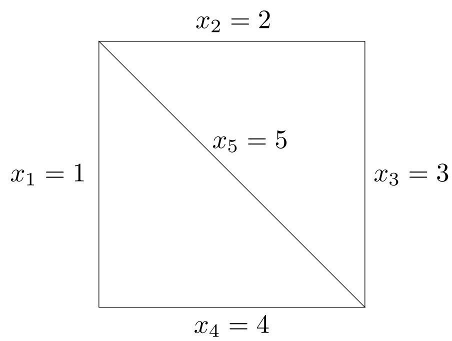
Exercise 1 (Lesson 16, 5 points). Let $f : \mathbb{R}^5 \to \mathbb{R}$ be the function defined in Lesson 16. Find $\nabla f(1,2,3,4,5)$ and $\nabla f(5,1,3,4,2)$. In other words, find $\dfrac{\partial f}{\partial x_i}$ for $i = 1, \ldots, 5$.

First one: The diagram at $(1,2,3,4,5)$ is below.

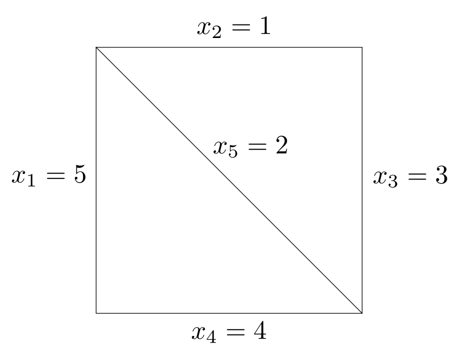


Its 1-dimensional persistence diagram is $I_{[4,\infty)} \oplus I_{[5,\infty)}$. Thus the output of the function is $f(1,2,3,4,5) = b_1^2 + b_2^2 = 4^2 + 5^2 = 41$. Note that $b_1 = x_4$ and $b_2 = x_5$, so the only x_i affecting the output of f at $(1,2,3,4,5)$ are x_4 and x_5. Thus the partial derivatives are

$$\frac{\partial f}{\partial x_i} = \begin{cases} 2b_1 = 8 & \text{if } i = 4, \\ 2b_2 = 10 & \text{if } i = 5, \\ 0 & \text{else.} \end{cases}$$

We can write this in gradient form as $\nabla f(1,2,3,4,5) = (0,0,0,8,10)$.

Second one: The diagram at $(5,1,3,4,2)$ is below.



The 1-dimensional persistence diagram is $I_{[3,\infty)} \oplus I_{[5,\infty)}$ this time, with $b_1 = x_3 = 3$ and $b_2 = x_1 = 5$. By a similar argument, the partial derivatives at this point are

$$\frac{\partial f}{\partial x_i} = \begin{cases} 2b_1 = 6 & \text{if } i = 3, \\ 2b_2 = 10 & \text{if } i = 1, \\ 0 & \text{else.} \end{cases}$$

We can write this in gradient form as $\nabla f(5,1,3,4,2) = (10,0,6,0,0)$.