Contents

Chapter 1

Calculus on Euclidean Space

1.1 Tangent Vectors

Definition 1: Tangent Space

Let $p \in \mathbb{R}^n$. Then the set of all tangent vectors in $\mathbb{R}^n$ originating at p is the **tangent space** of $\mathbb{R}^n$ at p. Denote this by $T_p(\mathbb{R}^n)$.

If we define addition by $v_p + w_p \doteq (v + w)_p$ and scalar multiplication by $\lambda w_p \doteq (\lambda w)_p$, then $T_p(\mathbb{R}^n)$ becomes a vector space..

Proposition 1. *$T_p(\mathbb{R}^n)$ is isomorphic to $\mathbb{R}^n$.*

Proof. Consider the function $v \mapsto v_p$. This is clearly a one-to-one function from $\mathbb{R}^n$ onto $T_p(\mathbb{R}^n)$. Additionally, it is clearly a homomorphism from the way we defined addition and scalar multiplication. $\square$

Definition 2: Vector Field

A **vector field** V on $\mathbb{R}^n$ is a function that maps $p \in \mathbb{R}^n$ to a tangent vector $V(p) \in T_p(\mathbb{R}^n)$.

Definition 3: Natural Frame Field

Let $U_1, \ldots, U_n$ be vector fields on $\mathbb{R}^n$ such that for each i, $U_i(p) = e_i$ for all $p \in \mathbb{R}^n$. Then $U_1, \ldots, U_n$ collectively are called the **natural frame field** on $\mathbb{R}^n$.

Note that U_i is a unit vector field in the positive x_i direction.

Proposition 2. *Let V be a vector field on $\mathbb{R}^n$, then there are unique real-valued functions $v_1, \ldots, v_n$ on $\mathbb{R}^n$ such that*

$$V = \sum_{i=1}^{n} v_i U_i.$$

Proof. Let $p \in \mathbb{R}^n$ be arbitrary, then by definition, $V(p) \in T_p(\mathbb{R}^n)$, so for some functions $v_1, \ldots, v_n$, we have

$$\begin{aligned} V(p) &= (v_1(p), \ldots, v_n(p)) \\ &= v_1(p)e_1 + \ldots v_n(p)e_n \\ &= v_1(p)U_1(p) + \ldots v_n(p)U_n(p). \end{aligned}$$

Since p was arbitrary, $V = \sum_{i=1}^{n} v_i U_i$. $\square$

Definition 4: Euclidean Coordinate Function

The v_i in the above proposition are the **Euclidean coordinate functions** of V.

The identity from the last proposition is important, so here it is again in a slightly different form.

$$(x_1, \ldots, x_n)_p = \sum_{i=1}^{n} x_i U_i(p).$$

Note 1

We say that a vector field is differentiable if its Euclidean coordinate functions are themselves differentiable. From now on, assume vector fields are differentiable.

1.2 Directional Derivatives

Definition 5: Directional Derivative

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable, and let $v \in T_p(\mathbb{R}^3)$. Then

$$v[f] \doteq \frac{d}{dt} f(p + tv)\Big|_{t=0}$$

is the derivative of f with respect to v. It is called the **directional derivative** of f at p in the direction of v.

Proposition 3. *Let $v \in T_p(\mathbb{R}^n)$, then*

$$v[f] = \sum_i v_i \frac{\partial f}{\partial x_i}(p).$$

Proof. Since $\frac{d}{dt}(p_i + tv_i) = v_i$, we can use the chain rule to get

$$\begin{aligned} \frac{d}{dt} f(p + tv)\Big|_{t=0} &= \sum_i v_i \frac{\partial f}{\partial x_i}(p + tv)\Big|_{t=0} \\ &= \sum_i v_i \frac{\partial f}{\partial x_i}(p). \end{aligned}$$

$\square$

Theorem 1

Let $f, g : \mathbb{R}^n \to \mathbb{R}$, let $v, w \in T_p(\mathbb{R}^n)$, and $\alpha, \beta \in \mathbb{R}$, then

1. $(\alpha v + \beta w)[f] = \alpha v[f] + \beta w[f]$,
2. $v[\alpha f + \beta g] = \alpha v[f] + \beta v[g]$, and
3. $v[fg] = v[f] \cdot g(p) + f(p) \cdot v[g]$.

Proof. We prove each of these by using Proposition 3.

1. We have

$$\begin{aligned} (\alpha v + \beta w)[f] &= \sum_i (\alpha v_i + \beta w_i) \frac{\partial f}{\partial x_i}(p) \\ &= \alpha \sum_i v_i \frac{\partial f}{\partial x_i}(p) + \beta \sum_i w_i \frac{\partial f}{\partial x_i}(p) \\ &= \alpha v[f] + \beta w[f]. \end{aligned}$$

2. We have

$$\begin{aligned} v[\alpha f + \beta g] &= \sum_i v_i \left[\alpha \frac{\partial f}{\partial x_i}(p) + \beta \frac{\partial g}{\partial x_i}(p) \right] \\ &= \alpha \sum_i v_i \frac{\partial f}{\partial x_i}(p) + \beta \sum_i v_i \frac{\partial g}{\partial x_i}(p) \\ &= \alpha v[f] + \beta v[g]. \end{aligned}$$

3. We have

$$\begin{aligned} v[fg] &= \sum_i v_i \frac{\partial (fg)}{\partial x_i}(p) \\ &= \sum_i v_i \frac{\partial f}{\partial x_i}(p) g(p) + \sum_i v_i f(p) \frac{\partial g}{\partial x_i}(p) \\ &= v[f] \cdot g(p) + f(p) \cdot v[g]. \end{aligned}$$

□

Parts (1) and (2) of this theorem say that $v[f]$ is linear in both v and f. Part (3) is just the Leibniz rule.

Definition 6: Operator of Vector Field on a Function

The **operator** of a vector field V on a function f is a function $V[f] : \mathbb{R}^n \to \mathbb{R}$ given by $p \mapsto V(p)[f]$. This is the derivative of f at the point p in the direction of $V(p)$.

Note that if U_i is part of the natural frame field on V, then $U_i[f] = \frac{\partial f}{\partial x_i}$.

Corollary 1. *Let V, W be vector fields on $\mathbb{R}^n$, let $f, g, h : \mathbb{R}^n \to \mathbb{R}$, and let $\alpha, \beta \in \mathbb{R}$, then*

1. $(fV + gW)[h] = fV[h] + gW[h]$,
2. $V[\alpha f + \beta g] = \alpha V[f] + \beta V[g]$, *and*
3. $V[fg] = V[f] \cdot g + f \cdot V[g]$.

Proof. We prove each of these by using the corresponding part of Theorem 1.

1. Fix p, then we have

$$\begin{aligned} (fV + gW)(p)[h] &= (f(p)V(p) + g(p)W(p))[h] \\ &= f(p)V(p)[h] + g(p)W(p)[h]. \end{aligned}$$

2. Fix p, then we have $V(p)[\alpha f + \beta g] = \alpha V(p)[f] + \beta V(p)[g]$.
3. Fix p, then we have $V(p)[fg] = V(p)[f] \cdot g(p) + f(p) \cdot V(p)[g]$.

$\square$

Note that a "scalar" in part (1) of this corollary can be a function, but the scalars must be actual numbers in part (2).

1.3 Parameterized Curves

Definition 7: Curve

A **curve** in $\mathbb{R}^n$ is a differentiable function $\alpha : I \to \mathbb{R}^n$, where I is an open interval in $\mathbb{R}$.

Example 1: Helix

To draw a helix, we can parameterize a curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ by

$$\alpha(t) = (a \cos t, a \sin t, bt),$$

where $a > 0$ and $b \neq 0$.

Definition 8: Velocity Vector

Let $\alpha : I \to \mathbb{R}^n$ be a curve. Then for every $t \in I$, the **velocity vector** of α at t is the tangent vector

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \ldots, \frac{d\alpha_n}{dt}(t) \right)_{\alpha(t)}$$

at the point $\alpha(t) \in \mathbb{R}^n$.

We can write the velocity vector alternatively as $\alpha'(t) = \sum_i \frac{d\alpha_i}{dt}(t) U_i(\alpha(t))$.

Example 2: Velocity Vector of a Helix

Using the parameterization of a helix from the previous example, its velocity vector is

$$\alpha'(t) = (-a \sin t, a \cos t, b)_{\alpha(t)}.$$

Definition 9: Reparameterization

Let $\alpha : I \to \mathbb{R}^n$ be a curve, and let $h : J \to I$ be differentiable, where J is an open interval. Then the function $\beta : J \to \mathbb{R}^n$ given by the composition $\beta = \alpha \circ h$ is called a **reparameterization** of α by h.

Proposition 4. *Let β be a reparameterization of α by h. Then its velocity vector is*

$$\beta'(s) = h'(s)\alpha'(h(s)).$$

Proof. To clarify notation, by $\alpha'(h(s))$ we mean $\frac{d}{dt}\alpha'(t)\big|_{t=h(s)}$. With that out of the way, this proof is just a straightforward application of the chain rule.

Since $\beta(s) = \alpha(h(s)) = \big(\dots, \alpha_i(h(s)), \dots\big)$, its derivative is given by $\beta'(s) = \big(\dots, h'(s)\alpha_i'(h(s)), \dots\big) = h'(s)\alpha'(h(s))$. □

Proposition 5. *Let α be a curve in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then*

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$$

Proof. By definition,

$$\alpha' = \left(\frac{d\alpha_1}{dt}, \dots, \frac{d\alpha_n}{dt}\right)_{\alpha(t)},$$

so by Proposition 3,

$$\alpha'(t)[f] = \sum_i \alpha_i'(t)\frac{\partial f}{\partial x_i}(\alpha(t)).$$

Noticing that the above expression is just an application of the chain rule, we can "undo" the chain rule to get

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$$

□

Is one-to-one-ness of a curve necessary?

Definition 10: Period

A curve $\alpha : \mathbb{R} \to \mathbb{R}^n$ is **periodic** if there is some $p > 0$ such that $\alpha(t + p) = \alpha(t)$ for all t. The smallest such p is then called the **period** of α.

Definition of regular curve.

1.4 1-Forms

Definition 11: 1-Form

A **1-form** on $\mathbb{R}^n$ is a linear function $\phi : T_p(\mathbb{R}^n) \to \mathbb{R}$, where p is some point in $\mathbb{R}^n$.

Something about ϕ being in dual space of $T_p(\mathbb{R}^n)$. In this sense it's dual to the notion of a vector field, whatever that means.

Addition of 1-forms is defined pointwise. We can also define a sort of scalar multiplication with functions. If $\phi : T_p(\mathbb{R}^n) \to \mathbb{R}$ is a 1-form and $f : \mathbb{R}^n \to \mathbb{R}$ is just your everyday real-valued function, define

$$(f\phi)(v) \doteq f(p)\phi(v)$$

for all $v \in T_p(\mathbb{R}^n)$.

Given a vector field V, we can naturally define an operation on it by a 1-form by

$$(\phi(V))(p) \doteq \phi(V(p)).$$

Thus we can view 1-forms as operators that convert vector fields into real-valued functions.

If $\phi(V)$ is differentiable whenever V is differentiable, then we say that ϕ itself is differentiable.

Note 2

From now on, assume any given 1-form is differentiable, unless otherwise stated.

It is easy to show that 1-forms are linear over vector fields. To be explicit, given a vector field V, functions f and g, and 1-forms ϕ and ψ, we have

$$\phi(fV + gV) = f\phi(V) + g\phi(V)$$

and

$$(f\phi + g\psi)(V) = f\phi(V) + g\psi(V).$$

Definition 12: Differential

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then the **differential** df of f is the 1-form such that

$$df(v) = v[f]$$

for all tangent vectors v of some point $p \in \mathbb{R}^n$.

Since $v[f]$ is real-valued, and since we proved earlier that it is linear for all p, $v[f]$ is in fact a 1-form.

Example 3

Consider the differentials $dx_1, \ldots, dx_n$ of the natural coordinate functions on $\mathbb{R}^n$. For a tangent vector v of a point p, we have

$$dx_i(v) = v[x_i] = \sum_j v_j \frac{\partial x_i}{\partial x_j}(p) = \sum_j v_j \delta_{ij} = v_i,$$

where δ_{ij} is the Kronecker delta. Thus the value of dx_i does *not* depend on the point of application p.