0.1 PDEs

At every step we choose some finite collection of vertices $\{v_i\}_{i=1}^m$. Let κ_i denote the size of the cluster to which v_i belongs. We'll use the following quantities a lot (all probabilities are implicitly taken at time t):

$$\begin{aligned}
X_m(k,t) &\doteq \mathbb{P}\left(\min\{\kappa_1,\ldots,\kappa_m\} = k\right); \\
\hat{X}_m(k,t) &\doteq \mathbb{P}\left(\min\{\kappa_1,\ldots,\kappa_m\} \geq k\right) \\
&= 1 - \sum_{j=1}^{k-1} X_m(j,t); \\
R(k,t) &= \mathbb{P}\left(\kappa_1 + \kappa_2 = k\right); \\
\hat{R}(k,t) &= \mathbb{P}\left(\kappa_1 + \kappa_2 \geq k\right).
\end{aligned}$$

A common case for X_m is $m = 1$ or 2, so we can abbreviate those as

$$P \doteq X_1, \qquad Q \doteq X_2.$$

Note that we can express X_m as

$$X_m(k,t) = \hat{P}(k-1,t)^m - \hat{P}(k,t)^m,$$

(go over why) so every X_m is a function of P. As a final note, I will frequently suppress t from now on.

We're interested in how P changes throughout the percolation process. The following table gives the value of $\partial_t P$, written in terms of the proper X_m, for each of our rules.

Rule	$\partial_t P(s,t)$
Erdős Rényi	$\frac{s}{2}\sum_{u+v=s} P(u,t)P(v,t) - sP(s,t)$
Adjacent Edge	$s\sum_{u+v=s} P(u,t)Q(v,t) - sP(s,t) - sQ(s,t)$
DaCosta	$s\sum_{u+v=s} X_m(u,t)X_m(v,t) - 2sX_m(s,t)$
Sum	**Do this.**
Product	**Do this.**

0.2 Consequences

Let $S(t)$ denote the relative size (i.e. divided by n) of the percolation cluster at time t, and let $X_m(k,t) \doteq \mathbb{P}\left(\min\{\kappa_1,\ldots,\kappa_m\} = k\right)$.

Proposition 1.

$$\sum_k X_m(k,t) = 1 - S^m(t).$$

To justify this, we can interpret $\sum_k X_m(k,t)$ as the probability that, at time t, the minimum cluster size of m vertex choices is finite (this is in the limit as $n \to \infty$). S is then the probability that a single choice is from an "infinite" cluster size. **I kinda want to do this more rigorously, but that's not too important right now...**

Differentiating this identity for $X_1 = P$ gives

$$\partial_t S = -\sum_s \partial_t P,$$

so we can track the size of the percolation cluster by knowing $P(s)$ for all s. In the following computations, we'll express $\partial_t S$ in terms of the moments of various X_m, which we denote by

$$\langle s^k \rangle_{X_m} \doteq \sum_s s^k X_m(s).$$

Sometimes I might denote $\langle \cdot \rangle_{X_m}$ by $\langle \cdot \rangle_m$. The below table gives $\partial_t S$ for each of our rules. A derivation of this quantity is given afterwards for the Erdős Rényi rule; the other quantities are derived similarly. **We don't really do anything with this information, so it could be fun to figure out what it tells us.**

Rule	$\partial_t S$
ER	$S\langle s \rangle_P$
AE	$\langle s \rangle_P S^2 + S\langle s \rangle_Q$
DC	$2S^m \langle s \rangle_{X_m}$
Sum	**Do this.**
Product	**Do this.**

Proposition 2. For the Erdős Rényi rule, $\partial_t S = S\langle s \rangle_P$.

Proof. In the below computation, I suppress the time t for clarity.

$$\begin{aligned}
\partial_t S &= -\sum_s \partial_t P \\
&= -\frac{1}{2}\sum_s s \sum_{u+v=s} P(u)P(v) + \sum_s sP(s) \\
&= -\frac{1}{2}\sum_u \sum_v (u+v)P(u)P(v) + \langle s \rangle_P \\
&= -\frac{1}{2}\left[\sum_u uP(u)\sum_v P(v) + \sum_u P(u)\sum_v vP(v)\right] + \langle s \rangle_P \\
&= -\frac{1}{2}\left[2\langle s \rangle_P (1-S)\right] + \langle s \rangle_P \\
&= -\langle s \rangle_P (1-S) + \langle s \rangle_P \\
&= S\langle s \rangle_P.
\end{aligned}$$

□

We're similarly able to calculate $\partial_t \langle s \rangle_P$ for these rules, as summarized in the below table. As before, I incluce the derivation for the Erdős Rényi rule afterwards, and the other derivations are similar.

Rule	$\partial_t \langle s \rangle_P$
ER	$\langle s\rangle_P^2 - \langle s^2\rangle_P S$
AE	$2\langle s\rangle_P\langle s\rangle_Q - \langle s^2\rangle_P S^2 - \langle s^2\rangle_Q S$
DC	$2\langle s\rangle_{X_m}^2 - 2\langle s^2\rangle_{X_m} S^m$
Sum	**Do this.**
Product	**Do this.**

Proposition 3. For the Erdős Rényi rule, $\partial_t \langle s\rangle_P = \langle s\rangle_P^2 - \langle s^2\rangle_P S$.

Proof. Once again, I suppress the time t for clarity.

$$\begin{aligned}
\partial_t \langle s\rangle_P &= \sum_s s\,\partial_t P(s) \\
&= \frac{1}{2}\sum_s s^2 \sum_{u+v=s} P(u)P(v) - \sum_s sP(s) \\
&= \frac{1}{2}\sum_u\sum_v (u+v)^2 P(u)P(v) - \langle s^2\rangle_P \\
&= \frac{1}{2}\left[\sum_u u^2P(u)\sum_v P(v) + 2\sum_u uP(u)\sum_v vP(v) + \sum_u P(u)\sum_v v^2P(v)\right] - \langle s^2\rangle_P \\
&= \frac{1}{2}\left[2\langle s^2\rangle_P(1-S) + 2\langle s\rangle_P^2\right] - \langle s^2\rangle_P \\
&= \langle s\rangle_P^2 - \langle s^2\rangle_P S.
\end{aligned}$$

□

NEED TO DEFINE $\sim$.

If $\delta \doteq |t - t_c|$ is very small, then we have the scaling relationship

$$\langle s\rangle_{X_m} \sim \delta^{-\gamma}$$

for some γ dependent on X_m **(Include lots more details about this)**. Differentiating gives us the relation

$$\partial_t \langle s\rangle_{X_m} \sim \delta^{-\gamma-1}.$$

Given a particular rule, we can take these two relations and subsitute them into our earlier calculation of $\partial_t \langle s\rangle_P$ to find out how the various γ are related. The below table sumarizes this relationship for all our rules.

Right now, I'm using the fact that $S = 0$ when $t < t_c$. I don't think it's necessary to be symmetric, though, since the behavior of the system seems to change after t_c anyway.

Rule	Scaling Relationship
ER	$\gamma_P = 1$
AE	$\gamma_Q = 1$
DaCosta	$\gamma_P + 1 = 2\gamma_{X_m}$
Sum	**Do this.**
Product	**Do this.**

Do we get any special information when $\gamma = 1$? At least in the Adjacent Edge case, I hope it gives us more since this method didn't give me γ_P.

0.3 Generalizations

I had some fun generalizing this somewhat. This particular version isn't useful whatsoever, but I'm keeping it here as a reminder to think about patterns in our rules that could actually be useful to generalize.

Proposition 4. If

$$\partial_t P(s) = \zeta_0 \left[s \sum_{u_1+\cdots+u_m=s} \prod_i X_{m_i}(u_i) \right] - \sum_i \zeta_i s X_{m_i}(u_i),$$

then the time derivative of S is

$$\partial_t S = \sum_i \langle s \rangle_{m_i} \left[-\zeta_0 \prod_{j \neq i} (1 - S^{m_j}) + \zeta_i \right].$$

Proof.

$$\begin{aligned}
\partial_t S &= -\sum_s \partial_t P(s) \\
&= -\zeta_0 \left[\sum_s s \sum_{\sum_i u_i = s} \prod_i X_{m_i}(u_i) \right] + \sum_i \zeta_i \langle s \rangle_{m_i} \\
&= -\zeta_0 \left[\sum_{u_1,\ldots,u_m} \left(\sum_i u_i \right) \prod_i X_{m_i}(u_i) \right] + \sum_i \zeta_i \langle s \rangle_{m_i} \\
&= -\zeta_0 \left[\sum_i \langle s \rangle_{m_i} \prod_{j \neq i} (1 - S^{m_j}) \right] + \sum_i \zeta_i \langle s \rangle_{m_i} \\
&= \sum_i \langle s \rangle_{m_i} \left[-\zeta_0 \prod_{j \neq i} (1 - S^{m_j}) + \zeta_i \right].
\end{aligned}$$

□