Contents

0.1 Categories

Definition 1: Category

A **category** $\mathscr{C}$ is a class of **objects** $\text{ob}(\mathscr{C})$ along with sets of **morphisms** between those objects. The set of morphisms A to B is denoted $\text{Hom}_{\mathscr{C}}(A, B)$. There must be a law of composition of morphisms, i.e. for all objects A, B, and C, there is a map

$$\text{Hom}_{\mathscr{C}}(A, B) \times \text{Hom}_{\mathscr{C}}(B, C) \to \text{Hom}_{\mathscr{C}}(A, C)$$

that sends the pair of morphisms (f, g) to their composition gf. Finally, the objects and morphisms satisfy:

1. If $A \neq C$ or $B \neq D$, then $\text{Hom}_{\mathscr{C}}(A, B)$ and $\text{Hom}_{\mathscr{C}}(C, D)$ are disjoint sets.
2. Morphism composition is associative.
3. Each object has an identity morphism, i.e. for object A, there is a map $1_A \in \text{Hom}_{\mathscr{C}}(A, A)$ such that $1_A g = g$ and $f 1_A = f$ for all $f \in \text{Hom}_{\mathscr{C}}(A, B)$ and $g \in \text{Hom}_{\mathscr{C}}(B, A)$, where B is arbitrary.

We will drop the subscript $\mathscr{C}$ in $\text{Hom}_{\mathscr{C}}$ if the category is clear.

Definition 2: Subcategory

$\mathscr{C}$ is a subcategory of $\mathscr{D}$ if

1. every object of $\mathscr{C}$ is an object of $\mathscr{D}$; and
2. for all objects A, B in $\mathscr{C}$, $\text{Hom}_{\mathscr{C}}(A, B) \subset \text{Hom}_{\mathscr{D}}(A, B)$.

Proposition 1. *The identity morphism of an object is unique.*

Proof. Suppose 1_A and $1'_A$ are both identity morphisms of A. Then by the two equalities in condition (3) above, $1_A = 1_A 1'_A = 1'_A$. □

Definition 3: Endomorphism

An **endomorphism** of A is a morphism from A to itself.

Definition 4: Isomorphism

An isomorphism $f : A \to B$ is an invertible morphism, i.e. there exists a morphism $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$.

Proposition 2. *Inverses of morphisms are unique.*

Proof. Suppose $f : A \to B$ is a morphism and $g, g' : B \to A$ are both inverses of it. Then by associativity of morphism composition, $g = g1_B = g(fg') = (gf)g' = 1_A g' = g'$. □

Now for some examples to make this *somewhat* less abstract.

1. **Set**: the category of all sets. The category of all finite sets is a subcategory of this.
   - $\mathrm{Hom}(A, B)$ is the set of all functions from A to B.
   - Morphism composition is the usual composition of functions.
   - The identity morphism sends $a \in A$ to itself.
2. **Grp**: the category of all groups. **Ab**, the category of all abelian groups, is a subcategory of this. Morphisms are group homomorphisms, and isomorphisms are, well, group isomorphisms.
3. **Ring**: the category of all nonzero rings with 1. The morphisms are ring homomorphisms that send 1 to 1.
4. **R-mod**: the category of all left R-modules. The morphisms are R-module homomorphisms.
5. **Top**: the category of all topological spaces. The morphisms are continuous maps between spaces, and the isomorphisms are homeomorphisms.

Definition 5: Discrete Category

A **discrete category** is a category in which all the morphisms are identities, i.e. every object is isolated.

Definition 6: Opposite/Dual Category

Given a category $\mathscr{C}$, its **opposite** or **dual** category $\mathscr{C}^{\mathrm{op}}$ is the category gotten by reversing the morphisms of $\mathscr{C}$. Formally, $\mathrm{ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{ob}(\mathscr{C})$, but

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A).$$

Note that the identities in a category and its dual are the same. Compositions, on the other hand, are reversed.

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "gf"'] & \bullet \arrow[d, "g"] \\ & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet & \bullet \arrow[l, "f'"'] \\ & \bullet \arrow[u, "g'"'] \arrow[ul, "f'g'"] \end{tikzcd}$$

Figure 1: A category and its dual. Since every object must have an identity morphism, I usually won't include them in a diagram unless necessary.

Definition 7: Product Category

Given categories $\mathscr{C}$ and $\mathscr{D}$, we can define their **product category** $\mathscr{C} \times \mathscr{D}$ as having the objects

$$\mathrm{ob}(\mathscr{C} \times \mathscr{D}) = \mathrm{ob}(\mathscr{C}) \times \mathrm{ob}(\mathscr{D})$$

and the morphisms

$$\mathrm{Hom}_{\mathscr{C} \times \mathscr{D}}((A, B), (A', B')) = \mathrm{Hom}_{\mathscr{C}}(A, A') \times \mathrm{Hom}_{\mathscr{D}}(B, B').$$

It is straightforward to define the identity morphisms and the composition of morphisms in product categories in a piecewise fashion, building off the identities and composition laws of $\mathscr{C}$ and $\mathscr{D}$

0.2 Functors

Definition 8: Functor

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **(covariant) functor** $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ associates every object A in $\mathscr{C}$ to an object $\mathcal{F}A$ in $\mathscr{D}$, and it also associates every morphism in $\mathrm{Hom}_{\mathscr{C}}(A, B)$ to a morphism in $\mathrm{Hom}_{\mathscr{D}}(\mathcal{F}A, \mathcal{F}B)$ such that identity morphisms and morphism compositions are preserved.
Contravariant functors are the same, except they reverse functors (and thus also morphism composition).

The formal definitions of both types of functions are given below, with the differences between them highlighted. Covariant functors satisfy:

1. For every object A in $\mathscr{C}$, $\mathcal{F}A$ is an object in $\mathscr{D}$.
2. For every $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $\mathcal{F}(f)$ is a morphism in $\mathrm{Hom}_{\mathscr{D}}(\mathcal{F}A, \mathcal{F}B)$ such that

(a) $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$, and
(b) $\mathcal{F}(1_A) = 1_{\mathcal{F}A}$.

Contravariant functors satisfy:

1. For every object A in $\mathscr{C}$, $\mathcal{F}A$ is an object in $\mathscr{D}$.
2. For every $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $\mathcal{F}(f)$ is a morphism in $\mathrm{Hom}_{\mathscr{D}}(\mathcal{F}B, \mathcal{F}A)$ such that
   (a) $\mathcal{F}(gf) = \mathcal{F}(f)\mathcal{F}(g)$, and
   (b) $\mathcal{F}(1_A) = 1_{\mathcal{F}A}$.

$$
\begin{tikzcd}
& B \arrow[rr, dashed] \arrow[dd, "g"] & & \mathcal{F}B \arrow[dd, "\mathcal{F}(g)"] \\
A \arrow[ur, "f"] \arrow[rr, dashed] & & \mathcal{F}A \arrow[ur, "\mathcal{F}(f)"] & \\
& C \arrow[ul, "h"] \arrow[rr, dashed] & & \mathcal{F}C \arrow[ul, "\mathcal{F}(h)"]
\end{tikzcd}
$$

Figure 2: A functor $\mathcal{F}$ between two categories.

Example 1: Category Inception

The category **CAT** has objects that are themselves categories, and its morphisms are functors.

Example 2: Forgetful Functors

A very basic type of functor might take a category and strip its objects of some kind of complexity, i.e. a functor from **Grp** to **Set**. A “forgetful functor” doesn’t have to just map objects to plain sets, though. We could also map **Ab** to **Grp**, forgetting the abelian nature of the groups in our category.

Definition 9: Domain/Codomain

Given a functor $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, A is the **domain** and B is the **codomain** of f.

There are tons of examples of functors, so here are some that aren’t too complicated.

1. The **identity functor** $\mathcal{I}_{\mathscr{C}}$ maps $\mathscr{C}$ to $\mathscr{C}$ by sending objects and morphims to themselves.

2. If $\mathscr{C}$ is a subcategory of $\mathscr{D}$, then the **inclusion functor** maps C to D by sending objects and morphisms to themselves, except now as members of $\mathscr{D}$ instead of $\mathscr{C}$.

More examples.