CONTENTS

1 THE BASICS

1.1 CATEGORIES

> **Definition 1.** A **category C** is a collection of **objects** ob(**C**) and **morphisms** mor(**C**), where $\mathrm{Hom}(A, B)$ denotes the morphsisms from object A to object B. There are several requirements:
>
> 1. Morphisms must compose: $(f, g) \mapsto gf$.
>
> 2. Morphism composition is associative.
>
> 3. If $A \neq C$ or $B \neq D$, then $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(C, D)$ are disjoint.
>
> 4. Each object has an identity morphism, which is a two-sided identity.
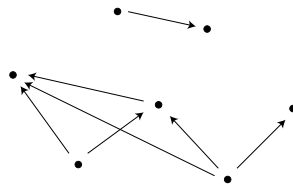
A category is **concrete** if, informally, its objects are underlying sets and its morphisms are functions between them, e.g. **Set**, **Top**, **Grp**. By contrast, **abstract** categories don't have this structure, e.g. BG for a group G.

A category is **discrete** if all its morphisms are identities, i.e. all its objects are isolated.

Because of set-theoretical issues, it's useful to denote when a category is "small enough". We say a category is **small** if it has only a set's worth of morphisms. Since

$$\text{identity morphisms} \leftrightarrow \text{objects},$$

small categories also have a set's worth of objects. We can loosen this somewhat: if $\mathrm{Hom}(X, Y)$ is always a set, the category is **locally small**.

> **Proposition 1.** Identity morphisms and morphism inverses are unique.

> **Definition 2.** An **isomorphism** is an invertible morphism.
>
> $$X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} Y$$

Isomorphisms (isos) generalize bijective functions, which are both injective and surjective. Injective functions generalize to monomorphisms (monos), and surjective functions to epimorphisms (epis).

   Include split monos/epis.

> **Definition 3.** A morphism f is a **monomorphism** if for all parallel (between same objects) morphisms g, h with the proper domains,
>
> $$fg = fh \implies g = h.$$
>
> Similarly, f is an **epimorphism** if
>
> $$gf = hf \implies g = h.$$

There's some fun vocab and symbols to go along with these. Monos are monic and denoted by $\rightarrowtail$, and epis are epic and denoted by $\twoheadrightarrow$. An isomorphism is necessarily both monic and epic, although the converse doesn't hold in general.

   Special types of morphisms get their own special names sometimes too. An **endomorphism** is a morphism $X \to X$. An isomorphic endomorphism is called an **automorphism**.

> **Definition 4.** A category S is a **subcategory** of C if
>
> 1. $\mathrm{ob}(\mathsf{S})$ is a subcollection of $\mathrm{ob}(\mathsf{C})$; and
>
> 2. for all $A, B \in \mathrm{ob}(\mathsf{S})$, $\mathrm{Hom}_{\mathsf{S}}(A, B)$ is a subcollection of $\mathrm{Hom}_{\mathsf{C}}(A, B)$ with identity.

A **full** subcategory doesn't remove any morphisms between the remaining objects, i.e.

$$\mathrm{Hom}_{\mathsf{S}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(A, B).$$

> **Definition 5.** A **groupoid** is a category whose morphisms are all isomorphisms.

Every category contains a subcategory called the **maximal groupoid**, which is all of the objects along with only the morphisms that are isomorphisms.

Example 1. We can define a **group** as a groupoid that has only one object. The group elements are the morphisms. The properties of a group follow from the properties of categories and the fact that our morphisms are all isomorphisms.

Given a group G, its representation as a single-object category is denoted BG.

1.2 DUALITY

> **Definition 6.** Given a category $\mathbf{C}$, its **opposite** or **dual** category $\mathbf{C}^{\mathrm{op}}$ is the category gotten by "reversing" the morphisms of $\mathbf{C}$. This means
>
> $$\mathrm{ob}(\mathbf{C}^{\mathrm{op}}) = \mathrm{ob}(\mathbf{C}),$$
> $$\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(B, A).$$

My biggest misconception of this at first was that we were actually reversing each morphism, but this is clearly impossible. For example, if we're working in **Set**, we physically can't reverse all the morphisms since not all functions are invertible.

> **Note 1.** We aren't actually changing any of the morphisms. The "reversal" of a morphism is a completely formal process. In fact, we can't even compare f and f^{op} since they live in different categories! At the end of the day, a category's dual has the same information, but the notation is just all reversed.

Note that the identities in a category and its dual are the same. Compositions, on the other hand, are necessarily reversed:

$$f^{\mathrm{op}} g^{\mathrm{op}} \doteq (gf)^{\mathrm{op}}.$$

Duals are important because they make universal quantifications twice as valuable: if a theorem applies "for all categories", then it certainly applies to the opposites of all categories. We can then reinterpret the theorem in the opposite case to get a dual theorem, and to prove it we just reverse all the morphisms in our original proof.
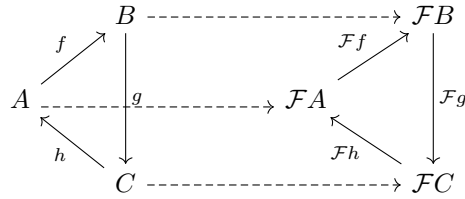
1.3 **FUNCTORS**

Functors are the morphisms associated with categories: they map categories to categories in ways that respect categorical structure.

Definition 7. A (**covariant**) **functor** $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ satisfies:

- If $A \in \mathbf{C}$, then $\mathcal{F}A \in \mathbf{D}$.

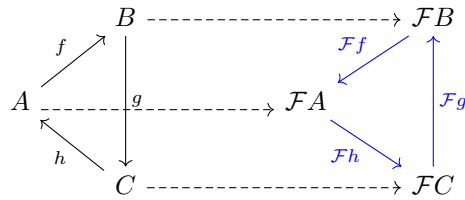- If $f : A \to B$, then $\mathcal{F}f : \mathcal{F}A \to \mathcal{F}B$.

These are subject to the functoriality axioms:

- $\mathcal{F}(fg) = \mathcal{F}f \cdot \mathcal{F}g$ for all f, g.

- $\mathcal{F}1_A = 1_{\mathcal{F}A}$ for all A.

A **contravariant functor** is the same but with the morphisms $\mathcal{F}f$ reversed. This is just a covariant functor in disguise, though: we can represent it by a covariant functor with domain $\mathbf{C}^{\mathrm{op}}$.

$$\mathcal{F} : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}.$$

Example 2. Some fun functors :)

1. Forgetful functors.

2. **Top** $\to$ **Htpy** is the identity on objects (topological spaces) and sends morphisms (continuous functions) to their homotopy class.

3. π_1 is a functor **Top**$_*$ $\to$ **Grp**.

Proposition 2. Functors preserve isos and split monos/epis.

Definition 8. A functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ is **faithful** if for all objects A, B of $\mathbf{C}$, the map

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathcal{F}A, \mathcal{F}B)$$
$$f \mapsto \mathcal{F}f$$

is one-to-one. $\mathcal{F}$ is **full** if this map is onto.

Note that the fixed A and B above are important. The injective/surjective conditions don't apply to arbitrary morphisms in $\mathbf{C}$ since they might connect different objects.
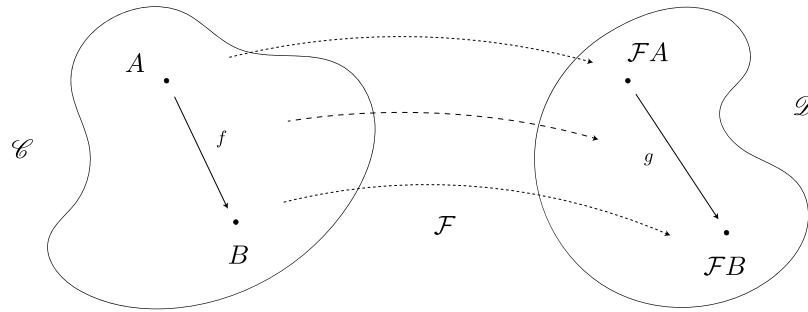

Figure 1.1: For all A, B, and g, a faithful functor sends at *most* one solid arrow in $\mathbf{C}$ to g. A full functor sends at *least* one solid arrow in $\mathbf{C}$ to g.

Example 3. The inclusion functor from $\mathbf{S}$ to $\mathbf{C}$ is always faithful, and it's full if and only if $\mathbf{S}$ is a full subcategory.

1.4   NATURAL TRANSFORMATIONS

Natural transformations change one functor into another in a way that respects the underlying structure of the categories involved. It's kinda like a homotopy between $\mathcal{F}$ and $\mathcal{G}$ in the sense that for all $C \in \mathbf{C}$, it gives a morphism from $\mathcal{F}C$ to $\mathcal{G}C$.
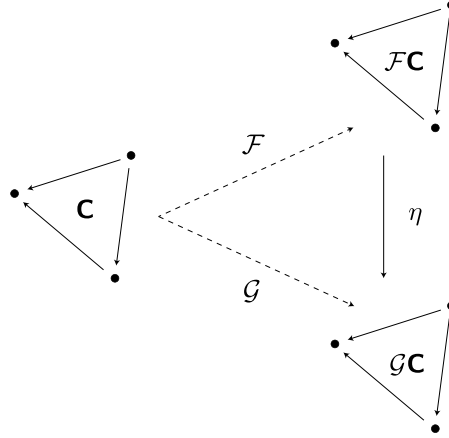
Definition 9. Suppose $\mathcal{F}, \mathcal{G} : \mathbf{C} \to \mathbf{D}$ are functors. Then a **natural transformation** $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$ is a family of **components**

$$\{\eta_X : \mathcal{F}X \to \mathcal{G}X\}_X$$

such that the following diagram commutes for any $f : X \to Y$ in $\mathbf{C}$.

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{\eta_X} & \mathcal{G}X \\ \mathcal{F}f \downarrow & & \downarrow \mathcal{G}f \\ \mathcal{F}Y & \xrightarrow{\eta_Y} & \mathcal{G}Y \end{array}$$

If every η_X is an isomorphism, then η is a **natural isomorphism** and we write $\eta : \mathcal{F} \cong \mathcal{G}$.

2 UNIVERSAL PROPERTIES

2.1 COMMON EXAMPLES

Definition 10. $(X, \{\pi_\alpha\}_\alpha)$ is a **product** of $\{X_\alpha\}_\alpha$ if for all Y and morphisms $f_\alpha : Y \to X_\alpha$, there is a unique morphism $f : Y \to X$ lifting each f_α.

$$
\begin{array}{ccc}
 & & X \\
 & \overset{\exists! \, f}{\nearrow} & \downarrow \pi_\alpha \\
Y & \underset{f_\alpha}{\longrightarrow} & X_\alpha
\end{array}
$$

Definition 11. $(X, \{\pi_\alpha\}_\alpha)$ is a **coproduct** of $\{X_\alpha\}_\alpha$ if for all Y and morphisms $f_\alpha : X_\alpha \to Y$, there is a unique morphism $f : X \to Y$ extending each f_α.

$$
\begin{array}{ccc}
 & & X \\
 & \overset{\exists! \, f}{\swarrow} & \uparrow i_\alpha \\
Y & \underset{f_\alpha}{\longleftarrow} & X_\alpha
\end{array}
$$

Proposition 3. If $(X, \{\pi_\alpha\})$ is a product, then each π_α is epic. If $(X, \{i_\alpha\})$ is a coproduct, then each i_α is monic.

Definition 12. (F, i) is free on the set B if for all objects X and maps $f : B \to X$, there is a unique morphism $F \to X$ extending f.

$$
\begin{array}{ccc}
F & & \\
i \uparrow & \overset{\exists!}{\searrow} & \\
B & \underset{f}{\longrightarrow} & X
\end{array}
$$