1 INTRODUCTION

Bounded size rules are a very general category of rule that eventually start acting like the Erdős-Rényi rule in the following sense: fix a constant $K \in \mathbb{N}$, then $\mathcal{R}$ is a bouded size rule if it treats all sampled points with $\kappa_i > K$ identically.

This stuff has already been proven by Riordan and Warnke, albeit in a very complicated manner.

Intuitively, it makes sense that a bounded size rule eventually "becomes" Erdős-Rényi. Eventually the number of clusters of size $\leq K$ will be so small that it won't have much of an effect on the growth of the giant component. The key to noticing the complexity here is actually twofold:

1. for any $k \leq K$, the number of clusters of size k will be *nonzero* at percolation; (**can I prove this for more than just BF?**)

2. after all clusters of size $\leq K$ have stopped having a real effect on the giant component, we're in a position much different than that of Erdős-Rényi. Instead of lots of isolated nodes, we have lots of finite size clusters.

We can use a simple bounded size rule to examine these problems, which we do using a variant of the well known Bohman-Frieze rule.

2 BOHMAN-FRIEZE

The original Bohman-Frieze rule is as follows:

1. At each step, pick two edges $e_i = \{u_i, v_i\}$.

2. If both u_1 and v_1 are isolated, pick e_1.

3. Otherwise, pick e_2.

Although well understood, this rule does not fit nicely into our frameworks. We can instead work with a slight variant (and also generalization) that makes this into a bona fide bounded size 2-choice rule:

1. At each step, pick m vertices. These are the group 1 vertices.

2. If any of the m vertices are isolated, pick the first such one as the group 1 representative. If not, sample one more random vertex and pick it no matter what.

3. Repeat this for group 2, and connect the 2 group representatives with an edge.

We can explicitly write out the probability $\phi(s)$ that a group representative cluster size is s:

$$\phi(s) = \begin{cases} 1 - (1-P_1)^{m+1} & s = 1, \\ (1-P_1)^m P_s & s > 1. \end{cases}$$

All our 2-choice analysis was based on the quantities $\langle 1 \rangle_\phi$ and $\langle s \rangle_\phi$, so we should compute those.

$$\begin{aligned} \langle 1 \rangle_\phi &= \phi(1) + \sum_{s>1} \phi(s) \\ &= 1 - (1-P_1)^{m+1} + \sum_{s>1} (1-P_1)^m P_s \\ &= 1 - (1-P_1)^m \left[1 - P_1 - \sum_{s>1} P_s \right] \\ &= 1 - (1-P_1)^m (1 - \langle 1 \rangle_P) \\ &= 1 - (1-P_1)^m S. \end{aligned}$$

Since $t \mapsto (1-P_1)^m$ is continuous at t_c, this has the same critical exponents as $\langle 1 \rangle_P = 1 - S$. Similarly,

$$\begin{aligned} \langle s \rangle_\phi &= \phi_1(1) + \sum_{s>1} s\phi(s) \\ &= 1 - (1-P_1)^{m+1} + \sum_{s>1} s(1-P_1)^m P_s \\ &= 1 - (1-P_1)^m \left[1 - P_1 - \sum_{s>1} sP_s \right] \\ &= 1 - (1-P_1)^m \left[1 - \sum_{s} sP_s \right] \\ &= 1 - (1-P_1)^m (1 - \langle s \rangle_P). \end{aligned}$$

For the same reason, this also has the same critical exponents as $\langle s \rangle_P$. The upshot of these two calculations is that our analysis of 2-choice rules relied only on the critical values of $\langle 1 \rangle_\phi$ and $\langle s \rangle_\phi$, so this Bohman-Frieze variant has all the same critical exponents as Erdős-Rényi.

For this particular rule, we can also explicitly track how the number of isolated clusters is changing through time, which will highlight problem 1 from the introduction. Let $m = 1$, then the Smoluchowski equation gives

$$\begin{aligned} \partial_t P_1 &= -2\phi(1) \\ &= -2 + 2(1-P_1)^{m+1} \\ &= -2 + 2(1-P_1)^2 \\ &= 2P_1^2 - 4P_1. \end{aligned}$$

Solving this differential equation and using the initial condition $P_1(0) = 1$ gives us the solution

$$P_1(t) = \frac{2}{e^{4t} + 1}.$$

We know that percolation occurs at $t = 1$ at the very latest. But plugging in $t = 1$ gives $P_1(1) = \frac{2}{e^4+1} \approx 0.024$, so at criticality, a strictly positive proportion of our nodes are still isolated.

3 GREEDY BOUNDED SIZE RULES

We can generalize our Bohman-Frieze variant, and consequently get closer to the case of a general bounded size rule. Let's still sample m points for each group, but now we'll take the first one whose cluster size is $\leq K$. If each of the m cluster sizes are greater than K, we'll sample a new random vertex and choose it no matter what.

Calculating $\phi(s)$ for $s \leq K$ would be pretty messy, but we definitely know

$$\sum_{s \leq K} \phi(s) = 1 - (1 - P_{\leq K})^{m+1}$$

and

$$\phi(s) = (1 - P_{\leq K})^m P_s \qquad \text{when } s > K.$$

This is enough to mimic the earlier calculation of $\langle 1 \rangle_\phi$ for the Bohman-Frieze rule:

$$\begin{aligned}
\langle 1 \rangle_\phi &= \sum_{s \leq K} \phi(s) + \sum_{s > K} \phi(s) \\
&= 1 - (1 - P_{\leq K})^{m+1} + \sum_{s > K} (1 - P_{\leq K})^m P_s \\
&= 1 - (1 - P_{\leq K})^m \left[1 - P_{\leq K} - \sum_{s > K} P_s \right] \\
&= 1 - (1 - P_{\leq K})^m \left[1 - \sum_{s} P_s \right] \\
&= 1 - (1 - P_{\leq K})^m S.
\end{aligned}$$

And since $t \mapsto (1 - P_{\leq K})^m$ is continuous at t_c, this has the same critical exponents as $\langle 1 \rangle_P = 1 - S$. Where I'm currently stuck is calculating $\langle s \rangle_\phi$. It seems like I'll need to come up with an expression for $\phi(s)$ when $s \leq K$, or at least for $\sum_{s \leq K} s\phi(s)$.