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# Chapter 1

## Topological Spaces

### 1.1 TOPOLOGICAL SPACES

**Definition 1.** Let  $X$  be a set, then a **topology** on  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  such that

1.  $\emptyset, X \in \mathcal{T}$ ,
2.  $\bigcup_{\alpha \in \mathcal{G}} U_{\alpha} \in \mathcal{T}$ , and
3.  $\bigcap_{i=1}^N U_i \in \mathcal{T}$ .

Elements of a topology are called **open sets**.

**Example 1.** 1. “Indiscrete” topology:  $\mathcal{T}_i = \{\emptyset, X\}$

2. “Discrete” topology:  $\mathcal{T}_d = \{\text{all subsets of } X\}$

**Definition 2.** Let  $\mathcal{T}, \mathcal{T}'$  be topologies on a set  $X$ , then  $\mathcal{T}$  is **finer** than  $\mathcal{T}'$  if  $\mathcal{T}' \subset \mathcal{T}$ .  $\mathcal{T}$  is **coarser** than  $\mathcal{T}'$  if  $\mathcal{T} \subset \mathcal{T}'$ . The notions of **strictly finer** and **strictly coarser** follow.

From this we see that “fine” is a notion of a large topology, and “coarse” is a notion of a small topology.

**Example 2.** The **lower limit topology** on  $\mathbb{R}$  is given by the basis

$$\mathcal{B} = \{[a, b) \mid a < b\}.$$

It is strictly finer than the standard topology on  $\mathbb{R}$ : since  $\bigcup_{n \in \mathbb{N}} [a + 1/n, b) = (a, b)$ , it contains the standard topology, but  $[a, b)$  is not open in the standard topology, so it is strictly finer.

**Example 3.** Let  $X$  be any set, then the **finite complement topology** is defined

$$\mathcal{T}_f = \{U \subset X \mid X - U \text{ is finite}\} \cup \{\emptyset\},$$

where  $X - U$  denotes the complement of  $U$  in  $X$ , i.e.  $X \setminus U$ . Checking that this is a topology boils down to just using DeMorgan's Laws.

## 1.2 CLOSED SETS AND LIMIT POINTS

**Definition 3.** A set  $A \subset (X, \mathcal{T})$  is closed if  $X - A$  is open in  $X$ .

**Theorem 1.** Let  $(X, \mathcal{T})$  be a topological space, and let  $F$  denote a closed set of  $X$ , then

1.  $\emptyset$  and  $X$  are closed,
2.  $\bigcap_{\alpha \in \mathcal{G}} F_\alpha$  is closed, and
3.  $\bigcup_{i=1}^N F_i$  is closed.

*Proof.* This is a straightforward application of DeMorgan's Laws. □

Properties of a closed set  $A$  in a subspace  $Y$  of  $X$ :

- $A$  is the intersection of  $Y$  and a closed set in  $X$ .
- If  $Y$  is closed in  $X$ , then  $A$  is closed in  $X$ .

**Proposition 1.** Let  $Y$  be a subspace of  $X$ . Then  $A$  is closed in  $Y$  if and only if it is equal to the intersection of a closed set of  $X$  with  $Y$ .

**Proposition 2.** Let  $Y$  be a subspace of  $X$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$ , then  $A$  is closed in  $X$ .

**Definition 4.** The **interior** of a set  $A$ , denoted  $A^\circ$ , is the union of all open sets contained in  $A$ .

The **closure** of a set  $A$ , denoted  $\overline{A}$  is the intersection of all closed sets containing  $A$ .

The closure of a set is clearly closed, and the interior of a set is clearly open. It is also clear that if  $A$  is open, then  $A^\circ = A$ , and if  $A$  is closed, then  $\overline{A} = A$ . We also have the obvious relation  $A^\circ \subset A \subset \overline{A}$ .

We have to be careful when describing closures. Given a subspace  $Y$  of  $X$ , the closure of  $A$  in  $X$  is generally not the same as the closure of  $A$  in  $Y$ . In this case, we use  $\overline{A}$  to denote the closure of  $A$  in  $X$  (the overall space). We relate this to the closure of  $A$  in  $Y$  (the subspace) with the following proposition.

**Proposition 3.** Let  $Y$  be a subspace of  $X$ , and let  $A \subset Y$ . Then  $\overline{A}_Y = \overline{A}_X \cap Y$ .

**Definition 5.** A **neighborhood** of a point  $x$  is an open set containing  $x$ .

**Theorem 2.** Let  $A$  be a subset of a topological space  $X$ , then

1.  $x \in \overline{A}$  if and only if every neighborhood of  $x$  intersects  $A$ , and
2. Supposing the topology of  $X$  is given by a basis, then  $x \in \overline{A}$  if and only if every basis element  $B$  containing  $x$  intersects  $A$ .

Make sure you have an intuitive understanding of why this is true.

**Definition 6.** Let  $A \subset (X, \mathcal{T})$ , then  $x \in X$  is a **limit point** of  $A$  if every open neighborhood of  $x$  intersects  $A$  at some point *other than*  $x$ .

Equivalently,  $x$  belongs to the closure of  $A - \{x\}$ . Note that  $x$  need not lie in  $A$ .

Think about this.

**Theorem 3.** Let  $A \subset (X, \mathcal{T})$ , and denote the set of limit points of  $A$  by  $A'$ . Then  $\overline{A} = A \cup A'$ .

**Corollary 1.** A subset of a topological space is closed if and only if it contains all its limit points.

*Proof.* Let  $A \subset (X, \mathcal{T})$ . Then  $A$  is closed if and only if  $A = \overline{A} = A \cup A'$ , and  $A = A \cup A'$  if and only if  $A' \subset A$ . □

### 1.3 BASES

**Definition 7.** Let  $\mathcal{T}$  be a topology on  $X$ , and let  $\mathcal{B} \subset \mathcal{T}$ . Then  $\mathcal{B}$  is a **basis** for  $\mathcal{T}$  if every open set of  $\mathcal{T}$  can be written as the union of elements of  $\mathcal{B}$ .

**Proposition 4.** Let  $\mathcal{T}$  be a topology on  $X$ , and let  $\mathcal{B}$  be a collection of subsets of  $X$ . Then  $\mathcal{B}$  is a basis for  $\mathcal{T}$  if and only if

1.  $\mathcal{B} \subset \mathcal{T}$ ; and
2. for each  $U \in \mathcal{T}$  and  $p \in U$ , there is a  $B \in \mathcal{B}$  such that  $p \in B \subset U$ .

*Proof.* The forward direction follows from every open set of  $\mathcal{T}$  being the union of elements of  $\mathcal{B}$ . For the backward direction, since  $p \in B_p \subset U$  for all  $p \in U$ , we have  $U = \bigcup_{p \in U} B_p$ , so every open set of  $\mathcal{T}$  is the union of elements of  $\mathcal{B}$ .  $\square$

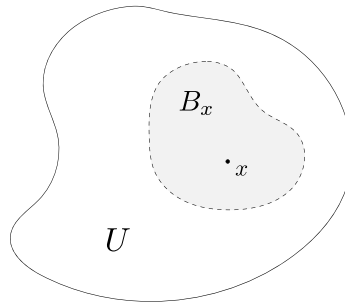


Figure 1.1: For any  $U \in \mathcal{T}$ , each  $x \in U$  lies in some  $B_x \in \mathcal{B}$  for  $B_x \subset U$ .

Not every set of subsets of  $X$  will generate a topology, so we need conditions for a collection  $\mathcal{B}$  to be a basis for *any* topology.

**Proposition 5.** Let  $\mathcal{B}$  be a collection of subsets of  $X$ . Then  $\mathcal{B}$  generates a topology if and only if

1.  $\bigcup_{B \in \mathcal{B}} B = X$ .
2. given  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there is a  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subset B_1 \cap B_2$ .

*Proof. Forward:* (1)  $X$  must be open, so  $X$  is the union of the elements of  $\mathcal{B}$ . (2) Since  $B_1$  and  $B_2$  are both open in the topology generated by  $\mathcal{B}$ , their intersection is, as well. Then since  $\mathcal{B}$  is a basis for this topology, we can find a satisfactory  $B_3$ .

**Backward:** The topology generated by a set  $\mathcal{B}$  is the collection of all unions of elements of  $\mathcal{B}$ . It is clear that  $\emptyset$  is in it, and condition (1) implies that  $X$  is, as well. Arbitrary unions are in the topology by definition. Induction on condition (2) shows that the topology also contains finite intersections.  $\square$

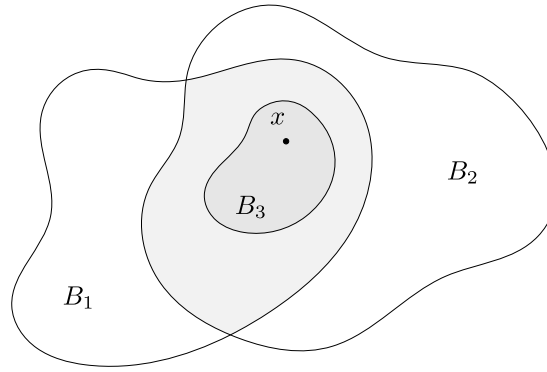


Figure 1.2: Condition (2) in Proposition 5.

**Note 1.** Since  $\mathcal{B}$  exists independently from any topology, it doesn't make sense to describe its members as "open" until after we've generated a topology from it. Once we've done so, though, it should be clear that every basis element is open in the generated topology.

We can also get a notion of how relatively fine or coarse a topology is by using its basis.

**Proposition 6.** Let  $\mathcal{B}, \mathcal{B}'$  be bases for the topologies  $\mathcal{T}, \mathcal{T}'$  on  $X$ , respectively. Then  $\mathcal{T}'$  is finer than  $\mathcal{T}$  if and only if for all  $B \in \mathcal{B}$  and  $x \in B$ , there is a  $B' \in \mathcal{B}'$  such that  $x \in B' \subset B$ .

*Proof.* First we show the backward implication. Let  $U \in \mathcal{T}$ , and let  $x \in U$ . Since  $\mathcal{B}$  generates  $\mathcal{T}$ , there is a  $B \in \mathcal{B}$  such that  $x \in B \subset U$ . By assumption, there is then a  $B' \in \mathcal{B}'$  such that  $x \in B' \subset B \subset U$ . Thus  $U \in \mathcal{T}'$ , so  $\mathcal{T}'$  is finer than  $\mathcal{T}$ .

Now we show the forward implication. Let  $B \in \mathcal{B}$ , and let  $x \in B$ , then  $B \in \mathcal{T}$ . By assumption,  $\mathcal{T} \subset \mathcal{T}'$ , so  $B \in \mathcal{T}'$  as well. Then by the definition of a generated topology, there is a  $B' \in \mathcal{B}'$  such that  $x \in B' \subset B$ .  $\square$

**Proposition 7.** The topology generated by a basis is the smallest topology containing that basis.

## 1.4 SUBBASIS

**Definition 8.** A **subbasis**  $\mathcal{S}$  for a topology  $\mathcal{T}$  on  $X$  is a collection of subsets of  $X$  whose finite intersections form a basis for  $\mathcal{T}$ .

Subbases are easier to construct than bases, but the construction of a topology from a subbasis involves an extra step, namely the finite intersections. What we are doing is creating a basis  $\mathcal{B}$  from  $\mathcal{S}$  by taking finite intersections of the subbasis elements. Then we are taking  $\mathcal{B}$  and constructing  $\mathcal{T}$  by taking arbitrary unions, as is usual.

$$\mathcal{S} \xrightarrow{\cap_{i=1}^N} \mathcal{B} \xrightarrow{\cup_{\alpha \in \mathcal{G}}} \mathcal{T}$$

Figure 1.3: The process for constructing a topology using a subbasis  $\mathcal{S}$ .

**Proposition 8.** Let  $\mathcal{T}$  be a topology on  $X$ , and let  $\mathcal{S}$  be a collection of subsets of  $X$ . Then  $\mathcal{S}$  is a subbasis for  $\mathcal{T}$  if and only if

1.  $\mathcal{S} \subset \mathcal{T}$ ; and
2. for each  $U \in \mathcal{T}$  and  $p \in U$ , there is a finite intersection  $\cap_{i=1}^n S_i$  of elements of  $\mathcal{S}$  such that  $p \in \cap_{i=1}^n S_i \subset U$ .

*Proof.* This follows from Proposition 4 (the analogue of this proposition for bases). When proving both directions, there's just an extra step to go from a generic basis element to a finite intersection of elements of  $\mathcal{S}$ .  $\square$

**Proposition 9.** Let  $\mathcal{S}$  be a collection of subsets of  $X$ . Then  $\mathcal{S}$  generates a topology if and only if  $\mathcal{S}$  covers  $X$ .

**Proposition 10.** The topology generated by a subbasis is the smallest topology containing that subbasis.



## 1.5 CONTINUOUS FUNCTIONS

The category **Top** has topological spaces as objects and continuous functions as morphisms.

**Definition 9.** Let  $X, Y$  be topological spaces, then  $f : X \rightarrow Y$  is **continuous** if for all  $U$  open in  $Y$ ,  $f^{-1}(U)$  is open in  $X$ .

**Proposition 11.** If  $Y$  has basis  $\mathcal{B}$  and  $f^{-1}(B)$  is open in  $X$  for all  $B \in \mathcal{B}$ , then  $f : X \rightarrow Y$  is continuous. Similarly, if  $Y$  has subbasis  $\mathcal{S}$  and  $f^{-1}(S)$  is open in  $X$  for all  $S \in \mathcal{S}$ , then  $f : X \rightarrow Y$  is continuous.

*Proof.* The preimage of any open set is the union of preimages of basis elements. The preimage of any basis element is the finite intersection of preimages of subbasis elements.  $\square$

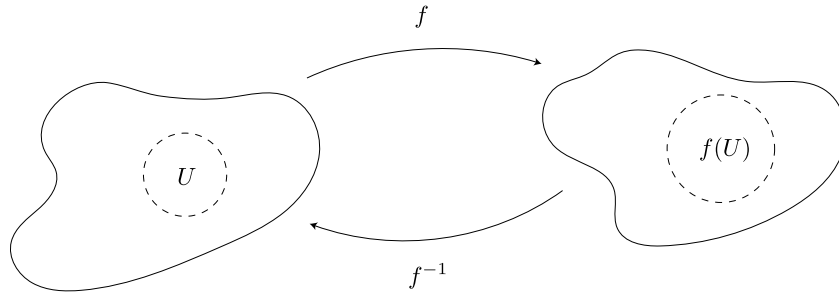
**Theorem 4.** Let  $X$  and  $Y$  be topological spaces, and let  $f : X \rightarrow Y$ , then the following are equivalent:

1.  $f$  is continuous.
2. For all  $A \subset X$ ,  $f(\overline{A}) \subset \overline{f(A)}$ .
3. For all  $B$  closed in  $Y$ ,  $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ .
4. For all  $B$  closed in  $Y$ ,  $f^{-1}(B)$  is closed in  $X$ .
5. For all  $x \in X$  and for each neighborhood  $V$  of  $f(x)$ , there is a neighborhood  $U$  of  $x$  such that  $f(U) \subset V$ .

**Example 4.** If  $X$  has the discrete topology, then any function *out* of  $X$  is continuous. If  $X$  has the indiscrete topology, then any function *into*  $X$  is continuous.

**Definition 10.** A **homeomorphism** is a continuous function with continuous inverse (an isomorphism in **Top**).

Equivalently, a homeomorphism is a bijective function  $f : X \leftrightarrow Y$  such that  $U$  is open in  $X$  if and only if  $f(U)$  is open in  $Y$ .

Figure 1.4: A homeomorphism  $f$ .

**Theorem 5** (The Pasting Lemma). *Let  $X = A \cup B$ , where  $A$  and  $B$  are either both closed or both open in  $X$ . Let  $f : A \rightarrow Y$  and  $g : B \rightarrow Y$  be continuous. If  $f(x) = g(x)$  for all  $x \in A \cap B$ , then the function  $h : X \rightarrow Y$  given by*

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

*is continuous.*

*Proof.* Suppose  $A$  and  $B$  are both closed. Let  $C$  be closed in  $Y$ , then  $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ . Since  $f$  and  $g$  are continuous, both  $f^{-1}(C)$  and  $g^{-1}(C)$  are closed in  $A$  and  $B$ , respectively. Since both  $A$  and  $B$  are closed in  $X$ , both preimages are also closed in  $X$ . Thus  $h^{-1}(C)$  is closed in  $X$  and  $h$  is subsequently continuous.

To show this when  $A$  and  $B$  are both open, replace the word “closed” with the word “open” in the above paragraph.  $\square$

Note that the condition  $f(x) = g(x)$  for all  $x \in A \cap B$  is not needed in this proof. It is only necessary to make  $h$  an actual function.

**Note 2.** If  $f : A \times B \rightarrow X$  instead, there is *no* useful criterion for the continuity of  $f$ .

The following maps are easily checked to be continuous:

- Constant maps.
- Inclusion maps.
- Restrictions of continuous maps.
- Compositions of continuous maps.

## Chapter 2

# Special Topologies

### 2.1 THE SUBSPACE TOPOLOGY

There is a natural way of a subset inheriting the topology of the set it lies in. The following definition is easily checked to actually be a topology.

**Definition 11.** Let  $(X, \mathcal{T})$  be a topological space. If  $Y \subset X$ , then

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is the **subspace topology** on  $Y$ . With this topology,  $Y$  is called a **subspace** of  $X$ .

**Proposition 12.** Let  $\mathcal{B}$  be a basis for the topology of  $X$ , then

$$\mathcal{B}_Y \doteq \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on  $Y$ .

*Proof.* Let  $y \in U \cap Y$ , where  $U$  is open in  $X$ . There exists  $B \in \mathcal{B}$  such that  $y \in B \subset U$ , so  $y \in B \cap Y \subset U \cap Y$ .  $\square$

**Proposition 13.** Let  $Y$  be a subspace of  $X$ , and let  $U$  be open in  $Y$  and  $Y$  be open in  $X$ . Then  $U$  is open in  $X$ .

*Proof.*  $U$  is open in  $Y$ , so  $U = Y \cap V$  for some  $V$  open in  $X$ . Both sets  $Y$  and  $V$  are open in  $X$ , so their intersection  $U$  must be as well.  $\square$

## 2.2 THE INITIAL TOPOLOGY

**Definition 12.** Let  $X$  be a set and  $\{Y_i\}_{i \in \mathcal{I}}$  a collection of topological spaces, and suppose we have functions  $f_i : X \rightarrow Y_i$ . The **initial topology** on  $X$  for these  $f_i$  is the coarsest topology on  $X$  such that each  $f_i$  is continuous.

**Proposition 14.** *The initial topology on  $X$  is generated by the subbasis*

$$\mathcal{S} = \{f_i^{-1}(U) \mid i \in \mathcal{I}; U \text{ open in } Y_i\}.$$

Think about this...

This is a nice generalization of the subspace and product topologies. In the next section, we'll derive the product topology as the initial topology on a Cartesian product that makes the canonical projections continuous. The initial topology on a subset such that the inclusion function is continuous is actually the subspace topology.

**Example 5.** Suppose  $Y \subset X$ , and consider the inclusion function  $\iota : Y \hookrightarrow X$ . The initial topology is generated by

$$\{\iota^{-1}(U) \mid U \text{ open in } X\} = \{Y \cap U \mid U \text{ open in } X\},$$

but this is just the subspace topology.

Why bother with saying "generated" if it's equal? Are there counterexamples?

## 2.3 THE PRODUCT TOPOLOGY

It would be natural to define the product topology as

$$\mathcal{P} = \{U \times V \mid U \text{ open in } X, V \text{ open in } Y\},$$

but this isn't enough to give a topology since you can construct examples where the union of elements in this set don't lie in the set.

This set *is*, however, perfectly valid as a basis, since  $\bigcup_{U,V}(U \times V) = X \times Y$  and  $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{P}$ .

**Definition 13.** The topology generated by  $\mathcal{P}$  is the **product topology** on  $X \times Y$ .

**Proposition 15.** If  $\mathcal{B}_X$  is a basis for  $X$  and  $\mathcal{B}_Y$  is a basis for  $Y$ , then  $\mathcal{B}_X \times \mathcal{B}_Y$  is a basis for the product topology.

**Proposition 16.** The product and subspace topologies “commute”.

*Proof.* It's straightforward to show that the product of two subspaces and the subspace of a product both have the same basis.  $\square$

Where to put the above stuff?

**Definition 14.** The **Cartesian product** of  $\{X_\alpha\}_{\alpha \in \mathcal{A}}$  is the set

$$\prod_{\alpha \in \mathcal{A}} X_\alpha = \left\{ f : \mathcal{A} \rightarrow \bigcup_{\alpha \in \mathcal{A}} X_\alpha \mid f(\alpha) \in X_\alpha \right\}.$$

Each function  $f$  represents a single “point” in the product.

**Example 6.** Suppose  $\mathcal{A} = \{1, \dots, n\}$  and  $X_\alpha = \mathbb{R}$  for all  $\alpha$ . Then each  $f$  in the Cartesian product is a function

$$f : \{1, \dots, n\} \rightarrow \mathbb{R}.$$

Since there are only a finite number of  $X_\alpha$ 's, we can write each  $f$  as a tuple

$$(f(1), f(2), \dots, f(n)).$$

Thus there is a clear bijection between  $\prod_{\alpha=1}^n X_\alpha$  and  $\mathbb{R}^n$ .

Extending the product topology to the case of a general Cartesian product is tricky. Given  $\prod_\alpha X_\alpha$ , we could naively say that the topology on it should be given by a basis

$$\mathcal{B} = \left\{ \prod_\alpha B_\alpha \mid B_\alpha \in \mathcal{B}_\alpha \right\},$$

where  $\mathcal{B}_\alpha$  is a basis for just  $X_\alpha$ . If we have a finite number of  $\alpha$ 's, this basis is just every possible ordered combination of basis elements from each  $X_\alpha$ :

$$\begin{aligned} &(B_{11}, B_{21}, \dots, B_{n1}), \\ &(B_{11}, B_{22}, \dots, B_{n2}), \\ &\vdots \\ &(B_{11}, B_{22}, \dots, B_{nn}), \\ &\vdots \end{aligned}$$

The topology generated by this basis is the **box topology**, and although simple, ends up not being the best notion of a topology on infinite products because it's actually too fine. This ends up making some "obviously" continuous functions discontinuous.

**Example 7.** Define

$$\mathbb{R}^\infty = \prod_{i \in \mathbb{Z}^+} \mathbb{R},$$

then the function

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R}^\infty \\ x &\mapsto (x, x, \dots) \end{aligned}$$

seems like it should be continuous; however, if  $\mathbb{R}^\infty$  has the box topology, then the preimage under  $f$  of the open set  $U = \prod_{i \in \mathbb{Z}^+} (-1/i, 1/i)$  is  $f^{-1}(U) = \{0\}$ . This isn't open in  $\mathbb{R}$ , so  $f$  is discontinuous.

We want the product topology to, in a sense, be continuous in each of its components. Unlike the box topology, though, we don't want it to be *too* fine. The way we formalize this is by saying that we want to find the coarsest topology on  $\prod X_\alpha$  such that the canonical projections

$$\begin{aligned} \pi_\beta : \prod_{\alpha \in \mathcal{A}} X_\alpha &\rightarrow X_\beta \\ (f : \mathcal{A} \rightarrow \bigcup X_\alpha) &\mapsto f(\beta) \end{aligned}$$

are continuous. This is just the initial topology on  $\prod X_\alpha$  with respect to the projections.

**Definition 15.** The **product topology** is generated by the subbasis

$$\left\{ \pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \text{ open in } X_\alpha \right\}.$$

The basis for the product topology is then of the form  $\prod U_\alpha$ , where only finitely many of the  $U_\alpha$  satisfy  $U_\alpha \neq X_\alpha$ .

**Proposition 17.** *The function  $f : Y \rightarrow \prod X_\alpha$  is continuous if and only if  $f_\alpha$  is continuous for all  $\alpha$ .*

*Proof.* If  $f$  is continuous, then  $f_\alpha = \pi_\alpha \circ f$  is the composition of continuous functions and so is itself continuous. Conversely, for any subbasis element  $S = \pi_\alpha(U_\alpha)$  for  $U_\alpha$  open in  $X_\alpha$ , we have

$$f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = (\pi_\alpha \circ f)^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha),$$

which is open since  $f_\alpha$  is continuous. □

## Chapter 3

# Special Spaces

### 3.1 HAUSDORFF SPACES

We say that a sequence  $\{x_n\}$  is **eventually** in  $U$  if there is some  $N$  such that  $x_n \in U$  when  $n \geq N$ .

**Definition 16.**  $\{x_n\}$  **converges** to  $x$  if it's eventually in every open neighborhood of  $x$ .

**Proposition 18.**  $\{x_n\}$  converges to  $x$  if and only if it's eventually in every basis/subbasis element containing  $x$ .

**Example 8.** In the discrete topology,  $x_n \rightarrow x$  if  $\{x_n\}$  eventually equals  $x$ .  
In the indiscrete topology, every sequence converges to every point.

If we want limits to be unique, we have to enforce certain conditions on our spaces.

**Definition 17.** A space is  $T_1$  if every pair of distinct points have neighborhoods not containing the other point. The space is **Hausdorff** if these neighborhoods are disjoint.

**Proposition 19.** A space is  $T_1$  if and only if all single points are closed.

*Proof.* **Forward:** Suppose  $X$  is  $T_1$ , then fix  $x \in X$ . Then for  $y \in X - \{x\}$ , there is an open  $U_y$  such that  $y \in U_y \subset X - \{x\}$ , so  $X - \{x\} = \bigcup_y U_y$ . Then  $X - \{x\}$  is open so  $\{x\}$  is closed.

**Backward:** Suppose all single points in  $X$  are closed. Fix  $x, y \in X$ , then  $X - \{x\}$  and  $X - \{y\}$  are the open sets we need to show that  $X$  is  $T_1$ .  $\square$



**Corollary 2.** *A space is  $T_1$  if and only if all finite point sets are closed.*

*Proof.* Do I even need one? Kinda obvious. □

**Proposition 20.** *Every finite set in a Hausdorff space is closed.*

*Proof.* Hausdorff spaces are  $T_1$ . □

**Proposition 21.** *Sequences converge to unique points in Hausdorff spaces.*

*Proof.* Suppose  $\{x_n\} \subset X$  such that  $x_n \rightarrow x \in X$ . If  $y \neq x$ , then since  $X$  is Hausdorff we can find disjoint open neighborhoods  $U$  and  $V$  of  $x$  and  $y$ , respectively. The set  $U$  contains all but finitely many of the points in  $\{x_n\}$ , so  $V$  can only contain finitely many of the points in  $\{x_n\}$ . Thus  $x_n$  cannot converge to  $y$ . □

**Proposition 22.** *The product of two Hausdorff spaces is a Hausdorff space.*

*Proof.* Do this. □

**Proposition 23.** *A subspace of a Hausdorff space is Hausdorff.*

*Proof.* Suppose  $X$  is Hausdorff and that  $Y$  is a subspace of  $X$  with distinct points  $u$  and  $v$ . Then  $u$  and  $v$  are also distinct points of  $X$ , so by the regularity of  $X$ , they are separated by disjoint open sets  $U$  and  $V$  in  $X$ . Then  $Y \cap U$  and  $Y \cap V$  are the desired open sets of  $Y$ . □

### 3.2 QUOTIENT SPACES

**Definition 18.** Suppose  $X$  has a partition  $\mathcal{P}$ . The **quotient space**  $X^*$  is  $\mathcal{P}$  equipped with the **quotient topology**:

$$U \text{ is open in } X^* \iff \pi_p^{-1}(U) \text{ is open in } X,$$

where

$$\begin{aligned} \pi_p : X &\rightarrow X^* \\ x &\mapsto [x]. \end{aligned}$$

**Note 3.** The quotient topology is the finest topology such that  $\pi_p$  is continuous. It is the **final topology** with respect to  $\pi_p$ . [Section about final topology?](#)

We can equivalently define quotient spaces in terms of images of certain functions.

**Definition 19.** A **quotient map** is a surjective map  $p : X \rightarrow Y$  such that

$$U \text{ is open in } Y \iff p^{-1}(U) \text{ is open in } X.$$

A quotient map is a homeomorphism that isn't necessarily one-to-one. Thus if we partition  $X$  based on the equivalence relation induced by  $p$ , we get injectivity and  $p$  then induces a homeomorphism between  $X^*$  and  $Y$  (see the next theorem).

**Quotient map is not necessarily open or closed map. This is subtle.**

**Proposition 24.** Suppose  $p : X \rightarrow Y$  is a quotient map and  $Z$  is any space. Then  $f : Y \rightarrow Z$  is continuous if and only if  $f \circ p : X \rightarrow Z$  is continuous.

$$\begin{array}{ccc} X & & \\ \downarrow p & \searrow f \circ p & \\ Y & \xrightarrow{f} & Z \end{array}$$

**Theorem 6.** Suppose  $p : X \rightarrow Y$  is a quotient map. Then the quotient space  $X^*$  induced by the equivalence relation induced by  $p$  is homeomorphic to  $Y$ .

**Note 4.** Given a surjective map  $p$ , the quotient topology is the final topology with respect to  $p$ .

**Definition 20.** A map  $f : X \rightarrow Y$  is **open** if it maps open sets to open sets, and it's **closed** if it maps closed sets to closed sets.

**Proposition 25.** Suppose  $f : X \rightarrow Y$  is surjective and continuous. If it's open or closed, then it's a quotient map.

**Corollary 3.** If  $f : X \rightarrow Y$  is continuous and surjective,  $X$  is compact, and  $Y$  is Hausdorff, then  $f$  is a quotient map.

*Proof.* Suppose  $A$  is closed in  $X$ , then since  $X$  is compact, so is  $A$ . Since  $f$  is continuous,  $f(A)$  is a compact subset of Hausdorff  $Y$ , so it is closed. Thus  $f$  is a closed continuous surjection, so it's a quotient map.  $\square$

If  $X = A \cup B$ , then we can make the union disjoint in a sense by introducing more dimensions. Define

$$A \sqcup B \doteq (A \times \{0\}) \cup (B \times \{1\})$$

(which is a subset of  $X \times \{0, 1\}$ ) with canonical projection

$$\begin{aligned} j : A \sqcup B &\rightarrow X \\ (x, i) &\mapsto x. \end{aligned}$$

**Theorem 7 (Gluing Maps).** Suppose  $X = A \cup B$  and  $f : A \rightarrow Z$ ,  $g : B \rightarrow Z$  agree on  $A \cap B$ . If the canonical projection  $j : A \sqcup B \rightarrow X$  is a quotient map, then the obvious concatenation of  $f$  and  $g$  is continuous.

The pasting lemma is a corollary of this theorem. **Go over this, I guess.**

### 3.3 METRIC SPACES

**Definition 21.** The **metric topology**  $\mathcal{T}_d$  on  $X$  induced by  $d$  is generated by the basis

$$\{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}.$$

**Proposition 26.** *The following give the same topologies on  $\mathbb{R}^n$ :*

1.  $d_2(x, y) = \|x - y\|_2$ ,
2.  $d_1(x, y) = \sum_i |x_i - y_i|$ ,
3.  $d_\infty(x, y) = \max_i |x_i - y_i|$ , and
4. *the product topology.*

We say a topological space is **metrizable** if there is some metric such that induces its topology.

**Proposition 27.** *Metrizable spaces are Hausdorff.*

*Proof.* **Do this. Should rely on metric space being Haus.** □

A metric space  $X$  is **bounded** if there is an  $x \in X$  and  $R > 0$  such that  $B(x, R)$  contains all of  $X$ . Equivalently, we can say that there is some  $R'$  such that  $d(a, b) < R'$  for all  $a, b \in X$ .

## Chapter 4

# Topological Properties

### 4.1 SEPARABILITY

**Definition 22.** A topological space is **separable** if it has a countable dense subset.

**Example 9.**  $\mathbb{R}$  is separable because  $\overline{\mathbb{Q}} = \mathbb{R}$ .

## 4.2 CONNECTEDNESS

**Definition 23.** A space is **connected** if it's *not* the union of 2 nonempty disjoint open subsets.

Since the whole space is the disjoint union of the 2 sets, the sets are also both closed. Thus we can separate a space with 2 nonempty disjoint *closed* sets, too.

**Theorem 8.** *The following are equivalent:*

1.  $X$  is connected.
2.  $A \subset X$  is both open and closed  $\iff A = X$  or  $X = \emptyset$ .
3. Whenever  $X = A \sqcup B$  with  $A, B$  nonempty and disjoint, one of  $A, B$  contains a limit point of the other.

**Proposition 28.**  $Y \subset X$  is disconnected if and only if there are disjoint nonempty  $A, B$  such that  $A \sqcup B = Y$  and neither  $A$  nor  $B$  contains a limit point of the other.

**Proposition 29.** Suppose  $U, V$  are disjoint and open in  $X$ . If  $Y$  is connected and  $Y \subset U \cup V$ , then  $Y$  lies entirely in one or the other.

*Proof.* If not,  $U$  and  $V$  separate  $Y$ , contradicting its connectedness.  $\square$

**Theorem 9.** A subset of  $\mathbb{R}$  is connected if and only if it's an interval.

**Proposition 30.** The continuous images of connected spaces are connected.

**Corollary 4.** If  $X \cong Y$ , then  $X$  is connected if and only if  $Y$  is.

**Theorem 10 (Intermediate Value Theorem).** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $f(a) \leq c \leq f(b)$ , then there is some  $x \in [a, b]$  such that  $f(x) = c$ .

**Lemma 1.** If  $\{A_\alpha\}$  is a collection of connected subspaces of  $X$  that all intersect. Then their union  $\bigcup_\alpha A_\alpha$  is connected.

**Proposition 31.** *If  $X$  and  $Y$  are connected, then  $X \times Y$  is connected.*

**Proposition 32.** *If  $A$  is connected and  $A \subset B \subset \overline{A}$ , then  $B$  is also connected.*

**Definition 24.** Define an equivalence relation by saying  $x \sim y$  if there is a connected component  $A$  containing  $x$  and  $y$ . Then the equivalence classes of  $\sim$  are the **(connected) components** of the space.

**Theorem 11.** *The components of  $X$  are closed, disjoint, connected, and union to  $X$ . Every connected subset is a subset of a component.*

### 4.3 PATH CONNECTEDNESS

**Definition 25.** A **path** in  $X$  from  $x$  to  $y$  is a continuous function  $\gamma : [0, 1] \rightarrow X$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$ . We say  $X$  is **path connected** if we can find paths between all points in  $X$ .

**Proposition 33.** *The continuous images of path connected spaces are path connected.*

**Proposition 34.** *Path connected spaces are connected.*

**Example 10** (The Topologist's Sine Curve). Let

$$T = \{(x, \sin(1/x)) \mid x > 0\} \cup \{(0, 0)\},$$

then  $T$  is connected but not path connected.



## 4.4 COMPACTNESS

**Definition 26.** A topological space is **compact** if every open cover has a finite subcover.

A finite union of compact spaces is compact since the finite union of finite subcovers is still finite.

**Proposition 35.** *Continuous images of compact spaces are compact.*

**Theorem 12** (Extreme Value Theorem). *If  $X$  is compact and  $f : X \rightarrow \mathbb{R}$ , then  $f$  attains its infimum and supremum on  $X$ .*

**Proposition 36.** 1. *Compact subsets of Hausdorff spaces are closed.*

2. *Closed subsets of compact spaces are closed.*

**Corollary 5.** *If  $f : X \rightarrow Y$  is a continuous bijection,  $X$  is compact, and  $Y$  is Hausdorff, then  $f$  is a homeomorphism.*

*Add quotient map version here?*

*Proof.* Let  $A$  be closed in  $X$ , then since  $X$  is compact,  $A$  must also be compact. Then  $f(A)$  is a compact subset of Hausdorff  $Y$ , so it is closed. Thus  $f$  is a closed map, so its inverse is also continuous.  $\square$

**Lemma 2** (Tube Lemma). *Suppose  $Y$  is compact and  $x_0 \in X$ . If  $N$  is a neighborhood of  $x_0 \times Y$ , then there is some neighborhood  $U$  of  $x_0$  such that  $U \times Y \subset N$ .*

**Theorem 13** (Tychonoff). *Arbitrary products of compact spaces are compact.*

**Theorem 14** (Heine-Borel).  *$K \subset \mathbb{R}^n$  (with the standard topology) is compact if and only if it's closed and bounded.*

**Proposition 37.** *If  $X$  is compact then every infinite subset  $A \subset X$  has a limit point.*

**Theorem 15.** *If  $X$  is a compact metric space, then every sequence has a convergence subsequence.*

**Theorem 16 (Lebesgue Number).** *If  $X$  is a compact metric space with open cover  $\mathcal{U}$ , then there is some  $\delta > 0$  such that all subsets  $S \subset X$  with  $\text{diam}(S) < \delta$  lie entirely in some  $U \in \mathcal{U}$ .*

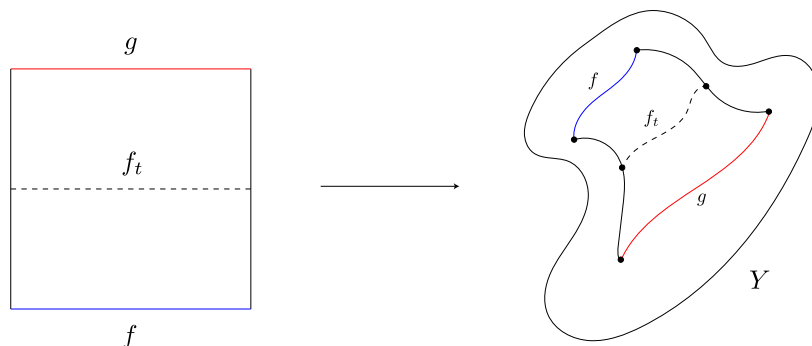
## Chapter 5

# The Fundamental Group

### 5.1 HOMOTOPIES

**Definition 27.** Maps  $f, g$  are **homotopic**, written  $f \simeq g$ , if there is some continuous map  $F : X \times I \rightarrow Y$  such that  $F(x, 0) = f(x)$  and  $F(x, 1) = g(x)$ .

We can denote  $F(\cdot, t)$  by  $f_t$ . With this notation,  $f_0 = f$  and  $f_1 = g$ .



We say that a map is **nullhomotopic** if it is homotopic to a constant map. Suppose that  $f$  and  $g$  agree on  $A$ , then  $f$  and  $g$  are **homotopic rel  $A$**  if there is a homotopy between them that fixes  $A$ . Note that if  $f$  and  $g$  agree on  $A$  and are homotopic via the straight line homotopy, then they are homotopic rel  $A$ .

**Example 11.** The **straight line homotopy** between  $f, g : X \rightarrow \mathbb{R}^n$  is defined

$$F(x, t) = (1 - t)f(x) + tg(x).$$

If  $X$  is convex, then the straight line homotopy can be used to show that any two maps are homotopic.

**Definition 28.** Two paths  $f, g$  are **path homotopic**, written  $f \simeq_p g$ , if they are homotopic rel  $\{0, 1\}$ .

We can “multiply” paths  $\alpha$  and  $\beta$  by first traveling along  $\alpha$ , then  $\beta$ , both at double speed. We denote the product of  $\alpha$  and  $\beta$  by  $\alpha\beta$ . Note that although this is similar to function composition, we read  $\alpha\beta$  left to right instead of right to left.

**Proposition 38.** *Path multiplication respects path homotopy, i.e. if  $\alpha \simeq_p \alpha'$  and  $\beta \simeq_p \beta'$ , then  $\alpha\beta \simeq_p \alpha'\beta'$ .*

A **loop** at  $p$  is a path that starts and ends at  $p$ . Path homotopies easily extend to work with loops instead, since loops are just a special type of path.

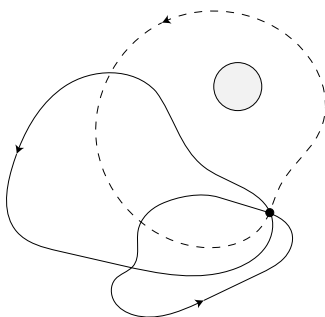


Figure 5.1: The solid loops are homotopic, but the dashed loop is not homotopic to either of the others because of the hole in the space.

## 5.2 THE FUNDAMENTAL GROUP

**Lemma 3.** *Homotopy and homotopy rel  $A$  are both equivalence relations.*

**Definition 29.** The **fundamental group** of  $X$  at  $p$  is

$$\pi_1(X, p) \doteq \{[\alpha] \mid \alpha \text{ a loop at } p\}$$

with group operation  $[\alpha][\beta] \doteq [\alpha\beta]$ .

Since path multiplication respects path homotopy, the above group operation is well-defined for all representative  $\alpha, \beta$ .

**Proposition 39.** *If  $p, q$  are in the same path component of  $X$ , then  $\pi_1(X, p) \cong \pi_1(X, q)$ .*

*Proof.* There must be a path  $\gamma$  from  $p$  to  $q$ , so if  $\alpha$  is a loop at  $p$ , then  $\gamma^{-1}\alpha\gamma$  is a loop at  $q$ . Then it's easy to check that

$$\begin{aligned} \phi : \pi_1(X, p) &\rightarrow \pi_1(X, q), \\ [\alpha] &\mapsto [\gamma^{-1}\alpha\gamma] \end{aligned}$$

is a well-defined homomorphism with homomorphic inverse, i.e. an isomorphism.  $\square$

**Corollary 6.** *If  $X$  is path connected, then all points in  $X$  have isomorphic fundamental groups.*

If we have a continuous map

$$\begin{aligned} f : X &\rightarrow Y \\ x &\mapsto y, \end{aligned}$$

then this induces a homomorphism between fundamental groups

$$\begin{aligned} f_* : \pi_1(X, x) &\rightarrow \pi_1(Y, y) \\ [\alpha] &\mapsto [f \circ \alpha]. \end{aligned}$$

This being a homomorphism follows from the distributivity of function composition.

**Proposition 40.** *If we have a sequence of continuous maps*

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

*then their induced homomorphisms satisfy  $(g \circ f)_* = g_* \circ f_*$ .*

**Theorem 17.** If  $X \cong Y$ , then  $\pi_1(X, p) \cong \pi_1(Y, q)$ .

*Proof.* Suppose  $X \cong Y$  via  $f$ , then we can picture the situation as below.

$$\begin{array}{ccc} & f & \\ (X, p) & \xrightarrow{\quad} & (Y, q) \\ & g=f^{-1} & \end{array}$$

These two maps then induce homomorphisms between the fundamental groups.

$$\begin{array}{ccc} & f_* & \\ \pi_1(X, p) & \xrightarrow{\quad} & \pi_1(Y, q) \\ & g_* & \end{array}$$

Since  $f_*g_* = (f \circ g)_* = (1_Y)_*$  and  $g_*f_* = (1_X)_*$ , the two fundamental groups are isomorphic.  $\square$

**Proposition 41.**  $\pi_1(X, p) \times \pi_1(Y, q) \cong \pi_1(X \times Y, (p, q))$ .

**Definition 30.**  $X$  is **simply connected** if it is path connected and  $\pi_1(X, p)$  is trivial for some  $p$  (and thus for all  $p$ ).

**Theorem 18.**  $\pi_1(S^1) \cong \mathbb{Z}$ .

**Theorem 19 (Brouwer Fixed Point).** If  $X \cong D^2$  (the closed unit disk), then every continuous map  $f : X \rightarrow X$  has a fixed point.

### 5.3 COVERING SPACES

**Definition 31.** A **covering space** of  $B$  is a space  $E$  and a **covering map**  $p : E \rightarrow B$ . For all  $x \in B$ , there is a neighborhood  $U$  of  $x$  such that  $p^{-1}(U)$  is a disjoint union of homeomorphic copies of  $U$ . Such neighborhoods are called **evenly covered**.

Equivalently, a covering space of is a fiber bundle with discrete fibers. Note that  $p$  must be continuous and surjective.

This is more about open sets than points. Once we get  $U$  from  $x$ , we forget about  $x$  in the definition.

Figure.

**Example 12.**  $\mathbb{R}$  is a covering space of  $S^1$ . One possible covering map is

$$\begin{aligned} p : \mathbb{R} &\rightarrow S^1 \\ t &\mapsto (\cos 2\pi t, \sin 2\pi t). \end{aligned}$$

**Definition 32.** If  $p : E \rightarrow B$  is a covering map and  $f : X \rightarrow B$  is continuous, then we say that  $\tilde{f}$  **lifts**  $f$  if  $p \circ \tilde{f} = f$ .

$$\begin{array}{ccc} & & E \\ & \nearrow \tilde{f} & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

**Proposition 42.** *Covering maps are open.*

Do this.

**Corollary 7.** *Covering maps are quotient maps.*

Why? Also show that the converse isn't necessarily true.

Lifting theorems.