MANIFOLDS

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Based on Tu's *An Introduction to Manifolds*.

CONTENTS

1 EUCLIDEAN SPACE

1.1 REMINDERS

We say f is **real analytic** at p if it's equal to its Taylor series at p in some neighborhood of p. Note that if f is real analytic, then it's also C^∞ (the converse isn't true in general, though).

Proposition 1 (Baby Taylor's Theorem with Remainder)**.** Let U be open in $\mathbb{R}^n$ and star-convex wrt p. If f is C^∞ on U, then there are C^∞ functions $g_1, \ldots, g_n$ on U such that

$$f(x) = f(p) + \sum_i (x^i - p^i) g_i(x)$$

and $g_i(p) = \frac{\partial f}{\partial x^i}(p)$ for all i.

Proof. Since U is star-convex wrt p, we can draw a straight line from p to any $x \in U$. Intuitively, $f(x)$ should be $f(p)$ plus all the changes in f along this line. We can use the FToC to formalize this:

$$f(x) - f(p) = \int_0^1 \frac{d}{dt} f(p + t(x-p))\, dt.$$

We can use the chain rule to evaluate $\frac{d}{dt} f(p + t(x-p))$, giving

$$f(x) - f(p) = \sum (x^i - p^i) \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x-p))\, dt.$$

Set $g_i(x)$ to be its respective integral in the above sum. $\square$

Definition 1. A vector space map $L : \mathcal{V} \to \mathcal{W}$ is **linear** if

1. $L(u + v) = Lu + Lv$; and
2. $L(\lambda v) = \lambda L v$.

If $\mathcal{V}$ and $\mathcal{W}$ are both over K, then we might say "K-linear".

Note that since an algebra *is* a vector space, a map of algebras can be linear.

Definition 2. An **algebra** over a field K is a K-vector space A with bilinear associative multiplication map:

1. $(ab)c = a(bc)$;
2. $(a+b)c = ac + bc$ and $a(b+c) = ab + ac$; and
3. $\lambda(ab) = (\lambda a)b = a(\lambda b)$.

Definition 3. A map of algebras $D : A \to A$ is a **derivation** of A if it's linear and satisfies the Leibniz rule

$$D(ab) = Da \cdot b + a \cdot Db.$$

1.2 TANGENT VECTORS AS POINT DERIVATIONS

Definition 4. Let $v \in T_p(\mathbb{R}^n)$, then define

$$D_v := \sum v^i \frac{\partial}{\partial x^i}\Big|_p.$$

If f is C^∞ in some neighborhood of p, then $D_v f$ is the **directional derivative** of f at p in the direction of v. The "p" is implicit in the notation since $v \in T_p$. Note that $D_v f$ is an actual number, not a function.

A useful equivalence class if all functions that have the same directional derivative at a fixed point, since these functions will all look the same locally (from the PoV of calculus, these shouldn't be distinguished).

Definition 5. Suppose f, g are C^∞ on open sets $U, V \subset \mathbb{R}^n$, respectively. Then we say $(f, U) \sim (g, V)$ if there's another open set $W \subset U \cap V$ such that $f = g$ on W. The **germ** of f at p is then just $[(f, U)]$. Denote the set of all germs at p by C_p^∞.

We can turn C_p^∞ into an $\mathbb{R}$-algebra by equipping it with the operations

$$\begin{aligned}
[(f,U)] + [(g,V)] &:= [(f+g, U \cap V)],\\
[(f,U)][(g,V)] &:= [(fg, U \cap V)],\\
\lambda[(f,U)] &:= [(\lambda f, U)].
\end{aligned}$$

Now note that $D_v : C_p^\infty \to \mathbb{R}$ is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$D_v(fg) = D_v f \cdot g(p) + f(p) \cdot D_v g,$$

so it's a kind of pseudo-derivation (it has the right properties, but is $C_p^\infty \to \mathbb{R}$ instead of $C_p^\infty \to C_p^\infty$). This motivates the following definition.

Definition 6. A **point derivation** at p is a linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule

$$D(fg) = Df \cdot g(p) + f(p) \cdot Dg.$$

Denote the real vector space of all point derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$.

Note the abuse of notation above: f and g are germs, not functions, but I can use normal function notation since any functions in the same germ at p must agree at p. This is the same reason we can treat D_v as a map on C_p^∞ instead of just a map of C^∞ functions.

Lemma 1. If D is a point derivation, then $Dc = 0$ for all constant functions c.

Proof. By the Leibniz rule, $D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) \implies D(1) = 0$. Then by linearity, $Dc = cD(1) = 0$. □

Theorem 1. $T_p(\mathbb{R}^n) \cong \mathcal{D}_p(\mathbb{R}^n)$ as vector spaces via the map

$$\begin{aligned} \phi : T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n) \\ v &\mapsto D_v. \end{aligned}$$

Proof. ϕ is clearly linear, so we only need to check bijectivity. To check injectivity, suppose $D_v = 0$, then apply each coordinate function x^j to show $0 = D_v x^j = v^j$. To show surjectivity, suppose $D \in \mathcal{D}_p$ and $[(f, U)] \in C_p^\infty$. We can always restrict U to an open ball and stay within the same equivalence class, so we can assume U is star-shaped wrt p. By Proposition 1, there are g_i such that

$$f = f(p) + \sum (x^i - p^i) g_i.$$

Now we apply D to f. Using linearity, the Leibniz rule, and $D(f(p)) = D(p^i) = 0$ (by Lemma 1),

$$Df = \sum Dx^i g_i(p) = \sum Dx^i \frac{\partial f}{\partial x^i}(p).$$

Thus $Df = D_v = \phi(v)$, where $v = (Dx^1, \ldots, Dx^n)$. □

Note 1. The big takeaway from all this is that we can *define* $T_p(\mathbb{R}^n)$ to be the real vector space $\mathcal{D}_p(\mathbb{R}^n)$ instead of the usual one spanned by $\{e_1, \ldots, e_n\}$. Since $\phi(e^j) = \frac{\partial}{\partial x^j}\Big|_p$, we know

$$\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$$

is a basis for $\mathcal{D}_p(\mathbb{R}^n)$. Thus any tangent vector at p can be written

$$v = \sum v^i \frac{\partial}{\partial x^i}\Big|_p.$$

1.3 THE WEDGE PRODUCT

Do this.

Discuss dx^i, and how to interpret it when used with a wedge product.

1.4 DIFFERENTIAL FORMS

A differential form is something that can be integrated. In n dimensions, the following are the bases of the spaces of 1-forms and 2-forms:

$$\begin{aligned} k=1: &\qquad \left\{dx^1, \ldots, dx^n\right\}, \\ k=2: &\qquad \left\{dx^1 \wedge dx^2, dx^1 \wedge dx^3, \ldots, dx^1 dx^n, dx^2 dx^3, dx^4, \ldots\right\}. \end{aligned}$$

The pattern is clear: the basis of the space of k-forms in an n-dimensional space is

$$\left\{dx^{i_1} \wedge \cdots \wedge dx^{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\right\}.$$

Thus a k-form takes in a k-dimensional parallelepiped sitting in the tangent space and spits out a real number proportional to the area of that parallelepiped. This lets us perform the usual integration procedure: suppose we have a k-dimensional region $U \subset \mathbb{R}^n$ that we want to integrate a k-form over, then

could have multiple collections of tangent vectors if $k < n$.

1. Approximate U by selecting a bunch of points.
2. At each of these points p, the collection of tangent basis vectors

   $$\left\{\frac{\partial}{\partial x^{i_1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^{i_k}}\bigg|_p\right\}$$

   represents a parallelepiped that roughly matches up with the local volume of the surface. Since this is infinitesimal, it doesn't really matter that it's just an approximation.
3. At every point p,

   $$\left(dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right)\left(\frac{\partial}{\partial x^{i_1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^{i_k}}\bigg|_p\right)$$

   gives the volume of this parallelepiped, allowing us to approximate the local volume of our surface.

show $dx^i(\partial_i|_p) = \delta_i^j$.
$dx^1 \wedge \cdots \wedge dx^n$ when evaluated on a bunch of vectors gives the determinant of those vectors or something like that?

Since the wedge product produces an alternating linear map, the space of k-forms on an open set $U \subset \mathbb{R}^n$ is **the same** as the space of alternating k-linear maps on $\prod_{i=1}^k T_p(U)$.

Still don't understand how this generalization helps if we just use the defintion of the Lebesgue integral anyway... does this help with change of variables or something?