Topology

Point-set and introductory algebraic topology

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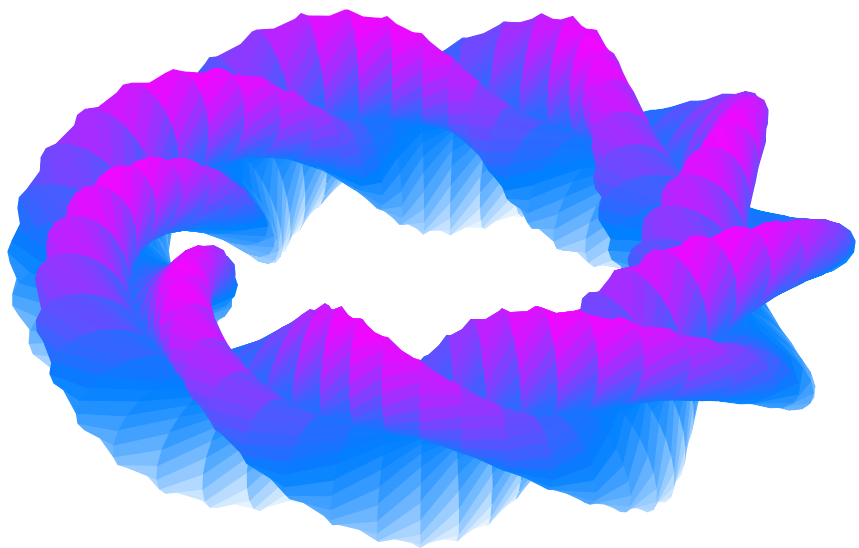

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Chapter 1

Topological Spaces

1.1 Topological Spaces

Need to redo structure of these notes b/c rn, there's stuff about continuity (in the product and initial topologies sections) that refers to continuous functions.

Definition 1. Let X be a set, then a **topology** on X is a collection $\mathscr{T}$ of subsets of X such that

1. $\varnothing, X \in \mathscr{T}$,
2. $\bigcup_{\alpha \in \mathscr{J}} U_\alpha \in \mathscr{T}$, and
3. $\bigcap_{i=1}^{N} U_i \in \mathscr{T}$.

Elements of a topology are called **open sets**.

Example 1.

1. "Indiscrete" topology: $\mathscr{T}_i = \{\varnothing, X\}$
2. "Discrete" topology: $\mathscr{T}_d = \{\text{all subsets of } X\}$

Definition 2. Let $\mathscr{T}, \mathscr{T}'$ be topologies on a set X, then $\mathscr{T}$ is **finer** than $\mathscr{T}'$ if $\mathscr{T}' \subset \mathscr{T}$. $\mathscr{T}$ is **coarser** than $\mathscr{T}'$ if $\mathscr{T} \subset \mathscr{T}'$. The notions of **strictly finer** and **strictly coarser** follow.

From this we see that "fine" is a notion of a large topology, and "coarse" is a notion of a small topology.

Example 2. The **lower limit topology** on $\mathbb{R}$ is given by the basis

$$\mathcal{B} = \{[a,b) \mid a < b\}.$$

It is strictly finer than the standard topology on $\mathbb{R}$: since $\bigcup_{n\in\mathbb{N}}[a+1/n, b) = (a,b)$, it contains the standard topology, but $[a,b)$ is not open in the standard topology, so it is strictly finer.

Example 3. Let X be any set, then the **finite complement topology** is defined

$$\mathcal{T}_f = \{U \subset X \mid X - U \text{ is finite}\} \cup \{\varnothing\},$$

where $X - U$ denotes the complement of U in X, i.e. $X \backslash U$. Checking that this is a topology boils down to just using DeMorgan's Laws.

1.2 Bases

Definition 3. Let $\mathcal{T}$ be a topoloy on X, and let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a **basis** for $\mathcal{T}$ if every open set of $\mathcal{T}$ can be written as the union of elements of $\mathcal{B}$.

Proposition 1. *Let $\mathcal{T}$ be a topology on X, and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if*

1. *$\mathcal{B} \subset \mathcal{T}$; and*
2. *for each $U \in \mathcal{T}$ and $p \in U$, there is a $B \in \mathcal{B}$ such that $p \in B \subset U$.*

Proof. The forward direction follows from every open set of $\mathcal{T}$ being the union of elements of $\mathcal{B}$. For the backward direction, since $p \in B_p \subset U$ for all $p \in U$, we have $U = \bigcup_{p\in U} B_p$, so every open set of $\mathcal{T}$ is the union of elements of $\mathcal{B}$. □

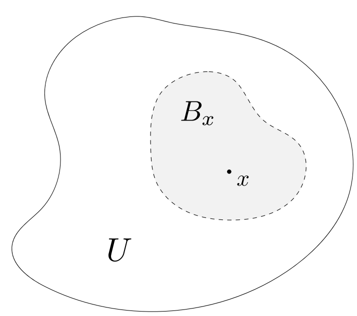

Figure 1.1: For any $U \in \mathcal{T}$, each $x \in U$ lies in some $B_x \in \mathcal{B}$ for $B_x \subset U$.

Not every set of subsets of X will generate a topology, so we need conditions for a collection $\mathscr{B}$ to be a basis for *any* topology.

Proposition 2. *Let $\mathscr{B}$ be a collection of subsets of X. Then $\mathscr{B}$ generates a topology if and only if*

1. $\bigcup_{B \in \mathscr{B}} = X$.
2. *given $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there is a $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.*

Proof. **Forward:** (1) X must be open, so X is the union of the elements of $\mathscr{B}$. (2) Since B_1 and B_2 are both open in the topology generated by $\mathscr{B}$, their intersection is, as well. Then since $\mathscr{B}$ is a basis for this topology, we can find a satisfactory B_3.

Backward: The topology generated by a set $\mathscr{B}$ is the collection of all unions of elements of $\mathscr{B}$. It is clear that $\varnothing$ is in it, and condition (1) implies that X is, as well. Arbitrary unions are in the topology by definition. Induction on condition (2) shows that the topology also contains finite intersections. □

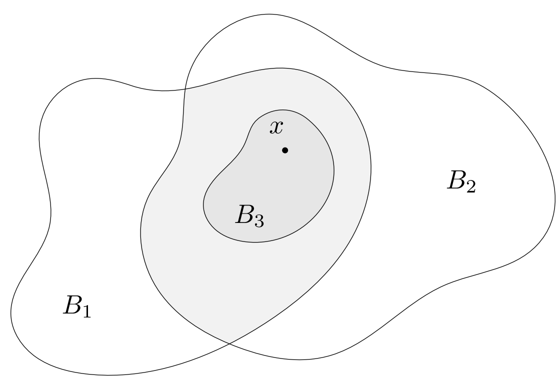

Figure 1.2: Condition (2) in Proposition 2.

Note 1. Since $\mathscr{B}$ exists independently from any topology, it doesn't make sense to describe its members as "open" until after we've generated a topology from it. Once we've done so, though, it should be clear that every basis element is open in the generated topology.

We can also get a notion of how relatively fine or coarse a topology is by using its basis.

Proposition 3. *Let $\mathcal{B}, \mathcal{B}'$ be bases for the topologies $\mathcal{T}, \mathcal{T}'$ on X, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for all $B \in \mathcal{B}$ and $x \in B$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subset \mathcal{B}$.*

Proof. First we show the backward implication. Let $U \in \mathcal{T}$, and let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is a $B \subset \mathcal{B}$ such that $x \in B \subset U$. By assumption, there is then a $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$. Thus $U \in \mathcal{T}'$, so $\mathcal{T}'$ is finer than $\mathcal{T}$.

Now we show the forward implication. Let $B \in \mathcal{B}$, and let $x \in B$, then $B \in \mathcal{T}$. By assmption, $\mathcal{T} \subset \mathcal{T}'$, so $B \in \mathcal{T}'$ as well. Then by the definition of a generated topology, there is a $B' \subset \mathcal{B}'$ such that $x \in B' \subset B$. $\square$

Proposition 4. *The topology generated by a basis is the smallest topology containing that basis.*

1.3 Subbases

Definition 4. A **subbasis** $\mathcal{S}$ for a topology $\mathcal{T}$ on X is a collection of subsets of X whose finite intersections form a basis for $\mathcal{T}$.

Subbases are easier to construct than bases, but the construction of a topology from a subbasis involves an extra step, namely the finite intersections. What we are doing is creating a basis $\mathcal{B}$ from $\mathcal{S}$ by taking finite intersections of the subbasis elements. Then we are taking $\mathcal{B}$ and constructing $\mathcal{T}$ by taking arbitrary unions, as is usual.

$$\mathcal{S} \xrightarrow{\bigcap_{i=1}^{N}} \mathcal{B} \xrightarrow{\bigcup_{\alpha \in \mathcal{J}}} \mathcal{T}$$

Figure 1.3: The process for constructing a topology using a subbasis $\mathcal{S}$.

Proposition 5. *Let $\mathcal{T}$ be a topology on X, and let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a subbasis for $\mathcal{T}$ if and only if*

1. *$\mathcal{S} \subset \mathcal{T}$; and*
2. *for each $U \in \mathcal{T}$ and $p \in U$, there is a finite intersection $\bigcap_{i=1}^{n} S_i$ of elements of $\mathcal{S}$ such that $p \in \bigcap_{i=1}^{n} S_i \subset U$.*

Proof. This follows from Proposition 1 (the analogue of this proposition for bases). When proving both directions, there's just an extra step to go from a genric basis element to a finite intersection of elements of $\mathcal{S}$. $\square$

Just as with bases, we have a condition for when an arbitrary collection of subsets of X can be a valid subbasis.

Proposition 6. *Let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ generates a topology if and only if $\mathcal{S}$ covers X.*

Proof. Sketch. □

Proposition 7. *The topology generated by a subbasis is the smallest topology containing that subbasis.*

1.4 The Subspace Topology

There is a natural way of a subset inheriting the topology of the set it lies in. The following definition is easily checked to actually be a topology.

Definition 5. Let $(X, \mathcal{T})$ be a topological space. If $Y \subset X$, then

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is the **subspace topology** on Y. With this topology, Y is called a **subspace** of X.

Proposition 8. *Let $\mathcal{B}$ be a basis for the topology of X, then*

$$\mathcal{B}_Y \doteq \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let $y \in U \cap Y$, where U is open in X. There exists $B \in \mathcal{B}$ such that $y \in B \subset U$, so $y \in B \cap Y \subset U \cap Y$. □

Proposition 9. *Let Y be a subspace of X, and let U be open in Y and Y be open in X. Then U is open in X.*

Proof. U is open in Y, so $U = Y \cap V$ for some V open in X. Both sets Y and V are open in X, so their intersection U must be as well. □

1.5 The Initial Topology

Definition 6. Let X be a set and $\{Y_i\}_{i\in\mathcal{J}}$ a collection of topological spaces, and suppose we have functions $f_i : X \to Y_i$. The **initial topology** on X for these f_i is the coarsest topology on X such that each f_i is continuous.

Proposition 10. *The initial topology on X is generated by the subbasis*

$$\mathcal{S} = \left\{ f_i^{-1}(U) \mid i \in \mathcal{J}; U \text{ open in } Y_i \right\}.$$

Think about this...

This is a nice generalization of the subspace and product topologies. In the next section, we'll derive the product topology as the initial topology on a Cartesian product that makes the canonical projection continuous. The initial topology on a subset such that the inclusion function is continuous is actually the subsapce topology.

Example 4. Suppose $Y \subset X$, and consider the inclusion function $\iota : S \hookrightarrow X$. The initial topology is generated by

$$\left\{ i^{-1}(U) \mid U \text{ open in } X \right\} = \{ Y \cap U \mid U \text{ open in } X \},$$

but this is just the subspace topology.

Why bother with saying "generated" if it's equal? Are there counterexamples?

1.6 The Product Topology

It would be natural to define the product topology as

$$\mathcal{P} = \{ U \times V \mid U \text{ open in } X, V \text{ open in } Y \},$$

but this isn't enough to give a topology since you can construct examples where the union of elements in this set don't lie in the set.

This set *is*, however, perfectly valid as a basis, since $\bigcup_{U,V}(U \times V) = X \times Y$ and $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{P}$.

Definition 7. The topology generated by $\mathcal{P}$ is the **product topology** on $X \times Y$.

Proposition 11. *If $\mathcal{B}_X$ is a basis for X and $\mathcal{B}_Y$ is a basis for Y, then $\mathcal{B}_X \times \mathcal{B}_Y$ is a basis for the product topology.*

Proposition 12. *The product and subspace topologies "commute".*

Proof. It's straightforward to show that the product of two subspaces and the subspace of a product both have the same basis. ☐

Where to put the above stuff?

Definition 8. The **Cartesian product** of $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is the set

$$\prod_{\alpha\in\mathcal{A}} X_\alpha = \left\{ f : \mathcal{A} \to \bigcup_{\alpha\in\mathcal{A}} X_\alpha \;\middle|\; f(\alpha) \in X_\alpha \right\}.$$

Each function f represents a single "point" in the product.

Example 5. Suppose $\mathcal{A} = \{1, \ldots, n\}$ and $X_\alpha = \mathbb{R}$ for all α. Then each f in the Cartesian product is a function

$$f : \{1, \ldots, n\} \to \mathbb{R}.$$

Since there are only a finite number of X_α's, we can write each f as a tuple

$$(f(1), f(2), \ldots, f(n)).$$

Thus there is a clear bijection between $\prod_{\alpha=1}^n X_\alpha$ and $\mathbb{R}^n$.

Extending the product topology to the case of a general Cartesian product is tricky. Given $\prod_\alpha X_\alpha$, we could naively say that the topology on it should be given by a basis

$$\mathcal{B} = \left\{ \prod_\alpha B_\alpha \;\middle|\; B_\alpha \in \mathcal{B}_\alpha \right\},$$

where $\mathcal{B}_\alpha$ is a basis for just X_α. If we have a finite number of α's, this basis is just every possible ordered combination of basis elements from each X_α:

$$\begin{gathered} (B_{11}, B_{21}, \ldots, B_{n1}), \\ (B_{11}, B_{22}, \ldots, B_{n2}), \\ \vdots \\ (B_{11}, B_{22}, \ldots, B_{nn}), \\ \vdots \end{gathered}$$

The topology generated by this basis is the **box topology**, and although simple, ends up not being the best notion of a topology on infinite products because it's actually too fine. This ends up making some "obviously" continuous functions discontinuous.

Example 6. Define

$$\mathbb{R}^\infty = \prod_{i\in\mathbb{Z}^+} \mathbb{R},$$

then the function

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^\infty \\ x &\mapsto (x, x, \dots) \end{aligned}$$

seems like it should be continuous; however, if $\mathbb{R}^\infty$ has the box topology, then the preimage under f of the open set $U = \prod_{i\in\mathbb{Z}^+}(-1/i, 1/i)$ is $f^{-1}(U) = \{0\}$. This isn't open in $\mathbb{R}$, so f is discontinuous.

We want the product topology to, in a sense, be continuous in each of its components. Unlike the box topology, though, we don't want it to be *too* fine. The way we formalize this is by saying that we want to find the coarsest topology on $\prod X_\alpha$ such that the canonical projection

$$\begin{aligned} \pi_\beta : \prod_{\alpha\in\mathcal{A}} X_\alpha &\to X_\beta \\ (f : \mathcal{A} \to \bigcup X_\alpha) &\mapsto f(\beta) \end{aligned}$$

is continuous. This is just the initial topology on $\prod X_\alpha$ with respect to the projections.

Definition 9. The **product topology** is generated by the subbasis

$$\left\{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \text{ open in } X_\alpha\right\}.$$

The basis for the product topology is then of the form $\prod U_\alpha$, where only finitely many of the U_α satisfy $U_\alpha \neq X_\alpha$.

Proposition 13. *The function $f : Y \to \prod X_\alpha$ is continuous if and only if f_α is continuous for all α.*

Proof. If f is continuous, then $f_\alpha = \pi_\alpha \circ f$ is the composition of continuous functions and so is itself continuous. Conversely, for any subbasis element $S = \pi_\alpha(U_\alpha)$ for U_α open in X_α, we have

$$f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = (\pi_\alpha \circ f)^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha),$$

which is open since f_α is continuous. $\square$

1.7 Closed Sets and Limit Points

Definition 10. A set $A \subset (X, \mathcal{T})$ is closed if $X - A$ is open in X.

Theorem 1. *Let $(X, \mathcal{T})$ be a topological space, and let F denote a closed set of X, then*

1. *$\varnothing$ and X are closed,*
2. *$\bigcap_{\alpha \in \mathcal{J}} F_\alpha$ is closed, and*
3. *$\bigcup_{i=1}^{N} F_i$ is closed.*

Proof. This is a straightforward application of DeMorgan's Laws. □

Proposition 14. *Let Y be a subspace of X. Then A is closed in Y if and only if it is equal to the intersection of a closed set of X with Y.*

Proof. First we show the forward implication. Assume A is closed in Y, then $Y - A$ is open in Y, so be definition $Y - A = U \cap Y$ for some U open in X. $X - U$ is closed in X, and $A = Y \cap (X - U)$, so A is the intersection of a closed set of X with Y.

Now we show the backward implication. Assume $A = C \cap Y$ for C closed in X. Then $X - C$ is open in X, so $(X - C) \cap Y$ is open in Y by the definition of the subspace topology. But $(X - C) \cap Y = Y - A$, so $Y - A$ is open in Y. Thus A is closed in Y. □

Proposition 15. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. $A = F \cap Y$ for some F closed in X. A is then the intersection of closed sets of X, so it is itself closed in X. □

Definition 11. The **interior** of a set A, denoted A^o, is the union of all open sets contained in A.

The **closure** of a set A, denoted $\overline{A}$ is the intersection of all closed sets containing A.

The closure of a set is clearly closed, and the interior of a set is clearly open. It is also clear that if A is open, then $A^o = A$, and if A is closed, then $\overline{A} = A$. We also have the obvious relation $A^o \subset A \subset \overline{A}$.

We have to be careful when describing closures. Given a subspace Y of X, the closure of A in X is generally not the same as the closure of A in Y. In this case, we use $\overline{A}$ to denote the closure of A in X (the overall space). We relate this to the closure of A in Y (the subspace) with the following proposition.

Proposition 16. *Let Y be a subspace of X, and let $A \subset Y$. Denote the closure of A in X by $\overline{A}$. Then the closure of A in Y is equal to $\overline{A} \cap Y$.*

Proof. Let B denote the closure of A in Y. $\overline{A}$ is closed in X, so by Proposition 14, $\overline{A} \cap Y$ is closed in Y. Since $\overline{A} \cap Y$ contains A, and since by definition B is the intersection of all closed subsets of Y containing A, we have $B \subset \overline{A} \cap Y$.

On the other hand, we know B is closed in Y. Again by Proposition 14, $B = C \cap Y$ for some C closed in X. Then C is a closed set of X containing A. Since $\overline{A}$ is the intersection of all such closed sets, we have $\overline{A} \subset C$, so $(\overline{A} \cap Y) \subset (C \cap Y) = B$. □

Definition 12. A **neighborhood** of a point X is an open set containing x.

Theorem 2. *Let A be a subset of a topological space X, then*

1. *$x \in \overline{A}$ if and only if every neighborhood of x intersects A, and*
2. *Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.*

Make sure you have an intuitive understanding of why this is true.

Definition 13. Let $A \subset (X, \mathcal{T})$, then $x \in X$ is a **limit point** of A if every open neighborhood of x intersects A at some point *other than* x.

Equivalently, x belongs to the closure of $A - \{x\}$. Note that x need not lie in A. Think about this.

Theorem 3. *Let $A \subset (X, \mathcal{T})$, and denote the set of limit points of A by A'. Then $\overline{A} = A \cup A'$.*

Proof. First we show $A' \cup A \subset \overline{A}$. Let $x \in A'$, then every open neighborhood of x intersects A (at a point other than x). Thus by Theorem 2, $x \in \overline{A}$, so $A' \subset \overline{A}$. Since $A \subset \overline{A}$ by definition, we have $A \subset A' \subset \overline{A}$.

Now we show $\overline{A} \subset A' \cup A$. Let $x \in \overline{A}$. If $x \in A$, then this is trivial, so assume $x \notin A$. Since $x \in \overline{A}$, every neighborhood of x intersects A. Since $x \notin A$, this must be at a point other than x. Thus $x \in A' \subset A' \cup A$, so $\overline{A} \subset A' \cup A$. □

Corollary 1. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. Let $A \subset (X, \mathcal{T})$. Then A is closed if and only if $A = \overline{A} = A \cup A'$, and $A = A \cup A'$ if and only if $A' \subset A$. □

1.8 Metric Spaces

Definition 14. The **metric topology** $\mathcal{T}_d$ on X induced by d is generated by the basis

$$\{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}.$$

Proposition 17. *The following give the same topologies on $\mathbb{R}^n$:*

1. $d_2(x, y) = \|x - y\|_2$,
2. $d_1(x, y) = \sum_i |x_i - y_i|$,
3. $d_\infty(x, y) = \max_i |x_i - y_i|$, *and*
4. *the product topology.*

Proof. Do this. □

We say a topological space is **metrizable** if there is some metric such that induces its topology.

Proposition 18. *Metrizable spaces are Hausdorff.*

Proof. Do this. Should rely on metric space being Haus. □

Chapter 2

Topological Properties

2.1 Separability

Definition 15. A topological space is **separable** if it has a countable dense subset.

Example 7. $\mathbb{R}$ is separable because $\overline{\mathbb{Q}} = \mathbb{R}$.

2.2 Connectedness

Definition 16. A space X is **disconnected** if there are disjoint, nonempty, open A, B that cover X.

Proposition 19. *If Y is a subspace of X, it is disconnected if there are disjoint, nonempty A, B whose union is Y without containing a limit point of each other.*

Proof. Do this. □

Lemma 1. *Suppose U, V are disjoint and open in X. If Y is a connected subspace and $Y \subset U \cup V$, then it's entirely contained in one or the other.*

Proof. If not, then U and V separate Y, contradicting its connectedness. □

Definition 17. $X \subset \mathbb{R}$ is an **interval** if $[a, b] \subset X$ for all $a, b \in X$.

Chapter 3

Separation Properties

3.1 Convergence

We say that a sequence $\{x_n\}$ is **eventually** in U if there is some N such that $x_n \in U$ when $n \geq N$.

Definition 18. $\{x_n\}$ **converges** to x if it's eventually in every open neighborhood of x.

Put that you only need to check basis/subbasis elements actually...

Example 8. In the discrete topology, $x_n \to x$ if $\{x_n\}$ eventually equals x.

In the indiscrete topology, every sequence converges to every point.

If we want limits to be unique, we have to enforce certain separation axioms.

3.2 The Separation Axioms

Turns out that separating points yields some nice properties of topological spaces. Who'da thunk?

Stuff about T_1-T_4. Theorems about how they're related.

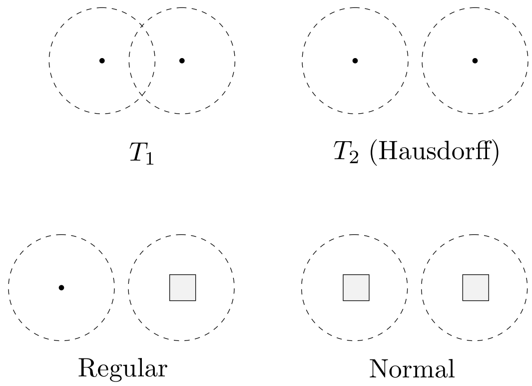

Figure 3.1: The four main types of separation.

Proposition 20. *Let X be T_1, and let A be a subset of X. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. **Backward:** Every neighborhood of x intersects A at infinitely many points, so it they surely all intersect A at a point other than x. Thus x is a limit point.

Forward: Let x be a limit point of A, and suppose some neighborhood U of x intersects A at only finitely many points. Let $\{x_1, \dots, x_m\} = U \cap (A - \{x\})$, then $X - \{x_1, \dots, x_m\}$ is open in X since X satisfies the T_1 axiom. Then $U \cap (X - \{x_1, \dots, x_m\})$ is a neighborhood of x that doesn't intersect $A - \{x\}$ at all. This contradicts x being a limit point of A, so every neighborhood of x must intersect A at infinitely many points. $\square$

Proposition 21. *A space is T_1 if and only if all single points are closed.*

Proof. **Forward:** Suppose X is T_1, then fix $x \in X$. Then for $y \in X - \{x\}$, there is an open U_y such that $y \in U_y \subset X - \{x\}$, so $X - \{x\} = \bigcup_y U_y$. Then $X - \{x\}$ is open so $\{x\}$ is closed.

Backward: Suppose all single points in X are closed. Fix $x, y \in X$, then $X - \{x\}$ and $X - \{y\}$ are the open sets we need to show that X is T_1. $\square$

Corollary 2. *A space is T_1 if and only if all finite point sets are closed.*

Proof. Do I even need one? Kinda obvious. $\square$

3.3 Hausdorff Spaces

Proposition 22. *Every finite set in a Hausdorff space is closed.*

Proof. Hausdorff spaces are T_1. □

Proposition 23. *Let X be a Hausdorff space, then a sequence of points in X converges to at most one point in X.*

Proof. Suppose $\{x_n\} \subset X$ such that $x_n \to x \in X$. If $y \neq x$, then since X is Hausdorff we can find disjoint open neighborhoods U and V of x and y, respectively. The set U contains all but finitely many of the points in $\{x_n\}$, so V can only contain finitely many of the points in $\{x_n\}$. Thus x_n cannot converge to y. □

Proposition 24. *Every simply ordered set is Hausdorff in the order topology.*

Proof. Do this. □

Proposition 25. *The product of two Hausdorff spaces is a Hausdorff space.*

Proof. Do this. □

Proposition 26. *A subspace of a Hausdorff space is Hausdorff.*

Proof. Suppose X is Hausdorff and that Y is a subspace of X with distinct points u and v. Then u and v are also distinct points of X, so by the regularity of X, they are separated by disjoint open sets U and V in X. Then $Y \cap U$ and $Y \cap V$ are the desired open sets of Y. □

3.4 Regular Spaces

Proposition 27. *X is regular if and only if for all $p \in X$ and open neighborhood U of p, there is an open set V such that $p \in V$ and $\overline{V} \subset U$.*

Proof. Do this. Forward direction already in written notes so don't throw out that paper yet. □

Proposition 28. *Every subspace of a regular space is regular.*

Proof. Let Y be a subspace of a regular space X, let A be closed in Y, and let $y \in Y - A$. Then $A = \mathrm{cl}_X(A) \cap Y$, and $\mathrm{cl}_X(A)$ is a closed set in X not containing y. Then the regularity of Y follows from the regularity of X. □

3.5 Normal Spaces

Proposition 29. *X is normal if and only if for all closed sets A and open sets U containing U, there exists an open set V such that $A \subset V$ and $\overline{V} \subset U$.*

Proof. Hey there. □

Proposition 30. *X is normal if and only if for all pairs of disjoint closed sets A and B, there are disjoint open containing A and B whose closures are also disjoint.*

Proof. These really be piling up. □

Suppose a space is normal and covered by 2 open sets. Then we can actually find two smaller patches (smaller in the sense that their closures are contained in the original patches) that cover the space too.

Theorem 4 (The Shrinking Theorem)**.** *X is a normal topological space if and only if for all open covers $\{U, V\}$ of X, there exist open sets U' and V' such that $\overline{U'} \subset U$, $\overline{V'} \subset V$, and U' and V' also cover X.*

Proof. **Forward:** Suppose X is normal and is covered by open sets U and V. Note that $X - U$ is closed and is a subset of V. Then by Proposition 29, there exists an open set V' such that $X - U \subset V' \subset \overline{V'} \subset V$.

Then $U \cup V'$ contains $U \cup X - U = X$, so $\{U, V'\}$ is an open cover of X. We can repeat this argument to replace U with the desired U'.

Backward: Let A be closed in X and contained in some open set U of X. Again by Proposition 29, we need to find an open set U' such that $A \subset U'$ and $\overline{U'} \subset U$.

Since $X - A$ is open and $X - A$ and U cover X, by assumption, there exist open U' and V' such that $\overline{U'} \subset U$, $\overline{V'} \subset X - A$, and $U' \cup V' = X$. We claim that U' is our desired open set. We already have $\overline{U'} \subset U$, so we only need to show $A \subset U'$.

Now U' and V' cover X, so $X - V' \subset U'$. Additionally, $V' \subset X - A$. Together, these give $A \subset X - V' \subset U'$. □

Extenstion of this theorem to more than 2 open sets.

A somewhat surprising result is that not all subspaces of normal spaces are normal. To see why, we can modify the earlier proof that subspaces of regular spaces are themselves regular. Let Y be a subspace of a normal space X, and let A and B be closed in Y. Then

$$A = \mathrm{cl}_X(A) \cap Y, B = \mathrm{cl}_X(B) \cap Y.$$

In order to use the regularity of X, we would need $\mathrm{cl}_X(A)$ and $\mathrm{cl}_X(B)$ to be disjoint. Unfortunately, this is not true in general (see the figure below). It *is* true, though, if Y is a closed subspace of X.

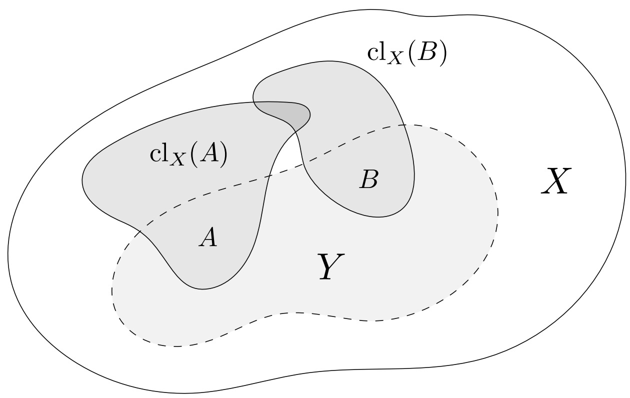

Figure 3.2: An exaggerated example of what could happen if Y is open in X.

Proposition 31. *Closed subspaces of normal spaces are normal.*

Proof. Let Y be a closed subspace of a normal space X, and let A and B be closed in Y. Then since Y is closed in X, both A and B are also closed in X. Then the normality of Y follows from the normality of X. □

Definition 19. $A, B \subset X$ are **separated** if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty, i.e. they don't contain each other's limit points.

I need a line break here.

Definition 20. X is **completely normal** if for any two separated sets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. A T_5 space is any space that is both T_1 and completely normal.

Lemma 2. *Let Y be a subspace of X, and let A and B be disjoint closed sets in Y. Then A and B are separated in X.*

Proof. Denote the closure in X with a bar, then $A = \overline{A} \cap Y$ and $B = \overline{B} \cap Y$. Then $\overline{A} \cap B = A \cap \overline{B} = \overline{A} \cap \overline{B} \cap Y = A \cap B = \varnothing$. □

Proposition 32. *X is completely normal if and only if every subspace of X is normal.*

Proof. **Forward:** Let A and B be closed in a subspace Y of X, then by the previous lemma, they are separated in X. Since X is completely normal, there are disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$. Then $U \cap Y$ and $V \cap Y$ are the desired open sets to show that the subspace Y is normal.

Backward: Let A and B be separated in X, then $A \in X - \overline{B}$ and $B \in X - \overline{A}$. Consider the subspace $Y \doteq (X - \overline{B}) \cup (X - \overline{A}) = X - (\overline{B} \cap \overline{A})$. Then since $\mathrm{cl}_Y(A) = \overline{A} \cap Y$ and $\mathrm{cl}_y(B) = \overline{B} \cap Y$, the definition of Y gives $\mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(B) = \overline{A} \cap \overline{B} \cap Y = \varnothing$.

Then since $\mathrm{cl}_Y(A)$ and $\mathrm{cl}_Y(B)$ are disjoint and closed in Y, and since Y is normal by assumption, there are disjoint open sets U and V of Y such that $\mathrm{cl}_Y(A) \subset U$ and $\mathrm{cl}_Y(B) \subset V$. Then $A \subset U$ and $B \subset V$, so X is completely normal. □

Chapter 4

Continuity

4.1 Continuous Functions

Definition 21. Let X, Y be topological spaces, then $f : X \to Y$ is **continuous** if for all U open in Y, $f^{-1}(U)$ is open in X.

Proposition 33. *If Y has basis $\mathcal{B}$ and $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$, then $f : X \to Y$ is continuous. Similarly, if Y has subbasis $\mathcal{S}$ and $f^{-1}(S)$ is open in X for all $S \in \mathcal{S}$, then $f : X \to Y$ is continuous.*

Proof. The preimage of any open set if the union of preimages of basis elements. The preimage of any basis element is the finite intersection of preimages of subbasis elements. $\square$

Theorem 5. *Let X and Y be topological spaces, and let $f : X \to Y$, then the following are equivalent:*

1. *f is continuous.*
2. *For all $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.*
3. *For all B closed in Y, $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.*
4. *For all B closed in Y, $f^{-1}(B)$ is closed in X.*
5. *For all $x \in X$ and for each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

Example 9. If X has the discrete topology, then any function *out* of X is continuous. If X has the indiscrete topology, then any function *into* X is continuous.

Definition 22. A **homeomorphism** is a continuous function with continuous inverse (an isomorphism in **Top**).

Equivalently, a homeomorphism is a bijective function $f : X \leftrightarrow Y$ such that U is open in X if and only if $f(U)$ is open in Y.

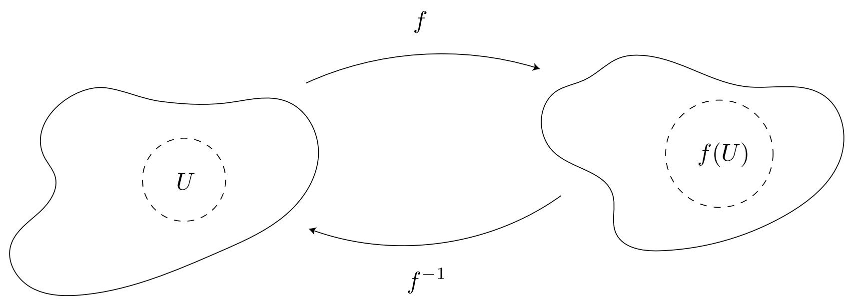

Figure 4.1: A homeomorphism f.

Definition 23. A **topological property** is a property of topological space X expressed entirely in terms of the topology on X (the open sets of X).

If $f : X \to Y$ is a homeomorphism, then Y has the topological properties of X.

Definition 24. Suppose $f : X \to Y$ is one-to-one and continuous. Then $f' : X \to f(X)$ (obtained by restricting the range of f) is bijective. If f' is a homeomorphism of X with $f(X)$, we say $f : X \hookrightarrow Y$ is a **(topological) embedding** of X in Y.

Theorem 6 (The Pasting Lemma)**.** *Let $X = A \cup B$, where A and B are either both closed or both open in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ given by*

$$H(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous.

Proof. Suppose A and B are both closed. Let C be closed in Y, then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f and g are continuous, both $f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B, respectively. Since both A and B are closed in X, both preimages are also closed in X. Thus $h^{-1}(C)$ is closed and h is subsequently continuous.

To show this when A and B are both open, replace the word "closed" with the word "open" in the above paragraph. □

Note that the condition $f(x) = g(x)$ for all $x \in A \cap B$ is not needed in this proof. It is only necessary to make h an actual function.

> **Theorem 7** (Maps into Products)**.** *Define $f : A \to X \times Y$ by $f(a) = (f_1(a), f_2(a))$, for $f_1 : A \to X$ and $f_2 : A \to Y$. Then f is continuous if and only if f_1 and f_2 are both continuous.*

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the obvious projections, which we know to be continuous.

We begin with the forward implication. Suppose f is continuous, then $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$, so f_1 and f_2 are compositions of continuous functions. Thus they are both continuous themselves.

Now we show the backward implication. Suppose f_1 and f_2 are continuous, then we'll show that for each basis element $U \times V$ of the topology on $X \times Y$, the preimage $f^{-1}(U \times V)$ is open. Take a point in the preimage $a \in f^{-1}(U \times V)$, then $f(a) \in U \times V$, so $f_1(a) \in U$ and $f_2(a) \in V$. Thus we have

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V).$$

Since U and V are open and f_1 and f_2 are continuous, both of the sets in the above intersection are open. The finite intersection of open sets is open, so $f^{-1}(U \times V)$ is open, so f is continuous. □

> **Note 2.** If $f : A \times B \to X$ instead, there is *no* useful criterion for the continuity of f.

4.2 Constructing Continuous Functions

Proposition 34. *Constant functions are continuous.*

Proof. Let $f : X \to Y$ be given by $f(x) = y$ for some constant y. If V is open in Y, then $f^{-1}(V)$ is either $\varnothing$ or X, depending on if $y \in V$. In either case, $f^{-1}(V)$ is open. □

Proposition 35. *Inclusion maps are continuous.*

Proof. Let $X \subset Y$, and define $\iota : X \to Y$ by $\iota(x) = x$. For V open in Y, $\iota^{-1}(V) = X \cap V$, which is open in X by definition. $\square$

Proposition 36. *Restrictions of continuous maps are continuous.*

Proof. Let $f : X \to Y$ be continuous, and let $A \subset X$. Define $f|_A : A \to Y$ by $f|_A(a) = f(a)$. For V open in Y, $f|_A^{-1}(V) = f^{-1}(V) \cap A$, which is open in A since $f^{-1}(V)$ is open in X. $\square$

Proposition 37. *Compositions of continuous functions are continuous.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For V open in Z, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$, which is open in X since f and g are both continuous. $\square$