0.1 Erdos Renyi

We've assumed that this scaling behavior exists near t_c, but it's natural to ask what is meant by "near". In the case of the Erdős Rényi rule, we can describe the size of this window using our earlier differential equation for $\partial_t P(s,t)$. I use the following two relationships, which hold when $t < t_c$ and s is large:

- $P(x) = \delta^{(\tau-1)/\sigma} \tilde{f}(x\delta^{1/\sigma})$.
- $\tilde{f}(x) \propto x^{\lambda} \exp\left(-Cx^{1+\log_2 m}\right)$, where $\lambda = (1+\log_2 m)\left(1 + \frac{1}{4m-2}\right) - \frac{2m}{2m-1}$.

In the below computation, I represent the constant of proportionality for $\tilde{f}$ by $\tilde{C}_f$. And to clean up notation (cause there's a lot of it), I use the following shorthand:

- $\mathcal{E}_x \doteq \exp\left(-Cx\delta^{1/\sigma}\right)$.

Note that $1 + \log_2 m = 1$ when $m = 1$, so our exponential law for $\tilde{f}$ becomes pretty simple. I'll also only be considering the case when $t < t_c$, so δ becomes just $t_c - c$ (this makes derivatives nicer). When s is large, our ODE gives

$$\partial_t P(s) = \frac{s}{2}\int_0^s P(u)P(s-u)\,du - sP(s)$$
$$\partial_t \left\{\delta^{(\tau-1)/\sigma}\tilde{f}\left(s\delta^{1/\sigma}\right)\right\} = \frac{s}{2}\int_0^s \delta^{2(\tau-1)/\sigma}\tilde{f}(u\delta^{1/\sigma})\tilde{f}(v\delta^{1/\sigma}) - s\delta^{(\tau-1)/\sigma}\tilde{f}(s\delta^{1/\sigma}).$$

We can pull out a $\mathcal{E}_s$, $\tilde{C}_f$, s, and $\delta^{(\tau-1+\lambda)/\sigma}$ from each side, leaving us with

$$s^{\lambda-1}\left[\frac{Cs}{\sigma}\delta^{(1-\sigma)/\sigma} - \frac{\tau-1+\lambda}{\sigma\delta}\right] = \frac{\tilde{C}_f}{2}\delta^{(\tau-1+\lambda)/\sigma}\int_0^s (us-u^2)^{\lambda}\,du - s^{\lambda}.$$

Now for Erdős Rényi, we know $\sigma = 1/2, \tau = 5/2$, so we get $\lambda = -1/2$. Plugging these in yields

$$4s^{-3/2}\left(Cs\delta - \frac{1}{\delta}\right) = \tilde{C}_f\delta^2\int_0^s (us-u^2)^{-1/2}\,du - s^{-1/2}.$$

Now **(in the limit as $s \to \infty$)** this integral evaluates to π. So letting s be very large, we can take only the nonvanishing terms to get

$$0 \approx \tilde{C}_f\delta^3\pi\sqrt{s} - \delta - 4C\delta^2.$$

Cancelling out a δ gives a quadratic, whose solution gives

$$\delta = \Theta(1/\sqrt{s}).$$

So the scaling window gets smaller as s gets bigger, which seems reasonable.

to be completely rigorous, should probably have integral from ε to s instead of 0 to s.

0.2 DaCosta

The DaCosta rule isn't as easy, mainly because there are a lot of extra exponents that make things a pain. One big issue is that the exponential terms no longer cancel out. At one point in the Erdős Rényi computation, we get $\mathcal{E}_{u+v} = \mathcal{E}_s$, but for DaCosta with $m = 2$, this would be $\mathcal{E}_{u^2+v^2}$, which doesn't really simplify.