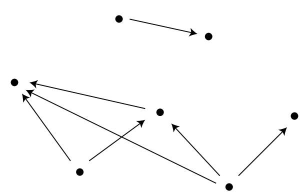
CONTENTS

1 THE BASICS

1.1 CATEGORIES

Definition 1. A **category $\mathbf{C}$** is a collection of **objects** $\mathrm{ob}(\mathbf{C})$ and **morphisms** $\mathrm{mor}(\mathbf{C})$, where $\mathrm{Hom}(A, B)$ denotes the morphsisms from object A to object B. There are several requirements:

1. Morphisms must compose: $(f, g) \mapsto gf$.
2. Morphism composition is associative.
3. If $A \neq C$ or $B \neq D$, then $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(C, D)$ are disjoint.
4. Each object has an identity morphism, which is a two-sided identity.



A category is **concrete** if, informally, its objects are underlying sets and its morphisms are functions between them, e.g. **Set**, **Top**, **Grp**. By contrast, **abstract** categories don't have this structure, e.g. BG for a group G.

A category is **discrete** if all its morphisms are identities, i.e. all its objects are isolated.

Because of set-theoretical issues, it's useful to denote when a category is "small enough". We say a category is **small** if it has only a set's worth of morphisms. Since

$$\text{identity morphisms} \leftrightarrow \text{objects},$$

small categories also have a set's worth of objects. We can loosen this somewhat: if $\mathrm{Hom}(X, Y)$ is always a set, the category is **locally small**.

Proposition 1. Identity morphisms and morphism inverses are unique.

Definition 2. An **isomorphism** is an invertible morphism.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Isomorphisms (isos) generalize bijective functions, which are both injective and surjective. Injective functions generalize to monomorphisms (monos), and surjective functions to epimorphisms (epis).

Include split monos/epis.

Definition 3. A morphism f is a **monomorphism** if for all parallel (between same objects) morphisms g, h with the proper domains,

$$fg = fh \implies g = h.$$

Similarly, f is an **epimorphism** if

$$gf = hf \implies g = h.$$

There's some fun vocab and symbols to go along with these. Monos are monic and denoted by $\rightarrowtail$, and epis are epic and denoted by $\twoheadrightarrow$. An isomorphism is necessarily both monic and epic, although the converse doesn't hold in general.

Special types of morphisms get their own special names sometimes too. An **endomorphism** is a morphism $X \to X$. An isomorphic endomorphism is called an **automorphism**.

Definition 4. A category $\mathbf{S}$ is a **subcategory** of $\mathbf{C}$ if

1. $\mathrm{ob}(\mathbf{S})$ is a subcollection of $\mathrm{ob}(\mathbf{C})$; and
2. for all $A, B \in \mathrm{ob}(\mathbf{S})$, $\mathrm{Hom}_{\mathbf{S}}(A, B)$ is a subcollection of $\mathrm{Hom}_{\mathbf{C}}(A, B)$ with identity.

A **full** subcategory doesn't remove any morphisms between the remaining objects, i.e.

$$\mathrm{Hom}_{\mathbf{S}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(A, B).$$

Definition 5. A **groupoid** is a category whose morphisms are all isomorphisms.

Every category contains a subcategory called the **maximal groupoid**, which is all of the objects along with only the morphisms that are isomorphisms.

Example 1. We can define a **group** as a groupoid that has only one object. The group elements are the morphisms. The properties of a group follow from the properties of categories and the fact that our morphisms are all isomorphisms.

Given a group G, its representation as a single-object category is denoted BG.

1.2 DUALITY

Definition 6. Given a category C, its **opposite** or **dual** category C^{op} is the category gotten by "reversing" the morphisms of C. This means

$$\mathrm{ob}(\mathsf{C}^{\mathrm{op}}) = \mathrm{ob}(\mathsf{C}),$$
$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(B, A).$$

My biggest misconception of this at first was that we were actually reversing each morphism, but this is clearly impossible. For example, if we're working in **Set**, we physically can't reverse all the morphisms since not all functions are invertible.

Note 1. We aren't actually changing any of the morphisms. The "reversal" of a morphism is a completely formal process. In fact, we can't even compare f and f^{op} since they live in different categories! At the end of the day, a category's dual has the same information, but the notation is just all reversed.

Note that the identities in a category and its dual are the same. Compositions, on the other hand, are necessarily reversed:

$$f^{\mathrm{op}} g^{\mathrm{op}} \doteq (gf)^{\mathrm{op}}.$$

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "gf"'] & \bullet \arrow[d, "g"] \\ & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet & \bullet \arrow[l, "f'"'] \\ & \bullet \arrow[u, "g'"'] \arrow[ul, "f'g'"] \end{tikzcd}$$

Duals are important because they make universal quantifications twice as valuable: if a theorem applies "for all categories", then it certainly applies to the opposites of all categories. We can then reinterpret the theorem in the opposite case to get a dual theorem, and to prove it we just reverse all the morphisms in our original proof.

1.3 FUNCTORS

Functors are the morphisms associated with categories: they map categories to categories in ways that respect categorical structure.

Definition 7. A **(covariant) functor** $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ satisfies:

- If $A \in \mathbf{C}$, then $\mathcal{F}A \in \mathbf{D}$.
- If $f : A \to B$, then $\mathcal{F}f : \mathcal{F}A \to \mathcal{F}B$.

These are subject to the functoriality axioms:

- $\mathcal{F}(fg) = \mathcal{F}f \cdot \mathcal{F}g$ for all f, g.
- $\mathcal{F}1_A = 1_{\mathcal{F}A}$ for all A.

$$\begin{tikzcd}
 & B \arrow[d, "g"] \arrow[rr, dashed] & & \mathcal{F}B \arrow[d, "\mathcal{F}g"] \\
A \arrow[ur, "f"] \arrow[rr, dashed] & & \mathcal{F}A \arrow[ur, "\mathcal{F}f"] & \mathcal{F}C \arrow[l, phantom] \\
 & C \arrow[ul, "h"] \arrow[rr, dashed] & & \mathcal{F}C \arrow[ul, "\mathcal{F}h"]
\end{tikzcd}$$

A **contravariant functor** is the same but with the morphisms $\mathcal{F}f$ reversed. This is just a covariant functor in disguise, though: we can represent it by a covariant functor with domain $\mathbf{C}^{\text{op}}$.

$$\mathcal{F} : \mathbf{C}^{\text{op}} \to \mathbf{D}.$$

$$\begin{tikzcd}
 & B \arrow[d, "g"] \arrow[rr, dashed] & & \mathcal{F}B \arrow[dl, "\mathcal{F}f"'] \\
A \arrow[ur, "f"] \arrow[rr, dashed] & & \mathcal{F}A \arrow[dr, "\mathcal{F}h"] & \\
 & C \arrow[ul, "h"] \arrow[rr, dashed] & & \mathcal{F}C \arrow[uu, "\mathcal{F}g"']
\end{tikzcd}$$

Example 2. Some fun functors :)

1. Forgetful functors.
2. $\mathbf{Top} \to \mathbf{Htpy}$ is the identity on objects (topological spaces) and sends morphisms (continuous functions) to their homotopy class.
3. π_1 is a functor $\mathbf{Top}_* \to \mathbf{Grp}$.

Proposition 2. Functors preserve isos and split monos/epis.

Definition 8. A functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ is **faithful** if for all objects A, B of $\mathbf{C}$, the map

$$\begin{aligned} \operatorname{Hom}(A, B) &\to \operatorname{Hom}(\mathcal{F}A, \mathcal{F}B) \\ f &\mapsto \mathcal{F}f \end{aligned}$$

is one-to-one. $\mathcal{F}$ is **full** if this map is onto.

Note that the fixed A and B above are important. The injective/surjective conditions don't apply to arbitrary morphisms in $\mathbf{C}$ since they might connect different objects.

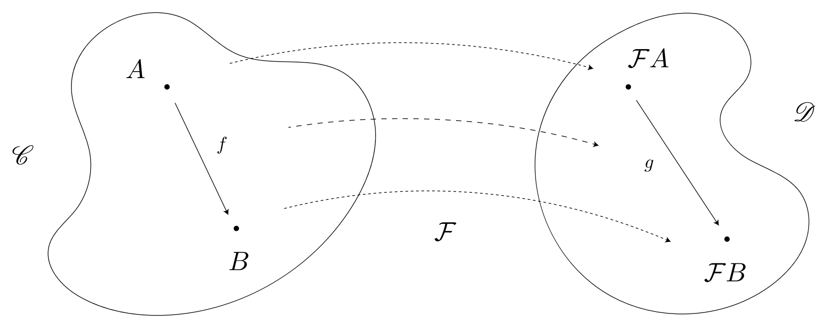


Figure 1.1: For all A, B, and g, a faithful functor sends at *most* one solid arrow in $\mathbf{C}$ to g. A full functor sends at *least* one solid arrow in $\mathbf{C}$ to g.

Example 3. The inclusion functor from $\mathbf{S}$ to $\mathbf{C}$ is always faithful, and it's full if and only if $\mathbf{S}$ is a full subcategory.

Definition 9. The following definitions apply for a covariant functor $\mathcal{F}$ if, given any short exact $0 \to A \to B \to C \to 0$, the given induced sequences are also exact.

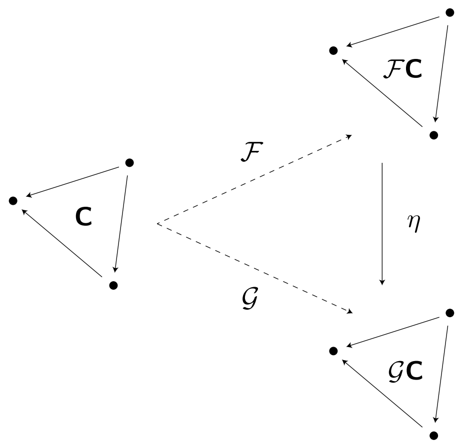
exact	$0 \to \mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C \to 0$
left exact	$0 \to \mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C$
right exact	$\mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C \to 0$

There are similar definitions for a contravariant functor $\mathcal{G}$.

$$
\begin{array}{rl}
\textbf{exact} & 0 \to \mathcal{G}C \to \mathcal{G}B \to \mathcal{G}A \to 0 \\
\textbf{left exact} & 0 \to \mathcal{G}C \to \mathcal{G}B \to \mathcal{G}A \\
\textbf{right exact} & \quad\ \mathcal{G}C \to \mathcal{G}B \to \mathcal{G}A \to 0
\end{array}
$$

1.4 NATURAL TRANSFORMATIONS

Natural transformations change one functor into another in a way that respects the underlying structure of the categories involved. It's kinda like a homotopy between $\mathcal{F}$ and $\mathcal{G}$ in the sense that for all $C \in \mathbf{C}$, it gives a morphism from $\mathcal{F}C$ to $\mathcal{G}C$.



Definition 10. Suppose $\mathcal{F}, \mathcal{G} : \mathbf{C} \to \mathbf{D}$ are functors. Then a **natural transformation** $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$ is a family of **components**

$$\{\eta_X : \mathcal{F}X \to \mathcal{G}X\}_X$$

such that the following diagram commutes for any $f : X \to Y$ in $\mathbf{C}$.

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{\eta_X} & \mathcal{G}X \\ {\scriptstyle \mathcal{F}f}\downarrow & & \downarrow{\scriptstyle \mathcal{G}f} \\ \mathcal{F}Y & \xrightarrow{\eta_Y} & \mathcal{G}Y \end{array}$$

If every η_X is an isomorphism, then η is a **natural isomorphism** and we write $\eta : \mathcal{F} \cong \mathcal{G}$.

2 UNIVERSAL PROPERTIES

2.1 COMMON EXAMPLES

Definition 11. $(X, \{\pi_\alpha\}_\alpha)$ is a **product** of $\{X_\alpha\}_\alpha$ if for all Y and morphisms $f_\alpha : Y \to X_\alpha$, there is a unique morphism $f : Y \to X$ lifting each f_α.

$$\begin{tikzcd} & X \arrow[d, "\pi_\alpha"] \\ Y \arrow[ur, dashed, "\exists!\, f"] \arrow[r, "f_\alpha"'] & X_\alpha \end{tikzcd}$$

Definition 12. $(X, \{\pi_\alpha\}_\alpha)$ is a **coproduct** of $\{X_\alpha\}_\alpha$ if for all Y and morphisms $f_\alpha : X_\alpha \to Y$, there is a unique morphism $f : X \to Y$ extending each f_α.

$$\begin{tikzcd} & X \arrow[dl, dashed, "\exists!\, f"'] \\ Y & X_\alpha \arrow[l, "f_\alpha"] \arrow[u, "i_\alpha"'] \end{tikzcd}$$

Proposition 3. If $(X, \{\pi_\alpha\})$ is a product, then each π_α is epic. If $(X, \{i_\alpha\})$ is a coproduct, then each i_α is monic.

Definition 13. (F, i) is free on the set B if for all objects X and maps $f : B \to X$, there is a unique morphism $F \to X$ extending f.

$$\begin{tikzcd} F \arrow[dr, dashed, "\exists!"] & \\ B \arrow[u, "i"] \arrow[r, "f"'] & X \end{tikzcd}$$