0.1 Chain Complexes

Definition 1. A **chain complex** is a sequence of abelian group homomorphisms (change to modules?)

$$\cdots \xrightarrow{d_{i+2}} C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all i.

We can also consider **cochain complexes**, which are the same except that the maps take you up a level instead of down.

$$\cdots \xleftarrow{d_{i+1}} C_{i+1} \xleftarrow{d_i} C_i \xleftarrow{d_{i-1}} C_{i-1} \xleftarrow{d_{i-2}} \cdots$$

The map d_i is the **boundary operator**, as it is a generalization of the geometric concept of a boundary (note $d^2 = 0$). Thus an element in the image of d is a **boundary**. Since usual geometric cycles have no boundary, we call the elements of the kernel of d **cycles**.

Example 1. Chain complexes generalize the concept of boundaries to objects that don't necessarily have clear cyclic geometric properties. Let $\mathfrak{X}(\mathbb{R}^3)$ denote the smooth vector fields on $\mathbb{R}^3$, then consider the chain complex

$$0\mathbb{R} \hookrightarrow C^\infty(\mathbb{R}^3) \xrightarrow{\text{grad}} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\text{curl}} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\mathbb{R}^3) \longrightarrow 0$$

If we consider the grad map, we see that its "cycles" are actually just constant functions.

A map $f : C \to D$ between chain complexes is a sequence of maps

$$f_i : C_i \to D_i$$

that respect the boundar operators. This means the following diagram commutes.

$$\begin{tikzcd}
C_i \arrow[r, "d_C"] \arrow[d, "f_i"'] & C_{i-1} \arrow[d, "f_{i-1}"] \\
D_i \arrow[r, "d_D"'] & D_{i-1}
\end{tikzcd}$$

Definition 2. The n-th **homology group** of a chain complex $\mathcal{C}$ is

$$H_n(\mathcal{C}) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

Similarly, the n-th **cohomology group** is $H^n(C) = \ker d_{n+1} / \operatorname{im} d_n$. In both cases the quotient represents the n-th dimensional holes in the complex, as it is the cycles that do not arise as boundaries of higher dimensional objects.

A chain complex C is exact if and only if $H_n(C)$ is trivial for all n.

Given a map $f : C \to D$ of chain complexes, the commutativity of the above diagram shows that f sends cycles to cycles and boundaries to boundaries, thus inducing a map $H_n(C) \to H_n(D)$.

Definition 3. A **quasi-isomorphism** is a map $f : C \to D$ of chain complexes where the induced map $H_n(C) \to H_n(D)$ is an isomorphism.