1. (i) The j-th point in the sequence $\{S+I\}_j$ is $S_j+I_j = S_{j-1}+\sigma I_{j-1}$. We can then define the difference between subsequent points in the sequence to be $\Delta(S+I)_j = (S_j+I_j)+(S_{j-1}+I_{j-1}) = (\sigma-1)I_{j-1}$. Since $\sigma = e^{-\alpha}$, it is always true that $0 < \sigma < 1$ when $\alpha < 0$. Since I is always non-negative, this implies that $\Delta(S+I)_j \leq 0$ at all points in time. This makes intuitive sense, as this model does not allow for indidivduals to leave the recovered/removed state.

   Since $\{S+I\}_j$ is bounded below by 0 and was just shown to be monotone non-increasing, it must converge to a limit $S_\infty + I_\infty \geq 0$.

   (ii) Since $(S+I)_j$ converges to some limit, it must be the case that $\Delta(S+I)_j$ converges to 0. Since $\Delta(S+I)_j = (\sigma-1)I_{j-1}$ and $\sigma - 1$ is strictly nonzero for $\alpha > 0$, this implies $I_j \to 0$.

   (iii) We start by noting that $S_{j+1} = S_j G_j = S_j e^{-\beta I_j/N}$, so $\frac{S_{j+1}}{S_j} = e^{-\beta I_j/N}$ and its logarithm is $\log \frac{S_{j+1}}{S_j} = -\beta \frac{I_j}{N}$. By the laws of logarithms, this is equivalent to $\log \frac{S_j}{S_{j+1}} = \beta \frac{I_j}{N}$.

   We can find the sum of this term up to some index m

$$\begin{aligned}\beta \sum_{j=0}^{m} \frac{I_j}{N} &= \sum_{j=0}^{m} \log \frac{S_j}{S_{j+1}} \\ &= \sum_{j=0}^{m} (\log S_j - \log S_{j+1}) \\ &= \log \frac{S_0}{S_m}\end{aligned}$$

   Taking the limit as $m \to \infty$, this becomes

$$\log \frac{S_0}{S_\infty} = \beta \sum_{j=0}^{\infty} \frac{I_j}{N}$$

   which is the desired result.

   (iv) We can find an alternative equality for $\beta \sum_{j=1}^{\infty} \frac{I_j}{N}$ and subsitute it into the previous equation. We can find this with the $\Delta(S+I)_j$ terms we defined earlier.

$$\begin{aligned}\Delta(S+I)_j &= (\sigma-1)I_{j-1} \\ \sum_{j=1}^{m+1} \Delta(S+I)_j &= -(1-\sigma)\sum_{j=0}^{m} I_j \\ -\frac{\beta}{N(1-\sigma)} \sum_{j=1}^{m+1} \Delta(S+I)_j &= \beta \sum_{j=0}^{m} \frac{I_j}{N}\end{aligned}$$

   Since we are defining $\mathscr{R}_0 = \frac{\beta}{1-\sigma}$ and since $\sum_{j=1}^{m+1} \Delta(S+I)_j = (S+I)_{m+1} - (S+I)_0$, this simplifies to

$$-\frac{\mathscr{R}_0}{N}\left((S+I)_{m+1} - (S+I)_0\right) = \beta \sum_{j=0}^{m} \frac{I_j}{N}$$

   By assumption, $S_0 + I_0 = N$, so this can be reduced further to

$$\mathscr{R}_0 \left(1 - \frac{(S+I)_{m+1}}{N}\right) = \beta \sum_{j=0}^{m} \frac{I_j}{N}$$

   Finally, we can take the limit as $m \to \infty$ and use the fact that $(S+I)_\infty = S_\infty$ to get

$$\mathscr{R}_0 \left(1 - \frac{S_\infty}{N}\right) = \beta \sum_{j=0}^{m} \frac{I_j}{N}$$

Substituting this into the final equality from part (iii) gives

$$\log \frac{S_0}{S_\infty} = \mathscr{R}_0 \left(1 - \frac{S_\infty}{N}\right)$$

as desired.

2. (i) The function $I(t) = e^{-\gamma t}$ satisfies the ODE since $I'(t) = -\gamma e^{-\gamma t} = -\gamma I(t)$.

   (ii) Using the tail sum formula, we can compute the expectation of T_R as

$$\begin{aligned}\mathbb{E}[T_R] &= \int_0^\infty \mathbb{P}(T_R > t)dt \\ &= \int_0^\infty I(t)dt \\ &= \int_0^\infty e^{-\gamma t}dt \\ &= \left[-\frac{1}{\gamma}e^{-\gamma t}\right]_0^\infty \\ &= \frac{1}{\gamma}\end{aligned}$$

3. (i) Since each $I_k(t)$ can be interpreted as the probability that someone is in stage k and has not yet moved to the next stage at time t, the set $\{I_1, \dots, I_n\}$ represent a set of disjoint probabilities. Thus the probability of their union of all these events (which would be someone being in any stage of the disease at time t) is just their sum.

$$\mathbb{P}(T_R > t) = \sum_{k=1}^n I_k(t)$$

   (ii) The density $\rho(t)$ of T_R is then

$$\begin{aligned}\rho_(t) &= \frac{d}{dt}\mathbb{P}(T_R \le t) \\ &= -\frac{d}{dt}\mathbb{P}(T_R > t) \\ &= -\sum_{k=1}^n I_k(t) \\ &= r\left[I_1 - I_1 + I_2 - I_2 + \cdots + I_{n-1} - I_{n-1} + I_n\right] \\ &= rI_n(t)\end{aligned}$$

   (iii) We claim that the solution of this system is of the form

$$I_k(t) = \frac{r^{k-1}}{(k-1)!}t^{k-1}e^{-rt}$$

which we can prove by induction. The first equation of the system is satisfied by this solution since

$$\begin{aligned}I_1(t) &= e^{-rt} \\ I_1'(t) &= -re^{-rt} \\ &= -rI_1(t)\end{aligned}$$

Assuming this holds for stage $k-1$, we can show that it also holds for stage k.

$$\begin{aligned}
I_k'(t) &= \frac{d}{dt}\left(\frac{r^{k-1}}{(k-1)!}t^{k-1}e^{-rt}\right) \\
&= \frac{r^{k-1}}{(k-2)!}t^{k-2}e^{-rt} - \frac{r^k}{(k-1)!}t^{k-1}e^{-rt} \\
&= r\left(\frac{r^{k-2}}{(k-2)!}t^{k-2}e^{-rt}\right) - r\left(\frac{r^{k-1}}{(k-1)!}t^{k-1}e^{-rt}\right) \\
&= rI_{k-1}(t) - rI_k(t)
\end{aligned}$$

where the last step follows from the inductive hypothesis and our proposed definition of I_k. This proves the claim.

(iv) Based on the given definition of the gamma distribution, it is clear that in our model, $\alpha = n$ and $\beta = r$. Thus $\mathbb{E}[T_R] = \frac{n}{r}$.

This makes intuitive sense, as larger n means more stages (and thus more time) are required to recover from the disease, and a higher transition rate r means that the stages are passed through more quickly (which decreases the expected recovery time).

(v) If $r = \gamma a$, then $\mathbb{E}[T_R] = \frac{1}{\gamma}$. The mean is then the same as with one stage; however, the mode of the distribution shifts to the right as n increases. This seems more realistic, as it will take some time for almost all people to recover from any given disease.