1 Simple growth model

a. This is false. If y is negative, then y' will also be negative. This means that y will be decreasing in t. In fact, whether or not the solution is increasing or decreasing in t depends on the initial condition $y(0)$. Since we know the solution to be $y(0)e^{rt}$, a negative $y(0$ will result in a solution that decreases monotonically in t.

b. If $y = c$ is a solution, then then $y' = 0$, meaning $ry = rc = 0$. Since $r \neq 0$, this means $c = 0$.

c.

$$\begin{aligned}\frac{dy}{dt} &= ry(t)\\ \frac{1}{y(t)}\frac{dy}{dt} &= r\\ \frac{d}{dt}\ln y(t) &= r\end{aligned}$$

Thus $G = \ln y(t)$.

d. The FTC says that for function f with anti-derivative F, $\int_a^b f(x)dx = F(b) - F(a)$. This gives

$$\begin{aligned}\frac{d}{dt}\ln y(t) &= r\\ \int_0^T \frac{d}{dt}\ln y(t)dt &= \int_0^T rdt\\ \ln y(T) - \ln y(0) &= rT\\ \ln\left(\frac{y(T)}{y(0)}\right) &= rt\\ y(T) &= y(0)e^{rT}\end{aligned}$$

e. All solutions to the ODE are exponentials, differing only by the initial condition $y(0)$ and rate r.

f. As r increases linearly, $y(t)$ increases exponentially.

g.

$$\begin{aligned}y(t) &= 2y_0\\ y_0e^{rt} &= 2y_0\\ t &= \frac{\ln 2}{r}\end{aligned}$$

At $t = 1/r$, y evaluates to $y(1/r) = y_0e$.

h. The precision at time T is the length of the interval $\left[\left(b - \frac{\varepsilon}{2}\right)e^{rT}, \left(b + \frac{\varepsilon}{2}\right)e^{rT}\right]$, which is εe^{rT}. To have precision δ, we must set ε to be $\varepsilon = \frac{\delta}{e^{rT}}$.

2 Logistic Growth

a. Since $r > 0$, we can divide both sides of the inequality by it to get

$$\begin{aligned}r\left(1 - \frac{y}{K}\right)y &> 0\\ y - \frac{y^2}{K} &> 0\end{aligned}$$

This only holds when $0 < y < K$. Thus $f(y) > 0$ when $0 < y < K$. When $f(y) < 0$, the inequality is flipped. Thus $f(y) < 0$ when $y < 0$ or $y > K$.

b. A constant function is a solution to the ODE when $y' = 0$, so

$$r\left(1 - \frac{y}{K}\right)y = 0$$
$$r\left(1 - \frac{c}{K}\right)c = 0$$

which is clearly true only when $c = 0$ or $c = K$. Interpreted in the context of logistic growth, this says that a population stagnates in size when there are either no people left or when the maximum resource consumption limit K has been reached exactly.

c. When $0 < y(t^*) < K$, the conditions from part (a) show that y is increasing.

d. Since $0 < y(0) < K$, y will initially increase. As this occurs, the value of y will approach K, at which point we know that y' becomes 0. Thus we expect the rate of change of y to slow down as $y(t)$ approaches K, convering to K as $t \to \infty$.

e. When $y(0) = 2K > K$, y will initially decrease. Based on similar reasoning as before, we expect the magnitude of the rate of change of y to decrease as y approaches K. As $t \to \infty$, we expect $y(t) \to K$.

   When $y(0) < 0$, the conditions from part (a) show that y' is always negative. Thus as $t \to \infty$, we expect $y(t)$ to diverge to $-\infty$.

3 Explosive Growth

a. As t increases linearly, $y(t)$ increases very rapidly. This rate increases quadratically as time increases linearly. Unlike with the solution of (0.1), it is impossible for this ODE to have negative growth. Another, more obvious, comparison is that this new ODE has much faster growth than the solution of (0.1).

b.

$$\frac{dy}{dt} = y^2$$
$$\frac{1}{y^2}\frac{dy}{dt} = 1$$
$$\frac{d}{dt}\left(-\frac{1}{y}\right) = 1$$

Thus $G(y(t)) = -\frac{1}{y(t)}$.

c. Applying the FTC gives

$$\frac{d}{dt}\left(-\frac{1}{y}\right) = 1$$
$$-\int_0^T \left(\frac{d}{dt}\frac{1}{y(t)}\right)dt = \int_0^T dt$$
$$-\frac{1}{y(T)} + \frac{1}{y(0)} = T$$
$$y(T) = \frac{y(0)}{1 - Ty(0)}$$

This cannot be a solution for all $t > 0$ since this function is undefined at $T = \frac{1}{y(0)}$. As T approaches this value, we have a vertical asymptote as the function diverges to ∞.

d. If $y(0) = 1$, there is no solution for all $t > 0$. In this case, the potential solution

$$y(t) = \frac{1}{1-t}$$

has a discontinuity at $t = 1$. After this point in time, the function suddenly becomes negative, which we know to be incorrect behavior since the rate of change of y is always positive.

4 Linear Inhomogeneous

a. The function $w(t) = y(t) - z(t)$ has derivative

$$\begin{aligned} w' &= y' - z' \\ &= ry + k - rz - k \\ &= r(y - z) \\ &= rw \end{aligned}$$

which is the desired form.

b. When trying to solve the ODE with the strategy from (0.1), we find that the y-dependent terms cannot be expressed as the derivative of any function. What we must do instead is introduce an integrating factor.

c.

$$\begin{aligned} \frac{d}{dt}(e^{-rt}y(t)) &= e^{-rt}y' - re^{-rt}y \\ &= e^{-rt}(y' - ry) \\ &= e^{-rt}(ry + k(t) - ry) \\ &= k(t)e^{-rt} \end{aligned}$$

d. Applying the FTC to this relation gives

$$\begin{aligned} \int_0^T \left(\frac{d}{dt}(e^{-rt}y(t))\right) dt &= \int_0^T k(t)e^{-rt} dt \\ e^{-rT}y(T) - y(0) &= \int_0^T k(t)e^{-rt} dt \\ y(T) &= e^{rT}\left[y(0) + \int_0^T k(t)e^{-rt} dt\right] \end{aligned}$$

5 Simple linear system

a. A simple system that these equations might describe is two friendly populations, where the first population supports the growth of the second. A toy example might be two small villages, where one village produces resources that benefit the second (perhaps the first village is a colony of the second). Both villages have their own base growth rates (r and α), but the first village's resources allow the second village to grow faster (the additional β term). Another interpretation from this two-village example might be the reassignment of people. If the first vilage in fact grows at a rate of $r + \beta$ but ships $\frac{\beta}{r+\beta}$ of its newborns to the second village, then the first village will have the proper growth rate in the equation.

b. The general solution for y is $y(t) = y(0)e^{rt}$.

c. We can use a similar strategy as with the linear inhomogeneous system. Let $k(t) \doteq \beta y_0 e^{rt}$, then we have

$$\begin{aligned}
z' &= \alpha z + \beta y_0 e^{rt} \\
z' - \alpha z &= k(t) \\
\frac{d}{dt} e^{-\alpha t} z &= k(t) e^{-\alpha t} \\
\int_0^T \left(\frac{d}{dt} e^{-\alpha t} z(t) \right) dt &= \int_0^T k(t) e^{-\alpha t} dt \\
e^{-\alpha t} z(T) - z(0) &= \beta y(0) \int_0^T e^{rt} e^{-\alpha t} dt \\
z(T) &= e^{\alpha T} \left[z(0) + \beta y(0) \int_0^T e^{(r-\alpha)t} dt \right]
\end{aligned}$$

d. Setting $z(0) = 0$, this becomes

$$z(T) = \beta y(0) e^{\alpha T} \int_0^T e^{(r-\alpha)t} dt$$

When $y(0) > 0$, the parameters have the following qualitative effects on $z(t)$

- β : As β increases linearly, so does $z(t)$.
- r, α : r and α balance each other out in this final equation. If r is larger than α, then we see an exponential increase in $z(t)$. As α increases, this growth becomes smaller. The function cannot be made arbitrarily small by increasing α, however, due to the presence of the $e^{\alpha T}$ term outside of the integral. In fact, fixing the other parameters and increasing α still results in exponential growth, just at a lower rate than if we had just increased r.