2019 AMC 10A Problems/Problem 1

Problem 1

What is the value of

$$2^{\left(0^{\left(1^9\right)}\right)}+\left(\left(2^0\right)^1\right)^9?$$

$\textbf{(A) } 0 \qquad\textbf{(B) } 1 \qquad\textbf{(C) } 2 \qquad\textbf{(D) } 3 \qquad\textbf{(E) } 4$

Solution

$2^{\left(0^{\left(1^9\right)}\right)}+\left(\left(2^0\right)^1\right)^9$

$=1+1=\boxed{2}$ which corresponds to $\boxed{\text{C}}$.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 2

Problem

What is the hundreds digit of $(20! - 15!)?$

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 4 \qquad \textbf{(E) } 5$

Solution

The last three digits of $n!$ for all $n \geq 15$ are 000, because there are at least three 2 s and three 5s in its prime factorization. Because $0 - 0 = 0$, the answer is $\boxed{\textbf{(A) } 0}$.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 3

Contents

Problem

Ana and Bonita were born on the same date in different years, n years apart. Last year Ana was 5 times as old as Bonita. This year Ana's age is the square of Bonita's age. What is $n?$

$\textbf{(A) } 3 \qquad\textbf{(B) } 5 \qquad\textbf{(C) } 9 \qquad\textbf{(D) } 12 \qquad\textbf{(E) } 15$

Solution

Solution 1

Let A be the age of Ana and B be the age of Bonita. Then,

and

$$A = B^2.$$

Substituting the second equation into the first gives us

$$B^2 - 1 = 5(B - 1).$$

By using difference of squares and dividing, $B = 4.$ Moreover, $A = B^2 = 16.$

The answer is

Solution 2 (Guess and Check)

Simple guess and check works. Start with all the square numbers - $1, 4, 9, 16, 25,$, etc. (probably stop at around 100 since at that point it wouldn't make sense). If Ana is 9, then Bonita is 3, so in the previous year, Ana's age was 4 times greater than Bonita's. If Ana is 16, then Bonita is 4, and Ana's age was 5 times greater than Bonita's in the previous year, as required. The difference in the ages is $16 - 4 = 12. \boxed{(D)}$

Solution 3 (Answer Choices)

The second sentence of the problem says that Ana's age was once 5 times Bonita's age. Therefore, the difference of the ages n must be divisible by $4.$ The only answer choice which is divisible by 4 is $12 \rightarrow \boxed{(D)}$.

- awesome_weisur

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 4

The following problem is from both the 2019 AMC 10A #4 and 2019 AMC 12A #3, so both problems redirect to this page.

Problem

A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn$?$

$\textbf{(A) } 75 \qquad \textbf{(B) } 76 \qquad \textbf{(C) } 79 \qquad \textbf{(D) } 84 \qquad \textbf{(E) } 91$

Solution

By choosing the maximum number of balls while getting < 15 of each color, we could have chosen 14 red balls, 14 green balls, 14 yellow balls, 13 blue balls, 11 white balls, and 9 black balls, for a total of 75 balls. Picking one more ball guarantees that we will get 15 balls of a color -- either red, green, or yellow. Thus the answer is $75 + 1 = \boxed{\textbf{(B) } 76}$.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
Preceded by **Problem 3**	Followed by **Problem 5**
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2019 AMC 12A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2019))	
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2019 AMC 10A Problems/Problem 5

The following problem is from both the 2019 AMC 10A #5 and 2019 AMC 12A #4, so both problems redirect to this page.

Contents

Problem

What is the greatest number of consecutive integers whose sum is $45?$

$\textbf{(A) } 9 \qquad\textbf{(B) } 25 \qquad\textbf{(C) } 45 \qquad\textbf{(D) } 90 \qquad\textbf{(E) } 120$

Solution 1

We might at first think that the answer would be 9, because $1 + 2 + 3 \dots + n = 45$ when $n = 9$. But note that the problem says that they can be integers, not necessarily positive. Observe also that every term in the sequence $-44, -43, \cdots, 44, 45$ cancels out except 45. Thus, the answer is, intuitively, $\boxed{\textbf{(D) } 90}$ integers.

Though impractical, a proof of maximality can proceed as follows: Let the desired sequence of consecutive integers be $a, a+1, \cdots, a+(N-1)$, where there are N terms, and we want to maximize N. Then the sum of the terms in this sequence is $aN + \frac{(N-1)(N)}{2} = 45$. Rearranging and factoring, this reduces to $N(2a+N-1) = 90$. Since N must divide 90, and we know that 90 is an attainable value of the sum, 90 must be the maximum.

Solution 2

To maximize the number of integers, we need to make the average of them as low as possible while still being positive. The average can be $\frac{1}{2}$ if the middle two numbers are 0 and 1, so the answer is $\frac{45}{\frac{1}{2}} = \boxed{\textbf{(D) } 90}$.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 12A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2019))	
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2019 AMC 10A Problems/Problem 6

Contents

Problem

For how many of the following types of quadrilaterals does there exist a point in the plane of the quadrilateral that is equidistant from all four vertices of the quadrilateral?

- a square
- a rectangle that is not a square
- a rhombus that is not a square
- a parallelogram that is not a rectangle or a rhombus
- an isosceles trapezoid that is not a parallelogram

$\textbf{(A) } 0 \qquad\textbf{(B) } 2 \qquad\textbf{(C) } 3 \qquad\textbf{(D) } 4 \qquad\textbf{(E) } 5$

Solution 1

This question is simply asking how many of the listed quadrilaterals are cyclic (since the point equidistant from all four vertices would be the center of the circumscribed circle). A square, a rectangle, and an isosceles trapezoid (that isn't a parallelogram) are all cyclic, and the other two are not. Thus, the answer is $\boxed{\textbf{(C) } 3}$.

We just have to see if opposite angles add up to 180 degrees~Williamgolly

Solution 2

We can use a process of elimination. Going down the list, we can see a square obviously works. A rectangle that is not a square works as well. Both rhombi and parallelograms don't have a point that is equidistant, but isosceles trapezoids do have such a point, so the answer is $\boxed{\textbf{(C) } 3}$.

Solution 3

The perpendicular bisector of a line segment is the locus of all points that are equidistant from the endpoints. The question then boils down to finding the shapes where the perpendicular bisectors of the sides all intersect at a point. This is true for a square, rectangle, and isosceles trapezoid, so the answer is $\boxed{\textbf{(C) } 3}$.

Solution 4

Only cyclic quadrilaterals will fit this description (they're R away). Going through, we see that the answer is $\boxed{\textbf{(C) } 3}$.

-JoshBrother32

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 7

The following problem is from both the 2019 AMC 10A #7 and 2019 AMC 12A #5, so both problems redirect to this page.

Contents

Problem

Two lines with slopes $\dfrac{1}{2}$ and 2 intersect at $(2,2)$. What is the area of the triangle enclosed by these two lines and the line $x+y=10?$

$\textbf{(A) } 4 \qquad\textbf{(B) } 4\sqrt{2} \qquad\textbf{(C) } 6 \qquad\textbf{(D) } 8 \qquad\textbf{(E) } 6\sqrt{2}$

Solution 1

Let's first work out the slope-intercept form of all three lines: $(x,y)=(2,2)$ and $y=\frac{x}{2}+b$ implies $2=\frac{2}{2}+b=1+b$ so $b=1$, while $y=2x+c$ implies $2=2\cdot2+c=4+c$ so $c=-2$. Also, $x+y=10$ implies $y=-x+10$. Thus the lines are $y=\frac{x}{2}+1, y=2x-2,$ and $y=-x+10$. Now we find the intersection points between each of the lines with $y=-x+10$, which are $(6,4)$ and $(4,6)$. Using the distance formula and then the Pythagorean Theorem, we see that we have an isosceles triangle with base $2\sqrt{2}$ and height $3\sqrt{2}$, whose area is $\boxed{\textbf{(C) } 6}$.

Solution 2

Like in Solution 1, we determine the coordinates of the three vertices of the triangle. The coordinates that we get are: $(2,2)\,(6,4)\,(4,6)$. Now, using the Shoelace Theorem, we can directly find that the area is $\boxed{\textbf{(C)}\ 6}$.

Solution 3

Like in the other solutions, solve the systems of equations to see that the triangle's two other vertices are at $(4,6)$ and $(6,4)$. Then apply Heron's Formula: the semi-perimeter will be $s=\sqrt{2}+\sqrt{20}$, so the area reduces nicely to a difference of squares, making it $\boxed{\textbf{(C)}\ 6}$.

Solution 4

Like in the other solutions, we find, either using algebra or simply by drawing the lines on squared paper, that the three points of intersection are $(2,2)$, $(4,6)$, and $(6,4)$. We can now draw the bounding square with vertices $(2,2)$, $(2,6)$, $(6,6)$ and $(6,2)$, and deduce that the triangle's area is $16-4-2-4=\boxed{\textbf{(C)}\ 6}$.

Solution 5

Like in other solutions, we find that the three points of intersection are $(2,2)$, $(4,6)$, and $(6,4)$. Using graph paper, we can see that this triangle has 6 boundary lattice points and 4 interior lattice points. By Pick's Theorem, the area is $\frac{6}{2}+4-1=\boxed{\textbf{(C)}\ 6}$.

Solution 6

Like in other solutions, we find the three points of intersection. Label these $A(2,2)$, $B(4,6)$, and $C(6,4)$. By the Pythagorean Theorem, $AB=AC=2\sqrt{5}$ and $BC=2\sqrt{2}$. By the Law of Cosines,

$$\cos A=\frac{(2\sqrt{5})^2+(2\sqrt{5})^2-(2\sqrt{2})^2}{2\cdot 2\sqrt{5}\cdot 2\sqrt{5}}=\frac{20+20-8}{40}=\frac{32}{40}=\frac{4}{5}$$

Therefore, $\sin A=\sqrt{1-\cos^2 A}=\frac{3}{5}$, so the area is $\frac{1}{2}bc\sin A=\frac{1}{2}\cdot 2\sqrt{5}\cdot 2\sqrt{5}\cdot\frac{3}{5}=\boxed{\textbf{(C)}\ 6}$.

Solution 7

Like in other solutions, we find that the three points of intersection are $(2,2)$, $(4,6)$, and $(6,4)$. The area of the triangle is half the absolute value of the determinant of the matrix determined by these points.

$$\frac{1}{2}\left\|\begin{matrix}2 & 2 & 1\\ 4 & 6 & 1\\ 6 & 4 & 1\end{matrix}\right\| = \frac{1}{2}|-12| = \frac{1}{2}\cdot 12 = \boxed{\textbf{(C) }6}$$

Solution 8

Like in other solutions, we find the three points of intersection. Label these $A(2,2)$, $B(4,6)$, and $C(6,4)$. Then vectors $\overrightarrow{AB} = \langle 2,4\rangle$ and $\overrightarrow{AC} = \langle 4,2\rangle$. The area of the triangle is half the magnitude of the cross product of these two vectors.

$$\frac{1}{2}\left\|\begin{matrix}i & j & k\\ 2 & 4 & 0\\ 4 & 2 & 0\end{matrix}\right\| = \frac{1}{2}|-12k| = \frac{1}{2}\cdot 12 = \boxed{\textbf{(C) }6}$$

Solution 9

Like in other solutions, we find that the three points of intersection are $(2,2)$, $(4,6)$, and $(6,4)$. By the Pythagorean theorem, this is an isosceles triangle with base $2\sqrt{2}$ and equal length $2\sqrt{5}$. The area of an isosceles triangle with base b and equal length l is $\dfrac{b\sqrt{4l^2-b^2}}{4}$. Plugging in $b=2\sqrt{2}$ and $l=2\sqrt{5}$,

$$\frac{2\sqrt{2}\cdot\sqrt{80-8}}{4} = \frac{\sqrt{576}}{4} = \frac{24}{4} = \boxed{\textbf{(C) }6}$$

Solution 10 (Trig)

Like in other solutions, we find the three points of intersection. Label these $A(2,2)$, $B(4,6)$, and $C(6,4)$. By the Pythagorean Theorem, $AB = AC = 2\sqrt{5}$ and $BC = 2\sqrt{2}$. By the Law of Cosines,

$$\cos A = \frac{(2\sqrt{5})^2 + (2\sqrt{5})^2 - (2\sqrt{2})^2}{2\cdot 2\sqrt{5}\cdot 2\sqrt{5}} = \frac{20+20-8}{40} = \frac{32}{40} = \frac{4}{5}$$

Therefore, $\sin A=\sqrt{1-\cos^2 A}=\frac{3}{5}$. By the extended Law of Sines,

$$2R=\frac{a}{\sin A}=\frac{2\sqrt{2}}{\frac{3}{5}}=\frac{10\sqrt{2}}{3}$$

$$R=\frac{5\sqrt{2}}{3}$$

Then the area is $\frac{abc}{4R}=\frac{2\sqrt{2}\cdot 2\sqrt{5}^2}{4\cdot\frac{5\sqrt{2}}{3}}=\boxed{\textbf{(C) }6}$.

Solution 11

The area of a triangle formed by three lines,

$$a_1x+a_2y+a_3=0$$

$$b_1x+b_2y+b_3=0$$

$$c_1x+c_2y+c_3=0$$

is the absolute value of

$$\frac{1}{2}\cdot\frac{1}{(b_1c_2-b_2c_1)(a_1c_2-a_2c_1)(a_1b_2-a_2b_1)}\cdot\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}^2$$

Plugging in the three lines,

$$-x+2y-2=0$$

$$-2x+y+2=0$$

$$x+y-10=0$$

the area is the absolute value of

$$\frac{1}{2}\cdot\frac{1}{(-2-1)(-1-2)(-1+4)}\cdot\begin{vmatrix}-1 & 2 & -2\\ -2 & 1 & 2\\ 1 & 1 & -10\end{vmatrix}^2=\frac{1}{2}\cdot\frac{1}{27}\cdot 18^2=\boxed{\textbf{(C) }6}$$

Source: Orrick, Michael L. “THE AREA OF A TRIANGLE FORMED BY THREE LINES.” Pi Mu Epsilon Journal, vol. 7, no. 5, 1981, pp. 294–298. JSTOR, www.jstor.org/stable/24336991.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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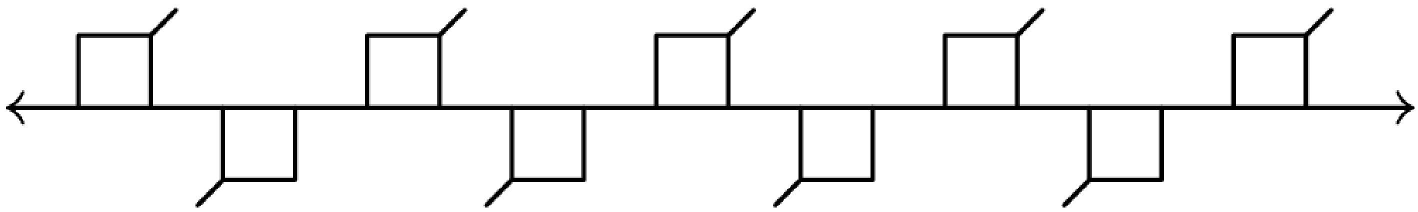
2019 AMC 10A Problems/Problem 8

The following problem is from both the 2019 AMC 10A #8 and 2019 AMC 12A #6, so both problems redirect to this page.

AMC 10 Problems

The figure below shows line ℓ with a regular, infinite, recurring pattern of squares and line segments.



How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?

- some rotation around a point of line ℓ
- some translation in the direction parallel to line ℓ
- the reflection across line ℓ
- some reflection across a line perpendicular to line ℓ

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 3 \qquad \textbf{(E) } 4$

Solution

Statement 1 is true. A 180° rotation about the point half way between an up-facing square and a down-facing square will yield the same figure.

Statement 2 is also true. A translation to the left or right will place the image onto itself when the figures above and below the line realign (the figure goes on infinitely in both directions).

Statement 3 is false. A reflection across line ℓ will change the up-facing squares to down-facing squares and vice versa.

Finally, statement 4 is also false because it will cause the diagonal lines extending from the squares to switch direction. Thus, only $\boxed{\textbf{(C) } 2}$ statements are true.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 9

Contents

Problem

What is the greatest three-digit positive integer n for which the sum of the first n positive integers is $\underline{not}$ a divisor of the product of the first n positive integers?

$\textbf{(A) } 995 \qquad\textbf{(B) } 996 \qquad\textbf{(C) } 997 \qquad\textbf{(D) } 998 \qquad\textbf{(E) } 999$

Solution 1

The sum of the first n positive integers is $\frac{(n)(n+1)}{2}$, and we want this not to be a divisor of $n!$ (the product of the first n positive integers). Notice that if and only if $n+1$ were composite, all of its factors would be less than or equal to n, which means they would be able to cancel with the factors in $n!$. Thus, the sum of n positive integers would be a divisor of $n!$ when $n+1$ is composite. (Note: This is true for all positive integers except for 1 because 2 is not a divisor/factor of 1.) Hence in this case, $n+1$ must instead be prime. The greatest three-digit integer that is prime is 997, so we subtract 1 to get $n = \boxed{\textbf{(B) } 996}$.

Solution 2

As in Solution 1, we deduce that $n+1$ must be prime. If we can't immediately recall what the greatest three-digit prime is, we can instead use this result to eliminate answer choices as possible values of n. Choices A, C, and E don't work because $n+1$ is even, and all even numbers are divisible by two, which makes choices A, C, and E composite and not prime. Choice D also does not work since 999 is divisible by 9, which means it's a composite number and not prime. Thus, the correct answer must be $\boxed{\textbf{(B) } 996}$.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 10

Contents

Problem

A rectangular floor that is 10 feet wide and 17 feet long is tiled with 170 one-foot square tiles. A bug walks from one corner to the opposite corner in a straight line. Including the first and the last tile, how many tiles does the bug visit?

$\textbf{(A) } 17 \qquad\textbf{(B) } 25 \qquad\textbf{(C) } 26 \qquad\textbf{(D) } 27 \qquad\textbf{(E) } 28$

Solution 1

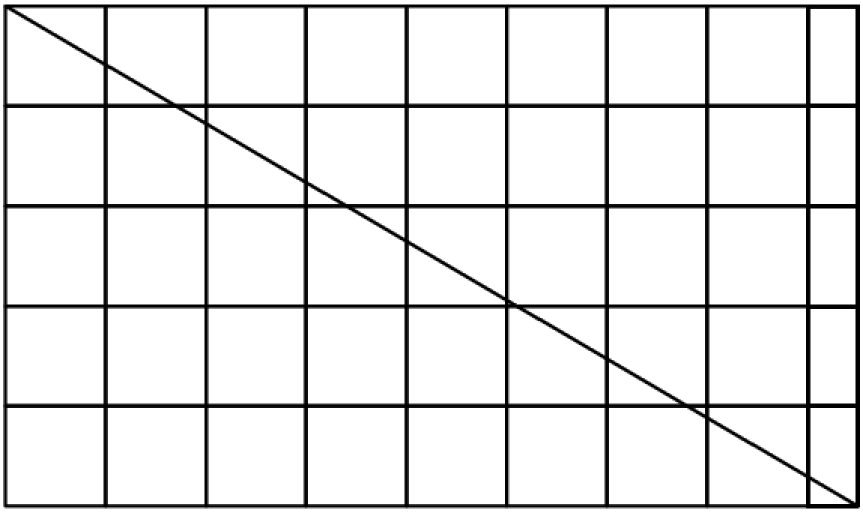
The number of tiles the bug visits is equal to 1 plus the number of times it crosses a horizontal or vertical line. As it must cross 16 horizontal lines and 9 vertical lines, it must be that the bug visits a total of $16 + 9 + 1 = \boxed{\textbf{(C) } 26}$ squares.

Note: The general formula for this is $a + b - \gcd(a, b)$, because it is the number of vertical/horizontal lines crossed minus the number of corners crossed (to avoid double counting). In this particular problem, it was $16 + 9 - 1$ (since $\gcd(16, 9) = 1$), which is 24, but then you add 2 because the first tile and the last tile are counted, which in the general formula are not counted.

One can see why it is gcd(a,b) due to slope ~Williamgolly

Solution 2 (drawing)

We can also draw a diagram or scale model of the entire rectangular floor (optionally with grid paper and/or a ruler so it will be to scale), then simply count the number of tiles the path crosses. To make this slightly easier, we can divide the full grid into 4 sections, and just draw one of these 5 feet by 8.5 feet sections.



Though it may appear that the line we drew comes very close to several points, we know that since 10 and 17 are relatively prime (numbers where the only positive integer that divides both of them is 1, a.k.a. numbers with a gcd of 1), the line will not actually pass through any of these points, so the total number of squares crossed will be the same regardless of which side we count. If we count the number of squares the line passes through using the diagram, we get 13 squares. We can then multiply this by 2 to find out the total number of squares the bug passes through on the rectangular floor giving us a total of $2 \cdot 13 = \boxed{\textbf{(C) } 26}$.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 11

Contents

Problem

How many positive integer divisors of 201^9 are perfect squares or perfect cubes (or both)?

$\textbf{(A) }32 \qquad \textbf{(B) }36 \qquad \textbf{(C) }37 \qquad \textbf{(D) }39 \qquad \textbf{(E) }41$

Solution 1

Prime factorizing 201^9, we get $3^9 \cdot 67^9$. A perfect square must have even powers of its prime factors, so our possible choices for our exponents of a perfect square are $0, 2, 4, 6, 8$ for both 3 and 67. This yields $5 \cdot 5 = 25$ perfect squares.

Perfect cubes must have multiples of 3 for each of their prime factors' exponents, so we have either $0, 3, 6$, or 9 for both 3 and 67, which yields $4 \cdot 4 = 16$ perfect cubes, for a total of $25 + 16 = 41$.

Subtracting the overcounted powers of 6 ($3^0 \cdot 67^0$, $3^0 \cdot 67^6$, $3^6 \cdot 67^0$, and $3^6 \cdot 67^6$), we get $41 - 4 = \boxed{\textbf{(C) }37}$.

Solution 2

Observe that $201 = 67 \cdot 3$. Now divide into cases:

Case 1: The factor is 3^n. Then we can have $n = 2$, 3, 4, 6, 8, or 9.

Case 2: The factor is 67^n. This is the same as Case 1.

Case 3: The factor is some combination of 3s and 67s.

This would be easy if we could just have any combination, as that would simply give $6 \cdot 6$. However, we must pair the numbers that generate squares with the numbers that generate squares and the same for cubes. In simpler terms, let's organize our values for n.

$n = 2$ is a "square" because it would give a factor of this number that is a perfect square. More generally, it is even.

$n = 3$ is a "cube" because it would give a factor of this number that is a perfect cube. More generally, it is a multiple of 3.

$n = 4$ is a "square".

$n = 6$ is interesting, since it's both a "square" and a "cube". Don't count this as either because this would double-count, so we will count this in another case.

$n = 8$ is a "square"

$n = 9$ is a "cube".

Now let's consider subcases:

Subcase 1: The squares are with each other.

Since we have 3 square terms, and they would pair with 3 other square terms, we get $3 \cdot 3 = 9$ possibilities.

Subcase 2: The cubes are with each other.

Since we have 2 cube terms, and they would pair with 2 other cube terms, we get $2 \cdot 2 = 4$ possibilities.

Subcase 3: A number pairs with $n = 6$.

Since any number can pair with $n = 6$ (as it gives both a square and a cube), there would be 6 possibilities. Remember however that there can be two different bases (3 and 67), and they would produce different results. Thus, there are in fact $6 \cdot 2 = 12$ possibilities.

Finally, summing the cases gives $6 + 6 + 9 + 4 + 12 = \boxed{\textbf{(C) } 37}$.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 12

The following problem is from both the 2019 AMC 10A #12 and 2019 AMC 12A #7, so both problems redirect to this page.

Contents

Problem

Melanie computes the mean μ, the median M, and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, $\ldots$, 12 28s, 11 29s, 11 30s, and 7 31s. Let d be the median of the modes. Which of the following statements is true?

$\textbf{(A) } \mu < d < M \qquad\textbf{(B) } M < d < \mu \qquad\textbf{(C) } d = M =\mu \qquad\textbf{(D) } d < M < \mu \qquad\textbf{(E) } d < \mu$

Solution 1

First of all, d obviously has to be smaller than M, since when calculating M, we must take into account the 29s, 30s, and 31s. So we can eliminate choices B and C. Since there are 365 total entries, the median, M, must be the 183rd one, at which point we note that $12 \cdot 15$ is 180, so 16 has to be the median (because 183 is between $12 \cdot 15 + 1 = 181$ and $12 \cdot 16 = 192$). Now, the mean, μ, must be smaller than 16, since there are many fewer 29s, 30s, and 31s. d is less than μ, because when calculating μ, we would include $29, 30$, and 31. Thus the answer is $\boxed{\textbf{(E) } d < \mu < M}$.

Solution 2

As in Solution 1, we find that the median is 16. Then, looking at the modes $(1-28)$, we realize that even if we were to have 12 of each, their median would remain the same, being 14.5. As for the mean, we note that the mean of the first 28 is simply the same as the median of them, which is 14.5. Hence, since we in fact have 29's, 30's, and 31's, the mean has to be higher than 14.5. On the other hand, since there are fewer 29's, 30's, and 31's than the rest of the numbers, the mean has to be lower than 16 (the median). By comparing these values, the answer is $\boxed{\textbf{(E) } d < \mu < M}$.

Solution 3 (direct calculation)

We can solve this problem simply by carefully calculating each of the values, which turn out to be $M = 16, d = 14.5$, and $\mu \approx 15.7$. Thus the answer is $\boxed{\textbf{(E) } d < \mu < M}$.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 13

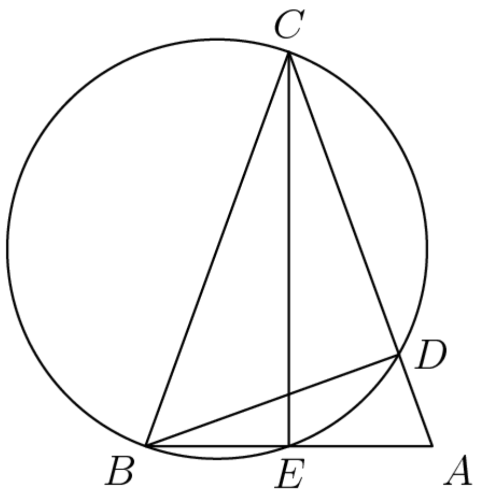
Contents

Problem

Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40^{\circ}$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC?$

$\textbf{(A) } 90 \qquad \textbf{(B) } 100 \qquad \textbf{(C) } 105 \qquad \textbf{(D) } 110 \qquad \textbf{(E) } 120$

Solution 1



Drawing it out, we see $\angle BDC$ and $\angle BEC$ are right angles, as they are inscribed in a semicircle. Using the fact that it is an isosceles triangle, we find $\angle ABC = 70^{\circ}$. We can find $\angle ECB = 20^{\circ}$ and $\angle DBC = 50^{\circ}$ by the triangle angle sum on $\triangle ECB$ and $\triangle DBC$.

$$\angle BDC + \angle DCB + \angle DBC = 180^{\circ} \implies 90^{\circ} + 40^{\circ} + \angle DBC = 180^{\circ} \implies \angle DBC = 50^{\circ}$$

$$\angle BEC + \angle EBC + \angle ECB = 180^{\circ} \implies 90^{\circ} + 70^{\circ} + \angle ECB = 180^{\circ} \implies \angle ECB = 20^{\circ}$$

Then, we take triangle BFC, and find $\angle BFC = 180^{\circ} - 50^{\circ} - 20^{\circ} = \boxed{\textbf{(D) } 110^{\circ}}$.

Solution 2

Alternatively, we could have used similar triangles. We start similarly to Solution 1.

Drawing it out, we see $\angle BDC$ and $\angle BEC$ are right angles, as they are inscribed in a semicircle. Therefore,

$$\angle BDA = 180^\circ - \angle BDC = 180^\circ - 90^\circ = 90^\circ.$$

So, $\triangle BEF \sim BDA$ by AA Similarity, since $\angle EBF = \angle DBA$ and $\angle BEC = 90^\circ = \angle BDA$. Thus, we know

$$\angle EFB = \angle DAB = \angle CAB = 70^\circ.$$

Finally, we deduce

$$\angle BFC = 180^\circ - \angle EFB = 180^\circ - 70^\circ = \boxed{\textbf{(D) } 110^\circ}.$$

Solution 3 (outside angles)

Through the property of angles formed by intersecting chords, we find that

$$m\angle BFC = \frac{m\overset{\frown}{\text{BC}} + m\overset{\frown}{\text{DE}}}{2}$$

Through the Outside Angles Theorem, we find that

$$m\angle CAB = \frac{m\overset{\frown}{\text{BC}} - m\overset{\frown}{\text{DE}}}{2}$$

Adding the two equations gives us

$$m\angle BFC - m\angle CAB = m\overset{\frown}{\text{BC}} \implies m\angle BFC = m\overset{\frown}{\text{BC}} - m\angle CAB$$

Since $\overset{\frown}{\text{BC}}$ is the diameter, $m\overset{\frown}{\text{BC}} = 180^\circ$, and because $\triangle ABC$ is isosceles and $m\angle ACB = 40^\circ$, we have $m\angle CAB = 70^\circ$. Thus

$$m\angle BFC = 180^\circ - 70^\circ = \boxed{\textbf{(D) } 110^\circ}$$

Solution 4

Notice that if $\angle BEC = 90^\circ$, then $\angle BCE$ and $\angle ACE$ must be 20°. Using cyclic quadrilateral properties (or the properties of a subtended arc), we can find that $\angle EBD \cong \angle ECD = 20^\circ$. Thus $\angle CBF = 70 - 20 = 50^\circ$, and so $\angle BFC = 180 - 20 - 50 = 110^\circ$, which is $\boxed{\textbf{(D)}}$.

Solution 5 (CHEATING)

If you can't see how to solve it, you could simply draw an accurate diagram and measure the angle using a protractor as 110 - $\boxed{\textbf{(D)}}$.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 14

The following problem is from both the 2019 AMC 10A #14 and 2019 AMC 12A #8, so both problems redirect to this page.

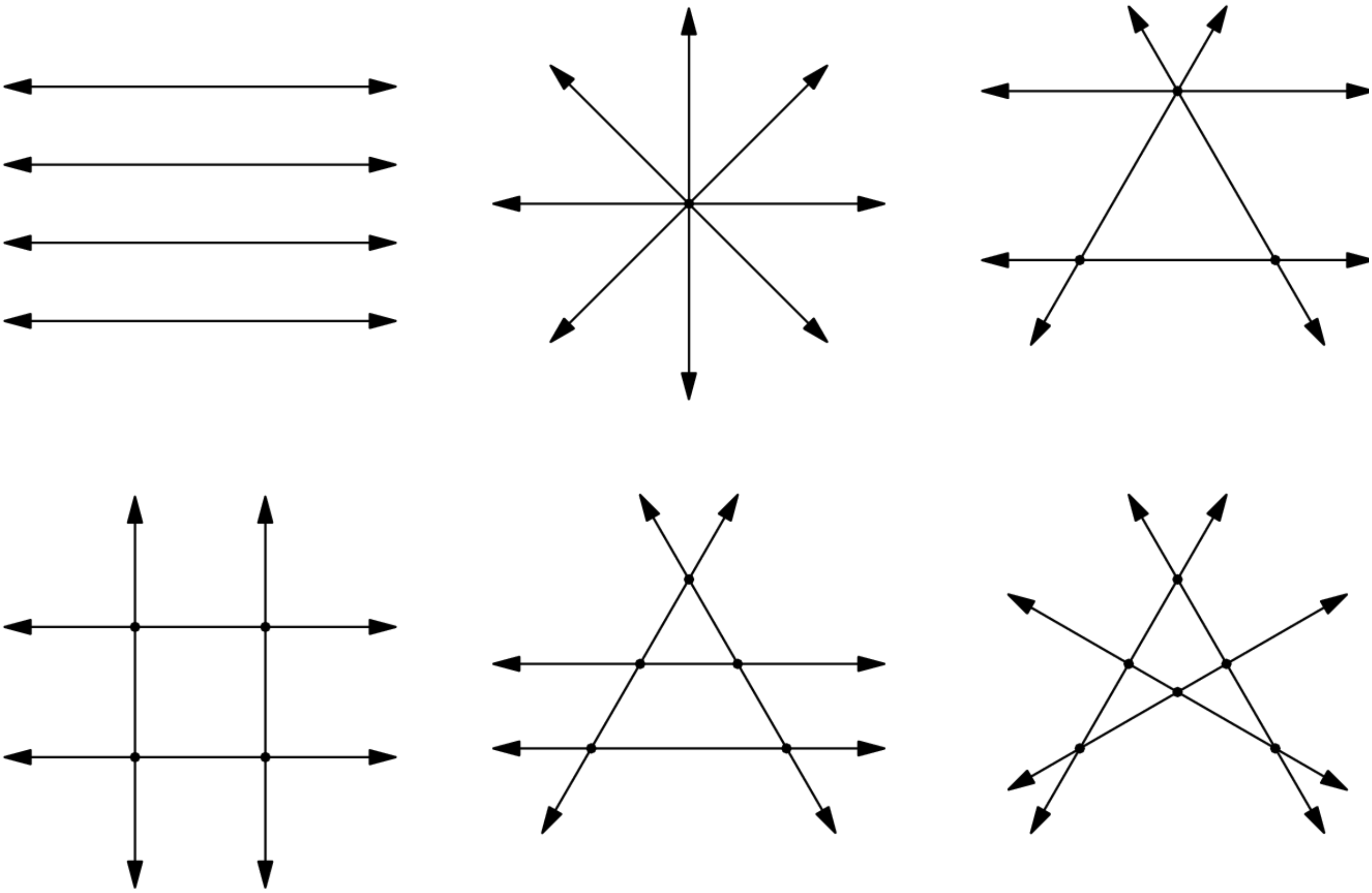
Problem

For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

$\textbf{(A) } 14 \qquad \textbf{(B) } 16 \qquad \textbf{(C) } 18 \qquad \textbf{(D) } 19 \qquad \textbf{(E) } 21$

Solution

It is possible to obtain 0, 1, 3, 4, 5, and 6 points of intersection, as demonstrated in the following figures:



It is clear that the maximum number of possible intersections is $\binom{4}{2} = 6$, since each pair of lines can intersect at most once. We now prove that it is impossible to obtain two intersections.

We proceed by contradiction. Assume a configuration of four lines exists such that there exist only two intersection points. Let these intersection points be A and B. Consider two cases:

Case 1: No line passes through both A and B

Then, since an intersection is obtained by an intersection between at least two lines, two lines pass through each of A and B. Then, since there can be no additional intersections, no line that passes through A can intersect a line that passes through B, and so each line that passes through A must be parallel to every line that passes through B. Then the two lines passing through B are parallel to each other by transitivity of parallelism, so they coincide, contradiction.

Case 2: There is a line passing through A and B

Then there must be a line l_a passing through A, and a line l_b passing through B. These lines must be parallel. The fourth line l must pass through either A or B. Without loss of generality, suppose l passes through A. Then since l and l_a cannot coincide, they cannot be parallel. Then l and l_b cannot be parallel either, so they intersect, contradiction.

All possibilities have been exhausted, and thus we can conclude that two intersections is impossible. Our answer is given by the sum $0+1+3+4+5+6=\boxed{\textbf{(D) } 19}$.

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 15

The following problem is from both the 2019 AMC 10A #15 and 2019 AMC 12A #9, so both problems redirect to this page.

Contents

Problem

A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$ Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p+q$?

$\textbf{(A) } 2020 \qquad \textbf{(B) } 4039 \qquad \textbf{(C) } 6057 \qquad \textbf{(D) } 6061 \qquad \textbf{(E) } 8078$

Solution 1 (Induction)

Using the recursive formula, we find $a_3 = \frac{3}{11}$, $a_4 = \frac{3}{15}$, and so on. It appears that $a_n = \frac{3}{4n-1}$, for all n. Setting $n = 2019$, we find $a_{2019} = \frac{3}{8075}$, so the answer is $\boxed{\textbf{(E) } 8078}$.

To prove this formula, we use induction. We are given that $a_1 = 1$ and $a_2 = \frac{3}{7}$, which satisfy our formula. Now assume the formula holds true for all $n \leq m$ for some positive integer m. By our assumption, $a_{m-1} = \frac{3}{4m-5}$ and $a_m = \frac{3}{4m-1}$. Using the recursive formula,

$$a_{m+1} = \frac{a_{m-1} \cdot a_m}{2a_{m-1} - a_m} = \frac{\frac{3}{4m-5} \cdot \frac{3}{4m-1}}{2 \cdot \frac{3}{4m-5} - \frac{3}{4m-1}} = \frac{\left(\frac{3}{4m-5} \cdot \frac{3}{4m-1}\right)(4m-5)(4m-1)}{\left(2 \cdot \frac{3}{4m-5} - \frac{3}{4m-1}\right)(4m-5)(4m-1)} = \frac{\quad}{6(4m -}$$

so our induction is complete.

Solution 2

Since we are interested in finding the sum of the numerator and the denominator, consider the sequence defined by $b_n = \frac{1}{a_n}$.

We have $\frac{1}{a_n} = \frac{2a_{n-2} - a_{n-1}}{a_{n-2} \cdot a_{n-1}} = \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}}$, so in other words,
$b_n = 2b_{n-1} - b_{n-2} = 3b_{n-2} - 2b_{n-3} = 4b_{n-3} - 3b_{n-4} = \ldots$.

By recursively following this pattern, we can see that $b_n = (n-1) \cdot b_2 - (n-2) \cdot b_1$.

By plugging in 2019, we thus find $b_{2019} = 2018 \cdot \frac{7}{3} - 2017 = \frac{8075}{3}$. Since the numerator and the denominator are relatively prime, the answer is $\boxed{\textbf{(E) } 8078}$.

-eric2020

Solution 3

It seems reasonable to transform the equation into something else. Let $a_n = x, a_{n-1} = y$, and $a_{n-2} = z$. Therefore, we have

$$x = \frac{zy}{2z - y}$$

$$2xz - xy = zy$$

$$2xz = y(x + z)$$

$$y = \frac{2xz}{x + z}$$

Thus, y is the harmonic mean of x and z. This implies a_n is a harmonic sequence or equivalently $b_n = \frac{1}{a_n}$ is arithmetic. Now, we have $b_1 = 1, b_2 = \frac{7}{3}, b_3 = \frac{11}{3}$, and so on. Since the common difference is $\frac{4}{3}$, we can express b_n explicitly as $b_n = \frac{4}{3}(n-1) + 1$. This gives $b_{2019} = \frac{4}{3}(2019 - 1) + 1 = \frac{8075}{3}$ which implies $a_{2019} = \frac{3}{8075} = \frac{p}{q}$. $p + q = \boxed{\textbf{(E) } 8078}$ ~jakeg314

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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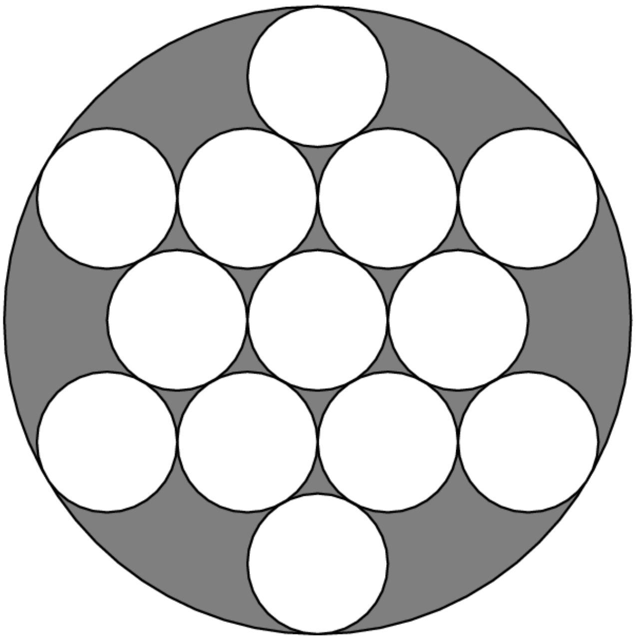
2019 AMC 10A Problems/Problem 16

The following problem is from both the 2019 AMC 10A #16 and 2019 AMC 12A #10, so both problems redirect to this page.

Contents

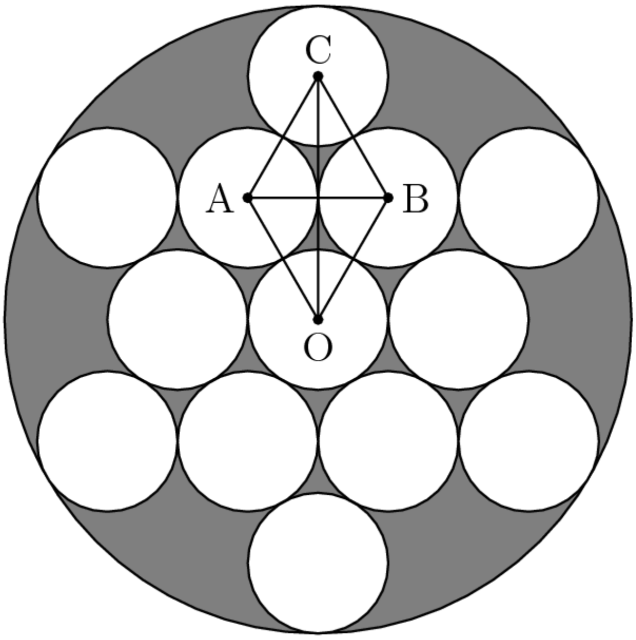
Problem

The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius $1?$



$\textbf{(A) }4\pi\sqrt{3}\qquad\textbf{(B) }7\pi\qquad\textbf{(C) }\pi\left(3\sqrt{3}+2\right)\qquad\textbf{(D) }10\pi\left(\sqrt{3}-1\right)\qquad\textbf{(E) }\pi\left(\sqrt{3}+6\right)$

Solution 1



In the diagram above, notice that triangle OAB and triangle ABC are congruent and equilateral with side length 2. We can see the radius of the larger circle is two times the altitude of OAB plus 1 (the distance from point C to the edge of the circle). Using $30^\circ-60^\circ-90^\circ$ triangles, we know the altitude is $\sqrt{3}$. Therefore, the radius of the larger circle is $2\sqrt{3}+1$.

The area of the larger circle is thus $\left(2\sqrt{3}+1\right)^2\pi=\left(13+4\sqrt{3}\right)\pi$, and the sum of the areas of the smaller circles is 13π, so the area of the dark region is $\left(13+4\sqrt{3}\right)\pi-13\pi=\boxed{\textbf{(A)}\ 4\pi\sqrt{3}}$.

Solution 2

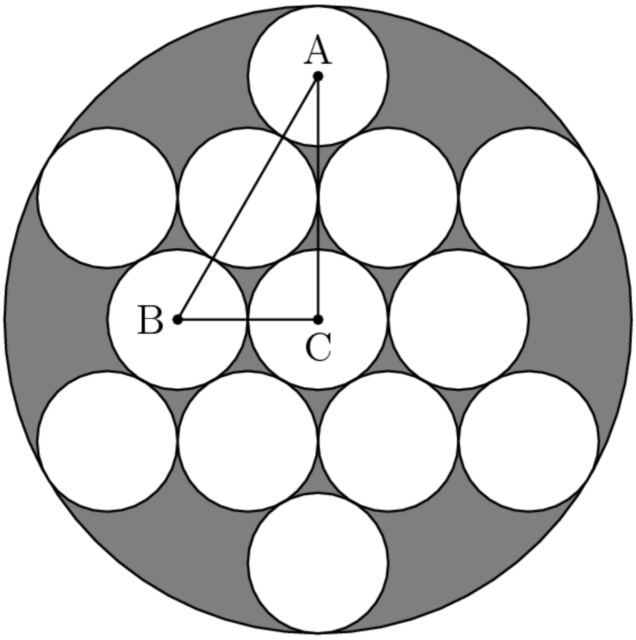
We can form an equilateral triangle with side length 6 from the centers of three of the unit circles tangent to the outer circle. The radius of the outer circle is the circumradius of the triangle plus 1. By using $R=\frac{abc}{4A}$ or $R=\frac{a}{2\sin A}$, we get the radius as $\frac{6}{\sqrt{3}}+1$.

The shaded area is thus $\pi((\frac{6}{\sqrt{3}}+1)^2-13)=\boxed{\textbf{(A)}\ 4\pi\sqrt{3}}$.

Solution 3

Like in Solution 2, we can form an equilateral triangle with side length 6 from the centers of three of the unit circles tangent to the outer circle. We can find the height of this triangle to be $3\sqrt{3}$. Then, we can form another equilateral triangle from the centers of the second and third circles in the third row and the center of the bottom circle with side length 2. The height of this triangle is clearly $\sqrt{3}$. Therefore the diameter of the large circle is $4\sqrt{3}+2$ and the radius is $\frac{4\sqrt{3}+2}{2}=2\sqrt{3}+1$. The area of the large circle is thus $\pi\left(2\sqrt{3}+1\right)^2=\pi\cdot\left(13+4\sqrt{3}\right)=13\pi+4\pi\sqrt{3}$. The total area of the 13 smaller circles is 13π, so the shaded area is $\left(13\pi+4\pi\sqrt{3}\right)-13\pi=\boxed{\textbf{(A)}\ 4\pi\sqrt{3}}$.

Solution 4



In the diagram above, $AB = 4$ and $BC = 2$, so $AC = \sqrt{4^2 - 2^2} = 2\sqrt{3}$. The larger circle's radius is $AC + 1 = 2\sqrt{3} + 1$, so the larger circle's area is $\pi\left(2\sqrt{3}+1\right)^2 = \pi\left(13 + 4\sqrt{3}\right) = 13\pi + 4\pi\sqrt{3}$. Now, subtracting the combined area of the smaller circles gives $13\pi + 4\pi\sqrt{3} - 13\pi = \boxed{\textbf{(A) } 4\pi\sqrt{3}}$.

See Also

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2019 AMC 10A Problems/Problem 17

Contents

Problem

A child builds towers using identically shaped cubes of different color. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

$\textbf{(A) } 24 \qquad\textbf{(B) } 288 \qquad\textbf{(C) } 312 \qquad\textbf{(D) } 1,260 \qquad\textbf{(E) } 40,320$

Solution 1

Arranging eight cubes is the same as arranging the nine cubes first, and then removing the last cube. In other words, there is a one-to-one correspondence between every arrangement of nine cubes, and every actual valid arrangement. Thus, we initially get $9!$. However, we have overcounted, because the red cubes can be permuted to have the same overall arrangement, and the same applies with the blue and green cubes. Thus, we have to divide by the $2!$ ways to arrange the red cubes, the $3!$ ways to arrange the blue cubes, and the $4!$ ways to arrange the green cubes. Thus we have $\frac{9!}{2!\cdot 3!\cdot 4!} = \boxed{\textbf{(D) } 1,260}$ different possible towers.

Note: this can be written more compactly as

$$\binom{9}{2,3,4} = \binom{9}{2}\binom{9-2}{3}\binom{9-(2+3)}{4} = \boxed{1,260}$$

Solution 2

We can divide the problem into three cases, each representing one cube to be excluded:

Case 1: The red cube is excluded. This gives us the problem of arranging one red cube, three blue cubes, and four green cubes. The number opossible arrangements is $\frac{8!}{4! \cdot 3!} = 280$. Note that we do not need to multiply by the number of red cubes because there is no way to distinguish between the first red cube and the second.

Case 2: The blue cube is excluded. This gives us the problem of arranging two red cubes, two blue cubes, and four green cubes. The number of possible arrangements is $\frac{8!}{2! \cdot 2! \cdot 4!} = 420$.

Case 3: The green cube is excluded. This gives us the problem of arranging two red cubes, three blue cubes, and three green cubes. The number of possible arrangements is $\frac{8!}{2! \cdot 3! \cdot 3!} = 560$.

Adding up the individual cases from above gives the answer as $280 + 420 + 560 = \boxed{\textbf{(D) } 1,260}$.

Solution 3 (guessing)

If you're running out of time, notice that choices A, B, and C are way too small, and choice E would make no sense since it would simply be $8!$, as if there were no restrictions. Thus, by educated guessing and elimination, the correct answer must be $\boxed{\textbf{(D) } 1,260}$.

Note: this strategy is NOT recommended!

See Also

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2019 AMC 10A Problems/Problem 18

The following problem is from both the 2019 AMC 10A #18 and 2019 AMC 12A #11, so both problems redirect to this page.

Contents

Problem

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323..._k$. What is k?

$\textbf{(A)}\ 13 \qquad \textbf{(B)}\ 14 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 16 \qquad \textbf{(E)}\ 17$

Solution 1

We can expand the fraction $0.\overline{23}_k$ as follows: $0.\overline{23}_k = 2\cdot k^{-1} + 3\cdot k^{-2} + 2\cdot k^{-3} + 3\cdot k^{-4} + \ldots$ Notice that this is equivalent to

$$2(k^{-1} + k^{-3} + k^{-5} + ...) + 3(k^{-2} + k^{-4} + k^{-6} + ...)$$

By summing the geometric series and simplifying, we have $\frac{2k+3}{k^2-1} = \frac{7}{51}$. Solving this quadratic equation (or simply testing the answer choices) yields the answer $k = \boxed{\textbf{(D)}\ 16}$.

Solution 2

Let $a = 0.2323\ldots_k$. Therefore, $k^2a = 23.2323\ldots_k$.

From this, we see that $k^2a - a = 23_k$, so $a = \frac{23_k}{k^2-1} = \frac{2k+3}{k^2-1} = \frac{7}{51}$.

Now, similar to in Solution 1, we can either test if $2k+3$ is a multiple of 7 with the answer choices, or actually solve the quadratic, so that the answer is $\boxed{\textbf{(D) }16}$.

Solution 3 (bash)

We can simply plug in all the answer choices as values of k, and see which one works. After legendary, amazingly, historically great calculations, this eventually gives us $\boxed{\textbf{(D) }16}$ as the answer.

Solution 4

Just as in Solution 1, we arrive at the equation $\frac{2k+3}{k^2-1} = \frac{7}{51}$.

We can now rewrite this as $\frac{2k+3}{(k-1)(k+1)} = \frac{7}{51} = \frac{7}{3\cdot 17}$. Notice that $2k+3 = 2(k+1)+1 = 2(k-1)+5$. As 17 is a prime, we therefore must have that one of $k-1$ and $k+1$ is divisible by 17. Now, checking each of the answer choices, this gives $\boxed{\textbf{(D) }16}$.

Solution 5

Assuming you are familiar with the rules for basic repeating decimals, $0.232323... = \frac{23}{99}$. Now we want our base, k, to conform to $23 = 7\,(mod\,k)$ and $99 = 51\,(mod\,k)$, the reason being that we wish to convert the number from base 10 to base k. Given the first equation, we know that k must equal 9, 16, 23, or generally, $7n+2$. The only number in this set that is one of the multiple choices is 16. When we test this on the second equation, $99 = 51\,(mod\,k)$, it comes to be true. Therefore, our answer is $\boxed{\textbf{(D) }16}$.

Video Solution

For those who want a video solution: https://www.youtube.com/watch?v=DFfRJolhwN0

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10A Problems/Problem 19

Contents

Problem

What is the least possible value of

$$(x+1)(x+2)(x+3)(x+4)+2019$$

where x is a real number?

$\textbf{(A) } 2017 \qquad \textbf{(B) } 2018 \qquad \textbf{(C) } 2019 \qquad \textbf{(D) } 2020 \qquad \textbf{(E) } 2021$

Solution 1

Grouping the first and last terms and two middle terms gives $(x^2+5x+4)(x^2+5x+6)+2019$, which can be simplified to $(x^2+5x+5)^2-1+2019$. Noting that squares are nonnegative, and verifying that $x^2+5x+5=0$ for some real x, the answer is $\boxed{\textbf{(B) } 2018}$.

Better to understand solution by Williamgolly: After we factor into $(x^2+5x+4)(x^2+5x+6)+2019$, let $y=x^2+5x+5$ and the forner expression turns to $(y-1)(y+1)+2019=y^2+2018$. Clearly, y^2 is non negative, so the minimum is achieved when $y=0$ and the expression equals 2018

Solution 2

Let $a = x + \frac{5}{2}$. Then the expression $(x+1)(x+2)(x+3)(x+4)$ becomes $\left(a - \frac{3}{2}\right)\left(a - \frac{1}{2}\right)\left(a + \frac{1}{2}\right)\left(a + \frac{3}{2}\right)$.

We can now use the difference of two squares to get $\left(a^2 - \frac{9}{4}\right)\left(a^2 - \frac{1}{4}\right)$, and expand this to get $a^4 - \frac{5}{2}a^2 + \frac{9}{16}$.

Refactor this by completing the square to get $\left(a^2 - \frac{5}{4}\right)^2 - 1$, which has a minimum value of -1. The answer is thus $2019 - 1 = \boxed{\textbf{(B) } 2018}$.

Solution 3 (calculus)

Similar to Solution 1, grouping the first and last terms and the middle terms, we get $(x^2 + 5x + 4)(x^2 + 5x + 6) + 2019$.

Letting $y = x^2 + 5x$, we get the expression $(y+4)(y+6) + 2019$. Now, we can find the critical points of $(y+4)(y+6)$ to minimize the function:

$$\frac{d}{dx}(y^2 + 10y + 24) = 0$$

$$2y + 10 = 0$$

$$2y(y+5) = 0$$

$$y = -5, 0$$

To minimize the result, we use $y = -5$. Hence, the minimum is $(-5+4)(-5+6) = -1$, so $-1 + 2019 = \boxed{\textbf{(B) } 2018}$.

Note: We could also have used the result that minimum/maximum point of a parabola $y = ax^2 + bx + c$ occurs at $x = -\dfrac{b}{2a}$.

Solution 4

The expression is negative when an odd number of the factors are negative. This happens when $-2 < x < -1$ or $-4 < x < -3$. Plugging in $x = -\dfrac{3}{2}$ or $x = -\dfrac{7}{2}$ yields $-\dfrac{15}{16}$, which is very close to -1. Thus the answer is $-1 + 2019 = \boxed{\textbf{(B) } 2018}$.

Solution 5 (using the answer choices)

Answer choices C, D, and E are impossible, since $(x+1)(x+2)(x+3)(x+4)$ can be negative (as seen when e.g. $x=-\frac{3}{2}$). Plug in $x=-\frac{3}{2}$ to see that it becomes $2019-\frac{15}{16}$, so round this to $\boxed{\textbf{(B) } 2018}$.

We can also see that the limit of the function is at least -1 since at the minimum, two of the numbers are less than 1, but two are between 1 and 2.

Video Solution

For those who want a video solution: https://www.youtube.com/watch?v=Mfa7j2BoNjI

See Also

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2019 AMC 10A Problems/Problem 20

The following problem is from both the 2019 AMC 10A #20 and 2019 AMC 12A #16, so both problems redirect to this page.

Contents

Problem

The numbers $1, 2, \dots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

$$\textbf{(A) }\frac{1}{21}\qquad\textbf{(B) }\frac{1}{14}\qquad\textbf{(C) }\frac{5}{63}\qquad\textbf{(D) }\frac{2}{21}\qquad\textbf{(E) }\frac{1}{7}$$

Solution 1

Note that odd sums can only be formed by (e,e,o) or (o,o,o), so we focus on placing the evens: we need to have each even be with another even in each row/column. It can be seen that there are 9 ways to do this. There are then $5!$ ways to permute the odd numbers, and $4!$ ways to permute the even numbers, thus giving the answer as $\frac{5!\cdot 4!\cdot 9}{9!} = \boxed{\textbf{(B) }\frac{1}{14}}$.

Solution 2

By the Pigeonhole Principle, there must be at least one row with 2 or more odd numbers in it. Therefore, that row must contain 3 odd numbers in order to have an odd sum. The same thing can be done with the columns. Thus we simply have to choose one row and one column to be filled with odd numbers, so the number of valid odd/even configurations (without regard to which particular odd and even numbers are placed where) is $3 \cdot 3 = 9$. The denominator will be $\binom{9}{4}$, the total number of ways we could choose which 4 of the 9 squares will contain an even number. Hence the answer is

$$\frac{9}{\binom{9}{4}} = \boxed{\textbf{(B) } \frac{1}{14}}$$

- The Pigeonhole Principle isn't really necessary here: After noting from the first solution that any row that contains evens must contain two evens, the result follows that the four evens must form the corners of a rectangle.

Video Solution

For those who want a video solution: https://www.youtube.com/watch?v=uJgS-q3-1JE

See Also

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2019 AMC 10A Problems/Problem 21

The following problem is from both the 2019 AMC 10A #21 and 2019 AMC 12A #18, so both problems redirect to this page.

Contents

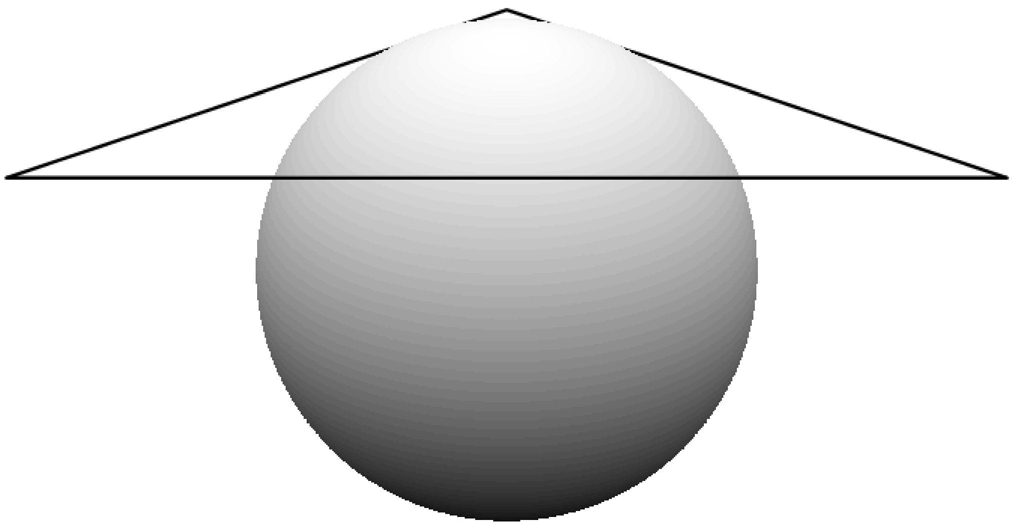
Problem

A sphere with center O has radius 6. A triangle with sides of length $15, 15,$ and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle?

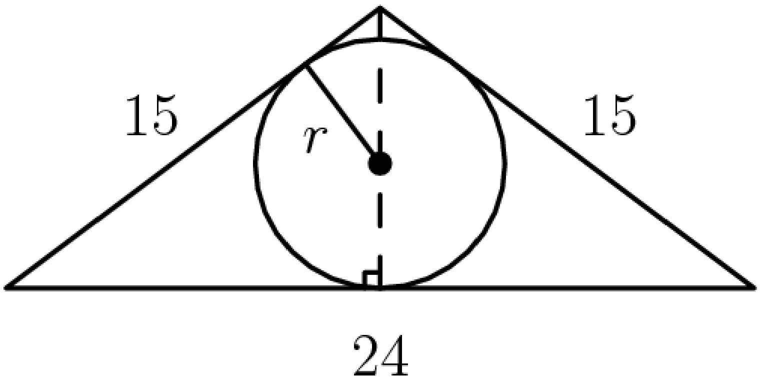
$\textbf{(A) }2\sqrt{3}\qquad \textbf{(B) }4\qquad \textbf{(C) }3\sqrt{2}\qquad \textbf{(D) }2\sqrt{5}\qquad \textbf{(E) }5$

Diagram

3D:



Plane through triangle:



Solution 1

The triangle is placed on the sphere so that its three sides are tangent to the sphere. The cross-section of the sphere created by the plane of the triangle is also the incircle of the triangle. To find the inradius, use $\text{area} = \text{inradius} \cdot \text{semiperimeter}$. The area of the triangle can be found by drawing an altitude from the vertex between sides with length 15 to the midpoint of the side with length 24. The Pythagorean triple 9 - 12 - 15 allows us easily to determine that the base is 24 and the height is 9. The formula $\frac {\text{base} \cdot \text{height}} {2}$ can also be used to find the area of the triangle as 108, while the semiperimeter is simply $\frac{15+15+24}{2} = 27$. After plugging into the equation, we thus get $108 = \text{inradius} \cdot 27$, so the inradius is 4. Now, let the distance between O and the triangle be x. Choose a point on the incircle and denote it by A. The distance OA is 6, because it is just the radius of the sphere. The distance from point A to the center of the incircle is 4, because it is the radius of the incircle. By using the Pythagorean Theorem, we thus find $x = \sqrt{6^2 - 4^2} = \sqrt{20} = \boxed{\textbf{(D) } 2\sqrt5}$.

Solution 2 (No Inradius)

Drop an altitude on the isosceles triangle. Let the resulting 3-4-5 right triangle ABC have $AB = 15$ and $BC = 12$. By special triangle, $AC = 9$. Let r be the circle's radius. Let the circle's center be O and D be the closest point on AB to O.

Then, $OD = r$. Obviously, $ODBC$ is a kite. Thus, $BC = DB = 12$, and $AD = 15 - DB = 3$. $AO = AC - r = 9 - r$. By Pythagoras, $AD^2 + OD^2 = AO^2$, so $3^2 + r^2 = (9-r)^2$. The r^2 terms cancel out, and $r = 4$.

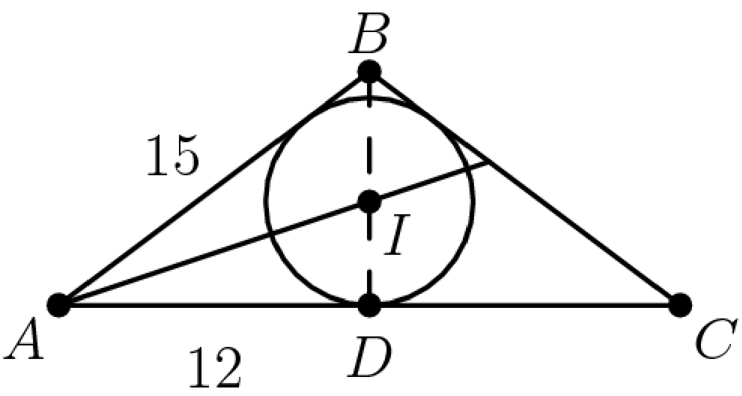
As before, using Pythagoras again, the distance is $\sqrt{6^2-4^2}=\boxed{(\textbf{D}) 2\sqrt{5}}$

(Solution by BJHHar)

Solution 3

As in Solution 1, we note that by the Pythagorean Theorem, the height of the triangle is 9, and that the three sides of the triangle are tangent to the sphere, so the circle in the cross-section of the sphere is the incenter of the triangle.

Recall that the inradius is the intersection of the angle bisectors. To find the inradius of the incircle, we use the Angle Bisector Theorem.

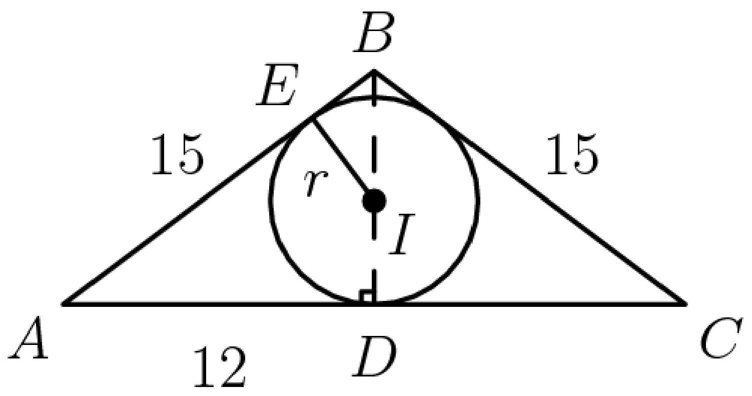


$$\begin{aligned} \frac{AB}{BI} &= \frac{AD}{DI} \\ \Rightarrow \frac{15}{BI} &= \frac{12}{DI} \\ \Rightarrow \frac{BI}{5} &= \frac{DI}{4} \end{aligned}$$

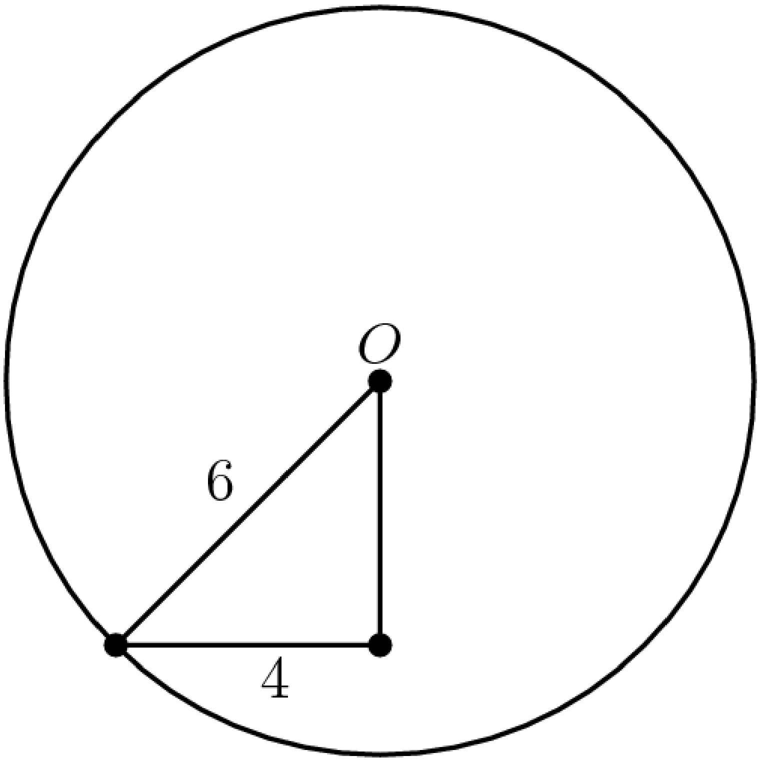
Since we know that $BI + DI$ (the height) is equal to 9, DI (the inradius) is 4. From here, the problem can be solved in the same way as in Solution 1. The answer is $\boxed{\textbf{(D) } 2\sqrt{5}}$.

Solution 4 (similar triangles)

First, we label a few points:



We have that $\triangle BDC$ is a $3-4-5$ triangle, so, as in Solution 1, $BD=9$. From this, we know that $\overline{BI}=9-r$. Since AB is tangent to circle I, we also know IEB is a right triangle. $\triangle BIE$ and $\triangle BDA$ share angle DBA, so $\triangle BIE \sim \triangle BDA$ since they have two equal angles. Hence, by this similarity, $\frac{9-r}{5}=\frac{r}{4}$. Cross-multiplying, we get $36-4r=5r$, which gives $r=4$. We now take another cross section of the sphere, perpendicular to the plane of the triangle.



Using the Pythagorean Theorem, we find that the distance from the center to the plane is $\boxed{\textbf{(D) } 2\sqrt{5}}$.

solution by woofle628 and GeniusKid1221

Community Discussion

https://artofproblemsolving.com/community/c3h1988124

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2019 AMC 10A Problems/Problem 22

The following problem is from both the 2019 AMC 10A #22 and 2019 AMC 12A #20, so both problems redirect to this page.

Problem

Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads, and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0, 1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x - y| > \tfrac{1}{2}$?

$$\textbf{(A) } \frac{1}{3} \qquad \textbf{(B) } \frac{7}{16} \qquad \textbf{(C) } \frac{1}{2} \qquad \textbf{(D) } \frac{9}{16} \qquad \textbf{(E) } \frac{2}{3}$$

Solution

There are several cases depending on what the first coin flip is when determining x and what the first coin flip is when determining y.

The four cases are:

Case 1: x is either 0 or 1, and y is either 0 or 1.

Case 2: x is either 0 or 1, and y is chosen from the interval $[0, 1]$.

Case 3: x is is chosen from the interval $[0, 1]$, and y is either 0 or 1.

Case 4: x is is chosen from the interval $[0, 1]$, and y is also chosen from the interval $[0, 1]$.

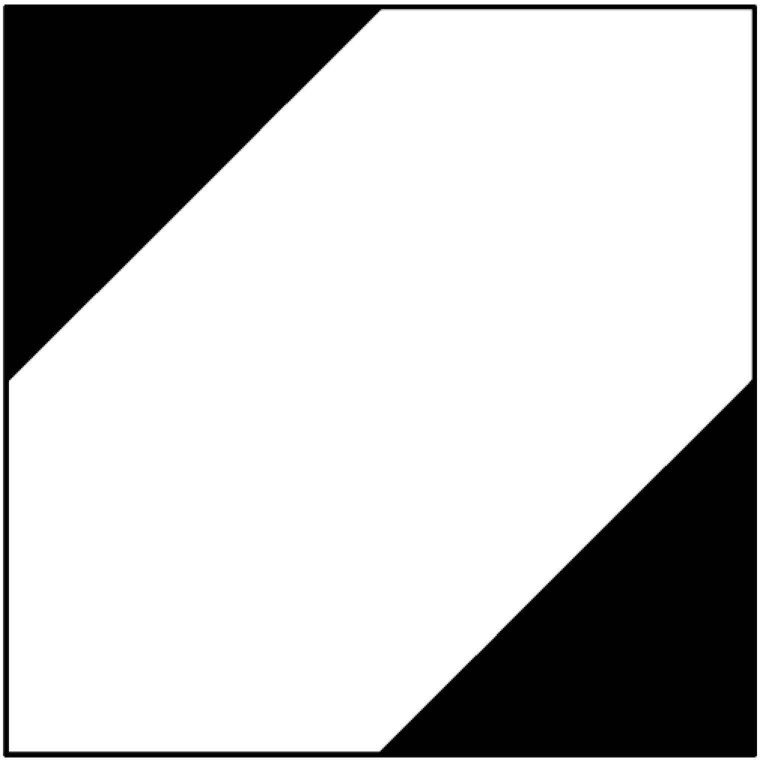
Each case has a $\dfrac{1}{4}$ chance of occurring (as it requires two coin flips).

For Case 1, we need x and y to be different. Therefore, the probability for success in Case 1 is .

For Case 2, if x is 0, we need y to be in the interval . If x is 1, we need y to be in the interval $\left[0, \frac{1}{2}\right)$. Regardless of what x is, the probability for success for Case 2 is .

By symmetry, Case 3 has the same success rate as Case 2.

For Case 4, we must use geometric probability because there are an infinite number of pairs that can be selected, whether they satisfy the inequality or not. Graphing $|x - y| > \frac{1}{2}$ gives us the following picture where the shaded area is the set of all the points that fulfill the inequality:



The shaded area is $\frac{1}{4}$, which means the probability for success for case 4 is $\frac{1}{4}$ (since the total area of the bounding square, containing all possible pairs, is 1).

Adding up the success rates from each case, we get:

$$\left(\frac{1}{4}\right) \cdot \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}\right) = \boxed{\textbf{(B) } \frac{7}{16}}.$$

See Also

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2019 AMC 10A Problems/Problem 23

Contents

Problem

Travis has to babysit the terrible Thompson triplets. Knowing that they love big numbers, Travis devises a counting game for them. First Tadd will say the number 1, then Todd must say the next two numbers (2 and 3), then Tucker must say the next three numbers (4, 5, 6), then Tadd must say the next four numbers (7, 8, 9, 10), and the process continues to rotate through the three children in order, each saying one more number than the previous child did, until the number $10,000$ is reached. What is the 2019th number said by Tadd?

$\textbf{(A)}\ 5743 \qquad \textbf{(B)}\ 5885 \qquad \textbf{(C)}\ 5979 \qquad \textbf{(D)}\ 6001 \qquad \textbf{(E)}\ 6011$

Solution 1

Define a *round* as one complete rotation through each of the three children.

We create a table to keep track of what numbers each child says for each round.

Round	Tadd	Todd	Tucker
1	1	2-3	4-6
2	7-10	11-15	16-21
3	22-28	29-36	37-45
4	46-55	56-66	67-78

Notice that at the end of each n^{th} round, the last number said is the $3n^{\text{th}}$ triangular number.

Tadd says 1 number in round 1, 4 numbers in round 2, 7 numbers in round 3, and in general $3n-2$ numbers in round n. At the end of round n, the number of numbers Tadd has said so far is $1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}$, by the arithmetic series sum formula.

We therefore want the smallest positive integer k such that $2019 \leq \frac{k(3k-1)}{2}$. The value of k will tell us in which round Tadd says his 2019^{th} number. Through guess and check (or by actually solving the quadratic inequality), $k=37$.

Now, using our formula $\frac{n(3n-1)}{2}$, Tadd says 1926 numbers in the first 36 rounds, so we are looking for the $(2019-1926)=93^{\text{rd}}$ number Tadd says in the 37^{th} round.

We found that the last number said at the very end of the n^{th} round is the $3n^{\text{th}}$ triangular number. For $n=36$, the 108^{th} triangular number is 5886. Thus the answer is $5886+93=\boxed{\textbf{(C)}\ 5979}$.

Solution 2

Firstly, as in Solution 1, we list how many numbers Tadd says, Todd says, and Tucker says in each round.

Tadd: $1, 4, 7, 10, 13 \cdots$

Todd: $2, 5, 8, 11, 14 \cdots$

Tucker: $3, 6, 9, 12, 15 \cdots$

We can find a general formula for the number of numbers each of the kids say after the nth round. For Tadd, we can either use the arithmetic series sum formula (like in Solution 1) or standard summation results to get $\sum_{i=1}^{n} 3n-2=-2n+3\sum_{i=1}^{n} n=-2n+\frac{3n(n+1)}{2}=\frac{3n^2-n}{2}$.

Now, to find the number of rotations Tadd and his siblings go through before Tadd says his 2019th number, we know the inequality $\frac{3n^2-n}{2}<2019$ must be satisfied, and testing numbers gives the maximum integer value of n as 36.

The next main insight, in order to simplify the computation process, is to notice that the 2019th number Tadd says is simply the number of numbers Todd and Tucker say plus the 2019 Tadd says, which will be the answer since Tadd goes first.

Carrying out the calculation thus becomes quite simple:

$$\left(\sum_{i=1}^{36} 3n+\sum_{i=1}^{36} 3n-1\right)+2019=\left(\sum_{i=1}^{36} 6n-1\right)+2019=(5+11+17...+215)+2019=\frac{36(220}{2}$$

At this point, we can note that the last digit of the answer is 9, which gives $\boxed{\textbf{(C) } 5979}$. (Completing the calculation will confirm the answer, if you have time.)

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2019 AMC 10A Problems/Problem 24

Contents

Problem

Let p, q, and r be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. It is given that there exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \not\in \{p, q, r\}$. What is $\tfrac1A+\tfrac1B+\tfrac1C$?

$\textbf{(A) }243\qquad\textbf{(B) }244\qquad\textbf{(C) }245\qquad\textbf{(D) }246\qquad\textbf{(E) } 247$

Solution 1

Multiplying both sides by $(s-p)(s-q)(s-r)$ yields

$$1 = A(s-q)(s-r) + B(s-p)(s-r) + C(s-p)(s-q)$$

As this is a polynomial identity, and it is true for infinitely many s, it must be true for all s (since a polynomial with infinitely many roots must in fact be the constant polynomial 0). This means we can plug in $s = p$ to find that $\frac1A = (p-q)(p-r)$. Similarly, we can find $\frac1B = (q-p)(q-r)$ and $\frac1C = (r-p)(r-q)$. Summing them up, we get that

$$\frac1A + \frac1B + \frac1C = p^2 + q^2 + r^2 - pq - qr - pr$$

By Vieta's Formulas, we know that $p^2 + q^2 + r^2 = (p+q+r)^2 - 2(pq + qr + pr) = 324$ and $pq + qr + pr = 80$. Thus the answer is $324 - 80 = \boxed{\textbf{(B) } 244}$.

Note: this process of substituting in the 'forbidden' values in the original identity is a standard technique for partial fraction decomposition, as taught in calculus classes.

Solution 2 (limits)

Multiplying by $(s-p)$ on both sides, we find that

$$\frac{s-p}{s^3 - 22s^2 + 80s - 67} = A + \frac{B(s-p)}{s-q} + \frac{C(s-p)}{s-r}$$

As $s \to p$, notice that the B and C terms on the right will cancel out and we will be left with only A. Hence, $A = \lim_{s \to p}\frac{s-p}{s^3 - 22s^2 + 80s - 67}$, which by L'Hôpital's rule becomes $\lim_{s \to p}\frac{1}{3s^2 - 44s + 80} = \frac{1}{3p^2 - 44p + 80}$. We can reason similarly to find B and C. Adding up the reciprocals and using Vieta's Formulas, we have that

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C} = 3(p^2+q^2+r^2)-44(p+q+r)+240 = 3(22^2-2(80))-44(22)+240 = \boxed{\textbf{(B) } 244}$$

See Also

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2019 AMC 10A Problems/Problem 25

The following problem is from both the 2019 AMC 10A #25 and 2019 AMC 12A #24, so both problems redirect to this page.

Contents

Problem

For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? (Recall that $0! = 1$.)

$\textbf{(A) } 31 \qquad \textbf{(B) } 32 \qquad \textbf{(C) } 33 \qquad \textbf{(D) } 34 \qquad \textbf{(E) } 35$

Solution 1

The main insight is that

$$\frac{(n^2)!}{(n!)^{n+1}}$$

is always an integer. This is true because it is precisely the number of ways to split up n^2 objects into n unordered groups of size n. Thus,

$$\frac{(n^2-1)!}{(n!)^n} = \frac{(n^2)!}{(n!)^{n+1}} \cdot \frac{n!}{n^2}$$

is an integer if $n^2 \mid n!$, or in other words, if $n \mid (n-1)!$. This condition is false precisely when $n=4$ or n is prime, by Wilson's Theorem. There are 15 primes between 1 and 50, inclusive, so there are 15 + 1 = 16 terms for which

$$\frac{(n^2-1)!}{(n!)^n}$$

is potentially not an integer. It can be easily verified that the above expression is not an integer for $n=4$ as there are more factors of 2 in the denominator than the numerator. Similarly, it can be verified that the above expression is not an integer for any prime $n=p$, as there are more factors of p in the denominator than the numerator. Thus all 16 values of n make the expression not an integer and the answer is $50-16=\boxed{\textbf{(D) } 34}$.

Solution 2

We can use the P-Adic Valuation of n to solve this problem (recall the P-Adic Valuation of 'n' is denoted by $v_p(n)$ and is defined as the greatest power of some prime 'p' that divides n. For example, $v_2(6)=1$ or $v_7(245)=2$.) Using Legendre's formula, we know that :

$$v_p(n!)=\sum_{i=1}^{\infty}\lfloor\frac{n}{p^i}\rfloor$$

Seeing factorials involved in the problem, this prompts us to use Legendre's formula where n is a power of a prime.

We also know that , $v_p(m^n)=n\cdot v_p(m)$. Knowing that $a \mid b$ if $v_p(a)\leq v_p(b)$, we have that :

$$n\cdot v_p(n!)\leq v_p((n^2-1)!)$$

and we must find all n for which this is true.

If we plug in $n=p$, by Legendre's we get two equations:

$$v_p((n^2-1)!) = \sum_{i=1}^{\infty} \lfloor \frac{n^2-1}{p^i} \rfloor = (p-1) + 0 + ... + 0 = p-1$$

And we also get :

$$v_p((n!)^n) = n \cdot v_p(n!) = n \cdot \sum_{i=1}^{\infty} \lfloor \frac{n}{p^i} \rfloor = p \cdot (1 + 0 + ...0) = p$$

But we are asked to prove that $n \cdot v_p(n!) \le v_p((n^2-1)!) \implies p \le p-1$ which is false for all 'n' where n is prime.

Now we try the same for $n = p^2$, where p is a prime. By Legendre we arrive at:

$$v_p((p^4-1)!) = p^3 + p^2 + p - 3$$

and

$$p^2 \cdot v_p(p^2!) = p^3 + p^2$$

Then we get:

$$p^2 \cdot v_p(p!) \le v_p((n^4-1)!) \implies p^3 + p^2 \le p^3 + p^2 + p - 3$$

Which is true for all primes except for 2, so $2^2 = 4$ doesn't work. It can easily be verified that for all $n = p^i$ where i is an integer greater than 2, satisfies the inequality :

$$n \cdot v_p(n!) \le v_p((n^2-1)!)$$

.

Therefore, there are 16 values that don't work and $50 - 16 = \boxed{\textbf{(D) } 34}$ values that work.

~qwertysri987

See Also

2019 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
Preceded by **Problem 24**	Followed by **Last Problem**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2019 AMC 12A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2019))	
Preceded by **Problem 23**	Followed by **Problem 25**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	