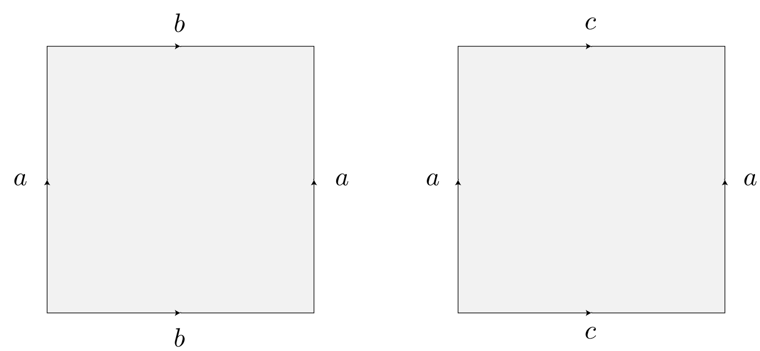
Exercise 1 (1.2: 8). π_1 of two tori with coinciding circle.



The situation is depicted above. We have two tori that share a common circle (in this case a). We fill these two 1-skeletons with 2-cells along the boundaries $aba^{-1}b^{-1}$ and $aca^{-1}c^{-1}$. Thus the fundamental group is

$$\langle a, b, c \mid [a, b], [a, c] \rangle \quad \cong \quad \mathbb{Z} \times (\mathbb{Z} * \mathbb{Z}).$$

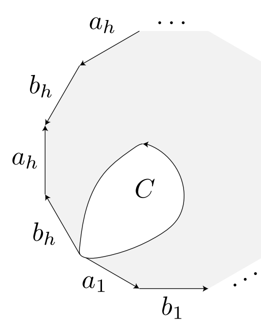
Exercise 2 (1.2: 9)**.** M_g doesn't retract to C, but it does retract to C'.

No retraction onto C: Suppose M_h' retracts onto C via r, then since ab is a covariant functor **Grp** $\to$ **Ab**, we have the following sequence of induced maps.

$$\begin{tikzcd} M_h' \arrow[d, shift left, two heads, "r"] \\ C \arrow[u, shift left, hook, "i"] \end{tikzcd} \qquad \begin{tikzcd} \pi_1(M_h') \arrow[d, shift left, two heads, "r_*"] \\ \pi_1(C) \cong \mathbb{Z} \arrow[u, shift left, hook, "i_*"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{ab}(\pi_1(M_h')) \arrow[d, shift left, two heads, "\tilde{r}_*"] \\ \mathbb{Z} \arrow[u, shift left, hook, "\tilde{i}_*"] \end{tikzcd}$$

Functoriality of π_1 and ab implies that $\tilde{r}_* \circ \tilde{i}_* = \mathrm{id}$; thus why the injectivity and surjectivity are preserved.

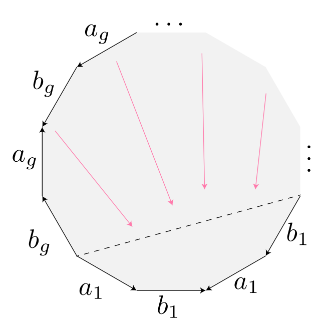
Now we have to derive a contradiction of some sort using these induced maps. We can depict M_h' as below.



From this figure, C is clearly homotopic to $[a_1, b_1] \cdots [a_h, b_h]$, so $1 \in \pi_1(C) = \mathbb{Z}$ is mapped to $[a_1, b_1] \cdots [a_h, b_h] \in \pi(M_h')$ by i_*. But this is trivial once $\pi_1(M_h')$ is abelianized, so $\tilde{i}_*$ is a constant map, contradicting its injectivity. Thus M_h' cannot retract onto C.

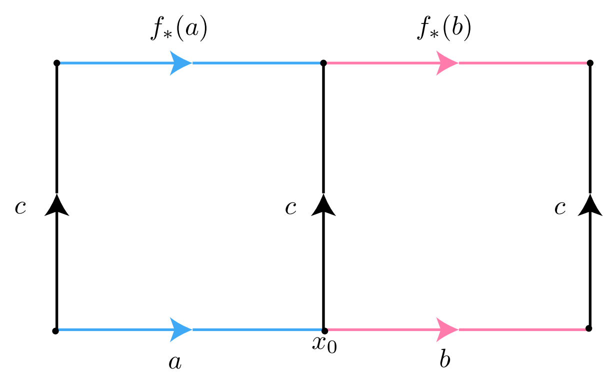
But if M_g retracts onto C via r, then M_h' retracts onto C via $r|_{M_h'}$, so this also implies that M_g cannot retract onto C.

Retraction onto C': The following image shows a retraction $M_g \to M_1 = S^1 \times S^1$. We can compose this with the projection map onto one coordinate of $S^1 \times S^1$, giving a retraction onto C'.



Exercise 3 (1.2: 11)**.** Mapping tori of $S_1 \vee S_1$ and $S_1 \times S_1$.

First part: Suppose x_0 is the basepoint connecting the two circles in $X = S^1 \vee S^1$, then since f is basepoint-preserving, we get the following picture.

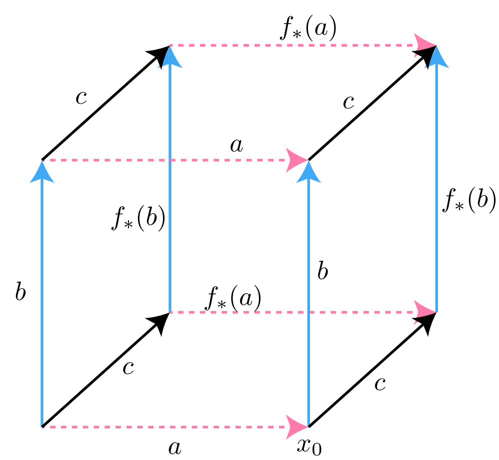


Each of the 6 vertices is identified into one 0-cell, and there are three distinct lines (1-cells) going out of and into this point, so the 1-skeleton is $S^1 \vee S^1 \vee S^1$.

Then T_f is then recovered from the 1-skeleton by gluing on 2-cells along $acf_*(a)^{-1}c^{-1}$ and $bcf_*(b)^{-1}c^{-1}$. Thus the fundamental group of T_f is

$$\pi_1(T_f) = \langle a, b, c \mid acf_*(a)^{-1}c^{-1},\ bcf_*(b)^{-1}c^{-1} \rangle.$$

Second part: Suppose instead that $X = S' \times S'$, then the new 1-skeleton is pictured below.



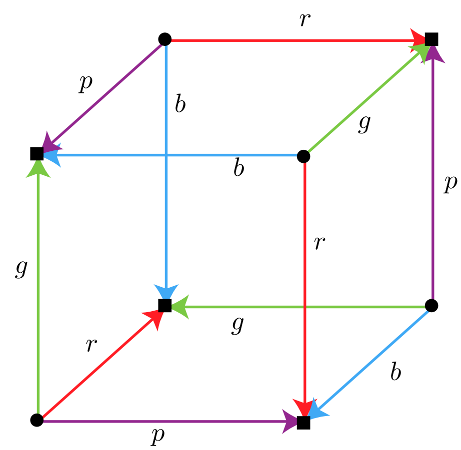
Note that all of the 8 vertices are identified with x_0.

Note that since all the vertices are identified and since there are three distinct 1-cells, this is $S^1_1 \vee S^1 \vee S^1$. To fill it, attach three 2-cells along $aba^{-1}b$, $bcf_*(b)^{-1}c^{-1}$, and $acf_*(a)^{-1}c^{-1}$. Thus the fundamental group is

$$\pi_1(T_f) \cong \langle a, b, c \mid [a, b], bcf_*(b)^{-1}c^{-1}, acf_*(a)^{-1}c^{-1} \rangle.$$

Exercise 4 (1.2: 14)**.** Funky cube with π_1 the quaternion group.

The 1-skeleton of this space is pictured below, with 1-cells of the same color and 0-cells of the same shape identified.



Because of the identifications, this is just $S^1 \vee S^1 \vee S^1$. I found that pb^{-1}, pr^{-1}, and pg^{-1} generate all other loops by simply checking all cases. We can add in 2-cells along the right, back, and top faces to fill in the space. Thus the fundamental group is

$$\pi_1(X) \cong \langle pb^{-1}, pr^{-1}, pg^{-1} \mid pg^{-1}rb^{-1}, pr^{-1}bg^{-1}, pb^{-1}gr^{-1} \rangle.$$

Now we can define $i \doteq pb^{-1}$, $j \doteq pr^{-1}$, and $j \doteq pg^{-1}$, which makes this

$$\begin{aligned}
&= \langle i, j, k \mid kj^{-1}i, ji^{-1}k, ik^{-1}j \rangle \\
&= \langle i, j, k \mid j = ki, i = jk, k = ij \rangle \\
&= \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle.
\end{aligned}$$

This is exactly the quaternion group.