Client Oriented Portfolio Optimization: Backtest

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Abstract

Since Harry Markovitz established its foundations in the decade of the fifties, Modern Portfolio Theory (MPT) has been widely applied and used by professional and amateur investors as well taught to students in most universities and finance schools. In this study, under the view of the client and its necessities, we perform several back-test computations to see if the application of the Mean-Variance optimization could result in both short and long term profit for the non professional investor in the IBEX35 market, including fees and taxes.

1 Introduction

In the core of MPT resides the fundamental thought that the only statistics needed for describing the end-of-period wealth of an investment are mean and variance. This assumption is intimately related to the assumption that the (log) returns of assets follow a normal distribution.

Given this two intimately related assumptions the main purpose of MPT is to achieve, through diversification, a portfolio with minimal risk or a maximal return portfolio within a given risk level.

As almost all models, and as we have begun, MPT states several previous assumptions in order to further develop its theoretical grounds. We will share some of them but others not, for our purpose is to achieve a certain degree of reality in some of the assumptions that traditionally MPT considered.

- Asset returns are normally distributed random variables.
- The purpose of investors is to maximize economic market returns.
- Investors are rational and avoid risk when possible.
- All investors have access to the same sources of information for investment decisions.
- Taxes and brokerage fees are not considered.
- There are no large enough players in the market to influence the price.
- Unlimited access to borrow (and lend) money at the risk free rate is available.
- No long positions are allowed.

We will focus on the fact that, in the original MPT, fees and taxes are not considered. In our case we will establish a range of feasible fees (some of them real and found in popular brokerage companies) and also some taxes that will apply on the annual revenues obtained from the portfolio. The taxes considered will not be accurate with respect to reality but will be useful to display the effect that they have to the final result of a portfolio optimization strategy.

It is true that the main stated flaw of MPT is that it considers the log-returns of the assets as being normally distributed, which is usually found to be far from what is observed, but we will not focus on this point. Also usually stated and observed, is that not all investors have access to the same information, contrary to what MPT states, but this also lays outside the focus of this study.

Hence, the main point of this back-test study is: would a portfolio optimization strategy based on Mean-Variance theory have been useful to obtain revenues when considering fees and taxes?

1.1 Theory and concepts

In this section the basic theory and concepts needed for the further optimization procedure are stated.

Given a number of assets,N, that have, each of them, a return defined as $r_i = \log \frac{P_t}{P_{t'}}$, where P_t is the price of the asset at time t, and a set of corresponding weights, w_i, assigned to each of the asset, that accomplish $\sum_{i=1}^{N} = 1$, the *expected portfolio return* is:

$$\mu_p = \mathbb{E}(\sum_i w_i r_i) = \sum_i w_i \mathbb{E}(r_i) = \sum_i w_i \mu_i = w^T \mu$$

The other core concept of MPT is *expected portfolio variance* and to define it we have to consider covariances between assets. The covariance of a pair of securities is defined as $\sigma_{ij} = \sigma_{ji} = \mathbb{E}((r_i - \mu_i)(r_j - \mu_j))$, being the variance of an individual security a special case of covariance $\sigma_i = \mathbb{E}((r_i - \mu_i)^2)$. Then, the variance of the portfolio can be defined as

$$\sigma_p^2 = \mathbb{E}((r_p - \mu_p)^2) = \sum_i \sum_j w_i w_j \sigma_{ij} = w^T \Sigma w$$

where Σ is the *covariance matrix* of the portfolio:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_N^2 \end{bmatrix}$$

Important is also to define the *expected portfolio standard deviation*, which is simply the square root of the variance: $\sigma_p = \sqrt{\sigma_p^2}$.

Related to those previous measures and important for our optimization purpose, it is useful to define the *Sharpe Ratio*, $SR = \frac{\mu_p - r_f}{\sigma_p}$, where r_f is the risk free rate. In a short statement, what the Sharpe ratio is measuring is the excess return of the portfolio weighted by the variance of the same portfolio. For the sake of simpleness, for the rest of this study r_f is assumed to be 0. Obviously to state is that, given the fact that the investor prefers increasing return against reducing risk, i.e. the investor is not *risk averse*, what any investor desires is to achieve a Sharpe Ratio as higher as possible.

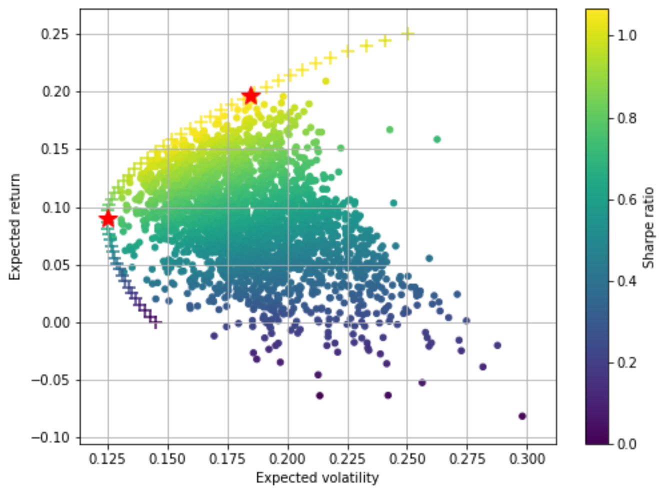

Figure 1: (Points) Universe of possible portfolios, and their expected return and volatility, built from a set of different assets N. (Crosses) Efficient frontier portfolios. (Red stars) Minimum variance and maximum Sharpe ratio efficient portfolios

To proceed further in our purpose of defining the important concepts for our portfolio optimization strategy what is needed to be considered now is the *universe of portfolios* in terms of mean and variance. Given a set of assets, N, like the one it was considered at the beginning of the section, each of one with its return and variance defined, we can consider all the combinations possible between them in terms of the weights, w_i, that they have inside the final portfolio. Then, for each of this combinations (i.e. portfolios), its return its plotted against its variance, as in Figure 1 . There, what is seen is an approximation (because not all combinations are not considered) of all the portfolios, with its final return and variance, that the investor can achieve with the assets defined. At the left, the color bar defines the Sharpe ratio value of each of the portfolios.

Once defined the universe of plausible portfolios, the efficient frontier of portfolios can be stated. An efficient portfolio is the portfolio with the maximum return given a certain variance value or the portfolio with minimum variance given a certain return value. Both definitions are equivalent. The efficient frontier representation can be observed at Figure 1. Among the efficient portfolios the ones with minimum variance and maximum Sharpe ratio can be selected, as seen also in 1. These portfolios are important since they have suitable properties for different kinds of investors. The minimum variance efficient portfolio is the one that has the minimum variance among all conceivable portfolios, useful for a risk averse investor, and the maximum Sharpe efficient portfolio is the one that has the maximum return weighted by variance, useful for a investor which desires maximum return but having into account the variability of the assets.

2 Methodology

In this section, both the strategy followed by the investor and the methodology for the backtest will be defined.

2.1 Investor characteristics

Given an investor, it is supposed that it has five parameters/characteristics that enable the strategy: initial capital, investment window, investment behavior, fees, taxes and portfolio optimization objective.

The initial capital is the quantity that the investor inputs into the portfolio at the beginning of the investment period. The investment window defines the temporal period of interest to have into account when optimizing the portfolio (calculating quantities of interest such as mean or variance) as well as when considering withdrawals or reinvestments in the portfolio. The behavior of the investor refers basically to its attitude to the withdrawal of funds when obtaining gains. It has to be clearly stated that the common behavior stated is the following: the client invests a quantity, never puts any more money in the portfolio, and only subtracts some quantity if the portfolio has acquired more value and if its value is beyond the initial investment. Hence, the investor behavior refers to the percentage withdrawn of the quantity won: it could be, for example, 50%. To be more clear on this, it is better to observe it in an example: if the value a t_{-1} of the portfolio was 1100 €, and at t_0 (now) is 1150€, and the withdrawal behavior is of 50%, the investor will cash 25€ and reinvest 1125€. The time parameter t is in units of the investment window, for example, 2 weeks.

Regarding taxes and fees the following is considered: one tax of 20% of the annual income is considered independently of the investment window and several pairs of absolute-relative fees are applied for each transaction. One example of pair of fees could be 2€ + 0.04% of the value of the transaction.

Regarding the objective of the optimization it considers if the investor wants to have portfolios with maximum Sharpe ratio, minimum variance or maximum return.

2.2 Backtest strategy

The process begins with half a year of data observation. Mean and variances values for the assets considered (IBEX35 data since 2007) are calculated for this period, which could be named *equilibrium period*. Once this period has end the investment strategy begins. With the mean and covariance values an optimized portfolio is established according to the objective of the investor. It has to be stated clearly that the optimization process outcome are exact quantities of assets that should be bought and from which assets. Then the resulting quantity is invested. The resting part of the initial capital is saved. Fees applied to the transactions made are subtracted from the previous saved quantity. Once the investment period has passed, the following steps are followed:

- The portfolio from previous step is revalued.
- If the portfolio has gained value, gains, according to the withdrawal policy of the investor, are subtracted.
- With the final value (counting withdrawals) a new portfolio is optimized having into account only the previous period.
- Fees are paid for the transactions.

This is repeated at each step.

Has to be noted that if the portfolio has not gained any value, no withdrawals are made. Also if the value of the portfolio is under the initial capital value no gains are subtracted. Taxes are only applied at the step placed at the end of each year.

Once the process has finished one main measure is calculated: mean benefits per year.

3 Results

To obtain the results the values stated in Table 1 have been taken into account. This accounts for initial investment quantity, temporal period of interest and percentage of drawback each time there is some benefit. Regarding fees, the values are stated in Table 2.

Time values (days)	Inv. Quantities	% drawback
10	1000	20
20	5000	40
60	10000	60
120	50000	80
240	100000	100

Table 1: Values for the temporal periods, drawback percentatge and initial investment quantities.

Absolute	Relative in %
2	0.4
4	1
8	2

Table 2: Values for the absolute fees in € and for the relative fees in percentage, which are applied to the values of eac transaction.

As it is clear, the absolute profit is proportional to the amount invested. Hence, the comparison is not made between different quantities of investment but between time intervals and drawback degrees.

To better understand the results that are going to be introduced the following has to be taken into account. The following values in the tables are the mean absolute profit per year obtained through the period December, 2007 and March 2018 (note that depending on the interval period the process may have finished earlier) by means of an initial investment. Hence, depending on the drawback rate the mean yearly absolute profit can be higher than the initial investment due to the growth of the latent capital. Also, the per year meaning has not been calculated each year and dong the mean but taking the full final absolute profit and dividing by the total length in natural years that the process has spanned among.

Also important to note is that the values that appear as a 0 mean that all the money invested has been lost and the process has been stopped (no short positions are allowed).

For the same initial quantity invested the absolute mean gain per year can be observed in tables 3 and 4 to see the influence of the diverse fees in the final results.

As can be seen when the value of the fees is higher the strategy does not deliver positive results for a low temporal period of investment. This is due to the fact that more transactions are made, then more fees are paid and the invested quantity cannot hold such big quantities paid in fees.

Another fact that can be observed is that, in table 3, the best result is achieved with a time period of 20 days and a drawback retirement rate of 20%. Meanwhile, in table 4, it can be seen that the best result is achieved also at a 20%, but in this case with a time period of 10 days. From this observation, which is going to be repeatedly observed through all the tables, two conclusions can be extracted.

The first one is that, if the initial investment quantity can hold a huge and expensive number of fees, the strategy will deliver better results using a smaller temporal period of investment than a larger one. This is mainly due to the fact that the optimization process is done at a definite point in time, and as the stock values and their behavior changes the more it is repeated the best it will cover those changes. Nevertheless, this results has to take into account the fees since they will affect the optimum value of the investment time interval. As higher are the fees higher should be the optimum time period.

This can be extended to all the results obtained for each time period. Once one has gone beyond the optimum time interval, the results tend to diminish proportionally to the nearness to the optimum interval. That is, if the optimum interval is 20, the result using a time period of 40 will be higher than a result using a time period of 240. This phenomena can be observed through all tables 3,4, 5 and 6, and is related to the previously commented fact of optimization rearrangement: the less time it is left between optimization the more accurate they will be.

The second result obtained, also can be seen through all the tables, is that the best results are always obtained with the lowest drawback rate. This is due to the fact that leaving most part of revenue as an investment allows the total value invested to grow higher and return higher profits in the future. This result, as it is logical, is not affected by fees. The more we let grow our investment the higher will be the mean revenue obtained each year.

This does not mean that having fixed a time interval the best result is the one with the lower drawback rate. It usually is but with the exception of high time intervals, like yearly operations. In this cases, the reason could be that, as the portfolio is optimized only once a year it is not useful to retire a little to let it grow since it has not been optimized with enough regularity. Instead, it is better to withdraw more to obtain more to obtain more cash.

DR/Δt	10	20	60	120	240
20	0.0	80.10	1504.77	922.19	222.22
40	0.0	42.74	1189.80	901.75	301.35
60	0.0	46.34	726.69	669.53	306.26
80	0.0	60.64	432.86	474.76	277.87
100	0.0	69.08	272.03	330.42	243.46

Table 3: Values of the mean absolute profit per year for an intial investment quantity of 1000€ and an absolute fee of 8€ and 2%. DR referes to drawback rate and Δt to the time interval.

DR/Δt	10	20	60	120	240
20	139470.54	64945.48	5406.22	1664.19	310.05
40	60462.47	22482.26	3632.66	1529.44	416.72
60	11690.21	5943.02	1859.32	1073.46	419.50
80	2175.91	1607.13	950.32	63.86	375.71
100	620.04	666.89	526.65	459.38	315.28

Table 4: Values of the mean absolute profit per year for an intial investment quantity of 1000€ and an absolute fee of 2€ and 0.04%. DR referes to drawback rate and Δt to the time interval.

DR/Δt	10	20	60	120	240
20	6968.77	38602.35	23271.53	11193.51	3147.36
40	5020.97	24446.68	18748.14	11232.70	4281.93
60	2800.07	12452.55	11881.24	8543.03	4194.43
80	2279.74	6339.09	7147.46	5924.45	3724.76
100	2154.68	4449.84	4479.87	4186.00	3242.48

Table 5: Values of the mean absolute profit per year for an intial investment quantity of 10000€ and an absolute fee of 8€ and 2%. DR referes to drawback rate and Δt to the time interval.

DR/Δt	10	20	60	120	240
20	2949601.19	792272.10	54590.19	17250.73	3797.39
40	736137.31	261745.58	39322.71	15972.24	5067.80
60	139895.66	71412.25	20509.37	11295.87	4939.35
80	29301.68	20045.16	10364.95	7294.44	4282.55
100	8838.12	7845.74	5620.30	4780.04	3530.30

Table 6: Values of the mean absolute profit per year for an intial investment quantity of 10000€ and an absolute fee of 2€ and 0.04%. DR referes to drawback rate and Δt to the time interval.

Also it can be observed that the results obtained in tables 3 and 5 are not related by a factor of 10, which is the relating factor between the initial investment values. It is obvious that this change is due to the absolute part of the fees. Hence, as higher the initial investment higher will be the results but the growth is not linear.

Two main works could be further developed in this previous sense. One would be the study of the growth of investments regarding the initial investment (i.e. which nature it has? Exponential, quadratic,..?) and another to really find a function to determine which is the optimum time interval by which invest, and not treat as configuration parameter.

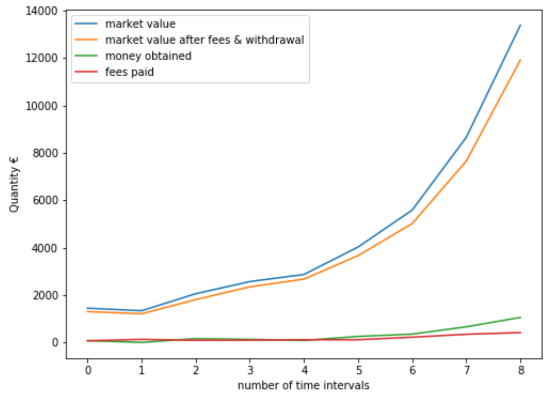

Figure 2: Development of portfoli with an initial investment of 1000€, absolute fee of 2€ and relative of 0.04%, time interval of 240 days, and drawback rate of 20%.

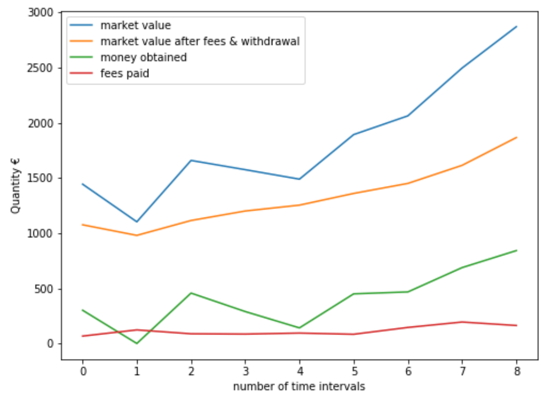

Figure 3: Development of portfoli with an initial investment of 1000€, absolute fee of 2€ and relative of 0.04%, time interval of 240 days, and drawback rate of 80%.

It has to be established that previous tables 3,4, 5 and 6, are obtained using a waiting configuration for the investment behavior. That is, if the value of the investments fall below the initial investment value the investor does not receive or withdraws anything from the obtained. As another behavior type, greedy behavior refers to the fact that, even if the value of the portfolio is under the initial investment, when the portfolio makes a gain compared to the last value, the investor withdraws some quantity regarding its withdrawal policy.

Although the results are not shown here the usual result is below the waiting behaviors. With an exception, that is when the portfolio crashes that some small benefits are collected.

As another important comment, it has to be stated that the overall results are strikingly good and that, being this experiment a back-test, an on-real-time experiment should be conducted to confirm the absolutely good results of the backtest.

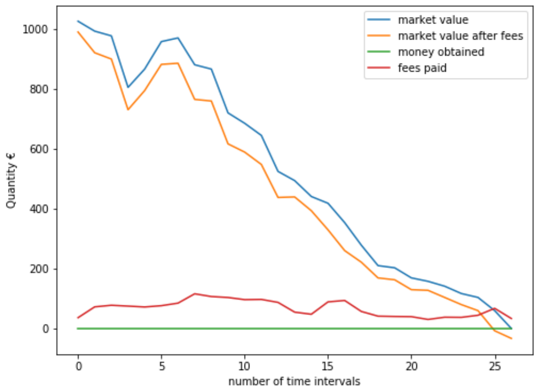

Figure 4: Development of portfoli with an initial investment of 1000€, absolute fee of 8€ and relative of 2%, time interval of 10 days, and drawback rate of 20%.

In figure 2, we can observe the evolution for a portfolio of initial investment as described in the caption. Has to be noted the behavior commented previously: as the portfolio has a low drawback rate its value is allowed to grow more. Its counterpart is figure 3, where the drawback rate has been augmented and the portfolio cannot grow a lot. Nevertheless, and as seen in table 4, the mean absolute yearly profit is more or less the same due to, as said before, the fact that being the time interval too high the portfolio cannot perform the best as it doe snot capture well the stock changes.

Finally, in figure 4 we can observe a portfolio that has totally crashed reaping any benefits and losing all the money.

4 Conclusion

The optimization method has provided positive back-test results. The importance of all the important variables and parameters (i.e. time interval and withdrawal rate) has been established to be able to determine which combination generates more profits depending on the initial investment and fees. Nevertheless, results might seem too strikingly good and hence hands on experimentation will be required to definitively accept this method.

As a main drawback of this method has to be stated that adjusted closing values for stocks have been used. Practically, is difficult to assert this task working on perfect closing prices due to the procedures of the apertures and closures of markets.

This work however, is not a complete task and other objectives remain undone to obtain a full working and profitable product:

- Develop local application based on functions and classes written.
- Develop module for update retrieval of market values.
- Extend to other markets such as: CAC40, DAX30, EUROSTOXX50, FOOTSIE100 and others.
- Include fixed-income products and develop model accordingly.

5 References

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