Exercise 1. (Hatcher 1.2: 1). Show that the free product $G * H$ of nontrivial groups G and H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.

Trivial center: Let $x = x_1 \dots x_n$ with $x_n \in G - \{1\}$ be an element of $G * H$. Let $h \in H - \{1\}$ be arbitrary, then hx in reduced form ends with $x_n \in G$, but xh in reduced form ends in $h \in H$. Thus $hx \neq xh$, so x cannot be in the center of $G * H$. A similar argument holds if $x_n \in H - \{1\}$ instead, so no nontrivial element of $G * H$ can be in the center.

Now the empty word, as the identity element, commutes with all other elements of $G * H$, so it is an element of the center. By the previous argument, it is the only such element, so the center of $G * H$ is trivial.

Finite order $\implies$ conjugate: First we show that any finite order element of $G * H$ is a conjugate of finite order elements of G and H. Suppose $x = x_1 \dots x_m \in G * H$ has finite order, i.e. x^n is the empty word for some $n \in \mathbb{N}$. This necessarily means

$$\underbrace{(x_1 \dots x_m) \dots (x_1 \dots x_m)}_{n \text{ times}} = \text{ the empty word.}$$

In order for this to reduce to the empty word, we need $x_{n-i} x_{i+1} = e$ for all $0 \leq i \leq m - 1$. Then since left and right inverses coincide in groups, $x_{m-1} = x_{i+1}^{-1}$ for all i. We then have two cases:

$$\begin{cases} (x_m^{-1} \dots x_{\lfloor m/2-1 \rfloor}^{-1}) \; x_{\lfloor m/2 \rfloor} \; (x_{\lfloor m/2+1 \rfloor} \dots x_m) & \text{m is odd} \\ (x_m^{-1} \dots x_{m/2-1}) \; e \; (x_{m/2+1} \dots x_m) & \text{m is even.} \end{cases}$$

Since e is clearly of finite order, x is the conjugate of a finite order element when m is even. When m is odd,

$$e = x^n = x_{\lfloor m/2 \rfloor},$$

so x is a conjugate of finite order $x_{\lfloor m/2 \rfloor}$.

conjugate $\implies$ finite order: Suppose xgx^{-1} is an element of $G * H$, where g is a finite order element of G or H, i.e. $g^n = e$ for some $n \in \mathbb{N}$. Then

$$\underbrace{(xgx^{-1}) \dots (xgx^{-1})}_{n \text{ times}} = xg^n x^{-1} = xex^{-1} = x^{-1} = \text{ the empty word}$$

since all pairs $x^{-1}x$ give the identity and then are reduced away. Thus xgx^{-1} has finite order n.

Exercise 2.

1. (Hatcher 1.2: 4). Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.
2. (Hatcher 1.2: 6). Use Proposition 1.26 to show that the complement of a closed discrete subspace of $\mathbb{R}^n$ is simply connected if $n \geq 3$.

1. This argument assumes that all n lines are disjoint. Obviously, if any two lines coincide, we can treat them as the same line and then we're working with $n-1$ lines instead of n.

   The map $F(x,t) : (1-t)x + t\frac{x}{\|x\|}$ is a deformation retraction from $\mathbb{R}^3 - X$ to S^2 minus $2n$ points, which shows that the two spaces are homotopy equivalent. Then since stereographic projection is a homeomorphism from $S^n - \{\text{pt}\}$ to $\mathbb{R}^n$, the sphere minus $2n$ points is homeomorphic to $\mathbb{R}^2$ minutes $2n-1$ points. Then this space clearly deformation retracts onto the bouquet of $2n-1$ circles, so we have the sequence

   $$\mathbb{R}^3 - X \quad \simeq \quad S^2 - \{2n \text{ points}\} \quad \cong \quad \mathbb{R}^2 - \{2n-1 \text{ points}\} \quad \simeq \quad \underbrace{S^1 \vee \cdots \vee S^1}_{2n-1 \text{ times}}.$$

   Then since the fundamental group of m circles wedged together is the free product on m generators, $\pi_1(\mathbb{R}^3 - X)$ is the free group on $2n-1$ generators.

2. **Trivial fundamental group:** Proposition 1.26 says that attaching any number of 3-cells to a space does not affect the fundamental group. So suppose $X = \{x_\alpha\}_\alpha$ is any closed discrete subspace of $\mathbb{R}^n$. Since it's discrete, by definition of the subspace topology, there must be open balls $B_{r_\alpha}(x_\alpha)$ of $\mathbb{R}^n$ that intersect X only at x_α.

   Then the balls $B_{r_\alpha/2}(x_\alpha)$ with radii halved are all disjoint. The space $\mathbb{R}^n - X$ is clearly a deformation retract of $\mathbb{R}^n - \{B_{r_\alpha/2}(x_\alpha)\}_\alpha$, so they have isomorphic fundamental groups. But we can recover $\mathbb{R}^n$ from this space by filling in the holes with 3-cells, so by Proposition 1.26,

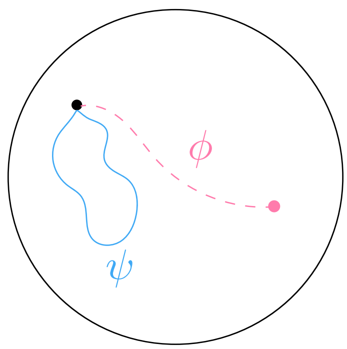
   $$1 \quad \cong \quad \pi_1(\mathbb{R}^n) \quad \cong \quad \pi_1\left(\mathbb{R}^n - \{B_{r_\alpha/2}(x_\alpha)\}_\alpha\right) \quad \cong \quad \pi_1(\mathbb{R}^n - X).$$

   Path connected: Suppose γ is a path between two point in $\mathbb{R}^n$ that intersects X at some points $\{x_\beta\}_\beta$. As argued earlier, we can find disjoint open balls in $\mathbb{R}^n$ containing a single x_β each. Then we can perturb γ away from x_β while staying in the open ball, giving us a path $\tilde{\gamma}$ between the same points that now doesn't intersect X. Thus $\mathbb{R}^n - X$ is path connected.

Exercise 3. (Hatcher 0: 14). Given positive integers v, e, and f satisfying $v - e + f = 2$, construct a cell structure on S^2 having v 0-cells, e 1-cells, and f 2-cells.

We can construct S^2 by taking a 0-cell and gluing the boundary of D^2 to it. Thus the sphere has $(v, e, f) = (1, 0, 1)$. Now consider the following two operations on S^2:

- ϕ: designate a point on S^2 to be a new 0-cell, and add a 1-cell between it and some other pre-existing 0-cell.
- ψ: Add a loop at some 0-cell.



Note that ϕ adds a vertex and an edge, and ψ adds an edge and a face, so both preserve the identity $v - e + f = 2$. If ϕ is performed a times and ψ is performed b times on the sphere, then we can represent the number of 0-cells, 1-cells, and 2-cells as

$$(1, 0, 1) + a(1, 1, 0) + b(0, 1, 1).$$

Suppose (v, e, f) is an arbitrary triple satisying $v - e + f = 2$, then if we let $a = v - 1$ and $b = e - v - 1$, this becomes

$$(v, e, 2 + e - v) = (v, e, f).$$

Since ϕ and ψ both preserve $v - e + f = 2$, we end up with our desired cell structure on S^2.

Exercise 4. Suppose Γ is a 1-dimensional cell complex and let E be an edge of Γ connecting two different vertices (0-cells) of Γ, where E includes both of its endpoints. Show that Γ is homotopy equivalent to the quotient space Γ/E obtained by shrinking E to a point (don't use Hatcher's Proposition 0.17 or the homotopy extension property, etc).

Since E connects two different endpoints, it is clearly contractible. Thus there is a homotopy $F : E \times I \to E$ between id_E and q, the quotient map $\Gamma \twoheadrightarrow \Gamma/E$. Now add some 1-cell to E to get a space $\tilde{E}$.

We can extend F to all of $\tilde{E} \times I$ as follows:

- First note that there is a retraction $r : D^1 \times I \to (\partial D^1 \times I) \cup (D^n \times \{0\})$ given by radially projecting from the point $(0, 2)$.
- We can let $F|_{D^1 \times 0}$ just be the identity.
- Now define $F|_{D^1 \times I}(x, t) = (F_t \circ r)(x)$. This is well-defined since it agrees with our original F on $E \times I$ and since r maps $D^1 \times I$ into $(\partial D^1 \times I) \cup (D^1 \times \{0\})$. This is a subset of $(E \times I) \cup (D^1 \times \{0\})$, where F is already defined.

Now extend this result by induction to all of Γ. This gives us a homotopy $G : \Gamma \times I \to \Gamma$. Since G_1 is constant on E, it induces a map $\psi : \Gamma/E \to \Gamma$ such that the following diagram commutes.

$$\begin{tikzcd} \Gamma \arrow[r, "G_1"] \arrow[d, two heads, "q"] & \Gamma \\ \Gamma/E \arrow[ur, "\psi"'] & \end{tikzcd}$$

We claim that ψ and q are maps showing $\Gamma \simeq \Gamma/E$. The homotopy G_t shows $\mathrm{id}_\Gamma \simeq \psi q$, but the other direction is less straightforward.

Note that since $G_t(E) \subset E$ for all t, we have $qG_t(e_1) = qG_t(e_2)$ whenever $e_1, e_2 \in E$. Being in E is exactly the equivalence relation determining the quotient space Γ/E, so by the universal property of quotient spaces, there is a unique $\tilde{G}_t$ making the following diagram commute.

$$\begin{tikzcd} \Gamma \arrow[d, two heads, "q"] \arrow[dr, "qG_t"] & \\ \Gamma/E \arrow[r, dashed, "\exists!\, \tilde{G}_t"'] & \Gamma/E. \end{tikzcd}$$

Putting these diagrams together, we get

$$\begin{tikzcd} \Gamma \arrow[r, "G_t"] \arrow[d, two heads, "q"] & \Gamma \arrow[d, two heads, "q"] \\ \Gamma/E \arrow[r, "\tilde{G}_t"'] & \Gamma/E \end{tikzcd} \qquad \begin{tikzcd} \Gamma \arrow[r, "G_1"] \arrow[d, two heads, "q"] & \Gamma \arrow[d, two heads, "q"] \\ \Gamma/E \arrow[r, "\tilde{G}_1"'] \arrow[ur, "\psi"] & \Gamma/E \end{tikzcd}$$

Now $\tilde{G}_t$ is a homotopy between $\mathrm{id}_{\Gamma/E}$ and $q\psi$: $\tilde{G}_0 q = qG_0 = q$, which implies $\tilde{G}_0 = \mathrm{id}$ since q is epic; $\tilde{G}_0 q = qG_1 = q\psi q$, which similarly implies $\tilde{G}_1 = q\psi$; and $\tilde{G}_t$ is continuous with respect to t since qG_t is.

Thus the quotient map q is a homotopy equivalence showing $\Gamma \simeq \Gamma/E$.