Exercises completed: All.

Exercise 1. Munkres §23, pg. 152 #4.

Collaborators: None.

Suppose Y is a space with the finite complement topology and X is an infinite set in Y. Note that since $X \subset Y$ is infinite, this forces Y to also be infinite. Suppose X is not connected, then there exist nonempty disjoint open sets U and V whose union is X.

Since $U \cap V = \varnothing$, by DeMorgan's laws the complement of their intersection is

$$Y - (U \cap V) = (Y - U) \cup (Y - V) = Y.$$

Since Y is infinite, this means at least one of $Y - U$ and $Y - V$ is infinite. This is a contradiction, though, since open sets in the finite complement topology have finite complements. Thus X must be connected.

Exercise 2. Munkres §23, pg. 152 #9.

Collaborators: None.

Let $Z \doteq (X \times Y) - (A \times B)$, and fix $(a, b) \in Z$ such that $a \in X - A$ and $b \in Y - B$. Then the segments $\{a\} \times Y$ and $X \times \{b\}$ are both connected subsets of Z (they are connected because they are homeomorphic to Y and X, respectively). Since they share the point (a, b), their union

$$T \doteq (X \times \{b\}) \cup (\{a\} \times Y)$$

is also connected. Now let (x, y) be an arbitrary point of Z, and define

$$T_{x,y} \doteq \begin{cases} \{x\} \times Y & \text{if } x \in X - A, \\ X \times \{y\} & \text{if } y \in Y - B. \end{cases}$$

In the first case, $T_{x,y}$ intersects T at the point (x, b). In the second case, it intersects T at the point (a, y). Since $T_{x,y}$ is homeomorphic to either X or Y, it is connected, and by definition it lies entirely in Z and intersects T. Thus

$$\tilde{T}_{x,y} \doteq T_{x,y} \cup T$$

is a connected subset of Z that contains the points (x, y) and (a, b). Finally,

$$\bigcup_{(x,y) \in Z} \tilde{T}_{x,y} = Z$$

is the union of connected sets that all contain the points (a, b), so Z is connected.

Exercise 3. Let $T = \{(x, sin(1/x) \mid x > 0\} \subset \mathbb{R}^2$. Prove that T is connected. You may assume the sine function is continuous.

Collaborators: Saloni Bulchandani, Rahul Ramesh.

Since $(0, \infty)$ is connected and the function $x \mapsto 1/x$ is continuous on $(0, \infty)$, the set $\{1/x \mid x > 0\}$ is connected. Then since the sine function is continuous, the image of this set under the sine function $\{\sin(1/x) \mid x > 0\}$ is connected. Then since the finite Cartesian product of connected spaces is connected, the set

$$T_1 \doteq \{(x, \sin(1/x) \mid x > 0\}$$

is connected. Thus if $T = T_1 \cup \{(0,0)\}$ is disconnected, then it is because $(0,0)$ can be separated from T_1.

Consider any neighborhood U of $(0,0)$, which is of the form $B((0,0), \varepsilon) \cap T$ for some $\varepsilon > 0$. There is some $n \in \mathbb{N}$ such that $\frac{1}{\pi n} < \varepsilon$, then for $x = \frac{1}{\pi n}$, we have

$$\sin(1/x) = \sin(\pi n) = 0.$$

The point $(1/\pi n, 0)$ is then in T_1, and its distance from the origin is

$$\|(1/\pi n, 0) - (0,0)\| = \|(1/\pi n, 0)\| = |1/\pi n| < \varepsilon,$$

so it is also in U. Since U was arbitrary, this shows that every neighborhood of the origin intersects T_1, so $(0,0)$ is a limit point of T_1. This gives

$$T_1 \subset T \subset \overline{T_1},$$

so since T_1 is connected, T is also connected.

Exercise 4. Munkres §24, pg. 158 #2.

Collaborators: Saloni Bulchandani, Rahul Ramesh.

If f is constant, then this is trivial, so assume f is not constant. Let $g(x) = f(x) - f(-x)$. Then $g(-x) = f(-x) - f(x) = -g(x)$. Since f is not constant, there is some x such that $g(x) \neq 0$. Then $g(-x)$ has the opposite sign. Then since S^1 is connected, by the intermediate value theorem there is some $\tilde{x} \in S^1$ such that $g(\tilde{x}) = 0$, i.e. $f(\tilde{x}) = f(-\tilde{x})$.

Exercise 5. Munkres §24, pg. 158 #9.

Collaborators: Saloni Bulchandani, Rahul Ramesh.

Fix arbitrary $x, y \in \mathbb{R}^2 - A$, then we must show that there is a continuous path between them. Consider the collection of straight lines extending from x

$$\mathcal{R}_x \doteq \{t \mapsto x + t(\cos\theta, \sin\theta) \mid \theta \in [0, \pi)\}.$$

Because of how we restricted θ, all of these lines are disjoint. We claim that an uncountable number of these lines never intersect A.

Suppose that an uncountable number of the lines intersect A at some point. Then since $\mathcal{R}_x$ is uncountable and the lines are all disjoint, A contains an uncountable number of points. Since A is countable, this is impossible, so there must be an uncountable number of lines in $\mathcal{R}_x$ that never intersect A.

Any two non-parallel straight lines in $\mathbb{R}^2$ will eventually intersect. Since we have uncountably many lines in both $\mathcal{R}_x$ and $\mathcal{R}_y$ that never intersect A, there must be two that are non-parallel and also never intersect A.

Take these two lines, then we can easily use them to construct a continuous path from x to y: Suppose L_x is the line extending from x and L_y is the line extending from y, and suppose they intersect at a point z. Then the path from x to z is continuous, and the path from z to y is continuous. Since they agree at z, by the pasting lemma, they can be combined into a continuous path from x to y. Since this path never intersects A, the space $\mathbb{R}^2 - A$ is connected.