Exercise 1 (0: 4)**.** Show that if X deformation retracts to A in the weak sense, then the inclusion $i : A \hookrightarrow X$ is a homotopy equivalence.

Suppose X deformation retracts (weakly) onto A via F, then this determines a map $r : X \to A$ by $x \mapsto F_1(x)$ (as an element of A). Then F is the homotopy showing $i \circ r \simeq \mathrm{id}_X$. We can also consider the map $r \circ i : A \to A$, which is homotopic to id_A via $F|_A$. Thus i is a homotopy equivalence.

Exercise 2 (0: 5)**.** Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X, there is a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

Suppose X deformation retracts onto x via F, and fix a neighborhood U of x. Now consider the open set $F^{-1}(U)$ in $X \times I$. We'd like to apply the tube lemma to find a nice neighborhood V of x now, but we have to check its two conditions:

a. $F^{-1}(U)$ contains $\{x\} \times I$ since $F_t(x) = x$ for all t and since t ranges over all of I;

b. I is compact.

Thus applying the tube lemma, we get a neighborhood V of x such that $V \times I \subset F^{-1}(U)$. Since $V \times I$ lies entirely in $F^{-1}(U)$, the map $F \circ i$ (where i is the inclusion $V \hookrightarrow U$) always maps into U. Thus $F \circ i : V \times I \to U$ is a well-defined homotopy showing $i \simeq x$.

Exercise 3 (0: 6). a. Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r rational in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point.

b. Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.

c. Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$. Show that there is a deformation retraction in the weak sense of Y onto Z, but no true deformation retraction.

a. Given a point $(x, 0)$, we can form an explicit deformation retraction

$$F((q,s),t) = \begin{cases} (q, s(1-2t)) & 0 \le t \le 1/2, \\ ((2t-1)x + (2-2t)q, 0) & 1/2 \le t \le 1. \end{cases}$$

Now we want to show that we cannot deformation retract X onto any point on one of the verticle prongs. To do so, we'll use the previous exercise (0: 5). Let $x = (q, s)$, where $q \in \mathbb{Q} \cap [0,1]$ and $s \in [0, 1-q]$, and suppose X deformation retracts to x. Let U be the open ball of radius s centered at x, so that it does not intersect the bottom line of X. By the previous exercise, there is some $V \subseteq U$ such that $V \hookrightarrow U$ is nullhomotopic, say via F. In particular, for all $v \in V$, the map $F(v)$ is a path from v to some constant point.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is some $r \in \mathbb{Q}$ such that the prong $\{r\} \times [0, 1-r]$ intersects V. Then since $V \subseteq U$ doesn't intersect the bottom line of X, the open sets $\{x < r\} \cap V$ and $\{x > r\} \cap V$ disconnect V. This contradicts the result of the previous paragraph (where each $v \in V$ could follow a path to x while staying entirely in U), so X cannot deformation retract onto x.

b. To show that Y is contractible, we assume the result from part (c) that Y deformation retracts in the weak sense to its zigzag subspace Z. Then by Exercise (0: 4), $Y \simeq Z$. This gives the sequence $Y \simeq Z \cong \mathbb{R}$. Since $\mathbb{R}$ is contractible, so is Y.

Now we show that Y does not deformation retract onto any point. Suppose y is on any of the prongs of any of the combs making up Y. Then by the same argument as in part (a), Y cannot deformation retract onto y. If y is instead on the bottom line of any of the combs, we have a similar situation. Any neighborhood of y intersects the prongs of another comb, so by a disconnection argument similar to that in (a), Y cannot deformation retract onto y.

c. Fix a point $y \in Y$, then consider the following path F_y (with everything done at the same constant speed):

- if y is on a prong, move toward the base of the current comb;
- if y is in Z, move to the right along Z.

The map F given by having every $y \in Y$ follow its path F_y is then the weak deformation retraction between Y and Z.

The fact that Y doesn't have any true deformation retraction onto Z is essentially the same argument as in parts (a) and (b). If we replace x in the argument in part (a) with some path connected space Z, we get that any neighborhood U of a point in Z must be path connected, which, as discussed in part (b), is not true.

Exercise 4 (0: 9)**.** Show that a retract of a contractible space is contractible.

Suppose X is contractible to a point x via F, and suppose $r : X \to A$ is a retraction, i.e. $r \circ i = \mathrm{id}_A$. Then consider the map

$$\begin{aligned} G : A \times I &\to A \\ (a,t) &\mapsto r(F_t(i(a))). \end{aligned}$$

At $t = 0$, we get $a \mapsto r(i(a)) = a$, so $G_0 = \mathrm{id}_A$. And at $t = 1$, we get $a \mapsto r(x)$. As the composition of continuous maps, G is also continuous. This shows that A is contractible to a single point, namely $r(x)$.

Exercise 5 (0: 13)**.** Show that any two deformation retractions r_t^0 and r_t^1 of X onto a subspace A can be joined by a continuous family of deformation retractions r_t^s.

All the indices were messing with me, so I let $F \doteq r^0$ and $G \doteq r^1$. Now define a homotopy $H : X \times I \times I \to X$ by

$$H(\,\cdot\,, s, t) \doteq F_{\varphi_2(s,t)} \circ G_{(\varphi_1(s,t))}$$

where $\varphi : I \times I \to I \times I$ is given by

$$\varphi(s,t) = \begin{cases} (2st, t) & 0 \le s \le 1/2, \\ (t, 2(1-s)t) & 1/2 \le s \le 1. \end{cases}$$

H is continuous since it's the composition of continuous functions. It transitions from F to G as s goes from 0 to 1:

- $H(\,\cdot\,, 0, t) = F_t \circ G_0 = F_t \circ \mathrm{id}_X = F_t$;
- $H(\,\cdot\,, 1, t) = F_0 \circ G_t = \mathrm{id}_X \circ G_t = G_t$.

Each H_s is itself a deformation retraction:

- When $t = 0$, H_s is the identity on X since $H(\,\cdot\,, s, 0) = F_0 \circ G_0 = \mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$;
- Let $t = 1$, then
$$H(\,\cdot\,, s, 1) = \begin{cases} F_1 \circ G_{2s} \\ F_{2(1-s)} \circ G_1. \end{cases}$$
In both cases, H_s maps X into A since F_1, G_1 map X into A and all F_t fix A.
- Each H_s fixes A since every F_t and G_t fixes A.

Exercise 6 (1.1: 6)**.** We can regard $\pi_1(X, x_0)$ as the basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that ϕ is onto if X is path-connected, and that $\phi([f]) = \phi([g]) \iff [f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence ϕ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

a. First we show that ϕ is onto if X is path-connected. Fix $[\tilde{g}] \in [S^1, X]$. Considering S^1 as $[0, 1]/0 \sim 1$, there is clearly some $[g] \in \pi_1(X, x_1)$ for some x_1 such that $\phi([g]) = [\tilde{g}]$. Since X is path-connected, there is an isomorphism $\pi_1(X, x_0) \to \pi(X, x_1)$ given by the change of basepoint map. Thus there is some $[f] \in \pi_1(X, x_0)$ such that

$$[f] \mapsto [\overline{h} f h] = [g],$$

where h is a path from x_0 to x_1. Now on S^1, we can form a homotopy from $\overline{h} f h$ to f by

$$F(x, t) = (\overline{h} f h) \left(\frac{1}{3} t x \right)$$

(note that this does not preserve basepoints). Thus in $[S^1, X]$, we have $[g] = [\overline{h} f h] = [f h \overline{h}] = [f]$. This implies $\phi([f]) = \phi([g]) = [\tilde{g}]$, so ϕ is onto.

b. Now we show that $\phi([f]) = \phi([g]) \iff [f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

Backward: Suppose $[h]^{-1}[f][h] = [\overline{h} f h] = [g]$ for soe $[h] \in \pi_1(X, x_0)$. In part (a) we showed that $[\overline{h} f h] = [f]$ in $[S^1, X]$, so $\phi([f]) = \phi([g])$.

Forward: Suppose $\phi([f]) = \phi([g])$ for $[f], [g] \in \pi_1(X, x_0)$, then $f \simeq g$ by some homotopy F that doesn't necessarily preserve the basepoint. Then by Lemma 1.19 in Hatcher, the following diagram commutes,

$$\begin{tikzcd} \pi_1(S^1, s_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(X, x_0) \arrow[d, "\beta_h"] \\ & \pi_1(X, x_0)) \end{tikzcd}$$

where β_h is the change of basepoint isomorphism induced by the path $h \doteq F(s_0)$ (note that since f and g are both loops at x_0, so is h). The commutativity of the diagram then implies $[g] = [\overline{h} f h] = [h]^{-1}[f][h]$, as desired (note that the second equality is valid since h is a loop at x_0, and so $[h] \in \pi_1(X, x_0)$).

Exercise 7 (1.1: 7)**.** Define $f : S^1 \times I \to S^1 \times I$ by

$$f(\theta, s) = (\theta + 2\pi s, s),$$

so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles. [What does f do to the path $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$?]

Homotopy that fixes one boundary circle: Consider the map

$$\begin{aligned} F : S^1 \times I \times I &\to S \times I \\ (\theta, s, t) &\mapsto (\theta + 2\pi st, s). \end{aligned}$$

Since $F_0 : (\theta, s) \mapsto (\theta, s)$ is the identity and $F_1 : (\theta, s) \mapsto (\theta + 2\pi s, s)$ is f, this is a homotopy between the identity and f. Furthermore, $F_t : (\theta, 0) \mapsto (\theta, 0)$ for all t, so this homotopy fixes the boundary circle $S^1 \times \{0\}$. It doesn't fix the other boundary circle $S^1 \times \{1\}$, though, since $F_t : (\theta, 1) \mapsto (\theta + 2\pi t, 1)$.

No homotopy exists that fixes both boundary circles: Suppose we have a homotopy G between f and the identity that fixes both boundary circles. Now fix θ_0 and consider the path

$$\varphi(s) \doteq (\theta_0, s).$$

Since G fixes the boundary circles, $G\varphi$ is a path homotopy showing $\varphi \simeq f\varphi$. Visually, at the very least, this is clearly impossible: φ is a straight line along the length of the cylinder, while $f\varphi$ is a helix, so they can't possibly be homotopic rel their endpoints.

To formalize this, we can post-compose $G\varphi$ with the natural projection $p : S^1 \times I \twoheadrightarrow S^1$ to get a new homotopy $pG\varphi$ showing $p\varphi \simeq pf\varphi$. Since the endpoints of φ and $f\phi$ all have the same θ_0 coordinate, and since $G\varphi$ fixes the boundary circles throughout the homotopy, $pG\varphi$ is a homotopy of loops with the same basepoint.

Now $p\varphi$ is a constant loop and $pf\varphi$ is a full loop around the circle, so we have that any loop going around the circle once is nullhomotopic. But a loop going around the circle n times can be decomposed into n single loops around the circle. And since loop homotopies respect path multiplication, this implies that all n-loops around the circle are nullhomotopic, i.e. $\pi_1(S^1) \cong 1$. But is this a contradiction, since we know $\pi_1(S^1) \cong \mathbb{Z}$. Thus our original homotopy G could not have existed in the first place.

Exercise 8 (1.1: 16 (a,b,c,f))**.** Show that there are no retractions in the following cases:

a. $\mathbb{R}^3$ onto any subspace $A \cong S^1$.

b. $S^1 \times D^2$ onto its boundary torus $S^1 \times S^1$.

c. $S^1 \times D^2$ onto the circle in the figure.

f. The Möbius band onto its boundary circle.

These problems use the proposition that given a retraction

$$A \underset{i}{\overset{r}{\leftrightarrows}} X,$$

r_* is surjective and i_* is injective.

a. An existence of a retraction $\mathbb{R}^3 \to A$ means there is an injection $i_* : \pi_1(A) \to \pi(\mathbb{R}^3)$. But since $A \cong S^1$ and $\mathbb{R}^n$ is simply connected, this is an injection from $\pi_1(A) \cong \pi_1(S^1) \cong \mathbb{Z}$ into $\pi_1(\mathbb{R}^3) \cong 1$, which is clearly impossible.

b. The argument here is the same. The existence of a retraction means we have an injection from $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ into $\pi_1(S^1 \times D^2) \cong \mathbb{Z}$. But this is impossible: suppose the injective homomorphism is ϕ, then the first isomorphism theorem gives $\mathbb{Z}^2 \cong \phi(\mathbb{Z}^2) \leq \mathbb{Z}$, but all subgroups of $\mathbb{Z}$ are cyclic and $\mathbb{Z}^2$ is not cyclic.

c. Once again i_* is an injection. But any loop in the circle from the figure becomes nullhomotopic when treated as an element of $S^1 \times D^2$ due to this torus being filled in, so i_* is the zero map, which contradicts its injectivity.

f. Since the Möbius band M deformation retracts to its center circle, $\pi_1(M) \cong \mathbb{Z}$. Its boundary circle A is a circle, so $\pi_1(A) \cong \mathbb{Z}$ as well. Now a loop that goes around A once ends up going around M twice, so i_* can be described explicitly by

$$\begin{aligned} i_* : \mathbb{Z} &\to \mathbb{Z} \\ 1 &\mapsto 2. \end{aligned}$$

This map is simply multiplication by 2. But since $r \circ i = \mathrm{id}_A$, we know $r_* \circ i_* = \mathrm{id}_{\mathbb{Z}}$. Thus we get the following commutative diagram.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\times 2"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbb{Z} \arrow[r, "r_*"] & \mathbb{Z} \end{tikzcd}$$

But this is impossible: the only map undoing multiplication by 2 is division by 2, but this is not a homomorphism $\mathbb{Z} \to \mathbb{Z}$.

Exercise 9 (1.1: 20)**.** If $F : X \to X$ is a homotopy such that $F_0 = F_1 = \mathrm{id}_X$, then for any x_0, the loop $F(x_0)$ represents an element of the center of $\pi_1(X, x_0)$.

By lemma 1.19, we get the following commutative diagram,

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & \pi_1(X,x_0) \arrow[d, "\beta_{F(x_0)}"] \\
& \pi_1(X,x_0)
\end{tikzcd}$$

where $\beta_{F(x_0)}$ is the change of basepoint isomorphism $[\alpha] \mapsto [F(x_0) \cdot \alpha \cdot \overline{F(x_0)}]$. This says that $\beta_{F(x_0)}$ is just the identity, i.e.

$$[F(x_0) \cdot \alpha \cdot \overline{F(x_0)}] = [\alpha]$$

for all $[\alpha]$. Then since $F(x_0)$ is itself a loop at x_0, we can decompose the change of basepoint:

$$\begin{aligned}
[F(x_0) \cdot \alpha \cdot \overline{F(x_0)}] &= [\alpha] \\
[F(x_0)][\alpha][F(x_0)]^{-1} &= [\alpha] \\
[F(x_0)][\alpha] &= [\alpha][F(x_0)].
\end{aligned}$$

Since this statement holds for all $[\alpha]$, the homotopy class $[F(x_0)]$ is in the center of $\pi_1(X, x_0)$.