Exercise 1 (1.2.4)**.** *If $V = y^2U_1 - x^2U_3$ and $W = x^2U_1 - zU_2$, find functions f and g such that the vector field $fV + gW$ can be expressed in terms of U_2 and U_3 only.*

If we let $f = x^2$ and $g = -y^2$, then the vector field $fV + gW$ becomes

$$\begin{aligned} fV + gW &= (fy^2 + gx^2)U_1 - gzU_2 - fx^2U_3 \\ &= (x^2y^2 - x^2y^2)U_1 + y^2zU_2 - x^4U_3 \\ &= y^2zU_2 - x^4U_3. \end{aligned}$$

Exercise 2 (1.2.5)**.** *Let $V_1 = U_1 - xU_3, V_2 = U_2$, and $V_3 = xU_1 + U_3$.*

a. Prove that the vectors $V_1(\mathbf{p}), V_2(\mathbf{p})$, and $V_3(\mathbf{p})$ are linearly independent at each point of $\mathbb{R}^3$.

b. Express the vector field $xU_1 + yU_2 + zU_3$ as a linear combination of V_1, V_2, and V_3.

a. We must show that if α_i are scalars, then for all $\mathbf{p} \in \mathbb{R}^3$,

$$\alpha_1V_1(\mathbf{p}) + \alpha_2V_2(\mathbf{p}) + \alpha_3V_3(\mathbf{p}) = \mathbf{0}$$

implies that each α_i is zero. Substituting in the definition of each V_i gives

$$(\alpha_1 + \alpha_3x(\mathbf{p}))U_1(\mathbf{p}) + \alpha_2U_2(\mathbf{p}) + (\alpha_3 - \alpha_1x(\mathbf{p}))U_3(\mathbf{p}) = \mathbf{0}.$$

Since the coefficients of each $U_i(\mathbf{p})$ must be 0 in order for their sum to be the zero vector, this clearly shows that $\alpha_2 = 0$. Additionally, we have the system

$$\begin{aligned} \alpha_1 + \alpha_3x(\mathbf{p}) &= 0 \\ \alpha_3 - \alpha_1x(\mathbf{p}) &= 0. \end{aligned}$$

Solving for α_1 in terms of α_3 in the first equation and substituting into the second yields

$$\alpha_3(1 + x(\mathbf{p})^2) = 0.$$

Since $(1 + x(\mathbf{p})^2)$ can never be 0, this shows $\alpha_3 = 0$, from which $\alpha_1 = 0$ follows. Since $\mathbf{p}$ was arbitrary, $\mathbf{p}$, $V_1(\mathbf{p}), V_2(\mathbf{p})$, and $V_3(\mathbf{p})$ are linearly independent for all $\mathbf{p} \in \mathbb{R}^3$.

b. If β_i is a scalar field, then we must solve the system

$$\begin{aligned} \beta_1 + \beta_3x &= x \\ \beta_2 &= y \\ \beta_3 - \beta_1x &= z. \end{aligned}$$

This has the solution

$$\beta_1 = \frac{x(1-z)}{1+x^2}$$
$$\beta_2 = y$$
$$\beta_3 = \frac{x^2+z}{1+x^2}.$$

Exercise 3 (1.3.3)**.** *Let $V = y^2U_1 - xU_3$, and let $f = xy, g = z^3$. Compute the following functions.*

a. $V[f] = y^2U_1[xy] - xU_3[xy] = y^3 - 0 = y^3$.

b. $V[g] = y^2U_1[z^3] - xU_3[z^3] = 0 - 3xz^2 = -3xz^2$.

c. $V[fg] = V[f]g = fV[g] = y^3z^3 - 3x^2yz^2$.

d. $fV[g] - gV[f] = -3x^2yz^2 - y^3z^3$ (this is just the above expression but with the first term negated).

e. $V[f^2+g^2] = V[f^2]+V[g^2]$. To reuse old computations, we can notice that since the reals commute, so do scalar fields. Thus by the Leibniz rule, this is $2fV[f] + 2gV[g] = 2xy^4 - 6xz^5$.

f. $V[V[f]] = V[y^3]$, but V is not in terms of U_2 at all, so this is equal to 0.

Exercise 4 (1.3.4)**.** *Prove the identity $V = \sum_i V[x_i]U_i$, where the x_i are the natural coordinate functions.*

Any vector space can be written in terms of its Euclidean coordinate functions as $V = \sum_i v_iU_i$, so evaluating V on the natural coordinate function x_j gives

$$\begin{aligned} V[x_j] &= \sum_i v_iU_i[x_j] \\ &= \sum_i v_i \frac{\partial x_j}{\partial x_i} \\ &= \sum_i v_i\delta_{ij} \\ &= v_j. \end{aligned}$$

Thus $\sum_i V[x_i]U_i = \sum_i v_iU_i = V$.

Exercise 5 (1.5.1)**.** *Let* $\mathbf{v} = (1, 2, -3)$ *and* $\mathbf{p} = (0, -2, 1)$*. Evaluate the following 1-forms on the tangent vector* $\mathbf{v_p}$*:*

a. $y^2\ dx$

b. $z\ dy - y\ dz$

c. $(z^2 - 1)\ dx - dy + x^2\ dz.$

a. $(y^2\ dx)(\mathbf{v_p}) = y^2(\mathbf{p})dx(\mathbf{v_p}) = 4\mathbf{v_p}[x] = 4v_1 = 4.$

b. $(z\ dy - y\ dz)(\mathbf{v_p}) = z(\mathbf{p})dy(\mathbf{v_p}) - y(\mathbf{p})dz(\mathbf{v_p}) = v_2 + 2v_3 = -4.$

c. $((z^2-1)dx-dy+x^2\ dz)(\mathbf{v_p}) = (z^2-1)(\mathbf{p})v_1-v_2+x^2(\mathbf{p})v_3 = 0-2+0 = -2.$

Exercise 6 (1.5.2)**.** *If* $\phi = \sum_i f_i dx_i$ *and* $V = \sum_i v_i U_i$*, show that the 1-form* ϕ *evaluated on the vector field* V *is the function* $\phi(V) = \sum_i f_i v_i$*.*

This relies on the fact that if $\phi = \sum_i f_i\ dx_i$, then $f_i = \phi(U_i)$. Then since ϕ is linear,

$$\begin{aligned} \phi(V) &= \phi\left(\sum_i v_i U_i\right) \\ &= \sum_i v_i \phi(U_i) \\ &= \sum_i v_i f_i. \end{aligned}$$

Exercise 7 (1.5.4a)**.** *Express the differential* $d(f^5)$ *in terms of* df*.*

This is a straightforward application of the identity $d(h(f)) = h'(f)\ df$.

$$d(f^5) = 5f^4\ df.$$

Exercise 8 (1.5.6a)**.** *For* $f = xy^2 - yz^2$*, compute the differential of* f *and find the directional derivative* $\mathbf{v_p}[f]$*, for* $\mathbf{v_p}$ *as in Exercise 1.5.1.*

For general f, the differential is $df = \sum_i \frac{\partial f}{\partial x_i}\ dx_i$. Then the differential for our particular f is

$$df = y^2\ dx + (2xy - z^2)\ dy - 2yz\ dz.$$

The directional derivative of f with respect to the tangent vector from Exercise 1.5.1 is then

$$\begin{aligned}\mathbf{v_p}[f] &= df(\mathbf{v_p}) \\ &= p_2^2 v_1 + (2p_1p_2 - p_3^2)v_2 - 2p_2p_3v_3 \\ &= 4 - 2 - 12 \\ &= -10.\end{aligned}$$

Exercise 9 (1.5.9)**.** *A 1-form of ϕ is **zero** at a point $\mathbf{p}$ provided $\phi(\mathbf{v_p}) = 0$ for all tangent vectors at $\mathbf{p}$. A point whose differential df is zero is called a **critical point** of the function f. Prove that $\mathbf{p}$ is a critical point of f if and only if*

$$\frac{\partial f}{\partial x}(\mathbf{p}) = \frac{\partial f}{\partial y}(\mathbf{p}) = \frac{\partial f}{\partial z}(\mathbf{p}) = 0.$$

Find all critical points of $f = (1-x^2)y + (1-y^2)z$.

Forward: Let $\mathbf{p}$ be a critical point of f, then for all tangent vectors $\mathbf{v_p}$ of $\mathbf{p}$,

$$df(\mathbf{v_p}) = \sum \frac{\partial f}{\partial x_i}(\mathbf{p}) dx_i(\mathbf{v_p}) = 0.$$

Evaluating this on the tangent vector $(1,0,0)_\mathbf{p}$ yields

$$\frac{\partial f}{\partial x_1}(\mathbf{p}) = 0.$$

Similarly, evaluating on $(0,1,0)_\mathbf{p}$ and $(0,0,1)_\mathbf{p}$ show that the other two partial derivatives are also 0.

Backward: Suppose the three given partial derivatives of f are 0 at p, then

$$df(\mathbf{v_p}) = \sum \frac{\partial f}{\partial x_i}(\mathbf{p}) dx_i(\mathbf{v_p}) = \sum 0 dx_i(\mathbf{v_p}) = 0$$

for all tangent vectors $\mathbf{v_p}$ of $\mathbf{p}$. Thus p is a critical point of f.

Finding critical points: For the given function f, the three partial derivatives are

$$\begin{aligned}\frac{\partial f}{\partial x} &= -2xy, \\ \frac{\partial f}{\partial y} &= 1 - x^2 - 2yz, \text{ and} \\ \frac{\partial f}{\partial z} &= 1 - y^2.\end{aligned}$$

Setting these to 0 gives a system whose solutions are the critical points of f. Using $\frac{\partial f}{\partial z} = 0$, we see that $y(\mathbf{p})$ must be ± 1. Then by $\frac{\partial f}{\partial x} = 0$, since $y(\mathbf{p})$ is nonzero, it must be the case that $x(\mathbf{p})$ is zero. We can then solve for $z(\mathbf{p})$ using $\frac{\partial f}{\partial y} = 0$, giving the two critical points

$$(0, 1, 1/2) \text{ and } (0, -1, -1/2).$$

Exercise 10 (1.5.10)**.** *Prove that the local maxima and local minima of f are critical points of f.*

It suffices to show only the case of local maxima, as the local minima of f are the local maxima of $-f$. Thus we will show that if $\mathbf{p}$ is a local max of f, then $df(\mathbf{v_p}) = 0$ for all tangent vectors $\mathbf{v_p}$ of $\mathbf{p}$. By Exercise 1.5.9, this is equivalent to showing that the partial derivatives of f with respect to the Euclidean coordinate functions are all 0.

We first consider

$$\frac{\partial f}{\partial x}(\mathbf{p}) = \lim_{h \to 0} \frac{f(\mathbf{p} + h\mathbf{e}_1) - f(\mathbf{p})}{h}.$$

Assuming f is differentiable, this quantitiy exists. First note that since $\mathbf{p}$ is a local maximum of f, there is some $\delta > 0$ such that

$$f(\mathbf{p} + he_1) \leq f(\mathbf{p})$$

when $0 < h < \delta$. Then the limit from above, since h is always positive, satisfies

$$\lim_{h \searrow 0} \frac{f(\mathbf{p} + he_1) - f(\mathbf{p})}{h} \leq 0.$$

When h approaches 0 from below, however, the denominator is negative, so the limit from below satisfies

$$\lim_{h \nearrow 0} \frac{f(\mathbf{p} + he_1) - f(\mathbf{p})}{h} \geq 0.$$

Since the limit exists, these two quantities must be equal, meaning that $\frac{\partial f}{\partial x}(\mathbf{p}) = 0$.

Similarly, the partial derivaties $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are also 0 at $\mathbf{p}$. Thus $\mathbf{p}$ is a critical point of f.

Exercise 11 (1.6.3)**.** *For any function f, show that $d(df) = 0$. Deduce that $d(f\,dg) = df \wedge dg$.*

Since $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$, we can write $d(df)$ as

$$\begin{aligned} d(df) &= \sum_i d\left(\frac{\partial f}{\partial x_i}\right) dx_i \\ &= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i. \end{aligned}$$

Since f is continuous,
$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$
for all pairs i, j. Then for all pairs i, j, there are two terms in the sum
$$\frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i \text{ and } \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j$$
whose coefficients are equivalent. Then by the anti-commutativity of the wedge product, the sum of each of these pairs is 0, so the whole sum is 0. Thus $d(df) = 0$.

Then $d(f\ dg) = df \wedge dg + f\ d(dg) = df \wedge dg$.

Exercise 12 (1.6.4)**.** *Simplify the following forms:*

a. $d(f\ dg + g\ df)$.

b. $d((f - g)(df + dg))$.

c. $d(f\ dg + g\ df)$.

d. $d(gf\ df) + d(f\ dg)$.

All of these calculations at some point or another use the identities $d^2 = 0$ and $d(f\ dg) = df \wedge dg$, as derived in the previous problem.

a.
$$\begin{aligned} d(f\ dg + gdf) &= df \wedge dg + dg \wedge gf \\ &= df \wedge dg - df \wedge dg \\ &= 0. \end{aligned}$$

b.
$$\begin{aligned} d((f - g)(df + dg)) &= d(f - g) \wedge (df + dg) + (f - g)d(df + dg) \\ &= (df - dg) \wedge (df + dg) + (f - g)0 \\ &= (df \wedge dg) - (dg \wedge df) \\ &= 2(df \wedge dg). \end{aligned}$$

c.
$$\begin{aligned} d(f\ dg \wedge g\ df) &= (d(f\ dg) \wedge g\ df) - (f\ dg \wedge d(g\ df)) \\ &= (df \wedge dg \wedge g\ df) - (f\ dg \wedge dg \wedge df) \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

d.

$$\begin{aligned} d(gf\ df) + d(f\ dg) &= (d(gf) \wedge df) + (df \wedge dg) \\ &= ((dg)f + g\ df) \wedge df + (df \wedge dg) \\ &= (f\ dg \wedge df) + (df \wedge dg) \\ &= (1-f)(df \wedge dg). \end{aligned}$$

Exercise 13 (1.6.6)**.** *If r, θ, z are the cylindrical coordinate functions on $\mathbb{R}^3$, then $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. Compute the* ***volume element*** *$dx\ dy\ dz$ of $\mathbb{R}^3$ in cylindrical coordinates (that is, express $dx\ dy\ dz$ in terms of the functions r, θ, z, and their differentials).*

In cylindrical coordinates, dx, dy, and dz become

- $dx = d(r\cos\theta) = dr\ \cos\theta + r\ d(\cos\theta) = \cos\theta\ dr - r\sin\theta\ d\theta$,
- $dy = dr\ \sin\theta + rd(\sin\theta) = \sin\theta\ dr + r\cos\theta\ d\theta$, and
- $dz = dz$.

Their wedge product simplifies considerably, as the $dr \wedge dr$ and $d\theta \wedge d\theta$ terms become 0.

$$\begin{aligned} dx \wedge dy \wedge dz &= (\cos\theta\ dr - r\sin\theta\ d\theta) \wedge (\sin\theta\ dr + r\cos\theta\ d\theta) \wedge dz \\ &= (r\cos^2\theta)\ dr\ d\theta\ dz - (r\sin^2\theta)\ d\theta\ dr\ dz \\ &= r(\cos^2\theta + \cos^2\theta)\ dr\ d\theta\ dz \\ &= r\ dr\ d\theta\ dz, \end{aligned}$$

where the last line follows from the identity $\cos^2\theta + \sin^2\theta = 1$.

Exercise 14 (1.6.7)**.** *Prove that for any 1-form ϕ, $d(d\phi) = 0$.*

$\phi = \sum f_i\ dx_i$, so $d\phi$ is the 2-form $\sum df_i \wedge dx_i$. By the definition of d on forms, applying d to just the scalar components of this 2-form is the same result as applying d to each df_i. Then by Exercise 1.6.3,

$$d\phi = \sum d(df_i) \wedge dx_i = 0.$$

Exercise 15 (1.6.8)**.** *Prove that the gradient, curl, and divergence can be expressed as exterior derivatives.*

a. The differential of f can be written

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

and the gradient of f is

$$\text{grad } f = \sum \frac{\partial f}{\partial x_i} U_i.$$

Then corresponding every dx_i and U_i gives $df \overset{(1)}{\leftrightarrow} \text{grad } f$.

b. Assume $\phi \overset{(1)}{\leftrightarrow} V$. Applying d to ϕ then yields

$d\phi = df_1\ dx + df_2\ dy + df_3\ dz.$

Expanding each df_i and removing instances of $dx_i \wedge dx_i$ gives

$$\begin{aligned}
&= \left(\frac{\partial f_1}{\partial x_2} dy + \frac{\partial f_1}{\partial x_3} dz\right) dx + \left(\frac{\partial f_2}{\partial x_1} dx + \frac{\partial f_2}{\partial x_3} dz\right) dy + \\
&\left(\frac{\partial f_3}{\partial x_1} dx + \frac{\partial f_3}{\partial x_2} dy\right) dz \\
&= \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx\ dy - \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) dx\ dz + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dy\ dz.
\end{aligned}$$

The given (2) correspondence then allows us to associate this with

$$\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) U_1 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) U_2 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) U_3,$$

which, since $\phi \overset{(1)}{\leftrightarrow} V$, corresponds to curl V.

c. Assume $\mu \overset{(1)}{\leftrightarrow} V$. We can write any 2-form μ as

$$\mu = f_3\ dx\ dy + f_2\ dz\ dx + f_1\ dy\ dz,$$

so applying d yields

$$d\mu = df_3\ dx\ dy + df_2\ dz\ dx + df_1\ dy\ dz.$$

Expanding each df_i and removing $dx_i \wedge dx_i$ terms gives

$$\begin{aligned}
&= \frac{\partial f_3}{\partial x_3} dz\ dx\ dy + \frac{\partial f_2}{\partial x_2} dy\ dz\ dx + \frac{\partial f_1}{\partial x_1} dx\ dy\ dz \\
&= \left(\sum_i \frac{\partial f_i}{\partial x_i}\right) dx\ dy\ dz.
\end{aligned}$$

Since $\mu \overset{(1)}{\leftrightarrow} V$, this is the same as (div V)$dx\ dy\ dz$.