BIG IDEAS MATH®

TEXAS EDITION

Algebra 1

Ron Larson and Laurie Boswell

BIG IDEAS LEARNING®

Erie, Pennsylvania

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Ron Larson, Ph.D., is well known as the lead author of a comprehensive program for mathematics that spans middle school, high school, and college courses. He holds the distinction of Professor Emeritus from Penn State Erie, The Behrend College, where he taught for nearly 40 years. He received his Ph.D. in mathematics from the University of Colorado. Dr. Larson's numerous professional activities keep him actively involved in the mathematics education community and allow him to fully understand the needs of students, teachers, supervisors, and administrators.

Ron Larson

Laurie Boswell, Ed.D., is the Head of School and a mathematics teacher at the Riverside School in Lyndonville, Vermont. Dr. Boswell is a recipient of the Presidential Award for Excellence in Mathematics Teaching and has taught mathematics to students at all levels, from elementary through college. Dr. Boswell was a Tandy Technology Scholar and served on the NCTM Board of Directors from 2002 to 2005. She currently serves on the board of NCSM and is a popular national speaker.

Laurie Boswell

Dr. Ron Larson and **Dr. Laurie Boswell** began writing together in 1992. Since that time, they have authored over two dozen textbooks. In their collaboration, Ron is primarily responsible for the student edition while Laurie is primarily responsible for the teaching edition.

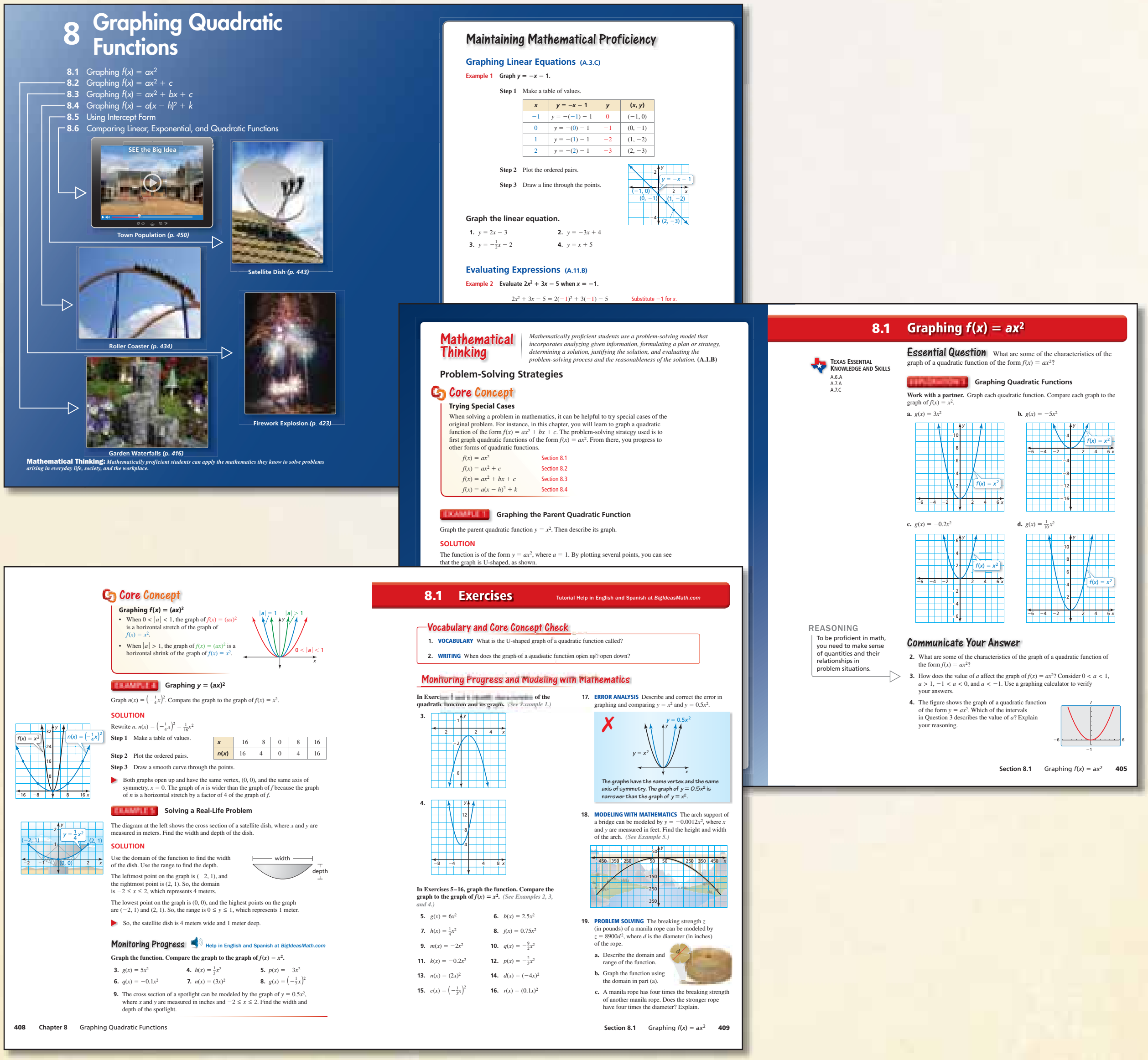
For the Student

Welcome to ***Big Ideas Math Algebra 1***. From start to finish, this program was designed with you, the learner, in mind.

As you work through the chapters in your Algebra 1 course, you will be encouraged to think and to make conjectures while you persevere through challenging problems and exercises. You will make errors—and that is ok! Learning and understanding occur when you make errors and push through mental roadblocks to comprehend and solve new and challenging problems.

In this program, you will also be required to explain your thinking and your analysis of diverse problems and exercises. Being actively involved in learning will help you develop mathematical reasoning and use it to solve math problems and work through other everyday challenges.

We wish you the best of luck as you explore Algebra 1. We are excited to be a part of your preparation for the challenges you will face in the remainder of your high school career and beyond.



Big Ideas Math High School Research

Big Ideas Math Algebra 1, Geometry, and Algebra 2 is a research-based program providing a rigorous, focused, and coherent curriculum for high school students. Ron Larson and Laurie Boswell utilized their expertise as well as the body of knowledge collected by additional expert mathematicians and researchers to develop each course. The pedagogical approach to this program follows the best practices outlined in the most prominent and widely-accepted educational research and standards, including:

Achieve, ACT, and The College Board

Adding It Up: Helping Children Learn Mathematics
National Research Council ©2001

Curriculum Focal Points and the *Principles and Standards for School Mathematics* ©2000
National Council of Teachers of Mathematics (NCTM)

Project Based Learning
The Buck Institute

Rigor/Relevance Framework™
International Center for Leadership in Education

Universal Design for Learning Guidelines
CAST ©2011

We would also like to express our gratitude to the experts who served as consultants for ***Big Ideas Math Algebra 1, Geometry, and Algebra 2***. Their input was an invaluable asset to the development of this program.

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We would also like to thank all of our reviewers who provided feedback during the final development phases. For a complete list of the ***Big Ideas Math*** program reviewers, please visit *www.BigIdeasLearning.com*.

Texas Mathematical Process Standards

Apply mathematics to problems arising in everyday life, society, and the workplace.

- Real-life scenarios are utilized in *Explorations*, *Examples*, *Exercises*, and *Assessments* so students have opportunities to apply the mathematical concepts they have learned to realistic situations.
- Real-world problems help students use the structure of mathematics to break down and solve more difficult problems.

Modeling Real-Life Problems

EXAMPLE 5 **Modeling with Mathematics**

Water fountains are usually designed to give a specific visual effect. For example, the water fountain shown consists of streams of water that are shaped like parabolas. Notice how the streams are designed to land on the underwater spotlights. Write and graph a quadratic function that models the path of a stream of water with a maximum height of 5 feet, represented by a vertex of (3, 5), landing on a spotlight 6 feet from the water jet, represented by (6, 0).

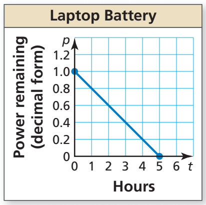
SOLUTION

1. **Understand the Problem** You know the vertex and another point on the graph that represents the parabolic path. You are asked to write and graph a quadratic function that models the path.
2. **Make a Plan** Use the given points and the vertex form to write a quadratic function. Then graph the function.

Use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution.

- *Reasoning*, *Critical Thinking*, *Abstract Reasoning*, and *Problem Solving* exercises challenge students to apply their acquired knowledge and reasoning skills to solve each problem.
- Students are continually encouraged to evaluate the reasonableness of their solutions and their steps in the problem-solving process.

29. **PROBLEM SOLVING** The graph shows the percent p (in decimal form) of battery power remaining in a laptop computer after t hours of use. A tablet computer initially has 75% of its battery power remaining and loses 12.5% per hour. Which computer's battery will last longer? Explain. *(See Example 5.)*



Select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems.

- Students are provided opportunities for selecting and utilizing the appropriate mathematical tool in *Using Tools* exercises. Students work with graphing calculators, dynamic geometry software, models, and more.
- A variety of tool papers and manipulatives are available for students to use in problems as strategically appropriate.

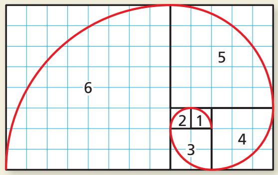
USING TOOLS **In Exercises 21–26, solve the inequality. Use a graphing calculator to verify your answer.**

21. $36 < 3y$ **22.** $17v \geq 51$

Communicate mathematical ideas, reasoning, and their implications using multiple representations, including symbols, diagrams, graphs, and language as appropriate.

- Students are asked to construct arguments, critique the reasoning of others, and evaluate multiple representations of problems in specialized exercises, including *Making an Argument*, *How Do You See It?*, *Drawing Conclusions*, *Reasoning*, *Error Analysis*, *Problem Solving*, and *Writing*.
- Real-life situations are translated into diagrams, tables, equations, and graphs to help students analyze relationships and draw conclusions.

54. HOW DO YOU SEE IT? Consider Squares 1–6 in the diagram.



a. Write a sequence in which each term a_n is the side length of square n.

b. What is the name of this sequence? What is the next term of this sequence?

c. Use the term in part (b) to add another square to the diagram and extend the spiral.

38. MODELING WITH MATHEMATICS You start a chain email and send it to six friends. The next day, each of your friends forwards the email to six people. The process continues for a few days.

a. Write a function that represents the number of people who have received the email after n days.

b. After how many days will 1296 people have received the email?

Create and use representations to organize, record, and communicate mathematical ideas.

- *Modeling with Mathematics* exercises allow students to interpret a problem in the context of a real-life situation, while utilizing tables, graphs, visual representations, and formulas.
- Multiple representations are presented to help students move from concrete to representative and into abstract thinking.

Analyze mathematical relationships to connect and communicate mathematical ideas.

- *Using Structure* exercises provide students with the opportunity to explore patterns and structure in mathematics.
- Stepped-out *Examples* encourage students to maintain oversight of their problem-solving process and pay attention to the relevant details in each step.

EXAMPLE 3 **Inequalities with Special Solutions**

Solve (a) $8b - 3 > 4(2b + 3)$ and (b) $2(5w - 1) \le 7 + 10w$.

SOLUTION

a.

$8b - 3 > 4(2b + 3)$		Write the inequality.
$8b - 3 > 8b + 12$		Distributive Property
$\underline{-\,8b} \qquad \underline{-\,8b}$		Subtract $8b$ from each side.
$-3 > 12$		Simplify.

▶ The inequality $-3 > 12$ is false. So, there is no solution.

Display, explain, and justify mathematical ideas and arguments using precise mathematical language in written or oral communication.

- *Vocabulary and Core Concept Check* exercises require students to use clear, precise mathematical language in their solutions and explanations.
- *Performance Tasks* for every chapter allow students to apply their skills to comprehensive problems and utilize precise mathematical language when analyzing, interpreting, and communicating their answers.

Performance Task

Any Beginning

With so many ways to represent a linear relationship, where do you start? Use what you know to move between equations, graphs, tables, and contexts.

To explore the answer to this question and more, go to *BigIdeasMath.com*.

1 Solving Linear Equations

See the Big Idea
Learn how boat navigators use dead reckoning to calculate their distance covered in a single direction.

Solving Linear Inequalities 2

See the Big Idea
Determine how designers decide on the number of electrical circuits needed in a house.

3 Graphing Linear Functions

See the Big Idea
Discover why unlike almost any other natural phenomenon, light travels at a constant speed.

4 Writing Linear Functions

See the Big Idea
Explore wind power and discover where the future of wind power will take us.

5 Solving Systems of Linear Equations

See the Big Idea
Learn how fisheries manage their complex ecosystems.

Exponential Functions and Sequences 6

See the Big Idea
Explore the variety of recursive sequences in language, art, music, nature, and games.

7 Polynomial Equations and Factoring

See the Big Idea
Explore whether seagulls and crows use the optimal height while dropping hard-shelled food to crack it open.

8 Graphing Quadratic Functions

See the Big Idea
Investigate the link between population growth and the classic exponential pay doubling application.

9 Solving Quadratic Equations

See the Big Idea
Explore the Parthenon and investigate how the use of the golden rectangle has evolved since its discovery.

How to Use Your Math Book

Get ready for each chapter by **Maintaining Mathematical Proficiency** and sharpening your **Mathematical Thinking**. Begin each section by working through the **EXPLORATIONS** to **Communicate Your Answer** to the **Essential Question**. Each **Lesson** will explain **What You Will Learn** through **EXAMPLES**, **Core Concepts**, and **Core Vocabulary**.

Answer the **Monitoring Progress** questions as you work through each lesson. Look for STUDY TIPS, COMMON ERRORS, and suggestions for looking at a problem ANOTHER WAY throughout the lessons. We will also provide you with guidance for accurate mathematical READING and concept details you should REMEMBER.

Sharpen your newly acquired skills with **Exercises** at the end of every section. Halfway through each chapter you will be asked **What Did You Learn?** and you can use the Mid-Chapter **Quiz** to check your progress. You can also use the **Chapter Review** and **Chapter Test** to review and assess yourself after you have completed a chapter.

Apply what you learned in each chapter to a **Performance Task** and build your confidence for taking standardized tests with each chapter's **Standards Assessment**.

For extra practice in any chapter, use your *Online Resources*, *Skills Review Handbook*, or your *Student Journal*.

1 Solving Linear Equations

1.1 Solving Simple Equations
1.2 Solving Multi-Step Equations
1.3 Solving Equations with Variables on Both Sides
1.4 Rewriting Equations and Formulas

Density of Pyrite *(p. 33)*

Ski Club *(p. 32)*

Boat *(p. 24)*

Biking *(p. 14)*

Average Speed *(p. 6)*

Mathematical Thinking: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

Maintaining Mathematical Proficiency

Adding and Subtracting Integers (6.3.D)

Example 1 **Evaluate $4 + (-12)$.**

$|-12| > |4|$. So, subtract $|4|$ from $|-12|$.

$$4 + (-12) = -8$$

Use the sign of -12.

Example 2 **Evaluate $-7 - (-16)$.**

$$-7 - (-16) = -7 + 16 \qquad \text{Add the opposite of } -16.$$
$$= 9 \qquad \text{Add.}$$

Add or subtract.

1. $-5 + (-2)$
2. $0 + (-13)$
3. $-6 + 14$
4. $19 - (-13)$
5. $-1 - 6$
6. $-5 - (-7)$
7. $17 + 5$
8. $8 + (-3)$
9. $11 - 15$

Multiplying and Dividing Integers (6.3.D)

Example 3 **Evaluate $-3 \cdot (-5)$.**

The integers have the same sign.

$$-3 \cdot (-5) = 15$$

The product is positive.

Example 4 **Evaluate $15 \div (-3)$.**

The integers have different signs.

$$15 \div (-3) = -5$$

The quotient is negative.

Multiply or divide.

10. $-3(8)$
11. $-7 \cdot (-9)$
12. $4 \cdot (-7)$
13. $-24 \div (-6)$
14. $-16 \div 2$
15. $12 \div (-3)$
16. $6 \cdot 8$
17. $36 \div 6$
18. $-3(-4)$
19. **ABSTRACT REASONING** Summarize the rules for (a) adding integers, (b) subtracting integers, (c) multiplying integers, and (d) dividing integers. Give an example of each.

Mathematical Thinking

Mathematically proficient students display, explain, and justify mathematical ideas and arguments using precise mathematical language in written or oral communication. **(A.1.G)**

Specifying Units of Measure

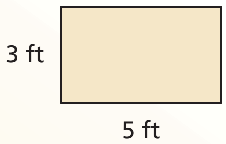
Core Concept

Operations and Unit Analysis

Addition and Subtraction

When you add or subtract quantities, they must have the same units of measure. The sum or difference will have the *same* unit of measure.

Example



Perimeter of rectangle

$= (3 \text{ ft}) + (5 \text{ ft}) + (3 \text{ ft}) + (5 \text{ ft})$

$= 16 \text{ feet}$

When you add **feet,** you get **feet.**

Multiplication and Division

When you multiply or divide quantities, the product or quotient will have a *different* unit of measure.

Example Area of rectangle $= (3 \text{ ft}) \times (5 \text{ ft})$

$= 15 \text{ square feet}$

When you multiply **feet,** you get feet squared, or **square feet.**

EXAMPLE 1 Specifying Units of Measure

You work 8 hours and earn \$72. What is your hourly wage?

SOLUTION

$$\overbrace{\text{Hourly wage (\$ per h)}}^{\text{dollars per hour}} = \overbrace{\$72 \div 8 \text{ h}}^{\text{dollars per hour}}$$

$$= \$9 \text{ per hour}$$

The units on each side of the equation balance. Both are specified in dollars per hour.

▶ Your hourly wage is \$9 per hour.

Monitoring Progress

Solve the problem and specify the units of measure.

1. The population of the United States was about 280 million in 2000 and about 310 million in 2010. What was the annual rate of change in population from 2000 to 2010?
2. You drive 240 miles and use 8 gallons of gasoline. What was your car's gas mileage (in miles per gallon)?
3. A bathtub is in the shape of a rectangular prism. Its dimensions are 5 feet by 3 feet by 18 inches. The bathtub is three-fourths full of water and drains at a rate of 1 cubic foot per minute. About how long does it take for all the water to drain?

1.1 Solving Simple Equations

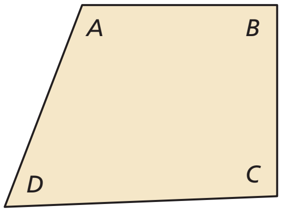
Essential Question How can you use simple equations to solve real-life problems?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.5.A

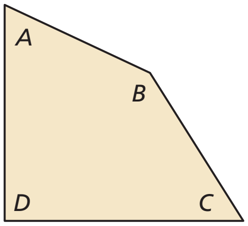
EXPLORATION 1 Measuring Angles

Work with a partner. Use a protractor to measure the angles of each quadrilateral. Copy and complete the table to organize your results. (The notation $m\angle A$ denotes the measure of angle A.) How precise are your measurements?

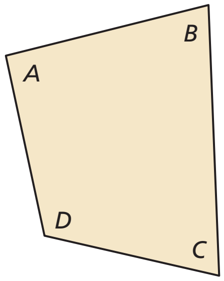
a.



b.



c.



Quadrilateral	$m\angle A$ (degrees)	$m\angle B$ (degrees)	$m\angle C$ (degrees)	$m\angle D$ (degrees)	$m\angle A + m\angle B + m\angle C + m\angle D$
a.					
b.					
c.					

UNDERSTANDING MATHEMATICAL TERMS

A **conjecture** is an unproven statement about a general mathematical concept. After the statement is proven, it is called a **rule** or a **theorem**.

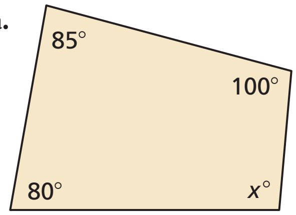
EXPLORATION 2 Making a Conjecture

Work with a partner. Use the completed table in Exploration 1 to write a conjecture about the sum of the angle measures of a quadrilateral. Draw three quadrilaterals that are different from those in Exploration 1 and use them to justify your conjecture.

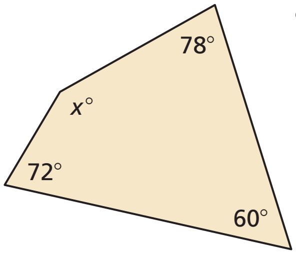
EXPLORATION 3 Applying Your Conjecture

Work with a partner. Use the conjecture you wrote in Exploration 2 to write an equation for each quadrilateral. Then solve the equation to find the value of x. Use a protractor to check the reasonableness of your answer.

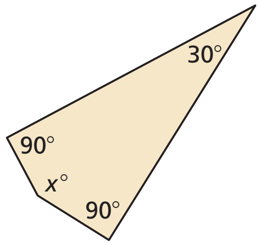
a.



b.



c.



Communicate Your Answer

4. How can you use simple equations to solve real-life problems?

5. Draw your own quadrilateral and cut it out. Tear off the four corners of the quadrilateral and rearrange them to affirm the conjecture you wrote in Exploration 2. Explain how this affirms the conjecture.

1.1 Lesson

Core Vocabulary

conjecture, *p. 3*
rule, *p. 3*
theorem, *p. 3*
equation, *p. 4*
linear equation in one variable, *p. 4*
solution, *p. 4*
inverse operations, *p. 4*
equivalent equations, *p. 4*

Previous
expression

What You Will Learn

- Solve linear equations using addition and subtraction.
- Solve linear equations using multiplication and division.
- Use linear equations to solve real-life problems.

Solving Linear Equations by Adding or Subtracting

An **equation** is a statement that two expressions are equal. A **linear equation in one variable** is an equation that can be written in the form $ax + b = 0$, where a and b are constants and $a \neq 0$. A **solution** of an equation is a value that makes the equation true.

Inverse operations are two operations that undo each other, such as addition and subtraction. When you perform the same inverse operation on each side of an equation, you produce an equivalent equation. **Equivalent equations** are equations that have the same solution(s).

Core Concept

Addition Property of Equality

Words Adding the same number to each side of an equation produces an equivalent equation.

Algebra If $a = b$, then $a + c = b + c$.

Subtraction Property of Equality

Words Subtracting the same number from each side of an equation produces an equivalent equation.

Algebra If $a = b$, then $a - c = b - c$.

EXAMPLE 1 Solving Equations by Addition or Subtraction

Solve each equation. Justify each step. Check your answer.

a. $x - 3 = -5$ **b.** $0.9 = y + 2.8$

SOLUTION

a. $x - 3 = -5$ Write the equation.

Addition Property of Equality: $\underline{+3}\quad\underline{+3}$ Add 3 to each side.

$x = -2$ Simplify.

▶ The solution is $x = -2$.

Check

$x - 3 = -5$

$-2 - 3 \stackrel{?}{=} -5$

$-5 = -5$ ✓

b. $0.9 = y + 2.8$ Write the equation.

Subtraction Property of Equality: $\underline{-2.8}\quad\underline{-2.8}$ Subtract 2.8 from each side.

$-1.9 = y$ Simplify.

▶ The solution is $y = -1.9$.

Check

$0.9 = y + 2.8$

$0.9 \stackrel{?}{=} -1.9 + 2.8$

$0.9 = 0.9$ ✓

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Solve the equation. Justify each step. Check your solution.

1. $n + 3 = -7$ **2.** $g - \frac{1}{3} = -\frac{2}{3}$ **3.** $-6.5 = p + 3.9$

Solving Linear Equations by Multiplying or Dividing

Core Concept

Multiplication Property of Equality

Words Multiplying each side of an equation by the same nonzero number produces an equivalent equation.

Algebra If $a = b$, then $a \cdot c = b \cdot c$, $c \neq 0$.

Division Property of Equality

Words Dividing each side of an equation by the same nonzero number produces an equivalent equation.

Algebra If $a = b$, then $a \div c = b \div c$, $c \neq 0$.

REMEMBER

Multiplication and division are inverse operations.

EXAMPLE 2 Solving Equations by Multiplication or Division

Solve each equation. Justify each step. Check your answer.

a. $-\frac{n}{5} = -3$ **b.** $\pi x = -2\pi$ **c.** $1.3z = 5.2$

SOLUTION

a. $-\frac{n}{5} = -3$ Write the equation.

Multiplication Property of Equality → $-5 \cdot \left(-\frac{n}{5}\right) = -5 \cdot (-3)$ Multiply each side by -5.

$n = 15$ Simplify.

The solution is $n = 15$.

Check

$-\frac{n}{5} = -3$

$-\frac{15}{5} \stackrel{?}{=} -3$

$-3 = -3$ ✓

b. $\pi x = -2\pi$ Write the equation.

Division Property of Equality → $\frac{\pi x}{\pi} = \frac{-2\pi}{\pi}$ Divide each side by π.

$x = -2$ Simplify.

The solution is $x = -2$.

Check

$\pi x = -2\pi$

$\pi(-2) \stackrel{?}{=} -2\pi$

$-2\pi = -2\pi$ ✓

c. $1.3z = 5.2$ Write the equation.

Division Property of Equality → $\frac{1.3z}{1.3} = \frac{5.2}{1.3}$ Divide each side by 1.3.

$z = 4$ Simplify.

The solution is $z = 4$.

Check

$1.3z = 5.2$

$1.3(4) \stackrel{?}{=} 5.2$

$5.2 = 5.2$ ✓

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Solve the equation. Justify each step. Check your solution.

4. $\frac{y}{3} = -6$ **5.** $9\pi = \pi x$ **6.** $0.05w = 1.4$

Solving Real-Life Problems

APPLYING MATHEMATICS

Mathematically proficient students routinely check that their solutions make sense in the context of a real-life problem.

Core Concept

Four-Step Approach to Problem Solving

1. **Understand the Problem** What is the unknown? What information is being given? What is being asked?
2. **Make a Plan** This plan might involve one or more of the problem-solving strategies shown on the next page.
3. **Solve the Problem** Carry out your plan. Check that each step is correct.
4. **Look Back** Examine your solution. Check that your solution makes sense in the original statement of the problem.

EXAMPLE 3 Modeling with Mathematics

In the 2012 Olympics, Usain Bolt won the 200-meter dash with a time of 19.32 seconds. Write and solve an equation to find his average speed to the nearest hundredth of a meter per second.

REMEMBER

The formula that relates distance d, rate or speed r, and time t is

$$d = rt.$$

REMEMBER

The symbol $\approx$ means "approximately equal to."

SOLUTION

1. **Understand the Problem** You know the winning time and the distance of the race. You are asked to find the average speed to the nearest hundredth of a meter per second.
2. **Make a Plan** Use the Distance Formula to write an equation that represents the problem. Then solve the equation.
3. **Solve the Problem**

$$d = r \cdot t \qquad \text{Write the Distance Formula.}$$

$$200 = r \cdot 19.32 \qquad \text{Substitute 200 for } d \text{ and 19.32 for } t.$$

$$\frac{200}{19.32} = \frac{19.32r}{19.32} \qquad \text{Divide each side by 19.32.}$$

$$10.35 \approx r \qquad \text{Simplify.}$$

▶ Bolt's average speed was about 10.35 meters per second.

4. **Look Back** Round Bolt's average speed to 10 meters per second. At this speed, it would take

$$\frac{200 \text{ m}}{10 \text{ m/sec}} = 20 \text{ seconds}$$

to run 200 meters. Because 20 is close to 19.32, your solution is reasonable.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

7. Suppose Usain Bolt ran 400 meters at the same average speed that he ran the 200 meters. How long would it take him to run 400 meters? Round your answer to the nearest hundredth of a second.

Core Concept

Common Problem-Solving Strategies

Use a verbal model.	Guess, check, and revise.
Draw a diagram.	Sketch a graph or number line.
Write an equation.	Make a table.
Look for a pattern.	Make a list.
Work backward.	Break the problem into parts.

EXAMPLE 4 Modeling with Mathematics

On January 22, 1943, the temperature in Spearfish, South Dakota, fell from 54°F at 9:00 A.M. to -4°F at 9:27 A.M. How many degrees did the temperature fall?

SOLUTION

1. **Understand the Problem** You know the temperature before and after the temperature fell. You are asked to find how many degrees the temperature fell.
2. **Make a Plan** Use a verbal model to write an equation that represents the problem. Then solve the equation.
3. **Solve the Problem**

Words Temperature at 9:27 A.M. = Temperature at 9:00 A.M. − Number of degrees the temperature fell

Variable Let T be the number of degrees the temperature fell.

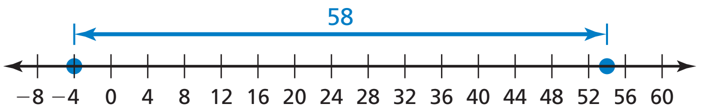
Equation $-4 \quad = \quad 54 \quad - \quad T$

$$-4 = 54 - T \qquad \text{Write the equation.}$$
$$-4 - 54 = 54 - 54 - T \qquad \text{Subtract 54 from each side.}$$
$$-58 = -T \qquad \text{Simplify.}$$
$$58 = T \qquad \text{Divide each side by } -1.$$

▶ The temperature fell 58°F.

REMEMBER
The distance between two points on a number line is always positive.

4. **Look Back** The temperature fell from 54 degrees *above* 0 to 4 degrees *below* 0. You can use a number line to check that your solution is reasonable.



Monitoring Progress

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8. You thought the balance in your checking account was $68. When your bank statement arrives, you realize that you forgot to record a check. The bank statement lists your balance as $26. Write and solve an equation to find the amount of the check that you forgot to record.

1.1 Exercises

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Vocabulary and Core Concept Check

1. **VOCABULARY** Which of the operations $+$, $-$, $\times$, and $\div$ are inverses of each other?
2. **VOCABULARY** Are the equations $-2x = 10$ and $-5x = 25$ equivalent? Explain.
3. **WRITING** Which property of equality would you use to solve the equation $14x = 56$? Explain.
4. **WHICH ONE DOESN'T BELONG?** Which expression does not belong with the other three? Explain your reasoning.

$8 = \frac{x}{2}$ $\qquad 3 = x \div 4$ $\qquad x - 6 = 5$ $\qquad \frac{x}{3} = 9$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–14, solve the equation. Justify each step. Check your solution. *(See Example 1.)*

5. $x + 5 = 8$
6. $m + 9 = 2$
7. $y - 4 = 3$
8. $s - 2 = 1$
9. $w + 3 = -4$
10. $n - 6 = -7$
11. $-14 = p - 11$
12. $0 = 4 + q$
13. $r + (-8) = 10$
14. $t - (-5) = 9$

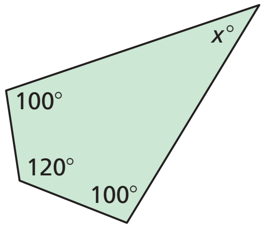
15. **MODELING WITH MATHEMATICS** A discounted amusement park ticket costs \$12.95 less than the original price p. Write and solve an equation to find the original price.

16. **MODELING WITH MATHEMATICS** You and a friend are playing a board game. Your final score x is 12 points less than your friend's final score. Write and solve an equation to find your final score.

	ROUND 9	ROUND 10	FINAL SCORE
Your Friend	22	12	195
You	9	25	?

USING TOOLS **The sum of the angle measures of a quadrilateral is 360°. In Exercises 17–20, write and solve an equation to find the value of x. Use a protractor to check the reasonableness of your answer.**

17.



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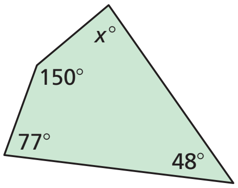

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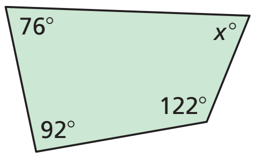

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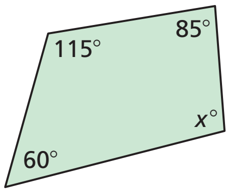


In Exercises 21–30, solve the equation. Justify each step. Check your solution. *(See Example 2.)*

21. $5g = 20$
22. $4q = 52$
23. $p \div 5 = 3$
24. $y \div 7 = 1$
25. $-8r = 64$
26. $x \div (-2) = 8$
27. $\frac{x}{6} = 8$
28. $\frac{w}{-3} = 6$
29. $-54 = 9s$
30. $-7 = \frac{t}{7}$

In Exercises 31–38, solve the equation. Check your solution.

31. $\frac{3}{2} + t = \frac{1}{2}$

32. $b - \frac{3}{16} = \frac{5}{16}$

33. $\frac{3}{7}m = 6$

34. $-\frac{2}{5}y = 4$

35. $5.2 = a - 0.4$

36. $f + 3\pi = 7\pi$

37. $-108\pi = 6\pi j$

38. $x \div (-2) = 1.4$

ERROR ANALYSIS **In Exercises 39 and 40, describe and correct the error in solving the equation.**

39.

$$-0.8 + r = 12.6$$
$$r = 12.6 + (-0.8)$$
$$r = 11.8$$

40.

$$-\frac{m}{3} = -4$$
$$3 \cdot \left(-\frac{m}{3}\right) = 3 \cdot (-4)$$
$$m = -12$$

41. **ANALYZING RELATIONSHIPS** A baker orders 162 eggs. Each carton contains 18 eggs. Which equation can you use to find the number x of cartons? Explain your reasoning and solve the equation.

Ⓐ $162x = 18$

Ⓑ $\frac{x}{18} = 162$

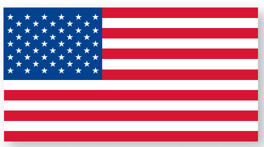
Ⓒ $18x = 162$

Ⓓ $x + 18 = 162$

MODELING WITH MATHEMATICS **In Exercises 42–44, write and solve an equation to answer the question.** *(See Examples 3 and 4.)*

42. The temperature at 5 P.M. is 20°F. The temperature at 10 P.M. is −5°F. How many degrees did the temperature fall?

43. The length of an American flag is 1.9 times its width. What is the width of the flag?

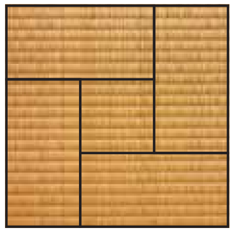


44. The balance of an investment account is \$308 more than the balance 4 years ago. The current balance of the account is \$4708. What was the balance 4 years ago?

45. **REASONING** Identify the property of equality that makes Equation 1 and Equation 2 equivalent.

Equation 1	$x - \frac{1}{2} = \frac{x}{4} + 3$
Equation 2	$4x - 2 = x + 12$

46. **PROBLEM SOLVING** Tatami mats are used as a floor covering in Japan. One possible layout uses four identical rectangular mats and one square mat, as shown. The area of the square mat is half the area of one of the rectangular mats.



a. Write and solve an equation to find the area of one rectangular mat.

b. The length of a rectangular mat is twice the width. Use Guess, Check, and Revise to find the dimensions of one rectangular mat.

47. **PROBLEM SOLVING** You spend \$30.40 on 4 CDs. Each CD costs the same amount and is on sale for 80% of the original price.

a. Write and solve an equation to find how much you spend on each CD.

b. The next day, the CDs are no longer on sale. You have \$25. Will you be able to buy 3 more CDs? Explain your reasoning.

48. **ANALYZING RELATIONSHIPS** As c increases, does the value of x *increase*, *decrease*, or *stay the same* for each equation? Assume c is positive.

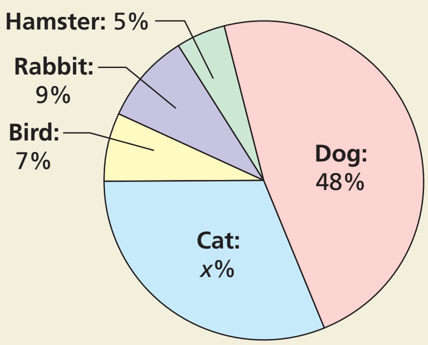
Equation	Value of x
$x - c = 0$	
$cx = 1$	
$cx = c$	
$\frac{x}{c} = 1$	

49. USING STRUCTURE Use the values -2, 5, 9, and 10 to complete each statement about the equation $ax = b - 5$.

a. When $a =$ ___ and $b =$ ___, x is a positive integer.

b. When $a =$ ___ and $b =$ ___, x is a negative integer.

50. HOW DO YOU SEE IT? The circle graph shows the percents of different animals sold at a local pet store in 1 year.



a. What percent is represented by the entire circle?

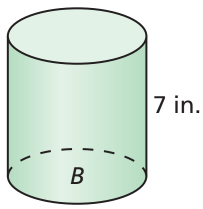
b. How does the equation $7 + 9 + 5 + 48 + x = 100$ relate to the circle graph? How can you use this equation to find the percent of cats sold?

51. REASONING One-sixth of the girls and two-sevenths of the boys in a school marching band are in the percussion section. The percussion section has 6 girls and 10 boys. How many students are in the marching band? Explain.

52. THOUGHT PROVOKING Write a real-life problem that can be modeled by an equation equivalent to the equation $5x = 30$. Then solve the equation and write the answer in the context of your real-life problem.

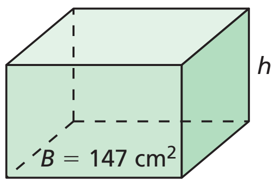
MATHEMATICAL CONNECTIONS In Exercises 53–56, find the height h or the area of the base B of the solid.

53.



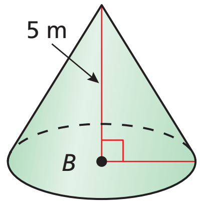
Volume $= 84\pi$ in.^{3}

54.



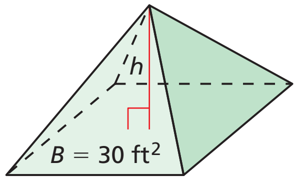
Volume $= 1323$ cm^3

55.



Volume $= 15\pi$ m^3

56.



Volume $= 35$ ft^3

57. MAKING AN ARGUMENT In baseball, a player's batting average is calculated by dividing the number of hits by the number of at-bats. The table shows Player A's batting average and number of at-bats for three regular seasons.

Season	Batting average	At-bats
2010	.312	596
2011	.296	446
2012	.295	599

a. How many hits did Player A have in the 2011 regular season? Round your answer to the nearest whole number.

b. Player B had 33 fewer hits in the 2011 season than Player A but had a greater batting average. Your friend concludes that Player B had more at-bats in the 2011 season than Player A. Is your friend correct? Explain.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Use the Distributive Property to simplify the expression. *(Skills Review Handbook)*

58. $8(y + 3)$

59. $\frac{5}{6}\left(x + \frac{1}{2} + 4\right)$

60. $5(m + 3 + n)$

61. $4(2p + 4q + 6)$

Copy and complete the statement. Round to the nearest hundredth, if necessary. *(Skills Review Handbook)*

62. $\dfrac{5\text{ L}}{\text{min}} = \dfrac{\square\text{ L}}{\text{h}}$

63. $\dfrac{68\text{ mi}}{\text{h}} \approx \dfrac{\square\text{ mi}}{\text{sec}}$

64. $\dfrac{7\text{ gal}}{\text{min}} \approx \dfrac{\square\text{ qt}}{\text{sec}}$

65. $\dfrac{8\text{ km}}{\text{min}} \approx \dfrac{\square\text{ mi}}{\text{h}}$

1.2 Solving Multi-Step Equations

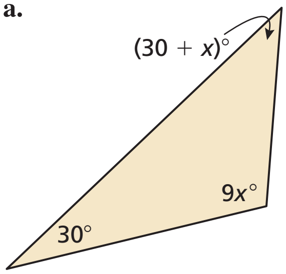
Essential Question How can you use multi-step equations to solve real-life problems?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.5.A
A.10.D

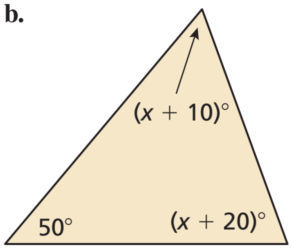
EXPLORATION 1 Solving for the Angle Measures of a Polygon

Work with a partner. The sum S of the angle measures of a polygon with n sides can be found using the formula $S = 180(n - 2)$. Write and solve an equation to find each value of x. Justify the steps in your solution. Then find the angle measures of each polygon. How can you check the reasonableness of your answers?

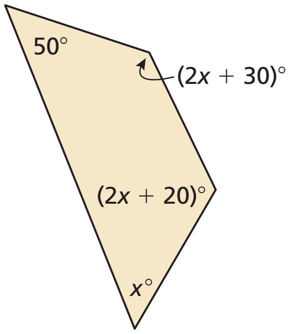
a.



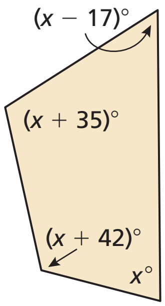
b.



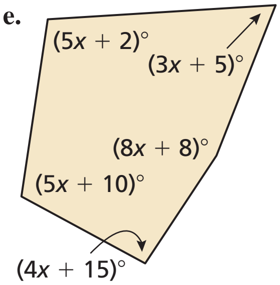
c.



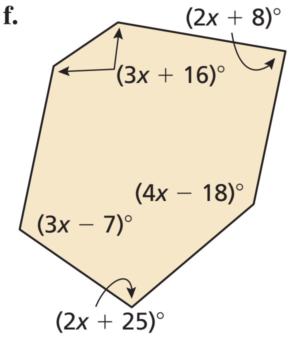
d.



e.



f.



JUSTIFYING THE SOLUTION

To be proficient in math, you need to be sure your answers make sense in the context of the problem. For instance, if you find the angle measures of a triangle, and they have a sum that is not equal to 180°, then you should check your work for mistakes.

EXPLORATION 2 Writing a Multi-Step Equation

Work with a partner.

a. Draw an irregular polygon.

b. Measure the angles of the polygon. Record the measurements on a separate sheet of paper.

c. Choose a value for x. Then, using this value, work backward to assign a variable expression to each angle measure, as in Exploration 1.

d. Trade polygons with your partner.

e. Solve an equation to find the angle measures of the polygon your partner drew. Do your answers seem reasonable? Explain.

Communicate Your Answer

3. How can you use multi-step equations to solve real-life problems?

4. In Exploration 1, you were given the formula for the sum S of the angle measures of a polygon with n sides. Explain why this formula works.

5. The sum of the angle measures of a polygon is 1080º. How many sides does the polygon have? Explain how you found your answer.

1.2 Lesson

What You Will Learn

- Solve multi-step linear equations using inverse operations.
- Use multi-step linear equations to solve real-life problems.
- Use unit analysis to model real-life problems.

Core Vocabulary

Previous
inverse operations
mean

Solving Multi-Step Linear Equations

Core Concept

Solving Multi-Step Equations

To solve a multi-step equation, simplify each side of the equation, if necessary. Then use inverse operations to isolate the variable.

EXAMPLE 1 Solving a Two-Step Equation

Solve $2.5x - 13 = 2$. Check your solution.

SOLUTION

$2.5x - 13 = 2$	Write the equation.
Undo the subtraction. → $\underline{+13} \quad \underline{+13}$	Add 13 to each side.
$2.5x = 15$	Simplify.
Undo the multiplication. → $\dfrac{2.5x}{2.5} = \dfrac{15}{2.5}$	Divide each side by 2.5.
$x = 6$	Simplify.

Check

$2.5x - 13 = 2$

$2.5(6) - 13 \stackrel{?}{=} 2$

$2 = 2$ ✓

- The solution is $x = 6$.

EXAMPLE 2 Combining Like Terms to Solve an Equation

Solve $-12 = 9x - 6x + 15$. Check your solution.

SOLUTION

$-12 = 9x - 6x + 15$	Write the equation.
$-12 = 3x + 15$	Combine like terms.
Undo the addition. → $\underline{-15} \quad \underline{-15}$	Subtract 15 from each side.
$-27 = 3x$	Simplify.
Undo the multiplication. → $\dfrac{-27}{3} = \dfrac{3x}{3}$	Divide each side by 3.
$-9 = x$	Simplify.

Check

$-12 = 9x - 6x + 15$

$-12 \stackrel{?}{=} 9(-9) - 6(-9) + 15$

$-12 = -12$ ✓

- The solution is $x = -9$.

Monitoring Progress

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Solve the equation. Check your solution.

1. $-2n + 3 = 9$

2. $-21 = \frac{1}{2}c - 11$

3. $-2x - 10x + 12 = 18$

EXAMPLE 3 Using Structure to Solve a Multi-Step Equation

Solve $2(1 - x) + 3 = -8$. Check your solution.

SOLUTION

Method 1 One way to solve the equation is by using the Distributive Property.

$2(1 - x) + 3 = -8$	Write the equation.
$2(1) - 2(x) + 3 = -8$	Distributive Property
$2 - 2x + 3 = -8$	Multiply.
$-2x + 5 = -8$	Combine like terms.
$\underline{-5} \quad \underline{-5}$	Subtract 5 from each side.
$-2x = -13$	Simplify.
$\frac{-2x}{-2} = \frac{-13}{-2}$	Divide each side by -2.
$x = 6.5$	Simplify.

▶ The solution is $x = 6.5$.

Check

$2(1 - x) + 3 = -8$

$2(1 - 6.5) + 3 \stackrel{?}{=} -8$

$-8 = -8$ ✓

Method 2 Another way to solve the equation is by interpreting the expression $1 - x$ as a single quantity.

ANALYZING MATHEMATICAL RELATIONSHIPS

First solve for the expression $1 - x$, and then solve for x.

$2(1 - x) + 3 = -8$	Write the equation.
$\underline{-3} \quad \underline{-3}$	Subtract 3 from each side.
$2(1 - x) = -11$	Simplify.
$\frac{2(1 - x)}{2} = \frac{-11}{2}$	Divide each side by 2.
$1 - x = -5.5$	Simplify.
$\underline{-1} \quad \underline{-1}$	Subtract 1 from each side.
$-x = -6.5$	Simplify.
$\frac{-x}{-1} = \frac{-6.5}{-1}$	Divide each side by -1.
$x = 6.5$	Simplify.

▶ The solution is $x = 6.5$, which is the same solution obtained in Method 1.

Monitoring Progress

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Solve the equation. Check your solution.

4. $3(x + 1) + 6 = -9$

5. $15 = 5 + 4(2d - 3)$

6. $13 = -2(y - 4) + 3y$

7. $2x(5 - 3) - 3x = 5$

8. $-4(2m + 5) - 3m = 35$

9. $5(3 - x) + 2(3 - x) = 14$

Solving Real-Life Problems

EXAMPLE 4 Modeling with Mathematics

Use the table to find the number of miles x you need to bike on Friday so that the mean number of miles biked per day is 5.

Day	Miles
Monday	3.5
Tuesday	5.5
Wednesday	0
Thursday	5
Friday	x

SOLUTION

1. **Understand the Problem** You know how many miles you biked Monday through Thursday. You are asked to find the number of miles you need to bike on Friday so that the mean number of miles biked per day is 5.
2. **Make a Plan** Use the definition of mean to write an equation that represents the problem. Then solve the equation.
3. **Solve the Problem** The mean of a data set is the sum of the data divided by the number of data values.

$$\frac{3.5 + 5.5 + 0 + 5 + x}{5} = 5 \qquad \text{Write the equation.}$$

$$\frac{14 + x}{5} = 5 \qquad \text{Combine like terms.}$$

$$5 \cdot \frac{14 + x}{5} = 5 \cdot 5 \qquad \text{Multiply each side by 5.}$$

$$14 + x = 25 \qquad \text{Simplify.}$$

$$\underline{-14} \qquad \underline{-14} \qquad \text{Subtract 14 from each side.}$$

$$x = 11 \qquad \text{Simplify.}$$

▶ You need to bike 11 miles on Friday.

4. **Look Back** Notice that on the days that you did bike, the values are close to the mean. Because you did not bike on Wednesday, you need to bike about twice the mean on Friday. Eleven miles is about twice the mean. So, your solution is reasonable.

Monitoring Progress

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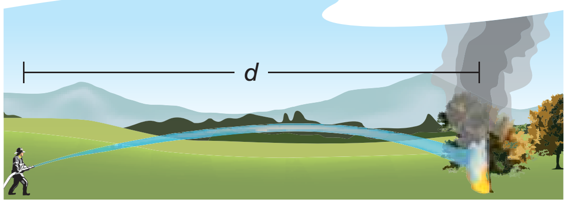
10. The formula $d = \frac{1}{2}n + 26$ relates the nozzle pressure n (in pounds per square inch) of a fire hose and the maximum horizontal distance the water reaches d (in feet). How much pressure is needed to reach a fire 50 feet away?



REMEMBER

When you add **miles** to **miles**, you get **miles**. But, when you divide **miles** by **days**, you get **miles per day**.

Using Unit Analysis to Model Real-Life Problems

When you write an equation to model a real-life problem, you should check that the units on each side of the equation balance. For instance, in Example 4, notice how the units balance.

miles miles per day

$$\frac{3.5 + 5.5 + 0 + 5 + x}{5} = 5 \qquad \frac{\text{mi}}{\text{day}} = \frac{\text{mi}}{\text{day}} \ \checkmark$$

per days

EXAMPLE 5 Solving a Real-Life Problem

Your school's drama club charges \$4 per person for admission to a play. The club borrowed \$400 to pay for costumes and props. After paying back the loan, the club has a profit of \$100. How many people attended the play?

SOLUTION

1. **Understand the Problem** You know how much the club charges for admission. You also know how much the club borrowed and its profit. You are asked to find how many people attended the play.
2. **Make a Plan** Use a verbal model to write an equation that represents the problem. Then solve the equation.
3. **Solve the Problem**

REMEMBER

When you multiply **dollars per person** by **people**, you get **dollars**.

Words Ticket price • Number of people who attended − Amount of loan = Profit

Variable Let x be the number of people who attended.

Equation $\dfrac{\$4}{\cancel{\text{person}}} \cdot x\ \cancel{\text{people}} - \$400 = \$100 \qquad \$ = \$ \ \checkmark$

$4x - 400 = 100$	Write the equation.
$4x - 400 + 400 = 100 + 400$	Add 400 to each side.
$4x = 500$	Simplify.
$\dfrac{4x}{4} = \dfrac{500}{4}$	Divide each side by 4.
$x = 125$	Simplify.

▶ So, 125 people attended the play.

4. **Look Back** To check that your solution is reasonable, multiply \$4 per person by 125 people. The result is \$500. After paying back the \$400 loan, the club has \$100, which is the profit.

Monitoring Progress

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11. You have 96 feet of fencing to enclose a rectangular pen for your dog. To provide sufficient running space for your dog to exercise, the pen should be three times as long as it is wide. Find the dimensions of the pen.

1.2 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** To solve the equation $2x + 3x = 20$, first combine $2x$ and $3x$ because they are __________.

2. **WRITING** Describe two ways to solve the equation $2(4x - 11) = 10$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–14, solve the equation. Check your solution. *(See Examples 1 and 2.)*

3. $3w + 7 = 19$

4. $2g - 13 = 3$

5. $11 = 12 - q$

6. $10 = 7 - m$

7. $5 = \frac{z}{-4} - 3$

8. $\frac{a}{3} + 4 = 6$

9. $\frac{h + 6}{5} = 2$

10. $\frac{d - 8}{-2} = 12$

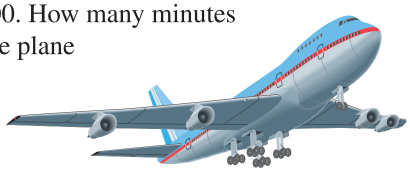
11. $8y + 3y = 44$

12. $36 = 13n - 4n$

13. $12v + 10v + 14 = 80$

14. $6c - 8 - 2c = -16$

15. **MODELING WITH MATHEMATICS** The altitude a (in feet) of a plane t minutes after liftoff is given by $a = 3400t + 600$. How many minutes after liftoff is the plane at an altitude of 21,000 feet?



16. **MODELING WITH MATHEMATICS** A repair bill for your car is \$553. The parts cost \$265. The labor cost is \$48 per hour. Write and solve an equation to find the number of hours of labor spent repairing the car.

In Exercises 17–24, solve the equation. Check your solution. *(See Example 3.)*

17. $4(z + 5) = 32$

18. $-2(4g - 3) = 30$

19. $6 + 5(m + 1) = 26$

20. $5h + 2(11 - h) = -5$

21. $27 = 3c - 3(6 - 2c)$

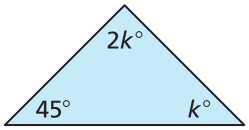
22. $-3 = 12y - 5(2y - 7)$

23. $-3(3 + x) + 4(x - 6) = -4$

24. $5(r + 9) - 2(1 - r) = 1$

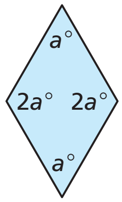
USING TOOLS **In Exercises 25–28, find the value of the variable. Then find the angle measures of the polygon. Use a protractor to check the reasonableness of your answer.**

25.



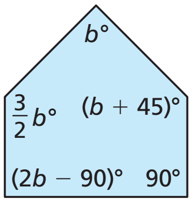
Sum of angle measures: 180°

26.



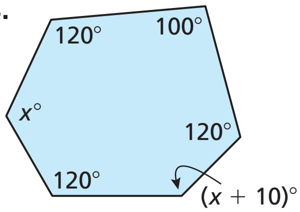
Sum of angle measures: 360°

27.



Sum of angle measures: 540°

28.



Sum of angle measures: 720°

In Exercises 29–34, write and solve an equation to find the number.

29. The sum of twice a number and 13 is 75.

30. The difference of three times a number and 4 is -19.

31. Eight plus the quotient of a number and 3 is -2.

32. The sum of twice a number and half the number is 10.

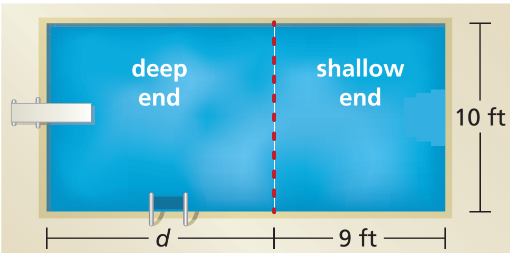
33. Six times the sum of a number and 15 is -42.

34. Four times the difference of a number and 7 is 12.

USING EQUATIONS **In Exercises 35–37, write and solve an equation to answer the question. Check that the units on each side of the equation balance.** *(See Examples 4 and 5.)*

35. During the summer, you work 30 hours per week at a gas station and earn \$8.75 per hour. You also work as a landscaper for \$11 per hour and can work as many hours as you want. You want to earn a total of \$400 per week. How many hours must you work as a landscaper?

36. The area of the surface of the swimming pool is 210 square feet. What is the length d of the deep end (in feet)?



37. You order two tacos and a salad. The salad costs \$2.50. You pay 8% sales tax and leave a \$3 tip. You pay a total of \$13.80. How much does one taco cost?

JUSTIFYING STEPS **In Exercises 38 and 39, justify each step of the solution.**

38. $-\frac{1}{2}(5x - 8) - 1 = 6$ Write the equation.

$-\frac{1}{2}(5x - 8) = 7$

$5x - 8 = -14$

$5x = -6$

$x = -\frac{6}{5}$

39. $2(x + 3) + x = -9$ Write the equation.

$2(x) + 2(3) + x = -9$

$2x + 6 + x = -9$

$3x + 6 = -9$

$3x = -15$

$x = -5$

ERROR ANALYSIS **In Exercises 40 and 41, describe and correct the error in solving the equation.**

40.

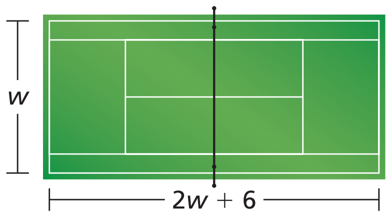
$$-2(7 - y) + 4 = -4$$
$$-14 - 2y + 4 = -4$$
$$-10 - 2y = -4$$
$$-2y = 6$$
$$y = -3$$

41.

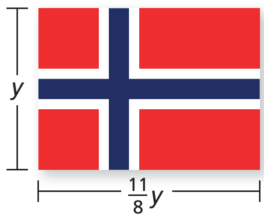
$$\frac{1}{4}(x - 2) + 4 = 12$$
$$\frac{1}{4}(x - 2) = 8$$
$$x - 2 = 2$$
$$x = 4$$

MATHEMATICAL CONNECTIONS **In Exercises 42–44, write and solve an equation to answer the question.**

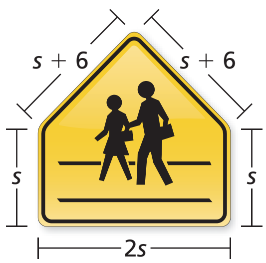
42. The perimeter of the tennis court is 228 feet. What are the dimensions of the court?



43. The perimeter of the Norwegian flag is 190 inches. What are the dimensions of the flag?



44. The perimeter of the school crossing sign is 102 inches. What is the length of each side?



45. **COMPARING METHODS** Solve the equation $2(4 - 8x) + 6 = -1$ using (a) Method 1 from Example 3 and (b) Method 2 from Example 3. Which method do you prefer? Explain.

46. **PROBLEM SOLVING** An online ticket agency charges the amounts shown for basketball tickets. The total cost for an order is \$220.70. How many tickets are purchased?

Charge	Amount
Ticket price	\$32.50 per ticket
Convenience charge	\$3.30 per ticket
Processing charge	\$5.90 per order

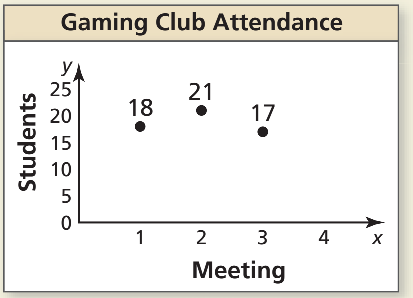
47. **MAKING AN ARGUMENT** You have quarters and dimes that total \$2.80. Your friend says it is possible that the number of quarters is 8 more than the number of dimes. Is your friend correct? Explain.

48. **THOUGHT PROVOKING** You teach a math class and assign a weight to each component of the class. You determine final grades by totaling the products of the weights and the component scores. Choose values for the remaining weights and find the necessary score on the final exam for a student to earn an A (90%) in the class, if possible. Explain your reasoning.

Component	Student's score	Weight	Score × Weight
Class Participation	92%	0.20	92% × 0.20 = 18.4%
Homework	95%		
Midterm Exam	88%		
Final Exam			
Total		1	

49. **REASONING** An even integer can be represented by the expression $2n$, where n is any integer. Find three consecutive even integers that have a sum of 54. Explain your reasoning.

50. **HOW DO YOU SEE IT?** The scatter plot shows the attendance for each meeting of a gaming club.



a. The mean attendance for the first four meetings is 20. Is the number of students who attended the fourth meeting greater than or less than 20? Explain.

b. Estimate the number of students who attended the fourth meeting.

c. Describe a way you can check your estimate in part (b).

REASONING **In Exercises 51–56, the letters *a*, *b*, and *c* represent nonzero constants. Solve the equation for *x*.**

51. $bx = -7$

52. $x + a = \frac{3}{4}$

53. $ax - b = 12.5$

54. $ax + b = c$

55. $2bx - bx = -8$

56. $cx - 4b = 5b$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

57. $4m + 5 - 3m$

58. $9 - 8b + 6b$

59. $6t + 3(1 - 2t) - 5$

Determine whether (a) $x = -1$ or (b) $x = 2$ is a solution of the equation. *(Skills Review Handbook)*

60. $x - 8 = -9$

61. $x + 1.5 = 3.5$

62. $2x - 1 = 3$

63. $3x + 4 = 1$

64. $x + 4 = 3x$

65. $-2(x - 1) = 1 - 3x$

1.1–1.2 What Did You Learn?

Core Vocabulary

conjecture, *p. 3*
rule, *p. 3*
theorem, *p. 3*
equation, *p. 4*
linear equation in one variable, *p. 4*
solution, *p. 4*
inverse operations, *p. 4*
equivalent equations, *p. 4*

Core Concepts

Section 1.1

Addition Property of Equality, *p. 4*
Subtraction Property of Equality, *p. 4*
Multiplication Property of Equality, *p. 5*
Division Property of Equality, *p. 5*
Four-Step Approach to Problem Solving, *p. 6*
Common Problem-Solving Strategies, *p. 7*

Section 1.2

Solving Multi-Step Equations, *p. 12*
Unit Analysis, *p. 15*

Mathematical Thinking

1. How did you make sense of the relationships between the quantities in Exercise 46 on page 9?

2. What is the limitation of the tool you used in Exercises 25–28 on page 16?

Study Skills

Completing Homework Efficiently

Before doing homework, review the core concepts and examples. Use the tutorials at *BigIdeasMath.com* for additional help.

Complete homework as though you are also preparing for a quiz. Memorize different types of problems, vocabulary, rules, and so on.

1.1–1.2 Quiz

Solve the equation. Justify each step. Check your solution. *(Section 1.1)*

1. $x + 9 = 7$

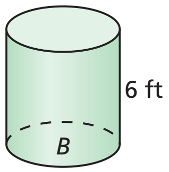
2. $8.6 = z - 3.8$

3. $60 = -12r$

4. $\frac{3}{4}p = 18$

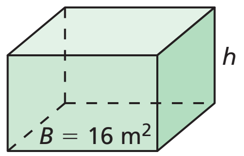
Find the height *h* or the area of the base *B* of the solid. *(Section 1.1)*

5.



Volume $= 24\pi$ ft^3

6.



Volume $= 48$ m^3

Solve the equation. Check your solution. *(Section 1.2)*

7. $2m - 3 = 13$

8. $5 = 10 - v$

9. $5 = 7w + 8w + 2$

10. $-21a + 28a - 6 = -10.2$

11. $2k - 3(2k - 3) = 45$

12. $68 = \frac{1}{5}(20x + 50) + 2$

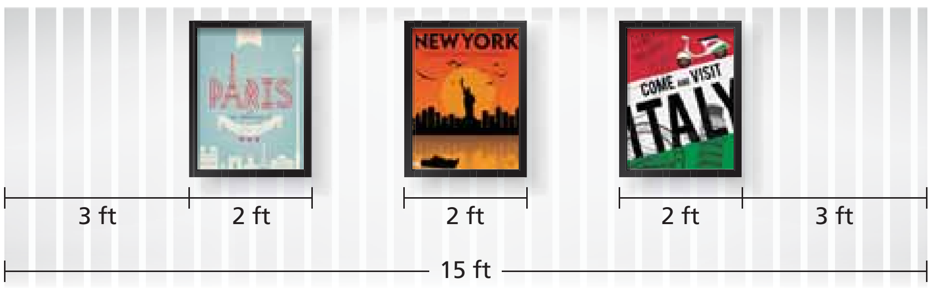
13. To estimate how many miles you are from a thunderstorm, count the seconds between when you see lightning and when you hear thunder. Then divide by 5. Write and solve an equation to determine how many seconds you would count for a thunderstorm that is 2 miles away. *(Section 1.1)*

14. A jar contains red, white, and blue marbles. One-fifth of the marbles are red and one-fourth of the marbles are white. There are 20 red marbles in the jar. How many blue marbles are in the jar? Explain. *(Section 1.1)*

15. Find three consecutive odd integers that have a sum of 45. Explain your reasoning. *(Section 1.2)*

16. You work as a salesperson for a marketing company. The position pays \$300 per week plus 10% of your total weekly sales. You normally earn \$550 each week. Write and solve an equation to determine how much your total weekly sales are normally. *(Section 1.2)*

17. You want to hang three equally-sized travel posters on a wall so that the posters on the ends are each 3 feet from the end of the wall. You want the spacing between posters to be equal. Write and solve an equation to determine how much space you should leave between the posters. *(Section 1.2)*



1.3 Solving Equations with Variables on Both Sides

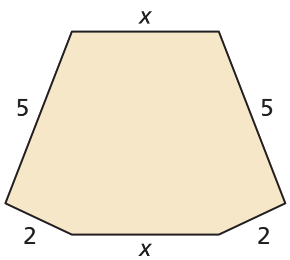
TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.5.A
A.10.D

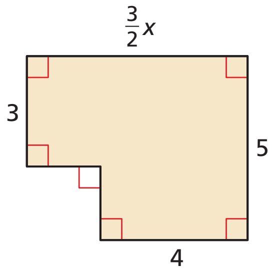
Essential Question

How can you solve an equation that has variables on both sides?

EXPLORATION 1 Perimeter

Work with a partner. The two polygons have the same perimeter. Use this information to write and solve an equation involving x. Explain the process you used to find the solution. Then find the perimeter of each polygon.





EXPLORATION 2 Perimeter and Area

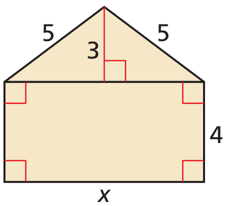
Work with a partner.

- Each figure has the unusual property that the value of its perimeter (in feet) is equal to the value of its area (in square feet). Use this information to write an equation for each figure.
- Solve each equation for x. Explain the process you used to find the solution.
- Find the perimeter and area of each figure.

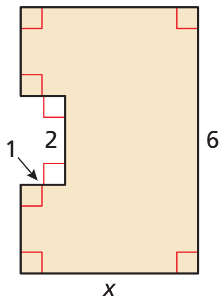
ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to visualize complex things, such as composite figures, as being made up of simpler, more manageable parts.

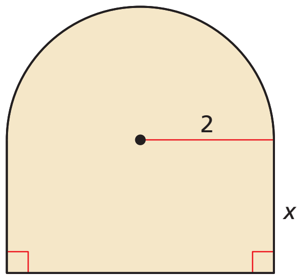
a.



b.



c.



Communicate Your Answer

3. How can you solve an equation that has variables on both sides?

4. Write three equations that have the variable x on both sides. The equations should be different from those you wrote in Explorations 1 and 2. Have your partner solve the equations.

1.3 Lesson

Core Vocabulary

identity, *p. 23*

Previous

inverse operations

What You Will Learn

- Solve linear equations that have variables on both sides.
- Identify special solutions of linear equations.
- Use linear equations to solve real-life problems.

Solving Equations with Variables on Both Sides

Core Concept

Solving Equations with Variables on Both Sides

To solve an equation with variables on both sides, simplify one or both sides of the equation, if necessary. Then use inverse operations to collect the variable terms on one side, collect the constant terms on the other side, and isolate the variable.

EXAMPLE 1 Solving an Equation with Variables on Both Sides

Solve $10 - 4x = -9x$. Check your solution.

SOLUTION

$10 - 4x = -9x$	Write the equation.
$\underline{+\,4x} \quad \underline{+\,4x}$	Add $4x$ to each side.
$10 = -5x$	Simplify.
$\dfrac{10}{-5} = \dfrac{-5x}{-5}$	Divide each side by -5.
$-2 = x$	Simplify.

Check

$10 - 4x = -9x$

$10 - 4(-2) \stackrel{?}{=} -9(-2)$

$18 = 18$ ✓

- The solution is $x = -2$.

EXAMPLE 2 Solving an Equation with Grouping Symbols

Solve $3(3x - 4) = \frac{1}{4}(32x + 56)$.

SOLUTION

$3(3x - 4) = \dfrac{1}{4}(32x + 56)$	Write the equation.
$9x - 12 = 8x + 14$	Distributive Property
$\underline{+\,12} \quad \underline{+\,12}$	Add 12 to each side.
$9x = 8x + 26$	Simplify.
$\underline{-\,8x} \quad \underline{-\,8x}$	Subtract $8x$ from each side.
$x = 26$	Simplify.

- The solution is $x = 26$.

Monitoring Progress

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Solve the equation. Check your solution.

1. $-2x = 3x + 10$ **2.** $\frac{1}{2}(6h - 4) = -5h + 1$ **3.** $-\frac{3}{4}(8n + 12) = 3(n - 3)$

Identifying Special Solutions of Linear Equations

Core Concept

Special Solutions of Linear Equations

Equations do not always have one solution. An equation that is true for all values of the variable is an **identity** and has *infinitely many solutions*. An equation that is not true for any value of the variable has *no solution*.

EXAMPLE 3 Identifying the Number of Solutions

Solve each equation.

a. $3(5x + 2) = 15x$ **b.** $-2(4y + 1) = -8y - 2$

SOLUTION

a.

$3(5x + 2) = 15x$	Write the equation.
$15x + 6 = 15x$	Distributive Property
$\underline{-\ 15x} \qquad \underline{-\ 15x}$	Subtract $15x$ from each side.
$6 = 0$ ✗	Simplify.

▶ The statement $6 = 0$ is never true. So, the equation has no solution.

b.

$-2(4y + 1) = -8y - 2$	Write the equation.
$-8y - 2 = -8y - 2$	Distributive Property
$\underline{+\ 8y} \qquad \underline{+\ 8y}$	Add $8y$ to each side.
$-2 = -2$	Simplify.

▶ The statement $-2 = -2$ is always true. So, the equation is an identity and has infinitely many solutions.

REASONING

The equation $15x + 6 = 15x$ is not true because the number $15x$ cannot be equal to 6 more than itself.

READING

All real numbers are solutions of an identity.

Monitoring Progress

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Solve the equation.

4. $4(1 - p) = -4p + 4$

5. $6m - m = \frac{5}{6}(6m - 10)$

6. $10k + 7 = -3 - 10k$

7. $3(2a - 2) = 2(3a - 3)$

Concept Summary

Steps for Solving Linear Equations

Here are several steps you can use to solve a linear equation. Depending on the equation, you may not need to use some steps.

Step 1 Use the Distributive Property to remove any grouping symbols.

Step 2 Simplify the expression on each side of the equation.

Step 3 Collect the variable terms on one side of the equation and the constant terms on the other side.

Step 4 Isolate the variable.

Step 5 Check your solution.

STUDY TIP

To check an identity, you can choose several different values of the variable.

Solving Real-Life Problems

EXAMPLE 4 Modeling with Mathematics

A boat leaves New Orleans and travels upstream on the Mississippi River for 4 hours. The return trip takes only 2.8 hours because the boat travels 3 miles per hour faster downstream due to the current. How far does the boat travel upstream?

SOLUTION

1. **Understand the Problem** You are given the amounts of time the boat travels and the difference in speeds for each direction. You are asked to find the distance the boat travels upstream.
2. **Make a Plan** Use the Distance Formula to write expressions that represent the problem. Because the distance the boat travels in both directions is the same, you can use the expressions to write an equation.
3. **Solve the Problem** Use the formula (distance) = (rate)(time).

Words Distance upstream = Distance downstream

Variable Let x be the speed (in miles per hour) of the boat traveling upstream.

Equation $\dfrac{x \text{ mi}}{1 \cancel{\text{h}}} \cdot 4 \cancel{\text{h}} = \dfrac{(x+3) \text{ mi}}{1 \cancel{\text{h}}} \cdot 2.8 \cancel{\text{h}}$ (mi = mi) ✓

$$4x = 2.8(x + 3) \qquad \text{Write the equation.}$$

$$4x = 2.8x + 8.4 \qquad \text{Distributive Property}$$

$$\underline{-2.8x} \quad \underline{-2.8x} \qquad \text{Subtract } 2.8x \text{ from each side.}$$

$$1.2x = 8.4 \qquad \text{Simplify.}$$

$$\frac{1.2x}{1.2} = \frac{8.4}{1.2} \qquad \text{Divide each side by 1.2.}$$

$$x = 7 \qquad \text{Simplify.}$$

▶ So, the boat travels 7 miles per hour upstream. To determine how far the boat travels upstream, multiply 7 miles per hour by 4 hours to obtain 28 miles.

4. **Look Back** To check that your solution is reasonable, use the formula for distance. Because the speed upstream is 7 miles per hour, the speed downstream is 7 + 3 = 10 miles per hour. When you substitute each speed into the Distance Formula, you get the same distance for upstream and downstream.

Upstream

$$\text{Distance} = \frac{7 \text{ mi}}{1 \cancel{\text{h}}} \cdot 4 \cancel{\text{h}} = 28 \text{ mi}$$

Downstream

$$\text{Distance} = \frac{10 \text{ mi}}{1 \cancel{\text{h}}} \cdot 2.8 \cancel{\text{h}} = 28 \text{ mi}$$

Monitoring Progress

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8. A boat travels upstream on the Mississippi River for 3.5 hours. The return trip only takes 2.5 hours because the boat travels 2 miles per hour faster downstream due to the current. How far does the boat travel upstream?

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Vocabulary and Core Concept Check

1. **VOCABULARY** Is the equation $-2(4 - x) = 2x + 8$ an identity? Explain your reasoning.

2. **WRITING** Describe the steps in solving the linear equation $3(3x - 8) = 4x + 6$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–16, solve the equation. Check your solution. *(See Examples 1 and 2.)*

3. $15 - 2x = 3x$
4. $26 - 4s = 9s$
5. $5p - 9 = 2p + 12$
6. $8g + 10 = 35 + 3g$
7. $5t + 16 = 6 - 5t$
8. $-3r + 10 = 15r - 8$
9. $7 + 3x - 12x = 3x + 1$
10. $w - 2 + 2w = 6 + 5w$
11. $10(g + 5) = 2(g + 9)$
12. $-9(t - 2) = 4(t - 15)$
13. $\frac{2}{3}(3x + 9) = -2(2x + 6)$
14. $2(2t + 4) = \frac{3}{4}(24 - 8t)$
15. $10(2y + 2) - y = 2(8y - 8)$
16. $2(4x + 2) = 4x - 12(x - 1)$

17. **MODELING WITH MATHEMATICS** You and your friend drive toward each other. The equation $50h = 190 - 45h$ represents the number h of hours until you and your friend meet. When will you meet?

18. **MODELING WITH MATHEMATICS** The equation $1.5r + 15 = 2.25r$ represents the number r of movies you must rent to spend the same amount at each movie store. How many movies must you rent to spend the same amount at each movie store?

Membership Fee: $15

Membership Fee: Free

In Exercises 19–24, solve the equation. Determine whether the equation has *one solution*, *no solution*, or *infinitely many solutions*. *(See Example 3.)*

19. $3t + 4 = 12 + 3t$
20. $6d + 8 = 14 + 3d$
21. $2(h + 1) = 5h - 7$
22. $12y + 6 = 6(2y + 1)$
23. $3(4g + 6) = 2(6g + 9)$
24. $5(1 + 2m) = \frac{1}{2}(8 + 20m)$

ERROR ANALYSIS **In Exercises 25 and 26, describe and correct the error in solving the equation.**

25. ✗

$$5c - 6 = 4 - 3c$$
$$2c - 6 = 4$$
$$2c = 10$$
$$c = 5$$

26. ✗

$$6(2y + 6) = 4(9 + 3y)$$
$$12y + 36 = 36 + 12y$$
$$12y = 12y$$
$$0 = 0$$

The equation has no solution.

27. **MODELING WITH MATHEMATICS** Write and solve an equation to find the month when you would pay the same total amount for each Internet service.

	Installation fee	Price per month
Company A	$60.00	$42.95
Company B	$25.00	$49.95

28. **PROBLEM SOLVING** One serving of granola provides 4% of the protein you need daily. You must get the remaining 48 grams of protein from other sources. How many grams of protein do you need daily?

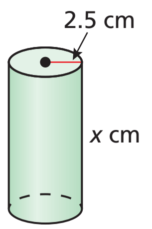
USING STRUCTURE In Exercises 29 and 30, find the value of *r*.

29. $8(x + 6) - 10 + r = 3(x + 12) + 5x$

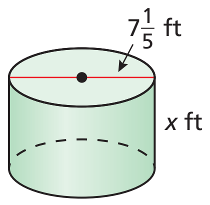
30. $4(x - 3) - r + 2x = 5(3x - 7) - 9x$

MATHEMATICAL CONNECTIONS In Exercises 31 and 32, the value of the surface area of the cylinder is equal to the value of the volume of the cylinder. Find the value of *x*. Then find the surface area and volume of the cylinder.

31.



32.



33. **MODELING WITH MATHEMATICS** A cheetah that is running 90 feet per second is 120 feet behind an antelope that is running 60 feet per second. How long will it take the cheetah to catch up to the antelope? *(See Example 4.)*

34. **MAKING AN ARGUMENT** A cheetah can run at top speed for only about 20 seconds. If an antelope is too far away for a cheetah to catch it in 20 seconds, the antelope is probably safe. Your friend claims the antelope in Exercise 33 will not be safe if the cheetah starts running 650 feet behind it. Is your friend correct? Explain.

REASONING In Exercises 35 and 36, for what value of *a* is the equation an identity? Explain your reasoning.

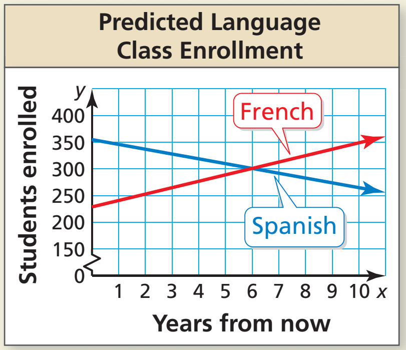
35. $a(2x + 3) = 9x + 15 + x$

36. $8x - 8 + 3ax = 5ax - 2a$

37. **REASONING** Two times the greater of two consecutive integers is 9 less than three times the lesser integer. What are the integers?

38. **HOW DO YOU SEE IT?** The table and the graph show information about students enrolled in Spanish and French classes at a high school.

	Students enrolled this year	Average rate of change
Spanish	355	9 fewer students each year
French	229	12 more students each year

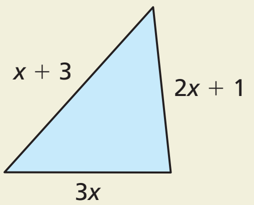


a. Use the graph to determine after how many years there will be equal enrollment in Spanish and French classes.

b. How does the equation $355 - 9x = 229 + 12x$ relate to the table and the graph? How can you use this equation to determine whether your answer in part (a) is reasonable?

39. **WRITING EQUATIONS** Give an example of a linear equation that has (a) no solution and (b) infinitely many solutions. Justify your answers.

40. **THOUGHT PROVOKING** Draw a different figure that has the same perimeter as the triangle shown. Explain why your figure has the same perimeter.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Order the values from least to greatest. *(Skills Review Handbook)*

41. $9, |-4|, -4, 5, |2|$

42. $|-32|, 22, -16, -|21|, |-10|$

43. $-18, |-24|, -19, |-18|, |22|$

44. $-|-3|, |0|, -1, |2|, -2$

1.4 Rewriting Equations and Formulas

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.12.E

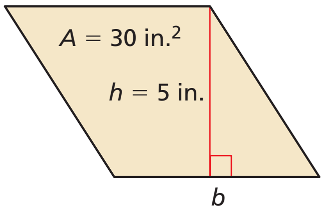
Essential Question

How can you use a formula for one measurement to write a formula for a different measurement?

EXPLORATION 1 Using an Area Formula

Work with a partner.

a. Write a formula for the area A of a parallelogram.



b. Substitute the given values into the formula. Then solve the equation for b. Justify each step.

c. Solve the formula in part (a) for b without first substituting values into the formula. Justify each step.

d. Compare how you solved the equations in parts (b) and (c). How are the processes similar? How are they different?

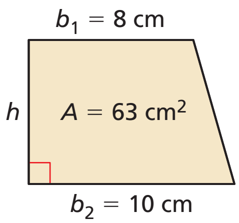
REASONING

To be proficient in math, you need to consider the given units. For instance, in Exploration 1, the area A is given in square inches and the height h is given in inches. A unit analysis shows that the units for the base b are also inches, which makes sense.

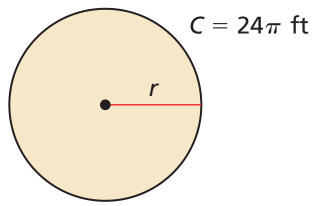
EXPLORATION 2 Using Area, Circumference, and Volume Formulas

Work with a partner. Write the indicated formula for each figure. Then write a new formula by solving for the variable whose value is not given. Use the new formula to find the value of the variable.

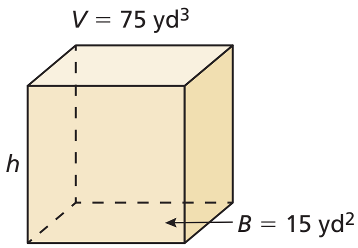
a. Area A of a trapezoid



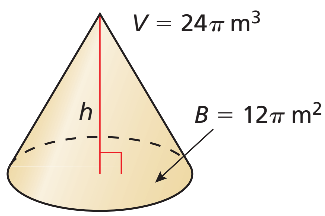
b. Circumference C of a circle



c. Volume V of a rectangular prism



d. Volume V of a cone



Communicate Your Answer

3. How can you use a formula for one measurement to write a formula for a different measurement? Give an example that is different from those given in Explorations 1 and 2.

1.4 Lesson

Core Vocabulary

literal equation, *p. 28*
formula, *p. 29*

Previous
surface area

What You Will Learn

- Rewrite literal equations.
- Rewrite and use formulas for area.
- Rewrite and use other common formulas.

Rewriting Literal Equations

An equation that has two or more variables is called a **literal equation**. To rewrite a literal equation, solve for one variable in terms of the other variable(s).

EXAMPLE 1 Rewriting a Literal Equation

Solve the literal equation $3y + 4x = 9$ for y.

SOLUTION

$3y + 4x = 9$	Write the equation.
$3y + 4x - 4x = 9 - 4x$	Subtract $4x$ from each side.
$3y = 9 - 4x$	Simplify.
$\dfrac{3y}{3} = \dfrac{9 - 4x}{3}$	Divide each side by 3.
$y = 3 - \dfrac{4}{3}x$	Simplify.

▶ The rewritten literal equation is $y = 3 - \dfrac{4}{3}x$.

EXAMPLE 2 Rewriting a Literal Equation

Solve the literal equation $y = 3x + 5xz$ for x.

SOLUTION

$y = 3x + 5xz$	Write the equation.
$y = x(3 + 5z)$	Distributive Property
$\dfrac{y}{3 + 5z} = \dfrac{x(3 + 5z)}{3 + 5z}$	Divide each side by $3 + 5z$.
$\dfrac{y}{3 + 5z} = x$	Simplify.

▶ The rewritten literal equation is $x = \dfrac{y}{3 + 5z}$.

REMEMBER
Division by 0 is undefined.

In Example 2, you must assume that $z \neq -\frac{3}{5}$ in order to divide by $3 + 5z$. In general, if you have to divide by a variable or variable expression when solving a literal equation, you should assume that the variable or variable expression does not equal 0.

Monitoring Progress

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Solve the literal equation for y.

1. $3y - x = 9$ **2.** $2x - 2y = 5$ **3.** $20 = 8x + 4y$

Solve the literal equation for x.

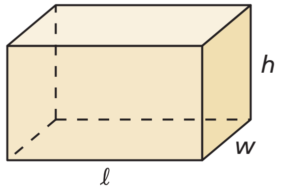
4. $y = 5x - 4x$ **5.** $2x + kx = m$ **6.** $3 + 5x - kx = y$

Rewriting and Using Formulas for Area

A **formula** shows how one variable is related to one or more other variables. A formula is a type of literal equation.

EXAMPLE 3 Rewriting a Formula for Surface Area

The formula for the surface area S of a rectangular prism is $S = 2\ell w + 2\ell h + 2wh$. Solve the formula for the length ℓ.



SOLUTION

$S = 2\ell w + 2\ell h + 2wh$	Write the equation.
$S - 2wh = 2\ell w + 2\ell h + 2wh - 2wh$	Subtract $2wh$ from each side.
$S - 2wh = 2\ell w + 2\ell h$	Simplify.
$S - 2wh = \ell(2w + 2h)$	Distributive Property
$\dfrac{S - 2wh}{2w + 2h} = \dfrac{\ell(2w + 2h)}{2w + 2h}$	Divide each side by $2w + 2h$.
$\dfrac{S - 2wh}{2w + 2h} = \ell$	Simplify.

▶ When you solve the formula for ℓ, you obtain $\ell = \dfrac{S - 2wh}{2w + 2h}$.

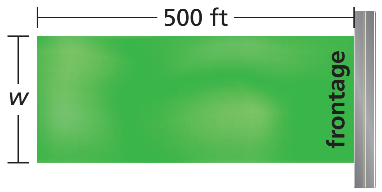
EXAMPLE 4 Using a Formula for Area

You own a rectangular lot that is 500 feet deep. It has an area of 100,000 square feet. To pay for a new water system, you are assessed \$5.50 per foot of lot frontage.

a. Find the frontage of your lot.

b. How much are you assessed for the new water system?

SOLUTION



a. In the formula for the area of a rectangle, let the width w represent the lot frontage.

$A = \ell w$	Write the formula for area of a rectangle.
$\dfrac{A}{\ell} = w$	Divide each side by ℓ to solve for w.
$\dfrac{100{,}000}{500} = w$	Substitute 100,000 for A and 500 for ℓ.
$200 = w$	Simplify.

▶ The frontage of your lot is 200 feet.

b. Each foot of frontage costs \$5.50, and $\dfrac{\$5.50}{1\ \cancel{\text{ft}}} \cdot 200\ \cancel{\text{ft}} = \$1100$.

▶ So, your total assessment is \$1100.

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Solve the formula for the indicated variable.

7. Area of a triangle: $A = \frac{1}{2}bh$; Solve for h.

8. Surface area of a cone: $S = \pi r^2 + \pi r\ell$; Solve for ℓ.

Rewriting and Using Other Common Formulas

Core Concept

Common Formulas

Temperature F = degrees Fahrenheit, C = degrees Celsius

$$C = \frac{5}{9}(F - 32)$$

Simple Interest I = interest, P = principal,
r = annual interest rate (decimal form),
t = time (years)

$$I = Prt$$

Distance d = distance traveled, r = rate, t = time

$$d = rt$$

EXAMPLE 5 Rewriting the Formula for Temperature

Solve the temperature formula for F.

SOLUTION

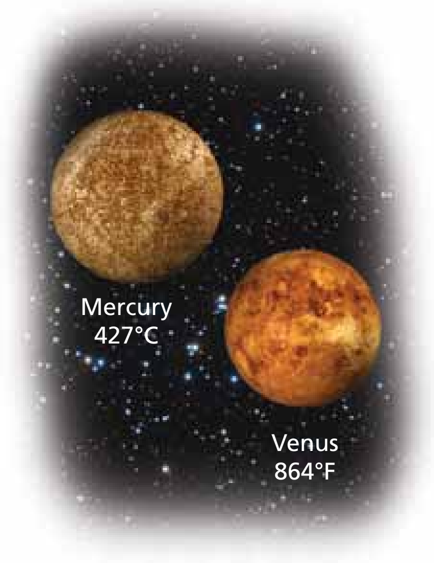
$C = \frac{5}{9}(F - 32)$ Write the temperature formula.

$\frac{9}{5}C = F - 32$ Multiply each side by $\frac{9}{5}$.

$\frac{9}{5}C + 32 = F - 32 + 32$ Add 32 to each side.

$\frac{9}{5}C + 32 = F$ Simplify.

▶ The rewritten formula is $F = \frac{9}{5}C + 32$.



EXAMPLE 6 Using the Formula for Temperature

Which has the greater surface temperature: Mercury or Venus?

SOLUTION

Convert the Celsius temperature of Mercury to degrees Fahrenheit.

$F = \frac{9}{5}C + 32$ Write the rewritten formula from Example 5.

$= \frac{9}{5}(427) + 32$ Substitute 427 for C.

$= 800.6$ Simplify.

▶ Because 864°F is greater than 800.6°F, Venus has the greater surface temperature.

Monitoring Progress

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9. A fever is generally considered to be a body temperature greater than 100°F. Your friend has a temperature of 37°C. Does your friend have a fever?

EXAMPLE 7 Using the Formula for Simple Interest

You deposit \$5000 in an account that earns simple interest. After 6 months, the account earns \$162.50 in interest. What is the annual interest rate?

COMMON ERROR

The unit of t is years. Be sure to convert months to years.

$\frac{1 \text{ yr}}{12 \cancel{\text{mo}}} \cdot 6 \cancel{\text{mo}} = 0.5 \text{ yr}$

SOLUTION

To find the annual interest rate, solve the simple interest formula for r.

$I = Prt$ — Write the simple interest formula.

$\frac{I}{Pt} = r$ — Divide each side by Pt to solve for r.

$\frac{162.50}{(5000)(0.5)} = r$ — Substitute 162.50 for I, 5000 for P, and 0.5 for t.

$0.065 = r$ — Simplify.

▶ The annual interest rate is 0.065, or 6.5%.

EXAMPLE 8 Solving a Real-Life Problem

A truck driver averages 60 miles per hour while delivering freight to a customer. On the return trip, the driver averages 50 miles per hour due to construction. The total driving time is 6.6 hours. How long does each trip take?

SOLUTION

Step 1 Rewrite the Distance Formula to write expressions that represent the two trip times. Solving the formula $d = rt$ for t, you obtain $t = \frac{d}{r}$. So, $\frac{d}{60}$ represents the delivery time, and $\frac{d}{50}$ represents the return trip time.

Step 2 Use these expressions and the total driving time to write and solve an equation to find the distance one way.

$\frac{d}{60} + \frac{d}{50} = 6.6$ — The sum of the two trip times is 6.6 hours.

$\frac{11d}{300} = 6.6$ — Add the left side using the LCD.

$11d = 1980$ — Multiply each side by 300 and simplify.

$d = 180$ — Divide each side by 11 and simplify.

The distance one way is 180 miles.

Step 3 Use the expressions from Step 1 to find the two trip times.

▶ So, the delivery takes $180 \text{ mi} \div \frac{60 \text{ mi}}{1 \text{ h}} = 3$ hours, and the return trip takes $180 \text{ mi} \div \frac{50 \text{ mi}}{1 \text{ h}} = 3.6$ hours.

Monitoring Progress

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10. How much money must you deposit in a simple interest account to earn \$500 in interest in 5 years at 4% annual interest?

11. A truck driver averages 60 miles per hour while delivering freight and 45 miles per hour on the return trip. The total driving time is 7 hours. How long does each trip take?

1.4 Exercises

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Vocabulary and Core Concept Check

1. **VOCABULARY** Is $9r + 16 = \frac{\pi}{5}$ a literal equation? Explain.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

Solve $3x + 6y = 24$ for x.	Solve $24 - 3x = 6y$ for x.
Solve $6y = 24 - 3x$ for y in terms of x.	Solve $24 - 6y = 3x$ for x in terms of y.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, solve the literal equation for y. *(See Example 1.)*

3. $y - 3x = 13$
4. $2x + y = 7$
5. $2y - 18x = -26$
6. $20x + 5y = 15$
7. $9x - y = 45$
8. $6 - 3y = -6$
9. $4x - 5 = 7 + 4y$
10. $16x + 9 = 9y - 2x$
11. $2 + \frac{1}{6}y = 3x + 4$
12. $11 - \frac{1}{2}y = 3 + 6x$

In Exercises 13–22, solve the literal equation for x. *(See Example 2.)*

13. $y = 4x + 8x$
14. $m = 10x - x$
15. $a = 2x + 6xz$
16. $y = 3bx - 7x$
17. $y = 4x + rx + 6$
18. $z = 8 + 6x - px$
19. $sx + tx = r$
20. $a = bx + cx + d$
21. $12 - 5x - 4kx = y$
22. $x - 9 + 2wx = y$

23. **MODELING WITH MATHEMATICS** The total cost C (in dollars) to participate in a ski club is given by the literal equation $C = 85x + 60$, where x is the number of ski trips you take.

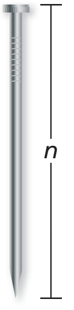
    a. Solve the equation for x.

    b. How many ski trips do you take if you spend a total of \$315? \$485?

24. **MODELING WITH MATHEMATICS** The penny size of a nail indicates the length of the nail. The penny size d is given by the literal equation $d = 4n - 2$, where n is the length (in inches) of the nail.

    a. Solve the equation for n.

    b. Use the equation from part (a) to find the lengths of nails with the following penny sizes: 3, 6, and 10.



ERROR ANALYSIS **In Exercises 25 and 26, describe and correct the error in solving the equation for x.**

25.
$$12 - 2x = -2(y - x)$$
$$-2x = -2(y - x) - 12$$
$$x = (y - x) + 6$$

26.
$$10 = ax - 3b$$
$$10 = x(a - 3b)$$
$$\frac{10}{a - 3b} = x$$

In Exercises 27–30, solve the formula for the indicated variable. *(See Examples 3 and 5.)*

27. Profit: $P = R - C$; Solve for C.
28. Surface area of a cylinder: $S = 2\pi r^2 + 2\pi rh$; Solve for h.
29. Area of a trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$; Solve for b_2.
30. Average acceleration of an object: $a = \frac{v_1 - v_0}{t}$; Solve for v_1.

31. REWRITING A FORMULA A common statistic used in professional football is the quarterback rating. This rating is made up of four major factors. One factor is the completion rating given by the formula

$$R = 5\left(\frac{C}{A} - 0.3\right)$$

where C is the number of completed passes and A is the number of attempted passes. Solve the formula for C.

32. REWRITING A FORMULA Newton's law of gravitation is given by the formula

$$F = G\left(\frac{m_1 m_2}{d^2}\right)$$

where F is the force between two objects of masses m_1 and m_2, G is the gravitational constant, and d is the distance between the two objects. Solve the formula for m_1.

33. MODELING WITH MATHEMATICS The sale price S (in dollars) of an item is given by the formula $S = L - rL$, where L is the list price (in dollars) and r is the discount rate (in decimal form). *(See Examples 4 and 6.)*

a. Solve the formula for r.

b. The list price of the shirt is \$30. What is the discount rate?

34. MODELING WITH MATHEMATICS The density d of a substance is given by the formula $d = \frac{m}{V}$, where m is its mass and V is its volume.

Pyrite

Density: $5.01 g/cm^3$ Volume: $1.2 cm^3$

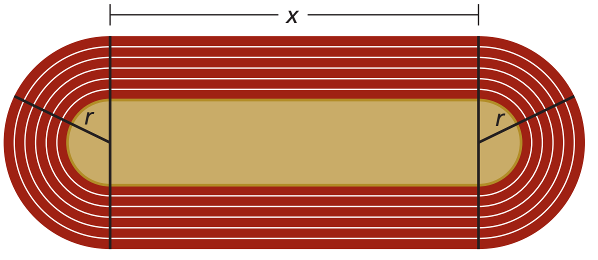
a. Solve the formula for m.

b. Find the mass of the pyrite sample.

35. PROBLEM SOLVING You deposit \$2000 in an account that earns simple interest at an annual rate of 4%. How long must you leave the money in the account to earn \$500 in interest? *(See Example 7.)*

36. PROBLEM SOLVING A flight averages 460 miles per hour. The return flight averages 500 miles per hour due to a tailwind. The total flying time is 4.8 hours. How long is each flight? Explain. *(See Example 8.)*

37. USING STRUCTURE An athletic facility is building an indoor track. The track is composed of a rectangle and two semicircles, as shown.

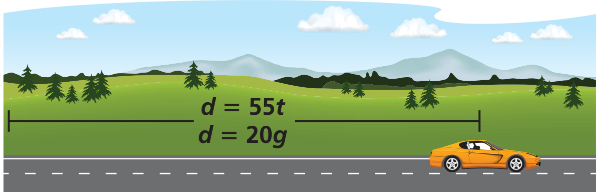


a. Write a formula for the perimeter of the indoor track.

b. Solve the formula for x.

c. The perimeter of the track is 660 feet, and r is 50 feet. Find x. Round your answer to the nearest foot.

38. MODELING WITH MATHEMATICS The distance d (in miles) you travel in a car is given by the two equations shown, where t is the time (in hours) and g is the number of gallons of gasoline the car uses.

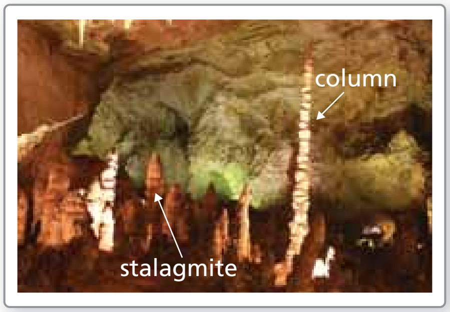


a. Write an equation that relates g and t.

b. Solve the equation for g.

c. You travel for 6 hours. How many gallons of gasoline does the car use? How far do you travel? Explain.

39. MODELING WITH MATHEMATICS One type of stone formation found in Carlsbad Caverns in New Mexico is called a column. This cylindrical stone formation connects to the ceiling and the floor of a cave.

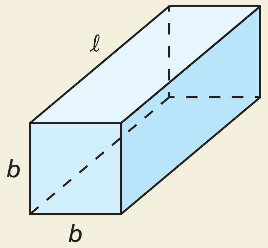


a. Rewrite the formula for the circumference of a circle, so that you can easily calculate the radius of a column given its circumference.

b. What is the radius (to the nearest tenth of a foot) of a column that has a circumference of 7 feet? 8 feet? 9 feet?

c. Explain how you can find the area of a cross section of a column when you know its circumference.

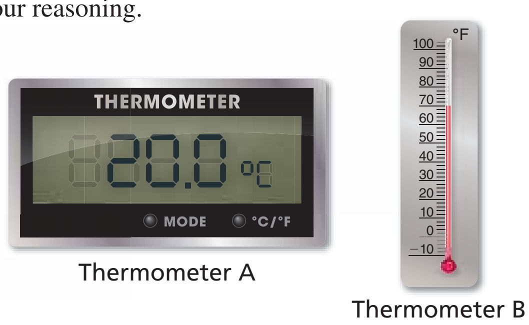
40. HOW DO YOU SEE IT? The rectangular prism shown has bases with equal side lengths.



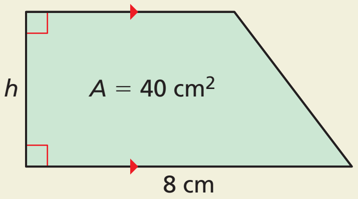
a. Use the figure to write a formula for the surface area S of the rectangular prism.

b. Your teacher asks you to rewrite the formula by solving for one of the side lengths, b or ℓ. Which side length would you choose? Explain your reasoning.

41. MAKING AN ARGUMENT Your friend claims that Thermometer A displays a greater temperature than Thermometer B. Is your friend correct? Explain your reasoning.

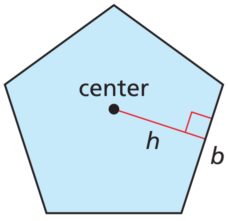


42. THOUGHT PROVOKING Give a possible value for h. Justify your answer. Draw and label the figure using your chosen value of h.

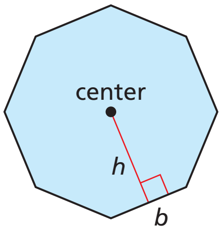


MATHEMATICAL CONNECTIONS In Exercises 43 and 44, write a formula for the area of the regular polygon. Solve the formula for the height h.

43.



44.



REASONING In Exercises 45 and 46, solve the literal equation for a.

45. $x = \dfrac{a + b + c}{ab}$

46. $y = x\left(\dfrac{ab}{a - b}\right)$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the expression. *(Skills Review Handbook)*

47. $15 - 5 + 5^2$

48. $18 \cdot 2 - 4^2 \div 8$

49. $3^3 + 12 \div 3 \cdot 5$

50. $2^5(5 - 6) + 9 \div 3$

Solve the equation. *(Section 1.2)*

51. $-15 = 6x + 3(4 - 3x)$

52. $-2(4 + m) + 6(m - 3) = 14$

1.3–1.4 What Did You Learn?

Core Vocabulary

identity, *p. 23*
literal equation, *p. 28*
formula, *p. 29*

Core Concepts

Section 1.3

Solving Equations with Variables on Both Sides, *p. 22*
Special Solutions of Linear Equations, *p. 23*

Section 1.4

Rewriting Literal Equations, *p. 28*
Common Formulas, *p. 30*

Mathematical Thinking

1. What definition did you use in your reasoning in Exercises 35 and 36 on page 26?

2. What entry points did you use to answer Exercises 43 and 44 on page 34?

Performance Task

Magic of Mathematics

Have you ever watched a magician perform a number trick? You can use algebra to explain how these types of tricks work.

To explore the answer to this question and more, go to ***BigIdeasMath.com***.

1 Chapter Review

1.1 Solving Simple Equations *(pp. 3–10)*

a. Solve $x - 5 = -9$. Justify each step.

	$x - 5 = -9$	Write the equation.
Addition Property of Equality →	$\underline{+5} \quad \underline{+5}$	Add 5 to each side.
	$x = -4$	Simplify.

▶ The solution is $x = -4$.

b. Solve $4x = 12$. Justify each step.

	$4x = 12$	Write the equation.
Division Property of Equality →	$\frac{4x}{4} = \frac{12}{4}$	Divide each side by 4.
	$x = 3$	Simplify.

▶ The solution is $x = 3$.

c. The *boiling point* of a liquid is the temperature at which the liquid becomes a gas. The boiling point of mercury, 357°C, is about $\frac{41}{200}$ of the boiling point of lead. Write and solve an equation to find the boiling point of lead.

Let x be the boiling point of lead.

	$\frac{41}{200}x = 357$	Write the equation.
Multiplication Property of Equality →	$\frac{200}{41} \cdot \left(\frac{41}{200}x\right) = \frac{200}{41} \cdot 357$	Multiply each side by $\frac{200}{41}$.
	$x \approx 1741$	Simplify.

▶ The boiling point of lead is about 1741°C.

Solve the equation. Justify each step. Check your solution.

1. $z + 3 = -6$ **2.** $2.6 = -0.2t$ **3.** $-\frac{n}{5} = -2$

1.2 Solving Multi-Step Equations *(pp. 11–18)*

a. Solve $-4p - 9 = 3$.

$-4p - 9 = 3$	Write the equation.
$-4p = 12$	Add 9 to each side.
$p = -3$	Divide each side by -4.

▶ The solution is $p = -3$.

b. Solve $-6x + 23 + 2x = 15$.

$-6x + 23 + 2x = 15$ Write the equation.

$-4x + 23 = 15$ Combine like terms.

$-4x = -8$ Subtract 23 from each side.

$x = 2$ Divide each side by -4.

▶ The solution is $x = 2$.

Solve the equation. Check your solution.

4. $3y + 11 = -16$

5. $6 = 1 - b$

6. $n + 5n + 7 = 43$

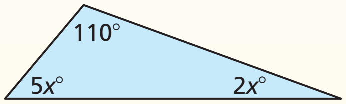
7. $-4(2z + 6) - 12 = 4$

8. $\frac{3}{2}(x - 2) - 5 = 19$

9. $6 = \frac{1}{5}w + \frac{7}{5}w - 4$

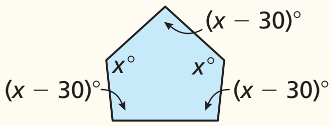
Find the value of x. Then find the angle measures of the polygon.

10.



Sum of angle measures: 180°

11.



Sum of angle measures: 540°

12. Use the table to write and solve an equation to find the number of points p you need to score in the fourth game so that the mean number of points scored per game is 20.

Game	1	2	3	4
Points	25	15	18	p

1.3 Solving Equations with Variables on Both Sides (pp. 21–26)

Solve $2(y - 4) = -4(y + 8)$.

$2(y - 4) = -4(y + 8)$ Write the equation.

$2y - 8 = -4y - 32$ Distributive Property

$6y - 8 = -32$ Add $4y$ to each side.

$6y = -24$ Add 8 to each side.

$y = -4$ Divide each side by 6.

▶ The solution is $y = -4$.

Solve the equation.

13. $3n - 3 = 4n + 1$

14. $5(1 + x) = 5x + 5$

15. $3(n + 4) = \frac{1}{2}(6n + 4)$

16. You are biking at a speed of 18 miles per hour. You are 3 miles behind your friend who is biking at a speed of 12 miles per hour. Write and solve an equation to find the amount of time it takes for you to catch up to your friend.

1.4 Rewriting Equations and Formulas (pp. 27–34)

a. The slope-intercept form of a linear equation is $y = mx + b$. Solve the equation for m.

$$y = mx + b$$ Write the equation.

$$y - b = mx + b - b$$ Subtract b from each side.

$$y - b = mx$$ Simplify.

$$\frac{y-b}{x} = \frac{mx}{x}$$ Divide each side by x.

$$\frac{y-b}{x} = m$$ Simplify.

▶ When you solve the equation for m, you obtain $m = \dfrac{y-b}{x}$.

b. The formula for the surface area S of a cylinder is $S = 2\pi r^2 + 2\pi rh$. Solve the formula for the height h.

$$S = 2\pi r^2 + 2\pi rh$$ Write the equation.

$$-2\pi r^2 \qquad -2\pi r^2$$ Subtract $2\pi r^2$ from each side.

$$S - 2\pi r^2 = 2\pi rh$$ Simplify.

$$\frac{S - 2\pi r^2}{2\pi r} = \frac{2\pi rh}{2\pi r}$$ Divide each side by $2\pi r$.

$$\frac{S - 2\pi r^2}{2\pi r} = h$$ Simplify.

▶ When you solve the formula for h, you obtain $h = \dfrac{S - 2\pi r^2}{2\pi r}$.

Solve the literal equation for y.

17. $2x - 4y = 20$

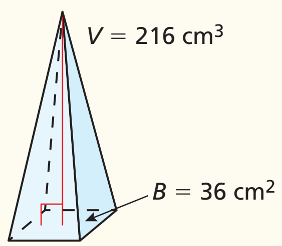
18. $8x - 3 = 5 + 4y$

19. $a = 9y + 3yx$

20. The volume V of a pyramid is given by the formula $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height.

a. Solve the formula for h.

b. Find the height h of the pyramid.



21. The formula $F = \frac{9}{5}(K - 273.15) + 32$ converts a temperature from kelvin K to degrees Fahrenheit F.

a. Solve the formula for K.

b. Convert 180°F to kelvin K. Round your answer to the nearest hundredth.

1 Chapter Test

Solve the equation. Justify each step. Check your solution.

1. $x - 7 = 15$

2. $\frac{2}{3}x + 5 = 3$

3. $11x + 1 = -1 + x$

Solve the equation.

4. $-2 + 5x - 7 = 3x - 9 + 2x$

5. $3(x + 4) - 1 = -7$

6. $\frac{1}{3}(6x + 12) - 2(x - 7) = 19$

Describe the values of *c* for which the equation has no solution. Explain your reasoning.

7. $3x - 5 = 3x - c$

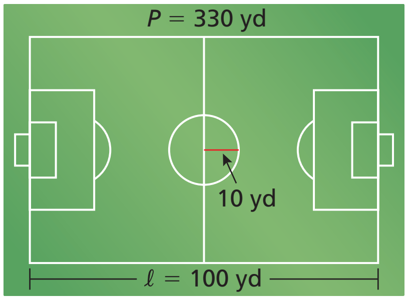
8. $4x + 1 = 2x + c$

9. The perimeter P (in yards) of a soccer field is represented by the formula $P = 2\ell + 2w$, where ℓ is the length (in yards) and w is the width (in yards).

a. Solve the formula for w.

b. Find the width of the field.

c. About what percent of the field is inside the circle?



10. Your car needs new brakes. You call a dealership and a local mechanic for prices.

	Cost of parts	Labor cost per hour
Dealership	\$24	\$99
Local mechanic	\$45	\$89

a. After how many hours are the total costs the same at both places? Justify your answer.

b. When do the repairs cost less at the dealership? at the local mechanic? Explain.

11. You want to paint a piece of pottery at an art studio. The total cost is the cost of the piece plus an hourly studio fee. There are two studios to choose from. *(Section 1.3)*

a. After how many hours of painting are the total costs the same at both studios? Justify your answer.

b. Studio B increases the hourly studio fee by \$2. How does this affect your answer in part (a)? Explain.

12. Your friend was solving the equation shown and was confused by the result "$-8 = -8$." Explain what this result means.

$$4(y - 2) - 2y = 6y - 8 - 4y$$
$$4y - 8 - 2y = 6y - 8 - 4y$$
$$2y - 8 = 2y - 8$$
$$-8 = -8$$

1 Standards Assessment

1. You are ordering concert tickets. You pay a processing fee of \$5, plus \$18 per ticket. How many tickets can you buy for \$100? *(TEKS A.5.A)*

Ⓐ 3 Ⓑ 4

Ⓒ 5 Ⓓ 6

2. Which equation is equivalent to $2(x + 1) + 1 = y$? *(TEKS A.10.D)*

Ⓕ $x + 7 = y + 4$ Ⓖ $2x - 3 = y - 3$

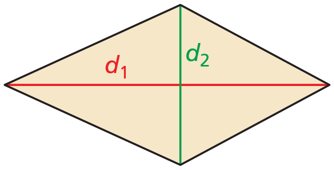
Ⓗ $3x + 3 = y + x$ Ⓙ $4x + 3 = 2y$

3. You are renting a canoe. It costs \$35 to rent the canoe, plus \$6 per hour. You have \$60. How long can you rent the canoe? Round to the nearest hour. *(TEKS A.5.A)*

Ⓐ 1 h Ⓑ 3 h

Ⓒ 4 h Ⓓ 5 h

4. The area A of a rhombus is given by the formula $A = \frac{1}{2}d_1d_2$, where d_1 and d_2 are the lengths of the diagonals. Which equation could you use to find the value of d_1 for different values of d_2 and A? *(TEKS A.12.E)*



Ⓕ $d_1 = A\left(\frac{d_2}{2}\right)$ Ⓖ $d_1 = \frac{2A}{d_2}$

Ⓗ $d_1 = \frac{A}{2d_2}$ Ⓙ $d_1 = A - d_2 - \frac{1}{2}$

5. Your friend's age is 5 years less than half her mother's age. Your friend is 15 years old. How old is her mother? *(TEKS A.5.A)*

Ⓐ 35 yr Ⓑ 40 yr

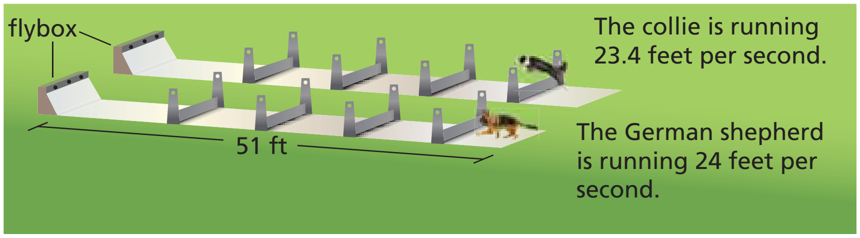
Ⓒ 45 yr Ⓓ 50 yr

6. In biology, the surface-area-to-volume quotient Q of a single spherical cell is given by the formal $Q = \frac{3}{r}$, where r is the radius of the cell. A biologist needs to calculate the diameter d of a cell in terms of Q. Because $d = 2r$, which formula could the biologist use to find d? *(TEKS A.12.E)*

Ⓕ $d = \frac{6}{r}$ Ⓖ $d = 2\left(\frac{3}{Q}\right)$

Ⓗ $d = 2Q$ Ⓙ $d = \frac{3}{2r}$

7. *Flyball* is a relay race for dogs. In each of the four legs of the relay, a dog jumps over hurdles, retrieves a ball from a flybox, and runs back over the hurdles. The collie starts the course 0.3 second before the German shepherd. How many seconds does it take for the German shepherd to catch up with the collie? *(TEKS A.5.A)*



Ⓐ 0.3 sec Ⓑ 1.8 sec
Ⓒ 11.7 sec Ⓓ 13.3 sec

8. A ski resort offers a super-saver pass for \$90 that allows you to buy lift tickets at half price. If you buy the pass, the cost (in dollars) of buying t tickets is $90 + 22.5t$. Otherwise, the cost (in dollars) is $45t$. How many lift tickets would a skier have to buy for the two costs to be equal? *(TEKS A.5.A)*

Ⓕ 2 Ⓖ 4
Ⓗ 6 Ⓙ 8

9. **GRIDDED ANSWER** What is the value of x in the equation $-7x = -\frac{3}{4}x - 5$? *(TEKS A.5.A)*

10. A music store offers a finance plan where you make a \$50 down payment on a guitar and pay the remaining balance in 6 equal monthly payments. You have \$50 and can afford to pay between \$60 and \$90 per month for a guitar. What is a reasonable price that you can afford to pay for a guitar? *(TEKS A.5.A)*

Ⓐ \$542 Ⓑ \$591
Ⓒ \$645 Ⓓ \$718

11. What is the solution of $7v - (6 - 2v) = 12$? *(TEKS A.5.A)*

Ⓕ $v = -3.6$ Ⓖ $v = -2$
Ⓗ $v = 2$ Ⓙ $v = 3.6$

12. An investor puts x dollars in Fund A and \$2000 in Fund B. The total amount invested in the two funds is $4x - 1000$ dollars. How much money did the investor put in Fund A? *(TEKS A.5.A)*

Ⓐ \$333 Ⓑ \$1000
Ⓒ \$3000 Ⓓ \$3333

2 Solving Linear Inequalities

Withdraw Money *(p. 71)*

Mountain Plant Life *(p. 77)*

Digital Camera *(p. 62)*

Microwave Electricity *(p. 56)*

Natural Arch *(p. 51)*

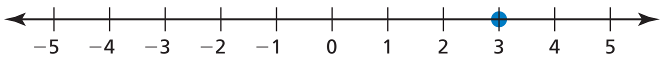
Mathematical Thinking: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

Maintaining Mathematical Proficiency

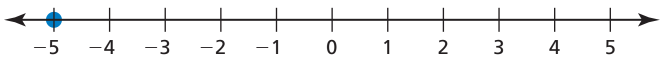
Graphing Numbers on a Number Line (6.2.C)

Example 1 Graph each number.

a. 3

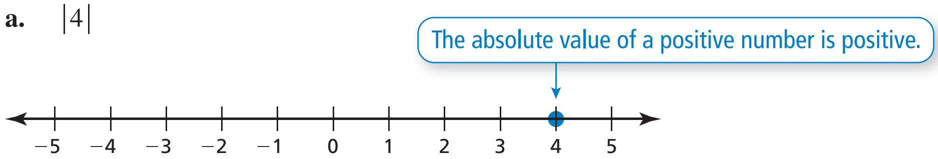


b. -5

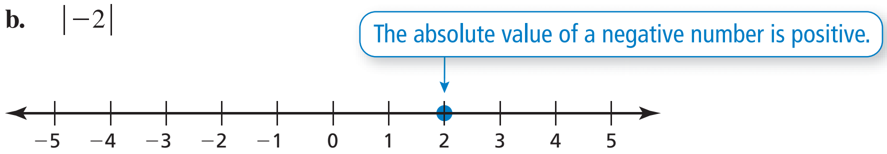


Example 2 Graph each number.

a. $|4|$



b. $|-2|$



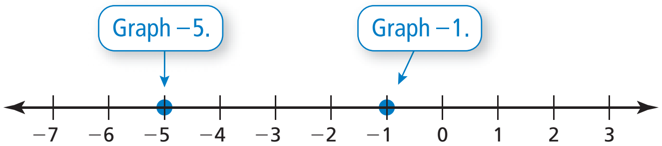
Graph the number.

1. 6 **2.** $|2|$ **3.** $|-1|$

4. $2 + |-2|$ **5.** $1 - |-4|$ **6.** $-5 + |3|$

Comparing Real Numbers (6.2.C)

Example 3 Complete the statement -1 ☐ -5 with $<$, $>$, or $=$.



▶ -1 is to the right of -5. So, $-1 > -5$.

Complete the statement with <, >, or =.

7. 2 ☐ 9 **8.** -6 ☐ 5 **9.** -12 ☐ -4

10. -7 ☐ -13 **11.** $|-8|$ ☐ $|8|$ **12.** -10 ☐ $|-18|$

13. ABSTRACT REASONING A number a is to the left of a number b on the number line. How do the numbers $-a$ and $-b$ compare?

Mathematical Thinking

Mathematically proficient students select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems. **(A.1.C)**

Using a Graphing Calculator

Core Concept

Solving an Inequality in One Variable

You can use a graphing calculator to solve an inequality.

1. Enter the inequality into a graphing calculator.

2. Graph the inequality.

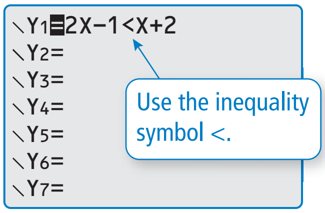
3. Use the graph to write the solution.

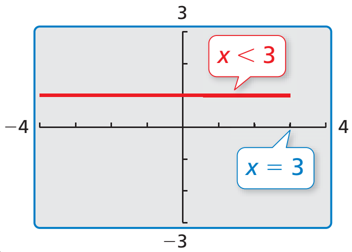
EXAMPLE 1 **Using a Graphing Calculator**

Use a graphing calculator to solve (a) $2x - 1 < x + 2$ and (b) $2x - 1 \leq x + 2$.

SOLUTION

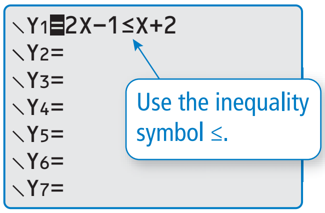
a. Enter the inequality $2x - 1 < x + 2$ into a graphing calculator. Press *graph*.

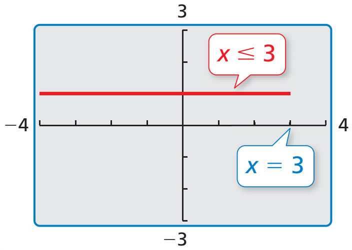




▶ The solution of the inequality is $x < 3$.

b. Enter the inequality $2x - 1 \leq x + 2$ into a graphing calculator. Press *graph*.





▶ The solution of the inequality is $x \leq 3$.

Notice that the graphing calculator does not distinguish between the solutions $x < 3$ and $x \leq 3$. You must distinguish between these yourself, based on the inequality symbol used in the original inequality.

Monitoring Progress

Use a graphing calculator to solve the inequality.

1. $2x + 3 < x - 1$ **2.** $-x - 1 > -2x + 2$ **3.** $\frac{1}{2}x + 1 \leq \frac{3}{2}x + 3$

2.1 Writing and Graphing Inequalities

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
Preparing for A.5.B

Essential Question

How can you use an inequality to describe a real-life statement?

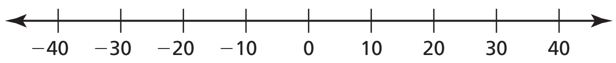
EXPLORATION 1 Writing and Graphing Inequalities

Work with a partner. Write an inequality for each statement. Then sketch the graph of the numbers that make each inequality true.

a. Statement The temperature t in Sweden is at least -10°C.

Inequality

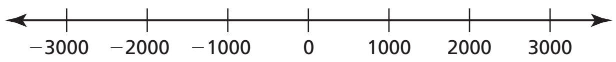
Graph



b. Statement The elevation e of Alabama is at most 2407 feet.

Inequality

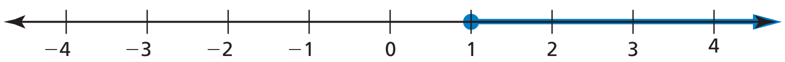
Graph



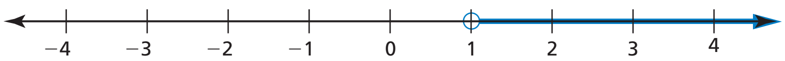
EXPLORATION 2 Writing Inequalities

Work with a partner. Write an inequality for each graph. Then, in words, describe all the values of x that make each inequality true.

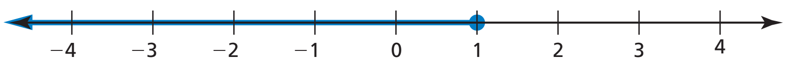
a.



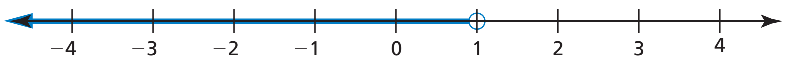
b.



c.



d.



USING PRECISE MATHEMATICAL LANGUAGE

To be proficient in math, you need to communicate precisely. You also need to state the meanings of the symbols you use.

Communicate Your Answer

3. How can you use an inequality to describe a real-life statement?

4. Write a real-life statement that involves each inequality.

a. $x < 3.5$

b. $x \leq 6$

c. $x > -2$

d. $x \geq 10$

2.1 Lesson

Core Vocabulary

inequality, *p. 46*
solution of an inequality, *p. 47*
solution set, *p. 47*
graph of an inequality, *p. 48*

Previous
expression

What You Will Learn

- Write linear inequalities.
- Sketch the graphs of linear inequalities.
- Write linear inequalities from graphs.

Writing Linear Inequalities

An **inequality** is a mathematical sentence that compares expressions. An inequality contains the symbol $<$, $>$, $\leq$, or $\geq$. To write an inequality, look for the following phrases to determine what inequality symbol to use.

Inequality Symbols				
Symbol	$<$	$>$	$\leq$	$\geq$
Key Phrases	• is less than • is fewer than	• is greater than • is more than	• is less than or equal to • is at most • is no more than	• is greater than or equal to • is at least • is no less than

EXAMPLE 1 Writing Inequalities

Write each sentence as an inequality.

a. A number w minus 3.5 is less than or equal to -2.

b. Three is less than a number n plus 5.

c. Zero is greater than or equal to twice a number x plus 1.

SOLUTION

a. A $\underbrace{\text{number } w \text{ minus } 3.5}_{w - 3.5}$ $\underbrace{\text{is less than or equal to}}_{\leq}$ $\underbrace{-2}_{-2}$.

- An inequality is $w - 3.5 \leq -2$.

b. $\underbrace{\text{Three}}_{3}$ $\underbrace{\text{is less than}}_{<}$ $\underbrace{\text{a number } n \text{ plus } 5}_{n + 5}$.

- An inequality is $3 < n + 5$.

READING

The inequality $3 < n + 5$ is the same as $n + 5 > 3$.

c. $\underbrace{\text{Zero}}_{0}$ $\underbrace{\text{is greater than or equal to}}_{\geq}$ $\underbrace{\text{twice a number } x \text{ plus } 1}_{2x + 1}$.

- An inequality is $0 \geq 2x + 1$.

Monitoring Progress

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Write the sentence as an inequality.

1. A number b is fewer than 30.4.

2. $-\frac{7}{10}$ is at least twice a number k minus 4.

Sketching the Graphs of Linear Inequalities

A **solution of an inequality** is a value that makes the inequality true. An inequality can have more than one solution. The set of all solutions of an inequality is called the **solution set**.

Value of x	$x + 5 \geq -2$	Is the inequality true?
-6	$-6 + 5 \overset{?}{\geq} -2$ $-1 \geq -2$ ✓	yes
-7	$-7 + 5 \overset{?}{\geq} -2$ $-2 \geq -2$ ✓	yes
-8	$-8 + 5 \overset{?}{\geq} -2$ $-3 \not\geq -2$ ✗	no

Recall that a diagonal line through an inequality symbol means the inequality is *not* true. For instance, the symbol $\not\geq$ means "is not greater than or equal to."

EXAMPLE 2 Checking Solutions

Tell whether -4 is a solution of each inequality.

a. $x + 8 < -3$

b. $-4.5x > -21$

SOLUTION

a.

$x + 8 < -3$ — Write the inequality.

$-4 + 8 \overset{?}{<} -3$ — Substitute -4 for x.

$4 \not< -3$ ✗ — Simplify.

4 is *not* less than -3.

▶ So, -4 is *not* a solution of the inequality.

b.

$-4.5x > -21$ — Write the inequality.

$-4.5(-4) \overset{?}{>} -21$ — Substitute -4 for x.

$18 > -21$ ✓ — Simplify.

18 is greater than -21.

▶ So, -4 is a solution of the inequality.

Monitoring Progress

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Tell whether -6 is a solution of the inequality.

3. $c + 4 < -1$

4. $10 \leq 3 - m$

5. $21 \div x \geq -3.5$

6. $4x - 25 > -2$

The **graph of an inequality** shows the solution set of the inequality on a number line. An open circle, ○, is used when a number is *not* a solution. A closed circle, ●, is used when a number is a solution. An arrow to the left or right shows that the graph continues in that direction.

EXAMPLE 3 Graphing Inequalities

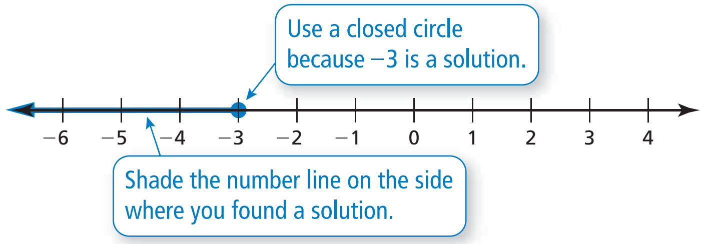
Graph each inequality.

a. $y \leq -3$ **b.** $2 < x$ **c.** $x > 0$

SOLUTION

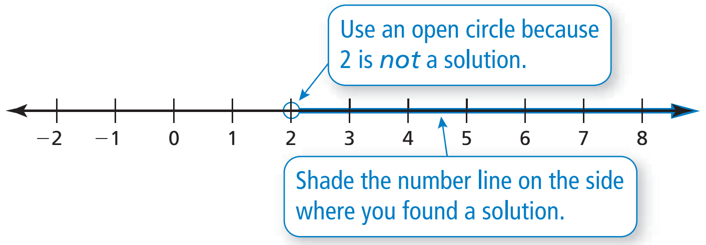
a. Test a number to the left of -3. $y = -4$ is a solution.

Test a number to the right of -3. $y = 0$ is *not* a solution.

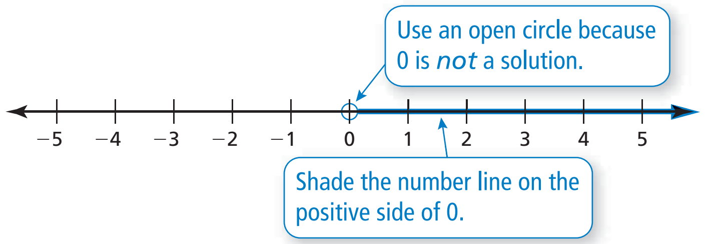


b. Test a number to the left of 2. $x = 0$ is *not* a solution.

Test a number to the right of 2. $x = 4$ is a solution.



c. Just by looking at the inequality, you can see that it represents the set of all positive numbers.



ANOTHER WAY

Another way to represent the solutions of an inequality is to use *set-builder notation*. In Example 3b, the solutions can be written as $\{x \mid x > 2\}$, which is read as "the set of all numbers x such that x is greater than 2."

Monitoring Progress

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Graph the inequality.

7. $b > -8$

8. $1.4 \geq g$

9. $r < \frac{1}{2}$

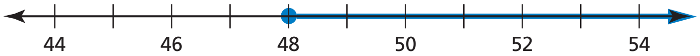
10. $v \geq \sqrt{36}$

Writing Linear Inequalities from Graphs

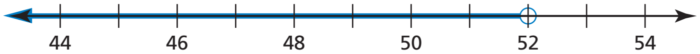
EXAMPLE 4 Writing Inequalities from Graphs

The graphs show the height restrictions h (in inches) for two rides at an amusement park. Write an inequality that represents the height restriction of each ride.

Ride A

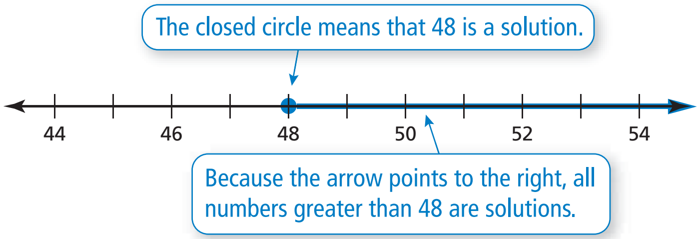


Ride B

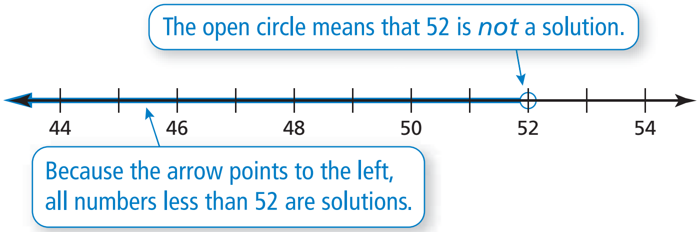


SOLUTION

Ride A



Ride B



So, $h \geq 48$ represents the height restriction for Ride A, and $h < 52$ represents the height restriction for Ride B.

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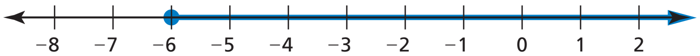
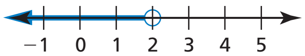
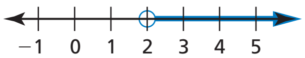
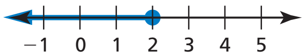
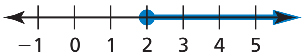
11. Write an inequality that represents the graph.



Concept Summary

Representing Linear Inequalities

Words	Algebra	Graph
x is less than 2	$x < 2$	
x is greater than 2	$x > 2$	
x is less than or equal to 2	$x \leq 2$	
x is greater than or equal to 2	$x \geq 2$	

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A mathematical sentence using the symbols <, >, ≤, or ≥ is called a(n)________.
2. **VOCABULARY** Is 5 in the solution set of $x + 3 > 8$? Explain.
3. **ATTENDING TO PRECISION** Describe how to graph an inequality.
4. **DIFFERENT WORDS, SAME QUESTION** Which is different? Write "both" inequalities.

   w is greater than or equal to -7.

   w is no less than -7.

   w is no more than -7.

   w is at least -7.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, write the sentence as an inequality. *(See Example 1.)*

5. A number x is greater than 3.
6. A number n plus 7 is less than or equal to 9.
7. Fifteen is no more than a number t divided by 5.
8. Three times a number w is less than 18.
9. One-half of a number y is more than 22.
10. Three is less than the sum of a number s and 4.
11. Thirteen is at least the difference of a number v and 1.
12. Four is no less than the quotient of a number x and 2.1.

13. **MODELING WITH MATHEMATICS** On a fishing trip, you catch two fish. The weight of the first fish is shown. The second fish weighs at least 0.5 pound more than the first fish. Write an inequality that represents the possible weights of the second fish.

14. **MODELING WITH MATHEMATICS** There are 430 people in a wave pool. Write an inequality that represents how many more people can enter the pool.

In Exercises 15–24, tell whether the value is a solution of the inequality. *(See Example 2.)*

15. $r + 4 > 8;\ r = 2$
16. $5 - x < 8;\ x = -3$
17. $3s \le 19;\ s = -6$
18. $17 \ge 2y;\ y = 7$
19. $-1 > -\frac{x}{2};\ x = 3$
20. $\frac{4}{z} \ge 3;\ z = 2$
21. $14 \ge -2n + 4;\ n = -5$
22. $-5 \div (2s) < -1;\ s = 10$
23. $20 \le \frac{10}{2z} + 20;\ z = 5$
24. $\frac{3m}{6} - 2 > 3;\ m = 8$

25. **MODELING WITH MATHEMATICS** The tallest person who ever lived was approximately 8 feet 11 inches tall.
    a. Write an inequality that represents the heights of every other person who has ever lived.
    b. Is 9 feet a solution of the inequality? Explain.

26. **DRAWING CONCLUSIONS** The winner of a weight-lifting competition bench-pressed 400 pounds. The other competitors all bench-pressed at least 23 pounds less.

a. Write an inequality that represents the weights that the other competitors bench-pressed.

b. Was one of the other competitors able to bench-press 379 pounds? Explain.

ERROR ANALYSIS **In Exercises 27 and 28, describe and correct the error in determining whether 8 is in the solution set of the inequality.**

27. ✗

$$-y + 7 < -4$$
$$-8 + 7 \overset{?}{<} -4$$
$$-1 < -4$$

8 is in the solution set.

28. ✗

$$\frac{1}{2}x + 2 \le 6$$
$$\frac{1}{2}(8) + 2 \overset{?}{\le} 6$$
$$4 + 2 \overset{?}{\le} 6$$
$$6 \le 6$$

8 is *not* in the solution set.

In Exercises 29–36, graph the inequality. *(See Example 3.)*

29. $x \ge 2$

30. $z \le 5$

31. $-1 > t$

32. $-2 < w$

33. $v \le -4$

34. $s < 1$

35. $\frac{1}{4} < p$

36. $r \ge -|5|$

In Exercises 37–40, write and graph an inequality for the given solution set.

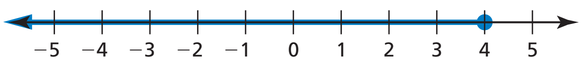
37. $\{x \mid x < 7\}$

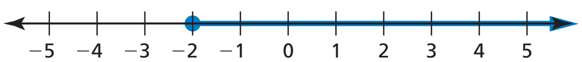
38. $\{n \mid n \ge -2\}$

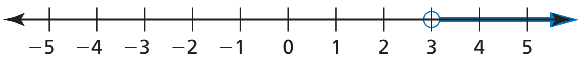
39. $\{z \mid 1.3 \le z\}$

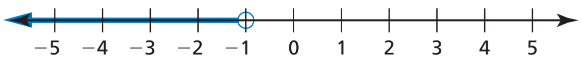
40. $\{w \mid 5.2 > w\}$

In Exercises 41–44, write an inequality that represents the graph. *(See Example 4.)*

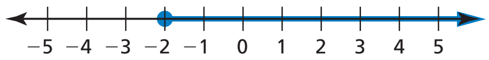
41.


42.


43.


44.


45. **ANALYZING RELATIONSHIPS** The water temperature of a swimming pool must be no less than 76°F. The temperature is currently 74°F. Which graph correctly shows how much the temperature needs to increase? Explain your reasoning.

Ⓐ


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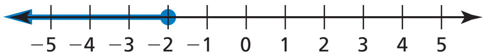

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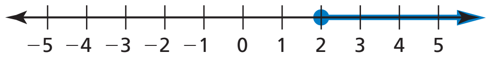

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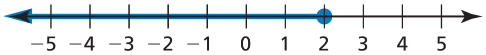
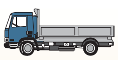
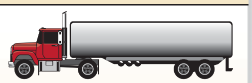


46. **MODELING WITH MATHEMATICS** According to a state law for vehicles traveling on state roads, the maximum total weight of a vehicle and its contents depends on the number of axles on the vehicle. For each type of vehicle, write and graph an inequality that represents the possible total weights w (in pounds) of the vehicle and its contents.

Maximum Total Weights

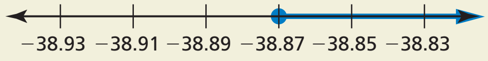
		
2 axles, 40,000 lb	3 axles, 60,000 lb	4 axles, 80,000 lb

47. **PROBLEM SOLVING** The Xianren Bridge is located in Guangxi Province, China. This arch is the world's longest natural arch, with a length of 400 feet. Write and graph an inequality that represents the lengths ℓ (in *inches*) of all other natural arches.

48. **THOUGHT PROVOKING** A student works no more than 25 hours each week at a part-time job. Write an inequality that represents how many hours the student can work each day.

49. **WRITING** Describe a real-life situation modeled by the inequality $23 + x \leq 31$.

50. **HOW DO YOU SEE IT?** The graph represents the known melting points of all metallic elements (in degrees Celsius).



a. Write an inequality represented by the graph.

b. Is it possible for a metallic element to have a melting point of -38.87°C? Explain.

51. **DRAWING CONCLUSIONS** A one-way ride on a subway costs \$0.90. A monthly pass costs \$24. Write an inequality that represents how many one-way rides you can buy before it is cheaper to buy the monthly pass. Is it cheaper to pay the one-way fare for 25 rides? Explain.

Subway Prices	
One-way ride	\$0.90
Monthly pass	\$24.00

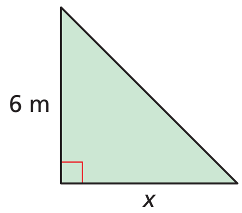
52. **MAKING AN ARGUMENT** The inequality $x \leq 1324$ represents the weights (in pounds) of all mako sharks ever caught using a rod and reel. Your friend says this means no one using a rod and reel has ever caught a mako shark that weighs 1324 pounds. Your cousin says this means someone using a rod and reel *has* caught a mako shark that weighs 1324 pounds. Who is correct? Explain your reasoning.

53. **CRITICAL THINKING** Describe a real-life situation that can be modeled by more than one inequality.

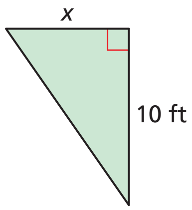
54. **MODELING WITH MATHEMATICS** In 1997, Superman's cape from the 1978 movie *Superman* was sold at an auction. The winning bid was \$17,000. Write and graph an inequality that represents the amounts all the losing bids.

MATHEMATICAL CONNECTIONS **In Exercises 55–58, write an inequality that represents the missing dimension *x*.**

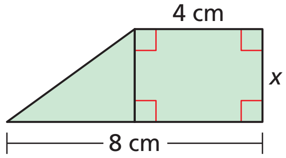
55. The area is less than 42 square meters.



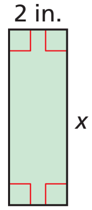
56. The area is greater than or equal to 8 square feet.



57. The area is less than 18 square centimeters.



58. The area is greater than 12 square inches.



59. **WRITING** A runner finishes a 200-meter dash in 35 seconds. Let r represent any speed (in meters per second) faster than the runner's speed.

a. Write an inequality that represents r. Then graph the inequality.

b. Every point on the graph represents a speed faster than the runner's speed. Do you think every point could represent the speed of a runner? Explain.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Section 1.1)*

60. $x + 2 = 3$

61. $y - 9 = 5$

62. $6 = 4 + y$

63. $-12 = y - 11$

Identify the property of equality that makes Equation 1 and 2 equivalent. *(Section 1.1)*

64.
Equation 1 $3x + 8 = x - 1$
Equation 2 $3x + 9 = x$

65.
Equation 1 $4y = 28$
Equation 2 $y = 7$

2.2 Solving Inequalities Using Addition or Subtraction

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.5.B

Essential Question

How can you use addition or subtraction to solve an inequality?

EXPLORATION 1 Quarterback Passing Efficiency

Work with a partner. The National Collegiate Athletic Association (NCAA) uses the following formula to rank the passing efficiencies P of quarterbacks.

$$P = \frac{8.4Y + 100C + 330T - 200N}{A}$$

Y = total length of all completed passes (in Yards)

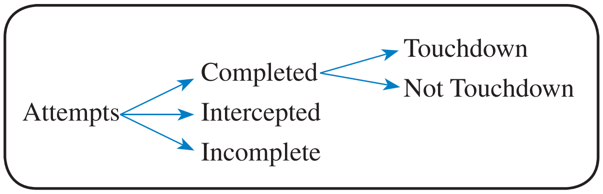
C = Completed passes

T = passes resulting in a Touchdown

N = iNtercepted passes

A = Attempted passes

M = incoMplete passes



APPLYING MATHEMATICS

To be proficient in math, you need to identify and analyze important relationships and then draw conclusions, using tools such as diagrams, flowcharts, and formulas.

Determine whether each inequality must be true. Explain your reasoning.

a. $T < C$ **b.** $C + N \leq A$ **c.** $N < A$ **d.** $A - C \geq M$

EXPLORATION 2 Finding Solutions of Inequalities

Work with a partner. Use the passing efficiency formula to create a passing record that makes each inequality true. Record your results in the table. Then describe the values of P that make each inequality true.

Attempts	Completions	Yards	Touchdowns	Interceptions

a. $P < 0$

b. $P + 100 \geq 250$

c. $P - 250 > -80$

Communicate Your Answer

3. How can you use addition or subtraction to solve an inequality?

4. Solve each inequality.

a. $x + 3 < 4$ **b.** $x - 3 \geq 5$

c. $4 > x - 2$ **d.** $-2 \leq x + 1$

2.2 Lesson

What You Will Learn

- Solve inequalities using addition.
- Solve inequalities using subtraction.
- Use inequalities to solve real-life problems.

Core Vocabulary

equivalent inequalities, *p. 54*

Previous

inequality

Solving Inequalities Using Addition

Just as you used the properties of equality to produce equivalent equations, you can use the properties of inequality to produce equivalent inequalities. **Equivalent inequalities** are inequalities that have the same solutions.

Core Concept

Addition Property of Inequality

Words Adding the same number to each side of an inequality produces an equivalent inequality.

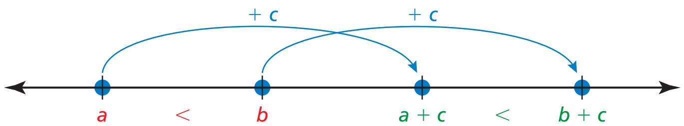
Numbers

$$\begin{array}{rcr} -3 & < & 2 \\ \underline{+\,4} & & \underline{+\,4} \\ 1 & < & 6 \end{array} \qquad \begin{array}{rcr} -3 & \geq & -10 \\ \underline{+\,3} & & \underline{+\,3} \\ 0 & \geq & -7 \end{array}$$

Algebra If $a > b$, then $a + c > b + c$. If $a \geq b$, then $a + c \geq b + c$.

If $a < b$, then $a + c < b + c$. If $a \leq b$, then $a + c \leq b + c$.

The diagram shows one way to visualize the Addition Property of Inequality when $c > 0$.



EXAMPLE 1 Solving an Inequality Using Addition

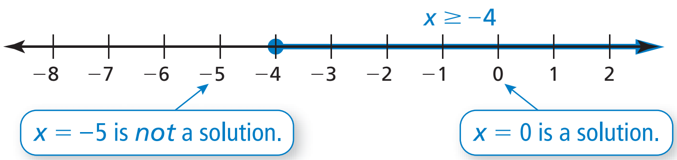
Solve $x - 6 \geq -10$. Graph the solution.

SOLUTION

$$\begin{array}{rcrl} x - 6 & \geq & -10 & \text{Write the inequality.} \\ \underline{+\,6} & & \underline{+\,6} & \text{Add 6 to each side.} \\ x & \geq & -4 & \text{Simplify.} \end{array}$$

Addition Property of Inequality

▶ The solution is $x \geq -4$.



REMEMBER

To check this solution, substitute a few numbers to the left and right of −4 into the original inequality.

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Solve the inequality. Graph the solution.

1. $b - 2 > -9$ **2.** $m - 3 \leq 5$ **3.** $\frac{1}{4} > y - \frac{1}{4}$

Solving Inequalities Using Subtraction

Core Concept

Subtraction Property of Inequality

Words Subtracting the same number from each side of an inequality produces an equivalent inequality.

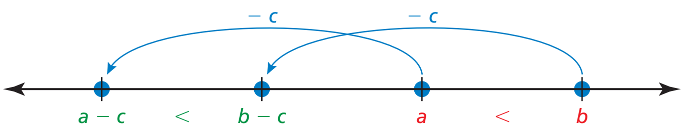
Numbers

$$\begin{array}{rcr} -3 & \le & 1 \\ \underline{-\ 5} & & \underline{-\ 5} \\ -8 & \le & -4 \end{array} \qquad \begin{array}{rcr} 7 & > & -20 \\ \underline{-\ 7} & & \underline{-\ 7} \\ 0 & > & -27 \end{array}$$

Algebra If $a > b$, then $a - c > b - c$. If $a \ge b$, then $a - c \ge b - c$.

If $a < b$, then $a - c < b - c$. If $a \le b$, then $a - c \le b - c$.

The diagram shows one way to visualize the Subtraction Property of Inequality when $c > 0$.



EXAMPLE 2 **Solving an Inequality Using Subtraction**

Solve each inequality. Graph the solution.

a. $y + 8 \le 5$

b. $-8 < 1.4 + m$

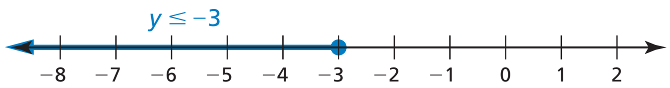
SOLUTION

a.

$y + 8 \le 5$	Write the inequality.
$\underline{-\ 8} \quad \underline{-\ 8}$	Subtract 8 from each side.
$y \le -3$	Simplify.

Subtraction Property of Inequality

▶ The solution is $y \le -3$.

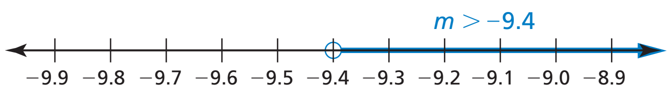


b.

$-8 < 1.4 + m$	Write the inequality.
$\underline{-\ 1.4} \quad \underline{-\ 1.4}$	Subtract 1.4 from each side.
$-9.4 < m$	Simplify.

Subtraction Property of Inequality

▶ The solution is $m > -9.4$.



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Solve the inequality. Graph the solution.

4. $k + 5 \le -3$

5. $\frac{5}{6} \le z + \frac{1}{6}$

6. $p + 0.7 > -2.3$

Solving Real-Life Problems

EXAMPLE 3 **Modeling with Mathematics**

A circuit overloads at 1800 watts of electricity. You plug a microwave oven that uses 1100 watts of electricity into the circuit.

a. Write and solve an inequality that represents how many watts you can add to the circuit without overloading the circuit.

b. In addition to the microwave oven, which of the following appliances can you plug into the circuit at the same time without overloading the circuit?

Appliance	Watts
Clock radio	50
Blender	300
Hot plate	1200
Toaster	800

SOLUTION

1. Understand the Problem You know that the microwave oven uses 1100 watts out of a possible 1800 watts. You are asked to write and solve an inequality that represents how many watts you can add without overloading the circuit. You also know the numbers of watts used by four other appliances. You are asked to identify the appliances you can plug in at the same time without overloading the circuit.

2. Make a Plan Use a verbal model to write an inequality. Then solve the inequality and identify other appliances that you can plug into the circuit at the same time without overloading the circuit.

3. Solve the Problem

Words Watts used by microwave oven + Additional watts < Overload wattage

Variable Let w be the additional watts you can add to the circuit.

Inequality $1100 \quad + \quad w \quad < \quad 1800$

$$1100 + w < 1800$$ Write the inequality.

Subtraction Property of Inequality

$$\underline{-\ 1100} \qquad \underline{-\ 1100}$$ Subtract 1100 from each side.

$$w < 700$$ Simplify.

▶ You can add up to 700 watts to the circuit, which means that you can also plug in the clock radio and the blender.

4. Look Back You can check that your answer is correct by adding the numbers of watts used by the microwave oven, clock radio, and blender.

$$1100 + 50 + 300 = 1450$$

The circuit will not overload because the total wattage is less than 1800 watts.

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7. The microwave oven uses only 1000 watts of electricity. Does this allow you to have both the microwave oven and the toaster plugged into the circuit at the same time? Explain your reasoning.

2.2 Exercises

Vocabulary and Core Concept Check

1. **VOCABULARY** Why is the inequality $x \le 6$ equivalent to the inequality $x - 5 \le 6 - 5$?

2. **WRITING** Compare solving equations using addition with solving inequalities using addition.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, tell which number you would add to or subtract from each side of the inequality to solve it.

3. $k + 11 < -3$
4. $v - 2 > 14$
5. $-1 \ge b - 9$
6. $-6 \le 17 + p$

In Exercises 7–20, solve the inequality. Graph the solution. *(See Examples 1 and 2.)*

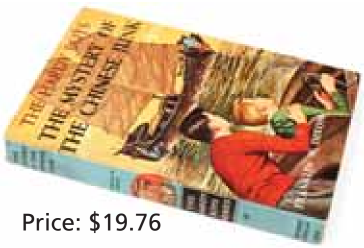
7. $x - 4 < -5$
8. $1 \le s - 8$
9. $6 \ge m - 1$
10. $c - 12 > -4$
11. $r + 4 < 5$
12. $-8 \le 8 + y$
13. $9 + w > 7$
14. $15 \ge q + 3$
15. $h - (-2) \ge 10$
16. $-6 > t - (-13)$
17. $j + 9 - 3 < 8$
18. $1 - 12 + y \ge -5$
19. $10 \ge 3p - 2p - 7$
20. $18 - 5z + 6z > 3 + 6$

In Exercises 21–24, write the sentence as an inequality. Then solve the inequality.

21. A number plus 8 is greater than 11.
22. A number minus 3 is at least -5.
23. The difference of a number and 9 is fewer than 4.
24. Six is less than or equal to the sum of a number and 15.

25. **MODELING WITH MATHEMATICS** You are riding a train. Your carry-on bag can weigh no more than 50 pounds. Your bag weighs 38 pounds. *(See Example 3.)*
    a. Write and solve an inequality that represents how much weight you can add to your bag.
    b. Can you add both a 9-pound laptop and a 5-pound pair of boots to your bag without going over the weight limit? Explain.

26. **MODELING WITH MATHEMATICS** You order the hardcover book shown from a website that offers free shipping on orders of \$25 or more. Write and solve an inequality that represents how much more you must spend to get free shipping.

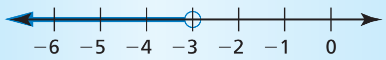


Price: \$19.76

ERROR ANALYSIS **In Exercises 27 and 28, describe and correct the error in solving the equation or graphing the solution.**

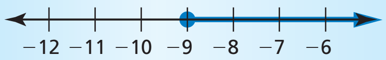
27. ✗

$$-17 < x - 14$$
$$-17 + 14 < x - 14 + 14$$
$$-3 < x$$



28. ✗

$$-10 + x \ge -9$$
$$-10 + 10 + x \ge -9$$
$$x \ge -9$$



29. **PROBLEM SOLVING** An NHL hockey player has 59 goals so far in a season. What are the possible numbers of additional goals the player can score to match or break the NHL record of 92 goals in a season?

30. MAKING AN ARGUMENT In an aerial ski competition, you perform two acrobatic ski jumps. The scores on the two jumps are then added together.

Ski jump	Competitor's score	Your score
1	117.1	119.5
2	119.8	

a. Describe the score that you must earn on your second jump to beat your competitor.

b. Your coach says that you will beat your competitor if you score 118.4 points. A teammate says that you only need 117.5 points. Who is correct? Explain.

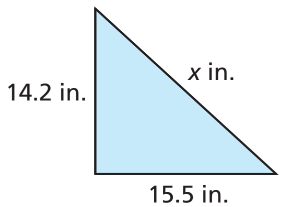
31. REASONING Which of the following inequalities are equivalent to the inequality $x - b < 3$, where b is a constant? Justify your answer.

(A) $x - b - 3 < 0$ (B) $0 > b - x + 3$

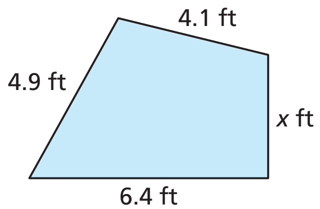
(C) $x < 3 - b$ (D) $-3 < b - x$

MATHEMATICAL CONNECTIONS **In Exercises 32 and 33, write and solve an inequality to find the possible values of x.**

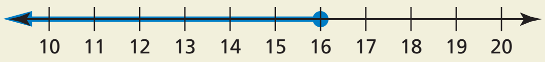
32. Perimeter $<$ 51.3 inches



33. Perimeter $\leq$ 18.7 feet

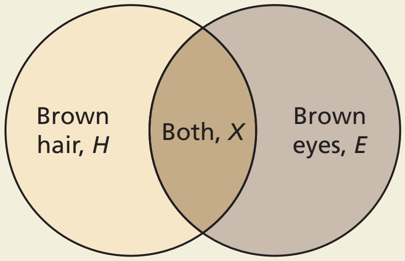


34. THOUGHT PROVOKING Write an inequality that has the solution shown in the graph. Describe a real-life situation that can be modeled by the inequality.



35. WRITING Is it possible to check all the numbers in the solution set of an inequality? When you solve the inequality $x - 11 \geq -3$, which numbers can you check to verify your solution? Explain your reasoning.

36. HOW DO YOU SEE IT? The diagram represents the numbers of students in a school with brown eyes, brown hair, or both.



Determine whether each inequality must be true. Explain your reasoning.

a. $H \geq E$ b. $H + 10 \geq E$

c. $H \geq X$ d. $H + 10 \geq X$

e. $H > X$ f. $H + 10 > X$

37. REASONING Write and graph an inequality that represents the numbers that are *not* solutions of each inequality.

a. $x + 8 < 14$

b. $x - 12 \geq 5.7$

38. PROBLEM SOLVING Use the inequalities $c - 3 \geq d$, $b + 4 < a + 1$, and $a - 2 \leq d - 7$ to order a, b, c, and d from least to greatest.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the product or quotient. *(Skills Review Handbook)*

39. $7 \cdot (-9)$ **40.** $-11 \cdot (-12)$ **41.** $-27 \div (-3)$ **42.** $20 \div (-5)$

Solve the equation. Check your solution. *(Section 1.1)*

43. $6x = 24$ **44.** $-3y = -18$ **45.** $\frac{s}{-8} = 13$ **46.** $\frac{n}{4} = -7.3$

2.3 Solving Inequalities Using Multiplication or Division

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.5.B

Essential Question

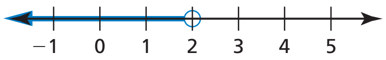
How can you use division to solve an inequality?

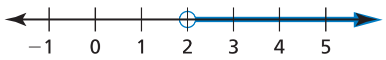
EXPLORATION 1 Writing a Rule

Work with a partner.

a. Copy and complete the table. Decide which graph represents the solution of the inequality $6 < 3x$. Write the solution of the inequality.

x	-1	0	1	2	3	4	5
$3x$	-3						
$6 \overset{?}{<} 3x$	No						





b. Use a table to solve each inequality. Then write a rule that describes how to use division to solve the inequalities.

i. $2x < 4$ **ii.** $3 \geq 3x$ **iii.** $2x < 8$ **iv.** $6 \geq 3x$

ANALYZING MATHEMATICAL RELATIONSHIPS

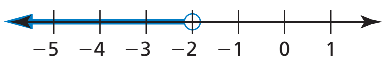
To be proficient in math, you need to investigate relationships, observe patterns, and use your observations to write general rules.

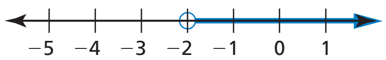
EXPLORATION 2 Writing a Rule

Work with a partner.

a. Copy and complete the table. Decide which graph represents the solution of the inequality $6 < -3x$. Write the solution of the inequality.

x	-5	-4	-3	-2	-1	0	1
$-3x$							
$6 \overset{?}{<} -3x$							





b. Use a table to solve each inequality. Then write a rule that describes how to use division to solve the inequalities.

i. $-2x < 4$ **ii.** $3 \geq -3x$ **iii.** $-2x < 8$ **iv.** $6 \geq -3x$

Communicate Your Answer

3. How can you use division to solve an inequality?

4. Use the rules you wrote in Explorations 1(b) and 2(b) to solve each inequality.

a. $7x < -21$ **b.** $12 \leq 4x$ **c.** $10 < -5x$ **d.** $-3x \leq 0$

2.3 Lesson

What You Will Learn

- Solve inequalities by multiplying or dividing by *positive* numbers.
- Solve inequalities by multiplying or dividing by *negative* numbers.
- Use inequalities to solve real-life problems.

Multiplying or Dividing by Positive Numbers

Core Concept

Multiplication and Division Properties of Inequality ($c > 0$)

Words Multiplying or dividing each side of an inequality by the same *positive* number produces an equivalent inequality.

Numbers

$$-6 < 8 \qquad\qquad 6 > -8$$

$$2 \cdot (-6) < 2 \cdot 8 \qquad\qquad \frac{6}{2} > \frac{-8}{2}$$

$$-12 < 16 \qquad\qquad 3 > -4$$

Algebra If $a > b$ and $c > 0$, then $ac > bc$. If $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$.

If $a < b$ and $c > 0$, then $ac < bc$. If $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$.

These properties are also true for $\leq$ and $\geq$.

EXAMPLE 1 Multiplying or Dividing by Positive Numbers

Solve (a) $\frac{x}{8} > -5$ and (b) $-24 \geq 3x$. Graph each solution.

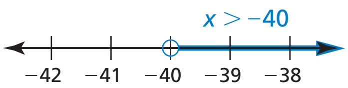
SOLUTION

a. $\frac{x}{8} > -5$ Write the inequality.

Multiplication Property of Inequality → $8 \cdot \frac{x}{8} > 8 \cdot (-5)$ Multiply each side by 8.

$x > -40$ Simplify.

▶ The solution is $x > -40$.

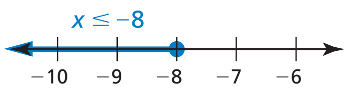


b. $-24 \geq 3x$ Write the inequality.

Division Property of Inequality → $\frac{-24}{3} \geq \frac{3x}{3}$ Divide each side by 3.

$-8 \geq x$ Simplify.

▶ The solution is $x \leq -8$.



Monitoring Progress

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Solve the inequality. Graph the solution.

1. $\frac{n}{7} \geq -1$ **2.** $-6.4 \geq \frac{1}{5}w$ **3.** $4b \geq 36$ **4.** $-18 > 1.5q$

Multiplying or Dividing by Negative Numbers

Core Concept

Multiplication and Division Properties of Inequality ($c < 0$)

Words When multiplying or dividing each side of an inequality by the same *negative* number, the direction of the inequality symbol must be reversed to produce an equivalent inequality.

Numbers

$$-6 < 8 \qquad\qquad 6 > -8$$

$$-2 \cdot (-6) > -2 \cdot 8 \qquad\qquad \frac{6}{-2} < \frac{-8}{-2}$$

$$12 > -16 \qquad\qquad -3 < 4$$

Algebra If $a > b$ and $c < 0$, then $ac < bc$. If $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$.

If $a < b$ and $c < 0$, then $ac > bc$. If $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$.

These properties are also true for $\le$ and $\ge$.

COMMON ERROR

A negative sign in an inequality does not necessarily mean you must reverse the inequality symbol, as shown in Example 1.

Only reverse the inequality symbol when you multiply or divide each side by a negative number.

EXAMPLE 2 Multiplying or Dividing by Negative Numbers

Solve each inequality. Graph each solution.

a. $2 < \frac{y}{-3}$

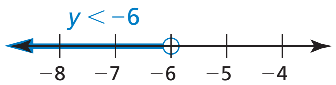
b. $-7y \le -35$

SOLUTION

a.

	$2 < \frac{y}{-3}$	Write the inequality.
Multiplication Property of Inequality	$-3 \cdot 2 > -3 \cdot \frac{y}{-3}$	Multiply each side by -3. Reverse the inequality symbol.
	$-6 > y$	Simplify.

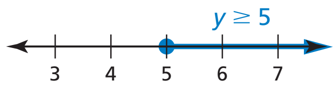
▶ The solution is $y < -6$.



b.

	$-7y \le -35$	Write the inequality.
Division Property of Inequality	$\frac{-7y}{-7} \ge \frac{-35}{-7}$	Divide each side by -7. Reverse the inequality symbol.
	$y \ge 5$	Simplify.

▶ The solution is $y \ge 5$.



Monitoring Progress

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Solve the inequality. Graph the solution.

5. $\frac{p}{-4} < 7$

6. $\frac{x}{-5} \le -5$

7. $1 \ge -\frac{1}{10}z$

8. $-9m > 63$

9. $-2r \ge -22$

10. $-0.4y \ge -12$

Solving Real-Life Problems

EXAMPLE 3 **Modeling with Mathematics**

You earn $9.50 per hour at your summer job. Write and solve an inequality that represents the numbers of hours you need to work to buy a digital camera that costs $247.

SOLUTION

1. **Understand the Problem** You know your hourly wage and the cost of the digital camera. You are asked to write and solve an inequality that represents the numbers of hours you need to work to buy the digital camera.
2. **Make a Plan** Use a verbal model to write an inequality. Then solve the inequality.
3. **Solve the Problem**

Words Hourly wage • Hours worked ≥ Cost of camera

Variable Let n be the number of hours worked.

Inequality $9.5 \cdot n \geq 247$

$$9.5n \geq 247 \quad \text{Write the inequality.}$$

Division Property of Inequality

$$\frac{9.5n}{9.5} \geq \frac{247}{9.5} \quad \text{Divide each side by 9.5.}$$

$$n \geq 26 \quad \text{Simplify.}$$

▶ You need to work at least 26 hours for your gross pay to be at least $247. If you have payroll deductions, such as Social Security taxes, you need to work more than 26 hours.

REMEMBER
Compatible numbers are numbers that are easy to compute mentally.

4. **Look Back** You can use estimation to check that your answer is reasonable.

$$\begin{array}{ccc} \$247 & \div & \$9.50/\text{h} \\ \downarrow & & \downarrow \\ \$250 & \div & \$10/\text{h} = 25 \text{ h} \end{array} \quad \text{Use compatible numbers.}$$

Your hourly wage is about $10 per hour. So, to earn about $250, you need to work about 25 hours.

Unit Analysis Each time you set up an equation or inequality to represent a real-life problem, be sure to check that the units balance.

$$\frac{\$9.50}{\cancel{\text{h}}} \times 26\cancel{\text{h}} = \$247$$

Monitoring Progress

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11. You have at most $3.65 to make copies. Each copy costs $0.25. Write and solve an inequality that represents the numbers of copies you can make.

12. The maximum speed limit for a school bus is 55 miles per hour. Write and solve an inequality that represents the numbers of hours it takes to travel 165 miles in a school bus.

2.3 Exercises

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Vocabulary and Core Concept Check

1. **WRITING** Explain how solving $2x < -8$ is different from solving $-2x < 8$.

2. **OPEN-ENDED** Write an inequality that is solved using the Division Property of Inequality where the inequality symbol needs to be reversed.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, solve the inequality. Graph the solution. *(See Example 1.)*

3. $4x < 8$

4. $3y \leq -9$

5. $-20 \leq 10n$

6. $35 < 7t$

7. $\frac{x}{2} > -2$

8. $\frac{a}{4} < 10.2$

9. $20 \geq \frac{4}{5}w$

10. $-16 \leq \frac{8}{3}t$

In Exercises 11–18, solve the inequality. Graph the solution. *(See Example 2.)*

11. $-6t < 12$

12. $-9y > 9$

13. $-10 \geq -2z$

14. $-15 \leq -3c$

15. $\frac{n}{-3} \geq 1$

16. $\frac{w}{-5} \leq 16$

17. $-8 < -\frac{1}{4}m$

18. $-6 > -\frac{2}{3}y$

19. **MODELING WITH MATHEMATICS** You have \$12 to buy five goldfish for your new fish tank. Write and solve an inequality that represents the prices you can pay per fish. *(See Example 3.)*

20. **MODELING WITH MATHEMATICS** A weather forecaster predicts that the temperature in Antarctica will decrease 8°F each hour for the next 6 hours. Write and solve an inequality to determine how many hours it will take for the temperature to drop at least 36°F.

USING TOOLS In Exercises 21–26, solve the inequality. Use a graphing calculator to verify your answer.

21. $36 < 3y$

22. $17v \geq 51$

23. $2 \leq -\frac{2}{9}x$

24. $4 > \frac{n}{-4}$

25. $2x > \frac{3}{4}$

26. $1.1y < 4.4$

ERROR ANALYSIS In Exercises 27 and 28, describe and correct the error in solving the inequality.

27.

$$-6 > \frac{2}{3}x$$

$$\frac{3}{2} \cdot (-6) < \frac{3}{2} \cdot \frac{2}{3}x$$

$$-\frac{18}{2} < x$$

$$-9 < x$$

The solution is $x > -9$.

28.

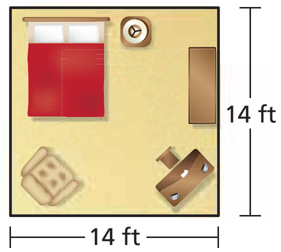
$$-4y \leq -32$$

$$\frac{-4y}{-4} \leq \frac{-32}{-4}$$

$$y \leq 8$$

The solution is $y \leq 8$.

29. **ATTENDING TO PRECISION** You have \$700 to buy new carpet for your bedroom. Write and solve an inequality that represents the costs per square foot that you can pay for the new carpet. Specify the units of measure in each step.



30. **HOW DO YOU SEE IT?** Let $m > 0$. Match each inequality with its graph. Explain your reasoning.

a. $\frac{x}{m} < -1$ **b.** $\frac{x}{m} > 1$

c. $\frac{x}{m} < 1$ **d.** $-\frac{x}{m} < 1$

A.


B.


C.


D.


31. **MAKING AN ARGUMENT** You run for 2 hours at a speed no faster than 6.3 miles per hour.

a. Write and solve an inequality that represents the possible numbers of miles you run.

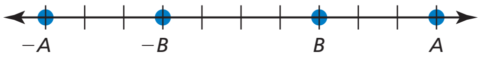
b. A marathon is approximately 26.2 miles. Your friend says that if you continue to run at this speed, you will not be able to complete a marathon in less than 4 hours. Is your friend correct? Explain.

32. **THOUGHT PROVOKING** The inequality $\frac{x}{4} \leq 5$ has a solution of $x = p$. Write a second inequality that also has a solution of $x = p$.

33. **PROBLEM SOLVING** The U.S. Mint pays \$0.02 to produce every penny. How many pennies are produced when the U.S. Mint pays more than \$6 million in production costs?

34. **REASONING** Are $x \leq \frac{2}{3}$ and $-3x \leq -2$ equivalent? Explain your reasoning.

35. **ANALYZING RELATIONSHIPS** Consider the number line shown.



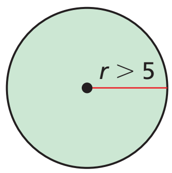
a. Write an inequality relating A and B.

b. Write an inequality relating $-A$ and $-B$.

c. Use the results from parts (a) and (b) to explain why the direction of the inequality symbol must be reversed when multiplying or dividing each side of an inequality by the same negative number.

36. **REASONING** Why might solving the inequality $\frac{4}{x} \geq 2$ by multiplying each side by x lead to an error? (*Hint:* Consider $x > 0$ and $x < 0$.)

37. **MATHEMATICAL CONNECTIONS** The radius of a circle is represented by the formula $r = \frac{C}{2\pi}$. Write and solve an inequality that represents the possible circumferences C of the circle.



38. **CRITICAL THINKING** A water-skiing instructor recommends that a boat pulling a beginning skier has a speed less than 18 miles per hour. Write and solve an inequality that represents the possible distances d (in miles) that a beginner can travel in 45 minutes of practice time.

39. **CRITICAL THINKING** A local zoo employs 36 people to take care of the animals each day. At most, 24 of the employees work full time. Write and solve an inequality that represents the fraction of employees who work part time. Graph the solution.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Section 1.2 and Section 1.3)*

40. $5x + 3 = 13$

41. $\frac{1}{2}y - 8 = -10$

42. $-3n + 2 = 2n - 3$

43. $\frac{1}{2}z + 4 = \frac{5}{2}z - 8$

Tell which number is greater. *(Skills Review Handbook)*

44. 0.8, 85%

45. $\frac{16}{30}$, 50%

46. 120%, 0.12

47. 60%, $\frac{2}{3}$

2.1–2.3 What Did You Learn?

Core Vocabulary

inequality, *p. 46*
solution of an inequality, *p. 47*
solution set, *p. 47*
graph of an inequality, *p. 48*
equivalent inequalities, *p. 54*

Core Concepts

Section 2.1

Representing Linear Inequalities, *p. 49*

Section 2.2

Addition Property of Inequality, *p. 54*
Subtraction Property of Inequality, *p. 55*

Section 2.3

Multiplication and Division Properties of Inequality ($c > 0$), *p. 60*
Multiplication and Division Properties of Inequality ($c < 0$), *p. 61*

Mathematical Thinking

1. Explain the meaning of the inequality symbol in your answer to Exercise 47 on page 51. How did you know which symbol to use?
2. In Exercise 30 on page 58, why is it important to check the reasonableness of your answer in part (a) before answering part (b)?
3. Explain how considering the units involved in Exercise 29 on page 63 helped you answer the question.

Study Skills

Analyzing Your Errors

Application Errors

What Happens: You can do numerical problems, but you struggle with problems that have context.

How to Avoid This Error: Do not just mimic the steps of solving an application problem. Explain out loud what the question is asking and why you are doing each step. After solving the problem, ask yourself, "Does my solution make sense?"

2.1–2.3 Quiz

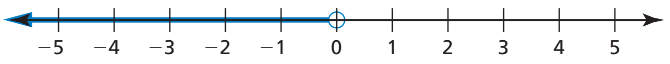
Write the sentence as an inequality. *(Section 2.1)*

1. A number z minus 6 is greater than or equal to 11.

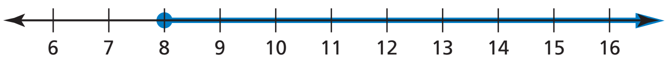
2. Twelve is no more than the sum of -1.5 times a number w and 4.

Write an inequality that represents the graph. *(Section 2.1)*

3.



4.



Solve the inequality. Graph the solution. *(Section 2.2 and Section 2.3)*

5. $9 + q \leq 15$

6. $z - (-7) < 5$

7. $-3 < y - 4$

8. $3p \geq 18$

9. $6 > \dfrac{w}{-2}$

10. $-20x > 5$

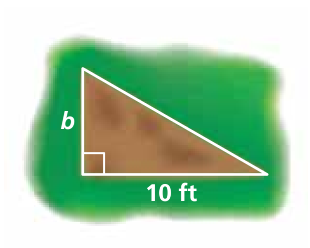
11. Three requirements for a lifeguard training course are shown. *(Section 2.1)*

a. Write and graph three inequalities that represent the requirements.

b. You can swim 250 feet, tread water for 6 minutes, and swim 35 feet underwater without taking a breath. Do you satisfy the requirements of the course? Explain.

12. The maximum volume of an American white pelican's bill is about 700 cubic inches. A pelican scoops up 100 cubic inches of water. Write and solve an inequality that represents the additional volumes the pelican's bill can contain. *(Section 2.2)*

13. The solution of $x - a > 4$ is $x > 11$. What is the value of a? *(Section 2.2)*



14. The area of the triangular garden must be less than 35 square feet. Write and solve an inequality that represents the value of b. *(Section 2.3)*

15. A candidate for class president receives 57 votes, which is at least 30% of the total number of votes. Write and solve an inequality that represents the numbers of students who cast votes. *(Section 2.3)*

2.4 Solving Multi-Step Inequalities

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.5.B

Essential Question

How can you solve a multi-step inequality?

EXPLORATION 1 Solving a Multi-Step Inequality

Work with a partner.

- Use what you already know about solving equations and inequalities to solve each multi-step inequality. Justify each step.
- Match each inequality with its graph. Use a graphing calculator to check your answer.

a. $2x + 3 \leq x + 5$

b. $-2x + 3 > x + 9$

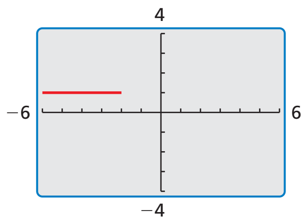
c. $27 \geq 5x + 4x$

d. $-8x + 2x - 16 < -5x + 7x$

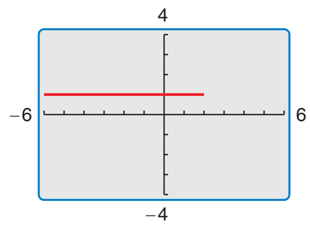
e. $3(x - 3) - 5x > -3x - 6$

f. $-5x - 6x \leq 8 - 8x - x$

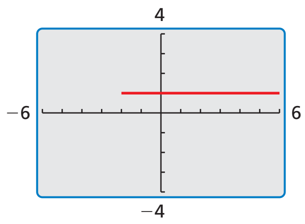
A.



B.



C.



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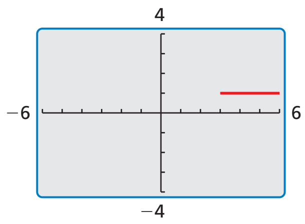

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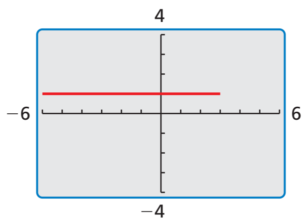

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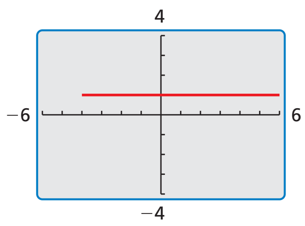


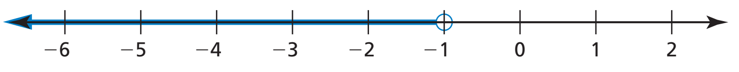
JUSTIFYING STEPS

To be proficient in math, you need to justify each step in a solution and communicate your justification to others.

Communicate Your Answer

2. How can you solve a multi-step inequality?

3. Write two different multi-step inequalities whose solutions are represented by the graph.



2.4 Lesson

What You Will Learn

- Solve multi-step inequalities.
- Use multi-step inequalities to solve real-life problems.

Solving Multi-Step Inequalities

To solve a multi-step inequality, simplify each side of the inequality, if necessary. Then use inverse operations to isolate the variable. Be sure to reverse the inequality symbol when multiplying or dividing by a negative number.

EXAMPLE 1 **Solving Multi-Step Inequalities**

Solve each inequality. Graph each solution.

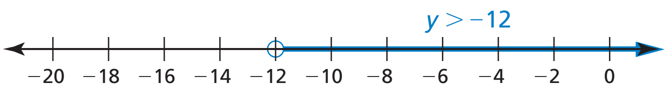
a. $\frac{y}{-6} + 7 < 9$ **b.** $2v - 4 \geq 8$

SOLUTION

a.

$\frac{y}{-6} + 7 < 9$	Write the inequality.
$\underline{-7} \quad \underline{-7}$	Subtract 7 from each side.
$\frac{y}{-6} < 2$	Simplify.
$-6 \cdot \frac{y}{-6} > -6 \cdot 2$	Multiply each side by –6. Reverse the inequality symbol.
$y > -12$	Simplify.

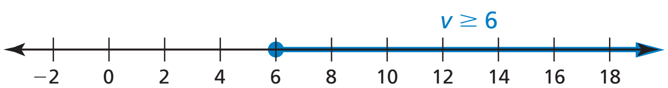
- The solution is $y > -12$.



b.

$2v - 4 \geq 8$	Write the inequality.
$\underline{+4} \quad \underline{+4}$	Add 4 to each side.
$2v \geq 12$	Simplify.
$\frac{2v}{2} \geq \frac{12}{2}$	Divide each side by 2.
$v \geq 6$	Simplify.

- The solution is $v \geq 6$.



Monitoring Progress

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Solve the inequality. Graph the solution.

1. $4b - 1 < 7$

2. $8 - 9c \geq -28$

3. $\frac{n}{-2} + 11 > 12$

4. $6 \geq 5 - \frac{v}{3}$

EXAMPLE 2 Solving an Inequality with Variables on Both Sides

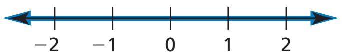
Solve $6x - 5 < 2x + 11$.

SOLUTION

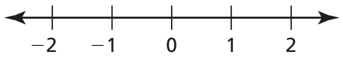
$$
\begin{array}{rcll}
6x - 5 & < & 2x + 11 & \text{Write the inequality.} \\
\underline{+\ 5} & & \underline{+\ 5} & \text{Add 5 to each side.} \\
6x & < & 2x + 16 & \text{Simplify.} \\
\underline{-\ 2x} & & \underline{-\ 2x} & \text{Subtract } 2x \text{ from each side.} \\
4x & < & 16 & \text{Simplify.} \\
\dfrac{4x}{4} & < & \dfrac{16}{4} & \text{Divide each by 4.} \\
x & < & 4 & \text{Simplify.}
\end{array}
$$

▶ The solution is $x < 4$.

When solving an inequality, if you obtain an equivalent inequality that is true, such as $-5 < 0$, the solutions of the inequality are *all real numbers*. If you obtain an equivalent inequality that is false, such as $3 \le -2$, the inequality has *no solution*.



Graph of an inequality whose solutions are all real numbers



Graph of an inequality that has no solution

EXAMPLE 3 Inequalities with Special Solutions

Solve (a) $8b - 3 > 4(2b + 3)$ and (b) $2(5w - 1) \le 7 + 10w$.

SOLUTION

a.

$$
\begin{array}{rcll}
8b - 3 & > & 4(2b + 3) & \text{Write the inequality.} \\
8b - 3 & > & 8b + 12 & \text{Distributive Property} \\
\underline{-\ 8b} & & \underline{-\ 8b} & \text{Subtract } 8b \text{ from each side.} \\
-3 & > & 12 \ ✗ & \text{Simplify.}
\end{array}
$$

▶ The inequality $-3 > 12$ is false. So, there is no solution.

b.

$$
\begin{array}{rcll}
2(5w - 1) & \le & 7 + 10w & \text{Write the inequality.} \\
10w - 2 & \le & 7 + 10w & \text{Distributive Property} \\
\underline{-\ 10w} & & \underline{-\ 10w} & \text{Subtract } 10w \text{ from each side.} \\
-2 & \le & 7 & \text{Simplify.}
\end{array}
$$

▶ The inequality $-2 \le 7$ is true. So, all real numbers are solutions.

ANALYZING MATHEMATICAL RELATIONSHIPS

When the variable terms on each side of an inequality are the same, the constant terms will determine whether the inequality is true or false.

Monitoring Progress

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Solve the inequality.

5. $5x - 12 \le 3x - 4$

6. $2(k - 5) < 2k + 5$

7. $-4(3n - 1) > -12n + 5.2$

8. $3(2a - 1) \ge 10a - 11$

Solving Real-Life Problems

EXAMPLE 4 Modeling with Mathematics

You need a mean score of at least 90 points to advance to the next round of the touch-screen trivia game. What scores in the fifth game will allow you to advance?

SOLUTION

1. Understand the Problem You know the scores of your first four games. You are asked to find the scores in the fifth game that will allow you to advance.

2. Make a Plan Use the definition of the mean of a set of numbers to write an inequality. Then solve the inequality and answer the question.

3. Solve the Problem Let x be your score in the fifth game.

$$\frac{95 + 91 + 77 + 89 + x}{5} \geq 90 \qquad \text{Write an inequality.}$$

$$\frac{352 + x}{5} \geq 90 \qquad \text{Simplify.}$$

$$5 \cdot \frac{352 + x}{5} \geq 5 \cdot 90 \qquad \text{Multiply each side by 5.}$$

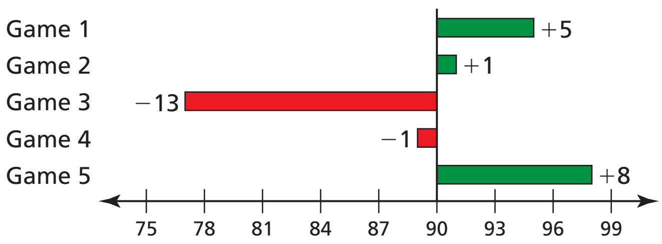
$$352 + x \geq 450 \qquad \text{Simplify.}$$

$$\underline{-352} \qquad \underline{-352} \qquad \text{Subtract 352 from each side.}$$

$$x \geq 98 \qquad \text{Simplify.}$$

▶ A score of at least 98 points will allow you to advance.

4. Look Back You can draw a diagram to check that your answer is reasonable. The horizontal bar graph shows the differences between the game scores and the desired mean of 90.



To have a mean of 90, the sum of the differences must be zero.

$5 + 1 - 13 - 1 + 8 = 0$ ✓

REMEMBER

The mean in Example 4 is equal to the sum of the game scores divided by the number of games.

Monitoring Progress

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9. **WHAT IF?** You need a mean score of at least 85 points to advance to the next round. What scores in the fifth game will allow you to advance?

2.4 Exercises

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Vocabulary and Core Concept Check

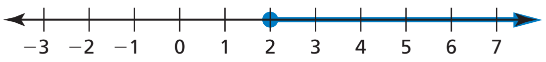
1. **WRITING** Compare solving multi-step inequalities and solving multi-step equations.
2. **WRITING** Without solving, how can you tell that the inequality $4x + 8 \le 4x - 3$ has no solution?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, match the inequality with its graph.

3. $7b - 4 \le 10$
4. $4p + 4 \ge 12$
5. $-6g + 2 \ge 20$
6. $3(2 - f) \le 15$

A.



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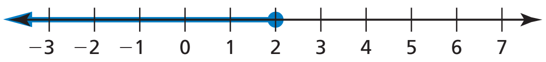

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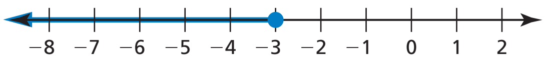

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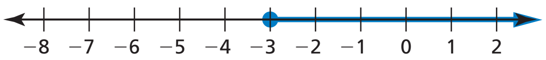


In Exercises 7–16, solve the inequality. Graph the solution. *(See Example 1.)*

7. $2x - 3 > 7$
8. $5y + 9 \le 4$
9. $-9 \le 7 - 8v$
10. $2 > -3t - 10$
11. $\frac{w}{2} + 4 > 5$
12. $1 + \frac{m}{3} \le 6$
13. $\frac{p}{-8} + 9 > 13$
14. $3 + \frac{r}{-4} \le 6$
15. $6 \ge -6(a + 2)$
16. $18 \le 3(b - 4)$

In Exercises 17–28, solve the inequality. *(See Examples 2 and 3.)*

17. $4 - 2m > 7 - 3m$
18. $8n + 2 \le 8n - 9$
19. $-2d - 2 < 3d + 8$
20. $8 + 10f > 14 - 2f$
21. $8g - 5g - 4 \le -3 + 3g$
22. $3w - 5 > 2w + w - 7$
23. $6(\ell + 3) < 3(2\ell + 6)$
24. $2(5c - 7) \ge 10(c - 3)$
25. $4\left(\frac{1}{2}t - 2\right) > 2(t - 3)$
26. $15\left(\frac{1}{3}b + 3\right) \le 6(b + 9)$
27. $9j - 6 + 6j \ge 3(5j - 2)$
28. $6h - 6 + 2h < 2(4h - 3)$

ERROR ANALYSIS **In Exercises 29 and 30, describe and correct the error in solving the inequality.**

29. ✗

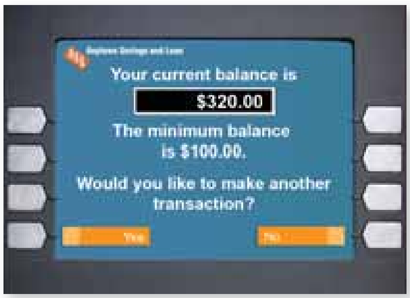
$$\frac{x}{4} + 6 \ge 3$$
$$x + 6 \ge 12$$
$$x \ge 6$$

30. ✗

$$-2(1 - x) \le 2x - 7$$
$$-2 + 2x \le 2x - 7$$
$$-2 \le -7$$

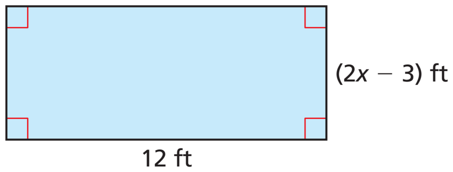
All real numbers are solutions.

31. **MODELING WITH MATHEMATICS** Write and solve an inequality that represents how many \$20 bills you can withdraw from the account without going below the minimum balance. *(See Example 4.)*



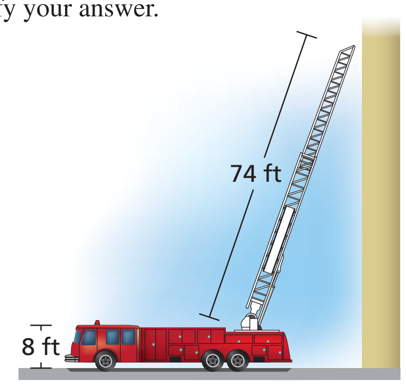
32. **MODELING WITH MATHEMATICS** A woodworker wants to earn at least \$25 an hour making and selling cabinets. He pays \$125 for materials. Write and solve an inequality that represents how many hours the woodworker can spend building the cabinet.

33. **MATHEMATICAL CONNECTIONS** The area of the rectangle is greater than 60 square feet. Write and solve an inequality to find the possible values of x.

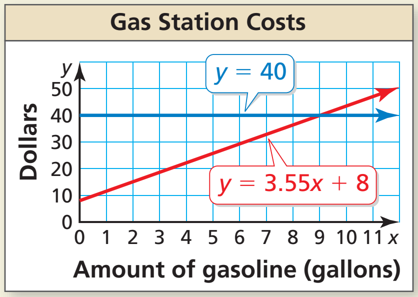


34. **MAKING AN ARGUMENT** Forest Park Campgrounds charges a \$100 membership fee plus \$35 per night. Woodland Campgrounds charges a \$20 membership fee plus \$55 per night. Your friend says that if you plan to camp for four or more nights, then you should choose Woodland Campgrounds. Is your friend correct? Explain.

35. **PROBLEM SOLVING** The height of one story of a building is about 10 feet. The bottom of the ladder on the fire truck must be at least 24 feet away from the building. How many stories can the ladder reach? Justify your answer.



36. **HOW DO YOU SEE IT?** The graph shows your budget, and the total cost of x gallons of gasoline and a car wash. You want to determine the possible amounts (in gallons) of gasoline you can buy within your budget.



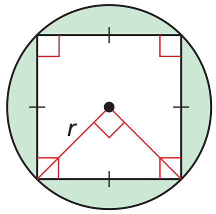
a. What is your budget?

b. How much does a gallon of gasoline cost? How much does a car wash cost?

c. Write an inequality that represents the possible amounts of gasoline you can buy.

d. Use the graph to estimate the solution of your inequality in part (c).

37. **PROBLEM SOLVING** For what values of r will the area of the shaded region be greater than or equal to $9(\pi - 2)$?



38. **THOUGHT PROVOKING** A runner's times (in minutes) in the four races he has completed are 25.5, 24.3, 24.8, and 23.5. The runner plans to run at least one more race and wants to have an average time less than 24 minutes. Write and solve an inequality to show how the runner can achieve his goal.

REASONING **In Exercises 39 and 40, find the value of a for which the solution of the inequality is all real numbers.**

39. $a(x + 3) < 5x + 15 - x$

40. $3x + 8 + 2ax \geq 3ax - 4a$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the sentence as an inequality. *(Section 2.1)*

41. Six times a number y is less than or equal to 10.

42. A number p plus 7 is greater than 24.

43. The quotient of a number r and 7 is no more than 18.

2.5 Solving Compound Inequalities

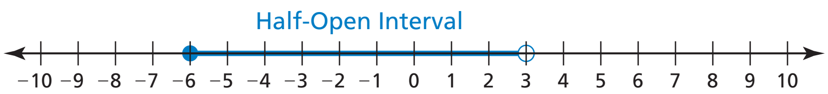
TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.5.B

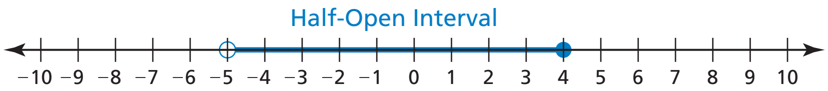
Essential Question

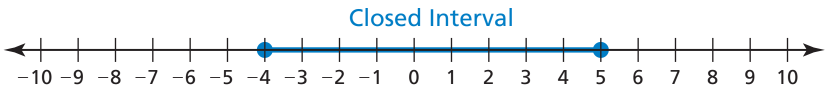
How can you use inequalities to describe intervals on the real number line?

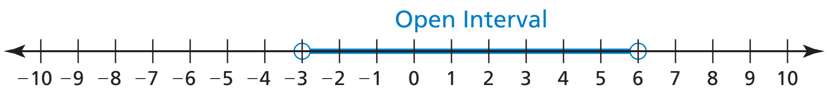
EXPLORATION 1 Describing Intervals on the Real Number Line

Work with a partner. In parts (a)–(d), use two inequalities to describe the interval.

a.


b.


c.


d.


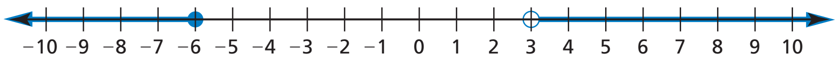
e. Do you use "and" or "or" to connect the two inequalities in parts (a)–(d)? Explain.

REASONING

To be proficient in math, you need to create a clear representation of the problem at hand.

EXPLORATION 2 Describing Two Infinite Intervals

Work with a partner. In parts (a)–(d), use two inequalities to describe the interval.

a.


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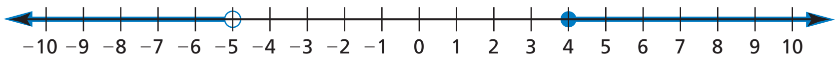
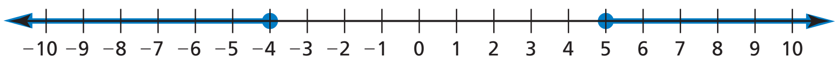
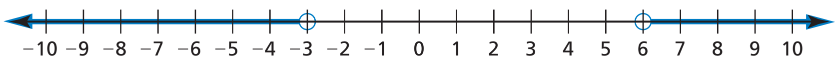


c.


d.


e. Do you use "and" or "or" to connect the two inequalities in parts (a)–(d)? Explain.

Communicate Your Answer

3. How can you use inequalities to describe intervals on the real number line?

2.5 Lesson

Core Vocabulary

compound inequality, *p. 74*

What You Will Learn

- Write and graph compound inequalities.
- Solve compound inequalities.
- Use compound inequalities to solve real-life problems.

Writing and Graphing Compound Inequalities

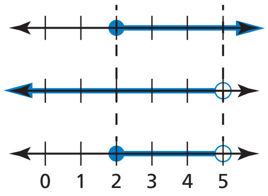
A **compound inequality** is an inequality formed by joining two inequalities with the word "and" or the word "or."

The graph of a compound inequality with "and" is the *intersection* of the graphs of the inequalities. The graph shows numbers that are solutions of *both* inequalities.

$x \geq 2$

$x < 5$

$2 \leq x$ *and* $x < 5$
$2 \leq x < 5$

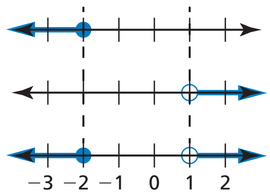


The graph of a compound inequality with "or" is the *union* of the graphs of the inequalities. The graph shows numbers that are solutions of *either* inequality.

$y \leq -2$

$y > 1$

$y \leq -2$ *or* $y > 1$



EXAMPLE 1 Writing and Graphing Compound Inequalities

Write each sentence as an inequality. Graph each inequality.

a. A number x is greater than -8 and less than or equal to 4.

b. A number y is at most 0 or at least 2.

REMEMBER

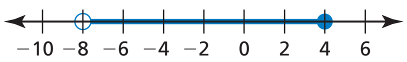
A compound inequality with "and" can be written as a single inequality. For example, you can write $x > -8$ and $x \leq 4$ as $-8 < x \leq 4$.

SOLUTION

a. A number x is greater than -8 and less than or equal to 4.

$x > -8$ *and* $x \leq 4$

▶ An inequality is $-8 < x \leq 4$.

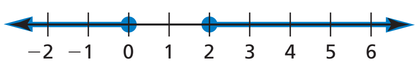


Graph the intersection of the graphs of $x > -8$ and $x \leq 4$.

b. A number y is at most 0 or at least 2.

$y \leq 0$ *or* $y \geq 2$

▶ An inequality is $y \leq 0$ *or* $y \geq 2$.



Graph the union of the graphs of $y \leq 0$ and $y \geq 2$.

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Write the sentence as an inequality. Graph the inequality.

1. A number d is more than 0 and less than 10.

2. A number a is fewer than -6 or no less than -3.

Solving Compound Inequalities

You can solve a compound inequality by solving two inequalities separately. When a compound inequality with "and" is written as a single inequality, you can solve the inequality by performing the same operation on each expression.

ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to see complicated things as single objects or as being composed of several objects.

EXAMPLE 2 Solving Compound Inequalities with "And"

Solve each inequality. Graph each solution.

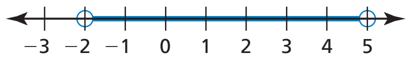
a. $-4 < x - 2 < 3$ **b.** $-3 < -2x + 1 \leq 9$

SOLUTION

a. Separate the compound inequality into two inequalities, then solve.

$$\begin{array}{lclr}
-4 < x - 2 & \textit{and} & x - 2 < 3 & \text{Write two inequalities.} \\
\underline{+2} \quad \underline{+2} & & \underline{+2} \quad \underline{+2} & \text{Add 2 to each side.} \\
-2 < x & \textit{and} & x < 5 & \text{Simplify.}
\end{array}$$

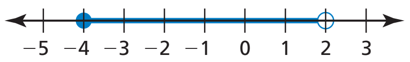
▶ The solution is $-2 < x < 5$.



b.

$$\begin{array}{lr}
-3 < -2x + 1 \leq 9 & \text{Write the inequality.} \\
\underline{-1} \quad \underline{-1} \quad \underline{-1} & \text{Subtract 1 from each expression.} \\
-4 < -2x \leq 8 & \text{Simplify.} \\
\dfrac{-4}{-2} > \dfrac{-2x}{-2} \geq \dfrac{8}{-2} & \text{Divide each expression by } -2. \text{ Reverse each inequality symbol.} \\
2 > x \geq -4 & \text{Simplify.}
\end{array}$$

▶ The solution is $-4 \leq x < 2$.



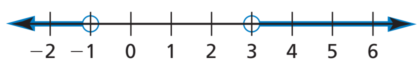
EXAMPLE 3 Solving a Compound Inequality with "Or"

Solve $3y - 5 < -8$ *or* $2y - 1 > 5$. Graph the solution.

SOLUTION

$$\begin{array}{lclr}
3y - 5 < -8 & \textit{or} & 2y - 1 > 5 & \text{Write the inequality.} \\
\underline{+5} \quad \underline{+5} & & \underline{+1} \quad \underline{+1} & \text{Addition Property of Inequality} \\
3y < -3 & & 2y > 6 & \text{Simplify.} \\
\dfrac{3y}{3} < \dfrac{-3}{3} & & \dfrac{2y}{2} > \dfrac{6}{2} & \text{Division Property of Inequality} \\
y < -1 & \textit{or} & y > 3 & \text{Simplify.}
\end{array}$$

▶ The solution is $y < -1$ *or* $y > 3$.



Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Solve the inequality. Graph the solution.

3. $5 \leq m + 4 < 10$

4. $-3 < 2k - 5 < 7$

5. $4c + 3 \leq -5$ *or* $c - 8 > -1$

6. $2p + 1 < -7$ *or* $3 - 2p \leq -1$

Solving Real-Life Problems

Operating temperature:
0°C to 35°C

EXAMPLE 4 Modeling with Mathematics

Electrical devices should operate effectively within a specified temperature range. Outside the operating temperature range, the device may fail.

a. Write and solve a compound inequality that represents the possible operating temperatures (in degrees Fahrenheit) of the smartphone.

b. Describe one situation in which the surrounding temperature could be below the operating range and one in which it could be above.

SOLUTION

1. Understand the Problem You know the operating temperature range in degrees Celsius. You are asked to write and solve a compound inequality that represents the possible operating temperatures (in degrees Fahrenheit) of the smartphone. Then you are asked to describe situations outside this range.

2. Make a Plan Write a compound inequality in degrees Celsius. Use the formula $C = \frac{5}{9}(F - 32)$ to rewrite the inequality in degrees Fahrenheit. Then solve the inequality and describe the situations.

3. Solve the Problem Let C be the temperature in degrees Celsius, and let F be the temperature in degrees Fahrenheit.

STUDY TIP

You can also solve the inequality by multiplying each expression by $\frac{9}{5}$.

$$0 \le C \le 35 \qquad \text{Write the inequality using } C.$$

$$0 \le \tfrac{5}{9}(F - 32) \le 35 \qquad \text{Substitute } \tfrac{5}{9}(F - 32) \text{ for } C.$$

$$9 \cdot 0 \le 9 \cdot \tfrac{5}{9}(F - 32) \le 9 \cdot 35 \qquad \text{Multiply each expression by 9.}$$

$$0 \le 5(F - 32) \le 315 \qquad \text{Simplify.}$$

$$0 \le 5F - 160 \le 315 \qquad \text{Distributive Property}$$

$$\underline{+\,160} \quad \underline{+\,160} \quad \underline{+\,160} \qquad \text{Add 160 to each expression.}$$

$$160 \le 5F \le 475 \qquad \text{Simplify.}$$

$$\frac{160}{5} \le \frac{5F}{5} \le \frac{475}{5} \qquad \text{Divide each expression by 5.}$$

$$32 \le F \le 95 \qquad \text{Simplify.}$$

▶ The solution is $32 \le F \le 95$. So, the operating temperature range of the smartphone is 32°F to 95°F. One situation when the surrounding temperature could be below this range is winter in Alaska. One situation when the surrounding temperature could be above this range is daytime in the Mojave Desert of the American Southwest.

4. Look Back You can use the formula $C = \frac{5}{9}(F - 32)$ to check that your answer is correct. Substitute 32 and 95 for F in the formula to verify that 0°C and 35°C are the minimum and maximum operating temperatures in degrees Celsius.

−40°C to 15°C

Monitoring Progress

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7. Write and solve a compound inequality that represents the temperature rating (in degrees Fahrenheit) of the winter boots.

2.5 Exercises

Tutorial Help in English and Spanish at *BigIdeasMath.com*

Vocabulary and Core Concept Check

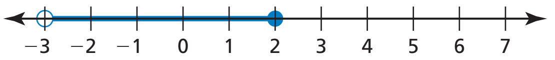
1. **WRITING** Compare the graph of $-6 \leq x \leq -4$ with the graph of $x \leq -6$ *or* $x \geq -4$.

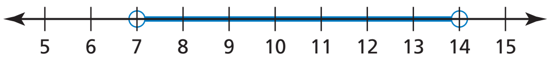
2. **WHICH ONE DOESN'T BELONG?** Which compound inequality does *not* belong with the other three? Explain your reasoning.

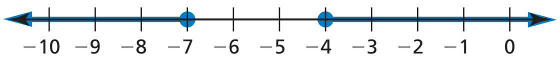
$a > 4$ *or* $a < -3$	$a < -2$ *or* $a > 8$	$a > 7$ *or* $a < -5$	$a < 6$ *or* $a > -9$

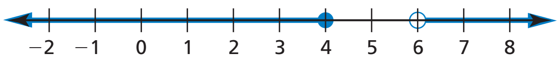
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, write a compound inequality that is represented by the graph.

3. 

4. 

5. 

6. 

In Exercises 7–10, write the sentence as an inequality. Graph the inequality. *(See Example 1.)*

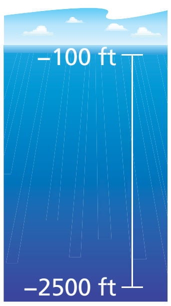
7. A number p is less than 6 and greater than 2.

8. A number n is less than or equal to -7 or greater than 12.

9. A number m is more than $-7\frac{2}{3}$ or at most -10.

10. A number r is no less than -1.5 and fewer than 9.5.

11. **MODELING WITH MATHEMATICS** Slitsnails are large mollusks that live in deep waters. They have been found in the range of elevations shown. Write and graph a compound inequality that represents this range.



12. **MODELING WITH MATHEMATICS** The life zones on Mount Rainier, a mountain in Washington, can be approximately classified by elevation, as follows.

Low-elevation forest: above 1700 feet to 2500 feet
Mid-elevation forest: above 2500 feet to 4000 feet
Subalpine: above 4000 feet to 6500 feet
Alpine: above 6500 feet to the summit

Elevation of Mount Rainier: 14,410 ft

Write a compound inequality that represents the elevation range for each type of plant life.

a. trees in the low-elevation forest zone

b. flowers in the subalpine and alpine zones

In Exercises 13–20, solve the inequality. Graph the solution. *(See Examples 2 and 3.)*

13. $6 < x + 5 \leq 11$

14. $24 > -3r \geq -9$

15. $v + 8 < 3$ *or* $-8v < -40$

16. $-14 > w + 3$ *or* $3w \geq -27$

17. $2r + 3 < 7$ *or* $-r + 9 \leq 2$

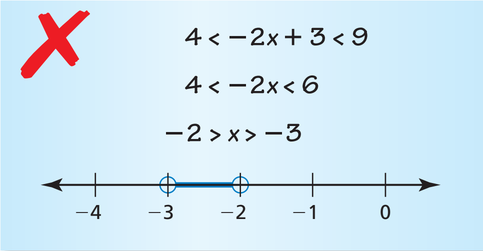
18. $-6 < 3n + 9 < 21$

19. $-12 < \frac{1}{2}(4x + 16) < 18$

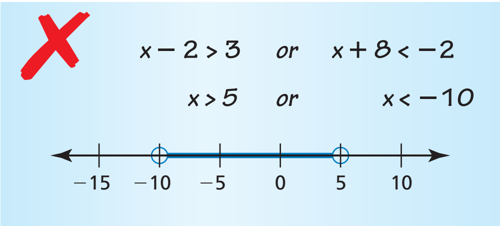
20. $35 < 7(2 - b)$ *or* $\frac{1}{3}(15b - 12) \geq 21$

ERROR ANALYSIS In Exercises 21 and 22, describe and correct the error in solving the inequality or graphing the solution.

21.



22.



23. MODELING WITH MATHEMATICS Write and solve a compound inequality that represents the possible temperatures (in degrees Fahrenheit) of the interior of the iceberg. *(See Example 4.)*

24. PROBLEM SOLVING A ski shop sells skis with lengths ranging from 150 centimeters to 220 centimeters. The shop says the length of the skis should be about 1.16 times a skier's height (in centimeters). Write and solve a compound inequality that represents the heights of skiers the shop does *not* provide skis for.

In Exercises 25–30, solve the inequality. Graph the solution, if possible.

25. $22 < -3c + 4 < 14$

26. $2m - 1 \geq 5$ *or* $5m > -25$

27. $-y + 3 \leq 8$ *and* $y + 2 > 9$

28. $x - 8 \leq 4$ *or* $2x + 3 > 9$

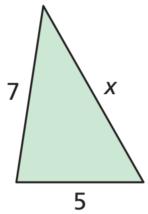
29. $2n + 19 \leq 10 + n$ *or* $-3n + 3 < -2n + 33$

30. $3x - 18 < 4x - 23$ *and* $x - 16 < -22$

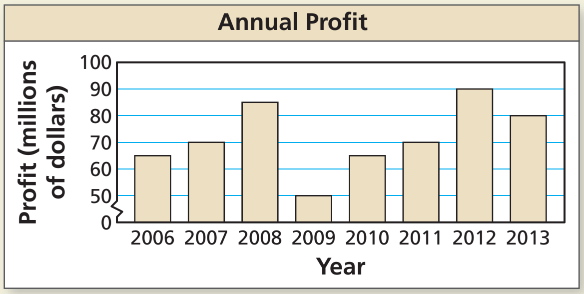
31. REASONING Fill in the compound inequality $4(x - 6)$ ▢ $2(x - 10)$ *and* $5(x + 2) \geq 2(x + 8)$ with $<$, $\leq$, $>$, or $\geq$ so that the solution is only one value.

32. THOUGHT PROVOKING Write a real-life story that can be modeled by the graph.

33. MAKING AN ARGUMENT The sum of the lengths of any two sides of a triangle is greater than the length of the third side. Use the triangle shown to write and solve three inequalities. Your friend claims the value of x can be 1. Is your friend correct? Explain.



34. HOW DO YOU SEE IT? The graph shows the annual profits of a company from 2006 to 2013.



a. Write a compound inequality that represents the annual profits from 2006 to 2013.

b. You can use the formula $P = R - C$ to find the profit P, where R is the revenue and C is the cost. From 2006 to 2013, the company's annual cost was about \$125 million. Is it possible the company had an annual revenue of \$160 million from 2006 to 2013? Explain.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Plot the ordered pair in a coordinate plane. Describe the location of the point. *(Skills Review Handbook)*

35. $A(1, 3)$ **36.** $B(0, -3)$ **37.** $C(-4, -2)$ **38.** $D(-1, 2)$

Find and interpret the mean absolute deviation of the data. *(Skills Review Handbook)*

39. 1, 1, 2, 5, 6, 8, 10, 12, 12, 13

40. 24, 26, 28, 28, 30, 30, 32, 32, 34, 36

2.4–2.5 What Did You Learn?

Core Vocabulary

compound inequality, *p. 74*

Core Concepts

Section 2.4

Solving Multi-Step Inequalities, *p. 68*
Special Solutions of Linear Inequalities, *p. 69*

Section 2.5

Writing and Graphing Compound Inequalities, *p. 74*
Solving Compound Inequalities, *p. 75*

Mathematical Thinking

1. How can you use a diagram to help you solve Exercise 12 on page 77?
2. In Exercises 13 and 14 on page 77, how can you use structure to break down the compound inequality into two inequalities?

Performance Task

Grading Calculations

You are not doing as well as you had hoped in one of your classes. So, you want to figure out the minimum grade you need on the final exam to receive the semester grade that you want. Is it still possible to get an A? How would you explain your calculations to a classmate?

To explore the answers to these questions and more, go to ***BigIdeasMath.com***.

2 Chapter Review

2.1 Writing and Graphing Inequalities *(pp. 45–52)*

a. A number x plus 36 is no more than 40. Write this sentence as an inequality.

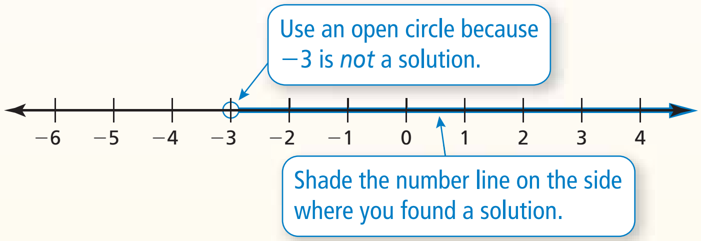
$$\text{A}\ \underbrace{\text{number } x \text{ plus } 36}_{x\,+\,36}\ \underbrace{\text{is no more than}}_{\leq}\ \underbrace{40}_{40}.$$

▶ An inequality is $x + 36 \leq 40$.

b. Graph $w > -3$.

Test a number to the left of -3. $w = -4$ is *not* a solution.

Test a number to the right of -3. $w = 0$ is a solution.



Write the sentence as an inequality.

1. A number d minus 2 is less than -1.
2. Ten is at least the product of a number h and 5.

Graph the inequality.

3. $x > 4$ 4. $y \leq 2$ 5. $-1 \geq z$

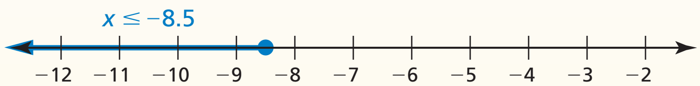
2.2 Solving Inequalities Using Addition or Subtraction *(pp. 53–58)*

Solve $x + 2.5 \leq -6$. Graph the solution.

$$
\begin{aligned}
x + 2.5 &\leq -6 && \text{Write the inequality.}\\
\underline{-\,2.5} \quad &\quad \underline{-\,2.5} && \text{Subtract 2.5 from each side.}\\
x &\leq -8.5 && \text{Simplify.}
\end{aligned}
$$

Subtraction Property of Inequality

▶ The solution is $x \leq -8.5$.



Solve the inequality. Graph the solution.

6. $p + 4 < 10$ 7. $r - 4 < -6$ 8. $2.1 \geq m - 6.7$

2.3 Solving Inequalities Using Multiplication or Division (pp. 59–64)

Solve $\frac{n}{-10} > 5$. Graph the solution.

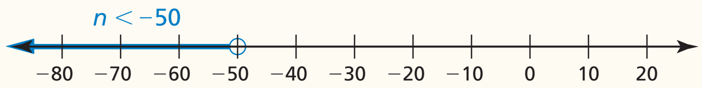
$$\frac{n}{-10} > 5 \qquad \text{Write the inequality.}$$

Multiplication Property of Inequality

$$-10 \cdot \frac{n}{-10} < -10 \cdot 5 \qquad \text{Multiply each side by } -10. \text{ Reverse the inequality symbol.}$$

$$n < -50 \qquad \text{Simplify.}$$

▶ The solution is $n < -50$.



Solve the inequality. Graph the solution.

9. $3x > -21$

10. $-4 \le \frac{g}{5}$

11. $-\frac{3}{4}n \le 3$

12. $\frac{s}{-8} \ge 11$

13. $36 < 2q$

14. $-1.2k > 6$

2.4 Solving Multi-Step Inequalities (pp. 67–72)

Solve $22 + 3y \ge 4$. Graph the solution.

$$22 + 3y \ge 4 \qquad \text{Write the inequality.}$$

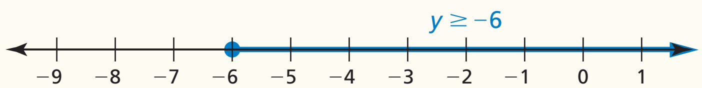
$$\underline{-22} \qquad \underline{-22} \qquad \text{Subtract 22 from each side.}$$

$$3y \ge -18 \qquad \text{Simplify.}$$

$$\frac{3y}{3} \ge \frac{-18}{3} \qquad \text{Divide each side by 3.}$$

$$y \ge -6 \qquad \text{Simplify.}$$

▶ The solution is $y \ge -6$.



Solve the inequality. Graph the solution, if possible.

15. $3x - 4 > 11$

16. $-4 < \frac{b}{2} + 9$

17. $7 - 3n \le n + 3$

18. $2(-4s + 2) \ge -5s - 10$

19. $6(2t + 9) \le 12t - 1$

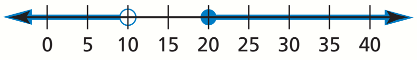
20. $3r - 8 > 3(r - 6)$

2.5 Solving Compound Inequalities (pp. 73–78)

a. A number m is less than 10 or at least 20. Write this sentence as an inequality. Graph the inequality.

A $\underbrace{\text{number } m \text{ is less than } 10}_{m < 10}$ $\underbrace{\text{or}}_{or}$ $\underbrace{\text{at least } 20}_{m \geq 20}$.

▶ An inequality is $m < 10$ *or* $m \geq 20$.



b. Solve $-1 \leq -2d + 7 \leq 9$. Graph the solution.

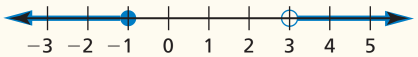
$-1 \leq -2d + 7 \leq 9$	Write the inequality.
$\underline{-7} \quad \underline{-7} \quad \underline{-7}$	Subtract 7 from each expression.
$-8 \leq -2d \leq 2$	Simplify.
$\dfrac{-8}{-2} \geq \dfrac{-2d}{-2} \geq \dfrac{2}{-2}$	Divide each expression by -2. Reverse each inequality symbol.
$4 \geq d \geq -1$	Simplify.

▶ The solution is $-1 \leq d \leq 4$.

c. Solve $2y - 3 \leq -5$ *or* $3y - 1 > 8$. Graph the solution.

$2y - 3 \leq -5$	*or*	$3y - 1 > 8$	Write the inequality.
$\underline{+3} \quad \underline{+3}$		$\underline{+1} \quad \underline{+1}$	Addition Property of Inequality
$2y \leq -2$		$3y > 9$	Simplify.
$\dfrac{2y}{2} \leq \dfrac{-2}{2}$		$\dfrac{3y}{3} > \dfrac{9}{3}$	Division Property of Inequality
$y \leq -1$	*or*	$y > 3$	Simplify.

▶ The solution is $y \leq -1$ *or* $y > 3$.



21. A number x is more than -6 and at most 8. Write this sentence as an inequality. Graph the inequality.

Solve the inequality. Graph the solution.

22. $4 > x - 7 > -6$

23. $2x + 2 \leq 4$ *or* $x + 2 \geq 5$

24. $19 \geq 3z + 1 \geq -5$

25. $\dfrac{r}{4} < -5$ *or* $-2r - 7 \leq 3$

2 Chapter Test

Write the sentence as an inequality.

1. The sum of a number y and 9 is at least -1.

2. A number r is more than 0 or less than or equal to -8.

3. A number k is less than 3 units from 10.

Solve the inequality. Graph the solution, if possible.

4. $\frac{x}{2} - 5 \geq -9$

5. $-4s < 6s + 1$

6. $4p + 3 \geq 2(2p + 1)$

7. $3y - 7 \geq 17$

8. $8(3g - 2) \leq 12(2g + 1)$

9. $6(2x - 1) \geq 3(4x + 1)$

10. $-7 < 2c - 1 < 10$

11. $-2 \leq 4 - 3a \leq 13$

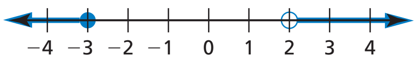
12. $-5 < 2 - h$ *or* $6h + 5 > 71$

13. You start a small baking business, and you want to earn a profit of at least \$250 in the first month. The expenses in the first month are \$155. What are the possible revenues that you need to earn to meet the profit goal?

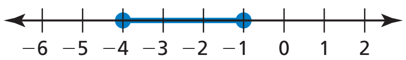
14. Let a, b, c, and d be constants. Describe the possible solution sets of the inequality $ax + b < cx + d$.

Write and graph a compound inequality that represents the numbers that are *not* solutions of the inequality represented by the graph shown. Explain your reasoning.

15.



16.



17. You save \$15 per week to purchase one of the bikes shown.

a. Write and solve an inequality to find the number of weeks you need to save to purchase a bike.

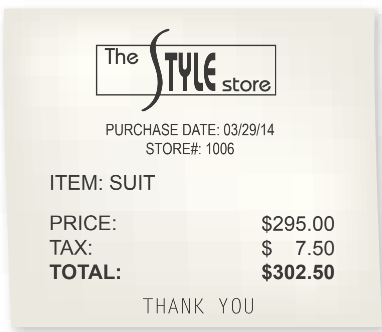
b. Your parents give you \$65 to help you buy the new bike. How does this affect your answer in part (a)? Use an inequality to justify your answer.

18. A state imposes a sales tax on items of clothing that cost more than \$175. The tax applies only to the difference of the price of the item and \$175.

a. Use the receipt shown to find the tax rate (as a percent).

b. A shopper has \$430 to spend on a winter coat. Write and solve an inequality to find the prices p of coats that the shopper can afford. Assume that $p \geq 175$.

c. Another state imposes a 5% sales tax on the entire price of an item of clothing. For which prices would paying the 5% tax be cheaper than paying the tax described above? Write and solve an inequality to find your answer and list three prices that are solutions.


The STYLE store

PURCHASE DATE: 03/29/14
STORE#: 1006

ITEM: SUIT

PRICE:	\$295.00
TAX:	\$ 7.50
TOTAL:	**\$302.50**

THANK YOU

2 Standards Assessment

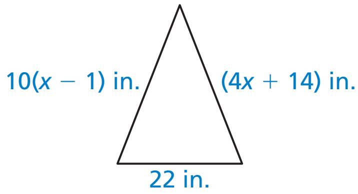
1. The triangle shown has a perimeter of 82 inches. What is the value of x? *(TEKS A.5.A)*

Ⓐ 3.4

Ⓑ 4.0

Ⓒ 7.1

Ⓓ 42.0



2. The sum of 5 times a number b and 8 is no less than b. Which inequality describes b? *(TEKS A.5.B)*

Ⓕ $b > -2$ Ⓖ $b < -2$

Ⓗ $b \geq -2$ Ⓙ $b \leq -2$

3. A skateboarding park charges \$7 per session to skate and \$4 per session to rent safety equipment. Your friend rents safety equipment every time he skates. Last year, he spent \$99 total for skating charges and equipment rentals. How much did he pay to rent safety equipment last year? *(TEKS A.5.A)*

Ⓐ \$4 Ⓑ \$9

Ⓒ \$11 Ⓓ \$36

4. **GRIDDED ANSWER** The cost C of parking in a parking garage is given by the equation $C = 2(x - 2) + 3$, where x is the amount of time in hours. You need to spend less than \$10 for parking. What is the maximum whole number of hours you can park in the garage? *(TEKS A.5.B)*

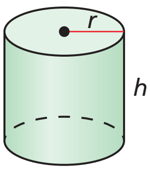
5. The volume V of a cylinder is given by the formula $V = \pi r^2 h$. Solve the formula for h. *(TEKS A.12.E)*

Ⓕ $h = V - \pi r^2$

Ⓖ $h = -\dfrac{V}{\pi r^2}$

Ⓗ $h = \dfrac{V}{\pi r}$

Ⓙ $h = \dfrac{V}{\pi r^2}$



6. The area of a rectangle is given by $3(x - 17) - x + 19$. Which inequality represents the possible values of x? *(TEKS A.5.B)*

Ⓐ $x \geq 16$ Ⓑ $x > -16$

Ⓒ $x > 16$ Ⓓ $-16 < x < 16$

7. What is the solution of $8x + 2x = 15x - 10$? ***(TEKS A.5.A)***

Ⓕ $x = -2$ Ⓖ $x = 0.4$

Ⓗ $x = 2$ Ⓙ $x = 5$

8. In 1862, the United States imposed a tax on annual income to pay for the expenses of the Civil War. The table shows the tax rates for different incomes. Which compound inequality describes the range of incomes x (in dollars) for which the tax was between \$450 and \$600? ***(TEKS A.5.B)***

Annual income	Tax rate
\$600 to \$10,000	3% of income
Greater than \$10,000	3% of first \$10,000 plus 5% of income over \$10,000

Ⓐ $3000 \le x \le 6000$ Ⓑ $9000 \le x \le 12{,}000$

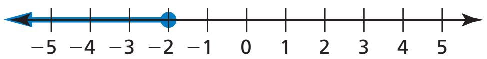
Ⓒ $13{,}000 \le x \le 16{,}000$ Ⓓ $15{,}000 \le x \le 20{,}000$

9. What is the solution of $4y + y + 1 = 7(y - 1)$? ***(TEKS A.5.A)***

Ⓕ $y = 4$ Ⓖ $y = 3$

Ⓗ $y = -3$ Ⓙ $y = -4$

10. The solution set of which inequality is represented by the graph? ***(TEKS A.5.B)***



Ⓐ $4x + 1 \le 2(x - 1) - 1$

Ⓑ $4x + 1 \ge 2(x - 1) - 1$

Ⓒ $4x - 1 \le 2(x + 1) + 1$

Ⓓ none of the above

11. A plumber charges \$64 per hour for labor and x dollars for replacement parts. The total bill is \$284 and includes 3 hours of labor. How much does the plumber charge for the replacement parts? ***(TEKS A.5.A)***

Ⓕ \$22 Ⓖ \$24

Ⓗ \$92 Ⓙ \$118

12. When $2(a + b) + a$ is negative, which statement must be true? ***(TEKS A.5.B)***

Ⓐ $a < 0, b < 0$ Ⓑ $a < 0, b > 0$

Ⓒ $a < -\frac{2}{3}b$ Ⓓ $a < b$

3 Graphing Linear Functions

Submersible *(p. 128)*

Basketball *(p. 122)*

Speed of Light *(p. 111)*

Coins *(p. 102)*

Taxi Ride *(p. 95)*

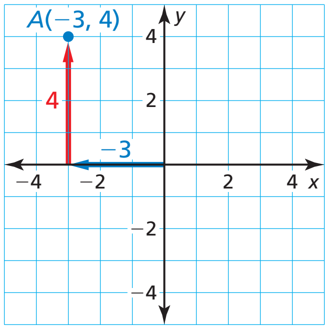
Mathematical Thinking: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

Maintaining Mathematical Proficiency

Plotting Points (6.11)

Example 1 **Plot the point $A(-3, 4)$ in a coordinate plane. Describe the location of the point.**

Start at the origin. Move 3 units left and 4 units up. Then plot the point.
The point is in Quadrant II.



Plot the point in a coordinate plane. Describe the location of the point.

1. $A(3, 2)$ **2.** $B(-5, 1)$ **3.** $C(0, 3)$

4. $D(-1, -4)$ **5.** $E(-3, 0)$ **6.** $F(2, -1)$

Evaluating Expressions (6.3.D)

Example 2 **Evaluate $4x - 5$ when $x = 3$.**

$$\begin{aligned} 4x - 5 &= 4(3) - 5 && \text{Substitute 3 for } x. \\ &= 12 - 5 && \text{Multiply.} \\ &= 7 && \text{Subtract.} \end{aligned}$$

Example 3 **Evaluate $-2x + 9$ when $x = -8$.**

$$\begin{aligned} -2x + 9 &= -2(-8) + 9 && \text{Substitute } -8 \text{ for } x. \\ &= 16 + 9 && \text{Multiply.} \\ &= 25 && \text{Add.} \end{aligned}$$

Evaluate the expression for the given value of *x*.

7. $3x - 4; x = 7$ **8.** $-5x + 8; x = 3$ **9.** $10x + 18; x = 5$

10. $-9x - 2; x = -4$ **11.** $24 - 8x; x = -2$ **12.** $15x + 9; x = -1$

13. **ABSTRACT REASONING** Let a and b be positive real numbers. Describe how to plot (a, b), $(-a, b)$, $(a, -b)$, and $(-a, -b)$.

Mathematical Thinking

Mathematically proficient students select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems. **(A.1.C)**

Using a Graphing Calculator

Core Concept

Standard and Square Viewing Windows

A typical graphing calculator screen has a height to width ratio of 2 to 3. This means that when you use the *standard viewing window* of -10 to 10 (on each axis), the graph will not be in its true perspective.

```
WINDOW
Xmin=-10
Xmax=10
Xscl=1
Ymin=-10
Ymax=10
Yscl=1
```

This is the standard viewing window.

To see a graph in its true perspective, you need to use a *square viewing window*, in which the tick marks on the x-axis are spaced the same as the tick marks on the y-axis.

```
WINDOW
Xmin=-9
Xmax=9
Xscl=1
Ymin=-6
Ymax=6
Yscl=1
```

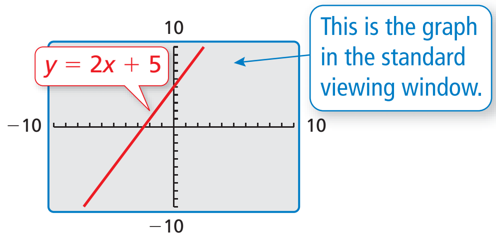
This is a square viewing window.

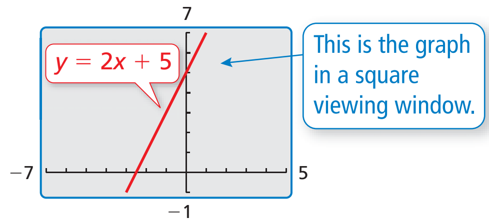
EXAMPLE 1 Using a Graphing Calculator

Use a graphing calculator to graph $y = 2x + 5$.

SOLUTION

Enter the equation $y = 2x + 5$ into your calculator. Then graph the equation. The standard viewing window does not show the graph in its true perspective. Notice that the tick marks on the y-axis are closer together than the tick marks on the x-axis. To see the graph in its true perspective, use a square viewing window.





Monitoring Progress

Determine whether the viewing window is square. Explain.

1. $-8 \le x \le 7,\ -3 \le y \le 7$ **2.** $-6 \le x \le 6,\ -9 \le y \le 9$ **3.** $-18 \le x \le 18,\ -12 \le y \le 12$

Use a graphing calculator to graph the equation. Use a square viewing window.

4. $y = x + 3$ **5.** $y = -x - 2$ **6.** $y = 2x - 1$

7. $y = -2x + 1$ **8.** $y = -\frac{1}{3}x - 4$ **9.** $y = \frac{1}{2}x + 2$

10. How does the appearance of the slope of a line change between a standard viewing window and a square viewing window?

3.1 Functions

Essential Question

What is a function?

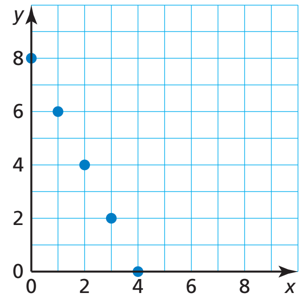
TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.12.A

A **relation** pairs inputs with outputs. When a relation is given as ordered pairs, the x-coordinates are inputs and the y-coordinates are outputs. A relation that pairs each input with *exactly one* output is a **function**.

EXPLORATION 1 Describing a Function

Work with a partner. Functions can be described in many ways.

- by an equation
- by an input-output table
- using words
- by a graph
- as a set of ordered pairs



a. Explain why the graph shown represents a function.

b. Describe the function in two other ways.

ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to analyze relationships mathematically to draw conclusions.

EXPLORATION 2 Identifying Functions

Work with a partner. Determine whether each relation represents a function. Explain your reasoning.

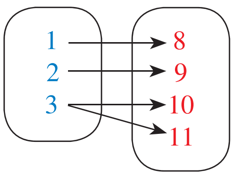
a.

Input, x	0	1	2	3	4
Output, y	8	8	8	8	8

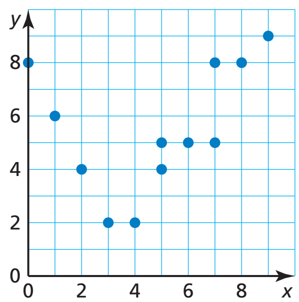
b.

Input, x	8	8	8	8	8
Output, y	0	1	2	3	4

c. **Input, x** **Output, y**



d.



e. $(-2, 5), (-1, 8), (0, 6), (1, 6), (2, 7)$

f. $(-2, 0), (-1, 0), (-1, 1), (0, 1), (1, 2), (2, 2)$

g. Each radio frequency x in a listening area has exactly one radio station y.

h. The same television station x can be found on more than one channel y.

i. $x = 2$

j. $y = 2x + 3$

Communicate Your Answer

3. What is a function? Give examples of relations, other than those in Explorations 1 and 2, that (a) are functions and (b) are not functions.

3.1 Lesson

Core Vocabulary

relation, *p. 90*
function, *p. 90*
domain, *p. 92*
range, *p. 92*
independent variable, *p. 93*
dependent variable, *p. 93*

Previous
ordered pair
mapping diagram

What You Will Learn

- Determine whether relations are functions.
- Find the domain and range of a function.
- Identify the independent and dependent variables of functions.

Determining Whether Relations Are Functions

A **relation** pairs inputs with outputs. When a relation is given as ordered pairs, the x-coordinates are inputs and the y-coordinates are outputs. A relation that pairs each input with *exactly one* output is a **function**.

EXAMPLE 1 Determining Whether Relations Are Functions

Determine whether each relation is a function. Explain.

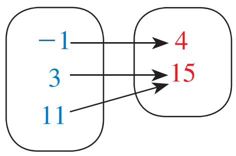
a. $(-2, 2), (-1, 2), (0, 2), (1, 0), (2, 0)$

b. $(4, 0), (8, 7), (6, 4), (4, 3), (5, 2)$

c.

Input, *x*	−2	−1	0	0	1	2
Output, *y*	3	4	5	6	7	8

d. **Input, *x*** **Output, *y***



REMEMBER

A relation can be represented by a mapping diagram.

SOLUTION

a. Every input has exactly one output. So, the relation is a function.

b. The input 4 has two outputs, 0 and 3. So, the relation is *not* a function.

c. The input 0 has two outputs, 5 and 6. So, the relation is *not* a function.

d. Every input has exactly one output. So, the relation is a function.

EXAMPLE 2 Determining Whether Relations Are Functions

Determine whether each relation is a function. Explain.

a. $y = 5x$ with inputs $x = 1$, $x = 2$, and $x = 3$

b. $x = y^2$ with inputs $x = 0$ and $x = 1$

SOLUTION

a. The input 1 has exactly one output, 5. The input 2 has exactly one output, 10. The input 3 has exactly one output, 15.

- So, the relation is a function.

b. The input 0 has exactly one output, 0, but the input 1 has two outputs, 1 and -1.

- So, the relation is *not* a function.

Monitoring Progress

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Determine whether the relation is a function. Explain.

1. $(-5, 0), (0, 0), (5, 0), (5, 10)$

2. $(-4, 8), (-1, 2), (2, -4), (5, -10)$

3.

Input, x	2	4	6
Output, y	2.6	5.2	7.8

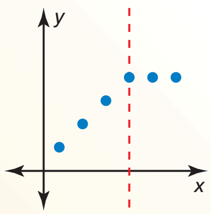
4. $y = 2x + 4$ with inputs $x = 2$ and $x = 4$

Core Concept

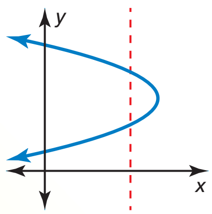
Vertical Line Test

Words A graph represents a function when no vertical line passes through more than one point on the graph.

Examples Function



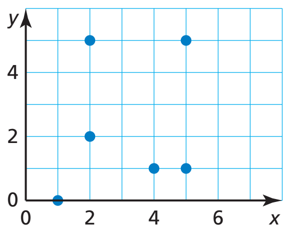
Not a function



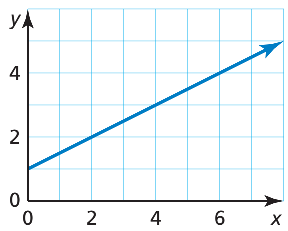
EXAMPLE 3 Using the Vertical Line Test

Determine whether each graph represents a function. Explain.

a.



b.



SOLUTION

a. You can draw a vertical line through (2, 2) and (2, 5).

▶ So, the graph does *not* represent a function.

b. No vertical line can be drawn through more than one point on the graph.

▶ So, the graph represents a function.

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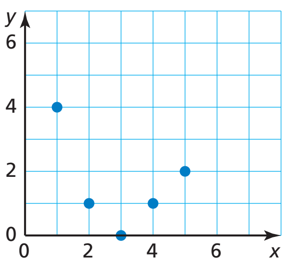
Determine whether the graph represents a function. Explain.

5.



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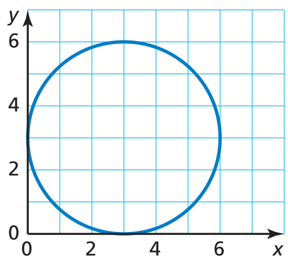


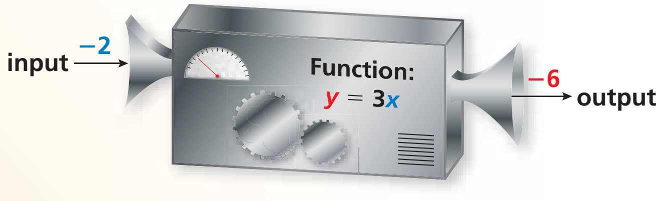
Finding the Domain and Range of a Function

Core Concept

The Domain and Range of a Function

The **domain** of a function is the set of all possible input values.

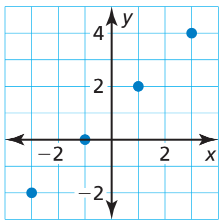
The **range** of a function is the set of all possible output values.



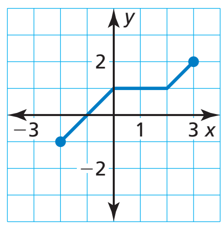
EXAMPLE 4 Finding the Domain and Range from a Graph

Find the domain and range of the function represented by the graph.

a.

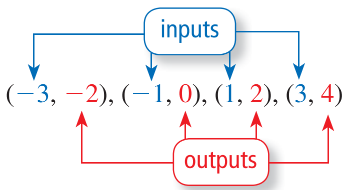


b.



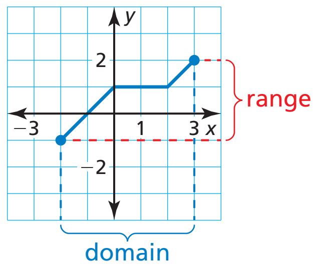
SOLUTION

a. Write the ordered pairs. Identify the inputs and outputs.



▶ The domain is −3, −1, 1, and 3.
The range is −2, 0, 2, and 4.

b. Identify the *x*- and *y*-values represented by the graph.



▶ The domain is $-2 \le x \le 3$.
The range is $-1 \le y \le 2$.

STUDY TIP

A relation also has a domain and a range.

Monitoring Progress

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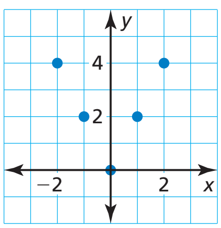
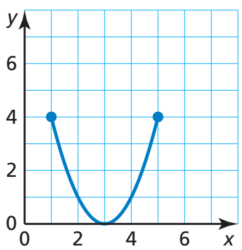
Find the domain and range of the function represented by the graph.

7.



8.



Identifying Independent and Dependent Variables

The variable that represents the input values of a function is the **independent variable** because it can be *any* value in the domain. The variable that represents the output values of a function is the **dependent variable** because it *depends* on the value of the independent variable. When an equation represents a function, the dependent variable is defined in terms of the independent variable. The statement "y is a function of x" means that y varies depending on the value of x.

$$y = -x + 10$$

dependent variable, y | independent variable, x

EXAMPLE 5 Identifying Independent and Dependent Variables

The function $y = -3x + 12$ represents the amount y (in fluid ounces) of juice remaining in a bottle after you take x gulps.

a. Identify the independent and dependent variables.

b. The domain is 0, 1, 2, 3, and 4. What is the range?

SOLUTION

a. The amount y of juice remaining depends on the number x of gulps.

► So, y is the dependent variable, and x is the independent variable.

b. Make an input-output table to find the range.

Input, x	$-3x + 12$	Output, y
0	$-3(0) + 12$	12
1	$-3(1) + 12$	9
2	$-3(2) + 12$	6
3	$-3(3) + 12$	3
4	$-3(4) + 12$	0

► The range is 12, 9, 6, 3, and 0.

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9. The function $a = -4b + 14$ represents the number a of avocados you have left after making b batches of guacamole.

a. Identify the independent and dependent variables.

b. The domain is 0, 1, 2, and 3. What is the range?

10. The function $t = 19m + 65$ represents the temperature t (in degrees Fahrenheit) of an oven after preheating for m minutes.

a. Identify the independent and dependent variables.

b. A recipe calls for an oven temperature of 350°F. Describe the domain and range of the function.

3.1 Exercises

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Vocabulary and Core Concept Check

1. **WRITING** How are independent variables and dependent variables different?

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

x	-1	0	1
y	7	5	-1

- Find the range of the function represented by the table.
- Find the inputs of the function represented by the table.
- Find the x-values of the function represented by $(-1, 7)$, $(0, 5)$, and $(1, -1)$.
- Find the domain of the function represented by $(-1, 7)$, $(0, 5)$, and $(1, -1)$.

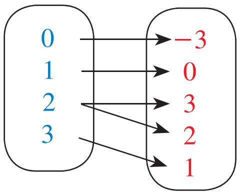
Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, determine whether the relation is a function. Explain. *(See Example 1.)*

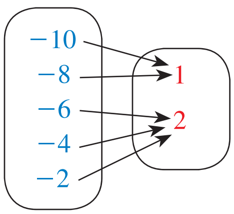
3. $(1, -2)$, $(2, 1)$, $(3, 6)$, $(4, 13)$, $(5, 22)$

4. $(7, 4)$, $(5, -1)$, $(3, -8)$, $(1, -5)$, $(3, 6)$

5. Input, x Output, y



6. Input, x Output, y



7.

Input, x	16	1	0	1	16
Output, y	-2	-1	0	1	2

8.

Input, x	-3	0	3	6	9
Output, y	11	5	-1	-7	-13

In Exercises 9–12, determine whether the relation is a function. Explain. *(See Example 2.)*

9. $y = -3x$ with inputs $x = 0$, $x = 1$, and $x = 2$

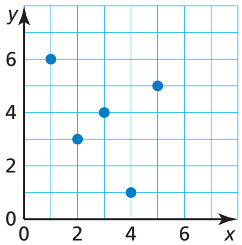
10. $y = 4x - 1$ with inputs $x = -3$, $x = -2$, and $x = -1$

11. $x = y^2$ with inputs $x = 4$ and $x = 9$

12. $x = |y|$ with inputs $x = 1$, $x = 2$, and $x = 3$

In Exercises 13–16, determine whether the graph represents a function. Explain. *(See Example 3.)*

13.



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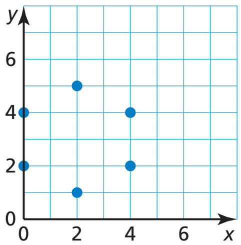

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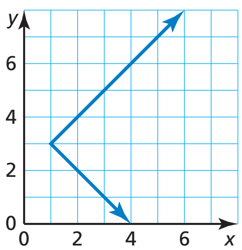

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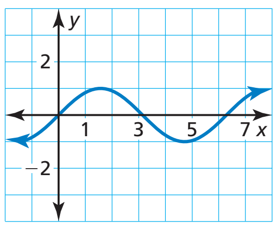
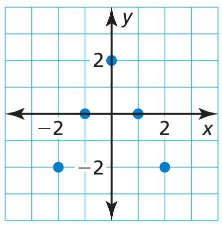


In Exercises 17–20, find the domain and range of the function represented by the graph. *(See Example 4.)*

17.



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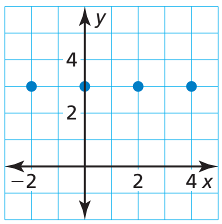

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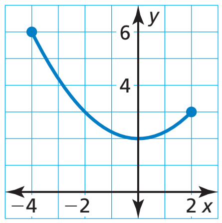

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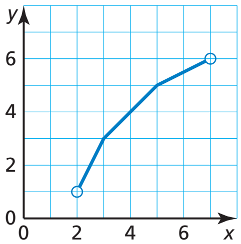


21. **MODELING WITH MATHEMATICS** The function $y = 25x + 500$ represents your monthly rent y (in dollars) when you pay x days late. *(See Example 5.)*

  a. Identify the independent and dependent variables.

  b. The domain is 0, 1, 2, 3, 4, and 5. What is the range?

22. **MODELING WITH MATHEMATICS** The function $y = 3.5x + 2.8$ represents the cost y (in dollars) of a taxi ride of x miles.

  a. Identify the independent and dependent variables.

  b. You have enough money to travel at most 20 miles in the taxi. Find the domain and range of the function.

ERROR ANALYSIS **In Exercises 23 and 24, describe and correct the error in the statement about the relation shown in the table.**

Input, *x*	1	2	3	4	5
Output, *y*	6	7	8	6	9

23. The relation is *not* a function. One output is paired with two inputs.

24. The relation is a function. The range is 1, 2, 3, 4, and 5.

ANALYZING RELATIONSHIPS **In Exercises 25–28, identify the independent and dependent variables.**

25. The length of your hair depends on the amount of time since your last haircut.

26. A baseball team's rank depends on the number of games the team wins.

27. The number of quarters you put into a parking meter affects the amount of time you have on the meter.

28. The battery power remaining on your MP3 player is based on the amount of time you listen to it.

29. **MULTIPLE REPRESENTATIONS** The balance y (in dollars) of your savings account is a function of the month x.

Month, *x*	0	1	2	3	4
Balance (dollars), *y*	100	125	150	175	200

  a. Describe this situation in words.

  b. Write the function as a set of ordered pairs.

  c. Plot the ordered pairs in a coordinate plane.

30. **MULTIPLE REPRESENTATIONS** The function $1.5x + 0.5y = 12$ represents the number of hardcover books x and softcover books y you can buy at a used book sale.

  a. Solve the equation for y.

  b. Make an input-output table to find ordered pairs for the function.

  c. Plot the ordered pairs in a coordinate plane.

31. **OPEN-ENDED** Graph a relation that fails the Vertical Line Test at exactly one value of x.

32. **OPEN-ENDED** List the inputs and outputs of a relation such that the relation is a function, but when the inputs and outputs are switched, the relation is not a function.

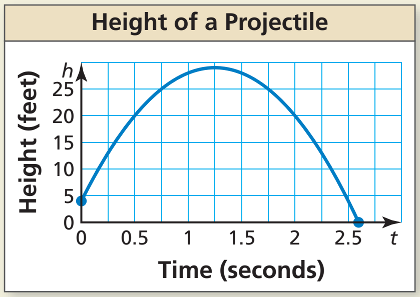
33. **ANALYZING RELATIONSHIPS** You select items in a vending machine by pressing one letter and then one number.

  a. Explain why the relation that pairs letter-number combinations with food or drink items is a function.

  b. Identify the independent and dependent variables.

  c. Find the domain and range of the function.

34. **HOW DO YOU SEE IT?** The graph represents the height h of a projectile after t seconds.



a. Explain why h is a function of t.

b. Approximate the height of the projectile after 0.5 second and after 1.25 seconds.

c. Approximate the domain of the function.

d. Is t a function of h? Explain.

35. **MAKING AN ARGUMENT** Your friend says that a line always represents a function. Is your friend correct? Explain.

36. **THOUGHT PROVOKING** Write a function in which the inputs and/or the outputs are not numbers. Identify the independent and dependent variables. Then find the domain and range of the function.

ATTENDING TO PRECISION In Exercises 37–40, determine whether the statement uses the word *function* in a way that is mathematically correct. Explain your reasoning.

37. The selling price of an item is a function of the cost of making the item.

38. The sales tax on a purchased item in a given state is a function of the selling price.

39. A function pairs each student in your school with a homeroom teacher.

40. A function pairs each chaperone on a school trip with 10 students.

REASONING In Exercises 41–44, tell whether the statement is true or false. If it is false, explain why.

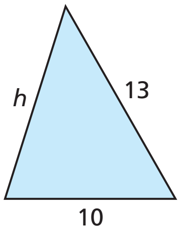
41. Every function is a relation.

42. Every relation is a function.

43. When you switch the inputs and outputs of any function, the resulting relation is a function.

44. When the domain of a function has an infinite number of values, the range always has an infinite number of values.

45. **MATHEMATICAL CONNECTIONS** Consider the triangle shown.



a. Write a function that represents the perimeter of the triangle.

b. Identify the independent and dependent variables.

c. Describe the domain and range of the function. (*Hint:* The sum of the lengths of any two sides of a triangle is greater than the length of the remaining side.)

REASONING In Exercises 46–49, find the domain and range of the function.

46. $y = |x|$

47. $y = -|x|$

48. $y = |x| - 6$

49. $y = 4 - |x|$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the sentence as an inequality. *(Section 2.1)*

50. A number y is less than 16.

51. Three is no less than a number x.

52. Seven is at most the quotient of a number d and -5.

53. The sum of a number w and 4 is more than -12.

Evaluate the expression. *(Skills Review Handbook)*

54. 11^2

55. $(-3)^4$

56. -5^2

57. 2^5

3.2 Linear Functions

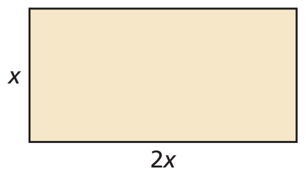
TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.2.A
A.3.C

Essential Question

How can you determine whether a function is linear or nonlinear?

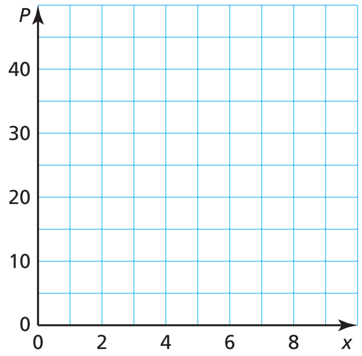
EXPLORATION 1 Finding Patterns for Similar Figures

Work with a partner. Copy and complete each table for the sequence of similar figures. (In parts (a) and (b), use the rectangle shown.) Graph the data in each table. Decide whether each pattern is linear or nonlinear. Justify your conclusion.



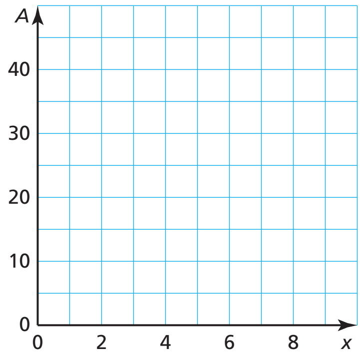
a. perimeters of similar rectangles

x	1	2	3	4	5
P					



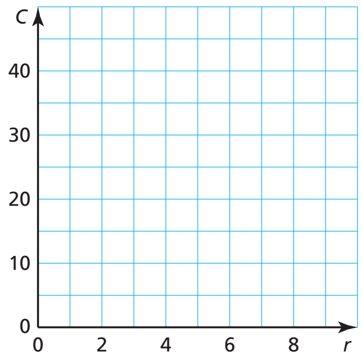
b. areas of similar rectangles

x	1	2	3	4	5
A					



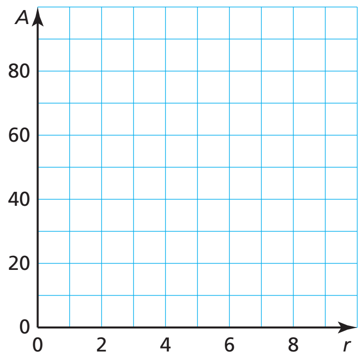
c. circumferences of circles of radius r

r	1	2	3	4	5
C					



d. areas of circles of radius r

r	1	2	3	4	5
A					



SELECTING TOOLS
To be proficient in math, you need to identify relationships using tools, such as tables and graphs.

Communicate Your Answer

2. How do you know that the patterns you found in Exploration 1 represent functions?

3. How can you determine whether a function is linear or nonlinear?

4. Describe two real-life patterns: one that is linear and one that is nonlinear. Use patterns that are different from those described in Exploration 1.

3.2 Lesson

Core Vocabulary

linear equation in two variables, *p. 98*
linear function, *p. 98*
nonlinear function, *p. 98*
solution of a linear equation in two variables, *p. 100*
discrete domain, *p. 100*
continuous domain, *p. 100*

Previous
whole number

What You Will Learn

- Identify linear functions using graphs, tables, and equations.
- Graph linear functions using discrete and continuous data.
- Write real-life problems to fit data.

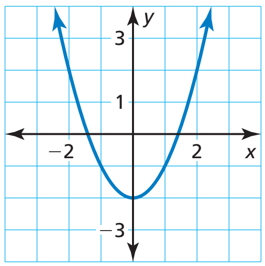
Identifying Linear Functions

A **linear equation in two variables**, x and y, is an equation that can be written in the form $y = mx + b$, where m and b are constants. The graph of a linear equation is a line. Likewise, a **linear function** is a function whose graph is a nonvertical line. A linear function has a constant rate of change and can be represented by a linear equation in two variables. A **nonlinear function** does not have a constant rate of change. So, its graph is *not* a line.

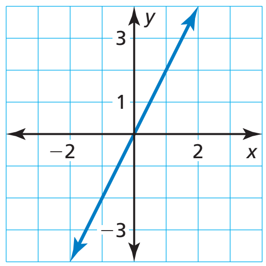
EXAMPLE 1 Identifying Linear Functions Using Graphs

Does the graph represent a *linear* or *nonlinear* function? Explain.

a.



b.



SOLUTION

a. The graph is *not* a line.

- So, the function is nonlinear.

b. The graph is a line.

- So, the function is linear.

EXAMPLE 2 Identifying Linear Functions Using Tables

Does the table represent a *linear* or *nonlinear* function? Explain.

a.

x	3	6	9	12
y	36	30	24	18

b.

x	1	3	5	7
y	2	9	20	35

SOLUTION

a.

+3 +3 +3

x	3	6	9	12
y	36	30	24	18

−6 −6 −6

As x increases by 3, y decreases by 6. The rate of change is constant.

- So, the function is linear.

b.

+2 +2 +2

x	1	3	5	7
y	2	9	20	35

+7 +11 +15

As x increases by 2, y increases by different amounts. The rate of change is *not* constant.

- So, the function is nonlinear.

REMEMBER

A constant rate of change describes a quantity that changes by equal amounts over equal intervals.

Monitoring Progress

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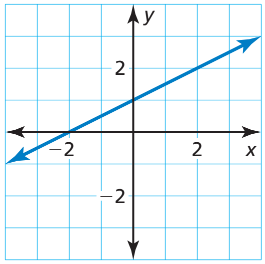
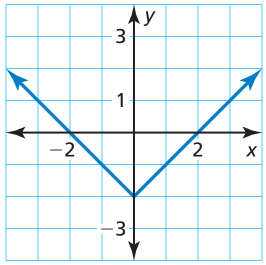
Does the graph or table represent a *linear* or *nonlinear* function? Explain.

1.



2.



3.

x	0	1	2	3
y	3	5	7	9

4.

x	1	2	3	4
y	16	8	4	2

EXAMPLE 3 Identifying Linear Functions Using Equations

Which of the following equations represent linear functions? Explain.

$$y = 3.8,\ y = \sqrt{x},\ y = 3^x,\ y = \frac{2}{x},\ y = 6(x - 1), \text{ and } x^2 - y = 0$$

SOLUTION

You cannot rewrite the equations $y = \sqrt{x}$, $y = 3^x$, $y = \frac{2}{x}$, and $x^2 - y = 0$ in the form $y = mx + b$. So, these equations cannot represent linear functions.

▶ You can rewrite the equation $y = 3.8$ as $y = 0x + 3.8$ and the equation $y = 6(x - 1)$ as $y = 6x - 6$. So, they represent linear functions.

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Does the equation represent a *linear* or *nonlinear* function? Explain.

5. $y = x + 9$

6. $y = \frac{3x}{5}$

7. $y = 5 - 2x^2$

Concept Summary

Representations of Functions

Words An output is 3 more than the input.

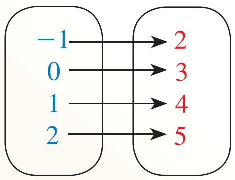
Equation $y = x + 3$

Input-Output Table

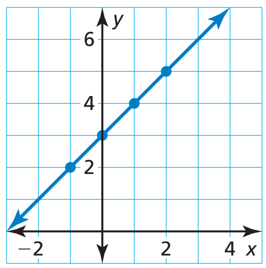
Input, x	Output, y
−1	2
0	3
1	4
2	5

Mapping Diagram

Input, x Output, y



Graph



Graphing Linear Functions

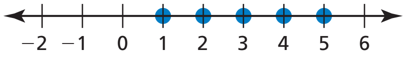
A **solution of a linear equation in two variables** is an ordered pair (x, y) that makes the equation true. The graph of a linear equation in two variables is the set of points (x, y) in a coordinate plane that represents all solutions of the equation. Sometimes the points are distinct, and other times the points are connected.

Core Concept

Discrete and Continuous Domains

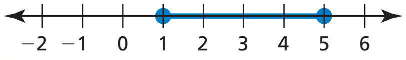
A **discrete domain** is a set of input values that consists of only certain numbers in an interval.

Example: Integers from 1 to 5



A **continuous domain** is a set of input values that consists of all numbers in an interval.

Example: All numbers from 1 to 5



EXAMPLE 4 Graphing Discrete Data

The linear function $y = 15.95x$ represents the cost y (in dollars) of x tickets for a museum. Each customer can buy a maximum of three tickets. (a) Find the domain of the function. Is the domain discrete or continuous? Explain. (b) Graph the function using its domain. (c) Find the range of the function.

STUDY TIP

The domain of a function depends on the real-life context of the function, not just the equation that represents the function.

SOLUTION

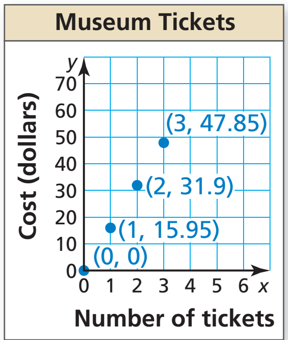
a. You cannot buy part of a ticket, only a certain number of tickets. Because x represents the number of tickets, it must be a whole number. The maximum number of tickets a customer can buy is three.

▶ So, the domain is 0, 1, 2, and 3, and it is discrete.

b. Step 1 Make an input-output table to find the ordered pairs.

Input, x	$15.95x$	Output, y	(x, y)
0	15.95(0)	0	(0, 0)
1	15.95(1)	15.95	(1, 15.95)
2	15.95(2)	31.9	(2, 31.9)
3	15.95(3)	47.85	(3, 47.85)

Step 2 Plot the ordered pairs. The domain is discrete. So, the graph consists of individual points.



c. Use the input-output table to find the range.

▶ The range is 0, 15.95, 31.9, and 47.85.

Monitoring Progress

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8. The linear function $m = 50 - 9d$ represents the amount m (in dollars) of money you have after buying d DVDs. (a) Find the domain of the function. Is the domain discrete or continuous? Explain. (b) Graph the function using its domain. (c) Find the range of the function.

EXAMPLE 5 Graphing Continuous Data

A cereal bar contains 130 calories. The number c of calories consumed is a function of the number b of bars eaten.

a. Does this situation represent a linear function? Explain.

b. Find the domain of the function. Is the domain discrete or continuous? Explain.

c. Graph the function using its domain.

d. Find the range of the function.

STUDY TIP

When the domain of a linear function is not specified or cannot be obtained from a real-life context, it is understood to be all real numbers.

SOLUTION

a. As b increases by 1, c increases by 130. The rate of change is constant.

▶ So, this situation represents a linear function.

b. You can eat part of a cereal bar. The number b of bars eaten can be any value greater than or equal to 0.

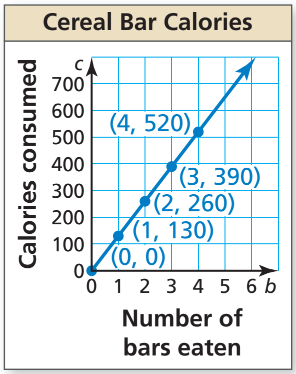
▶ So, the domain is $b \geq 0$, and it is continuous.

c. Step 1 Make an input-output table to find ordered pairs.

Input, b	Output, c	(b, c)
0	0	(0, 0)
1	130	(1, 130)
2	260	(2, 260)
3	390	(3, 390)
4	520	(4, 520)

Step 2 Plot the ordered pairs.

Step 3 Draw a line through the points. The line should start at (0, 0) and continue to the right. Use an arrow to indicate that the line continues without end, as shown. The domain is continuous. So, the graph is a line with a domain of $b \geq 0$.



d. From the graph or table, you can see that the output values are greater than or equal to 0.

▶ So, the range is $c \geq 0$.

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9. Is the domain discrete or continuous? Explain.

Input **Number of stories, x**	1	2	3
Output **Height of building (feet), y**	12	24	36

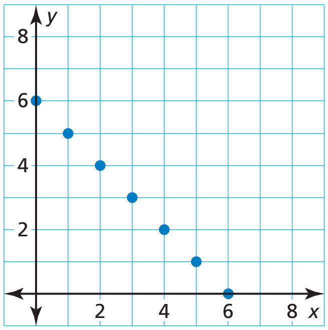
10. A 20-gallon bathtub is draining at a rate of 2.5 gallons per minute. The number g of gallons remaining is a function of the number m of minutes. (a) Does this situation represent a linear function? Explain. (b) Find the domain of the function. Is the domain discrete or continuous? Explain. (c) Graph the function using its domain. (d) Find the range of the function.

Writing Real-Life Problems

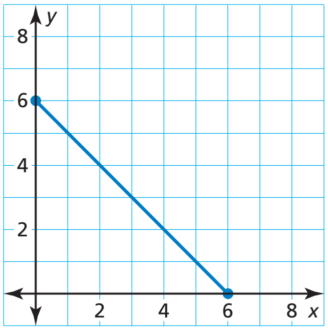
EXAMPLE 6 Writing Real-Life Problems

Write a real-life problem to fit the data shown in each graph. Is the domain of each function *discrete* or *continuous*? Explain.

a.



b.



SOLUTION

a. You want to think of a real-life situation in which there are two variables, x and y. Using the graph, notice that the sum of the variables is always 6, and the value of each variable must be a whole number from 0 to 6.

x	0	1	2	3	4	5	6
y	6	5	4	3	2	1	0

Discrete domain

▶ One possibility is two people bidding against each other on six coins at an auction. Each coin will be purchased by one of the two people. Because it is not possible to purchase part of a coin, the domain is discrete.

b. You want to think of a real-life situation in which there are two variables, x and y. Using the graph, notice that the sum of the variables is always 6, and the value of each variable can be any real number from 0 to 6.

$$x + y = 6 \quad \text{or} \quad y = -x + 6$$

Continuous domain

▶ One possibility is two people bidding against each other on 6 ounces of gold dust at an auction. All the dust will be purchased by the two people. Because it is possible to purchase any portion of the dust, the domain is continuous.

Monitoring Progress

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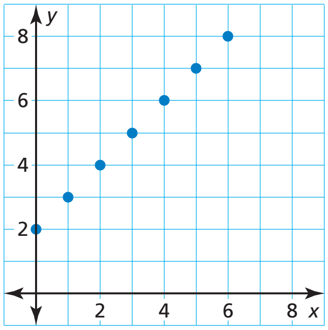
Write a real-life problem to fit the data shown in the graph. Is the domain of the function *discrete* or *continuous*? Explain.

11.



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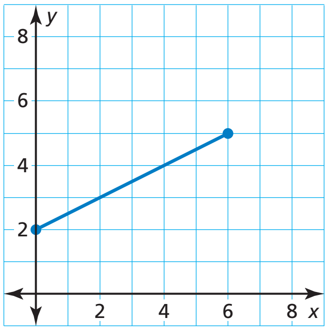


Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A linear equation in two variables is an equation that can be written in the form ________, where m and b are constants.

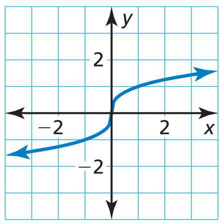
2. **VOCABULARY** Compare linear functions and nonlinear functions.

3. **VOCABULARY** Compare discrete domains and continuous domains.

4. **WRITING** How can you tell whether a graph shows a discrete domain or a continuous domain?

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, determine whether the graph represents a *linear* or *nonlinear* function. Explain. *(See Example 1.)*

5.


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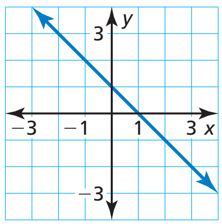

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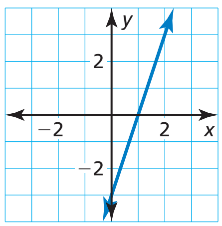

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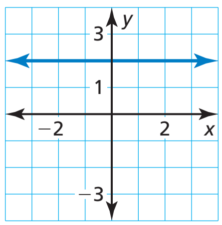

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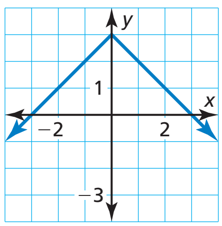

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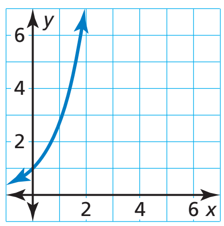


In Exercises 11–14, determine whether the table represents a *linear* or *nonlinear* function. Explain. *(See Example 2.)*

11.

x	1	2	3	4
y	5	10	15	20

12.

x	5	7	9	11
y	-9	-3	-1	3

13.

x	4	8	12	16
y	16	12	7	1

14.

x	-1	0	1	2
y	35	20	5	-10

ERROR ANALYSIS **In Exercises 15 and 16, describe and correct the error in determining whether the table or graph represents a linear function.**

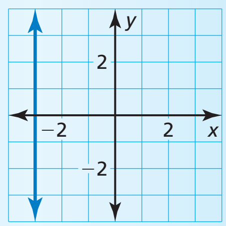
15.

$+2$ $+2$ $+2$

x	2	4	6	8
y	4	16	64	256

$\times 4$ $\times 4$ $\times 4$

As x increases by 2, y increases by a constant factor of 4. So, the function is linear.

16.


The graph is a line. So, the graph represents a linear function.

In Exercises 17–24, determine whether the equation represents a *linear* or *nonlinear* function. Explain. *(See Example 3.)*

17. $y = x^2 + 13$

18. $y = 7 - 3x$

19. $y = \sqrt[3]{8} - x$

20. $y = 4x(8 - x)$

21. $2 + \frac{1}{6}y = 3x + 4$

22. $y - x = 2x - \frac{2}{3}y$

23. $18x - 2y = 26$

24. $2x + 3y = 9xy$

25. CLASSIFYING FUNCTIONS Which of the following equations *do not* represent linear functions? Explain.

Ⓐ $12 = 2x^2 + 4y^2$

Ⓑ $y - x + 3 = x$

Ⓒ $x = 8$

Ⓓ $x = 9 - \frac{3}{4}y$

Ⓔ $y = \frac{5x}{11}$

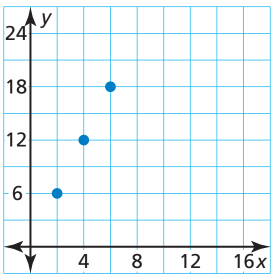
Ⓕ $y = \sqrt{x} + 3$

26. USING STRUCTURE Fill in the table so it represents a linear function.

x	5	10	15	20	25
y	−1				11

In Exercises 27 and 28, find the domain of the function represented by the graph. Determine whether the domain is *discrete* or *continuous*. Explain.

27.



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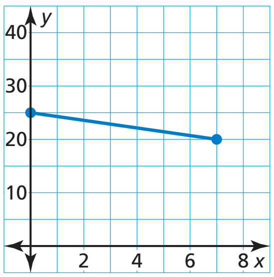


In Exercises 29–32, determine whether the domain is *discrete* or *continuous*. Explain.

29.

Input **Bags, *x***	2	4	6
Output **Marbles, *y***	20	40	60

30.

Input **Years, *x***	1	2	3
Output **Height of tree (feet), *y***	6	9	12

31.

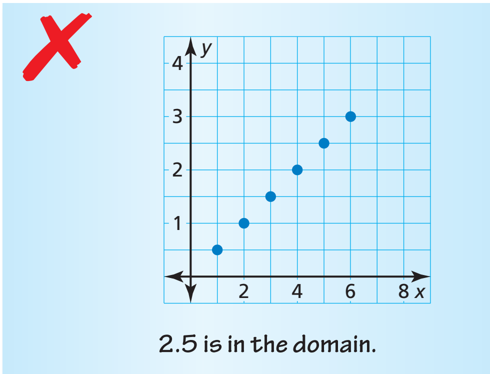
Input **Time (hours), *x***	3	6	9
Output **Distance (miles), *y***	150	300	450

32.

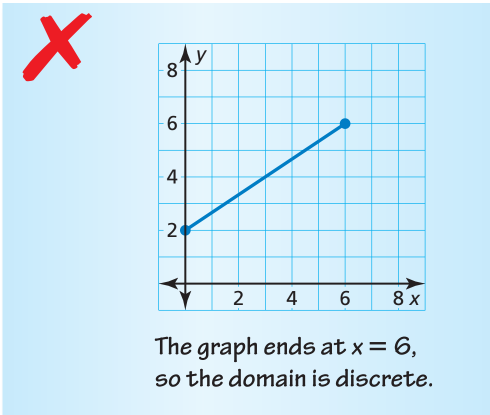
Input **Relay teams, *x***	0	1	2
Output **Athletes, *y***	0	4	8

ERROR ANALYSIS In Exercises 33 and 34, describe and correct the error in the statement about the domain.

33.



34.

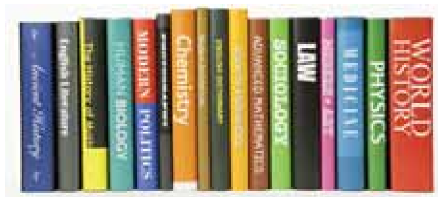


35. MODELING WITH MATHEMATICS The linear function $m = 55 - 8.5b$ represents the amount m (in dollars) of money that you have after buying b books. *(See Example 4.)*

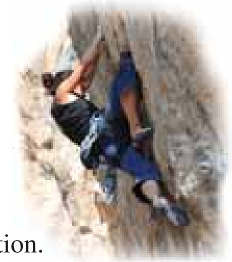
a. Find the domain of the function. Is the domain discrete or continuous? Explain.

b. Graph the function using its domain.

c. Find the range of the function.



36. **MODELING WITH MATHEMATICS** The number y of calories burned after x hours of rock climbing is represented by the linear function $y = 650x$.



a. Find the domain of the function. Is the domain discrete or continuous? Explain.

b. Graph the function using its domain.

c. Find the range of the function.

37. **MODELING WITH MATHEMATICS** You are researching the speed of sound waves in dry air at 86°F. The table shows the distances d (in miles) sound waves travel in t seconds. *(See Example 5.)*

Time (seconds), t	Distance (miles), d
2	0.434
4	0.868
6	1.302
8	1.736
10	2.170

a. Does this situation represent a linear function? Explain.

b. Find the domain of the function. Is the domain discrete or continuous? Explain.

c. Graph the function using its domain.

d. Find the range of the function.

38. **MODELING WITH MATHEMATICS** The function $y = 30 + 5x$ represents the cost y (in dollars) of having your dog groomed and buying x extra services.

a. Does this situation represent a linear function? Explain.

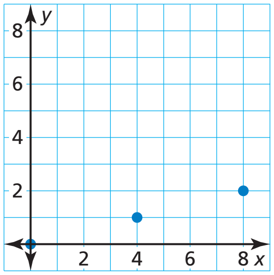
b. Find the domain of the function. Is the domain discrete or continuous? Explain.

c. Graph the function using its domain.

d. Find the range of the function.

WRITING In Exercises 39–42, write a real-life problem to fit the data shown in the graph. Determine whether the domain of the function is *discrete* or *continuous*. Explain. *(See Example 6.)*

39.



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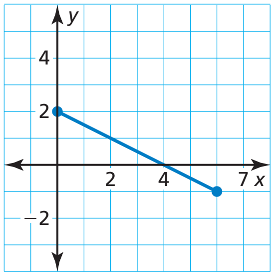

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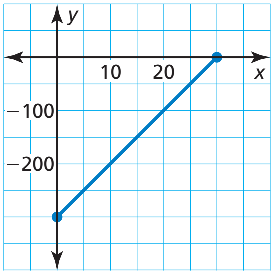

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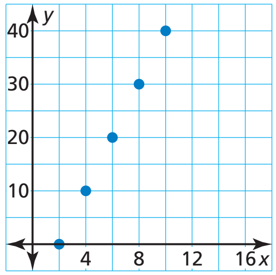


43. **USING STRUCTURE** The table shows your earnings y (in dollars) for working x hours.

a. What is the missing y-value that makes the table represent a linear function?

b. What is your hourly pay rate?

Time (hours), x	Earnings (dollars), y
4	40.80
5	
6	61.20
7	71.40

44. **MAKING AN ARGUMENT** The linear function $d = 50t$ represents the distance d (in miles) Car A is from a car rental store after t hours. The table shows the distances Car B is from the rental store.

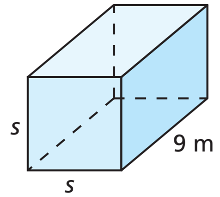
Time (hours), t	Distance (miles), d
1	60
3	180
5	310

a. Does the table represent a linear or nonlinear function? Explain.

b. Your friend claims Car B is moving at a faster rate. Is your friend correct? Explain.

MATHEMATICAL CONNECTIONS In Exercises 45–48, tell whether the volume of the solid is a linear or nonlinear function of the missing dimension(s). Explain.

45.



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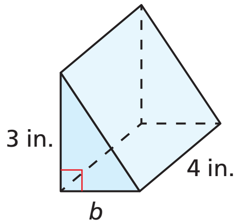

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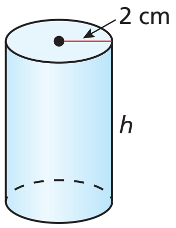

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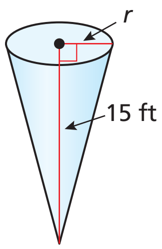


49. REASONING A water company fills two different-sized jugs. The first jug can hold x gallons of water. The second jug can hold y gallons of water. The company fills A jugs of the first size and B jugs of the second size. What does each expression represent? Does each expression represent a set of discrete or continuous values?

a. $x + y$

b. $A + B$

c. Ax

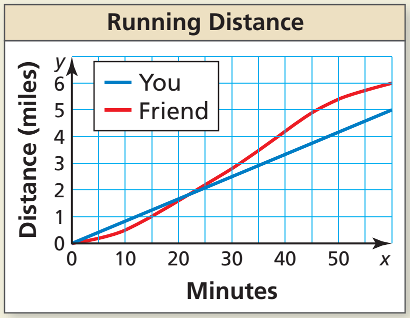
d. $Ax + By$

50. THOUGHT PROVOKING You go to a farmer's market to buy tomatoes. Graph a function that represents the cost of buying tomatoes. Explain your reasoning.

51. CLASSIFYING A FUNCTION Is the function represented by the ordered pairs linear or nonlinear? Explain your reasoning.

(0, 2), (3, 14), (5, 22), (9, 38), (11, 46)

52. HOW DO YOU SEE IT? You and your friend go running. The graph shows the distances you and your friend run.



a. Describe your run and your friend's run. Who runs at a constant rate? How do you know? Why might a person not run at a constant rate?

b. Find the domain of each function. Describe the domains using the context of the problem.

WRITING In Exercises 53 and 54, describe a real-life situation for the constraints.

53. The function has at least one negative number in the domain. The domain is continuous.

54. The function gives at least one negative number as an output. The domain is discrete.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the expression when $x = 2$. *(Skills Review Handbook)*

55. $6x + 8$

56. $10 - 2x + 8$

57. $4(x + 2 - 5x)$

58. $\frac{x}{2} + 5x - 7$

Solve the equation. Check your solution. *(Section 1.2)*

59. $2x + 10 = 16$

60. $8 = 5 - 3p$

61. $\frac{5t + 2}{7} = 6$

62. $12 = y - 7 - 3y$

Solve the inequality. Graph the solution. *(Section 2.3)*

63. $8x \geq 4$

64. $-3t < 24$

65. $\frac{r}{16} \leq -1$

66. $-\frac{1}{3}a > 6$

3.3 Function Notation

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS

A.3.C
A.12.B

Essential Question

How can you use function notation to represent a function?

The notation $f(x)$, called **function notation**, is another name for y. This notation is read as "the value of f at x" or "f of x." The parentheses do not imply multiplication. You can use letters other than f to name a function. The letters g, h, j, and k are often used to name functions.

USING PRECISE MATHEMATICAL LANGUAGE

To be proficient in math, you need to use clear definitions and state the meanings of the symbols you use.

EXPLORATION 1 Matching Functions with Their Graphs

Work with a partner. Match each function with its graph.

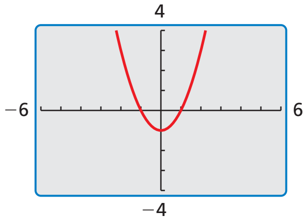
a. $f(x) = 2x - 3$

b. $g(x) = -x + 2$

c. $h(x) = x^2 - 1$

d. $j(x) = 2x^2 - 3$

A.



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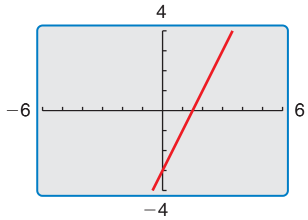

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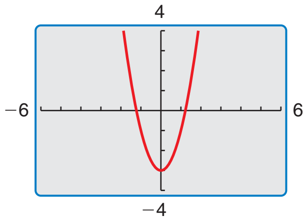
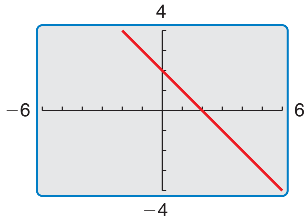


D.

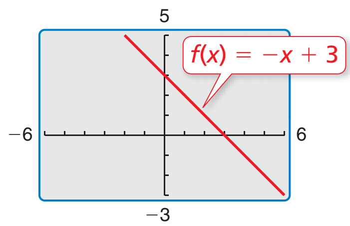


EXPLORATION 2 Evaluating a Function

Work with a partner. Consider the function

$f(x) = -x + 3.$

Locate the points $(x, f(x))$ on the graph. Explain how you found each point.



a. $(-1, f(-1))$

b. $(0, f(0))$

c. $(1, f(1))$

d. $(2, f(2))$

Communicate Your Answer

3. How can you use function notation to represent a function? How are standard notation and function notation similar? How are they different?

Standard Notation	*Function Notation*
$y = 2x + 5$	$f(x) = 2x + 5$

3.3 Lesson

Core Vocabulary

function notation, *p. 108*

Previous
linear function
quadrant

What You Will Learn

- Use function notation to evaluate and interpret functions.
- Use function notation to solve and graph functions.
- Solve real-life problems using function notation.

Using Function Notation to Evaluate and Interpret

You know that a linear function can be written in the form $y = mx + b$. By naming a linear function f, you can also write the function using **function notation**.

$f(x) = mx + b$ Function notation

The notation $f(x)$ is another name for y. If f is a function, and x is in its domain, then $f(x)$ represents the output of f corresponding to the input x. You can use letters other than f to name a function, such as g or h.

READING

The notation $f(x)$ is read as "the value of f at x" or "f of x." It does not mean "f times x."

EXAMPLE 1 Evaluating a Function

Evaluate $f(x) = -4x + 7$ when $x = 2$ and $x = -2$.

SOLUTION

$f(x) = -4x + 7$	Write the function.	$f(x) = -4x + 7$
$f(2) = -4(2) + 7$	Substitute for x.	$f(-2) = -4(-2) + 7$
$= -8 + 7$	Multiply.	$= 8 + 7$
$= -1$	Add.	$= 15$

▶ When $x = 2$, $f(x) = -1$, and when $x = -2$, $f(x) = 15$.

EXAMPLE 2 Interpreting Function Notation

Let $f(t)$ be the outside temperature (°F) t hours after 6 A.M. Explain the meaning of each statement.

a. $f(0) = 58$ **b.** $f(6) = n$ **c.** $f(3) < f(9)$

SOLUTION

a. The initial value of the function is 58. So, the temperature at 6 A.M. is 58°F.

b. The output of f when $t = 6$ is n. So, the temperature at noon (6 hours after 6 A.M.) is n°F.

c. The output of f when $t = 3$ is less than the output of f when $t = 9$. So, the temperature at 9 A.M. (3 hours after 6 A.M.) is less than the temperature at 3 P.M. (9 hours after 6 A.M.).

Monitoring Progress

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Evaluate the function when $x = -4$, 0, and 3.

1. $f(x) = 2x - 5$ **2.** $g(x) = -x - 1$

3. WHAT IF? In Example 2, let $f(t)$ be the outside temperature (°F) t hours after 9 A.M. Explain the meaning of each statement.

a. $f(4) = 75$ **b.** $f(m) = 70$ **c.** $f(2) = f(9)$ d. $f(6) > f(0)$

Using Function Notation to Solve and Graph

EXAMPLE 3 Solving for the Independent Variable

For $h(x) = \frac{2}{3}x - 5$, find the value of x for which $h(x) = -7$.

SOLUTION

$h(x) = \frac{2}{3}x - 5$	Write the function.
$-7 = \frac{2}{3}x - 5$	Substitute –7 for $h(x)$.
$\underline{+5} \qquad \underline{+5}$	Add 5 to each side.
$-2 = \frac{2}{3}x$	Simplify.
$\frac{3}{2} \cdot (-2) = \frac{3}{2} \cdot \frac{2}{3}x$	Multiply each side by $\frac{3}{2}$.
$-3 = x$	Simplify.

▶ When $x = -3$, $h(x) = -7$.

EXAMPLE 4 Graphing a Linear Function in Function Notation

Graph $f(x) = 2x + 5$.

SOLUTION

Step 1 Make an input-output table to find ordered pairs.

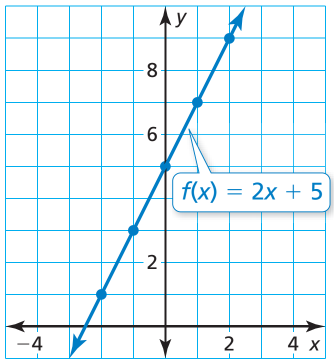
x	-2	-1	0	1	2
$f(x)$	1	3	5	7	9

Step 2 Plot the ordered pairs.

Step 3 Draw a line through the points.

STUDY TIP

The graph of $y = f(x)$ consists of the points $(x, f(x))$.



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Find the value of x so that the function has the given value.

4. $f(x) = 6x + 9$; $f(x) = 21$

5. $g(x) = -\frac{1}{2}x + 3$; $g(x) = -1$

Graph the linear function.

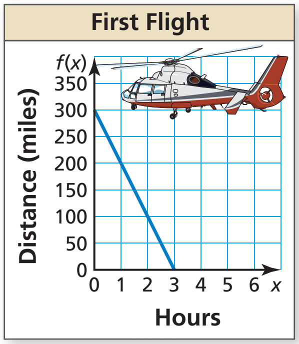
6. $f(x) = 3x - 2$

7. $g(x) = -x + 4$

8. $h(x) = -\frac{3}{4}x - 1$

Solving Real-Life Problems

EXAMPLE 5 Modeling with Mathematics



The graph shows the number of miles a helicopter is from its destination after x hours on its first flight. On its second flight, the helicopter travels 50 miles farther and increases its speed by 25 miles per hour. The function $f(x) = 350 - 125x$ represents the second flight, where $f(x)$ is the number of miles the helicopter is from its destination after x hours. Which flight takes less time? Explain.

SOLUTION

1. **Understand the Problem** You are given a graph of the first flight and an equation of the second flight. You are asked to compare the flight times to determine which flight takes less time.

2. **Make a Plan** Graph the function that represents the second flight. Compare the graph to the graph of the first flight. The x-value that corresponds to $f(x) = 0$ represents the flight time.

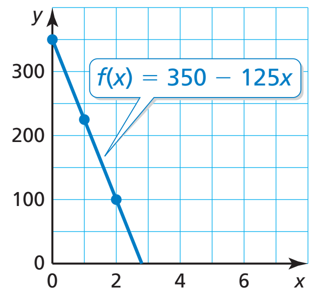
3. **Solve the Problem** Graph $f(x) = 350 - 125x$.

Step 1 Make an input-output table to find the ordered pairs.

x	0	1	2	3
$f(x)$	350	225	100	-25

Step 2 Plot the ordered pairs.

Step 3 Draw a line through the points. Note that the function only makes sense when x and $f(x)$ are positive. So, only draw the line in the first quadrant.



From the graph of the first flight, you can see that when $f(x) = 0$, $x = 3$. From the graph of the second flight, you can see that when $f(x) = 0$, x is slightly less than 3. So, the second flight takes less time.

4. **Look Back** You can check that your answer is correct by finding the value of x for which $f(x) = 0$.

$f(x) = 350 - 125x$ Write the function.

$0 = 350 - 125x$ Substitute 0 for $f(x)$.

$-350 = -125x$ Subtract 350 from each side.

$2.8 = x$ Divide each side by -125.

So, the second flight takes 2.8 hours, which is less than 3.

Monitoring Progress

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9. **WHAT IF?** Let $f(x) = 250 - 75x$ represent the second flight, where $f(x)$ is the number of miles the helicopter is from its destination after x hours. Which flight takes less time? Explain.

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** When you write the function $y = 2x + 10$ as $f(x) = 2x + 10$, you are using ______________.

2. **REASONING** Your height can be represented by a function h, where the input is your age. What does $h(14)$ represent?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, evaluate the function when $x = -2$, 0, and 5. *(See Example 1.)*

3. $f(x) = x + 6$
4. $g(x) = 3x$
5. $h(x) = -2x + 9$
6. $r(x) = -x - 7$
7. $p(x) = -3 + 4x$
8. $b(x) = 18 - 0.5x$
9. $v(x) = 12 - 2x - 5$
10. $n(x) = -1 - x + 4$

11. **INTERPRETING FUNCTION NOTATION** Let $c(t)$ be the number of customers in a restaurant t hours after 8 A.M. Explain the meaning of each statement. *(See Example 2.)*

    a. $c(0) = 0$ **b.** $c(3) = c(8)$

    c. $c(n) = 29$ **d.** $c(13) < c(12)$

12. **INTERPRETING FUNCTION NOTATION** Let $H(x)$ be the percent of U.S. households with Internet use x years after 1980. Explain the meaning of each statement.

    a. $H(23) = 55$ **b.** $H(4) = k$

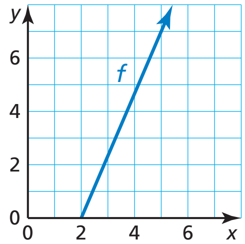
    c. $H(27) \geq 61$

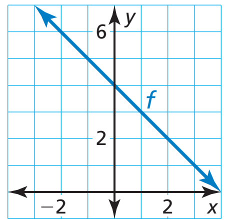
    d. $H(17) + H(21) \approx H(29)$

In Exercises 13–18, find the value of x so that the function has the given value. *(See Example 3.)*

13. $h(x) = -7x$; $h(x) = 63$
14. $t(x) = 3x$; $t(x) = 24$
15. $m(x) = 4x + 15$; $m(x) = 7$
16. $k(x) = 6x - 12$; $k(x) = 18$
17. $q(x) = \frac{1}{2}x - 3$; $q(x) = -4$
18. $j(x) = -\frac{4}{5}x + 7$; $j(x) = -5$

In Exercises 19 and 20, find the value of x so that $f(x) = 7$.

19. 

20. 

21. **MODELING WITH MATHEMATICS** The function $C(x) = 17.5x - 10$ represents the cost (in dollars) of buying x tickets to the orchestra with a \$10 coupon.

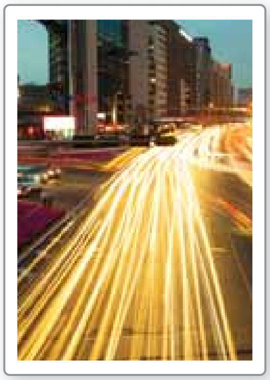
    a. How much does it cost to buy five tickets?

    b. How many tickets can you buy with \$130?

22. **MODELING WITH MATHEMATICS** The function $d(t) = 300{,}000t$ represents the distance (in kilometers) that light travels in t seconds.

    a. How far does light travel in 15 seconds?

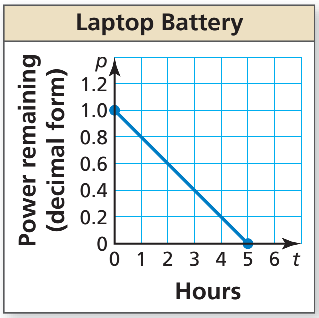
    b. How long does it take light to travel 12 million kilometers?



In Exercises 23–28, graph the linear function. *(See Example 4.)*

23. $p(x) = 4x$
24. $h(x) = -5$
25. $d(x) = -\frac{1}{2}x - 3$
26. $w(x) = \frac{3}{5}x + 2$
27. $g(x) = -4 + 7x$
28. $f(x) = 3 - 6x$

29. **PROBLEM SOLVING** The graph shows the percent p (in decimal form) of battery power remaining in a laptop computer after t hours of use. A tablet computer initially has 75% of its battery power remaining and loses 12.5% per hour. Which computer's battery will last longer? Explain. *(See Example 5.)*



30. **PROBLEM SOLVING** The function $C(x) = 25x + 50$ represents the labor cost (in dollars) for Certified Remodeling to build a deck, where x is the number of hours of labor. The table shows sample labor costs from its main competitor, Master Remodeling. The deck is estimated to take 8 hours of labor. Which company would you hire? Explain.

Hours	Cost
2	\$130
4	\$160
6	\$190

31. **MAKING AN ARGUMENT** Let $P(x)$ be the number of people in the U.S. who own a cell phone x years after 1990. Your friend says that $P(x + 1) > P(x)$ for any x because $x + 1$ is always greater than x. Is your friend correct? Explain.

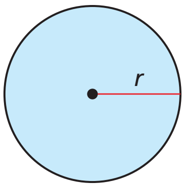
32. **THOUGHT PROVOKING** Let $B(t)$ be your bank account balance after t days. Describe a situation in which $B(0) < B(4) < B(2)$.

33. **MATHEMATICAL CONNECTIONS** Rewrite each geometry formula using function notation. Evaluate each function when $r = 5$ feet. Then explain the meaning of the result.

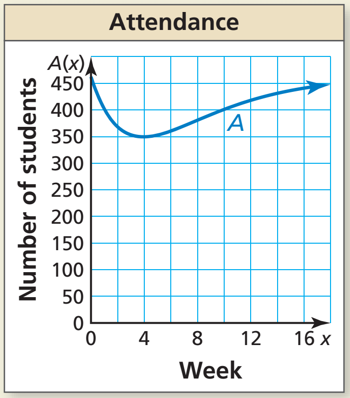
a. Diameter, $d = 2r$

b. Area, $A = \pi r^2$

c. Circumference, $C = 2\pi r$



34. **HOW DO YOU SEE IT?** The function $y = A(x)$ represents the attendance at a high school x weeks after a flu outbreak. The graph of the function is shown.



a. What happens to the school's attendance after the flu outbreak?

b. Estimate $A(13)$ and explain its meaning.

c. Use the graph to estimate the solution(s) of the equation $A(x) = 400$. Explain the meaning of the solution(s).

d. What was the least attendance? When did that occur?

e. How many students do you think are enrolled at this high school? Explain your reasoning.

35. **INTERPRETING FUNCTION NOTATION** Let f be a function. Use each statement to find the coordinates of a point on the graph of f.

a. $f(5)$ is equal to 9.

b. A solution of the equation $f(n) = -3$ is 5.

36. **REASONING** Given a function f, tell whether the statement

$$f(a + b) = f(a) + f(b)$$

is true or false for all inputs a and b. If it is false, explain why.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the inequality. Graph the solution. *(Section 2.5)*

37. $-2 \le x - 11 \le 6$

38. $5a < -35$ *or* $a - 14 > 1$

39. $-16 < 6k + 2 < 0$

40. $2d + 7 < -9$ *or* $4d - 1 > -3$

41. $5 \le 3y + 8 < 17$

42. $4v + 9 \le 5$ *or* $-3v \ge -6$

3.1–3.3 What Did You Learn?

Core Vocabulary

relation, *p. 90*
function, *p. 90*
domain, *p. 92*
range, *p. 92*
independent variable, *p. 93*
dependent variable, *p. 93*
linear equation in two variables, *p. 98*
linear function, *p. 98*
nonlinear function, *p. 98*
solution of a linear equation in two variables, *p. 100*
discrete domain, *p. 100*
continuous domain, *p. 100*
function notation, *p. 108*

Core Concepts

Section 3.1

Determining Whether Relations Are Functions, *p. 90*
Vertical Line Test, *p. 91*
The Domain and Range of a Function, *p. 92*
Independent and Dependent Variables, *p. 93*

Section 3.2

Linear and Nonlinear Functions, *p. 98*
Representations of Functions, *p. 99*
Discrete and Continuous Domains, *p. 100*

Section 3.3

Using Function Notation, *p. 108*

Mathematical Thinking

1. How can you use technology to confirm your answers in Exercises 46–49 on page 96?
2. How can you use patterns to solve Exercise 43 on page 105?
3. How can you make sense of the quantities in the function in Exercise 21 on page 111?

Study Skills

Staying Focused during Class

As soon as class starts, quickly review your notes from the previous class and start thinking about math.

Repeat what you are writing in your head.

When a particular topic is difficult, ask for another example.

3.1–3.3 Quiz

Determine whether the relation is a function. Explain. *(Section 3.1)*

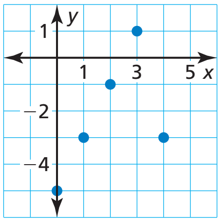
1.

Input, *x*	-1	0	1	2	3
Output, *y*	0	1	4	4	8

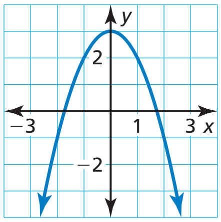
2. $(-10, 2), (-8, 3), (-6, 5), (-8, 8), (-10, 6)$

Find the domain and range of the function represented by the graph. *(Section 3.1)*

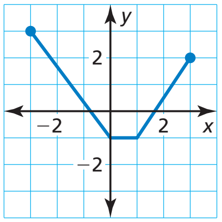
3.



4.

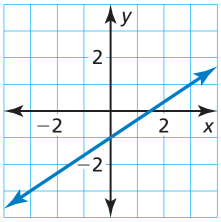


5.



Determine whether the graph, table, or equation represents a *linear* or *nonlinear* function. Explain. *(Section 3.2)*

6.



7.

x	***y***
-5	3
0	7
5	10

8. $y = x(2 - x)$

Determine whether the domain is *discrete* or *continuous*. Explain. *(Section 3.2)*

9.

Depth (feet), *x*	33	66	99
Pressure (ATM), *y*	2	3	4

10.

Hats, *x*	2	3	4
Cost (dollars), *y*	36	54	72

11. For $w(x) = -2x + 7$, find the value of x for which $w(x) = -3$. *(Section 3.3)*

Graph the linear function. *(Section 3.3)*

12. $g(x) = x + 3$

13. $p(x) = -3x - 1$

14. $m(x) = \frac{2}{3}x$

15. The function $m = 30 - 3r$ represents the amount m (in dollars) of money you have after renting r video games. *(Section 3.1 and Section 3.2)*

a. Identify the independent and dependent variables.

b. Find the domain of the function. Is the domain discrete or continuous? Explain.

c. Graph the function using its domain.

d. Find the range of the function.

16. The function $d(x) = 1375 - 110x$ represents the distance (in miles) a high-speed train is from its destination after x hours. *(Section 3.3)*

a. How far is the train from its destination after 8 hours?

b. How long does the train travel before reaching its destination?

3.4 Graphing Linear Equations in Standard Form

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.3.C

Essential Question

How can you describe the graph of the equation $Ax + By = C$?

EXPLORATION 1 Using a Table to Plot Points

Work with a partner. You sold a total of \$16 worth of tickets to a fundraiser. You lost track of how many of each type of ticket you sold. Adult tickets are \$4 each. Child tickets are \$2 each.

$$\frac{\square}{\text{adult}} \cdot \text{Number of adult tickets} + \frac{\square}{\text{child}} \cdot \text{Number of child tickets} = \square$$

a. Let x represent the number of adult tickets. Let y represent the number of child tickets. Use the verbal model to write an equation that relates x and y.

b. Copy and complete the table to show the different combinations of tickets you might have sold.

x					
y					

c. Plot the points from the table. Describe the pattern formed by the points.

d. If you remember how many adult tickets you sold, can you determine how many child tickets you sold? Explain your reasoning.

FORMULATING A PLAN

To be proficient in math, you need to find an entry point into the solution of a problem. Determining what information you know, and what you can do with that information, can help you find an entry point.

EXPLORATION 2 Rewriting and Graphing an Equation

Work with a partner. You sold a total of \$48 worth of cheese. You forgot how many pounds of each type of cheese you sold. Swiss cheese costs \$8 per pound. Cheddar cheese costs \$6 per pound.

$$\frac{\square}{\text{pound}} \cdot \text{Pounds of Swiss} + \frac{\square}{\text{pound}} \cdot \text{Pounds of cheddar} = \square$$

a. Let x represent the number of pounds of Swiss cheese. Let y represent the number of pounds of cheddar cheese. Use the verbal model to write an equation that relates x and y.

b. Solve the equation for y. Then use a graphing calculator to graph the equation. Given the real-life context of the problem, find the domain and range of the function.

c. The ***x*-intercept** of a graph is the x-coordinate of a point where the graph crosses the x-axis. The ***y*-intercept** of a graph is the y-coordinate of a point where the graph crosses the y-axis. Use the graph to determine the x- and y-intercepts.

d. How could you use the equation you found in part (a) to determine the x- and y-intercepts? Explain your reasoning.

e. Explain the meaning of the intercepts in the context of the problem.

Communicate Your Answer

3. How can you describe the graph of the equation $Ax + By = C$?

4. Write a real-life problem that is similar to those shown in Explorations 1 and 2.

3.4 Lesson

Core Vocabulary

standard form, *p. 116*
x-intercept, *p. 117*
y-intercept, *p. 117*
zero of a function, *p. 118*

Previous
ordered pair
quadrant

What You Will Learn

- Graph equations of horizontal and vertical lines.
- Graph linear equations in standard form using intercepts.
- Find zeros of functions.
- Use linear equations to solve real-life problems.

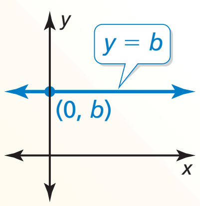
Horizontal and Vertical Lines

The **standard form** of a linear equation is $Ax + By = C$, where A, B, and C are real numbers and A and B are not both zero.

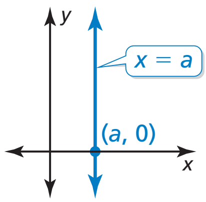
Consider what happens when $A = 0$ or when $B = 0$. When $A = 0$, the equation becomes $By = C$, or $y = \frac{C}{B}$. Because $\frac{C}{B}$ is a constant, you can write $y = b$. Similarly, when $B = 0$, the equation becomes $Ax = C$, or $x = \frac{C}{A}$, and you can write $x = a$.

Core Concept

Horizontal and Vertical Lines



The graph of $y = b$ is a horizontal line. The line passes through the point $(0, b)$.



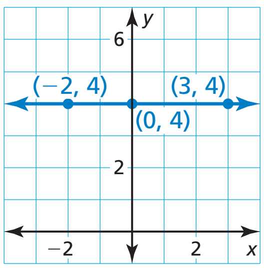
The graph of $x = a$ is a vertical line. The line passes through the point $(a, 0)$.

EXAMPLE 1 Horizontal and Vertical Lines

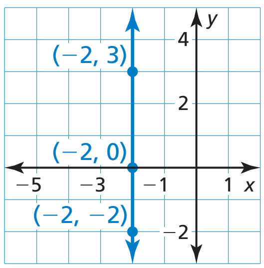
Graph (a) $y = 4$ and (b) $x = -2$.

SOLUTION

a. For every value of x, the value of y is 4. The graph of the equation $y = 4$ is a horizontal line 4 units above the x-axis.



b. For every value of y, the value of x is -2. The graph of the equation $x = -2$ is a vertical line 2 units to the left of the y-axis.



STUDY TIP

For every value of x, the ordered pair $(x, 4)$ is a solution of $y = 4$.

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Graph the linear equation.

1. $y = -2.5$

2. $x = 5$

Using Intercepts to Graph Linear Equations

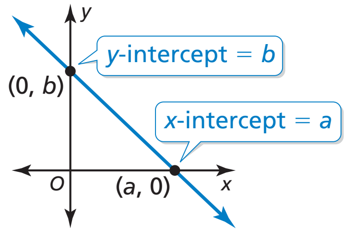
You can use the fact that two points determine a line to graph a linear equation. Two convenient points are the points where the graph crosses the axes.

Core Concept

Using Intercepts to Graph Equations

The **x-intercept** of a graph is the x-coordinate of a point where the graph crosses the x-axis. It occurs when $y = 0$.

The **y-intercept** of a graph is the y-coordinate of a point where the graph crosses the y-axis. It occurs when $x = 0$.



To graph the linear equation $Ax + By = C$, find the intercepts and draw the line that passes through the two intercepts.

- To find the x-intercept, let $y = 0$ and solve for x.
- To find the y-intercept, let $x = 0$ and solve for y.

EXAMPLE 2 Using Intercepts to Graph a Linear Equation

Use intercepts to graph the equation $3x + 4y = 12$.

SOLUTION

Step 1 Find the intercepts.

To find the x-intercept, substitute 0 for y and solve for x.

$3x + 4y = 12$ Write the original equation.

$3x + 4(0) = 12$ Substitute 0 for y.

$x = 4$ Solve for x.

To find the y-intercept, substitute 0 for x and solve for y.

$3x + 4y = 12$ Write the original equation.

$3(0) + 4y = 12$ Substitute 0 for x.

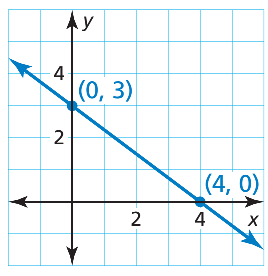
$y = 3$ Solve for y.

STUDY TIP

As a check, you can find a third solution of the equation and verify that the corresponding point is on the graph. To find a third solution, substitute any value for one of the variables and solve for the other variable.

Step 2 Plot the points and draw the line.

The x-intercept is 4, so plot the point (4, 0). The y-intercept is 3, so plot the point (0, 3). Draw a line through the points.



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Use intercepts to graph the linear equation. Label the points corresponding to the intercepts.

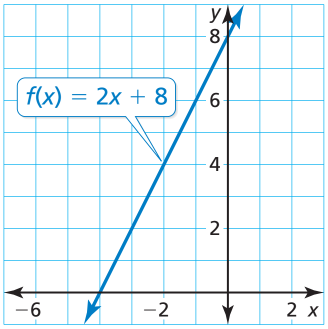
3. $2x - y = 4$

4. $x + 3y = -9$

Finding the Zeros of Functions

A **zero of a function** f is an x-value for which $f(x) = 0$ (or $y = 0$). A zero of a function is an x-intercept of the graph of the function.

EXAMPLE 3 Finding the Zero of a Function



Find the zero of the function $f(x) = 2x + 8$.

SOLUTION

Substitute 0 for $f(x)$ in the function and solve for x.

$f(x) = 2x + 8$ Write the function.

$0 = 2x + 8$ Substitute 0 for $f(x)$.

$x = -4$ Solve for x.

▶ The zero of the function is -4.

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Find the zero of the function.

5. $h(x) = 4x - 16$

6. $r(x) = -3x - 12$

7. $r(x) = -\frac{1}{2}x + 5$

8. $g(x) = 2x + \frac{2}{3}$

Solving Real-Life Problems

EXAMPLE 4 Modeling with Mathematics

An artist rents a booth at an art show for \$300. The function $f(x) = 50x - 300$ represents the artist's profit, where x is the number of paintings the artist sells. Find the zero of the function. Explain what the zero means in this situation.

SOLUTION

Substitute 0 for $f(x)$ in the function and solve for x.

$f(x) = 50x - 300$ Write the function.

$0 = 50x - 300$ Substitute 0 for $f(x)$.

$x = 6$ Solve for x.

▶ The zero of the function is 6. Because the function represents the artist's profit, the zero of the function represents the number of paintings the artist must sell to recover the cost of renting the booth. So, the artist must sell 6 paintings to recover the cost of renting the booth.

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9. The function $f(t) = -2t + 25$ represents the amount (in gallons) of water remaining in a tub after t seconds. Find the zero of the function. Explain what the zero means in this situation.

EXAMPLE 5 Modeling with Mathematics

You are planning an awards banquet for your school. You need to rent tables to seat 180 people. Tables come in two sizes. Small tables seat 6 people, and large tables seat 10 people. The equation $6x + 10y = 180$ models this situation, where x is the number of small tables and y is the number of large tables.

a. Graph the equation. Interpret the intercepts.

b. Find four possible solutions in the context of the problem.

SOLUTION

1. Understand the Problem You know the equation that models the situation. You are asked to graph the equation, interpret the intercepts, and find four solutions.

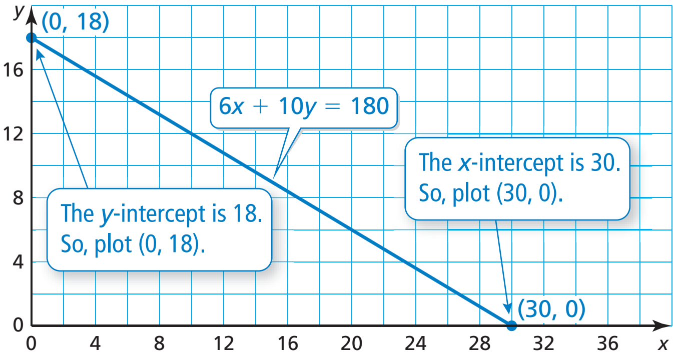
2. Make a Plan Use intercepts to graph the equation. Then use the graph to interpret the intercepts and find other solutions.

3. Solve the Problem

a. Use intercepts to graph the equation. Neither x nor y can be negative, so only graph the equation in the first quadrant.

STUDY TIP

Although x and y represent whole numbers, it is convenient to draw a line segment that includes points whose coordinates are not whole numbers.



▶ The x-intercept shows that you can rent 30 small tables when you do not rent any large tables. The y-intercept shows that you can rent 18 large tables when you do not rent any small tables.

b. Only whole-number values of x and y make sense in the context of the problem. Besides the intercepts, it appears that the line passes through the points (10, 12) and (20, 6). To verify that these points are solutions, check them in the equation, as shown.

Check

$$6x + 10y = 180$$
$$6(10) + 10(12) \stackrel{?}{=} 180$$
$$180 = 180 \checkmark$$

$$6x + 10y = 180$$
$$6(20) + 10(6) \stackrel{?}{=} 180$$
$$180 = 180 \checkmark$$

▶ So, four possible combinations of tables that will seat 180 people are 0 small and 18 large, 10 small and 12 large, 20 small and 6 large, and 30 small and 0 large.

4. Look Back The graph shows that as the number x of small tables increases, the number y of large tables decreases. This makes sense in the context of the problem. So, the graph is reasonable.

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10. **WHAT IF?** You decide to rent tables from a different company. The situation can be modeled by the equation $4x + 6y = 180$, where x is the number of small tables and y is the number of large tables. Graph the equation and interpret the intercepts.

3.4 Exercises

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Vocabulary and Core Concept Check

1. **WRITING** How are x-intercepts and y-intercepts alike? How are they different?

2. **WHICH ONE DOESN'T BELONG?** Which point does not belong with the other three? Explain your reasoning.

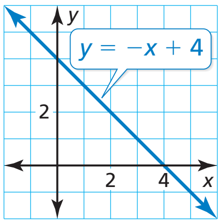
$(0, -3)$ $(0, 0)$ $(4, -3)$ $(4, 0)$

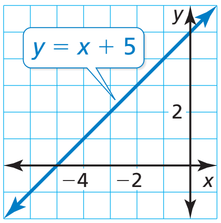
Monitoring Progress and Modeling with Mathematics

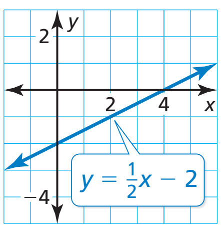
In Exercises 3–8, graph the linear equation. *(See Example 1.)*

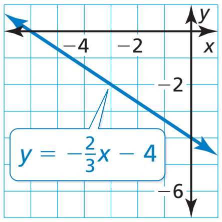
3. $x = 4$
4. $y = 2$
5. $y = -3$
6. $x = -1$
7. $x = -\frac{1}{3}$
8. $y = \frac{7}{2}$

In Exercises 9–12, find the x- and y-intercepts of the graph of the linear equation.

9. 

10. 

11. 

12. 

In Exercises 13–18, find the x- and y-intercepts of the graph of the linear equation.

13. $2x + 3y = 12$
14. $3x + 6y = 24$
15. $-4x + 8y = -16$
16. $-6x + 9y = -18$
17. $3x - 6y = 2$
18. $-x + 8y = 4$

In Exercises 19–28, use intercepts to graph the linear equation. Label the points corresponding to the intercepts. *(See Example 2.)*

19. $5x + 3y = 30$
20. $4x + 6y = 12$
21. $-12x + 3y = 24$
22. $-2x + 6y = 18$
23. $-4x + 3y = -30$
24. $-2x + 7y = -21$
25. $-x + 2y = 7$
26. $3x - y = -5$
27. $-\frac{5}{2}x + y = 10$
28. $-\frac{1}{2}x + y = -4$

ERROR ANALYSIS **In Exercises 29 and 30, describe and correct the error in finding the intercepts of the graph of the equation.**

29.

$3x + 12y = 24$	$3x + 12y = 24$
$3x + 12(0) = 24$	$3(0) + 12y = 24$
$3x = 24$	$12y = 24$
$x = 8$	$y = 2$

The intercept is at $(8, 2)$.

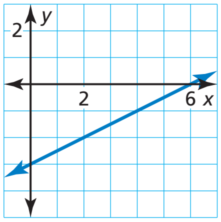
30.

$4x + 10y = 20$	$4x + 10y = 20$
$4x + 10(0) = 20$	$4(0) + 10y = 20$
$4x = 20$	$10y = 20$
$x = 5$	$y = 2$

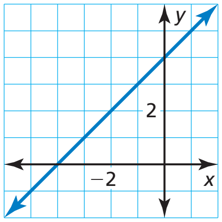
The x-intercept is at $(0, 5)$, and the y-intercept is at $(2, 0)$.

In Exercises 31–34, use the graph to find the zero of the function.

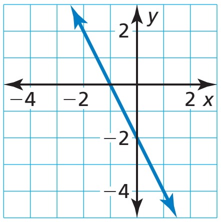
31. $f(x) = \frac{1}{2}x - 3$



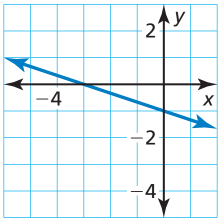
32. $f(x) = x + 4$



33. $f(x) = -2x - 2$



34. $f(x) = -\frac{1}{3}x - 1$



In Exercises 35–44, find the zero of the function. *(See Example 3.)*

35. $f(x) = x + 2$

36. $r(x) = x - 5$

37. $m(x) = 3x - 18$

38. $g(x) = 2x + 4$

39. $q(x) = -5x - 20$

40. $n(x) = -6x + 18$

41. $h(x) = 4x + 2$

42. $f(x) = -3x + 1$

43. $g(x) = -\frac{1}{2}x + 7$

44. $p(x) = \frac{1}{4}x - 3$

45. MODELING WITH MATHEMATICS The function $f(x) = -200x + 1000$ represents the balance (in dollars) in a checking account after x months. Find the zero of the function. Explain what the zero means in this situation. *(See Example 4.)*

46. MODELING WITH MATHEMATICS The function $f(t) = -10t + 3000$ represents the height (in feet) of a skydiver t seconds after opening the parachute. Find the zero of the function. Explain what the zero means in this situation.

47. REASONING The function $c(x) = 9 + 1.50x$ represents the total cost (in dollars) of a large pizza, where x is the number of additional toppings. Find the zero of the function. Does the zero make sense in this situation? Explain.

48. OPEN-ENDED Consider the equation $8 = 4x + 16$. Write a function so that the solution of the equation is the zero of the function. Explain your reasoning.

49. MODELING WITH MATHEMATICS A football team has an away game, and the bus breaks down. The coaches decide to drive the players to the game in cars and vans. Four players can ride in each car. Six players can ride in each van. There are 48 players on the team. The equation $4x + 6y = 48$ models this situation, where x is the number of cars and y is the number of vans. *(See Example 5.)*

a. Graph the equation. Interpret the intercepts.

b. Find four possible solutions in the context of the problem.

50. MODELING WITH MATHEMATICS You are ordering shirts for the math club at your school. Short-sleeved shirts cost \$10 each. Long-sleeved shirts cost \$12 each. You have a budget of \$300 for the shirts. The equation $10x + 12y = 300$ models the total cost, where x is the number of short-sleeved shirts and y is the number of long-sleeved shirts.

a. Graph the equation. Interpret the intercepts.

b. Twelve students decide they want short-sleeved shirts. How many long-sleeved shirts can you order?

51. MAKING AN ARGUMENT You overhear your friend explaining how to find intercepts to a classmate. Your friend says, "When you want to find the x-intercept, just substitute 0 for x and continue to solve the equation." Is your friend's explanation correct? Explain.

52. ANALYZING RELATIONSHIPS You lose track of how many 2-point baskets and 3-point baskets a team makes in a basketball game. The team misses all the 1-point baskets and still scores 54 points. The equation $2x + 3y = 54$ models the total points scored, where x is the number of 2-point baskets made and y is the number of 3-point baskets made.

a. Find and interpret the intercepts.

b. Can the number of 3-point baskets made be odd? Explain your reasoning.

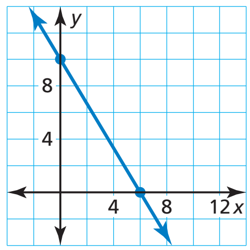
c. Graph the equation. Find two more possible solutions in the context of the problem.

MULTIPLE REPRESENTATIONS **In Exercises 53–56, match the equation with its graph.**

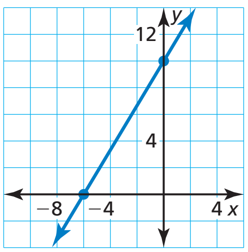
53. $5x + 3y = 30$ **54.** $5x + 3y = -30$

55. $5x - 3y = 30$ **56.** $5x - 3y = -30$

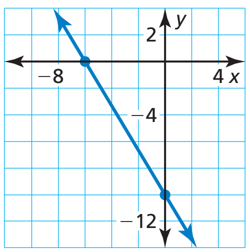
A.



B.



C.



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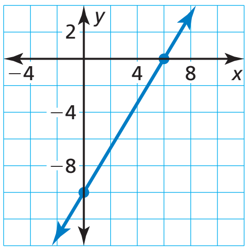
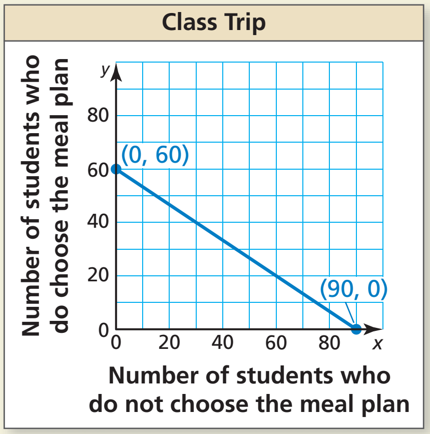


57. MATHEMATICAL CONNECTIONS Graph the equations $x = 5$, $x = 2$, $y = -2$, and $y = 1$. What enclosed shape do the lines form? Explain your reasoning.

58. HOW DO YOU SEE IT? You are organizing a class trip to an amusement park. The cost to enter the park is \$30. The cost to enter with a meal plan is \$45. You have a budget of \$2700 for the trip. The equation $30x + 45y = 2700$ models the total cost for the class to go on the trip, where x is the number of students who do not choose the meal plan and y is the number of students who do choose the meal plan.



a. Interpret the intercepts of the graph.

b. Describe the domain and range in the context of the problem.

59. REASONING Use the values to fill in the equation ☐x + ☐$y = 30$ so that the x-intercept of the graph is -10 and the y-intercept of the graph is 5.

-10 -3 1 5 6

60. THOUGHT PROVOKING Write an equation in standard form of a line whose intercepts are integers. Explain how you know the intercepts are integers.

61. WRITING Are the equations of horizontal and vertical lines written in standard form? Explain your reasoning.

62. ABSTRACT REASONING The x- and y-intercepts of the graph of the equation $3x + 5y = k$ are integers. Describe the values of k. Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

63. $\dfrac{2-(-2)}{4-(-4)}$ **64.** $\dfrac{14-18}{0-2}$

65. $\dfrac{-3-9}{8-(-7)}$ **66.** $\dfrac{12-17}{-5-(-2)}$

3.5 Graphing Linear Equations in Slope-Intercept Form

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.2.A
A.3.A
A.3.B
A.3.C

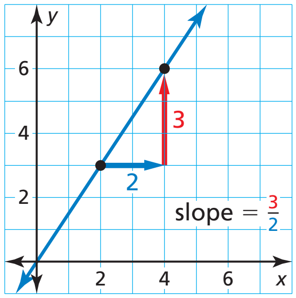
Essential Question

How can you describe the graph of the equation $y = mx + b$?

Slope is the rate of change between any two points on a line. It is the measure of the *steepness* of the line.

To find the slope of a line, find the ratio of the change in y (vertical change) to the change in x (horizontal change).

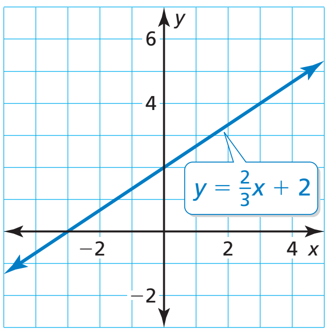
$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}$$



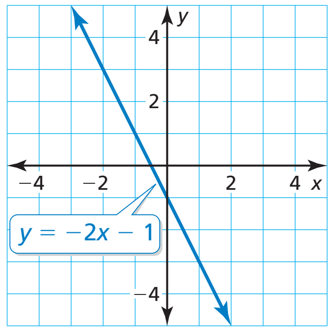
EXPLORATION 1 Finding Slopes and y-Intercepts

Work with a partner. Find the slope and y-intercept of each line.

a.



b.



ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you first need to collect and organize data. Then make conjectures about the patterns you observe in the data.

EXPLORATION 2 Writing a Conjecture

Work with a partner. Graph each equation. Then copy and complete the table. Use the completed table to write a conjecture about the relationship between the graph of $y = mx + b$ and the values of m and b.

Equation	Description of graph	Slope of graph	y-Intercept
a. $y = -\frac{2}{3}x + 3$	Line	$-\frac{2}{3}$	3
b. $y = 2x - 2$			
c. $y = -x + 1$			
d. $y = x - 4$			

Communicate Your Answer

3. How can you describe the graph of the equation $y = mx + b$?

a. How does the value of m affect the graph of the equation?

b. How does the value of b affect the graph of the equation?

c. Check your answers to parts (a) and (b) by choosing one equation from Exploration 2 and (1) varying only m and (2) varying only b.

3.5 Lesson

Core Vocabulary

slope, *p. 124*
rise, *p. 124*
run, *p. 124*
slope-intercept form, *p. 126*
constant function, *p. 126*

Previous
dependent variable
independent variable

What You Will Learn

- Find the slope of a line.
- Use the slope-intercept form of a linear equation.
- Use slopes and *y*-intercepts to solve real-life problems.

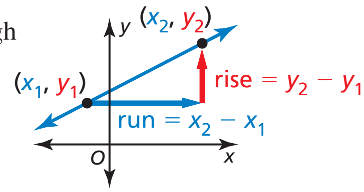
The Slope of a Line

Core Concept

Slope

The **slope** m of a nonvertical line passing through two points (x_1, y_1) and (x_2, y_2) is the ratio of the **rise** (change in y) to the **run** (change in x).

$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

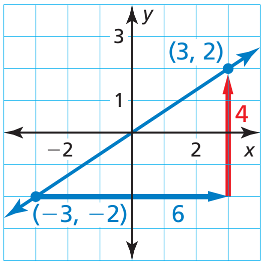


When the line rises from left to right, the slope is positive. When the line falls from left to right, the slope is negative.

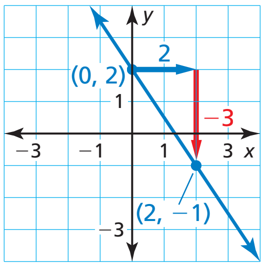
EXAMPLE 1 Finding the Slope of a Line

Describe the slope of each line. Then find the slope.

a.



b.



STUDY TIP

When finding slope, you can label either point as (x_1, y_1) and the other point as (x_2, y_2). The result is the same.

SOLUTION

a. The line rises from left to right. So, the slope is positive. Let $(x_1, y_1) = (-3, -2)$ and $(x_2, y_2) = (3, 2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-2)}{3 - (-3)} = \frac{4}{6} = \frac{2}{3}$$

b. The line falls from left to right. So, the slope is negative. Let $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (2, -1)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 2}{2 - 0} = \frac{-3}{2} = -\frac{3}{2}$$

READING

In the slope formula, x_1 is read as "*x* sub one" and y_2 is read as "*y* sub two." The numbers 1 and 2 in x_1 and y_2 are called *subscripts*.

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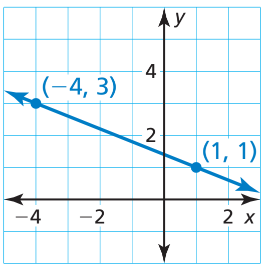
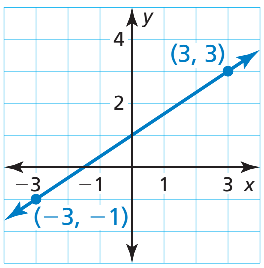
Describe the slope of the line. Then find the slope.

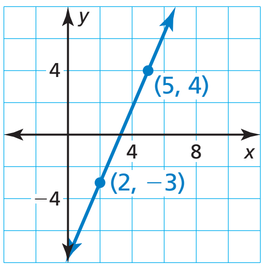
1.



2.



3.



EXAMPLE 2 Finding Slope from a Table

The points represented by each table lie on a line. How can you find the slope of each line from the table? What is the slope of each line?

a.

x	y
4	20
7	14
10	8
13	2

b.

x	y
-1	2
1	2
3	2
5	2

c.

x	y
-3	-3
-3	0
-3	6
-3	9

STUDY TIP

As a check, you can plot the points represented by the table to verify that the line through them has a slope of -2.

SOLUTION

a. Choose any two points from the table and use the slope formula. Use the points $(x_1, y_1) = (4, 20)$ and $(x_2, y_2) = (7, 14)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14 - 20}{7 - 4} = \frac{-6}{3}, \text{ or } -2$$

▶ The slope is -2.

b. Note that there is no change in y. Choose any two points from the table and use the slope formula. Use the points $(x_1, y_1) = (-1, 2)$ and $(x_2, y_2) = (5, 2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 2}{5 - (-1)} = \frac{0}{6}, \text{ or } 0$$

The change in y is 0.

▶ The slope is 0.

c. Note that there is no change in x. Choose any two points from the table and use the slope formula. Use the points $(x_1, y_1) = (-3, 0)$ and $(x_2, y_2) = (-3, 6)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 0}{-3 - (-3)} = \frac{6}{0} \quad ✗$$

The change in x is 0.

▶ Because division by zero is undefined, the slope of the line is undefined.

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The points represented by the table lie on a line. How can you find the slope of the line from the table? What is the slope of the line?

4.

x	2	4	6	8
y	10	15	20	25

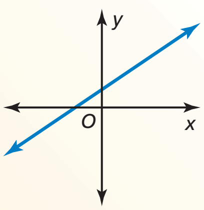
5.

x	5	5	5	5
y	-12	-9	-6	-3

Concept Summary

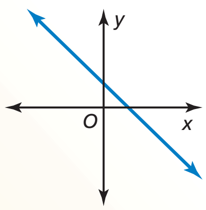
Slope

Positive slope



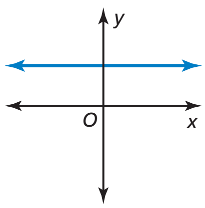
The line rises from left to right.

Negative slope



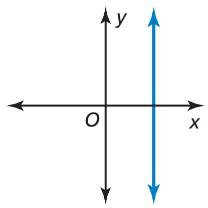
The line falls from left to right.

Slope of 0



The line is horizontal.

Undefined slope



The line is vertical.

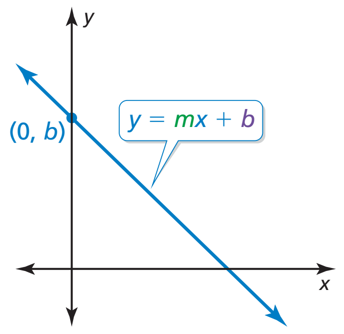
Using the Slope-Intercept Form of a Linear Equation

Core Concept

Slope-Intercept Form

Words A linear equation written in the form $y = mx + b$ is in **slope-intercept form**. The slope of the line is m, and the y-intercept of the line is b.

Algebra $y = \underbrace{m}_{\text{slope}}x + \underbrace{b}_{y\text{-intercept}}$



A linear equation written in the form $y = 0x + b$, or $y = b$, is a **constant function**. The graph of a constant function is a horizontal line.

EXAMPLE 3 Identifying Slopes and y-Intercepts

Find the slope and the y-intercept of the graph of each linear equation.

a. $y = 3x - 4$ **b.** $y = 6.5$ **c.** $-5x - y = -2$

SOLUTION

a. $y = mx + b$ Write the slope-intercept form.

$$y = \underbrace{3}_{\text{slope}}x + \underbrace{(-4)}_{y\text{-intercept}}$$ Rewrite the original equation in slope-intercept form.

▶ The slope is 3, and the y-intercept is -4.

b. The equation represents a constant function. The equation can also be written as $y = 0x + 6.5$.

▶ The slope is 0, and the y-intercept is 6.5.

c. Rewrite the equation in slope-intercept form by solving for y.

$$-5x - y = -2$$ Write the original equation.

$$\underline{+\,5x} \qquad \underline{+\,5x}$$ Add $5x$ to each side.

$$-y = 5x - 2$$ Simplify.

$$\frac{-y}{-1} = \frac{5x - 2}{-1}$$ Divide each side by -1.

$$y = -5x + 2$$ Simplify.

▶ The slope is -5, and the y-intercept is 2.

STUDY TIP

For a constant function, every input has the same output. For instance, in Example 3b, every input has an output of 6.5.

STUDY TIP

When you rewrite a linear equation in slope-intercept form, you are expressing y as a function of x.

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Find the slope and the y-intercept of the graph of the linear equation.

6. $y = -6x + 1$ **7.** $y = 8$ **8.** $x + 4y = -10$

EXAMPLE 4 Using Slope-Intercept Form to Graph

STUDY TIP

You can use the slope to find points on a line in either direction. In Example 4, note that the slope can be written as $\frac{2}{-1}$. So, you could move 1 unit left and 2 units up from (0, 2) to find the point (–1, 4).

Graph $2x + y = 2$. Identify the x-intercept.

SOLUTION

Step 1 Rewrite the equation in slope-intercept form.

$$y = -2x + 2$$

Step 2 Find the slope and the y-intercept.

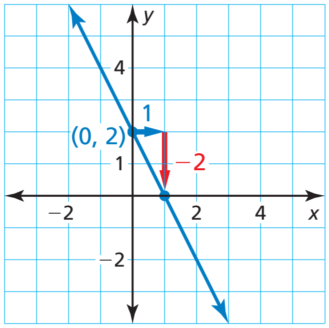
$$m = -2 \text{ and } b = 2$$

Step 3 The y-intercept is 2. So, plot (0, 2).

Step 4 Use the slope to find another point on the line.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-2}{1}$$

Plot the point that is 1 unit right and 2 units down from (0, 2). Draw a line through the two points.



▶ The line crosses the x-axis at (1, 0). So, the x-intercept is 1.

REMEMBER

You can also find the x-intercept by substituting 0 for y in the equation $2x + y = 2$ and solving for x.

EXAMPLE 5 Graphing from a Verbal Description

A linear function g models a relationship in which the dependent variable increases 3 units for every 1 unit the independent variable increases. Graph g when $g(0) = 3$. Identify the slope, y-intercept, and x-intercept of the graph.

SOLUTION

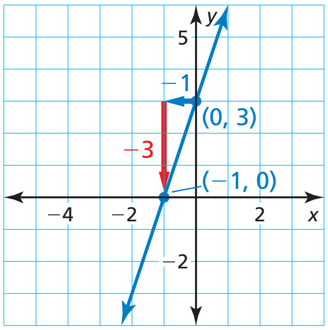
Because the function g is linear, it has a constant rate of change. Let x represent the independent variable and y represent the dependent variable.

Step 1 Find the slope. When the dependent variable increases by 3, the change in y is $+3$. When the independent variable increases by 1, the change in x is $+1$.

So, the slope is $\frac{3}{1}$, or 3.

Step 2 Find the y-intercept. The statement $g(0) = 3$ indicates that when $x = 0$, $y = 3$. So, the y-intercept is 3. Plot (0, 3).

Step 3 Use the slope to find another point on the line. A slope of 3 can be written as $\frac{-3}{-1}$. Plot the point that is 1 unit left and 3 units down from (0, 3). Draw a line through the two points. The line crosses the x-axis at $(-1, 0)$. So, the x-intercept is -1.



▶ The slope is 3, the y-intercept is 3, and the x-intercept is -1.

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Graph the linear equation. Identify the x-intercept.

9. $y = 4x - 4$

10. $3x + y = -3$

11. $x + 2y = 6$

12. A linear function h models a relationship in which the dependent variable decreases 2 units for every 5 units the independent variable increases. Graph h when $h(0) = 4$. Identify the slope, y-intercept, and x-intercept of the graph.

Solving Real-Life Problems

In most real-life problems, slope is interpreted as a rate, such as miles per hour, dollars per hour, or people per year.

EXAMPLE 6 Modeling with Mathematics

A submersible that is exploring the ocean floor begins to ascend to the surface. The elevation h (in feet) of the submersible is modeled by the function $h(t) = 650t - 13{,}000$, where t is the time (in minutes) since the submersible began to ascend.

a. Graph the function and identify its domain and range.

b. Interpret the slope and the intercepts of the graph.

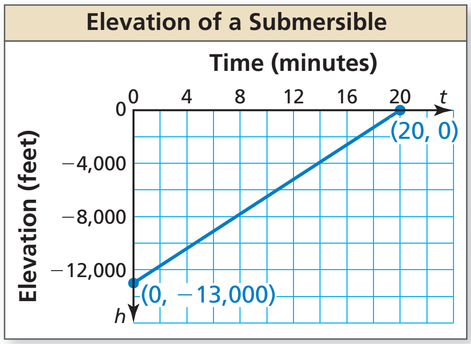
SOLUTION

1. Understand the Problem You know the function that models the elevation. You are asked to graph the function and identify its domain and range. Then you are asked to interpret the slope and intercepts of the graph.

2. Make a Plan Use the slope-intercept form of a linear equation to graph the function. Only graph values that make sense in the context of the problem. Examine the graph to interpret the slope and the intercepts.

3. Solve the Problem

a. The time t must be greater than or equal to 0. The elevation h is below sea level and must be less than or equal to 0. Use the slope of 650 and the h-intercept of $-13{,}000$ to graph the function in Quadrant IV.



STUDY TIP

Because t is the independent variable, the horizontal axis is the t-axis and the graph will have a "t-intercept." Similarly, the vertical axis is the h-axis and the graph will have an "h-intercept."

▶ The domain is $0 \le t \le 20$, and the range is $-13{,}000 \le h \le 0$.

b. The slope is 650. So, the submersible ascends at a rate of 650 feet per minute. The h-intercept is $-13{,}000$. So, the elevation of the submersible after 0 minutes, or when the ascent begins, is $-13{,}000$ feet. The t-intercept is 20. So, the submersible takes 20 minutes to reach an elevation of 0 feet, or sea level.

4. Look Back You can check that your graph is correct by substituting the t-intercept for t in the function. If $h = 0$ when $t = 20$, the graph is correct.

$h = 650(20) - 13{,}000$ Substitute 20 for t in the original equation.

$h = 0$ ✓   Simplify.

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13. WHAT IF? The elevation of the submersible is modeled by $h(t) = 500t - 10{,}000$. (a) Graph the function and identify its domain and range. (b) Interpret the slope and the intercepts of the graph.

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The ________ of a nonvertical line passing through two points is the ratio of the rise to the run.

2. **VOCABULARY** What is a constant function? What is the slope of a constant function?

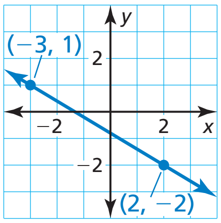
3. **WRITING** What is the slope-intercept form of a linear equation? Explain why this form is called the slope-intercept form.

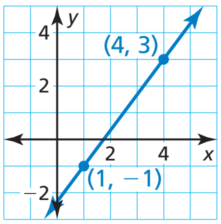
4. **WHICH ONE DOESN'T BELONG?** Which equation does *not* belong with the other three? Explain your reasoning.

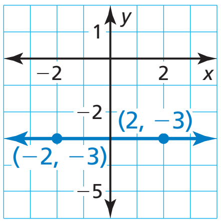
$y = -5x - 1$ $\quad$ $2x - y = 8$ $\quad$ $y = x + 4$ $\quad$ $y = -3x + 13$

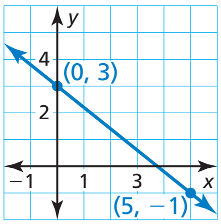
Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, describe the slope of the line. Then find the slope. *(See Example 1.)*

5. 

6. 

7. 

8. 

In Exercises 9–12, the points represented by the table lie on a line. Find the slope of the line. *(See Example 2.)*

9.

x	-9	-5	-1	3
y	-2	0	2	4

10.

x	-1	2	5	8
y	-6	-6	-6	-6

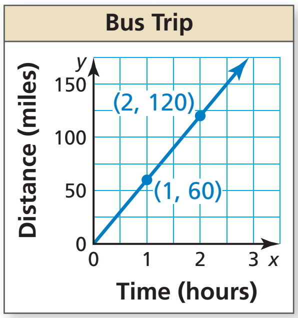
11.

x	0	0	0	0
y	-4	0	4	8

12.

x	-4	-3	-2	-1
y	2	-5	-12	-19

13. **ANALYZING A GRAPH** The graph shows the distance y (in miles) that a bus travels in x hours. Find and interpret the slope of the line.



14. **ANALYZING A TABLE** The table shows the amount x (in hours) of time you spend at a theme park and the admission fee y (in dollars) to the park. The points represented by the table lie on a line. Find and interpret the slope of the line.

Time (hours), x	Admission (dollars), y
6	54.99
7	54.99
8	54.99

In Exercises 15–22, find the slope and the y-intercept of the graph of the linear equation. *(See Example 3.)*

15. $y = -3x + 2$

16. $y = 4x - 7$

17. $y = 6x$

18. $y = -1$

19. $-2x + y = 4$

20. $x + y = -6$

21. $-5x = 8 - y$

22. $0 = 1 - 2y + 14x$

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in finding the slope and the y-intercept of the graph of the equation.

23.

$x = -4y$

The slope is -4, and the y-intercept is 0.

24.

$y = 3x - 6$

The slope is 3, and the y-intercept is 6.

In Exercises 25–32, graph the linear equation. Identify the x-intercept. *(See Example 4.)*

25. $y = -x + 7$

26. $y = \frac{1}{2}x + 3$

27. $y = 2x$

28. $y = -x$

29. $3x + y = -1$

30. $x + 4y = 8$

31. $-y + 5x = 0$

32. $2x - y + 6 = 0$

In Exercises 33 and 34, graph the function with the given description. Identify the slope, y-intercept, and x-intercept of the graph. *(See Example 5.)*

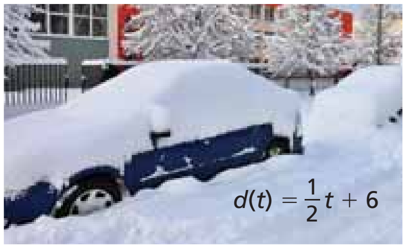
33. A linear function f models a relationship in which the dependent variable decreases 4 units for every 2 units the independent variable increases. The value of the function at 0 is -2.

34. A linear function h models a relationship in which the dependent variable increases 1 unit for every 5 units the independent variable decreases. The value of the function at 0 is 3.

35. GRAPHING FROM A VERBAL DESCRIPTION A linear function r models the growth of your right index fingernail. The length of the fingernail increases 0.7 millimeter every week. Graph r when $r(0) = 12$. Identify the slope and interpret the y-intercept of the graph.

36. GRAPHING FROM A VERBAL DESCRIPTION A linear function m models the amount of milk sold by a farm per month. The amount decreases 500 gallons for every $1 increase in price. Graph m when $m(0) = 3000$. Identify the slope and interpret the x- and y-intercepts of the graph.

37. MODELING WITH MATHEMATICS The function shown models the depth d (in inches) of snow on the ground during the first 9 hours of a snowstorm, where t is the time (in hours) after the snowstorm begins. *(See Example 6.)*



a. Graph the function and identify its domain and range.

b. Interpret the slope and the d-intercept of the graph.

38. MODELING WITH MATHEMATICS The function $c(x) = 0.5x + 70$ represents the cost c (in dollars) of renting a truck from a moving company, where x is the number of miles you drive the truck.

a. Graph the function and identify its domain and range.

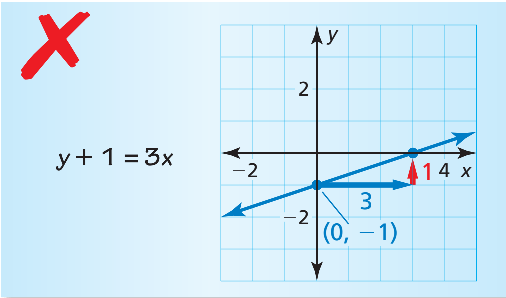
b. Interpret the slope and the c-intercept of the graph.

39. COMPARING FUNCTIONS A linear function models the cost of renting a truck from a moving company. The table shows the cost y (in dollars) when you drive the truck x miles. Graph the function and compare the slope and the y-intercept of the graph with the slope and the c-intercept of the graph in Exercise 38.

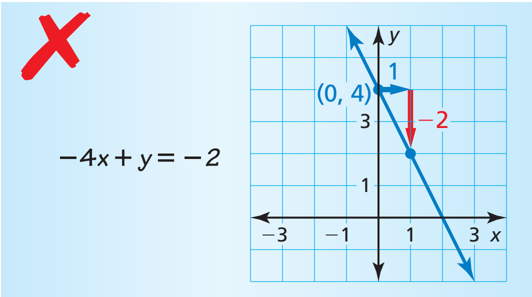
Miles, x	Cost (dollars), y
0	40
50	80
100	120

ERROR ANALYSIS **In Exercises 40 and 41, describe and correct the error in graphing the function.**

40.



41.



42. **MATHEMATICAL CONNECTIONS** Graph the four equations in the same coordinate plane.

$$3y = -x - 3$$

$$2y - 14 = 4x$$

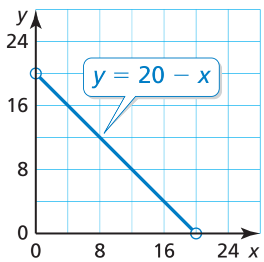
$$4x - 3 - y = 0$$

$$x - 12 = -3y$$

a. What enclosed shape do you think the lines form? Explain.

b. Write a conjecture about the equations of parallel lines.

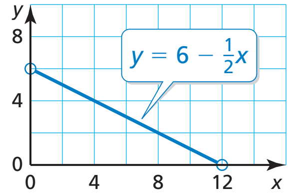
43. **MATHEMATICAL CONNECTIONS** The graph shows the relationship between the width y and the length x of a rectangle in inches. The perimeter of a second rectangle is 10 inches less than the perimeter of the first rectangle.



a. Graph the relationship between the width and length of the second rectangle.

b. How does the graph in part (a) compare to the graph shown?

44. **MATHEMATICAL CONNECTIONS** The graph shows the relationship between the base length x and the side length (of the two equal sides) y of an isosceles triangle in meters. The perimeter of a second isosceles triangle is 8 meters more than the perimeter of the first triangle.



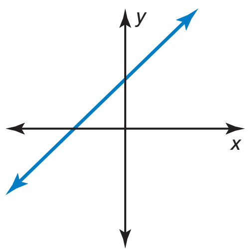
a. Graph the relationship between the base length and the side length of the second triangle.

b. How does the graph in part (a) compare to the graph shown?

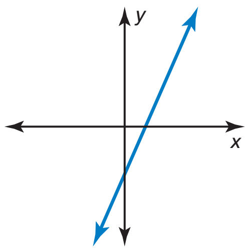
45. **ANALYZING EQUATIONS** Determine which of the equations could be represented by each graph.

$y = -3x + 8$	$y = -x - \frac{4}{3}$
$y = -7x$	$y = 2x - 4$
$y = \frac{7}{4}x - \frac{1}{4}$	$y = \frac{1}{3}x + 5$
$y = -4x - 9$	$y = 6$

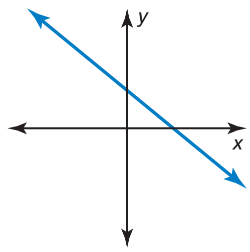
a.



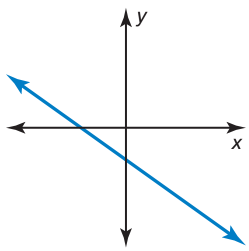
b.



c.



d.

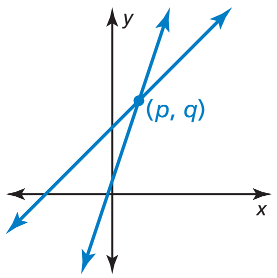


46. **MAKING AN ARGUMENT** Your friend says that you can write the equation of any line in slope-intercept form. Is your friend correct? Explain your reasoning.

47. WRITING Write the definition of the slope of a line in two different ways.

48. THOUGHT PROVOKING Your family goes on vacation to a beach 300 miles from your house. You reach your destination 6 hours after departing. Draw a graph that describes your trip. Explain what each part of your graph represents.

49. ANALYZING A GRAPH The graphs of the functions $g(x) = 6x + a$ and $h(x) = 2x + b$, where a and b are constants, are shown. They intersect at the point (p, q).

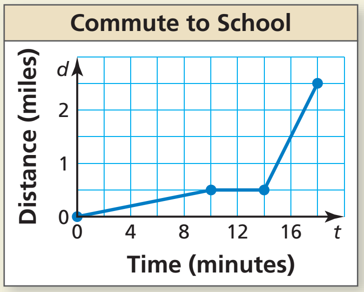


a. Label the graphs of g and h.

b. What do a and b represent?

c. Starting at the point (p, q), trace the graph of g until you get to the point with the x-coordinate $p + 2$. Mark this point C. Do the same with the graph of h. Mark this point D. How much greater is the y-coordinate of point C than the y-coordinate of point D?

50. HOW DO YOU SEE IT? You commute to school by walking and by riding a bus. The graph represents your commute.



a. Describe your commute in words.

b. Calculate and interpret the slopes of the different parts of the graph.

PROBLEM SOLVING In Exercises 51 and 52, find the value of k so that the graph of the equation has the given slope or y-intercept.

51. $y = 4kx - 5;\ m = \frac{1}{2}$

52. $y = -\frac{1}{3}x + \frac{5}{6}k;\ b = -10$

53. ABSTRACT REASONING To show that the slope of a line is constant, let (x_1, y_1) and (x_2, y_2) be any two points on the line $y = mx + b$. Use the equation of the line to express y_1 in terms of x_1 and y_2 in terms of x_2. Then use the slope formula to show that the slope between the points is m.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the inequality. *(Section 2.4)*

54. $8a - 7 \leq 2(3a - 1)$

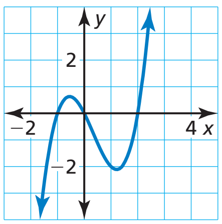
55. $-3(2p + 4) > -6p - 5$

56. $4(3h + 1.5) \geq 6(2h - 2)$

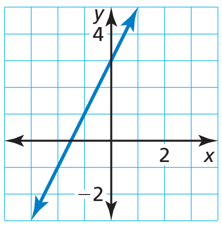
57. $-4(x + 6) < 2(2x - 9)$

Determine whether the graph or table represents a *linear* or *nonlinear* function. Explain. *(Section 3.2)*

58.



59.



60.

x	0	1	2	3	4
y	$\frac{1}{2}$	1	2	4	8

61.

x	2	4	6	8	10
y	0	−1	−2	−3	−4

3.6 Modeling Direct Variation

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.2.D

Essential Question

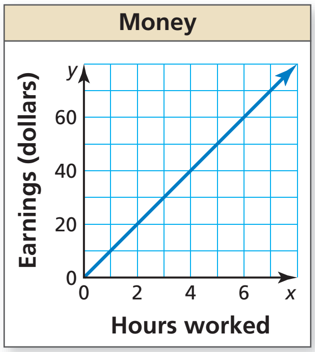
How can you describe the relationship between two quantities that vary directly?

Two quantities x and y show **direct variation** when $y = ax$ and $a \neq 0$.

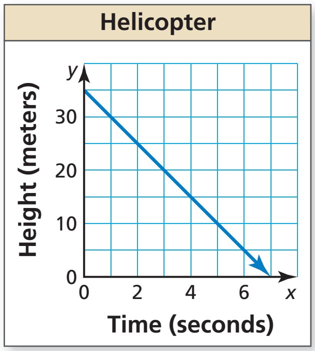
EXPLORATION 1 Identifying Direct Variation

Work with a partner. Determine whether x and y show direct variation. Explain your reasoning.

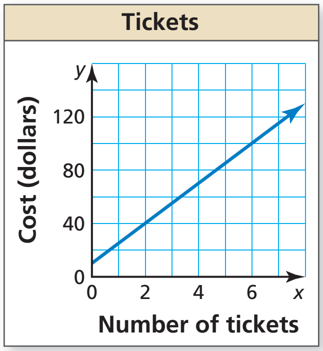
a.



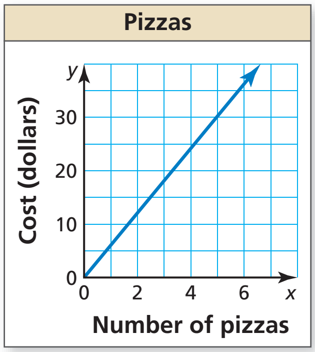
b.



c.



d.



e.

Laps, x	1	2	3	4
Time (seconds), y	90	200	325	480

f.

Cups of sugar, x	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
Cups of flour, y	1	2	3	4

REASONING

To be proficient in math, you need to reason abstractly. You also need to make sense of quantities and their relationships in problem situations.

EXPLORATION 2 Analyzing Relationships

Work with a partner. For the relationships that show direct variation in Exploration 1, do the following.

- Find the slope of the line.
- Find the value of y for the ordered pair $(1, y)$.

What do you notice? What does the value of y represent?

Communicate Your Answer

3. How can you describe the relationship between two quantities that vary directly?

4. Give a real-life example of two quantities that show direct variation. Write an equation that represents the relationship and sketch its graph.

3.6 Lesson

Core Vocabulary

direct variation, *p. 134*
constant of variation, *p. 134*

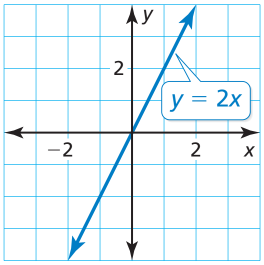
Previous
origin

What You Will Learn

- Determine whether two quantities show direct variation.
- Write direct variation equations.
- Use direct variation equations to solve real-life problems.

Direct Variation

Two quantities x and y show **direct variation** when $y = ax$ and $a \neq 0$. The number a is called the **constant of variation**, and y is said to *vary directly* with x. The equation $y = 2x$ is an example of direct variation, and the constant of variation is 2.



Notice that a direct variation equation $y = ax$ is a linear equation in slope-intercept form, $y = mx + b$, with $m = a$ and $b = 0$. The graph of a direct variation equation is a line with a slope of a that passes through the origin.

EXAMPLE 1 Identifying Direct Variation

Determine whether x and y show direct variation. If so, identify the constant of variation.

a. $2x - 3y = 0$ **b.** $-x + y = 4$

SOLUTION

a. Solve the equation for y.

$2x - 3y = 0$ Write the equation.

$-3y = -2x$ Subtract $2x$ from each side.

$y = \frac{2}{3}x$ Divide each side by -3 and simplify.

▶ The equation can be rewritten in the form $y = ax$. So, x and y show direct variation. The constant of variation is $\frac{2}{3}$.

b. Solve the equation for y.

$-x + y = 4$ Write the equation.

$y = x + 4$ Add x to each side.

▶ The equation *cannot* be rewritten in the form $y = ax$. So, x and y do *not* show direct variation.

Monitoring Progress

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Determine whether x and y show direct variation. If so, identify the constant of variation.

1. $-x + y = 1$ **2.** $2x + y = 0$ **3.** $-4x = -5y$

EXAMPLE 2 Identifying Direct Variation

Determine whether x and y show direct variation. Explain.

a.

x	1	2	3	4
y	-1	1	3	5

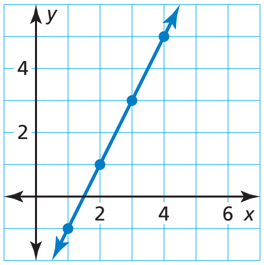
b.

x	-2	0	2	4
y	-1	0	1	2

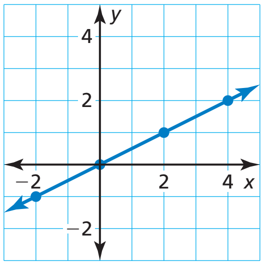
SOLUTION

Plot the ordered pairs. Then draw a line through the points.

a.



b.



▶ The line does *not* pass through the origin. So, x and y do *not* show direct variation.

▶ The line passes through the origin. So, x and y show direct variation.

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Determine whether x and y show direct variation. Explain.

4.

x	2	3	4	5
y	-4	-6	-8	-10

5.

x	0	1	2	3
y	-2	1	4	7

Writing Direct Variation Equations

The direct variation equation $y = ax$ can be rewritten as $\frac{y}{x} = a$ for $x \neq 0$. So, in a direct variation, the ratio of y to x is constant for all nonzero data pairs (x, y).

EXAMPLE 3 Writing a Direct Variation Equation

The table shows the costs C (in dollars) of downloading s songs from a music website.

Number of songs, s	Cost (dollars), C
3	2.97
5	4.95
7	6.93

a. Explain why C varies directly with s.

b. Write a direct variation equation that relates s and C.

SOLUTION

a. Find the ratio $\frac{C}{s}$ for each data pair (s, C).

$$\frac{2.97}{3} = 0.99, \quad \frac{4.95}{5} = 0.99, \quad \frac{6.93}{7} = 0.99$$

▶ All of the ratios are equal to 0.99. So, C varies directly with s.

b. Because $\frac{C}{s} = 0.99$, the constant of variation is 0.99. So, a direct variation equation is $C = 0.99s$.

APPLYING MATHEMATICS

For real-world data, the ratios may not be exactly equal. You may still be able to use direct variation when the ratios are approximately equal.

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6. **WHAT IF?** The website in Example 3 charges a total of \$1.99 for the first 5 songs you download and \$0.99 for each song after the first 5. Is it reasonable to use a direct variation model for this situation? Explain.

Solving Real-Life Problems

EXAMPLE 4 Modeling with Mathematics

The number s of tablespoons of sea salt needed in a saltwater fish tank varies directly with the number w of gallons of water in the tank. A pet shop owner recommends that you add 100 tablespoons of salt to a 20-gallon tank. How many tablespoons of salt should you add to a 30-gallon tank?

SOLUTION

1. **Understand the Problem** You know that a 20-gallon tank requires 100 tablespoons of salt and that the amount of salt varies directly with the capacity of the tank. You are asked to find the amount of salt you should add to a 30-gallon tank.
2. **Make a Plan** Use the direct variation equation $y = ax$ and the given values to write a direct variation equation for this situation. Then solve the equation when $w = 30$.
3. **Solve the Problem**

Step 1 Write a direct variation equation. Because s varies directly with w, you can use the equation $s = aw$. Also use the fact that $s = 100$ when $w = 20$.

$$s = aw \qquad \text{Write direct variation equation.}$$
$$(100) = a(20) \qquad \text{Substitute.}$$
$$5 = a \qquad \text{Divide each side by 20 and simplify.}$$

A direct variation equation is $s = 5w$.

Step 2 Find the number of tablespoons of salt that you should add to a 30-gallon tank. Use the direct variation equation from Step 1.

$$s = 5w \qquad \text{Write direct variation equation.}$$
$$s = 5(30) \qquad \text{Substitute 30 for } w.$$
$$s = 150 \qquad \text{Simplify.}$$

▶ You should add 150 tablespoons of salt to a 30-gallon tank.

4. **Look Back** Find the ratio $\frac{w}{s}$ for the given data pair and the data calculated in the solution.

$$\frac{100}{20} = 5, \frac{150}{30} = 5$$

Both ratios are equal to 5, so the solution makes sense.

USING PRECISE MATHEMATICAL LANGUAGE

The coefficient of variation in Example 4 is a rate of change: 5 tablespoons of sea salt per gallon of water.

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7. An object that weighs 100 pounds on Earth would weigh just 6 pounds on Pluto. Assume that weight P on Pluto varies directly with weight E on Earth. What would a boulder that weighs 45 pounds on Pluto weigh on Earth?

3.6 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** Two quantities x and y show ________ when $y = ax$ and $a \neq 0$.

2. **WRITING** A line has a slope of -3 and a y-intercept of 4. Does the equation of the line represent direct variation? Explain.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, determine whether x and y show direct variation. If so, identify the constant of variation. *(See Example 1.)*

3. $y = x$
4. $y = 2x$
5. $y = \frac{1}{2}x - 1$
6. $y = 3x + 2$
7. $4x + y = 1$
8. $-\frac{1}{4}x + y = 0$
9. $-x - 3y = 0$
10. $4 - 6x = 2y$

11. **ERROR ANALYSIS** Describe and correct the error in determining whether x and y show direct variation.

$$6x - y = 0$$

Because the equation is not in the form $y = ax$, it does not represent direct variation.

12. **ERROR ANALYSIS** Describe and correct the error in identifying the constant of variation for the direct variation equation.

$$-5x + 3y = 0$$
$$3y = 5x$$

The constant of variation is 5.

In Exercises 13–16, determine whether x and y show direct variation. Explain. *(See Example 2.)*

13.

x	1	2	3	4	6
y	5	10	15	20	30

14.

x	-3	-1	1	3	5
y	-2	0	2	4	6

15.

x	1	3	5	7	9
y	6	12	18	24	30

16.

x	2	4	6	8	10
y	1	2	4	8	16

In Exercises 17–22, the ordered pair is a solution of a direct variation equation. Write the equation and identify the constant of variation.

17. $(5, 6)$
18. $(2, 1)$
19. $(-1, 3)$
20. $(-3, -6)$
21. $(-2, -7)$
22. $(-5, 2)$

In Exercises 23 and 24, determine whether the situation shows direct variation. Explain your reasoning.

23. A canoe rental costs \$20 plus \$5 for each hour of the rental.

24. New carpet costs \$4 per square foot.

25. **MODELING WITH MATHEMATICS** At a recycling center, computers and computer accessories can be recycled for a fee f based on weight w, as shown in the table. *(See Example 3.)*

Weight (pounds), w	10	15	30
Fee (dollars), f	2.50	3.75	7.50

a. Explain why f varies directly with w.

b. Write a direct variation equation that relates w and f.

c. Find the total recycling fee for a computer that weighs 18 pounds and a printer that weighs 10 pounds.

26. MODELING WITH MATHEMATICS A jewelry store sells gold chain by the inch. The table shows the prices of various lengths of gold chain.

Length (inches), ℓ	7	9	16	18
Price (dollars), p	8.75	11.25	20.00	22.50

a. Explain why p varies directly with ℓ.

b. Write a direct variation equation that relates ℓ and p.

c. You have \$30. What is the longest chain that you can buy?

27. MODELING WITH MATHEMATICS At a company, the number h of vacation hours an employee earns varies directly with the number w of weeks the employee works. An employee who works 2 weeks earns 3 vacation hours. Find the number of vacation hours an employee earns for working 8 weeks. *(See Example 4.)*

28. MODELING WITH MATHEMATICS Landscapers plan to spread a layer of stone on a path. The number s of bags of stone needed varies directly with the depth d (in inches) of the layer. They need 20 bags to spread a layer of stone that is 2 inches deep. How deep will the layer of stone be when they use 15 bags of stone?

29. REASONING The slope of a line is $-\frac{1}{3}$ and the point $(-6, 2)$ lies on the line. Determine whether the equation of the line is a direct variation equation. Explain.

30. THOUGHT PROVOKING Two quantities x and y show *inverse variation* when $y = \frac{a}{x}$ and $a \neq 0$. Give an example of a real-life situation that shows inverse variation.

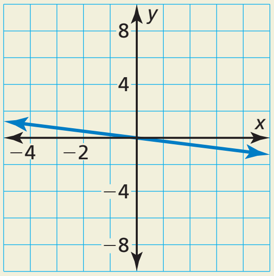
31. REASONING Consider the distance equation $d = rt$, where d is the distance (in feet), r is the rate (in feet per second), and t is the time (in seconds).

a. You run 6 feet per second. Do distance and time vary directly? Explain.

b. You run for 50 seconds. Do distance and rate vary directly? Explain.

c. You run 300 feet. Do rate and time vary directly? Explain.

32. HOW DO YOU SEE IT? Consider the graph shown.



a. Explain why x and y show direct variation.

b. The scale of the x-axis changes. Do x and y still show direct variation? Explain.

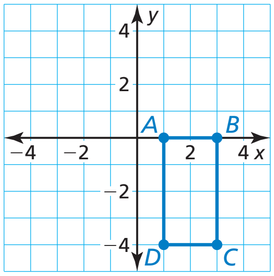
33. CRITICAL THINKING Consider an equation where y varies directly with x. Does x vary directly with y? If so, what is the relationship between the constants of variation?

Maintaining Mathematical Proficiency

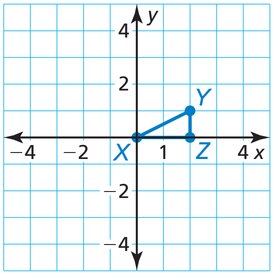
Reviewing what you learned in previous grades and lessons

Find the coordinates of the figure after the transformation. *(Skills Review Handbook)*

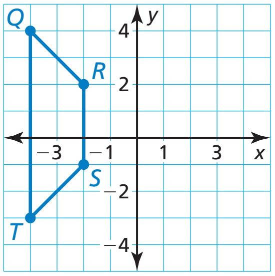
34. Translate the rectangle 4 units left.



35. Dilate the triangle with respect to the origin using a scale factor of 2.



36. Reflect the trapezoid in the y-axis.



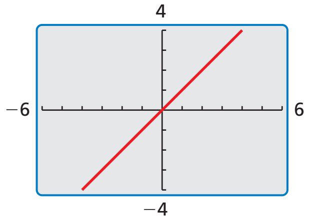
3.7 Transformations of Graphs of Linear Functions

Essential Question How does the graph of the linear function $f(x) = x$ compare to the graphs of $g(x) = f(x) + c$ and $h(x) = f(cx)$?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.3.E

EXPLORATION 1 Comparing Graphs of Functions

Work with a partner. The graph of $f(x) = x$ is shown. Sketch the graph of each function, along with f, on the same set of coordinate axes. Use a graphing calculator to check your results. What can you conclude?



a. $g(x) = x + 4$ **b.** $g(x) = x + 2$

c. $g(x) = x - 2$ **d.** $g(x) = x - 4$

SELECTING TOOLS
To be proficient in math, you need to use the appropriate tools, including graphs, tables, and technology, to check your results.

EXPLORATION 2 Comparing Graphs of Functions

Work with a partner. Sketch the graph of each function, along with $f(x) = x$, on the same set of coordinate axes. Use a graphing calculator to check your results. What can you conclude?

a. $h(x) = \frac{1}{2}x$ **b.** $h(x) = 2x$ **c.** $h(x) = -\frac{1}{2}x$ **d.** $h(x) = -2x$

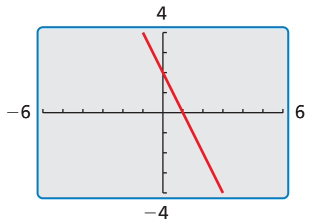
EXPLORATION 3 Matching Functions with Their Graphs

Work with a partner. Match each function with its graph. Use a graphing calculator to check your results. Then use the results of Explorations 1 and 2 to compare the graph of k to the graph of $f(x) = x$.

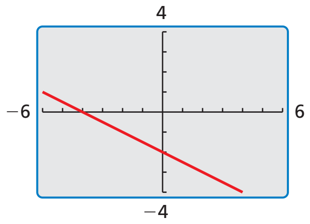
a. $k(x) = 2x - 4$ **b.** $k(x) = -2x + 2$

c. $k(x) = \frac{1}{2}x + 4$ **d.** $k(x) = -\frac{1}{2}x - 2$

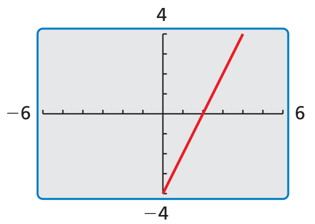
A.



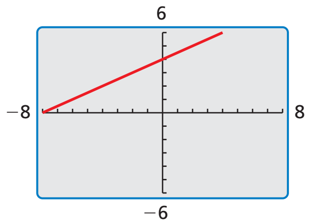
B.



C.



D.



Communicate Your Answer

4. How does the graph of the linear function $f(x) = x$ compare to the graphs of $g(x) = f(x) + c$ and $h(x) = f(cx)$?

3.7 Lesson

What You Will Learn

- Translate and reflect graphs of linear functions.
- Stretch and shrink graphs of linear functions.
- Combine transformations of graphs of linear functions.

Core Vocabulary

Previous
linear function

Translations and Reflections

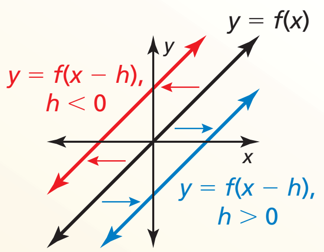
A **family of functions** is a group of functions with similar characteristics. The most basic function in a family of functions is the **parent function**. For nonconstant linear functions, the parent function is $f(x) = x$. The graphs of all other nonconstant linear functions are *transformations* of the graph of the parent function. A **transformation** changes the size, shape, position, or orientation of a graph.

Core Concept

A **translation** is a transformation that shifts a graph horizontally or vertically but does not change the size, shape, or orientation of the graph.

Horizontal Translations

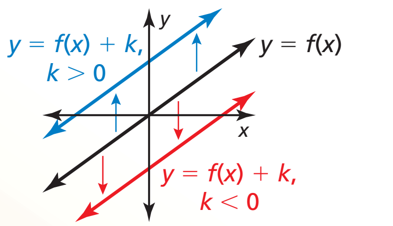
The graph of $y = f(x - h)$ is a horizontal translation of the graph of $y = f(x)$, where $h \neq 0$.



Subtracting h from the *inputs* before evaluating the function shifts the graph left when $h < 0$ and right when $h > 0$.

Vertical Translations

The graph of $y = f(x) + k$ is a vertical translation of the graph of $y = f(x)$, where $k \neq 0$.



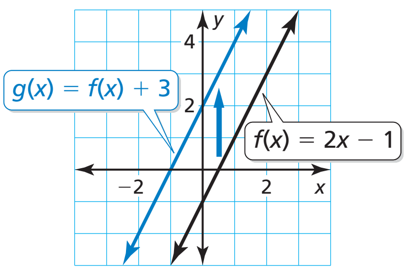
Adding k to the *outputs* shifts the graph down when $k < 0$ and up when $k > 0$.

EXAMPLE 1 Horizontal and Vertical Translations

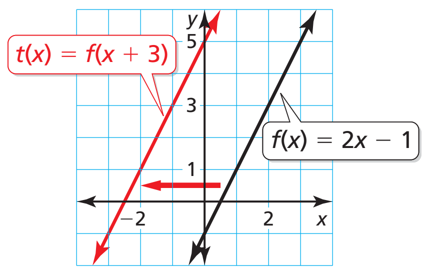
Let $f(x) = 2x - 1$. Graph (a) $g(x) = f(x) + 3$ and (b) $t(x) = f(x + 3)$. Describe the transformations from the graph of f to the graphs of g and t.

SOLUTION

a. The function g is of the form $y = f(x) + k$, where $k = 3$. So, the graph of g is a vertical translation 3 units up of the graph of f.



b. The function t is of the form $y = f(x - h)$, where $h = -3$. So, the graph of t is a horizontal translation 3 units left of the graph of f.



ANALYZING MATHEMATICAL RELATIONSHIPS

In part (a), the output of g is equal to the output of f plus 3.

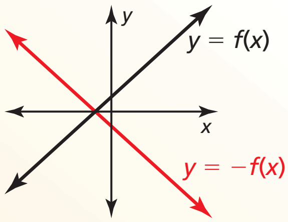
In part (b), the output of t is equal to the output of f when the input of f is 3 more than the input of t.

Core Concept

A **reflection** is a transformation that flips a graph over a line called the *line of reflection*.

Reflections in the x-axis

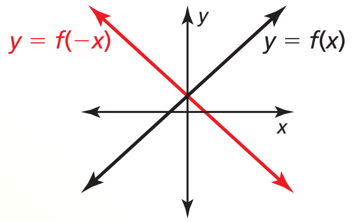
The graph of $y = -f(x)$ is a reflection in the x-axis of the graph of $y = f(x)$.



Multiplying the outputs by -1 changes their signs.

Reflections in the y-axis

The graph of $y = f(-x)$ is a reflection in the y-axis of the graph of $y = f(x)$.



Multiplying the inputs by -1 changes their signs.

STUDY TIP

A reflected point is the same distance from the line of reflection as the original point but on the opposite side of the line.

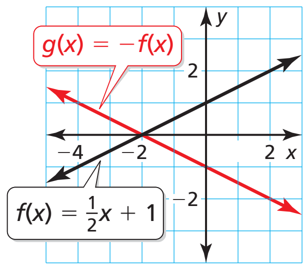
EXAMPLE 2 Reflections in the x-axis and the y-axis

Let $f(x) = \frac{1}{2}x + 1$. Graph (a) $g(x) = -f(x)$ and (b) $t(x) = f(-x)$. Describe the transformations from the graph of f to the graphs of g and t.

SOLUTION

a. To find the outputs of g, multiply the outputs of f by -1. The graph of g consists of the points $(x, -f(x))$.

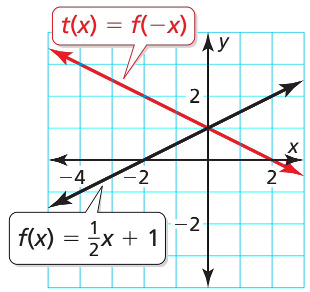
x	-4	-2	0
$f(x)$	-1	0	1
$-f(x)$	1	0	-1



The graph of g is a reflection in the x-axis of the graph of f.

b. To find the outputs of t, multiply the inputs by -1 and then evaluate f. The graph of t consists of the points $(x, f(-x))$.

x	-2	0	2
$-x$	2	0	-2
$f(-x)$	2	1	0



The graph of t is a reflection in the y-axis of the graph of f.

Monitoring Progress

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Using f, graph (a) g and (b) h. Describe the transformations from the graph of f to the graphs of g and h.

1. $f(x) = 3x + 1$; $g(x) = f(x) - 2$; $h(x) = f(x - 2)$

2. $f(x) = -4x - 2$; $g(x) = -f(x)$; $h(x) = f(-x)$

Stretches and Shrinks

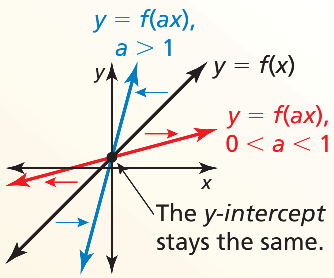
You can transform a function by multiplying all the x-coordinates (inputs) by the same factor a. When $a > 1$, the transformation is a **horizontal shrink** because the graph shrinks toward the y-axis. When $0 < a < 1$, the transformation is a **horizontal stretch** because the graph stretches away from the y-axis. In each case, the y-intercept stays the same.

You can also transform a function by multiplying all the y-coordinates (outputs) by the same factor a. When $a > 1$, the transformation is a **vertical stretch** because the graph stretches away from the x-axis. When $0 < a < 1$, the transformation is a **vertical shrink** because the graph shrinks toward the x-axis. In each case, the x-intercept stays the same.

Core Concept

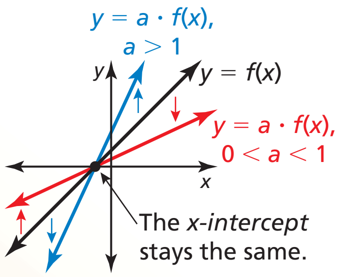
Horizontal Stretches and Shrinks

The graph of $y = f(ax)$ is a horizontal stretch or shrink by a factor of $\frac{1}{a}$ of the graph of $y = f(x)$, where $a > 0$ and $a \neq 1$.



Vertical Stretches and Shrinks

The graph of $y = a \cdot f(x)$ is a vertical stretch or shrink by a factor of a of the graph of $y = f(x)$, where $a > 0$ and $a \neq 1$.



STUDY TIP

The graphs of $y = f(-ax)$ and $y = -a \cdot f(x)$ represent a stretch or shrink *and* a reflection in the x- or y-axis of the graph of $y = f(x)$.

EXAMPLE 3 Horizontal and Vertical Stretches

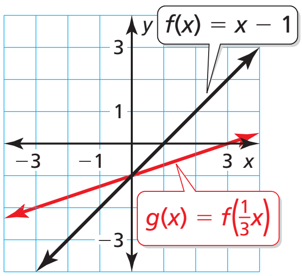
Let $f(x) = x - 1$. Graph (a) $g(x) = f\left(\frac{1}{3}x\right)$ and (b) $h(x) = 3f(x)$. Describe the transformations from the graph of f to the graphs of g and h.

SOLUTION

a. To find the outputs of g, multiply the inputs by $\frac{1}{3}$. Then evaluate f. The graph of g consists of the points $\left(x, f\left(\frac{1}{3}x\right)\right)$.

x	-3	0	3
$\frac{1}{3}(x)$	-1	0	1
$f\left(\frac{1}{3}x\right)$	-2	-1	0

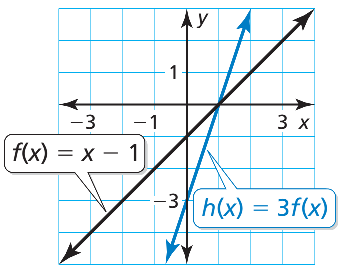
► The graph of g is a horizontal stretch of the graph of f by a factor of $1 \div \frac{1}{3} = 3$.



b. To find the outputs of h, multiply the outputs of f by 3. The graph of h consists of the points $(x, 3f(x))$.

x	0	1	2
$f(x)$	-1	0	1
$3f(x)$	-3	0	3

► The graph of h is a vertical stretch of the graph of f by a factor of 3.



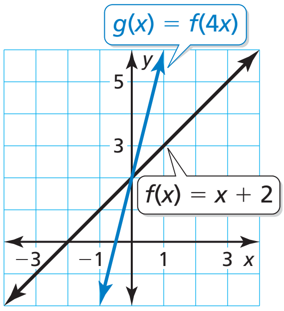
EXAMPLE 4 Horizontal and Vertical Shrinks

Let $f(x) = x + 2$. Graph (a) $g(x) = f(4x)$ and (b) $h(x) = \frac{1}{4}f(x)$. Describe the transformations from the graph of f to the graphs of g and h.

SOLUTION

a. To find the outputs of g, multiply the inputs by 4. Then evaluate f. The graph of g consists of the points $(x, f(4x))$.

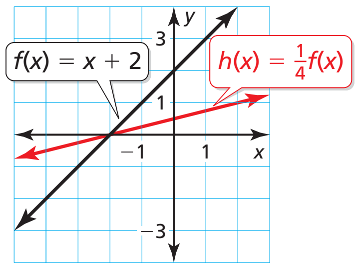
x	−1	0	1
4x	−4	0	4
f(4x)	−2	2	6



▶ The graph of g is a horizontal shrink of the graph of f by a factor of $\frac{1}{4}$.

b. To find the outputs of h, multiply the outputs of f by $\frac{1}{4}$. The graph of h consists of the points $\left(x, \frac{1}{4}f(x)\right)$.

x	−2	0	2
f(x)	0	2	4
$\frac{1}{4}$f(x)	0	$\frac{1}{2}$	1



▶ The graph of h is a vertical shrink of the graph of f by a factor of $\frac{1}{4}$.

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Using f, graph (a) g and (b) h. Describe the transformations from the graph of f to the graphs of g and h.

3. $f(x) = 4x - 2$; $g(x) = f\left(\frac{1}{2}x\right)$; $h(x) = 2f(x)$

4. $f(x) = -3x + 4$; $g(x) = f(2x)$; $h(x) = \frac{1}{2}f(x)$

Combining Transformations

STUDY TIP

You can perform transformations on the graph of *any* function f using these steps.

Core Concept

Transformations of Graphs

The graph of $y = a \cdot f(x - h) + k$ or the graph of $y = f(ax - h) + k$ can be obtained from the graph of $y = f(x)$ by performing these steps.

Step 1 Translate the graph of $y = f(x)$ horizontally h units.

Step 2 Use a to stretch or shrink the resulting graph from Step 1.

Step 3 Reflect the resulting graph from Step 2 when $a < 0$.

Step 4 Translate the resulting graph from Step 3 vertically k units.

EXAMPLE 5 Combining Transformations

Graph $f(x) = x$ and $g(x) = -2x + 3$. Describe the transformations from the graph of f to the graph of g.

SOLUTION

ANOTHER WAY

You could also rewrite g as $g(x) = f(-2x) + 3$. In this case, the transformations from the graph of f to the graph of g will be different from those in Example 5.

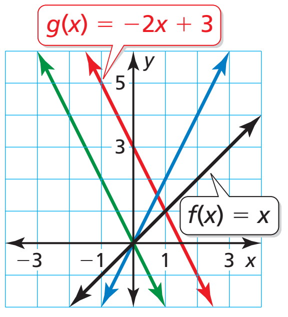
Note that you can rewrite g as $g(x) = -2f(x) + 3$.

Step 1 There is no horizontal translation from the graph of f to the graph of g.

Step 2 Stretch the graph of f vertically by a factor of 2 to get the graph of $h(x) = 2x$.

Step 3 Reflect the graph of h in the x-axis to get the graph of $r(x) = -2x$.

Step 4 Translate the graph of r vertically 3 units up to get the graph of $g(x) = -2x + 3$.

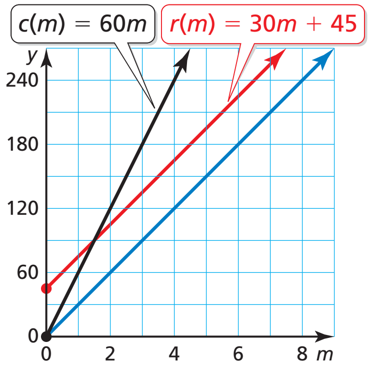


EXAMPLE 6 Solving a Real-Life Problem

A cable company charges customers \$60 per month for its service, with no installation fee. The cost to a customer is represented by $c(m) = 60m$, where m is the number of months of service. To attract new customers, the cable company reduces the monthly fee to \$30 but adds an installation fee of \$45. The cost to a new customer is represented by $r(m) = 30m + 45$, where m is the number of months of service. Describe the transformations from the graph of c to the graph of r.

SOLUTION

Note that you can rewrite r as $r(m) = \frac{1}{2}c(m) + 45$. In this form, you can use the order of operations to get the outputs of r from the outputs of c. First, multiply the outputs of c by $\frac{1}{2}$ to get $h(m) = 30m$. Then add 45 to the outputs of h to get $r(m) = 30m + 45$.



▶ The transformations are a vertical shrink by a factor of $\frac{1}{2}$ and then a vertical translation 45 units up.

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5. Graph $f(x) = x$ and $h(x) = \frac{1}{4}x - 2$. Describe the transformations from the graph of f to the graph of h.

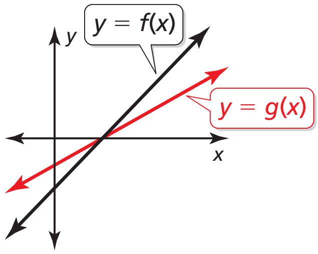
Vocabulary and Core Concept Check

1. **WRITING** Describe the relationship between $f(x) = x$ and all other nonconstant linear functions.

2. **VOCABULARY** Name four types of transformations. Give an example of each and describe how it affects the graph of a function.

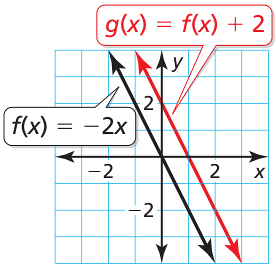
3. **WRITING** How does the value of a in the equation $y = f(ax)$ affect the graph of $y = f(x)$? How does the value of a in the equation $y = af(x)$ affect the graph of $y = f(x)$?

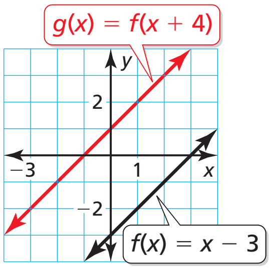
4. **REASONING** The functions f and g are linear functions. The graph of g is a vertical shrink of the graph of f. What can you say about the x-intercepts of the graphs of f and g? Is this always true? Explain.



Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, use the graphs of f and g to describe the transformation from the graph of f to the graph of g. *(See Example 1.)*

5. 

6. 

7. $f(x) = \frac{1}{3}x + 3$; $g(x) = f(x) - 3$

8. $f(x) = -3x + 4$; $g(x) = f(x) + 1$

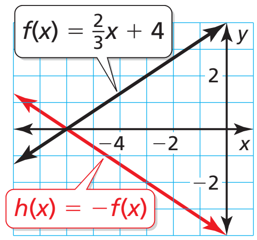
9. $f(x) = -x - 2$; $g(x) = f(x + 5)$

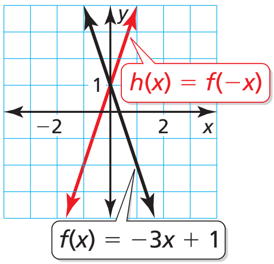
10. $f(x) = \frac{1}{2}x - 5$; $g(x) = f(x - 3)$

11. **MODELING WITH MATHEMATICS** You and a friend start biking from the same location. Your distance d (in miles) after t minutes is given by the function $d(t) = \frac{1}{5}t$. Your friend starts biking 5 minutes after you. Your friend's distance f is given by the function $f(t) = d(t - 5)$. Describe the transformation from the graph of d to the graph of f.

12. **MODELING WITH MATHEMATICS** The total cost C (in dollars) to cater an event with p people is given by the function $C(p) = 18p + 50$. The set-up fee increases by \$25. The new total cost T is given by the function $T(p) = C(p) + 25$. Describe the transformation from the graph of C to the graph of T.

In Exercises 13–16, use the graphs of f and h to describe the transformation from the graph of f to the graph of h. *(See Example 2.)*

13. 

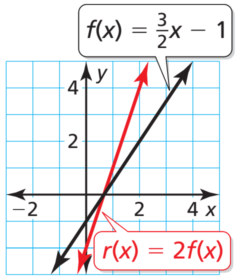
14. 

15. $f(x) = -5 - x$; $h(x) = f(-x)$

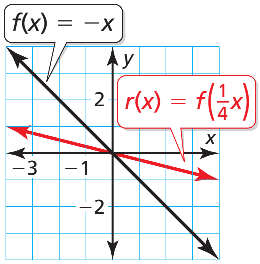
16. $f(x) = \frac{1}{4}x - 2$; $h(x) = -f(x)$

In Exercises 17–22, use the graphs of *f* and *r* to describe the transformation from the graph of *f* to the graph of *r*. *(See Example 3.)*

17.



18.



19. $f(x) = -2x - 4;\ r(x) = f\left(\frac{1}{2}x\right)$

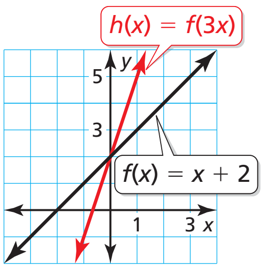
20. $f(x) = 3x + 5;\ r(x) = f\left(\frac{1}{3}x\right)$

21. $f(x) = \frac{2}{3}x + 1;\ r(x) = 3f(x)$

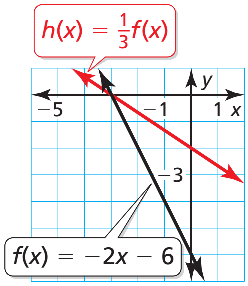
22. $f(x) = -\frac{1}{4}x - 2;\ r(x) = 4f(x)$

In Exercises 23–28, use the graphs of *f* and *h* to describe the transformation from the graph of *f* to the graph of *h*. *(See Example 4.)*

23.



24.



25. $f(x) = 3x - 12;\ h(x) = \frac{1}{6}f(x)$

26. $f(x) = -x + 1;\ h(x) = f(2x)$

27. $f(x) = -2x - 2;\ h(x) = f(5x)$

28. $f(x) = 4x + 8;\ h(x) = \frac{3}{4}f(x)$

In Exercises 29–34, use the graphs of *f* and *g* to describe the transformation from the graph of *f* to the graph of *g*.

29. $f(x) = x - 2;\ g(x) = \frac{1}{4}f(x)$

30. $f(x) = -4x + 8;\ g(x) = -f(x)$

31. $f(x) = -2x - 7;\ g(x) = f(x - 2)$

32. $f(x) = 3x + 8;\ g(x) = f\left(\frac{2}{3}x\right)$

33. $f(x) = x - 6;\ g(x) = 6f(x)$

34. $f(x) = -x;\ g(x) = f(x) - 3$

In Exercises 35–38, write a function *g* in terms of *f* so that the statement is true.

35. The graph of g is a horizontal translation 2 units right of the graph of f.

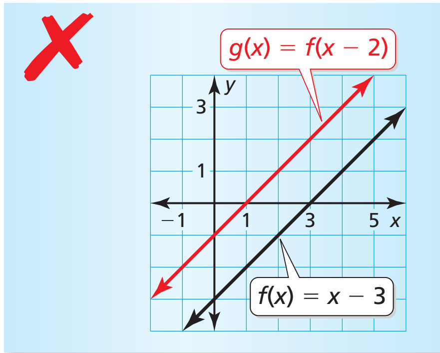
36. The graph of g is a reflection in the y-axis of the graph of f.

37. The graph of g is a vertical stretch by a factor of 4 of the graph of f.

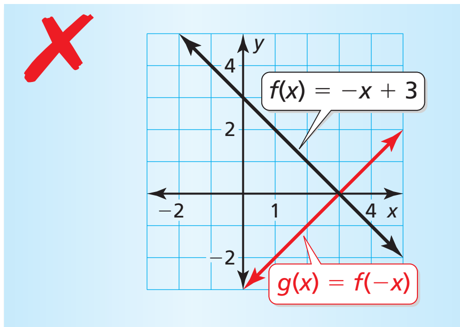
38. The graph of g is a horizontal shrink by a factor of $\frac{1}{5}$ of the graph of f.

ERROR ANALYSIS **In Exercises 39 and 40, describe and correct the error in graphing *g*.**

39.



40.



In Exercises 41–46, graph *f* and *h*. Describe the transformations from the graph of *f* to the graph of *h*. *(See Example 5.)*

41. $f(x) = x;\ h(x) = \frac{1}{3}x + 1$

42. $f(x) = x;\ h(x) = 4x - 2$

43. $f(x) = x;\ h(x) = -3x - 4$

44. $f(x) = x;\ h(x) = -\frac{1}{2}x + 3$

45. $f(x) = 2x;\ h(x) = 6x - 5$

46. $f(x) = 3x;\ h(x) = -3x - 7$

47. MODELING WITH MATHEMATICS The function $t(x) = -4x + 72$ represents the temperature from 5 P.M. to 11 P.M., where x is the number of hours after 5 P.M. The function $d(x) = 4x + 72$ represents the temperature from 10 A.M. to 4 P.M., where x is the number of hours after 10 A.M. Describe the transformation from the graph of t to the graph of d.

48. MODELING WITH MATHEMATICS A school sells T-shirts to promote school spirit. The school's profit is given by the function $P(x) = 8x - 150$, where x is the number of T-shirts sold. During the play-offs, the school increases the price of the T-shirts. The school's profit during the play-offs is given by the function $Q(x) = 16x - 200$, where x is the number of T-shirts sold. Describe the transformations from the graph of P to the graph of Q. *(See Example 6.)*

49. USING STRUCTURE The graph of $g(x) = a \cdot f(x - b) + c$ is a transformation of the graph of the linear function f. Select the word or value that makes each statement true.

reflection	translation	-1
stretch	shrink	0
left	right	1
y-axis	x-axis	

a. The graph of g is a vertical ______ of the graph of f when $a = 4$, $b = 0$, and $c = 0$.

b. The graph of g is a horizontal translation ______ of the graph of f when $a = 1$, $b = 2$, and $c = 0$.

c. The graph of g is a vertical translation 1 unit up of the graph of f when $a = 1$, $b = 0$, and $c =$ _____.

50. USING STRUCTURE The graph of $h(x) = a \cdot f(bx - c) + d$ is a transformation of the graph of the linear function f. Select the word or value that makes each statement true.

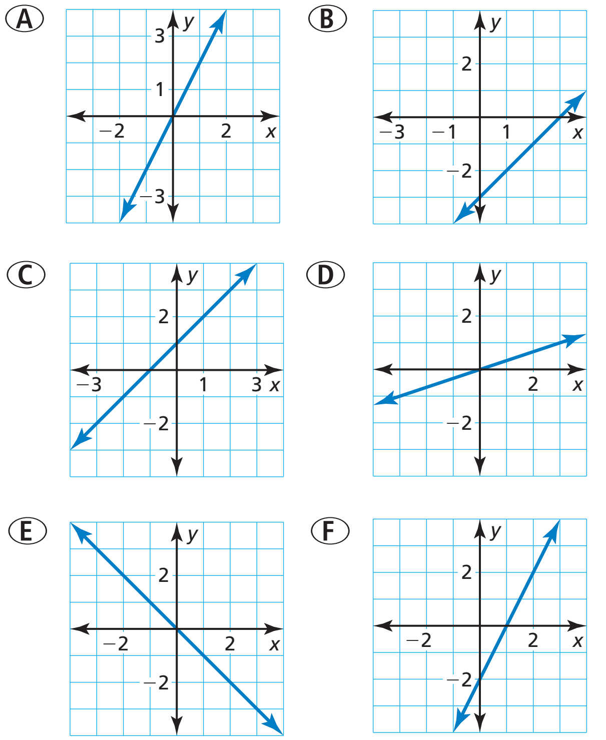
vertical	horizontal	0
stretch	shrink	$\frac{1}{5}$
y-axis	x-axis	5

a. The graph of h is a ______ shrink of the graph of f when $a = \frac{1}{3}$, $b = 1$, $c = 0$, and $d = 0$.

b. The graph of h is a reflection in the ______ of the graph of f when $a = 1$, $b = -1$, $c = 0$, and $d = 0$.

c. The graph of h is a horizontal stretch of the graph of f by a factor of 5 when $a = 1$, $b =$ _____, $c = 0$, and $d = 0$.

51. ANALYZING GRAPHS Which of the graphs are related by only a translation? Explain.



52. ANALYZING RELATIONSHIPS A swimming pool is filled with water by a hose at a rate of 1020 gallons per hour. The amount v (in gallons) of water in the pool after t hours is given by the function $v(t) = 1020t$. How does the graph of v change in each situation?

a. A larger hose is found. Then the pool is filled at a rate of 1360 gallons per hour.

b. Before filling up the pool with a hose, a water truck adds 2000 gallons of water to the pool.

53. ANALYZING RELATIONSHIPS You have \$50 to spend on fabric for a blanket. The amount m (in dollars) of money you have after buying y yards of fabric is given by the function $m(y) = -9.98y + 50$. How does the graph of m change in each situation?

a. You receive an additional \$10 to spend on the fabric.

b. The fabric goes on sale, and each yard now costs \$4.99.

54. THOUGHT PROVOKING Write a function g whose graph passes through the point (4, 2) and is a transformation of the graph of $f(x) = x$.

In Exercises 55–60, graph f and g. Write g in terms of f. Describe the transformation from the graph of f to the graph of g.

55. $f(x) = 2x - 5$; $g(x) = 2x - 8$

56. $f(x) = 4x + 1$; $g(x) = -4x - 1$

57. $f(x) = 3x + 9$; $g(x) = 3x + 15$

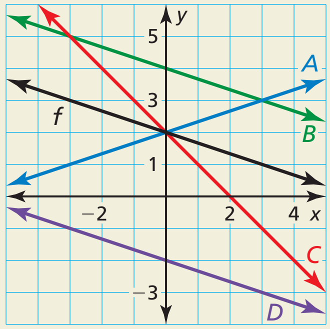
58. $f(x) = -x - 4$; $g(x) = x - 4$

59. $f(x) = x + 2$; $g(x) = \frac{2}{3}x + 2$

60. $f(x) = x - 1$; $g(x) = 3x - 3$

61. REASONING The graph of $f(x) = x + 5$ is a vertical translation 5 units up of the graph of $f(x) = x$. How can you obtain the graph of $f(x) = x + 5$ from the graph of $f(x) = x$ using a horizontal translation?

62. HOW DO YOU SEE IT? Match each function with its graph. Explain your reasoning.

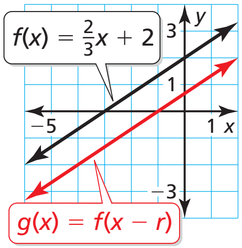


a. $a(x) = f(-x)$ **b.** $g(x) = f(x) - 4$

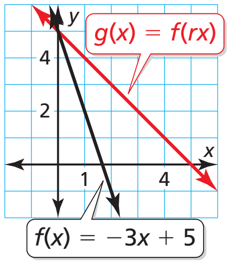
c. $h(x) = f(x) + 2$ **d.** $k(x) = f(3x)$

REASONING In Exercises 63–66, find the value of r.

63.



64.



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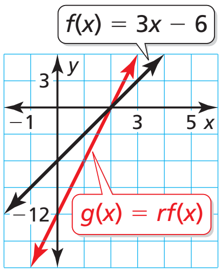
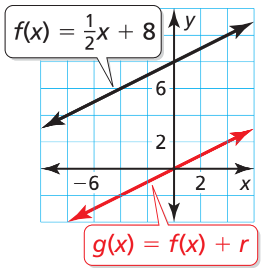


66.



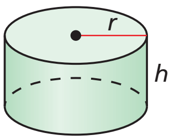
67. CRITICAL THINKING When is the graph of $y = f(x) + w$ the same as the graph of $y = f(x + w)$ for linear functions? Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

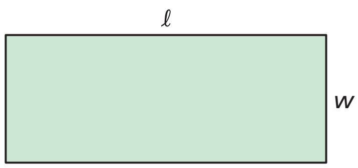
Solve the formula for the indicated variable. *(Section 1.4)*

68. Solve for h.



$V = \pi r^2 h$

69. Solve for w.



$P = 2\ell + 2w$

Solve the inequality. Graph the solution, if possible. *(Section 2.2)*

70. $x - 3 \le 14$ **71.** $16 < x + 4$ **72.** $-25 \ge x + 7$ **73.** $x - 5 > -15$

3.4–3.7 What Did You Learn?

Core Vocabulary

standard form, *p. 116*
x-intercept, *p. 117*
y-intercept, *p. 117*
zero of a function, *p. 118*
slope, *p. 124*
rise, *p. 124*
run, *p. 124*
slope-intercept form, *p. 126*
constant function, *p. 126*
direct variation, *p. 134*
constant of variation, *p. 134*
family of functions, *p. 140*
parent function, *p. 140*
transformation, *p. 140*
translation, *p. 140*
reflection, *p. 141*
horizontal shrink, *p. 142*
horizontal stretch, *p. 142*
vertical stretch, *p. 142*
vertical shrink, *p. 142*

Core Concepts

Section 3.4

Horizontal and Vertical Lines, *p. 116*
Using Intercepts to Graph Equations, *p. 117*
Finding Zeros of Functions, *p. 118*

Section 3.5

Slope, *p. 124*
Slope-Intercept Form, *p. 126*

Section 3.6

Identifying Direct Variation Equations, *p. 134*
Writing Direct Variation Equations, *p. 135*

Section 3.7

Horizontal Translations, *p. 140*
Vertical Translations, *p. 140*
Reflections in the *x*-axis, *p. 141*
Reflections in the *y*-axis, *p. 141*
Horizontal Stretches and Shrinks, *p. 142*
Vertical Stretches and Shrinks, *p. 142*
Transformations of Graphs, *p. 143*

Mathematical Thinking

1. Explain how you determined what units of measure to use for the horizontal and vertical axes in Exercise 37 on page 130.

2. Explain your plan for solving Exercise 48 on page 147.

Performance Task

The Cost of a T-Shirt

You receive bids for making T-shirts for your class fundraiser from four companies. To present the pricing information, one company uses a table, one company uses a written description, one company uses an equation, and one company uses a graph. How will you compare the different representations and make the final choice?

To explore the answer to this question and more, go to ***BigIdeasMath.com***.

3 Chapter Review

3.1 Functions (pp. 89–96)

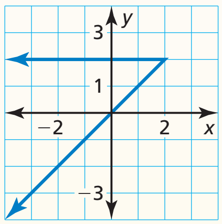
Determine whether the relation is a function. Explain.

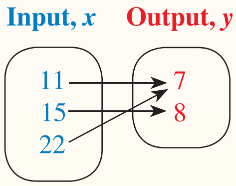
Input, x	2	5	7	9	14
Output, y	5	11	19	12	3

Every input has exactly one output.

▶ So, the relation is a function.

Determine whether the relation is a function. Explain.

1. 

2. 

3. $y = -x + 6$ with inputs $x = 0$ and $x = 3$

4. The function $y = 10x + 100$ represents the amount y (in dollars) of money in your bank account after you babysit for x hours.

a. Identify the independent and dependent variables.

b. You babysit for 4 hours. Find the domain and range of the function.

3.2 Linear Functions (pp. 97–106)

Does the table or equation represent a *linear* or *nonlinear* function? Explain.

a.

+4 +4 +4

x	6	10	14	18
y	5	9	14	20

+4 +5 +6

As x increases by 4, y increases by different amounts. The rate of change is *not* constant.

▶ So, the function is nonlinear.

b. $y = 3x - 4$

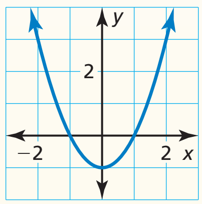
The equation is in the form $y = mx + b$.

▶ So, the equation represents a linear function.

Does the table or graph represent a *linear* or *nonlinear* function? Explain.

5.

x	2	7	12	17
y	2	−1	−4	−7

6. 

7. The function $y = 60 - 8x$ represents the amount y (in dollars) of money you have after buying x movie tickets. (a) Find the domain of the function. Is the domain discrete or continuous? Explain. (b) Graph the function using its domain. (c) Find the range of the function.

3.3 Function Notation (pp. 107–112)

a. Evaluate $f(x) = 3x - 9$ when $x = 2$.

$f(x) = 3x - 9$ — Write the function.

$f(2) = 3(2) - 9$ — Substitute 2 for x.

$= 6 - 9$ — Multiply.

$= -3$ — Subtract.

▶ When $x = 2$, $f(x) = -3$.

b. For $f(x) = 4x$, find the value of x for which $f(x) = 12$.

$f(x) = 4x$ — Write the function.

$12 = 4x$ — Substitute 12 for $f(x)$.

$3 = x$ — Divide each side by 4.

▶ When $x = 3$, $f(x) = 12$.

Evaluate the function when $x = -3$, 0, and 5.

8. $f(x) = x + 8$

9. $g(x) = 4 - 3x$

Find the value of x so that the function has the given value.

10. $k(x) = 7x;\ k(x) = 49$

11. $r(x) = -5x - 1;\ r(x) = 19$

3.4 Graphing Linear Equations in Standard Form (pp. 115–122)

Use intercepts to graph the equation $2x + 3y = 6$.

Step 1 Find the intercepts.

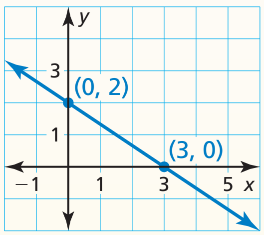
To find the x-intercept, substitute 0 for y and solve for x.

$$2x + 3y = 6$$
$$2x + 3(0) = 6$$
$$x = 3$$

To find the y-intercept, substitute 0 for x and solve for y.

$$2x + 3y = 6$$
$$2(0) + 3y = 6$$
$$y = 2$$

Step 2 Plot the points and draw the line.
The x-intercept is 3, so plot the point (3, 0).
The y-intercept is 2, so plot the point (0, 2).
Draw a line through the points.



Graph the linear equation.

12. $8x - 4y = 16$

13. $-12x - 3y = 36$

14. $y = -5$

15. $x = 6$

Find the zero of the function.

16. $p(x) = 2x - 10$

17. $v(x) = 0.25x + 3$

18. $b(x) = -8x + 4$

3.5 Graphing Linear Equations in Slope-Intercept Form (pp. 123–132)

a. The points represented by the table lie on a line. How can you find the slope of the line from the table? What is the slope of the line?

x	y
1	−7
4	2
7	11
10	20

Choose any two points from the table and use the slope formula. Use the points $(x_1, y_1) = (1, -7)$ and $(x_2, y_2) = (4, 2)$.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-7)}{4 - 1} = \frac{9}{3}, \text{ or } 3$$

▶ The slope is 3.

b. Graph $-\frac{1}{2}x + y = 1$. Identify the x-intercept.

Step 1 Rewrite the equation in slope-intercept form.

$$y = \frac{1}{2}x + 1$$

Step 2 Find the slope and the y-intercept.

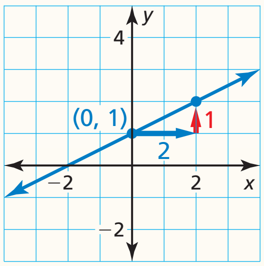
$$m = \frac{1}{2} \text{ and } b = 1$$

Step 3 The y-intercept is 1. So, plot (0, 1).

Step 4 Use the slope to find another point on the line.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{1}{2}$$

Plot the point that is 2 units right and 1 unit up from (0, 1). Draw a line through the two points.



▶ The line crosses the x-axis at (−2, 0). So, the x-intercept is −2.

The points represented by the table lie on a line. Find the slope of the line.

19.

x	y
6	9
11	15
16	21
21	27

20.

x	y
3	−5
3	−2
3	5
3	8

21.

x	y
−4	−1
−3	−1
1	−1
9	−1

Graph the linear equation. Identify the x-intercept.

22. $y = 2x + 4$

23. $-5x + y = -10$

24. $x + 3y = 9$

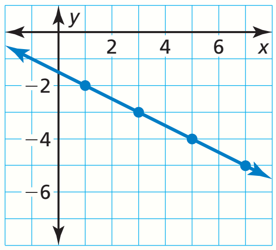
25. A linear function h models a relationship in which the dependent variable decreases 2 units for every 3 units the independent variable increases. Graph h when $h(0) = 2$. Identify the slope, y-intercept, and x-intercept of the graph.

3.6 Modeling Direct Variation (pp. 133–138)

a. Determine whether x and y show direct variation. Explain.

x	1	3	5	7
y	-2	-3	-4	-5

Plot the ordered pairs. Then draw a line through the points.



▶ The line does *not* pass through the origin. So, x and y do *not* show direct variation.

b. The table shows the costs C (in dollars) for h hours of repair work on your car. Explain why C varies directly with h. Then write a direct variation equation that relates h and C.

Number of hours, h	Cost (dollars), C
4	360
5	450
6	540

Find the ratio $\frac{C}{h}$ for each data pair (h, C).

$$\frac{360}{4} = 90, \frac{450}{5} = 90, \frac{540}{6} = 90$$

▶ All of the ratios are equal to 90. So, C varies directly with h.

Because $\frac{C}{h} = 90$, the constant of variation is 90. So, a direct variation equation is $C = 90h$.

Determine whether x and y show direct variation. Explain.

26. $-6x - y = 1$

27. $4x + y = 0$

28.

x	0	1	2	3
y	4	1	-2	-5

29.

x	-4	-2	0	2
y	-6	-3	0	3

30. The table shows the number p of gallons of paint needed to cover w square feet of wall space.

a. Explain why p varies directly with w. Then write a direct variation equation that relates w and p.

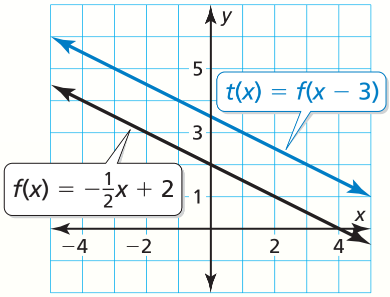
b. How much paint do you need to cover 1400 square feet of wall space?

Wall space (ft^2), w	Paint (gallons), p
700	2
2100	6
3500	10

3.7 Transformations of Graphs of Linear Functions (pp. 139–148)

a. **Let $f(x) = -\frac{1}{2}x + 2$. Graph $t(x) = f(x - 3)$. Describe the transformation from the graph of f to the graph of t.**

The function t is of the form $y = f(x - h)$, where $h = 3$. So, the graph of t is a horizontal translation 3 units right of the graph of f.



b. **Graph $f(x) = x$ and $g(x) = -3x - 2$. Describe the transformations from the graph of f to the graph of g.**

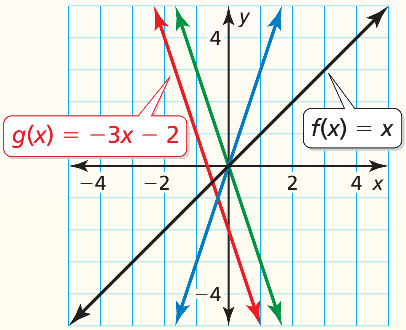
Note that you can rewrite g as $g(x) = -3f(x) - 2$.

Step 1 There is no horizontal translation from the graph of f to the graph of g.

Step 2 Stretch the graph of f vertically by a factor of 3 to get the graph of $h(x) = 3x$.

Step 3 Reflect the graph of h in the x-axis to get the graph of $r(x) = -3x$.

Step 4 Translate the graph of r vertically 2 units down to get the graph of $g(x) = -3x - 2$.



Let $f(x) = 3x + 4$. Graph f and h. Describe the transformation from the graph of f to the graph of h.

31. $h(x) = f(x + 3)$

32. $h(x) = f(x) + 1$

33. $h(x) = f(-x)$

34. $h(x) = -f(x)$

35. $h(x) = 3f(x)$

36. $h(x) = f(6x)$

37. Graph $f(x) = x$ and $g(x) = 5x + 1$. Describe the transformations from the graph of f to the graph of g.

3 Chapter Test

Determine whether the relation is a function. If the relation is a function, determine whether the function is *linear* or *nonlinear*. Explain.

1.

x	-1	0	1	2
y	6	5	9	14

2. $y = -2x + 3$

3. $x = -2$

Graph the equation and identify the intercept(s). Find the slope of the line.

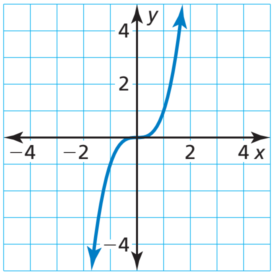
4. $2x - 3y = 6$

5. $y = 4.5$

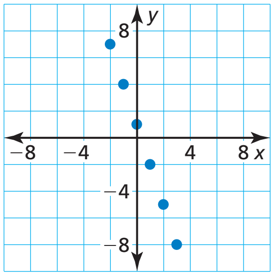
6. $y = -7x$

Find the domain and range of the function represented by the graph. Determine whether the domain is *discrete* or *continuous*. Explain.

7.



8.



Graph *f* and *g*. Describe the transformations from the graph of *f* to the graph of *g*.

9. $f(x) = 2x + 4$; $g(x) = \frac{1}{2}f(x)$

10. $f(x) = x$; $g(x) = -x + 3$

Find the zero of the function.

11. $h(x) = \frac{1}{3}x - 1$

12. $d(x) = -8x - 2$

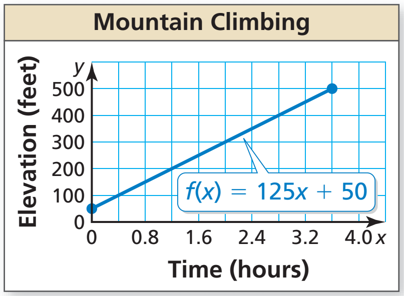
13. Function A represents the amount of money in a jar based on the number of quarters in the jar. Function B represents your distance from home over time. Compare the domains.

14. A mountain climber is scaling a 500-foot cliff. The graph shows the elevation of the climber over time.

a. Find and interpret the slope and the y-intercept of the graph.

b. Explain two ways to find $f(3)$. Then find $f(3)$ and interpret its meaning.

c. How long does it take the climber to reach the top of the cliff? Justify your answer.



15. Without graphing, compare the slopes and the intercepts of the graphs of the functions $f(x) = x + 1$ and $g(x) = f(2x)$.

16. You are making cranberry sauce. The number c of cups of fresh cranberries varies directly with the number s of servings of cranberry sauce. A recipe that serves 16 people calls for 4 cups of fresh cranberries. How many cups of fresh cranberries should you use to make 28 servings of cranberry sauce?

3 Standards Assessment

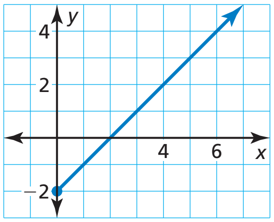
1. Which statement is true for the function whose graph is shown? *(TEKS A.2.A)*

Ⓐ The domain is $x \leq 0$.

Ⓑ The domain is $x \leq -2$.

Ⓒ The range is $y \leq -2$.

Ⓓ The range is $y \geq -2$.



2. You want to buy a jacket at a clothing store, and you can spend at most \$30. You have a coupon for 10% off any item in the store. Which inequality describes the original prices p (in dollars) of jackets you can buy? *(TEKS A.5.B)*

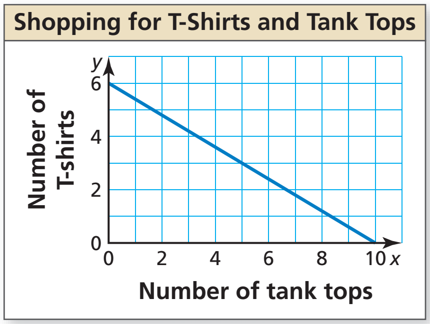
Ⓕ $p \geq 27.27$

Ⓖ $p \leq 27.27$

Ⓗ $p \leq 33.33$

Ⓙ $p \geq 33.33$

3. **GRIDDED ANSWER** The graph shows the numbers of T-shirts and tank tops you can buy with the amount of money you have. Suppose you buy only T-shirts. What is the greatest number you can buy? *(TEKS A.3.C)*



4. What is the solution of $3x + 24 = 18x - 5$? *(TEKS A.5.A)*

Ⓐ $x = \frac{15}{29}$

Ⓑ $x = \frac{29}{15}$

Ⓒ $x = -\frac{15}{29}$

Ⓓ $x = \frac{19}{21}$

5. The value of y varies directly with x. Which function represents the relationship between x and y if $y = \frac{10}{3}$ when $x = 60$? *(TEKS A.2.D)*

Ⓕ $y = 200x$

Ⓖ $y = \frac{1}{18}x$

Ⓗ $y = \frac{190}{3}x$

Ⓙ $y = 18x$

6. On the five science tests you have taken this semester, you received the following scores: 75, 82, 90, 84, and 71. You want a mean score of at least 80 after you take the sixth test. Which inequality describes the scores s you can earn on your sixth test to meet your goal? *(TEKS A.5.B)*

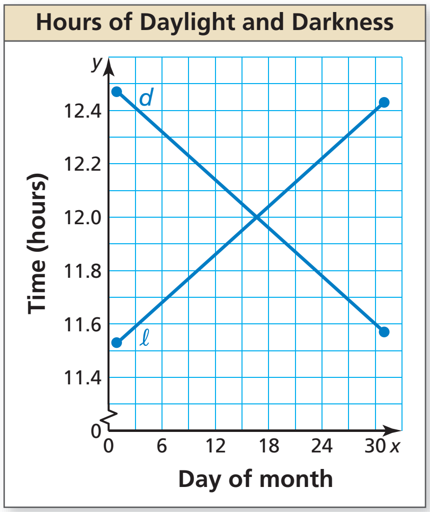
Ⓐ $s > 76$

Ⓑ $s > 78$

Ⓒ $s \geq 78$

Ⓓ $s \geq 80$

7. The number of hours of daylight in Austin, Texas, during the month of March can be modeled by the function $\ell(x) = 0.03x + 11.5$, where x is the day of the month. The number of hours of darkness can be modeled by d. The graphs of ℓ and d are shown. Which equation describes the relationship between ℓ and d? *(TEKS A.3.E)*



Ⓕ $d(x) = 24 - \ell(x)$

Ⓖ $\ell(x) = 24 + d(x)$

Ⓗ $\ell(x) = 12 - d(x)$

Ⓙ none of the above

8. Solve the literal equation $xy = 2z - 3y$ for y. *(TEKS A.12.E)*

Ⓐ $y = \frac{1}{3x}(2z)$

Ⓑ $y = \frac{2z}{x + 3}$

Ⓒ $y = \frac{1}{3}(2z - xy)$

Ⓓ $y = 2z - x - 3$

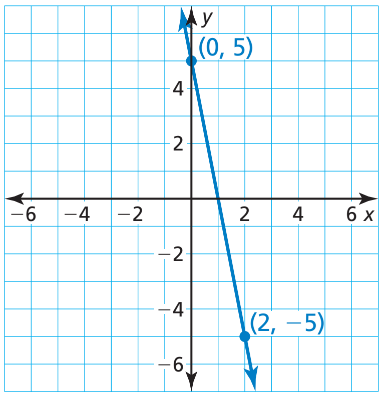
9. What is the slope of the line shown? *(TEKS A.3.A)*

Ⓕ 5

Ⓖ $\frac{1}{5}$

Ⓗ 0

Ⓙ -5



10. For which value of a does the solution of the compound inequality $a < 3x + 8$ *or* $a < -4x - 1$ consist of numbers greater than 5 *or* less than -6? *(TEKS A.5.B)*

Ⓐ 16

Ⓑ 19

Ⓒ 23

Ⓓ 26

11. For the function f, $f(-5) = 3$, and $f(3) = -2$. If $y = f(x)$, what is the value of y when $x = 3$? *(TEKS A.12.B)*

Ⓕ -5

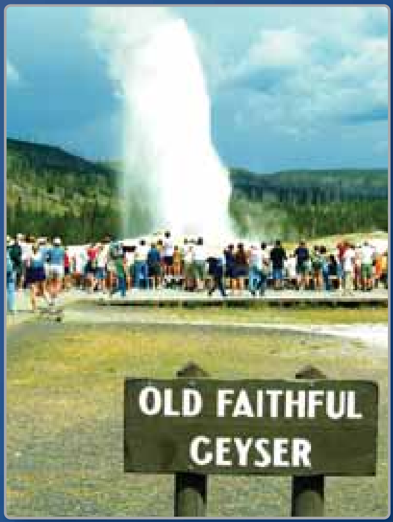
Ⓖ -2

Ⓗ 3

Ⓙ 5

4 Writing Linear Functions

Online Auction *(p. 205)*



Old Faithful Geyser *(p. 196)*

Helicopter Rescue *(p. 182)*

School Spirit *(p. 170)*

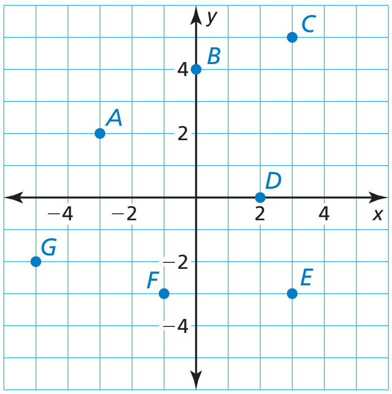
Renewable Energy *(p. 164)*

Mathematical Thinking: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

Maintaining Mathematical Proficiency

Using a Coordinate Plane (6.11)

Example 1 **What ordered pair corresponds to point A?**



Point A is 3 units to the left of the origin and 2 units up. So, the x-coordinate is -3 and the y-coordinate is 2.

▶ The ordered pair $(-3, 2)$ corresponds to point A.

Use the graph to answer the question.

1. What ordered pair corresponds to point G?

2. What ordered pair corresponds to point D?

3. Which point is located in Quadrant I?

4. Which point is located in Quadrant IV?

Rewriting Equations (A.12.E)

Example 2 **Solve the equation $3x - 2y = 8$ for y.**

$$3x - 2y = 8 \qquad \text{Write the equation.}$$

$$3x - 2y - 3x = 8 - 3x \qquad \text{Subtract } 3x \text{ from each side.}$$

$$-2y = 8 - 3x \qquad \text{Simplify.}$$

$$\frac{-2y}{-2} = \frac{8 - 3x}{-2} \qquad \text{Divide each side by } -2.$$

$$y = -4 + \frac{3}{2}x \qquad \text{Simplify.}$$

Solve the equation for y.

5. $x - y = 5$

6. $6x + 3y = -1$

7. $0 = 2y - 8x + 10$

8. $-x + 4y - 28 = 0$

9. $2y + 1 - x = 7x$

10. $y - 4 = 3x + 5y$

11. **ABSTRACT REASONING** Both coordinates of the point (x, y) are multiplied by a negative number. How does this change the location of the point? Be sure to consider points originally located in all four quadrants.

Mathematical Thinking

Mathematically proficient students use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solutions. **(A.1.B)**

Problem-Solving Strategies

Core Concept

Solve a Simpler Problem

When solving a real-life problem, if the numbers in the problem seem complicated, then try solving a simpler form of the problem. After you have solved the simpler problem, look for a general strategy. Then apply that strategy to the original problem.

EXAMPLE 1 Using a Problem-Solving Strategy

In the deli section of a grocery store, a half pound of sliced roast beef costs \$3.19. You buy 1.81 pounds. How much do you pay?

SOLUTION

Step 1 Solve a simpler problem.

Suppose the roast beef costs \$3 per half pound, and you buy 2 pounds.

$$\text{Total cost} = \frac{\$3}{1/2 \text{ lb}} \cdot 2 \text{ lb}$$ Use unit analysis to write a verbal model.

$$= \frac{\$6}{1 \cancel{\text{lb}}} \cdot 2 \cancel{\text{lb}}$$ Rewrite \$3 per $\frac{1}{2}$ pound as \$6 per pound.

$$= \$12$$ Simplify.

▶ In the simpler problem, you pay \$12.

Step 2 Apply the strategy to the original problem.

$$\text{Total cost} = \frac{\$3.19}{1/2 \text{ lb}} \cdot 1.81 \text{ lb}$$ Use unit analysis to write a verbal model.

$$= \frac{\$6.38}{1 \cancel{\text{lb}}} \cdot 1.81 \cancel{\text{lb}}$$ Rewrite \$3.19 per $\frac{1}{2}$ pound as \$6.38 per pound.

$$= \$11.55$$ Simplify.

▶ In the original problem, you pay \$11.55.

Your answer is reasonable because you bought about 2 pounds.

Monitoring Progress

1. You work $37\frac{1}{2}$ hours and earn \$352.50. What is your hourly wage?
2. You drive 1244.5 miles and use 47.5 gallons of gasoline. What is your car's gas mileage (in miles per gallon)?
3. You drive 236 miles in 4.6 hours. At the same rate, how long will it take you to drive 450 miles?

4.1 Writing Equations in Slope-Intercept Form

Texas Essential Knowledge and Skills
A.2.B
A.2.C
A.3.A

Essential Question

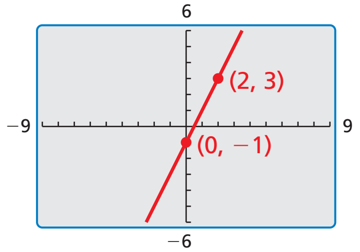
Given the graph of a linear function, how can you write an equation of the line?

EXPLORATION 1 Writing Equations in Slope-Intercept Form

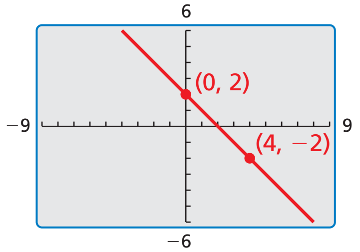
Work with a partner.

- Find the slope and y-intercept of each line.
- Write an equation of each line in slope-intercept form.
- Use a graphing calculator to verify your equation.

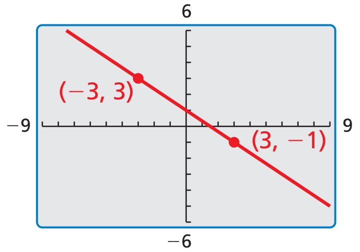
a.



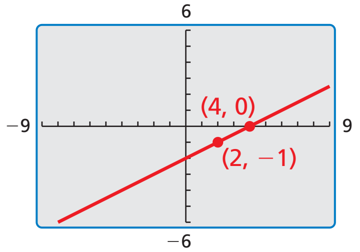
b.



c.



d.

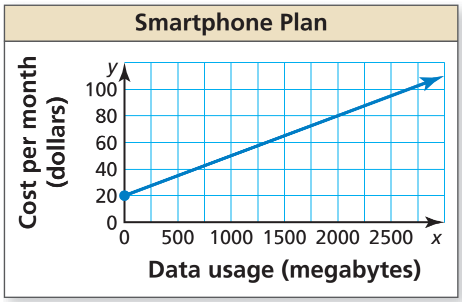


EXPLAINING MATHEMATICAL IDEAS

To be proficient in math, you need to routinely interpret your results in the context of the situation. The reason for studying mathematics is to enable you to model and solve real-life problems.

EXPLORATION 2 Mathematical Modeling

Work with a partner. The graph shows the cost of a smartphone plan.



a. What is the y-intercept of the line? Interpret the y-intercept in the context of the problem.

b. Approximate the slope of the line. Interpret the slope in the context of the problem.

c. Write an equation that represents the cost as a function of data usage.

Communicate Your Answer

3. Given the graph of a linear function, how can you write an equation of the line?

4. Give an example of a graph of a linear function that is different from those above. Then use the graph to write an equation of the line.

4.1 Lesson

What You Will Learn

- Write equations in slope-intercept form.
- Use linear equations to solve real-life problems.

Core Vocabulary

linear model, *p. 164*

Previous
slope-intercept form
function
rate

Writing Equations in Slope-Intercept Form

EXAMPLE 1 **Using Slopes and *y*-Intercepts to Write Equations**

Write an equation of each line with the given characteristics.

a. slope $= -3$; y-intercept $= \frac{1}{2}$ **b.** slope $= 4$; passes through $(-2, 5)$

SOLUTION

a. $y = mx + b$ Write the slope-intercept form.

$y = -3x + \frac{1}{2}$ Substitute -3 for m and $\frac{1}{2}$ for b.

▶ An equation is $y = -3x + \frac{1}{2}$.

b. Find the y-intercept.

$y = mx + b$ Write the slope-intercept form.

$5 = 4(-2) + b$ Substitute 4 for m, -2 for x, and 5 for y.

$13 = b$ Solve for b.

Write an equation.

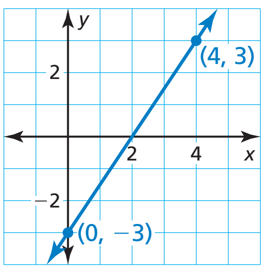
$y = mx + b$ Write the slope-intercept form.

$y = 4x + 13$ Substitute 4 for m and 13 for b.

▶ An equation is $y = 4x + 13$.

EXAMPLE 2 **Using a Graph to Write an Equation**

Write an equation of the line in slope-intercept form.



STUDY TIP

You can use any two points on a line to find the slope.

SOLUTION

Find the slope and y-intercept.

Let $(x_1, y_1) = (0, -3)$ and $(x_2, y_2) = (4, 3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-3)}{4 - 0} = \frac{6}{4}, \text{ or } \frac{3}{2}$$

Because the line crosses the y-axis at $(0, -3)$, the y-intercept is -3.

▶ So, the equation is $y = \frac{3}{2}x - 3$.

EXAMPLE 3 Using Points to Write Equations

Write an equation of each line that passes through the given points.

a. $(-3, 5), (0, -1)$

b. $(0, -5), (8, -5)$

SOLUTION

a. Find the slope and y-intercept.

$$m = \frac{-1 - 5}{0 - (-3)} = -2$$

Because the line crosses the y-axis at $(0, -1)$, the y-intercept is -1.

▶ So, an equation is $y = -2x - 1$.

b. Find the slope and y-intercept.

$$m = \frac{-5 - (-5)}{8 - 0} = 0$$

Because the line crosses the y-axis at $(0, -5)$, the y-intercept is -5.

▶ So, an equation is $y = -5$.

REMEMBER

If f is a function and x is in its domain, then $f(x)$ represents the output of f corresponding to the input x.

EXAMPLE 4 Writing a Linear Function

Write a linear function f with the values $f(0) = 10$ and $f(6) = 34$.

SOLUTION

Step 1 Write $f(0) = 10$ as $(0, 10)$ and $f(6) = 34$ as $(6, 34)$.

Step 2 Find the slope of the line that passes through $(0, 10)$ and $(6, 34)$.

$$m = \frac{34 - 10}{6 - 0} = \frac{24}{6}, \text{ or } 4$$

Step 3 Write an equation of the line. Because the line crosses the y-axis at $(0, 10)$, the y-intercept is 10.

$y = mx + b$ Write the slope-intercept form.

$y = 4x + 10$ Substitute 4 for m and 10 for b.

▶ A function is $f(x) = 4x + 10$.

Monitoring Progress

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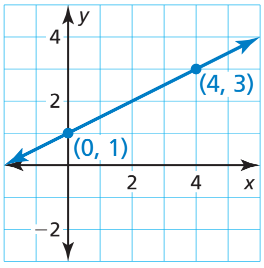
Write an equation of the line with the given characteristics.

1. slope $= 7$; y-intercept $= 2$

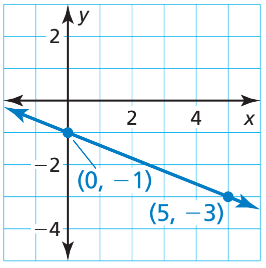
2. slope $= \frac{1}{3}$; passes through $(6, 1)$

Write an equation of the line in slope-intercept form.

3.



4.



5. Write an equation of the line that passes through $(0, -2)$ and $(4, 10)$.

6. Write a linear function g with the values $g(0) = 9$ and $g(8) = 7$.

Solving Real-Life Problems

A **linear model** is a linear function that models a real-life situation. When a quantity y changes at a constant rate with respect to a quantity x, you can use the equation $y = mx + b$ to model the relationship. The value of m is the constant rate of change, and the value of b is the initial, or starting, value of y.

EXAMPLE 5 Modeling with Mathematics

Excluding hydropower, U.S. power plants used renewable energy sources to generate 105 million megawatt hours of electricity in 2007. By 2012, the amount of electricity generated had increased to 219 million megawatt hours. Write a linear model that represents the number of megawatt hours generated by non-hydropower renewable energy sources as a function of the number of years since 2007. Use the model to predict the number of megawatt hours that will be generated in 2017.

SOLUTION

1. **Understand the Problem** You know the amounts of electricity generated in two distinct years. You are asked to write a linear model that represents the amount of electricity generated each year since 2007 and then predict a future amount.
2. **Make a Plan** Break the problem into parts and solve each part. Then combine the results to help you solve the original problem.

   Part 1 Define the variables. Find the initial value and the rate of change.

   Part 2 Write a linear model and predict the amount in 2017.
3. **Solve the Problem**

   Part 1 Let x represent the time (in years) since 2007 and let y represent the number of megawatt hours (in millions). Because time x is defined in years since 2007, 2007 corresponds to $x = 0$ and 2012 corresponds to $x = 5$. Let $(x_1, y_1) = (0, 105)$ and $(x_2, y_2) = (5, 219)$. The initial value is the y-intercept b, which is 105. The rate of change is the slope m.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{219 - 105}{5 - 0} = \frac{114}{5} = 22.8$$

   Part 2

Megawatt hours (millions)	=	Initial value	+	Rate of change	•	Years since 2007
y	=	105	+	22.8	•	x

$y = 105 + 22.8x$ Write the equation.

2017 corresponds to $x = 10$. → $y = 105 + 22.8(10)$ Substitute 10 for x.

$y = 333$ Simplify.

▶ The linear model is $y = 22.8x + 105$. The model predicts non-hydropower renewable energy sources will generate 333 million megawatt hours in 2017.

4. **Look Back** To check that your model is correct, verify that (0, 105) and (5, 219) are solutions of the equation.

Monitoring Progress

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7. The corresponding data for electricity generated by hydropower are 248 million megawatt hours in 2007 and 277 million megawatt hours in 2012. Write a linear model that represents the number of megawatt hours generated by hydropower as a function of the number of years since 2007.

4.1 Exercises

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Vocabulary and Core Concept Check

1. COMPLETE THE SENTENCE A linear function that models a real-life situation is called a __________.

2. WRITING Explain how you can use slope-intercept form to write an equation of a line given its slope and y-intercept.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, write an equation of the line with the given characteristics. *(See Example 1.)*

3. slope: 2
y-intercept: 9

4. slope: 0
passes through: $(-3, 5)$

5. slope: -3
passes through: $(2, -6)$

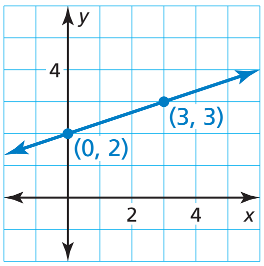
6. slope: -7
y-intercept: 1

7. slope: $\frac{2}{3}$
y-intercept: -8

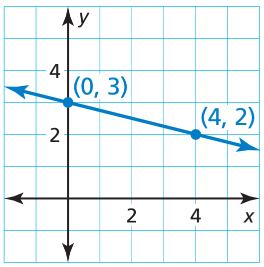
8. slope: $-\frac{3}{4}$
passes through: $(-8, 0)$

In Exercises 9–12, write an equation of the line in slope-intercept form. *(See Example 2.)*

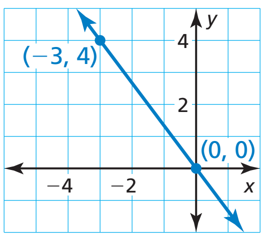
9.



10.



11.



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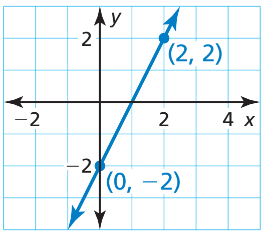


In Exercises 13–18, write an equation of the line that passes through the given points. *(See Example 3.)*

13. $(3, 1), (0, 10)$

14. $(2, 7), (0, -5)$

15. $(2, -4), (0, -4)$

16. $(-6, 0), (0, -24)$

17. $(0, 5), (-1.5, 1)$

18. $(0, 3), (-5, 2.5)$

In Exercises 19–24, write a linear function f with the given values. *(See Example 4.)*

19. $f(0) = 2, f(2) = 4$

20. $f(0) = 7, f(3) = 1$

21. $f(4) = -3, f(0) = -2$

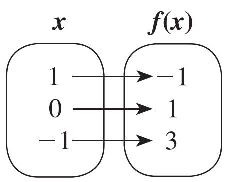
22. $f(5) = -1, f(0) = -5$

23. $f(-2) = 6, f(0) = -4$

24. $f(0) = 3, f(-6) = 3$

In Exercises 25 and 26, write a linear function f with the given values.

25.



26.

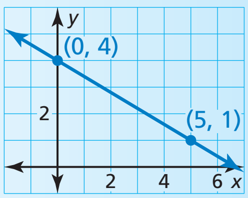
x	$f(x)$
-4	-2
-2	-1
0	0

27. ERROR ANALYSIS Describe and correct the error in writing an equation of the line with a slope of 2 and a y-intercept of 7.

$$y = 7x + 2$$

28. ERROR ANALYSIS Describe and correct the error in writing an equation of the line shown.

$$\text{slope} = \frac{1-4}{0-5}$$
$$= \frac{-3}{-5} = \frac{3}{5}$$
$$y = \frac{3}{5}x + 4$$



29. **MODELING WITH MATHEMATICS** In 1960, the world record for the men's mile was 3.91 minutes. In 1980, the record time was 3.81 minutes. *(See Example 5.)*

a. Write a linear model that represents the world record (in minutes) for the men's mile as a function of the number of years since 1960.

b. Use the model to estimate the record time in 2000 and predict the record time in 2020.

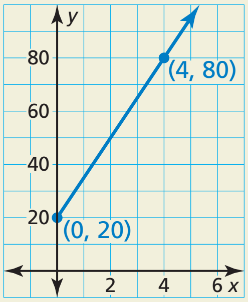
30. **MODELING WITH MATHEMATICS** A recording studio charges musicians an initial fee of \$50 to record an album. Studio time costs an additional \$75 per hour.

a. Write a linear model that represents the total cost of recording an album as a function of studio time (in hours).

b. Is it less expensive to purchase 12 hours of recording time at the studio or a \$750 music software program that you can use to record on your own computer? Explain.

31. **WRITING** A line passes through the points $(0, -2)$ and $(0, 5)$. Is it possible to write an equation of the line in slope-intercept form? Justify your answer.

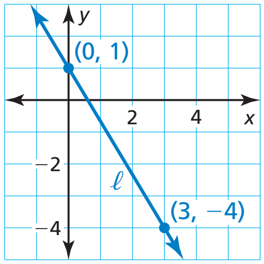
32. **THOUGHT PROVOKING** Describe a real-life situation involving a linear function whose graph passes through the points.



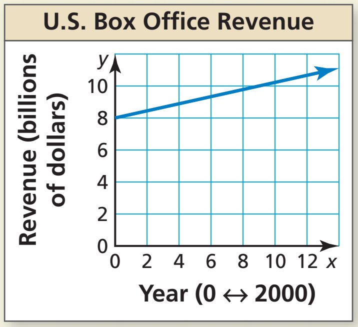
33. **REASONING** Recall that the standard form of a linear equation is $Ax + By = C$. Rewrite this equation in slope-intercept form. Use your answer to find the slope and y-intercept of the graph of the equation $-6x + 5y = 9$.

34. **MAKING AN ARGUMENT** Your friend claims that given $f(0)$ and any other value of a linear function f, you can write an equation in slope-intercept form that represents the function. Your cousin disagrees, claiming that the two points could lie on a vertical line. Who is correct? Explain.

35. **ANALYZING A GRAPH** Line ℓ is a reflection in the x-axis of line k. Write an equation that represents line k.



36. **HOW DO YOU SEE IT?** The graph shows the approximate U.S. box office revenues (in billions of dollars) from 2000 to 2012, where $x = 0$ represents the year 2000.



a. Estimate the slope and y-intercept of the graph.

b. Interpret your answers in part (a) in the context of the problem.

c. How can you use your answers in part (a) to predict the U.S. box office revenue in 2018?

37. **ABSTRACT REASONING** Show that the equation of the line that passes through the points $(0, b)$ and $(1, b + m)$ is $y = mx + b$. Explain how you can be sure that the point $(-1, b - m)$ also lies on the line.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. *(Section 1.3)*

38. $3(x - 15) = x + 11$ **39.** $2(3d + 3) = 7 + 6d$ **40.** $-5(4 - 3n) = 10(n - 2)$

Determine whether x and y show direct variation. If so, identify the constant of variation. *(Section 3.6)*

41. $y + 6x = 0$ **42.** $2y + 5x = 4$ **43.** $4y + x - 3 = 3$

4.2 Writing Equations in Point-Slope Form

Essential Question

How can you write an equation of a line when you are given the slope and a point on the line?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS

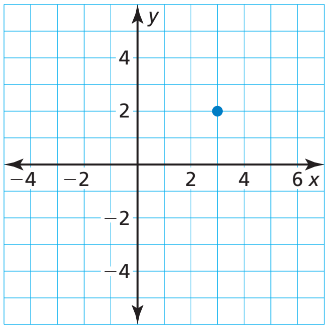
A.2.B
A.2.C
A.3.A

EXPLORATION 1 Writing Equations of Lines

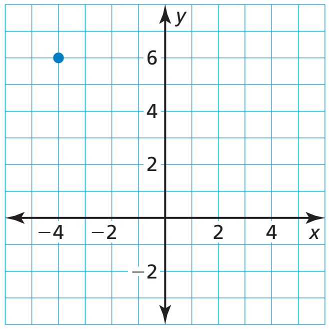
Work with a partner.

- Sketch the line that has the given slope and passes through the given point.
- Find the y-intercept of the line.
- Write an equation of the line.

a. $m = \frac{1}{2}$



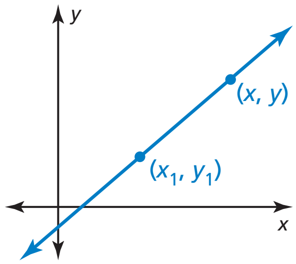
b. $m = -2$



EXPLORATION 2 Writing a Formula

Work with a partner.

The point (x_1, y_1) is a given point on a nonvertical line. The point (x, y) is any other point on the line. Write an equation that represents the slope m of the line. Then rewrite this equation by multiplying each side by the difference of the x-coordinates to obtain the **point-slope form** of a linear equation.



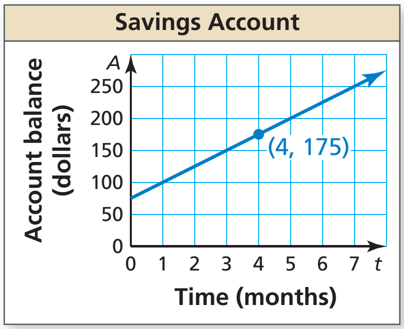
EXPLORATION 3 Writing an Equation

Work with a partner.

For four months, you have saved \$25 per month. You now have \$175 in your savings account.

a. Use your result from Exploration 2 to write an equation that represents the balance A after t months.

b. Use a graphing calculator to verify your equation.



USING TECHNOLOGY

To be proficient in math, you need to understand the feasibility, appropriateness, and limitations of the technological tools at your disposal. For instance, in real-life situations such as the one given in Exploration 3, it may not be feasible to use a square viewing window on a graphing calculator.

Communicate Your Answer

4. How can you write an equation of a line when you are given the slope and a point on the line?

5. Give an example of how to write an equation of a line when you are given the slope and a point on the line. Your example should be different from those above.

4.2 Lesson

Core Vocabulary

point-slope form, *p. 168*

Previous
slope-intercept form
function
linear model
rate

What You Will Learn

- Write an equation of a line given its slope and a point on the line.
- Write an equation of a line given two points on the line.
- Use linear equations to solve real-life problems.

Writing Equations of Lines in Point-Slope Form

Given a point on a line and the slope of the line, you can write an equation of the line. Consider the line that passes through (2, 3) and has a slope of $\frac{1}{2}$. Let (x, y) be another point on the line where $x \neq 2$. You can write an equation relating x and y using the slope formula with $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (x, y)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Write the slope formula.

$$\frac{1}{2} = \frac{y - 3}{x - 2}$$ Substitute values.

$$\frac{1}{2}(x - 2) = y - 3$$ Multiply each side by $(x - 2)$.

The equation in *point-slope form* is $y - 3 = \frac{1}{2}(x - 2)$.

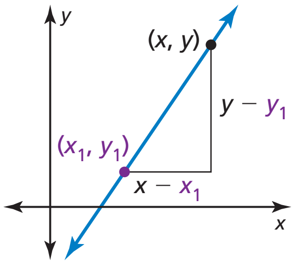
Core Concept

Point-Slope Form

Words A linear equation written in the form $y - y_1 = m(x - x_1)$ is in **point-slope form**. The line passes through the point (x_1, y_1), and the slope of the line is m.

Algebra $y - \underbrace{y_1}_{} = \underbrace{m}_{\text{slope}}(x - x_1)$

passes through (x_1, y_1)



EXAMPLE 1 Using Point-Slope Form

Identify the slope of the line $y + 2 = -3(x - 2)$. Then identify a point the line passes through.

SOLUTION

The equation is written in point-slope form, $y - y_1 = m(x - x_1)$, where $m = -3$, $x_1 = 2$, and $y_1 = -2$.

▶ So, the slope of the line is −3, and the line passes through the point (2, −2).

EXAMPLE 2 Using a Slope and a Point to Write an Equation

Write an equation in point-slope form of the line that passes through the point (8, 3) and has a slope of $\frac{1}{4}$.

SOLUTION

$$y - y_1 = m(x - x_1)$$ Write the point-slope form.

$$y - 3 = \frac{1}{4}(x - 8)$$ Substitute $\frac{1}{4}$ for m, 8 for x_1, and 3 for y_1.

▶ The equation is $y - 3 = \frac{1}{4}(x - 8)$.

Monitoring Progress

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1. Identify the slope of the line $y - \frac{1}{2} = -2\left(x - \frac{1}{2}\right)$. Then identify a point the line passes through.

Write an equation in point-slope form of the line that passes through the given point and has the given slope.

2. $(3, -1)$; $m = -2$

3. $(4, 0)$; $m = -\frac{2}{3}$

Writing Equations of Lines Given Two Points

When you are given two points on a line, you can write an equation of the line using the following steps.

Step 1 Find the slope of the line.

Step 2 Use the slope and one of the points to write an equation of the line in point-slope form.

ANOTHER WAY

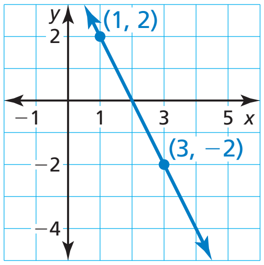
You can use either of the given points to write an equation of the line.

Use $m = -2$ and $(3, -2)$.

$$y - (-2) = -2(x - 3)$$
$$y + 2 = -2x + 6$$
$$y = -2x + 4$$

EXAMPLE 3 Using Two Points to Write an Equation

Write an equation in slope-intercept form of the line shown.



SOLUTION

Step 1 Find the slope of the line.

$$m = \frac{-2 - 2}{3 - 1} = \frac{-4}{2}, \text{ or } -2$$

Step 2 Use the slope $m = -2$ and the point $(1, 2)$ to write an equation of the line.

$y - y_1 = m(x - x_1)$ Write the point-slope form.

$y - 2 = -2(x - 1)$ Substitute -2 for m, 1 for x_1, and 2 for y_1.

$y - 2 = -2x + 2$ Distributive Property

$y = -2x + 4$ Write in slope-intercept form.

▶ The equation is $y = -2x + 4$.

EXAMPLE 4 Writing a Linear Function

Write a linear function f with the values $f(4) = -2$ and $f(12) = 10$.

SOLUTION

Note that you can rewrite $f(4) = -2$ as $(4, -2)$ and $f(12) = 10$ as $(12, 10)$.

Step 1 Find the slope of the line that passes through $(4, -2)$ and $(12, 10)$.

$$m = \frac{10 - (-2)}{12 - 4} = \frac{12}{8}, \text{ or } 1.5$$

Step 2 Use the slope $m = 1.5$ and the point $(12, 10)$ to write an equation of the line.

$y - y_1 = m(x - x_1)$ Write the point-slope form.

$y - 10 = 1.5(x - 12)$ Substitute 1.5 for m, 12 for x_1, and 10 for y_1.

$y = 1.5x - 8$ Write in slope-intercept form.

▶ A function is $f(x) = 1.5x - 8$.

Monitoring Progress

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Write an equation in slope-intercept form of the line that passes through the given points.

4. $(1, 4), (3, 10)$

5. $(-4, -1), (8, -4)$

Write a linear function g with the given values.

6. $g(2) = 3, g(6) = 5$

7. $g(-1) = 8, g(2) = -1$

Solving Real-Life Problems

EXAMPLE 5 Modeling with Mathematics

The student council is ordering customized foam hands to promote school spirit. The table shows the cost of ordering different numbers of foam hands. Can the situation be modeled by a linear equation? Explain. If possible, write a linear model that represents the cost as a function of the number of foam hands.

Number of foam hands	4	6	8	10	12
Cost (dollars)	34	46	58	70	82

SOLUTION

Step 1 Find the rate of change for consecutive data pairs in the table.

$$\frac{46-34}{6-4} = 6, \frac{58-46}{8-6} = 6, \frac{70-58}{10-8} = 6, \frac{82-70}{12-10} = 6$$

Because the rate of change is constant, the data are linear. So, use the point-slope form to write an equation that represents the data.

Step 2 Use the constant rate of change (slope) $m = 6$ and the data pair $(4, 34)$ to write an equation. Let C be the cost (in dollars) and n be the number of foam hands.

$C - C_1 = m(n - n_1)$ Write the point-slope form.

$C - 34 = 6(n - 4)$ Substitute 6 for m, 4 for n_1, and 34 for C_1.

$C = 6n + 10$ Write in slope-intercept form.

▶ Because the cost increases at a constant rate, the situation can be modeled by a linear equation. The linear model is $C = 6n + 10$.

Monitoring Progress

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8. You pay an installation fee and a monthly fee for Internet service. The table shows the total cost for different numbers of months. Can the situation be modeled by a linear equation? Explain. If possible, write a linear model that represents the total cost as a function of the number of months.

Number of months	Total cost (dollars)
3	176
6	302
9	428
12	554

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Vocabulary and Core Concept Check

1. **USING STRUCTURE** Without simplifying, identify the slope of the line given by the equation $y - 5 = -2(x + 5)$. Then identify one point on the line.

2. **WRITING** Explain how you can use the slope formula to write an equation of the line that passes through $(3, -2)$ and has a slope of 4.

Monitoring Progress and Modeling with Mathematics

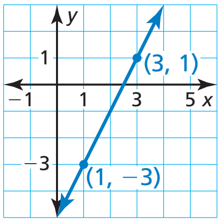
In Exercises 3–6, identify the slope of the line. Then identify a point the line passes through. *(See Example 1.)*

3. $y - 4 = \frac{2}{3}(x - 9)$
4. $y - 1 = -6(x - 3)$
5. $y - 10 = -8\left(x + \frac{1}{4}\right)$
6. $y + 6 = \frac{3}{5}x$

In Exercises 7–14, write an equation in point-slope form of the line that passes through the given point and has the given slope. *(See Example 2.)*

7. $(2, 1)$; $m = 2$
8. $(3, 5)$; $m = -1$
9. $(7, -4)$; $m = -6$
10. $(-8, -2)$; $m = 5$
11. $(9, 0)$; $m = -3$
12. $(0, 2)$; $m = 4$
13. $(-6, 6)$; $m = \frac{3}{2}$
14. $(5, -12)$; $m = -\frac{2}{5}$

In Exercises 15–18, write an equation in slope-intercept form of the line shown. *(See Example 3.)*

15.


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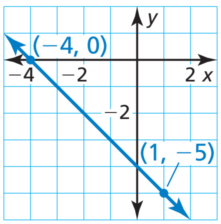

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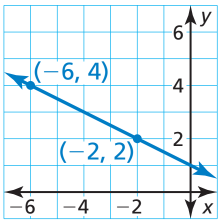

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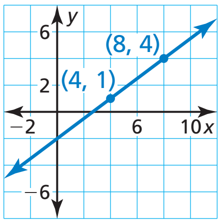


In Exercises 19–24, write an equation in slope-intercept form of the line that passes through the given points.

19. $(7, 2), (2, 12)$
20. $(6, -2), (12, 1)$
21. $(6, -1), (3, -7)$
22. $(-2, 5), (-4, -5)$
23. $(1, -9), (-3, -9)$
24. $(-5, 19), (5, 13)$

In Exercises 25–30, write a linear function f with the given values. *(See Example 4.)*

25. $f(2) = -2, f(1) = 1$
26. $f(5) = 7, f(-2) = 0$
27. $f(-4) = 2, f(6) = -3$
28. $f(-10) = 4, f(-2) = 4$
29. $f(-3) = 1, f(13) = 5$
30. $f(-9) = 10, f(-1) = -2$

In Exercises 31–34, tell whether the data in the table can be modeled by a linear equation. Explain. If possible, write a linear equation that represents y as a function of x. *(See Example 5.)*

31.

x	2	4	6	8	10
y	−1	5	15	29	47

32.

x	−3	−1	1	3	5
y	16	10	4	−2	−8

33.

x	y
0	1.2
1	1.4
2	1.6
4	2

34.

x	y
1	18
2	15
4	12
8	9

35. **ERROR ANALYSIS** Describe and correct the error in writing an equation of the line that passes through the point $(1, -5)$ and has a slope of -2.

$$y - y_1 = m(x - x_1)$$
$$y - 5 = -2(x - 1)$$

36. ERROR ANALYSIS Describe and correct the error in writing an equation of the line that passes through the points (1, 2) and (4, 3).

✗ $m = \frac{3 - 2}{4 - 1} = \frac{1}{3}$ $\quad y - 2 = \frac{1}{3}(x - 4)$

37. MODELING WITH MATHEMATICS You are designing a sticker to advertise your band. A company charges \$225 for the first 1000 stickers and \$80 for each additional 1000 stickers.

a. Write an equation that represents the total cost (in dollars) of the stickers as a function of the number (in thousands) of stickers ordered.

b. Find the total cost of 9000 stickers.

38. MODELING WITH MATHEMATICS You pay a processing fee and a daily fee to rent a beach house. The table shows the total cost of renting the beach house for different numbers of days.

Days	2	4	6	8
Total cost (dollars)	246	450	654	858

a. Can the situation be modeled by a linear equation? Explain.

b. What is the processing fee? the daily fee?

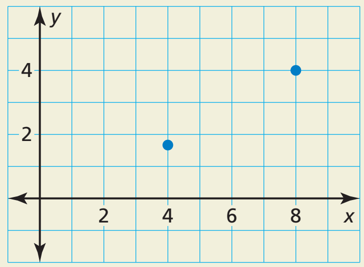
c. You can spend no more than \$1200 on the beach house rental. What is the maximum number of days you can rent the beach house?

39. WRITING Describe two ways to graph the equation $y - 1 = \frac{3}{2}(x - 4)$.

40. THOUGHT PROVOKING The graph of a linear function passes through the point $(12, -5)$ and has a slope of $\frac{2}{5}$. Represent this function in two other ways.

41. REASONING You are writing an equation of the line that passes through two points that are not on the y-axis. Would you use slope-intercept form or point-slope form to write the equation? Explain.

42. HOW DO YOU SEE IT? The graph shows two points that lie on the graph of a linear function.

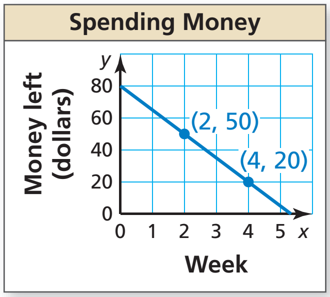


a. Does the y-intercept of the graph of the linear function appear to be positive or negative? Explain.

b. Estimate the coordinates of the two points. How can you use your estimates to confirm your answer in part (a)?

43. CONNECTION TO TRANSFORMATIONS Compare the graph of $y = 2x$ to the graph of $y - 1 = 2(x + 3)$. Make a conjecture about the graphs of $y = mx$ and $y - k = m(x - h)$.

44. COMPARING FUNCTIONS Three siblings each receive money for a holiday and then spend it at a constant weekly rate. The graph describes Sibling A's spending, the table describes Sibling B's spending, and the equation $y = -22.5x + 90$ describes Sibling C's spending. The variable y represents the amount of money left after x weeks.



Week, x	Money left, y
1	\$100
2	\$75
3	\$50
4	\$25

a. Which sibling received the most money? the least money?

b. Which sibling spends money at the fastest rate? the slowest rate?

c. Which sibling runs out of money first? last?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Use intercepts to graph the linear equation. *(Section 3.4)*

45. $-4x + 2y = 16$ **46.** $3x + 5y = -15$ **47.** $x - 6y = 24$ **48.** $-7x - 2y = -21$

4.3 Writing Equations in Standard Form

Essential Question How can you write the equation of a line in standard form?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.2.B
A.2.C
A.3.A

EXPLORATION 1 Writing Equations in Standard Form

Work with a partner. So far you have written equations of lines in slope-intercept form and point-slope form. Linear equations can also be written in standard form, $Ax + By = C$. Write each equation in standard form.

a. $y = 3x - 5$ — The equation is in slope-intercept form.

________ — Subtract $3x$ from each side.

The equation in standard form is ________.

b. $y - 4 = -2(x + 6)$ — The equation is in slope-intercept form.

________ — Distributive Property

________ — Add $2x$ to each side.

________ — Add 4 to each side.

The equation in standard form is ________.

c. $y = 4x + 2$

d. $y = -x - 7$

e. $y - 1 = 3(x + 4)$

f. $y + 8 = -4(x - 3)$

EXPLORATION 2 Finding the Slope and *y*-Intercept

Work with a partner. The slope and y-intercept of a line are not explicitly known from a linear equation written in standard form. Find the slope and y-intercept of the line represented by each equation.

a. $-6x + 3y = -15$ — The equation is in standard form.

________ — Add $6x$ to each side.

________ — Divide each side by 3.

The slope is ______, and the y-intercept is ______.

b. $5x - 10y = -40$

c. $-6x - 8y = 56$

MAKING MATHEMATICAL ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

Communicate Your Answer

3. How can you write the equation of a line in standard form?

4. How can you find the slope and y-intercept of a line given the equation of the line in standard form?

5. Consider the graph of $Ax + By = C$.

a. Does changing the value of A change the slope? Does changing the value of B change the slope? Explain your reasoning.

b. Does changing the value of A change the y-intercept? Does changing the value of B change the y-intercept? Explain your reasoning.

4.3 Lesson

Core Vocabulary

Previous
standard form
equivalent equation
point-slope form

What You Will Learn

- Write equations in standard form.
- Use linear equations to solve real-life problems.

Writing Equations in Standard Form

Recall that the linear equation $Ax + By = C$ is in standard form, where A, B, and C are real numbers and A and B are not both zero. All linear equations can be written in standard form.

REMEMBER

You can produce an equivalent equation by multiplying or dividing each side of an equation by the same nonzero number.

EXAMPLE 1 Writing Equivalent Equations in Standard Form

Write two equations in standard form that are equivalent to $2x - 6y = 4$.

SOLUTION

To write one equivalent equation, multiply each side of the original equation by 2.

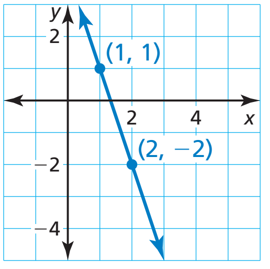
$$2(2x - 6y) = 2(4) \quad \Rightarrow \quad 4x - 12y = 8$$

To write another equivalent equation, divide each side of the original equation by 2.

$$\frac{2x - 6y}{2} = \frac{4}{2} \quad \Rightarrow \quad x - 3y = 2$$

EXAMPLE 2 Using Two Points to Write an Equation

Write an equation in standard form of the line shown.



SOLUTION

Step 1 Find the slope of the line.

$$m = \frac{1 - (-2)}{1 - 2} = \frac{3}{-1}, \text{ or } -3$$

Step 2 Use the slope $m = -3$ and the point $(1, 1)$ to write an equation in point-slope form.

$y - y_1 = m(x - x_1)$ Write the point-slope form.

$y - 1 = -3(x - 1)$ Substitute -3 for m, 1 for x_1, and 1 for y_1.

Step 3 Write the equation in standard form.

$y - 1 = -3(x - 1)$ Write the equation.

$y - 1 = -3x + 3$ Distributive Property

$3x + y - 1 = 3$ Add $3x$ to each side.

$3x + y = 4$ Add 1 to each side.

- An equation is $3x + y = 4$.

ANOTHER WAY

You can use either of the given points to write an equation of the line.

Use $m = -3$ and $(2, -2)$.

$$y - (-2) = -3(x - 2)$$
$$y + 2 = -3x + 6$$
$$3x + y + 2 = 6$$
$$3x + y = 4$$

Monitoring Progress

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1. Write two equations in standard form that are equivalent to $x - y = 3$.

2. Write an equation in standard form of the line that passes through $(3, -1)$ and $(2, -3)$.

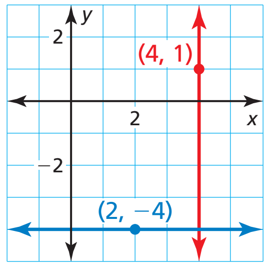
Recall that equations of horizontal lines have the form $y = b$ and equations of vertical lines have the form $x = a$. You cannot write an equation of a vertical line in slope-intercept form or point-slope form because the slope of a vertical line is undefined. However, you can write an equation of a vertical line in standard form.

EXAMPLE 3 Horizontal and Vertical Lines

Write an equation of the specified line.

a. blue line

b. red line



SOLUTION

a. The y-coordinate of the given point on the blue line is -4. This means that all points on the line have a y-coordinate of -4.

▶ So, an equation of the line is $y = -4$.

ANOTHER WAY

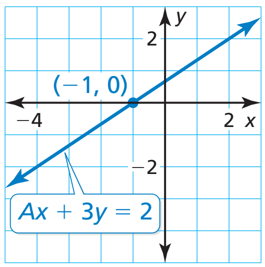
Using the slope-intercept form to find an equation of the horizontal line gives you $y = 0x - 4$, or $y = -4$.

b. The x-coordinate of the given point on the red line is 4. This means that all points on the line have an x-coordinate of 4.

▶ So, an equation of the line is $x = 4$.

EXAMPLE 4 Completing an Equation in Standard Form

Find the missing coefficient in the equation of the line shown. Write the completed equation.



SOLUTION

Step 1 Find the value of A. Substitute the coordinates of the given point for x and y in the equation. Then solve for A.

$Ax + 3y = 2$	Write the equation.
$A(-1) + 3(0) = 2$	Substitute -1 for x and 0 for y.
$-A = 2$	Simplify.
$A = -2$	Divide each side by -1.

Step 2 Complete the equation.

$-2x + 3y = 2$	Substitute -2 for A.

▶ An equation is $-2x + 3y = 2$.

Monitoring Progress

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Write equations of the horizontal and vertical lines that pass through the given point.

3. $(-8, -9)$

4. $(13, -5)$

Find the missing coefficient in the equation of the line that passes through the given point. Write the completed equation.

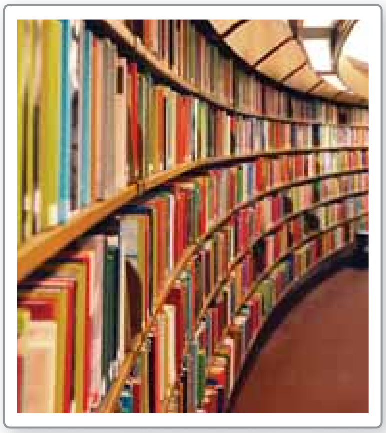
5. $-4x + By = 7$; $(-1, 1)$

6. $Ax + y = -3$; $(2, 11)$

Solving Real-Life Problems

EXAMPLE 5 **Modeling with Mathematics**

Your class is taking a trip to the public library. You can travel in small and large vans. A small van holds 8 people, and a large van holds 12 people. Your class can fill 15 small vans and 2 large vans.



a. Write an equation in standard form that models the possible combinations of small and large vans that your class can fill.

b. Graph the equation from part (a).

c. Find four possible combinations.

SOLUTION

a. Write a verbal model. Then write an equation.

Capacity of small van	•	Number of small vans	+	Capacity of large van	•	Number of large vans	=	People on trip
8	•	s	+	12	•	ℓ	=	p

Because your class can fill 15 small vans and 2 large vans, use (15, 2) to find the value of p.

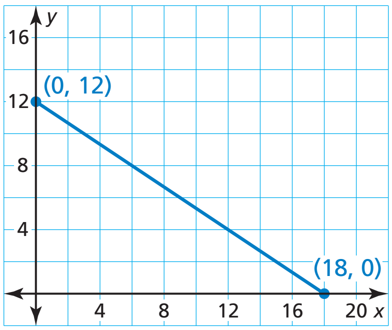
$$8s + 12\ell = p \qquad \text{Write the equation.}$$

$$8(15) + 12(2) = p \qquad \text{Substitute 15 for } s \text{ and 2 for } \ell.$$

$$144 = p \qquad \text{Simplify.}$$

So, the equation $8s + 12\ell = 144$ models the possible combinations.

b. Use intercepts to graph the equation.



Find the intercepts.

Substitute 0 for s.

$$8(0) + 12\ell = 144$$

$$\ell = 12$$

Substitute 0 for ℓ.

$$8s + 12(0) = 144$$

$$s = 18$$

Plot the points (0, 12) and (18, 0). Connect them with a line segment. For this problem, only whole-number values of s and ℓ make sense.

ANOTHER WAY

Another way to find possible combinations is to substitute values for s or ℓ in the equation and solve for the other variable.

c. The graph passes through (0, 12), (6, 8), (12, 4), and (18, 0). So, four possible combinations are 0 small and 12 large, 6 small and 8 large, 12 small and 4 large, and 18 small and 0 large.

Monitoring Progress

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7. WHAT IF? Eight students decide to not go on the class trip. Write an equation in standard form that models the possible combinations of small and large vans that your class can fill. Find four possible combinations.

4.3 Exercises

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Vocabulary and Core Concept Check

1. **WRITING** Explain how to write an equation in standard form of a line when two points on the line are given.

2. **WHICH ONE DOESN'T BELONG?** Which equation does *not* belong with the other three? Explain your reasoning.

$-3x + 8y = 1$	$y = 5x - 9$	$6x - 2y = 1$	$x + y = 4$

Monitoring Progress and Modeling with Mathematics

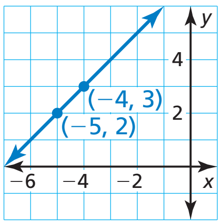
In Exercises 3–8, write two equations in standard form that are equivalent to the given equation. *(See Example 1.)*

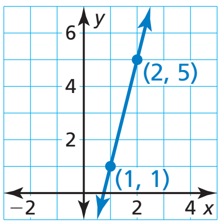
3. $x + y = -10$
4. $5x + 10y = 15$
5. $-x + 2y = 9$
6. $-9x - 12y = 6$
7. $9x - 3y = -12$
8. $-2x + 4y = -5$

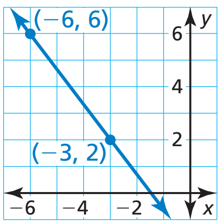
In Exercises 9–14, write an equation in standard form of the line that passes through the given point and has the given slope.

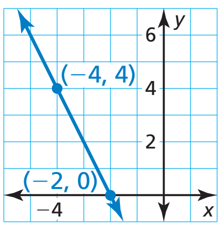
9. $(-3, 2)$; $m = 1$
10. $(4, -1)$; $m = 3$
11. $(0, 5)$; $m = -2$
12. $(-8, 0)$; $m = -4$
13. $(-4, -4)$; $m = -\frac{3}{2}$
14. $(-6, -10)$; $m = \frac{1}{6}$

In Exercises 15–18, write an equation in standard form of the line shown. *(See Example 2.)*

15. 

16. 

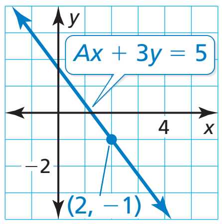
17. 

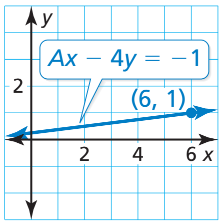
18. 

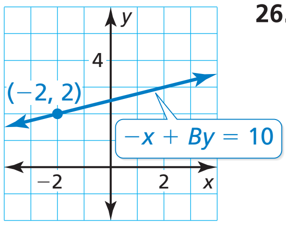
In Exercises 19–22, write equations of the horizontal and vertical lines that pass through the given point. *(See Example 3.)*

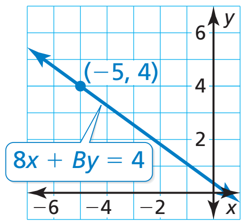
19. $(2, 3)$
20. $(-5, -4)$
21. $(8, -1)$
22. $(-6, 2)$

In Exercises 23–26, find the missing coefficient in the equation of the line shown. Write the completed equation. *(See Example 4.)*

23. 

24. 

25. 

26. 

27. **ERROR ANALYSIS** Describe and correct the error in finding the value of A for the equation $Ax - 3y = 5$, when the graph of the equation passes through the point $(1, -4)$.

✗

$$A(-4) - 3(1) = 5$$
$$-4A = 8$$
$$A = -2$$

28. **MAKING AN ARGUMENT** Your friend says that you can write an equation of a horizontal line in standard form but not in slope-intercept form or point-slope form. Is your friend correct? Explain.

29. **MODELING WITH MATHEMATICS** The diagram shows the prices of two types of ground cover plants. A gardener can afford to buy 125 vinca plants and 60 phlox plants. *(See Example 5.)*

a. Write an equation in standard form that models the possible combinations of vinca and phlox plants the gardener can afford to buy.

b. Graph the equation from part (a).

c. Find four possible combinations.

30. **MODELING WITH MATHEMATICS** One bus ride costs \$0.75. One subway ride costs \$1. A monthly pass for unlimited bus and subway rides costs the same as 36 bus rides plus 36 subway rides.

a. Write an equation in standard form that models the possible combinations of bus and subway rides with the same total cost as the pass.

b. Graph the equation from part (a).

c. You ride the bus 60 times in one month. How many times must you ride the subway for the total cost of the rides to equal the cost of the pass? Explain your reasoning.

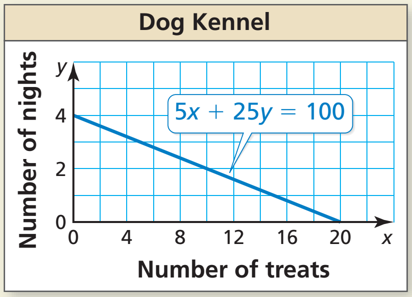
31. **WRITING** There are three forms of an equation of a line: *slope-intercept*, *point-slope*, and *standard* form. Which form would you prefer to use to do each of the following? Explain.

a. Graph the equation.

b. Find the x-intercept of the graph of the equation.

c. Write an equation of the line given two points on the line.

32. **HOW DO YOU SEE IT?** A dog kennel charges \$25 per night to board your dog. The kennel also sells dog treats for \$5 each. The graph shows the possible combinations of nights at the kennel and treats that you can buy for \$100.

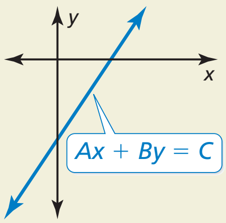


a. List two possible combinations.

b. Interpret the intercepts of the graph.

33. **ABSTRACT REASONING** Write an equation in standard form of the line that passes through $(a, 0)$ and $(0, b)$, where $a \neq 0$ and $b \neq 0$.

34. **THOUGHT PROVOKING** Use the graph shown.



a. What are the signs of B and C when A is positive? when A is negative?

b. Explain how to change the equation so that the graph is reflected in the x-axis.

c. Explain how to change the equation so that the graph is translated horizontally.

35. **MATHEMATICAL CONNECTIONS** Write an equation in standard form that models the possible lengths and widths (in feet) of a rectangle with the same perimeter as a rectangle that is 10 feet wide and 20 feet long. Make a table that shows five possible lengths and widths of the rectangle.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the reciprocal of the number. *(Skills Review Handbook)*

36. 5 **37.** -8 **38.** $-\frac{2}{7}$ **39.** $\frac{3}{2}$

4.4 Writing Equations of Parallel and Perpendicular Lines

Essential Question

How can you recognize lines that are parallel or perpendicular?

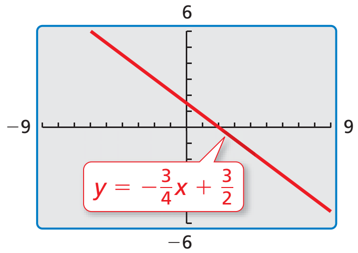
Texas Essential Knowledge and Skills

A.2.C
A.2.E
A.2.F
A.2.G
A.3.A

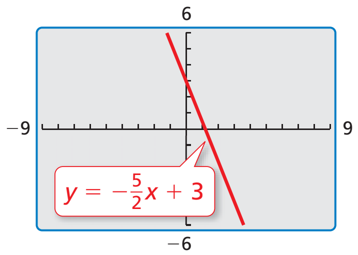
EXPLORATION 1 Recognizing Parallel Lines

Work with a partner. Write each linear equation in slope-intercept form. Then use a graphing calculator to graph the three equations in the same square viewing window. (The graph of the first equation is shown.) Which two lines appear parallel? How can you tell?

a. $3x + 4y = 6$
$3x + 4y = 12$
$4x + 3y = 12$



b. $5x + 2y = 6$
$2x + y = 3$
$2.5x + y = 5$



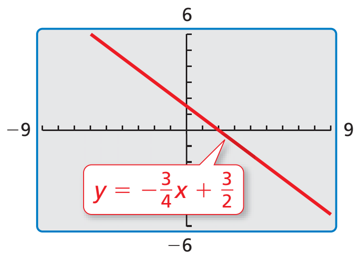
SELECTING TOOLS

To be proficient in math, you need to use a graphing calculator and other available technological tools, as appropriate, to help you explore relationships and deepen your understanding of concepts.

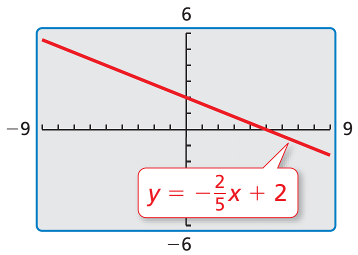
EXPLORATION 2 Recognizing Perpendicular Lines

Work with a partner. Write each linear equation in slope-intercept form. Then use a graphing calculator to graph the three equations in the same square viewing window. (The graph of the first equation is shown.) Which two lines appear perpendicular? How can you tell?

a. $3x + 4y = 6$
$3x - 4y = 12$
$4x - 3y = 12$



b. $2x + 5y = 10$
$-2x + y = 3$
$2.5x - y = 5$



Communicate Your Answer

3. How can you recognize lines that are parallel or perpendicular?

4. Compare the slopes of the lines in Exploration 1. How can you use slope to determine whether two lines are parallel? Explain your reasoning.

5. Compare the slopes of the lines in Exploration 2. How can you use slope to determine whether two lines are perpendicular? Explain your reasoning.

4.4 Lesson

What You Will Learn

- Identify and write equations of parallel lines.
- Identify and write equations of perpendicular lines.

Core Vocabulary

parallel lines, *p. 180*
perpendicular lines, *p. 181*

Previous
reciprocal

Identifying and Writing Equations of Parallel Lines

Core Concept

Parallel Lines and Slopes

Two lines in the same plane that never intersect are **parallel lines**. Nonvertical lines are parallel if and only if they have the same slope.

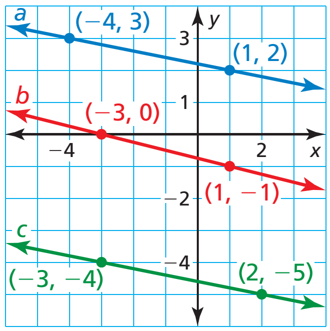
All vertical lines are parallel.

READING

The phrase "*A* if and only if *B*" is a way of writing two conditional statements at once. It means that if *A* is true, then *B* is true. It also means that if *B* is true, then *A* is true.

EXAMPLE 1 Identifying Parallel Lines

Determine which of the lines are parallel.



SOLUTION

Find the slope of each line.

Line a: $m = \dfrac{2-3}{1-(-4)} = -\dfrac{1}{5}$

Line b: $m = \dfrac{-1-0}{1-(-3)} = -\dfrac{1}{4}$

Line c: $m = \dfrac{-5-(-4)}{2-(-3)} = -\dfrac{1}{5}$

▶ Lines a and c have the same slope, so they are parallel.

EXAMPLE 2 Writing an Equation of a Parallel Line

Write an equation of the line that passes through $(5, -4)$ and is parallel to the line $y = 2x + 3$.

SOLUTION

Step 1 Find the slope of the parallel line. The graph of the given equation has a slope of 2. So, the parallel line that passes through $(5, -4)$ also has a slope of 2.

Step 2 Use the slope-intercept form to find the y-intercept of the parallel line.

$y = mx + b$ Write the slope-intercept form.

$-4 = 2(5) + b$ Substitute 2 for m, 5 for x, and -4 for y.

$-14 = b$ Solve for b.

▶ Using $m = 2$ and $b = -14$, an equation of the parallel line is $y = 2x - 14$.

ANOTHER WAY

You can also use the slope $m = 2$ and the point-slope form to write an equation of the line that passes through $(5, -4)$.

$$y - y_1 = m(x - x_1)$$
$$y - (-4) = 2(x - 5)$$
$$y = 2x - 14$$

Monitoring Progress

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1. Line a passes through $(-5, 3)$ and $(-6, -1)$. Line b passes through $(3, -2)$ and $(2, -7)$. Are the lines parallel? Explain.
2. Write an equation of the line that passes through $(-4, 2)$ and is parallel to the line $y = \frac{1}{4}x + 1$.

Identifying and Writing Equations of Perpendicular Lines

REMEMBER

The product of a nonzero number m and its negative reciprocal is -1:

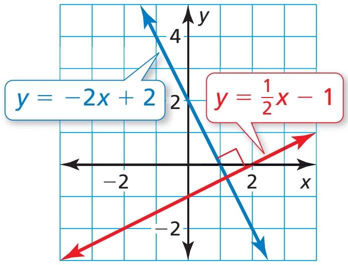
$$m\left(-\frac{1}{m}\right) = -1.$$

Core Concept

Perpendicular Lines and Slopes

Two lines in the same plane that intersect to form right angles are **perpendicular lines**. Nonvertical lines are perpendicular if and only if their slopes are negative reciprocals.

Vertical lines are perpendicular to horizontal lines.



EXAMPLE 3 Identifying Parallel and Perpendicular Lines

Determine which of the lines, if any, are parallel or perpendicular.

Line a: $y = 4x + 2$ Line b: $x + 4y = 3$ Line c: $-8y - 2x = 16$

SOLUTION

Write the equations in slope-intercept form. Then compare the slopes.

Line a: $y = 4x + 2$ Line b: $y = -\frac{1}{4}x + \frac{3}{4}$ Line c: $y = -\frac{1}{4}x - 2$

▶ Lines b and c have slopes of $-\frac{1}{4}$, so they are parallel. Line a has a slope of 4, the negative reciprocal of $-\frac{1}{4}$, so it is perpendicular to lines b and c.

EXAMPLE 4 Writing an Equation of a Perpendicular Line

Write an equation of the line that passes through $(-3, 1)$ and is perpendicular to the line $y = \frac{1}{2}x + 3$.

SOLUTION

Step 1 Find the slope of the perpendicular line. The graph of the given equation has a slope of $\frac{1}{2}$. Because the slopes of perpendicular lines are negative reciprocals, the slope of the perpendicular line that passes through $(-3, 1)$ is -2.

Step 2 Use the slope $m = -2$ and the point-slope form to write an equation of the perpendicular line that passes through $(-3, 1)$.

$y - y_1 = m(x - x_1)$ Write the point-slope form.

$y - 1 = -2[x - (-3)]$ Substitute -2 for m, -3 for x_1, and 1 for y_1.

$y - 1 = -2x - 6$ Simplify.

$y = -2x - 5$ Write in slope-intercept form.

▶ An equation of the perpendicular line is $y = -2x - 5$.

ANOTHER WAY

You can also use the slope $m = -2$ and the slope-intercept form to write an equation of the line that passes through $(-3, 1)$.

$y = mx + b$

$1 = -2(-3) + b$

$-5 = b$

So, $y = -2x - 5$.

Monitoring Progress

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3. Determine which of the lines, if any, are parallel or perpendicular. Explain.

   Line a: $2x + 6y = -3$ Line b: $y = 3x - 8$ Line c: $-6y + 18x = 9$

4. Write an equation of the line that passes through $(-3, 5)$ and is perpendicular to the line $y = -3x - 1$.

EXAMPLE 5 Horizontal and Vertical Lines

REMEMBER
The slope of a horizontal line is 0. The slope of a vertical line is undefined.

Write an equation of a line that is (a) parallel to the x-axis and (b) perpendicular to the x-axis. What is the slope of each line?

SOLUTION

a. The x-axis (the line $y = 0$) is a horizontal line and has a slope of 0. All horizontal lines are parallel. Equations of horizontal lines have the form $y = b$, where b is a constant. Let b equal any number other than 0, such as 5.

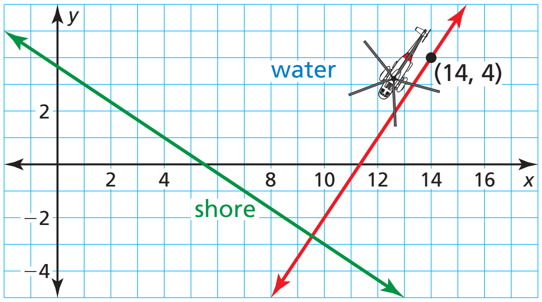
▶ An equation of a line parallel to the x-axis is $y = 5$. The slope of the line is 0.

b. The x-axis is a horizontal line. Vertical lines are perpendicular to horizontal lines. The slope of a vertical line is undefined. Equations of vertical lines have the form $x = a$, where a is a constant. Let a equal any number, such as 2.

▶ An equation of a line perpendicular to the x-axis is $x = 2$. The slope of the line is undefined.

EXAMPLE 6 Writing an Equation of a Perpendicular Line

The position of a helicopter search and rescue crew is shown in the graph. The shortest flight path to the shoreline is one that is perpendicular to the shoreline. Write an equation that represents this path.



SOLUTION

Step 1 Find the slope of the line that represents the shoreline. The line passes through points (1, 3) and (4, 1). So, the slope is

$$m = \frac{1 - 3}{4 - 1} = -\frac{2}{3}.$$

Because the shoreline and shortest flight path are perpendicular, the slopes of their respective graphs are negative reciprocals. So, the slope of the graph of the shortest flight path is $\frac{3}{2}$.

Step 2 Use the slope $m = \frac{3}{2}$ and the point-slope form to write an equation of the shortest flight path that passes through (14, 4).

$y - y_1 = m(x - x_1)$ — Write the point-slope form.

$y - 4 = \frac{3}{2}(x - 14)$ — Substitute $\frac{3}{2}$ for m, 14 for x_1, and 4 for y_1.

$y - 4 = \frac{3}{2}x - 21$ — Distributive Property

$y = \frac{3}{2}x - 17$ — Write in slope-intercept form.

▶ An equation that represents the shortest flight path is $y = \frac{3}{2}x - 17$.

Monitoring Progress

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5. Write an equation of a line that is (a) parallel to the line $x = 3$ and (b) perpendicular to the line $x = 3$. What is the slope of each line?

6. **WHAT IF?** In Example 6, a boat is traveling perpendicular to the shoreline and passes through (9, 3). Write an equation that represents the path of the boat.

4.4 Exercises

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Vocabulary and Core Concept Check

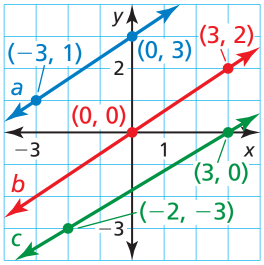
1. **COMPLETE THE SENTENCE** Nonvertical ________ lines have the same slope.

2. **VOCABULARY** Two lines are perpendicular. The slope of one line is $-\frac{5}{7}$. What is the slope of the other line? Justify your answer.

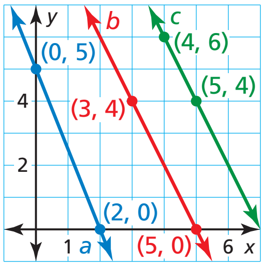
Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, determine which of the lines, if any, are parallel. Explain. *(See Example 1.)*

3.



4.



5. Line a passes through $(-1, -2)$ and $(1, 0)$.
Line b passes through $(4, 2)$ and $(2, -2)$.
Line c passes through $(0, 2)$ and $(-1, 1)$.

6. Line a passes through $(-1, 3)$ and $(1, 9)$.
Line b passes through $(-2, 12)$ and $(-1, 14)$.
Line c passes through $(3, 8)$ and $(6, 10)$.

7. Line a: $4y + x = 8$
Line b: $2y + x = 4$
Line c: $2y = -3x + 6$

8. Line a: $3y - x = 6$
Line b: $3y = x + 18$
Line c: $3y - 2x = 9$

In Exercises 9–12, write an equation of the line that passes through the given point and is parallel to the given line. *(See Example 2.)*

9. $(-1, 3)$; $y = 2x + 2$

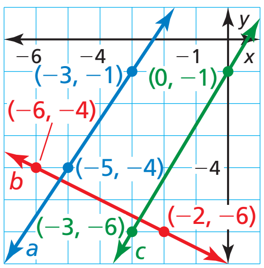
10. $(1, 2)$; $y = -5x + 4$

11. $(18, 2)$; $3y - x = -12$

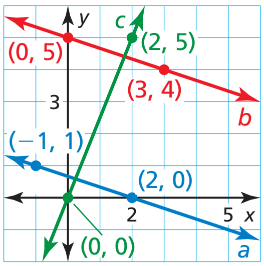
12. $(2, -5)$; $2y = 3x + 10$

In Exercises 13–18, determine which of the lines, if any, are parallel or perpendicular. Explain. *(See Example 3.)*

13.



14.



15. Line a passes through $(-2, 1)$ and $(0, 3)$.
Line b passes through $(4, 1)$ and $(6, 4)$.
Line c passes through $(1, 3)$ and $(4, 1)$.

16. Line a passes through $(2, 10)$ and $(4, 13)$.
Line b passes through $(4, 9)$ and $(6, 12)$.
Line c passes through $(2, 10)$ and $(4, 9)$.

17. Line a: $4x - 3y = 2$
Line b: $y = \frac{4}{3}x + 2$
Line c: $4y + 3x = 4$

18. Line a: $y = 6x - 2$
Line b: $6y = -x$
Line c: $y + 6x = 1$

In Exercises 19–22, write an equation of the line that passes through the given point and is perpendicular to the given line. *(See Example 4.)*

19. $(7, 10)$; $y = \frac{1}{2}x - 9$

20. $(-4, -1)$; $y = \frac{4}{3}x + 6$

21. $(-3, 3)$; $2y = 8x - 6$

22. $(8, 1)$; $2y + 4x = 12$

In Exercises 23–26, write an equation of a line that is (a) parallel to the given line and (b) perpendicular to the given line. *(See Example 5.)*

23. the y-axis

24. $x = -4$

25. $y = -2$

26. $y = 7$

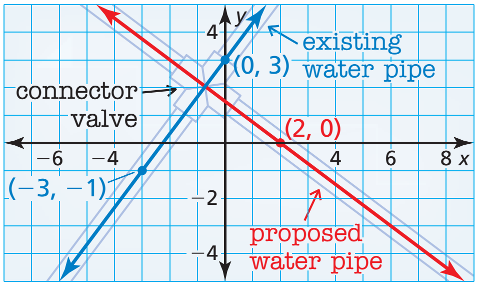
27. **ERROR ANALYSIS** Describe and correct the error in writing an equation of the line that passes through $(1, 3)$ and is parallel to the line $y = \frac{1}{4}x + 2$.

$$y - y_1 = m(x - x_1)$$
$$y - 3 = -4(x - 1)$$
$$y - 3 = -4x + 4$$
$$y = -4x + 7$$

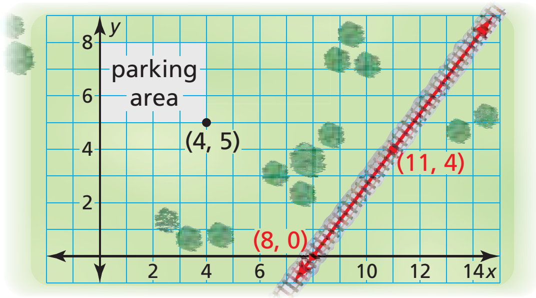
28. ERROR ANALYSIS Describe and correct the error in writing an equation of the line that passes through $(4, -5)$ and is perpendicular to the line $y = \frac{1}{3}x + 5$.

$$y - y_1 = m(x - x_1)$$
$$y - (-5) = 3(x - 4)$$
$$y + 5 = 3x - 12$$
$$y = 3x - 17$$

29. MODELING WITH MATHEMATICS A city water department is proposing the construction of a new water pipe, as shown. The new pipe will be perpendicular to the old pipe. Write an equation that represents the new pipe. *(See Example 6.)*



30. MODELING WITH MATHEMATICS A parks and recreation department is constructing a new bike path. The path will be parallel to the railroad tracks shown and pass through the parking area at the point $(4, 5)$. Write an equation that represents the path.



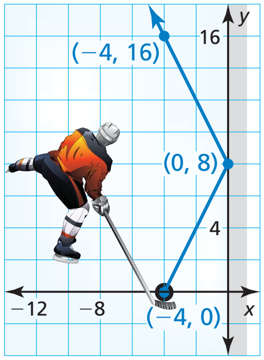
31. MATHEMATICAL CONNECTIONS The vertices of a quadrilateral are $A(2, 2)$, $B(6, 4)$, $C(8, 10)$, and $D(4, 8)$.

a. Is quadrilateral $ABCD$ a parallelogram? Explain.

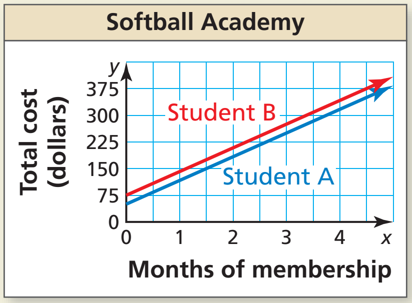
b. Is quadrilateral $ABCD$ a rectangle? Explain.

32. USING STRUCTURE For what value of a are the graphs of $6y = -2x + 4$ and $2y = ax - 5$ parallel? perpendicular?

33. MAKING AN ARGUMENT A hockey puck leaves the blade of a hockey stick, bounces off a wall, and travels in a new direction, as shown. Your friend claims the path of the puck forms a right angle. Is your friend correct? Explain.



34. HOW DO YOU SEE IT? A softball academy charges students an initial registration fee plus a monthly fee. The graph shows the total amounts paid by two students over a 4-month period. The lines are parallel.



a. Did one of the students pay a greater registration fee? Explain.

b. Did one of the students pay a greater monthly fee? Explain.

REASONING In Exercises 35–37, determine whether the statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

35. Two lines with positive slopes are perpendicular.

36. A vertical line is parallel to the y-axis.

37. Two lines with the same y-intercept are perpendicular.

38. THOUGHT PROVOKING You are designing a new logo for your math club. Your teacher asks you to include at least one pair of parallel lines and at least one pair of perpendicular lines. Sketch your logo in a coordinate plane. Write the equations of the parallel and perpendicular lines.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine whether the relation is a function. Explain. *(Section 3.1)*

39. (3, 6), (4, 8), (5, 10), (6, 9), (7, 14)

40. (−1, 6), (1, 4), (−1, 2), (1, 6), (−1, 5)

4.1–4.4 What Did You Learn?

Core Vocabulary

linear model, *p. 164*
point-slope form, *p. 168*
parallel lines, *p. 180*
perpendicular lines, *p. 181*

Core Concepts

Section 4.1

Using Slope-Intercept Form, *p. 162*

Section 4.2

Using Point-Slope Form, *p. 168*

Section 4.3

Writing Equations in Standard Form, *p. 174*

Section 4.4

Parallel Lines and Slopes, *p. 180*
Perpendicular Lines and Slopes, *p. 181*

Mathematical Thinking

1. How can you explain to yourself the meaning of the graph in Exercise 36 on page 166?
2. How did you use the structure of the equations in Exercise 43 on page 172 to make a conjecture?
3. How did you use the diagram in Exercise 33 on page 184 to determine whether your friend was correct?

Study Skills

Getting Actively Involved in Class

If you do not understand something at all and do not even know how to phrase a question, just ask for clarification. You might say something like, "Could you please explain the steps in this problem one more time?"

If your teacher asks for someone to go up to the board, volunteer. The student at the board often receives additional attention and instruction to complete the problem.

4.1–4.4 Quiz

Write an equation in slope-intercept form of the line with the given characteristics. *(Section 4.1)*

1. passes through: $(0, -2)$ and $(1, 3)$

2. slope: $-\frac{1}{3}$; passes through: $(3, 4)$

3. slope: 2; y-intercept: 3

Write an equation in point-slope form of the line that passes through the given points. *(Section 4.2)*

4. $(-2, 5), (1, -1)$

5. $(-3, -2), (2, -1)$

6. $(1, 0), (4, 4)$

Write a linear function *f* with the given values. *(Section 4.1 and Section 4.2)*

7. $f(0) = 2, f(5) = -3$

8. $f(-1) = -6, f(4) = -6$

9. $f(-3) = -2, f(-2) = 3$

Write an equation in standard form of the line with the given characteristics. *(Section 4.3)*

10. passes through: $(3, 0)$ and $(5, 4)$

11. slope: -1; passes through: $(-2, -7)$

12. passes through: $(3, 4)$ and $(3, 1)$

13. slope: 0; passes through: $(4, -1)$

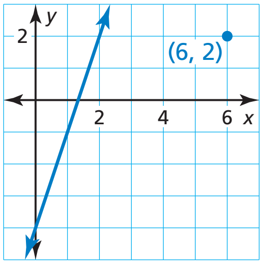
Determine which of the lines, if any, are parallel or perpendicular. Explain. *(Section 4.4)*

14. Line a passes through $(-2, 2)$ and $(2, 1)$.
Line b passes through $(1, -8)$ and $(3, 0)$.
Line c passes through $(-4, -3)$ and $(0, -2)$.

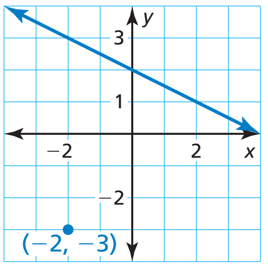
15. Line a: $2x + 6y = -12$
Line b: $y = \frac{3}{2}x - 5$
Line c: $3x - 2y = -4$

Write an equation of the line that passes through the given point and is (a) parallel to the given line and (b) perpendicular to the given line. *(Section 4.4)*

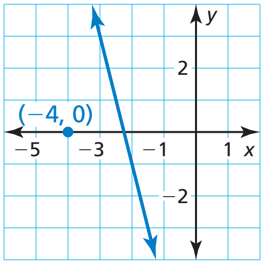
16.



17.



18.



19. A website hosting company charges an initial fee of \$48 to set up a website. The company charges \$44 per month to maintain the website. *(Section 4.1)*

a. Write a linear model that represents the total cost of setting up and maintaining a website as a function of the number of months it is maintained.

b. Find the total cost of setting up a website and maintaining it for 6 months.

c. A different website hosting company charges \$62 per month to maintain a website, but there is no initial set-up fee. You have \$620. At which company can you set up and maintain a website for the greatest amount of time? Explain.

20. The table shows the amount of water remaining in a water tank as it drains. Can the situation be modeled by a linear equation? Explain. If possible, write a linear model that represents the amount of water remaining in the tank as a function of time. *(Section 4.2)*

Time (minutes)	Water (gallons)
8	155
10	150
12	145
14	140
16	135

21. You have \$10 on a copy store gift card. The copy store charges \$0.10 for each black-and-white copy and \$0.50 for each color copy. (a) Write an equation in standard form that models the possible combinations of copies you can buy. (b) Graph the equation from part (a). (c) Find four possible combinations. *(Section 4.3)*

4.5 Scatter Plots and Lines of Fit

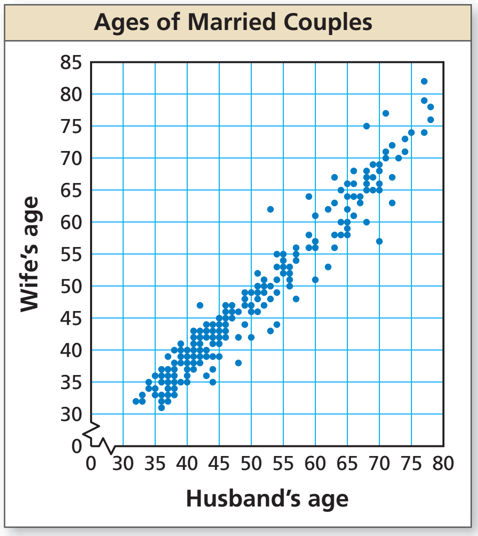
Essential Question How can you use a scatter plot and a line of fit to make conclusions about data?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.4.C

A **scatter plot** is a graph that shows the relationship between two data sets. The two data sets are graphed as ordered pairs in a coordinate plane.

EXPLORATION 1 Finding a Line of Fit

Work with a partner. A survey was taken of 179 married couples. Each person was asked his or her age. The scatter plot shows the results.



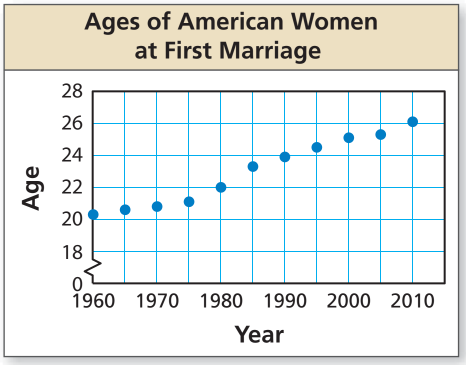
a. Draw a line that approximates the data. Write an equation of the line. Explain the method you used.

b. What conclusions can you make from the equation you wrote? Explain your reasoning.

REASONING
To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

EXPLORATION 2 Finding a Line of Fit

Work with a partner. The scatter plot shows the median ages of American women at their first marriage for selected years from 1960 through 2010.



a. Draw a line that approximates the data. Write an equation of the line. Let x represent the number of years since 1960. Explain the method you used.

b. What conclusions can you make from the equation you wrote?

c. Use your equation to predict the median age of American women at their first marriage in the year 2020.

Communicate Your Answer

3. How can you use a scatter plot and a line of fit to make conclusions about data?

4. Use the Internet or some other reference to find a scatter plot of real-life data that is different from those given above. Then draw a line that approximates the data and write an equation of the line. Explain the method you used.

4.5 Lesson

Core Vocabulary

scatter plot, *p. 188*
correlation, *p. 189*
line of fit, *p. 190*

What You Will Learn

- Interpret scatter plots.
- Identify correlations between data sets.
- Use lines of fit to model data.

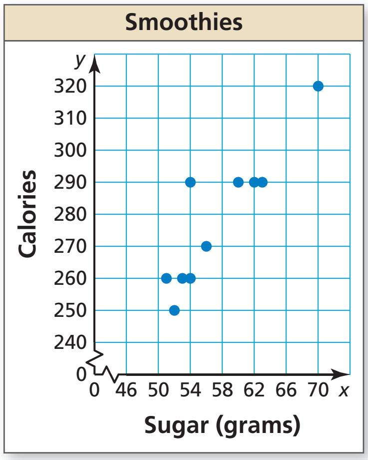
Interpreting Scatter Plots

Core Concept

Scatter Plot

A **scatter plot** is a graph that shows the relationship between two data sets. The two data sets are graphed as ordered pairs in a coordinate plane. Scatter plots can show trends in the data.

EXAMPLE 1 Interpreting a Scatter Plot



The scatter plot shows the amounts x (in grams) of sugar and the numbers y of calories in 10 smoothies.

a. How many calories are in the smoothie that contains 56 grams of sugar?

b. How many grams of sugar are in the smoothie that contains 320 calories?

c. What tends to happen to the number of calories as the number of grams of sugar increases?

SOLUTION

a. Draw a horizontal line from the point that has an x-value of 56. It crosses the y-axis at 270.

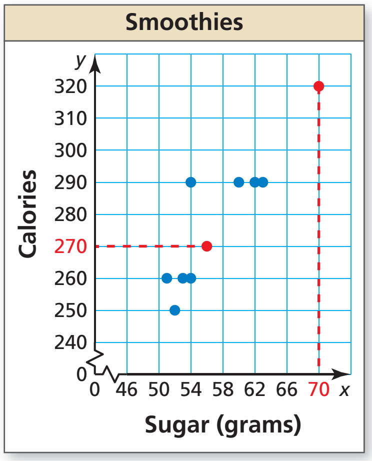
- So, the smoothie has 270 calories.

b. Draw a vertical line from the point that has a y-value of 320. It crosses the x-axis at 70.

- So, the smoothie has 70 grams of sugar.

c. Looking at the graph, the plotted points go up from left to right.

- So, as the number of grams of sugar increases, the number of calories increases.



Monitoring Progress

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1. How many calories are in the smoothie that contains 51 grams of sugar?

2. How many grams of sugar are in the smoothie that contains 250 calories?

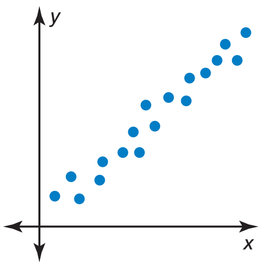
STUDY TIP

You can think of a positive correlation as having a positive slope and a negative correlation as having a negative slope.

Identifying Correlations between Data Sets

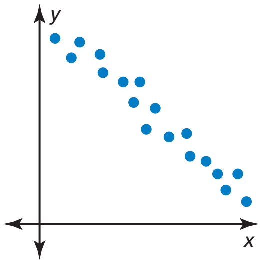
A **correlation** is a relationship between data sets. You can use a scatter plot to describe the correlation between data.

Positive Correlation



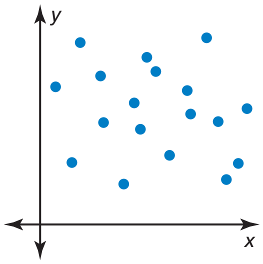
As *x* increases, *y* increases.

Negative Correlation



As *x* increases, *y* decreases.

No Correlation



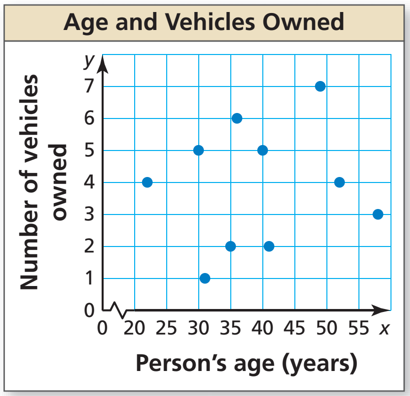
The points show no pattern.

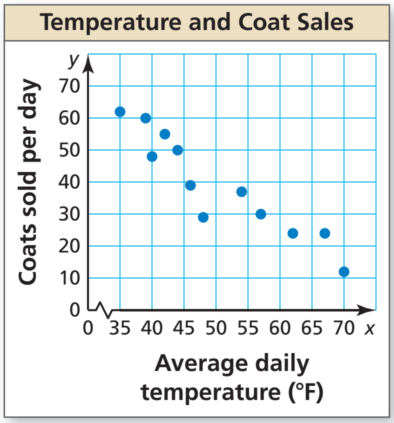
EXAMPLE 2 Identifying Correlations

Tell whether the data show a *positive*, a *negative*, or *no* correlation.

a. age and vehicles owned

b. temperature and coat sales at a store





SOLUTION

a. The points show no pattern. The number of vehicles owned does not depend on a person's age.

▶ So, the scatter plot shows no correlation.

b. As the average temperature increases, the number of coats sold decreases.

▶ So, the scatter plot shows a negative correlation.

Monitoring Progress

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Make a scatter plot of the data. Tell whether the data show a *positive*, a *negative*, or *no* correlation.

3.

Temperature (°F), *x*	82	78	68	87	75	71	92	84
Attendees (thousands), *y*	4.5	4.0	1.7	5.5	3.8	2.9	4.7	5.3

4.

Age of a car (years), *x*	1	2	3	4	5	6	7	8
Value (thousands), *y*	\$24	\$21	\$19	\$18	\$15	\$12	\$8	\$7

Using Lines of Fit to Model Data

When data show a positive or negative correlation, you can model the *trend* in the data using a line of fit. A **line of fit** is a line drawn on a scatter plot that is close to most of the data points.

STUDY TIP
A line of fit is also called a *trend line*.

Core Concept

Using a Line of Fit to Model Data

Step 1 Make a scatter plot of the data.

Step 2 Decide whether the data can be modeled by a line.

Step 3 Draw a line that appears to fit the data closely. There should be approximately as many points above the line as below it.

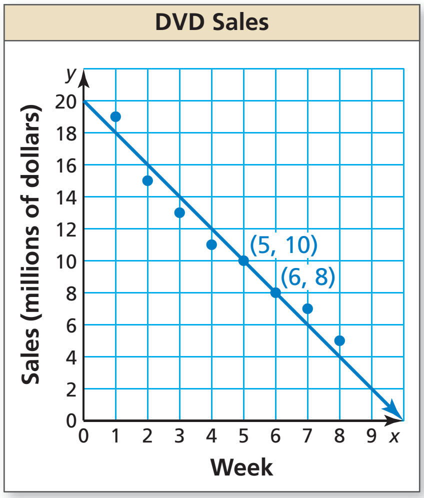
Step 4 Write an equation using two points on the line. The points do not have to represent actual data pairs, but they must lie on the line of fit.

EXAMPLE 3 Finding a Line of Fit

The table shows the weekly sales of a DVD and the number of weeks since its release. Write an equation that models the DVD sales as a function of the number of weeks since its release. Interpret the slope and y-intercept of the line of fit.

Week, x	1	2	3	4	5	6	7	8
Sales (millions), y	\$19	\$15	\$13	\$11	\$10	\$8	\$7	\$5

SOLUTION



Step 1 Make a scatter plot of the data.

Step 2 Decide whether the data can be modeled by a line. Because the scatter plot shows a negative correlation, you can fit a line to the data.

Step 3 Draw a line that appears to fit the data closely.

Step 4 Write an equation using two points on the line. Use (5, 10) and (6, 8).

The slope of the line is $m = \dfrac{8 - 10}{6 - 5} = -2$.

Use the slope $m = -2$ and the point (6, 8) to write an equation of the line.

$y - y_1 = m(x - x_1)$ Write the point-slope form.

$y - 8 = -2(x - 6)$ Substitute -2 for m, 6 for x_1, and 8 for y_1.

$y = -2x + 20$ Solve for y.

▶ An equation of the line of fit is $y = -2x + 20$. The slope of the line is -2. This means the sales are decreasing by about \$2 million each week. The y-intercept is 20. The y-intercept has no meaning in this context because there are no sales in week 0.

Monitoring Progress

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5. The following data pairs show the monthly income x (in dollars) and the monthly car payment y (in dollars) of six people: (2100, 410), (1650, 315), (1950, 405), (1500, 295), (2250, 440), and (1800, 375). Write an equation that models the monthly car payment as a function of the monthly income. Interpret the slope and y-intercept of the line of fit.

4.5 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** When data show a positive correlation, the dependent variable tends to ____________ as the independent variable increases.

2. **VOCABULARY** What is a line of fit?

Monitoring Progress and Modeling with Mathematics

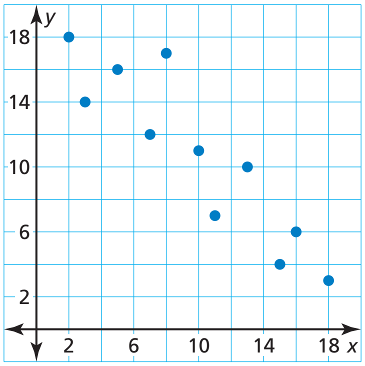
In Exercises 3–6, use the scatter plot to fill in the missing coordinate of the ordered pair.

3. (16, ____)

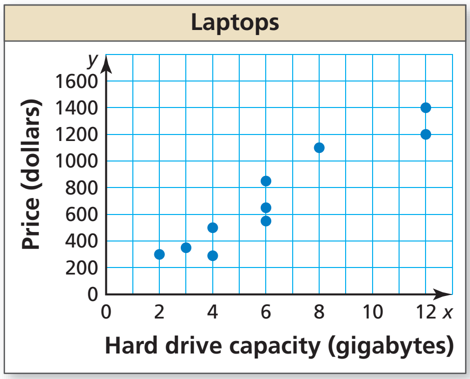
4. (3, ____)

5. (____, 12)

6. (____, 17)



7. **INTERPRETING A SCATTER PLOT** The scatter plot shows the hard drive capacities (in gigabytes) and the prices (in dollars) of 10 laptops. *(See Example 1.)*

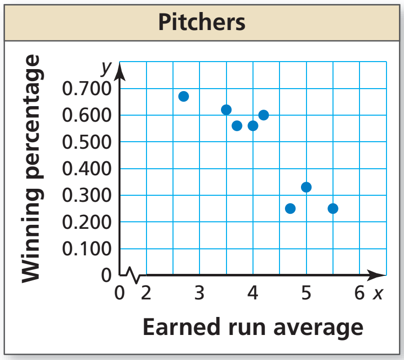


   a. What is the price of the laptop with a hard drive capacity of 8 gigabytes?

   b. What is the hard drive capacity of the $1200 laptop?

   c. What tends to happen to the price as the hard drive capacity increases?

8. **INTERPRETING A SCATTER PLOT** The scatter plot shows the earned run averages and the winning percentages of eight pitchers on a baseball team.

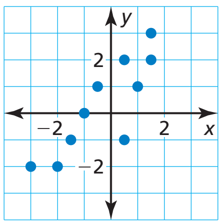


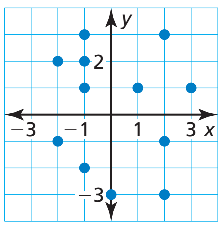
   a. What is the winning percentage of the pitcher with an earned run average of 4.2?

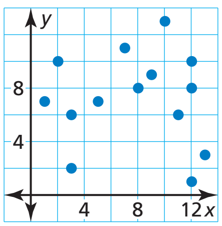
   b. What is the earned run average of the pitcher with a winning percentage of 0.33?

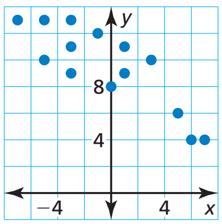
   c. What tends to happen to the winning percentage as the earned run average increases?

In Exercises 9–12, tell whether x and y show a *positive*, a *negative*, or *no* correlation. *(See Example 2.)*

9. 

10. 

11. 

12. 

In Exercises 13 and 14, make a scatter plot of the data. Tell whether x and y show a *positive*, a *negative*, or *no* correlation.

13.

x	3.1	2.2	2.5	3.7	3.9	1.5	2.7	2.0
y	1	0	1	2	0	2	3	2

14.

x	3	4	5	6	7	8	9	10
y	67	67	50	33	25	21	19	4

15. MODELING WITH MATHEMATICS The table shows the world birth rates y (number of births per 1000 people) x years since 1960. *(See Example 3.)*

x	0	10	20	30	40	50
y	35.4	33.6	28.3	27.0	22.4	20.0

a. Write an equation that models the birthrate as a function of the number of years since 1960.

b. Interpret the slope and y-intercept of the line of fit.

16. MODELING WITH MATHEMATICS The table shows the total earnings y (in dollars) of a food server who works x hours.

x	0	1	2	3	4	5	6
y	0	18	40	62	77	85	113

a. Write an equation that models the server's earnings as a function of the number of hours the server works.

b. Interpret the slope and y-intercept of the line of fit.

17. OPEN-ENDED Give an example of a real-life data set that shows a negative correlation.

18. MAKING AN ARGUMENT Your friend says that the data in the table show a negative correlation because the dependent variable y is decreasing. Is your friend correct? Explain.

x	14	12	10	8	6	4	2
y	4	1	0	−1	−2	−4	−5

19. USING TOOLS Use a ruler or a yardstick to find the heights and arm spans of five people.

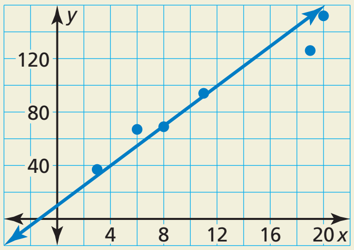
a. Make a scatter plot using the data you collected. Then draw a line of fit for the data.

b. Interpret the slope and y-intercept of the line of fit.

20. THOUGHT PROVOKING A line of fit for a scatter plot is given by the equation $y = 5x + 20$. Describe a real-life data set that could be represented by the scatter plot.

21. WRITING When is data best displayed in a scatter plot, rather than another type of display, such as a bar graph or circle graph?

22. HOW DO YOU SEE IT? The scatter plot shows part of a data set and a line of fit for the data set. Four data points are missing. Choose possible coordinates for these data points.



23. REASONING A data set has no correlation. Is it possible to find a line of fit for the data? Explain.

24. ANALYZING RELATIONSHIPS Make a scatter plot of the data in the tables. Describe the relationship between the variables. Is it possible to fit a line to the data? If so, write an equation of the line. If not, explain why.

x	−12	−9	−7	−4	−3	−1
y	150	76	50	15	10	1

x	2	5	6	7	9	15
y	5	22	37	52	90	226

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the function when $x = -3$, 0, and 4. *(Section 3.3)*

25. $g(x) = 6x$

26. $h(x) = -10x$

27. $f(x) = 5x - 8$

28. $v(x) = 14 - 3x$

4.6 Analyzing Lines of Fit

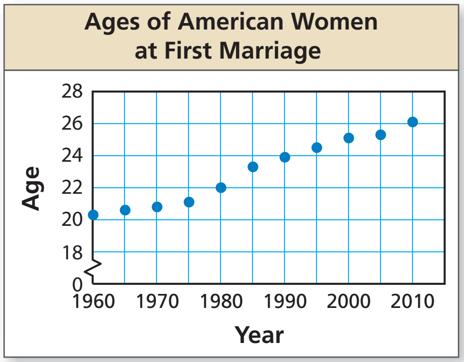
TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.4.A
A.4.B
A.4.C

Essential Question

How can you *analytically* find a line of best fit for a scatter plot?

EXPLORATION 1 Finding a Line of Best Fit

Work with a partner. The scatter plot shows the median ages of American women at their first marriage for selected years from 1960 through 2010. In Exploration 2 in Section 4.5, you approximated a line of fit graphically. To find the line of *best* fit, you can use a computer, spreadsheet, or graphing calculator that has a *linear regression* feature.



a. The data from the scatter plot is shown in the table. Note that 0, 5, 10, and so on represent the numbers of years since 1960. What does the ordered pair (25, 23.3) represent?

L1	L2	L3
0	20.3	
5	20.6	
10	20.8	
15	21.1	
20	22	
25	23.3	
30	23.9	
35	24.5	
40	25.1	
45	25.3	
50	26.1	

L1(55)=

b. Use the *linear regression* feature to find an equation of the line of best fit. You should obtain results such as those shown below.

```
LinReg
 y=ax+b
 a=.1261818182
 b=19.84545455
 r²=.9738676804
 r=.986847344
```

c. Write an equation of the line of best fit. Compare your result with the equation you obtained in Exploration 2 in Section 4.5.

MAKING MATHEMATICAL ARGUMENTS

To be proficient in math, you need to reason inductively about data.

Communicate Your Answer

2. How can you *analytically* find a line of best fit for a scatter plot?

3. The data set relates the number of chirps per second for striped ground crickets and the outside temperature in degrees Fahrenheit. Make a scatter plot of the data. Then find an equation of the line of best fit. Use your result to estimate the outside temperature when there are 19 chirps per second.

Chirps per second	20.0	16.0	19.8	18.4	17.1
Temperature (°F)	88.6	71.6	93.3	84.3	80.6

Chirps per second	14.7	15.4	16.2	15.0	14.4
Temperature (°F)	69.7	69.4	83.3	79.6	76.3

4.6 Lesson

Core Vocabulary

residual, *p. 194*
linear regression, *p. 195*
line of best fit, *p. 195*
correlation coefficient, *p. 195*
interpolation, *p. 197*
extrapolation, *p. 197*
causation, *p. 197*

What You Will Learn

- Use residuals to determine how well lines of fit model data.
- Use technology to find lines of best fit.
- Distinguish between correlation and causation.

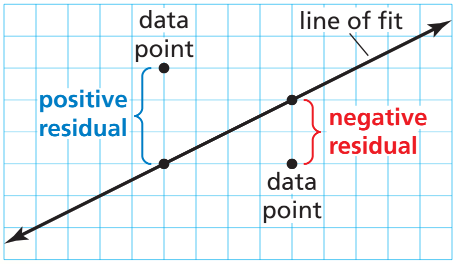
Analyzing Residuals

One way to determine how well a line of fit models a data set is to analyze *residuals*.

Core Concept

Residuals

A **residual** is the difference of the y-value of a data point and the corresponding y-value found using the line of fit. A residual can be positive, negative, or zero.



A scatter plot of the residuals shows how well a model fits a data set. If the model is a good fit, then the absolute values of the residuals are relatively small, and the residual points will be more or less evenly dispersed about the horizontal axis. If the model is not a good fit, then the residual points will form some type of pattern that suggests the data are not linear. Wildly scattered residual points suggest that the data might have no correlation.

EXAMPLE 1 Using Residuals

Week, x	Sales (millions), y
1	\$19
2	\$15
3	\$13
4	\$11
5	\$10
6	\$8
7	\$7
8	\$5

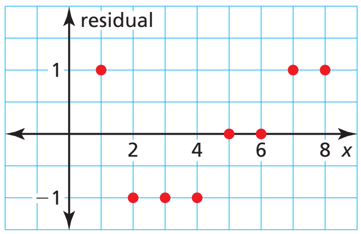
In Example 3 in Section 4.5, the equation $y = -2x + 20$ models the data in the table shown. Is the model a good fit?

SOLUTION

Step 1 Calculate the residuals. Organize your results in a table.

Step 2 Use the points (x, residual) to make a scatter plot.

x	y	y-Value from model	Residual
1	19	18	$19 - 18 = 1$
2	15	16	$15 - 16 = -1$
3	13	14	$13 - 14 = -1$
4	11	12	$11 - 12 = -1$
5	10	10	$10 - 10 = 0$
6	8	8	$8 - 8 = 0$
7	7	6	$7 - 6 = 1$
8	5	4	$5 - 4 = 1$



- The points are evenly dispersed about the horizontal axis. So, the equation $y = -2x + 20$ is a good fit.

EXAMPLE 2 Using Residuals

The table shows the ages x and salaries y (in thousands of dollars) of eight employees at a company. The equation $y = 0.2x + 38$ models the data. Is the model a good fit?

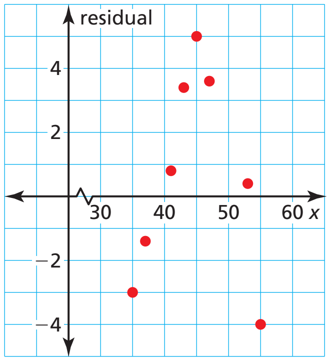
Age, x	35	37	41	43	45	47	53	55
Salary, y	42	44	47	50	52	51	49	45

SOLUTION

Step 1 Calculate the residuals. Organize your results in a table.

Step 2 Use the points (x, residual) to make a scatter plot.

x	y	y-Value from model	Residual
35	42	45.0	$42 - 45.0 = -3.0$
37	44	45.4	$44 - 45.4 = -1.4$
41	47	46.2	$47 - 46.2 = 0.8$
43	50	46.6	$50 - 46.6 = 3.4$
45	52	47.0	$52 - 47.0 = 5.0$
47	51	47.4	$51 - 47.4 = 3.6$
53	49	48.6	$49 - 48.6 = 0.4$
55	45	49.0	$45 - 49.0 = -4.0$



▶ The residual points form a ∩-shaped pattern, which suggests the data are not linear. So, the equation $y = 0.2x + 38$ does not model the data well.

Monitoring Progress

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1. The table shows the attendances y (in thousands) at an amusement park from 2005 to 2014, where $x = 0$ represents the year 2005. The equation $y = -9.8x + 850$ models the data. Is the model a good fit?

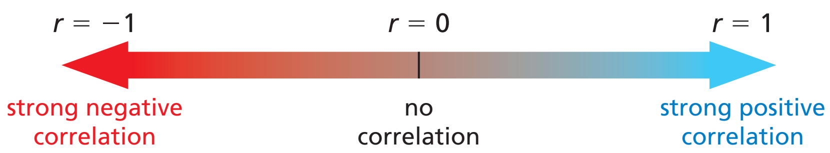
Year, x	0	1	2	3	4	5	6	7	8	9
Attendance, y	850	845	828	798	800	792	785	781	775	760

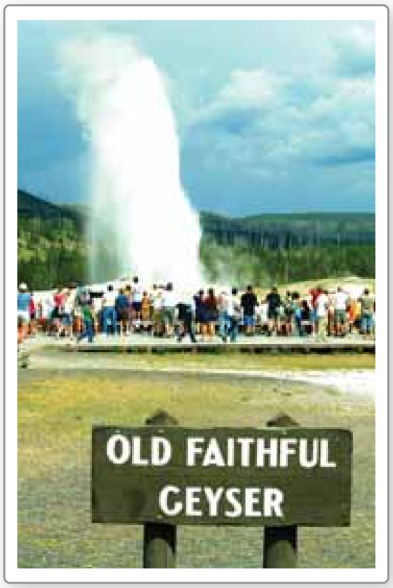
Finding Lines of Best Fit

STUDY TIP

You know how to use two points to find an equation of a line of fit. When finding an equation of the line of best fit, every point in the data set is used.

Graphing calculators use a method called **linear regression** to find a precise line of fit called a **line of best fit**. This line best models a set of data. A calculator often gives a value r, called the **correlation coefficient**. This value tells whether the correlation is positive or negative and how closely the equation models the data. Values of r range from -1 to 1. When r is close to 1 or -1, there is a strong correlation between the variables. As r, gets closer to 0, the correlation becomes weaker.





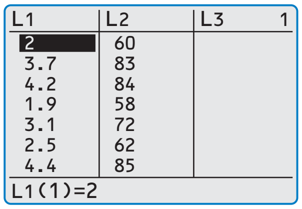
EXAMPLE 3 Finding a Line of Best Fit Using Technology

The table shows the durations x (in minutes) of several eruptions of the geyser Old Faithful and the times y (in minutes) until the next eruption. (a) Use a graphing calculator to find an equation of the line of best fit. Then plot the data and graph the equation in the same viewing window. (b) Identify and interpret the correlation coefficient. (c) Interpret the slope and y-intercept of the line of best fit.

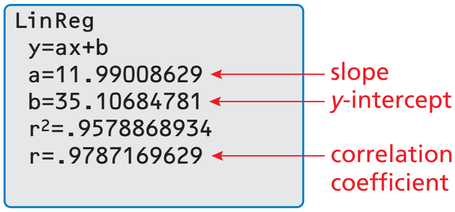
Duration, *x*	2.0	3.7	4.2	1.9	3.1	2.5	4.4	3.9
Time, *y*	60	83	84	58	72	62	85	85

SOLUTION

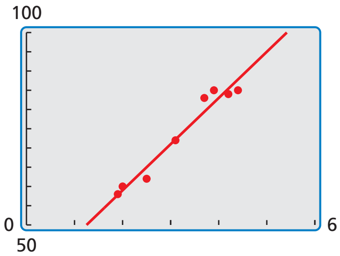
a. Step 1 Enter the data from the table into two lists.



Step 2 Use the *linear regression* feature. The values in the equation can be rounded to obtain $y = 12.0x + 35$.



Step 3 Enter the equation $y = 12.0x + 35$ into the calculator. Then plot the data and graph the equation in the same viewing window.



b. The correlation coefficient is about 0.979. This means that the relationship between the durations and the times until the next eruption has a strong positive correlation and the equation closely models the data, as shown in the graph.

c. The slope of the line is 12. This means the time until the next eruption increases by about 12 minutes for each minute the duration increases. The y-intercept is 35, but it has no meaning in this context because the duration cannot be 0 minutes.

PRECISION

Be sure to analyze the data values to select an appropriate viewing window for your graph.

Monitoring Progress

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2. Use the data in Monitoring Progress Question 1. (a) Use a graphing calculator to find an equation of the line of best fit. Then plot the data and graph the equation in the same viewing window. (b) Identify and interpret the correlation coefficient. (c) Interpret the slope and y-intercept of the line of best fit.

Using a graph or its equation to *approximate* a value between two known values is called **interpolation**. Using a graph or its equation to *predict* a value outside the range of known values is called **extrapolation**. In general, the farther removed a value is from the known values, the less confidence you can have in the accuracy of the prediction.

STUDY TIP

To approximate or predict an unknown value, you can evaluate the model algebraically or graph the model with a graphing calculator and use the *trace* feature.

EXAMPLE 4 Interpolating and Extrapolating Data

Refer to Example 3. Use the equation of the line of best fit.

a. Approximate the duration before a time of 77 minutes.

b. Predict the time after an eruption lasting 5.0 minutes.

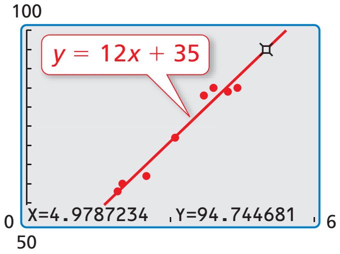
SOLUTION

a. $y = 12.0x + 35$ — Write the equation.

$77 = 12.0x + 35$ — Substitute 77 for y.

$3.5 = x$ — Solve for x.

▶ An eruption lasts about 3.5 minutes before a time of 77 minutes.



b. Use a graphing calculator to graph the equation. Use the *trace* feature to find the value of y when $x \approx 5.0$, as shown.

▶ A time of about 95 minutes will follow an eruption of 5.0 minutes.

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3. Refer to Monitoring Progress Question 2. Use the equation of the line of best fit to predict the attendance at the amusement park in 2017.

Correlation and Causation

When a change in one variable causes a change in another variable, it is called **causation**. Causation produces a strong correlation between the two variables. The converse is *not* true. In other words, correlation does not imply causation.

READING

A causal relationship exists when one variable causes a change in another variable.

EXAMPLE 5 Identifying Correlation and Causation

Tell whether a correlation is likely in the situation. If so, tell whether there is a causal relationship. Explain your reasoning.

a. time spent exercising and the number of calories burned

b. the number of banks and the population of a city

SOLUTION

a. There is a positive correlation and a causal relationship because the more time you spend exercising, the more calories you burn.

b. There may be a positive correlation but no causal relationship. Building more banks will not cause the population to increase.

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4. Is there a correlation between time spent playing video games and grade point average? If so, is there a causal relationship? Explain your reasoning.

4.6 Exercises

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Vocabulary and Core Concept Check

1. **VOCABULARY** When is a residual positive? When is it negative?

2. **WRITING** Explain how you can use residuals to determine how well a line of fit models a data set.

3. **VOCABULARY** Compare interpolation and extrapolation.

4. **WHICH ONE DOESN'T BELONG?** Which correlation coefficient does *not* belong with the other three? Explain your reasoning.

$r = -0.98$ $r = 0.96$ $r = -0.09$ $r = 0.97$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, use residuals to determine whether the model is a good fit for the data in the table. Explain. *(See Examples 1 and 2.)*

5. $y = 4x - 5$

x	−4	−3	−2	−1	0	1	2	3	4
y	−18	−13	−10	−7	−2	0	6	10	15

6. $y = 6x + 4$

x	1	2	3	4	5	6	7	8	9
y	13	14	23	26	31	42	45	52	62

7. $y = -1.3x + 1$

x	−8	−6	−4	−2	0	2	4	6	8
y	9	10	5	8	−1	1	−4	−12	−7

8. $y = -0.5x - 2$

x	4	6	8	10	12	14	16	18	20
y	−1	−3	−6	−8	−10	−10	−10	−9	−9

9. **ANALYZING RESIDUALS** The table shows the growth y (in inches) of an elk's antlers during week x. The equation $y = -0.7x + 6.8$ models the data. Is the model a good fit? Explain.

Week, x	1	2	3	4	5
Growth, y	6.0	5.5	4.7	3.9	3.3

10. **ANALYZING RESIDUALS** The table shows the approximate numbers y (in thousands) of movie tickets sold from January to June for a theater. In the table, $x = 1$ represents January. The equation $y = 1.3x + 27$ models the data. Is the model a good fit? Explain.

Month, x	Ticket sales, y
1	27
2	28
3	36
4	28
5	32
6	35

In Exercises 11–14, use a graphing calculator to find an equation of the line of best fit for the data. Identify and interpret the correlation coefficient.

11.

x	0	1	2	3	4	5	6	7
y	−8	−5	−2	−1	−1	2	5	8

12.

x	−4	−2	0	2	4	6	8	10
y	17	7	8	1	5	−2	2	−8

13.

x	−15	−10	−5	0	5	10	15	20
y	−4	2	7	16	22	30	37	43

14.

x	5	6	7	8	9	10	11	12
y	12	−2	8	3	−1	−4	6	0

ERROR ANALYSIS **In Exercises 15 and 16, describe and correct the error in interpreting the graphing calculator display.**

```
LinReg
 y=ax+b
 a=-4.47
 b=23.16
 r²=.9989451055
 r=-.9994724136
```

15. ✗ An equation of the line of best fit is $y = 23.16x - 4.47$.

16. ✗ The data have a strong positive correlation.

17. **MODELING WITH MATHEMATICS** The table shows the total numbers y of people who reported an earthquake x minutes after it ended. *(See Example 3.)*

Minutes, x	People, y
1	10
2	100
3	400
4	900
5	1400
6	1800
7	2100

a. Use a graphing calculator to find an equation of the line of best fit. Then plot the data and graph the equation in the same viewing window.

b. Identify and interpret the correlation coefficient.

c. Interpret the slope and y-intercept of the line of best fit.

18. **MODELING WITH MATHEMATICS** The table shows the numbers y of people who volunteer at an animal shelter on each day x.

Day, x	1	2	3	4	5	6	7	8
People, y	9	5	13	11	10	11	19	12

a. Use a graphing calculator to find an equation of the line of best fit. Then plot the data and graph the equation in the same viewing window.

b. Identify and interpret the correlation coefficient.

c. Interpret the slope and y-intercept of the line of best fit.

19. **MODELING WITH MATHEMATICS** The table shows the mileages x (in thousands of miles) and the selling prices y (in thousands of dollars) of several used automobiles of the same year and model. *(See Example 4.)*

Mileage, x	22	14	18	30	8	24
Price, y	16	17	17	14	18	15

a. Use a graphing calculator to find an equation of the line of best fit.

b. Identify and interpret the correlation coefficient.

c. Interpret the slope and y-intercept of the line of best fit.

d. Approximate the mileage of an automobile that costs \$15,500.

e. Predict the price of an automobile with 6000 miles.

20. **MODELING WITH MATHEMATICS** The table shows the lengths x and costs y of several sailboats.

Length (feet), x	Cost (thousands of dollars), y
27	94
18	56
25	58
32	123
18	60
26	87
36	145

a. Use a graphing calculator to find an equation of the line of best fit.

b. Identify and interpret the correlation coefficient.

c. Interpret the slope and y-intercept of the line of best fit.

d. Approximate the cost of a sailboat that is 20 feet long.

e. Predict the length of a sailboat that costs \$147,000.

In Exercises 21–24, tell whether a correlation is likely in the situation. If so, tell whether there is a causal relationship. Explain your reasoning. *(See Example 5.)*

21. the amount of time spent talking on a cell phone and the remaining battery life

22. the height of a toddler and the size of the toddler's vocabulary

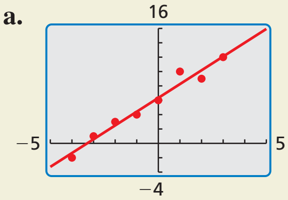
23. the number of hats you own and the size of your head

24. the weight of a dog and the length of its tail

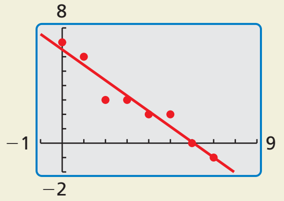
25. OPEN-ENDED Describe a data set that has a strong correlation but does not have a causal relationship.

26. HOW DO YOU SEE IT? Match each graph with its correlation coefficient. Explain your reasoning.

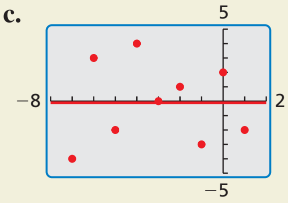
a.



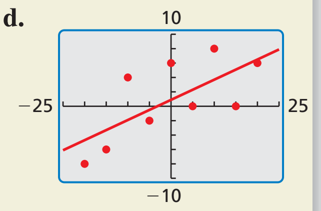
b.



c.



d.



A. $r = 0$ **B.** $r = 0.98$

C. $r = -0.97$ **D.** $r = 0.69$

27. ANALYZING RELATIONSHIPS The table shows the grade point averages y of several students and the numbers x of hours they spend watching television each week.

Hours, x	Grade point average, y
10	3.0
5	3.4
3	3.5
12	2.7
20	2.1
15	2.8
8	3.0
4	3.7
16	2.5

a. Use a graphing calculator to find an equation of the line of best fit. Identify and interpret the correlation coefficient.

b. Interpret the slope and y-intercept of the line of best fit.

c. Another student watches about 14 hours of television each week. Approximate the student's grade point average.

d. Do you think there is a causal relationship between time spent watching television and grade point average? Explain.

28. MAKING AN ARGUMENT A student spends 2 hours watching television each week and has a grade point average of 2.4. Your friend says including this information in the data set in Exercise 27 will weaken the correlation. Is your friend correct? Explain.

29. USING MODELS Refer to Exercise 17.

a. Predict the total numbers of people who reported an earthquake 9 minutes and 15 minutes after it ended.

b. The table shows the actual data. Describe the accuracy of your extrapolations in part (a).

Minutes, x	9	15
People, y	2750	3200

30. THOUGHT PROVOKING A data set consists of the numbers x of people at Beach 1 and the numbers y of people at Beach 2 recorded daily for 1 week. Sketch a possible graph of the data set. Describe the situation shown in the graph and give a possible correlation coefficient. Determine whether there is a causal relationship. Explain.

31. COMPARING METHODS The table shows the numbers y (in billions) of text messages sent each year in a five-year period, where $x = 1$ represents the first year in the five-year period.

Year, x	1	2	3	4	5
Text messages (billions), y	241	601	1360	1806	2206

a. Use a graphing calculator to find an equation of the line of best fit. Identify and interpret the correlation coefficient.

b. Is there a causal relationship? Explain your reasoning.

c. Calculate the residuals. Then make a scatter plot of the residuals and interpret the results.

d. Compare the methods you used in parts (a) and (c) to determine whether the model is a good fit. Which method do you prefer? Explain.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine whether the table represents a *linear* or *nonlinear* function. Explain. *(Section 3.2)*

32.

x	5	6	7	8
y	−4	4	−4	4

33.

x	2	4	6	8
y	13	8	3	−2

4.7 Arithmetic Sequences

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.12.D

Essential Question

How can you use an arithmetic sequence to describe a pattern?

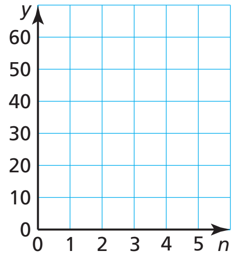
An **arithmetic sequence** is an ordered list of numbers in which the difference between each pair of consecutive **terms**, or numbers in the list, is the same.

EXPLORATION 1 Describing a Pattern

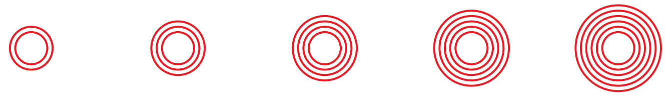
Work with a partner. Use the figures to complete the table. Plot the points given by your completed table. Describe the pattern of the *y*-values.

a. $n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$

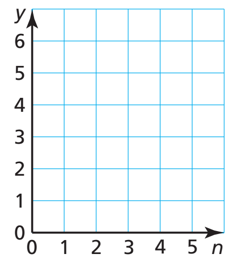
Number of stars, *n*	1	2	3	4	5
Number of sides, *y*					



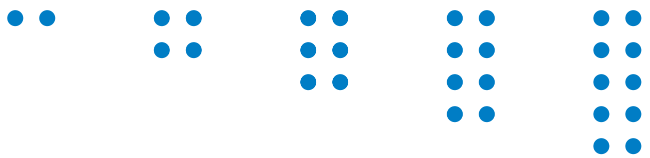
b. $n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$



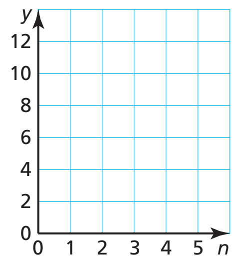
n	1	2	3	4	5
Number of circles, *y*					



c. $n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$



Number of rows, *n*	1	2	3	4	5
Number of dots, *y*					



ANALYZING MATHEMATICAL RELATIONSHIPS

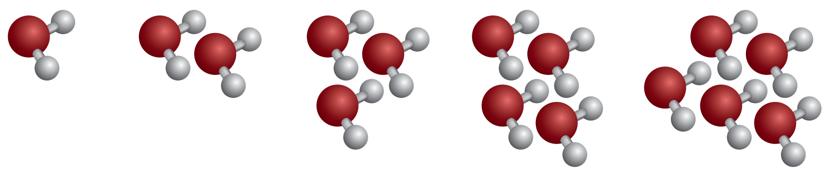
To be proficient in math, you need to look closely to discern patterns and structure.

Communicate Your Answer

2. How can you use an arithmetic sequence to describe a pattern? Give an example from real life.

3. In chemistry, water is called H_2O because each molecule of water has two hydrogen atoms and one oxygen atom. Describe the pattern shown below. Use the pattern to determine the number of atoms in 23 molecules.

$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$



4.7 Lesson

Core Vocabulary

sequence, *p. 202*
term, *p. 202*
arithmetic sequence, *p. 202*
common difference, *p. 202*

Previous
point-slope form
function notation

What You Will Learn

- Write the terms of arithmetic sequences.
- Graph arithmetic sequences.
- Write arithmetic sequences as functions.

Writing the Terms of Arithmetic Sequences

A **sequence** is an ordered list of numbers. Each number in a sequence is called a **term**. Each term a_n has a specific position n in the sequence.

$$5,\quad 10,\quad 15,\quad 20,\quad 25,\ \ldots,\ a_n,\ \ldots$$

1st position (5); 3rd position (15); *n*th position (a_n)

Core Concept

Arithmetic Sequence

In an **arithmetic sequence**, the difference between each pair of consecutive terms is the same. This difference is called the **common difference**. Each term is found by adding the common difference to the previous term.

$$\underbrace{5,\quad 10,}_{+5}\quad \underbrace{\quad 15,}_{+5}\quad \underbrace{\quad 20,}_{+5}\ \ldots$$

Terms of an arithmetic sequence

common difference

READING

An ellipsis (. . .) is a series of dots that indicates an intentional omission of information. In mathematics, the . . . notation means "and so forth." The ellipsis indicates that there are more terms in the sequence that are not shown.

EXAMPLE 1 Extending an Arithmetic Sequence

Write the next three terms of the arithmetic sequence.

$-7, -14, -21, -28, \ldots$

SOLUTION

Use a table to organize the terms and find the pattern.

Position	1	2	3	4
Term	−7	−14	−21	−28

+(−7) +(−7) +(−7)

Each term is 7 less than the previous term. So, the common difference is −7.

Add −7 to a term to find the next term.

Position	1	2	3	4	5	6	7
Term	−7	−14	−21	−28	−35	−42	−49

+(−7) +(−7) +(−7)

- The next three terms are −35, −42, and −49.

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Write the next three terms of the arithmetic sequence.

1. $-12, 0, 12, 24, \ldots$ **2.** $0.2, 0.6, 1, 1.4, \ldots$ **3.** $4, 3\frac{3}{4}, 3\frac{1}{2}, 3\frac{1}{4}, \ldots$

Graphing Arithmetic Sequences

To graph a sequence, let a term's position number n in the sequence be the x-value. The term a_n is the corresponding y-value. Plot the ordered pairs (n, a_n).

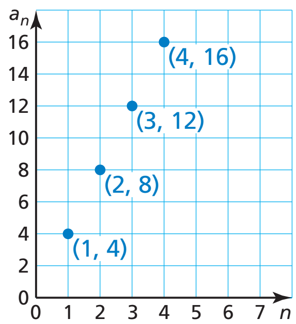
EXAMPLE 2 Graphing an Arithmetic Sequence

Graph the arithmetic sequence 4, 8, 12, 16, ... What do you notice?

SOLUTION

Make a table. Then plot the ordered pairs (n, a_n).

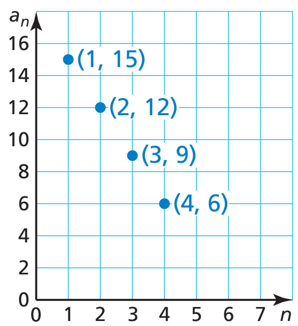
Position, n	Term, a_n
1	4
2	8
3	12
4	16



▶ The points lie on a line.

EXAMPLE 3 Identifying an Arithmetic Sequence from a Graph

Does the graph represent an arithmetic sequence? Explain.



SOLUTION

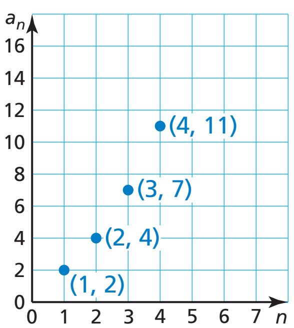
Make a table to organize the ordered pairs. Then determine whether there is a common difference.

Position, n	1	2	3	4
Term, a_n	15	12	9	6

$+(-3)$ $+(-3)$ $+(-3)$

Each term is 3 less than the previous term. So, the common difference is -3.

▶ Consecutive terms have a common difference of -3. So, the graph represents the arithmetic sequence 15, 12, 9, 6, ...



Monitoring Progress

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Graph the arithmetic sequence. What do you notice?

4. 3, 6, 9, 12, . . .
5. 4, 2, 0, -2, . . .
6. 1, 0.8, 0.6, 0.4, . . .

7. Does the graph shown represent an arithmetic sequence? Explain.

Writing Arithmetic Sequences as Functions

Because consecutive terms of an arithmetic sequence have a common difference, the sequence has a constant rate of change. So, the points represented by any arithmetic sequence lie on a line. You can use the first term and the common difference to write a linear function that describes an arithmetic sequence. Let $a_1 = 4$ and $d = 3$.

Position, n	Term, a_n	Written using a_1 and d	Numbers
1	first term, a_1	a_1	4
2	second term, a_2	$a_1 + d$	$4 + 3 = 7$
3	third term, a_3	$a_1 + 2d$	$4 + 2(3) = 10$
4	fourth term, a_4	$a_1 + 3d$	$4 + 3(3) = 13$
⋮	⋮	⋮	⋮
n	nth term, a_n	$a_1 + (n - 1)d$	$4 + (n - 1)(3)$

ANOTHER WAY

An *arithmetic sequence* is a linear function whose domain is the set of positive integers. You can think of d as the slope and $(1, a_1)$ as a point on the graph of the function. An equation in point-slope form for the function is

$$a_n - a_1 = d(n - 1).$$

This equation can be rewritten as

$$a_n = a_1 + (n - 1)d.$$

Core Concept

Equation for an Arithmetic Sequence

Let a_n be the nth term of an arithmetic sequence with first term a_1 and common difference d. The nth term is given by

$$a_n = a_1 + (n - 1)d.$$

EXAMPLE 4 Finding the nth Term of an Arithmetic Sequence

Write an equation for the nth term of the arithmetic sequence 14, 11, 8, 5, ... Then find a_{50}.

STUDY TIP

Notice that the equation in Example 4 is of the form $y = mx + b$, where y is replaced by a_n and x is replaced by n.

SOLUTION

The first term is 14, and the common difference is -3.

$a_n = a_1 + (n - 1)d$ — Equation for an arithmetic sequence

$a_n = 14 + (n - 1)(-3)$ — Substitute 14 for a_1 and -3 for d.

$a_n = -3n + 17$ — Simplify.

Use the equation to find the 50th term.

$a_n = -3n + 17$ — Write the equation.

$a_{50} = -3(50) + 17$ — Substitute 50 for n.

$= -133$ — Simplify.

▶ The 50th term of the arithmetic sequence is -133.

Monitoring Progress

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Write an equation for the nth term of the arithmetic sequence. Then find a_{25}.

8. 4, 5, 6, 7, . . .

9. 8, 16, 24, 32, . . .

10. 1, 0, -1, -2, . . .

You can rewrite the equation for an arithmetic sequence with first term a_1 and common difference d in function notation by replacing a_n with $f(n)$.

$$f(n) = a_1 + (n - 1)d$$

The domain of the function is the set of positive integers.

EXAMPLE 5 Writing Real-Life Functions

Online bidding for a purse increases by \$5 for each bid after the \$60 initial bid.

Bid number	1	2	3	4
Bid amount	\$60	\$65	\$70	\$75

a. Write a function that represents the arithmetic sequence.

b. Graph the function.

c. The winning bid is \$105. How many bids were there?

SOLUTION

a. The first term is 60, and the common difference is 5.

$f(n) = a_1 + (n - 1)d$ — Function for an arithmetic sequence

$f(n) = 60 + (n - 1)5$ — Substitute 60 for a_1 and 5 for d.

$f(n) = 5n + 55$ — Simplify.

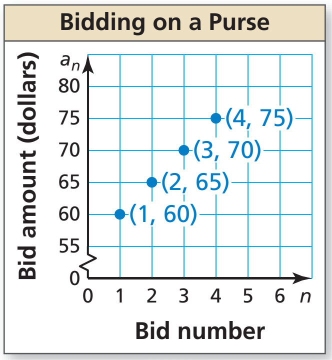
▶ The function $f(n) = 5n + 55$ represents the arithmetic sequence.

b. Make a table. Then plot the ordered pairs (n, a_n).

REMEMBER

The domain is the set of positive integers.

Bid number, n	Bid amount, a_n
1	60
2	65
3	70
4	75



c. Use the function to find the value of n for which $f(n) = 105$.

$f(n) = 5n + 55$ — Write the function.

$105 = 5n + 55$ — Substitute 105 for $f(n)$.

$10 = n$ — Solve for n.

▶ There were 10 bids.

Monitoring Progress

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11. A carnival charges \$2 for each game after you pay a \$5 entry fee.

Games	Total cost
1	\$7
2	\$9
3	\$11
4	\$13

a. Write a function that represents the arithmetic sequence.

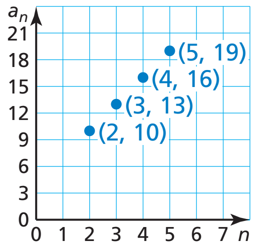
b. Graph the function.

c. How many games can you play when you take \$29 to the carnival?

Vocabulary and Core Concept Check

1. **WRITING** Describe the graph of an arithmetic sequence.

2. **DIFFERENT WORDS, SAME QUESTION** Consider the arithmetic sequence represented by the graph. Which is different? Find "both" answers.



- Find the slope of the linear function.
- Find the difference between consecutive terms of the arithmetic sequence.
- Find the difference between the terms a_2 and a_4.
- Find the common difference of the arithmetic sequence.

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, write the next three terms of the arithmetic sequence.

3. First term: 2
Common difference: 13

4. First term: 18
Common difference: –6

In Exercises 5–10, find the common difference of the arithmetic sequence.

5. 13, 18, 23, 28, . . .

6. 175, 150, 125, 100, . . .

7. $-16, -12, -8, -4, \ldots$

8. $4, 3\frac{2}{3}, 3\frac{1}{3}, 3, \ldots$

9. 6.5, 5, 3.5, 2, . . .

10. $-16, -7, 2, 11, \ldots$

In Exercises 11–16, write the next three terms of the arithmetic sequence. *(See Example 1.)*

11. 19, 22, 25, 28, . . .

12. 1, 12, 23, 34, . . .

13. 16, 21, 26, 31, . . .

14. $60, 30, 0, -30, \ldots$

15. 1.3, 1, 0.7, 0.4, . . .

16. $\frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \ldots$

In Exercises 17–22, graph the arithmetic sequence. *(See Example 2.)*

17. 4, 12, 20, 28, . . .

18. $-15, 0, 15, 30, \ldots$

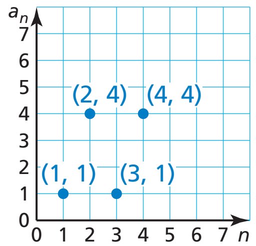
19. –1, –3, –5, –7, . . .

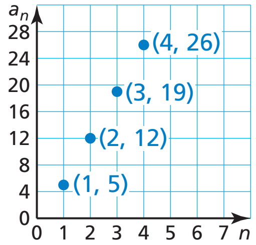
20. 2, 19, 36, 53, . . .

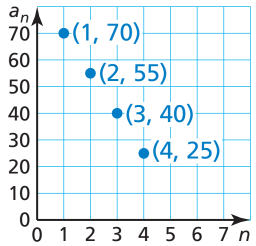
21. $0, 4\frac{1}{2}, 9, 13\frac{1}{2}, \ldots$

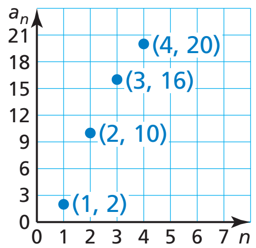
22. 6, 5.25, 4.5, 3.75, . . .

In Exercises 23–26, determine whether the graph represents an arithmetic sequence. Explain. *(See Example 3.)*

23.


24.


25.


26.


In Exercises 27–30, determine whether the sequence is arithmetic. If so, find the common difference.

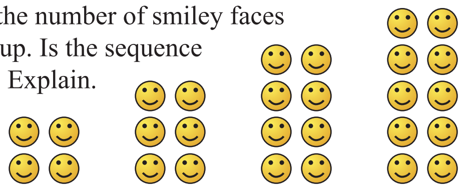
27. 13, 26, 39, 52, . . .

28. 5, 9, 14, 20, . . .

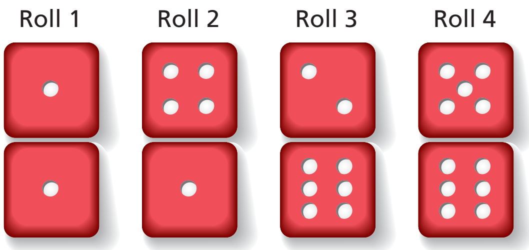
29. 48, 24, 12, 6, . . .

30. 87, 81, 75, 69, . . .

31. **FINDING A PATTERN** Write a sequence that represents the number of smiley faces in each group. Is the sequence arithmetic? Explain.



32. FINDING A PATTERN Write a sequence that represents the sum of the numbers in each roll. Is the sequence arithmetic? Explain.



In Exercises 33–38, write an equation for the *n*th term of the arithmetic sequence. Then find a_{10}. *(See Example 4.)*

33. $-5, -4, -3, -2, \ldots$

34. $-6, -9, -12, -15, \ldots$

35. $\frac{1}{2}, 1, 1\frac{1}{2}, 2, \ldots$

36. $100, 110, 120, 130, \ldots$

37. $10, 0, -10, -20, \ldots$

38. $\frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \ldots$

39. ERROR ANALYSIS Describe and correct the error in finding the common difference of the arithmetic sequence.

> ✗
>
> $2, \quad 1, \quad 0, \quad -1, \ldots$
>
> $-1 \quad -1 \quad -1$
>
> The common difference is 1.

40. ERROR ANALYSIS Describe and correct the error in writing an equation for the *n*th term of the arithmetic sequence.

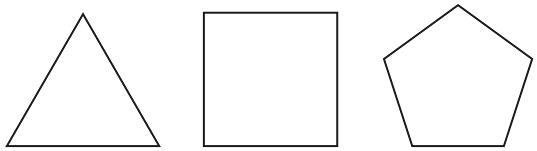
> ✗
>
> $14, 22, 30, 38, \ldots$
>
> $a_n = a_1 + nd$
>
> $a_n = 14 + 8n$

41. NUMBER SENSE The first term of an arithmetic sequence is 3. The common difference of the sequence is 1.5 times the first term. Write the next three terms of the sequence. Then graph the sequence.

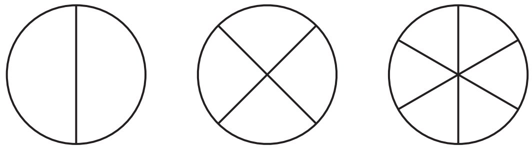
42. NUMBER SENSE The first row of a dominoes display has 10 dominoes. Each row after the first has two more dominoes than the row before it. Write the first five terms of the sequence that represents the number of dominoes in each row. Then graph the sequence.

REPEATED REASONING In Exercises 43 and 44, (a) draw the next three figures in the sequence and (b) describe the 20th figure in the sequence.

43.



44.



45. MODELING WITH MATHEMATICS The total number of babies born in a country each minute after midnight January 1st can be estimated by the sequence shown in the table. *(See Example 5.)*

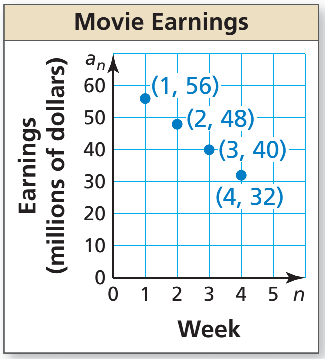
Minutes after midnight January 1st	1	2	3	4
Total babies born	5	10	15	20

a. Write a function that represents the arithmetic sequence.

b. Graph the function.

c. Estimate how many minutes after midnight January 1st it takes for 100 babies to be born.

46. MODELING WITH MATHEMATICS The amount of money a movie earns each week after its release can be approximated by the sequence shown in the graph.



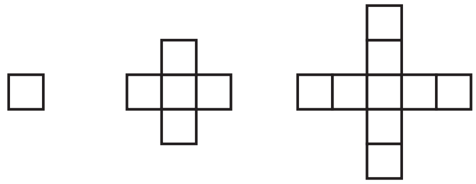
a. Write a function that represents the arithmetic sequence.

b. In what week does the movie earn \$16 million?

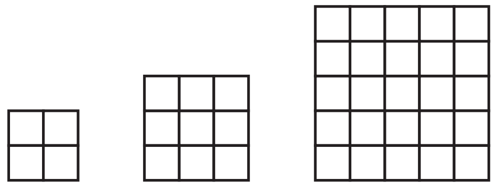
c. How much money does the movie earn overall?

MATHEMATICAL CONNECTIONS **In Exercises 47 and 48, each small square represents 1 square inch. Determine whether the areas of the figures form an arithmetic sequence. If so, write a function *f* that represents the arithmetic sequence and find *f*(30).**

47.



48.



49. **REASONING** Is the domain of an arithmetic sequence discrete or continuous? Is the range of an arithmetic sequence discrete or continuous?

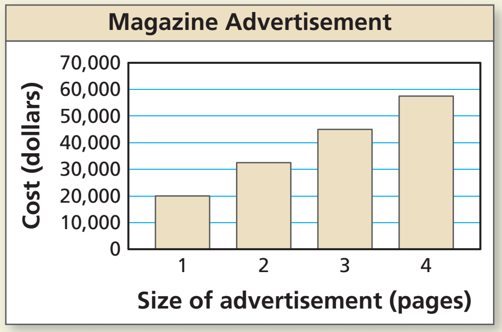
50. **MAKING AN ARGUMENT** Your friend says that the range of a function that represents an arithmetic sequence always contains only positive numbers or only negative numbers. Your friend claims this is true because the domain is the set of positive integers and the output values either constantly increase or constantly decrease. Is your friend correct? Explain.

51. **OPEN-ENDED** Write the first four terms of two different arithmetic sequences with a common difference of -3. Write an equation for the nth term of each sequence.

52. **THOUGHT PROVOKING** Describe an arithmetic sequence that models the numbers of people in a real-life situation.

53. **REPEATED REASONING** Firewood is stacked in a pile. The bottom row has 20 logs, and the top row has 14 logs. Each row has one more log than the row above it. How many logs are in the pile?

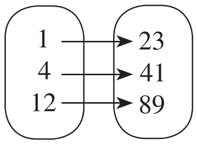
54. **HOW DO YOU SEE IT?** The bar graph shows the costs of advertising in a magazine.



a. Does the graph represent an arithmetic sequence? Explain.

b. Explain how you would estimate the cost of a six-page advertisement in the magazine.

55. **REASONING** Write a function f that represents the arithmetic sequence shown in the mapping diagram.



56. **PROBLEM SOLVING** A train stops at a station every 12 minutes starting at 6:00 A.M. You arrive at the station at 7:29 A.M. How long must you wait for the train?

57. **ABSTRACT REASONING** Let x be a constant. Determine whether each sequence is an arithmetic sequence. Explain.

a. $x + 6, 3x + 6, 5x + 6, 7x + 6, \ldots$

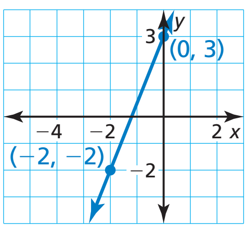
b. $x + 1, 3x + 1, 9x + 1, 27x + 1, \ldots$

Maintaining Mathematical Proficiency

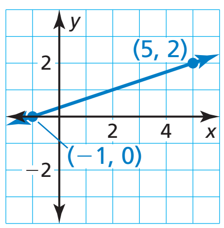
Reviewing what you learned in previous grades and lessons

Find the slope of the line. *(Section 3.5)*

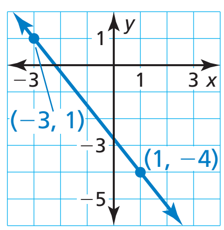
58.



59.



60.



4.5–4.7 What Did You Learn?

Core Vocabulary

scatter plot, *p. 188*
correlation, *p. 189*
line of fit, *p. 190*
residual, *p. 194*
linear regression, *p. 195*
line of best fit, *p. 195*
correlation coefficient, *p. 195*
interpolation, *p. 197*
extrapolation, *p. 197*
causation, *p. 197*
sequence, *p. 202*
term, *p. 202*
arithmetic sequence, *p. 202*
common difference, *p. 202*

Core Concepts

Section 4.5

Scatter Plot, *p. 188*
Identifying Correlations, *p. 189*
Using a Line of Fit to Model Data, *p. 190*

Section 4.6

Residuals, *p. 194*
Lines of Best Fit, *p. 195*
Correlation and Causation, *p. 197*

Section 4.7

Arithmetic Sequence, *p. 202*
Equation for an Arithmetic Sequence, *p. 204*

Mathematical Thinking

1. What resources can you use to help you answer Exercise 17 on page 192?
2. What calculations are repeated in Exercises 11–16 on page 206? When finding a term such as a_{50}, is there a general method or shortcut you can use instead of repeating calculations?

Performance Task

Any Beginning

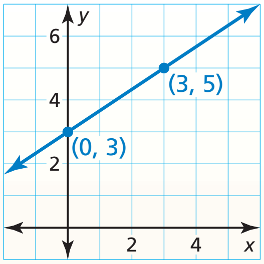
With so many ways to represent a linear relationship, where do you start? Use what you know to move between equations, graphs, tables, and contexts.

To explore the answer to this question and more, go to ***BigIdeasMath.com***.

4 Chapter Review

4.1 Writing Equations in Slope-Intercept Form (pp. 161–166)

Write an equation of the line in slope-intercept form.



Find the slope and y-intercept.

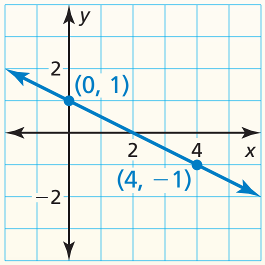
Let $(x_1, y_1) = (0, 3)$ and $(x_2, y_2) = (3, 5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{3 - 0} = \frac{2}{3}$$

Because the line crosses the y-axis at (0, 3), the y-intercept is 3.

▶ So, the equation is $y = \frac{2}{3}x + 3$.

1. Write an equation of the line in slope-intercept form.



4.2 Writing Equations in Point-Slope Form (pp. 167–172)

Write an equation in point-slope form of the line that passes through the point (−1, −8) and has a slope of 3.

$y - y_1 = m(x - x_1)$ — Write the point-slope form.

$y - (-8) = 3[x - (-1)]$ — Substitute 3 for m, −1 for x_1, and −8 for y_1.

$y + 8 = 3(x + 1)$ — Simplify.

▶ The equation is $y + 8 = 3(x + 1)$.

2. Write an equation in point-slope form of the line that passes through the point (4, 7) and has a slope of −1.

Write a linear function f with the given values.

3. $f(10) = 5, f(2) = -3$

4. $f(3) = -4, f(5) = -4$

5. $f(6) = 8, f(9) = 3$

4.3 Writing Equations in Standard Form (pp. 173–178)

Write an equation in standard form of the line that passes through the point (−1, 1) and has a slope of −2.

$y - y_1 = m(x - x_1)$ — Write the point-slope form.

$y - 1 = -2[x - (-1)]$ — Substitute −2 for m, −1 for x_1, and 1 for y_1.

$2x + y = -1$ — Simplify. Collect variable terms on one side and constants on the other.

▶ The equation is $2x + y = -1$.

6. Write two equations in standard form that are equivalent to $5x + y = -10$.

Write an equation in standard form of the line with the given characteristics.

7. slope: -4
passes through: $(-2, 7)$

8. passes through: $(-1, -5)$ and $(3, 7)$

Write equations of the horizontal and vertical lines that pass through the given point.

9. $(2, -12)$

10. $(-4, 3)$

4.4 Writing Equations of Parallel and Perpendicular Lines *(pp. 179–184)*

Determine which of the lines, if any, are parallel or perpendicular.

Line a: $y = 2x + 3$ Line b: $2y + x = 5$ Line c: $4y - 8x = -4$

Write the equations in slope-intercept form. Then compare the slopes.

Line a: $y = 2x + 3$ Line b: $y = -\frac{1}{2}x + \frac{5}{2}$ Line c: $y = 2x - 1$

▶ Lines a and c have slopes of 2, so they are parallel. Line b has a slope of $-\frac{1}{2}$, the negative reciprocal of 2, so it is perpendicular to lines a and c.

Determine which of the lines, if any, are parallel or perpendicular. Explain.

11. Line a passes through (0, 4) and (4, 3).
Line b passes through (0, 1) and (4, 0).
Line c passes through (2, 0) and (4, 4).

12. Line a: $2x - 7y = 14$
Line b: $y = \frac{7}{2}x - 8$
Line c: $2x + 7y = -21$

13. Write an equation of the line that passes through (1, 5) and is parallel to the line $y = -4x + 2$.

14. Write an equation of the line that passes through $(2, -3)$ and is perpendicular to the line $y = -2x - 3$.

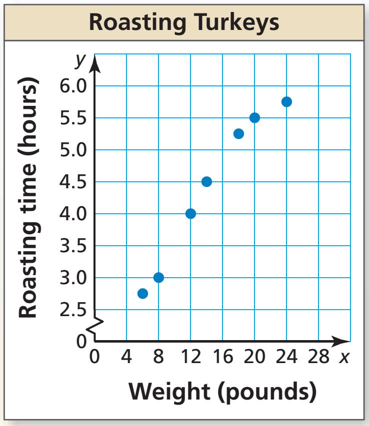
15. Write an equation of a line that is (a) parallel and (b) perpendicular to the line $x = 4$.

4.5 Scatter Plots and Lines of Fit *(pp. 187–192)*

The scatter plot shows the roasting times (in hours) and weights (in pounds) of seven turkeys. Tell whether the data show a *positive*, a *negative*, or *no* correlation.

As the weight of a turkey increases, the roasting time increases.

▶ So, the scatter plot shows a positive correlation.



Use the scatter plot in the example.

16. What is the roasting time for a 12-pound turkey?

17. Write an equation that models the roasting time as a function of the weight of a turkey. Interpret the slope and y-intercept of the line of fit.

4.6 Analyzing Lines of Fit (pp. 193–200)

The table shows the heights x (in inches) and shoe sizes y of several students. Use a graphing calculator to find an equation of the line of best fit. Identify and interpret the correlation coefficient.

Height, x	64	62	70	63	72	68	66	74	68	59
Shoe Size, y	9	7	12	8	13	9.5	9	13.5	10	6.5

Step 1 Enter the data from the table into two lists.

Step 2 Use the *linear regression* feature.

```
LinReg
 y=ax+b
 a=.4989919355
 b=-23.4828629
 r²=.9477256904
 r=.9735120392
```

▶ An equation of the line of best fit is $y = 0.50x - 23.5$. The correlation coefficient is about 0.974. This means that the relationship between the heights and the shoe sizes has a strong positive correlation and the equation closely models the data.

18. Make a scatter plot of the residuals to verify that the model in the example is a good fit.

19. Use the data in the example. (a) Approximate the height of a student whose shoe size is 9. (b) Predict the shoe size of a student whose height is 60 inches.

20. Is there a causal relationship in the data in the example? Explain.

4.7 Arithmetic Sequences (pp. 201–208)

Write an equation for the nth term of the arithmetic sequence $-3, -5, -7, -9, \ldots$. Then find a_{20}.

The first term is –3, and the common difference is –2.

$a_n = a_1 + (n - 1)d$ — Equation for an arithmetic sequence

$a_n = -3 + (n - 1)(-2)$ — Substitute -3 for a_1 and -2 for d.

$a_n = -2n - 1$ — Simplify.

Use the equation to find the 20th term.

$a_{20} = -2(20) - 1$ — Substitute 20 for n.

$= -41$ — Simplify.

▶ The 20th term of the arithmetic sequence is -41.

Write an equation for the nth term of the arithmetic sequence. Then find a_{30}.

21. 11, 10, 9, 8, . . .

22. 6, 12, 18, 24, . . .

23. $-9, -6, -3, 0, \ldots$

4 Chapter Test

Write an equation in standard form of the line with the given characteristics.

1. slope $= \frac{1}{2}$; y-intercept $= -6$

2. passes through $(1, 5)$ and $(3, -5)$

Write an equation in slope-intercept form of the line with the given characteristics.

3. slope $= \frac{2}{5}$; y-intercept $= -7$

4. slope $= -4$; passes through $(-2, 16)$

5. passes through $(0, 6)$ and $(3, -3)$

6. passes through $(5, -7)$ and $(10, -7)$

7. parallel to the line $y = 3x - 1$; passes through $(-2, -8)$

8. perpendicular to the line $y = \frac{1}{4}x - 9$; passes through $(1, 1)$

Write an equation in point-slope form of the line with the given characteristics.

9. slope $= 10$; passes through $(6, 2)$

10. passes through $(-3, 2)$ and $(6, -1)$

11. The first row of an auditorium has 42 seats. Each row after the first has three more seats than the row before it.

a. Find the number of seats in Row 25.

b. Which row has 90 seats?

12. The table shows the amount x (in dollars) spent on advertising for a neighborhood festival and the attendance y of the festival for several years.

a. Make a scatter plot of the data. Describe the correlation.

b. Write an equation that models the attendance as a function of the amount spent on advertising.

c. Interpret the slope and y-intercept of the line of fit.

Advertising (dollars), x	Yearly attendance, y
500	400
1000	550
1500	550
2000	800
2500	650
3000	800
3500	1050
4000	1100

13. Consider the data in the table in Exercise 12.

a. Use a graphing calculator to find an equation of the line of best fit.

b. Identify and interpret the correlation coefficient.

c. What would you expect the scatter plot of the residuals to look like?

d. Is there a causal relationship in the data? Explain your reasoning.

e. Predict the amount that must be spent on advertising to get 2000 people to attend the festival.

14. Let a, b, c, and d be constants. Determine which of the lines, if any, are parallel or perpendicular. Explain.

Line 1: $y - c = ax$ Line 2: $ay = -x - b$ Line 3: $ax + y = d$

15. You are buying ribbon to make costumes for a school play. Organza ribbon costs \$0.08 per yard. Satin ribbon costs \$0.04 per yard. (a) Write an equation in standard form that models the possible combinations of yards of ribbon you can buy for \$5. (b) Graph the equation from part (a). (c) Find four possible combinations.

4 Standards Assessment

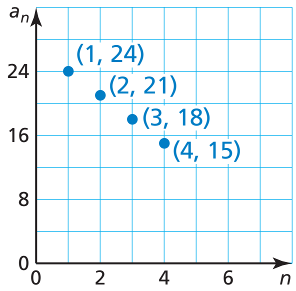
1. Which function represents the arithmetic sequence shown in the graph? *(TEKS A.12.D)*

Ⓐ $f(n) = 15 + 3n$

Ⓑ $f(n) = 4 - 3n$

Ⓒ $f(n) = 27 - 3n$

Ⓓ $f(n) = 24 - 3n$



2. In which situation is both a correlation and causation likely? *(TEKS A.4.B)*

Ⓕ the number of pairs of shoes a person owns and the person's shoe size

Ⓖ the number of cell phones and the number of taxis in a city

Ⓗ a person's IQ and the time it takes the person to run 50 meters

Ⓙ the price of a pair of pants and the number sold

3. Which inequality is *not* equivalent to $4x + 1 < 17$? *(TEKS A.5.B)*

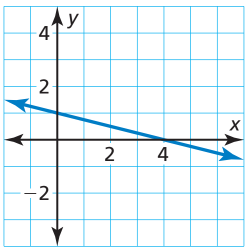
Ⓐ $3x + 6 < 2x + 10$

Ⓑ $-6x + 14 > -10$

Ⓒ $-x - 5 > 2x + 7$

Ⓓ $5x - 5 < x + 11$

4. **GRIDDED ANSWER** The equation of the line shown can be written in the standard form $-2x + By = -8$. What is the value of B? *(TEKS A.2.C)*



5. The cost to rent a movie at a video store can be modeled by a linear function. The table shows the total costs of renting a movie for different lengths of time. Based on the table, which statement is true? *(TEKS A.3.B)*

Ⓕ The slope is 1.5. It represents the initial cost.

Ⓖ The slope is 1.5. It represents the rate per night.

Ⓗ The slope is 1.25. It represents the initial cost.

Ⓙ The slope is 1.25. It represents the rate per night.

Nights rented, x	Total cost (dollars), y
1	1.50
2	2.75
3	4.00
4	5.25
5	6.50

6. Which equation represents the line that passes through (0, 0) and is parallel to the line that passes through (2, 3) and (6, 1)? *(TEKS A.2.E, TEKS A.3.A)*

Ⓐ $y = \frac{1}{2}x$　　Ⓑ $y = -\frac{1}{2}x$

Ⓒ $y = -2x$　　Ⓓ $y = 2x$

7. In which step below does a mistake first appear in solving the equation $-2(4 - x) + 5x = 6(2x - 3)$? *(TEKS A.5.A, TEKS A.10.D)*

Step 1 $-8 - 2x + 5x = 12x - 18$

Step 2 $-8 + 3x = 12x - 18$

Step 3 $10 = 9x$

Step 4 $x = \frac{10}{9}$

Ⓕ Step 1　　Ⓖ Step 2

Ⓗ Step 3　　Ⓙ Step 4

8. The table shows the daily high temperatures x (in degrees Fahrenheit) and the numbers y of frozen fruit bars sold on eight randomly selected days. Which of the following is a reasonable equation to model the data? *(TEKS A.4.C)*

Temperature (°F), *x*	54	60	68	72	78	84	92	98
Frozen fruit bars, *y*	40	120	180	260	280	260	220	180

Ⓐ $y = 3x - 50$　　Ⓑ $y = -3x + 50$

Ⓒ $y = x - 150$　　Ⓓ $y = -x + 150$

9. The ordered pairs $(-4, 7)$, $(-2, 6)$, $(0, 3)$, $(-2, 1)$, $(2, -1)$, and $(4, 1)$ represent a relation. Based on the relation, which statement is true? *(TEKS A.12.A)*

Ⓕ The relation is a function.

Ⓖ The relation is not a function because one output is paired with two inputs.

Ⓗ The relation is not a function because one input is paired with two outputs.

Ⓙ The relation is not a function because the rate of change is not constant.

10. Which of the following is enough information to write an equation of a line that does not pass through the origin? *(TEKS A.2.B)*

I. two points on the line

II. the slope of the line

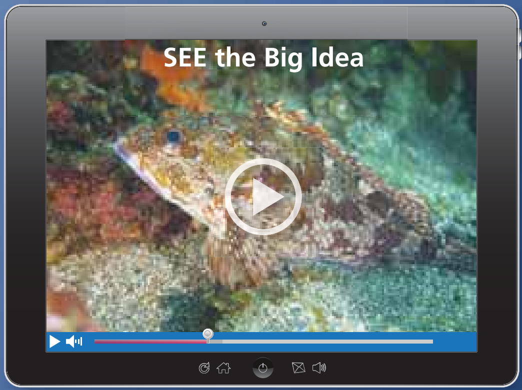
III. both intercepts of the line

IV. the slope and x-intercept of the line

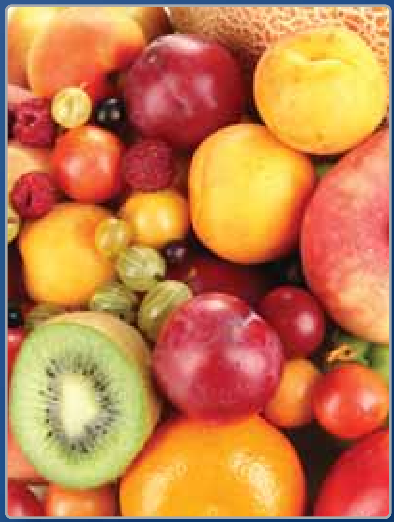
Ⓐ I and II only　　Ⓑ I, III, and IV only

Ⓒ I and IV only　　Ⓓ I, II, and IV only

5 Solving Systems of Linear Equations



Fishing *(p. 265)*



Fruit Salad *(p. 255)*

Pets *(p. 250)*

Drama Club *(p. 228)*

Roofing Contractor *(p. 222)*

Mathematical Thinking: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

Maintaining Mathematical Proficiency

Graphing Linear Functions (A.3.C)

Example 1 **Graph $3 + y = \frac{1}{2}x$.**

Step 1 Rewrite the equation in slope-intercept form.

$$y = \frac{1}{2}x - 3$$

Step 2 Find the slope and the y-intercept.

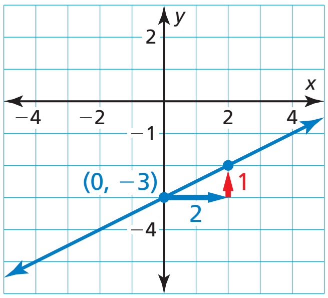
$$m = \frac{1}{2} \text{ and } b = -3$$

Step 3 The y-intercept is -3. So, plot $(0, -3)$.

Step 4 Use the slope to find another point on the line.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{1}{2}$$

Plot the point that is 2 units right and 1 unit up from $(0, -3)$. Draw a line through the two points.



Graph the equation.

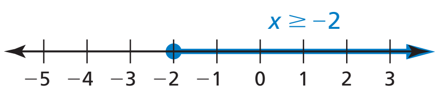
1. $y + 4 = x$ **2.** $6x - y = -1$ **3.** $4x + 5y = 20$ **4.** $-2y + 12 = -3x$

Solving and Graphing Linear Inequalities (7.10.B, A.5.B)

Example 2 **Solve $2x - 17 \leq 8x - 5$. Graph the solution.**

$2x - 17 \leq 8x - 5$	Write the inequality.
$\underline{+5} \quad \underline{+5}$	Add 5 to each side.
$2x - 12 \leq 8x$	Simplify.
$\underline{-2x} \quad \underline{-2x}$	Subtract $2x$ from each side.
$-12 \leq 6x$	Simplify.
$\frac{-12}{6} \leq \frac{6x}{6}$	Divide each side by 6.
$-2 \leq x$	Simplify.

▶ The solution is $x \geq -2$.



Solve the inequality. Graph the solution.

5. $m + 4 > 9$ **6.** $24 \leq -6t$ **7.** $2a - 5 \leq 13$

8. $-5z + 1 < -14$ **9.** $4k - 16 < k + 2$ **10.** $7w + 12 \geq 2w - 3$

11. **ABSTRACT REASONING** The graphs of the linear functions g and h have different slopes. The value of both functions at $x = a$ is b. When g and h are graphed in the same coordinate plane, what happens at the point (a, b)?

Mathematical Thinking

Mathematically proficient students select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems. **(A.1.C)**

Using a Graphing Calculator

Core Concept

Finding the Point of Intersection

You can use a graphing calculator to find the point of intersection, if it exists, of the graphs of two linear equations.

1. Enter the equations into a graphing calculator.
2. Graph the equations in an appropriate viewing window, so that the point of intersection is visible.
3. Use the *intersect* feature of the graphing calculator to find the point of intersection.

EXAMPLE 1 Using a Graphing Calculator

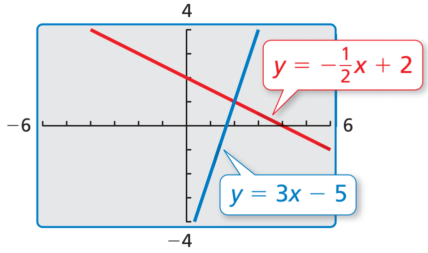
Use a graphing calculator to find the point of intersection, if it exists, of the graphs of the two linear equations.

$y = -\frac{1}{2}x + 2$ Equation 1

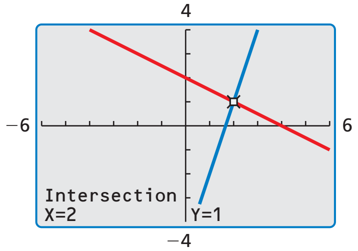
$y = 3x - 5$ Equation 2

SOLUTION

The slopes of the lines are not the same, so you know that the lines intersect. Enter the equations into a graphing calculator. Then graph the equations in an appropriate viewing window.



Use the *intersect* feature to find the point of intersection of the lines.



▶ The point of intersection is (2, 1).

Monitoring Progress

Use a graphing calculator to find the point of intersection of the graphs of the two linear equations.

1. $y = -2x - 3$
$y = \frac{1}{2}x - 3$

2. $y = -x + 1$
$y = x - 2$

3. $3x - 2y = 2$
$2x - y = 2$

5.1 Solving Systems of Linear Equations by Graphing

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.2.I
A.3.F
A.3.G
A.5.C

Essential Question

How can you solve a system of linear equations?

EXPLORATION 1 Writing a System of Linear Equations

Work with a partner. Your family opens a bed-and-breakfast. They spend $600 preparing a bedroom to rent. The cost to your family for food and utilities is $15 per night. They charge $75 per night to rent the bedroom.

a. Write an equation that represents the costs.

Cost, C (in dollars) = $15 per night • Number of nights, x + $600

b. Write an equation that represents the revenue (income).

Revenue, R (in dollars) = $75 per night • Number of nights, x

c. A set of two (or more) linear equations is called a **system of linear equations**. Write the system of linear equations for this problem.

APPLYING MATHEMATICS

To be proficient in math, you need to identify important quantities in real-life problems and map their relationships using tools such as diagrams, tables, and graphs.

EXPLORATION 2 Using a Table or Graph to Solve a System

Work with a partner. Use the cost and revenue equations from Exploration 1 to determine how many nights your family needs to rent the bedroom before recovering the cost of preparing the bedroom. This is the *break-even point*.

a. Copy and complete the table.

x **(nights)**	0	1	2	3	4	5	6	7	8	9	10	11
C **(dollars)**												
R **(dollars)**												

b. How many nights does your family need to rent the bedroom before breaking even?

c. In the same coordinate plane, graph the cost equation and the revenue equation from Exploration 1.

d. Find the point of intersection of the two graphs. What does this point represent? How does this compare to the break-even point in part (b)? Explain.

Communicate Your Answer

3. How can you solve a system of linear equations? How can you check your solution?

4. Solve each system by using a table or sketching a graph. Explain why you chose each method. Use a graphing calculator to check each solution.

a. $y = -4.3x - 1.3$
$y = 1.7x + 4.7$

b. $y = x$
$y = -3x + 8$

c. $y = -x - 1$
$y = 3x + 5$

5.1 Lesson

What You Will Learn

- Check solutions of systems of linear equations.
- Solve systems of linear equations by graphing.
- Use systems of linear equations to solve real-life problems.

Core Vocabulary

system of linear equations, *p. 220*
solution of a system of linear equations, *p. 220*

Previous
linear equation
ordered pair

Systems of Linear Equations

A **system of linear equations** is a set of two or more linear equations in the same variables. An example is shown below.

$x + y = 7$ Equation 1

$2x - 3y = -11$ Equation 2

A **solution of a system of linear equations** in two variables is an ordered pair that is a solution of each equation in the system.

EXAMPLE 1 Checking Solutions

Tell whether the ordered pair is a solution of the system of linear equations.

a. $(2, 5)$; $x + y = 7$ Equation 1; $2x - 3y = -11$ Equation 2

b. $(-2, 0)$; $y = -2x - 4$ Equation 1; $y = x + 4$ Equation 2

SOLUTION

a. Substitute 2 for x and 5 for y in each equation.

Equation 1

$x + y = 7$

$2 + 5 \stackrel{?}{=} 7$

$7 = 7$ ✓

Equation 2

$2x - 3y = -11$

$2(2) - 3(5) \stackrel{?}{=} -11$

$-11 = -11$ ✓

▶ Because the ordered pair $(2, 5)$ is a solution of each equation, it is a solution of the linear system.

READING

A system of linear equations is also called a *linear system*.

b. Substitute -2 for x and 0 for y in each equation.

Equation 1

$y = -2x - 4$

$0 \stackrel{?}{=} -2(-2) - 4$

$0 = 0$ ✓

Equation 2

$y = x + 4$

$0 \stackrel{?}{=} -2 + 4$

$0 \neq 2$ ✗

▶ The ordered pair $(-2, 0)$ is a solution of the first equation, but it is not a solution of the second equation. So, $(-2, 0)$ is *not* a solution of the linear system.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Tell whether the ordered pair is a solution of the system of linear equations.

1. $(1, -2)$; $2x + y = 0$, $-x + 2y = 5$

2. $(1, 4)$; $y = 3x + 1$, $y = -x + 5$

Solving Systems of Linear Equations by Graphing

The solution of a system of linear equations is the point of intersection of the graphs of the equations.

Core Concept

Solving a System of Linear Equations by Graphing

Step 1 Graph each equation in the same coordinate plane.

Step 2 Estimate the point of intersection.

Step 3 Check the point from Step 2 by substituting for x and y in each equation of the original system.

REMEMBER

Note that the linear equations are in slope-intercept form. You can use the method presented in Section 3.5 to graph the equations.

EXAMPLE 2 Solving a System of Linear Equations by Graphing

Solve the system of linear equations by graphing.

$y = -2x + 5$ Equation 1

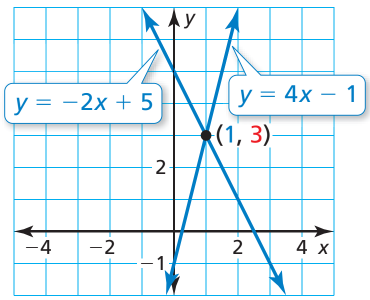
$y = 4x - 1$ Equation 2

SOLUTION

Step 1 Graph each equation.

Step 2 Estimate the point of intersection. The graphs appear to intersect at (1, 3).

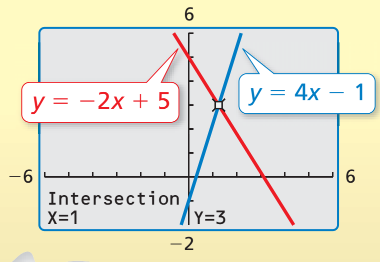
Step 3 Check your point from Step 2.



Equation 1	Equation 2
$y = -2x + 5$	$y = 4x - 1$
$3 \stackrel{?}{=} -2(1) + 5$	$3 \stackrel{?}{=} 4(1) - 1$
$3 = 3$ ✓	$3 = 3$ ✓

▶ The solution is (1, 3).

Check



Monitoring Progress

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Solve the system of linear equations by graphing.

3. $y = x - 2$
$y = -x + 4$

4. $y = \frac{1}{2}x + 3$
$y = -\frac{3}{2}x - 5$

5. $2x + y = 5$
$3x - 2y = 4$

Solving Real-Life Problems

EXAMPLE 3 Modeling with Mathematics

A roofing contractor buys 30 bundles of shingles and 4 rolls of roofing paper for \$1040. In a second purchase (at the same prices), the contractor buys 8 bundles of shingles for \$256. Find the price per bundle of shingles and the price per roll of roofing paper.

SOLUTION

1. **Understand the Problem** You know the total price of each purchase and how many of each item were purchased. You are asked to find the price of each item.
2. **Make a Plan** Use a verbal model to write a system of linear equations that represents the problem. Then solve the system of linear equations.
3. **Solve the Problem**

Words 30 • Price per bundle + 4 • Price per roll = 1040

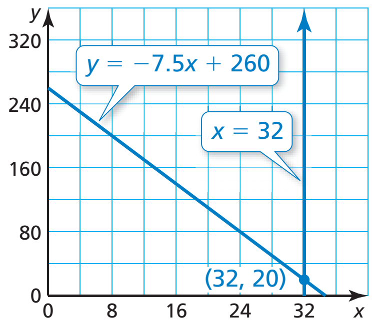
8 • Price per bundle + 0 • Price per roll = 256

Variables Let x be the price (in dollars) per bundle and let y be the price (in dollars) per roll.

System $30x + 4y = 1040$ Equation 1

$8x = 256$ Equation 2

Step 1 Graph each equation. Note that only the first quadrant is shown because x and y must be positive.



Step 2 Estimate the point of intersection. The graphs appear to intersect at (32, 20).

Step 3 Check your point from Step 2.

Equation 1

$$30x + 4y = 1040$$
$$30(32) + 4(20) \stackrel{?}{=} 1040$$
$$1040 = 1040 \checkmark$$

Equation 2

$$8x = 256$$
$$8(32) \stackrel{?}{=} 256$$
$$256 = 256 \checkmark$$

▶ The solution is (32, 20). So, the price per bundle of shingles is \$32, and the price per roll of roofing paper is \$20.

4. **Look Back** You can use estimation to check that your solution is reasonable. A bundle of shingles costs about \$30. So, 30 bundles of shingles and 4 rolls of roofing paper (at \$20 per roll) cost about $30(30) + 4(20) = \$980$, and 8 bundles of shingles costs about $8(30) = \$240$. These prices are close to the given values, so the solution seems reasonable.

Monitoring Progress

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6. You have a total of 18 math and science exercises for homework. You have six more math exercises than science exercises. How many exercises do you have in each subject?

Vocabulary and Core Concept Check

1. **VOCABULARY** Do the equations $5y - 2x = 18$ and $6x = -4y - 10$ form a system of linear equations? Explain.

2. **DIFFERENT WORDS, SAME QUESTION** Consider the system of linear equations $-4x + 2y = 4$ and $4x - y = -6$. Which is different? Find "both" answers.

Solve the system of linear equations.	Solve each equation for y.
Find the point of intersection of the graphs of the equations.	Find an ordered pair that is a solution of each equation in the system.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, tell whether the ordered pair is a solution of the system of linear equations. *(See Example 1.)*

3. $(2, 6)$; $\begin{aligned} x + y &= 8 \\ 3x - y &= 0 \end{aligned}$

4. $(8, 2)$; $\begin{aligned} x - y &= 6 \\ 2x - 10y &= 4 \end{aligned}$

5. $(-1, 3)$; $\begin{aligned} y &= -7x - 4 \\ y &= 8x + 5 \end{aligned}$

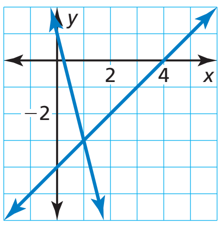
6. $(-4, -2)$; $\begin{aligned} y &= 2x + 6 \\ y &= -3x - 14 \end{aligned}$

7. $(-2, 1)$; $\begin{aligned} 6x + 5y &= -7 \\ 2x - 4y &= -8 \end{aligned}$

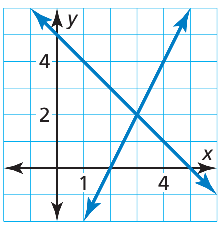
8. $(5, -6)$; $\begin{aligned} 6x + 3y &= 12 \\ 4x + y &= 14 \end{aligned}$

In Exercises 9–12, use the graph to solve the system of linear equations. Check your solution.

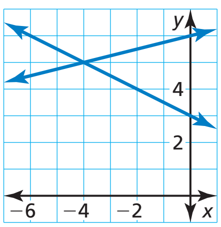
9. $x - y = 4$
   $4x + y = 1$



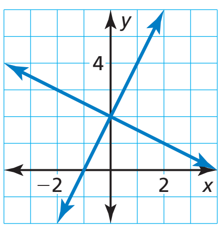
10. $x + y = 5$
    $y - 2x = -4$



11. $6y + 3x = 18$
    $-x + 4y = 24$



12. $2x - y = -2$
    $2x + 4y = 8$



In Exercises 13–20, solve the system of linear equations by graphing. *(See Example 2.)*

13. $y = -x + 7$
    $y = x + 1$

14. $y = -x + 4$
    $y = 2x - 8$

15. $y = \frac{1}{3}x + 2$
    $y = \frac{2}{3}x + 5$

16. $y = \frac{3}{4}x - 4$
    $y = -\frac{1}{2}x + 11$

17. $9x + 3y = -3$
    $2x - y = -4$

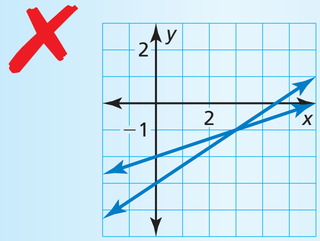
18. $4x - 4y = 20$
    $y = -5$

19. $x - 4y = -4$
    $-3x - 4y = 12$

20. $3y + 4x = 3$
    $x + 3y = -6$

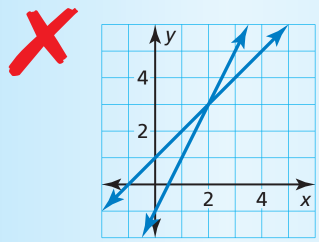
ERROR ANALYSIS **In Exercises 21 and 22, describe and correct the error in solving the system of linear equations.**

21.



The solution of the linear system $x - 3y = 6$ and $2x - 3y = 3$ is $(3, -1)$.

22.



The solution of the linear system $y = 2x - 1$ and $y = x + 1$ is $x = 2$.

USING TOOLS **In Exercises 23–26, use a graphing calculator to solve the system of linear equations.**

23. $0.2x + 0.4y = 4$
$-0.6x + 0.6y = -3$

24. $-1.6x - 3.2y = -24$
$2.6x + 2.6y = 26$

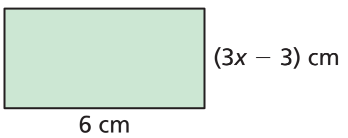
25. $-7x + 6y = 0$
$0.5x + y = 2$

26. $4x - y = 1.5$
$2x + y = 1.5$

27. **MODELING WITH MATHEMATICS** You have 40 minutes to exercise at the gym, and you want to burn 300 calories total using both machines. How much time should you spend on each machine? *(See Example 3.)*

28. **MODELING WITH MATHEMATICS** You sell small and large candles at a craft fair. You collect $144 selling a total of 28 candles. How many of each type of candle did you sell?

29. **MATHEMATICAL CONNECTIONS** Write a linear equation that represents the area and a linear equation that represents the perimeter of the rectangle. Solve the system of linear equations by graphing. Interpret your solution.



30. **THOUGHT PROVOKING** Your friend's bank account balance (in dollars) is represented by the equation $y = 25x + 250$, where x is the number of months. Graph this equation. After 6 months, you want to have the same account balance as your friend. Write a linear equation that represents your account balance. Interpret the slope and y-intercept of the line that represents your account balance.

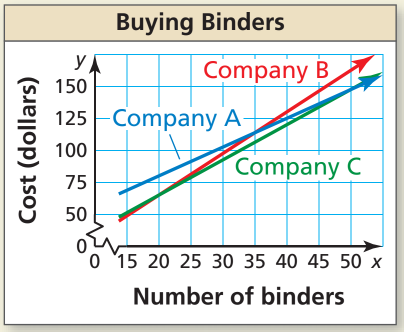
31. **COMPARING METHODS** Consider the equation $x + 2 = 3x - 4$.

a. Solve the equation using algebra.

b. Solve the system of linear equations $y = x + 2$ and $y = 3x - 4$ by graphing.

c. How is the linear system and the solution in part (b) related to the original equation and the solution in part (a)?

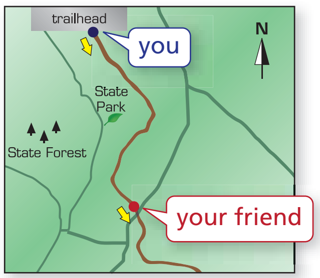
32. **HOW DO YOU SEE IT?** A teacher is purchasing binders for students. The graph shows the total costs of ordering x binders from three different companies.



a. For what numbers of binders are the costs the same at two different companies? Explain.

b. How do your answers in part (a) relate to systems of linear equations?

33. **MAKING AN ARGUMENT** You and a friend are going hiking but start at different locations. You start at the trailhead and walk 5 miles per hour. Your friend starts 3 miles from the trailhead and walks 3 miles per hour.



a. Write and graph a system of linear equations that represents this situation.

b. Your friend says that after an hour of hiking you will both be at the same location on the trail. Is your friend correct? Use the graph from part (a) to explain your answer.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the literal equation for y. *(Section 1.4)*

34. $10x + 5y = 5x + 20$

35. $9x + 18 = 6y - 3x$

36. $\frac{3}{4}x + \frac{1}{4}y = 5$

5.2 Solving Systems of Linear Equations by Substitution

Texas Essential Knowledge and Skills

A.2.I
A.5.C

Essential Question

How can you use substitution to solve a system of linear equations?

EXPLORATION 1 Using Substitution to Solve Systems

Work with a partner. Solve each system of linear equations using two methods.

Method 1 Solve for *x* first.

Solve for x in one of the equations. Substitute the expression for x into the other equation to find y. Then substitute the value of y into one of the original equations to find x.

Method 2 Solve for *y* first.

Solve for y in one of the equations. Substitute the expression for y into the other equation to find x. Then substitute the value of x into one of the original equations to find y.

Is the solution the same using both methods? Explain which method you would prefer to use for each system.

a. $x + y = -7$
$-5x + y = 5$

b. $x - 6y = -11$
$3x + 2y = 7$

c. $4x + y = -1$
$3x - 5y = -18$

EXPLORATION 2 Writing and Solving a System of Equations

Work with a partner.

a. Write a random ordered pair with integer coordinates. One way to do this is to use a graphing calculator. The ordered pair generated at the right is $(-2, -3)$.

Choose two random integers between -5 and 5.

```
randInt(-5,5,2)
            {-2 -3}
```

b. Write a system of linear equations that has your ordered pair as its solution.

c. Exchange systems with your partner and use one of the methods from Exploration 1 to solve the system. Explain your choice of method.

USING PRECISE MATHEMATICAL LANGUAGE

To be proficient in math, you need to communicate precisely with others.

Communicate Your Answer

3. How can you use substitution to solve a system of linear equations?

4. Use one of the methods from Exploration 1 to solve each system of linear equations. Explain your choice of method. Check your solutions.

a. $x + 2y = -7$
$2x - y = -9$

b. $x - 2y = -6$
$2x + y = -2$

c. $-3x + 2y = -10$
$-2x + y = -6$

d. $3x + 2y = 13$
$x - 3y = -3$

e. $3x - 2y = 9$
$-x - 3y = 8$

f. $3x - y = -6$
$4x + 5y = 11$

5.2 Lesson

What You Will Learn

- Solve systems of linear equations by substitution.
- Use systems of linear equations to solve real-life problems.

Core Vocabulary

Previous
system of linear equations
solution of a system of linear equations

Solving Linear Systems by Substitution

Another way to solve a system of linear equations is to use substitution.

Core Concept

Solving a System of Linear Equations by Substitution

Step 1 Solve one of the equations for one of the variables.

Step 2 Substitute the expression from Step 1 into the other equation and solve for the other variable.

Step 3 Substitute the value from Step 2 into one of the original equations and solve.

EXAMPLE 1 Solving a System of Linear Equations by Substitution

Solve the system of linear equations by substitution.

$y = -2x - 9$ Equation 1

$6x - 5y = -19$ Equation 2

SOLUTION

Step 1 Equation 1 is already solved for y.

Step 2 Substitute $-2x - 9$ for y in Equation 2 and solve for x.

$6x - 5y = -19$	Equation 2
$6x - 5(-2x - 9) = -19$	Substitute $-2x - 9$ for y.
$6x + 10x + 45 = -19$	Distributive Property
$16x + 45 = -19$	Combine like terms.
$16x = -64$	Subtract 45 from each side.
$x = -4$	Divide each side by 16.

Step 3 Substitute -4 for x in Equation 1 and solve for y.

$y = -2x - 9$	Equation 1
$= -2(-4) - 9$	Substitute -4 for x.
$= 8 - 9$	Multiply.
$= -1$	Subtract.

▶ The solution is $(-4, -1)$.

Check

Equation 1

$y = -2x - 9$

$-1 \stackrel{?}{=} -2(-4) - 9$

$-1 = -1$ ✓

Equation 2

$6x - 5y = -19$

$6(-4) - 5(-1) \stackrel{?}{=} -19$

$-19 = -19$ ✓

Monitoring Progress

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Solve the system of linear equations by substitution. Check your solution.

1. $y = 3x + 14$
$y = -4x$

2. $3x + 2y = 0$
$y = \frac{1}{2}x - 1$

3. $x = 6y - 7$
$4x + y = -3$

EXAMPLE 2 Solving a System of Linear Equations by Substitution

ANOTHER WAY

You could also begin by solving for x in Equation 1, solving for y in Equation 2, or solving for x in Equation 2.

Solve the system of linear equations by substitution.

$-x + y = 3$ Equation 1

$3x + y = -1$ Equation 2

SOLUTION

Step 1 Solve for y in Equation 1.

$y = x + 3$ Revised Equation 1

Step 2 Substitute $x + 3$ for y in Equation 2 and solve for x.

$3x + y = -1$ Equation 2

$3x + (x + 3) = -1$ Substitute $x + 3$ for y.

$4x + 3 = -1$ Combine like terms.

$4x = -4$ Subtract 3 from each side.

$x = -1$ Divide each side by 4.

Step 3 Substitute -1 for x in Equation 1 and solve for y.

$-x + y = 3$ Equation 1

$-(-1) + y = 3$ Substitute -1 for x.

$y = 2$ Subtract 1 from each side.

The solution is $(-1, 2)$.

Algebraic Check

Equation 1

$-x + y = 3$

$-(-1) + 2 \stackrel{?}{=} 3$

$3 = 3$ ✓

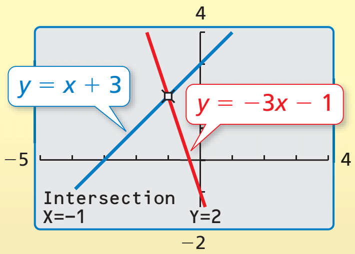
Equation 2

$3x + y = -1$

$3(-1) + 2 \stackrel{?}{=} -1$

$-1 = -1$ ✓

Graphical Check



Monitoring Progress

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Solve the system of linear equations by substitution. Check your solution.

4. $x + y = -2$

$-3x + y = 6$

5. $-x + y = -4$

$4x - y = 10$

6. $2x - y = -5$

$3x - y = 1$

7. $x - 2y = 7$

$3x - 2y = 3$

Solving Real-Life Problems

EXAMPLE 3 Modeling with Mathematics

A drama club earns \$1040 from a production. An adult ticket costs twice as much as a student ticket. Write a system of linear equations that represents this situation. What is the price of each type of ticket?

Tickets sold	
Type	**Number**
adult	64
student	132

SOLUTION

1. **Understand the Problem** You know the amount earned, the total numbers of adult and student tickets sold, and the relationship between the price of an adult ticket and the price of a student ticket. You are asked to write a system of linear equations that represents the situation and find the price of each type of ticket.
2. **Make a Plan** Use a verbal model to write a system of linear equations that represents the problem. Then solve the system of linear equations.
3. **Solve the Problem**

Words 64 • Adult ticket price + 132 • Student ticket price = 1040

Adult ticket price = 2 • Student ticket price

Variables Let x be the price (in dollars) of an adult ticket and let y be the price (in dollars) of a student ticket.

System $64x + 132y = 1040$ Equation 1

$x = 2y$ Equation 2

Step 1 Equation 2 is already solved for x.

Step 2 Substitute $2y$ for x in Equation 1 and solve for y.

$64x + 132y = 1040$ Equation 1

$64(2y) + 132y = 1040$ Substitute $2y$ for x.

$260y = 1040$ Simplify.

$y = 4$ Simplify.

Step 3 Substitute 4 for y in Equation 2 and solve for x.

$x = 2y$ Equation 2

$x = 2(4)$ Substitute 4 for y.

$x = 8$ Simplify.

The solution is (8, 4). So, an adult ticket costs \$8 and a student ticket costs \$4.

4. **Look Back** To check that your solution is correct, substitute the values of x and y into both of the original equations and simplify.

$64(8) + 132(4) = 1040$ $\qquad$ $8 = 2(4)$

$1040 = 1040$ ✓ $\qquad$ $8 = 8$ ✓

STUDY TIP

You can use either of the original equations to solve for x. However, using Equation 2 requires fewer calculations.

Monitoring Progress

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8. There are a total of 64 students in a drama club and a yearbook club. The drama club has 10 more students than the yearbook club. Write a system of linear equations that represents this situation. How many students are in each club?

5.2 Exercises

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Vocabulary and Core Concept Check

1. **WRITING** Describe how to solve a system of linear equations by substitution.

2. **NUMBER SENSE** When solving a system of linear equations by substitution, how do you decide which variable to solve for in Step 1?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, tell which equation you would choose to solve for one of the variables. Explain.

3. $x + 4y = 30$
$x - 2y = 0$

4. $3x - y = 0$
$2x + y = -10$

5. $5x + 3y = 11$
$5x - y = 5$

6. $3x - 2y = 19$
$x + y = 8$

7. $x - y = -3$
$4x + 3y = -5$

8. $3x + 5y = 25$
$x - 2y = -6$

In Exercises 9–16, solve the sytem of linear equations by substitution. Check your solution. *(See Examples 1 and 2.)*

9. $x = 17 - 4y$
$y = x - 2$

10. $6x - 9 = y$
$y = -3x$

11. $x = 16 - 4y$
$3x + 4y = 8$

12. $-5x + 3y = 51$
$y = 10x - 8$

13. $2x = 12$
$x - 5y = -29$

14. $2x - y = 23$
$x - 9 = -1$

15. $5x + 2y = 9$
$x + y = -3$

16. $11x - 7y = -14$
$x - 2y = -4$

17. **ERROR ANALYSIS** Describe and correct the error in solving for one of the variables in the linear system $8x + 2y = -12$ and $5x - y = 4$.

✗

Step 1 $\quad 5x - y = 4$
$-y = -5x + 4$
$y = 5x - 4$

Step 2 $\quad 5x - (5x - 4) = 4$
$5x - 5x + 4 = 4$
$4 = 4$

18. **ERROR ANALYSIS** Describe and correct the error in solving for one of the variables in the linear system $4x + 2y = 6$ and $3x + y = 9$.

✗

Step 1 $\quad 3x + y = 9$
$y = 9 - 3x$

Step 2 $\quad 4x + 2(9 - 3x) = 6$
$4x + 18 - 6x = 6$
$-2x = -12$
$x = 6$

Step 3 $\quad 3x + y = 9$
$3x + 6 = 9$
$3x = 3$
$x = 1$

19. **MODELING WITH MATHEMATICS** A farmer plants corn and wheat on a 180-acre farm. The farmer wants to plant three times as many acres of corn as wheat. Write a system of linear equations that represents this situation. How many acres of each crop should the farmer plant? *(See Example 3.)*

20. **MODELING WITH MATHEMATICS** A company that offers tubing trips down a river rents tubes for a person to use and "cooler" tubes to carry food and water. A group spends \$270 to rent a total of 15 tubes. Write a system of linear equations that represents this situation. How many of each type of tube does the group rent?

In Exercises 21–24, write a system of linear equations that has the ordered pair as its solution.

21. $(3, 5)$

22. $(-2, 8)$

23. $(-4, -12)$

24. $(15, -25)$

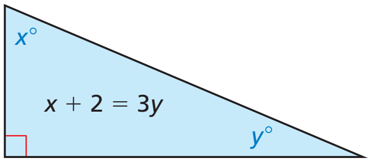
25. **PROBLEM SOLVING** A math test is worth 100 points and has 38 problems. Each problem is worth either 5 points or 2 points. How many problems of each point value are on the test?

26. **PROBLEM SOLVING** An investor owns shares of Stock A and Stock B. The investor owns a total of 200 shares with a total value of \$4000. How many shares of each stock does the investor own?

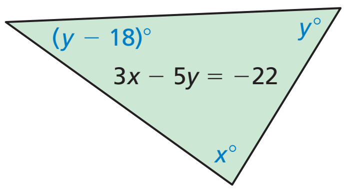
Stock	Price
A	\$9.50
B	\$27.00

MATHEMATICAL CONNECTIONS **In Exercises 27 and 28, (a) write an equation that represents the sum of the angle measures of the triangle and (b) use your equation and the equation shown to find the values of *x* and *y*.**

27.



28.



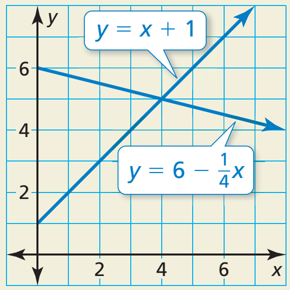
29. **REASONING** Find the values of a and b so that the solution of the linear system is $(-9, 1)$.

$ax + by = -31$ Equation 1
$ax - by = -41$ Equation 2

30. **MAKING AN ARGUMENT** Your friend says that given a linear system with an equation of a horizontal line and an equation of a vertical line, you cannot solve the system by substitution. Is your friend correct? Explain.

31. **OPEN-ENDED** Write a system of linear equations in which $(3, -5)$ is a solution of Equation 1 but not a solution of Equation 2, and $(-1, 7)$ is a solution of the system.

32. **HOW DO YOU SEE IT?** The graphs of two linear equations are shown.



a. At what point do the lines appear to intersect?

b. Could you solve a system of linear equations by substitution to check your answer in part (a)? Explain.

33. **REPEATED REASONING** A radio station plays a total of 272 pop, rock, and hip-hop songs during a day. The number of pop songs is 3 times the number of rock songs. The number of hip-hop songs is 32 more than the number of rock songs. How many of each type of song does the radio station play?

34. **THOUGHT PROVOKING** You have \$2.65 in coins. Write a system of equations that represents this situation. Use variables to represent the number of each type of coin.

35. **NUMBER SENSE** The sum of the digits of a two-digit number is 11. When the digits are reversed, the number increases by 27. Find the original number.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the sum or difference. (*Skills Review Handbook*)

36. $(x - 4) + (2x - 7)$

37. $(5y - 12) + (-5y - 1)$

38. $(t - 8) - (t + 15)$

39. $(6d + 2) - (3d - 3)$

40. $4(m + 2) + 3(6m - 4)$

41. $2(5v + 6) - 6(-9v + 2)$

5.3 Solving Systems of Linear Equations by Elimination

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.2.I
A.5.C

Essential Question

How can you use elimination to solve a system of linear equations?

EXPLORATION 1 Writing and Solving a System of Equations

Work with a partner. You purchase a drink and a sandwich for \$4.50. Your friend purchases a drink and five sandwiches for \$16.50. You want to determine the price of a drink and the price of a sandwich.

a. Let x represent the price (in dollars) of one drink. Let y represent the price (in dollars) of one sandwich. Write a system of equations for the situation. Use the following verbal model.

Number of drinks • Price per drink + Number of sandwiches • Price per sandwich = Total price

Label one of the equations Equation 1 and the other equation Equation 2.

b. Subtract Equation 1 from Equation 2. Explain how you can use the result to solve the system of equations. Then find and interpret the solution.

USING PROBLEM-SOLVING STRATEGIES

To be proficient in math, you need to monitor and evaluate your progress and change course using a different solution method, if necessary.

EXPLORATION 2 Using Elimination to Solve Systems

Work with a partner. Solve each system of linear equations using two methods.

Method 1 **Subtract.** Subtract Equation 2 from Equation 1. Then use the result to solve the system.

Method 2 **Add.** Add the two equations. Then use the result to solve the system.

Is the solution the same using both methods? Which method do you prefer?

a. $3x - y = 6$
$3x + y = 0$

b. $2x + y = 6$
$2x - y = 2$

c. $x - 2y = -7$
$x + 2y = 5$

EXPLORATION 3 Using Elimination to Solve a System

Work with a partner.

$2x + y = 7$ Equation 1
$x + 5y = 17$ Equation 2

a. Can you eliminate a variable by adding or subtracting the equations as they are? If not, what do you need to do to one or both equations so that you can?

b. Solve the system individually. Then exchange solutions with your partner and compare and check the solutions.

Communicate Your Answer

4. How can you use elimination to solve a system of linear equations?
5. When can you add or subtract the equations in a system to solve the system? When do you have to multiply first? Justify your answers with examples.
6. In Exploration 3, why can you multiply an equation in the system by a constant and not change the solution of the system? Explain your reasoning.

5.3 Lesson

What You Will Learn

- Solve systems of linear equations by elimination.
- Use systems of linear equations to solve real-life problems.

Core Vocabulary

Previous
coefficient

Solving Linear Systems by Elimination

Core Concept

Solving a System of Linear Equations by Elimination

Step 1 Multiply, if necessary, one or both equations by a constant so at least one pair of like terms has the same or opposite coefficients.

Step 2 Add or subtract the equations to eliminate one of the variables.

Step 3 Solve the resulting equation.

Step 4 Substitute the value from Step 3 into one of the original equations and solve for the other variable.

You can use elimination to solve a system of equations because replacing one equation in the system with the sum of that equation and a multiple of the other produces a system that has the same solution. Here is why.

System 1

$a = b$ Equation 1
$c = d$ Equation 2

System 2

$a + kc = b + kd$ Equation 3
$c = d$ Equation 2

Consider System 1. In this system, a and c are algebraic expressions, and b and d are constants. Begin by multiplying each side of Equation 2 by a constant k. By the Multiplication Property of Equality, $kc = kd$. You can rewrite Equation 1 as Equation 3 by adding kc on the left and kd on the right. You can rewrite Equation 3 as Equation 1 by subtracting kc on the left and kd on the right. Because you can rewrite either system as the other, System 1 and System 2 have the same solution.

EXAMPLE 1 Solving a System of Linear Equations by Elimination

Solve the system of linear equations by elimination.

$3x + 2y = 4$ Equation 1
$3x - 2y = -4$ Equation 2

SOLUTION

Step 1 Because the coefficients of the y-terms are opposites, you do not need to multiply either equation by a constant.

Step 2 Add the equations.

$3x + 2y = 4$ Equation 1
$\underline{3x - 2y = -4}$ Equation 2
$6x \quad = 0$ Add the equations.

Step 3 Solve for x.

$6x = 0$ Resulting equation from Step 2
$x = 0$ Divide each side by 6.

Step 4 Substitute 0 for x in one of the original equations and solve for y.

$3x + 2y = 4$ Equation 1
$3(0) + 2y = 4$ Substitute 0 for x.
$y = 2$ Solve for y.

▶ The solution is (0, 2).

Check

Equation 1

$3x + 2y = 4$
$3(0) + 2(2) \stackrel{?}{=} 4$
$4 = 4$ ✓

Equation 2

$3x - 2y = -4$
$3(0) - 2(2) \stackrel{?}{=} -4$
$-4 = -4$ ✓

EXAMPLE 2 Solving a System of Linear Equations by Elimination

Solve the system of linear equations by elimination.

$-10x + 3y = 1$ Equation 1

$-5x - 6y = 23$ Equation 2

SOLUTION

Step 1 Multiply Equation 2 by -2 so that the coefficients of the x-terms are opposites.

$-10x + 3y = 1$ → $-10x + 3y = 1$ Equation 1

$-5x - 6y = 23$ **Multiply by −2.** → $10x + 12y = -46$ Revised Equation 2

ANOTHER WAY

To use subtraction to eliminate one of the variables, multiply Equation 2 by 2 and then subtract the equations.

$$\begin{array}{r} -10x + 3y = 1 \\ \underline{-(-10x - 12y = 46)} \\ 15y = -45 \end{array}$$

Step 2 Add the equations.

$$\begin{array}{rl} -10x + 3y = 1 & \text{Equation 1} \\ \underline{10x + 12y = -46} & \text{Revised Equation 2} \\ 15y = -45 & \text{Add the equations.} \end{array}$$

Step 3 Solve for y.

$15y = -45$ Resulting equation from Step 2

$y = -3$ Divide each side by 15.

Step 4 Substitute -3 for y in one of the original equations and solve for x.

$-5x - 6y = 23$ Equation 2

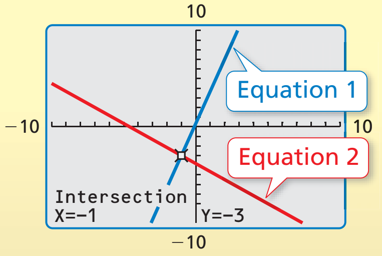
$-5x - 6(-3) = 23$ Substitute -3 for y.

$-5x + 18 = 23$ Multiply.

$-5x = 5$ Subtract 18 from each side.

$x = -1$ Divide each side by -5.

Check



▶ The solution is $(-1, -3)$.

Monitoring Progress

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Solve the system of linear equations by elimination. Check your solution.

1. $3x + 2y = 7$
$-3x + 4y = 5$

2. $x - 3y = 24$
$3x + y = 12$

3. $x + 4y = 22$
$4x + y = 13$

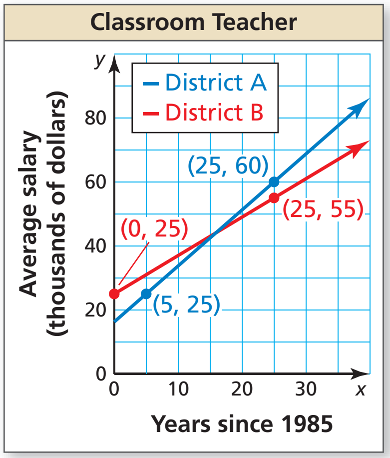
Concept Summary

Methods for Solving Systems of Linear Equations

Method	When to Use
Graphing *(Lesson 5.1)*	To estimate solutions
Substitution *(Lesson 5.2)*	When one of the variables in one of the equations has a coefficient of 1 or -1
Elimination *(Lesson 5.3)*	When at least one pair of like terms has the same or opposite coefficients
Elimination (Multiply First) *(Lesson 5.3)*	When one of the variables cannot be eliminated by adding or subtracting the equations

Solving Real-Life Problems

EXAMPLE 3 **Modeling with Mathematics**



The graph represents the average salaries of classroom teachers in two school districts. During what year were the average salaries in the two districts equal? What was the average salary in both districts in that year?

SOLUTION

1. **Understand the Problem** You know two points on each line in the graph. You are asked to determine the year in which the average salaries were equal and then determine the average salary in that year.
2. **Make a Plan** Use the points in the graph to write a system of linear equations. Then solve the system of linear equations.
3. **Solve the Problem** Find the slope of each line.

District A: $m = \dfrac{60 - 25}{25 - 5} = \dfrac{35}{20} = \dfrac{7}{4}$ District B: $m = \dfrac{55 - 25}{25 - 0} = \dfrac{30}{25} = \dfrac{6}{5}$

Use each slope and a point on each line to write equations of the lines.

District A		District B
$y - y_1 = m(x - x_1)$	Write the point-slope form.	$y - y_1 = m(x - x_1)$
$y - 25 = \frac{7}{4}(x - 5)$	Substitute for m, x_1, and y_1.	$y - 25 = \frac{6}{5}(x - 0)$
$-7x + 4y = 65$	Write in standard form.	$-6x + 5y = 125$

System $-7x + 4y = 65$ Equation 1

$-6x + 5y = 125$ Equation 2

STUDY TIP

In Example 3, both equations are multiplied by a constant so that the coefficients of the y-terms are opposites.

Step 1 Multiply Equation 1 by -5. Multiply Equation 2 by 4.

$-7x + 4y = 65$ **Multiply by −5.** $35x - 20y = -325$ Revised Equation 1

$-6x + 5y = 125$ **Multiply by 4.** $-24x + 20y = 500$ Revised Equation 2

Step 2 Add the equations.

$35x - 20y = -325$ Revised Equation 1

$-24x + 20y = \quad 500$ Revised Equation 2

$11x \quad = \quad 175$ Add the equations.

Step 3 Solving the equation $11x = 175$ gives $x = \frac{175}{11} \approx 15.9$.

Step 4 Substitute $\frac{175}{11}$ for x in one of the original equations and solve for y.

$-7x + 4y = 65$ Equation 1

$-7\left(\frac{175}{11}\right) + 4y = 65$ Substitute $\frac{175}{11}$ for x.

$y \approx 44$ Solve for y.

▶ The solution is about (15.9, 44). Because $x \approx 15.9$ corresponds to the year 2000, the average salary in both districts was about \$44,000 in 2000.

4. **Look Back** Using the graph, the point of intersection appears to be about (15, 45). So, the solution of (15.9, 44) is reasonable.

Monitoring Progress

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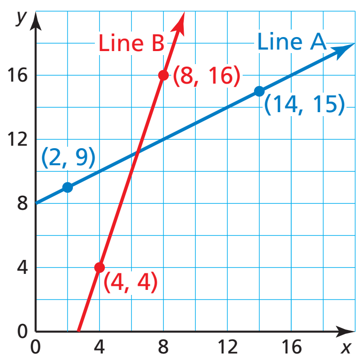

4. Write and solve a system of linear equations represented by the graph at the left.

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Vocabulary and Core Concept Check

1. **OPEN-ENDED** Give an example of a system of linear equations that can be solved by first adding the equations to eliminate one variable.

2. **WRITING** Explain how to solve the system of linear equations by elimination.

$2x - 3y = -4$ Equation 1
$-5x + 9y = 7$ Equation 2

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, solve the system of linear equations by elimination. Check your solution. *(See Example 1.)*

3. $x + 2y = 13$
   $-x + y = 5$

4. $9x + y = 2$
   $-4x - y = -17$

5. $5x + 6y = 50$
   $x - 6y = -26$

6. $-x + y = 4$
   $x + 3y = 4$

7. $-3x - 5y = -7$
   $-4x + 5y = 14$

8. $4x - 9y = -21$
   $-4x - 3y = 9$

9. $-y - 10 = 6x$
   $5x + y = -10$

10. $3x - 30 = y$
    $7y - 6 = 3x$

In Exercises 11–18, solve the system of linear equations by elimination. Check your solution. *(See Examples 2 and 3.)*

11. $x + y = 2$
    $2x + 7y = 9$

12. $8x - 5y = 11$
    $4x - 3y = 5$

13. $11x - 20y = 28$
    $3x + 4y = 36$

14. $10x - 9y = 46$
    $-2x + 3y = 10$

15. $4x - 3y = 8$
    $5x - 2y = -11$

16. $-2x - 5y = 9$
    $3x + 11y = 4$

17. $9x + 2y = 39$
    $6x + 13y = -9$

18. $12x - 7y = -2$
    $8x + 11y = 30$

19. **ERROR ANALYSIS** Describe and correct the error in solving for one of the variables in the linear system $5x - 7y = 16$ and $x + 7y = 8$.

$$\begin{aligned} 5x - 7y &= 16 \\ x + 7y &= 8 \\ \hline 4x \quad &= 24 \\ x &= 6 \end{aligned}$$

20. **ERROR ANALYSIS** Describe and correct the error in solving for one of the variables in the linear system $4x + 3y = 8$ and $x - 2y = -13$.

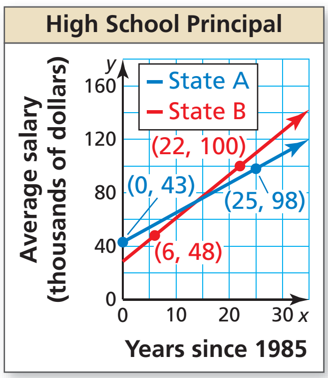
$4x + 3y = 8$
$x - 2y = -13$ Multiply by −4.

$$\begin{aligned} 4x + 3y &= 8 \\ -4x + 8y &= -13 \\ \hline 11y &= -5 \\ y &= \frac{-5}{11} \end{aligned}$$

21. **MODELING WITH MATHEMATICS** A service center charges a fee of x dollars for an oil change plus y dollars per quart of oil used. A sample of its sales record is shown. Write a system of linear equations that represents this situation. Find the fee and cost per quart of oil.

	A	B	C
1	Customer	Oil Tank Size (quarts)	Total Cost
2	A	5	$22.45
3	B	7	$25.45

22. **MODELING WITH MATHEMATICS** The graph represents the average salaries of high school principals in two states. During what year were the average salaries in the two states equal? What was the average salary in both states in that year?



In Exercises 23–26, solve the system of linear equations using any method. Explain why you chose the method.

23. $3x + 2y = 4$
$2y = 8 - 5x$

24. $-6y + 2 = -4x$
$y - 2 = x$

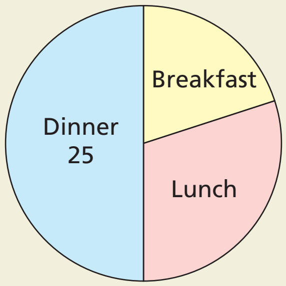
25. $y - x = 2$
$y = -\frac{1}{4}x + 7$

26. $3x + y = \frac{1}{3}$
$2x - 3y = \frac{8}{3}$

27. WRITING For what values of a can you solve the linear system $ax + 3y = 2$ and $4x + 5y = 6$ by elimination without multiplying first? Explain.

28. HOW DO YOU SEE IT? The circle graph shows the results of a survey in which 50 students were asked about their favorite meal.

Favorite Meal



a. Estimate the numbers of students who chose breakfast and lunch.

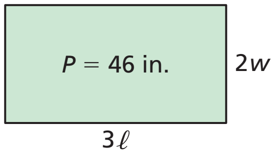
b. The number of students who chose lunch was 5 more than the number of students who chose breakfast. Write a system of linear equations that represents the numbers of students who chose breakfast and lunch.

c. Explain how you can solve the linear system in part (b) to check your answers in part (a).

29. MAKING AN ARGUMENT Your friend says that any system of equations that can be solved by elimination can be solved by substitution in an equal or fewer number of steps. Is your friend correct? Explain.

30. THOUGHT PROVOKING Write a system of linear equations that can be added to eliminate a variable *or* subtracted to eliminate a variable.

31. MATHEMATICAL CONNECTIONS A rectangle has a perimeter of 18 inches. A new rectangle is formed by doubling the width w and tripling the length ℓ, as shown. The new rectangle has a perimeter P of 46 inches.



a. Write and solve a system of linear equations to find the length and width of the original rectangle.

b. Find the length and width of the new rectangle.

32. CRITICAL THINKING Refer to the discussion of System 1 and System 2 on page 232. Without solving, explain why the two systems shown have the same solution.

System 1		System 2	
$3x - 2y = 8$	Equation 1	$5x = 20$	Equation 3
$x + y = 6$	Equation 2	$x + y = 6$	Equation 2

33. PROBLEM SOLVING You are making 6 quarts of fruit punch for a party. You have bottles of 100% fruit juice and 20% fruit juice. How many quarts of each type of juice should you mix to make 6 quarts of 80% fruit juice?

34. PROBLEM SOLVING A motorboat takes 40 minutes to travel 20 miles downstream. The return trip takes 60 minutes. What is the speed of the current?

35. CRITICAL THINKING Solve for x, y, and z in the system of equations. Explain your steps.

$x + 7y + 3z = 29$ Equation 1
$3z + x - 2y = -7$ Equation 2
$5y = 10 - 2x$ Equation 3

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Determine whether the equation has *one solution*, *no solution*, or *infinitely many solutions*. *(Section 1.3)*

36. $5d - 8 = 1 + 5d$

37. $9 + 4t = 12 - 4t$

38. $3n + 2 = 2(n - 3)$

39. $-3(4 - 2v) = 6v - 12$

Write an equation of the line that passes through the given point and is parallel to the given line. *(Section 4.4)*

40. $(4, -1)$; $y = -2x + 7$

41. $(0, 6)$; $y = 5x - 3$

42. $(-5, -2)$; $y = \frac{2}{3}x + 1$

5.4 Solving Special Systems of Linear Equations

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.2.I
A.3.F
A.5.C

Essential Question

Can a system of linear equations have no solution or infinitely many solutions?

EXPLORATION 1 Using a Table to Solve a System

Work with a partner. You invest \$450 for equipment to make skateboards. The materials for each skateboard cost \$20. You sell each skateboard for \$20.

a. Write the cost and revenue equations. Then copy and complete the table for your cost C and your revenue R.

x (skateboards)	0	1	2	3	4	5	6	7	8	9	10
C (dollars)											
R (dollars)											

b. When will your company break even? What is wrong?

APPLYING MATHEMATICS
To be proficient in math, you need to interpret mathematical results in real-life contexts.

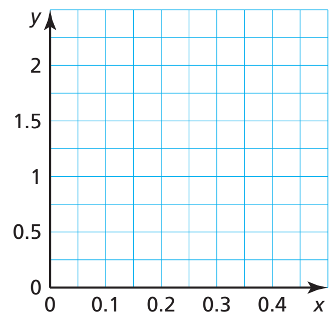
EXPLORATION 2 Writing and Analyzing a System

Work with a partner. A necklace and matching bracelet have two types of beads. The necklace has 40 small beads and 6 large beads and weighs 10 grams. The bracelet has 20 small beads and 3 large beads and weighs 5 grams. The threads holding the beads have no significant weight.

a. Write a system of linear equations that represents the situation. Let x be the weight (in grams) of a small bead and let y be the weight (in grams) of a large bead.

b. Graph the system in the coordinate plane shown. What do you notice about the two lines?

c. Can you find the weight of each type of bead? Explain your reasoning.

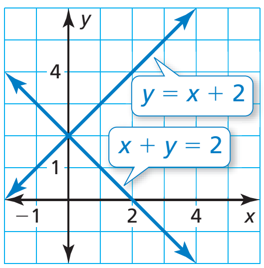


Communicate Your Answer

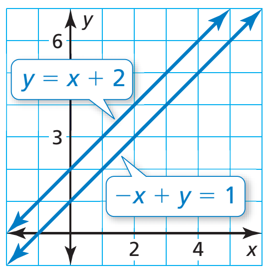
3. Can a system of linear equations have no solution or infinitely many solutions? Give examples to support your answers.

4. Does the system of linear equations represented by each graph have *no solution*, *one solution*, or *infinitely many solutions*? Explain.

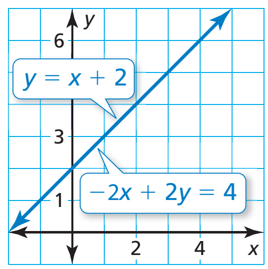
a.



b.



c.



5.4 Lesson

What You Will Learn

- Determine the numbers of solutions of linear systems.
- Use linear systems to solve real-life problems.

Core Vocabulary

Previous
parallel

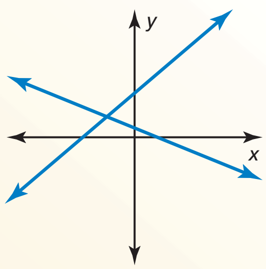
The Numbers of Solutions of Linear Systems

Core Concept

Solutions of Systems of Linear Equations

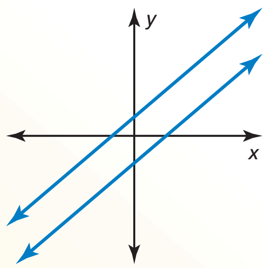
A system of linear equations can have *one solution*, *no solution*, or *infinitely many solutions*.

One solution



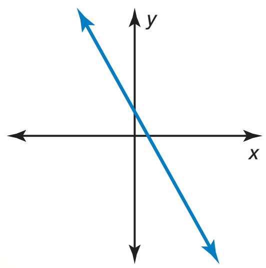
The lines intersect.

No solution



The lines are parallel.

Infinitely many solutions



The lines are the same.

ANOTHER WAY

You can solve some linear systems by inspection. In Example 1, notice you can rewrite the system as

$-2x + y = 1$
$-2x + y = -5.$

This system has no solution because $-2x + y$ cannot be equal to both 1 and -5.

EXAMPLE 1 Solving a System: No Solution

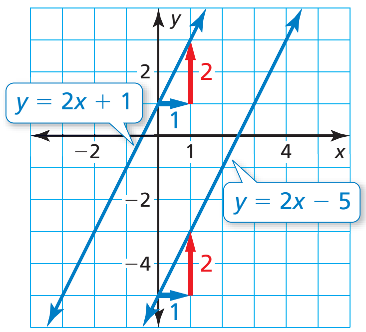
Solve the system of linear equations.

$y = 2x + 1$ Equation 1
$y = 2x - 5$ Equation 2

SOLUTION

Method 1 Solve by graphing.

Graph each equation.



The lines have the same slope and different y-intercepts. So, the lines are parallel.

Because parallel lines do not intersect, there is no point that is a solution of both equations.

- So, the system of linear equations has no solution.

Method 2 Solve by substitution.

Substitute $2x - 5$ for y in Equation 1.

$y = 2x + 1$ Equation 1
$2x - 5 = 2x + 1$ Substitute $2x - 5$ for y.
$-5 = 1$ ✗   Subtract $2x$ from each side.

- The equation $-5 = 1$ is never true. So, the system of linear equations has no solution.

STUDY TIP

A linear system with no solution is called an *inconsistent system*.

ANOTHER WAY

You can also solve the linear system by graphing. The lines have the same slope and the same y-intercept. So, the lines are the same, which means all points on the line are solutions of both equations.

EXAMPLE 2 Solving a System: Infinitely Many Solutions

Solve the system of linear equations.

$-2x + y = 3$ Equation 1

$-4x + 2y = 6$ Equation 2

SOLUTION

Solve by elimination.

Step 1 Multiply Equation 1 by -2.

$-2x + y = 3$ → Multiply by -2. → $4x - 2y = -6$ Revised Equation 1

$-4x + 2y = 6$ → $-4x + 2y = 6$ Equation 2

Step 2 Add the equations.

$4x - 2y = -6$ Revised Equation 1

$-4x + 2y = 6$ Equation 2

$0 = 0$ Add the equations.

STUDY TIP

A linear system with infinitely many solutions is called a *consistent dependent system*.

▶ The equation $0 = 0$ is always true. So, the solutions are all the points on the line $-2x + y = 3$. The system of linear equations has infinitely many solutions.

Monitoring Progress

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Solve the system of linear equations.

1. $x + y = 3$
$2x + 2y = 6$

2. $y = -x + 3$
$2x + 2y = 4$

Solving Real-Life Problems

EXAMPLE 3 Modeling with Mathematics

Cost (dollars)		
Hours	**Hall A**	**Hall B**
0	75	100
1	175	200
2	275	300
3	375	400

An athletic director is comparing the costs of renting two banquet halls for an awards banquet. Write a system of linear equations that represents this situation. If the cost patterns continue, will the cost of Hall A ever equal the cost of Hall B?

SOLUTION

Words Total cost = Cost per hour • Number of hours + Initial cost

Variables Let y be the cost (in dollars) and let x be the number of hours.

System $y = 100x + 75$ Equation 1 - Cost of Hall A

$y = 100x + 100$ Equation 2 - Cost of Hall B

The equations are in slope-intercept form. The graphs of the equations have the same slope but different y-intercepts. There is no solution because the lines are parallel.

▶ So, the cost of Hall A will never equal the cost of Hall B.

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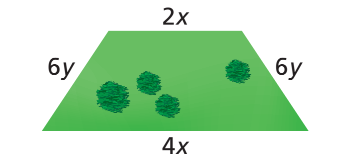
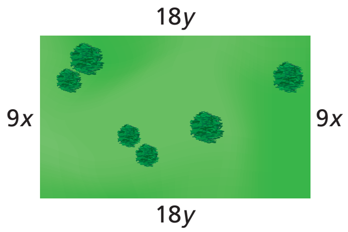
3. WHAT IF? What happens to the solution in Example 3 when the cost per hour for Hall A is \$125?

EXAMPLE 4 Modeling with Mathematics





The perimeter of the trapezoidal piece of land is 48 kilometers. The perimeter of the rectangular piece of land is 144 kilometers. Write and solve a system of linear equations to find the values of x and y.

SOLUTION

1. **Understand the Problem** You know the perimeter of each piece of land and the side lengths in terms of x or y. You are asked to write and solve a system of linear equations to find the values of x and y.
2. **Make a Plan** Use the figures and the definition of perimeter to write a system of linear equations that represents the problem. Then solve the system of linear equations.
3. **Solve the Problem**

Perimeter of trapezoid

$2x + 4x + 6y + 6y = 48$

$6x + 12y = 48$ Equation 1

Perimeter of rectangle

$9x + 9x + 18y + 18y = 144$

$18x + 36y = 144$ Equation 2

System $6x + 12y = 48$ Equation 1

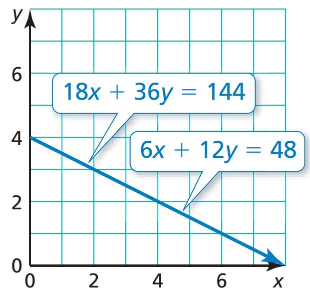
$18x + 36y = 144$ Equation 2

Method 1 Solve by graphing.

Graph each equation.

The lines have the same slope and the same y-intercept. So, the lines are the same.

In this context, x and y must be positive. Because the lines are the same, all the points on the line in Quadrant I are solutions of both equations.



▶ So, the system of linear equations has infinitely many solutions.

Method 2 Solve by elimination.

Multiply Equation 1 by -3 and add the equations.

$6x + 12y = 48$ **Multiply by −3.** → $-18x - 36y = -144$ Revised Equation 1

$18x + 36y = 144$ → $18x + 36y = 144$ Equation 2

$0 = 0$ Add the equations.

▶ The equation $0 = 0$ is always true. In this context, x and y must be positive. So, the solutions are all the points on the line $6x + 12y = 48$ in Quadrant I. The system of linear equations has infinitely many solutions.

4. **Look Back** Choose a few of the ordered pairs (x, y) that are solutions of Equation 1. You should find that no matter which ordered pairs you choose, they will also be solutions of Equation 2. So, *infinitely many solutions* seems reasonable.

Monitoring Progress

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4. **WHAT IF?** What happens to the solution in Example 4 when the perimeter of the trapezoidal piece of land is 96 kilometers? Explain.

5.4 Exercises

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Vocabulary and Core Concept Check

1. **REASONING** Is it possible for a system of linear equations to have exactly two solutions? Explain.

2. **WRITING** Compare the graph of a system of linear equations that has infinitely many solutions and the graph of a system of linear equations that has no solution.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, match the system of linear equations with its graph. Then determine whether the system has *one solution*, *no solution*, or *infinitely many solutions*.

3. $-x + y = 1$
   $x - y = 1$

4. $2x - 2y = 4$
   $-x + y = -2$

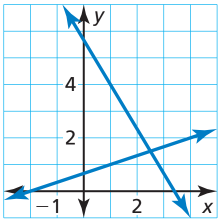
5. $2x + y = 4$
   $-4x - 2y = -8$

6. $x - y = 0$
   $5x - 2y = 6$

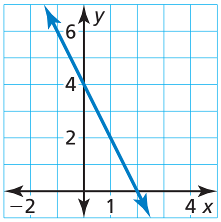
7. $-2x + 4y = 1$
   $3x - 6y = 9$

8. $5x + 3y = 17$
   $x - 3y = -2$

A.



B.



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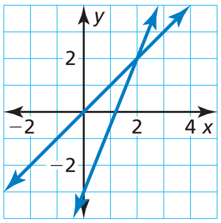

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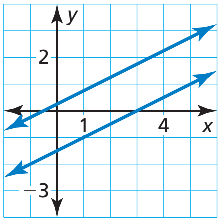

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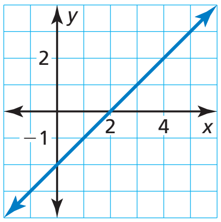
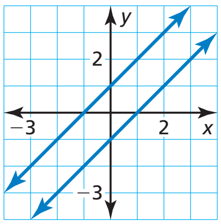


F.



In Exercises 9–16, solve the system of linear equations. *(See Examples 1 and 2.)*

9. $y = -2x - 4$
   $y = 2x - 4$

10. $y = -6x - 8$
    $y = -6x + 8$

11. $3x - y = 6$
    $-3x + y = -6$

12. $-x + 2y = 7$
    $x - 2y = 7$

13. $4x + 4y = -8$
    $-2x - 2y = 4$

14. $15x - 5y = -20$
    $-3x + y = 4$

15. $9x - 15y = 24$
    $6x - 10y = -16$

16. $3x - 2y = -5$
    $4x + 5y = 47$

In Exercises 17–22, use only the slopes and *y*-intercepts of the graphs of the equations to determine whether the system of linear equations has *one solution*, *no solution*, or *infinitely many solutions*. Explain.

17. $y = 7x + 13$
    $-21x + 3y = 39$

18. $y = -6x - 2$
    $12x + 2y = -6$

19. $4x + 3y = 27$
    $4x - 3y = -27$

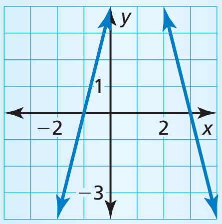
20. $-7x + 7y = 1$
    $2x - 2y = -18$

21. $-18x + 6y = 24$
    $3x - y = -2$

22. $2x - 2y = 16$
    $3x - 6y = 30$

ERROR ANALYSIS **In Exercises 23 and 24, describe and correct the error in solving the system of linear equations.**

23.



$-4x + y = 4$
$4x + y = 12$

The lines do not intersect. So, the system has no solution.

24.

$y = 3x - 8$
$y = 3x - 12$

The lines have the same slope. So, the system has infinitely many solutions.

25. MODELING WITH MATHEMATICS The table shows the distances two groups have traveled at different times during a canoeing excursion. The groups continue traveling at their current rates until they reach the same destination. Let d be the distance traveled and t be the time since 1 P.M. Write a system of linear equations that represents this situation. Will Group B catch up to Group A before reaching the destination? Explain. *(See Example 3.)*

	Distance Traveled (miles)			
	1 P.M.	**2 P.M.**	**3 P.M.**	**4 P.M.**
Group A	3	9	15	21
Group B	1	7	13	19

26. MODELING WITH MATHEMATICS A \$6-bag of trail mix contains 3 cups of dried fruit and 4 cups of almonds. A \$9-bag contains $4\frac{1}{2}$ cups of dried fruit and 6 cups of almonds. Write and solve a system of linear equations to find the price of 1 cup of dried fruit and 1 cup of almonds. *(See Example 4.)*

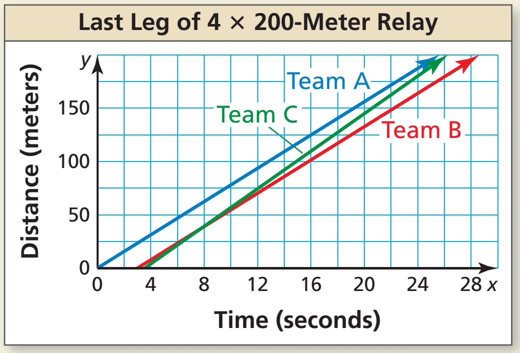
27. PROBLEM SOLVING A train travels from New York City to Washington, D.C., and then back to New York City. The table shows the number of tickets purchased for each leg of the trip. The cost per ticket is the same for each leg of the trip. Is there enough information to determine the cost of one coach ticket? Explain.

Destination	Coach tickets	Business class tickets	Money collected (dollars)
Washington, D.C.	150	80	22,860
New York City	170	100	27,280

28. THOUGHT PROVOKING Write a system of three linear equations in two variables so that any two of the equations have exactly one solution, but the entire system of equations has no solution.

29. REASONING In a system of linear equations, one equation has a slope of 2 and the other equation has a slope of $-\frac{1}{3}$. How many solutions does the system have? Explain.

30. HOW DO YOU SEE IT? The graph shows information about the last leg of a 4×200-meter relay for three relay teams. Team A's runner ran about 7.8 meters per second, Team B's runner ran about 7.8 meters per second, and Team C's runner ran about 8.8 meters per second.



a. Estimate the distance at which Team C's runner passed Team B's runner.

b. If the race was longer, could Team C's runner have passed Team A's runner? Explain.

c. If the race was longer, could Team B's runner have passed Team A's runner? Explain.

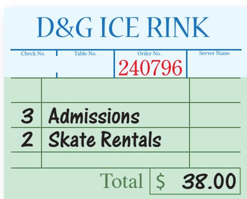
31. ABSTRACT REASONING Consider the system of linear equations $y = ax + 4$ and $y = bx - 2$, where a and b are real numbers. Determine whether each statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

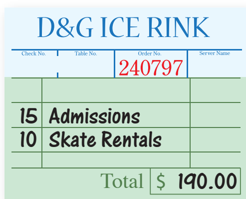
a. The system has infinitely many solutions.

b. The system has no solution.

c. When $a < b$, the system has one solution.

32. MAKING AN ARGUMENT One admission to an ice skating rink costs x dollars, and renting a pair of ice skates costs y dollars. Your friend says she can determine the exact cost of one admission and one skate rental. Is your friend correct? Explain.





Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the system of linear equations by graphing. *(Section 5.1)*

33. $y = x - 6$
$y = -x + 10$

34. $y = 2x + 3$
$y = -3x - 7$

35. $2x + y = 6$
$3x - 2y = 16$

5.1–5.4 What Did You Learn?

Core Vocabulary

system of linear equations, *p. 220*

solution of a system of linear equations, *p. 220*

Core Concepts

Section 5.1
Solving a System of Linear Equations by Graphing, *p. 221*

Section 5.2
Solving a System of Linear Equations by Substitution, *p. 226*

Section 5.3
Solving a System of Linear Equations by Elimination, *p. 232*

Section 5.4
Solutions of Systems of Linear Equations, *p. 238*

Mathematical Thinking

1. Describe the given information in Exercise 33 on page 230 and your plan for finding the solution.
2. Describe another real-life situation similar to Exercise 21 on page 235 and the mathematics that you can apply to solve the problem.
3. What question(s) can you ask your friend to help her understand the error in the statement she made in Exercise 32 on page 242?

Study Skills

Analyzing Your Errors

Study Errors

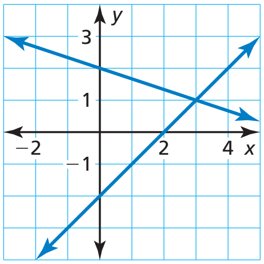
What Happens: You do not study the right material or you do not learn it well enough to remember it on a test without resources such as notes.

How to Avoid This Error: Take a practice test. Work with a study group. Discuss the topics on the test with your teacher. Do not try to learn a whole chapter's worth of material in one night.

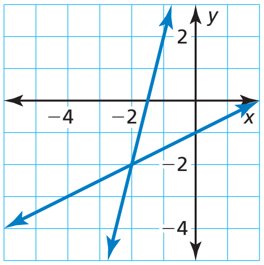
5.1–5.4 Quiz

Use the graph to solve the system of linear equations. Check your solution. *(Section 5.1)*

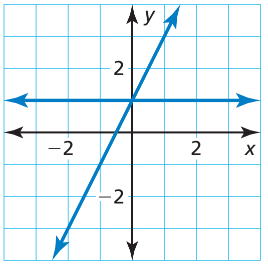
1. $y = -\frac{1}{3}x + 2$
$y = x - 2$



2. $y = \frac{1}{2}x - 1$
$y = 4x + 6$



3. $y = 1$
$y = 2x + 1$



Solve the system of linear equations by substitution. Check your solution. *(Section 5.2)*

4. $y = x - 4$
$-2x + y = 18$

5. $2y + x = -4$
$y - x = -5$

6. $3x - 5y = 13$
$x + 4y = 10$

Solve the system of linear equations by elimination. Check your solution. *(Section 5.3)*

7. $x + y = 4$
$-3x - y = -8$

8. $x + 3y = 1$
$5x + 6y = 14$

9. $2x - 3y = -5$
$5x + 2y = 16$

Solve the system of linear equations. *(Section 5.4)*

10. $x - y = 1$
$x - y = 6$

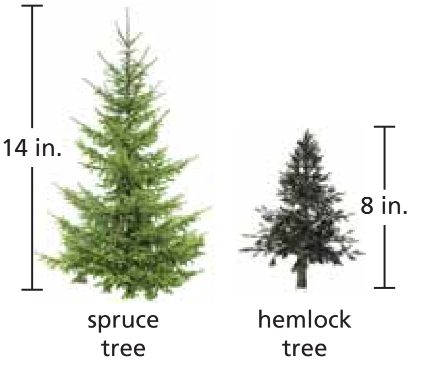
11. $6x + 2y = 16$
$2x - y = 2$

12. $3x - 3y = -2$
$-6x + 6y = 4$

13. You plant a spruce tree that grows 4 inches per year and a hemlock tree that grows 6 inches per year. The initial heights are shown. *(Section 5.1)*

a. Write a system of linear equations that represents this situation.

b. Solve the system by graphing. Interpret your solution.

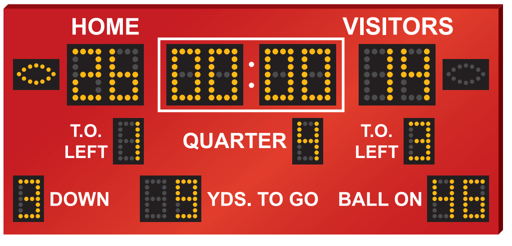


14. It takes you 3 hours to drive to a concert 135 miles away. You drive 55 miles per hour on highways and 40 miles per hour on the rest of the roads. *(Section 5.1, Section 5.2, and Section 5.3)*

a. How much time do you spend driving at each speed?

b. How many miles do you drive on highways? the rest of the roads?

15. In a football game, all of the home team's points are from 7-point touchdowns and 3-point field goals. The team scores six times. Write and solve a system of linear equations to find the numbers of touchdowns and field goals that the home team scores. *(Section 5.1, Section 5.2, and Section 5.3)*



5.5 Solving Equations by Graphing

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.5.A

Essential Question

How can you use a system of linear equations to solve an equation with variables on both sides?

Previously, you learned how to use algebra to solve equations with variables on both sides. Another way is to use a system of linear equations.

EXPLORATION 1 Solving an Equation by Graphing

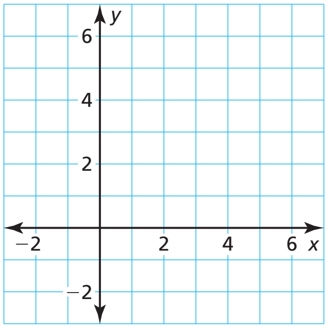
Work with a partner. Solve $2x - 1 = -\frac{1}{2}x + 4$ by graphing.

a. Use the left side to write a linear equation. Then use the right side to write another linear equation.

b. Graph the two linear equations from part (a). Find the x-value of the point of intersection. Check that the x-value is the solution of

$$2x - 1 = -\frac{1}{2}x + 4.$$

c. Explain why this "graphical method" works.



SELECTING TOOLS

To be proficient in math, you need to consider the available tools, which may include pencil and paper or a graphing calculator, when solving a mathematical problem.

EXPLORATION 2 Solving Equations Algebraically and Graphically

Work with a partner. Solve each equation using two methods.

Method 1 Use an algebraic method.

Method 2 Use a graphical method.

Is the solution the same using both methods?

a. $\frac{1}{2}x + 4 = -\frac{1}{4}x + 1$

b. $\frac{2}{3}x + 4 = \frac{1}{3}x + 3$

c. $-\frac{2}{3}x - 1 = \frac{1}{3}x - 4$

d. $\frac{4}{5}x + \frac{7}{5} = 3x - 3$

e. $-x + 2.5 = 2x - 0.5$

f. $-3x + 1.5 = x + 1.5$

Communicate Your Answer

3. How can you use a system of linear equations to solve an equation with variables on both sides?

4. Compare the algebraic method and the graphical method for solving a linear equation with variables on both sides. Describe the advantages and disadvantages of each method.

5.5 Lesson

What You Will Learn

- Solve linear equations by graphing.
- Use linear equations to solve real-life problems.

Core Vocabulary

Previous
absolute value equation

Solving Linear Equations by Graphing

You can use a system of linear equations to solve an equation with variables on both sides.

Core Concept

Solving Linear Equations by Graphing

Step 1 To solve the equation $ax + b = cx + d$, write two linear equations.

$$ax + b = cx + d$$

$y = ax + b$ and $y = cx + d$

Step 2 Graph the system of linear equations. The x-value of the solution of the system of linear equations is the solution of the equation $ax + b = cx + d$.

EXAMPLE 1 **Solving an Equation by Graphing**

Solve $-x + 1 = 2x - 5$ by graphing. Check your solution.

SOLUTION

Step 1 Write a system of linear equations using each side of the original equation.

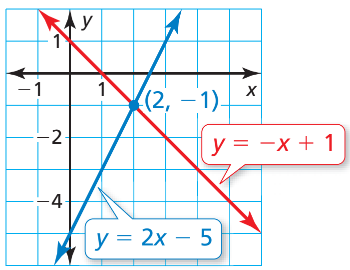
$$-x + 1 = 2x - 5$$

$y = -x + 1$ $\quad$ $y = 2x - 5$

Step 2 Graph the system.

$y = -x + 1$ Equation 1

$y = 2x - 5$ Equation 2



The graphs intersect at $(2, -1)$.

Check

$$-x + 1 = 2x - 5$$

$$-(2) + 1 \stackrel{?}{=} 2(2) - 5$$

$$-1 = -1 \checkmark$$

- So, the solution of the equation is $x = 2$.

Monitoring Progress

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Solve the equation by graphing. Check your solution.

1. $\frac{1}{2}x - 3 = 2x$ $\qquad$ **2.** $-4 + 9x = -3x + 2$

Solving Real-Life Problems

EXAMPLE 2 Modeling with Mathematics

You are an ecologist studying the populations of two types of fish in a lake. Use the information in the table to predict when the populations of the two types of fish will be equal.

Type	Current population	Change per year
A	5750	−250
B	2825	400

SOLUTION

1. **Understand the Problem** You know the current population of each type of fish and the annual change in each population. You are asked to determine when the populations of the two types of fish will be equal.

2. **Make a Plan** Use a verbal model to write an equation that represents the problem. Then solve the equation by graphing.

3. **Solve the Problem**

Words

$$\underbrace{\boxed{\text{Change per year}} \cdot \boxed{\text{Years}} + \boxed{\text{Current population}}}_{\text{Type A}} = \underbrace{\boxed{\text{Change per year}} \cdot \boxed{\text{Years}} + \boxed{\text{Current population}}}_{\text{Type B}}$$

Variable Let x be the number of years.

Equation $-250x + 5750 = 400x + 2825$

Solve the equation by graphing.

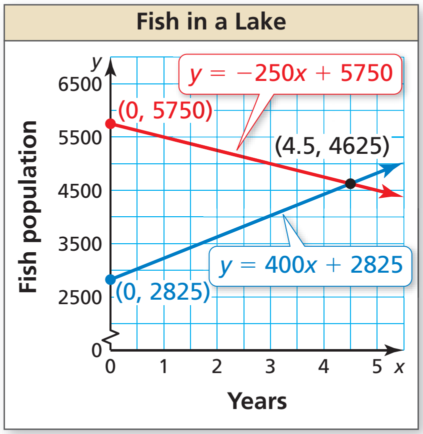
Step 1 Write a system of linear equations using each side of the original equation.

$$\underbrace{-250x + 5750}_{y = -250x + 5750} = \underbrace{400x + 2825}_{y = 400x + 2825}$$

Step 2 Graph the system.

$y = -250x + 5750$ Equation 1

$y = 400x + 2825$ Equation 2



The graphs appear to intersect at (4.5, 4625). Check this solution in each equation of the linear system, as shown.

▶ So, the populations of the two types of fish will be equal in 4.5 years.

Check

$y = -250x + 5750$

$4625 \stackrel{?}{=} -250(4.5) + 5750$

$4625 = 4625$ ✓

$y = 400x + 2825$

$4625 \stackrel{?}{=} 400(4.5) + 2825$

$4625 = 4625$ ✓

4. **Look Back** To check that your solution is correct, verify that $x = 4.5$ is the solution of the original equation.

$$-250(4.5) + 5750 = 400(4.5) + 2825$$

$$4625 = 4625 \checkmark$$

Monitoring Progress

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3. **WHAT IF?** Type C has a current population of 3500 and grows by 500 each year. Predict when the population of Type C will be equal to the population of Type A.

EXAMPLE 3 Modeling with Mathematics

Your family needs to rent a car for a week while on vacation. Company A charges \$3.25 per mile plus a flat fee of \$125 per week. Company B charges \$3 per mile plus a flat fee of \$150 per week. After how many miles of travel are the total costs the same at both companies?

SOLUTION

1. **Understand the Problem** You know the costs of renting a car from two companies. You are asked to determine how many miles of travel will result in the same total costs at both companies.
2. **Make a Plan** Use a verbal model to write an equation that represents the problem. Then solve the equation by graphing.
3. **Solve the Problem**

Words

$$\underbrace{\boxed{\text{Cost per mile}} \cdot \boxed{\text{Miles}} + \boxed{\text{Flat fee}}}_{\text{Company A}} = \underbrace{\boxed{\text{Cost per mile}} \cdot \boxed{\text{Miles}} + \boxed{\text{Flat fee}}}_{\text{Company B}}$$

Variable Let x be the number of miles traveled.

Equation $3.25x + 125 = 3x + 150$

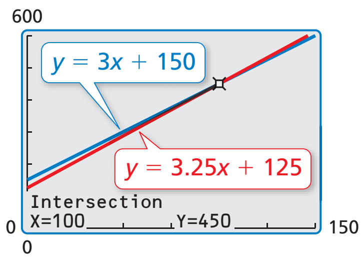
Solve the equation by graphing.

Step 1 Write a system of linear equations using each side of the original equation.

$$3.25x + 125 = 3x + 150$$

$y = 3.25x + 125$ $y = 3x + 150$

Step 2 Use a graphing calculator to graph the system.



Because the graphs intersect at (100, 450), the solution of the equation is $x = 100$.

▶ So, the total costs are the same after 100 miles.

4. **Look Back** One way to check your solution is to solve the equation algebraically, as shown.

Check

$$3.25x + 125 = 3x + 150$$
$$0.25x + 125 = 150$$
$$0.25x = 25$$
$$x = 100$$

Monitoring Progress

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4. **WHAT IF?** Company C charges \$3.30 per mile plus a flat fee of \$115 per week. After how many miles are the total costs the same at Company A and Company C?

5.5 Exercises

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Vocabulary and Core Concept Check

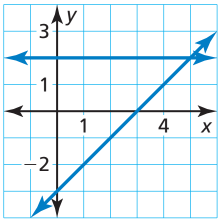
1. **REASONING** The graphs of the equations $y = 3x - 20$ and $y = -2x + 10$ intersect at the point $(6, -2)$. Without solving, find the solution of the equation $3x - 20 = -2x + 10$.

2. **WRITING** Consider the equation $ax + b = cx + d$, where $c = 0$. Can you solve the equation by graphing a system of linear equations? Explain.

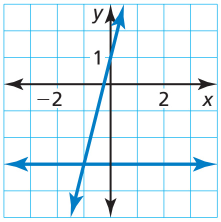
Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, use the graph to solve the equation. Check your solution.

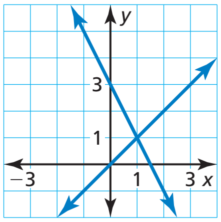
3. $x - 3 = 2$



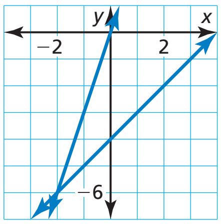
4. $-3 = 4x + 1$



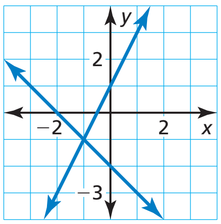
5. $-2x + 3 = x$



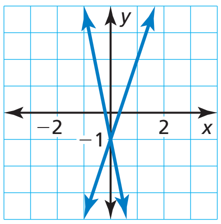
6. $x - 4 = 3x$



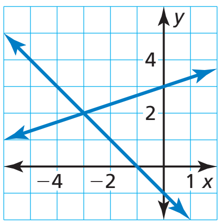
7. $2x + 1 = -x - 2$



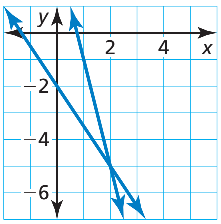
8. $-5x - 1 = 3x - 1$



9. $-x - 1 = \frac{1}{3}x + 3$



10. $-\frac{3}{2}x - 2 = -4x + 3$



In Exercises 11–18, solve the equation by graphing. Check your solution. *(See Example 1.)*

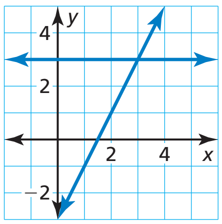
11. $x + 4 = -x$
12. $4x = x + 3$
13. $x + 5 = -2x - 4$
14. $-2x + 6 = 5x - 1$
15. $\frac{1}{2}x - 2 = 9 - 5x$
16. $-5 + \frac{1}{4}x = 3x + 6$
17. $5x - 7 = 2(x + 1)$
18. $-6(x + 4) = -3x - 6$

In Exercises 19–24, solve the equation by graphing. Determine whether the equation has *one solution*, *no solution*, or *infinitely many solutions*.

19. $3x - 1 = -x + 7$
20. $5x - 4 = 5x + 1$
21. $-4(2 - x) = 4x - 8$
22. $-2x - 3 = 2(x - 2)$
23. $-x - 5 = -\frac{1}{3}(3x + 5)$
24. $\frac{1}{2}(8x + 3) = 4x + \frac{3}{2}$

In Exercises 25–28, write an equation that has the same solution as the linear system represented by the graph.

25.



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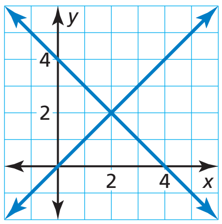
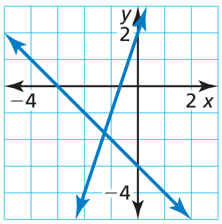


27.



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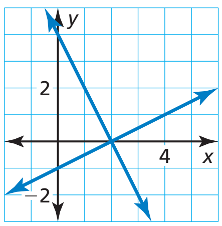


USING TOOLS **In Exercises 29 and 30, use a graphing calculator to solve the equation.**

29. $0.7x + 0.5 = -0.2x - 1.3$

30. $2.1x + 0.6 = -1.4x + 6.9$

31. **MODELING WITH MATHEMATICS** You need to hire a catering company to serve meals to guests at a wedding reception. Company A charges \$500 plus \$20 per guest. Company B charges \$800 plus \$16 per guest. For how many guests are the total costs the same at both companies? *(See Examples 2 and 3.)*

32. **MODELING WITH MATHEMATICS** Your dog is 16 years old in dog years. Your cat is 28 years old in cat years. For every human year, your dog ages by 7 dog years and your cat ages by 4 cat years. In how many human years will both pets be the same age in their respective types of years?

33. **MODELING WITH MATHEMATICS** You and a friend race across a field to a fence. Your friend has a 50-meter head start. The equations shown represent you and your friend's distances d (in meters) from the fence t seconds after the race begins. Find the time at which you catch up to your friend.

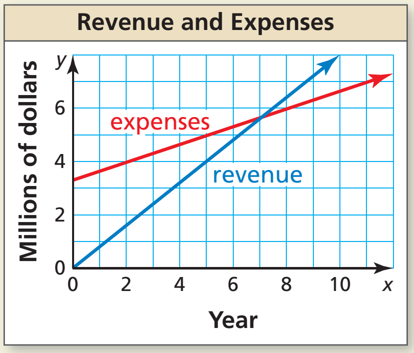
You: $d = -5t + 200$

Your friend: $d = -3\frac{1}{3}t + 150$

34. **MAKING AN ARGUMENT** The graphs of $y = -x + 4$ and $y = 2x - 8$ intersect at the point $(4, 0)$. So, your friend says the solution of the equation $-x + 4 = 2x - 8$ is $(4, 0)$. Is your friend correct? Explain.

35. **OPEN-ENDED** Find values for m and b so that the solution of the equation $mx + b = -2x - 1$ is $x = -3$.

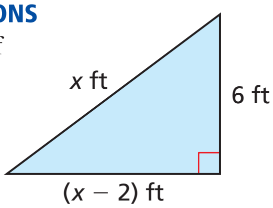
36. **HOW DO YOU SEE IT?** The graph shows the total revenue and expenses of a company x years after it opens for business.



a. Estimate the point of intersection of the graphs.

b. Interpret your answer in part (a).

37. **MATHEMATICAL CONNECTIONS** The value of the perimeter of the triangle (in feet) is equal to the value of the area of the triangle (in square feet). Use a graph to find x.



38. **THOUGHT PROVOKING** A car has an initial value of \$20,000 and decreases in value at a rate of \$1500 per year. Describe a different car that will be worth the same amount as this car in exactly 5 years. Specify the initial value and the rate at which the value decreases.

39. **ABSTRACT REASONING** Use a graph to determine the sign of the solution of the equation $ax + b = cx + d$ in each situation.

a. $0 < b < d$ and $a < c$ **b.** $d < b < 0$ and $a < c$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph the inequality. *(Section 2.1)*

40. $y > 5$ **41.** $x \leq -2$ **42.** $n \geq 9$ **43.** $c < -6$

Use the graphs of f and g to describe the transformation from the graph of f to the graph of g. *(Section 3.7)*

44. $f(x) = x - 5;\ g(x) = f(x + 2)$

45. $f(x) = 6x;\ g(x) = -f(x)$

46. $f(x) = -2x + 1;\ g(x) = f(4x)$

47. $f(x) = \frac{1}{2}x - 2;\ g(x) = f(x - 1)$

5.6 Linear Inequalities in Two Variables

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.2.H
A.3.D

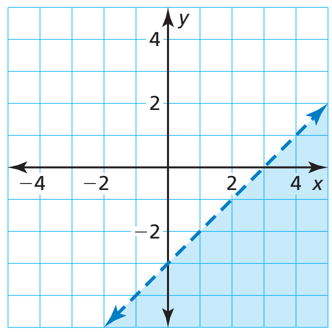
Essential Question

How can you write and graph a linear inequality in two variables?

A **solution of a linear inequality in two variables** is an ordered pair (x, y) that makes the inequality true. The **graph of a linear inequality** in two variables shows all the solutions of the inequality in a coordinate plane.

EXPLORATION 1 Writing a Linear Inequality in Two Variables

Work with a partner.



a. Write an equation represented by the dashed line.

b. The solutions of an inequality are represented by the shaded region. In words, describe the solutions of the inequality.

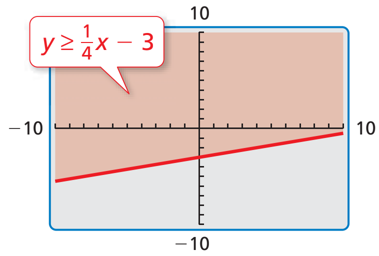
c. Write an inequality represented by the graph. Which inequality symbol did you use? Explain your reasoning.

EXPLORATION 2 Using a Graphing Calculator

Work with a partner. Use a graphing calculator to graph $y \geq \frac{1}{4}x - 3$.

a. Enter the equation $y = \frac{1}{4}x - 3$ into your calculator.

b. The inequality has the symbol $\geq$. So, the region to be shaded is above the graph of $y = \frac{1}{4}x - 3$, as shown. Verify this by testing a point in this region, such as $(0, 0)$, to make sure it is a solution of the inequality.



SELECTING TOOLS

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

Because the inequality symbol is *greater than or equal to*, the line is solid and not dashed. Some graphing calculators always use a solid line when graphing inequalities. In this case, you have to determine whether the line should be solid or dashed, based on the inequality symbol used in the original inequality.

EXPLORATION 3 Graphing Linear Inequalities in Two Variables

Work with a partner. Graph each linear inequality in two variables. Explain your steps. Use a graphing calculator to check your graphs.

a. $y > x + 5$ **b.** $y \leq -\frac{1}{2}x + 1$ **c.** $y \geq -x - 5$

Communicate Your Answer

4. How can you write and graph a linear inequality in two variables?

5. Give an example of a real-life situation that can be modeled using a linear inequality in two variables.

5.6 Lesson

What You Will Learn

- Check solutions of linear inequalities.
- Graph linear inequalities in two variables.
- Write linear inequalities in two variables.
- Use linear inequalities to solve real-life problems.

Core Vocabulary

linear inequality in two variables, *p. 252*
solution of a linear inequality in two variables, *p. 252*
graph of a linear inequality, *p. 252*
half-planes, *p. 252*

Previous
ordered pair

Linear Inequalities

A **linear inequality in two variables**, x and y, can be written as

$$ax + by < c \qquad ax + by \le c \qquad ax + by > c \qquad ax + by \ge c$$

where a, b, and c are real numbers. A **solution of a linear inequality in two variables** is an ordered pair (x, y) that makes the inequality true.

EXAMPLE 1 Checking Solutions

Tell whether the ordered pair is a solution of the inequality.

a. $2x + y < -3$; $(-1, 9)$ **b.** $x - 3y \ge 8$; $(2, -2)$

SOLUTION

a. $2x + y < -3$ Write the inequality.

$2(-1) + 9 \stackrel{?}{<} -3$ Substitute -1 for x and 9 for y.

$7 \nless -3$ ✗ Simplify. 7 is *not* less than -3.

▶ So, $(-1, 9)$ is *not* a solution of the inequality.

b. $x - 3y \ge 8$ Write the inequality.

$2 - 3(-2) \stackrel{?}{\ge} 8$ Substitute 2 for x and -2 for y.

$8 \ge 8$ ✓ Simplify. 8 is equal to 8.

▶ So, $(2, -2)$ is a solution of the inequality.

Monitoring Progress

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Tell whether the ordered pair is a solution of the inequality.

1. $x + y > 0$; $(-2, 2)$

2. $4x - y \ge 5$; $(0, 0)$

3. $5x - 2y \le -1$; $(-4, -1)$

4. $-2x - 3y < 15$; $(5, -7)$

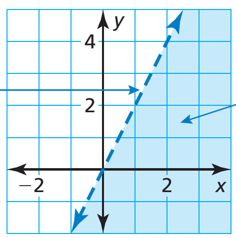
Graphing Linear Inequalities in Two Variables

The **graph of a linear inequality** in two variables shows all the solutions of the inequality in a coordinate plane.

READING

A dashed boundary line means that points on the line are *not* solutions. A solid boundary line means that points on the line are solutions.

All solutions of $y < 2x$ lie on one side of the *boundary line* $y = 2x$.



The boundary line divides the coordinate plane into two **half-planes**. The shaded half-plane is the graph of $y < 2x$.

Core Concept

Graphing a Linear Inequality in Two Variables

Step 1 Graph the boundary line for the inequality. Use a dashed line for $<$ or $>$. Use a solid line for $\leq$ or $\geq$.

Step 2 Test a point that is not on the boundary line to determine whether it is a solution of the inequality.

Step 3 When the test point is a solution, shade the half-plane that contains the point. When the test point is *not* a solution, shade the half-plane that does *not* contain the point.

EXAMPLE 2 Graphing a Linear Inequality in One Variable

Graph $y \leq 2$ in a coordinate plane.

SOLUTION

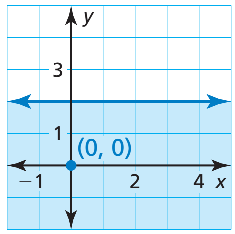
Step 1 Graph $y = 2$. Use a solid line because the inequality symbol is $\leq$.

Step 2 Test (0, 0).

$y \leq 2$ Write the inequality.

$0 \leq 2$ ✓ Substitute.

Step 3 Because (0, 0) is a solution, shade the half-plane that contains (0, 0).



STUDY TIP

It is often convenient to use the origin as a test point. However, you must choose a different test point when the origin is on the boundary line.

EXAMPLE 3 Graphing a Linear Inequality in Two Variables

Graph $-x + 2y > 2$ in a coordinate plane.

SOLUTION

Step 1 Graph $-x + 2y = 2$, or $y = \frac{1}{2}x + 1$. Use a dashed line because the inequality symbol is $>$.

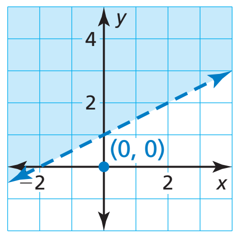
Step 2 Test (0, 0).

$-x + 2y > 2$ Write the inequality.

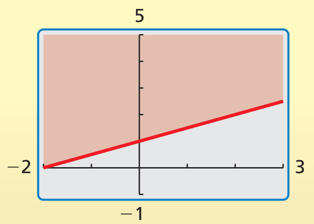
$-(0) + 2(0) \overset{?}{>} 2$ Substitute.

$0 \not> 2$ ✗ Simplify.

Step 3 Because (0, 0) is *not* a solution, shade the half-plane that does *not* contain (0, 0).



Check



Monitoring Progress

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Graph the inequality in a coordinate plane.

5. $y > -1$

6. $x \leq -4$

7. $x + y \leq -4$

8. $x - 2y < 0$

Writing Linear Inequalities in Two Variables

Core Concept

Writing a Linear Inequality in Two Variables Using a Graph

		Boundary Line	
		Dashed	**Solid**
Half-plane	**Above**	$>$	$\geq$
	Below	$<$	$\leq$

Write an equation in slope-intercept form of the boundary line.

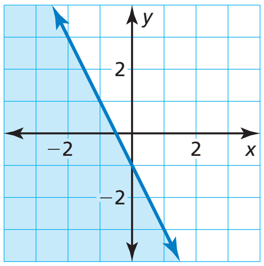
If the shaded half-plane is *above* the boundary line, then

- replace $=$ with $>$ when the boundary line is *dashed*.
- replace $=$ with $\geq$ when the boundary line is *solid*.

If the shaded half-plane is *below* the boundary line, then

- replace $=$ with $<$ when the boundary line is *dashed*.
- replace $=$ with $\leq$ when the boundary line is *solid*.

EXAMPLE 4 Writing a Linear Inequality Using a Graph



Write an inequality that represents the graph.

SOLUTION

The boundary line has a slope of -2 and a y-intercept of -1. So, an equation of the boundary line is $y = -2x - 1$. The shaded half-plane is *below* the boundary line and the boundary line is *solid*. So, replace $=$ with $\leq$.

▶ The inequality $y \leq -2x - 1$ represents the graph.

EXAMPLE 5 Writing a Linear Inequality Using a Table

Flavor	Quantity (gallons)
vanilla	20
chocolate	15
strawberry	x
banana	y
sherbet	18

An ice cream truck can carry at most 75 gallons of ice cream. The table shows the inventory on the truck. Write an inequality that represents the numbers of gallons of strawberry and banana ice cream on the truck.

SOLUTION

Use the table to write an inequality that represents the problem.

$$20 + 15 + x + y + 18 \leq 75$$ The total number of gallons is less than or equal to 75.

$$x + y \leq 22$$ Isolate the variable terms on one side.

▶ The inequality $x + y \leq 22$ represents the numbers of gallons of strawberry and banana ice cream on the truck.

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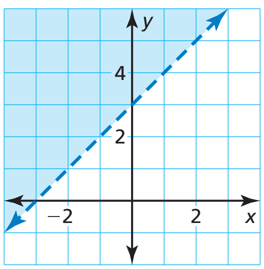


9. Write an inequality that represents the graph.

10. For a weight loss competition, the blue team needs to lose more than 55 pounds to win. Write an inequality that represents the amounts (in pounds) of weight Team Members B and E must lose so that the blue team wins.

Blue team member	A	B	C	D	E	F
Weight lost (pounds)	10	y	8	13	x	5

Solving Real-Life Problems

EXAMPLE 6 **Modeling with Mathematics**

You can spend at most \$10 on grapes and apples for a fruit salad. Grapes cost \$2.50 per pound, and apples cost \$1 per pound. Write and graph an inequality that represents the amounts of grapes and apples you can buy. Identify and interpret two solutions of the inequality.

SOLUTION

1. **Understand the Problem** You know the most that you can spend and the prices per pound for grapes and apples. You are asked to write and graph an inequality and then identify and interpret two solutions.

2. **Make a Plan** Use a verbal model to write an inequality that represents the problem. Then graph the inequality. Use the graph to identify two solutions. Then interpret the solutions.

3. **Solve the Problem**

Words Cost per pound of grapes • Pounds of grapes + Cost per pound of apples • Pounds of apples ≤ Amount you can spend

Variables Let x be pounds of grapes and y be pounds of apples.

Inequality $2.50 \cdot x + 1 \cdot y \leq 10$

Step 1 Graph $2.5x + y = 10$, or $y = -2.5x + 10$. Use a solid line because the inequality symbol is ≤. Restrict the graph to positive values of x and y because negative values do not make sense in this real-life context.

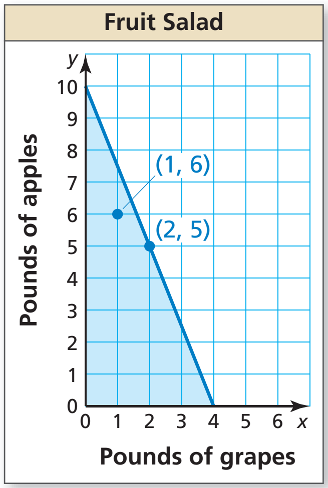
Step 2 Test (0, 0).

$$2.5x + y \leq 10 \qquad \text{Write the inequality.}$$

$$2.5(0) + 0 \overset{?}{\leq} 10 \qquad \text{Substitute.}$$

$$0 \leq 10 \ ✓ \qquad \text{Simplify.}$$

Step 3 Because (0, 0) is a solution, shade the half-plane that contains (0, 0).



▶ One possible solution is (1, 6) because it lies in the shaded half-plane. Another possible solution is (2, 5) because it lies on the solid line. So, you can buy 1 pound of grapes and 6 pounds of apples, or 2 pounds of grapes and 5 pounds of apples.

4. **Look Back** Check your solutions by substituting them into the original inequality, as shown.

Check

$$2.5x + y \leq 10$$

$$2.5(1) + 6 \overset{?}{\leq} 10$$

$$8.5 \leq 10 \ ✓$$

$$2.5x + y \leq 10$$

$$2.5(2) + 5 \overset{?}{\leq} 10$$

$$10 \leq 10 \ ✓$$

Monitoring Progress

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11. You can spend at most \$12 on red peppers and tomatoes for salsa. Red peppers cost \$4 per pound, and tomatoes cost \$3 per pound. Write and graph an inequality that represents the amounts of red peppers and tomatoes you can buy. Identify and interpret two solutions of the inequality.

5.6 Exercises

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Vocabulary and Core Concept Check

1. **VOCABULARY** How can you tell whether an ordered pair is a solution of a linear inequality?

2. **WRITING** Compare the graph of a linear inequality in two variables with the graph of a linear equation in two variables.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, tell whether the ordered pair is a solution of the inequality. *(See Example 1.)*

3. $x + y < 7$; $(2, 3)$

4. $x - y \leq 0$; $(5, 2)$

5. $x + 3y \geq -2$; $(-9, 2)$

6. $8x + y > -6$; $(-1, 2)$

7. $-6x + 4y \leq 6$; $(-3, -3)$

8. $3x - 5y \geq 2$; $(-1, -1)$

9. $-x - 6y > 12$; $(-8, 2)$

10. $-4x - 8y < 15$; $(-6, 3)$

In Exercises 11–16, tell whether the ordered pair is a solution of the inequality whose graph is shown.

11. $(0, -1)$

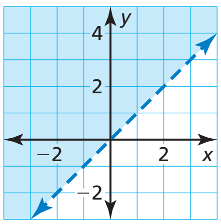
12. $(-1, 3)$

13. $(1, 4)$

14. $(0, 0)$

15. $(3, 3)$

16. $(2, 1)$



17. MODELING WITH MATHEMATICS A carpenter has at most \$250 to spend on lumber. The inequality $8x + 12y \leq 250$ represents the numbers x of 2-by-8 boards and the numbers y of 4-by-4 boards the carpenter can buy. Can the carpenter buy twelve 2-by-8 boards and fourteen 4-by-4 boards? Explain.

4 in. x 4 in. x 8 ft
\$12 each

2 in. x 8 in. x 8 ft
\$8 each

18. MODELING WITH MATHEMATICS The inequality $3x + 2y \geq 93$ represents the numbers x of multiple-choice questions and the numbers y of matching questions you can answer correctly to receive an A on a test. You answer 20 multiple-choice questions and 18 matching questions correctly. Do you receive an A on the test? Explain.

In Exercises 19–24, graph the inequality in a coordinate plane. *(See Example 2.)*

19. $y \leq 5$

20. $y > 6$

21. $x < 2$

22. $x \geq -3$

23. $y > -7$

24. $x < 9$

In Exercises 25–30, graph the inequality in a coordinate plane. *(See Example 3.)*

25. $y > -2x - 4$

26. $y \leq 3x - 1$

27. $-4x + y < -7$

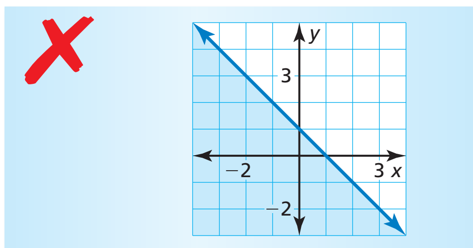
28. $3x - y \geq 5$

29. $5x - 2y \leq 6$

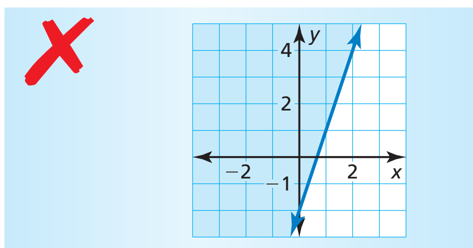
30. $-x + 4y > -12$

ERROR ANALYSIS In Exercises 31 and 32, describe and correct the error in graphing the inequality.

31. $y < -x + 1$

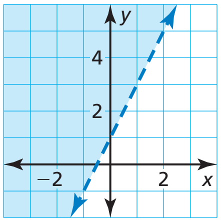


32. $y \leq 3x - 2$



In Exercises 33–38, write an inequality that represents the graph. *(See Example 4.)*

33.



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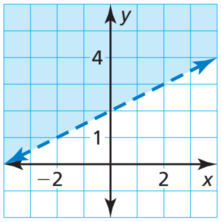

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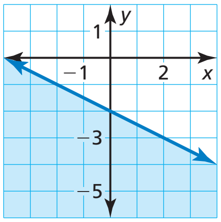

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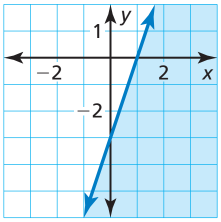

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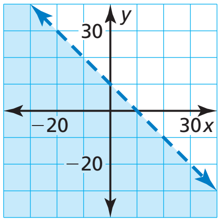

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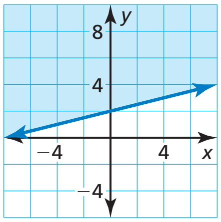


39. MODELING WITH MATHEMATICS A restaurant employee works at least 40 hours per week. The table shows the employee's weekly work schedule. Write an inequality that represents the numbers of hours the employee spends cooking and washing dishes. *(See Example 5.)*

Task	Hours
cleaning	6
cooking	x
ordering inventory	2
stocking food	3
washing dishes	y

40. MODELING WITH MATHEMATICS You are allowed to watch TV for less than 10 hours total on school nights. The table shows the numbers of hours you watch TV. Write an inequality that represents the numbers of hours you can watch TV on Sunday and Wednesday.

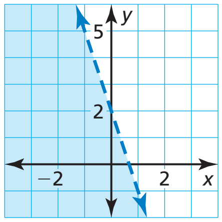
School night	Hours
Sunday	x
Monday	1.5
Tuesday	0.5
Wednesday	y
Thursday	2.5

41. MODELING WITH MATHEMATICS A department store offers a 10% discount on any purchase over \$200. The table shows the costs of items in a purchase at the store. Write an inequality that represents the costs of the shirt and shoes that will qualify the purchase for the discount.

Item	Cost
backpack	\$54.99
jeans	\$39.99
shirt	x
shoes	y
video game	\$49.95

ERROR ANALYSIS In Exercises 42 and 43, describe and correct the error in writing the inequality.

42.



✗ $y > -3x + 2$

43. You eat no more than 2500 calories a day. The table shows the numbers of calories you eat at each meal.

Meal	Calories
breakfast	770
lunch	x
snack	150
dinner	y
snack	250

✗ $x + y \leq 3670$

44. MODELING WITH MATHEMATICS You have at most \$20 to spend at an arcade. Arcade games cost \$0.75 each, and snacks cost \$2.25 each. Write and graph an inequality that represents the numbers of games you can play and snacks you can buy. Identify and interpret two solutions of the inequality. *(See Example 6.)*

45. MODELING WITH MATHEMATICS A drama club must sell at least \$1500 worth of tickets to cover the expenses of producing a play. Write and graph an inequality that represents how many adult and student tickets the club must sell. Identify and interpret two solutions of the inequality.

46. MODELING WITH MATHEMATICS A shipping company is delivering ceramic pots. The company earns \$1 for each unbroken ceramic pot and is fined \$4 for each broken ceramic pot. The company wants to earn at least \$2000 for this delivery. Write and graph an inequality that represents the numbers of unbroken and broken ceramic pots the company must deliver. Identify and interpret two solutions of the inequality.

47. MODELING WITH MATHEMATICS A clothing store sells T-shirts for \$20 and skirts for \$25. The store wants to sell a minimum of \$800 worth of T-shirts and skirts. Write and graph an inequality that represents the numbers of T-shirts and skirts the store must sell. Identify and interpret two solutions of the inequality.

48. PROBLEM SOLVING Large boxes weigh 75 pounds, and small boxes weigh 40 pounds.

a. Write and graph an inequality that represents the numbers of large and small boxes a 200-pound delivery person can take on the elevator.

b. Explain why some solutions of the inequality might not be practical in real life.

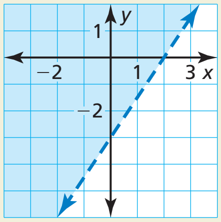
Weight limit:
2000 lb

49. WRITING Can you always use (0, 0) as a test point when graphing an inequality? Explain.

50. HOW DO YOU SEE IT? Match each inequality with its graph.

a. $3x - 2y \leq 6$ **b.** $3x - 2y < 6$

c. $3x - 2y > 6$ **d.** $3x - 2y \geq 6$

A.


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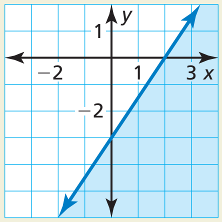
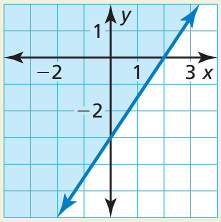
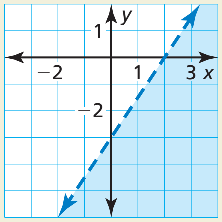


C.


D.


51. REASONING When graphing a linear inequality in two variables, why must you choose a test point that is *not* on the boundary line?

52. THOUGHT PROVOKING Write a linear inequality in two variables that has the following two properties.

- (0, 0), (0, −1), and (0, 1) are not solutions.
- (1, 1), (3, −1), and (−1, 3) are solutions.

CRITICAL THINKING In Exercises 53 and 54, write and graph an inequality whose graph is described by the given information.

53. The points (2, 5) and (−3, −5) lie on the boundary line. The points (6, 5) and (−2, −3) are solutions of the inequality.

54. The points (−7, −16) and (1, 8) lie on the boundary line. The points (−7, 0) and (3, 14) are *not* solutions of the inequality.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the next three terms of the arithmetic sequence. *(Section 4.7)*

55. 0, 8, 16, 24, 32, . . .

56. −5, −8, −11, −14, −17, . . .

57. 3.6, 2.8, 2, 1.2, 0.4, . . .

58. $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

5.7 Systems of Linear Inequalities

Essential Question

How can you graph a system of linear inequalities?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.3.H

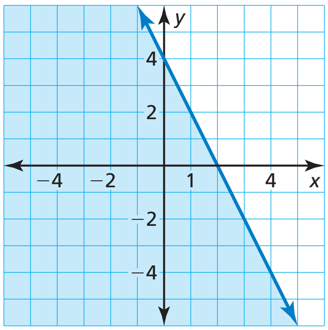
EXPLORATION 1 Graphing Linear Inequalities

Work with a partner. Match each linear inequality with its graph. Explain your reasoning.

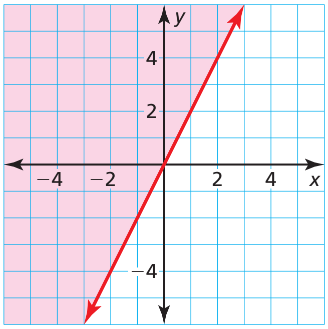
$2x + y \leq 4$ Inequality 1

$2x - y \leq 0$ Inequality 2

A.



B.



EXPLORATION 2 Graphing a System of Linear Inequalities

Work with a partner. Consider the linear inequalities given in Exploration 1.

$2x + y \leq 4$ Inequality 1

$2x - y \leq 0$ Inequality 2

a. Use two different colors to graph the inequalities in the same coordinate plane. What is the result?

b. Describe each of the shaded regions of the graph. What does the unshaded region represent?

USING PROBLEM-SOLVING STRATEGIES

To be proficient in math, you need to explain to yourself the meaning of a problem.

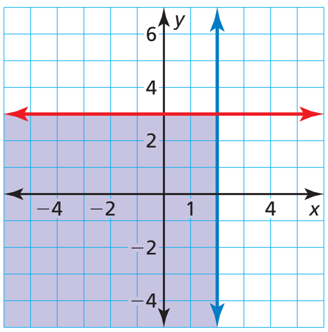
Communicate Your Answer

3. How can you graph a system of linear inequalities?

4. When graphing a system of linear inequalities, which region represents the solution of the system?

5. Do you think all systems of linear inequalities have a solution? Explain your reasoning.

6. Write a system of linear inequalities represented by the graph.



5.7 Lesson

Core Vocabulary

system of linear inequalities, *p. 260*
solution of a system of linear inequalities, *p. 260*
graph of a system of linear inequalities, *p. 261*

Previous
linear inequality in two variables

What You Will Learn

- Check solutions of systems of linear inequalities.
- Graph systems of linear inequalities.
- Write systems of linear inequalities.
- Use systems of linear inequalities to solve real-life problems.

Systems of Linear Inequalities

A **system of linear inequalities** is a set of two or more linear inequalities in the same variables. An example is shown below.

$y < x + 2$ Inequality 1

$y \geq 2x - 1$ Inequality 2

A **solution of a system of linear inequalities** in two variables is an ordered pair that is a solution of each inequality in the system.

EXAMPLE 1 Checking Solutions

Tell whether each ordered pair is a solution of the system of linear inequalities.

$y < 2x$ Inequality 1

$y \geq x + 1$ Inequality 2

a. (3, 5) **b.** (−2, 0)

SOLUTION

a. Substitute 3 for x and 5 for y in each inequality.

Inequality 1	Inequality 2
$y < 2x$	$y \geq x + 1$
$5 \overset{?}{<} 2(3)$	$5 \overset{?}{\geq} 3 + 1$
$5 < 6$ ✓	$5 \geq 4$ ✓

- Because the ordered pair (3, 5) is a solution of each inequality, it is a solution of the system.

b. Substitute −2 for x and 0 for y in each inequality.

Inequality 1	Inequality 2
$y < 2x$	$y \geq x + 1$
$0 \overset{?}{<} 2(-2)$	$0 \overset{?}{\geq} -2 + 1$
$0 \not< -4$ ✗	$0 \geq -1$ ✓

- Because (−2, 0) is not a solution of each inequality, it is *not* a solution of the system.

Monitoring Progress

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Tell whether the ordered pair is a solution of the system of linear inequalities.

1. $(-1, 5)$; $\begin{array}{l} y < 5 \\ y > x - 4 \end{array}$

2. $(1, 4)$; $\begin{array}{l} y \geq 3x + 1 \\ y > x - 1 \end{array}$

Graphing Systems of Linear Inequalities

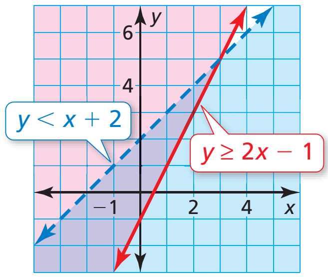
The **graph of a system of linear inequalities** is the graph of all the solutions of the system.

Core Concept

Graphing a System of Linear Inequalities

Step 1 Graph each inequality in the same coordinate plane.

Step 2 Find the intersection of the half-planes that are solutions of the inequalities. This intersection is the graph of the system.



Check

Verify that $(-3, 1)$ is a solution of each inequality.

Inequality 1

$y \leq 3$

$1 \leq 3$ ✓

Inequality 2

$y > x + 2$

$1 \stackrel{?}{>} -3 + 2$

$1 > -1$ ✓

EXAMPLE 2 Graphing a System of Linear Inequalities

Graph the system of linear inequalities.

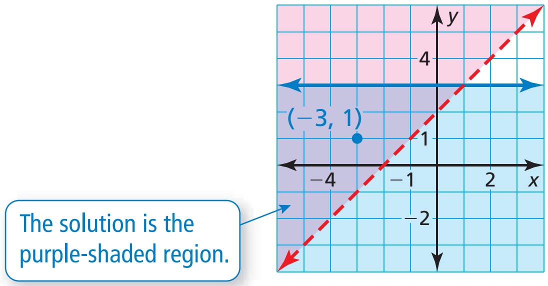
$y \leq 3$ Inequality 1

$y > x + 2$ Inequality 2

SOLUTION

Step 1 Graph each inequality.

Step 2 Find the intersection of the half-planes. One solution is $(-3, 1)$.



EXAMPLE 3 Graphing a System of Linear Inequalities: No Solution

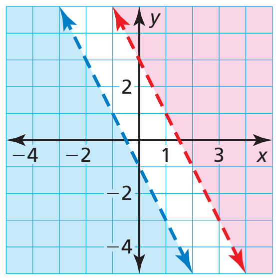
Graph the system of linear inequalities.

$2x + y < -1$ Inequality 1

$2x + y > 3$ Inequality 2

SOLUTION

Step 1 Graph each inequality.



Step 2 Find the intersection of the half-planes. Notice that the lines are parallel, and the half-planes do not intersect.

▶ So, the system has no solution.

Monitoring Progress

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Graph the system of linear inequalities.

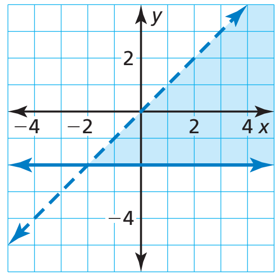
3. $y \geq -x + 4$
$x + y \leq 0$

4. $y > 2x - 3$
$y \geq \frac{1}{2}x + 1$

5. $-2x + y < 4$
$2x + y > 4$

Writing Systems of Linear Inequalities

EXAMPLE 4 Writing a System of Linear Inequalities



Write a system of linear inequalities represented by the graph.

SOLUTION

Inequality 1 The horizontal boundary line passes through $(0, -2)$. So, an equation of the line is $y = -2$. The shaded region is *above* the *solid* boundary line, so the inequality is $y \geq -2$.

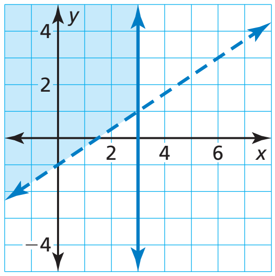
Inequality 2 The slope of the other boundary line is 1, and the y-intercept is 0. So, an equation of the line is $y = x$. The shaded region is *below* the *dashed* boundary line, so the inequality is $y < x$.

▶ The system of linear inequalities represented by the graph is

$y \geq -2$ Inequality 1

$y < x$. Inequality 2

EXAMPLE 5 Writing a System of Linear Inequalities



Write a system of linear inequalities represented by the graph.

SOLUTION

Inequality 1 The vertical boundary line passes through $(3, 0)$. So, an equation of the line is $x = 3$. The shaded region is to the *left* of the *solid* boundary line, so the inequality is $x \leq 3$.

Inequality 2 The slope of the other boundary line is $\frac{2}{3}$, and the y-intercept is -1. So, an equation of the line is $y = \frac{2}{3}x - 1$. The shaded region is *above* the *dashed* boundary line, so the inequality is $y > \frac{2}{3}x - 1$.

▶ The system of linear inequalities represented by the graph is

$x \leq 3$ Inequality 1

$y > \frac{2}{3}x - 1$. Inequality 2

Monitoring Progress

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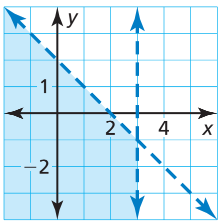
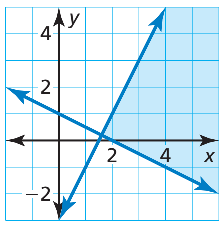
Write a system of linear inequalities represented by the graph.

6. 

7. 

Solving Real-Life Problems

EXAMPLE 6 **Modeling with Mathematics**

You have at most 8 hours to spend at the mall and at the beach. You want to spend at least 2 hours at the mall and more than 4 hours at the beach. Write and graph a system that represents the situation. How much time can you spend at each location?

SOLUTION

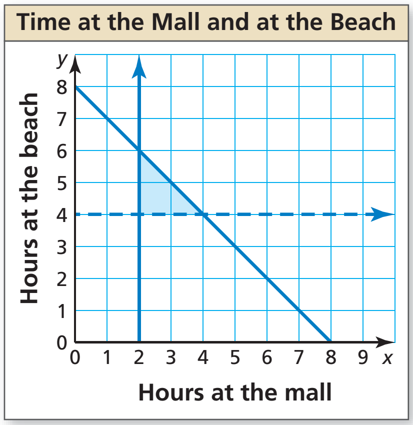
1. **Understand the Problem** You know the total amount of time you can spend at the mall and at the beach. You also know how much time you want to spend at each location. You are asked to write and graph a system that represents the situation and determine how much time you can spend at each location.

2. **Make a Plan** Use the given information to write a system of linear inequalities. Then graph the system and identify an ordered pair in the solution region.

3. **Solve the Problem** Let x be the number of hours at the mall and let y be the number of hours at the beach.

$x + y \leq 8$	at most 8 hours at the mall and at the beach
$x \geq 2$	at least 2 hours at the mall
$y > 4$	more than 4 hours at the beach

Graph the system.



One ordered pair in the solution region is (2.5, 5).

▶ So, you can spend 2.5 hours at the mall and 5 hours at the beach.

4. **Look Back** Check your solution by substituting it into the inequalities in the system, as shown.

Check

$x + y \leq 8$

$2.5 + 5 \overset{?}{\leq} 8$

$7.5 \leq 8$ ✓

$x \geq 2$

$2.5 \geq 2$ ✓

$y > 4$

$5 > 4$ ✓

Monitoring Progress

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8. Name another solution of Example 6.

9. **WHAT IF?** You want to spend at least 3 hours at the mall. How does this change the system? Is (2.5, 5) still a solution? Explain.

5.7 Exercises

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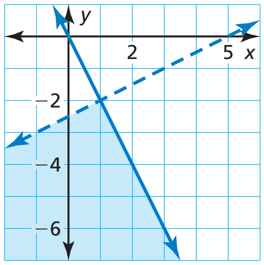
Vocabulary and Core Concept Check

1. **VOCABULARY** How can you verify that an ordered pair is a solution of a system of linear inequalities?

2. **WHICH ONE DOESN'T BELONG?** Use the graph shown. Which of the ordered pairs does *not* belong with the other three? Explain your reasoning.

$(1, -2)$	$(0, -4)$	$(-1, -6)$	$(2, -4)$



Monitoring Progress and Modeling with Mathematics

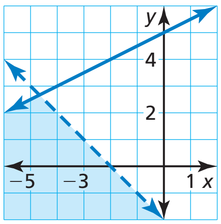
In Exercises 3–6, tell whether the ordered pair is a solution of the system of linear inequalities.

3. $(-4, 3)$

4. $(-3, -1)$

5. $(-2, 5)$

6. $(1, 1)$



In Exercises 7–10, tell whether the ordered pair is a solution of the system of linear inequalities. *(See Example 1.)*

7. $(-5, 2)$; $y < 4$, $y > x + 3$

8. $(1, -1)$; $y > -2$, $y > x - 5$

9. $(0, 0)$; $y \le x + 7$, $y \ge 2x + 3$

10. $(4, -3)$; $y \le -x + 1$, $y \le 5x - 2$

In Exercises 11–20, graph the system of linear inequalities. *(See Examples 2 and 3.)*

11. $y > -3$
    $y \ge 5x$

12. $y < -1$
    $x > 4$

13. $y < -2$
    $y > 2$

14. $y < x - 1$
    $y \ge x + 1$

15. $y \ge -5$
    $y - 1 < 3x$

16. $x + y > 4$
    $y \ge \frac{3}{2}x - 9$

17. $x + y > 1$
    $-x - y < -3$

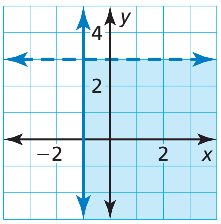
18. $2x + y \le 5$
    $y + 2 \ge -2x$

19. $x < 4$
    $y > 1$
    $y \ge -x + 1$

20. $x + y \le 10$
    $x - y \ge 2$
    $y > 2$

In Exercises 21–26, write a system of linear inequalities represented by the graph. *(See Examples 4 and 5.)*

21.



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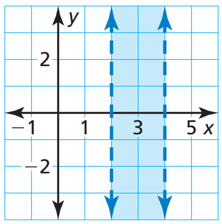

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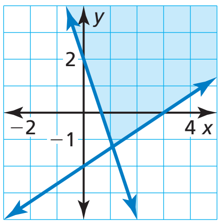

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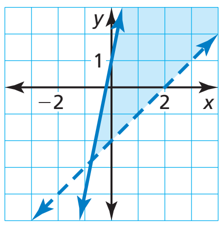

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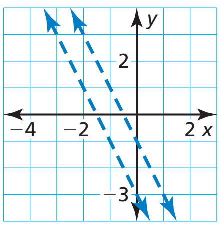

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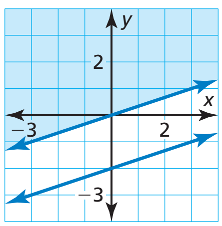
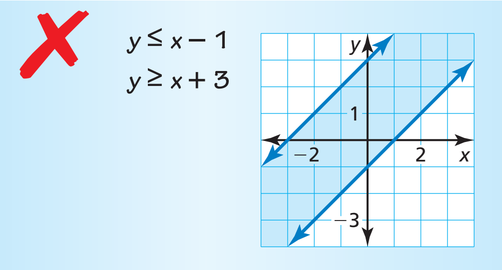


ERROR ANALYSIS **In Exercises 27 and 28, describe and correct the error in graphing the system of linear inequalities.**

27.



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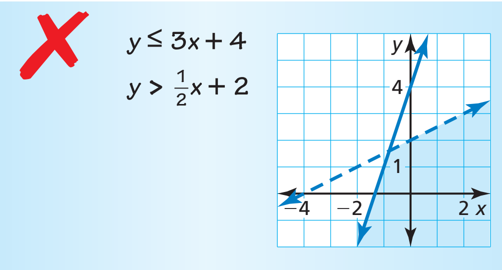


29. **MODELING WITH MATHEMATICS** You can spend at most \$21 on fruit. Blueberries cost \$4 per pound, and strawberries cost \$3 per pound. You need at least 3 pounds of fruit to make muffins. *(See Example 6.)*

a. Write and graph a system of linear inequalities that represents the situation.

b. Identify and interpret a solution of the system.

c. Use the graph to determine whether you can buy 4 pounds of blueberries and 1 pound of strawberries.

30. **MODELING WITH MATHEMATICS** You earn \$10 per hour working as a manager at a grocery store. You are required to work at the grocery store at least 8 hours per week. You also teach music lessons for \$15 per hour. You need to earn at least \$120 per week, but you do not want to work more than 20 hours per week.

a. Write and graph a system of linear inequalities that represents the situation.

b. Identify and interpret a solution of the system.

c. Use the graph to determine whether you can work 8 hours at the grocery store and teach 1 hour of music lessons.

31. **MODELING WITH MATHEMATICS** You are fishing for surfperch and rockfish, which are species of bottomfish. Gaming laws allow you to catch no more than 15 surfperch per day, no more than 10 rockfish per day, and no more than 20 total bottomfish per day.

a. Write and graph a system of linear inequalities that represents the situation.

b. Use the graph to determine whether you can catch 11 surfperch and 9 rockfish in 1 day.

surfperch

rockfish

32. **REASONING** Describe the intersection of the half-planes of the system shown.

$$x - y \leq 4$$
$$x - y \geq 4$$

33. **MATHEMATICAL CONNECTIONS** The following points are the vertices of a shaded rectangle.

$$(-1, 1), (6, 1), (6, -3), (-1, -3)$$

a. Write a system of linear inequalities represented by the shaded rectangle.

b. Find the area of the rectangle.

34. **MATHEMATICAL CONNECTIONS** The following points are the vertices of a shaded triangle.

$$(2, 5), (6, -3), (-2, -3)$$

a. Write a system of linear inequalities represented by the shaded triangle.

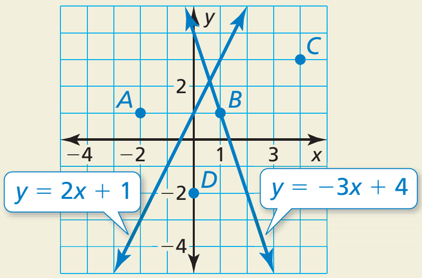
b. Find the area of the triangle.

35. **PROBLEM SOLVING** You plan to spend less than half of your monthly \$2000 paycheck on housing and savings. You want to spend at least 10% of your paycheck on savings and at most 30% of it on housing. How much money can you spend on savings and housing?

36. **PROBLEM SOLVING** On a road trip with a friend, you drive about 70 miles per hour, and your friend drives about 60 miles per hour. The plan is to drive less than 15 hours and at least 600 miles each day. Your friend will drive more hours than you. How many hours can you and your friend each drive in 1 day?

37. **WRITING** How are solving systems of linear inequalities and solving systems of linear equations similar? How are they different?

38. **HOW DO YOU SEE IT?** The graphs of two linear equations are shown.



Replace the equal signs with inequality symbols to create a system of linear inequalities that has point C as a solution, but not points A, B, and D. Explain your reasoning.

$$y \ \square \ -3x + 4$$

$$y \ \square \ 2x + 1$$

39. **OPEN-ENDED** Write a real-life problem that can be represented by a system of linear inequalities. Write the system of linear inequalities and graph the system.

40. **MAKING AN ARGUMENT** Your friend says that a system of linear inequalities in which the boundary lines are parallel must have no solution. Is your friend correct? Explain.

41. **CRITICAL THINKING** Is it possible for the solution set of a system of linear inequalities to be all real numbers? Explain your reasoning.

OPEN-ENDED In Exercises 42–44, write a system of linear inequalities with the given characteristic.

42. All solutions are in Quadrant I.

43. All solutions have one positive coordinate and one negative coordinate.

44. There are no solutions.

45. **OPEN-ENDED** One inequality in a system is $-4x + 2y > 6$. Write another inequality so the system has (a) no solution and (b) infinitely many solutions.

46. **THOUGHT PROVOKING** You receive a gift certificate for a clothing store and plan to use it to buy T-shirts and sweatshirts. Describe a situation in which you can buy 9 T-shirts and 1 sweatshirt, but you cannot buy 3 T-shirts and 8 sweatshirts. Write and graph a system of linear inequalities that represents the situation.

47. **CRITICAL THINKING** Write a system of linear inequalities that has exactly one solution.

48. **MODELING WITH MATHEMATICS** You make necklaces and key chains to sell at a craft fair. The table shows the amounts of time and money it takes to make a necklace and a key chain, and the amounts of time and money you have available for making them.

	Necklace	Key chain	Available
Time to make (hours)	0.5	0.25	20
Cost to make (dollars)	2	3	120

a. Write and graph a system of four linear inequalities that represents the number x of necklaces and the number y of key chains that you can make.

b. Find the vertices (corner points) of the graph of the system.

c. You sell each necklace for \$10 and each key chain for \$8. The revenue R is given by the equation $R = 10x + 8y$. Find the revenue corresponding to each ordered pair in part (b). Which vertex results in the maximum revenue?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the product using exponents. *(Skills Review Handbook)*

49. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

50. $(-13) \cdot (-13) \cdot (-13)$

51. $x \cdot x \cdot x \cdot x \cdot x \cdot x$

Write an equation of the line with the given slope and y-intercept. *(Section 4.1)*

52. slope: 1
y-intercept: -6

53. slope: -3
y-intercept: 5

54. slope: $-\frac{1}{4}$
y-intercept: -1

55. slope: $\frac{4}{3}$
y-intercept: 0

5.5–5.7 What Did You Learn?

Core Vocabulary

linear inequality in two variables, *p. 252*
solution of a linear inequality in two variables, *p. 252*
graph of a linear inequality, *p. 252*
half-planes, *p. 252*
system of linear inequalities, *p. 260*
solution of a system of linear inequalities, *p. 260*
graph of a system of linear inequalities, *p. 261*

Core Concepts

Section 5.5

Solving Linear Equations by Graphing, *p. 246*

Section 5.6

Graphing a Linear Inequality in Two Variables, *p. 253*
Writing a Linear Inequality in Two Variables, *p. 254*

Section 5.7

Graphing a System of Linear Inequalities, *p. 261*
Writing a System of Linear Inequalities, *p. 262*

Mathematical Thinking

1. Describe how to solve Exercise 39 on page 250 algebraically.
2. Why is it important to be precise when answering part (a) of Exercise 48 on page 258?
3. Describe the overall step-by-step process you used to solve Exercise 35 on page 265.

Performance Task

Prize Patrol

You have been selected to drive a prize patrol cart and place prizes on the competing teams' predetermined paths. You know the teams' routes and you can only make one pass. Where will you place the prizes so that each team will have a chance to find a prize on their route?

To explore the answer to this question and more, go to ***BigIdeasMath.com***.

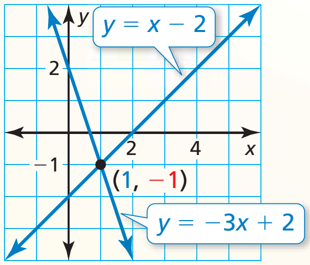
5 Chapter Review

5.1 Solving Systems of Linear Equations by Graphing (pp. 219–224)

Solve the system by graphing. $y = x - 2$ Equation 1
$y = -3x + 2$ Equation 2

Step 1 Graph each equation.

Step 2 Estimate the point of intersection.
The graphs appear to intersect at $(1, -1)$.



Step 3 Check your point from Step 2.

Equation 1	Equation 2
$y = x - 2$	$y = -3x + 2$
$-1 \stackrel{?}{=} 1 - 2$	$-1 \stackrel{?}{=} -3(1) + 2$
$-1 = -1$ ✓	$-1 = -1$ ✓

▶ The solution is $(1, -1)$.

Solve the system of linear equations by graphing.

1. $y = -3x + 1$
$y = x - 7$

2. $y = -4x + 3$
$4x - 2y = 6$

3. $5x + 5y = 15$
$2x - 2y = 10$

5.2 Solving Systems of Linear Equations by Substitution (pp. 225–230)

Solve the system by substitution. $-2x + y = -8$ Equation 1
$7x + y = 10$ Equation 2

Step 1 Solve for y in Equation 1.

$y = 2x - 8$ Revised Equation 1

Step 2 Substitute $2x - 8$ for y in Equation 2 and solve for x.

$7x + y = 10$	Equation 2
$7x + (2x - 8) = 10$	Substitute $2x - 8$ for y.
$9x - 8 = 10$	Combine like terms.
$9x = 18$	Add 8 to each side.
$x = 2$	Divide each side by 9.

Step 3 Substituting 2 for x in Equation 1 and solving for y gives $y = -4$.

▶ The solution is $(2, -4)$.

Solve the system of linear equations by substitution. Check your solution.

4. $3x + y = -9$
$y = 5x + 7$

5. $x + 4y = 6$
$x - y = 1$

6. $2x + 3y = 4$
$y + 3x = 6$

7. You spend \$20 total on tubes of paint and disposable brushes for an art project. Tubes of paint cost \$4.00 each and paintbrushes cost \$0.50 each. You purchase twice as many brushes as tubes of paint. How many brushes and tubes of paint do you purchase?

5.3 Solving Systems of Linear Equations by Elimination (pp. 231–236)

Solve the system by elimination. $\mathbf{4x + 6y = -8}$ Equation 1
$\mathbf{x - 2y = -2}$ Equation 2

Step 1 Multiply Equation 2 by 3 so that the coefficients of the y-terms are opposites.

$4x + 6y = -8$ → $4x + 6y = -8$ Equation 1

$x - 2y = -2$ Multiply by 3. → $3x - 6y = -6$ Revised Equation 2

Step 2 Add the equations.

$4x + 6y = -8$ Equation 1

$3x - 6y = -6$ Revised Equation 2

$7x = -14$ Add the equations.

Step 3 Solve for x.

$7x = -14$ Resulting equation from Step 2

$x = -2$ Divide each side by 7.

Step 4 Substitute -2 for x in one of the original equations and solve for y.

$4x + 6y = -8$ Equation 1

$4(-2) + 6y = -8$ Substitute -2 for x.

$-8 + 6y = -8$ Multiply.

$y = 0$ Solve for y.

▶ The solution is $(-2, 0)$.

Check

Equation 1

$4x + 6y = -8$

$4(-2) + 6(0) \stackrel{?}{=} -8$

$-8 = -8$ ✓

Equation 2

$x - 2y = -2$

$(-2) - 2(0) \stackrel{?}{=} -2$

$-2 = -2$ ✓

Solve the system of linear equations by elimination. Check your solution.

8. $9x - 2y = 34$
$5x + 2y = -6$

9. $x + 6y = 28$
$2x - 3y = -19$

10. $8x - 7y = -3$
$6x - 5y = -1$

5.4 Solving Special Systems of Linear Equations (pp. 237–242)

Solve the system. $\mathbf{4x + 2y = -14}$ Equation 1
$\mathbf{y = -2x - 6}$ Equation 2

Solve by substitution. Substitute $-2x - 6$ for y in Equation 1.

$4x + 2y = -14$ Equation 1

$4x + 2(-2x - 6) = -14$ Substitute $-2x - 6$ for y.

$4x - 4x - 12 = -14$ Distributive Property

$-12 = -14$ ✗ Combine like terms.

▶ The equation $-12 = -14$ is never true. So, the system has no solution.

Solve the system of linear equations.

11. $x = y + 2$
$-3x + 3y = 6$

12. $3x - 6y = -9$
$-5x + 10y = 10$

13. $-4x + 4y = 32$
$3x + 24 = 3y$

5.5 Solving Equations by Graphing (pp. 245–250)

Solve $3x - 1 = -2x + 4$ by graphing. Check your solution.

Step 1 Write a system of linear equations using each side of the original equation.

$$\underbrace{3x - 1}_{y = 3x - 1} = \underbrace{-2x + 4}_{y = -2x + 4}$$

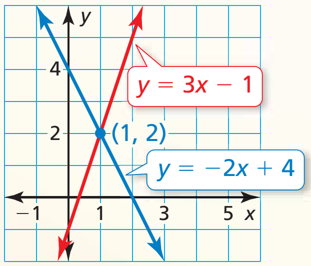
Step 2 Graph the system.

$y = 3x - 1$ Equation 1

$y = -2x + 4$ Equation 2

The graphs intersect at (1, 2).

▶ So, the solution of the equation is $x = 1$.



Check

$3x - 1 = -2x + 4$

$3(1) - 1 \stackrel{?}{=} -2(1) + 4$

$2 = 2$ ✓

Solve the equation by graphing. Check your solution(s).

14. $x + 1 = -x - 9$

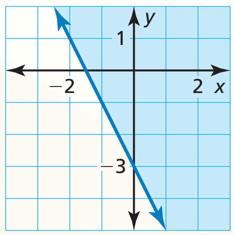
15. $2x - 8 = x + 5$

16. $\frac{1}{3}x + 5 = -2x - 2$

5.6 Linear Inequalities in Two Variables (pp. 251–258)

Graph $4x + 2y \geq -6$ in a coordinate plane.

Step 1 Graph $4x + 2y = -6$, or $y = -2x - 3$. Use a solid line because the inequality symbol is $\geq$.



Step 2 Test (0, 0).

$4x + 2y \geq -6$ Write the inequality.

$4(0) + 2(0) \stackrel{?}{\geq} -6$ Substitute.

$0 \geq -6$ ✓ Simplify.

Step 3 Because (0, 0) is a solution, shade the half-plane that contains (0, 0).

Graph the inequality in a coordinate plane.

17. $y > -4$

18. $-9x + 3y \geq 3$

19. $5x + 10y < 40$

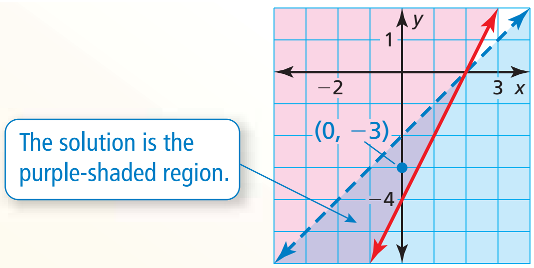
5.7 Systems of Linear Inequalities (pp. 259–266)

Graph the system. $y < x - 2$ Inequality 1

$y \geq 2x - 4$ Inequality 2

Step 1 Graph each inequality.

Step 2 Find the intersection of the half-planes. One solution is $(0, -3)$.



Graph the system of linear inequalities.

20. $y \leq x - 3$
$y \geq x + 1$

21. $y > -2x + 3$
$y \geq \frac{1}{4}x - 1$

22. $x + 3y > 6$
$2x + y < 7$

5 Chapter Test

Solve the system of linear equations using any method. Explain why you chose the method.

1. $8x + 3y = -9$
$-8x + y = 29$

2. $\frac{1}{2}x + y = -6$
$y = \frac{3}{5}x + 5$

3. $y = 4x + 4$
$-8x + 2y = 8$

4. $x = y - 11$
$x - 3y = 1$

5. $6x - 4y = 9$
$9x - 6y = 15$

6. $y = 5x - 7$
$-4x + y = -1$

7. Write a system of linear inequalities so the points (1, 2) and (4, -3) are solutions of the system, but the point (-2, 8) is not a solution of the system.

Graph the system of linear inequalities.

8. $y > -1$
$x < 3$

9. $y > \frac{1}{2}x + 4$
$2y \le x + 4$

10. $x + y < 1$
$5x + y > 4$

11. $y \ge -\frac{2}{3}x + 1$
$-3x + y > -2$

12. You pay $45.50 for 10 gallons of gasoline and 2 quarts of oil at a gas station. Your friend pays $22.75 for 5 gallons of the same gasoline and 1 quart of the same oil.

a. Is there enough information to determine the cost of 1 gallon of gasoline and 1 quart of oil? Explain.

b. The receipt shown is for buying the same gasoline and same oil. Is there now enough information to determine the cost of 1 gallon of gasoline and 1 quart of oil? Explain.

c. Determine the cost of 1 gallon of gasoline and 1 quart of oil.

13. Describe the advantages and disadvantages of solving a system of linear equations by graphing.

14. You have at most $60 to spend on trophies and medals to give as prizes for a contest.

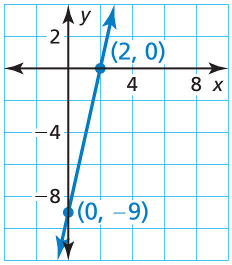
a. Write and graph an inequality that represents the numbers of trophies and medals you can buy. Identify and interpret a solution of the inequality.

b. You want to purchase at least 6 items. Write and graph a system that represents the situation. How many of each item can you buy?

15. Compare the slopes and y-intercepts of the graphs of the equations in the linear system $8x + 4y = 12$ and $3y = -6x - 15$ to determine whether the system has one solution, no solution, or infinitely many solutions. Explain.

5 Standards Assessment

1. The graph of which equation is shown? *(TEKS A.2.B)*



Ⓐ $9x - 2y = -18$

Ⓑ $-9x - 2y = 18$

Ⓒ $9x + 2y = 18$

Ⓓ $-9x + 2y = -18$

2. A van rental company rents out 6-, 8-, 12-, and 16-passenger vans. The function $C(x) = 100 + 5x$ represents the cost C (in dollars) of renting an x-passenger van for a day. Which of the following numbers is in the range of the function? *(TEKS A.2.A)*

Ⓕ 6

Ⓖ 100

Ⓗ 130

Ⓙ 150

3. **GRIDDED ANSWER** The students in the graduating classes at three high schools in a school district have to pay for their cap-and-gown sets and extra tassels. At one high school, students pay \$3262 for 215 cap-and-gown sets and 72 extra tassels. At another high school, students pay \$3346 for 221 cap-and-gown sets and 72 extra tassels. How much (in dollars) do students at the third high school pay for 218 cap-and-gown sets and 56 extra tassels? *(TEKS A.2.I, TEKS A.5.C)*

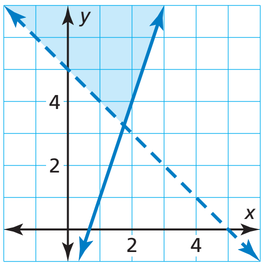
4. Which of the following points is *not* a solution of the system of linear inequalities represented by the graph? *(TEKS A.3.H)*

Ⓐ (4, 10)

Ⓑ (5, 8)

Ⓒ (−2, 8)

Ⓓ (0, 10)



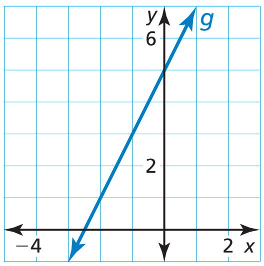
5. Consider the function $f(x) = 2x - 1$. Which of the following functions are represented by the graph? *(TEKS A.3.E)*

I. $g(x) = f(x) + 6$

II. $g(x) = f(x - 3)$

III. $g(x) = f(x + 3)$

IV. $g(x) = f(2x)$



Ⓕ I and II only

Ⓖ I and III only

Ⓗ I, II, and III only

Ⓙ I, II, III, and IV

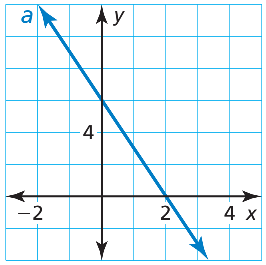
6. Line b is perpendicular to line a and passes through the point $(-9, 2)$. Which of the following is an equation for line b? *(TEKS A.2.F)*

Ⓐ $y = \frac{1}{3}x + 5$

Ⓑ $y = -\frac{1}{3}x - 1$

Ⓒ $y = \frac{1}{3}x - \frac{29}{3}$

Ⓓ $y = -\frac{1}{3}x + 1$



7. Which system of linear equations has no solution? *(TEKS A.5.C)*

Ⓕ $y - 2x = 5$
$4y - 10x = 20$

Ⓖ $y = -4x + 6$
$y = 8x - 12$

Ⓗ $2y = 6x + 4$
$y - \frac{1}{2} = 3x$

Ⓙ $y + 1 = 4x$
$y + 1 = \frac{1}{4}x$

8. A car dealership offers interest-free car loans for one day only. During this day, a salesperson at the dealership sells two cars. One of the clients decides to pay off a \$17,424 car in 36 monthly payments of \$484. The other client decides to pay off a \$15,840 car in 48 monthly payments of \$330. Which system of equations can you use to determine the number x of months after which both clients will have the same loan balance y (in dollars)? *(TEKS A.2.I)*

Ⓐ $y = -484x$
$y = -330x$

Ⓑ $y = -484x + 17{,}424$
$y = -330x + 15{,}840$

Ⓒ $y = -484x + 15{,}840$
$y = -330x + 17{,}424$

Ⓓ $y = 484x + 17{,}424$
$y = 330x + 15{,}840$

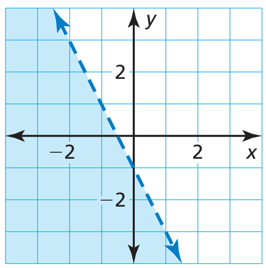
9. The graph of which linear inequality is shown? *(TEKS A.2.H)*

Ⓕ $y < -2x - 1$

Ⓖ $y \le -2x - 1$

Ⓗ $y > -2x - 1$

Ⓙ $y \ge -2x - 1$



10. Simplify the expression $4(2x + 4) - 6(x + 1)$. *(TEKS A.10.D)*

Ⓐ $2x + 10$

Ⓑ $2x + 15$

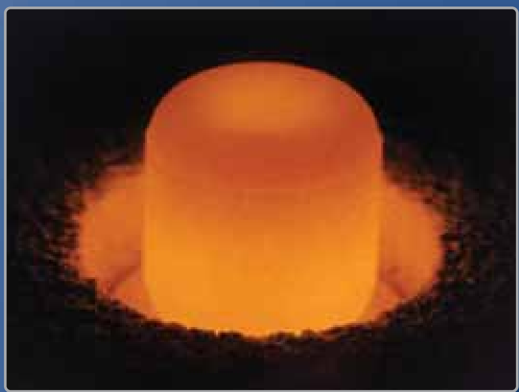
Ⓒ $2x + 17$

Ⓓ $2x + 22$

6 Exponential Functions and Sequences

Fibonacci and Flowers *(p. 323)*

Soup Kitchen *(p. 318)*



Plutonium Decay *(p. 307)*

Coyote Population *(p. 297)*

Jellyfish *(p. 281)*

Mathematical Thinking: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

Maintaining Mathematical Proficiency

Using Order of Operations (6.7.A)

Example 1 **Evaluate $10^2 \div (30 \div 3) - 4(3 - 9) + 5^1$.**

First:	Parentheses	$10^2 \div (30 \div 3) - 4(3 - 9) + 5^1 = 10^2 \div 10 - 4(-6) + 5^1$
Second:	Exponents	$= 100 \div 10 - 4(-6) + 5$
Third:	Multiplication and Division (from left to right)	$= 10 + 24 + 5$
Fourth:	Addition and Subtraction (from left to right)	$= 39$

Evaluate the expression.

1. $12\left(\frac{14}{2}\right) - 3^3 + 15 - 9^2$

2. $5^2 \cdot 8 \div 2^2 + 20 \cdot 3 - 4$

3. $-7 + 16 \div 2^4 + (10 - 4^2)$

Finding Square Roots (A.11.A)

Example 2 **Find $-\sqrt{81}$.**

▶ $-\sqrt{81}$ represents the negative square root. Because $9^2 = 81$, $-\sqrt{81} = -\sqrt{9^2} = -9$.

Find the square root(s).

4. $\sqrt{64}$

5. $-\sqrt{4}$

6. $-\sqrt{25}$

7. $\pm\sqrt{121}$

Writing Equations for Arithmetic Sequences (A.12.D)

Example 3 **Write an equation for the *n*th term of the arithmetic sequence 5, 15, 25, 35, ...**

The first term is 5, and the common difference is 10.

$a_n = a_1 + (n - 1)d$	Equation for an arithmetic sequence
$a_n = 5 + (n - 1)(10)$	Substitute 5 for a_1 and 10 for d.
$a_n = 10n - 5$	Simplify.

Write an equation for the *n*th term of the arithmetic sequence.

8. 12, 14, 16, 18, . . .

9. 6, 3, 0, −3, . . .

10. 22, 15, 8, 1, . . .

11. **ABSTRACT REASONING** Recall that a perfect square is a number with integers as its square roots. Is the product of two perfect squares always a perfect square? Is the quotient of two perfect squares always a perfect square? Explain your reasoning.

Mathematical Thinking

Mathematically proficient students use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution. **(A.1.B)**

Problem-Solving Strategies

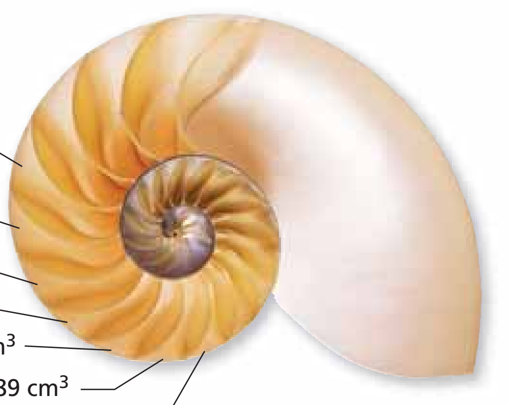
Core Concept

Finding a Pattern

When solving a real-life problem, look for a pattern in the data. The pattern could include repeating items, numbers, or events. After you find the pattern, describe it and use it to solve the problem.

EXAMPLE 1 Using a Problem-Solving Strategy

The volumes of seven chambers of a chambered nautilus are given. Find the volume of Chamber 10.



SOLUTION

To find a pattern, try dividing each volume by the volume of the previous chamber.

$$\frac{0.889}{0.836} \approx 1.063 \qquad \frac{0.945}{0.889} \approx 1.063 \qquad \frac{1.005}{0.945} \approx 1.063$$

$$\frac{1.068}{1.005} \approx 1.063 \qquad \frac{1.135}{1.068} \approx 1.063 \qquad \frac{1.207}{1.135} \approx 1.063$$

From this, you can see that the volume of each chamber is about 6.3% greater than the volume of the previous chamber. To find the volume of Chamber 10, multiply the volume of Chamber 7 by 1.063 three times.

$$\underbrace{1.207(1.063) \approx \mathbf{1.283}}_{\text{volume of Chamber 8}} \qquad \underbrace{1.283(1.063) \approx \mathbf{1.364}}_{\text{volume of Chamber 9}} \qquad \underbrace{1.364(1.063) \approx \mathbf{1.450}}_{\text{volume of Chamber 10}}$$

▶ The volume of Chamber 10 is about 1.450 cubic centimeters.

Monitoring Progress

1. A rabbit population over 8 consecutive years is given by 50, 80, 128, 205, 328, 524, 839, 1342. Find the population in the tenth year.
2. The sums of the numbers in the first eight rows of Pascal's Triangle are 1, 2, 4, 8, 16, 32, 64, 128. Find the sum of the numbers in the tenth row.

6.1 Properties of Exponents

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.11.B

Essential Question

How can you write general rules involving properties of exponents?

EXPLORATION 1 Writing Rules for Properties of Exponents

Work with a partner.

a. What happens when you multiply two powers with the same base? Write the product of the two powers as a single power. Then write a *general rule* for finding the product of two powers with the same base.

i. $(2^2)(2^3) =$ **ii.** $(4^1)(4^5) =$

iii. $(5^3)(5^5) =$ **iv.** $(x^2)(x^6) =$

b. What happens when you divide two powers with the same base? Write the quotient of the two powers as a single power. Then write a *general rule* for finding the quotient of two powers with the same base.

i. $\dfrac{4^3}{4^2} =$ **ii.** $\dfrac{2^5}{2^2} =$

iii. $\dfrac{x^6}{x^3} =$ **iv.** $\dfrac{3^4}{3^4} =$

c. What happens when you find a power of a power? Write the expression as a single power. Then write a *general rule* for finding a power of a power.

i. $(2^2)^4 =$ **ii.** $(7^3)^2 =$

iii. $(y^3)^3 =$ **iv.** $(x^4)^2 =$

d. What happens when you find a power of a product? Write the expression as the product of two powers. Then write a *general rule* for finding a power of a product.

i. $(2 \cdot 5)^2 =$ **ii.** $(5 \cdot 4)^3 =$

iii. $(6a)^2 =$ **iv.** $(3x)^2 =$

e. What happens when you find a power of a quotient? Write the expression as the quotient of two powers. Then write a *general rule* for finding a power of a quotient.

i. $\left(\dfrac{2}{3}\right)^2 =$ **ii.** $\left(\dfrac{4}{3}\right)^3 =$

iii. $\left(\dfrac{x}{2}\right)^3 =$ **iv.** $\left(\dfrac{a}{b}\right)^4 =$

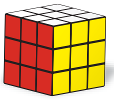
ANALYZING MATHEMATICAL RELATIONSHIPS

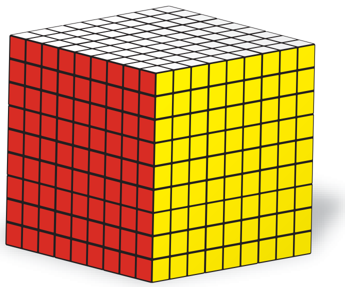
To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results in writing general rules.

Communicate Your Answer

2. How can you write general rules involving properties of exponents?

3. There are 3^3 small cubes in the cube below. Write an expression for the number of small cubes in the large cube at the right.





6.1 Lesson

Core Vocabulary

Previous
power
exponent
base
scientific notation

What You Will Learn

- Use zero and negative exponents.
- Use the properties of exponents.
- Solve real-life problems involving exponents.

Using Zero and Negative Exponents

Core Concept

Zero Exponent

Words For any nonzero number a, $a^0 = 1$. The power 0^0 is undefined.

Numbers $4^0 = 1$ **Algebra** $a^0 = 1$, where $a \neq 0$

Negative Exponents

Words For any integer n and any nonzero number a, a^{-n} is the reciprocal of a^n.

Numbers $4^{-2} = \dfrac{1}{4^2}$ **Algebra** $a^{-n} = \dfrac{1}{a^n}$, where $a \neq 0$

EXAMPLE 1 Using Zero and Negative Exponents

Evaluate each expression.

a. 6.7^0 **b.** $(-2)^{-4}$

SOLUTION

a. $6.7^0 = 1$ Definition of zero exponent

b. $(-2)^{-4} = \dfrac{1}{(-2)^4}$ Definition of negative exponent

$= \dfrac{1}{16}$ Simplify.

EXAMPLE 2 Simplifying an Expression

Simplify the expression $\dfrac{4x^0}{y^{-3}}$. Write your answer using only positive exponents.

SOLUTION

$\dfrac{4x^0}{y^{-3}} = 4x^0y^3$ Definition of negative exponent

$= 4y^3$ Definition of zero exponent

Monitoring Progress

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Evaluate the expression.

1. $(-9)^0$ **2.** 3^{-3} **3.** $\dfrac{-5^0}{2^{-2}}$

4. Simplify the expression $\dfrac{3^{-2}x^{-5}}{y^0}$. Write your answer using only positive exponents.

Using the Properties of Exponents

REMEMBER

The expression x^3 is called a *power*. The *base*, x, is used as a factor 3 times because the *exponent* is 3.

Core Concept

Product of Powers Property

Let a be a real number, and let m and n be integers.

Words To multiply powers with the same base, add their exponents.

Numbers $4^6 \cdot 4^3 = 4^{6+3} = 4^9$ **Algebra** $a^m \cdot a^n = a^{m+n}$

Quotient of Powers Property

Let a be a nonzero real number, and let m and n be integers.

Words To divide powers with the same base, subtract their exponents.

Numbers $\dfrac{4^6}{4^3} = 4^{6-3} = 4^3$ **Algebra** $\dfrac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$

Power of a Power Property

Let a be a real number, and let m and n be integers.

Words To find a power of a power, multiply the exponents.

Numbers $(4^6)^3 = 4^{6 \cdot 3} = 4^{18}$ **Algebra** $(a^m)^n = a^{mn}$

EXAMPLE 3 Using Properties of Exponents

Simplify each expression. Write your answer using only positive exponents.

a. $3^2 \cdot 3^6$ **b.** $\dfrac{(-4)^2}{(-4)^7}$ **c.** $(z^4)^{-3}$

SOLUTION

a. $3^2 \cdot 3^6 = 3^{2+6}$ Product of Powers Property

$= 3^8 = 6561$ Simplify.

b. $\dfrac{(-4)^2}{(-4)^7} = (-4)^{2-7}$ Quotient of Powers Property

$= (-4)^{-5}$ Simplify.

$= \dfrac{1}{(-4)^5} = -\dfrac{1}{1024}$ Definition of negative exponent

c. $(z^4)^{-3} = z^{4 \cdot (-3)}$ Power of a Power Property

$= z^{-12}$ Simplify.

$= \dfrac{1}{z^{12}}$ Definition of negative exponent

Monitoring Progress

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Simplify the expression. Write your answer using only positive exponents.

5. $10^4 \cdot 10^{-6}$ **6.** $x^9 \cdot x^{-9}$ **7.** $\dfrac{-5^8}{-5^4}$

8. $\dfrac{y^6}{y^7}$ **9.** $(6^{-2})^{-1}$ **10.** $(w^{12})^5$

Core Concept

Power of a Product Property

Let a and b be real numbers, and let m be an integer.

Words To find a power of a product, find the power of each factor and multiply.

Numbers $(3 \cdot 2)^5 = 3^5 \cdot 2^5$ **Algebra** $(ab)^m = a^m b^m$

Power of a Quotient Property

Let a and b be real numbers with $b \neq 0$, and let m be an integer.

Words To find the power of a quotient, find the power of the numerator and the power of the denominator and divide.

Numbers $\left(\frac{3}{2}\right)^5 = \frac{3^5}{2^5}$ **Algebra** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, where $b \neq 0$

EXAMPLE 4 Using Properties of Exponents

Simplify each expression. Write your answer using only positive exponents.

a. $(-1.5y)^2$ **b.** $\left(\frac{a}{-10}\right)^3$ **c.** $\left(\frac{3d}{2}\right)^4$ **d.** $\left(\frac{2x}{3}\right)^{-5}$

SOLUTION

a. $(-1.5y)^2 = (-1.5)^2 \cdot y^2$ Power of a Product Property

$= 2.25y^2$ Simplify.

b. $\left(\frac{a}{-10}\right)^3 = \frac{a^3}{(-10)^3}$ Power of a Quotient Property

$= -\frac{a^3}{1000}$ Simplify.

c. $\left(\frac{3d}{2}\right)^4 = \frac{(3d)^4}{2^4}$ Power of a Quotient Property

$= \frac{3^4 d^4}{2^4}$ Power of a Product Property

$= \frac{81d^4}{16}$ Simplify.

d. $\left(\frac{2x}{3}\right)^{-5} = \frac{(2x)^{-5}}{3^{-5}}$ Power of a Quotient Property

$= \frac{3^5}{(2x)^5}$ Definition of negative exponent

$= \frac{3^5}{2^5 x^5}$ Power of a Product Property

$= \frac{243}{32x^5}$ Simplify.

ANOTHER WAY

Because the exponent is negative, you could find the reciprocal of the base first. Then simplify.

$$\left(\frac{2x}{3}\right)^{-5} = \left(\frac{3}{2x}\right)^5 = \frac{243}{32x^5}$$

Monitoring Progress

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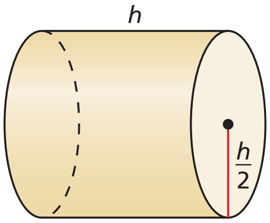
Simplify the expression. Write your answer using only positive exponents.

11. $(10y)^{-3}$ **12.** $\left(-\frac{4}{n}\right)^5$ **13.** $\left(\frac{1}{2k^2}\right)^5$ **14.** $\left(\frac{6c}{7}\right)^{-2}$

Solving Real-Life Problems



Volume = ?

$2\pi r^3$	$\pi h^3 2^{-2}$	$\pi h 4^{-1}$
$\dfrac{\pi h^2}{4}$	$\dfrac{\pi h^3}{4}$	$\dfrac{\pi h^3}{2}$

EXAMPLE 5 Simplifying a Real-Life Expression

Which of the expressions shown represent the volume of the cylinder, where r is the radius and h is the height?

SOLUTION

$$V = \pi r^2 h \qquad \text{Formula for the volume of a cylinder}$$

$$= \pi\left(\frac{h}{2}\right)^2(h) \qquad \text{Substitute } \frac{h}{2} \text{ for } r.$$

$$= \pi\left(\frac{h^2}{2^2}\right)(h) \qquad \text{Power of a Quotient Property}$$

$$= \frac{\pi h^3}{4} \qquad \text{Simplify.}$$

Any expression equivalent to $\dfrac{\pi h^3}{4}$ represents the volume of the cylinder.

- You can use the properties of exponents to write $\pi h^3 2^{-2}$ as $\dfrac{\pi h^3}{4}$.
- Note $h = 2r$. When you substitute $2r$ for h in $\dfrac{\pi h^3}{4}$, you can write $\dfrac{\pi(2r)^3}{4}$ as $2\pi r^3$.
- None of the other expressions are equivalent to $\dfrac{\pi h^3}{4}$.

▶ The expressions $2\pi r^3$, $\pi h^3 2^{-2}$, and $\dfrac{\pi h^3}{4}$ represent the volume of the cylinder.

REMEMBER

A number is written in scientific notation when it is of the form $a \times 10^b$, where $1 \le a < 10$ and b is an integer.

EXAMPLE 6 Solving a Real-Life Problem

A jellyfish emits about 1.25×10^8 particles of light, or photons, in 6.25×10^{-4} second. How many photons does the jellyfish emit each second? Write your answer in scientific notation and in standard form.

SOLUTION

Divide to find the unit rate.

$$\frac{1.25 \times 10^8 \leftarrow \text{photons}}{6.25 \times 10^{-4} \leftarrow \text{seconds}} \qquad \text{Write the rate.}$$

$$= \frac{1.25}{6.25} \times \frac{10^8}{10^{-4}} \qquad \text{Rewrite.}$$

$$= 0.2 \times 10^{12} \qquad \text{Simplify.}$$

$$= 2 \times 10^{11} \qquad \text{Write in scientific notation.}$$

▶ The jellyfish emits 2×10^{11}, or 200,000,000,000 photons per second.

Monitoring Progress

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15. Write two expressions that represent the area of a base of the cylinder in Example 5.

16. It takes the Sun about 2.3×10^8 years to orbit the center of the Milky Way. It takes Pluto about 2.5×10^2 years to orbit the Sun. How many times does Pluto orbit the Sun while the Sun completes one orbit around the center of the Milky Way? Write your answer in scientific notation.

6.1 Exercises

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Vocabulary and Core Concept Check

1. **VOCABULARY** Which definitions or properties would you use to simplify the expression $(4^8 \cdot 4^{-4})^{-2}$? Explain.

2. **WRITING** Explain when and how to use the Power of a Product Property.

3. **WRITING** Explain when and how to use the Quotient of Powers Property.

4. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

Simplify $3^3 \cdot 3^6$.	Simplify 3^{3+6}.	Simplify $3^{6 \cdot 3}$.	Simplify $3^6 \cdot 3^3$.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, evaluate the expression. *(See Example 1.)*

5. $(-7)^0$

6. 4^0

7. 5^{-4}

8. $(-2)^{-5}$

9. $\dfrac{2^{-4}}{4^0}$

10. $\dfrac{5^{-1}}{-9^0}$

11. $\dfrac{-3^{-3}}{6^{-2}}$

12. $\dfrac{(-8)^{-2}}{3^{-4}}$

In Exercises 13–22, simplify the expression. Write your answer using only positive exponents. *(See Example 2.)*

13. x^{-7}

14. y^0

15. $9x^0y^{-3}$

16. $15c^{-8}d^0$

17. $\dfrac{2^{-2}m^{-3}}{n^0}$

18. $\dfrac{10^0r^{-11}s}{3^2}$

19. $\dfrac{4^{-3}a^0}{b^{-7}}$

20. $\dfrac{p^{-8}}{7^{-2}q^{-9}}$

21. $\dfrac{2^2y^{-6}}{8^{-1}z^0x^{-7}}$

22. $\dfrac{13x^{-5}y^0}{5^{-3}z^{-10}}$

In Exercises 23–32, simplify the expression. Write your answer using only positive exponents. *(See Example 3.)*

23. $\dfrac{5^6}{5^2}$

24. $\dfrac{(-6)^8}{(-6)^5}$

25. $(-9)^2 \cdot (-9)^2$

26. $4^{-5} \cdot 4^5$

27. $(p^6)^4$

28. $(s^{-5})^3$

29. $6^{-8} \cdot 6^5$

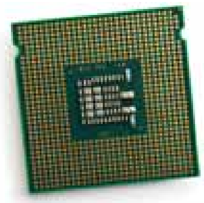
30. $-7 \cdot (-7)^{-4}$

31. $\dfrac{x^5}{x^4} \cdot x$

32. $\dfrac{z^8 \cdot z^2}{z^5}$

33. **USING PROPERTIES** A microscope magnifies an object 10^5 times. The length of an object is 10^{-7} meter. What is its magnified length?

34. **USING PROPERTIES** The area of the rectangular computer chip is $112a^3b^2$ square microns. What is the length?



width $= 8ab$ microns

ERROR ANALYSIS In Exercises 35 and 36, describe and correct the error in simplifying the expression.

35. ✗
$$2^4 \cdot 2^5 = (2 \cdot 2)^{4+5} = 4^9$$

36. ✗
$$\frac{x^5 \cdot x^3}{x^4} = \frac{x^8}{x^4} = x^{8/4} = x^2$$

In Exercises 37–44, simplify the expression. Write your answer using only positive exponents. *(See Example 4.)*

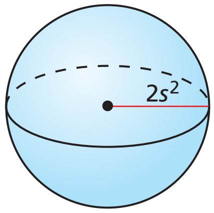
37. $(-5z)^3$ **38.** $(4x)^{-4}$

39. $\left(\frac{6}{n}\right)^{-2}$ **40.** $\left(\frac{-t}{3}\right)^2$

41. $(3s^8)^{-5}$ **42.** $(-5p^3)^3$

43. $\left(-\frac{w^3}{6}\right)^{-2}$ **44.** $\left(\frac{1}{2r^6}\right)^{-6}$

45. USING PROPERTIES Which of the expressions represent the volume of the sphere? Explain. *(See Example 5.)*



Ⓐ $\left(\frac{3s^2}{2^4\pi s^8}\right)^{-1}$ Ⓑ $(2^5\pi s^6)(3^{-1})$

Ⓒ $\frac{32\pi s^6}{3}$ Ⓓ $(2s)^5 \cdot \frac{\pi s}{3}$

Ⓔ $\left(\frac{3\pi s^6}{32}\right)^{-1}$ Ⓕ $\frac{32}{3}\pi s^5$

46. MODELING WITH MATHEMATICS Diffusion is the movement of molecules from one location to another. The time t (in seconds) it takes molecules to diffuse a distance of x centimeters is given by $t = \frac{x^2}{2D}$, where D is the diffusion coefficient. The diffusion coefficient for a drop of ink in water is about 10^{-5} square centimeters per second. How long will it take the ink to diffuse 1 micrometer (10^{-4} centimeter)?

In Exercises 47–50, simplify the expression. Write your answer using only positive exponents.

47. $\left(\frac{2x^{-2}y^3}{3xy^{-4}}\right)^4$ **48.** $\left(\frac{4s^5t^{-7}}{-2s^{-2}t^4}\right)^3$

49. $\left(\frac{3m^{-5}n^2}{4m^{-2}n^0}\right)^2 \cdot \left(\frac{mn^4}{9n}\right)^2$ **50.** $\left(\frac{3x^3y^0}{x^{-2}}\right)^4 \cdot \left(\frac{y^2x^{-4}}{5xy^{-8}}\right)^3$

In Exercises 51–54, evaluate the expression. Write your answer in scientific notation and standard form.

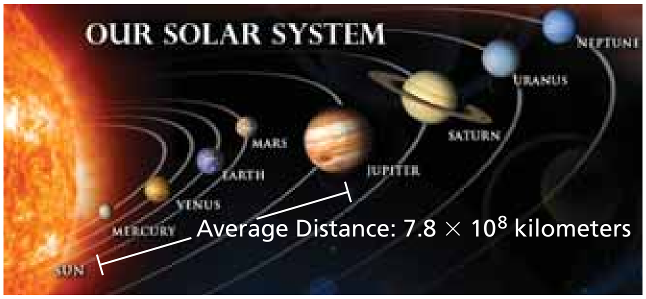
51. $(3 \times 10^2)(1.5 \times 10^{-5})$

52. $(6.1 \times 10^{-3})(8 \times 10^9)$

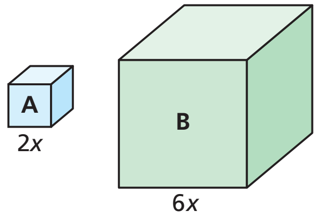
53. $\frac{(6.4 \times 10^7)}{(1.6 \times 10^5)}$ **54.** $\frac{(3.9 \times 10^{-5})}{(7.8 \times 10^{-8})}$

55. PROBLEM SOLVING In 2012, on average, about 9.46×10^{-1} pound of potatoes was produced for every 2.3×10^{-5} acre harvested. How many pounds of potatoes on average were produced for each acre harvested? Write your answer in scientific notation and in standard form. *(See Example 6.)*

56. PROBLEM SOLVING The speed of light is approximately 3×10^5 kilometers per second. How long does it take sunlight to reach Jupiter? Write your answer in scientific notation and in standard from.



57. MATHEMATICAL CONNECTIONS Consider Cube A and Cube B.



a. Which property of exponents should you use to simplify an expression for the volume of each cube?

b. How can you use the Power of a Quotient Property to find how many times greater the volume of Cube B is than the volume of Cube A?

58. PROBLEM SOLVING A byte is a unit used to measure a computer's memory. The table shows the numbers of bytes in several units of measure.

Unit	kilobyte	megabyte	gigabyte	terabyte
Number of bytes	2^{10}	2^{20}	2^{30}	2^{40}

a. How many kilobytes are in 1 terabyte?

b. How many megabytes are in 16 gigabytes?

c. Another unit used to measure a computer's memory is a bit. There are 8 bits in a byte. How can you convert the number of bytes in each unit of measure given in the table to bits? Can you still use a base of 2? Explain.

REWRITING EXPRESSIONS **In Exercises 59–62, rewrite the expression as a power of a product.**

59. $8a^3b^3$

60. $16r^2s^2$

61. $64w^{18}z^{12}$

62. $81x^4y^8$

63. **USING STRUCTURE** The probability of rolling a 6 on a number cube is $\frac{1}{6}$. The probability of rolling a 6 twice in a row is $\left(\frac{1}{6}\right)^2 = \frac{1}{36}$.

a. Write an expression that represents the probability of rolling a 6 n times in a row.

b. What is the probability of rolling a 6 four times in a row?

c. What is the probability of flipping heads on a coin five times in a row? Explain.

64. **HOW DO YOU SEE IT?** The shaded part of Figure n represents the portion of a piece of paper visible after folding the paper in half n times.

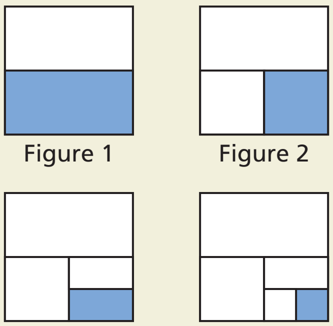


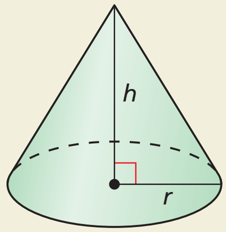
Figure 1 Figure 2 Figure 3 Figure 4

a. What fraction of the original piece of paper is each shaded part?

b. Rewrite each fraction from part (a) in the form 2^x.

65. **REASONING** Find x and y when $\frac{b^x}{b^y} = b^9$ and $\frac{b^x \cdot b^2}{b^{3y}} = b^{13}$. Explain how you found your answer.

66. **THOUGHT PROVOKING** Write expressions for r and h so that the volume of the cone can be represented by the expression $27\pi x^8$. Find r and h.



67. **MAKING AN ARGUMENT** One of the smallest plant seeds comes from an orchid, and one of the largest plant seeds comes from a double coconut palm. A seed from an orchid has a mass of 10^{-6} gram. The mass of a seed from a double coconut palm is 10^{10} times the mass of the seed from the orchid. Your friend says that the seed from the double coconut palm has a mass of about 1 kilogram. Is your friend correct? Explain.

68. **CRITICAL THINKING** Your school is conducting a survey. Students can answer the questions in either part with "agree" or "disagree."

Part 1: 13 questions		
Part 2: 10 questions		
Part 1: Classroom	Agree	Disagree
1. I come prepared for class.	○	○
2. I enjoy my assignments.	○	○

a. What power of 2 represents the number of different ways that a student can answer all the questions in Part 1?

b. What power of 2 represents the number of different ways that a student can answer all the questions on the entire survey?

c. The survey changes, and students can now answer "agree," "disagree," or "no opinion." How does this affect your answers in parts (a) and (b)?

69. **ABSTRACT REASONING** Compare the values of a^n and a^{-n} when $n < 0$, when $n = 0$, and when $n > 0$ for (a) $a > 1$ and (b) $0 < a < 1$. Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the square root(s). *(Skills Review Handbook)*

70. $\sqrt{25}$

71. $-\sqrt{100}$

72. $\pm\sqrt{\frac{1}{64}}$

Classify the real number in as many ways as possible. *(Skills Review Handbook)*

73. 12

74. $\frac{65}{9}$

75. $\frac{\pi}{4}$

6.2 Radicals and Rational Exponents

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.11.B

Essential Question

How can you write and evaluate an nth root of a number?

Recall that you cube a number as follows.

3rd power

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$ 2 cubed is 8.

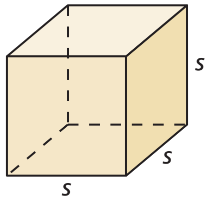
To "undo" cubing a number, take the cube root of the number.

Symbol for cube root is $\sqrt[3]{}$.

$$\sqrt[3]{8} = \sqrt[3]{2^3} = 2$$ The cube root of 8 is 2.

EXPLORATION 1 Finding Cube Roots

Work with a partner. Use a cube root symbol to write the side length of each cube. Then find the cube root. Check your answers by multiplying. Which cube is the largest? Which two cubes are the same size? Explain your reasoning.



a. Volume $= 27$ ft^3 **b.** Volume $= 125$ cm^3 **c.** Volume $= 3375$ in.^{3}

d. Volume $= 3.375$ m^3 **e.** Volume $= 1$ yd^3 **f.** Volume $= \frac{125}{8}$ mm^3

JUSTIFYING THE SOLUTION

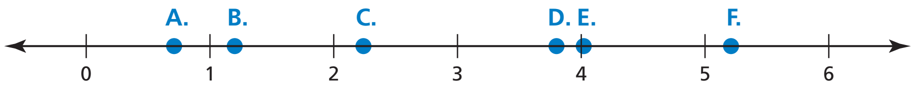
To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 2 Estimating nth Roots

Work with a partner. Estimate each positive nth root. Then match each nth root with the point on the number line. Justify your answers.

a. $\sqrt[4]{25}$ **b.** $\sqrt{0.5}$ **c.** $\sqrt[5]{2.5}$

d. $\sqrt[3]{65}$ **e.** $\sqrt[3]{55}$ **f.** $\sqrt[6]{20{,}000}$



Communicate Your Answer

3. How can you write and evaluate an nth root of a number?

4. The body mass m (in kilograms) of a dinosaur that walked on two feet can be modeled by

$$m = (0.00016)C^{2.73}$$

where C is the circumference (in millimeters) of the dinosaur's femur. The mass of a *Tyrannosaurus rex* was 4000 kilograms. Use a calculator to approximate the circumference of its femur.

6.2 Lesson

Core Vocabulary

*n*th root of *a*, *p. 286*
radical, *p. 286*
index of a radical, *p. 286*

Previous
square root

What You Will Learn

- Find *n*th roots.
- Evaluate expressions with rational exponents.
- Solve real-life problems involving rational exponents.

Finding *n*th Roots

You can extend the concept of a square root to other types of roots. For example, 2 is a cube root of 8 because $2^3 = 8$, and 3 is a fourth root of 81 because $3^4 = 81$. In general, for an integer n greater than 1, if $b^n = a$, then b is an ***n*th root of *a***. An nth root of a is written as $\sqrt[n]{a}$, where the expression $\sqrt[n]{a}$ is called a **radical** and n is the **index** of the radical.

You can also write an nth root of a as a power of a. If you assume the Power of a Power Property applies to rational exponents, then the following is true.

$$(a^{1/2})^2 = a^{(1/2) \cdot 2} = a^1 = a$$
$$(a^{1/3})^3 = a^{(1/3) \cdot 3} = a^1 = a$$
$$(a^{1/4})^4 = a^{(1/4) \cdot 4} = a^1 = a$$

Because $a^{1/2}$ is a number whose square is a, you can write $\sqrt{a} = a^{1/2}$. Similarly, $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[4]{a} = a^{1/4}$. In general, $\sqrt[n]{a} = a^{1/n}$ for any integer n greater than 1.

READING

$\pm\sqrt[n]{a}$ represents both the positive and negative nth roots of a.

Core Concept

Real *n*th Roots of *a*

Let n be an integer greater than 1, and let a be a real number.

- If n is odd, then a has one real nth root: $\sqrt[n]{a} = a^{1/n}$
- If n is even and $a > 0$, then a has two real nth roots: $\pm\sqrt[n]{a} = \pm a^{1/n}$
- If n is even and $a = 0$, then a has one real nth root: $\sqrt[n]{0} = 0$
- If n is even and $a < 0$, then a has no real nth roots.

The nth roots of a number may be real numbers or *imaginary numbers*. You will study imaginary numbers in a future course.

EXAMPLE 1 Finding *n*th Roots

Find the indicated real nth root(s) of a.

a. $n = 3, a = -27$ **b.** $n = 4, a = 16$

SOLUTION

a. The index $n = 3$ is odd, so -27 has one real cube root. Because $(-3)^3 = -27$, the cube root of -27 is $\sqrt[3]{-27} = -3$, or $(-27)^{1/3} = -3$.

b. The index $n = 4$ is even, and $a > 0$. So, 16 has two real fourth roots. Because $2^4 = 16$ and $(-2)^4 = 16$, the fourth roots of 16 are $\pm\sqrt[4]{16} = \pm 2$, or $\pm 16^{1/4} = \pm 2$.

Monitoring Progress

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Find the indicated real *n*th root(s) of *a*.

1. $n = 3, a = -125$ **2.** $n = 6, a = 64$

Evaluating Expressions with Rational Exponents

Recall that the radical $\sqrt{a}$ indicates the positive square root of a. Similarly, an nth root of a, $\sqrt[n]{a}$, with an *even* index indicates the positive nth root of a.

REMEMBER

The expression under the radical sign is the radicand.

EXAMPLE 2 **Evaluating *n*th Root Expressions**

Evaluate each expression.

a. $\sqrt[3]{-8}$ **b.** $-\sqrt[3]{8}$ **c.** $16^{1/4}$ **d.** $(-16)^{1/4}$

SOLUTION

a. $\sqrt[3]{-8} = \sqrt[3]{(-2) \cdot (-2) \cdot (-2)}$ Rewrite the expression showing factors.

$= -2$ Evaluate the cube root.

b. $-\sqrt[3]{8} = -\left(\sqrt[3]{2 \cdot 2 \cdot 2}\right)$ Rewrite the expression showing factors.

$= -(2)$ Evaluate the cube root.

$= -2$ Simplify.

c. $16^{1/4} = \sqrt[4]{16}$ Rewrite the expression in radical form.

$= \sqrt[4]{2 \cdot 2 \cdot 2 \cdot 2}$ Rewrite the expression showing factors.

$= 2$ Evaluate the fourth root.

d. $(-16)^{1/4}$ is not a real number because there is no real number that can be multiplied by itself four times to produce -16.

A rational exponent does not have to be of the form $1/n$. Other rational numbers such as $3/2$ can also be used as exponents. You can use the properties of exponents to evaluate or simplify expressions involving rational exponents.

STUDY TIP

You can rewrite $27^{2/3}$ as $27^{(1/3) \cdot 2}$ and then use the Power of a Power Property to show that

$27^{(1/3) \cdot 2} = (27^{1/3})^2$.

Core Concept

Rational Exponents

Let $a^{1/n}$ be an nth root of a, and let m be a positive integer.

Algebra $a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m$

Numbers $27^{2/3} = (27^{1/3})^2 = \left(\sqrt[3]{27}\right)^2$

EXAMPLE 3 **Evaluating Expressions with Rational Exponents**

Evaluate (a) $16^{3/4}$ and (b) $27^{4/3}$.

SOLUTION

a. $16^{3/4} = (16^{1/4})^3$ Rational exponents

$= 2^3$ Evaluate the nth root.

$= 8$ Evaluate the power.

b. $27^{4/3} = (27^{1/3})^4$

$= 3^4$

$= 81$

Monitoring Progress

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Evaluate the expression.

3. $\sqrt[3]{-125}$ **4.** $(-64)^{2/3}$ **5.** $9^{5/2}$ **6.** $256^{3/4}$

EXAMPLE 4 Simplifying Expressions with Rational Exponents

a. $27^{4/3} \cdot 27^{1/3} = 27^{4/3 + 1/3}$ — Product of Powers Property

$= 27^{5/3}$ — Simplify.

$= (27^{1/3})^5$ — Definition of rational exponent

$= 3^5$ — Evaluate the *n*th root.

$= 243$ — Evaluate the power.

b. $x^{1/2} \div x^{3/5} = x^{1/2 - 3/5}$ — Quotient of Powers Property

$= x^{5/10 - 6/10}$ — Rewrite fractions with common denominator.

$= x^{-1/10}$ — Simplify.

$= \dfrac{1}{x^{1/10}}$ — Definition of negative exponent

Solving Real-Life Problems

EXAMPLE 5 Solving a Real-Life Problem

Volume = 113 cubic feet

The radius r of a sphere is given by the equation $r = \left(\frac{3V}{4\pi}\right)^{1/3}$, where V is the volume of the sphere. Find the radius of the beach ball to the nearest foot. Use 3.14 for π.

SOLUTION

1. **Understand the Problem** You know the equation that represents the radius of a sphere in terms of its volume. You are asked to find the radius for a given volume.
2. **Make a Plan** Substitute the given volume into the equation. Then evaluate to find the radius.
3. **Solve the Problem**

$$r = \left(\frac{3V}{4\pi}\right)^{1/3}$$ Write the equation.

$$= \left(\frac{3(113)}{4(3.14)}\right)^{1/3}$$ Substitute 113 for V and 3.14 for π.

$$= \left(\frac{339}{12.56}\right)^{1/3}$$ Multiply.

$$\approx 3$$ Use a calculator.

▶ The radius of the beach ball is about 3 feet.

4. **Look Back** To check that your answer is reasonable, compare the size of the ball to the size of the woman pushing the ball. The ball appears to be slightly taller than the woman. The average height of a woman is between 5 and 6 feet. So, a radius of 3 feet, or height of 6 feet, seems reasonable for the beach ball.

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Simplify the expression. Write your answer using only positive exponents.

7. $12^{5/2} \div 12^{1/2}$ **8.** $16^{-3/4} \cdot 16^{1/4}$ **9.** $2y^{2/3} \cdot y^{5/6}$

10. **WHAT IF?** In Example 5, the volume of the beach ball is 17,000 cubic inches. Find the radius to the nearest inch. Use 3.14 for π.

6.2 Exercises

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Vocabulary and Core Concept Check

1. **WRITING** Explain how to evaluate $81^{1/4}$.

2. **WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$(\sqrt[3]{27})^2$	$27^{2/3}$	3^2	$(\sqrt[2]{27})^3$

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, rewrite the expression in rational exponent form.

3. $\sqrt{10}$

4. $\sqrt[5]{34}$

In Exercises 5 and 6, rewrite the expression in radical form.

5. $15^{1/3}$

6. $140^{1/8}$

In Exercises 7–10, find the indicated real *n*th root(s) of *a*. *(See Example 1.)*

7. $n = 2, a = 36$

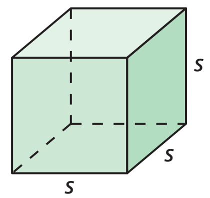
8. $n = 4, a = 81$

9. $n = 3, a = 1000$

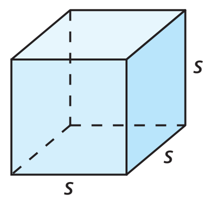
10. $n = 9, a = -512$

MATHEMATICAL CONNECTIONS **In Exercises 11 and 12, find the dimensions of the cube. Check your answer.**

11. Volume $= 64 \text{ in.}^3$



12. Volume $= 216 \text{ cm}^3$



In Exercises 13–18, evaluate the expression. *(See Example 2.)*

13. $\sqrt[4]{256}$

14. $\sqrt[3]{-216}$

15. $\sqrt[3]{-343}$

16. $-\sqrt[5]{1024}$

17. $128^{1/7}$

18. $(-64)^{1/2}$

In Exercises 19 and 20, rewrite the expression in rational exponent form.

19. $(\sqrt[5]{8})^4$

20. $(\sqrt[5]{-21})^6$

In Exercises 21 and 22, rewrite the expression in radical form.

21. $(-4)^{2/7}$

22. $9^{5/2}$

In Exercises 23–28, evaluate the expression. *(See Example 3.)*

23. $32^{3/5}$

24. $125^{2/3}$

25. $(-36)^{3/2}$

26. $(-243)^{2/5}$

27. $(-128)^{5/7}$

28. $343^{4/3}$

29. **ERROR ANALYSIS** Describe and correct the error in rewriting the expression in rational exponent form.

$$(\sqrt[3]{2})^4 = 2^{3/4}$$

30. **ERROR ANALYSIS** Describe and correct the error in evaluating the expression.

$$\begin{aligned}(-81)^{3/4} &= [(-81)^{1/4}]^3 \\ &= (-3)^3 \\ &= -27\end{aligned}$$

In Exercises 31–34, evaluate the expression.

31. $\left(\frac{1}{1000}\right)^{1/3}$

32. $\left(\frac{1}{64}\right)^{1/6}$

33. $(27)^{-2/3}$

34. $(9)^{-5/2}$

35. **PROBLEM SOLVING** A math club is having a bake sale. Find the area of the bake sale sign.

36. **PROBLEM SOLVING** The volume of a cube-shaped box is 27^5 cubic millimeters. Find the length of one side of the box.

In Exercises 37–44, simplify the expression. Write your answer using only positive exponents. *(See Example 4.)*

37. $5^{-3/2} \cdot 5^{7/2}$

38. $2^{5/3} \div 2^{-4/3}$

39. $125^3 \div 125^{8/3}$

40. $\left(\frac{1}{16}\right)^{1/2} \cdot \left(\frac{1}{16}\right)^{3/4}$

41. $d^{3/4} \div d^{1/6}$

42. $z^{-7/5} \cdot z^{2/3}$

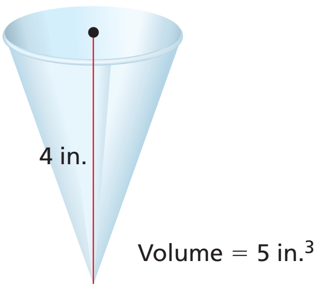
43. $4n^{1/8} \cdot n^{3/2}$

44. $a^{-3/4} \div 7a^{-1/12}$

45. **MODELING WITH MATHEMATICS** The radius r of the base of a cone is given by the equation

$$r = \left(\frac{3V}{\pi h}\right)^{1/2}$$

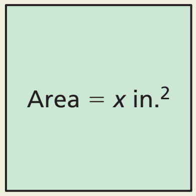
where V is the volume of the cone and h is the height of the cone. Find the radius of the paper cup to the nearest inch. Use 3.14 for π. *(See Example 5.)*



46. **MODELING WITH MATHEMATICS** The volume of a sphere is given by the equation $V = \frac{1}{6\sqrt{\pi}} S^{3/2}$, where S is the surface area of the sphere. Find the volume of a sphere, to the nearest cubic meter, that has a surface area of 60 square meters. Use 3.14 for π.

47. **WRITING** Explain how to write $(\sqrt[n]{a})^m$ in rational exponent form.

48. **HOW DO YOU SEE IT?** Write an expression in rational exponent form that represents the side length of the square.



49. **REASONING** For what values of x is $x = x^{1/5}$?

50. **MAKING AN ARGUMENT** Your friend says that for a real number a and a positive integer n, the value of $\sqrt[n]{a}$ is always positive and the value of $-\sqrt[n]{a}$ is always negative. Is your friend correct? Explain.

In Exercises 51–54, simplify the expression.

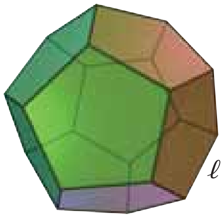
51. $(y^{1/6})^3 \cdot \sqrt{x}$

52. $(y \cdot y^{1/3})^{3/2}$

53. $x \cdot \sqrt[3]{y^6} + y^2 \cdot \sqrt[3]{x^3}$

54. $(x^{1/3} \cdot y^{1/2})^9 \cdot \sqrt{y}$

55. **PROBLEM SOLVING** The formula for the volume of a regular dodecahedron is $V \approx 7.66\ell^3$, where ℓ is the length of an edge. The volume of the dodecahedron is 20 cubic feet. Estimate the edge length.



56. **THOUGHT PROVOKING** Find a formula (for instance, from geometry or physics) that contains a radical. Rewrite the formula using rational exponents.

ABSTRACT REASONING **In Exercises 57–62, let x be a nonnegative real number. Determine whether the statement is *always*, *sometimes*, or *never* true. Justify your answer.**

57. $(x^{1/3})^3 = x$

58. $x^{1/3} = x^{-3}$

59. $x^{1/3} = \sqrt[3]{x}$

60. $x^{1/3} = x^3$

61. $\frac{x^{2/3}}{x^{1/3}} = \sqrt[3]{x}$

62. $x = x^{1/3} \cdot x^3$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the function when $x = -3$, 0, and 8. *(Section 3.3)*

63. $f(x) = 2x - 10$

64. $w(x) = -5x - 1$

65. $h(x) = 13 - x$

66. $g(x) = 8x + 16$

6.3 Exponential Functions

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.9.A
A.9.B
A.9.C
A.9.D
A.9.E

Essential Question

What are some of the characteristics of the graph of an exponential function?

EXPLORATION 1 Exploring an Exponential Function

Work with a partner. Copy and complete each table for the *exponential function* $y = 16(2)^x$. In each table, what do you notice about the values of x? What do you notice about the values of y?

x	$y = 16(2)^x$
0	
1	
2	
3	
4	
5	

x	$y = 16(2)^x$
0	
2	
4	
6	
8	
10	

JUSTIFYING THE SOLUTION

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 2 Exploring an Exponential Function

Work with a partner. Repeat Exploration 1 for the exponential function $y = 16\left(\frac{1}{2}\right)^x$. Do you think the statement below is true for *any* exponential function? Justify your answer.

> *"As the independent variable x changes by a constant amount, the dependent variable y is multiplied by a constant factor."*

EXPLORATION 3 Graphing Exponential Functions

Work with a partner. Sketch the graphs of the functions given in Explorations 1 and 2. How are the graphs similar? How are they different?

Communicate Your Answer

4. What are some of the characteristics of the graph of an exponential function?

5. Sketch the graph of each exponential function. Does each graph have the characteristics you described in Question 4? Explain your reasoning.

a. $y = 2^x$ **b.** $y = 2(3)^x$ **c.** $y = 3(1.5)^x$

d. $y = \left(\frac{1}{2}\right)^x$ **e.** $y = 3\left(\frac{1}{2}\right)^x$ **f.** $y = 2\left(\frac{3}{4}\right)^x$

6.3 Lesson

Core Vocabulary

exponential function, *p. 292*
asymptote, *p. 293*

Previous
independent variable
dependent variable
parent function

What You Will Learn

- Identify and evaluate exponential functions.
- Graph exponential functions.
- Solve real-life problems involving exponential functions.

Identifying and Evaluating Exponential Functions

An **exponential function** is a nonlinear function of the form $y = ab^x$, where $a \neq 0$, $b \neq 1$, and $b > 0$. As the independent variable x changes by a constant amount, the dependent variable y is multiplied by a constant factor, which means consecutive y-values form a constant ratio.

EXAMPLE 1 Identifying Functions

Does each table represent a *linear* or an *exponential* function? Explain.

a.

x	0	1	2	3
y	2	4	6	8

b.

x	0	1	2	3
y	4	8	16	32

SOLUTION

a.

+ 1 + 1 + 1

x	0	1	2	3
y	2	4	6	8

+ 2 + 2 + 2

As x increases by 1, y increases by 2. The rate of change is constant. So, the function is linear.

b.

+ 1 + 1 + 1

x	0	1	2	3
y	4	8	16	32

× 2 × 2 × 2

As x increases by 1, y is multiplied by 2. So, the function is exponential.

STUDY TIP

In Example 1b, consecutive y-values form a constant ratio.

$\frac{8}{4} = 2,\ \frac{16}{8} = 2,\ \frac{32}{16} = 2$

EXAMPLE 2 Evaluating Exponential Functions

Evaluate each function for the given value of x.

a. $y = -2(5)^x;\ x = 3$

b. $y = 3(0.5)^x;\ x = -2$

SOLUTION

a.
$y = -2(5)^x$ Write the function.
$= -2(5)^3$ Substitute for x.
$= -2(125)$ Evaluate the power.
$= -250$ Multiply.

b.
$y = 3(0.5)^x$
$= 3(0.5)^{-2}$
$= 3(4)$
$= 12$

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Does the table represent a *linear* or an *exponential* function? Explain.

1.

x	0	1	2	3
y	8	4	2	1

2.

x	−4	0	4	8
y	1	0	−1	−2

Evaluate the function when $x = -2$, 0, and $\frac{1}{2}$.

3. $y = 2(9)^x$

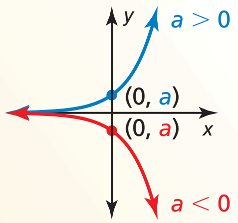
4. $y = 1.5(2)^x$

Graphing Exponential Functions

The graph of a function $y = ab^x$ is a vertical stretch or shrink by a factor of $|a|$ of the graph of the parent function $y = b^x$. When $a < 0$, the graph is also reflected in the x-axis. The y-intercept of the graph of $y = ab^x$ is a.

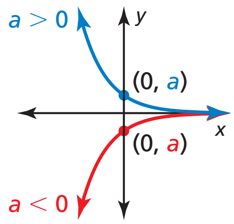
Core Concept

Graphing $y = ab^x$ When $b > 1$



The x-axis is an *asymptote* of the graph of $y = ab^x$. An **asymptote** is a line that a graph approaches but never intersects.

Graphing $y = ab^x$ When $0 < b < 1$



EXAMPLE 3 Graphing $y = ab^x$ When $b > 1$

Graph $f(x) = 4(2)^x$. Compare the graph to the graph of the parent function. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f.

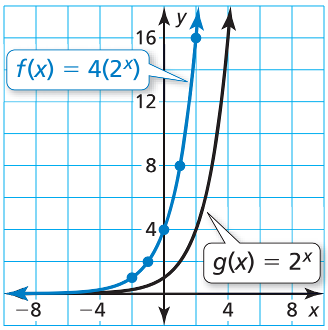
SOLUTION

x	-2	-1	0	1	2
$f(x)$	1	2	4	8	16

Step 1 Make a table of values.

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



▶ The parent function is $g(x) = 2^x$. The graph of f is a vertical stretch by a factor of 4 of the graph of g. The y-intercept of the graph of f, 4, is above the y-intercept of the graph of g, 1. The x-axis is an asymptote of both the graphs of f and g. From the graph of f, you can see that the domain is all real numbers and the range is $y > 0$.

EXAMPLE 4 Graphing $y = ab^x$ When $0 < b < 1$

Graph $f(x) = -\left(\frac{1}{2}\right)^x$. Compare the graph to the graph of the parent function. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f.

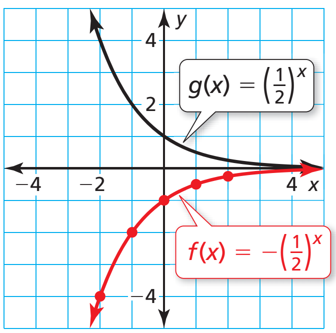
SOLUTION

x	-2	-1	0	1	2
$f(x)$	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$

Step 1 Make a table of values.

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



▶ The parent function is $g(x) = \left(\frac{1}{2}\right)^x$. The graph of f is a reflection in the x-axis of the graph of g. The y-intercept of the graph of f, -1, is below the y-intercept of the graph of g, 1. The x-axis is an asymptote of both the graphs of f and g. From the graph of f, you can see that the domain is all real numbers and the range is $y < 0$.

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Graph the function. Compare the graph to the graph of the parent function. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f.

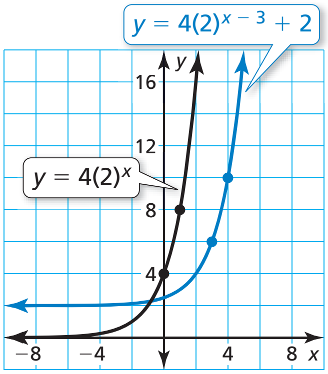
5. $f(x) = -2(4)^x$

6. $f(x) = 2\left(\frac{1}{4}\right)^x$

To graph a function of the form $y = ab^{x-h} + k$, begin by graphing $y = ab^x$. Then translate the graph horizontally h units and vertically k units.

EXAMPLE 5 Graphing $y = ab^{x-h} + k$

Graph $y = 4(2)^{x-3} + 2$. Identify the asymptote. Describe the domain and range.



SOLUTION

Step 1 Graph $y = 4(2)^x$. This is the same function that is in Example 3, which passes through (0, 4) and (1, 8).

Step 2 Translate the graph 3 units right and 2 units up. The graph passes through (3, 6) and (4, 10).

Notice that the graph approaches the line $y = 2$ but does not intersect it.

▶ So, the graph has an asymptote at $y = 2$. From the graph, you can see that the domain is all real numbers and the range is $y > 2$.

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Graph the function. Identify the asymptote. Describe the domain and range.

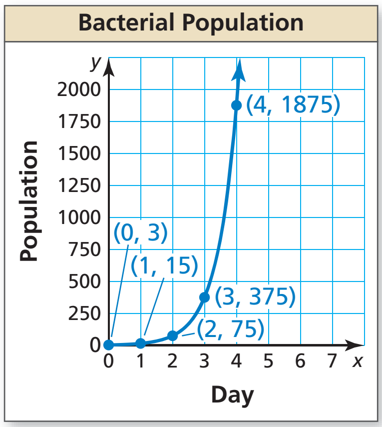
7. $y = -2(3)^{x+2} - 1$ **8.** $f(x) = (0.25)^x + 3$

Solving Real-Life Problems

For an exponential function of the form $y = ab^x$, the y-values change by a factor of b as x increases by 1. You can use this fact to write an exponential function when you know the y-intercept, a.

EXAMPLE 6 Modeling with Mathematics

The graph represents a bacterial population y after x days.



a. Write an exponential function that represents the population. Identify and interpret the y-intercept.

b. Describe the domain and range of the function.

c. Find the population after 5 days.

SOLUTION

a. Use the graph to make a table of values. The y-intercept is 3. The y-values increase by a factor of 5 as x increases by 1.

+1 +1 +1 +1

x	0	1	2	3	4
y	3	15	75	375	1875

×5 ×5 ×5 ×5

▶ So, the population can be modeled by $y = 3(5)^x$. The y-intercept of 3 means that the initial bacterial population is 3.

b. From the graph, you can see that the domain is $x \geq 0$ and the range is $y \geq 3$. Note that $x \geq 0$ because the number of days cannot be negative.

c. Substitute 5 for x in the function $y = 3(5)^x$.

▶ After 5 days, there are $y = 3(5)^5 = 3(3125) = 9375$ bacteria.

You learned to use the *linear regression* feature of a graphing calculator to find an equation of the line of best fit in Section 4.6. Similarly, you can use exponential regression to find an exponential function that fits a data set.

EXAMPLE 7 Writing an Exponential Function Using Technology

Time, x	Temperature, y
0	175
1	156
2	142
3	127
4	113
5	101
6	94
7	84
8	75

The table shows the time x (in minutes) since a cup of hot coffee was poured and the temperature y (in degrees Fahrenheit) of the coffee. (a) Use a graphing calculator to find an exponential function that fits the data. Then plot the data and graph the function in the same viewing window. (b) Predict the temperature of the coffee 10 minutes after it is poured.

SOLUTION

a. Step 1 Enter the data from the table into two lists.

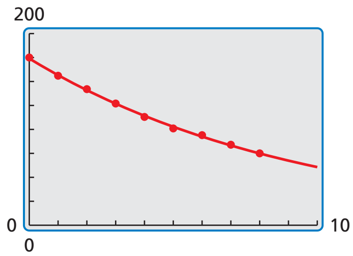
L1	L2	L3
0	175	------
1	156	
2	142	
3	127	
4	113	
5	101	
6	94	

L1(1)=0

Step 2 Use the *exponential regression* feature. The values in the equation can be rounded to obtain $y = 174(0.9)^x$.

```
ExpReg
 y=a*b^x
 a=173.9522643
 b=.9003179174
 r²=.9987302754
 r=-.999364936
```

Step 3 Enter the equation $y = 174(0.9)^x$ into the calculator. Then plot the data and graph the equation in the same viewing window.



b. After 10 minutes, the temperature of the coffee will be about $y = 174(0.9)^{10} \approx 60.7$ degrees Fahrenheit.

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9. A bacterial population y after x days can be represented by an exponential function whose graph passes through (0, 100) and (1, 200). (a) Write a function that represents the population. Identify and interpret the y-intercept. (b) Find the population after 6 days. (c) Does this bacterial population grow faster than the bacterial population in Example 6? Explain.

10. In Example 7, (a) identify and interpret the correlation coefficient and (b) predict the temperature of the coffee 15 minutes after it is poured.

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Vocabulary and Core Concept Check

1. **OPEN-ENDED** Sketch an increasing exponential function whose graph has a y-intercept of 2.

2. **REASONING** Why is a the y-intercept of the graph of the function $y = ab^x$?

3. **WRITING** Compare the graph of $y = 2(5)^x$ with the graph of $y = 5^x$.

4. **WHICH ONE DOESN'T BELONG?** Which equation does *not* belong with the other three? Explain your reasoning.

$y = 3^x$ $\qquad$ $f(x) = 2(4)^x$ $\qquad$ $f(x) = (-3)^x$ $\qquad$ $y = 5(3)^x$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, determine whether the equation represents an exponential function. Explain.

5. $y = 4(7)^x$
6. $y = -6x$
7. $y = 2x^3$
8. $y = -3^x$
9. $y = 9(-5)^x$
10. $y = \frac{1}{2}(1)^x$

In Exercises 11–14, determine whether the table represents a *linear* or an *exponential* function. Explain. *(See Example 1.)*

11.

x	y
1	−2
2	0
3	2
4	4

12.

x	y
1	6
2	12
3	24
4	48

13.

x	−1	0	1	2	3
y	0.25	1	4	16	64

14.

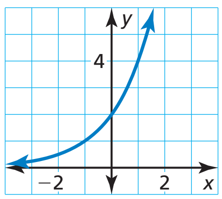
x	−3	0	3	6	9
y	10	1	−8	−17	−26

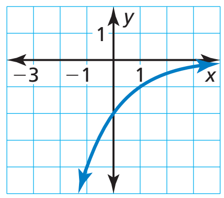
In Exercises 15–20, evaluate the function for the given value of x. *(See Example 2.)*

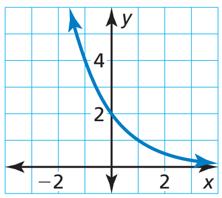
15. $y = 3^x;\ x = 2$
16. $f(x) = 3(2)^x;\ x = -1$
17. $y = -4(5)^x;\ x = 2$
18. $f(x) = 0.5^x;\ x = -3$
19. $f(x) = \frac{1}{3}(6)^x;\ x = 3$
20. $y = \frac{1}{4}(4)^x;\ x = \frac{3}{2}$

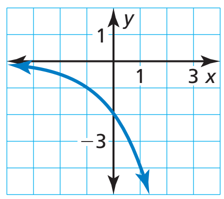
USING STRUCTURE In Exercises 21–24, match the function with its graph.

21. $f(x) = 2(0.5)^x$
22. $y = -2(0.5)^x$
23. $y = 2(2)^x$
24. $f(x) = -2(2)^x$

A.


B.


C.


D.


In Exercises 25–30, graph the function. Compare the graph to the graph of the parent function. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f. *(See Examples 3 and 4.)*

25. $f(x) = 3(0.5)^x$
26. $f(x) = -4^x$
27. $f(x) = -2(7)^x$
28. $f(x) = 6\left(\frac{1}{3}\right)^x$
29. $f(x) = \frac{1}{2}(8)^x$
30. $f(x) = \frac{3}{2}(0.25)^x$

In Exercises 31–36, graph the function. Identify the asymptote. Describe the domain and range. *(See Example 5.)*

31. $f(x) = 3^x - 1$
32. $f(x) = 4^{x+3}$

33. $y = 5^{x-2} + 7$

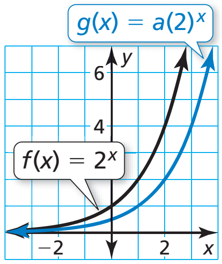
34. $y = -\left(\frac{1}{2}\right)^{x+1} - 3$

35. $y = -8(0.75)^{x+2} - 2$

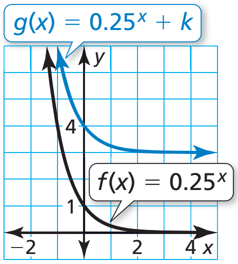
36. $f(x) = 3(6)^{x-1} - 5$

In Exercises 37–40, compare the graphs. Find the value of *h*, *k*, or *a*.

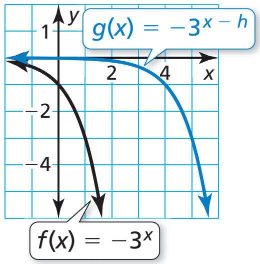
37.



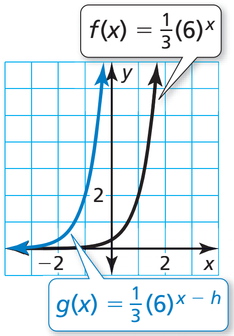
38.



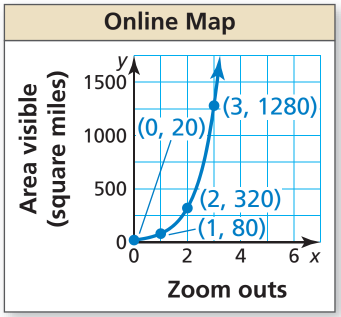
39.



40.



41. MODELING WITH MATHEMATICS The graph represents the amount of area visible *y* on an online map after you zoom out *x* times. *(See Example 6.)*

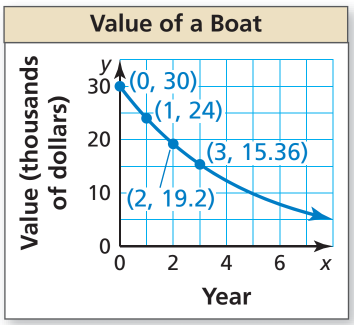


a. Write an exponential function that represents the area. Identify and interpret the *y*-intercept.

b. Describe the domain and range of the function.

c. Find the area visible after you zoom out 10 times.

42. MODELING WITH MATHEMATICS The graph represents the value *y* of a boat after *x* years.



a. Write an exponential function that represents the value. Identify and interpret the *y*-intercept.

b. Describe the domain and range of the function.

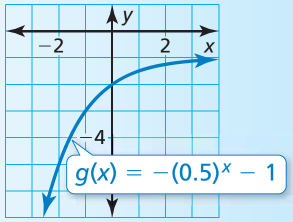
c. Find the value of the boat after 8 years.

43. ERROR ANALYSIS Describe and correct the error in evaluating the function.

$$g(x) = 6(0.5)^x;\ x = -2$$
$$g(-2) = 6(0.5)^{-2}$$
$$= 3^{-2}$$
$$= \frac{1}{9}$$

44. ERROR ANALYSIS Describe and correct the error in finding the domain and range of the function.

The domain is all real numbers, and the range is $y < 0$.



45. MODELING WITH MATHEMATICS The table shows the number *y* of views an online video receives after being online for *x* days. *(See Example 7.)*

Day, *x*	0	1	2	3	4	5
Views, *y*	12	68	613	3996	27,810	205,017

a. Use a graphing calculator to find an exponential function that fits the data. Then plot the data and graph the function in the same viewing window.

b. Predict the number of views the video receives after being online for 7 days.

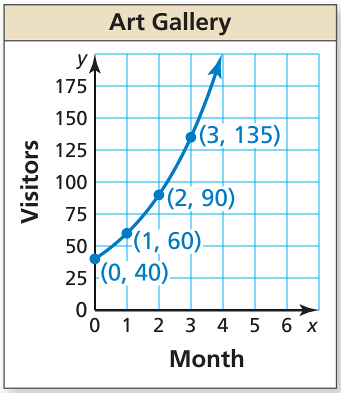
46. MODELING WITH MATHEMATICS The table shows the coyote population *y* in a national park after *t* decades.

Decade, *t*	0	1	2	3	4
Population, *y*	15	26	41	72	123

a. Use a graphing calculator to find an exponential function that fits the data. Then plot the data and graph the function in the same viewing window.

b. Predict the coyote population after 60 years.

47. **MODELING WITH MATHEMATICS** The graph represents the number y of visitors to a new art gallery after x months.



a. Write an exponential function that represents this situation.

b. Approximate the number of visitors after 5 months.

48. **PROBLEM SOLVING** A sales report shows that 3300 gas grills were purchased from a chain of hardware stores last year. The store expects grill sales to increase 6% each year. About how many grills does the store expect to sell in Year 6? Use an equation to justify your answer.

49. **WRITING** Graph the function $f(x) = -2^x$. Then graph $g(x) = -2^x - 3$. How are the y-intercept, domain, and range affected by the translation?

50. **MAKING AN ARGUMENT** Your friend says that the table represents an exponential function because y is multiplied by a constant factor. Is your friend correct? Explain.

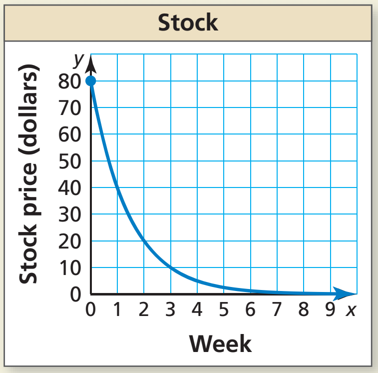
x	0	1	3	6
y	2	10	50	250

51. **WRITING** Describe the effect of a on the graph of $y = a \cdot 2^x$ when a is positive and when a is negative.

52. **OPEN-ENDED** Write a function whose graph is a horizontal translation of the graph of $h(x) = 4^x$.

53. **USING STRUCTURE** The graph of g is a translation 4 units up and 3 units right of the graph of $f(x) = 5^x$. Write an equation for g.

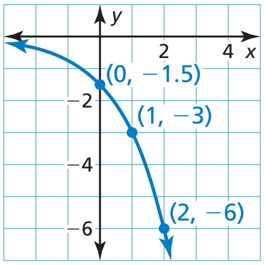
54. **HOW DO YOU SEE IT?** The exponential function $y = V(x)$ represents the projected value of a stock x weeks after a corporation loses an important legal battle. The graph of the function is shown.



a. After how many weeks will the stock be worth $20?

b. Describe the change in the stock price from Week 1 to Week 3.

55. **USING GRAPHS** The graph represents the exponential function f. Find $f(7)$.



56. **THOUGHT PROVOKING** Write a function of the form $y = ab^x$ that represents a real-life population. Explain the meaning of each of the constants a and b in the real-life context.

57. **REASONING** Let $f(x) = ab^x$. Show that when x is increased by a constant k, the quotient $\dfrac{f(x+k)}{f(x)}$ is always the same regardless of the value of x.

58. **PROBLEM SOLVING** A function g models a relationship in which the dependent variable is multiplied by 4 for every 2 units the independent variable increases. The value of the function at 0 is 5. Write an equation that represents the function.

59. **PROBLEM SOLVING** Write an exponential function f so that the slope from the point $(0, f(0))$ to the point $(2, f(2))$ is equal to 12.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the percent as a decimal. *(Skills Review Handbook)*

60. 4% 61. 35% 62. 128% 63. 250%

6.4 Exponential Growth and Decay

Essential Question What are some of the characteristics of exponential growth and exponential decay functions?

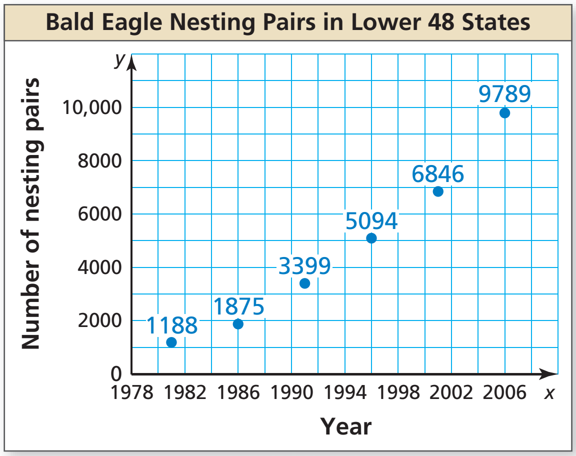
TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.9.B
A.9.C
A.9.D

EXPLORATION 1 Predicting a Future Event

Work with a partner. It is estimated, that in 1782, there were about 100,000 nesting pairs of bald eagles in the United States. By the 1960s, this number had dropped to about 500 nesting pairs. In 1967, the bald eagle was declared an endangered species in the United States. With protection, the nesting pair population began to increase. Finally, in 2007, the bald eagle was removed from the list of endangered and threatened species.

Describe the pattern shown in the graph. Is it exponential growth? Assume the pattern continues. When will the population return to that of the late 1700s? Explain your reasoning.

APPLYING MATHEMATICS
To be proficient in math, you need to apply the mathematics you know to solve problems arising in everyday life.



EXPLORATION 2 Describing a Decay Pattern

Work with a partner. A forensic pathologist was called to estimate the time of death of a person. At midnight, the body temperature was 80.5°F and the room temperature was a constant 60°F. One hour later, the body temperature was 78.5°F.

a. By what percent did the difference between the body temperature and the room temperature drop during the hour?

b. Assume that the original body temperature was 98.6°F. Use the percent decrease found in part (a) to make a table showing the decreases in body temperature. Use the table to estimate the time of death.

Communicate Your Answer

3. What are some of the characteristics of exponential growth and exponential decay functions?

4. Use the Internet or some other reference to find an example of each type of function. Your examples should be different than those given in Explorations 1 and 2.

a. exponential growth **b.** exponential decay

6.4 Lesson

Core Vocabulary

exponential growth, *p. 300*
exponential growth function, *p. 300*
exponential decay, *p. 301*
exponential decay function, *p. 301*
compound interest, *p. 303*

What You Will Learn

- Use and identify exponential growth and decay functions.
- Interpret and rewrite exponential growth and decay functions.
- Solve real-life problems involving exponential growth and decay.

Exponential Growth and Decay Functions

Exponential growth occurs when a quantity increases by the same factor over equal intervals of time.

Core Concept

Exponential Growth Functions

A function of the form $y = a(1 + r)^t$, where $a > 0$ and $r > 0$, is an **exponential growth function**.

$$\underbrace{y}_{\text{final amount}} = \overbrace{a}^{\text{initial amount}}\underbrace{(1 + \overbrace{r}^{\text{rate of growth (in decimal form)}})}_{\text{growth factor}}{}^{\overset{\text{time}}{t}}$$

STUDY TIP

Notice that an exponential growth function is of the form $y = ab^x$, where b is replaced by $1 + r$ and x is replaced by t.

EXAMPLE 1 Using an Exponential Growth Function

The inaugural attendance of an annual music festival is 150,000. The attendance y increases by 8% each year.

a. Write an exponential growth function that represents the attendance after t years.

b. How many people will attend the festival in the fifth year? Round your answer to the nearest thousand.

SOLUTION

a. The initial amount is 150,000, and the rate of growth is 8%, or 0.08.

$y = a(1 + r)^t$ — Write the exponential growth function.

$= 150{,}000(1 + 0.08)^t$ — Substitute 150,000 for a and 0.08 for r.

$= 150{,}000(1.08)^t$ — Add.

▶ The festival attendance can be represented by $y = 150{,}000(1.08)^t$.

b. The value $t = 4$ represents the fifth year because $t = 0$ represents the first year.

$y = 150{,}000(1.08)^t$ — Write the exponential growth function.

$= 150{,}000(1.08)^4$ — Substitute 4 for t.

$\approx 204{,}073$ — Use a calculator.

▶ About 204,000 people will attend the festival in the fifth year.

Monitoring Progress

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1. A website has 500,000 members in 2010. The number y of members increases by 15% each year. (a) Write an exponential growth function that represents the website membership t years after 2010. (b) How many members will there be in 2016? Round your answer to the nearest ten thousand.

Exponential decay occurs when a quantity decreases by the same factor over equal intervals of time.

STUDY TIP

Notice that an exponential decay function is of the form $y = ab^x$, where b is replaced by $1 - r$ and x is replaced by t.

Core Concept

Exponential Decay Functions

A function of the form $y = a(1 - r)^t$, where $a > 0$ and $0 < r < 1$, is an **exponential decay function**.

initial amount   rate of decay (in decimal form)

final amount → $y = a\underbrace{(1 - r)}_{\text{decay factor}}{}^t$ ← time

For exponential growth, the value inside the parentheses is greater than 1 because r is added to 1. For exponential decay, the value inside the parentheses is less than 1 because r is subtracted from 1.

EXAMPLE 2 Identifying Exponential Growth and Decay

Determine whether each table represents an *exponential growth function*, an *exponential decay function*, or *neither*.

a.

x	y
0	270
1	90
2	30
3	10

b.

x	0	1	2	3
y	5	10	20	40

SOLUTION

a.

x	y
0	270
1	90
2	30
3	10

$+1$ $+1$ $+1$

$\times\frac{1}{3}$ $\times\frac{1}{3}$ $\times\frac{1}{3}$

As x increases by 1, y is multiplied by $\frac{1}{3}$. So, the table represents an exponential decay function.

b.

$+1$ $+1$ $+1$

x	0	1	2	3
y	5	10	20	40

$\times 2$ $\times 2$ $\times 2$

As x increases by 1, y is multiplied by 2. So, the table represents an exponential growth function.

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Determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain.

2.

x	0	1	2	3
y	64	16	4	1

3.

x	1	3	5	7
y	4	11	18	25

Interpreting and Rewriting Exponential Functions

EXAMPLE 3 Interpreting Exponential Functions

Determine whether each function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor.

a. The function $y = 0.5(1.07)^t$ represents the value y (in dollars) of a baseball card t years after it is issued.

b. The function $y = 128(0.5)^x$ represents the number y of players left in a video game tournament after x rounds.

SOLUTION

a. The function is of the form $y = a(1 + r)^t$, where $1 + r > 1$, so it represents exponential growth. The initial amount is \$0.50, and the growth factor of 1.07 means that the value of the baseball card increases by 7% each year.

b. The function is of the form $y = a(1 - r)^t$, where $1 - r < 1$, so it represents exponential decay. The initial amount is 128 players, and the decay factor of 0.5 means that 50% of the players are left after each round.

STUDY TIP

You can rewrite exponential expressions and functions using the properties of exponents. Changing the form of an exponential function can reveal important attributes of the function.

EXAMPLE 4 Rewriting Exponential Functions

Rewrite each function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*.

a. $f(x) = 100(0.96)^{x/4}$ **b.** $f(x) = (1.1)^{x-3}$

SOLUTION

a.

$$\begin{aligned} f(x) &= 100(0.96)^{x/4} && \text{Write the function.} \\ &= 100(0.96^{1/4})^x && \text{Power of a Power Property} \\ &\approx 100(0.99)^x && \text{Evaluate the power.} \end{aligned}$$

▶ So, the function represents exponential decay.

b.

$$\begin{aligned} f(x) &= (1.1)^{x-3} && \text{Write the function.} \\ &= \frac{(1.1)^x}{(1.1)^3} && \text{Quotient of Powers Property} \\ &\approx 0.75(1.1)^x && \text{Evaluate the power and simplify.} \end{aligned}$$

▶ So, the function represents exponential growth.

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4. The function $y = 10(1.12)^n$ represents the average game attendance y (in thousands of people) of a professional baseball team in its nth season. Determine whether the function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor.

Rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*.

5. $f(x) = 3(1.02)^{10x}$ **6.** $f(x) = (0.95)^{x+2}$

Solving Real-Life Problems

Exponential growth functions are used in real-life situations involving *compound interest*. Although interest earned is expressed as an *annual* rate, the interest is usually compounded more frequently than once per year. So, the formula $y = a(1 + r)^t$ must be modified for compound interest problems.

STUDY TIP

For interest compounded yearly, you can substitute 1 for n in the formula to get $y = P(1 + r)^t$.

Core Concept

Compound Interest

Compound interest is the interest earned on the principal *and* on previously earned interest. The balance y of an account earning compound interest is

$$y = P\left(1 + \frac{r}{n}\right)^{nt}.$$

P = principal (initial amount)
r = annual interest rate (in decimal form)
t = time (in years)
n = number of times interest is compounded per year

EXAMPLE 5 Writing a Function

You deposit \$100 in a savings account that earns 6% annual interest compounded monthly. Write a function that represents the balance after t years.

SOLUTION

$y = P\left(1 + \frac{r}{n}\right)^{nt}$ — Write the compound interest formula.

$= 100\left(1 + \frac{0.06}{12}\right)^{12t}$ — Substitute 100 for P, 0.06 for r, and 12 for n.

$= 100(1.005)^{12t}$ — Simplify.

EXAMPLE 6 Solving a Real-Life Problem

The table shows the balance of a money market account over time.

a. Write a function that represents the balance after t years.

b. Graph the functions from part (a) and from Example 5 in the same coordinate plane. Compare the account balances.

Year, t	Balance
0	\$100
1	\$110
2	\$121
3	\$133.10
4	\$146.41
5	\$161.05

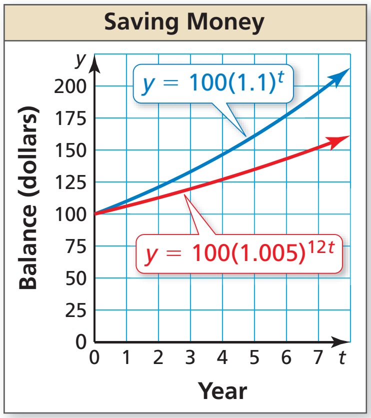
SOLUTION

a. From the table, you know the initial balance is \$100, and it increases 10% each year. So, $P = 100$ and $r = 0.1$.

$y = P(1 + r)^t$ — Write the compound interest formula when $n = 1$.

$= 100(1 + 0.1)^t$ — Substitute 100 for P and 0.1 for r.

$= 100(1.1)^t$ — Add.



b. The money market account earns 10% interest each year, and the savings account earns 6% interest each year. So, the balance of the money market account increases faster.

Monitoring Progress

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7. You deposit \$500 in a savings account that earns 9% annual interest compounded monthly. Write and graph a function that represents the balance y (in dollars) after t years.

EXAMPLE 7 Solving a Real-Life Problem

The value of a car is \$21,500. It loses 12% of its value every year. (a) Write a function that represents the value y (in dollars) of the car after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Identify and interpret any asymptotes of the graph. (d) Estimate the value of the car after 6 years.

STUDY TIP

In real life, the percent decrease in value of an asset is called the *depreciation rate*.

SOLUTION

1. **Understand the Problem** You know the value of the car and its annual percent decrease in value. You are asked to write a function that represents the value of the car over time and approximate the monthly percent decrease in value. Then graph the function and use the graph to estimate the value of the car in the future.
2. **Make a Plan** Use the initial amount and the annual percent decrease in value to write an exponential decay function. Rewrite the function using the properties of exponents to approximate the monthly percent decrease (rate of decay). Then graph the original function and use the graph to estimate the y-value when the t-value is 6.
3. **Solve the Problem**

a. The initial value is \$21,500, and the rate of decay is 12%, or 0.12.

$y = a(1 - r)^t$ — Write the exponential decay function.

$= 21{,}500(1 - 0.12)^t$ — Substitute 21,500 for a and 0.12 for r.

$= 21{,}500(0.88)^t$ — Subtract.

▶ The value of the car can be represented by $y = 21{,}500(0.88)^t$.

b. Use the fact that $t = \frac{1}{12}(12t)$ and the properties of exponents to rewrite the function in a form that reveals the monthly rate of decay.

$y = 21{,}500(0.88)^t$ — Write the original function.

$= 21{,}500(0.88)^{(1/12)(12t)}$ — Rewrite the exponent.

$= 21{,}500(0.88^{1/12})^{12t}$ — Power of a Power Property

$\approx 21{,}500(0.989)^{12t}$ — Evaluate the power.

Use the decay factor $1 - r \approx 0.989$ to find the rate of decay $r \approx 0.011$.

▶ So, the monthly percent decrease is about 1.1%.

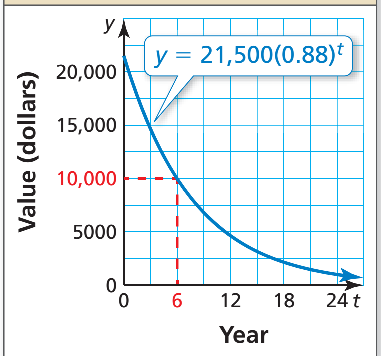
c. You can see that the graph approaches, but never intersects, the t-axis.

▶ So, the graph has an asymptote at $t = 0$. This makes sense because the car will never have a value of \$0.

d. From the graph, you can see that the y-value is about 10,000 when $t = 6$.

▶ So, the value of the car is about \$10,000 after 6 years.

Value of a Car



4. **Look Back** To check that the monthly percent decrease is reasonable, multiply it by 12 to see if it is close in value to the annual percent decrease of 12%.

$1.1\% \times 12 = 13.2\%$ — 13.2% is close to 12%, so 1.1% is reasonable.

When you evaluate $y = 21{,}500(0.88)^t$ for $t = 6$, you get about \$9985. So, \$10,000 is a reasonable estimation.

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8. **WHAT IF?** The car loses 9% of its value every year. (a) Write a function that represents the value y (in dollars) of the car after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Estimate the value of the car after 12 years. Round your answer to the nearest thousand.

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** In the exponential growth function $y = a(1 + r)^t$, the quantity r is called the ________.

2. **VOCABULARY** What is the decay factor in the exponential decay function $y = a(1 - r)^t$?

3. **VOCABULARY** Compare exponential growth and exponential decay.

4. **WRITING** When does the function $y = ab^x$ represent exponential growth? exponential decay?

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, identify the initial amount *a* and the rate of growth *r* (as a percent) of the exponential function. Evaluate the function when $t = 5$. Round your answer to the nearest tenth.

5. $y = 350(1 + 0.75)^t$

6. $y = 10(1 + 0.4)^t$

7. $y = 25(1.2)^t$

8. $y = 12(1.05)^t$

9. $f(t) = 1500(1.074)^t$

10. $h(t) = 175(1.028)^t$

11. $g(t) = 6.72(2)^t$

12. $p(t) = 1.8^t$

In Exercises 13–16, write a function that represents the situation.

13. Sales of \$10,000 increase by 65% each year.

14. Your starting annual salary of \$35,000 increases by 4% each year.

15. A population of 210,000 increases by 12.5% each year.

16. An item costs \$4.50, and its price increases by 3.5% each year.

17. **MODELING WITH MATHEMATICS** The population of a city has been increasing by 2% annually. The sign shown is from the year 2000. *(See Example 1.)*

    a. Write an exponential growth function that represents the population t years after 2000.

    b. What will the population be in 2020? Round your answer to the nearest thousand.

18. **MODELING WITH MATHEMATICS** A young channel catfish weighs about 0.1 pound. During the next 8 weeks, its weight increases by about 23% each week.

    a. Write an exponential growth function that represents the weight of the catfish after t weeks during the 8-week period.

    b. About how much will the catfish weigh after 4 weeks? Round your answer to the nearest thousandth.

In Exercises 19–26, identify the initial amount *a* and the rate of decay *r* (as a percent) of the exponential function. Evaluate the function when $t = 3$. Round your answer to the nearest tenth.

19. $y = 575(1 - 0.6)^t$

20. $y = 8(1 - 0.15)^t$

21. $g(t) = 240(0.75)^t$

22. $f(t) = 475(0.5)^t$

23. $w(t) = 700(0.995)^t$

24. $h(t) = 1250(0.865)^t$

25. $y = \left(\frac{7}{8}\right)^t$

26. $y = 0.5\left(\frac{3}{4}\right)^t$

In Exercises 27–30, write a function that represents the situation.

27. A population of 100,000 decreases by 2% each year.

28. A \$900 sound system decreases in value by 9% each year.

29. A stock valued at \$100 decreases in value by 9.5% each year.

30. A company profit of \$20,000 decreases by 13.4% each year.

31. ERROR ANALYSIS The growth rate of a bacterial culture is 150% each hour. Initially, there are 10 bacteria. Describe and correct the error in finding the number of bacteria in the culture after 8 hours.

> ✗
> $b(t) = 10(1.5)^t$
> $b(8) = 10(1.5)^8 \approx 256.3$
> After 8 hours, there are about 256 bacteria in the culture.

32. ERROR ANALYSIS You purchase a car in 2010 for \$25,000. The value of the car decreases by 14% annually. Describe and correct the error in finding the value of the car in 2015.

> ✗
> $v(t) = 25{,}000(1.14)^t$
> $v(4) = 25{,}000(1.14)^5 \approx 48{,}135$
> The value of the car in 2015 is about \$48,000.

In Exercises 33–38, determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain. *(See Example 2.)*

33.

x	y
−1	50
0	10
1	2
2	0.4

34.

x	y
0	32
1	28
2	24
3	20

35.

x	y
0	35
1	29
2	23
3	17

36.

x	y
1	17
2	51
3	153
4	459

37.

x	y
5	2
10	8
15	32
20	128

38.

x	y
3	432
5	72
7	12
9	2

39. ANALYZING RELATIONSHIPS The table shows the value of a camper t years after it is purchased.

t	Value
1	\$37,000
2	\$29,600
3	\$23,680
4	\$18,944

a. Determine whether the table represents an exponential growth function, an exponential decay function, or neither.

b. What is the value of the camper after 5 years?

40. ANALYZING RELATIONSHIPS The table shows the total numbers of visitors to a website t days after it is online.

t	42	43	44	45
Visitors	11,000	12,100	13,310	14,641

a. Determine whether the table represents an exponential growth function, an exponential decay function, or neither.

b. How many people will have visited the website after it is online 47 days?

In Exercises 41–44, determine whether the function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor. *(See Example 3.)*

41. The function $y = 22(0.94)^t$ represents the population y (in thousands of people) of a town after t years.

42. The function $y = 13(1.2)^x$ represents the number y of customers at a store x days after the store first opens.

43. The function $y = 2^n$ represents the number y of sprints required n days since the start of a training routine.

44. The function $y = 900(0.85)^d$ represents the number y of students who are healthy d days after a flu outbreak.

In Exercises 45–52, rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*. *(See Example 4.)*

45. $f(x) = (0.9)^{x-4}$

46. $f(x) = (1.4)^{x+8}$

47. $f(x) = 2(1.06)^{9x}$

48. $f(x) = 5(0.82)^{x/5}$

49. $f(x) = (1.45)^{x/2}$

50. $f(x) = 0.4(1.16)^{x-1}$

51. $f(x) = 4(0.55)^{x+3}$

52. $f(x) = (0.88)^{4x}$

In Exercises 53–56, write a function that represents the balance after t years. *(See Example 5.)*

53. \$2000 deposit that earns 5% annual interest compounded quarterly

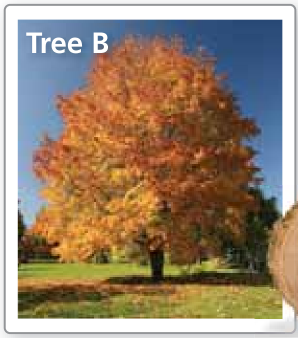
54. \$1400 deposit that earns 10% annual interest compounded semiannually

55. \$6200 deposit that earns 8.4% annual interest compounded monthly

56. \$3500 deposit that earns 9.2% annual interest compounded quarterly

57. PROBLEM SOLVING The cross-sectional area of a tree 4.5 feet from the ground is called its *basal area*. The table shows the basal areas (in square inches) of Tree A over time. *(See Example 6.)*

Year, t	0	1	2	3	4
Basal area, A	120	132	145.2	159.7	175.7



Growth rate: 6%

Initial basal area: 154 in.^{2}

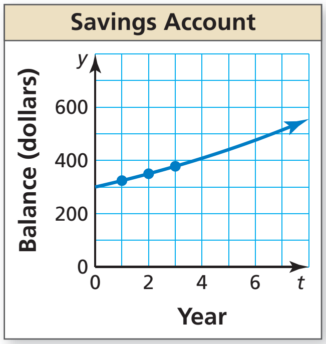
a. Write functions that represent the basal areas of the trees after t years.

b. Graph the functions from part (a) in the same coordinate plane. Compare the basal areas.

58. PROBLEM SOLVING You deposit \$300 into an investment account that earns 12% annual interest compounded quarterly. The graph shows the balance of a savings account over time.

a. Write functions that represent the balances of the accounts after t years.

b. Graph the functions from part (a) in the same coordinate plane. Compare the account balances.



59. PROBLEM SOLVING A city has a population of 25,000. The population is expected to increase by 5.5% annually for the next decade. *(See Example 7.)*

a. Write a function that represents the population y after t years.

b. Find the approximate monthly percent increase in population.

c. Graph the function from part (a). Estimate the population after 4 years.

60. PROBLEM SOLVING Plutonium-238 is a material that generates steady heat due to decay and is used in power systems for some spacecraft. The function $y = a(0.5)^{t/x}$ represents the amount y of a substance remaining after t years, where a is the initial amount and x is the length of the half-life (in years).

a. A scientist is studying a 3-gram sample. Write a function that represents the amount y of plutonium-238 after t years.

b. What is the yearly percent decrease of plutonium-238?

c. Graph the function from part (a). Identify and interpret any asymptotes of the graph.

d. Estimate the amount remaining after 12 years.

61. COMPARING FUNCTIONS The three given functions describe the amount y of ibuprofen (in milligrams) in a person's bloodstream t hours after taking the dosage.

$$y \approx 800(0.71)^t$$
$$y \approx 800(0.9943)^{60t}$$
$$y \approx 800(0.843)^{2t}$$

a. Show that these expressions are approximately equivalent.

b. Describe the information given by each of the functions.

62. **COMBINING FUNCTIONS** You deposit \$9000 in a savings account that earns 3.6% annual interest compounded monthly. You also save \$40 per month in a safe at home. Write a function $C(t) = b(t) + h(t)$, where $b(t)$ represents the balance of your savings account and $h(t)$ represents the amount in your safe after t years. What does $C(t)$ represent?

63. **NUMBER SENSE** During a flu epidemic, the number of sick people triples every week. What is the growth rate as a percent? Explain your reasoning.

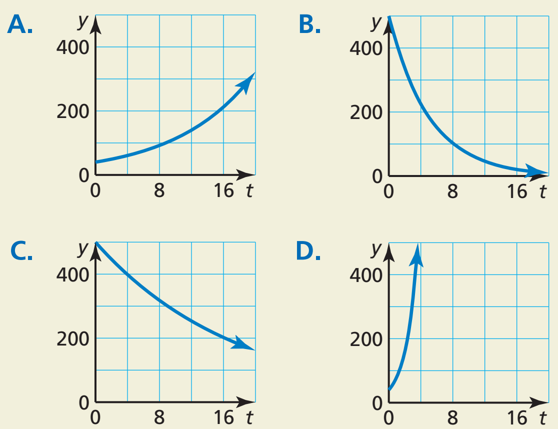
64. **HOW DO YOU SEE IT?** Match each situation with its graph. Explain your reasoning.

a. A bacterial population doubles each hour.

b. The value of a computer decreases by 18% each year.

c. A deposit earns 11% annual interest compounded yearly.

d. A radioactive element decays 5.5% each year.

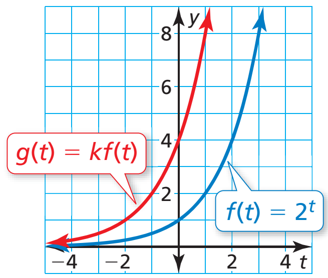


65. **WRITING** Give an example of an equation in the form $y = ab^x$ that does not represent an exponential growth function or an exponential decay function. Explain your reasoning.

66. **THOUGHT PROVOKING** Describe two account options into which you can deposit \$1000 and earn compound interest. Write a function that represents the balance of each account after t years. Which account would you rather use? Explain your reasoning.

67. **MAKING AN ARGUMENT** A store is having a sale on sweaters. On the first day, the prices of the sweaters are reduced by 20%. The prices will be reduced another 20% each day until the sweaters are sold. Your friend says the sweaters will be free on the fifth day. Is your friend correct? Explain.

68. **COMPARING FUNCTIONS** The graphs of f and g are shown.



a. Explain why f is an exponential growth function. Identify the rate of growth.

b. Describe the transformation from the graph of f to the graph of g. Determine the value of k.

c. The graph of g is the same as the graph of $h(t) = f(t + r)$. Use properties of exponents to find the value of r.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Section 1.3)*

69. $8x + 12 = 4x$

70. $5 - t = 7t + 21$

71. $6(r - 2) = 2r + 8$

Determine whether the sequence is arithmetic. If so, find the common difference. *(Section 4.7)*

72. $-20, -26, -32, -38, \ldots$

73. $9, 18, 36, 72, \ldots$

74. $-5, -8, -12, -17, \ldots$

75. $10, 20, 30, 40, \ldots$

6.1–6.4 What Did You Learn?

Core Vocabulary

*n*th root of *a*, *p. 286*
radical, *p. 286*
index of a radical, *p. 286*
exponential function, *p. 292*
asymptote, *p. 293*
exponential growth, *p. 300*
exponential growth function, *p. 300*
exponential decay, *p. 301*
exponential decay function, *p. 301*
compound interest, *p. 303*

Core Concepts

Section 6.1

Zero Exponent, *p. 278*
Negative Exponents, *p. 278*
Product of Powers Property, *p. 279*
Quotient of Powers Property, *p. 279*
Power of a Power Property, *p. 279*
Power of a Product Property, *p. 280*
Power of a Quotient Property, *p. 280*

Section 6.2

Real *n*th Roots of *a*, *p. 286*
Rational Exponents, *p. 287*

Section 6.3

Graphing $y = ab^x$ When $b > 1$, *p. 293*
Graphing $y = ab^x$ When $0 < b < 1$, *p. 293*

Section 6.4

Exponential Growth Functions, *p. 300*
Exponential Decay Functions, *p. 301*
Compound Interest, *p. 303*

Mathematical Thinking

1. How did you apply what you know to simplify the complicated situation in Exercise 56 on page 283?

2. How can you use previously established results to construct an argument in Exercise 50 on page 290?

3. How is the form of the function you wrote in Exercise 62 on page 308 related to the forms of other types of functions you have learned about in this course?

Study Skills

Analyzing Your Errors

Misreading Directions

- **What Happens:** You incorrectly read or do not understand directions.
- **How to Avoid This Error:** Read the instructions for exercises at least twice and make sure you understand what they mean. Make this a habit and use it when taking tests.

6.1–6.4 Quiz

Simplify the expression. Write your answer using only positive exponents. *(Section 6.1)*

1. $3^2 \cdot 3^4$ **2.** $(k^4)^{-3}$ **3.** $\left(\dfrac{4r^2}{3s^5}\right)^3$ **4.** $\left(\dfrac{2x^0}{4x^{-2}y^4}\right)^2$

Evaluate the expression. *(Section 6.2)*

5. $\sqrt[3]{27}$ **6.** $\left(\dfrac{1}{16}\right)^{1/4}$ **7.** $512^{2/3}$ **8.** $(\sqrt{4})^5$

Graph the function. Identify the asymptote. Describe the domain and range. *(Section 6.3)*

9. $y = 5^x$ **10.** $y = -2\left(\dfrac{1}{6}\right)^x$ **11.** $y = 6(2)^{x-4} - 1$

Determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain. *(Section 6.4)*

12.

x	0	1	2	3
y	7	21	63	189

13.

x	1	2	3	4
y	14,641	1331	121	11

Rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*. *(Section 6.4)*

14. $f(x) = 3(1.6)^{x-1}$ **15.** $f(x) = \dfrac{1}{3}(0.96)^{4x}$ **16.** $f(x) = 80\left(\dfrac{4}{5}\right)^{x/2}$

17. The table shows several units of mass. *(Section 6.1)*

Unit of mass	kilogram	hectogram	dekagram	decigram	centigram	milligram	microgram	nanogram
Mass (in grams)	10^3	10^2	10^1	10^{-1}	10^{-2}	10^{-3}	10^{-6}	10^{-9}

a. How many times larger is a kilogram than a nanogram? Write your answer using only positive exponents.

b. How many times smaller is a milligram than a hectogram? Write your answer using only positive exponents.

c. Which is greater, 10,000 milligrams or 1000 decigrams? Explain your reasoning.

18. You store blankets in a cedar chest. What is the volume of the cedar chest? *(Section 6.2)*

19. The function $f(t) = 5(4)^t$ represents the number of frogs in a pond after t years. *(Section 6.3 and Section 6.4)*

a. Does the function represent *exponential growth* or *exponential decay*? Explain.

b. Identify the initial amount and interpret the growth factor or decay factor.

c. Graph the function. Describe the domain and range.

d. What is the approximate monthly percent change?

e. How many frogs are in the pond after 4 years?

6.5 Geometric Sequences

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.12.D

Essential Question

How can you use a geometric sequence to describe a pattern?

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**.

EXPLORATION 1 Describing Calculator Patterns

Work with a partner. Enter the keystrokes on a calculator and record the results in the table. Describe the pattern.

a. Step 1 [2] [=]
Step 2 [×] [2] [=]
Step 3 [×] [2] [=]
Step 4 [×] [2] [=]
Step 5 [×] [2] [=]

Step	1	2	3	4	5
Calculator display					

b. Step 1 [6] [4] [=]
Step 2 [×] [.] [5] [=]
Step 3 [×] [.] [5] [=]
Step 4 [×] [.] [5] [=]
Step 5 [×] [.] [5] [=]

Step	1	2	3	4	5
Calculator display					

c. Use a calculator to make your own sequence. Start with any number and multiply by 3 each time. Record your results in the table.

Step	1	2	3	4	5
Calculator display					

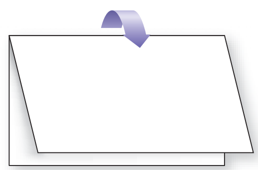
d. Part (a) involves a geometric sequence with a common ratio of 2. What is the common ratio in part (b)? part (c)?

ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to notice when calculations are repeated and look both for general methods and for shortcuts.

EXPLORATION 2 Folding a Sheet of Paper

Work with a partner. A sheet of paper is about 0.1 millimeter thick.

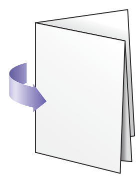


a. How thick will it be when you fold it in half once? twice? three times?

b. What is the greatest number of times you can fold a piece of paper in half? How thick is the result?

c. Do you agree with the statement below? Explain your reasoning.

"If it were possible to fold the paper in half 15 times, it would be taller than you."



Communicate Your Answer

3. How can you use a geometric sequence to describe a pattern?

4. Give an example of a geometric sequence from real life other than paper folding.

6.5 Lesson

Core Vocabulary

geometric sequence, *p. 312*
common ratio, *p. 312*

Previous
arithmetic sequence
common difference

What You Will Learn

- Identify geometric sequences.
- Extend and graph geometric sequences.
- Write geometric sequences as functions.

Identifying Geometric Sequences

Core Concept

Geometric Sequence

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**. Each term is found by multiplying the previous term by the common ratio.

$$1,\quad 5,\quad 25,\quad 125,\ \ldots$$

Terms of a geometric sequence

$\times 5 \quad \times 5 \quad \times 5$ ← common ratio

EXAMPLE 1 Identifying Geometric Sequences

Decide whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

a. 120, 60, 30, 15, . . .   **b.** 2, 6, 11, 17, . . .

SOLUTION

a. Find the ratio between each pair of consecutive terms.

$$120 \qquad 60 \qquad 30 \qquad 15$$

$$\frac{60}{120} = \frac{1}{2} \qquad \frac{30}{60} = \frac{1}{2} \qquad \frac{15}{30} = \frac{1}{2}$$

The ratios are the same. The common ratio is $\frac{1}{2}$.

▶ So, the sequence is geometric.

b. Find the ratio between each pair of consecutive terms.

$$2 \qquad 6 \qquad 11 \qquad 17$$

$$\frac{6}{2} = 3 \qquad \frac{11}{6} = 1\frac{5}{6} \qquad \frac{17}{11} = 1\frac{6}{11}$$

There is no common ratio, so the sequence is *not* geometric.

Find the difference between each pair of consecutive terms.

$$2 \qquad 6 \qquad 11 \qquad 17$$

$$6 - 2 = 4 \qquad 11 - 6 = 5 \qquad 17 - 11 = 6$$

There is no common difference, so the sequence is *not* arithmetic.

▶ So, the sequence is *neither* geometric nor arithmetic.

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Decide whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

1. 5, 1, −3, −7, . . .   **2.** 1024, 128, 16, 2, . . .   **3.** 2, 6, 10, 16, . . .

Extending and Graphing Geometric Sequences

EXAMPLE 2 Extending Geometric Sequences

Write the next three terms of each geometric sequence.

a. 3, 6, 12, 24, . . .   **b.** 64, -16, 4, -1, . . .

SOLUTION

Use tables to organize the terms and extend each sequence.

a.

Position	1	2	3	4	5	6	7
Term	3	6	12	24	48	96	192

$\times 2$ $\times 2$ $\times 2$ $\times 2$ $\times 2$ $\times 2$

Each term is twice the previous term. So, the common ratio is 2.

Multiply a term by 2 to find the next term.

The next three terms are 48, 96, and 192.

b.

Position	1	2	3	4	5	6	7
Term	64	-16	4	-1	$\frac{1}{4}$	$-\frac{1}{16}$	$\frac{1}{64}$

$\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$

Multiply a term by $-\frac{1}{4}$ to find the next term.

The next three terms are $\frac{1}{4}$, $-\frac{1}{16}$, and $\frac{1}{64}$.

ANALYZING MATHEMATICAL RELATIONSHIPS

When the terms of a geometric sequence alternate between positive and negative terms, or vice versa, the common ratio is negative.

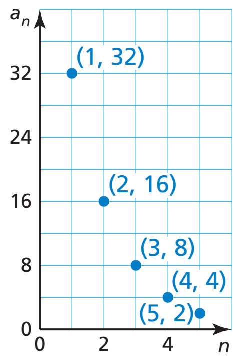
EXAMPLE 3 Graphing a Geometric Sequence

Graph the geometric sequence 32, 16, 8, 4, 2, ... What do you notice?

SOLUTION

Make a table. Then plot the ordered pairs (n, a_n).

Position, n	1	2	3	4	5
Term, a_n	32	16	8	4	2



The points appear to lie on an exponential curve.

STUDY TIP

The points of any geometric sequence with a *positive* common ratio lie on an exponential curve.

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Write the next three terms of the geometric sequence. Then graph the sequence.

4. 1, 3, 9, 27, . . .

5. 2500, 500, 100, 20, . . .

6. 80, -40, 20, -10, . . .

7. -2, 4, -8, 16, . . .

Writing Geometric Sequences as Functions

Because consecutive terms of a geometric sequence have a common ratio, you can use the first term a_1 and the common ratio r to write an exponential function that describes a geometric sequence. Let $a_1 = 1$ and $r = 5$.

Position, n	Term, a_n	Written using a_1 and r	Numbers
1	first term, a_1	a_1	1
2	second term, a_2	$a_1 r$	$1 \cdot 5 = 5$
3	third term, a_3	$a_1 r^2$	$1 \cdot 5^2 = 25$
4	fourth term, a_4	$a_1 r^3$	$1 \cdot 5^3 = 125$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	nth term, a_n	$a_1 r^{n-1}$	$1 \cdot 5^{n-1}$

STUDY TIP

Notice that the equation $a_n = a_1 r^{n-1}$ is of the form $y = ab^x$.

Core Concept

Equation for a Geometric Sequence

Let a_n be the nth term of a geometric sequence with first term a_1 and common ratio r. The nth term is given by

$$a_n = a_1 r^{n-1}.$$

EXAMPLE 4 Finding the nth Term of a Geometric Sequence

Write an equation for the nth term of the geometric sequence 2, 12, 72, 432, ... Then find a_{10}.

SOLUTION

The first term is 2, and the common ratio is 6.

$a_n = a_1 r^{n-1}$ Equation for a geometric sequence

$a_n = 2(6)^{n-1}$ Substitute 2 for a_1 and 6 for r.

Use the equation to find the 10th term.

$a_n = 2(6)^{n-1}$ Write the equation.

$a_{10} = 2(6)^{10-1}$ Substitute 10 for n.

$= 20{,}155{,}392$ Simplify.

▶ The 10th term of the geometric sequence is 20,155,392.

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Write an equation for the nth term of the geometric sequence. Then find a_7.

8. 1, −5, 25, −125, . . .

9. 13, 26, 52, 104, . . .

10. 432, 72, 12, 2, . . .

11. 4, 10, 25, 62.5, . . .

You can rewrite the equation for a geometric sequence with first term a_1 and common ratio r in function notation by replacing a_n with $f(n)$.

$$f(n) = a_1 r^{n-1}$$

The domain of the function is the set of positive integers.

EXAMPLE 5 Modeling with Mathematics

Clicking the *zoom-out* button on a mapping website doubles the side length of the square map. After how many clicks on the *zoom-out* button is the side length of the map 640 miles?

Zoom-out clicks	1	2	3
Map side length (miles)	5	10	20

SOLUTION

1. **Understand the Problem** You know that the side length of the square map doubles after each click on the *zoom-out* button. So, the side lengths of the map represent the terms of a geometric sequence. You need to find the number of clicks it takes for the side length of the map to be 640 miles.

2. **Make a Plan** Begin by writing a function f for the nth term of the geometric sequence. Then find the value of n for which $f(n) = 640$.

3. **Solve the Problem** The first term is 5, and the common ratio is 2.

$f(n) = a_1 r^{n-1}$ Function for a geometric sequence

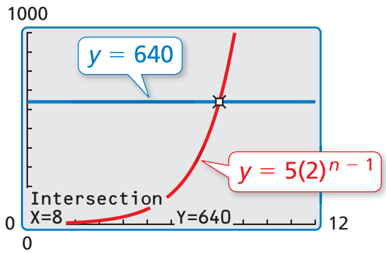
$f(n) = 5(2)^{n-1}$ Substitute 5 for a_1 and 2 for r.

The function $f(n) = 5(2)^{n-1}$ represents the geometric sequence. Use this function to find the value of n for which $f(n) = 640$. So, use the equation $640 = 5(2)^{n-1}$ to write a system of equations.

$y = 5(2)^{n-1}$ Equation 1

$y = 640$ Equation 2

Then use a graphing calculator to graph the equations and find the point of intersection. The point of intersection is (8, 640).



▶ So, after eight clicks, the side length of the map is 640 miles.

4. **Look Back** You can use the *table* feature of a graphing calculator to find the value of n for which $f(n) = 640$.

X	Y1	Y2
3	20	640
4	40	640
5	80	640
6	160	640
7	320	640
8	640	640
9	1280	640

X=8

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12. **WHAT IF?** After how many clicks on the *zoom-out* button is the side length of the map 2560 miles?

6.5 Exercises

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Vocabulary and Core Concept Check

1. **WRITING** Compare the two sequences.

$$2, 4, 6, 8, 10, \ldots \qquad 2, 4, 8, 16, 32, \ldots$$

2. **CRITICAL THINKING** Why do the points of a geometric sequence lie on an exponential curve only when the common ratio is positive?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the common ratio of the geometric sequence.

3. $4, 12, 36, 108, \ldots$

4. $36, 6, 1, \frac{1}{6}, \ldots$

5. $\frac{3}{8}, -3, 24, -192, \ldots$

6. $0.1, 1, 10, 100, \ldots$

7. $128, 96, 72, 54, \ldots$

8. $-162, 54, -18, 6, \ldots$

In Exercises 9–14, determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. *(See Example 1.)*

9. $-8, 0, 8, 16, \ldots$

10. $-1, 4, -7, 10, \ldots$

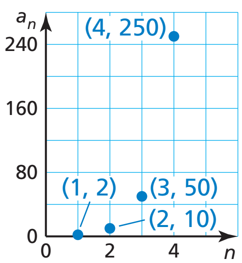
11. $9, 14, 20, 27, \ldots$

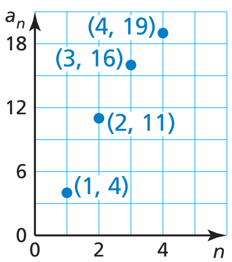
12. $\frac{3}{49}, \frac{3}{7}, 3, 21, \ldots$

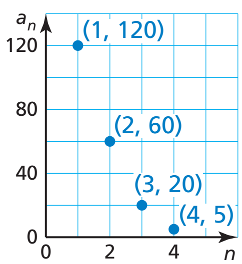
13. $192, 24, 3, \frac{3}{8}, \ldots$

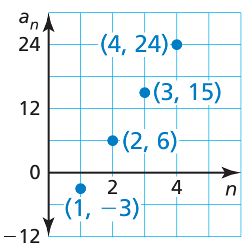
14. $-25, -18, -11, -4, \ldots$

In Exercises 15–18, determine whether the graph represents an *arithmetic sequence*, a *geometric sequence*, or *neither*. Explain your reasoning.

15.


16.


17.


18.


In Exercises 19–24, write the next three terms of the geometric sequence. Then graph the sequence. *(See Examples 2 and 3.)*

19. $5, 20, 80, 320, \ldots$

20. $-3, 12, -48, 192, \ldots$

21. $81, -27, 9, -3, \ldots$

22. $-375, -75, -15, -3, \ldots$

23. $32, 8, 2, \frac{1}{2}, \ldots$

24. $\frac{16}{9}, \frac{8}{3}, 4, 6, \ldots$

In Exercises 25–32, write an equation for the nth term of the geometric sequence. Then find a_6. *(See Example 4.)*

25. $2, 8, 32, 128, \ldots$

26. $0.6, -3, 15, -75, \ldots$

27. $-\frac{1}{8}, -\frac{1}{4}, -\frac{1}{2}, -1, \ldots$

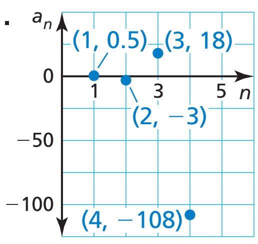
28. $0.1, 0.9, 8.1, 72.9, \ldots$

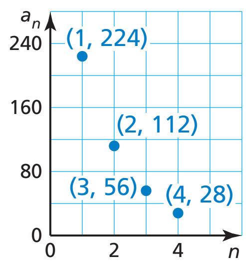
29.

n	1	2	3	4
a_n	7640	764	76.4	7.64

30.

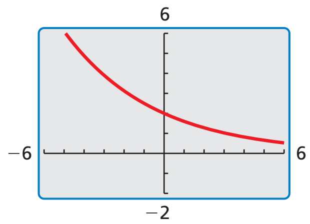
n	1	2	3	4
a_n	−192	48	−12	3

31.


32.


33. **PROBLEM SOLVING** A badminton tournament begins with 128 teams. After the first round, 64 teams remain. After the second round, 32 teams remain. How many teams remain after the third, fourth, and fifth rounds?

34. **PROBLEM SOLVING** A graphing calculator screen displays an area of 96 square units. After you zoom out once, the area is 384 square units. After you zoom out a second time, the area is 1536 square units. What is the screen area after you zoom out four times?



35. **ERROR ANALYSIS** Describe and correct the error in writing the next three terms of the geometric sequence.

$$-8, \quad 4, \quad -2, \quad 1, \ldots$$

$\times(-2)$ $\times(-2)$ $\times(-2)$

The next three terms are -2, 4, and -8.

36. **ERROR ANALYSIS** Describe and correct the error in writing an equation for the nth term of the geometric sequence.

$-2, -12, -72, -432, \ldots$

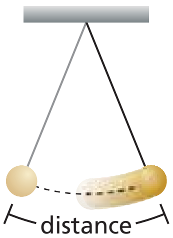
The first term is -2, and the common ratio is -6.

$a_n = a_1 r^{n-1}$

$a_n = -2(-6)^{n-1}$

37. **MODELING WITH MATHEMATICS** The distance (in millimeters) traveled by a swinging pendulum decreases after each swing, as shown in the table. *(See Example 5.)*

Swing	1	2	3
Distance (in millimeters)	625	500	400



a. Write a function that represents the distance the pendulum swings on its nth swing.

b. After how many swings is the distance 256 millimeters?

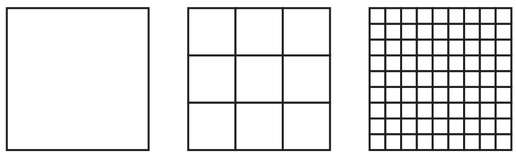
38. **MODELING WITH MATHEMATICS** You start a chain email and send it to six friends. The next day, each of your friends forwards the email to six people. The process continues for a few days.

a. Write a function that represents the number of people who have received the email after n days.

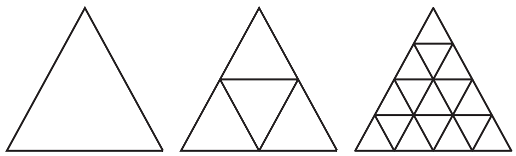
b. After how many days will 1296 people have received the email?

MATHEMATICAL CONNECTIONS **In Exercises 39 and 40, (a) write a function that represents the sequence of figures and (b) describe the 10th figure in the sequence.**

39.



40.



41. **REASONING** Write a sequence that represents the number of teams that have been eliminated after n rounds of the badminton tournament in Exercise 33. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

42. **REASONING** Write a sequence that represents the perimeter of the graphing calculator screen in Exercise 34 after you zoom out n times. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

43. **WRITING** Compare the graphs of arithmetic sequences to the graphs of geometric sequences.

44. **MAKING AN ARGUMENT** You are given two consecutive terms of a sequence.

$$\ldots, -8, 0, \ldots$$

Your friend says that the sequence is not geometric. A classmate says that is impossible to know given only two terms. Who is correct? Explain.

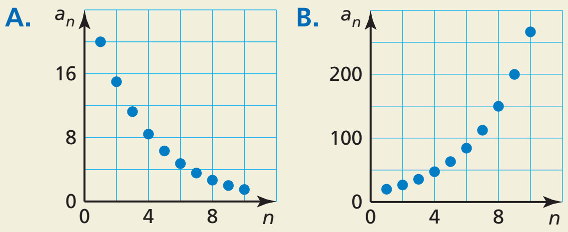
45. CRITICAL THINKING Is the sequence shown an *arithmetic* sequence? a *geometric* sequence? Explain your reasoning.

3, 3, 3, 3, . . .

46. HOW DO YOU SEE IT? Without performing any calculations, match each equation with its graph. Explain your reasoning.

a. $a_n = 20\left(\frac{4}{3}\right)^{n-1}$

b. $a_n = 20\left(\frac{3}{4}\right)^{n-1}$



47. REASONING What is the 9th term of the geometric sequence where $a_3 = 81$ and $r = 3$?

48. OPEN-ENDED Write a sequence that has a pattern but is not arithmetic or geometric. Describe the pattern.

49. ATTENDING TO PRECISION Are the terms of a geometric sequence independent or dependent? Explain your reasoning.

50. DRAWING CONCLUSIONS A college student makes a deal with her parents to live at home instead of living on campus. She will pay her parents \$0.01 for the first day of the month, \$0.02 for the second day, \$0.04 for the third day, and so on.

a. Write an equation that represents the nth term of the geometric sequence.

b. What will she pay on the 25th day?

c. Did the student make a good choice or should she have chosen to live on campus? Explain.

51. REPEATED REASONING A soup kitchen makes 16 gallons of soup. Each day, a quarter of the soup is served and the rest is saved for the next day.

a. Write the first five terms of the sequence of the number of fluid ounces of soup left each day.

b. Write an equation that represents the nth term of the sequence.

c. When is all the soup gone? Explain.

52. THOUGHT PROVOKING Find the sum of the terms of the geometric sequence.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{n-1}}, \ldots$$

Explain your reasoning. Write a different infinite geometric sequence that has the same sum.

53. OPEN-ENDED Write a geometric sequence in which $a_2 < a_1 < a_3$.

54. NUMBER SENSE Write an equation that represents the nth term of each geometric sequence shown.

n	1	2	3	4
a_n	2	6	18	54

n	1	2	3	4
b_n	1	5	25	125

a. Do the terms $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots$ form a geometric sequence? If so, how does the common ratio relate to the common ratios of the sequences above?

b. Do the terms $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ form a geometric sequence? If so, how does the common ratio relate to the common ratios of the sequences above?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Use residuals to determine whether the model is a good fit for the data in the table. Explain. *(Section 4.6)*

55. $y = 3x - 8$

x	0	1	2	3	4	5	6
y	−10	−2	−1	2	1	7	10

56. $y = -5x + 1$

x	−3	−2	−1	0	1	2	3
y	6	4	6	1	2	−4	−3

6.6 Recursively Defined Sequences

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.12.C

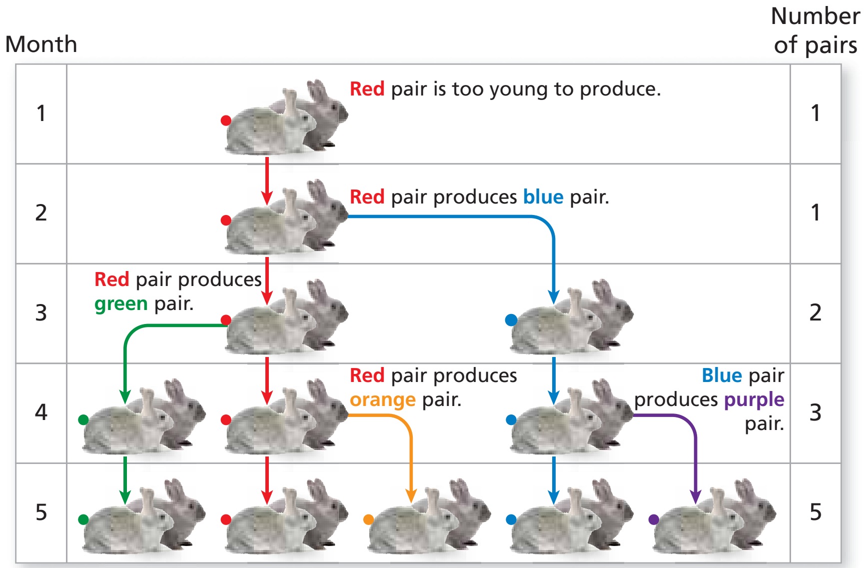
Essential Question

How can you define a sequence recursively?

A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

EXPLORATION 1 Describing a Pattern

Work with a partner. Consider a hypothetical population of rabbits. Start with one breeding pair. After each month, each breeding pair produces another breeding pair. The total number of rabbits each month follows the exponential pattern 2, 4, 8, 16, 32, ... Now suppose that in the first month after each pair is born, the pair is too young to reproduce. Each pair produces another pair after it is 2 months old. Find the total number of pairs in months 6, 7, and 8.



ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to look closely to discern a pattern or structure.

EXPLORATION 2 Using a Recursive Equation

Work with a partner. Consider the following recursive equation.

$$a_n = a_{n-1} + a_{n-2}$$

Each term in the sequence is the sum of the two preceding terms.

Copy and complete the table. Compare the results with the sequence of the number of pairs in Exploration 1.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1						

Communicate Your Answer

3. How can you define a sequence recursively?

4. Use the Internet or some other reference to determine the mathematician who first described the sequences in Explorations 1 and 2.

6.6 Lesson

Core Vocabulary

explicit rule, *p. 320*
recursive rule, *p. 320*

Previous
arithmetic sequence
geometric sequence

What You Will Learn

- Write terms of recursively defined sequences.
- Write recursive rules for sequences.
- Translate between recursive rules and explicit rules.
- Write recursive rules for special sequences.

Writing Terms of Recursively Defined Sequences

So far in this book, you have defined arithmetic and geometric sequences *explicitly*. An **explicit rule** gives a_n as a function of the term's position number n in the sequence. For example, an explicit rule for the arithmetic sequence 3, 5, 7, 9, . . . is $a_n = 3 + 2(n - 1)$, or $a_n = 2n + 1$.

Now, you will define arithmetic and geometric sequences *recursively*. A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

Core Concept

Recursive Equation for an Arithmetic Sequence

$a_n = a_{n-1} + d$, where d is the common difference

Recursive Equation for a Geometric Sequence

$a_n = r \cdot a_{n-1}$, where r is the common ratio

EXAMPLE 1 Writing Terms of Recursively Defined Sequences

Write the first six terms of each sequence. Then graph each sequence.

a. $a_1 = 2, a_n = a_{n-1} + 3$

b. $a_1 = 1, a_n = 3a_{n-1}$

SOLUTION

You are given the first term. Use the recursive equation to find the next five terms.

a. $a_1 = 2$

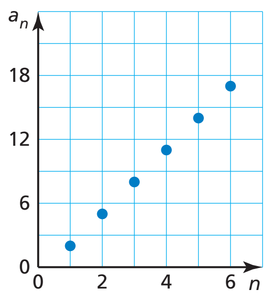
$a_2 = a_1 + 3 = 2 + 3 = 5$

$a_3 = a_2 + 3 = 5 + 3 = 8$

$a_4 = a_3 + 3 = 8 + 3 = 11$

$a_5 = a_4 + 3 = 11 + 3 = 14$

$a_6 = a_5 + 3 = 14 + 3 = 17$



b. $a_1 = 1$

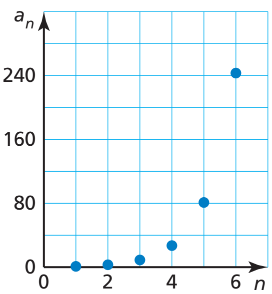
$a_2 = 3a_1 = 3(1) = 3$

$a_3 = 3a_2 = 3(3) = 9$

$a_4 = 3a_3 = 3(9) = 27$

$a_5 = 3a_4 = 3(27) = 81$

$a_6 = 3a_5 = 3(81) = 243$



STUDY TIP

A sequence is a discrete function. So, the points on the graph are not connected.

Monitoring Progress

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Write the first six terms of the sequence. Then graph the sequence.

1. $a_1 = 0,\ a_n = a_{n-1} - 8$

2. $a_1 = -7.5,\ a_n = a_{n-1} + 2.5$

3. $a_1 = -36,\ a_n = \frac{1}{2}a_{n-1}$

4. $a_1 = 0.7,\ a_n = 10a_{n-1}$

Writing Recursive Rules

EXAMPLE 2 Writing Recursive Rules

Write a recursive rule for each sequence.

a. $-30, -18, -6, 6, 18, \ldots$

b. $500, 100, 20, 4, 0.8, \ldots$

SOLUTION

Use a table to organize the terms and find the pattern.

a.

Position, n	1	2	3	4	5
Term, a_n	-30	-18	-6	6	18

$+12$ $+12$ $+12$ $+12$

The sequence is arithmetic, with first term $a_1 = -30$ and common difference $d = 12$.

$a_n = a_{n-1} + d$ Recursive equation for an arithmetic sequence

$a_n = a_{n-1} + 12$ Substitute 12 for d.

So, a recursive rule for the sequence is $a_1 = -30,\ a_n = a_{n-1} + 12$.

COMMON ERROR

When writing a recursive rule for a sequence, you need to write both the beginning term(s) and the recursive equation.

b.

Position, n	1	2	3	4	5
Term, a_n	500	100	20	4	0.8

$\times \frac{1}{5}$ $\times \frac{1}{5}$ $\times \frac{1}{5}$ $\times \frac{1}{5}$

The sequence is geometric, with first term $a_1 = 500$ and common ratio $r = \frac{1}{5}$.

$a_n = r \cdot a_{n-1}$ Recursive equation for a geometric sequence

$a_n = \frac{1}{5}a_{n-1}$ Substitute $\frac{1}{5}$ for r.

So, a recursive rule for the sequence is $a_1 = 500,\ a_n = \frac{1}{5}a_{n-1}$.

Monitoring Progress

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Write a recursive rule for the sequence.

5. $8, 3, -2, -7, -12, \ldots$

6. $1.3, 2.6, 3.9, 5.2, 6.5, \ldots$

7. $4, 20, 100, 500, 2500, \ldots$

8. $128, -32, 8, -2, 0.5, \ldots$

9. Write a recursive rule for the height of the sunflower over time.

Translating between Recursive and Explicit Rules

EXAMPLE 3 Translating from Recursive Rules to Explicit Rules

Write an explicit rule for each recursive rule.

a. $a_1 = 25,\ a_n = a_{n-1} - 10$

b. $a_1 = 19.6,\ a_n = -0.5a_{n-1}$

SOLUTION

a. The recursive rule represents an arithmetic sequence, with first term $a_1 = 25$ and common difference $d = -10$.

$a_n = a_1 + (n - 1)d$ — Explicit rule for an arithmetic sequence

$a_n = 25 + (n - 1)(-10)$ — Substitute 25 for a_1 and -10 for d.

$a_n = -10n + 35$ — Simplify.

▶ An explicit rule for the sequence is $a_n = -10n + 35$.

b. The recursive rule represents a geometric sequence, with first term $a_1 = 19.6$ and common ratio $r = -0.5$.

$a_n = a_1 r^{n-1}$ — Explicit rule for a geometric sequence

$a_n = 19.6(-0.5)^{n-1}$ — Substitute 19.6 for a_1 and -0.5 for r.

▶ An explicit rule for the sequence is $a_n = 19.6(-0.5)^{n-1}$.

EXAMPLE 4 Translating from Explicit Rules to Recursive Rules

Write a recursive rule for each explicit rule.

a. $a_n = -2n + 3$

b. $a_n = -3(2)^{n-1}$

SOLUTION

a. The explicit rule represents an arithmetic sequence, with first term $a_1 = -2(1) + 3 = 1$ and common difference $d = -2$.

$a_n = a_{n-1} + d$ — Recursive equation for an arithmetic sequence

$a_n = a_{n-1} + (-2)$ — Substitute -2 for d.

▶ So, a recursive rule for the sequence is $a_1 = 1,\ a_n = a_{n-1} - 2$.

b. The explicit rule represents a geometric sequence, with first term $a_1 = -3$ and common ratio $r = 2$.

$a_n = r \cdot a_{n-1}$ — Recursive equation for a geometric sequence

$a_n = 2a_{n-1}$ — Substitute 2 for r.

▶ So, a recursive rule for the sequence is $a_1 = -3,\ a_n = 2a_{n-1}$.

Monitoring Progress

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Write an explicit rule for the recursive rule.

10. $a_1 = -45,\ a_n = a_{n-1} + 20$

11. $a_1 = 13,\ a_n = -3a_{n-1}$

Write a recursive rule for the explicit rule.

12. $a_n = -n + 1$

13. $a_n = -2.5(4)^{n-1}$

Writing Recursive Rules for Special Sequences

You can write recursive rules for sequences that are neither arithmetic nor geometric. One way is to look for patterns in the sums of consecutive terms.

EXAMPLE 5 Writing Recursive Rules for Other Sequences

The sequence in Example 5 is called the *Fibonacci sequence*. This pattern is naturally occurring in many objects, such as flowers.

Use the sequence shown.

$$1, 1, 2, 3, 5, 8, \ldots$$

a. Write a recursive rule for the sequence.

b. Write the next three terms of the sequence.

SOLUTION

a. Find the difference and ratio between each pair of consecutive terms.

$$1 \quad 1 \quad 2 \quad 3$$

$1 - 1 = 0 \quad 2 - 1 = 1 \quad 3 - 2 = 1$

There is no common difference, so the sequence is *not* arithmetic.

$$1 \quad 1 \quad 2 \quad 3$$

$\frac{1}{1} = 1 \quad \frac{2}{1} = 2 \quad \frac{3}{2} = 1\frac{1}{2}$

There is no common ratio, so the sequence is *not* geometric.

Find the sum of each pair of consecutive terms.

$a_1 + a_2 = 1 + 1 = 2$ — 2 is the third term.

$a_2 + a_3 = 1 + 2 = 3$ — 3 is the fourth term.

$a_3 + a_4 = 2 + 3 = 5$ — 5 is the fifth term.

$a_4 + a_5 = 3 + 5 = 8$ — 8 is the sixth term.

Beginning with the third term, each term is the sum of the two previous terms. A recursive equation for the sequence is $a_n = a_{n-2} + a_{n-1}$.

▶ So, a recursive rule for the sequence is $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$.

b. Use the recursive equation $a_n = a_{n-2} + a_{n-1}$ to find the next three terms.

$a_7 = a_5 + a_6$	$a_8 = a_6 + a_7$	$a_9 = a_7 + a_8$
$= 5 + 8$	$= 8 + 13$	$= 13 + 21$
$= 13$	$= 21$	$= 34$

▶ The next three terms are 13, 21, and 34.

Monitoring Progress

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Write a recursive rule for the sequence. Then write the next three terms of the sequence.

14. 5, 6, 11, 17, 28, . . .

15. −3, −4, −7, −11, −18, . . .

16. 1, 1, 0, −1, −1, 0, 1, 1, . . .

17. 4, 3, 1, 2, −1, 3, −4, . . .

6.6 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A recursive rule gives the beginning term(s) of a sequence and a(n) _____________ that tells how a_n is related to one or more preceding terms.

2. **WHICH ONE DOESN'T BELONG?** Which rule does *not* belong with the other three? Explain your reasoning.

$a_1 = -1, a_n = 5a_{n-1}$	$a_n = 6n - 2$	$a_1 = -3, a_n = a_{n-1} + 1$	$a_1 = 9, a_n = 4a_{n-1}$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, determine whether the recursive rule represents an *arithmetic sequence* or a *geometric sequence*.

3. $a_1 = 2, a_n = 7a_{n-1}$

4. $a_1 = 18, a_n = a_{n-1} + 1$

5. $a_1 = 5, a_n = a_{n-1} - 4$

6. $a_1 = 3, a_n = -6a_{n-1}$

In Exercises 7–12, write the first six terms of the sequence. Then graph the sequence. *(See Example 1.)*

7. $a_1 = 0, a_n = a_{n-1} + 2$

8. $a_1 = 10, a_n = a_{n-1} - 5$

9. $a_1 = 2, a_n = 3a_{n-1}$

10. $a_1 = 8, a_n = 1.5a_{n-1}$

11. $a_1 = 80, a_n = -\frac{1}{2}a_{n-1}$

12. $a_1 = -7, a_n = -4a_{n-1}$

In Exercises 13–20, write a recursive rule for the sequence. *(See Example 2.)*

13.

n	1	2	3	4
a_n	7	16	25	34

14.

n	1	2	3	4
a_n	8	24	72	216

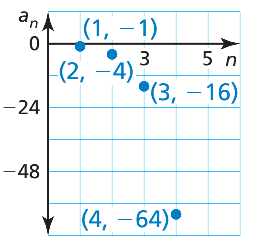
15. 243, 81, 27, 9, 3, . . .

16. 3, 11, 19, 27, 35, . . .

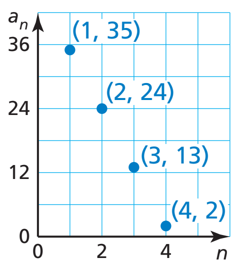
17. 0, −3, −6, −9, −12, . . .

18. 5, −20, 80, −320, 1280, . . .

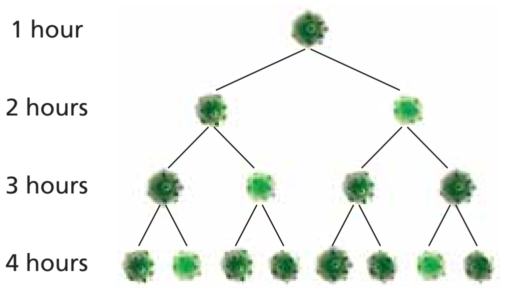
19.



20.



21. **MODELING WITH MATHEMATICS** Write a recursive rule for the number of bacterial cells over time.



22. **MODELING WITH MATHEMATICS** Write a recursive rule for the length of the deer antler over time.

In Exercises 23–28, write an explicit rule for the recursive rule. *(See Example 3.)*

23. $a_1 = -3, a_n = a_{n-1} + 3$

24. $a_1 = 8, a_n = a_{n-1} - 12$

25. $a_1 = 16, a_n = 0.5a_{n-1}$

26. $a_1 = -2, a_n = 9a_{n-1}$

27. $a_1 = 4, a_n = a_{n-1} + 17$

28. $a_1 = 5, a_n = -5a_{n-1}$

In Exercises 29–34, write a recursive rule for the explicit rule. *(See Example 4.)*

29. $a_n = 7(3)^{n-1}$

30. $a_n = -4n + 2$

31. $a_n = 1.5n + 3$

32. $a_n = 6n - 20$

33. $a_n = (-5)^{n-1}$

34. $a_n = -81\left(\frac{2}{3}\right)^{n-1}$

In Exercises 35–38, graph the first four terms of the sequence with the given description. Write a recursive rule and an explicit rule for the sequence.

35. The first term of a sequence is 5. Each term of the sequence is 15 more than the preceding term.

36. The first term of a sequence is 16. Each term of the sequence is half the preceding term.

37. The first term of a sequence is −1. Each term of the sequence is −3 times the preceding term.

38. The first term of a sequence is 19. Each term of the sequence is 13 less than the preceding term.

In Exercises 39–44, write a recursive rule for the sequence. Then write the next two terms of the sequence. *(See Example 5.)*

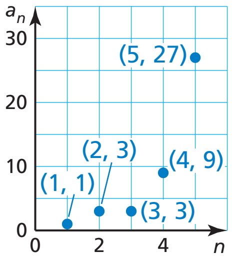
39. 1, 3, 4, 7, 11, . . .

40. 10, 9, 1, 8, −7, 15, . . .

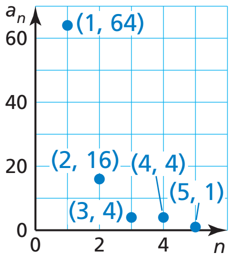
41. 2, 4, 2, −2, −4, −2, . . .

42. 6, 1, 7, 8, 15, 23, . . .

43.



44.



45. ERROR ANALYSIS Describe and correct the error in writing an explicit rule for the recursive rule $a_1 = 6$, $a_n = a_{n-1} - 12$.

✗

$a_n = a_1 + (n - 1)d$

$a_n = 6 + (n - 1)(12)$

$a_n = 6 + 12n - 12$

$a_n = -6 + 12n$

46. ERROR ANALYSIS Describe and correct the error in writing a recursive rule for the sequence 2, 4, 6, 10, 16, ...

✗

$2, \underbrace{\quad}_{+2} 4, \underbrace{\quad}_{+2} 6, \ldots$

The sequence is arithmetic, with first term $a_1 = 2$ and common difference $d = 2$.

$a_n = a_{n-1} + d$

$a_1 = 2, a_n = a_{n-1} + 2$

In Exercises 47–51, the function *f* represents a sequence. Find the 2nd, 5th, and 10th terms of the sequence.

47. $f(1) = 3, f(n) = f(n - 1) + 7$

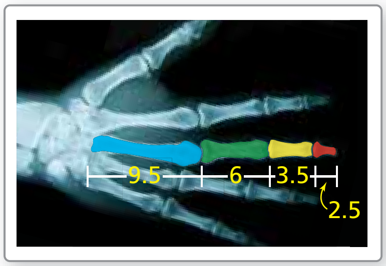
48. $f(1) = -1, f(n) = 6f(n - 1)$

49. $f(1) = 8, f(n) = -f(n - 1)$

50. $f(1) = 4, f(2) = 5, f(n) = f(n - 2) + f(n - 1)$

51. $f(1) = 10, f(2) = 15, f(n) = f(n - 1) - f(n - 2)$

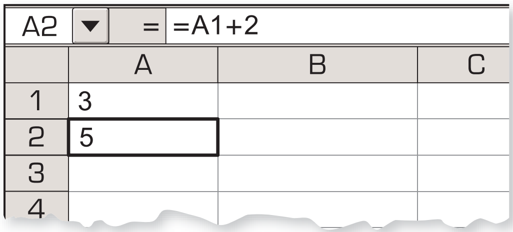
52. MODELING WITH MATHEMATICS The X-ray shows the lengths (in centimeters) of bones in a human hand.



a. Write a recursive rule for the lengths of the bones.

b. Measure the lengths of different sections of your hand. Can the lengths be represented by a recursively defined sequence? Explain.

53. **USING TOOLS** You can use a spreadsheet to generate the terms of a sequence.

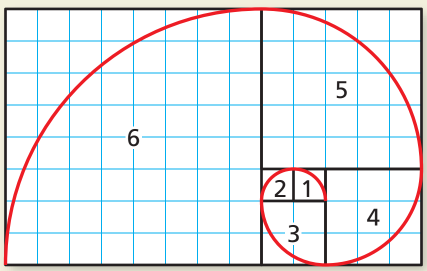


a. To generate the terms of the sequence $a_1 = 3$, $a_n = a_{n-1} + 2$, enter the value of a_1, 3, into cell A1. Then enter "=A1+2" into cell A2, as shown. Use the *fill down* feature to generate the first 10 terms of the sequence.

b. Use a spreadsheet to generate the first 10 terms of the sequence $a_1 = 3$, $a_n = 4a_{n-1}$. (*Hint:* Enter "=4*A1" into cell A2.)

c. Use a spreadsheet to generate the first 10 terms of the sequence $a_1 = 4$, $a_2 = 7$, $a_n = a_{n-1} - a_{n-2}$. (*Hint:* Enter "=A2-A1" into cell A3.)

54. **HOW DO YOU SEE IT?** Consider Squares 1–6 in the diagram.



a. Write a sequence in which each term a_n is the side length of square n.

b. What is the name of this sequence? What is the next term of this sequence?

c. Use the term in part (b) to add another square to the diagram and extend the spiral.

55. **REASONING** Write the first 5 terms of the sequence $a_1 = 5$, $a_n = 3a_{n-1} + 4$. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

56. **THOUGHT PROVOKING** Describe the pattern for the numbers in Pascal's Triangle, shown below. Write a recursive rule that gives the mth number in the nth row.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

57. **REASONING** The explicit rule $a_n = a_1 + (n - 1)d$ defines an arithmetic sequence.

a. Explain why $a_{n-1} = a_1 + [(n - 1) - 1]d$.

b. Justify each step in showing that a recursive equation for the sequence is $a_n = a_{n-1} + d$.

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ &= a_1 + [(n - 1) + 0]d \\ &= a_1 + [(n - 1) - 1 + 1]d \\ &= a_1 + [((n - 1) - 1) + 1]d \\ &= a_1 + [(n - 1) - 1]d + d \\ &= a_{n-1} + d \end{aligned}$$

58. **MAKING AN ARGUMENT** Your friend claims that the sequence

$$-5, 5, -5, 5, -5, \ldots$$

cannot be represented by a recursive rule. Is your friend correct? Explain.

59. **PROBLEM SOLVING** Write a recursive rule for the sequence.

$$3, 7, 15, 31, 63, \ldots$$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

60. $5x + 12x$ **61.** $9 - 6y - 14$ **62.** $2d - 7 - 8d$ **63.** $3 - 3m + 11m$

Write a linear function f with the given values. *(Section 4.2)*

64. $f(2) = 6, f(-1) = -3$ **65.** $f(-2) = 0, f(6) = -4$

66. $f(-3) = 5, f(-1) = 5$ **67.** $f(3) = -1, f(-4) = -15$

6.5–6.6 What Did You Learn?

Core Vocabulary

geometric sequence, *p. 312*
common ratio, *p. 312*
explicit rule, *p. 320*
recursive rule, *p. 320*

Core Concepts

Section 6.5

Geometric Sequence, *p. 312*
Equation for a Geometric Sequence, *p. 314*

Section 6.6

Recursive Equation for an Arithmetic Sequence, *p. 320*
Recursive Equation for a Geometric Sequence, *p. 320*

Mathematical Thinking

1. Explain how writing a function in Exercise 39 part (a) on page 317 created a shortcut for answering part (b).

2. How did you choose an appropriate tool in Exercise 52 part (b) on page 325?

Performance Task

The New Car

There is so much more to buying a new car than the purchase price. Interest rates, depreciation, and inflation are all factors. So, what is the real cost of your new car?

To explore the answer to this question and more, go to ***BigIdeasMath.com***.

6 Chapter Review

6.1 Properties of Exponents (pp. 277–284)

Simplify $\left(\frac{x}{4}\right)^{-4}$. Write your answer using only positive exponents.

$$\left(\frac{x}{4}\right)^{-4} = \frac{x^{-4}}{4^{-4}} \quad \text{Power of a Quotient Property}$$
$$= \frac{4^4}{x^4} \quad \text{Definition of negative exponent}$$
$$= \frac{256}{x^4} \quad \text{Simplify.}$$

Simplify the expression. Write your answer using only positive exponents.

1. $y^3 \cdot y^{-5}$ **2.** $\frac{x^4}{x^7}$ **3.** $(x^0y^2)^3$ **4.** $\left(\frac{2x^2}{5y^4}\right)^{-2}$

6.2 Radicals and Rational Exponents (pp. 285–290)

Evaluate $512^{1/3}$.

$$512^{1/3} = \sqrt[3]{512} \quad \text{Rewrite the expression in radical form.}$$
$$= \sqrt[3]{8 \cdot 8 \cdot 8} \quad \text{Rewrite the expression showing factors.}$$
$$= 8 \quad \text{Evaluate the cube root.}$$

Evaluate the expression.

5. $\sqrt[3]{8}$ **6.** $\sqrt[5]{-243}$ **7.** $625^{3/4}$ **8.** $(-25)^{1/2}$

6.3 Exponential Functions (pp. 291–298)

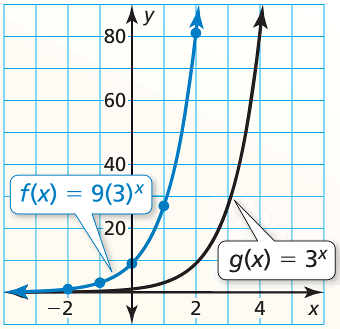
Graph $f(x) = 9(3)^x$. Compare the graph to the graph of the parent function. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f.

Step 1 Make a table of values.

x	−2	−1	0	1	2
$f(x)$	1	3	9	27	81

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



The parent function is $g(x) = 3^x$. The graph of f is a vertical stretch by a factor of 9 of the graph of g. The y-intercept of the graph of f, 9, is above the y-intercept of the graph of g, 1. The x-axis is an asymptote of both the graphs of f and g. From the graph of f, you can see that the domain is all real numbers and the range is $y > 0$.

Graph the function. Compare the graph to the graph of the parent function. Identify the *y*-intercepts and asymptotes of the graphs. Describe the domain and range of *f*.

9. $f(x) = -4\left(\frac{1}{4}\right)^x$

10. $f(x) = 2(3)^x$

11. Graph $f(x) = 2^{x-4} - 3$. Identify the asymptote. Describe the domain and range.

12. Use the table shown. (a) Use a graphing calculator to find an exponential function that fits the data. Then plot the data and graph the function in the same viewing window. (b) Identify and interpret the correlation coefficient.

x	0	1	2	3	4
y	200	96	57	25	10

6.4 Exponential Growth and Decay (pp. 299–308)

The function $f(x) = 75(0.5)^{x/10}$ represents the amount (in milligrams) of a radioactive substance remaining after x days. Rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor.

$$f(x) = 75(0.5)^{x/10} \qquad \text{Write the function.}$$

$$= 75(0.5^{1/10})^x \qquad \text{Power of a Power Property}$$

$$\approx 75(0.93)^x \qquad \text{Evaluate the power.}$$

The function is of the form $y = a(1 - r)^t$, where $1 - r < 1$, so it represents exponential decay. The initial amount is 75 milligrams, and the decay factor of 0.93 means that about 93% of the radioactive substance remains after each day.

Determine whether the table represents an *exponential growth function,* an *exponential decay function,* or *neither.* Explain.

13.

x	0	1	2	3
y	3	6	12	24

14.

x	1	2	3	4
y	162	108	72	48

15. The function $y = 1.75(1.02)^t$ represents the value y (in dollars) of a share of stock after t days. Determine whether the function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor.

Rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay.*

16. $f(x) = 4(1.25)^{x+3}$

17. $f(x) = (1.06)^{8x}$

18. $f(x) = 6(0.84)^{x-4}$

19. You deposit \$750 in a savings account that earns 5% annual interest compounded quarterly. (a) Write a function that represents the balance after t years. (b) What is the balance of the account after 4 years?

20. The value of a TV is \$1500. Its value decreases by 14% each year. (a) Write a function that represents the value y (in dollars) of the TV after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Identify and interpret any asymptotes of the graph. (d) Estimate the value of the TV after 3 years.

6.5 Geometric Sequences (pp. 311–318)

Write the next three terms of the geometric sequence 2, 6, 18, 54, ...

Use a table to organize the terms and extend the sequence.

Position	1	2	3	4	5	6	7
Term	2	6	18	54	162	486	1458

$\times 3$ $\times 3$ $\times 3$ $\times 3$ $\times 3$ $\times 3$

Each term is 3 times the previous term. So, the common ratio is 3.

Multiply a term by 3 to find the next term.

▶ The next three terms are 162, 486, and 1458.

Decide whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. If the sequence is geometric, write the next three terms and graph the sequence.

21. 3, 12, 48, 192, . . .
22. 9, −18, 27, −36, . . .
23. 375, −75, 15, −3, . . .

Write an equation for the *n*th term of the geometric sequence. Then find a_9.

24. 1, 4, 16, 64, . . .
25. 5, −10, 20, −40, . . .
26. 486, 162, 54, 18, . . .

6.6 Recursively Defined Sequences (pp. 319–326)

Write a recursive rule for the sequence 5, 12, 19, 26, 33, ...

Use a table to organize the terms and find the pattern.

Position, n	1	2	3	4	5
Term, a_n	5	12	19	26	33

$+7$ $+7$ $+7$ $+7$

The sequence is arithmetic, with first term $a_1 = 5$ and common difference $d = 7$.

$a_n = a_{n-1} + d$ Recursive equation for an arithmetic sequence

$a_n = a_{n-1} + 7$ Substitute 7 for d.

▶ So, a recursive rule for the sequence is $a_1 = 5$, $a_n = a_{n-1} + 7$.

Write the first six terms of the sequence. Then graph the sequence.

27. $a_1 = 4$, $a_n = a_{n-1} + 5$
28. $a_1 = -4$, $a_n = -3a_{n-1}$
29. $a_1 = 32$, $a_n = \frac{1}{4}a_{n-1}$

Write a recursive rule for the sequence.

30. 3, 8, 13, 18, 23, . . .
31. 3, 6, 12, 24, 48, . . .
32. 7, 6, 13, 19, 32, . . .

33. The first term of a sequence is 8. Each term of the sequence is 5 times the preceding term. Graph the first four terms of the sequence. Write a recursive rule and an explicit rule for the sequence.

6 Chapter Test

Evaluate the expression.

1. $-\sqrt[4]{16}$

2. $729^{1/6}$

3. $(-32)^{7/5}$

Simplify the expression. Write your answer using only positive exponents.

4. $z^{-1/4} \cdot z^{3/5}$

5. $\dfrac{b^{-5}}{a^0 b^{-8}}$

6. $\left(\dfrac{2c^4}{5}\right)^{-3}$

Write and graph a function that represents the situation.

7. Your starting annual salary of \$42,500 increases by 3% each year.

8. You deposit \$500 in an account that earns 6.5% annual interest compounded yearly.

Write an explicit rule and a recursive rule for the sequence.

9.

n	1	2	3	4
a_n	-6	8	22	36

10.

n	1	2	3	4
a_n	400	100	25	6.25

11. Graph $f(x) = 2(6)^x$. Compare the graph to the graph of $g(x) = 6^x$. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f.

12. The function $y = ab^t$ represents the population y (in millions of people) of a country after t years. Choose values of a and b so that the population is initially 13 million people, but is below 10 million people after 3 years.

Use the equation to complete the statement "a ▭ b" with the symbol $<$, $>$, or $=$. Do not attempt to solve the equation.

13. $\dfrac{5^a}{5^b} = 5^{-3}$

14. $9^a \cdot 9^{-b} = 1$

15. The first two terms of a sequence are $a_1 = 3$ and $a_2 = -12$. Let a_3 be the third term when the sequence is arithmetic and let b_3 be the third term when the sequence is geometric. Find $a_3 - b_3$.

16. At sea level, Earth's atmosphere exerts a pressure of 1 atmosphere. Atmospheric pressure P (in atmospheres) decreases with altitude. It can be modeled by $P = (0.99988)^a$, where a is the altitude (in meters).

a. Identify the initial amount, decay factor, and decay rate.

b. Use a graphing calculator to graph the function. Use the graph to estimate the atmospheric pressure at an altitude of 5000 feet.

17. You follow the training schedule from your coach.

a. Write an explicit rule and a recursive rule for the geometric sequence.

b. On what day do you run approximately 3 kilometers?

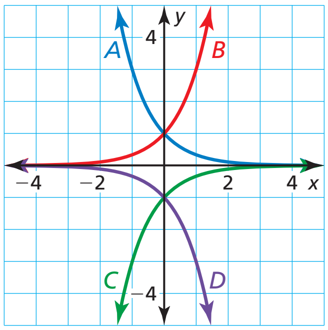
Training On Your Own

Day 1: Run 1 km.

Each day after Day 1: Run 20% farther than the previous day.

6 Standards Assessment

1. Which graph represents the function $f(x) = -3^x$? *(TEKS A.9.D)*



Ⓐ *A*　　Ⓑ *B*

Ⓒ *C*　　Ⓓ *D*

2. **GRIDDED ANSWER** What number should you use to complete the expression so that the statement is true? *(TEKS A.11.B)*

$$\frac{x^{5/3} \cdot x^{-1} \cdot \sqrt[3]{x}}{x^{-2} \cdot x^{0}} = x^{\square}$$

3. Which symbol should you use to complete Inequality 2 so that the system of linear inequalities has no solution? *(TEKS A.3.H)*

Inequality 1　$y - 2x \le 4$

Inequality 2　$6x - 3y \;\square\; -12$

Ⓕ $<$　　Ⓖ $>$

Ⓗ $\le$　　Ⓙ $\ge$

4. The second term of a sequence is 7. Each term of the sequence is 10 more than the preceding term. Which equation gives the *n*th term of the sequence? *(TEKS A.12.D)*

Ⓐ $a_n = -3n + 13$　　Ⓑ $a_n = 10n - 13$

Ⓒ $a_n = 10n - 3$　　Ⓓ $a_n = 10n + 7$

5. Which statement is true about the function $f(x) = -2(9)^x$? *(TEKS A.9.A)*

Ⓕ The domain is $x < 0$. The range is $y < 0$.

Ⓖ The domain is all real numbers. The range is $y < 0$.

Ⓗ The domain is all real numbers. The range is $y > 0$.

Ⓙ The domain is all real numbers. The range is all real numbers.

6. For 10 years, the population of a city increases in a pattern that is approximately exponential. Using exponential regression, you fit the function $p(t) = 15(1.064)^t$ to the population data, where p is the population (in thousands) of the city in year t. What is a reasonable prediction for the future population of the city? *(TEKS A.9.E)*

Ⓐ The population will be under 7000 in Year 12.

Ⓑ The population will be over 40,000 in Year 15.

Ⓒ The population will be under 50,000 in Year 18.

Ⓓ The population will be over 300,000 by Year 20.

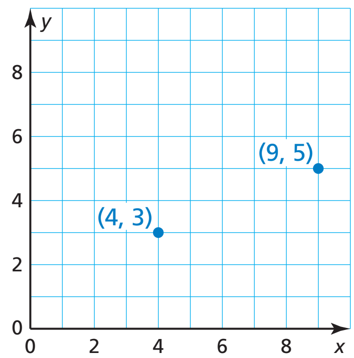
7. Which equation represents the line that passes through the two points shown? *(TEKS A.2.B)*

Ⓕ $y = \frac{5}{2}x - 13$

Ⓖ $y = \frac{2}{5}x - \frac{23}{5}$

Ⓗ $-5x + 2y = -14$

Ⓙ $-2x + 5y = 7$



8. Which of the relations is *not* a function? *(TEKS A.12.A)*

Ⓐ

x	−2	0	0	2
y	1	−3	3	5

Ⓑ

x	0	2	4	6
y	−5	−5	−5	−5

Ⓒ

x	1	2	3	4
y	−1	6	13	21

Ⓓ

x	−1	0	1	2
y	7	6	7	10

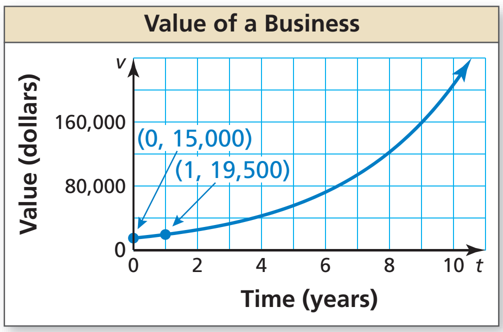
9. The graph shows the value of a business over time. Which equation models the value v (in dollars) of the business over time t (in years)? *(TEKS A.9.C)*

Ⓕ $v = 15{,}000(1.30)^t$

Ⓖ $v = 15{,}000(0.70)^t$

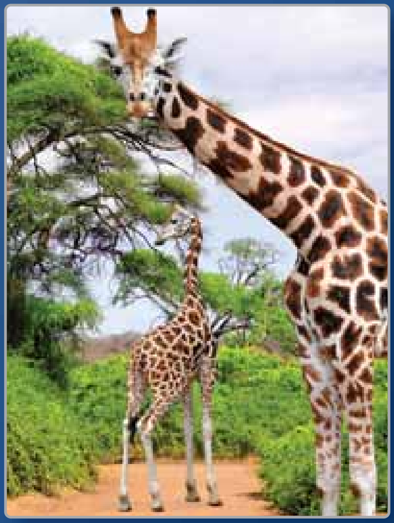
Ⓗ $v = 15{,}000(0.50)^t$

Ⓙ $v = 15{,}000(0.30)^t$



7 Polynomial Equations and Factoring

Height of a Falling Object *(p. 386)*



Game Reserve *(p. 380)*

Photo Cropping *(p. 376)*

Gateway Arch *(p. 368)*

Framing a Photo *(p. 350)*

Mathematical Thinking: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

Maintaining Mathematical Proficiency

Simplifying Algebraic Expressions (6.7.D)

Example 1 **Simplify $6x + 5 - 3x - 4$.**

$$6x + 5 - 3x - 4 = 6x - 3x + 5 - 4 \quad \text{Commutative Property of Addition}$$

$$= (6 - 3)x + 5 - 4 \quad \text{Distributive Property}$$

$$= 3x + 1 \quad \text{Simplify.}$$

Example 2 **Simplify $-8(y - 3) + 2y$.**

$$-8(y - 3) + 2y = -8(y) - (-8)(3) + 2y \quad \text{Distributive Property}$$

$$= -8y + 24 + 2y \quad \text{Multiply.}$$

$$= -8y + 2y + 24 \quad \text{Commutative Property of Addition}$$

$$= (-8 + 2)y + 24 \quad \text{Distributive Property}$$

$$= -6y + 24 \quad \text{Simplify.}$$

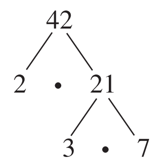
Simplify the expression.

1. $3x - 7 + 2x$
2. $4r + 6 - 9r - 1$
3. $-5t + 3 - t - 4 + 8t$
4. $3(s - 1) + 5$
5. $2m - 7(3 - m)$
6. $4(h + 6) - (h - 2)$

Writing Prime Factorizations (6.7.A)

Example 3 **Write the prime factorization of 42.**

Make a factor tree.



▶ The prime factorization of 42 is 2 • 3 • 7.

Write the prime factorization of the number.

7. 36
8. 63
9. 54
10. 72
11. 28
12. 30

13. **ABSTRACT REASONING** Is it possible for two integers to have the same prime factorization? Explain your reasoning.

Mathematical Thinking

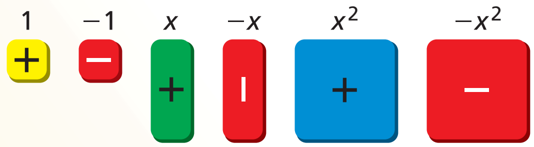
Mathematically proficient students select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems. **(A.1.C)**

Using Models

Core Concept

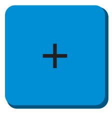
Using Algebra Tiles

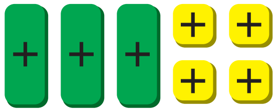
When solving a problem, it can be helpful to use a model. For instance, you can use algebra tiles to model algebraic expressions and operations with algebraic expressions.

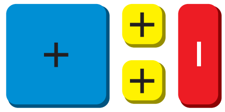


EXAMPLE 1 **Writing Expressions Modeled by Algebra Tiles**

Write the algebraic expression modeled by the algebra tiles.

a. 

b. 

c. 

SOLUTION

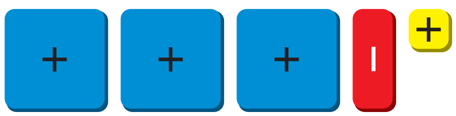
a. The algebraic expression is x^2.

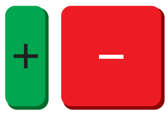
b. The algebraic expression is $3x + 4$.

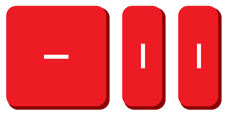
c. The algebraic expression is $x^2 - x + 2$.

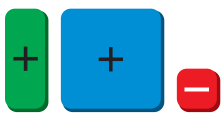
Monitoring Progress

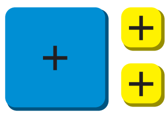
Write the algebraic expression modeled by the algebra tiles.

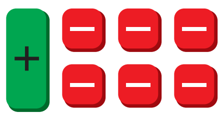
1. 

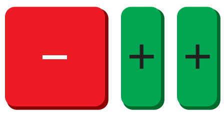
2. 

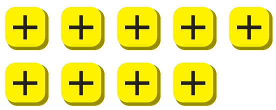
3. 

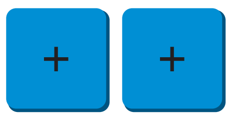
4. 

5. 

6. 

7. 

8. 

9. 

7.1 Adding and Subtracting Polynomials

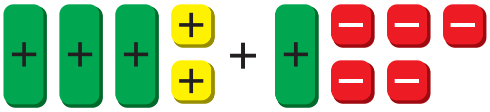
TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.10.A

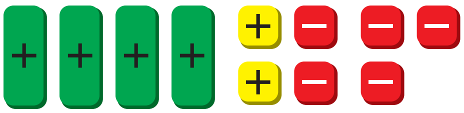
Essential Question

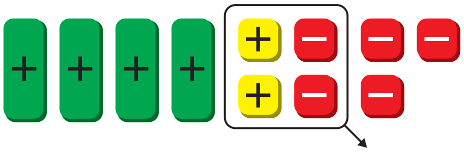
How can you add and subtract polynomials?

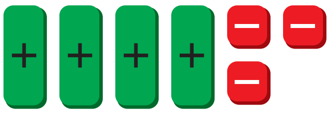
EXPLORATION 1 Adding Polynomials

Work with a partner. Write the expression modeled by the algebra tiles in each step.

Step 1

$(3x + 2) + (x - 5)$

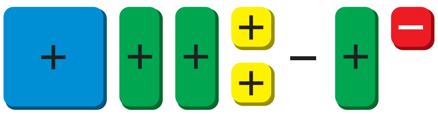
Step 2


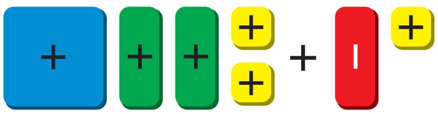
Step 3


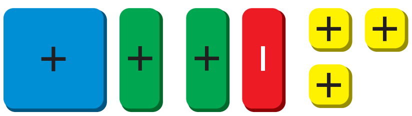
Step 4


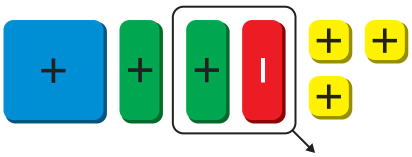
EXPLORATION 2 Subtracting Polynomials

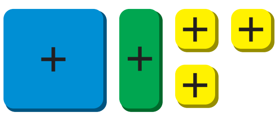
Work with a partner. Write the expression modeled by the algebra tiles in each step.

Step 1

$(x^2 + 2x + 2) - (x - 1)$

Step 2


Step 3


Step 4


Step 5


REASONING
To be proficient in math, you need to represent a given situation using symbols.

Communicate Your Answer

3. How can you add and subtract polynomials?

4. Use your methods in Question 3 to find each sum or difference.

a. $(x^2 + 2x - 1) + (2x^2 - 2x + 1)$ **b.** $(4x + 3) + (x - 2)$

c. $(x^2 + 2) - (3x^2 + 2x + 5)$ **d.** $(2x - 3x) - (x^2 - 2x + 4)$

7.1 Lesson

Core Vocabulary

monomial, *p. 338*
degree of a monomial, *p. 338*
polynomial, *p. 339*
binomial, *p. 339*
trinomial, *p. 339*
degree of a polynomial, *p. 339*
standard form, *p. 339*
leading coefficient, *p. 339*
closed, *p. 340*

What You Will Learn

- Find the degrees of monomials.
- Classify polynomials.
- Add and subtract polynomials.
- Solve real-life problems.

Finding the Degrees of Monomials

A **monomial** is a number, a variable, or the product of a number and one or more variables with whole number exponents.

The **degree of a monomial** is the sum of the exponents of the variables in the monomial. The degree of a nonzero constant term is 0. The constant 0 does not have a degree.

Monomial	Degree
10	0
$3x$	1
$\frac{1}{2}ab^2$	$1 + 2 = 3$
$-1.8m^5$	5

Not a monomial	Reason
$5 + x$	A sum is not a monomial.
$\frac{2}{n}$	A monomial cannot have a variable in the denominator.
4^a	A monomial cannot have a variable exponent.
x^{-1}	The variable must have a whole number exponent.

EXAMPLE 1 Finding the Degrees of Monomials

Find the degree of each monomial.

a. $5x^2$ **b.** $-\frac{1}{2}xy^3$ **c.** $8x^3y^3$ **d.** -3

SOLUTION

a. The exponent of x is 2.

▶ So, the degree of the monomial is 2.

b. The exponent of x is 1, and the exponent of y is 3.

▶ So, the degree of the monomial is $1 + 3$, or 4.

c. The exponent of x is 3, and the exponent of y is 3.

▶ So, the degree of the monomial is $3 + 3$, or 6.

d. You can rewrite -3 as $-3x^0$.

▶ So, the degree of the monomial is 0.

Monitoring Progress

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Find the degree of the monomial.

1. $-3x^4$ **2.** $7c^3d^2$ **3.** $\frac{5}{3}y$ **4.** -20.5

Classifying Polynomials

Core Concept

Polynomials

A **polynomial** is a monomial or a sum of monomials. Each monomial is called a *term* of the polynomial. A polynomial with two terms is a **binomial**. A polynomial with three terms is a **trinomial**.

Binomial	Trinomial
$5x + 2$	$x^2 + 5x + 2$

The **degree of a polynomial** is the greatest degree of its terms. A polynomial in one variable is in **standard form** when the exponents of the terms decrease from left to right. When you write a polynomial in standard form, the coefficient of the first term is the **leading coefficient**.

leading coefficient   degree   constant term

$$2x^3 + x^2 - 5x + 12$$

EXAMPLE 2 Writing a Polynomial in Standard Form

Write $15x - x^3 + 3$ in standard form. Identify the degree and leading coefficient of the polynomial.

SOLUTION

Consider the degree of each term of the polynomial.

Degree is 3.

Degree is 1. → $15x - x^3 + 3$ ← Degree is 0.

▶ You can write the polynomial in standard form as $-x^3 + 15x + 3$. The greatest degree is 3, so the degree of the polynomial is 3, and the leading coefficient is -1.

EXAMPLE 3 Classifying Polynomials

Write each polynomial in standard form. Identify the degree and classify each polynomial by the number of terms.

a. $-3z^4$ **b.** $4 + 5x^2 - x$ **c.** $8q + q^5$

SOLUTION

Polynomial	Standard Form	Degree	Type of Polynomial
a. $-3z^4$	$-3z^4$	4	monomial
b. $4 + 5x^2 - x$	$5x^2 - x + 4$	2	trinomial
c. $8q + q^5$	$q^5 + 8q$	5	binomial

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Write the polynomial in standard form. Identify the degree and leading coefficient of the polynomial. Then classify the polynomial by the number of terms.

5. $4 - 9z$ **6.** $t^2 - t^3 - 10t$ **7.** $2.8x + x^3$

Adding and Subtracting Polynomials

A set of numbers is **closed** under an operation when the operation performed on any two numbers in the set results in a number that is also in the set. For example, the set of integers is closed under addition, subtraction, and multiplication. This means that if a and b are two integers, then $a + b$, $a - b$, and ab are also integers.

The set of polynomials is closed under addition and subtraction. So, the sum or difference of any two polynomials is also a polynomial.

To add polynomials, add like terms. You can use a vertical or a horizontal format.

EXAMPLE 4 Adding Polynomials

Find the sum.

a. $(6x + 15) + (8x - 3)$ **b.** $(3x^2 + x - 6) + (x^2 + 4x + 10)$

SOLUTION

a. Vertical format: Align like terms vertically and add.

$$\begin{array}{rr} & 6x + 15 \\ + & 8x - 3 \\ \hline & 14x + 12 \end{array}$$

▶ The sum is $14x + 12$.

b. Horizontal format: Group like terms and simplify.

$$\begin{aligned}(3x^2 + x - 6) + (x^2 + 4x + 10) &= (3x^2 + x^2) + (x + 4x) + (-6 + 10) \\ &= 4x^2 + 5x + 4\end{aligned}$$

▶ The sum is $4x^2 + 5x + 4$.

To subtract a polynomial, add its opposite. To find the opposite of a polynomial, multiply each of its terms by -1.

EXAMPLE 5 Subtracting Polynomials

Find the difference.

a. $(4n + 5) - (-2n + 4)$ **b.** $(4x^2 - 3x + 5) - (3x^2 - x - 8)$

COMMON ERROR

Remember to multiply *each* term of the polynomial by -1 when you write the subtraction as addition.

SOLUTION

a. Vertical format: Align like terms vertically and subtract.

$$\begin{array}{rr} & 4n + 5 \\ - & (-2n + 4) \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{rr} & 4n + 5 \\ + & 2n - 4 \\ \hline & 6n + 1 \end{array}$$

▶ The difference is $6n + 1$.

b. Horizontal format: Group like terms and simplify.

$$\begin{aligned}(4x^2 - 3x + 5) - (3x^2 - x - 8) &= 4x^2 - 3x + 5 - 3x^2 + x + 8 \\ &= (4x^2 - 3x^2) + (-3x + x) + (5 + 8) \\ &= x^2 - 2x + 13\end{aligned}$$

▶ The difference is $x^2 - 2x + 13$.

Monitoring Progress

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Find the sum or difference.

8. $(b - 10) + (4b - 3)$

9. $(x^2 - x - 2) + (7x^2 - x + 8)$

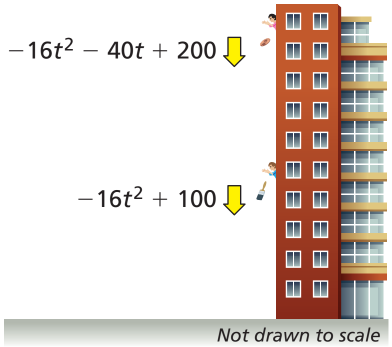
10. $(k + 5) - (3k - 6)$

11. $(p^2 + p + 3) - (-4p^2 - p + 3)$

Solving Real-Life Problems

EXAMPLE 6 Solving a Real-Life Problem

A penny is thrown straight down from a height of 200 feet. At the same time, a paintbrush is dropped from a height of 100 feet. The polynomials represent the heights (in feet) of the objects after t seconds.



a. Write a polynomial that represents the distance between the penny and the paintbrush after t seconds.

b. Interpret the coefficients of the polynomial in part (a).

STUDY TIP

When a power of the variable appears in one polynomial but not the other, leave a space in that column, or write the term with a coefficient of 0.

SOLUTION

a. To find the distance between the objects after t seconds, subtract the polynomials.

$$\begin{array}{lrcr}
\textbf{Penny} & -16t^2 - 40t + 200 & & -16t^2 - 40t + 200 \\
\textbf{Paintbrush} & \underline{-\ (-16t^2 \qquad\quad + 100)} & \rightarrow & \underline{+\quad 16t^2 \qquad\quad - 100} \\
 & & & -40t + 100
\end{array}$$

The polynomial $-40t + 100$ represents the distance between the objects after t seconds.

b. When $t = 0$, the distance between the objects is $-40(0) + 100 = 100$ feet. So, the constant term 100 represents the distance between the penny and the paintbrush when both objects begin to fall.

As the value of t increases by 1, the value of $-40t + 100$ decreases by 40. This means that the objects become 40 feet closer to each other each second. So, -40 represents the amount that the distance between the objects changes each second.

Monitoring Progress

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12. WHAT IF? The polynomial $-16t^2 - 25t + 200$ represents the height of the penny after t seconds.

a. Write a polynomial that represents the distance between the penny and the paintbrush after t seconds.

b. Interpret the coefficients of the polynomial in part (a).

7.1 Exercises

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Vocabulary and Core Concept Check

1. **VOCABULARY** When is a polynomial in one variable in standard form?

2. **OPEN-ENDED** Write a trinomial in one variable of degree 5 in standard form.

3. **VOCABULARY** How can you determine whether a set of numbers is closed under an operation?

4. **WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$a^3 + 4a$	$x^2 - 8^x$	$b - 2^{-1}$	$-\frac{\pi}{3} + 6y^8z$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, find the degree of the monomial. *(See Example 1.)*

5. $4g$
6. $23x^4$
7. $-1.75k^2$
8. $-\frac{4}{9}$
9. s^8t
10. $8m^2n^4$
11. $9xy^3z^7$
12. $-3q^4rs^6$

In Exercises 13–20, write the polynomial in standard form. Identify the degree and leading coefficient of the polynomial. Then classify the polynomial by the number of terms. *(See Examples 2 and 3.)*

13. $6c^2 + 2c^4 - c$
14. $4w^{11} - w^{12}$
15. $7 + 3p^2$
16. $8d - 2 - 4d^3$
17. $3t^8$
18. $5z + 2z^3 + 3z^4$
19. $\pi r^2 - \frac{5}{7}r^8 + 2r^5$
20. $\sqrt{7}n^4$

21. **MODELING WITH MATHEMATICS** The expression $\frac{4}{3}\pi r^3$ represents the volume of a sphere with radius r. Why is this expression a monomial? What is its degree?

22. **MODELING WITH MATHEMATICS** The amount of money you have after investing \$400 for 8 years and \$600 for 6 years at the same interest rate is represented by $400x^8 + 600x^6$, where x is the growth factor. Classify the polynomial by the number of terms. What is its degree?

In Exercises 23–30, find the sum. *(See Example 4.)*

23. $(5y + 4) + (-2y + 6)$
24. $(-8x - 12) + (9x + 4)$
25. $(2n^2 - 5n - 6) + (-n^2 - 3n + 11)$
26. $(-3p^2 + 5p - 2) + (-p^2 - 8p - 15)$
27. $(3g^2 - g) + (3g^2 - 8g + 4)$
28. $(9r^2 + 4r - 7) + (3r^2 - 3r)$
29. $(4a - a^3 - 3) + (2a^3 - 5a^2 + 8)$
30. $(s^3 - 2s - 9) + (2s^2 - 6s^3 + s)$

In Exercises 31–38, find the difference. *(See Example 5.)*

31. $(d - 9) - (3d - 1)$
32. $(6x + 9) - (7x + 1)$
33. $(y^2 - 4y + 9) - (3y^2 - 6y - 9)$
34. $(4m^2 - m + 2) - (-3m^2 + 10m + 4)$
35. $(k^3 - 7k + 2) - (k^2 - 12)$
36. $(-r - 10) - (-4r^3 + r^2 + 7r)$

37. $(t^4 - t^2 + t) - (12 - 9t^2 - 7t)$

38. $(4d - 6d^3 + 3d^2) - (10d^3 + 7d - 2)$

ERROR ANALYSIS **In Exercises 39 and 40, describe and correct the error in finding the sum or difference.**

39.

$$\begin{aligned}(x^2 + x) - (2x^2 - 3x) &= x^2 + x - 2x^2 - 3x \\ &= (x^2 - 2x^2) + (x - 3x) \\ &= -x^2 - 2x\end{aligned}$$

40.

$$\begin{array}{r} x^3 - 4x^2 + 3 \\ + \ -3x^3 + 8x \ - 2 \\ \hline -2x^3 + 4x^2 + 1 \end{array}$$

41. **MODELING WITH MATHEMATICS** The cost (in dollars) of making b bracelets is represented by $4 + 5b$. The cost (in dollars) of making b necklaces is represented by $8b + 6$. Write a polynomial that represents how much more it costs to make b necklaces than b bracelets.

42. **MODELING WITH MATHEMATICS** The number of individual memberships at a fitness center in m months is represented by $142 + 12m$. The number of family memberships at the fitness center in m months is represented by $52 + 6m$. Write a polynomial that represents the total number of memberships at the fitness center.

In Exercises 43–46, find the sum or difference.

43. $(2s^2 - 5st - t^2) - (s^2 + 7st - t^2)$

44. $(a^2 - 3ab + 2b^2) + (-4a^2 + 5ab - b^2)$

45. $(c^2 - 6d^2) + (c^2 - 2cd + 2d^2)$

46. $(-x^2 + 9xy) - (x^2 + 6xy - 8y^2)$

REASONING **In Exercises 47–50, complete the statement with *always*, *sometimes*, or *never*. Explain your reasoning.**

47. The terms of a polynomial are ________ monomials.

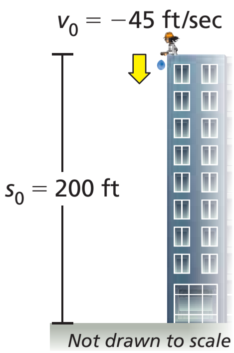
48. The difference of two trinomials is ________ a trinomial.

49. A binomial is ________ a polynomial of degree 2.

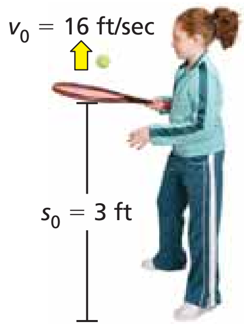
50. The sum of two polynomials is ________ a polynomial.

MODELING WITH MATHEMATICS **The polynomial $-16t^2 + v_0t + s_0$ represents the height (in feet) of an object, where v_0 is the initial vertical velocity (in feet per second), s_0 is the initial height of the object (in feet), and t is the time (in seconds). In Exercises 51 and 52, write a polynomial that represents the height of the object. Then find the height of the object after 1 second.**

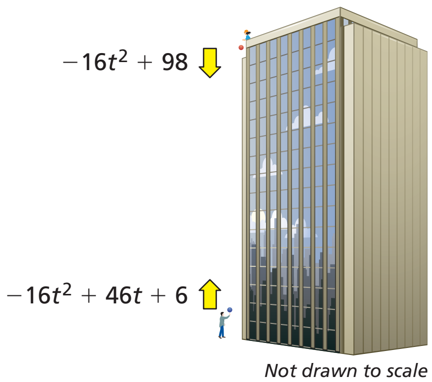
51. You throw a water balloon from a building.



52. You bounce a tennis ball on a racket.



53. **MODELING WITH MATHEMATICS** You drop a ball from a height of 98 feet. At the same time, your friend throws a ball upward. The polynomials represent the heights (in feet) of the balls after t seconds. *(See Example 6.)*



a. Write a polynomial that represents the distance between your ball and your friend's ball after t seconds.

b. Interpret the coefficients of the polynomial in part (a).

54. **MODELING WITH MATHEMATICS** During a 7-year period, the amounts (in millions of dollars) spent each year on buying new vehicles N and used vehicles U by United States residents are modeled by the equations

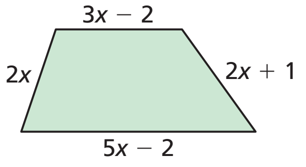
$$N = -0.028t^3 + 0.06t^2 + 0.1t + 17$$
$$U = -0.38t^2 + 1.5t + 42$$

where $t = 1$ represents the first year in the 7-year period.

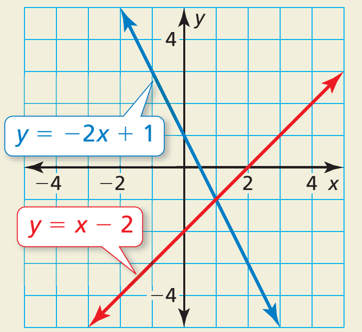
a. Write a polynomial that represents the total amount spent each year on buying new and used vehicles in the 7-year period.

b. How much is spent on buying new and used vehicles in the fifth year?

55. **MATHEMATICAL CONNECTIONS** Write the polynomial in standard form that represents the perimeter of the quadrilateral.



56. **HOW DO YOU SEE IT?** The right side of the equation of each line is a polynomial.



a. Explain how you can use the polynomials to find the vertical distance between points on the lines with the same x-value.

b. What do you know about the vertical distance when $x = 1$? How does this relate to your answer in part (a)?

57. **MAKING AN ARGUMENT** Your friend says that when adding polynomials, the order in which you add does not matter. Is your friend correct? Explain.

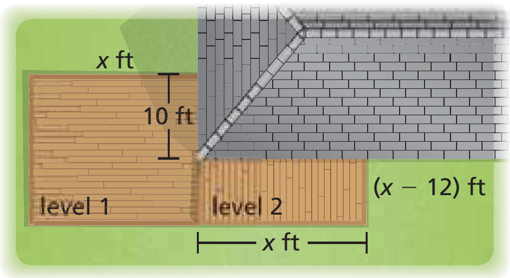
58. **THOUGHT PROVOKING** Write two polynomials whose sum is x^2 and whose difference is 1.

59. **REASONING** Determine whether the set is closed under the given operation. Explain.

a. the set of negative integers; multiplication

b. the set of whole numbers; addition

60. **PROBLEM SOLVING** You are building a multi-level deck.

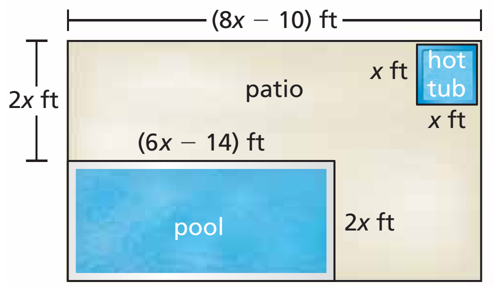


a. For each level, write a polynomial in standard form that represents the area of that level. Then write the polynomial in standard form that represents the total area of the deck.

b. What is the total area of the deck when $x = 20$?

c. A gallon of deck sealant covers 400 square feet. How many gallons of sealant do you need to cover the deck in part (b) once? Explain.

61. **PROBLEM SOLVING** A hotel installs a new swimming pool and a new hot tub.



a. Write the polynomial in standard form that represents the area of the patio.

b. The patio will cost \$10 per square foot. Determine the cost of the patio when $x = 9$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

62. $2(x - 1) + 3(x + 2)$

63. $8(4y - 3) + 2(y - 5)$

64. $5(2r + 1) - 3(-4r + 2)$

7.2 Multiplying Polynomials

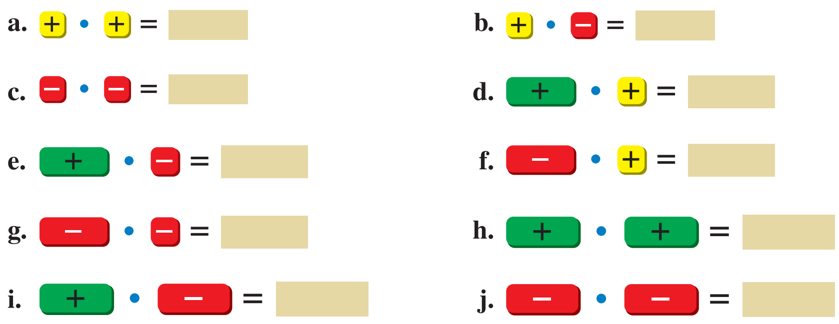
Texas Essential Knowledge and Skills
A.10.B
A.10.D

Essential Question

How can you multiply two polynomials?

EXPLORATION 1 Multiplying Monomials Using Algebra Tiles

Work with a partner. Write each product. Explain your reasoning.



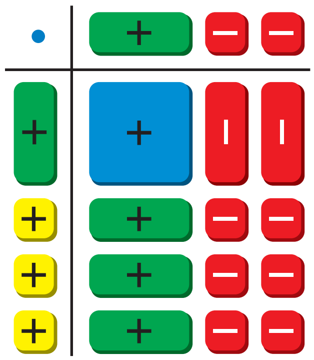
REASONING

To be proficient in math, you need to reason abstractly and quantitatively. You need to pause as needed to recall the meanings of the symbols, operations, and quantities involved.

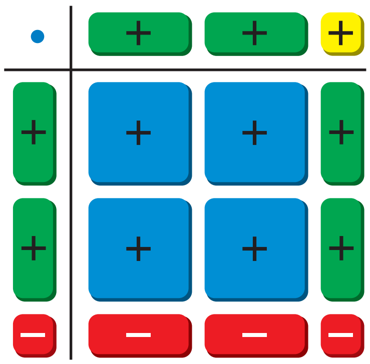
EXPLORATION 2 Multiplying Binomials Using Algebra Tiles

Work with a partner. Write the product of two binomials modeled by each rectangular array of algebra tiles. In parts (c) and (d), first draw the rectangular array of algebra tiles that models each product.

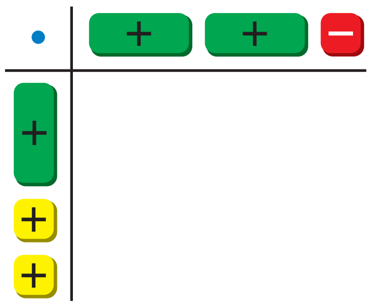
a. $(x + 3)(x - 2) =$



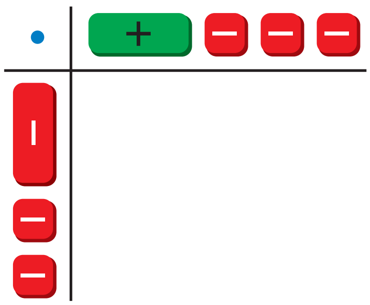
b. $(2x - 1)(2x + 1) =$



c. $(x + 2)(2x - 1) =$



d. $(-x - 2)(x - 3) =$



Communicate Your Answer

3. How can you multiply two polynomials?

4. Give another example of multiplying two binomials using algebra tiles that is similar to those in Exploration 2.

7.2 Lesson

Core Vocabulary

FOIL Method, *p. 347*

Previous
polynomial
closed
binomial
trinomial

What You Will Learn

- Multiply binomials.
- Use the FOIL Method.
- Multiply binomials and trinomials.

Multiplying Binomials

The product of two polynomials is always a polynomial. So, like the set of integers, the set of polynomials is closed under multiplication. You can use the Distributive Property to multiply two binomials.

EXAMPLE 1 Multiplying Binomials Using the Distributive Property

Find (a) $(x + 2)(x + 5)$ and (b) $(x + 3)(x - 4)$.

SOLUTION

a. Use the horizontal method.

$$(x + 2)(x + 5) = x(x + 5) + 2(x + 5)$$ Distribute $(x + 5)$ to each term of $(x + 2)$.

$$= x(x) + x(5) + 2(x) + 2(5)$$ Distributive Property

$$= x^2 + 5x + 2x + 10$$ Multiply.

$$= x^2 + 7x + 10$$ Combine like terms.

▶ The product is $x^2 + 7x + 10$.

b. Use the vertical method.

$$\begin{array}{rr} & x + 3 \\ \times & x - 4 \\ \hline & -4x - 12 \\ & x^2 + 3x \quad\; \\ \hline & x^2 - x - 12 \end{array}$$

Align like terms vertically.
Multiply $-4(x + 3)$. Distributive Property
Multiply $x(x + 3)$. Distributive Property
Combine like terms.

▶ The product is $x^2 - x - 12$.

EXAMPLE 2 Multiplying Binomials Using a Table

Find $(2x - 3)(x + 5)$.

SOLUTION

Step 1 Write each binomial as a sum of terms.

$$(2x - 3)(x + 5) = [2x + (-3)](x + 5)$$

Step 2 Make a table of products.

	$2x$	-3
x	$2x^2$	$-3x$
5	$10x$	-15

▶ The product is $2x^2 - 3x + 10x - 15$, or $2x^2 + 7x - 15$.

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Use the Distributive Property to find the product.

1. $(y + 4)(y + 1)$ **2.** $(z - 2)(z + 6)$

Use a table to find the product.

3. $(p + 3)(p - 8)$ **4.** $(r - 5)(2r - 1)$

Using the FOIL Method

The **FOIL Method** is a shortcut for multiplying two binomials.

Core Concept

FOIL Method

To multiply two binomials using the FOIL Method, find the sum of the products of the

First terms, $(x + 1)(x + 2)$ → $x(x) = x^2$

Outer terms, $(x + 1)(x + 2)$ → $x(2) = 2x$

Inner terms, and $(x + 1)(x + 2)$ → $1(x) = x$

Last terms. $(x + 1)(x + 2)$ → $1(2) = 2$

$$(x + 1)(x + 2) = x^2 + 2x + x + 2 = x^2 + 3x + 2$$

EXAMPLE 3 Multiplying Binomials Using the FOIL Method

Find each product.

a. $(x - 3)(x - 6)$

b. $(2x^2 + 1)(3x^2 - 5)$

SOLUTION

a. Use the FOIL Method.

First Outer Inner Last

$(x - 3)(x - 6) = x(x) + x(-6) + (-3)(x) + (-3)(-6)$ FOIL Method

$= x^2 + (-6x) + (-3x) + 18$ Multiply.

$= x^2 - 9x + 18$ Combine like terms.

▶ The product is $x^2 - 9x + 18$.

b. Use the FOIL Method.

First Outer Inner Last

$(2x^2 + 1)(3x^2 - 5) = 2x^2(3x^2) + 2x^2(-5) + 1(3x^2) + 1(-5)$ FOIL Method

$= 6x^4 + (-10x^2) + 3x^2 + (-5)$ Multiply.

$= 6x^4 - 7x^2 - 5$ Combine like terms.

▶ The product is $6x^4 - 7x^2 - 5$.

Monitoring Progress

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Use the FOIL Method to find the product.

5. $(m - 3)(m - 7)$

6. $(x - 4)(x + 2)$

7. $(n^2 + 2)(n^2 + 3)$

8. $\left(2u^2 + \frac{1}{2}\right)\left(4u^2 - \frac{3}{2}\right)$

Multiplying Binomials and Trinomials

EXAMPLE 4 Multiplying a Binomial and a Trinomial

Find $(x + 5)(x^2 - 3x - 2)$.

SOLUTION

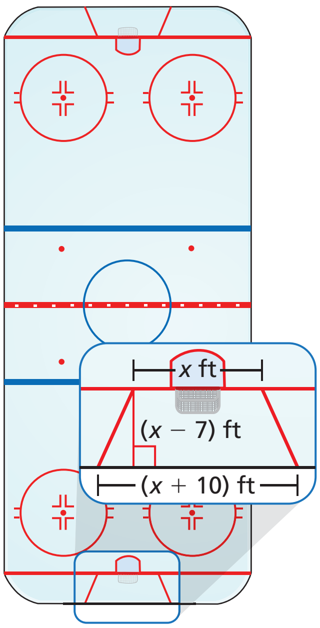
$$\begin{array}{rrrrl}
 & x^2 & -\ 3x & -\ 2 & \text{Align like terms vertically.} \\
\times & & x & +\ 5 & \\
\hline
 & 5x^2 & -\ 15x & -\ 10 & \text{Distributive Property} \\
x^3 & -\ 3x^2 & -\ 2x & & \text{Distributive Property} \\
\hline
x^3 & +\ 2x^2 & -\ 17x & -\ 10 & \text{Combine like terms.}
\end{array}$$

Multiply $5(x^2 - 3x - 2)$.

Multiply $x(x^2 - 3x - 2)$.

▶ The product is $x^3 + 2x^2 - 17x - 10$.

EXAMPLE 5 Solving a Real-Life Problem



In hockey, a goalie behind the goal line can only play a puck in the trapezoidal region.

a. Write a polynomial that represents the area of the trapezoidal region.

b. Find the area of the trapezoidal region when the shorter base is 18 feet.

SOLUTION

a. $\frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(x - 7)[x + (x + 10)]$ Substitute.

$= \frac{1}{2}(x - 7)(2x + 10)$ Combine like terms.

F O I L

$= \frac{1}{2}[2x^2 + 10x + (-14x) + (-70)]$ FOIL Method

$= \frac{1}{2}(2x^2 - 4x - 70)$ Combine like terms.

$= x^2 - 2x - 35$ Distributive Property

▶ A polynomial that represents the area of the trapezoidal region is $x^2 - 2x - 35$.

b. Find the value of $x^2 - 2x - 35$ when $x = 18$.

$x^2 - 2x - 35 = 18^2 - 2(18) - 35$ Substitute 18 for x.

$= 324 - 36 - 35$ Simplify.

$= 253$ Subtract.

▶ The area of the trapezoidal region is 253 square feet.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Find the product.

9. $(x + 1)(x^2 + 5x + 8)$

10. $(n - 3)(n^2 - 2n + 4)$

11. **WHAT IF?** In Example 5(a), how does the polynomial change when the longer base is extended by 1 foot? Explain.

7.2 Exercises

Vocabulary and Core Concept Check

1. **VOCABULARY** Describe two ways to find the product of two binomials.

2. **WRITING** Explain how the letters of the word FOIL can help you to remember how to multiply two binomials.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, use the Distributive Property to find the product. *(See Example 1.)*

3. $(x + 1)(x + 3)$
4. $(y + 6)(y + 4)$
5. $(z - 5)(z + 3)$
6. $(a + 8)(a - 3)$
7. $(g - 7)(g - 2)$
8. $(n - 6)(n - 4)$
9. $(3m + 1)(m + 9)$
10. $(5s + 6)(s - 2)$

In Exercises 11–18, use a table to find the product. *(See Example 2.)*

11. $(x + 3)(x + 2)$
12. $(y + 10)(y - 5)$
13. $(h - 8)(h - 9)$
14. $(c - 6)(c - 5)$
15. $(3k - 1)(4k + 9)$
16. $(5g + 3)(g + 8)$
17. $(-3 + 2j)(4j - 7)$
18. $(5d - 12)(-7 + 3d)$

ERROR ANALYSIS In Exercises 19 and 20, describe and correct the error in finding the product of the binomials.

19.
$$\begin{aligned}(t-2)(t+5) &= t - 2(t+5)\\ &= t - 2t - 10\\ &= -t - 10\end{aligned}$$

20. $(x - 5)(3x + 1)$

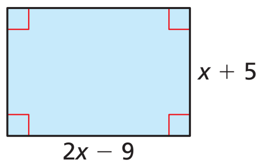
	$3x$	1
x	$3x^2$	x
5	$15x$	5

$(x - 5)(3x + 1) = 3x^2 + 16x + 5$

In Exercises 21–30, use the FOIL Method to find the product. *(See Example 3.)*

21. $(b + 3)(b + 7)$
22. $(w + 9)(w + 6)$
23. $\left(q - \frac{3}{4}\right)\left(q + \frac{1}{4}\right)$
24. $\left(z - \frac{5}{3}\right)\left(z - \frac{2}{3}\right)$
25. $(9 - r)(2 - 3r)$
26. $(8 - 4x)(2x + 6)$
27. $(6s^2 + 1)(2s^2 - 9)$
28. $(2x^2 - 4)(5x^2 + 3)$
29. $(w + 5)(w^2 + 3w)$
30. $(v - 3)(v^2 + 8v)$

MATHEMATICAL CONNECTIONS In Exercises 31–34, write a polynomial that represents the area of the shaded region.

31.


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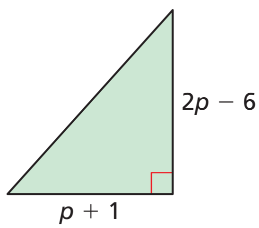

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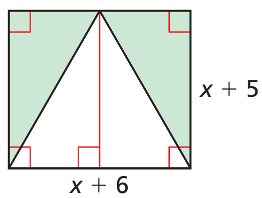
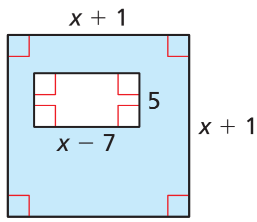
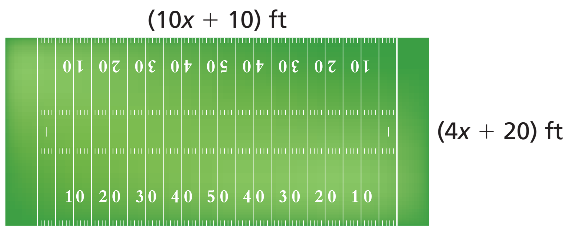


34.


In Exercises 35–42, find the product. *(See Example 4.)*

35. $(x + 4)(x^2 + 3x + 2)$
36. $(f + 1)(f^2 + 4f + 8)$
37. $(y + 3)(y^2 + 8y - 2)$
38. $(t - 2)(t^2 - 5t + 1)$
39. $(4 - b)(5b^2 + 5b - 4)$
40. $(d + 6)(2d^2 - d + 7)$
41. $(3e^2 - 5e + 7)(6e^2 + 1)$
42. $(6v^2 + 2v - 9)(4 - 5v^2)$

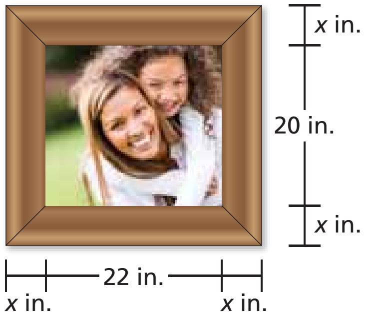
43. MODELING WITH MATHEMATICS The football field is rectangular. *(See Example 5.)*



a. Write a polynomial that represents the area of the football field.

b. Find the area of the football field when the width is 160 feet.

44. MODELING WITH MATHEMATICS You design a frame to surround a rectangular photo. The width of the frame is the same on every side, as shown.



a. Write a polynomial that represents the combined area of the photo and the frame.

b. Find the combined area of the photo and the frame when the width of the frame is 4 inches.

In Exercises 45–50, find the product.

45. $(y^3 + 2y^2)(y^2 - 3y)$

46. $(s^2 + 5s)(s^3 + 4s^2)$

47. $(x^2 - 2)(x^3 + 4x^2 + x)$

48. $(h^2 + 3)(h^3 - 2h^2 + h)$

49. $(r^2 + r + 1)(r^2 + 2r - 3)$

50. $(t^2 - 4t + 2)(t^2 + 5t + 1)$

51. WRITING When multiplying two binomials, explain how the degree of the product is related to the degree of each binomial.

52. THOUGHT PROVOKING Write two polynomials that are not monomials whose product is a trinomial of degree 3.

53. MAKING AN ARGUMENT Your friend says the FOIL Method can be used to multiply two trinomials. Is your friend correct? Explain your reasoning.

54. HOW DO YOU SEE IT? The table shows one method of finding the product of two binomials.

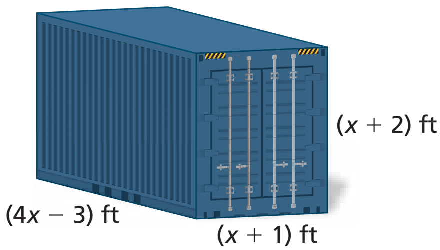
	$-4x$	3
$-8x$	a	b
-9	c	d

a. Write the two binomials being multiplied.

b. Determine whether a, b, c, and d will be positive or negative when $x > 0$.

55. COMPARING METHODS You use the Distributive Property to multiply $(x + 3)(x - 5)$. Your friend uses the FOIL Method to multiply $(x - 5)(x + 3)$. Should your answers be equivalent? Justify your answer.

56. USING STRUCTURE The shipping container is a rectangular prism. Write a polynomial that represents the volume of the container.



57. ABSTRACT REASONING The product of $(x + m)(x + n)$ is $x^2 + bx + c$.

a. What do you know about m and n when $c > 0$?

b. What do you know about m and n when $c < 0$?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the slope and the *y*-intercept of the graph of the linear equation. *(Section 3.5)*

58. $y = -6x + 7$

59. $y = \frac{1}{4}x + 7$

60. $3y = 6x - 12$

61. $2y + x = 8$

7.3 Special Products of Polynomials

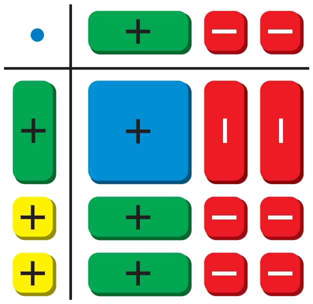
TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.10.B

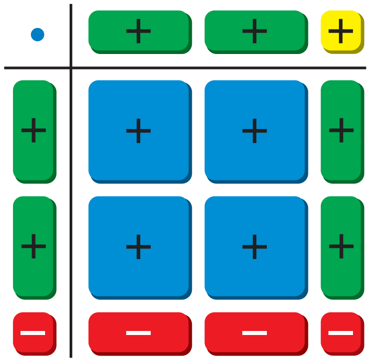
Essential Question What are the patterns in the special products $(a + b)(a - b)$, $(a + b)^2$, and $(a - b)^2$?

EXPLORATION 1 Finding a Sum and Difference Pattern

Work with a partner. Write the product of two binomials modeled by each rectangular array of algebra tiles.

a. $(x + 2)(x - 2) =$ ______ **b.** $(2x - 1)(2x + 1) =$ ______

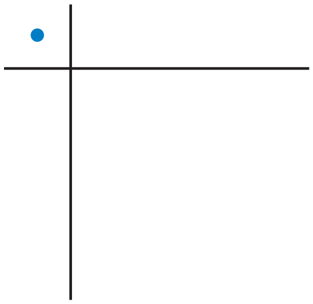


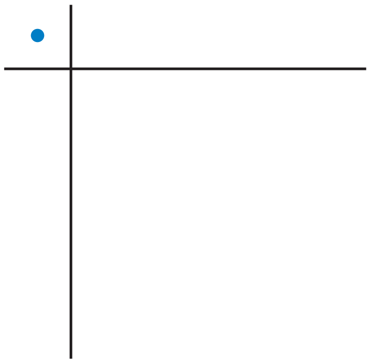


EXPLORATION 2 Finding the Square of a Binomial Pattern

Work with a partner. Draw the rectangular array of algebra tiles that models each product of two binomials. Write the product.

a. $(x + 2)^2 =$ ______ **b.** $(2x - 1)^2 =$ ______





ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to look closely to discern a pattern or structure.

Communicate Your Answer

3. What are the patterns in the special products $(a + b)(a - b)$, $(a + b)^2$, and $(a - b)^2$?

4. Use the appropriate special product pattern to find each product. Check your answers using algebra tiles.

a. $(x + 3)(x - 3)$ **b.** $(x - 4)(x + 4)$ **c.** $(3x + 1)(3x - 1)$

d. $(x + 3)^2$ **e.** $(x - 2)^2$ **f.** $(3x + 1)^2$

7.3 Lesson

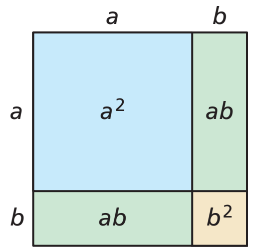
Core Vocabulary

Previous
binomial

What You Will Learn

- Use the square of a binomial pattern.
- Use the sum and difference pattern.
- Use special product patterns to solve real-life problems.

Using the Square of a Binomial Pattern



The diagram shows a square with a side length of $(a + b)$ units. You can see that the area of the square is

$$(a + b)^2 = a^2 + 2ab + b^2.$$

This is one version of a pattern called the square of a binomial. To find another version of this pattern, use algebra: replace b with $-b$.

$(a + (-b))^2 = a^2 + 2a(-b) + (-b)^2$ — Replace b with $-b$ in the pattern above.

$(a - b)^2 = a^2 - 2ab + b^2$ — Simplify.

Core Concept

Square of a Binomial Pattern

Algebra	Example
$(a + b)^2 = a^2 + 2ab + b^2$	$(x + 5)^2 = (x)^2 + 2(x)(5) + (5)^2$ $= x^2 + 10x + 25$
$(a - b)^2 = a^2 - 2ab + b^2$	$(2x - 3)^2 = (2x)^2 - 2(2x)(3) + (3)^2$ $= 4x^2 - 12x + 9$

ANALYZING MATHEMATICAL RELATIONSHIPS

When you use special product patterns, remember that a and b can be numbers, variables, or variable expressions.

EXAMPLE 1 Using the Square of a Binomial Pattern

Find each product.

a. $(3x + 4)^2$ **b.** $(5x - 2y)^2$

SOLUTION

a. $(3x + 4)^2 = (3x)^2 + 2(3x)(4) + 4^2$ — Square of a binomial pattern

$= 9x^2 + 24x + 16$ — Simplify.

The product is $9x^2 + 24x + 16$.

b. $(5x - 2y)^2 = (5x)^2 - 2(5x)(2y) + (2y)^2$ — Square of a binomial pattern

$= 25x^2 - 20xy + 4y^2$ — Simplify.

The product is $25x^2 - 20xy + 4y^2$.

Monitoring Progress

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Find the product.

1. $(x + 7)^2$ **2.** $(7x - 3)^2$ **3.** $(4x - y)^2$ **4.** $(3m + n)^2$

Using the Sum and Difference Pattern

To find the product $(x + 2)(x - 2)$, you can multiply the two binomials using the FOIL Method.

$$(x + 2)(x - 2) = x^2 - 2x + 2x - 4 \qquad \text{FOIL Method}$$
$$= x^2 - 4 \qquad \text{Combine like terms.}$$

This suggests a pattern for the product of the sum and difference of two terms.

Core Concept

Sum and Difference Pattern

Algebra	**Example**
$(a + b)(a - b) = a^2 - b^2$	$(x + 3)(x - 3) = x^2 - 9$

EXAMPLE 2 Using the Sum and Difference Pattern

Find each product.

a. $(t + 5)(t - 5)$ **b.** $(3x + y)(3x - y)$

SOLUTION

a. $(t + 5)(t - 5) = t^2 - 5^2$ Sum and difference pattern

$= t^2 - 25$ Simplify.

▶ The product is $t^2 - 25$.

b. $(3x + y)(3x - y) = (3x)^2 - y^2$ Sum and difference pattern

$= 9x^2 - y^2$ Simplify.

▶ The product is $9x^2 - y^2$.

The special product patterns can help you use mental math to find certain products of numbers.

EXAMPLE 3 Using Special Product Patterns and Mental Math

Use special product patterns to find the product $26 \cdot 34$.

SOLUTION

Notice that 26 is 4 less than 30, while 34 is 4 more than 30.

$26 \cdot 34 = (30 - 4)(30 + 4)$ Write as product of difference and sum.

$= 30^2 - 4^2$ Sum and difference pattern

$= 900 - 16$ Evaluate powers.

$= 884$ Simplify.

▶ The product is 884.

Monitoring Progress

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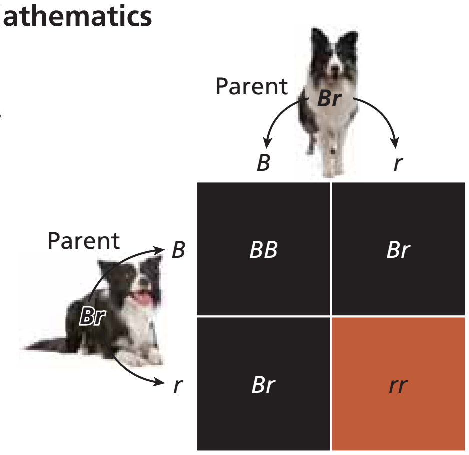
Find the product.

5. $(x + 10)(x - 10)$ **6.** $(2x + 1)(2x - 1)$ **7.** $(x + 3y)(x - 3y)$

8. Describe how to use special product patterns to find 21^2.

Solving Real-Life Problems

EXAMPLE 4 **Modeling with Mathematics**



A combination of two genes determines the color of the dark patches of a border collie's coat. An offspring inherits one patch color gene from each parent. Each parent has two color genes, and the offspring has an equal chance of inheriting either one.

The gene B is for black patches, and the gene r is for red patches. Any gene combination with a B results in black patches. Suppose each parent has the same gene combination Br. The Punnett square shows the possible gene combinations of the offspring and the resulting patch colors.

a. What percent of the possible gene combinations result in black patches?

b. Show how you could use a polynomial to model the possible gene combinations.

SOLUTION

a. Notice that the Punnett square shows four possible gene combinations of the offspring. Of these combinations, three result in black patches.

▶ So, 75% of the possible gene combinations result in black patches.

b. Model the gene from each parent with $0.5B + 0.5r$. There is an equal chance that the offspring inherits a black or a red gene from each parent.

You can model the possible gene combinations of the offspring with $(0.5B + 0.5r)^2$. Notice that this product also represents the area of the Punnett square.

Expand the product to find the possible patch colors of the offspring.

$$(0.5B + 0.5r)^2 = (0.5B)^2 + 2(0.5B)(0.5r) + (0.5r)^2$$

$$= 0.25B^2 + 0.5Br + 0.25r^2$$

Consider the coefficients in the polynomial.

$$0.25B^2 + 0.5Br + 0.25r^2$$

25% BB, black patches

50% Br, black patches

25% rr, red patches

The coefficients show that 25% + 50% = 75% of the possible gene combinations result in black patches.

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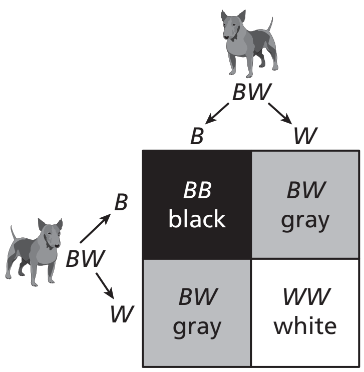

9. Each of two dogs has one black gene (B) and one white gene (W). The Punnett square shows the possible gene combinations of an offspring and the resulting colors.

a. What percent of the possible gene combinations result in black?

b. Show how you could use a polynomial to model the possible gene combinations of the offspring.

7.3 Exercises

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Vocabulary and Core Concept Check

1. WRITING Explain how to use the square of a binomial pattern.

2. WHICH ONE DOESN'T BELONG? Which expression does *not* belong with the other three? Explain your reasoning.

$(x + 1)(x - 1)$ $\quad$ $(3x + 2)(3x - 2)$ $\quad$ $(x + 2)(x - 3)$ $\quad$ $(2x + 5)(2x - 5)$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the product. *(See Example 1.)*

3. $(x + 8)^2$

4. $(a - 6)^2$

5. $(2f - 1)^2$

6. $(5p + 2)^2$

7. $(-7t + 4)^2$

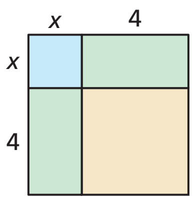
8. $(-12 - n)^2$

9. $(2a + b)^2$

10. $(6x - 3y)^2$

MATHEMATICAL CONNECTIONS **In Exercises 11–14, write a polynomial that represents the area of the square.**

11.



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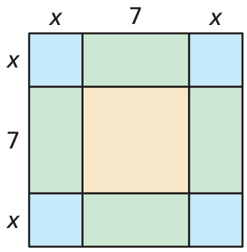

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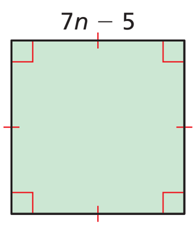
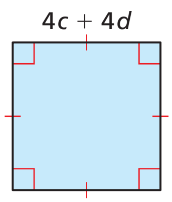


14.



In Exercises 15–24, find the product. *(See Example 2.)*

15. $(t - 7)(t + 7)$

16. $(m + 6)(m - 6)$

17. $(4x + 1)(4x - 1)$

18. $(2k - 4)(2k + 4)$

19. $(8 + 3a)(8 - 3a)$

20. $\left(\frac{1}{2} - c\right)\left(\frac{1}{2} + c\right)$

21. $(p - 10q)(p + 10q)$

22. $(7m + 8n)(7m - 8n)$

23. $(-y + 4)(-y - 4)$

24. $(-5g - 2h)(-5g + 2h)$

In Exercises 25–30, use special product patterns to find the product. *(See Example 3.)*

25. $16 \cdot 24$

26. $33 \cdot 27$

27. 42^2

28. 29^2

29. 30.5^2

30. $10\frac{1}{3} \cdot 9\frac{2}{3}$

ERROR ANALYSIS **In Exercises 31 and 32, describe and correct the error in finding the product.**

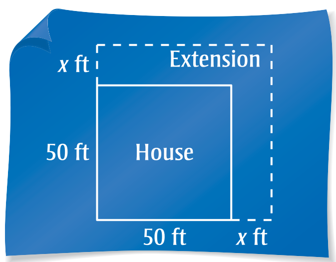
31.

$$\begin{aligned}(k + 4)^2 &= k^2 + 4^2 \\ &= k^2 + 16\end{aligned}$$

32.

$$\begin{aligned}(s + 5)(s - 5) &= s^2 + 2(s)(5) - 5^2 \\ &= s^2 + 10s - 25\end{aligned}$$

33. MODELING WITH MATHEMATICS A contractor extends a house on two sides.



a. The area of the house after the renovation is represented by $(x + 50)^2$. Find this product.

b. Use the polynomial in part (a) to find the area when $x = 15$. What is the area of the extension?

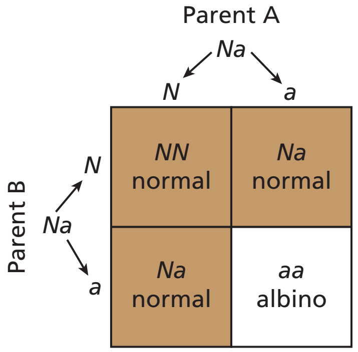
34. **MODELING WITH MATHEMATICS** A square-shaped parking lot with 100-foot sides is reduced by x feet on one side and extended by x feet on an adjacent side.

a. The area of the new parking lot is represented by $(100 - x)(100 + x)$. Find this product.

b. Does the area of the parking lot increase, decrease, or stay the same? Explain.

c. Use the polynomial in part (a) to find the area of the new parking lot when $x = 21$.

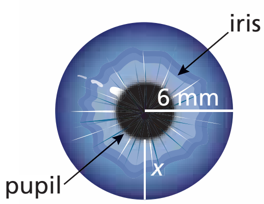
35. **MODELING WITH MATHEMATICS** In deer, the gene N is for normal coloring and the gene a is for no coloring, or albino. Any gene combination with an N results in normal coloring. The Punnett square shows the possible gene combinations of an offspring and the resulting colors from parents that both have the gene combination Na. *(See Example 4.)*



a. What percent of the possible gene combinations result in albino coloring?

b. Show how you could use a polynomial to model the possible gene combinations of the offspring.

36. **MODELING WITH MATHEMATICS** Your iris controls the amount of light that enters your eye by changing the size of your pupil.

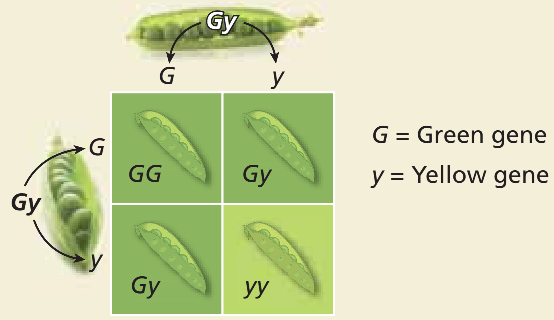


a. Write a polynomial that represents the area of your pupil. Write your answer in terms of π.

b. The width x of your iris decreases from 4 millimeters to 2 millimeters when you enter a dark room. How many times greater is the area of your pupil after entering the room than before entering the room? Explain.

37. **CRITICAL THINKING** Write two binomials that have the product $x^2 - 121$. Explain.

38. **HOW DO YOU SEE IT?** In pea plants, any gene combination with a green gene (G) results in a green pod. The Punnett square shows the possible gene combinations of the offspring of two Gy pea plants and the resulting pod colors.



A polynomial that models the possible gene combinations of the offspring is

$$(0.5G + 0.5y)^2 = 0.25G^2 + 0.5Gy + 0.25y^2.$$

Describe two ways to determine the percent of possible gene combinations that result in green pods.

In Exercises 39–42, find the product.

39. $(x^2 + 1)(x^2 - 1)$

40. $(y^3 + 4)^2$

41. $(2m^2 - 5n^2)^2$

42. $(r^3 - 6t^4)(r^3 + 6t^4)$

43. **MAKING AN ARGUMENT** Your friend claims to be able to use a special product pattern to determine that $\left(4\frac{1}{3}\right)^2$ is equal to $16\frac{1}{9}$. Is your friend correct? Explain.

44. **THOUGHT PROVOKING** The area (in square meters) of the surface of an artificial lake is represented by x^2. Describe three ways to modify the dimensions of the lake so that the new area can be represented by the three types of special product patterns discussed in this section.

45. **REASONING** Find k so that $9x^2 - 48x + k$ is the square of a binomial.

46. **REPEATED REASONING** Find $(x + 1)^3$ and $(x + 2)^3$. Find a pattern in the terms and use it to write a pattern for the cube of a binomial $(a + b)^3$.

47. **PROBLEM SOLVING** Find two numbers a and b such that $(a + b)(a - b) < (a - b)^2 < (a + b)^2$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. Write your answer using only positive exponents. *(Skills Review Handbook)*

48. $\dfrac{10^2}{10^5}$

49. $\dfrac{x^5 \cdot x}{x^8}$

50. $(3z^6)^{-3}$

51. $\left(\dfrac{2y^4}{y^3}\right)^{-2}$

7.4 Dividing Polynomials

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.10.C

Essential Question

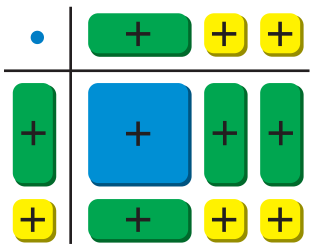
How can you use algebra tiles to divide two polynomials?

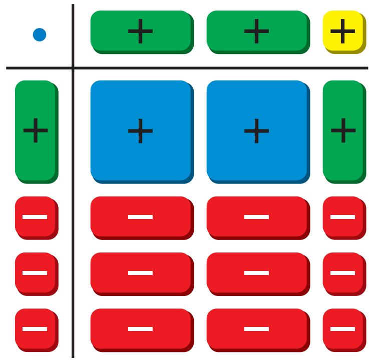
EXPLORATION 1 Dividing Polynomials Using Algebra Tiles

Work with a partner. Use algebra tiles to write each quotient.

a. $\dfrac{(x^2 + 3x + 2)}{(x + 1)} = $ ______

b. $\dfrac{(2x^2 - 5x - 3)}{(x - 3)} = $ ______



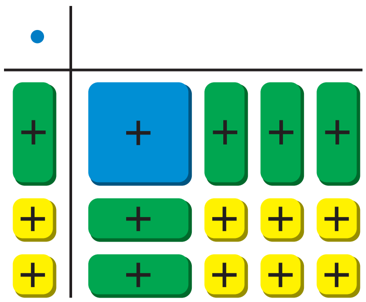


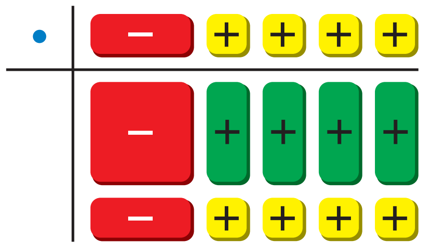
EXPLORATION 2 Dividing Polynomials Using Algebra Tiles

Work with a partner. Use algebra tiles to write each quotient.

a. $\dfrac{(x^2 + 5x + 6)}{(x + 2)} = $ ______

b. $\dfrac{(-x^2 + 3x + 4)}{(-x + 4)} = $ ______





c. $\dfrac{(x^2 + 4x - 5)}{(x - 1)} = $ ______

d. $\dfrac{(2x^2 - 6x + 4)}{(2x - 2)} = $ ______

ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to look closely to discern a pattern or structure.

Communicate Your Answer

3. How can you use algebra tiles to divide two polynomials?

4. Give another example of dividing two polynomials using algebra tiles that is similar to those in Explorations 1 and 2.

7.4 Lesson

What You Will Learn

- Divide polynomials by monomials.
- Use polynomial long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $x - k$.

Core Vocabulary

polynomial long division, *p. 358*
synthetic division, *p. 360*

Previous
polynomial
degree of a polynomial

Dividing Polynomials by Monomials

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

EXAMPLE 1 **Dividing a Polynomial by a Monomial**

Find $(6x^2 + 12x) \div (3x)$.

SOLUTION

$$\frac{6x^2 + 12x}{3x} = \frac{6x^2}{3x} + \frac{12x}{3x}$$

$$= 2x + 4$$

▶ $(6x^2 + 12x) \div (3x) = 2x + 4$

REMEMBER
When dividing terms with exponents, use the Quotient of Powers Property: $\frac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$.

Monitoring Progress

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Divide.

1. $(8x - 24) \div 4$

2. $(20x^2 + 5x) \div (-5x)$

Long Division of Polynomials

When you divide a polynomial $f(x)$ by a nonzero polynomial divisor $d(x)$, you get a quotient polynomial $q(x)$ and a remainder polynomial $r(x)$.

$$\underbrace{\frac{f(x)}{d(x)}}_{\text{dividend / divisor}} = \underbrace{q(x)}_{\text{quotient}} + \underbrace{\frac{r(x)}{d(x)}}_{\text{remainder}}$$

The degree of the remainder must be less than the degree of the divisor. When the remainder is 0, the divisor *divides evenly* into the dividend. Also, the degree of the divisor is less than or equal to the degree of the dividend $f(x)$. One way to divide polynomials is called **polynomial long division**.

Core Concept

Long Division of Polynomials

Step 1 Write the dividend and divisor in standard form.

Step 2 Insert placeholders with zero coefficients for missing powers of the variable.

Step 3 Perform the long division of the polynomials as you would with integers. At each stage, divide the term with the highest power in what is left of the dividend by the first term of the divisor. Line up the terms in the quotient with the like term in the dividend.

Step 4 Continue until the degree of the remainder is less than the degree of the divisor.

EXAMPLE 2 Using Polynomial Long Division

Find $(4x + 5) \div (x + 1)$. Determine whether the divisor evenly divides into the dividend.

SOLUTION

$$\begin{array}{r l}
\;4 & \\
x + 1\overline{)4x + 5} & \\
\underline{4x + 4} & \quad \text{Multiply divisor by } \frac{4x}{x} = 4. \\
1 & \quad \text{Subtract.}
\end{array}$$

▶ $(4x + 5) \div (x + 1) = 4 + \dfrac{1}{x + 1}$. Because the remainder does not equal 0, the divisor does not evenly divide into the dividend.

Check

You can check the result of a division problem by multiplying the quotient by the divisor and adding the remainder. The result should be the dividend.

$$\begin{aligned}(4)(x + 1) + 1 &= (4x + 4) + 1 \\ &= 4x + 5 \ \checkmark\end{aligned}$$

COMMON ERROR

The expression added to the quotient in the result of a long division problem is $\dfrac{r(x)}{d(x)}$, not $r(x)$.

EXAMPLE 3 Using Polynomial Long Division

Find $(6x^2 + 17x + 20) \div (3x + 4)$.

SOLUTION

$$\begin{array}{r l}
2x + \;\;3 & \\
3x + 4\overline{)6x^2 + 17x + 20} & \\
\underline{6x^2 + \;\;8x} \qquad & \quad \text{Multiply divisor by } \frac{6x^2}{3x} = 2x. \\
9x + 20 & \quad \text{Subtract. Bring down next term.} \\
\underline{9x + 12} & \quad \text{Multiply divisor by } \frac{9x}{3x} = 3. \\
8 & \quad \text{Subtract.}
\end{array}$$

▶ $(6x^2 + 17x + 20) \div (3x + 4) = 2x + 3 + \dfrac{8}{3x + 4}$

Check

$$\begin{aligned}&(2x + 3)(3x + 4) + 8 \\ &= (6x^2 + 17x + 12) + 8 \\ &= 6x^2 + 17x + 20 \ \checkmark\end{aligned}$$

EXAMPLE 4 Using Polynomial Long Division

Find $(3x^2 + 2x + 15) \div (1 + x^2)$.

SOLUTION

$$\begin{array}{r l}
3 & \\
x^2 + 0x + 1\overline{)3x^2 + 2x + 15} & \\
\underline{3x^2 + 0x + \;\;3} & \quad \text{Multiply divisor by } \frac{3x^2}{x^2} = 3. \\
2x + 12 & \quad \text{Subtract.}
\end{array}$$

▶ $(3x^2 + 2x + 15) \div (1 + x^2) = 3 + \dfrac{2x + 12}{x^2 + 1}$

COMMON ERROR

Be sure to write $1 + x^2$ in standard form and insert $0x$ as a placeholder.

Monitoring Progress

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Use polynomial long division to divide.

3. $(5x - 4) \div (x + 3)$

4. $(8x + 7) \div (-4x - 2)$

5. $(7x^2 + 4x + 11) \div (x - 2)$

6. $(10x^2 - 12x + 4) \div (-5x + 1)$

7. $(-6x^2 + 2x + 18) \div (x^2 + 5)$

8. $(4x^2 + 18 + 6x) \div (-2x^2 + x + 3)$

Synthetic Division

There is a shortcut for dividing polynomials by binomials of the form $x - k$. This shortcut is called **synthetic division**. The method is shown in the next example.

EXAMPLE 5 Using Synthetic Division

Find $(3x^2 - 10x + 20) \div (x - 2)$.

SOLUTION

Step 1 Write the coefficients of the dividend in order of descending exponents. Include a "0" for any missing powers of the variable. Because the divisor is $x - 2$, use $k = 2$. Write the k-value to the left of the vertical bar.

$$\begin{array}{r|rrr} k\text{-value} \longrightarrow 2 & 3 & -10 & 20 \longleftarrow \text{coefficients of } 3x^2 - 10x + 20 \\ & & & \\ \hline \end{array}$$

Step 2 Bring down the leading coefficient. Multiply the leading coefficient by the k-value. Write the product under the second coefficient. Add.

$$\begin{array}{r|rrr} 2 & 3 & -10 & 20 \\ & & 6 & \\ \hline & 3 & -4 & \end{array}$$

Step 3 Multiply the previous sum by the k-value. Write the product under the third coefficient. Add. The first two numbers in the bottom row are the coefficients of the quotient, and the last number is the remainder.

$$\begin{array}{r|rrr} 2 & 3 & -10 & 20 \\ & & 6 & -8 \\ \hline \text{coefficients of quotient} \longrightarrow & 3 & -4 & 12 \longleftarrow \text{remainder} \end{array}$$

▶ $(3x^2 - 10x + 20) \div (x - 2) = 3x - 4 + \dfrac{12}{x - 2}$

Monitoring Progress

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Use synthetic division to divide.

9. $(2x^2 + x + 3) \div (x - 3)$

10. $(-x^2 + 5x - 2) \div (x + 4)$

7.4 Exercises

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Vocabulary and Core Concept Check

1. **WRITING** Describe how you can check the result of a polynomial division problem. Give an example.

2. **VOCABULARY** Can you use synthetic division to find $(3x^2 + x - 2) \div (x + 1)$? Explain your reasoning.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, divide. *(See Example 1.)*

3. $(36x - 42) \div 6$
4. $(45x + 20) \div (-5)$
5. $(16x^2 + 8x) \div (-2x)$
6. $(-4x^2 - 28x) \div (4x)$

In Exercises 7–12, use polynomial long division to divide. Determine whether the divisor evenly divides into the dividend. *(See Example 2.)*

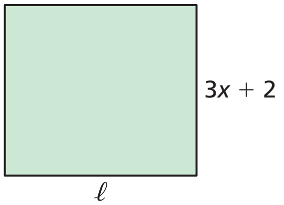
7. $(7x + 3) \div (x + 2)$
8. $(5x + 11) \div (x + 3)$
9. $(6x - 18) \div (2x - 6)$
10. $(8x - 5) \div (2x + 1)$
11. $(-8x - 10) \div (4x + 5)$
12. $(18x - 24) \div (-3x + 4)$

In Exercises 13–18, use polynomial long division to divide. *(See Example 3.)*

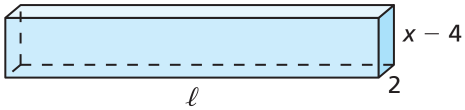
13. $(14x^2 - 23x + 6) \div (7x - 1)$
14. $(28x^2 + 36x - 1) \div (2x + 3)$
15. $(9x^2 + 24x + 15) \div (-3x - 2)$
16. $(8x^2 + 30x + 35) \div (4x + 3)$
17. $(15x + 14x^2 - 40) \div (-2x - 5)$
18. $(11x - 6x^2 - 8) \div (3x - 1)$

MATHEMATICAL CONNECTIONS **In Exercises 19 and 20, write an expression that represents the length ℓ of the figure.**

19. Area $= 9x^2 + 15x + 6$



20. Volume $= 4x^2 + 2x - 72$



In Exercises 21–28, use polynomial long division to divide. *(See Example 4.)*

21. $(4x^2 + 7x + 12) \div (x^2 + 2x + 1)$
22. $(3x^2 + 2x - 20) \div (x^2 - 8x + 12)$
23. $(16x^2 - 25 - 13x) \div (2x^2 + x - 10)$
24. $(7 - 8x^2 - 3x) \div (4x^2 + 5x - 3)$
25. $(5x^2 - 8x + 2) \div (x^2 - 3)$
26. $(9x^2 + 3x - 8) \div (3x^2 + 2)$
27. $(6x - 20x^2) \div (5x^2 - 3x + 2)$
28. $(-17 + 4x^2) \div (-2x^2 + 3x)$

In Exercises 29–36, use synthetic division to divide. *(See Example 5.)*

29. $(8x^2 + 4x - 19) \div (x - 1)$
30. $(2x^2 - 15x + 8) \div (x - 4)$
31. $(3x^2 + 17x + 15) \div (x + 5)$
32. $(9x^2 + 25x - 4) \div (x + 3)$
33. $(15x^2 - 48) \div (x - 2)$
34. $(2x^2 - 5x) \div (x - 3)$
35. $(3 - x^2 - x) \div (x + 4)$
36. $(12 - 7x^2) \div (x + 1)$

37. **ERROR ANALYSIS** Describe and correct the error in dividing $3x^2 + 10x + 6$ by $x + 3$.

$$
\begin{array}{r}
3x + 1 \\
x + 3 \overline{\smash{\big)}\, 3x^2 + 10x + 6} \\
\underline{3x^2 + 9x} \phantom{{}+6} \\
x + 6 \\
\underline{x + 3} \\
3
\end{array}
$$

$$(3x^2 + 10x + 6) \div (x + 3) = 3x + 1 + 3$$

38. **ERROR ANALYSIS** Describe and correct the error in using synthetic division to divide $2x^2 - 6x - 3$ by $x - 4$.

$$
\begin{array}{r|rrr}
4 & 2 & -6 & -3 \\
 & & 8 & 8 \\
\hline
 & 2 & 2 & 5
\end{array}
$$

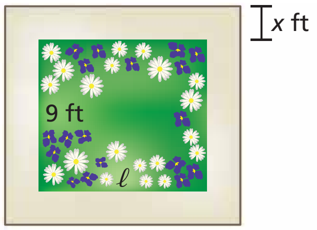
$$(2x^2 - 6x - 3) \div (x - 4) = 2x^2 + 2x + 5$$

39. **WRITING** Explain what it means for a divisor to divide evenly into a dividend.

40. **USING STRUCTURE** You divide two polynomials and obtain the result $3x + 2 - \frac{8}{x + 4}$. What is the divisor? What is the dividend? Explain how you found your answer.

41. **MODELING WITH MATHEMATICS** Your class is participating in a food drive. The total number of cans donated is represented by $6x + 4$, where x represents the number of students who participate. The total number of students in the class is represented by $x + 3$. Write an expression that represents the average number of cans donated by each student in the class. Use polynomial long division to rewrite the expression.

42. **MODELING WITH MATHEMATICS** A rectangular garden is bordered by a walkway. The width of the walkway is the same on every side. The total area (in square feet) of the garden and the walkway is represented by $4x^2 + 38x + 90$. Determine the length ℓ (in feet) of the garden without the walkway.



43. **MAKING AN ARGUMENT** Your friend says that you can use synthetic division to divide $3x^2 + x - 2$ by $x^2 + 1$ because the divisor is a binomial. Is your friend correct? Explain.

44. **HOW DO YOU SEE IT?** Write the divisor, dividend, quotient, and remainder represented by the synthetic division shown.

$$
\begin{array}{r|rrr}
-4 & 5 & 18 & 6 \\
 & & -20 & 8 \\
\hline
 & 5 & -2 & 14
\end{array}
$$

45. **COMPARING METHODS** Divide $6x^2 + 18x + 2$ by $3x + 6$ using polynomial long division. Then divide $6x^2 + 18x + 2$ by $x + 2$ using synthetic division. What do you notice about the results of the two division problems? Why do you think this is so?

46. **THOUGHT PROVOKING** Use polynomial long division to divide $3x^3 + 6x^2 - 12x + 13$ by $x + 4$. Can you also use synthetic division to divide the polynomials? Explain your reasoning.

47. **CRITICAL THINKING** Find a value for k so that $x + 8$ divides evenly into $x^2 + kx + 8$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Factor the expression using the GCF. *(Skills Review Handbook)*

48. $12y - 18$ 49. $9r + 27$ 50. $49s + 35t$ 51. $15x - 10y$

Solve the equation. Check your solution. *(Section 1.1)*

52. $p - 9 = 0$ 53. $z + 12 = -5$ 54. $6 = \frac{c}{-7}$ 55. $4k = 0$

7.5 Solving Polynomial Equations in Factored Form

Essential Question How can you solve a polynomial equation?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.8.A

EXPLORATION 1 Matching Equivalent Forms of an Equation

Work with a partner. An equation is considered to be in *factored form* when the product of the factors is equal to 0. Match each factored form of the equation with its equivalent standard form and nonstandard form.

	Factored Form		Standard Form		Nonstandard Form
a.	$(x-1)(x-3)=0$	**A.**	$x^2-x-2=0$	**1.**	$x^2-5x=-6$
b.	$(x-2)(x-3)=0$	**B.**	$x^2+x-2=0$	**2.**	$(x-1)^2=4$
c.	$(x+1)(x-2)=0$	**C.**	$x^2-4x+3=0$	**3.**	$x^2-x=2$
d.	$(x-1)(x+2)=0$	**D.**	$x^2-5x+6=0$	**4.**	$x(x+1)=2$
e.	$(x+1)(x-3)=0$	**E.**	$x^2-2x-3=0$	**5.**	$x^2-4x=-3$

SELECTING TOOLS
To be proficient in math, you need to consider using tools such as a table or a spreadsheet to organize your results.

EXPLORATION 2 Writing a Conjecture

Work with a partner. Substitute 1, 2, 3, 4, 5, and 6 for x in each equation and determine whether the equation is true. Organize your results in a table. Write a conjecture describing what you discovered.

a. $(x-1)(x-2)=0$ **b.** $(x-2)(x-3)=0$

c. $(x-3)(x-4)=0$ **d.** $(x-4)(x-5)=0$

e. $(x-5)(x-6)=0$ **f.** $(x-6)(x-1)=0$

EXPLORATION 3 Special Properties of 0 and 1

Work with a partner. The numbers 0 and 1 have special properties that are shared by no other numbers. For each of the following, decide whether the property is true for 0, 1, both, or neither. Explain your reasoning.

a. When you add ____ to a number n, you get n.

b. If the product of two numbers is ____, then at least one of the numbers is 0.

c. The square of ____ is equal to itself.

d. When you multiply a number n by ____, you get n.

e. When you multiply a number n by ____, you get 0.

f. The opposite of ____ is equal to itself.

Communicate Your Answer

4. How can you solve a polynomial equation?

5. One of the properties in Exploration 3 is called the Zero-Product Property. It is one of the most important properties in all of algebra. Which property is it? Why do you think it is called the Zero-Product Property? Explain how it is used in algebra and why it is so important.

7.5 Lesson

Core Vocabulary

factored form, *p. 364*
Zero-Product Property, *p. 364*
roots, *p. 364*
repeated roots, *p. 365*

Previous
polynomial
standard form
greatest common factor (GCF)
monomial

What You Will Learn

- Use the Zero-Product Property.
- Factor polynomials using the GCF.
- Use the Zero-Product Property to solve real-life problems.

Using the Zero-Product Property

A polynomial is in **factored form** when it is written as a product of factors.

Standard form	Factored form
$x^2 + 2x$	$x(x + 2)$
$x^2 + 5x - 24$	$(x - 3)(x + 8)$

When one side of an equation is a polynomial in factored form and the other side is 0, use the **Zero-Product Property** to solve the polynomial equation. The solutions of a polynomial equation are also called **roots**.

Core Concept

Zero-Product Property

Words If the product of two real numbers is 0, then at least one of the numbers is 0.

Algebra If a and b are real numbers and $ab = 0$, then $a = 0$ or $b = 0$.

EXAMPLE 1 Solving Polynomial Equations

Solve each equation.

a. $2x(x - 4) = 0$ **b.** $(x - 3)(x - 9) = 0$

SOLUTION

a. $2x(x - 4) = 0$ Write equation.

$2x = 0 \quad or \quad x - 4 = 0$ Zero-Product Property

$x = 0 \quad or \quad x = 4$ Solve for x.

- The roots are $x = 0$ and $x = 4$.

b. $(x - 3)(x - 9) = 0$ Write equation.

$x - 3 = 0 \quad or \quad x - 9 = 0$ Zero-Product Property

$x = 3 \quad or \quad x = 9$ Solve for x.

- The roots are $x = 3$ and $x = 9$.

Check

To check the solutions of Example 1(a), substitute each solution in the original equation.

$$2(0)(0 - 4) \stackrel{?}{=} 0$$
$$0(-4) \stackrel{?}{=} 0$$
$$0 = 0 \checkmark$$
$$2(4)(4 - 4) \stackrel{?}{=} 0$$
$$8(0) \stackrel{?}{=} 0$$
$$0 = 0 \checkmark$$

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Solve the equation. Check your solutions.

1. $x(x - 1) = 0$

2. $3t(t + 2) = 0$

3. $(z - 4)(z - 6) = 0$

When two or more roots of an equation are the same number, the equation has **repeated roots**.

EXAMPLE 2 **Solving Polynomial Equations**

Solve each equation.

a. $(2x + 7)(2x - 7) = 0$ **b.** $(x - 1)^2 = 0$ **c.** $(x + 1)(x - 3)(x - 2) = 0$

SOLUTION

a. $(2x + 7)(2x - 7) = 0$ Write equation.

$2x + 7 = 0$ *or* $2x - 7 = 0$ Zero-Product Property

$x = -\frac{7}{2}$ *or* $x = \frac{7}{2}$ Solve for x.

▶ The roots are $x = -\frac{7}{2}$ and $x = \frac{7}{2}$.

b. $(x - 1)^2 = 0$ Write equation.

$(x - 1)(x - 1) = 0$ Expand equation.

$x - 1 = 0$ *or* $x - 1 = 0$ Zero-Product Property

$x = 1$ *or* $x = 1$ Solve for x.

▶ The equation has repeated roots of $x = 1$.

c. $(x + 1)(x - 3)(x - 2) = 0$ Write equation.

$x + 1 = 0$ *or* $x - 3 = 0$ *or* $x - 2 = 0$ Zero-Product Property

$x = -1$ *or* $x = 3$ *or* $x = 2$ Solve for x.

▶ The roots are $x = -1$, $x = 3$, and $x = 2$.

STUDY TIP

You can extend the Zero-Product Property to products of more than two real numbers.

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Solve the equation. Check your solutions.

4. $(3s + 5)(5s + 8) = 0$ **5.** $(b + 7)^2 = 0$ **6.** $(d - 2)(d + 6)(d + 8) = 0$

Factoring Polynomials Using the GCF

To solve a polynomial equation using the Zero-Product Property, you may need to *factor* the polynomial, or write it as a product of other polynomials. Look for the *greatest common factor* (GCF) of the terms of the polynomial. This is a monomial that divides evenly into each term.

EXAMPLE 3 **Finding the Greatest Common Monomial Factor**

Factor out the greatest common monomial factor from $4x^4 + 24x^3$.

SOLUTION

The GCF of 4 and 24 is 4. The GCF of x^4 and x^3 is x^3. So, the greatest common monomial factor of the terms is $4x^3$.

▶ So, $4x^4 + 24x^3 = 4x^3(x + 6)$.

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7. Factor out the greatest common monomial factor from $8y^2 - 24y$.

EXAMPLE 4 Solving Equations by Factoring

Solve (a) $2x^2 + 8x = 0$ and (b) $6n^2 = 15n$.

SOLUTION

a.

$2x^2 + 8x = 0$	Write equation.
$2x(x + 4) = 0$	Factor left side.
$2x = 0 \quad or \quad x + 4 = 0$	Zero-Product Property
$x = 0 \quad or \quad x = -4$	Solve for x.

▶ The roots are $x = 0$ and $x = -4$.

b.

$6n^2 = 15n$	Write equation.
$6n^2 - 15n = 0$	Subtract $15n$ from each side.
$3n(2n - 5) = 0$	Factor left side.
$3n = 0 \quad or \quad 2n - 5 = 0$	Zero-Product Property
$n = 0 \quad or \quad n = \frac{5}{2}$	Solve for n.

▶ The roots are $n = 0$ and $n = \frac{5}{2}$.

Monitoring Progress

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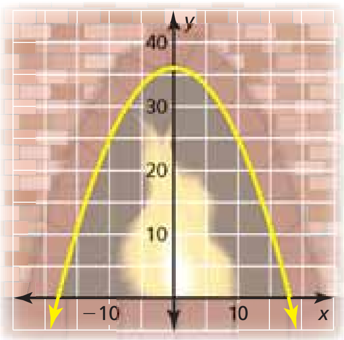
Solve the equation. Check your solutions.

8. $a^2 + 5a = 0$ **9.** $3s^2 - 9s = 0$ **10.** $4x^2 = 2x$

Solving Real-Life Problems

EXAMPLE 5 Modeling with Mathematics

You can model the arch of a fireplace using the equation $y = -\frac{1}{9}(x + 18)(x - 18)$, where x and y are measured in inches. The x-axis represents the floor. Find the width of the arch at floor level.



SOLUTION

Use the x-coordinates of the points where the arch meets the floor to find the width. At floor level, $y = 0$. So, substitute 0 for y and solve for x.

$y = -\frac{1}{9}(x + 18)(x - 18)$	Write equation.
$0 = -\frac{1}{9}(x + 18)(x - 18)$	Substitute 0 for y.
$0 = (x + 18)(x - 18)$	Multiply each side by -9.
$x + 18 = 0 \quad or \quad x - 18 = 0$	Zero-Product Property
$x = -18 \quad or \quad x = 18$	Solve for x.

The width is the distance between the x-coordinates, -18 and 18.

▶ So, the width of the arch at floor level is $|-18 - 18| = 36$ inches.

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11. You can model the entrance to a mine shaft using the equation $y = -\frac{1}{2}(x + 4)(x - 4)$, where x and y are measured in feet. The x-axis represents the ground. Find the width of the entrance at ground level.

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Vocabulary and Core Concept Check

1. **WRITING** Explain how to use the Zero-Product Property to find the solutions of the equation $3x(x - 6) = 0$.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find *both* answers.

Solve the equation $(2k + 4)(k - 3) = 0$.	Find the values of k for which $2k + 4 = 0$ *or* $k - 3 = 0$.
Find the value of k for which $(2k + 4) + (k - 3) = 0$.	Find the roots of the equation $(2k + 4)(k - 3) = 0$.

Monitoring Progress and Modeling with Mathematics

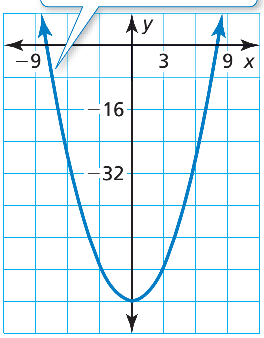
In Exercises 3–8, solve the equation. *(See Example 1.)*

3. $x(x + 7) = 0$
4. $r(r - 10) = 0$
5. $12t(t - 5) = 0$
6. $-2v(v + 1) = 0$
7. $(s - 9)(s - 1) = 0$
8. $(y + 2)(y - 6) = 0$

In Exercises 9–20, solve the equation. *(See Example 2.)*

9. $(2a - 6)(3a + 15) = 0$
10. $(4q + 3)(q + 2) = 0$
11. $(5m + 4)^2 = 0$
12. $(h - 8)^2 = 0$
13. $(3 - 2g)(7 - g) = 0$
14. $(2 - 4d)(2 + 4d) = 0$
15. $z(z + 2)(z - 1) = 0$
16. $5p(2p - 3)(p + 7) = 0$
17. $(r - 4)^2(r + 8) = 0$
18. $w(w - 6)^2 = 0$
19. $(15 - 5c)(5c + 5)(-c + 6) = 0$
20. $(2 - n)\left(6 + \frac{2}{3}n\right)(n - 2) = 0$

In Exercises 21–24, find the *x*-coordinates of the points where the graph crosses the *x*-axis.

21.


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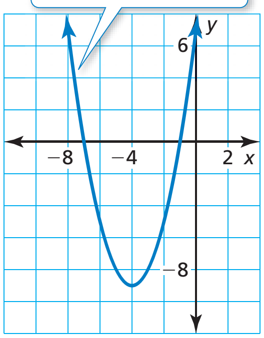

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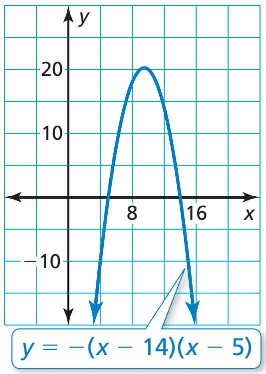
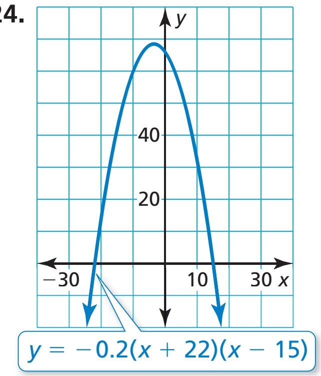


24.


In Exercises 25–30, factor the polynomial. *(See Example 3.)*

25. $5z^2 + 45z$
26. $6d^2 - 21d$
27. $3y^3 - 9y^2$
28. $20x^3 + 30x^2$
29. $5n^6 + 2n^5$
30. $12a^4 + 8a$

In Exercises 31–36, solve the equation. *(See Example 4.)*

31. $4p^2 - p = 0$
32. $6m^2 + 12m = 0$
33. $25c + 10c^2 = 0$
34. $18q - 2q^2 = 0$
35. $3n^2 = 9n$
36. $-28r = 4r^2$

37. **ERROR ANALYSIS** Describe and correct the error in solving the equation.

✗

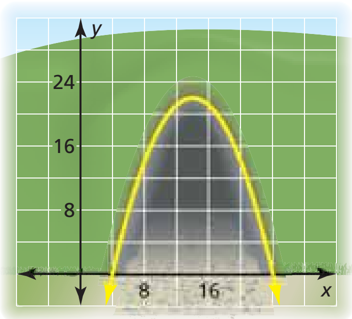
$$6x(x + 5) = 0$$
$$x + 5 = 0$$
$$x = -5$$

The root is $x = -5$.

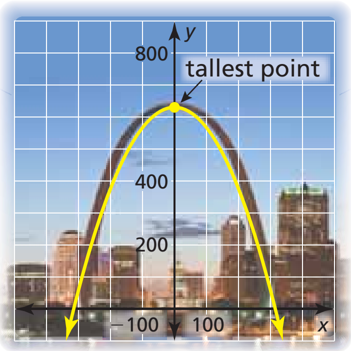
38. ERROR ANALYSIS Describe and correct the error in solving the equation.

$$3y^2 = 21y$$
$$3y = 21$$
$$y = 7$$
The root is $y = 7$.

39. MODELING WITH MATHEMATICS The entrance of a tunnel can be modeled by $y = -\frac{11}{50}(x - 4)(x - 24)$, where x and y are measured in feet. The x-axis represents the ground. Find the width of the tunnel at ground level. *(See Example 5.)*



40. MODELING WITH MATHEMATICS The Gateway Arch in St. Louis can be modeled by $y = -\frac{2}{315}(x + 315)(x - 315)$, where x and y are measured in feet. The x-axis represents the ground.

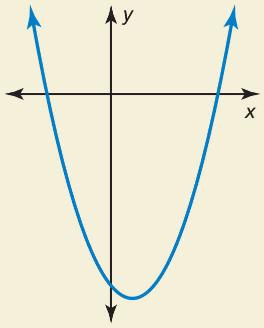


a. Find the width of the arch at ground level.

b. How tall is the arch?

41. MODELING WITH MATHEMATICS A penguin leaps out of the water while swimming. This action is called porpoising. The height y (in feet) of a porpoising penguin can be modeled by $y = -16x^2 + 4.8x$, where x is the time (in seconds) since the penguin leaped out of the water. Find the roots of the equation when $y = 0$. Explain what the roots mean in this situation.

42. HOW DO YOU SEE IT? Use the graph to fill in each blank in the equation with the symbol + or −. Explain your reasoning.



$$y = (x \;\square\; 5)(x \;\square\; 3)$$

43. CRITICAL THINKING How many x-intercepts does the graph of $y = (2x + 5)(x - 9)^2$ have? Explain.

44. MAKING AN ARGUMENT Your friend says that the graph of the equation $y = (x - a)(x - b)$ always has two x-intercepts for any values of a and b. Is your friend correct? Explain.

45. CRITICAL THINKING Does the equation $(x^2 + 3)(x^4 + 1) = 0$ have any real roots? Explain.

46. THOUGHT PROVOKING Write a polynomial equation of degree 4 whose only roots are $x = 1$, $x = 2$, and $x = 3$.

47. REASONING Find the values of x in terms of y that are solutions of each equation.

a. $(x + y)(2x - y) = 0$

b. $(x^2 - y^2)(4x + 16y) = 0$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

List the factor pairs of the number. *(Skills Review Handbook)*

48. 10

49. 18

50. 30

51. 48

52. 36

53. 52

7.1–7.5 What Did You Learn?

Core Vocabulary

monomial, *p. 338*
degree of a monomial, *p. 338*
polynomial, *p. 339*
binomial, *p. 339*
trinomial, *p. 339*
degree of a polynomial, *p. 339*
standard form, *p. 339*
leading coefficient, *p. 339*
closed, *p. 340*
FOIL Method, *p. 347*
polynomial long division, *p. 358*
synthetic division, *p. 360*
factored form, *p. 364*
Zero-Product Property, *p. 364*
roots, *p. 364*
repeated roots, *p. 365*

Core Concepts

Section 7.1
Polynomials, *p. 339*
Adding Polynomials, *p. 340*
Subtracting Polynomials, *p. 340*

Section 7.2
Multiplying Binomials, *p. 346*
FOIL Method, *p. 347*
Multiplying Binomials and Trinomials, *p. 348*

Section 7.3
Square of a Binomial Pattern, *p. 352*
Sum and Difference Pattern, *p. 353*

Section 7.4
Polynomial Long Division, *p. 358*
Synthetic Division, *p. 360*

Section 7.5
Zero-Product Property, *p. 364*
Factoring Polynomials Using the GCF, *p. 365*

Mathematical Thinking

1. Explain how you wrote the polynomial in Exercise 11 on page 355. Is there another method you can use to write the same polynomial?

2. Find a shortcut for exercises like Exercise 7 on page 367 when the variable has a coefficient of 1. Does your shortcut work when the coefficient is *not* 1?

Study Skills

Preparing for a Test

- Review examples of each type of problem that could appear on the test.
- Review the homework problems your teacher assigned.
- Take a practice test.

7.1–7.5 Quiz

Write the polynomial in standard form. Identify the degree and leading coefficient of the polynomial. Then classify the polynomial by the number of terms. *(Section 7.1)*

1. $-8q^3$

2. $-1.3z + 3z^4 + 7.4z^2$

Find the sum or difference. *(Section 7.1)*

3. $(2x + 5) + (-x + 4)$

4. $(-3n + 9) - (2n - 7)$

5. $(-p^2 + 4p) - (p^2 - 3p + 15)$

6. $(a^2 - 3ab + b^2) + (-a^2 + ab + b^2)$

Find the product. *(Section 7.2 and Section 7.3)*

7. $(w + 6)(w + 7)$

8. $(3 - 4d)(2d - 5)$

9. $(y + 9)(y^2 + 2y - 3)$

10. $(3z - 5)(3z + 5)$

11. $(t + 5)^2$

12. $(2q^2 - 6)^2$

Divide. *(Section 7.4)*

13. $(8x + 2) \div (2x - 1)$

14. $(9x^2 - 6x + 8) \div (3x + 5)$

15. $(7x^2 - 16) \div (x - 4)$

Solve the equation. *(Section 7.5)*

16. $5x^2 - 15x = 0$

17. $(8 - g)(8 - g) = 0$

18. $(3p + 7)(3p - 7)(p + 8) = 0$

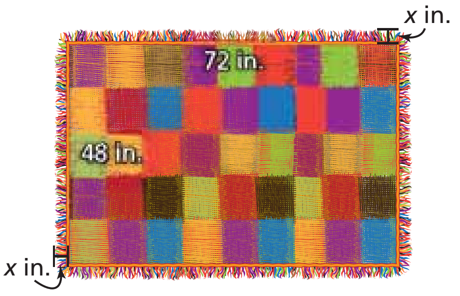
19. $-3y(y - 8)(2y + 1) = 0$

20. You are making a blanket with a fringe border of equal width on each side. *(Section 7.1 and Section 7.2)*

a. Write a polynomial that represents the perimeter of the blanket including the fringe.

b. Write a polynomial that represents the area of the blanket including the fringe.

c. Find the perimeter and the area of the blanket including the fringe when the width of the fringe is 4 inches.



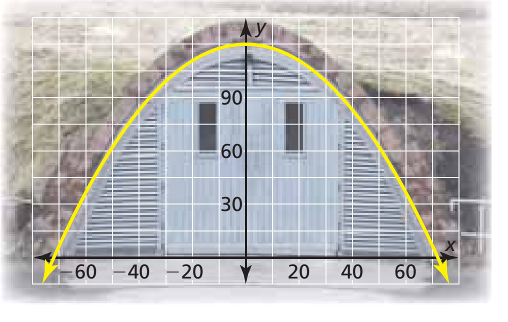
21. You are saving money to buy an electric guitar. You deposit \$1000 in an account that earns interest compounded annually. The expression $1000(1 + r)^2$ represents the balance after 2 years, where r is the annual interest rate in decimal form. *(Section 7.3)*

a. Write the polynomial in standard form that represents the balance of your account after 2 years.

b. The interest rate is 3%. What is the balance of your account after 2 years?

c. The guitar costs \$1100. Do you have enough money in your account *after 3 years*? Explain.

22. The front of a storage bunker can be modeled by $y = -\frac{5}{216}(x - 72)(x + 72)$, where x and y are measured in inches. The x-axis represents the ground. Find the width of the bunker at ground level. *(Section 7.5)*



7.6 Factoring $x^2 + bx + c$

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS

A.8.A
A.10.E

Essential Question

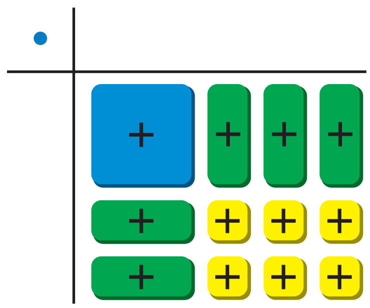
How can you use algebra tiles to factor the trinomial $x^2 + bx + c$ into the product of two binomials?

EXPLORATION 1 Finding Binomial Factors

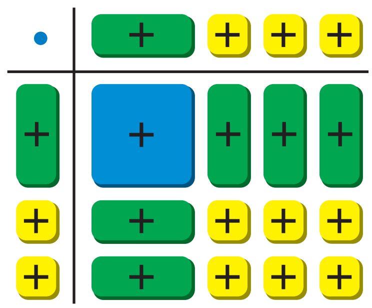
Work with a partner. Use algebra tiles to write each polynomial as the product of two binomials. Check your answer by multiplying.

Sample $x^2 + 5x + 6$

Step 1 Arrange algebra tiles that model $x^2 + 5x + 6$ into a rectangular array.



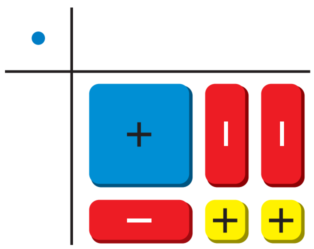
Step 2 Use additional algebra tiles to model the dimensions of the rectangle.



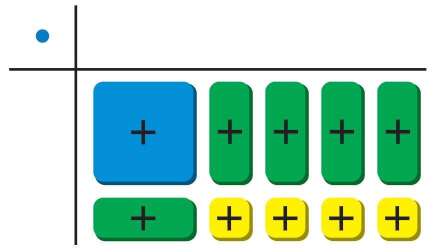
Step 3 Write the polynomial in factored form using the dimensions of the rectangle.

$$\text{Area} = x^2 + 5x + 6 = \overbrace{(x + 2)}^{\text{width}}\overbrace{(x + 3)}^{\text{length}}$$

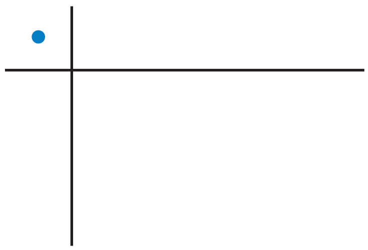
a. $x^2 - 3x + 2 =$ ______



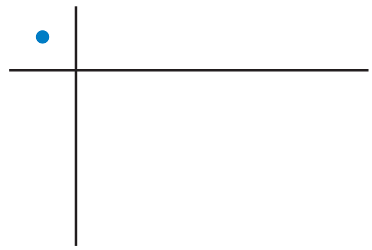
b. $x^2 + 5x + 4 =$ ______



c. $x^2 - 7x + 12 =$ ______



d. $x^2 + 7x + 12 =$ ______



REASONING

To be proficient in math, you need to understand a situation abstractly and represent it symbolically.

Communicate Your Answer

2. How can you use algebra tiles to factor the trinomial $x^2 + bx + c$ into the product of two binomials?

3. Describe a strategy for factoring the trinomial $x^2 + bx + c$ that does not use algebra tiles.

7.6 Lesson

Core Vocabulary

Previous
polynomial
FOIL Method
Zero-Product Property

What You Will Learn

- Factor $x^2 + bx + c$.
- Use factoring to solve real-life problems.

Factoring $x^2 + bx + c$

Writing a polynomial as a product of factors is called *factoring*. To factor $x^2 + bx + c$ as $(x + p)(x + q)$, you need to find p and q such that $p + q = b$ and $pq = c$.

$$\begin{aligned}(x + p)(x + q) &= x^2 + px + qx + pq \\ &= x^2 + (p + q)x + pq\end{aligned}$$

Core Concept

Factoring $x^2 + bx + c$ When c Is Positive

Algebra $x^2 + bx + c = (x + p)(x + q)$ when $p + q = b$ and $pq = c$.

When c is positive, p and q have the same sign as b.

Examples $x^2 + 6x + 5 = (x + 1)(x + 5)$

$x^2 - 6x + 5 = (x - 1)(x - 5)$

EXAMPLE 1 Factoring $x^2 + bx + c$ When b and c Are Positive

Factor $x^2 + 10x + 16$.

SOLUTION

Notice that $b = 10$ and $c = 16$.

- Because c is positive, the factors p and q must have the same sign so that pq is positive.
- Because b is also positive, p and q must each be positive so that $p + q$ is positive.

Find two positive integer factors of 16 whose sum is 10.

Factors of 16	Sum of factors
1, 16	17
2, 8	10
4, 4	8

The values of p and q are 2 and 8.

So, $x^2 + 10x + 16 = (x + 2)(x + 8)$.

Check

Use the FOIL Method.

$$\begin{aligned}&(x + 2)(x + 8) \\ &= x^2 + 8x + 2x + 16 \\ &= x^2 + 10x + 16 \checkmark\end{aligned}$$

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Factor the polynomial.

1. $x^2 + 7x + 6$

2. $x^2 + 9x + 8$

EXAMPLE 2 Factoring $x^2 + bx + c$ When b Is Negative and c Is Positive

Factor $x^2 - 8x + 12$.

SOLUTION

Notice that $b = -8$ and $c = 12$.

- Because c is positive, the factors p and q must have the same sign so that pq is positive.
- Because b is negative, p and q must each be negative so that $p + q$ is negative.

Find two negative integer factors of 12 whose sum is -8.

Factors of 12	$-1, -12$	$-2, -6$	$-3, -4$
Sum of factors	-13	-8	-7

The values of p and q are -2 and -6.

So, $x^2 - 8x + 12 = (x - 2)(x - 6)$.

Check

Use the FOIL Method.

$(x - 2)(x - 6)$

$= x^2 - 6x - 2x + 12$

$= x^2 - 8x + 12$ ✓

Core Concept

Factoring $x^2 + bx + c$ When c Is Negative

Algebra $x^2 + bx + c = (x + p)(x + q)$ when $p + q = b$ and $pq = c$.

When c is negative, p and q have different signs.

Example $x^2 - 4x - 5 = (x + 1)(x - 5)$

EXAMPLE 3 Factoring $x^2 + bx + c$ When c Is Negative

Factor $x^2 + 4x - 21$.

SOLUTION

Notice that $b = 4$ and $c = -21$. Because c is negative, the factors p and q must have different signs so that pq is negative.

Find two integer factors of -21 whose sum is 4.

Factors of -21	$-21, 1$	$-1, 21$	$-7, 3$	$-3, 7$
Sum of factors	-20	20	-4	4

The values of p and q are -3 and 7.

So, $x^2 + 4x - 21 = (x - 3)(x + 7)$.

Check

Use the FOIL Method.

$(x - 3)(x + 7)$

$= x^2 + 7x - 3x - 21$

$= x^2 + 4x - 21$ ✓

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Factor the polynomial.

3. $w^2 - 4w + 3$

4. $n^2 - 12n + 35$

5. $x^2 - 14x + 24$

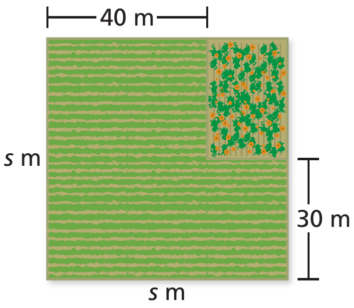
6. $x^2 + 2x - 15$

7. $y^2 + 13y - 30$

8. $v^2 - v - 42$

Solving Real-Life Problems

EXAMPLE 4 **Solving a Real-Life Problem**



A farmer plants a rectangular pumpkin patch in the northeast corner of a square plot of land. The area of the pumpkin patch is 600 square meters. What is the area of the square plot of land?

SOLUTION

1. **Understand the Problem** You are given the area of the pumpkin patch, the difference of the side length of the square plot and the length of the pumpkin patch, and the difference of the side length of the square plot and the width of the pumpkin patch.

2. **Make a Plan** The length of the pumpkin patch is $(s - 30)$ meters and the width is $(s - 40)$ meters. Write and solve an equation to find the side length s. Then use the solution to find the area of the square plot of land.

3. **Solve the Problem** Use the equation for the area of a rectangle to write and solve an equation to find the side length s of the square plot of land.

$600 = (s - 30)(s - 40)$	Write an equation.
$600 = s^2 - 70s + 1200$	Multiply.
$0 = s^2 - 70s + 600$	Subtract 600 from each side.
$0 = (s - 10)(s - 60)$	Factor the polynomial.
$s - 10 = 0$ *or* $s - 60 = 0$	Zero-Product Property
$s = 10$ *or* $s = 60$	Solve for s.

STUDY TIP

The diagram shows that the side length is more than 40 meters, so a side length of 10 meters does not make sense in this situation. The side length is 60 meters.

▶ So, the area of the square plot of land is $60(60) = 3600$ square meters.

4. **Look Back** Use the diagram to check that you found the correct side length. Using $s = 60$, the length of the pumpkin patch is $60 - 30 = 30$ meters and the width is $60 - 40 = 20$ meters. So, the area of the pumpkin patch is 600 square meters. This matches the given information and confirms the side length is 60 meters, which gives an area of 3600 square meters.

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9. **WHAT IF?** The area of the pumpkin patch is 200 square meters. What is the area of the square plot of land?

Concept Summary

Factoring $x^2 + bx + c$ as $(x + p)(x + q)$

The diagram shows the relationships between the signs of b and c and the signs of p and q.

$x^2 + bx + c = (x + p)(x + q)$

c is positive. b is positive.	c is positive. b is negative.	c is negative.
p and q are positive.	p and q are negative.	p and q have different signs.

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Vocabulary and Core Concept Check

1. **WRITING** You are factoring $x^2 + 11x - 26$. What do the signs of the terms tell you about the factors? Explain.

2. **OPEN-ENDED** Write a trinomial that can be factored as $(x + p)(x + q)$, where p and q are positive.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, factor the polynomial. *(See Example 1.)*

3. $x^2 + 8x + 7$

4. $z^2 + 10z + 21$

5. $n^2 + 9n + 20$

6. $s^2 + 11s + 30$

7. $h^2 + 11h + 18$

8. $y^2 + 13y + 40$

In Exercises 9–14, factor the polynomial. *(See Example 2.)*

9. $v^2 - 5v + 4$

10. $x^2 - 13x + 22$

11. $d^2 - 5d + 6$

12. $k^2 - 10k + 24$

13. $w^2 - 17w + 72$

14. $j^2 - 13j + 42$

In Exercises 15–24, factor the polynomial. *(See Example 3.)*

15. $x^2 + 3x - 4$

16. $z^2 + 7z - 18$

17. $n^2 + 4n - 12$

18. $s^2 + 3s - 40$

19. $y^2 + 2y - 48$

20. $h^2 + 6h - 27$

21. $x^2 - x - 20$

22. $m^2 - 6m - 7$

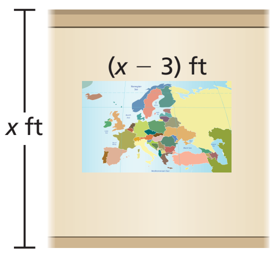
23. $-6t - 16 + t^2$

24. $-7y + y^2 - 30$

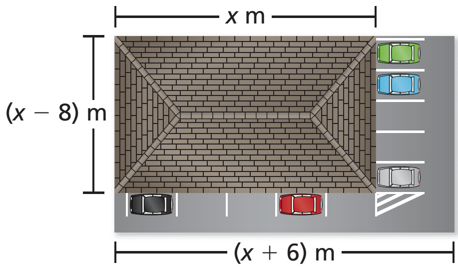
25. **MODELING WITH MATHEMATICS** A projector displays an image on a wall. The area (in square feet) of the projection is represented by $x^2 - 8x + 15$.

a. Write a binomial that represents the height of the projection.

b. Find the perimeter of the projection when the height of the wall is 8 feet.



26. **MODELING WITH MATHEMATICS** A dentist's office and parking lot are on a rectangular piece of land. The area (in square meters) of the land is represented by $x^2 + x - 30$.



a. Write a binomial that represents the width of the land.

b. Find the area of the land when the length of the dentist's office is 20 meters.

ERROR ANALYSIS **In Exercises 27 and 28, describe and correct the error in factoring the polynomial.**

27. ✗ $x^2 + 14x + 48 = (x + 4)(x + 12)$

28. ✗ $s^2 - 17s - 60 = (s - 5)(s - 12)$

In Exercises 29–38, solve the equation.

29. $m^2 + 3m + 2 = 0$

30. $n^2 - 9n + 18 = 0$

31. $x^2 + 5x - 14 = 0$

32. $v^2 + 11v - 26 = 0$

33. $t^2 + 15t = -36$

34. $n^2 - 5n = 24$

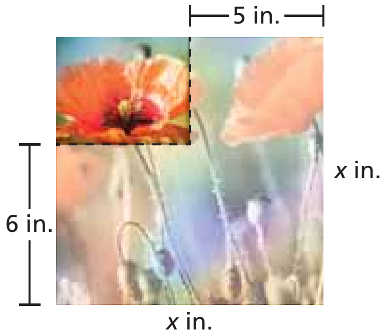
35. $a^2 + 5a - 20 = 30$

36. $y^2 - 2y - 8 = 7$

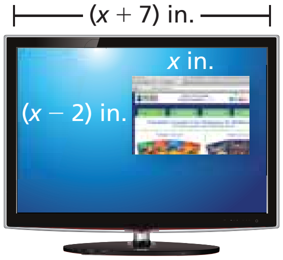
37. $m^2 + 10 = 15m - 34$

38. $b^2 + 5 = 8b - 10$

39. MODELING WITH MATHEMATICS You trimmed a large square picture so that you could fit it into a frame. The area of the cut picture is 20 square inches. What is the area of the original picture? *(See Example 4.)*



40. MODELING WITH MATHEMATICS A web browser is open on your computer screen.



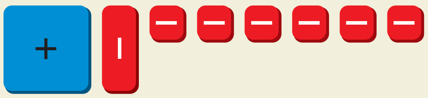
a. The area of the browser window is 24 square inches. Find the length of the browser window x.

b. The browser covers $\frac{3}{13}$ of the screen. What are the dimensions of the screen?

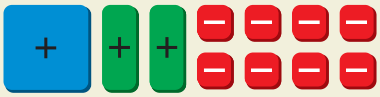
41. MAKING AN ARGUMENT Your friend says there are six integer values of b for which the trinomial $x^2 + bx - 12$ has two binomial factors of the form $(x + p)$ and $(x + q)$. Is your friend correct? Explain.

42. THOUGHT PROVOKING Use algebra tiles to factor each polynomial modeled by the tiles. Show your work.

a.

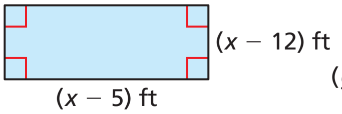


b.

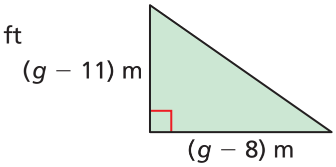


MATHEMATICAL CONNECTIONS In Exercises 43 and 44, find the dimensions of the polygon with the given area.

43. Area $= 44$ ft^2

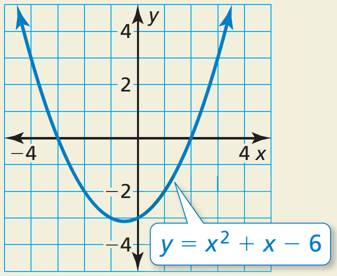


44. Area $= 35$ m^2



45. REASONING Write an equation of the form $x^2 + bx + c = 0$ that has the solutions $x = -4$ and $x = 6$. Explain how you found your answer.

46. HOW DO YOU SEE IT? The graph of $y = x^2 + x - 6$ is shown.



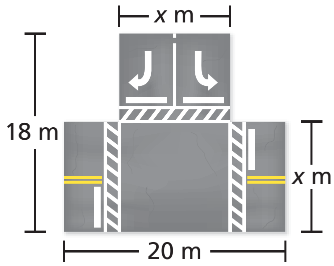
a. Explain how you can use the graph to factor the polynomial $x^2 + x - 6$.

b. Factor the polynomial.

47. PROBLEM SOLVING Road construction workers are paving the area shown.

a. Write an expression that represents the area being paved.

b. The area being paved is 280 square meters. Write and solve an equation to find the width of the road x.



USING STRUCTURE In Exercises 48–51, factor the polynomial.

48. $x^2 + 6xy + 8y^2$

49. $r^2 + 7rs + 12s^2$

50. $a^2 + 11ab - 26b^2$

51. $x^2 - 2xy - 35y^2$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write two equations in standard form that are equivalent to the given equation. *(Section 4.3)*

52. $x + y = -6$

53. $8x + 4y = 12$

54. $5x - 3y = -7$

55. $-2x + 6y = 10$

7.7 Factoring $ax^2 + bx + c$

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.8.A
A.10.D
A.10.E

Essential Question

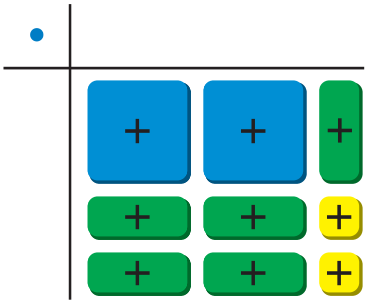
How can you use algebra tiles to factor the trinomial $ax^2 + bx + c$ into the product of two binomials?

EXPLORATION 1 Finding Binomial Factors

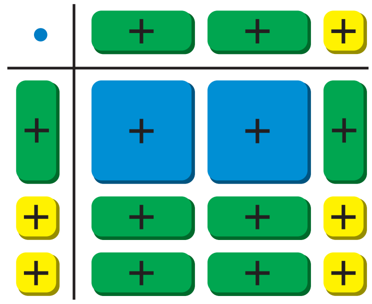
Work with a partner. Use algebra tiles to write each polynomial as the product of two binomials. Check your answer by multiplying.

Sample $2x^2 + 5x + 2$

Step 1 Arrange algebra tiles that model $2x^2 + 5x + 2$ into a rectangular array.



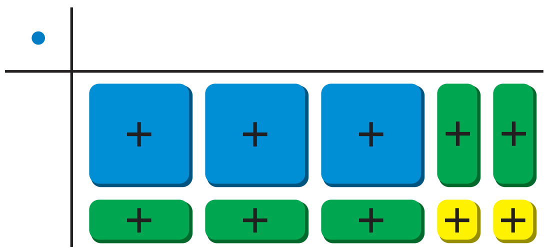
Step 2 Use additional algebra tiles to model the dimensions of the rectangle.



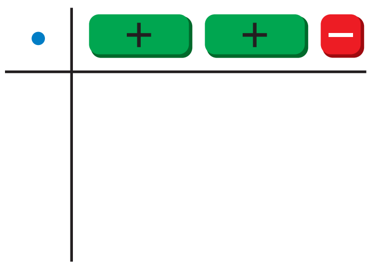
Step 3 Write the polynomial in factored form using the dimensions of the rectangle.

$$\text{Area} = 2x^2 + 5x + 2 = \underbrace{(x + 2)}_{\text{width}}\underbrace{(2x + 1)}_{\text{length}}$$

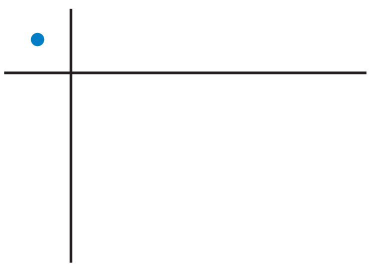
a. $3x^2 + 5x + 2 =$ ______



b. $4x^2 + 4x - 3 =$ ______



c. $2x^2 - 11x + 5 =$ ______



SELECTING TOOLS

To be proficient in math, you need to consider the available tools, including concrete models, when solving a mathematical problem.

Communicate Your Answer

2. How can you use algebra tiles to factor the trinomial $ax^2 + bx + c$ into the product of two binomials?

3. Is it possible to factor the trinomial $2x^2 + 2x + 1$? Explain your reasoning.

7.7 Lesson

What You Will Learn

- Factor $ax^2 + bx + c$.
- Use factoring to solve real-life problems.

Core Vocabulary

Previous
polynomial
greatest common factor (GCF)
Zero-Product Property

Factoring $ax^2 + bx + c$

In Section 7.6, you factored polynomials of the form $ax^2 + bx + c$, where $a = 1$. To factor polynomials of the form $ax^2 + bx + c$, where $a \neq 1$, first look for the GCF of the terms of the polynomial and then factor further if possible.

EXAMPLE 1 Factoring Out the GCF

Factor $5x^2 + 15x + 10$.

SOLUTION

Notice that the GCF of the terms $5x^2$, $15x$, and 10 is 5.

$5x^2 + 15x + 10 = 5(x^2 + 3x + 2)$ Factor out GCF.

$= 5(x + 1)(x + 2)$ Factor $x^2 + 3x + 2$.

▶ So, $5x^2 + 15x + 10 = 5(x + 1)(x + 2)$.

When there is no GCF, consider the possible factors of a and c.

EXAMPLE 2 Factoring $ax^2 + bx + c$ When ac Is Positive

Factor each polynomial.

a. $4x^2 + 13x + 3$ **b.** $3x^2 - 7x + 2$

SOLUTION

STUDY TIP
You must consider the order of the factors of 3, because the middle terms formed by the possible factorizations are different.

a. There is no GCF, so you need to consider the possible factors of a and c. Because b and c are both positive, the factors of c must be positive. Use a table to organize information about the factors of a and c.

Factors of 4	Factors of 3	Possible factorization	Middle term	
1, 4	1, 3	$(x + 1)(4x + 3)$	$3x + 4x = 7x$	✗
1, 4	3, 1	$(x + 3)(4x + 1)$	$x + 12x = 13x$	✓
2, 2	1, 3	$(2x + 1)(2x + 3)$	$6x + 2x = 8x$	✗

▶ So, $4x^2 + 13x + 3 = (x + 3)(4x + 1)$.

b. There is no GCF, so you need to consider the possible factors of a and c. Because b is negative and c is positive, both factors of c must be negative. Use a table to organize information about the factors of a and c.

Factors of 3	Factors of 2	Possible factorization	Middle term	
1, 3	−1, −2	$(x - 1)(3x - 2)$	$-2x - 3x = -5x$	✗
1, 3	−2, −1	$(x - 2)(3x - 1)$	$-x - 6x = -7x$	✓

▶ So, $3x^2 - 7x + 2 = (x - 2)(3x - 1)$.

EXAMPLE 3 Factoring $ax^2 + bx + c$ When ac Is Negative

Factor $2x^2 - 5x - 7$.

SOLUTION

There is no GCF, so you need to consider the possible factors of a and c. Because c is negative, the factors of c must have different signs. Use a table to organize information about the factors of a and c.

Factors of 2	Factors of −7	Possible factorization	Middle term	
1, 2	1, −7	$(x + 1)(2x - 7)$	$-7x + 2x = -5x$	✓
1, 2	7, −1	$(x + 7)(2x - 1)$	$-x + 14x = 13x$	✗
1, 2	−1, 7	$(x - 1)(2x + 7)$	$7x - 2x = 5x$	✗
1, 2	−7, 1	$(x - 7)(2x + 1)$	$x - 14x = -13x$	✗

▶ So, $2x^2 - 5x - 7 = (x + 1)(2x - 7)$.

STUDY TIP

When a is negative, factor -1 from each term of $ax^2 + bx + c$. Then factor the resulting trinomial as in the previous examples.

EXAMPLE 4 Factoring $ax^2 + bx + c$ When a Is Negative

Factor $-4x^2 - 8x + 5$.

SOLUTION

Step 1 Factor -1 from each term of the trinomial.

$$-4x^2 - 8x + 5 = -(4x^2 + 8x - 5)$$

Step 2 Factor the trinomial $4x^2 + 8x - 5$. Because c is negative, the factors of c must have different signs. Use a table to organize information about the factors of a and c.

Factors of 4	Factors of −5	Possible factorization	Middle term	
1, 4	1, −5	$(x + 1)(4x - 5)$	$-5x + 4x = -x$	✗
1, 4	5, −1	$(x + 5)(4x - 1)$	$-x + 20x = 19x$	✗
1, 4	−1, 5	$(x - 1)(4x + 5)$	$5x - 4x = x$	✗
1, 4	−5, 1	$(x - 5)(4x + 1)$	$x - 20x = -19x$	✗
2, 2	1, −5	$(2x + 1)(2x - 5)$	$-10x + 2x = -8x$	✗
2, 2	−1, 5	$(2x - 1)(2x + 5)$	$10x - 2x = 8x$	✓

▶ So, $-4x^2 - 8x + 5 = -(2x - 1)(2x + 5)$.

Monitoring Progress

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Factor the polynomial.

1. $8x^2 - 56x + 48$ **2.** $14x^2 + 31x + 15$ **3.** $2x^2 - 7x + 5$

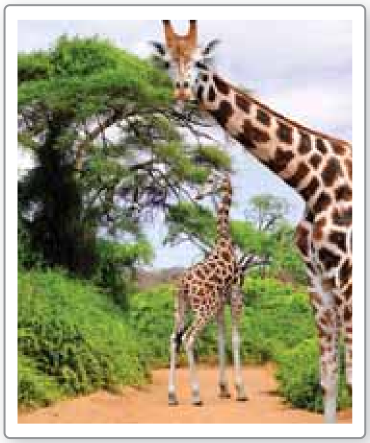
4. $3x^2 - 14x + 8$ **5.** $4x^2 - 19x - 5$ **6.** $6x^2 + x - 12$

7. $-2y^2 - 5y - 3$ **8.** $-5m^2 + 6m - 1$ **9.** $-3x^2 - x + 2$

Solving Real-Life Problems

EXAMPLE 5 Solving a Real-Life Problem

The length of a rectangular game reserve is 1 mile longer than twice the width. The area of the reserve is 55 square miles. What is the width of the reserve?



SOLUTION

Use the formula for the area of a rectangle to write an equation for the area of the reserve. Let w represent the width. Then $2w + 1$ represents the length. Solve for w.

$w(2w + 1) = 55$ Area of the reserve

$2w^2 + w = 55$ Distributive Property

$2w^2 + w - 55 = 0$ Subtract 55 from each side.

Factor the left side of the equation. There is no GCF, so you need to consider the possible factors of a and c. Because c is negative, the factors of c must have different signs. Use a table to organize information about the factors of a and c.

Factors of 2	Factors of −55	Possible factorization	Middle term	
1, 2	$1, -55$	$(w + 1)(2w - 55)$	$-55w + 2w = -53w$	✗
1, 2	$55, -1$	$(w + 55)(2w - 1)$	$-w + 110w = 109w$	✗
1, 2	$-1, 55$	$(w - 1)(2w + 55)$	$55w - 2w = 53w$	✗
1, 2	$-55, 1$	$(w - 55)(2w + 1)$	$w - 110w = -109w$	✗
1, 2	$5, -11$	$(w + 5)(2w - 11)$	$-11w + 10w = -w$	✗
1, 2	$11, -5$	$(w + 11)(2w - 5)$	$-5w + 22w = 17w$	✗
1, 2	$-5, 11$	$(w - 5)(2w + 11)$	$11w - 10w = w$	✓
1, 2	$-11, 5$	$(w - 11)(2w + 5)$	$5w - 22w = -17w$	✗

So, you can rewrite $2w^2 + w - 55$ as $(w - 5)(2w + 11)$. Write the equation with the left side factored and continue solving for w.

$(w - 5)(2w + 11) = 0$ Rewrite equation with left side factored.

$w - 5 = 0$ *or* $2w + 11 = 0$ Zero-Product Property

$w = 5$ *or* $w = -\frac{11}{2}$ Solve for w.

A negative width does not make sense, so you should use the positive solution.

▶ So, the width of the reserve is 5 miles.

Check

Use mental math.

The width is 5 miles, so the length is $5(2) + 1 = 11$ miles and the area is $5(11) = 55$ square miles. ✓

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10. **WHAT IF?** The area of the reserve is 136 square miles. How wide is the reserve?

7.7 Exercises

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Vocabulary and Core Concept Check

1. **REASONING** What is the greatest common factor of the terms of $3y^2 - 21y + 36$?

2. **WRITING** Compare factoring $6x^2 - x - 2$ with factoring $x^2 - x - 2$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, factor the polynomial. *(See Example 1.)*

3. $3x^2 + 3x - 6$
4. $8v^2 + 8v - 48$
5. $4k^2 + 28k + 48$
6. $6y^2 - 24y + 18$
7. $7b^2 - 63b + 140$
8. $9r^2 - 36r - 45$

In Exercises 9–16, factor the polynomial. *(See Examples 2 and 3.)*

9. $3h^2 + 11h + 6$
10. $8m^2 + 30m + 7$
11. $6x^2 - 5x + 1$
12. $10w^2 - 31w + 15$
13. $3n^2 + 5n - 2$
14. $4z^2 + 4z - 3$
15. $8g^2 - 10g - 12$
16. $18v^2 - 15v - 18$

In Exercises 17–22, factor the polynomial. *(See Example 4.)*

17. $-3t^2 + 11t - 6$
18. $-7v^2 - 25v - 12$
19. $-4c^2 + 19c + 5$
20. $-8h^2 - 13h + 6$
21. $-15w^2 - w + 28$
22. $-22d^2 + 29d - 9$

ERROR ANALYSIS **In Exercises 23 and 24, describe and correct the error in factoring the polynomial.**

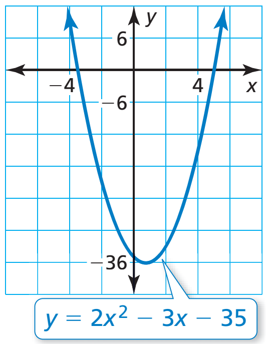
23. ✗
$$2x^2 - 2x - 24 = 2(x^2 - 2x - 24)$$
$$= 2(x - 6)(x + 4)$$

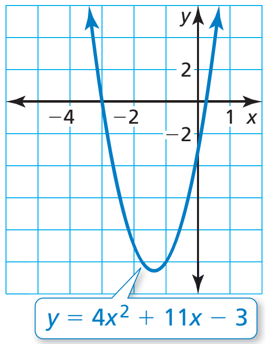
24. ✗
$$6x^2 - 7x - 3 = (3x - 3)(2x + 1)$$

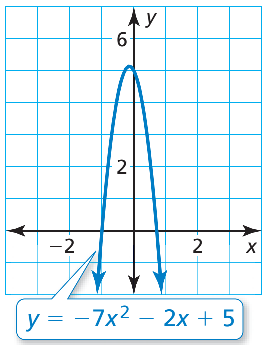
In Exercises 25–28, solve the equation.

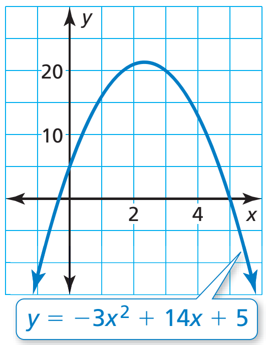
25. $5x^2 - 5x - 30 = 0$
26. $2k^2 - 5k - 18 = 0$
27. $-12n^2 - 11n = -15$
28. $14b^2 - 2 = -3b$

In Exercises 29–32, find the x-coordinates of the points where the graph crosses the x-axis.

29. 

30. 

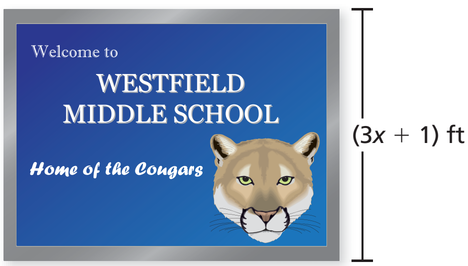
31. 

32. 

33. **MODELING WITH MATHEMATICS** The area (in square feet) of the school sign can be represented by $15x^2 - x - 2$.

 a. Write an expression that represents the length of the sign.

 b. Describe two ways to find the area of the sign when $x = 3$.



34. MODELING WITH MATHEMATICS The height h (in feet) above the water of a cliff diver is modeled by $h = -16t^2 + 8t + 80$, where t is the time (in seconds). How long is the diver in the air?

35. MODELING WITH MATHEMATICS The Parthenon in Athens, Greece, is an ancient structure that has a rectangular base. The length of the base of the Parthenon is 8 meters more than twice its width. The area of the base is about 2170 square meters. Find the length and width of the base. *(See Example 5.)*

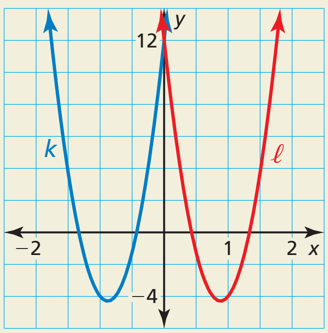
36. MODELING WITH MATHEMATICS The length of a rectangular birthday party invitation is 1 inch less than twice its width. The area of the invitation is 15 square inches. Will the invitation fit in the envelope shown without being folded? Explain.

$3\frac{5}{8}$ in.

$5\frac{1}{8}$ in.

37. OPEN-ENDED Write a binomial whose terms have a GCF of $3x$.

38. HOW DO YOU SEE IT? Without factoring, determine which of the graphs represents the function $g(x) = 21x^2 + 37x + 12$ and which represents the function $h(x) = 21x^2 - 37x + 12$. Explain your reasoning.



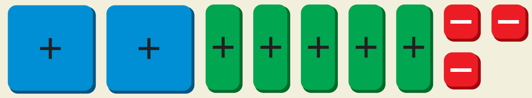
39. REASONING When is it not possible to factor $ax^2 + bx + c$, where $a \neq 1$? Give an example.

40. MAKING AN ARGUMENT Your friend says that to solve the equation $5x^2 + x - 4 = 2$, you should start by factoring the left side as $(5x - 4)(x + 1)$. Is your friend correct? Explain.

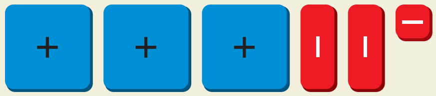
41. REASONING For what values of t can $2x^2 + tx + 10$ be written as the product of two binomials?

42. THOUGHT PROVOKING Use algebra tiles to factor each polynomial modeled by the tiles. Show your work.

a.

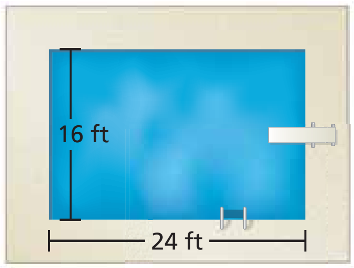


b.



43. MATHEMATICAL CONNECTIONS The length of a rectangle is 1 inch more than twice its width. The value of the area of the rectangle (in square inches) is 5 more than the value of the perimeter of the rectangle (in inches). Find the width.

44. PROBLEM SOLVING A rectangular swimming pool is bordered by a concrete patio. The width of the patio is the same on every side. The area of the surface of the pool is equal to the area of the patio. What is the width of the patio?



In Exercises 45–48, factor the polynomial.

45. $4k^2 + 7jk - 2j^2$

46. $6x^2 + 5xy - 4y^2$

47. $-6a^2 + 19ab - 14b^2$

48. $18m^3 + 39m^2n - 15mn^2$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the square root(s). *(Skills Review Handbook)*

49. $\pm\sqrt{64}$

50. $\sqrt{4}$

51. $-\sqrt{225}$

52. $\pm\sqrt{81}$

Solve the system of linear equations by substitution. Check your solution. *(Section 5.2)*

53. $y = 3 + 7x$
$y - x = -3$

54. $2x = y + 2$
$-x + 3y = 14$

55. $5x - 2y = 14$
$-7 = -2x + y$

56. $-x - 8 = -y$
$9y - 12 + 3x = 0$

7.8 Factoring Special Products

Essential Question

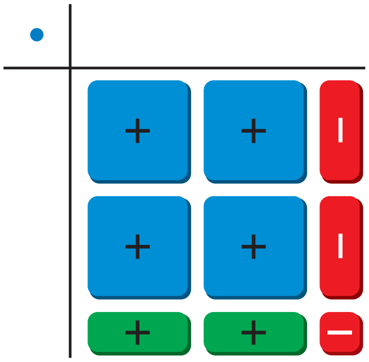
How can you recognize and factor special products?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.8.A
A.10.E
A.10.F

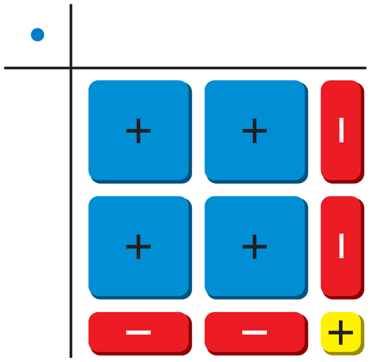
EXPLORATION 1 Factoring Special Products

Work with a partner. Use algebra tiles to write each polynomial as the product of two binomials. Check your answer by multiplying. State whether the product is a "special product" that you studied in Section 7.3.

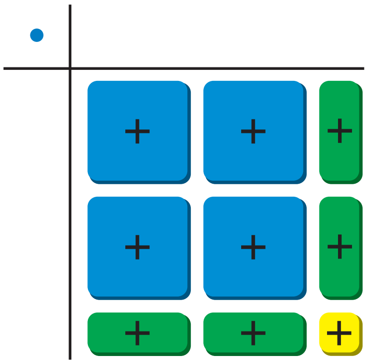
a. $4x^2 - 1 =$ ______



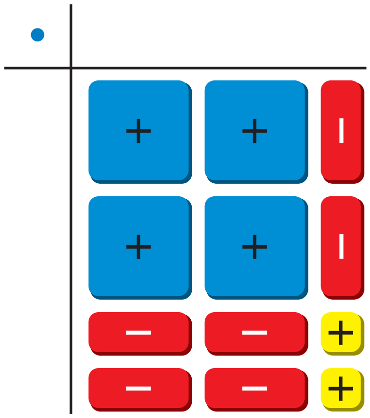
b. $4x^2 - 4x + 1 =$ ______



c. $4x^2 + 4x + 1 =$ ______



d. $4x^2 - 6x + 2 =$ ______

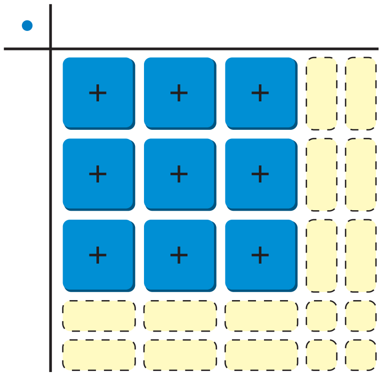


ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to see complicated things as single objects or as being composed of several objects.

EXPLORATION 2 Factoring Special Products

Work with a partner. Use algebra tiles to complete the rectangular array at the left in three different ways, so that each way represents a different special product. Write each special product in standard form and in factored form.



Communicate Your Answer

3. How can you recognize and factor special products? Describe a strategy for recognizing which polynomials can be factored as special products.

4. Use the strategy you described in Question 3 to factor each polynomial.

a. $25x^2 + 10x + 1$ **b.** $25x^2 - 10x + 1$ **c.** $25x^2 - 1$

7.8 Lesson

Core Vocabulary

Previous
polynomial
trinomial

What You Will Learn

- Factor the difference of two squares.
- Factor perfect square trinomials.
- Use factoring to solve real-life problems.

Factoring the Difference of Two Squares

You can use special product patterns to factor polynomials.

Core Concept

Difference of Two Squares Pattern

Algebra	**Example**
$a^2 - b^2 = (a + b)(a - b)$	$x^2 - 9 = x^2 - 3^2 = (x + 3)(x - 3)$

EXAMPLE 1 Factoring the Difference of Two Squares

Factor (a) $x^2 - 25$ and (b) $4z^2 - 1$.

SOLUTION

a. $x^2 - 25 = x^2 - 5^2$ — Write as $a^2 - b^2$.

$= (x + 5)(x - 5)$ — Difference of two squares pattern

▶ So, $x^2 - 25 = (x + 5)(x - 5)$.

b. $4z^2 - 1 = (2z)^2 - 1^2$ — Write as $a^2 - b^2$.

$= (2z + 1)(2z - 1)$ — Difference of two squares pattern

▶ So, $4z^2 - 1 = (2z + 1)(2z - 1)$.

EXAMPLE 2 Evaluating a Numerical Expression

Use a special product pattern to evaluate the expression $54^2 - 48^2$.

SOLUTION

Notice that $54^2 - 48^2$ is a difference of two squares. So, you can rewrite the expression in a form that it is easier to evaluate using the difference of two squares pattern.

$54^2 - 48^2 = (54 + 48)(54 - 48)$ — Difference of two squares pattern

$= 102(6)$ — Simplify.

$= 612$ — Multiply.

▶ So, $54^2 - 48^2 = 612$.

Monitoring Progress

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Factor the polynomial.

1. $x^2 - 36$ **2.** $100 - m^2$ **3.** $9n^2 - 16$ **4.** $16h^2 - 49$

Use a special product pattern to evaluate the expression.

5. $36^2 - 34^2$ **6.** $47^2 - 44^2$ **7.** $55^2 - 50^2$ **8.** $28^2 - 24^2$

Factoring Perfect Square Trinomials

Core Concept

Perfect Square Trinomial Pattern

Algebra	**Example**
$a^2 + 2ab + b^2 = (a + b)^2$	$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2$ $= (x + 3)^2$
$a^2 - 2ab + b^2 = (a - b)^2$	$x^2 - 6x + 9 = x^2 - 2(x)(3) + 3^2$ $= (x - 3)^2$

EXAMPLE 3 Factoring Perfect Square Trinomials

Factor each polynomial.

a. $n^2 + 8n + 16$ **b.** $4x^2 - 12x + 9$

SOLUTION

a. $n^2 + 8n + 16 = n^2 + 2(n)(4) + 4^2$ Write as $a^2 + 2ab + b^2$.

$= (n + 4)^2$ Perfect square trinomial pattern

▶ So, $n^2 + 8n + 16 = (n + 4)^2$.

b. $4x^2 - 12x + 9 = (2x)^2 - 2(2x)(3) + 3^2$ Write as $a^2 - 2ab + b^2$.

$= (2x - 3)^2$ Perfect square trinomial pattern

▶ So, $4x^2 - 12x + 9 = (2x - 3)^2$.

EXAMPLE 4 Solving a Polynomial Equation

Solve $x^2 + \frac{2}{3}x + \frac{1}{9} = 0$.

SOLUTION

$x^2 + \frac{2}{3}x + \frac{1}{9} = 0$ Write equation.

$9x^2 + 6x + 1 = 0$ Multiply each side by 9.

$(3x)^2 + 2(3x)(1) + 1^2 = 0$ Write left side as $a^2 + 2ab + b^2$.

$(3x + 1)^2 = 0$ Perfect square trinomial pattern

$3x + 1 = 0$ Zero-Product Property

$x = -\frac{1}{3}$ Solve for x.

▶ The solution is $x = -\frac{1}{3}$.

ANALYZING MATHEMATICAL RELATIONSHIPS

Equations of the form $(x + a)^2 = 0$ always have repeated roots of $x = -a$.

Monitoring Progress

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Factor the polynomial.

9. $m^2 - 2m + 1$ **10.** $d^2 - 10d + 25$ **11.** $9z^2 + 36z + 36$

Solve the equation.

12. $a^2 + 6a + 9 = 0$ **13.** $w^2 - \frac{7}{3}w + \frac{49}{36} = 0$ **14.** $n^2 - 81 = 0$

Solving Real-Life Problems

EXAMPLE 5 **Modeling with Mathematics**

A bird picks up a golf ball and drops it while flying. The function represents the height y (in feet) of the golf ball t seconds after it is dropped. The ball hits the top of a 32-foot-tall pine tree. After how many seconds does the ball hit the tree?

SOLUTION

1. **Understand the Problem** You are given the height of the golf ball as a function of the amount of time after it is dropped and the height of the tree that the golf ball hits. You are asked to determine how many seconds it takes for the ball to hit the tree.

2. **Make a Plan** Use the function for the height of the golf ball. Substitute the height of the tree for y and solve for the time t.

3. **Solve the Problem** Substitute 32 for y and solve for t.

$$y = 81 - 16t^2 \qquad \text{Write equation.}$$

$$32 = 81 - 16t^2 \qquad \text{Substitute 32 for } y.$$

$$0 = 49 - 16t^2 \qquad \text{Subtract 32 from each side.}$$

$$0 = 7^2 - (4t)^2 \qquad \text{Write as } a^2 - b^2.$$

$$0 = (7 + 4t)(7 - 4t) \qquad \text{Difference of two squares pattern}$$

$$7 + 4t = 0 \quad \text{or} \quad 7 - 4t = 0 \qquad \text{Zero-Product Property}$$

$$t = -\tfrac{7}{4} \quad \text{or} \quad t = \tfrac{7}{4} \qquad \text{Solve for } t.$$

A negative time does not make sense in this situation.

▶ So, the golf ball hits the tree after $\frac{7}{4}$, or 1.75 seconds.

4. **Look Back** Check your solution, as shown, by substituting $t = \frac{7}{4}$ into the equation $32 = 81 - 16t^2$. Then verify that a time of $\frac{7}{4}$ seconds gives a height of 32 feet.

Check

$$32 = 81 - 16t^2$$

$$32 \stackrel{?}{=} 81 - 16\left(\tfrac{7}{4}\right)^2$$

$$32 \stackrel{?}{=} 81 - 16\left(\tfrac{49}{16}\right)$$

$$32 \stackrel{?}{=} 81 - 49$$

$$32 = 32 \ ✓$$

Monitoring Progress

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15. **WHAT IF?** The golf ball does not hit the pine tree. After how many seconds does the ball hit the ground?

7.8 Exercises

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Vocabulary and Core Concept Check

1. **REASONING** Can you use the perfect square trinomial pattern to factor $y^2 + 16y + 64$? Explain.

2. **WHICH ONE DOESN'T BELONG?** Which polynomial does *not* belong with the other three? Explain your reasoning.

$n^2 - 4$	$g^2 - 6g + 9$	$r^2 + 12r + 36$	$k^2 + 25$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, factor the polynomial. *(See Example 1.)*

3. $m^2 - 49$
4. $z^2 - 81$
5. $64 - 81d^2$
6. $25 - 4x^2$
7. $225a^2 - 36b^2$
8. $16x^2 - 169y^2$

In Exercises 9–14, use a special product pattern to evaluate the expression. *(See Example 2.)*

9. $12^2 - 9^2$
10. $19^2 - 11^2$
11. $78^2 - 72^2$
12. $54^2 - 52^2$
13. $53^2 - 47^2$
14. $39^2 - 36^2$

In Exercises 15–22, factor the polynomial. *(See Example 3.)*

15. $h^2 + 12h + 36$
16. $p^2 + 30p + 225$
17. $y^2 - 22y + 121$
18. $x^2 - 4x + 4$
19. $a^2 - 28a + 196$
20. $m^2 + 24m + 144$
21. $25n^2 + 20n + 4$
22. $49a^2 - 14a + 1$

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in factoring the polynomial.

23. ✗
$$n^2 - 64 = n^2 - 8^2$$
$$= (n - 8)^2$$

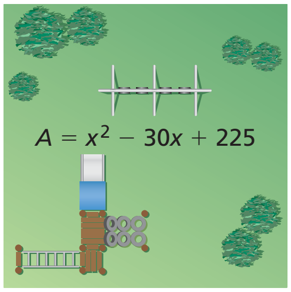
24. ✗
$$y^2 - 6y + 9 = y^2 - 2(y)(3) + 3^2$$
$$= (y - 3)(y + 3)$$

25. **MODELING WITH MATHEMATICS** The area (in square centimeters) of a square coaster can be represented by $d^2 + 8d + 16$.

   a. Write an expression that represents the side length of the coaster.

   b. Write an expression for the perimeter of the coaster.

26. **MODELING WITH MATHEMATICS** The polynomial represents the area (in square feet) of the square playground.



   a. Write a polynomial that represents the side length of the playground.

   b. Write an expression for the perimeter of the playground.

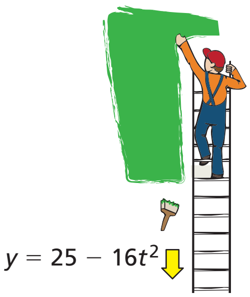
In Exercises 27–34, solve the equation. *(See Example 4.)*

27. $z^2 - 4 = 0$
28. $4x^2 = 49$
29. $k^2 - 16k + 64 = 0$
30. $s^2 + 20s + 100 = 0$
31. $n^2 + 9 = 6n$
32. $y^2 = 12y - 36$
33. $y^2 + \frac{1}{2}y = -\frac{1}{16}$
34. $-\frac{4}{3}x + \frac{4}{9} = -x^2$

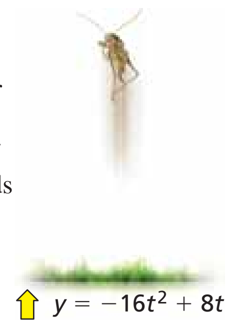
In Exercises 35–40, factor the polynomial.

35. $3z^2 - 27$
36. $2m^2 - 50$
37. $4y^2 - 16y + 16$
38. $8k^2 + 80k + 200$
39. $50y^2 + 120y + 72$
40. $27m^2 - 36m + 12$

41. MODELING WITH MATHEMATICS While standing on a ladder, you drop a paintbrush. The function represents the height y (in feet) of the paintbrush t seconds after it is dropped. After how many seconds does the paintbrush land on the ground? *(See Example 5.)*



42. MODELING WITH MATHEMATICS The function represents the height y (in feet) of a grasshopper jumping straight up from the ground t seconds after the start of the jump. After how many seconds is the grasshopper 1 foot off the ground?

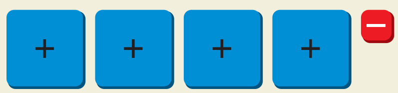


43. REASONING Tell whether the polynomial can be factored. If not, change the constant term so that the polynomial is a perfect square trinomial.

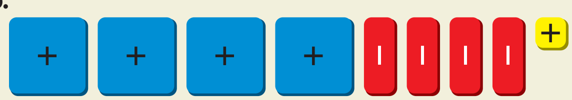
a. $w^2 + 18w + 84$ **b.** $y^2 - 10y + 23$

44. THOUGHT PROVOKING Use algebra tiles to factor each polynomial modeled by the tiles. Show your work.

a.

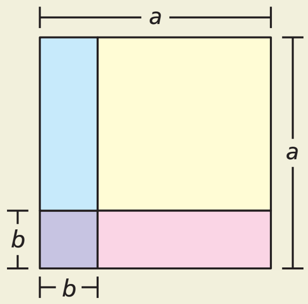


b.



45. COMPARING METHODS Describe two methods you can use to simplify $(2x - 5)^2 - (x - 4)^2$. Which one would you use? Explain.

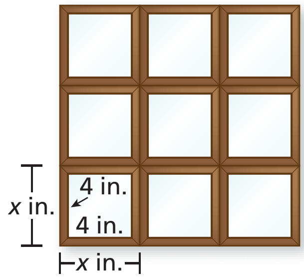
46. HOW DO YOU SEE IT? The figure shows a large square with an area of a^2 that contains a smaller square with an area of b^2.



a. Describe the regions that represent $a^2 - b^2$. How can you rearrange these regions to show that $a^2 - b^2 = (a + b)(a - b)$?

b. How can you use the figure to show that $(a - b)^2 = a^2 - 2ab + b^2$?

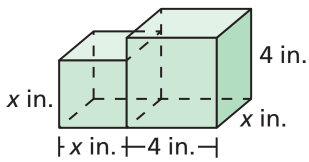
47. PROBLEM SOLVING You hang nine identical square picture frames on a wall.



a. Write a polynomial that represents the area of the picture frames, not including the pictures.

b. The area in part (a) is 81 square inches. What is the side length of one of the picture frames? Explain your reasoning.

48. MATHEMATICAL CONNECTIONS The composite solid is made up of a cube and a rectangular prism.



a. Write a polynomial that represents the volume of the composite solid.

b. The volume of the composite solid is equal to $25x$. What is the value of x? Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the prime factorization of the number. *(Skills Review Handbook)*

49. 50 **50.** 44 **51.** 85 **52.** 96

Graph the inequality in a coordinate plane. *(Section 5.6)*

53. $y \leq 4x - 1$ **54.** $y > -\frac{1}{2}x + 3$ **55.** $4y - 12 \geq 8x$ **56.** $3y + 3 < x$

7.9 Factoring Polynomials Completely

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.10.D
A.10.E

Essential Question How can you factor a polynomial completely?

EXPLORATION 1 Writing a Product of Linear Factors

Work with a partner. Write the product represented by the algebra tiles. Then multiply to write the polynomial in standard form.

a. 

b. 

c. 

d. 

e. 

f. 

REASONING

To be proficient in math, you need to know and flexibly use different properties of operations and objects.

EXPLORATION 2 Matching Standard and Factored Forms

Work with a partner. Match the standard form of the polynomial with the equivalent factored form. Explain your strategy.

a. $x^3 + x^2$	**A.** $x(x + 1)(x - 1)$
b. $x^3 - x$	**B.** $x(x - 1)^2$
c. $x^3 + x^2 - 2x$	**C.** $x(x + 1)^2$
d. $x^3 - 4x^2 + 4x$	**D.** $x(x + 2)(x - 1)$
e. $x^3 - 2x^2 - 3x$	**E.** $x(x - 1)(x - 2)$
f. $x^3 - 2x^2 + x$	**F.** $x(x + 2)(x - 2)$
g. $x^3 - 4x$	**G.** $x(x - 2)^2$
h. $x^3 + 2x^2$	**H.** $x(x + 2)^2$
i. $x^3 - x^2$	**I.** $x^2(x - 1)$
j. $x^3 - 3x^2 + 2x$	**J.** $x^2(x + 1)$
k. $x^3 + 2x^2 - 3x$	**K.** $x^2(x - 2)$
l. $x^3 - 4x^2 + 3x$	**L.** $x^2(x + 2)$
m. $x^3 - 2x^2$	**M.** $x(x + 3)(x - 1)$
n. $x^3 + 4x^2 + 4x$	**N.** $x(x + 1)(x - 3)$
o. $x^3 + 2x^2 + x$	**O.** $x(x - 1)(x - 3)$

Communicate Your Answer

3. How can you factor a polynomial completely?

4. Use your answer to Question 3 to factor each polynomial completely.

a. $x^3 + 4x^2 + 3x$ **b.** $x^3 - 6x^2 + 9x$ **c.** $x^3 + 6x^2 + 9x$

7.9 Lesson

Core Vocabulary

factoring by grouping, *p. 390*
factored completely, *p. 390*

Previous
polynomial
binomial

What You Will Learn

- Factor polynomials by grouping
- Factor polynomials completely.
- Use factoring to solve real-life problems.

Factoring Polynomials by Grouping

You have used the Distributive Property to factor out a greatest common monomial from a polynomial. Sometimes, you can factor out a common binomial. You may be able to use the Distributive Property to factor polynomials with four terms, as described below.

Core Concept

Factoring by Grouping

To factor a polynomial with four terms, group the terms into pairs. Factor the GCF out of each pair of terms. Look for and factor out the common binomial factor. This process is called **factoring by grouping**.

EXAMPLE 1 Factoring by Grouping

Factor each polynomial by grouping.

a. $x^3 + 3x^2 + 2x + 6$ **b.** $x^2 + y + x + xy$

SOLUTION

a. $x^3 + 3x^2 + 2x + 6 = (x^3 + 3x^2) + (2x + 6)$ Group terms with common factors.

Common binomial factor is $x + 3$.

$= x^2(x + 3) + 2(x + 3)$ Factor out GCF of each pair of terms.

$= (x + 3)(x^2 + 2)$ Factor out $(x + 3)$.

So, $x^3 + 3x^2 + 2x + 6 = (x + 3)(x^2 + 2)$.

b. $x^2 + y + x + xy = x^2 + x + xy + y$ Rewrite polynomial.

$= (x^2 + x) + (xy + y)$ Group terms with common factors.

Common binomial factor is $x + 1$.

$= x(x + 1) + y(x + 1)$ Factor out GCF of each pair of terms.

$= (x + 1)(x + y)$ Factor out $(x + 1)$.

So, $x^2 + y + x + xy = (x + 1)(x + y)$.

Monitoring Progress

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Factor the polynomial by grouping.

1. $a^3 + 3a^2 + a + 3$ **2.** $y^2 + 2x + yx + 2y$

Factoring Polynomials Completely

You have seen that the polynomial $x^2 - 1$ can be factored as $(x + 1)(x - 1)$. This polynomial is factorable. Notice that the polynomial $x^2 + 1$ cannot be written as the product of polynomials with integer coefficients. This polynomial is unfactorable. A factorable polynomial with integer coefficients is **factored completely** when it is written as a product of unfactorable polynomials with integer coefficients.

Concept Summary

Guidelines for Factoring Polynomials Completely

To factor a polynomial completely, you should try each of these steps.

1.	Factor out the greatest common monomial factor.	$3x^2 + 6x = 3x(x + 2)$
2.	Look for a difference of two squares or a perfect square trinomial.	$x^2 + 4x + 4 = (x + 2)^2$
3.	Factor a trinomial of the form $ax^2 + bx + c$ into a product of binomial factors.	$3x^2 - 5x - 2 = (3x + 1)(x - 2)$
4.	Factor a polynomial with four terms by grouping.	$x^3 + x - 4x^2 - 4 = (x^2 + 1)(x - 4)$

EXAMPLE 2 Factoring Completely

Factor (a) $3x^3 + 6x^2 - 18x$ and (b) $7x^4 - 28x^2$.

SOLUTION

a. $3x^3 + 6x^2 - 18x = 3x(x^2 + 2x - 6)$ — Factor out $3x$.

$x^2 + 2x - 6$ is unfactorable, so the polynomial is factored completely.

▶ So, $3x^3 + 6x^2 - 18x = 3x(x^2 + 2x - 6)$.

b.
$$
\begin{aligned}
7x^4 - 28x^2 &= 7x^2(x^2 - 4) && \text{Factor out } 7x^2.\\
&= 7x^2(x^2 - 2^2) && \text{Write as } a^2 - b^2.\\
&= 7x^2(x + 2)(x - 2) && \text{Difference of two squares pattern}
\end{aligned}
$$

▶ So, $7x^4 - 28x^2 = 7x^2(x + 2)(x - 2)$.

EXAMPLE 3 Solving an Equation by Factoring Completely

Solve $2x^3 + 8x^2 = 10x$.

SOLUTION

$$
\begin{aligned}
2x^3 + 8x^2 &= 10x && \text{Original equation}\\
2x^3 + 8x^2 - 10x &= 0 && \text{Subtract } 10x \text{ from each side.}\\
2x(x^2 + 4x - 5) &= 0 && \text{Factor out } 2x.\\
2x(x + 5)(x - 1) &= 0 && \text{Factor } x^2 + 4x - 5.
\end{aligned}
$$

$2x = 0$ *or* $x + 5 = 0$ *or* $x - 1 = 0$ — Zero-Product Property

$x = 0$ *or* $x = -5$ *or* $x = 1$ — Solve for x.

▶ The roots are $x = -5$, $x = 0$, and $x = 1$.

Monitoring Progress

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Factor the polynomial completely.

3. $3x^3 - 12x$ **4.** $2y^3 - 12y^2 + 18y$ **5.** $m^3 - 2m^2 - 8m$

Solve the equation.

6. $w^3 - 8w^2 + 16w = 0$ **7.** $x^3 - 25x = 0$ **8.** $c^3 - 7c^2 + 12c = 0$

Solving Real-Life Problems

EXAMPLE 4 **Modeling with Mathematics**

A terrarium in the shape of a rectangular prism has a volume of 4608 cubic inches. Its length is more than 10 inches. The dimensions of the terrarium in terms of its width are shown. Find the length, width, and height of the terrarium.

SOLUTION

1. **Understand the Problem** You are given the volume of a terrarium in the shape of a rectangular prism and a description of the length. The dimensions are written in terms of its width. You are asked to find the length, width, and height of the terrarium.

2. **Make a Plan** Use the formula for the volume of a rectangular prism to write and solve an equation for the width of the terrarium. Then substitute that value in the expressions for the length and height of the terrarium.

3. **Solve the Problem**

Volume $=$ length $\cdot$ width $\cdot$ height	Volume of a rectangular prism
$4608 = (36 - w)(w)(w + 4)$	Write equation.
$4608 = 32w^2 + 144w - w^3$	Multiply.
$0 = 32w^2 + 144w - w^3 - 4608$	Subtract 4608 from each side.
$0 = (-w^3 + 32w^2) + (144w - 4608)$	Group terms with common factors.
$0 = -w^2(w - 32) + 144(w - 32)$	Factor out GCF of each pair of terms.
$0 = (w - 32)(-w^2 + 144)$	Factor out $(w - 32)$.
$0 = -1(w - 32)(w^2 - 144)$	Factor -1 from $-w^2 + 144$.
$0 = -1(w - 32)(w - 12)(w + 12)$	Difference of two squares pattern
$w - 32 = 0$ *or* $w - 12 = 0$ *or* $w + 12 = 0$	Zero-Product Property
$w = 32$ *or* $w = 12$ *or* $w = -12$	Solve for w.

Disregard $w = -12$ because a negative width does not make sense. You know that the length is more than 10 inches. Test the solutions of the equation, 12 and 32, in the expression for the length.

length $= 36 - w = 36 - 12 = 24$ ✓ *or* length $= 36 - w = 36 - 32 = 4$ ✗

The solution 12 gives a length of 24 inches, so 12 is the correct value of w.

Use $w = 12$ to find the height, as shown.

height $= w + 4 = 12 + 4 = 16$

▶ The width is 12 inches, the length is 24 inches, and the height is 16 inches.

Check

$V = \ell wh$

$4608 \stackrel{?}{=} 24(12)(16)$

$4608 = 4608$ ✓

4. **Look Back** Check your solution. Substitute the values for the length, width, and height when the width is 12 inches into the formula for volume. The volume of the terrarium should be 4608 cubic inches.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

9. A box in the shape of a rectangular prism has a volume of 72 cubic feet. The box has a length of x feet, a width of $(x - 1)$ feet, and a height of $(x + 9)$ feet. Find the dimensions of the box.

7.9 Exercises

Tutorial Help in English and Spanish at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What does it mean for a polynomial to be factored completely?

2. **WRITING** Explain how to choose which terms to group together when factoring by grouping.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, factor the polynomial by grouping. *(See Example 1.)*

3. $x^3 + x^2 + 2x + 2$
4. $y^3 - 9y^2 + y - 9$
5. $3z^3 + 2z - 12z^2 - 8$
6. $2s^3 - 27 - 18s + 3s^2$
7. $x^2 + xy + 8x + 8y$
8. $q^2 + q + 5pq + 5p$
9. $m^2 - 3m + mn - 3n$
10. $2a^2 + 8ab - 3a - 12b$

In Exercises 11–22, factor the polynomial completely. *(See Example 2.)*

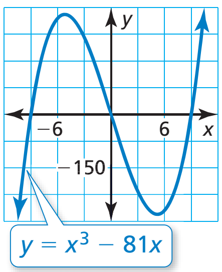
11. $2x^3 - 2x$
12. $36a^4 - 4a^2$
13. $2c^2 - 7c + 19$
14. $m^2 - 5m - 35$
15. $6g^3 - 24g^2 + 24g$
16. $-15d^3 + 21d^2 - 6d$
17. $3r^5 + 3r^4 - 90r^3$
18. $5w^4 - 40w^3 + 80w^2$
19. $-4c^4 + 8c^3 - 28c^2$
20. $8t^2 + 8t - 72$
21. $b^3 - 5b^2 - 4b + 20$
22. $h^3 + 4h^2 - 25h - 100$

In Exercises 23–28, solve the equation. *(See Example 3.)*

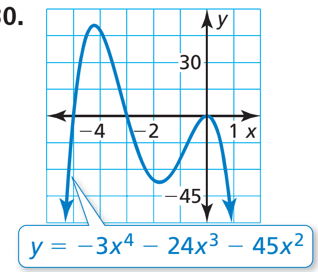
23. $5n^3 - 30n^2 + 40n = 0$
24. $k^4 - 100k^2 = 0$
25. $x^3 + x^2 = 4x + 4$
26. $2t^5 + 2t^4 - 144t^3 = 0$
27. $12s - 3s^3 = 0$
28. $4y^3 - 7y^2 + 28 = 16y$

In Exercises 29–32, find the *x*-coordinates of the points where the graph crosses the *x*-axis.

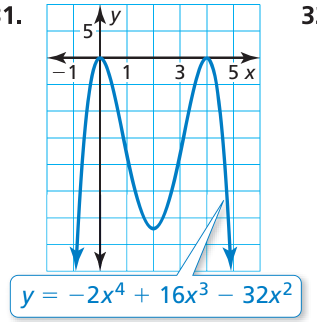
29.



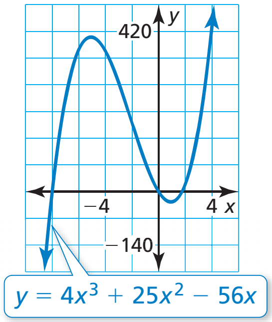
30.



31.



32.

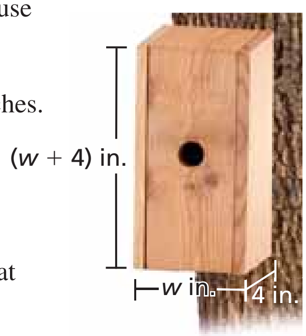


ERROR ANALYSIS In Exercises 33 and 34, describe and correct the error in factoring the polynomial completely.

33.
$$\begin{aligned} a^3 + 8a^2 - 6a - 48 &= a^2(a + 8) + 6(a + 8) \\ &= (a + 8)(a^2 + 6) \end{aligned}$$

34.
$$\begin{aligned} x^3 - 6x^2 - 9x + 54 &= x^2(x - 6) - 9(x - 6) \\ &= (x - 6)(x^2 - 9) \end{aligned}$$

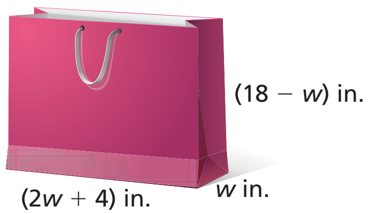
35. **MODELING WITH MATHEMATICS** You are building a birdhouse in the shape of a rectangular prism that has a volume of 128 cubic inches. The dimensions of the birdhouse in terms of its width are shown. *(See Example 4.)*



a. Write a polynomial that represents the volume of the birdhouse.

b. What are the dimensions of the birdhouse?

36. **MODELING WITH MATHEMATICS** A gift bag shaped like a rectangular prism has a volume of 1152 cubic inches. The dimensions of the gift bag in terms of its width are shown. The height is greater than the width. What are the dimensions of the gift bag?



In Exercises 37–40, factor the polynomial completely.

37. $x^3 + 2x^2y - x - 2y$

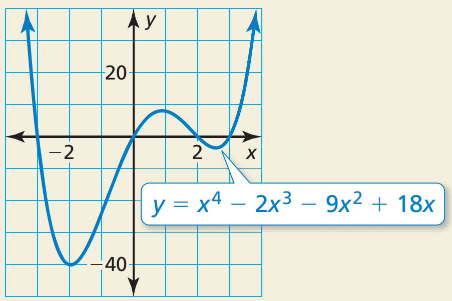
38. $8b^3 - 4b^2a - 18b + 9a$

39. $4s^2 - s + 12st - 3t$

40. $6m^3 - 12mn + m^2n - 2n^2$

41. **WRITING** Is it possible to find three real solutions of the equation $x^3 + 2x^2 + 3x + 6 = 0$? Explain your reasoning.

42. **HOW DO YOU SEE IT?** How can you use the factored form of the polynomial $x^4 - 2x^3 - 9x^2 + 18x = x(x - 3)(x + 3)(x - 2)$ to find the x-intercepts of the graph of the function?



43. **OPEN-ENDED** Write a polynomial of degree 3 that satsifies each of the given conditions.

a. is not factorable **b.** can be factored by grouping

44. **MAKING AN ARGUMENT** Your friend says that if a trinomial cannot be factored as the product of two binomials, then the trinomial is factored completely. Is your friend correct? Explain.

45. **PROBLEM SOLVING** The volume (in cubic feet) of a room is represented by $12z^3 - 27z$. Find expressions that could represent the dimensions of the room.

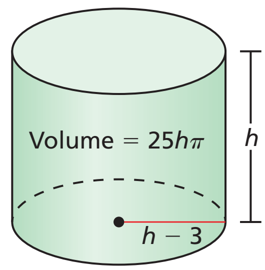
46. **MATHEMATICAL CONNECTIONS** The width of a box is 4 inches more than the height h. The length is the difference of 9 inches and the height.

a. Write a polynomial that represents the volume of the box in terms of its height (in inches).

b. The volume of the box is 180 cubic inches. What are the possible dimensions of the box?

c. Which dimensions result in a box with the least possible surface area? Explain your reasoning.

47. **MATHEMATICAL CONNECTIONS** The volume of a cylinder is given by $V = \pi r^2 h$, where r is the radius of the base of the cylinder and h is the height of the cylinder. Find the dimensions of the cylinder.



48. **THOUGHT PROVOKING** Factor the polynomial $x^5 - x^4 - 5x^3 + 5x^2 + 4x - 4$ completely.

49. **REASONING** Find a value for w so that the equation has (a) two solutions and (b) three solutions. Explain your reasoning.

$$5x^3 + wx^2 + 80x = 0$$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the system of linear equations by graphing. *(Section 5.1)*

50. $y = x - 4$
$y = -2x + 2$

51. $y = \frac{1}{2}x + 2$
$y = 3x - 3$

52. $5x - y = 12$
$\frac{1}{4}x + y = 9$

53. $x = 3y$
$y - 10 = 2x$

Graph the function. Describe the domain and range. *(Section 6.3)*

54. $f(x) = 5^x$

55. $y = 9\left(\frac{1}{3}\right)^x$

56. $y = -3(0.5)^x$

57. $f(x) = -3(4)^x$

7.6–7.9 What Did You Learn?

Core Vocabulary

factoring by grouping, *p. 390*
factored completely, *p. 390*

Core Concepts

Section 7.6

Factoring $x^2 + bx + c$ When c Is Positive, *p. 372*
Factoring $x^2 + bx + c$ When c Is Negative, *p. 373*

Section 7.7

Factoring $ax^2 + bx + c$ When ac Is Positive, *p. 378*
Factoring $ax^2 + bx + c$ When ac Is Negative, *p. 379*

Section 7.8

Difference of Two Squares Pattern, *p. 384*
Perfect Square Trinomial Pattern, *p. 385*

Section 7.9

Factoring by Grouping, *p. 390*
Factoring Polynomials Completely, *p. 390*

Mathematical Thinking

1. How are the solutions of Exercise 29 on page 375 related to the graph of $y = m^2 + 3m + 2$?
2. The equation in part (b) of Exercise 47 on page 376 has two solutions. Are both solutions of the equation reasonable in the context of the problem? Explain your reasoning.

Performance Task

The View Matters

The way an equation or expression is written can help you interpret and solve problems. Which representation would you rather have when trying to solve for specific information? Why?

To explore the answers to these questions and more, go to ***BigIdeasMath.com***.

7 Chapter Review

7.1 Adding and Subtracting Polynomials (pp. 337–344)

Find $(2x^3 + 6x^2 - x) - (-3x^3 - 2x^2 - 9x)$.

$$\begin{aligned}(2x^3 + 6x^2 - x) - (-3x^3 - 2x^2 - 9x) &= (2x^3 + 6x^2 - x) + (3x^3 + 2x^2 + 9x)\\ &= (2x^3 + 3x^3) + (6x^2 + 2x^2) + (-x + 9x)\\ &= 5x^3 + 8x^2 + 8x\end{aligned}$$

Write the polynomial in standard form. Identify the degree and leading coefficient of the polynomial. Then classify the polynomial by the number of terms.

1. $6 + 2x^2$ **2.** $-3p^3 + 5p^6 - 4$ **3.** $9x^7 - 6x^2 + 13x^5$ **4.** $-12y + 8y^3$

Find the sum or difference.

5. $(3a + 7) + (a - 1)$ **6.** $(x^2 + 6x - 5) + (2x^2 + 15)$

7. $(-y^2 + y + 2) - (y^2 - 5y - 2)$ **8.** $(p + 7) - (6p + 13)$

7.2 Multiplying Polynomials (pp. 345–350)

Find $(x + 7)(x - 9)$.

$(x + 7)(x - 9) = x(x - 9) + 7(x - 9)$ Distribute $(x - 9)$ to each term of $(x + 7)$.

$= x(x) + x(-9) + 7(x) + 7(-9)$ Distributive Property

$= x^2 + (-9x) + 7x + (-63)$ Multiply.

$= x^2 - 2x - 63$ Combine like terms.

Find the product.

9. $(x + 6)(x - 4)$ **10.** $(y - 5)(3y + 8)$

11. $(x^2 + 4)(x^2 + 7x)$ **12.** $(-3y + 1)(4y^2 - y - 7)$

7.3 Special Products of Polynomials (pp. 351–356)

Find each product.

a. $(6x + 4y)^2$

$(6x + 4y)^2 = (6x)^2 + 2(6x)(4y) + (4y)^2$ Square of a binomial pattern

$= 36x^2 + 48xy + 16y^2$ Simplify.

b. $(2x + 3y)(2x - 3y)$

$(2x + 3y)(2x - 3y) = (2x)^2 - (3y)^2$ Sum and difference pattern

$= 4x^2 - 9y^2$ Simplify.

Find the product.

13. $(x + 9)(x - 9)$ **14.** $(2y + 4)(2y - 4)$ **15.** $(p + 4)^2$ **16.** $(-1 + 2d)^2$

7.4 Dividing Polynomials (pp. 357–362)

Find $(20x^2 + 19x - 3) \div (-4x + 1)$.

$$\begin{array}{r l}
 -5x - 6 & \\
-4x + 1 \overline{) 20x^2 + 19x - 3} & \\
\underline{20x^2 - 5x} & \text{Multiply divisor by } \frac{20x}{-4x} = -5x. \\
24x - 3 & \text{Subtract. Bring down next term.} \\
\underline{24x - 6} & \text{Multiply divisor by } \frac{24x}{-4x} = -6. \\
3 & \text{Subtract.}
\end{array}$$

▶ $(20x^2 + 19x - 3) \div (-4x + 1) = -5x - 6 + \dfrac{3}{-4x + 1}$

Divide.

17. $(36x + 3) \div (9x + 4)$

18. $(-16x^2 - 26x - 13) \div (2x - 3)$

19. $(11x^2 - 3x + 16) \div (x - 3)$

20. $(7x + 14x^2 - 4) \div (-7x^2 + 5)$

7.5 Solving Polynomial Equations in Factored Form (pp. 363–368)

Solve $(x + 6)(x - 8) = 0$.

$(x + 6)(x - 8) = 0$ Write equation.

$x + 6 = 0$ *or* $x - 8 = 0$ Zero-Product Property

$x = -6$ *or* $x = 8$ Solve for x.

Solve the equation.

21. $x^2 + 5x = 0$

22. $(z + 3)(z - 7) = 0$

23. $(b + 13)^2 = 0$

24. $2y(y - 9)(y + 4) = 0$

7.6 Factoring $x^2 + bx + c$ (pp. 371–376)

Factor $x^2 + 6x - 27$.

Notice that $b = 6$ and $c = -27$. Because c is negative, the factors p and q must have different signs so that pq is negative.

Find two integer factors of -27 whose sum is 6.

Factors of −27	−27, 1	−1, 27	−9, 3	−3, 9
Sum of factors	−26	26	−6	6

The values of p and q are -3 and 9.

▶ So, $x^2 + 6x - 27 = (x - 3)(x + 9)$.

Factor the polynomial.

25. $p^2 + 2p - 35$

26. $b^2 + 18b + 80$

27. $z^2 - 4z - 21$

28. $x^2 - 11x + 28$

7.7 Factoring $ax^2 + bx + c$ (pp. 377–382)

Factor $5x^2 + 36x + 7$.

There is no GCF, so you need to consider the possible factors of a and c. Because b and c are both positive, the factors of c must be positive. Use a table to organize information about the factors of a and c.

Factors of 5	Factors of 7	Possible factorization	Middle term	
1, 5	1, 7	$(x + 1)(5x + 7)$	$7x + 5x = 12x$	✗
1, 5	7, 1	$(x + 7)(5x + 1)$	$x + 35x = 36x$	✓

▶ So, $5x^2 + 36x + 7 = (x + 7)(5x + 1)$.

Factor the polynomial.

29. $3t^2 + 16t - 12$ **30.** $-4y^2 + 14y - 12$ **31.** $6x^2 + 17x + 7$

7.8 Factoring Special Products (pp. 383–388)

Factor $25x^2 - 30x + 9$.

$$25x^2 - 30x + 9 = (5x)^2 - 2(5x)(3) + 3^2 \quad \text{Write as } a^2 - 2ab + b^2.$$

$$= (5x - 3)^2 \quad \text{Perfect square trinomial pattern}$$

Factor the polynomial.

32. $x^2 - 9$ **33.** $y^2 - 100$ **34.** $z^2 - 6z + 9$ **35.** $m^2 + 16m + 64$

7.9 Factoring Polynomials Completely (pp. 389–394)

Factor $x^3 + 4x^2 - 3x - 12$.

$$x^3 + 4x^2 - 3x - 12 = (x^3 + 4x^2) + (-3x - 12) \quad \text{Group terms with common factors.}$$

$$= x^2(x + 4) + (-3)(x + 4) \quad \text{Factor out GCF of each pair of terms.}$$

$$= (x + 4)(x^2 - 3) \quad \text{Factor out } (x + 4).$$

Factor the polynomial completely.

36. $n^3 - 9n$ **37.** $x^2 - 3x + 4ax - 12a$ **38.** $2x^4 + 2x^3 - 20x^2$

Solve the equation.

39. $3x^3 - 9x^2 - 54x = 0$ **40.** $16x^2 - 36 = 0$ **41.** $z^3 + 3z^2 - 25z - 75 = 0$

42. A box in the shape of a rectangular prism has a volume of 96 cubic feet. The box has a length of $(x + 8)$ feet, a width of x feet, and a height of $(x - 2)$ feet. Find the dimensions of the box.

7 Chapter Test

Find the sum or difference. Then identify the degree of the sum or difference and classify it by the number of terms.

1. $(-2p + 4) - (p^2 - 6p + 8)$

2. $(9c^6 - 5b^4) - (4c^6 - 5b^4)$

3. $(4s^4 + 2st + t) + (2s^4 - 2st - 4t)$

Find the product.

4. $(h - 5)(h - 8)$

5. $(2w - 3)(3w + 5)$

6. $(z + 11)(z - 11)$

7. Explain how you can determine whether a polynomial is a perfect square trinomial.

8. Is 18 a polynomial? Explain your reasoning.

Divide.

9. $(-3x^2 + 12x - 8) \div (x + 1)$

10. $(45x - 6) \div (15x - 5)$

11. $(5x^2 + 4 + 17x) \div (-x^2 + 7)$

Factor the polynomial completely.

12. $s^2 - 15s + 50$

13. $h^3 + 2h^2 - 9h - 18$

14. $-5k^2 - 22k + 15$

Solve the equation.

15. $(n - 1)(n + 6)(n + 5) = 0$

16. $d^2 + 14d + 49 = 0$

17. $6x^4 + 8x^2 = 26x^3$

18. The expression $\pi(r - 3)^2$ represents the area covered by the hour hand on a clock in one rotation, where r is the radius of the entire clock. Write a polynomial in standard form that represents the area covered by the hour hand in one rotation.

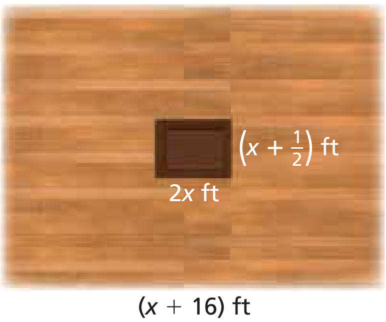
19. A magician's stage has a trapdoor.

a. The total area (in square feet) of the stage can be represented by $x^2 + 27x + 176$. Write an expression for the width of the stage.

b. Write an expression for the perimeter of the stage.

c. The area of the trapdoor is 10 square feet. Find the value of x.

d. The magician wishes to have the area of the stage be at least 20 times the area of the trapdoor. Does this stage satisfy his requirement? Explain.

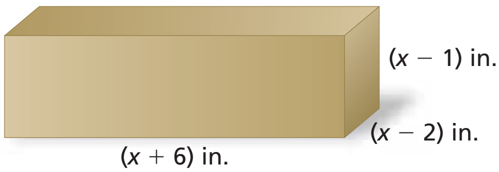


20. You are jumping on a trampoline. For one jump, your height y (in feet) above the trampoline after t seconds can be represented by $y = -16t^2 + 24t$. How many seconds are you in the air?

21. A cardboard box in the shape of a rectangular prism has the dimensions shown.

a. Write a polynomial that represents the volume of the box.

b. The volume of the box is 60 cubic inches. What are the length, width, and height of the box?



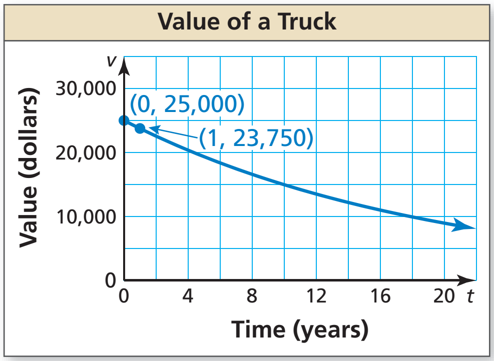
7 Standards Assessment

1. Which expression is equivalent to $-2x + 15x^2 - 8$? *(TEKS A.10.E)*

Ⓐ $(3x - 2)(5x - 4)$ Ⓑ $(5x + 4)(3x + 2)$

Ⓒ $(5x - 4)(3x + 2)$ Ⓓ $(3x - 2)(5x + 4)$

2. The graph shows the value v (in dollars) of a truck over time t (in years). Based on the graph, which statement is true? *(TEKS A.9.B, TEKS A.9.C)*



Ⓕ The value of the truck decreases by 5% each year.

Ⓖ The value of the truck decreases by 95% each year.

Ⓗ The value of the truck decreases by \$1250 each year.

Ⓙ The value of the truck decreases by \$2250 each year.

3. Which polynomial represents the product of $2x - 4$ and $x^2 + 6x - 2$? *(TEKS A.10.B)*

Ⓐ $2x^3 + 8x^2 - 4x + 8$ Ⓑ $2x^3 + 8x^2 - 28x + 8$

Ⓒ $2x^3 + 8$ Ⓓ $2x^3 - 24x - 2$

4. Which of the following correctly describes a line that is perpendicular to the x-axis? *(TEKS A.2.G)*

Ⓕ $y = -4$; slope $= 0$ Ⓖ $x = -4$; slope $= 0$

Ⓗ $y = -4$; The slope is undefined. Ⓙ $x = -4$; The slope is undefined.

5. Which of the following are solutions of the equation $x^3 + 6x^2 - 4x = 24$? *(TEKS A.10.F)*

I. -6

II. -2

III. 0

IV. 2

Ⓐ I and IV only Ⓑ I and II only

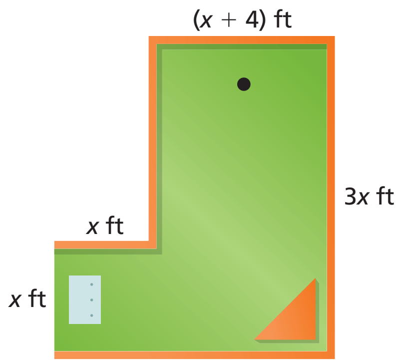
Ⓒ I, II, and IV only Ⓓ I, II, III, and IV

6. Which of the following is a direct variation equation? *(TEKS A.2.D)*

Ⓕ $4y - 2x = 6$ Ⓖ $5y = 5x + 10$

Ⓗ $3y + 12x = 0$ Ⓙ $y + 4 = 0$

7. **GRIDDED ANSWER** You are playing miniature golf on the hole shown. The area of the golf hole is 216 square feet. What is the perimeter (in feet) of the golf hole? *(TEKS A.8.A, TEKS A.10.E)*

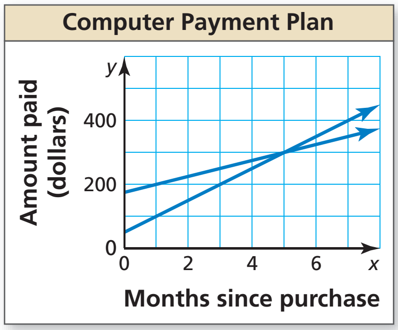


8. What is the 4th term of the geometric sequence $f(1) = 6, f(n) = -\frac{1}{2}f(n - 1)$? *(TEKS A.12.C)*

Ⓐ $-\frac{3}{2}$ Ⓑ $-\frac{3}{4}$

Ⓒ $\frac{3}{8}$ Ⓓ $\frac{3}{4}$

9. At an electronics store, two customers each buy a computer on the same day. Each customer arranges a payment plan. The graph shows the amount y (in dollars) paid for the computers after x months. What does the intersection of the lines represent? *(TEKS A.3.G)*



Ⓕ After 5 months, each customer will have paid the same amount of \$300.

Ⓖ After 5 months, each customer will have paid the same amount of \$350.

Ⓗ After 6 months, each customer will have paid the same amount of \$350.

Ⓙ none of the above

10. What is the remainder of $(4x^2 + 7x - 1) \div (4 + x)$? *(TEKS A.10.C)*

Ⓐ $-9x - 1$ Ⓑ $23x - 1$

Ⓒ 35 Ⓓ -37

8 Graphing Quadratic Functions

Town Population ***(p. 450)***

Satellite Dish ***(p. 443)***

Roller Coaster ***(p. 434)***

Firework Explosion ***(p. 423)***

Garden Waterfalls ***(p. 416)***

Mathematical Thinking: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

Maintaining Mathematical Proficiency

Graphing Linear Equations (A.3.C)

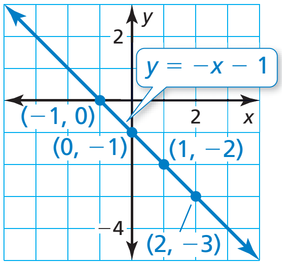
Example 1 **Graph $y = -x - 1$.**

Step 1 Make a table of values.

x	$y = -x - 1$	y	(x, y)
-1	$y = -(-1) - 1$	0	$(-1, 0)$
0	$y = -(0) - 1$	-1	$(0, -1)$
1	$y = -(1) - 1$	-2	$(1, -2)$
2	$y = -(2) - 1$	-3	$(2, -3)$

Step 2 Plot the ordered pairs.

Step 3 Draw a line through the points.



Graph the linear equation.

1. $y = 2x - 3$

2. $y = -3x + 4$

3. $y = -\frac{1}{2}x - 2$

4. $y = x + 5$

Evaluating Expressions (A.11.B)

Example 2 **Evaluate $2x^2 + 3x - 5$ when $x = -1$.**

$$
\begin{aligned}
2x^2 + 3x - 5 &= 2(-1)^2 + 3(-1) - 5 && \text{Substitute } -1 \text{ for } x.\\
&= 2(1) + 3(-1) - 5 && \text{Evaluate the power.}\\
&= 2 - 3 - 5 && \text{Multiply.}\\
&= -6 && \text{Subtract.}
\end{aligned}
$$

Evaluate the expression when $x = -2$.

5. $5x^2 - 9$

6. $3x^2 + x - 2$

7. $-x^2 + 4x + 1$

8. $x^2 + 8x + 5$

9. $-2x^2 - 4x + 3$

10. $-4x^2 + 2x - 6$

11. **ABSTRACT REASONING** Complete the table. Find a pattern in the differences of consecutive y-values. Use the pattern to write an expression for y when $x = 6$.

x	1	2	3	4	5
$y = ax^2$					

Mathematical Thinking

Mathematically proficient students use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution. **(A.1.B)**

Problem-Solving Strategies

Core Concept

Trying Special Cases

When solving a problem in mathematics, it can be helpful to try special cases of the original problem. For instance, in this chapter, you will learn to graph a quadratic function of the form $f(x) = ax^2 + bx + c$. The problem-solving strategy used is to first graph quadratic functions of the form $f(x) = ax^2$. From there, you progress to other forms of quadratic functions.

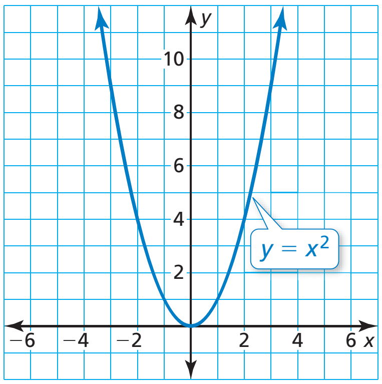
$f(x) = ax^2$	Section 8.1
$f(x) = ax^2 + c$	Section 8.2
$f(x) = ax^2 + bx + c$	Section 8.3
$f(x) = a(x - h)^2 + k$	Section 8.4

EXAMPLE 1 Graphing the Parent Quadratic Function

Graph the parent quadratic function $y = x^2$. Then describe its graph.

SOLUTION

The function is of the form $y = ax^2$, where $a = 1$. By plotting several points, you can see that the graph is U-shaped, as shown.



▶ The graph opens up, and the lowest point is at the origin.

Monitoring Progress

Graph the quadratic function. Then describe its graph.

1. $y = -x^2$
2. $y = 2x^2$
3. $f(x) = 2x^2 + 1$
4. $f(x) = 2x^2 - 1$
5. $f(x) = \frac{1}{2}x^2 + 4x + 3$
6. $f(x) = \frac{1}{2}x^2 - 4x + 3$
7. $y = -2(x + 1)^2 + 1$
8. $y = -2(x - 1)^2 + 1$
9. How are the graphs in Monitoring Progress Questions 1–8 similar? How are they different?

8.1 Graphing $f(x) = ax^2$

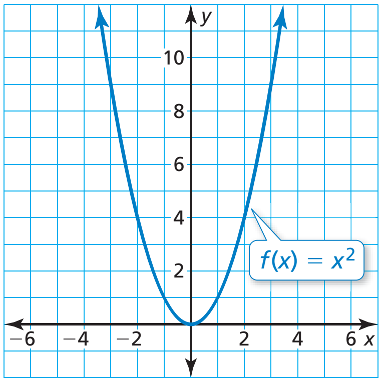
Essential Question What are some of the characteristics of the graph of a quadratic function of the form $f(x) = ax^2$?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.6.A
A.7.A
A.7.C

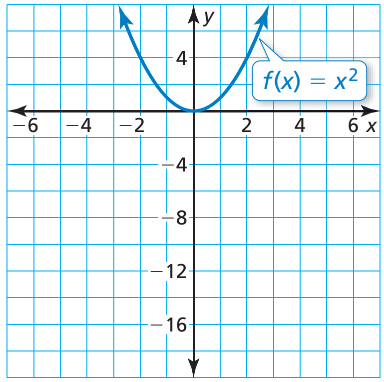
EXPLORATION 1 Graphing Quadratic Functions

Work with a partner. Graph each quadratic function. Compare each graph to the graph of $f(x) = x^2$.

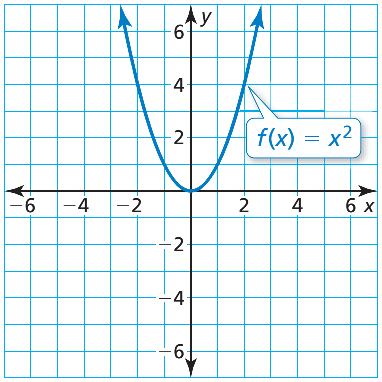
a. $g(x) = 3x^2$



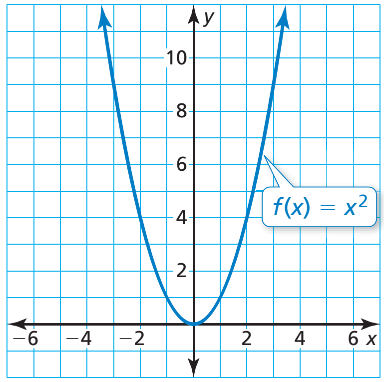
b. $g(x) = -5x^2$



c. $g(x) = -0.2x^2$



d. $g(x) = \frac{1}{10}x^2$



REASONING

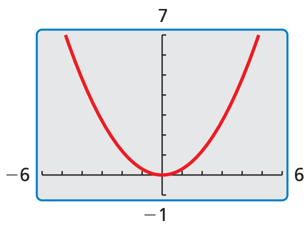
To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

Communicate Your Answer

2. What are some of the characteristics of the graph of a quadratic function of the form $f(x) = ax^2$?

3. How does the value of a affect the graph of $f(x) = ax^2$? Consider $0 < a < 1$, $a > 1$, $-1 < a < 0$, and $a < -1$. Use a graphing calculator to verify your answers.

4. The figure shows the graph of a quadratic function of the form $y = ax^2$. Which of the intervals in Question 3 describes the value of a? Explain your reasoning.



8.1 Lesson

Core Vocabulary

quadratic function, *p. 406*
parabola, *p. 406*
vertex, *p. 406*
axis of symmetry, *p. 406*

Previous
domain
range
vertical shrink
vertical stretch
reflection

What You Will Learn

- Identify characteristics of quadratic functions.
- Graph and use quadratic functions of the form $f(x) = ax^2$.

Identifying Characteristics of Quadratic Functions

A **quadratic function** is a nonlinear function that can be written in the standard form $y = ax^2 + bx + c$, where $a \neq 0$. The U-shaped graph of a quadratic function is called a **parabola**. In this lesson, you will graph quadratic functions, where b and c equal 0.

REMEMBER

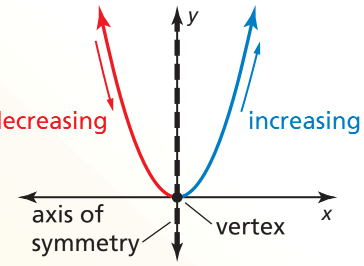
The notation $f(x)$ is another name for y.

Core Concept

Characteristics of Quadratic Functions

The *parent quadratic function* is $f(x) = x^2$. The graphs of all other quadratic functions are *transformations* of the graph of the parent quadratic function.

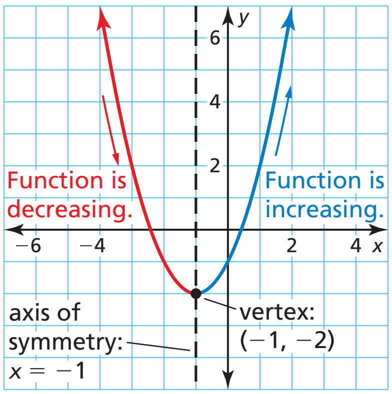
The lowest point on a parabola that opens up or the highest point on a parabola that opens down is the **vertex**. The vertex of the graph of $f(x) = x^2$ is $(0, 0)$.



The vertical line that divides the parabola into two symmetric parts is the **axis of symmetry**. The axis of symmetry passes through the vertex. For the graph of $f(x) = x^2$, the axis of symmetry is the y-axis, or $x = 0$.

EXAMPLE 1 Identifying Characteristics of a Quadratic Function

Consider the graph of the quadratic function.



Using the graph, you can identify characteristics such as the vertex, axis of symmetry, and the behavior of the graph, as shown.

You can also determine the following:

- The domain is all real numbers.
- The range is all real numbers greater than or equal to -2.
- When $x < -1$, y increases as x decreases.
- When $x > -1$, y increases as x increases.

Monitoring Progress

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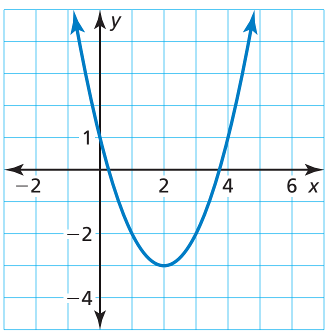
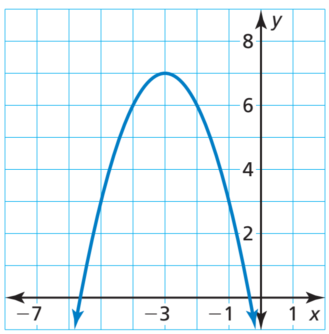
Identify characteristics of the quadratic function and its graph.

1.



2.



Graphing and Using $f(x) = ax^2$

REMEMBER

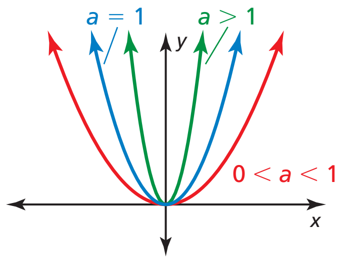
The graph of $y = a \cdot f(x)$ is a vertical stretch or shrink by a factor of a of the graph of $y = f(x)$.

The graph of $y = -f(x)$ is a reflection in the x-axis of the graph of $y = f(x)$.

Core Concept

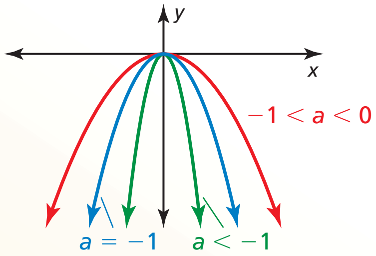
Graphing $f(x) = ax^2$ When $a > 0$

- When $0 < a < 1$, the graph of $f(x) = ax^2$ is a vertical shrink of the graph of $f(x) = x^2$.
- When $a > 1$, the graph of $f(x) = ax^2$ is a vertical stretch of the graph of $f(x) = x^2$.



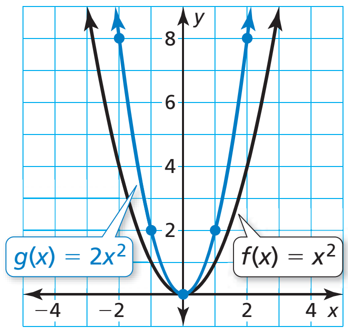
Graphing $f(x) = ax^2$ When $a < 0$

- When $-1 < a < 0$, the graph of $f(x) = ax^2$ is a vertical shrink with a reflection in the x-axis of the graph of $f(x) = x^2$.
- When $a < -1$, the graph of $f(x) = ax^2$ is a vertical stretch with a reflection in the x-axis of the graph of $f(x) = x^2$.



EXAMPLE 2 Graphing $y = ax^2$ When $a > 0$

Graph $g(x) = 2x^2$. Compare the graph to the graph of $f(x) = x^2$.



SOLUTION

Step 1 Make a table of values.

x	-2	-1	0	1	2
$g(x)$	8	2	0	2	8

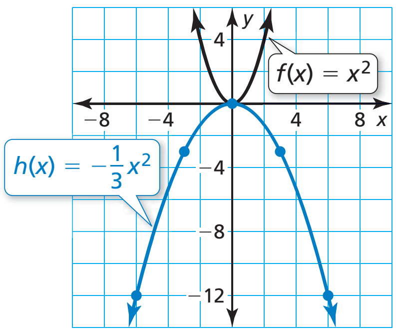
Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

▶ Both graphs open up and have the same vertex, (0, 0), and the same axis of symmetry, $x = 0$. The graph of g is narrower than the graph of f because the graph of g is a vertical stretch by a factor of 2 of the graph of f.

EXAMPLE 3 Graphing $y = ax^2$ When $a < 0$

Graph $h(x) = -\frac{1}{3}x^2$. Compare the graph to the graph of $f(x) = x^2$.



SOLUTION

Step 1 Make a table of values.

x	-6	-3	0	3	6
$h(x)$	-12	-3	0	-3	-12

Step 2 Plot the ordered pairs.

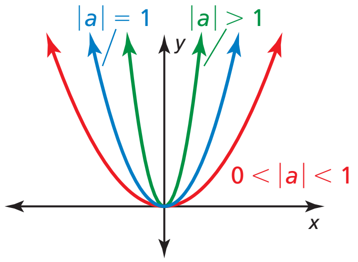
Step 3 Draw a smooth curve through the points.

▶ The graphs have the same vertex, (0, 0), and the same axis of symmetry, $x = 0$, but the graph of h opens down and is wider than the graph of f. So, the graph of h is a vertical shrink by a factor of $\frac{1}{3}$ and a reflection in the x-axis of the graph of f.

Core Concept

Graphing $f(x) = (ax)^2$

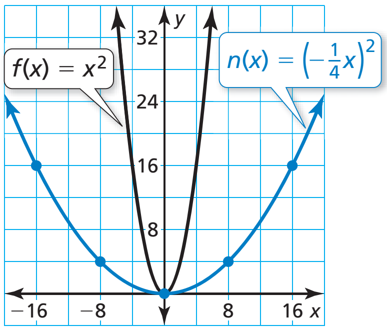
- When $0 < |a| < 1$, the graph of $f(x) = (ax)^2$ is a horizontal stretch of the graph of $f(x) = x^2$.
- When $|a| > 1$, the graph of $f(x) = (ax)^2$ is a horizontal shrink of the graph of $f(x) = x^2$.



EXAMPLE 4 Graphing $y = (ax)^2$

Graph $n(x) = \left(-\frac{1}{4}x\right)^2$. Compare the graph to the graph of $f(x) = x^2$.

SOLUTION



Rewrite n as $n(x) = \left(-\frac{1}{4}x\right)^2 = \frac{1}{16}x^2$.

Step 1 Make a table of values.

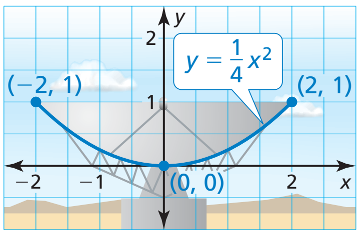
x	-16	-8	0	8	16
$n(x)$	16	4	0	4	16

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

▶ Both graphs open up and have the same vertex, (0, 0), and the same axis of symmetry, $x = 0$. The graph of n is wider than the graph of f because the graph of n is a horizontal stretch by a factor of 4 of the graph of f.

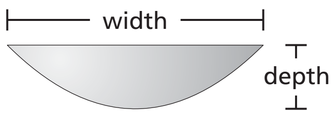
EXAMPLE 5 Solving a Real-Life Problem



The diagram at the left shows the cross section of a satellite dish, where x and y are measured in meters. Find the width and depth of the dish.

SOLUTION

Use the domain of the function to find the width of the dish. Use the range to find the depth.



The leftmost point on the graph is $(-2, 1)$, and the rightmost point is $(2, 1)$. So, the domain is $-2 \le x \le 2$, which represents 4 meters.

The lowest point on the graph is $(0, 0)$, and the highest points on the graph are $(-2, 1)$ and $(2, 1)$. So, the range is $0 \le y \le 1$, which represents 1 meter.

▶ So, the satellite dish is 4 meters wide and 1 meter deep.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

3. $g(x) = 5x^2$ **4.** $h(x) = \frac{1}{3}x^2$ **5.** $p(x) = -3x^2$

6. $q(x) = -0.1x^2$ **7.** $n(x) = (3x)^2$ **8.** $g(x) = \left(-\frac{1}{2}x\right)^2$

9. The cross section of a spotlight can be modeled by the graph of $y = 0.5x^2$, where x and y are measured in inches and $-2 \le x \le 2$. Find the width and depth of the spotlight.

8.1 Exercises

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Vocabulary and Core Concept Check

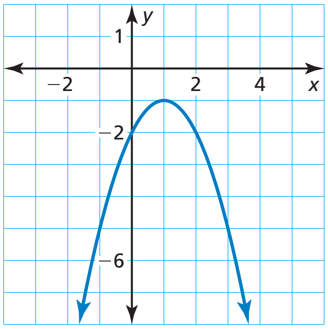
1. **VOCABULARY** What is the U-shaped graph of a quadratic function called?

2. **WRITING** When does the graph of a quadratic function open up? open down?

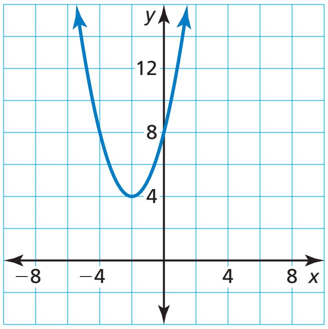
Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, identify characteristics of the quadratic function and its graph. *(See Example 1.)*

3.



4.



In Exercises 5–16, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Examples 2, 3, and 4.)*

5. $g(x) = 6x^2$

6. $b(x) = 2.5x^2$

7. $h(x) = \frac{1}{4}x^2$

8. $j(x) = 0.75x^2$

9. $m(x) = -2x^2$

10. $q(x) = -\frac{9}{2}x^2$

11. $k(x) = -0.2x^2$

12. $p(x) = -\frac{2}{3}x^2$

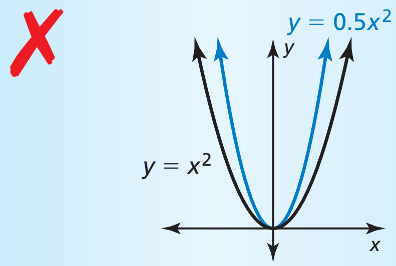
13. $n(x) = (2x)^2$

14. $d(x) = (-4x)^2$

15. $c(x) = \left(-\frac{1}{3}x\right)^2$

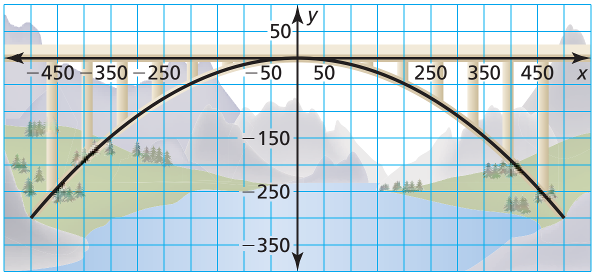
16. $r(x) = (0.1x)^2$

17. **ERROR ANALYSIS** Describe and correct the error in graphing and comparing $y = x^2$ and $y = 0.5x^2$.



The graphs have the same vertex and the same axis of symmetry. The graph of $y = 0.5x^2$ is narrower than the graph of $y = x^2$.

18. **MODELING WITH MATHEMATICS** The arch support of a bridge can be modeled by $y = -0.0012x^2$, where x and y are measured in feet. Find the height and width of the arch. *(See Example 5.)*



19. **PROBLEM SOLVING** The breaking strength z (in pounds) of a manila rope can be modeled by $z = 8900d^2$, where d is the diameter (in inches) of the rope.

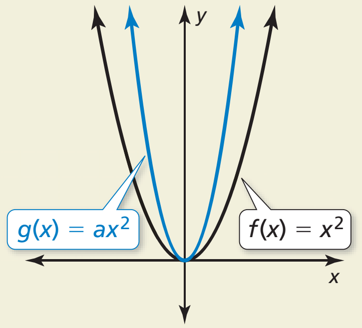
   a. Describe the domain and range of the function.

   b. Graph the function using the domain in part (a).

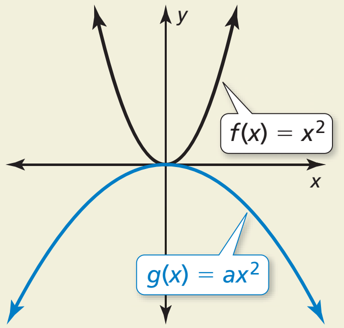
   c. A manila rope has four times the breaking strength of another manila rope. Does the stronger rope have four times the diameter? Explain.

20. HOW DO YOU SEE IT? Describe the possible values of a.

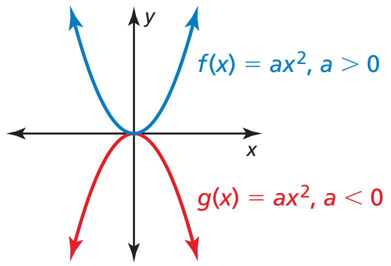
a.



b.



ANALYZING GRAPHS **In Exercises 21–23, use the graph.**



21. When is each function increasing?

22. When is each function decreasing?

23. Which function could include the point $(-2, 3)$? Find the value of a when the graph passes through $(-2, 3)$.

24. REASONING Is the x-intercept of the graph of $y = ax^2$ always 0? Justify your answer.

25. REASONING A parabola opens up and passes through $(-4, 2)$ and $(6, -3)$. How do you know that $(-4, 2)$ is not the vertex?

ABSTRACT REASONING **In Exercises 26–29, determine whether the statement is *always*, *sometimes*, or *never* true. Explain your reasoning.**

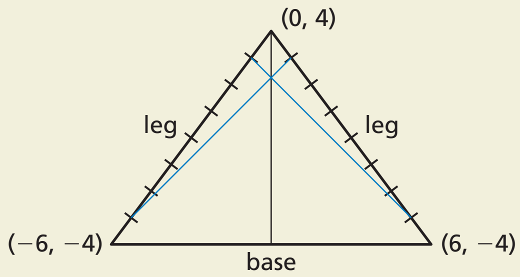
26. The graph of $f(x) = ax^2$ is narrower than the graph of $g(x) = x^2$ when $a > 0$.

27. The graph of $f(x) = ax^2$ is narrower than the graph of $g(x) = x^2$ when $|a| > 1$.

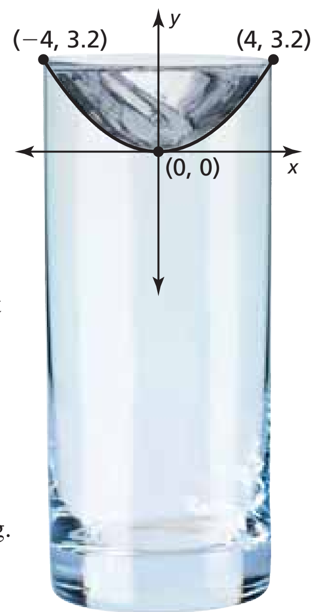
28. The graph of $f(x) = ax^2$ is wider than the graph of $g(x) = x^2$ when $0 < |a| < 1$.

29. The graph of $f(x) = ax^2$ is wider than the graph of $g(x) = dx^2$ when $|a| > |d|$.

30. THOUGHT PROVOKING Draw the isosceles triangle shown. Divide each leg into eight congruent segments. Connect the highest point of one leg with the lowest point of the other leg. Then connect the second highest point of one leg to the second lowest point of the other leg. Continue this process. Write a quadratic equation whose graph models the shape that appears.



31. MAKING AN ARGUMENT The diagram shows the parabolic cross section of a swirling glass of water, where x and y are measured in centimeters.



a. About how wide is the mouth of the glass?

b. Your friend claims that the rotational speed of the water would have to increase for the cross section to be modeled by $y = 0.1x^2$. Is your friend correct? Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the expression when $n = 3$ and $x = -2$. *(Skills Review Handbook)*

32. $n^2 + 5$ **33.** $3x^2 - 9$ **34.** $-4n^2 + 11$ **35.** $n + 2x^2$

8.2 Graphing $f(x) = ax^2 + c$

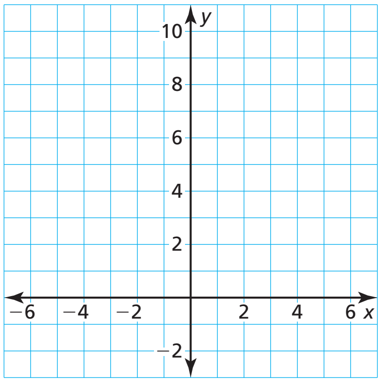
Essential Question How does the value of c affect the graph of $f(x) = ax^2 + c$?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.7.A
A.7.C

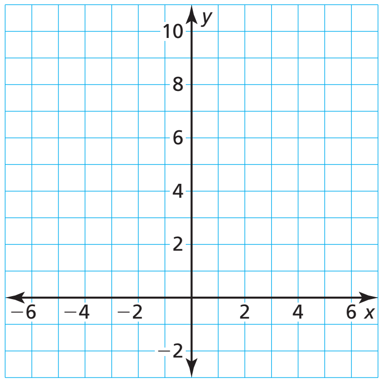
EXPLORATION 1 Graphing $y = ax^2 + c$

Work with a partner. Sketch the graphs of the functions in the same coordinate plane. What do you notice?

a. $f(x) = x^2$ and $g(x) = x^2 + 2$



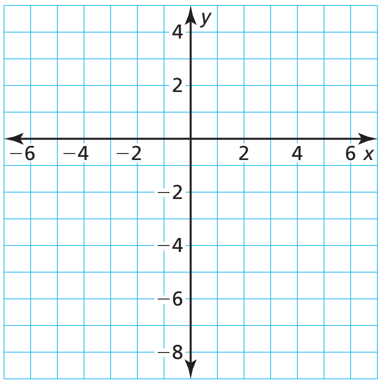
b. $f(x) = 2x^2$ and $g(x) = 2x^2 - 2$



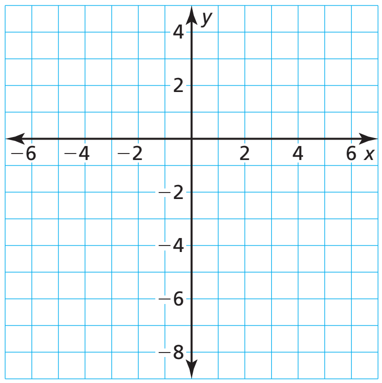
EXPLORATION 2 Finding x-Intercepts of Graphs

Work with a partner. Graph each function. Find the x-intercepts of the graph. Explain how you found the x-intercepts.

a. $y = x^2 - 7$



b. $y = -x^2 + 1$



SELECTING TOOLS

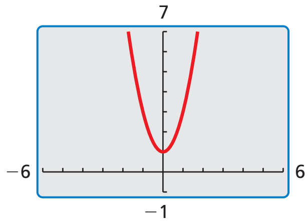
To be proficient in math, you need to consider the available tools, such as a graphing calculator, when solving a mathematical problem.

Communicate Your Answer

3. How does the value of c affect the graph of $f(x) = ax^2 + c$?

4. Use a graphing calculator to verify your answers to Question 3.

5. The figure shows the graph of a quadratic function of the form $y = ax^2 + c$. Describe possible values of a and c. Explain your reasoning.



8.2 Lesson

Core Vocabulary

Previous
translation
vertex of a parabola
axis of symmetry
vertical stretch
vertical shrink
zero of a function

What You Will Learn

- Graph quadratic functions of the form $f(x) = ax^2 + c$.
- Solve real-life problems involving functions of the form $f(x) = ax^2 + c$.

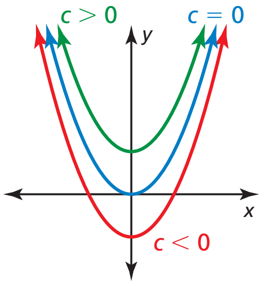
Graphing $f(x) = ax^2 + c$

Core Concept

Graphing $f(x) = ax^2 + c$

- When $c > 0$, the graph of $f(x) = ax^2 + c$ is a vertical translation c units up of the graph of $f(x) = ax^2$.
- When $c < 0$, the graph of $f(x) = ax^2 + c$ is a vertical translation $|c|$ units down of the graph of $f(x) = ax^2$.

The vertex of the graph of $f(x) = ax^2 + c$ is $(0, c)$, and the axis of symmetry is $x = 0$.



EXAMPLE 1 Graphing $y = x^2 + c$

Graph $g(x) = x^2 - 2$. Compare the graph to the graph of $f(x) = x^2$.

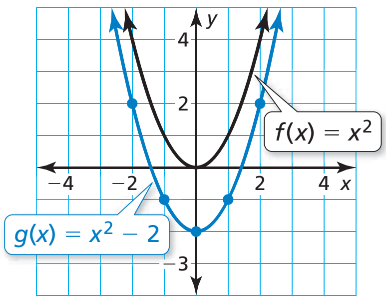
SOLUTION

Step 1 Make a table of values.

x	-2	-1	0	1	2
$g(x)$	2	-1	-2	-1	2

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



REMEMBER

The graph of $y = f(x) + k$ is a vertical translation, and the graph of $y = f(x - h)$ is a horizontal translation of the graph of f.

- Both graphs open up and have the same axis of symmetry, $x = 0$. The vertex of the graph of g, $(0, -2)$, is below the vertex of the graph of f, $(0, 0)$, because the graph of g is a vertical translation 2 units down of the graph of f.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

1. $g(x) = x^2 - 5$

2. $h(x) = x^2 + 3$

EXAMPLE 2 Graphing $y = ax^2 + c$

Graph $g(x) = 4x^2 + 1$. Compare the graph to the graph of $f(x) = x^2$.

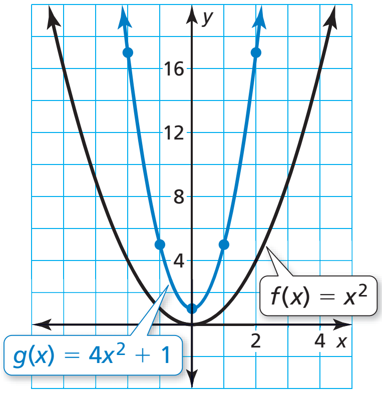
SOLUTION

Step 1 Make a table of values.

x	-2	-1	0	1	2
$g(x)$	17	5	1	5	17

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



Both graphs open up and have the same axis of symmetry, $x = 0$. The graph of g is narrower, and its vertex, $(0, 1)$, is above the vertex of the graph of f, $(0, 0)$. So, the graph of g is a vertical stretch by a factor of 4 and a vertical translation 1 unit up of the graph of f.

EXAMPLE 3 Translating the Graph of $y = ax^2 + c$

Let $f(x) = -0.5x^2 + 2$ and $g(x) = f(x) - 7$.

a. Describe the transformation from the graph of f to the graph of g. Then graph f and g in the same coordinate plane.

b. Write an equation that represents g in terms of x.

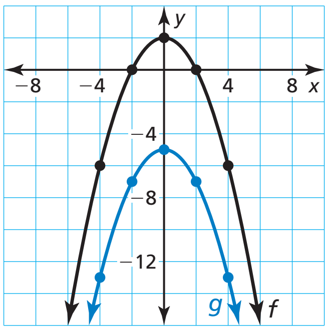
SOLUTION

a. The function g is of the form $y = f(x) + k$, where $k = -7$. So, the graph of g is a vertical translation 7 units down of the graph of f.

x	-4	-2	0	2	4
$f(x)$	-6	0	2	0	-6
$g(x)$	-13	-7	-5	-7	-13

$-0.5x^2 + 2$

$f(x) - 7$



b. $g(x) = f(x) - 7$ — Write the function g.

$= -0.5x^2 + 2 - 7$ — Substitute for $f(x)$.

$= -0.5x^2 - 5$ — Subtract.

So, the equation $g(x) = -0.5x^2 - 5$ represents g in terms of x.

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

3. $g(x) = 2x^2 - 5$

4. $h(x) = -\frac{1}{4}x^2 + 4$

5. Let $f(x) = 3x^2 - 1$ and $g(x) = f(x) + 3$.

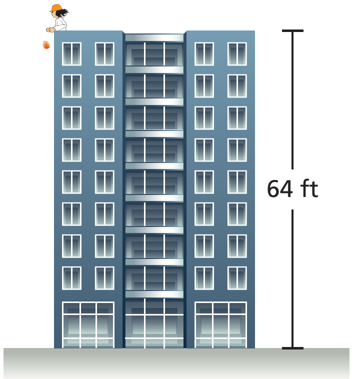
a. Describe the transformation from the graph of f to the graph of g. Then graph f and g in the same coordinate plane.

b. Write an equation that represents g in terms of x.

Solving Real-Life Problems

Recall that a zero of a function f is an x-value for which $f(x) = 0$. A zero of a function is an x-intercept of the graph of the function.

EXAMPLE 4 Solving a Real-Life Problem



The function $f(t) = -16t^2 + s_0$ represents the approximate height (in feet) of a falling object t seconds after it is dropped from an initial height s_0 (in feet). An egg is dropped from a height of 64 feet.

a. After how many seconds does the egg hit the ground?

b. Suppose the initial height is adjusted by k feet. How will this affect part (a)?

SOLUTION

1. **Understand the Problem** You know the function that models the height of a falling object and the initial height of an egg. You are asked to find how many seconds it takes the egg to hit the ground when dropped from the initial height. Then you need to describe how a change in the initial height affects how long it takes the egg to hit the ground.

2. **Make a Plan** Use the initial height to write a function that models the height of the egg. Use a table to graph the function. Find the zero(s) of the function to answer the question. Then explain how vertical translations of the graph affect the zero(s) of the function.

COMMON ERROR

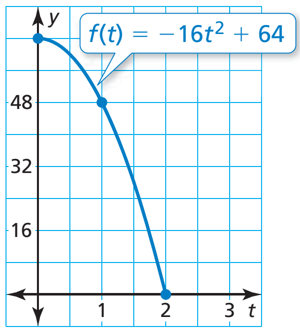
The graph in Step 1 shows the height of the object over time, not the path of the object.

3. **Solve the Problem**

a. The initial height is 64 feet. So, the function $f(t) = -16t^2 + 64$ represents the height of the egg t seconds after it is dropped. The egg hits the ground when $f(t) = 0$.

Step 1 Make a table of values and sketch the graph.

t	0	1	2
$f(t)$	64	48	0



Step 2 Find the positive zero of the function. When $t = 2$, $f(t) = 0$. So, the zero is 2.

▶ The egg hits the ground 2 seconds after it is dropped.

b. When the initial height is adjusted by k feet, the graph of f is translated up k units when $k > 0$ or down $|k|$ units when $k < 0$. So, the x-intercept of the graph of f will move right when $k > 0$ or left when $k < 0$.

▶ When $k > 0$, the egg will take more than 2 seconds to hit the ground. When $k < 0$, the egg will take less than 2 seconds to hit the ground.

4. **Look Back** To check that the egg hits the ground 2 seconds after it is dropped, you can solve $0 = -16t^2 + 64$ by factoring.

Monitoring Progress

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6. Explain why only nonnegative values of t are used in Example 4.

7. **WHAT IF?** The egg is dropped from a height of 100 feet. After how many seconds does the egg hit the ground?

8.2 Exercises

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Vocabulary and Core Concept Check

1. VOCABULARY State the vertex and axis of symmetry of the graph of $y = ax^2 + c$.

2. WRITING How does the graph of $y = ax^2 + c$ compare to the graph of $y = ax^2$?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Example 1.)*

3. $g(x) = x^2 + 6$

4. $h(x) = x^2 + 8$

5. $p(x) = x^2 - 3$

6. $q(x) = x^2 - 1$

In Exercises 7–12, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Example 2.)*

7. $g(x) = -x^2 + 3$

8. $h(x) = -x^2 - 7$

9. $s(x) = 2x^2 - 4$

10. $t(x) = -3x^2 + 1$

11. $p(x) = -\frac{1}{3}x^2 - 2$

12. $q(x) = \frac{1}{2}x^2 + 6$

In Exercises 13–16, describe the transformation from the graph of f to the graph of g. Then graph f and g in the same coordinate plane. Write an equation that represents g in terms of x. *(See Example 3.)*

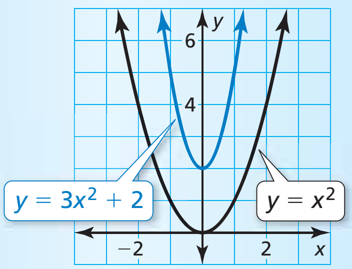
13. $f(x) = 3x^2 + 4$
$g(x) = f(x) + 2$

14. $f(x) = \frac{1}{2}x^2 + 1$
$g(x) = f(x) - 4$

15. $f(x) = -\frac{1}{4}x^2 - 6$
$g(x) = f(x) - 3$

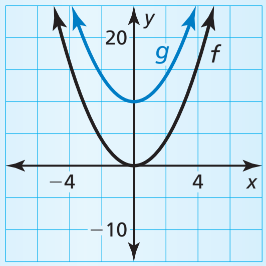
16. $f(x) = 4x^2 - 5$
$g(x) = f(x) + 7$

17. ERROR ANALYSIS Describe and correct the error in comparing the graphs.



The graph of $y = 3x^2 + 2$ is a vertical shrink by a factor of 3 and a translation 2 units up of the graph of $y = x^2$.

18. ERROR ANALYSIS Describe and correct the error in graphing and comparing $f(x) = x^2$ and $g(x) = x^2 - 10$.



Both graphs open up and have the same axis of symmetry. However, the vertex of the graph of g, $(0, 10)$, is 10 units above the vertex of the graph of f, $(0, 0)$.

In Exercises 19–26, find the zeros of the function.

19. $y = x^2 - 1$

20. $y = x^2 - 36$

21. $f(x) = -x^2 + 25$

22. $f(x) = -x^2 + 49$

23. $f(x) = 4x^2 - 16$

24. $f(x) = 3x^2 - 27$

25. $f(x) = -12x^2 + 3$

26. $f(x) = -8x^2 + 98$

27. MODELING WITH MATHEMATICS A water balloon is dropped from a height of 144 feet. *(See Example 4.)*

a. After how many seconds does the water balloon hit the ground?

b. Suppose the initial height is adjusted by k feet. How does this affect part (a)?

28. MODELING WITH MATHEMATICS The function $y = -16x^2 + 36$ represents the height y (in feet) of an apple x seconds after falling from a tree. Find and interpret the x- and y-intercepts.

In Exercises 29–32, sketch a parabola with the given characteristics.

29. The parabola opens up, and the vertex is (0, 3).

30. The vertex is (0, 4), and one of the x-intercepts is 2.

31. The related function is increasing when $x < 0$, and the zeros are -1 and 1.

32. The highest point on the parabola is (0, -5).

33. DRAWING CONCLUSIONS You and your friend both drop a ball at the same time. The function $h(x) = -16x^2 + 256$ represents the height (in feet) of your ball after x seconds. The function $g(x) = -16x^2 + 300$ represents the height (in feet) of your friend's ball after x seconds.

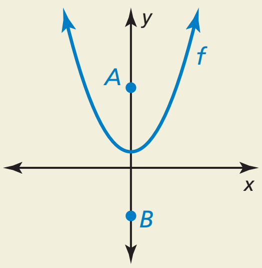
a. Write the function $T(x) = h(x) - g(x)$. What does $T(x)$ represent?

b. When your ball hits the ground, what is the height of your friend's ball? Use a graph to justify your answer.

34. MAKING AN ARGUMENT Your friend claims that in the equation $y = ax^2 + c$, the vertex changes when the value of a changes. Is your friend correct? Explain your reasoning.

35. MATHEMATICAL CONNECTIONS The area A (in square feet) of a square patio is represented by $A = x^2$, where x is the length of one side of the patio. You add 48 square feet to the patio, resulting in a total area of 192 square feet. What are the dimensions of the original patio? Use a graph to justify your answer.

36. HOW DO YOU SEE IT? The graph of $f(x) = ax^2 + c$ is shown. Points A and B are the same distance from the vertex of the graph of f. Which point is closer to the vertex of the graph of f as c increases?



37. REASONING Describe two algebraic methods you can use to find the zeros of the function $f(t) = -16t^2 + 400$. Check your answer by graphing.

38. PROBLEM SOLVING The paths of water from three different garden waterfalls are given below. Each function gives the height h (in feet) and the horizontal distance d (in feet) of the water.

Waterfall 1 $h = -3.1d^2 + 4.8$

Waterfall 2 $h = -3.5d^2 + 1.9$

Waterfall 3 $h = -1.1d^2 + 1.6$

a. Which waterfall drops water from the highest point?

b. Which waterfall follows the narrowest path?

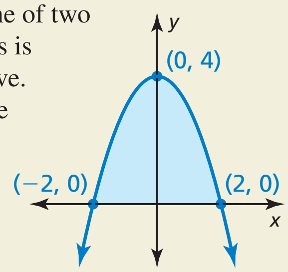
c. Which waterfall sends water the farthest?

39. WRITING EQUATIONS Two acorns fall to the ground from an oak tree. One falls 45 feet, while the other falls 32 feet.

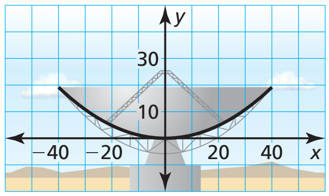
a. For each acorn, write an equation that represents the height h (in feet) as a function of the time t (in seconds).

b. Describe how the graphs of the two equations are related.

40. THOUGHT PROVOKING One of two classic problems in calculus is to find the area under a curve. Approximate the area of the region bounded by the parabola and the x-axis. Show your work.



41. CRITICAL THINKING A cross section of the parabolic surface of the antenna shown can be modeled by $y = 0.012x^2$, where x and y are measured in feet. The antenna is moved up so that the outer edges of the dish are 25 feet above the ground. Where is the vertex of the cross section located? Explain.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the expression when $a = 4$ and $b = -3$. *(Skills Review Handbook)*

42. $\dfrac{a}{4b}$

43. $-\dfrac{b}{2a}$

44. $\dfrac{a - b}{3a + b}$

45. $-\dfrac{b + 2a}{ab}$

8.3 Graphing $f(x) = ax^2 + bx + c$

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.6.A
A.7.A

Essential Question

How can you find the vertex of the graph of $f(x) = ax^2 + bx + c$?

EXPLORATION 1 Comparing x-Intercepts with the Vertex

Work with a partner.

a. Sketch the graphs of $y = 2x^2 - 8x$ and $y = 2x^2 - 8x + 6$.

b. What do you notice about the x-coordinate of the vertex of each graph?

c. Use the graph of $y = 2x^2 - 8x$ to find its x-intercepts. Verify your answer by solving $0 = 2x^2 - 8x$.

d. Compare the value of the x-coordinate of the vertex with the values of the x-intercepts.

EXPLORATION 2 Finding x-Intercepts

Work with a partner.

a. Solve $0 = ax^2 + bx$ for x by factoring.

b. What are the x-intercepts of the graph of $y = ax^2 + bx$?

c. Copy and complete the table to verify your answer.

x	$y = ax^2 + bx$
0	
$-\frac{b}{a}$	

MAKING MATHEMATICAL ARGUMENTS

To be proficient in math, you need to make conjectures and build a logical progression of statements.

EXPLORATION 3 Deductive Reasoning

Work with a partner. Complete the following logical argument.

The x-intercepts of the graph of $y = ax^2 + bx$ are 0 and $-\frac{b}{a}$.

The vertex of the graph of $y = ax^2 + bx$ occurs when $x =$ ______.

The vertices of the graphs of $y = ax^2 + bx$ and $y = ax^2 + bx + c$ have the same x-coordinate.

The vertex of the graph of $y = ax^2 + bx + c$ occurs when $x =$ ______.

Communicate Your Answer

4. How can you find the vertex of the graph of $f(x) = ax^2 + bx + c$?

5. Without graphing, find the vertex of the graph of $f(x) = x^2 - 4x + 3$. Check your result by graphing.

8.3 Lesson

Core Vocabulary

maximum value, *p. 419*
minimum value, *p. 419*

Previous
independent variable
dependent variable

What You Will Learn

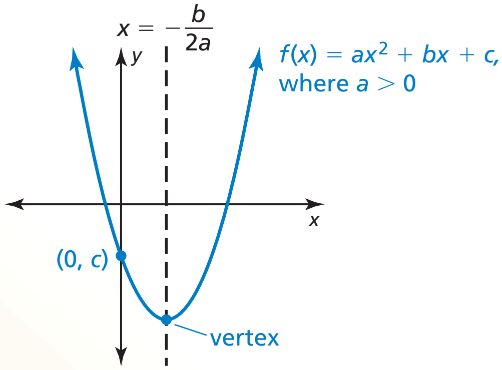
- Graph quadratic functions of the form $f(x) = ax^2 + bx + c$.
- Find maximum and minimum values of quadratic functions.

Graphing $f(x) = ax^2 + bx + c$

Core Concept

Graphing $f(x) = ax^2 + bx + c$

- The graph opens up when $a > 0$, and the graph opens down when $a < 0$.
- The y-intercept is c.
- The x-coordinate of the vertex is $-\frac{b}{2a}$.
- The axis of symmetry is $x = -\frac{b}{2a}$.



EXAMPLE 1 **Finding the Axis of Symmetry and the Vertex**

Find (a) the axis of symmetry and (b) the vertex of the graph of $f(x) = 2x^2 + 8x - 1$.

SOLUTION

a. Find the axis of symmetry when $a = 2$ and $b = 8$.

$x = -\frac{b}{2a}$ — Write the equation for the axis of symmetry.

$x = -\frac{8}{2(2)}$ — Substitute 2 for *a* and 8 for *b*.

$x = -2$ — Simplify.

▶ The axis of symmetry is $x = -2$.

b. The axis of symmetry is $x = -2$, so the x-coordinate of the vertex is -2. Use the function to find the y-coordinate of the vertex.

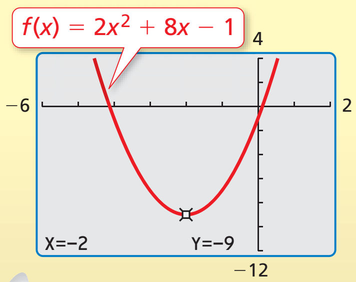
$f(x) = 2x^2 + 8x - 1$ — Write the function.

$f(-2) = 2(-2)^2 + 8(-2) - 1$ — Substitute -2 for *x*.

$= -9$ — Simplify.

▶ The vertex is $(-2, -9)$.

Check



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Find (a) the axis of symmetry and (b) the vertex of the graph of the function.

1. $f(x) = 3x^2 - 2x$ **2.** $g(x) = x^2 + 6x + 5$ **3.** $h(x) = -\frac{1}{2}x^2 + 7x - 4$

EXAMPLE 2 Graphing $f(x) = ax^2 + bx + c$

Graph $f(x) = 3x^2 - 6x + 5$. Describe the domain and range.

COMMON ERROR

Be sure to include the negative sign before the fraction when finding the axis of symmetry.

SOLUTION

Step 1 Find and graph the axis of symmetry.

$$x = -\frac{b}{2a} = -\frac{(-6)}{2(3)} = 1 \qquad \text{Substitute and simplify.}$$

Step 2 Find and plot the vertex.

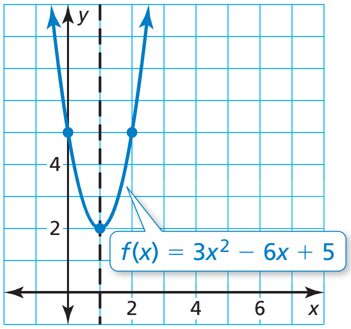
The axis of symmetry is $x = 1$, so the x-coordinate of the vertex is 1. Use the function to find the y-coordinate of the vertex.

$f(x) = 3x^2 - 6x + 5$ — Write the function.

$f(1) = 3(1)^2 - 6(1) + 5$ — Substitute 1 for x.

$= 2$ — Simplify.

So, the vertex is $(1, 2)$.



Step 3 Use the y-intercept to find two more points on the graph.

Because $c = 5$, the y-intercept is 5. So, $(0, 5)$ lies on the graph. Because the axis of symmetry is $x = 1$, the point $(2, 5)$ also lies on the graph.

Step 4 Draw a smooth curve through the points.

▶ The domain is all real numbers. The range is $y \geq 2$.

REMEMBER

The domain is the set of all possible input values of the independent variable x. The range is the set of all possible output values of the dependent variable y.

Monitoring Progress

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Graph the function. Describe the domain and range.

4. $h(x) = 2x^2 + 4x + 1$ **5.** $k(x) = x^2 - 8x + 7$ **6.** $p(x) = -5x^2 - 10x - 2$

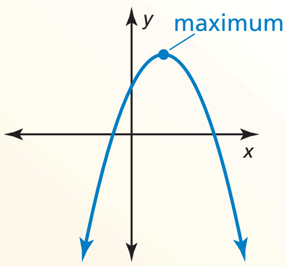
Finding Maximum and Minimum Values

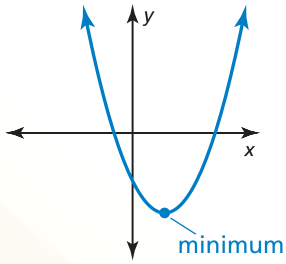
Core Concept

Maximum and Minimum Values

The y-coordinate of the vertex of the graph of $f(x) = ax^2 + bx + c$ is the **maximum value** of the function when $a < 0$ or the **minimum value** of the function when $a > 0$.

$f(x) = ax^2 + bx + c, a < 0$ $\qquad$ $f(x) = ax^2 + bx + c, a > 0$





EXAMPLE 3 Finding a Maximum or Minimum Value

Tell whether the function $f(x) = -4x^2 - 24x - 19$ has a minimum value or a maximum value. Then find the value.

SOLUTION

For $f(x) = -4x^2 - 24x - 19$, $a = -4$ and $-4 < 0$. So, the parabola opens down and the function has a maximum value. To find the maximum value, find the y-coordinate of the vertex.

First, find the x-coordinate of the vertex. Use $a = -4$ and $b = -24$.

$$x = -\frac{b}{2a} = -\frac{(-24)}{2(-4)} = -3 \qquad \text{Substitute and simplify.}$$

Then evaluate the function when $x = -3$ to find the y-coordinate of the vertex.

$$f(-3) = -4(-3)^2 - 24(-3) - 19 \qquad \text{Substitute } -3 \text{ for } x.$$

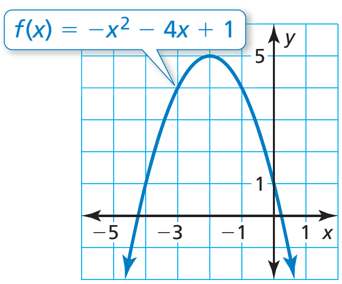
$$= 17 \qquad \text{Simplify.}$$

▶ The maximum value is 17.

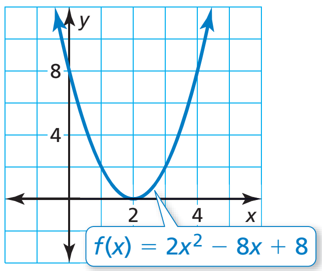
EXAMPLE 4 Finding a Maximum or Minimum Value

Estimate the y-intercept of the graph and the maximum or minimum value of the function represented by the graph.

a.



b.



Check

y-intercept:

$f(0) = 2(0)^2 - 8(0) + 8$

$= 8$ ✓

vertex at: $x = -\dfrac{(-8)}{2(2)} = 2$

minimum value:

$f(2) = 2(2)^2 - 8(2) + 8$

$= 0$ ✓

SOLUTION

a. From the graph, you can estimate that the y-intercept is 1 and the function has a maximum value of 5.

b. From the graph, you can estimate that the y-intercept is 8 and the function has a minimum value of 0. You can check your estimates as shown.

Monitoring Progress

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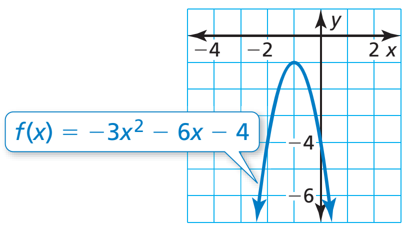
Tell whether the function has a minimum value or a maximum value. Then find the value.

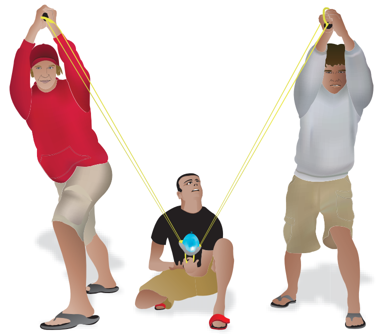
7. $g(x) = 8x^2 - 8x + 6$

8. $h(x) = -\frac{1}{4}x^2 + 3x + 1$

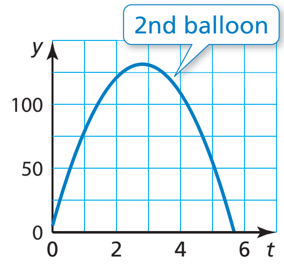
9. Estimate the y-intercept of the graph and the maximum or minimum value of the function represented by the graph.



EXAMPLE 5 Modeling with Mathematics



A group of friends is launching water balloons. The function $f(t) = -16t^2 + 80t + 5$ represents the height (in feet) of the first water balloon t seconds after it is launched. The height of the second water balloon t seconds after it is launched is shown in the graph. Which water balloon went higher?



APPLYING MATHEMATICS

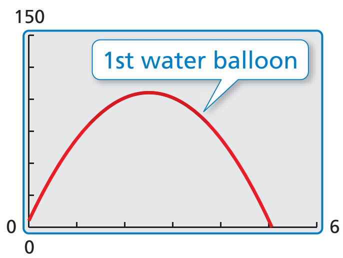
Because time cannot be negative, use only nonnegative values of t.

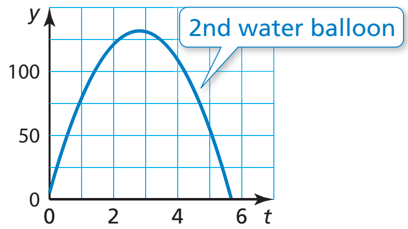
SOLUTION

1. **Understand the Problem** You are given a function that represents the height of the first water balloon. The height of the second water balloon is represented graphically. You need to find and compare the maximum heights of the water balloons.

2. **Make a Plan** To compare the maximum heights, represent both functions graphically. Use a graphing calculator to graph $f(t) = -16t^2 + 80t + 5$ in an appropriate viewing window. Then visually compare the heights of the water balloons.

3. **Solve the Problem** Enter the function $f(t) = -16t^2 + 80t + 5$ into your calculator and graph it. Compare the graphs to determine which function has a greater maximum value.

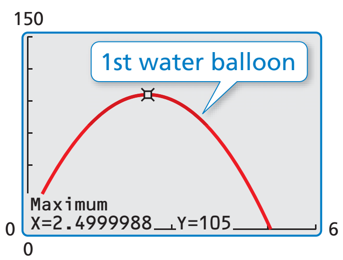


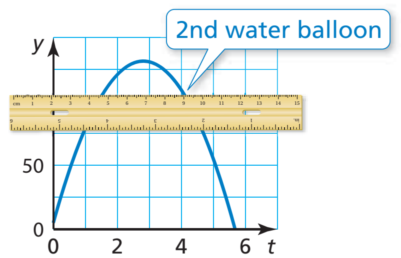


You can see that the second water balloon reaches a height of about 125 feet, while the first water balloon reaches a height of only about 100 feet.

▶ So, the second water balloon went higher.

4. **Look Back** Use the *maximum* feature to determine that the maximum value of $f(t) = -16t^2 + 80t + 5$ is 105. Use a straightedge to represent a height of 105 feet on the graph that represents the second water balloon to clearly see that the second water balloon went higher.





Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

10. Which balloon is in the air longer? Explain your reasoning.

11. Which balloon reaches its maximum height faster? Explain your reasoning.

8.3 Exercises

Tutorial Help in English and Spanish at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** Explain how you can tell whether a quadratic function has a maximum value or a minimum value without graphing the function.

2. **DIFFERENT WORDS, SAME QUESTION** Consider the quadratic function $f(x) = -2x^2 + 8x + 24$. Which is different? Find "both" answers.

   What is the maximum value of the function?

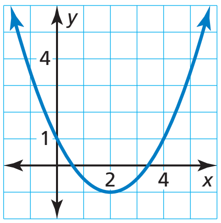
   What is the greatest number in the range of the function?

   What is the y-coordinate of the vertex of the graph of the function?

   What is the axis of symmetry of the graph of the function?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the vertex, the axis of symmetry, and the y-intercept of the graph.

3.


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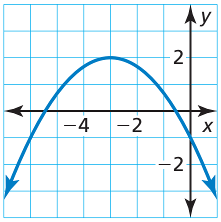

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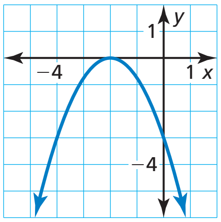

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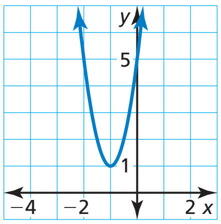


In Exercises 7–12, find (a) the axis of symmetry and (b) the vertex of the graph of the function. *(See Example 1.)*

7. $f(x) = 2x^2 - 4x$
8. $y = 3x^2 + 2x$
9. $y = -9x^2 - 18x - 1$
10. $f(x) = -6x^2 + 24x - 20$
11. $f(x) = \frac{2}{5}x^2 - 4x + 14$
12. $y = -\frac{3}{4}x^2 + 9x - 18$

In Exercises 13–18, graph the function. Describe the domain and range. *(See Example 2.)*

13. $f(x) = 2x^2 + 12x + 4$
14. $y = 4x^2 + 24x + 13$
15. $y = -8x^2 - 16x - 9$
16. $f(x) = -5x^2 + 20x - 7$
17. $y = \frac{2}{3}x^2 - 6x + 5$
18. $f(x) = -\frac{1}{2}x^2 - 3x - 4$

19. **ERROR ANALYSIS** Describe and correct the error in finding the axis of symmetry of the graph of $y = 3x^2 - 12x + 11$.

    ✗ $x = -\frac{b}{2a} = \frac{-12}{2(3)} = -2$

    The axis of symmetry is $x = -2$.

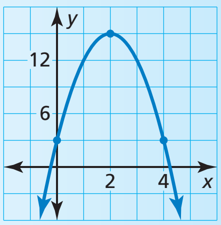
20. **ERROR ANALYSIS** Describe and correct the error in graphing the function $f(x) = x^2 + 4x + 3$.

    ✗ The axis of symmetry is $x = \frac{b}{2a} = \frac{4}{2(1)} = 2$.

    $f(2) = 2^2 + 4(2) + 3 = 15$

    So, the vertex is $(2, 15)$.

    The y-intercept is 3. So, the points $(0, 3)$ and $(4, 3)$ lie on the graph.



In Exercises 21–26, tell whether the function has a minimum value or a maximum value. Then find the value. *(See Example 3.)*

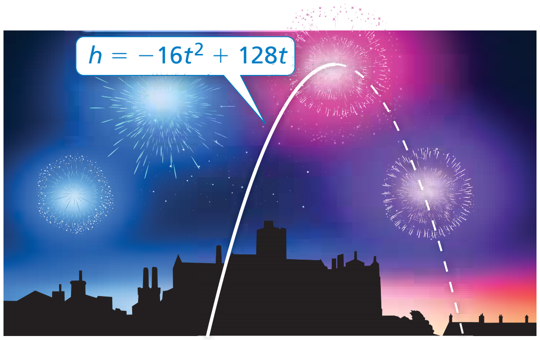
21. $y = 3x^2 - 18x + 15$
22. $f(x) = -5x^2 + 10x + 7$
23. $f(x) = -4x^2 + 4x - 2$

24. $y = 2x^2 - 10x + 13$

25. $y = -\frac{1}{2}x^2 - 11x + 6$

26. $f(x) = \frac{1}{5}x^2 - 5x + 27$

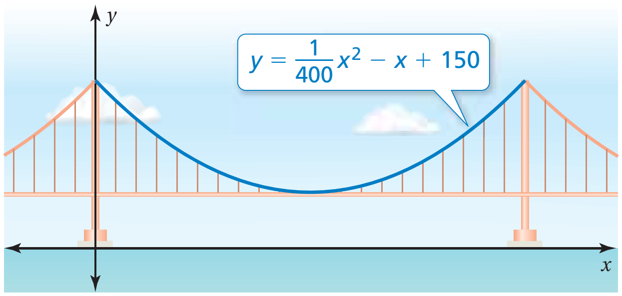
27. **MODELING WITH MATHEMATICS** The function shown represents the height h (in feet) of a firework t seconds after it is launched. The firework explodes at its highest point.



a. When does the firework explode?

b. At what height does the firework explode?

28. **MODELING WITH MATHEMATICS** The cable between two towers of a suspension bridge can be modeled by the function shown, where x and y are measured in feet. The cable is at road level midway between the towers.



a. How far from each tower shown is the lowest point of the cable?

b. How high is the road above the water?

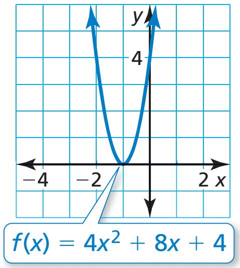
c. Describe the domain and range of the function shown.

29. **ATTENDING TO PRECISION** The vertex of a parabola is $(3, -1)$. One point on the parabola is $(6, 8)$. Find another point on the parabola. Justify your answer.

30. **MAKING AN ARGUMENT** Your friend claims that it is possible to draw a parabola through any two points with different x-coordinates. Is your friend correct? Explain.

In Exercises 31–34, estimate the y-intercept of the graph and the minimum or maximum value of the function represented by the graph. *(See Example 4.)*

31.



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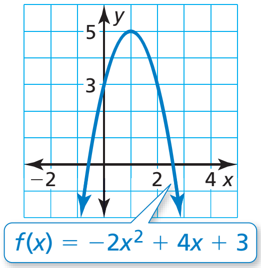

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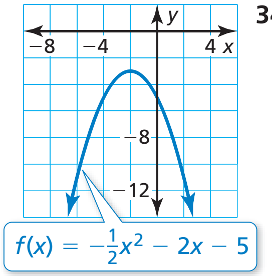

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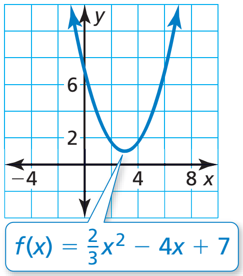
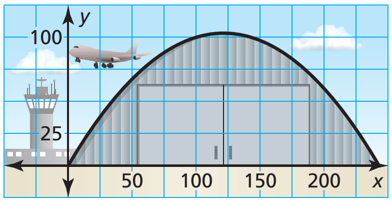


USING TOOLS **In Exercises 35 and 36, use the *minimum* or *maximum* feature of a graphing calculator to approximate the vertex of the graph of the function.**

35. $y = 0.5x^2 + \sqrt{2}x - 3$

36. $y = -\pi x^2 + 3x$

37. **MODELING WITH MATHEMATICS** The opening of one aircraft hangar is a parabolic arch that can be modeled by the equation $y = -0.006x^2 + 1.5x$, where x and y are measured in feet. The opening of a second aircraft hangar is shown in the graph. *(See Example 5.)*



a. Which aircraft hangar is taller?

b. Which aircraft hangar is wider?

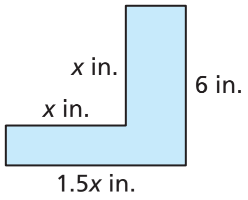
38. **MODELING WITH MATHEMATICS** An office supply store sells about 80 graphing calculators per month for \$120 each. For each \$6 decrease in price, the store expects to sell eight more calculators. The revenue from calculator sales is given by the function $R(n) = (\text{unit price})(\text{units sold})$, or $R(n) = (120 - 6n)(80 + 8n)$, where n is the number of \$6 price decreases.

a. How much should the store charge to maximize monthly revenue?

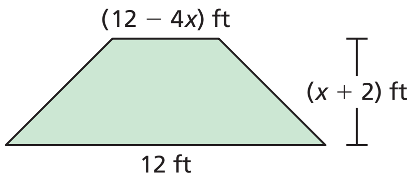
b. Using a different revenue model, the store expects to sell five more calculators for each \$4 decrease in price. Which revenue model results in a greater maximum monthly revenue? Explain.

MATHEMATICAL CONNECTIONS **In Exercises 39 and 40, (a) find the value of x that maximizes the area of the figure and (b) find the maximum area.**

39.

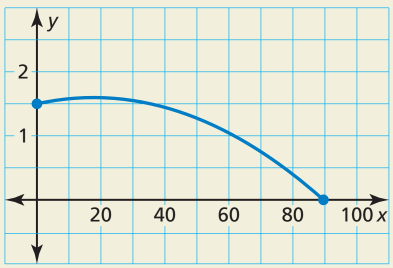


40.



41. **WRITING** Compare the graph of $g(x) = x^2 + 4x + 1$ with the graph of $h(x) = x^2 - 4x + 1$.

42. **HOW DO YOU SEE IT?** During an archery competition, an archer shoots an arrow. The arrow follows the parabolic path shown, where x and y are measured in meters.



a. What is the initial height of the arrow?

b. Estimate the maximum height of the arrow.

c. How far does the arrow travel?

43. **USING TOOLS** The graph of a quadratic function passes through (3, 2), (4, 7), and (9, 2). Does the graph open up or down? Explain your reasoning.

44. **REASONING** For a quadratic function f, what does $f\left(-\frac{b}{2a}\right)$ represent? Explain your reasoning.

45. **PROBLEM SOLVING** Write a function of the form $y = ax^2 + bx$ whose graph contains the points (1, 6) and (3, 6).

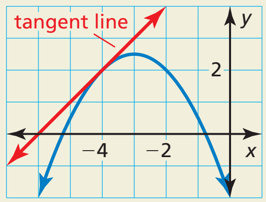
46. **CRITICAL THINKING** Parabolas A and B contain the points shown. Identify characteristics of each parabola, if possible. Explain your reasoning.

Parabola A	
x	y
2	3
6	4

Parabola B	
x	y
1	4
3	−4
5	4

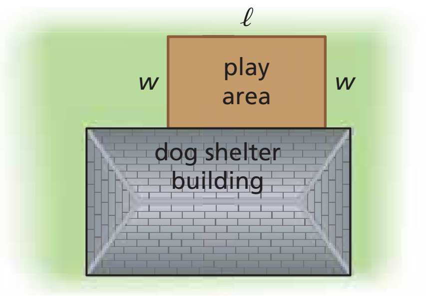
47. **MODELING WITH MATHEMATICS** At a basketball game, an air cannon launches T-shirts into the crowd. The function $y = -\frac{1}{8}x^2 + 4x$ represents the path of a T-shirt. The function $3y = 2x - 14$ represents the height of the bleachers. In both functions, y represents vertical height (in feet) and x represents horizontal distance (in feet). At what height does the T-shirt land in the bleachers?

48. **THOUGHT PROVOKING** One of two classic problems in calculus is finding the slope of a *tangent line* to a curve. An example of a tangent line, which just touches the parabola at one point, is shown.



Approximate the slope of the tangent line to the graph of $y = x^2$ at the point (1, 1). Explain your reasoning.

49. **PROBLEM SOLVING** The owners of a dog shelter want to enclose a rectangular play area on the side of their building. They have k feet of fencing. What is the maximum area of the outside enclosure in terms of k? (*Hint:* Find the y-coordinate of the vertex of the graph of the area function.)



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph f and h. Describe the transformations from the graph of f to the graph of h. *(Section 3.7)*

50. $f(x) = x$; $h(x) = 4x + 3$

51. $f(x) = x$; $h(x) = -x - 8$

52. $f(x) = x$; $h(x) = -\frac{1}{2}x + 5$

8.1–8.3 What Did You Learn?

Core Vocabulary

quadratic function, *p. 406*
parabola, *p. 406*
vertex, *p. 406*
axis of symmetry, *p. 406*
maximum value, *p. 419*
minimum value, *p. 419*

Core Concepts

Section 8.1

Characteristics of Quadratic Functions, *p. 406*
Graphing $f(x) = ax^2$ When $a > 0$, *p. 407*
Graphing $f(x) = ax^2$ When $a < 0$, *p. 407*
Graphing $f(x) = (ax)^2$, *p. 408*

Section 8.2

Graphing $f(x) = ax^2 + c$, *p. 412*

Section 8.3

Graphing $f(x) = ax^2 + bx + c$, *p. 418*
Maximum and Minimum Values, *p. 419*

Mathematical Thinking

1. Explain your plan for solving Exercise 18 on page 409.
2. How does graphing the function in Exercise 27 on page 415 help you answer the questions?
3. What definition and characteristics of the graph of a quadratic function did you use to answer Exercise 44 on page 424?

Study Skills

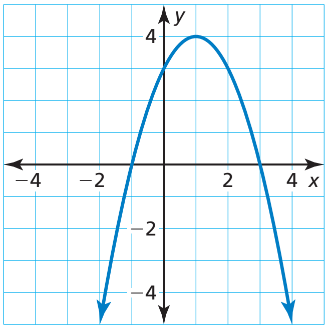
Learning Visually

- Draw a picture of a word problem before writing a verbal model. You do not have to be an artist.
- When making a review card for a word problem, include a picture. This will help you recall the information while taking a test.
- Make sure your notes are visually neat for easy recall.

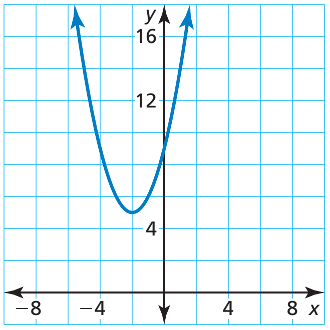
8.1–8.3 Quiz

Identify characteristics of the quadratic function and its graph. *(Section 8.1)*

1.



2.



Graph the function. Compare the graph to the graph of $f(x) = x^2$. *(Section 8.1 and Section 8.2)*

3. $h(x) = -x^2$

4. $p(x) = 2x^2 + 2$

5. $r(x) = 4x^2 - 16$

6. $b(x) = (5x)^2$

7. $g(x) = \frac{2}{5}x^2$

8. $m(x) = -\frac{1}{2}x^2 - 4$

Describe the transformation from the graph of f to the graph of g. Then graph f and g in the same coordinate plane. Write an equation that represents g in terms of x. *(Section 8.2)*

9. $f(x) = 2x^2 + 1;\ g(x) = f(x) + 2$

10. $f(x) = -3x^2 + 12;\ g(x) = f(x) - 9$

11. $f(x) = \frac{1}{2}x^2 - 2;\ g(x) = f(x) - 6$

12. $f(x) = 5x^2 - 3;\ g(x) = f(x) + 1$

Graph the function. Describe the domain and range. *(Section 8.3)*

13. $f(x) = -4x^2 - 4x + 7$

14. $f(x) = 2x^2 + 12x + 5$

15. $y = x^2 + 4x - 5$

16. $y = -3x^2 + 6x + 9$

Tell whether the function has a minimum value or a maximum value. Then find the value. *(Section 8.3)*

17. $f(x) = 5x^2 + 10x - 3$

18. $f(x) = -\frac{1}{2}x^2 + 2x + 16$

19. $y = -x^2 + 4x + 12$

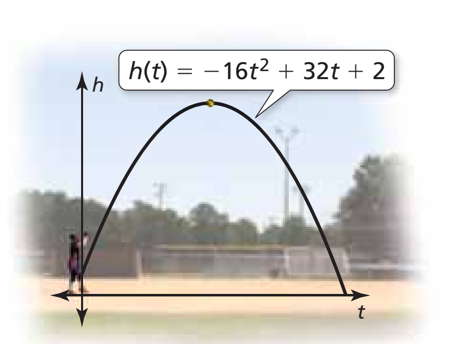
20. $y = 2x^2 + 8x + 3$

21. The distance y (in feet) that a coconut falls after t seconds is given by the function $y = 16t^2$. Use a graph to determine how many seconds it takes for the coconut to fall 64 feet. *(Section 8.1)*

22. The function $y = -16t^2 + 25$ represents the height y (in feet) of a pinecone t seconds after falling from a tree. *(Section 8.2)*

a. After how many seconds does the pinecone hit the ground?

b. A second pinecone falls from a height of 36 feet. Which pinecone hits the ground in the least amount of time? Explain.



23. The function shown models the height (in feet) of a softball t seconds after it is pitched in an underhand motion. Describe the domain and range. Find the maximum height of the softball. *(Section 8.3)*

8.4 Graphing $f(x) = a(x - h)^2 + k$

Essential Question

How can you describe the graph of $f(x) = a(x - h)^2$?

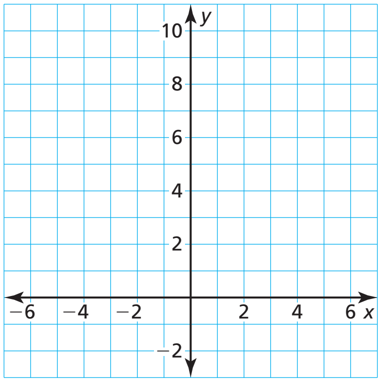
Texas Essential Knowledge and Skills

A.6.B
A.7.A
A.7.C

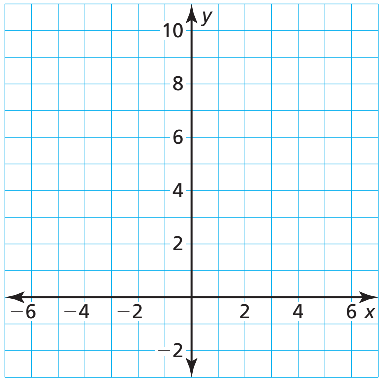
EXPLORATION 1 Graphing $y = a(x - h)^2$ When $h > 0$

Work with a partner. Sketch the graphs of the functions in the same coordinate plane. How does the value of h affect the graph of $y = a(x - h)^2$?

a. $f(x) = x^2$ and $g(x) = (x - 2)^2$



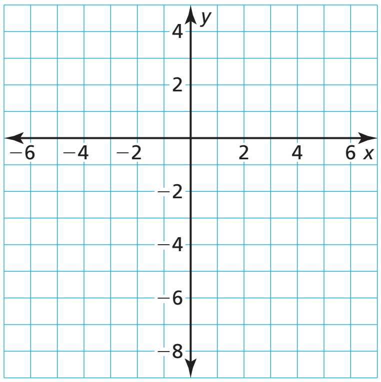
b. $f(x) = 2x^2$ and $g(x) = 2(x - 2)^2$



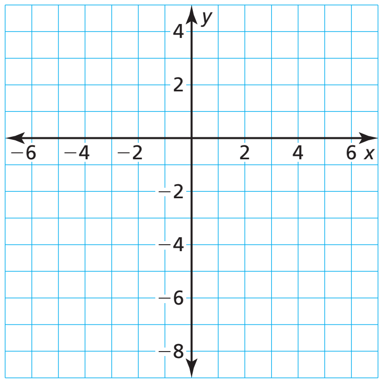
EXPLORATION 2 Graphing $y = a(x - h)^2$ When $h < 0$

Work with a partner. Sketch the graphs of the functions in the same coordinate plane. How does the value of h affect the graph of $y = a(x - h)^2$?

a. $f(x) = -x^2$ and $g(x) = -(x + 2)^2$



b. $f(x) = -2x^2$ and $g(x) = -2(x + 2)^2$



SELECTING TOOLS

To be proficient in math, you need to consider the available tools, such as a graphing calculator, when solving a mathematical problem.

Communicate Your Answer

3. How can you describe the graph of $f(x) = a(x - h)^2$?

4. Without graphing, describe the graph of each function. Use a graphing calculator to check your answer.

a. $y = (x - 3)^2$

b. $y = (x + 3)^2$

c. $y = -(x - 3)^2$

8.4 Lesson

Core Vocabulary

even function, *p. 428*
odd function, *p. 428*
vertex form (of a quadratic function), *p. 430*

Previous
reflection

What You Will Learn

- Identify even and odd functions.
- Graph quadratic functions of the form $f(x) = a(x - h)^2$.
- Graph quadratic functions of the form $f(x) = a(x - h)^2 + k$.
- Model real-life problems using $f(x) = a(x - h)^2 + k$.

Identifying Even and Odd Functions

Core Concept

Even and Odd Functions

A function $y = f(x)$ is **even** when $f(-x) = f(x)$ for each x in the domain of f. The graph of an even function is symmetric about the y-axis.

A function $y = f(x)$ is **odd** when $f(-x) = -f(x)$ for each x in the domain of f. The graph of an odd function is symmetric about the origin. A graph is *symmetric about the origin* when it looks the same after reflections in the x-axis and then in the y-axis.

STUDY TIP
The graph of an odd function looks the same after a 180° rotation about the origin.

EXAMPLE 1 Identifying Even and Odd Functions

Determine whether each function is *even*, *odd*, or *neither*.

a. $f(x) = 2x$ **b.** $g(x) = x^2 - 2$ **c.** $h(x) = 2x^2 + x - 2$

SOLUTION

a.

$f(x) = 2x$ Write the original function.

$f(-x) = 2(-x)$ Substitute $-x$ for x.

$= -2x$ Simplify.

$= -f(x)$ Substitute $f(x)$ for $2x$.

▶ Because $f(-x) = -f(x)$, the function is odd.

b.

$g(x) = x^2 - 2$ Write the original function.

$g(-x) = (-x)^2 - 2$ Substitute $-x$ for x.

$= x^2 - 2$ Simplify.

$= g(x)$ Substitute $g(x)$ for $x^2 - 2$.

▶ Because $g(-x) = g(x)$, the function is even.

c.

$h(x) = 2x^2 + x - 2$ Write the original function.

$h(-x) = 2(-x)^2 + (-x) - 2$ Substitute $-x$ for x.

$= 2x^2 - x - 2$ Simplify.

▶ Because $h(x) = 2x^2 + x - 2$ and $-h(x) = -2x^2 - x + 2$, you can conclude that $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$. So, the function is neither even nor odd.

STUDY TIP
Most functions are neither even nor odd.

Monitoring Progress

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Determine whether the function is *even*, *odd*, or *neither*.

1. $f(x) = 5x$ **2.** $g(x) = 2^x$ **3.** $h(x) = 2x^2 + 3$

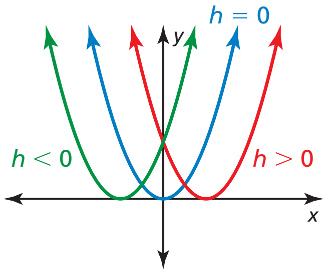
Graphing $f(x) = a(x - h)^2$

Core Concept

Graphing $f(x) = a(x - h)^2$

- When $h > 0$, the graph of $f(x) = a(x - h)^2$ is a horizontal translation h units right of the graph of $f(x) = ax^2$.
- When $h < 0$, the graph of $f(x) = a(x - h)^2$ is a horizontal translation $|h|$ units left of the graph of $f(x) = ax^2$.

The vertex of the graph of $f(x) = a(x - h)^2$ is $(h, 0)$, and the axis of symmetry is $x = h$.



EXAMPLE 2 **Graphing $y = a(x - h)^2$**

Graph $g(x) = \frac{1}{2}(x - 4)^2$. Compare the graph to the graph of $f(x) = x^2$.

SOLUTION

Step 1 Graph the axis of symmetry. Because $h = 4$, graph $x = 4$.

Step 2 Plot the vertex. Because $h = 4$, plot $(4, 0)$.

Step 3 Find and plot two more points on the graph. Choose two x-values less than the x-coordinate of the vertex. Then find $g(x)$ for each x-value.

When $x = 0$:

$$g(0) = \tfrac{1}{2}(0 - 4)^2 = 8$$

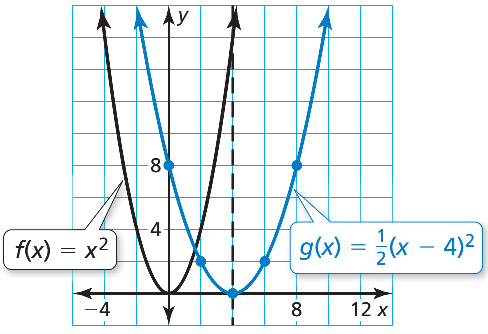
When $x = 2$:

$$g(2) = \tfrac{1}{2}(2 - 4)^2 = 2$$

So, plot $(0, 8)$ and $(2, 2)$.

Step 4 Reflect the points plotted in Step 3 in the axis of symmetry. So, plot $(8, 8)$ and $(6, 2)$.

Step 5 Draw a smooth curve through the points.



ANOTHER WAY

In Step 3, you could instead choose two x-values greater than the x-coordinate of the vertex.

STUDY TIP

From the graph, you can see that $f(x) = x^2$ is an even function. However, $g(x) = \frac{1}{2}(x - 4)^2$ is neither even nor odd.

▶ Both graphs open up. The graph of g is wider than the graph of f. The axis of symmetry $x = 4$ and the vertex $(4, 0)$ of the graph of g are 4 units right of the axis of symmetry $x = 0$ and the vertex $(0, 0)$ of the graph of f. So, the graph of g is a translation 4 units right and a vertical shrink by a factor of $\frac{1}{2}$ of the graph of f.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

4. $g(x) = 2(x + 5)^2$

5. $h(x) = -(x - 2)^2$

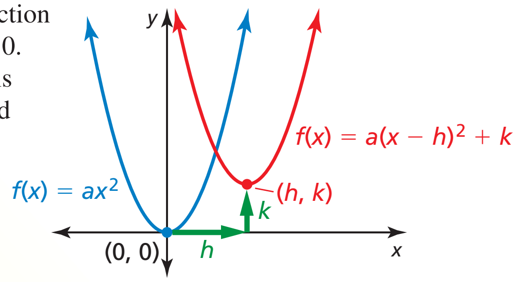
Graphing $f(x) = a(x - h)^2 + k$

Core Concept

Graphing $f(x) = a(x - h)^2 + k$

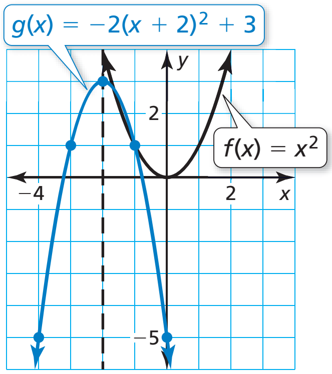
The **vertex form** of a quadratic function is $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The graph of $f(x) = a(x - h)^2 + k$ is a translation h units horizontally and k units vertically of the graph of $f(x) = ax^2$.

The vertex of the graph of $f(x) = a(x - h)^2 + k$ is (h, k), and the axis of symmetry is $x = h$.



EXAMPLE 3 Graphing $y = a(x - h)^2 + k$

Graph $g(x) = -2(x + 2)^2 + 3$. Compare the graph to the graph of $f(x) = x^2$.



SOLUTION

Step 1 Graph the axis of symmetry. Because $h = -2$, graph $x = -2$.

Step 2 Plot the vertex. Because $h = -2$ and $k = 3$, plot $(-2, 3)$.

Step 3 Find and plot two more points on the graph. Choose two x-values less than the x-coordinate of the vertex. Then find $g(x)$ for each x-value. So, plot $(-4, -5)$ and $(-3, 1)$.

x	-4	-3
$g(x)$	-5	1

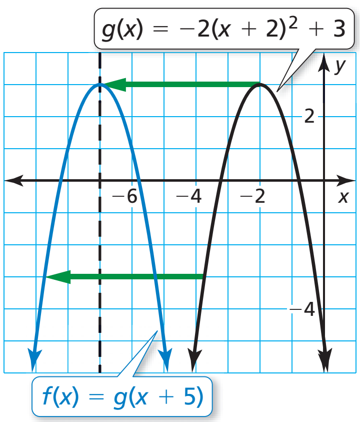
Step 4 Reflect the points plotted in Step 3 in the axis of symmetry. So, plot $(-1, 1)$ and $(0, -5)$.

Step 5 Draw a smooth curve through the points.

▶ The graph of g opens down and is narrower than the graph of f. The vertex of the graph of g, $(-2, 3)$, is 2 units left and 3 units up of the vertex of the graph of f, $(0, 0)$. So, the graph of g is a vertical stretch by a factor of 2, a reflection in the x-axis, and a translation 2 units left and 3 units up of the graph of f.

EXAMPLE 4 Transforming the Graph of $y = a(x - h)^2 + k$

Consider function g in Example 3. Graph $f(x) = g(x + 5)$.



SOLUTION

The function f is of the form $y = g(x - h)$, where $h = -5$. So, the graph of f is a horizontal translation 5 units left of the graph of g. To graph f, subtract 5 from the x-coordinates of the points on the graph of g.

Monitoring Progress

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Graph the function. Compare the graph to the graph of $f(x) = x^2$.

6. $g(x) = 3(x - 1)^2 + 6$

7. $h(x) = \frac{1}{2}(x + 4)^2 - 2$

8. Consider function g in Example 3. Graph $f(x) = g(x) - 3$.

Modeling Real-Life Problems

EXAMPLE 5 **Modeling with Mathematics**

Water fountains are usually designed to give a specific visual effect. For example, the water fountain shown consists of streams of water that are shaped like parabolas. Notice how the streams are designed to land on the underwater spotlights. Write and graph a quadratic function that models the path of a stream of water with a maximum height of 5 feet, represented by a vertex of (3, 5), landing on a spotlight 6 feet from the water jet, represented by (6, 0).

SOLUTION

1. **Understand the Problem** You know the vertex and another point on the graph that represents the parabolic path. You are asked to write and graph a quadratic function that models the path.

2. **Make a Plan** Use the given points and the vertex form to write a quadratic function. Then graph the function.

3. **Solve the Problem**
Use the vertex form, vertex (3, 5), and point (6, 0) to find the value of a.

$f(x) = a(x - h)^2 + k$	Write the vertex form of a quadratic function.
$f(x) = a(x - 3)^2 + 5$	Substitute 3 for h and 5 for k.
$0 = a(6 - 3)^2 + 5$	Substitute 6 for x and 0 for $f(x)$.
$0 = 9a + 5$	Simplify.
$-\frac{5}{9} = a$	Solve for a.

So, $f(x) = -\frac{5}{9}(x - 3)^2 + 5$ models the path of a stream of water. Now graph the function.

Step 1 Graph the axis of symmetry. Because $h = 3$, graph $x = 3$.

Step 2 Plot the vertex, (3, 5).

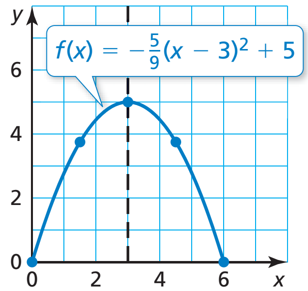
Step 3 Find and plot two more points on the graph. Because the x-axis represents the water surface, the graph should only contain points with nonnegative values of $f(x)$. You know that (6, 0) is on the graph. To find another point, choose an x-value between $x = 3$ and $x = 6$. Then find the corresponding value of $f(x)$.

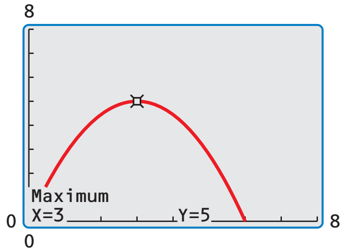
$$f(4.5) = -\frac{5}{9}(4.5 - 3)^2 + 5 = 3.75$$

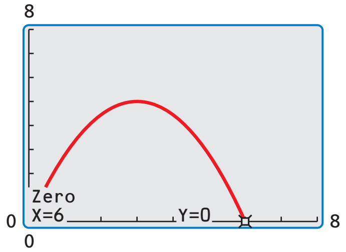
So, plot (6, 0) and (4.5, 3.75).

Step 4 Reflect the points plotted in Step 3 in the axis of symmetry. So, plot (0, 0) and (1.5, 3.75).

Step 5 Draw a smooth curve through the points.







4. **Look Back** Use a graphing calculator to graph $f(x) = -\frac{5}{9}(x - 3)^2 + 5$. Use the *maximum* feature to verify that the maximum value is 5. Then use the *zero* feature to verify that $x = 6$ is a zero of the function.

Monitoring Progress

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9. **WHAT IF?** The vertex is (3, 6). Write and graph a quadratic function that models the path.

8.4 Exercises

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Vocabulary and Core Concept Check

1. **VOCABULARY** Compare the graph of an even function with the graph of an odd function.

2. **OPEN-ENDED** Write a quadratic function whose graph has a vertex of (1, 2).

3. **WRITING** Describe the transformation from the graph of $f(x) = ax^2$ to the graph of $g(x) = a(x - h)^2 + k$.

4. **WHICH ONE DOESN'T BELONG?** Which function does *not* belong with the other three? Explain your reasoning.

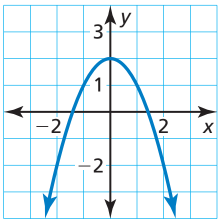
$f(x) = 8(x + 4)^2$ $\quad$ $f(x) = (x - 2)^2 + 4$ $\quad$ $f(x) = 2(x + 0)^2$ $\quad$ $f(x) = 3(x + 1)^2 + 1$

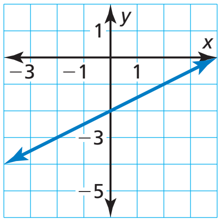
Monitoring Progress and Modeling with Mathematics

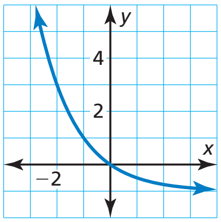
In Exercises 5–12, determine whether the function is *even*, *odd*, or *neither*. *(See Example 1.)*

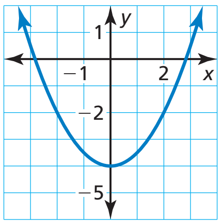
5. $f(x) = 4x + 3$
6. $g(x) = 3x^2$
7. $h(x) = 5^x + 2$
8. $m(x) = 2x^2 - 7x$
9. $p(x) = -x^2 + 8$
10. $f(x) = -\frac{1}{2}x$
11. $n(x) = 2x^2 - 7x + 3$
12. $r(x) = -6x^2 + 5$

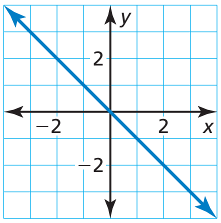
In Exercises 13–18, determine whether the function represented by the graph is *even*, *odd*, or *neither*.

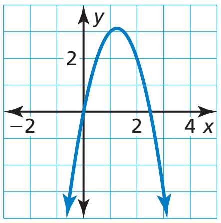
13. 

14. 

15. 

16. 

17. 

18. 

In Exercises 19–22, find the vertex and the axis of symmetry of the graph of the function.

19. $f(x) = 3(x + 1)^2$
20. $f(x) = \frac{1}{4}(x - 6)^2$
21. $y = -\frac{1}{8}(x - 4)^2$
22. $y = -5(x + 9)^2$

In Exercises 23–28, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Example 2.)*

23. $g(x) = 2(x + 3)^2$
24. $p(x) = 3(x - 1)^2$
25. $r(x) = \frac{1}{4}(x + 10)^2$
26. $n(x) = \frac{1}{3}(x - 6)^2$
27. $d(x) = \frac{1}{5}(x - 5)^2$
28. $q(x) = 6(x + 2)^2$

29. **ERROR ANALYSIS** Describe and correct the error in determining whether the function $f(x) = x^2 + 3$ is even, odd, or neither.

$$f(x) = x^2 + 3$$
$$f(-x) = (-x)^2 + 3$$
$$= x^2 + 3$$
$$= f(x)$$

So, $f(x)$ is an odd function.

30. **ERROR ANALYSIS** Describe and correct the error in finding the vertex of the graph of the function.

$y = -(x + 8)^2$

Because $h = -8$, the vertex is $(0, -8)$.

In Exercises 31–34, find the vertex and the axis of symmetry of the graph of the function.

31. $y = -6(x + 4)^2 - 3$ **32.** $f(x) = 3(x - 3)^2 + 6$

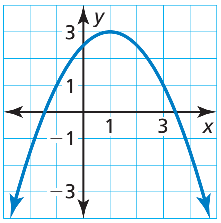
33. $f(x) = -4(x + 3)^2 + 1$ **34.** $y = -(x - 6)^2 - 5$

In Exercises 35–38, match the function with its graph.

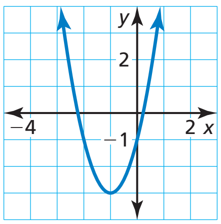
35. $y = -(x + 1)^2 - 3$ **36.** $y = -\frac{1}{2}(x - 1)^2 + 3$

37. $y = \frac{1}{3}(x - 1)^2 + 3$ **38.** $y = 2(x + 1)^2 - 3$

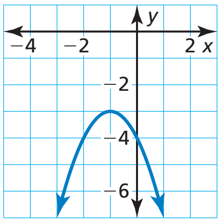
A.



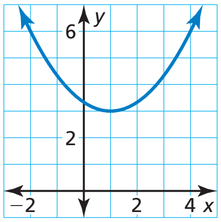
B.



C.



D.



In Exercises 39–44, graph the function. Compare the graph to the graph of $f(x) = x^2$. *(See Example 3.)*

39. $h(x) = (x - 2)^2 + 4$ **40.** $g(x) = (x + 1)^2 - 7$

41. $r(x) = 4(x - 1)^2 - 5$ **42.** $n(x) = -(x + 4)^2 + 2$

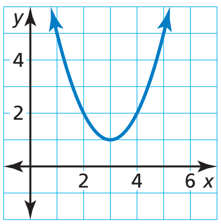
43. $g(x) = -\frac{1}{3}(x + 3)^2 - 2$ **44.** $r(x) = \frac{1}{2}(x - 2)^2 - 4$

In Exercises 45–48, let $f(x) = (x - 2)^2 + 1$. Match the function with its graph.

45. $g(x) = f(x - 1)$ **46.** $r(x) = f(x + 2)$

47. $h(x) = f(x) + 2$ **48.** $p(x) = f(x) - 3$

A.



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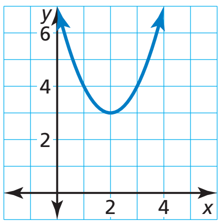
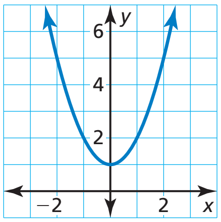
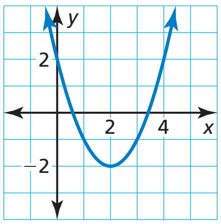


C.



D.



In Exercises 49–54, graph g. *(See Example 4.)*

49. $f(x) = 2(x - 1)^2 + 1$; $g(x) = f(x + 3)$

50. $f(x) = -(x + 1)^2 + 2$; $g(x) = \frac{1}{2}f(x)$

51. $f(x) = -3(x + 5)^2 - 6$; $g(x) = 2f(x)$

52. $f(x) = 5(x - 3)^2 - 1$; $g(x) = f(x) - 6$

53. $f(x) = (x + 3)^2 + 5$; $g(x) = f(x - 4)$

54. $f(x) = -2(x - 4)^2 - 8$; $g(x) = -f(x)$

55. MODELING WITH MATHEMATICS The height (in meters) of a bird diving to catch a fish is represented by $h(t) = 5(t - 2.5)^2$, where t is the number of seconds after beginning the dive.

a. Graph h.

b. Another bird's dive is represented by $r(t) = 2h(t)$. Graph r.

c. Compare the graphs. Which bird starts its dive from a greater height? Explain.

56. MODELING WITH MATHEMATICS A kicker punts a football. The height (in yards) of the football is represented by $f(x) = -\frac{1}{9}(x - 30)^2 + 25$, where x is the horizontal distance (in yards) from the kicker's goal line.

a. Graph f. Describe the domain and range.

b. On the next possession, the kicker punts the football. The height of the football is represented by $g(x) = f(x + 5)$. Graph g. Describe the domain and range.

c. Compare the graphs. On which possession does the kicker punt closer to his goal line? Explain.

In Exercises 57–62, write a quadratic function in vertex form whose graph has the given vertex and passes through the given point.

57. vertex: (1, 2); passes through (3, 10)

58. vertex: (−3, 5); passes through (0, −14)

59. vertex: (−2, −4); passes through (−1, −6)

60. vertex: (1, 8); passes through (3, 12)

61. vertex: (5, −2); passes through (7, 0)

62. vertex: (−5, −1); passes through (−2, 2)

63. **MODELING WITH MATHEMATICS** A portion of a roller coaster track is in the shape of a parabola. Write and graph a quadratic function that models this portion of the roller coaster with a maximum height of 90 feet, represented by a vertex of (25, 90), passing through the point (50, 0). *(See Example 5.)*

64. **MODELING WITH MATHEMATICS** A flare is launched from a boat and travels in a parabolic path until reaching the water. Write and graph a quadratic function that models the path of the flare with a maximum height of 300 meters, represented by a vertex of (59, 300), landing in the water at the point (119, 0).

In Exercises 65–68, rewrite the quadratic function in vertex form.

65. $y = 2x^2 - 8x + 4$

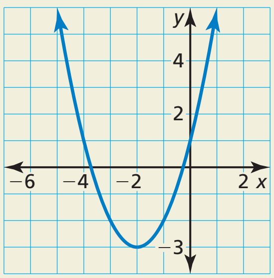
66. $y = 3x^2 + 6x - 1$

67. $f(x) = -5x^2 + 10x + 3$

68. $f(x) = -x^2 - 4x + 2$

69. **REASONING** Can a function be symmetric about the x-axis? Explain.

70. **HOW DO YOU SEE IT?** The graph of a quadratic function is shown. Determine which symbols to use to complete the vertex form of the quadratic function. Explain your reasoning.



$y = a(x \;\square\; 2)^2 \;\square\; 3$

In Exercises 71–74, describe the transformation from the graph of f to the graph of h. Write an equation that represents h in terms of x.

71. $f(x) = -(x + 1)^2 - 2$
$h(x) = f(x) + 4$

72. $f(x) = 2(x - 1)^2 + 1$
$h(x) = f(x - 5)$

73. $f(x) = 4(x - 2)^2 + 3$
$h(x) = 2f(x)$

74. $f(x) = -(x + 5)^2 - 6$
$h(x) = \frac{1}{3}f(x)$

75. **REASONING** The graph of $y = x^2$ is translated 2 units right and 5 units down. Write an equation for the function in vertex form and in standard form. Describe advantages of writing the function in each form.

76. **THOUGHT PROVOKING** Which of the following are true? Justify your answers.

  a. Any constant multiple of an even function is even.

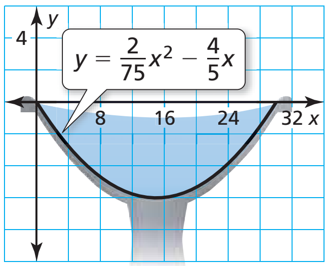
  b. Any constant multiple of an odd function is odd.

  c. The sum or difference of two even functions is even.

  d. The sum or difference of two odd functions is odd.

  e. The sum or difference of an even function and an odd function is odd.

77. **COMPARING FUNCTIONS** A cross section of a birdbath can be modeled by $y = \frac{1}{81}(x - 18)^2 - 4$, where x and y are measured in inches. The graph shows the cross section of another birdbath.



  a. Which birdbath is deeper? Explain.

  b. Which birdbath is wider? Explain.

78. **REASONING** Compare the graphs of $y = 2x^2 + 8x + 8$ and $y = x^2$ without graphing the functions. How can factoring help you compare the parabolas? Explain.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. *(Section 7.5)*

79. $x(x - 1) = 0$

80. $(x + 3)(x - 8) = 0$

81. $(3x - 9)(4x + 12) = 0$

8.5 Using Intercept Form

Essential Question What are some of the characteristics of the graph of $f(x) = a(x - p)(x - q)$?

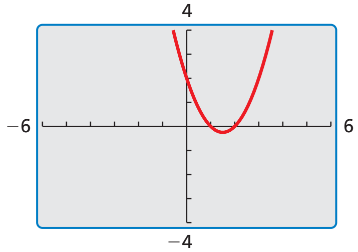
TEXAS ESSENTIAL KNOWLEDGE AND SKILLS

A.6.A
A.6.B
A.6.C
A.7.A
A.7.B

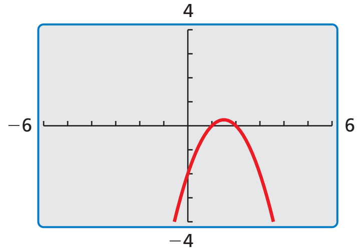
EXPLORATION 1 Using Zeros to Write Functions

Work with a partner. Each graph represents a function of the form $f(x) = (x - p)(x - q)$ or $f(x) = -(x - p)(x - q)$. Write the function represented by each graph. Explain your reasoning.

a.



b.



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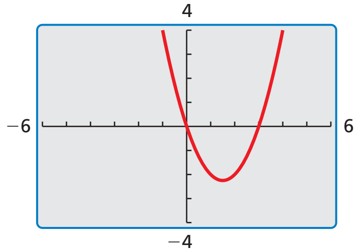

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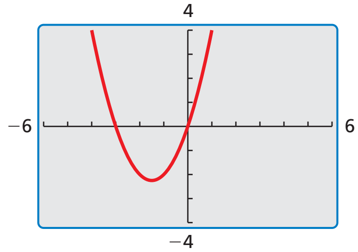

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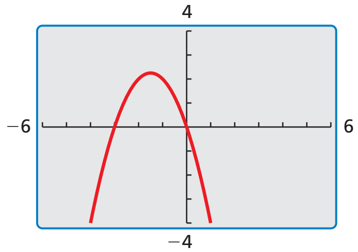

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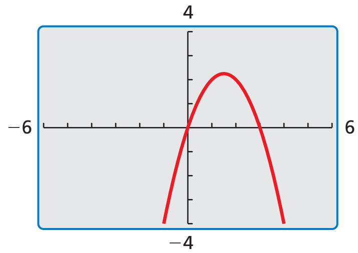

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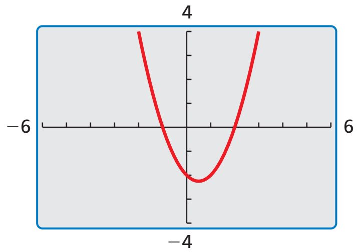

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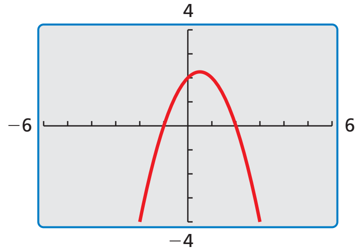


MAKING MATHEMATICAL ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

Communicate Your Answer

2. What are some of the characteristics of the graph of $f(x) = a(x - p)(x - q)$?

3. Consider the graph of $f(x) = a(x - p)(x - q)$.

a. Does changing the sign of a change the x-intercepts? Does changing the sign of a change the y-intercept? Explain your reasoning.

b. Does changing the value of p change the x-intercepts? Does changing the value of p change the y-intercept? Explain your reasoning.

8.5 Lesson

Core Vocabulary

intercept form, *p. 436*

What You Will Learn

- Graph quadratic functions of the form $f(x) = a(x - p)(x - q)$.
- Use intercept form to find zeros of functions.
- Graphing and writing quadratic functions.
- Graphing and writing cubic functions.

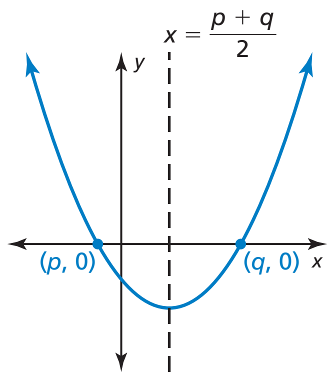
Graphing $f(x) = a(x - p)(x - q)$

You have already graphed quadratic functions written in several different forms, such as $f(x) = ax^2 + bx + c$ (standard form) and $g(x) = a(x - h)^2 + k$ (vertex form). Quadratic functions can also be written in **intercept form**, $f(x) = a(x - p)(x - q)$, where $a \neq 0$. In this form, the polynomial that defines a function is in factored form and the x-intercepts of the graph can be easily determined.

Core Concept

Graphing $f(x) = a(x - p)(x - q)$

- The x-intercepts are p and q.
- The axis of symmetry is halfway between $(p, 0)$ and $(q, 0)$. So, the axis of symmetry is $x = \dfrac{p + q}{2}$.
- The graph opens up when $a > 0$, and the graph opens down when $a < 0$.



EXAMPLE 1 **Graphing $f(x) = a(x - p)(x - q)$**

Graph $f(x) = -(x + 1)(x - 5)$. Describe the domain and range.

SOLUTION

Step 1 Identify the x-intercepts. Because the x-intercepts are $p = -1$ and $q = 5$, plot $(-1, 0)$ and $(5, 0)$.

Step 2 Find and graph the axis of symmetry.

$$x = \frac{p + q}{2} = \frac{-1 + 5}{2} = 2$$

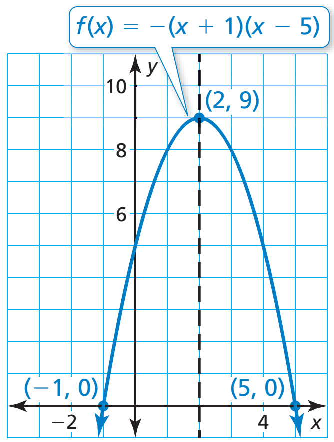
Step 3 Find and plot the vertex.

The x-coordinate of the vertex is 2. To find the y-coordinate of the vertex, substitute 2 for x and simplify.

$$f(2) = -(2 + 1)(2 - 5) = 9$$

So, the vertex is $(2, 9)$.

Step 4 Draw a parabola through the vertex and the points where the x-intercepts occur.



▶ The domain is all real numbers. The range is $y \leq 9$.

EXAMPLE 2 Graphing a Quadratic Function

Graph $f(x) = 2x^2 - 8$. Describe the domain and range.

SOLUTION

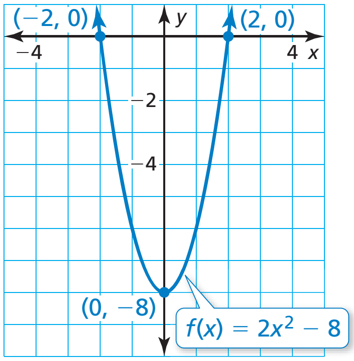
Step 1 Rewrite the quadratic function in intercept form.

$$f(x) = 2x^2 - 8 \quad \text{Write the function.}$$
$$= 2(x^2 - 4) \quad \text{Factor out common factor.}$$
$$= 2(x + 2)(x - 2) \quad \text{Difference of two squares pattern}$$

Step 2 Identify the x-intercepts. Because the x-intercepts are $p = -2$ and $q = 2$, plot $(-2, 0)$ and $(2, 0)$.

Step 3 Find and graph the axis of symmetry.

$$x = \frac{p+q}{2} = \frac{-2+2}{2} = 0$$



Step 4 Find and plot the vertex.

The x-coordinate of the vertex is 0.
The y-coordinate of the vertex is

$$f(0) = 2(0)^2 - 8 = -8.$$

So, the vertex is $(0, -8)$.

Step 5 Draw a parabola through the vertex and the points where the x-intercepts occur.

▶ The domain is all real numbers. The range is $y \geq -8$.

Monitoring Progress

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Graph the quadratic function. Label the vertex, axis of symmetry, and x-intercepts. Describe the domain and range of the function.

1. $f(x) = (x + 2)(x - 3)$ **2.** $g(x) = -2(x - 4)(x + 1)$ **3.** $h(x) = 4x^2 - 36$

REMEMBER

Functions have zeros, and *graphs* have x-intercepts.

Using Intercept Form to Find Zeros of Functions

In Section 3.4, you learned that a zero of a function is an x-value for which $f(x) = 0$. You can use the intercept form of a function to find the zeros of the function.

EXAMPLE 3 Finding Zeros of a Function

Find the zeros of $f(x) = (x - 1)(x + 2)$.

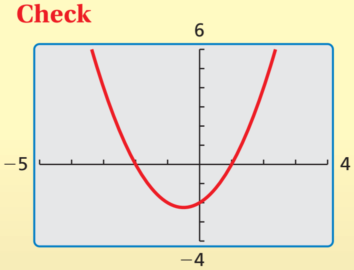
SOLUTION

To find the zeros, determine the x-values for which $f(x)$ is 0.

$$f(x) = (x - 1)(x + 2) \quad \text{Write the function.}$$
$$0 = (x - 1)(x + 2) \quad \text{Substitute 0 for } f(x).$$
$$x - 1 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{Zero-Product Property}$$
$$x = 1 \quad \text{or} \quad x = -2 \quad \text{Solve for } x.$$

▶ So, the zeros of the function are -2 and 1.

Check



Core Concept

Factors and Zeros

For any factor $x - n$ of a polynomial, n is a zero of the function defined by the polynomial.

EXAMPLE 4 Finding Zeros of Functions

Find the zeros of each function.

a. $f(x) = -2x^2 - 10x - 12$ **b.** $h(x) = (x - 1)(x^2 - 16)$

SOLUTION

Write each function in intercept form to identify the zeros.

a. $f(x) = -2x^2 - 10x - 12$ Write the function.

$= -2(x^2 + 5x + 6)$ Factor out common factor.

$= -2(x + 3)(x + 2)$ Factor the trinomial.

► So, the zeros of the function are -3 and -2.

b. $h(x) = (x - 1)(x^2 - 16)$ Write the function.

$= (x - 1)(x + 4)(x - 4)$ Difference of two squares pattern

► So, the zeros of the function are -4, 1, and 4.

ANALYZING MATHEMATICAL RELATIONSHIPS

The function in Example 4(b) is called a *cubic function*. You can extend the concept of intercept form to cubic functions. You will graph a cubic function in Example 8.

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Find the zero(s) of the function.

4. $f(x) = (x - 6)(x - 1)$ **5.** $g(x) = 3x^2 - 12x + 12$ **6.** $h(x) = x(x^2 - 1)$

Graphing and Writing Quadratic Functions

EXAMPLE 5 Graphing a Quadratic Function Using Zeros

Use zeros to graph $h(x) = x^2 - 2x - 3$.

SOLUTION

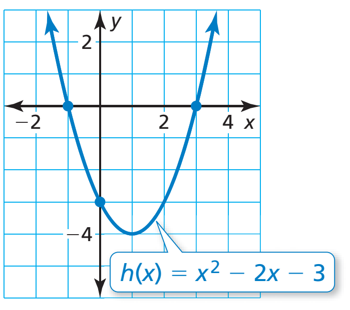
The function is in standard form. You know that the parabola opens up $(a > 0)$ and the y-intercept is -3. So, begin by plotting $(0, -3)$.

Notice that the polynomial that defines the function is factorable. So, write the function in intercept form and identify the zeros.

$h(x) = x^2 - 2x - 3$ Write the function.

$= (x + 1)(x - 3)$ Factor the trinomial.

The zeros of the function are -1 and 3. So, plot $(-1, 0)$ and $(3, 0)$. Draw a parabola through the points.



USING PRECISE MATHEMATICAL LANGUAGE

To sketch a more precise graph, make a table of values and plot other points on the graph.

STUDY TIP

In part (a), many possible functions satisfy the given condition. The value *a* can be *any* nonzero number. To allow easier calculations, let $a = 1$. By letting $a = 2$, the resulting function would be $f(x) = 2x^2 - 6x - 56$.

EXAMPLE 6 Writing Quadratic Functions

Write a quadratic function in standard form whose graph satisfies the given condition(s).

a. x-intercepts: -4 and 7 **b.** vertex: $(-3, 4)$

SOLUTION

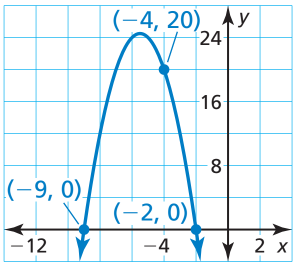
a. Because you know the x-intercepts, use intercept form to write a function.

$f(x) = a(x - p)(x - q)$	Intercept form
$= 1(x + 4)(x - 7)$	Substitute for *a*, *p*, and *q*.
$= x^2 - 7x + 4x - 28$	FOIL Method
$= x^2 - 3x - 28$	Combine like terms.

b. Because you know the vertex, use vertex form to write a function.

$f(x) = a(x - h)^2 + k$	Vertex form
$= 1(x + 3)^2 + 4$	Substitute for *a*, *h*, and *k*.
$= x^2 + 6x + 9 + 4$	Find the product $(x + 3)^2$.
$= x^2 + 6x + 13$	Combine like terms.

EXAMPLE 7 Writing a Quadratic Function



The graph represents a quadratic function. Write the function.

SOLUTION

From the graph, you can see that the x-intercepts are -9 and -2. Use intercept form to write a function.

$f(x) = a(x - p)(x - q)$	Intercept form
$= a(x + 9)(x + 2)$	Substitute for *p* and *q*.

Use the other given point, $(-4, 20)$, to find the value of a.

$20 = a(-4 + 9)(-4 + 2)$	Substitute -4 for *x* and 20 for *f*(*x*).
$20 = a(5)(-2)$	Simplify.
$-2 = a$	Solve for *a*.

Use the value of a to write the function.

$f(x) = -2(x + 9)(x + 2)$	Substitute -2 for *a*.
$= -2x^2 - 22x - 36$	Simplify.

▶ The function represented by the graph is $f(x) = -2x^2 - 22x - 36$.

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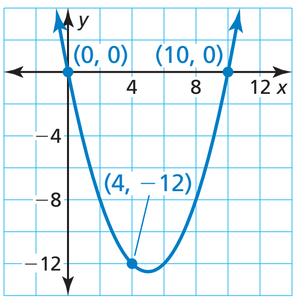
Use zeros to graph the function.

7. $f(x) = (x - 1)(x - 4)$ **8.** $g(x) = x^2 + x - 12$

Write a quadratic function in standard form whose graph satisfies the given condition(s).

9. x-intercepts: -1 and 1 **10.** vertex: $(8, 8)$



11. The graph at the left represents a quadratic function. Write the function.

Graphing and Writing Cubic Functions

In Example 4, you extended the concept of intercept form to cubic functions.

$f(x) = a(x - p)(x - q)(x - r),\ a \neq 0$ — Intercept form of a cubic function

The x-intercepts of the graph of f are p, q, and r.

EXAMPLE 8 Graphing a Cubic Function Using Zeros

Use zeros to graph $f(x) = x^3 - 4x$.

SOLUTION

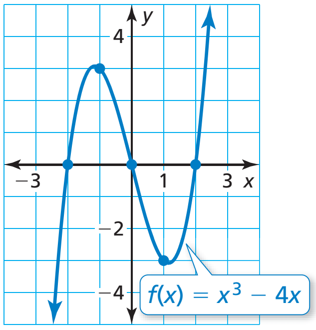
Notice that the polynomial that defines the function is factorable. So, write the function in intercept form and identify the zeros.

$f(x) = x^3 - 4x$ — Write the function.

$= x(x^2 - 4)$ — Factor out x.

$= x(x + 2)(x - 2)$ — Difference of two squares pattern

The zeros of the function are -2, 0, and 2. So, plot $(-2, 0)$, $(0, 0)$, and $(2, 0)$.

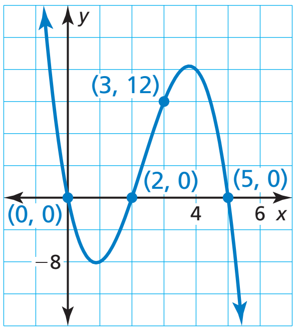


To help determine the shape of the graph, find points between the zeros. Plot $(-1, 3)$ and $(1, -3)$. Draw a smooth curve through the points.

x	-1	1
$f(x)$	3	-3

EXAMPLE 9 Writing a Cubic Function

The graph represents a cubic function. Write the function.



SOLUTION

From the graph, you can see that the x-intercepts are 0, 2, and 5. Use intercept form to write a function.

$f(x) = a(x - p)(x - q)(x - r)$ — Intercept form

$= a(x - 0)(x - 2)(x - 5)$ — Substitute for p, q, and r.

$= a(x)(x - 2)(x - 5)$ — Simplify.

Use the other given point, (3, 12), to find the value of a.

$12 = a(3)(3 - 2)(3 - 5)$ — Substitute 3 for x and 12 for $f(x)$.

$-2 = a$ — Solve for a.

Use the value of a to write the function.

$f(x) = -2(x)(x - 2)(x - 5)$ — Substitute -2 for a.

$= -2x^3 + 14x^2 - 20x$ — Simplify.

▶ The function represented by the graph is $f(x) = -2x^3 + 14x^2 - 20x$.

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Use zeros to graph the function.

12. $g(x) = (x - 1)(x - 3)(x + 3)$

13. $h(x) = x^3 - 6x^2 + 5x$

14. The zeros of a cubic function are -3, -1, and 1. The graph of the function passes through the point $(0, -3)$. Write the function.

8.5 Exercises

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Vocabulary and Core Concept Check

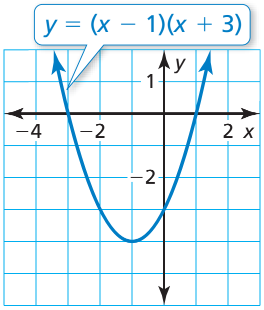
1. **COMPLETE THE SENTENCE** The values p and q are __________ of the graph of the function $f(x) = a(x - p)(x - q)$.

2. **WRITING** Explain how to find the maximum value or minimum value of a quadratic function when the function is given in intercept form.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the x-intercepts and axis of symmetry of the graph of the function.

3.



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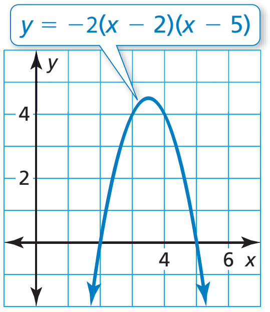


5. $f(x) = -5(x + 7)(x - 5)$

6. $g(x) = \frac{2}{3}x(x + 8)$

In Exercises 7–12, graph the quadratic function. Label the vertex, axis of symmetry, and x-intercepts. Describe the domain and range of the function. *(See Example 1.)*

7. $f(x) = (x + 4)(x + 1)$

8. $y = (x - 2)(x + 2)$

9. $y = -(x + 6)(x - 4)$

10. $h(x) = -4(x - 7)(x - 3)$

11. $g(x) = 5(x + 1)(x + 2)$

12. $y = -2(x - 3)(x + 4)$

In Exercises 13–20, graph the quadratic function. Label the vertex, axis of symmetry, and x-intercepts. Describe the domain and range of the function. *(See Example 2.)*

13. $y = x^2 - 9$

14. $f(x) = x^2 - 8x$

15. $h(x) = -5x^2 + 5x$

16. $y = 3x^2 - 48$

17. $q(x) = x^2 + 9x + 14$

18. $p(x) = x^2 + 6x - 27$

19. $y = 4x^2 - 36x + 32$

20. $y = -2x^2 - 4x + 30$

In Exercises 21–30, find the zero(s) of the function. *(See Examples 3 and 4.)*

21. $y = -2(x - 2)(x - 10)$

22. $f(x) = \frac{1}{3}(x + 5)(x - 1)$

23. $g(x) = x^2 + 5x - 24$

24. $y = x^2 - 17x + 52$

25. $y = 3x^2 - 15x - 42$

26. $g(x) = -4x^2 - 8x - 4$

27. $f(x) = (x + 5)(x^2 - 4)$

28. $h(x) = (x^2 - 36)(x - 11)$

29. $y = x^3 - 49x$

30. $y = x^3 - x^2 - 9x + 9$

In Exercises 31–36, match the function with its graph.

31. $y = (x + 5)(x + 3)$

32. $y = (x + 5)(x - 3)$

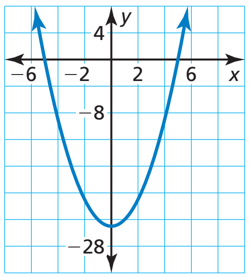
33. $y = (x - 5)(x + 3)$

34. $y = (x - 5)(x - 3)$

35. $y = (x + 5)(x - 5)$

36. $y = (x + 3)(x - 3)$

A.



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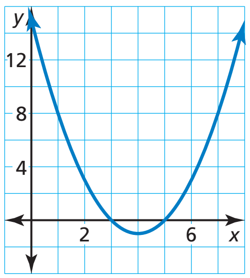

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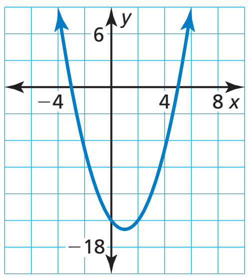

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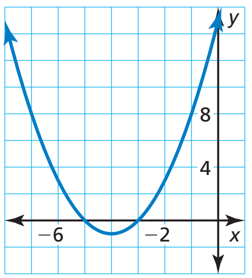

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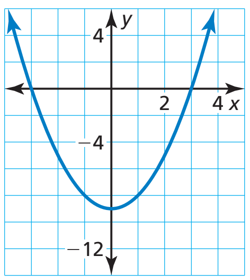

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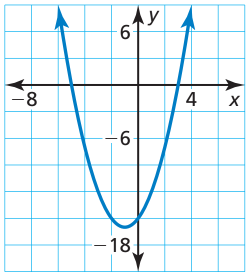


In Exercises 37–42, use zeros to graph the function. *(See Example 5.)*

37. $f(x) = (x + 2)(x - 6)$ **38.** $g(x) = -3(x + 1)(x + 7)$

39. $y = x^2 - 11x + 18$ **40.** $y = x^2 - x - 30$

41. $y = -5x^2 - 10x + 40$ **42.** $h(x) = 8x^2 - 8$

ERROR ANALYSIS **In Exercises 43 and 44, describe and correct the error in finding the zeros of the function.**

43.

$y = 5(x + 3)(x - 2)$

The zeros of the function are 3 and -2.

44.

$y = (x + 4)(x^2 - 9)$

The zeros of the function are -4 and 9.

In Exercises 45–56, write a quadratic function in standard form whose graph satisfies the given condition(s). *(See Example 6.)*

45. vertex: $(7, -3)$ **46.** vertex: $(4, 8)$

47. x-intercepts: 1 and 9 **48.** x-intercepts: -2 and -5

49. passes through $(-4, 0)$, $(3, 0)$, and $(2, -18)$

50. passes through $(-5, 0)$, $(-1, 0)$, and $(-4, 3)$

51. passes through $(7, 0)$

52. passes through $(0, 0)$ and $(6, 0)$

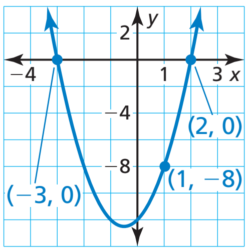
53. axis of symmetry: $x = -5$

54. y increases as x increases when $x < 4$; y decreases as x increases when $x > 4$.

55. range: $y \geq -3$ **56.** range: $y \leq 10$

In Exercises 57–60, write the quadratic function represented by the graph. *(See Example 7.)*

57.



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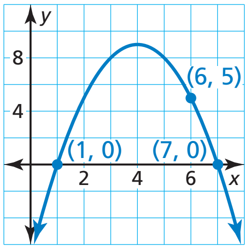

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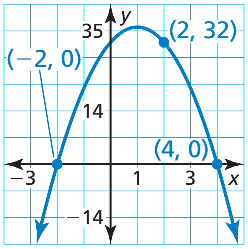

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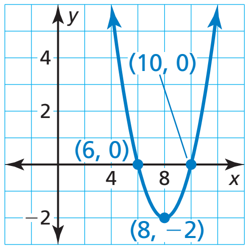


In Exercises 61–68, use zeros to graph the function. *(See Example 8.)*

61. $y = 5x(x + 2)(x - 6)$ **62.** $f(x) = -x(x + 9)(x + 3)$

63. $h(x) = (x - 2)(x + 2)(x + 7)$

64. $y = (x + 1)(x - 5)(x - 4)$

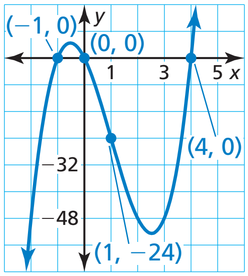
65. $f(x) = 3x^3 - 48x$ **66.** $y = -2x^3 + 20x^2 - 50x$

67. $y = -x^3 - 16x^2 - 28x$

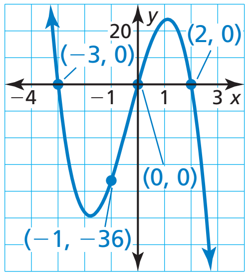
68. $g(x) = 6x^3 + 30x^2 - 36x$

In Exercises 69–72, write the cubic function represented by the graph. *(See Example 9.)*

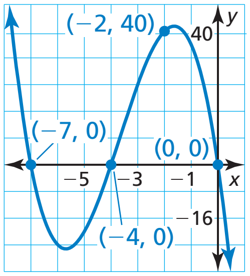
69.



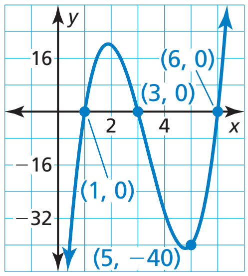
70.



71.



72.



In Exercises 73–76, write a cubic function whose graph satisfies the given condition(s).

73. x-intercepts: -2, 3, and 8

74. x-intercepts: -7, -5, and 0

75. passes through $(1, 0)$ and $(7, 0)$

76. passes through $(0, 6)$

In Exercises 77–80, all the zeros of a function are given. Use the zeros and the other point given to write a quadratic or cubic function represented by the table.

77.

x	y
0	0
2	30
7	0

78.

x	y
-3	0
1	-72
4	0

79.

x	y
-4	0
-3	0
0	-180
3	0

80.

x	y
-8	0
-6	-36
-3	0
0	0

In Exercises 81–84, sketch a parabola that satisfies the given conditions.

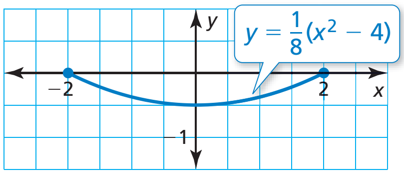
81. x-intercepts: -4 and 2; range: $y \geq -3$

82. axis of symmetry: $x = 6$; passes through $(4, 15)$

83. range: $y \leq 5$; passes through $(0, 2)$

84. x-intercept: 6; y-intercept: 1; range: $y \geq -4$

85. **MODELING WITH MATHEMATICS** Satellite dishes are shaped like parabolas to optimally receive signals. The cross section of a satellite dish can be modeled by the function shown, where x and y are measured in feet. The x-axis represents the top of the opening of the dish.

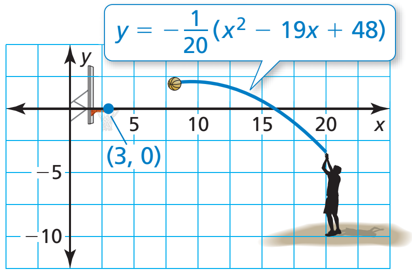


a. How wide is the satellite dish?

b. How deep is the satellite dish?

c. Write a quadratic function in standard form that models the cross section of a satellite dish that is 6 feet wide and 1.5 feet deep.

86. **MODELING WITH MATHEMATICS** A professional basketball player's shot is modeled by the function shown, where x and y are measured in feet.



a. Does the player make the shot? Explain.

b. The basketball player releases another shot from the point $(13, 0)$ and makes the shot. The shot also passes through the point $(10, 1.4)$. Write a quadratic function in standard form that models the path of the shot.

USING STRUCTURE **In Exercises 87–90, match the function with its graph.**

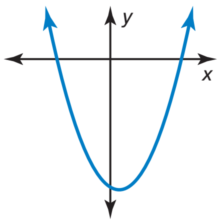
87. $y = -x^2 + 5x$

88. $y = x^2 - x - 12$

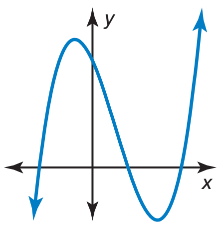
89. $y = x^3 - 2x^2 - 8x$

90. $y = x^3 - 4x^2 - 11x + 30$

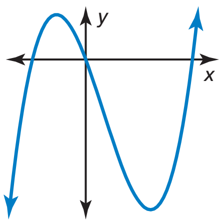
A.



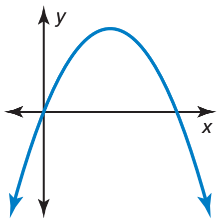
B.



C.



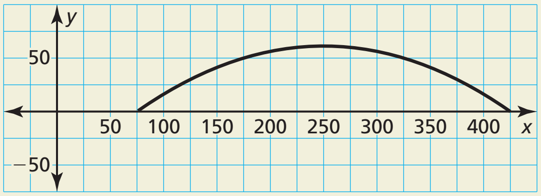
D.



91. **CRITICAL THINKING** Write a quadratic function represented by the table, if possible. If not, explain why.

x	-5	-3	-1	1
y	0	12	4	0

92. HOW DO YOU SEE IT? The graph shows the parabolic arch that supports the roof of a convention center, where x and y are measured in feet.



a. The arch can be represented by a function of the form $f(x) = a(x - p)(x - q)$. Estimate the values of p and q.

b. Estimate the width and height of the arch. Explain how you can use your height estimate to calculate a.

ANALYZING EQUATIONS **In Exercises 93 and 94, (a) rewrite the quadratic function in intercept form and (b) graph the function using any method. Explain the method you used.**

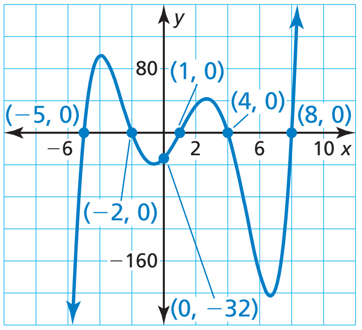
93. $f(x) = -3(x + 1)^2 + 27$

94. $g(x) = 2(x - 1)^2 - 2$

95. WRITING Can a quadratic function with exactly one real zero be written in intercept form? Explain.

96. MAKING AN ARGUMENT Your friend claims that any quadratic function can be written in standard form and in vertex form. Is your friend correct? Explain.

97. PROBLEM SOLVING Write the function represented by the graph in intercept form.



98. THOUGHT PROVOKING Sketch the graph of each function. Explain your procedure.

a. $f(x) = (x^2 - 1)(x^2 - 4)$

b. $g(x) = x(x^2 - 1)(x^2 - 4)$

99. REASONING Let k be a constant. Find the zeros of the function $f(x) = kx^2 - k^2x - 2k^3$ in terms of k.

PROBLEM SOLVING **In Exercises 100 and 101, write a system of two quadratic equations whose graphs intersect at the given points. Explain your reasoning.**

100. $(-4, 0)$ and $(2, 0)$

101. $(3, 6)$ and $(7, 6)$

Maintaining Mathematical Proficiency

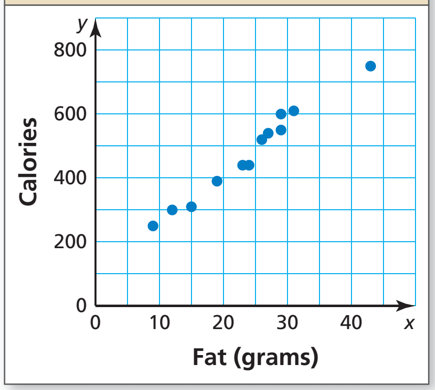
Reviewing what you learned in previous grades and lessons

The scatter plot shows the amounts x (in grams) of fat and the numbers y of calories in 12 burgers at a fast-food restaurant. *(Section 4.5)*

102. How many calories are in the burger that contains 12 grams of fat?

103. How many grams of fat are in the burger that contains 600 calories?

104. What tends to happen to the number of calories as the number of grams of fat increases?



Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. *(Section 6.5)*

105. 3, 11, 21, 33, 47, . . .

106. −2, −6, −18, −54, . . .

107. 26, 18, 10, 2, −6, . . .

108. 4, 5, 9, 14, 23, . . .

8.6 Comparing Linear, Exponential, and Quadratic Functions

Essential Question How can you compare the growth rates of linear, exponential, and quadratic functions?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.2.C
A.6.C
A.9.C

EXPLORATION 1 Comparing Speeds

Work with a partner. Three cars start traveling at the same time. The distance traveled in t minutes is y miles. Complete each table and sketch all three graphs in the same coordinate plane. Compare the speeds of the three cars. Which car has a constant speed? Which car is accelerating the most? Explain your reasoning.

t	$y = t$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

t	$y = 2^t - 1$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

t	$y = t^2$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

APPLYING MATHEMATICS

To be proficient in math, you need to visualize the results of varying assumptions, explore consequences, and compare predictions with data.

EXPLORATION 2 Comparing Speeds

Work with a partner. Analyze the speeds of the three cars over the given time periods. The distance traveled in t minutes is y miles. Which car eventually overtakes the others?

t	$y = t$
1	
2	
3	
4	
5	
6	
7	
8	
9	

t	$y = 2^t - 1$
1	
2	
3	
4	
5	
6	
7	
8	
9	

t	$y = t^2$
1	
2	
3	
4	
5	
6	
7	
8	
9	

Communicate Your Answer

3. How can you compare the growth rates of linear, exponential, and quadratic functions?

4. Which function has a growth rate that is eventually much greater than the growth rates of the other two functions? Explain your reasoning.

8.6 Lesson

Core Vocabulary

average rate of change, *p. 448*

Previous

zero of a function

slope

What You Will Learn

- Choose functions to model data.
- Write functions to model data.
- Compare functions using average rates of change.
- Solve real-life problems involving different function types.

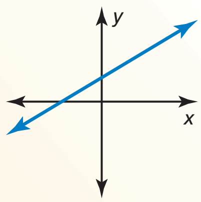
Choosing Functions to Model Data

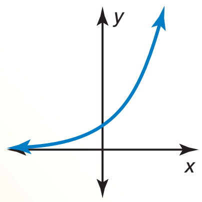
So far, you have studied linear functions, exponential functions, and quadratic functions. You can use these functions to model data.

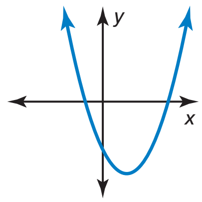
Core Concept

Linear, Exponential, and Quadratic Functions

Linear Function	Exponential Function	Quadratic Function
$y = mx + b$	$y = ab^x$	$y = ax^2 + bx + c$







EXAMPLE 1 **Using Graphs to Identify Functions**

Plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

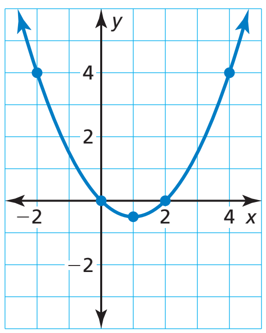
a. $(4, 4), (2, 0), (0, 0), \left(1, -\frac{1}{2}\right), (-2, 4)$

b. $(0, 1), (2, 4), (4, 7), (-2, -2), (-4, -5)$

c. $(0, 2), (2, 8), (1, 4), (-1, 1), \left(-2, \frac{1}{2}\right)$

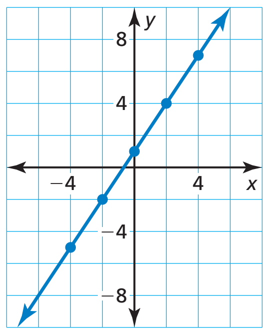
SOLUTION

a.



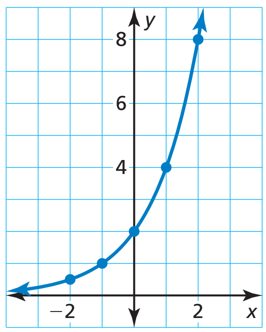
quadratic

b.



linear

c.



exponential

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Plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

1. $(-1, 5), (2, -1), (0, -1), (3, 5), (1, -3)$

2. $(-1, 2), (-2, 8), (-3, 32), \left(0, \frac{1}{2}\right), \left(1, \frac{1}{8}\right)$

3. $(-3, 5), (0, -1), (2, -5), (-4, 7), (1, -3)$

Core Concept

Differences and Ratios of Functions

You can use patterns between consecutive data pairs to determine which type of function models the data. The differences of consecutive y-values are called *first differences*. The differences of consecutive first differences are called *second differences*.

- **Linear Function** The first differences are constant.
- **Exponential Function** Consecutive y-values have a common *ratio*.
- **Quadratic Function** The second differences are constant.

In all cases, the differences of consecutive x-values need to be constant.

STUDY TIP

The first differences for exponential and quadratic functions are *not* constant.

EXAMPLE 2 Using Differences or Ratios to Identify Functions

STUDY TIP

First determine that the differences of consecutive x-values are constant. Then check whether the first differences are constant or consecutive y-values have a common ratio. If neither of these is true, check whether the second differences are constant.

Tell whether each table of values represents a *linear*, an *exponential*, or a *quadratic* function.

a.

x	−3	−2	−1	0	1
y	11	8	5	2	−1

b.

x	−2	−1	0	1	2
y	1	2	4	8	16

c.

x	−2	−1	0	1	2
y	−1	−2	−1	2	7

SOLUTION

a.

+1 +1 +1 +1

x	−3	−2	−1	0	1
y	11	8	5	2	−1

−3 −3 −3 −3

▶ The first differences are constant. So, the table represents a linear function.

b.

+1 +1 +1 +1

x	−2	−1	0	1	2
y	1	2	4	8	16

×2 ×2 ×2 ×2

▶ Consecutive y-values have a common ratio. So, the table represents an exponential function.

c.

+1 +1 +1 +1

x	−2	−1	0	1	2
y	−1	−2	−1	2	7

first differences → −1 +1 +3 +3

second differences → +2 +2 +2

▶ The second differences are constant. So, the table represents a quadratic function.

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4. Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function.

x	−1	0	1	2	3
y	1	3	9	27	81

Writing Functions to Model Data

EXAMPLE 3 Writing a Function to Model Data

x	2	4	6	8	10
y	12	0	−4	0	12

Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Then write the function.

SOLUTION

Step 1 Determine which type of function the table of values represents.

x	2	4	6	8	10
y	12	0	−4	0	12

+2 +2 +2 +2

first differences → −12 −4 +4 +12

second differences → +8 +8 +8

The second differences are constant. So, the table represents a quadratic function.

Step 2 Write an equation of the quadratic function. Using the table, notice that the x-intercepts are 4 and 8. So, use intercept form to write a function.

$y = a(x - 4)(x - 8)$ Substitute for p and q in intercept form.

Use another point from the table, such as (2, 12), to find a.

$12 = a(2 - 4)(2 - 8)$ Substitute 2 for x and 12 for y.

$1 = a$ Solve for a.

Use the value of a to write the function.

$y = (x - 4)(x - 8)$ Substitute 1 for a.

$= x^2 - 12x + 32$ Use the FOIL Method and combine like terms.

STUDY TIP

To check your function in Example 3, substitute the other points from the table to verify that they satisfy the function.

So, the quadratic function is $y = x^2 - 12x + 32$.

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x	−1	0	1	2	3
y	16	8	4	2	1

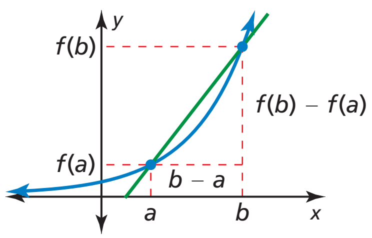
5. Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Then write the function.

Comparing Functions Using Average Rates of Change

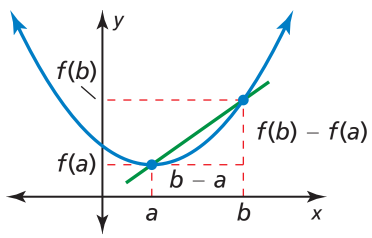
For nonlinear functions, the rate of change is not constant. You can compare two nonlinear functions over the same interval using their *average rates of change*. The **average rate of change** of a function $y = f(x)$ between $x = a$ and $x = b$ is the slope of the line through $(a, f(a))$ and $(b, f(b))$.

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

Exponential Function



Quadratic Function



Core Concept

Comparing Functions Using Average Rates of Change

- Over the same interval, the average rate of change of a function increasing quadratically eventually exceeds the average rate of change of a function increasing linearly. So, the value of the quadratic function eventually exceeds the value of the linear function.
- Over the same interval, the average rate of change of a function increasing exponentially eventually exceeds the average rate of change of a function increasing linearly or quadratically. So, the value of the exponential function eventually exceeds the value of the linear or quadratic function.

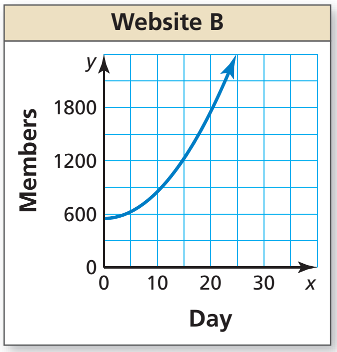
STUDY TIP

You can explore these concepts using a graphing calculator.

EXAMPLE 4 Using and Interpreting Average Rates of Change

Two social media websites open their memberships to the public. (a) Compare the websites by calculating and interpreting the average rates of change from Day 10 to Day 20. (b) Predict which website will have more members after 50 days. Explain.

Website A	
Day, x	**Members, y**
0	650
5	1025
10	1400
15	1775
20	2150
25	2525



SOLUTION

a. Calculate the average rates of change by using the points whose x-coordinates are 10 and 20.

Website A: Use (10, 1400) and (20, 2150).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} = \frac{2150 - 1400}{20 - 10} = \frac{750}{10} = 75$$

Website B: Use the graph to estimate the points when $x = 10$ and $x = 20$. Use (10, 850) and (20, 1800).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} \approx \frac{1800 - 850}{20 - 10} = \frac{950}{10} = 95$$

From Day 10 to Day 20, Website A membership increases at an average rate of 75 people per day, and Website B membership increases at an average rate of about 95 people per day. So, Website B membership is growing faster.

b. After 25 days, the memberships of both websites are about equal. Using the table, the average rates of change of Website A are constant, so membership will increase linearly. Using the graph, the average rates of change of Website B are increasing, so membership appears to increase quadratically or exponentially. So, Website B will have more members after 50 days.

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6. Compare the websites in Example 4 by calculating and interpreting the average rates of change from Day 0 to Day 10.

Solving Real-Life Problems

EXAMPLE 5 Comparing Different Function Types

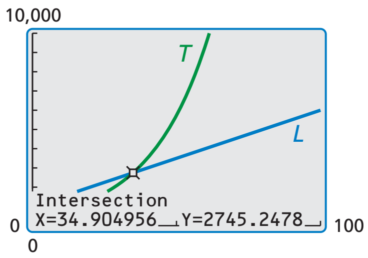
In 1900, Littleton had a population of 1000 people. Littleton's population increased by 50 people each year. In 1900, Tinyville had a population of 500 people. Tinyville's population increased by 5% each year.

a. In what year were the populations about equal?

b. Suppose Littleton's initial population doubled to 2000 and maintained a constant rate of increase of 50 people each year. Did Tinyville's population still catch up to Littleton's population? If so, in which year?

c. Suppose Littleton's rate of increase doubled to 100 people each year, in addition to doubling the initial population. Did Tinyville's population still catch up to Littleton's population? Explain.

SOLUTION



a. Let x represent the number of years since 1900. Write a function to model the population of each town.

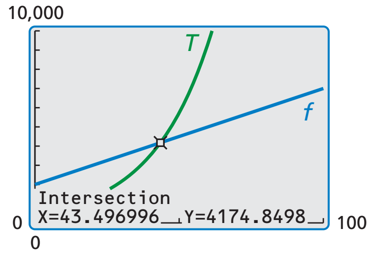
Littleton: $L(x) = 50x + 1000$ Linear function

Tinyville: $T(x) = 500(1.05)^x$ Exponential function

Use a graphing calculator to graph each function in the same viewing window. Use the *intersect* feature to find the value of x for which $L(x) \approx T(x)$. The graphs intersect when $x \approx 34.9$.

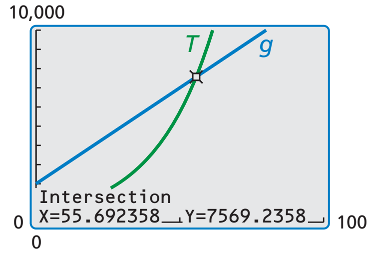
▶ So, the populations were about equal in 1934.

b. Littleton's new population function is $f(x) = 50x + 2000$. Use a graphing calculator to graph f and T in the same viewing window. Use the *intersect* feature to find the value of x for which $f(x) \approx T(x)$. The graphs intersect when $x \approx 43.5$.



▶ So, Tinyville's population caught Littleton's population in 1943.

c. Littleton's new population function is $g(x) = 100x + 2000$. Use a graphing calculator to graph g and T in the same viewing window. Use the *intersect* feature to find the value of x for which $g(x) \approx T(x)$. The graphs intersect when $x \approx 55.7$.



▶ So, Tinyville's population caught Littleton's population in 1955. Because Littleton's population increased linearly and Tinyville's population increased exponentially, Tinyville's population eventually exceeded Littleton's regardless of Littleton's constant rate or initial value.

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7. WHAT IF? Tinyville's population increased by 8% each year. In what year were the populations about equal?

8.6 Exercises

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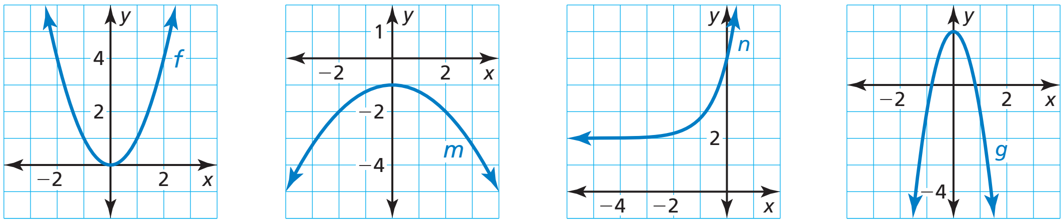
Vocabulary and Core Concept Check

1. **WRITING** Name three types of functions that you can use to model data. Describe the equation and graph of each function.

2. **WRITING** How can you decide whether to use a linear, an exponential, or a quadratic function to model a data set?

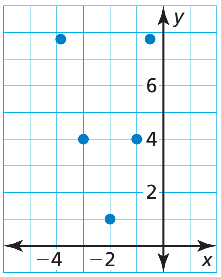
3. **VOCABULARY** Describe how to find the average rate of change of a function $y = f(x)$ between $x = a$ and $x = b$.

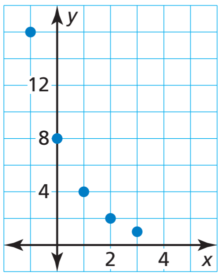
4. **WHICH ONE DOESN'T BELONG?** Which graph does *not* belong with the other three? Explain your reasoning.

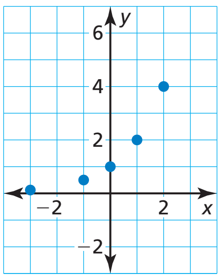


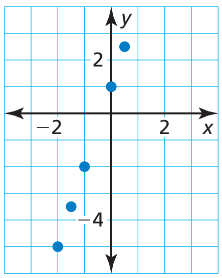
Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

5. 

6. 

7. 

8. 

In Exercises 9–14, plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function. *(See Example 1.)*

9. $(-2, -1), (-1, 0), (1, 2), (2, 3), (0, 1)$

10. $\left(0, \frac{1}{4}\right), (1, 1), (2, 4), (3, 16), \left(-1, \frac{1}{16}\right)$

11. $(0, -3), (1, 0), (2, 9), (-2, 9), (-1, 0)$

12. $(-1, -3), (-3, 5), (0, -1), (1, 5), (2, 15)$

13. $(-4, -4), (-2, -3.4), (0, -3), (2, -2.6), (4, -2)$

14. $(0, 8), (-4, 0.25), (-3, 0.4), (-2, 1), (-1, 3)$

In Exercises 15–18, tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. *(See Example 2.)*

15.

x	−2	−1	0	1	2
y	0	0.5	1	1.5	2

16.

x	−1	0	1	2	3
y	0.2	1	5	25	125

17.

x	2	3	4	5	6
y	2	6	18	54	162

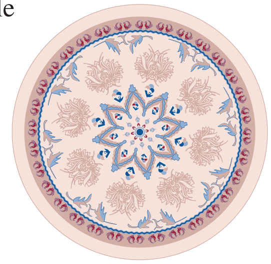
18.

x	−3	−2	−1	0	1
y	2	4.5	8	12.5	18

19. **MODELING WITH MATHEMATICS** A student takes a subway to a public library. The table shows the distances d (in miles) the student travels in t minutes. Let the time t represent the independent variable. Tell whether the data can be modeled by a *linear*, an *exponential*, or a *quadratic* function. Explain.

Time, t	0.5	1	3	5
Distance, d	0.335	0.67	2.01	3.35

20. **MODELING WITH MATHEMATICS** A store sells custom circular rugs. The table shows the costs c (in dollars) of rugs that have diameters of d feet. Let the diameter d represent the independent variable. Tell whether the data can be modeled by a *linear*, an *exponential*, or a *quadratic* function. Explain.



Diameter, d	3	4	5	6
Cost, c	63.90	113.60	177.50	255.60

In Exercises 21–26, tell whether the data represent a *linear*, an *exponential*, or a *quadratic* function. Then write the function. *(See Example 3.)*

21. $(-2, 8), (-1, 0), (0, -4), (1, -4), (2, 0), (3, 8)$

22. $(-3, 8), (-2, 4), (-1, 2), (0, 1), (1, 0.5)$

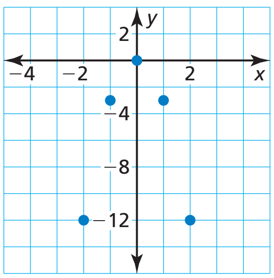
23.

x	-2	-1	0	1	2
y	4	1	-2	-5	-8

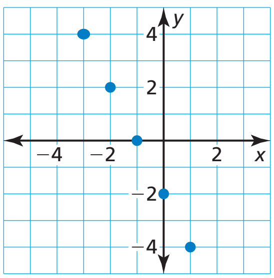
24.

x	-1	0	1	2	3
y	2.5	5	10	20	40

25.



26.



27. **ERROR ANALYSIS** Describe and correct the error in determining whether the table represents a linear, an exponential, or a quadratic function.

$+1 \quad +1 \quad +1 \quad +1$

x	1	2	3	4	5
y	3	9	27	81	243

$\times 3 \quad \times 3 \quad \times 3 \quad \times 3$

Consecutive y-values change by a constant amount. So, the table represents a linear function.

28. **ERROR ANALYSIS** Describe and correct the error in writing the function represented by the table.

$+1 \quad +1 \quad +1 \quad +1$

x	-3	-2	-1	0	1
y	4	0	-2	-2	0

first differences → $-4 \quad -2 \quad +0 \quad +2$

second differences → $+2 \quad +2 \quad +2$

The table represents a quadratic function.

$$f(x) = a(x-2)(x-1)$$
$$4 = a(-3-2)(-3-1)$$
$$\frac{1}{5} = a$$
$$f(x) = \frac{1}{5}(x-2)(x-1)$$
$$= \frac{1}{5}x^2 - \frac{3}{5}x + \frac{2}{5}$$

So, the function is $f(x) = \frac{1}{5}x^2 - \frac{3}{5}x + \frac{2}{5}$.

29. **REASONING** The table shows the numbers of people attending the first five football games at a high school.

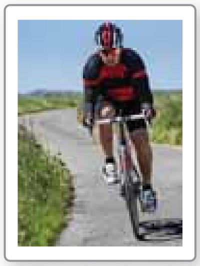
Game, g	1	2	3	4	5
People, p	252	325	270	249	310

a. Plot the points. Let the game g represent the independent variable.

b. Can a linear, an exponential, or a quadratic function represent this situation? Explain.

30. **MODELING WITH MATHEMATICS** The table shows the breathing rates y (in liters of air per minute) of a cyclist traveling at different speeds x (in miles per hour).

Speed, x	20	21	22	23	24
Breathing rate, y	51.4	57.1	63.3	70.3	78.0



a. Plot the points. Let the speed x represent the independent variable. Then determine the type of function that best represents this situation.

b. Write a function that models the data.

c. Find the breathing rate of a cyclist traveling 18 miles per hour. Round your answer to the nearest tenth.

31. **ANALYZING RATES OF CHANGE** The function $f(t) = -16t^2 + 48t + 3$ represents the height (in feet) of a volleyball t seconds after it is hit into the air.

a. Copy and complete the table.

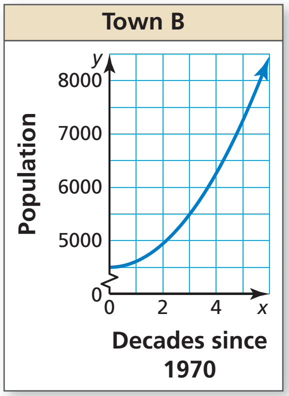
t	0	0.5	1	1.5	2	2.5	3
$f(t)$							

b. Plot the ordered pairs and draw a smooth curve through the points.

c. Describe where the function is increasing and decreasing.

d. Find the average rate of change for each 0.5-second interval in the table. What do you notice about the average rates of change when the function is increasing? decreasing?

32. **ANALYZING RELATIONSHIPS** The population of Town A in 1970 was 3000. The population of Town A increased by 20% every decade. Let x represent the number of decades since 1970. The graph shows the population of Town B. *(See Example 4.)*

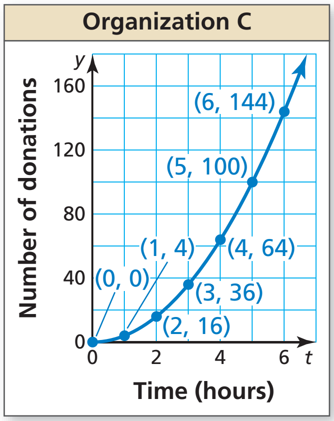


a. Compare the populations of the towns by calculating and interpreting the average rates of change from 1990 to 2010.

b. Predict which town will have a greater population after 2020. Explain.

33. **ANALYZING RELATIONSHIPS** Three organizations are collecting donations for a cause. Organization A begins with one donation, and the number of donations quadruples each hour. The table shows the numbers of donations collected by Organization B. The graph shows the numbers of donations collected by Organization C.

Time (hours), t	Number of donations, y
0	0
1	4
2	8
3	12
4	16
5	20
6	24



a. What type of function represents the numbers of donations collected by Organization A? B? C?

b. Find the average rates of change of each function for each 1-hour interval from $t = 0$ to $t = 6$.

c. For which function does the average rate of change increase most quickly? What does this tell you about the numbers of donations collected by the three organizations?

34. **COMPARING FUNCTIONS** The room expenses for two different resorts are shown. *(See Example 5.)*

a. For what length of vacation does each resort cost about the same?

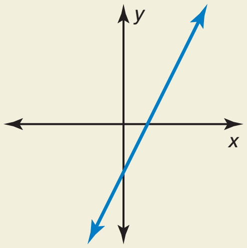
b. Suppose Blue Water Resort charges \$1450 for the first three nights and \$105 for each additional night. Would Sea Breeze Resort ever be more expensive than Blue Water Resort? Explain.

c. Suppose Sea Breeze Resort charges \$1200 for the first three nights. The charge increases 10% for each additional night. Would Blue Water Resort ever be more expensive than Sea Breeze Resort? Explain.

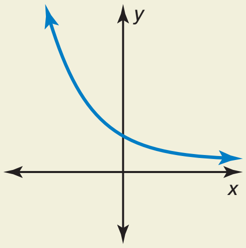
35. REASONING Explain why the average rate of change of a linear function is constant and the average rate of change of a quadratic or exponential function is not constant.

36. HOW DO YOU SEE IT? Match each graph with its function. Explain your reasoning.

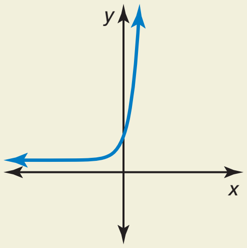
a.



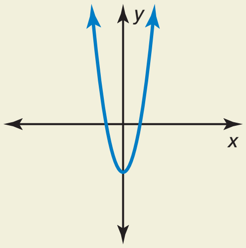
b.



c.



d.



A. $y = 2x^2 - 4$ **B.** $y = 2(4)^x + 1$

C. $y = 2\left(\frac{3}{4}\right)^x + 1$ **D.** $y = 2x - 4$

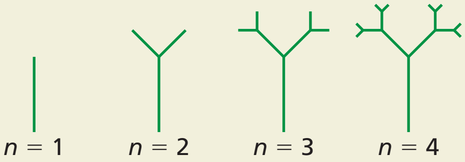
37. CRITICAL THINKING In the ordered pairs below, the y-values are given in terms of n. Tell whether the ordered pairs represent a *linear*, an *exponential*, or a *quadratic* function. Explain.

$(1, 3n - 1)$, $(2, 10n + 2)$, $(3, 26n)$,
$(4, 51n - 7)$, $(5, 85n - 19)$

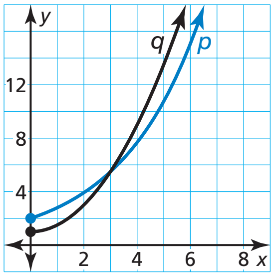
38. USING STRUCTURE Write a function that has constant second differences of 3.

39. CRITICAL THINKING Is the graph of a set of points enough to determine whether the points represent a linear, an exponential, or a quadratic function? Justify your answer.

40. THOUGHT PROVOKING Find four different patterns in the figure. Determine whether each pattern represents a *linear*, an *exponential*, or a *quadratic* function. Write a model for each pattern.



41. MAKING AN ARGUMENT Function p is an exponential function and function q is a quadratic function. Your friend says that after about $x = 3$, function q will always have a greater y-value than function p. Is your friend correct? Explain.



42. USING TOOLS The table shows the amount a (in billions of dollars) United States residents spent on pets or pet-related products and services each year for a 5-year period. Let the year x represent the independent variable. Using technology, find a function that models the data. How did you choose the model? Predict how much residents will spend on pets or pet-related products and services in Year 7.

Year, x	1	2	3	4	5
Amount, a	53.1	56.9	61.8	65.7	67.1

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the expression. *(Section 6.2)*

43. $\sqrt{121}$

44. $\sqrt[3]{125}$

45. $\sqrt[3]{512}$

46. $\sqrt[5]{243}$

Find the product. *(Section 7.3)*

47. $(x + 8)(x - 8)$

48. $(4y + 2)(4y - 2)$

49. $(3a - 5b)(3a + 5b)$

50. $(-2r + 6s)(-2r - 6s)$

8.4–8.6 What Did You Learn?

Core Vocabulary

even function, *p. 428*
odd function, *p. 428*
vertex form (of a quadratic function), *p. 430*
intercept form, *p. 436*
average rate of change, *p. 448*

Core Concepts

Section 8.4

Even and Odd Functions, *p. 428*
Graphing $f(x) = a(x - h)^2$, *p. 429*
Graphing $f(x) = a(x - h)^2 + k$, *p. 430*
Writing Quadratic Functions of the Form $f(x) = a(x - h)^2 + k$, *p. 431*

Section 8.5

Graphing $f(x) = a(x - p)(x - q)$, *p. 436*
Factors and Zeros, *p. 438*
Graphing and Writing Quadratic Functions, *p. 438*
Graphing and Writing Cubic Functions, *p. 440*

Section 8.6

Linear, Exponential, and Quadratic Functions, *p. 446*
Differences and Ratios of Functions, *p. 447*
Writing Functions to Model Data, *p. 448*
Comparing Functions Using Average Rates of Change, *p. 449*

Mathematical Thinking

1. How can you use technology to confirm your answer in Exercise 64 on page 434?
2. How did you use the structure of the equation in Exercise 85 on page 443 to solve the problem?
3. Describe why your answer makes sense considering the context of the data in Exercise 20 on page 452.

Performance Task

Asteroid Aim

Apps take a long time to design and program. One app in development is a game in which players shoot lasers at asteroids. They score points based on the number of hits per shot. The designer wants your feedback. Do you think students will like the game and want to play it? What changes would improve it?

To explore the answers to these questions and more, go to ***BigIdeasMath.com***.

8 Chapter Review

8.1 Graphing $f(x) = ax^2$ (pp. 405–410)

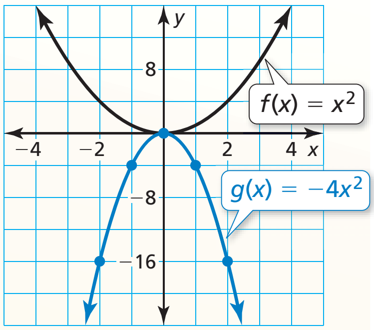
Graph $g(x) = -4x^2$. Compare the graph to the graph of $f(x) = x^2$.

Step 1 Make a table of values.

x	-2	-1	0	1	2
$g(x)$	-16	-4	0	-4	-16

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

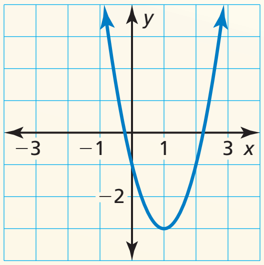


▶ The graphs have the same vertex, (0, 0), and the same axis of symmetry, $x = 0$, but the graph of g opens down and is narrower than the graph of f. So, the graph of g is a vertical stretch by a factor of 4 and a reflection in the x-axis of the graph of f.

Graph the function. Compare the graph to the graph of $f(x) = x^2$.

1. $p(x) = 7x^2$ **2.** $g(x) = -\frac{3}{4}x^2$ **3.** $h(x) = -6x^2$ **4.** $q(x) = \left(\frac{3}{2}x\right)^2$

5. Identify characteristics of the quadratic function and its graph.



8.2 Graphing $f(x) = ax^2 + c$ (pp. 411–416)

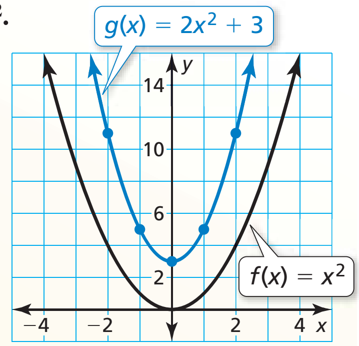
Graph $g(x) = 2x^2 + 3$. Compare the graph to the graph of $f(x) = x^2$.

Step 1 Make a table of values.

x	-2	-1	0	1	2
$g(x)$	11	5	3	5	11

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



▶ Both graphs open up and have the same axis of symmetry, $x = 0$. The graph of g is narrower, and its vertex, (0, 3), is above the vertex of the graph of f, (0, 0). So, the graph of g is a vertical stretch by a factor of 2 and a vertical translation 3 units up of the graph of f.

Graph the function. Compare the graph to the graph of $f(x) = x^2$.

6. $g(x) = x^2 + 5$ **7.** $h(x) = -x^2 - 4$ **8.** $m(x) = -2x^2 + 6$ **9.** $n(x) = \frac{1}{3}x^2 - 5$

8.3 Graphing $f(x) = ax^2 + bx + c$ (pp. 417–424)

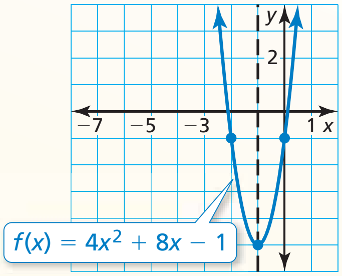
Graph $f(x) = 4x^2 + 8x - 1$. Describe the domain and range.

Step 1 Find and graph the axis of symmetry: $x = -\frac{b}{2a} = -\frac{8}{2(4)} = -1$.

Step 2 Find and plot the vertex. The axis of symmetry is $x = -1$. So, the x-coordinate of the vertex is -1. The y-coordinate of the vertex is $f(-1) = 4(-1)^2 + 8(-1) - 1 = -5$. So, the vertex is $(-1, -5)$.

Step 3 Use the y-intercept to find two more points on the graph. Because $c = -1$, the y-intercept is -1. So, $(0, -1)$ lies on the graph. Because the axis of symmetry is $x = -1$, the point $(-2, -1)$ also lies on the graph.

Step 4 Draw a smooth curve through the points.



▶ The domain is all real numbers. The range is $y \geq -5$.

Graph the function. Describe the domain and range.

10. $y = x^2 - 2x + 7$

11. $f(x) = -3x^2 + 3x - 4$

12. $y = \frac{1}{2}x^2 - 6x + 10$

13. The function $f(t) = -16t^2 + 88t + 12$ represents the height (in feet) of a pumpkin t seconds after it is launched from a catapult. When does the pumpkin reach its maximum height? What is the maximum height of the pumpkin?

8.4 Graphing $f(x) = a(x - h)^2 + k$ (pp. 427–434)

Determine whether $f(x) = 2x^2 + 4$ is *even*, *odd*, or *neither*.

$f(x) = 2x^2 + 4$ — Write the original function.

$f(-x) = 2(-x)^2 + 4$ — Substitute $-x$ for x.

$= 2x^2 + 4$ — Simplify.

$= f(x)$ — Substitute $f(x)$ for $2x^2 + 4$.

▶ Because $f(-x) = f(x)$, the function is even.

Determine whether the function is *even*, *odd*, or *neither*.

14. $w(x) = 5^x$

15. $r(x) = -8x$

16. $h(x) = 3x^2 - 2x$

Graph the function. Compare the graph to the graph of $f(x) = x^2$.

17. $h(x) = 2(x - 4)^2$

18. $g(x) = \frac{1}{2}(x - 1)^2 + 1$

19. $q(x) = -(x + 4)^2 + 7$

20. Consider the function $g(x) = -3(x + 2)^2 - 4$. Graph $h(x) = g(x - 1)$.

21. Write a quadratic function whose graph has a vertex of (3, 2) and passes through the point (4, 7).

8.5 Using Intercept Form (pp. 435–444)

Use zeros to graph $h(x) = x^2 - 7x + 6$.

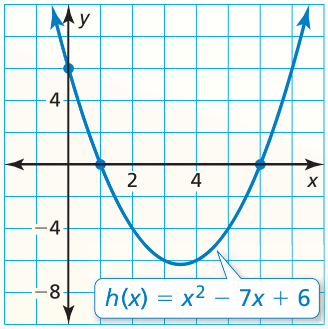
The function is in standard form. The parabola opens up ($a > 0$), and the y-intercept is 6. So, plot (0, 6).

The polynomial that defines the function is factorable. So, write the function in intercept form and identify the zeros.

$$h(x) = x^2 - 7x + 6 \qquad \text{Write the function.}$$

$$= (x - 6)(x - 1) \qquad \text{Factor the trinomial.}$$

The zeros of the function are 1 and 6. So, plot (1, 0) and (6, 0). Draw a parabola through the points.



Graph the quadratic function. Label the vertex, axis of symmetry, and x-intercepts. Describe the domain and range of the function.

22. $y = (x - 4)(x + 2)$

23. $f(x) = -3(x + 3)(x + 1)$

24. $y = x^2 - 8x + 15$

Use zeros to graph the function.

25. $y = -2x^2 + 6x + 8$

26. $f(x) = x^2 + x - 2$

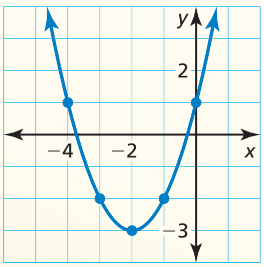
27. $f(x) = 2x^3 - 18x$

28. Write a quadratic function in standard form whose graph passes through (4, 0) and (6, 0).

8.6 Comparing Linear, Exponential, and Quadratic Functions (pp. 445–454)

Tell whether the data represent a *linear*, an *exponential*, or a *quadratic* function.

a. $(-4, 1), (-3, -2), (-2, -3)$
$(-1, -2), (0, 1)$



▶ The points appear to represent a quadratic function.

b.

x	−1	0	1	2	3
y	15	8	1	−6	−13

+1 +1 +1 +1

x	−1	0	1	2	3
y	15	8	1	−6	−13

−7 −7 −7 −7

▶ The first differences are constant. So, the table represents a linear function.

29. Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Then write the function.

x	−1	0	1	2	3
y	512	128	32	8	2

30. The balance y (in dollars) of your savings account after t years is represented by $y = 200(1.1)^t$. The beginning balance of your friend's account is \$250, and the balance increases by \$20 each year. (a) Compare the account balances by calculating and interpreting the average rates of change from $t = 2$ to $t = 7$. (b) Predict which account will have a greater balance after 10 years. Explain.

8 Chapter Test

Graph the function. Compare the graph to the graph of $f(x) = x^2$.

1. $h(x) = 2x^2 - 3$

2. $g(x) = -\frac{1}{2}x^2$

3. $p(x) = \frac{1}{2}(x + 1)^2 - 1$

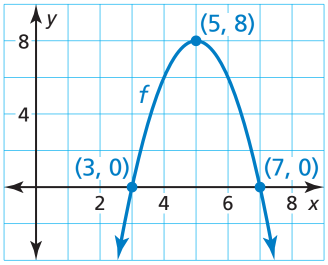
4. Consider the graph of the function f.

a. Find the domain, range, and zeros of the function.

b. Write the function f in standard form.

c. Compare the graph of f to the graph of $g(x) = x^2$.

d. Graph $h(x) = f(x - 6)$.



Use zeros to graph the function. Describe the domain and range of the function.

5. $f(x) = 2x^2 - 8x + 8$

6. $y = -(x + 5)(x - 1)$

7. $h(x) = 16x^2 - 4$

Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Explain your reasoning. Then write the function.

8.

x	-1	0	1	2	3
y	4	8	16	32	64

9.

x	-2	-1	0	1	2
y	-8	-2	0	-2	-8

Write a quadratic function in standard form whose graph satisfies the given conditions. Explain the process you used.

10. passes through $(-8, 0)$, $(-2, 0)$, and $(-6, 4)$

11. passes through $(0, 0)$, $(10, 0)$, and $(9, -27)$

12. is even and has a range of $y \geq 3$

13. passes through $(4, 0)$ and $(1, 9)$

14. The table shows the distances d (in miles) that Earth moves in its orbit around the Sun after t seconds. Let the time t be the independent variable. Tell whether the data can be modeled by a *linear*, an *exponential*, or a *quadratic* function. Explain. Then write a function that models the data.

Time, t	1	2	3	4	5
Distance, d	19	38	57	76	95

15. You are playing tennis with a friend. The path of the tennis ball after you return a serve can be modeled by the function $y = -0.005x^2 + 0.17x + 3$, where x is the horizontal distance (in feet) from where you hit the ball and y is the height (in feet) of the ball.

a. What is the maximum height of the tennis ball?

b. You are standing 30 feet from the net, which is 3 feet high. Will the ball clear the net? Explain your reasoning.

16. Find values of a, b, and c so that the function $f(x) = ax^2 + bx + c$ is (a) even, (b) odd, and (c) neither even nor odd.

17. Consider the function $f(x) = x^2 + 4$. Find the average rate of change from $x = 0$ to $x = 1$, from $x = 1$ to $x = 2$, and from $x = 2$ to $x = 3$. What do you notice about the average rates of change when the function is increasing?

8 Standards Assessment

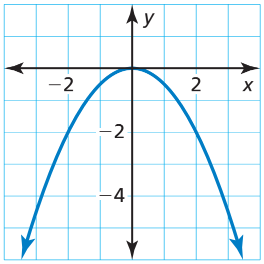
1. Which function is represented by the graph? *(TEKS A.6.C)*

Ⓐ $y = \frac{1}{2}x^2$

Ⓑ $y = 2x^2$

Ⓒ $y = -\frac{1}{2}x^2$

Ⓓ $y = -2x^2$



2. Which of the following expressions are equivalent to $(b^{-5})^{-4}$? *(TEKS A.11.B)*

I. b^{-20}

II. $\dfrac{b^6}{b^{-14}}$

III. $b^{12}b^8$

IV. $(b^{-4})^{-5}$

Ⓕ I and IV only

Ⓖ II and III only

Ⓗ III and IV only

Ⓙ II, III, and IV only

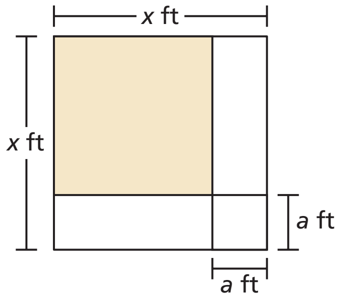
3. Which polynomial represents the area (in square feet) of the shaded region of the figure? *(TEKS A.10.B)*

Ⓐ $a^2 - x^2$

Ⓑ $x^2 - a^2$

Ⓒ $x^2 - 2ax + a^2$

Ⓓ $x^2 + 2ax + a^2$



4. **GRIDDED ANSWER** A quadratic function whose graph has a vertex of $(-2, 8)$ and passes through the point $(-4, -4)$ can be written in the standard form $f(x) = -3x^2 - 12x + c$. What is the value of c? *(TEKS A.6.B)*

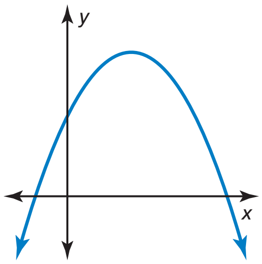
5. Which function could be represented by the graph? *(TEKS A.7.B)*

Ⓕ $y = -(x + 1)(x - 5)$

Ⓖ $y = 3(x + 1)(x - 5)$

Ⓗ $y = \frac{1}{2}(x - 1)(x - 5)$

Ⓙ $y = -2(x + 1)(x + 5)$



6. Which expression represents the sum of $3x^2 + 9x - 8$ and $-5(x^2 + 8 - 3x)$? *(TEKS A.10.A)*

Ⓐ $-2x^2 + 6x$

Ⓑ $-2x^2 + 24x - 48$

Ⓒ $-2x^2 - 6x + 32$

Ⓓ $8x^2 - 6x + 32$

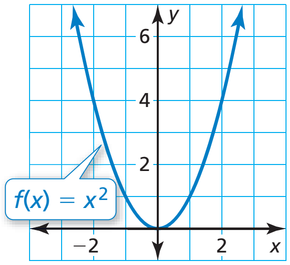
7. The graph of $f(x) = x^2$ is shown. Which statement describes the graph of $g(x) = -2f(x)$? *(TEKS A.7.C)*

Ⓕ The graph of g opens down and is narrower than the graph of f.

Ⓖ The graph of g opens down and is wider than the graph of f.

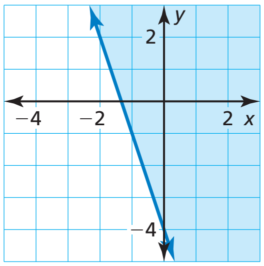
Ⓗ The graph of g opens up and is narrower than the graph of f.

Ⓙ The graph of g opens up and is wider than the graph of f.

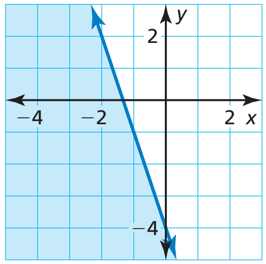


8. Which graph represents the solution of $-3y - 9x \leq 12$? *(TEKS A.3.D)*

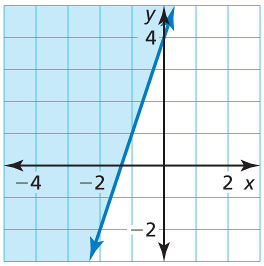
Ⓐ



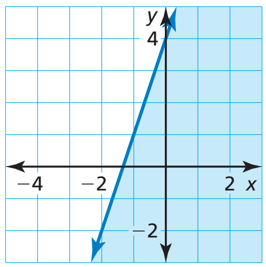
Ⓑ



Ⓒ



Ⓓ



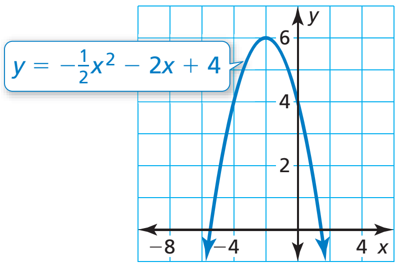
9. What is the domain and range of the quadratic function represented by the graph? *(TEKS A.6.A)*

Ⓕ domain: all real numbers; range: $y \leq 6$

Ⓖ domain: all real numbers; range: $y < 6$

Ⓗ domain: $x > -2$; range: all real numbers

Ⓙ none of the above



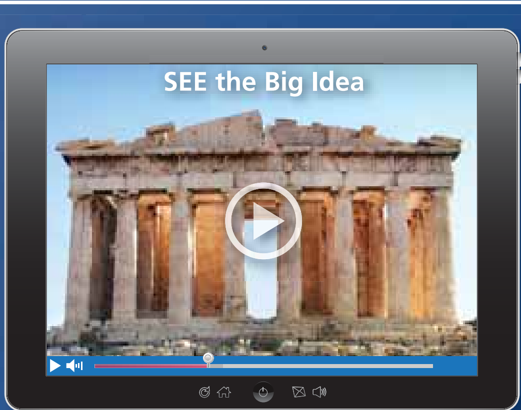
9 Solving Quadratic Equations

Dolphin *(p. 509)*

Half-pipe *(p. 501)*

Pond *(p. 489)*

Kicker *(p. 479)*



Parthenon *(p. 469)*

Mathematical Thinking: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

Maintaining Mathematical Proficiency

Factoring Perfect Square Trinomials (A.10.E)

Example 1 **Factor $x^2 + 14x + 49$.**

$$x^2 + 14x + 49 = x^2 + 2(x)(7) + 7^2 \quad \text{Write as } a^2 + 2ab + b^2.$$

$$= (x + 7)^2 \quad \text{Perfect square trinomial pattern}$$

Factor the trinomial.

1. $x^2 + 10x + 25$ **2.** $x^2 - 20x + 100$ **3.** $x^2 + 12x + 36$

4. $x^2 - 18x + 81$ **5.** $x^2 + 16x + 64$ **6.** $x^2 - 30x + 225$

Solving Systems of Linear Equations by Graphing (A.5.C)

Example 2 **Solve the system of linear equations by graphing.**

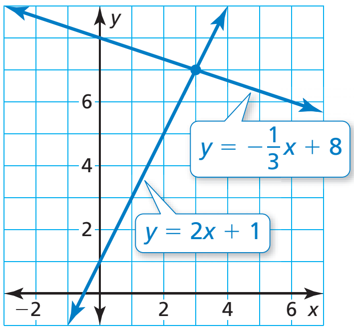
$y = 2x + 1$ Equation 1

$y = -\frac{1}{3}x + 8$ Equation 2

Step 1 Graph each equation.

Step 2 Estimate the point of intersection. The graphs appear to intersect at (3, 7).

Step 3 Check your point from Step 2.



Equation 1	Equation 2
$y = 2x + 1$	$y = -\frac{1}{3}x + 8$
$7 \stackrel{?}{=} 2(3) + 1$	$7 \stackrel{?}{=} -\frac{1}{3}(3) + 8$
$7 = 7$ ✓	$7 = 7$ ✓

▶ The solution is (3, 7).

Solve the system of linear equations by graphing.

7. $y = -5x + 3$
$y = 2x - 4$

8. $y = \frac{3}{2}x - 2$
$y = -\frac{1}{4}x + 5$

9. $y = \frac{1}{2}x + 4$
$y = -3x - 3$

10. **ABSTRACT REASONING** What value of c makes $x^2 + bx + c$ a perfect square trinomial?

Mathematical Thinking

Mathematically proficient students use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution. **(A.1.B)**

Problem-Solving Strategies

Core Concept

Guess, Check, and Revise

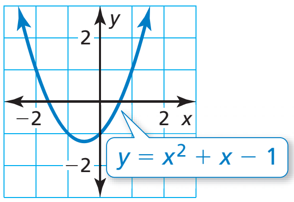
When solving a problem in mathematics, it is often helpful to estimate a solution and then observe how close that solution is to being correct. For instance, you can use the guess, check, and revise strategy to find a decimal approximation of the square root of 2.

	Guess	Check	How to revise
1.	1.4	$1.4^2 = 1.96$	Increase guess.
2.	1.41	$1.41^2 = 1.9881$	Increase guess.
3.	1.415	$1.415^2 = 2.002225$	Decrease guess.

By continuing this process, you can determine that the square root of 2 is approximately 1.4142.

EXAMPLE 1 Approximating a Solution of an Equation

The graph of $y = x^2 + x - 1$ is shown. Approximate the positive solution of the equation $x^2 + x - 1 = 0$ to the nearest thousandth.



SOLUTION

Using the graph, you can make an initial estimate of the positive solution to be $x = 0.65$.

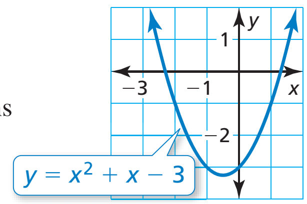
	Guess	Check	How to revise
1.	0.65	$0.65^2 + 0.65 - 1 = 0.0725$	Decrease guess.
2.	0.62	$0.62^2 + 0.62 - 1 = 0.0044$	Decrease guess.
3.	0.618	$0.618^2 + 0.618 - 1 = -0.000076$	Increase guess.
4.	0.6181	$0.6181^2 + 0.6181 - 1 \approx 0.00015$	The solution is between 0.618 and 0.6181.

▶ So, to the nearest thousandth, the positive solution of the equation is $x = 0.618$.

Monitoring Progress

1. Use the graph in Example 1 to approximate the negative solution of the equation $x^2 + x - 1 = 0$ to the nearest thousandth.

2. The graph of $y = x^2 + x - 3$ is shown. Approximate both solutions of the equation $x^2 + x - 3 = 0$ to the nearest thousandth.



9.1 Properties of Radicals

Essential Question

How can you multiply and divide square roots?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.11.A

EXPLORATION 1 Operations with Square Roots

Work with a partner. For each operation with square roots, compare the results obtained using the two indicated orders of operations. What can you conclude?

a. Square Roots and Addition

Is $\sqrt{36} + \sqrt{64}$ equal to $\sqrt{36 + 64}$?

In general, is $\sqrt{a} + \sqrt{b}$ equal to $\sqrt{a + b}$? Explain your reasoning.

b. Square Roots and Multiplication

Is $\sqrt{4} \cdot \sqrt{9}$ equal to $\sqrt{4 \cdot 9}$?

In general, is $\sqrt{a} \cdot \sqrt{b}$ equal to $\sqrt{a \cdot b}$? Explain your reasoning.

c. Square Roots and Subtraction

Is $\sqrt{64} - \sqrt{36}$ equal to $\sqrt{64 - 36}$?

In general, is $\sqrt{a} - \sqrt{b}$ equal to $\sqrt{a - b}$? Explain your reasoning.

d. Square Roots and Division

Is $\dfrac{\sqrt{100}}{\sqrt{4}}$ equal to $\sqrt{\dfrac{100}{4}}$?

In general, is $\dfrac{\sqrt{a}}{\sqrt{b}}$ equal to $\sqrt{\dfrac{a}{b}}$? Explain your reasoning.

REASONING

To be proficient in math, you need to recognize and use counterexamples.

EXPLORATION 2 Writing Counterexamples

Work with a partner. A **counterexample** is an example that proves that a general statement is *not* true. For each general statement in Exploration 1 that is not true, write a counterexample different from the example given.

Communicate Your Answer

3. How can you multiply and divide square roots?

4. Give an example of multiplying square roots and an example of dividing square roots that are different from the examples in Exploration 1.

5. Write an algebraic rule for each operation.

a. the product of square roots

b. the quotient of square roots

9.1 Lesson

Core Vocabulary

counterexample, *p. 465*
radical expression, *p. 466*
simplest form, *p. 466*
rationalizing the denominator, *p. 468*
conjugates, *p. 468*
like radicals, *p. 470*

Previous
radicand
perfect cube

What You Will Learn

- Use properties of radicals to simplify expressions.
- Simplify expressions by rationalizing the denominator.
- Perform operations with radicals.

Using Properties of Radicals

A **radical expression** is an expression that contains a radical. A radical with index n is in **simplest form** when these three conditions are met.

- No radicands have perfect nth powers as factors other than 1.
- No radicands contain fractions.
- No radicals appear in the denominator of a fraction.

You can use the property below to simplify radical expressions involving square roots.

Core Concept

Product Property of Square Roots

Words The square root of a product equals the product of the square roots of the factors.

Numbers $\sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$

Algebra $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, where $a, b \geq 0$

STUDY TIP

There can be more than one way to factor a radicand. An efficient method is to find the greatest perfect square factor.

EXAMPLE 1 Using the Product Property of Square Roots

a. $\sqrt{108} = \sqrt{36 \cdot 3}$ Factor using the greatest perfect square factor.

$= \sqrt{36} \cdot \sqrt{3}$ Product Property of Square Roots

$= 6\sqrt{3}$ Simplify.

b. $\sqrt{9x^3} = \sqrt{9 \cdot x^2 \cdot x}$ Factor using the greatest perfect square factor.

$= \sqrt{9} \cdot \sqrt{x^2} \cdot \sqrt{x}$ Product Property of Square Roots

$= 3x\sqrt{x}$ Simplify.

STUDY TIP

In this course, whenever a variable appears in the radicand, assume that it has only *nonnegative* values.

Monitoring Progress

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Simplify the expression.

1. $\sqrt{24}$ **2.** $-\sqrt{80}$ **3.** $\sqrt{49x^3}$ **4.** $\sqrt{75n^5}$

Core Concept

Quotient Property of Square Roots

Words The square root of a quotient equals the quotient of the square roots of the numerator and denominator.

Numbers $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$

Algebra $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, where $a \geq 0$ and $b > 0$

EXAMPLE 2 Using the Quotient Property of Square Roots

a. $\sqrt{\dfrac{15}{64}} = \dfrac{\sqrt{15}}{\sqrt{64}}$ — Quotient Property of Square Roots

$= \dfrac{\sqrt{15}}{8}$ — Simplify.

b. $\sqrt{\dfrac{81}{x^2}} = \dfrac{\sqrt{81}}{\sqrt{x^2}}$ — Quotient Property of Square Roots

$= \dfrac{9}{x}$ — Simplify.

You can extend the Product and Quotient Properties of Square Roots to other radicals, such as cube roots. When using these *properties of cube roots*, the radicands may contain negative numbers.

EXAMPLE 3 Using Properties of Cube Roots

STUDY TIP

To write a cube root in simplest form, find factors of the radicand that are perfect cubes.

a. $\sqrt[3]{-128} = \sqrt[3]{-64 \cdot 2}$ — Factor using the greatest perfect cube factor.

$= \sqrt[3]{-64} \cdot \sqrt[3]{2}$ — Product Property of Cube Roots

$= -4\sqrt[3]{2}$ — Simplify.

b. $\sqrt[3]{125x^7} = \sqrt[3]{125 \cdot x^6 \cdot x}$ — Factor using the greatest perfect cube factors.

$= \sqrt[3]{125} \cdot \sqrt[3]{x^6} \cdot \sqrt[3]{x}$ — Product Property of Cube Roots

$= 5x^2\sqrt[3]{x}$ — Simplify.

c. $\sqrt[3]{\dfrac{y}{216}} = \dfrac{\sqrt[3]{y}}{\sqrt[3]{216}}$ — Quotient Property of Cube Roots

$= \dfrac{\sqrt[3]{y}}{6}$ — Simplify.

d. $\sqrt[3]{\dfrac{8x^4}{27y^3}} = \dfrac{\sqrt[3]{8x^4}}{\sqrt[3]{27y^3}}$ — Quotient Property of Cube Roots

$= \dfrac{\sqrt[3]{8 \cdot x^3 \cdot x}}{\sqrt[3]{27 \cdot y^3}}$ — Factor using the greatest perfect cube factors.

$= \dfrac{\sqrt[3]{8} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{x}}{\sqrt[3]{27} \cdot \sqrt[3]{y^3}}$ — Product Property of Cube Roots

$= \dfrac{2x\sqrt[3]{x}}{3y}$ — Simplify.

Monitoring Progress

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Simplify the expression.

5. $\sqrt{\dfrac{23}{9}}$ **6.** $-\sqrt{\dfrac{17}{100}}$ **7.** $\sqrt{\dfrac{36}{z^2}}$ **8.** $\sqrt{\dfrac{4x^2}{64}}$

9. $\sqrt[3]{54}$ **10.** $\sqrt[3]{16x^4}$ **11.** $\sqrt[3]{\dfrac{a}{-27}}$ **12.** $\sqrt[3]{\dfrac{25c^7d^3}{64}}$

Rationalizing the Denominator

When a radical is in the denominator of a fraction, you can multiply the fraction by an appropriate form of 1 to eliminate the radical from the denominator. This process is called **rationalizing the denominator**.

EXAMPLE 4 Rationalizing the Denominator

a. $\dfrac{\sqrt{5}}{\sqrt{3n}} = \dfrac{\sqrt{5}}{\sqrt{3n}} \cdot \dfrac{\sqrt{3n}}{\sqrt{3n}}$ — Multiply by $\dfrac{\sqrt{3n}}{\sqrt{3n}}$.

$= \dfrac{\sqrt{15n}}{\sqrt{9n^2}}$ — Product Property of Square Roots

$= \dfrac{\sqrt{15n}}{\sqrt{9} \cdot \sqrt{n^2}}$ — Product Property of Square Roots

$= \dfrac{\sqrt{15n}}{3n}$ — Simplify.

b. $\dfrac{2}{\sqrt[3]{9}} = \dfrac{2}{\sqrt[3]{9}} \cdot \dfrac{\sqrt[3]{3}}{\sqrt[3]{3}}$ — Multiply by $\dfrac{\sqrt[3]{3}}{\sqrt[3]{3}}$.

$= \dfrac{2\sqrt[3]{3}}{\sqrt[3]{27}}$ — Product Property of Cube Roots

$= \dfrac{2\sqrt[3]{3}}{3}$ — Simplify.

STUDY TIP

Rationalizing the denominator works because you multiply the numerator and denominator by the same nonzero number a, which is the same as multiplying by $\frac{a}{a}$, or 1.

The binomials $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where a, b, c, and d are rational numbers, are called **conjugates**. You can use conjugates to simplify radical expressions that contain a sum or difference involving square roots in the denominator.

EXAMPLE 5 Rationalizing the Denominator Using Conjugates

Simplify $\dfrac{7}{2 - \sqrt{3}}$.

SOLUTION

$\dfrac{7}{2 - \sqrt{3}} = \dfrac{7}{2 - \sqrt{3}} \cdot \dfrac{2 + \sqrt{3}}{2 + \sqrt{3}}$ — The conjugate of $2 - \sqrt{3}$ is $2 + \sqrt{3}$.

$= \dfrac{7(2 + \sqrt{3})}{2^2 - (\sqrt{3})^2}$ — Sum and difference pattern

$= \dfrac{14 + 7\sqrt{3}}{1}$ — Simplify.

$= 14 + 7\sqrt{3}$ — Simplify.

ANALYZING MATHEMATICAL RELATIONSHIPS

Notice that the product of two conjugates $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$ does not contain a radical and is a *rational* number.

$(a\sqrt{b} + c\sqrt{d})(a\sqrt{b} - c\sqrt{d})$

$= (a\sqrt{b})^2 - (c\sqrt{d})^2$

$= a^2b - c^2d$

Monitoring Progress

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Simplify the expression.

13. $\dfrac{1}{\sqrt{5}}$ **14.** $\dfrac{\sqrt{10}}{\sqrt{3}}$ **15.** $\dfrac{7}{\sqrt{2x}}$ **16.** $\sqrt{\dfrac{2y^2}{3}}$

17. $\dfrac{5}{\sqrt[3]{32}}$ **18.** $\dfrac{8}{1 + \sqrt{3}}$ **19.** $\dfrac{\sqrt{13}}{\sqrt{5} - 2}$ **20.** $\dfrac{12}{\sqrt{2} + \sqrt{7}}$

EXAMPLE 6 Solving a Real-Life Problem

The distance d (in miles) that you can see to the horizon with your eye level h feet above the water is given by $d = \sqrt{\frac{3h}{2}}$. How far can you see when your eye level is 5 feet above the water?

SOLUTION

$$d = \sqrt{\frac{3(5)}{2}} \qquad \text{Substitute 5 for } h.$$

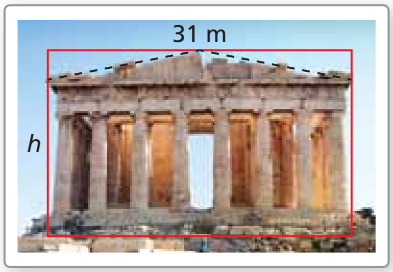
$$= \frac{\sqrt{15}}{\sqrt{2}} \qquad \text{Quotient Property of Square Roots}$$

$$= \frac{\sqrt{15}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \qquad \text{Multiply by } \frac{\sqrt{2}}{\sqrt{2}}.$$

$$= \frac{\sqrt{30}}{2} \qquad \text{Simplify.}$$

▶ You can see $\frac{\sqrt{30}}{2}$, or about 2.74 miles.

EXAMPLE 7 Modeling with Mathematics



The ratio of the length to the width of a *golden rectangle* is $(1 + \sqrt{5}) : 2$. The dimensions of the face of the Parthenon in Greece form a golden rectangle. What is the height h of the Parthenon?

SOLUTION

1. **Understand the Problem** Think of the length and height of the Parthenon as the length and width of a golden rectangle. The length of the rectangular face is 31 meters. You know the ratio of the length to the height. Find the height h.

2. **Make a Plan** Use the ratio $(1 + \sqrt{5}) : 2$ to write a proportion and solve for h.

3. **Solve the Problem**

$$\frac{1 + \sqrt{5}}{2} = \frac{31}{h} \qquad \text{Write a proportion.}$$

$$h(1 + \sqrt{5}) = 62 \qquad \text{Cross Products Property}$$

$$h = \frac{62}{1 + \sqrt{5}} \qquad \text{Divide each side by } 1 + \sqrt{5}.$$

$$h = \frac{62}{1 + \sqrt{5}} \cdot \frac{1 - \sqrt{5}}{1 - \sqrt{5}} \qquad \text{Multiply the numerator and denominator by the conjugate.}$$

$$h = \frac{62 - 62\sqrt{5}}{-4} \qquad \text{Simplify.}$$

$$h \approx 19.16 \qquad \text{Use a calculator.}$$

▶ The height is about 19 meters.

4. **Look Back** $\frac{1 + \sqrt{5}}{2} \approx 1.62$ and $\frac{31}{19.16} \approx 1.62$. So, your answer is reasonable.

Monitoring Progress

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21. WHAT IF? In Example 6, how far can you see when your eye level is 35 feet above the water?

22. The dimensions of a dance floor form a golden rectangle. The shorter side of the dance floor is 50 feet. What is the length of the longer side of the dance floor?

Performing Operations with Radicals

Radicals with the same index and radicand are called **like radicals**. You can add and subtract like radicals the same way you combine like terms by using the Distributive Property.

STUDY TIP

Do not assume that radicals with different radicands cannot be added or subtracted. Always check to see whether you can simplify the radicals. In some cases, the radicals will become like radicals.

EXAMPLE 8 Adding and Subtracting Radicals

a. $5\sqrt{7} + \sqrt{11} - 8\sqrt{7} = 5\sqrt{7} - 8\sqrt{7} + \sqrt{11}$ — Commutative Property of Addition

$= (5 - 8)\sqrt{7} + \sqrt{11}$ — Distributive Property

$= -3\sqrt{7} + \sqrt{11}$ — Subtract.

b. $10\sqrt{5} + \sqrt{20} = 10\sqrt{5} + \sqrt{4 \cdot 5}$ — Factor using the greatest perfect square factor.

$= 10\sqrt{5} + \sqrt{4} \cdot \sqrt{5}$ — Product Property of Square Roots

$= 10\sqrt{5} + 2\sqrt{5}$ — Simplify.

$= (10 + 2)\sqrt{5}$ — Distributive Property

$= 12\sqrt{5}$ — Add.

c. $6\sqrt[3]{x} + 2\sqrt[3]{x} = (6 + 2)\sqrt[3]{x}$ — Distributive Property

$= 8\sqrt[3]{x}$ — Add.

EXAMPLE 9 Multiplying Radicals

Simplify $\sqrt{5}\left(\sqrt{3} - \sqrt{75}\right)$.

SOLUTION

Method 1 $\sqrt{5}\left(\sqrt{3} - \sqrt{75}\right) = \sqrt{5} \cdot \sqrt{3} - \sqrt{5} \cdot \sqrt{75}$ — Distributive Property

$= \sqrt{15} - \sqrt{375}$ — Product Property of Square Roots

$= \sqrt{15} - 5\sqrt{15}$ — Simplify.

$= (1 - 5)\sqrt{15}$ — Distributive Property

$= -4\sqrt{15}$ — Subtract.

Method 2 $\sqrt{5}\left(\sqrt{3} - \sqrt{75}\right) = \sqrt{5}\left(\sqrt{3} - 5\sqrt{3}\right)$ — Simplify $\sqrt{75}$.

$= \sqrt{5}\left[(1 - 5)\sqrt{3}\right]$ — Distributive Property

$= \sqrt{5}\left(-4\sqrt{3}\right)$ — Subtract.

$= -4\sqrt{15}$ — Product Property of Square Roots

Monitoring Progress

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Simplify the expression.

23. $3\sqrt{2} - \sqrt{6} + 10\sqrt{2}$

24. $4\sqrt{7} - 6\sqrt{63}$

25. $4\sqrt[3]{5x} - 11\sqrt[3]{5x}$

26. $\sqrt{3}\left(8\sqrt{2} + 7\sqrt{32}\right)$

27. $\left(2\sqrt{5} - 4\right)^2$

28. $\sqrt[3]{-4}\left(\sqrt[3]{2} - \sqrt[3]{16}\right)$

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The process of eliminating a radical from the denominator of a radical expression is called ________________.

2. **VOCABULARY** What is the conjugate of the binomial $\sqrt{6} + 4$?

3. **WRITING** Are the expressions $\frac{1}{3}\sqrt{2x}$ and $\sqrt{\frac{2x}{9}}$ equivalent? Explain your reasoning.

4. **WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$-\frac{1}{3}\sqrt{6}$	$6\sqrt{3}$	$\frac{1}{6}\sqrt{3}$	$-3\sqrt{3}$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, determine whether the expression is in simplest form. If the expression is not in simplest form, explain why.

5. $\sqrt{19}$
6. $\sqrt{\frac{1}{7}}$
7. $\sqrt{48}$
8. $\sqrt{34}$
9. $\frac{5}{\sqrt{2}}$
10. $\frac{3\sqrt{10}}{4}$
11. $\frac{1}{2 + \sqrt[3]{2}}$
12. $6 - \sqrt[3]{54}$

In Exercises 13–20, simplify the expression. *(See Example 1.)*

13. $\sqrt{20}$
14. $\sqrt{32}$
15. $\sqrt{128}$
16. $-\sqrt{72}$
17. $\sqrt{125b}$
18. $\sqrt{4x^2}$
19. $-\sqrt{81m^3}$
20. $\sqrt{48n^5}$

In Exercises 21–28, simplify the expression. *(See Example 2.)*

21. $\sqrt{\frac{4}{49}}$
22. $-\sqrt{\frac{7}{81}}$
23. $-\sqrt{\frac{23}{64}}$
24. $\sqrt{\frac{65}{121}}$
25. $\sqrt{\frac{a^3}{49}}$
26. $\sqrt{\frac{144}{k^2}}$
27. $\sqrt{\frac{100}{4x^2}}$
28. $\sqrt{\frac{25v^2}{36}}$

In Exercises 29–36, simplify the expression. *(See Example 3.)*

29. $\sqrt[3]{16}$
30. $\sqrt[3]{-108}$
31. $\sqrt[3]{-64x^5}$
32. $-\sqrt[3]{343n^2}$
33. $\sqrt[3]{\frac{6c}{-125}}$
34. $\sqrt[3]{\frac{8h^4}{27}}$
35. $-\sqrt[3]{\frac{81y^2}{1000x^3}}$
36. $\sqrt[3]{\frac{21}{-64a^3b^6}}$

ERROR ANALYSIS In Exercises 37 and 38, describe and correct the error in simplifying the expression.

37. ✗
$$\begin{aligned}\sqrt{72} &= \sqrt{4 \cdot 18}\\ &= \sqrt{4} \cdot \sqrt{18}\\ &= 2\sqrt{18}\end{aligned}$$

38. ✗
$$\begin{aligned}\sqrt[3]{\frac{128y^3}{125}} &= \frac{\sqrt[3]{128y^3}}{125}\\ &= \frac{\sqrt[3]{64 \cdot 2 \cdot y^3}}{125}\\ &= \frac{\sqrt[3]{64} \cdot \sqrt[3]{2} \cdot \sqrt[3]{y^3}}{125}\\ &= \frac{4y\sqrt[3]{2}}{125}\end{aligned}$$

In Exercises 39–44, write a factor that you can use to rationalize the denominator of the expression.

39. $\dfrac{4}{\sqrt{6}}$

40. $\dfrac{1}{\sqrt{13z}}$

41. $\dfrac{2}{\sqrt[3]{x^2}}$

42. $\dfrac{3m}{\sqrt[3]{4}}$

43. $\dfrac{\sqrt{2}}{\sqrt{5} - 8}$

44. $\dfrac{5}{\sqrt{3} + \sqrt{7}}$

In Exercises 45–54, simplify the expression. *(See Example 4.)*

45. $\dfrac{2}{\sqrt{2}}$

46. $\dfrac{4}{\sqrt{3}}$

47. $\dfrac{\sqrt{5}}{\sqrt{48}}$

48. $\sqrt{\dfrac{4}{52}}$

49. $\dfrac{3}{\sqrt{a}}$

50. $\dfrac{1}{\sqrt{2x}}$

51. $\sqrt{\dfrac{3d^2}{5}}$

52. $\dfrac{\sqrt{8}}{\sqrt{3n^3}}$

53. $\dfrac{4}{\sqrt[3]{25}}$

54. $\sqrt[3]{\dfrac{1}{108y^2}}$

In Exercises 55–60, simplify the expression. *(See Example 5.)*

55. $\dfrac{1}{\sqrt{7} + 1}$

56. $\dfrac{2}{5 - \sqrt{3}}$

57. $\dfrac{\sqrt{10}}{7 - \sqrt{2}}$

58. $\dfrac{\sqrt{5}}{6 + \sqrt{5}}$

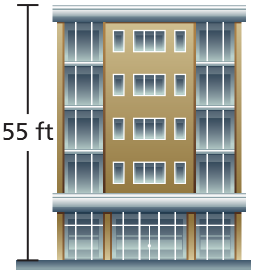
59. $\dfrac{3}{\sqrt{5} - \sqrt{2}}$

60. $\dfrac{\sqrt{3}}{\sqrt{7} + \sqrt{3}}$

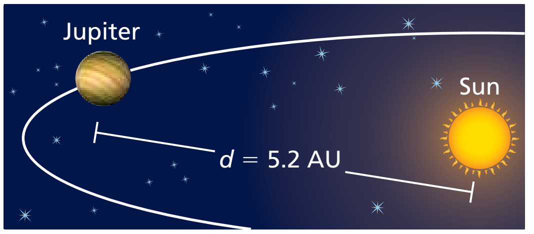
61. MODELING WITH MATHEMATICS The time t (in seconds) it takes an object to hit the ground is given by $t = \sqrt{\dfrac{h}{16}}$, where h is the height (in feet) from which the object was dropped. *(See Example 6.)*

a. How long does it take an earring to hit the ground when it falls from the roof of the building?

b. How much sooner does the earring hit the ground when it is dropped from two stories (22 feet) below the roof?



62. MODELING WITH MATHEMATICS The orbital period of a planet is the time it takes the planet to travel around the Sun. You can find the orbital period P (in Earth years) using the formula $P = \sqrt{d^3}$, where d is the average distance (in astronomical units, abbreviated AU) of the planet from the Sun.



a. Simplify the formula.

b. What is Jupiter's orbital period?

63. MODELING WITH MATHEMATICS The electric current I (in amperes) an appliance uses is given by the formula $I = \sqrt{\dfrac{P}{R}}$, where P is the power (in watts) and R is the resistance (in ohms). Find the current an appliance uses when the power is 147 watts and the resistance is 5 ohms.

64. MODELING WITH MATHEMATICS You can find the average annual interest rate r (in decimal form) of a savings account using the formula $r = \sqrt{\dfrac{V_2}{V_0}} - 1$, where V_0 is the initial investment and V_2 is the balance of the account after 2 years. Use the formula to compare the savings accounts. In which account would you invest money? Explain.

Account	Initial investment	Balance after 2 years
1	\$275	\$293
2	\$361	\$382
3	\$199	\$214
4	\$254	\$272
5	\$386	\$406

In Exercises 65–68, evaluate the function for the given value of x. Write your answer in simplest form and in decimal form rounded to the nearest hundredth.

65. $h(x) = \sqrt{5x};\ x = 10$

66. $g(x) = \sqrt{3x};\ x = 60$

67. $r(x) = \sqrt{\dfrac{3x}{3x^2 + 6}};\ x = 4$

68. $p(x) = \sqrt{\dfrac{x - 1}{5x}};\ x = 8$

In Exercises 69–72, evaluate the expression when $a = -2$, $b = 8$, and $c = \frac{1}{2}$. Write your answer in simplest form and in decimal form rounded to the nearest hundredth.

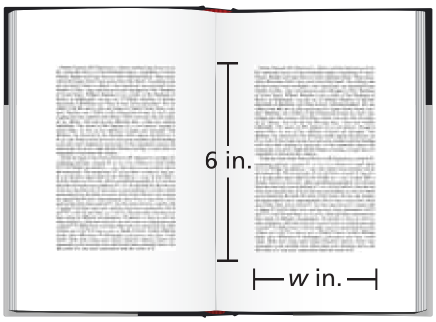
69. $\sqrt{a^2 + bc}$

70. $-\sqrt{4c - 6ab}$

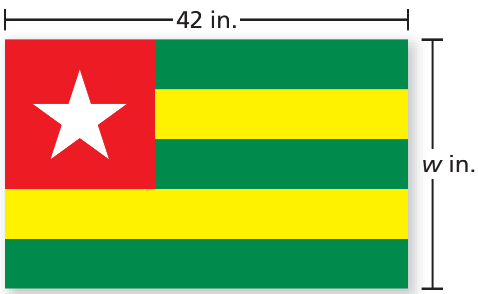
71. $-\sqrt{2a^2 + b^2}$

72. $\sqrt{b^2 - 4ac}$

73. MODELING WITH MATHEMATICS The text in the book shown forms a golden rectangle. What is the width w of the text? *(See Example 7.)*



74. MODELING WITH MATHEMATICS The flag of Togo is approximately the shape of a golden rectangle. What is the width w of the flag?



In Exercises 75–82, simplify the expression. *(See Example 8.)*

75. $\sqrt{3} - 2\sqrt{2} + 6\sqrt{2}$

76. $\sqrt{5} - 5\sqrt{13} - 8\sqrt{5}$

77. $2\sqrt{6} - 5\sqrt{54}$

78. $9\sqrt{32} + \sqrt{2}$

79. $\sqrt{12} + 6\sqrt{3} + 2\sqrt{6}$

80. $3\sqrt{7} - 5\sqrt{14} + 2\sqrt{28}$

81. $\sqrt[3]{-81} + 4\sqrt[3]{3}$

82. $6\sqrt[3]{128t} - 2\sqrt[3]{2t}$

In Exercises 83–90, simplify the expression. *(See Example 9.)*

83. $\sqrt{2}(\sqrt{45} + \sqrt{5})$

84. $\sqrt{3}(\sqrt{72} - 3\sqrt{2})$

85. $\sqrt{5}(2\sqrt{6x} - \sqrt{96x})$

86. $\sqrt{7y}(\sqrt{27y} + 5\sqrt{12y})$

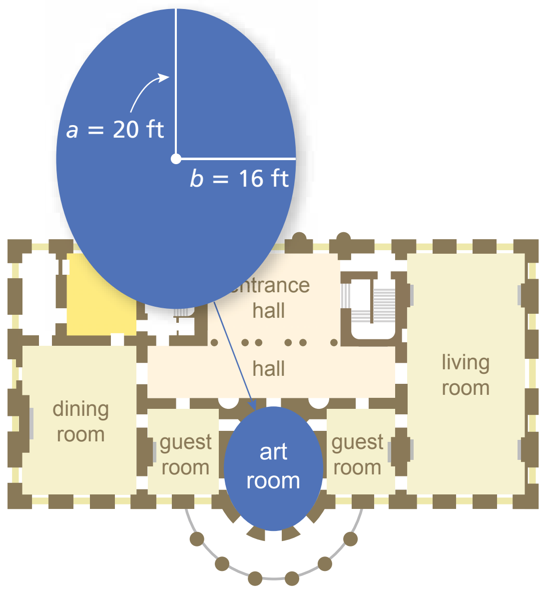
87. $(4\sqrt{2} - \sqrt{98})^2$

88. $(\sqrt{3} + \sqrt{48})(\sqrt{20} - \sqrt{5})$

89. $\sqrt[3]{3}(\sqrt[3]{4} + \sqrt[3]{32})$

90. $\sqrt[3]{2}(\sqrt[3]{135} - 4\sqrt[3]{5})$

91. MODELING WITH MATHEMATICS The circumference C of the art room in a mansion is approximated by the formula $C \approx 2\pi\sqrt{\dfrac{a^2 + b^2}{2}}$. Approximate the circumference of the room.



92. CRITICAL THINKING Determine whether each expression represents a *rational* or an *irrational* number. Justify your answer.

a. $4 + \sqrt{6}$

b. $\dfrac{\sqrt{48}}{\sqrt{3}}$

c. $\dfrac{8}{\sqrt{12}}$

d. $\sqrt{3} + \sqrt{7}$

e. $\dfrac{a}{\sqrt{10} - \sqrt{2}}$, where a is a positive integer

f. $\dfrac{2 + \sqrt{5}}{2b + \sqrt{5b^2}}$, where b is a positive integer

In Exercises 93–98, simplify the expression.

93. $\sqrt[5]{\dfrac{13}{5x^5}}$

94. $\sqrt[4]{\dfrac{10}{81}}$

95. $\sqrt[4]{256y}$

96. $\sqrt[5]{160x^6}$

97. $6\sqrt[4]{9} - \sqrt[5]{9} + 3\sqrt[4]{9}$

98. $\sqrt[5]{2}(\sqrt[4]{7} + \sqrt[5]{16})$

REASONING In Exercises 99 and 100, use the table shown.

	2	$\frac{1}{4}$	0	$\sqrt{3}$	$-\sqrt{3}$	π
2						
$\frac{1}{4}$						
0						
$\sqrt{3}$						
$-\sqrt{3}$						
π						

99. Copy and complete the table by (a) finding each sum $\left(2 + 2, 2 + \frac{1}{4}, \text{etc.}\right)$ and (b) finding each product $\left(2 \cdot 2, 2 \cdot \frac{1}{4}, \text{etc.}\right)$.

100. Use your answers in Exercise 99 to determine whether each statement is *always*, *sometimes*, or *never* true. Justify your answer.

a. The sum of a rational number and a rational number is rational.

b. The sum of a rational number and an irrational number is irrational.

c. The sum of an irrational number and an irrational number is irrational.

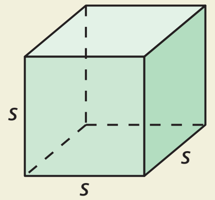
d. The product of a rational number and a rational number is rational.

e. The product of a nonzero rational number and an irrational number is irrational.

f. The product of an irrational number and an irrational number is irrational.

101. REASONING Let m be a positive integer. For what values of m will the simplified form of the expression $\sqrt{2^m}$ contain a radical? For what values will it *not* contain a radical? Explain.

102. HOW DO YOU SEE IT? The edge length s of a cube is an irrational number, the surface area is an irrational number, and the volume is a rational number. Give a possible value of s.



103. REASONING Let a and b be positive numbers. Explain why $\sqrt{ab}$ lies between a and b on a number line. (*Hint:* Let $a < b$ and multiply each side of $a < b$ by a. Then let $a < b$ and multiply each side by b.)

104. MAKING AN ARGUMENT Your friend says that you can rationalize the denominator of the expression $\dfrac{2}{4 + \sqrt[3]{5}}$ by multiplying the numerator and denominator by $4 - \sqrt[3]{5}$. Is your friend correct? Explain.

105. PROBLEM SOLVING The ratio of consecutive terms $\dfrac{a_n}{a_{n-1}}$ in the Fibonacci sequence gets closer and closer to the golden ratio $\dfrac{1 + \sqrt{5}}{2}$ as n increases. Find the term that precedes 610 in the sequence.

106. THOUGHT PROVOKING Use the golden ratio $\dfrac{1 + \sqrt{5}}{2}$ and the golden ratio conjugate $\dfrac{1 - \sqrt{5}}{2}$ for each of the following.

a. Show that the golden ratio and golden ratio conjugate are both solutions of $x^2 - x - 1 = 0$.

b. Construct a geometric diagram that has the golden ratio as the length of a part of the diagram.

107. CRITICAL THINKING Use the special product pattern $(a + b)(a^2 - ab + b^2) = a^3 + b^3$ to simplify the expression $\dfrac{2}{\sqrt[3]{x} + 1}$. Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph the linear equation. Identify the *x*-intercept. *(Section 3.5)*

108. $y = x - 4$ **109.** $y = -2x + 6$ **110.** $y = -\frac{1}{3}x - 1$ **111.** $y = \frac{3}{2}x + 6$

Solve the equation by graphing. Check your solution. *(Section 5.5)*

112. $-3x = -2x + 1$ **113.** $5x + 3 = \frac{3}{2}x - 4$ **114.** $4x - 3 = 8 - \frac{2}{5}x$ **115.** $6x - 1 = -4(x - 1)$

9.2 Solving Quadratic Equations by Graphing

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.7.A
A.8.B

Essential Question

How can you use a graph to solve a quadratic equation in one variable?

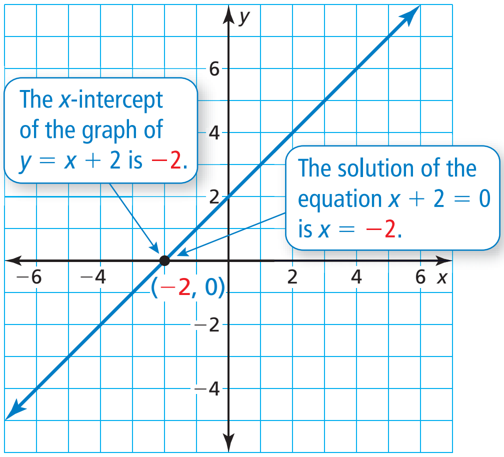
Based on what you learned about the x-intercepts of a graph in Section 3.4, it follows that the x-intercept of the graph of the linear equation

$y = ax + b$ 2 variables

is the same value as the solution of

$ax + b = 0.$ 1 variable

You can use similar reasoning to solve *quadratic equations*.



EXPLORATION 1 Solving a Quadratic Equation by Graphing

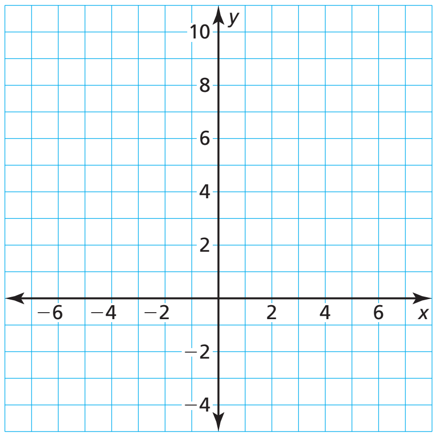
Work with a partner.

a. Sketch the graph of $y = x^2 - 2x$.

b. What is the definition of an x-intercept of a graph? How many x-intercepts does this graph have? What are they?

c. What is the definition of a solution of an equation in x? How many solutions does the equation $x^2 - 2x = 0$ have? What are they?

d. Explain how you can verify the solutions you found in part (c).



EXPLORATION 2 Solving Quadratic Equations by Graphing

Work with a partner. Solve each equation by graphing.

a. $x^2 - 4 = 0$

b. $x^2 + 3x = 0$

c. $-x^2 + 2x = 0$

d. $x^2 - 2x + 1 = 0$

e. $x^2 - 3x + 5 = 0$

f. $-x^2 + 3x - 6 = 0$

USING PROBLEM-SOLVING STRATEGIES

To be proficient in math, you need to check your answers to problems using a different method and continually ask yourself, "Does this make sense?"

Communicate Your Answer

3. How can you use a graph to solve a quadratic equation in one variable?

4. After you find a solution graphically, how can you check your result algebraically? Check your solutions for parts (a)–(d) in Exploration 2 algebraically.

5. How can you determine graphically that a quadratic equation has no solution?

9.2 Lesson

Core Vocabulary

quadratic equation, *p. 476*

Previous

x-intercept
root
zero of a function

What You Will Learn

- Solve quadratic equations by graphing.
- Use graphs to find and approximate the zeros of functions.
- Solve real-life problems using graphs of quadratic functions.

Solving Quadratic Equations by Graphing

A **quadratic equation** is a nonlinear equation that can be written in the standard form $ax^2 + bx + c = 0$, where $a \neq 0$.

In Chapter 7, you solved quadratic equations by factoring. You can also solve quadratic equations by graphing.

Core Concept

Solving Quadratic Equations by Graphing

Step 1 Write the equation in standard form, $ax^2 + bx + c = 0$.

Step 2 Graph the related function $y = ax^2 + bx + c$.

Step 3 Find the *x*-intercepts, if any.

The solutions, or *roots*, of $ax^2 + bx + c = 0$ are the *x*-intercepts of the graph.

EXAMPLE 1 Solving a Quadratic Equation: Two Real Solutions

Solve $x^2 + 2x = 3$ by graphing.

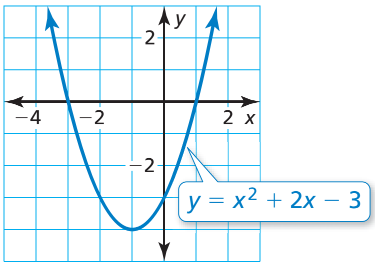
SOLUTION

Step 1 Write the equation in standard form.

$$x^2 + 2x = 3 \quad \text{Write original equation.}$$

$$x^2 + 2x - 3 = 0 \quad \text{Subtract 3 from each side.}$$

Step 2 Graph the related function $y = x^2 + 2x - 3$.



Step 3 Find the *x*-intercepts. The *x*-intercepts are -3 and 1.

- So, the solutions are $x = -3$ and $x = 1$.

Check

$x^2 + 2x = 3$	Original equation	$x^2 + 2x = 3$
$(-3)^2 + 2(-3) \stackrel{?}{=} 3$	Substitute.	$1^2 + 2(1) \stackrel{?}{=} 3$
$3 = 3$ ✓	Simplify.	$3 = 3$ ✓

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Solve the equation by graphing. Check your solutions.

1. $x^2 - x - 2 = 0$ **2.** $x^2 + 7x = -10$ **3.** $x^2 + x = 12$

EXAMPLE 2 Solving a Quadratic Equation: One Real Solution

Solve $x^2 - 8x = -16$ by graphing.

SOLUTION

Step 1 Write the equation in standard form.

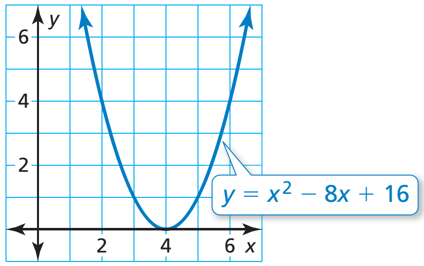
$$x^2 - 8x = -16 \qquad \text{Write original equation.}$$

$$x^2 - 8x + 16 = 0 \qquad \text{Add 16 to each side.}$$

Step 2 Graph the related function $y = x^2 - 8x + 16$.

Step 3 Find the x-intercept. The only x-intercept is at the vertex, $(4, 0)$.

▶ So, the solution is $x = 4$.



ANOTHER WAY

You can also solve the equation in Example 2 by factoring.

$$x^2 - 8x + 16 = 0$$

$$(x - 4)(x - 4) = 0$$

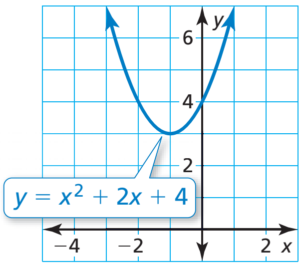
So, $x = 4$.

EXAMPLE 3 Solving a Quadratic Equation: No Real Solutions

Solve $-x^2 = 2x + 4$ by graphing.

SOLUTION

Method 1 Write the equation in standard form, $x^2 + 2x + 4 = 0$. Then graph the related function $y = x^2 + 2x + 4$, as shown at the left.



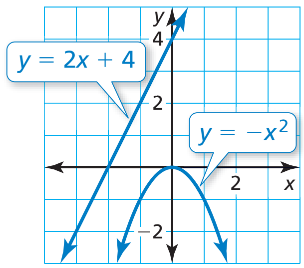
▶ There are no x-intercepts. So, $-x^2 = 2x + 4$ has no real solutions.

Method 2 Graph each side of the equation.

$$y = -x^2 \qquad \text{Left side}$$

$$y = 2x + 4 \qquad \text{Right side}$$

▶ The graphs do not intersect. So, $-x^2 = 2x + 4$ has no real solutions.



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Solve the equation by graphing.

4. $x^2 + 36 = 12x$ **5.** $x^2 + 4x = 0$ **6.** $x^2 + 10x = -25$

7. $x^2 = 3x - 3$ **8.** $x^2 + 7x = -6$ **9.** $2x + 5 = -x^2$

Concept Summary

Number of Solutions of a Quadratic Equation

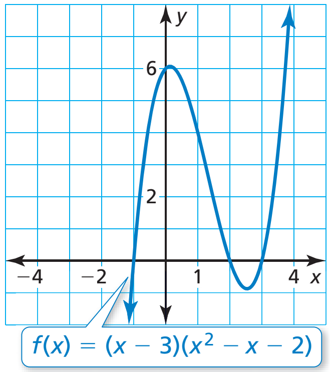
A quadratic equation has:

- two real solutions when the graph of its related function has two x-intercepts.
- one real solution when the graph of its related function has one x-intercept.
- no real solutions when the graph of its related function has no x-intercepts.

Finding Zeros of Functions

Recall that a zero of a function is an x-intercept of the graph of the function.

EXAMPLE 4 Finding the Zeros of a Function



The graph of $f(x) = (x - 3)(x^2 - x - 2)$ is shown. Find the zeros of f.

SOLUTION

The x-intercepts are -1, 2, and 3.

▶ So, the zeros of f are -1, 2, and 3.

Check

$f(-1) = (-1 - 3)[(-1)^2 - (-1) - 2] = 0$ ✓

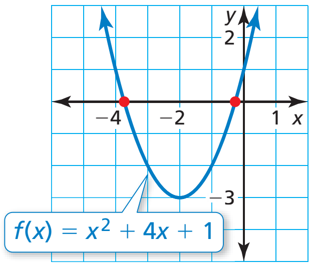
$f(2) = (2 - 3)(2^2 - 2 - 2) = 0$ ✓

$f(3) = (3 - 3)(3^2 - 3 - 2) = 0$ ✓

The zeros of a function are not necessarily integers. To approximate zeros, analyze the signs of function values. When two function values have different signs, a zero lies between the x-values that correspond to the function values.

EXAMPLE 5 Approximating the Zeros of a Function

The graph of $f(x) = x^2 + 4x + 1$ is shown. Approximate the zeros of f to the nearest tenth.



SOLUTION

There are two x-intercepts: one between -4 and -3, and another between -1 and 0.

Make tables using x-values between -4 and -3, and between -1 and 0. Use an increment of 0.1. Look for a change in the signs of the function values.

x	−3.9	−3.8	−3.7	−3.6	−3.5	−3.4	−3.3	−3.2	−3.1
$f(x)$	0.61	0.24	−0.11	−0.44	−0.75	−1.04	−1.31	−1.56	−1.79

change in signs

x	−0.9	−0.8	−0.7	−0.6	−0.5	−0.4	−0.3	−0.2	−0.1
$f(x)$	−1.79	−1.56	−1.31	−1.04	−0.75	−0.44	−0.11	0.24	0.61

change in signs

ANOTHER WAY

You could approximate one zero using a table and then use the axis of symmetry to find the other zero.

The function values that are closest to 0 correspond to x-values that best approximate the zeros of the function.

▶ In each table, the function value closest to 0 is -0.11. So, the zeros of f are about -3.7 and -0.3.

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10. Graph $f(x) = x^2 + x - 6$. Find the zeros of f.

11. Graph $f(x) = -x^2 + 2x + 2$. Approximate the zeros of f to the nearest tenth.

Solving Real-Life Problems

Seconds, t	Height, h
0	2
1	61
2	88
3	83
4	46
5	-23

EXAMPLE 6 Real-Life Application

A football player kicks a football 2 feet above the ground with an initial vertical velocity of 75 feet per second. The function $h = -16t^2 + 75t + 2$ represents the height h (in feet) of the football after t seconds. (a) Find the height of the football each second after it is kicked. (b) Use the results of part (a) to estimate when the height of the football is 50 feet. (c) Using a graph, after how many seconds is the football 50 feet above the ground?

SOLUTION

a. Make a table of values starting with $t = 0$ seconds using an increment of 1. Continue the table until a function value is negative.

▶ The height of the football is 61 feet after 1 second, 88 feet after 2 seconds, 83 feet after 3 seconds, and 46 feet after 4 seconds.

b. From part (a), you can estimate that the height of the football is 50 feet between 0 and 1 second and between 3 and 4 seconds.

▶ Based on the function values, it is reasonable to estimate that the height of the football is 50 feet slightly less than 1 second and slightly less than 4 seconds after it is kicked.

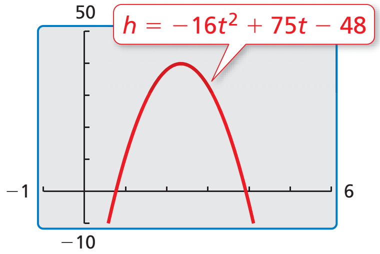
c. To determine when the football is 50 feet above the ground, find the t-values for which $h = 50$. So, solve the equation $-16t^2 + 75t + 2 = 50$ by graphing.

Step 1 Write the equation in standard form.

$$-16t^2 + 75t + 2 = 50 \qquad \text{Write the equation.}$$

$$-16t^2 + 75t - 48 = 0 \qquad \text{Subtract 50 from each side.}$$

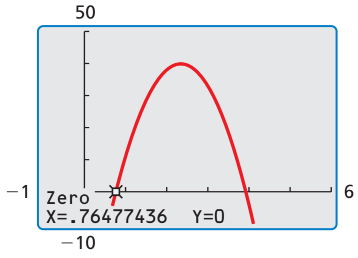
Step 2 Use a graphing calculator to graph the related function $h = -16t^2 + 75t - 48$.

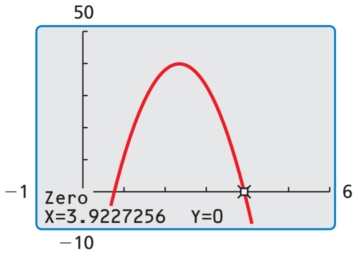


REMEMBER

Equations have *solutions*, or *roots*. Graphs have *x-intercepts*. Functions have *zeros*.

Step 3 Use the *zero* feature to find the zeros of the function.





▶ The football is 50 feet above the ground after about 0.8 second and about 3.9 seconds, which supports the estimates in part (b).

Monitoring Progress

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12. WHAT IF? After how many seconds is the football 65 feet above the ground?

In Section 4.6, you used a graphing calculator to perform linear regression on a set of data to find a linear model for the data. You can also perform *quadratic regression*.

EXAMPLE 7 Finding a Quadratic Model Using Technology

The table shows the recorded temperatures (in degrees Fahrenheit) for a portion of a day. (a) Use a graphing calculator to find a quadratic model for the data. Then determine whether the model is a good fit. (b) At what time(s) during the day is the temperature 77°F?

Time	Temperature (°F)
6 A.M.	58
8 A.M.	68
10 A.M.	76
12 P.M.	82
2 P.M.	84
4 P.M.	81
6 P.M.	75

SOLUTION

a. Step 1 Enter the data from the table into two lists. Let x represent the number of hours after midnight.

```
L1      L2      L3     1
6       58      ------
8       68
10      76
12      82
14      84
16      81
18      75
L1(1)=6
```

Step 2 Use the *quadratic regression* feature. The values in the equation can be rounded to obtain $y = -0.43x^2 + 11.7x + 2$.

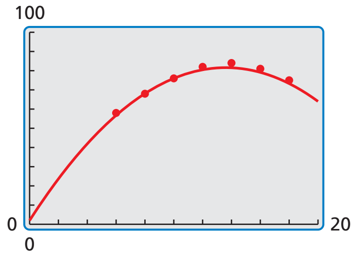
```
QuadReg
 y=ax²+bx+c
 a=-.4255952381
 b=11.73214286
 c=2.166666667
 R²=.9933031503
```

STUDY TIP

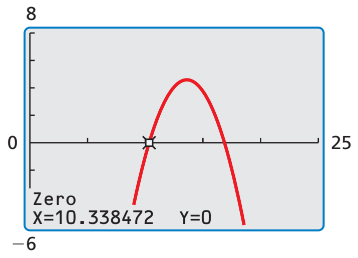
Notice that the graphing calculator does not calculate the correlation coefficient r, but it does calculate R^2, which is called the *coefficient of determination*. An R^2 value that is close to 1 also indicates that the model is a good fit for the data.

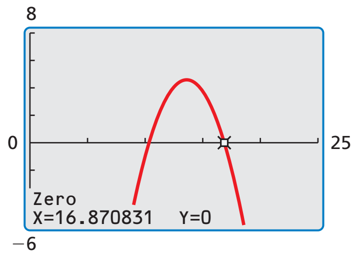
Step 3 Enter the equation $y = -0.43x^2 + 11.7x + 2$ into the calculator. Then plot the data and graph the equation in the same viewing window.

▶ The graph of the equation passes through or is close to all of the data points. So, the model is a good fit.



b. Find the x-values for which $y = 77$ by writing $-0.43x^2 + 11.7x + 2 = 77$ in standard form, graphing the related function $y = -0.43x^2 + 11.7x - 75$, and finding its zeros.





▶ The temperature is 77°F at about 10.3, or 10:18 A.M., and at about 16.9, or 4:54 P.M.

JUSTIFYING THE SOLUTION

From the table, you can estimate that the temperature is 77°F between 10 A.M. and 12 P.M. and between 4 P.M. and 6 P.M. So, your answers are reasonable.

Monitoring Progress

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13. After a break, two students come to school with the flu. The table shows the total numbers of students infected with the flu x days after the break. (a) Use a graphing calculator to find a quadratic model for the data. Then determine whether the model is a good fit. (b) How many days after the break are 26 students infected?

Days after break	0	7	14	21	28	35	42	49
Students with flu	2	3	5	8	18	30	42	55

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Vocabulary and Core Concept Check

1. **VOCABULARY** What is a quadratic equation?

2. **WHICH ONE DOESN'T BELONG?** Which equation does *not* belong with the other three? Explain your reasoning.

$x^2 + 5x = 20$	$x^2 + x - 4 = 0$	$x^2 - 6 = 4x$	$7x + 12 = x^2$

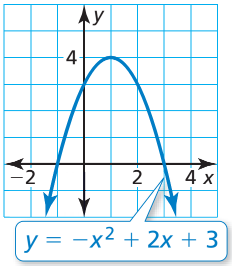
3. **WRITING** How can you use a graph to find the number of solutions of a quadratic equation?

4. **WRITING** How are solutions, roots, x-intercepts, and zeros related?

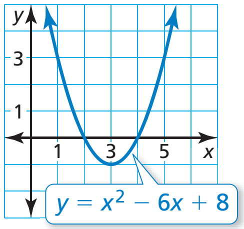
Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, use the graph to solve the equation.

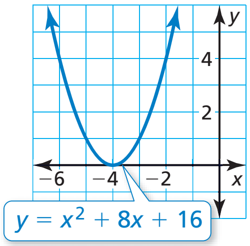
5. $-x^2 + 2x + 3 = 0$



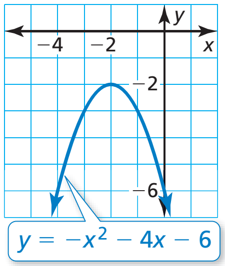
6. $x^2 - 6x + 8 = 0$



7. $x^2 + 8x + 16 = 0$



8. $-x^2 - 4x - 6 = 0$



In Exercises 9–12, write the equation in standard form.

9. $4x^2 = 12$

10. $-x^2 = 15$

11. $2x - x^2 = 1$

12. $5 + x = 3x^2$

In Exercises 13–24, solve the equation by graphing. *(See Examples 1, 2, and 3.)*

13. $x^2 - 5x = 0$

14. $x^2 - 4x + 4 = 0$

15. $x^2 - 2x + 5 = 0$

16. $x^2 - 6x - 7 = 0$

17. $x^2 = 6x - 9$

18. $-x^2 = 8x + 20$

19. $x^2 = -1 - 2x$

20. $x^2 = -x - 3$

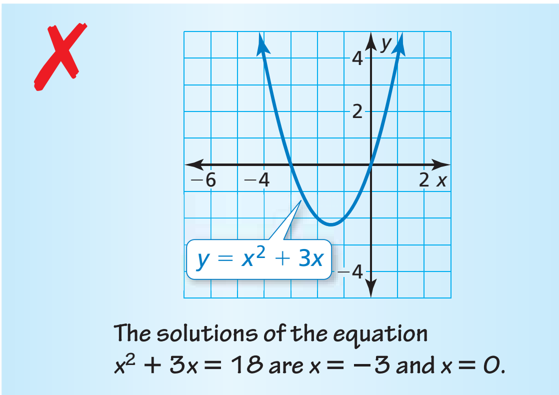
21. $4x - 12 = -x^2$

22. $5x - 6 = x^2$

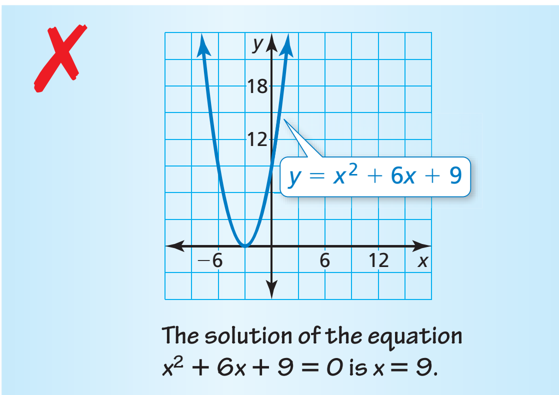
23. $x^2 - 2 = -x$

24. $16 - x^2 = -8x$

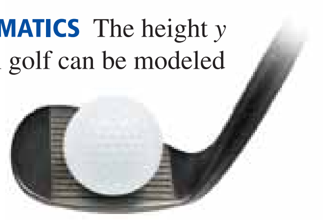
25. **ERROR ANALYSIS** Describe and correct the error in solving $x^2 + 3x = 18$ by graphing.



26. **ERROR ANALYSIS** Describe and correct the error in solving $x^2 + 6x + 9 = 0$ by graphing.



27. **MODELING WITH MATHEMATICS** The height y (in yards) of a flop shot in golf can be modeled by $y = -x^2 + 5x$, where x is the horizontal distance (in yards).



a. Interpret the x-intercepts of the graph of the equation.

b. How far away does the golf ball land?

28. **MODELING WITH MATHEMATICS** The height h (in feet) of an underhand volleyball serve can be modeled by $h = -16t^2 + 30t + 4$, where t is the time (in seconds).

a. Do both t-intercepts of the graph of the function have meaning in this situation? Explain.

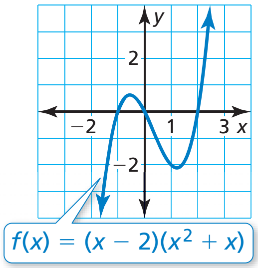
b. No one receives the serve. After how many seconds does the volleyball hit the ground?

In Exercises 29–36, solve the equation by using Method 2 from Example 3.

29. $x^2 = 10 - 3x$	30. $2x - 3 = x^2$
31. $5x - 7 = x^2$	32. $x^2 = 6x - 5$
33. $x^2 + 12x = -20$	34. $x^2 + 8x = 9$
35. $-x^2 - 5 = -2x$	36. $-x^2 - 4 = -4x$

In Exercises 37–42, find the zero(s) of f. *(See Example 4.)*

37.



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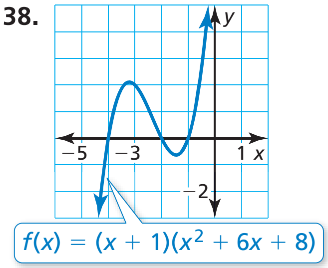

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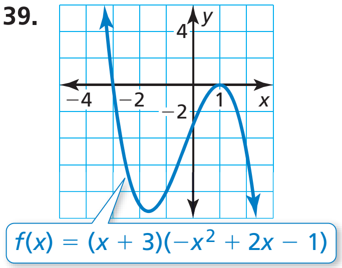

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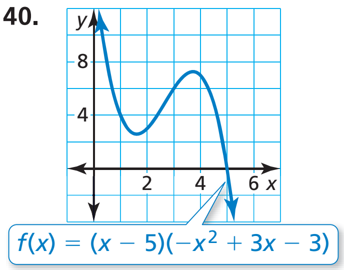

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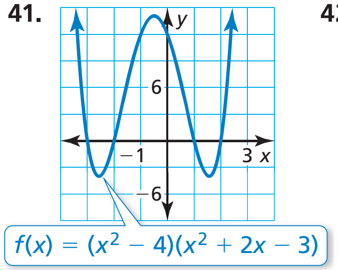

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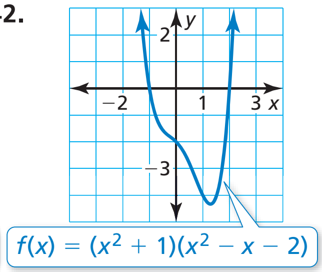
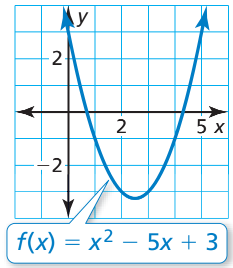


In Exercises 43–46, approximate the zeros of f to the nearest tenth. *(See Example 5.)*

43.



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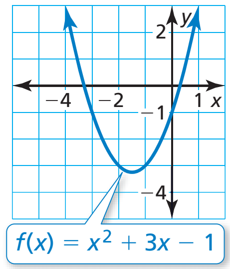

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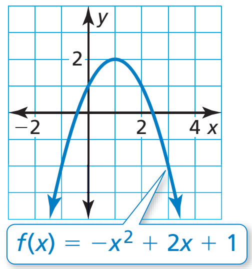
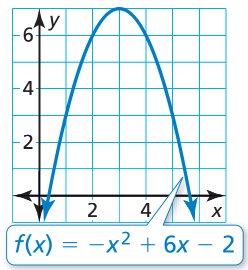


46.



In Exercises 47–52, graph the function. Approximate the zeros of the function to the nearest tenth, if necessary.

47. $f(x) = x^2 + 6x + 1$	48. $f(x) = x^2 - 3x + 2$
49. $y = -x^2 + 4x - 2$	50. $y = -x^2 + 9x - 6$
51. $f(x) = \frac{1}{2}x^2 + 2x - 5$	52. $f(x) = -3x^2 + 4x + 3$

53. **MODELING WITH MATHEMATICS** At a Civil War reenactment, a cannonball is fired into the air with an initial vertical velocity of 128 feet per second. The release point is 6 feet above the ground. The function $h = -16t^2 + 128t + 6$ represents the height h (in feet) of the cannonball after t seconds. *(See Example 6.)*

a. Find the height of the cannonball each second after it is fired.

b. Use the results of part (a) to estimate when the height of the cannonball is 150 feet.

c. Using a graph, after how many seconds is the cannonball 150 feet above the ground?

54. **MODELING WITH MATHEMATICS** You throw a softball straight up into the air with an initial vertical velocity of 40 feet per second. The release point is 5 feet above the ground. The function $h = -16t^2 + 40t + 5$ represents the height h (in feet) of the softball after t seconds.

a. Find the height of the softball each second after it is released.

b. Use the results of part (a) to estimate when the height of the softball is 15 feet.

c. Using a graph, after how many seconds is the softball 15 feet above the ground?

55. **MODELING WITH MATHEMATICS** The table shows the temperatures (in degrees Fahrenheit) of a cup of hot chocolate over time. *(See Example 7.)*

Time (minutes)	Temperature (°F)
0	200
10	157
20	128
30	109
40	99
50	92
60	90

a. Use a graphing calculator to find a quadratic model for the data. Then determine whether the model is a good fit.

b. After how many minutes is the temperature of the hot chocolate 120°F? Round your answer to the nearest tenth.

c. Should you use the quadratic model you found in part (a) to predict the temperature of the hot chocolate after 60 minutes? Explain.

56. **MODELING WITH MATHEMATICS** The table shows the values (in dollars) of a car over time.

Age (years)	0	3	6	9	12
Value (dollars)	18,900	12,275	7972	5178	3363

a. Use a graphing calculator to find a quadratic model for the data. Then determine whether the model is a good fit.

b. After how many years is the value of the car \$10,000? Round your answer to the nearest tenth.

c. Should you use the quadratic model you found in part (a) to predict the value of the car after it is 12 years old? Explain your reasoning.

57. **MATHEMATICAL CONNECTIONS** The table shows the numbers of line segments that you can draw whose endpoints are chosen from x points, no three of which are collinear.

Number of points, x	2	3	4	5	6
Number of line segments, y	1	3			

a. Copy and complete the table. Use diagrams to support your answers.

b. Use a graphing calculator to find a quadratic model for the data. Then determine whether the model is a good fit.

c. Predict the number of line segments that you can draw whose endpoints are chosen from 9 points.

d. How many points are chosen when you can draw 66 line segments? Explain how you found your answer.

58. **MODELING WITH MATHEMATICS** The table shows the numbers of cellular telephone sites (in thousands) in the U.S. for selected years from 1990 to 2012.

Year	Cellular sites (thousands)
1992	10.3
1996	30.0
2000	104.3
2004	175.7
2008	242.1
2012	301.8

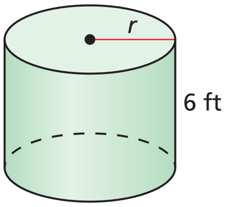
a. Use a graphing calculator to find a linear model and a quadratic model for the data. Let $x = 0$ represent 1990. Is either model a better fit for the data? Explain.

b. Use each model in part (a) to determine in what year the number of cellular sites reached 200,000. Do you get the same result? Justify your answer.

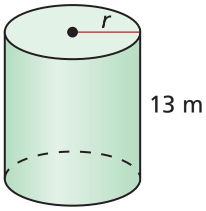
c. Use each model in part (a) to predict in what year the number of cellular sites will reach 500,000. Do you get the same result? Justify your answer.

MATHEMATICAL CONNECTIONS **In Exercises 59 and 60, use the given surface area S of the cylinder to find the radius r to the nearest tenth.**

59. $S = 225 \text{ ft}^2$

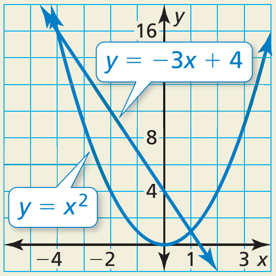


60. $S = 750 \text{ m}^2$



61. **WRITING** Explain how to approximate zeros of a function when the zeros are not integers.

62. **HOW DO YOU SEE IT?** Consider the graph shown.



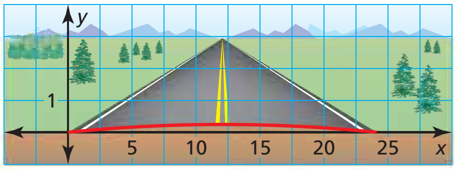
a. How many solutions does the quadratic equation $x^2 = -3x + 4$ have? Explain.

b. Without graphing, describe what you know about the graph of $y = x^2 + 3x - 4$.

63. **COMPARING METHODS** Example 3 shows two methods for solving a quadratic equation. Which method do you prefer? Explain your reasoning.

64. **THOUGHT PROVOKING** How many different parabolas have -2 and 2 as x-intercepts? Sketch examples of parabolas that have these two x-intercepts.

65. **MODELING WITH MATHEMATICS** To keep water off a road, the surface of the road is shaped like a parabola. A cross section of the road is shown in the diagram. The surface of the road can be modeled by $y = -0.0017x^2 + 0.041x$, where x and y are measured in feet. Find the width of the road to the nearest tenth of a foot.



66. **MAKING AN ARGUMENT** A stream of water from a fire hose can be modeled by $y = -0.003x^2 + 0.58x + 3$, where x and y are measured in feet. A firefighter is standing 57 feet from a building and is holding the hose 3 feet above the ground. The bottom of a window of the building is 26 feet above the ground. Your friend claims the stream of water will pass through the window. Is your friend correct? Explain.

REASONING **In Exercises 67–69, determine whether the statement is *always*, *sometimes*, or *never* true. Justify your answer.**

67. The graph of $y = ax^2 + c$ has two x-intercepts when a is negative.

68. The graph of $y = ax^2 + c$ has no x-intercepts when a and c have the same sign.

69. The graph of $y = ax^2 + bx + c$ has more than two x-intercepts when $a \neq 0$.

70. **WRITING** You want to find a model for a set of data. How do you determine whether to perform linear regression or quadratic regression on the set of data?

71. **REASONING** Show how you can use a system of equations to solve the problem in Example 7(b).

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain. *(Section 6.4)*

72.

x	-1	0	1	2
y	18	3	$\frac{1}{2}$	$\frac{1}{12}$

73.

x	0	1	2	3
y	2	8	32	128

Simplify the expression. *(Section 9.1)*

74. $\dfrac{5}{\sqrt{13}}$

75. $\dfrac{\sqrt{10}}{\sqrt{6x}}$

76. $\dfrac{8}{7 + \sqrt{5}}$

77. $\dfrac{3}{\sqrt{3} - 2}$

9.3 Solving Quadratic Equations Using Square Roots

Texas Essential Knowledge and Skills
A.8.A

Essential Question

How can you determine the number of solutions of a quadratic equation of the form $ax^2 + c = 0$?

EXPLORATION 1 The Number of Solutions of $ax^2 + c = 0$

Work with a partner. Solve each equation by graphing. Explain how the number of solutions of $ax^2 + c = 0$ relates to the graph of $y = ax^2 + c$.

a. $x^2 - 4 = 0$

b. $2x^2 + 5 = 0$

c. $x^2 = 0$

d. $x^2 - 5 = 0$

EXPLORATION 2 Estimating Solutions

Work with a partner. Complete each table. Use the completed tables to estimate the solutions of $x^2 - 5 = 0$. Explain your reasoning.

a.

x	$x^2 - 5$
2.21	
2.22	
2.23	
2.24	
2.25	
2.26	

b.

x	$x^2 - 5$
-2.21	
-2.22	
-2.23	
-2.24	
-2.25	
-2.26	

USING PRECISE MATHEMATICAL LANGUAGE

To be proficient in math, you need to calculate accurately and express numerical answers with a level of precision appropriate for the problem's context.

EXPLORATION 3 Using Technology to Estimate Solutions

Work with a partner. Two equations are equivalent when they have the same solutions.

a. Are the equations $x^2 - 5 = 0$ and $x^2 = 5$ equivalent? Explain your reasoning.

b. Use the square root key on a calculator to estimate the solutions of $x^2 - 5 = 0$. Describe the accuracy of your estimates in Exploration 2.

c. Write the exact solutions of $x^2 - 5 = 0$.

Communicate Your Answer

4. How can you determine the number of solutions of a quadratic equation of the form $ax^2 + c = 0$?

5. Write the exact solutions of each equation. Then use a calculator to estimate the solutions.

a. $x^2 - 2 = 0$

b. $3x^2 - 18 = 0$

c. $x^2 = 8$

9.3 Lesson

Core Vocabulary

Previous
square root
zero of a function

What You Will Learn

- Solve quadratic equations using square roots.
- Approximate the solutions of quadratic equations.

Solving Quadratic Equations Using Square Roots

Earlier in this chapter, you studied properties of square roots. Now you will use square roots to solve quadratic equations of the form $ax^2 + c = 0$. First isolate x^2 on one side of the equation to obtain $x^2 = d$. Then solve by taking the square root of each side.

Core Concept

Solutions of $x^2 = d$

- When $d > 0$, $x^2 = d$ has two real solutions, $x = \pm\sqrt{d}$.
- When $d = 0$, $x^2 = d$ has one real solution, $x = 0$.
- When $d < 0$, $x^2 = d$ has no real solutions.

ANOTHER WAY

You can also solve $3x^2 - 27 = 0$ by factoring.

$$3(x^2 - 9) = 0$$
$$3(x - 3)(x + 3) = 0$$
$$x = 3 \text{ or } x = -3$$

EXAMPLE 1 Solving Quadratic Equations Using Square Roots

a. Solve $3x^2 - 27 = 0$ using square roots.

$3x^2 - 27 = 0$	Write the equation.
$3x^2 = 27$	Add 27 to each side.
$x^2 = 9$	Divide each side by 3.
$x = \pm\sqrt{9}$	Take the square root of each side.
$x = \pm 3$	Simplify.

▶ The solutions are $x = 3$ and $x = -3$.

b. Solve $x^2 - 10 = -10$ using square roots.

$x^2 - 10 = -10$	Write the equation.
$x^2 = 0$	Add 10 to each side.
$x = 0$	Take the square root of each side.

▶ The only solution is $x = 0$.

c. Solve $-5x^2 + 11 = 16$ using square roots.

$-5x^2 + 11 = 16$	Write the equation.
$-5x^2 = 5$	Subtract 11 from each side.
$x^2 = -1$	Divide each side by -5.

▶ The square of a real number cannot be negative. So, the equation has no real solutions.

STUDY TIP

Each side of the equation $(x-1)^2 = 25$ is a square. So, you can still solve by taking the square root of each side.

EXAMPLE 2 Solving a Quadratic Equation Using Square Roots

Solve $(x-1)^2 = 25$ using square roots.

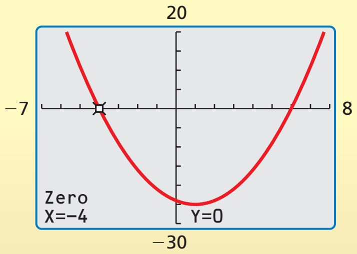
SOLUTION

$(x-1)^2 = 25$	Write the equation.
$x - 1 = \pm 5$	Take the square root of each side.
$x = 1 \pm 5$	Add 1 to each side.

► So, the solutions are $x = 1 + 5 = 6$ and $x = 1 - 5 = -4$.

Check

Use a graphing calculator to check your answer. Rewrite the equation as $(x-1)^2 - 25 = 0$. Graph the related function $f(x) = (x-1)^2 - 25$ and find the zeros of the function. The zeros are -4 and 6.



Monitoring Progress

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Solve the equation using square roots.

1. $-3x^2 = -75$ **2.** $x^2 + 12 = 10$ **3.** $4x^2 - 15 = -15$

4. $(x+7)^2 = 0$ **5.** $4(x-3)^2 = 9$ **6.** $(2x+1)^2 = 36$

Approximating Solutions of Quadratic Equations

EXAMPLE 3 Approximating Solutions of a Quadratic Equation

Solve $4x^2 - 13 = 15$ using square roots. Round the solutions to the nearest hundredth.

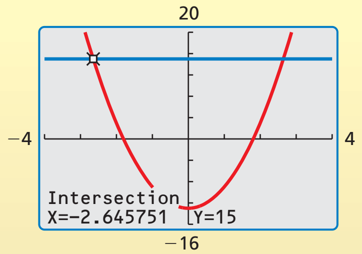
SOLUTION

$4x^2 - 13 = 15$	Write the equation.
$4x^2 = 28$	Add 13 to each side.
$x^2 = 7$	Divide each side by 4.
$x = \pm\sqrt{7}$	Take the square root of each side.
$x \approx \pm 2.65$	Use a calculator.

► The solutions are $x \approx -2.65$ and $x \approx 2.65$.

Check

Graph each side of the equation and find the points of intersection. The x-values of the points of intersection are about -2.65 and 2.65.



Monitoring Progress

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Solve the equation using square roots. Round your solutions to the nearest hundredth.

7. $x^2 + 8 = 19$ **8.** $5x^2 - 2 = 0$ **9.** $3x^2 - 30 = 4$

EXAMPLE 4 Solving a Real-Life Problem

A touch tank has a height of 3 feet. Its length is three times its width. The volume of the tank is 270 cubic feet. Find the length and width of the tank.

SOLUTION

The length ℓ is three times the width w, so $\ell = 3w$. Write an equation using the formula for the volume of a rectangular prism.

$V = \ell wh$ — Write the formula.

$270 = 3w(w)(3)$ — Substitute 270 for V, $3w$ for ℓ, and 3 for h.

$270 = 9w^2$ — Multiply.

$30 = w^2$ — Divide each side by 9.

$\pm\sqrt{30} = w$ — Take the square root of each side.

The solutions are $\sqrt{30}$ and $-\sqrt{30}$. Use the positive solution.

▶ So, the width is $\sqrt{30} \approx 5.5$ feet and the length is $3\sqrt{30} \approx 16.4$ feet.

EXPLAINING MATHEMATICAL IDEAS

Use the positive square root because negative solutions do not make sense in this context. Length and width cannot be negative.

EXAMPLE 5 Rearranging and Evaluating a Formula

The area A of an equilateral triangle with side length s is given by the formula $A = \frac{\sqrt{3}}{4}s^2$. Solve the formula for s. Then approximate the side length of the traffic sign that has an area of 390 square inches.

SOLUTION

Step 1 Solve the formula for s.

$A = \frac{\sqrt{3}}{4}s^2$ — Write the formula.

$\frac{4A}{\sqrt{3}} = s^2$ — Multiply each side by $\frac{4}{\sqrt{3}}$.

$\sqrt{\frac{4A}{\sqrt{3}}} = s$ — Take the positive square root of each side.

Step 2 Substitute 390 for A in the new formula and evaluate.

$$s = \sqrt{\frac{4A}{\sqrt{3}}} = \sqrt{\frac{4(390)}{\sqrt{3}}} = \sqrt{\frac{1560}{\sqrt{3}}} \approx 30$$ Use a calculator.

▶ The side length of the traffic sign is about 30 inches.

ANOTHER WAY

Notice that you can rewrite the formula as $s = \frac{2}{3^{1/4}}\sqrt{A}$, or $s \approx 1.52\sqrt{A}$. This can help you efficiently find the value of s for various values of A.

Monitoring Progress

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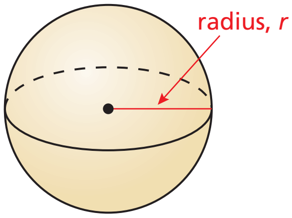
10. WHAT IF? In Example 4, the volume of the tank is 315 cubic feet. Find the length and width of the tank.

11. The surface area S of a sphere with radius r is given by the formula $S = 4\pi r^2$. Solve the formula for r. Then find the radius of a globe with a surface area of 804 square inches.



9.3 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The equation $x^2 = d$ has ____ real solutions when $d > 0$.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

Solve $x^2 = 144$ using square roots.	Solve $x^2 - 144 = 0$ using square roots.
Solve $x^2 + 146 = 2$ using square roots.	Solve $x^2 + 2 = 146$ using square roots.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, determine the number of real solutions of the equation. Then solve the equation using square roots.

3. $x^2 = 25$
4. $x^2 = -36$
5. $x^2 = -21$
6. $x^2 = 400$
7. $x^2 = 0$
8. $x^2 = 169$

In Exercises 9–18, solve the equation using square roots. *(See Example 1.)*

9. $x^2 - 16 = 0$
10. $x^2 + 6 = 0$
11. $3x^2 + 12 = 0$
12. $x^2 - 55 = 26$
13. $2x^2 - 98 = 0$
14. $-x^2 + 9 = 9$
15. $-3x^2 - 5 = -5$
16. $4x^2 - 371 = 29$
17. $4x^2 + 10 = 11$
18. $9x^2 - 35 = 14$

In Exercises 19–24, solve the equation using square roots. *(See Example 2.)*

19. $(x + 3)^2 = 0$
20. $(x - 1)^2 = 4$
21. $(2x - 1)^2 = 81$
22. $(4x + 5)^2 = 9$
23. $9(x + 1)^2 = 16$
24. $4(x - 2)^2 = 25$

In Exercises 25–30, solve the equation using square roots. Round your solutions to the nearest hundredth. *(See Example 3.)*

25. $x^2 + 6 = 13$
26. $x^2 + 11 = 24$
27. $2x^2 - 9 = 11$
28. $5x^2 + 2 = 6$
29. $-21 = 15 - 2x^2$
30. $2 = 4x^2 - 5$

31. **ERROR ANALYSIS** Describe and correct the error in solving the equation $2x^2 - 33 = 39$ using square roots.

$$2x^2 - 33 = 39$$
$$2x^2 = 72$$
$$x^2 = 36$$
$$x = 6$$

The solution is $x = 6$.

32. **MODELING WITH MATHEMATICS** An in-ground pond has the shape of a rectangular prism. The pond has a depth of 24 inches and a volume of 72,000 cubic inches. The length of the pond is two times its width. Find the length and width of the pond. *(See Example 4.)*

33. **MODELING WITH MATHEMATICS** A person sitting in the top row of the bleachers at a sporting event drops a pair of sunglasses from a height of 24 feet. The function $h = -16x^2 + 24$ represents the height h (in feet) of the sunglasses after x seconds. How long does it take the sunglasses to hit the ground?

34. MAKING AN ARGUMENT Your friend says that the solution of the equation $x^2 + 4 = 0$ is $x = 0$. Your cousin says that the equation has no real solutions. Who is correct? Explain your reasoning.

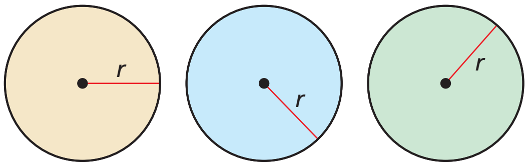
35. MODELING WITH MATHEMATICS The design of a square rug for your living room is shown. You want the area of the inner square to be 25% of the total area of the rug. Find the side length x of the inner square.

6 ft

36. MATHEMATICAL CONNECTIONS The area A of a circle with radius r is given by the formula $A = \pi r^2$. *(See Example 5.)*

a. Solve the formula for r.

b. Use the formula from part (a) to find the radius of each circle.



$A = 113\text{ ft}^2$ $A = 1810\text{ in.}^2$ $A = 531\text{ m}^2$

c. Explain why it is beneficial to solve the formula for r before finding the radius.

37. WRITING How can you approximate the roots of a quadratic equation when the roots are not integers?

38. WRITING Given the equation $ax^2 + c = 0$, describe the values of a and c so the equation has the following number of solutions.

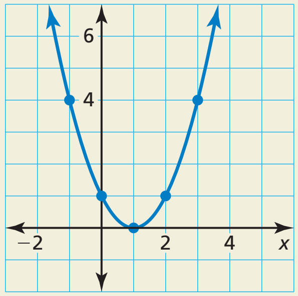
a. two real solutions

b. one real solution

c. no real solutions

39. REASONING Without graphing, where do the graphs of $y = x^2$ and $y = 9$ intersect? Explain.

40. HOW DO YOU SEE IT? The graph represents the function $f(x) = (x - 1)^2$. How many solutions does the equation $(x - 1)^2 = 0$ have? Explain.



41. REASONING Solve $x^2 = 1.44$ without using a calculator. Explain your reasoning.

42. THOUGHT PROVOKING The quadratic equation

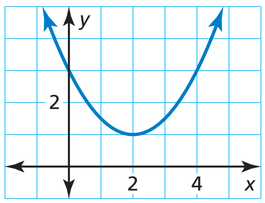
$$ax^2 + bx + c = 0$$

can be rewritten in the following form.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Use this form to write the solutions of the equation.

43. REASONING An equation of the graph shown is $y = \frac{1}{2}(x - 2)^2 + 1$. Two points on the parabola have y-coordinates of 9. Find the x-coordinates of these points.



44. CRITICAL THINKING Solve each equation without graphing.

a. $x^2 - 12x + 36 = 64$

b. $x^2 + 14x + 49 = 16$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Factor the polynomial. *(Section 7.8)*

45. $x^2 + 8x + 16$

46. $x^2 - 4x + 4$

47. $x^2 - 14x + 49$

48. $x^2 + 18x + 81$

49. $x^2 + 12x + 36$

50. $x^2 - 22x + 121$

9.1–9.3 What Did You Learn?

Core Vocabulary

counterexample, *p. 465*
radical expression, *p. 466*
simplest form, *p. 466*
rationalizing the denominator, *p. 468*
conjugates, *p. 468*
like radicals, *p. 470*
quadratic equation, *p. 476*

Core Concepts

Section 9.1

Product Property of Square Roots, *p. 466*
Quotient Property of Square Roots, *p. 466*
Rationalizing the Denominator, *p. 468*
Performing Operations with Radicals, *p. 470*

Section 9.2

Solving Quadratic Equations by Graphing, *p. 476*
Number of Solutions of a Quadratic Equation, *p. 477*
Finding Zeros of Functions, *p. 478*

Section 9.3

Solutions of $x^2 = d$, *p. 486*
Approximating Solutions of Quadratic Equations, *p. 487*

Mathematical Thinking

1. For each part of Exercise 100 on page 474 that is *sometimes* true, list all examples and counterexamples from the table that represent the sum or product being described.
2. Which Examples can you use to help you solve Exercise 54 on page 483?
3. Describe how solving a simpler equation can help you solve the equation in Exercise 41 on page 490.

Study Skills

Keeping a Positive Attitude

Do you ever feel frustrated or overwhelmed by math? You're not alone. Just take a deep breath and assess the situation. Try to find a productive study environment, review your notes and the examples in the textbook, and ask your teacher or friends for help.

9.1–9.3 Quiz

Simplify the expression. *(Section 9.1)*

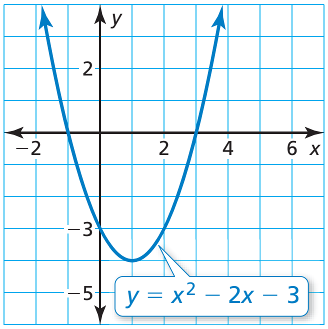
1. $\sqrt{112x^3}$ **2.** $\sqrt{\frac{18}{81}}$ **3.** $\sqrt[3]{-625}$ **4.** $\frac{12}{\sqrt{32}}$

5. $\frac{4}{\sqrt{11}}$ **6.** $\sqrt{\frac{144}{13}}$ **7.** $\sqrt[3]{\frac{54x^4}{343y^6}}$ **8.** $\sqrt{\frac{4x^2}{28y^4z^5}}$

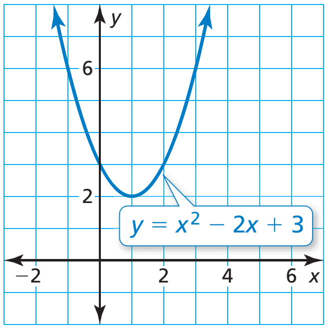
9. $\frac{6}{5+\sqrt{3}}$ **10.** $2\sqrt{5} + 7\sqrt{10} - 3\sqrt{20}$ **11.** $\frac{10}{\sqrt{8}-\sqrt{10}}$ **12.** $\sqrt{6}(7\sqrt{12} - 4\sqrt{3})$

Use the graph to solve the equation. *(Section 9.2)*

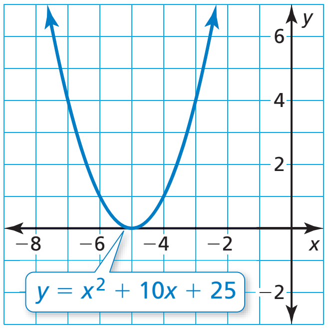
13. $x^2 - 2x - 3 = 0$



14. $x^2 - 2x + 3 = 0$



15. $x^2 + 10x + 25 = 0$



Solve the equation by graphing. *(Section 9.2)*

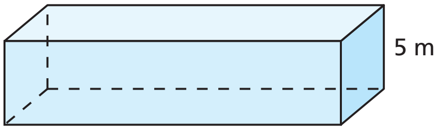
16. $x^2 + 9x + 14 = 0$ **17.** $x^2 - 7x = 8$ **18.** $x + 4 = -x^2$

Solve the equation using square roots. *(Section 9.3)*

19. $4x^2 = 64$ **20.** $-3x^2 + 6 = 10$ **21.** $(x - 8)^2 = 1$

22. Explain how to determine the number of real solutions of $x^2 = 100$ without solving. *(Section 9.3)*

23. The length of a rectangular prism is four times its width. The volume of the prism is 380 cubic meters. Find the length and width of the prism. *(Section 9.3)*



24. You cast a fishing lure into the water from a height of 4 feet above the water. The height h (in feet) of the fishing lure after t seconds can be modeled by the equation $h = -16t^2 + 24t + 4$. *(Section 9.2)*

a. After how many seconds does the fishing lure reach a height of 12 feet?

b. After how many seconds does the fishing lure hit the water?

9.4 Solving Quadratic Equations by Completing the Square

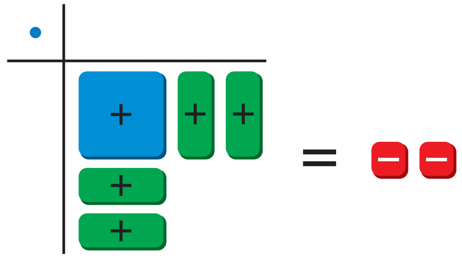
Essential Question How can you use "completing the square" to solve a quadratic equation?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.8.A

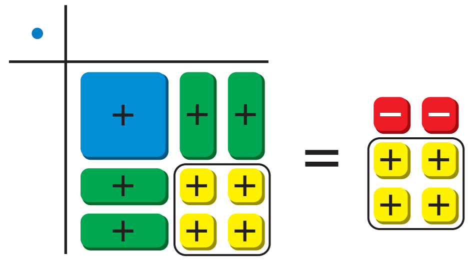
EXPLORATION 1 Solving by Completing the Square

Work with a partner.

a. Write the equation modeled by the algebra tiles. This is the equation to be solved.

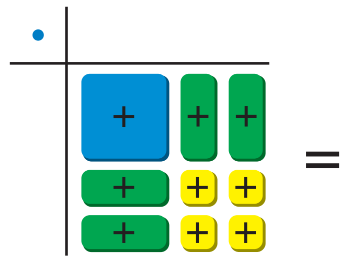


b. Four algebra tiles are added to the left side to "complete the square." Why are four algebra tiles also added to the right side?



c. Use algebra tiles to label the dimensions of the square on the left side and simplify on the right side.

d. Write the equation modeled by the algebra tiles so that the left side is the square of a binomial. Solve the equation using square roots.



USING PROBLEM-SOLVING STRATEGIES

To be proficient in math, you need to explain to yourself the meaning of a problem. After that, you need to look for entry points to its solution.

EXPLORATION 2 Solving by Completing the Square

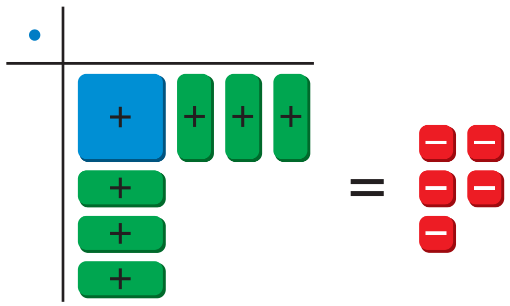
Work with a partner.

a. Write the equation modeled by the algebra tiles.

b. Use algebra tiles to "complete the square."

c. Write the solutions of the equation.

d. Check each solution in the original equation.



Communicate Your Answer

3. How can you use "completing the square" to solve a quadratic equation?

4. Solve each quadratic equation by completing the square.

a. $x^2 - 2x = 1$ **b.** $x^2 - 4x = -1$ **c.** $x^2 + 4x = -3$

9.4 Lesson

Core Vocabulary

completing the square, *p. 494*

Previous
perfect square trinomial
coefficient
maximum value
minimum value
vertex form of a quadratic function

What You Will Learn

- Complete the square for expressions of the form $x^2 + bx$.
- Solve quadratic equations by completing the square.
- Find and use maximum and minimum values.
- Solve real-life problems by completing the square.

Completing the Square

For an expression of the form $x^2 + bx$, you can add a constant c to the expression so that $x^2 + bx + c$ is a perfect square trinomial. This process is called **completing the square**.

Core Concept

Completing the Square

Words To complete the square for an expression of the form $x^2 + bx$, follow these steps.

Step 1 Find one-half of b, the coefficient of x.

Step 2 Square the result from Step 1.

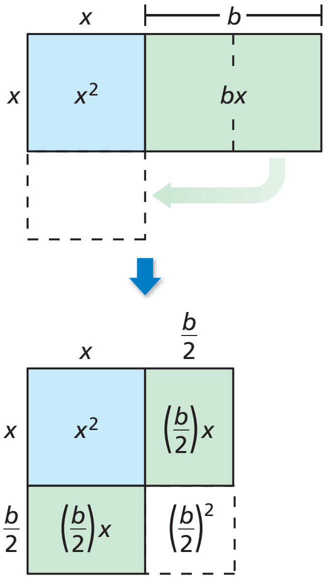
Step 3 Add the result from Step 2 to $x^2 + bx$.

Factor the resulting expression as the square of a binomial.

Algebra $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

JUSTIFYING STEPS

In each diagram below, the combined area of the shaded regions is $x^2 + bx$. Adding $\left(\frac{b}{2}\right)^2$ completes the square in the second diagram.



EXAMPLE 1 Completing the Square

Complete the square for each expression. Then factor the trinomial.

a. $x^2 + 6x$ **b.** $x^2 - 9x$

SOLUTION

a. Step 1 Find one-half of b. $\quad \frac{b}{2} = \frac{6}{2} = 3$

Step 2 Square the result from Step 1. $\quad 3^2 = 9$

Step 3 Add the result from Step 2 to $x^2 + bx$. $\quad x^2 + 6x + 9$

▶ $x^2 + 6x + 9 = (x + 3)^2$

b. Step 1 Find one-half of b. $\quad \frac{b}{2} = \frac{-9}{2}$

Step 2 Square the result from Step 1. $\quad \left(\frac{-9}{2}\right)^2 = \frac{81}{4}$

Step 3 Add the result from Step 2 to $x^2 + bx$. $\quad x^2 - 9x + \frac{81}{4}$

▶ $x^2 - 9x + \frac{81}{4} = \left(x - \frac{9}{2}\right)^2$

Monitoring Progress

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Complete the square for the expression. Then factor the trinomial.

1. $x^2 + 10x$ **2.** $x^2 - 4x$ **3.** $x^2 + 7x$

Solving Quadratic Equations by Completing the Square

The method of completing the square can be used to solve any quadratic equation. To solve a quadratic equation by completing the square, you must write the equation in the form $x^2 + bx = d$.

COMMON ERROR

When completing the square to solve an equation, be sure to add $\left(\frac{b}{2}\right)^2$ to each side of the equation.

EXAMPLE 2 **Solving a Quadratic Equation: $x^2 + bx = d$**

Solve $x^2 - 16x = -15$ by completing the square.

SOLUTION

$x^2 - 16x = -15$	Write the equation.
$x^2 - 16x + (-8)^2 = -15 + (-8)^2$	Complete the square by adding $\left(\frac{-16}{2}\right)^2$, or $(-8)^2$, to each side.
$(x - 8)^2 = 49$	Write the left side as the square of a binomial.
$x - 8 = \pm 7$	Take the square root of each side.
$x = 8 \pm 7$	Add 8 to each side.

▶ The solutions are $x = 8 + 7 = 15$ and $x = 8 - 7 = 1$.

Check

$x^2 - 16x = -15$	Original equation	$x^2 - 16x = -15$
$15^2 - 16(15) \stackrel{?}{=} -15$	Substitute.	$1^2 - 16(1) \stackrel{?}{=} -15$
$-15 = -15$ ✓	Simplify.	$-15 = -15$ ✓

EXAMPLE 3 **Solving a Quadratic Equation: $ax^2 + bx + c = 0$**

Solve $2x^2 + 20x - 8 = 0$ by completing the square.

COMMON ERROR

Before you complete the square, be sure that the coefficient of the x^2-term is 1.

SOLUTION

$2x^2 + 20x - 8 = 0$	Write the equation.
$2x^2 + 20x = 8$	Add 8 to each side.
$x^2 + 10x = 4$	Divide each side by 2.
$x^2 + 10x + 5^2 = 4 + 5^2$	Complete the square by adding $\left(\frac{10}{2}\right)^2$, or 5^2, to each side.
$(x + 5)^2 = 29$	Write the left side as the square of a binomial.
$x + 5 = \pm\sqrt{29}$	Take the square root of each side.
$x = -5 \pm \sqrt{29}$	Subtract 5 from each side.

▶ The solutions are $x = -5 + \sqrt{29} \approx 0.39$ and $x = -5 - \sqrt{29} \approx -10.39$.

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Solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary.

4. $x^2 - 2x = 3$ **5.** $m^2 + 12m = -8$ **6.** $3g^2 - 24g + 27 = 0$

Finding and Using Maximum and Minimum Values

One way to find the maximum or minimum value of a quadratic function is to write the function in vertex form by completing the square. Recall that the vertex form of a quadratic function is $y = a(x - h)^2 + k$, where $a \neq 0$. The vertex of the graph is (h, k).

EXAMPLE 4 Finding a Minimum Value

Find the minimum value of $y = x^2 + 4x - 1$.

SOLUTION

Write the function in vertex form.

$$y = x^2 + 4x - 1 \qquad \text{Write the function.}$$

$$y + 1 = x^2 + 4x \qquad \text{Add 1 to each side.}$$

$$y + 1 + 4 = x^2 + 4x + 4 \qquad \text{Complete the square for } x^2 + 4x.$$

$$y + 5 = x^2 + 4x + 4 \qquad \text{Simplify the left side.}$$

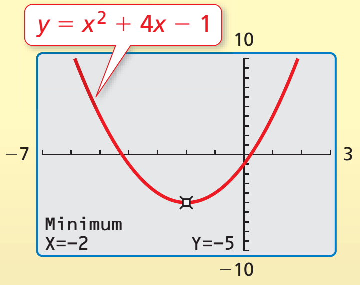
$$y + 5 = (x + 2)^2 \qquad \text{Write the right side as the square of a binomial.}$$

$$y = (x + 2)^2 - 5 \qquad \text{Write in vertex form.}$$

The vertex is $(-2, -5)$. Because a is positive $(a = 1)$, the parabola opens up and the y-coordinate of the vertex is the minimum value.

▶ So, the function has a minimum value of -5.

Check



EXAMPLE 5 Finding a Maximum Value

Find the maximum value of $y = -x^2 + 2x + 7$.

SOLUTION

Write the function in vertex form.

$$y = -x^2 + 2x + 7 \qquad \text{Write the function.}$$

$$y - 7 = -x^2 + 2x \qquad \text{Subtract 7 from each side.}$$

$$y - 7 = -(x^2 - 2x) \qquad \text{Factor out } -1.$$

$$y - 7 - 1 = -(x^2 - 2x + 1) \qquad \text{Complete the square for } x^2 - 2x.$$

$$y - 8 = -(x^2 - 2x + 1) \qquad \text{Simplify the left side.}$$

$$y - 8 = -(x - 1)^2 \qquad \text{Write } x^2 - 2x + 1 \text{ as the square of a binomial.}$$

$$y = -(x - 1)^2 + 8 \qquad \text{Write in vertex form.}$$

STUDY TIP

Adding 1 inside the parentheses results in subtracting 1 from the right side of the equation.

The vertex is $(1, 8)$. Because a is negative $(a = -1)$, the parabola opens down and the y-coordinate of the vertex is the maximum value.

▶ So, the function has a maximum value of 8.

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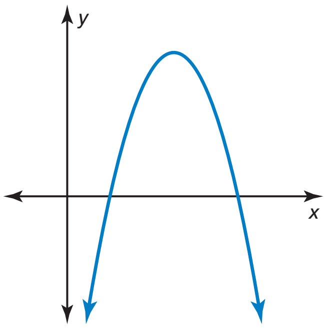
Determine whether the quadratic function has a maximum or minimum value. Then find the value.

7. $y = -x^2 - 4x + 4$ **8.** $y = x^2 + 12x + 40$ **9.** $y = x^2 - 2x - 2$

EXAMPLE 6 Interpreting Forms of Quadratic Functions



$f(x) = -\frac{1}{2}(x + 4)^2 + 8$

$g(x) = -(x - 5)^2 + 9$

$m(x) = (x - 3)(x - 12)$

$p(x) = -(x - 2)(x - 8)$

Which of the functions could be represented by the graph? Explain.

SOLUTION

You do not know the scale of either axis. To eliminate functions, consider the characteristics of the graph and information provided by the form of each function. The graph appears to be a parabola that opens down, which means the function has a maximum value. The vertex of the graph is in the first quadrant. Both x-intercepts are positive.

- The graph of f opens down because $a < 0$, which means f has a maximum value. However, the vertex $(-4, 8)$ of the graph of f is in the second quadrant. So, the graph does not represent f.
- The graph of g opens down because $a < 0$, which means g has a maximum value. The vertex $(5, 9)$ of the graph of g is in the first quadrant. By solving $0 = -(x - 5)^2 + 9$, you see that the x-intercepts of the graph of g are 2 and 8. So, the graph could represent g.
- The graph of m has two positive x-intercepts. However, its graph opens up because $a > 0$, which means m has a minimum value. So, the graph does not represent m.
- The graph of p has two positive x-intercepts, and its graph opens down because $a < 0$. This means that p has a maximum value and the vertex must be in the first quadrant. So, the graph could represent p.

▶ The graph could represent function g or function p.

EXAMPLE 7 Real-Life Application

The function $y = -16x^2 + 96x$ represents the height y (in feet) of a model rocket x seconds after it is launched. (a) Find the maximum height of the rocket. (b) Find and interpret the axis of symmetry.

SOLUTION

a. To find the maximum height, identify the maximum value of the function.

$y = -16x^2 + 96x$	Write the function.
$y = -16(x^2 - 6x)$	Factor out -16.
$y - 144 = -16(x^2 - 6x + 9)$	Complete the square for $x^2 - 6x$.
$y = -16(x - 3)^2 + 144$	Write in vertex form.

STUDY TIP

Adding 9 inside the parentheses results in subtracting 144 from the right side of the equation.

▶ Because the maximum value is 144, the model rocket reaches a maximum height of 144 feet.

b. The vertex is $(3, 144)$. So, the axis of symmetry is $x = 3$. On the left side of $x = 3$, the height increases as time increases. On the right side of $x = 3$, the height decreases as time increases.

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Determine whether the function could be represented by the graph in Example 6. Explain.

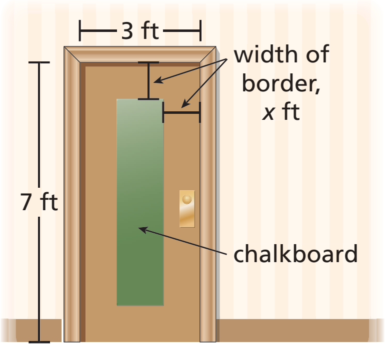
10. $h(x) = (x - 8)^2 + 10$

11. $n(x) = -2(x - 5)(x - 20)$

12. WHAT IF? Repeat Example 7 when the function is $y = -16x^2 + 128x$.

Solving Real-Life Problems

EXAMPLE 8 **Modeling with Mathematics**



You decide to use chalkboard paint to create a chalkboard on a door. You want the chalkboard to cover 6 square feet and to have a uniform border, as shown. Find the width of the border to the nearest inch.

SOLUTION

1. **Understand the Problem** You know the dimensions (in feet) of the door from the diagram. You also know the area (in square feet) of the chalkboard and that it will have a uniform border. You are asked to find the width of the border to the nearest inch.

2. **Make a Plan** Use a verbal model to write an equation that represents the area of the chalkboard. Then solve the equation.

3. **Solve the Problem**

Let x be the width (in feet) of the border, as shown in the diagram.

Area of chalkboard (square feet)	=	Length of chalkboard (feet)	•	Width of chalkboard (feet)
6	=	$(7 - 2x)$	•	$(3 - 2x)$

$6 = (7 - 2x)(3 - 2x)$ Write the equation.

$6 = 21 - 20x + 4x^2$ Multiply the binomials.

$-15 = 4x^2 - 20x$ Subtract 21 from each side.

$-\frac{15}{4} = x^2 - 5x$ Divide each side by 4.

$-\frac{15}{4} + \frac{25}{4} = x^2 - 5x + \frac{25}{4}$ Complete the square for $x^2 - 5x$.

$\frac{5}{2} = x^2 - 5x + \frac{25}{4}$ Simplify the left side.

$\frac{5}{2} = \left(x - \frac{5}{2}\right)^2$ Write the right side as the square of a binomial.

$\pm\sqrt{\frac{5}{2}} = x - \frac{5}{2}$ Take the square root of each side.

$\frac{5}{2} \pm \sqrt{\frac{5}{2}} = x$ Add $\frac{5}{2}$ to each side.

The solutions of the equation are $x = \frac{5}{2} + \sqrt{\frac{5}{2}} \approx 4.08$ and $x = \frac{5}{2} - \sqrt{\frac{5}{2}} \approx 0.92$.

It is not possible for the width of the border to be 4.08 feet because the width of the door is 3 feet. So, the width of the border is about 0.92 foot.

$0.92\ \cancel{\text{ft}} \cdot \frac{12 \text{ in.}}{1\ \cancel{\text{ft}}} = 11.04 \text{ in.}$ Convert 0.92 foot to inches.

▶ The width of the border should be about 11 inches.

4. **Look Back** When the width of the border is slightly less than 1 foot, the length of the chalkboard is slightly more than 5 feet and the width of the chalkboard is slightly more than 1 foot. Multiplying these dimensions gives an area close to 6 square feet. So, an 11-inch border is reasonable.

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13. **WHAT IF?** You want the chalkboard to cover 4 square feet. Find the width of the border to the nearest inch.

9.4 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The process of adding a constant c to the expression $x^2 + bx$ so that $x^2 + bx + c$ is a perfect square trinomial is called ________________.

2. **VOCABULARY** Explain how to complete the square for an expression of the form $x^2 + bx$.

3. **WRITING** Is it more convenient to complete the square for $x^2 + bx$ when b is odd or when b is even? Explain.

4. **WRITING** Describe how you can use the process of completing the square to find the maximum or minimum value of a quadratic function.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, find the value of c that completes the square.

5. $x^2 - 8x + c$

6. $x^2 - 2x + c$

7. $x^2 + 4x + c$

8. $x^2 + 12x + c$

9. $x^2 - 15x + c$

10. $x^2 + 9x + c$

In Exercises 11–16, complete the square for the expression. Then factor the trinomial. *(See Example 1.)*

11. $x^2 - 10x$

12. $x^2 - 40x$

13. $x^2 + 16x$

14. $x^2 + 22x$

15. $x^2 + 5x$

16. $x^2 - 3x$

In Exercises 17–22, solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary. *(See Example 2.)*

17. $x^2 + 14x = 15$

18. $x^2 - 6x = 16$

19. $x^2 - 4x = -2$

20. $x^2 + 2x = 5$

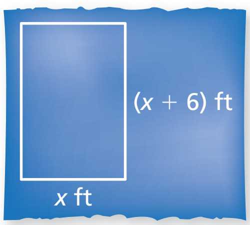
21. $x^2 - 5x = 8$

22. $x^2 + 11x = -10$

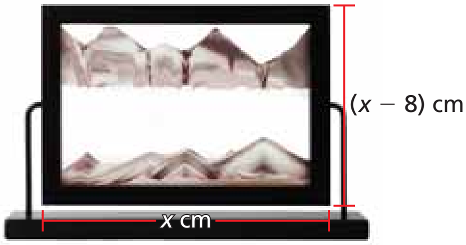
23. **MODELING WITH MATHEMATICS** The area of the patio is 216 square feet.

    a. Write an equation that represents the area of the patio.

    b. Find the dimensions of the patio by completing the square.



24. **MODELING WITH MATHEMATICS** Some sand art contains sand and water sealed in a glass case, similar to the one shown. When the art is turned upside down, the sand and water fall to create a new picture. The glass case has a depth of 1 centimeter and a volume of 768 cubic centimeters.



    a. Write an equation that represents the volume of the glass case.

    b. Find the dimensions of the glass case by completing the square.

In Exercises 25–32, solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary. *(See Example 3.)*

25. $x^2 - 8x + 15 = 0$

26. $x^2 + 4x - 21 = 0$

27. $2x^2 + 20x + 44 = 0$

28. $3x^2 - 18x + 12 = 0$

29. $-3x^2 - 24x + 17 = -40$

30. $-5x^2 - 20x + 35 = 30$

31. $2x^2 - 14x + 10 = 26$

32. $4x^2 + 12x - 15 = 5$

33. **ERROR ANALYSIS** Describe and correct the error in solving $x^2 + 8x = 10$ by completing the square.

$$\begin{aligned} x^2 + 8x &= 10 \\ x^2 + 8x + 16 &= 10 \\ (x+4)^2 &= 10 \\ x + 4 &= \pm\sqrt{10} \\ x &= -4 \pm \sqrt{10} \end{aligned}$$

34. **ERROR ANALYSIS** Describe and correct the error in the first two steps of solving $2x^2 - 2x - 4 = 0$ by completing the square.

$$\begin{aligned} 2x^2 - 2x - 4 &= 0 \\ 2x^2 - 2x &= 4 \\ 2x^2 - 2x + 1 &= 4 + 1 \end{aligned}$$

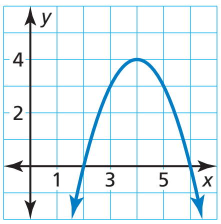
35. **NUMBER SENSE** Find all values of b for which $x^2 + bx + 25$ is a perfect square trinomial. Explain how you found your answer.

36. **REASONING** You are completing the square to solve $3x^2 + 6x = 12$. What is the first step?

In Exercises 37–40, write the function in vertex form by completing the square. Then match the function with its graph.

37. $y = x^2 + 6x + 3$ 38. $y = -x^2 + 8x - 12$

39. $y = -x^2 - 4x - 2$ 40. $y = x^2 - 2x + 4$

A.


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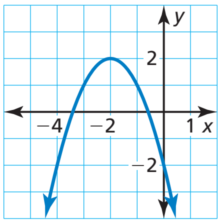

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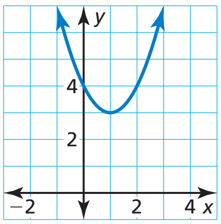
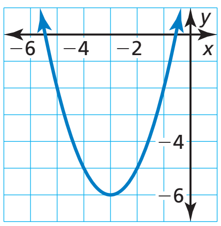


D.


In Exercises 41–46, determine whether the quadratic function has a maximum or minimum value. Then find the value. *(See Examples 4 and 5.)*

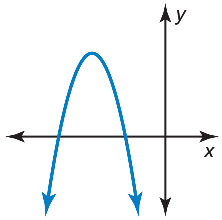
41. $y = x^2 - 4x - 2$ 42. $y = x^2 + 6x + 10$

43. $y = -x^2 - 10x - 30$ 44. $y = -x^2 + 14x - 34$

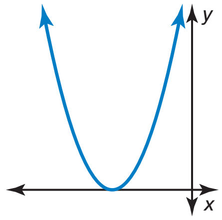
45. $f(x) = -3x^2 - 6x - 9$ 46. $f(x) = 4x^2 - 28x + 32$

In Exercises 47–50, determine whether the graph could represent the function. Explain.

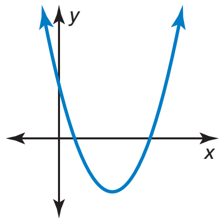
47. $y = -(x + 8)(x + 3)$



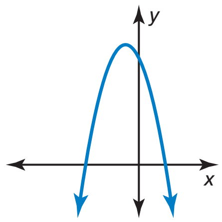
48. $y = (x - 5)^2$



49. $y = \frac{1}{4}(x + 2)^2 - 4$

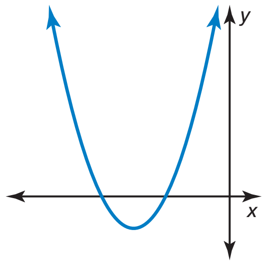


50. $y = -2(x - 1)(x + 2)$



In Exercises 51 and 52, determine which of the functions could be represented by the graph. Explain. *(See Example 6.)*

51.



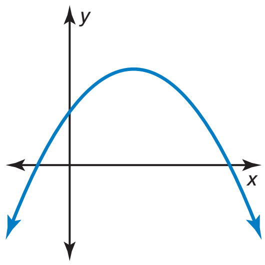
$h(x) = (x + 2)^2 + 3$

$f(x) = 2(x + 3)^2 - 2$

$g(x) = -\frac{1}{2}(x - 8)(x - 4)$

$m(x) = (x + 2)(x + 4)$

52.



$r(x) = -\frac{1}{3}(x - 5)(x + 1)$

$p(x) = -2(x - 2)(x - 6)$

$q(x) = (x + 1)^2 + 4$

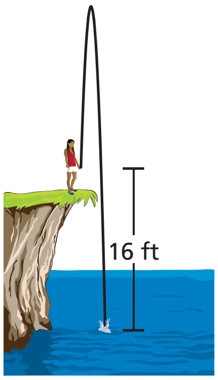
$n(x) = -(x - 2)^2 + 9$

53. **MODELING WITH MATHEMATICS** The function $h = -16t^2 + 48t$ represents the height h (in feet) of a kickball t seconds after it is kicked from the ground. *(See Example 7.)*

a. Find the maximum height of the kickball.

b. Find and interpret the axis of symmetry.

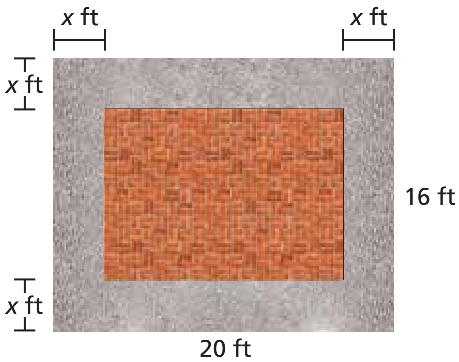
54. MODELING WITH MATHEMATICS You throw a stone from a height of 16 feet with an initial vertical velocity of 32 feet per second. The function $h = -16t^2 + 32t + 16$ represents the height h (in feet) of the stone after t seconds.



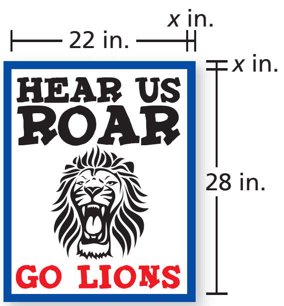
a. Find the maximum height of the stone.

b. Find and interpret the axis of symmetry.

55. MODELING WITH MATHEMATICS You are building a rectangular brick patio surrounded by a crushed stone border with a uniform width, as shown. You purchase patio bricks to cover 140 square feet. Find the width of the border. *(See Example 8.)*

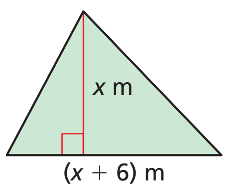


56. MODELING WITH MATHEMATICS You are making a poster that will have a uniform border, as shown. The total area of the poster is 722 square inches. Find the width of the border to the nearest inch.

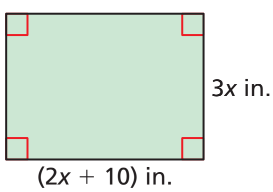


MATHEMATICAL CONNECTIONS **In Exercises 57 and 58, find the value of x. Round your answer to the nearest hundredth, if necessary.**

57. $A = 108 \text{ m}^2$



58. $A = 288 \text{ in.}^2$



In Exercises 59–62, solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary.

59. $0.5x^2 + x - 2 = 0$

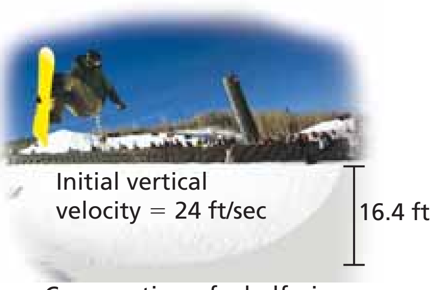
60. $0.75x^2 + 1.5x = 4$

61. $\frac{8}{3}x - \frac{2}{3}x^2 = -\frac{5}{6}$

62. $\frac{1}{4}x^2 + \frac{1}{2}x - \frac{5}{4} = 0$

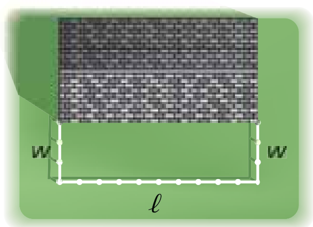
63. PROBLEM SOLVING The distance d (in feet) that it takes a car to come to a complete stop can be modeled by $d = 0.05s^2 + 2.2s$, where s is the speed of the car (in miles per hour). A car has 168 feet to come to a complete stop. Find the maximum speed at which the car can travel.

64. PROBLEM SOLVING During a "big air" competition, snowboarders launch themselves from a half-pipe, perform tricks in the air, and land back in the half-pipe. The height h (in feet) of a snowboarder above the bottom of the half-pipe can be modeled by $h = -16t^2 + 24t + 16.4$, where t is the time (in seconds) after the snowboarder launches into the air. The snowboarder lands 3.2 feet lower than the height of the launch. How long is the snowboarder in the air? Round your answer to the nearest tenth of a second.



Cross section of a half-pipe

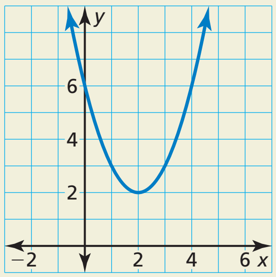
65. PROBLEM SOLVING You have 80 feet of fencing to make a rectangular horse pasture that covers 750 square feet. A barn will be used as one side of the pasture, as shown.



a. Write equations for the amount of fencing to be used and the area enclosed by the fencing.

b. Use substitution to solve the system of equations from part (a). What are the possible dimensions of the pasture?

66. HOW DO YOU SEE IT? The graph represents the quadratic function $y = x^2 - 4x + 6$.



a. Use the graph to estimate the x-values for which $y = 3$.

b. Explain how you can use the method of completing the square to check your estimates in part (a).

67. COMPARING METHODS Consider the quadratic equation $x^2 + 12x + 2 = 12$.

a. Solve the equation by graphing.

b. Solve the equation by completing the square.

c. Compare the two methods. Which do you prefer? Explain.

68. THOUGHT PROVOKING Sketch the graph of the equation $x^2 - 2xy + y^2 - x - y = 0$. Identify the graph.

69. REASONING The product of two consecutive even integers that are positive is 48. Write and solve an equation to find the integers.

70. REASONING The product of two consecutive odd integers that are negative is 195. Write and solve an equation to find the integers.

71. MAKING AN ARGUMENT You purchase stock for \$16 per share. You sell the stock 30 days later for \$23.50 per share. The price y (in dollars) of a share during the 30-day period can be modeled by $y = -0.025x^2 + x + 16$, where x is the number of days after the stock is purchased. Your friend says you could have sold the stock earlier for \$23.50 per share. Is your friend correct? Explain.

72. REASONING You are solving the equation $x^2 + 9x = 18$. What are the advantages of solving the equation by completing the square instead of using other methods you have learned?

73. PROBLEM SOLVING You are knitting a rectangular scarf. The pattern results in a scarf that is 60 inches long and 4 inches wide. However, you have enough yarn to knit 396 square inches. You decide to increase the dimensions of the scarf so that you will use all your yarn. The increase in the length is three times the increase in the width. What are the dimensions of your scarf?

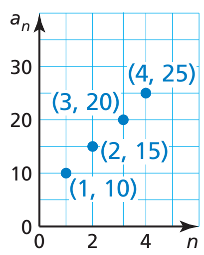
74. WRITING How many solutions does $x^2 + bx = c$ have when $c < -\left(\frac{b}{2}\right)^2$? Explain.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write a recursive rule for the sequence. *(Section 6.6)*

75.



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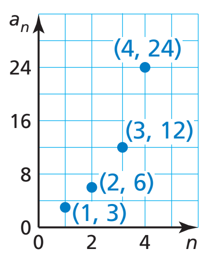

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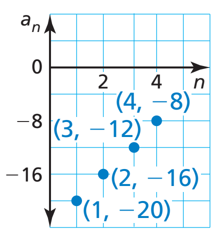

Simplify the expression $\sqrt{b^2 - 4ac}$ for the given values. *(Section 9.1)*

78. $a = 3, b = -6, c = 2$

79. $a = -2, b = 4, c = 7$

80. $a = 1, b = 6, c = 4$

9.5 Solving Quadratic Equations Using the Quadratic Formula

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.8.A

Essential Question

How can you derive a formula that can be used to write the solutions of any quadratic equation in standard form?

EXPLORATION 1 Deriving the Quadratic Formula

Work with a partner. The following steps show a method of solving $ax^2 + bx + c = 0$. Explain what was done in each step.

$$ax^2 + bx + c = 0$$ ← 1. Write the equation.

$$4a^2x^2 + 4abx + 4ac = 0$$ ← 2. What was done?

$$4a^2x^2 + 4abx + 4ac + b^2 = b^2$$ ← 3. What was done?

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$ ← 4. What was done?

$$(2ax + b)^2 = b^2 - 4ac$$ ← 5. What was done?

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$ ← 6. What was done?

$$2ax = -b \pm \sqrt{b^2 - 4ac}$$ ← 7. What was done?

Quadratic Formula: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ ← 8. What was done?

EXPLORATION 2 Deriving the Quadratic Formula by Completing the Square

Work with a partner.

a. Solve $ax^2 + bx + c = 0$ by completing the square. (*Hint:* Subtract c from each side, divide each side by a, and then proceed by completing the square.)

b. Compare this method with the method in Exploration 1. Explain why you think $4a$ and b^2 were chosen in Steps 2 and 3 of Exploration 1.

Communicate Your Answer

3. How can you derive a formula that can be used to write the solutions of any quadratic equation in standard form?

4. Use the Quadratic Formula to solve each quadratic equation.

a. $x^2 + 2x - 3 = 0$ **b.** $x^2 - 4x + 4 = 0$ **c.** $x^2 + 4x + 5 = 0$

5. Use the Internet to research *imaginary numbers*. How are they related to quadratic equations?

SELECTING TOOLS

To be proficient in math, you need to identify relevant external mathematical resources.

9.5 Lesson

Core Vocabulary

Quadratic Formula, *p. 504*
discriminant, *p. 506*

What You Will Learn

- Solve quadratic equations using the Quadratic Formula.
- Interpret the discriminant.
- Choose efficient methods for solving quadratic equations.

Using the Quadratic Formula

By completing the square for the quadratic equation $ax^2 + bx + c = 0$, you can develop a formula that gives the solutions of any quadratic equation in standard form. This formula is called the **Quadratic Formula**.

Core Concept

Quadratic Formula

The real solutions of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

where $a \neq 0$ and $b^2 - 4ac \geq 0$.

EXAMPLE 1 Using the Quadratic Formula

Solve $2x^2 - 5x + 3 = 0$ using the Quadratic Formula.

SOLUTION

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(3)}}{2(2)} \qquad \text{Substitute 2 for } a\text{, } -5 \text{ for } b\text{, and 3 for } c.$$

$$= \frac{5 \pm \sqrt{1}}{4} \qquad \text{Simplify.}$$

$$= \frac{5 \pm 1}{4} \qquad \text{Evaluate the square root.}$$

So, the solutions are $x = \frac{5+1}{4} = \frac{3}{2}$ and $x = \frac{5-1}{4} = 1$.

STUDY TIP

You can use the roots of a quadratic equation to factor the related expression. In Example 1, you can use 1 and $\frac{3}{2}$ to factor $2x^2 - 5x + 3$ as $(x - 1)(2x - 3)$.

Check

$2x^2 - 5x + 3 = 0$	Original equation	$2x^2 - 5x + 3 = 0$
$2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3 \stackrel{?}{=} 0$	Substitute.	$2(1)^2 - 5(1) + 3 \stackrel{?}{=} 0$
$\frac{9}{2} - \frac{15}{2} + 3 \stackrel{?}{=} 0$	Simplify.	$2 - 5 + 3 \stackrel{?}{=} 0$
$0 = 0$ ✓	Simplify.	$0 = 0$ ✓

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Solve the equation using the Quadratic Formula. Round your solutions to the nearest tenth, if necessary.

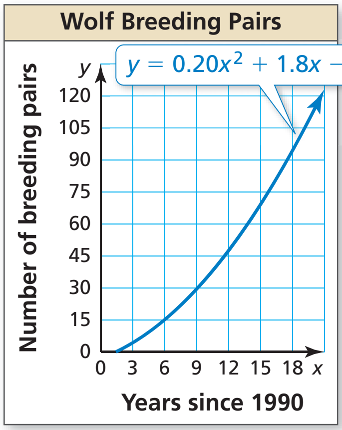
1. $x^2 - 6x + 5 = 0$

2. $\frac{1}{2}x^2 + x - 10 = 0$

3. $-3x^2 + 2x + 7 = 0$

4. $4x^2 - 4x = -1$

EXAMPLE 2 Modeling With Mathematics



The number y of Northern Rocky Mountain wolf breeding pairs x years since 1990 can be modeled by the function $y = 0.20x^2 + 1.8x - 3$. When were there about 35 breeding pairs?

SOLUTION

1. **Understand the Problem** You are given a quadratic function that represents the number of wolf breeding pairs for years after 1990. You need to use the model to determine when there were 35 wolf breeding pairs.

2. **Make a Plan** To determine when there were 35 wolf breeding pairs, find the x-values for which $y = 35$. So, solve the equation $35 = 0.20x^2 + 1.8x - 3$.

3. **Solve the Problem**

$$35 = 0.20x^2 + 1.8x - 3 \qquad \text{Write the equation.}$$

$$0 = 0.20x^2 + 1.8x - 38 \qquad \text{Write in standard form.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$= \frac{-1.8 \pm \sqrt{1.8^2 - 4(0.2)(-38)}}{2(0.2)} \qquad \text{Substitute 0.2 for } a\text{, 1.8 for } b\text{, and } -38 \text{ for } c.$$

$$= \frac{-1.8 \pm \sqrt{33.64}}{0.4} \qquad \text{Simplify.}$$

$$= \frac{-1.8 \pm 5.8}{0.4} \qquad \text{Simplify.}$$

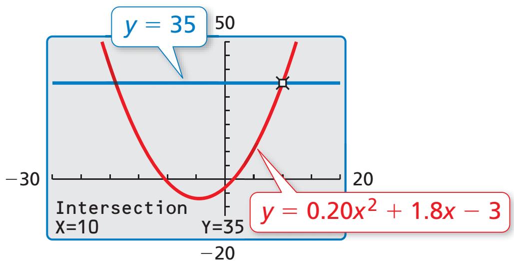
The solutions are $x = \dfrac{-1.8 + 5.8}{0.4} = 10$ and $x = \dfrac{-1.8 - 5.8}{0.4} = -19$.

EXPLAINING MATHEMATICAL IDEAS

You can ignore the solution $x = -19$ because -19 represents the year 1971, which is not in the given time period.

▶ Because x represents the number of years since 1990, x is greater than or equal to zero. So, there were about 35 breeding pairs 10 years after 1990, in 2000.

4. **Look Back** Use a graphing calculator to graph the equations $y = 0.20x^2 + 1.8x - 3$ and $y = 35$. Then use the *intersect* feature to find the point of intersection. The graphs intersect at (10, 35).



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5. **WHAT IF?** When were there about 60 wolf breeding pairs?

6. The number y of bald eagle nesting pairs in a state x years since 2000 can be modeled by the function $y = 0.34x^2 + 13.1x + 51$.

   a. When were there about 160 bald eagle nesting pairs?

   b. How many bald eagle nesting pairs were there in 2000?

Interpreting the Discriminant

The expression $b^2 - 4ac$ in the Quadratic Formula is called the **discriminant**.

$$x = \frac{-b \pm \sqrt{\underbrace{b^2 - 4ac}_{\text{discriminant}}}}{2a}$$

Because the discriminant is under the radical symbol, you can use the value of the discriminant to determine the number of real solutions of a quadratic equation and the number of x-intercepts of the graph of the related function.

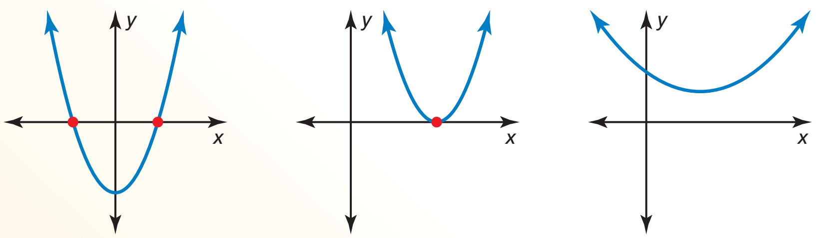
STUDY TIP

The solutions of a quadratic equation may be real numbers or *imaginary numbers*. You will study imaginary numbers in a future course.

Core Concept

Interpreting the Discriminant

$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$



• two real solutions • two x-intercepts	• one real solution • one x-intercept	• no real solutions • no x-intercepts

EXAMPLE 3 Determining the Number of Real Solutions

a. Determine the number of real solutions of $x^2 + 8x - 3 = 0$.

$$\begin{aligned} b^2 - 4ac &= 8^2 - 4(1)(-3) && \text{Substitute 1 for } a\text{, 8 for } b\text{, and } -3 \text{ for } c. \\ &= 64 + 12 && \text{Simplify.} \\ &= 76 && \text{Add.} \end{aligned}$$

▶ The discriminant is greater than 0. So, the equation has two real solutions.

b. Determine the number of real solutions of $9x^2 + 1 = 6x$.

Write the equation in standard form: $9x^2 - 6x + 1 = 0$.

$$\begin{aligned} b^2 - 4ac &= (-6)^2 - 4(9)(1) && \text{Substitute 9 for } a\text{, } -6 \text{ for } b\text{, and 1 for } c. \\ &= 36 - 36 && \text{Simplify.} \\ &= 0 && \text{Subtract.} \end{aligned}$$

▶ The discriminant is 0. So, the equation has one real solution.

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Determine the number of real solutions of the equation.

7. $-x^2 + 4x - 4 = 0$

8. $6x^2 + 2x = -1$

9. $\frac{1}{2}x^2 = 7x - 1$

EXAMPLE 4 Finding the Number of x-Intercepts of a Parabola

Find the number of x-intercepts of the graph of $y = 2x^2 + 3x + 9$.

SOLUTION

Determine the number of real solutions of $0 = 2x^2 + 3x + 9$.

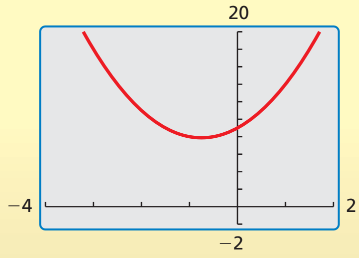
$$b^2 - 4ac = 3^2 - 4(2)(9) \quad \text{Substitute 2 for } a\text{, 3 for } b\text{, and 9 for } c.$$
$$= 9 - 72 \quad \text{Simplify.}$$
$$= -63 \quad \text{Subtract.}$$

Because the discriminant is less than 0, the equation has no real solutions.

▶ So, the graph of $y = 2x^2 + 3x + 9$ has no x-intercepts.

Check

Use a graphing calculator to check your answer. Notice that the graph of $y = 2x^2 + 3x + 9$ has no x-intercepts.



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Find the number of x-intercepts of the graph of the function.

10. $y = -x^2 + x - 6$ **11.** $y = x^2 - x$ **12.** $f(x) = x^2 + 12x + 36$

Choosing an Efficient Method

The table shows five methods for solving quadratic equations. For a given equation, it may be more efficient to use one method instead of another. Some advantages and disadvantages of each method are shown.

Core Concept

Methods for Solving Quadratic Equations

Method	Advantages	Disadvantages
Factoring *(Lessons 7.6–7.9)*	• Straightforward when the equation can be factored easily	• Some equations are not factorable.
Graphing *(Lesson 9.2)*	• Can easily see the number of solutions • Use when approximate solutions are sufficient. • Can use a graphing calculator	• May not give exact solutions
Using Square Roots *(Lesson 9.3)*	• Use to solve equations of the form $x^2 = d$.	• Can only be used for certain equations
Completing the Square *(Lesson 9.4)*	• Best used when $a = 1$ and b is even	• May involve difficult calculations
Quadratic Formula *(Lesson 9.5)*	• Can be used for any quadratic equation • Gives exact solutions	• Takes time to do calculations

EXAMPLE 5 Choosing a Method

Solve the equation using any method. Explain your choice of method.

a. $x^2 - 10x = 1$ **b.** $2x^2 - 13x - 24 = 0$ **c.** $x^2 + 8x + 12 = 0$

SOLUTION

a. The coefficient of the x^2-term is 1, and the coefficient of the x-term is an even number. So, solve by completing the square.

$$x^2 - 10x = 1 \quad \text{Write the equation.}$$

$$x^2 - 10x + 25 = 1 + 25 \quad \text{Complete the square for } x^2 - 10x.$$

$$(x - 5)^2 = 26 \quad \text{Write the left side as the square of a binomial.}$$

$$x - 5 = \pm\sqrt{26} \quad \text{Take the square root of each side.}$$

$$x = 5 \pm \sqrt{26} \quad \text{Add 5 to each side.}$$

▶ So, the solutions are $x = 5 + \sqrt{26} \approx 10.1$ and $x = 5 - \sqrt{26} \approx -0.1$.

b. The equation is not easily factorable, and the numbers are somewhat large. So, solve using the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Quadratic Formula}$$

$$= \frac{-(-13) \pm \sqrt{(-13)^2 - 4(2)(-24)}}{2(2)} \quad \text{Substitute 2 for } a, -13 \text{ for } b, \text{ and } -24 \text{ for } c.$$

$$= \frac{13 \pm \sqrt{361}}{4} \quad \text{Simplify.}$$

$$= \frac{13 \pm 19}{4} \quad \text{Evaluate the square root.}$$

▶ So, the solutions are $x = \frac{13 + 19}{4} = 8$ and $x = \frac{13 - 19}{4} = -\frac{3}{2}$.

c. The equation is easily factorable. So, solve by factoring.

$$x^2 + 8x + 12 = 0 \quad \text{Write the equation.}$$

$$(x + 2)(x + 6) = 0 \quad \text{Factor the polynomial.}$$

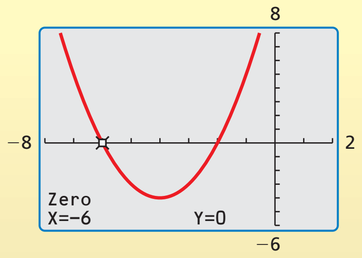
$$x + 2 = 0 \quad \text{or} \quad x + 6 = 0 \quad \text{Zero-Product Property}$$

$$x = -2 \quad \text{or} \quad x = -6 \quad \text{Solve for } x.$$

▶ The solutions are $x = -2$ and $x = -6$.

Check

Graph the related function $f(x) = x^2 + 8x + 12$ and find the zeros. The zeros are -6 and -2.



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Solve the equation using any method. Explain your choice of method.

13. $x^2 + 11x - 12 = 0$

14. $9x^2 - 5 = 4$

15. $5x^2 - x - 1 = 0$

16. $x^2 = 2x - 5$

9.5 Exercises

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Vocabulary and Core Concept Check

1. **VOCABULARY** What formula can you use to solve any quadratic equation? Write the formula.

2. **VOCABULARY** In the Quadratic Formula, what is the discriminant? What does the value of the discriminant determine?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, write the equation in standard form. Then identify the values of *a*, *b*, and *c* that you would use to solve the equation using the Quadratic Formula.

3. $x^2 = 7x$
4. $x^2 - 4x = -12$
5. $-2x^2 + 1 = 5x$
6. $3x + 2 = 4x^2$
7. $4 - 3x = -x^2 + 3x$
8. $-8x - 1 = 3x^2 + 2$

In Exercises 9–22, solve the equation using the Quadratic Formula. Round your solutions to the nearest tenth, if necessary. *(See Example 1.)*

9. $x^2 - 12x + 36 = 0$
10. $x^2 + 7x + 16 = 0$
11. $x^2 - 10x - 11 = 0$
12. $2x^2 - x - 1 = 0$
13. $2x^2 - 6x + 5 = 0$
14. $9x^2 - 6x + 1 = 0$
15. $6x^2 - 13x = -6$
16. $-3x^2 + 6x = 4$
17. $1 - 8x = -16x^2$
18. $x^2 - 5x + 3 = 0$
19. $x^2 + 2x = 9$
20. $5x^2 - 2 = 4x$
21. $2x^2 + 9x + 7 = 3$
22. $8x^2 + 8 = 6 - 9x$

23. **MODELING WITH MATHEMATICS** A dolphin jumps out of the water, as shown in the diagram. The function $h = -16t^2 + 26t$ models the height h (in feet) of the dolphin after t seconds. After how many seconds is the dolphin at a height of 5 feet? *(See Example 2.)*

24. **MODELING WITH MATHEMATICS** The amount of trout y (in tons) caught in a lake from 1995 to 2014 can be modeled by the equation $y = -0.08x^2 + 1.6x + 10$, where x is the number of years since 1995.

    a. When were about 15 tons of trout caught in the lake?

    b. Do you think this model can be used to determine the amounts of trout caught in future years? Explain your reasoning.

In Exercises 25–30, determine the number of real solutions of the equation. *(See Example 3.)*

25. $x^2 - 6x + 10 = 0$
26. $x^2 - 5x - 3 = 0$
27. $2x^2 - 12x = -18$
28. $4x^2 = 4x - 1$
29. $-\frac{1}{4}x^2 + 4x = -2$
30. $-5x^2 + 8x = 9$

In Exercises 31–36, find the number of *x*-intercepts of the graph of the function. *(See Example 4.)*

31. $y = x^2 + 5x - 1$
32. $y = 4x^2 + 4x + 1$
33. $y = -6x^2 + 3x - 4$
34. $y = -x^2 + 5x + 13$
35. $f(x) = 4x^2 + 3x - 6$
36. $f(x) = 2x^2 + 8x + 8$

In Exercises 37–44, solve the equation using any method. Explain your choice of method. *(See Example 5.)*

37. $-10x^2 + 13x = 4$
38. $x^2 - 3x - 40 = 0$
39. $x^2 + 6x = 5$
40. $-5x^2 = -25$
41. $x^2 + x - 12 = 0$
42. $x^2 - 4x + 1 = 0$
43. $4x^2 - x = 17$
44. $x^2 + 6x + 9 = 16$

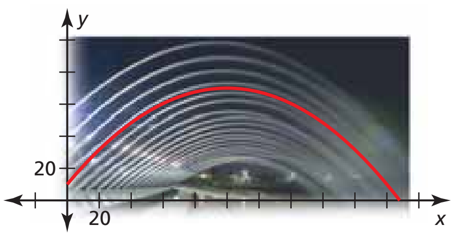
45. ERROR ANALYSIS Describe and correct the error in solving the equation $3x^2 - 7x - 6 = 0$ using the Quadratic Formula.

$$x = \frac{-7 \pm \sqrt{(-7)^2 - 4(3)(-6)}}{2(3)}$$
$$= \frac{-7 \pm \sqrt{121}}{6}$$
$$x = \frac{2}{3} \text{ and } x = -3$$

46. ERROR ANALYSIS Describe and correct the error in solving the equation $-2x^2 + 9x = 4$ using the Quadratic Formula.

$$x = \frac{-9 \pm \sqrt{9^2 - 4(-2)(4)}}{2(-2)}$$
$$= \frac{-9 \pm \sqrt{113}}{-4}$$
$$x \approx -0.41 \text{ and } x \approx 4.91$$

47. MODELING WITH MATHEMATICS A fountain shoots a water arc that can be modeled by the graph of the equation $y = -0.006x^2 + 1.2x + 10$, where x is the horizontal distance (in feet) from the river's north shore and y is the height (in feet) above the river. Does the water arc reach a height of 50 feet? If so, about how far from the north shore is the water arc 50 feet above the water?



48. MODELING WITH MATHEMATICS Between the months of April and September, the number y of hours of daylight per day in Seattle, Washington, can be modeled by $y = -0.00046x^2 + 0.076x + 13$, where x is the number of days since April 1.

a. Do any of the days between April and September in Seattle have 17 hours of daylight? If so, how many?

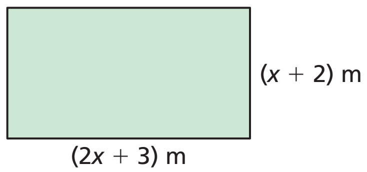
b. Do any of the days between April and September in Seattle have 14 hours of daylight? If so, how many?

49. MAKING AN ARGUMENT Your friend uses the discriminant of the equation $2x^2 - 5x - 2 = -11$ and determines that the equation has two real solutions. Is your friend correct? Explain your reasoning.

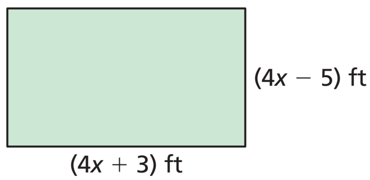
50. MODELING WITH MATHEMATICS The frame of the tent shown is defined by a rectangular base and two parabolic arches that connect the opposite corners of the base. The graph of $y = -0.18x^2 + 1.6x$ models the height y (in feet) of one of the arches x feet along the diagonal of the base. Can a child who is 4 feet tall walk under one of the arches without having to bend over? Explain.

MATHEMATICAL CONNECTIONS In Exercises 51 and 52, use the given area A of the rectangle to find the value of x. Then give the dimensions of the rectangle.

51. $A = 91 \text{ m}^2$



52. $A = 209 \text{ ft}^2$



COMPARING METHODS In Exercises 53 and 54, solve the equation by (a) graphing, (b) factoring, and (c) using the Quadratic Formula. Which method do you prefer? Explain your reasoning.

53. $x^2 + 4x + 4 = 0$

54. $3x^2 + 11x + 6 = 0$

55. REASONING How many solutions does the equation $ax^2 + bx + c = 0$ have when a and c have different signs? Explain your reasoning.

56. REASONING When the discriminant is a perfect square, are the solutions of $ax^2 + bx + c = 0$ rational or irrational? (Assume a, b, and c are integers.) Explain your reasoning.

REASONING In Exercises 57–59, give a value of c for which the equation has (a) two solutions, (b) one solution, and (c) no solutions.

57. $x^2 - 2x + c = 0$

58. $x^2 - 8x + c = 0$

59. $4x^2 + 12x + c = 0$

60. REPEATED REASONING You use the Quadratic Formula to solve an equation.

a. You obtain solutions that are integers. Could you have used factoring to solve the equation? Explain your reasoning.

b. You obtain solutions that are fractions. Could you have used factoring to solve the equation? Explain your reasoning.

c. Make a generalization about quadratic equations with rational solutions.

61. MODELING WITH MATHEMATICS The fuel economy y (in miles per gallon) of a car can be modeled by the equation $y = -0.013x^2 + 1.25x + 5.6$, where $5 \le x \le 75$ and x is the speed (in miles per hour) of the car. Find the speed(s) at which you can travel and have a fuel economy of 32 miles per gallon.

62. MODELING WITH MATHEMATICS The depth d (in feet) of a river can be modeled by the equation $d = -0.25t^2 + 1.7t + 3.5$, where $0 \le t \le 7$ and t is the time (in hours) after a heavy rain begins. When is the river 6 feet deep?

ANALYZING EQUATIONS In Exercises 63–68, tell whether the vertex of the graph of the function lies above, below, or on the *x*-axis. Explain your reasoning without using a graph.

63. $y = x^2 - 3x + 2$

64. $y = 3x^2 - 6x + 3$

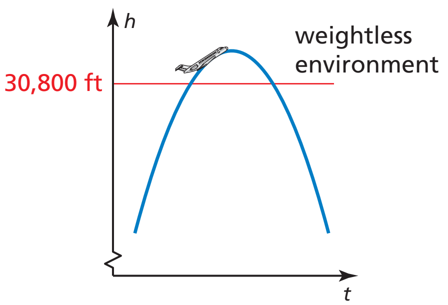
65. $y = 6x^2 - 2x + 4$

66. $y = -15x^2 + 10x - 25$

67. $f(x) = -3x^2 - 4x + 8$

68. $f(x) = 9x^2 - 24x + 16$

69. REASONING NASA creates a weightless environment by flying a plane in a series of parabolic paths. The height h (in feet) of a plane after t seconds in a parabolic flight path can be modeled by $h = -11t^2 + 700t + 21{,}000$. The passengers experience a weightless environment when the height of the plane is greater than or equal to 30,800 feet. For approximately how many seconds do passengers experience weightlessness on such a flight? Explain.

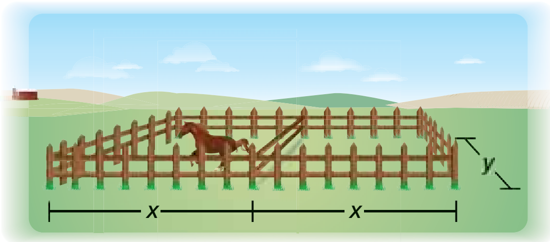


70. WRITING EQUATIONS Use the numbers to create a quadratic equation with the solutions $x = -1$ and $x = -\frac{1}{4}$.

$$\_\_\_x^2 + \_\_\_x + \_\_\_ = 0$$

−5	−4	−3	−2	−1
1	2	3	4	5

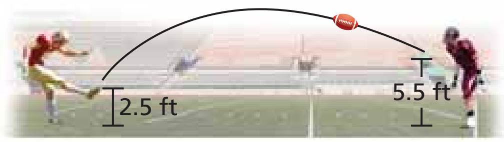
71. PROBLEM SOLVING A rancher constructs two rectangular horse pastures that share a side, as shown. The pastures are enclosed by 1050 feet of fencing. Each pasture has an area of 15,000 square feet.



a. Show that $y = 350 - \frac{4}{3}x$.

b. Find the possible lengths and widths of each pasture.

72. PROBLEM SOLVING A kicker punts a football from a height of 2.5 feet above the ground with an initial vertical velocity of 45 feet per second.



Not drawn to scale

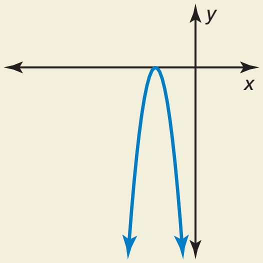
a. Write an equation that models this situation using the function $h = -16t^2 + v_0t + s_0$, where h is the height (in feet) of the football, t is the time (in seconds) after the football is punted, v_0 is the initial vertical velocity (in feet per second), and s_0 is the initial height (in feet).

b. The football is caught 5.5 feet above the ground, as shown in the diagram. Find the amount of time that the football is in the air.

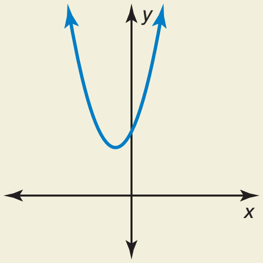
73. CRITICAL THINKING The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. Find the mean of the solutions. How is the mean of the solutions related to the graph of $y = ax^2 + bx + c$? Explain.

74. **HOW DO YOU SEE IT?** Match each graph with its discriminant. Explain your reasoning.

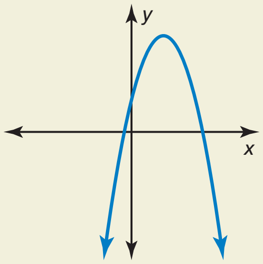
A.



B.



C.



a. $b^2 - 4ac > 0$

b. $b^2 - 4ac = 0$

c. $b^2 - 4ac < 0$

75. **CRITICAL THINKING** You are trying to hang a tire swing. To get the rope over a tree branch that is 15 feet high, you tie the rope to a weight and throw it over the branch. You release the weight at a height s_0 of 5.5 feet. What is the minimum initial vertical velocity v_0 needed to reach the branch? (*Hint*: Use the equation $h = -16t^2 + v_0t + s_0$.)

76. **THOUGHT PROVOKING** Consider the graph of the standard form of a quadratic function $y = ax^2 + bx + c$. Then consider the Quadratic Formula as given by

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Write a graphical interpretation of the two parts of this formula.

77. **ANALYZING RELATIONSHIPS** Find the sum and product of $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. Then write a quadratic equation whose solutions have a sum of 2 and a product of $\frac{1}{2}$.

78. **WRITING A FORMULA** Derive a formula that can be used to find solutions of equations that have the form $ax^2 + x + c = 0$. Use your formula to solve $-2x^2 + x + 8 = 0$.

79. **MULTIPLE REPRESENTATIONS** If p is a solution of a quadratic equation $ax^2 + bx + c = 0$, then $(x - p)$ is a factor of $ax^2 + bx + c$.

a. Copy and complete the table for each pair of solutions.

Solutions	Factors	Quadratic equation
3, 4	$(x - 3), (x - 4)$	$x^2 - 7x + 12 = 0$
$-1, 6$		
0, 2		
$-\frac{1}{2}, 5$		

b. Graph the related function for each equation. Identify the zeros of the function.

CRITICAL THINKING **In Exercises 80–82, find all values of k for which the equation has (a) two solutions, (b) one solution, and (c) no solutions.**

80. $2x^2 + x + 3k = 0$

81. $x^2 - 4kx + 36 = 0$

82. $kx^2 + 5x - 16 = 0$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the system of linear equations using any method. Explain why you chose the method. *(Section 5.1, Section 5.2, and Section 5.3)*

83. $y = -x + 4$
$y = 2x - 8$

84. $x = 16 - 4y$
$3x + 4y = 8$

85. $2x - y = 7$
$2x + 7y = 31$

86. $3x - 2y = -20$
$x + 1.2y = 6.4$

9.4–9.5 What Did You Learn?

Core Vocabulary

completing the square, *p. 494*
Quadratic Formula, *p. 504*
discriminant, *p. 506*

Core Concepts

Section 9.4

Completing the Square, *p. 494*

Section 9.5

Quadratic Formula, *p. 504*
Interpreting the Discriminant, *p. 506*

Mathematical Thinking

1. How does your answer to Exercise 74 on page 502 help create a shortcut when solving some quadratic equations by completing the square?
2. What logical progression led you to your answer in Exercise 55 on page 510?

Performance Task

Form Matters

Each form of a quadratic function has its pros and cons. Using one form, you can easily find the vertex, but the zeros are more difficult to find. Using another form, you can easily find the y-intercept, but the vertex is more difficult to find. Which form would you use in different situations? How can you convert one form into another?

To explore the answers to these questions and more, go to ***BigIdeasMath.com***.

9 Chapter Review

9.1 Properties of Radicals (pp. 465–474)

a. Simplify $\sqrt{\frac{19}{169}}$.

$$\sqrt{\frac{19}{169}} = \frac{\sqrt{19}}{\sqrt{169}} \qquad \text{Quotient Property of Square Roots}$$

$$= \frac{\sqrt{19}}{13} \qquad \text{Simplify.}$$

b. Simplify $\sqrt[3]{27x^{10}}$.

$$\sqrt[3]{27x^{10}} = \sqrt[3]{27 \cdot x^9 \cdot x} \qquad \text{Factor using the greatest perfect cube factors.}$$

$$= \sqrt[3]{27} \cdot \sqrt[3]{x^9} \cdot \sqrt[3]{x} \qquad \text{Product Property of Cube Roots}$$

$$= 3x^3\sqrt[3]{x} \qquad \text{Simplify.}$$

c. Simplify $\frac{12}{3 + \sqrt{5}}$.

$$\frac{12}{3+\sqrt{5}} = \frac{12}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}} \qquad \text{The conjugate of } 3+\sqrt{5} \text{ is } 3-\sqrt{5}.$$

$$= \frac{12(3-\sqrt{5})}{3^2 - (\sqrt{5})^2} \qquad \text{Sum and difference pattern}$$

$$= \frac{36 - 12\sqrt{5}}{4} \qquad \text{Simplify.}$$

$$= 9 - 3\sqrt{5} \qquad \text{Simplify.}$$

d. Simplify $9\sqrt{6} + \sqrt{10} + 7\sqrt{6}$.

$$9\sqrt{6} + \sqrt{10} + 7\sqrt{6} = 9\sqrt{6} + 7\sqrt{6} + \sqrt{10} \qquad \text{Commutative Property of Addition}$$

$$= (9+7)\sqrt{6} + \sqrt{10} \qquad \text{Distributive Property}$$

$$= 16\sqrt{6} + \sqrt{10} \qquad \text{Add.}$$

Simplify the expression.

1. $\sqrt{124}$ **2.** $\sqrt{72p^7}$ **3.** $\sqrt{\frac{45}{7y}}$

4. $\sqrt[3]{\frac{125x^{11}}{4}}$ **5.** $\frac{\sqrt{15}}{\sqrt{10}}$ **6.** $\frac{4}{\sqrt{6x}}$

7. $\frac{8}{\sqrt{6}+2}$ **8.** $4\sqrt{3} + 5\sqrt{12}$ **9.** $\sqrt{5} + 2\sqrt{7} - 2\sqrt{5}$

10. $15\sqrt[3]{2} - 2\sqrt[3]{54}$ **11.** $(3\sqrt{7} + 5)^2$ **12.** $\sqrt{6}(\sqrt{18} + \sqrt{8})$

9.2 Solving Quadratic Equations by Graphing (pp. 475–484)

Solve $x^2 + 3x = 4$ by graphing.

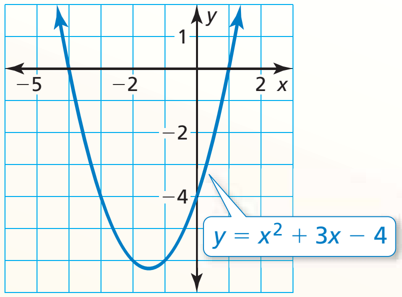
Step 1 Write the equation in standard form.

$$x^2 + 3x = 4 \quad \text{Write original equation.}$$

$$x^2 + 3x - 4 = 0 \quad \text{Subtract 4 from each side.}$$

Step 2 Graph the related function $y = x^2 + 3x - 4$.

Step 3 Find the x-intercepts. The x-intercepts are -4 and 1.



▶ So, the solutions are $x = -4$ and $x = 1$.

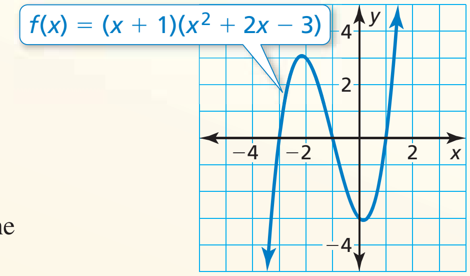
Solve the equation by graphing.

13. $x^2 - 9x + 18 = 0$

14. $x^2 - 2x = -4$

15. $-8x - 16 = x^2$

16. The graph of $f(x) = (x + 1)(x^2 + 2x - 3)$ is shown. Find the zeros of f.



17. Graph $f(x) = x^2 + 2x - 5$. Approximate the zeros of f to the nearest tenth.

9.3 Solving Quadratic Equations Using Square Roots (pp. 485–490)

A sprinkler sprays water that covers a circular region of 90π square feet. Find the diameter of the circle.

Write an equation using the formula for the area of a circle.

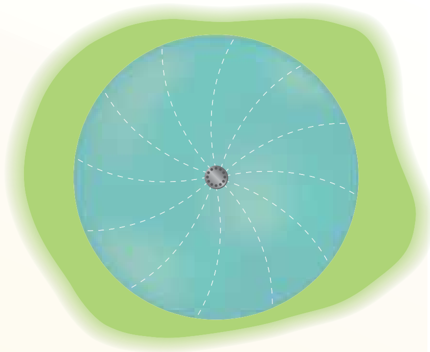
$$A = \pi r^2 \quad \text{Write the formula.}$$

$$90\pi = \pi r^2 \quad \text{Substitute } 90\pi \text{ for } A.$$

$$90 = r^2 \quad \text{Divide each side by } \pi.$$

$$\pm\sqrt{90} = r \quad \text{Take the square root of each side.}$$

$$\pm 3\sqrt{10} = r \quad \text{Simplify.}$$



A diameter cannot be negative, so use the positive square root. The diameter is twice the radius. So, the diameter is $6\sqrt{10}$.

▶ The diameter of the circle is $6\sqrt{10} \approx 19$ feet.

Solve the equation using square roots. Round your solutions to the nearest hundredth, if necessary.

18. $x^2 + 5 = 17$

19. $x^2 - 14 = -14$

20. $(x + 2)^2 = 64$

21. $4x^2 + 25 = -75$

22. $(x - 1)^2 = 0$

23. $19 = 30 - 5x^2$

9.4 Solving Quadratic Equations by Completing the Square (pp. 493–502)

Solve $x^2 - 6x + 4 = 11$ by completing the square.

$x^2 - 6x + 4 = 11$	Write the equation.
$x^2 - 6x = 7$	Subtract 4 from each side.
$x^2 - 6x + (-3)^2 = 7 + (-3)^2$	Complete the square by adding $\left(\frac{-6}{2}\right)^2$, or $(-3)^2$, to each side.
$(x - 3)^2 = 16$	Write the left side as the square of a binomial.
$x - 3 = \pm 4$	Take the square root of each side.
$x = 3 \pm 4$	Add 3 to each side.

▶ The solutions are $x = 3 + 4 = 7$ and $x = 3 - 4 = -1$.

Solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary.

24. $x^2 + 6x - 40 = 0$ **25.** $x^2 + 2x + 5 = 4$ **26.** $2x^2 - 4x = 10$

Determine whether the quadratic function has a maximum or minimum value. Then find the value.

27. $y = -x^2 + 6x - 1$ **28.** $f(x) = x^2 + 4x + 11$ **29.** $y = 3x^2 - 24x + 15$

30. The width w of a credit card is 3 centimeters shorter than the length ℓ. The area is 46.75 square centimeters. Find the perimeter.

9.5 Solving Quadratic Equations Using the Quadratic Formula (pp. 503–512)

Solve $-3x^2 + x = -8$ using the Quadratic Formula.

$-3x^2 + x = -8$	Write the equation.
$-3x^2 + x + 8 = 0$	Write in standard form.
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic Formula
$x = \dfrac{-1 \pm \sqrt{1^2 - 4(-3)(8)}}{2(-3)}$	Substitute -3 for a, 1 for b, and 8 for c.
$x = \dfrac{-1 \pm \sqrt{97}}{-6}$	Simplify.

▶ So, the solutions are $x = \dfrac{-1 + \sqrt{97}}{-6} \approx -1.5$ and $x = \dfrac{-1 - \sqrt{97}}{-6} \approx 1.8$.

Solve the equation using the Quadratic Formula. Round your solutions to the nearest tenth, if necessary.

31. $x^2 + 2x - 15 = 0$ **32.** $2x^2 - x + 8 = 16$ **33.** $-5x^2 + 10x = 5$

Find the number of x-intercepts of the graph of the function.

34. $y = -x^2 + 6x - 9$ **35.** $y = 2x^2 + 4x + 8$ **36.** $y = -\frac{1}{2}x^2 + 2x$

9 Chapter Test

Simplify the expression.

1. $-\sqrt{117}$

2. $\sqrt{98b^5}$

3. $\sqrt[3]{40w^3}$

4. $\sqrt[3]{\dfrac{27y^4}{1000z^3}}$

5. $\dfrac{10}{\sqrt{7}}$

6. $\dfrac{13}{\sqrt{3}-4}$

7. $(2\sqrt{5}-1)^2$

8. $12\sqrt{8}+3\sqrt{18}$

Solve the equation using any method. Explain your choice of method.

9. $x^2 - 121 = 0$

10. $x^2 - 6x = 10$

11. $-2x^2 + 3x + 7 = 0$

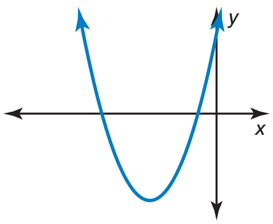
12. $x^2 - 7x + 12 = 0$

13. $5x^2 + x - 4 = 0$

14. $(4x+3)^2 = 16$

15. Explain how you can determine, without graphing, whether the quadratic function $y = -x^2 - 6x - 1$ has a maximum or minimum value. Then find the value.

16. Describe how you can use the method of completing the square to determine whether the function $f(x) = 2x^2 + 4x - 6$ can be represented by the graph shown.



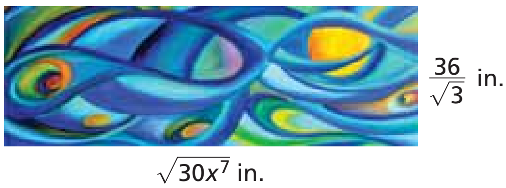
17. The dimensions of a park form a golden rectangle. The shorter side of the park is 84 meters. What is the length of the longer side of the park?

18. A skier leaves an 8-foot-tall ramp with an initial vertical velocity of 28 feet per second. The function $h = -16t^2 + 28t + 8$ represents the height h (in feet) of the skier after t seconds. The skier has a perfect landing. How many points does the skier earn?

Criteria	Scoring
Maximum height	1 point per foot
Time in air	5 points per second
Perfect landing	25 points

19. An amusement park ride lifts seated riders 265 feet above the ground. The riders are then dropped and experience free fall until the brakes are activated 105 feet above the ground. The function $h = -16t^2 + 265$ represents the height h (in feet) of the riders t seconds after they are dropped. How long do the riders experience free fall? Round your solution to the nearest hundredth.

20. Write an expression in simplest form that represents the area of the painting shown.



21. Explain how you can determine the number of times the graph of $y = 5x^2 - 10x + 5$ intersects the x-axis without graphing or solving an equation.

22. Consider the quadratic equation $ax^2 + bx = c$. Find values of a, b, and c so that the graph of its related function has (a) two x-intercepts, (b) one x-intercept, and (c) no x-intercepts.

9 Standards Assessment

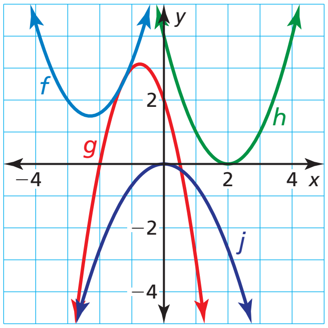
1. The graphs of four quadratic functions are shown. Which equation has a negative discriminant? *(TEKS A.7.A)*

Ⓐ $f(x) = 0$

Ⓑ $g(x) = 0$

Ⓒ $h(x) = 0$

Ⓓ $j(x) = 0$



2. The population p (in thousands) of Phoenix, Arizona, can be modeled by the function $p(t) = 0.08t^2 + 17.4t + 421$, where $t = 0$ represents the year 1960. Using the model, what will the population of Phoenix be in the year 2030? *(TEKS A.8.B)*

Ⓕ 1015

Ⓖ 2031

Ⓗ 1,015,000

Ⓙ 2,031,000

3. The table represents the numbers of bottles of sports drink sold at a concession stand on days with different average temperatures. You find an equation of the line of best fit. Which of the following is a reasonable correlation coefficient? *(TEKS A.4.A)*

Temperature (°F), *x*	14	27	32	41	48	62	73
Bottles of sports drink, *y*	8	12	13	16	19	27	29

Ⓐ −0.99

Ⓑ −0.12

Ⓒ 0.12

Ⓓ 0.99

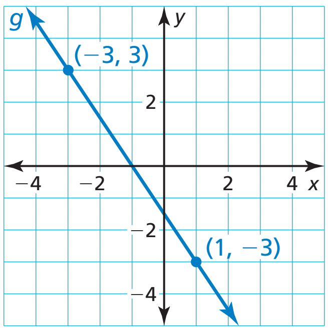
4. Which equation represents the line that passes through $(6, -1)$ and is perpendicular to line g? *(TEKS A.2.F)*

Ⓕ $y = -\frac{3}{2}x + \frac{9}{2}$

Ⓖ $y = -\frac{3}{2}x + 8$

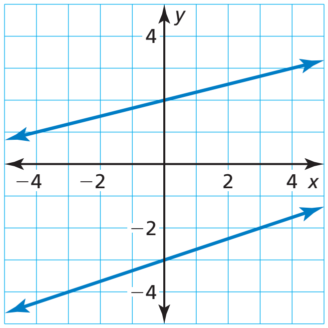
Ⓗ $y = \frac{2}{3}x - 5$

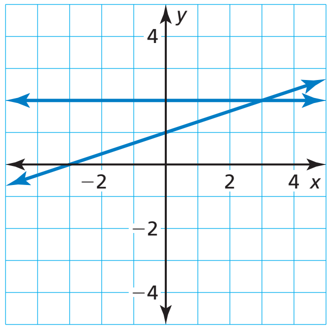
Ⓙ $y = \frac{2}{3}x + \frac{20}{3}$

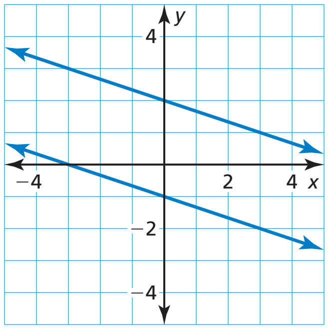


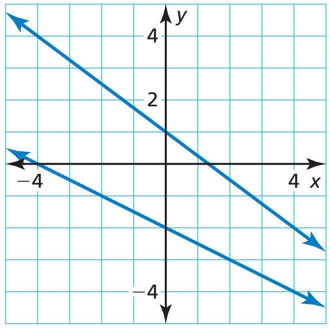
5. **GRIDDED ANSWER** What is the radius (in centimeters) of a cylinder with a height of 12 centimeters and a surface area of 320π square centimeters? *(TEKS A.8.A)*

6. Which graph represents a system of equations that has no solution? *(TEKS A.3.F)*

Ⓐ 

Ⓑ 

Ⓒ 

Ⓓ 

7. Which of the following expressions are in simplest form? *(TEKS A.11.A)*

I. $x\sqrt{45x}$

II. $\sqrt[3]{\frac{4}{9}}$

III. $3x\sqrt{5x}$

IV. $2\sqrt[3]{x^2}$

Ⓕ I and II only

Ⓖ II and III only

Ⓗ I, III, and IV only

Ⓙ II, III, and IV only

8. The domain of the function $f(x) = 4x - 5$ is all integers in the interval $-3 < x \leq 3$. What is the range of the function? *(TEKS A.2.A)*

Ⓐ $-17, -13, -9, -5, -1, 3, 7$

Ⓑ $-13, -9, -5, -1, 3, 7$

Ⓒ $-17 < x \leq 7$

Ⓓ $-13 \leq x \leq 7$

Selected Answers

Chapter 1

Chapter 1 Maintaining Mathematical Proficiency *(p. 1)*

1. -7 **2.** -13 **3.** 8 **4.** 32 **5.** -7 **6.** 2
7. 22 **8.** 5 **9.** -4 **10.** -24 **11.** 63
12. -28 **13.** 4 **14.** -8 **15.** -4 **16.** 48
17. 6 **18.** 12

19. **a.** If the signs are the same, add the absolute values and attach the sign. If the signs are different, subtract the absolute values and attach the sign of the number with the greatest absolute value; *Sample answer:* $-6 + 2 = -4$

b. Add the opposite; *Sample answer:* $5 - (-3) = 5 + 3 = 8$

c. Multiply the absolute values.
If the signs are the same, then the product is positive.
If the signs are different, then the product is negative;
Sample answer: $(-6)(-4) = 24$

d. Divide the absolute values. If the signs are the same, then the quotient is positive. If the signs are different, then the quotient is negative; *Sample answer:* $-15 \div 3 = -5$

1.1 Vocabulary and Core Concept Check *(p. 8)*

1. $+$ and $-$; $\times$ and $\div$

3. Division Property of Equality; Divide each side by 14.

1.1 Monitoring Progress and Modeling with Mathematics *(pp. 8–10)*

5. $x = 3$; Subtract 5 from each side.

7. $y = 7$; Add 4 to each side.

9. $w = -7$; Subtract 3 from each side.

11. $p = -3$; Add 11 to each side.

13. $r = 18$; Add 8 to each side.

15. $p - 12.95 = 44$; \$56.95

17. $x + 100 + 120 + 100 = 360$; $x = 40$

19. $x + 76 + 92 + 122 = 360$; $x = 70$

21. $g = 4$; Divide each side by 5.

23. $p = 15$; Multiply each side by 5.

25. $r = -8$; Divide each side by -8.

27. $x = 48$; Multiply each side by 6.

29. $s = -6$; Divide each side by 9. **31.** $t = -1$

33. $m = 14$ **35.** $a = 5.6$ **37.** $j = -18$

39. Subtract -0.8 from each side, not add; $r = 12.6 - (-0.8)$; $r = 13.4$

41. *C*; Multiplying the number of eggs in each carton by the number of cartons will give the total number of eggs; 9 cartons

43. $9.5 = 1.9w$; 5 ft **45.** Multiplication Property of Equality

47. **a.** $4p = 30.40$; \$7.60

b. no; Each CD costs \$9.50 at the regular price, so 3 CDs would cost \$28.50 which is greater than \$25.

49. **a.** 5; 10 **b.** -2; 9

51. 71; Because $\frac{1}{6}$ of the girls is 6, there are 36 girls. Because $\frac{2}{7}$ of the boys is 10, there are 35 boys; $36 + 35 = 71$

53. $B = 12\pi \text{ in.}^2$ **55.** $B = 9\pi \text{ m}^2$

57. **a.** 132 hits

b. no; Dividing by a larger number of at-bats decreases the value of the average.

1.1 Maintaining Mathematical Proficiency *(p. 10)*

59. $\frac{5}{6}x + \frac{15}{4}$ **61.** $8p + 16q + 24$ **63.** 0.02

65. 298.26

1.2 Vocabulary and Core Concept Check *(p. 16)*

1. like terms

1.2 Monitoring Progress and Modeling with Mathematics *(pp. 16–18)*

3. $w = 4$ **5.** $q = 1$ **7.** $z = -32$ **9.** $h = 4$

11. $y = 4$ **13.** $v = 3$ **15.** 6 min **17.** $z = 3$

19. $m = 3$ **21.** $c = 5$ **23.** $x = 29$

25. $k = 45$; 45°, 90°, 45° **27.** $b = 90$; 90°, 135°, 90°, 90°, 135°

29. $2n + 13 = 75$; $n = 31$ **31.** $8 + \dfrac{n}{3} = -2$; $n = -30$

33. $6(n + 15) = -42$; $n = -22$

35. $30(8.75) + 11t = 400$; 12.5 h

37. $1.08(2t + 2.50) + 3 = 13.80$; \$3.75

39. Distributive Property; Simplify; Combine like terms; Subtract 6 from each side; Divide each side by 3.

41. In the third step, the right side should be 8×4, not $8 \div 4$; $x - 2 = 32$; $x = 34$

43. $2y + 2 \cdot \frac{11}{8}y = 190$, $y = 40$; 55 in. by 40 in.

45. $x = \frac{15}{16}$; *Sample answer:* method 1; There are no fractions until the last step.

47. no; Solving the equation $0.25(d + 8) + 0.10d = 2.80$ results in the number of dimes not being a whole number.

49. 16, 18, 20; The next consecutive even integers after $2n$ are $2n + 2$ and $2n + 4$. Solve the equation $2n + (2n + 2) + (2n + 4) = 54$. Then substitute the solution into the expressions for the integers.

51. $x = -\dfrac{7}{b}$ **53.** $x = \dfrac{12.5 + b}{a}$ **55.** $x = -\dfrac{8}{b}$

1.2 Maintaining Mathematical Proficiency *(p. 18)*

57. $m + 5$ **59.** -2 **61.** b **63.** a **65.** a

1.3 Vocabulary and Core Concept Check *(p. 25)*

1. no; Solving the equation gives a statement that is never true, not one that is always true.

1.3 Monitoring Progress and Modeling with Mathematics *(pp. 25–26)*

3. $x = 3$ **5.** $p = 7$ **7.** $t = -1$ **9.** $x = \frac{1}{2}$

11. $g = -4$ **13.** $x = -3$ **15.** $y = -12$ **17.** 2 h

19. no solution **21.** $h = 3$; one solution

23. infinitely many solutions

25. In the second step, you should add $3c$ to each side; $8c - 6 = 4$, $8c = 10$, $c = \frac{5}{4}$

27. $60 + 42.95x = 25 + 49.95x$; 5th month **29.** $r = -2$

31. $x = 10$; $S = 62.5\pi\ \text{cm}^2$ or about $196.35\ \text{cm}^2$; $V = 62.5\pi\ \text{cm}^3$ or about $196.35\ \text{cm}^3$

33. 4 sec **35.** $a = 5$; Both sides simplify to $10x + 15$.

37. 11, 12

39. **a.** *Sample answer:* $3x + 12 = 2x + x$; simplifies to a statement that is never true

b. *Sample answer:* $5x + 3 = 2x + 3 + 3x$; simplifies to a statement that is always true

1.3 Maintaining Mathematical Proficiency *(p. 26)*

41. $-4, |2|, |-4|, 5, 9$ **43.** $-19, -18, |-18|, |22|, |-24|$

1.4 Vocabulary and Core Concept Check *(p. 32)*

1. no; It only has one variable.

1.4 Monitoring Progress and Modeling with Mathematics *(pp. 32–34)*

3. $y = 13 + 3x$ **5.** $y = -13 + 9x$ **7.** $y = 9x - 45$

9. $y = x - 3$ **11.** $y = 18x + 12$ **13.** $x = \frac{1}{12}y$

15. $x = \dfrac{a}{2 + 6z}$ **17.** $x = \dfrac{y - 6}{4 + r}$ **19.** $x = \dfrac{r}{s + t}$

21. $x = \dfrac{y - 12}{-5 - 4k}$ **23.** **a.** $x = \dfrac{C - 60}{85}$ **b.** 3 trips; 5 trips

25. The equation is not solved for x because there is still a term with x on both sides; $x = y - x + 6$; $2x = y + 6$; $x = \dfrac{y + 6}{2}$

27. $C = R - P$ **29.** $b_2 = \dfrac{2A}{h} - b_1$ **31.** $C = A\left(\dfrac{R}{5} + 0.3\right)$

33. **a.** $r = \dfrac{L - S}{L}$ **b.** 0.4 **35.** 6.25 yr

37. **a.** $P = 2x + 2\pi r$ **b.** $x = \frac{1}{2}P - \pi r$ **c.** 173 ft

39. **a.** $r = \dfrac{C}{2\pi}$ **b.** 1.1 ft; 1.3 ft; 1.4 ft

c. First find the radius using the formula from part (a), then substitute this into the formula for the area of a circle.

41. no; 70°F is about 21.1°C, which is greater than 20°C.

43. $A = \dfrac{5}{2}bh$; $h = \dfrac{2A}{5b}$ **45.** $a = \dfrac{b + c}{bx - 1}$

1.4 Maintaining Mathematical Proficiency *(p. 34)*

47. 35 **49.** 47 **51.** $x = 9$

Chapter 1 Review *(pp. 36–38)*

1. $z = -9$; Subtract 3 from each side.

2. $t = -13$; Divide each side by -0.2.

3. $n = 10$; Multiply each side by -5. **4.** $y = -9$

5. $b = -5$ **6.** $n = 6$ **7.** $z = -5$ **8.** $x = 18$

9. $w = \frac{25}{4}$ **10.** $x = 10$; 110°, 50°, 20°

11. $x = 126$; 126°, 96°, 126°, 96°, 96°

12. $\dfrac{25 + 15 + 18 + p}{4} = 20$; 22 points **13.** $n = -4$

14. all real numbers **15.** no solution

16. $18x - 3 = 12x$; 0.5 h **17.** $y = \frac{1}{2}x - 5$ **18.** $y = 2x - 2$

19. $y = \dfrac{a}{9 + 3x}$ **20.** **a.** $h = \dfrac{3V}{B}$ **b.** 18 cm

21. **a.** $K = \frac{5}{9}(F - 32) + 273.15$ **b.** about 355.37 K

Chapter 2

Chapter 2 Maintaining Mathematical Proficiency *(p. 43)*

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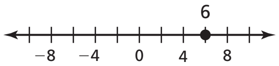

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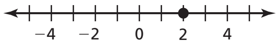

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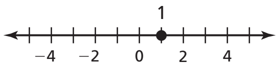

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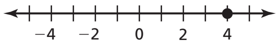

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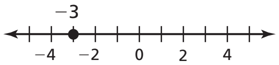

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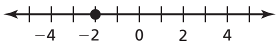

7. $<$ **8.** $<$ **9.** $<$ **10.** $>$ **11.** $=$ **12.** $<$

13. $-b < -a$

2.1 Vocabulary and Core Concept Check *(p. 50)*

1. inequality

3. Draw an open circle when a number is not part of the solution. Draw a closed circle when a number is part of the solution. Draw an arrow to the left or right to show that the graph continues in that direction.

2.1 Monitoring Progress and Modeling with Mathematics *(pp. 50–52)*

5. $x > 3$ **7.** $15 \le \dfrac{t}{5}$ **9.** $\frac{1}{2}y > 22$ **11.** $13 \ge v - 1$

13. $w \ge 1.7$ **15.** no **17.** yes **19.** yes

21. yes **23.** yes

25. **a.** $h < 107$ in.

b. no; A height of 9 feet is equal to 108 inches, which is not less than 107 inches.

27. Because -1 is not less than -4, the final result is not true; $-1 \nless -4$; 8 is not in the solution set.

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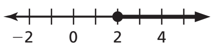

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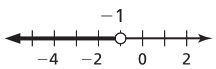

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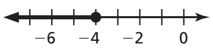

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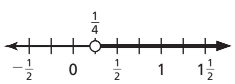
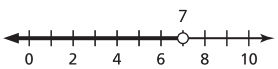
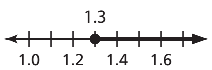


37. $x < 7$



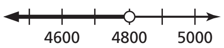
39. $1.3 \le z$



41. $x \le 4$ **43.** $x > 3$

45. C; The temperature must be at least 2°F warmer, so the increase is represented by $x \ge 2$.

47. $\ell < 4800$



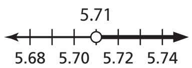
49. *Sample answer:* You spend \$23 on admission and x dollars on snacks, and you can spend no more than \$31 total.

51. $0.90x \le 24$; yes; Because $0.9(25) = \$22.50$, which is less than \$24, the inequality is true.

53. *Sample answer:* A temperature above the freezing point of water can be represented by $T > 0$ if the temperature is in degrees Celsius, or by $T > 32$ if the temperature is in degrees Fahrenheit.

55. $x < 14$ **57.** $x < 3$

59. **a.** $r > \frac{40}{7}$ (about 5.71)



b. no; The graph includes speeds beyond the maximum speed a human can run.

2.1 Maintaining Mathematical Proficency *(p. 52)*

61. $y = 14$ **63.** $y = -1$

65. Division Property of Equality

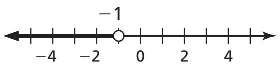
2.2 Vocabulary and Core Concept Check *(p. 57)*

1. Subtraction Property of Inequality

2.2 Monitoring Progress and Modeling with Mathematics *(pp. 57–58)*

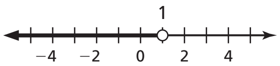
3. subtract 11 **5.** add 9

7. $x < -1$

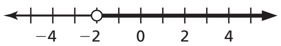


9. $m \le 7$

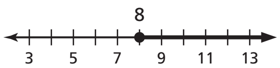
11. $r < 1$



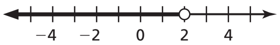
13. $w > -2$



15. $h \ge 8$



17. $j < 2$



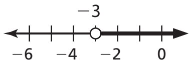
19. $p \le 17$

21. $n + 8 > 11; n > 3$ **23.** $n - 9 < 4; n < 13$

25. **a.** $38 + w \le 50; w \le 12$

b. no; The total being added is 14 pounds, which is not a solution of the inequality found in part (a).

27. The graph is going in the wrong direction.



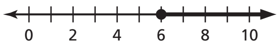
29. 33 or more goals

31. A; D; Subtract 3 from each side; order of inequality reverses for opposites

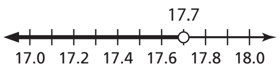
33. $6.4 + 4.9 + 4.1 + x \le 18.7; x \le 3.3$

35. no; *Sample answer:* 3, 7, 8, 9, and 12; There are infinitely many solutions. Check 8 and a few numbers greater than and less than 8.

37. **a.** $x \ge 6$



b. $x < 17.7$



2.2 Maintaining Mathematical Proficiency *(p. 58)*

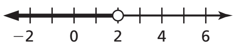
39. -63 **41.** 9 **43.** $x = 4$ **45.** $s = -104$

2.3 Vocabulary and Core Concept Check *(p. 63)*

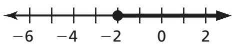
1. When solving $2x < -8$, the inequality symbol is not reversed when dividing each side by 2. When solving $-2x < 8$, the inequality is reversed when dividing each side by -2.

2.3 Monitoring Progress and Modeling with Mathematics *(p. 63–64)*

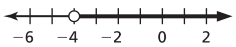
3. $x < 2$



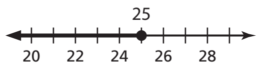
5. $n \ge -2$



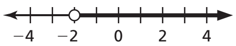
7. $x > -4$



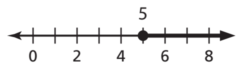
9. $w \le 25$



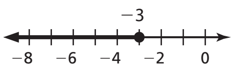
11. $t > -2$



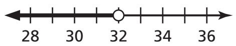
13. $z \ge 5$



15. $n \le -3$



17. $m < 32$



19. $5p \le 12, p \le 2.4$ **21.** $y > 12$ **23.** $x \le -9$

25. $x > \frac{3}{8}$

27. The inequality should not be reversed when multiplying each side by $\frac{3}{2}$; $\frac{3}{2} \cdot (-6) > \frac{3}{2} \cdot \frac{2}{3}x$; $-\frac{18}{2} > x$; $-9 > x$; $x < -9$; The solution is $x < -9$.

29. $(14 \cdot 14)c \le 700$; ft, ft, dollars;

$196c \le 700$; ft^2, dollars;

$c \le 3.57$; dollars/ft^2

31. **a.** $d \le 6.3(2)$, $d \le 12.6$

b. yes; The distance traveled in 4 hours would be no more than 25.2 miles, which is less than the distance required for a marathon.

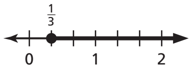
33. more than 300 million pennies

35. **a.** $A > B$ or $B < A$ **b.** $-A < -B$ or $-B > -A$

c. As numbers move farther away from zero, their absolute value becomes larger. $A > B$ and $|A| > |B|$. $-A < -B$ and $|A| > |B|$.

37. $\frac{C}{2\pi} > 5$, $C > 10\pi$

39. $36p \geq 12$, $p \geq \frac{1}{3}$



2.3 Maintaining Mathematical Proficiency *(p. 64)*

41. $y = -4$ **43.** $z = 6$ **45.** $\frac{16}{30}$ **47.** $\frac{2}{3}$

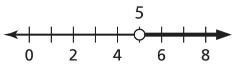
2.4 Vocabulary and Core Concept Check *(p. 71)*

1. *Sample answer:* The same steps can be applied when solving multi-step inequalities and multi-step equations, except that when each side of an inequality is divided by a negative number, the inequality must be reversed.

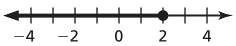
2.4 Monitoring Progress and Modeling with Mathematics *(pp. 71–72)*

3. B **5.** C

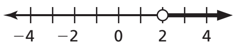
7. $x > 5$



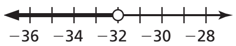
9. $v \leq 2$



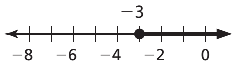
11. $w > 2$



13. $p < -32$



15. $a \geq -3$



17. $m > 3$ **19.** $d > -2$ **21.** all real numbers

23. no solution **25.** no solution **27.** all real numbers

29. In the first step, you need to use the Distributive Property on the left side; $x + 24 \geq 12$; $x \geq -12$

31. $20n + 100 \leq 320$, $n \leq 11$ **33.** $12(2x - 3) > 60$, $x > 4$

35. 7 stories; Using the Pythagorean Theorem, the 74-foot ladder can reach at most 70 feet. Solving the inequality $10n - 8 \leq 70$ gives $n \leq 7.8$, so the ladder cannot quite reach the 8th story.

37. $r \geq 3$ **39.** $a = 4$

2.4 Maintaining Mathematical Proficiency *(p. 72)*

41. $6y \leq 10$ **43.** $\frac{r}{7} \leq 18$

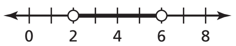
2.5 Vocabulary and Core Concept Check *(p. 77)*

1. The graph of $-6 \leq x \leq -4$ shows a single segment between -6 and -4. The graph of $x \leq -6$ or $x \geq -4$ shows two opposite rays with endpoints at -6 and -4.

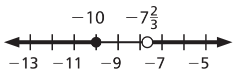
2.5 Monitoring Progress and Modeling with Mathematics *(pp. 77–78)*

3. $-3 < x \leq 2$ **5.** $x \leq -7$ or $x \geq -4$

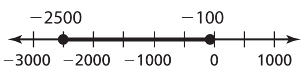
7. $2 < p < 6$



9. $m > -7\frac{2}{3}$ or $m \leq -10$

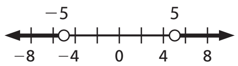


11. $-2500 \leq e \leq -100$

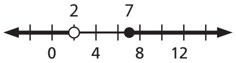


13. $1 < x \leq 6$

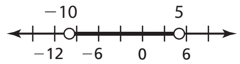
15. $v < -5$ or $v > 5$



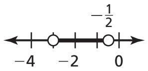
17. $r < 2$ or $r \geq 7$



19. $-10 < x < 5$



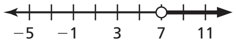
21. In the second step, 3 should have been subtracted from 4 on the left side;
$1 < -2x < 6$; $-\frac{1}{2} > x > -3$



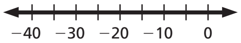
23. $-20 \leq \frac{5}{9}(F - 32) \leq -15$, $-4 \leq F \leq 5$

25. no solution

27. $y > 7$



29. all real numbers

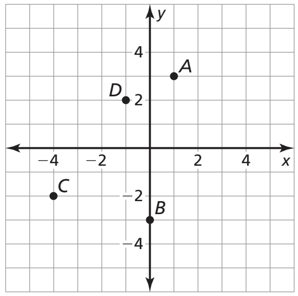


31. $\geq$

33. $7 + 5 > x$; $x < 12$; $7 + x > 5$; $x > -2$; $5 + x > 7$; $x > 2$; no; A value of 1 does not make the inequality $x > 2$ true.

2.5 Maintaining Mathematical Proficiency *(p. 78)*

35–37.



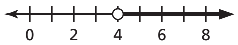
Point A is one unit left and 3 units up from the origin. Point B is 3 units down from the origin. Point C is 4 units left and 2 units down from the origin. Point D is 1 unit left and 2 units up from the origin.

39. 4; The data values are clustered close together.

Chapter 2 Review *(pp. 80–82)*

1. $d - 2 < -1$ **2.** $5h \leq 10$

3.



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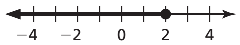

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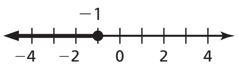
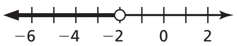
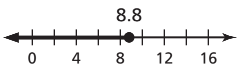


6. $p < 6$

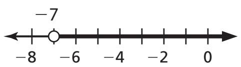
7. $r < -2$



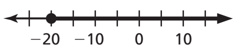
8. $m \leq 8.8$



9. $x > -7$

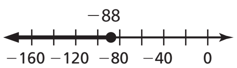


10. $g \geq -20$

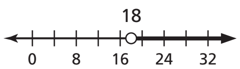


11. $n \geq -4$

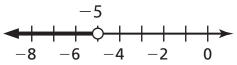
12. $s \leq -88$



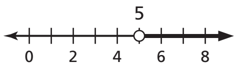
13. $q > 18$



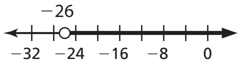
14. $k < -5$



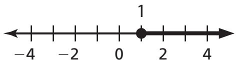
15. $x > 5$



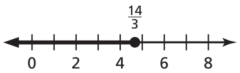
16. $b > -26$



17. $n \geq 1$

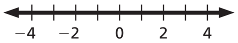


18. $s \leq \frac{14}{3}$



19. no solution

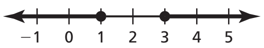
20. all real numbers



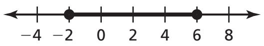
21. $-6 < x \leq 8$

22. $1 < x < 11$

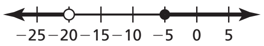
23. $x \leq 1$ or $x \geq 3$



24. $-2 \leq z \leq 6$



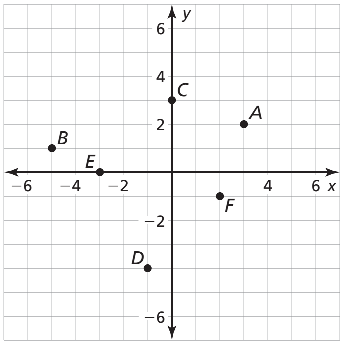
25. $r < -20$ or $r \geq -5$



Chapter 3

Chapter 3 Maintaining Mathematical Proficiency *(p. 87)*

1–6.



1. in Quadrant I **2.** in Quadrant II

3. on positive y-axis **4.** in Quadrant III

5. on negative x-axis **6.** in Quadrant IV **7.** 17

8. −7 **9.** 68 **10.** 34 **11.** 40 **12.** −6

13. Start at the origin. Move a units right and b units up. Plot the point; Start at the origin. Move a units left and b units up. Plot the point; Start at the origin. Move a units right and b units down. Plot the point; Start at the origin. Move a units left and b units down. Plot the point.

3.1 Vocabulary and Core Concept Check *(p. 94)*

1. The independent variable can be any value in the domain, but the dependent variable depends on the value of the independent variable.

3.1 Monitoring Progress and Modeling with Mathematics *(pp. 94–96)*

3. function; Every input has exactly one output.

5. not a function; The input 2 has two outputs, 3 and 2.

7. not a function; The input 16 has two outputs, −2 and 2, and the input 1 has two outputs, −1 and 1.

9. function; Every input has exactly one output.

11. not a function; The input 4 has two outputs, −2 and 2, and the input 9 has two outputs, −3 and 3.

13. function; No vertical line can be drawn through more than one point on the graph.

15. not a function; A vertical line can be drawn through more than one point on the graph in many places, such as (4, 0) and (4, 6).

17. domain: −2, −1, 0, 1, 2; range: −2, 0, 2

19. domain: $-4 \leq x \leq 2$; range: $2 \leq y \leq 6$

21. a. y is the dependent variable and x is the independent variable.

b. 500, 525, 550, 575, 600, 625

23. It is not a function if one input is paired with more than one output; The relation is a function. No input is paired with more than one output.

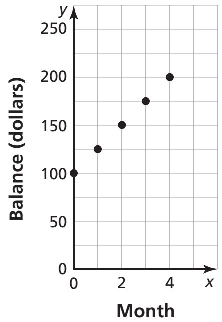
25. The length of your hair is the dependent variable and the amount of time since your last haircut is the independent variable.

27. The amount of time is the dependent variable and the number of quarters is the independent variable.

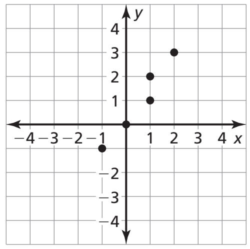
29. a. *Sample answer:* The balance of the savings account is \$100 in month 0 and increases by \$25 per month through month 4.

b. (0, 100), (1, 125), (2, 150), (3, 175), (4, 200)

c.



31. *Sample answer:*



33. a. Each letter-number combination is paired with exactly one food or drink item.

b. The food item is the dependent variable and the letter-number combination is the independent variable.

c. domain: A1, A2, A3, B1, B2, B3, B4, C1, C2, C3, C4; range: popcorn, nuts, pretzels, protein bar, granola bar, cereal, energy bar, orange juice, water, milk

35. no; A vertical line does not represent a function.

37. no; Items that cost the same to make could be sold for different prices.

39. yes; Each student has exactly one homeroom teacher.

41. true

43. false; More than one input can have the same output in a function, so reversing the values may not produce a function.

45. **a.** $P = h + 23$

b. P is the dependent variable and h is the independent variable.

c. domain: $3 < h < 23$; range: $26 < P < 46$

47. domain: all real numbers; range: $y \leq 0$

49. domain: all real numbers; range: $y \leq 4$

3.1 Maintaining Mathematical Proficiency *(p. 96)*

51. $3 \geq x$ **53.** $w + 4 > 12$ **55.** 81 **57.** 32

3.2 Vocabulary and Core Concept Check *(p. 103)*

1. $y = mx + b$

3. Discrete domains consist of only certain numbers in an interval. Continuous domains consist of all numbers in an interval.

3.2 Monitoring Progress and Modeling with Mathematics *(pp. 103–106)*

5. nonlinear; The graph is not a line.

7. linear; The graph is a line.

9. nonlinear; The graph is not a line.

11. linear; As x increases by 1, y increases by 5. The rate of change is constant.

13. nonlinear; As x increases by 4, y decreases by different amounts. The rate of change is not constant.

15. The increase in y needs to be done by adding or subtracting the same amount to be linear, not multiplying; As x increases by 2, y increases by different amounts. The rate of change is not constant. So, the function is nonlinear.

17. nonlinear; It cannot be rewritten in the form $y = mx + b$.

19. linear; It can be rewritten as $y = -1x + 2$.

21. linear; It can be rewritten as $y = 18x + 12$.

23. linear; It can be rewritten as $y = 9x - 13$.

25. A, C, F; None of these can be rewritten in the form $y = mx + b$.

27. 2, 4, 6; discrete; The graph consists of individual points.

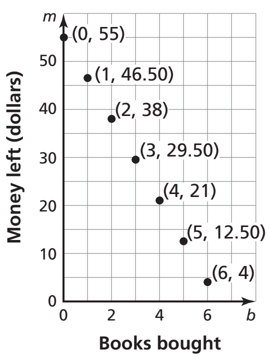
29. discrete; The number of bags must be a whole number.

31. continuous; The time can be any value greater than or equal to 0.

33. There is no point with an x-value of 2.5; 2.5 is not in the domain.

35. **a.** 0, 1, 2, 3, 4, 5, 6; discrete; The number of books must be a whole number.

b.

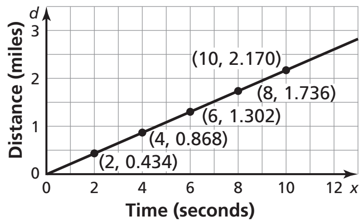


c. 4, 12.50, 21, 29.50, 38, 46.50, 55

37. **a.** yes; As t increases by 2, d increases by 0.434. The rate of change is constant.

b. $t \geq 0$; continuous; The time can be any value greater than or equal to 0.

c.



d. $d \geq 0$

39. *Sample answer:* The number of hours on a parking meter is a function of the number of tokens used, 4 tokens for 1 hour and a maximum time of 2 hours; discrete; The number of tokens used must be 0, 4, or 8.

41. *Sample answer:* The depth (in feet) of a scuba diver returning to the surface of an ocean as a function of the time; continuous; The time can be any value from 0 to 30.

43. **a.** 51.00 **b.** \$10.20

45. nonlinear; $V = 9s^2$ cannot be written in linear form.

47. linear; The formula can be written in the form $V = (4\pi)h + 0$.

49. **a.** the total gallons of water in one jug of each type; continuous

b. the total number of jugs of both types; discrete

c. the total gallons of water in all the jugs of the first type; continuous

d. the total gallons of water in all the jugs of both types; continuous

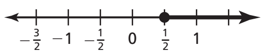
51. linear; As x increases by 1, y increases by 4. The rate of change is constant.

53. *Sample answer:* how long it takes an ice cube to melt as a function of its temperature in degrees Celsius

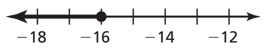
3.2 Maintaining Mathematical Proficiency *(p. 106)*

55. 20 **57.** -24 **59.** $x = 3$ **61.** $t = 8$

63. $x \geq \frac{1}{2}$



65. $r \leq -16$



3.3 Vocabulary and Core Concept Check *(p. 111)*

1. function notation

3.3 Monitoring Progress and Modeling with Mathematics *(pp. 111–112)*

3. 4; 6; 11 **5.** 13; 9; -1 **7.** -11; -3; 17

9. 11; 7; -3

11. **a.** There are no customers in the restaurant at 8 A.M.

b. There are the same number of customers in the restaurant at 11 A.M. as there are at 4 P.M.

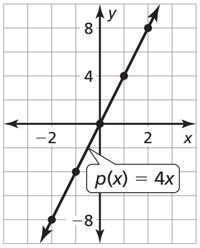
c. There are 29 customers in the restaurant n hours after 8 A.M.

d. There are fewer customers in the restaurant at 9 P.M. than there are at 8 P.M.

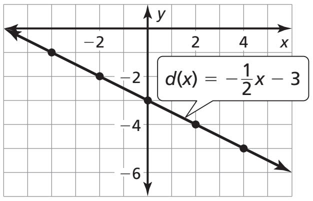
13. $x = -9$ **15.** $x = -2$ **17.** $x = -2$ **19.** $x = 5$

21. **a.** \$77.50 **b.** 8 tickets

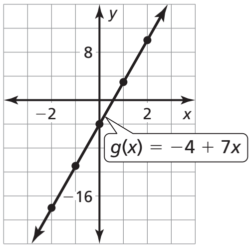
23.



25.



27.



29. the tablet computer; The graph of the tablet computer shows that when $p(t) = 0$, t is 6, which is greater than 5.

31. no; Because the function rule is unknown, it is unknown whether an increase in x would result in an increase or decrease in the value of the function.

33. **a.** $d(r) = 2r$; 10; The diameter of the circle is 10 feet.

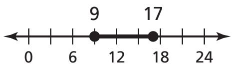
b. $A(r) = \pi r^2$; 25π or about 78.5; The area of the circle is 25π square feet.

c. $C(r) = 2\pi r$; 10π or about 31.4; The circumference of the circle is 10π feet.

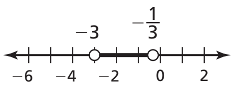
35. **a.** (5, 9) **b.** (5, −3)

3.3 Maintaining Mathematical Proficiency *(p. 112)*

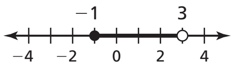
37. $9 \leq x \leq 17$;



39. $-3 < k < -\frac{1}{3}$;



41. $-1 \leq y < 3$;

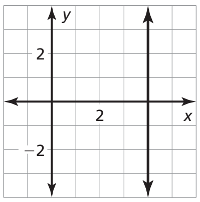


3.4 Vocabulary and Core Concept Check *(p. 120)*

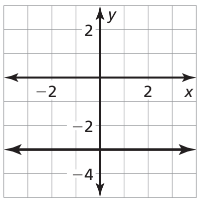
1. They are both places where a graph crosses an axis; The x-intercept is where a graph crosses the x-axis. The y-intercept is where a graph crosses the y-axis.

3.4 Monitoring Progress and Modeling with Mathematics *(p. 120–122)*

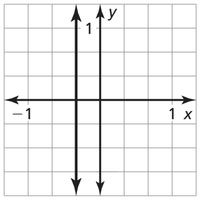
3.



5.



7.



9. x-intercept: 4, y-intercept: 4

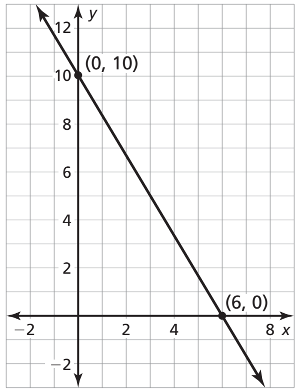
11. x-intercept: 4, y-intercept: -2

13. x-intercept: 6, y-intercept: 4

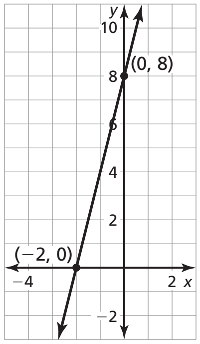
15. x-intercept: 4, y-intercept: -2

17. x-intercept: $\frac{2}{3}$, y-intercept: $-\frac{1}{3}$

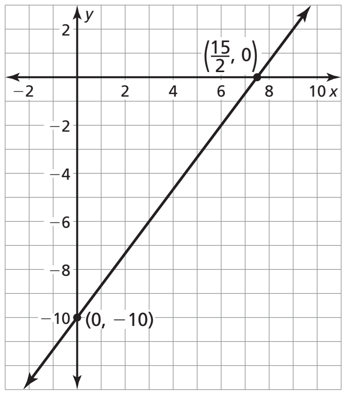
19.



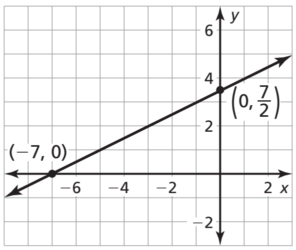
21.



23.



25.



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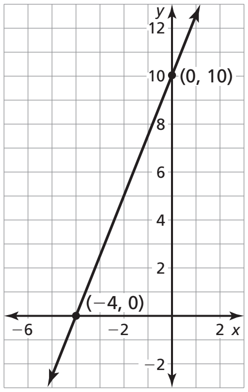


29. The equations locate 2 separate intercepts, not 1; The intercepts are 8 and 2.

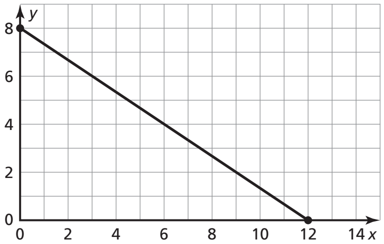
31. 6 **33.** -1 **35.** -2 **37.** 6 **39.** -4

41. $-\frac{1}{2}$ **43.** 14

45. 5; The balance of the account will be \$0 after 5 months.

47. -6; no; The number of additional toppings cannot be a negative number.

49. **a.**



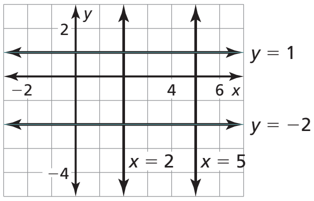
The x-intercept shows they can take 12 cars and 0 vans. The y-intercept shows they can take 8 vans and 0 cars.

b. *Sample answer:* 12 cars, 0 vans; 9 cars, 2 vans; 6 cars, 4 vans; 0 cars, 8 vans

51. no; You have to substitute 0 for y to find the x-intercept, not for x.

53. A **55.** D

57.



square; The graphs are 2 horizontal lines and 2 vertical lines, which will intersect at right angles. The length of each side is 3 units.

59. -3; 6

61. yes; $x = a$ can be written as $1x + 0y = a$, and $y = b$ can be written as $0x + 1y = b$.

3.4 Maintaining Mathematical Proficiency *(p. 122)*

63. $\frac{1}{2}$ **65.** $-\frac{4}{5}$

3.5 Vocabulary and Core Concept Check *(p. 129)*

1. slope

3. $y = mx + b$; The equation gives the slope m and the y-intercept b.

3.5 Monitoring Progress and Modeling with Mathematics *(pp. 129–132)*

5. negative; $-\frac{3}{5}$ **7.** zero; 0 **9.** $\frac{1}{2}$ **11.** undefined

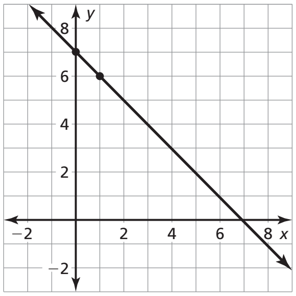
13. $m = 60$; The bus is traveling at a speed of 60 miles per hour.

15. slope: -3; y-intercept: 2 **17.** slope: 6; y-intercept: 0

19. slope: 2; y-intercept: 4 **21.** slope: 5; y-intercept: 8

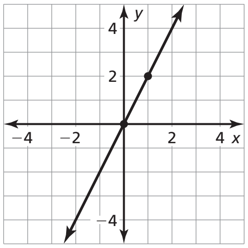
23. To be in slope-intercept form the equation needs to be solved for y, not x; $y = -\frac{1}{4}x$; The slope is $-\frac{1}{4}$ and the y-intercept is 0.

25.



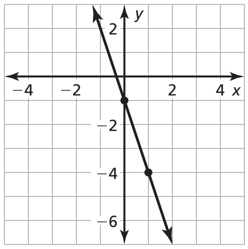
x-intercept: 7

27.



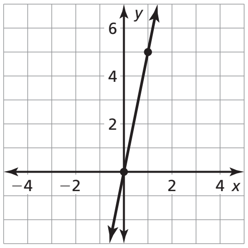
x-intercept: 0

29.



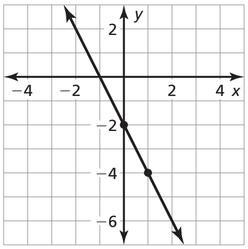
x-intercept: $-\frac{1}{3}$

31.



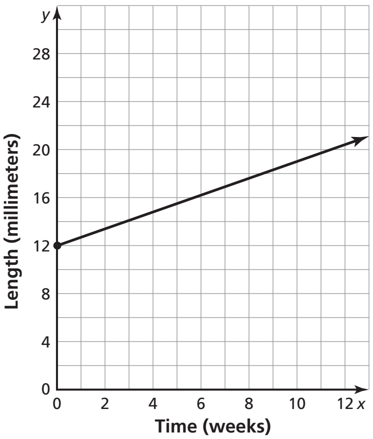
x-intercept: 0

33.



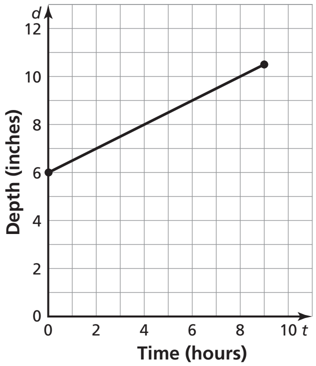
slope: -2; y-intercept: -2; x-intercept: -1

35.



slope: $\frac{7}{10}$; y-intercept: 12; So, the right index fingernail is initially 12 millimeters long.

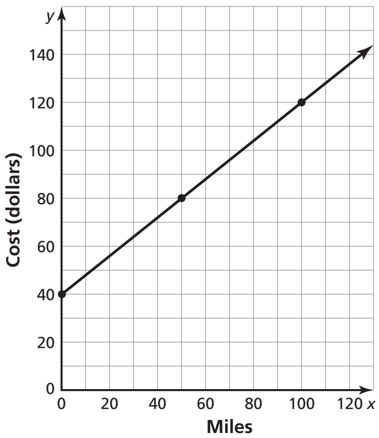
37. a.



domain: $0 \le t \le 9$; range: $6 \le d \le 10\frac{1}{2}$

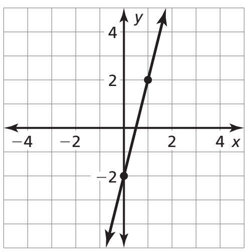
b. slope: $\frac{1}{2}$; So, $\frac{1}{2}$ inch of snow falls every hour during the storm; d-intercept: 6; So, there were 6 inches of snow already on the ground at the start of the storm.

39.

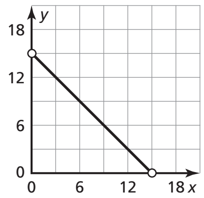


The slope of $\frac{4}{5}$ is greater than the slope in Exercise 38. The y-intercept of 40 is less than the c-intercept in Exercise 38.

41. The slope was interpreted as the y-intercept, and the y-intercept was interpreted as the slope.



43. a.



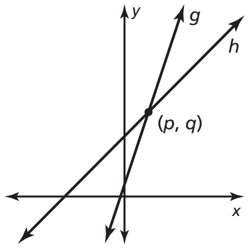
b. The slope of each graph is the same. The open circles in the graph in part (a) are closer to the origin.

45. a. $y = \frac{1}{3}x + 5$ **b.** $y = \frac{7}{4}x - \frac{1}{4}$; $y = 2x - 4$

c. $y = -3x + 8$ **d.** $y = -4x - 9$; $y = -x - \frac{4}{3}$

47. *Sample answer:* the ratio of the rise to the run; the ratio of the change in y to the change in x

49. a.



b. a is the y-intercept of g; b is the y-intercept of h

c. 8 units greater

51. $k = \frac{1}{8}$

53.
$$\begin{aligned}\frac{y_2 - y_1}{x_2 - x_1} &= \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1}\\ &= \frac{mx_2 + b - mx_1 - b}{x_2 - x_1}\\ &= \frac{mx_2 - mx_1}{x_2 - x_1}\\ &= \frac{m(x_2 - x_1)}{x_2 - x_1}\\ &= m\end{aligned}$$

3.5 Maintaining Mathematical Proficiency *(p. 132)*

55. no solution **57.** $x > -\frac{3}{4}$

59. linear; The graph is a line.

61. linear; As x increases by 2, y decreases by 1. The rate of change is constant.

3.6 Vocabulary and Core Concept Check *(p. 137)*

1. direct variation

3.6 Monitoring Progress and Modeling with Mathematics *(pp. 137–138)*

3. yes; 1 **5.** no **7.** no **9.** yes; $-\frac{1}{3}$

11. The equation needs to be solved for y; $y = 6x$; x and y show direct variation because the equation can be written in the form $y = ax$.

13. yes; The ordered pairs lie on a line that passes through the origin.

15. no; The ordered pairs lie on a line that does not pass through the origin.

17. $y = \frac{6}{5}x$; $\frac{6}{5}$ **19.** $y = -3x$; -3 **21.** $y = \frac{7}{2}x$; $\frac{7}{2}$

23. no; The equation that represents the situation is not of the form $y = ax$.

25. a. All of the ratios $\frac{f}{w}$ are equal to 0.25.

b. $f = 0.25w$ c. \$4.50; \$2.50

27. 12 h

29. yes; The equation is of the form $y = ax$.

31. a. yes; Distance varies directly with time and the constant of variation is 6.

b. yes; Distance varies directly with rate and the constant of variation is 50.

c. no; The graph of $300 = rt$ is not a line.

33. yes; The constants of variation are reciprocals of each other.

3.6 Maintaining Mathematical Proficiency *(p. 138)*

35. $X'(0, 0)$, $Y'(4, 2)$, $Z'(4, 0)$

3.7 Vocabulary and Core Concept Check *(p. 145)*

1. The graphs of all other nonconstant linear functions are transformations of the graph of $f(x) = x$.

3. causes a horizontal stretch or shrink; causes a vertical stretch or shrink

3.7 Monitoring Progress and Modeling with Mathematics *(pp. 145–148)*

5. The graph of g is a vertical translation 2 units up of the graph of f.

7. The graph of g is a vertical translation 3 units down of the graph of f.

9. The graph of g is a horizontal translation 5 units left of the graph of f.

11. The graph of f is a horizontal translation 5 units right of the graph of d.

13. The graph of h is a reflection in the x-axis of the graph of f.

15. The graph of h is a reflection in the y-axis of the graph of f.

17. The graph of r is a vertical stretch of the graph of f by a factor of 2.

19. The graph of r is a horizontal stretch of the graph of f by a factor of 2.

21. The graph of r is a vertical stretch of the graph of f by a factor of 3.

23. The graph of h is a horizontal shrink of the graph of f by a factor of $\frac{1}{3}$.

25. The graph of h is a vertical shrink of the graph of f by a factor of $\frac{1}{6}$.

27. The graph of h is a horizontal shrink of the graph of f by a factor of $\frac{1}{5}$.

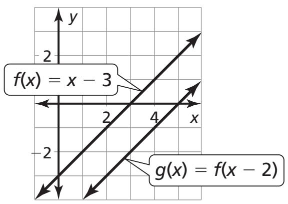
29. The graph of g is a vertical shrink of the graph of f by a factor of $\frac{1}{4}$.

31. The graph of g is a horizontal translation 2 units right of the graph of f.

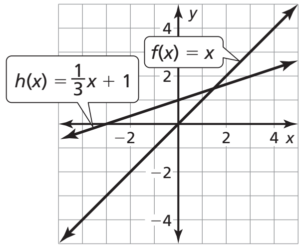
33. The graph of g is a vertical stretch of the graph of f by a factor of 6.

35. $g(x) = f(x - 2)$ 37. $g(x) = 4f(x)$

39. $f(x - 2)$ is a translation to the right, not to the left.

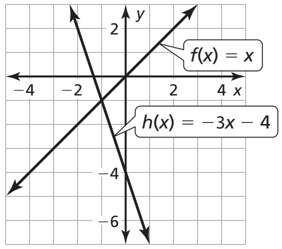


41.



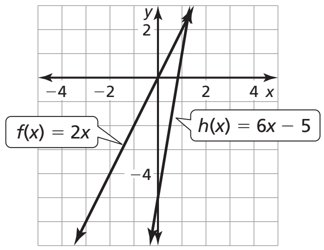
The transformations are a vertical shrink by a factor of $\frac{1}{3}$ then a vertical translation 1 unit up.

43.



The transformations are a vertical stretch by a factor of 3, then a reflection in the x-axis, then a vertical translation 4 units down.

45.



The transformations are a vertical stretch by a factor of 3 then a vertical translation 5 units down.

47. The graph of d is a reflection in the y-axis of the graph of t.

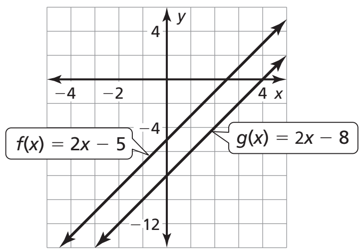
49. a. stretch b. right c. 1

51. B and C, A and F; Adding 4 to the y-coordinate of each point on B gives the corresponding y-coordinate on C. Adding 2 to the y-coordinate of each point on F gives you the corresponding y-coordinate on A.

53. a. The new graph is a vertical translation 10 units up of the old graph.

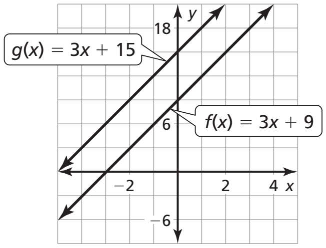
b. The new graph is a horizontal stretch of the old graph by a factor of 2.

55.



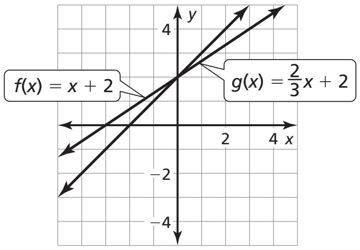
$g(x) = f(x) - 3$; The graph of g is a vertical translation 3 units down of the graph of f.

57.



$g(x) = f(x) + 6$; The graph of g is a vertical translation 6 units up of the graph of f.

59.



$g(x) = f\left(\frac{2}{3}x\right)$; The graph of g is a horizontal stretch of the graph of f by a factor of $\frac{3}{2}$.

61. Translate the graph of $f(x) = x$ horizontally 5 units left.

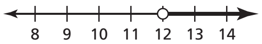
63. $r = 2$ **65.** $r = 2$

67. when the slope is 1; A slope of 1 occurs when the ratio of the vertical change to the horizontal change is 1, meaning the vertical change and horizontal change are the same.

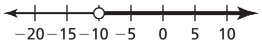
3.7 Maintaining Mathematical Proficiency *(p. 148)*

69. $w = \dfrac{P - 2\ell}{2}$ or $w = \frac{1}{2}P - \ell$

71. $x > 12$



73. $x > -10$



Chapter 3 Review *(pp. 150–154)*

1. not a function; A vertical line can be drawn through more than one point on the graph in many places.

2. function; Every input has exactly one output.

3. function; Every input has exactly one output.

4. **a.** The amount of money in the bank account is the dependent variable and the hours you babysat is the independent variable.

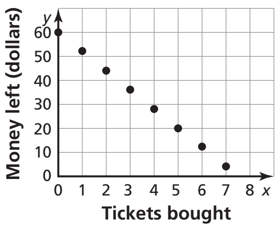
b. domain: $0 \le x \le 4$; range: $100 \le y \le 140$

5. linear; As x increases by 5, y decreases by 3. The rate of change is constant.

6. nonlinear; The graph is not a line.

7. **a.** 0, 1, 2, 3, 4, 5, 6, 7; discrete; The number of tickets bought must be a whole number.

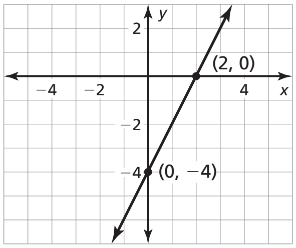
b.



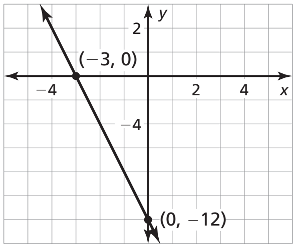
c. 60, 52, 44, 36, 28, 20, 12, 4

8. 5, 8, 13 **9.** 13, 4, −11 **10.** $x = 7$ **11.** $x = -4$

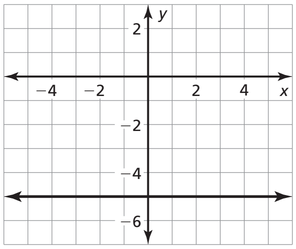
12.



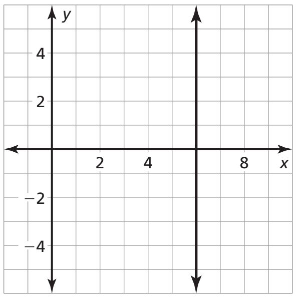
13.



14.



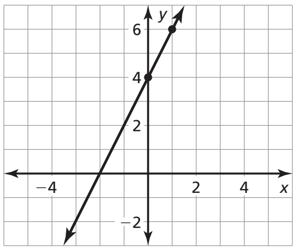
15.



16. 5 **17.** −12 **18.** $\frac{1}{2}$ **19.** $\frac{6}{5}$ **20.** undefined

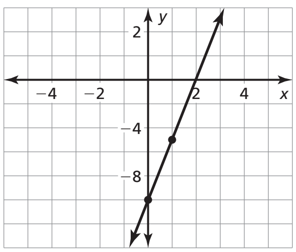
21. 0

22.



x-intercept: −2

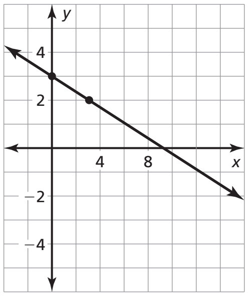
23.



x-intercept: 2

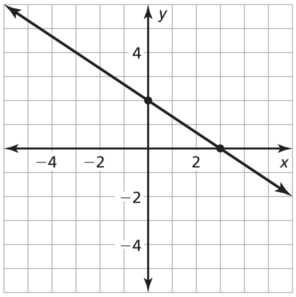
Selected Answers

24.



x-intercept: 9

25.



slope: $-\frac{2}{3}$; y-intercept: 2; x-intercept: 3

26. no; The equation cannot be written in the form $y = ax$.

27. yes; The equation can be written in the form $y = ax$.

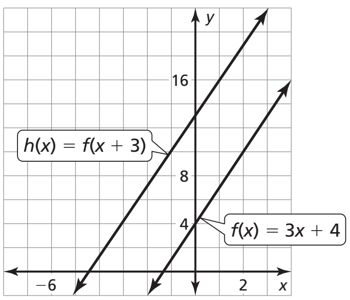
28. no; The ordered pairs lie on a line that does not pass through the origin.

29. yes; The ordered pairs lie on a line that passes through the origin.

30. **a.** All of the ratios $\frac{p}{w}$ are equal to $\frac{1}{350}$; $p = \frac{1}{350}w$

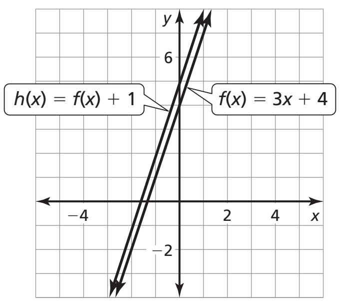
b. 4 gal

31.



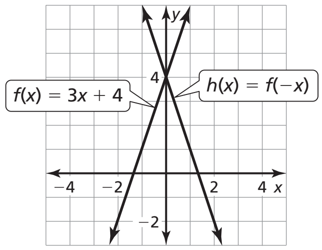
The graph of h is a horizontal translation 3 units left of the graph of f.

32.



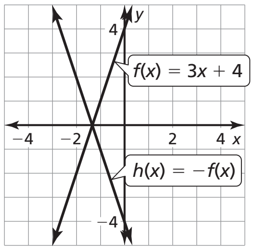
The graph of h is a vertical translation 1 unit up of the graph of f.

33.



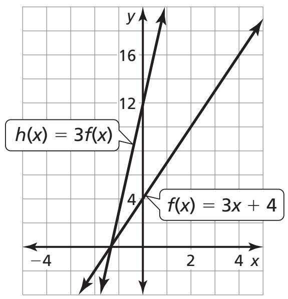
The graph of h is a reflection in the y-axis of the graph of f.

34.



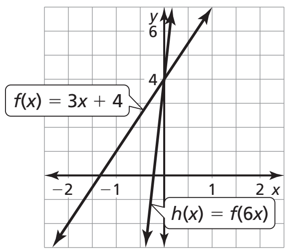
The graph of h is a reflection in the x-axis of the graph of f.

35.



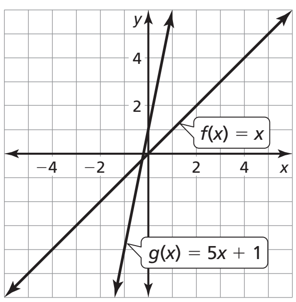
The graph of h is a vertical stretch of the graph of f by a factor of 3.

36.



The graph of h is a horizontal shrink of the graph of f by a factor of $\frac{1}{6}$.

37.



The transformations are a vertical stretch by a factor of 5, then a vertical translation 1 unit up.

Chapter 4

Chapter 4 Maintaining Mathematical Proficiency *(p. 159)*

1. $(-5, -2)$ **2.** $(2, 0)$ **3.** C **4.** E

5. $y = -5 + x$ **6.** $y = -\frac{1}{3} - 2x$ **7.** $y = 4x - 5$

8. $y = \frac{1}{4}x + 7$ **9.** $y = 4x - \frac{1}{2}$ **10.** $y = -\frac{3}{4}x - 1$

11. Points in Quadrant I move to Quadrant III, points in Quadrant II move to Quadrant IV, points in Quadrant III move to Quadrant I, and points in Quadrant IV move to Quadrant II.

4.1 Vocabulary and Core Concept Check *(p. 165)*

1. linear model

4.1 Monitoring Progress and Modeling with Mathematics *(pp. 165–166)*

3. $y = 2x + 9$ **5.** $y = -3x$ **7.** $y = \frac{2}{3}x - 8$

9. $y = \frac{1}{3}x + 2$ **11.** $y = -\frac{4}{3}x$ **13.** $y = -3x + 10$

15. $y = -4$ **17.** $y = \frac{8}{3}x + 5$ **19.** $f(x) = x + 2$

21. $f(x) = -\frac{1}{4}x - 2$ **23.** $f(x) = -5x - 4$

25. $f(x) = -2x + 1$

27. The slope and the y-intercept were substituted incorrectly; $y = 2x + 7$

29. **a.** $y = -0.005t + 3.91$ **b.** 3.71 min; 3.61 min

31. no; Because the x-coordinates are the same, the line is vertical and has an undefined slope.

33. $y = -\frac{A}{B}x + \frac{C}{B}$; slope is $\frac{6}{5}$; y-intercept is $\frac{9}{5}$

35. $y = \frac{5}{3}x - 1$

37. slope $= \frac{(b + m) - b}{1 - 0} = m$; y-intercept $= b$; $y = mx + b$;

Substitute -1 for x in $y = mx + b$ and verify that $y = b - m$.

4.1 Maintaining Mathematical Proficiency *(p. 166)*

39. no solution **41.** yes; -6 **43.** no

4.2 Vocabulary and Core Concept Check *(p. 171)*

1. -2; $(-5, 5)$

4.2 Monitoring Progress and Modeling with Mathematics *(pp. 171–172)*

3. $\frac{2}{3}$; *Sample answer:* $(9, 4)$

5. -8; *Sample answer:* $\left(-\frac{1}{4}, 10\right)$ **7.** $y - 1 = 2(x - 2)$

9. $y + 4 = -6(x - 7)$ **11.** $y = -3(x - 9)$

13. $y - 6 = \frac{3}{2}(x + 6)$ **15.** $y = 2x - 5$ **17.** $y = -\frac{1}{2}x + 1$

19. $y = -2x + 16$ **21.** $y = 2x - 13$ **23.** $y = -9$

25. $f(x) = -3x + 4$ **27.** $f(x) = -\frac{1}{2}x$ **29.** $f(x) = \frac{1}{4}x + \frac{7}{4}$

31. no; y does not increase at a constant rate.

33. yes; y increases at a constant rate; $y = 0.2x + 1.2$

35. The value substituted for y_1 was 5 instead of -5; $y + 5 = -2(x - 1)$

37. **a.** $C = 80n + 145$ **b.** \$865

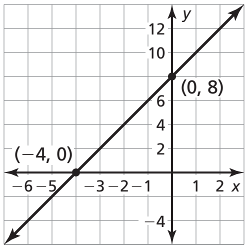
39. *Sample answer:* Plot the point $(4, 1)$, then use the slope of $\frac{2}{3}$ to find a second point on the graph and draw a line through the points; Rewrite the equation in slope-intercept form, then use the y-intercept $(0, 5)$ and the slope of $\frac{2}{3}$ to graph the equation.

41. *Sample answer:* point-slope form; The value of the y-intercept is unknown.

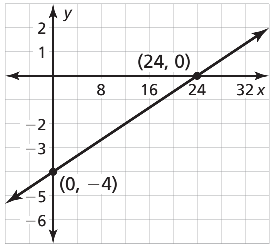
43. *Sample answer:* The graph of $y - k_1 = m(x - h)$ is a translation h units to the right and k units upward of the graph of $y = mx$.

4.2 Maintaining Mathematical Proficiency *(p. 172)*

45.



47.



4.3 Vocabulary and Core Concept Check *(p. 177)*

1. Find the slope of the line. Use the slope and one of the given points to write an equation in point-slope form. Then, rewrite the equation in standard form.

4.3 Monitoring Progress and Modeling with Mathematics *(pp. 177–178)*

3. *Sample answer:* $2x + 2y = -20$, $3x + 3y = -30$

5. *Sample answer:* $-2x + 4y = 18$, $-3x + 6y = 27$

7. *Sample answer:* $3x - y = -4$, $18x - 6y = -24$

9. $-x + y = 5$ **11.** $2x + y = 5$ **13.** $\frac{3}{2}x + y = -10$

15. $-x + y = 7$ **17.** $\frac{4}{3}x + y = -2$ **19.** $y = 3$, $x = 2$

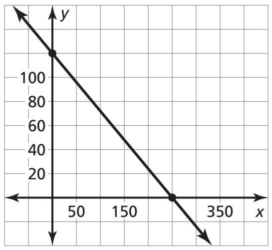
21. $y = -1$, $x = 8$ **23.** 4; $4x + 3y = 5$

25. 4; $-x + 4y = 10$

27. -4 was substituted for x instead of y and 1 was substituted for y instead of x; $A = -7$

29. **a.** $1.20x + 2.50y = 300$

b.



c. *Sample answer:* 0 Vinca plants and 120 Phlox plants, 50 Vinca plants and 96 Phlox plants, 150 Vinca plants and 48 Phlox plants, and 250 Vinca plants and 0 Phlox plants

31. **a.** *Sample answer:* slope-intercept form; The slope and y-intercept can be easily determined and used to graph the equation.

b. *Sample answer:* standard form; Substituting 0 for y allows you to solve the equation for x easily.

c. *Sample answer:* point-slope form; One point can easily be substituted into an equation of this form.

33. $\frac{b}{a}x + y = b$

35. $2\ell + 2w = 60$; *Sample answer:*

Length	Width
28	2
26	4
25	5
22	8
18	12

4.3 Maintaining Mathematical Proficiency *(p. 178)*

37. $-\frac{1}{8}$ **39.** $\frac{2}{3}$

4.4 Vocabulary and Core Concept Check *(p. 183)*

1. parallel

4.4 Monitoring Progress and Modeling with Mathematics *(pp. 183–184)*

3. lines a and b; They have the same slope.

5. lines a and c; They have the same slope.

7. none; None of the lines have the same slope.

9. $y = 2x + 5$ **11.** $y = \frac{1}{3}x - 4$

13. None are parallel or perpendicular; None of the lines have the same slope or slopes that are negative reciprocals of each other.

15. None are parallel; Lines b and c are perpendicular; None of the lines have the same slope and the slope of line b is the negative reciprocal of the slope of line c.

17. Lines a and b are parallel; Line c is perpendicular to lines a and b; Lines a and b have the same slope and the slope of line c is the negative reciprocal of the slopes of lines a and b.

19. $y = -2x + 24$ **21.** $y = -\frac{1}{4}x + \frac{9}{4}$

23. **a.** *Sample answer:* $x = 1$ **b.** *Sample answer:* $y = -3$

25. **a.** *Sample answer:* $y = -6$ **b.** *Sample answer:* $x = 2$

27. Parallel lines have the same slope, not negative reciprocal slopes; $y - 3 = \frac{1}{4}(x - 1)$; $y - 3 = \frac{1}{4}x - \frac{1}{4}$; $y = \frac{1}{4}x + \frac{11}{4}$

29. $y = -\frac{3}{4}x + \frac{3}{2}$

31. **a.** yes; Opposite sides are parallel.

b. no; Adjacent sides are not perpendicular.

33. no; The lines that form the angle are not perpendicular.

35. never; Perpendicular lines have opposite reciprocal slopes, so one must be positive and the other must be negative.

37. sometimes; They are perpendicular when the slopes are negative reciprocals, otherwise they will not be perpendicular.

4.4 Maintaining Mathematical Proficiency *(p. 184)*

39. function; Each input value is paired with exactly one output value.

4.5 Vocabulary and Core Concept Check *(p. 191)*

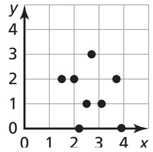
1. increase

4.5 Monitoring Progress and Modeling with Mathematics *(pp. 191–192)*

3. 6 **5.** 7 **7.** **a.** \$1100 **b.** 12 GB **c.** increases

9. positive **11.** no

13.



no

15. **a.** *Sample answer:* $y = -0.3x + 35$

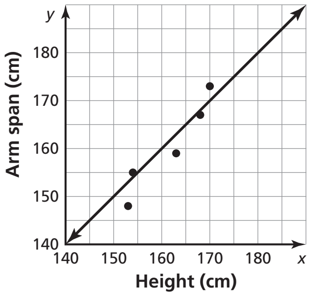
b. *Sample answer:* The slope of -0.3 means the birthrate is decreasing by about 3 births per 1000 people every 10 years. The y-intercept of 35 means in 1960 the birth rate was about 35 births per 1000 people.

17. *Sample answer:*

Weight of car (pounds), x	2400	2500	2900	3000
Gas mileage (mpg), y	39	38	25	32

Weight of car (pounds), x	3400	3500	3700	5100
Gas mileage (mpg), y	30	24	21	16

19. **a.** *Sample answer:*



b. *Sample answer:* The slope of 1 means a person's arm span increases by about 1 centimeter for every 1 centimeter increase in height. The y-intercept of 0 has no meaning in this context because the height cannot be 0.

21. *Sample answer:* When the data are from two sets such as age and time.

23. no; The data points do not have a linear trend.

4.5 Maintaining Mathematical Proficiency *(p. 192)*

25. $-18, 0, 24$ **27.** $-23, -8, 12$

4.6 Vocabulary and Core Concept Check *(p. 198)*

1. when the actual y-value is greater than the y-value from the model; when the actual y-value is less than the y-value from the model.

3. Interpolation is using a graph or its equation to approximate a value between two known values, and extrapolation is using a graph or its equation to predict a value outside the range of known values.

4.6 Monitoring Progress and Modeling with Mathematics *(pp. 198–200)*

5. no; The residual points are not evenly dispersed about the horizontal axis.

7. yes; The residual points are evenly dispersed about the horizontal axis.

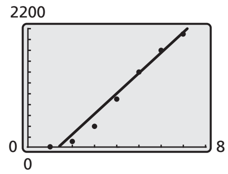
9. yes; The residual points are evenly dispersed about the horizontal axis.

11. $y = 2.1x - 8$; $r = 0.980$; strong positive correlation

13. $y = 1.4x + 16$; $r = 0.999$; strong positive correlation

15. The slope and y-intercept were reversed; $y = -4.47x + 23.16$

17. a. $y = 381x - 566$



b. $r = 0.989$; strong positive correlation

c. The slope of 381 means the number of people who reported an earthquake increased by about 381 each minute after the earthquake ended. The y-intercept of -566 has no meaning in this context because the number of people cannot be negative.

19. a. $y = -0.2x + 20$

b. $r = -0.968$; strong negative correlation

c. The slope of -0.2 means the selling price decreases by about \$200 for every increase in mileage of 1000 miles. The y-intercept of 19.7 has no meaning in this context because a used car cannot have 0 mileage.

d. 22,500 mi **e.** \$18,800

21. There is a negative correlation and a causal relationship because the more time you spend talking on the phone, the less charge there is left in the battery.

23. A correlation is unlikely. The number of hats you own is not related to the size of your head.

25. *Sample answer:* ACT math score and SAT math score

27. a. $y = -0.08x + 3.8$; $r = -0.965$; strong negative correlation

b. The slope of -0.08 means the GPA decreases by about 0.08 for every hour spent watching television in a week. The y-intercept of 3.8 means that a student who watches no television has a GPA of about 3.8.

c. 2.7

d. *Sample answer:* no; The time spent watching television does not determine GPA.

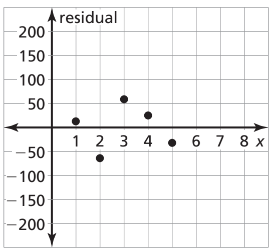
29. a. 2863 people; 5149 people

b. relatively accurate at 9 min, but not accurate at 15 min.

31. a. $y = 513.5x - 298$; $r = 0.986$; strong positive correlation

b. no; The year does not determine the number of text messages sent.

c. 25.5; -128; 117.5; 50; -63.5



The equation $y = 513.5x - 298$ is a good fit.

d. *Sample answer:* part (a); The correlation coefficient is a single value, which is easily interpreted where as interpreting the scatter plot of the residuals is more subjective.

4.6 Maintaining Mathematical Proficiency *(p. 200)*

33. linear; The rate of change is constant.

4.7 Vocabulary and Core Concept Check *(p. 206)*

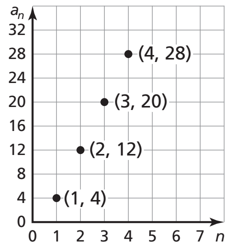
1. The graph of an arithmetic sequence is a graph of a linear function whose domain is the set of positive integers.

4.7 Monitoring Progress and Modeling with Mathematics *(pp. 206–208)*

3. 15, 28, 41 **5.** 5 **7.** 4 **9.** -1.5

11. 31, 34, 37 **13.** 36, 41, 46 **15.** 0.1, -0.2, -0.5

17.



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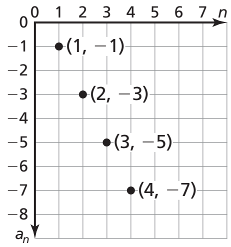

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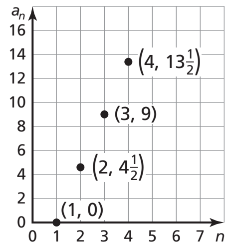


23. not an arithmetic sequence; Consecutive terms do not have a common difference.

25. arithmetic sequence; Consecutive terms have a common difference of -15.

27. arithmetic sequence; 13 **29.** not an arithmetic sequence

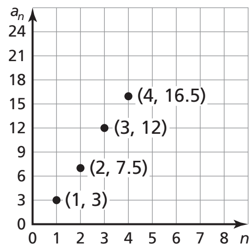
31. 4, 6, 8, 10; yes; Consecutive terms have a common difference of 2.

33. $a_n = n - 6$; 4 **35.** $a_n = \frac{1}{2}n$; 5

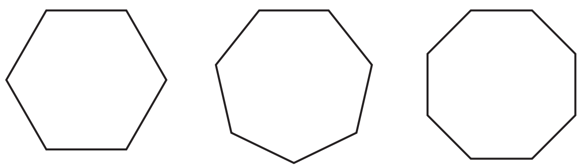
37. $a_n = -10n + 20$; -80

39. -1 is added each time, not 1; The common difference is -1.

41. 7.5, 12, 16.5



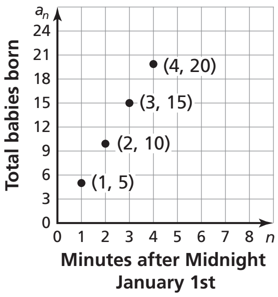
43.



a regular 22-sided polygon

45. **a.** $f(n) = 5n$

b.



c. 20 min

47. arithmetic sequence; $f(n) = 4n - 3$; 117

49. discrete; discrete

51. *Sample answer:* $-12, -15, -18, -21$; $a_n = -3n - 9$; 12, 9, 6, 3; $a_n = -3n + 15$

53. 119 **55.** $f(n) = 6n + 17$

57. **a.** arithmetic sequence; Consecutive terms have a common difference of $2x$.

b. not an arithmetic sequence; Consecutive terms do not have a common difference.

4.7 Maintaining Mathematical Proficiency *(p. 208)*

59. $\frac{1}{3}$

Chapter 4 Review *(pp. 210–212)*

1. $y = -\frac{1}{2}x + 1$ **2.** $y - 7 = -(x - 4)$

3. $f(x) = x - 5$ **4.** $f(x) = -4$ **5.** $f(x) = -\frac{5}{3}x + 18$

6. $10x + 2y = -20$, $x + \frac{1}{5}y = -2$ **7.** $4x + y = -1$

8. $3x - y = 2$ **9.** $y = -12$, $x = 2$ **10.** $y = 3$, $x = -4$

11. Lines a and b are parallel; None of the lines are perpendicular; Lines a and b have the same slope and none of them have negative reciprocal slopes.

12. None of the lines are parallel; Lines b and c are perpendicular; None of the lines have the same slope and the slope of line b is the negative reciprocal of the slope of line c.

13. $y = -4x + 9$ **14.** $y = \frac{1}{2}x - 4$

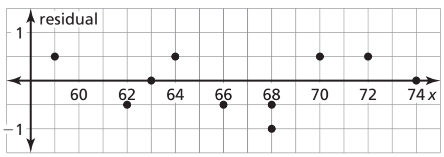
15. **a.** *Sample answer:* $x = 6$

b. *Sample answer:* $y = 4$

16. $4h$

17. *Sample answer:* $y = \frac{1}{5}x + 2$; The slope of $\frac{1}{5}$ means the roasting time increases by about $\frac{1}{5}$ hour for each pound the weight of the turkey increases. The y-intercept of 2 has no meaning in this context because the weight of the turkey cannot be 0 pounds.

18.



19. **a.** 65 in. **b.** 6.5

20. no; Height does not determine shoe size.

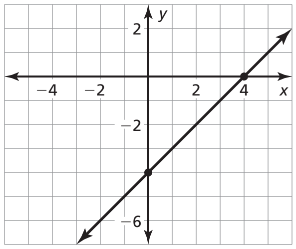
21. $a_n = -n + 12$; -18 **22.** $a_n = 6n$; 180

23. $a_n = 3n - 12$; 78

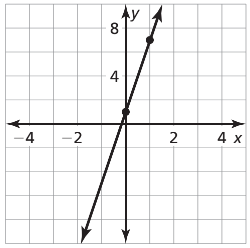
Chapter 5

Chapter 5 Maintaining Mathematical Proficiency *(p. 217)*

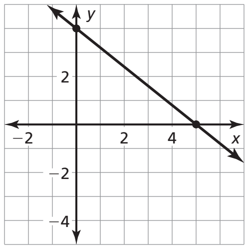
1.



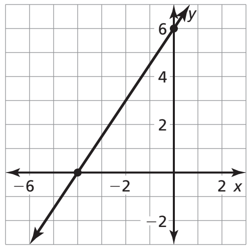
2.



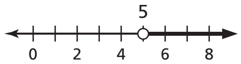
3.



4.

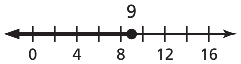


5. $m > 5$

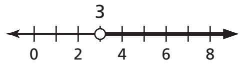


6. $t \leq -4$

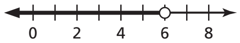
7. $a \leq 9$



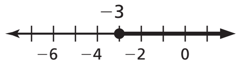
8. $z > 3$



9. $k < 6$



10. $w \geq -3$



11. The two lines intersect.

5.1 Vocabulary and Core Concept Check *(p. 223)*

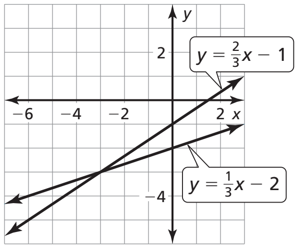
1. yes; They are two equations in the same variables.

5.1 Monitoring Progress and Modeling with Mathematics *(pp. 223–224)*

3. yes **5.** no **7.** yes **9.** $(1, -3)$ **11.** $(-4, 5)$

13. $(3, 4)$ **15.** $(-9, -1)$ **17.** $(-1, 2)$ **19.** $(-4, 0)$

21. The graph of the second equation should be $y = \frac{2}{3}x - 1$.



$(-3, -3)$

23. $(10, 5)$ **25.** $(1.2, 1.4)$

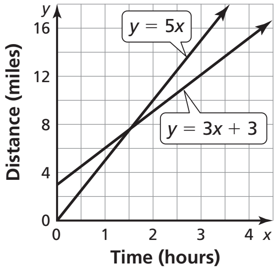
27. 30 min on the elliptical trainer, 10 min on the stationary bike

29. $A = 18x - 18$; $P = 6x + 6$; $(2, 18)$; The value of the perimeter and the area of the rectangle will both be 18 when $x = 2$.

31. **a.** $x = 3$ **b.** $(3, 5)$

c. The equation in part (a) shows that the 2 values of y in the system in part (b) are equal. The x-coordinate of the solution of the system in part (b) is the solution of the equation in part (a).

33. **a.** $y = 5x$ and $y = 3x + 3$



b. no; The point of intersection does not occur at $x = 1$.

5.1 Maintaining Mathematical Proficiency *(p. 224)*

35. $y = 2x + 3$

5.2 Vocabulary and Core Concept Check *(p. 229)*

1. Solve one of the equations for one of the variables. Substitute the expression for that variable into the other equation to find the value of the other variable. Substitute this value into one of the original equations to find the value of the remaining variable.

5.2 Monitoring Progress and Modeling with Mathematics *(pp. 229–230)*

3. *Sample answer:* $x - 2y = 0$; This equation can be solved for x easily.

5. *Sample answer:* $5x - y = 5$; This equation can be solved for y easily.

7. *Sample answer:* $x - y = -3$; This equation can be solved for x easily.

9. $(5, 3)$ **11.** $(-4, 5)$ **13.** $(6, 7)$ **15.** $(5, -8)$

17. In Step 2, the expression for y needs to be substituted in the other equation; $8x + 2(5x - 4) = -12$, $8x + 10x - 8 = -12$, $18x - 8 = -12$, $18x = -4$, $x = -\frac{2}{9}$

19. $x + y = 180$ and $x = 3y$; 135 acres of corn, 45 acres of wheat

21. *Sample answer:* $y = x + 2$ and $x + 2y = 13$

23. *Sample answer:* $x = \frac{1}{3}y$ and $x + y = -16$

25. 8 five-point problems, 30 two-point problems

27. **a.** $x + y + 90 = 180$ **b.** $x = 67, y = 23$

29. $a = 4, b = 5$

31. *Sample answer:* $y = -3x + 4$ and $x + y = 6$

33. 144 pop songs, 48 rock songs, 80 hip-hop songs **35.** 47

5.2 Maintaining Mathematical Proficiency *(p. 230)*

37. -13 **39.** $3d + 5$ **41.** $64v$

5.3 Vocabulary and Core Concept Check *(p. 235)*

1. *Sample answer:* $x + 3y = 5$ and $-x + 4y = 10$

5.3 Monitoring Progress and Modeling with Mathematics *(pp. 235–236)*

3. $(1, 6)$ **5.** $(4, 5)$ **7.** $(-1, 2)$ **9.** $(0, -10)$

11. $(1, 1)$ **13.** $(8, 3)$ **15.** $(-7, -12)$ **17.** $(5, -3)$

19. $5x + x \neq 4x$; $6x = 24$, $x = 4$

21. $x + 5y = 22.45$ and $x + 7y = 25.45$; \$14.95 fee, \$1.50 per quart of oil

23. $(2, -1)$; *Sample answer:* elimination because y has the same coefficient in both equations

25. $(4, 6)$; *Sample answer:* substitution because the second equation is already solved for y

27. 4 and –4; These values yield x terms with either the same or opposite coefficients.

29. no; *Sample answer:* The system $8x - 5y = 11$ and $4x - 3y = 5$ can be solved by elimination in fewer steps than it can be solved by substitution.

31. **a.** 2ℓ and $2w = 18$ and $6\ell + 4w = 46$; length: 5 in.; width: 4 in.

b. length: 15 in.; width: 8 in.

33. 4.5 qt of 100% fruit juice, 1.5 qt of 20% fruit juice

35. $(-5, 4, 2)$; *Sample answer:* Subtract Equation 2 from Equation 1. The resulting equation only has 1 variable, y, so use it to solve for y. Substitute this result in Equation 3 and solve for x. Substitute the values of x and y in Equation 2 and solve for z.

5.3 Maintaining Mathematical Proficiency *(p. 236)*

37. $t = \frac{3}{8}$; one solution

39. all real numbers; infinitely many solutions

41. $y = 5x + 6$

5.4 Vocabulary and Core Concept Check *(p. 241)*

1. no; Two lines cannot intersect at exactly two points.

5.4 Monitoring Progress and Modeling with Mathematics *(pp. 241–242)*

3. F; no solution **5.** B; infinitely many solutions

7. D; no solution **9.** $(0, -4)$

11. infinitely many solutions **13.** infinitely many solutions

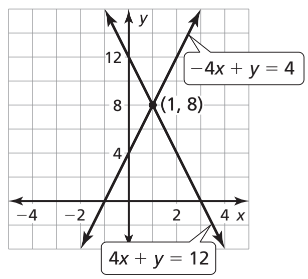
15. no solution

17. infinitely many solutions; The lines have the same slope and the same y-intercept, so they are the same line.

19. one solution; The lines have different slopes, so they will intersect.

21. no solution; The lines have the same slope but different y-intercepts, so they are parallel.

23. The lines are not parallel, so they must intersect.



$(1, 8)$

25. $d = 6t + 3$ and $d = 6t + 1$; no; The system has no solution.

27. yes; The system of equations $150c + 80b = 22{,}860$ and $170c + 100b = 27{,}280$ has a solution.

29. one; The lines have different slopes, so they will intersect at exactly one point.

31. **a.** never; The y-intercepts are different, so they can never be equations for the same line.

b. sometimes; When $a = b$, the lines are parallel and there is no solution.

c. always; When $a < b$, the slopes are different and the lines intersect at one point.

5.4 Maintaining Mathematical Proficiency *(p. 242)*

33. (8, 2) **35.** (4, −2)

5.5 Vocabulary and Core Concept Check *(p. 249)*

1. $x = 6$

5.5 Monitoring Progress and Modeling with Mathematics *(pp. 249–250)*

3. $x = 5$ **5.** $x = 1$ **7.** $x = -1$ **9.** $x = -3$

11. $x = -2$ **13.** $x = -3$ **15.** $x = 2$ **17.** $x = 3$

19. $x = 2$; one solution

21. all real numbers; infinitely many solutions

23. no solution **25.** $2x - 3 = 3$ **27.** $-x - 3 = 3x + 2$

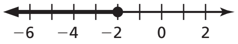
29. $x = -2$ **31.** 75 guests **33.** 30 sec

35. *Sample answer:* $m = 1, b = 8$ **37.** $x = 10$

39. **a.** negative **b.** positive

5.5 Maintaining Mathematical Proficiency *(p. 250)*

41.



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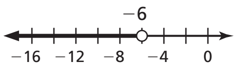


45. The graph of g is a reflection in the x-axis of the graph of f.

47. The graph of g is a horizontal translation 1 unit right of the graph of f.

5.6 Vocabulary and Core Concept Check *(p. 256)*

1. Substitute the values into the inequality and verify that the statement is true.

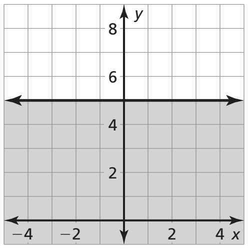
5.6 Monitoring Progress and Modeling with Mathematics *(pp. 256–258)*

3. yes **5.** no **7.** yes **9.** no **11.** no

13. yes **15.** no

17. no; (12, 14) is not a solution of the inequality.

19.



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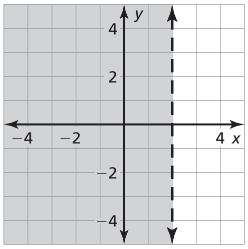

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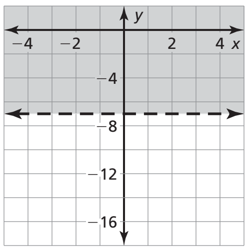

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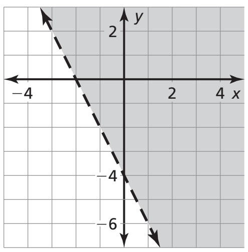

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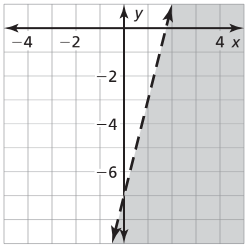

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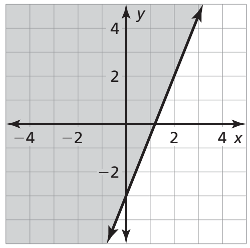
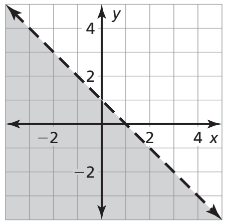


31. The line should be dashed.

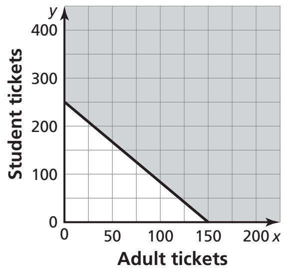


33. $y > 2x + 1$ **35.** $y \leq -\frac{1}{2}x - 2$ **37.** $y < -x + 10$

39. $x + y \geq 29$ **41.** $x + y > 55.07$

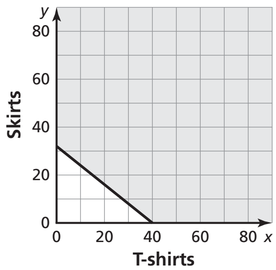
43. The calories for breakfast and both snacks were added to 2500 instead of subtracted; $x + y \leq 1330$

45. $10x + 6y \geq 1500$



Sample answer: (150, 0), 150 adult tickets were sold and no student tickets were sold; (100, 300), 100 adult tickets were sold and 300 student tickets were sold.

47. $20x + 25y \geq 800$

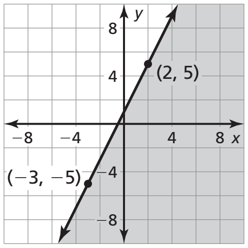


Sample answer: (40, 10), The store can sell 40 T-shirts and 10 skirts; (20, 30), The store can sell 20 T-shirts and 30 skirts.

49. no; You cannot use (0, 0) as a test point when it is on the boundary line.

51. A test point on the boundary line is not in either half-plane, so it will not indicate which half-plane to shade.

53. $y \le 2x + 1$



5.6 Maintaining Mathematical Proficiency *(p. 258)*

55. 40, 48, 56 **57.** −0.4, −1.2, −2

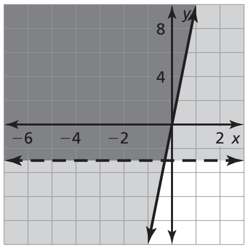
5.7 Vocabulary and Core Concept Check *(p. 264)*

1. Substitute the values into both inequalities and verify that both inequalities are true.

5.7 Monitoring Progress and Modeling with Mathematics *(pp. 264–266)*

3. no **5.** no **7.** yes **9.** no

11.



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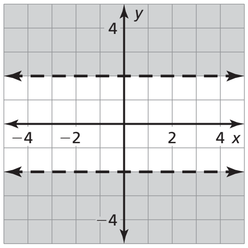

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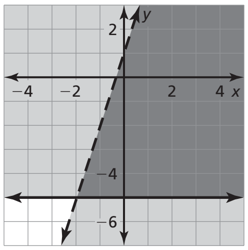

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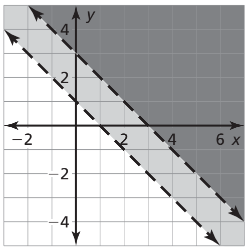
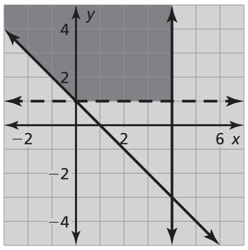


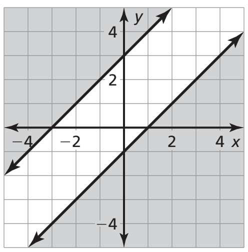
19.



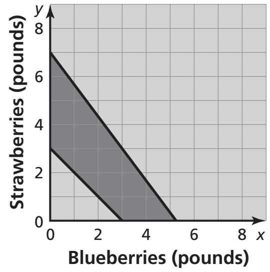
21. $x \ge -1$ and $y < 3$ **23.** $y \ge \frac{2}{3}x - 2$ and $y \ge -3x + 2$

25. $y > -2x - 1$ and $y < -2x - 3$

27. The wrong regions are shaded for both inequalities.



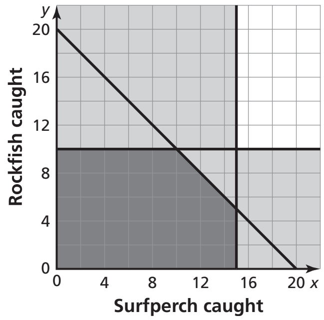
29. a. $4x + 3y \le 21$ and $x + y \ge 3$



b. *Sample answer:* (2, 4), You can buy 2 pounds of blueberries and 4 pounds of strawberries.

c. yes

31. a. $x \le 15$, $y \le 10$, and $x + y \le 20$



b. yes

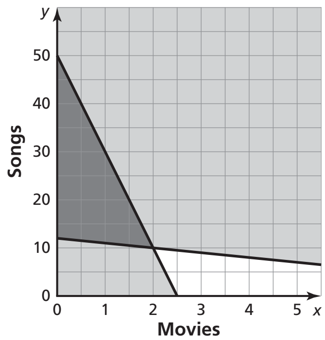
33. a. $x \ge -1$, $x \le 6$, $y \ge -3$, and $y \le 1$

b. 28 square units

35. *Sample answer:* \$300 on savings, \$500 on housing

37. Both can be done by graphing; Solving a system of linear inequalities requires finding overlapping half-planes, solving a system of linear equations does not.

39. *Sample answer:* You receive a \$50 gift card to an online retail store. Movies cost \$20 each and songs cost \$1 each. You want to purchase at least 12 items. How many of each item can you buy?; $20x + y \le 50$ and $x + y \ge 12$



41. no; The solution of each inequality is a half-plane, and so the intersection can be at most a half-plane.

43. *Sample answer:* $x > 2$ and $y < -4$

45. a. *Sample answer:* $-4x + 2y < 6$

b. *Sample answer:* $-2x + y > 3$

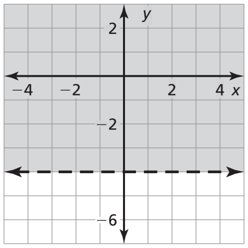
47. *Sample answer:* $x \ge 4$, $x \le 4$, $y \ge 5$, and $y \le 5$

5.7 Maintaining Mathematical Proficiency *(p. 266)*

49. 4^5 **51.** x^6 **53.** $y = -3x + 5$ **55.** $y = \frac{4}{3}x$

Chapter 5 Review *(pp. 268–270)*

1. (2, −5) **2.** (1, −1) **3.** (4, −1) **4.** (−2, −3)
5. (2, 1) **6.** (2, 0) **7.** 8 brushes, 4 tubes of paint
8. (2, −8) **9.** (−2, 5) **10.** (4, 5) **11.** no solution
12. no solution **13.** infinitely many solutions
14. $x = -5$ **15.** $x = 13$ **16.** $x = -3$

17.


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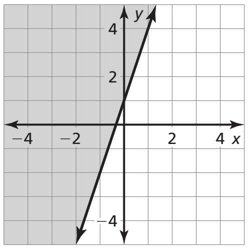

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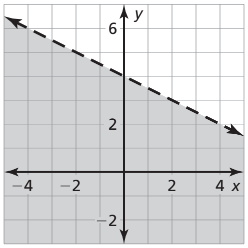

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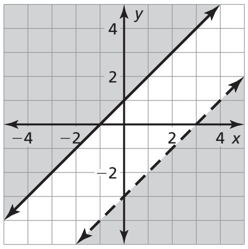

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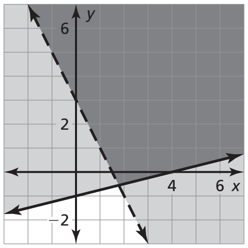

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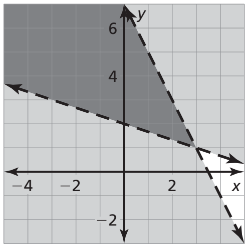


Chapter 6

Chapter 6 Maintaining Mathematical Proficiency *(p. 275)*

1. −9 **2.** 106 **3.** −12 **4.** 8 **5.** −2
6. −5 **7.** ±11 **8.** $a_n = 2n + 10$
9. $a_n = -3n + 9$ **10.** $a_n = -7n + 29$
11. yes; no; The product of two perfect squares can be represented by $m^2n^2 = (mm)(nn) = (mn)(mn) = (mn)^2$. If m and n are integers, their product is also an integer, so $(mn)^2$ is an integer. There are many counterexamples illustrating that the quotient of two perfect squares does not have to be a perfect square, such as $9 \div 4$.

6.1 Vocabulary and Core Concept Check *(p. 282)*

1. Product of Powers Property, Power of a Power Property, definition of negative exponents; Use the Product of Powers Property to simplify the expression inside the parentheses to 4^4. Then, use the Power of a Power Property to simplify the entire expression to 4^{-8}. Then, use the definition of negative exponents to produce the final answer, $\frac{1}{4^8} = \frac{1}{65{,}536}$.

3. The Quotient of Powers Property is used when dividing powers that have the same base. The answer is the common base raised to the difference of the exponents of the numerator and denominator.

6.1 Monitoring Progress and Modeling with Mathematics *(pp. 282–284)*

5. 1 **7.** $\frac{1}{625}$ **9.** $\frac{1}{16}$ **11.** $-\frac{4}{3}$ **13.** $\frac{1}{x^7}$

15. $\frac{9}{y^3}$ **17.** $\frac{1}{4m^3}$ **19.** $\frac{b^7}{64}$ **21.** $\frac{32x^7}{y^6}$ **23.** 625

25. 6561 **27.** p^{24} **29.** $\frac{1}{216}$ **31.** x^2 **33.** 10^{-2} m

35. The product has a base of 2, not $2 \cdot 2$; $2^4 \cdot 2^5 = 2^9$

37. $-125z^3$ **39.** $\frac{n^2}{36}$ **41.** $\frac{1}{243s^{40}}$ **43.** $\frac{36}{w^6}$

45. B, C, D; These expressions simplify to be the volume of the sphere, which is $\frac{32\pi s^6}{3}$.

47. $\frac{16y^{28}}{81x^{12}}$ **49.** $\frac{n^{10}}{144m^4}$ **51.** 4.5×10^{-3}; 0.0045

53. 4×10^2; 400

55. about 4.113×10^4 lb/acre; about 41,130 lb/acre

57. a. Power of a Product Property

b. Express $\frac{(6x)^3}{(2x)^3}$ as $\left(\frac{6x}{2x}\right)^3$. Simplify the expression inside the parentheses to produce $(3)^3$, so the volume is 27 times greater.

59. $(2ab)^3$ **61.** $(2w^3z^2)^6$

63. a. $\left(\frac{1}{6}\right)^n$ **b.** $\frac{1}{1296}$

c. $\frac{1}{32}$; The probability of flipping heads once is $\frac{1}{2}$, and $\left(\frac{1}{2}\right)^5$ is $\frac{1}{32}$.

65. $x = 8$, $y = -1$; Using the Quotient of Powers Property, you can conclude from the first equation that $x - y = 9$. Using the Product of Powers Property and the Quotient of Powers Property, you can conclude from the second equation that $x + 2 - 3y = 13$. Use these equations to solve a system of linear equations.

67. no; The mass of the seed from the double coconut palm is 10 kilograms.

69. a. When $a > 1$ and $n < 0$, $a^n < a^{-n}$ because a^n will be less than 1 and a^{-n} will be greater than 1. When $a > 1$ and $n = 0$, $a^n = a^{-n} = 1$, because any number to the zero power is 1. When $a > 1$ and $n > 0$, $a^n > a^{-n}$ because a^n will be greater than 1 and a^{-n} will be less than 1.

b. When $0 < a < 1$ and $n < 0$, $a^n > a^{-n}$, because a^n will be greater than 1 and a^{-n} will be less than 1. When $0 < a < 1$ and $n = 0$, $a^n = a^{-n} = 1$, because any number to the zero power is 1. When $0 < a < 1$ and $n > 0$, $a^n < a^{-n}$ because a^n will be less than 1 and a^{-n} will be greater than 1.

6.1 Maintaining Mathematical Proficiency *(p. 284)*

71. −10

73. natural number, whole number, integer, rational number, real number

75. irrational number, real number

6.2 Vocabulary and Core Concept Check *(p. 289)*

1. Find the fourth root of 81, or what real number multiplied by itself four times produces 81.

6.2 Monitoring Progress and Modeling with Mathematics *(pp. 289–290)*

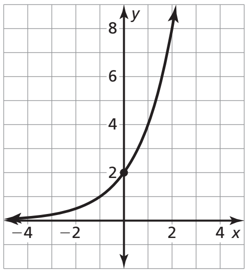
3. $10^{1/2}$ **5.** $\sqrt[3]{15}$ **7.** ± 6 **9.** 10 **11.** $s = 4$ in.
13. 4 **15.** -7 **17.** 2 **19.** $8^{4/5}$ **21.** $\left(\sqrt[7]{-4}\right)^2$
23. 8 **25.** not a real number **27.** -32
29. The numerator and denominator are reversed; $\left(\sqrt[3]{2}\right)^4 = 2^{4/3}$
31. $\frac{1}{10}$ **33.** $\frac{1}{9}$ **35.** 6 ft^2 **37.** 25 **39.** 5
41. $d^{7/12}$ **43.** $4n^{13/8}$ **45.** about 1 in.
47. Write the radicand, a, as the base and write the exponent as a fraction with the power, m, as the numerator and the index, n, as the denominator.
49. -1, 0, and 1 **51.** $(xy)^{1/2}$ **53.** $2xy^2$
55. about 1.38 ft **57.** always; Power of a Power Property
59. always; definition of rational exponent
61. sometimes; false when $x = 0$ because division by 0 is undefined

6.2 Maintaining Mathematical Proficiency *(p. 290)*

63. $f(-3) = -16$; $f(0) = -10$; $f(8) = 6$
65. $h(-3) = 16$; $h(0) = 13$; $h(8) = 5$

6.3 Vocabulary and Core Concept Check *(p. 296)*

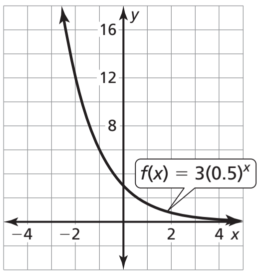
1. *Sample answer:*



3. The graph of $y = 2(5)^x$ is a vertical stretch by a factor of 2 of the graph of $y = 5^x$. The y-intercept of $y = 2(5)^x$, 2, is above the y-intercept of $y = 5^x$, 1.

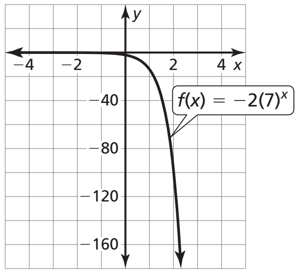
6.3 Monitoring Progress and Modeling with Mathematics *(pp. 296–298)*

5. yes; It fits the pattern $y = ab^x$.
7. no; The exponent is a constant.
9. no; Although it fits the pattern $y = ab^x$, b cannot be negative.
11. linear; As x increases by 1, y increases by 2. The rate of change is constant.
13. exponential; As x increases by 1, y is multiplied by 4.
15. 9 **17.** -100 **19.** 72 **21.** C **23.** A
25.



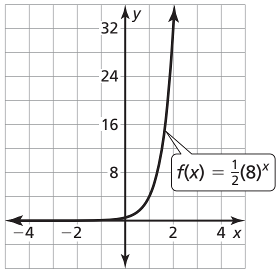
The graph of f is a vertical stretch by a factor of 3 of the graph of $g(x) = 0.5^x$. The y-intercept of the graph of f, 3, is above the y-intercept of the graph of g, 1. The x-axis is an asymptote of both the graphs of f and g; domain: all real numbers, range: $y > 0$

27.



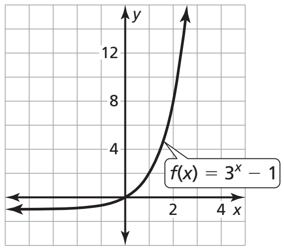
The graph of f is a vertical stretch by a factor of 2 and a reflection in the x-axis of the graph of $g(x) = 7^x$. The y-intercept of the graph of f, -2, is below the y-intercept of the graph of g, 1. The x-axis is an asymptote of both the graphs of f and g; domain: all real numbers, range: $y < 0$

29.



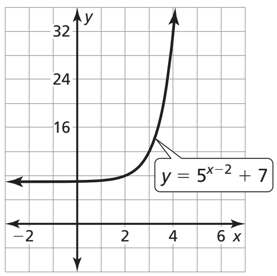
The graph of f is a vertical shrink by a factor of $\frac{1}{2}$ of the graph of $g(x) = 8^x$. The y-intercept of the graph of f, $\frac{1}{2}$, is below the y-intercept of the graph of g, 1. The x-axis is an asymptote of both the graphs of f and g; domain: all real numbers, range: $y > 0$

31.



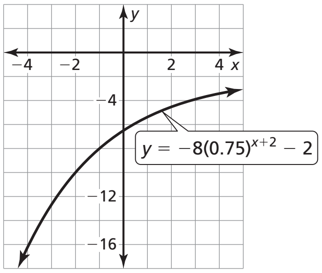
asymptote: $y = -1$; domain: all real numbers, range: $y > -1$

33.



asymptote: $y = 7$; domain: all real numbers, range: $y > 7$

35.



asymptote: $y = -2$; domain: all real numbers, range: $y < -2$

37. The graph of g is a vertical shrink by a factor of $\frac{1}{2}$ of the graph of f; $a = \frac{1}{2}$

39. The graph of g is a horizontal translation 4 units right of the graph of f; $h = 4$

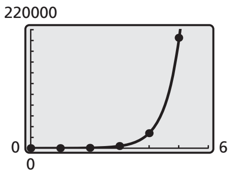
41. **a.** $y = 20(4)^x$; $y = 20$; The initial area visible is 20 square miles.

b. domain: all whole numbers, range: $y \geq 20$

c. 20,971,520 mi^2

43. need to simplify the power before multiplying; $6(0.5)^{-2} = 6(4) = 24$

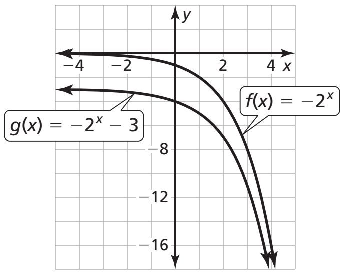
45. **a.** $y = 11(7.1)^x$



b. about 10.1 million views

47. **a.** $y = 40\left(\frac{3}{2}\right)^x$ **b.** about 304 visitors

49.



The y-intercept of g is 3 units below the y-intercept of f. The domain of both functions is all real numbers. The range of g is $y < -3$ and the range of f is $y < 0$.

51. When a is positive, it causes a vertical stretch or shrink on the graph. When a is negative, it causes a vertical stretch or shrink and a reflection in the x-axis on the graph.

53. $g(x) = 5^{x-3} + 4$ **55.** -192

57. $\dfrac{f(x+h)}{f(x)} = \dfrac{ab^{x+h}}{ab^x} = \dfrac{b^{x+h}}{b^x} = b^{(x+h)-x} = b^h$

59. *Sample answer:* $y = 8(2)^x$

6.3 Maintaining Mathematical Proficiency *(p. 298)*

61. 0.35 **63.** 2.5

6.4 Vocabulary and Core Concept Check *(p. 305)*

1. rate of growth

3. Exponential growth occurs when a quantity increases by the same factor over equal intervals of time. Exponential decay occurs when a quantity decreases by the same factor over equal intervals of time.

6.4 Monitoring Progress and Modeling with Mathematics *(pp. 305–308)*

5. $a = 350$, $r = 75\%$; about 5744.6

7. $a = 25$, $r = 20\%$; about 62.2

9. $a = 1500$, $r = 7.4\%$; about 2143.4

11. $a = 6.72$, $r = 100\%$; about 215.0 **13.** $y = 10{,}000(1.65)^t$

15. $y = 210{,}000(1.125)^t$

17. **a.** $y = 315{,}000(1.02)^t$ **b.** about 468,000

19. $a = 575$, $r = 60\%$; about 36.8

21. $a = 240$, $r = 25\%$; about 101.3

23. $a = 700$, $r = 0.5\%$; about 689.6

25. $a = 1$, $r = 12.5\%$; about 0.7 **27.** $y = 100{,}000(0.98)^t$

29. $y = 100(0.905)^t$

31. The growth rate is $1 + 1.5$, not just 1.5; $b(t) = 10(2.5)^t$; $b(8) = 10(2.5)^8 \approx 15{,}258.8$; After 8 hours, there are about 15,259 bacteria in the culture.

33. exponential decay; As x increases by 1, y is multiplied by $\frac{1}{5}$.

35. neither; As x increases by 1, y decreases by 6.

37. exponential growth; As x increases by 5, y is multiplied by 4.

39. **a.** exponential decay **b.** about \$15,155

41. exponential decay; 22,000 people; The population decreases by 6% each year.

43. exponential growth; 1 sprint; The number of sprints increases by 100% each day.

45. $f(x) = 1.52(0.9)^x$; exponential decay

47. $f(x) = 2(1.69)^x$; exponential growth

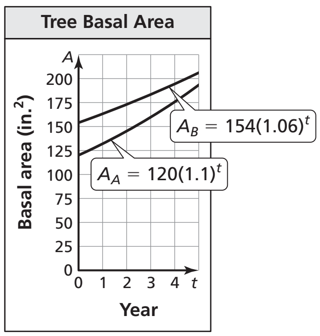
49. $f(x) = (1.20)^x$; exponential growth

51. $f(x) = 0.67(0.55)^x$; exponential decay

53. $y = 2000(1.0125)^{4t}$ **55.** $y = 6200(1.007)^{12t}$

57. **a.** $A_A = 120(1.1)^t$; $A_B = 154(1.06)^t$

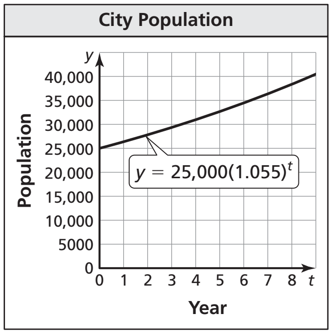
b.



The basal area of Tree B is larger than the basal area of Tree A, but the difference between the basal areas is decreasing.

59. **a.** $y = 25{,}000(1.055)^t$ **b.** about 0.45%

c.



about 30,971 people

61. **a.** $y \approx 800(0.9943)^{60t} \approx 800(0.9943^{60})^t \approx 800(0.7097)^t$;
$y \approx 800(0.843)^{2t} \approx 800(0.843^2)^t \approx 800(0.7106)^t$

b. All three functions indicate the initial amount of ibuprofen in a person's bloodstream is 800 mg. The first function indicates the amount of ibuprofen in a person's bloodstream decreases by about 29% each hour. The second function indicates the amount of ibuprofen in a person's bloodstream decreases by about 0.57% each minute. The third function indicates the amount of ibuprofen in a person's bloodstream decreases by about 15.7% each half-hour.

63. 200%; The growth factor is 3, which is also $r + 1$, so r, the growth rate, is 2, or 200%.

65. *Sample answer:* $y = 5(1)^x$; Any power of one is 1, so this is a constant function equivalent to $y = 5$.

67. no; The discount is 20% of the preceding day's price, not always the original price, so the amount of the discount is less each day.

6.4 Maintaining Mathematical Proficiency *(p. 308)*

69. $x = -3$ **71.** $r = 5$

73. not an arithmetic sequence; Consecutive terms do not have a common difference.

75. arithmetic sequence; Consecutive terms have a common difference of 10.

6.5 Vocabulary and Core Concept Check *(p. 316)*

1. The first sequence is an arithmetic sequence with a common difference of 2. The second sequence is a geometric sequence with a common ratio of 2.

6.5 Monitoring Progress and Modeling with Mathematics *(pp. 316–318)*

3. 3 **5.** -8 **7.** $\frac{3}{4}$

9. arithmetic; There is a common difference of 8.

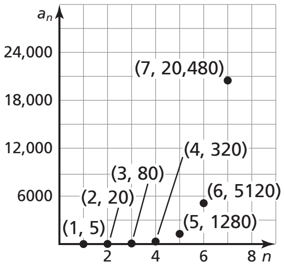
11. neither; There is neither a common difference nor a common ratio.

13. geometric; There is a common ratio of $\frac{1}{8}$.

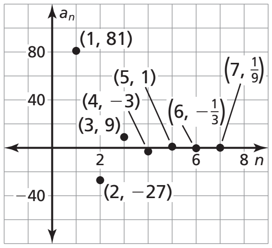
15. geometric; There is a common ratio of 5.

17. neither; There is neither a common difference nor a common ratio.

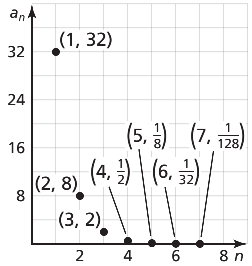
19. 1280; 5120; 20,480



21. $1, -\frac{1}{3}, \frac{1}{9}$



23. $\frac{1}{8}, \frac{1}{32}, \frac{1}{128}$



25. $a_n = 2(4)^{n-1}$; 2048 **27.** $a_n = -\frac{1}{8}(2)^{n-1}$; -4

29. $a_n = 7640(0.1)^{n-1}$; 0.0764

31. $a_n = 0.5(-6)^{n-1}$; -3888

33. 16 teams; 8 teams; 4 teams

35. The common factor is $-\frac{1}{2}$, not -2;

$$-8 \xrightarrow{\times\left(-\frac{1}{2}\right)} 4 \xrightarrow{\times\left(-\frac{1}{2}\right)} -2 \xrightarrow{\times\left(-\frac{1}{2}\right)} 1, \ldots$$

The next three terms are $-\frac{1}{2}$, $\frac{1}{4}$, and $-\frac{1}{8}$.

37. **a.** $a_n = 625\left(\frac{4}{5}\right)^{n-1}$ **b.** 5 swings

39. **a.** $a_n = 9^{n-1}$

b. a large square containing 387,420,489 small squares

41. $a_n = 64\left(\frac{1}{2}\right)^{n-1}$; geometric; There is a common ratio of $\frac{1}{2}$.

43. Graphs of arithmetic sequences form a linear pattern. Graphs of geometric sequences form an exponential pattern when the common ratio is positive, and a pattern of points alternating between Quadrants I and IV when the common ratio is negative.

45. yes; yes; It is an arithmetic sequence with a common difference of 0, and a geometric sequence with a common ratio of 1.

47. 59,049

49. dependent; Each term is calculated from the preceding term.

51. **a.** 2048, 1536, 1152, 864, 648 **b.** $a_n = 2048\left(\frac{3}{4}\right)^{n-1}$

c. never; An exponential function approaches 0, but never actually equals 0.

53. *Sample answer:* $1, -2, 4, -8, \ldots$

6.5 Maintaining Mathematical Proficiency *(p. 318)*

55. yes; The residual points are evenly dispersed about the horizontal axis.

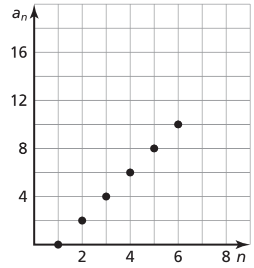
6.6 Vocabulary and Core Concept Check *(p. 324)*

1. recursive equation

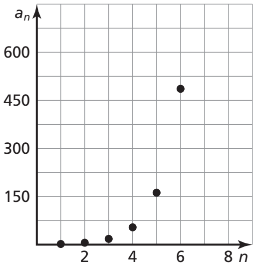
6.6 Monitoring Progress and Modeling with Mathematics *(pp. 324–326)*

3. geometric **5.** arithmetic

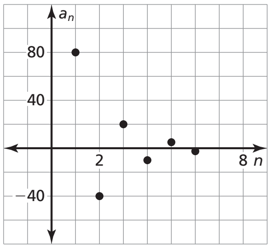
7. 0, 2, 4, 6, 8, 10



9. 2, 6, 18, 54, 162, 486



11. 80, -40, 20, -10, 5, $-\frac{5}{2}$



13. $a_1 = 7, a_n = a_{n-1} + 9$ **15.** $a_1 = 243, a_n = \frac{1}{3}a_{n-1}$

17. $a_1 = 0, a_n = a_{n-1} - 3$ **19.** $a_1 = -1, a_n = 4a_{n-1}$

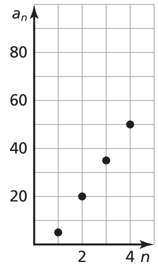
21. $a_1 = 1$ cell, $a_n = (2a_{n-1})$ cells **23.** $a_n = 3n - 6$

25. $a_n = 16(0.5)^{n-1}$ **27.** $a_n = 17n - 13$

29. $a_1 = 7, a_n = 3a_{n-1}$ **31.** $a_1 = 4.5, a_n = a_{n-1} + 1.5$

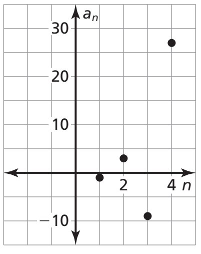
33. $a_1 = 1, a_n = -5a_{n-1}$

35.



$a_1 = 5, a_n = a_{n-1} + 15; a_n = 15n - 10$

37.



$a_1 = -1, a_n = -3a_{n-1}; a_n = -(-3)^{n-1}$

39. $a_1 = 1, a_2 = 3, a_n = a_{n-2} + a_{n-1}$; 18, 29

41. $a_1 = 2, a_2 = 4, a_n = a_{n-1} - a_{n-2}$; 2, 4

43. $a_1 = 1, a_2 = 3, a_n = (a_n - 1)(a_{n-2})$; 243, 6561

45. The common difference is -12, not 12;
$a_n = 6 + (n - 1)(-12)$; $a_n = 6 - 12n + 12$; $a_n = 18 - 12n$

47. 10; 31; 66 **49.** -8; 8; -8 **51.** 15; -15; -10

53. **a.** 3, 5, 7, 9, 11, 13, 15, 17, 19, 21

b. 3; 12; 48; 192; 768; 3072; 12,288; 49,152; 196,608; 786,432

c. 4, 7, 3, -4, -7, -3, 4, 7, 3, -4

55. 5, 19, 61, 187, 565; neither; There is no common difference or common ratio.

57. **a.** Substituting $n - 1$ for n in the explicit rule that defines an arithmetic sequence results in this expression.

b. Write the equation; Identity Property of Addition; Additive Inverse Property; Associative Property of Addition; Distributive Property; Substitution Property of Equality

59. $a_1 = 3, a_n = a_{n-1} + 2^n$

6.6 Maintaining Mathematical Proficiency *(p. 326)*

61. $-6y - 5$ **63.** $8m + 3$ **65.** $f(x) = -\frac{1}{2}x - 1$

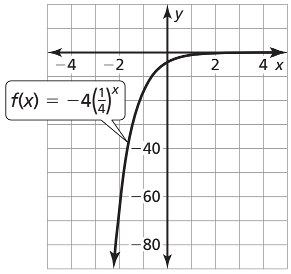
67. $f(x) = 2x - 7$

Chapter 6 Review *(pp. 328–330)*

1. $\frac{1}{y^2}$ **2.** $\frac{1}{x^3}$ **3.** y^6 **4.** $\frac{25y^8}{4x^4}$ **5.** 2 **6.** -3

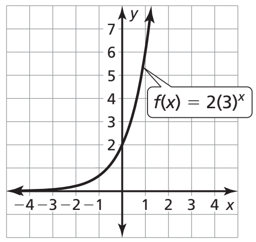
7. 125 **8.** not a real number

9.



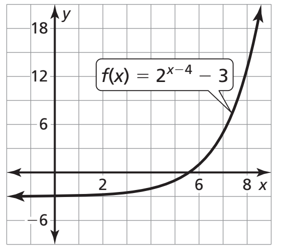
The graph of f is a vertical stretch by a factor of 3 and a reflection in the x-axis of the graph of $g(x) = \left(\frac{1}{4}\right)^x$. The y-intercept of the graph of f, -4, is below the y-intercept of the graph of g, 1. The x-axis is the asymptote of both the graphs of f and g; domain: all real numbers, range: $y < 0$

10.



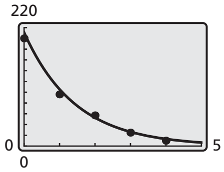
The graph of f is a vertical stretch by a factor of 2 of the graph of $g(x) = 3^x$. The y-intercept of the graph of f, 2, is above the y-intercept of the graph of g, 1. The x-axis is the asymptote of both the graphs of f and g; domain: all real numbers, range: $y > 0$

11.



asymptote: $y = -3$; domain: all real numbers, range: $y > -3$

12. **a.** $y = 211(0.5)^x$



b. $r = -0.996$; The relationship between x and y has a strong negative correlation and the equation closely models the data.

13. exponential growth; As x increases by 1, y is multiplied by 2.

14. exponential decay; As x increases by 1, y is multiplied by $\frac{2}{3}$.

15. exponential growth; \$1.75; The value of the stock increases by 2% each day.

16. $f(x) = 7.81(1.25)^x$, exponential growth; 25%

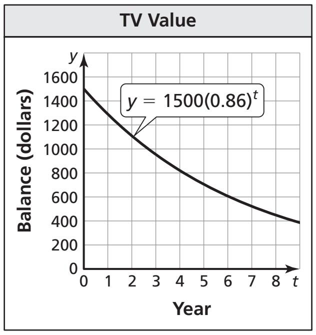
17. $f(x) = 1.59^x$, exponential growth; 59%

18. $f(x) = 12.05(0.84)^x$, exponential decay; 16%

19. **a.** $y = 750(1.0125)^{4t}$ **b.** \$914.92

20. **a.** $y = 1500(0.86)^t$ **b.** about 1.2%

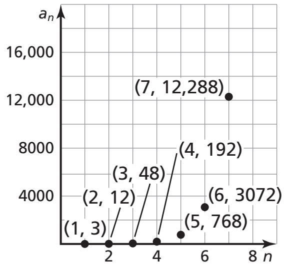
c.



asymptote: $y = 0$; The value of the TV approaches but never equals \$0.

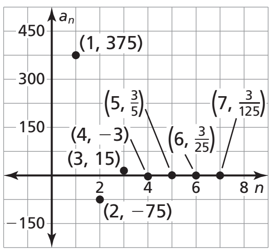
d. about \$950

21. geometric; There is a common ratio of 4; 768, 3072, 12,288



22. neither; There is neither a common ratio nor a common difference.

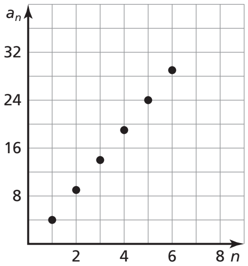
23. geometric; There is a common ratio of $-\frac{1}{5}$; $\frac{3}{5}$, $-\frac{3}{25}$, $\frac{3}{125}$



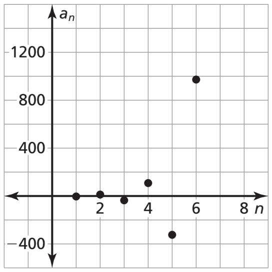
24. $a_n = 4^{n-1}$; 65,536 **25.** $a_n = 5(-2)^{n-1}$; 1280

26. $a_n = 486\left(\frac{1}{3}\right)^{n-1}$; $\frac{2}{27}$

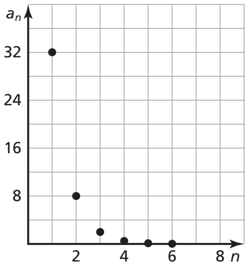
27. 4, 9, 14, 19, 24, 29



28. −4, 12, −36, 108, −324, 972



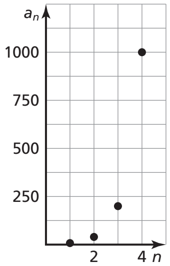
29. 32, 8, 2, $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{32}$



30. $a_1 = 3$, $a_n = a_{n-1} + 5$ **31.** $a_1 = 3$, $a_n = 2a_{n-1}$

32. $a_1 = 7$, $a_2 = 6$, $a_n = a_{n-2} + a_{n-1}$

33.



$a_1 = 8$, $a_n = 5a_{n-1}$; $a_n = 8(5)^{n-1}$

Chapter 7

Chapter 7 Maintaining Mathematical Proficiency *(p. 335)*

1. $5x - 7$ **2.** $-5r + 5$ **3.** $2t - 1$ **4.** $3s + 2$

5. $9m - 21$ **6.** $3h + 26$ **7.** $2^2 \cdot 3^2$ **8.** $3^2 \cdot 7$

9. $2 \cdot 3^3$ **10.** $2^3 \cdot 3^2$ **11.** $2^2 \cdot 7$ **12.** $2 \cdot 3 \cdot 5$

13. no; The product of any factorization is a unique number.

7.1 Vocabulary and Core Concept Check *(p. 342)*

1. when the exponents of the terms decrease from left to right

3. Determine if performing the operation on any two numbers in the set always results in a number that is also in the set.

7.1 Monitoring Progress and Modeling with Mathematics (pp. 342–344)

5. 1 **7.** 2 **9.** 9 **11.** 11

13. $2c^4 + 6c^2 - c$; 4; 2; trinomial

15. $3p^2 + 7$; 2; 3; binomial **17.** $3t^8$; 8; 3; monomial

19. $-\frac{5}{7}r^8 + 2r^5 + \pi r^2$; 8; $-\frac{5}{7}$; trinomial

21. It is the product of a number, $\frac{4}{3}\pi$, and a variable with a whole number exponent, r^3; 3

23. $3y + 10$ **25.** $n^2 - 8n + 5$ **27.** $6g^2 - 9g + 4$

29. $a^3 - 5a^2 + 4a + 5$ **31.** $-2d - 8$

33. $-2y^2 + 2y + 18$ **35.** $k^3 - k^2 - 7k + 14$

37. $t^4 + 8t^2 + 8t - 12$

39. When writing the subtraction as addition, the last term of the polynomial was not multiplied by -1;
$= (x^2 + x) + (-2x^2 + 3x) = (x^2 - 2x^2) + (x + 3x)$
$= -x^2 + 4x$

41. $3b + 2$ **43.** $s^2 - 12st$ **45.** $2c^2 - 2cd - 4d^2$

47. always; A polynomial is a monomial or a sum of monomials, and each monomial is a term of the polynomial.

49. sometimes; The two terms in the binomial can be of any degree.

51. $-16t^2 - 45t + 200$; 139 ft

53. a. $-46t + 92$

b. The constant term 92 indicates that the distance between the two balls is 92 feet when they begin. The coefficient of the linear term -46 indicates that the two balls become 46 feet closer to each other each second.

55. $12x - 3$

57. yes; Addition is commutative and associative, so you can add in any order.

59. a. no; The product of two negative integers is always a positive integer.

b. yes; The sum of two whole numbers is always a whole number.

61. a. $19x^2 - 12x$ **b.** $14,310

7.1 Maintaining Mathematical Proficiency (p. 344)

63. $34y - 34$

7.2 Vocabulary and Core Concept Check (p. 349)

1. *Sample answer:* Distribute one of the binomials over each term in the other binomial and simplify; Write each binomial as a sum of terms and make a table of products.

7.2 Monitoring Progress and Modeling with Mathematics (pp. 349–350)

3. $x^2 + 4x + 3$ **5.** $z^2 - 2z - 15$ **7.** $g^2 - 9g + 14$

9. $3m^2 + 28m + 9$ **11.** $x^2 + 5x + 6$ **13.** $h^2 - 17h + 72$

15. $12k^2 + 23k - 9$ **17.** $8j^2 - 26j + 21$

19. t also should be multiplied by $t + 5$; $= t(t + 5) - 2(t + 5)$
$= t^2 + 5t - 2t - 10 = t^2 + 3t - 10$

21. $b^2 + 10b + 21$ **23.** $q^2 - \frac{1}{2}q - \frac{3}{16}$

25. $3r^2 - 29r + 18$ **27.** $12s^4 - 52s^2 - 9$

29. $w^3 + 8w^2 + 15w$ **31.** $2x^2 + x - 45$

33. $\frac{1}{2}x^2 + \frac{11}{2}x + 15$ **35.** $x^3 + 7x^2 + 14x + 8$

37. $y^3 + 11y^2 + 22y - 6$ **39.** $-5b^3 + 15b^2 + 24b - 16$

41. $18e^4 - 30e^3 + 45e^2 - 5e + 7$

43. a. $(40x^2 + 240x + 200)$ ft^2 **b.** 57,600 ft^2

45. $y^5 - y^4 - 6y^3$ **47.** $x^5 + 4x^4 - x^3 - 8x^2 - 2x$

49. $r^4 + 3r^3 - r - 3$

51. The degree of the product is the sum of the degrees of each binomial.

53. no; FOIL would leave out the products that include the middle terms of the two trinomials.

55. yes; You are both multiplying the same binomials, and neither the order in which you multiply nor the method used will make a difference.

57. a. They have the same signs.

b. They have opposite signs.

7.2 Maintaining Mathematical Proficiency (p. 350)

59. slope: $\frac{1}{4}$; y-intercept: 7 **61.** slope: $-\frac{1}{2}$; y-intercept: 4

7.3 Vocabulary and Core Concept Check (p. 355)

1. Substitute the first term of the binomial for a and the second term of the binomial for b in the square of a binomial pattern, then simplify.

7.3 Monitoring Progress and Modeling with Mathematics (pp. 355–356)

3. $x^2 + 16x + 64$ **5.** $4f^2 - 4f + 1$ **7.** $49t^2 - 56t + 16$

9. $4a^2 + 4ab + b^2$ **11.** $x^2 + 8x + 16$

13. $49n^2 - 70n + 25$ **15.** $t^2 - 49$ **17.** $16x^2 - 1$

19. $64 - 9a^2$ **21.** $p^2 - 100q^2$ **23.** $y^2 - 16$

25. 384 **27.** 1764 **29.** 930.25

31. The middle term in the square of a binomial pattern was not included; $= k^2 + 2(k)(4) + 4^2 = k^2 + 8k + 16$

33. a. $(x^2 + 100x + 2500)$ ft^2 **b.** 4225 ft^2; 1725 ft^2

35. a. 25% **b.** $(0.5N + 0.5a)^2 = 0.25N^2 + 0.5Na + 0.25a^2$

37. $(x + 11)(x - 11)$; $x^2 - 121$ fits the product side of the sum and difference pattern, so working backwards, a and b are the square roots of a^2 and b^2.

39. $x^4 - 1$ **41.** $4m^4 - 20m^2n^2 + 25n^4$

43. no; $\left(4\frac{1}{3}\right)^2$ can be written as $\left(4 + \frac{1}{3}\right)^2$, however using the square of a binomial pattern results in $16 + \frac{8}{3} + \frac{1}{9}$, which is $18\frac{7}{9}$, not $16\frac{1}{9}$.

45. $k = 64$ **47.** *Sample answer:* $a = 3, b = 4$

7.3 Maintaining Mathematical Proficiency (p. 356)

49. $\dfrac{1}{x^2}$ **51.** $\dfrac{1}{4y^2}$

7.4 Vocabulary and Core Concept Check (p. 361)

1. Multiply the quotient by the divisor and add the remainder. The result should be the dividend; *Sample answer:* Check the result of Example 5:

$(3x - 4)(x - 2) + 12 = (3x^2 - 10x + 8) + 12$
$= 3x^2 - 10x + 20$ ✓

7.4 Monitoring Progress and Modeling with Mathematics (pp. 361–362)

3. $6x - 7$ **5.** $-8x - 4$ **7.** $7 + \dfrac{-11}{x + 2}$; no

9. 3; yes **11.** -2; yes **13.** $2x - 3 + \dfrac{3}{7x - 1}$

15. $-3x - 6 + \dfrac{3}{-3x - 2}$ **17.** $-7x + 10 + \dfrac{10}{-2x - 5}$

19. $3x + 3$ **21.** $4 + \dfrac{-x + 8}{x^2 + 2x + 1}$

23. $8 + \frac{-21x + 55}{2x^2 + x - 10}$ **25.** $5 + \frac{-8x + 17}{x^2 - 3}$

27. $-4 + \frac{-6x + 8}{5x^2 - 3x + 2}$ **29.** $8x + 12 + \frac{-7}{x - 1}$

31. $3x + 2 + \frac{5}{x + 5}$ **33.** $15x + 30 + \frac{12}{x - 2}$

35. $-x + 3 - \frac{9}{x + 4}$

37. The remainder should be divided by the divisor; $(3x^2 + 10x + 6) \div (x + 3) = 3x + 1 + \frac{3}{x + 3}$

39. When dividing the dividend by the divisor, the remainder is 0.

41. $\frac{6x + 4}{x + 3}$; $6 + \frac{-14}{x + 3}$

43. no; The divisor $x^2 + 1$ is not of the form $x - k$.

45. $2x + 2 + \frac{-10}{3x + 6}$; $6x + 6 + \frac{-10}{x + 2}$; The result of the long division problem is $\frac{1}{3}$ of the result of the synthetic division problem; The divisor of the long division problem, $3x + 6$, is 3 times the divisor of the synthetic division problem, $x + 2$.

47. 9

7.4 Maintaining Mathematical Proficiency *(p. 362)*

49. $9(r + 3)$ **51.** $5(3x - 2y)$ **53.** $z = -17$

55. $k = 0$

7.5 Vocabulary and Core Concept Check *(p. 367)*

1. Set $3x = 0$ and $x - 6 = 0$, then solve both of the equations to get the solutions $x = 0$ and $x = 6$.

7.5 Monitoring Progress and Modeling with Mathematics *(pp. 367–368)*

3. $x = 0, x = -7$ **5.** $t = 0, t = 5$ **7.** $s = 9, s = 1$

9. $a = 3, a = -5$ **11.** $m = -\frac{4}{5}$ **13.** $g = \frac{3}{2}, g = 7$

15. $z = 0, z = -2, z = 1$ **17.** $r = 4, r = -8$

19. $c = 3, c = -1, c = 6$ **21.** $x = 8, x = -8$

23. $x = 14, x = 5$ **25.** $5z(z + 9)$ **27.** $3y^2(y - 3)$

29. $n^5(5n + 2)$ **31.** $p = 0, p = \frac{1}{4}$ **33.** $c = 0, c = -\frac{5}{2}$

35. $n = 0, n = 3$

37. also need to set $6x = 0$ and solve; $6x = 0$ or $x + 5 = 0$; $x = 0$ or $x = -5$; The roots are $x = 0$ and $x = -5$.

39. 20 ft

41. $x = 0, x = 0.3$ sec; The roots represent the times when the penguin is at water level. $x = 0$ is when it leaves the water, and $x = 0.3$ second is when it returns to the water after the leap.

43. 2; x-intercepts occur when $y = 0$ and the equation has 2 roots when $y = 0$.

45. no; Roots will occur if $x^2 + 3 = 0$ or $x^4 + 1 = 0$. However, solving these equations results in $x^2 = -3$ or $x^4 = -1$, and even powers of any number cannot be negative.

47. **a.** $x = -y, x = \frac{1}{2}y$ **b.** $x = \pm y, x = -4y$

7.5 Maintaining Mathematical Proficiency *(p. 368)*

49. 1, 18; 2, 9; 3, 6 **51.** 1, 48; 2, 24; 3, 16; 4, 12; 6, 8

53. 1, 52; 2, 26, 4, 13

7.6 Vocabulary and Core Concept Check *(p. 375)*

1. They have opposite signs; When factoring $x^2 + bx + c = (x + p)(x + q)$, if c is negative, p and q must have opposite signs.

7.6 Monitoring Progress and Modeling with Mathematics *(pp. 375–376)*

3. $(x + 1)(x + 7)$ **5.** $(n + 4)(n + 5)$ **7.** $(h + 2)(h + 9)$

9. $(v - 1)(v - 4)$ **11.** $(d - 2)(d - 3)$

13. $(w - 8)(w - 9)$ **15.** $(x - 1)(x + 4)$

17. $(n - 2)(n + 6)$ **19.** $(y - 6)(y + 8)$

21. $(x + 4)(x - 5)$ **23.** $(t + 2)(t - 8)$

25. **a.** $(x - 5)$ ft **b.** 16 ft

27. $4 + 12$ is not 14; $= (x + 6)(x + 8)$

29. $m = -1, m = -2$ **31.** $x = 2, x = -7$

33. $t = -3, t = -12$ **35.** $a = 5, a = -10$

37. $m = 4, m = 11$ **39.** 100 in.^{2}

41. yes; p and q must be factors of -12 that have a sum of b, and -12 has 6 sets of integer factors, -1 and 12, -2 and 6, -3 and 4, -4 and 3, -6 and 2, and -12 and 1.

43. length: 11 ft, width: 4 ft

45. $x^2 - 2x - 24 = 0$; Multiply $(x + 4)(x - 6)$.

47. **a.** $-x^2 + 38x$ **b.** $-x^2 + 38x = 280$; 10 m

49. $(r + 3s)(r + 4s)$ **51.** $(x + 5y)(x - 7y)$

7.6 Maintaining Mathematical Proficiency *(p. 376)*

53. *Sample answer:* $4x + 2y = 6, 2x + y = 3$

55. *Sample answer:* $-x + 3y = 5, -4x + 12y = 20$

7.7 Vocabulary and Core Concept Check *(p. 381)*

1. 3

7.7 Monitoring Progress and Modeling with Mathematics *(pp. 381–382)*

3. $3(x - 1)(x + 2)$ **5.** $4(k + 3)(k + 4)$

7. $7(b - 4)(b - 5)$ **9.** $(3h + 2)(h + 3)$

11. $(2x - 1)(3x - 1)$ **13.** $(n + 2)(3n - 1)$

15. $2(g - 2)(4g + 3)$ **17.** $-(t - 3)(3t - 2)$

19. $-(c - 5)(4c + 1)$ **21.** $-(3w - 4)(5w + 7)$

23. need to factor 2 out of every term; $= 2(x^2 - x - 12) = 2(x + 3)(x - 4)$

25. $x = -2, x = 3$ **27.** $n = -\frac{5}{3}, n = \frac{3}{4}$

29. $x = -\frac{7}{2}, x = 5$ **31.** $x = -1, x = \frac{5}{7}$

33. **a.** $(5x - 2)$ ft

b. Substitute 3 for x into the expression for the area $15x^2 - x - 2$, then simplify; Substitute 3 for x into the expressions for the length $(5x - 2)$ and width $(3x + 1)$, simplify each, then multiply these two numbers.

35. length: 70 m, width: 31 m

37. *Sample answer:* $6x^2 + 3x$

39. when no combination of factors of a and c produce the correct middle term; *Sample answer:* $2x^2 + x + 1$

41. $\pm 9, \pm 12, \pm 21$ **43.** 3.5 in. **45.** $(k + 2j)(4k - j)$

47. $-(a - 2b)(6a - 7b)$

7.7 Maintaining Mathematical Proficiency *(p. 382)*

49. ± 8 **51.** -15 **53.** $(-1, -4)$ **55.** $(0, -7)$

7.8 Vocabulary and Core Concept Check *(p. 387)*

1. yes; The square roots of the first and last terms are y and 8, and the middle term is $2 \cdot y \cdot 8$, so it fits the pattern.

7.8 Monitoring Progress and Modeling with Mathematics *(pp. 387–388)*

3. $(m + 7)(m - 7)$ **5.** $(8 + 9d)(8 - 9d)$

7. $(15a + 6b)(15a - 6b)$ **9.** 63 **11.** 900 **13.** 600

15. $(h + 6)^2$ **17.** $(y - 11)^2$ **19.** $(a - 14)^2$

21. $(5n + 2)^2$

23. should follow the difference of two squares pattern; $= (n + 8)(n - 8)$

25. **a.** $(d + 4)$ cm **b.** $(4d + 16)$ cm

27. $z = -2, z = 2$ **29.** $k = 8$ **31.** $n = 3$

33. $y = -\frac{1}{4}$ **35.** $3(z + 3)(z - 3)$ **37.** $4(y - 2)^2$

39. $2(5y + 6)^2$ **41.** 1.25 sec

43. **a.** no; $w^2 + 18w + 81$ **b.** no; $y^2 - 10y + 25$

45. Square each binomial, then combine like terms; Use the difference of two squares pattern with each binomial as one of the terms, then simplify; *Sample answer:* The difference of two squares pattern; You do not need to square any binomials.

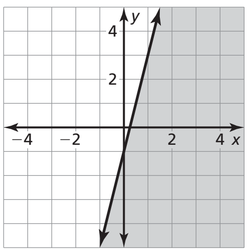
47. **a.** $(9x^2 - 144)$ in.^{2}

b. 5 in.; Setting the polynomial in part (a) equal to 81 and solving results in $x = -5$ or $x = 5$. Length cannot be negative, so 5 is the solution.

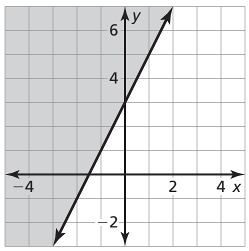
7.8 Maintaining Mathematical Proficiency *(p. 388)*

49. $2 \cdot 5^2$ **51.** $5 \cdot 17$

53.



55.



7.9 Vocabulary and Core Concept Check *(p. 393)*

1. It is written as a product of unfactorable polynomials with integer coefficients.

7.9 Monitoring Progress and Modeling with Mathematics *(pp. 393–394)*

3. $(x + 1)(x^2 + 2)$ **5.** $(z - 4)(3z^2 + 2)$

7. $(x + y)(x + 8)$ **9.** $(m - 3)(m + n)$

11. $2x(x + 1)(x - 1)$ **13.** unfactorable **15.** $6g(g - 2)^2$

17. $3r^3(r + 6)(r - 5)$ **19.** $-4c^2(c^2 - 2c + 7)$

21. $(b - 5)(b + 2)(b - 2)$ **23.** $n = 0, n = 2, n = 4$

25. $x = -1, x = -2, x = 2$ **27.** $s = 0, s = 2, s = -2$

29. $x = 0, x = -9, x = 9$ **31.** $x = 0, x = 4$

33. In the second group, factor out -6 instead of 6; $= a^2(a + 8) - 6(a + 8) = (a + 8)(a^2 - 6)$

35. **a.** $(4w^2 + 16w)$ in.^{3}

b. length: 4 in., width: 4 in., height: 8 in.

37. $(x + 2y)(x + 1)(x - 1)$ **39.** $(4s - 1)(s + 3t)$

41. no; The factors of the polynomial are $x^2 + 3$ and $x + 2$. Using the Zero Product Property, $x + 2 = 0$ will give 1 real solution, but $x^2 + 3 = 0$ has no real solutions.

43. **a.** *Sample answer:* $x^3 + x^2 + x + 2$

b. *Sample answer:* $x^3 + x^2 + x + 1$

45. $3z, 2z + 3, 2z - 3$ **47.** radius: 5, height: 8

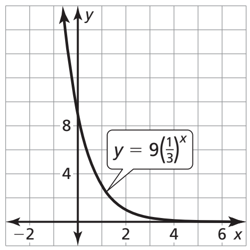
49. **a.** *Sample answer:* $w = 40$; When $w = 40$, factoring out $5x$ will leave a perfect square trinomial, so there will be 2 factors.

b. *Sample answer:* $w = 50$; When $w = 50$, factoring out $5x$ will leave a factorable trinomial that is not a perfect square, so there will be 3 factors.

7.9 Maintaining Mathematical Proficiency *(p. 394)*

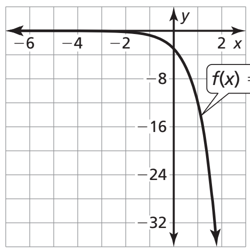
51. $(2, 3)$ **53.** $(-6, -2)$

55.



domain: all real numbers, range: $y > 0$

57.



domain: all real numbers, range: $y < 0$

Chapter 7 Review *(pp. 396–398)*

1. $2x^2 + 6$; 2; 2; binomial **2.** $5p^6 - 3p^3 - 4$; 6; 5; trinomial

3. $9x^7 + 13x^5 - 6x^2$; 7; 9; trinomial

4. $8y^3 - 12y$; 3; 8; binomial **5.** $4a + 6$

6. $3x^2 + 6x + 10$ **7.** $-2y^2 + 6y + 4$ **8.** $-5p - 6$

9. $x^2 + 2x - 24$ **10.** $3y^2 - 7y - 40$

11. $x^4 + 7x^3 + 4x^2 + 28x$ **12.** $-12y^3 + 7y^2 + 20y - 7$

13. $x^2 - 81$ **14.** $4y^2 - 16$ **15.** $p^2 + 8p + 16$

16. $4d^2 - 4d + 1$ **17.** $4 + \frac{-13}{9x + 4}$

18. $-8x - 25 + \frac{-88}{2x - 3}$ **19.** $11x + 30 + \frac{106}{x - 3}$

20. $-2 + \frac{7x + 6}{-7x^2 + 6}$ **21.** $x = 0, x = -5$

22. $z = -3, z = 7$ **23.** $b = -13$

24. $y = 0, y = 9, y = -4$ **25.** $(p + 7)(p - 5)$

26. $(b + 8)(b + 10)$ **27.** $(z + 3)(z - 7)$

28. $(x - 7)(x - 4)$ **29.** $(t + 6)(3t - 2)$

30. $-2(2y - 3)(y - 2)$ **31.** $(2x + 1)(3x + 7)$

32. $(x + 3)(x - 3)$ **33.** $(y + 10)(y - 10)$ **34.** $(z - 3)^2$

35. $(m + 8)^2$ **36.** $n(n + 3)(n - 3)$ **37.** $(x - 3)(x + 4a)$

38. $2x^2(x^2 + x - 10)$ **39.** $x = 0, x = 6, x = -3$

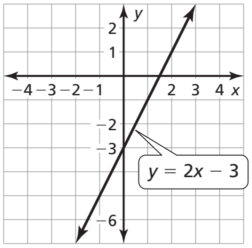
40. $x = -\frac{3}{2}, x = \frac{3}{2}$ **41.** $z = -3, z = 5, z = -5$

42. length: 12 ft, width: 4 ft, height: 2 ft

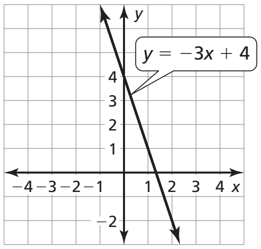
Chapter 8

Chapter 8 Maintaining Mathematical Proficiency *(p. 403)*

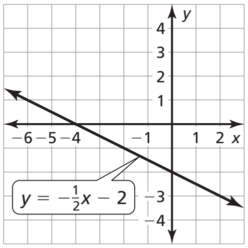
1.



2.



3.



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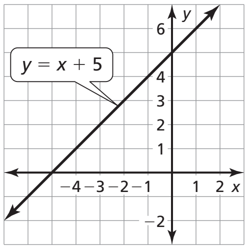


5. 11 **6.** 8 **7.** -11 **8.** -7 **9.** 3 **10.** -26

11. a, $4a$, $9a$, $16a$, $25a$; *Sample answer:* The coefficient of each difference is the next consecutive odd integer; $36a$

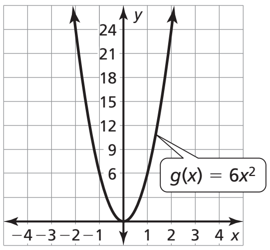
8.1 Vocabulary and Core Concept Check *(p. 409)*

1. parabola

8.1 Monitoring Progress and Modeling with Mathematics *(pp. 409–410)*

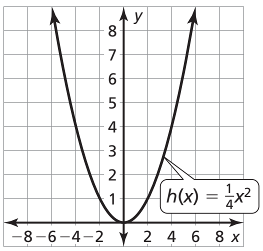
3. The vertex is $(1, -1)$. The axis of symmetry is $x = 1$. The domain is all real numbers. The range is $y \leq -1$. When $x < 1$, y increases as x increases. When $x > 1$, y increases as x decreases.

5.



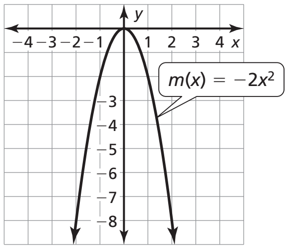
The graph of g is a vertical stretch by a factor of 6 of the graph of f.

7.



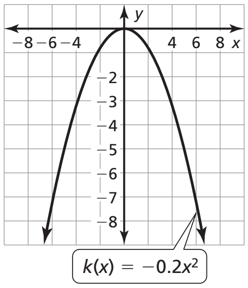
The graph of h is a vertical shrink by a factor of $\frac{1}{4}$ of the graph of f.

9.



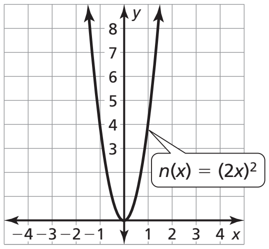
The graph of m is a vertical stretch by a factor of 2 and a reflection in the x-axis of the graph of f.

11.



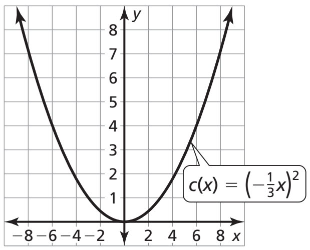
The graph of k is a vertical shrink by a factor of 0.2 and a reflection in the x-axis of the graph of f.

13.



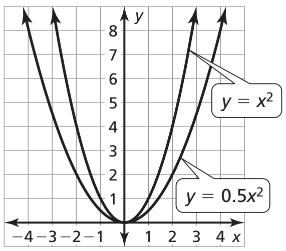
The graph of n is a horizontal shrink by a factor of $\frac{1}{2}$ of the graph of f.

15.



The graph of c is a horizontal stretch by a factor of 3 of the graph of f.

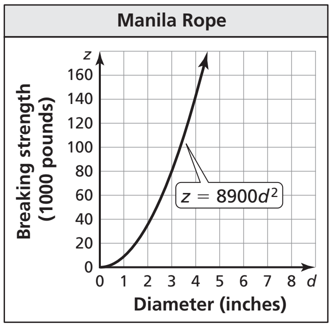
17. The graph of $y = 0.5x^2$ should be wider than the graph of $y = x^2$.



The graphs have the same vertex and the same axis of symmetry. The graph of $y = 0.5x^2$ is wider than the graph of $y = x^2$.

19. **a.** domain: $d \geq 0$, range: $z \geq 0$

b.



c. no; *Sample answer:* The relationship is quadratic, so a rope with 4 times the diameter will have $4^2 = 16$ times the breaking strength.

21. f is increasing when $x > 0$. g is increasing when $x < 0$.

23. f; $a = \frac{3}{4}$

25. *Sample answer:* The vertex of a parabola that opens up is the minimum point, so its y-coordinate is the minimum value of y. The graph passes through $(6, -3)$, so 2 is not the minimum value of y.

27. always; *Sample answer:* When $|a| > 1$, the graph of f will be a vertical stretch of the graph of g, so it will be narrower.

29. never; *Sample answer:* When $|a| > |d|$, the graph of f will be a vertical stretch of the graph of g, so it will be narrower, not wider.

31. **a.** 8 cm

b. no; *Sample answer:* A faster rotational speed would increase the depth. The diagram shown has a depth of 3.2 centimeters. A model of $y = 0.1x^2$ would only have a depth of 1.6 centimeters, so it would have a slower rotational speed.

8.1 Maintaining Mathematical Proficiency *(p. 410)*

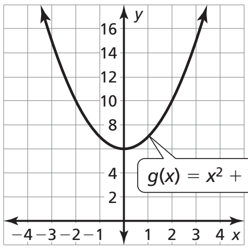
33. 3 **35.** 11

8.2 Vocabulary and Core Concept Check *(p. 415)*

1. $(0, c)$, $x = 0$

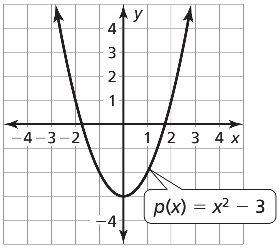
8.2 Monitoring Progress and Modeling with Mathematics *(pp. 415–416)*

3.



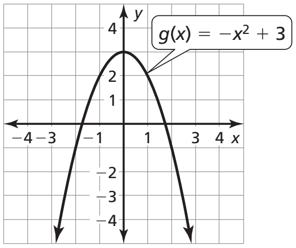
The graph of g is a vertical translation 6 units up of the graph of f.

5.



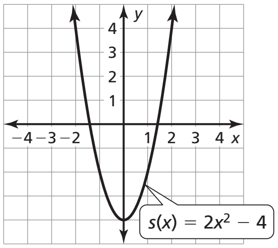
The graph of p is a vertical translation 3 units down of the graph of f.

7.



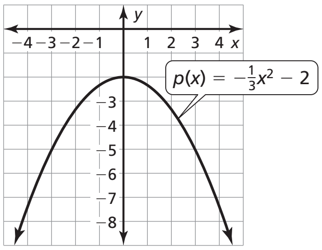
The graph of g is a reflection in the x-axis, and a vertical translation 3 units up of the graph of f.

9.



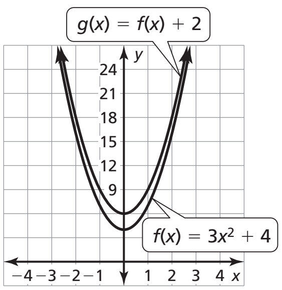
The graph of s is a vertical stretch by a factor of 2 and a vertical translation 4 units down of the graph of f.

11.



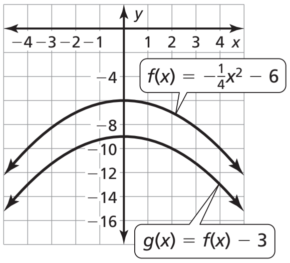
The graph of p is a vertical shrink by a factor of $\frac{1}{3}$, a reflection in the x-axis, and a vertical translation 2 units down of the graph of f.

13. The graph of g is a vertical translation 2 units up of the graph of f.



$g(x) = 3x^2 + 6$

15. The graph of g is a vertical translation 3 units down of the graph of f.



$g(x) = -\frac{1}{4}x^2 - 9$

17. The graph of $y = 3x^2 + 2$ is narrower, so it should be a stretch not a shrink; The graph of $y = 3x^2 + 2$ is a vertical stretch by a factor of 3 and a translation 2 units up of the graph of $y = x^2$.

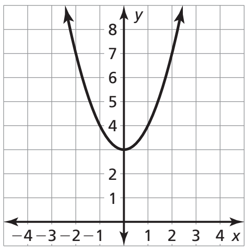
19. $x = 1, x = -1$ **21.** $x = 5, x = -5$

23. $x = 2, x = -2$ **25.** $x = \frac{1}{2}, x = -\frac{1}{2}$

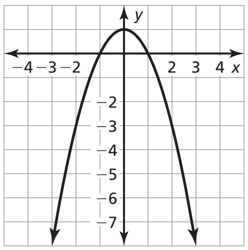
27. **a.** 3 sec

b. When $k > 0$, the water balloon will take more than 3 seconds to hit the ground. When $k < 0$, the water balloon will take less than 3 seconds to hit the ground.

29. *Sample answer:*

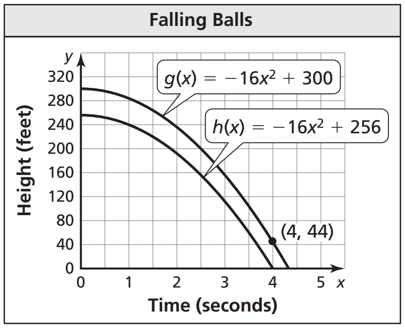


31. *Sample answer:*

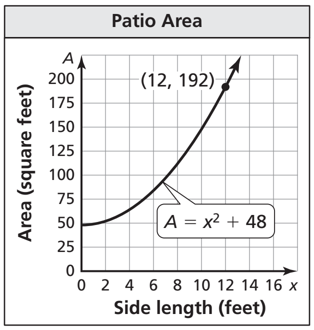


33. **a.** $T(x) = -44$; the distance between the balls

b. 44 ft



35. 12 m by 12 m



37. Graph the function and determine the x-intercepts; Set the function equal to 0, factor $-16t^2 + 400$, and apply the Zero-Product Property.

39. **a.** $h = -16t^2 + 45$; $h = -16t^2 + 32$

b. The graph of $h = -16t^2 + 32$ is a vertical translation 13 units down of the graph of $h = -16t^2 + 45$.

41. (0, 5.8); *Sample answer:* The outer edges are located 40 feet from the center. Substituting this into $y = 0.012x^2$ indicates they are 19.2 feet above the ground. To be 25 feet above the ground, they must be vertically translated up 5.8 feet.

8.2 Maintaining Mathematical Proficiency *(p. 416)*

43. $\frac{3}{8}$ **45.** $\frac{5}{12}$

8.3 Vocabulary and Core Concept Check *(p. 422)*

1. *Sample answer:* If the leading coefficient is positive, the graph has a minimum value. If the leading coefficient is negative, the graph has a maximum value.

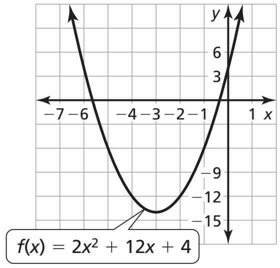
8.3 Monitoring Progress and Modeling with Mathematics *(pp. 422–424)*

3. $(2, -1)$; $x = 2$; 1 **5.** $(-2, 0)$; $x = -2$; -3

7. **a.** $x = 1$ **b.** $(1, -2)$ **9.** **a.** $x = -1$ **b.** $(-1, 8)$

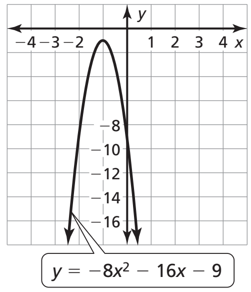
11. **a.** $x = 5$ **b.** $(5, 4)$

13.



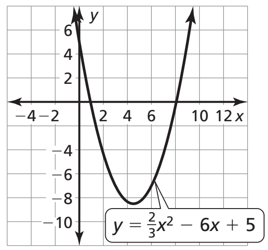
domain: all real numbers, range: $y \geq -14$

15.



domain: all real numbers, range: $y \leq -1$

17.



domain: all real numbers, range: $y \geq -\frac{17}{2}$

19. There should be two negatives in the substitution, one from the formula and one because b is -12;

$x = -\dfrac{b}{2a} = -\dfrac{-12}{2(3)} = 2$; The axis of symmetry is $x = 2$.

21. minimum value; -12 **23.** maximum value; -1

Selected Answers

25. maximum value; $66\frac{1}{2}$ **27. a.** 4 sec **b.** 256 ft

29. (0, 8); *Sample answer:* Because the axis of symmetry is $x = 3$, the point (0, 8) would also lie on the graph.

31. 4; minimum value: 0 **33.** -5; maximum value: -3

35. $(-1.41, -4)$

37. a. second aircraft hangar **b.** first aircraft hangar

39. a. $x = 4.5$ **b.** 20.25 in.^{2}

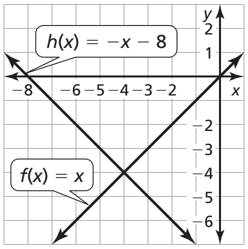
41. The graph of g is a reflection in the y-axis of the graph of h.

43. down; *Sample answer:* Because (3, 2) and (9, 2) have the same y-coordinate, any point with an x-coordinate between 3 and 9 lies on the part of the parabola between these two points that passes through the vertex. Because 7 is greater than 2, the vertex must be above these 2 points, so the parabola opens down.

45. $y = -2x^2 + 8x$ **47.** 14 ft **49.** $\frac{k^2}{8}$ ft^2

8.3 Maintaining Mathematical Proficiency *(p. 424)*

51.



The graph of h is a reflection in the x-axis and a vertical translation 8 units down of the graph of f.

8.4 Vocabulary and Core Concept Check *(p. 432)*

1. *Sample answer:* The graph of an even function is symmetric about the y-axis. The graph of an odd function is symmetric about the origin.

3. The graph of g is a horizontal translation h units right if h is positive or $|h|$ units left if h is negative, and a vertical translation k units up if k is positive or $|k|$ units down if k is negative of the graph of f.

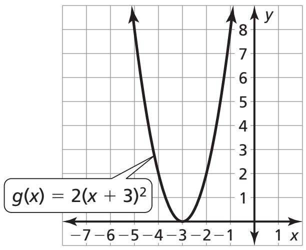
8.4 Monitoring Progress and Modeling with Mathematics *(pp. 432–434)*

5. neither **7.** neither **9.** even **11.** neither

13. even **15.** neither **17.** odd **19.** $(-1, 0)$; $x = -1$

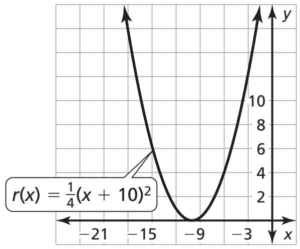
21. $(4, 0)$; $x = 4$

23.



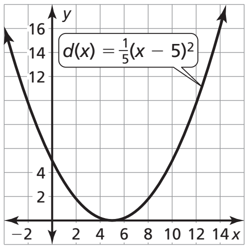
The graph of g is a horizontal translation 3 units left and a vertical stretch by a factor of 2 of the graph of f.

25.



The graph of r is a horizontal translation 10 units left and a vertical shrink by a factor of $\frac{1}{4}$ of the graph of f.

27.



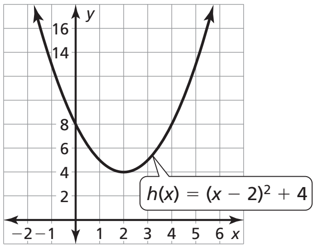
The graph of d is a horizontal translation 5 units right and a vertical shrink by a factor of $\frac{1}{5}$ of the graph of f.

29. If $f(-x) = f(x)$ the function is even; So, $f(x)$ is an even function.

31. $(-4, -3)$; $x = -4$ **33.** $(-3, 1)$; $x = -3$ **35.** C

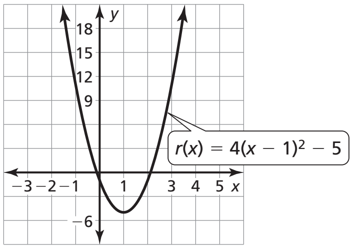
37. D

39.



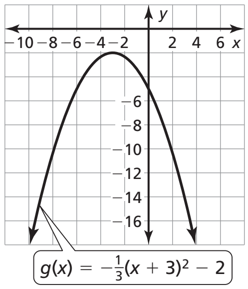
The graph of h is a translation 2 units right and 4 units up of the graph of f.

41.



The graph of r is a vertical stretch by a factor of 4, and a translation 1 unit right and 5 units down of the graph of f.

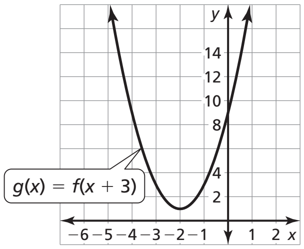
43.



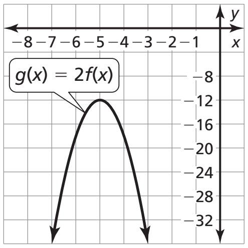
The graph of g is a vertical shrink by a factor of $\frac{1}{3}$, a reflection in the x-axis, and a translation 3 units left and 2 units down of the graph of f.

45. A **47.** B

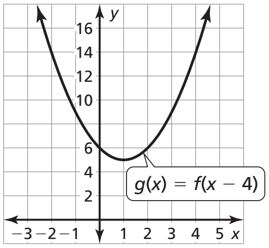
49.



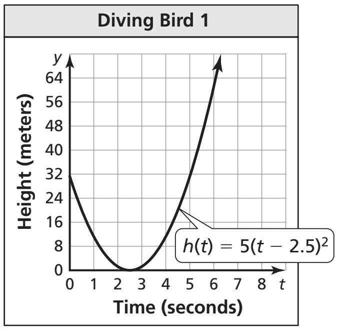
51.



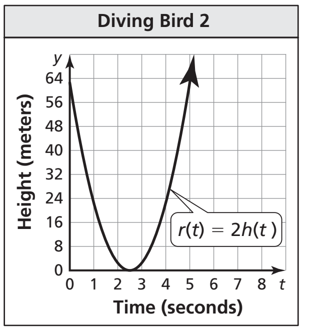
53.



55. a.



b.

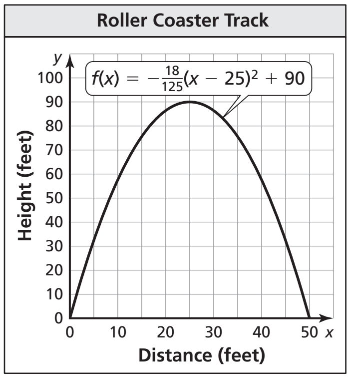


c. The graph of r is a vertical stretch by a factor of 2 of the graph of h; the second bird; *Sample answer:* Because $r(t)$ is twice $h(t)$, the second bird starts at a height twice as high as the first bird.

57. $f(x) = 2(x - 1)^2 + 2$ **59.** $f(x) = -2(x + 2)^2 - 4$

61. $f(x) = \frac{1}{2}(x - 5)^2 - 2$

63. $f(x) = -\frac{18}{125}(x - 25)^2 + 90$



65. $y = 2(x - 2)^2 - 4$ **67.** $f(x) = -5(x - 1)^2 + 8$

69. no; *Sample answer:* The graph would not pass the vertical line test.

71. The graph of h is a vertical translation 4 units up of the graph of f; $h(x) = -(x + 1)^2 + 2$

73. The graph of h is a vertical stretch by a factor of 2 of the graph of f; $h(x) = 8(x - 2)^2 + 6$

75. $y = (x - 2)^2 - 5$; $y = x^2 - 4x - 1$; *Sample answer:* The vertex, $(2, -5)$, can be quickly determined from the vertex form; The y-intercept, -1, can be quickly determined from the standard form.

77. a. the second birdbath; *Sample answer:* The first birdbath has a depth of 4 inches and the second birdbath has a depth of 6 inches.

b. the first birdbath; *Sample answer:* The first birdbath has a width of 36 inches and the second birdbath has a width of 30 inches.

8.4 Maintaining Mathematical Proficiency *(p. 434)*

79. $x = 0, x = 1$ **81.** $x = 3, x = -3$

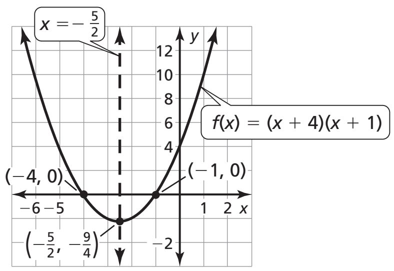
8.5 Vocabulary and Core Concept Check *(p. 441)*

1. x-intercepts

8.5 Monitoring Progress and Modeling with Mathematics *(pp. 441–444)*

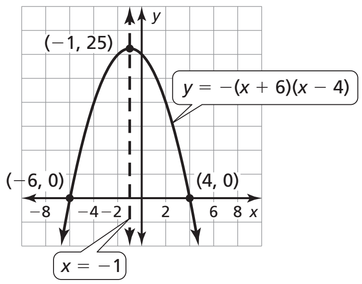
3. $-3, 1; x = -1$ **5.** $-7, 5; x = -1$

7.



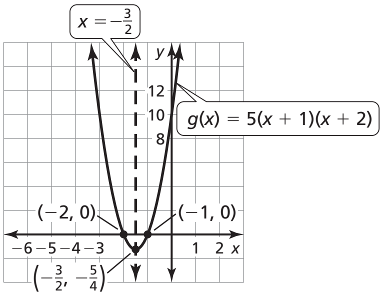
domain: all real numbers, range: $y \geq -\frac{9}{4}$

9.



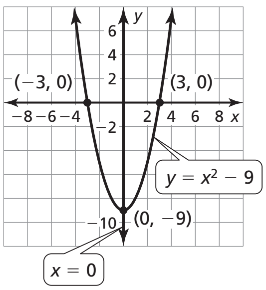
domain: all real numbers, range: $y \le 25$

11.



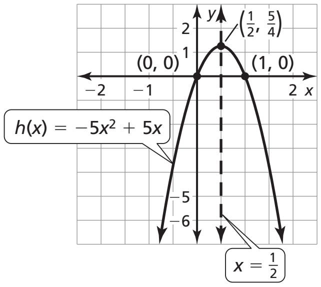
domain: all real numbers, range: $y \ge -\frac{5}{4}$

13.



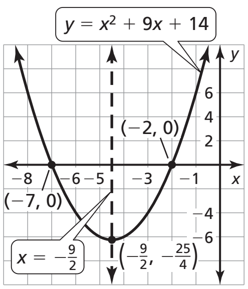
domain: all real numbers, range: $y \ge -9$

15.



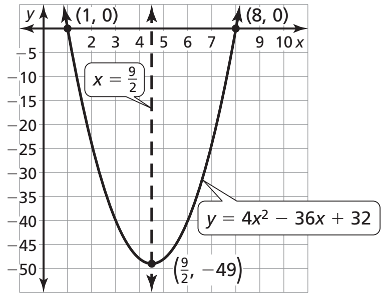
domain: all real numbers, range: $y \le \frac{5}{4}$

17.



domain: all real numbers, range: $y \ge -\frac{25}{4}$

19.

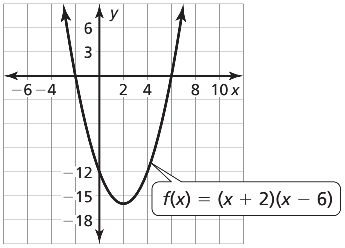


domain: all real numbers, range: $y \ge -49$

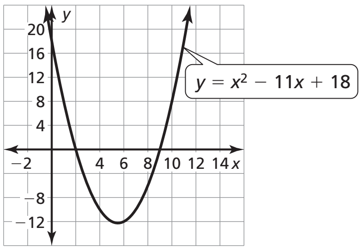
21. 2, 10 **23.** −8, 3 **25.** −2, 7 **27.** −5, −2, 2

29. −7, 0, 7 **31.** D **33.** C **35.** A

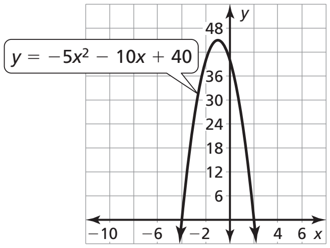
37.



39.



41.



43. The factors need to be set equal to 0 and solved to find the zeros; $x + 3 = 0$ or $x - 2 = 0$; $x = -3$ or $x = 2$; The zeros of the function are −3 and 2.

45. *Sample answer:* $f(x) = x^2 - 14x + 46$

47. *Sample answer:* $f(x) = x^2 - 10x + 9$

49. $f(x) = 3x^2 + 3x - 36$

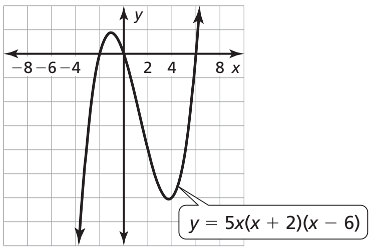
51. *Sample answer:* $f(x) = x^2 - 7x$

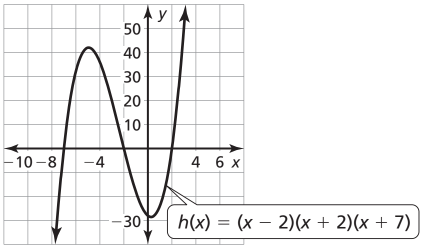
53. *Sample answer:* $f(x) = x^2 + 10x + 25$

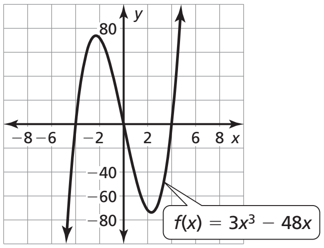
55. *Sample answer:* $f(x) = x^2 - 3$

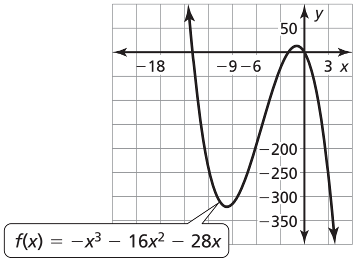
57. $f(x) = 2x^2 + 2x - 12$ **59.** $f(x) = -4x^2 + 8x + 32$

61.



63. 

65. 

67. 

69. $f(x) = 4x^3 - 12x^2 - 16x$

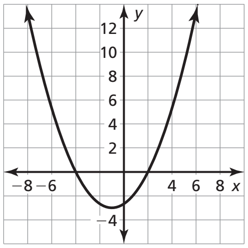
71. $f(x) = -2x^3 - 22x^2 - 56x$

73. *Sample answer:* $f(x) = x^3 - 9x^2 + 2x + 48$

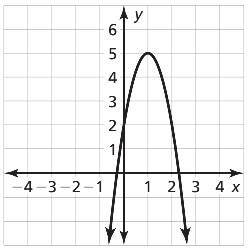
75. *Sample answer:* $f(x) = x^3 - 8x^2 + 7x$

77. $f(x) = -3x^2 + 21x$

79. $f(x) = 5x^3 + 20x^2 - 45x - 180$

81. 

83. *Sample answer:*

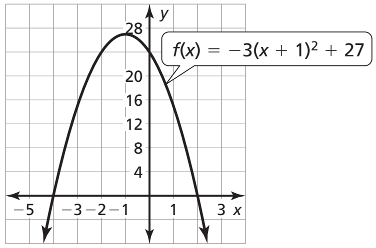


85. **a.** 4 ft **b.** $\frac{1}{2}$ ft **c.** $y = \frac{1}{6}(x^2 - 9)$ **87.** D

89. C

91. not possible; *Sample answer:* Because -5 and 1 are the x-intercepts, the axis of symmetry is $x = -2$. The points $(-3, 12)$ and $(-1, 4)$ are the same horizontal distance from the axis of symmetry, so for both of them to lie on the parabola they would have to have the same y-coordinate.

93. **a.** $f(x) = -3(x + 4)(x - 2)$

b. 

Sample answer: Plot the vertex $(-1, 27)$, which can be determined from the vertex form. Then plot the x-intercepts $(-4, 0)$ and $(2, 0)$, which can be determined from the intercept form. Draw a smooth curve through these points.

95. yes; *Sample answer:* If a quadratic function has exactly one real zero, then the vertex must lie on the x-axis. This means the zero is one of the intercepts and the x-coordinate of the vertex. $x = \dfrac{p + q}{2}$ can only be true if $p = q$, so the function can be written in intercept form with p and q having the same values.

97. $f(x) = \frac{1}{10}(x + 5)(x + 2)(x - 1)(x - 4)(x - 8)$ **99.** $-k$, $2k$

101. *Sample answer:* $y = (x - 5)^2 + 2$ and $y = -(x - 5)^2 + 10$; The two given points have the same y-coordinate, so the axis of symmetry of any parabola passing through them would be halfway between, which is $x = 5$. The vertex form can be used with $h = 5$ and any selected value of k, where $k \neq 6$, to find quadratic equations that would pass through these points. Any two of these equations would form a system with these two points as solutions.

8.5 Maintaining Mathematical Proficiency *(p. 444)*

103. 29 g

105. neither; There is no common difference or common ratio.

107. arithmetic; There is a common difference of -8.

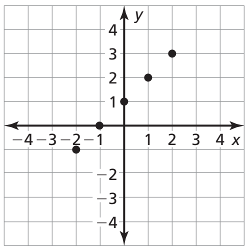
8.6 Vocabulary and Core Concept Check *(p. 451)*

1. linear, exponential, quadratic; $y = mx + b$, $y = ab^x$, $y = ax^2 + bx + c$; a straight line, a continuously increasing or decreasing curve, a parabola

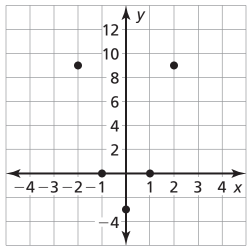
3. Find the slope of the line through $(a, f(a))$ and $(b, f(b))$.

8.6 Monitoring Progress and Modeling with Mathematics *(pp. 451–454)*

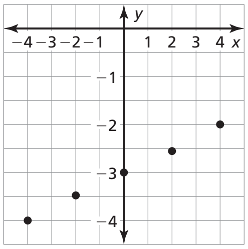
5. quadratic **7.** exponential

9. 

linear

11. 

quadratic

13. 

linear

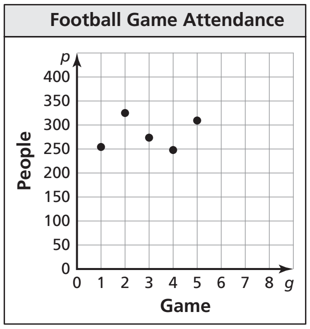
15. linear **17.** exponential

19. linear; The first differences are constant.

21. quadratic; $y = 2x^2 - 2x - 4$ **23.** linear; $y = -3x - 2$

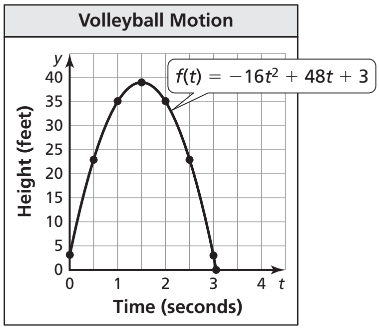
25. quadratic; $y = -3x^2$

27. Consecutive y-values have a constant ratio. They do not change by a constant amount; Consecutive y-values change by a constant ratio. So, the table represents an exponential function.

29. **a.** 

b. no; The points do not appear to follow the shape of any of these types of functions.

31. **a.** 3, 23, 35, 39, 35, 23, 3

b. 

c. The function is increasing between 0 and 1.5 seconds, and the function is decreasing between 1.5 seconds and about 3 seconds.

d. 40, 24, 8, -8, -24, -40; The average rate of change decreases when the function is increasing, and the average rate of change increases in the negative direction when the function is decreasing.

33. **a.** exponential; linear; quadratic

b. Organization A: 3, 12, 48, 192, 768, 3072; Organization B: 4, 4, 4, 4, 4, 4; Organization C: 4, 12, 20, 28, 36, 44

c. Organization A; Organization A will have the most donations, followed by Organization C, then Organization B.

35. The average rate of change of a linear function is constant because the dependent variable of a linear function increases by the same amount for each constant change in the independent variable. The average rate of change of a quadratic or exponential function is not constant because the dependent variable of a quadratic or exponential function changes by a different amount for each constant change in the independent variable.

37. quadratic; The second differences have a constant value of $9n - 5$.

39. no; *Sample answer:* There may not be enough points to clearly determine the shape of the graph.

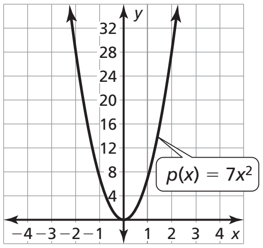
exponential; $y = (2)^{n-1}$, $y = \left(\frac{1}{2}\right)^{n-1}$, $y = n$, $y = 2^n - 1$

41. no; The graph of an exponential function will always eventually have greater y-values than the graph of a quadratic function.

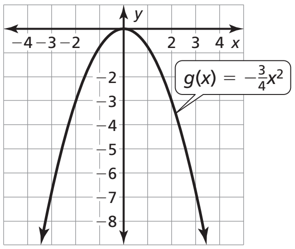
8.6 Maintaining Mathematical Proficiency *(p. 454)*

43. 11 **45.** 8 **47.** $x^2 - 64$ **49.** $9a^2 - 25b^2$

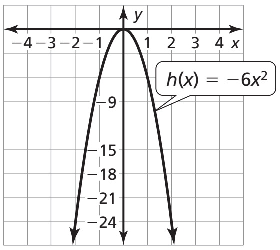
Chapter 8 Review *(pp. 456–458)*

1. 

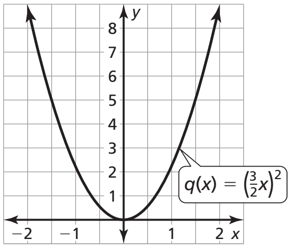
The graph of p is a vertical stretch by a factor of 7 of the graph of f.

2. 

The graph of g is a vertical shrink by a factor of $\frac{3}{4}$ and a reflection in the x-axis of the graph of f.

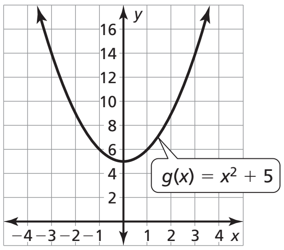
3. 

The graph of h is a vertical stretch by a factor of 6 and a reflection in the x-axis of the graph of f.

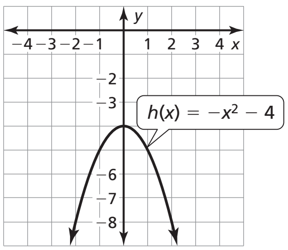
4. 

The graph of q is a horizontal shrink by a factor of $\frac{2}{3}$ of the graph of f.

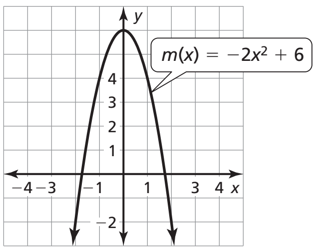
5. The vertex is $(1, -3)$. The axis of symmetry is $x = 1$. The domain is all real numbers. The range is $y \geq -3$. When $x < 1$, y increases as x decreases. When $x > 1$, y increases as x increases.

6. 

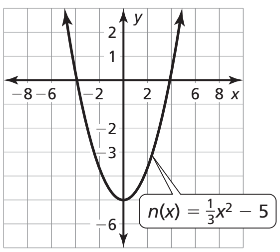
The graph of g is a vertical translation 5 units up of the graph of f.

7. 

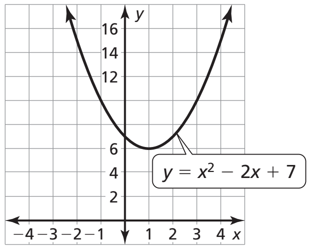
The graph of h is a reflection in the x-axis, and a vertical translation 4 units down of the graph of f.

8. 

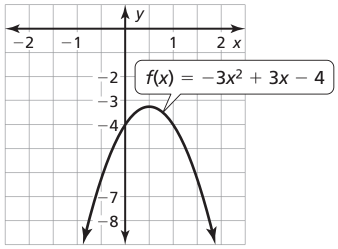
The graph of m is a vertical stretch by a factor of 2, a reflection in the x-axis, and a vertical translation 6 units up of the graph of f.

9. 

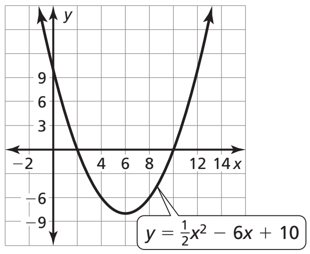
The graph of n is a vertical shrink by a factor of $\frac{1}{3}$ and a vertical translation 5 units down of the graph of f.

10. 

domain: all real numbers, range: $y \geq 6$

11. 

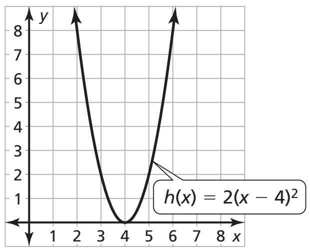
domain: all real numbers, range: $y \leq -\frac{13}{4}$

12. 

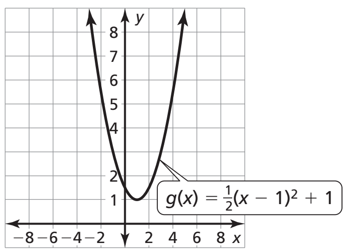
domain: all real numbers, range: $y \geq -8$

13. 2.75 sec; 133 ft **14.** neither **15.** odd

16. neither

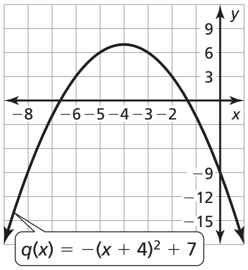
17. 

The graph of h is a vertical stretch by a factor of 2 and a horizontal translation 4 units right of the graph of f.

18. 

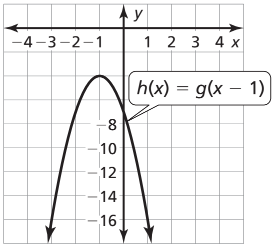
The graph of g is a vertical shrink by a factor of $\frac{1}{2}$, and a translation 1 unit right and 1 unit up of the graph of f.

19.



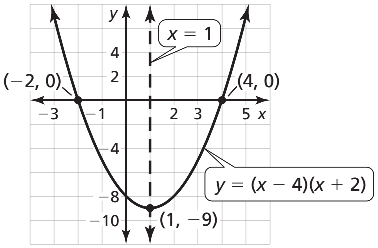
The graph of q is a reflection in the x-axis, and a translation 4 units left and 7 units up of the graph of f.

20.



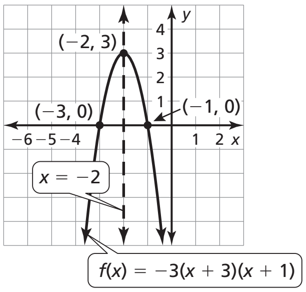
21. $f(x) = 5(x - 3)^2 + 2$

22.



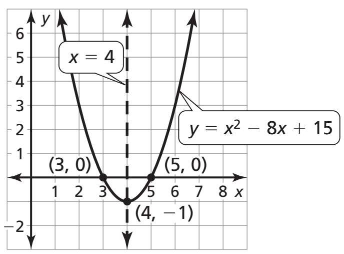
domain: all real numbers, range: $y \geq -9$

23.



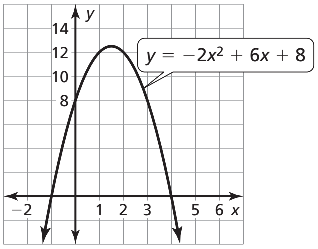
domain: all real numbers, range: $y \leq 3$

24.

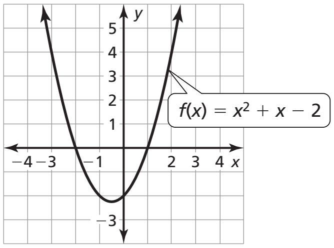


domain: all real numbers, range: $y \geq -1$

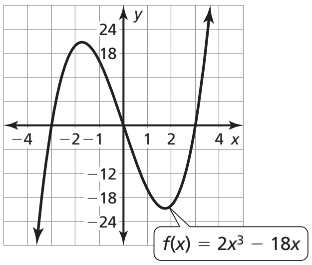
25.



26.



27.



28. *Sample answer:* $x^2 - 10x + 24$

29. exponential; $y = 128\left(\frac{1}{4}\right)^x$

30. **a.** From year 2 to year 7, your account increased at an average rate of about \$29.55 per year and your friend's account increased at an average rate of \$20 per year. So, your account is growing faster.

b. your account; The exponential growth of your account will continue to have a higher balance than your friend's account which has linear growth.

Chapter 9

Chapter 9 Maintaining Mathematical Proficiency *(p. 463)*

1. $(x + 5)^2$ **2.** $(x - 10)^2$ **3.** $(x + 6)^2$ **4.** $(x - 9)^2$

5. $(x + 8)^2$ **6.** $(x - 15)^2$ **7.** $(1, -2)$ **8.** $(4, 4)$

9. $(-2, 3)$ **10.** $\left(\frac{b}{2}\right)^2$

9.1 Vocabulary and Core Concept Check *(p. 471)*

1. rationalizing the denominator

3. yes; $\sqrt{\frac{2x}{9}} = \sqrt{\frac{1}{9} \cdot 2x}, = \sqrt{\frac{1}{9}} \cdot \sqrt{2x} = \frac{1}{3}\sqrt{2x}$

9.1 Monitoring Progress and Modeling with Mathematics *(pp. 471–474)*

5. yes

7. no; The radicand has a perfect square factor of 16.

9. no; A radical appears in the denominator of a fraction.

11. no; A radical appears in the denominator of a fraction.

13. $2\sqrt{5}$ **15.** $8\sqrt{2}$ **17.** $5\sqrt{5b}$ **19.** $-9m\sqrt{m}$

21. $\frac{2}{7}$ **23.** $-\frac{\sqrt{23}}{8}$ **25.** $\frac{a\sqrt{a}}{7}$ **27.** $\frac{5}{x}$ **29.** $2\sqrt[3]{2}$

31. $-4x\sqrt[3]{x^2}$ **33.** $-\frac{\sqrt[3]{6c}}{5}$ **35.** $-\frac{3\sqrt[3]{3y^2}}{10x}$

37. The radicand 18 has a perfect square factor of 9; $\sqrt{72} = \sqrt{36 \bullet 2} = 6\sqrt{2}$

39. $\frac{\sqrt{6}}{\sqrt{6}}$ **41.** $\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$ **43.** $\frac{\sqrt{5}+8}{\sqrt{5}+8}$ **45.** $\sqrt{2}$ **47.** $\frac{\sqrt{15}}{12}$

49. $\frac{3\sqrt{a}}{a}$ **51.** $\frac{d\sqrt{15}}{5}$ **53.** $\frac{4\sqrt[3]{5}}{5}$ **55.** $\frac{\sqrt{7}-1}{6}$

57. $\frac{7\sqrt{10}+2\sqrt{5}}{47}$ **59.** $\sqrt{5}+\sqrt{2}$

61. **a.** about 1.85 sec **b.** about 0.68 sec

63. about 5.42 amperes **65.** $5\sqrt{2}$, about 7.07

67. $\frac{\sqrt{2}}{3}$, about 0.47 **69.** $2\sqrt{2}$, about 2.83

71. $-6\sqrt{2}$, about -8.49 **73.** about 3.71 in.

75. $\sqrt{3}+4\sqrt{2}$ **77.** $-13\sqrt{6}$ **79.** $8\sqrt{3}+2\sqrt{6}$

81. $\sqrt[3]{3}$ **83.** $4\sqrt{10}$ **85.** $-2\sqrt{30x}$ **87.** 18

89. $3\sqrt[3]{12}$ **91.** about 114 ft **93.** $\frac{\sqrt[5]{8125}}{5x}$ **95.** $4\sqrt[4]{y}$

97. $9\sqrt[4]{9} - \sqrt[5]{9}$

99. **a.** $4, 2\frac{1}{4}, 2, 2+\sqrt{3}, 2-\sqrt{3}, 2+\pi$;

$2\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}+\sqrt{3}, \frac{1}{4}-\sqrt{3}, \frac{1}{4}+\pi$;

$2, \frac{1}{4}, 0, \sqrt{3}, -\sqrt{3}, \pi$;

$2+\sqrt{3}, \frac{1}{4}+\sqrt{3}, \sqrt{3}, 2\sqrt{3}, 0, \pi+\sqrt{3}$;

$2-\sqrt{3}, \frac{1}{4}-\sqrt{3}, -\sqrt{3}, 0, -2\sqrt{3}, \pi-\sqrt{3}$;

$2+\pi, \frac{1}{4}+\pi, \pi, \pi+\sqrt{3}, \pi-\sqrt{3}, 2\pi$

b. $4, \frac{1}{2}, 0, 2\sqrt{3}, -2\sqrt{3}, 2\pi$;

$\frac{1}{2}, \frac{1}{16}, 0, \frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{4}, \frac{\pi}{4}$;

$0, 0, 0, 0, 0, 0$;

$2\sqrt{3}, \frac{\sqrt{3}}{4}, 0, 3, -3, \pi\sqrt{3}$;

$-2\sqrt{3}, -\frac{\sqrt{3}}{4}, 0, -3, 3, -\pi\sqrt{3}$;

$2\pi, \frac{\pi}{4}, 0, \pi\sqrt{3}, -\pi\sqrt{3}, \pi^2$

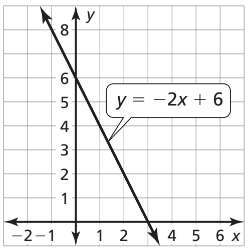
101. odd; even; When m is even, 2^m is a perfect square.

103. $a^2 < ab < b^2$ when $a < b$. **105.** 377

107. $\frac{2\sqrt[3]{x^2} - 2\sqrt[3]{x} + 2}{x+1}$; Multiplying the numerator and denominator by $\sqrt[3]{x^2} - \sqrt[3]{x} + 1$ rationalizes the denominator.

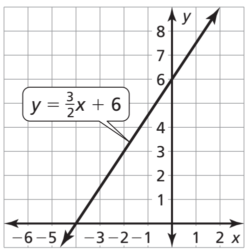
9.1 Maintaining Mathematical Proficiency *(p. 474)*

109.



3

111.



-4

113. $x = -2$ **115.** $x = \frac{1}{2}$

9.2 Vocabulary and Core Concept Check *(p. 481)*

1. an equation that can be written in the standard form $ax^2 + bx + c = 0$, where $a \neq 0$

3. The number of x-intercepts is the number of solutions.

9.2 Monitoring Progress and Modeling with Mathematics *(pp. 481–484)*

5. $x = 3, x = -1$ **7.** $x = -4$

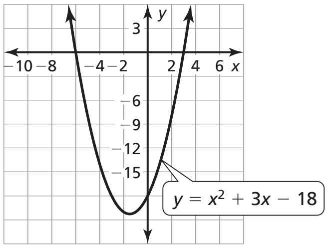
9. $4x^2 - 12 = 0$ or $-4x^2 + 12 = 0$

11. $x^2 - 2x + 1 = 0$ or $-x^2 + 2x - 1 = 0$

13. $x = 0, x = 5$ **15.** no solution **17.** $x = 3$

19. $x = -1$ **21.** $x = -6, x = 2$ **23.** $x = -2, x = 1$

25. The equation needs to be in standard form; $x^2 + 3x - 18 = 0$



The solutions are $x = -6$ and $x = 3$.

27. **a.** The x-intercepts are the distances at which the height is 0 yards.

b. 5 yd

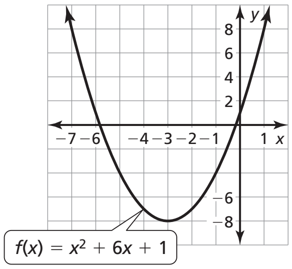
29. $x = -5, x = 2$ **31.** no solution

33. $x = -10, x = -2$ **35.** no solution **37.** $2, 0, -1$

39. $-3, 1$ **41.** $-2, 2, -3, 1$ **43.** about 4.3, about 0.7

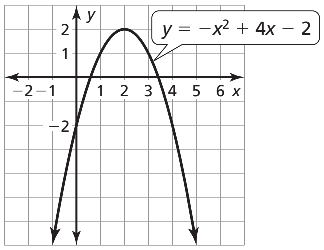
45. about 2.4, about -0.4

47.



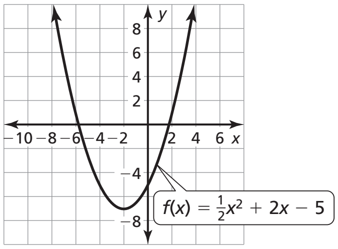
about -0.2, about -5.8

49.



about 3.4, about 0.6

Selected Answers

51. 

about 1.7, about −5.7

53. **a.** 118 ft, 198 ft, 246 ft, 262 ft, 246 ft, 198 ft, 118 ft, 6 ft
b. about 1.5 sec, about 6.5 sec
c. about 1.4 sec, about 6.6 sec

55. **a.** $y = 0.04x^2 - 4.1x + 197$; Yes, the graph of the equation passes through or is close to all of the data points.
b. about 24.8 min
c. no; The quadratic model suggests that the temperature will increase.

57. **a.** 6; 10; 15; Check students' work.
b. $y = 0.5x^2 - 0.5x$; Yes, the graph of the equation passes through all of the data points.
c. 36
d. 12; Set the equation from part (b) equal to 66. Write the equation in standard form and graph the equation. Find the zeros for $x > 0$.

59. about 3.7 ft

61. Graph the function to determine which integers the solutions are between. Then make tables using x-values between the integers with an interval of 0.1. Look for a change of sign in the function values, then select the value closest to zero.

63. *Sample answer:* Method 1; Only one graph needs to be drawn.

65. about 24.1 ft

67. sometimes; $y = -2x^2 + 1$ has two x-intercepts, but $y = -2x^2 + (-1)$ has no x-intercepts.

69. never; The graph of $y = ax^2 + bx + c$ has at most two x-intercepts.

71. Graph the system of equations $y = -0.43x^2 + 11.7x + 2$ and $y = 77$. Find the points of intersection, which are about (10.3, 77) and about (16.9, 77).

9.2 Maintaining Mathematical Proficiency *(p. 484)*

73. exponential growth function; As x increases by 1, y is multiplied by 4.

75. $\dfrac{\sqrt{15x}}{3x}$ **77.** $-3\sqrt{3} - 6$

9.3 Vocabulary and Core Concept Check *(p. 489)*

1. two

9.3 Monitoring Progress and Modeling with Mathematics *(pp. 489–490)*

3. 2; $x = 5, x = -5$ **5.** 0; no real solutions
7. 1; $x = 0$ **9.** $x = 4, x = -4$ **11.** no real solutions
13. $x = 7, x = -7$ **15.** $x = 0$ **17.** $x = \frac{1}{2}, x = -\frac{1}{2}$
19. $x = -3$ **21.** $x = 5, x = -4$ **23.** $x = \frac{1}{3}, x = -\frac{7}{3}$
25. $x \approx 2.65, x \approx -2.65$ **27.** $x \approx 3.16, x \approx -3.16$
29. $x \approx 4.24, x \approx -4.24$
31. The number 36 has both a positive and negative square root; $x = \pm 6$
33. about 1.2 sec **35.** 3 ft
37. *Sample answer:* Use a calculator.
39. (3, 9), (−3, 9); When $y = 9$, $x = \pm 3$.
41. $x = 1.2, x = -1.2$; $1.2^2 = 1.44$ **43.** $x = 6, x = -2$

9.3 Maintaining Mathematical Proficiency *(p. 490)*

45. $(x + 4)^2$ **47.** $(x - 7)^2$ **49.** $(x + 6)^2$

9.4 Vocabulary and Core Concept Check *(p. 499)*

1. completing the square

3. even; When b is even, $\dfrac{b}{2}$ is an integer.

9.4 Monitoring Progress and Modeling with Mathematics *(pp. 499–502)*

5. 16 **7.** 4 **9.** $\frac{225}{4}$ **11.** $x^2 - 10x + 25$; $(x - 5)^2$
13. $x^2 + 16x + 64$; $(x + 8)^2$ **15.** $x^2 + 5x + \frac{25}{4}$; $\left(x + \frac{5}{2}\right)^2$
17. $x = 1, x = -15$ **19.** $x \approx 3.41, x \approx 0.59$
21. $x \approx 6.27, x \approx -1.27$
23. **a.** $x^2 + 6x = 216$ **b.** width: 12 ft, length: 18 ft
25. $x = 5, x = 3$ **27.** $x \approx -3.27, x \approx -6.73$
29. $x \approx 1.92, x \approx -9.92$ **31.** $x = 8, x = -1$
33. The number 16 should be added to each side of the equation; $x^2 + 8x + 16 = 10 + 16$; $x = -4 \pm \sqrt{26}$
35. $b = 10, b = -10$; In a perfect square trinomial $c = \left(\dfrac{b}{2}\right)^2$, so $b = \pm 2\sqrt{c}$.
37. $y = (x + 3)^2 - 6$; D **39.** $y = -(x + 2)^2 + 2$; B
41. minimum value; −6 **43.** maximum value; −5
45. maximum value; −6
47. yes; The graph has two negative x-intercepts and it opens down.
49. no; The x-intercepts are both positive.
51. f, m; The graph has two negative x-intercepts and it opens up.
53. **a.** 36 ft
b. $x = \frac{3}{2}$; On the left side of $x = \frac{3}{2}$, the height increases as time increases. On the right side of $x = \frac{3}{2}$, the height decreases as time increases.
55. 3 ft **57.** 12 **59.** $x \approx 1.24, x \approx -3.24$
61. $x \approx 4.29, x \approx -0.29$ **63.** 40 mi/h
65. **a.** $\ell + 2w = 80$, $\ell w = 750$
b. length: 30 ft, width: 25 ft; length: 50 ft, width: 15 ft
67. **a.** $x \approx 0.8, x \approx -12.8$ **b.** $x \approx 0.78, x \approx -12.78$
c. *Sample answer:* completing the square; The result is more accurate.
69. $x(x + 2) = 48$, 6 and 8
71. yes; Substituting 23.50 for y in the model, the stock is worth \$23.50 ten days and thirty days after the stock is purchased.
73. length: 66 in., width: 6 in.

9.4 Maintaining Mathematical Proficiency *(p. 502)*

75. $a_1 = 10, a_n = a_{n+1} + 5$ **77.** $a_1 = -20, a_n = a_{n-1} + 4$
79. $6\sqrt{2}$

9.5 Vocabulary and Core Concept Check *(p. 509)*

1. the quadratic formula; $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

9.5 Monitoring Progress and Modeling with Mathematics *(pp. 509–512)*

3. $x^2 - 7x = 0$; $a = 1$, $b = -7$, $c = 0$ or $-x^2 + 7x = 0$; $a = -1$, $b = 7$, $c = 0$

5. $-2x^2 - 5x + 1 = 0$; $a = -2$, $b = -5$, $c = 1$ or $2x^2 + 5x - 1 = 0$; $a = 2$, $b = 5$, $c = -1$

7. $x^2 - 6x + 4 = 0$; $a = 1$, $b = -6$, $c = 4$ or $-x^2 + 6x - 4 = 0$; $a = -1$, $b = 6$, $c = -4$

9. $x = 6$ **11.** $x = 11$, $x = -1$ **13.** no real solutions

15. $x = \frac{3}{2}$, $x = \frac{2}{3}$ **17.** $x = \frac{1}{4}$ **19.** $x \approx 2.2$, $x \approx -4.2$

21. $x = -\frac{1}{2}$, $x = -4$ **23.** about 0.2 sec, about 1.4 sec

25. no real solutions **27.** one real solution

29. two real solutions **31.** two x-intercepts

33. no x-intercepts **35.** two x-intercepts

37. $x = \frac{1}{2}$, $x = \frac{4}{5}$; *Sample answer:* The equation is not easily factorable and $a \neq 1$, so solve using the quadratic formula.

39. $x \approx 0.74$, $x \approx -6.74$; *Sample answer:* $a = 1$ and b is even, so solve by completing the square.

41. $x = -4$, $x = 3$; *Sample answer:* The equation is easily factorable, so solve by factoring.

43. $x \approx 2.19$, $x \approx -1.94$; *Sample answer:* The equation cannot be factored and $a \neq 1$, so solve using the quadratic formula.

45. $-b$ should be $-(-7)$, not -7;
$x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-6)}}{2(3)}$; $x = 3$ and $x = -\dfrac{2}{3}$

47. yes; about 42 ft, about 158 ft

49. no; The discriminant is -47, so the equation has no real solutions.

51. 5; length: 13 m, width: 7 m

53. **a–c.** $x = -2$
Sample answer: factoring; The equation is easily factorable.

55. 2; When a and c have different signs, ac is negative, so the discriminant is positive.

57. **a.** *Sample answer:* $\frac{1}{2}$ **b.** 1 **c.** *Sample answer:* 2

59. **a.** *Sample answer:* 1 **b.** 9 **c.** *Sample answer:* 10

61. about 31 mi/h, about 65 mi/h

63. below the x-axis; The discriminant is positive and $a > 0$.

65. above the x-axis; The discriminant is negative and $a > 0$.

67. above the x-axis; The discriminant is positive and $a < 0$.

69. about 22 sec; The height is 30,800 feet after about 20.8 seconds and after about 42.8 seconds.

71. **a.** $4x + 3y = 1050$; $3y = 1050 - 4x$; $y = 350 - \frac{4}{3}x$
b. length: about 54 ft, width: about 278 ft; length: about 209 ft, width: about 72 ft

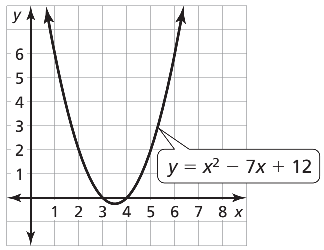
73. $-\dfrac{b}{2a}$; The mean of the solutions is the x-coordinate of the vertex; The mean of the solutions is equal to the graph's axis of symmetry, which is where the vertex lies.

75. about 24.7 ft/sec

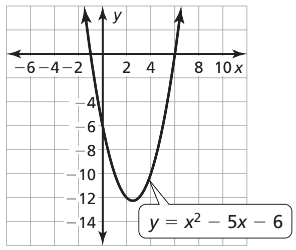
77. $\dfrac{-b}{a}$; $\dfrac{c}{a}$; *Sample answer:* $2x^2 - 4x + 1 = 0$

79. **a.** $(x + 1)$, $(x - 6)$; $x^2 - 5x - 6 = 0$;
x, $(x - 2)$; $x^2 - 2x = 0$;
$\left(x + \frac{1}{2}\right)$, $(x - 5)$; $x^2 - \frac{9}{2}x - \frac{5}{2} = 0$

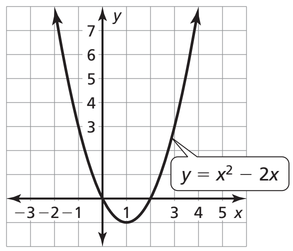
b.



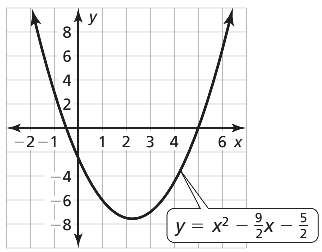
3, 4



-1, 6



0, 2



$-\frac{1}{2}$, 5

81. **a.** $k < -3$ or $k > 3$ **b.** $k = -3$ or $k = 3$
c. $-3 < k < 3$

9.5 Maintaining Mathematical Proficiency *(p. 512)*

83. (4, 0); *Sample answer:* substitution because both equations are solved for y

85. (5, 3); *Sample answer:* elimination because one pair of like terms has the same coefficient

Selected Answers

Chapter 9 Review *(pp. 514–516)*

1. $2\sqrt{31}$ **2.** $6p^3\sqrt{2p}$ **3.** $\dfrac{3\sqrt{35y}}{7y}$ **4.** $\dfrac{5x^3\sqrt[3]{2x^2}}{2}$

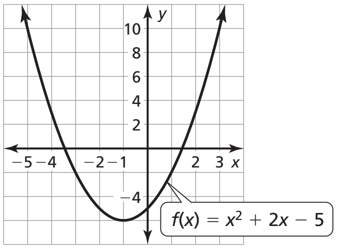
5. $\dfrac{\sqrt{6}}{2}$ **6.** $\dfrac{2\sqrt{6x}}{3x}$ **7.** $4\sqrt{6} - 8$ **8.** $14\sqrt{3}$

9. $2\sqrt{7} - \sqrt{5}$ **10.** $9\sqrt[3]{2}$ **11.** $88 + 30\sqrt{7}$

12. $10\sqrt{3}$ **13.** $x = 6, x = 3$ **14.** no solutions

15. $x = -4$ **16.** $-3, -1, 1$

17.



$-3.4, 1.4$

18. $x \approx 3.46, x \approx -3.46$ **19.** $x = 0$

20. $x = 6, x = -10$ **21.** no solutions **22.** $x = 1$

23. $x \approx 1.48, x \approx -1.48$ **24.** $x = 4, x = -10$

25. $x = -1$ **26.** $x \approx 3.45, x \approx -1.45$

27. maximum value; 8 **28.** minimum value; 7

29. minimum value; -33 **30.** 28 cm

31. $x = 3, x = -5$ **32.** $x \approx 2.3, x \approx -1.8$ **33.** $x = 1$

34. one x-intercept **35.** no x-intercepts

36. two x-intercepts