

# CONTENTS

<b>1</b>	<b>Chain Complexes</b>	<b>1</b>
1.1	Chain Complexes . . . . .	1
1.2	Chain Homotopies . . . . .	2
1.3	Homology . . . . .	3
<b>2</b>	<b>Derived Functors</b>	<b>5</b>
2.1	Resolutions . . . . .	5
2.2	Derived Functors . . . . .	7
2.3	The Ext Functor . . . . .	8

# 1 CHAIN COMPLEXES

## 1.1 CHAIN COMPLEXES

**Definition 1.** A **chain complex**  $\mathcal{C}$  is a sequence of  $R$ -morphisms

$$\cdots \xrightarrow{d_{i+2}} C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots$$

such that  $d^2 = 0$  for all  $i$ . **Cochain complexes** are the same, except the boundary maps take you up a level instead of down.

$$\cdots \xrightarrow{d_{i-1}} C^{i-1} \xrightarrow{d_i} C^i \xrightarrow{d_{i+1}} C^{i+1} \xrightarrow{d_{i+2}} \cdots$$

The map  $d_i$  is the **boundary operator**, as it is a generalization of the geometric concept of a boundary (note  $d^2 = 0$ ). Thus an element of  $\text{Im } d$  is a **boundary**. Since usual geometric cycles have no boundary, we call the elements of  $\text{Ker } d$  **cycles**.

**Example 1.** Chain complexes generalize the concept of boundaries to objects that don't necessarily have clear cyclic geometric properties. Let  $\Omega_n(M)$  denote the space of differential  $n$ -forms on a manifold  $M$ , then we have a cochain complex

$$\Omega_0(M) \xrightarrow{d} \Omega_1(M) \xrightarrow{d} \Omega_2(M) \xrightarrow{d} \cdots$$

where  $d$  is the exterior derivative. From this we see that the cycles of  $\Omega_0(M)$  (the space of differentiable functions on  $M$ ) are the constant functions.

A **morphism of complexes/chain morphism**  $f : \mathcal{C} \rightarrow \mathcal{D}$  is a sequence of morphisms  $f_i : C_i \rightarrow D_i$  respecting the boundary map, i.e. making the following diagram commute.

$$\begin{array}{ccc} C_i & \xrightarrow{d_{\mathcal{C}}} & C_{i-1} \\ f_i \downarrow & & \downarrow f_{i-1} \\ D_i & \xrightarrow{d_{\mathcal{D}}} & D_{i-1} \end{array}$$

## 1.2 CHAIN HOMOTOPIES

**Definition 2.** Given two chain complexes  $\mathcal{A}, \mathcal{B}$ , two chain morphisms  $f, g : \mathcal{A} \rightarrow \mathcal{B}$  are **(chain) homotopic**, written  $f \simeq g$ , if there are morphisms  $s_i : A_i \rightarrow B_{i-1}$  such that

$$d's + sd = f - g.$$

If  $\mathcal{A}, \mathcal{B}$  are cochain complexes instead, then  $s_i : A_i \rightarrow B_{i+1}$ .

$$\begin{array}{ccccc} A_{i-1} & \xrightarrow{d} & A_i & \xrightarrow{d} & A_{i+1} \\ & \swarrow s_i & \downarrow f_i & \downarrow g_i & \nwarrow s_{i+1} \\ B_{i-1} & \xrightarrow{d'} & B_i & \xrightarrow{d'} & B_{i+1} \end{array}$$

**Motivation for this?**

**Proposition 1.** Additive functors preserve homotopy equivalence.

*Proof.* Let  $f \simeq g$ . If  $\mathcal{F}$  is additive and covariant, then  $d's + sd = f - g \implies \mathcal{F}(d')\mathcal{F}(s) + \mathcal{F}(s)\mathcal{F}(d) = \mathcal{F}f - \mathcal{F}g$ . Thus  $\mathcal{F}f \simeq \mathcal{F}g$ . If  $\mathcal{G}$  is additive and contravariant, then  $\mathcal{G}(d)\mathcal{G}(s) = \mathcal{G}(s)\mathcal{G}(d') = \mathcal{G}f - \mathcal{G}g$ . Since all the arrows are reversed, the LHS is the right form, so  $\mathcal{G}f \simeq \mathcal{G}g$ .  $\square$

## 1.3 HOMOLOGY

**Definition 3.** The  $n$ -th **homology group** of a chain complex  $\mathcal{C}$  is

$$H_n(\mathcal{C}) = \frac{\text{Ker } d_n}{\text{Im } d_{n+1}}.$$

The  $n$ -th **cohomology group** is

$$H^n(\mathcal{C}) = \frac{\text{Ker } d_{n+1}}{\text{Im } d_n}.$$

**Proposition 2.** A chain/cochain complex is exact  $\iff$  all its homology/cohomology groups are trivial.

Thus the (co)homology groups of a (co)chain complex measure how much it fails to be exact.

**Proposition 3.** A morphism of complexes  $f : \mathcal{A} \rightarrow \mathcal{B}$  induces group morphisms between the complexes' homology/cohomology groups given by  $[a] \mapsto [f_n(a)]$ .

$$\begin{array}{ccccccc} A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & & H^n(\mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow & \rightsquigarrow & \downarrow \\ B^{n-1} & \longrightarrow & B^n & \longrightarrow & B^{n+1} & & H^n(\mathcal{B}) \end{array}$$

*Proof.* This follows from the morphism of complexes respecting the boundary map and thus mapping the kernels and images of the first complex to the kernels and images of the second.  $\square$

**Proposition 4.** If  $f_*$  is the induced (co)homology map of  $f$ , then  $(gf)_* = g_*f_*$ .

**Definition 4.**  $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$  is a **short exact sequence of complexes** if each  $0 \rightarrow A^n \rightarrow B^n \rightarrow C^n \rightarrow 0$  is short exact.

**Maybe build up to this with snake lemma?**

**Theorem 1** (Long Exact Sequence in Cohomology). If  $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$  is a short exact sequence of complexes, then there is a long exact sequence of cohomologies

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{A}) &\rightarrow H^0(\mathcal{B}) \rightarrow H^0(\mathcal{C}) \\ &\rightarrow H^1(\mathcal{A}) \rightarrow H^1(\mathcal{B}) \rightarrow H^1(\mathcal{C}) \\ &\rightarrow H^2(\mathcal{A}) \rightarrow \cdots \end{aligned}$$

where the morphisms  $H^n(\mathcal{C}) \rightarrow H^{n+1}(\mathcal{A})$  are the **connecting morphisms**.

*Proof.* **Construct connecting morphisms.**

**Intuition?** □

**Corollary 1.** If  $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$  is exact and any 2 of the complexes are exact themselves, then so is the third.

*Proof.* The LES of cohomologies becomes all 0, except for each  $H^n(\mathcal{X})$ , where  $\mathcal{X}$  is the third complex. Now  $0 \rightarrow H^n(\mathcal{X}) \rightarrow 0$  exact  $\implies H^n(\mathcal{X}) \cong 0$ , so  $\mathcal{X}$  is exact. □

**Definition 5.** A morphism of complexes is a **quasi-isomorphism** if the (co)homology maps it induces are all iso.

**Lemma 1.** If  $f \simeq g$  then they induce the same (co)homology maps, i.e.  $f_* = g_*$ .

*Proof.* Suppressing subscripts, suppose  $f = d's + sd$ , then the induced map is

$$[a] \mapsto [f(a)] = [(d's)(a) + (sd)(a)] = [d'(s(a)) + s(0)] = [0].$$

Then if  $f \simeq g$ , we have  $[f(a)] = [(g + d's + sd)(a)] = [g(a)]$ . □

**Proposition 5.** A homotopy equivalence is a quasi-iso.

*Proof.* Suppose  $f$  and  $g$  are inverse chain homotopies, then by the lemma,  $f_*g_* = (fg)_* = (1_B)_* = 1_{H(\mathcal{B})}$  and, similarly,  $g_*f_* = 1_{H(\mathcal{A})}$ . Thus  $H^n(\mathcal{A}) \cong H^n(\mathcal{B})$  for all  $n$ . □

## 2 DERIVED FUNCTORS

### 2.1 RESOLUTIONS

**Definition 6.** Suppose  $A$  is an  $R$ -module, then a **projective resolution** over  $A$  is an exact sequence of projective  $R$ -modules

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

and a **injective resolution** over  $A$  is an exact sequence of injective  $R$ -modules

$$0 \longrightarrow A \xrightarrow{\varepsilon} I_0 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} I_{n-1} \xrightarrow{d_n} I_n \longrightarrow \cdots$$

**Theorem 2** (Existence). Every  $R$ -module has a projective **and injective** resolution.

*Proof.* Let  $A$  be an  $R$ -module, and let  $P_0$  be free on  $A$ . Then there's a unique  $R$ -morphism  $\varepsilon : P_0 \rightarrow A$  extending  $1_A$ . It's clearly epic, so  $P_0 \xrightarrow{\varepsilon} A \rightarrow 0$  is exact. Now let  $P_1$  be free on  $\text{Ker } \varepsilon$ , then there's a unique  $R$ -morphism  $d_1 : P_1 \rightarrow \text{Ker } \varepsilon$ , so  $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A$  is exact. Continue inductively, with  $P_{n+1}$  free on  $\text{Ker } d_n$ . Since each  $P_i$  is free, each is projective.  $\square$

**Proposition 6.**  $R$ -morphisms lift to morphisms of projective/injective resolutions that are unique up to chain homotopy.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\ \cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & A' \longrightarrow 0 \\ \\ 0 & \longrightarrow & A & \longrightarrow & I_0 & \longrightarrow & I_1 \longrightarrow \cdots \\ & & \downarrow g & & \downarrow g_0 & & \downarrow g_1 \\ 0 & \longrightarrow & A' & \longrightarrow & I'_0 & \longrightarrow & I'_1 \longrightarrow \cdots \end{array}$$

*Proof.* Inductively use the fact that all the  $P_i, P'_i$  are projective and  $I_i, I'_i$  are injective to get each  $f_n$  and  $g_n$ . **Chain homotopy bit.**  $\square$

**Corollary 2.** Any two projective/injective resolutions of the same module are homotopy equivalent.

*Proof.* Consider the following lifts to projective resolutions (the case of injective resolutions is similar).

$$\begin{array}{ccccc}
 \mathcal{P} & \longrightarrow & A & \longrightarrow & 0 \\
 \downarrow f_\bullet & & \downarrow 1_A & & \\
 \mathcal{P}' & \longrightarrow & A & \longrightarrow & 0 \\
 \downarrow g_\bullet & & \downarrow 1_A & & \\
 \mathcal{P} & \longrightarrow & A & \longrightarrow & 0
 \end{array}$$

The identity map is a valid candidate for each  $g_n f_n$ , so  $g_n f_n \simeq 1$ . Now flip the diagram upside down and use the same  $f_n$  and  $g_n$  maps, yielding  $f_n g_n \simeq 1$ .  $\square$

**Corollary 3.** Any two projective/injective resolutions have isomorphic (co)homology groups.

*Proof.* They're homotopy equivalent, so they're quasi-isomorphic by Proposition 5.  $\square$

## 2.2 DERIVED FUNCTORS

Suppose we apply an additive functor  $\mathcal{F}$  to some projective/injective resolution of a module  $A$ . The (co)homology groups of the resulting complex are unique up to isomorphism:

- Let  $\mathcal{X}, \mathcal{Y}$  be projective/injective resolutions of  $A$ , then they're homotopy equivalent by Corollary 2.
- By Proposition 1, additive functors preserve homotopy equivalence, so  $\mathcal{F}\mathcal{X}$  and  $\mathcal{F}\mathcal{Y}$  are also homotopy equivalent.
- By Proposition 5, homotopy equivalent complexes have isomorphic (co)homology groups.

Thus we can define the (co)homology groups of the sequence gotten by applying an additive functor  $\mathcal{F}$  to a projective/injective resolution of  $A$  using *any* resolution.

**Define left and right derived functors.**

**Make sure to include chopping off the first term.**

**Proposition 7.** If  $\mathcal{F}$  is left exact, then  $R_0\mathcal{F} = \mathcal{F}$ .

*Proof.* **Covariant:** If  $0 \rightarrow A \xrightarrow{f} I_0 \xrightarrow{g} I_1$  is exact, then so is  $0 \rightarrow \mathcal{F}A \xrightarrow{\mathcal{F}f} \mathcal{F}I_0 \xrightarrow{\mathcal{F}g} \mathcal{F}I_1$ . Then  $R^0\mathcal{F}(A) = \text{Ker}(\mathcal{F}g) = \text{Im}(\mathcal{F}f) \cong \mathcal{F}(A)$  (since  $\mathcal{F}f$  monic).

**Contravariant:** Use a projective resolution instead. The process is the same.  $\square$

**left exact  $\rightarrow$  right derived / right exact  $\rightarrow$  left derived???**

**When to use proj/inj resolutions?**

**LES of left/right derived functors**

**If  $A$  projective, then can determine derived functors very easily with projective resolution  $0 \rightarrow A \rightarrow A \rightarrow 0$ .**



## 2.3 THE EXT FUNCTOR

**Note 1.** Big idea: the hom functors are left exact, but we can use cohomology to measure how much they fail to be right exact.

**Definition 7.** The **Ext functors** are the (right) derived functors of the hom functors.

Given an  $R$ -module  $A$ , there are two equivalent ways to construct them:

1. **Using  $\text{Hom}(-, M)$  (contravariant):** Take a projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

and apply  $\text{Hom}(-, M)$  to it. Removing  $\text{Hom}(A, M)$  from the sequence gives

$$0 \longrightarrow \text{Hom}(P_0, M) \xrightarrow{d_1^*} \text{Hom}(P_1, M) \xrightarrow{d_2^*} \cdots$$

This is a cochain complex since for any map  $f$ , applying  $d^*$  twice gives  $d^{*2}(f) = f d^2 = 0$ . Then  $\text{Ext}_R^n(A, M)$  is the  $n$ -th cohomology group of this complex.

2. **Using  $\text{Hom}(M, -)$  (covariant):** Take an injective resolution, apply  $\text{Hom}(M, -)$ , remove  $\text{Hom}(M, A)$ , then take homology.

**Proposition 8.**  $\text{Ext}_R^0(A, M) \cong \text{Hom}_R(A, M)$ .

*Proof.* The hom functors are left exact, so apply Proposition 7. □