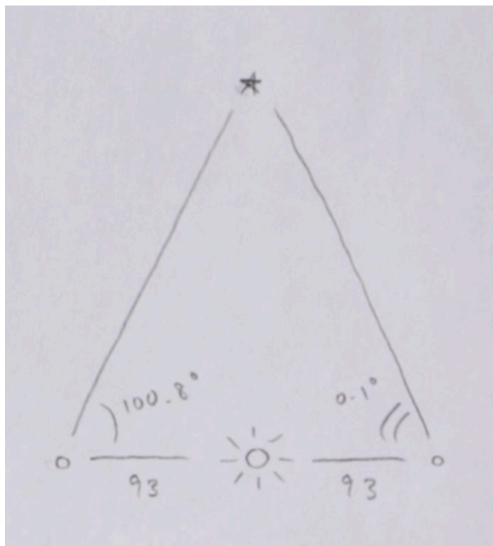
Exercise 1 (1.1)**.** Distance between earth and star.

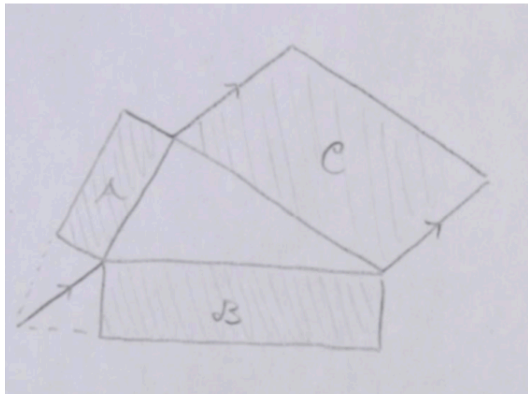
Assuming Euclidean geometry, the angles in the below diagram sum to $180\,^\circ$. Thus the unmarked angle is $0.1\,^\circ$. Then by the law of sines,

$$\frac{a}{\sin(100.8\,^\circ)} = \frac{b}{\sin(79.1\,^\circ)} = \frac{186}{\sin(0.1\,^\circ)}.$$

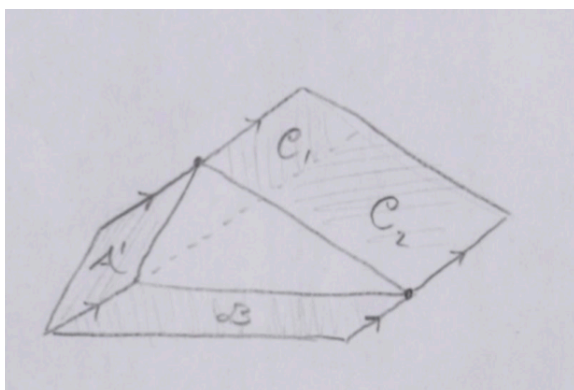
Solving yields $a \approx 104682$ and $b \approx 104647$. Everything here was done in terms of millions of miles, so the distance between the star and the earth is approximately 100,000,000,000 miles.



Exercise 2 (1.7)**.** Pappus' Variation on the Pythagorean Theorem.



Since a sheared image of a parallelogram has the same base and height, it has the same area, and thus any shear preserves area. Thus we can shear both $\mathcal{A}$ and $\mathcal{B}$ onto the bottom-left copy of $\mathbf{v}$ to get the following image.



Now shear $\mathcal{A}'$ onto $\mathcal{C}_1$ and shear $\mathcal{B}'$ onto $\mathcal{C}_2$. Since we have completely filled $\mathcal{C}$ with parallelograms of the same area as $\mathcal{A}$ and $\mathcal{B}$, we have

$$\text{Area}(\mathcal{A}) + \text{Area}(\mathcal{B}) = \text{Area}(\mathcal{C}).$$

Exercise 3 (1.15)**.** Isometry with 2 fixed points is either the identity or a reflection.

Suppose f is an isometry with fixed points P, Q.

All points in the line through P and Q are also fixed points: Let R be on the line ℓ through P and Q. Draw two circles: one at P with radius $|PR|$ and the second at Q with radius $|QR|$. By lemma 1.3.2, since R is on ℓ, these two circles intersect at only one point (R itself). But then since f preserves distances, this singular point is the only possible destination for R, i.e. $f(R) = R$.

Either identity or reflection: Now we show that f must be either the identity map or a reflection through ℓ. Let R be any point not on ℓ. Again we draw two circles, one at P with radius $|PR|$ and the second at Q with radius $|QR|$. Since R is not on ℓ, these circles intersect at two points (one of which must be R). Thus f can either map R to itself or to that second point R'.

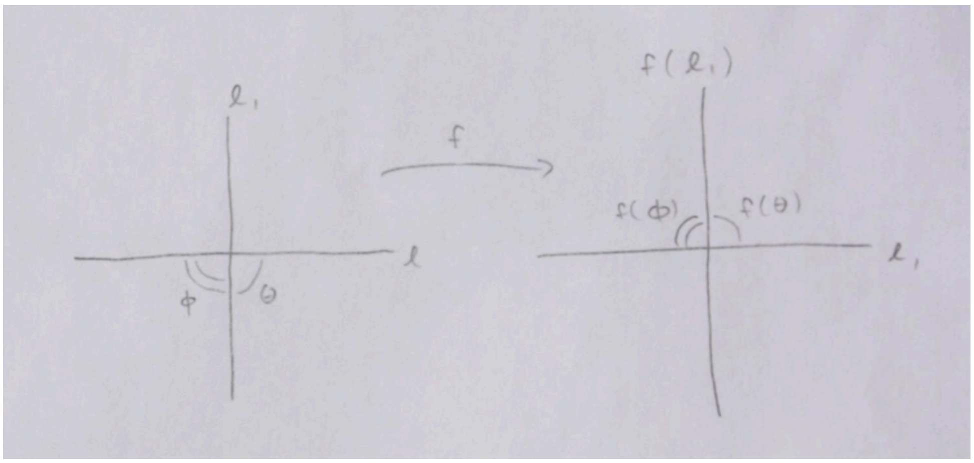
Now fix a point $S \neq R$ not on ℓ. This point S has a similar situation, in that it can either be mapped to itself or to one other point S'. We claim that R is a fixed point if and only if S is a fixed point.

- Suppose R is a fixed point. If S is mapped to S', then its distance to R is different, contradicting that f is an isometry.
- Suppose S is not a fixed point, then by a similar argument, f must map R to R' in order to preserve distance.

Thus in one case, all of $\mathbb{R}^2 - \ell$ is mapped to itself, i.e. $f = \mathrm{id}$. In the other case, no point in $\mathbb{R}^2 - \ell$ is fixed, i.e. f is a reflection.

Exercise 4 (1.17)**.** If $\ell_1 \neq \ell$ is sent to itself under a reflection through ℓ, then ℓ_1 and ℓ intersect at right angles.

Suppose f is the reflection through ℓ, and fix an angle θ at the intersection of ℓ and ℓ_1. Since isometries preserve angles, $f(\theta)$, one of the adjacent angles of θ, is congruent to θ. Thus ℓ and ℓ_1 intersect at right angles.



Exercise 5 (1.22)**.** Show that the interior angles in a quadrilateral sum to $360\,^\circ$. Generalize this to n-gons.

We claim that for any n-gon, the sum of the interior angles is

$$(n-2)\cdot 180\,^\circ\,.$$

We begin with the simple case of a quadrilateral.

Suppose we have a quadrilateral $ABCD$, then we can decompose this into two triangles ABD and BCD. Since the interior angles of a triangle sum to $180\,^\circ$, the interior angles of $ABCD$ must sum to $2\cdot 180\,^\circ = 360\,^\circ$. Note that this satisfies the original claim.

We now extend this result through induction. Suppose the hypothesis holds for all n-gons, then we must show it holds for all $(n+1)$-gons. Let $X_1\cdots X_{n+1}$ is an $(n+1)$-gon with points labeled clockwise, then we can decompose it into the n-gon $X_1\cdots X_n$ and the triangle $X_nX_{n+1}X_1$. By our inductive hypothesis and the fact that the interior angles of a triangles sum to $180\,^\circ$, the sum of our $(n+1)$-gon's interior angles is

$$(n-2)\cdot 180\,^\circ + 180\,^\circ = ((n+1)-2)\cdot 180\,^\circ\,.$$

Thus all n-gons satisfy the original claim.

Exercise 6 (1.23)**.** What is the sum of the exterior angles of an n-gon?

Let E_i denote the i-th exterior angle of our n-gon. By definition, it is adjacent to the i-th interior angle I_i. By the previous exercise, the sum of all the E_i is

$$\sum_{i=1}^{n} E_i = \sum_{i=1}^{n}(180\,^\circ - I_i) = n \cdot 180\,^\circ - (n-2) \cdot 180\,^\circ = 360\,^\circ\,.$$

Thus the sum of the exterior angles of all n-gons is $360\,^\circ$.

Exercise 7 (1.28)**.** If ABC is a right inscribed angle, then AC is a diameter.

Suppose ABC is a right inscribed angle in a circle of origin O and radius r as pictured below. Then by the Star Trek Lemma, the angle AOC is $2 \cdot 90\,^\circ = 180\,^\circ$. Then AC is a straight line through the center of the circle of length $2r$, i.e. a diameter.