MATH 531 HOMEWORK 2

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Exercise i. *Prove that every real number has an additive inverse and every nonzero real number has a multiplicative inverse.*

Lemma 0.1. *Let $\{x_n\}$ be a non-increasing function bounded below, then we can find $\{s_n\}$ non-decreasing and bounded above such that $\{x_n\} - \{s_n\} \sim i(0)$.*

Proof. Let m_0 be the largest integer such that $m_0 \leq x_i$ for all i. Inductively define m_i to be the largest element of $\{0, 1, \ldots, 9\}$ such that $\sum_{k=0}^{i} \frac{m_k}{10^k} \leq x_i$ for all i. Now let the sequence $\{s_n\}$ be defined by $s_n = \sum_{k=0}^{n} \frac{m_k}{10^k}$. We are essentially constructing a non-decreasing sequence that "converges" to the same point as x_n (we have not yet constructed the real numbers, so any notion of convergence is just intended to informally explain the general method of this proof).

We claim that $\{x_n\} - \{s_n\} \sim i(0)$. First we must show that if L is an upper bound of $i(0)$, then L is also an upper bound of $\{x_n\} - \{s_n\}$ (which from now on we abbreviate with $\{z_n\}$). Since $s_j \leq x_j$ by construction for all j, it follows that $s_j - x_j \leq 0$ for all j as well. Since L is at least 0, it is clear that $z_n \leq L$.

Now we must show that if L is an upper bound of $\{z_n\}$, then it is also an upper bound of $i(0)$, which we will prove by contradiction. If L is *not* an upper bound of $i(0)$, then there is some j such that $L + \frac{1}{10^j} < 0$. We also know that $z_j \leq L$ (by definition of an upper bound). Putting these two statements together yields

$$z_j + \frac{1}{10^j} \leq L + \frac{1}{10^j} < 0$$

However, by construction we know $z_j + \frac{1}{10^j}$ is an upper bound for 0. This is a contradiction, so it must be the case that L is actually an upper bound of $i(0)$. □

Additive Inverse Let $\{x_n\}$ be a non-decreasing sequence bounded above, then the negation of this sequence, $\{-x_n\}$, is non-increasing and bounded below by $-L$. Thus by Lemma 0.1, we can construct a non-decreasing sequence $\{z_n\}$ such that $\{-x_n\} - \{z_n\} \sim i(0) \implies \{x_n\} + \{z_n\} \sim i(0)$. This means that the sequence $\{z_n\}$ is the additive inverse of $\{x_n\}$.

Multiplicative Inverse For any nonzero $\{x_n\}$, we need to find $\{x_n\}^{-1}$ such that $\{x_n\} \cdot \{x_n\}^{-1} \sim i(1)$. This is straightforward if we use additive inverses, which we just proved exist in our construction.

The sequence $\{1/x_n\}$ clearly multiplies with $\{x_n\}$ to yield $i(1)$, but we cannot use it since it is a monotonically decreasing function. However, Lemma 0.1 tells us that since $\{1/x_n\}$ is bounded below by $\min\{0, 1/L\}$, we can find a sequence $\{b_n\}$ such that $\{1/x_n\} - \{b_n\} \sim i(0) \implies \{1\} - \{x_n b_n\} \sim i(0) \implies \{x_n b_n\} \sim i(1)$. This means that $\{b_n\}$ is the mlutiplicative inverse of $\{x_n\}$.

Exercise ii. *The naive definition of multiplication does not always take monotone increasing sequences to monotone increasing sequences. Fix the definition so that it does.*

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The problem in the naive definition of multiplication arises when a negative sequence "flips" the product sequence, making it non-increasing instead of non-decreasing. We can use the existence of additive inverses to fix this. Suppose the product $\{x_ny_n\}$ produces a non-increasing sequence, then we can negate one of the sequences to create a non-decreasing sequence $-\{x_ny_n\}$. Since we have proven that additive inverses exist, then there must be some sequence $- - \{x_ny_n\} = \{x_ny_n\}$ that is non-decreasing. In cases where the naive definition of $\{x_ny_n\}$ would result in a non-increasing sequences, we can replace the product with this new sequence instead and stay within the set of non-decreasing sequences.

Exercise 1.3.3. *If $P \subset Q \subset \mathbb{R}$, $P \neq \emptyset$, and P and Q are bounded aboved, show $\sup P \leq \sup Q$.*

Suppose $\sup Q < \sup P$, then $\sup Q$ is not an upper bound of P. This is because by definition, no upper bound of P can be strictly less than the supremum of P. However, $P \subset Q$ implies that if $p \in P$, then $p \in Q$ as well. Then by the definition of the supremum, $p \leq \sup Q$. This is a contradiction, so our original assumption that $\sup Q < \sup P$ must be incorrect. By the law of trichotomy for the real numbers, it must then be the case that $\sup P \leq \sup Q$.

Exercise 1.3.5. *Let $S \subset [0, 1]$ consist of all infinite decimal expansions $x = 0.a_1a_2a_3 \cdots$ where all but finitely many digits are 5 or 6. Find $\sup S$.*

The supremum of this set is 1. Fist we can show that 1 is an upper bound of the set, then we can show that any number less than 1 cannot be an upper bound.

To show that 1 is an upper bound, note that 1 can be written as

$$\sum_{i=0}^{\infty} \frac{n_i}{10^i}$$

where $n_0 = 1$ and all other $n_i = 0$ (all $n_i \in \{0, 1, \ldots, 9\}$). From this it is clear that if $x \sum_{i=0}^{\infty} \frac{m_i}{10^i} < 1$, then m_0 must be 0. Since all elements of the set S satisfy this, 1 is an upper bound for the set.

Now we must show that any $x < 1$ cannot be an upper bound of S. Take an arbitary $x < 1$ and write it in the decimal expansion notation

$$x = \sum_{i=0}^{\infty} \frac{m_i}{10^i}$$

Find the first i such that $m_i \neq 9$. Set this to 9 to get a new number x'. Now set all $m_j = 6$ for $j > i$. Then $x' \in S$ and $x' > x$. The only number for which this process does not work is $0.999\ldots$, since there are no digits not equal to 9. In this case, we can note that given a point x such that $0.999 \cdots \leq x \leq 1$, it must be the case that $|1 - x| \leq \frac{1}{10^n}$ for any n. By the Archimedean property, we can always find n such that $|1 - x| > \frac{1}{10^n}$ when x is fixed and $1 - x$ is nonzero. Thus the only solution is $x = 1$, so this in fact does not invalidate our earlier process.

This shows that no $x < 1$ can be an upper bound of S. Since $x < 1$ implies that x is not an upper bound and since 1 is an upper bound itself, it must be the least upper bound.

Exercise 1.4.1. *Let x_n satisfy $|x_n - x_{n+1}| < 1/3^n$. Show that x_n converges.*

We need to show that for some ε, there exists N such that $|x_n - x_m| < \varepsilon$ for all $n, m > N$. We can use the algebraic identity $1 + r + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$ to form the following inequality

$$\begin{aligned} r^n + r^{n+1} + \cdots + r^{n+k-1} &= \frac{1-r^{n+1}}{1-r} - \frac{1-r^n}{1-r} \\ &= \frac{r^n - r^{n+k}}{1-r} \\ &< \frac{r^n}{1-r} \end{aligned}$$

where the final inequality holds if $0 < r < 1$.

Now we can bound the difference between two points in the sequence separated by k points.

$$\begin{aligned} |x_n - x_{n+k}| &\le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{n+k-1} - x_{n+k}| \\ &< \frac{1}{3^n} + \frac{1}{3^{n+1}} + \cdots + \frac{1}{3^{n+k-1}} \end{aligned}$$

Using our derived inequality with $r = 1/3$, we can bound this with

$$\begin{aligned} &< \frac{(1/3)^n}{2/3} \\ &= \frac{1}{2 \cdot 3^{n-1}} \end{aligned}$$

We claim that the sequence defined by $y_n = \frac{1}{2} \cdot \frac{1}{3^{n-1}}$ converges to 0. If this is true, then by definition, we can find an N such that $|x_n - x_{n+k}| < \frac{1}{2} \cdot \frac{1}{3^{n-1}} = |\frac{1}{2} \cdot \frac{1}{3^{n-1}}| < \varepsilon$ for any $\varepsilon > 0$. Since this is not dependent on the choice of k, this holds for any arbitrary choice of k and thus the sequence x_n is a Cauchy sequence and, subsequently, converges.

To show that y_n converges to 0, we can derive the inequality $y_n < \frac{1}{n}$ and use the fact that $\frac{1}{n} \to 0$ in $\mathbb{R}$. The inequality clearly holds when $n = 1$ since $\frac{1}{2} < 1$. Assuming the inequality holds for some given n, we can then show that it holds for $n + 1$.

$$\frac{1}{2} \cdot \frac{1}{3^n} < \frac{1}{3^n} = \frac{1}{3 \cdot 3^{n-1}} < \frac{1}{3n} = \frac{1}{n+n+n} < \frac{1}{n+1}$$

Thus by induction, $y_n < \frac{1}{n}$. Since $\frac{1}{n} \to 0$ as $n \to \infty$ and $y_n > 0$, we know $y_n \to 0$ by the squeeze theorem. By the argument above, it immediately follows that x_n is a Cauchy sequence and converges to some limit.

Exercise 1.4.2. *Show that the sequence $x_n = e^{\sin(5n)}$ has a convergent subsequence.*

The sin function is defined on the codomain $[-1, 1]$, so it is bounded by definition. Since e^x is monotonically increasing in x ($x \le y \implies e^x \le e^y$), it must be true that $\frac{1}{e} \le e^x \le e$ when $x \in [-1, 1]$. This means that the entire sequence $x_n = e^{sin(5n)}$ is bounded. Then by the Bolzano-Weierstrass property, this sequence is guaranteed to have a convergent subsequence.

Exercise 1.4.3. *Find a bounded sequence with three subsequences converging to three different numbers.*

Let the sequence $\{x_n\}$ be defined by the pattern $\{1, 2, 3, 1, 2, 3, \dots\}$. More formally, this sequence can be defined by

$$x_n = \begin{cases} 1 & x = 3n+1 \text{ for some } n \in \mathbb{N} \\ 2 & x = 3n+2 \text{ for some } n \in \mathbb{N} \\ 3 & x = 3n \text{ for some } n \in \mathbb{N} \end{cases}$$

This sequence is clearly bounded above by 3 and bounded below by 1. We can define a subsequence of this sequence with the map $\phi_p(n) = 3n + p$, where $n \in \mathbb{N}, p \in \mathbb{N}_0$.

Now take the three subsequences $\left\{x_{\phi_0(n)}\right\}_{n=1}^{\infty} = \{1, 1, \dots\}$, $\left\{x_{\phi_1(n)}\right\}_{n=1}^{\infty} = \{2, 2, \dots\}$, and $\left\{x_{\phi_2(n)}\right\}_{n=1}^{\infty} = \{3, 3, \dots\}$. These clearly converge to 1, 2, and 3, respectively.

Exercise 1.4.4. *Let x_n be a Cauchy sequence. Suppose that for every $\varepsilon > 0$, there is some $n > 1/\varepsilon$ such that $|x_n| < \varepsilon$. Prove that $x_n \to 0$.*

Let $\varepsilon > 0$, then by assumption there exists some $N \in \mathbb{N}$ such that $|x_j - x_{j+k}| < \frac{\varepsilon}{2}$ for all $j > N, k > 0$. There are two possible relationships between ε and N. The first is $\frac{1}{\varepsilon} \geq N$, and the second is $\frac{1}{\varepsilon} < N$. In the first case, set an additional variable m equal to 1. In the second case, find m such that $\frac{2m}{\varepsilon} > N$. The existence of such an m is guaranteed by the definition of an Archimedean ordered field ($\mathbb{R}$ is such a field since it is complete).

Using this value $\frac{2m}{\varepsilon} > N$, we can guarantee two things:

(1) By assumption, there exists $n > \frac{2m}{\varepsilon}$ such that $|x_n| < \frac{\varepsilon}{2m}$.
(2) Since this n satisfies $n > N$, it is true that $|x_n - x_{n+k}| < \frac{\varepsilon}{2}$ for any natural number $k > 0$.

With these two statements, we can bound the distance of x_{n+k} from 0 for any k.

$$\begin{aligned}
|x_{n+k}| &= |x_k + x_{n+k} - x_k| \\
&\leq |x_n| + |x_{n+k} - x_k| \\
&< \frac{\varepsilon}{2m} + \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Since the choice of k was arbitrary, this holds for any point in the sequence after x_n. Thus given $\varepsilon > 0$, we can find an n such that for all $\tilde{n} > n$, $|x_{\tilde{n}} - 0| = |x_{\tilde{n}}| < \varepsilon$. By the definition of convergence, x_n then converges to 0.