# MATH 531 HOMEWORK 11

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**Pg. 286, Ex. 2** Suppose that $p_n$ is a sequence of polynomials converging uniformly to $f$ on $[0,1]$ and $f$ is *not* a polynomial. Prove that the degrees of the $p_n$ are not bounded. [Hint: An $N$-th degree polynomial $p$ is uniquely determined y its values at $N+1$ points $x_0, \dots, x_N$ via Lagrange's interpolation formula

$$p(x) = \sum_{i=0}^{N} \pi_i(x) \frac{p(x_i)}{\pi_i(x_i)},$$

where $\pi_i(x) = (x - x_0)(x - x_1) \cdots (x - x_N)/(x - x_i)$.]

We will prove this statement by contrapositive, i.e. given a sequence of polynomials $p_k$ converging uniformly to some function $f$, we will show that if the $p_k$'s have uniformly bounded degree, then $f$ must be a polynomial.

Since we are assuming that our sequence of polynomials have uniformly bounded degree, there is some $N$ such that $\deg(p_k) \leq N$ for all $k$. Take an arbitrary collection of distinct points $\mathcal{X} \doteq \{x_0, \dots, x_N\} \subset [0,1]$.

We claim that $p_k$ converges to the polynomial uniquely defined by the $N+1$ points $f(x_0), \dots f(x_N)$. We denote this polynomial by $f_p$. Fix $\varepsilon > 0$, then we use Lagrange's interpolation formula to get

$$\begin{aligned} |f_p(x) - p_k(x)| &= \left| \sum_{i=0}^{N} \frac{\pi_i(x)}{\pi_i(x_i)} (f(x_i) - p_k(x_i)) \right| \\ &\leq \sum_{i=0}^{N} \left| \frac{\pi_i(x)}{\pi_i(x_i)} \right| |f(x_i) - p_k(x_i)| . \end{aligned}$$

Note that $|x - x_i| \leq 1$ for all $i$ since $p$ and $f$ are defined on $[0,1]$. Additionally, since $\mathcal{X}$ is finite, we can define the minimum distance bewteen points in $\mathcal{X}$ as $d \doteq \min_{i,j} |x_i - x_j|$, from which we clearly have $|x_i - x_j| \geq d$ for all $i \neq j$. This means we have

$$\left| \frac{\pi_i(x)}{\pi_i(x_i)} \right| \leq \frac{1}{d^N}.$$

Thus the distance between $f_p$ and $p_k$ is bounded by

$$|f_p(x) - p_k(x)| \leq \frac{1}{d^N} \sum_{i=0}^{N} |f(x_i) - p_k(x_i)|.$$

Now since $p_k$ converges uniformly to $f$, we can find a $K$ such that

$$|f(x) - p_k(x)| < \frac{d^N \varepsilon}{N+1}$$

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when $k > K$. Thus for $k > K$, we have

$$\begin{aligned} |f_p(x) - p_k(x)| &< \frac{1}{d^N} \sum_{i=0}^{N} \frac{d^N \varepsilon}{N+1} \\ &= \frac{d^N}{d^N} \frac{N+1}{N+1} \varepsilon \\ &= \varepsilon. \end{aligned}$$

Thus $p_k$ converges to the polynomial $f_p$. We have shown that the contrapositive of the desired implication is true, so we are done.

**Pg. 286, Ex. 4** Consider the set of all polynomials $p(x, y)$ in two variables $x, y \in [0,1] \times [0,1]$. Prove that this set is dense in $\mathcal{C}([0,1] \times [0,1], \mathbb{R})$.

We will show this using the Stone-Weierstrass theorem. Denote the set of polynomials of the two variables $x$ and $y$ by

$$\mathcal{P}(x, y) \doteq \left\{ \text{all functions of the form } \sum_{i=1}^{n} a_i x^{b_i} y^{c_i} \mid a_i \in \mathbb{R} \text{ and } b_i, c_i, n \in \mathbb{Z} \right\}.$$

The set $[0,1] \times [0,1]$ is compact in $\mathbb{R}^2$ since it is closed and bounded, and the set of two-variable polynomials on $[0,1] \times [0,1]$ is a subset of $\mathcal{C}([0,1] \times [0,1], \mathbb{R})$, so we need only show the three main conditions of the Stone-Weierstrass Theorem to conclude that $\mathcal{P}(x, y)$ is dense in $\mathcal{C}([0,1] \times [0,1], \mathbb{R})$.

First we show that $\mathcal{P}(x, y)$ is an algebra. Suppose we have $p_1, p_2 \in \mathcal{P}(x, y)$, where $p_1 = \sum_{i=1}^{n} a_i x^{b_i} y^{c_i}$ and $p_2 = \sum_{i=1}^{m} \alpha_i x^{\beta_i} y^{\gamma_i}$. The product of these two polynomials will have $n + m$ terms, the coefficients will be of the form $a_i \alpha_j$, and the $x$ and $y$ exponents will be of the form $b_i \beta_j$ and $c_i \gamma_j$, respectively. Thus $p_1 p_2 \in \mathcal{P}(x, y)$. The sum $p_1 + p_2$ is similarly in $\mathcal{P}(x, y)$. Finally, for constant $\rho$, $\rho p_1$ has form $\sum_{i=1}^{n} \rho a_i x^{b_i} y^{c_i}$, so it too is in $\mathcal{P}(x, y)$. Thus $\mathcal{P}(x, y)$ is an algebra.

Now we show $1 \in \mathcal{P}(x, y)$. Let $n = 1, a_1 = 1, b_1 = 0$, and $c_1 = 0$, then

$$\sum_{i=1}^{n} a_i x^{b_i} y^{c_i} = 1,$$

so $1 \in \mathcal{P}(x, y)$.

Now we show that $\mathcal{P}(x, y)$ separates points. If $(x_1, y_1) \neq (x_2, y_2)$, then either $x_1 \neq x_2$ or $y_1 \neq y_2$ (or both simultaneously, which does not merit its own case since it is superceded by the two given cases). If $x_1 \neq x_2$, then for $p(x, y) = x$, we have $p(x_1, y_1) \neq p(x_2, y_2)$. Similarly, if $y_1 \neq y_2$, then for $p(x, y) = y$, we have $p(x_1, y_1) \neq p(x_2, y_2)$.

Thus by the Stone-Weierstrass theorem, $\mathcal{P}(x, y)$ is dense in $\mathcal{C}([0,1] \times [0,1], \mathbb{R})$.

**Pg. 286, Ex. 5** Consider the set of all functions on $[0,1]$ of the form

$$h(x) = \sum_{j=1}^{n} a_j e^{b_j x}, \quad \text{where } a_j, b_j \in \mathbb{R}.$$

Is this set dense in $\mathcal{C}([0,1],\mathbb{R})$?

We will show that the set of all $h(x)$ is dense in $\mathcal{C}([0,1],\mathbb{R})$ using the Stone-Weierstrass theorem. Let $\mathcal{H}$ denote the set of all possible $h(x)$.

First we show that $\mathcal{H}$ is an algebra. Suppose we have $h_1(x) = \sum_{j=1}^{n} a_j e^{b_j x}$ and $h_2(x) = \sum_{j=1}^{m} \alpha_j e^{\beta_j x}$. Since $e^{bx}e^{\beta x} = e^{(b+\beta)x}$, the product $h_1(x)h_2(x)$ is in $\mathcal{H}$. The sum $h_1(x)+h_2(x)$ is more straightforwardly in $\mathcal{H}$, as the sum is just the concatenation of the two individual summations. Finally, given constant $\rho$, $\rho h_1(x) = \sum_{j=1}^{n} (\rho a_j) e^{b_j x}$ is clearly in $\mathcal{H}$. Thus $\mathcal{H}$ is an algebra.

Now we show that $1 \in \mathcal{H}$. Let $n = 1, a_1 = 1$, and $b_1 = 0$, then

$$\sum_{j=1}^{n} a_j e^{b_j x} = 1,$$

so $1 \in \mathcal{H}$.

Now we show that $\mathcal{H}$ separates points. Suppose $x \neq y$, then the function $h(x) = e^x \in \mathcal{H}$ yields $e^x \neq e^y$ since it is strictly monotonically increasing, so $\mathcal{H}$ separates points.

Thus by the Stone-Weierstrass theorem, $\mathcal{H}$ is dense in $\mathcal{C}([0,1],\mathbb{R})$.

**Pg. 322, Ex. 51** Consider a double series

$$\sum_{m,n=0}^{\infty} a_{mn}, \quad \text{where } a_{mn} \in \mathbb{R}, \quad m,n = 0,1,2,\ldots.$$

Say that it **converges to** $S$ if for any $\varepsilon > 0$, there is an $N$ such that $n, m > N$ implies

$$\left| \sum_{k,l=0}^{m,n} a_{kl} - S \right| < \varepsilon.$$

Define **absolute convergence** and prove that if $\sum_{m,n=0}^{\infty} a_{nm}$ is absolutely convergent, then the sum can be rearranged as follows:

$$\sum_{m,n=0}^{\infty} a_{nm} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} a_{nm} \right) = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} a_{nm} \right).$$

Interpret this result in terms of summing entries in an infinite matrix by rows and columns.

We say a double series $\sum_{k,l=0}^{\infty} a_{mn}$ is **absolutely convergent** if for any $\varepsilon > 0$, there is an $N$ such that

$$\left| S - \sum_{k,l=0}^{m,n} |a_{mn}| \right| < \varepsilon$$

when $m, n > N$.

Let $\sum_{m,n} a_{mn}$ be such an absolutely convergent double series, and let

$$s_{mn} \doteq \sum_{k,l=0}^{m,n} a_{kl} = \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} = \sum_{l=0}^{n} \sum_{k=0}^{m} a_{kl}$$

denote the partial sums of the double series (note that we can exchange the summations in this way since each summation has a finite number of terms).

Since $\sum_{k,l=0}^{m,n} |a_{kl}|$ converges, we know that $\sum_{k,l=0}^{m,n} a_{kl}$ converges, so $s_{mn}$ converges as well. Let $S \doteq \lim_{n\to\infty, m\to\infty} s_{mn}$, then for all $\varepsilon > 0$, there is an $N$ such that $|s_{mn} - S| < \varepsilon$ when $m, n > N$. We will now show that both desired rearrangements of our double series converge to this $S$.

Fix $m$, then consider the subseries $\sum_{n=0}^{\infty} a_{mn}$. Since the full series is absolutely convergent, any of its subsequences converge. Thus $\sum_{n=0}^{\infty} a_{mn}$ must converge. Denote its limit by $b_m$, then we claim that $\sum_{m=0}^{\infty} b_m$ converges to $S$. Fix $\varepsilon > 0$, then we have

$$\begin{aligned} \left| S - \sum_{k=0}^{m} b_k \right| &= \left| S - \sum_{k=0}^{m} b_k + \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} - \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} \right| \\ &\leq \left| S - \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} \right| + \left| \sum_{k=0}^{m} \left( \sum_{l=0}^{n} a_{kl} - b_k \right) \right|. \end{aligned}$$

Since $\sum_k \sum_l a_{kl}$ converges to $S$ by definition, we can find $N$ such that the first absolute value term is less than $\varepsilon/2$ when $m, n > N_1$. Additionally, for fixed $m$ and $k$, since $\sum_l a_{kl}$ converges to $b_k$, we can find $N_k$ such that $|\sum_k a_{kl} - b_k| < \varepsilon/(2m)$ when $n > N_k$. Thus if we take $m > N$ and $n$ such that

$$n > \max\{N, N_0, N_1, \ldots, N_m\},$$

we have

$$\begin{aligned} \left| S - \sum_{k=0}^{m} b_k \right| &\leq \frac{\varepsilon}{2} + \sum_{k=0}^{m} \left| \frac{\varepsilon}{2m} \right| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus $\sum_k b_k$ converges to $S$, so

$$S = \sum_{k,l=0}^{\infty} a_{kl} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} a_{mn} \right).$$

By an analogous argument, we can show

$$S = \sum_{k,l=0}^{\infty} a_{kl} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} a_{mn} \right).$$

In terms of summing the entries of an infinite matrix, this result means that we can either

(a) sum each row, then add each row sum; or
(b) sum each column, then add each column sum;

and the final sum of each will be the same.

**Pg. 324, Ex. 59**

(a) Let $p > 1$ with $1/p + 1/q = 1$. For $a, b, t > 0$, prove that
$$ab \leq \frac{a^p t^p}{p} + \frac{b^q t^{-q}}{q}$$
and that $ab$ is the minimum value of the right side (One way to prove this is to use elementary calculus).

(b) Prove **Hölder's inequality:** If $a_k, b_k \geq 0$, $p > 1$, and $1/p + 1/q = 1$, then
$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}.$$
[Hint: Imitate the proof of the Cauchy-Schwarz inequality, using part **a**.]

(c) Prove **Minkowski's inequality:** If $a_k, b_k \geq 0$ and $p > 1$, then
$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}.$$

(a) We need only show that $ab$ is the minimum value of the given expression, as the inequality follows from it. Fix $a, b, p$, and $q$, then consider
$$f(t) = \frac{a^p t^p}{p} + \frac{b^q t^{-q}}{q}.$$
Its derivative is
$$f'(t) = a^p t^{p-1} - b^q t^{-(q+1)}.$$
We know that if $f(t)$ achieves a local minimum or maximum at some $t^*$, then $f(t^*) = 0$. This gives us a necessary condition for finding the global minimum of $f$. Setting $f'(t) = 0$ and solving for $t$ yields
$$t = \left(\frac{b^q}{a^p}\right)^{\frac{1}{p+q}} = \left(\frac{b^q}{a^p}\right)^{\frac{1}{pq}},$$
where the second equality follows from
$$\frac{1}{p} + \frac{1}{q} = 1 \implies p + q = pq.$$
Denote this value of $t$ by $t^*$, then the value of $f$ at this point is
$$\begin{aligned} f(t^*) &= \frac{a^p \frac{b}{a^{p/q}}}{p} + \frac{b^q \frac{a}{b^{q/p}}}{q} \\ &= \frac{a^{p-p/q} b}{p} + \frac{b^{q-q/p} a}{q}. \end{aligned}$$
Now since $p + q = pq$, we have
$$p - \frac{p}{q} = \frac{pq - p}{q} = \frac{p + q - p}{q} = 1.$$
Similarly, $q - q/p = 1$. Thus we can simplify our expression for $f(t^*)$ to
$$f(t^*) = \frac{ab}{p} + \frac{ba}{q} = ab\left(\frac{1}{p} + \frac{1}{q}\right) = ab.$$

Thus the only possible extrumum of $f$ is the value $ab$. To check that this is a minimum, we can use the second derivative test. We have

$$f''(t) = (p-1)a^p t^{p-2} + (q+1)b^q t^{-(q+2)}.$$

Since $a, b, t > 0$ and $p > 1$, this will always be positive. Thus $ab$ is in fact the minimum value of $f$.

(b) Let $\tilde{a} \doteq (\sum_{k=1}^n a_k^p)^{1/p}$ and $\tilde{b} \doteq (\sum_{k=1}^n b_k^q)^{1/q}$. Then our goal is to show

$$\sum_{k=1}^n a_k b_k \leq \tilde{a}\tilde{b}.$$

If $\tilde{a} = 0$, then since each $a_k$ is non-negative, each $a_k$ must be 0, so the inequality is trivial. The case is similar if $\tilde{b} = 0$. Thus if either is 0, the inequality holds (it is actually an equality). We now assume that neither is 0.

Since neither $\tilde{a}$ nor $\tilde{b}$ is 0, our desired inequality is equivalent to

$$\sum_{k=1}^n \frac{a_k}{\tilde{a}} \frac{b_k}{\tilde{b}} \leq 1.$$

Since $a_k/\tilde{a}, b_k/\tilde{b} > 0$ for each $k$, by part **a** we have

$$\begin{aligned}
\sum_{k=1}^n \frac{a_k}{\tilde{a}} \frac{b_k}{\tilde{b}} &\leq \sum_{k=1}^n \left( \frac{ak^p t^p}{\tilde{a}^p p} + \frac{b_k^q t^{-q}}{\tilde{b}^q q} \right) \\
&= \frac{t^p}{p\tilde{a}^p} \sum_{k=1}^n a_k^p + \frac{t^{-q}}{q\tilde{b}^q} \sum_{k=1}^n b_k^q \\
&= \frac{t^p \sum_{k=1}^n a_k^p}{p \sum_{k=1}^n a_k^p} + \frac{t^{-q} \sum_{k=1}^n b_k^q}{q \sum_{k=1}^n b_k^q} \\
&= \frac{t^p}{p} + \frac{t^{-q}}{q}
\end{aligned}$$

for any $t > 0$. Since this holds for any $t$, it certainly holds for $t = 1$. Thus our inequality is

$$\sum_{k=1}^n \frac{a_k}{\tilde{a}} \frac{b_k}{\tilde{b}} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

and we are done.

(c) By part **b**, we have

$$\begin{aligned}
\sum_{k=1}^n (a_k + b_k)^p &= \sum_{k=1}^n (a_k + b_k)^{p-1} a_k + \sum_{k=1}^n (a_k + b_k)^{p-1} b_k \\
&\leq \left( \sum_{k=1}^n (a_k + b_k)^{q(p-1)} \right)^{1/q} \left[ \left( \sum_{k=1}^n a_k^p \right)^{1/p} + \left( \sum_{k=1}^n b_k^p \right)^{1/p} \right].
\end{aligned}$$

But $q(p-1) = pq - q = p + q - q = 1$, so this becomes

$$\sum_{k=1}^{n}(a_k+b_k)^p \leq \left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{1/q}\left[\left(\sum_{k=1}^{n}a_k^p\right)^{1/p}+\left(\sum_{k=1}^{n}b_k^p\right)^{1/p}\right]$$

$$\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{1-1/q} \leq \left(\sum_{k=1}^{n}a_k^p\right)^{1/p}+\left(\sum_{k=1}^{n}b_k^p\right)^{1/p}$$

$$\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{1/p} \leq \left(\sum_{k=1}^{n}a_k^p\right)^{1/p}+\left(\sum_{k=1}^{n}b_k^p\right)^{1/p},$$

which is the desired result.

**Pg. 334, Ex. 3** Let $L$ be a linear map of $\mathbb{R}^n \to \mathbb{R}^m$, let $g : \mathbb{R}^n \to \mathbb{R}^m$ be such that $\|g(x)\| \leq M\|x\|^2$, and let $f(x) = L(x) + g(x)$.Prove that $\mathbf{D}f_0 = L$.

First we show that $\mathbf{D}g_0 = 0$, then that $\mathbf{D}L_x = L$ for all $x$, then that $\mathbf{D}f = \mathbf{D}g + \mathbf{D}L$, from which the conclusion follows.

**Part 1:** To begin, note that by assumption, $\|g(0)\| \leq 0$, so $g(0) = 0$. Then taking $0(x - x_0)$ to mean "the zero function evaluated at $x - x_0$", we have

$$\begin{aligned}\lim_{x\to 0}\frac{\|g(x)-g(0)-0(x-0)\|}{\|x-0\|} &= \lim_{x\to 0}\frac{\|g(x)\|}{\|x\|}\\ &\leq \lim_{x\to 0}\frac{M\|x\|^2}{\|x\|}\\ &= \lim_{x\to 0} M\|x\|\\ &= 0,\end{aligned}$$

Thus $g$ is differentiable at 0 and the derivative of $g$ at 0 is the zero function.

**Part 2:** Now let $x_0$ be arbitrary, then since $L$ is linear, we have

$$\begin{aligned}\lim_{x\to x_0}\frac{\|L(x)-L(x_0)-L(x-x_0)\|}{\|x-x_0\|} &= \lim_{x\to x_0}\frac{\|L(x)-L(x_0)-L(x)+L(x_0)\|}{\|x-x_0\|}\\ &= \lim_{x\to x_0} 0\\ &= 0.\end{aligned}$$

Since $x_0$ was arbitrary, this shows that for all $x$, $L$ is differentiable at $x$ and $\mathbf{D}L_x = L$.

**Part 3:** For the final major part of the proof, we show that two differentiable functions $\phi, \psi : \mathcal{V} \to \mathcal{W}$ (where $\mathcal{V}$ and $\mathcal{W}$ are normed vector spaces) satisfy $\mathbf{D}(\phi + \psi) = \mathbf{D}\phi + \mathbf{D}\psi$. By the triangle inequality, for all $x_0 \in \mathcal{W}$ we have

$$\begin{aligned}&\lim_{x\to x_0}\frac{\|\phi(x)+\psi(x)-\phi(x_0)-\psi(x_0)-\mathbf{D}\phi_{x_0}(x-x_0)-\mathbf{D}\psi_{x_0}(x-x_0)\|}{\|x-x_0\|}\\ &\leq \lim_{x\to x_0}\frac{\|\phi(x)-\phi(x_0)-\mathbf{D}\phi_{x_0}(x-x_0)\|}{\|x-x_0\|}+\lim_{x\to x_0}\frac{\|\psi(x)-\psi(x_0)-\mathbf{D}\psi_{x_0}(x-x_0)\|}{\|x-x_0\|}\\ &= 0+0=0.\end{aligned}$$

Thus $\mathbf{D}(\phi + \psi) = \mathbf{D}\phi + \mathbf{D}\psi$.

**Conclusion:** We can apply these three facts to show our desired result. We have

$$\mathbf{D}f_0 = \mathbf{D}(g+L)_0 = \mathbf{D}g_0 + \mathbf{D}L_0 = 0 + L = L.$$

**Pg. 344, Ex. 2** Investigate the differentiability of

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$

at $(0,0)$ if $f(0,0) = 0$.

If $f$ is differentiable at $(0,0)$, then

$$\lim_{x,y\to 0} \frac{\|f(x,y) - f(0,0) - \mathbf{D}f_{(0,0)}(x,y)\|}{\|(x,y)\|} = \lim_{x,y\to 0} \frac{\|f(x,y) - \partial_x f(0,0)x - \partial_y f(0,0)y\|}{\|(x,y)\|}$$

must equal 0. We can evaluate the partial derivatives of $f$. We have

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h\cdot 0}{h}}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0.\end{aligned}$$

Similarly, $\partial_y f(0,0) = 0$ as well. Thus our original limit is

$$\lim_{x,y\to 0} \frac{\|f(x,y) - \partial_x f(0,0)x - \partial_y f(0,0)y\|}{\|(x,y)\|} = \lim_{x,y\to 0} \frac{\|f(x,y)\|}{\|(x,y)\|} = \lim_{x,y\to 0} \frac{|xy|}{x^2+y^2}.$$

We claim that this limit does not exist. We consider the limit as we approach $(0,0)$ along the $x$-axis (i.e. $y$ is fixed at 0). We have

$$\lim_{x,y\to 0} \frac{|xy|}{x^2+y^2} = \lim_{x\to 0} \frac{|x\cdot 0|}{x^2} = \lim_{x\to 0} 0 = 0.$$

We now consider the limit as we approach $(0,0)$ along the line $y = x$. We have

$$\lim_{x,y\to 0} \frac{|xy|}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

These two values do not agree, so the limit does not exist. Since it does not exist, it surely cannot equal 0, so $f$ is *not* differentiable at $(0,0)$.