0.1 Notation

Any probability p is implicitly a function of n, so keep this in mind when taking limits as $n \to \infty$. You can't just treat p like a constant.

The $O, \Omega, \Theta, o, \omega$ notation is all standard. If

$$\lim_{n\to\infty} \frac{A_n}{B_n} = 1$$

then we say $A_n \approx B_n$. We abuse notation a bit by saying $A_n \le B_n$ if there is some constant c such that $A_n \le cB_n$ for all n.

0.2 Erdos-Renyi Random Clique Complexes

Important. In the Erdos-Renyi clique complex, if p is in a certain regime, the betti numbers will be nonzero with high probability.

Lemma 1 (Morse Inequalities)**.** Change name? I dont think "Morse inequalities" is standard. Let f_k denote the number of k-dimensional faces of a simplicial complex Δ, and let β_k denote the k-th Betti number of Δ. Then

$$-f_{k-1} + f_k - f_{k+1} \le \beta_k \le f_k.$$

Proof. Do this. Uses definition of simplicial homology and the rank nullity theorem. □

Theorem 1. *Suppose $p = \omega(n^{-1/k})$ and $p = o(n^{-1/(k+1)})$, then*

$$\lim_{n\to\infty} \frac{\mathbb{E}[\beta_k]}{n^{k+1} p^{\binom{k+1}{2}}} = \frac{1}{(k+1)!}.$$

Proof. The desired limit relation is actually straightforward to show for f_k instead of β_k. So we'll do that, then use our assumptions on p and the Morse inequalities to show that β_k has the same property.

Note that f_k also represents the number of $(k+1)$-cliques of our complex. Since the complex is Erdos-Renyi, each of the $\binom{n}{k+1}$ possible $(k+1)$-cliques occur with the same probability. Since a $(k+1)$-clique has $\binom{k+1}{2}$ distinct edges, this probability is $p^{\binom{k+1}{2}}$.

But f_k is really just the sum of $\binom{n}{k+1}$ indicator functions, each tracking whether or not a particular $(k+1)$-clique is present in the complex. Since each clique has equal probability of forming, this means the expectation of f_k is

$$\mathbb{E}[f_k] = \binom{n}{k+1} p^{\binom{k+1}{2}} = \frac{n!}{(n-k-1)!(k+1)!} p^{\binom{k+1}{2}}.$$

The limit of our desired quantity, but with f_k substituted in place of β_k, is then

$$\lim_{n\to\infty} \frac{\mathbb{E}[f_k]}{n^{k+1} p^{\binom{k+1}{2}}} = \lim_{n\to\infty} \frac{n \cdots (n-k)}{n^{k+1}(k+1)!} = \frac{1}{(k+1)!}.$$

We can then use our assumption on the regime of p and the Morse inequalities to show that β_k has the same property. Since $p = \omega(n^{-1/k})$,

$$\lim_{n\to\infty} \frac{\mathbb{E}[f_{k-1}]}{\mathbb{E}[f_k]} = \lim_{n\to\infty} \frac{k+1}{np^k} = 0.$$

And since $p = o(n^{-1/(k+1)})$, we similarly have $\lim_{n\to\infty} \mathbb{E}\left[f_{k+1}\right] / \mathbb{E}\left[f_k\right] = 0$. Thus in this regime of p,

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[-f_{k-1} + f_k - f_{k+1}\right]}{\mathbb{E}\left[f_k\right]} = 1.$$

Then by the Morse inqualities, β_k must satisfy the desired limit property. □

Now we know that β_k is very likely nonzero (at least in our specific regime of p). Since it's not just some trivial quantity, proving a central limit theorem for it is a nontrivial statement.

Theorem 2. *Suppose $p = \omega(n^{-1/k})$ and $p = o(n^{-1/(k+1)})$, then*

$$\frac{\beta_k - \mathbb{E}\left[\beta_k\right]}{\sqrt{\mathrm{Var}(\beta_k)}}.$$

Proof. Do this. □