Problems completed: All.

Exercise 1. Munkres §19, pg. 118 #6.

Collaborators: None.

> **Lemma 1.** *In a general topological space, a sequence x_n converges to x if and only if x_n is eventually in every subbasis element containing x.*

Proof. Let X be a topological space with topology generated by subbasis $\mathcal{S}$. Suppose $x_n \to x$ in X, then since each subbasis element is itself an open set, it follows from the definition of convergence that x_n is eventually in each $S \in \mathcal{S}$.

Conversely, suppose that x_n is eventually in each $S \in \mathcal{S}$ containing x. Let U be an arbitrary neighborhood of x, then

$$U = \bigcup_{\alpha} \bigcap_{i=1}^{N} S_{\alpha,i},$$

where each $S_{\alpha,i}$ is a subbasis element containing x. Fix arbitrary α, and let N_i be the point at which the sequence crosses into $S_{\alpha,i}$ permanently. Then for $n > \max_i N_i$, the sequence lies in the entire intersection. Thus x_n is eventually in every intersection, meaning that it's eventually in their union U. $\square$

With this lemma, the desired characterization of convergence in the product topology almost follows directly from just the definitions.

Forward: Suppose $\mathbf{x}_n \to \mathbf{x}$, then $\mathbf{x}_n$ is eventually in every subbasis element $\pi_\alpha^{-1}(U_\alpha)$ containing $\mathbf{x}$, so $\pi_\alpha(\mathbf{x}_n)$ is eventually in all neighborhoods $\pi_\alpha(\pi_\alpha^{-1}(U_\alpha)) = U_\alpha$ of $\pi_\alpha(\mathbf{x})$. Thus $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ for all α.

Backward: Suppose $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ for all α, then $\pi_\alpha(\mathbf{x}_n)$ is eventually in every neighborhood U_α of $\pi_\alpha(\mathbf{x})$. Then $\mathbf{x}_n$ is eventually in every subbasic neighborhood $\pi_\alpha^{-1}(U_\alpha)$ of $\mathbf{x}$, so $\mathbf{x}_n \to \mathbf{x}$.

Box topology part: The box topology doesn't exhibit this behavior. Consider the sequence in $\prod_{i\in\mathbb{Z}^+} \mathbb{R}$ with components $\pi_i(\mathbf{x}_n) = 1/n$. Each component $\pi_\alpha(\mathbf{x}_n)$ converges to 0, but we claim that $\mathbf{x}_n$ does *not* converge to the zero sequence. To see this, take the open set

$$U = \prod_{i\in\mathbb{Z}^+} \left(-\frac{1}{i}, \frac{1}{i}\right),$$

which clearly contains the zero sequence. Fix n, then note that $\mathbf{x}_n$ is not contained in U since $1/n$ is not in any of the intervals $(-1/i, 1/i)$ for $i \geq n$. Since n was arbitrary, this means that $\mathbf{x}_n$ is never contained in U for any n, so $\mathbf{x}_n$ cannot converge to the zero sequence.

Exercise 2. Munkres §19, pg. 118 #7.

Collaborators: None.

Box Topology: In the box topology, $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$. It suffices to show that $\mathbb{R}^\infty$ is closed, which is equivalent to showing that $\mathbb{R}^\omega - \mathbb{R}^\infty$ is open.

Let $x \in \mathbb{R}^\omega - \mathbb{R}^\infty$, then x is not eventually 0. We can define an open set U by $U = \prod U_i$, where

$$U_i = \begin{cases} B(x_i, |x_i|/2) & x_i \neq 0, \\ \mathbb{R} & x_i = 0 \end{cases}.$$

Note that since U has infinitely many U_i that are not $\mathbb{R}$, it is open in the box topology but not the product topology. It clearly contains x, and we claim that it lies entirely in $\mathbb{R}^\omega - \mathbb{R}^\infty$. Since any $y \in \mathbb{R}^\infty$ is eventually 0 and our x isn't, there must be some i such that $U_i = B(x_i, |x_i|/2)$ (which doesn't contain 0) and $y_i = 0$. Since y was arbitrary, U does not contain any elements of $\mathbb{R}^\infty$. Thus $\mathbb{R}^\omega - \mathbb{R}^\infty$ is open, so $\mathbb{R}^\infty$ is closed, so $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$.

Product Topology: In the product topology, $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$. Let $x \in \mathbb{R}^\omega$ be arbitrary, then we will show that any neighborhood of x intersects $\mathbb{R}^\infty$, making x a limit point of $\mathbb{R}^\infty$.

Let U be any neighborhood of x, then $U = \prod U_i$, where only finitely many of the U_i are not $\mathbb{R}$. This means U_i is eventually $\mathbb{R}$, so the sequence y given by

$$y_i = \begin{cases} \text{any } u_i \in U_i & U_i \neq \mathbb{R}, \\ 0 & U_i = \mathbb{R} \end{cases}$$

is contained in U and is eventually 0, i.e. in $\mathbb{R}^\infty$. Thus every neighborhood of x intersects $\mathbb{R}^\infty$, and since x was arbitrary, this means all points of $\mathbb{R}^\omega$ are limit points of $\mathbb{R}^\infty$, i.e. $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

Exercise 3. For any sequence of real numbers $x = \{x_n\}_{n=1}^{\infty}$, define $\|x\|_{\infty} = \sup_n |x_n|$. Let ℓ^{∞} be the collection of all sequences x satisfying $\|x\|_{\infty} < \infty$ and define $d(x, y) = \|x - y\|_{\infty}$. Prove (ℓ^{∞}, d) is a metric space.

Collaborators: None.

To begin with, since ℓ^{∞} has sequences with bounded supremum norm, we know that d is a function into $\mathbb{R}$ and not $\mathbb{R} \cup \{\infty\}$. Now we show that d is a metric, which will make (ℓ^{∞}, d) a metric space.

1. d is non-negative: $d(x, y) = \sup_i |x_i - y_i| \geq 0$.
2. $d(x, y) = \sup_i |x_i - y_i| = 0$ if and only if $x_i - y_i = 0$ for all i, which implies $x = y$.
3. d is symmetric: $d(x, y) = \sup_i |x_i - y_i| = \sup_i |y_i - x_i| = d(y, x)$.
4. Triangle inequality: For any sequence z, we have $d(x, y) = \sup_i |x_i - y_i| = \sup_i |x_i - z_i + z_i - y_i| \leq \sup_i |x_i - z_i| + \sup_i |z_i - y_i| = d(x, z) + d(z, y)$.

Exercise 4. Munkres, §21, pg. 133 #1.

Collaborators: None.

By definition of the metric topology, the topology on X is given by the basis

$$\mathcal{B}_X = \{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}.$$

Then by definition of the subspace topology, the topology on A as a subspace of X is given by the basis

$$\mathcal{B}_A = \{B_d(x, \varepsilon) \cap A \mid x \in X, \varepsilon > 0\}.$$

If $x \in A \subset X$, then $B_d(x, \varepsilon) \cap A$ can be written $B_{d|A \times A}(x, \varepsilon)$. And if $x \in X - A$, then there are two cases. If a particular ε-ball around x does not intersect A, then since we've already shown that any ε-ball can be expressed as the union of balls contained in it, we can express $B_d(x, \varepsilon) \cap A$ as the union of balls with centers in A based on the metric $d|A \times A$. Thus we can express $\mathcal{B}_A$ as

$$\mathcal{B}_A = \left\{D_{d|A \times A}(x, \varepsilon) \mid x \in A, \varepsilon > 0\right\},$$

so the topology on A as a subspace of X is induced by the metric $d|A \times A$.

Exercise 5. Wasserstein video.

I watched the video.