Exercise 1 (Lesson 10, 5 points). Let $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$ be a simplex-wise filtration of K and σ_i a negative $(p+1)$-simplex. Prove $\beta_p(K^i) = \beta_p(K^{i-1}) - 1$.

Since

$$\beta_p(K^i) = \dim H_p(K^i) = \dim Z_p(K^i) - \dim B_p(K^i),$$
$$\beta_p(K^{i-1}) = \dim H_p(K^{i-1}) = \dim Z_p(K^{i-1}) - \dim B_p(K^{i-1}),$$

we can show the desired relationship by calculating the dimensions of $Z_p(K^i), Z_p(K^{i-1}), B_p(K^i)$, and $B_p(K^{i-1})$.

Cycles: Since $\sigma_i \in C_{p+1}$, we know $C_p(K^{i-1}) = C_p(K^i)$. This implies $Z_p(K^i) = Z_p(K^{i-1})$, so their dimensions are the same.

Boundaries: Since σ_i is negative, there is no cycle in $Z_{p+1}(K^i)$ containing it. This implies that $\partial\sigma_i$ cannot be in $B_p(K^{i-1})$: if it were, there would be some $(p+1)$-chain $c \in C_{p+1}(K^{i-1})$ such that $\partial c = \partial\sigma_i$; then by linearity, $\partial(c + \sigma_i) = 0$, so $c + \sigma_i$ would be a cycle in K^{i+1} containing σ_i, contradicting σ_i being negative.

Now $\partial\sigma_i$ is clearly in $B_p(K^i)$, so we know that $B_p(K^i)$ must have a higher dimension than $B_p(K^{i-1})$. We must show that the dimension only increases by 1.

Suppose $\{\partial\tau_j\}_j$ is a basis for $B_p(K^i)$. Since we add 1 simplex every timestep, at *most* 1 of these basis elements can be missing from $B_p(K^{i-1})$. But we just argued that at *least* one of them must be missing (otherwise $\partial\sigma_i$ would be an element of $B_p(K^{i-1})$). Thus only 1 of them is added at time i, i.e. $\dim B_p(K^i) = \dim B_p(K^{i-1}) + 1$.

Conclusion: Putting this all together, we get

$$\begin{aligned}\beta_p(K^i) &= \dim Z_p(K^i) - \dim B_p(K^i) \\ &= \dim Z_p(K^{i-1}) - \dim B_p(K^i) - 1 \\ &= \beta_p(K^{i-1}) - 1.\end{aligned}$$

Exercise 2 (Lesson 10, 5 points)**.** Let $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$ be a simplex-wise filtration of K and σ_i a positive p-simplex. In class, we proved there exists $c_i \in Z_p(K^i)$ such that $\sigma_i \in c_i$, $c_i \notin B_p(K^i)$, and c_i does not contain any other positive simplices besides σ_i. Prove c_i is the unique cycle satisfying these properties.

Suppose σ_i is a positive p-simplex, and suppose there exist $c, c' \in Z_p(K^i)$ both containing σ_i and not containing any other positive simplices. Consider $c + c'$ and note that it does not contain σ_i. By linearity,

$$\partial(c + c') = \partial c + \partial c' = 0,$$

so it is a cycle. Take the simplex in $c + c'$ added at the latest time, then $c + c'$ is the cycle showing that this simplex is positive. But this is a contradiction, as σ_i was the *only* positive simplex in both c and c'. Thus by contradiction, c is unique.