MATH 531 HOMEWORK 7

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Exercise 7. *Consider a compact set $B \subset \mathbb{R}^n$ and let $f : B \to \mathbb{R}^m$ be continuous and one-to-one. Then prove that $f^{-1} : f(B) \to B$ is continuous. Show be example that this may fail if B is connected but not compact.*

Let $U \subset B$ be open in B, then we must show that $(f^{-1})^{-1}(U) = f(U)$ is open. Since U is open in B, $B - U$ is closed in B. A closed subset of a compact set is itself compact, so $B - U$ is compact. Now since f is continuous, $f(B - U)$ is also compact and, subsequently, closed. Then its complement $f(B) - f(B - U) = f(U)$ is open. Thus f^{-1} is continuous.

As a counterexample when B is connected but not compact, take $f : [0, 2\pi) \to \mathbb{R}$ defined by $f(t) = (\sin x, \cos x)$. This maps the non-open interval $[0, 2\pi)$ to the entire unit circle, which is compact since it is closed and bounded in $\mathbb{R}^2$; however, since the unit circle is compact, any continuous map with the unit circle as its domain would have a compact image. Since $[0, 2\pi)$ is not closed in $\mathbb{R}$, it is not compact, so the inverse map of f cannot be continuous.

Exercise 13. *Let f be a bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$. Prove that $f(U)$ is open for all open sets $U \subset \mathbb{R}^n$ if and only if for all nonempty open sets $V \subset \mathbb{R}^n$,*

$$\inf_{x \in V} f(x) < f(y) < \sup_{x \in V} f(x)$$

for all $y \in V$.

Forward: Assume that for all open $U \subset \mathbb{R}^n$, the set $f(U)$ is open in $\mathbb{R}$. Let $V \subset \mathbb{R}^n$ be nonempty and open in V, then by assumption, $f(V)$ is open in $\mathbb{R}$. Let $y \in V$, then becuase $f(V)$ is open, there is $\varepsilon > 0$ such that $(f(y) - \varepsilon, f(y) + \varepsilon) \subset f(V)$. Thus there is $y_1 \in V$ such that $f(y_1) \in (f(y) - \varepsilon, f(y))$, so

$$\inf_{x \in V} f(x) \leq f(y_1) < f(y).$$

Similarly, there is a point $y_2 \in V$ such that

$$f(y) < f(y_2) \leq \sup_{x \in V} f(x).$$

Chaining these inequalities together gives the desired result.

Backward: Let V be an open set in $\mathbb{R}^n$, then by assumption $\inf_{x \in V} f(x) < f(y) < \sup_{x \in V} f(x)$ for arbitrary $y \in V$. To avoid the case where either the infimum or supremum is unbounded, we can extend our assumption to say that there exist y_1, $y_2 \in V$ such that

$$\inf_{x} f(x) \leq f(y_1) < f(y) < f(y_2) \leq \sup_{x} f(x).$$

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There are two cases we must consider: when V is connected and when V is disconnected.

When V is connected, $f(V)$ is also connected since f is continuous. Then the open ball $D(y, \min\{d(y, y_1), d(y, y_2)\})$ clearly lies in $f(V)$.

When V is disconnected, it must be made up of connected open components. Each of these components falls into the previous category, so $f(V)$ is the union of open sets and is thus itself open.

Exercise 14. (1) *Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \text{ and } \lim_{y\to 0}\lim_{x\to 0} f(x,y)$$

exist but are not equal.

(2) *Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the two limits in **(a)** exist and are equal but f is not continuous.*

(3) *Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous on every line through the origin but is not continuous.*

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(1) Let $f(x,y) = x^y$, then

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} 1 = 1,$$

but

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} 0 = 0,$$

so we have found a satisfactory function.

(2) Let $f(x,y) = xy/(x^2+y^2)$, with $f(0,0) = 0$. The two limits in question are

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} 0 = 0$$

and

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} 0 = 0$$

so the limits exist and are equal. We claim, however, that f is not continuous. Let $z \neq 0$, then $f(z,z) = 1/2$. Then the limit of $f(z,z)$ as z approaches 0 along the line $y = x$ is $1/2$, not 0. Thus f is not continuous.

(3) A line through the origin can be either vertical (the y-axis) or of the form $y = mx$. In the former case, restrict f to the y-axis gives

$$f(0,y) = \frac{0}{y^2} = 0,$$

which is constant everywhere and, subsequently, continuous. In the latter case, restricting f to a line $y = mx$ gives

$$f(x,mx) = \frac{mx^2}{x^2+m^2x^2} = \frac{m}{m^2+1},$$

which is also constant and, subsequently, continuous. Thus for any line through the origin, f is continuous. We have already shown, though, that f is not continuous.

Exercise 23. *Let X be a compact metric space and $f : X \to X$ an isometry; that is, $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that f is a bijection.*

First we show that f is injective. Let $x, y \in X$ such that $x \neq y$, then $d(x, y) > 0$. By assumption, $d(f(x), f(y)) = d(x, y) > 0$, so $f(x) \neq f(y)$.

Now suppose that f is not surjective, i.e. $X - f(X)$ is nonempty. Let $x_0 \in X - f(X)$. Since X is compact, $f(X)$ is closed and $X - f(X)$ is open. Thus there exists $\varepsilon > 0$ such that $D(x_0, \varepsilon) \in X - f(X)$. This implies that any element of $f(X)$ is at least ε away from x_0.

Now let $x_1 = f(x_0)$, $x_2 = f(x_1)$, and continue inductively to construct a sequence $\{x_n\}_{n=0}^{\infty} \subset X$. For x_k in this sequence and $l > 0$, the distance between points x_k and x_{k+l} is

$$\begin{aligned} d(x_k, x_{k+l}) &= d(f(x_{k-1}), f(x_{k+l-1})) \\ &= d(x_{k-1}, x_{k+l-1}) \\ &\ \ \vdots \\ &= d(x_0, x_l). \end{aligned}$$

Since $x_l = f(x_{l-1}) \in f(X)$, it is at least ε away from x_0. Thus $d(x_k, x_{k+l}) \geq \varepsilon$ for any k with $l > 0$. Since every point in our sequence $\{x_n\}$ is at least ε apart, we cannot find any convergent subsequence. This is a contradiction, as X being compact implies that X is sequentially compact. Thus our original assumption must have been false, so $X - f(X)$ is empty. Equivalently, $f(X)$ is surjective.

Exercise 24. *Let $f : A \subset M \to N$.*

(1) *Prove that f is uniformly continuous on A if and only if for every pair of sequences x_k, y_k of A such that $d(x_k, y_k) \to 0$, we have $\rho(f(x_k), f(y_k)) \to 0$.*
(2) *Let f be uniformly continuous, and let x_k be a Cauchy sequence of A. Show that $f(x_k)$ is a Cauchy sequence.*
(3) *Let f be uniformly continuous and N be complete. Show that f has a unique extension to a continuous function on $\overline{A}$.*

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(1) **Forward:** Assume f is uniformly continuous on A. Let $\{x_k\}, \{y_k\}$ be sequences such that $d(x_k, y_k) \to 0$. Fix $\varepsilon > 0$, then there is a $\delta > 0$ such that $\rho(f(x_k), f(y_k)) < \varepsilon$ when $d(x_k, y_k) < \delta$. Since $d(x_k, y_k) \to 0$, there exists N such that $d(x_k, y_k) < \delta$ when $k > N$. Thus for $k > N$, $\rho(f(x_k), f(y_k)) < \varepsilon$, so $\rho(f(x_k), f(y_k)) \to 0$.

    Backward: We will prove this by contrapositive. Assume f is not uniformly continuous, then there exists $\varepsilon > 0$ such that for all $\delta > 0$, there are x_k and y_k such that $d(x_k, y_k) < \delta$ but $\rho(f(x_k), f(y_k)) \geq \varepsilon$. In particular such x_k and y_k exist for $\delta = 1/n$ for all $n \in \mathbb{N}$. We can take these x_k and y_k to form a sequence that, by construction, satisfies $d(x_k, y_k) \to 0$. However, also by construction, $\rho(f(x_k), f(y_k))$ does not converge to 0. This shows the contrapositive, so the original statement must also be true.

(2) Since f is uniformly continuous, for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all x, y satisfying $d(x, y) < \delta$, we have $\rho(f(x), f(y)) < \varepsilon$. Since $\{x_k\}$ is a Cauchy sequence,

there exists N such that $d(x_m, x_n) < \delta$ when $m, n > N$. Putting these together, when $k, l > N$, we have $\rho(f(x_n), (x_m)) < \varepsilon$. Thus $\{f(x_k)\}$ is a Cauchy sequence.

(3) Let $a \in \overline{A}$, then $a_n \to a$ for some sequence $\{a_n\} \subset A$. Since this sequence converges, it is Cauchy. Then by part (2), $\{f(x_k)\}$ is also Cauchy. Since N is complete, this implies that $\{f(x_k)\}$ converges to some point which we define to be $f(a)$.

We claim that this extension is continuous. Let $a \in \overline{A}$. If $a \in A$, then we know by assumption that f is continuous already, so consider the case when $a \notin A$. In this case, we have by definition $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(a)$, so f is continuous on $\overline{A}$.

We now show that this extension of f is unique. Let $\{a_n\}$ and $\{b_n\}$ both converge to $a \in \overline{A}$. Then it is clear that $d(a_n, b_n) \to 0$. By part (1), this implies $\rho(f(a_n), f(b_n)) \to 0$ as well, so $\{f(a_n)\}$ and $\{f(b_n)\}$ both converge to the same element of $\overline{A}$, namely $f(a)$. This shows that $f(a)$ is independent of the convergent sequence chosen, so our extension of f is unique.

Exercise 25. *Let $f : (0,1) \to \mathbb{R}$ be differentiable and let $f'(x)$ be bounded. Show that $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ exist. Do this both directly and by applying exercise 24c. Give a counterexample if $f'(x)$ is not bounded.*

Direct proof: We start by showing the existence of $\lim_{x\to 0^+} f(x)$. Since f' is bounded, $|f'(x)| \le M$ for all $x \in (0,1)$. Let $x \in (0, 1/n)$, then by the mean value theorem, for some $c \in (x, 1/n)$ we have

$$\begin{aligned} |f(x) - f(1/n)| &= |f'(c)(x - 1/n)| \\ &\le M|x - 1/n| \\ &< M/n. \end{aligned}$$

Since $1/m \in (0, 1/n)$ when $m > n$, we apply this inequality to show

$$|f(1/m) - f(1/n)| < M/n.$$

For any $\varepsilon > 0$, when $m > n > M/\varepsilon$, we have $|f(1/m) - f(1/n)| < \varepsilon$. Thus $\{f(1/n)\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, any Cauchy sequence converges, so $f(1/n) \to L$ for some $L \in \mathbb{R}$ as $n \to \infty$. Since $1/n$ converges to 0 from the right as $n \to \infty$, this is equivalent to saying that $f(x) \to L$ as $x \to 0^+$. Thus $\lim_{x\to 0^+} f(x)$ exists.

Using a similar argument and the sequence $1 - (1/n)$ instead of $1/n$, we can show that $\lim_{x\to 1^-} f(x)$ exists.

Using (24c): Since f is differentiable on $(0,1)$ and f' is bounded, say $|f'| \le M$, by the Mean Value Theorem we have $|f(y) - f(x)| \le M|y - x|$ for all $x, y \in (0,1)$. Fix $\varepsilon > 0$, then set $\delta = \varepsilon/M$. If $|x - y| < \delta$, then $|f(y) - f(x)| \le M|y - x| < \varepsilon$. Since δ was independent of x and y, this shows that f is uniformly continuous on $(0,1)$.

Since f is uniformly continuous and $\mathbb{R}$ is complete, we can apply (24c) to show that f has a unique continuous extension to $\overline{(a,b)} = [a,b]$. Since a function that is continuous on a set is continuous at each of the points in that set, we know that $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ exist.

Counterexample: Now we show a counterexample when f' is not bounded. Let $f : (0,1) \to \mathbb{R}$ be defined by $f(x) = 1/x$, which has unbounded derivative $f'(x) = -1/x^2$. Assume $\lim_{x\to 0^+} f(x) = L$, then for any sequence $\{x_n\}$ satisfying $x_n \to 0$ and $x_n \neq 0$, the sequence $\{f(x_n)\} = \{1/x_n\}$ converges to L. If we let $x_n = 1/n$, then we have $f(x_n) = n$, so the sequence $\{f(x_n)\}$ is unbounded and thus cannot converge. Then by contradiction, no such L exists.

Exercise 26. *Let $f : (a, b] \to \mathbb{R}$ be continuous such that $f'(x)$ exists on (a, b) and the limit $\lim_{x \to a^+} f'(x)$ exists. Prove that f is uniformly continuous.*

Since $\lim_{x \to a^+} f'(x) = L$ for some L, we have that for $\varepsilon = 1$, there exists some $\delta > 0$ such that $|f'(x) - L| < \varepsilon$ when $x \in (a, \delta)$. This allows us to bound f' over this interval:

$$|f'(x)| - |L| \leq |f'(x) - L| < 1$$

so

$$|f'(x)| < |L| + 1$$

when $x \in (a, \delta)$. Then since f' exists at δ, we can bound f' over the interval $(a, \delta]$ by

$$|f'(x)| < \mathcal{L} \doteq \max \{|L| + 1, |f'(\delta)\}.$$

Then by the mean value theorem, for all $x, y \in (a, \delta]$, we have

$$|f(y) - f(x)| \leq \mathcal{L}|y - x|.$$

Fix $\varepsilon' > 0$, then set $\delta' = \varepsilon'/\mathcal{L}$. Then for $|x - y| < \delta'$, we have $|f(y) - f(x)| \leq \mathcal{L}|y - x| < \varepsilon'$. Thus f is uniformly continuous on $(a, \delta]$.

The function f is also uniformly continuous on $[\delta, b]$, since this is a compact interval and f is continuous to begin with. Then since f is uniformly continuous over both segments of $(a, b]$, it is uniformly continuous over the whole interval, which we now prove.

Fix $\varepsilon > 0$, then we can find δ_l such that for $x, y \in (a, \delta]$, $|x - y| < \delta_l$ implies $|f(x) - f(y)| < \varepsilon/2$. Let δ_r be the corresponding value for the interval $[\delta, b]$ instead. Now set $\delta = \min\{\delta_l, \delta_r\}$, and let the pair x, y be in the entire interval $(a, b]$. If both points are in $(a, \delta]$ or if both are in $[\delta, b]$, then clearly $|f(x) - f(y)| < \varepsilon/2 < \varepsilon$.

If one point is in $(a, \delta]$ and the other is in $[\delta, b]$, then assume without loss of generality that x is in the former and y is in the latter. In this case δ lies between x and y, so $|x - \delta| \leq |x - y| < \delta$ and $|y - \delta| \leq |y - x| < \delta$. Then we have

$$|f(x) - f(y)| \leq |f(x) - f(\delta)| + |f(\delta) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus f is uniformly continuous over $(a, b]$.

Exercise 29. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq |x - y|^2$. Prove that f is a constant.*

We will show that the derivative of this function is 0. Using the given bound and the fact that $x \mapsto |x|$ is a continuous map, we have

$$\begin{aligned} |f'(x)| &= \left| \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right| \\ &= \lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} \\ &\leq \lim_{h \to 0} \frac{|h|^2}{|h|} \\ &= \lim_{h \to 0} |h| \\ &= 0. \end{aligned}$$

This implies $f'(x) = 0$, which is only true if f is a constant function.

Exercise 34. *Assuming that the temperature on the surface of the earth is a continuous function, prove that on any great circle of the earth there are two antipodal points with the same temperature.*

Let C denote any great circle of the earth. If $x \in C$, then denote its antipodal point by x'. Finally denote the temperature of the earth by the continuous function $T : C \to \mathbb{R}$, and define the continuous function $f(x) = T(x) - T(x')$.

Let $x \in C$ be arbitrary, then we have two cases: $f(x) = 0$ or $f(x) \neq 0$. If the former case holds, the result is trivial, so assume $f(x) \neq 0$. Then we have $f(x') = T(x') - T(x) = -f(x)$, so $f(x)$ and $f(x')$ have opposite signs.

Since C is connected and f is continuous, we then can apply the intermediate value theorem to show that there is some point $z \in C$ such that $f(z) = T(z) - T(z') = 0$, which proves the result.

Exercise 38. *A real-valued function defined on (a, b) is called **convex** when the following inequality holds for x, y in (a, b) and t in $[0, 1]$:*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

If f has a continuous second derivative and $f'' > 0$, show that f is convex.

Let $z = tf(x) + (1-t)f(y)$, and without loss of generality, assume $y \geq x$. Then we wish to show $tf(x) + (1-t)f(y) \geq f(z)$. We will do so by showing that $tf(x) + (1-t)f(y) - f(z)$ is non-negative. We have

$$\begin{aligned} tf(x) + (1-t)f(y) - f(z) &= tf(x) + (1-t)f(y) - tf(z) + (1-t)f(z) \\ &= t[f(x) - f(z)] + (1-t)[f(y) - f(z)]. \end{aligned}$$

Let $\alpha \in (x, z)$, $\beta \in (z, y)$, then by the mean value theorem this becomes

$$= t[f'(\alpha)(x - z)] + (1-t)[f'(\beta)(y - z)].$$

Finally, expand z to get

$$\begin{aligned} &= t(1-t)f'(\alpha)(x - y) + t(1-t)f'(\beta)(y - x) \\ &= t(1-t)(y - x)[f'(\beta) - f'(\alpha)]. \end{aligned}$$

Since $t \in [0,1]$, we know $t(1-t) \geq 0$. By assumption, $y - x \geq 0$ as well. Since $f''(x) > 0$ for all $x \in (a,b)$, this means f' is always increasing. Then since $\beta > \alpha$, $f'(\beta) - f'(\alpha)$ must also be non-negative. Thus we have

$$tf(x) + (1-t)f(y) - f(z) = t(1-t)(y-x)[f'(\beta) - f'(\alpha)] \geq 0,$$

so

$$tf(x) + (1-t)f(y) \geq f(z),$$

as desired.