PERCOLATION PROCESSES ON DYNAMICALLY GROWN GRAPHS

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Math 493: Research Independent Study

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Abstract

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1 INTRODUCTION

Intro to random graphs

1.1 PERCOLATION

General percolation
History of explosive percolation

1.2 THE SCALING ASSUMPTION

scaling assumption
critical exponents
induced coefficient maps

2 2-CHOICE RULES

To begin, we will need to define some basic terms that will be used over and over again. Let S be the relative size of the dominating cluster as $n \to \infty$. If x_i is a vertex, then we denote its absolute cluster size by κ_i. Denote the probability that the minimum of m i.i.d. sampled vertices is s by

$$Q_m(s) := \mathbb{P}\left(\min\left\{\kappa_1, \ldots, \kappa_m\right\} = s\right).$$

Consider the sum $\langle 1\rangle_m := \sum_s Q_m(s)$, where s implicitly ranges over only finite values. Since S is the probability that a randomly chosen vertex belongs to an infinite cluster, we can write this sum as $\langle 1\rangle_m = 1 - S^m$. Since they frequently show up in common examples, we give $m = 1$ and $m = 2$ shorthands:

$$P := Q_1, \qquad Q := Q_2.$$

We also define

$$\langle s^k\rangle_m := \sum_{s=1}^{\infty} s^k Q_m(s).$$

We will use $\langle \cdot \rangle_P$ and $\langle \cdot \rangle_Q$ instead of $\langle \cdot \rangle_1$ and $\langle \cdot \rangle_2$, respectively.

With these definitions in hand, we can turn our attention to the main attraction. We will be discussing rules that add a single edge every $t = 1/n$ units of time, gotten by randomly sampling two groups of vertices i.i.d. from the graph, then choosing an endpoint vertex from each group. The following definition is a straightforward generalization of this idea, but we will only concern ourselves in this work with the case $\ell = 2$.

Definition 1. Define a rule $\mathcal{R}$ as follows:

- Every $t = 1/n$ units of time, choose ℓ groups of vertices $\mathcal{V}_1, \ldots, \mathcal{V}_\ell$ (of potentially different sizes) by sampling vertices i.i.d. from the graph.

- For each i, follow some rule $\mathcal{F}_i$ to choose a vertex x_i with cluster size κ_w from group $\mathcal{V}_i$, subject to the condition that $\mathcal{F}_i$ induces a function $\phi_i(s) = \mathbb{P}(\kappa_i = s)$ that does not depend on any other ϕ_j for $j \neq i$.

We call $\mathcal{R}$ an **ℓ-choice rule**.

If $\phi_i = Q_{m_i}$ for each i, then $\mathcal{R}$ is **minimizing**. We can similarly define **maximizing** rules. If each ϕ_i is the same, $\mathcal{R}$ is **symmetric**. We would like to restrict the vertex selection processes in each group as little as possible in order to get a more general theory, but for the aforementioned special rules, we can conduct more illuminating analysis.

In the following sections, we will derive scaling relations for general 2-choice rules, then use these to analyze minimizing rules and several other commonly studied rules. **make sure I talk about scaling assumption before this and say that I'm assuming it everywhere...**

2.1 SCALING RELATIONS

Suppose $\mathcal{R}$ is some general 2-choice rule, then, since none of the ϕ_i depend on each other, it satisfies the Smoluchowski equation

$$\partial_t P(s) = s \sum_{u+v=s} \phi_1(u)\phi_2(v) - s\phi_1(s) - s\phi_2(s).$$

The summation term represents two components merging into a new component of size s, and the last two terms each represent a component of size s joining with another component. We can use this to calculate the growth rate of the giant component, which allows us to prove an important computational theorem.

Lemma 1. For any 2-choice rule $\mathcal{R}$, the relative size of the giant component S satisfies

$$\partial_t S = \langle s \rangle_{\phi_1} (1 - \langle 1 \rangle_{\phi_2}) + \langle s \rangle_{\phi_2} (1 - \langle 1 \rangle_{\phi_1}).$$

Proof. Using the identity $\sum_s P(s) = 1 - S$, we calculate

$$\begin{aligned}
\partial_t S &= -\sum_s \partial_t P(s) \\
&= -\sum_s s \sum_{u+v=s} \phi_1(u)\phi_2(v) + \sum_s s\phi_1(s) \sum_s + s\phi_2(s) \\
&= -\sum_u \sum_v (u+v)\phi_1(u)\phi_2(v) + \langle s \rangle_{\phi_1} + \langle s \rangle_{\phi_2} \\
&= -\sum_u u\phi_1(u) \sum_v \phi_2(v) - \sum_u \phi_1(u) \sum_v v\phi_2(v) + \langle s \rangle_{\phi_1} + \langle s \rangle_{\phi_2} \\
&= -\langle s \rangle_{\phi_1} \langle 1 \rangle_{\phi_2} - \langle 1 \rangle_{\phi_1} \langle s \rangle_{\phi_2} + \langle s \rangle_{\phi_1} + \langle s \rangle_{\phi_2} \\
&= \langle s \rangle_{\phi_1} (1 - \langle 1 \rangle_{\phi_2}) + \langle s \rangle_{\phi_2} (1 - \langle 1 \rangle_{\phi_1}).
\end{aligned}$$

□

Theorem 1. For any 2-choice rule $\mathcal{R}$, there are nonnegative functions ζ_1 and ζ_2 such that

$$\partial_t S = \langle s \rangle_{\phi_1} \zeta_2(S) + \langle s \rangle_{\phi_2} \zeta_1(S).$$

Furthermore, the two terms above have the same order $F_1(\beta) + F_2(\beta) - \beta + 1$ when the expression is put into scaling form.

Proof. **use previous lemma.**

Now for both i, consider the probability $\zeta_i(S)$ that a vertex chosen from group i belongs to an infinite cluster. This satisfies the relation $\langle 1 \rangle_{\phi_i} = 1 - \zeta_i(S)$, which allows us to recover the desired form of $\partial_t S$.

Same order. □

To determine scaling relations for 2-choice rules, we'll need one last result, a consequence of Theorem 1.

Corollary 1. For any 2-choice rule $\mathcal{R}$, the average cluster size $\langle s \rangle_P$ satisfies

$$\partial_t \langle s \rangle_P = 2\langle s \rangle_{\phi_1} \langle s \rangle_{\phi_2} - \langle s^2 \rangle_{\phi_1} \zeta_2(S) - \langle s^2 \rangle_{\phi_2} \zeta_1(S),$$

where ζ_1 and ζ_2 are the functions from Theorem 1.

Proof. Recall from the proof of Theorem 1 that there are functions ζ_1 and ζ_2 satisfying the relation $\langle 1 \rangle_{\phi_i} = 1 - \zeta_i(S)$. We can then explicitly compute $\partial_t \langle s \rangle_P$.

$$\begin{aligned}
\partial_t \langle s \rangle_P &= \sum_s s \partial_t P(s) \\
&= \sum_s s^2 \sum_{u+v=s} \phi_1(u)\phi_2(v) - \sum_s s^2 \phi_1(s) - \sum_s s^2 \phi_2(s) \\
&= \sum_u \sum_v (u+v)^2 \phi_1(u)\phi_2(v) - \langle s^2 \rangle_{\phi_1} - \langle s^2 \rangle_{\phi_2} \\
&= \langle s^2 \rangle_{\phi_1} \langle 1 \rangle_{\phi_2} + 2\langle s \rangle_{\phi_1} \langle s \rangle_{\phi_2} + \langle 1 \rangle_{\phi_1} \langle s^2 \rangle_{\phi_2} - \langle s^2 \rangle_{\phi_1} - \langle s^2 \rangle_{\phi_2} \\
&= 2\langle s \rangle_{\phi_1} \langle s \rangle_{\phi_2} + \langle s^2 \rangle_{\phi_1} (\langle 1 \rangle_{\phi_2} - 1) + \langle s^2 \rangle_{\phi_2} (\langle 1 \rangle_{\phi_1} - 1) \\
&= 2\langle s \rangle_{\phi_1} \langle s \rangle_{\phi_2} - \langle s^2 \rangle_{\phi_1} \zeta_2(S) - \langle s^2 \rangle_{\phi_2} \zeta_1(S).
\end{aligned}$$

□

Theorem 2. **summarize scaling relations in theorem, then show them. Redo the calculations to figure out the best way to present them.**

Suppose we are instead working with a minimizing 2-choice rule, i.e. $\phi_1 = Q_a$ and $\phi_2 = Q_b$ for two positive integers a and b. In this case, we have simpler forms for the scaling relations.

Corollary 2. For minimizing 2-choice rules, the scaling relations from Theorem 2 become

$$\begin{aligned}\gamma_a &= 1 + (b-1)\beta,\\ \gamma_b &= 1 + (a-1)\beta,\\ \gamma_P &= 1 + (a+b-2)\beta,\\ \frac{1}{\sigma} &= 1 + (a+b-1)\beta,\\ \tau &= \frac{\beta}{1+(a+b-1)\beta} + 2.\end{aligned}$$

Proof. The relation $\langle 1 \rangle_m = 1 - S^m$ holds for all m, so the induced coefficient map for Q_m is $\beta \mapsto m\beta$. The result then follows from Theorem 2. □

Include examples here and compare different rules

2.2 LIMITING BEHAVIOR

2.3 CLUSTER SIZE VARIANCE

3 ERDŐS-RÉNYI

The Erdős-Rényi rule is the simplest of all the 2-choice rules. For each group, simply pick one vertex at random to be the group representative. Thus an equivalent way of defining the Erdős-Rényi rule is to add one of the $\binom{n}{2}$ possible edges in the graph at random at each time step.

It is clear that Erdős-Rényi is both minimizing and symmetric, so plugging in $a = b = 1$ into the scaling relations from Corollary 2 gives

$$\begin{aligned}\gamma_P &= 1,\\ \frac{1}{\sigma} &= 1 + \beta,\\ \tau &= \frac{\beta}{1+\beta} + 2.\end{aligned}$$

With such a simple rule, we can in fact determine much more. **talk about $\beta = 1$.**

Thus the scaling relations for Erdős-Rényi are

$$\begin{aligned}\gamma_P &= 1,\\ \frac{1}{\sigma} &= 2,\\ \tau &= 5/2.\end{aligned}$$

A critical assumption of this work was the scaling assumption, and for Erdős-Rényi, we can find an explicit bound on the size of the *scaling window*, the region where the scaling assumption is accurate.

Theorem 3. In the limit as $s \to \infty$, **finish...**

Proof. We use the following two relationships, which hold for symmetric minimizing 2-choice rules when $t < t_c$ and s is large: **where did they come from?**

- $P(x) = \delta^{(\tau-1)/\sigma}\tilde{f}(x\delta^{1/\sigma})$;
- $\tilde{f}(x) \propto x^\lambda \exp\left(-Cx^{1+\log_2 m}\right)$, where $\lambda = (1+\log_2 m)\left(1+\frac{1}{4m-2}\right) - \frac{2m}{2m-1}$;

where m is the group size. In the case of Erdős-Rényi, we know $m = 1$, $\tau = 5/2$, and $1/\sigma = 2$, so these reduce to

$$P(x) = \tilde{C}\delta^2 x^{-1/2}\exp\left(-Cx\delta^2\right),$$

where $\tilde{C}$ is the constant of propotionality from $\tilde{f}$. To clean up notation, we use the shorthand $\mathcal{E}_x := \exp\left(-Cx\delta^2\right)$. Also note that $\delta = t_c - t$ when $t < t_c$, so $\partial_t\delta = -1$. With these observations, we can begin the computation. When s is large, we know P satisfies the ODE **(switching over to integrals...)**

$$\begin{aligned}
\partial_t P(s) &= \frac{s}{2}\int_0^s P(u)P(s-u)\,du - sP(s)\\
\partial_t\left\{\tilde{C}\delta^2 s^{-1/2}\mathcal{E}_s\right\} &= \frac{s}{2}\int_0^s \tilde{C}^2\delta^4(su-u^2)^{-1/2}\mathcal{E}_u\mathcal{E}_{s-u}\,du \quad - \quad \tilde{C}\delta^2 s^{1/2}\mathcal{E}_s\\
\tilde{C}\left[2\delta^3 C s^{1/2}\mathcal{E}_s - 2\delta s^{-1/2}\mathcal{E}_s\right] &= \frac{s}{2}\int_0^s \tilde{C}^2\delta^4(su-u^2)^{-1/2}\mathcal{E}_s\,du \quad - \quad \tilde{C}\delta^2 s^{1/2}\mathcal{E}_s\\
2\delta^2 C s^{1/2} - 2s^{-1/2} &= \frac{1}{2}s\tilde{C}\delta^3\int_0^s (su-u^2)^{-1/2}\,du \quad - \quad \delta s^{1/2}.
\end{aligned}$$

integral goes to π as $s \to \infty$. Thus in the limit as $s \to \infty$, this becomes

$$2\delta^2 C s^{1/2} - 2s^{-1/2} = \frac{1}{2}s\tilde{C}\delta^3\pi - \delta s^{1/2}.$$

Note that the two terms on the left hand side also go to zero in the limit as $s \to \infty$, so this is really

$$\begin{aligned}
\frac{1}{2}s\tilde{C}\delta^3\pi &= \delta s^{1/2}\\
\delta^2 &= \frac{2}{s^{-1/2}\tilde{C}\pi}\\
\delta &= \Theta(s^{-1/4}).
\end{aligned}$$

□

Should I turn this into a "various result" subsection? I'll have to add some of Rick's other results anyway...

3.1 BOUNDED SIZE RULES

bounded size rules and how they relate to Erdős-Rényi