MATH 531 HOMEWORK 10

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Page 274, Ex. 4. *Let $\mathcal{B} \subset \mathcal{C}([0,1], \mathbb{R})$ be closed, bounded, and equicontinuous. Let $I : \mathcal{B} \to \mathbb{R}$ be defined by $I(f) = \int_0^1 f(x)\, dx$. Show that there is an $f_0 \in \mathcal{B}$ at which the value of I is maximized.*

We first show that $\mathcal{B}$ is compact. Consider $\mathcal{B}_x = \{f(x) \mid f \in \mathcal{B}\}$. Since $\mathcal{B} \subset \mathbb{R}$, it is compact if and only if it is closed and bounded. Since $\mathcal{B}$ is bounded, $\mathcal{B}_x$ must also be bounded. Now for fixed x, consider $f_n(x) \to f(x)$ in $\mathcal{B}_x$. Since $\mathcal{B}$ is closed, f is in $\mathcal{B}$, so $f(x)$ is in $\mathcal{B}_x$. Thus $\mathcal{B}_x$ is closed. This shows that $\mathcal{B}$ is pointwise compact. Since we are given that it is also closed and equicontinuous, and since $[0,1]$ is compact in $\mathbb{R}$, $\mathcal{B}$ compact by the Arzela-Ascoli theorem.

Now we show that I is continuous. Fix $\varepsilon > 0$, and let f and g be functions in $\mathcal{B}$ such that $|f - g| < \varepsilon$, as measured by the supremum norm. Then

$$\begin{aligned}
|I(f) - I(g)| &= \left| \int_0^1 (f(x) - g(x))\, dx \right| \\
&\leq \int_0^1 |f(x) - g(x)|\, dx \\
&< \int_0^1 \varepsilon\, dx \\
&= \varepsilon.
\end{aligned}$$

Thus I is continuous.

Since $\mathcal{B}$ is compact and I is continuous, by the minimum-maximum theorem we know that there exists $f_0 \in \mathcal{B}$ such that $I(f_0) = \sup_f I(f)$.

Page 275, Ex. 5. *Let the functions $f_n : [a,b] \to \mathbb{R}$ be uniformly bounded continuous functions. Set*

$$F_n(x) = \int_a^x f_n(t)\, dt, \quad a \leq x \leq b.$$

Prove that F_n has a uniformly convergent subsequence.

We will show that the set $\mathcal{B} \doteq \{F_n\}_{n=1}^{\infty}$ is equicontinuous and pointwise bounded, then we can use the same proof as for Corollary 5.6.3 in the textbook to show our desired result.

First we show that $\mathcal{B}$ is equicontinuous. Let G_n be any antiderivative of f_n, then $F_n(x) = G_n(x) - G_n(a)$, so $F_n'(x) = f_n(x)$. Then F_n is an antiderivative of f_n, so the intermediate value theorem gives us

$$|F_n(x) - F_n(y)| = |f_n(c)||x - y|$$

Date: November 5, 2020.

for some $c \in [x, y]$. Since $\{f_n\}$ is uniformly bounded, for all $n \in \mathbb{N}, x \in [a, b]$, we have $|f_n(x)| \leq M$ for some $M \geq 0$. Thus we have the inequality

$$|F_n(x) - F_n(y)| \leq M|x - y|.$$

Now fix $\varepsilon > 0$. If $|x-y| < \varepsilon/M$, then for all $n \in \mathbb{N}$ and $x, y \in [a, b]$, we have $|F_n(x)-F_n(y)| < \varepsilon$, so $\mathcal{B}$ is equicontinuous.

Now we show that $\mathcal{B}$ is pointwise bounded. Fix x, then

$$\begin{aligned} |F_n(x)| &= \left| \int_a^x f_n(t)\ dt \right| \\ &\leq \int_a^x |f_n(t)|\ dt \\ &\leq \int_a^x M\ dt \\ &\leq M(x - a). \end{aligned}$$

Thus $\mathcal{B}$ is pointwise bounded.

Then by Corollary 5.6.3 in the textbook, every sequence in $\mathcal{B}$ has a uniformly convergent subsequence.

Page 282, Ex. 4. *Show that the system of equations*

$$\begin{aligned} x_1 &= \frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{2}{15}x_3 + 3 \\ x_2 &= \frac{1}{4}x_1 + \frac{1}{5}x_2 + \frac{1}{2}x_3 - 1 \\ x_3 &= -\frac{1}{4}x_1 + \frac{1}{3}x_2 - \frac{1}{3}x_3 + 2 \end{aligned}$$

has a unique solution, using the contraction mapping principle. [Hint: Either choose a clever norm on $\mathbb{R}^3$, or estimate using the Schwarz inequality.]

The given system defines a map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$. We will show that Φ is a contraction in the taxicab norm $\|x\| = \sum_i |x_i|$, from which the result for the usual Euclidean norm follows (since the two norms are equivalent). We have

$$\begin{aligned} d(\Phi(x), \Phi(y)) &= \|\Phi(x) - \Phi(y)\| \\ &\leq \left|\frac{1}{4}(x_1 - y_1)\right| - \left|\frac{1}{4}(x_2 - y_2)\right| + \left|\frac{2}{15}(x_3 - y_3)\right| \\ &\quad + \left|\frac{1}{4}(x_1 - y_1)\right| + \left|\frac{1}{5}(x_2 - y_2)\right| + \left|\frac{1}{2}(x_3 - y_3)\right| \\ &\quad + \left|-\frac{1}{4}(x_1 - y_1)\right| + \left|\frac{1}{3}(x_2 - y_2)\right| - \left|\frac{1}{3}(x_3 - y_3)\right| \\ &= \frac{3}{4}|x_1 - y_1| + \frac{47}{60}|x_2 - y_2| + \frac{29}{30}|x_3 - y_3| \\ &\leq \frac{29}{30}\left(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|\right) \\ &= \frac{29}{30}d(x, y). \end{aligned}$$

Since $0 < 29/30 < 1$, we have found a k such that $d(\Phi(x), \Phi(y)) \leq kd(x, y)$ for $0 \leq k < 1$. Thus by the contraction mapping principle, this system has a unique solution (i.e. fixed point).

Page 283, Ex. 8. *Let M be a compact metric space and $\Phi : M \to M$ be such that $d(\Phi(x), \Phi(y)) < d(x, y)$ for all $x, y \in M, x \neq y$.*

(a) *Show that Φ has a unique fixed point [Hint: Minimize $d(\Phi(x), y)$.]*
(b) *Show that **a** is false if M is not compact (find a counterexample).*

(a) Consider the map $f : M \to \mathbb{R}$ given by $f(x) = d(\Phi(x), x)$. We claim that f is continuous. By the triangle inequality and our assumption on Φ, we have

$$\begin{aligned} d(x, \Phi(x)) &\leq d(x, y) + d(y, \Phi(y)) + d(\Phi(y), \Phi(x)) \\ d(x, \Phi(x)) - d(y, \Phi(y)) &\leq d(x, y) + d(\Phi(y), \Phi(x)) \\ f(x) - f(y) &< 2d(x, y). \end{aligned}$$

Similarly, we also have $f(y)-f(x) < 2d(x, y)$. Putting these two inequalites together gives

$$|f(x) - f(y)| \leq 2d(x, y).$$

Now fix $\varepsilon > 0$. When $d(x, y) < \varepsilon/2$, $|f(x) - f(y)| < \varepsilon$, so f is continuous.

Since M is compact and f is continuous, it attains its infimum, i.e. there exists $x_0 \in M$ such that $f(x_0) = \inf_x f(x) = \inf_x d(x, \Phi(x))$. Denote this infimum by I. Consider the case when $I > 0$. In this case we have a contradiction, as our assumption on Φ gives

$$d(\Phi(\Phi(x_0)), \Phi(x_0)) < d(\Phi(x_0), x_0) = I,$$

which cannot be true if I is an infimum. Thus $I = 0$, meaning $d(x_0, \Phi(x_0)) = 0$, so $x_0 = \Phi(x_0)$. Thus x_0 is a fixed point of Φ.

We now show that x_0 is a unique fixed point. Let x_0 and x_1 be distinct fixed points of Φ, then our assumption on Φ gives

$$\begin{aligned} d(\Phi(x_0), \Phi(x_1)) &< d(x_0, x_1) \\ d(x_0, x_1) &< d(x_0, x_1). \end{aligned}$$

This is a contradiction, so x_0 and x_1 must be equal. Thus the fixed point x_0 of Φ is unique.

(b) Let $M = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Consider $\Phi : \mathbb{R} \to \mathbb{R}$ given by $\Phi(x) = \sqrt{x^2 + 2}$. Note that

$$\left|\sqrt{x^2 + 2} - \sqrt{y^2 + 2}\right| < |x - y|$$

for all $x, y \in \mathbb{R}$, so Φ satisfies the conditions from part **a**.

If x_0 were a fixed point of Φ, then it would satisfy $\sqrt{x_0^2 + 2} = x_0$; however, trying to solve this yields

$$\begin{aligned} \sqrt{x_0^2 + 2} &= x_0 \\ x_0^2 + 2 &= x_0^2 \\ 2 &= 0, \end{aligned}$$

so no such x_0 can exist. Thus Φ has no fixed points.

Page 286, Ex. 3. *Prove that the set of polynomials in $\mathcal{C}([a,b],\mathbb{R})$ is not open. Can a subset of a metric space ever be both open and dense?*

In the last homework we showed that the sequence of functions $\{f_k\}$ given by

$$f_k = \frac{\sin x}{k}$$

converges uniformly to the zero function. We claim that this is a sequence of non-polynomial functions. If $(\sin x)/k$ were a polynomial, then we could write it

$$\frac{\sin x}{k} = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0$$

for some $n \in \mathbb{N}$; however, note that the n-th derivative of the RHS is 0 for all x while the n-th derivative of the LHS is nonzero when x is nonzero and not a multiple of 2π. Thus $f_k = (\sin x)/k$ is not a polynomial.

Since f_k converges uniformly to the zero function, for all $\varepsilon > 0$, we can find an f_k such that $\sup_x |f_k(x)| < \varepsilon$. Since 0 is a polynomial and all f_k are non-polynomials, this means the set of polynomials in $\mathcal{C}([a,b],\mathbb{R})$ is not open.

In general, it is possible for a dense subset of a metric space to be open. Consider $\mathbb{R}$ equipped with the usual metric. Then the subset $\mathbb{R} - \{0\}$ is open and dense. It is open since its complement $\{0\}$ is closed, and it is dense since its closure is all of $\mathbb{R}$.

Page 317, Ex. 11. (a) *Must a contraction on any metric space have a fixed point? Discuss.*

(b) *let $f : X \to X$, where X is a complete metric space (such as $\mathbb{R}$), satisfy*

$$d(f(x), f(y)) < d(x,y)$$

for all $x, y \in X$. Must f have a fixed point? What if X is compact?

(a) A contraction on a metric space need not have a fixed point. In order to guarantee that a fixed point exists, the metric space needs to be complete.

Suppose we are working in the space $\mathbb{R} - \{0\}$, which is not complete. Consider the map $\Phi(x) = x/2$, which is a contraction since

$$d(x,y) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x-y| = \frac{1}{2}d(x,y).$$

However, solving $x = x/2$ yields $x = 0$, so the only possible fixed point of Φ is 0, which is not in our metric space. Thus Φ has no fixed point.

(b) f need not have a fixed point. Consider $f(x) = x + 1/x$ on $[2, \infty)$. For all distinct $x, y \in [2, \infty)$, we have

$$\begin{aligned} d(f(x), f(y)) &= \left| x + \frac{1}{x} - y - \frac{1}{y} \right| \\ &= \left| (x - y) + \left(\frac{y - x}{xy} \right) \right| \\ &= \left| (x - y) \left(1 - \frac{1}{xy} \right) \right| \\ &\leq |x - y| \left| 1 - \frac{1}{xy} \right| \\ &< |x - y|, \end{aligned}$$

where the last inequality follows from x and y both being greater than or equal to 2. This shows that that f satisfies the given condition; however, solving $x = x + 1/x$ yields

$$\begin{aligned} x &= x + \frac{1}{x} \\ 0 &= \frac{1}{x} \\ 0 &= 1, \end{aligned}$$

so there are no fixed points of f.

When we are working in a *compact* metric space, the given condition is enough to guarantee that f has a unique fixed point. This was proven earlier in Exercise 8 (Page 283).

Page 322, Ex. 46. *Let $f(t, x)$ be defined and continuous for $a \leq t \leq b$ and $x \in \mathbb{R}^n$. The purpose of this exercise is to show that the problem $dx/dt = f(t, x), x(a) = x_0$, has a solution on an interval $t \in [a, c]$ for some $c > a$ (it is unique only under more stringent conditions). Perform the operations as follows: Divide $[a, b]$ into n equal parts $t_0 = a \ldots, t_n = b$, and define a continuous function x_n inductively by*

$$\begin{cases} x_n'(t) = f(t_i, x_n(t_i)), & t_i < t < t_{i+1}, \\ x_n(a) = x_0. \end{cases}$$

Put $\Delta_n(t) = x_n'(t) - f(t, x_n(t))$, so that

$$x_n(t) = x_0 + \int_a^t f(s, x_n(s)) + \Delta_n(s) \, ds.$$

Use the Arzela-Ascoli theorem to find a convergent subsequence of the x_n. Show that the limit satisfies $dx/dt = f(t, x)$ and $x(a) = x_0$.

This method is called ***polygonal approximation****.*

First we note that f is bounded (by, say, M) since it is a continuous function on a compact domain. Additionally, since $|t_i - t| \leq 2/n$ and f is continuous, we know $\Delta_n(s)$ is also always bounded (by, say, N). Then

$$\begin{aligned} |x_n(t)| &\leq |x_0| + \int_a^t |M + N| \, ds \\ &= |x_0| + (t - a)|M + N|, \end{aligned}$$

so each x_n is pointwise bounded.

Then by the mean value theorem,

$$\begin{aligned} |x_n(t_1) - x_n(t_2)| &= |x_n'(t_3)||t_1 - t_2| \\ &= |f(t_i, x_n(t_i))||t_1 - t_2| \\ &\le M|t_1 - t_2|. \end{aligned}$$

Since this Lipschitz property holds for all n, we have equicontinuity of the space containing the x_n's.

Now we can apply Corollary 5.6.3 from the textboook to show that $\{x_n\}$ has a uniformly convergent subsequence $\{x_{\sigma(n)}\}$, and we denote its limit function by x. We must now show that its limit satisfies the given ODE.

Since this sequence converges uniformly, it certainly converges pointwise, so consider the sequence $\{x_{\sigma(n)}(a)\}$. Since $x_n(a) = x_0$ for all n, the limit $x(a)$ of this sequence must also equal x_0. Now we must show that the derivative of x is equal to $f(t, x)$.

Since f is continuous and $x_{\sigma(n)}$ converges uniformly to x, then $f(s, x_{\sigma(n)}(s))$ converges uniformly to $f(s, x(s))$. Thus $\Delta_{\sigma(n)}(s) = f(t_i, x_{\sigma(n)}(t_i)) - f(t, x_{\sigma(n)}(t))$ converges uniformly to

$$\Delta(s) = f(t_i, x(t_i)) - f(t, x(t)).$$

Since $|t - t_i| < 2/n$ by construction, $t_i \to t$ as $n \to \infty$. Since f is continuous this means $f(t_i, x(t_i)) \to f(t, x(t))$. Thus we have

$$\Delta(s) = 0.$$

Putting this all together, we can write $x(t)$ as

$$\begin{aligned} x(t) = \lim_{n\to\infty} x_n(t) &= x_0 + \lim_{n\to\infty} \int_a^t f(s, x_n(s))\, ds + \lim_{n\to\infty} \int_a^t \Delta_n(s)\, ds \\ &= x_0 + \int_a^t f(s, x(s))\, ds + \int_a^t 0\, ds \\ &= x_0 + \int_a^t f(s, x(s))\, ds. \end{aligned}$$

And taking its derivative gives

$$x'(t) = f(t, x(t)),$$

as desired.

Page 324, Ex. 58b. *Prove that if* $u_n > 0$, $\frac{u_{n+1}}{u_n} \ge 1 - \frac{1}{n} - \frac{1}{n \log n}$, *then* $\sum u_n$ *diverges.*

We are given $u_{n+1} \ge (1 - 1/n - 1/(n \log n))$, and we can expand this to

$$\begin{aligned} u_{n+1} &\ge u_2 \prod_{k=2}^{n} \left(1 - \frac{1}{k} - \frac{1}{k \log k}\right) \\ &= u_2 \exp\left(\sum_{k=2}^{n} \log\left(1 - \frac{1}{k} - \frac{1}{k \log k}\right)\right). \end{aligned}$$

We can expand $\log(1 - x)$ into

$$\log(1 - x) = -\sum_{m=1}^{\infty} \frac{x^m}{m},$$

so we have

$$\log\left(1-\frac{1}{k}-\frac{1}{k\log k}\right)=\frac{1}{k}+\frac{1}{k\log k}+\rho(k),$$

where

$$\rho(k)=-\sum_{m=2}^{\infty}\frac{\left(-\frac{1}{k}-\frac{1}{k\log k}\right)^m}{m}.$$

Thus our bound on u_{n+1} becomes

$$\begin{aligned}
u_{n+1} &\geq u_2\exp\left(\sum_{k=2}^{n}\frac{1}{k}+\frac{1}{k\log k}+\rho(k)\right)\\
&\geq u_2\exp\left(\sum_{k=2}^{n}\frac{1}{k}\right)\exp\left(\sum_{k=2}^{n}\frac{1}{k\log k}\right)\exp\left(\sum_{k=2}^{n}\rho(k)\right)\\
&\geq u_2\exp\left(\sum_{k=2}^{n}\frac{1}{k}\right)\exp\left(\sum_{k=2}^{n}\frac{1}{k\log k}\right)\prod_{k=2}^{n}e^{\rho}(k).
\end{aligned}$$

We will now find lower bounds for each of the exponential terms. The sequence $1/k$ is monotonically decreasing, so

$$\log n-\log 2=\int_2^n\frac{1}{t}\,dt\leq\sum_{k=2}^{n}\frac{1}{k}.$$

Similarly, the sequence $1/(k\log k)$ is also monotonically decreasing, so

$$\log|\log n|-\log|\log 2|=\int_2^n\frac{1}{k\log k}\,dt\leq\sum_{k=2}^{n}\frac{1}{k\log k}.$$

In the case of $k=2$, $\rho(k)$ diverges to $-\infty$, so $e^{\rho}(2)\geq e^0=1$. For $k\geq 3$, $\rho(k)$ converges to 0 from above (as the first term in the summation is positive, the signs of each term alternate, and the absolute value of each subsequent term decreases). So in this case, $e^{\rho(k)}=0$. Thus we have $e^{\rho(k)}\geq 0$ for all $k\geq 2$.

Using these derived inequalities, our bound on u_{n+1} becomes

$$\begin{aligned}
u_{n+1} &\geq u_2\exp\left(\sum_{k=2}^{n}\frac{1}{k}\right)\exp\left(\sum_{k=2}^{n}\frac{1}{k\log k}\right)\prod_{k=2}^{n}e^{\rho}(k)\\
&\geq u_2\exp(\log n-\log 2)\cdot\exp(\log|\log n|-\log|\log 2|)\cdot 1\cdots 1\\
&=\frac{u_2}{2\log 2}n\log n.
\end{aligned}$$

This grows unbouded as $n\to\infty$, so the series $\sum_n u_n$ must diverge.