0.1 PDEs

At every step we choose some finite collection of vertices $\{v_i\}_{i=1}^m$. Let κ_i denote the size of the cluster to which v_i belongs. We'll use the following quantities a lot (all probabilities are implicitly taken at time t):

$$
\begin{aligned}
X_m(k,t) &\doteq \mathbb{P}\left(\min\{\kappa_1,\dots,\kappa_m\} = k\right); \\
\hat{X}_m(k,t) &\doteq \mathbb{P}\left(\min\{\kappa_1,\dots,\kappa_m\} \geq k\right) \\
&= 1 - \sum_{j=1}^{k-1} X_m(j,t); \\
R(k,t) &= \mathbb{P}\left(\kappa_1 + \kappa_2 = k\right); \\
\hat{R}(k,t) &= \mathbb{P}\left(\kappa_1 + \kappa_2 \geq k\right).
\end{aligned}
$$

Would it be useful to generalize R? A common case for X_m is $m = 1$ or 2, so we can abbreviate those as

$$P \doteq X_1, \qquad Q \doteq X_2.$$

Note that we can express X_m as

$$X_m(k,t) = \hat{P}(k-1,t)^m - \hat{P}(k,t)^m,$$

(go over why) so every X_m is a function of P. As a final note, I will frequently suppress t from now on.

We're interested in how P changes throughout the percolation process. The following table gives the value of $\partial_t P$, written in terms of the proper X_m, for each of our rules.

Rule	$\partial_t P(s,t)$
Erdős Rényi	$\frac{s}{2}\sum_{u+v=s} P(u,t)P(v,t) - sP(s,t)$
Adjacent Edge	$s\sum_{u+v=s} P(u,t)Q(v,t) - sP(s,t) - sQ(s,t)$
DaCosta	$s\sum_{u+v=s} X_m(u,t)X_m(v,t) - 2sX_m(s,t)$
Sum	**Do this.**
Product	**Do this.**

0.2 Consequences

Let $S(t)$ denote the relative size (i.e. divided by n) of the percolation cluster at time t, and let $X_m(k,t) \doteq \mathbb{P}\left(\min\{\kappa_1,\dots,\kappa_m\} = k\right)$.

Proposition 1.

$$\sum_k X_m(k,t) = 1 - S^m(t).$$

To justify this, we can interpret $\sum_k X_m(k,t)$ as the probability that, at time t, the minimum cluster size of m vertex choices is finite (this is in the limit as $n \to \infty$). S is then the probability that a single choice is from an "infinite" cluster size. **I kinda want to do this more rigorously...**

Differentiating this identity for $X_1 = P$ gives

$$\partial_t S = -\sum_s \partial_t P,$$

so we can track the size of the percolation cluster by knowing $P(s)$ for all s. In the following computations, we'll express $\partial_t S$ in terms of the moments of various X_m, which we denote by

$$\langle s^k \rangle_{X_m} \doteq \sum_s s^k X_m(s).$$

Sometimes I might denote $\langle \cdot \rangle_{X_m}$ by $\langle \cdot \rangle_m$. The below table gives $\partial_t S$ for each of our rules. A derivation of this quantity is given afterwards for the Erdős Rényi rule; the other quantities are derived similarly.

Rule	$\partial_t S$
ER	$S\langle s\rangle_P$
AE	$\langle s\rangle_P S^2 + S\langle s\rangle_Q$
DC	$2S^m\langle s\rangle_{X_m}$
Sum	**Do this.**
Product	**Do this.**

Proposition 2. *For the Erdős Rényi rule, $\partial_t S = S\langle s\rangle_P$.*

Proof. In the below computation, I suppress the time t for clarity.

$$\begin{aligned}
\partial_t S &= -\sum_s \partial_t P \\
&= -\frac{1}{2}\sum_s s \sum_{u+v=s} P(u)P(v) + \sum_s sP(s) \\
&= -\frac{1}{2}\sum_u \sum_v (u+v)P(u)P(v) + \langle s\rangle_P \\
&= -\frac{1}{2}\left[\sum_u uP(u)\sum_v P(v) + \sum_u P(u)\sum_v vP(v)\right] + \langle s\rangle_P \\
&= -\frac{1}{2}\left[2\langle s\rangle_P (1-S)\right] + \langle s\rangle_P \\
&= -\langle s\rangle_P(1-S) + \langle s\rangle_P \\
&= S\langle s\rangle_P.
\end{aligned}$$

□

We're similarly able to calculate $\partial_t \langle s\rangle_P$ for these rules, as summarized in the below table. As before, I incluce the derivation for the Erdős Rényi rule afterwards, and the other derivations are similar.

Rule	$\partial_t \langle s \rangle_P$
ER	$\langle s \rangle_P^2 - \langle s^2 \rangle_P S$
AE	$2\langle s \rangle_P \langle s \rangle_Q - \langle s^2 \rangle_P S^2 - \langle s^2 \rangle_Q S$
DC	$2\langle s \rangle_{X_m}^2 - 2\langle s^2 \rangle_{X_m} S^m$
Sum	**Do this.**
Product	**Do this.**

Proposition 3. *For the Erdős Rényi rule, $\partial_t \langle s \rangle_P = S\langle s \rangle_P$.*

Proof. Once again, I suppress the time t for clarity.

$$\begin{aligned}
\partial_t \langle s \rangle_P &= \sum_s s\, \partial_t P(s) \\
&= \frac{1}{2}\sum_s s^2 \sum_{u+v=s} P(u)P(v) - \sum_s sP(s) \\
&= \frac{1}{2}\sum_u \sum_v (u+v)^2 P(u)P(v) - \langle s^2 \rangle_P \\
&= \frac{1}{2}\left[\sum_u u^2 P(u) \sum_v P(v) + 2\sum_u uP(u) \sum_v vP(v) + \sum_u P(u) \sum_v v^2 P(v)\right] - \langle s^2 \rangle_P \\
&= \frac{1}{2}\left[2\langle s^2 \rangle_P (1-S) + 2\langle s \rangle_P^2\right] - \langle s^2 \rangle_P \\
&= \langle s \rangle_P^2 - \langle s^2 \rangle_P S.
\end{aligned}$$

□

NEED TO DEFINE $\sim$.

If $\delta \doteq |t - t_c|$ is very small, then **(at least for DaCosta)** we have the scaling relationship

$$\langle s \rangle_{X_m} \sim \delta^{-\gamma}$$

for some γ dependent on X_m **(Go over why. This is also under the assumption that P has a scaling form near t_c)**. Differentiating gives us the relation

$$\partial_t \langle s \rangle_{X_m} \sim \delta^{-\gamma-1}.$$

Given a particular rule, we can take these two relations and subsitute them into our earlier calculation of $\partial_t \langle s \rangle_P$ to find out how γ_P and γ_Q are related. The below table sumarizes this relationship for all our rules.

Rule	Scaling Relationship
ER	**Nonsense right now.**
AE	**Do this.**
DaCosta $(m = 2)$	$\gamma_P + 1 = 2\gamma_Q$

I need to go over the definition of $\sim$ for this to make sense. Right now I'm getting nonsense for some of these.

Note that this holds because when $t < t_c$, $S = 0$. Not sure what to do for times past t_c.

0.3 Generalizations

Try to generalize a lot of this. When will these expressions be nice and simple, i.e. in terms of the $\langle s \rangle_{k,X_m}$?

Proposition 4. *If*

$$\partial_t P(s) = \zeta_0 \left[\sum_{u_1+\cdots+u_m=s} \prod_i X_{m_i}(u_i) \right] - \sum_i \zeta_i s X_{m_i}(u_i),$$

then the time derivative of S is

$$\partial_t S = \sum_i \langle s \rangle_{m_i} \left[-\zeta_0 \prod_{j \neq i} (1 - S^{m_j}) + \zeta_i \right].$$

Proof.

$$\begin{aligned}
\partial_t S &= -\sum_s \partial_t P(s) \\
&= -\zeta_0 \left[\sum_s s \sum_{\sum_i u_i = s} \prod_i X_{m_i}(u_i) \right] + \sum_i \zeta_i \langle s \rangle_{m_i} \\
&= -\zeta_0 \left[\sum_{u_1,\ldots,u_m} \left(\sum_i u_i \right) \prod_i X_{m_i}(u_i) \right] + \sum_i \zeta_i \langle s \rangle_{m_i} \\
&= -\zeta_0 \left[\sum_i \langle s \rangle_{m_i} \prod_{j \neq i} (1 - S^{m_j}) \right] + \sum_i \zeta_i \langle s \rangle_{m_i} \\
&= \sum_i \langle s \rangle_{m_i} \left[-\zeta_0 \prod_{j \neq i} (1 - S^{m_j}) + \zeta_i \right].
\end{aligned}$$

□

This is a start, but the class of ODEs is very, very restricted at the moment.

Also, probably only want to care about when we add one edge at a time. I think the above ODE is adding $m-1$ edges at once, which isn't what we want.

0.4 Scaling Behavior

I use the following two relationships:

- $P(s) = \delta^{(\tau-1)/\sigma} \tilde{f}(s\delta^{1/\sigma})$ for large s.
- $\tilde{f}(x) \propto \exp\left(-Cx^{1+\log_2 m}\right)$ when $t < t_c$.

In the below computation, I represent the constant of proportionality for $\tilde{f}$ by $\tilde{C}_f$. And to clean up notation (cause there's a lot of it), I use the following shorthands

- $\tilde{f}_x \doteq \tilde{f}(x\delta^{1/\sigma})$.
- $\mathcal{E}_x \doteq \exp\left(-Cx\delta^{1/\sigma}\right)$.

We've assumed that this scaling behavior exists near t_c, but it's natural to ask how big this window is. In the case of the Erdős Rényi rule, we can describe the size of the window using our earlier differential equation for $\partial_t P(s,t)$.

Note that $1 + \log_2 m = 1$ when $m = 1$, so our exponential law for $\tilde{f}$ becomes pretty simple. I'll also only be considering the case when $t < t_c$, so δ becomes just $t_c - c$ (this makes derivatives nicer). When s is large, our ODE gives

$$\begin{aligned}
\partial_t P(s) &= \frac{s}{2} \sum_{u+v=s} P(u)P(v) - sP(s) \\
\partial_t \left[s^{\frac{\tau-1}{\sigma}} \tilde{f}_s \right] &= \frac{s}{2} \sum_{u+v=s} \delta^{\frac{2(\tau-1)}{\sigma}} \tilde{f}_u \tilde{f}_v - s\delta^{\frac{\tau-1}{\sigma}} \tilde{f}_s \\
\delta^{\frac{\tau-1}{\sigma}} \tilde{C}_f \mathcal{E}_s \left[\frac{\tau-1}{\sigma\delta} + \frac{Cs}{\sigma}\delta^{\frac{1-\sigma}{\sigma}} \right] &= \delta^{\frac{\tau-1}{\sigma}} \tilde{C}_f \mathcal{E}_s \left[\frac{s}{2}\delta^{\frac{\tau-1}{\sigma}} \tilde{C}_f \left(\sum_{u+v=1} 1 \right) - s \right] \\
\frac{\tau-1}{\sigma\delta} + \frac{Cs}{\sigma}\delta^{\frac{1-\sigma}{\sigma}} &= \frac{s}{2}\delta^{\frac{\tau-1}{\sigma}} \tilde{C}_f (s-1) - s.
\end{aligned}$$

Now δ is bounded between 0 and 1, so as $s \to \infty$, the dominating terms give the relation

$$\begin{aligned}
\delta^{\frac{2-\tau-\sigma}{\sigma}} &\approx \frac{\tilde{C}_f \sigma s}{2C} \\
&\propto \sigma s.
\end{aligned}$$

We know that for Erdős Rényi, $\sigma = 1/2$ and $\tau = 5/2$, so this relation becomes

$$\begin{aligned}
\delta &\approx \sqrt{\frac{4C}{\tilde{C}_f s}} \\
&\propto \sqrt{\frac{1}{s}}.
\end{aligned}$$

So the scaling window is smaller when s is larger, which seems reasonable.