1. Taking the derivative of the proposed solution $\mathbf{y}(t) = \sum_{i=1}^{k} c_i e^{\lambda}$ gives

$$\mathbf{y}'(t) = \sum_{i=1}^{k} c_i \lambda_i \mathbf{v}_i e^{\lambda_i t}$$

Since $\mathbf{v}_i$ is the eigenvector corresponding to eigenvalue λ_i, this becomes

$$\begin{aligned} &= \sum_{i=1}^{k} c_i A \mathbf{v}_i e^{\lambda_i t} \\ &= A \sum_{i=1}^{k} c_i \mathbf{v}_i e^{\lambda_i t} \\ &= A\mathbf{y}(t) \end{aligned}$$

Thus $\mathbf{y}$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

2. (i) The differential of $\mathbf{F}$ is

$$D\mathbf{F} = \begin{pmatrix} \frac{d\mathbf{F}_1}{dS} & \frac{d\mathbf{F}_1}{dE} & \frac{d\mathbf{F}_1}{dI} & \frac{d\mathbf{F}_1}{dR} \\ \frac{d\mathbf{F}_2}{dS} & \ddots & & \vdots \\ \frac{d\mathbf{F}_3}{dS} & & \ddots & \vdots \\ \frac{d\mathbf{F}_4}{dS} & \frac{d\mathbf{F}_4}{dE} & \frac{d\mathbf{F}_4}{dI} & \frac{d\mathbf{F}_4}{dR} \end{pmatrix} = \begin{pmatrix} -\beta I & 0 & -\beta S & 0 \\ \beta I & -a & \beta S & 0 \\ 0 & a & -\gamma & 0 \\ 0 & 0 & \gamma & 0 \end{pmatrix}$$

(ii) The particular matrix $D\mathbf{F}(y_0)$, where $y_0 = (1-v, 0, 0, v)$, is

$$D\mathbf{F}(y_0) = \begin{pmatrix} 0 & 0 & -\beta(1-v) & 0 \\ 0 & -a & \beta(1-v) & 0 \\ 0 & a & -\gamma & 0 \\ 0 & 0 & \gamma & 0 \end{pmatrix}$$

(iii) When $v = 0$, the submatrix M_0 corresponding to the E and I rows is

$$M_0 = \begin{pmatrix} -a & \beta \\ a & -\gamma \end{pmatrix}$$

The trace of M_0 is $\mathrm{tr}(M_0) = \lambda_1 + \lambda_2 = -a - \gamma$, and the determinant is $\det(M_0) = \lambda_1 \lambda_2 = a(\gamma - \beta)$. Since a and γ are both positive, the trace must be negative. There are then two possible cases

- If the determinant is positive, then both eigenvalues have the same sign. Since two positives cannot add together to yield a negative, both eigenvalues must be positive.
- If the determinant is negative, then the eigenvalues are opposite signs. This means one is positive and one is negative.

If $\beta < \gamma$, then the determinant $a(\gamma - \beta)$ is positive. Thus both eigenvalues are negative. If $\beta > \gamma$ instead, then the determinant is negative and we have one positive and one negative eigenvalue.

(iv) For a linear homogeneous ODE $\mathbf{y}' = A\mathbf{y}$, the origin is unstable if $\mathrm{Re}(\lambda_i) > 0$ for any eigenvalue λ_i of A. For the given two-dimensional system $\mathbf{z}' = M_v \mathbf{z}$, the matrix

$$M_v = \begin{pmatrix} -a & \beta(1-v) \\ a & -\gamma \end{pmatrix}$$

has trace $\lambda_1 + \lambda_2 = -a - \gamma < 0$ and determinant $\lambda_1 \lambda_2 = a(\gamma - \beta(1-v))$. We can now use the cases from the previous question to determine the signs of the eigenvalues of M_v.

As before, since the trace is negative, a negative determinant implies eigenvalues with opposite signs, which is the only way to a positive eigenvalue for this matrix. Thus we have

$$\begin{aligned} \mathrm{Re}(\lambda_i) \text{ for some } i &\iff \det(M_v) \\ &\iff a(\gamma - \beta(1-v)) < 0 \end{aligned}$$

Since $a > 0$, this becomes

$$\iff \gamma - \beta(1-v) < 0$$

If $v = 1$, then we have the inequality $\gamma < 0$, which is always false. This means that if $v = 1$, then the origin is stable. If $v \neq 1$, then the inequality becomes $\beta > \gamma/(1-v)$. So if β is greater than this threshold, i.e. $\beta \in (\gamma/(1-v), \infty)$, the origin is unstable.

(v) If the flow rate from S to E is high enough, i.e. enough people get exposed to the disease on a regular basis, then the disease will persist. This threshold increases as γ (the flow rate from I to R) increases. This makes sense, as a faster recovery rate would lead to fewer overall cases). The threshold also increases as v increases. This also makes sense, as an increased proportion of vaccinated individuals will lead to fewer people being susceptible.

The threshold does *not* depend on the latency parameter a. Thus when determining if a disease will die out (stable equilibrium at 0 cases) or continue to grow (unstable equilibrium at 0 cases), what matters is the rate of exposure, not the rate of infection for those already exposed.