

## 0.1 UNIQUENESS OF THE GIANT COMPONENT

Lemmas 4 and 5 and Theorem 1 used general  $\ell$ -vertex rules, but these cannot guarantee uniqueness of the giant component: consider a rule that only adds an edge if all  $\ell$  points come from different components, and it will merge the two smallest components in this case. Then  $\ell - 1$  large components are able to grow, and they will never be merged together.

The underlying issue here is that the probability of our large components merging is 0, an issue that is remedied by forcing our rule to be merging.

**Definition 1.** Suppose  $C_1$  and  $C_2$  are distinct components with

$$|C_1|, |C_2| \geq \varepsilon n.$$

An  $\ell$ -vertex rule is **merging** if the probability of  $C_1$  and  $C_2$  merging after a single step is at least  $\varepsilon^\ell$ .

So merging rules have a lower bound on this merging probability that grows based on the size of the components, ensuring that they will always be likely to merge. We will show in Theorem 2 that merging rules have unique giant components. But before we get there, we'll need two technical lemmas about merging rules. The first is a standard half life bound.

**Lemma 1.** Suppose  $|C_1|, |C_2| \geq \varepsilon n$ . Using a merging  $\ell$ -vertex rule, the probability that  $C_1$  and  $C_2$  are *not* merged after  $m$  steps is at least  $e^{-\varepsilon^\ell m}$ .

**Corollary 1.** If  $|C_1|, |C_2| \geq \frac{\varepsilon n}{2^\ell}$ , then the probability they don't merge after  $\Delta/2$  steps is at least

$$\exp\left(-\frac{\varepsilon n}{\ell^\ell k}\right).$$

*Proof.* By the previous lemma,

$$\mathbb{P}(C_i, C_j \text{ don't merge}) \leq \exp\left(-\left(\frac{\varepsilon}{2^\ell}\right)^\ell \frac{\Delta}{2}\right) = \exp\left(-\frac{\varepsilon^\ell}{2^\ell \ell^\ell} \left\lceil \frac{2^\ell n}{\varepsilon^{\ell-1} k} \right\rceil\right) \leq \exp\left(-\frac{\varepsilon n}{\ell^\ell k}\right).$$

□

We can now prove an analogue of Lemma 4 for merging rules. Let  $V_{\geq k}(m)$  be the union of all components with  $k$  or more vertices, so  $|V_{\geq k}(m)| = N_{\geq k}(m)$ .

**Lemma 6.** Suppose we have a merging  $\ell$ -vertex rule, and set  $\Delta = 2 \left\lceil \frac{2^\ell n}{\varepsilon^{\ell-1} k} \right\rceil$ . With probability at least  $1 - \ell e^{-cn/k}$ , there is a component of  $G(m + \Delta)$  with at least  $N_{\geq k}(m) - \varepsilon n$  vertices from  $V_{\geq k}(m)$ , where  $c > 0$  is a function of  $\varepsilon$  and  $\ell$ .

*Proof.* Let  $W = V_{\geq k}(m)$ . Assume  $|W| \geq \varepsilon n$ , and let  $\alpha = |W|/n \geq \varepsilon$ . Suppose we haven't yet gotten to the point where  $\ell - 1$  components together contain at least  $(\alpha - \varepsilon/2)n$

vertices from  $W$ , then the probability that we choose  $\ell$  vertices all in  $W$  and all in different components is at least  $\alpha(\varepsilon/2)^{\ell-1}$ :

1. Pick some  $v_1$ , say in component  $C_1$ . The probability that  $v_1 \in W$  is  $\alpha$ .
2. Pick some  $v_2$ . If  $\mathbb{P}(v_2 \in W - C_1) < \varepsilon/2$ , then  $|C_1| \geq (\alpha - \varepsilon/2)n$ , so by contradiction,  $\mathbb{P}(v_2 \in W - C_1) \geq \varepsilon/2$ .
3. Continue picking vertices until we get to  $v_\ell$ . If  $\mathbb{P}(v_\ell \in W - C_1 - \dots - C_{\ell-1}) < \varepsilon/2$ , then  $|C_1 \cup \dots \cup C_{\ell-1}| \geq (\alpha - \varepsilon/2)n$ , so by contradiction,  $\mathbb{P}(v_\ell \in W - C_1 - \dots - C_{\ell-1}) \geq \varepsilon/2$ .

Thus the probability of picking  $\ell$  vertices all in  $W$  and all in different components is at least  $\alpha(\varepsilon/2)^{\ell-1}$ . In this case, since all the components of the chosen vertices are distinct, at least 2 components must merge, so the number of components intersecting  $W$  must decrease by at least 1.

**Appeal to Lemma 4.**

Thus after  $\Delta/2$  steps, we have (with high probability?) components  $C_1, \dots, C_{\ell-1}$  that together contain at least  $(\alpha - \varepsilon/2)n$  vertices of  $W$ .

Ignore any  $C_i$  with fewer than  $\frac{\varepsilon n}{2\ell}$  vertices in  $W$ , then by the previous corollary, the probability that any two sufficiently large  $C_i$  merge after  $\Delta/2$  steps is at least  $\exp\left(-\frac{\varepsilon n}{\ell^2 k}\right)$ .

**Need to take this statement about *some* pair of components and make it a bound for *all* the component pairs, which the paper coincidentally doesn't do. This will show why that all components merge, giving one large component of the proper size. There's also the issue that we have two bounds: one for the first  $\Delta/2$  steps and one for the last  $\Delta/2$  steps). Combining these bounds should give the desired  $1 - \ell \exp\left(-\frac{\varepsilon n}{\ell^2 k}\right)$ .**

To make this fit with the statement of the lemma, simply set  $c(\varepsilon, \ell) = \varepsilon/\ell^\ell$ .  $\square$

This lemma lets us prove Theorem 2 from the paper, which essentially says that with high probability, there will be *no* time throughout our graph construction process where we have lots of vertices in equally large, yet still separate, components. Thus the bad example that I started these notes with can't happen with merging rules.

**Theorem 2.** Suppose we have a merging  $\ell$ -vertex rule for some  $\ell \geq 2$  that generates a random sequence of graphs  $\{G(m)\}_{m \geq 0}$ . For each  $\varepsilon > 0$ , there is a  $K = K(\varepsilon, \ell)$  such that

$$\mathbb{P}(\forall m, N_{\geq K}(m) < L_1(m) + \varepsilon n) \rightarrow 1.$$

as  $n \rightarrow \infty$ .

*Proof.* I haven't worked through this entirely yet.  $\square$

## 0.2 No JUMPS

The following theorem says that with high probability, a merging rule won't result in a linear jump in the size of the giant component after a sublinear amount of time.

**Theorem 7.** *Suppose we have a merging  $\ell$ -vertex rule that generates a random sequence of graphs  $\{G(m)\}_{m \geq 0}$ . Let  $h(n)$  be any  $o(n)$  function and let  $0 \leq a < b$  be any constants. Let  $\mathcal{J}$  be the event that there exist  $m_1, m_2$  such that all three of the following conditions hold:*

1.  $L_1(G(m_1)) \leq an$ ;
2.  $L_1(G(m_2)) \geq bn$ ;
3.  $m_2 \leq m_1 + h(n)$ .

*Then  $\mathbb{P}(\mathcal{J}) \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* As with the earlier theorems, we'll use a proof by contradiction. To derive the contradiction, we'll use the uniqueness of the giant component that we got from Theorem 2. Suppose  $\mathcal{J}$  holds, i.e. we can find  $m_1$  and  $m_2$  satisfying all three conditions. This means that the quantities

$$\begin{aligned} m^- &= \max \{m \mid L_1(m) \leq an\}, \\ m^+ &= \min \{m \mid L_1(m) \geq bn\}. \end{aligned}$$

both exist and  $m^+ - m^-$  differ by an  $o(n)$  function. Set  $\varepsilon = (b - a)/2$ , then Theorem 2 gives us (with high probability) a  $K = K(\varepsilon, \ell)$  such that

$$N_{\geq K}(m) < L_1(m) + \varepsilon n$$

for all times  $m$  (which of course includes  $m^-$  and  $m^+$ ). Now set

$$m^* = m^+ - \frac{\varepsilon n}{2\ell^2 K}.$$

Similar to what we did in the proof of Theorem 2,

$$N_{\geq K}(m^*) \geq L_1(m^+) - \ell^2 K(m^+ - m^*) > (b - \varepsilon)n = (a + \varepsilon)n.$$

However, our assumption that  $m^+ - m^-$  is  $o(n)$  means that we can take  $n$  large enough so that  $m^* < m^-$ . Then by Theorem 2 and the fact that  $m^-$  satisfies condition 1,

$$N_{\geq K}(m^*) \leq N_{\geq K}(m^-) < L_1(m^-) + \varepsilon n \leq (a + \varepsilon)n,$$

which is a contradiction to the previous inequality. Thus when Theorem 2 holds,  $\mathcal{J}$  cannot occur. Since Theorem 2 holds with high probability, this means  $\mathcal{J}$  fails with high probability, as desired.  $\square$

### 0.3 LOCAL AND GLOBAL CONVERGENCE

$X_n$  converges to  $X$  in probability, denoted  $X_n \xrightarrow{p} X$ , if

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0.$$

We say that a rule is locally convergent if there are functions  $\rho_k : [0, \infty) \rightarrow [0, 1]$  such that for all fixed  $k \geq 1$  and  $t \geq 0$ ,

$$\frac{N_k(\lfloor tn \rfloor)}{n} \xrightarrow{p} \rho_k(t).$$

We say that a rule is globally convergent if there is a nondecreasing function  $\rho : [0, \infty) \rightarrow [0, 1]$  such that for all  $t$  at which  $\rho$  is continuous,

$$\frac{L_1(\lfloor tn \rfloor)}{n} \xrightarrow{p} \rho(t).$$

A corollary of Theorem 7 is that  $\rho$ , if it exists, is continuous everywhere, i.e. this limiting behavior exists at all timesteps throughout the graph creation process.

**Corollary 8.** *If a merging  $\ell$ -vertex rule is globally convergent, then  $\rho$  is continuous on  $[0, \infty)$ .*

*Proof.* Since any  $\ell$ -vertex rule can add at most  $\binom{\ell}{2}$  edges per step,  $0 \leq \rho(t) \leq \binom{\ell}{2}t$ . Thus

$$0 \leq \lim_{t \searrow 0} \rho(t) \leq \lim_{t \searrow 0} \binom{\ell}{2}t = 0.$$

Note that the only possible value for  $\rho(0)$  is 0, since  $L_1(0) \rightarrow 0$  as  $n \rightarrow \infty$ , so  $\rho$  must be continuous at 0. Now suppose  $\rho$  is discontinuous at some other time  $t > 0$ . Since  $\rho$  is nondecreasing, this means that  $\sup_{t' < t} \rho(t') < \inf_{t' > t} \rho(t')$ , so we can take  $a < b$  such that

$$\sup_{t' < t} \rho(t') < a < b < \inf_{t' > t} \rho(t').$$

Fix  $\varepsilon > 0$ , then **by the definition of global convergence**, we can take  $n$  large enough such that

$$\mathbb{P}(L_1(\lfloor (t - \varepsilon)n \rfloor) \leq an \text{ and } L_1(\lfloor (t + \varepsilon)n \rfloor) \geq bn) \geq 1 - \varepsilon.$$

Since  $\varepsilon$  was arbitrary, we can let  $\varepsilon \rightarrow 0$ . But then we contradict Theorem 7, since our times  $\lfloor (t - \varepsilon)n \rfloor$  and  $\lfloor (t + \varepsilon)n \rfloor$  become too close to allow this event to happen almost surely. **(should make precise)** □

**Theorem 3.** *Suppose we have a locally convergent merging  $\ell$ -vertex rule, then it is also globally convergent. Furthermore,  $\rho$  is continuous and satisfies*

$$\rho(t) = 1 - \sum_{k \geq 1} \rho_k(t).$$

*Proof.* When a single edge is added to a graph,  $N_k$  changes by at most  $2k$  (when two components of size  $k$  are joined), so it is Lipschitz continuous. Then since the definition of local convergence doesn't depend on any specific  $t$ , the convergence of  $N_k(\lfloor tn \rfloor)/n$  to  $\rho_k(t)$  is uniform as  $n \rightarrow \infty$ . Thus each  $\rho_k$  is also continuous.

It should be clear that the function  $\rho_{\leq k}(t) = \sum_{j \leq k} \rho_j(t)$  is nonincreasing for each  $k$ . Then the function

$$\rho(t) = 1 - \lim_{k \rightarrow \infty} \rho_{\leq k}(t)$$

must be increasing. Fix  $t$  and  $\varepsilon$ , then we claim that the probability that

$$\sup_{0 \leq t' < t} \rho(t') - \varepsilon \leq \frac{L_1(\lfloor tn \rfloor)}{n} \leq \rho(t) + \varepsilon$$

goes to 1 as  $n \rightarrow \infty$  (can we just use  $\rho(t)$  in the lower bound? Is that not the same thing?).

If this is true, then  $L_1(\lfloor tn \rfloor)/n \xrightarrow{P} \rho(t)$  whenever  $\rho$  is continuous at  $t$ , i.e. our rule is globally convergent. Then by Corollary 8,  $\rho$  must also be continuous everywhere.

Now all that's left to do is prove the claim. We'll start with the upper bound. By the definition of  $\rho$ , there is some  $K$  such that  $\rho_{\leq K}(t) \geq 1 - \rho(t) - \varepsilon/4$ . Then by the local convergence of our rule, with high probability,

$$N_{\leq K}(\lfloor tn \rfloor)/n \geq \rho_{\leq K}(t) - \varepsilon/4 \geq 1 - \rho(t) - \varepsilon/2.$$

So for  $n$  large enough,

$$L_1(\lfloor tn \rfloor)/n \leq 1 - N_{\leq K}(\lfloor tn \rfloor)/n \leq \rho(t) + \varepsilon.$$

Now we show the lower bound. Choose  $t' < t$  such that  $\rho(t')$  is within  $\varepsilon/2$  of the supremum. (there's one step in the proof I dont understand. I highlighted it on my paper copy) □