Spectral Sequences

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Math 502: Algebraic Structures II

Homology

- Chain complex:

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

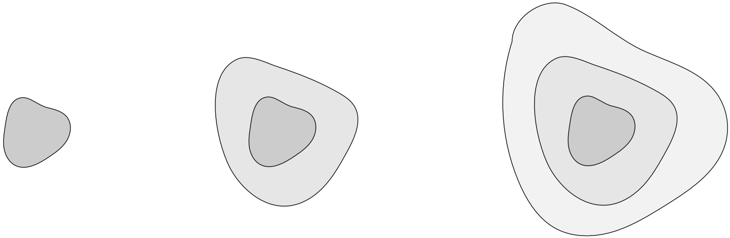
  such that $d^2 = 0$.
- n-th homology: $H_n(A) = \ker d_n / \operatorname{im} d_{n+1}$.

Preliminaries

Make this about motivating SS's instead.

Filtered Complexes

$$\cdots \subset F_{p-1}C \subset F_pC \subset F_{p+1}C \subset \cdots$$



$F_{p-1}C$ $\qquad$ F_pC $\qquad$ $F_{p+1}C$

Filtered Complexes

Induces filtration on each element of the complex.

$$\begin{array}{ccc} F_{p+1}C_{n+1} & F_{p+1}C_{n} & F_{p+1}C_{n-1} \\ \\ F_{p}C_{n+1} & F_{p}C_{n} & F_{p}C_{n-1} \\ \\ F_{p-1}C_{n+1} & F_{p-1}C_{n} & F_{p-1}C_{n-1} \end{array}$$

Filtered Complexes

Induces filtration on each element of the complex.

$$
\begin{tikzcd}
F_{p+1}C_{n+1} \arrow[r] \arrow[dr] \arrow[ddr] & F_{p+1}C_n & F_{p+1}C_{n-1} \\
F_pC_{n+1} & F_pC_n \arrow[r] \arrow[dr] & F_pC_{n-1} \\
F_{p-1}C_{n+1} & F_{p-1}C_n & F_{p-1}C_{n-1}
\end{tikzcd}
$$

Filtered complex: $d(F_pC_n) \subset F_pC_{n-1}$.

Calculating Homology

- Suppose calculating $H_*(C)$ directly is diffiult.
- We can try a "divide and conquer" strategy to make the computation easier.

Calculating Homology

Idea 1: Calculate the homology row by row, then sum them.

$$F_{p+1}C_{n+1} \longrightarrow F_{p+1}C_n \longrightarrow F_{p+1}C_{n-1}$$

$$F_pC_{n+1} \longrightarrow F_pC_n \longrightarrow F_pC_{n-1}$$

$$F_{p-1}C_{n+1} \longrightarrow F_{p-1}C_n \longrightarrow F_{p-1}C_{n-1}$$

Fails because each row is a subset of the rows above it.

Calculating Homology

Idea 2: Quotient each row by the rows below it, then calculate homology row by row and sum them.

$$
\begin{array}{ccccc}
\frac{F_{p+1}C_{n+1}}{F_pC_{n+1}} & \longrightarrow & \frac{F_{p+1}C_n}{F_pC_n} & \longrightarrow & \frac{F_{p+1}C_{n-1}}{F_pC_{n-1}} \\
\\
\frac{F_pC_{n+1}}{F_{p-1}C_{n+1}} & \longrightarrow & \frac{F_pC_n}{F_{p_1}C_n} & \longrightarrow & \frac{F_pC_{n-1}}{F_{p-1}C_{n-1}} \\
\\
\frac{F_{p-1}C_{n+1}}{F_{p-2}C_{n+1}} & \longrightarrow & \frac{F_{p-1}C_n}{F_{p-2}C_n} & \longrightarrow & \frac{F_{p-1}C_{n-1}}{F_{p-2}C_{n-1}}
\end{array}
$$

Calculating Homology

Idea 2: Quotient each row by the rows below it, then calculate homology row by row and sum them.

$$
\begin{tikzcd}
\frac{F_{p+1}C_{n+1}}{F_pC_{n+1}} \arrow[r] \arrow[dr] \arrow[ddr] & \frac{F_{p+1}C_n}{F_pC_n} & \frac{F_{p+1}C_{n-1}}{F_pC_{n-1}} \\
\frac{F_pC_{n+1}}{F_{p-1}C_{n+1}} & \frac{F_pC_n}{F_{p_1}C_n} \arrow[r] \arrow[dr] & \frac{F_pC_{n-1}}{F_{p-1}C_{n-1}} \\
\frac{F_{p-1}C_{n+1}}{F_{p-2}C_{n+1}} & \frac{F_{p-1}C_n}{F_{p-2}C_n} & \frac{F_{p-1}C_{n-1}}{F_{p-2}C_{n-1}}
\end{tikzcd}
$$

Still fails. The rows aren't subsets of each other anymore, but d still travels between rows.

Construct sequence and show intuition behind "convergence"

Convergence

Definition

A spectral sequence $\{E^r\}_{r\geq 0}$ *converges* to a graded module H if there is a filtration F on H such that

$$E^{\infty}_{n,p} \cong F_p H_n / F_{p-1} H_n.$$

Convergence

Theorem

The spectral sequence induced by a filtered complex C with bounded filtration converges to $H_*(C)$.

Since each column in our grid is finite, our earlier process eventually terminates.

Indexing Convention

Most authors use a different indexing notation. Instead of

$$E^0_{n,p} = F_p C_n / F_{p-1} C_n,$$

we could use complimentary degrees instead:

$$E^0_{p,q} = F_p C_{p+q} / F_{p-1} C_{p+q}.$$

Indexing Convention

So instead of

$$\begin{tikzcd}
C_{n+1,p} & C_{n,p} \arrow[r, "d^0"] \arrow[dr, "d^1"] \arrow[ddr, "d^2"] \arrow[dddr, "d^3"] & E_{n-1,p} \\
C_{n+1,p-1} & C_{n,p-1} & C_{n-1,p-1} \\
C_{n+1,p-2} & C_{n,p-2} & C_{n-1,p-2} \\
C_{n+1,p-3} & C_{n,p-3} & C_{n-1,p-3}
\end{tikzcd}$$

Indexing Convention

We have

$$\begin{tikzcd}
C_{p-3,q+2} & C_{p-2,q+2} & C_{p-1,q+2} & C_{p,q+2} \\
C_{p-3,q+1} & C_{p-2,q+1} & C_{p-1,q+1} & C_{p,q+1} \\
C_{p-3,q} & C_{p-2,q} & C_{p-1,q} & C_{p,q} \arrow[l, "d^1"] \arrow[ull, "d^2"'] \arrow[uulll, "d^3"'] \arrow[d, "d^0"] \\
C_{p-3,q-1} & C_{p-2,q-1} & C_{p-1,q-1} & C_{p,q-1}
\end{tikzcd}$$

Change color of the d^i.

Homological Spectral Sequences

Define Homological SS's based on this new bidegree.