MATH 531 HOMEWORK 4

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Exercise 2.12. *Prove the following properties for subsets A and B of a metric space:*

(1) $(A^o)^o = A^o$
(2) $(A \cup B)^o \supset A^o \cup B^o$
(3) $(A \cap B)^o = A^o \cap B^o$

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(1) For any set X, by definition $X^o = X$ if and only if X is open. Now for any $a \in A^o$, there exists open neighborhood $U \subset A$ such that $a \in U$. This shows A^o is open, so $(A^o)^o = A^o$.
(2) Let $x \in A^o \cup B^o$. If $x \in A^o$, then there exists an open neighborhood U of x that lies within A and, by extension, $A \cup B$. Thus $x \in (A \cup B)^o$. A similar argument holds for when $x \in B^o$, so we conclude $A^o \cup B^o \subset (A \cup B)^o$.
(3) Let $x \in A^o \cap B^o$, then there exist open neighborhoods $U_A \subset A$ and $U_B \subset B$ of x. Their intersection $U_A \cap U_B \subset A \cap B$ is still an open neighborhood of x, so $x \in (A \cap B)^o$. Thus $A^o \cap B^o \subset (A \cap B)^o$.
Now let $x \in (A \cap B)^o$, then there exists open neighborhood $U \subset A \cap B$ of x. Since $U \subset A$ and $U \subset B$, x is in both A^o and B^o. Thus $(A \cap B)^o \subset A^o \cap B^o$.
These two inclusions show $(A \cap B)^o = A^o \cap B^o$.

Exercise 2.15. *Prove the following for subsets of a metric space M:*

(1) $\partial A = \partial(A^c)$
(2) $\partial(\partial A) \subset \partial A$
(3) $\partial(A \cup B) \subset \partial A \cup \partial B \subset \partial(A \cup B) \cup A \cup B$
(4) $\partial(\partial(\partial A)) = \partial(\partial A)$

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(1) $\partial A = \overline{A} \cap \overline{A^c} = \overline{A^c} \cap \overline{A} = \partial(A^c)$.
(2) Let $a \in \partial(\partial A) = \overline{\partial A} \cap \overline{(\partial A)^c} = \partial A \cap \overline{(\partial A)^c}$, then it is clear that $a \in \partial A$ as well. Thus $\partial(\partial A) \subset \partial A$.
(3) **First inclusion:** $\partial(A \cup B) = \overline{A \cup B} \cap \overline{(A \cup B)^c} = (\overline{A} \cup \overline{B}) \cap \overline{A^c \cap B^c}$. Let $x \in \partial(A \cup B)$. If $x \in \overline{A}$, then it must also be an element of $\overline{A^c \cap B^c} \subset \overline{A^c}$. This means $x \in \overline{A} \implies x \in \overline{A} \cap \overline{A} = \partial A$. Similarly, if $x \in \overline{B}$, then it is also in ∂B. Thus $\partial(A \cup B) \subset \partial A \cup \partial B$.
Second inclusion: We start by proving a helpful implication. We can show by contrapositive that $x \in \partial A \cup \partial B \implies x \in \overline{A \cup B}$. Assume $x \notin \overline{A \cup B} = \overline{A} \cup \overline{B}$, then x is not in $\overline{A}$ or $\overline{B}$, which further implies that x is not in ∂A or ∂B. This shows $x \in \partial A \cup \partial B \implies x \in \overline{A \cup B}$.
Using this implication, we can prove the desired inclusion. Suppose $x \in \partial A \cup \partial B$ such that $x \notin \partial(A \cup B)$ (if this is not possible, then $\partial(A \cup B) = \partial A \cup \partial B$, in which case the desired inclusion is trivial). Then by using the identity $\partial(A \cup B) = \overline{A \cup B} \cap \overline{(A \cup B)^c}$ and the implication we just proved, we know that the only way

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x is not in $\partial(A \cup B)$ is if $x \notin \overline{(A \cup B)^c}$. This means $x \in \overline{(A \cup B)^c}^c = (A \cup B)^o \subset A \cup B \subset \partial(A \cup B) \cup A \cup B$. This is the desired result.

(4) For any closed set A, $\overline{A} = A$. We can use this to simplify the definition of the boundary of a boundary, since the boundary is defined to be the intersection of two closed sets and is thus also closed.

$$\partial(\partial A) = \overline{\partial A} \cap \overline{(\partial A)^c} = \partial A \cap \overline{(\partial A)^c}$$

Moreover, since $\partial\partial A$ is also closed, we can do something similar for $\partial\partial\partial A$ and show that it reduces to this same quantity.

$$\begin{aligned}
\partial(\partial(\partial A)) &= \overline{\partial(\partial A)} \cap \overline{(\partial(\partial A))^c} \\
&= \partial(\partial A) \cap \overline{(\partial A)^c \cup \partial A} \\
&= \partial A \cap \overline{(\partial A)^c} \cap \left(\overline{(\partial A)^c} \cup \overline{\partial A}\right) \\
&= \partial A \cap \overline{(\partial A)^c} \cap \left(\overline{(\partial A)^c} \cup \partial A\right) \\
&= \partial A \cap \overline{(\partial A)^c} \\
&= \partial(\partial A)
\end{aligned}$$

which is the desired equality.

Exercise 2.20. *For a set A in a metric space M and $x \in M$, let*

$$d(x, A) = \inf \{d(x, y) \mid y \in A\},$$

and for $\varepsilon > 0$, let $D(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\}$.

(1) *Show that $D(A, \varepsilon)$ is open.*
(2) *Let $A \subset M$ and $N_\varepsilon = \{x \in M \mid d(x, A) \leq \varepsilon\}$, where $\varepsilon > 0$. Show that N_ε is closed and that A is closed if and only if $A = \cap \{N_\varepsilon \mid \varepsilon > 0\}$.*

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(1) The ε-ball around a point $a \in A$ is $D(a, \varepsilon) = \{x \in M \mid d(x, a) < \varepsilon\}$, so we can write $D(A, \varepsilon)$ as

$$D(A, \varepsilon) = \bigcup_{a \in A} D(a, \varepsilon).$$

Since the union of an arbitrary colleciton of open sets is itself open, it suffices to prove that $D(a, \varepsilon)$ is open. It is known that ε-balls are open, and $D(a, \varepsilon)$ is a specific instance of an ε-ball around a, so the desired result immediately follows.

(2) First we show that N_ε^c is open. Let $x \in N_\varepsilon^c$, then the ε-ball $D(x, \varepsilon)$ lies entirely in N_ε^c. Then by definition, N_ε is closed. Now we can show that A is closed if and only if $A = \cap \{N_\varepsilon \mid \varepsilon > 0\}$.

Backward: Assume $A = \cap \{N_\varepsilon \mid \varepsilon > 0\}$. Since each N_ε is closed and the intersection of an arbitrary collection of closed sets is itself closed, A must be closed.

Forward: Assume A is closed, and let $a \in A$ be arbitrary. Then $a \in \cap_\varepsilon \{N_\varepsilon\}$ since $d(a, A) = 0 \leq \varepsilon$ for every $\varepsilon > 0$. This shows $A \subset \cap_\varepsilon \{N_\varepsilon\}$. We now show the reverse inclusion.

Let $n \in \cap_\varepsilon \{N_\varepsilon\}$, then we can show $n \in A$ by contradiction. Assume $n \notin A$. Since A is closed, it contains all its accumulation points, so $d(n, A) > 0$. Now let $\varepsilon < \delta$. If $n \in N_\varepsilon$, then $n \notin \{N_\varepsilon\}$ and, subsequently, $n \notin \cap_\varepsilon \{N_\varepsilon\}$. This is a contradiction, so n must be in A. Thus $\cap_\varepsilon \{N_\varepsilon\} \subset A$.

These two inclusions show $A = \cap_\varepsilon \{N_\varepsilon\}$.

Exercise 2.21. *Prove that a sequence $\{x_k\}$ in a normed vector space is a Cauchy sequence if and only iff for every neighborhood U of 0, there is an N such that $k, l \geq N$ implies $x_k - x_l \in U$.*

Backward: Fix $\varepsilon > 0$, then consider the ε-ball $D(0, \varepsilon)$. Since every ε-ball is open, then by assumption there exists N such that if $k, l \geq N$, then $x_k - x_l \in D(0, \varepsilon)$. This implies $\|x_k - x_l\| < \varepsilon$, which shows that $\{x_n\}$ is a Cauchy sequence.

Forward: Since by assumption $\{x_n\}$ is a Cauchy sequence, we know that for every $\varepsilon > 0$, there exists N such that if $k, l > N$, then $\|x_k - x_l\| < \varepsilon$. This implies that the element $x_k - x_l$ is contained in $D(0, \varepsilon)$.

Now let U be an open set containing 0, then by definition there exists an $\varepsilon > 0$ such that $D(0, \varepsilon) \subset U$. We just showed that there exists some N such that if $k, l > N$, then $x_k - x_l \in D(0, \varepsilon)$ for all $\varepsilon > 0$, so clearly $x_k - x_l \in U$.

Exercise 2.28. *Give examples of:*

(1) *An infinite set in $\mathbb{R}$ with no accumulation points.*
(2) *A nonempty subset of $\mathbb{R}$ that is contained in its set of accumulation points.*
(3) *A subset of $\mathbb{R}$ that has infinitely many accumulation points but contains none of them.*
(4) *A set A such that $\partial A = \bar{A}$.*

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(1) An example is $\mathbb{Z}$. Take $D(z, 1/2)$ for any $z \in \mathbb{Z}$ to show this.
(2) An example is $[0, 1]$, since $\operatorname{acc}([0, 1]) = [0, 1]$.
(3) Let $A_b = \{b + 1/n \mid n \in \mathbb{N}\}$ and $A = \bigcup_{b \in \mathbb{Z}} A_b$. Each A_b has one accumulation point, namely b; however $b \notin A_b$ for any b. There are an infinite number of A_b's, so there are an infinite number of accumulation points in A, meaning we have a set with infinite accumulation points that contains none of them.
(4) An example is $\varnothing$. Let $A = \varnothing$, then $\overline{A} = \varnothing$ and $\partial A = \overline{A} \cap \overline{A^c} = \varnothing \cap \overline{A^c} = \varnothing$, so $\partial A = \overline{A}$.

Exercise 2.38. *Let $x_k \in \mathbb{R}^n$ satisfy $\|x_k - x_l\| \leq \frac{1}{k} + \frac{1}{l}$. Prove that x_k converges.*

Note that $\|x_n + x_{n+k}\| \leq \frac{1}{n} + \frac{1}{n+k} \leq \frac{2}{n}$ for any $k \geq 0$. Then to make $\|x_k + x_{n+k}\| < \varepsilon$ for some given $\varepsilon > 0$, we can find n such that $n > 2/\varepsilon$. Let $N > 2/\varepsilon$ be some integer, then for all $n > N$, we have $\|x_n - x_{n+k}\| < \varepsilon$ for all $k \geq 0$. Since this holds for arbitrary k, we have $\|x_k - x_l\| < \varepsilon$ for all $k, l > N$. This shows that $\{x_k\}$ is a Cauchy sequence. Since we are operating in $\mathbb{R}^n$, this is sufficient to show that $\{x_k\}$ converges.

Exercise 2.40. *Suppose in $\mathbb{R}$ that for all $n, a_n \leq b_n, a_n \leq a_{n+1}$, and $b_{n+1} \leq b_n$. Prove that a_n converges.*

Since $b_1 \geq b_n \geq a_n$ for all n, the sequence $\{a_n\}$ is bounded above by b_1. Since $\mathbb{R}$ satisfies the monotone convergence property and since $\{a_n\}$ is a monotone non-decreasing sequence, $\{a_n\}$ must converge.

Exercise 2.42. *Let $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Define $d(x, A) = \inf\{d(x, y) \mid y \in A\}$. Must there be a $z \in A$ such that $d(x, A) = d(x, z)$?*

No. Consider the open interval $A = (0, \infty) \subset \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ and the point $x = -1$. We claim $d(x, A) = 1$.

Since $a > 0$ for all $a \in A$, the distance between $x = -1$ and any point in A satisfies $d(-1, a) = |a + 1| \geq 1$. Thus 1 is a lower bound of $d(x, a)$. Now take $1 + \varepsilon$ for some

$\varepsilon > 0$. This cannot be a lower bound on $d(x, a)$ since for $\varepsilon/2 \in A$, the distance from x is $d(-1, \varepsilon/2) = |1 + \varepsilon/2| < 1 + \varepsilon$. Thus $d(x, A) = 1$.

However, the only points z that satisfy $d(x, z) = d(-1, z) = 1$ are -2 and 0, and neither of these points are in A.

Exercise 2.44. *A set $A \subset \mathbb{R}^n$ is said to be **dense** in $B \subset \mathbb{R}^n$ if $B \subset \bar{A}$. If A is dense in $\mathbb{R}^n$ and U is open, prove that $A \cap U$ is dense in U. Is this true if U is not open?*

Let $u \in U$ and let V be any open neighborhood of u. Then since U and V are both open, their intersection $U \cap V$ is also open. Then there eixsts $\varepsilon > 0$ such that $D(u, \varepsilon) \subset U \cap V$. Assume $A \cap (D(u, \varepsilon) \setminus \{u\}) = \varnothing$ for some ε, then let $y \in D(u, \varepsilon)$ be any member of this disk that is not equal to u. Since $D(u, \varepsilon)$ has an empty intersection with A, the point y is clearly not in A. It is also not an accumulation point of A, as the disk $D(y, \varepsilon - d(u, y))$ does not contain any points of A. This implies $y \notin \overline{A}$, which is a contradiction since we know $\mathbb{R}^n \subset \overline{A}$.

Thus by contradiction, $A \cap (D(u, \varepsilon) \setminus \{u\}) \neq \varnothing$ for any $\varepsilon > 0$. This implies $A \cap (U \cap V \setminus \{u\})$ is nonempty. Rearranged, this says $(A \cap U) \cap (V \setminus \{u\})$ is nonempty for any open neighborhood V of u. Then by definition, u is an accumulation point of $A \cap U$, meaning $u \in \overline{A \cap U}$. This implies $U \subset \overline{A \cap U}$, so $A \cap U$ is dense in U.

The crux of this proof relied on the observation that no open holes can exist in A. Thus if U had *not* been open, it could have been a subset of the holes that A does contain. Then the original claim "$A \cap U$ is dense in U" would become "the empty set is dense in a nonempty set U", which is clearly false. Thus the requirement that U be open was necessary.

Exercise 2.51.

(1) *If $u_n > 0, n = 1, 2, \ldots,$ show that*

$$\liminf \frac{u_{n+1}}{u_n} \leq \liminf \sqrt[n]{u_n} \leq \limsup \sqrt[n]{u_n} \leq \limsup \frac{u_{n+1}}{u_n}.$$

(2) *Deduce that if* $\lim(u_{n+1}/u_n) = A$, *then* $\limsup \sqrt[n]{u_n} = A$.

(3) *Show that the converse of part (b) is false by use of the seuqence $u_{2n} = u_{2n+1} = 2^{-n}$.*

(4) *Calculate* $\limsup \frac{\sqrt[n]{n!}}{n}$.

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(1) Let $L = \limsup \frac{u_{n+1}}{u_n}$. If $L = \infty$, then the desired inequality is trivial, so assume $L < \infty$. Now fix $\varepsilon > 0$, then there exists N such that $u_n < L + \varepsilon$ whenever $n \geq N$. Then for $n \geq N$,

$$\frac{u_n}{u_N} = \frac{u_n}{u_{n-1}} \frac{u_{n-1}}{u_{n-2}} \cdots \frac{u_{N+1}}{u_N} \leq (L + \varepsilon)^{n-N}$$

Rearranging gives

$$u_n \leq (L + \varepsilon)^n \frac{u_N}{(L + \varepsilon)^N}$$

$$\sqrt[n]{u_n} \leq (L + \varepsilon) \left(\frac{u_N}{(L + \varepsilon)^N} \right)^{1/n}$$

Since the fraction on the RHS is strictly greater than 0 and strictly less than 1, we get the final inequality

$$\sqrt[n]{u_n} \leq (L + \varepsilon)$$

Taking the limit superior of both sides then gives

$$\limsup \sqrt[n]{u_n} \leq (L + \varepsilon) = \limsup \frac{u_{n+1}}{u_n}$$

Similarly, we can show

$$\liminf \frac{u_{n+1}}{u_n} \leq (L+\varepsilon) = \liminf \sqrt[n]{u_n}$$

Combining these two inequalities with the fact that $\liminf x_n \leq \limsup x_n$ for any sequence x_n, we get the desired inequality

$$\liminf \frac{u_{n+1}}{u_n} \leq \liminf \sqrt[n]{u_n} \leq \limsup \sqrt[n]{u_n} \leq \limsup \frac{u_{n+1}}{u_n}.$$

(2) $\lim(u_{n+1}/u_n) = A$ if and only if $\liminf(u_{n+1}/u_n) = \limsup(u_{n+1}/u_n) = A$, so we can use the inequality from part (a) to get

$$A \leq \limsup \sqrt[n]{u_n} \leq A$$

which implies $\limsup \sqrt[n]{u_n} = A$.

(3) For this series, $\sqrt[n]{u_n} = 1/2$ for all n, so clearly $\sup \sqrt[n]{u_n} = 1/2$. However, u_{n+1}/u_n alternates between $1/2$ and 1, so there is clearly no limit. Thus the converse of part (b) is false.

(4) Let $x_n = \frac{n!}{n^n}$, then

$$\begin{aligned}
\lim_{n\to\infty} \frac{x_{n+1}}{x_n} &= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} \\
&= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}
\end{aligned}$$

By definition, this is $1/e$. Then by the result from part (b), we know

$$\limsup \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$