<u>Lecture:</u> How to derive Runge-Kutta Methods (Second Order) Jan 1, 2018

<u>Part 1)</u>

$$x(t_0+h) = x(t_0) + \int_{t_0}^{t_0+h} f(\tau, x(\tau))\, d\tau$$

$$x(t_0+h) \approx x(t_0) + h \sum_{i=1}^{N} \omega_i \underbrace{f(\tau_i, x(\tau_i))}_{K_i \equiv f(\tau_i, x(\tau_i))}$$

<u>Let</u> $N = 2$ = Second-Order Method

<u>Let</u> $\tau_1 = t_0 \longrightarrow K_1 = f(t_0, \overbrace{x(t_0)}^{x_0}) \equiv f_0$

<u>Let</u> $\tau_2 = t_0 + \alpha h \longrightarrow K_2 = f(t_0 + \alpha h, \underbrace{x(t_0+\alpha h)}_{\substack{\text{Use Euler's Method!} \\ x_0 + \alpha h f_0}})$

$$x_0(t_0+h) \approx x_0 + h(\omega_1 K_1 + \omega_2 K_2)$$

$$\approx x_0 + h\Big(\omega_1 f_0 + \omega_2 \underbrace{f(t_0+\alpha h, x_0 + \alpha h f_0)}_{\text{what about this?}}\Big)$$

$$f(t_0+h, x_0+\Delta x) = \overbrace{f(t_0, x_0)}^{f_0} + \overbrace{\frac{\partial}{\partial t} f_0}^{\dot{f}_0} \cdot h + \overbrace{\frac{\partial}{\partial x} f_0}^{f_0'} \cdot \Delta x$$

$$f(t_0+\alpha h, x_0+\alpha h f_0) = f_0 + \dot{f}_0 \alpha h + f_0' \alpha h f_0$$

$$x_0(t_0+h) \approx x_0 + h\Big(\omega_1 f_0 + \overbrace{\omega_2 f_0 + \omega_2 \dot{f}_0 \alpha h + \omega_2 f_0' \alpha h f_0}\Big)$$

(continued) Derivation for 2nd-Order Runge Kutta Jan 1, 2018

Part 2) $x(t_0+h) = x_0 + h\dot{x}_0 + \frac{1}{2}h^2\ddot{x}_0 + \mathcal{O}(3)\ldots$

Sub into

$$\dot{x}(t) \equiv f(t, x(t))$$

$$\ddot{x}(t) = \frac{\partial}{\partial t}f() + \frac{\partial}{\partial x}f()\cdot f() = \dot{f} + f'f \quad \begin{pmatrix}\text{chain}\\ \text{Rule}\end{pmatrix}$$

$$x(t_0+h) = x_0 + hf_0 + \frac{1}{2}h^2\left(\dot{f} + f'f\right) + \mathcal{O}(3)\ldots$$

Part 3) Set result of ① and ② equal

$$① \quad x(t_0+h) = x_0 + h\left(w_1 f_0 + w_2 f_0\right) + h^2\left(w_2 \dot{f}_0 \alpha + w_2 f_0' \alpha f_0\right)$$

$$||$$

$$② \quad x(t_0+h) = x_0 + h\left(f_0\right) + h^2\left(\frac{1}{2}\dot{f} + \frac{1}{2}f'f\right)$$

Solve by powers of h...

$$w_1 f_0 + w_2 f_0 = f_0 \longrightarrow \boxed{w_1 + w_2 = 1}$$

$$w_2 \dot{f}_0 \alpha + w_2 f_0' \alpha f_0 = \frac{1}{2}\dot{f}_0 + \frac{1}{2}f_0' f_0$$

$$w_2 \alpha\left(\dot{f}_0 + f_0' f_0\right) = \frac{1}{2}\left(\dot{f}_0 + f_0' f_0\right) \longrightarrow \boxed{w_2 \alpha = \frac{1}{2}}$$

Solve for quadrature weights:

$$\boxed{w_2 = \frac{1}{2\alpha} \qquad w_1 = 1 - \frac{1}{2\alpha}}$$

$$\boxed{x(t_0+h) \approx x_0 + h\left(w_1 K_1 + w_2 K_2\right)}$$

$$\boxed{\begin{aligned} K_1 &= f_0 \\ K_2 &= f(t_0+\alpha h,\ x_0 + \alpha h K_1)\end{aligned}}$$