Contents

Chapter 1

Topological Spaces

1.1 Topological Spaces

Definition 1: Topology

Let X be a set, then a **topology** on X is a collection $\mathcal{T}$ of subsets of X such that

1. $\varnothing, X \in \mathcal{T}$,

2. $\bigcup_{\alpha \in \mathcal{G}} U_\alpha \in \mathcal{T}$, and

3. $\bigcap_{i=1}^{N} U_i \in \mathcal{T}$.

Elements of a topology are called **open sets**.

Example 1: Topological Spaces

1. "Indiscrete" topology: $\mathcal{T}_i = \{\varnothing, X\}$

2. "Discrete" topology: $\mathcal{T}_d = \{\text{all subsets of } X\}$

Definition 2: Finer/Coarser

Let $\mathcal{T}, \mathcal{T}'$ be topologies on a set X, then $\mathcal{T}$ is **finer** than $\mathcal{T}'$ if $\mathcal{T}' \subset \mathcal{T}$. It is **strictly finer** if, in addition, $\mathcal{T}' \neq \mathcal{T}$.
$\mathcal{T}$ is **coarser** than $\mathcal{T}'$ if $\mathcal{T} \subset \mathcal{T}'$. It is **strictly coarser** if, in addition, $\mathcal{T}' \neq \mathcal{T}$.

From this we see that "fine" is a notion of a large topology, and "coarse" is a notion of a small topology.

Example 2: Finite Complement Toplogy

Let X be any set, then the **finite complement topology** is defined

$$\mathcal{T}_f = \{U \subset X \mid X - U \text{ is finite}\} \cup \{\varnothing\},$$

where $X - U$ denotes the complement of U in X, i.e. $X \backslash U$. Checking that this is a topology boils down to just using DeMorgan's Laws.

1.2   Bases

Definition 3: Basis

Let $\mathcal{T}$ be a topoloy on X, and let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a **basis** for $\mathcal{T}$ if every open set of $\mathcal{T}$ can be written as the union of elements of $\mathcal{B}$.

Proposition 1. *Let $\mathcal{T}$ be a topology on X, and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if*

1. *$\mathcal{B} \subset \mathcal{T}$; and*

2. *for each $U \in \mathcal{T}$ and $p \in U$, there is a $B \in \mathcal{B}$ such that $p \in B \subset U$.*

Proof. The forward direction follows from every open set of $\mathcal{T}$ being the union of elements of $\mathcal{B}$. For the backward direction, since $p \in B_p \subset U$ for all $p \in U$, we have $U = \bigcup_{p \in U} B_p$, so every open set of $\mathcal{T}$ is the union of elements of $\mathcal{B}$. $\square$
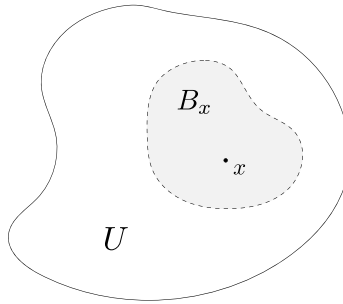

Figure 1.1: For any $U \in \mathcal{T}$, each $x \in U$ lies in some $B_x \in \mathcal{B}$ for $B_x \subset U$.

Not every set of subsets of X will generate a topology, so we need conditions for a collection $\mathcal{B}$ to be a basis for *any* topology.

> **Proposition 2.** *Let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ generates a topology if and only if*
>
> *1. for each $x \in X$, there is a $B \in \mathcal{B}$ containing x; and*
>
> *2. given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.*

Proof. **Forward:** (1) X must be open, so X is the union of elements of $\mathcal{B}$, so any $x \in X$ must lie in at least one of them. (2) Since B_1 and B_2 are both open in the topology generated by $\mathcal{B}$, their intersection is, as well. Then since $\mathcal{B}$ is a basis for this topology, we can find a satisfactory B_3.

Backward: The topology generated by a set $\mathcal{B}$ is the collection of all unions of elements of $\mathcal{B}$. It is clear that $\varnothing$ is in it, and condition (1) implies that X is, as well. Arbitrary unions are in the topology by definition. Induction on condition (2) shows that the topology also contains finite intersections. $\square$
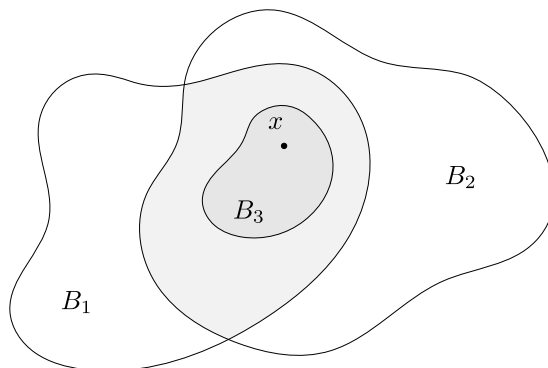

Figure 1.2: Condition (2) in Proposition 2.

Condition (1) in the above proposition is saying that X is the union of all elements of $\mathcal{B}$. Condition (2) is used to ensure that finite intersections of open elements remain open.

Additionally, note that since $\mathcal{B}$ exists independently from any topology, it doesn't make sense to describe its members as "open" until after we've generated a topology from it. Once we've done so, though, it should be clear that every basis element is open in the generated topology.

We can also get a notion of how relatively fine or coarse a topology is by using its basis.

Proposition 3. *Let $\mathcal{B}, \mathcal{B}'$ be bases for the topologies $\mathcal{T}, \mathcal{T}'$ on X, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for all $B \in \mathcal{B}$ and $x \in B$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subset \mathcal{B}$.*

Proof. First we show the backward implication. Let $U \in \mathcal{T}$, and let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is a $B \subset \mathcal{B}$ such that $x \in B \subset U$. By assumption, there is then a $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$. Thus $U \in \mathcal{T}'$, so $\mathcal{T}'$ is finer than $\mathcal{T}$.

Now we show the forward implication. Let $B \in \mathcal{B}$, and let $x \in B$, then $B \in \mathcal{T}$. By assmption, $\mathcal{T} \subset \mathcal{T}'$, so $B \in \mathcal{T}'$ as well. Then by thedefinition of a generated topology, there is a $B' \subset \mathcal{B}'$ such that $x \in B' \subset B$. $\qquad\square$

Definition 4: Topologies on $\mathbb{R}$

If $\mathcal{B}$ is the collection of all open intervals of the real line, the topology generated by $\mathcal{B}$ is the **standard topology** on the real line.

If $\mathcal{B}$ is the collection of half-open intervals of the form $[a, b)$, it generates the **lower limit topology** on $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_l$.

Let K denote the set of all numbers of the form $1/n$ for $n \in \mathbb{Z}_+$, and let $\mathcal{B}$ be the collection of all open intervals (a, b) along with all sets of the form $(a, b) - K$. Then $\mathcal{B}$ generates the **K-topology** on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_K$.

Proposition 4. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with each other.*

Proof. Let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_l$, and $\mathbb{R}_K$, respectively. Given a basis element (a, b) of $\mathcal{T}$ and $x \in (a, b)$, the basis element $[x, b)$ of $\mathcal{T}'$ contains x and lies in (a, b). However, given the basis element $[x, d)$ of $\mathcal{T}'$, there is no open interval that contains x and lies in $[x, d)$. Thus $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

Similarly, given a basis element (a, b) of $\mathcal{T}$ and $x \in (a, b)$, this same interval is a basis element for $\mathcal{T}''$ that contains x. However, given the basis element $B = (-1, 1) - K$ for $\mathcal{T}''$ and the point $0 \in B$, there is no open interval that contains 0 and lies in B (since $\mathbb{Q}$ is dense in $\mathbb{R}$). Thus $\mathcal{T}''$ is strictly finer than $\mathcal{T}$.

Show $\mathcal{T}'$ and $\mathcal{T}''$ aren't comparable. $\qquad\square$

1.3   Subbases

Definition 5: Subbasis

A **subbasis** $\mathcal{S}$ for a topology $\mathcal{T}$ on X is a collection of subsets of X whose finite intersections form a basis for $\mathcal{T}$.

Subbases are easier to construct than bases, but the construction of a topology from a subbasis involves an extra step, namely the finite intersections. What we are doing is creating a basis $\mathcal{B}$ from $\mathcal{S}$ by taking finite intersections of the subbasis elements. Then we are taking $\mathcal{B}$ and constructing $\mathcal{T}$ by taking arbitrary unions, as is usual.

$$\mathcal{S} \xrightarrow{\cap_{i=1}^{N}} \mathcal{B} \xrightarrow{\cup_{\alpha \in \mathcal{G}}} \mathcal{T}$$

Figure 1.3: The process for constructing a topology using a subbasis $\mathcal{S}$.

Subbasis is smallest set contained in topology or something like that?

Proposition 5. *Let $\mathcal{T}$ be a topology on X, and let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a subbasis for $\mathcal{T}$ if and only if*

1. *$\mathcal{S} \subset \mathcal{T}$; and*

2. *for each $U \in \mathcal{T}$ and $p \in U$, there is a finite intersection $\bigcap_{i=1}^{n} S_i$ of elements of $\mathcal{S}$ such that $p \in \bigcap_{i=1}^{n} S_i \subset U$.*

Proof. This follows from Proposition 1 (the analogue of this proposition for bases). When proving both directions, there's just an extra step to go from a genric basis element to a finite intersection of elements of $\mathcal{S}$. □

Just as with bases, we have a condition for when an arbitrary collection of subsets of X can be a valid subbasis.

Proposition 6. *Let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ generates a topology if and only if every point of X is in some element of $\mathcal{S}$.*

Proof. Sketch. □

1.4   The Order Topology

Definition 6: Total Order

A binary relation $\leq$ is a total order on a set X if

1. $a \leq b, b \leq a \implies a = b$,

2. $a \leq b, b \leq c \implies a \leq c$, and

3. $a \leq b$ or $b \leq a$.

A totally ordered set X can be divided into intervals of the form (a, b), $(a, b]$, $[a, b)$, and $[a, b]$. Note that even though this is reminiscent of the real number line, this is for any arbitrary ordered set.

Definition 7: The Order Topology

Let X be a totally ordered set with more than one element, and let $\mathcal{B}$ be a collection of

1. All open intervals (a, b) in X,

2. All half-open intervals $[a_0, b)$, where a_0 is the smallest element of X, and

3. All intervals of the form $(a, b_0]$, where b_0 is the largest element of X.

If X has no smallest element there are no sets of type (2), and if X has no largest elemtn, there are no sets of type (3). Then $\mathcal{B}$ is a basis for the **order topology** on X.

Example 3

The standard topology on $\mathbb{R}$, as defined in the previous section, is just the order topology derived from the usual order on $\mathbb{R}$.

Definition 8: Ray

Let a be an element of an ordered set X, then the **rays** of X determined by a are the subsets $(a, +\infty)$, $(-\infty, a)$, $[a, +\infty)$, and $(-\infty, a]$. The first two are **open rays** and the latter two are **closed rays**.

Based on the name, we expect open rays to be open sets in the order topology, and so they are. Consider (a, ∞). If X has a largest element b_0, then $(a, \infty) = (a, b_0]$, which is in the topology. If X has no largest element, then $(a, \infty) =$

$\bigcup_{x>a}(a,x)$, so it is open. The argument is similar for $(-\infty, a)$.

In fact, the open rays form a subbasis for the order topology on X. Since the open rays are open in the order topology, the topology they generate is contained in the order topology. On the other hand, every basis element for the order topology is a finite intersection of open rays. The interval (a, b) is the intersection of $(-\infty, b)$ and (a, ∞). If $[a_0, b)$ or $(a, b_0]$ exist, they are themselves open rays. Thus the topology generated by the open rays contains the order topology.

1.5   The (2D) Product Topology

Move general subspace stuff in here, up to but not including the stuff about Hausdorff, etc.

Definition 9: Product Topology

If X and Y are topological spaces, then the **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is a open subset of X and V is an open susbet of Y.

Since $X \times Y$ is itself a basis element, the first condition for a basis is trivial. Note that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Since each of these is a finite intersection of open sets, they are themselves open. Thus the intersection is itself a basis element, and the second condition for a basis is satisfied.

Theorem 1

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Proof. Given an open set W of $X \times Y$ and $x \times y \in W$, by definition of the product topology there is a basis element $U \times V$ such that $x \times y \in U \times V \subset W$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, respectively, we can choose an element $B \in \mathcal{B}$ such that $x \in B \subset U$ and an element $C \in \mathcal{C}$ such that $y \in C \subset V$. Then $x \times y \in B \times C \subset W$. Since $B \times C \in \mathcal{D}$, we have that $\mathcal{D}$ is a basis for $X \times Y$. Redo. $\square$

We can also express the product topology in terms of subbases, but we must first define projections.

Definition 10: Projection

Let $\pi_1 : X \times Y \to X$ be defined by

$$\pi_1(x, y) = x,$$

and let $\pi_2 : X \times Y \to Y$ be defined by

$$\pi_2(x, y) = y,$$

then the maps π_1 and π_2 are called the **projections** of $X \times Y$ onto its first and second factors, respectively.

Note that π_1 and π_2 are surjective (unless X or Y is empty, in which case $X \times Y$ is empty, but this is a meaningless case).

If U is an open subset of X, then $\pi_1^{-1}(U) = U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then $\pi_2^{-1}(V) = X \times V$, which is open in $X \times Y$ as well. The intersection of these two sets is $U \times V$, which leads to the following theorem.
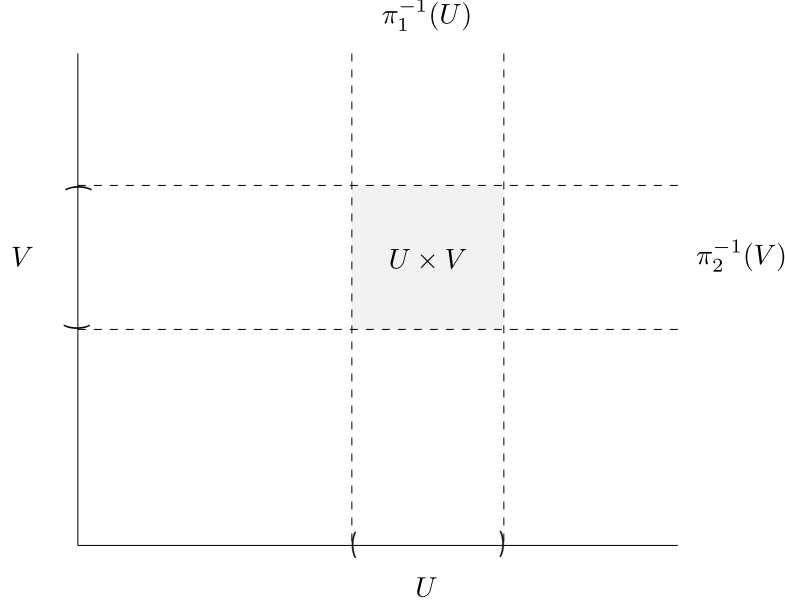

Figure 1.4: The open set in $X \times Y$ formed by the intersection of the preimages of the projections onto open sets $U \subset X$ and $V \subset Y$.

> **Theorem 2**
>
> The collection
> $$\mathcal{S} = \left\{\pi_1^{-1}(U) \mid U \text{ open in } X\right\} \cup \left\{\pi_2^{-1}(V) \mid V \text{ open in } Y\right\}$$
> is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Every element of $\mathcal{S}$ is in $\mathcal{T}$, so arbitrary unions of finite intersections of elements of $\mathcal{S}$ are also in $\mathcal{T}$. Thus $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite inersectino of elements of $\mathcal{S}$, since
$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$
Then $U \times V \in \mathcal{T}'$, so $\mathcal{T} \subset \mathcal{T}'$. Thus $\mathcal{T} = \mathcal{T}'$. □

1.6   The Subspace Topology

> **Definition 11: Subspace Topology**
>
> Let $(X, \mathcal{T})$ be a topological space. If $Y \subset X$, then
> $$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$
> is the **subspace topology** on Y. With this topology, Y is called a **subspace** of X.

> **Proposition 7.** $\mathcal{T}_Y$ *is a topology on* Y.

Proof. $\varnothing \cap Y = \varnothing \in \mathcal{T}_Y$. Additionally, $Y \subset X$ implies $Y = Y \cap X$, so $Y \in \mathcal{T}_Y$. Arbitrary unions are open since

$$\bigcup_{\alpha \in \mathcal{G}} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in \mathcal{G}} U_\alpha\right) \cap Y \in \mathcal{T}_Y.$$

Finite intersections are also open since

$$\bigcap_{i=1}^{N} (U_i \cap Y) = \left(\bigcap_{i=1}^{N} U_i\right) \cap Y \in \mathcal{T}_Y.$$

□

Proposition 8. *Let $\mathscr{B}$ be a basis for the topology of X, then*

$$\mathscr{B}_Y \doteq \{B \cap Y \mid B \in \mathscr{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let $y \in U \cap Y$, where U is open in X. There exists $B \in \mathscr{B}$ such that $y \in B \subset U$, so $y \in B \cap Y \subset U \cap Y$. $\qquad\square$

Proposition 9. *Let Y be a subspace of X, and let U be open in Y and Y be open in X. Then U is open in X.*

Proof. U is open in Y, so $U = Y \cap V$ for some V open in X. Both sets Y and V are open in X, so their intersection U must be as well. $\qquad\square$

Theorem 3

Let A be a subspace of X and B be a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof. $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y. Thus $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ is the general basis element for the subspace topology on $A \times B$.

Since $U \cap A$ and $V \cap B$ are the general open sets for the suspace topology on A and B, respectively, $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$.

This shows that the two topologies have the same bases, so the topologies themselves must be the same. $\qquad\square$

This equivalence of topologies holds for the product topology, but not in general for the order topology, however.

Example 4

Let $Y = [0,1] \cap \{2\}$ be a subset of $\mathbb{R}$. $\{2\}$ is open in the subspace topology since $(3/2, 5/2) \cap Y = \{2\}$, but $\{2\}$ is not open in the order topology since no sets of the form (a,b), $(a,b]$, or $[a,b)$ contain $\{2\}$ without also containing points not in Y.

It *is* possible to have equivalence with the ordered topology, though. We can define a condition in which this equivalence holds, but we'll need another definition first.

Definition 12: Convex

Let X be an ordered set, then $Y \subset X$ is **convex** in X if for all pairs a and b in Y such that $a < b$, the interval $(a, b) \subset X$ also lies in Y.

Note that intervals and rays in X are convex in X.

Theorem 4

Let X be an ordered set in the order topology, and let $Y \subset X$ be convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Proof. Consider $(a, \infty) \subset X$. If $a \in Y$, then $(a, \infty) \cap Y = \{x \mid x \in Y, x > a\}$. This is an open ray of Y. If $a \notin Y$, then a is either a lower bound or upper bound of Y. If a is a lower bound of Y, then $(a, \infty) \cap Y = Y$. If a is an upper bound of Y, then $(a, \infty) \cap Y = \varnothing$.

Similarly, $(-\infty, a) \cap Y$ is either $\varnothing$, Y, or an open ray of Y.

Since $(a, \infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y, and since each is open in the order topology, the order topology contains the subspace topology.

Any open ray of Y is the intersection of open rays of X with Y, so it is open in the subspace topology on Y. Since the open rays of Y are a subbasis for the order topology on Y, the order topology is contained in the subspace topology. Thus the order topology is the same as the subspace topology in this case. $\square$

From here on out, assume that if X is an ordered set in the order topology and $Y \subset X$, then Y has the subspace topology. If Y is convex in X, this is the same as the order topology on Y, otherwise it might not be.

1.7   Closed Sets and Limit Points

Definition 13: Closed Set

A set $A \subset (X, \mathcal{T})$ is closed if $X - A$ is open in X.

> ### Theorem 5
>
> Let $(X, \mathcal{T})$ be a topological space, and let F denote a closed set of X, then
>
> 1. $\varnothing$ and X are closed,
>
> 2. $\bigcap_{\alpha \in \mathcal{G}} F_\alpha$ is closed, and
>
> 3. $\bigcup_{i=1}^{N} F_i$ is closed.

The proof of this is a straightforward application of DeMorgan's Laws.

> **Proposition 10.** *Let Y be a subspace of X. Then A is closed in Y if and only if it is equal to the intersection of a closed set of X with Y.*

Proof. First we show the forward implication. Assume A is closed in Y, then $Y - A$ is open in Y, so be definition $Y - A = U \cap Y$ for some U open in X. $X - U$ is closed in X, and $A = Y \cap (X - U)$, so A is the intersection of a closed set of X with Y.

Now we show the backward implication. Assume $A = C \cap Y$ for C closed in X. Then $X - C$ is open in X, so $(X - C) \cap Y$ is open in Y by the definition of the subspace topology. But $(X - C) \cap Y = Y - A$, so $Y - A$ is open in Y. Thus A is closed in Y. $\qquad \square$

> **Proposition 11.** *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. $A = F \cap Y$ for some F closed in X. A is then the intersection of closed sets of X, so it is itself closed in X. $\qquad \square$
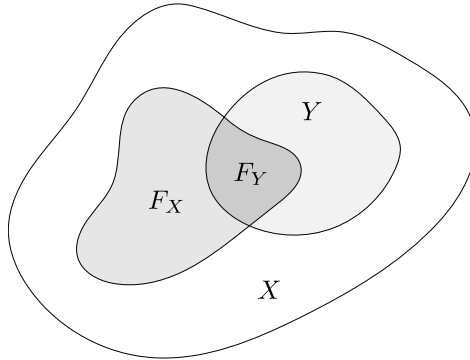

Figure 1.5: A set F_Y closed in Y, where $F_Y = Y \cap F_X$ for F_X closed in X.

Definition 14: Interior of a Set

The **interior** of a set A, denoted A^o, is the union of all open sets contained in A.

Definition 15: Closure of a Set

The **closure** of a set A, denoted $\overline{A}$ is the intersection of all closed sets containing A.

The closure of a set is clearly closed, and the interior of a set is clearly open. It is also clear that if A is open, then $A^o = A$, and if A is closed, then $\overline{A} = A$. We also have the obvious relation $A^o \subset A \subset \overline{A}$.

We have to be careful when describing closures. Given a subspace Y of X, the closure of A in X is generally not the same as the closure of A in Y. In this case, we use $\overline{A}$ to denote the closure of A in X (the overall space). We relate this to the closure of A in Y (the subspace) with the following proposition.

Proposition 12. *Let Y be a subspace of X, and let $A \subset Y$. Denote the closure of A in X by $\overline{A}$. Then the closure of A in Y is equal to $\overline{A} \cap Y$.*

Proof. Let B denote the closure of A in Y. $\overline{A}$ is closed in X, so by Proposition 10, $\overline{A} \cap Y$ is closed in Y. Since $\overline{A} \cap Y$ contains A, and since by definition B is the intersection of all closed subsets of Y containing A, we have $B \subset \overline{A} \cap Y$.

On the other hand, we know B is closed in Y. Again by Proposition 10, $B = C \cap Y$ for some C closed in X. Then C is a closed set of X containing A. Since $\overline{A}$ is the intersection of all such closed sets, we have $\overline{A} \subset C$, so $(\overline{A} \cap Y) \subset (C \cap Y) = B$. $\qquad\square$

Definition 16: Neighborhood

A **neighborhood** of a point X is an open set containing x.

Theorem 6

Let A be a subset of a topological space X, then

1. $x \in \overline{A}$ if and only if every neighborhood of x intersects A, and

2. Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Proof.   1. We'll prove the contrapositive: $x \notin \overline{A}$ if and only if there exists an

open neighborhood of x that does not intersect A. We will first show the forward implication. Assume $x \notin \overline{A}$, then $U = X - \overline{A}$ is open, contains x, and does not intersect A.

Now we show the backward implication. Assume there exists an open neighborhood U of x that does not intersect A, then $X - U$ is a closed set containing A. By the definition of $\overline{A}$, $X - U$ must contain $\overline{A}$. Thus $x \notin \overline{A}$.

2. First we show the forward implication. By (1) we know if $x \in \overline{A}$, then every open neighborhood of x intersects A. Since basis elements are open, this implies that every basis element containing x intersects A.

Now we show the backward implication. Assume every basis element containing x intersects A. Since every open neighborhood U of x contains some basis element, every such U also intersects A. Again by (1), this implies $x \in \overline{A}$.

$\square$

Definition 17: Limit Point

Let $A \subset (X, \mathcal{T})$, then $x \in X$ is a **limit point**/cluster point/accumulation point of A if every open neighborhood of x intersects A at some point *other than x* .
Equivalently, x belongs to the closure of $A - \{x\}$. Note that x need not lie in A.

Theorem 7

Let $A \subset (X, \mathcal{T})$, and denote the set of limit points of A by A'. Then $\overline{A} = A \cup A'$.

Proof. First we show $A' \cup A \subset \overline{A}$. Let $x \in A'$, then every open neighborhood of x intersects A (at a point other than x). Thus by Theorem 6, $x \in \overline{A}$, so $A' \subset \overline{A}$. Since $A \subset \overline{A}$ by definition, we have $A \subset A' \subset \overline{A}$.

Now we show $\overline{A} \subset A' \cup A$. Let $x \in \overline{A}$. If $x \in A$, then this is trivial, so assume $x \notin A$. Since $x \in \overline{A}$, every neighborhood of x intersects A. Since $x \notin A$, this must be at a point other than x. Thus $x \in A' \subset A' \cup A$, so $\overline{A} \subset A' \cup A$. $\square$

Corollary 1. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. Let $A \subset (X, \mathcal{T})$. Then A is closed if and only if $A = \overline{A} = A \cup A'$, and $A = A \cup A'$ if and only if $A' \subset A$. $\square$

Chapter 2

Separation Properties

2.1 The Separation Axioms

Turns out that separating points yields some nice properties of topological spaces. Who'da thunk?

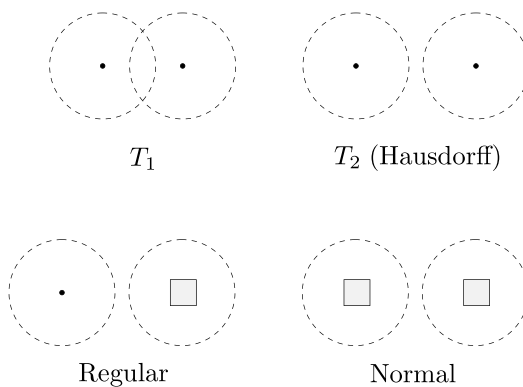Stuff about T_1-T_4. Theorems about how they're related.

Figure 2.1: The four main types of separation.

Proposition 13. *Let X be T_1, and let A be a subset of X. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. **Backward:** Every neighborhood of x intersects A at infinitely many points, so it they surely all intersect A at a point other than x. Thus x is a limit point.

Forward: Let x be a limit point of A, and suppose some neighborhood U of x intersects A at only finitely many points. Let $\{x_1, \ldots, x_m\} = U \cap (A - \{x\})$, then $X - \{x_1, \ldots, x_m\}$ is open in X since X satisfies the T_1 axiom. Then $U \cap (X - \{x_1, \ldots, x_m\})$ is a neighborhood of x that doesn't intersect $A - \{x\}$ at all. This contradicts x being a limit point of A, so every neighborhood of x must intersect A at infinitely many points. $\qquad\square$

Proposition 14. *A space is T_1 if and only if all single points are closed.*

Proof. **Forward:** Suppose X is T_1, then fix $x \in X$. Then for $y \in X - \{x\}$, there is an open U_y such that $y \in U_y \subset X - \{x\}$, so $X - \{x\} = \bigcup_y U_y$. Then $X - \{x\}$ is open so $\{x\}$ is closed.

Backward: Suppose all single points in X are closed. Fix $x, y \in X$, then $X - \{x\}$ and $X - \{y\}$ are the open sets we need to show that X is T_1. $\qquad\square$

Corollary 2. *A space is T_1 if and only if all finite point sets are closed.*

Proof. Do I even need one? Kinda obvious. $\qquad\square$

2.2   Hausdorff Spaces

In a Hausdorff space, any two distinct points can be separated by disjoint open sets.
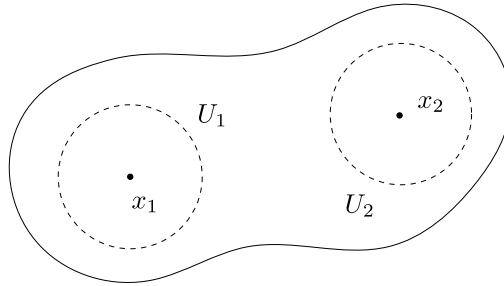

Figure 2.2: Two points in a Hausdorff space.

Proposition 15. *Every finite set in a Hausdorff space is closed.*

Proof. This follows from T_1, so move this proof up to T_1 proposition.

Let X be a Hausdorff space. It suffices to show the arbitrary single point set $\{x_0\}$ is closed, as any finite set is the finite union of single points. Let

$x \in X$ such that $x \neq x_0$, then x and x_0 have disjoint neighborhoods. Thus x's neighborhood does not intersect $\{x_0\}$, so x is not in the closure of $\{x_0\}$. Since x was arbitrary, the closure of $\{x_0\}$ is just $\{x_0\}$ itself, so it is closed. $\qquad \square$

Proposition 16. *Let X be a Hausdorff space, then a sequence of points in X converges to at most one point in X.*

Proof. Suppose $\{x_n\} \subset X$ such that $x_n \to x \in X$. If $y \neq x$, then since X is Hausdorff we can find disjoint open neighborhoods U and V of x and y, respectively. The set U contains all but finitely many of the points in $\{x_n\}$, so V can only contain finitely many of the points in $\{x_n\}$. Thus x_n cannot converge to y. $\qquad \square$

Proposition 17. *Every simply ordered set is Hausdorff in the order topology.*

Proof. Do this. $\qquad \square$

Proposition 18. *The product of two Hausdorff spaces is a Hausdorff space.*

Proof. Do this. $\qquad \square$

Proposition 19. *A subspace of a Hausdorff space is Hausdorff.*

Proof. Suppose X is Hausdorff and that Y is a subspace of X with distinct points u and v. Then u and v are also distinct points of X, so by the regularity of X, they are separated by disjoint open sets U and V in X. Then $Y \cap U$ and $Y \cap V$ are the desired open sets of Y. $\qquad \square$

2.3   Regular Spaces

Proposition 20. *X is regular if and only if for all $p \in X$ and open neighborhood U of p, there is an open set V such that $p \in V$ and $\overline{V} \subset U$.*

Proof. Do this. Forward direction already in written notes so don't throw out that paper yet. $\qquad \square$

Proposition 21. *Every subspace of a regular space is regular.*

Proof. Let Y be a subspace of a regular space X, let A be closed in Y, and let $y \in Y - A$. Then $A = \mathrm{cl}_X(A) \cap Y$, and $\mathrm{cl}_X(A)$ is a closed set in X not containing y. Then the regularity of Y follows from the regularity of X. $\qquad\square$

2.4   Normal Spaces

Proposition 22. *X is normal if and only if for all closed sets A and open sets U containing U, there exists an open set V such that $A \subset V$ and $\overline{V} \subset U$.*

Proof. Hey there. $\qquad\square$

Proposition 23. *X is normal if and only if for all pairs of disjoint closed sets A and B, there are disjoint open containing A and B whose closures are also disjoint.*

Proof. These really be piling up. $\qquad\square$

Suppose a space is normal and covered by 2 open sets. Then we can actually find two smaller patches (smaller in the sense that their closures are contained in the original patches) that cover the space too.

Theorem 8: The Shrinking Theorem

X is a normal topological space if and only if for all open covers $\{U, V\}$ of X, there exist open sets U' and V' such that $\overline{U'} \subset U$, $\overline{V'} \subset V$, and U' and V' also cover X.

Proof. **Forward:** Suppose X is normal and is covered by open sets U and V. Note that $X - U$ is closed and is a subset of V. Then by Proposition 22, there exists an open set V' such that $X - U \subset V' \subset \overline{V'} \subset V$.

Then $U \cup V'$ contains $U \cup X - U = X$, so $\{U, V'\}$ is an open cover of X. We can repeat this argument to replace U with the desired U'.

Backward: Let A be closed in X and contained in some open set U of X. Again by Proposition 22, we need to find an open set U' such that $A \subset U'$ and $\overline{U'} \subset U$.

Since $X - A$ is open and $X - A$ and U cover X, by assumption, there exist open U' and V' such that $\overline{U'} \subset U$, $\overline{V'} \subset X - A$, and $U' \cup V' = X$. We claim

that U' is our desired open set. We already have $\overline{U'} \subset U$, so we only need to show $A \subset U'$.

Now U' and V' cover X, so $X - V' \subset U'$. Additionally, $V' \subset X - A$. Together, these give $A \subset X - V' \subset U'$. $\qquad\square$

Extenstion of this theorem to more than 2 open sets.

A somewhat surprising result is that not all subspaces of normal spaces are normal. To see why, we can modify the earlier proof that subspaces of regular spaces are themselves regular. Let Y be a subspace of a normal space X, and let A and B be closed in Y. Then

$$A = \mathrm{cl}_X(A) \cap Y, B = \mathrm{cl}_X(B) \cap Y.$$

In order to use the regularity of X, we would need $\mathrm{cl}_X(A)$ and $\mathrm{cl}_X(B)$ to be disjoint. Unfortunately, this is not true in general (see the figure below). It *is* true, though, if Y is a closed subspace of X.
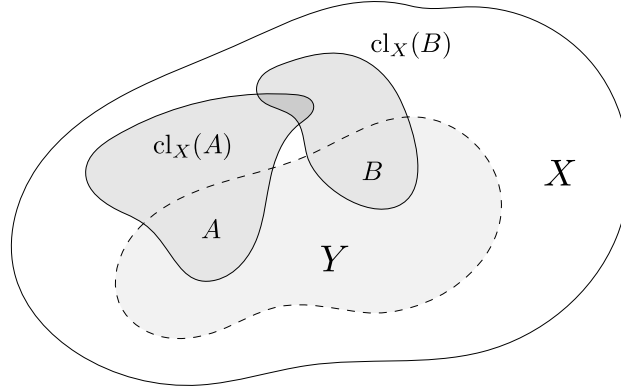

Figure 2.3: An exaggerated example of what could happen if Y is open in X.

Proposition 24. *Closed subspaces of normal spaces are normal.*

Proof. Let Y be a closed subspace of a normal space X, and let A and B be closed in Y. Then since Y is closed in X, both A and B are also closed in X. Then the normality of Y follows from the normality of X. $\qquad\square$

Definition 18: Separated

$A, B \subset X$ are **separated** if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty, i.e. they don't contain each other's limit points.

Definition 19: Completely Normal

X is **completely normal** if for any two separated sets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. A T_5 space is any space that is both T_1 and completely normal.

Lemma 1. *Let Y be a subspace of X, and let A and B be disjoint closed sets in Y. Then A and B are separated in X.*

Proof. Denote the closure in X with a bar, then $A = \overline{A} \cap Y$ and $B = \overline{B} \cap Y$. Then $\overline{A} \cap B = A \cap \overline{B} = \overline{A} \cap \overline{B} \cap Y = A \cap B = \varnothing$. $\qquad\square$

Proposition 25. *X is completely normal if and only if every subspace of X is normal.*

Proof. **Forward:** Let A and B be closed in a subspace Y of X, then by the previous lemma, they are separated in X. Since X is completely normal, there are disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$. Then $U \cap Y$ and $V \cap Y$ are the desired open sets to show that the subspace Y is normal.

Backward: Let A and B be separated in X, then $A \in X - \overline{B}$ and $B \in X - \overline{A}$. Consider the subspace $Y \doteq (X - \overline{B}) \cup (X - \overline{A}) = X - (\overline{B} \cap \overline{A})$. Then since $\mathrm{cl}_Y(A) = \overline{A} \cap Y$ and $\mathrm{cl}_y(B) = \overline{B} \cap Y$, the definition of Y gives $\mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(B) = \overline{A} \cap \overline{B} \cap Y = \varnothing$.

Then since $\mathrm{cl}_Y(A)$ and $\mathrm{cl}_Y(B)$ are disjoint and closed in Y, and since Y is normal by assumption, there are disjoint open sets U and V of Y such that $\mathrm{cl}_Y(A) \subset U$ and $\mathrm{cl}_Y(B) \subset V$. Then $A \subset U$ and $B \subset V$, so X is completely normal. $\qquad\square$

2.5   Separation and Products

Really gotta reorganize this. Put the product stuff earlier and put the separation stuff in the proper section in this chapter.

Definition 20: $\mathcal{J}$-tuple

Let $\mathcal{J}$ be an arbitrary index set, and let X be a set. The $\mathcal{J}$**-tuple** of elements of X is a function $\mathbf{x} : \mathcal{J} \to X$.
If $\alpha \in \mathcal{J}$, denote $\mathbf{x}(\alpha)$ by x_α. We call this the α-th coordinate of $\mathbf{x}$. We often denote $\mathbf{x}$ itself by $(x_\alpha)_{\alpha \in \mathcal{J}}$.
Denote the set of all $\mathcal{J}$-tuples of X by $X^{\mathcal{J}}$.

Definition 21: Cartesian Product

Let $\{A_\alpha\}_{\alpha \in \mathcal{G}}$ be a family of sets, and let $X = \bigcup_{\alpha \in \mathcal{G}} A_\alpha$. The **Cartesian product**, denoted

$$\prod_{\alpha \in \mathcal{G}} A_\alpha,$$

is the set of all $\mathcal{G}$-tuples of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in \mathcal{G}$.

In other words, it is the set of all functions $\mathbf{x} : \mathcal{G} \to X$ such that $\mathbf{x}(\alpha) = x_\alpha \in A_\alpha$ for each $\alpha \in \mathcal{G}$.

Definition 22: Box Topology

Let $\{X_\alpha\}_{\alpha \in \mathcal{G}}$ be a family of topological spaces. Take as a basis for a topology on the product space $\prod_{\alpha \in \mathcal{G}} X_\alpha$ the collection of all sets of the form

$$\prod_{\alpha \in \mathcal{G}} U_\alpha,$$

where U_α is open in X_α for each $\alpha \in \mathcal{G}$. The topology generated by this basis is called the **box topology**.

Definition 23: Projection Mapping

Let $\pi_\beta : \prod_{\alpha \in \mathcal{G}} X_\alpha \to X_\beta$ be the function that assigns each element of the product space to its β-th coordinate, defined by

$$\pi_\beta(\mathbf{x}) = \pi_\beta((x_\alpha)_{\alpha \in \mathcal{G}}) = x_\beta.$$

Then x_β is the **projection mapping** associated with index β.

Definition 24: Product Topology

Define $\mathcal{S}_\beta = \left\{ \pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta \right\}$, and let $\mathcal{S} = \bigcup_{\beta \in \mathcal{G}} \mathcal{S}_\beta$. The topology generated by the subbasis $\mathcal{S}$ is the **product topology.**

We now examine the typical basis element of the product topology. Let $\beta_1, \ldots, \beta_N$ be a finite set of distinct indices from $\mathcal{G}$, and let U_{β_i} be open in X_{β_i} for each i. Then the typical basis element is of the form

$$B = \bigcup_{i=1}^{N} \pi_{\beta_i}^{-1}(U_{\beta_i}).$$

Note that

1. $\mathbf{x} = (x_\alpha) \in B$ if and only if the β_i-th coordinate of $\mathbf{x}$ is in U_{β_i} for each i; and

2. if $\alpha \notin \{\beta_1, \ldots, \beta_N\}$, then there is no restriction on the α-th coordinate of $\mathbf{x}$ if we want $\mathbf{x}$ to lie in B.

As a result, we can write

$$B = \prod_{\alpha \in \mathcal{I}} U_\alpha,$$

where $U_\alpha = X_\alpha$ if $\alpha \notin \{\beta_1, \ldots, \beta_N\}$.

Note 1: Box vs. Product Topologies

The box topology on $\prod X_\alpha$ has as a basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α.

The product topology on $\prod X_\alpha$ has as a basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ except for finitely many values of α.

Note that for finite product spaces, the box topology is equivalent to the product topology. Additionally, the box topology is in general finer than the product topology.

Theorem 9

Suppose a topology on each X_α is given by a basis $\mathcal{B}_\alpha$. Then

1. the collection of all sets of the form $\prod_\alpha B_\alpha$, where $B_\alpha \in \mathcal{B}_\alpha$ for each α, is a basis for the box topology on $\prod X_\alpha$; and

2. the collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices and $B_\alpha = X_\alpha$ for all other indices, is a basis for the product topology on $\prod X_\alpha$.

Proof. Do this.   □

Proposition 26. *Let A_α be a subspace of X_α for each α. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ (in either the box or product topology).*

Proof. Do this.   □

Proposition 27. *If each X_α is Hausdorff, then so $\prod X_\alpha$ (in either the box or product topology).*

Proof. Do this.   □

Proposition 28. *If $A_\alpha \subset X_\alpha$ for each α, then $\prod \overline{A}_\alpha = \overline{\prod A_\alpha}$ (in either the box or product topology).*

Proof. First we show $\prod \overline{A}_\alpha \subset \overline{\prod A_\alpha}$. Let $\mathbf{x} \in \prod \overline{A}_\alpha$, and let $U = \prod U_\alpha$ be a basis element for either topology that contains $\mathbf{x}$. For each α, $x_\alpha \in \overline{A}_\alpha$, so we can choose $y_\alpha \in U_\alpha \cap A_\alpha$. Then $\mathbf{y} = (y_\alpha)$ is in U and in $\prod A_\alpha$. Since U was arbitrary, $\mathbf{x} \in \overline{\prod A_\alpha}$.

Now we show $\overline{\prod A_\alpha} \subset \prod \overline{A}_\alpha$. Let $\mathbf{x} \in \overline{\prod A_\alpha}$ (in either topology), and let β be an arbitrary index. Let V_β be an open neighborhood of x_β in X_β. Since each $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in both topologies, it contains a point $\mathbf{y} \in \prod A_\alpha$. Then $y_\beta \in V_\beta \cap A_\beta$. Since V_β was arbitrary, $x_\beta \in \overline{A}_\beta$. Since β was arbitrary, this means $\mathbf{x} \in \prod \overline{A}_\alpha$.   □

Note 2

Whenever we consider the product space $\prod X_\alpha$, assume it has the product topology unless otherwise stated. This is because the product topology will end up being much more useful than the box topology.

An example of this is shown in the next theorem, which *only* holds in the product topology, not the box topology.

Theorem 10

Let $f : A \to \prod_{\alpha \in \mathcal{J}} X_\alpha$ be given by

$$f(a) = (f_\alpha(a))_{\alpha \in \mathcal{J}},$$

where $f_\alpha : A \to X_\alpha$ for each α. Suppose $\prod X_\alpha$ has the product topology, then f is continuous if and only if each f_α is continuous.

Proof. Do this.   □

Chapter 3

Continuity

3.1 Continuous Functions

Let X, Y be topological spaces, then $f : X \to Y$ is **continuous** if for all U open in Y, $f^{-1}(U)$ is open in X.

Continuity obviously depends on f, but it also depends on the topologies of X and Y. We can emphasize this by saying that f is continuous *relative* to the topologies on X and Y.

Proposition 29. *Let Y be given by the basis $\mathcal{B}$. If $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$, then $f : X \to Y$ is continuous.*

Proof. Any U open in Y can be written $U = \bigcup_{\alpha \in \mathcal{J}} B_\alpha$ for $B_\alpha \in \mathcal{B}$, so the preimage of U is $f^{-1}(U) = \bigcup_{\alpha \in \mathcal{J}} f^{-1}(B_\alpha)$. Since each $f^{-1}(B_\alpha)$ is open, $f^{-1}(U)$ is also open. $\square$

Proposition 30. *Let Y be given by the subbasis $\mathcal{S}$. If $f^{-1}(S)$ is open in X for all $S \in \mathcal{S}$, then $f : X \to Y$ is continuous.*

Proof. Any basis B of Y can be written $B = S_1 \cap \cdots \cap S_n$ for $S_i \in \mathcal{S}$. Then $f^{-1}(B) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$, so if each $f^{-1}(S_i)$ is open, then $f^{-1}(B)$ is open. Then by the previous proposition, f is continuous. $\square$

Theorem 11

Let X and Y be topological spaces, and let $f : X \to Y$, then the following are equivalent:

1. f is continuous.

2. For all $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.

3. For all B closed in Y, $f^{-1}(B)$ is closed in X.

4. For all $x \in X$ and for each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Proof. First we will show (1) $\implies$ (2) $\implies$ (3) $\implies$ (1), then we will show (1) $\implies$ (4) $\implies$ (1).

1 implies 2: Assume f is continuous, and let $A \subset X$. We will show that if $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$, then by assumption $f^{-1}(V)$ is open in X and contains x. Since x is a limit point of A, $f^{-1}(V)$ must intersect A at some point y, so V intersects $f(A)$ at $f(y)$. This was for arbitrary V, so $f(x)$ is a limit point of $f(A)$. Thus $f(x) \in \overline{f(A)}$.

2 implies 3: Let B be closed in Y, and let $A = f^{-1}(B)$. We want to show that A is closed in X, which we accomplish by showing $A = \overline{A}$. Now $f(A) = f(f^{-1}(B)) \subset B$, so if $x \in \overline{A}$, then

$$f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B.$$

Thus $x \in f^{-1}(B) = A$, so $\overline{A} \subset A$. Clearly $A \subset \overline{A}$, so $A = \overline{A}$.

3 implies 1: Let V be open in Y, and let $B = Y - V$. Since B is closed in y, by assumption $f^{-1}(B)$ is closed in X. But

$$f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V),$$

so $X - f^{-1}(V)$ is closed in X, so $f^{-1}(V)$ is open in X.

1 implies 4: Let $x \in X$ and let V be a neighborhood of $f(x)$, then $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subset V$.

4 implies 1: Let V be open in Y, and let $x \in f^{-1}(V)$, then $f(x) \in V$. By assumption, there is a neighborhood U_x of x such that $f(U_x) \subset V$, so $U_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ is the union of the open sets $\{U_x\}$, so $f^{-1}(V)$ is open in X. $\square$

Definition 26: Continuous at a Point

If condition (4) above holds for the point $x \in X$, then we say that f is **continuous at** x.

Definition 27: Homeomorphism

Let X and Y be topological spaces, and let $f : X \leftrightarrow Y$ be bijective. If f and f^{-1} are continuous, then f is a **homeomorphism**.

Equivalently, a homeomorphism is a bijective function $f : X \leftrightarrow Y$ such that U is open in X if and only if $f(U)$ is open in Y.
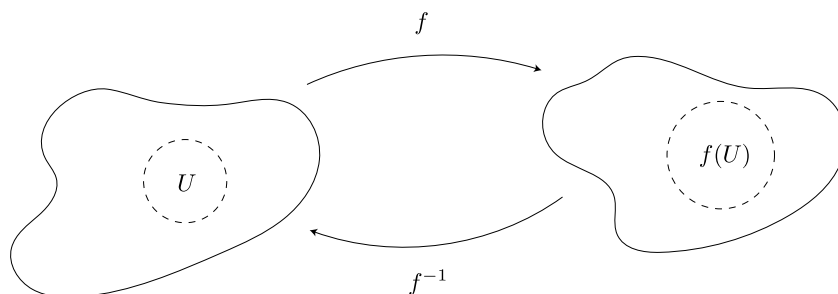
Figure 3.1: A homeomorphism f.

Definition 28: Topological Property

A **topological property** is a property of topological space X expressed entirely in terms of the topology on X (the open sets of X).

If $f : X \to Y$ is a homeomorphism, then Y has the topological properties of X. From this viewpoint, a homeomorphism is a bijective correspondence that preserves topological structure. This can be thought of as the topological analogue to isomorphisms in algebra.

Definition 29: Topological Embedding

Suppose $f : X \hookrightarrow Y$ is one-to-one and continuous. Then $f' : X \leftrightarrow f(X)$ (obtained by restricting the range of f) is bijective. If f' is a homeomorphism of X with $f(X)$, we say $f : X \hookrightarrow Y$ is a **(topological) embedding** of X in Y.

Theorem 12: The Pasting Lemma

Let $X = A \cup B$, where A and B are either both closed or both open in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ given by

$$H(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous.

Proof. Suppose A and B are both closed. Let C be closed in Y, then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f and g are continuous, both $f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B, respectively. Since both A and B are closed in X, both preimages are also closed in X. Thus $h^{-1}(C)$ is closed and h is subsequently continuous.

To show this when A and B are both open, replace the word "closed" with the word "open" in the above paragraph. $\square$

Note that the condition $f(x) = g(x)$ for all $x \in A \cap B$ is not needed in this proof. It is only necessary to make h an actual function.

Theorem 13: Maps into Products

Define $f : A \to X \times Y$ by $f(a) = (f_1(a), f_2(a))$, for $f_1 : A \to X$ and $f_2 : A \to Y$. Then f is continuous if and only if f_1 and f_2 are both continuous.

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the obvious projections, which we know to be continuous.

We begin with the forward implication. Suppose f is continuous, then $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$, so f_1 and f_2 are compositions of continuous functions. Thus they are both continuous themselves.

Now we show the backward implication. Suppose f_1 and f_2 are continuous, then we'll show that for each basis element $U \times V$ of the topology on $X \times Y$, the preimage $f^{-1}(U \times V)$ is open. Take a point in the preimage $a \in f^{-1}(U \times V)$, then $f(a) \in U \times V$, so $f_1(a) \in U$ and $f_2(a) \in V$. Thus we have

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V).$$

Since U and V are open and f_1 and f_2 are continuous, both of the sets in the above intersection are open. The finite intersection of open sets is open, so $f^{-1}(U \times V)$ is open, so f is continuous. $\square$

> **Note 3**
>
> If $f : A \times B \to X$ instead, there is *no* useful criterion for the continuity of f.

3.2   Constructing Continuous Functions

> **Proposition 31.** *Constant functions are continuous.*

Proof. Let $f : X \to Y$ be given by $f(x) = y$ for some constant y. If V is open in Y, then $f^{-1}(V)$ is either $\varnothing$ or X, depending on if $y \in V$. In either case, $f^{-1}(V)$ is open. $\square$

> **Proposition 32.** *Inclusion maps are continuous.*

Proof. Let $X \subset Y$, and define $\iota : X \to Y$ by $\iota(x) = x$. For V open in Y, $\iota^{-1}(V) = X \cap V$, which is open in X by definition. $\square$

> **Proposition 33.** *Restrictions of continuous maps are continuous.*

Proof. Let $f : X \to Y$ be continuous, and let $A \subset X$. Define $f|_A : A \to Y$ by $f|_A(a) = f(a)$. For V open in Y, $f|_A^{-1}(V) = f^{-1}(V) \cap A$, which is open in A since $f^{-1}(V)$ is open in X. $\square$

> **Proposition 34.** *Compositions of continuous functions are continuous.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For V open in Z, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$, which is open in X since f and g are both continuous. $\square$