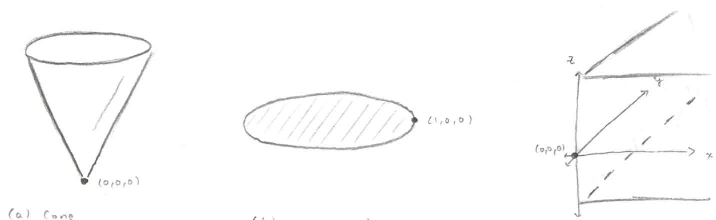
Exercise 1. §4.1: #1.



(a) Cone (b) closed disk (c) folded plane

a. The point $(0, 0, 0)$ on the cone is not contained in a proper patch since the cone is not differentiable at that point.

b. Any point along the boundary of the closed disk, for example $(1, 0, 0)$, is not contained in a proper patch, as the derivative does not exist in all directions at these points.

c. We again take a boundary point, for example $(0, 0, 0)$, as the derivative is not defined in all directions at this point.

Exercise 2. §4.1: #6.

The intersection of the monkey saddle with the xy-plane is given by

$$y^3 - 3yx^2 = y(y^2 - 3x^2) = 0.$$

This is true when either $y = 0$ or $y^2 = 3x^2$, so the intersection is composed of the lines

$$y = 0,$$
$$y = \pm\sqrt{3}x.$$

Assuming we take some point (x, y) that does not lie in the intersection of the monkey saddle with the xy-plane, $f(x, y)$ will be positive when y and $y^2 - 3x^2$ both have the same signs, and $f(x, y)$ will be negative when y and $y^2 - 3x^2$ have different signs.

Exercise 3. §4.1: #9.

To show that $\mathbf{x}$ is a proper patch, we must show that it is one-to-one, regular, and has a continuous inverse.

- **One-to-one:** If $\mathbf{x}(u,v) = \mathbf{x}(a,b)$, then we have the system

$$\begin{aligned} u + v &= a + b \\ u - v &= a - b \\ uv &= ab. \end{aligned}$$

Solving this system yields $u = a$ and $v = b$, so $\mathbf{s}$ is one-to-one.

- **Regular:** The Jacobian of $\mathbf{x}$ is

$$\begin{pmatrix} 1 & 1 & v \\ 1 & -1 & u \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} 1 & 0 & (u+v)/2 \\ 0 & 1 & (v-u)/2 \end{pmatrix}.$$

Thus the Jacobian is full rank, which implies that $\mathbf{x}$ is regular.

- **Continuous Inverse:** We can solve for u and v in the system

$$\begin{aligned} x &= u + v \\ y &= u - v \\ z &= uv \end{aligned}$$

to show that the inverse of $\mathbf{x}$ is

$$\mathbf{x}^{-1}(x,y,z) = \left(\frac{x+y}{2}, \frac{x-y}{2} \right).$$

This is clearly continuous, so $\mathbf{x}$ is proper.

Finally, since $z = uv$, we can use our expressions for $u(x,y)$ and $v(x,y)$ calculated just above to get

$$z = u(x,y)\, v(x,y) = \left(\frac{x+y}{2} \right) \left(\frac{x-y}{2} \right) = \frac{x^2 - y^2}{4},$$

so the image of $\mathbf{x}$ is the desired surface.

Exercise 4. §4.3: #1.

For a sphere with radius r, we know

$$\mathbf{x}(u,v) = (r \cos v \cos u, r \cos v \sin u, r \sin v).$$

We then have:

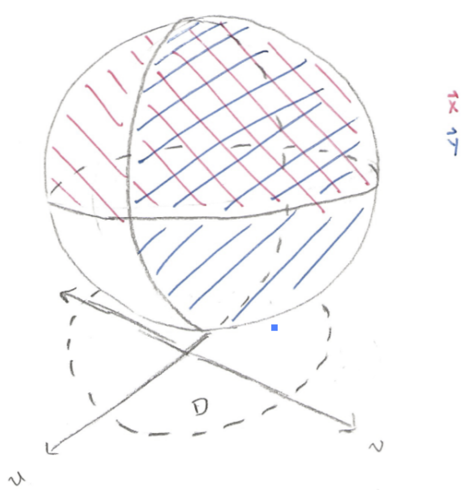
a.

$$\begin{aligned} f(\mathbf{x}(u,v)) &= r^2\cos^2 v\cos^2 u + r^2\cos^2 v\sin^2 u \\ &= r^2\cos^2 v. \end{aligned}$$

b.

$$\begin{aligned} f(\mathbf{x}(u,v)) &= [r\cos v(\cos u - \sin u)]^2 + r^2\sin^2 v \\ &= r^2\cos^2 v\left[\cos^2 u - 2\cos u\sin u + \sin^2 u\right] + r^2\sin^2 v \\ &= r^2\cos^2 v\,[1 - 2\cos u\sin u] + r^2\sin^2 v \\ &= [r^2\cos^2 v + r^2\sin^2 v] - 2r^2\cos^2 v\cos u\sin u \\ &= r^2(1 - 2\cos^2 v\cos u\sin u). \end{aligned}$$

Exercise 5. §4.3: #6.



a. In the next two parts of this problem, we use the fact that the inverses of $\mathbf{x}$ and $\mathbf{y}$ are

$$\mathbf{x}^{-1}(p_1,p_2,p_3) = (p_1,p_2), \mathbf{y}^{-1}(p_1,p_2,p_3) \qquad = (p_3,p_1).$$

b. The function $\mathbf{y}^{-1}\mathbf{x}$ is defined on

$$\{(u,v)\in\mathcal{D} \mid \mathbf{x}(u,v)\in\mathbf{y}(\mathcal{D})\} = \{(u,v)\in\mathcal{D} \mid v \geq 0\},$$

and it is given by

$$(\mathbf{y}^{-1}\mathbf{x})(u,v) = \mathbf{y}^{-1}(u,v,f(u,v)) = (f(u,v),u).$$

c. The function $\mathbf{x}^{-1}\mathbf{y}$ is defined on

$$\{(u,v) \in \mathcal{D} \mid \mathbf{y}(u,v) \in \mathbf{x}(\mathcal{D})\} = \{(u,v) \in \mathcal{D} \mid u \geq 0\},$$

and it is given by

$$(\mathbf{x}^{-1}\mathbf{y})(u,v) = \mathbf{x}^{-1}(v, f(u,v), u) = (v, f(u,v)).$$

Exercise 6. §4.3: #8.

a. In the proof of Lemma 3.6, we see that any $\alpha'(t)$ can be written

$$\alpha'(t) = \frac{d\alpha_1}{dt}\mathbf{x}_u + \frac{d\alpha_2}{dt}\mathbf{x}_v.$$

Then we have

$$\alpha'(t) = \sqrt{2}\mathbf{x}_u(\alpha(t)) + e^t\mathbf{x}_v(\alpha(t).$$

b. We manually calculate

$$\begin{aligned}\mathbf{x}_u &= (-v\sin u, v\cos u, 0)\\ \mathbf{x}_v &= (\cos u, \sin u, 1),\end{aligned}$$

which implies

$$\begin{aligned}\|\mathbf{x}_u\| &= v = e^t\\ \|\mathbf{x}_v\| &= \sqrt{2}.\end{aligned}$$

We now show that $\alpha' \cdot (\mathbf{x}_u/\|\mathbf{x}_u\|) = \alpha' \cdot (\mathbf{x}_v/\|\mathbf{x}_v\|)$. Since

$$\mathbf{x}_u \cdot \mathbf{x}_v = v\cos u \sin u - v\cos u\sin u = 0,$$

we have

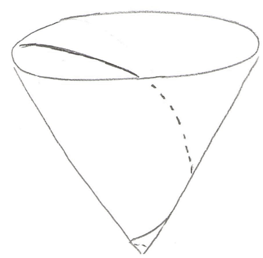
$$\alpha' \cdot \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} = \sqrt{2}\|\mathbf{x}_u\| + e^t\frac{\mathbf{x}_u \cdot \mathbf{x}_v}{\|\mathbf{x}_u\|} = \sqrt{2}\|\mathbf{x}_u\| = \sqrt{2}e^t$$

and

$$\alpha' \cdot \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|} = e^t\|\mathbf{x}_v\| = \sqrt{2}e^t.$$

Since they are equal, α' has the same angle with both $\mathbf{v}_u$ and $\mathbf{x}_v$, i.e. it bisects them.

c. A sketch of the cone and the curve α is below.



Exercise 7. §4.3: #9.

a. The Euclidean tangent plane is the collection

$$\overline{T}_{\mathbf{p}}(\mathcal{M}) = \{\mathbf{r} \mid (\mathbf{r} - \mathbf{p}) \cdot \mathbf{z} = 0\}.$$

Thus $\mathbf{v_p}$ is a tangent point of M at $\mathbf{p}$, i.e. $\mathbf{v} \cdot \mathbf{z} = 0$, if and only if $(\mathbf{v} + \mathbf{p} - \mathbf{p}) \cdot \mathbf{z} = 0$ if and only if $\mathbf{v} + \mathbf{p} \in \overline{T}_{\mathbf{p}}(\mathcal{M})$.

b. Every tangent vector at $\mathbf{x}(u,v)$ is a linear combination of $\mathbf{x}_u$ and $\mathbf{x}_v$, so $\overline{T}_{x(u,v)}(\mathcal{M})$ is spanned by $\mathbf{X}_u$ and $\mathbf{x}_v$. This means that both are orthogonal to $\mathbf{z}$, and since we're operating in only 3 dimensions, we can take their cross product to yield a vector parallel to $\mathbf{z}$. This means

$$(\mathbf{r} - (\mathbf{x}(u,v)) \cdot \mathbf{z} = 0 \iff (\mathbf{r} - \mathbf{x}(u,v)) \cdot (\mathbf{x}_u \times \mathbf{x}_v) = 0.$$

c. ∇g is normal to $\mathcal{M}$, so $(\nabla g)(\mathbf{p})$ is normal to $\mathcal{M}$ at $\mathbf{p}$, i.e. parallel to $\mathbf{z}$. Thus

$$(\mathbf{r} - \mathbf{p}) \cdot \mathbf{z} = 0 \iff (\mathbf{r} - \mathbf{p}) \cdot (\nabla g)(\mathbf{p}) = 0.$$

Exercise 8. §4.4: #2.

a. For property (1), suppose $\mathbf{v}$ is tangent to $\mathbf{p}$, then

$$\begin{aligned}(f_1\, du_1 + f_2\, du_2)(\mathbf{v}) &= \phi(U_1(\mathbf{p}))du_1(\mathbf{v}) + \phi(U_2(\mathbf{p}))du_2(\mathbf{v}) \\ &= \phi(U_1(\mathbf{p})v_1 + U_2(\mathbf{p})v_2) \\ &= \phi(\mathbf{v}).\end{aligned}$$

For property (2), suppose $\mathbf{w}$ is also tangent to $\mathbf{p}$, then

$$\begin{aligned}(g\, du_1\, du_2)(\mathbf{v}, \mathbf{w}) &= \eta(U_1(\mathbf{p}), U_2(\mathbf{p}))(v_1 w_2 - w_1 v_2) \\ &= \eta(U_1(\mathbf{p}), U_2(\mathbf{p}))(v_1 w_2 - v_2 w_1).\end{aligned}$$

Then by Lemma 4.2, this becomes

$$\begin{aligned} &= \eta(v_1 U_1(\mathbf{p}) + v_2 U_2(\mathbf{p}), w_1 U_1(\mathbf{p}) + w_2 U_2(\mathbf{p})) \\ &= \eta(\mathbf{v}, \mathbf{w}). \end{aligned}$$

b. For property (3), we have

$$\begin{aligned} \phi \wedge \psi &= (f_1\ du_1 + f_2\ du_2) \wedge (g_1\ du_1 + g_2\ du_2) \\ &= f_1 g_1\ du_1\ du_1 + f_1 g_2\ du_1\ du_2 + f_2 g_1\ du_2\ du_1 + f_2 g_2\ du_2\ du_2 \\ &= (f_1 g_2 - f_2 g_1) du_1\ du_2. \end{aligned}$$

For property (4), we have

$$\begin{aligned} df(\mathbf{v}) = \mathbf{v}[f] &= (\sum v_i U_i(\mathbf{p}))[f] \\ &= \sum v_i (U_i(\mathbf{p})[f]) \\ &= \sum v_i \frac{\partial f}{\partial u_i}(\mathbf{p}) \\ &= \sum du_i(\mathbf{v}) \frac{\partial f}{\partial u_i}(\mathbf{p}). \end{aligned}$$

For property (5), we can use property (4) to get

$$\begin{aligned} d\phi &= df_1 \wedge du_1 + df_2 \wedge du_2 \\ &= \frac{\partial f_1}{\partial u_1} du_1\ du_1 + \frac{\partial f_1}{\partial u_2} du_2\ du_1 + \frac{\partial f_2}{\partial u_1} du_1\ du_2 + \frac{\partial f_2}{\partial u_2} du_2\ du_2 \\ &= \left(\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2} \right) du_1\ du_2. \end{aligned}$$

Exercise 9. §4.4: #4.

a.

$$\begin{aligned} d(fgh) &= (df)(gh) + f\ d(gh) \\ &= gh\ df + f\,[(dg)h + g\ dh] \\ &= gh\ df + fh\ dg + fg\ dh. \end{aligned}$$

b.

$$\begin{aligned} d(\phi f) = d\,(f\phi) &= f\ d\phi + df \wedge \phi \\ &= f\ d\phi - \phi \wedge df. \end{aligned}$$

c.

$$\begin{aligned} (df \wedge dg)(\mathbf{v}, \mathbf{w}) &= df(\mathbf{v}) dg(\mathbf{w}) - df(\mathbf{w}) dg(\mathbf{v}) \\ &= \mathbf{v}[f]\mathbf{w}[g] - \mathbf{v}[g]\mathbf{w}[f]. \end{aligned}$$

Exercise 10. §4.4: #7.

a. By definition,

$$(u, v) = \mathbf{x}^{-1}(\mathbf{x}(u, v)) = (\tilde{u}(\mathbf{x}(u, v)), \tilde{v}(\mathbf{x}(u, v))).$$

b. We have

$$d\tilde{u}(\mathbf{x}_u) = \mathbf{x}_u[\tilde{u}] = \frac{\partial(\tilde{u}(\mathbf{x}))}{\partial u} = \frac{\partial u}{\partial u} = 1.$$

Similarly,

$$\begin{aligned} d\tilde{u}(\mathbf{x}_v) &= \frac{\partial u}{\partial v} = 0, \\ d\tilde{v}(\mathbf{x}_u) &= \frac{\partial v}{\partial u} = 0, \\ d\tilde{v}(\mathbf{x}_v) &= \frac{\partial v}{\partial v} = 1. \end{aligned}$$

c. Let $\mathbf{v}$ be a tangent vector of $\mathbf{x}(u, v)$, then it can be written $\mathbf{v} = a\mathbf{x}_u + b\mathbf{x}_v$ for some a, b. Then using linearity and the properties from part (b), we have

$$\begin{aligned} (f\ d\tilde{u} + g\ d\tilde{v})(\mathbf{v}) &= (f\ d\tilde{u} + g\ d\tilde{v})(a\mathbf{x}_u + b\mathbf{x}_v) \\ &= a\phi(\mathbf{x}_u) + b\phi(\mathbf{x}_v) \\ &= \phi(a\mathbf{x}_u + b\mathbf{x}_v) \\ &= \phi(\mathbf{v}). \end{aligned}$$

Now suppose $\mathbf{w}$ is another tangent vector of $\mathbf{x}(u, v)$ given by $\mathbf{w} = c\mathbf{x}_u + d\mathbf{x}_v$, then by the definition of the wedge product,

$$\begin{aligned} (h\ d\tilde{u}\ d\tilde{v})(\mathbf{v}, \mathbf{w}) &= (h\ d\tilde{u}\ d\tilde{v})(a\mathbf{x}_u + b\mathbf{x}_v, c\mathbf{x}_u + d\mathbf{x}_v) \\ &= \eta(\mathbf{x}_u, \mathbf{x}_v)(ad - bc). \end{aligned}$$

Then by Lemma 4.2, this becomes

$$\begin{aligned} &= \eta(a\mathbf{x}_u + b\mathbf{x}_v, c\mathbf{x}_u + d\mathbf{x}_v) \\ &= \eta(\mathbf{v}, \mathbf{w}). \end{aligned}$$

Exercise 11. §4.5: #1.

By Theorem 3.2, $F : \mathbb{R}^3 \to N$ is differentiable. Then $F|M$ is a differentiable function from M to N. Since $\mathbf{y}^{-1}(F|M)\mathbf{x}$ is then the composition of differentiable functions, it is itself differentiable, so $F|M$ is a mapping of surfaces.

Exercise 12. §4.5: #2.

I took "meridian" to mean lines extending from pole to pole over only one half of the sphere, and I took "parallel" to mean lines wrapping around the entire sphere horizontally.

a. The meridians are moved 180 degrees around the sphere, and the parallels are reflected over the xy-plane.

b. The meridians are made horizontal, then revolved around the x-axis. The parallels rotate around the center of the earth, with motion parallel to the x-axis.

c. The parallels are reflected over the xy-plane.