Contents

Chapter 1

The Fundamental Group

1.1 BASICS

Proposition 1. $\pi_1(S^n) = 0$ when $n \geq 2$.

Proof. **Do this.** $\square$

Corollary 1. $\mathbb{R}^2 \not\cong \mathbb{R}^n$ when $n \neq 2$.

Proof. When $n = 1$, removing a point destroys the path connectivity of $\mathbb{R}$ but not of $\mathbb{R}^2$. For $n > 2$, use the relation $\mathbb{R}^n - \{0\} \cong S^{n-1} \times \mathbb{R}$ and compare fundamental groups, using the above proposition. $\square$

You might expect that if we can deformation retract one space onto another, they'll have the same fundamental groups (with fixed basepoint, of course). Although this doesn't hold with plain retracts, we still get some information about the relationship between the fundamental groups.

Proposition 2. *If X retracts onto A, then*

$$i_* : \pi_1(A, x) \to \pi_1(X, x)$$

is surjective. If A is a deformation retract of X, then i_* is an isomorphism, i.e.

$$\pi_1(A, x) \cong \pi_1(X, x).$$

Proof. Given a retraction $r : X \to A$, we have $ri = 1_A$, so $r_* i_* = 1_{\pi_1(A)}$, so i_* is injective. If r_t is a deformation retraction of X onto A, then $r_0 = 1_X$, $r_t|A = 1_A$, and $r_1(X) \subset A$. Let f be any loop in X, then $r_t f$ gives a homotopy to some loop in A. Thus i_* is surjective, so it's an isomorphism. $\square$

Consider homotopic functions $f \simeq g$. It seems reasonable that the fundamental groups of the codomain at $f(x)$ and at $g(x)$ will be isomorphic, as f and g are related by some continuous deformation. We can in fact show more than this: the isomorphism in question plays nicely with the induced maps f_* and g_*.

Lemma 1. Let $f \simeq g$ and let h be the map between $f(x)$ and $g(x)$ induced by the homotopy. Then
$$\pi_1(Y, f(x)) \cong \pi_1(Y, g(x))$$
and the following diagram commutes.

$$\pi_1(X, x) \xrightarrow{g_*} \pi_1(Y, g(x))$$

$$\downarrow \beta_h$$

$$f_* \searrow$$

$$\pi_1(Y, f(x))$$

Here β_h denotes the change of basepoint isomorphism induced by h.

Proof. **Do this.**

Should this even be a lemma? □

Proposition 3. *Suppose $X \simeq Y$ via ϕ, then $\pi_1(X, x) \cong \pi_1(Y, \phi(x))$ via ϕ_*.*

Proof. **Do this.** □

1.2 Van Kampen's Theorem

The problem: how to calculate π_1 of complicated spaces. The idea: decompose a complicated space into simpler spaces with known π_1, then patch these together somehow to construct π_1 for the entire space.

Our object of interest will be the map ϕ in the following property of free products.

Proposition 4. *Any collection of group morphisms $\{\phi_\alpha : G_\alpha \to H\}_\alpha$ extends uniquely to a group morphism $\phi : *_\alpha G_\alpha \to H$.*

Proof. **Do this.** □

Suppose $X = \bigcup_\alpha A_\alpha$, where each A_α is open, path connected, and contains some fixed basepoint $x_0 \in X$. By the previous proposition, the family of morphisms

$$\{i_{\alpha*} : \pi_1(A_\alpha) \to \pi_1(X)\}_\alpha$$

induced by the inclusions extends uniquely to the morphism

$$\phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$$

(the path connectedness and shared basepoint ensure that the above statments, which don't ever reference a basepoint for any π_1, aren't nonsensical). Van Kampen's Theorem is then, informally, "ϕ is usually surjective, but it's usually not injective."

This means $*_\alpha \pi_1(A_\alpha)$ will usually be "too big", but we can fix this by simply modding out the extra stuff. This will give some subgroup of $*_\alpha \pi_1(A_\alpha)$ that $\pi_1(X)$ is isomorphic to.

Theorem 1 (van Kampen)**.** *Suppose $X = \bigcup_\alpha A_\alpha$, where each A_α is open, path connected, and contains some fixed basepoint $x_0 \in X$.*

  1. *If each $A_\alpha \cap A_\beta$ is path connected, then*

$$\phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$$

  is surjective.

  2. *If each $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then $\ker \phi$ is a normal subgroup of $*_\alpha \pi_1(A_\alpha)$ generated by all elements of the form*

$$i_{\alpha\beta*}(\omega)i_{\beta\alpha*}(\omega)^{-1},$$

  where $i_{\alpha\beta*}$ is the group morphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$. Thus ϕ induces an isomorphim

$$\pi_1(X) \cong \frac{*_\alpha \pi_1(A_\alpha)}{\ker \phi}.$$