Exercise 1 (7.2: 1)**.** The Poincare half-plane has $K = -1$.

Since $\langle \mathbf{v}, \mathbf{w} \rangle = (\mathbf{v} \cdot \mathbf{w})/v^2$,

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1/v^2, \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0, \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1/v^2. \end{aligned}$$

Then

$$\begin{aligned} (\sqrt{G})_u &= (1/v)_u = 0 \\ (\sqrt{E})_v &= (1/v)_v = -1/v^2, \end{aligned}$$

so by the formula on page 297,

$$\begin{aligned} K &= -\frac{1}{\sqrt{EG}} \left[\left(\frac{(\sqrt{G})_u}{\sqrt{E}} \right)_u + \left(\frac{(\sqrt{E})_v}{\sqrt{G}} \right)_v \right] \\ &= -v^2 \left[0 + (-1/v)_v \right] \\ &= -v^2 (1/v^2) \\ &= -1. \end{aligned}$$

> **Exercise 2** (7.2: 2)**.** Find the dual forms, connection forms, and K for the conformal structure on the entire plane with $h = \operatorname{sech}(uv)$.

If we let $T = \tanh(uv)$ and $S = \operatorname{sech}(uv)$, then $\theta_1 = du/S$ and $\theta_2 = dv/S$.

$$d\theta_1 = d(1/S) \wedge du = \frac{Tu}{S} du \wedge dv,$$
$$d\theta_2 = d(1/S) \wedge dv = -\frac{Tv}{S} du \wedge dv.$$

Then by the first structural equations,

$$\begin{aligned} d\theta_1 &= \omega_{12} \wedge \theta_2 \\ (Tu/S) du \wedge dv &= \omega_{12} dv/S \end{aligned}$$

and

$$\begin{aligned} d\theta_2 &= -\omega_{12} \wedge \theta_1 \\ (-Tv/S) du \wedge dv &= du/S \wedge \omega_{12}. \end{aligned}$$

This implies

$$\omega_{12} = Tu\ du - Tv\ dv.$$

Then by the second structural equation,

$$\begin{aligned} d\omega_{12} &= -K\theta_1 \wedge \theta_2 \\ -S^2(u^2+v^2)\ du \wedge dv &= -\frac{K}{S^2}\ du \wedge dv, \end{aligned}$$

so $K = S^4(u^2+v^2)$. Using Corollary 2.3 gives the same result: we calculate

$$h_u = -TSv, \quad h_v = -TSu, \quad h_{uu} = -Sv^2(S^2 - T^2), \quad h_{vv} = -Su^2(S^2 - T^2).$$

Then by the corollary,

$$\begin{aligned} K &= h(h_{uu} + h_{vv}) - (h_u^2 + h_v^2) \\ &= -S^2\left[(u^2+v^2)(S^2 - T^2) - (T^2v^2 + t^2u^2)\right] \\ &= S^2\left[-v^2S^2 - u^2S^2\right] \\ &= -S^4(u^2+v^2). \end{aligned}$$

> **Exercise 3** (7.2: 3)**.** Find the area of the disk $u^2+v^2 \leq a^2$ in the hyperbolic plane.

With the frame hU_1, hU_2, we have the dual frame $\theta_1 = du/h, \theta_2 = dv/h$. The area form is then given by $\theta_1 \wedge \theta_2 = (1/h^2)\ du \wedge dv$. Converting to polar coordinates, the area form becomes

$$\frac{r}{h^2}\ dr \wedge d\theta,$$

so the area of the disk is

$$\begin{aligned}\int_0^{2\pi} \int_0^a \frac{r}{h^2}\ dr\ d\theta &= \int_0^{2\pi} \left[\frac{2}{h}\right]_{r=0}^{r=a}\ dv \\ &= 4\pi \left(\frac{1}{1-a^2/4} - 1\right) \\ &= \frac{\pi a^2}{1-\frac{a^2}{4}}.\end{aligned}$$

Then since the entire hyperbolic disk has radius 2, its area is the limit of this expression as $a \to 2$, which is ∞.

Exercise 4 (7.2: 4)**.** $H(r)$ has constant Gaussian curvature $K = -1/r^2$.

Since $h = 1 - \frac{u^2+v^2}{4r^2}$, we calculate

$$\begin{aligned} h_u &= -\frac{u}{2r^2}, \quad h_{uu} = -\frac{1}{2r^2}, \\ h_v &= -\frac{v}{2r^2}, \quad h_{vv} = -\frac{1}{2r^2}. \end{aligned}$$

Then by Corollary 2.3,

$$\begin{aligned} K &= h(h_{uu} + h_{vv}) - (h_u^2 + h_v^2) \\ &= \left(1 - \frac{u^2+v^2}{4r^2}\right)\left(-\frac{1}{r^2}\right) - \frac{u^2+v^2}{4r^4} \\ &= \frac{u^2+v^2-4r^2}{4r^4} - \frac{u^2+v^2}{4r^4} \\ &= -\frac{1}{r^2}. \end{aligned}$$

Exercise 5 (7.2: 7)**.** Scale changes.

a. The norm on the scaled surface is

$$\|\mathbf{v}\|^{-} = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle^{-}} = \sqrt{c^2 \langle \mathbf{v}, \mathbf{v} \rangle} = c\|\mathbf{v}\|.$$

Then if θ is the angle between $\mathbf{v}, \mathbf{w} \in M$ and $\overline{\theta}$ is the corresponding angle in $\overline{M}$,

$$\cos \overline{\theta} = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|^{-} \|\mathbf{w}\|^{-}} = \frac{c^2 \langle \mathbf{v}, \mathbf{w} \rangle}{c^2 \|\mathbf{v}\| \|\mathbf{w}\|} = \cos \theta.$$

Thus angles are preserved.

b. The length of α in $\overline{M}$ is

$$\overline{L}(\alpha) = \int_\alpha \|\alpha'\|^{-} = c \int_\alpha \|\alpha'\| = cL(\alpha).$$

c. We need $\overline{\theta}_i(\overline{E}_j) = \overline{\theta}_i(E_j)/c = \delta_{ij}$, so $\overline{\theta}_i = c\theta_i$. Then by the first structural equations,

$$c(\overline{\omega}_{12} \wedge \theta_2) = \overline{\omega}_{12} \wedge \overline{\theta}_2 = d\overline{\theta}_1 = c \, d\theta_1 = c(\omega_{12} \wedge \theta_2),$$

so $\overline{\omega}_{12} = \omega_{12}$.

d. The area form on $\overline{M}$ is

$$d\overline{M} = \overline{\theta}_1 \wedge \overline{\theta}_2 = c^2 \, \theta_1 \wedge \theta_2 = c^2 \, dM,$$

so the area of $\mathscr{R}$ is

$$\int_{\mathscr{R}} d\overline{M} = c^2 \int_{\mathscr{R}} dM = c^2 A.$$

Thus $\mathscr{R}$ has area A in M if and only if it has area $c^2 A$ in $\overline{M}$.

e. By the second structural equation and part (c),

$$-c^2 \overline{K} \, \theta_1 \wedge \theta_2 = -\overline{K} \, \overline{\theta}_1 \wedge \overline{\theta}_2 = d\overline{\omega}_{12} = d\omega_{12} = -K \, \theta_1 \wedge \theta_2.$$

Thus $\overline{K} = K/c^2$.

Exercise 6 (7.2: 8)**.** a. $S(r)$ and Σ scaled by r are isometric.

b. $H(r)$ and $H(1)$ scalred by r are isometric.

a. Based on the standard parameterization of $S(r)$

$$\mathbf{x} = (r\sin u, \cos v, r\sin u \sin v, r\sin u),$$

we get partials

$$\begin{aligned}\mathbf{x}_u &= (r\cos u\cos v, r\cos u\sin v, r\cos u)\\ \mathbf{x}_v &= (r\sin u\sin v, r\sin u\cos v, 0).\end{aligned}$$

We can then calculate

$$\mathbf{x}_u\cdot\mathbf{x}_u = 2r^2\cos^2 u,\quad \mathbf{x}_u\cdot\mathbf{x}_v = 0,\quad \mathbf{x}_v\cdot\mathbf{x}_v = r^2\sin^2 u.$$

From this we see that the dot product between *any* two tangent vectors on $S(r)$ is a linear combination of these terms. Now define the map

$$\begin{aligned}F : S(r) &\to \Sigma \text{ scaled by } r\\ \mathbf{p} &\mapsto \mathbf{p}/r.\end{aligned}$$

This is clearly bijective. Additionally, in Σ scaled by r we have

$$\begin{aligned}F_*\mathbf{x}_u\cdot F_*\mathbf{x}_u &= r^2(2\cos^2 u),\\ F_*\mathbf{x}_v\cdot F_*\mathbf{x}_v &= r^2(\sin^2 u),\end{aligned}$$

which are equivalent to $\mathbf{x}_u\cdot\mathbf{x}_u$ and $\mathbf{x}_v\cdot\mathbf{x}_v$ in $S(r)$. Thus F is an isometry.

b. We claim that the map

$$\begin{aligned}F : H(r) &\to H(1) \text{ scaled by } r\\ \mathbf{p} &\mapsto \mathbf{p}/r\end{aligned}$$

is an isometry. It is clearly bijective, so we must show that it is metric preserving. Note that F_* maps $\mathbf{v}\mapsto\mathbf{v}/r$. Then for all $\mathbf{v},\mathbf{w}$,

$$\begin{aligned}\langle F_*\mathbf{v}, F_*\mathbf{w}\rangle &= r^2\frac{(1/r^2)\mathbf{v}\cdot\mathbf{w}}{\left(1-\frac{(u^2+v^2)(1/r^2)}{4}\right)^2}\\ &= \frac{\mathbf{v}\cdot\mathbf{w}}{\left(1-\frac{(u^2+v^2)}{4r^2}\right)^2}\\ &= \langle \mathbf{v},\mathbf{w}\rangle,\end{aligned}$$

where the inner product on the first line is in $H(1)$ scaled by r and the inner product on the last line is in $H(r)$. Thus F is an isometry.

Exercise 7 (7.2: 9)**.** Classical tensor formula for Gaussian curvature.

a. We have

$$\begin{aligned}
\langle E_1, E_1\rangle &= \frac{\langle \mathbf{x}_u, \mathbf{x}_u\rangle}{E} = \frac{E}{E} = 1, \\
\langle E_2, E_2\rangle &= \frac{1}{W^2 E}\left(E^2 \langle \mathbf{x}_v, \mathbf{x}_v\rangle - 2EF\langle \mathbf{x}_u, \mathbf{x}_v\rangle + F^2\langle \mathbf{x}_u, \mathbf{x}_u\rangle\right) \\
&= \frac{(EG - F^2)E}{(EG - F^2)E} = 1, \\
\langle E_1, E_2\rangle &= \frac{1}{WE}\left(E\langle \mathbf{x}_u, \mathbf{x}_v\rangle - F\langle \mathbf{x}_u, \mathbf{x}_u\rangle\right) \\
&= \frac{1}{WE}(EF - EF) = 0,
\end{aligned}$$

so E_1, E_2 are orthonormal.

b. Since $\theta_i = \langle E_i, \mathbf{x}_u\rangle\, du + \langle E_i, \mathbf{x}_v\rangle\, dv$,

$$\begin{aligned}
\theta_1 &= \sqrt{E} du + \frac{F}{\sqrt{E}} dv \\
\theta_2 &= \frac{1}{W\sqrt{E}}\left[\langle E\mathbf{x}_v - F\mathbf{x}_u, \mathbf{x}_u\rangle\, du + \langle E\mathbf{x}_v - F\mathbf{x}_u, \mathbf{x}_v\rangle\, dv\right] \\
&= \frac{1}{W\sqrt{E}}\left[(EF - EF)du + (EG - F^2)dv\right] \\
&= \frac{W}{\sqrt{E}} dv.
\end{aligned}$$

c. By part (b) and the first structural equations,

$$\begin{aligned}
d\theta_1 &= (P\, du + Q\, dv) \wedge \frac{W}{\sqrt{E}}\, dv \\
d\theta_2 &= (-P\, du - Q\, dv) \wedge (\sqrt{E}\, du + \frac{F}{\sqrt{E}}\, dv).
\end{aligned}$$

Then manually calculating $d\theta_1$ and $d\theta_2$ and solving for P and Q yields

$$\begin{aligned}
P &= \frac{2EF_u - FE_u - EE_v}{2EW} \\
Q &= -\frac{FE_v - EG_u}{2EW}.
\end{aligned}$$

d. By the second structural equation,

$$\begin{aligned}
d\omega_{12} &= -K\theta_1 \wedge \theta_2 \\
(P_v - Q_u) du \wedge dv &= (KW)\, du \wedge dv,
\end{aligned}$$

which implies

$$\begin{aligned}
K &= \frac{P_v - Q_u}{W} \\
&= \frac{1}{2W}\left[\frac{\partial}{\partial v}\left(\frac{2EF_u - FE_u - EE_v}{EW}\right) + \frac{\partial}{\partial u}\left(\frac{FE_v - EG_u}{EW}\right)\right].
\end{aligned}$$

e. If $\mathbf{x}$ is orthogonal, then $F = 0$ (and by extension, $W = \sqrt{EG}$), so

$$\begin{aligned}
K &= -\frac{1}{2W}\left[\left(\frac{E_v}{W}\right)_v + \left(\frac{G_u}{W}\right)_u\right] \\
&= -\frac{1}{2\sqrt{EG}}\left[\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right].
\end{aligned}$$

But $(2\sqrt{f})_x = f_x/\sqrt{f}$ for any function f, so this becomes

$$K = -\frac{1}{\sqrt{EG}}\left[\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u + \left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v\right],$$

which matches proposition 6.3 from chapter 6.

f. Since $E = 1 + v^2, F = uv, G = 1 + u^2$, we calculate $E_u = 0, E_v = 2v, F_u = v, G_u = 2u$, and $W = \sqrt{EG - F^2} = \sqrt{1 + u^2 + v^2}$. Then substituting into part (d) yields

$$\begin{aligned}
K &= \frac{1}{2W}\left[(0)_v - \left(\frac{2u}{(1 + v^2)W}\right)_u\right] \\
&= \frac{1}{(1 + v^2)W}\left[\frac{W = u^2W^{-1}}{W^2}\right] \\
&= \frac{1}{1 + v^2}\left[\frac{W^2 - u^2}{W^4}\right] \\
&= \frac{1}{W^4} \\
&= \frac{1}{(1 + u^2 + v^2)^2},
\end{aligned}$$

as desired.