CONTENTS

1 DEFINITIONS

To begin, we'll need to define some basic functions that we'll use over and over again. If x_i is a vertex, then we denote its cluster size by κ_i. Denote the probability that the minimum of m i.i.d. sampled vertices is s by

$$Q_m(s) \doteq \mathbb{P}\left(\min\{\kappa_1, \ldots, \kappa_m\} = s\right).$$

Note that Q_m satisfies the identity $\sum_{s=1}^{\infty} Q_m(s) = 1 - S^m$. Since they frequently show up in common examples, we give $m = 1$ and $m = 2$ shorthands:

$$P \doteq Q_1, \qquad Q \doteq Q_2.$$

We also define

$$\langle s^k \rangle_m \doteq \sum_{s=1}^{\infty} s^k Q_m(s).$$

I'll use $\langle \cdot \rangle_P$ and $\langle \cdot \rangle_Q$ instead of $\langle \cdot \rangle_1$ and $\langle \cdot \rangle_2$, respectively.

Now for the main attraction. In these notes, we'll be discussing rules that add a single edge every $t = 1/n$ units of time, gotten by selecting two vertices total from two separate groups of vertices that are sampled i.i.d. from the graph.

Definition 1. Define a rule $\mathcal{R}$ as follows:

- Every $t = 1/n$ units of time, choose two groups of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ by sampling vertices i.i.d. from the graph.
- For both i, follow some rule $\mathcal{F}_i$ to choose a vertex x_i with cluster size κ_w from group $\mathcal{V}_i$, subject to the condition that $\mathcal{F}_i$ induces a function $\phi_i(s) = \mathbb{P}(\kappa_i = s)$.

We call $\mathcal{R}$ a **2-choice rule**.

We'd like to restrict this vertex selection processes in each group as little as possible in order to get a more general theory, but in some cases we can perform much greater analysis if some information is known about them.

Definition 2. A 2-choice rule $\mathcal{R}$ is **minimizing** if $\phi_1 = Q_a$ and $\phi_2 = Q_b$ for some a, b.

Such a rule exhibits "explosive" behavior in the sense that the critical time is significantly delayed and the giant component emerges incredibly quickly. Under the assumption that P exhibits scaling behavior, minimizing 2-choice rules can be analyzed in a straightforward manner.

2 THE SCALING ASSUMPTION

Most of the results in these notes follow from the assumption that near the critical time t_c, P has the form

$$P(s) = s^{1-\tau} f(s\delta^{1/\sigma})$$

for constants τ, σ and scaling function f. **Motivation for this.** The following theorem gives relations between these constants if some regularity conditions hold for the scaling function f.

Theorem 1. Suppose a rule $\mathcal{R}$ has a scaling function f such that

1. $\lim_{x\to\infty} x^{2-\tau} f(x) = 0$; and
2. $\int_0^\infty x^{2-\tau} f'(x)\, dx$ is finite.

Then there are **critical exponents**

$$\begin{aligned} \beta &= (\tau - 2)/\sigma, \\ \gamma_m &= (m(2-\tau)+1)/\sigma. \end{aligned}$$

such that $S \sim \delta^\beta$ and $\langle s^k \rangle_m \sim \delta^{-\gamma_m - (k-1)/\sigma}$.

Proof. We'll begin by deriving β. Since

$$S \approx \int_0^\infty s^{1-\tau}(f(0) - f(s\delta^{1/\sigma}))\, ds,$$

we can make the change of variable $s = x\delta^{-1/\sigma}$ to get

$$= \delta^{(\tau-2)/\sigma} \int_0^\infty x^{1-\tau}(f(0) - f(x))\, dx.$$

Integrating by parts gives

$$= \frac{\delta^{(\tau-2)/\sigma}}{\tau - 2}\left[\left[-x^{2-\tau}(f(0) - f(x))\right]_{x=0}^{x=\infty} - \int_0^\infty x^{2-\tau} f'(x)\, dx\right].$$

So by our assumptions on f, we have $S \sim \delta^\beta$, where $\beta = (\tau - 2)/\sigma$. **Type up the rest of this. Go over your concerns with the assumptions on f and the relations between f and g in Appendex E of da Costa.** □

3 UNIFORM SCALING

It would be nice to express all these critical exponents in terms of just one (in our case, we'll express everything in terms of β). This has two main applications:

1. if we determine a single critical exponent, then we automatically know all others; and
2. **we can determine the limiting behavior of the critical exponents** as $m \to \infty$ (which drives $\beta \to 0$).

The following property will be critical in establishing systems of equations that we can use to solve for the critical exponents in terms of β. As we will see later on, it always holds for minimizing 2-choice rules, and a partial version of it holds for general 2-choice rules.

Definition 3. We say that a rule $\mathcal{R}$ that exhibits scaling behavior **scales uniformly** if for S and all $\langle s \rangle_m$, every δ term comprising it has the same order. **Can I fix this so that it's more formal, i.e. is the result of one thing that's more readily definable?**

Note that since $\langle s^k \rangle_m \sim \delta^{-(\gamma_m + \frac{k-1}{\sigma})}$ differs from $\langle s \rangle_m \sim \delta^{-\gamma_m}$ by only an added constant, this property also applies to all k-th moments. Uniform scaling ends up giving us a systematic way of solving for all crtical exponents in terms of β when we're working with minimizing 2-choice rules.

4 SCALING RELATIONS FOR MINIMIZING RULES

4.1 UNIFORM SCALING

To start, note that $\partial_t P(s)$ for any minimizing 2-choice rule has the form

$$\partial_t P(s) = s \sum_{u+v=s} Q_a(u) Q_b(v) - s Q_a(s) - s Q_b(s).$$

As we've done in earlier notes, we can solve for $\partial_t S$ and $\partial_t \langle s \rangle_P$:

$$\partial_t S = S^b \langle s \rangle_a + S^a \langle s \rangle_b, \tag{$\star$}$$

$$\partial_t \langle s \rangle_P = 2 \langle s \rangle_a \langle s \rangle_b - S^b \langle s^2 \rangle_a - S^a \langle s^2 \rangle_b. \tag{$\star\star$}$$

Using these calculations, we can show that all minimizing 2-choice rules scale uniformly.

Proposition 1. All minimizing 2-choice rules scale uniformly.

Proof. The primary tool in this proof is the following identities that we proved earlier for rules with scaling behavior:

$$\beta = \frac{\tau - 2}{\sigma},$$

$$\gamma_m = \frac{m(2-\tau)+1}{\sigma}.$$

We'll start with $\partial_t S$. The terms on the RHS of $(\star)$ have orders

$$\beta b - \gamma_a, \qquad \beta a - \gamma_b.$$

But plugging in our identities for β and γ_m yields

$$\frac{(\tau-2)(a+b)-1}{\sigma}$$

in both cases. Now we have to check $\partial_t \langle s \rangle_P$. The three terms on the RHS of $(\star\star)$ have orders

$$-\gamma_a - \gamma_b, \qquad \beta b - \gamma_a - \frac{1}{\sigma}, \qquad \beta a - \gamma_b - \frac{1}{\sigma}.$$

As before, plugging in our identities for β and γ_m shows that all three of these are equal to

$$\frac{(\tau-2)(a+b)-2}{\sigma}.$$

Thus all 2-choice rules scale uniformly. $\square$

4.2 SCALING RELATIONS

Now, given any minimizing 2-choice rule $\mathcal{R}$, we can follow the same systematic approach for expressing its critical exponents in terms of just β. Since $S \sim \delta^{\beta}$, $(\star)$ gives

$$\beta - 1 \quad = \quad \beta b - \gamma_a \quad = \quad \beta a - \gamma_b,$$

which gives us the two relations

$$\gamma_a = 1 + (b-1)\beta, \tag{1}$$
$$\gamma_b = 1 + (a-1)\beta. \tag{2}$$

Since $\langle s \rangle_P \sim \delta^{-\gamma_P}$, our expression $(\star\star)$ gives

$$-\gamma_p - 1 \quad = \quad -\gamma_a - \gamma_b \quad = \quad \beta b - \frac{1}{\sigma} - \gamma_a \quad = \quad \beta a - \frac{1}{\sigma} - \gamma_b,$$

which gives us the relations

$$\gamma_P = \gamma_a + \gamma_b - 1, \tag{3}$$
$$\gamma_P = -\beta b + \frac{1}{\sigma} + \gamma_a - 1, \tag{4}$$
$$\gamma_P = -\beta a + \frac{1}{\sigma} + \gamma_b - 1, \tag{5}$$
$$\gamma_b = \frac{1}{\sigma} - \beta b, \tag{6}$$
$$\gamma_a = \frac{1}{\sigma} - \beta a, \tag{7}$$
$$\gamma_a - \gamma_b = \beta(b-a). \tag{8}$$

There's a bit of redundant information in this system, though, so we won't end up using all the relations. I'm including them all so I don't forget any of them, though. By (1), (2), and (3), we get

$$\gamma_P = 1 + (a + b - 2)\beta. \tag{9}$$

By (1) and (7), or also by (2) and (6), we get

$$\frac{1}{\sigma} = 1 + (a + b - 1)\beta. \tag{10}$$

And finally, by (10) and the relation $\beta = (\tau - 2)/\sigma$, we get

$$\tau = \frac{\beta}{1 + (a + b - 1)\beta} + 2. \tag{11}$$

Formulas (1), (2), (9), (10), and (11) are our desired relations. As a sanity check, plugging in the values of a and b for our three understood rules gives the relations we previously derived for those.

4.3 OBSERVATIONS

Suppose that $a = 1$, then $\gamma_b = 1$, no matter what b is. A symmetric statement holds if $b = 1$ instead. This matches what we saw with the adjacent edge rule, and reveals a somewhat surprising (at least to me) relationship. Here are some more scattered thoughts:

- Unless we're using Erdős Rényi, γ_P will always have a dependence on β.
- σ and τ will always depend on β.

So in summary, if neither of our groups has size 1, we can't know *any* of the critical exponents until we've calculated β, which stinks.

5 SCALING RELATIONS FOR GENERAL 2-CHOICE RULES

5.1 UNIFORM SCALING

General 2-choice rules have an ODE of the form

$$\partial_t P(s) = s \sum_{u+v=s} \phi_1(u)\phi_1(v) - s\phi_1(s) - s\phi_2(s).$$

This lets us calculate $\partial_t S$.

Proposition 2. For 2-choice rules,

$$\partial_t S = \langle s \rangle_{\phi_1} (1 - \langle 1 \rangle_{\phi_2}) + \langle s \rangle_{\phi_2} (1 - \langle 1 \rangle_{\phi_1}).$$

Example 1. If our rule is minimizing, then $\phi_1 = Q_a$ and $\phi_2 = Q_b$. Then using the identity $\sum_s Q_m(s) = 1 - S^m$, this reduces to

$$\partial_t S = \langle s\rangle_a S^b + \langle s\rangle_b S^a.$$

Lemma 1. For any ϕ_i, there is an associated function ζ_i such that

$$\langle 1\rangle_{\phi_i} = 1 - \zeta_i(S).$$

Proof. **Note that ζ should have some kind of non-negativity condition or something like that.**

I'd like to find a method for determining ζ_i explicitly. The existence of such a ζ_i is straightforward. Since ϕ_i is a probability measure, we necessarily have $\sum_s \phi_i(s) \leq 1$. **Seems like it has to be a function of S since we're picking things from the space of finite clusters, but idk how to formalize that.** □

Note that $\zeta_i(S)$ induces another function $F_i : \beta \mapsto \alpha_i \beta$ that scales β by the dominating (minimum) order α_i of S in $\zeta_i(S)$.

Example 2. Consider the rule where in each of the 2 groups, we do the following:

1. Pick 3 verties v_1, v_2, v_3 i.i.d.
2. Of v_1 and v_2, choose the one with the smaller cluster size and label it $\tilde{v}$.
3. Choose between $\tilde{v}$ and v_3 randomly to get the vertex for the group.

In terms of ϕ_i, this rule is defined by $\phi_1 = \phi_2 = \frac{1}{2}(P+Q)$. A straightforward computation gives $\langle 1\rangle_{\phi_i} = 1 - \frac{1}{2}(S+S^2)$, so $\zeta_i(S) = \frac{1}{2}(S+S^2)$. The minimum order of S here is 1, so the induced map is $F_i : \beta \mapsto \beta$ for both i.

Theorem 2. If $\mathcal{R}$ is a 2-choice rule, then it has two dominating terms with the same order.

Proof. By Proposition 2 and the preceding lemma,

$$\partial_t S = \langle s\rangle_{\phi_1} \zeta_2(S) + \langle s\rangle_{\phi_2} \zeta_1(S).$$

Suppose $F_i(\beta)$ is induced from $\zeta_i(S)$ **(Check to make sure that the two terms this applies to actually dominate)**. Then there are two dominating terms, and both have order $(\alpha_1+\alpha_2)\beta - 1/\sigma$. □

Since we know $\partial_t S \sim \delta^{\beta-1}$, this immediately implies

$$\frac{1}{\sigma} = (\alpha_1 + \alpha_2 - 1)\beta + 1,$$

so we can always determine σ in terms of β if we know both the α_i.

Corollary 1. If $\mathcal{R}$ is a 2-choice rule such that $\zeta_i(S)$ is a single term for both i, then $\mathcal{R}$ scales uniformly.

Proof. If $\zeta_i(S)$ is a single term for both i, then $\partial_t S$ has 2 total terms, which must necessarily have the same order. □

Example 3 (Uniform scaling). For minimizing rules, $\zeta_1(S) = S^a$, which induces $F_1(\beta) = a\beta$. Similarly, $\zeta_2(S) = S^b$ and $F_2(\beta) = b\beta$. So the terms in $\partial_t S$ have order

$$(a+b)\beta - \frac{1}{\sigma},$$

which we can verify as true.

Example 4 (Partial uniform scaling). Recall from Example 2 the rule that was defined by $\phi_1 = \phi_2 = \frac{1}{2}(P+Q)$, and how the induced scaling map for both i was $F_i : \beta \mapsto \beta$. This means $\alpha_i = 1$ for both i, so

$$\frac{1}{\sigma} = \beta + 1.$$

Through the usual methods (just with a lot more algebra), we can derive

$$\partial_t S = \frac{1}{2}(S^2+2)(\langle s\rangle_P + \langle s\rangle_Q).$$

Expanding this out and substituting in scaling forms gives

$$\delta^{\beta-1} \sim \delta^{2\beta-\gamma_P} + \delta^{2\beta-\gamma_Q} + \delta^{\beta-\gamma_P} + \delta^{\beta-\gamma_Q}.$$

This clearly cannot scale uniformly, since that would imply that $\beta = 0$, i.e. the giant component doesn't grow at all near criticality; however, our theory tells us that since this is a 2-choice rule, the two dominating terms have the same order. Thus we have

$$\beta - 1 \quad = \quad \beta - \gamma_P \quad = \quad \beta - \gamma_Q.$$

This system implies

$$\gamma_P = \gamma_Q = 1.$$

With all this in place, we can give a simpler proof that all minimizing 2-choice rules scale uniformly.

Proposition 3. All minimizing 2-choice rules scale uniformly.

Proof. For both i, $\phi_i = Q_m$ for some m. Then $\langle 1\rangle_{\phi_i} = 1 - S^m$, so $\zeta_i(S) = S^m$. Since this is a single term, the rule must scale uniformly. □

5.2 SCALING RELATIONS

We just saw that

$$\frac{1}{\sigma} = (\alpha_1 + \alpha_2 - 1)\beta + 1, \tag{12}$$

but we can derive other scaling relations for general 2-choice rules, too.

Since $\langle 1 \rangle_{\phi_i} = 1 - \zeta_i(S)$, it will have scaling behavior based on β. **I'd like to somehow show this formally with the integral stuff, but I'm having trouble.** Thus it makes sense to define γ_{ϕ_i} as the constant satisfying

$$\langle s \rangle_{\phi_i} \sim \delta^{-\gamma_{\phi_i}}.$$

Then since $\partial_t S = \langle s \rangle_{\phi_1} \zeta_2(S) + \langle s \rangle_{\phi_2} \zeta_1(S)$, the two dominating terms near criticality give us the system

$$\beta - 1 \quad = \quad -\gamma_{\phi_1} + \alpha_2\beta \quad = \quad -\gamma_{\phi_2} + \alpha_1\beta.$$

This system implies

$$\gamma_{\phi_1} = (\alpha_2 - 1)\beta + 1, \tag{13}$$

$$\gamma_{\phi_2} = (\alpha_1 - 1)\beta + 1. \tag{14}$$

Example 5 (da Costa)**.** If $\phi_1 = \phi_2 = Q_m$, then $\gamma_{\phi_1} = \gamma_{\phi_2} = \gamma_m$. We know that for da Costa $\alpha = m$, so $\gamma_m = (m-1)\beta + 1$.

One last constant that we care about is γ_P, which tells us how the average finite cluster size changes. In order to determine it, we need to differentiate $\langle s \rangle_P$.

Proposition 4. For 2-choice rules,

$$\partial_t \langle s \rangle_P = 2\langle s \rangle_{\phi_1} \langle s \rangle_{\phi_2} - \langle s^2 \rangle_{\phi_1} \zeta_2(S) - \langle s^2 \rangle_{\phi_2} \zeta_1(S).$$

Check that the three dominating terms scale uniformly... although they definitely should.

This gives us the system

$$-\gamma_P - 1 \quad = \quad -\gamma_{\phi_1} - \frac{1}{\sigma} + \alpha_2\beta \quad = \quad -\gamma_{\phi_2} - \frac{1}{\sigma} + \alpha_1\beta \quad = \quad -\gamma_{\phi} - \gamma_{\phi_2}.$$

Using (12), this system gives us

$$\gamma_P = (\alpha_1 + \alpha_2 - 2)\beta + 1. \tag{15}$$

Based on (12), we see

$$\gamma_P = \frac{1}{\sigma} - \beta,$$

which coincidentally agrees with (6) and (7) (with $a = b = 1$).

Is it possible to get similar statements for all γ_m?

Example 6 (da Costa)**.** Since $\alpha_i = m$, we have $\gamma_P = 2(m-1)\beta + 1$.