Problems completed: All.

Exercise 1 (6 points)**.** Find a counterexample to the following statement. If every sequence $\{x_n\}_{n=1}^{\infty}$ converges to at most 1 point in X, then X is Hausdorff.

Collaborators: Saloni Bulchandani.

We consider the topology on $\mathbb{R}$ given by

$$\mathcal{T} = \{U \mid \mathbb{R} - U \text{ countable }\} \cup \{\varnothing\}\,.$$

We first show that this is actually a topology. It contains $\varnothing$ by definition, and $\mathbb{R} \in \mathcal{T}$ since $\mathbb{R} - \mathbb{R} = \varnothing$ is countable. It is closed under arbitrary unions, since for $U_\alpha \in \mathcal{T}$, $\mathbb{R} - \bigcup_\alpha U_\alpha = \bigcap_\alpha (\mathbb{R} - U_\alpha)$, which is the intersection of countable sets and so is also countable. It is also closed under finite intersections, since for $U_i \in \mathcal{T}$, $\mathbb{R} - \bigcap_{i=1}^{N} U_i = \bigcup_{i=1}^{N} (\mathbb{R} - U_i)$, which is the finite union of countable sets and so is also countable.

Unique Limits: Now we show that $\mathcal{T}$ has unique limits by showing that any convergent sequence in $\mathcal{T}$ must eventually be constant. Suppose $x_n \to x$ with respect to $\mathcal{T}$. Define the set U_x by

$$U_x \doteq (\mathbb{R} - \{x_n\}_{n=1}^{\infty}) \cup \{x\}\,,$$

then its complement is

$$\mathbb{R} - U_x = \{x_n \mid x_n \neq x\}\,.$$

Since sequences are countable and this is a subset of the sequence $\{x_n\}_{n=1}^{\infty}$, this means U_x is an open set. In particular, it is a neighborhood of x. So if $\{x_n\}_{n=1}^{\infty}$ is not eventually x, then it is never eventually in this neighborhood of x, so it cannot converge to x. Thus any convergent sequence in this topology must eventually be constant, meaning that it cannot converge to more than one point.

Not Hausdorff: Now we show that $\mathbb{R}$ with this topology is not Hausdorff. Suppose $x_1 \neq x_2$ and U_1 and U_2 are neighborhoods of x_1 and x_2, respectively. Note that since $\mathbb{R}$ is uncountable and the complements of U_1 and U_2 are bouth countable, then both U_1 and U_2 must be uncountable. Now suppose U_1 and U_2 do not intersect, then $U_1 \subset \mathbb{R} - U_2$, but this is impossible, as $\mathbb{R} - U_2$ is countable and U_2 is uncountable. Thus U_1 and U_2 must intersect, so this space is not Hausdorff.

Exercise 2 (6 points). a. Munkres §13, pg. 83 #5.

b. Equip $\mathbb{R}^\infty = \Pi_{i\in\mathbb{Z}^+}\mathbb{R}$ with the product topology. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}^\infty$ defined by $f(x) = (x, x, \dots)$ is continuous.

Collaborators: Saloni Bulchandani.

a. Denote the topology generated by $\mathcal{T}_A$, and denote the intersection of all topologies containing $\mathcal{A}$ by $\bigcap_\beta \mathcal{T}_\beta$.

   First we show that $\mathcal{T}_A$ is contained in $\bigcap_\beta \mathcal{T}_\beta$. Since each $\mathcal{T}_\beta$ is a topology containing $\mathcal{A}$, each contains arbitrary unions of finite intersections of elements of $\mathcal{A}$. If $\mathcal{A}$ is a basis, then $\mathcal{T}_A$ is the collection of all arbitrary unions of elements of $\mathcal{A}$, and if $\mathcal{A}$ is a subbasis, then $\mathcal{T}_A$ is the collection of all arbitrary unions of finite intersections of elements of $\mathcal{A}$. So in either case, each $\mathcal{T}_\beta$ contains $\mathcal{T}_A$, so $\mathcal{T}_A \subset \bigcap_\beta \mathcal{T}_\beta$.

   Now we show that $\bigcap_\beta \mathcal{T}_\beta$ is contained in $\mathcal{T}_A$. Whether $\mathcal{A}$ is a basis or subbasis, $\mathcal{T}_A$ is itself a topology containing $\mathcal{A}$, so it is one of the $\mathcal{T}_\beta$ in the intersection. Then if $U \in \bigcap_\beta \mathcal{T}_\beta$, U must be in $\mathcal{T}_A$, so $\bigcap_\beta \mathcal{T}_\beta \subset \mathcal{T}_A$.

b. It suffices to show that for all subbasis elements S of the space $\mathbb{R}^\infty$, the set $f^{-1}(S)$ is open in $\mathbb{R}$. Let $\mathbb{R}_i$ denote the i-th copy of $\mathbb{R}$ in the cartesian product, then any S is of the form

   $$\left\{\pi_i^{-1}(U_i) \mid U_i \text{ open in } \mathbb{R}_i\right\}$$

   for some fixed i. But this is just the usual cartesian product with the single restriction that the functions that compose it must map i into U_i instead of all of $\mathbb{R}_i$.

   The preimage of S under f clearly contains U_i. It also can't contain any additional elements of $\mathbb{R}$, since f being the identity map means that the i-th coordinate of the cartesian product would contain elements outside of U_i, which we know cannot be the case. Thus

   $$f^{-1}(S) = U_i \in \mathbb{R}.$$

   Since U_i is open by definition, we have shown that f is continuous.

Exercise 3 (7 points). a. Munkres §18, pg. 111 #2.

b. Munkres §18, pg. 111 #6.

Collaborators: Saloni Bulchandani.

a. Let X be any topological space containing a limit point x of some subset $A \subset X$, and let Y be the singleton $\{0\}$ endowed with the indiscrete topology.

   The only possible function $f : X \to Y$ is the zero function. It is continuous since it is onto a space that has the indiscrete topology: $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$ are both open in X; however, $f(x)$ is *not* a limit point of Y. Since $f(x) = 0$ and 0 is the only element of Y, it is impossible for a neighborhood of $f(x)$ to intersect any subset of Y at a point other than $f(x)$.

b. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

   We first show that f is continuous at 0. Fix an arbitrary neighborhood U of $f(0) = 0$. Since U is open and contains 0, there must be some other neighborhood V of 0 that is contained in U. Then

$$f(V) = (V \cap \mathbb{Q}) \cup \{0\} = V \cap \mathbb{Q} \ \subset \ U \cap \mathbb{Q} \ \subset \ U,$$

   so f is continuous at 0.

   Now we show that f is not continuous anywhere else. Let $x \neq 0$ be a rational number, and consider the neighborhood $U = B(x, x/2)$ of $f(x) = x$, which does not contain 0. If f is continuous, then we can find some neighborhood V of x such that $f(V) \subset U$; however, any such V contains an irrational number, so $f(V)$ will contain 0 and as such not be a subset of U. Thus f is not continuous at any nonzero rational.

   Now suppose $x \neq 0$ is an irrational number, then consider the neighborhood $U = B(0, x/2)$ of $f(x) = 0$. If we take any neighborhood V of x, then it contains a rational number between $x/2$ and x, so $f(V)$ contains the same number and thus cannot be a subset of U. This shows that f is not continuous at any nonzero irrational number, so it can only be continuous at 0.

Exercise 4 (4 points)**.** Munkres §18, pg. 111 #7a.

Collaborators: Saloni Bulchandani.

Fix arbitrary $a \in \mathbb{R}$. By assumption, for all $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$ such that $f(x) \in B(f(a), \varepsilon)$ when $x \in [a, a + \delta)$, i.e.

$$f([a, a + \delta_\varepsilon)) \subset B(f(a), \varepsilon).$$

Let U be an arbitrary neighborhood of $f(a)$, then there is some $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subset U$. Then the open set $V = [a, a + \delta_\varepsilon)$ in $\mathbb{R}_l$ contains a and satisfies $f(V) \subset U$, so f is continuous.