2023 AMC 8 Solutions

Problem1

What is the value of $(8 \times 4 + 2) - (8 + 4 \times 2)$?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 10 \qquad \textbf{(D)}\ 18 \qquad \textbf{(E)}\ 24$

Solution 1

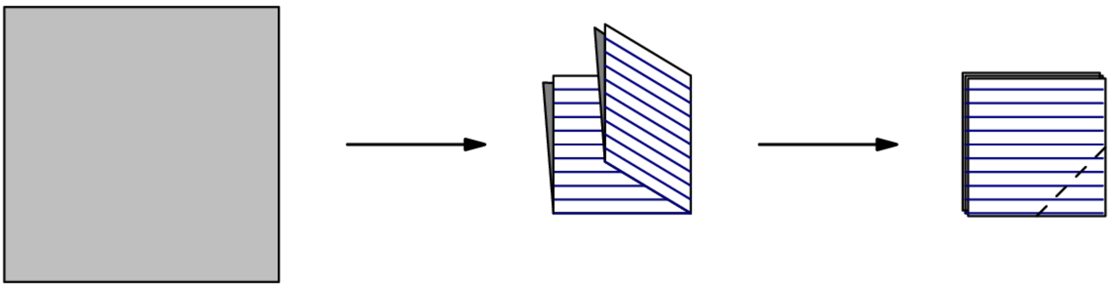
By the order of operations, we have
$(8\times4+2)-(8+4\times2) = (32+2)-(8+8) = 34-16 = \boxed{\textbf{(D)}\ 18}$.

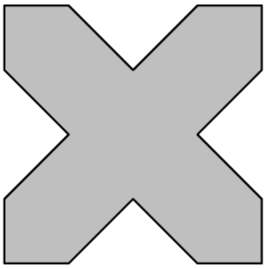
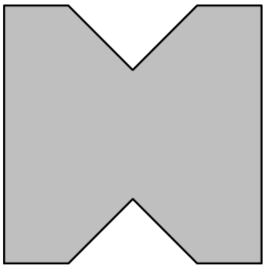
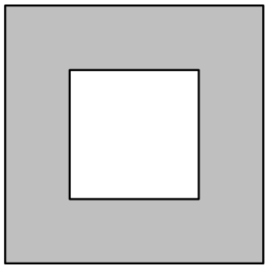
Solution 2

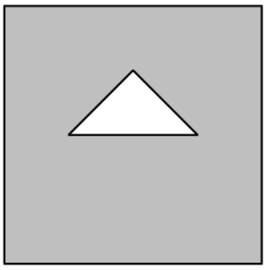
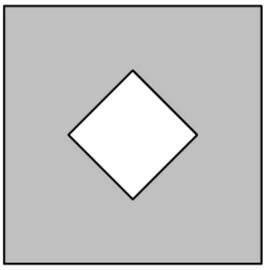
We can simplify the expression above in another way:
$(8\times4+2)-(8+4\times2) = 8\times4+2-8-4\times2 = 32+2-8-8 = 34-16 = \boxed{\textbf{(D)}\ 18}$.

Problem2

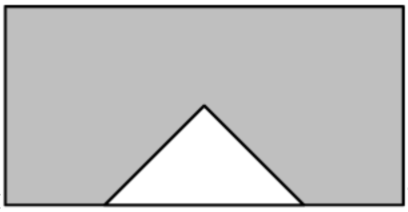
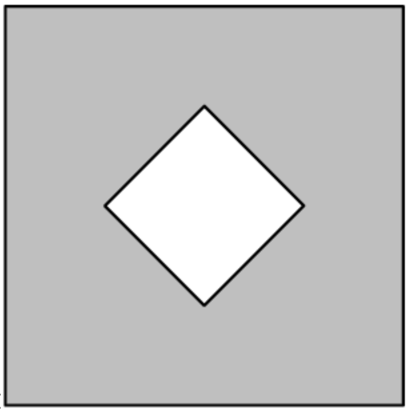
A square piece of paper is folded twice into four equal quarters, as shown below, then cut along the dashed line. When unfolded, the paper will match which of the following figures?



(A)  (B)  (C) 

(D)  (E) 

Solution

Notice that when we unfold the paper along the vertical fold line, we get the following shape:  Then, if we unfold the paper along the horizontal fold line, we get the following shape: 

It is clear that the answer is $\boxed{\textbf{(E)}}$.

Problem3

Wind chill is a measure of how cold people feel when exposed to wind outside. A good estimate for wind chill can be found using this
calculation

$$(\text{wind chill}) = (\text{air temperature}) - 0.7 \times (\text{wind speed}),$$

where temperature is measured in degrees Fahrenheit $(^\circ\text{F})$ and the wind speed is measured in miles per hour (mph). Suppose the air temperature is 36°F and the wind speed is 18 mph. Which of the following is closest to the approximate wind chill?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 23 \qquad \textbf{(C)}\ 28 \qquad \textbf{(D)}\ 32 \qquad \textbf{(E)}\ 35$

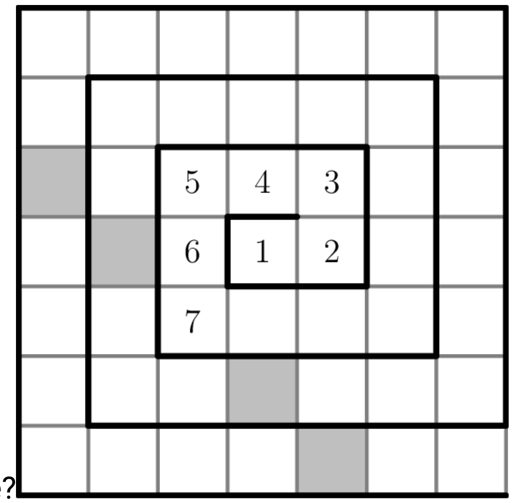
Solution

By substitution, we have

$$\begin{aligned}(\text{wind chill}) &= 36 - 0.7 \times 18 \\ &= 36 - 12.6 \\ &= 23.4 \\ &\approx \boxed{\textbf{(B)}\ 23}.\end{aligned}$$

Problem4

The numbers from 1 to 49 are arranged in a spiral pattern on a square grid, beginning at the center. The first few numbers have been entered into the grid below. Consider the four numbers that will appear in the shaded squares, on the same diagonal as the number $7.$ How many of these four numbers are prime?



$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 4$

Solution 1

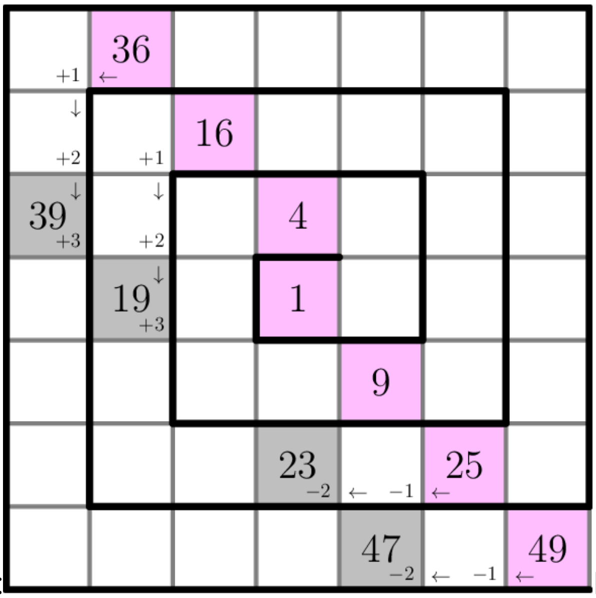
We fill out the grid, as shown below:

37	36	35	34	33	32	31
38	17	16	15	14	13	30
39	18	5	4	3	12	29
40	19	6	1	2	11	28
41	20	7	8	9	10	27
42	21	22	23	24	25	26
43	44	45	46	47	48	49

From the four numbers that appear in the shaded squares, $\boxed{\textbf{(D) } 3}$ of them are prime: $19, 23,$ and $47.$

Solution 2

Note that given time constraint, it's better to only count from perfect squares (in pink), as shown below:



From the four numbers that appear in the shaded squares, $\boxed{\textbf{(D) } 3}$ of them are prime: $19, 23,$ and $47.$

Problem5

A lake contains 250 trout, along with a variety of other fish. When a marine biologist catches and releases a sample of 180 fish from the lake, 30 are identified as trout. Assume that the ratio of trout to the total number of fish is the same in both the sample and the lake. How many fish are there in the lake?

$\textbf{(A)}\ 1250 \qquad \textbf{(B)}\ 1500 \qquad \textbf{(C)}\ 1750 \qquad \textbf{(D)}\ 1800 \qquad \textbf{(E)}\ 2000$

Solution

Note that $\frac{\text{number of trout}}{\text{total number of fish}} = \frac{30}{180} = \frac{1}{6}.$ So, the total number of fish is 6 times the number of trout. Since the lake contains 250 trout, there are $250 \cdot 6 = \boxed{\textbf{(B)}\ 1500}$ fish in the lake.

Problem6

The digits $2, 0, 2,$ and 3 are placed in the expression below, one digit per box. What is the maximum possible value of the expression?

$$\square^{\square} \times \square^{\square}$$

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 9 \qquad \textbf{(D)}\ 16 \qquad \textbf{(E)}\ 18$

Solution 1

First, let us consider the case where 0 is a base: This would result in the entire expression being 0. Contrastingly, if 0 is an exponent, we will get a value greater than 0. As $3^2 \times 2^0 = 9$ is greater than $2^3 \times 2^0 = 8$ and $2^2 \times 3^0 = 4,$ the answer is $\boxed{\textbf{(C)}\ 9}$.

Solution 2

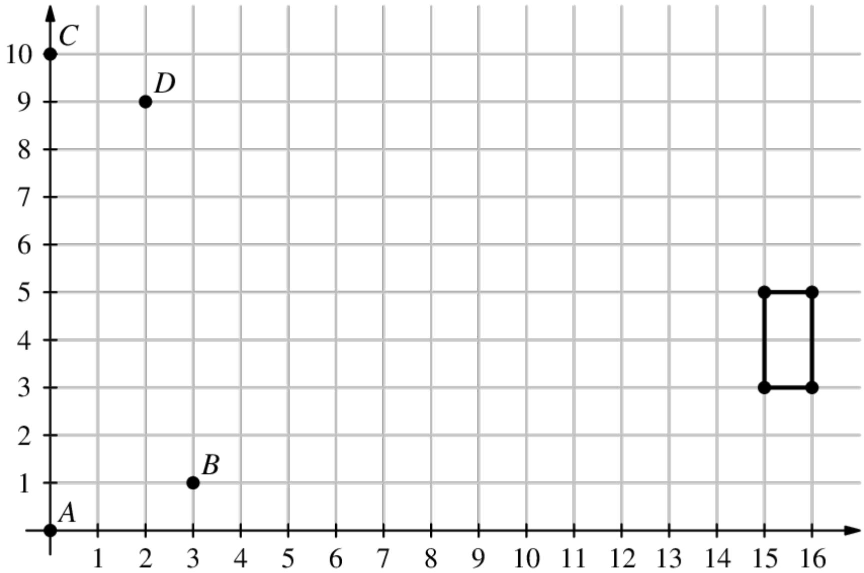
The maximum possible value of using the digits $2, 0, 2,$ and 3: We can maximize our value by keeping the 3 and 2 together in one power (the biggest with the biggest and the smallest with the smallest). This shows $3^2 \times 2^0 = 9 \times 1 = 9.$ (We don't want 0^2 because that is 0.) It is going to be $\boxed{\textbf{(C) } 9}$.

Solution 3

Trying all 12 distinct orderings, we see that the only possible values are $0, 4, 8,$ and $9,$ the greatest of which is $\boxed{\textbf{(C) } 9}$.

Problem7

A rectangle, with sides parallel to the x-axis and y-axis, has opposite vertices located at $(15, 3)$ and $(16, 5)$. A line drawn through points $A(0, 0)$ and $B(3, 1)$. Another line is drawn through points $C(0, 10)$ and $D(2, 9)$. How many points on the rectangle lie on at least one of the two

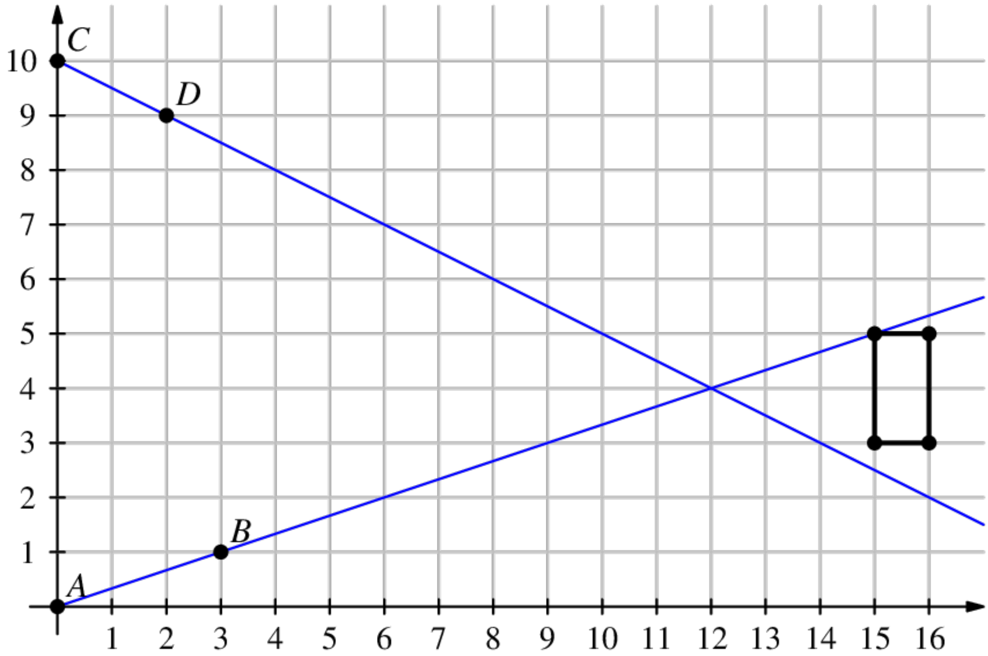


lines?

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 3 \qquad \textbf{(E) } 4$

Solution 1

If we extend the lines, we have the following diagram:



Hence, we see that the answer is $\boxed{\textbf{(B) } 1}$.

Solution 2

Note that the y-intercepts of line AB and line CD are 0 and 10. If the analytic expression for line AB is $y = k_1x$, and the analytic expression for line CD is $y = k_2x + 10$, we have equations: $3k_1 = 1$ and $2k_2 + 10 = 9$. Solving these equations, we can find out that $k_1 = \frac{1}{3}$ and $k_2 = -\frac{1}{2}$. Therefore, we can determine that the expression for line AB is $y = \frac{1}{3}x$ and the expression for line CD is $y = -\frac{1}{2}x + 10$. When $x = 15$, the coordinates that line AB and line CD pass through are $(15, 5)$ and $\left(15, \frac{5}{2}\right)$,

and $(15, 5)$ lies perfectly on one vertex of the rectangle while the y coordinate of $\left(15, \frac{5}{2}\right)$ is out of the range $3 \leq y \leq 5$ (lower than the bottom left corner of the rectangle $(15, 3)$). Considering that the y value of the line CD will only decrease, and the y value of the line AB will only increase, there will not be another point on the rectangle that lies on either of the two lines.

Thus, we can conclude that the answer is $\boxed{\textbf{(B) } 1}$.

Problem8

Lola, Lolo, Tiya, and Tiyo participated in a ping pong tournament. Each player competed against each of the other three players exactly twice. Shown below are the win-loss records for the players. The numbers 1 and 0 represent a win or loss, respectively. For example, Lola won five matches and lost the fourth match. What was Tiyo's win-loss record?

Player	Result
Lola	111011
Lolo	101010
Tiya	010100
Tiyo	??????

$\textbf{(A)}$ 000101 $\qquad$ $\textbf{(B)}$ 001001 $\qquad$ $\textbf{(C)}$ 010000 $\qquad$ $\textbf{(D)}$ 010101 $\qquad$ $\textbf{(E)}$ 011000

Solution 1

We can calculate the total number of wins $(1$'s) by seeing how many matches were players, which is 12 matches played. Then, we can calculate the # of wins already on the table, which is $5+3+2=10$, so there are $12-10=2$ wins left in the mystery player. Now, we will make the key observation that there is only 2 wins $(1$'s) per column as there are 2 winners and 2 losers in each round. Strategically looking through the columns counting the 1's and putting our own 2 1's when the column isn't already full yields $\boxed{\textbf{(A) }000101}$.

Solution 2 (Similar to #1 but More Detailed)

In total, there will be $\binom{4}{2} \cdot 2 = 6 \cdot 2 = 12$ games because there are $\binom{4}{2}$ ways to choose a pair of people from the four players. And, each player will play each other player exactly twice. Each of these 12 games will have 1 winner and 1 loser, so there will be a total of 12 1's and 12 0's in the win-loss table. Therefore, Tiyo will have $12-10=2$ 1's and $12-8=4$ 0's in his record.

Now, all we have to do is figure out the order of these 1's and 0's. In every round, there are two games; the players are split into two pairs and the people in those pairs play each other. Thus, every round should have 2 winners and 2 losers which means that every column of the win-loss table should have 2 1's and 2 0's. Looking at the filled-in table so far, we see that columns 4 and 6 need one more 1, so Tiyo must have 1's in those columns and 0's gone in the others.

Therefore, our answer is $\boxed{\textbf{(A) }000101}$.

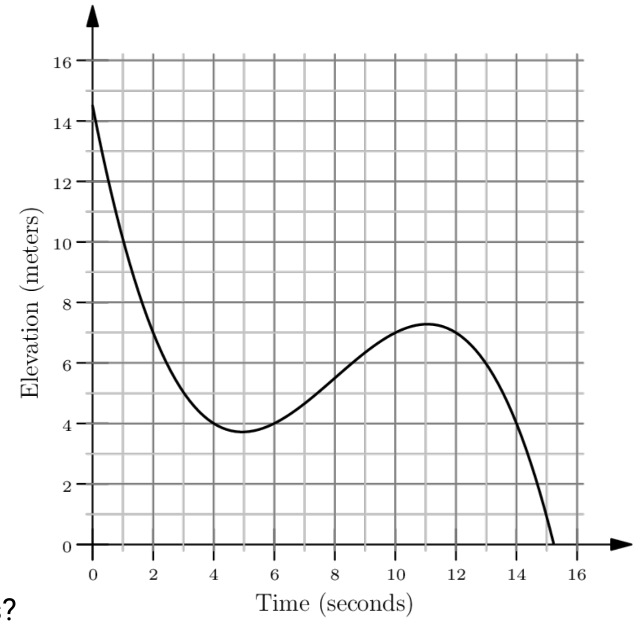
Solution 3 (Faster)

We can look one by one. We see that Lola and Lolo won the first game and Tiya lost. This shows that Tiyo must have lost as well because the results must be 2 wins and 2 loses. We use the same logic for games 2 and 3, giving us 0's again. We look at the choices, and we see A is the only one that has 3 0's

This shows our answer is $\boxed{\textbf{(A) }000101}$.

Problem9

Malaika is skiing on a mountain. The graph below shows her elevation, in meters, above the base of the mountain as she skis along a trail. In total, how many seconds does she spend at an elevation

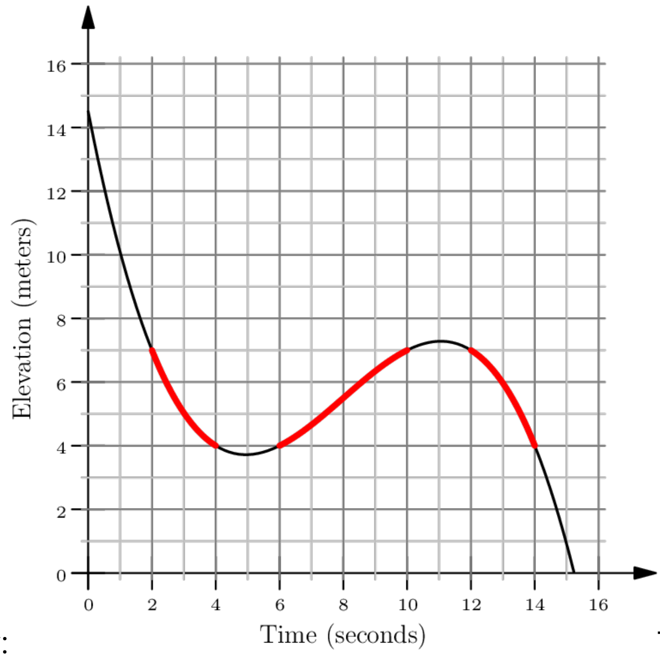


between 4 and 7 meters?

$\textbf{(A)}\ 6 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 10 \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 14$

Solution 1

We mark the time intervals in which Malaika's elevation is between 4 and 7 meters in red, as shown



below: The requested time intervals are:

- from the 2nd to the 4th seconds
- from the 6th to the 10th seconds
- from the 12th to the 14th seconds

In total, Malaika spends $(4-2)+(10-6)+(14-12)=\boxed{\textbf{(B)}\ 8}$ seconds at such elevation.

Solution 2

Notice that the entire section between the 2 second mark and the 14 second mark is between the 4 and 7 feet elevation level except the 2seconds where she skis just under the 4 feet mark and when she skis just above the 7 feet mark, making the answer $14-2-2-2=\boxed{\textbf{(B)}\ 8}$.

Problem10

Harold made a plum pie to take on a picnic. He was able to eat only $\frac{1}{4}$ of the pie, and he left the rest for his friends. A moose came by and ate $\frac{1}{3}$of what Harold left behind. After that, a porcupine ate $\frac{1}{3}$ of what the moose left behind. How much of the original pie still remained after the porcupine left?

$$\textbf{(A)}\ \frac{1}{12} \qquad \textbf{(B)}\ \frac{1}{6} \qquad \textbf{(C)}\ \frac{1}{4} \qquad \textbf{(D)}\ \frac{1}{3} \qquad \textbf{(E)}\ \frac{5}{12}$$

Solution

Note that:

- Harold ate $\frac{1}{4}$ of the pie. After that, $1-\frac{1}{4}=\frac{3}{4}$ of the pie was left behind.
- The moose ate $\frac{1}{3}\cdot\frac{3}{4}=\frac{1}{4}$ of the pie. After that, $\frac{3}{4}-\frac{1}{4}=\frac{1}{2}$ of the pie was left behind.
- The porcupine ate $\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}$ of the pie. After that, $\frac{1}{2}-\frac{1}{6}=\boxed{\textbf{(D)}\ \frac{1}{3}}$ of the pie was left behind.

Alternatively, we can condense the solution above into the following equation: $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{3}\right)=\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{2}{3}=\frac{1}{3}.$

Problem11

NASA's Perseverance Rover was launched on July $30, 2020.$ After traveling $292{,}526{,}838$ miles, it landed on Mars in Jezero Crater about 6.5 months later. Which of the following is closest to the Rover's average interplanetary speed in miles per hour?

$\textbf{(A)}\ 6{,}000 \qquad \textbf{(B)}\ 12{,}000 \qquad \textbf{(C)}\ 60{,}000 \qquad \textbf{(D)}\ 120{,}000 \qquad \textbf{(E)}\ 600{,}000$

Solution 1

Note that 6.5 months is approximately $6.5\cdot30\cdot24$ hours. Therefore, the speed (in miles per hour) is

$$\frac{292{,}526{,}838}{6.5\cdot30\cdot24}\approx\frac{300{,}000{,}000}{6.5\cdot30\cdot24}=\frac{10{,}000{,}000}{6.5\cdot24}\approx\frac{10{,}000{,}000}{6.4\cdot25}=\frac{10{,}000{,}000}{160}=62500\approx\boxed{\textbf{(C)}\ 60{,}000}.$$

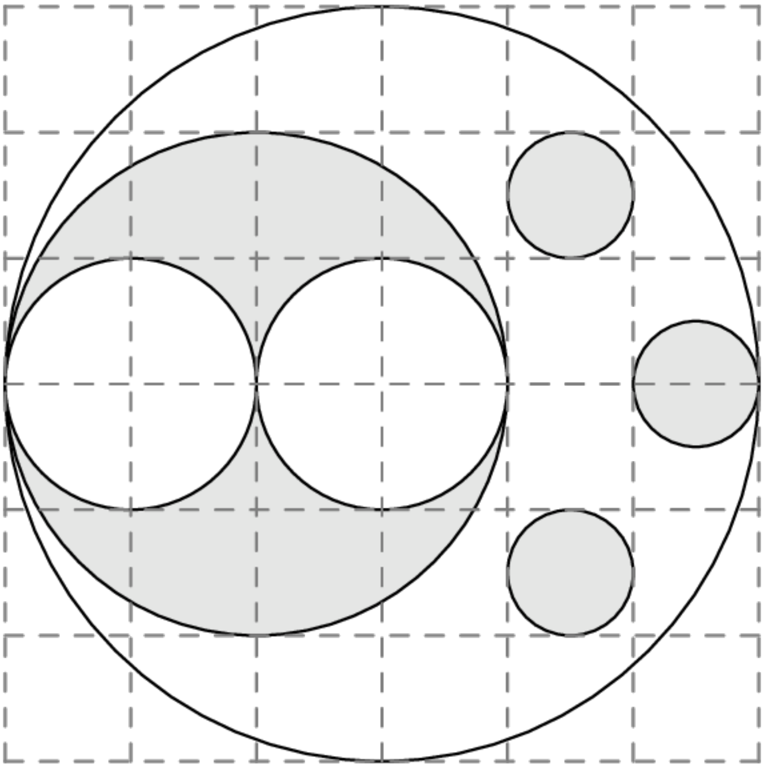
As the answer choices are far apart from each other, we can ensure that the approximation is correct.

Solution 2

Note that $292{,}526{,}838 \approx 300{,}000{,}000$ miles. We also know that 6.5 months is approximately $6.5 \cdot 30 \cdot 24$ hours. Now, we can calculate the speed in miles per hour, which we find is about $\frac{300{,}000{,}000}{6.5 \cdot 30 \cdot 24} = \frac{10{,}000{,}000}{6.5 \cdot 24} = \frac{10{,}000{,}000}{13 \cdot 12} = \frac{10{,}000{,}000}{156} \approx \frac{10{,}000{,}000}{150} \approx \frac{200{,}000}{3} \approx \boxed{\textbf{(C)}\ 60{,}000}$.

Problem12

The figure below shows a large white circle with a number of smaller white and shaded circles in its interior. What fraction of the interior of the large white circle is shaded?



$\textbf{(A)}\ \frac{1}{4} \qquad \textbf{(B)}\ \frac{11}{36} \qquad \textbf{(C)}\ \frac{1}{3} \qquad \textbf{(D)}\ \frac{19}{36} \qquad \textbf{(E)}\ \frac{5}{9}$

Solution 1

First, the total area of the radius 3 circle is simply just $9 * \pi$ when using our area of a circle formula.

Now from here, we have to find our shaded area. This can be done by adding the areas of the 3 $\frac{1}{2}$-radius circles and add; then, take the area of the 2 radius circle and subtract that from the area of the 2 radius 1 circles to get our resulting complex area shape. Adding these up, we will get $3 * \frac{1}{4}\pi + 4\pi - \pi - \pi = \frac{3}{4}\pi + 2\pi = \frac{11}{4}$.

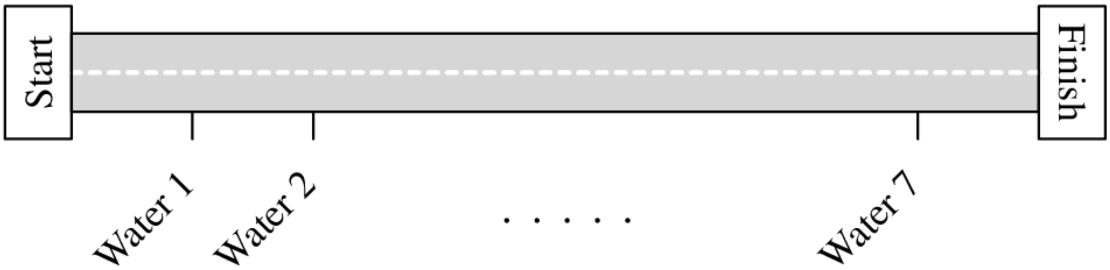
So, our answer is $\frac{\frac{11}{4}\pi}{9\pi} = \boxed{\textbf{(B) } \frac{11}{36}}$.

Solution 2

Pretend each circle is a square. The second largest circle is a square with area 16 units^2 and there are two squares in that square that each has areas of 4 units^2 which add up to 8 units^2. Subtracting the medium-sized squares' areas from the second-largest square's area, we have 8 units^2. The largest circle becomes a square that has area 36 units^2, and the three smallest circles become three squares with area 8 units^2 and add up to 3 units^2. Adding the areas of the shaded regions, we get 11 units^2, so our answer is $\boxed{\textbf{(B) } \frac{11}{36}}$.

Problem13

Along the route of a bicycle race, 7 water stations are evenly spaced between the start and finish lines, as shown in the figure below. There are also 2 repair stations evenly spaced between the start and finish lines. The 3rd water station is located 2 miles after the 1st repair station. How long is the race in miles?



$\textbf{(A) } 8 \qquad \textbf{(B) } 16 \qquad \textbf{(C) } 24 \qquad \textbf{(D) } 48 \qquad \textbf{(E) } 96$

Solution

Suppose that the race is d miles long. The water stations are located at $\frac{d}{8}, \frac{2d}{8}, \ldots, \frac{7d}{8}$ miles from the start, and the repair stations are located at $\frac{d}{3}, \frac{2d}{3}$ miles from the start.

We are given that $\frac{3d}{8} = \frac{d}{3} + 2,$ from which $$\begin{aligned} \frac{9d}{24} &= \frac{8d}{24} + 2 \\ \frac{d}{24} &= 2 \\ d &= \boxed{\textbf{(D) } 48}. \end{aligned}$$

Problem14

Nicolas is planning to send a package to his friend Anton, who is a stamp collector. To pay for the postage, Nicolas would like to cover the package with a large number of stamps. Suppose he has a collection of 5-cent, 10-cent, and 25-cent stamps, with exactly 20 of each type. What is the greatest number of stamps Nicolas can use to make exactly $\$7.10$ in postage? (Note: The amount $\$7.10$ corresponds to 7 dollars and 10 cents. One dollar is worth 100 cents.)

$\textbf{(A) } 45 \qquad \textbf{(B) } 46 \qquad \textbf{(C) } 51 \qquad \textbf{(D) } 54 \qquad \textbf{(E) } 55$

Solution 1

Let's use the most stamps to make $7.10.$ We have 20 of each stamp, 5-cent (like nickels), 10-cent (like dimes), and 25-cent (like quarters).

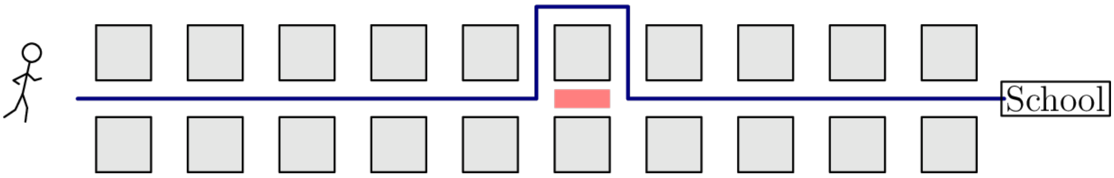
If we want to have the highest number of stamps, we have to have the highest number of the smaller value stamps (like the coins above). We can use 20 nickels and 20 dimes to bring our total cost to $7.10-3.00=4.10$. However, when we try to use quarters, the 25 cents don't fit evenly, so we have to give back 15 cents in order to make the quarter amount 4.25. The most efficient way to do this is to give back a 10-cent (dime) stamp and a 5-cent (nickel) stamp to have 38 stamps (coins) used so far. Now, we just use $\dfrac{425}{25}=17$ quarters to get a grand total of $38+17=\boxed{\textbf{(E) } 55}$.

Solution 2

The value of his entire stamp collect is 8 dollars. To make $\$7.10$ with stamps, he should remove 90 cents worth of stamps with as few stamps as possible. To do this, he should start by removing as many 25 cent stamps as possible as they have the greatest denomination. He can remove at most 3 of these stamps. He still has to remove $90-25\cdot 3=15$ cents worth of stamps. This can be done with one 5 and 10 cent stamp. In total, he has $20\cdot 3=60$ stamps in his entire collect. As a result, the maximum number of stamps he can use is $20\cdot 3-5=\boxed{\textbf{(E) } 55}$.

Problem15

Viswam walks half a mile to get to school each day. His route consists of 10 city blocks of equal length and he takes 1 minute to walk each block. Today, after walking 5 blocks, Viswam discovers he has to make a detour, walking 3 blocks of equal length instead of 1 block to reach the next corner. From the time he starts his detour, at what speed, in mph, must he walk, in order to get to school at his usual
time?



$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 4.2 \qquad \textbf{(C)}\ 4.5 \qquad \textbf{(D)}\ 4.8 \qquad \textbf{(E)}\ 5$

Solution 1

Note that Viswam walks at a constant speed of 60 blocks per hour as he takes 1 minute to walk each block. After walking 5 blocks, he has taken 5 minutes, and he has 5 minutes remaining, to walk 7 blocks. Therefore, he must walk at a speed of $7 \cdot 60 \div 5 = 84$ blocks per hour to get to school on time, from the time he starts his detour. Since he normally walks $\frac{1}{2}$ mile, which is equal to 10 blocks, 1 mile is equal to 20 blocks. Therefore, he must walk at $84 \div 20 = 4.2$ mph from the time he starts his detour to get to school on time, so the answer is $\boxed{\textbf{(B)}\ 4.2}$.

Solution 2

Viswam walks 10 blocks, or half a mile, in 10 minutes. Therefore, he walks at a rate of 3 mph. From the time he takes his detour, he must travel 7blocks instead of 5. Our final equation is $7/5 \times 3 = 21/5 = \boxed{\textbf{(B)}\ 4.2}$.

Solution 3 (Cheap)

Notice that Viswam will need to walk 7 blocks during the second half as opposed to his normal 5 blocks. Since rate is equal to distance over time, this implies that the final answer will likely be a multiple of 7, since you will need to convert 7 blocks to miles. The only answer choice that is a multiple of 7 is $\boxed{\textbf{(B) } 4.2}$.

Problem16

The letters P, Q, and R are entered into a 20×20 table according to the pattern shown below. How many Ps, Qs, and Rs will appear in the completed table?

⋮	⋮	⋮	⋮	⋮	⋰
Q	R	P	Q	R	⋯
P	Q	R	P	Q	⋯
R	P	Q	R	P	⋯
Q	R	P	Q	R	⋯
P	Q	R	P	Q	⋯

$\textbf{(A) } 132 \text{ Ps}, 134 \text{ Qs}, 134 \text{ Rs}$

$\textbf{(B) } 133 \text{ Ps}, 133 \text{ Qs}, 134 \text{ Rs}$

$\textbf{(C) } 133 \text{ Ps}, 134 \text{ Qs}, 133 \text{ Rs}$

$\textbf{(D) } 134 \text{ Ps}, 132 \text{ Qs}, 134 \text{ Rs}$

$\textbf{(E) } 134 \text{ Ps}, 133 \text{ Qs}, 133 \text{ Rs}$

Solution 1

In our 5×5 grid we can see there are $8, 9$ and 8 of the letters $\mathrm{P}, \mathrm{Q},$ and R's respectively. We can see our pattern between each is $x, x+1,$ and x for the $\mathrm{P}, \mathrm{Q},$ and R's respectively. This such pattern will follow in our bigger example, so we can see that the only answer choice which satisfies this condition is $\boxed{\textbf{(C) } 133 \text{ Ps}, 134 \text{ Qs}, 133 \text{ Rs}}$.

(Note: you could also "cheese" this problem by listing out all of the letters horizontally in a single line and looking at the repeating pattern.)

Solution 2

We think about which letter is in the diagonal with 20 of a letter. We find that it is $2(2+5+8+11+14+17)+20=134.$ The rest of the grid with the P's and R's is symmetric. Therefore, the answer is $\boxed{\textbf{(C) } 133 \text{ Ps}, 134 \text{ Qs}, 133 \text{ Rs}}$.

Solution 3

Notice that rows x and $x+3$ are the same, for any $1 \leq x \leq 17.$ Additionally, rows $1, 2,$ and 3 collectively contain the same number of Ps, Qs, and Rs, because the letters are just substituted for one another. Therefore, the number of Ps, Qs, and Rs in the first 18 rows is 120. The first row has 7 Ps, 7 Qs, and 6 Rs, and the second row has 6 Ps, 7 Qs, and 7 Rs. Adding these up, we obtain $\boxed{\textbf{(C) } 133 \text{ Ps}, 134 \text{ Qs}, 133 \text{ Rs}}$.

Solution 4

From the full diagram below, the answer

is $\boxed{\textbf{(C) } 133 \text{ Ps, } 134 \text{ Qs, } 133 \text{ Rs}}$.

Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R
P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q
R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P
Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R
P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q
R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P
Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R
P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q
R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P
Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R
P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q
R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P
Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R
P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q
R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P
Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R
P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q
R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P
Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R
P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q

This solution is extremely time-consuming and error-prone. Please do not try it in a real competition unless you have no other options.

Solution 5

This solution refers to the full diagram in Solution 4.

Note the Q diagonals are symmetric. The R and P diagonals are not symmetric, but are reflections of each other about the Q diagonals:

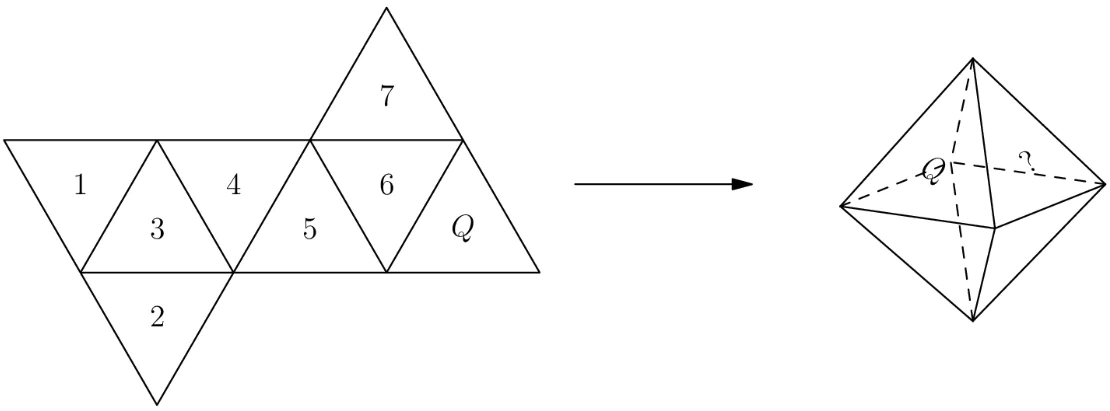
- The upper Q diagonal of length 2 is surrounded by a P diagonal of length 3 and an R diagonal of length $1.$
- The lower Q diagonal of length 2 is surrounded by a P diagonal of length 1 and an R diagonal of length $3.$

When looking at a pair of Q diagonals of the same length x, there is a total of $2x$ Rs and Ps next to these 2 diagonals.

The main diagonal of 20 Qs has 19 Ps and 19 Rs next to it. Thus, the total is $x+1$ Qs, x Ps, x Rs. Therefore, the answer is $\boxed{\textbf{(C)}\ 133\ \mathrm{Ps},\ 134\ \mathrm{Qs},\ 133\ \mathrm{Rs}}$.

Problem17

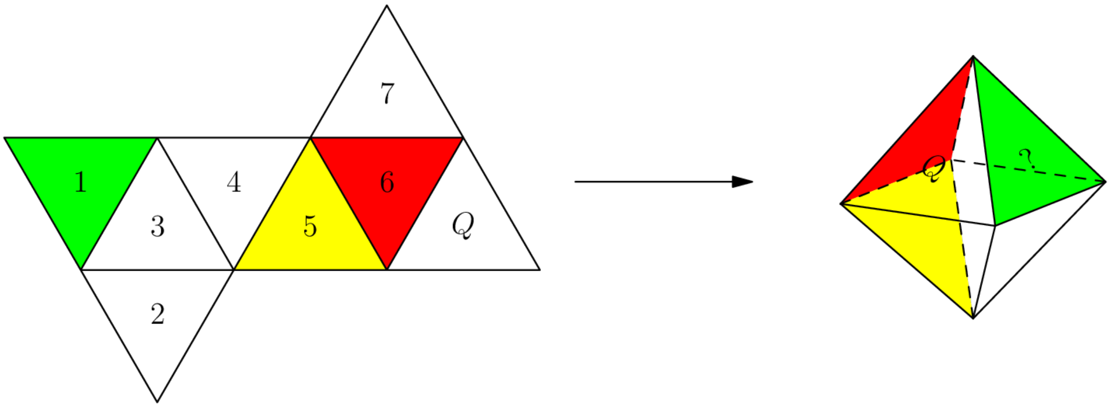
A *regular octahedron* has eight equilateral triangle faces with four faces meeting at each vertex. Jun will make the regular octahedron shown on the right by folding the piece of paper shown on the left. Which numbered face will end up to the right of Q?



$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution 1

We color face 6 red and face 5 yellow. Note that from the octahedron, face 5 and face $?$ do not share anything in common. From the net, face 5shares at least one vertex with all other faces except face $1,$ which is shown in green:



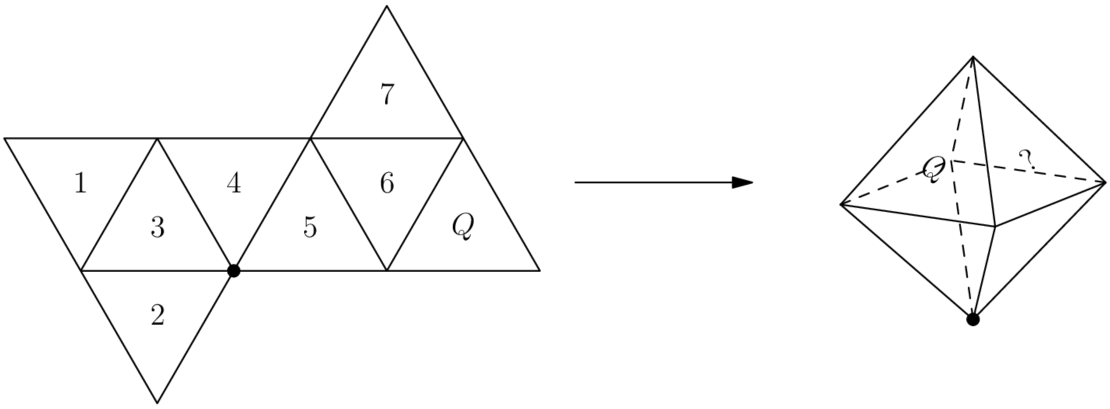
Therefore, the answer is $\boxed{\textbf{(A) } 1}$.

Solution 2

We label the octohedron going triangle by triangle until we reach the $?$ triangle. The triangle to the left of the Q should be labeled with a 6. Underneath triangle 6 is triangle 5. The triangle to the right of triangle 5 is triangle 4 and further to the right is triangle 3. Finally, the side of triangle 3 under triangle Q is 2, so the triangle to the right of Q is $\boxed{\textbf{(A) } 1}$.

Solution 3 (Fast and Cheap)

Notice that the triangles labeled $2, 3, 4,$ and 5 make the bottom half of the octahedron, as shown below:



Therefore, $\textbf{(B)}$, $\textbf{(C)}$, $\textbf{(D)}$, and $\textbf{(E)}$ are clearly not the correct answer.

Thus, the only choice left is $\boxed{\textbf{(A) }1}$.

Problem18

Greta Grasshopper sits on a long line of lily pads in a pond. From any lily pad, Greta can jump 5 pads to the right or 3 pads to the left. What is the fewest number of jumps Greta must make to reach the lilly pad located 2023 pads to the right of her starting position?

$\textbf{(A) } 405 \qquad \textbf{(B) } 407 \qquad \textbf{(C) } 409 \qquad \textbf{(D) } 411 \qquad \textbf{(E) } 413$

Solution 1

We have 2 directions going 5 right or 3 left. We can assign a variable to each of these directions. We can call going right 1 direction X and we can call going 1 left Y. We can build a equation of $5X - 3Y = 2023$. Where we have to limit the number of moves we do. We can do this by making more of our moves the 5 move turn then the 3 move turn. The first obvious step is to go some amount of moves in the right direction then subtract off in the left direction to land on 2023. The least amount of 3's added to 2023 to make a multiple of 5 is 4 as $2023 + 4(3) = 2035$. So now we have solved the problem as we just go $\dfrac{2035}{5} = 407$ hops right, and just do 4 more hops left.

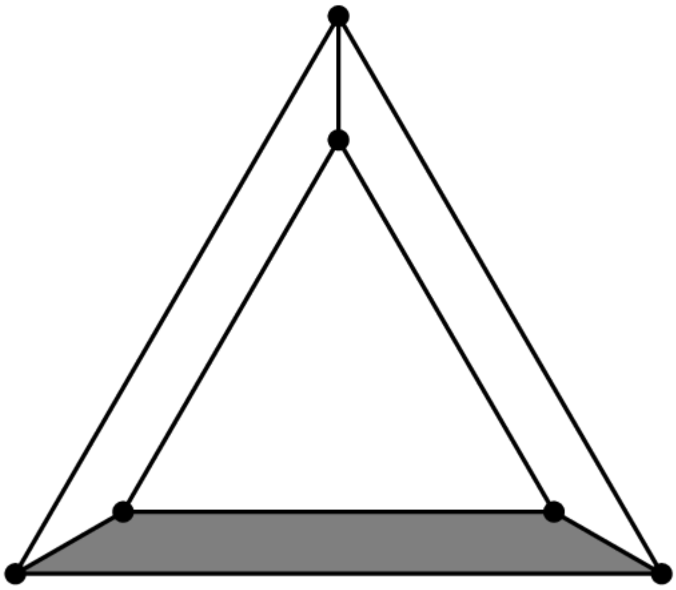
Yielding $407 + 4 = \boxed{\textbf{(D) } 411}$ as our answer.

Solution 2

Notice that $2023 = 3 \pmod 5$, and jumping to the left increases the value of Greta's position $\pmod 5$ by 2. Therefore, the number of jumps to the left must be $4 \pmod 5$. As the number of jumps to the left increases, so does the number of jumps to the right, so therefore we must minimize both, which occurs when we jump 4 to the left and 407 to the right. The answer is $\boxed{\textbf{(D)}\ 411}$.

Problem19

An equilateral triangle is placed inside a larger equilateral triangle so that the region between them can be divided into three congruent trapezoids, as shown below. The side length of the inner triangle is $\frac{2}{3}$ the side length of the larger triangle. What is the ratio of the area of one trapezoid to the area of the inner triangle?



$\textbf{(A)}\ 1:3 \qquad \textbf{(B)}\ 3:8 \qquad \textbf{(C)}\ 5:12 \qquad \textbf{(D)}\ 7:16 \qquad \textbf{(E)}\ 4:9$

Solution 1

All equilateral triangles are similar. For the outer equilateral triangle to the inner equilateral triangle, since their side-length ratio is $\frac{3}{2},$ their area ratio is $\left(\frac{3}{2}\right)^2=\frac{9}{4}.$ It follows that the area ratio of three trapezoids to the inner equilateral triangle is $\frac{9}{4}-1=\frac{5}{4}$, so the area ratio of one trapezoid to the inner equilateral triangle is $\frac{5}{4}\cdot\frac{1}{3}=\frac{5}{12}=\boxed{\textbf{(C) } 5:12}$.

Solution 2

Subtracting the larger equilateral triangle from the smaller one yields the sum of the three trapezoids. Since the ratio of the side lengths of the larger to the smaller one is $3:2$, we can set the side lengths as 3 and 2, respectively. So, the sum of the trapezoids is $\frac{9\sqrt{3}}{4}-\frac{4\sqrt{3}}{4}=\frac{5}{4}\sqrt{3}$. We are also told that the three trapezoids are congruent, thus the area of each of them is $\frac{1}{3}\cdot\frac{5}{4}\sqrt{3}=\frac{5}{12}\sqrt{3}$. Hence, the ratio is $\frac{\frac{5}{12}\sqrt{3}}{\sqrt{3}}=\frac{5}{12}=\boxed{\textbf{(C) } 5:12}$.

Problem20

Two integers are inserted into the list $3, 3, 8, 11, 28$ to double its range. The mode and median remain unchanged. What is the maximum possible sum of the two additional numbers?

$\textbf{(A) } 56 \qquad \textbf{(B) } 57 \qquad \textbf{(C) } 58 \qquad \textbf{(D) } 60 \qquad \textbf{(E) } 61$

Solution

To double the range we must find the current range, which is $28-3=25$, to then the double is $2(25)=50$ Since we dont want to change the median we need to get a value greater than 8 (as 8 would change the mode) for the smaller and 53 is fixed for the larger as anything less than $3$is not beneficial to the optimization. So taking our optimal values of 53 and 7 we have an answer of $53+7=\boxed{\textbf{(D) }60}$.

Problem21

Alina writes the numbers $1, 2, \dots , 9$ on separate cards, one number per card. She wishes to divide the cards into 3 groups of 3 cards so that the sum of the numbers in each group will be the same. In how many ways can this be done?

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 3 \qquad \textbf{(E) } 4$

Solution 1

First we need to find the sum of each group when split. This is the total sum of all the elements divided by the # of groups. $1+2\dots+9=\frac{9(10)}{2}=45$. Then dividing by 3 we have $\frac{45}{3}=15$ so each group of 3 must have a sum of 15. To make the counting easier we we will just see the possible groups 9 can be with. The posible groups 9 can be with with 2 distinct numbers are $(9,2,4)$ and $(9,1,5)$. Going down each of these avenues we will repeat the same process for 8 using the remaining elements in the list. Where there is only 1 set of elements getting the sum of 7, 8 needs in both cases. After 8 is decided the remaining 3 elements are forced in a group. Yielding us an answer of $\boxed{\textbf{(C) }2}$ as our sets are $(9,1,5)(8,3,4)(7,2,6)$ and $(9,2,4)(8,1,6)(7,3,5)$

Solution 2

The group with 5 must have the two other numbers adding up to 10, since the sum of all the numbers is $(1+2\dots+9)=\frac{9(10)}{2}=45$. The sum of the numbers in each group must therefore be $\frac{45}{3} = 15$. We can have $(1, 5, 9)$, $(2, 5, 8)$, $(3, 5, 7)$, or $(4, 5, 6)$. With the first group, we have $(2, 3, 4, 6, 7, 8)$ left over. The only way to form a group of 3 numbers that add up to 15 is with $(3, 4, 8)$ or $(2, 6, 7)$. One of the possible arrangements is therefore $(1, 5, 9)(3, 4, 8)(2, 6, 7)$. Then, with the second group, we have $(1, 3, 4, 6, 7, 9)$ left over. With these numbers, there is no way to form a group of 3 numbers adding to 15. Similarly, with the third group there is $(1, 2, 4, 6, 8, 9)$ left over and we can make a group of 3 numbers adding to 15 with $(1, 6, 8)$ or $(2, 4, 9)$. Another arrangement is $(3, 5, 7)(1, 6, 8)(2, 4, 9)$. Finally, the last group has $(1, 2, 3, 7, 8, 9)$ left over. There is no way to make a group of 3 numbers adding to 15 with this, so the arrangements are $(1, 5, 9)(3, 4, 8)(2, 6, 7)$ and $(3, 5, 7)(1, 6, 8)(2, 4, 9)$.

There are $\boxed{\textbf{(C) } 2}$ sets that can be formed.

Problem22

In a sequence of positive integers, each term after the second is the product of the previous two terms. The sixth term is 4000. What is the first term?

$$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 4 \qquad \textbf{(D) } 5 \qquad \textbf{(E) } 10$$

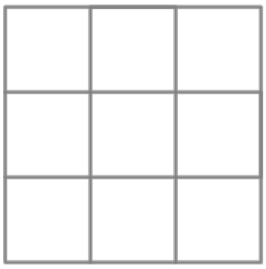
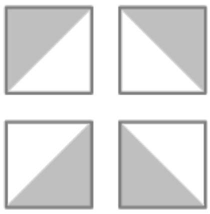
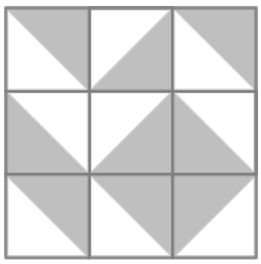
Solution 1

Suppose the first 2 terms were x and y. Then, the next proceeding terms would be xy, xy^2, x^2y^3, and x^3y^5. Since x^3y^5 is the 6th term, this must be equal to 4000. So, $x^3y^5 = 4000$. If we prime factorize 4000 we get $4000 = 5^3 \cdot 2^5$. We conclude $x = 5$ and $y = 2$, which means that the answer is $\boxed{\textbf{(D)}\ 5}$

Solution 2

In this solution, we will use trial and error to solve. 4000 can be expressed as 200×20. We divide 200 by 20 and get 10, divide 20 by 10 and get 2, and divide 10 by 2 to get $\boxed{\textbf{(D)}\ 5}$. No one said that they have to be in ascending order!

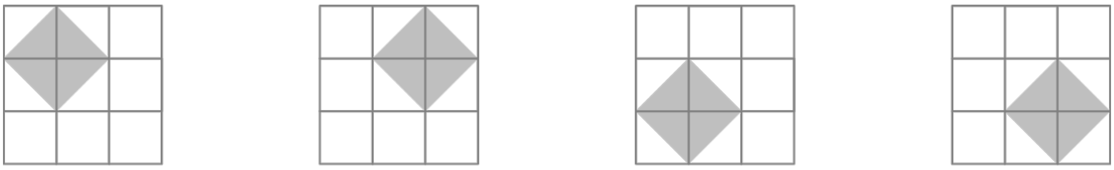
Problem23

Each square in a 3×3 grid is randomly filled with one of the 4 gray and white tiles shown below on the right.   What is the probability that the tiling will contain a large gray diamond in one of the smaller 2×2 grids? Below is an example of such tiling. 

$\textbf{(A)}\ \frac{1}{1024} \qquad \textbf{(B)}\ \frac{1}{256} \qquad \textbf{(C)}\ \frac{1}{64} \qquad \textbf{(D)}\ \frac{1}{16} \qquad \textbf{(E)}\ \frac{1}{4}$

Solution 1

There are 4 cases that the tiling will contain a large gray diamond in one of the smaller 2×2 grids, as shown below:



There are 4^5 ways to decide the 5 white squares for each case, and the cases do not have any overlap.

So, the requested probability is $\frac{4\cdot 4^5}{4^9}=\frac{4^6}{4^9}=\frac{1}{4^3}=\boxed{\textbf{(C) } \frac{1}{64}}$.

Solution 2

Note that the middle tile can be any of the four tiles. The gray part of the middle tile points towards one of the corners, and for the gray diamond to appear the three adjacent tiles must all be perfect. Thus, the solution is $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \boxed{\textbf{(C) } \frac{1}{64}}$.

Solution 3 (Linearity of Expectation)

Let S_1, S_2, S_3, and S_4 denote the 4 smaller 2×2 squares within the 3×3 square in some order. For each S_i, let $X_i = 1$ if it contains a large gray diamond tiling and $X_i = 0$ otherwise. This means that $\mathbb{E}[X_i]$ is the probability that square S_i has a large gray diamond, so $\mathbb{E}[X_1 + X_2 + X_3 + X_4]$ is our desired probability. However, since there is only one possible way to arrange the squares within every 2×2square to form such a tiling, we have $\mathbb{E}[X_i] = (\frac{1}{4})^2 = \frac{1}{256}$ for all i (as each of the smallest tiles has 4 possible arrangements), and from the linearity of expectation we get $\mathbb{E}[X_1 + X_2 + X_3 + X_4] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4] = \frac{1}{256} + \frac{1}{256} + \frac{1}{256} + \frac{1}{256} = \boxed{\textbf{(C) } \frac{1}{64}}$.

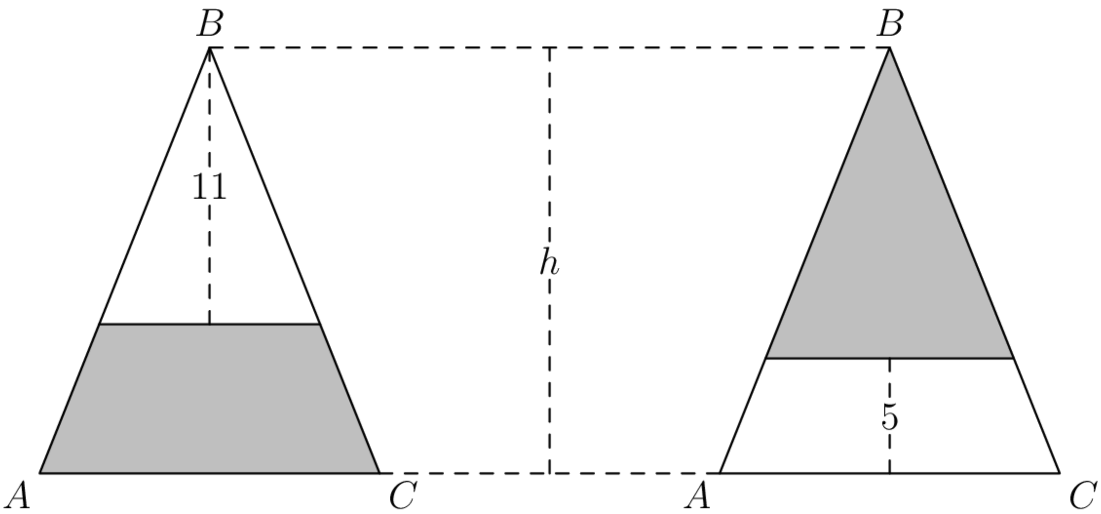
Remark 1: This method might be too advanced for the AMC 8, and is probably unnecessary (refer to the other solutions for simpler techniques).

Remark 2: Note that Probability and Expected Value are equivalent in this problem since there will never be two diamonds on one tiling.

i.e. $X_1 + X_2 + X_3 + X_4 \leq 1$.

Problem24

Isosceles $\triangle ABC$ has equal side lengths AB and BC. In the figure below, segments are drawn parallel to $\overline{AC}$ so that the shaded portions of $\triangle ABC$ have the same area. The heights of the two unshaded portions are 11 and 5 units, respectively. What is the height of h of $\triangle ABC$?



(note: diagrams are not necessarily drawn to scale)

$\textbf{(A)}\ 14.6 \qquad \textbf{(B)}\ 14.8 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 15.2 \qquad \textbf{(E)}\ 15.4$

Solution 1

First, we notice that the smaller isosceles triangles are similar to the larger isosceles triangles. We can find that the area of the gray area in the first triangle is $[ABC]\cdot\left(1-\left(\frac{11}{h}\right)^2\right)$. Similarly, we can find that the area of the gray part in the second triangle is $[ABC]\cdot\left(\frac{h-5}{h}\right)^2$. These areas are equal, so $1 - \left(\frac{11}{h}\right)^2 = \left(\frac{h-5}{h}\right)^2$. Simplifying yields $10h = 146$ so $h = \boxed{\textbf{(A) } 14.6}$.

Solution 2 (Thorough)

We can call the length of AC as x. Therefore, the length of the base of the triangle with height 11 is $11/h = a/x$. Therefore, the base of the smaller triangle is $11x/h$. We find that the area of the trapezoid is $(hx)/2 - 11^2x/2h$.

Using similar triangles once again, we find that the base of the shaded triangle is $(h-5)/h = b/x$. Therefore, the area is $(h-5)(hx-5x)/h$.

Since the areas are the same, we find that $(hx)/2 - 121x/2h = (h-5)(hx-5x)/h$. Multiplying each side by $2h$, we get $h^2x - 121x = h^2x - 5hx - 5hx + 25x$. Therefore, we can subtract $25x + h^2x$ from both sides, and get $-146x = -10hx$. Finally, we divide both sides by $-x$ and get $10h = 146$. h is $\boxed{\textbf{(A)} 14.6}$.

Solution 3 (Faster)

Since the length of AC does not matter, we can assume the base of triangle ABC is h. Therefore, the area of the trapezoid in the first diagram is $h^2/2 - 11^2/2$.

The area of the triangle in the second diagram is now $(h-5)^2/2$.

Therefore, $h^2/2 - 11^2/2 = (h-5)^2/2$. Multiplying both sides by 2, we get $h^2 - 121 = h^2 - 10h + 25$. Subtracting $h^2 + 25$ from both sides, we get $-146 = -10h$ and h is $\boxed{\textbf{(A)} 14.6}$.

Problem25

Fifteen integers $a_1, a_2, a_3, \dots, a_{15}$ are arranged in order on a number line. The integers are equally spaced and have the property that $1 \le a_1 \le 10, \; 13 \le a_2 \le 20, \quad \text{and} \quad 241 \le a_{15} \le 250.$

What is the sum of digits of a_{14}?

$\textbf{(A)}\ 8 \qquad \textbf{(B)}\ 9 \qquad \textbf{(C)}\ 10 \qquad \textbf{(D)}\ 11 \qquad \textbf{(E)}\ 12$

Solution 1

We can find the possible values of the common difference by finding the numbers which satisfy the conditions. To do this, we find the minimum of the last two–$241 - 20 = 221$, and the maximum–$250 - 13 = 237$. There is a difference of 13 between them, so only 17 and 18 work, as $17 \cdot 13 = 221$, so 17 satisfies $221 \le 13x \le 237$. The number 18 is similarly found. 19, however, is too much.

Now, we check with the first and last equations using the same method. We know $241 - 10 \le 14x \le 250 - 1$.

Therefore, $231 \le 14x \le 249$. We test both values we just got, and we can realize that 18 is too large to satisfy this inequality. On the other hand, we can now find that the difference will be 17, which satisfies this inequality. The last step is to find the first term. We know that the first term can only be from 1 to 3, since any larger value would render the second inequality invalid. Testing these three, we find that only $a_1 = 3$ will satisfy all the inequalities.

Therefore, $a_{14} = 13 \cdot 17 + 3 = 224$. The sum of the digits is therefore $\boxed{\textbf{(A) } 8}$.

Solution 2

Let the common difference between consecutive a_i be d. Then, since $a_{15} - a_1 = 14d$, we find from the first and last inequalities that $231 \leq 14d \leq 249$. As d must be an integer, this means $d = 17$. Plugging this into all of the given inequalities so we may extract information about a_1 gives

$$1 \leq a_1 \leq 10,\ 13 \leq a_1 + 17 \leq 20,\ 241 \leq a_1 + 238 \leq 250.$$

The second inequality tells us that $a_1 \leq 3$, while the last inequality tells us $3 \leq a_1$, so we must have $a_1 = 3$. Finally, to solve for a_{14}, we simply have $a_{14} = a_1 + 13d = 3 + 221 = 224$, so our answer is $\boxed{\textbf{(A) } 8}$.