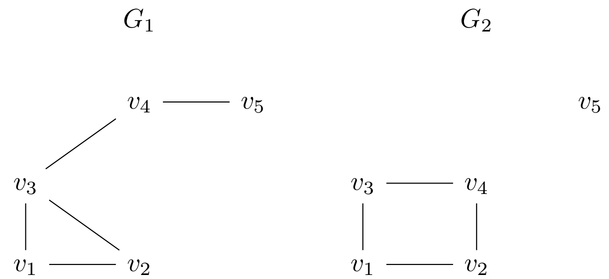
Exercise 1 (Lesson 3, 5 points)**.** Compute β_0 and β_1 of the graphs G_1 and G_2 given in the Lesson 3 notes.



Since our simplices are both 1-dimensional, our chain complex in both cases will be

$$0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0,$$

so the homology groups will be

$$\begin{aligned} H_0 &= \frac{\operatorname{Ker}\partial_0}{\operatorname{Im}\partial_1} \cong \frac{C_0}{\operatorname{Im}\partial_1}, \\ H_1 &= \frac{\operatorname{Ker}\partial_1}{\operatorname{Im}\partial_0} \cong \operatorname{Ker}\partial_1. \end{aligned}$$

For our two graphs G_1 and G_2, we have $\dim C_0 = \dim C_1 = 5$ since each graph has 5 vertices and 5 edges each. Thus the betti numbers are

$$\begin{aligned} \beta_0 &= \dim H_0 \\ &= \dim C_0 - \dim(\operatorname{Im}\partial_1) \\ &= \dim C_0 - (\dim C_1 - \dim(\operatorname{Ker}\partial_1)) \\ &= \dim(\operatorname{Ker}\partial_1) \end{aligned}$$

and

$$\beta_1 = \dim(\operatorname{Ker}\partial_1).$$

So both these graphs 0th and 1st betti numbers are both just the dimension of the kernel of ∂_1. We compute these kernels below.

1. For G_1, an element of the kernel of ∂_1 satisfies

$$\begin{aligned} \partial_1\left(\alpha[v_1, v_2] + \beta[v_2, v_3] + \gamma[v_1, v_3] + \delta[v_3, v_4] + \varepsilon[v_4, v_5]\right) &= 0 \\ \alpha(v_1 + v_2) + \beta(v_2 + v_3) + \gamma(v_1 + v_3) + \delta(v_3 + v_4) + \varepsilon(v_4 + v_5) &= 0 \\ (\alpha + \gamma)v_1 + (\alpha + \beta)v_2 + (\beta + \gamma + \delta)v_3 + (\delta + \varepsilon)v_4 + \varepsilon v_5 &= 0. \end{aligned}$$

Each coefficient must then be zero, giving us a system that we can represent in matrix form. Performing Gaussian elimination gives

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\delta = \varepsilon = 0$ and $\alpha = \beta = \gamma$. The space of all viable tuples $(\alpha, \beta, \gamma, \delta, \varepsilon)$ is then spanned by $(1,1,1,0,0)\gamma$ (the single triangle in the diagram), so $\dim(\operatorname{Ker} p_1) = 1$. Thus $\beta_0 = \beta_1 = 1$.

2. For G_2, we can similarly derive that any element of the kernel of ∂_1 satisfies

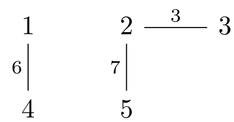
$$\partial_1 \left(\alpha[v_1, v_2] + \beta[v_2, v_4] + \gamma[v_1, v_4] + \delta[v_1, v_3] + \varepsilon[v_3, v_4]\right) = 0$$
$$(\alpha + \gamma + \delta)v_1 + (\alpha + \beta)v_2 + (\delta + \varepsilon)v_3 + (\beta + \gamma + \varepsilon)v_4 = 0.$$

Once again, we perform Gaussian elimination on the matrix form of this system to get

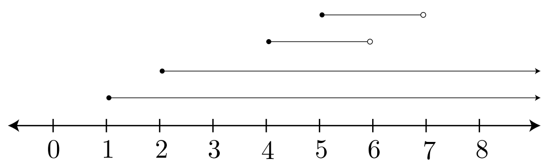
$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\alpha = \beta = \gamma + \delta$ and $\delta = \varepsilon$. Then the whole space of viable $(\alpha, \beta, \gamma, \delta, \varepsilon)$ is spanned by $(1,1,1,0,0)\gamma + (1,1,0,1,1)\delta$, the square and one of the triangles in the diagram. Thus $\dim(\operatorname{Ker} \partial_1) = 2$, so $\beta_0 = \beta_1 = 2$.

Exercise 2 (Lesson 4, 5 points)**.** Compute the zero dimensional persistence diagram of the filtered graph (i.e. graph and associated monotonic function) shown in the Lesson 4 notes.



The persistence diagram is below.



As a set, this is

$$\{[1,\infty),[2,\infty),[4,6),[5,7)\}\,.$$