MATH 531 HOMEWORK 9

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Page 316, Ex. 2. *Determine which of the following sequences converge (pointwise or uniformly) as $k \to \infty$. Check the continuity of the limit in each case.*

(1) $(\sin x)/k$ *on* $\mathbb{R}$
(2) $1/(kx+1)$ *on* $(0,1)$
(3) $x/(kx+1)$ *on* $(0,1)$
(4) $x/(1+kx^2)$ *on* $\mathbb{R}$
(5) $(1, (\cos x)/k^2)$, *a sequence of functions from* $\mathbb{R}$ *to* $\mathbb{R}^2$

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(1) This converges uniformly to the zero function. We know $|\sin x| \leq 1$, so for all x,
$$\left|\frac{\sin x}{k}\right| = \frac{|\sin x|}{k} \leq \frac{1}{k}.$$
Fix $\varepsilon > 0$, then set $K = 1/\varepsilon$. Then by the inequality we just derived, $|f_k(x)| < \varepsilon$ for all x when $k > K$. Thus f_k converges uniformly to the zero function, which is continuous.

(2) Fix x, then $1/(kx+1)$ clearly converges to 0, which is a continuous limit. The convergence is only pointwise. To see this, fix $0 < \varepsilon < 1$. Then $1/(kx+1) \geq \varepsilon$ when
$$x \leq \frac{\frac{1}{\varepsilon} - 1}{k}.$$
Note that this value is in $(0,1)$ since $0 < \varepsilon < 1$, so we can find an x satisfying this inequality.

(3) This converges uniformly to the zero function. For $0 < x < 1$, we have the inequality
$$\left|\frac{x}{kx+1}\right| = \frac{x}{kx+1} = \frac{1}{k + \frac{1}{x}} < \frac{1}{k+1}.$$
Fix $\varepsilon > 0$, then set $K = (1/\varepsilon) - 1$. Then by the above inequality, $|f_k(x)| < \varepsilon$ for all x when $k > K$. Thus f_k converges uniformly to the zero function, which is continuous.

(4) Fix x, then since
$$\frac{x}{1+kx^2} = \frac{1}{\frac{1}{x} + kx},$$
we clearly have pointwise convergence to 0, which is a continuous limit. Now since
$$\frac{d}{dx}\left(\frac{x}{1+kx^2}\right) = \frac{1-kx^2}{(1+kx^2)^2},$$

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the maximum and minimum values of this function satisfy $x = \pm 1/\sqrt{k}$. Then for all x, we have

$$\left|\frac{x}{1+kx^2}\right| \leq \frac{1}{2\sqrt{k}}.$$

Fix $\varepsilon > 0$. If we choose any $k > (1/(2\varepsilon))^2$, then

$$\left|\frac{x}{1+kx^2}\right| < \varepsilon,$$

for all x, so we also have uniform convergence to the zero function.

(5) We claim that this converges uniformly to the constant function $(1,0)$, which is a continuous limit. For all $x \in \mathbb{R}$, we have

$$\|(1, \frac{\cos x}{k^2}) - (1,0)\| = \left|\frac{\cos x}{k^2}\right| \leq \frac{1}{k^2}.$$

Fix $\varepsilon > 0$. If we choose $k > \sqrt{1/\varepsilon}$, then the above norm is less than ε for all x. Thus we have uniform convergence.

Page 317, ex. 3. *Determine which of the following real series $\sum_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.*

(1) $g_k(x) = \begin{cases} 0, & x \leq k \\ (-1)^k, & x > k. \end{cases}$

(2) $g_k(x) = \begin{cases} 1/k^2, & |x| \leq k \\ 1/x^2, & |x| > k. \end{cases}$

(3) $g_k(x) = \left(\frac{(-1)^k}{\sqrt{k}}\right) \cos(kx)$ *on* $\mathbb{R}$.

(4) $g_k(x) = x^k$ *on* $(0,1)$.

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(1) Fix x, then the series is

$$\sum_{k=1}^{\infty} g_k(x) = \sum_{k=1}^{K} (-1)^k + \sum_{k=K+1}^{\infty} 0,$$

where K is the largest natural number less than x (0 if no such natural number exists). If K is even, then this sum is 0, but if K is odd, then this sum is -1. Thus for different values of x, g_k converges to different functions. Thus $\sum_{k=1}^{\infty} g_k$ does not converge anywhere.

(2) Clearly $g_k(x) \leq 1/k^2$, and $\sum_{k=1}^{\infty} 1/k^2$ converges because it is a p-series with $p = 2 > 1$. Thus by the Weierstrass-M test, $\sum_{k=1}^{\infty} g_k$ converges uniformly.

Each g_k is clearly continuous, so each partial sum $\sum_{k=1}^{n} g_k$ is also continuous. Then since the convergence of the partial sums is uniform, the limit $\sum_{k=1}^{\infty} g_k$ is itself continuous.

(3) The alternating sum $\sum_{k=1}^{\infty} (-1)^k$ is clearly bounded, and the sequence $\cos(kx)/\sqrt{k}$ converges uniformly to the zero function, so by the Dirichlet test we have that $\sum_k g_k(x)$ converges uniformly. Since each g_k is continuous and the convergence is uniform, the limit function must be continuous.

(4) Since $x \in (0,1)$, then $|x| < 1$, so

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

This shows convergence to a continuous limit, but we can show that the convergence is *not* uniform. Fix $n \in \mathbb{N}$, then

$$\left| \frac{1}{1-x} - \sum_{k=0}^{n} x^k \right| = \left| \frac{1}{1-x} - \frac{1-x^{n+1}}{1-x} \right| = \left| \frac{x^{n+1}}{1-x} \right| = \frac{x^{n+1}}{1-x}.$$

If $x > 1/2$, then the error between the partial sum and the limit $x^{n+1}/(1-x)$ is greater than $2x^{n+1}$. So for fixed n, we can make the error larger than any $\varepsilon < 2$ by selecting $x \in (1/2, 1)$ such that $s > (\varepsilon/2)^{1/(n+1)}$. Thus the convergence is not uniform.

Page 317, Ex. 4. *Let $f_n : [1,2] \to \mathbb{R}$ be defined by $f_n(x) = x/(1+x)^n$.*

(1) *Prove that $\sum_{n=1}^{\infty} f_n(x)$ is convergent for $x \in [1,2]$.*
(2) *Is it uniformly convergent?*
(3) *Is $\int_1^2 \left(\sum_1^{\infty} f_n(x)\right)\, dx = \sum_1^{\infty} \int_1^2 f_n(x)\, dx$?*

(1,2) We will use the Weierstrass-M test to show (2), from which (1) clearly follows. Let $M_n = 1/(2^{n-1})$, then over $[1,2]$ we have the inequality

$$|f_n(x)| = f_n(x) = \frac{x}{(1+x)^n} \leq \frac{2}{2^n} = \frac{1}{2^{n-1}} = M_n.$$

Since the series $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} 1/(2^{n-1}) = \sum_{n=0}^{\infty} (1/2)^n$ is a geometric series with $|r| = 1/2 < 1$, it converges. Thus by the Weierstrass-M test, the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

(3) Since each f_n is bounded and continuous on $[1,2]$, each is Riemann integrable. Then the desired equality is a direct consequence of the uniform convergence of f_n.

Page 318, Ex. 12. *A function $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}^n$, is called* ***lower semicontinuous*** *if whenever $x_0 \in A$ and $\lambda < f(x_0)$, there is a neighborhood U of x_0 such that $\lambda < f(x)$ for all $x \in U \cap A$.* ***Upper semicontinuity*** *is defined similarly.*

(1) *Show that f is continuous if and only if it is both upper and lower semicontinuous.*
(2) *If the functions f_k are lower semicontinuous, $f_k \to f$ pointwise, and $f_{k+1}(x) \geq f_k(x)$, then prove that f is lower semicontinuous.*
(3) *In **b**, show that f need not be continuous even if the f_k are continuous.*
(4) *Let $f : [0,1] \to \mathbb{R}$, and let $g(x) = \sup_{\delta > 0} \inf_{|y-x|<\delta} f(y)$. Prove that g is lower semicontinuous.*

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(1) **Forward:** Assume f is continuous, then for all U open in $f(A)$, the preimage $f^{-1}(U)$ is open in A. Fix $x_0 \in A$ and take $\lambda < f(x_0)$, then $f(x_0) = \lambda + \varepsilon$ for some $\varepsilon > 0$. Now consider the epsilon ball around $f(x_0)$, which we denote by

$$V \doteq D(f(x_0), \varepsilon).$$

Since V is open and f is continuous, the preimage $f^{-1}(V)$ is open in A. Since all points x in $f^{-1}(V)$ satisfy $\lambda < f(x)$, f is lower continuous. Similarly, f is upper continuous as well.

Backward: Assume f is both lower and upper semicontinuous. Consider any open set in $f(A) \subset \mathbb{R}$. Since the open sets in $\mathbb{R}$ are just open intervals, we consider the arbitrary open set (a, b).

Let $f(x)$ be any element of (a, b). Since f is lower semicontinuous, there is an open neighborhood U_a of x such that a is less than all elements of $f(U_a)$. Similarly, since f is upper semicontinuous, there is an open neighborhood U_b of x such that b is greater than all elements of $f(U_b)$.

Now consider $U \doteq U_a \cap U_b$. This new set is also open, since it is the finite intersection of open sets. Moreover, if y is in U, then $a < f(y) < b$, so $y \in f^{-1}((a, b))$. Thus U is an open neighborhood of x that lies in $f^{-1}((a, b))$. Since the original $f(x)$ that we considered was arbitrary, this holds for all $x \in f^{-1}((a, b))$. Thus $f^{-1}((a, b))$ is open and, subsequently, f is continuous.

(2) Let $x_0 \in A$ and let $\lambda < f(x_0)$. This means that $\lambda = f(x_0) - \varepsilon$ for some $\varepsilon > 0$. Now since f_k converges pointwise to f, we can find a $K \in \mathbb{N}$ such that $|f_k(x_0) - f(x_0)| < \varepsilon$ when $k > K$. Take any such $k > K$, then $\lambda < f_k(x_0)$ as well. Then since each f_k is lower semicontinuous, this means we have a neighborhood U of x_0 such that every point $x \in U$ satisfies $\lambda < f_k(x)$.

Furthermore, since $f_k(x) < f_{k+1}(x)$ for all x and f_k converges pointwise to f, we know $f_k(x) < f(x)$ for all k and for all x. Thus for every point x in our previously mentioned neighborhood U of x_0, we have

$$\lambda < f_k(x) < f(x).$$

We have found a satisfactory neighborhood for the limit function f, so f is lower semicontinuous.

(3) Consider $f_k : [0, \infty) \to \mathbb{R}$, a modification of the well-known sigmoid function given by

$$f_k(x) \doteq \frac{1}{1 + e^{-kx}}.$$

The function f_k is continuous for all k, so each f_k is also necessarily lower semicontinuous. Additionally, since x is nonnegative, we have

$$f_{k+1}(x) = \frac{1}{1 + e^{-(k+1)x}} \geq \frac{1}{1 + e^{-kx}} = f_k(x).$$

All that's left is to show that f_k converges to a discontinuous function. When $x = 0$, $f_k(x) = 1/2$ for all k, so the pointwise limit of f_k at the point 0 is the constant function $f(x) = 1/2$.

When x is nonzero, we claim that f_k converges to the constant function 1. Fix $x \neq 0$, then

$$|1 - f_k(x)| = \left|1 - \frac{1}{1 + e^{-kx}}\right| = \left|\frac{e^{-kx}}{1 + e^{-kx}}\right| = \left|\frac{1}{e^{kx} + 1}\right| = \frac{1}{e^{kx} + 1},$$

so for fixed x, we can choose k large enough to make $f_k(x)$ arbitrarily close to 1. This shows that f_k converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0. \end{cases}$$

(4) Fix $x_0 \in [0,1]$ and take any λ such that $\lambda < g(x_0)$, then for some $\varepsilon > 0$ we have

$$\begin{aligned} \lambda + \varepsilon &= g(x_0) \\ &= \sup_{\delta > 0} \inf_{y \in D(x_0, \delta)} f(y). \end{aligned}$$

Since the supremum is necessarily a limit point, we can construct a sequence $\{\delta_n\}$ such that

$$g_n \doteq \inf_{y \in D(x_0, \delta_n)} f(y)$$

converges to $g(x_0)$. Since this sequence converges to $g(x_0)$, we can find $N \in \mathbb{N}$ such that $|g_n(x_0) - g(x_0)| < \varepsilon$ when $n > N$. Take any $n < N$, then since $g(x_0)$ is ε away from λ, we must have $\lambda < g_n(x_0)$ as well. Expanding $g_n(x_0)$ shows that

$$\lambda < g_n(x_0) = \inf_{y \in D(x_0, \delta_n)} f(y),$$

so $\lambda < g_n(x)$ for all $x \in D(x_0, \delta_n)$.

Now since $g(x_0)$ is the supremum of the sequence $\{g_n(x_0)\}$, we know $g_n(x) \leq g(x)$ for all n and for all x in the epsilon balls $\{D(x_0, \delta_n)\}_n$. Combining this with the previous inequality gives

$$\lambda < g_n(x) \leq g(x)$$

for all $x \in D(x_0, \delta_n)$, so we have found a satisfactory open set and g is consequently lower semicontinuous.

Page 318, Ex. 15. *Let $g_k \in \mathbb{R}^n$ and let f_k be a subsequence of g_k. Prove that if $\sum g_k$ converges absolutely, then $\sum f_k$ converges absolutely as well. Find a counterexample if $\sum g_k$ is just convergent.*

Since $\sum g_k$ converges absolutely, we know $\sum |g_k| \to L$ for some L. Since $|f_k|$ is a subsequence of $|g_k|$ and each term of $|g_k|$ is non-negative, we know

$$0 \leq \sum_{k=1}^{n} |f_k| \leq \sum_{k=1}^{n} |g_k|$$

for all n. Then by the comparison test, $|f_k|$ also converges, so f_k is absolutely convergent.

Now consider the subsequence $f_k = 1/(2k)$ of the sequence $\{g_k\} = \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \cdots\}$. The series $\sum_k g_k$ converges by the alternating series test, but it does **not** converge absolutely, as the series $\sum_k |g_k| = \sum_k 1/k$ is the harmonic series, which is known not to converge.

The series $\sum_k f_k$ is

$$\sum_{k=1}^{\infty} f_k = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k},$$

which is the harmonic series multiplied by a constant. Since the harmonic series diverges, $\sum_k f_k$ also diverges, so $\sum_k g_k$ being convergent but not absolutely convergent is not enough to guarantee that $\sum_k f_k$ is absolutely convergent. In fact, we have shown that $\sum_k f_k$ need not converge at all.

Page 318, Ex. 17. *Let $\sum_{n=0}^{\infty} a_n$ be a convergent, not absolutely convergent, real series. Given any number x, show that there is a rearrangement $\sum b_n$ of the series that converges to x.*

First we deconstruct $\sum_n a_n$ into its positive and negative terms, then we use these two new series to construct a rearrangement of $\sum_n a_n$ that converges to a given, arbitrary real number x. Since zero terms do not affect the convergence of a series, we assume that $a_n \neq 0$ for all n.

Deconstruction of $\sum_n a_n$: First we show that $\{a_n\}$ has infinite positive and infinite negative terms. Assume that $\{a_n\}$ has only finite positive terms, then the sum of all its positive terms is finite. Since the series $\sum_n a_n$ converges, this means that the sum of all its negative terms must converge. However, if $\{q_n\}$ is a sequence of negative terms and $\sum_n q_n$ converges to some value q, then

$$\sum_{n=1}^{\infty} |q_n| = -\sum_{n=1}^{\infty} q_n = -q,$$

so $\sum_n a_n$ is absolutely convergent. This is a contradiction, so $\{a_n\}$ must have infinitely many positive terms. Similarly, it must also have infinitely many negative terms.

Denote the positive elements of $\{a_n\}$ by $\{p_n\}$, and the negative elements by $\{q_n\}$. We now show that the two series $\sum_n p_n$ and $\sum_n q_n$ both diverge, which we can do by case analysis.

If both series converge, i.e. $\sum_n p_n \to p$ and $\sum_n q_n \to q$, then

$$\sum_n |a_n| = \sum_n p_n + \sum_n |q_n| = \sum_n p_n - \sum_n q_n$$

converges to $p-q$. This shows that $\sum_n a_n$ is absolutely convergent, which is a contradiction. The next case to consider is either series diverging. In this case, $\sum_n a_n$ would also diverge, so this cannot be possible either. The only remaining possibility is that both series diverge.

Rearrangement into $\sum_n b_n$: Let x be any real number, then take just enough terms (in order) from $\{p_n\}$ such that their sum is greater than x, i.e. find k_1 such that

$$\sum_{i=1}^{k_1-1} p_i \leq x < \sum_{i=1}^{k_1} p_i.$$

Denote the sum up to p_{k_1} by b_1, and note that b_1 differs from x by at most p_{k_1}. Now add terms from $\{q_n\}$ (in order) to this summation until it is less than x, i.e. find k_2 such that

$$\sum_{i=1}^{k_1} p_i + \sum_{i=1}^{k_2} q_i < x < \sum_{i=1}^{k_1} p_i + \sum_{i=1}^{k_2-1} q_i.$$

Denote this second sum up through q_{k_2} by b_2, and note that b_2 differs from x by at most $|q_{k_2}|$. Continuing this process indefinitely, we construct a sequence of sums $\{b_n\}$ such that b_n differs from x by at most either p_n or $|q_n|$.

Now since $\sum_{n=0}^{\infty} a_n$ converges in the first place, we know $a_n \to 0$, so we have $p_n \to 0$ and $q_n \to 0$. Fix $\varepsilon > 0$, then we can find N such that $p_{k_N} < \varepsilon/2$ and $|q_{k_N}| < \varepsilon/2$. Thus for $n > N$, when n is odd we have

$$|b_n - x| \leq |b_n - p_n| + |p_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and similarly when n is even we have

$$|b_n - x| \leq |b_n - q_n| + |q_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus b_n converges to x. Since b_n was constructed using terms from $\{p_n\}$ and $\{q_n\}$ in order, we know that we use all terms of $\{a_n\}$ in this process, i.e. this is a proper rearrangement of $\{a_n\}$.

Page 318, Ex. 18. *Give an example of a sequence of discontinuous functions f_k converging uniformly to a limit function f that is continuous.*

Let $f(x) = 0$ for all $x \in \mathbb{R}$, and for $k \in \mathbb{N}$, define $f_k(x)$ by

$$f_k(x) = \begin{cases} \frac{1}{k} & \text{if } x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $1/k > 0$ for all $k \in \mathbb{N}$, every $f_k(x)$ has a discontinuity at the point $x = 1$, but we claim that f_k converges uniformly to the continuous function f.

We must show that for all $\varepsilon > 0$, there is a $K \in \mathbb{N}$ such that $|f(x) - f_k(x)| < \varepsilon$ for all $x \in \mathbb{R}$ when $k > K$. Note that when $x \leq 1$, $f_k(x) = f(x)$ for all k, so the inequality is trivial in this case. Thus we consider only the case when $x > 1$, and the K that we find in this case will clearly also apply when $x \leq 1$.

Fix $\varepsilon > 0$, then for all $x \in \mathbb{R}$, $|f(x) - f_k(x)| = 1/k$. Let K be any natural number larger than $1/\varepsilon$, then $|f(x) - f_k(x)| < \varepsilon$ when $k > K$. Since this holds for all x, f_k converges to f uniformly. Thus we have a sequence of discontinuous functions that converges uniformly to a continuous function.

Page 319, Ex. 20. *Construct the function $g(x)$ by letting $g(x) = |x|$ if $x \in [-1/2, 1/2]$ and extending g so that it becomes periodic. Define*

$$f(x) = \sum_{n=1}^{\infty} \frac{g(4^{n-1}x)}{4^{n-1}}.$$

(1) *Sketch g and the first few terms in the sum.*
(2) *use the Weierstrass M test to show that f is continuous.*
(3) *Prove that f is differentiable at **no** point.*

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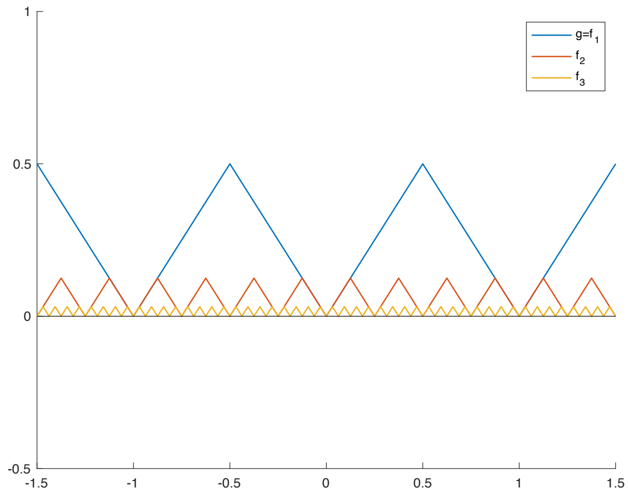
(1) Define a sequence of functions $\{f_k\}$ by

$$f_k = \frac{g(4^{n-1}x)}{4^{n-1}},$$

then $f(x) = \sum_{n=1}^{\infty} f_n(x)$. The first three functions in this sequence are sketched below. Note that $g(x)$ is equal to $f_1(x)$. The pattern in this image continues for further f_n.



(2) In order to apply the Weierstrass-M test, we need to find M_n such that
  (a) $|f_n(x)| \leq M_n$ for all $x \in \mathbb{R}$ and
  (b) $\sum_{n=1}^{\infty} M_n$ converges.

Since $0 \leq g(x) \leq 1/2$ for all x, we can derive the bound

$$|f_n(x)| = \left|\frac{g(4^{n-1}x)}{4^{n-1}}\right| \leq \frac{1}{2 \cdot 4^{n-1}} \leq \frac{1}{4^{n-1}}.$$

Then we if let $M_n = \frac{1}{4^{n-1}}$, condition (a) is satisfied. Now the infinite series $\sum_{n=1}^{\infty} M_n$ evalutes to

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{4^{n-1}} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n,$$

which converges since it is a geometric series with $|r| = 1/4 < 1$. Thus by the Weierstrass-M test, $\sum_{n=1}^{\infty} f_n$ converges uniformly to f. Since each f_n is continuous, this means f is also continuous.

(3) Fix x, then we can find an integer k such that

$$\frac{k}{4^{m-1}} \leq x \leq \frac{k+1}{4^{m-1}}.$$

Denote the left fraction by α_{m-1} and the right fraction by β_{m-1}, then we have $\beta_{m-1} - \alpha_{m-1} = 1/4^m$. Since this approaches 0 as m increases and since x is sandwiched between the sequence of α_m and β_m, the derivative of f at x can be written

$$\begin{aligned} f'(x) &= \lim_{m\to\infty} \frac{f(\beta_{m-1}) - f(\alpha_{m-1})}{\beta_{m-1} - \alpha_{m-1}} \\ &= \lim_{m\to\infty} \frac{\sum_{n=1}^{\infty} (1/4)^{n-1} \left[g(4^{n-1}\frac{k+1}{4^{m-1}}) - g(4^{n-1}\frac{k}{4^{m-1}})\right]}{\beta_{m-1} - \alpha_{m-1}} \\ &= \lim_{m\to\infty} \frac{\sum_{n=1}^{\infty} (1/4)^{n-1} \left[g(4^{n-m}(k+1)) - g(4^{n-m}k)\right]}{\beta_{m-1} - \alpha_{m-1}}. \end{aligned}$$

In order to simplify this rather unwieldy expression, we consider how the term $G_{n,m} \doteq g(4^{n-m}(k+1)) - g(4^{n-m}k)$ changes for different values of n and m.

When $n \geq m$, $4^{n-m}(k+1)$ and $4^{n-m}k$ are both integers, i.e. they are roots of g, so $G_{n,m} = 0$. When $n < m$, $4^{n-m}(k+1)$ and $4^{n-m}k$ lie within consecutive integers, so their difference is just $G_{n,m} = 4^{n-m}$.

Thus we can rewrite our expression for the derivative as

$$\begin{aligned} f'(x) &= \lim_{m\to\infty} \frac{\sum_{n=1}^{m}(1/4)^{n-1}4^{n-m} + \sum_{n=m+1}^{\infty} 0}{\beta_{m-1} - \alpha_{m-1}} \\ &= \lim_{m\to\infty} \frac{\sum_{n=1}^{m} 4^{-m+1}}{1/4^m} \\ &= \lim_{m\to\infty} \frac{4^{-m+1}m}{1/4^m} \\ &= \lim_{m\to\infty} 4m. \end{aligned}$$

This clearly diverges, so f cannot be differentiable at x. Since x was arbitrary, this shows that f is nowhere differentiable.