Exercise 1. §6.3 #2.

Suppose $k_1 = 0$ and k_2 is never 0, and suppose $\{E_i\}$ is a principal frame field of M. If α is a principal curve of k_1, then by Theorem 1.4, $\alpha'' = \nabla_{E_1} E_1 = \sum_{j=1}^{3} \omega_{ij}(E_1)E_j$. We know $\omega_{11} = 0$, and since $\{E_i\}$ is principal, $\omega_{13}(E_1) = k_1 = 0$. Also, by Theorem 2.6, $\omega_{12}(E_1) = E_2[k_1]/(k_1 - k_2)$. But since $k_1 = 0$, this reduces to 0, so $\alpha'' = 0$. Thus α is a straight line.

Exercise 2. §6.3 #5.

Suppose k_1, k_2 are constant on a surface M.

- **Case 1:** $k_1 = k_2$. If both are 0, then M is part of a plane. If both are nonzero, then since M must be everywhere umbilic, then by Theorem 3.3, M is part of a sphere.
- **Case 2:** $k_1 \neq k_2$. Assume there is a principal frame field $\{E_i\}$ on M. Since k_1, k_2 are constant, Theorem 2.6 says

  $$E_1[k_2] = (k_1 - k_2)\omega_{12}(E_2) = 0$$
  $$E_2[k_1] = (k_1 - k_2)\omega_{12}(E_1) = 0.$$

  Since $k_1 \neq k_2$, the only way to have these expressions equal 0 is if $k_i = 0$ or $\omega_{12}(E_i) = 0$. Since $\omega_{12}(E_i)$ cannot be 0 for both $i = 1, 2$, this means that $k_i = 0$ for one of the i. Without loss of generality, assume $k_1 = 0$ and $\omega_{12}(E_2) = 0$. By the assumption that $k_1 \neq k_2$, k_2 can never be 0. Then by the previous exercise, the principal curves of k_1 are straight lines. They must also be parallel, as $\nabla_{E_2} E_1 = \omega_{12}(E_2)E_2 + \omega_{12}(E_2)E_3 = 0$. The principal curves in the k_2 direction have fixed nonzero curvature, so they are circles. Thus M is part of a circular cylinder.

We have gone through every case, showing that M must be part of a sphere, a plane, or a circular cylinder.

Exercise 3. §6.8 #2.

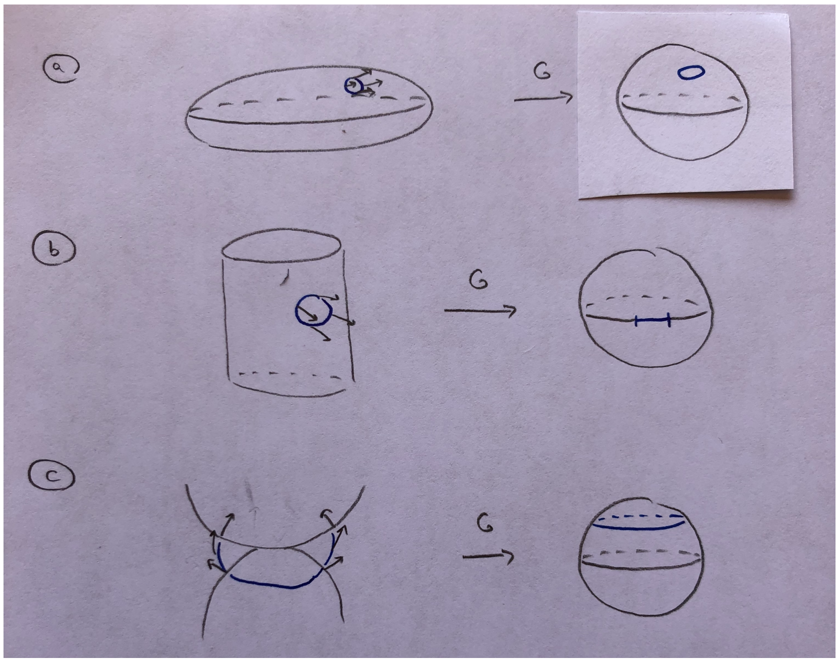

Figure 1:

Exercise 4. §6.8 #12.

a. By §5.4 #2 (from HW 6), the Gaussian curvature of S (before the Polar parameterization) is

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(f_u^2 + f_v^2 + 1)^2} = -\frac{1}{(u^2+v^2+1)^2}.$$

Then after converting to polar coordinates it is

$$K = -\frac{1}{(r^2+1)^2}.$$

We can also parametrize S with polar coordinates using the patch

$$\mathbf{x} : (r, \theta) \mapsto (r\cos\theta, r\sin\theta, r).$$

We then calculate

$$\mathbf{x}_r \times \mathbf{x}_\theta = \begin{vmatrix} U_1 & U_2 & U_3 \\ \cos\theta & \sin\theta & 2r\cos\theta\sin\theta \\ -r\sin\theta & r\cos\theta & r^2(\cos^2\theta - \sin^2\theta) \end{vmatrix} = (-r^2\sin\theta, -r^2\cos\theta, r),$$

so the area form dS is given by

$$dS = \|\mathbf{x}_r \times \mathbf{x}_\theta\| dr\ d\theta = r(r^2+1)^{1/2}\ dr\ d\theta.$$

We can now calculate the total curvature as

$$\begin{aligned}\iint_S K\ dS &= \int_0^{2\pi}\int_0^{\infty} (-r(r^2+1)^{-3/2})\ dr\ d\theta \\ &= \int_0^{2\pi} \Big[(r^2+1)^{-1/2}\Big]_{r=0}^{r=\infty}\ d\theta \\ &= \int_0^{2\pi} (-1)\ d\theta \\ &= -2\pi.\end{aligned}$$

b. We can find a normal vector field by calculating the gradient of $xy - z$, which is

$$\nabla(xy - z) = (y, x, -1)\,.$$

Converting to polar coordinates and normalizing gives unit normals

$$\pm U = \pm\frac{1}{(r^2+1)^{1/2}}(r\sin\theta, r\cos\theta, -1).$$

Over the range $0 < \theta \leq 2\pi$, the tuple $(\cos\theta, \sin\theta)$ never has repeated points, so $G(U)$ is one-to-one. Since K is also clearly always negative, by Corollary 8.6 the total curvature is $\pm$ the area of $G(S)$. Since we can sweep out all the points that lie on and within the circle S_1 using our parameterization, and since the z-coordinate is always negative, this means $G(S)$ is the Southern hemisphere of Σ, which has area 2π. This is the negative of our answer from part (a), so it agrees with the corollary.

Exercise 5. §7.1 #2.

Let $\langle \mathbf{v}, \mathbf{w}\rangle = (\mathbf{v}\cdot\mathbf{w})/v^2(\mathbf{p})$ and $\alpha(t) = (r\cos t, r\sin t)$ for $0 < t < \pi$.

a. The velocity of α is $\alpha'(t) = (-r\sin t, r\cos t)$, so its speed is

$$\|\alpha'(t)\| = \sqrt{\langle \alpha'(t), \alpha'(t)\rangle} = \frac{\sqrt{\alpha'(t)\cdot\alpha'(t)}}{v(\alpha(t))} = \frac{1}{\cos t} = \csc t.$$

b. The length of α is

$$\int_0^\pi \|\alpha'(t)\|\ dt = \int_0^\pi \csc t\ dt.$$

But $\csc t \to \infty$ as $t \to 0, \pi$, so the Poincare length of α goes to ∞. The Euclidean length, on the other hand, is just half of the circle's circumference, i.e. πr.

c. If we were working in the usual Euclidean plane, the area of this region would be
$$\int_0^\pi \int_0^r \rho \, d\rho \, d\theta = \frac{1}{2} r^2 \pi,$$
which is half the area of $S^1(r)$. If we're working in the Poincare half-plane, then the area form must be divided by $v^2(\mathbf{p})$. But as we approach the u-axis, $1/v \to \infty$, so the area under α is infinite in this case.

Exercise 6. §7.1 #4.

a. The expression given is a formal sum of bilinear functions of $\mathbf{x}_u$ and $\mathbf{x}_v$, so it is as a whole bilinear. It is also symmetric because $\mathbb{R}$ commutes over multiplication. We now show that $\langle \mathbf{x}_u, \mathbf{x}_v \rangle$ is positive definite.

   Suppose $\mathbf{v} \neq 0$, then
$$\begin{aligned} \langle \mathbf{v}, \mathbf{v} \rangle &= \alpha v_1^2 + 2b v_1 v_2 + v_2^2 \\ &= v_2^2 \left(\alpha \left(\frac{v_1}{v_2} \right)^2 + 2b \left(\frac{v_1}{v_2} \right) + c \right). \end{aligned}$$
   But since $b^2 < ac$, we have $(2b)^2 = 4b^2 < 4ac$, so the quadratic formula tells us that this quadratic polynomial has no roots, i.e. no nonzero $\mathbf{v}$ gives $\langle \mathbf{v}, \mathbf{v} \rangle = 0$.

   Now plugging in $\mathbf{v} = (1, 0)$ gives $\langle \mathbf{v}, \mathbf{v} \rangle = a^2 > 0$, where this last inequality follows from the assumption $a > 0$. Since this polynomial is continuous, doesn't ever equal 0, and is positive at some other point, we know that it is always positive for nonzero $\mathbf{v}$. Then since $\langle 0, 0 \rangle = 0$, we have
$$\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$$
   and
$$\langle \mathbf{v}, \mathbf{v} \rangle = 0 \iff \mathbf{v} = 0.$$
   Thus this satisfies the properties of an inner product.

b. We can use the definition of $\langle \, , \, \rangle$ from part (a). The uniqueness of $\langle \, , \, \rangle$ follows from extending linearly to all tangent vectors instead of just $\mathbf{x}_u, \mathbf{x}_v$.

Exercise 7. §7.1 #7.

We will use the fact that

$$\begin{aligned} F_*(U_1) &= f_u U_1 + g_u U_2 \\ F_*(U_2) &= f_v U_1 + g_v U_2. \end{aligned}$$

Forward: Suppose F is conformal and orientation-preserving. Then

$$\begin{aligned} f_v U_1 + g_v U_2 &= F_* U_2 \\ &= F_*(J U_1). \end{aligned}$$

Since F is conformal and orientation-preserving, F_* commutes with J, so this becomes

$$\begin{aligned} &= J(F_* U_1) \\ &= J(f_u U_1 + g_u U_2) \\ &= f_u J U_1 + g_u J U_2 \\ &= f_u U_2 - g_u U_1. \end{aligned}$$

This implies $f_u = g_v$ and $f_v = -g_u$.

Backward: Suppose $f_u = g_v$ and $f_v = -g_u$. Then

$$\langle F_* U_1, F_* U_2 \rangle = f_u f_v + g_u g_v = f_u f_v - f_u f_v = 0$$

and

$$\langle F_* U_1, F_* U_1 \rangle = f_u^2 + g_u^2 = \left| \frac{dF}{dz} \right|^2 = f_v^2 + g_v^2 = \langle F_* U_2, F_* U_2 \rangle .$$

Thus F is conformal with scale factor $|dF/dz|$. It's orientation preserving since

$$J_F = f_u g_v - f_v g_u = f_u^2 + f_v^2 > 0.$$

Note that the inequality is strict because F is regular.