Stability Analysis: Euler's Method

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$$\dot{x} = f(t, x)$$

difference between sol'n is bounded

Solution is stable if $\|x(t) - \hat{x}(t)\| \leq \varepsilon$ $\quad t > t_0$

$$x_{k+1} = x_k + h f(t_k, x_k) \qquad \|x(t_0) - \hat{x}(t_0)\| \leq \delta$$

difference in initial condition is bounded.

$$\hat{x}_{k+1} = \hat{x}_k + h f(t_k, \hat{x}_k)$$

Let $\begin{cases} e_k \equiv \hat{x}_k - x_k \\ e_0 \equiv \hat{x}_0 - x_0 \end{cases} \Rightarrow e_{k+1} = e_k + h\, g(t_k, e_k)$

$$g(t_k, e_k) = f(t_k, \hat{x}_k) - f(t_k, x_k)$$

Taylor Series:

$$f(t_k, x_k + \Delta x) = f(t_k, x_k) + \underbrace{\frac{d}{dx} f}_{\equiv F} \cdot \Delta x$$

$$g(t, e_k) = \left[f_k + F \cdot \overbrace{(\hat{x}_k - x_k)}^{e_k}\right] - f_k = F \cdot e_k$$

$$e_{k+1} = e_k + hF \cdot e_k = (1 + hF) \cdot e_k$$

$$e_k = \underbrace{(1 - hF)^k}_{\text{Value as } k \to \infty \,?} e_0$$

→ Non-Recursive

$$A^\infty \to \begin{cases} 0 & \|A\| < 1 \\ 1 & \|A\| = 1 \\ \infty & \|A\| > 1 \end{cases}$$

$$\|1 - hF\| \leq 1$$

$\hookrightarrow \boxed{0 \leq hF \leq 2}$

→ h = Step Size

→ $F = \frac{d}{dx} f(t, x)$