MATH 531 HOMEWORK 3

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Exercise 1.7. *For nonempty sets $A, B \subset \mathbb{R}$, let $a + B = \{x + y \mid x \in A, y \in B\}$. Show that* $\sup(A + B) = \sup(A) + \sup(B)$.

Let $L_A \doteq \sup A$ and $L_B \doteq \sup B$. For any $a \in A$, we have $a + b \leq L_A + L_B$, which implies that $L_A + L_B$ is an upper bound of $A + B$. By definition, $\sup(A + B) \leq L_A + L_B$.

Now fix $\varepsilon > 0$, then L_A is a least upper bound of A, we can find $a \in A$ such that $a > L_A - \varepsilon/2$. Similarly, we can find $b \in B$ such that $b > L_B - \varepsilon/2$. Then $a + b > L_A + L_B - \varepsilon$.

Suppose $a' + b' < L_A + L_B$ is an upper bound of $A + B$, then $a + b \leq a' + b' < L_A + L_B - \varepsilon'$ for all $a \in A, b \in B$ and for some $\varepsilon' > 0$. But we just showed that for any $\varepsilon' > 0$, we can always find a and b such that $a + b > L_A + L_B - \varepsilon'$. Thus by contradiction, nothing strictly less than $L_A + L_B$ can be an upper bound of $A + B$.

This allows us to conclude that $L_A + L_B$ is the least upper bound of $A + B$, also written $\sup(A + B) = \sup A + \sup B$.

Exercise 1.8. *For nonempty sets $A, B \subset \mathbb{R}$, determine which of the following statements are true. Prove the true statements and give a counterexample for those that are false:*

(1) $\sup(A \cap B) \leq \inf \{\sup(A), \sup(B)\}$
(2) $\sup(A \cap B) = \inf \{\sup(A), \sup(B)\}$
(3) $\sup(A \cup B) \geq \sup\{\sup(A), \sup(B)\}$
(4) $\sup(A \cup B) = \sup\{\sup(A), \sup(B)\}$

(1) False. Let A and B be disjoint sets, then $A \cap B = \varnothing$ and $\sup(A \cap B) = \infty$. The infimum of a set is defined to be either finite or $-\infty$, so the inequality fails.
(2) False, by the same counterexample as above.
(3) True, which we show by evaluating two possible cases.

Case 1: *A and B are both bounded above.* Without loss of generality, let $\sup A \leq \sup B < \infty$ then $\sup\{\sup A, \sup B\} = \sup B$. Let $c \in A \cup B$. If $c \in A$, then $c \leq \sup A \leq \sup B$, and if $c \in B$, then $c \leq \sup B$. Thus $\sup B$ is clearly an upper bound of any element of $A \cup B$. We must now show that it is the least upper bound.

For any $L < \sup B$, we can find $b \in B$ such that $b > L$ (by the definition of the supremum). Since $b \in B \implies b \in A \cup B$, L cannot be an upper bound of $A \cup B$. Thus $\sup(A \cup B) = \sup\{\sup A, \sup B\} = \sup B$.

Case 2: *Either A or B is unbounded above.* Then $A \cup B$ is unbounded above, which implies that $\sup(A \cup B) = \infty$. Since $\sup\{\sup A, \sup B\}$ is clearly also ∞, we have $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

We have proven a stronger condition that what was required.

(4) True, by the same proof as above.

Exercise 1.10. *Verify that the bounded metric is indeed a metric.*

We must verify the four properties of a metric.

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(1) **Non-negativity:** Here we can use the fact that $d(x,y) \geq 0 \implies 1 + d(x,y) > 0$. This allows us to form the fraction $d(x,y)/(1+d(x,y))$ with no possibility of division by zero, and it also allows us to multiply both sides of an inequality by $1 + d(x,y)$ without reversing the inequality.

$$\rho(x,y) \geq 0 \iff \frac{d(x,y)}{1+d(x,y)} \geq 0 \iff d(x,y) \geq 0$$

(2) **Zero distance implies equal points:** We can use the same property as before.

$$\rho(x,y) = 0 \iff \frac{d(x,y)}{1+d(x,y)} = 0 \iff d(x,y) = 0 \iff x = y$$

(3) **Symmetry:**

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \rho(y,x)$$

(4) **Triangle Inequality:**

$$\begin{aligned} \rho(x,y) &= \frac{d(x,y)}{1+d(x,y)} \\ &= 1 - \frac{1}{1+d(x,y)} \end{aligned}$$

Using the triangle inequality for the metric d, $d(x,y) \leq d(x,z) + d(z,y)$, we can turn this into the inequality

$$\begin{aligned} &\leq 1 - \frac{1}{1+d(x,z)+d(z,y)} \\ &= \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)} \\ &\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\ &= \rho(x,z) + \rho(z,y) \end{aligned}$$

Since all four properties are satisfied, ρ is a metric.

Exercise 1.12. *In an inner product space show that*

(1) $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$ *(parallelogram law)*
(2) $\|x+y\|\|x-y\| \leq \|x\|^2 + \|y\|^2$
(3) $4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2$ *(polarization identity)*

Interpret these results geometrically in terms of the parallelogram formed by x and y.

These identities will rely on the same expansions of $\|x+y\|^2$ and $\|x-y\|^2$, namely

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$$
$$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$$

(1) We can add these two identities to get the desired result

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Geometrically, this can be interpreted as the sum of the squared edge lengths of the parallelogram (formed by x and y) being equal to the sum of the squared lengths of the diagonals.

(2) We can show this by expanding the norms of $x+y$ and $x-y$ and then simplifying.

$$\begin{aligned}
\|x+y\|\|x-y\| &= \left[\left(\|x\|^2 + 2\langle x,y\rangle + \|y\|^2\right)\left(\|x\|^2 - 2\langle x,y\rangle + \|y\|^2\right)\right]^{1/2} \\
&= \left[\|x\|^4 + 2\|x\|^2\|y\|^2 - 4\langle x,y\rangle^2 + \|y\|^4\right]^{1/2} \\
&\le \left[\|x\|^4 + 2\|x\|^2\|y\|^2 + \|y\|^4\right]^{1/2} \\
&= \|x\|^2 + \|y\|^2
\end{aligned}$$

Geometrically, the product of the lenghts of the diagonals of the parallelogram formed by x and y is equal to the sum of the squared lengths of x and y.

(3) Subtracting the second of our two original expansions gives the desired result

$$\|x+y\|^2 - \|x-y\|^2 = 4\langle x,y\rangle$$

Geometrically, the difference in the lengths of the diagonals of the parallelogram formed by x and y is equal to 4 times the inner product of x and y.

Exercise 1.15. *Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such taht $d(x_n, x_{n+1}) \le d(x_{n-1}, x_n)/2$. Show that $\{x_n\}$ is a Cauchy sequence.*

We can show that $\{x_n\}$ is a Cauchy sequence by first unraveling the recursion and then finding a useful bound on the distance between a point in the sequence and any future point in the sequence. We start by proving the lemma

Lemma 0.1. *If $d(x_1, y_2) = \ell/2$, then $d(x_n, x_{n+1}) \le \frac{\ell}{2^n}$.*

Proof. We can prove this inductively. The base case is trivial since $\ell/2 \le \ell/2$. Assuming the hypothesis holds for $d(x_n, x_{n+1})$, we can show it also holds for $d(x_{n+1}, x_{n+2})$.

$$d(x_{n+1}, x_{n+2}) \le \frac{d(x_n, x_{n+1})}{2} \le \frac{\ell/2^n}{2} = \frac{\ell}{2^{n+1}}.$$

Thus the inequality holds for all $n \ge 1$. □

With this inequality in place, we can unravel the recursion of the distance between a point x_n and x_{n+k}.

$$\begin{aligned}
d(x_n, x_{n+k}) &\le d(x_n, x_{x+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\
&\le \frac{\ell}{2^n} + \frac{\ell}{2^{n+1}} + \cdots + \frac{\ell}{2^{n+k-1}} \\
&= \ell\left[\frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+k-1}}\right]
\end{aligned}$$

Lemma 0.2. $\frac{1}{2^n} + \cdots + \frac{1}{2^{n+k-1}} \le \frac{1}{2^{n-1}}$

Proof. We first show that the sequence $s_k \doteq \sum_{i=1}^{k} \frac{1}{2^i} \le 1$, and we will then extend it to the more general case that is desired. Fix k, then we have

$$2s_k = 2\sum_{i=1}^{k} \frac{1}{2^i} = \sum_{i=0}^{k-1} \frac{1}{2^i} = 1 + s_k - \frac{1}{2^k}$$

Slight re-arranging then gives $s_k = 1 - 1/2^k$. Since $1/2^k$ is positive for any $k \ge 0$, $s_k \le 1$ for any $k \ge 0$. The extension to the more general case is straightforward. Note that

$$\frac{1}{2^n} + \cdots \frac{1}{2^{n+k-1}} \le \frac{1}{2^{n-1}} \iff \frac{1}{2} + \cdots \frac{1}{2^k} \le 1$$

Since we have already proven the latter inequality, we know that the desired general inequality holds. □

This inequality shows that $d(x_n, x_{n+k}) \leq \frac{\ell}{2^{n-1}}$. Since this does not depend on k, this holds for any $m \geq n$. Then for any given $\varepsilon > 0$, choose N such that $\frac{\ell}{2^{N-1}} < \varepsilon$, then for all $n, m > N$, we have $d(x_n, x_m) < \frac{\ell}{2^{N-1}} < \varepsilon$. This means the sequence $\{x_n\}$ is a Cauchy sequence.

Exercise 1.31. *Let $A, B \subset \mathbb{R}$ and $f : a \times B \to \mathbb{R}$ be bounded. Is it true that*

$$\sup_{(x,y)\in A\times B} f(x,y) = \sup_{y\in B}\left(\sup_{x\in A} f(x,y)\right)$$

Let $L = \sup_{(x,y)\in A\times B} f(x,y)$. Then by the definition of the supremum, for fixed y', $\sup_x f(x,y') \leq L$. Now since $f(x',y') \leq L$ for all $x' \in A, y' \in B$, taking the supremum over B of $\sup_x f(x,y')$ does not affect the inequality.

$$\sup_y\left(\sup_x f(x,y)\right) \leq L = \sup_{x,y} f(x,y)$$

Now we show that flipping the inequality is still valid. For every $x' \in A, y' \in B$, we know $f(x',y') \leq \sup_y f(x',y)$. Thus for $(\tilde{x},\tilde{y})$ in the pre-image of L under f,

$$f(\tilde{x},\tilde{y}) = L \leq \sup_x f(x,\tilde{y}) \leq \sup_y\left(\sup_x f(x,y)\right)$$

Based on these two inequalities, it must be the case that

$$\sup_{x,y} f(x,y) = \sup_y\left(\sup_x f(x,y)\right)$$

which is the desired result.

Exercise 2.1.2. *Let $S = \left\{(x,y) \in \mathbb{R}^2 \mid xy > 1\right\}$. Show that S is open.*

The set S can be written $S = S_+ \cup S_-$, where $S_+ = \{(x,y) \mid xy > 1, x, y > 0\}$ and $S_- = \{(x,y) \mid xy > 1, x, y < 0\}$. We will show that S_+ is an open set, then a similar argument will show that S_- is also open. Since the union of a finite number of open sets is also open, it must then be the case that S is open.

To find the greatest possible radius of an open disk around an arbitrary point $(x,y) \in S_+$, we can find the roots of $(x-\varepsilon)(y-\varepsilon) - 1 = 0$. The smaller of the two possible roots is

$$\varepsilon = \frac{x + y - \sqrt{x^2 - 2xy + y^2 + 4}}{2}$$

We claim that $\varepsilon > 0$ and $x - \varepsilon, y - \varepsilon > 0$. Since

$$\sqrt{x^2 - 2xy + y^2 + 4} > \sqrt{(x-y)^2} = |x - y|$$

we cound bound $x - \varepsilon$.

$$\begin{aligned}
x - \varepsilon &= x - \frac{x + y - \sqrt{x^2 - 2xy + y^2 + 4}}{2} \\
&= \frac{x - y + \sqrt{x^2 - 2xy + y^2 + 4}}{2} \\
&> \frac{x - y + |x - y|}{2} \\
&\geq 0
\end{aligned}$$

A similar argument holds for $y - \varepsilon$, so we know $x - \varepsilon, y - \varepsilon > 0$. Since x and y are known to be positive, ε is as well.

We now claim that all points in $D((x,y),\varepsilon)$ are also in S_+. To show this, we can take any point in the ε ball $(\tilde{x},\tilde{y})$. Since by the definition of the ball, $(x-\tilde{x})^2+(y-\tilde{y})^2<\varepsilon^2$, we have

$$\begin{aligned}(x-\tilde{x})^2+(y-\tilde{y})^2 \leq (x-\tilde{x})^2 < \varepsilon^2 &\implies |x-\tilde{x}|<\varepsilon \\ &\implies x-\varepsilon<\tilde{x}<x+\varepsilon\end{aligned}$$

Similarly, $y-\varepsilon<\tilde{y}<y+\varepsilon$. From this it immediately follows that

$$\tilde{x}\tilde{y} > (x-\varepsilon)(y-\varepsilon) = 1$$

Since we already showed $\tilde{x},\tilde{y}>0$, it must be the case that $(\tilde{x},\tilde{y})\in S_+$. Similarly, S_- is open. Then $S=S_+\cup S_-$ must be open.

Exercise 2.1.4. *Let $B\subset\mathbb{R}^n$ be any set. Define $C=\{x\in\mathbb{R}^n \mid d(x,y)<1$ for some $y\in B\}$. Show that C is open.*

Let $D_y=\{x\in\mathbb{R}^n \mid d(x,y)<1\}$, then $C=\cup_{y\in B}D_y$. It is known that the ε-ball $D(z,\varepsilon)$ is open for any z in a metric space M. Letting $M=\mathbb{R}^n$, we have $D_y=D(y,1)$, so D_y must be open. Then since the union of an arbitrary collection of open sets is open, it must also be the case that $C=\cup_{y\in B}D_y$ is open.

Exercise 2.2.3. *If $A\subset B$, is $A^o\subset B^o$?*

Let $a\in A^o$, then there exists an open set U such that $a\in U\subset A\subset B$. Since U lies entirely in B, this implies that $a\in B^o$ as well. Since a was arbitrary, this implies $A^o\subset B^o$.