MATH 531 HOMEWORK 6

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Scalar Multiplication. *Prove scalar multiplication is a continuous function from $\mathbb{R}\times\mathcal{V}\to\mathcal{V}$, for $\mathcal{V}$ a normed vector space and $\mathbb{R}\times\mathcal{V}$ equipped with the product metric space structure.*

Given open $U\subset\mathcal{V}$, we must show that $\cdot^{-1}(U)$ is open. Let $(k,v)\in\cdot^{-1}(U)$, i.e. $kv\in U$, then there is an $\varepsilon>0$ such that $D(kv,\varepsilon)\subset U$. If we find δ such that $\cdot(D((k,v),\delta))\subset D(kv,\varepsilon)$, then we are done. Since $\mathbb{R}\times\mathcal{V}$ is equipped with the product metric, if $(k',v')\in D((k,v),\delta)$, then

$$d_1((k,v),(k',v'))=|k-k'|+\|v-v'\|<\delta.$$

Since both of these terms are nonnegative, this implies $|k-k'|<\delta$ and $\|v-v'\|<\delta$. Then by the triangle inequality,

$$\begin{aligned}d_2(kv,k'v')&=\|kv-k'v'\|\\&=\|kn-k'v+k'v-k'v'\|\\&\leq\|v\||k-k'|+|k'|\|v-v'\|.\end{aligned}$$

Since $|k'|=|k'-k+k|\leq\delta+|k|$, this becomes

$$d_2(kv,k'v')<\|v\|\delta+(\delta+|k|)\delta.$$

Surely we can let $\delta\leq 1$, since δ need not be “optimal” in any sense. Then we have

$$d_2(kv,k'v')<\|v\|+|k|+\delta.$$

So if we take $\delta=\min\{1,\varepsilon-\|v\|-|k|\}$, then $\cdot(D((k,v),\delta))\subset D(kv,\varepsilon)$, which implies that $\cdot^{-1}(U)$ is open. Thus scalar multiplication is a continuous function on a normed vector space.

Page 182, 4.1.3. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be continuous. Show $A=\big\{(x,y)\in\mathbb{R}^2\mid 0\leq f(x,y)\leq 1\big\}$ is closed.*

Since f is continuous, we know that for any closed set $F\subset\mathbb{R}$, $f^{-1}(F)$ is closed in $\mathbb{R}^2$. Thus it suffices to find a closed set $B\subset\mathbb{R}$ such that $f^{-1}(B)=A$.

Let $B=\{z\in\mathbb{R}\mid 0\leq z\leq 1\}$, then we claim that B is closed in $\mathbb{R}$ and that $f^{-1}(B)=A$. Since $B^c=\{z<0\}\cup\{z>1\}$ is clearly open (it is the finite intersection of open sets), B is closed by definition. Also by definition, $f^{-1}(B)=\{(x,y)\mid f(x,y)\in B\}=\{(x,y)\mid 0\leq f(x,y)\leq 1\}=A$.

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Page 184, 4.2.4. *Let $A, B \subset \mathbb{R}$, and suppose $A \times B \subset \mathbb{R}^2$ is connected.*

(1) *Prove that A is connected.*
(2) *Generalize to metric spaces.*

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(1) Define $f : A \times B \to A$ by $f((a,b)) = a$. We claim that f is continuous. Let $U \subset A$ be open, then we must show $f^{-1}(U) = \{(a,b) \mid a \in U\}$ is also open. Let $(a,b) \in f^{-1}(U)$, and let $D(a,\varepsilon) \subset U$ (we know this ball exists for some $\varepsilon > 0$ since U is open). We now claim that $D((a,b),\varepsilon) \subset f^{-1}(U)$.

If $(a',b') \in D((a,b),\varepsilon)$, then since $A \times B$ is equipped with the product metric, $|a-a'|^2 \le |a-a'|^2 + |b-b'|^2 < \varepsilon^2$ This implies $a' \in D(a,\varepsilon) \subset U$, so $(a',b') \in f^{-1}(U)$. Thus $f^{-1}(U)$ is open and, subsequently, f is continuous. Since the image of a connected set under a continuous map is itself continuous, we have that A is continuous.

(2) To generalize this to metric spaces, note that the proof only uses a property of $\mathbb{R}^n$ when defining the ε-balls around a and (a,b). Let $A \times B$ be equipped with the product metric, d_A be the metric for A, and d_B be the metric for B. Then instead of

$$|a-a'|^2 \le |a-a'|^2 + |b-b'|^2 < \varepsilon^2,$$

we write

$$d_A(a,a') \le d_A(a,a') + d_B(b,b') < \varepsilon.$$

The rest of the proof is valid without any changes, so the result holds for general metric spaces.

Page 184, 4.2.5. *Let $A, B \subset \mathbb{R}$, and suppose $A \times B \subset \mathbb{R}^2$ is open. Must A be open?*

Yes. Let $a \in A$ and $b \in B$, then there is an $\varepsilon > 0$ such that $D((a,b),\varepsilon) \subset A \times B$. Consider $D(a,\varepsilon)$. For $a' \in D(a,\varepsilon)$, we have $|a'-a| + |b-b| = |a'-a| < \varepsilon$, which implies $(a',b) \in D((a,b),\varepsilon) \subset A \times B$. Thus $a' \in A$, so $D(a,\varepsilon) \subset A$, so A is open.

Page 191, 4.4.3. *Let $f : K \subset \mathbb{R}^n \to \mathbb{R}$ be continuous on a compact set K and let $M = \{x \in K \mid f(x)$ is the maximum of f on $K\}$. Show that M is a compact set.*

Since K is compact and f is continuous, K contains all x' such that $f(x') = \sup_x f(x)$. Thus $M = f^{-1}(\{\sup f(x)\})$. The set $\{\sup f(x)\}$ is a single point in $\mathbb{R}$, so it is closed. Since f is continuous, M must then be closed too. Furthermore, $M \subset K$ and K is bounded (since it is compact), so M must also be bounded. Then by the Heine-Borel theorem, M is compact in $\mathbb{R}^n$.

Page 193, 4.5.1. *What happens when you apply the method used in Example 4.5.4 to quadratic polynomials? To quintic polynomials?*

Let $f(x) = x^2 + 10$ be a quadratic polynomial. It is clear that $f(x)$ is always positive, so it need not have a real root. Thus the method from the example fails.

Let $f(x) = a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6$ for $a_1 > 0$. Then we can rewrite this as

$$f(x) = a_1x^5\left(1 + \frac{a_2}{a_1x} + \frac{a_3}{a_1x^2} + \frac{a_4}{a_1x^3} + \frac{a_5}{a_1x^4} + \frac{a_6}{a_1x^5}\right).$$

The sum in parentheses tends to 1 as $x \to \infty$, so $f(x) > 0$ for some large and positive x. Similarly, $f(x) < 0$ for some large and negative x. Since $\mathbb{R}$ is connected and f is continuous, by the intermediate value theorem we know that f has a real root.

In general, the largest exponent in a polynomial being odd guarantees that we can find points satisfying $f(x) < 0$ and $f(x) > 0$, in which case the method from the example is valid. As seen with the quadratic example, when the largest exponent is even, this is not always possible.

Page 193, 4.5.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Let $\Gamma = \{(x, f(x)) \mid x \in \mathbb{R}^n\}$ be the graph of f in $\mathbb{R}^n \times \mathbb{R}^m$. Prove that Γ is closed and connected. Generalize your result to metric spaces.*

Γ is the image of the map $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ defined by $g(x) = (x, f(x))$. We claim that g is continuous. Fix $x \in \mathbb{R}^n$ and $\varepsilon > 0$, then we wish to show that the preimage of $D((x, f(x)), \varepsilon)$ under g is open. Since f is continuous, for any $\varepsilon > 0$ we can find δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon/2$. Let $\delta' = \min\{\varepsilon/2, \delta\}$, then for any $y \in D(x, \delta')$ we have

$$|x - y| + |f(x) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\mathbb{R}^n \times \mathbb{R}^m$ is equipped with the product metric, this shows that $D(x, \delta')$ is a subset of $g^{-1}\left(D((x, f(x)), \varepsilon)\right)$. Thus g is continuous. Since g is continuous and $\mathbb{R}^n$ is connected, Γ must also be connected.

Now we show that Γ is closed. Let $\{(x_n, f(x_n)\}_{n=1}^{\infty}$ converge, i.e. $(x_n, f(x_n)) \to (x, y)$ for some $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. The map f is given to be continuous, so $\lim_{n\to\infty} f(x_n) = f(x)$. Since by assumption, $\lim_{n\to\infty} f(x_n) = y$, this implies $f(x) = y$. Thus our sequence converges to $(x, f(x))$, which clearly lies in Γ. This shows that Γ is closed.

The proof of connectedness relied on $\mathbb{R}^n$ being connected. If we replace $\mathbb{R}^n$ with any connected metric space, the result still holds. The proof of closedness did not rely on the structure of $\mathbb{R}^n$ at all, so this result holds in any metric space. This gives us a more general proposition:

Let X and Y be metric spaces, and let $f : X \to X \times Y$ be continuous. Then $\Gamma = \{(x, f(x)) \mid x \in X\}$ is closed. If X is connected, then Γ is connected.

Page 193, 4.5.3. *Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that f has a fixed point.*

Since $[0, 1]$ is connected and f is continuous, the intermediate value theorem holds. Define another continuous function $g : [0, 1] \to [-1, 1]$ by $g(x) = f(x) - x$, then note that $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. If $g(0) = 0$, then $f(0) = 0$ and we have a fixed point. Similarly, if $g(1) = 0$, then $f(1) = 1$ and we have a fixed point. So assuming that these trivial cases are false, we have $g(0) < 0 < g(1)$. Then by the intermediate value theorem, there is some $z \in [0, 1]$ such that $g(z) = f(z) - z = 0$, which implies $f(z) = z$, so we have a fixed point.

Page 194, 4.5.4. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that the range of f is a bounded closed interval.*

The interval $[a, b]$ is compact, so $f([a, b])$ contains its minimum and maximum, denoted by $f(c)$ and $f(d)$, respectively. Then since $[a, b]$ is also connected, by the intermediate value theorem, the image also contains all points between $f(c)$ and $f(d)$. Thus the image is $f([a, b]) = [f(c), f(d)]$, a closed and bounded interval in $\mathbb{R}$.

Page 194, 4.5.5. *Prove that there is no continuous map taking* $[0, 1]$ *onto* $(0, 1)$.

Let $f : [0, 1] \to (0, 1)$ be continuous. Then since $[0, 1]$ is compact, so is $f([0, 1]) \subset (0, 1)$, i.e. $f([0, 1]) = [a, b]$ for some $a > 0$, $b < 0$. But since $a > 0$, this means $a/2$, which is in $(0, 1)$, is not in $f([a, b])$. Thus f is not onto.