MATH 531 HOMEWORK 8

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Page 210, Exercise 4.8.4. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be integrable and $f \leq M$. Prove that*

$$\int_a^b f(x)\,dx \leq (b-a)M.$$

Since f is Riemann integrable, we know $\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx$. Expanding the definition of the upper intergral gives

$$\begin{aligned}\int_a^b f(x)\,dx &= \overline{\int_a^b} f(x)\,dx \\ &= \inf_P \{U(f,P)\}.\end{aligned}$$

Replacing the infimum over partitions with any fixed partition P, we get the bound

$$\begin{aligned}&\leq U(f,P) \\ &= \sum_{i=0}^{n-1} \sup_{x\in[x_i,x_{i+1}]} f(x)(x_{i+1}-x_i).\end{aligned}$$

Finally, we use the fact that f is bounded above by M to get

$$\begin{aligned}&\leq \sum_{i=0}^{n-1} M(x_{i+1}-x_i) \\ &= (b-a)M.\end{aligned}$$

This is the desired bound.

Page 211, Exercise 4.8.7. *Let $f : [0,1] \to \mathbb{R}$, $f(x) = 1$ if $x = 1/n$, n an integer, and $f(x) = 0$ otherwise.*

1. *Prove that f is integrable.*
2. *Show that $\int_0^1 f(x)\,dx = 0$.*

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1. To show that f is Riemann integrable, we must show that its upper and lower integrals are equal. First, note that any subinterval of $[0,1]$ contains an irrational number since the irrationals are dense in $\mathbb{R}$. Since $1/n$ is the form of a rational

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number, this means that every subinterval of $[0,1]$ contains at least one point x satisfying $f(x)=0$. Thus for any partition P of $[0,1]$, the lower sum is

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n-1} \inf_{x\in[x_i,x_{i+1}]} f(x)(x_{i+1}-x_i) \\ &= \sum_{i=1}^{n-1} 0x_{i+1}-x_i) \\ &= 0. \end{aligned}$$

Thus $\sup_P \{L(f,P)\} = \underline{\int_0^1} f(x)\,dx = 0$.

Now fix $n\in\mathbb{N}$, and let $[x_0,x_1]=[0,\sqrt{2}/n]$. Construct a partition P_n of $[0,1]$ containing x_0 and x_1 as its first two points such that all subsequent intervals have length no more than $1/n^2$. Then we can bound the upper sum as follows.

$$\begin{aligned} U(f,P_n) &= \sum_{i=0}^{n-1} \sup_{x\in[x_i,x_{i+1}]} f(x)(x_{i+1}-x_i) \\ &= \sup_{x\in[x_0,x_1]}(x_1-x_0) + \sum_{i=1}^{n-1} \sup_{x\in[x_i,x_{i+1}]} f(x)(x_{i+1}-x_i). \end{aligned}$$

We can bound every supremum term by 1, resulting in the upper bound

$$\begin{aligned} &\le (x_1-x_0) + \sum_{i=1}^{n-1}(x_{i+1}-x_i) \\ &\le \frac{\sqrt{2}}{n} + \sum_{i=1}^{n-1}\frac{1}{n^2} \\ &= \frac{\sqrt{2}}{n} + \frac{n-1}{n^2} \\ &\le \frac{\sqrt{2}+1}{n}. \end{aligned}$$

Taking the limit as n increases, we get

$$\lim_{n\to\infty} U(f,P_n) \le \lim_{n\to\infty}\frac{\sqrt{2}+1}{n} = 0.$$

Thus $\inf_P \{U(f,P)\} = \overline{\int_0^1} f(x)\,dx = 0 = \underline{\int_0^1} f(x)\,dx$, so f is Riemann integrable.

(2) Since both the lower and upper integral are 0, the Riemann integral $\int_0^1 f(x)\,dx$ is also 0.

Page 211, Exercise 4.8.8. *Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable and $|f(x)|\le M$. Let $F(x)=\int_a^x f(t)\,dt$. Prove that $|F(y)-F(x)|\le M|y-x|$. Deduce that F is continuous. Does this check with Example 4.8.10?*

The case $x = y$ is trivial, so assume x and y are distinct. If $x < y$, then we have

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right| \\ &= \left| \int_x^y f(t)\,dt \right| \\ &= \left| \inf_P U(f,P) \right| \end{aligned}$$

where the infimum is over all partitions P of $[x, y]$. Selecting an arbitrary partition $\tilde{P}$ of the form $\{x_0 = x, x_1, \ldots, x_n = y\}$ gives the bound

$$\begin{aligned} &\le \left| U(f,\tilde{P}) \right| \\ &= \left| \sum_{i=0}^{n-1} \sup_{x \in [x_i, x_{i+1}]} f(x)(x_{i+1} - x_i) \right| \\ &\le \sum_{i=0}^{n-1} \sup_{x \in [x_i, x_{i+1}]} |f(x)|\,(x_{i+1} - x_i) \\ &\le \sum_{i=0}^{n-1} M(x_{i+1} - x_i) \\ &= M(y - x). \end{aligned}$$

So $|F(y) - F(x)| \le M(y - x)$ when $x < y$. Similarly, if $y < x$, then $|F(y) - F(x)| \le M(x - y)$. Putting these together yields the desired inequality

$$|F(y) - F(x)| \le M|y - x|.$$

Now fix $\varepsilon > 0$ and set $\delta = \varepsilon/M$. If $|y - x| < \delta$, then

$$|F(y) - F(x)| \le M|y - x| < M\frac{\varepsilon}{M} = \varepsilon,$$

so F is continuous on $[a, b]$. Furthermore, since δ does not depend on the specific x and y being used, F is uniformly continuous on $[a, b]$.

This checks with Example 4.8.10, as continuity of a function is not enough to guarantee differentiability.

Page 235, Exercise 4.41. *Prove that*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

and

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

We can rewrite $\sum_{k=1}^{n} k$ as

$$\begin{aligned}\sum_{k=1}^{n} k &= \frac{1}{2}\left(\sum_{k=1}^{n} k + \sum_{k=1}^{n} k\right)\\ &= \frac{1}{2}\left(\sum_{k=1}^{n} k + \sum_{k=1}^{n}(n-k+1)\right)\\ &= \frac{n(n+1)}{2},\end{aligned}$$

as desired.

We can prove the second identity by induction. When $n = 1$, we have $\sum_{k=1}^{n} k^2 = 1$ and $n(n+1)(2n+2) = 1(2)(3)/6 = 1$. Assuming the identity holds for some arbitrary n, we must show it holds for $n+1$. We have

$$\begin{aligned}\sum_{k=1}^{n=1} k^2 &= (n+1)^2 + \sum_{k=1}^{n} k^2\\ &= \frac{6(n+1)^2 + n(n+1)(2n+1)}{6}\\ &= \frac{(n+1)(2n^2+7n+6)}{6}\\ &= \frac{(n+1)(n+2)(2n+3)}{6}\\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6},\end{aligned}$$

so the identity holds for all n.

Page 236, Exercise 4.42. *For $x > 0$, define $L(x) = \int_1^x (1/t)\ dt$. Prove the following, using this definition:*

(1) *L is increasing in x.*
(2) *$L(xy) = L(x) + L(y)$.*
(3) *$L'(x) = 1/x$.*
(4) *$L(1) = 0$.*
(5) *Properties **c** and **d** uniquely determine L. What is L?*

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(1) Let $x' \geq x > 0$, then

$$\begin{aligned}L(x') - L(x) &= \int_1^{x'} \frac{1}{t}\ dt - \int_1^{x} \frac{1}{t}\ dt\\ &= \int_1^{x} \frac{1}{t}\ dt + \int_x^{x'} \frac{1}{t}\ dt - \int_1^{x} \frac{1}{t}\ dt\\ &= \int_x^{x'} \frac{1}{t}\ dt.\end{aligned}$$

Since $1/t \geq 0$ for all t and $x' \geq x$, this integral is nonzero, so L is increasing in x.

(2) We have

$$\begin{aligned} L(xy) &= \int_1^{xy} \frac{1}{t}\, dt \\ &= \int_1^{x} \frac{1}{t}\, dt + \int_x^{xy} \frac{1}{t}\, dt. \end{aligned}$$

Now let $u = t/x$, then this becomes

$$\begin{aligned} &= \int_1^{x} \frac{1}{t}\, dt + \int_1^{y} \frac{1}{u}\, du \\ &= L(x) + L(y), \end{aligned}$$

as desired.

(3) Let G be any antiderivative of $1/x$, then the derivative of L is

$$\begin{aligned} L'(x) &= \frac{d}{dx} \int_1^{x} \frac{1}{t}\, dt \\ &= \frac{d}{dx} [G(x) - G(1)] \\ &= \frac{1}{x} + 0, \end{aligned}$$

so $L'(x) = 1/x$.

(4) Let G again be any antiderivative of $1/x$, then $L(1)$ is

$$\begin{aligned} L(1) &= \int_1^{1} \frac{1}{t}\, dt \\ &= G(1) - G(1) \\ &= 0, \end{aligned}$$

so $L(1) = 0$.

(5) L is ln. In class we showed that the derivative of $\ln(x)$ is $1/x$, so by (c), $L(x) = \ln(x)+C(x)$, where $C'(x) = 0$. But by (d), we know $L(1) = \ln(1)+C(1) = C(1) = 0$. Since $C(1)$ is just a constant, this means all terms in C must be 0. Thus $L(x) = \ln(x)$.

Page 236, Exercise 4.44. *Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable and suppose for every a, b with $0 \le a < b \le 1$ there is a c with $a < c < b$ and $f(c) = 0$. Prove $\int_0^1 f = 0$. Must f be zero? What if f is continuous?*

Consider any partition P of $[0, 1]$ of the form $\{x_0 = 0, x_1, \ldots, x_n = 1\}$. By assumption, the interval $[x_i, x_{i+1}]$ contains a point c_i such that $f(c_i) = 0$. Thus the infimum of f on this interval is 0.

Since f is Riemann integrable, the value of the integral is equal to $\sup_P L(f, P)$. Expanding the lower sum into its full definition gives

$$\int_0^1 f(x)\, dx = \sup_P \sum_{i=0}^{n-1} \inf_{[x_i, x_{i+1}]} f(x)(x_{i+1} - x_i).$$

We just showed that each infimum term in this summation is 0, so $L(f, P) = 0$ for every partition P. Since it is true for every partition, $\sup_P L(f, P) = 0$ as well, so $\int_0^1 f$ must itself be 0.

With no further constraints, f need not be the zero function. Consider the function

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$

defined for all $x \in [0,1]$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, this function satisfies the conditions from the original problem, and thus $\int_0^1 f = 0$ even though f in this case is clearly not the zero function.

If f is continuous, however, it must be the zero function, which we show by contradiction. To begin, note that since f is continuous and it is defined on the compact set $[0,1]$, f is in fact uniformly continuous on $[0,1]$. Thus for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ when $|y - x| < \delta$.

Suppose $f(z) \neq 0$ for some $z \in [0,1]$, i.e. $|f(z)| = \varepsilon$ for some $\varepsilon > 0$. Now take $x \in [0,1]$ such that $|z-x| < \delta$, then $|f(x)-f(z)| < \varepsilon$. This means that $f(x)$ is not 0 either. Moreover, $f(x)$ must be the same sign as $f(z)$. If $x < z$, consider the interval $[x,z]$, and if $z < x$, consider the interval $[z,x]$. For all y in this interval, $|y - z| < \delta$, so $|f(x) - f(z)| < \varepsilon$. Since $f(x)$ and $f(z)$ are the same sign, $f(y)$ is also nonzero. We have found an interval with no roots of f, which contradicts the assumption that all intervals of $[0,1]$ contain a root of f. Thus $f(z) = 0$ for all $z \in [0,1]$, i.e. f is the zero function.

Page 236, Exercise 4.45. *Prove the following* ***second mean value theorem****. Let f and g be defined on $[a,b]$ with g continuous, $f \geq 0$, and f integrable. Then there is a point $x_0 \in (a,b)$ such that*

$$\int_a^b f(x)g(x)\,dx = g(x_0)\int_a^b f(x)\,dx.$$

Let $m = \inf\{g([a,b])\}$, and let $M = \sup\{g([a,b])\}$, then clearly $m \leq g(x) \leq M$ for all $x \in [a,b]$. Since $f \geq 0$, the integral $\int_a^b f$ is also non-negative. Thus we have

$$m\int_a^b f(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b f(x)\,dx.$$

Now consider $h(t) = t\int_a^b f(x)\,dx$. For fixed f, this is continuous with respect to t (it is just a linear function). Since $\int_a^b f(x)g(x)\,dx$ lies in the connected set $[h(m), h(M)]$, by the intermediate value theorem we know there exists $t_0 \in [m,M]$ such that

$$t_0\int_a^b f(x)\,dx = \int_a^b f(x)g(x)\,dx.$$

Since $[a,b]$ is connected, g is continuous, and $m \leq t_0 \leq M$, by the intermediate value theorem again we know there exists $x_0 \in [a,b]$ such that $g(x_0) = t_0$. Thus we have

$$t_0\int_a^b f(x)\,dx = g(x_0)\int_a^b f(x)\,dx = \int_a^b f(x)g(x)\,dx,$$

which gives us the desired equality.