Exercise 1 (Lesson 6, 5 points)**.** If $f : K \to L$ is a simplicial map, show

1. $f_\#$ maps cycles to cycles.
2. $f_\#$ maps boundaries to boundaries.
3. The map

$$\begin{aligned} f_* : H_\bullet(K) &\to H_\bullet(L) \\ [x] &\mapsto [f_\#(x)] \end{aligned}$$

is a well-defined linear map.

First two parts: In class we proved that $f_\#$ respects the boundary operator in the sense that $f_\# \circ \partial_K = \partial_L \circ f_\#$, as proven in class. This implies that the following diagram commutes.

$$\begin{tikzcd} C_{p+1}^K \arrow[r, "\partial_K"] \arrow[d, "f_\#"] & C_p^K \arrow[r, "\partial_K"] \arrow[d, "f_\#"] & C_{p-1}^L \arrow[d, "f_\#"] \\ C_{p+1}^L \arrow[r, "\partial_L"] & C_p^L \arrow[r, "\partial_L"] & C_{p-1}^L \end{tikzcd}$$

We can show (1) and (2) through a diagram chase. Suppose $z \in Z_p^K$ is a cycle, then $(\partial_L \circ f_\#)(z) = (f_\# \circ \partial_K)(z) = f_\#(0) = 0$. Thus $f_\#(z)$ is also a cycle.

Now suppose $b \in B_p^K$, then there is some $a \in C_{p+1}^K$ such that $\partial_K(a) = b$. Then $(\partial_L \circ f_\#)(a) = (f_\# \circ \partial_K)(a) = f_\#(b)$, i.e. $f_\#(b)$ is a boundary.

Third part: We now show that f_* is a well-defined linear map. By (1), $f_\#(x)$ is a cycle if x is a cycle; thus it makes sense to take the equivalence class $f_*(x) = [f_\#(x)]$. Also note that since $[x] = [y] \iff x - y \in \operatorname{Im} \partial$, then (2) says that $f_*(x-y) = [f_\#(x-y)] = 0$. Then since $f_\#$ is linear,

$$f_*([x]) - f_*([y]) = [f_\#(x)] - [f_\#(y)] = [f_\#(x) - f_\#(y)] = [f_\#(x-y)] = 0.$$

Thus $f_*([x]) = f_*([y])$, so f_* is well-defined. The linearity of $f_\#$ also implies

$$f_*(\mu[x] + \lambda[y]) = f_*([\mu x + \lambda y]) = [f_\#(\mu x + \lambda y)] = [\mu f_\#(x) + \lambda f_\#(y)] = \mu f_*([x]) + \lambda f_*([y]),$$

so f_* is also linear.

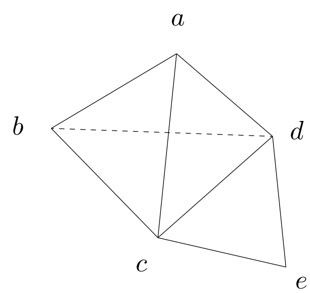
Alternative third part: Consider the following diagram,

$$\begin{tikzcd} Z_p^K \arrow[r, "f_\#"] \arrow[d, two heads, "\pi_K"] & Z_p^L \arrow[d, two heads, "\pi_L"] \\ H_p(K) \arrow[r, dashed, "\exists!\, f_*"'] & H_p(L) \end{tikzcd}$$

where π is the canonical projection. The map $f_\# : Z_p^K \to Z_p^L$ is well-defined since $f_\#$ sends cycles to cycles. Since it also maps boundaries to boundaries, $B_p^K \subseteq \operatorname{Ker}(\pi_L \circ f_\#)$. Then by the universal property of quotients, there is a unique linear map f_* making the diagram commute. By commutativity, this f_* is exactly the f_* defined in the problem.

Exercise 2 (Lesson 7, 5 points)**.** Compute β_1 and β_2 of the simplicial complex shown in the notes.

The simplicial complex is pictured below. The four faces of the empty tetradhedron are in fact filled in, but the tetrahedron itself is not filled in.



I wrote the attached code to put a matrix into Smith normal form. Since it uses only elementary row/column operations, rank is preserved, so $\text{rank}(\texttt{smith}(A)) = \text{rank}(A)$.

I start with the following matrices, where A represents ∂_2 and B represents ∂_1.

$$A = \begin{array}{c|cccc}
 & \text{abc} & \text{abd} & \text{acd} & \text{bcd} \\
\hline
\text{ab} & 1 & 1 & 0 & 0 \\
\text{ac} & 1 & 0 & 1 & 0 \\
\text{ad} & 0 & 1 & 1 & 0 \\
\text{bc} & 1 & 0 & 0 & 1 \\
\text{bd} & 0 & 1 & 0 & 1 \\
\text{cd} & 0 & 0 & 1 & 1 \\
\text{ce} & 0 & 0 & 0 & 0 \\
\text{de} & 0 & 0 & 0 & 0
\end{array}$$

$$B = \begin{array}{c|cccccccc}
 & \text{ab} & \text{ad} & \text{ac} & \text{bd} & \text{bc} & \text{cd} & \text{ce} & \text{de} \\
\hline
\text{a} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\text{b} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\text{c} & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
\text{d} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\text{e} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}$$

Their Smith normal forms were then computed, and their ranks were

$$\dim(\operatorname{Im} \partial_2) = \text{rank}(\texttt{smith}(A)) = 3,$$
$$\dim(\operatorname{Im} \partial_1) = \text{rank}(\texttt{smith}(B)) = 4.$$

Since there are 8 edges and 4 faces, $\dim C_2 = 4$ and $\dim C_1 = 8$. Then by rank-nullity,

$$\dim(\operatorname{Ker} \partial_2) = 4 - 3 = 1$$
$$\dim(\operatorname{Ker} \partial_1) = 8 - 4 = 4.$$

Finally, since our chain complex of homologies is

$$\begin{tikzcd}
0 \arrow[r] & C_2 \arrow[r] \arrow[d, "H"] & C_1 \arrow[r] \arrow[d, "H"] & C_0 \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker} \partial_2 \arrow[r] & \frac{\operatorname{Ker} \partial_1}{\operatorname{Im} \partial_2} \arrow[r] & \cdots
\end{tikzcd}$$

the 1st and 2nd Betti numbers are

$$\begin{aligned}
\beta_2 &= \dim(\operatorname{Ker} \partial_2) = 1, \\
\beta_1 &= \dim(\operatorname{Ker} \partial_1) - \dim(\operatorname{Im} p_2) = 1.
\end{aligned}$$

This makes sense visually, as there is 1 missing triangle and 1 missing tetrahedron in the given simplicial complex.