Exercise 1. Show that $[X, Y]$ satisfies a Leibniz rule.

There are two different Leibniz-esque rules that I thought fit here. The first for $[X, Y](fg)$ the second is for $[X, fY](g)$.

Version 1: We'll need the fact that

$$\begin{aligned} X(Y(fg)) &= X(Y(f)g + fY(g)) \\ &= X(Y(f))g + fX(Y(g)). \end{aligned}$$

Similarly,

$$Y(X(fg)) = Y(X(f))g + fY(X(g)).$$

Then we have

$$\begin{aligned} [X, Y] &= X(Y(fg)) - Y(X(fg)) \\ &= (X(Y(f)) - Y(X(f)))\, g + f\,(X(Y(g)) - Y(X(g))) \\ &= [X, Y](f)g + f[X, Y](g). \end{aligned}$$

Version 2:

$$\begin{aligned} [X, fY](g) &= X(fY(g)) - fY(X(g)) \\ &= X(f)Y(g) + fX(Y(g)) - fY(X(g)) \\ &= X(f)Y(g) = f[X, Y](g). \end{aligned}$$

Exercise 2. What are the components of $[X, Y]$?

We have

$$
\begin{aligned}
[X, Y] &= X\left(w^i \frac{\partial}{\partial x^i}\right) - Y\left(v^i \frac{\partial}{\partial x^i}\right) \\
&= v^j \frac{\partial}{\partial x^j} w^i \frac{\partial}{\partial x^i} - w^j \frac{\partial}{\partial x^j} v^i \frac{\partial}{\partial x^i} \\
&= \left(v^j \frac{\partial}{\partial x^j} w^i - w^j \frac{\partial}{\partial x^j} v^i\right) \frac{\partial}{\partial x^i}.
\end{aligned}
$$

Thus the components of $[X, Y]$ are $v^j \frac{\partial}{\partial x^j} w^i - w^j \frac{\partial}{\partial x^j} v^i$.

Exercise 3. Show $R(X,Y,Z)$ is a tensor.

If we show that R is linear in each variable, then it'll be a tensor. For showing linearity in the first two terms, we use the result from Exercise 1 that

$$[X, fY] = X(f)Y + f[X,Y],$$

which we can also apply to $[fX, Y]$ since $[X,Y] = -[Y,X]$.

Linear in X:

$$\begin{aligned} R(fX,Y,Z) &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z \\ &= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X} Z \\ &= f\nabla_X\nabla_Y Z - \cancel{Y(f)\nabla_X Z} - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]}Z + \cancel{Y(f)\nabla_X Z} \\ &= fR(X,Y,Z). \end{aligned}$$

Linear in Y:

$$\begin{aligned} R(X,fY,Z) &= \nabla_X\nabla_{fY} Z - \nabla_{fY}\nabla_X Z - \nabla_{[X,Y]} Z \\ &= \cancel{X(f)\nabla_Y Z} + f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - \cancel{X(f)\nabla_Y Z} - f\nabla_{[X,Y]} Z \\ &= fR(X,Y,Z). \end{aligned}$$

Linear in Z:

$$\begin{aligned} R(X,Y,fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) \\ &= \nabla_X(Y(f)Z + f\nabla_Y Z) - \nabla_Y(X(f)Z + f\nabla_X Z) - ([X,Y](f) + f\nabla_{[X,Y]}Z) \\ &= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]} Z \\ &= fR(X,Y,Z). \end{aligned}$$

Thus $R(X,Y,Z)$ is a tensor.

Exercise 4. Compute the Levi-Civita connection Γ^i_{jk} and the Riemann curvature tensor R^i_{jkl}, then show that

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}).$$

The metric tensor has matrix

$$(g_{ij}) = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2 \theta \end{pmatrix}$$

and inverse matrix

$$(g^{ij}) = \begin{pmatrix} r^{-2} & 0 \\ 0 & r^{-2} \sin^{-2} \theta \end{pmatrix}.$$

Note that since $x^1 = \theta$ and $x^2 = \phi$,

$$\partial_2 \, g_{xy} = \partial_2 \, g^{xy} = \partial_1 \, g_{11} = \partial_1 \, g_{12} = \partial_1 \, g_{21} = 0.$$

We can then plug these into the definition of the Levi-Civita connection

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j \, g_{kl} + \partial_k \, g_{jl} - \partial_l \, g_{jk}).$$

Because so many of the partial derivatives are 0, the computations end up being relatively simple. We get

$$\begin{aligned} \Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = \Gamma^2_{22} &= 0, \\ \Gamma^1_{22} &= -\sin\theta\cos\theta, \\ \Gamma^2_{12} = \Gamma^2_{21} &= \sin^{-1}\theta\cos\theta. \end{aligned}$$

Then we can plug these into the relation

$$R^i_{jkl} = \partial_k \, \Gamma^i_{lj} - \partial_l \, \Gamma^i_{kj} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}$$

to recover the Riemann curvature tensor. The computations come out to

$$\begin{aligned} R^1_{111} = R^1_{112} = R^1_{121} = R^1_{211} = R^1_{122} = R^1_{222} &= 0, \\ R^2_{111} = R^2_{211} = R^2_{221} = R^2_{212} = R^2_{122} = R^2_{222} &= 0, \\ R^1_{221} &= -\sin^2\theta, \\ R^1_{212} &= \sin^2\theta, \\ R^2_{112} &= -1, \\ R^2_{121} &= 1. \end{aligned}$$

Then using the formula

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}),$$

we can plug in these values of R^i_{jkl} along with $K = 1/r^2$, which we know to be the Gaussian curvature for this surface. In every case, we get equality, so the formula holds.