$$= \binom{n + \frac{m}{2}}{n} + \binom{n + \frac{m}{2} - 1}{n} \quad \text{for } m \text{ even.}$$

This conjecture has in fact been proved for "most" values of m and n, specifically for all $m \leqq 8$ and for $n \leqq 3$ and $n \geqq (m/2)^2 - 2$. To see what this means, the first unsolved cases are

$$m = 9 \qquad 4 \leqq n \leqq 9.$$

In general for each $m > 8$ there is an interval of values of n for which the conjecture has not been verified.

This strange situation together with the one described in the introduction concerning the number of replacements required to solve an $m \times n$ system are perhaps the most interesting features of what might superficially appear to be a dull and routine problem. To mix metaphors a little, they indicate how close to the surface the so-called frontiers of mathematics sometimes lie.

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References

1. G. B. Dantzig, A. Orden and P. Wolfe, Generalized simplex method for minimizing a linear form under linear inequality restraints, Pacific J. Math., 5 (1955) 183–195.
2. G. B. Dantzig, Constructive proof of the min-max theorem, Pacific J. Math., 6 (1959) 25–33.
3. Gerard Debreu, Nonnegative solutions of linear inequalities, Internat. Economic Review, 5 (1964).
4. B. Grünbaum, Convex Polytopes, Wiley, New York 1967.
5. V. L. Klee, A class of linear programming problems requiring a large number of iterations, Numer. Math., 7 (1965) 313–321.

SPECTRAL SEQUENCES FOR THE LAYMAN

BARRY MITCHELL, Bowdoin College

There are two methods for obtaining the usual facts relating the iterated homology of a double complex to the total homology. The first is to regard the double complex as a filtered complex and consider the resulting spectral sequence. This requires a mathematical maturity which many mathematicians never attain. The second method, which is accessible to anyone who knows how to take the quotient of two modules, is to chase diagrams. The purpose of this paper is to show that this alternative is not only possible, but furthermore quite simple.

As a student (and since) I had a very difficult time learning spectral sequences, and I felt that certain results in homological algebra and sheaf theory would forever remain inaccessible to me. It came as an immense relief when I found that, in these fields anyway, one does not often need an involved theory of spectral sequences, but rather only an elementary theory of double complexes. The proofs involved in the latter make excellent exercises for someone who is learning how to chase diagrams. At first I thought that this aversion to spectral sequences was a peculiarity of my own. However, over the years I have found that many students have the same problem, without realizing, unfortunately, that in order to handle double complexes one does not need a confusing involved theory. I am by no means claiming of course that there are no honest spectral sequences. But a spectral sequence which arises from a double complex has always struck me as being some sort of a fraud, and the present article is intended to show this.

The list of properties we shall deal with is at least complete enough to obtain all the properties stated in [1, Chapter 15, Section 6]. Since the arguments are always the same, the reader may regard any omissions as exercises in the technique of diagram chasing. I don't think that I can make the paper any simpler, since one man can only confuse another by trying to do his diagram chasing for him.

1. Edge morphisms. A *double complex* is a family of (left) R-modules $\{X^{ij} \mid (i,j) \in Z \times Z\}$, equipped with morphisms $d'_{ij}: X^{ij} \to X^{i+1,j}$, $d''_{ij}: X^{ij} \to X^{i,j+1}$ satisfying the rules

$$d'_{i+1,j} d'_{ij} = 0, \tag{1}$$

$$d''_{i,j+1} d''_{ij} = 0, \tag{2}$$

$$d''_{i+1,j} d'_{ij} + d'_{i,j+1} d''_{ij} = 0. \tag{3}$$

Because of (1), we can define

$$H_I^{ij} = \text{kernel } d'_{ij}/\text{image } d'_{i-1,j}.$$

Then using (3), we find that the morphisms d'' induce morphisms $\overline{d''_{ij}}: H_I^{ij} \to H_I^{i,j+1}$, and because of (2) we have $\overline{d''_{i,j+1}}\,\overline{d''_{ij}} = 0$. Consequently we can define

$$H_{II} H_I^{ij} = \text{kernel } \overline{d''_{ij}}/\text{image } \overline{d''_{i,j-1}}.$$

This module may be defined alternatively as follows. First define Z^{ij} to be the submodule of X^{ij} consisting of all elements x_{ij} such that

$$d' x_{ij} = 0 \quad \text{and} \quad d'' x_{ij} = d' x_{i-1,j+1} \tag{4}$$

for some $x_{i-1,j+1} \in X^{i-1,j+1}$. That is, Z^{ij} is the set of all elements of X^{ij} which go into 0 on the right, and which go into something above which comes from something on the left. Also define B^{ij} to be the submodule of X^{ij} consisting of all elements x_{ij} such that

$$x_{ij} = d''x_{i,j-1} + d'x_{i-1,j} \quad \text{where} \quad d'x_{i,j-1} = 0 \tag{5}$$

for some $x_{i,j-1} \in X^{i,j-1}$ and $x_{i-1,j} \in X^{i-1,j}$. Thus B^{ij} is the set of all elements of X^{ij} which can be written as the sum of an element from underneath with an element from the left, where the element underneath goes into 0 on its right. Then using (1), (2), and (3), we see that $B^{ij} \subset Z^{ij}$, and so we can define $H^{ij} = Z^{ij}/B^{ij}$. Then it is easily seen that $H_{II}H_I^{ij}$ is isomorphic to H^{ij}.

For each integer n, we define $X^n = \oplus_{i+j=n} X^{ij}$, and we define $d_n: X^n \to X^{n+1}$ by the rule

$$d_n x_{ij} = d'x_{ij} + d''x_{ij}.$$

Then using (1), (2), and (3), we find $d_{n+1}d_n = 0$ and so we can define $H^n = Z^n/B^n$, where $Z^n = \text{kernel } d_n$ and $B^n = \text{image } d_{n-1}$. For some fixed pair of integers (p, q) we shall be interested in the relationship between $H^{p,q}$ and H^{p+q}. By a translation, we may assume that $(p, q) = (0, 0)$.

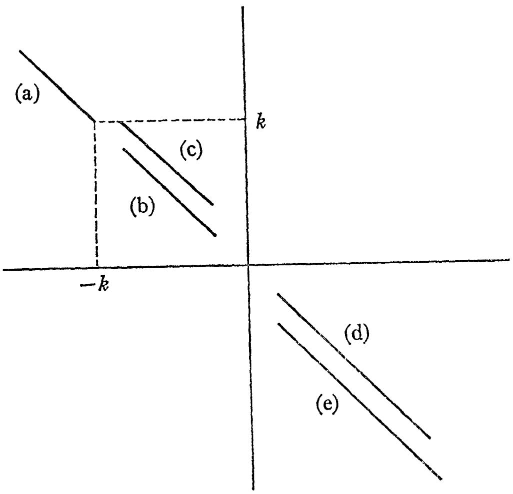
First we define an element of X^0 of the form $x = \sum_{i=0}^{k} x_{-i,i}$ to be a *second quadrant element of* X^0. If x is a second quadrant element and if dx has 0 entries in positions (1, 0) and (0, 1), then $x_{00} \in Z^{00}$. In particular this is true if $x \in Z^0$.

For some fixed integer $k > 0$, we specialize the following properties of the double complex X.

(a) $X^{-i,i} = 0$ for all $i \geqq k$.
(b) $H^{-i,i} = 0$ for $0 < i < k$.
(c) $H^{-i,i+1} = 0$ for $0 < i < k$.
(d) $H^{i,-i} = 0$ for all $i > 0$.
(e) $H^{i,-i-1} = 0$ for all $i > 0$.

The sets of integral points in the plane involved in the above conditions are indicated in the following diagram:

(6)



LEMMA 1.1. (i) *If X satisfies* (a) *and* (c), *then any element of Z^{00} can be extended to a second quadrant element of Z^0.*

(ii) *If X satisfies* (a) *and* (b), *and if z is a second quadrant element of Z^0 such that $z_{00} \in B^{00}$, then $z \in B^0$.*

(iii) *If X satisfies* (e), *and if z is a second quadrant element of B^0, then $z_{00} \in B^{00}$.*

(iv) *If X satisfies* (d), *then any element of Z^0 is congruent* mod B^0 *to a second quadrant element.*

Proof. All points are verified by looking hard at the diagram (6). As examples of the type of diagram chasing involved, we prove parts (i) and (ii). Suppose that X satisfies (a) and (c). Let $z_{00} \in Z^{00}$, so that we can write

$$d'z_{00} = 0, \qquad d''z_{00} + d'x_{-1,1} = 0. \tag{7}$$

Then using (3) and (2) we find $d'd''x_{-1,1} = 0$, and so since also $d''d''x_{-1,1} = 0$, we see that $d''x_{-1,1}$ is an element of $Z^{-1,2}$. By condition (c), the latter is $B^{-1,2}$, and so we can write

$$d''x_{-1,1} + d''y_{-1,1} + d'x_{-2,2} = 0 \quad \text{where} \quad d'y_{-1,1} = 0.$$

Inductively we can produce elements $x_{-i-1,i+1}$ and $y_{-i,i}$ for $0 < i < k$ satisfying

$$d''x_{-i,i} + d''y_{-i,i} + d'x_{-i-1,i+1} = 0, \quad \text{where} \quad d'y_{-i,i} = 0. \tag{8}$$

By condition (a) we have $x_{-k,k} = 0$. Hence adding equations (7) and (8), we find that $z_{00} + \sum_{i=1}^{k-1} (x_{-i,i} + y_{-i,i})$ is an element of Z^0. This proves part (i).

Now assume (a) and (b), and suppose that z is a second quadrant element of Z^0 where $z_{00} \in B^{00}$. Then we can write $z_{00} + d''x_{0,-1} + d'x_{-1,0} = 0$ where $d'x_{0,-1} = 0$. We can add $d(x_{-1,0} + x_{0,-1})$ to z without changing the class of z mod B^0 to obtain an element $z' = z'_{-1,1} + \sum_{i=2}^{k-1} z_{-i,i}$. Using (b) we can repeat this process until we are left with an element of $X^{-k,k}$, and by condition (a) the latter is 0. This proves that $z \in B^0$.

In verifying part (iii) it is useful to keep in mind that $dd = 0$, so that if $dx = z$ where $x \in X^{-1}$, then we may alter x by anything of the form dy without changing z. Thus, starting with the lowest term we can reduce one by one the elements of x in the fourth quadrant to 0. The proof of (iv) is similar.

The various parts of the lemma may now be put together as follows.

PROPOSITION 1.2. *If X satisfies conditions* (a), (b), *and* (c), *then there is a natural morphism α: $H^{00} \to H^0$. If X satisfies* (d), *then α is an epimorphism. If X satisfies* (e), *then α is a monomorphism.*

If X satisfies conditions (d) *and* (e), *then there is a natural morphism β: $H^0 \to H^{00}$. If X satisfies* (a) *and* (b), *then β is a monomorphism. If X satisfies* (a) *and* (c), *then β is an epimorphism.*

Proof. The morphism α is defined by assigning to the class of $z_{00} \in Z^{00}$ the class of any second quadrant extension of z_{00} to an element of Z^0. That such an extension exists is guaranteed by (i) of the lemma. That α is well defined follows from (ii).

The morphism β is defined by representing an element of H^0 by a second quadrant element z of Z^0, and assigning to it the class of z_{00}. This is justified by (iv). That β is well defined follows from (iii).

The other assertions follow directly from the various parts of the lemma.

The morphisms α and β are called *edge morphisms*. They are usually defined under stronger conditions. The condition under which they are isomorphism is sometimes referred to as the *maximal term principle* for double complexes.

In view of 1.2, we shall say that "α_{00} is defined" if X satisfies conditions (a), (b), and (c) (all relative to the same positive integer k), and that "β_{00} is defined" if X satisfies conditions (d) and (e). The conditions "α_{ij} is defined" and "β_{ij} is defined" are obtained by translation.

2. Exact sequences. Let $x_{1,0}\in Z^{1,0}$, and let $x_{0,1}$ be as in equation (4) of Section 1. Then it is easily seen that $d''x_{0,1}\in Z^{0,2}$, and that this element is independent mod $B^{0,2}$ of the choice of $x_{0,1}$. Furthermore if $x_{1,0}\in B^{1,0}$, then $d''x_{0,1}\in B^{0,2}$. There results a morphism $\delta\colon H^{1,0}\to H^{0,2}$ (which is known to chess players as the *knight's morphism*).

PROPOSITION 2.1. *If $\alpha_{0,1}$ and $\beta_{1,0}$ are defined, then*

$$H^{0,1}\xrightarrow{\alpha} H^1 \xrightarrow{\beta} H^{1,0} \text{ is exact.}$$

If $\beta_{1,0}$ is defined, and if for some $k>0$ we have $H^{-i,i+2}=0$ for $0<i<k$ and $X^{-i,i+1}=0$ for $i\geqq k$, then

$$H^1\xrightarrow{\beta} H^{1,0} \xrightarrow{\delta} H^{0,2} \text{ is exact.}$$

If $\alpha_{0,2}$ is defined, and if $H^{i,-i+1}=0$ for $i>1$, then

$$H^{1,0}\xrightarrow{\delta} H^{0,2} \xrightarrow{\alpha} H^{2} \text{ is exact.}$$

Proof. It is trivial to show that the three compositions are zero. The relations kernel $\subset$ image are verified using arguments similar to those used in the proof of 1.1. The reader should draw a diagram, and should not write anything down.

REMARK. If one combines the three sets of conditions and adds the condition $H^{1,-1}=0$ (so that since $\beta_{1,0}$ is defined, $H^{i,-i}=0$ for all $i>0$, and consequently by 1.2, $\alpha_{0,1}$ is a monomorphism), one obtains the "exact sequence for terms of low degree" of [**1**, Chap. 15, Section 6].

COROLLARY 2.2. *If $H^{ij}=0$ for all $j\neq 0$, 1, and if there is a positive k such that $X^{ij}=0$ for $j\geqq k$, then for each integer n we have an exact sequence*

$$0\to H^{n-1,1}\to H^n\to H^{n,0}\to 0.$$

Proof. Make a horizontal translation and apply 2.1.

COROLLARY 2.3. *If $H^{ij}=0$ for $i\neq 0$, 1, and if there is a negative k such that $X^{ij}=0$ for $i\leqq k$, then there is an exact sequence*

$$\cdots\to H^{0,n}\to H^n\to H^{1,n-1}\to H^{0,n+1}\to H^{n+1}\to\cdots.$$

Proof. Make a vertical translation and apply 2.1.

3. Morphisms of complexes. Let (X, d', d'') and (Y, e', e'') denote double complexes of R-modules. A *morphism* $f\colon X\to Y$ of double complexes is a family of morphisms $f_{ij}\colon X^{ij}\to Y^{ij}$ satisfying

$$e'_{ij}f_{ij} = f_{i+1,j}\,d'_{ij},\qquad e''_{ij}f_{ij} = f_{i,j+1}\,d''_{ij}. \tag{1}$$

Using conditions (1), we obtain induced morphisms

$$H^{ij}(f)\colon H^{ij}(X)\to H^{ij}(Y) \tag{2}$$

$$H^{n}(f)\colon H^{n}(X)\to H^{n}(Y). \tag{3}$$

We are interested in finding sufficient conditions on the morphisms (2) in order that the morphisms (3) be monomorphisms (epimorphisms).

PROPOSITION 3.1. (a) *Suppose that $X^{i,-i}=0$ for $i<0$. If $H^{i,-i}(f)$ is a monomorphism and $H^{i,-i-1}(f)$ is an epimorphism for all $i\geqq 0$, then $H^0(f)$ is a monomorphism.*

(b) *Suppose that $X^{i,-i}=0=Y^{i,-i}$ for $i<0$. If $H^{i,-i}(f)$ is an epimorphism and $H^{i,-i+1}(f)$ is a monomorphism for all $i\geqq 0$, then $H^0(f)$ is an epimorphism.*

Proof. The diagram chasing involved here is slightly more complicated than that we have encountered so far owing to the fact that there are two complexes instead of one, and so we shall give a proof of part (a). The reader is again advised, however, that the proof is much easier if one does not try to write anything down.

Suppose $x=\sum_{i=0}^{n}x_{i,-i}\in Z^0(X)$, and that $fx=ey$ where $y=\sum_{i=k}^{m}y_{i,-i-1}$. We may assume $m=n$ by adding 0 terms to x or y if necessary. Now $x_{n,-n}\in Z^{n,-n}(X)$, and $fx_{n,-n}\in B^{n,-n}(Y)$. Consequently since $H^{n,-n}(f)$ is a monomorphism, we can write

$$x_{n,-n} = d''x_{n,-n-1} + d'x_{n-1,-n}\quad\text{where}\quad d'x_{n,-n-1} = 0. \tag{4}$$

But then we find that $y_{n,-n-1}-fx_{n,-n-1}\in Z^{n,-n-1}(Y)$, and so since $H^{n,-n-1}(f)$ is an epimorphism, we can write

$$y_{n,-n-1} - fx_{n,-n-1} - f\bar{x}_{n,-n-1} = e''y_{n,-n-2} + e'y_{n-1,-n-1} \tag{5}$$

where $e'y_{n,-n-2}=0$, and where

$$d''\bar{x}_{n,-n-1} + d'\bar{x}_{n-1,-n} = 0,\ d'\bar{x}_{n,-n-1} = 0. \tag{6}$$

If we alter y by subtracting $e(y_{n,-n-2}+y_{n-1,-n-1})$ from it, we may assume the right side of (5) is 0, and this does not change the fact that $fx=ey$ (remember that $ee=0$). Also if we alter $x_{n,-n-1}$ and $x_{n-1,-n}$ by adding to them $\bar{x}_{n,-n-1}$ and $\bar{x}_{n-1,-n}$ respectively, we see from (6) that this does not change (4). But then equation (5) is reduced to $y_{n,-n-1}=fx_{n,-n-1}$, and so combining this with $fx=ey$, we obtain

$$f(x - d(x_{n,-n-1} + x_{n-1,-n})) = e(y - y_{n,-n-1} - fx_{n-1,-n}). \tag{7}$$

Since the term in outer parenthesis on the left of (7) has 0 at position $(n, -n)$ and below, and the term in parenthesis on the right has 0 at position $(n, -n-1)$ and below, we may now apply induction on n and the fact that $X^{i,-i}=0$ for $i<0$ to deduce that $x\in B^0(X)$.

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References

1. H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, N. J., 1956.
2. S. MacLane, Homology, Springer, Berlin, 1963.
3. S. Eilenberg and J. Moore, Limits and spectral sequences, Topology, 1 (1961) 1–23.

THE JORDAN CURVE THEOREM FOR PIECEWISE SMOOTH CURVES

R. N. PEDERSON, Stanford University

1. Introduction. It is the purpose of this note to provide an elementary proof of the Jordan Curve Theorem for the class of piecewise smooth curves. The only tools which we require are the notions of compactness, continuity, and the concept of the index of a closed curve relative to a point. Since these topics are included in a standard advanced undergraduate or beginning graduate course in complex analysis, it is our hope that the proof will fit in well with such a course.

We begin with an informal outline of the proof as it would apply to a polygon. In order to prove the Jordan Arc Theorem for a simple polygon, it suffices to demonstrate that its complement is arcwise connected. Suppose this is true for all simple polygons having at most n segments. A simple polygon P_{n+1} having $n+1$ segments is obtained by adjoining a single segment σ to a simple polygon P_n having n-segments. Any two points in the complement of P_{n+1} can be joined by a polygonal arc C in the complement of P_n. If C does not intersect σ, then it clearly lies in the complement of P_{n+1}. If it does intersect σ, then by drawing parallel lines on either side of σ, it is easily seen that C may be replaced by a polygon which does not intersect P_{n+1}. Hence, the complement of P_{n+1} is connected. In order to obtain a valid induction proof, it suffices to note that the complement of a single segment is indeed connected.

Now let P be a simple closed polygon. The Jordan Curve Theorem for P asserts that the complement of P is comprised of two nonempty components E and I. Let Γ be the simple polygon obtained by removing from P a segment σ. Choose ζ to be a point which lies outside of a disk containing P in its interior. Denote by E the set of points in the complement of P which can be joined, in