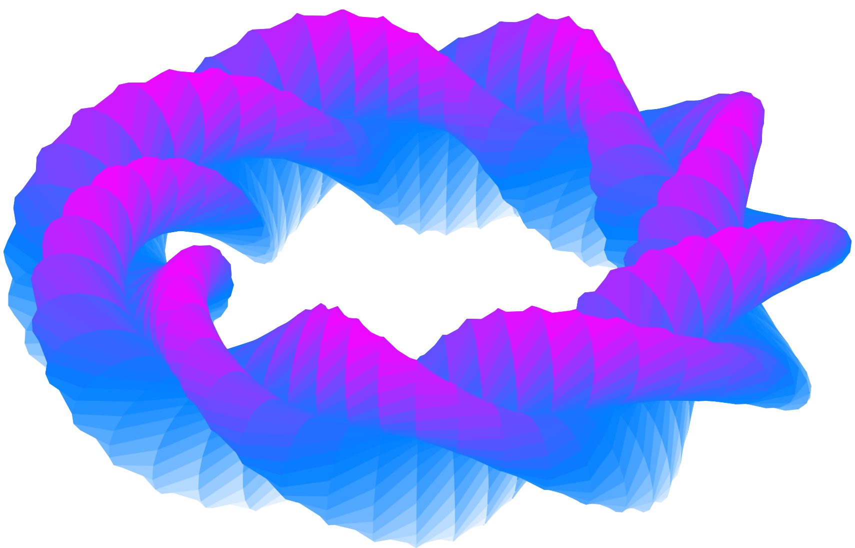
Differential Geometry

Calculus on Surfaces and Riemannian Geometry

Braden Hoagland



Contents

Chapter 1

Calculus on Euclidean Space

1.1 Differentiable Functions

The **coordinate functions** are maps from $\mathbb{R}^n$ to $\mathbb{R}$, essentially picking out a single coordinate. They are given by

$$\begin{aligned} x_i : \mathbb{R}^n &\to \mathbb{R} \\ \mathbf{p} &\mapsto \mathbf{p}_i. \end{aligned}$$

If I'm working in $\mathbb{R}^3$, I'll probably use x, y, and z instead of x_1, x_2, and x_3.

We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** if f can be written as $f(x_1, x_2, \ldots, x_n)$ and all partial derivatives of all orders exist and are continuous. A differentiable function is also called a **scalar field**.

The set of all differentiable functions/scalar fields forms a ring, so we can perform basic arithmetic with them.

1.2 Tangent Vectors and Vector Fields

Definition 1. *Let $\mathbf{p} \in \mathbb{R}^n$. Then the **tangent space** $T_{\mathbf{p}}(\mathbb{R}^n)$ of $\mathbb{R}^n$ at $\mathbf{p}$ is the set of all tangent vectors in $\mathbb{R}^n$ originating at $\mathbf{p}$. The collection of all tangent spaces of $\mathbb{R}^n$ is the **tangent bundle** $T(\mathbb{R}^n)$ of $\mathbb{R}^n$.*

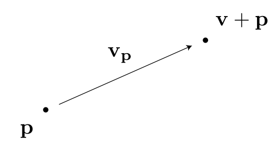


Figure 1.1: A tangent vector $\mathbf{v_p}$ of $\mathbf{p}$ is an element of $T_\mathbf{p}(\mathbb{R}^n)$, and it's just the arrow from $\mathbf{p}$ to $\mathbf{p}+\mathbf{v}$.

If we define addition by $\mathbf{v_p} + \mathbf{w_p} \doteq (\mathbf{v}+\mathbf{w})_\mathbf{p}$ and scalar multiplication by $\lambda \mathbf{w_p} \doteq (\lambda \mathbf{w})_\mathbf{p}$, then $T_\mathbf{p}(\mathbb{R}^n)$ becomes a vector space.

We say two tangent vectors $\mathbf{v_p}$ and $\mathbf{w_q}$ are equal if $\mathbf{v} = \mathbf{w}$ and $\mathbf{p} = \mathbf{q}$. We say they are parallel if $\mathbf{v} = \mathbf{w}$.

Proposition 1. *$T_\mathbf{p}(\mathbb{R}^n)$ is isomorphic to $\mathbb{R}^n$.*

Proof. Consider the function $\mathbf{v} \mapsto \mathbf{v_p}$. This is clearly a one-to-one function from $\mathbb{R}^n$ onto $T_\mathbf{p}(\mathbb{R}^n)$. Additionally, it is clearly a homomorphism from the way we defined addition and scalar multiplication. □

Definition 2. *A **vector field** V on $\mathbb{R}^n$ is a function*

$$V : \mathbb{R}^n \to T_\mathbf{p}(\mathbb{R}^n).$$

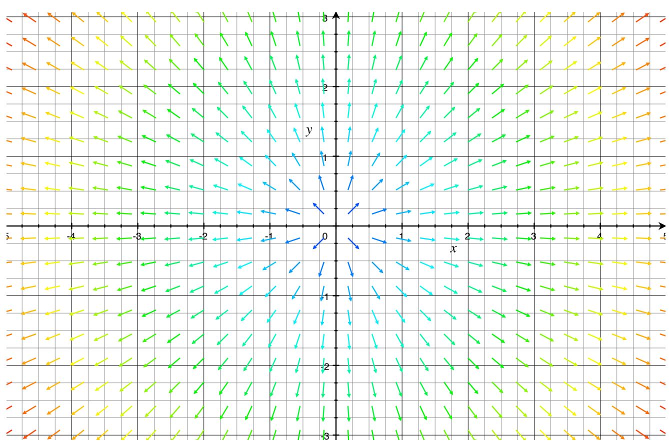


Figure 1.2: A vector field on $\mathbb{R}^2$ given by $\mathbf{p} \mapsto \mathbf{p_p}$.

Vector fields can be added and scalar multiplied in the usual way for functions.

Let $U_1, \dots, U_n$ be vector fields on $\mathbb{R}^n$ such that

$$U_1(\mathbf{p}) = (1, 0, 0 \dots)_{\mathbf{p}}$$
$$U_2(\mathbf{p}) = (0, 1, 0, \dots)_{\mathbf{p}},$$

etc. for all $\mathbf{p} \in \mathbb{R}^n$. Then $U_1, \dots, U_n$ collectively are called the **natural frame field** on $\mathbb{R}^n$. Note that U_i is a unit vector field in the positive x_i direction.

We can decompose a vector field, in a sense, into multiple scalar fields. The scalar fields act as coordinates for our vector field, and the basis is the natural frame field.

Proposition 2. *Let V be a vector field on $\mathbb{R}^n$, then there are unique real-valued functions $v_1, \dots, v_n$ on $\mathbb{R}^n$ such that*

$$V = \sum_{i=1}^{n} v_i U_i.$$

Proof. Let $\mathbf{p} \in \mathbb{R}^n$ be arbitrary, then by definition, $V(\mathbf{p}) \in T_{\mathbf{p}}(\mathbb{R}^n)$, so for some scalar fields $v_1, \dots, v_n$, we have

$$\begin{aligned} V(\mathbf{p}) &= (v_1(\mathbf{p}), \dots, v_n(\mathbf{p})) \\ &= v_1(\mathbf{p})e_1 + \dots v_n(\mathbf{p})e_n \\ &= v_1(\mathbf{p})U_1(\mathbf{p}) + \dots v_n(\mathbf{p})U_n(\mathbf{p}). \end{aligned}$$

Since $\mathbf{p}$ was arbitrary, $V = \sum_{i=1}^{n} v_i U_i$. The uniqueness of the v_i follows from the fact that we got them directly from the components of the map V. □

Definition 3. *The v_i in the above proposition are the **Euclidean coordinate functions** of V.*

Note 1. *We say that a vector field is differentiable if its Euclidean coordinate functions are themselves differentiable. From now on, assume vector fields are differentiable.*

1.3 Directional Derivatives

Vector fields can naturally be thought of as machines that transform scalar fields into scalar fields. Given some scalar field f and a vector field V, the most obvious way to act on f with V would be to use the tangent vectors given by V to calculate derivatives of f. This requires us to first define the derivative of a scalar field at a point $\mathbf{p}$ with respect to a tangent vector $\mathbf{v}_{\mathbf{p}}$.

Definition 4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar field, and let $\mathbf{v_p} \in T_p(\mathbb{R}^n)$. Then*

$$\mathbf{v_p}[f] \doteq \frac{d}{dt} f(\mathbf{p} + t\mathbf{v})\Big|_{t=0}$$

*is the **directional derivative** of f at $\mathbf{p}$ in the direction of $\mathbf{v}$. Note that it is a real number.*

Proposition 3. *Let $\mathbf{v_p} \in T_\mathbf{p}(\mathbb{R}^n)$, then*

$$\mathbf{v_p}[f] = \sum_i v_i \frac{\partial f}{\partial x_i}(\mathbf{p}).$$

Proof. Since $\frac{d}{dt}(p_i + tv_i) = v_i$, we can use the chain rule to get

$$\begin{aligned} \frac{d}{dt} f(\mathbf{p} + t\mathbf{v})\Big|_{t=0} &= \sum_i v_i \frac{\partial f}{\partial x_i}(\mathbf{p} + t\mathbf{v})\Big|_{t=0} \\ &= \sum_i v_i \frac{\partial f}{\partial x_i}(\mathbf{p}). \end{aligned}$$

□

Theorem 1. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be scalar functions, let $\mathbf{v}, \mathbf{w} \in T_\mathbf{p}(\mathbb{R}^n)$, and let $\lambda, \eta \in \mathbb{R}$, then*

1. $(\lambda\mathbf{v} + \eta\mathbf{w})[f] = \lambda\mathbf{v}[f] + \eta\mathbf{w}[f]$,
2. $\mathbf{v}[\lambda f + \eta g] = \lambda\mathbf{v}[f] + \eta\mathbf{v}[g]$, *and*
3. $\mathbf{v}[fg] = \mathbf{v}[f]g(\mathbf{p}) + f(\mathbf{p})\mathbf{v}[g]$.

Proof. It's straightforward to prove these by using Proposition 3. □

Parts (1) and (2) of this theorem say that $\mathbf{v}[f]$ is linear in both $\mathbf{v}$ and f. Part (3) is just the Leibniz rule.

It's easy to extend this idea to use a vector field instead of a fixed tangent vector. We simply use the vector field to map to a tangent vector, then use that to construct a directional derivative.

Note 2. *In this sense, vector fields map scalar fields to scalar fields.*

Definition 5. *The **operator** of a vector field V on a scalar field f is itself a scalar field*

$$\begin{aligned} V[f] : \mathbb{R}^n &\to \mathbb{R} \\ \mathbf{p} &\mapsto V(\mathbf{p})[f]. \end{aligned}$$

This is the derivative of f at the point $\mathbf{p}$ in the direction of $V(\mathbf{p})$.

Example 1. *If $\{U_i\}_{i=1}^n$ is the natural frame field on V, then $U_i[f] = \frac{\partial f}{\partial x_i}$.*

Corollary 1. *Let V, W be vector fields on $\mathbb{R}^n$, let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ be scalar fields, and let $\lambda, \eta \in \mathbb{R}$, then*

1. $(fV + gW)[h] = fV[h] + gW[h]$,
2. $V[\lambda f + \eta g] = \lambda V[f] + \eta V[g]$, *and*
3. $V[fg] = V[f]g + fV[g]$.

Proof. It's straightforward to prove these by using the corresponding part of Theorem 1. □

Note that a "scalar" in part (1) of this corollary can be a function, but the scalars must be actual numbers in part (2). That's because in part (1), the functions f and g get evaluated at some point $\mathbf{p}$, which yields a real number. In part (2), the only thing that gets evaluated at $\mathbf{p}$ is V, not anything in brackets.

1.4 Parameterized Curves

Definition 6. *A **curve** in $\mathbb{R}^n$ is a differentiable function $\alpha : I \to \mathbb{R}^n$, where I is an open interval in $\mathbb{R}$.*

Although curves could be defined without the condition that I is open, it makes defining derivatives (which we'll do soon) easier. Without open intervals, we'd have edge cases where some derivatives need to be defined with one-sided limits, and we'd rather just avoid that altogether.

Example 2. *If $\alpha_i = p_i + tq_i$ for some points $\mathbf{p}$ and $\mathbf{q}$, then α is a line.*

Example 3. *To draw a helix, we can parameterize a curve* $\alpha : \mathbb{R} \to \mathbb{R}^3$ *by*

$$\alpha(t) = (a\cos t, a\sin t, bt),$$

where $a, b \neq 0$.

Definition 7. *Let* $\alpha : I \to \mathbb{R}^n$ *be a curve. Then for every* $t \in I$*, the* ***velocity*** *of* α *at* t *is the tangent vector*

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \ldots, \frac{d\alpha_n}{dt}(t)\right)_{\alpha(t)}$$

at the point $\alpha(t) \in \mathbb{R}^n$.

We can write the velocity vector alternatively as $\alpha'(t) = \sum_i \frac{d\alpha_i}{dt}(t)U_i(\alpha(t))$.

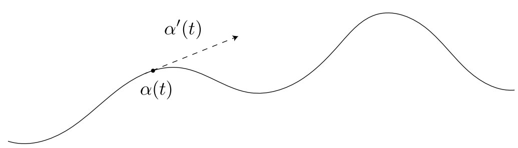


Figure 1.3: The velocity $\alpha'(t)$ is a vector tangent to $\alpha(t)$.

Example 4. *Using the parameterization of a helix from the previous example, its velocity vector is*

$$\alpha'(t) = (-a\sin t, a\cos t, b)_{\alpha(t)}.$$

The velocity of a curve isn't determined by the shape of the curve, but rather by how quickly you travel the curve. To illuminate this, we consider reparameterizations of the same curve.

Definition 8. *Let* I *and* J *be open intervals, let* $\alpha : I \to \mathbb{R}^n$ *be a curve, and let* $h : J \to I$ *be differentiable. Then the curve* $\beta : J \to \mathbb{R}^n$ *given by the composition* $\beta = \alpha \circ h$ *is called a* ***reparameterization*** *of* α *by* h.

Let β be a reparameterization of α by h. Then by the chain rule, its velocity vector is

$$\beta'(s) = h'(s)\alpha'(h(s)).$$

From this we see that unless h has a constant derivative of 1, the velocities of α and β will be different at the same point, even though they describe the same curve.

Proposition 4. *Let α be a curve in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then*
$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$$

Proof. Go over notation of this with prof... By definition,
$$\alpha' = \left(\frac{d\alpha_1}{dt}, \ldots, \frac{d\alpha_n}{dt}\right)_{\alpha(t)},$$
so by Proposition 3,
$$\alpha'(t)[f] = \sum_i \frac{d\alpha}{dt}\frac{\partial f}{\partial x_i}(\alpha(t)).$$
Noticing that the above expression is just an application of the chain rule, we can "undo" the chain rule to get
$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$$
□

A curve $\alpha : \mathbb{R} \to \mathbb{R}^n$ is **periodic** if there is some $p > 0$ such that $\alpha(t+p) = \alpha(t)$ for all t. The smallest such p is then called the **period** of α.

A curve whose velocity is nonzero at all points is called **regular**.

1.5 1-Forms

Definition 9. *A **1-form** on $\mathbb{R}^n$ is a linear function*
$$\phi : T(\mathbb{R}^n) \to \mathbb{R},$$
where $T(\mathbb{R}^n)$ is the tangent bundle of $\mathbb{R}^n$. For all $\mathbf{p}$, we can limit ϕ to
$$\phi_{\mathbf{p}} : T_{\mathbf{p}}(\mathbb{R}^n) \to \mathbb{R}.$$

Example 5. *The map $(v_1, v_2, v_3)_{\mathbf{p}} \mapsto v_1$ for all $\mathbf{p} \in \mathbb{R}^3$ is a 1-form.*

There's an equivalent way of formulating this that uses fancier vocab. If $\mathcal{V}$ is a vector space, then the set of all linear maps from $\mathcal{V}$ to $\mathbb{R}$ is itself a vector space. We call it the **dual space** of $\mathcal{V}$ and denote it by $\mathcal{V}^*$, and its elements are called **covectors**.

Thus a 1-form ϕ is an element of $T^*(\mathbb{R}^n)$. It can be limited to elements $\phi_{\mathbf{p}}$ of the **cotangent space** $T^*_{\mathbf{p}}(\mathbb{R}^n)$, in which cae we can also call $\phi_{\mathbf{p}}$ a **tangent covector**.

Note 3. *Since vector fields produce tangent vectors and 1-forms map tangent vectors to real numbers, we can think of 1-forms as being dual to the notion of vector fields.*

Addition of 1-forms is defined pointwise. We can also define a sort of scalar multiplication with scalar fields. If ϕ is a 1-form and f is a scalar field, define

$$(f\phi)(\mathbf{v_p}) \doteq f(\mathbf{p})\phi(\mathbf{v_p})$$

for all $\mathbf{v_p} \in T_{\mathbf{p}}(\mathbb{R}^n)$.

Given a vector field V, we can naturally act on it with a 1-form ϕ by

$$\begin{aligned} \phi(V) : \mathbb{R}^n &\to \mathbb{R} \\ \mathbf{p} &\mapsto \phi(V(\mathbf{p})). \end{aligned}$$

Thus we can view 1-forms as operators that convert vector fields into scalar fields. Additionally, it is easy to show that 1-forms act linearly on vector fields.

If $\phi(V)$ is differentiable whenever V is differentiable, then we say that ϕ itself is differentiable.

Note 4. *From now on, assume any given 1-form is differentiable.*

Definition 10. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then the* ***differential*** *df of f is the 1-form such that*

$$df(\mathbf{v_p}) = \mathbf{v_p}[f]$$

for all tangent vectors $\mathbf{v_p}$ of some point $\mathbf{p} \in \mathbb{R}^n$.

Since $\mathbf{v_p}[f]$ is a map from the tangent space to $\mathbb{R}$, and since we proved earlier that it is linear for all p, df is in fact a 1-form.

Example 6. *Consider the differentials $dx_1, \ldots, dx_n$ of the natural coordinate functions on $\mathbb{R}^n$. For a tangent vector $\mathbf{v_p}$ of a point $\mathbf{p}$, we have*

$$dx_i(\mathbf{v_p}) = \mathbf{v_p}[x_i] = \sum_j v_j \frac{\partial x_i}{\partial x_j}(\mathbf{p}) = \sum_j v_j \delta_{ij} = v_i,$$

where δ_{ij} is the Kronecker delta. Thus dx_i just extracts the i-th coordinate of a tangent vector, regardless of its point of application.

We can now go about classifying all possible 1-forms. Since every dx_i is a one-form, any function of the form

$$\sum_i f_i \, dx_i$$

is also a 1-form. As it turns out, this describes every possible 1-form. The f_i for a particular 1-form are determined by where that form sends the natural frame field, similar to how linear functions are determined by where they send basis elements.

Proposition 5. *Let ϕ be a 1-form, then*

$$\phi = \sum_i f_i \, dx_i,$$

where $f_i = \phi(U_i)$.

Proof. Since $\mathbf{v_p} = \sum_i v_i U_i(\mathbf{p})$ and since ϕ is linear, we have

$$\begin{aligned}
\phi(\mathbf{v_p}) &= \phi\left(\sum v_i U_i(\mathbf{p})\right) \\
&= \sum \phi(U_i(\mathbf{p})) v_i \\
&= \sum f_i(\mathbf{p}) v_i \\
&= \left(\sum f_i \, dx_i\right)(\mathbf{v_p}).
\end{aligned}$$

□

Definition 11. *The f_i above are called the **Euclidean coordinate functions** of ϕ.*

Corollary 2.

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

Proof. $df(U_i) = U_i[f] = \frac{\partial f}{\partial x_i}$. □

Bottom of pg 25 to top of pg 27.

1.6 Differential Forms

1-forms are part of a larger family of differential forms. 0-forms are just scalar fields, 1-forms are of the form $\sum f_i \, dx_i$, and we can get all other n-forms through multiplication of other forms.

Define the wedge product ... (do this.) The only unusual rule that the wedge product must follow is anti-commutativity:

$$\phi \wedge \psi = -\psi \wedge \phi.$$

Example 7. *A direct consequence of anti-commutativity is that two 1-forms wedged together is the zero map:*

$$\phi \wedge \phi = 0.$$

Note 5. *I might write $dx_i \wedge dx_j$ as $dx_i\, dx_j$ for simplicity.*

We can generalize the differential to apply to more than just scalar fields (0-forms). Given any form

$$\varphi = \sum_I f_I \; dx_I,$$

we define $d\varphi$) by

$$d\varphi \doteq \sum_I d(f_I) \wedge \; dx_I.$$

Thus we can view the d operator as converting an n-form to an $(n+1)$-form, and we call it the **exterior derivative**.

Stuff about how this generalizes div, grad, and curl...

1.7 Mappings

In general, we can have functions $F : \mathbb{R}^n \to \mathbb{R}^m$, which are given by m real-valued functions of n variables each. We call differentiable such functions **mappings**.

Definition 12. *The* ***tangent map*** *F_* of a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is a map*

$$F_* : T(\mathbb{R}^n) \to T(\mathbb{R}^m)$$

between the tangent bundles of $\mathbb{R}^n$ and $\mathbb{R}^m$. In particular, at any point $\mathbf{p}$, the tangent map can be limited to the function

$$F_{*\mathbf{p}} : T_{\mathbf{p}}(\mathbb{R}^n) \to T_{F(\mathbf{p})}(\mathbb{R}^m)$$

given by mapping $\mathbf{v_p}$ to the initial velocity of the curve $t \mapsto F(\mathbf{p} + t\mathbf{v})$.

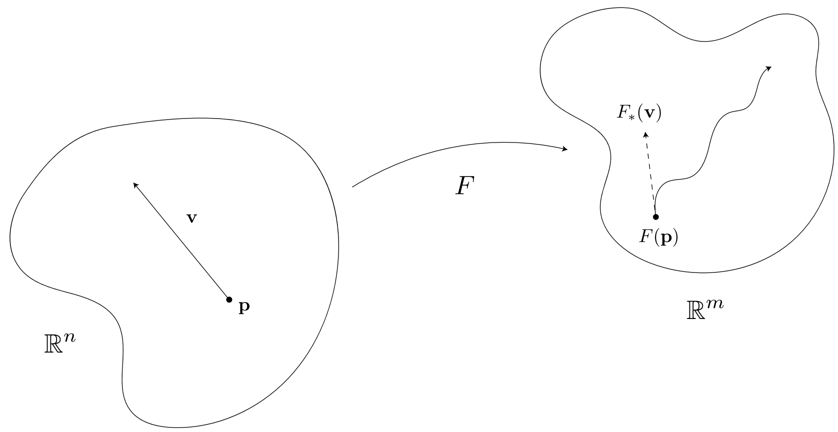


Figure 1.4: The tangent map F_* maps tangent spaces to tangent spaces by using the velocity of a curve.

The defintion requires us to use the *initial* velocity because even though we're starting with a line in $\mathbb{R}^n$, its image under F might be all curvy.

Proposition 6. *Let* $F = (f_1, \ldots, f_m)$*. If* $\mathbf{v} \in T_{\mathbf{p}}(\mathbb{R}^n)$*, then*

$$F_*(\mathbf{v}) = (\mathbf{v}[f_1], \ldots, \mathbf{v}[f_m])_{F(\mathbf{p})}.$$

Proof. Consider the curve β traced out by $F(\mathbf{p} + t\mathbf{v})$, then

$$\beta(t) = (\ldots, f_i(\mathbf{p} + t\mathbf{v}), \ldots).$$

$F_*(\mathbf{v})$ is just $\beta'(0)$, so we calculate it as

$$\beta'(0) = \left(\ldots, \frac{d}{dt} f_i(\mathbf{p} + t\mathbf{v}) \Big|_{t=0}, \ldots \right)_{\beta(0)}.$$

But each component is straight up the definition of the directional derivative, and $\beta(0) = F(\mathbf{p})$, so this becomes

$$F_*(\mathbf{v}) = (\ldots, \mathbf{v}[f_i], \ldots)_{F(\mathbf{p})}.$$

□

This shows F_* is linear, but it's more than just a linear map between tangent spaces. It's very deeply connected to the more subtle than this, I think...Jacobian... in the sense that it actually is the Jacobian. The corollary below might look complicated, but all it's saying is that F_* is the linear map given by the matrix $[\frac{\partial f_i}{\partial f_j}]_{ij}$.

Actually calculate $F_{*\mathbf{p}}(\mathbf{v_p})$. Does it involve multiplying a tangent vector by the Jacobian maybe?

1-1 iff onto when linear transformation is between two vec spaces of same dim. Useful for showing when $F_{*\mathbf{p}}$ is an isomorphism.

Corollary 3.

$$F_*(U_j(\mathbf{p})) = \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j}(\mathbf{p})\overline{U}_i(F(\mathbf{p})),$$

where $\{\overline{U}_i\}_{i=1}^{m}$ *is the natural frame field on* $\mathbb{R}^m$.

Proof. $U_j[f_i] = \frac{\partial f_i}{\partial x_j}$. □

Note 6. *We can interpret* F_* *at* $\mathbf{p}$ *as the best linear approximation to* F *at* $\mathbf{p}$.

Definition 13. *A mapping* F *is* ***regular*** *if* F_* *is one-to-one at all* $\mathbf{p}$.

This isn't the only way to characterize a regular mapping. Since F_* is linear, the following are equivalent:

1. F_* is everywhere one-to-one.
2. $F_*(\mathbf{v_p}) = 0$ implies $\mathbf{v_p} = 0$.
3. The Jacobian of F is full rank.

Definition 14. *A* ***diffeomorphism*** *is a mapping that has a differentiable inverse.*

Theorem 2 (Inverse Function Theorem)**.** *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a mapping between Euclidean spaces of the same dimension. If* F_* *is one-to-one at* $\mathbf{p} \in \mathbb{R}^n$*, then there is an open neighborhood* U *of* $\mathbf{p}$ *such that* F *restricted to* U *is a diffeomorphism onto some open set* V*.*

This is saying that if we have a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is one-to-one at a given point, then that mapping is locally invertible at that point.

Chapter 2

Frame Fields

2.1 Frames

Earlier we showed that $\mathbf{v} \mapsto \mathbf{v_p}$ was an isomoprhism, so we can naturally extend the usual dot product to work in tangent spaces. Given $\mathbf{v_p}, \mathbf{w_p} \in T_{\mathbf{p}}(\mathbb{R}^n)$, we define their dot product to be

$$\mathbf{v_p} \cdot \mathbf{w_p} = \mathbf{v} \cdot \mathbf{w}.$$

Get rid of use of e_i earlier on?

Definition 15. *Let $\{\mathbf{e}_i\}_{i=1}^n$ be a set of mutually orthogonal unit vectors tangent to $\mathbf{p}$. Then it is a **frame** of $\mathbb{R}^n$ as $\mathbf{p}$.*

Example 8. *The natural frame field evaluated at a point $\mathbf{p}$*

$$U_1(\mathbf{p}), \ldots, U_n(\mathbf{p})$$

is a frame of $\mathbb{R}^n$ at $\mathbf{p}$.

Lemma 1. *A frame of $\mathbb{R}^n$ is a basis for $T_{\mathbf{p}}(\mathbb{R}^n)$.*

Proof. If we dot $\sum \alpha_i \mathbf{e}_i = \mathbf{0}$ with $\mathbf{e}_j$, then the result is $\alpha_j = 0$. Thus our frame is linearly independent, so it's a basis for $T_{\mathbf{p}}(\mathbb{R}^n)$. □

Theorem 3. *For $\mathbf{v} \in T_{\mathbf{p}}$,$\mathbf{v} = \sum(\mathbf{v} \cdot \mathbf{e}_i)\mathbf{e}_i$.*

Proof. Since frames are bases of the tangent space, we can write $\mathbf{v} = \sum c_i \mathbf{e}_i$ for some c_i, and $\mathbf{v} \cdot \mathbf{e}_j = (\sum_i c_i \mathbf{e}_i) \cdot \mathbf{e}_j = c_j$. □

The dot product has the same form in any frame:

$$\mathbf{v} \cdot \mathbf{w} = \sum (\mathbf{v} \cdot \mathbf{e}_i)(\mathbf{w} \cdot \mathbf{e}_i).$$

Definition 16. *The* ***attitude matrix*** *of a frame* $\{\mathbf{e}_i\}_{i=1}^n$ *is the matrix*

$$A = \begin{pmatrix} — & \mathbf{e}_1 & — \\ & \vdots & \\ — & \mathbf{e}_n & — \end{pmatrix}.$$

Technically, each $\mathbf{e}_i$ is in the tangent space, but A's rows are just the coordinates of each $\mathbf{e}_i$.

Note that the rows are orthonormal, so $AA^T = I_n$, so $A^{-1} = A^T$, i.e. A is an **orthogonal** matrix.

2.2 The Cross Product

For $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}(\mathbb{R}^3)$, we define the cross product as

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} U_1(\mathbf{p}) & U_2(\mathbf{p}) & U_3(\mathbf{p}) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Interpretation as signed area?

The cross product is linear in $\mathbf{v}$ and $\mathbf{w}$, and it satisfies the alternation rule

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v},$$

so any vector crossed with itself is $\mathbf{0}$.

The cross product produces a vector that is orthogonal to $\mathbf{v}$ and $\mathbf{w}$, so

$$\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = 0.$$

ε stuff and experssion in terms of sin.

2.3 Curves

We've already defined curves $\alpha : I \to \mathbb{R}^n$ and their velocity $\alpha'(t)$. We can easily define the **speed** of a curve by

$$v(t) = \|\alpha'(t)\|.$$

The **arc length** function (starting from a constant $t = a$) is then defined

$$s(t) = \int_a^t \|\alpha'(\tau)\| \, d\tau.$$

From this we have $ds/dt = \|\alpha\| = v$ and $\int_a^b ds = \int_a^b \|\alpha'(t)\| \, dt$.

$ds = \|\alpha'(t)\| \, dt$?

Theorem 4. *We can reparameterize any regular curve α to have unit speed.*

Proof. Since α is regular, $ds/dt = \|\alpha'\| > 0$ everywhere. See notes... □

The curve β above has the **arc length parameterization**, i.e. it has unit speed.

We can define a notion of vector fields on curves.

Definition 17. *A* ***vector field*** *on a curve $\alpha : I \to \mathbb{R}^n$ is a function*

$$Y : I \to T(\mathbb{R}^n)$$

given by mapping t to a point tangent to $\alpha(t)$.

A vector field $Y = (y_1, \dots, y_n)$ on a curve α can be decomposed into the natural frame field by

$$Y = \sum y_i U_i(\alpha).$$

We can then define the derivative of a vector field on a curve in a pointwise fashion:

$$Y' = \sum y_i' U_i.$$

A vector field on a curve is **parallel** if it products parallel vectors. This means that all of it's y_i are constant.

Lemma 2.
1. *α is constant $\iff \alpha' = 0$.*
2. *α is a nonconstant straight line $\iff \alpha'' = 0$.*
3. *Y is parallel $\iff Y' = 0$.*

2.4 The Frenet Formulas

We start out by describing a particular frame on a curve that ends up being very helpful. The core idea behind this is that we can take derivatives to find tangent and normal components of the frame, then take their cross product to get a third orthogonal component. This constitutes a frame that changes based on the curve's geometry at each point.

Suppose β is a unit speed curve in $\mathbb{R}^3$. Then $T \doteq \beta'$ is the **unit tangent vector field**. Since β is unit speed, $\|T\| = 1$ everywhere, so it needs no further modifications to be a part of our frame.

Since $\|T\|^2 = T \cdot T = 1$, we differentiate to get $2(T' \cdot T) = 0$, so its derivative is orthogonal to it. The normalized version of this field will be the next component of our frame. Define the **curvature** of β by $\kappa(s) = \|T'(s)\| \geq 0$, then when $\kappa > 0$, define $N = T'/\kappa$ to be the **principal normal vector field**.

Finally, define $\beta = T \times N$ to be the **binormal vector field**. Cross two units = unit. In summary, the **Frenet apparatus** is

- $T = \alpha'$,
- $\kappa = \|\alpha'\|$,
- $N = T'/\kappa$, and
- $B = T \times N$.

Definition 18. *For a unit speed curve β with $\kappa > 0$ everywhere, B, N, and T form the* ***Frenet frame field*** *on β. They constitute a frame at each point on β.*

Picture.

Stuff about deriving torsion.

Theorem 5 (The Frenet Formulas)**.** *Let $\beta : I \to \mathbb{R}^3$ be a unit speed curve with nonzero κ everywhere and torsion τ, then*

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N. \end{aligned}$$

Proof. Do expansion of N'. □

From this we see that the geometry of a curve is encoded in its curvature and torsion.

Examples...

Proposition 7. *A curve is a straight line if and only if $\kappa = 0$.*

Proof?

Proposition 8. *A unit speed curve with $\kappa > 0$ is planar if and only if $\tau = 0$.*

Proof. Do this. $\square$

Lemma 3. *If $\kappa > 0$ and $\kappa' = \tau = 0$, then β is (part of) a circle of radius $1/\kappa$.*

Proof. Do this. $\square$

2.5 Arbitrary Speed Curves

Let $\overline{\alpha} : I \to \mathbb{R}^3$ be a unit speed curve, and let $\alpha = \overline{\alpha}(s)$ be an arbitrary reparameterization.

We can "correct" the old Frenet formulas to give formulas that work for our arbitrary curve by simplying multiplying by the speed function v. To see this, note that for our new curve α,

$$T' = \frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = \overline{T}'v.$$

The cases for B' and N' are similar.

Theorem 6. *Let α be an arbitrary regular ($\kappa > 0$)) curve, then*

$$\begin{aligned} T' &= \kappa v N, \\ N' &= -\kappa v T + \tau v B, \\ B' &= -\tau v N. \end{aligned}$$

If β has constant speed $\|\beta'\|$, then differentiating gives $2(\beta'' \cdot \beta') = 0$, so T and N are orthogonal. This might not be the case if the speed of β isn't constant, though. Does this mean the Frenet "frame" isn't a frame when α isn't constant speed?

Are $\beta' and \beta''$ always in correspondence with T and N? I assumed so above.

Lemma 4. *Let α be a regular curve with speed function v, then it's velocity and acceleration are given by*

$$\begin{aligned} \alpha' &= vT, \\ \alpha'' &= v'T + \kappa v^2 N. \end{aligned}$$

Proof. The velocity is $\alpha' = \overline{\alpha}'(s)v = \overline{T}(s)v = Tv$. Differentiating gives $\alpha'' = v'T + vT'$, which is equal to $v'T + \kappa v^2 N$ by the updated Frenet formulas. ☐

Theorem 7. *We can express the Frenet apparatus as*

$$T = \frac{\alpha'}{\|\alpha'\|}, \qquad N = B \times T, \qquad B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|},$$
$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2}.$$

Proof. Do this. ☐

Picture.

2.6 The Covariant Derivative

The covariant derivative is how we measure the rate of change of a vector field in a particular direction. Comparison to Y' from vector field on curve?

Definition 19. *Let* $\mathbf{v} \in T_{\mathbf{p}}$*, then the* ***covariant derivative*** *of a vector field* W *with respect to* $\mathbf{v}$ *is*

$$\nabla_{\mathbf{v}} W = \frac{d}{dt} W(\mathbf{p} + t\mathbf{v})\Big|_{t=0}.$$

If $W = \sum w_i U_i$, then this can be written $\nabla_{\mathbf{v}} W = \sum \mathbf{v}[w_i] U_i(\mathbf{p})$.

Linearity and all that.

We can easily extend this to being based on another vector field instead of a single tangent vector. If V and W are vector fields, then

$$\nabla_V W(\mathbf{p}) = \nabla_{V(\mathbf{p})} W.$$

If $W = \sum w_i U_i$, then $\nabla_V W = \sum V[w_i] U_i$.

Since $\nabla_V W$ maps from $\mathbb{R}^n$ to $T_{\mathbf{p}}(\mathbb{R}^n)$, it is itself a vector field.

Linearity and all that stuff.