2008 AMC 12B Problems/Problem 1

Problem

A basketball player made 5 baskets during a game. Each basket was worth either 2 or 3 points. How many different numbers could represent the total points scored by the player?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 6$

Solution

If the basketball player makes x three-point shots and $5-x$ two-point shots, he scores $3x+2(5-x)=10+x$ points. Clearly every value of x yields a different number of total points. Since he can make any number of three-point shots between 0 and 5 inclusive, the number of different point totals is $6 \Rightarrow E$.

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by First question	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 12B Problems/Problem 2

The following problem is from both the 2008 AMC 12B #2 and 2008 AMC 10B #2, so both problems redirect to this page.

Problem

A 4×4 block of calendar dates is shown. The order of the numbers in the second row is to be reversed. Then the order of the numbers in the fourth row is to be reversed. Finally, the numbers on each diagonal are to be added. What will be the positive difference between the two diagonal sums?

1	2	3	4
8	9	10	11
15	16	17	18
22	23	24	25

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 10$

Solution

After reversing the numbers on the second and fourth rows, the block will look like this:

1	2	3	4
11	10	9	8
15	16	17	18
25	24	23	22

The difference between the two diagonal sums is:
$(4+9+16+25)-(1+10+17+22)=3-1-1+3=4 \Rightarrow B.$

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2008))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2008 AMC 12B Problems/Problem 3

Problem 3

A semipro baseball league has teams with 21 players each. League rules state that a player must be paid at least $15,000$ dollars, and that the total of all players' salaries for each team cannot exceed $700,000$ dollars. What is the maximum possiblle salary, in dollars, for a single player?

$\textbf{(A)}\ 270,000 \qquad \textbf{(B)}\ 385,000 \qquad \textbf{(C)}\ 400,000 \qquad \textbf{(D)}\ 430,000 \qquad \textbf{(E)}\ 700,000$

Solution

We want to find the maximum any player could make, so assume that everyone else makes the minimum possible and that the combined salaries total the maximum of $700,000$

$$700,000 = 20 * 15,000 + x$$

$$x = 400,000$$

The maximum any player could make is $400,000$ dollars $\Rightarrow C$

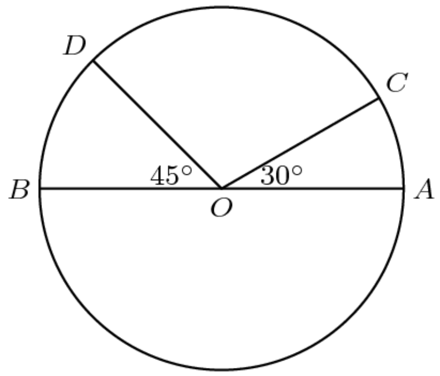
See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 12B Problems/Problem 4

Problem

On circle O, points C and D are on the same side of diameter $\overline{AB}$, $\angle AOC = 30^\circ$, and $\angle DOB = 45^\circ$. What is the ratio of the area of the smaller sector COD to the area of the circle?



$\textbf{(A)}\ \frac{2}{9} \qquad \textbf{(B)}\ \frac{1}{4} \qquad \textbf{(C)}\ \frac{5}{18} \qquad \textbf{(D)}\ \frac{7}{24} \qquad \textbf{(E)}\ \frac{3}{10}$

Solution

$\angle COD = \angle AOB - \angle AOC - \angle BOD = 180^\circ - 30^\circ - 45^\circ = 105^\circ.$

Since a circle has 360°, the desired ratio is $\dfrac{105^\circ}{360^\circ} = \dfrac{7}{24} \Rightarrow D.$

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 12B Problems/Problem 5

Problem 5

A class collects 50 dollars to buy flowers for a classmate who is in the hospital. Roses cost 3 dollars each, and carnations cost 2 dollars each. No other flowers are to be used. How many different bouquets could be purchased for exactly 50 dollars?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 7 \qquad \textbf{(C)}\ 9 \qquad \textbf{(D)}\ 16 \qquad \textbf{(E)}\ 17$

Solution

The class could send 25 carnations and no roses, 22 carnations and 2 roses, 19 carnations and 4 roses, and so on, down to 1 carnation and 16 roses. There are 9 total possibilities (from 0 to 16 roses, incrementing by 2 at each step), $\Rightarrow \boxed{C}$

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 12B Problems/Problem 6

Problem 6

Postman Pete has a pedometer to count his steps. The pedometer records up to 99999 steps, then flips over to 00000 on the next step. Pete plans to determine his mileage for a year. On January 1 Pete sets the pedometer to 00000. During the year, the pedometer flips from 99999 to 00000 forty-four times. On December 31 the pedometer reads 50000. Pete takes 1800 steps per mile. Which of the following is closest to the number of miles Pete walked during the year?

$\textbf{(A)}\ 2500 \qquad \textbf{(B)}\ 3000 \qquad \textbf{(C)}\ 3500 \qquad \textbf{(D)}\ 4000 \qquad \textbf{(E)}\ 4500$

Solution

Every time the pedometer flips, Pete has walked $100,000$ steps. Therefore, Pete has walked a total of $100,000 * 44 + 50,000 = 4,450,000$ steps, which is $4,450,000/1,800 = 2472.2$ miles, which is closest to answer choice $\boxed{A}$.

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Combinatorics Problems

2008 AMC 12B Problems/Problem 7

Problem 7

For real numbers a and b, define $a\$b = (a-b)^2$. What is $(x-y)^2\$(y-x)^2$?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ x^2+y^2 \qquad \textbf{(C)}\ 2x^2 \qquad \textbf{(D)}\ 2y^2 \qquad \textbf{(E)}\ 4xy$

Solution

$\left[(x-y)^2-(y-x)^2\right]^2$

$\left[(x-y)^2-(x-y)^2\right]^2$

$[0]^2$

$0 \Rightarrow \textbf{(A)}$

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 12B Problems/Problem 8

Problem

Points B and C lie on $\overline{AD}$. The length of $\overline{AB}$ is 4 times the length of $\overline{BD}$, and the length of $\overline{AC}$ is 9 times the length of $\overline{CD}$. The length of $\overline{BC}$ is what fraction of the length of $\overline{AD}$?

$\textbf{(A)}\ \frac{1}{36} \qquad \textbf{(B)}\ \frac{1}{13} \qquad \textbf{(C)}\ \frac{1}{10} \qquad \textbf{(D)}\ \frac{5}{36} \qquad \textbf{(E)}\ \frac{1}{5}$

Solution

Since $\overline{AB} = 4\overline{BD}$ and $\overline{AB} + \overline{BD} = \overline{AD}$, $\overline{AB} = \frac{4}{5}\overline{AD}$.

Since $\overline{AC} = 9\overline{CD}$ and $\overline{AC} + \overline{CD} = \overline{AD}$, $\overline{AC} = \frac{9}{10}\overline{AD}$.

Thus, $\overline{BC} = \overline{AC} - \overline{AB} = \left(\frac{9}{10} - \frac{4}{5}\right)\overline{AD} = \frac{1}{10}\overline{AD} \Rightarrow C$.

See Also

<table>
<tr><td colspan="2">2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))</td></tr>
<tr><td>Preceded by
Problem 7</td><td>Followed by
Problem 9</td></tr>
<tr><td colspan="2">1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25</td></tr>
<tr><td colspan="2">All AMC 12 Problems and Solutions</td></tr>
</table>

2008 AMC 12B Problems/Problem 9

Contents

Problem 9

Points A and B are on a circle of radius 5 and $AB = 6$. Point C is the midpoint of the minor arc AB. What is the length of the line segment AC?

$\textbf{(A)}\ \sqrt{10} \qquad \textbf{(B)}\ \frac{7}{2} \qquad \textbf{(C)}\ \sqrt{14} \qquad \textbf{(D)}\ \sqrt{15} \qquad \textbf{(E)}\ 4$

Solutions

Solution 1

Let α be the angle that subtends the arc AB. By the law of cosines, $6^2=5^2+5^2-2\cdot 5\cdot 5\cos(\alpha)$ implies $\cos(\alpha) = 7/25$.

The half-angle formula says that $\cos(\alpha/2) = \frac{\sqrt{1+\cos(\alpha)}}{2} = \sqrt{\frac{32/25}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$. The law of cosines tells us $AC = \sqrt{5^2+5^2-2\cdot 5\cdot 5\cdot \frac{4}{5}} = \sqrt{50-50\frac{4}{5}} = \sqrt{10}$, which is answer choice $\boxed{\text{A}}$.

Solution 2

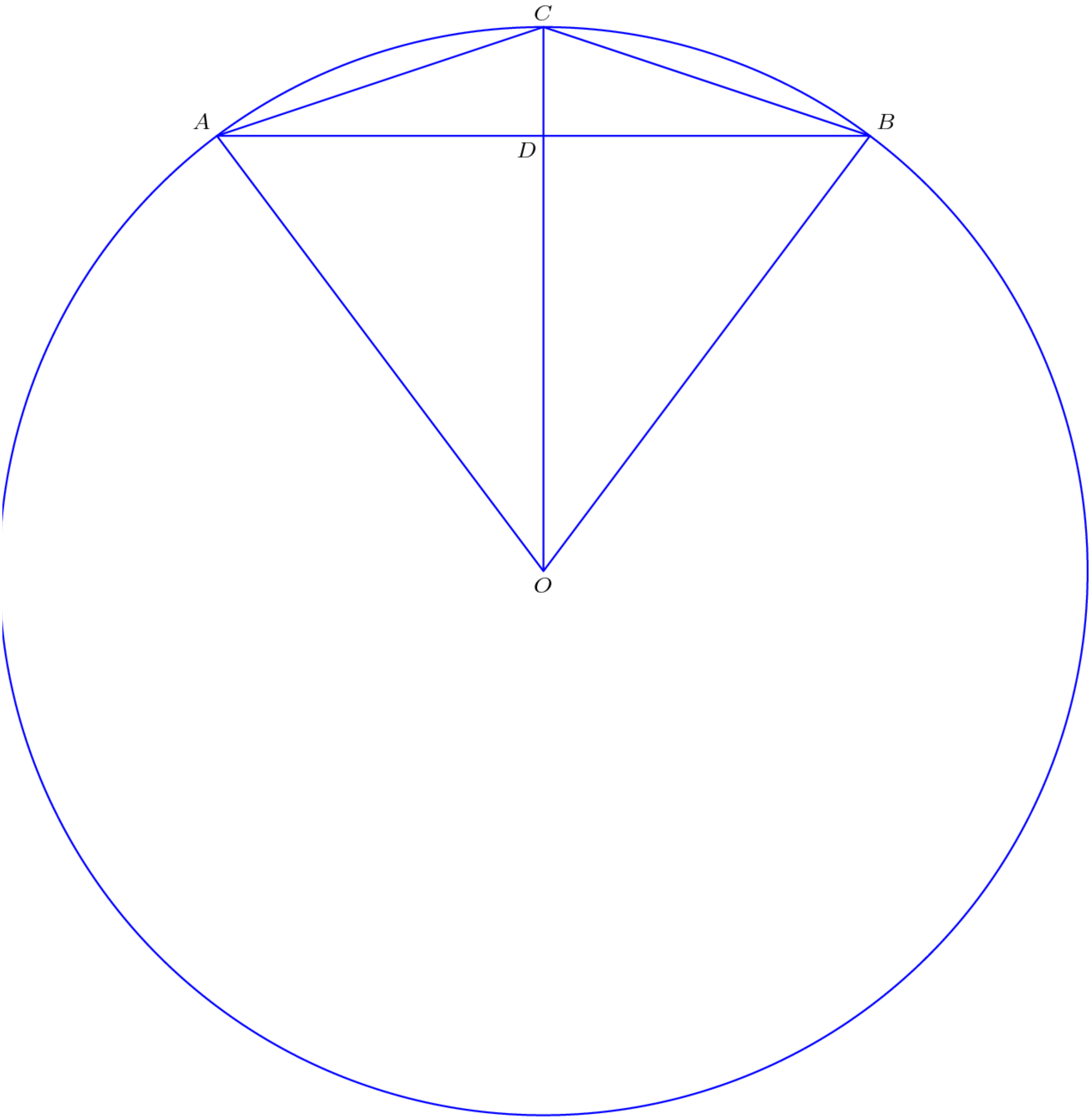


Figure 1

Define D as the midpoint of line segment $\overline{AB}$, and O the center of the circle. Then O, C, and D are collinear, and since D is the midpoint of AB, $m\angle ODA = 90\deg$ and so $OD = \sqrt{5^2 - 3^2} = 4$. Since $OD = 4$, $CD = 5 - 4 = 1$, and so $AC = \sqrt{3^2 + 1^2} = \sqrt{10} \rightarrow \boxed{\text{A}}$.

See Also

2008 AMC 12B Problems/Problem 10

Problem

Bricklayer Brenda would take 9 hours to build a chimney alone, and bricklayer Brandon would take 10 hours to build it alone. When they work together they talk a lot, and their combined output is decreased by 10 bricks per hour. Working together, they build the chimney in 5 hours. How many bricks are in the chimney?

$\textbf{(A)}\ 500 \qquad \textbf{(B)}\ 900 \qquad \textbf{(C)}\ 950 \qquad \textbf{(D)}\ 1000 \qquad \textbf{(E)}\ 1900$

Solution

Let h be the number of bricks in the chimney.

Without talking, Brenda and Brandon lay $\frac{h}{9}$ and $\frac{h}{10}$ bricks per hour respectively, so together they lay $\frac{h}{9} + \frac{h}{10} - 10$ per hour together.

Since they finish the chimney in 5 hours, $h = 5\left(\frac{h}{9} + \frac{h}{10} - 10\right)$. Thus, $h = 900 \Rightarrow B$.

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2008 AMC 12B Problems/Problem 11

Problem 11

A cone-shaped mountain has its base on the ocean floor and has a height of 8000 feet. The top $\frac{1}{8}$ of the volume of the mountain is above water. What is the depth of the ocean at the base of the mountain in feet?

$\textbf{(A)}\ 4000 \qquad \textbf{(B)}\ 2000(4-\sqrt{2}) \qquad \textbf{(C)}\ 6000 \qquad \textbf{(D)}\ 6400 \qquad \textbf{(E)}\ 7000$

Solution

In a cone, radius and height each vary inversely with increasing height (i.e. the radius of the cone formed by cutting off the mountain at $4,000$ feet is half that of the original mountain). Therefore, volume varies as the inverse cube of increasing height (expressed as a percentage of the total height of cone):

$V_I \times \text{Height}^3 = V_N$

Plugging in our given condition, $\frac{1}{8} = \text{Height}^3 \Rightarrow \text{Height} = \frac{1}{2}$.

$8000 \cdot \frac{1}{2} = 4000 \Rightarrow \boxed{\textbf{A}}$.

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Categories: Introductory Geometry Problems | 3D Geometry Problems

2008 AMC 12B Problems/Problem 12

Contents

Problem 12

For each positive integer n, the mean of the first n terms of a sequence is n. What is the 2008th term of the sequence?

$\textbf{(A)}\ 2008 \qquad \textbf{(B)}\ 4015 \qquad \textbf{(C)}\ 4016 \qquad \textbf{(D)}\ 4030056 \qquad \textbf{(E)}\ 4032064$

Solution

Letting S_n be the nth partial sum of the sequence:

$$\frac{S_n}{n} = n$$

$$S_n = n^2$$

The only possible sequence with this result is the sequence of odd integers.

$$a_n = 2n - 1$$

$$a_{2008} = 2(2008) - 1 = 4015 \Rightarrow \textbf{(B)}$$

Alternate Solution

Letting the sum of the sequence equal $a_1 + a_2 + \cdots + a_n$ yields the following two equations:

$\dfrac{a_1 + a_2 + \cdots + a_{2008}}{2008} = 2008$ and

$\dfrac{a_1 + a_2 + \cdots + a_{2007}}{2007} = 2007$.

Therefore:

$a_1 + a_2 + \cdots + a_{2008} = 2008^2$ and $a_1 + a_2 + \cdots + a_{2007} = 2007^2$

Hence, by substitution,

$$a_{2008} = 2008^2 - 2007^2 = (2008 + 2007)(2008 - 2007) = 4015(1) = 4015 \implies \boxed{\textbf{B}}$$

See Also

2008 AMC 12B Problems/Problem 13

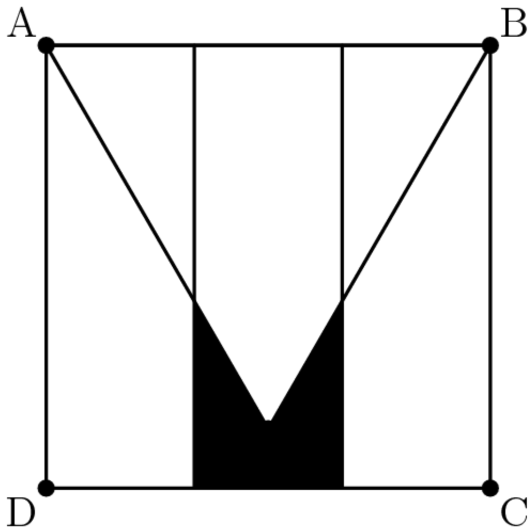
Problem

Vertex E of equilateral $\triangle ABE$ is in the interior of unit square $ABCD$. Let R be the region consisting of all points inside $ABCD$ and outside $\triangle ABE$ whose distance from AD is between $\frac{1}{3}$ and $\frac{2}{3}$. What is the area of R?

$\textbf{(A)}\ \frac{12-5\sqrt3}{72} \qquad \textbf{(B)}\ \frac{12-5\sqrt3}{36} \qquad \textbf{(C)}\ \frac{\sqrt3}{18} \qquad \textbf{(D)}\ \frac{3-\sqrt3}{9} \qquad \textbf{(E)}\ \frac{\sqrt3}{12}$

Solution

The region is the shaded area:



We can find the area of the shaded region by subtracting the pentagon from the middle third of the square. The area of the middle third of the square is $\left(\frac{1}{3}\right)(1)=\frac{1}{3}$. The pentagon can be split into a rectangle and an equilateral triangle.

The base of the equilateral triangle is $\frac{1}{3}$ and the height is $\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)(\sqrt{3})=\frac{\sqrt{3}}{6}$. Thus, the area is $\left(\frac{\sqrt{3}}{6}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)=\frac{\sqrt{3}}{36}$.

The base of the rectangle is $\frac{1}{3}$ and the height is the height of the equilateral triangle minus the height of the smaller equilateral triangle. This is: $\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{6}=\frac{\sqrt{3}}{3}$ Therefore, the area of the shaded region is

$$\frac{1}{3}-\frac{\sqrt{3}}{9}-\frac{\sqrt{3}}{36}=\boxed{\text{(B)}\ \frac{12-5\sqrt{3}}{36}}.$$

See Also

2008 AMC 12B Problems/Problem 14

Problem

A circle has a radius of $\log_{10}{(a^2)}$ and a circumference of $\log_{10}{(b^4)}$. What is $\log_{a}{b}$?

$\textbf{(A)}\ \frac{1}{4\pi} \qquad \textbf{(B)}\ \frac{1}{\pi} \qquad \textbf{(C)}\ \pi \qquad \textbf{(D)}\ 2\pi \qquad \textbf{(E)}\ 10^{2\pi}$

Solution

Let C be the circumference of the circle, and let r be the radius of the circle.

Using log properties, $C=\log_{10}{(b^4)}=4\log_{10}{(b)}$ and $r=\log_{10}{(a^2)}=2\log_{10}{(a)}$.

Since $C=2\pi r$, $4\log_{10}{(b)}=2\pi\cdot2\log_{10}{(a)} \Rightarrow \log_{a}{b}=\frac{\log_{10}{(b)}}{\log_{10}{(a)}}=\pi \Rightarrow C$.

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

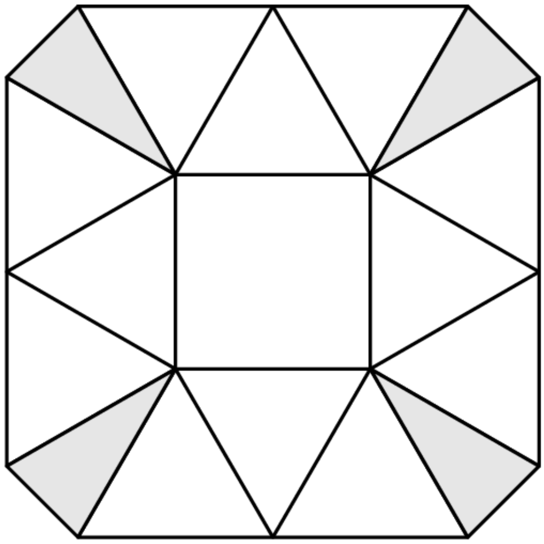
2008 AMC 12B Problems/Problem 15

Problem

On each side of a unit square, an equilateral triangle of side length 1 is constructed. On each new side of each equilateral triangle, another equilateral triangle of side length 1 is constructed. The interiors of the square and the 12 triangles have no points in common. Let R be the region formed by the union of the square and all the triangles, and S be the smallest convex polygon that contains R. What is the area of the region that is inside S but outside R?

$\textbf{(A)} \; \frac{1}{4} \qquad \textbf{(B)} \; \frac{\sqrt{2}}{4} \qquad \textbf{(C)} \; 1 \qquad \textbf{(D)} \; \sqrt{3} \qquad \textbf{(E)} \; 2\sqrt{3}$

Solution



The equilateral triangles form trapezoids with side lengths $1,1,1,2$ (half a unit hexagon) on each face of the unit square. The four triangles "in between" these trapezoids are isosceles triangles with two side lengths 1 and an angle of 30° in between them, so the total area of these triangles (which is the area of $S-R$) is, by the Law of Sines, $4\left(\frac{1}{2}\sin 30^{\circ}\right) = 1$ which makes the answer $\boxed{C}$.

See Also

Region with Squares and Triangles (http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1051031#p1051031)

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 12B Problems/Problem 16

Problem

A rectangular floor measures a by b feet, where a and b are positive integers with $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half of the area of the entire floor. How many possibilities are there for the ordered pair (a, b)?

$\textbf{(A)}\ 1\qquad\textbf{(B)}\ 2\qquad\textbf{(C)}\ 3\qquad\textbf{(D)}\ 4\qquad\textbf{(E)}\ 5$

Solution

$A_{outer} = ab$

$A_{inner} = (a-2)(b-2)$

$A_{outer} = 2A_{inner}$

$ab = 2(a-2)(b-2) = 2ab - 4a - 4b + 8$

$0 = ab - 4a - 4b + 8$

By Simon's Favorite Factoring Trick:

$8 = ab - 4a - 4b + 16 = (a-4)(b-4)$

Since $8 = 1 \times 8$ and $8 = 2 \times 4$ are the only positive factorings of 8.

$(a, b) = (5, 12)$ or $(a, b) = (6, 8)$ yielding $2 \Rightarrow \textbf{(B)}$ solutions. Notice that because $b > a$, the reversed pairs are invalid.

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 15	Followed by Problem 17
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 12B Problems/Problem 17

Problem

Let A, B and C be three distinct points on the graph of $y = x^2$ such that line AB is parallel to the x-axis and $\triangle ABC$ is a right triangle with area 2008. What is the sum of the digits of the y-coordinate of C?

$\textbf{(A)}\ 16\qquad\textbf{(B)}\ 17\qquad\textbf{(C)}\ 18\qquad\textbf{(D)}\ 19\qquad\textbf{(E)}\ 20$

Solution

Supposing $\angle A = 90^\circ$, AC is perpendicular to AB and, it follows, to the x-axis, making AB a segment of the line x=m. But that would mean that the coordinates of C are (m, m^2), contradicting the given that points A and C are distinct. So $\angle A$ is not 90°. By a similar logic, neither is $\angle B$.

This means that $\angle C = 90^\circ$ and AC is perpendicular to BC. Let C be the point (n, n^2). So the slope of BC is the negative reciprocal of the slope of AC, yielding $m + n = \frac{1}{m - n} \Rightarrow$ $m^2 - n^2 = 1$.

Because $m^2 - n^2$ is the length of the altitude of triangle ABC from AB, and $2m$ is the length of AB, the area of $\triangle ABC = m(m^2 - n^2) = 2008$. Since $m^2 - n^2 = 1$, $m = 2008$. Substituting, $2008^2 - n^2 = 1 \Rightarrow$ $n^2 = 2008^2 - 1 = (2000 + 8)^2 - 1 = 4000000 + 32000 + 64 - 1 = 4032063$, whose digits sum to $18 \Rightarrow \textbf{(C)}$.

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 12B Problems/Problem 18

Problem

A pyramid has a square base $ABCD$ and vertex E. The area of square $ABCD$ is 196, and the areas of $\triangle ABE$ and $\triangle CDE$ are 105 and 91, respectively. What is the volume of the pyramid?

$\textbf{(A)}\ 392 \qquad \textbf{(B)}\ 196\sqrt{6} \qquad \textbf{(C)}\ 392\sqrt{2} \qquad \textbf{(D)}\ 392\sqrt{3} \qquad \textbf{(E)}\ 784$

Solution

Let h be the height of the pyramid and a be the distance from h to CD. The side length of the base is 14. The heights of $\triangle ABE$ and $\triangle CDE$ are $2 \cdot 105 \div 14 = 15$ and $2 \cdot 91 \div 14 = 13$, respectively. Consider a side view of the pyramid from $\triangle BCE$. We have a systems of equations through the Pythagorean Theorem:

$$13^2 - (14-a)^2 = h^2$$
$$15^2 - a^2 = h^2$$

Setting them equal to each other and simplifying gives $-27 + 28a = 225 \implies a = 9$.

Therefore, $h = \sqrt{15^2 - 9^2} = 12$, and the volume of the pyramid is $\frac{bh}{3} = \frac{12 \cdot 196}{3} = \boxed{784 \Rightarrow E}$

.

See also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Categories: Introductory Geometry Problems | 3D Geometry Problems

2008 AMC 12B Problems/Problem 19

Problem 19

A function f is defined by $f(z) = (4+i)z^2 + \alpha z + \gamma$ for all complex numbers z, where α and γ are complex numbers and $i^2 = -1$. Suppose that $f(1)$ and $f(i)$ are both real. What is the smallest possible value of $|\alpha| + |\gamma|$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ \sqrt{2} \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 2\sqrt{2} \qquad \textbf{(E)}\ 4$

Solution

We need only concern ourselves with the imaginary portions of $f(1)$ and $f(i)$ (both of which must be 0). These are:

$$\text{Im}(f(1)) = i + i\text{Im}(\alpha) + i\text{Im}(\gamma)$$
$$\text{Im}(f(i)) = -i + i\text{Re}(\alpha) + i\text{Im}(\gamma)$$

Let $p = \text{Im}(\gamma)$ and $q = \text{Re}(\gamma)$, then we know $\text{Im}(\alpha) = -p - 1$ and $\text{Re}(\alpha) = 1 - p$. Therefore

$$|\alpha| + |\gamma| = \sqrt{(1-p)^2 + (-1-p)^2} + \sqrt{q^2 + p^2} = \sqrt{2p^2 + 2} + \sqrt{p^2 + q^2},$$

which reaches its minimum $\sqrt{2}$ when $p = q = 0$. So the answer is $\boxed{B}$.

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 12B Problems/Problem 20

The following problem is from both the 2008 AMC 12B #20 and 2008 AMC 10B #25, so both problems redirect to this page.

Problem

Michael walks at the rate of 5 feet per second on a long straight path. Trash pails are located every 200 feet along the path. A garbage truck traveling at 10 feet per second in the same direction as Michael stops for 30 seconds at each pail. As Michael passes a pail, he notices the truck ahead of him just leaving the next pail. How many times will Michael and the truck intersect?

$\mathrm{(A)}\ 4\qquad\mathrm{(B)}\ 5\qquad\mathrm{(C)}\ 6\qquad\mathrm{(D)}\ 7\qquad\mathrm{(E)}\ 8$

Solution

Pick a coordinate system where Michael's starting pail is 0 and the one where the truck starts is 200. Let $M(t)$ and $T(t)$ be the coordinates of Michael and the truck after t seconds. Let $D(t)=T(t)-M(t)$ be their (signed) distance after t seconds. Meetings occur whenever $D(t)=0$. We have $D(0)=200$.

The truck always moves for 20 seconds, then stands still for 30. During the first 20 seconds of the cycle the truck moves by 200 feet and Michael by 100, hence during the first 20 seconds of the cycle $D(t)$ increases by 100. During the remaining 30 seconds $D(t)$ decreases by 150.

From this observation it is obvious that after four full cycles, i.e. at $t=200$, we will have $D(t)=0$ for the first time.

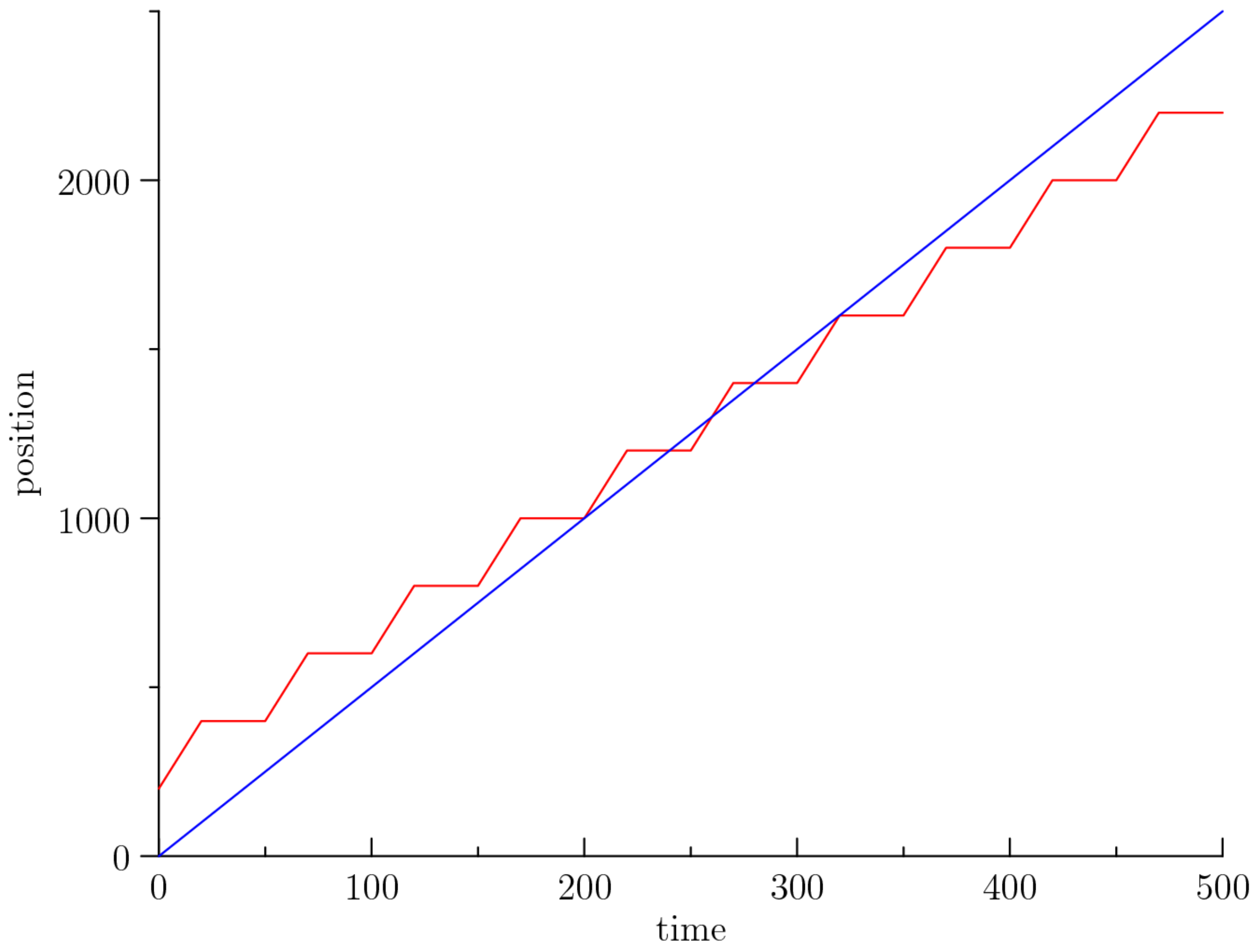
During the fifth cycle, $D(t)$ will first grow from 0 to 100, then fall from 100 to -50. Hence Michael overtakes the truck while it is standing at the pail.

During the sixth cycle, $D(t)$ will first grow from -50 to 50, then fall from 50 to -100. Hence the truck starts moving, overtakes Michael on their way to the next pail, and then Michael overtakes the truck while it is standing at the pail.

During the seventh cycle, $D(t)$ will first grow from -100 to 0, then fall from 0 to -150. Hence the truck meets Michael at the moment when it arrives to the next pail.

Obviously, from this point on $D(t)$ will always be negative, meaning that Michael is already too far ahead. Hence we found all $\boxed{5 \Longrightarrow B}$ meetings.

The movement of Michael and the truck is plotted below: Michael in blue, the truck in red.



See also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2008))	
Preceded by Problem 24	Followed by Last Question
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2008 AMC 12B Problems/Problem 21

Contents

Problem

Two circles of radius 1 are to be constructed as follows. The center of circle A is chosen uniformly and at random from the line segment joining $(0,0)$ and $(2,0)$. The center of circle B is chosen uniformly and at random, and independently of the first choice, from the line segment joining $(0,1)$ to $(2,1)$. What is the probability that circles A and B intersect?

$\textbf{(A)} \ \frac {2 + \sqrt {2}}{4} \qquad \textbf{(B)} \ \frac {3\sqrt {3} + 2}{8} \qquad \textbf{(C)} \ \frac {2\sqrt {2} - 1}{2} \qquad \textbf{(D)} \ \frac {2 + \sqrt {3}}{4} \qquad \textbf{(E)} \ \frac {4\sqrt {3} - 3}{4}$

Solution 1

Circles centered at A and B will overlap if A and B are closer to each other than if the circles were tangent. The circles are tangent when the distance between their centers is equal to the sum of their radii. Thus, the distance from A to B will be 2. Since A and B are separated by 1 vertically, they must be separated by $\sqrt{3}$ horizontally. Thus, if $|A_x-B_x|<\sqrt{3}$, the circles intersect.

Now, plot the two random variables A_x and B_x on the coordinate plane. Each variable ranges from 0 to 2. The circles intersect if the variables are within $\sqrt{3}$ of each other. Thus, the area in which the circles don't intersect is equal to the total area of two small triangles on opposite corners, each of area $\frac{(2-\sqrt{3})^2}{2}$. We conclude the probability the circles intersect is:

$$1-\frac{(2-\sqrt{3})^2}{4}=\boxed{\textbf{(E)} \frac{4\sqrt{3}-3}{4}}.$$

Solution 2

Two circles intersect if the distance between their centers is less than the sum of their radii. In this problem, A and B intersect iff

$$\sqrt{(A_X-B_X)^2+(A_Y-B_Y)^2} \le 2 \Rightarrow (A_X-B_X)^2+1 \le 4 \Rightarrow (A_X-B_X) \le \sqrt{3}.$$

In other words, the two chosen x-coordinates must differ by no more than $\sqrt{3}$. To find this probability, we divide the problem into cases:

1) A_X is on the interval $(0,2-\sqrt{3})$. The probability that B_X falls within the desired range for a given A_X is A_X (on the left) $+\sqrt{3}$ (on the right) all over 2 (the range of possible values). The total probability for this range is the sum of all these probabilities of B_X (over the range of A_X) divided by the total range of A_X (which is 2). Thus, the total probability for this interval is

$$\frac{1}{2}\left(\int_0^{2-\sqrt{3}} \frac{x+\sqrt{3}}{2}\,dx\right) = \frac{1}{2}\left(x^2/4+\frac{x\sqrt{3}}{2}\Big|_0^{2-\sqrt{3}}\right)$$

$$= \frac{1}{2}(\frac{4 - 4\sqrt{3} + 3}{4} + \sqrt{3} - \frac{3}{2}) = \frac{1}{8}.$$

2) A_X is on the interval $(2 - \sqrt{3}, \sqrt{3})$. In this case, any value of B_X will do, so the probability for the interval is simply $\frac{\sqrt{3} - (2 - \sqrt{3})}{2} = \sqrt{3} - 1$.

3) A_X is on the interval $(\sqrt{3}, 2)$. This is identical, by symmetry, to case 1.

The total probability is therefore $\sqrt{3} - 1 + \frac{1}{4} = \frac{4\sqrt{3} - 3}{4} \Rightarrow \boxed{E}$

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 20	Followed by Problem 22
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2008 AMC 12B Problems/Problem 22

Problem 22

A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

$\textbf{(A)} \; \frac {11}{20} \qquad \textbf{(B)} \; \frac {4}{7} \qquad \textbf{(C)} \; \frac {81}{140} \qquad \textbf{(D)} \; \frac {3}{5} \qquad \textbf{(E)} \; \frac {17}{28}$

Solution

Auntie Em won't be able to park only when none of the four available spots touch. We can form a bijection between all such cases and the number of ways to pick four spots out of 13: since none of the spots touch, remove a spot from between each of the cars. From the other direction, given four spots out of 13, simply add a spot between each. So the probability she can park is

$$1-\frac{\binom{13}{4}}{\binom{16}{4}}=1-\frac{13\cdot12\cdot11\cdot10}{16\cdot15\cdot14\cdot13}=1-\frac{11}{28}=\textbf{(E)}\frac{17}{28}.$$

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 21	Followed by Problem 23
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Combinatorics Problems

2008 AMC 12B Problems/Problem 23

Problem 23

The sum of the base-10 logarithms of the divisors of 10^n is 792. What is n?

$\text{(A)}\ 11\qquad \text{(B)}\ 12\qquad \text{(C)}\ 13\qquad \text{(D)}\ 14\qquad \text{(E)}\ 15$

Contents

Solution

Solution 1

Every factor of 10^n will be of the form $2^a \times 5^b, a \leq n, b \leq n$. Using the logarithmic property $\log(a \times b) = \log(a) + \log(b)$, it suffices to count the total number of 2's and 5's running through all possible (a, b). For every factor $2^a \times 5^b$, there will be another $2^b \times 5^a$, so it suffices to count the total number of 2's occurring in all factors (because of this symmetry, the number of 5's will be equal). And since $\log(2) + \log(5) = \log(10) = 1$, the final sum will be the total number of 2's occurring in all factors of 10^n.

There are $n+1$ choices for the exponent of 5 in each factor, and for each of those choices, there are $n+1$ factors (each corresponding to a different exponent of 2), yielding $0 + 1 + 2 + 3... + n = \frac{n(n+1)}{2}$ total 2's. The total number of 2's is therefore $\frac{n \cdot (n+1)^2}{2} = \frac{n^3 + 2n^2 + n}{2}$. Plugging in our answer choices into this formula yields 11 (answer choice (A)) as the correct answer.

Solution 2

We are given

$$\log_{10}d_1 + \log_{10}d_2 + \ldots + \log_{10}d_k = 792$$

The property $\log(ab) = \log(a) + \log(b)$ now gives

$$\log_{10}(d_1 d_2 \cdot \ldots d_k) = 792$$

The product of the divisors is (from elementary number theory) $a^{d(n)/2}$ where $d(n)$ is the number of divisors. Note that $10^n = 2^n \cdot 5^n$, so $d(n) = (n + 1)^2$. Substituting these values with $a = 10^n$ in our equation above, we get $n(n + 1)^2 = 1584$, from whence we immediately obtain $\boxed{\text{(A)}}$ as the correct answer.

Solution 3

For every divisor d of 10^n, $d \le \sqrt{10^n}$, we have $\log d + \log \frac{10^n}{d} = \log 10^n = n$. There are $\left\lfloor \frac{(n+1)^2}{2} \right\rfloor$ divisors of $10^n = 2^n \times 5^n$ that are $\le \sqrt{10^n}$. After casework on the parity of n, we find that the answer is given by $n \times \frac{(n+1)^2}{2} = 792 \implies n = 11 \text{ (A)}$.

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2008 AMC 12B Problems/Problem 24

Problem 24

Let $A_0 = (0,0)$. Distinct points $A_1, A_2, \dots$ lie on the x-axis, and distinct points $B_1, B_2, \dots$ lie on the graph of $y = \sqrt{x}$. For every positive integer n, $A_{n-1}B_nA_n$ is an equilateral triangle. What is the least n for which the length $A_0A_n \geq 100$?

$\textbf{(A)}\ 13\qquad \textbf{(B)}\ 15\qquad \textbf{(C)}\ 17\qquad \textbf{(D)}\ 19\qquad \textbf{(E)}\ 21$

Solution

Let $a_n = |A_{n-1}A_n|$. We need to rewrite the recursion into something manageable. The two strange conditions, B's lie on the graph of $y = \sqrt{x}$ and $A_{n-1}B_nA_n$ is an equilateral triangle, can be compacted as follows:

$$\left(a_n\frac{\sqrt{3}}{2}\right)^2 = \frac{a_n}{2} + a_{n-1} + a_{n-2} + \cdots + a_1$$

which uses $y^2 = x$, where x is the height of the equilateral triangle and therefore $\frac{\sqrt{3}}{2}$ times its base.

The relation above holds for $n = k$ and for $n = k-1$ $(k > 1)$, so

$$\left(a_k\frac{\sqrt{3}}{2}\right)^2 - \left(a_{k-1}\frac{\sqrt{3}}{2}\right)^2 =$$

$$= \left(\frac{a_k}{2} + a_{k-1} + a_{k-2} + \cdots + a_1\right) - \left(\frac{a_{k-1}}{2} + a_{k-2} + a_{k-3} + \cdots + a_1\right)$$

Or,

$$a_k - a_{k-1} = \frac{2}{3}$$

This implies that each segment of a successive triangle is $\frac{2}{3}$ more than the last triangle. To find a_1, we merely have to plug in $k = 1$ into the aforementioned recursion and we have $a_1 - a_0 = \frac{2}{3}$. Knowing that a_0 is 0, we can deduce that $a_1 = 2/3$. Thus, $a_n = \frac{2n}{3}$, so $A_0A_n = a_n + a_{n-1} + \cdots + a_1 = \frac{2}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)}{3}$. We want to find n so that $n^2 < 300 < (n+1)^2$. $n = \boxed{17}$ is our answer.

See Also

2008 AMC 12B Problems/Problem 25

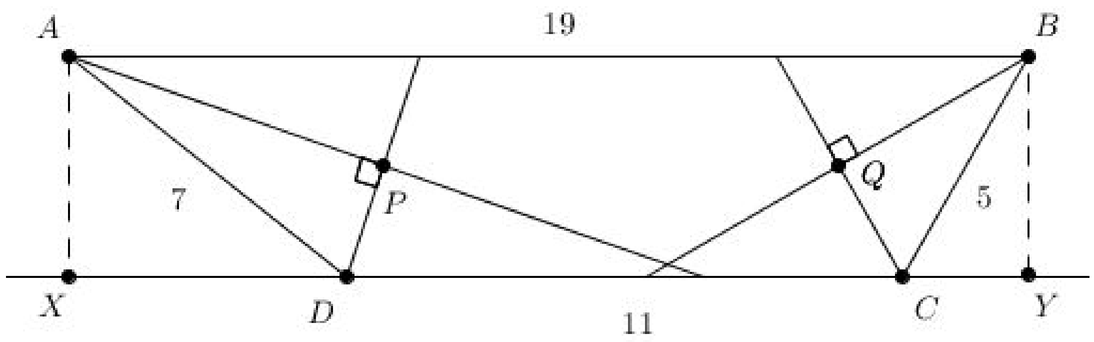
Contents

Problem 25

Let $ABCD$ be a trapezoid with $AB||CD, AB=11, BC=5, CD=19,$ and $DA=7$. Bisectors of $\angle A$ and $\angle D$ meet at P, and bisectors of $\angle B$ and $\angle C$ meet at Q. What is the area of hexagon $ABQCDP$?

$\textbf{(A)}\ 28\sqrt{3}\qquad \textbf{(B)}\ 30\sqrt{3}\qquad \textbf{(C)}\ 32\sqrt{3}\qquad \textbf{(D)}\ 35\sqrt{3}\qquad \textbf{(E)}\ 36\sqrt{3}$

Solution



Note: In the image AB and CD have been swapped from their given lengths in the problem. However, this doesn't affect any of the solving.

Drop perpendiculars to CD from A and B, and call the intersections X, Y respectively. Now, $DA^2 - BC^2 = (7-5)(7+5) = DX^2 - CY^2$ and $DX + CY = 19 - 11 = 8$. Thus, $DX - CY = 3$. We conclude $DX = \frac{11}{2}$ and $CY = \frac{5}{2}$. To simplify things even more, notice that $90^\circ = \frac{\angle D + \angle A}{2} = 180^\circ - \angle APD$, so $\angle P = \angle Q = 90^\circ$.

Also,

$$\sin(\angle PAD) = \sin(\frac{1}{2}\angle XDA) = \sqrt{\frac{1-\cos(\angle XDA)}{2}} = \sqrt{\frac{3}{28}}$$

So the area of $\triangle APD$ is:

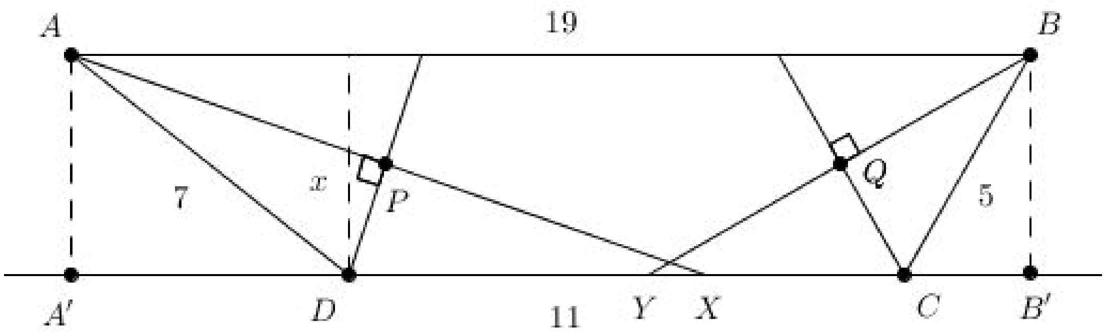
$$R \cdot c \sin a \sin b = \frac{7 \cdot 7}{2}\sqrt{\frac{3}{28}}\sqrt{1 - \frac{3}{28}} = \frac{35}{8}\sqrt{3}$$

Over to the other side: $\triangle BCY$ is $30-60-90$, and is therefore congruent to $\triangle BCQ$. So $[BCQ] = \frac{5 \cdot 5\sqrt{3}}{8}$.

The area of the hexagon is clearly $[ABCD] - ([BCQ] + [APD])$

$$= \frac{15 \cdot 5\sqrt{3}}{2} - \frac{60\sqrt{3}}{8} = 30\sqrt{3} \implies \boxed{B}$$

Alternate Solution



Let AP and BQ meet CD at X and Y, respectively.

Since $\angle APD = 90^\circ$, $\angle ADP = \angle XDP$, and they share DP, triangles APD and XPD are congruent.

By the same reasoning, we also have that triangles BQC and YQC are congruent.

Hence, we have

$$[ABQCDP] = [ABYX] + \frac{[ABCD] - [ABYX]}{2} = \frac{[ABCD] + [ABYX]}{2}.$$

If we let the height of the trapezoid be x, we have $[ABQCDP] = \frac{\frac{11+19}{2} \cdot x + \frac{11+7}{2} \cdot x}{2} = 12x.$

Thusly, if we find the height of the trapezoid and multiply it by 12, we will be done.

Let the projections of A and B to CD be A' and B', respectively.

We have $DA' + CB' = 19 - 11 = 8$, $DA' = \sqrt{DA^2 - AA'^2} = \sqrt{49 - x^2}$, and $CB' = \sqrt{CB^2 - BB'^2} = \sqrt{25 - x^2}$.

Therefore, $\sqrt{49 - x^2} + \sqrt{25 - x^2} = 8$. Solving this, we easily get that $x = \frac{5\sqrt{3}}{2}$.

Multiplying this by 12, we find that the area of hexagon $ABQCDP$ is $30\sqrt{3}$, which corresponds to answer choice $\boxed{B}$.

See Also

2008 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
Preceded by Problem 24	Followed by Last Question
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	