1. The completed matrix M is

	I^h	I^u	I_c^e	I_c^i
I^h	$\beta ST_h(1-\phi)\rho$	$\beta ST_h(1-\phi)(1-\rho)$	$\beta ST_h\phi p_e$	$\beta ST_h\phi(1-p_e)$
I^u	$\beta ST_u\rho$	$\beta ST_u(1-\rho)$	0	0
I_c^e	$\beta_m ST_m(1-\phi)\rho$	$\beta_m ST_m(1-\phi)(1-\rho)$	$\beta_m ST_m\phi p_e$	$\beta_m ST_m\phi(1-p_e)$
I_c^i	$\beta ST_i(1-\phi)\rho$	$\beta ST_i(1-\phi)(1-\rho)$	$\beta ST_i\phi p_e$	$\beta ST_i\phi(1-p_e)$

2. If $p_e=1$, then M becomes

	I^h	I^u	I_c^e	I_c^i
I^h	$\beta ST_h(1-\phi)\rho$	$\beta ST_h(1-\phi)(1-\rho)$	$\beta ST_h\phi$	0
I^u	$\beta ST_u\rho$	$\beta ST_u(1-\rho)$	0	0
I_c^e	$\beta_m ST_m(1-\phi)\rho$	$\beta_m ST_m(1-\phi)(1-\rho)$	$\beta_m ST_m\phi$	0
I_c^i	$\beta ST_i(1-\phi)\rho$	$\beta ST_i(1-\phi)(1-\rho)$	$\beta ST_i\phi$	0

which satisfies the $M_{j,4}=0$ for $j=1,2,3$. Inductively, let M^n be of the form $M^n=(N,\mathbf{0})$, where $N\in\mathbb{R}^{4\times 3}$ has arbitrary elements and $0\in\mathbb{R}^4$. Then

$$\begin{aligned}(M^{n+1})_{j,4} &= (j\text{-th row of } M^n)\cdot(4\text{th column of } M^n)\\ &= (j\text{-th row of } M^n)\cdot\mathbf{0}\\ &= 0.\end{aligned}$$

So $(M^n)_{j,4}=0$ for $j=1,2,3$ for all $n\in\mathbb{N}$.

3. If $p_e=1$ and $\beta_m=0$, then M becomes

	I^h	I^u	I_c^e	I_c^i
I^h	$\beta ST_h(1-\phi)\rho$	$\beta ST_h(1-\phi)(1-\rho)$	$\beta ST_h\phi$	0
I^u	$\beta ST_u\rho$	$\beta ST_u(1-\rho)$	0	0
I_c^e	0	0	0	0
I_c^i	$\beta ST_i(1-\phi)\rho$	$\beta ST_i(1-\phi)(1-\rho)$	$\beta ST_i\phi$	0

To find the eigenvalues of M, we can find the determinant of this $M-\lambda I$ and set it equal to 0, then solve for λ. We have

$$\begin{aligned}\det(M-\lambda I) &= -\lambda\det\begin{pmatrix}\beta ST_h(1-\phi)\rho-\lambda & \beta ST_h(1-\phi)(1-\rho) & 0\\ \beta ST_u\rho & \beta ST_u(1-\rho)-\lambda & 0\\ \beta ST_i(1-\phi)\rho & \beta ST_i(1-\phi)(1-\rho) & -\lambda\end{pmatrix}\\ &= -\lambda^2\det\begin{pmatrix}\beta ST_h(1-\phi)\rho-\lambda & \beta ST_h(1-\phi)(1-\rho)\\ \beta ST_u\rho & \beta ST_u(1-\rho)-\lambda\end{pmatrix}\\ &= -\lambda^2\left[\left(\beta ST_h(1-\phi)\rho-\lambda\right)\left(\beta ST_u(1-\rho)-\lambda\right)-\beta^2S^2T_hT_u(1-\phi)(1-\rho)\rho\right].\end{aligned}$$

Since we want $\det(M-\lambda I)=0$, this implies that either $\lambda=0$ or the rest of the expression is 0. Assuming $\lambda\neq 0$, we can solve for the latter case.

$$\begin{aligned}\left(\beta ST_h(1-\phi)\rho-\lambda\right)\left(\beta ST_u(1-\rho)-\lambda\right)-\beta^2S^2T_hT_u(1-\phi)(1-\rho)\rho &= 0\\ \beta^2S^2T_hT_u(1-\phi)(1-\rho)\rho-\lambda\left[\beta ST_h(1-\phi)\rho+\beta ST_u(1-\rho)\right]+\lambda^2-\beta^2S^2T_hT_u(1-\phi)(1-\rho)\rho &= 0.\end{aligned}$$

The first and last terms on the LHS cancel out, leaving

$$\begin{aligned}-\lambda\left[\beta ST_h(1-\phi)\rho+\beta ST_u(1-\rho)\right]+\lambda^2 &= 0\\ \beta ST_h(1-\phi)\rho+\beta ST_u(1-\rho) &= \lambda\\ \beta ST_u\left[\rho\frac{T_h}{T_u}(1-\phi)+1-\rho\right] &= \lambda.\end{aligned}$$

Since $S = S_0$ and the paper defined $\mathscr{R}_0$ to be $\beta S_0 T_u$, this becomes

$$\lambda = \mathscr{R}_0 \left[\rho \frac{T_h}{T_u}(1-\phi) + 1 - \rho \right],$$

which matches the expression for $\mathscr{R}_e$ in the paper.