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Chapter 1

Vector Bundles

1.1 VECTOR BUNDLES

Note 1. Throughout these notes, a **map** is a continuous function.

A fiber bundle is a space that looks locally like a product space. It's a generalization of a covering space. Vector bundles are just fiber bundles whose fibers are vector spaces.

Definition 1. A **fiber bundle** is a surjective map $p : E \rightarrow B$ along with a **fiber** F . For each $x \in B$, there is a neighborhood U of x such that there is a homeomorphism ϕ making the following diagram commute.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi} & U \times F \\ \downarrow p & \swarrow \pi_1 & \\ U & & \end{array}$$

We call $p^{-1}(x)$ the **fiber over x** . **Note that it must be isomorphic to F** , so it makes sense to refer to the “fibers” of a fiber bundle even if we defined it in terms of the single fiber F . Each fiber in the bundle is its own object, but they're each homeomorphic to F .

Definition 2. A **vector bundle** is a fiber bundle whose fibers are vector spaces and where ϕ is a linear isomorphism on each $p^{-1}(U)$.

If the fibers of a vector bundle are over \mathbb{R} , then we call it a **real vector bundle**. If they're over \mathbb{C} , then it's a **complex vector bundle**. In the case of real vector bundles with finite-dimensional fibers, we can specialize the definition a bit more. Since any finite-dimensional real vector space is isomorphic to \mathbb{R}^n for some n , we can equivalently write the diagram as

follows.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi} & U \times \mathbb{R}^n \\ \downarrow p & \swarrow \pi_1 & \\ U & & \end{array}$$

Once we define maps between bundles, we can use them as morphisms in the category of all fiber bundles. If we add a condition to make them respect vector space structure, then we can extend them to work with vector bundles.

Definition 3. A **bundle map** is a map $\phi : E_1 \rightarrow E_2$ that induces another map $f : B_1 \rightarrow B_2$ making the following diagram commute.

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow p_1 & & \downarrow p_2 \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

If ϕ is a linear map (isomorphism) between fibers $p_1^{-1}(x)$ and $p_2^{-1}(f(x))$, then it is a **vector bundle morphism (isomorphism)**.

Confused about what an inverse of a bundle map would be, and thus confused about how to define bundle isomorphisms.

If $B_1 = B_2 = B$, then $f = 1_B$, so we don't have to worry about f at all and defining isomorphisms becomes more straightforward. In this case, a bundle isomorphism is a homeomorphism that maps $p_1^{-1}(x) \rightarrow p_2^{-1}(x)$, and a vector bundle isomorphism additionally restricts to a linear isomorphism $p_1^{-1}(x) \rightarrow p_2^{-1}(x)$.

Definition 4. A **section** of $p : E \rightarrow B$ is a map $f : B \rightarrow E$ such that $pf = 1_B$.

The image of a section in E is homeomorphic to B via p : consider the section $f(x)$, then $fp(f(x)) = f(pf(x)) = f(x)$ by definition, so $fp = 1_{f(x)}$. **Image.**

Example 1. The **zero section** of a vector bundle maps every $x \in B$ to the 0 element in the corresponding fiber $f^{-1}(x)$.