MANIFOLDS

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Based on *An Introduction to Manifolds* by L. Tu.

CONTENTS

1 FOUNDATIONS: EUCLIDEAN SPACE

1.1 REMINDERS

We say f is **real analytic** at p if it's equal to its Taylor series at p in some neighborhood of p. Note that if f is real analytic, then it's also C^∞ (the converse isn't true in general, though).

Proposition 1 (Baby Taylor's Theorem with Remainder)**.** Let U be open in $\mathbb{R}^n$ and star-convex wrt p. If f is C^∞ on U, then there are C^∞ functions $g_1, \ldots, g_n$ on U such that

$$f(x) = f(p) + \sum_i (x^i - p^i) g_i(x)$$

and $g_i(p) = \frac{\partial f}{\partial x^i}(p)$ for all i.

Proof. Since U is star-convex wrt p, we can draw a straight line from p to any $x \in U$. Intuitively, $f(x)$ should be $f(p)$ plus all the changes in f along this line. We can use the FToC to formalize this:

$$f(x) - f(p) = \int_0^1 \frac{d}{dt} f(p + t(x-p))\, dt.$$

We can use the chain rule to evaluate $\frac{d}{dt} f(p + t(x-p))$, giving

$$f(x) - f(p) = \sum (x^i - p^i) \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x-p))\, dt.$$

Set $g_i(x)$ to be its respective integral in the above sum. □

1.2 TANGENT VECTORS AS DERIVATIONS