**Exercises Solved:** All.

**Exercise 1** (3 points)**.** *Munkres exercise* 1.9*, p.*15*.*

Collaborators: None.

Let $\mathcal{A}$ be a nonempty collection of subsets of $X$, and let $\mathcal{J}$ be an index set for $\mathcal{A}$. Then DeMorgan's Laws are

$$X - \bigcup_{\alpha\in\mathcal{J}} A_\alpha = \bigcap_{\alpha\in\mathcal{J}} (X - A_\alpha) \text{ and}$$
$$X - \bigcap_{\alpha\in\mathcal{J}} A_\alpha = \bigcup_{\alpha\in\mathcal{J}} (X - A_\alpha).$$

**Law 1:** Let $x \in X - \bigcup_\alpha A_\alpha$, then $x$ is in none of the $A_\alpha$, so $x \in X - A_\alpha$ for all $\alpha$. Thus $x \in \bigcap_\alpha (X - A_\alpha)$, so $X - \bigcup_\alpha A_\alpha \subset \bigcap_\alpha (X - A_\alpha)$. Conversely, let $x \in \bigcap_\alpha (X - A_\alpha)$, then $x$ is not in $A_\alpha$ for any $\alpha$, so $x$ cannot be in their union, i.e. $x \in X - \bigcup_\alpha A_\alpha$. Thus $\bigcap_\alpha (X - A_\alpha) \subset X - \bigcup_\alpha A_\alpha$. Since we have proven both inclusions, this means the two sets are equal.

**Law 2:** This uses the same strategy of proving both inclusions. Let $x \in X - \bigcap_\alpha A_\alpha$, then $x \notin A_\beta$ for some $\beta$, i.e. $x \in X - A_\beta$. Then it is certainly in the union of all possible $X - A_\alpha$, i.e. $x \in \bigcup_\alpha (X - A_\alpha)$. Conversely, let $x \in \bigcup_{(X - A_\alpha)}$, then $x \in X - A_\beta$ for some $\beta$, so $x \notin A_\beta$. Thus $x \notin \bigcap_\alpha A_\alpha$, or equivalently, $x \in X - \bigcap_\alpha A_\alpha$.

**Exercise 2** (6 points)**.** *Munkres exercise* 2.4*, p.*21*.*

Collaborators: None.

a. By the associativity of function composition, $(f \circ g) \circ (g^{-1} \circ f^{-1}) = f \circ (g \circ g^{-1}) \circ f^{-1} = f \circ f^{-1}$, which is just the identity map. Thus $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, so a set $C_0 \subset C$ has the same image under both.

b. Let $f, g$ be injective, and suppose $g(f(x)) = g(f(y))$. Since $g$ is injective, this implies that $f(x) = f(y)$, and since $f$ is injective, this implies that $x = y$. Thus $g \circ f$ is injective.

c. If $g \circ f$ is injective, then we claim that $f$ must be injective but $g$ need not be. Suppose $f(a_1) = f(a_2)$, then since $g$ is a function, $g(f(a_1)) = g(f(a_2))$. Then by the injectivity of $g \circ f$, we have $a_1 = a_2$, so $f$ is injective.

Now consider the maps $f : \{0\} \to \{0,1\}$ and $g : \{0,1\} \to \{0\}$ given by $f(0) = 0$ and $g(0) = g(1) = 0$. Since there is only one element of $C$ and one element of $A$, the composition $g \circ f$ is necessarily injective; however, the map $g$ is not injective.

d. Let $f, g$ be surjective, and suppose $c \in C$. Since $g$ is surjective onto $C$, there is some $b \in B$ such that $g(b) = c$. And since $f$ is surjective onto $B$, there is some $a \in A$ such that $f(a) = b$. Then $g(f(a)) = c$, so $g \circ f$ is surjective onto $C$.

e. If $g \circ f$ is surjective, then we claim that $g$ must be surjective but $f$ need not be. Since $g \circ f$ is surjective, then for all $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$. Then $g$ maps $f(a) \in B$ to $c$. Thus $g$ maps elements of $B$ onto every element of $C$.

   Now consider the counterexample maps $f$ and $g$ from part (c). The composition $g \circ f$ is surjective, but $f$ is not.

f. The composition of injective functions is injective, and the composition of surjective functions is surjective. Conversely, the second map of a surjective composition must be surjective while the first map of an injective composition must be injective.

**Exercise 3** (5 points)**.** *Find a countable basis that generates the standard topology on* $\mathbb{R}^n$.

Collaborators: Lucas Fagan, Michael Liu.

We claim that

$$\mathcal{C} = \{B\,(p,q) \mid p \in \mathbb{Q}^n, q \in \mathbb{Q}\}$$

is a countable basis for the standard topology on $\mathbb{R}^n$. We must first show that this set is countable. Since $\mathbb{Q}$ is countable, so is each set

$$\mathcal{C}_p = \{B(p,q) \mid q \in \mathbb{Q}\}.$$

Additionally, since the finite product of countable sets is countable, $\mathbb{Q}^n$ is countable. Then since the countable union of countable sets is countable, this means $\mathcal{C} = \bigcup_{p \in \mathbb{Q}^n} \mathcal{C}_p$ is countable.

Denote the topology generated by $\mathcal{C}$ by $\mathcal{T}_C$, and denote the standard topology on $\mathbb{R}^n$ by $\mathcal{T}_S$. To show that $\mathcal{C}$ generates the standard topology, we will show that $\mathcal{T}_S \subset \mathcal{T}_C$ and $\mathcal{T}_C \subset \mathcal{T}_S$.

Since $\mathbb{Q}^n$ is a subset of $\mathbb{R}^n$, the usual basis $\mathcal{B} = \{B(x, \varepsilon) \mid x \in \mathbb{R}^n, \varepsilon > 0\}$ for the standard topology contains $\mathcal{C}$. Thus $\mathcal{T}_C \subset \mathcal{T}_S$.

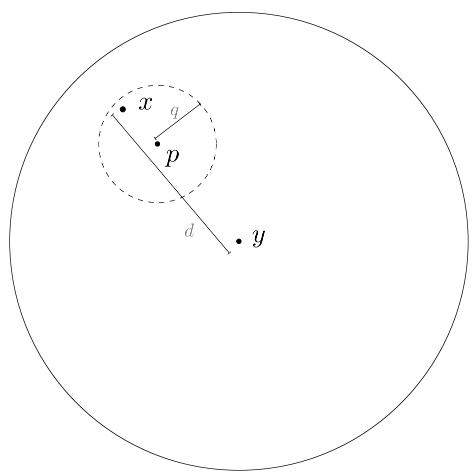
To show that $\mathcal{T}_S \subset \mathcal{T}_C$, we show that for all $B(y, \varepsilon) \in \mathcal{B}$ and $x \in B(y, \varepsilon)$, there is some $C \in \mathcal{C}$ such that $x \in C \subset B(y, \varepsilon)$. Let $B(y, \varepsilon)$ be an arbitrary element of $\mathcal{B}$, and let $x \in B(y, \varepsilon)$, then $\|x - y\| = d$ for some $d < \varepsilon$. Also, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a $p \in \mathbb{Q}^n$ such that

$$\|y - p\| < \|y - x\| = d \text{ and } \|x - p\| < \varepsilon - d,$$

i.e. $p$ is closer to $y$ than $x$ is and $p$ is closer to $x$ than $x$ is to the border of $B(y, \varepsilon)$. Let $q$ be any rational number smaller than $\varepsilon - d$, and choose $C = B(p, q)$, then $x$ is in $C$ and for all $z \in C$,

$$\|z - y\| \leq \|z - q\| + \|q - y\| < \varepsilon - d + d = \varepsilon.$$

Thus $C$ is contained in $B(y, \varepsilon)$, so $\mathcal{T}_S \subset \mathcal{T}_C$.




Since we have shown both inclusions, it follows that the topology generated by $\mathcal{C}$ is the same as the standard topology on $\mathbb{R}^n$.

**Exercise 4** (5 points). *Let $T$ be the collection of subsets of $\mathbb{R}$ consisting of $\emptyset$ and every set $U$ such that $\mathbb{R} \setminus U$ is finite. Show that $T$ is a topology for $\mathbb{R}$. If $T_S$ is the standard topology for $\mathbb{R}$, is $T = T_S$, $T \subseteq T_S$ or $T_S \subseteq T$?*

Collaborators: None.

$T$ **is a topology:** To show that $T$ is a topology on $\mathbb{R}$, we must show that it contains $\varnothing$ and $\mathbb{R}$ and is closed under arbitrary unions and finite intersections.

a. By definition, $T$ contains $\varnothing$. Then $\mathbb{R} - \mathbb{R} = \varnothing$, which is finite, so $\mathbb{R} \in T$.

b. Consider the arbitrary union $\bigcup_\alpha U_\alpha$ of elements of $T$. Its complement $\mathbb{R} - \bigcup_\alpha U_\alpha$ is equivalent to, by DeMorgan's Laws, $\bigcap_\alpha (\mathbb{R} - U_\alpha)$. Each of the $(\mathbb{R} - U_\alpha)$ is finite by assumption, so their intersection must also be finite. Thus $T$ is closed under arbitrary unions.

c. Consider the finite intersection $\bigcap_{i=1}^N U_i$ of elements of $T$. Then again by DeMorgan's Laws, $\mathbb{R} - \bigcap_{i=1}^N U_i = \bigcup_{i=1}^N (\mathbb{R} - U_i)$. Each $(\mathbb{R} - U_i)$ is finite, and the finite union of finite sets is itself finite. Thus $T$ is closed under finite intersections.

$T$ **is strictly coarser than** $T_S$**:** We claim that $T$ is a proper subset of $T_S$. Any element of $T$ has a complement of the form $\{x_i\}_{i=1}^N$, so any element of $T$ is of the form $\mathbb{R} - \{x_i\}_{i=1}^N$. If we order the $x_i$ in increasing order, this coincides with the set

$$(-\infty, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_{N-1}, x_N) \cup (x_N, \infty),$$

which is open in $T_S$ since it is the union of open intervals of $\mathbb{R}$. Thus $T \subset T_S$.

This is in fact a strict inequality. Take arbitrary $a < b$, then the interval $(a, b)$ is open in $\mathbb{R}$, yet its complement is $\mathbb{R} - (a, b) = (-\infty, a] \cup [b, \infty)$ is infinite. Thus $(a, b)$ is in $T_S$ but not in $T$, so $T$ is a strict subset of $T_S$.

**Exercise 5** (5 points)**.** *Let $C$ denote the unit circle $\{(x, y) \mid x^2+y^2 = 1\}$ in $\mathbb{R}^2$ and let $[0, 1)$ denote the half-open interval $\{t \mid 0 \leq t < 1\}$ in $\mathbb{R}$. Endow $C$ and $[0, 1)$ with the subspace topology from $\mathbb{R}^2$ and $\mathbb{R}$, respectively. Define $f : [0, 1) \to C$ by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Is $f$ a homeomorphism?*

Collaborators: None.

The map $f$ is *not* a homeomorphism, as its inverse is not continuous. We demonstrate this by finding an open set $U$ in $[0, 1)$ such that $(f^{-1})^{-1}(U) = f(U)$ is not open in $C$.

Let $U = [0, 1/2)$. This is open in $[0, 1)$, as it is the intersection of $[0, 1)$ and an open set, say $(-1, 1/2)$, of $\mathbb{R}$. Then $f(U)$ is the upper half of the unit circle, including the point $(1, 0)$ and excluding the point $(-1, 0)$. This is *not* open in $C$, though, since for any $\varepsilon > 0$, the ball $B((1, 0), \varepsilon)$ intersected with $C$ is not entirely contained in $C$.