Single Shooting

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Example Problem:

Given: $\vec{x}_0$, $\vec{x}_F$, T, $\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{u})$

Find: $\vec{u}(t)$

Minimize: $J(\vec{x}(), \vec{u}()) = \int_0^T g(\vec{x}(t), \vec{u}(t))\, dt$

Subject to: $\vec{x}(0) = \vec{x}_0 \qquad \vec{x}(T) = x_F$

Transcription via Single Shooting: Euler's Method

uniform grid.

N = number of steps ; $h = T/N$

Dec. Vars: $\vec{u}_0, \vec{u}_1, \ldots \vec{u}_{N-1} \rightarrow \vec{z} = \begin{bmatrix} \vec{u}_0 \\ \vec{u}_1 \\ \vdots \\ \vec{u}_{N-1} \end{bmatrix}$ = Column vector

Dynamics:

$$\left.\begin{aligned} \vec{x}(0) &= \vec{x}_0 = \text{Given} \\ \vec{x}(h) &= \vec{x}_1 = \vec{x}_0 + h \cdot \vec{f}(\vec{x}_0, \vec{u}_0) \\ \vec{x}(2h) &= \vec{x}_2 = \vec{x}_1 + h f(\vec{x}_1, \vec{u}_1) \\ &\vdots \\ \vec{x}(T) &= \vec{x}_N = \vec{x}_{N-1} + h f(\vec{x}_{N-1}, \vec{u}_{N-1}) \end{aligned}\right\} \text{Simulation via Euler's Method}$$

Bound. Constraint: $\vec{x}_N - \vec{x}_F = \vec{C} = 0$

computed using simulation

Objective: $J = h \sum_{k=0}^{N-1} g(\vec{x}_k, \vec{u}_k)$

* this is now a non-linear program - can be solved with FMINCON in Matlab.

Single Shooting Gradients (Jacobian)

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- $$\frac{\partial \vec{c}}{\partial \vec{z}} = \frac{\partial}{\partial \vec{z}}\left[\underset{\text{Final State (from Simulation)}}{\vec{x}_N} - \overset{\text{Constant (Given)}}{\vec{x}_F}\right]$$ = Partial Derivatives of the final boundary constraint with respect to the control inputs.

$$\frac{\partial \vec{c}}{\partial \vec{z}} = \frac{\partial \vec{x}_N}{\partial \vec{z}} - \cancel{\frac{\partial \vec{x}_F}{\partial \vec{z}}}^{\to 0 \text{ (constant)}}$$

* Drop vector $\vec{x} \to x$ notation to improve readability *

$$\frac{\partial x_N}{\partial z} = \left[\frac{\partial x_N}{\partial u_0}, \frac{\partial x_N}{\partial u_1}, \dots, \frac{\partial x_N}{\partial u_{N-1}}\right]$$

How to write this in terms of known functions and decision variables?

Start w/ simple toy problem → Let $N=1$

- $$\frac{\partial x_1}{\partial u_0} = \frac{\partial}{\partial u_0}\left(x_0 + h f(x_0, u_0)\right)$$

$$= \frac{\partial x_0}{\partial u_0} + h \frac{\partial f}{\partial u_0}(x_0, u_0)$$

$$= 0 + h\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x_0}{\partial u_0} + \frac{\partial f}{\partial u}\cdot\frac{\partial u_0}{\partial u_0}\right)$$

$$= 0 + h\left(0 + \frac{\partial f}{\partial u}\cdot 1\right) = h f_u(x_0, u_0)$$

$$\frac{\partial x_1}{\partial u_0} = h \frac{\partial f}{\partial u}(x_0, u_0) = \underbrace{h f_u(0)}_{\text{simple notation}}$$

- $f_u = \frac{\partial f}{\partial u}$ $\qquad f(0) = f(x_0, u_0)$

Single Shooting Gradients (continued)

A harder toy problem $\rightarrow$ Let $N=2$

$$\frac{\partial x_2}{\partial u_0} = \frac{\partial}{\partial u_0}\left(x_1 + h f(x_1, u_1)\right)$$

Problem: x_1 is the result of a calculation...

$$\frac{\partial x_2}{\partial u_0} = \frac{\partial}{\partial u_0}\left(\left(x_0 + h f(x_0, u_0)\right) + h \cdot f\left(\left(x_0 + h f(x_0, u_0)\right), u_1\right)\right)$$

Now it is in terms of decision variables and constants

$$= \frac{\partial x_0}{\partial u_0} + h \frac{\partial f}{\partial u_0} f(x_0, u_0) + h \underbrace{\frac{\partial f}{\partial u_0}\left(x_0 + h f(x_0, u_0), u_1\right)}_{\text{Need to be careful here}}$$

$$= 0 + \overbrace{h f_u(0)}^{\text{Notation from prev. page}} + h \cdot \left[\frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial u_0}\left(x_0 + h f(x_0, u_0)\right) + \frac{\partial f}{\partial u} \cdot \frac{\partial u_1}{\partial u_0}\right]$$

$$= 0 + h f_u(0) + h \cdot \left[f_x(1) \cdot \left(0 + h f_u(0)\right) + f_u(1) \cdot 0\right]$$

$$= h f_u(0) + h f_x(1) \cdot h f_u(0) = \left(1 + h f_x(1)\right) \cdot \left(h f_u(0)\right)$$

$$\frac{\partial x_2}{\partial u_0} = \left(1 + h f_x(1)\right) \cdot \frac{\partial x_1}{\partial u_0}$$

$$\frac{\partial x_{k+1}}{\partial u_0} = \frac{\partial}{\partial u_0}\left(x_k + h f(x_k, u_k)\right)$$

$$= \frac{\partial}{\partial x_k}\left(x_k + h f(x_k, u_k)\right) \frac{\partial x_k}{\partial u_0} + \frac{\partial}{\partial u_k}\left(x_k + h f(x_k, u_k)\right) \frac{\partial u_k}{\partial u_0}$$

$$\frac{\partial x_{k+1}}{\partial u_0} = \left(1 + h f_x(k)\right) \frac{\partial x_k}{\partial u_0} + h f_u(k) \cdot \frac{\partial u_k}{\partial u}$$

Single Shooting Gradients (continued)

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$$\boxed{\frac{\partial x_{k+1}}{\partial u_p} = \left(1 + h f_x(k)\right)\frac{\partial x_k}{\partial u_p} + h f_u(k) \cdot \frac{\partial u_k}{\partial u_p}}$$

Recursive Equation while $k > p$

$\underline{k \le p} \longrightarrow \dfrac{\partial x_k}{\partial u_p} = 0$ current state is not affected by current or future controls

$\underline{k = p} \longrightarrow \dfrac{\partial u_k}{\partial u_p} = 1$ current control is only dependent on itself.

$$\frac{\partial J}{\partial u_p} = h \sum_{k=0}^{N-1} \left[\frac{\partial}{\partial u_p} g(x_k, u_k)\right]$$

$$= h \sum_{k=0}^{N-1} \left[\frac{\partial g(k)}{\partial x_k} \cdot \frac{\partial x_k}{\partial u_p} + \frac{\partial g(k)}{\partial u_k} \cdot \frac{\partial u_k}{\partial u_p}\right]$$

$$\boxed{\frac{\partial J}{\partial u_p} = h \sum_{k=0}^{N-1} \left(g_x(k) \cdot \frac{\partial x_k}{\partial u_p} + g_u(k) \cdot \frac{\partial u_k}{\partial u_p} \right)}$$
