Exercise 1 (1.3: 16)**.** If $q : Y \to Z$ and $qp : X \to Z$ are covering maps (in the sense of Hatcher, i.e. not surjective), then so is $p : X \to Y$. It's normal if $qp : X \to Z$ is normal.

Covering map: Let $y \in Y$, let U_1 be a neighborhood of y that's evenly covered by q, and let U_2 be a neighborhood of y evenly covered by pq. Then $U_1 \cap U_2$ is evenly covered by both. Since Z is locally path connected, there is some $U \subset U_1 \cap U_2$ that's path connected and still evenly covered.

We know $q^{-1}(U) = \bigsqcup_{i\in\mathcal{J}} V_i$, where each $V_i \cong U$ via q. Let V be the V_i that contains y, then we claim that V is evenly covered by p. First off, note that since $V \cong U$ and U is path connected, so is V. Also note that $(qp)^{-1}(U) = \bigsqcup_{\alpha\in\mathcal{A}} W_\alpha$, where each $W_\alpha \cong U$ via pq (and is thus path connected, too). We then know

$$p^{-1}(V) \subset p^{-1}(q^{-1}(U)) = (qp)^{-1}(U) = \bigsqcup_{\alpha\in\mathcal{A}} W_\alpha.$$

We now must show that $p^{-1}(V) = \bigsqcup_{\beta\in\mathcal{B}} W_\beta$ for $\mathcal{B} \subset \mathcal{A}$, but this ends up being a consequence of path connectedness. Suppose there's some W_α intersects more than one $p^{-1}(V_i)$, then we've disconnected that particular W_α, a contradiction. Thus each W_a lies inside of 1 and only 1 $p^{-1}(V_i)$. This implies that $p^{-1}(V) = \bigsqcup_{\beta\in\mathcal{B}} W_\beta$, as desired. This shows that p is a covering map (in the sense of Hatcher, i.e. missing the surjectivity requirement).

Normal: Suppose $y \in Y$ and $x_0, x_1 \in p^{-1}(y)$, then we want to find some $\tau \in G(X)$ such that $\tau x_0 = x_1$. Now x_0, x_1 are both lifts of $q(y)$ under the covering map qp, and that covering map is normal, so there is some homeomorphism $\sigma : X \to X$ sending $x_0 \mapsto x_1$ and making the following commute.

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "qp"'] & X \arrow[dl, "qp"] \\ Z & \end{tikzcd}$$

We can now transform this deck transformation for pq into one for p. Like any space, X can be be partitioned into its path components, i.e. $X = \bigcup_i P_i$ for each P_i the path component of X containing x_i. We'll define τ in terms of these path components.

Since σ is a homeomorphism, it must map path components onto path components, i.e. $\sigma : P_i \mapsto P_j$ In particular, we know $\sigma(P_0) = P_1$. Define $\tau|P_0 = \sigma|P_0$. If $P_0 = P_1$, then define $\tau|P_k = \text{id}$ for all $k \neq 0$. If $P_0 \neq P_1$, then define $\tau|P_1 = \sigma^{-1}|P_1$ and $\tau|P_k = \text{id}$ for all $k \neq 0, 1$. With this definition, τ is bijective, both τ, τ^{-1} are continuous, and τ maps $x_0 \mapsto x_1$. All that's left to check is that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[d, "p"'] & X \arrow[dl, "p"] \\ Y & \end{tikzcd}$$

Let $x \in P_0$, then there is a path h from x_0 to x. The two paths $p\tau h$ and ph in Y both start at the same point, so by unique path lifting, $p\tau h = ph$. Thus $(p\tau)(x) = (p\tau h)(1) = (ph)(1) = p(x)$, so the diagram commutes in this case. The argument is similar if $x \in P_1$ instead. If $x \in P_k$ for $k \neq 0, 1$, then the diagram definitely commutes since we defined τ to be the identity in this case. Thus $X \xrightarrow{p} Y$ is a normal covering map.

Exercise 2 (1.3: 18)**.** X has an abelian covering space that covers all other abelian covers. It is unique up to isomorphism. Describe it explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.

Consider the covering $\tilde{X} \xrightarrow{p} X$ corresponding to the subgroup $G := [\pi_1(X), \pi_1(X)]$ of $\pi_1(X)$. We claim that it's the desired covering.

Abelian: (This section is essentially one big application of Proposition 1.39 from the textbook). Commutator subgroups are normal, so G is normal, so $\tilde{X}$ is normal. Thus $G(\tilde{X}) \cong \pi_1(X)/G$, but modding a group by its commutator gives an abelian group, so $G(\tilde{X})$ is abelian. Thus $\tilde{X}$ is an abelian cover.

Covers every other abelian cover: Suppose $\tilde{Y} \xrightarrow{q} X$ is another abelian cover, i.e. $H := p_*(\pi_1(\tilde{Y}))$ is normal and $G(\tilde{Y})$ is abelian. Now

$$G(\tilde{Y}) \cong \pi_1(X)/H \text{ is abelian } \iff G \subset H$$

since G is the smallest normal subgroup making its quotient with $\pi_1(X)$ abelian. This means

$$p_*(\pi_1(\tilde{X})) = G \subset H = q_*(\pi_1(\tilde{Y})),$$

so we can apply the lifting criterion to get the following commutative diagram.

$$\begin{tikzcd}
& \tilde{Y} \arrow[d, "q"] \\
\tilde{X} \arrow[ur, dashed, "\exists\, \phi"] \arrow[r, "p"] & X
\end{tikzcd}$$

We claim that ϕ is a covering map. Fix $\tilde{y} \in \tilde{Y}$ and consider $x := q(\tilde{y})$. This has neighborhoods U (evenly covered by p) and V (evenly covered by q). Then $U \cap V$ is evenly covered by both. Now consider the unique set W in $q^{-1}(U \cap V)$ containing $\tilde{y}$. We claim that this is an evenly covered neighborhood of $\tilde{y}$ (which would make $\tilde{X}$ a covering of $\tilde{Y}$).

Since the above diagram commutes, we know there is some subset of $\tilde{X}$ that ϕ maps to W. Since p is a homeomorphism on each homeomorphic copy of $U \cap V$ inside $\tilde{X}$, we know $\phi = pq^{-1}$ on each copy as well, meaning that ϕ itself is a homeomorphism. Thus ϕ is a covering map that evenly covers W.

Unique up to iso: Suppose $\tilde{Z}$ is an abelian covering space with the same properties as $\tilde{X}$. By a similar argument as above, we have the following commutative diagram.

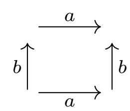
$$\begin{tikzcd}
& \tilde{Z} \arrow[d, "q"] \arrow[dl, dashed, shift left, "\exists\, \psi"] \\
\tilde{X} \arrow[ur, dashed, shift left, "\exists\, \phi"] \arrow[r, "p"'] & X
\end{tikzcd}$$

We claim that ϕ and ψ are mutually inverse (and thus isomorphisms). Now consider the following diagram.

$$\begin{tikzcd}
& \tilde{X} \arrow[d, "p"] \\
\tilde{X} \arrow[ur, "\psi\phi"] \arrow[r, "p"] & X
\end{tikzcd}$$

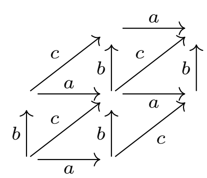
It commutes since $p\psi\phi = q\phi = p$. A possible lift of p is $\mathrm{id}_{\tilde{X}}$, so by uniqueness of lifts, $\psi\phi = \mathrm{id}_{\tilde{X}}$. Using similar logic, we get $\phi\psi = \mathrm{id}_{\tilde{Z}}$. Thus ϕ and ψ are isomorphisms.

Examples: When $X = S^1 \vee S^1$, we have $\mathrm{ab}(\pi_1(X)) \cong \mathrm{ab}(\mathbb{Z} * \mathbb{Z}) \cong \mathbb{Z}^2$. Thus we need a covering space $\tilde{X} \xrightarrow{p} X$ with $p_*(\pi_1(\tilde{X})) = \mathbb{Z}^2$. Let $\tilde{X}$ be the lattice in $\mathbb{R}^2$ with squares as shown below.



Note that p_* maps $a \mapsto a$ and $b \mapsto b$, so $p_*(\pi_1(\tilde{X})) = p_*(\langle a, b\rangle) = \langle a, b\rangle \cong \mathbb{Z}^2$, as desired.

Similarly, when $X = S^1 \vee S^1$, we have $\mathrm{ab}(\pi_1(X)) \cong \mathbb{Z}^3$. Then we can use the lattice in $\mathbb{R}^3$ with cubes as shown below.



Again, we have $p_*(\pi_1(\tilde{X})) \cong \langle a, b, c\rangle \cong \mathbb{Z}^3$, as desired.

Exercise 3 (1.3: 19)**.** Use 1.3: 18 to show that a closed orientable surface M_g has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ iff $n \leq 2g$. For $n = 3$ and $g \geq 3$, describe such a covering explicitly as a subspace of $\mathbb{R}^3$ with translations of $\mathbb{R}^3$ as deck transformations.

First part: We know that the fundamental group of a closed genus g surface is

$$\pi_1(M_g) \cong \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle,$$

and its abelianization is

$$\mathrm{ab}(\pi_1(M_g)) \cong \langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle \cong \mathbb{Z}^{2g}.$$

We also know that if $\tilde{X}$ is a normal covering space of $\tilde{X}$, then $G(\tilde{X}) \cong \pi_1(X)/H$, where is the subgroup of $\pi_1(X)$ corresponding to $\tilde{X}$. Now by the previous exercise, we know M_g has an abelian covering space $\tilde{X}$ that covers all others, and we showed that it corresponds to $H = [\pi_1(M_g), \pi_1(M_g)]$. Its deck transformation group is then $\mathrm{ab}(\pi_1(M_g)) \cong \mathbb{Z}^{2g}$.

The subgroups of $\mathbb{Z}^{2g}$ are exactly the $\mathbb{Z}^n$ for $n \leq 2g$, i.e. for all $n \leq 2g$, there exist G_n such that

$$\frac{\pi_1(M_g)}{[\pi_1(M_g), \pi_1(M_g)]} \Big/ \frac{G_n}{[\pi_1(M_g), \pi_1(M_g)]} \cong \frac{\pi_1(M_g)}{G_n} \cong \mathbb{Z}^n.$$

Then the deck transformation group of the covering space corresponding to G_n is $\mathbb{Z}^n$. Since there are no other subgroups of $\mathbb{Z}^{2g}$, these are no other coverings with this form of deck transformation group.

Covering in $\mathbb{R}^3$: In 3 dimensions we can take the integer lattice in $\mathbb{R}^3$ (as in the previous problem), widen each line, and make each tube hollow to get a lattice with elements as below.

Each of the three "axes" pictured above in the tube corresponds to the three main loops in M_3.

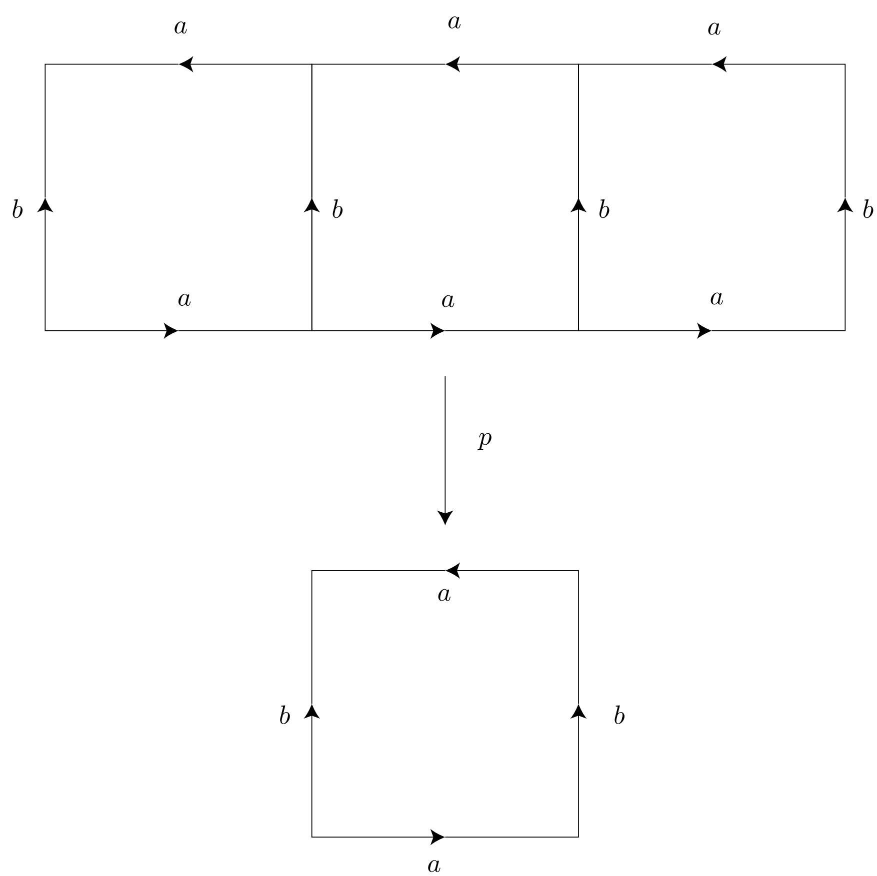
Thus every copy of this figure in the lattice is itself a covering of M_3, so the whole lattice is as well. The deck transformations of this lattice are the translations

$$
\begin{gathered}
x+1, y, z; \\
x, y+1, z; \\
x, y, z+1.
\end{gathered}
$$

Thus the deck transformation group is given by $\mathbb{Z}^3$, as desired.

Exercise 4 (1.3: 20)**.** Construct nonnormal covering spaces of the Klein bottle by a Klein bottle and by a torus.

Cover with Klein bottle: Consider the following covering map $p : K \to K$, where K is the Klein bottle. We divide K into three identical parts and map each part onto K in the natural way.



The induced map p_* then maps $a \mapsto a^3$ and $b \mapsto b^2$. Since we know by van Kampen that

$$\pi_1(K) \cong \langle a, b \mid abab^{-1} \rangle,$$

this means

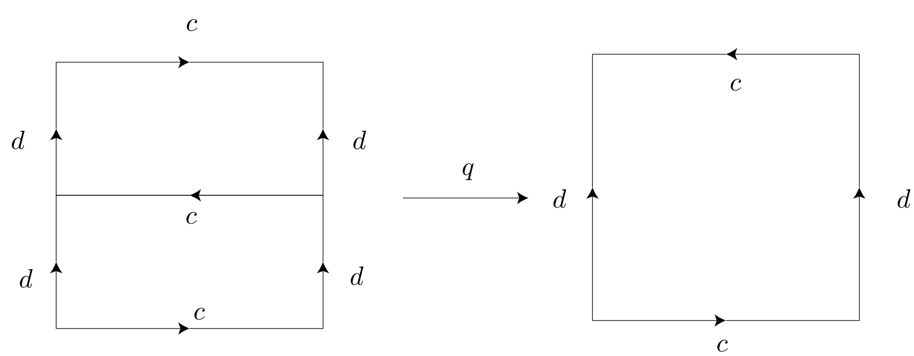
$$H := p_*(\pi_1(K)) \cong \langle a^2, b \mid abab^{-1} \rangle.$$

We claim that this is not a normal subgroup of $\pi_1(K)$. Consider $g = a \in \pi_1(K)$ and $n = b \in H$. If H is normal in $\pi_1(K)$, then

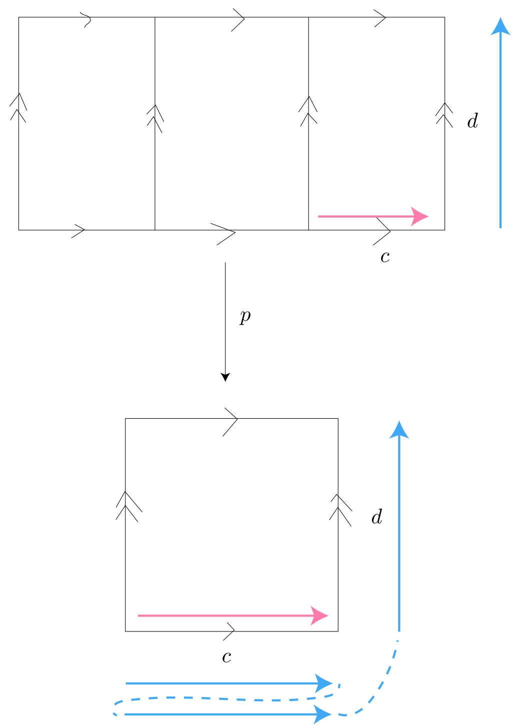
$$gng^{-1} = aba^{-1} = a^2 b \in H,$$

where the last equality follows from the relation in H. But this isn't true since a^2 cannot be generated by a^3 and b. Thus H is not normal in $\pi_1(K)$, so the covering space is not normal.

Cover with torus: Consider the following covering map $q : T^2 \to K$. We divide the torus T^2 into two separate compartments, then map onto K in the obvious way for each compartment.



Note that q_* maps $c \mapsto a$ and $d \mapsto b^2$. Now consider the subgroup of $\pi_1(T^2)$ generated by c^3 and c^2d. This corresponds to a covering of the torus by itself $p : T^2 \to T^2$. This covering is pictured below, where we send each horizontal component of the boundary to c and we send each vertical component to c^2d (essentially looping it around the other circle composing T^2 twice before sending it to the natural place).



Then the composition $qp : T^2 \to K$ is a covering of the Klein bottle by a torus such that $(qp)_*$ maps $c \mapsto a^3$ and $d \mapsto a^2b^2$. By a similar argument as above, though, $aba^{-1} = a^2b$ is not in this subgroup, so it is not normal and thus the covering is not normal.