Exercises completed: None.

Exercise 1. §24 #8 a,b,d.

Collaborators: None.

a. Yes (if we're using the product topology). Suppose X_α is path connected for all α in some indexing set, and let $\mathbf{x}, \mathbf{y} \in \prod X_\alpha$. Since each X_α is path connected, we can find a continuous path γ_α from $\mathbf{x}_\alpha$ to $\mathbf{y}_\alpha$ for all α. Define a new function $\gamma = (\gamma_\alpha)_\alpha$, then by Munkres Theorem 19.6, γ is continuous since each γ_α is continuous. Since γ is then a continuous path from $\mathbf{x}$ to $\mathbf{y}$, the space $\prod X_\alpha$ is path connected.

b. No. If we let

$$S \doteq \{(x, \sin(1/x) \mid 0 < x \leq 1\},$$

then $\overline{S}$ is the topologist's sine curve. S is the graph of a continuous function over a path connected domain, so it is itself path connected; however, $\overline{S}$ is known to not be path connected.

d. Yes. Suppose $x, y \in \bigcup A_\alpha$, then $x \in A_{\alpha_x}$ and $y \in A_{\alpha_y}$ for some α_x, α_y. Since the intersection of all A_α is nonempty, we know that A_{α_x} and A_{α_y} have at least one common point. Let z be a point in their intersection, then we can find paths γ_1 from x to z and γ_2 from z to y. By the pasting lemma, we can use γ_1 and γ_2 to construct a single continuous path from x to y. Thus $\bigcup A_\alpha$ is path connected.

Exercise 2. §25 #1.

Collaborators: None.'

The connected components of $\mathbb{R}_l$ are its individual points. Suppose $x < y$ are in the same connected component C of $\mathbb{R}_l$, then they can be separated by the disjoint open sets $C \cap (-\infty, y)$ and $C \cap [y, \infty)$. Thus no connected component of $\mathbb{R}_l$ can have more than 1 distinct point.

By Theorem 25.5, every path component lies in a component of $\mathbb{R}_l$. Thus every path component is also just a single point in $\mathbb{R}_l$.

Suppose $f : \mathbb{R} \to \mathbb{R}_l$ is continuous. Since $\mathbb{R}$ is connected and continuous maps preserve connectedness, f must map $\mathbb{R}$ to a connected subset of $\mathbb{R}_l$. But the connected subsets of $\mathbb{R}_l$ are just single points, so f must be constant. Thus the continuous maps from $\mathbb{R}$ to $\mathbb{R}_l$ are the constant maps.

Exercise 3. §26 #5.

Collaborators: None.

Fix $b \in B$. Since X is Hausdorff, for all $a \in A$ we can find disjoint neighborhoods $U_{a,b}, V_{a,b}$ of a, b, respectively. Since $\{U_{a,b}\}_{a\in A}$ covers A and A is compact, there is a finite subcover $\{U_{a_i,b}\}_{i=1}^N$. Then $U_b \doteq \bigcup_{i=1}^N U_{a_i,b}$ contains A and does not intersect $V_b \doteq \bigcap_{i=1}^N V_{a_i,b}$, which is a neighborhood of b since the intersection is finite.

Now $\{V_b\}_{b\in B}$ is an open cover of B and B is compact, we can find a finite subcover $\left\{V_{b_j}\right\}_{j=1}^M$. Since U_b doesn't intersect V_b for all b, we can define two disjoint open sets

$$U \doteq \bigcap_{j=1}^{M} U_{b_j}, \quad V \doteq \bigcup_{j=1}^{M} V_{b_j}.$$

Since each U_{b_j} contains A, so does their intersection U, and V is a cover of B by definition. Thus we have found disjoint open sets containing A and B.

Exercise 4. §27 #2.

Collaborators: None.

a. **Forward:** Suppose $d(x, A) = 0$. If $x \notin \overline{A}$, then there is a neighborhood U of x such that U does not intersect A. Now there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$, which implies $d(x, A) \geq \varepsilon$, but this contradicts the assumption that $d(x, A) = 0$, so $x \in \overline{A}$.

   Backward: Supose $x \in \overline{A}$. If $x \in A$, then $d(x, A)$ is clearly 0, so assume x is a limit point of A but not in A. Since it's a limit point, for all $\varepsilon > 0$, $B(x, \varepsilon)$ intersects A. Since ε was arbitrary, the only possibility for $d(x, A)$ is 0.

b. Suppose A is compact. Fix $x \in X$, then we will show that $f(a) \doteq d(x, a)$ is continuous, from which the desired result will follow. Fix $\varepsilon > 0$ and set $\delta = \varepsilon$. If $d(a_1, a_2) < \delta = \varepsilon$, then by the triangle inequality,

$$|f(a_1) - f(a_2)| = |d(a_1, x) - d(a_2, x)| \leq d(a_1, a_2) < \delta = \varepsilon.$$

   Thus f is continuous. Then by the extreme value theorem, f attains its infimum on A. This means there is some $a \in A$ such that $f(a) = d(x, a) = d(x, A)$.

c. Fix $\varepsilon > 0$. Let $x \in \bigcup_{a\in A} B(a, \varepsilon)$, then $x \in B(\tilde{a}, \varepsilon)$ for some $\tilde{a} \in A$. Then $d(x, \tilde{a}) < \varepsilon$, so $d(x, A) < \varepsilon$, so $x \in U(A, \varepsilon)$. Thus $\bigcup_{a\in A} B(a, \varepsilon) \subset U(a, \varepsilon)$.

   Conversely, let $x \in U(a, \varepsilon)$, then $d(x, A) < \varepsilon$. Then there is some $\tilde{a} \in A$ such that $d(x, \tilde{a}) < \varepsilon$, so $x \in B(\tilde{a}, \varepsilon)$. Thus $U(A, \varepsilon) \subset \bigcup_{a\in A} B(a, \varepsilon)$.

d. Suppose A is compact and is contained in an open set U. Then for all $a \in A$, there is some $\varepsilon_a > 0$ such that $B_a \doteq B(a, \varepsilon_a/2) \subset U$. Since $\{B_a\}_{a \in A}$ is an open cover of A and A is compact, we can find a finite subcover $\{B_{a_i}\}_{i=1}^N$ of A (which still lies entirely in U).

Let $\varepsilon \doteq \min_i \varepsilon_{a_i}$, then we claim that $U(A, \varepsilon/2)$ is contained U. Let $x \in U(A, \varepsilon/2)$, then by part (c), x is in some ball $B(\tilde{a}, \varepsilon/2)$. Since $\tilde{a} \in A$, we can use our finite open cover of A to find some ball $B(a_i, \varepsilon_{a_i}/2)$ that contains $\tilde{a}$. Then by the triangle inequality,

$$d(x, a_i) \leq d(x, \tilde{a}) + d(\tilde{a}, a_i) < \frac{\varepsilon}{2} + \frac{\varepsilon_{a_i}}{2} \leq \varepsilon_{a_i}.$$

Thus $x \in B(a, \varepsilon_{a_i}) \subset U$, so $U(A, \varepsilon/2)$ is an ε-neighborhood of A that is contained in U.

e. Let $A = \mathbb{Z} \subset \mathbb{R}$, which is closed since its complement $\mathbb{R} - \mathbb{Z} = \bigcup_{z \in \mathbb{Z}} (n, n+1)$ is open. Since it's an unbounded set in $\mathbb{R}$, it cannot be compact. Now consider the open set

$$U = \bigcup_{n \in \mathbb{Z}} \left(n - \frac{1}{|n|}, n + \frac{1}{|n|} \right).$$

Fix $\varepsilon > 0$, then since we can always find $n \in \mathbb{Z}$ such that $1/n < \varepsilon$, the ε-neighborhood $U(A, \varepsilon)$ can never be fully contained in U. Since ε was arbitrary, no $U(A, \varepsilon)$ can be fully contained in U. Thus (d) does not necessarily hold if A is not compact.

Exercise 5. §28 #7 b.

Collaborators: None.

I couldn't fill in all the gaps in the proof outlined in the book's hint, so I did something else entirely that's similar to what I did in Exercise 4(b). We claim that the function

$$\begin{aligned} g : X &\to \mathbb{R} \\ x &\mapsto d(x, f(x)) \end{aligned}$$

is continuous. To show this, note that by the triangle inequality and the fact that f is a shrinking map,

$$\begin{aligned} d(x, f(x)) &\leq d(x, y) + d(y, f(y)) + d(f(y), f(x)) \\ g(x) - g(y) &\leq d(x, y) + d(f(x), f(y)) \\ g(x) - g(y) &< 2d(x, y). \end{aligned}$$

Swapping x and y and using the symmetry of d gives the same inequality, so

$$|g(x) - g(y)| < 2d(x, y).$$

Now fix $\varepsilon > 0$ and let $\delta = \varepsilon/2$. When $d(x, y) < \delta$, the inequality we just derived gives

$$|g(x) - g(y)| < 2d(x, y) < \varepsilon,$$

so g is continuous. Since X is compact, the extreme value theorem says that g attains its infimum I on X, say at a point x. Suppose $I > 0$, then since f is a shrinking map,

$$g(f(x)) = d(f(x), f^2(x)) < d(x, f(x)) = g(x) = I.$$

This contradicts the fact that I is an infimum, so I must be 0, i.e. x is a fixed point of f. To show that x is unique, suppose $y \neq x$ is also a fixed point of f. Then

$$d(x, y) = d(f(x), f(y)) < d(x, y),$$

which is a contradiction, so $x = y$. Thus we've found a unique fixed point of f.