Spectral Sequences

Braden Hoagland

Math 502: Algebraic Structures II

Homology

- Chain complex:

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

  such that $d^2 = 0$.
- n-th homology: $H_n(A) = \ker d_n / \operatorname{im} d_{n+1}$.

Preliminaries

Theorem

A homomorphism of complexes induces a homomorphism of homologies.

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & A_n \arrow[r] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r] & B_n \arrow[r] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Preliminaries

Theorem

A homomorphism of complexes induces a homomorphism of homologies.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_{n+1}(A) & \longrightarrow & H_n(A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_{n+1}(B) & \longrightarrow & H_n(B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & \cdots
\end{array}$$

The homomorphism of homologies is given by

$$\begin{aligned} \phi_n : H_n(A) &\to H_n(B) \\ [a] &\mapsto [f_n(a)]. \end{aligned}$$

Can check it's well-defined (sends kernels to kernels and images to images) with a diagram chase.

Preliminaries

Theorem

Suppose $0 \to A \to B \to C \to 0$ *is a short exact sequence of complexes, then there is a long exact sequence of homologies*

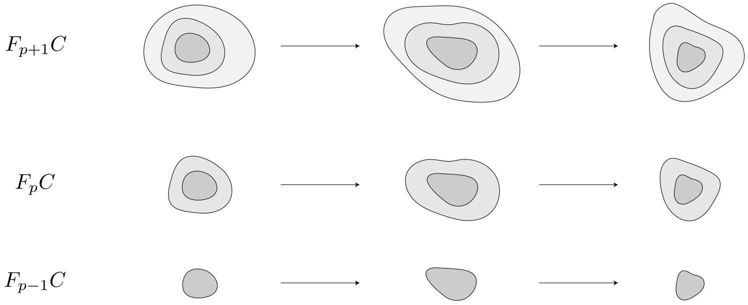
$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to \cdots .$$

Filtered Complexes

- Now for the main problem: suppose we want to calculate the homology of a *filtered complex*.
- We can build our complex as a sequence of subcomplexes.

Filtered Complexes

$$\cdots \subset F_{p-1}C \subset F_pC \subset F_{p+1}C \subset \cdots$$



Filtered Complexes

Induces a bigrading on the complex.

$$\begin{array}{ccc}
F_{p+1}C_{n+1} & F_{p+1}C_{n} & F_{p+1}C_{n-1} \\
F_{p}C_{n+1} & F_{p}C_{n} & F_{p}C_{n-1} \\
F_{p-1}C_{n+1} & F_{p-1}C_{n} & F_{p-1}C_{n-1}
\end{array}$$

Filtered Complexes

Induces a bigrading on the complex.

$$
\begin{tikzcd}
F_{p+1}C_{n+1} \arrow[r] \arrow[dr] \arrow[ddr] & F_{p+1}C_n & F_{p+1}C_{n-1} \\
F_pC_{n+1} & F_pC_n \arrow[r] \arrow[dr] & F_pC_{n-1} \\
F_{p-1}C_{n+1} & F_{p-1}C_n & F_{p-1}C_{n-1}
\end{tikzcd}
$$

Filtered complex: $d(F_pC_n) \subset F_pC_{n-1}$.

F is *bounded* if it has a finite number of levels.

Calculating Homology

- Suppose calculating $H_*(C)$ directly is diffiult.
- We can try a “divide and conquer” strategy to make the computation easier.

Calculating Homology

Idea 1: Calculate the homology row by row, then sum them.

$$
\begin{array}{ccccc}
F_{p+1}C_{n+1} & \longrightarrow & F_{p+1}C_n & \longrightarrow & F_{p+1}C_{n-1} \\
\\
F_pC_{n+1} & \longrightarrow & F_pC_n & \longrightarrow & F_pC_{n-1} \\
\\
F_{p-1}C_{n+1} & \longrightarrow & F_{p-1}C_n & \longrightarrow & F_{p-1}C_{n-1}
\end{array}
$$

Calculating Homology

Idea 1: Calculate the homology row by row, then sum them.

$$F_{p+1}C_{n+1} \longrightarrow F_{p+1}C_n \longrightarrow F_{p+1}C_{n-1}$$

$$F_pC_{n+1} \longrightarrow F_pC_n \longrightarrow F_pC_{n-1}$$

$$F_{p-1}C_{n+1} \longrightarrow F_{p-1}C_n \longrightarrow F_{p-1}C_{n-1}$$

Fails because each row is a subset of the rows above it.

Calculating Homology

Idea 2: Quotient each row by the rows below it, then calculate homology row by row and sum them.

$$\begin{array}{ccccc}
\dfrac{F_{p+1}C_{n+1}}{F_pC_{n+1}} & \longrightarrow & \dfrac{F_{p+1}C_n}{F_pC_n} & \longrightarrow & \dfrac{F_{p+1}C_{n-1}}{F_pC_{n-1}} \\[2em]
\dfrac{F_pC_{n+1}}{F_{p-1}C_{n+1}} & \longrightarrow & \dfrac{F_pC_n}{F_{p-1}C_n} & \longrightarrow & \dfrac{F_pC_{n-1}}{F_{p-1}C_{n-1}} \\[2em]
\dfrac{F_{p-1}C_{n+1}}{F_{p-2}C_{n+1}} & \longrightarrow & \dfrac{F_{p-1}C_n}{F_{p-2}C_n} & \longrightarrow & \dfrac{F_{p-1}C_{n-1}}{F_{p-2}C_{n-1}}
\end{array}$$

Calculating Homology

Idea 2: Quotient each row by the rows below it, then calculate homology row by row and sum them.

$$\begin{tikzcd}
\frac{F_{p+1}C_{n+1}}{F_pC_{n+1}} \arrow[r] \arrow[dr, dashed] \arrow[ddr, dashed] & \frac{F_{p+1}C_n}{F_pC_n} & \frac{F_{p+1}C_{n-1}}{F_pC_{n-1}} \\
\frac{F_pC_{n+1}}{F_{p-1}C_{n+1}} & \frac{F_pC_n}{F_{p-1}C_n} \arrow[r] \arrow[dr, dashed] & \frac{F_pC_{n-1}}{F_{p-1}C_{n-1}} \\
\frac{F_{p-1}C_{n+1}}{F_{p-2}C_{n+1}} & \frac{F_{p-1}C_n}{F_{p-2}C_n} & \frac{F_{p-1}C_{n-1}}{F_{p-2}C_{n-1}}
\end{tikzcd}$$

Still fails. The rows aren't subsets of each other anymore, but d still travels between rows.

Calculating Homology

We can remove inter-level dependencies by taking homology again.

$$
\begin{tikzcd}
\frac{F_{p+1}C_{n+1}}{F_pC_{n+1}} \arrow[r] \arrow[dr, dashed] \arrow[ddr, dashed] & \frac{F_{p+1}C_n}{F_pC_n} & \frac{F_{p+1}C_{n-1}}{F_pC_{n-1}} \\
\frac{F_pC_{n+1}}{F_{p-1}C_{n+1}} \arrow[r, blue] & \frac{F_pC_n}{F_{p-1}C_n} \arrow[r, blue] \arrow[dr, dashed] & \frac{F_pC_{n-1}}{F_{p-1}C_{n-1}} \\
\frac{F_{p-1}C_{n+1}}{F_{p-2}C_{n+1}} & \frac{F_{p-1}C_n}{F_{p-2}C_n} & \frac{F_{p-1}C_{n-1}}{F_{p-2}C_{n-1}}
\end{tikzcd}
$$

Note that the diagonal maps of bidegree $(-1,-1)$ induce maps on these homologies.

Calculating Homology

- We repeat this process, getting rid of another inter-level dependency every time we take another homology.
- If our filtration is finite, eventually we run out of dependencies.

Calculating Homology

Let $E^\infty_{n,p}$ denote the stablized homology of homology of ... of $F_pC_n/F_{p-q}C_n$.

If C_n is finite dimensional and F is a bounded filtration, then

$$H_n(C) \cong \bigoplus_p E^\infty_{n,p} \cong \bigoplus_p F_pH_n/F_{p-1}H_n.$$

Spectral Sequences

Spectral sequences are a generalization of what we just did.

Definition

A *spectral sequence* is a collection of R-modules $\{E^r\}_{r\geq 0}$ called *pages* with endomorphisms

$$d^r : E^r \to E^r$$

called *differentials* such that $d^r \circ d^r = 0$. Subsequent pages are related by

$$E^{r+1} \cong H_*(E^r).$$

Convergence

Definition

A spectral sequence $\{E^r\}_{r\geq 0}$ *converges* to a graded R-module H if there is a filtration F on H such that

$$E^{\infty}_{n,p} \cong F_p H_n / F_{p-1} H_n,$$

where $E^{\infty}_{n,p}$ is a stable limiting term of $E^r_{n,p}$.

Convergence

Definition

A spectral sequence $\{E^r\}_{r\geq 0}$ *converges* to a graded R-module H if there is a filtration F on H such that

$$E^{\infty}_{n,p} \cong F_p H_n / F_{p-1} H_n,$$

where $E^{\infty}_{n,p}$ is a stable limiting term of $E^r_{n,p}$.

Theorem

The spectral sequence induced by a filtered complex C with bounded filtration converges to $H_*(C)$.

Indexing Convention

Most authors use a different indexing notation. Instead of

$$E^0_{n,p} = F_p C_n / F_{p-1} C_n,$$

we could use complimentary degrees instead:

$$E^0_{p,q} = F_p C_{p+q} / F_{p-1} C_{p+q}.$$

Note that for convergent spectral sequences, we now sum down diagonals to get $H_{p+q}(C)$ instead of summing down a column to get $H_n(C)$.

Indexing Convention

So instead of

$$
\begin{tikzcd}
C_{n+1,p} & C_{n,p} \arrow[r, "d^0"] \arrow[dr, "d^1"] \arrow[ddr, "d^2"] \arrow[dddr, "d^3"'] & E_{n-1,p} \\
C_{n+1,p-1} & C_{n,p-1} & C_{n-1,p-1} \\
C_{n+1,p-2} & C_{n,p-2} & C_{n-1,p-2} \\
C_{n+1,p-3} & C_{n,p-3} & C_{n-1,p-3}
\end{tikzcd}
$$

Indexing Convention

We have

$$
\begin{tikzcd}
C_{p-3,q+2} & C_{p-2,q+2} & C_{p-1,q+2} & C_{p,q+2} \\
C_{p-3,q+1} & C_{p-2,q+1} & C_{p-1,q+1} & C_{p,q+1} \\
C_{p-3,q} & C_{p-2,q} & C_{p-1,q} & C_{p,q} \arrow[l, "d^1"] \arrow[ull, "d^2"'] \arrow[uulll, "d^3"'] \arrow[d, "d^0"] \\
C_{p-3,q-1} & C_{p-2,q-1} & C_{p-1,q-1} & C_{p,q-1}
\end{tikzcd}
$$

Homological Spectral Sequences

Definition

A *homological spectral sequence* is a spectral sequence where each differential d^r has bidegree $(-r, r-1)$.

Bockstein Spectral Sequence

Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{\text{mod } p} \mathbb{Z}_p \longrightarrow 0.$$

Bockstein Spectral Sequence

Suppose C is a torsion-free complex over $\mathbb{Z}$, then

$$0 \longrightarrow C \xrightarrow{\ p\ } C \xrightarrow{\text{mod } p} C \otimes \mathbb{Z}_p \longrightarrow 0$$

is also exact. Go over why.

Bockstein Spectral Sequence

The associated long exact sequence of homologies is

$$\cdots \to H_n(C) \to H_n(C) \to H_n(C \otimes \mathbb{Z}_p) \to H_{n-1}(C) \to \cdots$$

Bockstein Spectral Sequence

This is in the form of an "exact couple".

$$\begin{tikzcd}
H_*(C) \arrow[rr] & & H_*(C) \arrow[dl] \\
& H_*(C \otimes \mathbb{Z}_p) \arrow[ul] &
\end{tikzcd}$$

Theorem

This exact couple induces a homological spectral sequence.

Bockstein Spectral Sequence

This is in the form of an "exact couple".

$$\begin{tikzcd}
H_*(C) \arrow[rr] & & H_*(C) \arrow[dl] \\
& H_*(C \otimes \mathbb{Z}_p) \arrow[ul] &
\end{tikzcd}$$

Theorem

This exact couple induces a homological spectral sequence.

Cool application: the relations between the elements on each page give a generalization of the *universal coefficient theorem* for homologies with coefficients in $\mathbb{Z}_p$.

Homology of the 3-Sphere

If the know the homologies of the 1 and 2-spheres, plus a few facts from algebraic topology, then we can calculate the homology of the 3-sphere using the *Serre spectral sequence*.

Things We'll Need

Theorem

Suppose $F \to X \to B$ is a fibration with B a path connected space. If $\pi_1(B)$ acts trivially on $H_*(F)$, then there is a homological spectral sequence with

$$E^2_{p,q} \cong H_p(B; H_q(F)),$$

that converges to $H_*(X)$.

We can use the well-known Hopf fibration

$$S^1 \to S^3 \to S^2.$$

Things We'll Need

Theorem (Hurewicz)

Given a path connected topological space X, the abelianization of $\pi_1(X)$ is isomorphic to $H_1(X)$.

Things We'll Need

The homologies of the 1 and 2-spheres are

$$H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Homology of the 3-Sphere

The second page of the Serre spectral sequence is

$$\begin{array}{c|ccc} 1 & \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 & \mathbb{Z} \\ \hline & 0 & 1 & 2 \end{array}.$$

Homology of the 3-Sphere

Only one differential is nontrivial.

$$
\begin{tikzcd}
\mathbb{Z} & 0 & \mathbb{Z} \\
\mathbb{Z} & 0 & \mathbb{Z} \arrow[ull, "d"']
\end{tikzcd}
$$

Homology of the 3-Sphere

Only one differential is nontrivial.

$$\begin{tikzcd} \mathbb{Z} & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} \arrow[ull, "d"'] \end{tikzcd}$$

S^3 is path connected, so $\pi_1(S^3)$ is trivial.

Hurewicz: $H_1(S^3)$ is trivial, so the top-left $\mathbb{Z}$ must become 0.

Homology of the 3-Sphere

Only one differential is nontrivial.

$$\begin{tikzcd}
\mathbb{Z} & 0 & \mathbb{Z} \\
\mathbb{Z} & 0 & \mathbb{Z} \arrow[ull, "d"']
\end{tikzcd}$$

S^3 is path connected, so $\pi_1(S^3)$ is trivial.

Hurewicz: $H_1(S^3)$ is trivial, so the top-left $\mathbb{Z}$ must become 0.

d must be surjective $\implies$ d must be injective $\implies$ the bottom-right $\mathbb{Z}$ also becomes 0.

Homology of the 3-Sphere

The third page is then

$$\begin{array}{c|ccc} 1 & 0 & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 & 0 \\ \hline & 0 & 1 & 2 \end{array}.$$

This has fully stabilized, so we take the direct sum over the diagonals to get

$$H_k(S^3) = \begin{cases} \mathbb{Z} & k = 0, 3 \\ 0 & \text{otherwise} \end{cases}.$$