Exercise 1 (1.58)**.** Area of arbelos.

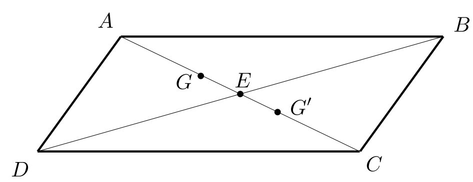
By symmetry, doubling CD gives a chord of the circle. Then by power of the point,

$$|AC||CB| = |CD|^2.$$

Then the area of the arbelos is

$$\begin{aligned}
\text{Area(arbelos)} &= \frac{\pi}{2}\left[\left(\frac{|AC| + |CB|}{2}\right)^2 - \left(\frac{|AC|}{2}\right)^2 - \left(\frac{|CB|}{2}\right)^2\right] \\
&= \frac{\pi}{4}|AC||CB| \\
&= \frac{\pi}{4}|CD|^2 \\
&= \pi\left(\frac{|CD|}{2}\right)^2 \\
&= \text{Area(circle with diameter } CD\text{)}.
\end{aligned}$$

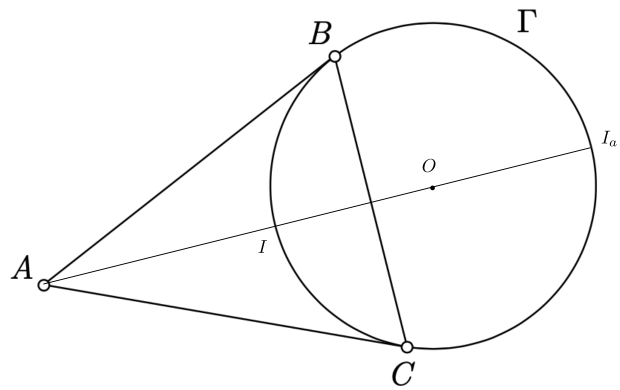
Exercise 2 (1.71)**.** Diagonals of parallelogram bisect each other.



Let E be the midpoint of BD, and let G be the centroid of ΔABD. Then by Theorem 1.9.1, $|AE| = |AG| + |GE| = 3|GE|$. By SSS, $\Delta ABD \cong \Delta CDB$. Thus if G' is the centroid of ΔCDB, we have $|G'E| = |GE|$. Then $|CE| = 3|G'E| = 3|GE| = |AE|$.

By a similar argument, $|BE| = |ED|$, so the diagonals bisect each other.

Exercise 3 (1.79)**.** I and I_a both lie on Γ.

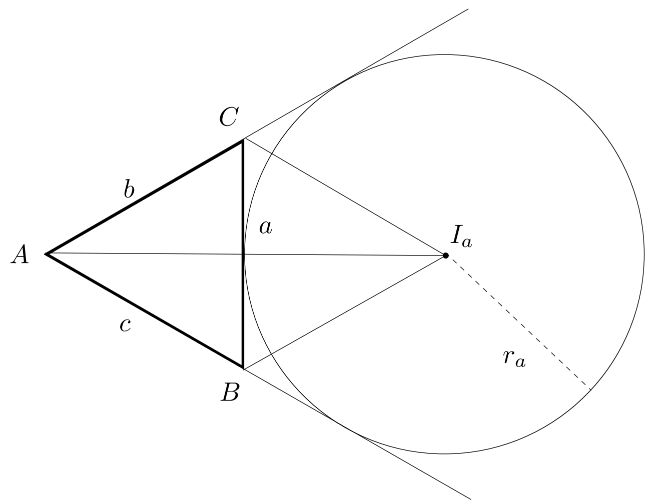


Let O be the center of Γ. By symmetry, AO bisects $\angle BAC$.

Incenter: Since $\angle CBA$ subtends the arc BC, BC's measure is $2\angle CBA$. Then since I is the midpoint of BC, $\angle IBA = \frac{1}{2}\angle CBA$. Thus I is the intersection point of lines bisecting $\angle BAC$ and $\angle CBA$, so I is the incenter.

Excenter: The exterior angle at C subtends the large arc BC (passing through I_a). By symmetry again, I_a is the midpoint of that arc. Then BI_a bisects the exterior angle at B. Since I is a point of A's angle bisector and B, C's exterior angle bisectors, I_a is an excenter.

Exercise 4 (1.80)**.** $|\Delta ABC| = (s-a)r_a$.



Consider the quadrilateral ACI_aB. Its area is

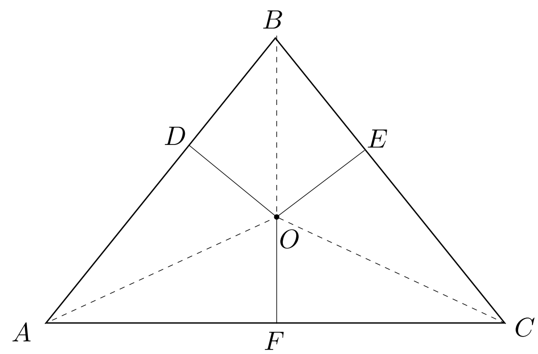
$$|ACI_aB| = |\Delta ABC| + |\Delta BCI_a| = |\Delta ABI_a| + |\Delta ACI_a|,$$

so

$$\begin{aligned}|\Delta ABC| &= |\Delta ABI_a| + |\Delta ACI_a| - |\Delta BCI_a| \\ &= \frac{1}{2}cr_a + \frac{1}{2}br_a - \frac{1}{2}ar_a \\ &= \frac{1}{2}(-a+b+c)r_a \\ &= (s-a)r_a.\end{aligned}$$

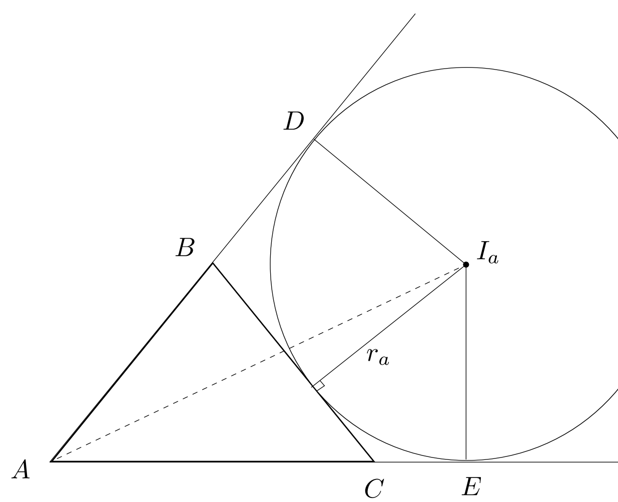
Exercise 5 (1.81)**.** Distance from A to a bunch of tangent things.

In all three problems, the diagrams are set up so that we have to find $|AD|$.



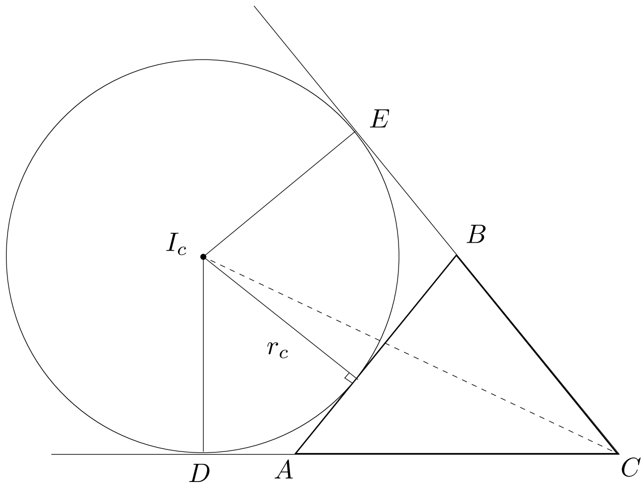
1. By Theorem 1.10.3, $|\Delta ABC| = sr$. But after drawing altitudes from O, we get three pairs of congruent triangles. Thus we can also calculate $|\Delta ABC| = |AD|r + |BE|r + |CE|r = (|AD| + a)r$, so

$$sr = (|AD| + a)r \implies |AD| = s - a.$$



2. By Exercise 1.80, $|\Delta ABC| = (s - a)r_a$. But the area of the quadrilateral $BDEC$ is $r_a a$, so $|\Delta ABC| = \frac{1}{2}|AD|r_a + \frac{1}{2}|AE|r_a - r_a a$. Since $|AD| = |AE|$ by symmetry, these two expressions for $|\Delta ABC|$ give

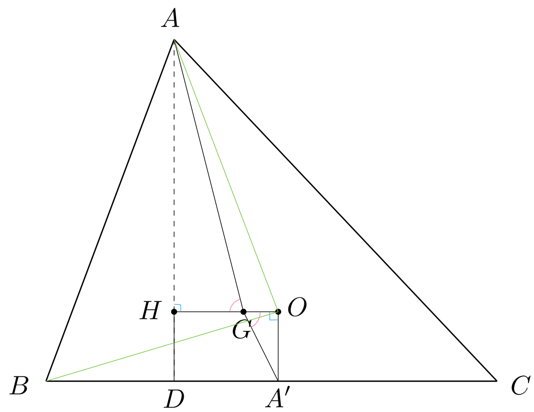
$$(s - a)r_a = (|AD| - a)r_a \implies |AD| = s.$$



3. A clear extension of Exercise 1.80 is $|\Delta ABC| = (s-c)r_c$. But we also have $|\Delta ABC| = (b+|AD|)r_c - cr_c$ via an argument similar to that in part (b). This gives

$$(b+|AD|-c)r_c = (s-c)r_c \implies |AD| = s-b.$$

Exercise 6 (1.100)**.** Putnam problem.



By the Euler Line theorem, the centroid G lies on HO and gives the ratio

$$\frac{|OG|}{|GH|} = \frac{1}{2}.$$

Note that since the two pink angles and the two blue angles are equal, $\Delta AGH \sim \Delta A'GO$. Then we can use the above ratio, along with the given fact $|A'D| = 5$, to get

$$\frac{|A'O|}{|AH|} = \frac{|OG|}{|GH|} \implies |AH| = 10.$$

Then by the Pythagorean Theorem, $|AO|^2 = |AH|^2 + |HO|^2 = 221$. Since O is the circumcenter, $|AO| = |BO|$. Then by the Pythaogrean Theorem again,

$$\begin{aligned} |BA'|^2 + |A'O|^2 &= |BO|^2 \\ |BA'|^2 + |A'O|^2 &= |AO|^2 \\ |BA'|^2 + 5^2 &= 221 \\ |BA'| &= 14. \end{aligned}$$

Since A' is the midpoint of BC, this implies $|BC| = 28$.

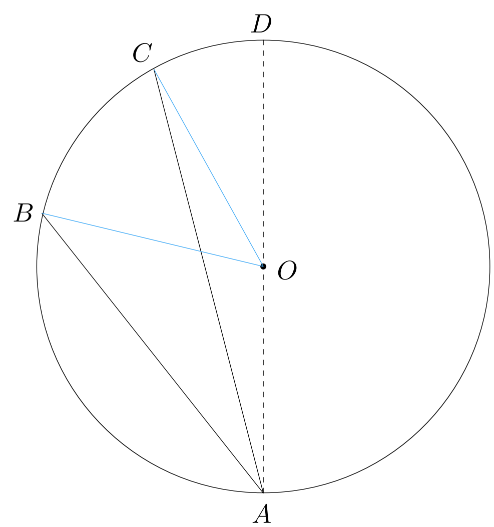
Exercise 7 (1.107)**.** Star Trek Lemma with oriented angles.

Star Trek: Suppose $\angle CAB$ is an oriented angle inscribed in a circle with center O, then

$$\angle COB = 2\angle CAB,$$

where $\angle COB$ is also oriented.

Works in all cases: When $\angle CAB$ is acute or obtuse and contains O, the proof in the textbook is clearly valid. When $\angle CAB$ is acute and does not contain O, the only non-straightforward part of the proof is the statement $\angle COB = \angle COD + \angle COB$. But based on the image below, we see that with oriented angles, this is true since $\angle COD = -\angle DOC$.



In the tangential case, we simply have to define the angle $\angle TAB$, where T is a point outside the circle tangent to A, to be the angle inscribed by A and B, then the proof is straightforward.