Summation Algorithms

Suppose we want to numerically compute $s = \sum_{i=1}^{N} x_i$. In certain cases, we may witness an intolerable loss of precision.

In MATLAB, $\beta = 2$ denotes we use the base-2 format and $t = 16$ denotes the number of digits of precision. Unit roundoff is then given by $u = \frac{1}{2}\beta^{1-t} = 1 \times 10^{-15}$.

1 Kahan's Compensated Summation - Iterated Error

1.1 Algorithm

```
sort x by descending absolute value;
s = 0; e = 0;
for i = 1:N
    temp = s;
    y = x(i) + e; %Error-compensated summand
    s = temp + y;
    e = (temp - s) + y; %Error for next iteration
end
```

1.2 Error Analysis

$|E_N| \leq (2u + O(Nu^2)) \sum_{i=1}^{N} |x_i|$

2 Kahan's Compensated Summation - Cumulative Error

2.1 Algorithm

```
sort x by descending absolute value;
s = 0; e = 0;

for i = 1:N
    temp = s;
    y = x(i);
    s = temp + y;
```

```
    e = e + (temp - s) + y;
end
s = s + e;
```

2.2 Error Analysis

$|E_N| \leq (2u + N^2u^2)\sum_{i=1}^{N}|x_i|$ when $Nu \leq 0.1$

3 Priest's Doubly Compensated Summation

3.1 Algorithm

```
sort x by descending absolute value;
s = x(1); c = 0;
for i = 2:l
    xi = x(i);
    y = c + xi; u = xi - (y - c);
    t = y + s; v = y - (t - s);
    z = v + u;
    s = t + z;
    c = z - (s - t);
end
```

3.2 Error Analysis

When $N \leq \beta^{t-3}$, then we have that $|E_N| \leq 2u|s|$.

4 Neumaier's Compensated Summation

4.1 Algorithm

```
sort x by descending absolute value;
s = 0; e = 0;
for i = 1:N
    temp = s + x(i);
    if abs(s) >= abs(x(i))
        e = e + (s - temp) + x(i);
    else
        e = e + (x(i) - temp) + s;
    end
    s = temp;
end
s = s + e;
```

4.2 Error Analysis

$$|E_N| \leq ((A+B)u + Cu^2) \sum_{i=1}^{N} |x_i|.$$

5 Pairwise Summation

5.1 Algorithm

```
y = x;
while length(y)>1
     y = [y1 + y2,...];
end
s = y;
```

5.2 Error Analysis

$$|E_N| \leq \gamma \sum_{i=1}^{N} |x_i| \text{ where } \gamma = \left|1 - (1+u)^{\log_2 N}\right|.$$