Problems completed: All.

Exercise 1 (§51 #1)**.** If $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Collaborators: None.

Suppose $h \simeq h'$ via H and $k \simeq k'$ via K, then we claim that $k \circ h \simeq k' \circ h'$ via the map

$$\begin{aligned} F : X \times I &\to Z \\ (x, t) &\mapsto K(H(x, t), t). \end{aligned}$$

At $t = 0$, $F(x, 0) = K(H(x, 0), 0) = K(h(x), 0) = (k \circ h)(x)$. Similarly, at $t = 1$, $F(x, 1) = K(H(x, 1), 1) = K(h'(x), 1) = (k' \circ h')(x)$. Finally, F is continuous since it is the composition of continuous functions: Let $H' = (H, \pi_2)$, then H' is continuous and $F = K \circ H'$.

Exercise 2 (§51 #3)**.** X is **contractible** if i_X is nulhomotopic.

a. $I = [0,1]$ and $\mathbb{R}$ are contractible.

b. A contractible space is path connected.

c. If Y is contractible, then for any X, the set $[X,Y]$ (the set of homotopy classes of maps of X into Y) has a single element.

d. If X is contractible and Y is path connected, then $[X,Y]$ has a single element.

Collaborators: None.

a. If a space is convex, then any two paths are homotopic via the straight line homotopy. Since I and $\mathbb{R}$ are both convex, they are both contractible.

b. Suppose X is contractible and $x, y \in X$. Since X is contractible, i_X and some constant function $\tilde{f}(x) = c$ are homotopic via some F. Since F is continuous by definition,

$$f_x \cdot F(x, \cdot)$$
$$f_y \cdot F(y, \cdot)$$

are both continuous. Then f_x is a path from x to c, and f_y is a path from y to c. Then $f : [0,2] \to X$ given by

$$f(t) \doteq \begin{cases} f_x(t) & t \leq 1 \\ f_y(2-t) & t \geq 1 \end{cases}$$

is a path from x to y. Thus X is path connected.

c. Since Y is contractible, i_Y and some constant function $y \mapsto c$ are homotopic via some G. Let $f : X \to Y$ be arbitrary, then $F(x,t) = G(f(x),t)$ is a homotopy from f to $\tilde{f}$, where $\tilde{f}(x) = c$. Since f was arbitrary, this means that all maps from X to Y are homotopic to $\tilde{f}$. Since homotopy is an equivalence relation, this means all such maps are homotpic to each other, i.e. $[X,Y]$ has one element.

d. Fix $y \in Y$. Since X is contractible, i_X and some constant function $x \mapsto c$ are homotopic via some G. Then for all maps $f : X \to Y$, the map $F(x,t) = f(G(x,t))$ is a homotopy from f to the constant function $x \mapsto f(c)$. Now since Y is path connected, there is a path γ from $f(c)$ to y. Then $H(x,t) = \gamma(t)$ is a homotopy from $x \mapsto f(c)$ to $x \mapsto y$. Composing these two homotopies shows $f \simeq x \mapsto y$. But since f was arbitrary, this means all maps from X to Y are homotopic to the same constant function. Since homotopy is an equivalence relation, this means $[X,Y]$ has one element.