Exercise 1 (1.7: 4)**.** *Let $F(u,v) = (u^2 - v^2, 2uv)$. Find a formula for the Jacobian matrix of F at all points, and deduce that $F_{*\mathbf{p}}$ is a linear isomorphism at every point of $\mathbb{R}^2$ except the origin.*

The Jacobian matrix of F at a point $\mathbf{p} = (p_1, p_2)$ is

$$\begin{pmatrix} \frac{\partial f_1}{\partial u}(\mathbf{p}) & \frac{\partial f_1}{\partial v}(\mathbf{p}) \\ \frac{\partial f_2}{\partial u}(\mathbf{p}) & \frac{\partial f_2}{\partial v}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} 2p_1 & -2p_2 \\ 2p_2 & 2p_1 \end{pmatrix}.$$

We already know that $F_{*\mathbf{p}}$ is linear, so it is a vector space homomorphism. If $\mathbf{p} = \mathbf{0}$, then the Jacobian has rank 0; however, if $\mathbf{p} \neq \mathbf{0}$, then the Jacobian reduces to I_2 by Gaussian elimination, meaning that it has full rank. Thus when $\mathbf{p}$ is nonzero, $F_{*\mathbf{p}}$ is one-to-one. Since $F_{*\mathbf{p}}$ is a linear map between vector spaces that are both dimension 2 ($T_{\mathbf{p}}(\mathbb{R}^2)$ and $T_{F(\mathbf{p})}(\mathbb{R}^2)$ are both isomorphic to $\mathbb{R}^2$), it is automatically also onto. Thus $F_{*\mathbf{p}}$ is linear isomorphism at all points other than the origin.

Exercise 2 (2.1: 3)**.** *Prove that the tangent vectors*

$$\mathbf{e}_1 = \frac{(1,2,1)}{\sqrt{6}}, \mathbf{e}_2 = \frac{(-2,0,2)}{\sqrt{8}}, \mathbf{e}_3 = \frac{(1,-1,1)}{\sqrt{3}}$$

constitute a frame. Express $\mathbf{v} = (6,1,-1)$ as a linear combination of these vectors.

Since $\|\mathbf{e}_1\| = \sqrt{1+4+1}/\sqrt{6} = 1$, $\|\mathbf{e}_2\| = \sqrt{4+0+4}/\sqrt{8} = 1$, and $\|\mathbf{e}_3\| = \sqrt{1+1+1}/\sqrt{3} = 1$, all three vectors are unit vectors. Additionally, their dot products are

$$\begin{aligned} \mathbf{e}_1 \cdot \mathbf{e}_2 &= \frac{-2+0+2}{\sqrt{6}\sqrt{8}} = 0 \\ \mathbf{e}_1 \cdot \mathbf{e}_3 &= \frac{1-2+1}{\sqrt{6}\sqrt{3}} = 0 \\ \mathbf{e}_2 \cdot \mathbf{e}_3 &= \frac{-2+0+2}{\sqrt{8}\sqrt{3}} = 0, \end{aligned}$$

so they are mutually orthogonal. Thus $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ forms a frame.

We can express $\mathbf{v}$ as a linear combination of this frame by

$$\begin{aligned} \mathbf{v} &= \frac{7\sqrt{6}}{6}\mathbf{e}_1 - \frac{7\sqrt{8}}{4}\mathbf{e}_2 + \frac{4\sqrt{3}}{3}\mathbf{e}_3 \\ &= \left(\frac{7}{6}, \frac{14}{6}, \frac{7}{6}\right) - \left(-\frac{14}{4}, 0, \frac{14}{4}\right) + \left(\frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right) \\ &= (6, 1, -1). \end{aligned}$$

Exercise 3 (2.1: 9)**.** *Prove, using ε-neighborhoods, that each of the following subsets of $\mathbb{R}^3$ is open:*

a. All points $\mathbf{p}$ such that $\|\mathbf{p}\| < 1$.

b. All $\mathbf{p}$ such that $p_3 > 0$.

a. Let A denote the set of all points with norm less than 1, and let $\mathbf{p} \in A$. Then $\|\mathbf{p}\| = d$ for some $d < 1$. We claim that the ball $B(\mathbf{p}, 1-d)$ is contained in A. Let $\mathbf{q} \in B(\mathbf{p}, 1-d)$, then

$$\|\mathbf{q}\| = \|\mathbf{q} - \mathbf{p} + \mathbf{p}\| \leq \|\mathbf{q} - \mathbf{p}\| + \|\mathbf{p}\| < 1 - d + d = 1.$$

Thus $B(\mathbf{p}, 1-d) \subset A$. Since $\mathbf{p}$ was arbitrary, this shows that A is open.

b. Let B denote the set of all points whose 3rd coordinate is positive. Let $\mathbf{p} \in B$, then $p_3 = d > 0$. We claim that the ball $B(\mathbf{p}, d)$ is contained in B. Let $\mathbf{q} \in B(\mathbf{p}, d)$, then

$$|q_3 - p_3| \leq \|\mathbf{q} - \mathbf{p}\| < d,$$

so $q_3 > 0$. Since $\mathbf{p}$ was arbitrary, this shows that B is open.

Exercise 4 (2.3: 1)**.** *Compute the* ***Frenet apparatus*** κ, τ, T, N, B *of the unit-speed curve*

$$\beta(s) = \left(\frac{4}{5}\cos s, 1 - \sin s, -\frac{3}{5}\cos s\right).$$

Show that this curve is a circle; find its center and radius.

The tangent vector field is

$$T(s) = \beta'(s) = \left(-\frac{4}{5}\sin s, -\cos s, \frac{3}{5}\sin s\right).$$

The curvative is then

$$\kappa(s) = \|T'(s)\| = \frac{16}{25}\cos^2 s + \sin^2 s + \frac{9}{25}\cos^2 s = 1.$$

The principal normal vector field is

$$N(s) = T'(s)/\kappa(s) = T'(s) = \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right).$$

The binormal vector field is

$$\begin{aligned} B = T \times N &= \begin{vmatrix} U_1 & U_2 & U_3 \\ -4/5\ \sin & -\cos & 3/5\ \sin \\ -4/5 \cos & \sin & 3/5\ \cos \end{vmatrix} \\ &= \left(-\frac{3}{5}\cos^2 - \frac{3}{5}\sin^2 \right) U_1 + \left(-\frac{4}{5}\sin^2 - \frac{4}{5}\cos^2 \right) U_3 \\ &= \left(-\frac{3}{5}, 0, -\frac{4}{5} \right). \end{aligned}$$

Then since $B' = -\tau N$, $B' = 0$, and N is not everywhere 0, the torsion τ must be 0.

Since κ is a positive constant and $\tau = 0$, β is part of a circle of radius $1/\kappa = 1$. Its center is

$$\beta(s) + \frac{1}{\kappa(s)} N(s) = \beta(s) + N(s) = (0, 1, 0).$$

Exercise 5 (2.3: 5)**.** *If A is a vector field $\tau T + \kappa B$ on a unit-speed curve β, show that the Frenet formulas become*

$$\begin{aligned} T' &= A \times T, \\ N' &= A \times N, \\ B' &= A \times B. \end{aligned}$$

Since $N = B \times T$,

$$A \times T = (T \times T) + \kappa(B \times T) = 0 + \kappa N = T'.$$

Since $B = T \times N$, $T = N \times B$, and the cross product is antisymmetric,

$$A \times N = \tau(T \times N) + \kappa(B \times N) = \tau B - \kappa T = N'.$$

Finally, since $N = B \times T = -T \times B$,

$$A \times B = \tau(T \times B) + \kappa(B \times B) = -\tau N = B'.$$

Exercise 6 (2.3: 6)**.** *A unit-speed parameterization of a circle may be written*

$$\gamma(s) = \mathbf{c} + r \cos \frac{s}{r} \mathbf{e}_1 + r \sin \frac{s}{r} \mathbf{e}_2,$$

where $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.

If β is a unit-speed curve with $\kappa(0) > 0$, prove that there is one and only one circle γ that approximates β near $\beta(0)$ in the sense that

$$\gamma(0) = \beta(0), \quad \gamma'(0) = \beta'(0), \quad \textit{and } \gamma''(0) = \beta''(0).$$

Show that γ lies in the osculating plane of β at $\beta(0)$ and find its center $\mathbf{c}$ *and radius* r.

Assuming that γ matches β and its first two derivatives at $s = 0$, we can take the derivative of γ and evaluate at $s = 0$ to get

$$\begin{aligned}\gamma'(s) &= -\sin\left(\frac{s}{r}\right)\mathbf{e}_1 + \cos\left(\frac{s}{r}\right)\mathbf{e}_2 \\ \gamma'(0) &= -\sin(0)\mathbf{e}_1 + \cos(0)\mathbf{e}_2 \\ T(0) = \beta'(0) &= \mathbf{e}_2,\end{aligned}$$

which gives us the first component of the osculating circle's frame. Differentiating again gives

$$\begin{aligned}\gamma''(s) &= -\frac{1}{r}\cos\left(\frac{s}{r}\right)\mathbf{e}_1 - \frac{1}{r}\sin\left(\frac{s}{r}\right)\mathbf{e}_2 \\ \gamma''(0) &= -\frac{1}{r}\cos(0)\mathbf{e}_1 - \frac{1}{r}\sin(0)\mathbf{e}_2 \\ \kappa(0)N(0) = T'(0) = \beta''(0) &= -\frac{1}{r}\mathbf{e}_1.\end{aligned}$$

Since $\kappa(s)$ is strictly greater than 0 for all s, this implies that $\mathbf{e}_1 = -N(0)$ and $r = 1/\kappa(0)$. Plugging these into the original form of the circle at $s = 0$ gives

$$\begin{aligned}\beta(0) = \gamma(0) &= \mathbf{c} - \frac{1}{\kappa(0)}\cos(0)N(0) + \frac{1}{\kappa(0)}\sin(0)T \\ &= \mathbf{c} - \frac{1}{\kappa(0)}N(0).\end{aligned}$$

Simply rearranging this shows that the center of the circle is $\mathbf{c} = \beta(0) + \frac{1}{\kappa(0)}N(0)$. We have found a circle that matches β and its first two derivatives at $s = 0$, and since it is defined totally in terms of the unique values $T(0), N(0)$, and $\kappa(0)$, we know the circle itself is unique. Finally, we note that since our circle is defined in terms of $N(0)$ and $T(0)$, it lies in the osculating plane of β at $\beta(0)$.

Exercise 7 (2.3: 7)**.** *If α and a reparameterization $\overline{\alpha} = \alpha(h)$ are both unit-speed curves, show that*

a. $h(s) = \pm s + s_0$ for some number s_0;

b. $\overline{T} = \pm T(h)$, $\overline{N} = N(h), \overline{\kappa} = \kappa(h)$, $\overline{\tau} = \tau(h)$, and $\overline{B} = \pm B(h)$,

where the sign ($\pm$) is the same as that in (a), and we assume $\kappa > 0$.

a. The speed of the reparameterization is $\|\overline{\alpha}'\| = \|\alpha'(h)\|\|h'\| = 1$, but we are also given that $\|\alpha\| = 1$, so it must be the case that $h' = \pm 1$. The only functions h that satisfy this everywhere are of the form $h(s) = \pm s + s_0$, since the added constant term does not affect the derivative.

b. These derivations are straightforward applications of the fact that $h' = \pm 1$. The unit tangent vector field is

$$\overline{T} = \overline{\alpha}' = \alpha'(h)h' = T(h)h' = \pm T(h).$$

Then the curvature is

$$\overline{\kappa} = \|\overline{T}'\| = \|\pm T'(h)h'\| = \|T'(h)\| = \kappa(h),$$

so the principal normal vector field is

$$\overline{N} = \frac{\overline{T}'}{\overline{\kappa}} = \frac{\pm T'(h)h'}{\kappa(h)} = \frac{T'(h)}{\kappa(h)} = N(h).$$

The binormal vector field is

$$\overline{B} = \overline{T} \times \overline{N} = \pm T(h) \times N(h) = \pm B(h),$$

which means the torsion is

$$\overline{\tau} = -\frac{\overline{B}'}{\overline{N}} = -\frac{\pm B'(h)h'}{N(h)} = -\frac{B'(h)}{N(h)} = \tau(h).$$

Exercise 8 (2.3: 10)**.** *Let α be a unit-speed curve with $\kappa > 0$, $\tau \neq 0$.*

a. If α lies on a sphere of center $\mathbf{c}$ and radius r, show that

$$\alpha - \mathbf{c} = -\rho N - \rho' \sigma B,$$

where $\rho = 1/\kappa$ and $\sigma = 1/\tau$. Thus $r^2 = \rho^2 + (\rho'\sigma)^2$.

b. Conversely, if $\rho^2 + (\rho'\sigma)^2$ has constant value r^2 and $\rho' \neq 0$, show that α lies on a sphere of radius r.

a. We can express any vector $\mathbf{v}$ in terms of the Frenet frame by

$$\mathbf{v} = (\mathbf{v} \cdot T)T + (\mathbf{v} \cdot N)N + (\mathbf{v} \cdot B)B,$$

so need to calculate each of these dot products for $\mathbf{v} = \alpha - \mathbf{c}$. Since α lies on a sphere with center $\mathbf{c}$ and radius r, we know

$$r^2 = \|\alpha - \mathbf{c}\|^2 = (\alpha - \mathbf{c}) \cdot (\alpha - \mathbf{c}).$$

We can differentiate this several times and use the Frenet formulas to derive our desired dot product expressions. Taking the first derivative yields

$$\begin{aligned} 0 &= 2(\alpha - \mathbf{c})' \cdot (\alpha - \mathbf{c}) \\ &= 2T \cdot (\alpha - c), \end{aligned}$$

from which we get $T\cdot(\alpha-\mathbf{c})=0$. Differentiating this and using the Frenet formula for T' gives

$$\begin{aligned}0&=T'\cdot(\alpha-\mathbf{c})+T\cdot T\\&=\kappa N\cdot(\alpha-\mathbf{c})+1,\end{aligned}$$

from which we get $N\cdot(\alpha-\mathbf{c})=-1/\kappa=-\rho$. Differentiating this and using the Frenet formula for N' gives

$$\begin{aligned}0&=\kappa' N\cdot(\alpha-\mathbf{c})+kN'\cdot(\alpha-\mathbf{c})+\kappa N\cdot T\\&=-\kappa'\frac{1}{\kappa}+\kappa(\tau B-\kappa T)\cdot(\alpha-\mathbf{c})+0.\end{aligned}$$

using the just-derived fact that T and $(\alpha-\mathbf{c})$ are orthogonal, this simplifies to

$$0=-\frac{\kappa'}{\kappa}+\kappa\tau B\cdot(\alpha-\mathbf{c})$$

which can be rearranged into

$$B\cdot(\alpha-\mathbf{c})=\frac{\kappa'}{\kappa^2}\frac{1}{\tau}=-\left(\frac{1}{\kappa}\right)'\frac{1}{\tau}=-\rho'\sigma.$$

Now that we've calculated each of the dot products that we needed, we can write $\alpha-\mathbf{c}$ in terms of the Frenet frame as

$$\alpha-\mathbf{c}=-\rho N-\rho'\sigma B.$$

b. We are given that $\rho^2+(\rho'\sigma)^2=r^2$, so we can differentiate this to gain more information:

$$\begin{aligned}2\rho\rho'+2\rho'\sigma[\rho'\sigma'+\rho''\sigma]0\\ \rho'\sigma'+\rho''\sigma&=-\frac{\rho}{\sigma}.\end{aligned}$$

We can use this identity to show that the curve $\gamma=\alpha+\rho N+\rho'\sigma B$ is constant. Differentiating γ and substituting in the Frenet formulas yields

$$\begin{aligned}\gamma'&=T+\rho' N+\rho N'+\rho''\sigma B+\rho'\sigma' B+\rho'\sigma B'\\&=[1-\rho\kappa]T+[\rho'-\rho'\sigma\tau]N+[\rho\tau+\rho''\sigma+\rho'\sigma']B\\&=[1-1]T+[\rho'-\rho']N+[\rho/\sigma-\rho/\sigma]B\\&=0.\end{aligned}$$

Thus γ is a constant curve, i.e. a point, which we can choose to call $\mathbf{c}$. Then

$$\|\alpha-\mathbf{c}\|^2=\|-\rho N+\rho'\sigma B\|^2=\rho^2+(\rho'\sigma)^2=r^2,$$

so α lies on the sphere centered at $\mathbf{c}$ of radius r.

Exercise 9 (2.4: 4)**.** *Show that the curvature of a regular curve in* $\mathbb{R}^3$ *is given by*

$$\kappa^2 v^4 = \|\alpha''\|^2 - \left(\frac{dv}{dt}\right)^2.$$

Since $\alpha'' = \frac{dv}{dt}T + \kappa v^2 N$, we have

$$\begin{aligned}\|\alpha''\|^2 - \left(\frac{dv}{dt}\right)^2 &= \left\|\frac{dv}{dt}T + \kappa v^2 N\right\| - \left(\frac{dv}{dt}\right)^2 \\ &= \left(\frac{dv}{dt}\right)^2 + \kappa^2 v^4 - \left(\frac{dv}{dt}\right)^2 \\ &= \kappa_{62} v^4.\end{aligned}$$

Exercise 10 (2.4: 16)**.** *Let*

$$f(t) = \begin{cases} 0 & t \leq 0, \\ e^{-1/t^2} & t > 0 \end{cases}$$

and let

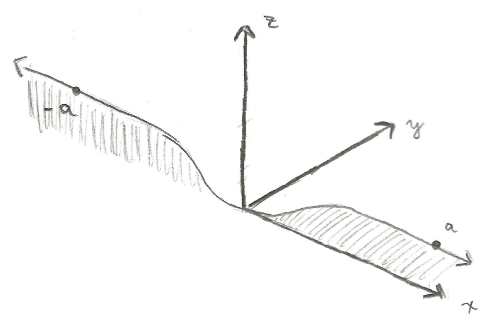
$$\alpha(t) = (t, f(t), f(-t)).$$

a. *Sketch* α *on an interval* $-a \leq t \leq a$.

b. *Show that the curvature of* α *is zero only at* $t = 0$.

c. *What are the osculating planes of* α *for* $t < 0$ *and* $t > 0$*?*

a. Below is a sketch of α over the interval $-a \leq t \leq a$. As I've tried to indicate with shading, the portion of the curve with $t > 0$ lies entirely in the $x - y$ plane, and the portion of the curve with $t < 0$ lies entirely in the $x - z$ plane.



b. The curvature can be calculated as $\kappa = \|\alpha' \times \alpha''\|/\|\alpha'\|^3$. Since

$$\|\alpha'(t)\| = \sqrt{1 + (f'(t))^2 + (f'(-t))^2} \geq \sqrt{1} = 1,$$

there can never be division by 0. Thus this question reduces to showing that $\|\alpha' \times \alpha''\| = 0$ only when $t = 0$.

We have

$$\begin{aligned} \alpha'(t) \times \alpha''(t) &= \begin{vmatrix} U_1 & U_2 & U_3 \\ 1 & f'(t) & -f'(-t) \\ 0 & f''(t) & f''(-t) \end{vmatrix} \\ &= (f'(t)f''(-t) + f'(-t)f''(t))\, U_1 - f''(-t)U_2 + f''(t)U_3, \end{aligned}$$

which has norm

$$\|\alpha'(t) \times \alpha''(t)\| = \sqrt{[f'(t)f''(-t) + f'(-t)f''(t)]^2 + f''(-t)^2 + f''(t)^2}.$$

We calculate that when $t > 0$,

$$\begin{aligned} f'(t) &= 2t^{-3}e^{-1/t^2} \\ f''(t) &= e^{-1/t^2}(4t^{-6} - et^{-4}). \end{aligned}$$

When $t \leq 0$, both f' and f'' evaluate to 0. The important part of these calculations is not the formulas themselves, but rather the fact that if t is strictly positive, then $f'(t)$ and $f''(t)$ are both nonzero.

Now for $t = 0$, every term in norm becomes 0, so the norm overall is 0. If $t > 0$, then $f''(-t) = f'(-t) = 0$, so the norm reduces to $|f''(t)|$. Similarly, when $t < 0$, the norm reduces to $|f''(-t)|$. In these latter two cases, t and $-t$ are both strictly positive, so f'' is nonzero. Thus the curvature is zero only when $t = 0$.

c. The osculating plane is spanned by vectors parallel to T and N, so we find such vectors by calculating α' (parallel to T) and α'' (parallel to N). We can also describe the planes with a single vector that is orthogonal to the plane.

When $t > 0$, we have

$$\begin{aligned} \alpha' &= (1, f', 0) \\ \alpha'' &= (0, f'', 0), \end{aligned}$$

so these two vectors span the osculating plane at $\alpha(t)$. Since neither vector has a nonzero third component, the osculating plane is clearly perpendicular to $(0, 0, 1)$, i.e. it is the $x - y$ plane.

When $t < 0$, we have

$$\begin{aligned} \alpha' &= (1, 0, -f') \\ \alpha'' &= (0, 0, -f''). \end{aligned}$$

With similar reasoning as before, the osculating plane is perpendicular to $(0, 1, 0)$, i.e. it is the $x - z$ plane.