CONTENTS

1 CHAIN COMPLEXES

1.1 CHAIN COMPLEXES

Definition 1. A **chain complex** $\mathcal{C}$ is a sequence of R-morphisms

$$\cdots \xrightarrow{d_{i+2}} C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots$$

such that $d^2 = 0$ for all i. **Cochain complexes** are the same, except the boundary maps take you up a level instead of down.

$$\cdots \xrightarrow{d_{i-1}} C^{i-1} \xrightarrow{d_i} C^i \xrightarrow{d_{i+1}} C^{i+1} \xrightarrow{d_{i+2}} \cdots$$

The map d_i is the **boundary operator**, as it is a generalization of the geometric concept of a boundary (note $d^2 = 0$). Thus an element of $\operatorname{Im} d$ is a **boundary**. Since usual geometric cycles have no boundary, we call the elements of $\operatorname{Ker} d$ **cycles**.

Example 1. Chain complexes generalize the concept of boundaries to objects that don't necessarily have clear cyclic geometric properties. Let $\Omega_n(M)$ denote the space of differential n-forms on a manifold M, then we have a cochain complex

$$\Omega_0(M) \xrightarrow{d} \Omega_1(M) \xrightarrow{d} \Omega_2(M) \xrightarrow{d} \cdots$$

where d is the exterior derivative. From this we see that the cycles of $\Omega_0(M)$ (the space of differentiable functions on M) are the constant functions.

A **morphism of complexes/chain morphism** $f : \mathcal{C} \to \mathcal{D}$ is a sequence of morphisms $f_i : C_i \to D_i$ respecting the boundary map, i.e. making the following diagram commute.

$$\begin{tikzcd} C_i \arrow[r, "d_{\mathcal{C}}"] \arrow[d, "f_i"'] & C_{i-1} \arrow[d, "f_{i-1}"] \\ D_i \arrow[r, "d_{\mathcal{D}}"'] & D_{i-1} \end{tikzcd}$$

1.2 CHAIN HOMOTOPIES

Definition 2. Given two chain complexes $\mathcal{A}, \mathcal{B}$, two chain morphisms $f, g : \mathcal{A} \to \mathcal{B}$ are **(chain) homotopic**, written $f \simeq g$, if there are morphisms $s_i : A_i \to B_{i-1}$ such that

$$d's + sd = f - g.$$

If $\mathcal{A}, \mathcal{B}$ are cochain complexes instead, then $s_i : A_i \to B_{i+1}$.

$$\begin{tikzcd}
A_{i-1} \arrow[r, "d"] & A_i \arrow[r, "d"] \arrow[dl, dashed, "s_i"'] \arrow[d, shift right, "f_i"'] \arrow[d, shift left, "g_i"] & A_{i+1} \arrow[dl, dashed, "s_{i+1}"] \\
B_{i-1} \arrow[r, "d'"'] & B_i \arrow[r, "d'"'] & B_{i+1}
\end{tikzcd}$$

Motivation for this?

Definition 3. A chain morphism $f : \mathcal{A} \to \mathcal{B}$ is a **homotopy equivalence** if there's another chain morphism $g : \mathcal{B} \to \mathcal{A}$ such that $fg \simeq 1_B$ and $gf \simeq 1_A$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "f"] & \mathcal{B} \arrow[l, bend left, "g"]
\end{tikzcd}$$

Proposition 1. Additive functors preserve homotopy equivalence.

Proof. Let $f \simeq g$. If $\mathcal{F}$ is additive and covariant, then $d's + sd = f - g \implies \mathcal{F}(d')\mathcal{F}(s) + \mathcal{F}(s)\mathcal{F}(d) = \mathcal{F}f - \mathcal{F}g$. Thus $\mathcal{F}f \simeq \mathcal{F}g$. If $\mathcal{G}$ is additive and contravariant, then $\mathcal{G}(d)\mathcal{G}(s) = \mathcal{G}(s)\mathcal{G}(d') = \mathcal{G}f - \mathcal{G}g$. Since all the arrows are reversed, the LHS is the right form, so $\mathcal{G}f \simeq \mathcal{G}g$. $\square$

1.3 HOMOLOGY

Definition 4. The n-th **homology group** $H_n(\mathcal{C})$ of a chain complex $\mathcal{C}$ is the kernel of the map going out of C_n quotiented by the image of the map coming into C_n. Cochain complexes have **cohomology groups** $H^n(\mathcal{C})$ instead.

Proposition 2. A chain/cochain complex is exact $\iff$ all its homology/cohomology groups are trivial.

Thus the (co)homology groups of a (co)chain complex measure how much it fails to be exact.

Proposition 3. A morphism of complexes $f : \mathcal{A} \to \mathcal{B}$ induces group morphisms between the complexes' homology/cohomology groups given by $[a] \mapsto [f_n(a)]$.

$$\begin{tikzcd} A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d] & \rightsquigarrow & H^n(\mathcal{A}) \arrow[d] \\ B^{n-1} \arrow[r] & B^n \arrow[r] & B^{n+1} & & H^n(\mathcal{B}) \end{tikzcd}$$

Proof. This follows from the morphism of complexes respecting the boundary map and thus mapping the kernels and images of the first complex to the kernels and images of the second. □

Proposition 4. If f_* is the induced (co)homology map of f, then $(gf)_* = g_* f_*$.

Definition 5. $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a **short exact sequence of complexes** if each $0 \to A^n \to B^n \to C^n \to 0$ is short exact.

Lemma 1 (Snake Lemma)**.** If the following diagram has exact rows,

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r, two heads] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, tail] & B' \arrow[r] & C' & \end{tikzcd}$$

then there is an induced exact sequence

$$\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma.$$

Theorem 1 (Long Exact Sequence in Cohomology)**.** If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence of complexes, then there is a long exact sequence of cohomologies

$$\begin{aligned} 0 \to H^0(\mathcal{A}) &\to H^0(\mathcal{B}) \to H^0(\mathcal{C}) \\ \to H^1(\mathcal{A}) &\to H^1(\mathcal{B}) \to H^1(\mathcal{C}) \\ \to H^2(\mathcal{A}) &\to \cdots \end{aligned}$$

where the morphisms $H^n(\mathcal{C}) \to H^{n+1}(\mathcal{A})$ are the **connecting morphisms**.

Proof. **Intuition? Use snake lemma (have proof of this in spectral sequences paper).** □

Corollary 1. If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is exact and any 2 of the complexes are exact themselves, then so is the third.

Proof. The LES of cohomologies becomes all 0, except for each $H^n(\mathcal{X})$, where $\mathcal{X}$ is the third complex. Now $0 \to H^n(\mathcal{X}) \to 0$ exact $\implies H^n(\mathcal{X}) \cong 0$, so $\mathcal{X}$ is exact. □

Definition 6. A morphism of complexes is a **quasi-isomorphism** if the (co)homology maps it induces are all iso.

Lemma 2. If $f \simeq g$, then they induce the same (co)homology maps, i.e. $f_* = g_*$.

Proof. Suppressing subscripts, suppose $f = d's + sd$, then the induced map is

$$[a] \mapsto [f(a)] = [(d's)(a) + (sd)(a)] = [d'(s(a)) + s(0)] = [0].$$

Then if $f \simeq g$, we have $[f(a)] = [(g + d's + sd)(a)] = [g(a)]$. □

Proposition 5. A homotopy equivalence is a quasi-iso.

Proof. Suppose f and g are inverse chain homotopies, then by the lemma, $f_* g_* = (fg)_* = (1_B)_* = 1_{H(\mathcal{B})}$ and, similarly, $g_* f_* = 1_{H(\mathcal{A})}$. Thus $H^n(\mathcal{A}) \cong H^n(\mathcal{B})$ for all n. □

2 DERIVED FUNCTORS

2.1 RESOLUTIONS

Definition 7. Suppose A is an R-module, then a **projective resolution** over A is an exact sequence of projective R-modules

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

and a **injective resolution** over A is an exact sequence of injective R-modules

$$0 \longrightarrow A \xrightarrow{\varepsilon} I_0 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} I_{n-1} \xrightarrow{d_n} I_n \longrightarrow \cdots.$$

Theorem 2 (Existence)**.** Every R-module has a projective **and injective** resolution.

Proof. Let A be an R-module, and let P_0 be free on A. Then there's a unique R-morphism $\varepsilon : P_0 \to A$ extending 1_A. It's clearly epic, so $P_0 \overset{\varepsilon}{\to} A \to 0$ is exact. Now let P_1 be free on $\operatorname{Ker}\varepsilon$, then there's a unique R-morphism $d_1 : P_1 \twoheadrightarrow \operatorname{Ker}\varepsilon$, so $P_1 \overset{d_1}{\to} P_0 \overset{\varepsilon}{\to} A$ is exact. Continue inductively, with P_{n+1} free on $\operatorname{Ker} d_n$. Since each P_i is free, each is projective. $\square$

Proposition 6. R-morphisms lift to morphisms of projective/injective resolutions that are unique up to chain homotopy.

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dashed, "f_1"] & P_0 \arrow[r] \arrow[d, dashed, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "g"] & I_0 \arrow[r] \arrow[d, dashed, "g_0"] & I_1 \arrow[r] \arrow[d, dashed, "g_1"] & \cdots \\
0 \arrow[r] & A' \arrow[r] & I_0' \arrow[r] & I_1' \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Inductively use the fact that all the P_i, P_i' are projective and I_i, I_i' are injective to get each f_n and g_n. **Chain homotopy bit.** □

Corollary 2. Any two projective/injective resolutions of the same module are homotopy equivalent.

Proof. Consider the following lifts to projective resolutions (the case of injective resolutions is similar).

$$\begin{tikzcd}
\mathcal{P} \arrow[r] \arrow[d, dashed, "f_\bullet"] & A \arrow[r] \arrow[d, "1_A"] & 0 \\
\mathcal{P}' \arrow[r] \arrow[d, dashed, "g_\bullet"] & A \arrow[r] \arrow[d, "1_A"] & 0 \\
\mathcal{P} \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

The identity map is a valid candidate for each $g_n f_n$, so $g_n f_n \simeq 1$. Now flip the diagram upside down and use the same f_n and g_n maps, yielding $f_n g_n \simeq 1$. □

Corollary 3. Any two projective/injective resolutions have isomorphic (co)homology groups.

Proof. They're homotopy equivalent, so they're quasi-isomorphic by Proposition 5. □

2.2 DERIVED FUNCTORS

Suppose we apply an additive functor $\mathcal{F}$ to some projective/injective resolution of a module A. The (co)homology groups of the resulting complex are unique up to isomorphism:

- Let $\mathcal{X}, \mathcal{Y}$ be projective/injective resolutions of A, then they're homotopy equivalent by Corollary 2.
- By Proposition 1, additive functors preserve homotopy equivalence, so $\mathcal{F}\mathcal{X}$ and $\mathcal{F}\mathcal{Y}$ are also homotopy equivalent.
- By Proposition 5, homotopy equivalent complexes have isomorphic (co)homology groups.

Thus we can define the (co)homology groups of the sequence gotten by applying an additive functor $\mathcal{F}$ to a projective/injective resolution of A using *any* resolution.

Definition 8. Let $\mathcal{F}$ be a functor and A an R-module, then choose a resolution of A from the following chart based on $\mathcal{F}$.

Left exact, covariant	injective
Left exact, contravariant	projective
Right exact, covariant	projective
Right exact, contravariant	injective

Apply $\mathcal{F}$ to the resolution, remove the $\mathcal{F}A$ term from it, then take (co)homologies. If $\mathcal{F}$ is left exact, the cohomologies $R^i\mathcal{F}$ are the **right derived functors** of $\mathcal{F}$. If $\mathcal{F}$ is right exact, the homologies $L_i\mathcal{F}$ are the **left derived functors** of $\mathcal{F}$.

I don't think the derived functors depend on A at all, they can just be applied to A, etc. to get new objects...

With left exact functors, we end up with induced sequences of the form

$$0 \to \mathcal{F}X_0 \to \mathcal{F}X_1 \to \mathcal{F}X_2 \to \cdots,$$

thus why the derived functors are "right". Similarly, for right exact functors, we end up with induced sequences of the form

$$\mathcal{F}X_2 \to \mathcal{F}X_1 \to \mathcal{F}X_0 \to 0,$$

thus why the derived functors are "left". As examples, see the next two propositions.

Can you get both sets of sequences at once if $\mathcal{F}$ is exact?

They're actually functors.... b/c (co)homology is a functor.

More detail about *why* we choose inj or proj resolution.

Proposition 7. If $\mathcal{F}$ is left exact, then $R^0\mathcal{F} = \mathcal{F}$.

Proof. **Covariant:** If $0 \to A \overset{f}{\rightarrowtail} I_0 \xrightarrow{g} I_1$ is exact, then so is $0 \to \mathcal{F}A \overset{\mathcal{F}f}{\rightarrowtail} \mathcal{F}I_0 \xrightarrow{\mathcal{F}g} \mathcal{F}I_1$. Then $R^0\mathcal{F}(A) = \operatorname{Ker}(\mathcal{F}g) = \operatorname{Im}(\mathcal{F}f) \cong \mathcal{F}(A)$ (by the 1st iso theorem since $\mathcal{F}f$ monic).

Contravariant: Use a projective resolution instead. The process is the same. ☐

Proposition 8. If $\mathcal{F}$ is right exact, then $L_0\mathcal{F} = \mathcal{F}$.

Proof. **Covariant:** If $P_1 \xrightarrow{f} P_0 \overset{g}{\twoheadrightarrow} A \to 0$ is exact, then so is $\mathcal{F}P_1 \xrightarrow{\mathcal{F}f} \mathcal{F}P_0 \overset{\mathcal{F}g}{\twoheadrightarrow} \mathcal{F}A \to 0$. Then $L_0\mathcal{F}(A) = \frac{\mathcal{F}P_0}{\operatorname{Im}\mathcal{F}f} = \frac{\mathcal{F}P_0}{\operatorname{Ker}\mathcal{F}g} \cong \mathcal{F}A$ (by the 1st iso theorem since $\mathcal{F}g$ epic).

Contravariant: Use an injective resolution instead. The process is the same. ☐

Does "left derived functor of left exact functor" make sense? Are they all just trivial or something? To prove something like that, would you use a left exact variant of "exact functors preserves LES's"?

Theorem 3 (LES of derived functors)**.** **Do this.**

If derived functors measure the extent to which a functor fails to be exact, then an exact functor should have trivial derived functors. This turns out to be true.

Proposition 9. If $\mathcal{F}$ is exact, then $R^i\mathcal{F} = L_i\mathcal{F} = 0$ for all $i > 0$.

Proof. **Covariant:** Exact functors preserve exactness, so $0 \to A \rightarrowtail I_0 \to I_1 \to \cdots$ exact implies $0 \to \mathcal{F}A \to \mathcal{F}I_0 \to \mathcal{F}I_1 \to \cdots$ exact. Chopping off the $\mathcal{F}A$ term and taking cohomologies gives $L_i\mathcal{F} = 0$ when $i > 0$. Now repeat the argument with a projective resolution for the R^i.

Contravariant: Similar argument. ☐

Proposition 10. Fix a functor $\mathcal{F}$. If A is projective/injective (depending on the type of $\mathcal{F}$), then $R^i\mathcal{F}(A)$ or $L_i\mathcal{F}(A)$ (whichever is correct for $\mathcal{F}$) is trivial when $i > 0$.

Proof. We consider the case when $\mathcal{F}$ is left exact and covariant, but the other three cases are similar. Suppose A is injective, then $0 \to A \xrightarrow{\text{id}} A \to 0$ is an injective resolution of A. This induces the exact sequence $0 \to \mathcal{F}A \xrightarrow{\text{id}} \mathcal{F}A \to 0$. Chopping off the first $\mathcal{F}A$ term and taking cohomologies gives $R^i\mathcal{F}(A) = 0$. ☐

2.3 THE EXT FUNCTOR

Note 1. Big idea: the hom functors are left exact, but we can use cohomology to measure how much they fail to be right exact.

Definition 9. The **Ext functors** are the (right) derived functors of the hom functors.

Given an R-module A, there are two equivalent ways to construct them:

1. **Using** $\mathrm{Hom}(-, M)$ **(contravariant):** Take a projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

   and apply $\mathrm{Hom}(-, M)$ to it. Removing $\mathrm{Hom}(A, M)$ from the sequence gives

$$0 \longrightarrow \mathrm{Hom}(P_0, M) \xrightarrow{d_1^*} \mathrm{Hom}(P_1, M) \xrightarrow{d_2^*} \cdots .$$

   This is a cochain complex since for any map f, applying d^* twice gives $d^{*2}(f) = fd^2 = 0$. Then $\mathrm{Ext}^n_R(A, M)$ is the n-th cohomology group of this complex.

2. **Using** $\mathrm{Hom}(M, -)$ **(covariant):** Take an injective resolution, apply $\mathrm{Hom}(M, -)$, remove $\mathrm{Hom}(M, A)$, then take homology. **Is this actually equivalent?**

Proposition 11. $\mathrm{Ext}^0_R(A, M) \cong \mathrm{Hom}_R(A, M)$.

Proof. The hom functors are left exact, so apply Proposition 7. □

Ext(A,B) is contra in A, cov in B.