Exercise 1 (2.5: 2)**.** *Let* $V = -yU_1 + xU_3$ *and* $W = \cos xU_1 + \sin xU_2$*. Express the following covariant derivatives in terms of* U_1, U_2, U_3*.*

1. $V[\cos x] = -yU_1[\cos x] = y \sin x$ and $V[\sin x] = -yU_1[\sin x] = -y \cos x$, so

$$\begin{aligned}\nabla_V W &= V[\cos x]U_1 + V[\sin x]U_2 \\ &= y \sin xU_1 - y \cos xU_2.\end{aligned}$$

2. $V[-y]$ is 0 since V has no U_2 component, so

$$\begin{aligned}\nabla_V V &= V[-y]U_1 + V[x]U_3 \\ &= -yU_1[x]U_3 \\ &= -yU_3.\end{aligned}$$

3. $V[z^2 \cos x] = yz^2 \sin x + 2xz \cos x$ and $V[z^2 \sin x] = -yz^2 \cos x + 2xz \sin x$, so

$$\begin{aligned}\nabla_V(z^2W) &= V[z^2 \cos x]U_1 + V[z^2 \sin x]U_2 \\ &= (yz^2 \sin x + 2xz \cos x)U_1 + (-yz^2 \cos x + 2xz \sin x)U_2.\end{aligned}$$

4. Since $W[-y] = -\sin x$ and $W[x] = \cos x$,

$$\begin{aligned}\nabla_W V &= W[-y]U_1 + W[x]U_3 \\ &= -\sin xU_1 + \cos xU_3.\end{aligned}$$

5. Since $V[y \sin x] = -y^2 \cos x$ and $V[-y \cos x] = -y^2 \sin x$,

$$\begin{aligned}\nabla_V(\nabla_V W)) &= \nabla_V(y \sin x\ U_1 - y \cos x\ U_2) \\ &= V[y \sin x]U_1 + V[-y \cos x]U_2 \\ &= -y^2 \cos x\ U_1 - y^2 \sin x\ U_2.\end{aligned}$$

6. This is

$$\begin{aligned}\nabla_V(xV - zW) &= \nabla_V\big((-xy - z\cos x)\ U_1 - z \sin x\ U_2 + x^2\ U_3\big) \\ &= V[-xy - z \cos x]U_1 + V[-z \sin x]U_2 + V[x^2]U_3 \\ &= (y^2 - yz \sin x - x \cos x)U_1 + (yz \cos x - x \sin x)U_2 \\ &\quad + (-2xy)U_3.\end{aligned}$$

Exercise 2 (2.5: 3)**.** *If W is a vector field with constant length $\|W\|$, prove that for any vector field V, the covariant derivative $\nabla_V W$ is everywhere orthogonal to W.*

Since $\|W\|$ is constant, $\|W\|^2$ is constant, so $\nabla_V \|W\|^2 = \nabla_V (W \cdot W) = 0$; however, we can manually calculate this derivative to be

$$\begin{aligned} \nabla_V \|W\|^2 &= \nabla_V (W \cdot W) \\ &= \nabla_V W \cdot W + W \cdot \nabla_V W \\ &= 2(\nabla_V W \cdot W), \end{aligned}$$

so $\nabla_V W \cdot W = 0$. Thus for any vector field V, $\nabla_V W$ is everywhere orthogonal to W.

Exercise 3 (2.5: 5)**.** *Let W be a vector field defined on a region containing a regular curve α. Then $t \to W(\alpha(t))$ is a vector field on α called the restriction of W to α and is denoted by W_α.*

1. *Prove that $\nabla_{\alpha'(t)} W = (W_\alpha)'(t)$.*
2. *Deduce that the straight line in Definition 5.1 may be replaced by any curve with initial velocity $\mathbf{v}$.*

1. The vector field W can be written $W = \sum w_i U_i$. Using this we have

$$\nabla_{\alpha'(t)} W = \sum \alpha'(t)[w_i] U_i(\alpha(t))$$

and

$$W_\alpha(t) = W(\alpha(t)) = \sum w_i(\alpha(t)) U_i(\alpha(t)),$$

from which it follows that

$$(W_\alpha)'(t) = \sum w_i'(\alpha(t)) \alpha'(t) U_i(\alpha(t)).$$

Thus $\nabla_{\alpha'(t)} W = (W_\alpha)'(t)$ if $\alpha'(t)[w_i] = w_i'(\alpha(t))\alpha'(t)$. This is true, since

$$\begin{aligned} \alpha'(t)[w_i] &= \frac{d}{ds} w_i(\alpha(t) + s\alpha'(t)) \Big|_{s=0} \\ &= w_i'(\alpha(t) + s\alpha'(t)) a'(t) \Big|_{s=0} \\ &= w_i'(\alpha(t))\alpha'(t). \end{aligned}$$

2. Then if $\alpha'(0) = \mathbf{v}$, then

$$\begin{aligned} \nabla_{\mathbf{v}} W = \nabla_{\alpha'(0)} W &= (W_\alpha)'(0) \\ &= W'(\alpha(0))\alpha'(0) \\ &= \frac{d}{ds} W(\alpha(t) + s\alpha'(t)) \Big|_{s=0}. \end{aligned}$$

This is exactly the covariant derivative, except now we have $\alpha(t)$ instead of $\mathbf{p}$ and $\alpha'(t)$ instead of $\mathbf{v}$.

Exercise 4 (2.7: 2)**.** *Find the connection forms of the natural frame field U_1, U_2, U_3.*

The attitude matrix for the natural frame field is simply I_3, so $dA = 0$. Then the matrix of connection forms is $\omega = dA\ A^T = 0$. Thus every connection form is the zero function.

Exercise 5 (2.7: 4)**.** *Prove that the connection forms of the spherical frame field are*

$$\omega_{12} = \cos\varphi\ d\theta, \quad \omega_{13} = d\varphi, \quad \omega_{23} = \sin\varphi\ d\theta.$$

Given the spherical frame fields

$$\begin{aligned}
F_1 &= \cos\varphi(\cos\theta U_1 + \sin\theta U_2) + \sin\varphi U_3, \\
F_2 &= -\sin\theta U_1 + \cos\theta U_2, \\
F_3 &= -\sin\varphi(\cos\theta U_1 + \sin\theta U_2) + \cos\varphi U_3,
\end{aligned}$$

we can form the attitude matrix

$$A = \begin{pmatrix} \cos\varphi\cos\theta & \cos\varphi\sin\theta & \sin\varphi \\ -\sin\theta & \cos\theta & 0 \\ -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & \cos\varphi \end{pmatrix}.$$

Applying d to the entries of this matrix yields $dA =$

$$\begin{pmatrix} -\sin\varphi\cos\theta d\varphi - \cos\varphi\sin\theta d\theta & -\sin\varphi\sin\theta d\varphi + \cos\varphi\cos\theta d\theta & \cos\varphi d\varphi \\ -\cos\theta d\theta & -\sin\theta d\theta & 0 \\ -\cos\varphi\cos\theta d\varphi + \sin\varphi\sin\theta d\theta & -\cos\varphi\sin\theta d\varphi - \sin\varphi\cos\theta d\theta & -\sin\varphi d\varphi \end{pmatrix}.$$

We then find the connection forms by $\omega = dA\ A^T$. The entries for ω_{12}, ω_{13}, and ω_{23} are then

$$\begin{aligned}
\omega_{12} &= \sin\varphi\sin\theta\cos\theta\ d\varphi + \cos\varphi\sin^2\theta\ d\theta \\
&\quad - \sin\varphi\sin\theta\cos\theta\ d\varphi + \cos\varphi\cos^2\theta\ d\theta \\
&= \cos\varphi(\sin^2\theta + \cos^2\theta)d\theta \\
&= \cos\varphi\ d\theta. \\
\omega_{13} &= \sin^2\varphi\cos^2\theta\ d\varphi + \sin\varphi\cos\varphi\sin\theta\cos\theta\ d\theta \\
&\quad + \sin^2\varphi\sin^2\theta\ d\varphi - \sin\varphi\cos\varphi\sin\theta\cos\theta\ d\theta + \cos^2\varphi\ d\varphi \\
&= (\sin^2\varphi(\cos^2\theta + \sin^2\theta) + \cos^2\varphi)d\varphi \\
&= d\varphi. \\
\omega_{23} &= \sin\varphi\cos^2\theta\ d\theta + \sin\varphi\sin^2\theta\ d\theta \\
&= \sin\varphi(\sin^2\theta + \cos^2\theta)\ d\theta \\
&= \sin\varphi\ d\theta.
\end{aligned}$$

Exercise 6 (2.7: 5)**.** *If E_1, E_2, E_3 is a frame field and $W = \sum f_i E_i$, prove the covariant derivative formula*

$$\nabla_V W = \sum_j \left\{ V[f_j] + \sum_i f_i \omega_{ij}(V) \right\} E_j.$$

Using the linearity of the covariant derivative, its Leibniz property, and the decomposition

$$\nabla_V E_i = \sum_j \omega_{ij}(V) E_j,$$

we have

$$\begin{aligned}
\nabla_V W &= \nabla_V \left(\sum_i f_i E_i \right) \\
&= \sum_i \nabla_V f_i E_i \\
&= \sum_i \left\{ V[f_i] E_i + f_i \nabla_V E_i \right\} \\
&= \sum_i \left\{ V[f_i] E_i + f_i \sum_j \omega_{ij}(V) E_j \right\} \\
&= \sum_i V[f_i] E_i + \sum_{i,j} f_i \omega_{ij}(V) E_j.
\end{aligned}$$

Now we can change the first summation to use j instead of i, since it's just a symbol and doesn't change what we're actually summing over. This then becomes

$$= \sum_j \left\{ V[f_j] + \sum_i f_i \omega_{ij}(V) \right\} E_j.$$

Exercise 7 (2.7: 8)**.** *Let β be a unit-speed curve in $\mathbb{R}^3$ with $\kappa > 0$, and suppose that E_1, E_2, E_3 is a frame field on $\mathbb{R}^3$ such that the restriction of these vector fields to β gives the Frenet frame field T, N, B of β. Prove that*

$$\omega_{12}(T) = \kappa, \quad \omega_{13}(T) = 0, \quad \omega_{23}(T) = \tau.$$

Then deduce the Frenet formulas from the connection equations.

In §2.5:5, we showed that $\nabla_{\alpha'} V = \nabla_T V = \frac{d}{dt} V_\alpha$ for any vector field V. The Frenet formulas then allow us to write the covariant derivative of our frame field with respect to T as

$$\nabla_T \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Thus $\omega_{12}(T), \omega_{13}(T) = 0$, and $\omega_{23}(T)$. The Frenet formulas are clear from the above matrix.

Exercise 8 (2.8: 1)**.** *For a 1-form $\phi = \sum f_i \theta_i$, prove*

$$d\phi = \sum_j \left\{ df_j + \sum_i f_i \omega_{ij} \right\} \wedge \theta_j.$$

Applying d to ϕ gives

$$\begin{aligned} d\phi &= d \sum f_i \theta_i \\ &= \sum d(f_i \theta_i) \\ &= \sum \left\{ df_i \wedge \theta_i + f_i d\theta_i \right\}. \end{aligned}$$

Then by the Cartan structure equation for $d\theta_i$, this becomes

$$= \sum_i df_i \wedge \theta_i + \sum_{i,j} f_i \omega_{ij} \wedge \theta_j.$$

If we use j instead of i as the indexing variable in the leftmost sum, this becomes

$$= \sum_j \left\{ df_j + \sum_i f_i \omega_{ij} \right\} \wedge \theta_j.$$