

0.1 CHAIN COMPLEXES

Definition 1. A **chain complex** is a sequence of abelian group homomorphisms
(change to modules?)

$$\cdots \xrightarrow{d_{i+2}} C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all i .

We can also consider **cochain complexes**, which are the same except that the maps take you up a level instead of down.

$$\cdots \xleftarrow{d_{i+1}} C_{i+1} \xleftarrow{d_i} C_i \xleftarrow{d_{i-1}} C_{i-1} \xleftarrow{d_{i-2}} \cdots$$

The map d_i is the **boundary operator**, as it is a generalization of the geometric concept of a boundary (note $d^2 = 0$). Thus an element in the image of d is a **boundary**. Since usual geometric cycles have no boundary, we call the elements of the kernel of d **cycles**.

Example 1. Chain complexes generalize the concept of boundaries to objects that don't necessarily have clear cyclic geometric properties. Let $\Omega_n(M)$ denote the space of differential n -forms on a manifold M , then we have a cochain complex

$$\Omega_0(M) \xrightarrow{d} \Omega_1(M) \xrightarrow{d} \Omega_2(M) \xrightarrow{d} \cdots$$

where d is the exterior derivative. From this we see that the cycles of $\Omega_0(M)$ (the space of differentiable functions on M) are the constant functions.

A map $f : C \rightarrow D$ between chain complexes is a sequence of maps

$$f_i : C_i \rightarrow D_i$$

that respect the boundary operators. This means the following diagram commutes.

$$\begin{array}{ccc} C_i & \xrightarrow{d_C} & C_{i-1} \\ f_i \downarrow & & \downarrow f_{i-1} \\ D_i & \xrightarrow{d_D} & D_{i-1} \end{array}$$

Definition 2. The n -th **homology group** of a chain complex C is

$$H_n(C) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

Similarly, the n -th **cohomology group** is $H^n(C) = \ker d_{n+1} / \operatorname{im} d_n$. In both cases the quotient represents the n -th dimensional holes in the complex, as it is the cycles that do not arise as boundaries of higher dimensional objects.

A chain complex C is exact if and only if $H_n(C)$ is trivial for all n .

Given a map $f : C \rightarrow D$ of chain complexes, the commutativity of the above diagram shows that f sends cycles to cycles and boundaries to boundaries, thus inducing a map $H_n(C) \rightarrow H_n(D)$.

Definition 3. A **quasi-isomorphism** is a map $f : C \rightarrow D$ of chain complexes where the induced map $H_n(C) \rightarrow H_n(D)$ is an isomorphism.