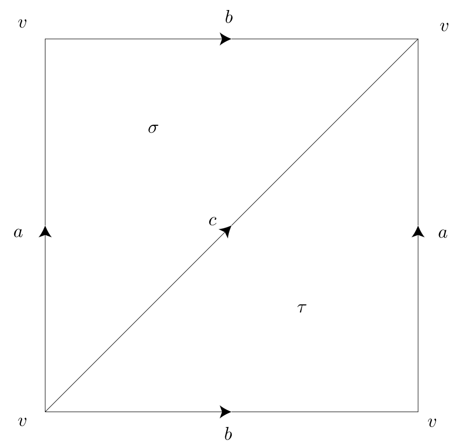
$$
\begin{tikzcd}
\tilde{H}_3(A) \arrow[r] & \tilde{H}_3(X) \arrow[r] & H_3(X,A) \arrow[dll] \\
\tilde{H}_2(A) \arrow[r] & \tilde{H}_2(X) \arrow[r] & H_2(X,A) \arrow[dll] \\
\tilde{H}_1(A) \arrow[r] & \tilde{H}_1(X) \arrow[r] & H_1(X,A) \arrow[dll] \\
\tilde{H}_0(A) \arrow[r] & \tilde{H}_0(X) \arrow[r] & H_0(X,A) \arrow[dll] \\
0 & &
\end{tikzcd}
$$

Plugging in our calculated values of $\tilde{H}_n$, this becomes the following.

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & H_3(X,A) \arrow[dll] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & H_2(X,A) \arrow[dll] \\
0 \arrow[r] & 0 \arrow[r] & H_1(X,A) \arrow[dll] \\
\mathbb{Z}^{m-1} \arrow[r] & 0 \arrow[r] & H_0(X,A) \arrow[dll] \\
0 & &
\end{tikzcd}
$$

All rows above this that aren't shown are identical to the top row, except with different $n$. Since an exact sequence $0 \to Y \to 0$ implies that $Y = 0$, we see that for $n \geq 3$ and $n = 0$, $H_n(X,A) = 0$. Since $0 \to X \to Y \to 0$ exact implies $X \cong Y$, we have $H_2(X,A) \cong \mathbb{Z}$ and $H_1(X,A) \cong \mathbb{Z}^{m-1}$.

**Torus:** The strategy here is the same, except the homology groups $H_n(X)$ are different. To start, we can draw the torus as a simplicial complex as follows.




The homology groups of the torus are then

$$
\begin{aligned}
&H_n(X) = 0 \text{ when } n \geq 3,\\
&H_2(X) = \operatorname{Ker}\partial_2 = \langle \sigma - \tau \rangle \cong \mathbb{Z},\\
&H_1(X) = \frac{\operatorname{Ker}\partial_1}{\operatorname{Im}\partial_2} = \frac{\langle a, b, c\rangle}{\langle a + b - c\rangle} \cong \langle a, b\rangle \cong \mathbb{Z}^2,\\
&H_0(X) = \frac{C_0(X)}{\operatorname{Im}\partial_1} = \frac{\langle v \rangle}{0} \cong \mathbb{Z}.
\end{aligned}
$$

Note that the reduced homologies are all the same, except now $\tilde{H}_0(X) = 0$. The long exact sequence of the pair $(X, A)$ in reduced homology is then as follows.

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & H_3(X,A) \arrow[dll] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & H_2(X,A) \arrow[dll] \\
0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "\alpha"] & H_1(X,A) \arrow[dll, "\delta"] \\
\mathbb{Z}^{m-1} \arrow[r] & 0 \arrow[r] & H_0(X,A) \arrow[dll] \\
0 & &
\end{tikzcd}
$$

By the same arguments as for the 2-sphere, $H_n(X, A) = 0$ when $n \geq 3$ and $n = 0$, and $H_2(X, A) \cong \mathbb{Z}$. To calculate $H_1(X, A)$, note that we have a short exact sequence

$$0 \to \mathbb{Z}^2 \to H_1(X, A) \to \mathbb{Z}^{m-1} \to 0,$$

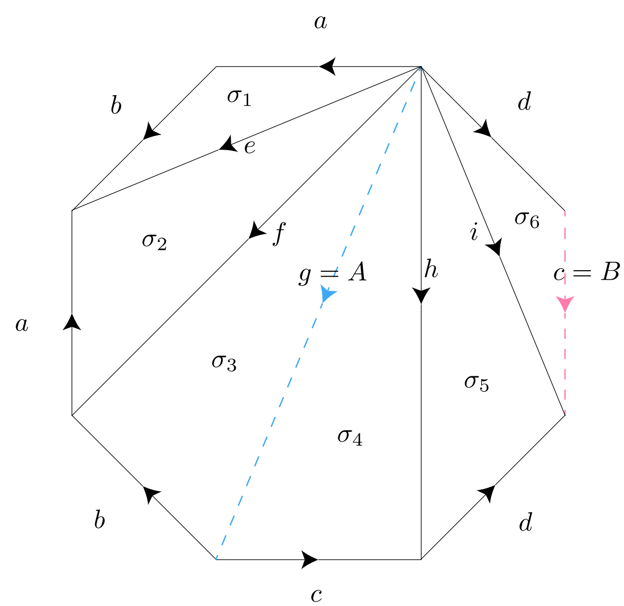
so by the first isomorphism theorem,

$$\mathbb{Z}^{m-1} \cong \frac{H_1(X,A)}{\operatorname{Ker}\delta} = \frac{H_1(X,A)}{\operatorname{Im}\alpha} \cong \frac{H_1(X,A)}{\mathbb{Z}^2}.$$

Thus $H_1(X, A) \cong \mathbb{Z}^{m+1}$.

b. We'll once again figure out what the relative homology groups are via the long exact sequence of a pair in reduce homology. This time, though, $A$ and $B$ will induced *different* maps in the long exact sequence, leading to different relative homology groups.

To start, we'll calculate the homology of the genus 2 surface $X$. We can turn its fundamental polygon into a simplicial complex as below. In the figure, I've noted which edges give $A$ and $B$.




All vertices are identified to the same point $v$, so $C_0(X) = \langle v \rangle \cong \mathbb{Z}$ and $\operatorname{Im} \partial_1 = 0$. Thus $H_0(X) = C_0(x)/\operatorname{Im} p_1 \cong \mathbb{Z}$, which implies $\tilde{H}_0(X) = 0$. There are 9 edges $a, b, \ldots, h, i$ that are all cycles, so $\operatorname{Ker} \partial_2 = \langle a, b, \ldots, h, i \rangle \cong \mathbb{Z}^9$. The image of $\partial_2$ is generated by

$$\partial\sigma_1 = a + b - e, \quad \partial\sigma_2 = a + f - e, \quad \partial\sigma_3 = b + g - f,$$
$$\partial\sigma_4 = c + g - h, \quad \partial\sigma_5 = d + h - i, \quad \partial\sigma_6 = c + d - i.$$

To compute $H_1(X) = \frac{\langle a,b,\ldots,i \rangle}{\partial\sigma_1,\ldots,\partial\sigma_6}$, we can set each of the 6 equations above to 0 and solve the system to get

$$i = c + d, \quad h = c, \quad g = 0, \quad f = b, \quad e = a + b.$$

Thus modding out by our 6 relations gets rid of all of the generators except those on the boundary of the polygon, i.e. $H_1(X) \cong \langle a, b, c, d \rangle \cong \mathbb{Z}^4$. Finally, $\operatorname{Ker} \partial_2 = \langle \sigma_1 - \sigma_2 - \sigma_3 + \sigma_4 + \sigma_5 - \sigma_6 \rangle \cong \mathbb{Z}$, so $H_2(X) = \operatorname{Ker} \partial_2 \cong \mathbb{Z}$. There are no higher dimensional cells, so $H_n(X) = 0$ for all $n \geq 0$.

Now note that both $A$ and $B$ are just $S^1$, so $\tilde{H}_n(A) = \tilde{H}_n(B) = \mathbb{Z}$ when $n = 1$ and 0 otherwise. Thus the long exact sequence at first glance looks the same for both.

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & H_3(X,A) \arrow[dll, out=-5, in=175] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\alpha"] & H_2(X,A) \arrow[dll, out=-5, in=175, "\delta"] \\
\mathbb{Z} \arrow[r, "\beta"] & \mathbb{Z}^4 \arrow[r, "\gamma"] & H_1(X,A) \arrow[dll, out=-5, in=175, "\delta'"] \\
0 \arrow[r] & 0 \arrow[r] & H_0(X,A) \arrow[dll, out=-5, in=175] \\
0 & &
\end{tikzcd}
$$

We can use the same argument as in part (a) to show that

$$H_n(X,A) = H_n(X,B) = 0 \quad \text{for } n \geq 3 \text{ and } n = 0.$$

The only difference between $A$ and $B$ is what the 1st and 2nd relative homology groups are.

**A:** Since $A = g = \delta(\sigma_3 - \sigma_1 - \sigma_2)$, we know it is 0 in homology. Thus for $A$, the map $\beta$ is the 0 map. Then $\operatorname{Ker}\gamma = \operatorname{Im}\beta = 0$, so $\gamma$ is injective. And since $\gamma$ is necessarily also surjective since $\mathbb{Z}^4 \to H_1(X,A) \to 0$ is exact, it's an isomorphism, i.e. $H_1(X,A) \cong \mathbb{Z}^4$.

Finally, $\operatorname{Im}\delta = \operatorname{Ker}\beta = \mathbb{Z}$ since $\beta = 0$, so $\delta$ is surjective. We also know $\alpha$ is injective since $0 \to \mathbb{Z} \overset{\alpha}{\to} H_2(X,A)$ is exact. By the first isomorphism theorem,

$$\mathbb{Z} \cong \frac{H_2(X,A)}{\operatorname{Ker}\delta} = \frac{H_2(X,A)}{\operatorname{Im}\alpha} \cong \frac{H_2(X,A)}{\mathbb{Z}},$$

so $H_2(X,A) \cong \mathbb{Z}^2$.

**B:** We can use the same diagram for the long exact sequence of the pair $(X,B)$ as we did for $(X,A)$, just replacing every $A$ with a $B$. I'll keep all the names of the maps the same. The main difference in this case is that $\beta$ is a nonzero map.

When computing the homology of $X$, we showed that $B = c$ was a generator of $H_1(X)$, so $\beta$ maps $1 \mapsto (0,0,1,0)$. Then $\operatorname{Ker}\gamma = \operatorname{Im}\beta = \langle(0,0,1,0)\rangle$, so by the first isomorphism theorem and the fact that $\gamma$ is surjective,

$$H_1(X,B) \cong \frac{\mathbb{Z}^4}{\langle(0,0,1,0)\rangle} \cong \mathbb{Z}^3.$$

We also know $\operatorname{Im}\delta = \operatorname{Ker}\beta = 0$ since $\beta$ is injective, so $\delta$ is the 0 map. Then $\operatorname{Im}\alpha = \operatorname{Ker}\delta = H_2(X,B)$, so $\alpha$ is surjective. Since $\alpha$ is already necessarily injective, it's an isomorphism, i.e. $H_2(X,B) \cong \mathbb{Z}$.

**Exercise 5** (2.1: 26)**.** Show that $H_1(X, A)$ is not isomorphic to $\tilde{H}_1(X/A)$ if $X = [0, 1]$ and $A$ is the sequence $1, 1/2, 1/3, \ldots$ together with its limit 0.

The strategy here will be to show that $H_1(X, A)$ is countable while $\tilde{H}_1(X/A)$ is uncountable, making it impossible to find a bijection (let alone an isomorphism) between them. Since $A \subset X$, we can calculate $H_1(X, A)$ using the long exact sequence of the pair $(X, A)$ in reduced homology. To do this, we'll need the homologies of $A$ and $X$ individually.

Since $[0, 1]$, it has the same homotopy type as a point. Then since homology is a homotopy invariant, $H_n(X) \cong H_n(\mathrm{pt}) = \mathbb{Z}$ if $n = 0$ and 0 otherwise. Then $\tilde{H}_n(X) = 0$ for all $n$.

$A$ is the union of a bunch of isolated points, and we know that we can decompose the homology of a space into the direct sum of the homologies of its path components. Each isolated point is its own path component, so $H_1(A) = 0$ and $H_0(A) \cong \bigoplus_{n\in\mathbb{N}_0} \mathbb{Z}$. Taking reduced homology gives $\tilde{H}_0(A) \cong \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$. We then have the following exact sequence.

$$\tilde{H}_1(A) \to \tilde{H}_1(X) \to H_1(X, A) \to \tilde{H}_0(A) \to \tilde{H}_0(X)$$

Based on the above computations, this becomes the following.

$$0 \to 0 \to H_1(X, A) \to \bigoplus_{n\in\mathbb{N}} \mathbb{Z} \to 0$$

This implies $H_1(X, A) \cong \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$. Since the direct sum of countable sets is itself countable, this shows that $H_1(X, A)$ is countable. To show that $\tilde{H}_1(X/A)$ is uncountable, we'll follow a strategy similar to that in Example 1.25 in the text.

First off, note that $X/A$ is the Hawaiian earring space composed of circles $\{C_n\}_{n\in\mathbb{N}}$ all intersecting at a single common point. For all $n$, there is a retraction $r_n : X/A \to C_n$ fixing $C_n$ and sending all other $C_i$ to their common intersection point. Then since $H_1$ is a covariant functor, we can apply it to get induced maps $(r_n)_* : H_1(X/A) \to H_1(C_n) = H_1(S^1) \cong \mathbb{Z}$.

$$X/A \underset{i}{\overset{r_n}{\rightleftarrows}} C_n \qquad\qquad H_1(X/A) \underset{i_*}{\overset{(r_n)_*}{\rightleftarrows}} \mathbb{Z}$$

Note that $r_n$ being surjective is equivalent to $r_n i = \mathrm{id}$. Then since $(r_n)_* i_* = (r_n i)_* = \mathrm{id}_* = \mathrm{id}$, the induced map $(r_n)_*$ is also surjective. The product of the many $(r_n)_*$ maps is then a surjective map $H_1(X/A) \twoheadrightarrow \prod_{n\in\mathbb{N}} \mathbb{Z}$. But $\prod_{n\in\mathbb{N}} \mathbb{Z}$ is uncountable, so this map being surjective implies that $H_1(X/A) \cong \tilde{H}_1(X/A)$ is also uncountable. Thus $\tilde{H}_1(X/A)$ and $H_1(X, A)$ cannot possibly be isomorphic.