Contents

Chapter 1

Topological Spaces

1.1 Topological Spaces

Definition 1. Let X be a set, then a **topology** on X is a collection $\mathcal{T}$ of subsets of X such that

1. $\varnothing, X \in \mathcal{T}$,
2. $\bigcup_{\alpha \in \mathcal{J}} U_\alpha \in \mathcal{T}$, and
3. $\bigcap_{i=1}^{N} U_i \in \mathcal{T}$.

Elements of a topology are called **open sets**.

Example 1. 1. "Indiscrete" topology: $\mathcal{T}_i = \{\varnothing, X\}$

2. "Discrete" topology: $\mathcal{T}_d = \{\text{all subsets of } X\}$

Definition 2. Let $\mathcal{T}, \mathcal{T}'$ be topologies on a set X, then $\mathcal{T}$ is **finer** than $\mathcal{T}'$ if $\mathcal{T}' \subset \mathcal{T}$. $\mathcal{T}$ is **coarser** than $\mathcal{T}'$ if $\mathcal{T} \subset \mathcal{T}'$. The notions of **strictly finer** and **strictly coarser** follow.

From this we see that "fine" is a notion of a large topology, and "coarse" is a notion of a small topology.

Example 2. The **lower limit topology** on $\mathbb{R}$ is given by the basis

$$\mathcal{B} = \{[a, b) \mid a < b\}.$$

It is strictly finer than the standard topology on $\mathbb{R}$: since $\bigcup_{n \in \mathbb{N}} [a + 1/n, b) = (a, b)$, it contains the standard topology, but $[a, b)$ is not open in the standard topology, so it is strictly finer.

Example 3. Let X be any set, then the **finite complement topology** is defined

$$\mathcal{T}_f = \{U \subset X \mid X - U \text{ is finite}\} \cup \{\varnothing\},$$

where $X - U$ denotes the complement of U in X, i.e. $X \backslash U$. Checking that this is a topology boils down to just using DeMorgan's Laws.

1.2 Closed Sets and Limit Points

Definition 3. A set $A \subset (X, \mathcal{T})$ is closed if $X - A$ is open in X.

Theorem 1. *Let $(X, \mathcal{T})$ be a topological space, and let F denote a closed set of X, then*

1. *$\varnothing$ and X are closed,*
2. *$\bigcap_{\alpha \in \mathcal{J}} F_\alpha$ is closed, and*
3. *$\bigcup_{i=1}^{N} F_i$ is closed.*

Proof. This is a straightforward application of DeMorgan's Laws. □

Properties of a closed set A in a subspace Y of X:

- A is the intersection of Y and a closed set in X.
- If Y is closed in X, then A is closed in X.

Proposition 1. *Let Y be a subspace of X. Then A is closed in Y if and only if it is equal to the intersection of a closed set of X with Y.*

Proposition 2. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Definition 4. The **interior** of a set A, denoted A^o, is the union of all open sets contained in A.

The **closure** of a set A, denoted $\overline{A}$ is the intersection of all closed sets containing A.

The closure of a set is clearly closed, and the interior of a set is clearly open. It is also clear that if A is open, then $A^o = A$, and if A is closed, then $\overline{A} = A$. We also have the obvious relation $A^o \subset A \subset \overline{A}$.

We have to be careful when describing closures. Given a subspace Y of X, the closure of A in X is generally not the same as the closure of A in Y. In this case, we use $\overline{A}$ to denote the closure of A in X (the overall space). We relate this to the closure of A in Y (the subspace) with the following proposition.

Proposition 3. *Let Y be a subspace of X, and let $A \subset Y$. Then $\overline{A}_Y = \overline{A}_X \cap Y$.*

Definition 5. A **neighborhood** of a point X is an open set containing x.

Theorem 2. *Let A be a subset of a topological space X, then*

1. *$x \in \overline{A}$ if and only if every neighborhood of x intersects A, and*
2. *Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.*

Make sure you have an intuitive understanding of why this is true.

Definition 6. Let $A \subset (X, \mathcal{T})$, then $x \in X$ is a **limit point** of A if every open neighborhood of x intersects A at some point *other than* x.

Equivalently, x belongs to the closure of $A - \{x\}$. Note that x need not lie in A. Think about this.

Theorem 3. *Let $A \subset (X, \mathcal{T})$, and denote the set of limit points of A by A'. Then $\overline{A} = A \cup A'$.*

Corollary 1. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. Let $A \subset (X, \mathcal{T})$. Then A is closed if and only if $A = \overline{A} = A \cup A'$, and $A = A \cup A'$ if and only if $A' \subset A$. $\square$

1.3 Bases

Definition 7. Let $\mathcal{T}$ be a topoloy on X, and let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a **basis** for $\mathcal{T}$ if every open set of $\mathcal{T}$ can be written as the union of elements of $\mathcal{B}$.

Proposition 4. *Let $\mathcal{T}$ be a topology on X, and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if*

1. *$\mathcal{B} \subset \mathcal{T}$; and*
2. *for each $U \in \mathcal{T}$ and $p \in U$, there is a $B \in \mathcal{B}$ such that $p \in B \subset U$.*

Proof. The forward direction follows from every open set of $\mathcal{T}$ being the union of elements of $\mathcal{B}$. For the backward direction, since $p \in B_p \subset U$ for all $p \in U$, we have $U = \bigcup_{p \in U} B_p$, so every open set of $\mathcal{T}$ is the union of elements of $\mathcal{B}$. □

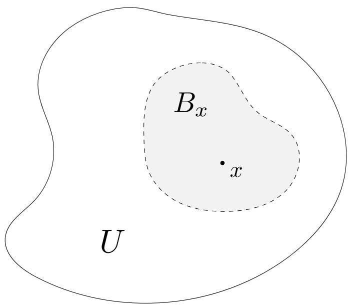


Figure 1.1: For any $U \in \mathcal{T}$, each $x \in U$ lies in some $B_x \in \mathcal{B}$ for $B_x \subset U$.

Not every set of subsets of X will generate a topology, so we need conditions for a collection $\mathcal{B}$ to be a basis for *any* topology.

Proposition 5. *Let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ generates a topology if and only if*

1. *$\bigcup_{B \in \mathcal{B}} = X$.*
2. *given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.*

Proof. **Forward:** (1) X must be open, so X is the union of the elements of $\mathcal{B}$. (2) Since B_1 and B_2 are both open in the topology generated by $\mathcal{B}$, their intersection is, as well. Then since $\mathcal{B}$ is a basis for this topology, we can find a satisfactory B_3.

Backward: The topology generated by a set $\mathcal{B}$ is the collection of all unions of elements of $\mathcal{B}$. It is clear that $\varnothing$ is in it, and condition (1) implies that X is, as well. Arbitrary unions are in the topology by definition. Induction on condition (2) shows that the topology also contains finite intersections. □

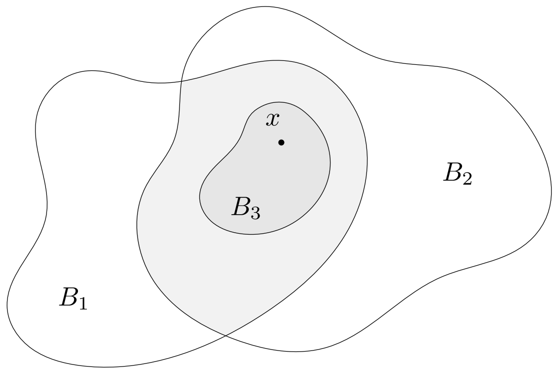


Figure 1.2: Condition (2) in Proposition 5.

Note 1. Since $\mathcal{B}$ exists independently from any topology, it doesn't make sense to describe its members as "open" until after we've generated a topology from it. Once we've done so, though, it should be clear that every basis element is open in the generated topology.

We can also get a notion of how relatively fine or coarse a topology is by using its basis.

Proposition 6. *Let $\mathcal{B}, \mathcal{B}'$ be bases for the topologies $\mathcal{T}, \mathcal{T}'$ on X, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for all $B \in \mathcal{B}$ and $x \in B$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subset \mathcal{B}$.*

Proof. First we show the backward implication. Let $U \in \mathcal{T}$, and let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is a $B \subset \mathcal{B}$ such that $x \in B \subset U$. By assumption, there is then a $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$. Thus $U \in \mathcal{T}'$, so $\mathcal{T}'$ is finer than $\mathcal{T}$.

Now we show the forward implication. Let $B \in \mathcal{B}$, and let $x \in B$, then $B \in \mathcal{T}$. By assmption, $\mathcal{T} \subset \mathcal{T}'$, so $B \in \mathcal{T}'$ as well. Then by the definition of a generated topology, there is a $B' \subset \mathcal{B}'$ such that $x \in B' \subset B$. $\square$

Proposition 7. *The topology generated by a basis is the smallest topology containing that basis.*

1.4 SUBBASES

Definition 8. A **subbasis** $\mathcal{S}$ for a topology $\mathcal{T}$ on X is a collection of subsets of X whose finite intersections form a basis for $\mathcal{T}$.

Subbases are easier to construct than bases, but the construction of a topology from a subbasis involves an extra step, namely the finite intersections. What we are doing is creating a basis $\mathcal{B}$ from $\mathcal{S}$ by taking finite intersections of the subbasis elements. Then we are taking $\mathcal{B}$ and constructing $\mathcal{T}$ by taking arbitrary unions, as is usual.

$$\mathcal{S} \xrightarrow{\bigcap_{i=1}^{N}} \mathcal{B} \xrightarrow{\bigcup_{\alpha \in \mathcal{J}}} \mathcal{T}$$

Figure 1.3: The process for constructing a topology using a subbasis $\mathcal{S}$.

Proposition 8. *Let $\mathcal{T}$ be a topology on X, and let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a subbasis for $\mathcal{T}$ if and only if*

1. *$\mathcal{S} \subset \mathcal{T}$; and*
2. *for each $U \in \mathcal{T}$ and $p \in U$, there is a finite intersection $\bigcap_{i=1}^{n} S_i$ of elements of $\mathcal{S}$ such that $p \in \bigcap_{i=1}^{n} S_i \subset U$.*

Proof. This follows from Proposition 4 (the analogue of this proposition for bases). When proving both directions, there's just an extra step to go from a genric basis element to a finite intersection of elements of $\mathcal{S}$. □

Proposition 9. *Let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ generates a topology if and only if $\mathcal{S}$ covers X.*

Proposition 10. *The topology generated by a subbasis is the smallest topology containing that subbasis.*

1.5 Continuous Functions

The category **Top** has topological spaces as objects and continuous functions as morphisms.

Definition 9. Let X, Y be topological spaces, then $f : X \to Y$ is **continuous** if for all U open in Y, $f^{-1}(U)$ is open in X.

Proposition 11. *If Y has basis $\mathcal{B}$ and $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$, then $f : X \to Y$ is continuous. Similarly, if Y has subbasis $\mathcal{S}$ and $f^{-1}(S)$ is open in X for all $S \in \mathcal{S}$, then $f : X \to Y$ is continuous.*

Proof. The preimage of any open set if the union of preimages of basis elements. The preimage of any basis element is the finite intersection of preimages of subbasis elements. □

Theorem 4. *Let X and Y be topological spaces, and let $f : X \to Y$, then the following are equivalent:*

1. *f is continuous.*
2. *For all $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.*
3. *For all B closed in Y, $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.*
4. *For all B closed in Y, $f^{-1}(B)$ is closed in X.*
5. *For all $x \in X$ and for each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

Example 4. If X has the discrete topology, then any function *out* of X is continuous. If X has the indiscrete topology, then any function *into* X is continuous.

Definition 10. A **homeomorphism** is a continuous function with continuous inverse (an isomorphism in **Top**).

Equivalently, a homeomorphism is a bijective function $f : X \leftrightarrow Y$ such that U is open in X if and only if $f(U)$ is open in Y.

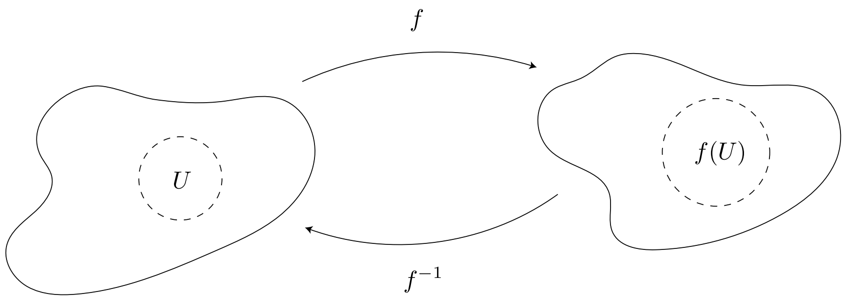


Figure 1.4: A homeomorphism f.

Theorem 5 (The Pasting Lemma). *Let $X = A \cup B$, where A and B are either both closed or both open in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ given by*

$$H(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous.

Proof. Suppose A and B are both closed. Let C be closed in Y, then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f and g are continuous, both $f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B, respectively. Since both A and B are closed in X, both preimages are also closed in X. Thus $h^{-1}(C)$ is closed in X and h is subsequently continuous.

To show this when A and B are both open, replace the word "closed" with the word "open" in the above paragraph. □

Note that the condition $f(x) = g(x)$ for all $x \in A \cap B$ is not needed in this proof. It is only necessary to make h an actual function.

Note 2. If $f : A \times B \to X$ instead, there is *no* useful criterion for the continuity of f.

The following maps are easily checked to be continuous:

- Constant maps.
- Inclusion maps.
- Restrictions of continuous maps.
- Compositions of continuous maps.

Chapter 2

Special Topologies

2.1 The Subspace Topology

There is a natural way of a subset inheriting the topology of the set it lies in. The following definition is easily checked to actually be a topology.

Definition 11. Let $(X, \mathcal{T})$ be a topological space. If $Y \subset X$, then

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is the **subspace topology** on Y. With this topology, Y is called a **subspace** of X.

Proposition 12. *Let $\mathcal{B}$ be a basis for the topology of X, then*

$$\mathcal{B}_Y \doteq \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let $y \in U \cap Y$, where U is open in X. There exists $B \in \mathcal{B}$ such that $y \in B \subset U$, so $y \in B \cap Y \subset U \cap Y$. □

Proposition 13. *Let Y be a subspace of X, and let U be open in Y and Y be open in X. Then U is open in X.*

Proof. U is open in Y, so $U = Y \cap V$ for some V open in X. Both sets Y and V are open in X, so their intersection U must be as well. □

2.2 The Initial Topology

Definition 12. Let X be a set and $\{Y_i\}_{i \in \mathcal{J}}$ a collection of topological spaces, and suppose we have functions $f_i : X \to Y_i$. The **initial topology** on X for these f_i is the coarsest topology on X such that each f_i is continuous.

Proposition 14. *The initial topology on X is generated by the subbasis*

$$\mathcal{S} = \left\{ f_i^{-1}(U) \mid i \in \mathcal{J}; U \text{ open in } Y_i \right\}.$$

Think about this...

This is a nice generalization of the subspace and product topologies. In the next section, we'll derive the product topology as the initial topology on a Cartesian product that makes the canonical projections continuous. The initial topology on a subset such that the inclusion function is continuous is actually the subsapce topology.

Example 5. Suppose $Y \subset X$, and consider the inclusion function $\iota : S \hookrightarrow X$. The initial topology is generated by

$$\left\{ i^{-1}(U) \mid U \text{ open in } X \right\} = \{ Y \cap U \mid U \text{ open in } X \},$$

but this is just the subspace topology.

Why bother with saying "generated" if it's equal? Are there counterexamples?

2.3 The Product Topology

It would be natural to define the product topology as

$$\mathcal{P} = \{U \times V \mid U \text{ open in } X, V \text{ open in } Y\},$$

but this isn't enough to give a topology since you can construct examples where the union of elements in this set don't lie in the set.

This set *is*, however, perfectly valid as a basis, since $\bigcup_{U,V}(U \times V) = X \times Y$ and $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{P}$.

Definition 13. The topology generated by $\mathcal{P}$ is the **product topology** on $X \times Y$.

Proposition 15. *If $\mathcal{B}_X$ is a basis for X and $\mathcal{B}_Y$ is a basis for Y, then $\mathcal{B}_X \times \mathcal{B}_Y$ is a basis for the product topology.*

Proposition 16. *The product and subspace topologies "commute".*

Proof. It's straightforward to show that the product of two subspaces and the subspace of a product both have the same basis. □

Where to put the above stuff?

Definition 14. The **Cartesian product** of $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is the set

$$\prod_{\alpha \in \mathcal{A}} X_\alpha = \left\{ f : \mathcal{A} \to \bigcup_{\alpha \in \mathcal{A}} X_\alpha \;\middle|\; f(\alpha) \in X_\alpha \right\}.$$

Each function f represents a single "point" in the product.

Example 6. Suppose $\mathcal{A} = \{1, \ldots, n\}$ and $X_\alpha = \mathbb{R}$ for all α. Then each f in the Cartesian product is a function

$$f : \{1, \ldots, n\} \to \mathbb{R}.$$

Since there are only a finite number of X_α's, we can write each f as a tuple

$$(f(1), f(2), \ldots, f(n)).$$

Thus there is a clear bijection between $\prod_{\alpha=1}^n X_\alpha$ and $\mathbb{R}^n$.

Extending the product topology to the case of a general Cartesian product is tricky. Given $\prod_\alpha X_\alpha$, we could naively say that the topology on it should be given by a basis

$$\mathcal{B} = \left\{ \prod_\alpha B_\alpha \;\middle|\; B_\alpha \in \mathcal{B}_\alpha \right\},$$

where $\mathcal{B}_\alpha$ is a basis for just X_α. If we have a finite number of α's, this basis is just every possible ordered combination of basis elements from each X_α:

$$
\begin{gathered}
(B_{11}, B_{21}, \dots, B_{n1}), \\
(B_{11}, B_{22}, \dots, B_{n2}), \\
\vdots \\
(B_{11}, B_{22}, \dots, B_{nn}), \\
\vdots
\end{gathered}
$$

The topology generated by this basis is the **box topology**, and although simple, ends up not being the best notion of a topology on infinite products because it's actually too fine. This ends up making some "obviously" continuous functions discontinuous.

Example 7. Define

$$\mathbb{R}^\infty = \prod_{i \in \mathbb{Z}^+} \mathbb{R},$$

then the function

$$
\begin{aligned}
f : \mathbb{R} &\to \mathbb{R}^\infty \\
x &\mapsto (x, x, \dots)
\end{aligned}
$$

seems like it should be continuous; however, if $\mathbb{R}^\infty$ has the box topology, then the preimage under f of the open set $U = \prod_{i \in \mathbb{Z}^+}(-1/i, 1/i)$ is $f^{-1}(U) = \{0\}$. This isn't open in $\mathbb{R}$, so f is discontinuous.

We want the product topology to, in a sense, be continuous in each of its components. Unlike the box topology, though, we don't want it to be *too* fine. The way we formalize this is by saying that we want to find the coarsest topology on $\prod X_\alpha$ such that the canonical projections

$$
\begin{aligned}
\pi_\beta : \prod_{\alpha \in \mathcal{A}} X_\alpha &\to X_\beta \\
(f : \mathcal{A} \to \bigcup X_\alpha) &\mapsto f(\beta)
\end{aligned}
$$

are continuous. This is just the initial topology on $\prod X_\alpha$ with respect to the projections.

Definition 15. The **product topology** is generated by the subbasis

$$\left\{ \pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \text{ open in } X_\alpha \right\}.$$

The basis for the product topology is then of the form $\prod U_\alpha$, where only finitely many of the U_α satisfy $U_\alpha \neq X_\alpha$.

Proposition 17. *The function $f : Y \to \prod X_\alpha$ is continuous if and only if f_α is continuous for all α.*

Proof. If f is continuous, then $f_\alpha = \pi_\alpha \circ f$ is the composition of continuous functions and so is itself continuous. Conversely, for any subbasis element $S = \pi_\alpha(U_\alpha)$ for U_α open in X_α, we have

$$f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = (\pi_\alpha \circ f)^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha),$$

which is open since f_α is continuous. $\square$

Chapter 3

Special Spaces

3.1 Hausdorff Spaces

We say that a sequence $\{x_n\}$ is **eventually** in U if there is some N such that $x_n \in U$ when $n \geq N$.

Definition 16. $\{x_n\}$ **converges** to x if it's eventually in every open neighborhood of x.

Proposition 18. *$\{x_n\}$ converges to x if and only if it's eventually in every basis/subbasis element containing x.*

Example 8. In the discrete topology, $x_n \to x$ if $\{x_n\}$ eventually equals x.

In the indiscrete topology, every sequence converges to every point.

If we want limits to be unique, we have to enforce certain conditions on our spaces.

Definition 17. A space is T_1 if every pair of distinct points have neighborhoods not containing the other point. The space is **Hausdorff** if these neighborhoods are disjoint.

Proposition 19. *A space is T_1 if and only if all single points are closed.*

Proof. **Forward:** Suppose X is T_1, then fix $x \in$ X. Then for $y \in X - \{x\}$, there is an open U_y such that $y \in U_y \subset X - \{x\}$, so $X - \{x\} = \bigcup_y U_y$. Then $X - \{x\}$ is open so $\{x\}$ is closed.

Backward: Suppose all single points in X are closed. Fix $x, y \in X$, then $X - \{x\}$ and $X - \{y\}$ are the open sets we need to show that X is T_1. □

Corollary 2. *A space is T_1 if and only if all finite point sets are closed.*

Proof. Do I even need one? Kinda obvious. ☐

Proposition 20. *Every finite set in a Hausdorff space is closed.*

Proof. Hausdorff spaces are T_1. ☐

Proposition 21. *Sequences converge to unique points in Hausdorff spaces.*

Proof. Suppose $\{x_n\} \subset X$ such that $x_n \to x \in X$. If $y \neq x$, then since X is Hausdorff we can find disjoint open neighborhoods U and $V$of x and y, respectively. The set U contains all but finitely many of the points in $\{x_n\}$, so V can only contain finitely many of the points in $\{x_n\}$. Thus x_n cannot converge to y. ☐

Proposition 22. *The product of two Hausdorff spaces is a Hausdorff space.*

Proof. Do this. ☐

Proposition 23. *A subspace of a Hausdorff space is Hausdorff.*

Proof. Suppose X is Hausdorff and that Y is a subspace of X with distinct points u and v. Then u and v are also distinct points of X, so by the regularity of X, they are separated by disjoint open sets U and V in X. Then $Y \cap U$ and $Y \cap V$ are the desired open sets of Y. ☐

3.2 Quotient Spaces

Definition 18. Suppose X has a partition $\mathscr{P}$. The **quotient space** X^* is $\mathscr{P}$ equipped with the **quotient topology**:

$$U \text{ is open in } X^* \iff \pi_p^{-1}(U) \text{ is open in } X,$$

where

$$\begin{aligned} \pi_p : X &\to X^* \\ x &\mapsto [x]. \end{aligned}$$

Note 3. The quotient topology is the finest topology such that π_p is continuous. It is the **final topology** with respect to π_p. Section about final topology?

We can equivalently define quotient spaces in terms of images of certain functions.

Definition 19. A **quotient map** is a surjective map $p : X \to Y$ such that

$$U \text{ is open in } Y \iff p^{-1}(U) \text{ is open in } X.$$

A quotient map is a homeomorphism that isn't necessarily one-to-one. Thus if we partition X based on the equivalence relation induced by p, we get injectivity and p then induces a homeomorphism between X^* and Y (see the next theorem).

Quotient map is not necessarily open or closed map. This is subtle.

Proposition 24. *Suppose $p : X \to Y$ is a quotient map and Z is any space. Then $f : Y \to Z$ is continuous if and only if $f \circ p : X \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[d, "p"] \arrow[dr, "f \circ p"] & \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

Theorem 6. *Suppose $p : X \to Y$ is a quotient map. Then the quotient space X^* induced by the equivalence relation induced by p is homeomorphic to Y.*

Note 4. Given a surjective map p, the quotient topology is the final topology with respect to p.

Definition 20. A map $f : X \to Y$ is **open** if it maps open sets to open sets, and it's **closed** if it maps closed sets to closed sets.

Proposition 25. *Suppose $f : X \to Y$ is surjective and continuous. If it's open or closed, then it's a quotient map.*

Corollary 3. *If $f : X \to Y$ is continuous and surjective, X is compact, and Y is Hausdorff, then f is a quotient map.*

Proof. Suppose A is closed in X, then since X is compact, so is A. Since f is continuous, $f(A)$ is a compact subset of Hausdorff Y, so it is closed. Thus f is a closed continuous surjection, so it's a quotient map. □

If $X = A \cup B$, then we can make the union disjoint in a sense by introducing more dimensions. Define

$$A \sqcup B \doteq (A \times \{0\}) \cup (B \times \{1\})$$

(which is a subset of $X \times \{0, 1\}$) with canonical projection

$$\begin{aligned} j : A \sqcup B &\to X \\ (x, i) &\mapsto x. \end{aligned}$$

Theorem 7 (Gluing Maps)**.** *Suppose $X = A \cup B$ and $f : A \to Z$, $g : B \to Z$ agree on $A \cap B$. If the canonical projection $j : A \sqcup B \to X$ is a quotient map, then the obvious concatenation of f and g is continuous.*

The pasting lemma is a corollary of this theorem. Go over this, I guess.

3.3 Metric Spaces

Definition 21. The **metric topology** $\mathcal{T}_d$ on X induced by d is generated by the basis

$$\{B_d(x,\varepsilon) \mid x \in X, \varepsilon > 0\}.$$

Proposition 26. *The following give the same topologies on $\mathbb{R}^n$:*

1. $d_2(x,y) = \|x-y\|_2$,
2. $d_1(x,y) = \sum_i |x_i - y_i|$,
3. $d_\infty(x,y) = \max_i |x_i - y_i|$, *and*
4. *the product topology.*

We say a topological space is **metrizable** if there is some metric such that induces its topology.

Proposition 27. *Metrizable spaces are Hausdorff.*

Proof. Do this. Should rely on metric space being Haus. □

A metric space X is **bounded** if there is an $x \in X$ and $R > 0$ such that $B(x,R)$ contains all of X. Equivalently, we can say that there is some R' such that $d(a,b) < R$ for all $a, b \in X$.

Chapter 4

Topological Properties

4.1 Separability

Definition 22. A topological space is **separable** if it has a countable dense subset.

Example 9. $\mathbb{R}$ is separable because $\overline{\mathbb{Q}} = \mathbb{R}$.

4.2 Connectedness

Definition 23. A space is **connected** if it's *not* the union of 2 nonempty disjoint open subsets.

Since the whole space is the disjoint union of the 2 sets, the sets are also both closed. Thus we can separate a space with 2 nonempty disjoint *closed* sets, too.

Theorem 8. *The following are equivalent:*

1. *X is connected.*
2. *$A \subset X$ is both open and closed $\iff$ $A = X$ or $X = \varnothing$.*
3. *Whenever $X = A \sqcup B$ with A, B nonempty and disjoint, one of A, B contains a limit point of the other.*

Proposition 28. *$Y \subset X$ is disconnected if and only if there are disjoint nonempty A, B such that $A \sqcup B = Y$ and neither A nor B contains a limit point of the other.*

Proposition 29. *Suppose U, V are disjoint and open in X. If Y is connected and $Y \subset U \cup V$, then Y lies entirely in one or the other.*

Proof. If not, U and V separate Y, contradicting its connectedness. □

Theorem 9. *A subset of $\mathbb{R}$ is connected if and only if it's an interval.*

Proposition 30. *The continuous images of connected spaces are connected.*

Corollary 4. *If $X \cong Y$, then X is connected if and only if Y is.*

Theorem 10 (Intermediate Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \leq c \leq f(b)$, then there is some $x \in [a, b]$ such that $f(x) = c$.*

Lemma 1. *If $\{A_\alpha\}$ is a collection of connected subspaces of X that all intersect. Then their union $\bigcup_\alpha A_\alpha$ is connected.*

Proposition 31. *If X and Y are connected, then $X \times Y$ is connected.*

Proposition 32. *If A is connected and $A \subset B \subset \overline{A}$, then B is also connected.*

Definition 24. Define an equivalence relation by saying $x \sim y$ if there is a connected component A containing x and y. Then the equivalence classes of $\sim$ are the **(connected) components** of the space.

Theorem 11. *The components of X are closed, disjoint, connected, and union to X. Every connected subset is a subset of a component.*

4.3 Path Connectedness

Definition 25. A **path** in X from x to Y is a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. We say X is **path connected** if we can find paths between all points in X.

Proposition 33. *The continuous images of path connected spaces are path connected.*

Proposition 34. *Path connected spaces are connected.*

Example 10 (The Topologist's Sine Curve)**.** Let

$$T = \{(x, \sin(1/x) \mid x > 0\} \cup \{(0,0)\},$$

then T is connected but not path connected.

4.4 Compactness

Definition 26. A topological space is **compact** if every open cover has a finite subcover.

A finite union of compact spaces is compact since the finite union of finite subcovers is still finite.

Proposition 35. *Continuous images of compact spaces are compact.*

Theorem 12 (Extreme Value Theorem)**.** *If X is compact and $f : X \to \mathbb{R}$, then f attains its infimum and supremum on X.*

Proposition 36. 1. *Compact subsets of Hausdorff spaces are closed.*

2. *Closed subsets of compact spaces are closed.*

Corollary 5. *If $f : X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Add quotient map version here?

Proof. Let A be closed in X, then since X is compact, A must also be compact. Then $f(X)$ is a compact subset of Hausdorff Y, so it is closed. Thus f is a closed map, so its inverse is also continuous. □

Lemma 2 (Tube Lemma)**.** *Suppose Y is compact and $x_0 \in X$. If N is a neighborhood of $x_0 \times Y$, then there is some neighborhood U of x_0 such that $U \times Y \subset N$.*

Theorem 13 (Tychonoff)**.** *Arbitrary products of compact spaces are compact.*

Theorem 14 (Heine-Borel)**.** *$K \subset \mathbb{R}^n$ (with the standard topology) is compact if and only if it's closed and bounded.*

Proposition 37. *If X is compact then every infinite subset $A \subset X$ has a limit point.*

Theorem 15. *If X is a compact metric space, then every sequence has a convergence subsequence.*

Theorem 16 (Lebesgue Number)**.** *If X is a compact metric space with open cover $\mathcal{U}$, then there is some $\delta > 0$ such that all subsets $S \subset X$ with $diam(S) < \delta$ lie entirely in some $U \in \mathcal{U}$.*

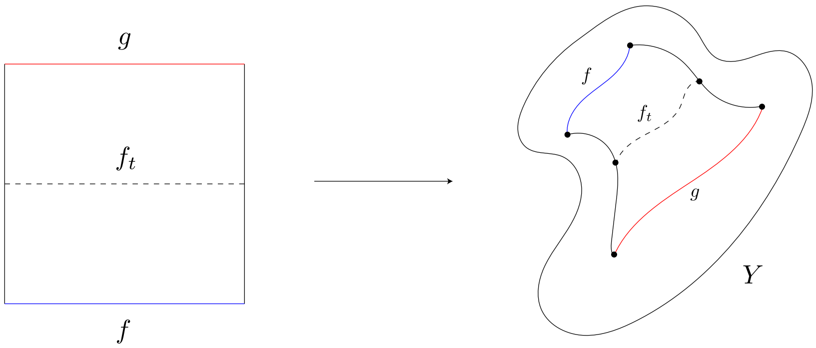
Chapter 5

The Fundamental Group

5.1 Homotopies

Definition 27. Maps f, g are **homotopic**, written $f \simeq g$, if there is some continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

We can denote $F(\cdot, t)$ by f_t. With this notation, $f_0 = f$ and $f_1 = g$.



We say that a map is **nullhomotopic** if it is homotopic to a constant map. Suppose that f and g agree on A, then f and g are **homotopic rel** A if there is a homotopy between them that fixes A. Note that if f and g agree on A and are homotopic via the straight line homotopy, then they are homotopic rel A.

Example 11. The **straight line homotopy** between $f, g : X \to \mathbb{R}^n$ is defined

$$F(x, t) = (1 - t)f(x) + tg(x).$$

If X is convex, then the straight line homotopy can be used to show that any two maps are homotopic.

Definition 28. Two paths f, g are **path homotopic**, written $f \simeq_p g$, if they are homotopic rel $\{0, 1\}$.

We can "multiply" paths α and β by first traveling along α, then β, both at double speed. We denote the product of α and β by $\alpha\beta$. Note that although this is similar to function composition, we read $\alpha\beta$ left to right instead of right to left.

Proposition 38. *Path multiplication respects path homotopy, i.e. if $\alpha \simeq_p \alpha'$ and $\beta \simeq_p \beta'$, then $\alpha\beta \simeq_p \alpha'\beta'$.*

A **loop** at p is a path that starts and ends at p. Path homotopies easily extend to work with loops instead, since loops are just a special type of path.

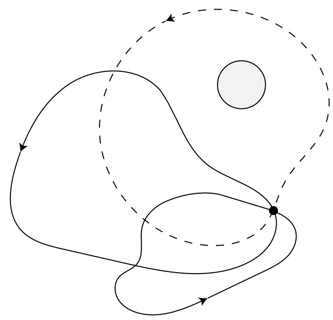


Figure 5.1: The solid loops are homotopic, but the dashed loop is not homotopic to either of the others because of the hole in the space.

5.2 The Fundamental Group

Lemma 3. *Homotopy and homotopy rel A are both equivalence relations.*

Definition 29. The **fundamental group** of X at p is

$$\pi_1(X, p) \doteq \{[\alpha] \mid \alpha \text{ a loop at } p\}$$

with group operation $[\alpha][\beta] \doteq [\alpha\beta]$.

Since path multiplication respects path homotopy, the above group operation is well-defined for all representative α,β.

Proposition 39. *If p, q are in the same path component of X, then $\pi_1(X, p) \cong \pi_1(X, q)$.*

Proof. There must be a path γ from p to q, so if α is a loop at p, then $\gamma^{-1}\alpha\gamma$ is a loop at q. Then it's easy to check that

$$\begin{aligned} \phi : \pi_1(X, p) &\to \pi_1(X, q), \\ [\alpha] &\mapsto [\gamma^{-1}\alpha\gamma] \end{aligned}$$

is a well-defined homomorphism with homomorphic inverse, i.e. an isomorphism. □

Corollary 6. *If X is path connected, then all points in X have isomorphic fundamental groups.*

If we have a continuous map

$$\begin{aligned} f : X &\to Y \\ x &\mapsto y, \end{aligned}$$

then this induces a homomorphism between fundamental groups

$$\begin{aligned} f_* : \pi_1(X, x) &\to \pi_1(Y, y) \\ [\alpha] &\mapsto [f \circ \alpha]. \end{aligned}$$

This being a homomorphism follows from the distributivity of function composition.

Proposition 40. *If we have a sequence of continuous maps*

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

then their induced homomorphisms satisfy $(g \circ f)_* = g_* \circ f_*$.

Theorem 17. *If $X \cong Y$, then $\pi_1(X,p) \cong \pi_1(Y,q)$.*

Proof. Suppose $X \cong Y$ via f, then we can picture the situation as below.

$$\begin{tikzcd} (X,p) \arrow[r, bend left, "f"] & (Y,q) \arrow[l, bend left, "g=f^{-1}"] \end{tikzcd}$$

These two maps then induce homomorphisms between the fundamental groups.

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, bend left, "f_*"] & \pi_1(Y,q) \arrow[l, bend left, "g_*"] \end{tikzcd}$$

Since $f_*g_* = (f \circ g)_* = (1_Y)_* = 1_{\pi_1(X,p)}$ and $g_*f_* = (1_X)_* = 1_{\pi_1(Y,q)}$, the two fundamental groups are isomorphic. □

Proposition 41. $\pi_1(X,p) \times \pi_1(Y,q) \cong \pi_1(X \times Y, (p,q))$.

Definition 30. X if **simply connected** if it is path connected and $\pi_1(X,p)$ is trivial for some p (and thus for all p).

Theorem 18. $\pi_1(S^1) \cong \mathbb{Z}$.

Theorem 19 (Brouwer Fixed Point). *If $X \cong D^2$ (the closed unit disk), then every continuous map $f : X \to X$ has a fixed point.*

5.3 Covering Spaces

Definition 31. A **covering space** of B is a space E and a **covering map** $p : E \to B$. For all $x \in B$, there is a neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of homeomorphic copies of U. Such neighborhoods are called **evenly covered**.

Equivalently, a covering space of is a fiber bundle with discrete fibers. Note that p must be continuous and surjective.

This is more about open sets than points. Once we get U from x, we forget about x in the definition.

Figure.

Example 12. $\mathbb{R}$ is a covering space of S^1. One possible covering map is

$$\begin{aligned} p : \mathbb{R} &\to S^1 \\ t &\mapsto (\cos 2\pi t, \sin 2\pi t). \end{aligned}$$

Definition 32. If $p : E \to B$ is a covering map and $f : X \to B$ is continuous, then we say that $\tilde{f}$ **lifts** f if $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

Proposition 42. *Covering maps are open.*

Do this.

Corollary 7. *Covering maps are quotient maps.*

Why? Also show that the converse isn't necessarily true.

Lifting theorems.