Problems completed: All.

Exercise 1. S generates G if every element of G can be written as a finite product of elements of $S \cup S^{-1}$. Suppose $\phi, \psi : G \to H$ are group homomorphisms that agree on S. Prove $\phi = \psi$.

Collaborators: None.

Let $s^{-1} \in S^{-1}$, then since ϕ, ψ are homomorphisms,

$$\phi(s^{-1}) = \phi(s)^{-1} = \psi(s)^{-1} = \psi(s^{-1}),$$

so ϕ and ψ agree on $S \cup S^{-1}$. Now since S generates G, any $g \in G$ can be written $g = \prod_{i=1}^{n} s_{k_i}$, where the k_i's index into $S \cup S^{-1}$. Then again by properties of group homomorphisms,

$$\phi(g) = \phi\left(\prod_{i=1}^{n} s_{k_i}\right) = \prod_{i=1}^{n} \phi(s_{k_i}) = \prod_{i=1}^{n} \psi(s_{k_i}) = \psi\left(\prod_{i=1}^{n} s_{k_i}\right) = \psi(g).$$

Thus $\phi = \psi$.

Exercise 2 (§52 pg. 334 #1)**.** $A \subset \mathbb{R}^n$ is **star convex**if there is some point a_0 such that all line segments joining a_0 to other points of A lie in A.

a. Find a star convex set that is not convex.

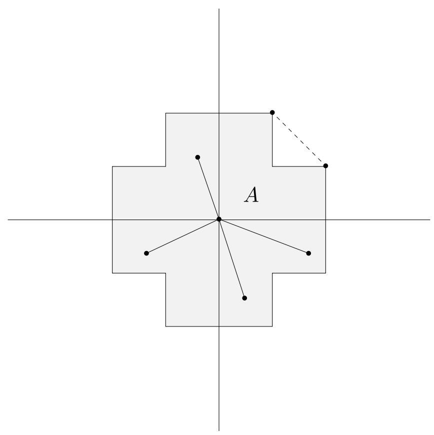
b. Show that if A is star convex, A is simply connected.

Collaborators: None.

a. Consider the set

$$A = \{(x, y) \mid -2 \leq x \leq 2, -1 \leq y \leq 1\} \\ \cup \{(x, y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\},$$

pictured below. Every line segment from the origin to another point in A lies entirely in A, but the line from $(1, 2)$ to $(2, 1)$ does not.



b. Since A is star convex, for all $x, y \in A$, we can find a line segment from a_0 to x and from a_0 to y. Pasting these two lines together gives a path form x to y, so A is path connected.

Now suppose f, g are loops based at a_0. Then since every point on f, g has a line segment to a_0 lying entirely in A, we can use the straight line homotopy to send f and g to the constant map $x \mapsto a_0$. Thus $f, g \simeq_p x \mapsto a_0$, which implies $f \simeq_p g$. Since f and g were arbitrary, this means all loops at a_0 are path homotopic, so $\pi_1(A, a_0)$ is trivial.

Since A is path connected and has a trivial fundamental group at some point, A is simply connected.

Exercise 3 (§52 pg. 335 #5). If A is a subspace of $\mathbb{R}^n$ and $H : (A, a_0) \to (Y, y_0)$ can be extended to a continuous map of $\mathbb{R}^n$ into Y, then h_* is the trivial homomorphism.

Collaborators: None.

We're given that there is some continuous $\tilde{h} : \mathbb{R}^n \to Y$ such that $h = \tilde{h} \circ i$, where $i : A \hookrightarrow \mathbb{R}^n$ is the usual inclusion map. Now $\mathbb{R}^n$ is simply connected, so $\pi_1(\mathbb{R}^n, a_0)$ is the trivial group. Since homomorphisms map identities to identities, the induced map $\tilde{h}_*$ must be the trivial homomorphism.

Since the homomorphism induced by h is $h_* = (\tilde{h} \circ i)_* = \tilde{h}_* \circ i_*$, this means h_* must also be the trivial homomorphism.

Exercise 4 (§53 pg. 341 #3)**.** Let $p : E \to B$ be a covering map, and let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elemenets for all $b \in B$.

Collaborators: None.

First we show that all points in any evenly covered neighbhorbood have the same number of elements in their preimage under p, then we use the connectedness of B to make this local property global.

Fix some $\tilde{b} \in B$ and consider an evenly covered neighborhood U of $\tilde{b}$. If $|p^{-1}(\tilde{b})| = k$, then we claim that $p^{-1}(U)$ has k exactly homeomorphic copies of U. Since p restricted to each $V_i \in p^{-1}(U)$ is a homeomorphism, we know that each V_i maps at most 1 point to $\tilde{b}$ (one-to-one) and at least 1 point to $\tilde{b}$ (onto). Thus each V_i maps exactly one point to $\tilde{b}$, so there must be k such V_i. Then by a similar argument, any $b \in U$ is mapped to by a single point in each V_i, so the inverse image of any point in U has k elements.

We now extend this local property to all of B. For each $b \in B$, choose an evenly covered neighborhood U_b of b. None of these neighborhoods is disjoint from all others: if there were some $U_{b'}$ disjoint from all others, then $U_{b'}, \bigcup_{b \neq b'} U_b$ would separate B, contradicting the connectedness of B. Thus for any $b \in B$, we can find some chain of evenly covered neighborhoods linking U_b to U_{b_0}. Then all points in U_b (in particular, the point b) have k elements in their preimage under p. Thus $|p^{-1}(b)| = k$ for all $b \in B$.

Exercise 5. Suppose $f, g : X \to S^2 = \{(x, y, z) \mid z^2 + y^2 + z^2 = 1\}$ satisfy $f(x) \neq -g(x)$ for all $x \in X$. Prove that f and g are homotopic.

Collaborators: None.

Let $F(x,t) = (1-t)f(x) + tg(x)$. This is the form of the usual straight line homotopy, but it doesn't work in our case since it doesn't lie entirely in S^2; however, we can modify it to remain on S^2 by

$$\tilde{F}(x,t) = \frac{F(x,t)}{\|F(x,t)\|}.$$

Since $f(x)$ and $g(x)$ are assumed to never be antipodal, we know that $F(x,t)$ never crosses the origin. This means $\|F(x,t)\|$ is never 0, so $\tilde{F}$ is well-defined. It has norm $\|\tilde{F}(x,t)\| = 1$, and since we've defined S^2 to be all points in $\mathbb{R}^3$ with norm 1, $\tilde{F}$ lies entirely in S^2. Thus $\tilde{F}$ is a homotopy from f to g.