Notes on *Topics in Functional Analysis and Applications*

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Questions

- Page 112. For proof of corollary in Section 3.1. Proof of $(x_1 - \pi_K x_1, \pi_K x_2 - \pi_K x_1) \leq 0$ where K is a closed convex subset of a Hilbert space H.

1 Distributions

1.1 Introduction

Example 1.1.1 The following equation

$$u_t + uu_x = 0,\ x \in \mathbb{R},\ t > 0$$

is called **Burger's Equations** and is related to a class of partial differential equations known as hyperbolic conservation laws.

Example 1.1.2 The **Laplace operator** is defined by

$$\Delta u = u_{xx} + u_{yy}$$

Notation 1.1.3 A **multi-index** is an $n-$tuple $\alpha = (\alpha_1, ..., \alpha_n)$ where $\alpha_i \in \mathbb{Z}^+$.

Associated with a multi-index α, we have the notation

- $|\alpha| = \alpha_1 + ... + \alpha_n$
- $\alpha! = \alpha_1! \cdot ... \cdot \alpha_n!$
- $x^\alpha = x_1^{\alpha_1} \cdot ... \cdot x_n^{\alpha_n}$ $x \in \mathbb{R}^n$
- $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \ \partial x_n^{\alpha_n}}$

Multi-index notation is important for describing higher order derivatives in $\mathbb{R}^n$, where the derivatives of order m are given by D^α for all possibilities and permutations of multi-indices α such that $|\alpha| = m$.

For example, for $n = 2$ and $\alpha = (2, 1)$, we have $D^\alpha = \frac{\partial^3}{\partial x_1^2 \partial x_2}$.

1.2 Test Functions and Distributions

Definition 1.2.1 Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. The **support** of ϕ is defined as

$$supp(\phi) = \{x \in \mathbb{R}^n | \bar{\phi}(x) \neq 0\}$$

Furthermore, if $supp(\phi)$ is a compact set, ϕ is said to have **compact support**.

Definition 1.2.2 Let $\Omega \subseteq \mathbb{R}^n$ and let ϕ have compact support on Ω. If $\phi \in C^\infty(\Omega)$, then ϕ is a **test function** of Ω. The space of test functions of Ω is a vector space over $\mathbb{R}$ and is denoted $\mathscr{D}(\Omega)$. As convention, $\mathscr{D}(\mathbb{R}^n) = \mathscr{D}$.

Remark $\mathscr{D}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, which is a Hilbert space.

Theorem 1.2.3 (Locally Finite C^∞ Partition of Unity) Let $\Omega \subseteq \mathbb{R}^n$ be an open set such that

$$\Omega = \cup_{i \in I} \Omega_i \text{ where each } \Omega_i \text{ is open.}$$

Then there exists $\phi_i \in C^\infty(\Omega)$ such that

(i) $supp(\phi_i) \subset \Omega_i$

(ii) $\{supp(\phi_i)\}_{i \in I}$ is a locally finite collection

(iii) $0 \leq \phi_i(x) \leq 1$ for all $i \in I$

(iv) $\sum_{i \in I} \phi_i = 1$

Corollary 1.2.4 Given $K \subset \mathbb{R}^n$ compact, there exists $\phi \in \mathscr{D}(\mathbb{R}^n)$ such that $\phi = 1$ on K.

Definition 1.2.5 A sequence $\{\phi_n\}_{n \geq 1}$ **converges** to ϕ in $\mathscr{D}(\Omega)$ if

(i) For all α the sequence $\{D^\alpha \phi_n\}_{n \geq 1}$ converges uniformly to $D^\alpha \phi$

(ii) There exists $K \subset \mathbb{R}^n$ compact such that $supp(\phi_n) \subset K$ for all n.

Definition 1.2.6 A **distribution** is a linear functional T with the property that

$$\phi_n \to 0 \Rightarrow T(\phi_n) \to 0$$

for any such convergent sequence $\{\phi_n\}_{n \geq 1} \subset \mathscr{D}(\Omega)$. The space of distributions is denoted $\mathscr{D}'(\Omega)$ and, as convention, $\mathscr{D}'(\mathbb{R}^n) = \mathscr{D}'$

Definition 1.2.7 A function $f : \Omega \to \mathbb{R}$ is **locally integrable** if

$$\int_K |f| < \infty$$

for every $K \subset \Omega$ compact.

Every locally integrable function induces a distribution by

$$T_f(\phi) = \int_\Omega f\phi$$

Example 1.2.8 For any $x \in \mathbb{R}^n$, define the distribution

$$\delta_x(\phi) = \phi(x), \ \phi \in \mathscr{D}$$

When $x = 0$, we define $\delta = \delta_0$ which is called the **Dirac $\delta-$ function**.

This distribution is not induced by any locally integrable function.

Example 1.2.9 (Measures as distributions) Let μ be a Borel measure or positive measure such that $\mu(K) < \infty$ for all $K \in \mathbb{R}^n$ compact. The assignment

$$\phi \mapsto \int_{\mathbb{R}^n} \phi d\mu, \ \phi \in \mathscr{D}(\mathbb{R}^n)$$

defines a distribution. Note that every locally integrable function induces such a measure.

Example 1.2.10 The **doublet or dipole distribution**, denoted $\delta^{(1)}$ is defined by

$$\phi \mapsto \phi'(0), \ \phi \in \mathscr{D}(\mathbb{R})$$

More generally, we can define $\delta^{(n)}$

$$\phi \mapsto \phi^{(n)}(0), \ \phi \in \mathscr{D}(\mathbb{R})$$

Note that the space of distributions is "larger" than the space of measures, which is larger than the space of locally integrable functions.

Using integration by parts, $\int_\mathbb{R} f'\phi = -\int_\mathbb{R} f\phi'$ for f differentiable and $\phi \in \mathscr{D}(\mathbb{R})$. Analogously, $\delta'(\phi) = -\delta(\phi') = -\phi'(0) = -\delta^{(1)}(\phi)$.

Theorem 1.2.11 Given $\Omega \subseteq \mathbb{R}^n$ open, then $T \in \mathscr{D}'(\Omega) \iff$ for every $K \subset \Omega$ compact, there exists a constant $C_K > 0$ and an integer N_K such that

$$|T(\phi)| \leq C_K \|\phi\|_{N_K} \text{ for all } \phi \in \mathscr{D}(\Omega) \text{ with } supp(\phi) \subset K$$

Here $\|\phi\|_{N_K}$ is defined as the maximum value attained on Ω by the absolute values of ϕ and all its derivatives up to order N_K.

If for $T \in \mathscr{D}'(\Omega)$ there exists some smallest finite order N that satisfies the condition for all K compact, then T has order N. If not, then T has infinite order.

1.3 Operations with Distributions

Definition 1.3.1 Given a distribution $T \in \mathscr{D}'(\Omega)$, the **derivative** is given by

$\Omega = \mathbb{R}$

- If $T = T_f$ for $f \in C^1$ (and thus locally integrable), then for $\phi \in \mathscr{D}(\mathbb{R})$

$$T_f'(\phi) = \int_{\mathbb{R}} f'\phi = -\int_{\mathbb{R}} f\phi' = -T_f(\phi')$$

- For more general distributions, we generalize this notion

$$T'(\phi) = -T(\phi')$$

- Iterating this operation, we have

$$T^{(k)}(\phi) = (-1)^k T(\phi^{(k)})$$

$\Omega \subseteq \mathbb{R}^n$ For open subsets of $\mathbb{R}^n$ we use multi-index notation

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

Consider that $T \in \mathscr{D}'(\mathbb{R})$ implies that $T' \in \mathscr{D}'(\mathbb{R})$ since $\phi_n \to 0$ implies $\phi_n' \to 0$. More generally, the space of distributions is closed under differentiation.

Example 1.3.2 Consider the **Heaviside function**

$$H(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Since this function is locally integrable, this defines a distribution $T_H(\phi) = \int_{\mathbb{R}} H\phi$. Then consider that

$$T_H'(\phi) = -T_H(\phi') = \int_0^\infty \phi' = \phi(0) = \delta(\phi)$$

So $T_H' = \delta$.

Remark 1.3.3 Given $\psi \in C^\infty(\Omega)$ and $T \in \mathscr{D}'(\Omega)$ we have

$$(\psi T)(\phi) = T(\psi\phi) \text{ for all } \phi \in \mathscr{D}(\Omega)$$

If T corresponds to a C^∞ function f so that $T = T_f$, then $\psi T = T_{\psi f}$.

Theorem 1.3.4 (Product Rule)

$$(\psi T)' = \psi T' + \psi' T$$

Theorem 1.3.5 (Leibniz's Formula) Let $\Omega \subseteq \mathbb{R}^n$ be open and $\psi \in C^\infty(\Omega)$, $T \in \mathscr{D}'(\Omega)$. Then for any multi-index α

$$D^\alpha(\psi T) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta \psi D^{\alpha-\beta} T$$

Theorem 1.3.6 Let $T_m \to T$ in $\mathscr{D}'(\Omega)$. Then for any multi-index α, we have

$$D^\alpha T_m \to D^\alpha T \text{ in } \mathscr{D}'(\Omega)$$

1.4 Supports and Singular Supports of Distributions

Definition 1.4.1 The **support** of a distribution is the complement of the largest open set on which the distribution vanishes.

1.5 Convolution of Functions

Definiton 1.5.1 Given $f, g \in L^1(\mathbb{R}^n)$, the **convolution** of f and g, $h = f * g$ is defined as

$$h(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

Theorem 1.5.2 Convolution is a commutative and associative binary operation on $L^1(\mathbb{R}^n)$.

Theorem 1.5.3 Fix $p \in (1, \infty)$. Given $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $f * g$ is well-defined, $f * g \in L^p(\mathbb{R}^n)$, and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

1.6 Convolution of Distributions

Notation 1.6.1 Let $u : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$. Define

(Translation Operator $\tau_x u(y) = u(y-x)$

(Sign Operator) $u^\vee(y) = u(-y)$

Using this notation we can verify that

$$(\tau_x u)^\vee = \tau_{-x}(u^\vee)$$
$$\tau_x \tau_y = \tau_{x+y}$$

1.7 Fundamental Solutions

1.8 Fourier Transform

Definition 1.8.1 Let $f \in L^1(\mathbb{R}^n)$. The **Fourier Transform** of f, denoted $\hat{f}$, is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)dx$$

where $x\xi = \sum_{j=1}^n x_j \xi_j$ is the usual inner product on $\mathbb{R}^n$.

Theorem 1.8.2 Given $f \in L^1(\mathbb{R}^n)$, $\hat{f}$ is uniformly continuous.

Theorem 1.8.3

2 Sobolev Spaces

2.1 Definition and Basic Properties

Definition 2.1.1 An integer $m > 0$ and $p \in [1, \infty]$ define the Sobolev space defined by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) |\ D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq m\}$$

In other words, $W^{m,p}$ is the vector space of functions in $L^p(\Omega)$ such that all distribtuion derivatives up to order m are also in $L^p(\Omega)$.

$W^{m,p}$ is equipped with the norm

$$\|u\|_{m,p,\Omega} = \textstyle\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}$$

When $p < \infty$, this norm is

$$\|u\|_{m,p,\Omega} = (\textstyle\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^p)^{1/p}$$

Remark 2.1.2

(i) Of particular importance is $L^2(\Omega)$, so we denote

$$H^m(\Omega) = W^{m,2}(\Omega)$$

with the norm $\|u\|_{m,\Omega} = \|u\|_{m,2,\Omega}$.

(ii) The semi-norm on Sobolev spaces is defined by

$$|u|_{m,p,\Omega} = \textstyle\sum_{|\alpha| = m} \|D^\alpha u\|_{L^p(\Omega)}.$$

For $H^m(\Omega)$, the semi-norm is denoted $|.|_{m,\Omega}$.

(iii) For the case when $m = 0$, $W^{m,p}(\Omega) = L^p(\Omega)$

Definition 2.1.3 Given $u, v \in\ H^m(\Omega)$, we define the inner product

$$(u, v)_{m,\Omega} = \textstyle\sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D\alpha v$$

Remark 2.1.4 In the case $\Omega = \mathbb{R}^n$, $H^m(\mathbb{R}^n)$ can be defined by the Fourier transform.

Fact 2.1.5 For $\Omega \subset \mathbb{R}^n$, the map $\phi : W^{1,p}(\Omega) \to (L^p(\Omega))^{n+1}$ defined by

$$u \mapsto (u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n})$$

is an isometry when provided with the $(L^p(\Omega))^{n+1}$ norm

$$\|u\| = \sum_{i=1}^{n+1} |u_i|_{0,p,\Omega} \text{ or } \|u\| = (\sum_{i=1}^{n+1} |u_i|_{0,p,\Omega}^p)^{1/p}$$

Theorem 2.1.6 Consider $W^{1,p}(\Omega)$

- If $p \in [1, \infty]$, then $W^{1,p}(\Omega)$ is a Banach space.
- If $p \in (1, \infty)$ then $W^{1,p}(\Omega)$ is reflexive.
- If $p \in [1, \infty)$ then $W^{1,p}(\Omega)$ is separable.
- $H^1(\Omega)$ is a separable Hilbert space.

Proof. Let $p \in [1, \infty]$. Let $\{u_m\}_{m\geq 1} \subset W^{1,p}(\Omega)$ be a Cauchy sequence. It is enough to show that $\{u_m\}_{m\geq 1}, \{\frac{\partial u_m}{\partial x_i}\}_{m\geq 1} \subset L^p(\Omega)$ converge Considering the norm on the Sobolev space, it is enough to show that $\{u_m\}_{m\geq 1}, \{\frac{\partial u_m}{\partial x_i} \subset L^p(\Omega)$ converge in $L^p(\Omega)$ for all $1 \leq i \leq n$, show that differentiation holds through convergence, and extend to higher derivatives. Using this norm, it is clear that $\{u_m\}_{m\geq 1}$ being Cauchy in $W^{1,p}(\Omega)$ implies that $\{u_m\}_{m\geq 1}, \{\frac{\partial u_m}{\partial x_i}$ are Cauchy in $L^p(\Omega)$. Since $L^p(\Omega)$ is a Banach space, these sequences converge to, say, u and v_i in $L^p(\Omega)$.

We then want to show that $v_i = \frac{\partial u}{\partial x_i}$ a.e. For any $\phi \in \mathscr{D}(\Omega)$,

$$\int_\Omega \frac{\partial u_m}{\partial x_i}\phi = -\int_\Omega u_m \frac{\partial \phi}{\partial x_i}$$

Since $u_m, \frac{\partial u_m}{\partial x_i}$ converge in $L^p(\Omega)$ and $\phi \in \mathscr{D}(\Omega)$, we have

$$\lim_{m\to\infty} |\int_\Omega \frac{\partial u_m}{\partial x_i}\phi - \frac{\partial u}{\partial x_i}\phi| \leq \lim_{m\to\infty} \|(\frac{\partial u_m}{\partial x_i} - \frac{\partial u}{\partial x_i})\phi\|_1 \leq$$
$$\lim_{m\to\infty} \|\frac{\partial u_m}{\partial x_i} - \frac{\partial u}{\partial x_i}\|_p \cdot \|\phi\|_{\frac{p}{p-1}} = 0$$

Similarly, $\lim_{m\to\infty} \int_\Omega u_m \frac{\partial u_m}{\partial x_i} = \int_\Omega u \frac{\partial u}{\partial x_i}$. Then

$$\int_\Omega \frac{\partial u}{\partial x_i}\phi = -\int_\Omega u \frac{\partial \phi}{\partial x_i} = \lim_{m\to\infty} -\int_\Omega u_m \frac{\partial \phi}{\partial x_i} = \lim_{m\to\infty} \int_\Omega \frac{\partial u_m}{\partial x_i}\phi = \int_\Omega v_i \phi$$

So $W^{1,p}(\Omega)$ is complete. □

This Theorem extends for $m \geq 2$.

Definition 2.1.7 A function $f : [a, b] \to \mathbb{R}$ is **absolutely continuous** if, for all $x \in [a, b]$,

$$f(x) = f(a) + \int_a^x f'(t)dt$$

Theorem 2.1.8 Let $I \subset \mathbb{R}$ be open. If $u \in W^{1,p}(I)$, then u is absolutely continuous.

Theorem 2.1.9 Given $p \in [1, \infty)$, for any $m \geq 0$, we have that $W^{m,p}(\mathbb{R}^n) = W_0^{m,p}(\mathbb{R}^n)$.

2.2 Approximation by Smooth Functions

Theorem 2.2.1 (Friedrichs) Let $p \in [1, \infty)$. Given $u \in W^{1,p}(\Omega)$,

- There exists a sequence $\{u_m\}_{m \geq 1} \subset \mathscr{D}(\mathbb{R}^n)$ such that $u_m \to u$ in $L^p(\Omega)$
- For every $R \subset \Omega$ relatively compact and every $i \in [1, n]$, we have convergence

$$\tfrac{\partial u_m}{\partial x_i}|_R \to \tfrac{\partial u}{\partial x_i}|_R \text{ in } L^p(R)$$

Proof. (Step 1) Let $\{p_\epsilon\}$ be a family of mollifiers. Define

$$\bar{u} = \begin{cases} u \; x \in \Omega \\ 0 \; x \notin \Omega \end{cases}$$

Then $p_\epsilon * \bar{u}$ converges to u in $L^p(\Omega)$.

(Step 2)

$\square$

Definition 2.2.2 An **extension operator** P for $W^{1,p}(\Omega)$ is a bounded linear operator $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that $Pu|_\Omega = u$ for all $u \in W^{1,p}(\Omega)$.

A sufficient condition for the existence of P is that the boundary of Ω is smooth.

Theorem 2.2.3 Let Ω be such that an extension operator P exists. Then, given $u \in W^{1,p}(\Omega)$, there exists a sequence $\{u_m\}_{m \geq 1} \subset \mathscr{D}(\mathbb{R}^n)$ such that $u_m|_\Omega$ converges to u in $W^{1,p}(\Omega)$.

Remark 2.2.4 $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for $p \in [1, \infty)$.

Theorem 2.2.5 (Chain Rule) Let $G \in C^1(\mathbb{R})$ such that $G(0) = 0$ and G' is bounded. Given $u \in W^{1,p}(\Omega)$, $G \circ u \in W^{1,p}(\Omega)$ and

$$\tfrac{\partial}{\partial x_i}(G \circ u) = (G' \circ u)\tfrac{\partial u}{\partial x_i} \text{ for } 1 \leq i \leq n$$

Theorem 2.2.6 Let $p \in [1, \infty)$ and $u \in W^{1,p}(\Omega)$ be such that u vanishes outside a compact set contained in Ω. Then $u \in W_0^{1,p}(\Omega)$.

Theorem 2.2.7 (Stampacchia) Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $G(0) = 0$. If Ω is bounded, $p \in (1, \infty)$, and $u \in W_0^{1,p}(\Omega)$, then $G \circ u \in W_0^{1,p}(\Omega)$.

Corollary 2.2.8 Let $\Omega \in \mathbb{R}^n$ be bounded and open. If $u \in H_0^1(\Omega)$, then

$$\max\{u(x), 0\}, \max\{-u(x), 0\}, |u(x)| \in H_0^1(\Omega)$$

Theorem 2.2.9 Let $p \in [1, \infty]$ and $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. If $u = 0$ on $\partial\Omega$, then $u \in W_0^{1,p}(\Omega)$.

2.3 Extension Theorems

One of the fundamental methods of providing extensions is the **method of reflection**. This can be used to show that the half-space has the extension property.

Notation 2.3.1 Let $x \in \mathbb{R}^n$, $x = (x_1, ..., x_n)$. We set $x' = (x_1, ..., x_{n-1})$ and write $x = (x', x_n)$. Define

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_n > 0\}$$

Theorem 2.3.2 Let $u \in W^{1,p}(\mathbb{R}^n_+)$. Define u^* on $\mathbb{R}^n$ by

$$u^*(x) = \begin{cases} u(x', x_n), & x_n > 0 \\ u(x', -x_n), & x_n < 0 \end{cases}$$

Then $u^* \in W^{1,p}(\mathbb{R}^n)$ and we have

- $|u^*|_{0,p,\mathbb{R}^n} \leq 2|u|_{0,p,\mathbb{R}^n_+}$
- $|u^*|_{1,p,\mathbb{R}^n} \leq 2|u|_{1,p,\mathbb{R}^n_+}$

Finally, we have that the map $u \mapsto u^*$ defines an extension operator $W^{1,p}(\mathbb{R}^n_+) \to W^{1,p}(\mathbb{R}^n)$

Corollary 2.3.3 For $p \in [1, \infty)$, we have $\mathscr{D}(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n_+)$ is dense in $W^{1,p}(\mathbb{R}^n_+)$.

This is equally valid for sets of the form

$$Q_+ = \{x \in \mathbb{R}^n | \ \|x'\| < 1, \ 0 < x_n < 1\}$$

where the method of reflection gives an extension operator $W^{1,p}(Q_+) \to W^{1,p}(Q)$.

Definition 2.3.4 An open set Ω is **of class** C^k, $k \in \mathbb{Z}^+$, if for every $x \in \partial\Omega$, there exists a neighborhood U of x in $\mathbb{R}^n$ and a map $T : Q \to U$ such that

- T is a bijection
- $T \in C^k(\bar{Q})$, $T^{-1} \in C^k(\bar{U})$
- $T(Q_+) = U \cap Q$, $T(Q_0) = U \cap \partial\Omega$

where $Q_0 = \{x \in Q| \ x_n = 0\}$.

Lemma 2.3.5 Let $u \in W^{1,p}(\Omega)$. If $K \subset \Omega$ is closed and u vanishes outside K, then the function

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \backslash Q \end{cases}$$

is in $W^{1,p}(\mathbb{R}^n)$.

Theorem 2.3.6 Let Ω be of class C^1 with bounded boundary. Then there exists an extension operator $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$.

Corollary 2.3.6 Given, $p \in [1, \infty)$, if Ω is of class C^1 and has $\partial\Omega$ bounded, then $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$.

Theorem 2.3.7 Let $p \in (1, \infty)$ and $u \in W_0^{1,p}(\Omega)$. Then $\tilde{u}$, the extension of u by 0 outside Ω, is in $W^{1,p}(\mathbb{R}^n)$. Additionally, for any integer $1 \leq i \leq n$

$$\tfrac{\partial \tilde{u}}{\partial x_i} = \widetilde{\left(\tfrac{\partial u}{\partial x_i}\right)}$$

Theorem 2.3.8 (Poincare's Inequality) Let Ω be bounded. Then there exists a positive constant C, depending on Ω, p, such that

$$|u|_{0,p,\Omega} \leq C|u|_{1,p,\Omega} \text{ for every } u \in W_0^{1,p}(\Omega)$$

Additionally, $u \mapsto |u|_{0,p,\Omega}$ defines a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the norm $\|.\|_{1,p,\Omega}$. On $H_0^1(\Omega)$, the bilinear form $(u,v) \mapsto \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}$ defines an inner product which defines a norm equivalent to $\|.\|_{1,\Omega}$.

Proof. First let $\Omega_a = (-a, a)^n$ and let $u \in \mathscr{D}(\Omega)$. Then

$$u(x) = \textstyle\int_{-a}^{x_n} \frac{\partial u}{\partial x_n}(x_1, ..., x_{n-1}, t)dt \text{ where } x = (x_1, ..., x_n)$$

This is true since $u \in W_0^{1,p}(\Omega)$, so u vanishes on $\partial\Omega$ and $(x_1, ..., x_{n-1}, -a) \in \partial\Omega$.

Now, let q be the conjugate of p. By Holder's inequality

$$|u(x)| \leq (\textstyle\int_{-a}^{x_n} |\frac{\partial u}{\partial x_n}(x', t)|^p dt)^{1/p} \cdot |x_n + a|^{1/q}$$

(Norm)

(Inner Product)

$\square$

2.4 Imbedding Theorems

Lemma 2.4.1 (Gagliardo) First, some notation: for $x \in \mathbb{R}^n$, define

$$\hat{x}_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in \mathbb{R}^{n-1}$$

Assume $n \geq 2$. Given $f_1, ..., f_n \in L^{n-1}(\mathbb{R}^{n-1})$, define

$$f(x) = f_1(\hat{x}_1) \cdot ... \cdot f_n(\hat{x}_n) \text{ for } x \in \mathbb{R}^n$$

Then $f \in L^1(\mathbb{R}^n)$ and $|f|_{0,1\mathbb{R}^n} \leq \Pi_{i=1}^n |f_i|_{0,n-1,\mathbb{R}^{n-1}}$.

Proof. We proceed by induction

($n = 3$) Let $n = 3$, then by the Cauchy-Schwarz inequality,

$$\int_{\mathbb{R}} |f(x)| dx_3 = |f_3(x_1, x_2)| \big(\int_{\mathbb{R}} |f_1(x_2, x_3)||f_2(x_1, x_3)| dx_3 \big) \leq$$
$$|f_3(x_1, x_3)| \big(\int_{\mathbb{R}} |f_1(x_2, x_3)|^2 dx_3 \big)^{1/2} \big(\int_{\mathbb{R}} |f_2(x_1, x_3)|^2 dx_3 \big)^{1/2}$$

By integrating the above inequality with respect to x_1 and x_2 and applying the Cauchy-Schwarz inequality again,

$$\int_{\mathbb{R}^3} |f(x)| dx \leq \big(\int |f_3(x_1, x_2)|^2 dx_1 dx_2 \big)^{1/2} \cdot \big(\int |f_1(x_2, x_3)|^2 dx_3 dx_2 \big)^{1/2} \cdot$$
$$\big(\int |f_2(x_1, x_3)|^2 dx_1 dx_3 \big)^{1/2}$$

(General Case) We assume the result for n. Let m be the conjugate of n. If we fix x_{n+1}, then by Holder's Inequality,

$$\int_{\mathbb{R}^n} |f(x)| dx_1 ... dx_n \leq |f_{n+1}|_{0,n,\mathbb{R}^n} \big(\int |f_1 ... f_n|^m dx_1 ... dx_n \big)^{1/m}$$

□

Theorem 2.4.2 (Sobolev's Inequality) Given $p \in [1, n)$, there exists a constant $C > 0$, depending on p, n such that

$$|u|_{0,p^*,\mathbb{R}^n} \leq C |u|_{1,p,\mathbb{R}^n} \text{ for all } u \in W^{1,p}(\mathbb{R}^n)$$

where p^* is such that $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ and is called the **Sobolev conjugate**.

Additionally, the inclusion map $W^{1,p}(\mathbb{R}^n) \to L^{p^*}(\mathbb{R}^n)$ is continuous.

Corollary 2.4.3 Let $p \in [1, n)$. Then the inclusion maps $W^{1,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, where $q \in [p, p^*]$, are continuous.

Corollary 2.4.4 Given $u \in W_0^{1,p}(\Omega)$, then $u \in L^q(\Omega)$ for $q \in [p, p^*]$ and there exists a constant $C > 0$, depending on p, n, such that

$$\left.\begin{array}{l} |u|_{0,p^*,\Omega} \leq C|u|_{1,p,\Omega} \\ |u|_{0,q,\Omega} \leq C\|u\|_{1,p,\Omega} \end{array}\right\} \text{ for all } u \in W_0^{1,p}(\Omega)$$

Theorem 2.4.5 $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for all $q \in [n, \infty)$.

Theorem 2.4.6 Let $p > n$, $\Omega \subseteq \mathbb{R}^n$. Then the inclusion map $W^{1,p}(\Omega) \to L^\infty(\Omega)$ is continuous.

Further, there exists a constant $C > 0$ depending on p, n such that

$$|u(x) - u(y)| \leq C|x - y|^\alpha |u|_{1,p,\Omega} \text{ a.e. in } \Omega \text{ for every } u \in W^{1,p}(\Omega)$$

where $\alpha = 1 - n/p$.

Theorem 2.4.7 Let Ω be R^n_+ or an open set of class C^1 with bounded boundary $\partial\Omega$. Then the inclusion maps

(i) For $p \in [1, n)$, $W^{1,p}(\Omega) \to L^{p^*}(\Omega)$

(ii) For $p = n$, $W^{1,n}(\Omega) \to L^q(\Omega)$ for $q \in [n, \infty)$

(iii) For $p > n$, $W^{1,p}(\Omega) \to L^\infty(\Omega)$

are all continuous.

Further, in case (iii), u is Holder continuous of exponent $\alpha = 1 - n/p$. In particular,

$$W^{1,p}(\Omega) \subset C(\bar{\Omega}) \text{ for } p > n$$

Example 2.4.8 Let $\Omega = B(0, \frac{1}{2}) \subset \mathbb{R}^2$. Let $r = |x| = (\sum_{i=1}^n |x_i|^2)^{1/2}$. Define

$$u(x) = \log(\log(2/r)),\ x \in \Omega$$

Then $u \notin L^\infty(\Omega)$ because of the singularity at the origin. However, we will show that $u \in H^1(\Omega)$ (so that p=2=n). First of all, $u \in L^2(\Omega)$, for

$$\int_\Omega |u|^2 = \int_0^{2\pi} d\theta \int_0^{1/2} r \log(\log(2/r))^2 dr$$

and an application of L'Hopital's rule will show that the integrand is a bounded and continuous function on $(0, \frac{1}{2})$ and thus the integral is finite.

Now we will show that the distribution derivatives are the same as the classical derivatives on $\Omega \backslash \{0\}$. To see this, let $\Omega_\epsilon = \{x |\ \epsilon < r < \frac{1}{2}\}$. If $\phi \in \mathscr{D}(\Omega)$, we have

$$\frac{\partial u}{\partial x_i}(\phi) = -\int_\Omega u \frac{\partial \phi}{\partial x_i} = -\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} u \frac{\partial \phi}{\partial x_i}$$

If we define u_{x_i} to be the classical partial derivative on Ω_ϵ, then by Green's theorem (???),

$$-\int_{\Omega_\epsilon} u \frac{\partial \phi}{\partial x_i}$$

Theorem 2.4.9 Let $m \geq 1$ be an integer and $p \in [1, \infty)$, then

(i) $\frac{1}{p} - \frac{m}{n} > 0$ implies $W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ where q satisfies $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$

(ii) $\frac{1}{p} - \frac{m}{n} = 0$ implies $W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$, $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$ for $q \in [p, \infty)$

(iii) $\frac{1}{p} - \frac{m}{n} < 0$ implies $W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$

2.5 Compactness Theorems

Theorem 2.5.1 Let $\Omega' \subset \Omega$ be a relatively compact subset. For $p \in [1, \infty)$, let $\mathscr{F} \subset L^p(\Omega)$. Suppose for every $\epsilon > 0$ that there exists $\delta > 0$ such that

(i) $\delta < dist(\Omega', \mathbb{R}^n \backslash \Omega)$

(ii) For every $h \in \mathbb{R}^n$ with $|h| < \delta$, we have for all $f \in \mathscr{F}$

$$|f(\cdot - h) - f|_{0,p,\Omega'} < \epsilon$$

Then the set $\mathscr{F}|_{\Omega'} = \{f|_{\Omega'} | \ f \in \mathscr{F}\}$ is relatively compact in $L^p(\Omega')$.

Theorem 2.5.2 For $p \in [1, \infty)$, let $\mathscr{F} \subset L^p(\Omega)$ be bounded. Assume that

(i) for every $\epsilon > 0$ and every relatively compact subset $\Omega' \subset \Omega$ that there exists a $\delta > 0$ such that $\delta < dist(\Omega', \mathbb{R}^n \backslash \Omega)$ and that

$$|f(\cdot - h) - f|_{0,p,\Omega'} < \epsilon$$

for all $h \in B_\delta(0)$ and all $f \in \mathscr{F}$

(ii) For every ϵ there exists a relatively compact subset $\Omega' \subset \Omega$ such that

$$|f|_{0,p,\Omega \backslash \Omega'} < \epsilon \text{ for every } f \in \mathscr{F}$$

Then $\mathscr{F}$ is relatively compact in $L^p(\Omega)$.

Lemma 2.5.3 Let $p \in [1, \infty]$. Let $u \in W^{1,p}(\Omega)$. Then for every $\Omega' \subset\subset \Omega$ and every $h \in \mathbb{R}^n$ with $|h| < dist(\Omega', \mathbb{R}^n \backslash \Omega$, we have

$$|u(\cdot - h) - u|_{0,p,\Omega'} \leq |h||u|_{1,p,\Omega}$$

Remark 2.5.4 If $p \in (1, \infty]$, the converse is also true. If there exists a constant $C > 0$ such that, for a given $u \in L^p(\Omega)$

$$|u(\cdot - h) - u|_{0,p,\Omega'} \leq C|h|$$

for every $\Omega' \subset\subset \Omega$, every $h \in \mathbb{R}^n$ with $|h| < dist(\Omega', \mathbb{R}^n \backslash \Omega)$, then $u \in W^{1,p}(\Omega)$ and $|u|_{1,p,\Omega} \leq C$. Functions satisfying this condition for $p = 1$ belong to class known as functions of bounded variation.

Theorem 2.5.5 (Rellich-Kondrasov) Let Ω be bounded of class C^1. Then the inclusion maps

(i) for $p < n$, $W^{1,p}(\Omega) \to L^q(\Omega)$ where $q \in [1, p^*)$

(ii) for $p = n$, $W^{1,n}(\Omega) \to L^q(\Omega)$ where $q \in [1, \infty)$

(iii) for $p > n$, $W^{1,p}(\Omega) \to C(\bar{\Omega})$

are compact.

As a remark the inclusion $W^{1,p}(\Omega) \to L^{p^*}(\Omega)$ is never compact.

Theorem 2.5.6 Let Ω be bounded and connected of class C^1. Let $P_m(\Omega)$ denote the space of polynomials with $deg \leq m$. Let $p \in [1, \infty]$, for $v \in W^{m+1,p}(\Omega)$, we denote its equivalence class in $W^{m+1,p}(\Omega)/P_m(\Omega)$ by $\bar{v}$. Let the norm for this quotient space be defined by

$$\|\bar{v}\|_{m+1,p,\Omega} = \inf_{f \in P_m(\Omega)} \|v + f\|_{m,p,\Omega}$$

Then this norm is equivalent to $|v|_{m+1,p,\Omega}$.

Corollary 2.5.7 Let V be a Banach space containing $W^{m+1,p}(\Omega)$ and $\Pi : W^{m+1,p}(\Omega) \to V$ a continuous linear operator. Assume that $\Pi(f) = f$ for all $f \in P_m(\Omega)$. Then there exists a constant $C > 0$ such that

$$\|u - \Pi(u)\|_V \leq C|u|_{m+1,p,\Omega}$$

for all $u \in W^{m+1,p}(\Omega)$.

Theorem 2.5.8 (Poincare-Wirtinger Inequality) There exists a constant $C > 0$ such that for every $u \in W^{1,p}(\Omega)$, $p \in [1, \infty]$

$$|u - \bar{u}|_{0,p,\Omega} \leq C|u|_{1,p\Omega} \text{ where } \bar{u} = \frac{1}{meas(\Omega)} \int_\Omega u$$

Furthermore, if $p < n$, then

$$|u - \bar{u}|_{0,p^*,\Omega} \leq C|u|_{1,p,\Omega}$$

2.6 Dual Spaces, Fractional Order Spaces, and Trace Spaces

Definition 2.6.1 Let $p \in [1, \infty)$ and q be the conjugate of p. The **dual space of** $W_0^{m,p}(\Omega)$, where $m \geq 1$ is an integer, is denoted by $W^{-m,q}(\Omega)$. If $p = 2$, then $H^{-m}(\Omega)$ is the dual space of $H_0^m(\Omega)$.

Remark 2.6.2 $H_0^m(\Omega)$ is a Hilbert space and so, by Riesz Representation, there is a bijection between it and its dual space. However, only when $m = 0$, so for $H_0^0(\Omega)$ (i.e. $L^2(\Omega)$), is the space equivalent to its dual. We have the following dense and continuous inclusions:

$$H_0^1(\Omega) \to L^2(\Omega) \to H^{-1}(\Omega)$$

Theorem 2.6.3 Let $F \in W^{-1,q}(\Omega)$. Then there exist functions $f_0, f_1, ..., f_n \in L^q(\Omega)$ such that

$$F(v) = \int_\Omega f_0 v + \sum_{i=1}^n \int_\Omega f_i \frac{\partial v}{\partial x_i}, \text{ for } v \in W_0^{1,p}$$

and

$$\|F\| = \max_{0 \geq i \geq n} |f_i|_{0,q,\Omega}$$

Further, if Ω is bounded, we may assume $f_0 = 0$.

2.7 Trace Theory

Theorem 2.7.1 Let $\Omega = \mathbb{R}_+^n$. Then there exists a continuous linear map

$$\gamma_0 : H^1(\mathbb{R}_+^n) \to L^2(\mathbb{R}^{n-1})$$

with the property that, if v is continuous on $\bar{\mathbb{R}_+^n}$ then

$$\gamma_0(v) = v|_{\mathbb{R}^{n-1}}$$

This map is called the **trace map** of order 0.

Theorem 2.7.2 The range of the map γ_0 is the space $H^{1/2}(\mathbb{R}^{n-1})$.

Lemma 2.7.3 (Green's Formula) Let $u, v \in H^1(\mathbb{R}_+^n)$. Then

$$\int_{\mathbb{R}_+^n} u\frac{\partial v}{\partial x_i} = -\int_{\mathbb{R}_+^n} \frac{\partial u}{\partial x_i} v \text{ for } 1 \leq i \leq n-1$$
$$\int_{\mathbb{R}_+^n} u\frac{\partial v}{\partial x_n} = -\int_{\mathbb{R}_+^n} \frac{\partial u}{\partial x_n} v - \int_{\mathbb{R}^{n-1}} \gamma_0(u)\gamma_0(v)$$

Corollary 2.7.4 If $u, v \in H^1(\mathbb{R}_+^n)$ and u or $v \in Ker(\gamma_0)$, then

$$\int_{\mathbb{R}_+^n} u\frac{\partial v}{\partial x_i} = -\int_{\mathbb{R}_+^n} \frac{\partial u}{\partial x_i} v \text{ for } 1 \leq i \leq n$$

Lemma 2.7.5 Let $v \in Ker(\gamma_0)$. Then its extension by zero outside $\mathbb{R}_+^n$, denoted $\tilde{v}$, is in $H^1(\mathbb{R}^n)$ and

$$\frac{\partial \tilde{v}}{\partial x_i} = \left(\frac{\partial v}{\partial x_i}\right)^\sim, \text{ for } 1 \leq i \leq n$$

Lemma 2.7.6 Let $p \in [1, \infty)$ and $h \in \mathbb{R}^n$. Then for $f \in L^p(\mathbb{R}^n)$

$$\lim_{h \to 0} |f(\cdot + h) - f|_{0,p,\mathbb{R}^n} = 0$$

Corollary 2.7.7 If $v \in H^1(\mathbb{R}^n)$, then

$$\lim h \to 0 \|v(\cdot + h) - v\| + 1, \mathbb{R}^n = 0$$

Theorem 2.7.8 $Ker(\gamma_0) = H_0^1(\mathbb{R}_+^n)$.

Theorem 2.7.9 (Trace Theorem) Let Ω be bounded of class C^{n+1}. Then there exists a trace map $\gamma = (\gamma_0, ..., \gamma_{m-1}) : H^m)(\Omega) \to (L^2(\Omega))^m$ such that

(i) If $v \in C^\infty(\bar{\Omega})$, then $\gamma_0(v) = v|_{\partial\Omega}, \gamma_1(v) = \frac{\partial}{\partial \eta}(v)|_{\partial\Omega}, ..., \gamma_{m-1}(v) = \frac{\partial}{\partial \eta^{m-1}}(v)|_{\partial\Omega}$ where η is the unit exterior normal to the boundary of Ω

(ii) The range of γ is the space $\Pi_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$

(iii) The kernel of γ is $H_0^m(\Omega)$

Theorem 2.7.10 (Green's Theorem) Let Ω be bounded of class C^1 lying on the same side of its boundary $\partial\Omega$ (???). Let $u, v \in H^1(\Omega)$. Then for $1 \leq i \leq n$, we have

$$\int_\Omega u \frac{\partial v}{\partial x_i} = -\int_\Omega \frac{\partial u}{\partial x_i} v + \int_{\partial\Omega} (\gamma_0 u)(\gamma_0 v) \nu_i$$

3 Weak Solutions of Elliptic Boundary Value Problems

3.1 Abstract Variational Problems

Theorem 3.1.1 Let H be a Hilbert space and $K \subset H$ be closed, convex. Given $x \in H$, there exists a unique $y \in K$ such that

$$\|x - y\| = \min_{z \in K} \|x - z\|$$

In other words, $(x - y, z - y) \leq 0$ for all $z \in K$.

Proof. Let $d = \inf_{z \in K} \|x - z\|$ and let $\{y_m\}_{m \geq 1}$ be a sequence such that $\|x - y_m\| \leq d + \frac{1}{m}$. Since H is a Hilbert space, there exists a convergent subsequence $\{y_{m_k}\}_{k \geq 1}$ that converges to $y \in H$. Since K is closed, this implies that $y \in K$. So $\|x - y\| \geq d$. Using convexity and the parallelogram identity, we have $\|x - y\| \leq d$. So $\|x - y\| = d$. Another application of the parallelogam identity shows uniqueness. □

Corollary 3.1.2 Let H be a Hilbert space and $K \subset H$ be closed, convex. Define π_K to be the projection map that exists due to the previous theorem. Then for $x, y \in H$,

$$\|\pi_K x - \pi_K y\| \leq \|x - y\|$$

Proof. Let $x_1, x_2 \in H$ and $\pi_K : H \to K$ the standard projection map. First, we must show

$$(x_1 - \pi_K x_1, \pi_K x_2 - \pi_K x_1) \leq 0$$

and

$$(x_2 - \pi_K x_2, \pi_K x_1 - \pi_K x_2) \leq 0$$

Without loss of generality, we will prove the first inequality. These two inequalities imply the following

$$(\pi_K x_1 - \pi_K x_2, \pi_K x_1 - \pi_K x_2) \leq (x_1 - x_2, \pi_K x_1 - \pi_K x_2)$$

This implies, by the Cauchy-Schwarz Inequality

$$\|\pi_K x_1 - \pi_K x_2\|^2 \leq (x_1 - x_2, \pi_K x_1 - \pi_K x_2) \leq \|x_1 - x_2\| \cdot \|\pi_K x_1 - \pi_K x_2\|$$

This implies that $\|\pi_K x - \pi_K y\| \leq \|x - y\|$. □

Definition 3.1.3 Let $a : H \times H \to \mathbb{R}$ be a bilinear form.

- a is **continuous** if there exists a constant $M > 0$ such that

$$\|a(u, v)\| \leq M\|u\|\|v\| \text{ for all } u, v \in H$$

- a is $H-$elliptic if there exists a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha\|v\|^2 \text{ for all } v \in H$$

Theorem 3.1.4 Let a be a continuous, symmetric, $H-$elliptic bilinear form on a Hilbert space H and $K \subset H$ a closed convex subset.

- Given $f \in H$, there exists a unique $u \in K$ such that

$$a(u, v - u) \geq (f, v - u) \text{ for all } v \in K.$$

- Further, define $J(v) = \frac{1}{2}a(v, v) - (f, v)$. Then we have

$$J(u) = \min_{v \in K} J(v) = \min_{v \in K} \tfrac{1}{2}a(v, v) - (f, v) = \tfrac{1}{2}a(u, u) - (f, u)$$

Proof.

- First, consider that since a is symmetric and bilinear, that $a(\cdot,\cdot)$ forms an inner product on H. Since this is an inner product, it induces a norm, say $|.|_a$.

  Since a is continuous and $H-$elliptic, then $\alpha\|u\|^2 \le |u|_a^2 \le M\|u\|^2$. This implies that $|\cdot|_a$ and $\|\cdot\|$ are equivalent norms.

  Now, define $T \in H^*$ by $T(v) = (f, v)$. Then by the Riesz Representation Theorem, there exists $\tilde{f} \in H$ such that $a(\tilde{f}, \cdot) = (f, \cdot)$.

  By Theorem 3.1.1, there exists a unique $u \in K$ such that

  $$a(\tilde{f}-u, \tilde{f}-u) = \min_{v\in K} a(\tilde{f}-v, \tilde{f}-v)$$

  and, furthermore, that

  $$a(\tilde{f}-u, \tilde{f}-u) - a(\tilde{f}-v, \tilde{f}-v) \le 0$$

  This results in the conclusion

  $$a(\tilde{f}-u, v-u) \le 0 \implies a(u, v-u) \ge (f, v-u) \text{ for all } v \in K$$

- Now consider

  $$J(v) = \tfrac{1}{2}a(v,v) - (f,v) = \tfrac{1}{2}a(v,v) - a(v,\tilde{f}) + \tfrac{1}{2}a(\tilde{f},\tilde{f}) - \tfrac{1}{2}|\tilde{f}|_a^2 = \tfrac{1}{2}a(v-\tilde{f}, v-\tilde{f}) - \tfrac{1}{2}|\tilde{f}|_a^2$$

  Since $\frac{1}{2}|\tilde{f}|_a^2$ is a constant, the minimum of $J(v)$ is attained by minimizing $a(v-\tilde{f}, v-\tilde{f})$. So $\min_{v\in K} J(v) = J(u)$.

□

Theorem 3.1.5 (Stampacchia) Let H be a Hilbert space and a be a continuous, $H-$elliptic bilinear form on H. Given $f \in H$, there exists a unique $u \in K$ such that

$$a(u, v-u) \ge (f, v-u) \text{ for all } v \in K$$

Proof. Given $u \in H$, let $A : H \to H$ be a map such that $a(u,v) = (Au, v)$ for all $v \in H$. Since a is bilinear, it follows that A is linear. Since $\|a(u,v)\| \le M\|u\| \cdot \|v\|$, it follows that $\|Au\| \le M\|u\|$. Finally, it is clear that $(Au, u) \ge \|u\|^2$. Then we must find a unique $u \in K$ such that

$$(Au, v-u) \ge (f, v-u) \text{ for all } v \in K$$

We can manipulate this by multiplying by some, for now, arbitrary constant $\rho > 0$ and to arrive at

$$(\rho f - \rho Au, v-u) \le 0 \text{ for all } v \in K$$

it is importnat to remember that these two statements are equivalent since ρ is positive. Then we can add zero to have

$$(\rho f - \rho Au + u - v, v - u) \leq 0$$

Using the statement from Theorem 3.1.1, this is equivalent to saying

$$\|u - (-\rho f + \rho Au + u)\| = \min_{v \in K} \|v - u\|$$

However, since we are looking for u in K, we are then looking for u such that $u = \rho f - \rho Au - u$. Therefore, we need to find $\rho > 0$ so that the function $F_\rho : H \to H$ defined by $F_\rho(h) = \pi_K(\rho f - \rho Au - u)$ is a contraction map. We have by Corollary 3.1.2

$$\|F_\rho(h_1) - F_\rho(h_2)\| = \|\pi_K(\rho f - \rho A h_1 - h_1) - (\pi_K(\rho f - \rho A h_2 - h_2)) = \\ \|\pi_K(h_1 - h_2) - \pi_K(\rho A(h_1 - h_2)\| \leq \|(h_1 - h_2) - \rho A(h_1 - h_2)\|$$

This implies that

$$\|F_\rho(h_1) - F_\rho(h_2)\|^2 \leq \|h_1 - h_2\|^2 - 2\rho(A(h_2 - h_1), h_1 - h_2) + \rho^2 \|A(h_1 - h_2)\|^2 \leq \\ (1 - 2\alpha\rho + M^2\rho)\|h_1 - h_2\|^2$$

So, if we choose ρ so that $\rho < \frac{2\alpha}{M^2}$, this guarantees that F_ρ will be a contraction map. Then F_ρ has the desired unique fixed point $u \in K$ such that $(\rho f - \rho Au, v - u) = \rho(f - Au, v - u) \leq 0$. Then this implies $(f - Au, v - u) \leq 0$, so we have the desired unique $u \in K$. $\square$

Theorem 3.1.6 (Lax-Milgram) Let H be a Hilbert space and a a continuous, $H-$elliptic bilinear form. Given $f \in H$ there exists a unique $u \in H$ such that

$$a(u, v) = (f, v) \text{ for all } v \in H$$

If a is also symmetric, then

$$\tfrac{1}{2}a(v, v) - (f, v) \text{ attains a minimum value at } u$$

Proof. Let K be a closed, convex subspace. Using the Stampacchia Theorem, given $f \in H$, we have a unique $u \in K$ such that

$$a(u, w - u) \geq (f, w - u) \text{ for all } w \in K$$

In particular, we have that $a(u, (u + v) - u) \geq (f, (u + v) - u)$ since $u + v \in K$ for all $v \in K$. this implies that $a(u, v) \geq (f, v)$.

Now, we also have that $a(u, (u - v) - u) \geq (f, (u - v) - u)$. So $a(u, -v) \geq (f, -v)$ which implies that $a(u, v) \leq (f, v)$.

So $a(u, v) = (f, v)$. Since H is automatically a closed, convex subspace, considering the case where $H = K$ proves the theorem. $\square$

Theorem 3.1.7 (Babuska-Brezzi) Let Σ, V be Hilbert spaces. Let $b : \Sigma \times V \to \mathbb{R}$ be a continuous bilinear form with the condition that there exists a constant $\beta > 0$ such that

$$\sup_{\tau\in\Sigma} \frac{b(\tau,v)}{\|\tau\|_\Sigma} \geq \beta\|v\| \text{ for all } v \in V$$

Let $a : \Sigma \times \Sigma \to \mathbb{R}$ be a continuous and $Z-$elliptic where

$$Z = \{\sigma \in \Sigma |\ b(\sigma, v) = 0 \text{ for every } v \in V\}$$

Then, given $i \in \Sigma, j \in V$, there exists a unique $x \in \Sigma, y \in V$ such that

$$\begin{aligned} a(x, \sigma) + b(\sigma, y) &= (i, \sigma) \text{ for all } \sigma \in \Sigma \\ b(x, v) &= (j, v) \text{ for all } v \in V \end{aligned}$$

Proof. (Existence) Let $A : \Sigma \to \Sigma$ be such that $(A(\cdot), \sigma) = a(\cdot, \sigma)$ for all $\sigma \in \Sigma$

$B : \Sigma \to V$ be such that $(B(\cdot), v) = b(\cdot, v)$ for all $v \in V$.

Then $B^* : V \to \Sigma$ is such that $(\sigma, B^*(\cdot)) = b(\sigma, \cdot)$ for all $\sigma \in \Sigma$.

To show that $\|B^*(v)\| \geq \beta\|v\|$ for all $v \in V$, consider

$$\sup_{\sigma\in\Sigma} \frac{b(\sigma,v)}{\|\sigma\|} = \sup_{\sigma\in\Sigma} \frac{(\sigma,B^*v)}{\|\sigma\|} \leq \sup_{\sigma\in\Sigma} \frac{\|\sigma\|\cdot\|B^*v\|}{\|\sigma\|} = \|B^*v\|$$

Using this fact, we can show that $Im(B^*)$ is closed and that $Ker(B^*) = \{0\}$.

Let $\{a_n\}_{n\geq1} \subset Im(B^*)$ be a convergent sequence. Then we can define $\{v_n\}_{n\geq1} \subset V$ so that $B^*(v_n) = a_n$. Using the inequality, we have that $\{v_n\}_{n\geq1}$ is Cauchy. Since V is a Hilbert space, then $v_n \to v_0 \in V$. Then, since B^* is continuous, $\lim_{n\to\infty} B*(v_n) = B^*(v_0)$. So $Im(B^*)$ is closed.

The inequality clearly shows that B^* is surjective. So, B injective and $Im(B) = V$, meaning that there exists $x_0 \in \Sigma$ so that $Bx_0 = j$.

Then, by the Lax-Milgram Theorem, there exists a unique $x_1 \in Z$ such that $a(x_1, \sigma) = (i - Ax_0, \sigma)$ for all $\sigma \in\ Z$.

Therefore, we choose $x = x_0 + x_1$. Since $x_1 \in Z = Ker(B)$, it is clear that $Bx = j$.

Now, since $(i - Ax, \sigma) = 0$ for all $\sigma \in Z$, we see that $i - Ax \in Z^\perp$. We have that $Im(B^*) = Ker(B)^\perp$ since, given $\sigma \in Im(B^*)$ and $v \in Ker(B)$, we let $\sigma = B^*g$

$$(\sigma, v) = (B^*g, v) = (g, Bv) = 0$$

Therefore, there exists some $y \in V$ so that $B^*y = i-Ax$. This along with $Bx = j$ imply the desired result.

(Uniqueness) Suppose (x, y) and (x', y') both satisfy the given conditions. Then

$$a(x-x',\sigma)+b(\sigma,y-y')=0$$
$$b(x-x',v)=0$$

This implies that $x-x'\in Z$. In particular, we have $a(x-x',x-x')=0$. Since a is $Z-$elliptic, this implies that $x-x'=0$. Then we are left with $b(\sigma,y-y')=0$, where $\|B^*(v)\|\geq\beta\|v\|$ implies that $y-y'=0$. So x,y are unique.

□

Remark 3.1.8 The condition on b in the previous theorem is called the **inf-sup condition**.

3.2 Examples of Elliptic Boundary Value Problems

1. *Dirichlet Problem for Second Order Elliptic Operators* Let $\Omega\subset\mathbb{R}^n$ be bounded and open with boundary Γ.

$$-\delta u=f \text{ in } \Omega$$
$$u=0 \text{ in } \Gamma$$

where $f:\Omega\to\mathbb{R}$.

Definition 3.2.1 **classical solution** to such a problem is a function $u\in C^2(\bar\Omega)$ which satisfies the problem conditions pointwise.

If we assume $u\in C^2(\bar\Omega)$ is a classical solution, then we multiply both sides of $-\nabla u=f$ by $\phi\in\mathscr{D}(\Omega)$ and integrate to get

$$-\int_\Omega \delta u\cdot\phi=\int_\Omega f\phi$$

Since ϕ vanishes on $\partial\Omega$, when we apply Green's Theorem we have

$$\int_\Omega \nabla u\cdot\nabla\phi=\int_\Omega f\phi$$

Definition 3.2.2 A **weak solution** to such a problem is a function $u\in H_0^1(\Omega)$ that satisfies

$$\textstyle\int_\Omega \nabla u\cdot\nabla v=\int_\Omega fv \text{ for every } v\in H_0^1(\Omega)$$

Theorem 3.2.3 Let Ω be bounded, open and $f\in L^2(\Omega)$. Then there exists a unique weak solution $u\in H_0^1(\Omega)$ (satisfying 3.2.2) characterized by

$$J(u)=\min_{v\in H_0^1(\Omega)} J(v)$$

where $J(v)=\frac{1}{2}\int_\Omega\nabla v\cdot\nabla v-\int_\Omega fv$.