Percolation Phase Transitions on Dynamically Grown Graphs

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April 15, 2022

Background

Dynamically grown graphs and percolation

Dynamically Grown Graphs

Start with a graph with n vertices and 0 edges

Add edges randomly every $1/n$ units of time

We'll work in the limit as $n \to \infty$

Percolation

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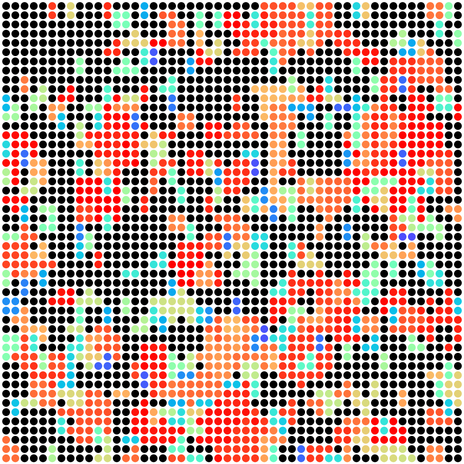
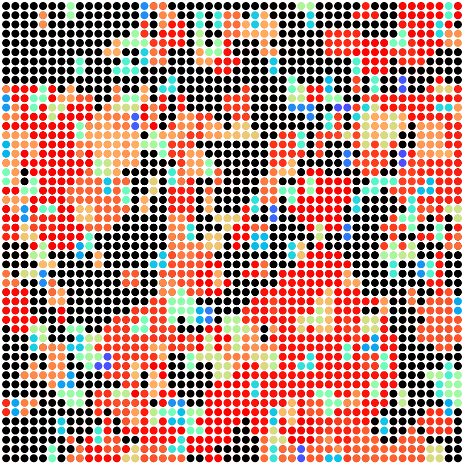
This emergence has lots of different behaviors

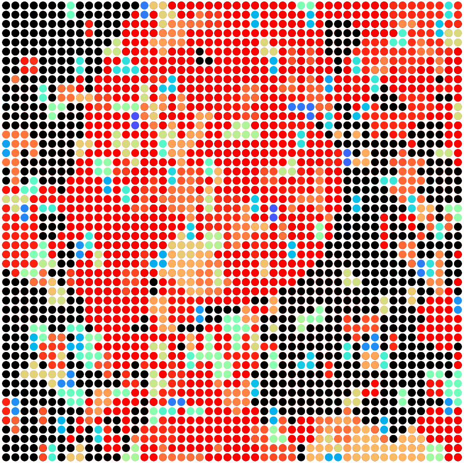
Erdős Rényi $(t = 2/5)$



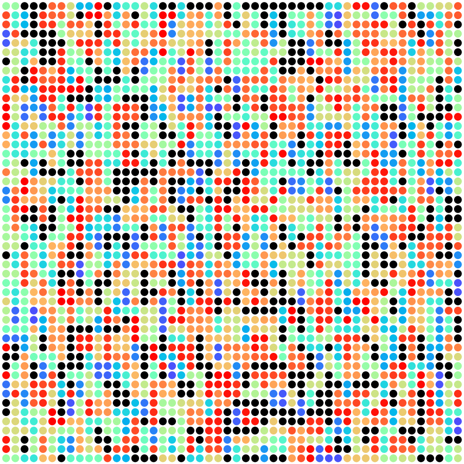
Erdős Rényi $(t = 1/2)$



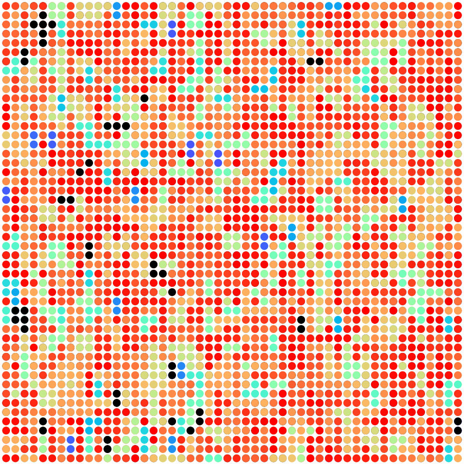
Erdős Rényi $(t = 3/5)$



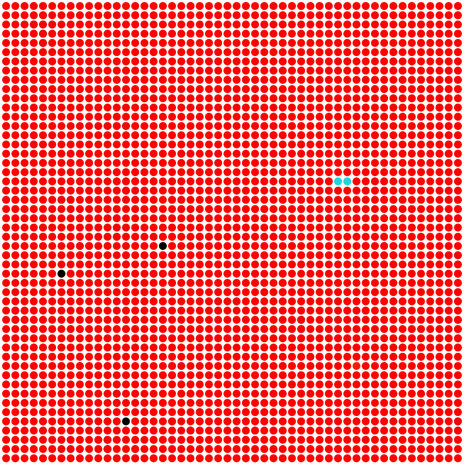
da Costa $(t = 1/2)$



da Costa $(t = 3/4)$



da Costa $(t = 1)$



Explosive Percolation

Explosive Percolation is a sudden, seemingly discontinuous emergence of the giant component

Basic Results

Continuous phase transition and scaling behavior

Continuous phase transition

Achlioptas rule: select two edges at random, then choose one of them to keep via some rule based on the cluster sizes of the edges' vertices

Achlioptas (2009): claimed to have found a discontinuous emergence of a giant component based on simulations

Continuous phase transition

Riordan and Warnke (2012)

ℓ-vertex rule: choose ℓ vertices i.i.d., and you're only required to add an edge if all ℓ of them are in distinct clusters (generalizes Achlioptas processes)

All ℓ-vertex rules have a continuous phase transition

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Proof by contradiction...

Scaling behavior

Because of the continuous phase transition, the distribution of vertices belonging to a cluster of size s follows a power law

$$P(s,t) = s^{1-\tau} f(s\delta^{1/\sigma})$$

where $\delta = t - t_c$ and f is a scaling function.

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Noticing scaling behavior in rules with explosive percolation was motivation for proving their continuity

Critical Exponents

Let S be the relative size of the giant component, and let

$$\langle s^k \rangle_P = \sum_{s \text{ finite}} s^k P(s).$$

Then

$$S \sim \delta^{\beta}, \qquad \langle 1 \rangle_P \sim \delta^{-\gamma}, \qquad \frac{\langle s^{k+1} \rangle_P}{\langle s^k \rangle_P} \sim \delta^{-1/\sigma}$$

These exponents are called *critical exponents*.

Two-Choice Rules

Our Results

Two-Choice Rules

Pick two finite groups of i.i.d. vertices

Follow a deterministic method to choose a representative vertex from each group (can be a different rule for each group)

Add an edge between the two representatives

Two-Choice Rules

Erdős Rényi: Both groups are size 1, so this is the same as sampling edges randomly

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da Costa: both groups are of size m, and pick the vertex with the smallest cluster size from each group.

Scaling Relations

Consider

$$1 - \langle 1 \rangle_\phi = 1 - \sum_{s \text{ finite}} \phi(s),$$

where $\phi(s) = \mathbb{P}(\text{representative has cluster size } s)$

Under certain regularity conditions, this quantity is $\sim \delta^{F(\beta)}$

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Can express critical exponents in terms of β and $F(\beta)$

Scaling Relations

For a two-choice rule with $1 - \langle 1 \rangle_\phi \sim \delta^{F(\beta)}$,

$$
\begin{aligned}
\gamma_\phi &= F(\beta) - \beta + 1 \\
\gamma_P &= 2(F(\beta) - \beta) + 1 \\
\frac{1}{\sigma} &= 2F(\beta) - \beta + 1 \\
\tau &= \frac{\beta}{2F(\beta) - \beta + 1} + 2
\end{aligned}
$$

da Costa

Since $1 - \langle 1 \rangle_\phi$ is the probability that a vertex chosen by our rule is in an infinite cluster,

$$1 - \langle 1 \rangle_\phi = S^m \sim (\delta^\beta)^m \sim \delta^{m\beta},$$

so $F(\beta) = m\beta$

Plugging this into our earlier equations gives scaling relations for the da Costa rule

da Costa

Matches with da Costa's own results

$$\gamma_\phi = 1 + (m-1)\beta$$
$$\gamma_P = 1 + 2(m-1)\beta$$
$$\frac{1}{\sigma} = 1 + (2m-1)\beta$$
$$\tau = 2 + \frac{\beta}{1 + (2m-1)\beta}$$

Erdős Rényi

This is the da Costa rule with $m = 1$, so

$$\begin{aligned} \gamma_\phi = \gamma_P &= 1 \\ \frac{1}{\sigma} &= \beta + 1 \\ \tau &= \frac{\beta}{\beta + 1} + 2 \end{aligned}$$

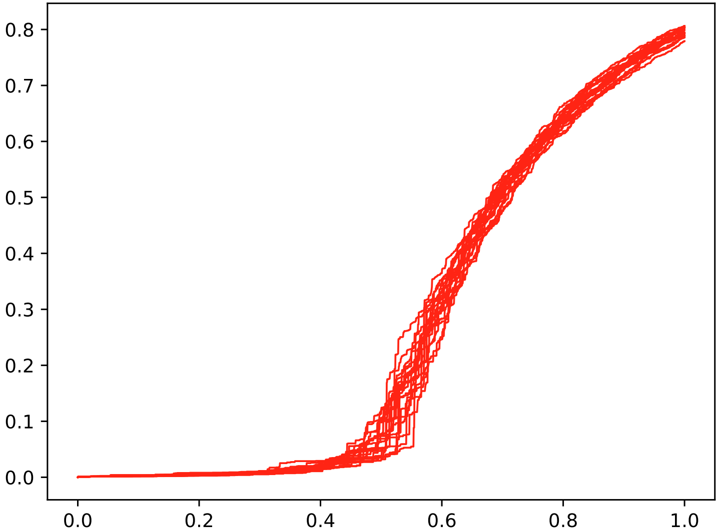
Erdős Rényi

Much more is possible since Erdős Rényi is such a simple rule

Known result: $\beta = 1$, i.e. giant component grows linearly near t_c

This fixes $\gamma_P = 1, \quad \frac{1}{\sigma} = 2, \quad \tau = 5/2$

Erdős Rényi $(n = 2500)$



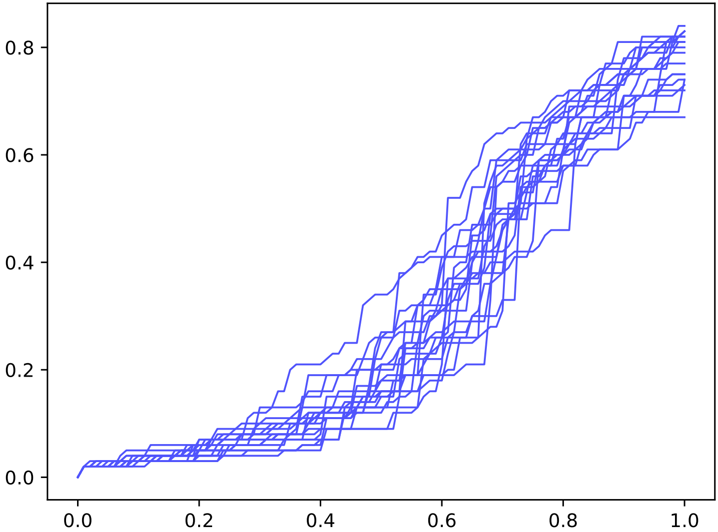
Erdős Rényi

Scaling behavior occurs in region of order $\Theta(s^{-1/2})$ around t_c

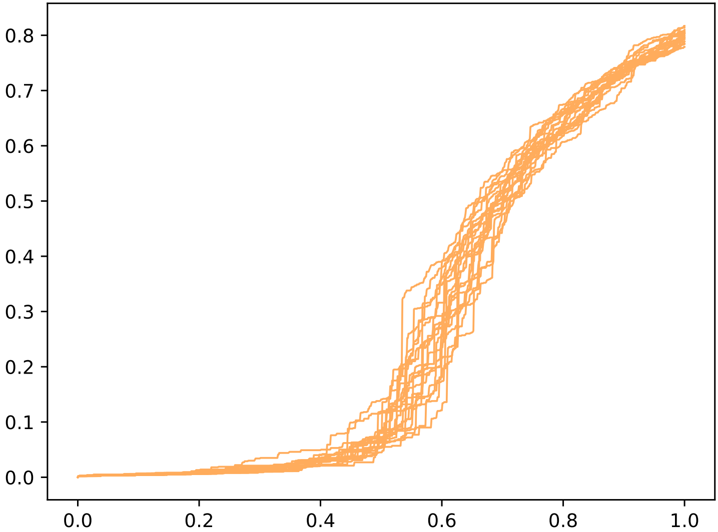
As $s \to \infty$, the scaling window shrinks

In particular, the region of linear giant component growth shrinks as $n \to \infty$

Erdős Rényi ($n = 100$)



Erdős Rényi ($n = 1000$)



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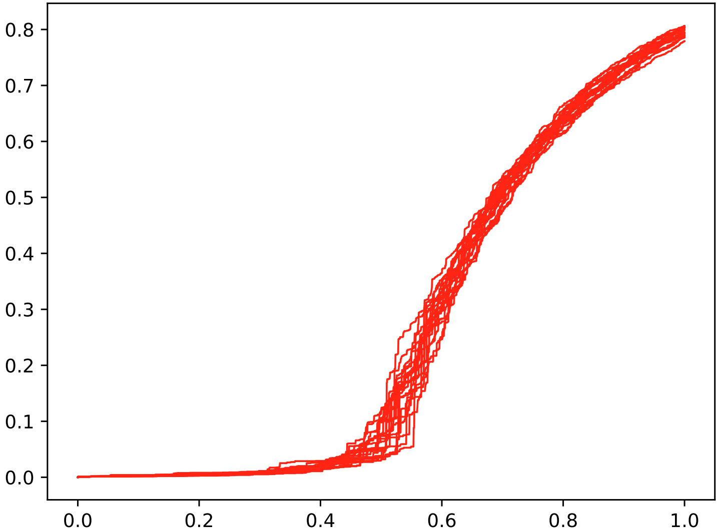

Bounded Size Rules

Rules that are almost Erdős Rényi

Bounded Size Rules

A *bounded size rule* with size threshold K treats all clusters of size $> K$ the same

Intuition: eventually there will be so few clusters of size $\leq K$ that it starts acting like Erdős Rényi

Bohman-Frieze

For each group of vertices, do the following:

- Pick $m + 1$ vertices
- If any of the first m vertices are isolated, pick one at random as the representative
- If not, pick the $(m + 1)$-st vertex instead

Bounded size rule with size threshold $K = 1$

Scaling Relations

For any bounded size rule, $1 - \langle 1 \rangle_\phi \sim \delta^\beta$

Thus $F(\beta) = \beta$ (the same as Erdős Rényi!)

Same scaling relations as Erdős Rényi (with potentially different β)

$$\gamma_\phi = \gamma_P = 1$$
$$\frac{1}{\sigma} = \beta + 1$$
$$\tau = \frac{\beta}{\beta + 1} + 2$$

Thank you!