2009 AMC 10B Problems/Problem 1

The following problem is from both the 2009 AMC 10B #1 and 2009 AMC 12B #1, so both problems redirect to this page.

Problem

Each morning of her five-day workweek, Jane bought either a 50-cent muffin or a 75-cent bagel. Her total cost for the week was a whole number of dollars, How many bagels did she buy?

$\mathrm{(A)}\ 1\qquad\mathrm{(B)}\ 2\qquad\mathrm{(C)}\ 3\qquad\mathrm{(D)}\ 4\qquad\mathrm{(E)}\ 5$

Solution

The only combination of five items with total cost a whole number of dollars is 3 muffins and $\boxed{2}$ bagels. The answer is $\mathrm{(B)}$.

See also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by First Question	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by First Question	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 10B Problems/Problem 3

The following problem is from both the 2009 AMC 10B #3 and 2009 AMC 12B #2, so both problems redirect to this page.

Problem

Paula the painter had just enough paint for 30 identically sized rooms. Unfortunately, on the way to work, three cans of paint fell off her truck, so she had only enough paint for 25 rooms. How many cans of paint did she use for the 25 rooms?

$\mathrm{(A)}\ 10 \qquad \mathrm{(B)}\ 12 \qquad \mathrm{(C)}\ 15 \qquad \mathrm{(D)}\ 18 \qquad \mathrm{(E)}\ 25$

Solution

Losing three cans of paint corresponds to being able to paint five fewer rooms. So $\frac{3}{5} \cdot 25 = \boxed{15}$. The answer is $\mathrm{(C)}$.

See also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 10B Problems/Problem 5

The following problem is from both the 2009 AMC 10B #5 and 2009 AMC 12B #3, so both problems redirect to this page.

Problem

Twenty percent off 60 is one-third more than what number?

$\mathrm{(A)}\ 16 \qquad \mathrm{(B)}\ 30 \qquad \mathrm{(C)}\ 32 \qquad \mathrm{(D)}\ 36 \qquad \mathrm{(E)}\ 48$

Solution

Twenty percent less than 60 is $\frac{4}{5} \cdot 60 = 48$. One-third more than a number n is $\frac{4}{3}n$. Therefore $\frac{4}{3}n = 48$ and the number is $\boxed{36}$. The answer is $\mathrm{(D)}$.

See also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 10B Problems/Problem 4

The following problem is from both the 2009 AMC 10B #4 and 2009 AMC 12B #4, so both problems redirect to this page.

Problem

A rectangular yard contains two flower beds in the shape of congruent isosceles right triangles. The remainder of the yard has a trapezoidal shape, as shown. The parallel sides of the trapezoid have lengths 15 and 25 meters. What fraction of the yard is occupied by the flower beds?

$\mathrm{(A)}\frac18 \qquad \mathrm{(B)}\frac16 \qquad \mathrm{(C)}\frac15 \qquad \mathrm{(D)}\frac14 \qquad \mathrm{(E)}\frac13$

Solution

Each triangle has leg length $\frac12 \cdot (25 - 15) = 5$ meters and area $\frac12 \cdot 5^2 = \frac{25}{2}$ square meters. Thus the flower beds have a total area of 25 square meters. The entire yard has length 25 m and width 5 m, so its area is 125 square meters. The fraction of the yard occupied by the flower beds is $\frac{25}{125} = \boxed{\frac15}$. The answer is $\mathrm{(C)}$.

See also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 10B Problems/Problem 6

> The following problem is from both the 2009 AMC 10B #6 and 2009 AMC 12B #5, so both problems redirect to this page.

Problem

Kiana has two older twin brothers. The product of their three ages is 128. What is the sum of their three ages?

$\mathrm{(A)}\ 10\qquad \mathrm{(B)}\ 12\qquad \mathrm{(C)}\ 16\qquad \mathrm{(D)}\ 18\qquad \mathrm{(E)}\ 24$

Solution

The age of each person is a factor of $128 = 2^7$. So the twins could be $2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8$ years of age and, consequently Kiana could be 128, 32, 8 or 2 years old, respectively. Because Kiana is younger than her brothers, she must be 2 years old. So the sum of their ages is $2 + 8 + 8 = \boxed{18}$. The answer is $\mathrm{(D)}$.

See also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 10B Problems/Problem 7

The following problem is from both the 2009 AMC 10B #7 and 2009 AMC 12B #6, so both problems redirect to this page.

Problem

By inserting parentheses, it is possible to give the expression

$$2 \times 3 + 4 \times 5$$

several values. How many different values can be obtained?

$\text{(A) } 2 \qquad \text{(B) } 3 \qquad \text{(C) } 4 \qquad \text{(D) } 5 \qquad \text{(E) } 6$

Solution

The three operations can be performed on any of $3! = 6$ orders. However, if the addition is performed either first or last, then multiplying in either order produces the same result. So at most four distinct values can be obtained. It is easy to check that the values of the four expressions

$$\begin{aligned}(2 \times 3) + (4 \times 5) &= 26,\\ (2 \times 3 + 4) \times 5 &= 50,\\ 2 \times (3 + (4 \times 5)) &= 46,\\ 2 \times (3 + 4) \times 5 &= 70\end{aligned}$$

are in fact all distinct. So the answer is $\boxed{4}$, which is choice (C).

See also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 10B Problems/Problem 8

The following problem is from both the 2009 AMC 10B #8 and 2009 AMC 12B #7, so both problems redirect to this page.

Problem

In a certain year the price of gasoline rose by 20% during January, fell by 20% during February, rose by 25% during March, and fell by $x\%$ during April. The price of gasoline at the end of April was the same as it had been at the beginning of January. To the nearest integer, what is x

$\mathrm{(A)}\ 12\qquad \mathrm{(B)}\ 17\qquad \mathrm{(C)}\ 20\qquad \mathrm{(D)}\ 25\qquad \mathrm{(E)}\ 35$

Solution

Let p be the price at the beginning of January. The price at the end of March was $(1.2)(0.8)(1.25)p = 1.2p.$ Because the price at the of April was p, the price decreased by $0.2p$ during April, and the percent decrease was

$$x = 100 \cdot \frac{0.2p}{1.2p} = \frac{100}{6} \approx 16.7.$$

So to the nearest integer x is $\boxed{17}$. The answer is $\mathrm{(B)}$.

See also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 7	Followed by Problem 9
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 10B Problems/Problem 15

The following problem is from both the 2009 AMC 10B #15 and 2009 AMC 12B #8, so both problems redirect to this page.

Contents

Problem

When a bucket is two-thirds full of water, the bucket and water weigh a kilograms. When the bucket is one-half full of water the total weight is b kilograms. In terms of a and b, what is the total weight in kilograms when the bucket is full of water?

$\mathrm{(A)}\ \frac{2}{3}a + \frac{1}{3}b\qquad \mathrm{(B)}\ \frac{3}{2}a - \frac{1}{2}b\qquad \mathrm{(C)}\ \frac{3}{2}a + b\qquad \mathrm{(D)}\ \frac{3}{2}a + 2b\qquad \mathrm{(E)}\ 3a - 2b$

Solution

Solution 1

Let x be the weight of the bucket and let y be the weight of the water in a full bucket. Then we are given that $x + \frac{2}{3}y = a$ and $x + \frac{1}{2}y = b$. Hence $\frac{1}{6}y = a - b$, so $y = 6a - 6b$. Thus $x = b - \frac{1}{2}(6a - 6b) = -3a + 4b$. Finally $x + y = \boxed{3a - 2b}$. The answer is $\mathrm{(E)}$.

Solution 2

Imagine that we take three buckets of the first type, to get rid of the fraction. We will have three buckets and two buckets' worth of water.

On the other hand, if we take two buckets of the second type, we will have two buckets and enough water to fill one bucket.

The difference between these is exactly one bucket full of water, hence the answer is $3a - 2b$.

Solution 3

We are looking for an expression of the form $xa + yb$.

We must have $x + y = 1$, as the desired result contains exactly one bucket. Also, we must have $\frac{2}{3}x + \frac{1}{2}y = 1$, as the desired result contains exactly one bucket of water.

At this moment, it is easiest to check that only the options (A), (B), and (E) satisfy $x + y = 1$, and out of these only (E) satisfies the second equation.

Alternately, we can directly solve the system, getting $x = 3$ and $y = -2$.

See also

2009 AMC 12B Problems/Problem 9

Contents

Problem

Triangle ABC has vertices $A = (3,0)$, $B = (0,3)$, and C, where C is on the line $x + y = 7$. What is the area of $\triangle ABC$?

$\mathrm{(A)}\ 6\qquad \mathrm{(B)}\ 8\qquad \mathrm{(C)}\ 10\qquad \mathrm{(D)}\ 12\qquad \mathrm{(E)}\ 14$

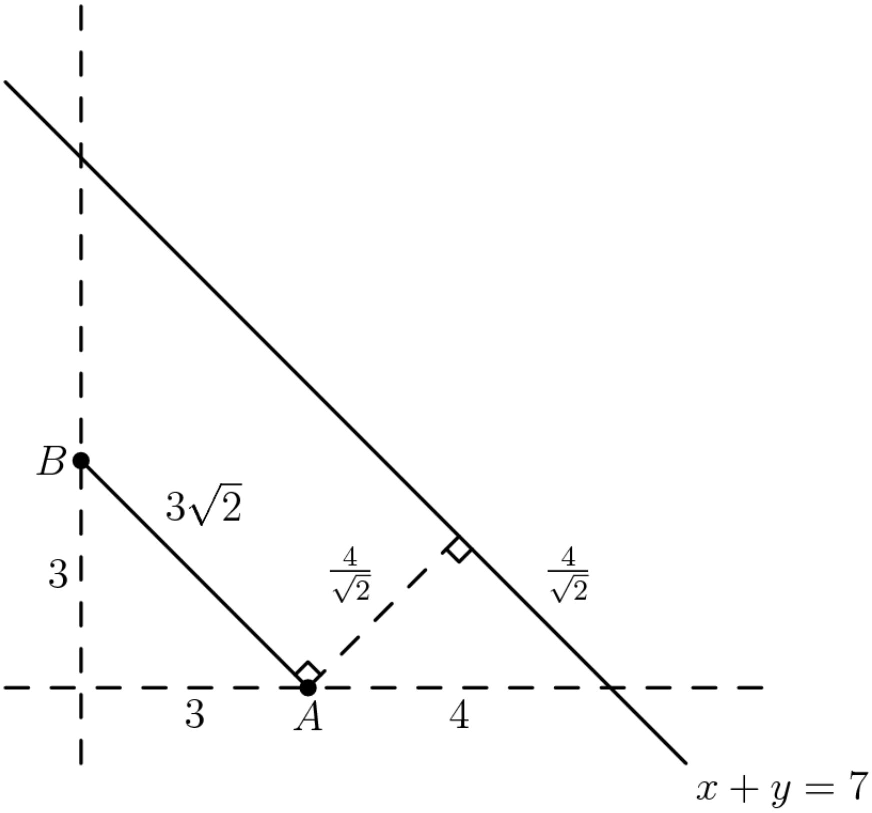
Solution

Solution 1

Because the line $x + y = 7$ is parallel to $\overline{AB}$, the area of $\triangle ABC$ is independent of the location of C on that line. Therefore it may be assumed that C is $(7,0)$. In that case the triangle has base $AC = 4$ and altitude 3, so its area is $\frac{1}{2} \cdot 4 \cdot 3 = \boxed{6}$.

Solution 2

The base of the triangle is $AB = \sqrt{3^2 + 3^2} = 3\sqrt{2}$. Its altitude is the distance between the point A and the parallel line $x + y = 7$, which is $\frac{4}{\sqrt{2}} = 2\sqrt{2}$. Therefore its area is $\frac{1}{2} \cdot 3\sqrt{2} \cdot 2\sqrt{2} = \boxed{6}$. The answer is $\mathrm{(A)}$.



See also

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Categories: Introductory Geometry Problems | Area Problems

2009 AMC 10B Problems/Problem 19

The following problem is from both the 2009 AMC 10B #19 and 2009 AMC 12B #10, so both problems redirect to this page.

Contents

Problem

A particular 12-hour digital clock displays the hour and minute of a day. Unfortunately, whenever it is supposed to display a 1, it mistakenly displays a 9. For example, when it is 1:16 PM the clock incorrectly shows 9:96 PM. What fraction of the day will the clock show the correct time?

$$\mathrm{(A)}\ \frac 12\qquad \mathrm{(B)}\ \frac 58\qquad \mathrm{(C)}\ \frac 34\qquad \mathrm{(D)}\ \frac 56\qquad \mathrm{(E)}\ \frac {9}{10}$$

Solution

Solution 1

The clock will display the incorrect time for the entire hours of $1, 10, 11$ and 12. So the correct hour is displayed $\frac 23$ of the time. The minutes will not display correctly whenever either the tens digit or the ones digit is a 1, so the minutes that will not display correctly are $10, 11, 12, \dots, 19$ and $01, 21, 31, 41,$ and 51. This amounts to fifteen of the sixty possible minutes for any given hour. Hence the fraction of the day that the clock shows the correct time is $\frac 23 \cdot \left(1 - \frac {15}{60}\right) = \frac 23 \cdot \frac 34 = \boxed{\frac 12}$. The answer is $\mathrm{(A)}$.

Solution 2

The required fraction is the number of correct times divided by the total times. There are 60 minutes in an hour and 12 hours on a clock, so there are 720 total times.

We count the correct times directly; let a correct time be $x:yz$, where x is a number from 1 to 12 and y and z are digits, where $y < 6$. There are 8 values of x that will display the correct time: 2, 3, 4, 5, 6, 7, 8, and 9. There are five values of y that will display the correct time: 0, 2, 3, 4, and 5. There are nine values of z that will display the correct time: 0, 2, 3, 4, 5, 6, 7, 8, and 9. Therefore there are $8\cdot 5\cdot 9=40\cdot 9=360$ correct times.

Therefore the required fraction is $\frac{360}{720}=\frac{1}{2}\Rightarrow \boxed{\mathrm{(A)}}$.

See also

2009 AMC 10B Problems/Problem 14

The following problem is from both the 2009 AMC 10B #14 and 2009 AMC 12B #11, so both problems redirect to this page.

Problem

On Monday, Millie puts a quart of seeds, 25% of which are millet, into a bird feeder. On each successive day she adds another quart of the same mix of seeds without removing any seeds that are left. Each day the birds eat only 25% of the millet in the feeder, but they eat all of the other seeds. On which day, just after Millie has placed the seeds, will the birds find that more than half the seeds in the feeder are millet?

$\textbf{(A)}\ \text{Tuesday}\qquad \textbf{(B)}\ \text{Wednesday}\qquad \textbf{(C)}\ \text{Thursday}\qquad \textbf{(D)}\ \text{Friday}\qquad \textbf{(E)}\ \text{Saturday}$

Solution

On Monday, day 1, the birds find $\frac14$ quart of millet in the feeder. On Tuesday they find

$$\frac14 + \frac34 \cdot \frac14$$

quarts of millet. On Wednesday, day 3, they find

$$\frac14 + \frac34 \cdot \frac14 + \left(\frac34\right)^2 \cdot \frac14$$

quarts of millet. The number of quarts of millet they find on day n is

$$\frac14+\frac34\cdot\frac14+\left(\frac34\right)^2\cdot\frac14+\cdots+\left(\frac34\right)^{n-1}\cdot\frac14 = \frac{(\frac14)(1-(\frac34)^n)}{1-\frac34} = 1-\left(\frac34\right)^n.$$

The birds always find $\frac34$ quart of other seeds, so more than half the seeds are millet if $1-\left(\frac34\right)^n > \frac34$, that is, when $\left(\frac34\right)^n < \frac14$. Because $\left(\frac34\right)^4 = \frac{81}{256} > \frac14$ and $\left(\frac34\right)^5 = \frac{243}{1024} < \frac14$, this will first occur on day 5 which is $\boxed{\text{Friday}}$. The answer is (D).

See also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B Problems/Problem 12

Problem

The fifth and eighth terms of a geometric sequence of real numbers are $7!$ and $8!$ respectively. What is the first term?

$\text{(A) } 60 \qquad \text{(B) } 75 \qquad \text{(C) } 120 \qquad \text{(D) } 225 \qquad \text{(E) } 315$

Solution

Let the nth term of the series be ar^{n-1}. Because

$$\frac{8!}{7!} = \frac{ar^7}{ar^4} = r^3 = 8,$$

it follows that $r = 2$ and the first term is $a = \frac{7!}{r^4} = \frac{7!}{16} = \boxed{315}$. The answer is (E).

See also

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 12B Problems/Problem 13

Problem

Triangle ABC has $AB = 13$ and $AC = 15$, and the altitude to $\overline{BC}$ has length 12. What is the sum of the two possible values of BC?

$\mathrm{(A) \ } 15\qquad \mathrm{(B) \ } 16\qquad \mathrm{(C) \ } 17\qquad \mathrm{(D) \ } 18\qquad \mathrm{(E) \ } 19$

Solution

Let D be the foot of the altitude to $\overline{B}C$. Then $BD = \sqrt{13^2 - 12^2} = 5$ and $DC = \sqrt{15^2 - 12^2} = 9$. Thus $BC = BD + BC = 5 + 9 = 14$ or $BC = DC - BD = 9 - 5 = 4$. The sum of the two possible values is $14 + 4 = \boxed{18}$. The answer is $\mathrm{(D)}$.

See also

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 12	Followed by Problem 14
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

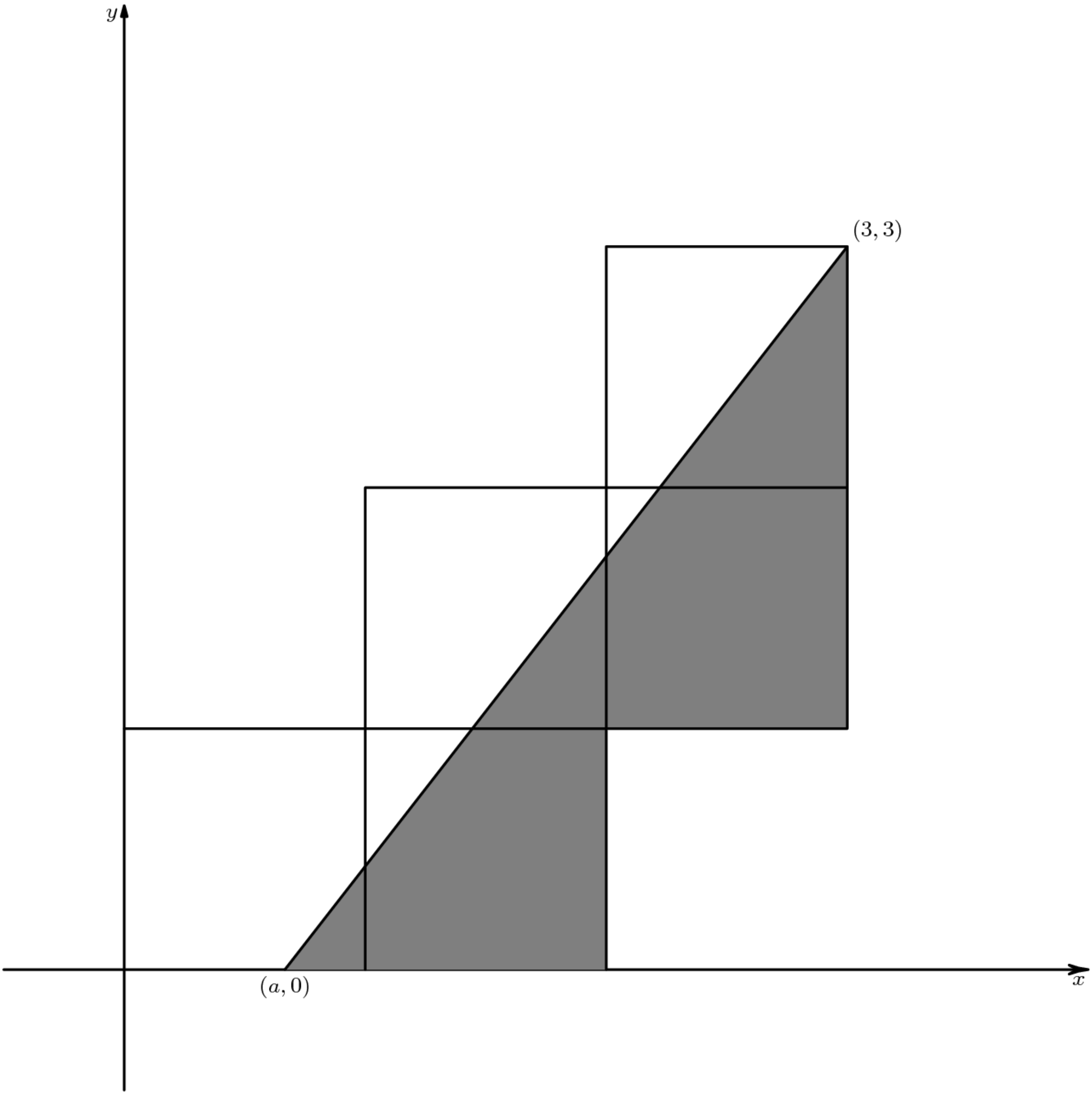
2009 AMC 12B Problems/Problem 14

The following problem is from both the 2009 AMC 10B #17 and 2009 AMC 12B #14, so both problems redirect to this page.

Contents

Problem

Five unit squares are arranged in the coordinate plane as shown, with the lower left corner at the origin. The slanted line, extending from $(a,0)$ to $(3,3)$, divides the entire region into two regions of equal area. What is a?



(A) $\frac{1}{2}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{4}{5}$

Solution

For $a \geq 1.5$ the shaded area is at most 1.5, which is too little. Hence $a < 1.5$, and therefore the point $(2,1)$ is indeed inside the shaded part, as shown in the picture.

Then the area of the shaded part is one less than the area of the triangle with vertices $(a,0)$, $(3,0)$, and $(3,3)$. Its area is obviously $\frac{3(3-a)}{2}$, therefore the area of the shaded part is $\frac{7-3a}{2}$.

The entire figure has area 5, hence we want the shaded part to have area $\frac{5}{2}$. Solving for a, we get $a = \boxed{\frac{2}{3}}$. The answer is (C).

Solution 2

The unit square is of area 1, so the five unit squares have area 5. Therefore the shaded space must occupy 2.5. The missing unit square is of area 1, and if reconstituted the original triangle would be of are 3.5. It can then be inferred: (3-a) * 3 = 7.

3-a = $\frac{7}{3}$, so 3 − $\frac{7}{3}$ = a.

3 − $\frac{7}{3}$ = $\frac{9-7}{3}$ = $\frac{2}{3}$. (C).

See Also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2009 AMC 12B Problems/Problem 15

Problem

Assume $0 < r < 3$. Below are five equations for x. Which equation has the largest solution x?

$\textbf{(A)}\ 3(1 + r)^x = 7\qquad \textbf{(B)}\ 3(1 + r/10)^x = 7\qquad \textbf{(C)}\ 3(1 + 2r)^x = 7$
$\textbf{(D)}\ 3(1 + \sqrt {r})^x = 7\qquad \textbf{(E)}\ 3(1 + 1/r)^x = 7$

Solution

(B) Intuitively, x will be largest for that option for which the value in the parentheses is smallest.

Formally, first note that each of the values in parentheses is larger than 1. Now, each of the options is of the form $3f(r)^x = 7$. This can be rewritten as $x\log f(r) = \log\frac 73$. As $f(r)>1$, we have $\log f(r)>0$. Thus x is the largest for the option for which $\log f(r)$ is smallest. And as $\log f(r)$ is an increasing function, this is the option for which $f(r)$ is smallest.

We now get the following easier problem: Given that $0<r<3$, find the smallest value in the set $\{ 1+r, 1+r/10, 1+2r, 1+\sqrt r, 1+1/r\}$.

Clearly $1+r/10$ is smaller than the first and the third option.

We have $r^2 < 10$, dividing both sides by $10r$ we get $r/10 < 1/r$.

And finally, $r/100 < 1$, therefore $r^2/100 < r$, and as both sides are positive, we can take the square root and get $r/10 < \sqrt r$.

Thus the answer is $\boxed{\textbf{(B)} \ 3\left(1 + \frac{r}{10}\right)^x = 7}$.

See Also

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 12B Problems/Problem 16

Contents

Problem

Trapezoid $ABCD$ has $AD||BC$, $BD = 1$, $\angle DBA = 23^{\circ}$, and $\angle BDC = 46^{\circ}$. The ratio $BC: AD$ is $9: 5$. What is CD?

$$\mathrm{(A)}\ \frac 79 \qquad \mathrm{(B)}\ \frac 45 \qquad \mathrm{(C)}\ \frac {13}{15} \qquad \mathrm{(D)}\ \frac 89 \qquad \mathrm{(E)}\ \frac {14}{15}$$

Solution

Solution 1

Extend $\overline{AB}$ and $\overline{DC}$ to meet at E. Then

$$\begin{aligned} \angle BED &= 180^\circ - \angle EDB - \angle DBE \\ &= 180^\circ - 134^\circ - 23^\circ = 23^\circ. \end{aligned}$$

Thus $\triangle BDE$ is isosceles with $DE = BD$. Because $\overline{AD} \parallel \overline{BC}$, it follows that the triangles BCE and ADE are similar. Therefore

$$\frac{9}{5} = \frac{BC}{AD} = \frac{CD + DE}{DE} = \frac{CD}{BD} + 1 = CD + 1,$$

so $CD = \boxed{\frac 45}$.

Solution 2

Let E be the intersection of $\overline{BC}$ and the line through D parallel to $\overline{AB}$. By constuction $BE=AD$ and $\angle BDE=23^\circ$; it follows that DE is the bisector of the angle BDC. So by the Angle Bisector Theorem we get

$$CD = \frac{CD}{BD} = \frac{EC}{BE} = \frac{BC-BE}{BE} = \frac{BC}{AD} - 1 = \frac 95 - 1 = \boxed{\frac 45}.$$

The answer is $\mathrm{(B)}$.

See also

2009 AMC 10B Problems/Problem 25

The following problem is from both the 2009 AMC 10B #25 and 2009 AMC 12B #17, so both problems redirect to this page.

Contents

Problem

Each face of a cube is given a single narrow stripe painted from the center of one edge to the center of the opposite edge. The choice of the edge pairing is made at random and independently for each face. What is the probability that there is a continuous stripe encircling the cube?

$\mathrm{(A)}\frac{1}{8}\qquad\mathrm{(B)}\frac{3}{16}\qquad\mathrm{(C)}\frac{1}{4}\qquad\mathrm{(D)}\frac{3}{8}\qquad\mathrm{(E)}\frac{1}{2}$

Solution

Solution 1

There are two possible stripe orientations for each of the six faces of the cube, so there are $2^6=64$ possible stripe combinations. There are three pairs of parallel faces so, if there is an encircling stripe, then the pair of faces that do not contribute uniquely determine the stripe orientation for the remaining faces. In addition, the stripe on each face that does not contribute may be oriented in either of two different ways. So a total of $3\cdot2\cdot2=12$ stripe combinations on the cube result in a continuous stripe around the cube. The required probability is $\frac{12}{64}=\boxed{\frac{3}{16}}$.

Solution 2

Without loss of generality, orient the cube so that the stripe on the top face goes from front to back. There are two mutually exclusive ways for there to be an encircling stripe: either the front, bottom and back faces are painted to complete an encircling stripe with the top face's stripe or the front, right, back and left faces are painted to form an encircling stripe. The probability of the first case is $\left(\frac{1}{2}\right)^3=\frac{1}{8}$, and the probability of the second case is $(\frac{1}{2})^4=\frac{1}{16}$. The cases are disjoint, so the probabilities sum $\frac{1}{8}+\frac{1}{16}=\boxed{\frac{3}{16}}$.

Solution 3

There are three possible orientations of an encircling stripe. For any one of these to appear, the stripes on the four faces through which the continuous stripe is to pass must be properly aligned. The probability of each such stripe alignment is $\left(\frac{1}{2}\right)^4=\frac{1}{16}$. Since there are three such possibilities and they are disjoint, the total probability is $3 \cdot \frac{1}{16} = \boxed{\frac{3}{16}}$. The answer is (B).

Solution 4

Consider a vertex on the cube and the three faces that are adjacent to that vertex. If no two stripes on those three faces are aligned, then there is no stripe encircling the cube. The probability that the stripes aren't aligned is $\frac{1}{4}$, since for each alignment of one stripe, there is one and only one way to align the other two stripes out of four total possibilities. therefore there is a probability of $\frac{3}{4}$ that two stripes are aligned.

Now consider the opposing vertex and the three sides adjacent to it. Given the two connected stripes next to our first vertex, we have two more that must be connected to make a continuous stripe. There is a probability of $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ that they are aligned, so there is a probability of $\frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$ that there is a continuous stripe.

See also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 24	Followed by Last question
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 10B Problems/Problem 23

The following problem is from both the 2009 AMC 10B #23 and 2009 AMC 12B #18, so both problems redirect to this page.

Problem

Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the same line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture?

$\mathrm{(A)}\frac{1}{16} \qquad \mathrm{(B)}\frac{1}{8} \qquad \mathrm{(C)}\frac{3}{16} \qquad \mathrm{(D)}\frac{1}{4} \qquad \mathrm{(E)}\frac{5}{16}$

Solution

After 10 minutes (600 seconds), Rachel will have completed 6 laps and be 30 seconds from completing her seventh lap. Because Rachel runs one-fourth of a lap in 22.5 seconds, she will be in the picture between 18.75 seconds and 41.25 seconds of the tenth minute. After 10 minutes Robert will have completed 7 laps and will be 40 seconds past the starting line. Because Robert runs one-fourth of a lap in 20 seconds, he will be in the picture between 30 and 50 seconds of the tenth minute. Hence both Rachel and Robert will be in the picture if it is taken between 30 and 41.25 seconds of the tenth minute. So the probability that both runners are in the picture is $\frac{41.25-30}{60} = \boxed{\frac{3}{16}}$. The answer is $\mathrm{(C)}$.

See also

2009 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Combinatorics Problems

2009 AMC 12B Problems/Problem 19

Contents

Problem

For each positive integer n, let $f(n) = n^4 - 360n^2 + 400$. What is the sum of all values of $f(n)$ that are prime numbers?

$\textbf{(A)}\ 794\qquad \textbf{(B)}\ 796\qquad \textbf{(C)}\ 798\qquad \textbf{(D)}\ 800\qquad \textbf{(E)}\ 802$

Solution

Solution 1

To find the answer it was enough to play around with f. One can easily find that $f(1)=41$ is a prime, then f becomes negative for n between 2 and 18, and then $f(19)=761$ is again a prime number. And as $41 + 761 = 802$ is already the largest option, the answer must be $\boxed{802}$.

Solution 2

We will now show a complete solution, with a proof that no other values are prime.

Consider the function $g(x) = x^2 - 360x + 400$, then obviously $f(x) = g(x^2)$.

The roots of g are:

$$x_{1,2} = \frac{ 360 \pm \sqrt{ 360^2 - 4\cdot 400 } }2 = 180 \pm 80 \sqrt 5$$

We can then write $g(x) = (x - 180 - 80\sqrt 5)(x - 180 + 80\sqrt 5)$, and thus $f(x) = (x^2 - 180 - 80\sqrt 5)(x^2 - 180 + 80\sqrt 5)$.

We would now like to factor the right hand side further, using the formula $(x^2 - y^2) = (x-y)(x+y)$. To do this, we need to express both constants as squares of some other constants. Luckily, we have a pretty good idea how they look like.

We are looking for rational a and b such that $(a+b\sqrt 5)^2 = 180 + 80\sqrt 5$. Expanding the left hand side and comparing coefficients, we get $ab=40$ and $a^2 + 5b^2 = 180$. We can easily guess (or compute) the solution $a=10$, $b=4$.

Hence $180 + 80\sqrt 5 = (10 + 4\sqrt 5)^2$, and we can easily verify that also $180 - 80\sqrt 5 = (10 - 4\sqrt 5)^2$.

We now know the complete factorization of $f(x)$:

$$f(x) = (x-10-4\sqrt{5})(x+10+4\sqrt{5})(x-10+4\sqrt{5})(x+10-4\sqrt{5})$$

As the final step, we can now combine the factors in a different way, in order to get rid of the square roots.

We have $(x-10-4\sqrt{5})(x-10+4\sqrt{5}) = (x-10)^2 - (4\sqrt{5})^2 = x^2 - 20x + 20$, and $(x+10-4\sqrt{5})(x+10+4\sqrt{5}) = x^2 + 20x + 20$.

Hence we obtain the factorization $f(x) = (x^2 - 20x + 20)(x^2 + 20x + 20)$.

For $x \geq 20$ both terms are positive and larger than one, hence $f(x)$ is not prime. For $1 < x < 19$ the second factor is positive and the first one is negative, hence $f(x)$ is not a prime. The remaining cases are $x = 1$ and $x = 19$. In both cases, $f(x)$ is indeed a prime, and their sum is $f(1) + f(19) = 41 + 761 = \boxed{802}$.

Solution 3

Instead of doing the hard work, we can try to guess the factorization. One good approach:

We can make the observation that $f(x)$ looks similar to $(x^2+20)^2$ with the exception of the x^2 term. In fact, we have $(x^2+20)^2 = x^4 + 40x^2 + 400$. But then we notice that it differs from the desired expression by a square: $f(x) = (x^2+20)^2 - 400x^2 = (x^2+20)^2 - (20x)^2$.

Now we can use the formula $(x^2 - y^2) = (x-y)(x+y)$ to obtain the same factorization as in the previous solution, without all the work.

Solution 4

After arriving at the factorization $f(x) = (x^2 - 20x + 20)(x^2 + 20x + 20)$, a more mathematical approach would be to realize that the second factor is always positive when x is a positive integer. Therefore, in order for $f(x)$ to be prime, the first factor has to be 1.

We can set it equal to 1 and solve for x:

$$x^2 - 20x + 20 = 1$$

$$x^2 - 20x + 19 = 0$$

$$(x-1)(x-19) = 0$$

$$x = 1, x = 19$$

Substituting these values into the second factor and adding would give the answer.

See Also

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 12B Problems/Problem 20

Contents

Problem

A convex polyhedron Q has vertices $V_1, V_2, \ldots, V_n$, and 100 edges. The polyhedron is cut by planes $P_1, P_2, \ldots, P_n$ in such a way that plane P_k cuts only those edges that meet at vertex V_k. In addition, no two planes intersect inside or on Q. The cuts produce n pyramids and a new polyhedron R. How many edges does R have?

$\mathrm{(A)}\ 200\qquad \mathrm{(B)}\ 2n\qquad \mathrm{(C)}\ 300\qquad \mathrm{(D)}\ 400\qquad \mathrm{(E)}\ 4n$

Solution

Solution 1

Each edge of Q is cut by two planes, so R has 200 vertices. Three edges of R meet at each vertex, so R has $\frac{1}{2}\cdot 3\cdot 200 = \boxed{300}$ edges.

Solution 2

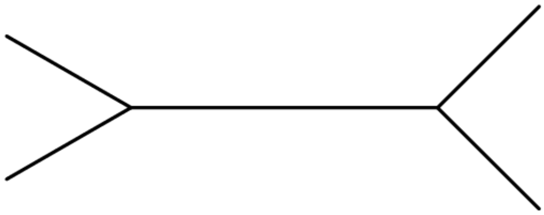
At each vertex, as many new edges are created by this process as there are original edges meeting at that vertex. Thus the total number of new edges is the total number of endpoints of the original edges, which is 200. A middle portion of each original edge is also present in R, so R has $100+200=\boxed{300}$ edges.

Solution 3

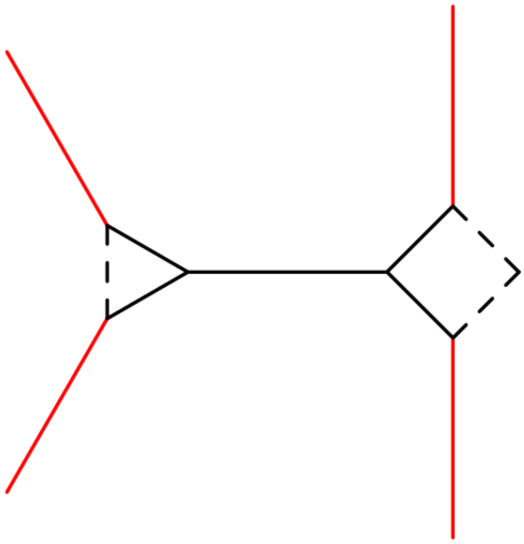
Euler's Polyhedron Formula applied to Q gives $n-100+F=2$, where F is the number of faces of Q. Each edge of Q is cut by two planes, so R has 200 vertices. Each cut by a plane P_k creates an additional face on R, so Euler's Polyhedron Formula applied to R gives $200-E+(F+n)=2$, where E is the number of edges of R. Subtracting the first equation from the second gives $300-E=0$, whence $E=\boxed{300}$. The answer is $\mathrm{(C)}$.

Solution 4

Each edge connects two points. The plane cuts that edge so it splits into 2 at each end (like two legs) for a total of 4 new edges.



But because each new edge is shared by an adjacent original edge cut similarly, the additional edges are overcounted $\times 2$.



Since there are 100 edges to start with, $400/2 = 200$ new edges result. So there are $100 + 200 = \boxed{\textbf{(C)}\ 300}$ edges in the figure.

Solution 5

The question specifies the slices create as many pyramids as there are vertices, implying each vertex owns 4 edge ends. There are twice as many edge-ends as there are edges, and 2 * 100 = 200.

$\frac{200}{4}$ = 50, so there are 50 vertices.

The base of a pyramid has 4 edges, so each sliced vertex would add four edges to R.

100 + 4 * 50 = $\boxed{\textbf{(C)}\ 300}$

See also

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 12B Problems/Problem 21

Contents

Problem

Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can the women be reseated?

$\textbf{(A)}\ 89\qquad \textbf{(B)}\ 90\qquad \textbf{(C)}\ 120\qquad \textbf{(D)}\ 2^{10}\qquad \textbf{(E)}\ 2^2 3^8$

Solution 1

Notice that either a woman stays in her own seat after the rearrangement, or two adjacent women swap places. Thus, our answer is counting the number of ways to arrange 1x1 and 2x1 blocks to form a 1x10 rectangle. This can be done via casework depending on the number of 2x1 blocks. The cases of 0, 1, 2, 3, 4, 5 2x1 blocks correspond to 10, 8, 6, 4, 2, 0 1x1 blocks, and so the sum of the cases is

$$\binom{10}{0}+\binom{9}{1}+\binom{8}{2}+\binom{7}{3}+\binom{6}{4}+\binom{5}{5} = 1+9+28+35+15+1 = \boxed{89}.$$

Solution 2

Let S_n be the number of possible seating arrangements with n women. Consider $n\geq 3,$ and focus on the rightmost woman. If she returns back to her seat, then there are S_{n-1} ways to seat the remaining $n-1$ women. If she sits in the second to last seat, then the woman who previously sat there must now sit at the rightmost seat. This gives us S_{n-2} ways to seat the other $n-2$ women, so we obtain the recursion

$$S_n = S_{n-1}+S_{n-2}.$$

Starting with $S_1 = 1$ and $S_2 = 2,$ we can calculate $S_{10} = \boxed{89}.$

See Also

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 20	Followed by Problem 22
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 12B Problems/Problem 22

Contents

Problem

Parallelogram $ABCD$ has area $1{,}000{,}000$. Vertex A is at $(0,0)$ and all other vertices are in the first quadrant. Vertices B and D are lattice points on the lines $y=x$ and $y=kx$ for some integer $k>1$, respectively. How many such parallelograms are there?

$\textbf{(A)}\ 49\qquad \textbf{(B)}\ 720\qquad \textbf{(C)}\ 784\qquad \textbf{(D)}\ 2009\qquad \textbf{(E)}\ 2048$

Solution

Solution 1

The area of any parallelogram $ABCD$ can be computed as the size of the vector product of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

In our setting where $A=(0,0)$, $B=(s,s)$, and $D=(t,kt)$ this is simply $s\cdot kt - s\cdot t = (k-1)st$.

In other words, we need to count the triples of integers (k,s,t) where $k>1$, $s,t>0$ and $(k-1)st = 1{,}000{,}000 = 2^6 5^6$.

These can be counted as follows: We have 6 identical red balls (representing powers of 2), 6 blue balls (representing powers of 5), and three labeled urns (representing the factors $k-1$, s, and t). The red balls can be distributed in $\binom{8}{2} = 28$ ways, and for each of these ways, the blue balls can then also be distributed in 28 ways. (See Distinguishability for a more detailed explanation.)

Thus there are exactly $28^2 = 784$ ways how to break $1{,}000{,}000$ into three positive integer factors, and for each of them we get a single parallelogram. Hence the number of valid parallelograms is $\boxed{784}$.

Solution 2

Without the vector product the area of $ABCD$ can be computed for example as follows: If $B=(s,s)$ and $D=(t,kt)$, then clearly $C=(s+t,s+kt)$. Let $B'=(s,0)$, $C'=(s+t,0)$ and $D'=(t,0)$ be the orthogonal projections of B, C, and D onto the x axis. Let $[P]$ denote the area of the polygon P. We can then compute:

$$\begin{align*}
[ABCD] & = [ADD'] + [DD'C'C] - [BB'C'C] - [ABB'] \\
& = \frac{kt^2}{2} + \frac{s(s+2kt)}{2} - \frac{t(2s+kt)}{2} - \frac{s^2}{2} \\
& = kst - st \\
& = (k-1)st.
\end{align*}$$

The remainder of the solution is the same as the above.

2009 AMC 12B Problems/Problem 23

Problem

A region S in the complex plane is defined by

$$S = \{x + iy : -1 \le x \le 1, -1 \le y \le 1\}.$$

A complex number $z = x + iy$ is chosen uniformly at random from S. What is the probability that $\left(\frac{3}{4} + \frac{3}{4}i\right) z$ is also in S?

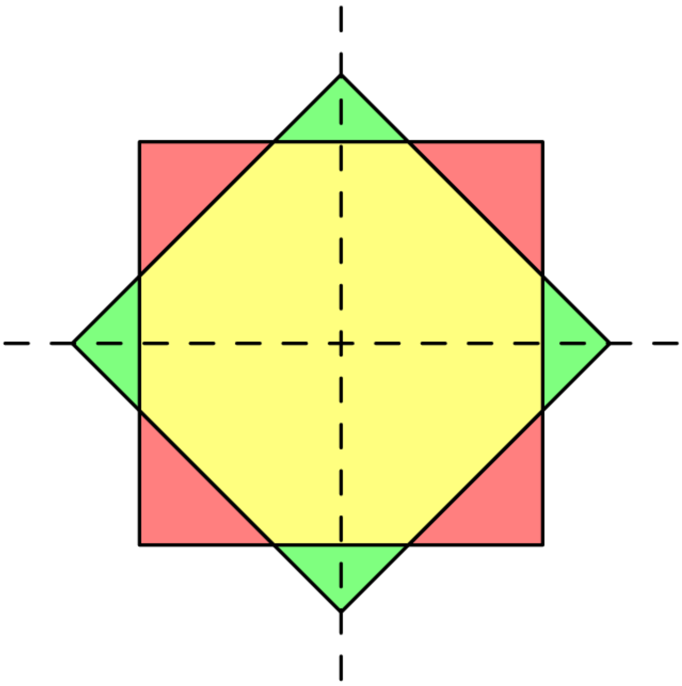
$\textbf{(A)}\ \frac{1}{2} \qquad \textbf{(B)}\ \frac{2}{3} \qquad \textbf{(C)}\ \frac{3}{4} \qquad \textbf{(D)}\ \frac{7}{9} \qquad \textbf{(E)}\ \frac{7}{8}$

Solution

We can directly compute $\left(\frac{3}{4} + \frac{3}{4}i\right) z = \left(\frac{3}{4} + \frac{3}{4}i\right)(x + iy) = \frac{3(x-y)}{4} + \frac{3(x+y)}{4} \cdot i$.

This number is in S if and only if $-1 \le \frac{3(x-y)}{4} \le 1$ and at the same time $-1 \le \frac{3(x+y)}{4} \le 1$. This simplifies to $|x-y| \le \frac{4}{3}$ and $|x+y| \le \frac{4}{3}$.

Let $T = \{x + iy : |x-y| \le \frac{4}{3} \ \land \ |x+y| \le \frac{4}{3}\}$, and let $[X]$ denote the area of the region X. Then obviously the probability we seek is $\frac{[S \cap T]}{[S]} = \frac{[S \cap T]}{4}$. All we need to do is to compute the area of the intersection of S and T. It is easiest to do this graphically:



Coordinate axes are dashed, S is shown in red, T in green and their intersection is yellow. The intersections of the boundary of S and T are obviously at $(\pm 1, \pm 1/3)$ and at $(\pm 1/3, \pm 1)$.

Hence each of the four red triangles is an isosceles right triangle with legs long $\frac{2}{3}$, and hence the area of a single red triangle is $\frac{1}{2} \cdot \left(\frac{2}{3}\right)^2 = \frac{2}{9}$. Then the area of all four is $\frac{8}{9}$, and therefore the area of $S \cap T$ is $4 - \frac{8}{9}$. Then the probability we seek is $\frac{[S \cap T]}{4} = \frac{4 - \frac{8}{9}}{4} = 1 - \frac{2}{9} = \boxed{\frac{7}{9}}$.

(Alternately, when we got to the point that we know that a single red triangle is $\frac{2}{9}$, we can directly note that the picture is symmetric, hence we can just consider the first quadrant and there the probability is $1 - \frac{2}{9} = \frac{7}{9}$. This saves us the work of first multiplying and then dividing by 4.)

See Also

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2009 AMC 12B Problems/Problem 24

Problem

For how many values of x in $[0, \pi]$ is $\sin^{-1}(\sin 6x) = \cos^{-1}(\cos x)$? Note: The functions $\sin^{-1} = \arcsin$ and $\cos^{-1} = \arccos$ denote inverse trigonometric functions.

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 7$

Solution

First of all, we have to agree on the range of $\sin^{-1}$ and $\cos^{-1}$. This should have been a part of the problem statement -- but as it is missing, we will assume the most common definition: $\forall x : -\pi/2 \le \sin^{-1}(x) \le \pi/2$ and $\forall x : 0 \le \cos^{-1}(x) \le \pi$.

Hence we get that $\forall x \in [0, \pi] : \cos^{-1}(\cos x) = x$, thus our equation simplifies to $\sin^{-1}(\sin 6x) = x$.

Consider the function $f(x) = \sin^{-1}(\sin 6x) - x$. We are looking for roots of f on $[0, \pi]$.

By analyzing properties of $\sin$ and $\sin^{-1}$ (or by computing the derivative of f) one can discover the following properties of f:

- $f(0) = 0$.
- f is increasing and then decreasing on $[0, \pi/6]$.
- f is decreasing and then increasing on $[\pi/6, 2\pi/6]$.
- f is increasing and then decreasing on $[2\pi/6, 3\pi/6]$.

For $x = \pi/6$ we have $f(x) = \sin^{-1}(\sin \pi) - \pi/6 = -\pi/6 < 0$. Hence f has exactly one root on $(0, \pi/6)$.

For $x = 2\pi/6$ we have $f(x) = \sin^{-1}(\sin 2\pi) - 2\pi/6 = -2\pi/6 < 0$. Hence f is negative on the entire interval $[\pi/6, 2\pi/6]$.

Now note that $\forall t : \sin^{-1}(t) \le \pi/2$. Hence for $x > 3\pi/6$ we have $f(x) < 0$, and we can easily check that $f(3\pi/6) < 0$ as well.

Thus the only unknown part of f is the interval $(2\pi/6, 3\pi/6)$. On this interval, f is negative in both endpoints, and we know that it is first increasing and then decreasing. Hence there can be zero, one, or two roots on this interval.

To prove that there are two roots, it is enough to find any x from this interval such that $f(x) > 0$.

A good guess is its midpoint, $x = 5\pi/12$, where the function $\sin^{-1}(\sin 6x)$ has its local maximum. We can evaluate: $f(x) = \sin^{-1}(\sin 5\pi/2) - 5\pi/12 = \pi/2 - 5\pi/12 = \pi/12 > 0$.

Summary: The function f has $\boxed{4}$ roots on $[0, \pi]$: the first one is 0, the second one is in $(0, \pi/6)$, and the last two are in $(2\pi/6, 3\pi/6)$.

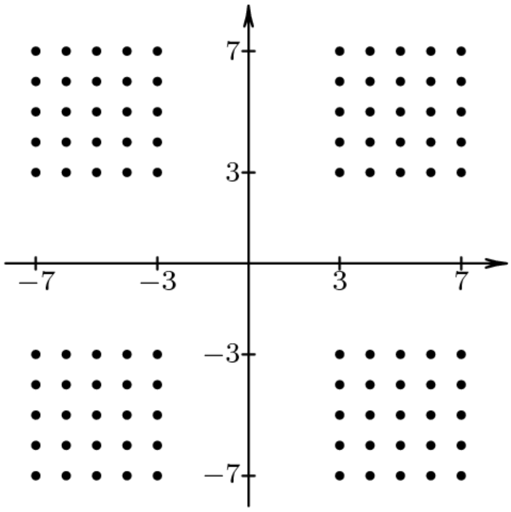
See Also

2009 AMC 12B Problems/Problem 25

Contents

Problem

The set G is defined by the points (x,y) with integer coordinates, $3 \le |x| \le 7$, $3 \le |y| \le 7$. How many squares of side at least 6 have their four vertices in G?



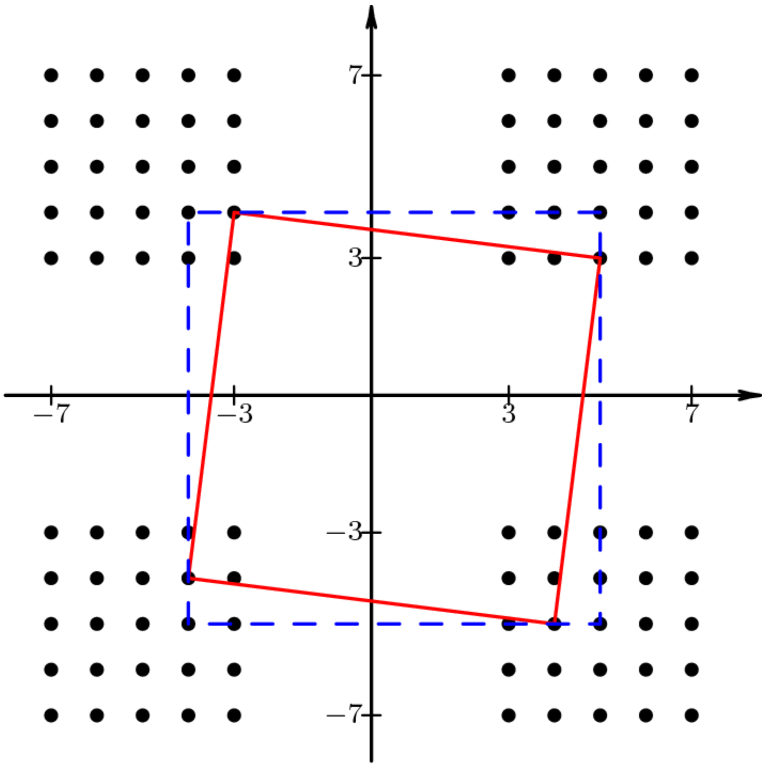
$\textbf{(A)}\ 125\qquad \textbf{(B)}\ 150\qquad \textbf{(C)}\ 175\qquad \textbf{(D)}\ 200\qquad \textbf{(E)}\ 225$

Solution

We need to find a reasonably easy way to count the squares.

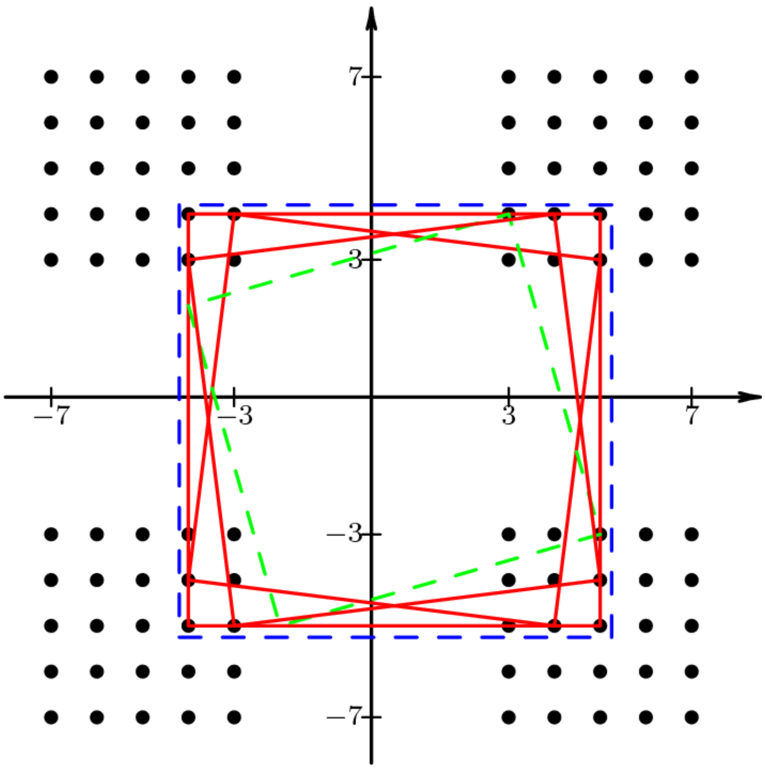
First, obviously the maximum distance between two points in the same quadrant is $4\sqrt{2} < 6$, hence each square has exactly one vertex in each quadrant.

Given any square, we can circumscribe another axes-parallel square around it. In the picture below, the original square is red and the circumscribed one is blue.



Let's now consider the opposite direction. Assume that we picked the blue square, how many different red squares do share it?

Answering this question is not as simple as it may seem. Consider the picture below. It shows all three red squares that share the same blue square. In addition, the picture shows a green square that is not valid, as two of its vertices are in bad locations.



The size of the blue square can range from 6×6 to 14×14, and for the intermediate sizes there is more than one valid placement. We will now examine the cases one after another. Also, we can use symmetry to reduce the number of cases.

size	upper_right	solutions	symmetries	total
6	$(3, 3)$	1	1	1

7	$(3,3)$	1	4	4
8	$(3,3)$	1	4	4
8	$(3,4)$	1	4	4
8	$(4,4)$	3	1	3
9	$(3,3)$	1	4	4
9	$(3,4)$	1	8	8
9	$(4,4)$	3	4	12
10	$(3,3)$	1	4	4
10	$(3,4)$	1	8	8
10	$(3,5)$	1	4	4
10	$(4,4)$	3	4	12
10	$(4,5)$	3	4	12
10	$(5,5)$	5	1	5
11	$(4,4)$	3	4	12
11	$(4,5)$	3	8	24
11	$(5,5)$	5	4	20
12	$(5,5)$	5	4	20
12	$(5,6)$	5	4	20
12	$(6,6)$	7	1	7
13	$(6,6)$	7	4	28
14	$(7,7)$	9	1	9

Summing the last column, we get that the answer is $\boxed{225}$.

Solution 2

This is based on a clever bijection given in this page (http://answers.yahoo.com/question/index?qid=20110101175843AAhdbwT).

Consider any square $ABCD$ where all four vertices are in G, and the side length is at least 6, so the four vertices must lie in distinct quadrants (Same proof as in solution 1). Without loss of generality, assume that A, B, C, D are in the first, second, third, fourth quadrant. Then we consider the following mapping:

$$A \to A' = A$$

$$B \to B' = B + (10, 0)$$

$$C \to C' = C + (10, 10)$$

$$D \to D' = D + (0, 10)$$

Then the new points A', B', C', D' are either being the same point or forming a square in $G_1 = G \cap \{x > 0, y > 0\}$, a 5x5 grid.

Conversely, for any point in G_1, it can be reversed to a square $ABCD$; however, for any square in G_1, there are four possible squares $ABCD$ that were mapped to them. Therefore the number of possible squares $ABCD$ is equal to $25 + 4N$, where N is the number of squares inscribed in G_1.

Moreover by the same idea in solution 1, each square (with sides parallel or slanted to the axes) in a G_1 can be inscribed in a square in G_1, with sides parallel to one of the axes, call it "standard square". Noticing that each standard square of side length a corresponds to a inscribed squares, and that there are $(5-a)^2$ number of standard squares of side length a, we have

$$N = \sum_{a=1}^{4} a(5-a)^2 = 1\cdot16+2\cdot9+3\cdot4+4\cdot1 = 16+18+12+4 = 50$$

So the answer is $25 + 4 \cdot 50 = 225$

See Also

2009 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 24	Followed by Last Question
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
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