A BRIEF INTRODUCTION TO GEOMETRIC ALGEBRA

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Math 323S: Geometry

1 INTRODUCTION

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2 DEFINITIONS

Although the intuition is straightforward, actually defining a geometric algebra is less so. Several axioms are necessary to ensure that the algebra behaves in a proper manner, and a few of the axioms depend on definitions that in turn depend on previous axioms. As such, the definition of a geometric given below will be somewhat lengthy and frequently interrupted by other necessary definitions.

Without further ado, a **geometric algebra** is an associative unital algebra $\mathbb{G}$ whose multiplication operation is called the **geometric product**. An element of $\mathbb{G}$ is called a **multivector**. It is subject to the following axioms.

Axiom 1. *$\mathbb{G}$ contains a characteristic zero field $\mathbb{G}_0$ of **scalars**, whose additive and multiplicative identities coincide with those of $\mathbb{G}$.*

Axiom 2. *$\mathbb{G}$ contains a vector space $\mathbb{G}_1$ over $\mathbb{G}_0$. Perhaps unsurprisingly, we call the elements of $\mathbb{G}_1$ **vectors**.*

Axiom 3. *The square of any vector is a scalar.*

At this point we arrive at our first promised interruption of the definition. Consider the following expression, which is true because of the previous axiom:

$$uv = \frac{1}{2}(uv + vu) + \frac{1}{2}(uv - vu).$$

If we define

$$u \cdot v \doteq \frac{1}{2}(uv + vu),$$
$$u \wedge v \doteq \frac{1}{2}(uv - vu),$$

then we can clearly decompose the geometric product of vectors into

$$uv = u \cdot v + u \wedge v.$$

The first operation $\cdot$ is the **inner product**, and the second operation $\wedge$ is the **outer product**. Although this decomposition seems rather arbitrary at first glance, it is in fact incredibly useful. As we will see later, the inner product generalizes the dot product in $\mathbb{R}^n$, and the outer product generalizes the cross product of $\mathbb{R}^3$.

With this in mind, we say that two vectors u and v are **orthogonal** if their inner product is 0, which is true if and only if they anticommute. Thus when u and v are orthogonal, $uv = u \wedge v$. Also note that $u \wedge v = -v \wedge u$ for all u and v. In a similar vein, since every vector u necessarily commutes with itself, we have $uu = u \cdot u$. These two special cases will prove very handy in cleaning up otherwise messy algebraic expressions later on.

The wedge product is actually an important tool in constructing higher-dimensional vectors. An r**-blade** is the wedge product of r orthogonal vectors, and an r**-vector** is a finite sum of r-blades. Although this definition is presently pretty opaque, these are precisely the higher-dimensional geometric objects that we will study. We denote the space of all r-vectors by $\mathbb{G}_r$.

Note that since $\mathbb{G}_1$ is closed under scalar multiplication and since each $\mathbb{G}_r$ is built up from vectors, each $\mathbb{G}_r$ is necessarily also closed under scalar multiplication. In particular, let the scalar be 0, then we see that 0 is an r-vector for all r. With all this in mind, the following axioms should actually make sense.

Axiom 4. *The only vector orthogonal to all other vectors is 0.*

Axiom 5. *If $\mathbb{G}_1 = \mathbb{G}_0$, then $\mathbb{G} = \mathbb{G}_0$. Otherwise, $\mathbb{G} = \bigoplus_r \mathbb{G}_r$.*

This final axiom implies an incredibly important aspect of the structure of a geometric algebra. Suppose that $\{e_1, e_3\}$ is an orthonomal basis for $\mathbb{G}_0$, then we have a **canonical basis** for $\mathbb{G}$ given by

$$\begin{gathered} 1 \\ e_1, \quad e_2, \quad e_3 \\ e_1e_2, \quad e_1e_3, \quad e_2e_3 \\ e_1e_2e_3. \end{gathered}$$

The extension to when $\mathbb{G}_0$ is n-dimensional is straightforward. In general, this basis will have 2^n elements, so $\mathbb{G}$ is 2^n-dimensional. The n-th row of the above basis is itself a basis for the space of n-vectors.

Proposition 1. *The canonical basis for $\mathbb{G}$ is in fact a basis.*

Proof. **Do this.** □

Proposition 2. $\{e_{i_1} \dots e_{i_n} \mid i_j < i_k$ *when* $j < k\}$ *is a basis for the space of n-vectors.*

Proof. **Do this.** □

Show existence of GA.
Everything in GA is coordinate free (unless working with a specific example, ofc).
norm of blade.

3 THE INNER AND OUTER PRODUCT

r-blade is outer product of arbitrary vectors.

Example 1 (The outer product generalizes the cross product)**.** *Suppose we have two vectors* $\mathbf{a} = a_1e_1 + a_2e_2 + a_3e_3$ *and* $\mathbf{b} = b_1e_1 + b_2e_2 + b_3e_3$*, then*

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= (a_2b_3 - a_3b_2)e_2e_3 + (a_1b_3 - a_3b_1)e_1e_3 + (a_1b_2 - a_2b_1)e_1e_2, \\ \mathbf{a} \times \mathbf{b} &= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}. \end{aligned}$$

Thus when working with vectors in particular, the wedge product is in one-to-one correspondence with the cross product. Also note that the norm of both expressions is the same, so they represent geometric objects with the same area.

OP changes sign when pair is swapped (b/c it's associative). It's also linear.

4 GEOMETRIC INTERPRETATIONS

4.1 THE INNER PRODUCT

Do this section.

4.2 THE OUTER PRODUCT

We begin by interpreting the outerproduct in terms of geometric subspaces of $\mathbb{R}^n$. The simplest subspaces are represented by nonzero r-blades. This is formalized in the next proposition.

Proposition 3. *$a_1 \wedge \cdots \wedge a_r = 0$ if and only if $\{a_1, \ldots, a_r\}$ is linearly dependent.*

Proof. **Do forward direction. See Thrm 25 in $\mathcal{G}$ paper.** Conversely, if $\{a_1, \ldots, a_n\}$ is linearly dependent, then some factor is repeated in $a_1 \wedge \cdots \wedge a_n$. Then since the outer product is antisymmetric, this forces the outer product to be 0. □

Corollary 1. *Suppose $\mathbf{A}$ is a nonzero r-blade, then a vector a lies in the span of the factors of $\mathbf{A}$ if and only if $a \wedge \mathbf{A} = 0$.* □

This is enough for us to finally get a geometric interpretation of the outer product. Although it is a generalization of the cross product, with arbitrary blades, it actually represents the direct sum.

Notation $\mathbf{A}_r$ below.

Theorem 1. *$\mathbf{A}_r \wedge \mathbf{B}_s = 0$ if and only if $\mathbf{A}_r$ and $\mathbf{B}_s$ share a nonzero vector. If the outer product is nonzero, it represents the direct sum of the subspaces represented by $\mathbf{A}_r$ and $\mathbf{B}_s$.*

Proof. By a clear extension of Proposition 3, $\mathbf{A}_r \wedge \mathbf{B}_s \iff$ they form a linearly dependent set $\iff$ they share a nonzero vector. Now suppose the outer product is nonzero, then $\mathbf{A}_r$ and $\mathbf{B}_s$ only coincide at 0. **If v is any vector, $v \wedge \mathbf{A}_r \wedge \mathbf{B}_s = 0 \iff v$ is a linear combination of the factors of $\mathbf{A}_r$ and $\mathbf{B}_s$.** Thus $\mathbf{A}_r \wedge \mathbf{B}_s$ represents the direct sum. □