Chapter 1

Basics

variation, quadratic variation
filtration

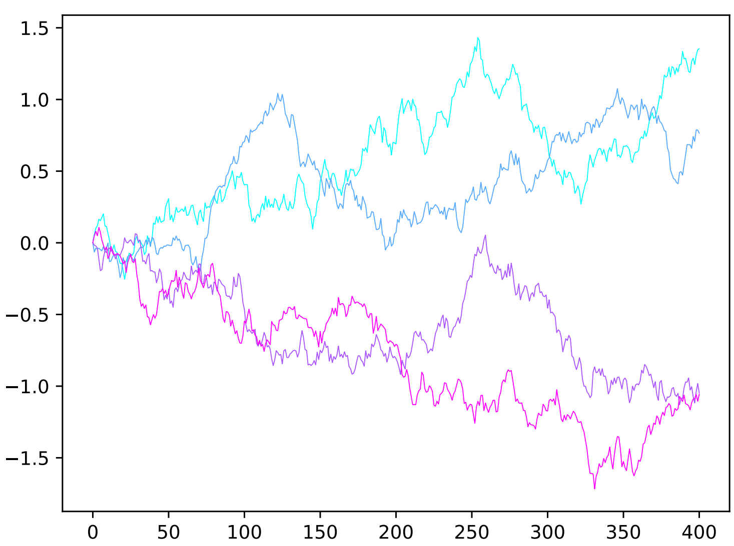
Chapter 2

Brownian Motion

Definition 1. A standard **Brownian motion** $B(t,\omega)$ is a continuous time $\mathbb{R}$-valued stochastic process over some $(\Omega, \mathcal{F}, \mathbb{P})$ such that

1. $B_t - B_s \sim \mathcal{N}(0, t-s)$;
2. Disjoint increments are independent;
3. The sample path $t \mapsto B_t(\omega)$ is continuous with probability 1.

At all times, a Brownian motion receives an infinitesimal Gaussian kick. The intuition here is that "dB" is then a Gaussian random variable. Of course, dB is meaningless right now since B is nowhere differentiable with probability 1, but we will give it meaning later in terms of Itô integrals, and the interpretation will be the same.



Proposition 1. If B_t is a Brownian motion, then so are the following two processes:

- $X_t := \frac{1}{\sqrt{\alpha}} B_{\alpha t}$ for fixed $\alpha > 0$;
- $Y_t := B_{s+t} - B_s$ for fixed $s > 0$;
- $Z_t := t B_{1/t}$.

A useful fact for proving that disjoint intervals are independent: two Gaussians are independent $\iff$ they have 0 covariance.

Proposition 2. If B_t is a Brownian motion, then $\mathrm{Cov}(B_t, B_s) = \min(t, s)$.

Construct a BM using Wiener measure and $B_t(\omega) = \omega_t$.
Let $A := \{\omega \mid B_{t_k}(\omega) \in (a_k, b_k) \text{ for } k = 1, \ldots, N\}$. If

$$\phi(s, y) := \frac{\exp\left(-y^2/(2s)\right)}{\sqrt{2\pi s^2}},$$

then the probability of A is

$$\mathbb{P}(A) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \phi(t_1, x_1) \prod_{i=2}^{N} \phi(t_i - t_{i-1}, x_i - x_{i-1})\, dx_1 \, \cdots \, dx_n.$$

The idea here is that $\phi(t_i - t_{i-1}, x_i - x_{i-1})$ is the conditional density for B_{t_k} given $B_{t_{k-1}} = x_{k-1}$.

Definition 2. A function $f : I \to \mathbb{R}$ is γ-**Hölder continuous** if there is a $C < \infty$ such that

$$|f(t) - f(s)| \leq C\, |t - s|^{\gamma}$$

for all $s, t \in I$. Functions with $\gamma = 1$ are **Lipschitz continuous**.

Theorem 1 (Kolmogorov Continuity Theorem)**.** Let $\{X_t\}$ be a stochastic process on $[0, 1]$. If there are $\alpha, \beta, C > 0$ such that

$$\mathbb{E}\left(|X_t - X_s|^{\alpha}\right) \leq C |t - s|^{1+\beta},$$

then there is a version $\tilde{X}_t$ of X_t with sample paths that are almost surely γ-Hölder continuous for $\gamma \in (0, \beta/\alpha)$.

version means $\mathbb{P}\left(\tilde{X}_t = X_t\right) = 1$ for all t.
This implies that the sample paths of Brownian motion are almost surely Hölder continuous for $\gamma \in (0, 1/2)$.

Proposition 3. If B is a Brownian motion on $[0, T]$, then with probability 1,

- $V^p(B, [0, T]) < \infty$ for $p > 2$;
- $V^p(B, [0, T]) = \infty$ for $p < 2$.

The quadratic variation of B is $[B, B](t) = t$.