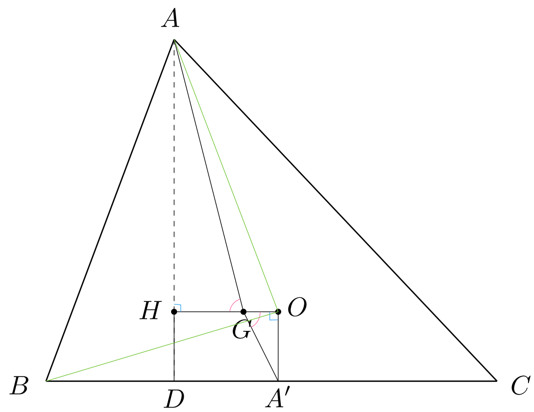
Exercise 1 (1.138)**.** Putnam problem.



By the Euler Line theorem, the centroid G lies on HO and gives the ratio

$$\frac{|OG|}{|GH|} = \frac{1}{2}.$$

Note that since the two pink angles and the two blue angles are equal, $\Delta AGH \sim \Delta A'GO$. Then we can use the above ratio, along with the given fact $|A'D| = 5$, to get

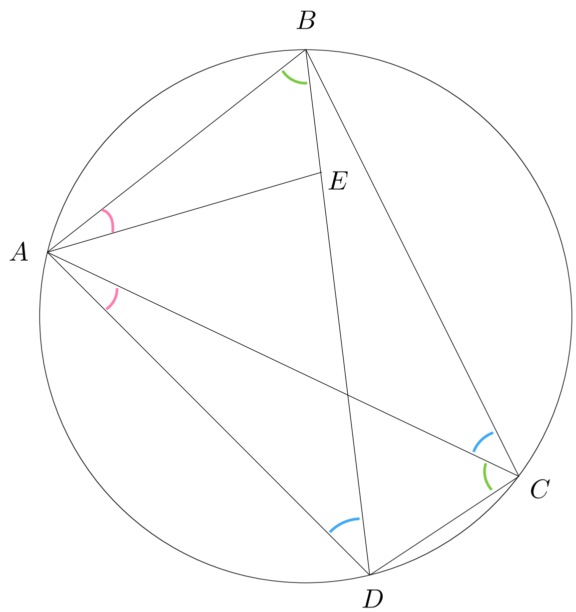
$$\frac{|A'O|}{|AH|} = \frac{|OG|}{|GH|} \implies |AH| = 10.$$

Then by the Pythagorean Theorem, $|AO|^2 = |AH|^2 + |HO|^2 = 221$. Since O is the circumcenter, $|AO| = |BO|$. Then by the Pythaogrean Theorem again,

$$\begin{aligned}
|BA'|^2 + |A'O|^2 &= |BO|^2 \\
|BA'|^2 + |A'O|^2 &= |AO|^2 \\
|BA'|^2 + 5^2 &= 221 \\
|BA'| &= 14.
\end{aligned}$$

Since A' is the midpoint of BC, this implies $|BC| = 28$.

Exercise 2 (1.151)**.** Ptolemy's Theorem.



Choose E on BD such that $\angle BAE = \angle CAD$. By the Star Trek lemma, since $\angle ABD, \angle ACD$ subtend the same arc, they're equal. Then since they have two equal angles, $\Delta ABE \sim \Delta ACD$. Thus

$$\frac{|AB|}{|AC|} = \frac{|BE|}{|CD|} \implies |AB||CD| = |AC||BE|.$$

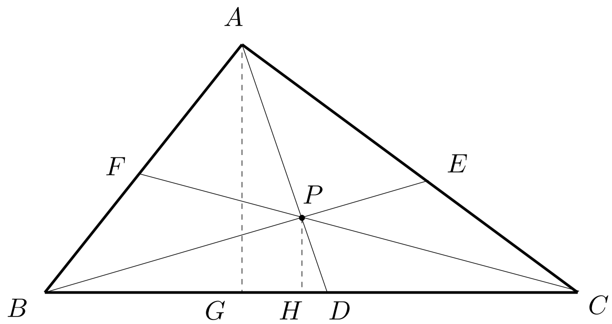
Similarly, $\Delta ABC \sim \Delta AED$, so

$$\frac{|AB|}{|AE|} = \frac{|BC|}{|ED|} \implies |BC||AD| = |AC||ED|.$$

Adding these two equalities gives

$$\begin{aligned} |AB||CD| + |BC||AD| &= |AC|(|BE| + |ED|) \\ &= |AC||DB|. \end{aligned}$$

Exercise 3 (1.165)**.** Show $\frac{|\Delta ABP|}{|\Delta APC|} = \frac{|BD|}{|DC|}$, then use this to give an alternative proof of Ceva's Theorem without using Menelaus.



First part: Draw the altitudes down from A and P as shown, then we have

$$\begin{aligned} |\Delta ABD| &= |\Delta ABP| + |\Delta BPD| \\ \frac{1}{2}|AG||BD| &= |\Delta ABP| + \frac{1}{2}|PH||BD| \\ \frac{1}{2}|BD|(|AG| - |PH|) &= |\Delta ABP|. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |\Delta ACD| &= |\Delta APC| + |\Delta CDP| \\ \frac{1}{2}|AG||DC| &= |\Delta APC| + \frac{1}{2}|PH||DC| \\ \frac{1}{2}|DC|(|AG| - |PH|) &= |\Delta APC|. \end{aligned}$$

Thus their quotient is

$$\frac{|\Delta ABP|}{|\Delta APC|} = \frac{|BD|}{|DC|}.$$

Second part: Using a similar strategy as above, we can show

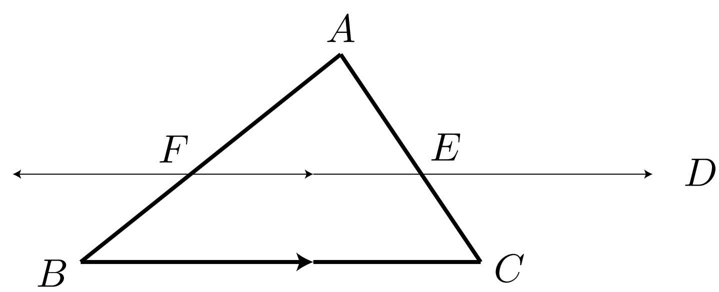
$$\frac{|\Delta APC|}{|\Delta BPC|} = \frac{|AF|}{|FB|} \quad \text{and} \quad \frac{|\Delta BPC|}{|\Delta ABP|} = \frac{|CE|}{|EA|}.$$

Then

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = \frac{|\Delta APC|}{|\Delta BPC|}\frac{|\Delta ABP|}{|\Delta APC|}\frac{|\Delta BPC|}{|\Delta ABP|} = 1.$$

The converse direction of Ceva's Theorem from the book requires no change, as it doesn't rely on Menelaus' Theorem.

Exercise 4 (1.166)**.** Analogue of Menelaus' Theorem when D is at infinity?



Note that D is at infinitely if and only if FE is parallel to BC. Also note that in this case, we have the signed ratio

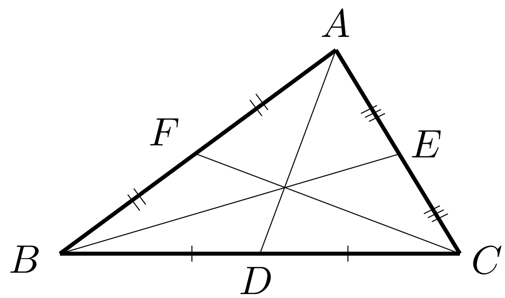
$$\frac{|BD|}{|DC|} = -1$$

since D is outside of B and C. Thus when D is at infinity, Menelaus' theorem becomes

$$FE \text{ is parallel to } BC \iff \frac{|AF|}{|FB|} = \frac{|AE|}{|EC|}.$$

This is precisely Theorem 1.64 in the text, which we've used before.

Exercise 5 (1.167)**.** Medians intersect at common point.

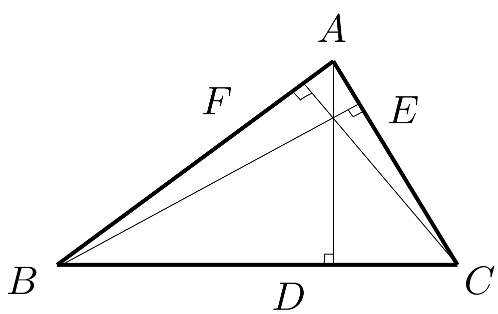


Since the medians bisect the sides of the ΔABC,

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1 \cdot 1 \cdot 1 = 1.$$

Thus by Ceva's theorem, the medians intersect at a common point.

Exercise 6 (1.168)**.** Altitudes intersect at a common point.



Since ΔBFC and ΔBDA share the angle $\angle ABC$ and since both have a right angle, $\Delta BFC \sim \Delta BDA$. Thus

$$\frac{|BD|}{|FB|} = \frac{|AB|}{|BC|}.$$

Similarly, $\Delta AFC \sim \Delta AEB$ and $\Delta CDA \sim \Delta CEB$, so

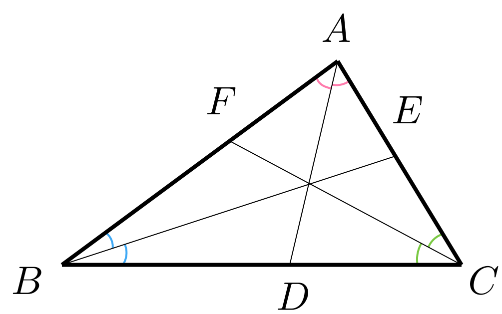
$$\frac{|AF|}{|EA|} = \frac{|AC|}{|AB|} \quad \text{and} \quad \frac{|CE|}{|DC|} = \frac{|BC|}{|AC|}.$$

Thus

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = \frac{|AC|}{|AB|}\frac{|AB|}{|BC|}\frac{|BC|}{|AC|} = 1,$$

so by Ceva's theorem, the altitudes intersect at a common point.

Exercise 7 (1.169)**.** Angle bisectors intersect at a social point.



In Homework 2 we proved the angle bisector theorem, which gives

$$\frac{|BD|}{|DC|} = \frac{|AB|}{|AC|}, \qquad \frac{|CE|}{|EA|} = \frac{|BC|}{|AB|}, \quad \text{and} \quad \frac{|AF|}{|FB|} = \frac{|AC|}{|BC|}.$$

Thus

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = \frac{|AC|}{|BC|}\frac{|AB|}{|AC|}\frac{|BC|}{|AB|} = 1,$$

so by Ceva's theorem, the angle bisectors intersect at a common point.