Notes on *Applications of Lie Groups to Differential Equations*

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1 Introduction to Lie Groups

1.1 Manifolds

Definition 1.7 Let $F : M \to N$ be a smooth mapping from M $m-$dim to N $n-$dim. The *rank* of F at a point $x \in M$ is the rank of the $n \times m$ Jacobian matrix at x. F is of *maximal rank* on a subset $S \subset M$ if, for all $x \in S$, $rank F = \min\{n, m\}$.

Theorem 1.8 Let $F : M \to N$ be of maximal rank at $x_0 \in M$. Then there are local coordinates $x = (x^1, ..., x^m)$ around x_0 and $y = (y^1, ..., y^n)$ around $F(x_0)$ such that F has has the simple form $y = (x^1, ..., x^m, 0, ..., 0)$ if $n > m$ or $y = (x^1, ..., x^n)$ if $n \leq m$.

Definition 1.9 Let M be a smooth manifold. A *submanifold* of M is a subset $N \subset M$, together with a smooth injective map $\phi : \tilde{N} \to N$ satisfying the maximal rank condition everywhere, where the *parameter space* $\tilde{N}$ is some other manifold such that $N = \phi(\tilde{N})$. In particular, $dim N = dim \tilde{N} \leq dim M$.

Definition 1.11 A *regular submanifold* N of a manifold M is a submanifold parameterized by $\phi : \tilde{N} \to M$ with the property that, for every $x \in N$, there exists an open nbd $U \subset M$ such that $\phi^{-1}(U \cap N)$ is a connected open subset of $\tilde{N}$.

Lemma 1.12 An n-dimensional submanifold $N \subset M$ is regular if and only if, for each $x_0 \in N$, there exist local coordinates $x = (x^1, ..., x^m)$ defined on a nbd U such that $N \cap U = \{x : x^{n+1} = ... = x^m = 0\}$. Such a coordinate chart is called flat on M.

Theorem 1.13 Let M be a smooth $m-$dim manifold and $F : M \to \mathbb{R}^n$, $n \leq m$, be a smooth map. If F is of maxiaml rank on the subset $N = \{x : F(x) = 0\}$, then N is a regular $(m - n)$-dim submanifold of M.

1.2 Lie Groups

Definition 1.16 An *r-parameter Lie group* is a group G which has the structure of an $r-$dim manifold. so that the group operation and inversion are smooth maps. A Lie group homomorphism is a group homomorphism that is also a smooth map.

Definition 1.18 A *Lie subgroup* of a Lie group G is a submanifold H with $\phi : \tilde{H} \to G$, $H = \phi(\tilde{H})$ where $\tilde{H}$ is a Lie group and ϕ a Lie group homomorphism.

Theorem 1.19 Let G be a Lie group. If H is a closed subgroup of G, then H is a regular submanifold of G and hence a Lie group in its own right. Conversely, any regular Lie subgroup of G is a closed subgroup.

Definition 1.20 An *r-parameter local Lie group* consists of connected subsets $V_0 \subset V \subset \mathbb{R}^r$ containing the origin 0, and smooth maps $m : V \times V \to \mathbb{R}^r$ defining the group operation, and $i : V_0 \to V$, defining the group inversion, with the following properties

a) Associativity: If $x, y, z \in V$, and also $m(x, y)$ and $m(y, z)$ are in V, then $m(x, m(y, z)) = m(m(x, y), z)$.

b) Identity: For all $x \in V$, $m(0, x) = x = m(x, 0)$.

c) Inverse: For each $x \in V_0$, $m(x, i(x)) = 0 = m(i(x), x)$.

Proposition 1.24 Let G be a connected Lie group and U be some neighborhood of e. Then any $g \in G$ can be written as $g = g_1 ... g_k$ for $g_i \in U$.

Proof: Let S be the set of products of elements of U. Since $gU \subset S$ for any $g \in S$, then S is open. If $g \notin S$, then gU is disjoint with S. Otherwise, if $gu \in S$, then since $u^{-1} \in S$, then $guu^{-1} = g \in S$. Since G is connected, then $S = G$.

Group Actions

Definition 1.25 Let M be a smooth manifold. A *local group of transformations* acting on M is given by a (local) Lie group G, an open subset U such that $\{e\} \times M \subset U \subset G \times M$, and a smooth map $\Phi : U \to M$ with the following properties:

a) If (h, x) and $(g, \Phi(h, x))$ are in U, and also $(gh, x) \in U$, then $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$.

b) For all $x \in M$, $\Phi(e, x) = x$.

c) If $(g, x) \in U$, then $(g^{-1}, \Phi(g, x)) \in U$ and $\Phi(g^{-1}, \Phi(g, x)) = x$.

Definition An *orbit* of a local transformation group is a minimal nonempty group-invariant subset of the manifold M. In other words, $O \subset M$ is an orbit if

a) For all $x \in M$, $g \in O$ such that gx is defined, then $gx \in O$.

b) For all $\tilde{O} \subset O$ satisfying a), either $\tilde{O} = O$ or $\tilde{O} = \emptyset$.

Definition 1.26 Let G be a local group of transformations acting on M.

a) G acts semi-regularly if all orbits O are of the same dimension as submanifolds of M.

b) G acts regularly if the action is semi-regular and, for every $x \in M$, there exist nbds $U \subset M$ such that $U \cap O$ is pathwise connected for every orbit O.

c) G acts transitively if there is only one orbit, the manifold M itself.

1.3 Vector Fields

Definition Given a curve C in a manifold M parameterized in local coordinates $(x^1, ..., x^m)$ by a function $\phi : I \to M$ so that $\phi(t) = (\phi^1(t), ..., \phi^m(t))$. At each point $x = \phi(t)$, the vector tangent to the curve is given by

$$v|_x = \dot{\phi}(t) = \dot{\phi}^1(t) \frac{\partial}{\partial x^1}\bigg|_{\phi(t)} + ... + \dot{\phi}^m(t) \frac{\partial}{\partial x^m}\bigg|_{\phi(t)}$$

Definition A vector field V on M is a map $M \to TM$ with the form

$$V(p) = f^1(p) \tfrac{\partial}{\partial x^1}|_p + ... + f^m(p) \tfrac{\partial}{\partial x^m}|_p$$

Flows

Definition Given a vector field V, an integral curve is a smooth parameterized curve $\phi : I \to M$ such that $\dot{\phi}(t) = V(\phi(t))$ for all $t \in I$. Given local coordinates $x = (x^1, ..., x^m)$, $\phi(t) = (\phi^1(t), ..., \phi^m(t))$ is solves the system of differential equations

$$\tfrac{dx^i}{dt} = f^i(x) \text{ for } i = 1, ..., m$$

For a smooth vector field, there exists a unique maximal integral curve, i.e., if $\phi : I \to M$ is maximal and $\psi : I' \to M$ is another integral curve, then $I' \subset I$.

Definition Given a vector field V, let $\Psi(t,x) : I \to M$ denote the maximal integral curve at x. Varying x, $\Psi(t,x) : I \times M \to M$ defines the flow of V. The flow has the following properties:

- $\Psi(\tau, \Psi(t,x)) = \Psi(\tau + t, x)$ for t, τ such that both sides are defined.
- $\frac{d}{dt}\Psi(t,x) = V(\Psi(t,x))$ for all t where defined.

The flow of a vector field defines a local group action of $\mathbb{R}$ on M, often called the one-parameter group of transformations. The vector field is called the infinitesimal generator of the action. We can see that each maximal integral curve is an orbit of the group action. Using Taylor's Theorem, in local coordinates we have

$$\Psi(t,x) = (x^1, ..., x^m) + t(f^1(x), ..., f^m(x)) + O(t^2)$$

Remark Conversely, given a one-parameter group of transformations on M, let the group action of $\mathbb{R}$ be defined by $\Psi(t,x) : I \times M \to M$. This determines a vector field, its infinitesimal generator, by

$$V(x) = \frac{d}{dt}|_{t=0}\Psi(t,x)$$

Definition Mapping a vector field to its flow is referred to as exponentiation and is denoted by $\exp(tV)x = \Psi(t,x)$. Following from the above, this has the following properties:

- For all x, $\exp((t+\tau)V)x = \exp(tV)\exp(\tau V)x$ for all t, τ when defined.
- $\exp(0v)x = x$
- $\frac{d}{dt}(\exp(tV)x) = V(\exp(tV)x)$

Proposition 1.29 Let V be a vector field so that $V(x_0) \neq 0$ for $x_0 \in M$. Then there exists a local coordinate chart containing x_0 with coordinates $y = (y^1, ..., y^m)$ such that $V = \frac{\partial}{\partial y^1}$.

Actions on Functions Let V be a vector field and we can use this to define a group action on $C^\infty(M)$. Let $g \in C^\infty(M)$ and, in local coordinates, $V = \sum f^i \frac{\partial}{\partial x^i}$. Consider the function $g(\exp(tV)x) : I \times M \to \mathbb{R}$. Then, by the chain rule, we have

$$\frac{d}{dt}g(\exp(tV)x) = \sum_{i=1}^m f^i(\exp(tV)x)\frac{\partial g}{\partial x^i}(\exp(tV)x) = [V(g)](\exp(tV)x)$$

In particular, when $t = 0$, we have $\frac{d}{dt}|_{t=0}g(\exp(tV)x) = [V(g)](x)$. Therefore, vector fields act as first order partial differential operators on smooth functions.

Definition The Lie series is given, assuming convergence, by the Taylor series in t:

$$g(\exp(tV)x) = \sum_{k=0}^{\infty} \frac{t^k}{k!}[V^k(g)](x) \text{ where, for example, } V^2(g) = V(V(g)).$$

Therefore, each tangent vector $V(x)$ defines a derivation on $C^\infty(M)$ so that $f \mapsto [V(f)](x) \in \mathbb{R}$. This has the following properties:

- (Linearity) $V(cf + g) = cV(f) + V(g)$
- (Leibniz' Rule) $V(fg) = V(f)g + fV(g)$

Differentials

Remark Given a smooth map $F : M \to N$, F maps parameterized curves in M to ones in N. Moreover, the differential defines a map $dF_p : T_pM \to T_{F(p)}N$.

Lemma 1.31 Given smooth functions $F : M \to N$ and $H : N \to P$, then

$$d(H \circ F) = dH \circ dF$$

Lie Brackets

Definition Given vector fields V, W, their *Lie bracket* is the unique vector field satisfying

$$[V, W](f) = V(W(f)) - W(V(f)) \text{ for all } f \in C^\infty(M)$$

Proposition 1.32 The Lie bracket has the following properties:

- Bilinearity
- Skew-symmetry: $[V, W] = -[W, V]$
- Jacobi Identity: $[U, [V, W]] + [W, [U, V]] + [V, [W, U]] = 0$

Theorem 1.33 Let V, W be smooth vector fields on M. For every $x \in M$ and for sufficiently small $t \geq 0$, the commutator

$$\psi(t, x) = \exp(-\sqrt{t}W)\exp(-\sqrt{t}V)\exp(\sqrt{t}W)\exp(\sqrt{t}V)x$$

defines a smooth curve. Furthermore, the Lie bracket gives a tangent vector $[V, W]|_x$ to the curve at the point $\psi(0, x) = x$, i.e.,

$$[V, W]|_x = \frac{d}{dt}|_{t=0}\psi(t, x)$$

Theorem 1.34 Let V, W be vector fields on M. Then, for all x, t, θ where defined,

$$\exp(tV)\exp(\theta W)x = \exp(\theta W)\exp(tV)x$$

if and only if

$$[V, W] = [W, V] = 0$$

Tangent Spaces and Vector Fields on Submanifolds

Proposition 1.35 Let $F : M \to \mathbb{R}^n$, where $n \leq m$, be of maximal rank on $M' = \{x : F(x) = 0\}$ so that M' is a regular $(m-n)$-dim submanifold. Given $y \in M'$, the tangent space to M' at y is the kernel of the differential of F at y, i.e., $T_yM' = \{V_y \in T_yM : d_yF(V_y) = 0\}$.

Lemma 1.37 If V and W are tangent to a submanifold N, then so is $[V, W]$.

Frobenius' Theorem

Definition 1.38 Let $V_1, ..., V_R$ be vector fields on a smooth manifold M. An *integral submanifold* of $\{V_1, ..., V_r\}$ is a submanifold $N \subset M$ whose tangent space T_yN is spanned by the vectors $\{V_1|_y, ..., V_r|_y\}$ for each $y \in N$. The system of vector fields $\{V_1, ..., V_r\}$ is *integrable* if through every point $x_0 \in M$ there passes an integral submanifold.

Definition 1.39 A system of vector fields $\{V_1, ..., V_r\}$ on M is an involution if there functions $h_{ij}^k \in C^\infty(M)$, $i, j, k = 1, ..., r$, such that $[V_i, V_j] = \sum_{k=1}^r h_{ij}^k V_k$.

Theorem 1.40 Let $V_1, ..., V_r$ be smooth vector fields on M. Then $\{V_1, ..., V_r\}$ is an integrable system if and only if it is an involution.

Theorem 1.41 Let $\mathscr{H}$ be a system of vector fields on a manifold M. Then $\mathscr{H}$ is integrable if and only if it is an involution and rank-invariant.

Theorem 1.43 Let $\{V_1, ..., V_r\}$ be an integrable system of vector fields such that the dimension of the span of $\{V_1|_x, ..., V_r|_x\}$ in T_xM is a constant s, independent of $x \in M$. Then for each $x_0 \in M$ there exist flat local coordinates $y = (y^1, ..., y^m)$ near x_0 such that the integral submanifolds intersect the given coordinate chart in the "slices" $\{y :^1 = c_1, ..., y^{m-s} = c_{m-s}\}$ where $c_1, ..., c_{m-s}$ are arbitrary constants. If, in addition, the system is regular, then the coordinate chart can be chosen so that each integral submanifold intersects it in at most one such slice.

1.4 Lie Algebras

Definition Let G be a Lie group. For $g \in G$, the *right multiplication map* $R_g(h) = h * g$ is a diffeomorphism with inverse $R_{g^{-1}}$. A vector field V on G is *right-invariant* if $dR_g(V_h) = V_{R_g(h)} = V_{hg}$ for all $g, h \in G$. If V, W are right-invariant, then so are any of their linear combinations.

Definition 1.44 Given a Lie group G, its *Lie algebra*, denoted $\mathfrak{g}$ is the vector space of right-invariant vector fields on G. Note: Any right-invariant vector field is determined by its value at the identity since $V_g = dR_g(V_e)$. Conversely, any tangent vector to G at e determines a right-invariant vector field. Therefore we can make the identification $\mathfrak{g} \cong T_eG$.

Definition 1.45 In more generality, a *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear operation, the Lie bracket:

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

Satisfying the following:

(a) Biliinearity

(b) Skew-symmetry: $[V, W] = -[W, V]$

(c) Jacobi Identity

$$[U, [V, W]] + [W, [U, V]] + [V, [W, U]] = 0$$

Examples Let $G = (\mathbb{R}, +)$. Let x be the coordinate on G. Then for $h \in \mathbb{R}$, $R_h(x) = x + h$. Then $dR_h(\frac{\partial}{\partial x}) = \frac{\partial R_h}{\partial x}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}$. This implies $\frac{\partial}{\partial x}$ is right invariant, and therefore in the Lie algebra. However, $x\frac{\partial}{\partial x}$ is not right invariant. Therefore, the Lie algebra is $\mathfrak{g} = span_{\mathbb{R}}\{\frac{\partial}{\partial x}\}$.

Let $G = (\mathbb{R}^+, \cdot)$. Let x be the coordinate on G. Then, $R_h(x) = x \cdot h$. Consider $V = \frac{\partial}{\partial x}$ Then for $g \in \mathbb{R}^+$, consider that $dR_h|_g(x\frac{\partial}{\partial x}) = \frac{\partial R_h}{\partial x}(g)\frac{\partial}{\partial x}|_{gh} = gh\frac{\partial}{\partial x}|_{gh} = V(gh)$. Therefore, $dR_h(V(g)) = V(gh)$, so this is right invariant. So $\mathfrak{g} = span_{\mathbb{R}}\{x\frac{\partial}{\partial x}\}$.

Let $G = GL_n(\mathbb{R})$. This is an n^2-parameter Lie group, so $\mathfrak{g} \cong \mathbb{R}^{n\times n}$ as vector spaces and they are isomorphic as Lie algebras with $\mathbb{R}^{n\times n}$ having the matrix commutator $AB - BA$ as its Lie bracket. Consider $A \in \mathbb{R}^{n\times n}$. Then the vector field induced by it is $v_A|_I = a_{ij}\frac{\partial}{\partial x^{ij}}$. For any $Y \in GL(n)$, we have

$$v_A|_Y = dR_Y(v_A|_I) = \sum_{i,j,m} a_{ij}y_{jm}\frac{\partial}{\partial x^{im}} = v_{AY}|_I$$

One can check that, for the Lie bracket, $[v_A, v_B] = v_{AB-BA}$.

One-Parameter Subgroups

Proposition 1.48 Let $V \neq 0$ be a right-invariant vector field on a Lie group G. Then the flow generated by V through the identity, namely

$$g_\epsilon = \exp(\epsilon V)e$$

is defined for all $\epsilon \in \mathbb{R}$ and forms a one-parameter subgroup of G, with

$$g_{\epsilon+\delta} = g_\epsilon g_\delta,\ g_0 = e,\ g_\epsilon^{-1} = g_{\epsilon^{-1}}$$

isomorphic to either $\mathbb{R}$ itself or the circle group $SO(2)$. Conversely, any connected one-dimensional subgroup of G is generated by such a right-invariant vector field in the above manner.

Example 1. Consider, again, $G = GL(n)$. For $A \in \mathfrak{gl}(n)$, we can express its vector field by $v_A|_X = \sum_{i,j} \big(\sum_k a_{ik}x^{kj}\big)\frac{\partial}{\partial x^{ij}}$. The one parameter subgroup of $\exp(tv_A)e$ is found by integrating the $n \times n$ differential equation system

$$\frac{dx^{ij}}{dt} = \sum_{k=1}^{n} a_{ik}x^{kj},\ x^{ij}(0) = \delta_{ij}$$

2. Consider the torus T^2 with group multiplication defined by

$$(\theta, \rho) \cdot (\theta', \rho') = (\theta + \theta', \rho + \rho') \mod 2\pi$$

Then T^2 is a Lie algebra, spanned by $\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\rho}$, with trivial Lie bracket, i.e., $[\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\rho}] = 0$. Consider the vector field $V = \partial_\theta + a\partial_\rho$. Then this corresponds to the one-parameter subgroup $\exp(tV)(0,0) = (t, ta) \mod 2\pi$. If a is rational, then this subgroup is closed and is isomorphic to $SO(2)$. If a is irrational, then it is a dense subset isomorphic to $\mathbb{R}$.

Subalgebras

Remark In general, a *subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a vector subspace which is closed under the Lie bracket, so $[V, W] \in \mathfrak{h}$ whenever $V, W \in \mathfrak{h}$. If H is a Lie subgroup of G, any right-invariant vector field V on H can be extended to a right-invariant vector field on G (Just set $V_g = dR_g(V_e), g \in G$). In this way, the Lie algebra $\mathfrak{h}$ of H is realized as a subalgebra of the the Lie algebra $\mathfrak{g}$ of G. The correspondence between one-parameter subgroups of Lie group G and one-dimensional subspaces (subalgebras) of its Lie algebra $\mathfrak{g}$ generalizes to provide a complete one-to-one correspondence between Lie subgroups of G and subalgebras of $\mathfrak{g}$.

Proposition 1.48 Let $V \neq 0$ be a right-invariant vector field on a Lie group G. Then the flow generated by V through the identity, namely,

$$g_\epsilon = \exp(\epsilon V)e = \exp(\epsilon V)$$

is defined for all $\epsilon \in \mathbb{R}$ and forms a one-parameter subgroup of G, with $g_{\epsilon+\delta} = g_\epsilon g_\delta$, $g_0 = e$, $g_\epsilon^{-1} = g_{\epsilon^{-1}}$. This subgroup is isomorphic to either $\mathbb{R}$ or $SO(2)$. Conversely, any connected one-dimensional subgroup of G is generated by such a right-invariant vector field in the above manner.

Example Suppose $G = GL(n)$ with Lie algebra $\mathfrak{gl}(n)$. Then $\mathfrak{gl}(n) \cong \mathbb{R}^{n\times n}$, the vector spaces of $n \times n$ matrices with the commutator, i.e., $AB - BA$ for all $A, B \in \mathfrak{gl}(n)$, as a Lie bracket. To see this, consider some matrix $A \in \mathbb{R}^{n\times n}$. Then at the identity $I \in GL(n)$, A induces the tangent vector $v_A|_I = a_{ij}\frac{\partial}{\partial x^{ij}}|_I$ for $1 \leq i, j \leq n$. To check this is right invariant, consider a matrix $Y \in GL(n)$. Then

$$v_A|_Y = dR_Y(v_A|_I) = \sum_{i,j,m} a_{i,j}y_{j,m}\frac{\partial}{\partial x^{im}} = v_{AY}|_I$$

Theorem 1.51 Let G be a Lie group with Lie algebra $\mathfrak{g}$ If $H \subset G$ is a Lie subgroup, its Lie algebra is a subalgebra of $\mathfrak{g}$. Conversely, if $\mathfrak{h}$ is any $s-$dimensional subalgebra of $\mathfrak{g}$, there is a unique connectec $s-$parameter Lie subgroup H of G with Lie algebra $\mathfrak{h}$.

1.5 Differential Forms

1.6 Exercises

1.27 Let G be a Lie group with Lie algebra $\mathfrak{g}$.

a) Prove that the exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of the identity in G.

b) Prove the normal coordinate formula (1.40).

$$g = \exp(\epsilon^1 V i_1) ... \exp(\epsilon^k V_{i_k}) \text{ for suitable } \epsilon^j \in \mathbb{R},\ 1 \leq i_j \leq r, j = 1, ..., k$$

Solution

a) Given $V \in \mathfrak{g}$, $\exp(V)$ is defined as $\exp(1V)e$. Since exp is a smooth function, we want to show that $d_0 \exp : T_0\mathfrak{g} \to T_eG$ is an isomorphism. In particular, since $T_0\mathfrak{g} \cong \mathfrak{g} \cong T_eG$, we want to show that it is the identity map. Let $(x^1, ..., x^n)$ be local coordinates in G and $\phi \in C^\infty(G)$, then we want to show that

$$[d_0 \exp(V)]\phi = [\exp_* V_0]\phi = V_0(\exp^* \phi) = V_0(\phi \circ \exp) = V_e(\phi)$$

b) By Proposition 1.24, we know that we can write $g = g_1, ..., g_k$ for elements in some neighborhood U. For each g_i we can define V_i so that $\exp(V_i)e = g_i$. Then $g = \exp(V_1)...\exp(V_k)$. Since $\mathfrak{g} \cong T_eG$, then there exists a basis $\{E_1, ..., E_r\}$, so that our k vector fields can be written as linear combinations of k basis elements. $V_i = \sum c_i^l E_{i_l}$. Then since we know that $[E_p, E_q] = 0$ for $p \neq q$, then by Theorem 1.34, $\exp(E_p)\exp(E_q) = \exp(E_q)\exp(E_p)$. Then we can rearrange $g = \exp(V_1)...\exp(V_k) = \exp(c_1 E_{i_1})...\exp(c_k E_{i_k})$.

2 Symmetry Groups of Differential Equations

2.1 Symmetries of Algebraic Equations

Definition 2.1 Let G be a local group of transformations acting on a manifold M. A subset $\mathscr{I} \subset M$ is called $G-$invariant, and G is called a symmetry group of $\mathscr{I}$, if whenever $x \in \mathscr{I}$, and $g \in G$ is such that gx is defined, then $gx \in \mathscr{I}$.

In most applications, the set $\mathscr{I}$ will be the set of solutions or subvariety determined by the common zeros of a collection of smooth functions $F = (F_1, ..., F_l)$,

$$\mathscr{I}_F = \{x : F(x) = 0\}.$$

If $\mathscr{I}_1, \mathscr{I}_2$ are $G-$invariant sets, then so are $\mathscr{I}_1 \cap \mathscr{I}_2$ and $\mathscr{I}_1 \cup \mathscr{I}_2$.

Invariant Functions

Definition 2.3 Let G be a local group of transformations acting on a manifold M. A function $F : M \to N$ is called a $G-$invariant function if, for all $x \in M$ and $g \in G$ such that gx is defined, then $F(gx) = F(x)$.

A real-valued $G-$invariant function $M \to \mathbb{R}$ is simply called an invariant of G.

Proposition 2.5 A smooth function $F : M \to \mathbb{R}^l$ is $G-$invariant if and only if every level set is a $G-$invariant set.

Infinitesimal Invariance The infinitesimal criterion is the key to transforming the nonlinear conditions for invariance of a subset or function into an equivalent linear condition of infinitesimal invariance under the corresponding infinitesimal generators of the group action. The criterion follows from the basic formula describing how functions change under the flow by a vector field.

Proposition 2.6 Let G be a connected group of transformations () acting on the manifold M. A smooth real-valued function $\xi : M \to \mathbb{R}$ is an invariant function for G if and only if

$$V(\xi) = 0 \text{ for all } x \in M \text{ and every infinitesimal generator } V \text{ of } G$$

Proof

If $V_1, ..., V_r$ form a basis of $\mathfrak{g}$, the Lie algebra of infinitesimal generators of G, then Proposition 2.6 says that $\xi(x)$ is an invariant if and only if $V_k(\xi) = 0$ for all k.

First, suppose ξ is $G-$invariant. Then every level set is $G-$invariant. We have that $\frac{d}{dt}\xi(\exp(tV)x) = V(\xi)[\exp(tV)x]$. When $t = 0$, $V(\xi)(x) = \frac{d}{dt}\xi(x) = 0$. Therefore, $V(\xi)[x] = 0$ for all $x \in M$.

Now, suppose $V(\xi) = 0$. Then, $\frac{d}{dt}\xi(\exp(tV)x) = 0$ where defined. Then let $H = \{g \in G : gx \text{ is defined for all } x\}$. Since any g can be written as a finite product of exponential generators, then $\xi(gx) = \xi(x)$ for all $g \in G_x$.

Essentially this proposition says that ξ is invariant if and only if ξ solves the differential equation system:

$$v_k(\xi) = \sum_{i=1}^{m} \xi_k^i \frac{\partial}{\partial x^i} = 0 \text{ for } k = 1, .., r$$

where $\{v_k : \; k = 1, .., r\}$ is a basis for the Lie algebra $\mathfrak{g}$ of infinitesimal generators.

Example Consider the translation group G_c, i.e., translations defined by $(x, y) \mapsto (x + ct, y + t), t \in \mathbb{R}$ for some fixed constant c. The invariant sets of this translation are unions of lines with slope c. Indeed, the function $\xi_c(x, y) = x - cy$ is a G_c-invariant function. Every invariant function of this translation can be expressed as a single-variable smooth function $f_c(T)$ where $T = x - cy$ is the variable. To obtain the infinitesimal generator, for each c in our Lie group, we must associate a vector field V_c such that $V_c(\xi_c) = 0$. Let $V_c = c\partial_x + \partial_y$.

Theorem 2.8 Let G be a connected local Lie group of transformations acting on the m-dim manifold M. Let $F : M \to \mathbb{R}^l$, $l \leq m$ define a system of algebraic equations $F_k(x) = 0$, $k = 1, ..., l$. Assuming the system is of maximal rank (i.e., the Jacobian has maximal rank at every solution), then G is a symmetry group of the system if and only if

$$V[F_k(x)] = 0, \; k = 1, ..., l \text{ whenever } F(x) = 0$$

for every infinitesimal generator V of G.

Proof

($\Longrightarrow$) Suppose G is a symmetry group of the system. Then for any infinitesimal generator V, we have $F(\exp(tV)x) = 0$. Differentiating and setting $t = 0$ results in $V[F(x)] = 0$.

($\Longleftarrow$) Now suppose $V[F_k(x)] = 0$ where $V = \xi^i(y)\frac{\partial}{\partial y^i}$.

Proposition 2.10 Let $F: M \to \mathbb{R}^l$ be of maximal rank on the subvariety $A = \{x \in M : F(x) = 0\}$. Then a function $h: M \to \mathbb{R}$ vanishes on A if and only if there exist smooth functions $Q_1, ..., Q_l : M \to \mathbb{R}$ such that

$$f(x) = F_1(x)Q_1(x) + ... + F_l(x)Q_l(x) \text{ for all } x \in M$$

A consequence of this is that the invariance criterion is equivalent to

$$V(F_k) = \sum_{i=1}^{l} Q_i^k(x)F_i(x)$$

for a yet undetermined set of functions Q_i^k. In general if we have $h(x) = \sum_k R_k(x)F_k(X) = \sum_k \tilde{R}_k(x)F_k(x)$, then the differences $Q_k(x) = R_k(x) - \tilde{R}_k(x)$ satisfy

$$\sum_{k=1}^{l} Q_k(x)F_k(x) = 0 \text{ for all } x \in M.$$

Proposition 2.11 Let $F: M \to \mathbb{R}^l$ be of maximal rank on $A = \{x \in M : F(x) = 0\}$. Suppose $Q_1, ..., Q_l$ satisfy the above condition for all $x \in M$. Then $Q_k(x) = 0$ for all $x \in A$. Equivalently, there exist functions $S_k^i(x)$ for $k, i = 1, ..., l$ such that

$$Q_k(x) \sum_{i=1}^{l} S_k^i(x)F_i(x),\ x \in M$$

Moreover the functions S_k^i can be chosen to be skew symmetric in their indices: $S_k^i = -S_i^k$.

Local Invariance

Local invariance requires functions to be invariant for a group of transformations sufficiently close to the identity.

Definition 2.12 Let G be a local group of transformations, acting on the manifold M. A subset $A \subset M$ is called *locally G-invariant* if, for every $x \in A$, there exists a neighborhood $U \subset G_x$ of the identity in G such that $g \cdot x \in A$ for every $g \in U$. A function $F : S \subset M \to N$ is locally $G-$invariant if, for every $x \in S$, there is some neighborhood $U \subset G$ of the identity such that $F(g \cdot x) = F(x)$ for all $g \in U$.

Proposition 2.14 A submanifold $N \subset M$ is locally $G-$invariant if and only if the infinitesimal generators of G are everywhere tangent to N.