Exercise 1. Prove a graph is connected according to Definition 1 (every pair has a path) if and only if it is connected according to Definition 2 (there is no separation).

1 implies 2: Suppose every pair of points in G has a path between them. Now suppose U, W separate G. Then for any $u \in U$, $w \in W$, there is a path connecting them. But this path must cross from U to W at some point, so there is an edge starting in U and ending in V. But this contradicts the definition of a separation, so no separation of G exists.

2 implies 1: Suppose there is no separation of G, and fix $x, y \in G$. Now suppose there is no path from x to y, then x and y are in different (necessarily nonempty) connected components X and Y, respectively. Suppose Z is the union of all other connected components, then $(Z \cup X)$ and Y separate G. This contradicts our original assumption, so there must be a path from x to y.

Exercise 2. Let V be a finite dimensional vector space with subspace. Let N be a subspace with basis $\mathcal{A} = \{a_1, \ldots, a_n\}$, and let $\mathcal{B} = \{[b_1], \ldots, [b_m]\}$ be a basis for V/N. Prove $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ is a basis for V.

We must show that this basis spans V and is linearly independent.

Spans: Fix $v \in V$. Since $[v] \in V/N$ and V/N has basis $\mathcal{B}$, we know

$$[v] = \sum_{j=1}^{m} \lambda_j [b_j] = \left[\sum_{j=1}^{m} \lambda_j b_j\right]$$

for some collection of scalars $\{\lambda_j\}$. In particular, this means $v - \sum_j \lambda_j b_j \in N$. Then since N has basis $\mathcal{A}$, this means

$$v - \sum_{j=1}^{m} \lambda_j b_j = \sum_{i=1}^{n} \mu_i a_i$$

for some collection of scalars $\{\mu_i\}$. Then $v = \sum_i \mu_i a_i + \sum_j \lambda_j b_j$, so the proposed basis spans V.

Linearly Independent: Suppose $\sum_i \mu_i a_i + \sum_j \lambda_j b_j = 0$, then we want to show that each μ_i and λ_j is 0. To start, note that since $\mathcal{B}$ is a basis (and is thus linearly independent),

$$\sum_j \lambda_j [b_j] = \left[\sum_j \lambda_j b_j\right] = [0] = N \quad \Longrightarrow \quad \lambda_i = 0 \text{ for all } i.$$

In particular, this means that if $\sum_j \lambda_j b_j \in N$, then each λ_j is 0. But by our original assumption, $\sum_j \lambda_j b_j = -\sum_i \mu_i a_i \in N$, so $\lambda_j = 0$ for all j. This leaves us with $\sum_i \mu_i a_i = 0$. Then since $\mathcal{A}$ is a basis, this implies that each $\mu_i = 0$ as well. Thus the proposed basis is also linearly independent.