Exercise 1 (2.1: 20)**.** Show that $\tilde{H}_n(X) \approx \tilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

First part: Suppose $A \subset B$ is contractible, then the long exact sequence of the pair (B, A) in reduced homology gives the following exact sequence for all n.

$$0 = \tilde{H}_n(A) \to \tilde{H}_n(B) \to H_n(B, A) \to \tilde{H}_{n-1}(A) = 0$$

Thus $H_n(B, A) \cong \tilde{H}_n(B)$ when A is contractible. We can apply this to the problem at hand; We know SX is the union of two cones CX and $C'X$. Then since $CX \subset SX$ is clearly contractible,

$$\tilde{H}_{n+1}(SX) \cong H_{n+1}(SX, CX) \cong H_{n+1}(C'X, X),$$

where the second isomorphism follows from Corollary 2.24 in the text with $A = CX$ and $B = C'X$ (a consequence of excision). The long exact sequence of the pair $(C'X, X)$ in reduced homology then gives the following exact sequence.

$$0 = \tilde{H}_{n+1}(C'X) \to H_{n+1}(C'X, X) \to \tilde{H}_n(X) \to \tilde{H}_n(C'X) = 0$$

Thus $H_{n+1}(C'X, X) \cong \tilde{H}_n(X)$. Composing all our isomorphisms gives

$$\tilde{H}_{n+1}(SX) \cong \tilde{H}_n(X),$$

as desired.

Second part: Let $S^k X$ denote k cones CX with bases all identified together. Since $S^k X$ is the union of $S^{k-1}X$ and CX and since CX is contractible, we have

$$\tilde{H}_{n+1}(S^k X) \cong H_{n+1}(S^k X, CX) \cong H_{n+1}(S^{k-1}, X),$$

where the second isomorphism once again comes from Corollary 2.24, with $A = CX$ and $B = S^{k-1}X$. We claim that $(S^{k-1}X, X)$ is a good pair: X is clearly closed in S^{k-1}, and if we remove all the points where the individual cones are identified to a point, then this open set deformation retracts onto X. Since it's a good pair, we have $H_{n+1}(S^{k-1}X, X) \cong \tilde{H}_{n+1}(S^{k-1}X/X)$.

But $S^i X/X \cong \bigvee_{j=1}^{i} SX$, as the following illustration illustrates in the case $i = 2$.

Then by Corollary 2.25 and the first part of the question, we have

$$\tilde{H}_{n+1}(S^{k-1}X/X) \cong \tilde{H}_{n+1}\left(\bigvee_{i=1}^{k-1} SX\right) \cong \bigoplus_{i=1}^{k-1} \tilde{H}_{n+1}(SX) \cong \bigoplus_{i=1}^{k-1} \tilde{H}_n(X).$$

Composing all our isomorphisms together, we get

$$\tilde{H}_{n+1}(S^k X) \cong \bigoplus_{i=1}^{k-1} \tilde{H}_n(X).$$

Exercise 2 (2.1: 22)**.** Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres:

a. If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.

b. $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.

c. If X has k n-cells, then $H_n(X)$ is generated by at most k elements.

All three parts of the question will use the following result: suppose α indexes the n-cells of a CW complex X, then $X^n/X^{n-1} \cong \bigvee_\alpha S^n$. We claim that (X^n, X^{n-1}) is a good pair: X^n is a union of D^n such that each $\text{int}D^n$ is disjoint, and X^{n-1} is the border of all the D^n. If we remove a point from each D^n, then we get an open set that clearly deformation retracts onto its boundary, i.e. onto X^{n-1}. Thus $X^n, X^{n-1})$ is a good pair for all n. Then by Proposition 2.22 and Corollary 2.25 in the text,

$$H_i(X^n, X^{n-1}) \cong \tilde{H}_i(X^n/X^{n-1}) \cong \tilde{H}_i\left(\bigvee_\alpha S^n\right) \cong \bigoplus_\alpha \tilde{H}_i(S^n)$$

for all i. We know the reduced homology groups of the n-sphere, so this chain of isomorphisms gives us

$$H_i(X^n, X^{n-1}) \cong \begin{cases} \bigoplus_\alpha \mathbb{Z} & \text{if } i = n, \\ 0 & \text{else,} \end{cases} \tag{$\star$}$$

where α indexes the n-cells. Additionally,

$$H_i(X) \cong H_i(X^n) \qquad \text{when } n > i. \tag{$\star\star$}$$

To see this, consider the long exact sequence of the pair (X^{n+1}, X^n), which gives the following exact sequence.

$$H_{i+1}(X^{n+1}, X^n) \to H_i(X^n) \to H_i X^{n+1} \to H_i(X^{n+1}, X^n)$$

When $n > i$, the first and last elements in the sequence are both 0 by ($\star$), so $H_i(X^n) \cong H_i(X^{n+1})$. Since X is finite dimensional, we can induct on n to get $H_i(X^n) \cong H_i(X)$.

a. Suppose $n = 0$, then $X = X^0$ is just a set of isolated points (0-cells). Since we can decompose the homology of a space into the direct sum of the homology of its path components, and since we know the homology of a point, this means

$$H_i(X) \cong \begin{cases} \bigoplus_\alpha \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{else,} \end{cases}$$

where α indexes the points (0-cells). This shows $H_i(X) = 0$ when $i > 0$. Since the direct sum of free modules is free, this also means $H_0(X)$ is free. Now we can induct on n: suppose this holds for $n-1$, then we want to show it is true for n.

From the long exact sequence of the pair (X^n, X^{n-1}), the following is exact for all i.

$$H_i(X^{n-1}) \to H_i(X^n) \to H_i(X^n, X^{n-1})$$

By $(\star)$ and our inductive hypothesis, this becomes the following when $i > n$.

$$0 \to H_i(X^n) \to 0$$

This implies $H_i(X^n) = 0$ when $i > n$. When $i = n$, this becomes the following instead.

$$0 \to H_n(X^n) \rightarrowtail \bigoplus_\alpha \mathbb{Z}$$

Since the second map is injective by exactness, $H_n(X^n)$ is isomorphic to a subgroup of a free group. But the subgroup of a free group is also free, so we can pass the basis of this subgroup to a basis of $H_n(X^n)$ via the isomorphism. Thus $H_n(X^n)$ is free.

b. We'll need two base cases for this induction. Suppose $n = 0$, then we know $H_0(X) \cong \bigoplus_\beta \mathbb{Z}$, where β indexes the path components of X. Since X has no 1-cells by assumption, each 0-cell must be isolated, i.e. there is a single 0-cell in each path component. Thus the basis of $H_0(X) \cong \bigoplus_\gamma \mathbb{Z}$ is in bijective correspondence with the 0-cells.

Now suppose $n = 1$, then X has no 0-cells by assumption. But a CW complex without any 0-cells cannot have any other cells: any 1-cell must have 0-cells at its boundary, so there cannot be any 1-cells; any 2-cell must have 1- and 0-cells at its boundary, so there cannot be any 2-cells. Since X is finite dimensional, we can repeat this argument up through the dimension of X to show that it is empty. The claim is then trivially true.

Now we induct on n. Suppose X has no $n-1$ or $n+1$ cells, then the long exact sequence of the pair (X^n, X^{n-1}) gives the following exact sequence.

$$H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$$

By our assumption that there are no $n-1$ or $n+1$ cells, $X^{n-1} = X^{n-2}$ and $X^{n+1} = X^n$, so this becomes the following.

$$H_n(X^{n-1}) \to H_n(X^{n+1}) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-2})$$

Then by $(\star)$, $(\star\star)$, and (a), this reduces to the following, where α indexes the n-cells.

$$0 \to H_n(X) \rightarrowtail \bigoplus_\alpha \mathbb{Z} \to 0$$

Thus $H_n(X) \cong \bigoplus_\alpha \mathbb{Z}$, so it is free with basis is in bijective correspondence with the n-cells.

c. The long exact sequence of the pair (X^n, X^{n-1}) gives the following exact sequence.

$$H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n, X^{n-1})$$

By part (a), the first term above is 0, and by ($\star$) and our assumption that there are k n-cells, the last term is $\mathbb{Z}^k$. Thus the exact sequence becomes the following.

$$0 \to H_n(X^n) \rightarrowtail \mathbb{Z}^k$$

By exactness, the last map is injective. Since $\mathbb{Z}^k$ has exactly k generators and $H_n(X^n)$ is identified with a subgroup of $\mathbb{Z}^k$, this means $H_n(X^n)$ has at most k generators.

The long exact sequence of the pair (X^{n+1}, X^n) gives the following exact sequence.

$$H_n(X^n) \twoheadrightarrow H_n(X^{n+1}) \to H_n(X^{n+1}, X^n) = 0.$$

Exactness makes the first map surjective. By ($\star\star$), $H_n(X^{n+1}) \cong H_n(X)$, so this becomes the following.

$$H_n(X^n) \twoheadrightarrow H_n(X) \to 0$$

Since we just argued that $H_n(X^n)$ has at most k generators, this means $H_n(X)$ must also have at most k generators.

Exercise 3 (2.1: 27)**.** Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences.

a. Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.

b. For the case of the inclusion $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs – there is no $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such taht fg and gf are homotopic to the identity through maps of pairs. [Observe that a homotopy equivalence of pairs $(X, A) \to (Y, B)$ is also a homotopy equivalence for the pairs obtained by replacing A and B by their closures.]

a. Consider the following diagram, where $f_\# : \mathcal{C}(X, A) \to \mathcal{C}(Y, B)$ is an abuse of notation: it's actually induced by the map $f_\#$ that sends $\mathcal{C}(X) \to \mathcal{C}(Y)$ and $\mathcal{C}(A) \to \mathcal{C}(B)$.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}(A) \arrow[r, "i_\#"] \arrow[d, "f_\#"] & \mathcal{C}(X) \arrow[r, "\pi_\#"] \arrow[d, "f_\#"] & \mathcal{C}(X,A) \arrow[r] \arrow[d, "f_\#"] & 0 \\ 0 \arrow[r] & \mathcal{C}(B) \arrow[r, "i_\#"] & \mathcal{C}(Y) \arrow[r, "\pi_\#"] & \mathcal{C}(Y,B) \arrow[r] & 0 \end{tikzcd}$$

Since $A \subset X$ and $B \subset Y$, each $i_\#$ is induced the natural inclusion. Since $\mathcal{C}(X, A)$ and $\mathcal{C}(Y, B)$ are quotients, each $\pi_\#$ is induced from the canonical projection.

It's straightforward to check that this diagram commutes. For any n and any $\sigma \in C_n(X)$, we have $(f_\#\pi_\#)(\sigma) = [f\sigma] = (\pi_\# f_\#)(\sigma)$. For any n and any $\sigma \in C_n(A)$, we have $(f_\# i_\#)(\sigma) = f\sigma = (i_\# f_\#)(\sigma)$. Thus the diagram commutes.

Then by naturality, we have the following commutative diagram for all n, where both rows are exact and the middle f_* is once again a similar abuse of notation.

$$\begin{tikzcd} H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(X) \arrow[d, "f_*"] \\ H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(Y) \end{tikzcd}$$

We'd like to apply the five lemma here, but to do this, we'll need to show that f_* is an isomorphism $H_n(X) \to H_n(Y)$ and $H_n(A) \to H_n(B)$ for all n. Since f is a homotopy equivalence $X \simeq Y$, we know there is a map $g : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$. Applying the homology functor for any n then gives the following diagram.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} H_n(X) \arrow[r, bend left, "f_*"] & H_n(Y) \arrow[l, bend left, "g_*"] \end{tikzcd}$$

By functoriality and the fact that homotopic maps have the same induced map on homology, $g_* f_* = (gf)_* = \mathrm{id}_* = \mathrm{id}$. Similarly, $f_* g_* = \mathrm{id}$ as well. Thus f_* is an isomorphism $H_n(X) \cong H_n(Y)$. Since f restricts to a homotopy equivalence $A \simeq B$ as well, we can repeat this argument to show $H_n(A) \simeq H_n(B)$.

Thus we can apply the five lemma, which gives $H_n(X, A) \cong H_n(Y, B)$ for all n.

b. First we'll show that any homotopy of pairs $(X, A) \to (Y, B)$ is also a homotopy of pairs $(X, \overline{A}) \to (Y, \overline{B})$. If we have a homotopy $H : X \times I \to Y$ such that $H_t : A \to B$ for all t, then all we need for this is for H_t to map $\overline{A}$ into $\overline{B}$. But since each H_t is necessarily continuous,

$$H_t(\overline{A}) \subset \overline{H_t(A)} \subset \overline{B}$$

for all t. Thus we have a homotopy of maps $(X, \overline{A}) \to (Y, \overline{B})$. Now we can show that the f in the problem statement isn't a homotopy equivalence of pairs.

If we have a homotopy equivalence of pairs $(X, \overline{A}) \to (Y, \overline{B})$, we can clearly restict the homotopy equivalence $X \simeq Y$ to get a homotopy equivalence $\overline{A} \simeq \overline{B}$. In the problem, we have $X = Y = D^n$, $\overline{A} = S^{n-1}$, and $\overline{B} = \overline{D^n - \{0\}} = D^n$, so by part (a), we must have

$$H_n(D^n, S^{n-1}) \cong H_n(D^n, D^n) \cong 0$$

for all n. By the long exact sequence of the pair (D^n, S^{n-1}) in reduced homology, we have the following exact sequence.

$$\tilde{H}_n(D^n) \to H_n(D^n, S^{n-1}) \to \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(D^n)$$

Since D^n is contractible, it has trivial reduced homology in all dimensions. We also know $\tilde{H}_{n-1}(S^{n-1}) \cong n-1$, so this exact sequence is actually the following.

$$0 \to H_n(D^n, S^{n-1}) \to \mathbb{Z} \to 0$$

This implies $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$, but $\mathbb{Z} \ncong 0 \cong H_n(D^n, D^n)$, so we've arrived at a contradiction. Thus f cannot be a homotopy equivalence of pairs.