Exercise 1 (10.15)**.** Spherical law of sines is compatible with Euclidean law of sines for very small a, b, c.

The Taylor expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots,$$

so when x is small, $\sin x \approx x$. Thus when a, b, c are small,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{B} = \frac{\sin c}{\sin C}$$

becomes approximately

$$\frac{a}{\sin A} = \frac{b}{B} = \frac{c}{\sin C},$$

which is exactly the Euclidean law of sines.

Exercise 2 (10.16). Spherical law of cosines is compatible with Euclidean law of cosines for very small a, b, c.

The Taylor expansion of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots ,$$

so when x is small, $\cos x \approx 1 - x^2/2$. Thus when a, b, c are small,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

becomes approximately

$$1 - \frac{c^2}{2} = \left(1 - \frac{a^2}{2}\right)\left(1 - \frac{b^2}{2}\right) + ab\cos C$$
$$c^2 = a^2 + b^2 + \frac{a^2b^2}{4} - 2ab\cos C.$$

But the $a^2b^2/4$ term goes to 0 much faster than the other terms, so this is approximately

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

which is exactly the Euclidean law of cosines.

Exercise 3 (10.17)**.** Spherical law of sines and cosines for sphere of radius ρ.

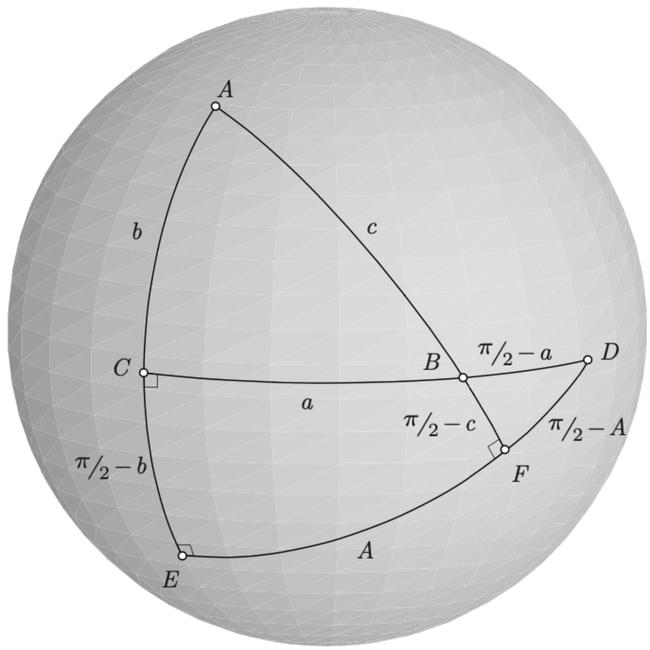
Suppose an arc on a sphere of radius ρ is subtended by an angle θ, then its arc length is $\theta\rho$. We can then transform it into an arc on the unit sphere viw the map $x \mapsto x/\rho$. Thus the modified laws of sines and cosines are

$$\frac{\sin(a/\rho)}{\sin A} = \frac{\sin(b/\rho)}{B} = \frac{\sin(c/\rho)}{\sin C}$$

and

$$\cos(c/\rho) = \cos(a/\rho)\cos(b/\rho) + \sin(a/\rho)\sin(b/\rho)\cos C$$

Exercise 4 (10.21)**.** Derive Pythagorean theorem using Menelaus on the triangle in Figure 7.



Since D, E, F are collinear, Menelaus' theorem says

$$\frac{\sin|AF|}{\sin|FB|}\frac{\sin|BD|}{\sin|DC|}\frac{\sin|CE|}{\sin|EA|} = -1.$$

But these are signed ratios, and D does not lie on BC, so the LHS of the above equation has an additional negative sign. Thus plugging in the lengths of each line gives

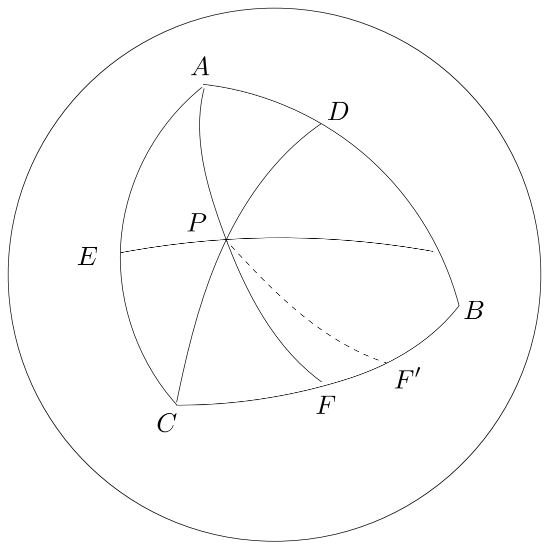
$$\begin{gathered}\sin|AF|\sin|BD|\sin|CE| = \sin|FB|\sin|DC|\sin|EA| \\ \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2} - a\right)\sin\left(\frac{\pi}{2} - b\right) = \sin\left(\frac{\pi}{2} - a\right)\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) \\ \cos a \cos b = \cos a,\end{gathered}$$

which is the spherical Pythagorean theorem.

Exercise 5 (10.26)**.** State and prove Ceva's theorem on the sphere.

Theorem 1 (Spherical Ceva)**.** Suppose ΔABC be a spherical triangle and let D, E, F be points on BC, AC, AB, respectively. Then AD, BE, CF are concurrent $\iff$

$$\frac{\sin|AF|}{\sin|FB|}\frac{\sin|BD|}{\sin|DC|}\frac{\sin|CE|}{\sin|EA|} = 1.$$



Forward: Suppose BC, AC, and AB are intersect at a common point P, then by Menelaus' theorem,

$$\frac{\sin|AF|}{\sin|FB|}\frac{\sin|BP|}{\sin|PE|}\frac{\sin|EC|}{\sin|CA|} = -1 = \frac{\sin|BD|}{\sin|DC|}\frac{\sin|CA|}{\sin|AE|}\frac{\sin|EP|}{\sin|PB|}.$$

Multiplying these two together and cancelling terms then gives

$$\frac{\sin|AF|}{\sin|FB|}\frac{\sin|BD|}{\sin|DC|}\frac{\sin|CE|}{\sin|EA|} = 1.$$

Backward: Suppose $\frac{\sin|AF|}{\sin|FB|}\frac{\sin|BD|}{\sin|DC|}\frac{\sin|CE|}{\sin|EA|} = 1$. Suppose AD and BE intersect at a point P, and let CP intersect AB at F'. By Menelaus' theorem and a similar argument as in the first half of the proof,

$$\frac{\sin|AF'|}{\sin|F'B|}\frac{\sin|BD|}{\sin|DC|}\frac{\sin|CE|}{\sin|EA|} = 1.$$

Then by our original assumption,

$$\begin{aligned}\frac{\sin|AF'|}{\sin|F'B|}\frac{\sin|BD|}{\sin|DC|}\frac{\sin|CE|}{\sin|EA|} &= \frac{\sin|AF|}{\sin|FB|}\frac{\sin|BD|}{\sin|DC|}\frac{\sin|CE|}{\sin|EA|}\\ \frac{\sin|AF'|}{\sin|F'B|} &= \frac{\sin|AF|}{\sin|FB|}.\end{aligned}$$

If $F' \neq F$, then $|AF'| = |AF| + x$ and $|F'B| = |FB| - x$. Plugging these into the above equality yields

$$\begin{aligned}
\frac{\sin|AF| + x}{\sin|FB| - x} &= \frac{\sin|AF|}{\sin|FB|} \\
(\sin|AF| + x)\ \sin|FB| &= (\sin|FB| - x)\ \sin|AF| \\
x(\sin|FB| + \sin|AF|) &= 0 \\
x &= 0.
\end{aligned}$$

This is a contradiction, so $F = F'$, meaning that BC, AC, and AB all intersect at P.

Exercise 6 (10.35)**.** Edge length of the polygons in the semiregular tiling $(3, 3, 5, 5)$ of the unit sphere.

There are two pentagons and 2 triangles per vertex, and their angles at each vertex must sum to 2π in order for it to be a tiling. Since the tiling is semiregular, by symmetry we know the angles of 1 pentagon and 1 triangle at each vertex sum to π, i.e. form a portion of a geodesic. We can repeat this argument to find a sequence of edges forming an entire eodesic on the sphere.

Thus this problem reduces to counting the number of edges in a geodesic along the "great circles" of a truncated icosahedron. There are 10 edges, so since the length of a geodesic on a unit sphere is 2π, each edge must have length

$$\frac{2\pi}{10} = \frac{\pi}{5}.$$

Exercise 7 (10.36)**.** Percentage of area of sphere covered by triangles in the semiregular tiling $(3, 3, 5, 5)$.

By the previous exercise, each triangle has sides of length $\pi/5$. Since they're regular triangles, each inner angle is the same; denote it by θ. Then by the spherical law of cosines,

$$\begin{aligned}
\cos\left(\frac{\pi}{5}\right) &= \cos^2\left(\frac{\pi}{5}\right) + \sin^2\left(\frac{\pi}{5}\right)\cos\theta \\
\frac{1+\sqrt{5}}{4} &= \frac{3+\sqrt{5}}{8} + \frac{5-\sqrt{5}}{8}\cos\theta \\
\cos\theta &= \frac{2+2\sqrt{5}-3-\sqrt{5}}{5-\sqrt{5}} \\
\cos\theta &= \frac{1}{\sqrt{5}}.
\end{aligned}$$

Then the area of each triangle in the tiling is $3\theta - \pi = 3\arccos(1/\sqrt{5}) - \pi$. Since the entire sphere has area 4π, the percentage of the sphere covered by triangles is

$$\frac{20\big(3\arccos(1/\sqrt{5}) - \pi\big)}{4\pi} \approx 28.62\%.$$