CONTENTS

1 DEFINITIONS

To begin, we'll need to define some basic terms that we'll use over and over again. Let S be the relative size of the dominating cluster as $n \to \infty$. If x_i is a vertex, then we denote its absolute cluster size by κ_i. Denote the probability that the minimum of m i.i.d. sampled vertices is s by

$$Q_m(s) \doteq \mathbb{P}\left(\min\{\kappa_1, \dots, \kappa_m\} = s\right).$$

Note that Q_m satisfies the identity $\sum_{s=1}^{\infty} Q_m(s) = 1 - S^m$. Since they frequently show up in common examples, we give $m = 1$ and $m = 2$ shorthands:

$$P \doteq Q_1, \qquad Q \doteq Q_2.$$

We also define

$$\langle s^k \rangle_m \doteq \sum_{s=1}^{\infty} s^k Q_m(s).$$

I'll use $\langle \cdot \rangle_P$ and $\langle \cdot \rangle_Q$ instead of $\langle \cdot \rangle_1$ and $\langle \cdot \rangle_2$, respectively.

Now for the main attraction. In these notes, we'll be discussing rules that add a single edge every $t = 1/n$ units of time, gotten by selecting two vertices total from two separate groups of vertices that are sampled i.i.d. from the graph.

Definition 1. Define a rule $\mathcal{R}$ as follows:

- Every $t = 1/n$ units of time, choose ℓ groups of vertices $\mathcal{V}_1, \dots, \mathcal{V}_\ell$ (of potentially different sizes) by sampling vertices i.i.d. from the graph.
- For each i, follow some rule $\mathcal{F}_i$ to choose a vertex x_i with cluster size κ_w from group $\mathcal{V}_i$, subject to the condition that $\mathcal{F}_i$ induces a function $\phi_i(s) = \mathbb{P}(\kappa_i = s)$ that is independent from all other ϕ_j for $j \neq i$.

We call $\mathcal{R}$ an **ℓ-choice rule**.

We'd like to restrict this vertex selection processes in each group as little as possible in order to get a more general theory, but in some cases we can perform much greater analysis if some information is known about them.

Definition 2. An ℓ-choice rule is **minimizing** if $\phi_i = Q_{m_i}$ for each i. It is **symmetric** if each ϕ_i is the same.

Minimizing rules exhibit "explosive" behavior in the sense that the critical time is significantly delayed and the giant component emerges incredibly quickly. Under the assumption that P exhibits scaling behavior, minimizing 2-choice rules in particular can be analyzed in a straightforward manner.

2 THE SCALING ASSUMPTION

NEED TO REDO THIS SECTION BASED ON GENERAL THINGS INSTEAD, AND BE VERY CLEAR ABOUT WHAT ALL THE DIFFERENT COEFFICIENTS ARE.

Most of the results in these notes follow from the assumption that near the critical time t_c, P has the form

$$P(s) = s^{1-\tau} f(s\delta^{1/\sigma})$$

for constants τ, σ and scaling function f. **Motivation for this.** The following theorem gives relations between these constants if some regularity conditions hold for the scaling function f.

Theorem 1. Suppose a rule $\mathcal{R}$ has a scaling function f such that

1. $\lim_{x\to\infty} x^{2-\tau} f(x) = 0$; and
2. $\int_0^\infty x^{2-\tau} f'(x)\, dx$ is finite.

Then there are **critical exponents**

$$\beta = (\tau - 2)/\sigma,$$
$$\gamma_m = (m(2-\tau)+1)/\sigma.$$

such that $S \sim \delta^\beta$ and $\langle s^k \rangle_m \sim \delta^{-\gamma_m - (k-1)/\sigma}$.

Proof. We'll begin by deriving β. Since

$$S \approx \int_0^\infty s^{1-\tau}(f(0) - f(s\delta^{1/\sigma}))\, ds,$$

we can make the change of variable $s = x\delta^{-1/\sigma}$ to get

$$= \delta^{(\tau-2)/\sigma} \int_0^\infty x^{1-\tau}(f(0) - f(x))\, dx.$$

Integrating by parts gives

$$= \frac{\delta^{(\tau-2)/\sigma}}{\tau - 2} \left[\left[-x^{2-\tau}(f(0) - f(x)) \right]_{x=0}^{x=\infty} - \int_0^\infty x^{2-\tau} f'(x)\, dx \right].$$

So by our assumptions on f, we have $S \sim \delta^\beta$, where $\beta = (\tau - 2)/\sigma$. **Type up the rest of this. Go over your concerns with the assumptions on f and the relations between f and g in Appendex E of da Costa.** □

2.1 INDUCED COEFFICENT MAPS

Definition 3. Suppose we have a function $\zeta(S)$, then this necessarily induces a map $F(\beta)$ by taking the exponent in the dominating (minimum order) terms of $\zeta(S)$ in its scaling form. This map F is called the **induced coefficient map** of ζ.

Example 1. Fix a, b, then let $\zeta(S) = S^a + S^b$. In scaling form, this is $\delta^{a\beta} + \delta^{b\beta}$. The induced coefficient map is $F(\beta) = \min a, b \cdot \beta$.

3 UNIFORM SCALING

It would be nice to express all these critical exponents in terms of just one (in our case, we'll express everything in terms of β). This has two main applications for ℓ-choice rules:

1. if we determine a single critical exponent, then we automatically know all others; and
2. we can determine the limiting behavior of the critical exponents as as the minimum group size goes to ∞ (which drives $\beta \to 0$).

The following property will be critical in establishing systems of equations that we can use to solve for the critical exponents in terms of β. As we will see later on, it always holds for minimizing 2-choice rules, and a partial version of it holds for general 2-choice rules.

Definition 4. We say that a rule $\mathcal{R}$ that exhibits scaling behavior **scales uniformly** if for S and all $\langle s\rangle_m$, every δ term comprising it has the same order. **Can I fix this so that it's more formal, i.e. is the result of one thing that's more readily definable?**

Note that since $\langle s^k\rangle_m \sim \delta^{-(\gamma_m + \frac{k-1}{\sigma})}$ differs from $\langle s\rangle_m \sim \delta^{-\gamma_m}$ by only an added constant, this property also applies to all k-th moments. Uniform scaling ends up giving us a systematic way of solving for all crtical exponents in terms of β when we're working with minimizing 2-choice rules.

4 SCALING RELATIONS FOR 2-CHOICE RULES

General 2-choice rules have an ODE of the form

$$\partial_t P(s) = s \sum_{u+v=s} \phi_1(u)\phi_1(v) - s\phi_1(s) - s\phi_2(s).$$

This lets us calculate $\partial_t S$.

Proposition 1. For 2-choice rules,

$$\partial_t S = \langle s \rangle_{\phi_1} (1 - \langle 1 \rangle_{\phi_2}) + \langle s \rangle_{\phi_2} (1 - \langle 1 \rangle_{\phi_1}).$$

Lemma 1. For any ϕ_i, there is an associated nonnegative function ζ_i such that

$$\langle 1 \rangle_{\phi_i} = 1 - \zeta_i(S).$$

Proof. ζ_i is just the probability that a vertex chosen from group i belongs to a cluster of inifinite size, so it must be a nonnegative function of S. (**Make more rigorous, perhaps making it explicit**) □

Need to establish scaling behavior of $\langle 1 \rangle_\phi$ using this.
Is $F_i(\beta)$ the scaling coefficient for $\langle 1 \rangle_{\phi_i}$?
Could determine classes of rules based on the form of the induced coefficient maps F_i...

Theorem 2. If $\mathcal{R}$ is a 2-choice rule, then it has two dominating terms with the same order.

Proof. By Proposition 1 and the preceding lemma,

$$\partial_t S = \langle s \rangle_{\phi_1} \zeta_2(S) + \langle s \rangle_{\phi_2} \zeta_1(S).$$

Suppose $F_i(\beta)$ is the induced coefficient map of $\zeta_i(S)$ (**Check to make sure that the two terms this applies to actually dominate**). Then there are two dominating terms, and both have order $F_1(\beta) + F_2(\beta) - 1/\sigma$. □

Corollary 1. If $\mathcal{R}$ is a 2-choice rule such that $\zeta_i(S)$ is a single term for both i, then $\mathcal{R}$ scales uniformly.

Proof. If $\zeta_i(S)$ is a single term for both i, then $\partial_t S$ has 2 total terms, which must necessarily have the same order. □

Corollary 2. All minimizing 2-choice rules scale uniformly.

Proof. For both i, $\phi_i = Q_m$ for some m. Then $\langle 1 \rangle_{\phi_i} = 1 - S^m$, so $\zeta_i(S) = S^m$. Since this is a single term, the rule must scale uniformly. □

4.1 SCALING RELATIONS

Since $\partial_t S \sim \delta^{\beta - 1}$, the proof of Theorem 2 shows that

$$\frac{1}{\sigma} = F_1(\beta) + F_2(\beta) - \beta + 1, \tag{1}$$

but we can derive other scaling relations for general 2-choice rules, too.

Since $\langle 1\rangle_{\phi_i} = 1-\zeta_i(S)$, it will have scaling behavior based on β **(right? see if you can derive this rigorously using a scaling form for ϕ similar to that of P)**. Thus it makes sense to define γ_{ϕ_i} as the constant satisfying

$$\langle s\rangle_{\phi_i} \sim \delta^{-\gamma_{\phi_i}}.$$

Then since $\partial_t S = \langle s\rangle_{\phi_1}\zeta_2(S) + \langle s\rangle_{\phi_2}\zeta_1(S)$, the two dominating terms near criticality give us the system

$$\beta - 1 \quad = \quad -\gamma_{\phi_1} + F_2(\beta) \quad = \quad -\gamma_{\phi_2} + F_1(\beta).$$

This system implies

$$\gamma_{\phi_1} = F_2(\beta) - \beta + 1, \tag{2}$$

$$\gamma_{\phi_2} = F_1(\beta) - \beta + 1. \tag{3}$$

One last constant that we care about is γ_P, which tells us how the average finite cluster size changes. In order to determine it, we need to differentiate $\langle s\rangle_P$.

Proposition 2. For 2-choice rules,

$$\partial_t \langle s\rangle_P = 2\langle s\rangle_{\phi_1}\langle s\rangle_{\phi_2} - \langle s^2\rangle_{\phi_1}\zeta_2(S) - \langle s^2\rangle_{\phi_2}\zeta_1(S).$$

Check that the three dominating terms scale uniformly... although they definitely should. This gives us the system

$$-\gamma_P - 1 \quad = \quad -\gamma_{\phi_1} - \gamma_{\phi_2} \quad = \quad -\gamma_{\phi_1} - \frac{1}{\sigma} + F_2(\beta) \quad = \quad -\gamma_{\phi_2} - \frac{1}{\sigma} + F_1(\beta).$$

Using (12)-(14), this system gives us

$$\gamma_P = F_1(\beta) + F_2(\beta) - 2\beta + 1. \tag{4}$$

Based on (12), we see

$$\gamma_P = \frac{1}{\sigma} - \beta,$$

which coincidentally agrees with (6) and (7) (with $a = b = 1$).

Is it possible to get similar statements for all γ_m?

Example 2 (da Costa)**.** Since $F_1(\beta) = F_2(\beta) = \beta m$, we have $\gamma_P = 2(m-1)\beta + 1$.

Finally, using the identity $\beta = (\tau - 2)/\sigma$, we get

$$\tau = \frac{\beta}{F_1(\beta) + F_2(\beta) - \beta + 1} + 2. \tag{5}$$

4.2 SUMMARY

The important equations for general 2-choice rules are

$$
\begin{aligned}
\partial_t P(s) &= s \sum_{u+v=s} \phi_1(u)\phi_1(v) - s\phi_1(s) - s\phi_2(s), \\
\partial_t S &= \langle s\rangle_{\phi_1}\left(1 - \langle 1\rangle_{\phi_2}\right) + \langle s\rangle_{\phi_2}\left(1 - \langle 1\rangle_{\phi_1}\right), \\
\partial_t \langle s\rangle_P &= 2\langle s\rangle_{\phi_1}\langle s\rangle_{\phi_2} - \langle s^2\rangle_{\phi_1}\zeta_2(S) - \langle s^2\rangle_{\phi_2}\zeta_1(S),
\end{aligned}
$$

and the scaling relations are

$$
\begin{aligned}
\gamma_{\phi_1} &= F_2(\beta) - \beta + 1, \\
\gamma_{\phi_2} &= F_1(\beta) - \beta + 1, \\
\gamma_P &= F_1(\beta) + F_2(\beta) - 2\beta + 1, \\
\frac{1}{\sigma} &= F_1(\beta) + F_2(\beta) - \beta + 1, \\
\tau &= \frac{\beta}{F_1(\beta) + F_2(\beta) - \beta + 1} + 2.
\end{aligned}
$$

4.3 RESULTS FOR MINIMIZING 2-CHOICE RULES

Suppose that $\phi_1 = Q$, $\phi_2 = b$. Then the scaling relations take on simpler forms. In particular, note that since $\langle 1\rangle_m = 1 - S^m$, the induced coefficient map for Q_m is $\beta \mapsto m\beta$. Thus the important equations for minimizing 2-choice rules are

$$
\begin{aligned}
\partial_t P(s) &= s \sum_{u+v=s} Q_a(u)Q_b(v) - sQ_a(s) - sQ_b(s), \\
\partial_t S &= S^b\langle s\rangle_a + S^a\langle s\rangle_b \\
\partial_t \langle s\rangle_P &= 2\langle s\rangle_a\langle s\rangle_b - S^b\langle s^2\rangle_a - S^a\langle s^2\rangle_b,
\end{aligned}
$$

and the scaling relations are

$$
\begin{aligned}
\gamma_a &= 1 + (b-1)\beta, \\
\gamma_b &= 1 + (a-1)\beta, \\
\gamma_P &= 1 + (a+b-2)\beta, \\
\frac{1}{\sigma} &= 1 + (a+b-1)\beta, \\
\tau &= \frac{\beta}{1 + (a+b-1)\beta} + 2.
\end{aligned}
$$

Suppose that $a = 1$, then $\gamma_b = 1$, no matter what b is. A symmetric statement holds if $b = 1$ instead. This matches what we saw with the adjacent edge rule, and reveals a somewhat surprising (at least to me) relationship. Here are some more scattered thoughts:

- Unless we're using Erdős Rényi, γ_P will always have a dependence on β.

- σ and τ will always depend on β.

So in summary, if neither of our groups has size 1, we can't know *any* of the critical exponents until we've calculated β, which stinks.

5 LIMITING BEHAVIOR OF 2-CHOICE RULES

5.1 SYMMETRIC RULES

Suppose $\mathcal{R}$ is a symmetric 2-choice rule with induced coefficient map F, then its scaling relations are

$$\begin{aligned}
\gamma_{\phi_1} = \gamma_{\phi_2} &= F(\beta) - \beta + 1, \\
\gamma_P &= 2F(\beta) - 2\beta + 1, \\
\frac{1}{\sigma} &= 2F(\beta) - \beta + 1, \\
\tau &= \frac{\beta}{2F(\beta) - \beta + 1} + 2.
\end{aligned}$$

Proposition 3. Suppose we have a symmetric 2-choice rule $\mathcal{R}$ with induced coefficient function F. If $F(\beta) \to 0$ as $a, b \to \infty$, then the scaling coefficients for $\mathcal{R}$ have limits

$$\begin{aligned}
\gamma_{\phi_1} = \gamma_{\phi_2} = \gamma_P = \frac{1}{\sigma} &= 1, \\
\tau &= 2.
\end{aligned}$$

Proof. We already know $\beta \to 0$, so if $F(\beta) \to 0$ too, then the above limits are straightforward computations. $\square$

Theorem 3. $a\beta, b\beta \to 0$, i.e. any minimizing 2-choice rule's scaling coefficients have the above limits.

Proof. **Do this.** $\square$

6 VARIANCE

The variance of the cluster size (at a fixed time) is $\mathrm{Var}_i(s) = \langle s^2\rangle_{\phi_i} - \langle s\rangle_{\phi_i}^2$. For 2-choice rules, we can use our scaling relations to get that near criticality,

$$\begin{aligned}\mathrm{Var}_i(s) &= \delta^{-[\gamma_{\phi_i}+1/\sigma]} - \delta^{-2\gamma_{\phi_i}}\\ &= \delta^{-[F_1(\beta)+2F_2(\beta)-2\beta+2]} + \delta^{-[2F_2(\beta)-2\beta+2]}.\end{aligned}$$

Then if $F_1(\beta) \geq 0$ **(which I think should be the case?)**, this is dominated by the first term. So near t_c, we have

$$\mathrm{Var}_i(s) \approx \delta^{-[F_1(\beta)+2F_2(\beta)-2\beta+2]}.$$

Example 3. Suppose we're working with a minimizing 2-choice rule, then $F_1(\beta) = a\beta$ and $F_2(\beta) = b\beta$. Then

$$\begin{aligned}\mathrm{Var}_1(s) &= \delta^{(2-a-2b)\beta-2},\\ \mathrm{Var}_2(s) &= \delta^{(2-2a-b)\beta-2}.\end{aligned}$$

By Theorem 3, $\mathrm{Var}_i(s) \to \delta^{-2}$ for both i. **(I wonder if the exponent is actually τ)**

Central limit theorem for cluster size?

7 IDEAS AND QUESTIONS

- The edge weighting idea seems very cool. If we have a graph whose vertices lie in some metric space, then we can weight them proportional to their distance from some fixed basepoint to have a stochastic process that grows "cracks" from that base point.
- Can F_i ever negate β, or does any induced map of the form $\beta \mapsto A\beta$ need $A > 0$?
- Can we reformulate $F(\beta) \to 0$ in terms of $\zeta(S)$?
- I don't think it's necessary to have something of the form $1-\zeta(S)$ to get the 2 dominating terms to scale uniformly, at least as long as the two unknown scaling coefficients are non-positive.
- This is more of just a fun thing. It could be cool to describe different group actions on these graphs that relate to adding edges, then describe some things algebraically.
- Can we give a sequence of graphs a product/coproduct structure? If we can turn a sequence of graphs into a coproduct, then that means that whatever property we're tracking has to appear in finite time.