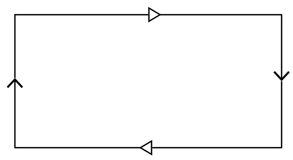
Exercise 1. What happens if you join two Möbius strips together along their circle boundaries? Show this in two ways:

1. By using Theorem 4.10.
2. By cutting and gluing squares with arrows on their edges.

1. Since the Klein bottle K can be represented by



we can calculate its Euler characteristic as $\chi(K) = 0$ (1 vertex, 2 edges, 1 face).

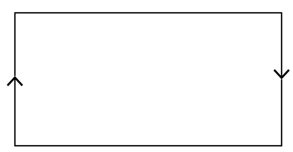
Note that in order for a Möbius strip to be a manifold, it cannot have a boundary. Thus by "Möbius strip" we mean the following diagram with open boundary.

Once we glue two strips together, though, this open bit goes away and we're left with a compact surface.

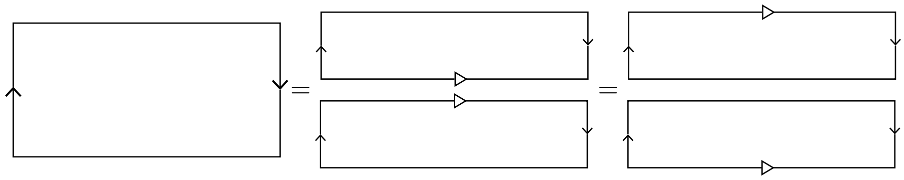
Using this image, we can calculate that the Euler characteristic of a Möbius strip is 0 (2 vertices, 3 edges, 1 face). Denote the glued-together Möbius strips by M, then the rectangular decomposition of M into Möbius strips shows that $\chi(M) = 0$.

Both K and M are connected, compact, and non-orientable. Then since both have Euler characteristic 0, they are diffeomorphic by Theorem 4.10. Thus the Klein bottle is just two Möbius strips glued together along their boundaries.

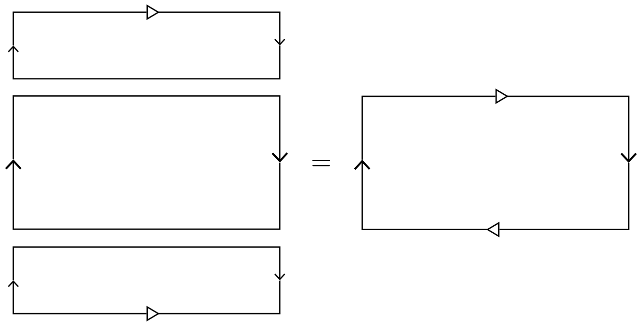
2. Given a Möbius strip



it's not visually clear how to glue another Möbius strip to this. But we can cut the strip as follows.



Then it's clear how to glue this to another (whole) Möbius strip.



This is exactly a Klein bottle.