1 GRAPHS

Definition 1. A **(simple undirected) graph** G is a set of vertices V and undirected edges E, where E has no self-loops or duplicate edges.

Definition 2. A **path** between vertices x and y is a sequences of vertices

$$x = u_0, \quad u_1, \quad \dots, \quad u_m = y$$

such that $[u_i, u_{i+1}]$ is an edge for all i.

Definition 3. A graph is **connected** if there is a path between every pair of vertices.

A **separation** of G is two nonempty subsets of G with no edges going between them. We can then equivalently define a graph to be connected if it has no separation.

Proposition 1. Let $x \sim_p y$ if there is a path from x to y. Then $\sim_p$ is an equivalence relation.

We call the equivalence classes of $\sim_p$ **connected components**. Since equivalence relations naturally form partitions, the connected components of a graph union to the entire graph.

Example 1. Let V be a vector space with subspace N, then $x \sim y \iff x - y \in N$ is an equivalence relation (since N has 0 and is closed under addition and additive inverse). The quotient V/N can then be defined as the equivalence classes of $\sim$, which is also a vector space with the operations $\alpha[x] = [\alpha x]$ and $[x] + [y] = [x + y]$.

2 SIMPLICIAL HOMOLOGY

Definition 4. Suppose X is a simplicial complex, then let $C_n(X)$ be the **vector space over $\mathbb{Z}_2$** with basis the n-simplices in X. Elements of $C_n(X)$ are called n**-chains**.

- C_0: vertices
- C_1: edges
- C_2: triangles

Definition 5. The **boundary map** ∂_n is the linear map

$$\begin{aligned} C_n(X) &\to C_{n-1}(X) \\ [v_0, \dots, v_n] &\mapsto \sum_i [v_0, \dots, \hat{v}_i, \dots, v_n], \end{aligned}$$

where $\hat{v}_i$ indicates that v_i has been removed from the simplex.

Proposition 2. $\partial^2 = 0$.

Proof. Applying the defintion of ∂ gives

$$\begin{aligned} \partial^2([v_0, \dots, v_n]) &= \sum_i \partial([v_0, \dots, \hat{v}_i, \dots, v_n]) \\ &= \sum_{j<i} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n] + \sum_{i<j} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]. \end{aligned}$$

Now we can swap the roles of i and j in the second sum to get a sum identical to the first. This gives

$$\begin{aligned} &= 2 \sum_{j<i} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n] \\ &= 0 \end{aligned}$$

since we're working over $\mathbb{Z}_2$. ∎

This result shows that

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

is a chain complex. Thus we call $Z_k(X) \doteq \operatorname{Ker} \partial_k$ the space of k**-cycles** and $B_k(X) \doteq \operatorname{Im} \partial_{k+1}$ the space of k**-boundaries**.

Definition 6. The **k-th homology** of X is $H_k(X) \doteq Z_k(X)/B_k(X)$, and its dimension β_k is the **k-th Betti number**.

Proposition 3. β_0 is the number of connected components of X. **Infinite case?**

Proof. Suppose X has connected components $X_1, \dots, X_\ell$. Then since the homology functor commutes with direct sums,

$$H_0(X) = H_0\left(\bigoplus_{i=1}^{\ell} X_i\right) = \bigoplus_{i=1}^{\ell} H_0(X_i).$$

Show that $\beta_0 = 1$ when X is connected. Then since β_0 of a connected complex is 1,

$$\beta_0 = \dim\left(\bigoplus_{i=1}^{\ell} H_0(X_i)\right) = \sum_{i=1}^{\ell} \dim H_0(X_i) = \sum_{i=1}^{\ell} 1 = \ell.$$

$\square$

3 PERSISTENT HOMOLOGY

Given a function $f : G \to \mathbb{R}$, we can think of $f(x)$ as the time at which x appears.

Definition 7. $F : G \to \mathbb{R}$ is **monotonic** if $f(v) \leq f(e)$ whenever e is an edge containing vertex v. **gen to complexes...**

Thus for monotonic functions, no edge will appear until both its vertices have also appeared.

0 dim Persistent Homology stuff...

Note that every birth-death pair is an element of

$$\overline{\mathbb{R}}^2_< \doteq \{(x, y) \mid x \in \mathbb{R},\ y \in \mathbb{R} \cup \{\infty\}\}.$$

Figure.

Definition 8. A **partial mapping** between multisets $P, Q \subset \overline{\mathbb{R}}^2_<$ is a bijection $\eta : P_0 \to Q_0$, where $P_0 \subset P$ and $Q_0 \subset Q$. We denote it

$$\eta : P \leftrightarrow Q.$$

We define the cost of a partial matching $\eta : P \leftrightarrow Q$