**Exercise 1** (Lesson 17, 5 points)**.** Let $X$ be a nonempty set and $\phi : X \to \mathcal{H}$ a feature map into a Hilbert space $\mathcal{H}$. Define $k : X \times X \to \mathbb{R}$ by $k(x, y) = \langle \phi(x), \varphi(y) \rangle$ for every $x, y \in X$. Prove $k$ is a kernel.

In order to be a kernel, $k$ must be symmetric and positive definite. These checks amount to just using the properties of inner products, which are

1. $\langle x, y\rangle = \langle x, y\rangle$;
2. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$; and
3. $\langle x, x\rangle \geq 0$, with $\langle x, x\rangle = 0 \iff x = 0$.

**Symmetric:** Since inner products are symmetric (property (1) above),

$$k(x, y) = \langle \phi(x), \phi(y)\rangle = \langle \phi(y), \phi(x)\rangle = k(y, x).$$

**Positive definite:** Fix arbitrary $a_1, \ldots, a_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in X$, then we want to show $\sum_{i,j} a_i a_j k(x_i, x_j) \geq 0$. From the properties of inner products,

$$\begin{aligned}
\sum_{i,j} a_i a_j k(x_i, x_j) &= \sum_{i,j} a_i a_j \langle \phi(x_i),\ \phi(x_j)\rangle \\
&= \sum_i a_i \sum_j a_j \ \langle \phi(x_i),\ \phi(x_j)\rangle \\
&= \sum_i a_i \left\langle \phi(x_i),\ \sum_j a_j \phi(x_j) \right\rangle && \text{(by property 2)} \\
&= \left\langle \sum_i a_i \phi(x_i),\ \sum_j a_j \phi(x_j) \right\rangle. && \text{(by property 2)}
\end{aligned}$$

Since both $i$ and $j$ range over the same values, both sums are the same. Thus this becomes

$$\begin{aligned}
&= \left\langle \sum_i a_i \phi(x_i),\ \sum_i a_i \phi(x_i) \right\rangle \\
&\geq 0. && \text{(by property 3)}
\end{aligned}$$

**Exercise 2** (Lesson 18, 5 points)**.** Prove using the Schoenberg Theorem that the function

$$k(x,y) = e^{\frac{-\|x-y\|^2}{2\sigma^2}},\ x, y \in \mathbb{R}^d$$

is a kernel on $\mathbb{R}^d$ for every $\sigma^2 > 0$.

For this problem, $f(x,y)$ from the Schenberg theorem is

$$f(x,y) = \|x-y\|^2.$$

We must show that this $f$ is symmetric and CNSD.

**Symmetric:** Our $f$ is symmetric since

$$f(x,y) = \|x-y\|^2 = (-1\,\|x-y\|)^2 = \|y-x\|^2 = f(y,x).$$

**CNSD:** Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $\sum_i \alpha_i = 0$, and let $x_1, \ldots x_n \in X$. Then

$$\begin{aligned}
\sum_{i,j} \alpha_i \alpha_j \|x_i - x_j\|^2 &= \sum_i \alpha_i \left( \sum_j \alpha_j \|x_i - x_j\|^2 \right) \\
&\leq \sum_i \alpha_i \left( \sum_j \alpha_j \left(\|x_i\| + \|x_j\|\right)^2 \right) \\
&\leq \sum_i \alpha_i \left( \sum_j a_j \left( \sum_k \|x_k\| \right)^2 \right) \\
&= \sum_i \alpha_i \left( \left( \sum_k \|x_k\| \right)^2 \left( \sum_j \alpha_j \right) \right) \\
&= 0,
\end{aligned}$$

where the last equality follows from $\sum_j \alpha_j = 0$.