Percolation Phase Transitions on Dynamically Grown Graphs

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Background

Dynamically grown graphs and percolation

Dynamically Grown Graphs

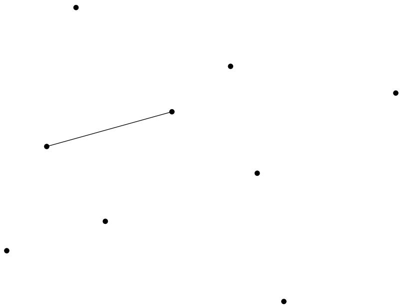
Start with a graph with n vertices and 0 edges

Add edges randomly every $1/n$ units of time

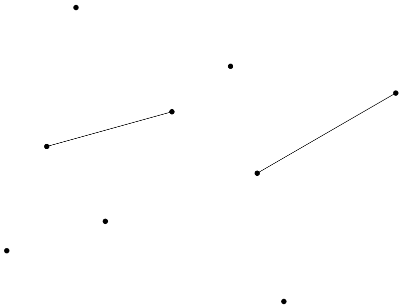
We'll work in the limit as $n \to \infty$

Dynamically Grown Graphs

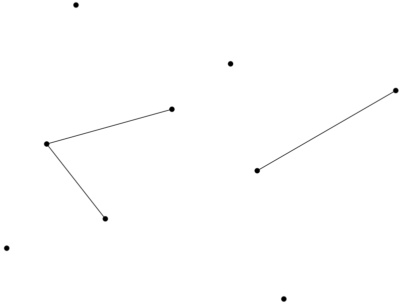
Dynamically Grown Graphs



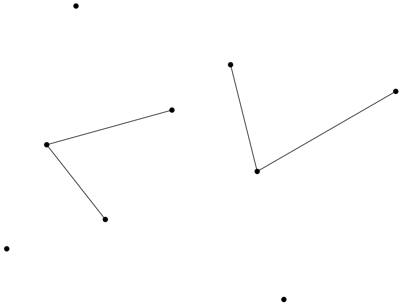
Dynamically Grown Graphs



Dynamically Grown Graphs



Dynamically Grown Graphs



Percolation

A *giant component* is a cluster that takes up a finite fraction of the graph

Percolation is when a giant component first emerges (call this time t_c)

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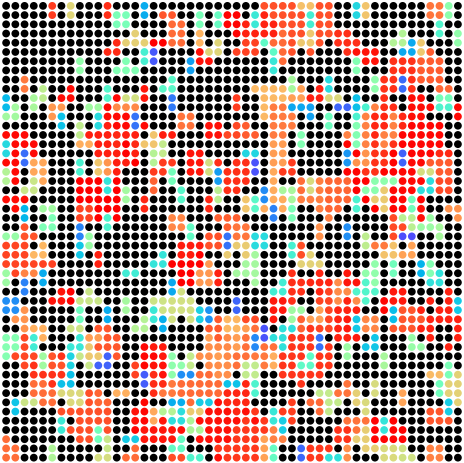
This emergence has lots of different behaviors

Erdős Rényi



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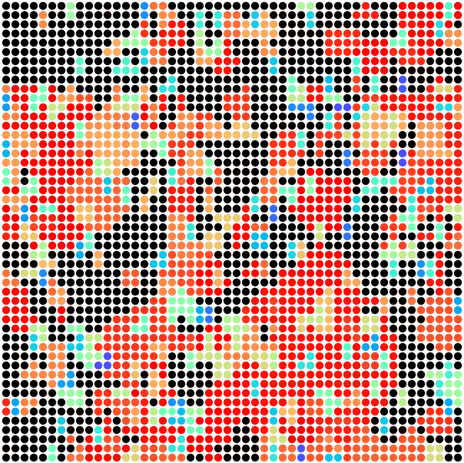

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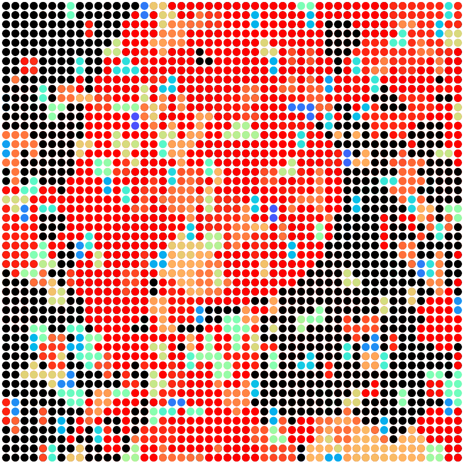
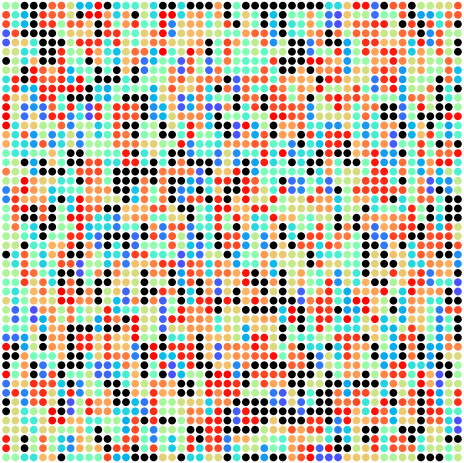


da Costa



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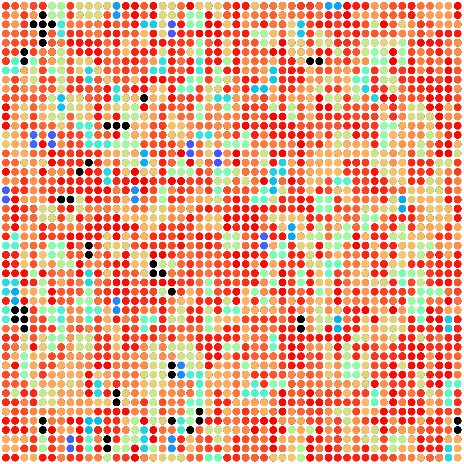

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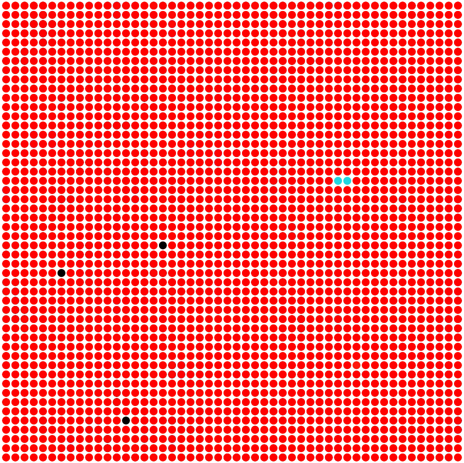

Explosive Percolation

Explosive Percolation is a sudden, seemingly discontinuous emergence of the giant component

Basic Results

Continuous phase transition and scaling behavior

Continuous phase transition

Define Achlioptas rule

Achlioptas claimed to have found a discontinuous emergence of a giant component based on simulations **When?**

Continuous phase transition

Riordan and Warnke (2012)

ℓ-vertex rule: choose ℓ vertices i.i.d., and you're only required to add an edge if all ℓ of them are in distinct clusters (generalizes Achlioptas processes)

All ℓ-vertex rules have a continuous phase transition

Continuous phase transition

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Proof by contradiction...

Scaling behavior

reference da Costa? who tf showed this?

The distribution of vertices belonging to a cluster of size s follows a power law

$$s^{1-\tau} f(s\delta^{1/\sigma})$$

where $\delta = t - t_c$ and f is a scaling function.

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Noticing scaling behavior in rules with explosive percolation was motivation for proving their continuity

Critical Exponents

Hi

Two-Choice Rules

Our Results

Two-Choice Rules

Pick two finite groups of i.i.d. vertices

Follow a deterministic method to choose a representative vertex from each group (can be a different rule for each group)

Add an edge between the two representatives

Two-Choice Rules

Erdős Rényi: Both groups are size 1, so this is the same as sampling edges randomly

Correspondence with Erdős Rényi random graph

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Correspondence with Erdős Rényi random graph

da Costa: both groups are of size m, and pick the vertex with the smallest cluster size from each group

Scaling Relations

Consider

$$1 - \langle 1 \rangle_\phi = 1 - \sum_{s \text{ finite}} \phi(s),$$

where $\phi(s) = \mathbb{P}(\text{representative has cluster size } s)$

Under certain regularity conditions, this quantity is $\sim \delta^{F(\beta)}$ **need to define $\sim$**

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Can express critical exponents in terms of β and $F(\beta)$

Scaling Relations

Put scaling relations for 2C rules here

Scaling Relations

put example for da Costa and then say that it agress with what he derived

Erdős Rényi

Erdős Rényi scaling relations

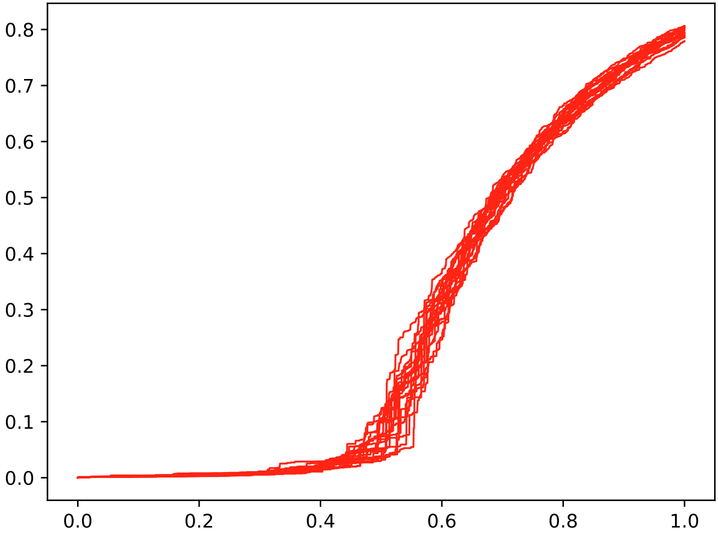
Erdős Rényi

Much more is possible since Erdős Rényi is such a simple rule

$\beta = 1$, i.e. giant component grows linearly near t_c

other crit exps

Erdős Rényi $(n = 2500)$



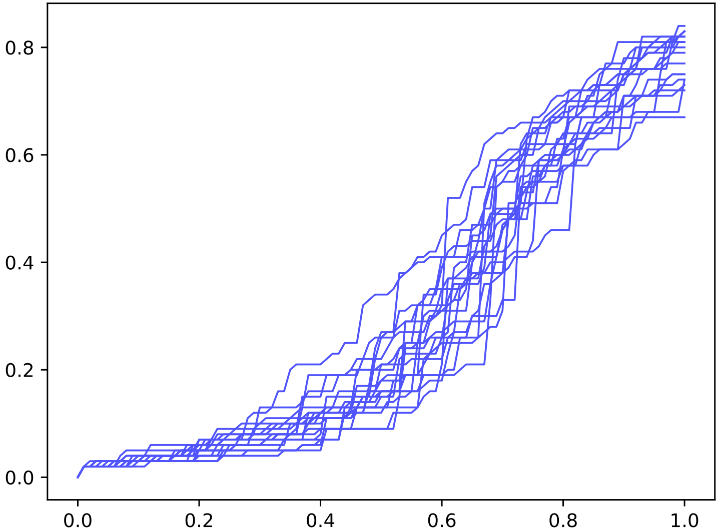
Erdős Rényi

Scaling behavior occurs in region of order $\Theta(s^{-1/2})$ around t_c

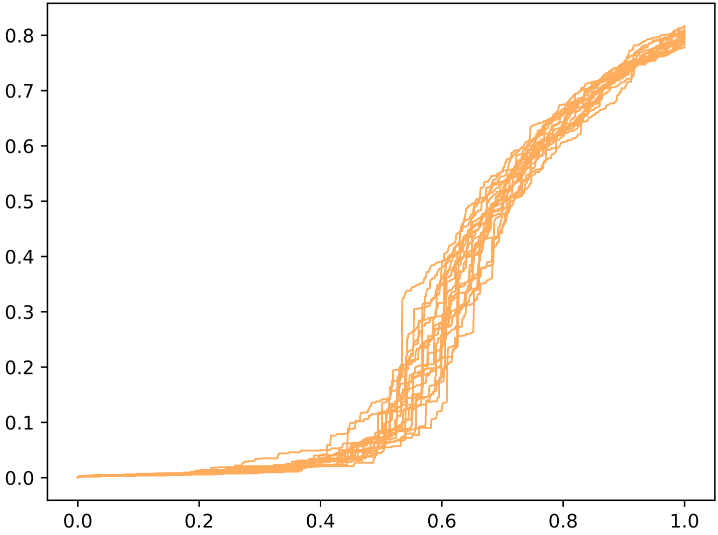
As $s \to \infty$, the scaling window shrinks

In particular, the region of linear giant component growth shrinks as $n \to \infty$

Erdős Rényi $(n = 100)$



Erdős Rényi $(n = 1000)$



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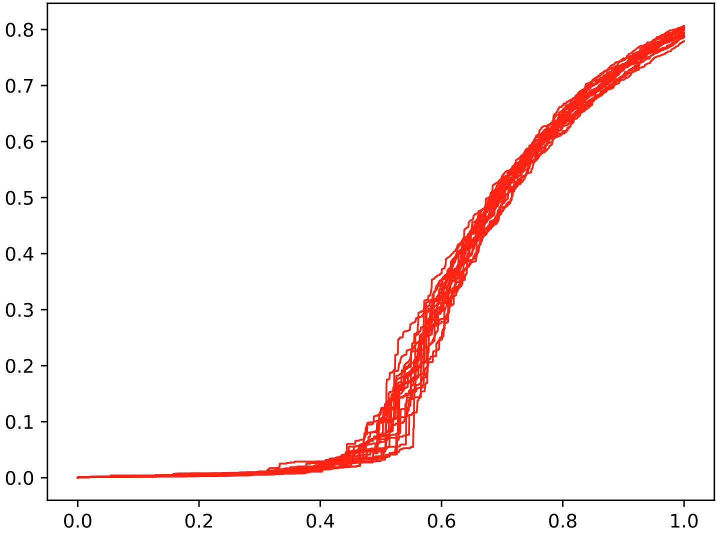

Bounded Size Rules

Rules that are almost Erdős Rényi

Bounded Size Rules

A *bounded size rule* with size threshold K treats all clusters of size $> K$ the same

Intuition: eventually there will be so few clusters of size $\leq K$ that it starts acting like Erdős Rényi

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Can have noticeable differences in behavior vs Erdős Rényi!

Bohman-Frieze

define and then show nonzero proportion of vertices are isolated (below size threshold) at percolation

Scaling Relations

Can still calculate scaling relations in terms of β

put them here

Same as Erdős Rényi (with potentially different β)

Need to finish up the presentation somehow