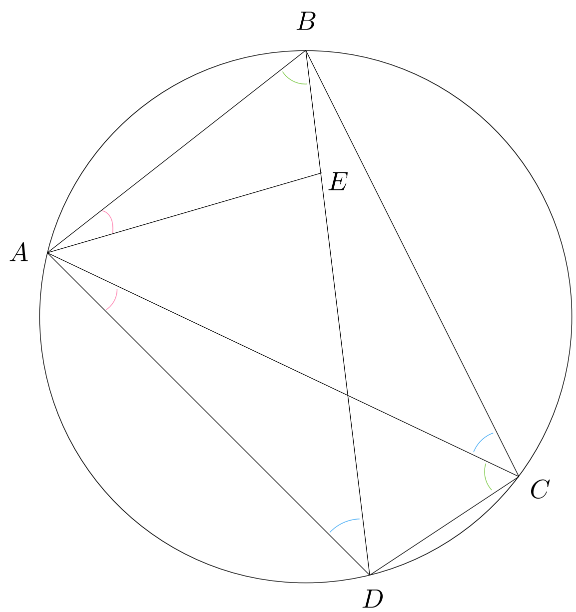
Exercise 1 (1.109)**.** Ptolemy's Theorem.



By the Star Trek lemma, since $\angle ABD, \angle ACD$ subtend the same arc, they're equal. Then since they have two equal angles, $\Delta ABE \sim \Delta ACD$. Thus

$$\frac{|AB|}{|AC|} = \frac{|BE|}{|CD|} \implies |AB||CD| = |AC||BE|.$$

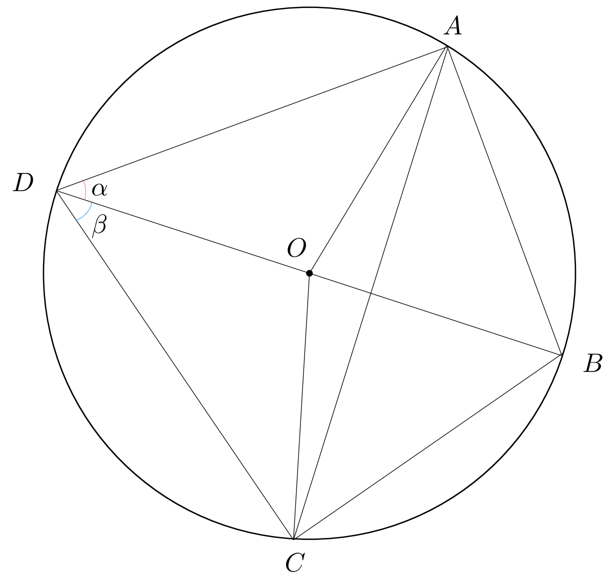
Similarly, $\Delta ABC \sim \Delta AED$, so

$$\frac{|AB|}{|AE|} = \frac{|BC|}{|ED|} \implies |BC||AD| = |AC||ED|.$$

Adding these two equalities gives

$$\begin{aligned} |AB||CD| + |BC||AD| &= |AC|(|BE| + |ED|) \\ &= |AC||DB|. \end{aligned}$$

Exercise 2 (1.111)**.** Use Ptolemy's Theorem to show the angle sum formula for sines.



In the figure, suppose BD is the diameter, and scale everything so that $|BD| = 2$. This means the radius of the circle is one, so the extended law of sines gives

$$\begin{aligned}|AC| &= 2\sin(\alpha+\beta),\\ |BC| &= 2\sin\alpha\\ |AB| &= 2\sin\beta.\end{aligned}$$

Now $\angle BAD, \angle DCB$ both subtend half of the circle since BD is the diameter, so both angles are right angles. This means ΔABD and ΔDBC are right triangles with hypotnuse length $|BD| = 2$, so

$$\begin{aligned}|CD| &= 2\cos\alpha,\\ |AD| &= 2\cos\beta.\end{aligned}$$

Then by Ptolemy's Theorem,

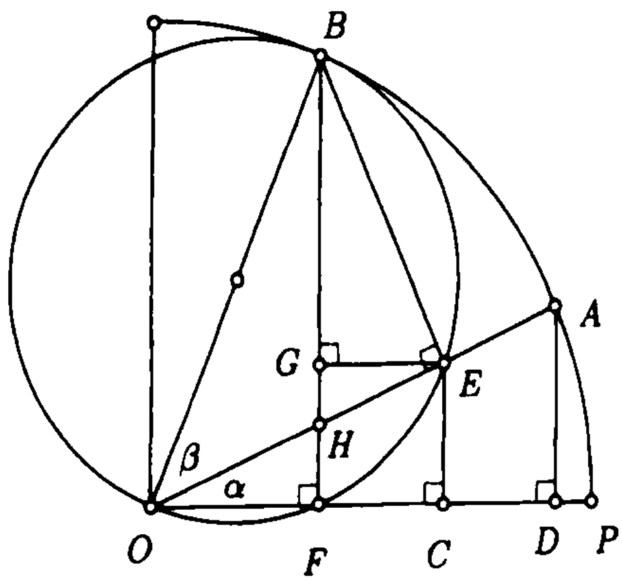
$$\begin{aligned}|AC||BD| &= |AB||CD| + |BC||AD|\\ \sin(\alpha+\beta) &= \sin\beta\cos\alpha + \sin\alpha\cos\beta.\end{aligned}$$

Exercise 3 (1.112)**.** Cosine formula using sine formula.

Let $\alpha' = \frac{\pi}{2} - \alpha$ and $\beta' = -\beta$, then by Exercise 1.111,

$$\begin{aligned}
\sin(\alpha' + \beta') &= \sin\alpha' \cos\beta' + \sin\beta' \cos\alpha' \\
\sin\left(\frac{\pi}{2} - (\alpha + \beta)\right) &= \sin\left(\frac{\pi}{2} - \alpha\right)\cos(-\beta) + \sin(-\beta)\cos\left(\frac{\pi}{2} - \alpha\right) \\
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\beta\sin\alpha.
\end{aligned}$$

Exercise 4 (1.114)**.** Angle sum formula for sines and cosines.



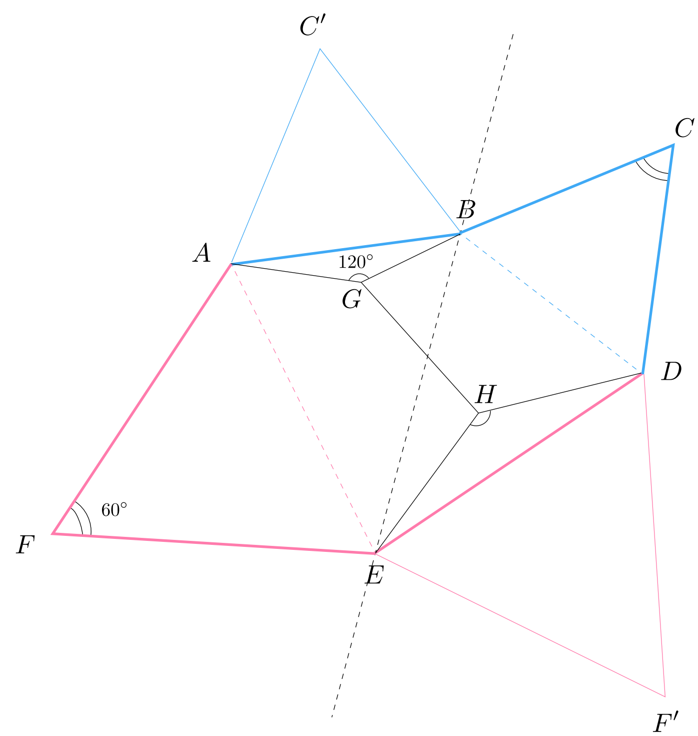
Scale everything so that $|OB| = 1$. Since they share 2 angles, $\Delta OHF \sim \Delta BHE$. In particular, $\angle EBF = \alpha$. Thus $|FC| = |GE| = |BE|\sin(\angle EBF) = \sin\beta\sin\alpha$ and $|OC| = |OE|\cos\alpha = \cos\beta\cos\alpha$. This implies

$$\cos(\alpha+\beta) = |OF| = |OC| - |FC| = \cos\beta\cos\alpha - \sin\beta\sin\alpha.$$

Similarly, $|BG| = |BE|\cos\alpha = \sin\beta\cos\alpha$ and $|GF| = |EC| = |OE|\sin\alpha = \cos\beta\sin\alpha$. This implies

$$\sin(\alpha+\beta) = |BF| = |BG| + |GF| = \sin\beta\cos\alpha + \cos\beta\sin\alpha.$$

Exercise 5 (1.118)**.** IMO hexagon inequality problem.



Note that ΔBCD is isosceles with a 60° angle, so it's equilateral. Similarly, ΔAEF is also equilateral (in the diagram all blue lines are the same length and all pink lines are the same length). Then by SSS, $\Delta BDE \cong \Delta BAE$. This means when reflecting over the line BE, each of these two triangles will become the other.

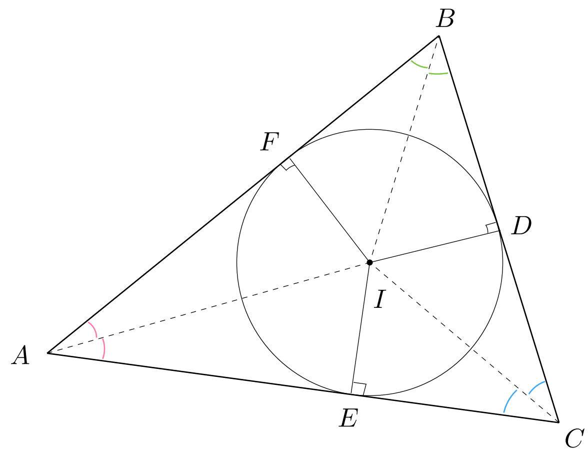
First reflect ΔBCD over BE, making the triangle $\Delta C'BA$ (by the previous comment, BD maps perfectly onto AB). Since we're given $\angle BGA = 120°$ and since we know that the interior angles of an equilateral triangle are all 60°, we have $\angle BGA + \angle AC'B = 180°$. Then by Theorem 1.14.1, $C'BGA$ is a cyclic quadrilateral. Now we can use Ptolemy's Theorem to get $|C'G||AB| = |C'B||AG| + |GB||AC'|$. But $|AB| = |C'B| = |AC'|$ since $\Delta C'BA$ is an equialateral triangle, so this simplifies to $|C'G| = |AG| + |GB|$.

Similarly, we can reflect ΔFAE over BE and follow the same steps to derive $|HF'| = |DH| + |HE|$. Adding these two identities together and adding an extra $|GH|$ on both sides yields

$$\begin{aligned} |AG| + |GB| + |DH| + |HE| + |GH| &= |HF'| + |C'G| + |GH| \\ &\geq |C'F'| \\ &= |CF|, \end{aligned}$$

where the final equality follows from reflections being isometries.

Exercise 6 (1.125)**.** Tangents of the incircle are concurrent.

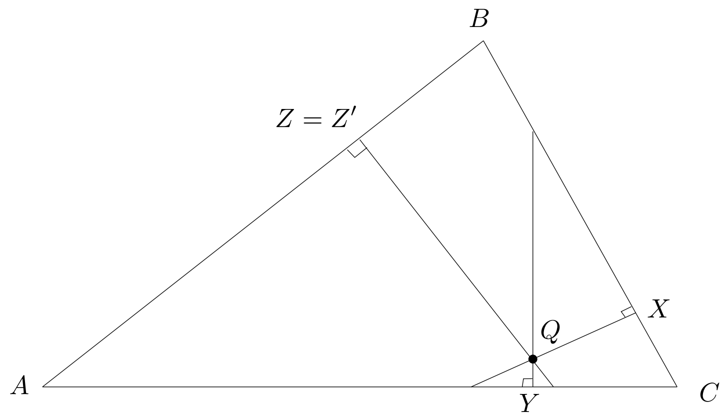


By Ceva's Theorem, AD, BE, CF are concurrent $\iff \frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1$. Now the incenter I is the intersection point of the interior angle bisectors, so all adjacent angles are equal (see the diagram).

Consider the triangles $\Delta IDC, \Delta IEC$. Due to the equal adjacent angles, $\Delta IDC \sim \Delta IEC$. But since both triangles share a side, they're actually congruent. In particular, $|DC| = |CE|$.

Similarly, we find $|FB| = |BD|$ and $|AF| = |EA|$. Thus $\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1$, so AD, BE, CF are concurrent.

Exercise 7 (1.129)**.** Intersecting perpendiculars.



Concurrency implies the equation: Suppose the perpendiculars from X, Y, and Z intersect at Q. By the Pythagorean Theorem,

- $|AQ|^2 = |AZ|^2 + |ZQ|^2 = |AY|^2 + |YQ|^2$,
- $|BQ|^2 = |BX|^2 + |XQ|^2 = |BZ|^2 + |ZQ|^2$,
- $|CQ|^2 = |CY|^2 + |YQ|^2 = |CX|^2 + |XQ|^2$.

Using these identities, $|AZ|^2 - |ZB|^2 + |BX|^2 - |XC|^2 + |CY|^2 - |YA|^2 = 0$.

The equation implies concurrency: Let

$$\mathcal{F}(X, Y, Z) = |AZ|^2 - |ZB|^2 + |BX|^2 - |XC|^2 + |CY|^2 - |YA|^2,$$

suppose the perpendiculars from X and Y intersect at a point Q, and suppose $\mathcal{F}(X, Y, Z) = 0$. Now drop a perpendicular from Q onto AB, and say it lands at a point Z'. Then by the other direction of the proof, X, Y, Z' all satisfy the equation.

Without loss of generality, suppose Z sits between A and Z', then

$$\begin{aligned}
\mathcal{F}(X, Y, Z) &= \mathcal{F}(X, Y, Z') \\
|AZ|^2 - |ZB|^2 &= |AZ'|^2 - |Z'B|^2 \\
|AZ|^2 - (|ZZ'| + |ZB|)^2 &= (|AZ| + |ZZ'|)^2 - |Z'B|^2 \\
|AZ|^2 - |ZZ'|^2 - 2|ZZ'||Z'B| - |Z'B|^2 &= |AZ|^2 + 2|AZ||ZZ'| + |ZZ'|^2 - |Z'B|^2 \\
|ZZ'|(|AZ| + |ZZ'| + |Z'B|) &= 0 \\
|ZZ'||AB| &= 0 \\
|ZZ'| &= 0,
\end{aligned}$$

where the last equality follows from $|AB|$ being nonzero. Thus $Z = Z'$, so the three perpendiculars intersect.