Exercise 1. 7.3: 1.

a. Since $E_i = vU_i$, the dual frame is given by

$$\theta_i(V) = \langle E_i, V \rangle = \langle vU_i, V \rangle = \frac{U_i \cdot V}{v},$$

which forces $\theta_1 = du/v, \theta_2 = dv/v$. Then by the first structural equations,

$$\begin{aligned} \frac{1}{v^2}\, du \wedge dv = d\theta_1 &= \omega_{12} \wedge \theta_2 = \omega_{12} \wedge dv/v, \\ 0 = d\theta_2 &= \omega_{21} \wedge \theta_1 = -\omega_{12} \wedge du/v, \end{aligned}$$

which implies $\omega_{12} = du/v = \theta_1$.

b. Since $\alpha = (r\cos t, r\sin t)$, our frame is given by

$$\begin{aligned} E_1 &= r\sin t\, U_1, \\ E_1 &= r\sin t\, U_2. \end{aligned}$$

W can then calculate the velocity of α by

$$\begin{aligned} \alpha &= r\cos t\, U_1 + r\sin t\, U_2, \\ \alpha' &= -r\sin t\, U_1 + r\cos t\, U_2 \\ &= -E_1 + \cot t\, E_2. \end{aligned}$$

Since $\omega_{12} = \theta_1$, the covariant derivative formula for curves becomes

$$\begin{aligned} \alpha'' = \nabla_{\alpha'}\alpha' &= [f_1' + f_2\omega_{21}(\alpha')]\, E_1 + [f_2' + f_1\omega_{12}(\alpha')]\, E_2 \\ &= [f_1' - f_1 f_2]\, E_1 + \left[f_2' + f_1^2\right] E_2 \\ &= \cot t\, E_1 + (1 - \csc^2 t) E_2 \\ &= \cot t\, E_1 - \cot^2 t\, E_2 \\ &= -\cot t\, \alpha'. \end{aligned}$$

c. Similarly, $v = st$, so $\beta' = (c, s)$ written in terms of our frame field is

$$\beta' = \frac{c}{st}\, E_1 + \frac{1}{t}\, E_2.$$

Then

$$\begin{aligned} \beta'' = \nabla_{\beta'}\beta' &= \left[-\frac{c}{st^2} - \frac{c}{st^2}\right] E_1 + \left[-\frac{1}{t^2} + \left(\frac{c}{st}\right)^2\right] E_2 \\ &= -\frac{2c}{st^2} E_1 + \frac{c^2 - s_2}{s^2 t^2} E_2. \end{aligned}$$

Now by our condition $c^2 + s^2 = 1$,

$$\langle \beta', \beta' \rangle = \frac{\beta' \cdot \beta'}{v^2} = \frac{c^2 + s^2}{s^2 t^2} = \frac{1}{s^2 t^2},$$

we get $\langle \beta', \beta' \rangle' = -2/s^2 t^3$. The same condition also gives

$$\langle \beta', \beta'' \rangle = \frac{-c^2 - s^2}{s^2 t^3 p} = \frac{-1}{s^2 t^3},$$

so $2 \langle \beta', \beta'' \rangle = -2/s^2 t^3 = \langle \beta', \beta' \rangle$, as desired.

Exercise 2. 7.3: 4.

Since $\mathbf{x}(u, v) = (r \cos v \cos u, r \cos v \sin u, r \sin v)$, our frame is

$$\begin{aligned} E_1 &= \frac{\mathbf{x}_u}{\sqrt{E}} = \frac{\mathbf{x}_u}{r \cos v}, \\ E_2 &= \frac{\mathbf{x}_v}{\sqrt{G}} = \frac{\mathbf{x}_v}{r}. \end{aligned}$$

The velocity of α is then

$$\begin{aligned} \alpha' &= (-r \cos v_0 \sin u, r \cos v_0 \cos u, 0) \\ &= r \cos v_0 \ E_1. \end{aligned}$$

To use the textbook's form of the covariant derivative formula for curves, we define $f_1 = r \cos v_0$, $f_2 = 0$. Note that $f_1' = f_2' = 0$. Then since the connection is worked out in the chapter as $\omega_{12} = \sin v \ du$, the covariant derivative becomes

$$\begin{aligned} \alpha'' &= f_1 \omega_{12}(\alpha') \ E_2 \\ &= r \cos v_0 \sin v_0 \ E_2. \end{aligned}$$

Exercise 3. 7.3: 5.

a. Suppose ω_{12} is the connection form on a frame field of $\mathcal{D}$. By the second structural equation,

$$d\omega_{12} = -K\theta_1 \wedge \theta_2 = -K\ dM.$$

Then by Stokes' Theorem,

$$\psi_\alpha = -\int_\alpha \omega_{12} = \iint_{\mathcal{D}} K\ dM.$$

b. Since

$$\frac{\psi_\alpha}{A(\mathcal{D})} = \frac{\iint_{\mathcal{D}} K\ dM}{\iint_{\mathcal{D}} dM},$$

as we take the limit as $\mathcal{D}$ is contracted to $\mathbf{p}$, we recover just $K(\mathbf{p})$.

Exercise 4. 7.4: 2.

At $t = 0$, $\gamma_{cv}(t) = \gamma_v(ct)$ since $c\mathbf{v}, \mathbf{v}$ have the same point of application. Also,

$$\gamma_v'(ct) = c\frac{d\gamma_v(ct)}{d(ct)} = c\gamma_v'(t) = c\mathbf{v}.$$

Since $\gamma_v(ct)$ and $\gamma_{cv}(t)$ agree at their initial position and velocity, and since initial position and velocity uniquely determine geodesics, these two geodesics must be the same.

Exercise 5. 7.4: 6.

In this question I use the fact that $F : \Sigma(r) \to P(r)$ is a local isometry. Since geodesics are isometric invariants, this means the geodesics of $P(r)$ are precisely the images of the geodesics of $\Sigma(r)$ under F.

a. The geodesics of $\Sigma(r)$ are the great circles of $\Sigma(r)$, which are simple closed curves of radius $2\pi r$. After identifying antipodal points, the radius of these curves becomes just πr. No self-intersections are introduced by F and the endpoints of the curves still equal the starting points, so they're still simple and closed.

b. Any two distinct points on the sphere that are not antipodal have a unique great circle C containing both of them. Then $F(C)$ is a geodesic in $P(r)$ containing both points (they're still distinct since they weren't antipodal and thus weren't identified by F), which induces a geodesic route between them.

c. On $\Sigma(r)$, two distinct geodesics meet at 2 antipodal points. Then in $P(r)$, since these two intersection points are identified, two distinct geodesics intersect at exactly 1 point.

Exercise 6. 7.5: 5.

Since $\mathbf{x}(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$, we can manually calculate $G = \mathbf{x}_v \cdot \mathbf{x}_v = (R + r\cos u)^2$. This means that the slant for the torus is

$$c = \sqrt{G}(a_1)\sin\phi = (R + r\cos a_1)\sin\phi.$$

a. If α is tangent to the top circle, then $a_1 = \phi = \pi/2$, so $\cos a_1 = \sin\phi = 1$, so $c = R$. Then by Theorem 5.3, we can't leave the region

$$\left\{G \geq c^2\right\} = \left\{(R + rc\cos u)^2 \geq R^2\right\} = \left\{-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}\right\}.$$

We know that the parellels of the circle (besides the inner and outer equators) are not geodesics, so α must leave the top circle, i.e. $\sin\phi$ decreases. But as $\sin\phi$ approaches 0, we approach the boundary of the restricted region, meaning that α has to level out and become tangent to the bottom circle. Then by a symmetric argument, α returns to the top circle, then repeats this cycle.

b. If α crosses the inner equator, then $a_1 = -\pi$, so the slant is

$$c = (R + r\cos a_1)\sin\phi = (R - r)\sin\phi < R - r,$$

where the last inequality follows from α not being tangent to the inner equator, i.e. $\sin\phi < 1$. Without loss of generality, suppose α is traveling upward, then $\sin\phi$ must decrease, so $R + r\cos a_1$ must increase, meaning that α approaches the top circle.

As α crosses the top circle (it cannot be tangent to it, as then c would equal R, which isn't less than $R - r$), $R + r\cos a_1$ continues to increase and $\sin\phi$ continues to decrease. Once α croses the outer equator, we're in a symmetric situation as to the one we started in, so α will continue to loop around the torus.

If α is a meridian, then it is just a closed loop that intersects the two equators at one point each. If α is not a meridiean, then it must twist around the torus.

c. Since $c = (R + r\cos a_1)\sin\phi$, then

$$c^2 = (R + r\cos a_1)^2\sin^2\phi \leq (R + r)^2,$$

so $|c| \leq R + r$. In particular, this means part (a) is the case $R - r < |c| < R + r$ and part (b) is the case $0 \leq |c| < R - r$. In the case $|c| = R - r$, we have the inner equator, and in the case $|c| = R + r$, we have the outer equator. This is all possible cases, so all geodesics besides the equators must cross the outer equator.

Exercise 7. 7.5: 8.

a. Since $\overline{iz+2} = 2 - \overline{z}i$,

$$\begin{aligned} F(z_0 &= \frac{z+2i}{iz+2} \\ &= \frac{z+2i}{iz+2}\frac{2-\overline{z}i}{2-\overline{z}i} \\ &= \frac{2(z-\overline{z}) + (4-|z|^2)i}{|iz+2|^2}. \end{aligned}$$

Thus the imaginary component of $F(z)$ is

$$\mathscr{I}F(z) = \frac{4-|z|^2}{|iz+2|^2}.$$

We also note that the real component is

$$\mathscr{R}F(z) = \frac{2(z-\overline{z})}{|iz+2|^2}.$$

b. We can manually solve for the inverse of F, which is

$$F^{-1}(z) = \frac{2(z-i)}{1-zi}.$$

We can check manually that $F(F^{-1}(z)) = F^{-1}(F(z)) = z$, so this is indeed the inverse. Since it's also differentiable, F is a diffeomorphism.

c. Since F is a diffeomorphism, it is necessarily regular. Then by §7.1 Exercise 7 (we did this in HW 8), $F = (f, g)$ is conformal if $f_u = g_v$ and $f_v = -g_u$ and has scale factor $|dF/dz|$.

We can write F in this form by $F = (\mathscr{I}F, \mathscr{R}F)$, then expanding into u and v components by $z = u + vi$, we get

$$f(u,v) = \frac{4u}{4-4v+v^2+u^2}, \quad g(u,v) = \frac{4-u^2-v^2}{4-4v+v^2+u^2}.$$

We can manually calculate all the partials, getting

$$\begin{aligned} f_u &= \frac{16-16v+4v^2-4u^2}{(4-4v+v^2+u^2)^2} = g_v, \\ f_v &= \frac{16u-8uv}{(4-4v+v^2+u^2)^2} = -g_u. \end{aligned}$$

Thus F is conformal, and its scale factor is

$$\lambda(z) = \left|\frac{dF}{dz}\right| = \left|\frac{(iz+2)-(z+2i)i}{(iz+2)^2}\right| = \left|\frac{4}{(iz+2)^2}\right| = \frac{4}{|iz+2|^2}.$$

d. For $\mathbf{v}, \mathbf{w}$ tangent to z, the metrics on H and P are

$$\begin{aligned}\langle \mathbf{v}, \mathbf{w} \rangle_H &= \frac{\mathbf{v} \cdot \mathbf{w}}{(1 - |z|^2/4)^2} = \frac{4^2(\mathbf{v} \cdot \mathbf{w})}{(4 - |z|^2)^2}, \\ \langle \mathbf{v}, \mathbf{w} \rangle_P &= \frac{\mathbf{v} \cdot \mathbf{w}}{(\mathscr{I} z)^2}.\end{aligned}$$

Since in HW 7 we proved that conformal maps preserve inner products up to λ^2,

$$\begin{aligned}\langle F_* \mathbf{v}, F_* \mathbf{w} \rangle_P &= \frac{F_* \mathbf{v} \cdot F_* \mathbf{w}}{(\mathscr{I} F(z))^2} \\ &= \frac{(4 - |z|^2)^2}{4^2} \frac{(|iz + 2|^2)^2}{(4 - |z|^2)^2} \langle F_* \mathbf{v}, F_* \mathbf{w} \rangle_H \\ &= \left(\frac{|iz + 2|^2}{4} \right)^2 \lambda^2(z) \langle \mathbf{v}, \mathbf{w} \rangle_H \\ &= \lambda^{-2}(z) \lambda^2(z) \langle \mathbf{v}, \mathbf{w} \rangle_H \\ &= \langle \mathbf{v}, \mathbf{w} \rangle_H .\end{aligned}$$

Thus F is an isometry.