STABLE DISTRIBUTIONS

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CONTENTS

1 STABLE DISTRIBUTIONS

1.1 DEFINITIONS

There are three main equivalent definitions of stable distributions.

Definition 1. A random variable X is **stable** if any linear combination of independent copies of X preserves the distribution of X up to scaling and shifting, i.e. for any a, b,

$$aX_1 + bX_2 \stackrel{d}{=} cX + d$$

for some c, d. If $d = 0$ for all a and b, then X is **strictly stable**. If X is stable and $X \stackrel{d}{=} -X$, then it is **symmetric stable**.

Note 1. This is **not** the same as saying "the sum of two Gaussians is itself a Gaussian." Instead, if we take two copies of the *same* Gaussian and take a linear combination of those, it's just a scale and/or shift of that *original* Gaussian.

Definition 2. X is **stable** if

$$X_1 + \cdots + X_n = c_n X + d_n$$

finish...

We're interested in these because by the **generalized central limit theorem**, stable distributions are the only limit distributions for properly normalized and centered sums of i.i.d. random variables.

1.2 EXPLICIT DISTRIBUTIONS

There are only three families of stable distributions with known closed form densities:

- Gaussian: $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$
- Cauchy (Lorentz): $\frac{1}{\pi} \frac{\gamma}{\gamma^2+(x-\delta)^2}$
- Lévy: $\sqrt{\frac{\gamma}{2\pi}} \frac{1}{(x-\delta)^{3/2}} \exp\left(-\frac{\gamma}{2(x-\delta)}\right)$ for $x > \delta$

summarize important properties of each
add in plots

1.3 PARAMETERIZATIONS

Proposition 1. X is stable $\iff X \stackrel{d}{=} \gamma Z + \delta$, where $\gamma \geq 0$ and δ are real numbers and Z is a random variable with **characteristic function**

$$\phi(u) = \mathbb{E}\left(e^{iuZ}\right) = \begin{cases} \exp\left(-|u|^{\alpha}\left(1 - i\beta\tan(\frac{\pi\alpha}{2}\right) \cdot \operatorname{sign}(u)\right) & \alpha \neq 1, \\ \exp\left(-|u|\left(1 + \frac{2}{\pi} i\beta \cdot \operatorname{sign}(u) \cdot \log|u|\right)\right) & \alpha = 1 \end{cases}$$

for $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$.

Thus we can parameterize a stable distribution with four parameters:

- α: index of stability / characteristic exponent
- β: skewness
- γ: scale
- δ: location

Different parameterizations

1.3.1 QUALITATIVE BEHAVIOR

α controls how fat the tails are.

- $\alpha = 2$: Gaussian
  - In this case, β becomes inconsequential since $\tan(\pi\alpha/2) = 0$, i.e. $Z(2, \beta) \stackrel{d}{=} Z(2, 0)$
- As α increases, it becomes more Gaussian (and changing β has less of an effect)
- As α decreases, you get a taller, thinner peak and fatter tails

β controls how one-sided a distribution is. If $\beta = \pm 1$, then the distribution is **totally skewed to the right (+1) / left (-1)**.

- $\beta = 0$: Symmetric
- $\beta > 0$: Right tail is fatter than left
- $\beta < 0$: Left tail is fatter than right

when totally skewed, can eliminate one of the tails... (lemma 1.1 in text)

plots for different β

Every stable distribution is unimodal, but there's no known formula for the mode; however, $m(\alpha, \beta) := \mathbb{S}(\alpha, \beta; 0)$ can be numerically computed.

1.4 TAILS

1.5 MOMENTS

1.6 QUANTILES

If $\beta \neq 0$, then the quantiles are not symmetric (unlike with Gaussians). In general, they depend on α and β.

Let $z_\lambda(\alpha, \beta)$ be the unique real number associated with the standardized (i.e. $\gamma = 1, \delta = 0$) distribution $Z \sim \mathbb{S}(\alpha, \beta; 0)$ such that $\mathbb{P}\left(X < z_\lambda\right) = \lambda$. By the reflection property, $z_\lambda(\alpha, \beta) = -z_{1-\lambda}(\alpha, -\beta)$.

Quantiles scale differently depending on the parameterization. For the 0-th parameterization, it's simply $\gamma z_\lambda + \delta$. For other parameterizations, it's probably easiest to just convert the parameters to their 0-th parameterization equivalents and then do the same thing.

1.7 SUMS OF α-STABLE DISTRIBUTIONS

Theorem 1. $X + Y$ is α-stable $\iff$ X and Y are both α-stable themselves.

reference for this in text is pretty late in the book

If the distributions in the sum are independent, there's an exact form for the sum. If not, it's more complicated and depends on the dependence structure.

Proposition 2. This is all for the 0-parameterization.

1. If X Im $\mathbb{S}(\alpha, \beta, \gamma, \delta; 0)$, then $aX + b \ \sim \ \mathbb{S}(\alpha, \text{sign}(a)\beta, |\alpha|\gamma, a\delta + b; 0)$.
2. The characteristic functions, densities, and CDFs are **jointly continuous**in all parameters and in **x**.
3. If $X_i \sim \mathbb{S}(\alpha, \beta_i, \gamma_i, \delta_i; 0)$ and $X_1 \perp X_2$, then $X_1 + X_2 \sim \mathbb{S}(\alpha, \beta, \gamma, \delta; 0)$, where

$$\beta = \frac{\beta_1\gamma_1^\alpha + \beta_2\gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha},$$
$$\gamma^\alpha = \gamma_1^\alpha + \gamma_2^\alpha,$$
$$\delta = \begin{cases} \delta_1 + \delta_2 + \tan\left(\frac{\pi\alpha}{2}\right)(\beta\gamma - \beta_1\gamma_1 - b_2\gamma_2) & \alpha \neq 1, \\ \delta_1 + \delta_2 + \frac{2}{\pi}(\beta\gamma\log\gamma - \beta_1\gamma_1\log\gamma_1 - \beta_2\gamma_2\log\gamma_2) & \alpha = 1. \end{cases}$$