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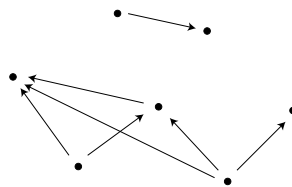
Chapter 1

The Basics

1.1 CATEGORIES

Definition 1. A **category** \mathcal{C} is a collection of **objects** $\text{ob}(\mathcal{C})$ and **morphisms** $\text{mor}(\mathcal{C})$. There are several requirements:

1. Morphisms must compose: $(f, g) \mapsto gf$.
2. Morphism composition is associative.
3. If $A \neq C$ or $B \neq D$, then $\text{Hom}_{\mathcal{C}}(A, B)$ and $\text{Hom}_{\mathcal{C}}(C, D)$ are disjoint sets.
4. Each object has an identity morphism, which is a two-sided identity.



A category is **concrete** if, informally, its objects are underlying sets and its morphisms are functions between them, e.g. **Set**, **Top**, **Grp**. **Abstract** categories don't have this structure, e.g. BG for a group G .

A category is **discrete** if all its morphisms are identities, i.e. all its objects are isolated.

Because of set-theoretical issues, it's useful to denote when a category is "small enough". We say a category is **small** if it has only a set's worth of morphisms. Since

$$\text{identity morphisms} \leftrightarrow \text{objects},$$

small categories also have a set's worth of objects. We can loosen this somewhat: if $\text{Hom}(X, Y)$ is always a set, the category is **locally small**.

Proposition 1. Identity morphisms and morphism inverses are unique.

Definition 2. An **isomorphism** is an invertible morphism.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \xleftarrow{f^{-1}} & \end{array}$$

Isomorphisms (isos) generalize bijective functions, which are both injective and surjective. Injective functions generalize to monomorphisms (monos), and surjective functions to epimorphisms (epis).

Definition 3. A morphism f is a **monomorphism** if for all parallel (between same objects) morphisms g, h with the proper domains,

$$fg = fh \implies g = h.$$

Similarly, f is an **epimorphism** if

$$gf = hf \implies g = h.$$

There's some fun vocab and symbols to go along with these. Monos are monic and denoted by \rightarrowtail , and epis are epic and denoted by \twoheadrightarrow . An isomorphism is necessarily both monic and epic, although the converse doesn't hold in general.

Special types of morphisms get their own special names sometimes too. An **endomorphism** is a morphism $X \rightarrow X$. An isomorphic endomorphism is called an **automorphism**.

Definition 4. A category \mathbf{S} is a **subcategory** of \mathbf{C} if

1. $\text{ob}(\mathbf{S})$ is a subcollection of $\text{ob}(\mathbf{C})$; and
2. for all $A, B \in \text{ob}(\mathbf{S})$, $\text{Hom}_{\mathbf{S}}(A, B)$ is a subcollection of $\text{Hom}_{\mathbf{C}}(A, B)$ with identity.

A **full** subcategory doesn't remove any morphisms between the remaining objects, i.e. $\text{Hom}_{\mathbf{S}}(A, B) = \text{Hom}_{\mathbf{C}}(A, B)$.

Definition 5. A **groupoid** is a category whose morphisms are all isomorphisms.

Every category contains a subcategory called the **maximal groupoid**, which is all of the objects along with only the morphisms that are isomorphisms.

Example 1. We can define a **group** as a groupoid that has only one object. The group elements are the morphisms. The properties of a group follow from the properties of categories and the fact that our morphisms are all isomorphisms.

Given a group G , its representation as a single-object category is denoted BG .

1.2 DUALITY

Definition 6. Given a category \mathbf{C} , its **opposite** or **dual** category \mathbf{C}^{op} is the category gotten by “reversing” the morphisms of \mathbf{C} . This means

$$\begin{aligned}\text{ob}(\mathbf{C}^{\text{op}}) &= \text{ob}(\mathbf{C}), \\ \text{Hom}_{\mathbf{C}^{\text{op}}}(A, B) &= \text{Hom}_{\mathbf{C}}(B, A).\end{aligned}$$

My biggest misconception of this at first was that we were actually reversing each morphism, but this is clearly impossible. For example, if we’re working in **Set**, we physically can’t reverse all the morphisms since not all functions are invertible.

Note 1. We aren’t actually changing any of the morphisms. The “reversal” of a morphism is a completely formal process. In fact, we can’t even compare f and f^{op} since they live in different categories! At the end of the day, a category’s dual has the same information, but the notation is just all reversed.

Note that the identities in a category and its dual are the same. Compositions, on the other hand, are necessarily reversed:

$$f^{\text{op}}g^{\text{op}} \doteq (gf)^{\text{op}}.$$



Duals are important because they make universal quantifications twice as valuable: if a theorem applies “for all categories”, then it certainly applies to the opposites of all categories. We can then reinterpret the theorem in the opposite case to get a dual theorem, and to prove it we just reverse all the morphisms in our original proof.

1.3 FUNCTORS

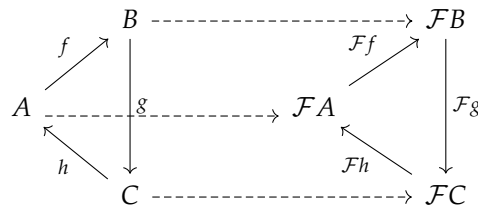
Functors are the morphisms associated with categories: they map categories to categories in ways that respect categorical structure.

Definition 7. A (covariant) functor $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{D}$ satisfies:

- If $A \in \mathbf{C}$, then $\mathcal{F}A \in \mathbf{D}$.
- If $f : A \rightarrow B$, then $\mathcal{F}f : \mathcal{F}A \rightarrow \mathcal{F}B$.

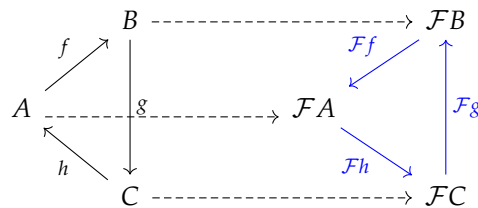
These are subject to the functoriality axioms:

- $\mathcal{F}(fg) = \mathcal{F}f \cdot \mathcal{F}g$ for all f, g .
- $\mathcal{F}1_A = 1_{\mathcal{F}A}$ for all A .



Definition 8. A contravariant functor is the same but with the morphisms $\mathcal{F}f$ reversed. We can represent this by a standard functor but with a different domain:

$$\mathcal{F} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}.$$



Example 2. Some functors :)

1. Forgetful functors.
2. $\mathbf{Top} \rightarrow \mathbf{Htpy}$ is the identity on objects (topological spaces) and sends morphisms (continuous functions) to their homotopy class.
3. π_1 is a functor $\mathbf{Top}_* \rightarrow \mathbf{Grp}$.

Proposition 2. Functors preserve isos and split monos/epis.

Definition 9. A functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is **faithful** if for all objects A, B of \mathcal{C} , the map

$$\begin{aligned} \text{Hom}(A, B) &\rightarrow \text{Hom}(\mathcal{F}A, \mathcal{F}B) \\ f &\mapsto \mathcal{F}f \end{aligned}$$

is one-to-one. \mathcal{F} is **full** if this map is onto.

Note that the fixed A and B above are important. The injective/surjective conditions don't apply to arbitrary morphisms in \mathcal{C} since they might connect different objects.

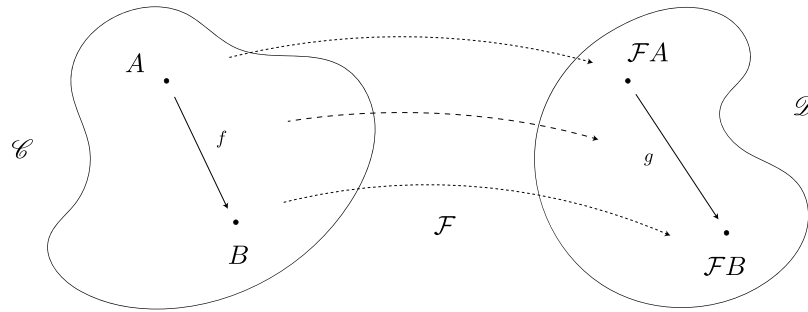
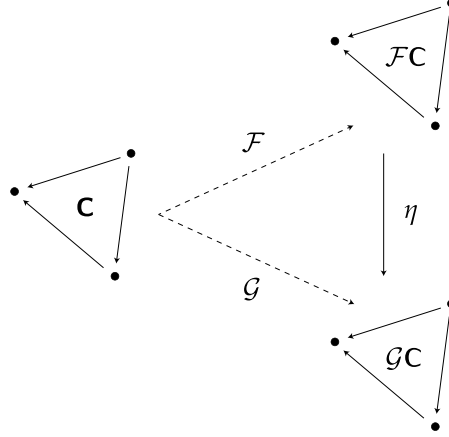


Figure 1.1: For all A, B , and g , a faithful functor sends at *most* one solid arrow in \mathcal{C} to g . A full functor sends at *least* one solid arrow in \mathcal{C} to g .

Example 3. The inclusion functor from \mathcal{S} to \mathcal{C} is always faithful, and it's full if and only if \mathcal{S} is a full subcategory.

1.4 NATURAL TRANSFORMATIONS

Natural transformations change one functor into another in a way that respects the underlying structure of the categories involved.



Definition 10. Suppose $\mathcal{F}, \mathcal{G} : \mathbf{C} \rightarrow \mathbf{D}$ are functors. Then a **natural transformation** $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$ is a family of **components**

$$\{\eta_X : \mathcal{F}X \rightarrow \mathcal{G}X\}_X$$

such that the following diagram commutes for any $f : X \rightarrow Y$ in \mathbf{C} .

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{\eta_X} & \mathcal{G}X \\ \mathcal{F}f \downarrow & & \downarrow \mathcal{G}f \\ \mathcal{F}Y & \xrightarrow{\eta_Y} & \mathcal{G}Y \end{array}$$

If every η_X is an isomorphism, then η is a **natural isomorphism** and we write $\eta : \mathcal{F} \cong \mathcal{G}$.