Exercises completed: All.

Exercise 1. Suppose X is path connected. Prove that the following are equivalent:

1. $\pi_1(X, x_0)$ is abelian.
2. For paths h_1, h_2 in X from x_0 to x_1, $\beta_{h_1} = \beta_{h_2}$, where β is the change-of-basepoint homomorphism $\beta_h([\alpha]) = [h \cdot \alpha \cdot \overline{h}]$ from $\pi_1(X, x_1)$ to $\pi_1(X, x_0)$.

Collaborators: None.

Forward: Suppose that for all $[a], [b] \in \pi_1(X, x_0)$, we have $[a][b] = [b][a]$, then

$$\beta_{h_1}([\alpha]) = [h_1 \cdot \alpha \cdot \overline{h_1}] = [h_1 \cdot \alpha \cdot \overline{h_2}][h_2 \cdot \overline{h_1}] = [h_2 \cdot \overline{h_1}][h_1 \cdot \alpha \cdot \overline{h_2}] = [h_2 \cdot \alpha \cdot \overline{h_2}] = \beta_{h_2}([\alpha]).$$

Thus $\beta_{h_1} = \beta_{h_2}$.

Backward: Suppose α, β are loops at x_0. Since X is path connected, we can find a path h from x_0 to x_1. Then $\alpha h, h$ are both paths from x_0 to x_1 and $\overline{h}\beta\alpha h$ is a loop at x_1. Note that $\overline{\alpha h} = \overline{h}\overline{\alpha}$.

$$\begin{tikzcd} x_0 \arrow[loop above, "\alpha"] \arrow[loop below, "\beta"] \arrow[r, bend left, "h"] & x_1 \arrow[l, bend left, "\overline{h}"] \end{tikzcd}$$

By assumption $\beta_{\alpha h}([\overline{h}\beta\alpha h]) = \beta_h([\overline{h}\beta\alpha h])$, and

$$\beta_{\alpha h}([\overline{h}\beta\alpha h]) = [\alpha h\overline{h}\beta\alpha h\overline{h}\overline{\alpha}] = [\alpha\beta],$$
$$\beta_h([\overline{h}\beta\alpha h]) = [h\overline{h}\beta\alpha h\overline{h}] = [\beta\alpha].$$

Thus $[\alpha][\beta] = [\beta][\alpha]$ for all loops α, β at x_0.

Exercise 2. Using the fact that $\mathbb{R}^2 - \{0\}$ is homeomorphic to $S^1 \times \mathbb{R}$, prove that $\mathbb{R}^2 - \{0\}$ is not homeomorphic to the torus.

Collaborators: None.

Since all spaces here are path connected, I drop the basepoint of each fundamental group.

We'll use two facts for this:

1. If $X \cong Y$, then $\pi_1(X) \cong \pi_1(Y)$.
2. $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Suppose $\mathbb{R}^2 - \{0\}$ is homeomorphic to the torus $S^1 \times S^1$. Then

$$\pi_1(\mathbb{R}^2 - \{0\}) \cong \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

But by assumption, $\mathbb{R}^2 - \{0\}$ being homeomorphic to $S^1 \times \mathbb{R}$ and $\mathbb{R}$ being simply connected gives

$$\pi_1(\mathbb{R}^2 - \{0\}) \cong \pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z} \times \{e\} \cong \mathbb{Z}.$$

But $\mathbb{Z} \times \mathbb{Z}$ is not isomorphic to $\mathbb{Z}$, so by contradiction, $\mathbb{R}^2 - \{0\}$ cannot be homeomorphic to the torus.

Exercise 3 (Munkres §55 #1). Show that if A is a retract of B^2, then every continuous map $f : A \to A$ has a fixed point.

Collaborators: None.

If A is a retract of B^2, then there is some continuous map $r : B^2 \to A$ that fixes A. Then if $f : A \to A$ is any continuous function, consider the map $g = ifr$. Since g is the composition of continuous functions, it is continuous. Then since it's a map from B^2 to B^2, it has a fixed point x by the Brouwer fixed point theorem. Then $r(x)$ is a fixed point of f, since

$$f(r(x)) = (rifr)(x) = r(g(x)) = r(x).$$

Exercise 4 (Munkres §55 #2)**.** Show that if $h : S^1 \to S^1$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Collaborators: None.

By Lemma 55.3, h extends to a continuous function $k : B^2 \to S^1$, i.e. $h = ki$, where i is the standard inclusion $i : S^1 \hookrightarrow B^2$. Then the composition

$$\tilde{k} : B^2 \xrightarrow{k} S^1 \xhookrightarrow{i} B^2$$

is a continuous map $B^2 \to B^2$. Then by the Brouwer fixed point theorem, there is some $y \in B^2$ such that $\tilde{k}(y) = (ik)(y) = y$. Note that since $y \in i(S^1)$, then $i^{-1}(y)$ is well-defined. This is in fact the fixed point of h, since

$$h(i^{-1}(y)) = (i^{-1}ikii^{-1})(y) = (i^{-1}\tilde{k})(y) = i^{-1}(y).$$

Consider $-h$, which is also a continuous function $S^1 \to S^1$. We now know that $-h$ has a fixed point, i.e. there is some $x \in S^1$ such that $-h(x) = x$. But this implies $h(x) = -x$, so h must map some point to its antipode.

Exercise 5. Show that if A is a nonsingular 3 by 3 matrix having nonnegative entries, then A has a positive real eigenvalue.

Collaborators: None.

Consider the intersection of S^2 and the first octant of R^3, and denote it by X. Then $x \in X$ has all nonnegative components and at least 1 positive component. Since A has all nonnegative entries, this means Ax has all nonnegative components. We claim that Ax is in fact nonzero.

Suppose $Ax = 0$, then since A is nonsingular (and thus invertible), $A^{-1}Ax = A^{-1}0$, which implies $x = 0$. But $0 \notin X$, so this is impossible. Thus $Ax \neq 0$ for all x.

This means that $x \mapsto Ax/\|Ax\|$ is a well-defined map from X to X. Then since $X \cong B^2$, the Brouwer fixed point theorem says that it has a fixed point, i.e. there is some x such that

$$\frac{Ax}{\|Ax\|} = x.$$

But this implies $Ax = \|Ax\|x$, so $\|Ax\|$ is an eigenvalue of A. Since $Ax \neq 0$, we know this eigenvalue is strictly positive.