Contents

Chapter 1

Vector Bundles

1.1 Vector Bundles

Note 1. Throughout these notes, a **map** is a continuous function.

A fiber bundle is a space that looks locally like a product space. It's a generalization of a covering space. Vector bundles are just fiber bundles whose fibers are vector spaces.

Definition 1. A **fiber bundle** is a surjective map $p : E \to B$ along with a **fiber** F. For each $x \in B$, there is a neighborhood U of x such that there is a homeomorphism ϕ making the following diagram commute.

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[d, "p"] & U \times F \arrow[dl, "\pi_1"] \\ U & \end{tikzcd}$$

We call $p^{-1}(x)$ the **fiber over** x. Note that it must be isomorphic to F, so it makes sense to refer to the "fibers" of a fiber bundle even if we defined it in terms of the single fiber F. Each fiber in the bundle is its own object, but they're each homeomorphic to F.

Definition 2. A **vector bundle** is a fiber bundle whose fibers are vector spaces and where ϕ is a linear isomorphism on each $p^{-1}(U)$.

If the fibers of a vector bundle are over $\mathbb{R}$, then we call it a **real vector bundle**. If they're over $\mathbb{C}$, then it's a **complex vector bundle**. In the case of real vector bundles with finite-dimensional fibers, we can specialize the definition a bit more. Since any finite-dimensional real vector space is isomorphic to $\mathbb{R}^n$ for some n, we can equivalently write the diagram as follows.

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[d, "p"] & U \times \mathbb{R}^n \arrow[dl, "\pi_1"] \\ U & \end{tikzcd}$$

Once we define maps between bundles, we can use them as morphisms in the category of all fiber bundles. If we add a condition to make them respect vector space structure, then we can extend them to work with vector bundles.

Definition 3. A **bundle map** is a map $\phi : E_1 \to E_2$ that induces another map $f : B_1 \to B_2$ making the following diagram commute.

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

If ϕ is a linear map (isomorphism) between fibers $p_1^{-1}(x)$ and $p_2^{-1}(f(x))$, then it is a **vector bundle morphism (isomorphism)**.

Confused about what an inverse of a bundle map would be, and thus confused about how to define bundle isomorphisms.

If $B_1 = B_2 = B$, then $f = 1_B$, so we don't have to worry about f at all and defining isomorhpisms becomes more straightforward. In this case, a bundle isomorphism is a homeomorphism that maps $p_1^{-1}(x) \to p_2^{-1}(x)$, and a vector bundle isomorphism additionally restricts to a linear isomorphism $p_1^{-1}(x) \to p_2^{-1}(x)$.

Definition 4. A **section** of $p : E \to B$ is a map $f : B \to E$ such that $pf = 1_B$.

The image of a section in E is homeomorphic to B via p: consider the section $f(x)$, then $fp(f(x)) = f(pf(x)) = f(x)$ by definition, so $fp = 1_{f(x)}$. Image.

Example 1. The **zero section** of a vector bundle maps every $x \in B$ to the 0 element in the corresponding fiber $f^{-1}(x)$.