Analysis of Target Velocity and Position Estimation via Doppler-Shift Measurements

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Abstract— **This paper outlines the problem of doppler-based target position and velocity estimation using a sensor network. The minimum number of doppler shift measurements at distinct generic sensor positions to have a finite number of solutions, and later, a unique solution for the unknown target position and velocity is stated analytically, for the case when no measurement noise is present. Furthermore, we study the same problem where not only doppler shift measurements are collected, but also other types of measurements are available, e.g. bearing or distance to the target from each of the sensors. Subsequently, allowing nonzero measurement noise, we present an optimization method to estimate the position and the velocity of the target. An illustrative example is presented to show the validity of the analysis and the performance of the estimation method proposed. Some concluding remarks and future work directions are presented in the end.**

Index Terms— **Doppler Measurements, Localization, Motion Estimation, Polynomial Optimization**

I. Introduction

Using doppler-shifts for position and velocity estimation has a long history; see e.g. [1]–[8]. Recently, the doppler effect has gained a renewed interest and it has been implemented for cooperative positioning in vehicular networks [9].

In this paper, we consider a scenario with n nodes with both transmitting and sensing capabilities, that are called sensors for the rest of this paper. The target has an unknown position and velocity $\mathbf{x} = [\mathbf{p}^\top \ \mathbf{v}^\top]^\top \in \mathbb{R}^4$. The position of the non-collocated sensors is given by $\mathbf{s}_i = [s_{i,1} \ s_{i,2}]^\top \in \mathbb{R}^2$, $\forall i \in \{1, \dots, n\}$.

The measured doppler-shift is $\hat{f}_i$ at the i^{th} sensor and is caused by a target reflection due to a signal generated earlier by the same sensor. This frequency shift can be approximated by

$$\hat{f}_i = f_i + w_i \tag{1a}$$

$$= 2\frac{f_{c,i}}{c}\left(\frac{(\mathbf{p}-\mathbf{s}_i)^\top}{\|\mathbf{p}-\mathbf{s}_i\|}\right)\mathbf{v} + w_i \tag{1b}$$

where c is the speed of light (or signal propagation) and $\|\cdot\|$ is the standard Euclidean vector norm and $f_{c,i}$ is the carrier frequency employed by this sensor. Finally, w_i is a zero mean Gaussian random variable with known variance σ_i^2. Note here that the localization is to be achieved instantaneously; we are not envisaging collecting information from agents at a number of successive instants of time and using them to infer position at a single instant of time (The connection with filtering methods is explored further below). There are many studies in the literature that try to solve a similar problem via collecting measurements over a time interval and feeding them into an estimator. For example, in [5] the problem of localization of a single aircraft using doppler measurements is studied. Other similar approaches can be found in [4], [6]–[8]. The analysis carried out in this paper along with the optimization method proposed can be considered as constituting a batch processing method for instantaneous estimation of the target location and velocity. This in turn is used to initialize and improve the updates of any other implemented filter which tracks the target position as the target moves in the environment. For example, Kalman-based filters are prone to errors when the target's motion model deviates significantly from the actual target motion. Our analysis can guard against such behavior and re-initialize the filter, e.g. see [10] and references therein.

The main contribution of this paper is that the minimum number of doppler shift measurements required to have a finite number of solutions for the unknown $\mathbf{x}$ is algebraically derived. In some scenarios, a separate piece of knowledge about the target will allow disambiguation. Later, it is extended to the case where having a unique solution is required. Moreover, the scenarios where different types of measurements, e.g. direction-of-arrival, or distance, are available in addition to doppler shift measurements are considered. The aforementioned conclusions assume zero measurement noise; following from this, an optimization method based on polynomial optimization methods is introduced to calculate the velocity and the position of a target where noisy doppler shift measurements are available.

The remaining sections of this paper are organized as follows. In the next section the main problem of interest in considered. In Section III the case where different types of measurement in addition to doppler shift measurement are available to the sensors is considered. A method based on polynomial optimization to estimate the position and the velocity of the target where doppler shift measurements are contaminated by noise is presented in Section IV. An illustrative example presenting the performance of the proposed optimization method is given in Section V. Concluding remarks and future directions come in Section VI.

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II. Required Minimum Number of Doppler Shift Measurements

Initially, it is assumed $f_{c,i}$ is the same for $i = 1, \ldots, n$ and that the measurements are noiseless, i.e. $w_i = 0$.

$$\begin{aligned} \hat{f}_i &= f_i \\ &= 2\frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i)}{\|\mathbf{p}-\mathbf{s}_i\|}\frac{f_c}{c}. \end{aligned} \tag{2}$$

Normalizing so that $2\frac{f_c}{c} \triangleq 1$, we obtain

$$f_i = \frac{\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i)}{\|\mathbf{p}-\mathbf{s}_i\|} \tag{3}$$

Now we are ready to pose the problem of interest in this section.

Problem 1. *Consider n stationary sensors at $\mathbf{s}_i \in \mathbb{R}^2$ capable of collecting noiseless doppler shift measurements from a target at position $\mathbf{p}$ moving with a nonzero velocity $\mathbf{v}$ of the form* (2)*.*

1) *What is the minimum value for n such that there is a finite number of solutions for $\mathbf{x}$?*
2) *What is the minimum value for n such that there is a unique solution for $\mathbf{x}$?*

We limit out analysis to the case where the nodes and the target are in $\mathbb{R}^2$, however, note that the analysis for the case that they are in $\mathbb{R}^3$ is much the same.

The answer to the first question posed in Problem 1 is formally presented in the following proposition for the case where $n = 4$ doppler shift measurements are available. This proposition states that with $n = 4$ measurements we have a finite number of solutions and one might be able to disambiguate the solutions using other measurements, e.g. range, or bearing measurements. Moreover, disambiguation may also be made possible due to prior measurements, or to a priori knowledge about the geographic constraints on targets in the area of interest. This particularly is important for the cases where the possible solutions are widely separated, e.g., see Fig. 1.

Proposition 1. *For $n = 4$ doppler measurements as described by* (2) *and generic positions of the sensors there is a finite number of solutions for the unknown $\mathbf{x}$.*

Proof: Denote the noiseless mapping from the agent position and velocity, $\mathbf{x} = [\mathbf{p}^\top \ \mathbf{v}^\top]^\top$ (a vector in $\mathbb{R}^4$) to measurements (another vector in $\mathbb{R}^n$) by F, where n is the number of sensors. More specifically $F(\mathbf{x}) = [f_1 \ f_2 \ f_3 \ f_4]^\top$. Denote J_F to be the Jacobian of F:

$$J_F = \begin{bmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial p_2} & \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_4}{\partial p_1} & \frac{\partial f_4}{\partial p_2} & \frac{\partial f_4}{\partial v_1} & \frac{\partial f_4}{\partial v_2} \end{bmatrix} \tag{4}$$

where

$$\frac{\partial f_i}{\partial p_1} = \frac{-(p_2 - s_{i,2})(s_{i,2}v_1 - s_{i,1}v_2 + p_1 v_2 - p_2 v_1)}{\|\mathbf{s}_i - \mathbf{p}\|^3} \tag{5}$$

$$\frac{\partial f_i}{\partial p_2} = \frac{(p_1 - s_{i,1})(s_{i,2}v_1 - s_{i,1}v_2 + p_1 v_2 - p_2 v_1)}{\|\mathbf{s}_i - \mathbf{p}\|^3} \tag{6}$$

$$\frac{\partial f_i}{\partial v_1} = \frac{(p_1 - s_{i,1})}{\|\mathbf{s}_i - \mathbf{p}\|} \tag{7}$$

$$\frac{\partial f_i}{\partial v_2} = \frac{(p_2 - s_{i,1})}{\|\mathbf{s}_i - \mathbf{p}\|} \tag{8}$$

It is easy to check that for generic values of $\mathbf{x}$, $\mathbf{s}_i$, $i = 1, \cdots, 4$, J_F is not singular. Moreover, we know that the set of solutions to (2) form an algebraic variety. The reason for this is that while (2) is not a polynomial equation, it is easy to see that its zeros are also the zeros of the following polynomial equation.

$$f_i^2\|\mathbf{p}-\mathbf{s}_i\|^2 - (\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i))^2 = 0 \tag{9}$$

The set of the solutions to the equations described by (9) is known to have at least one member; that is the solution corresponding to the physical setup. The nonsingularity of the Jacobian implies that for generic values for measurements we do not have a continuous set of solutions. As a result, the variety is a zero-dimensional variety. This means that there is a finite number of solutions for the localization problem using doppler measurements where $n \geq 4$. $\square$

Now we briefly consider the case where the Jacobian matrix J_F is singular. This corresponds to those sensor and target geometries where there is an infinite number of solutions for the unknown $\mathbf{p}$ and $\mathbf{v}$. We call these geometries as *bad geometries*. The following proposition characterizes one of these bad geometries.

Proposition 2. *The Jacobian matrix J_F* (4) *is singular if the sensors $\mathbf{s}_1, \cdots, \mathbf{s}_4$ and the target $\mathbf{p}$ are collinear.*

Proof: To prove this proposition it is enough to evaluate (4) for the case where $\mathbf{s}_1, \cdots, \mathbf{s}_4$, and $\mathbf{p}$ lie on the same line. The calculations are trivial and are omitted for brevity. $\square$

It is worthwhile to note that any geometry in which the sensors and the target are almost collinear will also be problematic.

After establishing that there is a finite number of solutions for generic positions of the sensors where four doppler shift measurements are available, we present an answer to the second question posed in Problem 1. Before we formally propose the answer, note that the number of unknowns is four, and that with four pieces of data you get four polynomials equations, which have multiple solutions (though they may not all be real). However, with five pieces of data, one expects that the associated equations to have a unique solution. This solution is the one solution common to two selections of four. In the next proposition we prove that the position and the velocity of the target can be uniquely calculated if there are five doppler measurement available.

Proposition 3. *For $n \geq 5$ doppler measurements as described by* (2) *and generic positions for the sensors there is a unique solution for the unknown $\mathbf{x}$.*

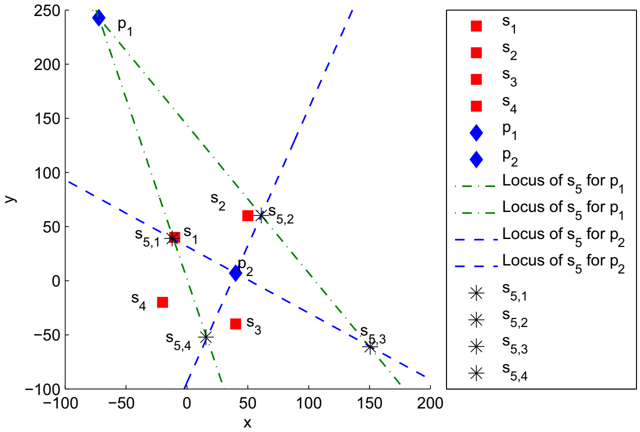

Fig. 1. Placing the fifth sensor at any of the positions indicated by * results in an ambiguous solution for the position of the target.

Proof: From Proposition 1 we know that for $n = 4$ there is a finite number of solutions for $\mathbf{x}$. Call the solutions for the position and the velocity of the target $\mathbf{x}_1, \cdots, \mathbf{x}_m$. Now, temporarily regard the position of sensor 5, $\mathbf{s}_5 = [s_{5,1}\ s_{5,2}]$ as an unknown. Consider the relationship between each of these solutions, $\mathbf{x}_i = [p_{i,1}\ p_{i,2}\ v_{i,1}\ v_{i,2}]^\top$, and the position of sensor 5, $\mathbf{s}_5$:

$$\begin{aligned} f_5^2\left((p_{i,1}-s_{5,1})^2+(p_{i,2}-s_{5,2})^2\right) - \\ (v_{i,1}(p_{i,1}-s_{5,1})+v_{i,2}(p_{i,2}-s_{5,2}))^2 = 0 \end{aligned} \tag{10}$$

This equation results in two straight lines intersecting at $\mathbf{p}_i$, and one at least of which the true $\mathbf{s}_5$ must lie, namely that associated with the correct target position. We claim that generically only one such equation can be satisfied by the true $\mathbf{s}_5$. To establish the claim, we argue by contradiction. Assume that $\mathbf{x}_j$ and $\mathbf{x}_k$ define two different loci on which $\mathbf{s}_5$ must lie. These loci at most intersect at four points. So for all positions of $\mathbf{s}_5$ except these four points, $\mathbf{x}_j$ and $\mathbf{x}_k$ cannot be simultaneously the solutions to the localization problem via doppler measurements. With similar arguments one can eliminate any multiplicity of solutions for generic values of doppler shift measurements for $n \geq 5$. □

The proof of Proposition 3 indicates that, given four sensors in generic positions there are isolated positions where placing a fifth sensor does not lead to a unique solution for $\mathbf{x}$. An example of this scenario is depicted in Fig. 1, where two possible solutions or the location of the target are shown. Moreover, four positions are identified such that the placement of a fifth sensor at any of these positions will not resolve the ambiguity in the target position. An interesting direction for future work would be to consider the notion of optimal sensor placement as discussed in, e.g., [11], [12] which not only considers ambiguities but also the likely estimator performance given the relative sensor-target geometry.

So far, we have considered the case where the carrier frequency, f_c, is known. In some cases this is not realistic; in what comes next we show that there is no way from a single set of instantaneous doppler only measurements one can separate f_c and $\mathbf{v}$. We conclude this section by formalizing this fact and presenting a result on the case where the carrier frequency is unknown.

Proposition 4. *For $n \geq 5$ doppler measurements as described by* (2) *and generic positions for the sensors, and unknown carrier frequency f_c, there is a unique solution for the unknown $\mathbf{p}$ and for the vector $f_c\mathbf{v}$.*

Proof: Define $\nu = [\nu_1\ \nu_2]^\top \triangleq f_c\mathbf{v}$. For (2) we have

$$f_i = 2\frac{\nu^\top(\mathbf{p}-\mathbf{s}_i)}{c\|\mathbf{p}-\mathbf{s}_i\|} \tag{11}$$

From Proposition 3 we know that using five equations of the form (11), $\mathbf{p}$ and ν can be calculated uniquely. It follows easily that if one does not know f_c nor $\mathbf{v}$ and one estimates $\nu \triangleq f_c\mathbf{v}$ that for any chosen f_c (or $\mathbf{v}$) there is a subsequent value of $\mathbf{v}$ (or f_c) that satisfies $\nu = f_c\mathbf{v}$. Thus, f_c and $\mathbf{v}$ cannot be calculated separately.

Remark 1. *By using the calculated value $\mathbf{p}$ in consecutive time steps, one can estimate the velocity of the agent. Using this estimated velocity and knowing ν, one can further estimate f_c.*

III. Required Minimum Number of Hybrid Measurements

In this section we study the effect of having other types of measurements additional to doppler shift measurements on calculating the velocity and the position of the target.

Before continuing further, we note that to have a unique solution for a set of polynomial equations there is usually a need to have more equations than unknowns., save in cases where the equations are linear. However, having more equations than unknowns does not of itself guarantee the existence of a unique solution; the extra equation must in some way be independent, and it may have this property in almost all circumstances, i.e. generically, but not always. In this section we establish the cases where it can be mathematically shown that a unique solution for $\mathbf{x}$ exists when a combination of doppler and other measurements is available.

First, we consider the case where in addition to the doppler shift measurements described earlier, the distances between each of the sensors and the target can be measured as well. Denote the distance between the sensor i and the target as d_i where

$$d_i = \|\mathbf{p}-\mathbf{s}_i\|, \quad i = 1, \cdots, m \tag{12}$$

We have the following results:

Proposition 5. *For $n \geq 2$ doppler measurements as described by* (2) *and $m \geq 3$ distance measurements* (12) *and generic positions of the sensors there is a unique solution for the unknown $\mathbf{x}$.*

Proof: With three or more distance measurements the position of the target can be determined uniquely; i.e. three generic circles have at most a single point of intersection.

Knowing the position, then the doppler equations are linear equations in the velocity of the target. Hence, a unique solution for the velocity follows. $\square$

Proposition 6. *For $n \geq 3$ doppler measurements as described by* (2) *and two distance measurements* (12) *and generic positions of the sensors there is a unique solution for the unknown* $\mathbf{x}$.

Proof: The two distance measurements pin down the target to a binary ambiguity; i.e. two generic circles have two points of intersection. Using each of these positions, then from the doppler equations we obtain two sets of three linear equations in the velocity of the target. Generically, only one of these sets forms a consistent system of linear equations. $\square$

We now consider the case that instead of distance measurements, bearing measurements to the target at each of the sensors are available, viz.

$$\rho_i = [\rho_{i,1} \quad \rho_{i,2}]^\top = \frac{\mathbf{p} - \mathbf{s}_i}{\|\mathbf{p} - \mathbf{s}_i\|} \tag{13}$$

is known at each sensor i. We formally consider this scenario in the next proposition.

Proposition 7. *For $n \geq 2$ doppler measurements as described by* (2) *and bearing measurements* (13) *there is a unique solution for the unknown* $\mathbf{x}$.

Proof: With at least two bearing measurements the position of the target can be determined uniquely. Knowing the position, then the doppler equations are linear equations in the velocity of the target. Hence, a unique solution for the velocity follows. $\square$

In what comes next we consider the case where only one of the sensors is equipped with the capability to measure the target bearing. Without loss of generality assume that only sensor 1 can collect a bearing measurements to the target in addition to the doppler shift measurement. The rest of the sensors can only measure the doppler shift. For this scenario we have the following result.

Proposition 8. *For $n \geq 4$ doppler measurements as described by* (2) *and only one measured bearing to the target there is a unique solution for the unknown* $\mathbf{x}$.

Proof: Without loss of generality assume that the bearing measurement is measured at $\mathbf{s}_1$. For the doppler measurement at 1 we have

$$\begin{aligned} f_1 &= 2\mathbf{v}^\top \rho_1 \frac{f_c}{c} \\ f_1 &= 2(v_1\rho_{1,1} + v_2\rho_{1,2})\frac{f_c}{c} \end{aligned} \tag{14}$$

that is a linear equation in $\mathbf{v}$. Moreover, we have

$$p_2 = s_{1,2} + \rho_{1,2}\frac{p_1 - s_{1,1}}{\rho_{1,1}} \tag{15}$$

Calculating v_2 in terms of v_1 from (14) and p_2 in terms of p_1 from (14) and replacing them in

$$f_i = 2\mathbf{v}^\top \frac{\mathbf{p} - \mathbf{s}_i}{\|\mathbf{p} - \mathbf{s}_i\|}\frac{f_c}{c} \quad i = 2, 3, 4 \tag{16}$$

we obtain three quadratic equations in v_1 and p_1 only. Furthermore, it is known that generically three quadratic equations in two variables have a unique solution. Hence, $\mathbf{x}$ can be determined uniquely. $\square$

In the next section, we introduce an algorithm to estimate the positon and the velocity of the target using the doppler-shift measurements measured at each of the sensor.

IV. An Algorithm to Estimate the Position and the Velocity of the Target

To calculate the position and the velocity of the target in the noiseless case it is enought to solve the following system of equations.

$$\delta_i - \frac{\mathbf{v}^\top(\mathbf{p} - \mathbf{s}_i)}{\|\mathbf{p} - \mathbf{s}_i\|} = 0 \quad i = 1, \ldots, n \tag{17}$$

where $\delta_i = \dfrac{cf_i}{2f_c}$. Equivalently due to the fact that the target and the sensors are not collocated (i.e. $||p - s_i||$ is nonzero), instead of solving (17) one can solve:

$$\delta_i\|\mathbf{p} - \mathbf{s}_i\| - \mathbf{v}^\top(\mathbf{p} - \mathbf{s}_i) = 0 \quad i = 1, \ldots, n, \tag{18}$$

or

$$\delta_i\sqrt{(\mathbf{p} - \mathbf{s}_i)^\top(\mathbf{p} - \mathbf{s}_i)} - \mathbf{v}^\top(\mathbf{p} - \mathbf{s}_i) = 0 \quad i = 1, \ldots, n. \tag{19}$$

Having the square-root in (19) makes it undesirable for solving numerically. Hence, instead we consider the following set of equations.

$$\delta_i^2(\mathbf{p} - \mathbf{s}_i)^\top(\mathbf{p} - \mathbf{s}_i) - (\mathbf{v}^\top(\mathbf{p} - \mathbf{s}_i))^2 = 0 \quad i = 1, \ldots, n. \tag{20}$$

Note that any solution of (19) is a solution of (20) but not vice versa. Assume $\mathbf{x}^\star = [\mathbf{p}^{\star\top} \quad \mathbf{v}^{\star\top}]^\top$ is a solution to (19); then both $\mathbf{x}_1^\star = [\mathbf{p}^{\star\top} \quad \mathbf{v}^{\star\top}]^\top$ and $\mathbf{x}_2^\star = [\mathbf{p}^{\star\top} \quad -\mathbf{v}^{\star\top}]^\top$ satisfy (20). From Proposition 3 we know that for $n \geq 5$ (19) has a unique solution. Then it easily follows that for $n \geq 5$, (20) has exactly two solutions. Replacing $\mathbf{p}$ by $\mathbf{p}^\star$ in (17) results in a set of linear equations in $\mathbf{v}$ where only one of the values of $\mathbf{v}^\star$ or $-\mathbf{v}^\star$ satisfies it. The aforementioned analysis of the solutions for $\mathbf{x}$ in the noiseless case form the basis of the algorithm proposed in this section.

Now we consider the case where the doppler shift measurement is corrupted by noise. That is, each sensor measures $\hat{f}_i = f_i + w_i$ where w_i corresponds to the noise in the measurement carried out by sensor i. Setting $\hat{\delta}_i = \dfrac{c\hat{f}_i}{2f_c}$, we have

$$\hat{\delta}_i = \frac{\mathbf{v}^\top(\mathbf{p} - \mathbf{s}_i)}{\|\mathbf{p} - \mathbf{s}_i\|}. \tag{21}$$

Note that $\hat{\delta}_i = \delta_i + \omega_i$, where $\omega_i = \dfrac{cw_i}{2f_c}$. In the noisy case neither (19) nor (20) has a solution for $\mathbf{x}$. Instead we propose solving the following minimization problem to calculate the position and the velocity of the target. Solution of similar minimization problems when range, range-difference, and bearing measurments are available are studied in [13], [14].

$$[\mathbf{p}^\star, \mathbf{v}^\star] = \underset{\mathbf{p},\mathbf{v}}{\operatorname{argmin}} \quad F(\mathbf{p}, \mathbf{v}), \tag{22}$$

where $F(\mathbf{p},\mathbf{v}) = \sum_{i=1}^{n} \left(\hat{\delta}_i^2\|\mathbf{p}-\mathbf{s}_i\|^2 - (\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i))^2\right)^2$. The advantage of having such a cost function is that it is a polynomial in the unknowns, and can be minimized using modern polynomial optimization methods, e.g. see [15], [16]. By solving this minimization problem we obtain two values for the position and the velocity of the target, viz. $(\mathbf{p}_1^\star, \mathbf{p}_2^\star)$ and $(\mathbf{v}_1^\star, \mathbf{v}_2^\star)$, where $\mathbf{p}_1^\star = \mathbf{p}_2^\star$ and $\mathbf{v}_1^\star = -\mathbf{v}_2^\star$. To find the correct value for the velocity as before we replace $\mathbf{p}$ by $\mathbf{p}^\star \triangleq \mathbf{p}_1^\star = \mathbf{p}_2^\star$ in (21) to obtain a set of linear equations in $\mathbf{v}$:

$$\hat{\delta}_i = \frac{\mathbf{v}^\top(\mathbf{p}^\star - \mathbf{s}_i)}{\|\mathbf{p}^\star - \mathbf{s}_i\|}. \tag{23}$$

It is easy to check that the linear system of equations (23) is over-determined and inconsistent, and cannot be solved. However, a least-square solution to this syetem of linear equations can be obtained. The least squares solution to this system of equations is the estimated value for the target velocity, $\mathbf{v}^\star$. This procedure is outlined in Algorithm 1.

Algorithm 1 Target Position and Velocity Estimation Using n Doppler-shift Measurements Collecting at Nodes $(1,\ldots,n)$ in a Sensor Network Via Polynomial Optimization.
Input: $\hat{\delta}_1,\ldots,\hat{\delta}_n$
Output: $\mathbf{p}^\star$, $\mathbf{v}^\star$

Require: At least 5 sensors at generic positions.
$F(\mathbf{p},\mathbf{v}) \leftarrow \sum_{i=1}^{n} \left(\hat{\delta}_i^2\|\mathbf{p}-\mathbf{s}_i\|^2 - (\mathbf{v}^\top(\mathbf{p}-\mathbf{s}_i))^2\right)^2$
$[\mathbf{p}^\star,\ \mathbf{v}^\star] \leftarrow \underset{\mathbf{p},\mathbf{v}}{\text{argmin}}\ F(\mathbf{p},\mathbf{v})$
for $i = 1 : n$ **do**
  $\hat{\rho}_i \leftarrow \frac{\mathbf{p}^\star - \mathbf{s}_i}{\|\mathbf{p}^\star - \mathbf{s}_i\|}$
end for
$\hat{A} \leftarrow [\hat{\rho}_1,\ldots,\hat{\rho}_1]^\top$
$\hat{\mathbf{b}} \leftarrow [\hat{\delta}_1,\ldots,\hat{\delta}_n]^\top$
$\mathbf{v}^\star \leftarrow \hat{A}^\dagger\hat{\mathbf{b}}$ {$\hat{A}^\dagger$ is the pseudo-inverse of $\hat{A}$.}
return $\mathbf{p}^\star$ and $\mathbf{v}^\star$

V. Illustrative Example

In this section we demonstrate the performance of the algorithm introduced in Section IV to estimate the position and the velocity of the target. We consider the setting depicted in Fig, 2. We consider the case where the measurements are corrupted by ten different levels of noise, where the noise is assumed to be gaussian with zero mean and variable variance (and independent at each sensor). The error in the estimates of position and the velocity for different noise levels after repeating the scenario for twenty times when six and seven measurements are used are depicted at Fig. 3 and Fig. 4 respectively. Moreover, note that the setting presented in Fig. 2 shows elements of a bad geometry as well: sensors at positions $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ and the target are nearly collinear. However, here due to the presence of other sensors at non-collinear positions the estimation can be carried out effecievely.

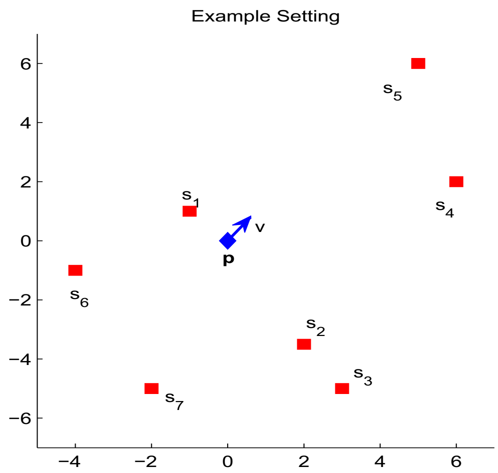

Fig. 2. The setting considered in the illustrative example. The squares denote the position of the sensors and the diamond is the target.

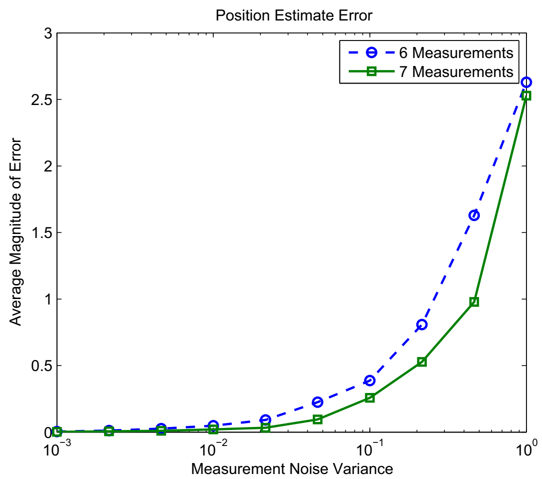

Fig. 3. The error in the estimate of the position of the target after repeating the scenario 20 times with variable noise variances when six and seven sensors are used.

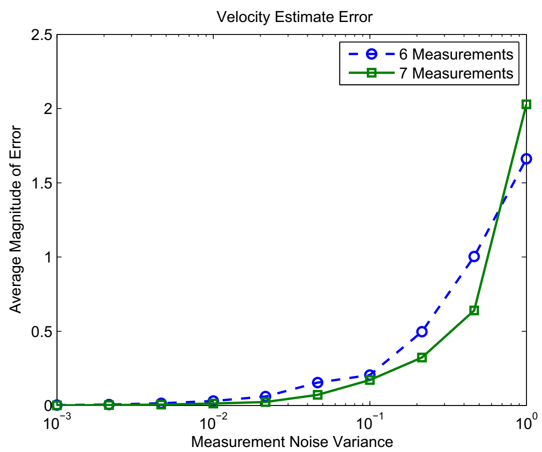

Fig. 4. The error in the estimate of the velocity of the target after repeating the scenario 20 times with variable noise variances when six and seven sensros are used.

VI. Concluding Remarks and Future Directions

The minimum number of doppler shift measurements necessary to have a finite number of solutions for the unknown target position and velocity is calculated analytically via algebraic arguments. Additionally, we stated the necessary and sufficient number of generic measurements to have a unique solution for the target parameters. Later, the same problem has been studied where in addition to doppler shift measurements, other types of measurements are available, e.g. bearing or distance to the target from each of the sensors. Finally, a method based on polynomial optimization is introduced to calculate an estimate for the position and the velocity of the target using noisy doppler shift measurements. A numerical example is presented to demonstrate the performance of this algorithm.

A possible future research direction is to design a dynamical estimator to estimate the position and the velocity of the target measuring the doppler shift measurements continuously. Alternatively, one could consider the notion of constraint-based optimization for localization as discussed in, e.g., [17]–[22].

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