2002 AMC 12B Problems/Problem 1

The following problem is from both the 2002 AMC 12B #1 and 2002 AMC 10B #3, so both problems redirect to this page.

Problem

The arithmetic mean of the nine numbers in the set $\{9, 99, 999, 9999, \ldots, 999999999\}$ is a 9-digit number M, all of whose digits are distinct. The number M does not contain the digit

$\mathrm{(A)}\ 0 \qquad\mathrm{(B)}\ 2 \qquad\mathrm{(C)}\ 4 \qquad\mathrm{(D)}\ 6 \qquad\mathrm{(E)}\ 8$

Solution

We wish to find $\frac{9+99+\cdots+999999999}{9}$, or $\frac{9(1+11+111+\cdots+111111111)}{9} = 123456789$. This does not have the digit 0, so the answer is $\boxed{\mathrm{(A)}\ 0}$

See also

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Category: Introductory Algebra Problems

2002 AMC 12B Problems/Problem 2

The following problem is from both the 2002 AMC 12B #2 and 2002 AMC 10B #4, so both problems redirect to this page.

Contents

- 1 Problem
- 2 Solution
- 3 Comment
- 4 See also

Problem

What is the value of $(3x-2)(4x+1)-(3x-2)4x+1$ when $x=4$?

$\mathrm{(A) \ 0 } \qquad \mathrm{(B) \ 1 } \qquad \mathrm{(C) \ 10 } \qquad \mathrm{(D) \ 11 } \qquad \mathrm{(E) \ 12 }$

Solution

By the distributive property,

$$(3x-2)[(4x+1)-4x]+1 = 3x-2+1 = 3x-1 = 3(4)-1 = \boxed{\mathrm{(D)}\ 11}$$

Comment

It would be nicer if the organizers chose a more difficult substitution, say $x=4.7$, to penalize those solutions that do not take advantage of the distributive property.

See also

2002 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2002))	
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2002 AMC 12B Problems/Problem 3

The following problem is from both the 2002 AMC 12B #3 and 2002 AMC 10B #6, so both problems redirect to this page.

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See also

Problem

For how many positive integers n is $n^2 - 3n + 2$ a prime number?

$\mathrm{(A)}\ \text{none} \qquad \mathrm{(B)}\ \text{one} \qquad \mathrm{(C)}\ \text{two} \qquad \mathrm{(D)}\ \text{more than two, but finitely many} \qquad \mathrm{(E)}\ \text{infinitely many}$

Solution 1

Factoring, we get $n^2 - 3n + 2 = (n-2)(n-1)$. Either $n-1$ or $n-2$ is odd, and the other is even. Their product must yield an even number. The only prime that is even is 2, which is when n is 3. The answer is $\boxed{\mathrm{(B)}\ \text{one}}$.

Solution 2

Considering parity, we see that $n^2 - 3n + 2$ is always even. The only even prime is 2, and so $n^2 - 3n = 0$ whence $n = 3 \Rightarrow \boxed{\mathrm{(B)}\ \text{one}}$.

See also

2002 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2002))	
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2002 AMC 12B Problems/Problem 4

The following problem is from both the 2002 AMC 12B #4 and 2002 AMC 10B #7, so both problems redirect to this page.

Problem

Let n be a positive integer such that $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}$ is an integer. Which of the following statements is not true:

$\mathrm{(A)}\ 2\ \text{divides}\ n \qquad\mathrm{(B)}\ 3\ \text{divides}\ n \qquad\mathrm{(C)}\ 6\ \text{divides}\ n \qquad\mathrm{(D)}\ 7\ \text{divides}\ n \qquad\mathrm{(E)}\ n > 84$

Solution

Since $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}=\frac{41}{42}$,

$$0 < \lim_{n\rightarrow\infty}\left(\frac{41}{42}+\frac{1}{n}\right) < \frac{41}{42}+\frac{1}{n} < \frac{41}{42}+\frac{1}{1} < 2$$

From which it follows that $\frac{41}{42}+\frac{1}{n}=1$ and $n=42$. The only answer choice that is not true is $\boxed{\mathrm{(E)}\ n > 84}$.

See also

2002 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2002))	
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2002 AMC 12B Problems/Problem 5

Problem

Let $v, w, x, y,$ and z be the degree measures of the five angles of a pentagon. Suppose that $v < w < x < y < z$ and $v, w, x, y,$ and z form an arithmetic sequence. Find the value of x.

$\mathrm{(A)}\ 72 \qquad\mathrm{(B)}\ 84 \qquad\mathrm{(C)}\ 90 \qquad\mathrm{(D)}\ 108 \qquad\mathrm{(E)}\ 120$

Solution

The sum of the degree measures of the angles of a pentagon (as a pentagon can be split into $5 - 2 = 3$ triangles) is $3 \cdot 180 = 540^{\circ}$. If we let $v = x - 2d, w = x - d, y = x + d, z = x + 2d$, it follows that

$$(x-2d)+(x-d)+x+(x+d)+(x+2d) = 5x = 540 \Longrightarrow x = 108\ \mathrm{(D)}$$

Note that since x is the middle term of an arithmetic sequence with an odd number of terms, it is simply the average of the sequence.

See also

2002 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2002))	
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2002 AMC 12B Problems/Problem 6

The following problem is from both the 2002 AMC 12B #6 and 2002 AMC 10B #10, so both problems redirect to this page.

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- 1 Problem
- 2 Solution
  - 2.1 Solution 1
  - 2.2 Solution 2
- 3 See also

Problem

Suppose that a and b are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. Then the pair (a, b) is

$$\mathrm{(A)}\ (-2,1) \qquad\mathrm{(B)}\ (-1,2) \qquad\mathrm{(C)}\ (1,-2) \qquad\mathrm{(D)}\ (2,-1) \qquad\mathrm{(E)}\ (4,4)$$

Solution

Solution 1

Since $(x-a)(x-b) = x^2 - (a+b)x + ab = x^2 + ax + b = 0$, it follows by comparing coefficients that $-a - b = a$ and that $ab = b$. Since b is nonzero, $a = 1$, and $-1 - b = 1 \Longrightarrow b = -2$. Thus $(a,b) = \boxed{\mathrm{(C)}\ (1,-2)}$.

Solution 2

Another method is to use Vieta's formulas. The sum of the solutions to this polynomial is equal to the opposite of the x coefficient, since the leading coefficient is 1; in other words, $a + b = -a$ and the product of the solutions is equal to the constant term (i.e, $a * b = b$). Since b is nonzero, it follows that $a = 1$ and therefore (from the first equation), $b = -2a = -2$. Hence, $(a,b) = \boxed{\mathrm{(C)}\ (1,-2)}$

See also

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2002 AMC 12B Problems/Problem 7

The following problem is from both the 2002 AMC 12B #7 and 2002 AMC 10B #11, so both problems redirect to this page.

Problem

The product of three consecutive positive integers is 8 times their sum. What is the sum of their squares?

$\mathrm{(A)}\ 50 \qquad\mathrm{(B)}\ 77 \qquad\mathrm{(C)}\ 110 \qquad\mathrm{(D)}\ 149 \qquad\mathrm{(E)}\ 194$

Solution

Let the three consecutive positive integers be $a-1$, a, and $a+1$. So, $a(a-1)(a+1)=24a \Rightarrow (a-1)(a+1)=24$. $24=4\times 6$, so $a=5$. Hence, the sum of the squares is $4^2+5^2+6^2=\boxed{\mathrm{(B)}\ 77}$.

See also

2002 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2002))	
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2002 AMC 12B Problems/Problem 8

> The following problem is from both the 2002 AMC 12B #8 and 2002 AMC 10B #8, so both problems redirect to this page.

Problem

Suppose July of year N has five Mondays. Which of the following must occurs five times in the August of year N? (Note: Both months have 31 days.)

$\text{(A)}\ \text{Monday} \qquad \text{(B)}\ \text{Tuesday} \qquad \text{(C)}\ \text{Wednesday} \qquad \text{(D)}\ \text{Thursday} \qquad \text{(E)}\ \text{Friday}$

Solution

If there are five Mondays, there are only three possibilities for their dates: $(1, 8, 15, 22, 29)$, $(2, 9, 16, 23, 30)$, and $(3, 10, 17, 24, 31)$.

In the first case August starts on a Thursday, and there are five Thursdays, Fridays, and Saturdays in August.

In the second case August starts on a Wednesday, and there are five Wednesdays, Thursdays, and Fridays in August.

In the third case August starts on a Tuesday, and there are five Tuesdays, Wednesdays, and Thursdays in August.

The only day of the week that is guaranteed to appear five times is therefore $\boxed{\text{(D)}\ \text{Thursday}}$.

See Also

2002 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2002))	
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2002 AMC 12B Problems/Problem 9

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  - 2.1 Solution 1
  - 2.2 Solution 2
  - 2.3 Solution 3
- 3 See also

Problem

If a,b,c,d are positive real numbers such that a,b,c,d form an increasing arithmetic sequence and a,b,d form a geometric sequence, then $\frac{a}{d}$ is

$\mathrm{(A)}\ \frac{1}{12} \qquad\mathrm{(B)}\ \frac{1}{6} \qquad\mathrm{(C)}\ \frac{1}{4} \qquad\mathrm{(D)}\ \frac{1}{3} \qquad\mathrm{(E)}\ \frac{1}{2}$

Solution

Solution 1

We can let a=1, b=2, c=3, and d=4. $\frac{a}{d}=\boxed{\frac{1}{4}}\Longrightarrow \mathrm{(C)}$

Solution 2

As a,b,d is a geometric sequence, let $b=ka$ and $d=k^2a$ for some $k>0$.

Now, a,b,c,d is an arithmetic sequence. Its difference is $b-a=(k-1)a$. Thus $d=a+3(k-1)a=(3k-2)a$.

Comparing the two expressions for d we get $k^2=3k-2$. The positive solution is $k=2$, and $\frac{a}{d}=\frac{a}{k^2a}=\frac{1}{k^2}=\boxed{\frac{1}{4}}\Longrightarrow \mathrm{(C)}$.

Solution 3

Letting n be the common difference of the arithmetic progression, we have $b=a+n$, $c=a+2n$, $d=a+3n$. We are given that b/a = d/b, or

$$\frac{a+n}{a}=\frac{a+3n}{a+n}.$$

Cross-multiplying, we get

$$a^2+2an+n^2=a^2+3an$$

$$n^2=an$$

$$n=a$$

So $\dfrac{a}{d} = \dfrac{a}{a+3n} = \dfrac{a}{4a} = \boxed{\dfrac{1}{4}} \Longrightarrow \text{(C)}$.

See also

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2002 AMC 12B Problems/Problem 10

Problem

How many different integers can be expressed as the sum of three distinct members of the set $\{1,4,7,10,13,16,19\}$? $\text{(A)}\ 13 \qquad \text{(B)}\ 16 \qquad \text{(C)}\ 24 \qquad \text{(D)}\ 30 \qquad \text{(E)}\ 35$

Solution

Subtracting 10 from each number in the set, and dividing the results by 3, we obtain the set $\{-3,-2,-1,0,1,2,3\}$. It is easy to see that we can get any integer between -6 and 6 inclusive as the sum of three elements from this set, for the total of $\boxed{(A) 13}$ integers.

See also

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2002 AMC 12B Problems/Problem 11

The following problem is from both the 2002 AMC 12B #11 and 2002 AMC 10B #15, so both problems redirect to this page.

Problem

The positive integers $A, B, A-B,$ and $A+B$ are all prime numbers. The sum of these four primes is

$\mathrm{(A)}\ \mathrm{even} \qquad \mathrm{(B)}\ \mathrm{divisible\ by\ }3 \qquad \mathrm{(C)}\ \mathrm{divisible\ by\ }5 \qquad \mathrm{(D)}\ \mathrm{divisible\ by\ }7 \qquad \mathrm{(E)}\ \mathrm{prime}$

Solution

Since $A-B$ and $A+B$ must have the same parity, and since there is only one even prime number, it follows that $A-B$ and $A+B$ are both odd. Thus one of A, B is odd and the other even. Since $A+B > A > A-B > 2$, it follows that A (as a prime greater than 2) is odd. Thus $B=2$, and $A-2, A, A+2$ are consecutive odd primes. At least one of $A-2, A, A+2$ is divisible by 3, from which it follows that $A-2=3$ and $A=5$. The sum of these numbers is thus 17, a prime, so the answer is $\boxed{\mathrm{(E)}\ \mathrm{prime}}$.

See also

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2002 AMC 12B Problems/Problem 12

The following problem is from both the 2002 AMC 12B #12 and 2002 AMC 10B #16, so both problems redirect to this page.

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- 1 Problem
- 2 Solution
  - 2.1 Solution 1
  - 2.2 Solution 2
- 3 See also

Problem

For how many integers n is $\frac{n}{20-n}$ the square of an integer?

$\mathrm{(A)}\ 1 \qquad\mathrm{(B)}\ 2 \qquad\mathrm{(C)}\ 3 \qquad\mathrm{(D)}\ 4 \qquad\mathrm{(E)}\ 10$

Solution

Solution 1

Let $x^2 = \frac{n}{20-n}$, with $x \geq 0$ (note that the solutions $x < 0$ do not give any additional solutions for n). Then rewriting, $n = \frac{20x^2}{x^2 + 1}$. Since $\text{gcd}(x^2, x^2 + 1) = 1$, it follows that $x^2 + 1$ divides 20. Listing the factors of 20, we find that $x = 0, 1, 2 , 3$ are the only $\boxed{\mathrm{(D)}\ 4}$ solutions (respectively yielding $n = 0, 10, 16, 18$).

Solution 2

For $n<0$ and $n>20$ the fraction is negative, for $n=20$ it is not defined, and for $n\in\{1,\dots,9\}$ it is between 0 and 1.

Thus we only need to examine $n=0$ and $n\in\{10,\dots,19\}$.

For $n=0$ and $n=10$ we obviously get the squares 0 and 1 respectively.

For prime n the fraction will not be an integer, as the denominator will not contain the prime in the numerator.

This leaves $n\in\{12,14,15,16,18\}$, and a quick substitution shows that out of these only $n=16$ and $n=18$ yield a square. Therefore, there are only $\boxed{\mathrm{(D)}\ 4}$ solutions (respectively yielding $n = 0, 10, 16, 18$).

See also

2002 AMC 12B Problems/Problem 13

Problem

The sum of 18 consecutive positive integers is a perfect square. The smallest possible value of this sum is

$$\text{(A)}\ 169 \qquad \text{(B)}\ 225 \qquad \text{(C)}\ 289 \qquad \text{(D)}\ 361 \qquad \text{(E)}\ 441$$

Solution

Let $a, a+1, \dots, a+17$ be the consecutive positive integers. Their sum, $18a + \frac{17(18)}{2} = 9(2a+17)$, is a perfect square. Since 9 is a perfect square, it follows that $2a+17$ is a perfect square. The smallest possible such perfect square is 25 when $a = 4$, and the sum is $225 \Rightarrow \mathrm{(B)}$.

See also

2002 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2002))	
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2002 AMC 12B Problems/Problem 14

The following problem is from both the 2002 AMC 12B #14 and 2002 AMC 10B #18, so both problems redirect to this page.

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See also

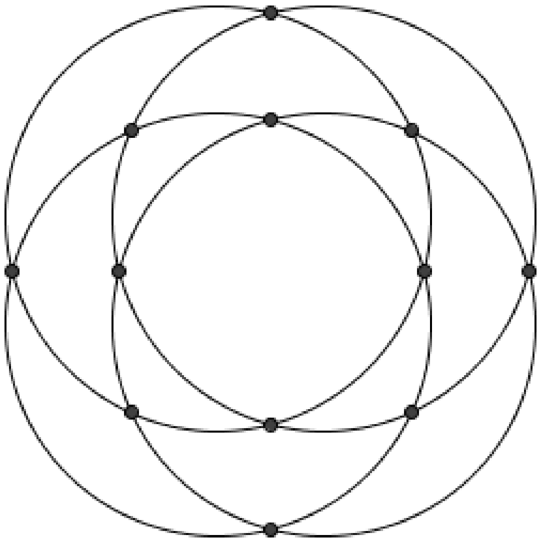
Problem

Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?

$\mathrm{(A)}\ 8\qquad\mathrm{(B)}\ 9\qquad\mathrm{(C)}\ 10\qquad\mathrm{(D)}\ 12\qquad\mathrm{(E)}\ 16$

Solution 1

For any given pair of circles, they can intersect at most 2 times. Since there are $\binom{4}{2} = 6$ pairs of circles, the maximum number of possible intersections is $6 \cdot 2 = 12$. We can construct such a situation as below, so the answer is $\boxed{\mathrm{(D)}\ 12}$.



Solution 2

Because a pair or circles can intersect at most 2 times, the first circle can intersect the second at 2 points, the third can intersect the first two at 4 points, and the fourth can intersect the first three at 6 points. This means that our answer is $2+4+6=\boxed{\mathrm{(D)}\ 12}$.

See also

2002 AMC 12B Problems/Problem 15

Problem

How many four-digit numbers N have the property that the three-digit number obtained by removing the leftmost digit is one ninth of N?

$\text{(A)}\ 4 \qquad \text{(B)}\ 5 \qquad \text{(C)}\ 6 \qquad \text{(D)}\ 7 \qquad \text{(E)}\ 8$

Solution

Let $N = \overline{abcd} = 1000a + \overline{bcd}$, such that $\frac{N}{9} = \overline{bcd}$. Then $1000a + \overline{bcd} = 9\overline{bcd} \Longrightarrow 125a = \overline{bcd}$. Since $100 \le \overline{bcd} < 1000$, from $a = 1, \ldots, 7$ we have 7 three-digit solutions, and the answer is $\mathrm{(D)}$.

See also

2002 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2002))	
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2002 AMC 12B Problems/Problem 16

Problem

Juan rolls a fair regular octahedral die marked with the numbers 1 through 8. Then Amal rolls a fair six-sided die. What is the probability that the product of the two rolls is a multiple of 3?

$$\text{(A) } \frac{1}{12} \qquad \text{(B) } \frac{1}{3} \qquad \text{(C) } \frac{1}{2} \qquad \text{(D) } \frac{7}{12} \qquad \text{(E) } \frac{2}{3}$$

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- 1 Problem
- 2 Solution
  - 2.1 Solution 1
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- 3 See also

Solution

Solution 1

On both dice, only the faces with the numbers $3, 6$ are divisible by 3. Let $P(a) = \frac{2}{8} = \frac{1}{4}$ be the probability that Juan rolls a 3 or a 6, and $P(b) = \frac{2}{6} = \frac{1}{3}$ that Amal does. By the Principle of Inclusion-Exclusion,

$$P(a \cup b) = P(a) + P(b) - P(a \cap b) = \frac{1}{4} + \frac{1}{3} - \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{2} \Rightarrow \text{(C)}$$

Alternatively, the probability that Juan rolls a multiple of 3 is $\frac{1}{4}$, and the probability that Juan does not roll a multiple of 3 but Amal does is $\left(1 - \frac{1}{4}\right) \cdot \frac{1}{3} = \frac{1}{4}$. Thus the total probability is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Solution 2

The probability that neither Juan nor Amal rolls a multiple of 3 is $\frac{6}{8} \cdot \frac{4}{6} = \frac{1}{2}$; using complementary counting, the probability that at least one does is $1 - \frac{1}{2} = \frac{1}{2} \Rightarrow \text{(C)}$.

See also

2002 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2002))	
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2002 AMC 12B Problems/Problem 17

The following problem is from both the 2002 AMC 12B #17 and 2002 AMC 10B #21, so both problems redirect to this page.

Problem

Andy's lawn has twice as much area as Beth's lawn and three times as much area as Carlos' lawn. Carlos' lawn mower cuts half as fast as Beth's mower and one third as fast as Andy's mower. If they all start to mow their lawns at the same time, who will finish first?

$\mathrm{(A)}\ \text{Andy} \qquad\mathrm{(B)}\ \text{Beth} \qquad\mathrm{(C)}\ \text{Carlos} \qquad\mathrm{(D)}\ \text{Andy and \ Carlos tie for first.} \qquad\mathrm{(E)}\ \text{All three tie.}$

Solution

We say Andy's lawn has an area of x. Beth's lawn thus has an area of $\frac{x}{2}$, and Carlos's lawn has an area of $\frac{x}{3}$.

We say Andy's lawn mower cuts at a speed of y. Carlos's cuts at a speed of $\frac{y}{3}$, and Beth's cuts at a speed $\frac{2y}{3}$.

Each person's lawn is cut at a speed of $\frac{\text{area}}{\text{rate}}$, so Andy's is cut in $\frac{x}{y}$ time, as is Carlos's. Beth's is cut in $\frac{3}{4}\times\frac{x}{y}$, so the first one to finish is $\boxed{\mathrm{(B)}\ \text{Beth}}$.

See also

2002 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2002))	
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Category: Introductory Algebra Problems

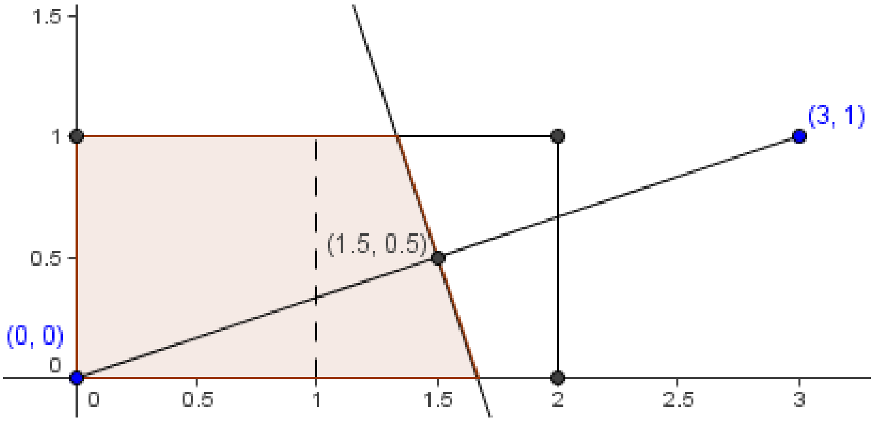
2002 AMC 12B Problems/Problem 18

Problem

A point P is randomly selected from the rectangular region with vertices $(0,0),(2,0),(2,1),(0,1)$. What is the probability that P is closer to the origin than it is to the point $(3,1)$?

$\text{(A)}\ \frac{1}{2} \qquad \text{(B)}\ \frac{2}{3} \qquad \text{(C)}\ \frac{3}{4} \qquad \text{(D)}\ \frac{4}{5} \qquad \text{(E)}\ 1$

Solution



The region containing the points closer to $(0,0)$ than to $(3,1)$ is bounded by the perpendicular bisector of the segment with endpoints $(0,0),(3,1)$. The perpendicular bisector passes through midpoint of $(0,0),(3,1)$, which is $\left(\frac{3}{2},\frac{1}{2}\right)$, the center of the unit square with coordinates $(1,0),(2,0),(2,1),(1,1)$. Thus, it cuts the unit square into two equal halves of area $1/2$. The total area of the rectangle is 2, so the area closer to the origin than to $(3,1)$ and in the rectangle is $2-\frac{1}{2}=\frac{3}{2}$. The probability is $\frac{3/2}{2}=\frac{3}{4}\Rightarrow \text{(C)}$.

See also

2002 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2002))	
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All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2002 AMC 12B Problems/Problem 19

Problem

If $a, b,$ and c are positive real numbers such that $a(b+c) = 152, b(c+a) = 162,$ and $c(a+b) = 170$, then abc is

$\mathrm{(A)}\ 672\qquad \mathrm{(B)}\ 688\qquad \mathrm{(C)}\ 704\qquad \mathrm{(D)}\ 720\qquad \mathrm{(E)}\ 750$

Solution

Adding up the three equations gives $2(ab + bc + ca) = 152 + 162 + 170 = 484 \Longrightarrow ab + bc + ca = 242$. Subtracting each of the above equations from this yields, respectively, $bc = 90, ca = 80, ab = 72$. Taking their product, $ab \cdot bc \cdot ca = a^2b^2c^2 = 90 \cdot 80 \cdot 72 = 720^2 \Longrightarrow abc = \boxed{720} \Rightarrow \mathrm{(D)}$.

See also

2002 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2002))	
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2002 AMC 12B Problems/Problem 20

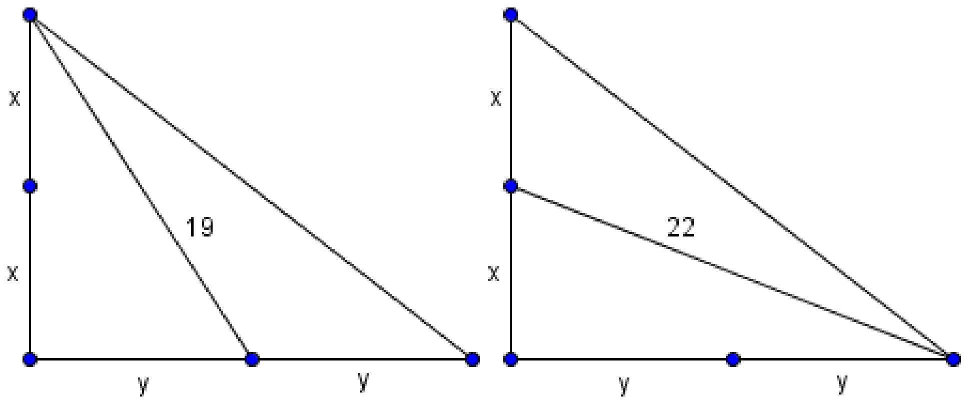
The following problem is from both the 2002 AMC 12B #20 and 2002 AMC 10B #22, so both problems redirect to this page.

Problem

Let $\triangle XOY$ be a right-angled triangle with $m\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.

$\mathrm{(A)}\ 24 \qquad \mathrm{(B)}\ 26 \qquad \mathrm{(C)}\ 28 \qquad \mathrm{(D)}\ 30 \qquad \mathrm{(E)}\ 32$

Solution



Let $OM = x$, $ON = y$. By the Pythagorean Theorem on $\triangle XON, MOY$ respectively,

$$(2x)^2 + y^2 = 19^2$$
$$x^2 + (2y)^2 = 22^2$$

Summing these gives $5x^2 + 5y^2 = 845 \Longrightarrow x^2 + y^2 = 169$.

By the Pythagorean Theorem again, we have

$$(2x)^2 + (2y)^2 = XY^2 \Longrightarrow XY = \sqrt{4(x^2+y^2)} = \sqrt{4(169)} = \sqrt{676} = \boxed{\mathrm{(B)}\ 26}$$

See also

2002 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2002))	
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2002 AMC 12B Problems/Problem 21

Problem

For all positive integers n less than 2002, let

$$a_n = \begin{cases} 11, & \text{if } n \text{ is divisible by } 13 \text{ and } 14; \\ 13, & \text{if } n \text{ is divisible by } 14 \text{ and } 11; \\ 14, & \text{if } n \text{ is divisible by } 11 \text{ and } 13; \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $\sum_{n=1}^{2001} a_n$.

$\mathrm{(A)}\ 448 \qquad\mathrm{(B)}\ 486 \qquad\mathrm{(C)}\ 1560 \qquad\mathrm{(D)}\ 2001 \qquad\mathrm{(E)}\ 2002$

Solution

Since $2002 = 11 \cdot 13 \cdot 14$, it follows that

$$a_n = \begin{cases} 11, & \text{if } n = 13 \cdot 14 \cdot k, & k = 1, 2, \cdots 10; \\ 13, & \text{if } n = 14 \cdot 11 \cdot k, & k = 1, 2, \cdots 12; \\ 14, & \text{if } n = 11 \cdot 13 \cdot k, & k = 1, 2, \cdots 13; \end{cases}$$

Thus $\sum_{n=1}^{2001} a_n = 11 \cdot 10 + 13 \cdot 12 + 14 \cdot 13 = 448 \Rightarrow \mathrm{(A)}$.

See also

2002 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2002))	
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2002 AMC 12B Problems/Problem 22

Problem

For all integers n greater than 1, define $a_n = \dfrac{1}{\log_n 2002}$. Let $b = a_2 + a_3 + a_4 + a_5$ and $c = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}$. Then $b - c$ equals

$\mathrm{(A)}\ -2 \qquad \mathrm{(B)}\ -1 \qquad \mathrm{(C)}\ \frac{1}{2002} \qquad \mathrm{(D)}\ \frac{1}{1001} \qquad \mathrm{(E)}\ \frac{1}{2}$

Solution

By the change of base formula, $a_n = \dfrac{1}{\frac{\log 2002}{\log n}} = \left(\dfrac{1}{\log 2002}\right) \log n$. Thus

$$\begin{aligned}
b - c &= \left(\frac{1}{\log 2002}\right)(\log 2 + \log 3 + \log 4 + \log 5 - \log 10 - \log 11 - \log 12 - \log 13 - \log 14) \\
&= \left(\frac{1}{\log 2002}\right)\left(\log \frac{2 \cdot 3 \cdot 4 \cdot 5}{10 \cdot 11 \cdot 12 \cdot 13 \cdot 14}\right) \\
&= \left(\frac{1}{\log 2002}\right) \log 2002^{-1} = -\left(\frac{\log 2002}{\log 2002}\right) = -1 \Rightarrow \mathrm{(B)}
\end{aligned}$$

See also

2002 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2002))	
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Category: Introductory Algebra Problems

2002 AMC 12B Problems/Problem 23

Contents

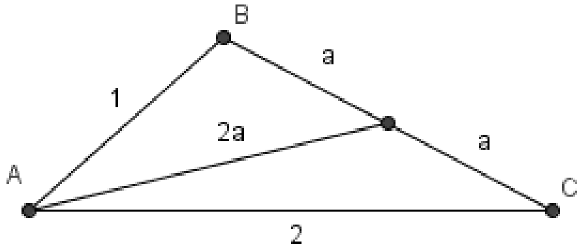
- 1 Problem
- 2 Solution
- 3 Alternate Solution
- 4 See also

Problem

In $\triangle ABC$, we have $AB = 1$ and $AC = 2$. Side $\overline{BC}$ and the median from A to $\overline{BC}$ have the same length. What is BC?

$$\mathrm{(A)}\ \frac{1+\sqrt{2}}{2} \qquad \mathrm{(B)}\ \frac{1+\sqrt{3}}{2} \qquad \mathrm{(C)}\ \sqrt{2} \qquad \mathrm{(D)}\ \frac{3}{2} \qquad \mathrm{(E)}\ \sqrt{3}$$

Solution



Let D be the foot of the median from A to $\overline{BC}$, and we let $AD = BC = 2a$. Then by the Law of Cosines on $\triangle ABD, \triangle ACD$, we have

$$1^2 = a^2 + (2a)^2 - 2(a)(2a)\cos ADB$$
$$2^2 = a^2 + (2a)^2 - 2(a)(2a)\cos ADC$$

Since $\cos ADC = \cos(180 - ADB) = -\cos ADB$, we can add these two equations and get

$$5 = 10a^2$$

Hence $a = \frac{1}{\sqrt{2}}$ and $BC = 2a = \sqrt{2} \Rightarrow \mathrm{(C)}$.

Alternate Solution

From Stewart's Theorem, we have $(2)(1/2)a(2) + (1)(1/2)a(1) = (a)(a)(a) + (1/2)a(a)(1/2)a$. Simplifying, we get $(5/4)a^3 = (5/2)a \implies (5/4)a^2 = 5/2 \implies a^2 = 2 \implies a = \boxed{\sqrt{2}}$.

See also

2002 AMC 12B Problems/Problem 24

Problem

A convex quadrilateral $ABCD$ with area 2002 contains a point P in its interior such that $PA = 24, PB = 32, PC = 28, PD = 45$. Find the perimeter of $ABCD$.

$\mathrm{(A)}\ 4\sqrt{2002} \qquad \mathrm{(B)}\ 2\sqrt{8465} \qquad \mathrm{(C)}\ 2(48+\sqrt{2002}) \qquad \mathrm{(D)}\ 2\sqrt{8633} \qquad \mathrm{(E)}\ 4(36 + \sqrt{113})$

Solution

We have

$$[ABCD] = 2002 \le \frac{1}{2}(AC \cdot BD)$$

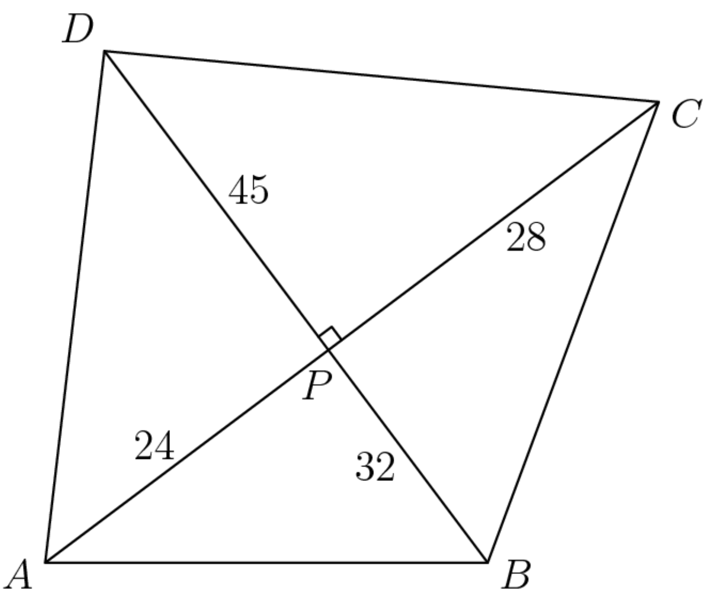
(This is true for any convex quadrilateral: split the quadrilateral along AC and then using the triangle area formula to evaluate $[ACB]$ and $[ACD]$), with equality only if $\overline{AC} \perp \overline{BD}$. By the triangle inequality,

$$\begin{aligned} AC &\le PA + PC = 52 \\ BD &\le PB + PD = 77 \end{aligned}$$

with equality if P lies on $\overline{AC}$ and $\overline{BD}$ respectively. Thus

$$2002 \le \frac{1}{2}AC \cdot BD \le \frac{1}{2} \cdot 52 \cdot 77 = 2002$$

Since we have the equality case, $\overline{AC} \perp \overline{BD}$ at point P, as shown below.



By the Pythagorean Theorem,

$$AB = \sqrt{PA^2 + PB^2} = \sqrt{24^2 + 32^2} = 40$$
$$BC = \sqrt{PB^2 + PC^2} = \sqrt{32^2 + 28^2} = 4\sqrt{113}$$
$$CD = \sqrt{PC^2 + PD^2} = \sqrt{28^2 + 45^2} = 53$$
$$DA = \sqrt{PD^2 + PA^2} = \sqrt{45^2 + 24^2} = 51$$

The perimeter of $ABCD$ is $AB + BC + CD + DA = 4(36 + \sqrt{113}) \Rightarrow \text{(E)}$.

See also

2002 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2002))	
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2002 AMC 12B Problems/Problem 25

Problem

Let $f(x) = x^2 + 6x + 1$, and let R denote the set of points (x, y) in the coordinate plane such that

$$f(x) + f(y) \le 0 \qquad \text{and} \qquad f(x) - f(y) \le 0$$

The area of R is closest to

$$\mathrm{(A)}\ 21 \qquad \mathrm{(B)}\ 22 \qquad \mathrm{(C)}\ 23 \qquad \mathrm{(D)}\ 24 \qquad \mathrm{(E)}\ 25$$

Solution

The first condition gives us that

$$x^2 + 6x + 1 + y^2 + 6y + 1 \le 0 \Longrightarrow (x+3)^2 + (y+3)^2 \le 16$$

which is a circle centered at $(-3, -3)$ with radius 4. The second condition gives us that

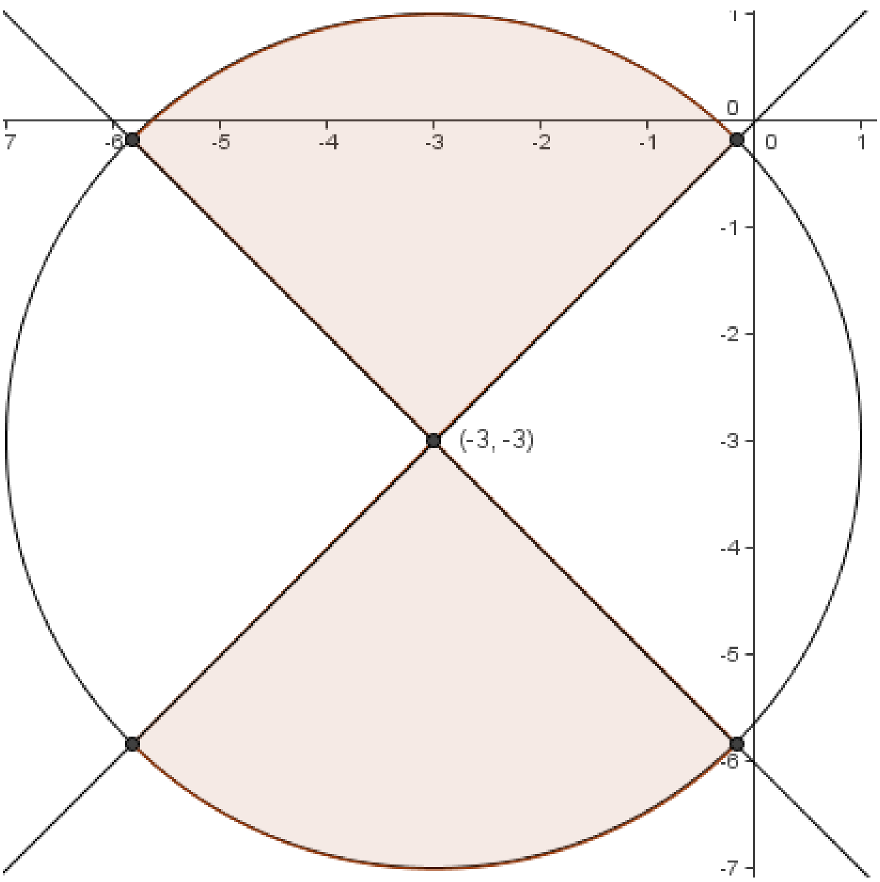
$$x^2+6x+1-y^2-6y-1 \le 0 \Longrightarrow (x^2-y^2)+6(x-y) \le 0 \Longrightarrow (x-y)(x+y+6) \le 0$$

Thus either

$$x - y \ge 0, \quad x + y + 6 \le 0$$

or

$$x - y \le 0, \quad x + y + 6 \ge 0$$



Each of those lines passes through $(-3,-3)$ and has slope ± 1, as shown above. Therefore, the area of R is half of the area of the circle, which is $\frac{1}{2}(\pi \cdot 4^2) = 8\pi \approx 25 \Rightarrow \mathrm{(E)}$.

See also

2002 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2002))	
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