MATH 531 HOMEWORK 5

BRADEN HOAGLAND

Exercise 3.1.5. *Let M be a set with the discrete metric. Show that any infinite subset of M is noncompact. Why does this not contradict the statement in Execise 4?*

Let A be an infinite subset of discrete metric space M, then $U = \{D(a, 1/2) \mid a \in A\}$ is clearly an infinite open cover of A. Now select arbitrary $a' \in A$ and remove its corresponding ball to yield $U' = \{D(a, 1/2) \mid a \in A, a \neq a'\}$. Then U' does *not* cover A since by definition of the discrete metric, none of the balls in U' cover a'. Since a' was arbitrary, we cannot remove any of the open sets from U, meaning that we cannot find a finite subcover for A. Thus A is noncompact.

Exercise 4 required that we find a convergent sequence $x_n \to x$. In a discrete metric space, the only convergent sequence is a sequence that eventually becomes constant. So even though the sequence has infinite terms, it is in fact still only a finite subset of M. Thus exercise 4 does not contradict the result of this problem.

Exercise 3.3.2. *Is the nested set property true if "compact nonempty" is replaced by "open bounded nonempty"?*

No. In $\mathbb{R}$, let $U_n = (0, 1/n)$, then $\{U_n\}_{n=1}^{\infty}$ is a sequence of open bounded nonempty decreasing sets. Assume $x \in \cap_{n=1}^{\infty} U_n$, then $0 < x < 1/n$ for all $n \in \mathbb{N}$. Since $\mathbb{R}$ is Archimedean, this is impossible, so no such x exists and $\cap_{n=1}^{\infty} U_n = \varnothing$.

Exercise 3.3.4. *Let $x_k \to x$ be a convergent sequence in a metric space. Let $\mathcal{A}$ be a family of closed sets with the property that for each $A \in \mathcal{A}$, there is an N such that $k \geq N$ implies $x_k \in A$. Prove that $x \in \cap \mathcal{A}$.*

Let $A \in \mathcal{A}$ be arbitrary. We know there exists some N such that if $k \geq N$, then $x_k \in A$. Thus we have a convergent sequence $\{x_k\}_{k=N}^{\infty} \subset A$. Since A is closed, it contains its limit points, in this case x. Since A was arbitrary, x must lie in all A. Thus $x \in \cap \mathcal{A}$.

Exercise 3.5.2. *Is $\big\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\big\} \cup \{(x, 0) \mid 1 < x < 2\}$ connected? Prove or disprove.*

Let $A = \big\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\big\}$ and $B = \{(x, 0) \mid 1 < x < 2\}$ We will show that $A \cup B$ is connected by showing that it is path-connected. Take two points $x, y \in \mathbb{R}^2$. There are three cases we must consider when constructing a continuous path from x to y that lies in $A \cup B$.

(1) Assume $v, w \in A$. Now let $\varphi_1 : [0, 1] \to A$ be defined by $\varphi(t) = (w - v)t + v$. This is a continuous mapping since for any sequence z_k and constant λ, $\lambda z_k \to \lambda z$ and $z_k + \lambda \to z + \lambda$, implying $\varphi_1(t_k) \to \varphi_1(t)$ if $t_k \to t$.

Note that $\varphi_1(0) = v$ and $\varphi_1(1) = w$. Now let x_v be the x-component of v and x_w be the x-component of w, then $x_{\varphi(t)} = (x_w - x_v)t + x_v = x_w t + (1 - t)x_v$. We can show that this is in A for any $t \in [0, 1]$ since

$$x_w t + (1 - t)x_v \geq 1 - t \geq 0$$

Date: September 23, 2020.

and

$$x_w t + (1-t)x_v \leq t + (1-t) = 1 \leq 1.$$

Thus if v and w are both in A, we can construct a continuous path between them that also lies in A.

(2) Assume $v, w \in B$. We can define $\varphi_2(t) = wt + (1-t)v$, the same as the previous case. Similarly, φ_2 is a continuous map from v to w that lies in B.

(3) Without loss of generality, assume $v \in A$ and $w \in B$. Let z be the point $(1, 0)$, then define $\varphi_3 : [0, 2] \to A \cup B$ by

$$\varphi_2(t) = \begin{cases} (z-v)t + v & \text{if } t \leq 1 \\ (w-z)(t-1) + z & \text{if } t \geq 1 \end{cases}$$

Informally, this can be thought of as a concatenation of the two continuous maps presented earlier. It is then itself a continuous map from v to w that lies entirely in $A \cup B$.

We have found continuous maps between any two points in $A \cup B$, so it is path-connected and, subsequently, connected.

Exercise 3.17. *Let K be a nonempty closed set in $\mathbb{R}^n$ and $x \in K^c$. Prove that there is a $y \in K$ such that $d(x, y) = \inf\{d(x, z) \mid z \in K\}$. Is this true for open sets? Is it true in general metric spaces?*

Let $L = \inf\{d(x, z) \mid z \in K\}$. Consider $K' = K \cap \{x' \mid d(x', x) \leq L + 1\}$. K' is the intersection of closed sets, so it is also closed. Moreover, it is bounded since the closed ball of radius $L + 1$ around x is clearly bounded. Since we are operating in $\mathbb{R}^n$, the Heine-Borel theorem shows that K' is then compact. This will allow us to construct a sequence with a convergent subsequence.

Let $y_1 \in K'$ such that $d(y_1, x) > L$. If no such y_1 exists, then the problem is trivial since K is nonempty, which implies that the only points of K' are distance L away from x. Now select $y_2 \in K'$ such that $d(y_1, x) > d(y_2, x) > L$. If no such y_2 exists, then the problem is once again trivial since L being an infimum of the distances implies that there must exist some point $\tilde{y}$ satsifying $d(\tilde{y}, x) = L$. Continuining in this manner and assuming we run into no trivial cases, construct the sequence $\{y_k\}_{k=1}^{\infty}$ such that $d(y_k, x) > d(y_{k+1}, x) > L$. Since K' is compact, this sequence has a convergent subsequence $y_{\sigma(k)} \to y \in K'$.

We have in fact created two sequences: a sequence of points and a sequence of distances. Denote the latter by $\left\{d(y_{\sigma(k)}, x)\right\}_{k=1}^{\infty}$. Since this lies in $\mathbb{R}$ and is strictly decreasing and bounded below, it must converge. If it converges to any point other than L, we contradict the fact that L is the infimum of the distances, so it must converge to L.

Since $y_{\sigma(k)} \to y$, for all $\varepsilon > 0$ there exists N_1 such that if $k > N_1$, then $d(y_{\sigma(k)}, y) < \varepsilon$. Similarly, for all $\varepsilon > 0$ there exists N_2 such that if $k > N_2$, then $|d(y_{\sigma(k)}, x) - L| < \varepsilon$. Since $d(y_{\sigma(k)}, x) > L$ by construction, this implies $d(y_{\sigma(k)}, x) < L + \varepsilon$.

Now let $\varepsilon' > 0$, then we know that for k large enough,

$$\begin{aligned} d(y, x) &\leq d(y, y_k) + d(y_k, x) \\ &< \frac{\varepsilon'}{2} + L + \frac{\varepsilon'}{2} \\ &= L + \varepsilon'. \end{aligned}$$

Since $d(y, x)$ is strictly lower than any $L + \varepsilon$, it must be the case that $d(y, x) \leq L$. Since L is the infimum of the distances, i.e. $d(y, x) \geq L$, this implies $d(y, x) = L$.

This would not work if K had been open, as y could have been an element of K^c instead. This will hold in any metric space in which closed balls are compact, as this requirement is enough to find a convergent subsequence in K'.

Exercise 3.29. *Let $A = \{(x,y) \in \mathbb{R}^2 \mid x^4 + y^4 = 1\}$. Show that A is compact. Is it connected?*

Since $A \subset \mathbb{R}^2$, by the Heine-Borel theorem it suffices to show that A is closd and bounded.

Closed: Let $(x_n, y_n) \to (x,y)$ for $(x_n, y_n) \in A$. Since convergence in $\mathbb{R}^n$ implies pointwise convergence, we know $x_n \to x$ and $y_n \to y$. Then by limit arithmetic in $\mathbb{R}$, $1 = x_n^4 + y_n^4 = x^4 + y^4$. Thus $(x,y) \in A$, so A is closed.

Bounded: If $|x| > 1$ or $|y| > 1$, then the sum $x^4 + y^4$ would be strictly greater than 1, since both x^4 and y^4 are non-negative. Thus any point in A satisfies $|x|, |y| \leq 1$. The distance from any point in A to the origin can then be bounded by

$$\|(x,y)\| = \sqrt{x^2 + y^2} \leq \sqrt{1+1} = \sqrt{2},$$

so clearly $A \subset D(0,2)$.

To show that A is connected, it suffices to show that it is path-connected. We will do so by constructing two maps: one from $[0, 2\pi]$ to an intermediate set, and a second from this intermediate set to A. Let $\tilde{A} = \{(x,y) \mid x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$, then define $\phi_1 : [0, 2\pi] \to \tilde{A}$ and $\varphi_2 : \tilde{A} \to A$ by

$$\varphi_1(t) = (\cos t, \sin t)$$
$$\varphi_2((x,y)) = \frac{1}{(x^4 + y^4)^{1/4}}(x,y),$$

then their composition $\varphi = \varphi_2 \circ \varphi_1$ is a continuous map $[0, 2\pi] \to A$. To see that φ_1 indeed maps to $\tilde{A}$, note that $\cos^2 t + \sin^2 t = 1$ by a trigonometric identity. To see that φ_2 indeed maps to A, note that

$$\left\| \frac{(x,y)}{\|(x,y)\|_4} \right\|_4 = \frac{\|(x,y)\|_4}{\|(x,y)\|_4} = 1.$$

Given two points in A, we can then find a continuous map between them. Let $x, y \in A$, then there exist $a, b \in [0, 2\pi]$ such that $\varphi(a) = x$ and $\varphi(b) = y$. Then define the continuous map $\varphi_3 : [0,1] \to [0, 2\pi]$ by

$$\varphi_3(t) = a + t(b-a).$$

Then $\varphi_2 \circ \varphi_1 \circ \varphi_3$ is a continuous path between x and y that lies entirely in A. Thus A is path-connected and, subsequently, connected.

Exercise 3.33. *A set S in a metric space is called **nowhere dense** if for any nonempty open set U, we have $\overline{S} \cap U \neq U$, or equivalently, $(\overline{S})^o = \varnothing$. Show that $\mathbb{R}^n$ cannot be written as the countable union of nowhere dense sets.*

Let $\mathbb{R}^n = \cup_{n=1}^{\infty} A_n$. Assume that each A_n is a nowhere dense set in $\mathbb{R}^n$, then none of them contain nonempty open subsets. This means we can find a nonempty open subset in A_1^c, so let

$$D_1 \doteq D(x_1, \varepsilon_1) \subset A_1^c$$

for some $x_1 \in A_1^c$ and $0 < \varepsilon_1 < 1$. Similarly, there must be a nonempty open subset in $D_1 \cap A_2^c$, so let

$$D_2 \doteq D(x_2, \varepsilon_2) \subset D_1 \cap A_2^c$$

for some $x_2 \in D_2 \cap A_2^c$. Inductively construct a sequence satisfying

$$D_{k+1} \subset D_n \cap A_{n+1}^c, \quad \varepsilon_n < \frac{1}{2^n}.$$

Then by construction, $\overline{D_1} \supset D_1 \supset \overline{D_2} \supset D_2 \supset \cdots$. Since we are working in $\mathbb{R}^n$ and each $\overline{D_k}$ is closed and bounded, they are compact. Then by the Nested Set Property, there exists some $x \in \mathbb{R}^n$ in the intersection $\cap_n \overline{D_n}$. Since x lies in every D_n, $x \notin A_n$ for any n. Thus $x \notin \cup_n A_n$. Then by contradiction, we have that $\mathbb{R}^n$ cannot be constructed as the countable union of nowhere dense sets.

Exercise 3.34. *Prove that any closed set $A \subset M$ is an intersection of a countable family of open sets.*

Let $U_n = \cup_{a\in A} D(a, 1/n)$ and let $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$, then we claim $\cap\,\mathcal{U} = A$. Clearly we have $A \subset \cap\,\mathcal{U}$ since by definition, every point of A lies in every U_n.

We can prove the reverse inclusion by contrapositive. Let $x \notin A$, then we must show $x \notin \cap\,\mathcal{U}$. Since $x \notin A$, we must have $x \in A^c$. Since A is open, A^c is closed, so there exists $\varepsilon > 0$ such that $D(x, \varepsilon) \subset A^c$. This implies that there exists $n \in \mathbb{N}$ such that $D(x, 1/n) \subset A^c$, meaning that $D(x, 1/n) \cap A = \varnothing$. This implies that every point of A is at least a distance of $1/n$ away from x, so $x \notin D(a, 1/n)$ for any $a \in A$. Thus $x \notin \cup_{a\in A} D(a, 1/n) = U_n$, so $x \notin \cap\,\mathcal{U}$.

Exercise 3.36. *Let $A \subset \mathbb{R}^n$ be uncountable. Prove that A has an accumulation point.*

We can first find a closed and bounded infinite subset of $\mathbb{R}^n$. Let F_l^k be the set of all points in A whose k-th coordinate lies in the closed interval $[l, l+1]$, then $\mathbb{R}^n = \cup_{k=1}^n \cup_{l\in\mathbb{Z}} F_l^k$. At least one F_l^k must be uncountable, otherwise $\mathbb{R}^n$ would be countable. We can thus find a closed, bounded, uncountable set $F_l^k \subset \mathbb{R}^n$. For simplicity, denote this set B.

$B \subset \mathbb{R}^n$ is closed and bounded, so it is compact by the Heine-Borel theorem. By Bolzano-Weierstrass, it is sequentially compact. Thus every infinite sequence $\{x_k\} \subset B$ has a subsequence $\left\{x_{\sigma(k)}\right\}$ that converges to some point $b \in B$.

Since there are infinite points in B, we can form our sequence $\{x_k\}$ using unique elements. By the definition of convergence, for every open neighborhood U of b, there exists $\sigma(k)$ such that $x_l \in U$ when $l > \sigma(k)$. Since every element of our sequence is unique, this implies that at least one such x_l is not equal to b. Thus $U \cap A \setminus \{b\}$ is nonempty for every U, so b is an accumulation point of A.

Exercise 3.37. *Let $A, B \subset M$ with A compact, B closed, and $A \cap B = \varnothing$.*

(1) *Show that there is an $\varepsilon > 0$ such that $d(x, y) > \varepsilon$ for all $x \in A$ and $y \in B$.*
(2) *Is this true if A and B are merely closed?*

---

(1) We first state two facts that we will use to prove this statement.
   (*) Since $A \cap B = \varnothing$, A must lie in B^c, which is open since B is closed. Then for each $a \in A$, there exists δ_a such that $D(a, \delta_a) \subset B^c$. Thus for any $a \in A$, $d(a, b) \ge \delta_a$ for any $b \in B$.
   (**) Additionally, we can take the open cover $\{D(a, \delta_a/2)\}$ of A and use the compactness of A to find a finite open subcover $\{D(a_k, \delta_{a_k}/2)\}_{k=1}^N$. By definition, any element in one of the balls in the subcover is at most $\delta_a/2$ away from its corresponding a_k.

   Let $a \in A$, then a lies in at least one of the sets in the finite subcover, i.e. $a \in D(a_k, \delta_{a_k}/2)$ for some $k \in \{1, \ldots, N\}$. Then by the triangle inequality we have

$$d(a_k, b) \le d(a, b) + d(a_k, a)$$
$$d(a, b) \ge d(a_k, b) - d(a_k, a)$$

Using facts $(*)$ and $(**)$ gives

$$\begin{aligned} &> \delta_{a_k} - \delta_{a_k}/2 \\ &= \delta_{a_k}/2 \end{aligned}$$

Set $\varepsilon = \min\{\delta_{a_1}/2, \ldots, \delta_{a_N}/2\}$, then $d(a,b) > \varepsilon$ for all $a \in A$ and $b \in B$.

(2) No. Let $A = \{(x,0) \mid x \in \mathbb{R}\}$ and $B = \{(x,1/x) \mid x \in \mathbb{R}, x > 0\}$ be closed subsets of $\mathbb{R}^2$. Fix $x > 0$, then consider $a = (x,0) \in A$ and $b = (x,1/x) \in B$. The distance between them is $d(a,b) = |1/x| = 1/x$ under the usual metric. Assume satisfactory $\varepsilon > 0$ exists, then $1/x > \varepsilon$ for all x. Since $\mathbb{R}$ is archimedean, however, we can find x such that $1/x < \varepsilon$ for any ε. Thus A and B just being closed is not enough to get the same result.