0.1 PDEs

At every step we choose some finite collection of vertices $\{v_i\}_{i=1}^m$. Let κ_i denote the size of the cluster to which v_i belongs. We'll use the following quantities a lot (all probabilities are implicitly taken at time t):

$$\begin{aligned} Q_m(k,t) &\doteq \mathbb{P}\left(\min\{\kappa_1,\ldots,\kappa_m\} = k\right); \\ \hat{Q}_m(k,t) &\doteq \mathbb{P}\left(\min\{\kappa_1,\ldots,\kappa_m\} \geq k\right) \\ &= 1 - \sum_{j=1}^{k-1} Q_m(j,t); \\ R(k,t) &= \mathbb{P}\left(\kappa_1 + \kappa_2 = k\right); \\ \hat{R}(k,t) &= \mathbb{P}\left(\kappa_1 + \kappa_2 \geq k\right). \end{aligned}$$

A common case for Q_m is $m = 1$ or 2, so we can abbreviate those as

$$P \doteq Q_1, \qquad Q \doteq Q_2.$$

Note that we can express Q_m as

$$Q_m(k,t) = \hat{P}(k-1,t)^m - \hat{P}(k,t)^m,$$

(go over why) so every Q_m is a function of P. Let $S(t)$ denote the relative size (i.e. divided by n) of the percolation cluster at time t. As a final note, I will frequently suppress the time t from now on.

We're interested in how P changes throughout the percolation process. The following table gives the value of $\partial_t P$, written in terms of the proper Q_m, for each of our rules.

Rule	$\partial_t P(s,t)$
Erdős Rényi	$\frac{s}{2}\sum_{u+v=s} P(u,t)P(v,t) - sP(s,t)$
Adjacent Edge	$s\sum_{u+v=s} P(u,t)Q(v,t) - sP(s,t) - sQ(s,t)$
DaCosta	$s\sum_{u+v=s} Q_m(u,t)Q_m(v,t) - 2sQ_m(s,t)$
Sum	**Do this.**
Product	**Do this.**

0.2 Consequences

Proposition 1. $\sum_k Q_m(k,t) = 1 - S^m(t)$.

To justify this, we can interpret $\sum_k Q_m(k,t)$ as the probability that, at time t, the minimum cluster size of m vertex choices is finite (this is in the limit as $n \to \infty$). S is then the probability that a single choice is from an "infinite" cluster size. **I kinda want to do this more rigorously, but that's not too important right now...**

Differentiating this identity for $X_1 = P$ gives

$$\partial_t S = -\sum_s \partial_t P,$$

so we can track the size of the percolation cluster by knowing $P(s)$ for all s. In the following computations, we'll express $\partial_t S$ in terms of the moments of various Q_m, which we denote by

$$\langle s^k \rangle_{Q_m} \doteq \sum_s s^k Q_m(s).$$

Sometimes I might denote $\langle \cdot \rangle_{Q_m}$ by $\langle \cdot \rangle_m$. The below table gives $\partial_t S$ for each of our rules. A derivation of this quantity is given afterwards for the Erdős Rényi rule; the other quantities are derived similarly.

Rule	$\partial_t S$
ER	$S\langle s \rangle_P$
AE	$\langle s \rangle_P S^2 + S\langle s \rangle_Q$
DC	$2S^m \langle s \rangle_{Q_m}$
Sum	**Do this.**
Product	**Do this.**

Using the assumption $S = \delta^\beta$ near t_c, these quantities can be used to relate β to the various other exponents.

Proposition 2. For the Erdős Rényi rule, $\partial_t S = S\langle s \rangle_P$.

Proof. In the below computation, I suppress the time t for clarity.

$$\begin{aligned}
\partial_t S &= -\sum_s \partial_t P \\
&= -\frac{1}{2}\sum_s s \sum_{u+v=s} P(u)P(v) + \sum_s sP(s) \\
&= -\frac{1}{2}\sum_u \sum_v (u+v)P(u)P(v) + \langle s \rangle_P \\
&= -\frac{1}{2}\left[\sum_u uP(u)\sum_v P(v) + \sum_u P(u)\sum_v vP(v)\right] + \langle s \rangle_P \\
&= -\frac{1}{2}\left[2\,\langle s \rangle_P\,(1-S)\right] + \langle s \rangle_P \\
&= -\langle s \rangle_P (1-S) + \langle s \rangle_P \\
&= S\langle s \rangle_P.
\end{aligned}$$

□

We're similarly able to calculate $\partial_t \langle s \rangle_P$ for these rules, as summarized in the below table. As before, I incluce the derivation for the Erdős Rényi rule afterwards, and the other derivations are similar.

Rule	$\partial_t \langle s \rangle_P$
ER	$\langle s\rangle_P^2 - \langle s^2\rangle_P S$
AE	$2\langle s\rangle_P \langle s\rangle_Q - \langle s^2\rangle_P S^2 - \langle s^2\rangle_Q S$
DC	$2\langle s\rangle_{Q_m}^2 - 2\langle s^2\rangle_{Q_m} S^m$
Sum	**Do this.**
Product	**Do this.**

Proposition 3. For the Erdős Rényi rule, $\partial_t \langle s \rangle_P = \langle s\rangle_P^2 - \langle s^2\rangle_P S$.

Proof. Once again, I suppress the time t for clarity.

$$\begin{aligned}
\partial_t \langle s \rangle_P &= \sum_s s\, \partial_t P(s) \\
&= \frac{1}{2}\sum_s s^2 \sum_{u+v=s} P(u)P(v) - \sum_s sP(s) \\
&= \frac{1}{2}\sum_u \sum_v (u+v)^2 P(u)P(v) - \langle s^2\rangle_P \\
&= \frac{1}{2}\left[\sum_u u^2 P(u) \sum_v P(v) + 2\sum_u uP(u)\sum_v vP(v) + \sum_u P(u)\sum_v v^2 P(v)\right] - \langle s^2\rangle_P \\
&= \frac{1}{2}\left[2\langle s^2\rangle_P (1-S) + 2\langle s\rangle_P^2\right] - \langle s^2\rangle_P \\
&= \langle s\rangle_P^2 - \langle s^2\rangle_P S.
\end{aligned}$$

□

NEED TO DEFINE $\sim$.

If $\delta \doteq |t - t_c|$ is very small, then we have the scaling relationship

$$\langle s\rangle_{Q_m} \sim \delta^{-\gamma}$$

for some γ dependent on Q_m. Differentiating gives us the relation

$$\partial_t \langle s\rangle_{Q_m} \sim \delta^{-\gamma-1}.$$

Given a particular rule, we can take these two relations and subsitute them into our earlier calculation of $\partial_t \langle s\rangle_P$ to find out how the various γ are related. The below table sumarizes this relationship for all our rules.

Right now, I'm using the fact that $S = 0$ when $t < t_c$. I don't think it's necessary to be symmetric, though, since the behavior of the system seems to change after t_c anyway.

Rule	Scaling Relationship
ER	$\gamma_P = 1$
AE	$\gamma_Q = 1$
DaCosta	$\gamma_P + 1 = 2\gamma_{Q_m}$
Sum	**Do this.**
Product	**Do this.**

Do we get any special information when $\gamma = 1$? I wonder if that makes any other computations elsewhere easier...

0.3 Scaling Relationships

There are relationships between the coefficients β, τ, and σ that all our models use, and we can use the γ relationships from the previous table to express everything in terms of just two of these.

Theorem 1. *Suppose a rule has a scaling function f such that*

$$\lim_{x\to\infty} x^{2-\tau} f(x) = 0$$

and

$$\int_0^\infty x^{2-\tau} f'(x)\,dx$$

is finite. Then

$$\begin{aligned}\beta &= (\tau-2)/\sigma,\\ \gamma_P &= (3-\tau)/\sigma,\\ \gamma_Q &= (2m - m\tau + 1)/\sigma.\end{aligned}$$

Proof. We'll begin with the relation for β. Since

$$S \approx \int_0^\infty s^{1-\tau}(f(0) - f(s\delta^{1/\sigma}))\,ds,$$

we can make the change of variable $s = x\delta^{-1/\sigma}$ to get

$$= \delta^{(\tau-2)/\sigma} \int_0^\infty x^{1-\tau}(f(0) - f(x))\,dx.$$

Integrating by parts gives

$$= \frac{\delta^{(\tau-2)/\sigma}}{\tau-2}\left[\left[-x^{2-\tau}(f(0)-f(x))\right]_{x=0}^{x=\infty} - \int_0^\infty x^{2-\tau} f'(x)\,dx\right].$$

So by our assumptions on f, we have $S = \Theta\left(\delta^{(\tau-2)/\sigma}\right)$. Since we're already assuming $S \approx \delta^\beta$, this means $\beta = (\tau-2)/\sigma$. The proof is similar for γ_P. Since

$$\langle s\rangle_P = \int_0^\infty s^{2-\tau} f(s\delta^{1/\sigma})\,ds,$$

we can once again make a change of variables and integrate by parts. **(Actually, at this point I'm confused since there's another x term floating around everywhere, but the paper doesn't seem to address this).** Then since $\langle s\rangle_P \approx \delta^{-\gamma_P}$, we get $\gamma_P = (3-\tau)/\sigma$. The derivation for γ_Q is similar, with the relations in Appendix E of DaCosta between f' and g' ensuring that the final integral will be finite with g' instead of f'. □

0.4 Computations for Adjacent Edge Rule

Results we've already shown:

$$\partial_t S = \langle s\rangle_P S^2 + S\langle s\rangle_Q,$$
$$\gamma_Q = 1.$$

If we have scaling behavior, then the previous theorem applies and we get

$$\beta = (\tau - 2)/\sigma \tag{1}$$
$$\gamma_P = (3 - \tau)/\sigma \tag{2}$$
$$\gamma_Q = (5 - 2\tau)/\sigma. \tag{3}$$

Since we know $\gamma_Q = 1$, we have three equations and four unknowns, so we can solve for all of them in terms of just one. Plugging $\gamma_Q = 1$ into (3) and rearranging gives $\sigma = 5 - 2\tau$, which we can plug into (1) to get $\tau = (5\beta + 2)/(2\beta + 1)$. Plug this new expression for τ into $\sigma = 5 - 2\tau$ to get $\sigma = 1/(2\beta + 1)$. Finally, plug these expressions for σ and τ into (2) to get $\gamma_P = \beta + 1$. In summary,

$$\sigma = \frac{1}{2\beta + 1},$$
$$\tau = \frac{5\beta + 2}{2\beta + 1},$$
$$\gamma_P = \beta + 1,$$
$$\gamma_Q = 1.$$