Problems Completed: All.

Exercise 1 (§22, #2).

a. $p : X \to Y$ continuous. If there is a continuous $f : Y \to X$ such that $p \circ f = 1_Y$, then p is a quotient map.

b. Show that a retraction r of X onto A is a quotient map.

Collaborators: None.

a. Let $p : X \to Y$ be continuous, and suppose there is some other continuous $f : Y \to X$ such that $p \circ f = 1_Y$.

   - Let $y \in Y$ be arbitrary. Then $(p \circ f)(y) = y$, so $p(f(y)) = y$, so p is surjective.
   - Suppose U is open in Y, then since p is continuous $p^{-1}(U)$ is open in X. Conversely, suppose $p^{-1}(U)$ is open in X, then

   $$f^{-1}(p^{-1}(U)) = (p \circ f)^{-1}(U) = 1_Y^{-1}(U) = U.$$

   Since f is continuous, this means U is open.

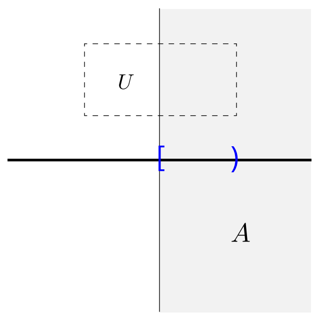
   Thus p is a quotient map.

b. Suppose $r : X \to A$ is a retraction onto A, then it is continuous and fixes A. We can define $\iota : A \to X$ to be the usual inclusion map, which we know to be continuous. Then $r \circ \iota = 1_A$, so by part (a), r is a quotient map.

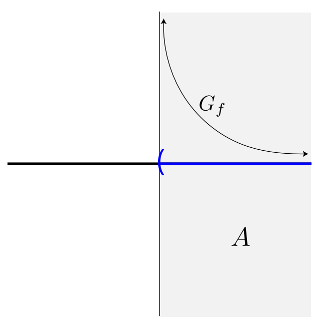
Exercise 2 (§22, #3)**.** $A \subset \mathbb{R}^2$ is all points for which either $x \geq 0$ or $y = 0$. Let $q = \pi_1|A$. Show that q is a quotient map that is neither open nor closed.

Collaborators: None. q **is a quotient map:** Define a continuous map $f : \mathbb{R} \to A$ by $f(x) = (x, 0)$, then $q \circ f = 1_{\mathbb{R}}$, so by Exercise 1 part a, q is a quotient map onto $\mathbb{R}$.

q **is not open:** Let U be the open rectangle $\{(x, y) \quad -1 < x < 1, 1 < y < 2\}$ in $\mathbb{R}^2$. Then $U \cap A = \{(x, y) \mid 0 \leq x < 1, 1 < y < 2\}$ is open in A. Its projection onto $\mathbb{R}$ is the interval $[0, 1)$, which is not open in $\mathbb{R}$, so q is not an open map.



q **is not closed:** As proved in Homework 6 Exercise 4, the graph of a continuous function whose codomain is Hausdorff must be closed. Thus the graph G_f of $f(x) = 1/x$ for $x > 0$ is closed in A. But $q(G_f) = (0, \infty)$ is not closed in $\mathbb{R}$, so q is not a closed map.



Exercise 3 (§22, #4)**.**

a. Let $X = \mathbb{R}^2$. Define $(x_0, y_0) \sim (x_1, y_1) \iff x_0 + y_0^2 = x_1 + y_1^2$. What is X^* homeomorphic to?

b. Repeat (a) for $(x_0, y_0) \sim (x_1, y_1) \iff x_0^2 + y_1^2 = x_1^2 + y_1^2$.

Collaborators: None.

a. Let $g(x, y) = x + y^2$, then g is a continuous map that induces $\sim$, so $g(X) \cong X^*$ if and only if g is a quotient map. Define a continuous map $f : \mathbb{R} \to X$ by $f(x) = (x, 0)$, then $g \circ f = 1_{\mathbb{R}}$, so by Exercise 1 part (a), g is a quotient map onto $\mathbb{R}$. Thus $X^* \cong g(X) = \mathbb{R}$.

b. Let $g(x, y) = x^2 + y^2$, then just as in part (a), $g(X) \cong X^*$ if and only if g is a quotient map. Define a continuous map $f : \mathbb{R}_{\geq 0} \to X$ by $f(x) = (\sqrt{x}, 0)$. Then $g \circ f = 1_{\mathbb{R}_{\geq 0}}$, so g is a quotient map onto $\mathbb{R}_{\geq 0}$. Thus $X^* \cong g(X) = \mathbb{R}_{\geq 0}$.

Exercise 4. Define $(x, y, z) \sim (-x, -y, -z)$ and denote the resulting quotient space by $\mathbb{RP}^2$. Consider

$$\begin{aligned} g : S^2 &\to \mathbb{R}^4 \\ (x, y, z) &\mapsto (x^2 - y^2, xy, xz, yz). \end{aligned}$$

a. Prove $g : S^2 \to g(S^2)$ is a quotient map.

b. Prove that $\mathbb{RP}^2 \cong g(S^2)$ with the subspace topology.

Collaborators: Saloni Bulchandani.

a. The function $g : S^2 \to g(S^2)$ is surjective because it's onto its image, and it's continuous since each of its components are continuous. Then since S^2 is compact (it's closed and bounded in $\mathbb{R}^n$) and $g(S^2)$ is Hausdorff (it's a subspace of $\mathbb{R}^4$, which is Hausdorff), g is a closed map. Since it's closed and continuous, it is a quotient map.

b. Now to show $\mathbb{RP}^2 \cong g(S^2)$, we can show that g induces the same partition of S^2 that $\sim$ does. Suppose $\mathbf{x} := (x, y, z) \sim (\tilde{x}, \tilde{y}, \tilde{z}) =: \mathbf{y}$, then we manually check that $f(\mathbf{x}) = f(\mathbf{y})$.

   Conversely, given $f(\mathbf{x}) = (a, b, c, d) = f(\mathbf{y})$, we wish to find all possible values of $\mathbf{y}$. We have the system

$$x^2 - y^2 = a, \quad xy = b, \quad xz = c, \quad yz = d.$$

   In the case that $x = 0$, the first equation gives $-y^2 = a$, which forces both to be 0. This then makes $b = c = d = 0$. Since we're on S^2, we know $x^2 + y^2 + z^2 = 0 + 0 + z^2 = 1$, so $z = \pm 1$. Then the only possible values for $\mathbf{y}$ are $\pm(0, 0, 1) = \pm(x, y, z)$.

   In the case that $x \neq 0$, the second equation gives $y = b/x$, and substituting into the first and multiplying both sides by x^2 gives the polynomial

$$(x^2)^2 - ax^2 - b^2 = 0.$$

   By the quadratic formula, this has solutions

$$x^2 = \frac{a \pm \sqrt{a^2 + 4b^2}}{2}.$$

   Substituting our expressions for a and b from our system of equations into this expression yields

$$x^2 = \frac{x^2 - y^2 \pm (x^2 + y^2)}{2}.$$

If we use $-(x^2 + y^2)$, then this becomes $x^2 = -y^2$, which is impossible since one side is always positive and the other side is always negative. Thus x^2 can only satisfy

$$x^2 = \frac{a + \sqrt{a^2 + 4b^2}}{2},$$

and similarly, $y^2 = (-a + \sqrt{a^2 + 4b^2})/2$, which means that $\tilde{x}$ and $\tilde{y}$ are determined up to their sign by the first equation in our system. Then by our second equation $xy = b$, we have two possibilities: $(\tilde{x}, \tilde{y}) = \pm(x, y)$.

Then by our last two equations $xz = c, yz = d$, we know that $\tilde{z}$ must match the sign of $\tilde{x}$ and $\tilde{y}$, i.e $\mathbf{x} = \pm\mathbf{y}$.

Thus g induces $\sim$, so $g(S^2) \sim \mathbb{RP}^2$.