# Percolation Phase Transitions on Dynamically Grown Graphs

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# Background

## Dynamically grown graphs and percolation

# Dynamically Grown Graphs

Start with a graph with $n$ vertices and no edges.

Every $t = 1/n$ units of time, add edges to the graph by sampling $m$ vertices i.i.d. and following some fixed rule.

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Every $t = 1/n$ units of time, add edges to the graph by sampling $m$ vertices i.i.d. and following some fixed rule.

Let $n \to \infty$.

# Percolation

A *giant component* is a cluster comprising a finite fraction $\varepsilon n$ of the graph.

Percolation is the emergence of a giant component.

Percolation can have lots of different qualitative behaviors.

# Explosive Percolation

For simple rules, the giant component might emerge is a predictable, linear manner.

If a rule prioritizes adding together smaller clusters, the giant component's emergence is delayed and happens very quickly (seemingly discontinuous). This is called *explosive percolation*.

# Basic Results

## Continuity of the phase transition and scaling behavior

# Continuity of the Phase Transition

$\ell$*-vertex rule*: choose $\ell$ vertices i.i.d., and you're only required to add an edge if all $\ell$ of them are in distinct clusters.

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Their proof is by contradiction, so it doesn't give us much quantitative information about the clusters' behavior.

# Scaling Behavior

For rules with continuous phase transitions, we see *scaling behavior*.

Let $\delta = t - t_c$ and let $P(s,t)$ be the probability that a randomly chosen vertex has cluster size $s$ at time $t$. Then near $t_c$, there are constants $\tau$ and $\sigma$ such that

$$P(s) = s^{1-\tau} f(s\delta^{1/\sigma}).$$

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**From now on, we assume scaling behavior.**

# Scaling Behavior

Let $S$ be the size of the giant component, and let

$$\chi_k(t) = \sum_s s^k P(s, t).$$

Then

$$S \approx \delta^{\beta}, \qquad \chi_1(t) \approx \delta^{-\gamma}, \qquad \frac{\chi_k(t)}{\chi_{k-1}(t)} \approx \delta^{-\Delta}$$

These unknowns are called *critical exponents*.

# Scaling Relations

Goal: determine all critical exponents in terms of one unknown.

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Why is this useful?

What kinds of rules can we do this for?

# 2-Choice Rules

## Generalizing rules with useful properties

# 2-Choice Rules

Pick two groups of vertices i.i.d.

Select one vertex from each and add an edge between them.

$\phi_i(s) = \mathbb{P}\,($vertex chosen from group $i$ has cluster size $s)$.

# Erdős Rényi

Pick two random vertices and add an edge between them.

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Percolation occurs after $t_c = 1/2$.

$\beta = 1$, so $S$ grows linearly near $t_c$.

# da Costa

Introduced by da Costa to disprove Achlioptas' discontinuity conjecture.

Pick two groups of vertices, both of size $m$. Pick the vertex with the smallest cluster size from the two groups and add an edge between them.

Same as Erdős Rényi when $m = 1$. As $m \to \infty$,

$$\beta \to 0, \qquad t_c \to 1.$$

# Uniform Scaling

For any 2-choice rule, the quantity $\partial_t S$ has a simple form that can be explicitly calculated.

Near $t_c$, $\partial_t S$ will look like

$$\delta^a + \delta^b + \delta^c + \cdots$$

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All 2-choice rules **almost** scale uniformly.

# Uniform Scaling

**Theorem**

*For any 2-choice rule, there will be two dominating terms of* $\partial_t S$ *with the same order. If some extra technical conditions hold, then the rule scales uniformly.*

# Uniform Scaling

Consequences:

- For all 2-choice rules, we can solve for all critical exponents, as well as the growth rate of the average cluster size, in terms of $\beta$.
- For a large family of 2-choice rules, we can do this algorithmically.

# Future Directions

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- Interaction between the groups?
- When does scaling behavior actually occur?
- What about $n$-choice rules?