# Contents

## 0.1 Categories

**Definition 1.** A **category** $\mathbf{C}$ is a class of **objects** $\mathrm{ob}(\mathbf{C})$ along with sets of **morphisms** between those objects. The set of morphisms $A$ to $B$ is denoted $\mathrm{Hom}_{\mathbf{C}}(A, B)$ or $\mathbf{C}(A, B)$. There must be a law of composition of morphisms

$$(f, g) \mapsto gf.$$

Finally, the objects and morphisms satisfy:

1. If $A \neq C$ or $B \neq D$, then $\mathrm{Hom}_{\mathbf{C}}(A, B)$ and $\mathrm{Hom}_{\mathbf{C}}(C, D)$ are disjoint sets.
2. Morphism composition is associative.
3. Each object has an identity morphism.

We will drop the subscript $\mathbf{C}$ in $\mathrm{Hom}_{\mathbf{C}}$ if the category is clear.

**Definition 2.** A category $\mathbf{S}$ is a **subcategory** of $\mathbf{C}$ if

1. $\mathrm{ob}(\mathbf{S})$ is a subclass of $\mathrm{ob}(\mathbf{C})$; and
2. for all $A, B \in \mathrm{ob}(\mathbf{S})$, $\mathrm{Hom}_{\mathbf{S}}(A, B)$ is a subclass of $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

A **full** subcategory maintains all morphisms from $\mathbf{C}$ among the objects that it maintains, i.e. for $A, B \in \mathrm{ob}(\mathbf{S})$, $\mathrm{Hom}_{\mathbf{S}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(A, B)$.

**Note 1.** The image of a category need not be a subcategory.

**Proposition 1.** *The identity morphism of an object is unique.*

*Proof.* Suppose $1_A$ and $1'_A$ are both identity morphisms of $A$. Then $1_A = 1_A 1'_A = 1'_A$. $\square$

**Definition 3.** An **endomorphism** of $A$ is a morphism from $A$ to itself.

**Definition 4.** An **isomorphism** $f : A \to B$ is an invertible morphism, i.e. there exists a morphism $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$.

**Proposition 2.** *Inverses of morphisms are unique.*

*Proof.* Suppose $f : A \to B$ is a morphism and $g, g' : B \to A$ are both inverses of it. Then by associativity of morphism composition, $g = g1_B = g(fg') = (gf)g' = 1_A g' = g'$. $\square$

**Definition 5.** A **groupoid** is a category whose morphisms are all isomorphisms. Every category contains a subcategory called the **maximal groupoid**, which is all of the objects along with only the morphisms that are isomorphisms.

**Example 1.** We can define a **group** as a groupoid that has only one object. The group elements are the morphisms. The properties of a group follow from the properties of categories and the fact that our morphisms are all isomorphisms.

Put picture of group and groupoid as category.

Now for some examples to make this *somewhat* less abstract.

1. **Set**: the category of all sets. The category of all finite sets is a subcategory of this.
   - $\mathrm{Hom}(A, B)$ is the set of all functions from $A$ to $B$.
   - Morphism composition is the usual composition of functions.
   - The identity morphism sends $a \in A$ to itself.
2. **Grp**: the category of all groups. **Ab**, the category of all abelian groups, is a subcategory of this. Morphisms are group homomorphisms, and isomorphisms are, well, group isomorphisms.
3. **Ring**: the category of all nonzero rings with 1. The morphisms are ring homomorphisms that send 1 to 1.
4. **R-mod**: the category of all left $R$-modules. The morphisms are $R$-module homomorphisms.
5. **Top**: the category of all topological spaces. The morphisms are continuous maps between spaces, and the isomorphisms are homeomorphisms.

**Definition 6.** A **discrete category** is a category in which all the morphisms are identities, i.e. every object is isolated.

**Definition 7.** Given a category $\mathbf{C}$, its **opposite** or **dual** category $\mathbf{C}^{\mathrm{op}}$ is the category gotten by reversing the morphisms of $\mathbf{C}$. Formally, $\mathrm{ob}(\mathbf{C}^{\mathrm{op}}) = \mathrm{ob}(\mathbf{C})$, but

$$\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(B, A).$$

Note that the identities in a category and its dual are the same. Compositions, on the other hand, are reversed.

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "gf"'] & \bullet \arrow[d, "g"] \\ & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet & \bullet \arrow[l, "f'"'] \\ & \bullet \arrow[u, "g'"'] \arrow[ul, "f'g'"] \end{tikzcd}$$

Figure 1: A category and its dual. Since every object must have an identity morphism, I usually won't include them in a diagram unless necessary.

**Definition 8.** Given categories $\mathsf{C}$ and $\mathsf{D}$, we can define their **product category** $\mathsf{C} \times \mathsf{D}$ as having the objects

$$\mathrm{ob}(\mathsf{C} \times \mathsf{D}) = \mathrm{ob}(\mathsf{C}) \times \mathrm{ob}(\mathsf{D})$$

and the morphisms

$$\mathrm{Hom}_{\mathsf{C} \times \mathsf{D}}((A, B), (A', B')) = \mathrm{Hom}_{\mathsf{C}}(A, A') \times \mathrm{Hom}_{\mathsf{D}}(B, B').$$

It is straightforward to define the identity morphisms and the composition of morphisms in product categories in a piecewise fashion, building off the identities and composition laws of $\mathsf{C}$ and $\mathsf{D}$.

## 0.2 Universal Properties

**Definition 9.** An object $A$ is **initial** if there is a unique morphism $A \to B$ for all objects $B$. An object $C$ is **final** if there is a unique morhpism $B \to C$ for all objects $C$. An object that is either initial or fianl is called **universal**, and it's a **zero object** if it's both.

**Example 2.** In **Grp**, the trivial group $\{1\}$ is a zero object.

**Proposition 3.** *All initial objects are isomorphic. All final objects are isomorphic.*

*Proof.* If $A, A'$ are both initial, there are unique morphisms $f : A \to A'$ and $g : A' \to A$. Then $gf$ is a morphism $A \to A$. Since $A$ is universal, it has only one endomorphism, namely the identity map. Thus $gf = 1_A$. Similarly, $fg = 1_{A'}$. Thus $A \cong A'$. The proof is similar for final objects. $\square$

**Example 3.** If we have a morphism $f : X \to Y$, we can define an equivalence relation

$$a \sim b \iff f(a) = f(b).$$

Then if $\pi$ is the canonical projection map, the following diagram commutes.

$$\begin{tikzcd}
X/\sim \arrow[r, dashed, "\exists! g"] & Y \\
X \arrow[u, "\pi"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Consider the category with objects the morphisms $X \to Y$ that respect $\sim$ and morphisms given by push-forwards. In this category, $\pi$ is initial (and thus universal). Colloquially, we say that $X/\sim$ has this universal property, although it's really $\pi$ that has the property. But $\pi$ is the only map $X \to X/\sim$ that makes sense, though, so we can say either and it's obvious what we mean.

Is $\operatorname{im} f \cong X/\sim$ for all categories? Seems like it should depend on what the canonical projection is in the particular category...

## 0.3 Functors

Functors map categories to categories by associating objects with objects and morphisms with morphisms in ways that respect morphism composition and identities.

$$\begin{tikzcd}
& B \arrow[d, "g"] \arrow[rr, dashed] & & \mathcal{F}B \arrow[d, "\mathcal{F}(g)"] \\
A \arrow[ur, "f"] \arrow[rr, dashed] & & \mathcal{F}A \arrow[ur, "\mathcal{F}(f)"] & \\
& C \arrow[ul, "h"] \arrow[rr, dashed] & & \mathcal{F}C \arrow[ul, "\mathcal{F}(h)"]
\end{tikzcd}$$

Figure 2: A functor $\mathcal{F}$ between two categories.

**Definition 10.** A **functor** $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ satisfies:

1. For every object $A$ in $\mathsf{C}$, $\mathcal{F}A$ is an object in $\mathsf{D}$.
2. For every $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, $\mathcal{F}(f)$ is a morphism in $\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}A, \mathcal{F}B)$ such that
   (a) $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$, and
   (b) $\mathcal{F}(1_A) = 1_{\mathcal{F}A}$.

Sometimes we call a functor a **covariant functor** to differentiate it from another type of functor, which we define in a bit.

**Example 4** (Category Inception)**.** The category **CAT** has objects that are themselves categories, and its morphisms are functors.

**Definition 11.** Given a functor $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, $A$ is the **domain** and $B$ is the **codomain** of $f$.

There are tons of examples of functors, so here are some that aren't too complicated.

1. The **identity functor** $\mathcal{I}_{\mathsf{C}}$ maps $\mathsf{C}$ to $\mathsf{C}$ by sending objects and morphims to themselves.

2. If $\mathsf{C}$ is a subcategory of $\mathsf{D}$, then the **inclusion functor** maps $C$ to $D$ by sending objects and morphisms to themselves, except now as members of $\mathsf{D}$ instead of $\mathsf{C}$.

3. **Forgetful functors** take a category and strip its objects of some kind of complexity, i.e. a functor from **Grp** to **Set**. A forgetful functor doesn't have to just map objects to plain sets, though. We could also map **Ab** to **Grp**, forgetting the abelian nature of the groups in our category.

More examples.

In order to "respect" morphisms, we might either keep the morphisms all in the same direction or flip them. If we decide to flip them all, we get a different type of functor.

**Proposition 4.** *Functors preserve isomorphisms.*

*Proof.* Suppose $\mathcal{F} : \mathsf{A} \to \mathsf{B}$ is a functor, and suppose $A \cong A'$ are isomorphic objects in $\mathsf{A}$. Since $A$ and $A'$ are isomorphic, there are inverses $f : A \to A'$ and $g : A' \to A$. By definition, $\mathcal{F}(f)$ and $\mathcal{F}(g)$ can be composed and

$$\mathcal{F}(f)\mathcal{F}(g) = \mathcal{F}(fg) = \mathcal{F}(1_{A'}) = 1_{\mathcal{F}A'}.$$

Similarly, $\mathcal{F}(g)\mathcal{F}(f) = 1_{\mathcal{F}A}$, so $\mathcal{F}A \cong \mathcal{F}A'$. $\square$

**Definition 12.** A **contravariant functor** from $\mathsf{C}$ to $\mathsf{D}$ is a functor from $\mathsf{C}^{\mathrm{op}}$ to $\mathsf{D}$.

**Definition 13.** A functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ is **faithful** if for all objects $A, B$ of $\mathsf{C}$, the map

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}(A, B) &\to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}A, \mathcal{F}B) \\ f &\mapsto \mathcal{F}(f) \end{aligned}$$

is one-to-one. $\mathcal{F}$ is **full** if this map is onto.

Note that the fixed $A$ and $B$ above are important. The injective/surjective conditions don't apply to arbitrary morphisms in $\mathsf{C}$ since they might connect different objects.

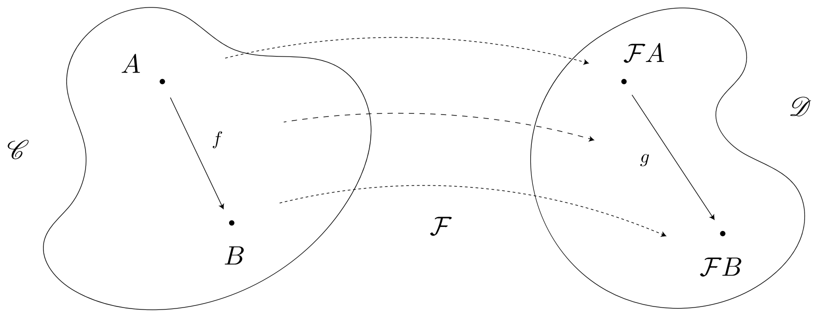


Figure 3: For all $A, B$, and $g$, a faithful functor sends at *most* one solid arrow in $\mathsf{C}$ to $g$. A full functor sends at *least* one solid arrow in $\mathsf{C}$ to $g$.

**Example 5.** The inclusion functor from $\mathsf{S}$ to $\mathsf{C}$ is always faithful, and it's full if and only if $\mathsf{S}$ is a full subcategory.

## 0.4 Natural Transformations

When functors have the same domain and codomain, we can define a map between them.

**Definition 14.** Suppose $\mathcal{F}, \mathcal{G} : \mathsf{A} \to \mathsf{B}$ are functors. Then a **natural transformation** $\alpha : \mathcal{F} \to \mathcal{G}$ is a family

$$(\alpha_A : \mathcal{F}A \to \mathcal{G}A)_{A \in \mathsf{A}}$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}A \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\alpha_A"'] & \mathcal{F}A' \arrow[d, "\alpha_{A'}"] \\
\mathcal{G}A \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}A'
\end{tikzcd}$$

The maps $\alpha_A$ are the **components** of $\alpha$.

The diagram above commuting means $\mathcal{G}(f) \circ \alpha_A = \alpha_{A'} \circ \mathcal{F}(f)$, but it's easier to understand the diagram, so you should remember it by that.

Also, the diagram commuting means that there's only one map from $\mathcal{F}A$ to $\mathcal{G}A'$, namely the diagonal of the diagram (which you can construct by taking the composition of either path).

$$\begin{tikzcd}
\mathsf{A} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathsf{B}
\end{tikzcd}$$

Figure 4: Because we love overloading notation, we'll denote natural transformations with a $\Longrightarrow$ , as in this diagram.

Given two natural transformations $\alpha$ and $\beta$, we can form the composition

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A.$$

Additionally, we can define the identity natural transformation $1_F$ by

$$(1_F)_A = 1_{F(A)}.$$

**Example 6** (Even more category inception)**.** We can form a category whose objects are functors from $\mathsf{A}$ to $\mathsf{B}$ and whose morphisms are natural transformations. This is called the **functor category** from $\mathsf{A}$ to $\mathsf{B}$, and we denote it by $[\mathsf{A}, \mathsf{B}]$ or $\mathsf{B}^{\mathsf{A}}$.

For funsies, we'll break down what's going on in the category functor $[2, \mathsf{C}]$, where 2 is the discrete category with 2 objects and $\mathsf{C}$ is arbitrary, by examining what two functors $\mathcal{F}, \mathcal{G} \in \text{ob}([2, \mathsf{C}])$ do.

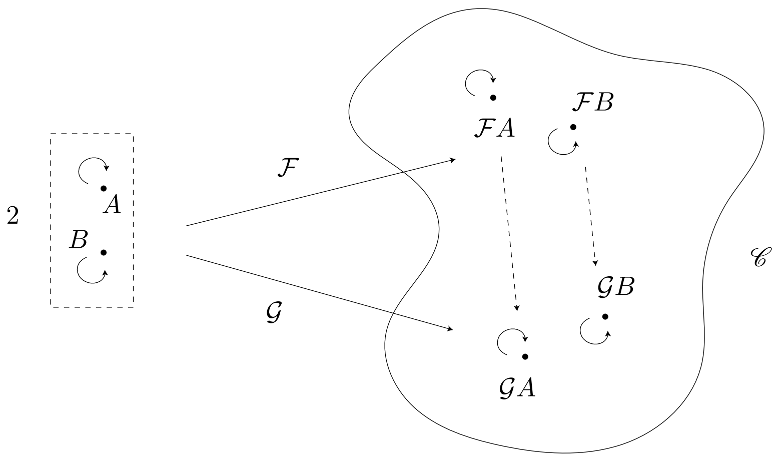


Figure 5: Two functors $\mathcal{F}$ and $\mathcal{G}$ in $[2, \mathbf{C}]$.

Although a functor is more than just an object, we can uniquely represent both functors by a pair of objects $(\mathcal{F}A, \mathcal{F}B)$ and $(\mathcal{G}A, \mathcal{G}B)$. A natural transformation between them can then be uniqely represented by a pair of morphisms in $\mathcal{C}$ that run from $\mathcal{F}A \to \mathcal{G}A$ and $\mathcal{F}B \to \mathcal{G}B$ (the dotted lines in the figure). So we've represented $[2, \mathbf{C}]$ using only 1) pairs objects in $\mathcal{C}$ and 2) pairs of morphisms in $\mathcal{C}$.

Thus structurally, this functor category is the same as $\mathbf{C} \times \mathbf{C}$, i.e. $[2, \mathbf{C}] \cong \mathbf{C} \times \mathbf{C}$. This particular case works nicely with the other notation for functor categories, i.e. $\mathbf{C}^2$.

**Definition 15.** A **natural isomorphism** between functors from $\mathbf{A}$ to $\mathbf{B}$ is an isomorphism in $[\mathbf{A}, \mathbf{B}]$.

**Proposition 5.** *Let $\mathcal{F}, \mathcal{G} : \boldsymbol{A} \to \boldsymbol{B}$ be functors, and let $\alpha : F \implies G$ be a natural transformation between them. Then $\alpha$ is a natural isomorphism if and only if $\alpha_A : \mathcal{F}A \to \mathcal{G}A$ is an isomorphism for all $A \in \boldsymbol{A}$.*

*Proof.* You should read over your proof again cause it seems a little trivial? This statement kinda is, though? $\square$