2022 AMC 10A Solutions

Problem1

What is the value of $$3+\frac{1}{3+\frac{1}{3+\frac13}}?$$

$\textbf{(A) }\frac{31}{10}\qquad\textbf{(B) }\frac{49}{15}\qquad\textbf{(C) }\frac{33}{10}\qquad\textbf{(D) }\frac{109}{33}\qquad\textbf{(E) }\frac{15}{4}$

Solution 1

We have $$\begin{aligned}3+\frac{1}{3+\frac{1}{3+\frac13}} &= 3+\frac{1}{3+\frac{1}{\left(\frac{10}{3}\right)}} \\ &= 3+\frac{1}{3+\frac{3}{10}} \\ &= 3+\frac{1}{\left(\frac{33}{10}\right)} \\ &= 3+\frac{10}{33} \\ &= \boxed{\textbf{(D) }\frac{109}{33}}.\end{aligned}$$

Solution 2

Continued fractions are expressed as $\dfrac{[q_0, q_1, q_2, \ldots, q_n]}{[q_1, q_2, \ldots, q_n]}$ where

$$\begin{aligned}
[q_0, q_1, q_2, \ldots, q_n] &= q_0[q_1, q_2, \ldots, q_n] + [q_2, \ldots, q_n] \\
[3] &= 3 \\
[3, 3] &= 3(3) + 1 = 10 \\
[3, 3, 3] &= 3(10) + 3 = 33 \\
[3, 3, 3, 3] &= 3(33) + 10 = 109 \\
\frac{[q_0, q_1, q_2, \ldots, q_n]}{[q_1, q_2, \ldots, q_n]} &= \frac{[3, 3, 3, 3]}{[3, 3, 3]} \\
&= \boxed{\textbf{(D) } \frac{109}{33}}
\end{aligned}$$

Problem2

Mike cycled 15 laps in 57 minutes. Assume he cycled at a constant speed throughout. Approximately how many laps did he complete in the first 27 minutes?

$\textbf{(A) } 5 \qquad \textbf{(B) } 7 \qquad \textbf{(C) } 9 \qquad \textbf{(D) } 11 \qquad \textbf{(E) } 13$

Solution 1

Mike's speed is $\dfrac{15}{57} = \dfrac{5}{19}$ laps per minute.

In the first 27 minutes, he completed approximately $\dfrac{5}{19} \cdot 27 \approx \dfrac{1}{4} \cdot 28 = \boxed{\textbf{(B) } 7}$ laps.

Solution 2

Mike runs 1 lap in $\frac{57}{15} = \frac{19}{5}$ minutes. So, in 27 minutes, Mike ran about $\frac{27}{\frac{19}{5}} \approx \boxed{\textbf{(B)}\ 7}$ laps.

Solution 3

Mike's rate is $\frac{15}{57} = \frac{x}{27}$, where x is the number of laps he can complete in 27 minutes. If you cross multiply, $57x = 405$.

So, $x = \frac{405}{57} \approx \boxed{\textbf{(B)}\ 7}$.

Solution 4 (Quick Estimate)

Note that 27 minutes is a little bit less than half of 57 minutes. Mike will therefore run a little bit less than $15/2 = 7.5$ laps, which is about $\boxed{\textbf{(B)}\ 7}$.

Problem3

The sum of three numbers is $96.$ The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 4 \qquad \textbf{(E) } 5$

Solution

Let x be the third number. It follows that the first number is $6x,$ and the second number is $x + 40.$

We have $6x + (x + 40) + x = 8x + 40 = 96$,from which $x = 7$.

Therefore, the first number is 42, and the second number is 47. Their absolute value of the difference is $|42 - 47| = \boxed{\textbf{(E) } 5}$.

Solution 2

Solve this using a system of equations. Let x, y, and z be the three numbers, respectively. We get three equations:$x + y + z = 96$$x = 6z$

$z = y - 40$Rewriting the third equation gives us $y = z + 40$, so we can substitute x as $6z$ and y as $z + 40$.

Therefore, we get$6z + (z + 40) + z = 96$$8z + 40 = 96$

$8z = 56$$z = 7$

Substituting 7 in for z gives us$x = 6z = 6(7) = 42$

and $y = z + 40 = 7 + 40 = 47$

So $|x - y| = |42 - 47| = \boxed{\textbf{(E) } 5}$

Problem4

In some countries, automobile fuel efficiency is measured in liters per 100 kilometers while other countries use miles per gallon. Suppose that 1 kilometer equals m miles, and 1 gallon equals l liters. Which of the following gives the fuel efficiency in liters per 100 kilometers for a car that gets x miles per gallon?

$\textbf{(A) } \frac{x}{100lm} \qquad \textbf{(B) } \frac{xlm}{100} \qquad \textbf{(C) } \frac{lm}{100x} \qquad \textbf{(D) } \frac{100}{xlm} \qquad \textbf{(E) } \frac{100lm}{x}$

Solution 1

The formula for fuel efficiency is $\frac{\text{Distance}}{\text{Gas Consumption}}.$ Note that 1 mile equals $\frac{1}{m}$ kilometers. We have
$$\frac{x\text{ miles}}{1\text{ gallon}} = \frac{\frac{x}{m}\text{ kilometers}}{l\text{ liters}} = \frac{1\text{ kilometer}}{\frac{lm}{x}\text{ liters}} = \frac{100\text{ kilometers}}{\frac{100lm}{x}\text{ liters}}.$$
Therefore, the answer is $\boxed{\textbf{(E)}\ \frac{100lm}{x}}.$

Solution 2

Since it can be a bit odd to think of "liters per 100 km", this statement's numerical value is equivalent to 100 km per 1 liter:

1 km requires l liters, so the numerator is simply l. Since l liters is 1 gallon, and x miles is 1 gallon, we have $1\text{ liter} = \frac{x}{l}$.

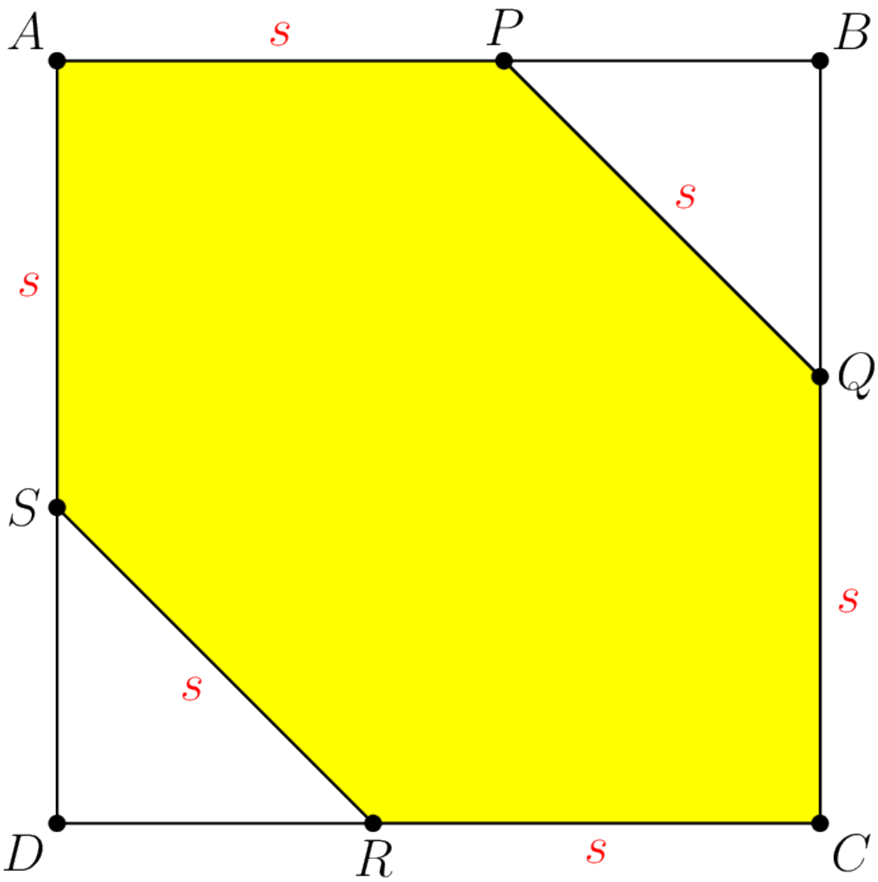
Therefore, the requested expression is $100 \cdot \frac{m}{\left(\frac{x}{l}\right)} = \boxed{\textbf{(E)}\ \frac{100lm}{x}}.$

Problem5

Square $ABCD$ has side length 1. Points P, Q, R, and S each lie on a side of $ABCD$ such that $APQCRS$ is an equilateral convex hexagon with side length S. What is S?

$\textbf{(A)}\ \frac{\sqrt{2}}{3} \qquad \textbf{(B)}\ \frac{1}{2} \qquad \textbf{(C)}\ 2-\sqrt{2} \qquad \textbf{(D)}\ 1-\frac{\sqrt{2}}{4} \qquad \textbf{(E)}\ \frac{2}{3}$

Diagram



Solution

Note that $BP = BQ = DR = DS = 1 - s.$ It follows that $\triangle BPQ$ and $\triangle DRS$ are isosceles right triangles.

In $\triangle BPQ$, we have $PQ = BP\sqrt{2}$, or

$$\begin{aligned} s &= (1-s)\sqrt{2} \\ s &= \sqrt{2} - s\sqrt{2} \\ \left(\sqrt{2}+1\right) s &= \sqrt{2} \\ s &= \frac{\sqrt{2}}{\sqrt{2}+1}. \end{aligned}$$

Therefore, the answer is

$$s = \frac{\sqrt{2}}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = \boxed{\textbf{(C) } 2-\sqrt{2}}.$$

Solution 2

Since it is an equilateral convex hexagon, all sides are the same, so we will call the side length x. Notice that $(1-x)^2 \cdot (1-x)^2 = x^2$. We can solve this equation which gives us our answer.

$$\begin{aligned} 1 + x^2 - 2x + 1 + x^2 - 2x &= x^2 \\ 2x^2 - 4x + 2 &= x^2 \\ x^2 - 4x + 2 &= 0 \end{aligned}$$

We then use the quadratic formula which gives us:

$$\begin{aligned} x &= \frac{4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} \\ &= \frac{4 \pm \sqrt{8}}{2} \\ &= \frac{4 \pm 2\sqrt{2}}{2} \end{aligned}$$

Then we simplify it by dividing and crossing out 2 which gives us $2 \pm \sqrt{2}$ and that gives us $\boxed{\textbf{(C) } 2-\sqrt{2}}$.

Solution 3 (Area)

We can find areas in terms of s. From the diagram, draw in segments SP and RQ. We then have two non-shaded triangles, two shaded triangles, and a rectangle.

The non-shaded triangles have leg lengths of $1-s,$ so they each have area $\frac{(1-s)^2}{2}.$ Therefore, the total area of the two triangles is $(1-s)^2.$

The shaded triangles have side lengths $s,$ so they each have area $\frac{s^2}{2}.$ Then, we get that their combined area is s^2.

Looking at the rectangle, we find that $SP=RQ=s\sqrt{2},$ from 45-45-90 triangles APS and CRQ. Multiplying this with the other side length $s,$ we see that the rectangle has area $s^2\sqrt{2}.$

These three expressions of area sum up to the big square, which has area 1. So, we add them up and solve:

$$\begin{aligned}(1-s)^2+s^2+s^2\sqrt{2}&=1\\ s^2-2s+1+s^2+s^2\sqrt{2}&=1\\ 2s^2+s^2\sqrt{2}-2s&=0\\ s((2+\sqrt{2})s-2)&=0\end{aligned}$$

s cannot be 0, since it represents a positive side length. This means that s satisfies $(2+\sqrt{2})s-2=0.$ Solving, we see that

$$s=\frac{2}{2+\sqrt{2}}=\frac{2}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}}=\frac{4-2\sqrt{2}}{2}=\boxed{\textbf{(C) } 2-\sqrt{2}}.$$

Problem6

Which expression is equal to $\left|a-2-\sqrt{(a-1)^2}\right|$ for $a<0?$

$\textbf{(A) } 3-2a \qquad \textbf{(B) } 1-a \qquad \textbf{(C) } 1 \qquad \textbf{(D) } a+1 \qquad \textbf{(E) } 3$

Solution 1

We have

$$\begin{aligned} \left|a-2-\sqrt{(a-1)^2}\right| &= |a-2-|a-1|| \\ &= |a-2-(1-a)| \\ &= |2a-3| \\ &= \boxed{\textbf{(A) } 3-2a}. \end{aligned}$$

Solution 2

Assume that $a=-1.$ Then, the given expression simplifies to 5:

$$\begin{aligned} \left|a-2-\sqrt{(a-1)^2}\right| &= \left|-1-2-\sqrt{(-1-1)^2}\right| \\ &= \left|-1-2-\sqrt{4}\right| \\ &= |-1-2-2| \\ &= 5. \end{aligned}$$

Then, we test each of the answer choices to see which one is equal to 5:

$\textbf{(A) } 3-2a = 3-2\cdot(-1) = 3+2 = 5.$

$\textbf{(B) } 1-a = 1-(-1) = 2 \neq 5.$

$\textbf{(C) } 1 \neq 5.$

$\textbf{(D)}\ a+1=-1+1=0\neq 5.$

$\textbf{(E)}\ 3\neq 5.$

The only answer choice equal to 5 for $a=-1$ is $\boxed{\textbf{(A)}\ 3-2a}$.

Problem7

The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 9 \qquad \textbf{(E)}\ 12$

Solution 1

$$\begin{aligned} 18 &= 2\cdot 3^2, \\ 180 &= 2^2\cdot 3^2\cdot 5, \\ 45 &= 3^2\cdot 5 \end{aligned}$$

Note that $15=3\cdot 5.$ From the least common multiple condition, we conclude that $n=2^2\cdot 3^k\cdot 5,$ where $k\in\{0,1,2\}.$

From the greatest common divisor condition, we conclude that $k=1.$

Therefore, we have $n=2^2\cdot 3^1\cdot 5=60.$ The sum of its digits is $6+0=\boxed{\textbf{(B)}\ 6}$.

Solution 2

Since the lcm contains only factors of $2, 3$, and 5, n cannot be divisible by any other prime. Let n = $2^a\ 3^b\ 5^c$, where a ,b, and c are nonnegative integers. We know that $lcm(n,18)$ = $lcm(n,3^2*2)$ =

$lcm(2^a*3^b*5^c,3^2*2)$ = $lcm(2^{max(a,1)}\cdot 3^{max(a,2)}\cdot 5^b)$

= 180 = $2^2\cdot 3^2\cdot 5$. Thus

(1) $max(a,1)$ = 2 so $a=2$
(2) $max(a,2)$ = 2 so $0\le a\le 2$
(3) $c=1$.

From the gcf information, $gcf(n,45)$ = gcf(n, $3^2\cdot 5$)

= $gcf(2^a\cdot 3^b\cdot 5^c,3^2\cdot 5)$ = $3^{min(b,2)}\cdot 5^{min(c,1)}=15$ This

means, that since $c=1$, $3^{min(b,2)}\cdot 5=15$,

so $min(b,2)$ = 1 and $b=1$. Hence, multiplying

using $a=2$, $b=1$, $c=1$ gives $n=60$ and the sum of digits is

hence $\boxed{\textbf{(B) } 6}$.

Problem8

A data set consists of 6 (not distinct) positive integers: 1, 7, 5, 2, 5, and X. The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all positive values of X?

$\textbf{(A) } 10 \qquad \textbf{(B) } 26 \qquad \textbf{(C) } 32 \qquad \textbf{(D) } 36 \qquad \textbf{(E) } 40$

Solution (Casework)

First, note that $1+7+5+2+5=20$. There are 3 possible cases:

Case 1: the mean is 5.

$X=5\cdot 6-20=10$.

Case 2: the mean is 7.

$$X = 7 \cdot 6 - 20 = 22.$$

Case 3: the mean is X.

$$\frac{20 + X}{6} = X \Rightarrow X = 4.$$

Hence, the answer is $10 + 22 + 4 = \boxed{\textbf{(D) } 36}$

Problem9

A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?

$\textbf{(A) } 120 \qquad \textbf{(B) } 270 \qquad \textbf{(C) } 360 \qquad \textbf{(D) } 540 \qquad \textbf{(E) } 720$

Solution 1

The top left rectangle can be 5 possible colors. Then the bottom left region can only be 4 possible colors, and the bottom middle can only be 3colors since it is next to the top left and bottom left. Similarly, we have 3 choices for the top right and 3 choices for the bottom right, which gives us a total of $5 \cdot 4 \cdot 3 \cdot 3 \cdot 3 = \boxed{\textbf{(D) } 540}$.

Solution 2 (casework)

Case 1: All the rectangles are different colors. It would be $5! = 120$ choices.

Case 2: Two rectangles that are the same color. Grouping these two rectangles as one gives us $5 \cdot 4 \cdot 3 \cdot 2 = 120$. But, you need to multiply this number by three because the same-colored rectangles can be chosen at the top left and bottom right, the top right and bottom left, or the bottom right and bottom left, which gives us a grand total of 360.

Case 3: We have two sets of rectangles chosen from these choices (top right & bottom left, top left & bottom right) that have the same color. However, the choice of the bottom left and bottom right does not work for this case, as the second pair would be chosen from two touching rectangles. Again, grouping the same-colored rectangles gives us $5 \cdot 4 \cdot 3 = 60$.

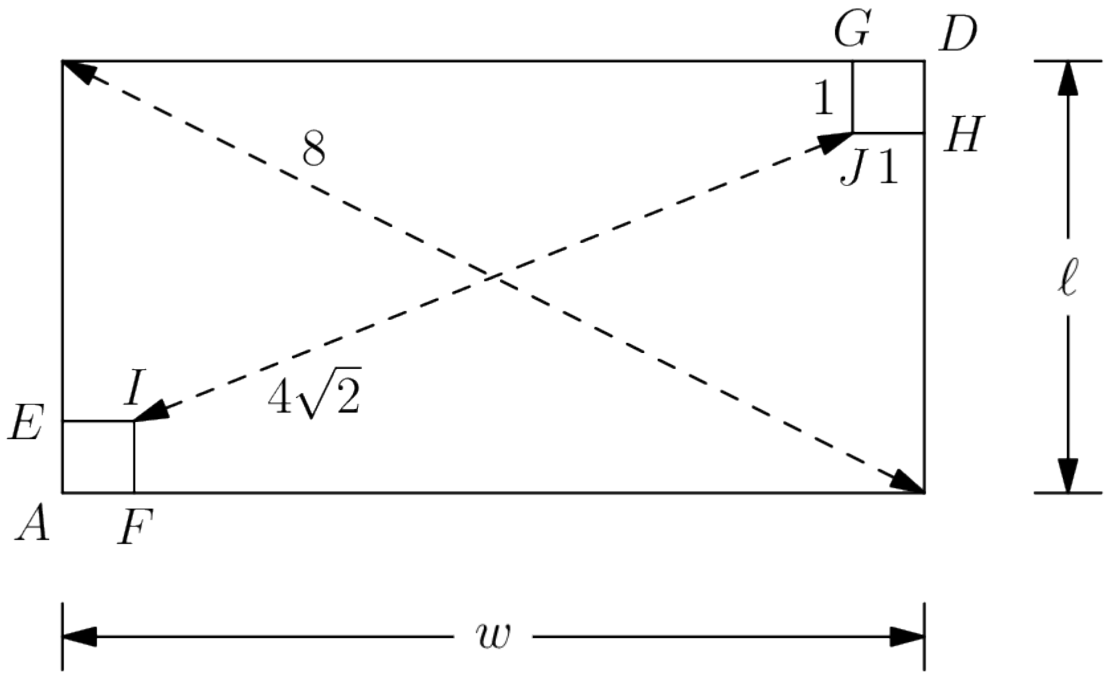
Therefore, we have $120 + 360 + 60 = \boxed{\textbf{(D) } 540}$.

Problem10

Daniel finds a rectangular index card and measures its diagonal to be 8 centimeters. Daniel then cuts out equal squares of side 1 cm at two opposite corners of the index card and measures the distance between the two closest vertices of these squares to be centimeters, as shown below. What is the area of the original index card?

$\textbf{(A) } 14 \qquad \textbf{(B) } 10\sqrt{2} \qquad \textbf{(C) } 16 \qquad \textbf{(D) } 12\sqrt{2} \qquad \textbf{(E) } 18$

Solution 1 (Coordinate Geometry)



We will use coordinates here. Label the bottom left corner of the larger rectangle(without the square cut out) as $A = (0, 0)$ and the top right as $D = (w, \ell)$, where w is the width of the rectangle and ℓ is the length. Now we have vertices $E = (0, 1), F = (1, 0), G = (w - 1, \ell)$, and $H = (w, \ell - 1)$ as vertices of the irregular octagon created by cutting out the squares. Label $I = (1, 1)$ and $J = (w - 1, \ell - 1)$ as the two closest vertices formed by the squares. The distance between the two closest vertices of the squares is thus $IJ = \left(4\sqrt{2}\right)^2$. Substituting, we get

$$IJ^2 = (w-2)^2 + (\ell-2)^2 = \left(4\sqrt{2}\right)^2 = 32 \implies w^2 + \ell^2 - 4w - 4\ell = 24.$$

Using the fact that the diagonal of the rectangle is 8, we get $w^2 + \ell^2 = 64.$ Subtracting the first equation from the second equation, we get $4w + 4\ell = 40 \implies w + \ell = 10.$ Squaring yields $w^2 + 2w\ell + \ell^2 = 100.$ Subtracting the second equation from this, we get $2w\ell = 36,$ and thus area of the original rectangle is $w\ell = \boxed{\textbf{(E) } 18}$.

Solution 2 (Algebra)

Let the dimensions of the index card be x and y. We have $x^2 + y^2 = 64$ and $(x-1)^2 + (y-1)^2 = 32$ by Pythagoras. Subtracting, we obtain

$$x^2 - (x^2 - 4x + 4) + y^2 - (y^2 - 4y + 4) = 4x - 4 + 4y - 4 = 32$$

. Thus, we have $x + y - 2 = 8$, so $x + y = 10$.

This means that $(x+y)^2 = x^2 + 2xy + y^2 = 100$.

Subtracting $x^2 + y^2$ from this, we get $2xy = 36$ and so $xy = 18$.

The area of the index card is thus $\boxed{\textbf{(E) } 18}$.

Problem11

Ted mistakenly wrote $2^m \cdot \sqrt{\frac{1}{4096}}$ as $2 \cdot \sqrt[m]{\frac{1}{4096}}$. What is the sum of all real numbers m for which these two expressions have the same value?

$\textbf{(A) } 5 \qquad \textbf{(B) } 6 \qquad \textbf{(C) } 7 \qquad \textbf{(D) } 8 \qquad \textbf{(E) } 9$

Solution 1

We are given that $2^m \cdot \sqrt{\frac{1}{4096}} = 2 \cdot \sqrt[m]{\frac{1}{4096}}$. Converting everything into powers of $2,$ we have

$$2^m \cdot (2^{-12})^{\frac{1}{2}} = 2 \cdot (2^{-12})^{\frac{1}{m}}$$
$$2^{m-6} = 2^{1-\frac{12}{m}}$$
$$m-6 = 1 - \frac{12}{m}.$$

We multiply both sides by m, then rearrange as $m^2 - 7m + 12 = 0$. By Vieta's Formulas, the sum of such values of m is $\boxed{\textbf{(C) } 7}$.

Note that $m = 3$ or $m = 4$ from the quadratic equation above.

Solution 2

Note that m can only be 1, 2, 3, 4, 6, and 12. $\sqrt{\frac{1}{4096}} = \frac{1}{64}$. Testing out m, we see that only 3 and 4 work. Hence, $3 + 4 = \boxed{\textbf{(C) } 7}$.

Problem12

On Halloween 31 children walked into the principal's office asking for candy.

They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

$\textbf{(A)}\ 7 \qquad \textbf{(B)}\ 12 \qquad \textbf{(C)}\ 21 \qquad \textbf{(D)}\ 27 \qquad \textbf{(E)}\ 31$

Solution

Consider when the principal asks "Are you a liar?": The truth tellers truthfully say no, and the liars lie and say no. This leaves only alternaters who lie on this question to answer yes. Thus, all 9 children that answered yes are alternaters that falsely answer question 1 and 3, and truthfully answer question 2. The rest of the alternaters, however many there are, have the opposite behavior.

Consider the second question, "Are you an alternater?": The truth tellers again answer no, the liars falsely answer yes, and alternaters that truthfully answer also say yes. From the previous part, we know that 9 alternaters truthfully answer here. Because only liars and 9 alternaters answer yes, we can deduce that there are $15-9=6$ liars.

Consider the first question, "Are you a truth teller?": Truth tellers say yes, liars also say yes, and alternaters that lie on this question also say yes. From the first part, we know that 9 alternaters lie here. From the previous part, we know that there are 6 liars. Because only the number of truth tellers is unknown here, we can deduce that there are $22-9-6=7$ truth tellers.

The final question is how many pieces of candy did the principal give to truth tellers. Because truth tellers only answer yes on the first question, we know that all 7 of them said yes once, resulting in $\boxed{\textbf{(A)}\ 7}$ pieces of candy.

Solution 2

On the first question, truth tellers would say yes. Liars would say yes because they are not truth tellers and thus will say the opposite of no. Some alternators may lie on this question too, meaning they say yes.

On the second question, Liars would say yes because they are not alternators and thus will say the opposite of no. Alternators only say yes to this question if and only if they said yes to the previous question. Thus, the difference between the amount of people that said yes first and that said yes second is the amount of truth tellers. $22-15=7$. Because it is obvious that truth tellers only say yes on the first question, our answer is $\boxed{\textbf{(A) } 7}$

Solution 3

You can write it into an equation, T as the truth tellers, L as the liars, x as the alternators who say T first, and y as the alternators who say L first. We have:
$$\begin{aligned} T+L+x+y&=22 \\ L+x+y&=15 \end{aligned}$$

We subtract the equations and get that $T=7$.

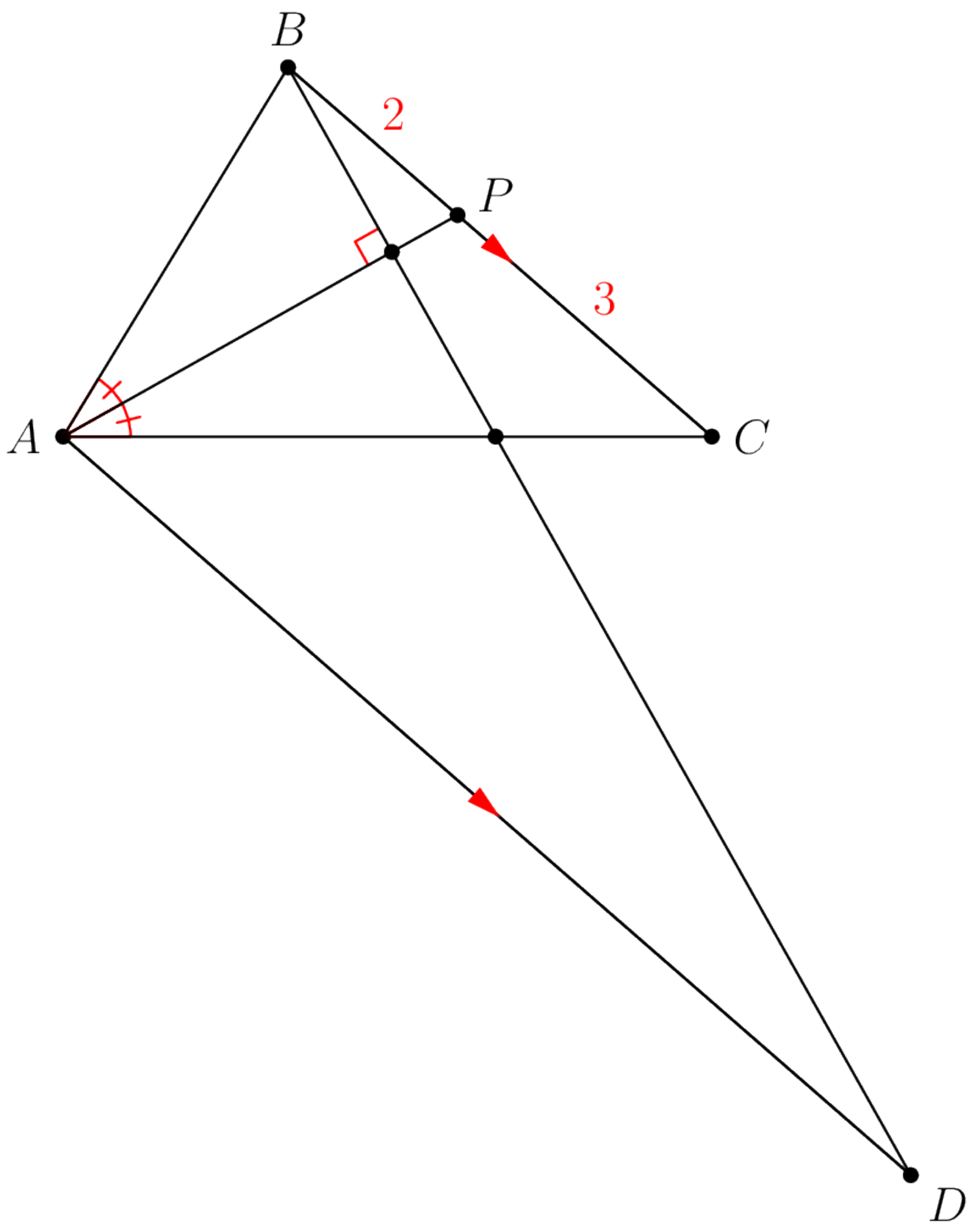
This is obviously the answer since truth tellers only tell the truth, therefore the answer is $\boxed{\textbf{(A) } 7}$.

Problem13

Let $\triangle ABC$ be a scalene triangle. Point P lies on $\overline{BC}$ so that $\overline{AP}$ bisects $\angle BAC.$ The line through B perpendicular to $\overline{AP}$ intersects the line through A parallel to $\overline{BC}$ at point $D.$ Suppose $BP=2$ and $PC=3.$ What is $AD?$

$\textbf{(A) } 8 \qquad \textbf{(B) } 9 \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 11 \qquad \textbf{(E) } 12$

Diagram



Solution 1 (Angle Bisector Theorem and Similar Triangles)

Suppose that $\overline{BD}$ intersect $\overline{AP}$ and $\overline{AC}$ at X and $Y,$ respectively. By Angle-Side-Angle, we conclude that $\triangle ABX \cong \triangle AYX.$

Let $AB = AY = 2x.$ By the Angle Bisector Theorem, we have $AC = 3x,$ or $YC = x.$

By alternate interior angles, we get $\angle YAD = \angle YCB$ and $\angle YDA = \angle YBC.$ Note that $\triangle ADY \sim \triangle CBY$ by the Angle-Angle Similarity, with the ratio of similitude $\dfrac{AY}{CY} = 2.$ It follows that $AD = 2CB = 2(BP + PC) = \boxed{\textbf{(C) } 10}.$

Solution 2 (Auxiliary Lines)

Let the intersection of AC and BD be M, and the intersection of AP and BD be N. Draw a line from M to BC, and label the point of intersection O.

By adding this extra line, we now have many pairs of similar triangles. We have $\triangle BPN \sim \triangle BOM$, with a ratio of 2, so $BO = 4$ and $OC = 1$. We also have $\triangle COM \sim \triangle CAP$ with ratio 3. Additionally, $\triangle BPN \sim \triangle ADN$ (with an unknown ratio). It is also true that $\triangle BAN \cong \triangle MAN$.

Suppose the area of $\triangle COM$ is x. Then, $[\triangle CAP] = 9x$.

Because $\triangle CAP$ and $\triangle BAP$ share the same height and have a base ratio of $3:2$, $[\triangle BAP] = 6x$.

Because $\triangle BOM$ and $\triangle COM$ share the same height and have a base ratio of $4:1$, $[\triangle BOM] = 4x$, $[\triangle BPN] = x$, and thus $[OMNP] = 4x - x = 3x$.

Thus, $[\triangle MAN] = [\triangle BAN] = 5x$.

Finally, we have $\frac{[\triangle BAN]}{[\triangle BPN]} = \frac{5x}{x} = 5$, and because these triangles share the same height $\frac{AN}{PN} = 5$. Notice that these side lengths are corresponding side lengths of the similar triangles BPN and ADN. This means that $AD = 5 \cdot BP = \boxed{\textbf{(C) } 10}$.

Solution 3 (Slopes)

Let point B be the origin, with C being on the positive x-axis and A being in the first quadrant.

By the Angle Bisector Theorem, $AB : AC = 2 : 3$. Thus, assume that $AB = 4$, and $AC = 6$.

Let the perpendicular from A to BC be AM.

Using Heron's formula,

$$[ABC] = \sqrt{\frac{15}{2}\left(\frac{15}{2} - 4\right)\left(\frac{15}{2} - 5\right)\left(\frac{15}{2} - 6\right)} = \frac{15}{4}\sqrt{7}.$$

Hence, $AM = \frac{\frac{15}{4}\sqrt{7}}{\frac{5}{2}} = \frac{3}{2}\sqrt{7}.$

Next, we have $BM^2 + AM^2 = AB^2$

$$\therefore BM = \sqrt{16 - \frac{63}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2} \text{ and } PM = \frac{3}{2}.$$

The slope of line AP is thus $\frac{-\frac{3}{2}\sqrt{7}}{\frac{3}{2}} = -\sqrt{7}.$

Therefore, since the slopes of perpendicular lines have a product of -1, the slope of line BD is $\frac{1}{\sqrt{7}}$. This means that we can solve for the coordinates of D:

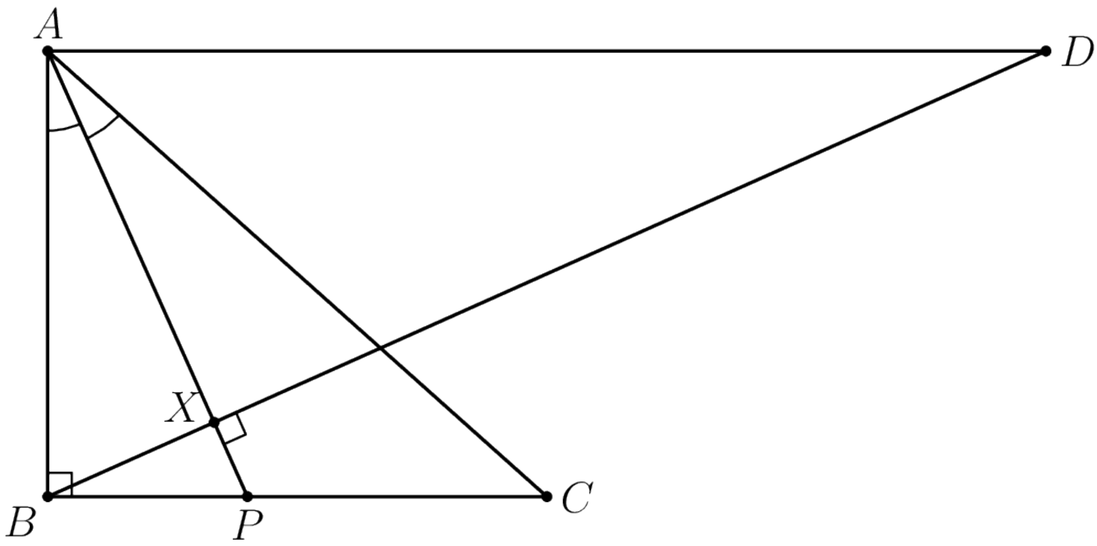
$$y = \frac{3}{2}\sqrt{7}y = \frac{1}{\sqrt{7}}x\frac{1}{\sqrt{7}}x = \frac{3}{2}\sqrt{7}x = \frac{7 \cdot 3}{2} = \frac{21}{2}$$
$$D = \left(\frac{21}{2}, \frac{3}{2}\sqrt{7}\right).$$

We also know that the coordinates of A are $\left(\frac{1}{2}, \frac{3}{2}\sqrt{7}\right)$,

because $BM = \frac{1}{2}$ and $AM = \frac{3}{2}\sqrt{7}$.

Since the y-coordinates of A and D are the same, and their x-coordinates differ by 10, the distance between them is 10. Our answer is $\boxed{\textbf{(C) } 10}$.

Solution 4 (Assumption)



Since there is only one possible value of AD, we assume $\angle B = 90^\circ$. By the angle bisector theorem, $\frac{AB}{AC} = \frac{2}{3}$,

so $AB = 2\sqrt{5}$ and $AC = 3\sqrt{5}$. Now observe that $\angle BAD = 90^\circ$. Let the intersection of BD and AP be X.

Then $\angle ABD = 90^\circ - \angle BAX = \angle APB$.

Consequently, $\triangle DAB \sim \triangle ABP$ and therefore $\frac{DA}{AB} = \frac{AB}{BP}$,

so $AD = \boxed{\textbf{(C) } 10}$, and we're done!

Problem14

How many ways are there to split the integers 1 through 14 into 7 pairs such that in each pair, the greater number is at least 2 times the lesser number?

$\textbf{(A) } 108 \qquad \textbf{(B) } 120 \qquad \textbf{(C) } 126 \qquad \textbf{(D) } 132 \qquad \textbf{(E) } 144$

Solution 1

Clearly, the integers from 8 through 14 must be in different pairs, and 7 must pair with $14.$

Note that 6 can pair with either 12 or $13.$ From here, we consider casework:

- If 6 pairs with $12,$ then 5 can pair with one of $10, 11, 13.$ After that, each of $1, 2, 3, 4$ does not have any restrictions. This case produces $3 \cdot 4! = 72$ ways.

- If 6 pairs with $13,$ then 5 can pair with one of $10, 11, 12.$ After that, each of $1, 2, 3, 4$ does not have any restrictions. This case produces $3 \cdot 4! = 72$ ways.

Together, the answer is $72 + 72 = \boxed{\textbf{(E) } 144}$.

Solution 2

As said above, clearly, the integers from 8 through 14 must be in different pairs. We know that 8 or 9 can pair with any integer from 1 to 4, 10 or 11 can pair with any integer from 1 to 5, and 12 or 13 can pair with any integer from 1 to 6. Thus, 8 will have 4 choices to pair with, 9 will then have 3 choices to pair with (9 cannot pair with the same number as the one 8 pairs with). 10 cannot pair with the numbers 8 and 9 has paired with but can also now pair with 5, so there are 3 choices. 11 cannot pair with 8's, 9's, or 10's paired numbers, so there will be 2 choices for 11. 12 can pair with an integer from 1 to 5 that hasn't been paired with already, or it can pair with 6. 13 will only have one choice left, and 7 must pair with 14.

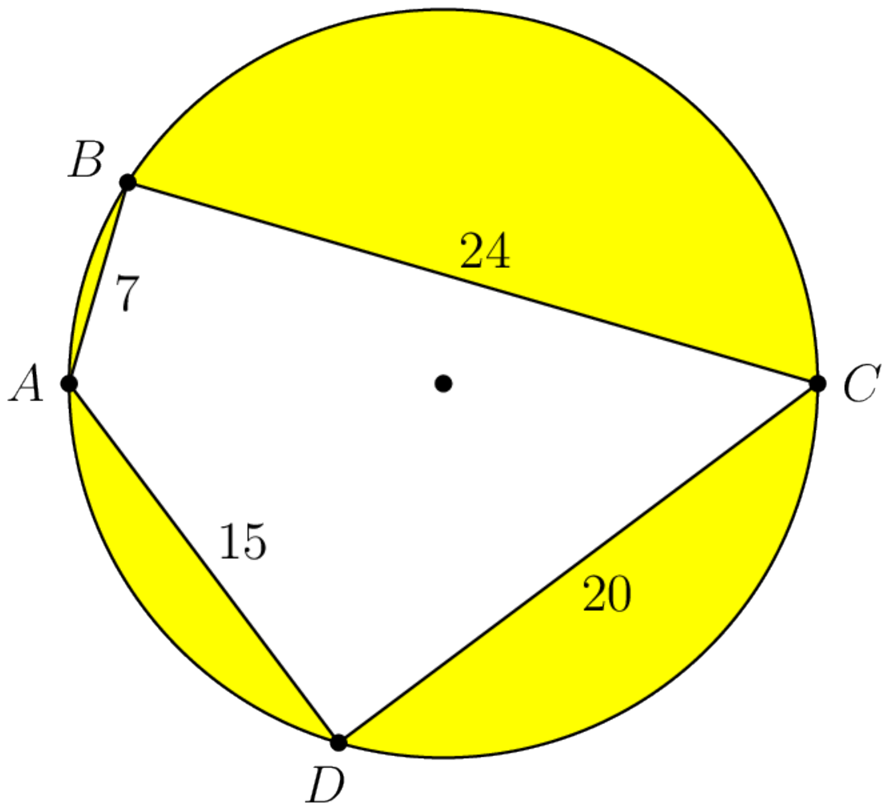
So, the answer is $4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = \boxed{\textbf{(E) } 144}$.

Problem15

Quadrilateral $ABCD$ with side lengths $AB=7, BC=24, CD=20, DA=15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi-b}{c},$ where $a, b,$ and c are positive integers such that a and c have no common prime factor. What is $a+b+c$?

$\textbf{(A)}\ 260 \qquad \textbf{(B)}\ 855 \qquad \textbf{(C)}\ 1235 \qquad \textbf{(D)}\ 1565 \qquad \textbf{(E)}\ 1997$

Diagram



Solution 1 (Inscribed Angle Theorem)

Opposite angles of every cyclic quadrilateral are supplementary, so $\angle B+\angle D=180^\circ$.We claim that $AC=25.$ We can prove it by contradiction:

- If $AC < 25$, then $\angle B$ and $\angle D$ are both acute angles. This arrives at a contradiction.
- If $AC > 25$, then $\angle B$ and $\angle D$ are both obtuse angles. This arrives at a contradiction.

By the Inscribed Angle Theorem, we conclude that $\overline{AC}$ is the diameter of the circle. So, the radius of the circle is $r = \frac{AC}{2} = \frac{25}{2}$.

The area of the requested region is

$$\pi r^2 - \frac{1}{2} \cdot AB \cdot BC - \frac{1}{2} \cdot AD \cdot DC = \frac{625\pi}{4} - \frac{168}{2} - \frac{300}{2} = \frac{625\pi - 936}{4}.$$

Therefore, the answer is $a + b + c = \boxed{\textbf{(D) } 1565}$.

Solution 2 (Brahmagupta's Formula)

When we look at the side lengths of the quadrilateral we see 7 and 24, which screams out 25 because of Pythagorean triplets. As a result, we can draw a line through points A and C to make a diameter of 25. See Solution 1 for a rigorous proof.

Since the diameter is 25, we can see the area of the circle is just $\frac{625\pi}{4}$ from the formula of the area of the circle with just a diameter.

Then we can use Brahmagupta Formula $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where a, b, c, d are side lengths, and s is semi-perimeter to find the area of the quadrilateral.

If we just plug the values in, we get $\sqrt{54756} = 234.$ So now the area of the region we are trying to find is $\frac{625\pi}{4} - 234 = \frac{625\pi - 936}{4}$.

Therefore, the answer is $a + b + c = \boxed{\textbf{(D) } 1565}$.

Solution 3 (Circumradius's Formula)

We can observe that this quadrilateral is actually made of two right triangles: $\triangle CDA$ has a 3-4-5 ratio in the side lengths, and $\triangle ABC$ is a 7-24-25 triangle.

Next, we can choose one of these triangles and use the circumradius formula to find the radius. Let's choose the $15-20-25$ triangle. The area of the triangle is equal to the product of the side lengths divided by 4 times the circumradius. Therefore, $150 = \frac{15 \cdot 20 \cdot 25}{4r}$. Solving this simple algebraic equation gives us $r = \frac{25}{2}$.

Plugging in the values, we have $\frac{25}{2}^2 \cdot \pi - \left(\frac{15 \cdot 20}{2} + \frac{7 \cdot 24}{2}\right) = \frac{625 \cdot \pi}{4} - 234$.

Rewriting this gives us $\frac{625\pi - 936}{4}$.

Therefore, adding these values gets us $\boxed{\textbf{(D) } 1565}$.

Problem16

The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

$\textbf{(A) } \frac{24}{5} \qquad \textbf{(B) } \frac{42}{5} \qquad \textbf{(C) } \frac{81}{5} \qquad \textbf{(D) } 30 \qquad \textbf{(E) } 48$

Solution 1 (Vieta's Formulas)

Let a, b, c be the three roots of the polynomial. The lengthened prism's volume is

$V = (a+2)(b+2)(c+2) = abc+2ac+2ab+2bc+4a+4b+4c+8 = abc + 2(ab+ac+bc) + 4(a+b+c) + 8$

.

By vieta's formulas, we know that:

$$abc = \frac{-D}{A} = \frac{6}{10}$$

$$ab+ac+bc = \frac{C}{A} = \frac{29}{10}$$

$$a+b+c = \frac{-B}{A} = \frac{39}{10}.$$

We can substitute these into the expression, obtaining

$$V = \frac{6}{10} + 2\left(\frac{29}{10}\right) + 4\left(\frac{39}{10}\right) + 8 = \boxed{\textbf{(D) } 30}.$$

Solution 2 (Guessing Roots)

From the answer choices, we can assume the roots are rational numbers, and therefore this polynomial should be easily factorable. The coefficients of x must multiply to 10, so these coefficients must be $5, 2, 1$ or $10, 1,$ in some order. We can try one at a time, and therefore write the factored form as follows: $(5x-p)(2x-q)(x-r) p, q, r$ have to multiply to 6, so they must be either $3, 2, 1$ or $6, 1, 1$ in some order. Again, we can try one at a time in different positions and see if they multiply correctly. We try $(5x-2)(2x-1)(x-3)$ and multiply the $x-$terms, and sure enough they add up to 29. You can try to add up the x^2 terms and they add up to -39. Therefore the roots are $\frac{2}{5}$, $\frac{1}{2}$ and 3. Now if you add 2 to each root, you get the volume is $\frac{12}{5} \cdot \frac{5}{2} \cdot 5 = 6 \cdot 5 = 30 = \boxed{\textbf{(D) } 30}$.

Solution 3 (Rational Root Theorem Bash)

We can find the roots of the cubic using the Rational Root Theorem, which tells us that the rational roots of the cubic must be in the form $\frac{p}{q}$, where p is a factor of the constant (-6) and q is a factor of the leading coefficient (10).

Therefore, p is $\pm(1,2,3,6)$ and q is $\pm(1,2,5,10)$.

Doing Synthetic Division, we find that 3 is a root of the cubic:

$$\begin{array}{c|cccc} & 10 & -39 & 29 & -6 \\ 3 & & 30 & -27 & 6 \\ \hline & 10 & -9 & 2 & 0 \end{array}.$$

Then, we have a quadratic $10x^2 - 9x + 2$. Using the Quadratic Formula, we can find the other two roots: $x = \frac{9 \pm \sqrt{(-9)^2 - 4(10)(2)}}{2 \cdot 10}$, which simplifies to $x = \frac{1}{2}, \frac{2}{5}$.

To find the new volume, we add 2 to each of the roots we found: $(3+2) \cdot (\frac{1}{2}+2) \cdot (\frac{2}{5}+2)$. Simplifying, we find that the new volume is $\boxed{\textbf{(D) } 30}$.

Solution 4

Let $P(x) = 10x^3 - 39x^2 + 29x - 6$, and let a, b, c be the roots of $P(x)$. The roots of $P(x-2)$ are then $a+2, b+2, c+2,$ so the product of the roots of $P(x-2)$ is the area of the desired rectangular prism.

$P(x-2)$ has leading coefficient 10 and constant term
$P(0-2) = P(-2) = 10(-2)^3 - 39(-2)^2 + 29(-2) - 6 = -300$
.

Thus, by Vieta's Formulas, the product of the roots of $P(x-2)$ is $\dfrac{-(-300)}{10} = \boxed{\textbf{(D) } 30}$.

Solution 5 (Quickest)

Let $P(x) = 10x^3 - 39x^2 + 29x - 6$. This can be rewritten in factored form as $P(x) = 10(x-a)(x-b)(x-c)$, where a, b, and c are the roots of $P(x)$. We want $V = (a+2)(b+2)(c+2)$, and luckily we can use $P(-2) = 10(-2-a)(-2-b)(-2-c) = -10V$ to figure this value out. It turns out $P(-2) = -300$, giving us $V = \boxed{\textbf{(D) } 30}$.

Solution 6 (Desperate Final Effort)

By Vieta's, we can see that $ABC = \dfrac{6}{10}$. Using this, we can see that if each side ABC is the same length, the length is between 0.8 (0.512)

and 0.9 (0.729). Adding 2 to these numbers would give us three numbers that are close to 3. Rounding up, we will just assume they are all three. If we multiply all of them, it gives us 27. The closest answer choice is $\boxed{\textbf{(D) } 30}$, as all of the other choices are far from this number (the second closest answer choice being 11 away).

Problem17

How many three-digit positive integers $\underline{a} \ \underline{b} \ \underline{c}$ are there whose nonzero digits $a, b,$ and c satisfy $0.\overline{\underline{a} \ \underline{b} \ \underline{c}} = \frac{1}{3}(0.\overline{a} + 0.\overline{b} + 0.\overline{c})?$ (The bar indicates repetition, thus $0.\overline{\underline{a} \ \underline{b} \ \underline{c}}$ in the infinite repeating decimal $0.\underline{a} \ \underline{b} \ \underline{c} \ \underline{a} \ \underline{b} \ \underline{c} \ \cdots$)

$\textbf{(A) } 9 \qquad \textbf{(B) } 10 \qquad \textbf{(C) } 11 \qquad \textbf{(D) } 13 \qquad \textbf{(E) } 14$

Solution 1

We rewrite the given equation, then rearrange:

$$\begin{aligned}\frac{100a+10b+c}{999} &= \frac{1}{3}\left(\frac{a}{9}+\frac{b}{9}+\frac{c}{9}\right) \\ 100a+10b+c &= 37a+37b+37c \\ 63a &= 27b+36c \\ 7a &= 3b+4c.\end{aligned}$$

Now, this problem is equivalent to counting the ordered triples (a,b,c) that satisfies the equation.

Clearly, the 9 ordered triples $(a,b,c)=(1,1,1),(2,2,2),\ldots,(9,9,9)$ are solutions to this equation.

The expression $3b+4c$ has the same value when:

- b increases by 4 as c decreases by $3.$
- b decreases by 4 as c increases by $3.$

We find 4 more solutions from the 9 solutions above: $(a,b,c)=(4,8,1),(5,1,8),(5,9,2),(6,2,9).$ Note that all solutions are symmetric about $(a,b,c)=(5,5,5).$

Together, we have $9+4=\boxed{\textbf{(D)}\ 13}$ ordered triples $(a,b,c).$

Problem18

Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \cdots, T_n$ returns the point $(1,0)$ back to itself?

$\textbf{(A)}\ 359 \qquad \textbf{(B)}\ 360 \qquad \textbf{(C)}\ 719 \qquad \textbf{(D)}\ 720 \qquad \textbf{(E)}\ 721$

Solution 1

Let $P=(r,\theta)$ be a point in polar coordinates, where θ is in degrees.

Rotating P by k° counterclockwise around the origin gives the transformation $(r,\theta)\rightarrow(r,\theta+k^\circ).$ Reflecting P across the y-axis gives the transformation $(r,\theta)\rightarrow(r,180^\circ-\theta).$ Note that

$$T_k(P)=(r,180^\circ-\theta-k^\circ),$$

$$T_{k+1}(T_k(P))=(r,\theta-1^\circ).$$

We start with $(1,0^\circ)$ in polar coordinates. For the sequence of transformations $T_1, T_2, T_3, \cdots, T_k,$ it follows that

- After T_1, we have $(1, 179^\circ)$.
- After T_2, we have $(1, -1^\circ)$.
- After T_3, we have $(1, 178^\circ)$.
- After T_4, we have $(1, -2^\circ)$.
- After T_5, we have $(1, 177^\circ)$.
- After T_6, we have $(1, -3^\circ)$.
- ...
- After T_{2k-1}, we have $(1, 180^\circ - k^\circ)$.
- After T_{2k}, we have $(1, -k^\circ)$.

The least such positive integer k is $180.$ Therefore, the least such positive integer n is $2k-1=\boxed{\textbf{(A) } 359}.$

Solution 2

Note that since we're reflecting across the y-axis, if the point ever makes it to $(-1,0)$ then it will flip back to the original point. Note that after $T_1$the point will be 1 degree clockwise from the negative x-axis. Applying T_2 will rotate it to be 1 degree counterclockwise from the negative x-axis, and then flip it so that it is 1 degree clockwise from the positive x-axis. Therefore, after every 2 transformations, the point rotates 1 degree clockwise. To rotate it so that it will rotate 179 degrees clockwise will require $179 \cdot 2 = 358$ transformations. Then finally on the last transformation, it will rotate on to $(-1, 0)$ and then flip back to it's original position. Therefore, the answer is $358 + 1 = 359 = \boxed{\textbf{(A) } 359}$.

Solution 3

Let A_n be the point $(\cos n^\circ, \sin n^\circ)$.

Starting with $n = 0$, the sequence goes
$A_0 \to A_{179} \to A_{359} \to A_{178} \to A_{358} \to A_{177} \to A_{357} \to \cdots$

We see that it takes 2 turns to downgrade the point by 1°. Since the fifth point in the sequence is A_{177}, the answer is $5 + 2(177) = \boxed{\textbf{(A) } 359}$

Problem19

Define L_n as the least common multiple of all the integers from 1 to n inclusive. There is a unique integer h such that $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{17} = \frac{h}{L_{17}}$ What is the remainder when h is divided by 17?

$\textbf{(A) } 1 \qquad \textbf{(B) } 3 \qquad \textbf{(C) } 5 \qquad \textbf{(D) } 7 \qquad \textbf{(E) } 9$

Solution

Notice that L_{17} contains the highest power of every prime below 17.

Thus, $L_{17} = 16 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$.

When writing the sum under a common fraction, we multiply the denominators by L_{17} divided by each denominator. However, since L_{17} is a multiple of 17, all terms will be a multiple of 17 until we divide out 17, and the only term that will do this is $\frac{1}{17}$. Thus, the remainder of all other terms when divided by 17 will be 0, so the problem is essentially asking us what the remainder of $\frac{L_{17}}{17}$ divided by 17 is. This is equivalent to finding the remainder of $16 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ divided by 17.

We use modular arithmetic to simplify our answer:

This is congruent to $-1 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \pmod{17}$

Evaluating, we get:
$$\begin{aligned}(-1) \cdot 9 \cdot 35 \cdot 11 \cdot 13 &\equiv (-1) \cdot 9 \cdot 1 \cdot 11 \cdot 13 \pmod{17} \\ &\equiv 9 \cdot 11 \cdot (-13) \pmod{17} \\ &\equiv 9 \cdot 11 \cdot 4 \pmod{17} \\ &\equiv 2 \cdot 11 \pmod{17} \\ &\equiv 5 \pmod{17}\end{aligned}$$

Therefore the remainder is $\boxed{\textbf{(C) } 5}$.

Solution 2

As in solution 1, we express the LHS as a sum under one common denominator. We note that $\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{17}=\frac{\frac{17!}{1}}{17!}+\frac{\frac{17!}{2}}{17!}+\frac{\frac{17!}{3}}{17!}+\cdots+\frac{\frac{17!}{17}}{17!}$

Now, we have $h=L_{17}\left(\frac{\frac{17!}{1}+\frac{17!}{2}+\frac{17!}{3}+\cdots+\frac{17!}{17}}{17!}\right)$. We'd like to find $h \pmod{17}$, so we can evaluate our expression $\pmod{17}$. Since $\frac{\frac{17!}{1}}{17!}, \frac{\frac{17!}{2}}{17!}, \ldots, \frac{\frac{17!}{16}}{17!}$ don't have a factor of 17 in their denominators, and since L_{17} is a multiple of 17, multiplying each of those terms and adding them will get a multiple of $17.$ $\pmod{17}$, that result is $0.$ Thus, we only need to consider $L_{17}\cdot\frac{\frac{17!}{17}}{17!}=\frac{L_{17}}{17}\pmod{17}.$ Proceed with solution 1 to get $\boxed{\textbf{(C) }5}$.

Problem20

A four-term sequence is formed by adding each term of a four-term arithmetic sequence of positive integers to the corresponding term of a four-term geometric sequence of positive integers. The first three terms of the resulting four-term sequence are 57, 60, and 91. What is the fourth term of this sequence?

$\textbf{(A) } 190 \qquad \textbf{(B) } 194 \qquad \textbf{(C) } 198 \qquad \textbf{(D) } 202 \qquad \textbf{(E) } 206$

Solution

Let the arithmetic sequence be $a, a+d, a+2d, a+3d$ and the geometric sequence be $b, br, br^2, br^3.$

$$a+b=57,$$
$$a+d+br=60,$$

We are given that $a+2d+br^2=91,$ and we wish to find $a+3d+br^3.$

Subtracting the first equation from the second and the second equation from the third, we get $d+b(r-1)=3,$ $d+br(r-1)=31.$ Subtract these results, we get $b(r-1)^2=28.$

Note that either $b=28$ or $b=7.$ We proceed with casework:

- If $b = 28,$ then $r = 2, a = 29,$ and $d = 25.$ The arithmetic sequence is $29, 4, -21, -46,$ arriving at a contradiction.

- If $b = 7,$ then $r = 3, a = 50,$ and $d = -11.$ The arithmetic sequence is $50, 39, 28, 17,$ and the geometric sequence is $7, 21, 63, 189.$ This case is valid.

Therefore, The answer is $a + 3d + br^3 = 17 + 189 = \boxed{\textbf{(E) } 206}.$

Solution 2

Start similarly to solution 1 and deduce the three equations
$$a + b = 57,$$
$$a + d + br = 60,$$
$$a + 2d + br^2 = 91,$$

Then, add the last two equations and take away the first equation to get $a + 3d + br^2 + br - b = 94$ We can solve for this in terms of what we want: $a + 3d = -br^2 - br + b + 94$ We're looking for $a + 3d + br^3.$ We can substitute our value of $a + 3d$ in here to get: $br^3 - br^2 - br + b + 94$ We can factor this to get: $b(r+1)(r-1)(r-1) + 94$. Since our sequence only has positive integers we can now check by the answer choices. For each answer choice, we can subtract 94 and factor it to see if it has a perfect square factor and at least one other factor and those should differ by

$$(A) 190 - 94 = 96 = 2^5 * 3,$$
$$(B) 194 - 94 = 100 = 2^2 * 5^2,$$
$$(C) 198 - 94 = 104 = 2^3 * 13,$$
$$(D) 202 - 94 = 108 = 2^2 * 3^3,$$

2. $(E) 206 - 94 = 112 = 2^4 * 7$ From this, we know that the only possible answer choices are A and E where r=3. To solve for b, we look back to our 3 equations:

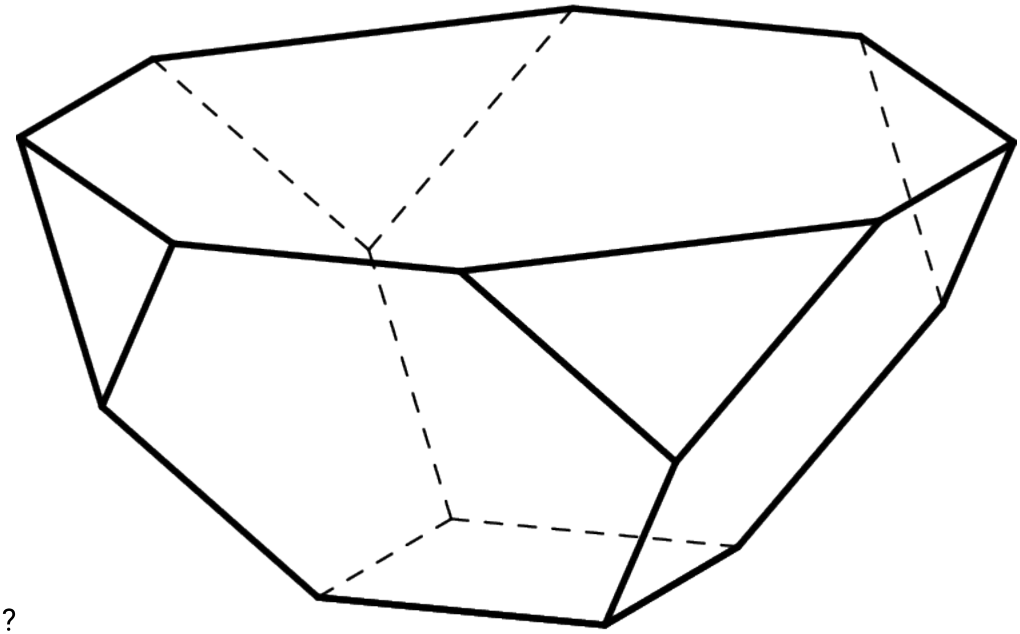
$$a + b = 57,$$
$$a + d + 3b = 60,$$
$$a + 2d + 9b = 91,$$

We are looking for $a + 3d + 27b$ If A were the answer, then we know that a would have to be divisible by 3 and b would equal 6. Looking at our second equation, if this were the case, then d would also have to be divisible by 3. But,this contradicts the third equation, as all variables are divisible by 3, but their sum isn't.

So, $\boxed{\textbf{(E)}\ 206}$ is our answer.

Problem21

A bowl is formed by attaching four regular hexagons of side 1 to a square of side 1. The edges of the adjacent hexagons coincide, as shown in the figure. What is the area of the octagon obtained by joining the top eight vertices of the four hexagons, situated on the rim of the

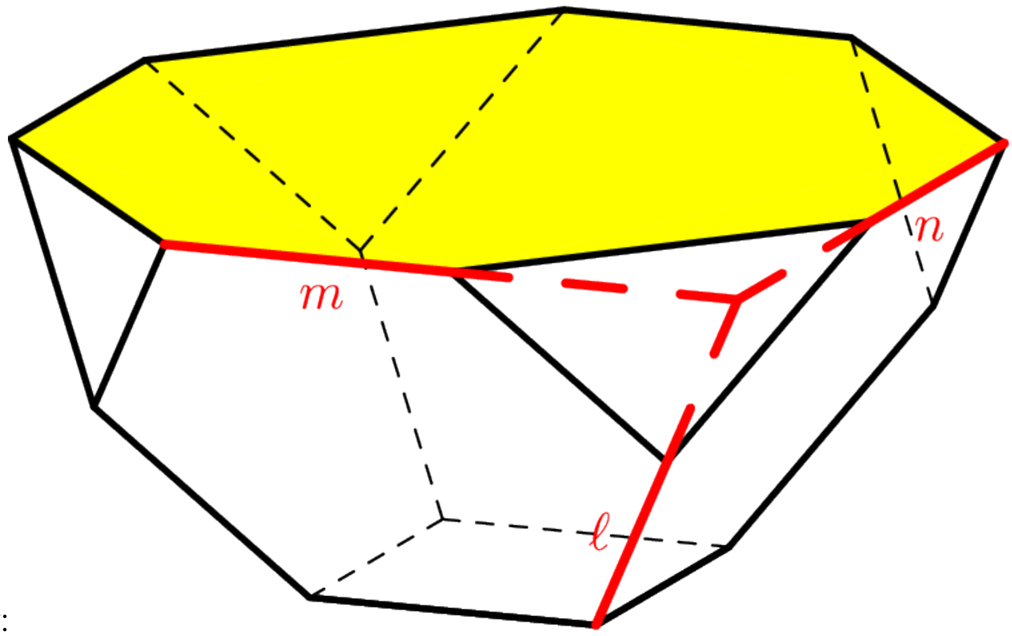


bowl?

$\textbf{(A) } 6 \qquad \textbf{(B) } 7 \qquad \textbf{(C) } 5+2\sqrt{2} \qquad \textbf{(D) } 8 \qquad \textbf{(E) } 9$

Solution 1

We extend line segments $\ell, m,$ and n to their point of concurrency, as shown

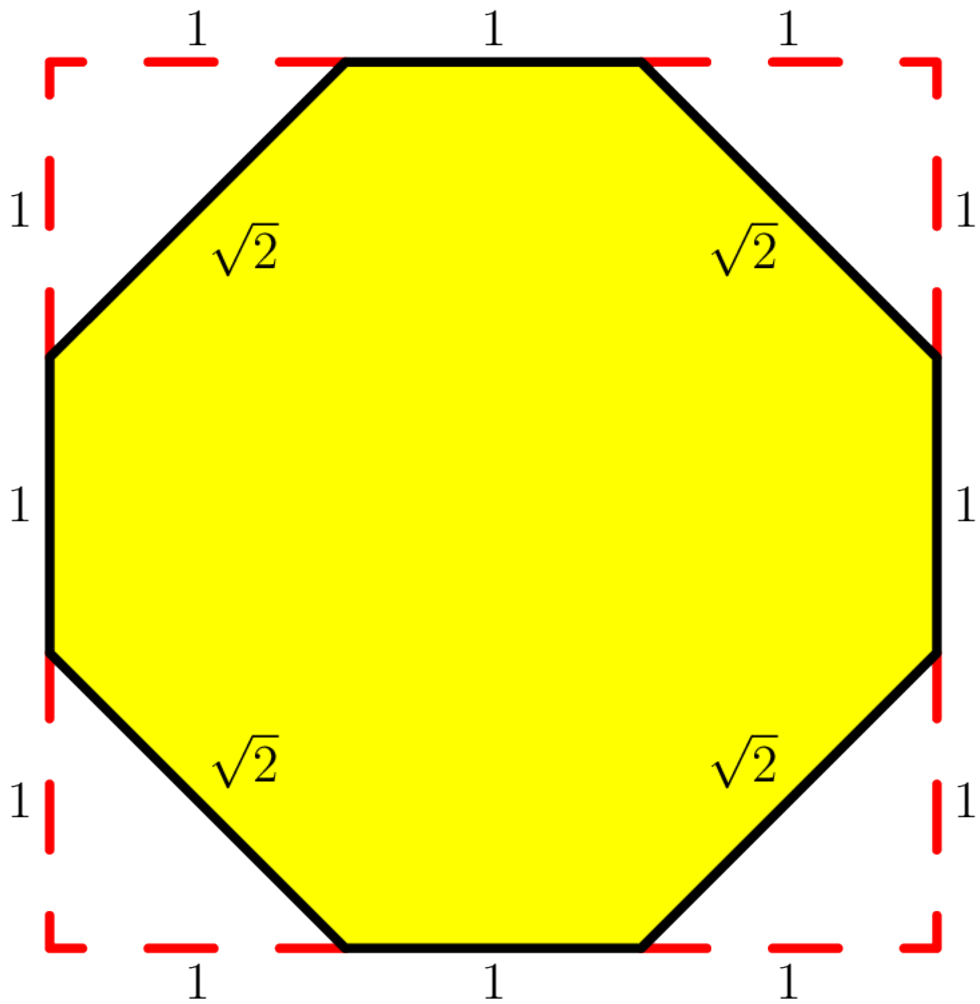


below:

We claim that lines $\ell, m,$ and n are concurrent: In the lateral faces of the bowl, we know that lines ℓ and m must intersect, and lines ℓ and $n$must intersect. In addition, line ℓ intersects the top plane of the bowl at exactly one point. Since lines m and n are both in the top plane of the bowl, we conclude that lines $\ell, m,$ and n are concurrent.

In the lateral faces of the bowl, the dashed red line segments create equilateral triangles. So, the dashed red line segments all have length $1.$ In the top plane of the bowl, we know that $\overleftrightarrow{m} \perp \overleftrightarrow{n}.$ So, the dashed red line segments create an isosceles triangle with leg-length $1.$

Note that octagon has four pairs of parallel sides, and the successive side-lengths are $1, \sqrt{2}, 1, \sqrt{2}, 1, \sqrt{2}, 1, \sqrt{2},$ as shown below:

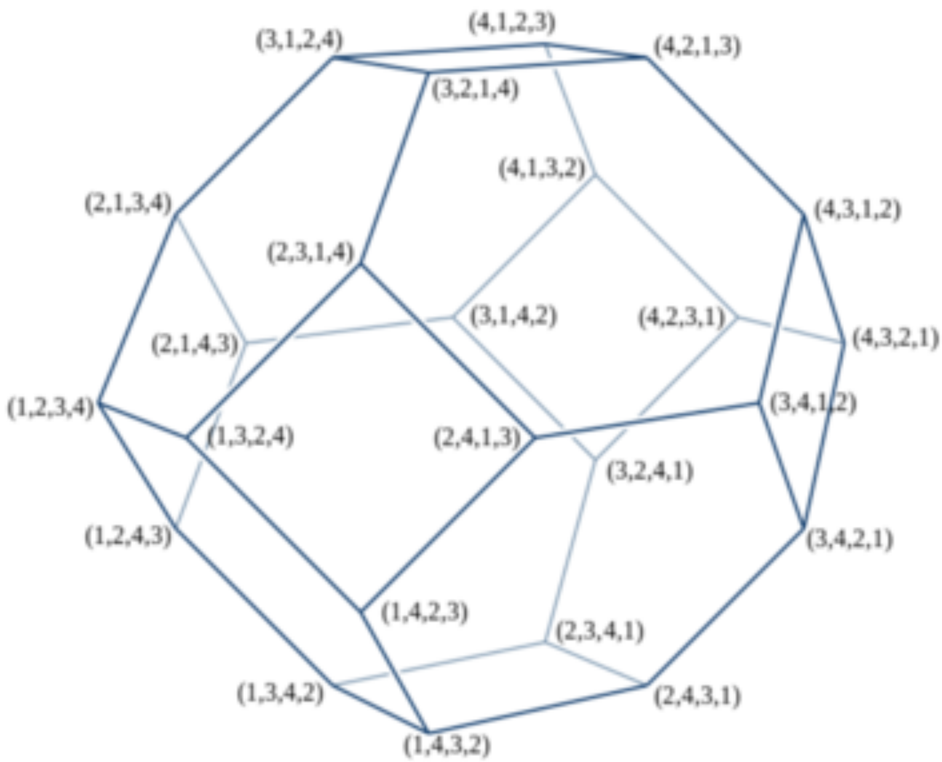


The area of the octagon is $3^2 - 4 \cdot \left(\frac{1}{2} \cdot 1^2\right) = \boxed{\textbf{(B) } 7}.$

Solution 2

Note that the octagon is equiangular by symmetry, but it is not equilateral. 4 of it's sides are shared with the hexagon's sides, so each of those sides have side length 1. However, the other 4 sides are touching the triangles, so we wish to find the length of these sides.

Notice that when two adjacent hexagons meet at a side, their planes make the same dihedral angle at the bottom-most point of intersection and at the top-most point of intersection by symmetry. Therefore, the triangle that is wedged between the two hexagons has the same angle as the square at the bottom wedged between the hexagons. Thus, the triangle is a $45-45-90$ isosceles triangle. This conclusion can also be reached by cutting the bottom square across a diagonal and noticing that each resulting triangle is congruent to each triangle wedged between the hexagons by symmetry. Furthermore, notice that if you take a copy of this bowl and invert it and place it on top of this bowl, you will get a polyhedron with faces of hexagons and squares, a truncated octahedron, and therefore this triangle has a 90 degree angle:



Now that we have come to this conclusion, by simple Pythagorean theorem, we have that the other 4 sides of the octagon are $\sqrt{2}$.

We can draw a square around the octagon so that the area of the octagon is the area of the square minus each corner triangle. The hypotenuse of these corner triangles are 1 and they are $45-45-90$ triangles because the octagon is equiangular, so each has dimensions $\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1$. The side length of the square is $\sqrt{2}$ for the larger sides of the octagon, and adding 2 of $\frac{\sqrt{2}}{2}$ for each width of the triangle. Therefore, the area of the square is: $\left(\sqrt{2} + 2 \cdot \frac{\sqrt{2}}{2}\right)^2 \implies (2\sqrt{2})^2 = 8$ The area of each triangle is $\frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{4}$ and there are 4 of them, so we subtract 1 from the area of the square. The area of the octagon is thus $7 = \boxed{B}$

Solution 3 (Not Rigorous)

Through observation, we can reasonably assume that each of the triangles on this shape is a right triangle. Since each side length of the hexagons is 1, the hypotenuse of the triangles would be $\sqrt{2}$. Now we know the side lengths of the octagon whose area we are solving for. The octagon can be broken into nine pieces. We have four triangles whose side lengths are 1, and their hypotenuse is a side whose length is $\sqrt{2}$. Next, we have 5 1 by 1 squares. The triangles each have an area of $\frac{1}{2}$, and the squares each have an area of 1.

Then, we add these up, so we get $\boxed{\textbf{(B) } 7}$.

Solution 4 (Truncated Icosahedron)

Notice that the shape is half a truncated icosahedron. It is then trivial that the angles in question at the top of the bowl are 90°, and we can then find the answer to be $\boxed{\textbf{(B) } 7}$ as in the previous solutions.

Problem22

Suppose that 13 cards numbered $1, 2, 3, \ldots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards $1, 2, 3$ are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, $7, 8, 9, 10$ on the fourth pass, and $11, 12, 13$ on the fifth pass. For how many of the $13!$ possible orderings of the cards will the 13 cards be picked up in exactly two passes?

$$\boxed{7}\ \boxed{11}\ \boxed{8}\ \boxed{6}\ \boxed{4}\ \boxed{5}\ \boxed{9}\ \boxed{12}\ \boxed{1}\ \boxed{13}\ \boxed{10}\ \boxed{2}\ \boxed{3}$$

$\textbf{(A)}\ 4082 \qquad \textbf{(B)}\ 4095 \qquad \textbf{(C)}\ 4096 \qquad \textbf{(D)}\ 8178 \qquad \textbf{(E)}\ 8191$

Solution 1 (Casework)

For $1 \leq n \leq 12,$ suppose that cards $1, 2, \ldots, n$ are picked up on the first pass. It follows that cards $n+1, n+2, \ldots, 13$ are picked up on the second pass.

Once we pick the spots for the cards on the first pass, there is only one way to arrange all 13 cards.

For each value of $n,$ there are $\binom{13}{n}-1$ ways to pick the n spots for the cards on the first pass: We exclude the arrangement

$$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}\ \boxed{9}\ \boxed{10}\ \boxed{11}\ \boxed{12}\ \boxed{13}$$

in which the first pass consists of all 13 cards.

Therefore, the answer is $\sum_{k=1}^{12}\left[\binom{13}{k}-1\right]=\left[\sum_{k=1}^{12}\binom{13}{k}\right]-12=\left[\sum_{k=0}^{13}\binom{13}{k}\right]-14=2^{13}-14=\boxed{\textbf{(D) } 8178}$.

Solution 2 (Casework)

Since the 13 cards are picked up in two passes, the first pass must pick up the first n cards and the second pass must pick up the remaining cards m through 13. Also note that if m, which is the card that is numbered one more than n, is placed before n, then m will not be picked up on the first pass since cards are picked up in order. Therefore we desire m to be placed before n to create a second pass, and that after the first pass, the numbers m through 13 are lined up in order from least to greatest.

To construct this, n cannot go in the nth position because all cards 1 to $n-1$ will have to precede it and there will be no room for m. Therefore n must be in slots $n+1$ to 13. Let's do casework on which slot n goes into to get a general idea for how the problem works.

$\textbf{Case 1:}$ With n in spot $n+1$, there are n available slots before n, and there are $n-1$ cards preceding n. Therefore the number of ways to reserve these slots for the $n-1$ cards is $\binom{n}{n-1}$. Then there is only 1 way to order these cards (since we want them in increasing order). Then card m goes into whatever slot is remaining, and the $13-m$ cards are ordered in increasing order after slot $n+1$, giving only 1 way. Therefore in this case there are $\binom{n}{n-1}$ possibilities.

$\textbf{Case 2:}$ With n in spot $n+2$, there are $n+1$ available slots before n, and there are $n-1$ cards preceding n. Therefore the number of ways to reserve slots for these cards are $\binom{n+1}{n-1}$. Then there is one way to order these cards. Then cards m and $m+1$ must go in the remaining two slots, and there is only one way to order them since they must be in increasing order. Finally, cards $m+2$ to 13 will be ordered in increasing order after slot $n+1$, which yields 1 way. Therefore, this case has $\binom{n+1}{n-1}$ possibilities.

I think we can see a general pattern now. With n in slot x, there are $x-1$ slots to distribute to the previous $n-1$ cards, which can be done in $\binom{x-1}{n-1}$ ways. Then the remaining cards fill in in just 1 way. Since the cases of n start in slot $n+1$ and end in slot 13, this sum amounts to: $\binom{n}{n-1}+\binom{n+1}{n-1}+\binom{n+2}{n-1}+\cdots+\binom{12}{n-1}$ for any n.

Hmmm ... where have we seen this before?

We use wishful thinking to add a term of $\binom{n-1}{n-1}$:

$$\binom{n-1}{n-1}+\binom{n}{n-1}+\binom{n+1}{n-1}+\binom{n+2}{n-1}+\cdots+\binom{12}{n-1}$$

This is just the hockey stick identity! Applying it, this expression is equal to $\binom{13}{n}$. However, we added an extra term, so subtracting it off, the total number of ways to order the 13 cards for any n is $\binom{13}{n} - 1$

Finally, to calculate the total for all n, we sum from $n = 0$ to 13. This yields us:

$$\sum_{n=0}^{13}\binom{13}{n} - 1 \implies \sum_{n=0}^{13}\binom{13}{n} - \sum_{n=0}^{13}1$$
$$\implies 2^{13} - 14 = 8192 - 14 = 8178 = \boxed{\textbf{(D) } 8178}.$$

Solution 3 (Recursion)

To solve this problem, we can use recursion on n. Let A_n be the number of arrangements for n numbers. Now, let's look at how these arrangements are formed by case work on the first number a_1.

If $a_1 = 1$, the remaining $n-1$ numbers from 2 to n are arranged in the same way just like number 1 to $n-1$ in the case of $n-1$ numbers. So there are A_{n-1} arrangements.

If $a_1 = 2$, then we need to choose 1 position from position 2 to $n-1$ to put 1, and all remaining numbers must be arranged in increasing order, so there are $\binom{n-1}{1}$ such arrangements.

If $a_1 = k$, then we need to choose $k-1$ positions from position 2 to $n-1$ to put $1, 2, \cdots k-1$, and all remaining numbers must be arranged in increasing order, so there are $\binom{n-1}{k-1}$ such arrangements.

So we can write

$$A_n = A_{n-1} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1}$$

which can be simplified to $A_n = A_{n-1} + 2^{n-1} - 1$ We can solve this recursive sequence by summing up $n-1$ lines of the recursive formula $A_n - A_{n-1} = 2^{n-1} - 1$ $A_{n-1} - A_{n-2} = 2^{n-2} - 1$ ... $A_2 - A_1 = 2^1 - 1$ to get

$$A_n - A_1 = \sum_{k=1}^{n-1}(2^k - 1) = 2^n - 2 - (n-1) = 2^n - n - 1$$

since $A_1 = 0$, we have $A_n = 2^n - n - 1$

and $A_{13} = 2^{13} - 14 = \boxed{\textbf{(D) } 8178}$.

Solution 4 (Engineer's Induction)

When we have 3 cards arranged in a row, after listing out all possible arrangements, we see that we have 4 ones: $(1,3,2), (2,1,3,)(2,3,1),$ and $(3,1,2)$. When we have 4 cards, we find 11 possible arrangements: $(1,2,4,3), (1,3,2,4), (1,3,4,2), (1,4,2,3), (2,1,3,4), (2,3,1,4), (2,3,4,1), (3,1,2,4), (3,1,4,2), (3,4,1,2),$ and $(4,1,2,3)$. Hence, we recognize the pattern that for n cards, we have $2^n - n - 1$ valid arrangements, so our answer is $2^{13} - 13 - 1 = \boxed{\textbf{(D) } 8178}$.

Solution 5 (Alternative)

Notice that for each card "position", we can choose for it to be picked up on the first or second pass, for a total of 2^{13} options. However, if all of the cards selected to be picked up first are before all of the cards to be picked up second, then this means that the list is in consecutive ascending order (and thus all cards will be picked up on the first pass instead). This can happen in 14 ways, so our answer is $2^{13} - 14 = \boxed{\textbf{(D) } 8178}$.

Problem23

Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1, PB = 2, PC = 3,$ and $PD = 4.$ What is $\frac{BC}{AD}$?

$$\textbf{(A) } \frac{1}{4} \qquad \textbf{(B) } \frac{1}{3} \qquad \textbf{(C) } \frac{1}{2} \qquad \textbf{(D) } \frac{2}{3} \qquad \textbf{(E) } \frac{3}{4}$$

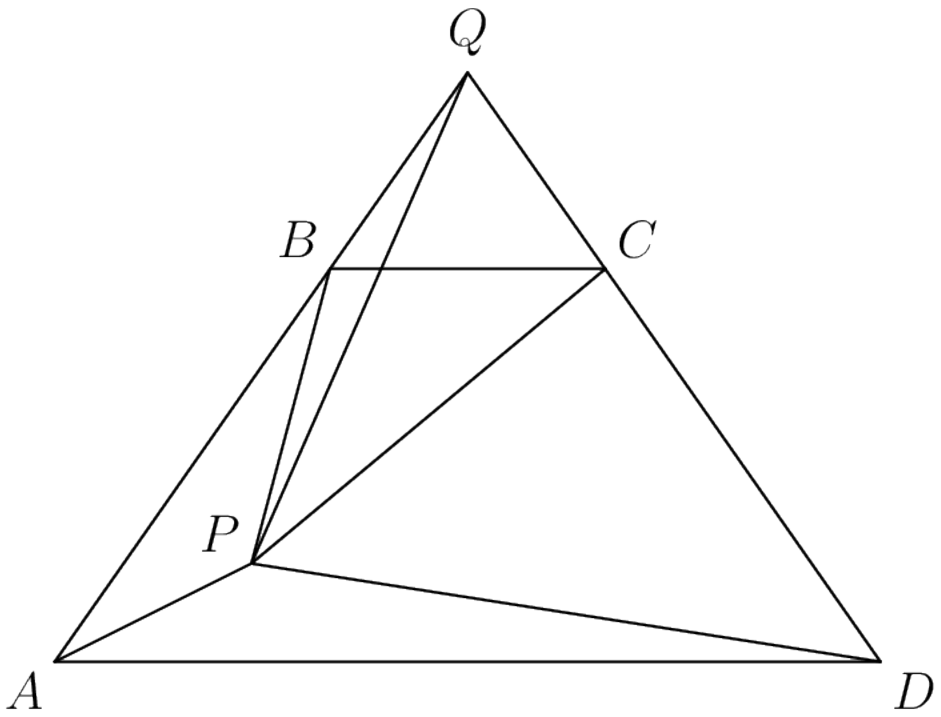
Solution 1 (Reflections + Ptolemy's Theorem)

Consider the reflection P' of P over the perpendicular bisector of $\overline{BC}$, creating two new isosceles trapezoids $DAPP'$ and $CBPP'$. Under this reflection, $P'A = PD = 4$, $P'D = PA = 1$, $P'C = PB = 2$, and $P'B = PC = 3$. By Ptolemy's theorem

$$PP' \cdot AD + 1 = 16$$
$$PP' \cdot BC + 4 = 9$$

Thus $PP' \cdot AD = 15$ and $PP' \cdot BC = 5$; dividing these two equations and taking the reciprocal yields $\frac{BC}{AD} = \boxed{\textbf{(B) } \frac{1}{3}}$.

Solution 2 (Extensions + Stewart's Theorem)



Extend AB and CD to a point Q as shown, and let $PQ = s$. Then let $BQ = CQ = b$ and $AQ = DQ = a$. Notice that $\frac{BC}{AD} = \frac{QC}{QD} = \frac{a}{a+b}$ by similar triangles.

By Stewart's theorem on APQ and DPQ, we have

$$ab(a+b) + 9(a+b) = 16a + s^2b$$
$$ab(a+b) + 4(a+b) = a + s^2b$$

Subtracting, $5(a+b) = 15a$, and

so $\frac{BC}{AD} = \frac{a}{a+b} = \frac{5}{15} = \boxed{\textbf{(B)}\ \frac{1}{3}}$.

Solution 3 (Coordinate Bashing)

Since we're given distances and nothing else, we can represent each point as a coordinate and use the distance formula to set up a series of systems and equations. Let the height of the trapezoid be h, and let the coordinates of A and D be at $(-a, 0)$ and $(a, 0)$, respectively. Then let B and C be at $(-b, h)$ and (b, h), respectively. This follows the rules that this is an isosceles trapezoid since the origin is centered on the middle of AD. Finally, let P be located at point (c, d).

The distance from P to A is 1, so by the distance formula:

$$\sqrt{(c+a)^2+(d-h)^2} = 1 \implies (c+a)^2+(d-h)^2 = 1$$

The distance from P to D is 4, so

$$\sqrt{(c-a)^2+(d-h)^2} = 1 \implies (c-a)^2+(d-h)^2 = 16$$

Looking at these two equations alone, notice that the second term is the same for both equations, so we can subtract the equations. This yields $-4ac = 15$

Next, the distance from P to B is 2, so

$$\sqrt{(c+b)^2+(d-h)^2} = 2 \implies (c+b)^2+(d-h)^2 = 4$$

The distance from P to C is 3, so

$$\sqrt{(c-b)^2+(d-h)^2} = 3 \implies (c-b)^2+(d-h)^2 = 9$$

Again, we can subtract these equations, yielding $-4bc = 5$

We can now divide the equations to eliminate c, yielding $\frac{b}{a} = \frac{5}{15} = \frac{1}{3}$

We wanted to find $\frac{BC}{AD}$. But since b is half of BC and a is half of AD, this ratio is equal to the ratio we want.

Therefore $\frac{BC}{AD} = \boxed{\textbf{(B)}\ \frac{1}{3}}$.

Solution 4 (Coordinate Bashing)

Let the point P be at the origin, and draw four concentric circles around P each with radius 1, 2, 3, and 4, respectively. The vertices of the trapezoid would be then on each of the four concentric circles. WLOG, let BC and AD be parallel to the x-axis. Assigning coordinates to each point, we have: $A = (x_1, y_1)$ $B = (x_2, y_2)$ $C = (x_3, y_2)$ $D = (x_4, y_1)$ which satisfy the following: $$x_1^2 + y_1^2 = 1 \qquad (1)$$
$$x_2^2 + y_2^2 = 4 \qquad (2)$$
$$x_3^2 + y_2^2 = 9 \qquad (3)$$
$$x_4^2 + y_1^2 = 16 \qquad (4)$$
In addition, because the trapezoid is isosceles ($AB = CD$), the midpoints of the two bases would then have the same x-coordinate, giving us $$x_1 + x_4 = x_2 + x_3 \qquad (5)$$
Subtracting Equation 2 from Equation 3, and Equation 1 from Equation 4, we have $$x_3^2 - x_2^2 = 5 \qquad (6)$$
$$x_4^2 - x_1^2 = 15 \qquad (7)$$
Dividing Equation 6 by Equation 7,

we have $\frac{x_3^2 - x_2^2}{x_4^2 - x_1^2} = \frac{1(x_3 - x_2)(x_3 + x_2)}{3(x_4 - x_1)(x_4 + x_1)} = \frac{1}{3}.$

Cancelling $(x_3 + x_2)$ and $(x_4 + x_1)$ with Equation 5, we get $\frac{(x_3 - x_2)}{(x_4 - x_1)} = \frac{1}{3}.$ In other words, $\frac{BC}{AD} = \frac{1}{3} = \boxed{\textbf{(B) } \frac{1}{3}}.$ ~G63566

Solution 5 (Cheese)

Notice that the question never says what the height of the trapezoid is; the only property we know about it is that $AC = BD$. Therefore, we can say WLOG that the height of the trapezoid is 0 and all 5 points, including P, lie on the same line with $PA = AB = BC = CD = 1$. Notice that this satisfies the problem requirements because $PA = 1, PB = 2, PC = 3, PD = 4$, and $AC = BD = 2$. Now all we have to find is $\frac{BC}{AD} = \boxed{\textbf{(B) } \frac{1}{3}}$

Problem24

How many strings of length 5 formed from the digits 0, 1, 2, 3, 4 are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j? (For example, 02214 satisfies this condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

$\textbf{(A) } 500 \qquad \textbf{(B) } 625 \qquad \textbf{(C) } 1089 \qquad \textbf{(D) } 1199 \qquad \textbf{(E) } 1296$

Solution 1 (Parking Functions)

For some n, let there be $n+1$ parking spaces counterclockwise in a circle.

Consider a string of n integers $c_1c_2\cdots c_n$ each between 0 and n, and let n cars come into this circle so that the ith car tries to park at spot c_i, but if it is already taken then it instead keeps going counterclockwise and takes the next avaliable spot. After this process, exactly one spot will remain empty. Then the strings of n numbers between 0 and $n-1$ that contain at least k integers $< k$ for $1 \leq k \leq n+1$ are exactly the set of strings that leave spot n empty. Also note for any string $c_1c_2\cdots c_n$, we can add 1 to each c_i (mod $n+1$) to shift the empty spot counterclockwise, meaning for each string there exists exactly one j with $0 \leq j \leq n$ so that $(c_1+j)(c_2+j)\cdots(c_n+j)$ leaves spot n empty. This gives there are $$\frac{(n+1)^n}{n+1} = (n+1)^{n-1}$$ such strings.

Plugging in $n=5$ gives $\boxed{\textbf{(E) } 1296}$ such strings.

Solution 2 (Casework)

Note that a valid string must have at least one 0.

We perform casework on the number of different digits such strings can have. For each string, we list the digits in ascending order, then consider permutations:

1. The string has 1 different digit.

   The only possibility is 00000.

   There is 1 string in this case.

2. The string has 2 different digits.

   We have the following table:

Digits	**01**	**02**	**03**	**04**	**Row's Count**
	00001	00002	00003	00004	$4 \cdot \frac{5!}{4!1!} = 20$
	00011	00022	00033		$3 \cdot \frac{5!}{3!2!} = 30$
	00111	00222			$2 \cdot \frac{5!}{2!3!} = 20$
	01111				$1 \cdot \frac{5!}{1!4!} = 5$

There are $\mathbf{20 + 30 + 20 + 5 = 75}$ strings in this case.

3. The string has 3 different digits.

   We have the following table:

Digits	**012**	**013**	**014**	**023**	**024**	**034**	**Row's Count**
	00012	00013	00014	00023	00024	00034	$6 \cdot \frac{5!}{3!1!1!} = 120$
	00112	00113	00114	00223	00224		$5 \cdot \frac{5!}{2!2!1!} = 150$
	00122	00133		00233			$3 \cdot \frac{5!}{2!1!2!} = 90$
	01112	01113	01114				$3 \cdot \frac{5!}{1!3!1!} = 60$
	01122	01133					$2 \cdot \frac{5!}{1!2!2!} = 60$
	01222						$1 \cdot \frac{5!}{1!1!3!} = 20$

There are $\mathbf{120 + 150 + 90 + 60 + 60 + 20 = 500}$ strings in this case.

4. The string has 4 different digits.

We have the following table:

Digits	0123	0124	0134	0234
	00123	00124	00134	00234
	01123	01124	01134	
	01223	01224		
	01233			

There are $\mathbf{10 \cdot \dfrac{5!}{2!1!1!1!} = 600}$ strings in this case.

5. The string has 5 different digits.

   There are $\mathbf{5! = 120}$ strings in this case.

Together, the answer is $1 + 75 + 500 + 600 + 120 = \boxed{\textbf{(E) } 1296}$.

Solution 3 (Recursive Equations Approach)

Denote by $N(p, q)$ the number of p-digit strings formed by using numbers $0, 1, \cdots, q$, where for each $j \in \{1, 2, \cdots, q\}$, at least j of the digits are less than j.

We have the following recursive equation:

$$N(p, q) = \sum_{i=0}^{p-q} \binom{p}{i} N(p-i, q-1), \ \forall \ p \geq q \text{ and } q \geq 1$$

and the boundary condition $N(p, 0) = 1$ for any $p \geq 0$.

By solving this recursive equation, for $q = 1$ and $p \geq q$, we get

$$\begin{aligned} N(p,1) &= \sum_{i=0}^{p-1} \binom{p}{i} N(p-i,0) \\ &= \sum_{i=0}^{p-1} \binom{p}{i} \\ &= \sum_{i=0}^{p} \binom{p}{i} - \binom{p}{p} \\ &= 2^p - 1. \end{aligned}$$

For $q = 2$ and $p \geq q$, we get

$$\begin{aligned} N(p,2) &= \sum_{i=0}^{p-2} \binom{p}{i} N(p-i,1) \\ &= \sum_{i=0}^{p-2} \binom{p}{i} \left(2^{p-i} - 1\right) \\ &= \sum_{i=0}^{p} \binom{p}{i} \left(2^{p-i} - 1\right) - \sum_{i=p-1}^{p} \binom{p}{i} \left(2^{p-i} - 1\right) \\ &= \sum_{i=0}^{p} \left(\binom{p}{i} 1^i 2^{p-i} - \binom{p}{i} 1^i 1^{p-i} \right) - p \\ &= (1+2)^p - (1+1)^p - p \\ &= 3^p - 2^p - p. \end{aligned}$$

For $q = 3$ and $p \geq q$, we get

$$\begin{aligned} N(p,3) &= \sum_{i=0}^{p-3} \binom{p}{i} N(p-i,2) \\ &= \sum_{i=0}^{p-3} \binom{p}{i} \left(3^{p-i} - 2^{p-i} - (p-i)\right) \\ &= \sum_{i=0}^{p} \binom{p}{i} \left(3^{p-i} - 2^{p-i} - (p-i)\right) - \sum_{i=p-2}^{p} \binom{p}{i} \left(3^{p-i} - 2^{p-i} - (p-i)\right) \\ &= \sum_{i=0}^{p} \left(\binom{p}{i} 1^i 3^{p-i} - \binom{p}{i} 1^i 2^{p-i} - \binom{p}{i} (p-i) \right) - \frac{3}{2} p (p-1) \\ &= (1+3)^p - (1+2)^p - \frac{d(1+x)^p}{dx}\bigg|_{x=1} - \frac{3}{2} p(p-1) \\ &= 4^p - 3^p - 2^{p-1} p - \frac{3}{2} p (p-1) . \end{aligned}$$

For $q = 4$ and $p = 5$, we get

$$\begin{aligned} N(5,4) &= \sum_{i=0}^{1} \binom{5}{i} N(5-i,3) \\ &= N(5,3) + 5N(4,3) \\ &= \boxed{\textbf{(E) } 1296}. \end{aligned}$$

Solution 4 (Answer Choices)

Let the set of all valid sequences be S. Notice that for any sequence $\{a_1, a_2, a_3, a_4, a_5\}$ in S, the

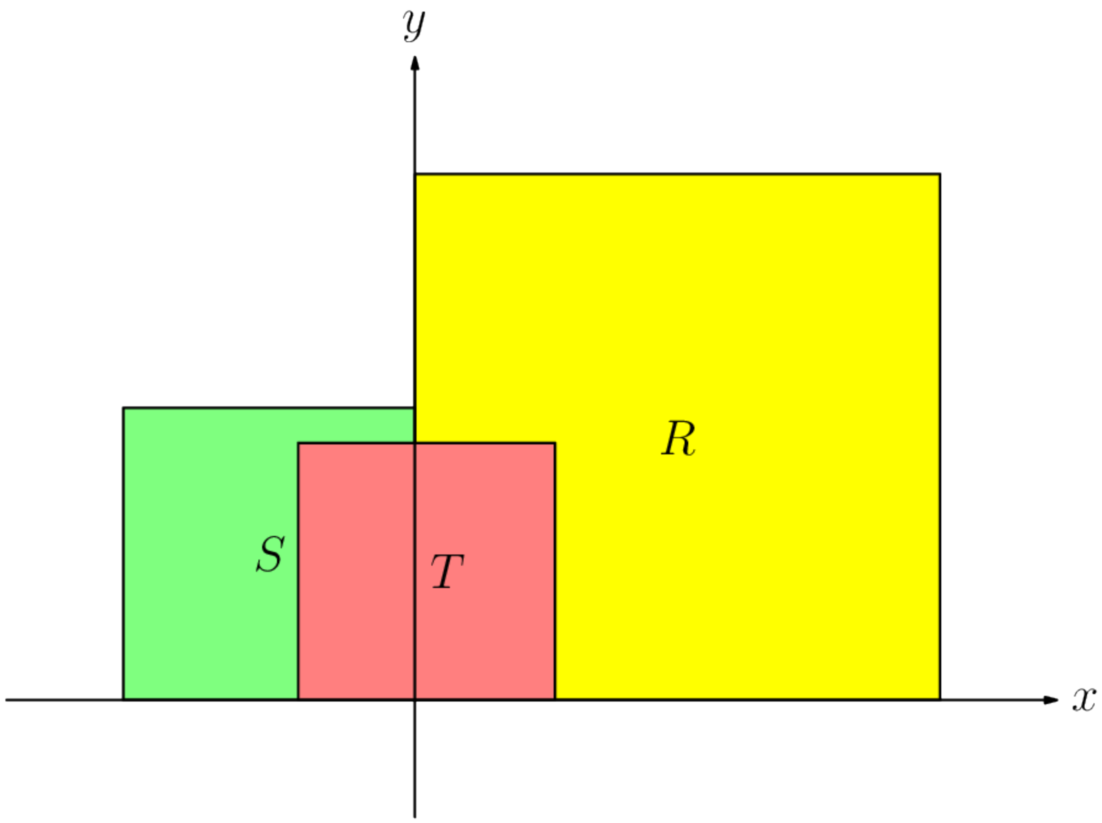
$$\{a_2, a_3, a_4, a_5, a_1\}$$
$$\{a_3, a_4, a_5, a_1, a_2\}$$
$$\{a_4, a_5, a_1, a_2, a_3\}$$

sequences $\{a_5, a_1, a_2, a_3, a_4\}$ must also belong in S. However, one must consider the edge case all 5 elements are the same (only $\{0, 0, 0, 0, 0\}$), in which case all sequences listed are equivalent. Then $|S| \equiv 1 \pmod 5$,

which yields $\boxed{\textbf{(E) } 1296}$ by inspection.

Problem25

Let R, S, and T be squares that have vertices at lattice points (i.e., points whose coordinates are both integers) in the coordinate plane, together with their interiors. The bottom edge of each square is on the x-axis. The left edge of R and the right edge of S are on the y-axis, and Rcontains $\frac{9}{4}$ as many lattice points as does S. The top two vertices of T are in $R \cup S$, and T contains $\frac{1}{4}$ of the lattice points contained in $R \cup S$. See the figure (not drawn to
scale).



The fraction of lattice points in S that are in $S \cap T$ is 27 times the fraction of lattice points in R that are in $R \cap T$. What is the minimum possible value of the edge length of R plus the edge length of S plus the edge length of T?

$\textbf{(A) }336\qquad\textbf{(B) }337\qquad\textbf{(C) }338\qquad\textbf{(D) }339\qquad\textbf{(E) }340$

Solution 1 (Generalized)

Let r be the number of lattice points on the side length of square R, s be the number of lattice points on the side length of square S, and t be the number of lattice points on the side length of square T. Note that the actual lengths of the side lengths are the number of lattice points minus 1, so we can work in terms of r, s, t and subtract 3 to get the actual answer at the end. Furthermore, note that the number of lattice points inside a rectangular region is equal to the number of lattice points in its width times the number of lattice points along its length.

Using this fact, the number of lattice points in R is r^2, the number of lattice points in S is s^2, and the number of lattice points in T is t^2.

Now, by the first condition, we have
$$r^2 = \frac{9}{4} \cdot s^2 \implies r = \frac{3}{2}s \qquad (1)$$

The second condition, the number of lattice points contained in T is a fourth of the number of lattice points contained in $R \cup S$. The number of lattice points in $R \cup S$ is equal to the sum of the lattice points in their individually bounded regions, but the lattice points along the y-axis for the full length of square S is shared by both of them, so we need to subtract that out.

In all, this condition yields us
$$t^2 = \frac{1}{4} \cdot (r^2 + s^2 - s) \implies t^2 = \frac{1}{4} \cdot \left(\frac{9}{4} \cdot s^2 + s^2 - s\right)$$
$$\implies t^2 = \frac{1}{4} \cdot \frac{13s^2 - 4s}{4} \implies 16t^2 = s(13s-4)$$

Note from (1) that s is a multiple of 2. We can write $s = 2j$ and substitute: $16t^2 = 2j(26j - 4) \implies 4t^2 = j(13j - 2)$. Note that j must be divisible by two for the product to be divisible by 4. Thus we make another substitution, $j = 2k$:

$$4t^2 = 2k(26k - 2) \implies t^2 = k(13k - 1) \qquad (2)$$

Finally we look at the last condition; that the fraction of the lattice points inside S that are inside $S \cap T$ is 27 times the fraction of lattice points inside r that are inside $R \cap T$.

Let x be the number of lattice points along the bottom of the rectangle formed by $S \cap T$, and y be the number of lattice points along the bottom of the the rectangle formed by $R \cap T$.

Therefore, the number of lattice points in $S \cap T$ is xt and the number of lattice points in $R \cap T$ is yt.

Thus by this condition,

$$\frac{xt}{s^2} = 27 \cdot \frac{yt}{r^2} \implies \frac{x}{s^2} = 27 \cdot \frac{y}{\frac{9}{4} \cdot s^2} \implies x = 12y$$

Finally, notice that $t = x + y - 1 = 12y + y - 1$ (subtracting overlap), and so we have $t = 13y - 1 \qquad (3)$

Now notice that by (3), $t \equiv -1 \pmod{13} \implies t^2 \equiv 1 \pmod{13}$.

However, by (2), $t^2 \equiv k \cdot -1 \pmod{13}$.

Therefore, $-k \equiv 1 \pmod{13} \implies k \equiv -1 \pmod{13}$

Also, by (2), we know k must be a perfect square since k is relatively prime to $13k-1$ (Euclids algorithm) and the two must multiply to a perfect square. Hence we know two conditions on k, and we can now guess and check to find the smallest that satisfies both.

We check $k=12$ first since its one less than a multiple of 13, but this does not work. Next, we have $k=25$ which works because 25 is a perfect square. Thus we have found the smallest k, and therefore the smallest r, s, t.

Now we just work backwards: $j=2k=50$ and $s=2j=100$.

Then $r=\frac{3}{2}\cdot 100=150$. Finally,

from (2),

$$t^2=25(13\cdot 25-1) \implies t^2=25\cdot 324 \implies t=5\cdot 18=90$$

.

Finally, the sum of each square's side lengths

is $r+s+t-3=340-3=337=\boxed{\textbf{(B) } 337}$.

Solution 2 (Answer Choices)

Notice that each answer choice has a different residue mod 13. Therefore, we can just find the residue of $r+s+t-3$ mod 13 and find the unique answer choice that fits, without actually finding r, s, t.

From Solution 1, we have $16t^2=s(13s-4)$ from the second condition. From the third condition, $t \equiv -1 \pmod{13} \implies t^2 \equiv 1 \pmod{13}$.

Substituting, we get $16\cdot 1 \equiv s\cdot -4 \pmod{13}$.

Therefore, $s \equiv -4 \pmod{13}$. From the first condition, we have $r = \frac{3}{2} \cdot s$, so $r \equiv -6 \pmod{13}$.

Therefore $r + s + t \equiv -6 - 4 - 1 \equiv -11 \equiv 2 \pmod{13}$.

We want to find $r + s + t - 3$, so our answer will have a remainder of -1 when divided by 13.

We divide 340 by 13 and find that the remainder is 2. Therefore the answer that will give us a remainder of -1 will be $340 - 3 = 337 = \boxed{\textbf{(B) } 337}$.

Solution 3 (Quick Solution)

Solution: Let r, s, t be the edge length of square R, S, and T respectively. Then we have

$$(r+1)^2 = \frac{9}{4}(s+1)^2 \qquad (t+1)^2 = \frac{1}{4}((s+1)^2 + (r+1)^2 - (s+1))$$

Therefore $r = \dfrac{3s+1}{2} \qquad t = \dfrac{1}{4}\sqrt{(s+1)(13s+9)} - 1$

Therefore

$$r + s + t = \frac{3s+1}{2} + s + \frac{1}{4}\sqrt{(s+1)(13s+9)} - 1$$

$$\approx \frac{5}{2}s + \frac{\sqrt{13}}{4}s - \frac{1}{2} \approx 3.4 \cdot s$$

Given that average of the answer choices is around 340, therefore $s \approx 100$.

Since t is an integer, therefore $(s+1)(13s+9)$ must be a perfect square divisible by 16. Plugging in $s = 99$, $t = 89$ and $s = 149$.

Therefore $r + s + t = 99 + 89 + 149 = 337$. So the answer is $\boxed{\text{(B) } 337}$.