Percolation Phase Transitions on Dynamically Grown Graphs

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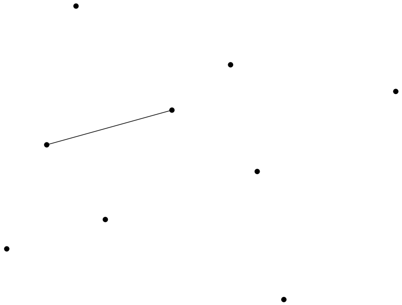
Background

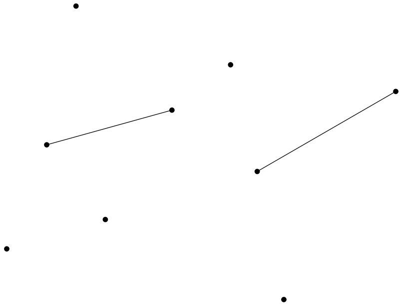
Dynamically grown graphs and percolation

Dynamically Grown Graphs

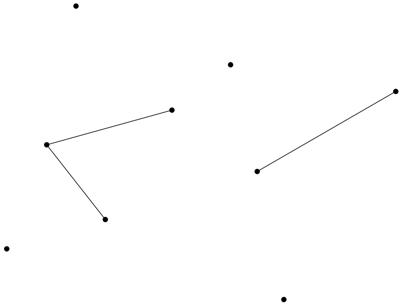
Dynamically Grown Graphs



Dynamically Grown Graphs



Dynamically Grown Graphs



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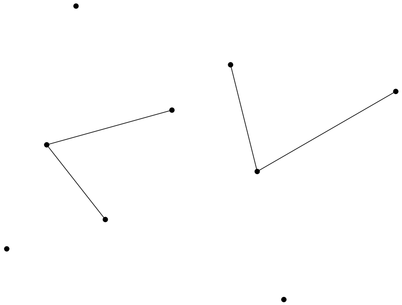

Dynamically Grown Graphs

Every $t = 1/n$ units of time, sample m vertices.

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Let $n \to \infty$.

Percolation

A *giant component*: finite fraction of graph.

Percolation is the emergence of a giant component.

Lots of different behaviors.

Explosive Percolation

Simple rules: linear.

Prioritize merging smaller clusters: *explosive percolation*.

Basic Results

Continuity of the phase transition and scaling behavior

Continuity of the Phase Transition

ℓ-*vertex rule*: choose ℓ vertices i.i.d., and you're only required to add an edge if all ℓ of them are in distinct clusters.

Riordan, R., and Warnke, L. (2012): continuous phase transition.

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Proof by contradiction...

Scaling Behavior

For rules with continuous phase transitions, we see *scaling behavior*.

Let $\delta = t - t_c$ and let $P(s, t)$ be the probability that a randomly chosen vertex has cluster size s at time t. Then near t_c, there are constants τ and σ such that

$$P(s) = s^{1-\tau} f(s\delta^{1/\sigma}).$$

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From now on, we assume scaling behavior.

Scaling Behavior

Let S be the relative size of the giant component, and let

$$\chi_k(t) = \sum_s s^k P(s,t).$$

Then

$$S \approx \delta^{\beta}, \qquad \chi_1(t) \approx \delta^{-\gamma}, \qquad \frac{\chi_k(t)}{\chi_{k-1}(t)} \approx \delta^{-\Delta}$$

These unknowns are called *critical exponents*.

Scaling Relations

Goal: determine all critical exponents in terms of one unknown.

Why is this useful?

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What kinds of rules can we do this for?

2-Choice Rules

Generalizing rules with useful properties

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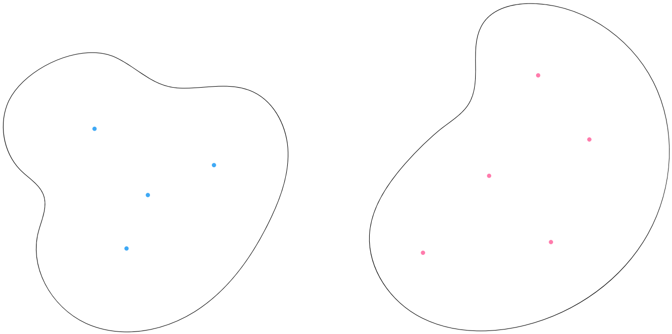
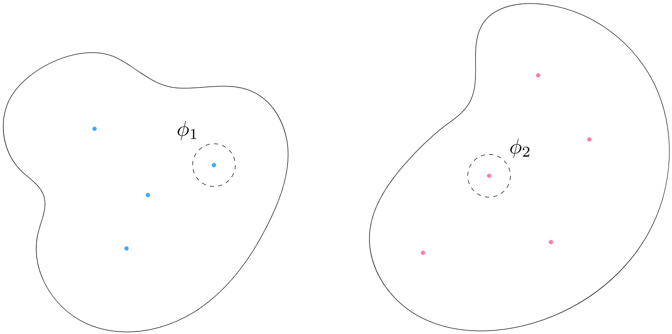


2-Choice Rules



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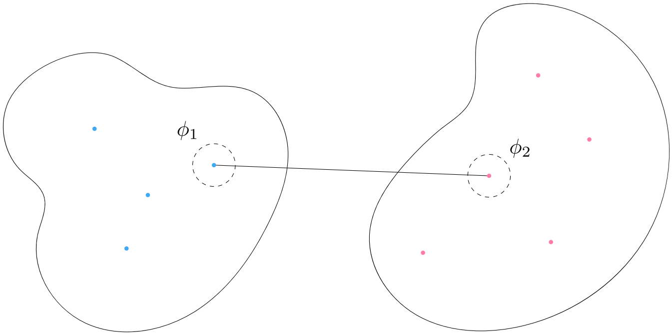

Erdős Rényi

Pick two random vertices and add an edge between them.

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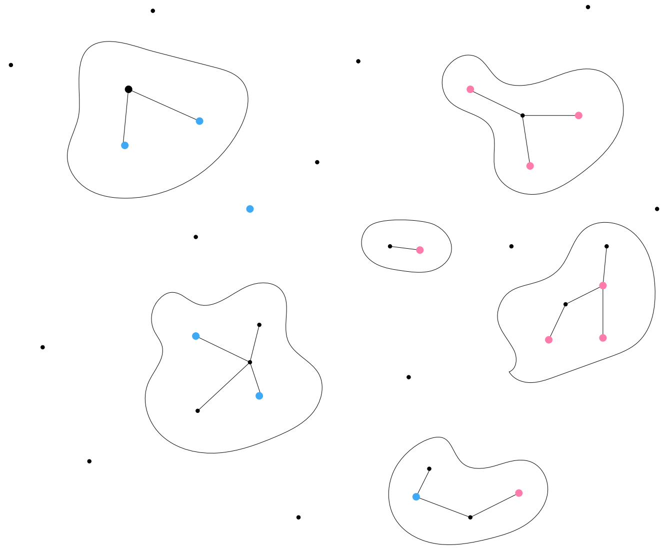
Percolation occurs after $t_c = 1/2$.

$\beta = 1$, so S grows linearly near t_c.

Minimizing Rules



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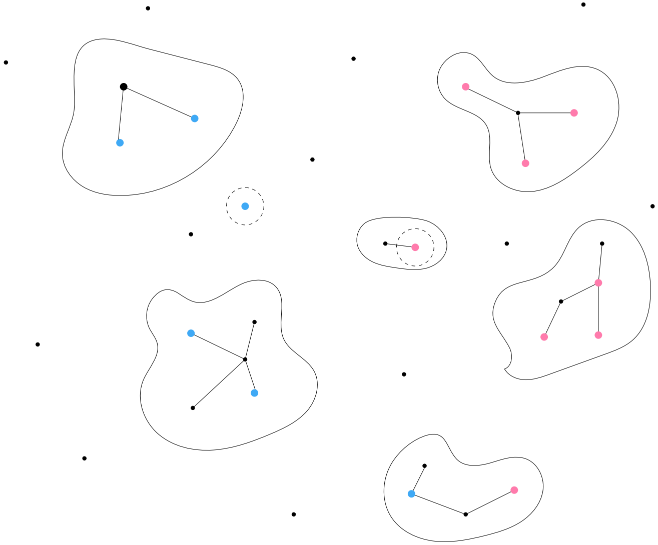

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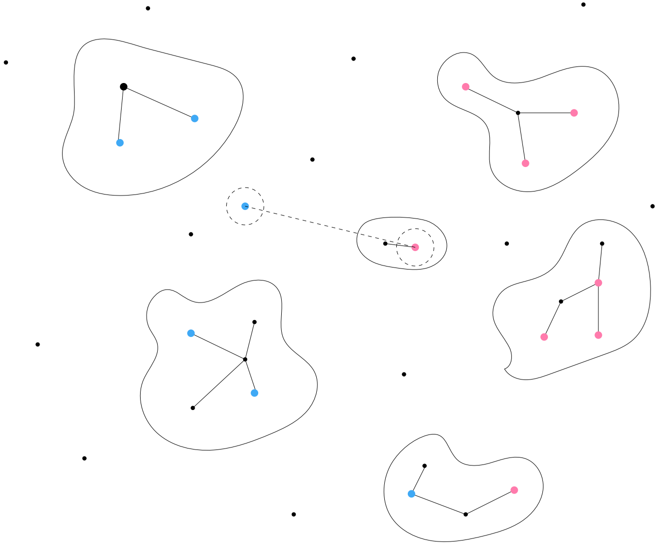

da Costa

Minimizing rule with equal size groups.

Originally introduced to disprove Achlioptas' discontinuity conjecture.

Same as Erdős Rényi when $m = 1$. As $m \to \infty$,

$$\beta \to 0, \qquad t_c \to 1.$$

Finding the Critical Exponents

For any 2-choice rule, the quantity $\partial_t S$ has a simple form that can be explicitly calculated.

Near t_c, it will look like

$$\delta^a + \delta^b + \delta^c + \cdots$$

Finding the Critical Exponents

Theorem

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For all 2-choice rules, we can solve for all critical exponents in terms of β.

We also get the growth rate of the average cluster size.

Minimizing Rules

$$
\begin{aligned}
\gamma_a &= 1 + (b-1)\beta, \\
\gamma_b &= 1 + (a-1)\beta, \\
\gamma_P &= 1 + (a+b-2)\beta, \\
\frac{1}{\sigma} &= 1 + (a+b-1)\beta, \\
\tau &= \frac{\beta}{1 + (a+b-1)\beta} + 2.
\end{aligned}
$$

Asymptotics for Minimizing Rules

$\beta \to 0$ as $a, b \to \infty$.

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$\mathrm{Var}(s) \to \delta^{-2}$.

Future Directions

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- Erdős Rényi is nice. Can we relate other rules to it?
  - bounded size rules
  - Universality classes
- When is t_c?
  - Bohman-Frieze variant
- How fast is convergence to the asymptotic case?
- When does scaling behavior actually occur?