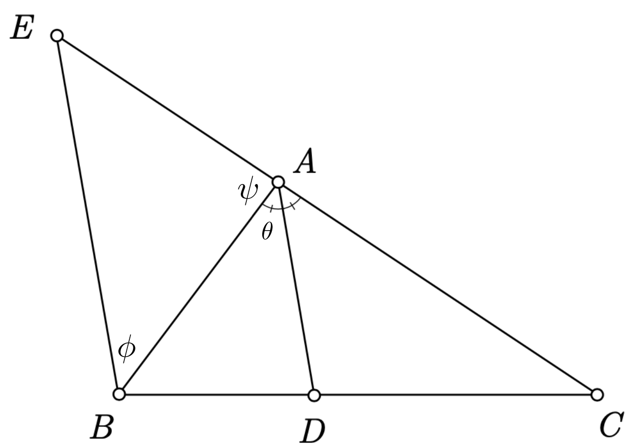
Exercise 1 (1.29)**.** Bow Tie Lemma.

Both angles subtend the same arc BC, so they are both equal to $\frac{1}{2}\angle BOC$ by the Star Trek lemma.

Exercise 2 (1.43). The Angle Bisector Theorem.



Construct the line ℓ_1 parallel to AD and going through B. Since AD and AC are not parallel, ℓ_1 intersects AC eventually. Call this intersection point E. Then by theorem 1.7.2,

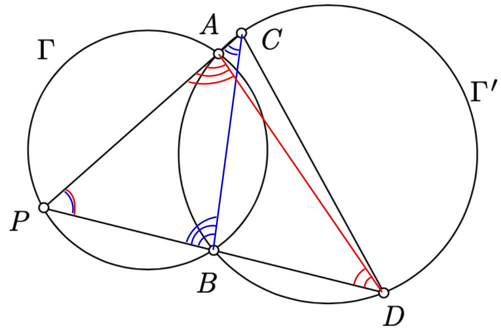
$$\frac{|EC|}{|AC|} = \frac{|DC|}{|BC|}.$$

Now note that the angles θ and ϕ_1 are opposite interior angles, so $\theta = \phi$. The angle ψ is then equal to $180\,^\circ - 2\theta = 180\,^\circ - 2\phi$, so the last unmarked angle in triangle ΔEAB is also ϕ. This means ΔEAB is an isosceles triangle, i.e. $|AE| = |AB|$.

Because of this, $|EC| = |EA| + |AC| = |AB| + |AC|$. Since $|BC| = |BD| + |DC|$, we can combine this with our earlier equivalent ratios to get

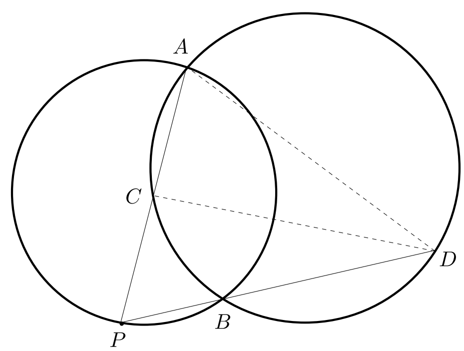
$$\frac{|AB| + |AC|}{|AC|} = \frac{|BD| + |DC|}{|DC|} \implies \frac{|AB|}{|AC|} = \frac{|BD|}{|DC|}.$$

Exercise 3 (1.49)**.** Show that $|CD|$ is independent of the choice of P.



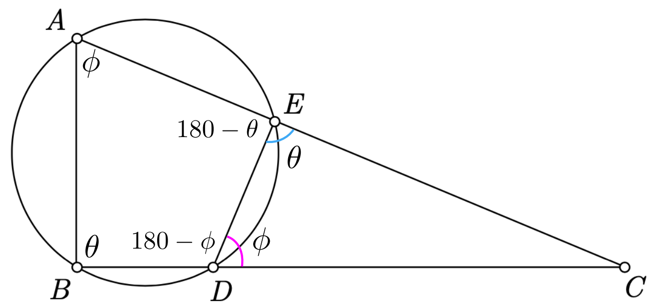
Case 1: Note that $\angle CPB = \angle APD$ must always pass through both A and B, which never change. Thus by the Star Trek lemma, $\angle CPB = \angle APD$ is constant when P is changed. Similarly, $\angle ACB = \angle ADB$ are also both constant.

Note that both pairs of angles are in triangles ΔPCB and ΔPAD, respectively, which implies that $\angle PBC = \angle PAD$ are both constant. This in turn implies $\angle CAD = \angle CBD$ are both constant. Since these both subtend the arc CD, this means $|CD|$ is constant.



Case 2: Without loss of generality, we consider just the case when C lies on AP, as the case when D lies on PB is symmetric. By a similar argument as in case 1, $\angle APD, \angle ADP$ are constant. Thus $\angle PAD = \angle CAD$, as the last angle in ΔPAD, is also constant. But this subtends the arc CD, so $|CD|$ is constant.

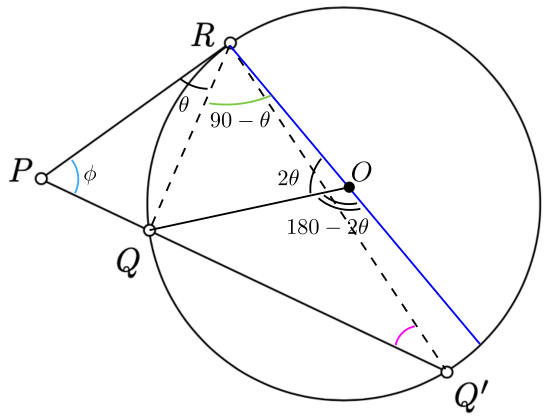
Exercise 4 (1.52)**.** What is $|DE|$?



By the Star Trek theorem, since opposing angles of $AEDB$ subtend disjoint arcs that span the whole circle, the opposing angles are supplementary. This implies that the blue angle is θ and the pink angle is ϕ. This further implies that $\Delta DEC \sim \Delta ABC$, which gives the ratio

$$\frac{|DE|}{|AB|} = \frac{|DC|}{|AC|} \implies \frac{|DE|}{5} = \frac{9}{13} \implies |DE| = \frac{45}{13}.$$

Exercise 5 (1.53)**.** Tangential version of Power of the Point.

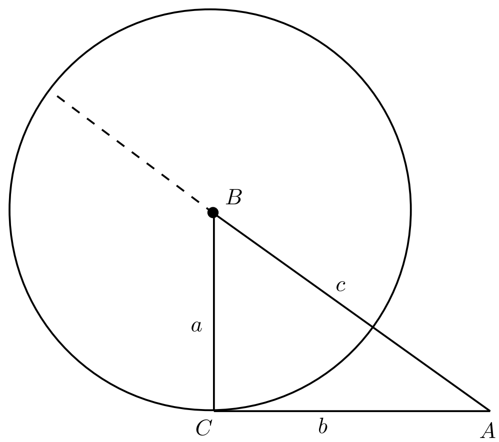


Draw the diameter through R and the origin O. Since PR is tangent to the circle, it forms a right angle with this diameter. Thus the pictured angle θ is complimentary to the green angle. Then by the Star Trek lemma, $\angle QOQ' = 180\,^{\circ} - 2\theta$. The adjacent angle to this, since it's a long a line, is then just 2θ. Applying the Star Trek lemma again, the pink angle $\angle QQ'R = \theta$.

Note that ΔPQR and $\Delta PRQ'$ already share the blue angle ϕ, so they're similar. This gives the ratio

$$\frac{|PR|}{|PQ'|} = \frac{|PQ|}{|PR|} \implies |PR|^2 = |PQ|\,|PQ'|.$$

Exercise 6 (1.54)**.** Use the tangential power of the point to prove the Pythagorean Theorem.

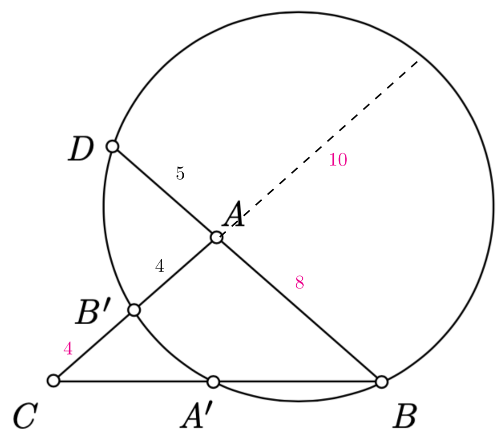


Fix the right triangle BAC, then construct the circle of radius a with center B. Since $\angle BCA = 90\,^{\circ}$, the line segement CA is tangent to the circle. Suppose the line extending AB intersects the circle as pictured, then by the tangential version of power of the point,

$$\begin{aligned} b^2 &= (c-a)(c+a) \\ b^2 &= c^2 - a^2, \end{aligned}$$

which implies $a^2 + b^2 = c^2$.

Exercise 7 (1.57)**.** Find the side lengths of the triangle ΔABC.



Since $|AB'| = 4$, B' is the midpoint of AC, and $|AC| = |AB|$, we know $|AB| = |AC| = 8$ and $|CB'| = 4$. Then by the power of the point (inside the circle), the dashed line has length d satisfying $5 \cdot 8 = 4d$, i.e. $d = 10$.

Then by the power of the point again (outside the circle), $18 \cdot 4 = |CA'|\ |CB| = 2|CA'|^2$, where the last equality follows from A' being a midpoint of CB. This implies $|CA'| = 6$, so $|CB| = 12$.

Thus the two equal sides of the triangle are length 8, and the base is length 12.