2018 AMC 10A Problems/Problem 1

Problem

What is the value of

$$\left(\left((2+1)^{-1}+1\right)^{-1}+1\right)^{-1}+1?$$

$\textbf{(A) } \frac{5}{8} \qquad \textbf{(B) } \frac{11}{7} \qquad \textbf{(C) } \frac{8}{5} \qquad \textbf{(D) } \frac{18}{11} \qquad \textbf{(E) } \frac{15}{8}$

Solution

We will start with $2+1=3$ and then apply the operation "invert and add one" three times. These iterations yield (after 3): $\frac{4}{3}, \frac{7}{4}$, and finally $\boxed{\textbf{(B) } \frac{11}{7}}$

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **First Problem**	Followed by **Problem 2**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 10A Problems/Problem 2

Contents

- 1 Problem
- 2 Solution
- 3 Solution 2
- 4 See Also

Problem

Liliane has 50% more soda than Jacqueline, and Alice has 25% more soda than Jacqueline. What is the relationship between the amounts of soda that Liliane and Alica have?

$\textbf{(A)}$ Liliane has 20% more soda than Alice.

$\textbf{(B)}$ Liliane has 25% more soda than Alice.

$\textbf{(C)}$ Liliane has 45% more soda than Alice.

$\textbf{(D)}$ Liliane has 75% more soda than Alice.

$\textbf{(E)}$ Liliane has 100% more soda than Alice.

Solution

Let's assume that Jacqueline has 1 gallon(s) of soda. Then Alice has 1.25 gallons and Liliane has 1.5 gallons. Doing division, we find out that $\frac{1.5}{1.25} = 1.2$, which means that Liliane has 20% more soda. Therefore, the answer is $\boxed{\textbf{(A)} \text{ } 20\%}$

Solution 2

If Jacqueline has x gallons of soda, Alice has $1.25x$ gallons, and Liliane has $1.5x$ gallons. Thus, the answer is $\frac{1.5}{1.25} = 1.2$ -> Liliane has 20% more soda. Our answer is $\boxed{\textbf{(A)} \text{ } 20\%}$.

```
~lakecomo224
```

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 1**	Followed by **Problem 3**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 10A Problems/Problem 3

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See Also

Problem

A unit of blood expires after $10! = 10 \cdot 9 \cdot 8 \cdots 1$ seconds. Yasin donates a unit of blood at noon of January 1. On what day does his unit of blood expire?

$\textbf{(A) }\text{January 2}\qquad\textbf{(B) }\text{January 12}\qquad\textbf{(C) }\text{January 22}\qquad\textbf{(D) }\text{February 11}\qquad\textbf{(E) }\text{February 12}$

Solution 1

There are $10!$ seconds that the blood has before expiring and there are $60 \cdot 60 \cdot 24$ seconds in a day. Dividing $\frac{10!}{60 \cdot 60 \cdot 24}$ gives 42 days. 42 days after January 1 is $\boxed{\textbf{(E) }\text{February 12}}$.

Solution 2

The problem says there are $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ seconds. Convert $10!$ seconds to minutes by dividing by 60: $9 \cdot 8 \cdot 7 \cdot 5 \cdot 4 \cdot 3 \cdot 2$ minutes. Convert minutes to hours by again, dividing by 60: $9 \cdot 8 \cdot 7 \cdot 2$ hours. Convert hours to days by dividing by 24: $3 \cdot 7 \cdot 2 = 42$ days.

Now we need to count 42 days after January 1. Since we start on Jan. 1, then we can't count that as a day itself. When we reach Jan. 31(The end of the month), we only have counted 30 days. $42 - 30 = 12$. Count 12 more days, resulting $\boxed{\textbf{(E) }\text{February 12}}$

```
~nosysnow and Max0815
```

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 2**	Followed by **Problem 4**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 10A Problems/Problem 4

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3
- 5 See Also

Problem

How many ways can a student schedule 3 mathematics courses -- algebra, geometry, and number theory -- in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)

$\textbf{(A) }3\qquad\textbf{(B) }6\qquad\textbf{(C) }12\qquad\textbf{(D) }18\qquad\textbf{(E) }24$

Solution 1

We must place the classes into the periods such that no two classes are in the same period or in consecutive periods.

Ignoring distinguishability, we can thus list out the ways that three periods can be chosen for the classes when periods cannot be consecutive:

Periods $1, 3, 5$

Periods $1, 3, 6$

Periods $1, 4, 6$

Periods $2, 4, 6$

There are 4 ways to place 3 nondistinguishable classes into 6 periods such that no two classes are in consecutive periods. For each of these ways, there are $3! = 6$ orderings of the classes among themselves.

Therefore, there are $4 \cdot 6 = \boxed{\textbf{(E) } 24}$ ways to choose the classes.

-Versailles15625

Solution 2

Realize that the number of ways of placing, regardless of order, the 3 mathematics courses in a 6-period day so that no two are consecutive is the same as the number of ways of placing 3 mathematics courses in a sequence of 4 periods regardless of order and whether or not they are consecutive.

To see that there is a one to one correlation, note that for every way of placing 3 mathematics courses in 4 total periods (as above) one can add a non-mathematics course between each pair (2 total) of consecutively occurring mathematics courses (not necessarily back to back) to ensure there will be no two consecutive mathematics courses in the resulting 6-period day. For example, where M denotes a math course and O denotes a non-math course:
$MOMM \rightarrow MOOMOM$

For each 6-period sequence consisting of Ms and Os, we have $3!$ orderings of the 3 distinct mathematics courses.

So, our answer is $\binom{4}{3}(3!) = \boxed{\textbf{(E) } 24}$

- Gregwwl

Solution 3

Counting what we don't want is another slick way to solve this problem. Use PIE to count two cases: 1. Two classes consecutive, 2. Three classes consecutive.

Case 1: Consider two consecutive periods as a "block" of which there are 5 places to put in(1,2; 2,3; 3,4; 4,5; 5,6). Then we simply need to place two classes within the block, $3 \cdot 2$. Finally we have 4 periods remaining to place the final math class. Thus there are $5 \cdot 3 \cdot 2 \cdot 4$ ways to place two consecutive math classes with disregard to the third.

Case 2: Now consider three consecutive periods as a "block" of which there are now 4 places to put in(1,2,3; 2,3,4; 3,4,5; 4,5,6). We now need to arrange the math classes in the block, $3 \cdot 2 \cdot 1$. Thus there are $4 \cdot 3 \cdot 2 \cdot 1$ ways to place all three consecutive math classes.

By PIE we subtract Case 1 by Case 2 in order to not overcount: $120 - 24$. Then we subtract that answer from the total ways to place the classes with no restrictions: $(6 \cdot 5 \cdot 4) - 96 = \boxed{\textbf{(E) } 24}$

-LitJamal

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 3**	Followed by **Problem 5**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Combinatorics Problems

2018 AMC 10A Problems/Problem 5

Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, "We are at least 6 miles away," Bob replied, "We are at most 5 miles away." Charlie then remarked, "Actually the nearest town is at most 4 miles away." It turned out that none of the three statements were true. Let d be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of d?

$\textbf{(A) } (0,4) \qquad \textbf{(B) } (4,5) \qquad \textbf{(C) } (4,6) \qquad \textbf{(D) } (5,6) \qquad \textbf{(E) } (5,\infty)$

Solution

From Alice and Bob, we know that $5 < d < 6.$ From Charlie, we know that $4 < d.$ We take the union of these two intervals to yield $\boxed{\textbf{(D) } (5,6)}$, because the nearest town is between 5 and 6 miles away.

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 4**	Followed by **Problem 6**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Algebra Problems

2018 AMC 10A Problems/Problem 6

Problem

Sangho uploaded a video to a website where viewers can vote that they like or dislike a video. Each video begins with a score of 0, and the score increases by 1 for each like vote and decreases by 1 for each dislike vote. At one point Sangho saw that his video had a score of 90, and that 65% of the votes cast on his video were like votes. How many votes had been cast on Sangho's video at that point?

$\textbf{(A) } 200 \qquad \textbf{(B) } 300 \qquad \textbf{(C) } 400 \qquad \textbf{(D) } 500 \qquad \textbf{(E) } 600$

Solution

If 65% of the votes were likes, then 35% of the votes were dislikes. $65\% - 35\% = 30\%$, so 90 votes is 30% of the total number of votes. Doing quick (maths) arithmetic shows that the answer is $\boxed{\textbf{(B) } 300}$

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 5**	Followed by **Problem 7**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 10A Problems/Problem 7

For how many (not necessarily positive) integer values of n is the value of $4000 \cdot \left(\frac{2}{5}\right)^n$ an integer?

$\textbf{(A) }3 \qquad \textbf{(B) }4 \qquad \textbf{(C) }6 \qquad \textbf{(D) }8 \qquad \textbf{(E) }9$

Solution

The prime factorization of 4000 is $2^5 \cdot 5^3$. Therefore, the maximum number for n is 3, and the minimum number for n is -5. Then we must find the range from -5 to 3, which is $3-(-5)+1 = 8+1 = \boxed{\textbf{(E) } 9}$.

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 6**	Followed by **Problem 8**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 6**	Followed by **Problem 8**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Number Theory Problems

2018 AMC 10A Problems/Problem 8

Joe has a collection of 23 coins, consisting of 5-cent coins, 10-cent coins, and 25-cent coins. He has 3 more 10-cent coins than 5-cent coins, and the total value of his collection is 320 cents. How many more 25-cent coins does Joe have than 5-cent coins?

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 3 \qquad \textbf{(E) } 4$

Contents

- 1 Solution 1
- 2 Solution 2
- 3 Solution 3
- 4 See Also

Solution 1

Let x be the number of 5-cent coins that Joe has. Therefore, he must have $(x+3)$ 10-cent coins and $(23-(x+3)-x)$ 25-cent coins. Since the total value of his collection is 320 cents, we can write

$$5x+10(x+3)+25(23-(x+3)-x) = 320 \Rightarrow 5x+10x+30+500-50x = 320 \Rightarrow 35x = 210 \Rightarrow$$

Joe has 6 5-cent coins, 9 10-cent coins, and 8 25-cent coins. Thus, our answer is $8-6 = \boxed{\textbf{(C) } 2}$

~Nivek

Solution 2

Let n be the number of 5 cent coins Joe has, d be the number of 10 cent coins, and q the number of 25 cent coins. We are solving for q - n.

We know that the value of the coins add up to 320 cents. Thus, we have 5n + 10d + 25q = 320. Let this be (1).

We know that there are 23 coins. Thus, we have n + d + q = 23. Let this be (2).

We know that there are 3 more dimes than nickels, which also means that there are 3 less nickels than dimes. Thus, we have d - 3 = n.

Plugging d-3 into the other two equations for n, (1) becomes 2d + q - 3 = 23 and (2) becomes 15d + 25q - 15 = 320. (1) then becomes 2d + q = 26, and (2) then becomes 15d + 25q = 335.

Multiplying (1) by 25, we have 50d + 25q = 650 (or 25^2 + 25). Subtracting (2) from (1) gives us 35d = 315, which means d = 9.

Plugging d into d - 3 = n, n = 6.

Plugging d and q into the (2) we had at the beginning of this problem, q = 8.

Thus, the answer is 8 - 6 = $\boxed{\textbf{(C) } 2}$.

Solution 3

So you set the number of 5-cent coins as x, the number of 10-cent coins as x+3, and the number of quarters y.

You make the two equations:

$$5x+10(x+3)+25y = 320 \Rightarrow 15x+25y+30 = 320 \Rightarrow 15x+25y = 290$$

$$x+x+3+y = 23 \Rightarrow 2x+3+y = 23 \Rightarrow 2x+y = 20$$

From there, you multiply the second equation by 25 to get

$$50x + 25y = 500$$

You subtract the first equation from the multiplied second equation to get

$$35x = 210 \Rightarrow x = 6$$

You can plug that value into one of the equations to get

$$y = 8$$

So, the answer is $8 - 6 = \boxed{\textbf{(C) } 2}$.

- mutinykids

See Also

2018 AMC 10A (Problems • Answer Key • Resources)	
Preceded by **Problem 7**	Followed by **Problem 9**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Algebra Problems

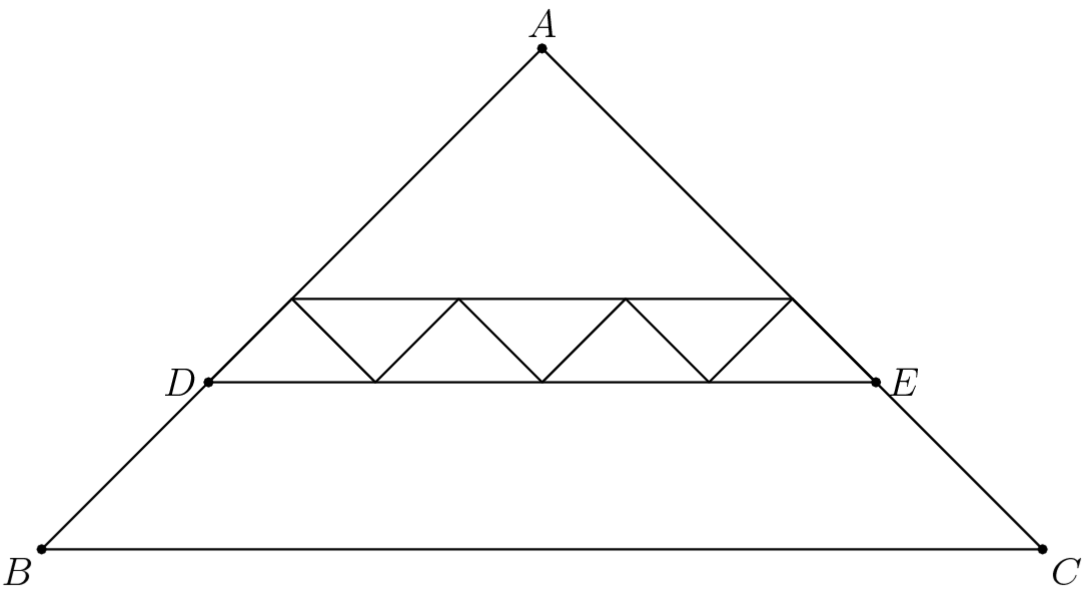
2018 AMC 10A Problems/Problem 9

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3
- 5 Solution 4
- 6 Solution 5
- 7 Solution 6
- 8 See Also

Problem

All of the triangles in the diagram below are similar to isosceles triangle ABC, in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?



$\textbf{(A) } 16 \qquad \textbf{(B) } 18 \qquad \textbf{(C) } 20 \qquad \textbf{(D) } 22 \qquad \textbf{(E) } 24$

Solution 1

Let x be the area of ADE. Note that x is comprised of the 7 small isosceles triangles and a triangle similar to ADE with side length ratio $3:4$ (so an area ratio of $9:16$). Thus, we have

$$x = 7 + \frac{9}{16}x$$

This gives $x = 16$, so the area of $DBCE = 40 - x = \boxed{24}$.

Solution 2

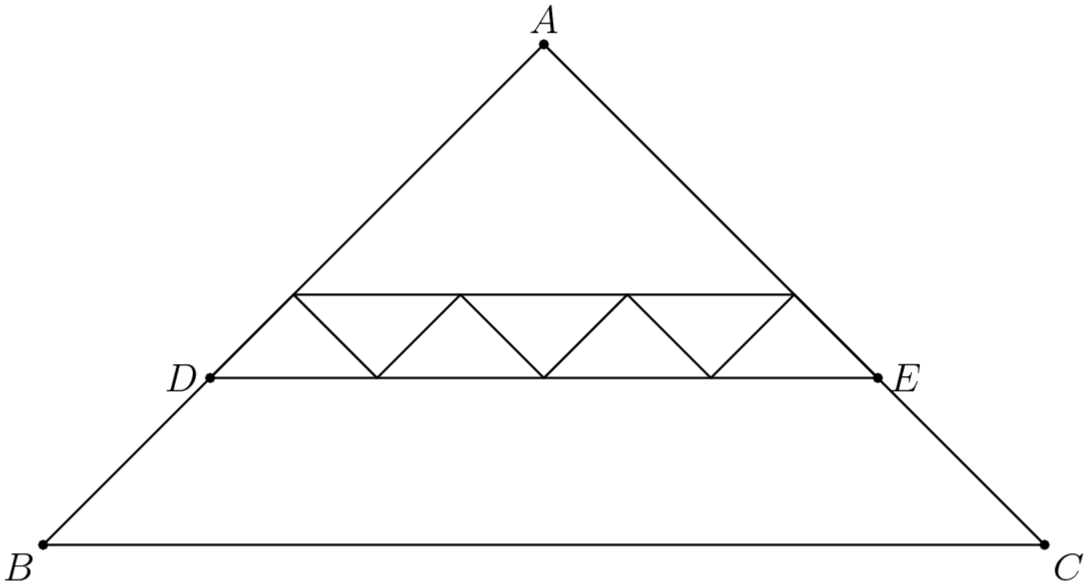
Let the base length of the small triangle be x. Then, there is a triangle ADE encompassing the 7 small triangles and sharing the top angle with a base length of $4x$. Because the area is proportional to the square of the side, let the base BC be $\sqrt{40}x$. Then triangle ADE has an area of 16. So the area is $40 - 16 = \boxed{24}$.

Solution 3

Notice $[DBCE] = [ABC] - [ADE]$. Let the base of the small triangles of area 1 be x, then the base length of $\Delta ADE = 4x$. Notice, $(\frac{DE}{BC})^2 = \frac{1}{40} \implies \frac{x}{BC} = \frac{1}{\sqrt{40}}$, then

$4x = \frac{4BC}{\sqrt{40}} \implies [ADE] = (\frac{4}{\sqrt{40}})^2 \cdot [ABC] = \frac{2}{5}[ABC]$ Thus,

$[DBCE] = [ABC] - [ADE] = [ABC](1 - \frac{2}{5}) = \frac{3}{5} \cdot 40 = \boxed{24}$



Solution by ktong

Solution 4

The area of ADE is 16 times the area of the small triangle, as they are similar and their side ratio is $4:1$. Therefore the area of the trapezoid is $40 - 16 = \boxed{24}$.

Solution 5

You can see that we can create a "stack" of 5 triangles congruent to the 7 small triangles shown here, arranged in a row above those 7, whose total area would be 5. Similarly, we can create another row of 3, and finally 1 more at the top, as follows. We know this cumulative area will be $7 + 5 + 3 + 1 = 16$, so to find the area of such trapezoid $BCED$, we just take $40 - 16 = \boxed{24}$, like so. ■ --anna0kear

Solution 6

The combined area of the small triangles is 7, and from the fact that each small triangle has an area of 1, we can deduce that the larger triangle above has an area of 9 (as the sides of the triangles are in a proportion of $\frac{1}{3}$, so will their areas have a proportion that is the square of the proportion of their sides, or $\frac{1}{9}$). Thus, the combined area of the top triangle and the trapezoid immediately below is $7 + 9 = 16$. The area of trapezoid $BCED$ is thus the area of triangle $ABC - 16 = \boxed{24}$. --lepetitmoulin

See Also

Category: Introductory Geometry Problems

2018 AMC 10A Problems/Problem 10

Contents

- 1 Problem
- 2 Solutions
  - 2.1 Solution 1
  - 2.2 Solution 2
  - 2.3 Solution 3
  - 2.4 Solution 4 (Geometric Interpretation)
- 3 See Also

Problem

Suppose that real number x satisfies

$$\sqrt{49-x^2}-\sqrt{25-x^2}=3$$

What is the value of $\sqrt{49-x^2}+\sqrt{25-x^2}$?

$\textbf{(A) }8\qquad\textbf{(B) }\sqrt{33}+8\qquad\textbf{(C) }9\qquad\textbf{(D) }2\sqrt{10}+4\qquad\textbf{(E) }12$

Solutions

Solution 1

In order to get rid of the square roots, we multiply by the conjugate. Its value is the solution.The x^2 terms cancel nicely.
$(\sqrt{49-x^2}+\sqrt{25-x^2})(\sqrt{49-x^2}-\sqrt{25-x^2})=49-x^2-25+x^2=24$

Given that $(\sqrt{49-x^2}-\sqrt{25-x^2})=3$, $(\sqrt{49-x^2}+\sqrt{25-x^2})=\frac{24}{3}=\boxed{\textbf{(A) } 8}$. -cookiemonster2004

Solution 2

Let $u=\sqrt{49-x^2}$, and let $v=\sqrt{25-x^2}$. Then $v=\sqrt{u^2-24}$. Substituting, we get $u-\sqrt{u^2-24}=3$. Rearranging, we get $u-3=\sqrt{u^2-24}$. Squaring both sides and solving, we get $u=\frac{11}{2}$ and $v=\frac{11}{2}-3=\frac{5}{2}$. Adding, we get that the answer is $\boxed{\textbf{(A) } 8}$.

Solution 3

Put the equations to one side. $\sqrt{49-x^2}-\sqrt{25-x^2}=3$ can be changed into $\sqrt{49-x^2}=\sqrt{25-x^2}+3$.

We can square both sides, getting us $49-x^2=(25-x^2)+(3^2)+2\cdot3\cdot\sqrt{25-x^2}$.

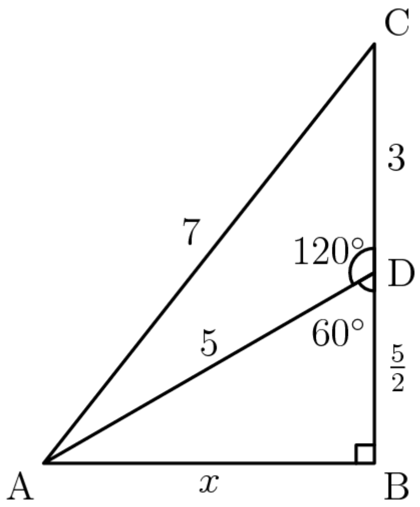
That simplifies out to $15=6\sqrt{25-x^2}$. Dividing both sides by 6 gets us $\frac{5}{2}=\sqrt{25-x^2}$.

Following that, we can square both sides again, resulting in the equation $\frac{25}{4}=25-x^2$. Simplifying that, we get $x^2=\frac{75}{4}$.

Substituting into the equation $\sqrt{49-x^2}+\sqrt{25-x^2}$, we get $\sqrt{49-\frac{75}{4}}+\sqrt{25-\frac{75}{4}}$. Immediately, we simplify into $\sqrt{\frac{121}{4}}+\sqrt{\frac{25}{4}}$. The two numbers inside the square roots are simplified to be $\frac{11}{2}$ and $\frac{5}{2}$, so you add them up: $\frac{11}{2}+\frac{5}{2}=\boxed{\textbf{(A) } 8}$.

Solution 4 (Geometric Interpretation)

Draw a right triangle ABC with a hypotenuse AC of length 7 and leg AB of length x. Draw D on BC such that $AD=5$. Note that $BC=\sqrt{49-x^2}$ and $BD=\sqrt{25-x^2}$. Thus, from the given equation, $BC-BD=DC=3$. Using Law of Cosines on triangle ADC, we see that $\angle ADC=120^\circ$ so $\angle ADB=60^\circ$. Since ADB is a $30-60-90$ triangle, $\sqrt{25-x^2}=BD=\frac{5}{2}$ and $\sqrt{49-x^2}=\frac{5}{2}+3=\frac{11}{2}$. Finally, $\sqrt{49-x^2}+\sqrt{25-x^2}=\frac{5}{2}+\frac{11}{2}=\boxed{\textbf{(A) } 8}$.



See Also

2018 AMC 10A (Problems • Answer Key • Resources)	
Preceded by **Problem 9**	Followed by **Problem 11**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Algebra Problems

2018 AMC 10A Problems/Problem 11

When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n?

$\textbf{(A) } 42 \qquad \textbf{(B) } 49 \qquad \textbf{(C) } 56 \qquad \textbf{(D) } 63 \qquad \textbf{(E) } 84$

Contents

- 1 Solutions
  - 1.1 Solution 1
  - 1.2 Solution 2
  - 1.3 Solution 3
  - 1.4 Solution 4 (overkill)
- 2 See Also

Solutions

Solution 1

The minimum number that can be shown on the face of a die is 1, so the least possible sum of the top faces of the 7 dies is 7.

In order for the sum to be exactly 10, 1 to 3 dices' number on the top face must be increased by a total of 3.

There are 3 ways to do so: 3, 2+1, and 1+1+1

There are 7 for Case 1, $7*6=42$ for Case 2, and $\frac{7*6*5}{3!}=35$ for Case 3.

Therefore, the answer is $7+42+35=\boxed{\textbf{(E) } 84}$

Solution by PancakeMonster2004

Solution 2

Rolling a sum of 10 with 7 dice can be represented with stars and bars, with 10 stars and 6 bars. Each star represents one of the dots on the die's faces and the bars represent separation between different dice. However, we must note that each die must have at least one dot on a face, so there must already be 7 stars predetermined. We are left with 3 stars and 6 bars, which we can rearrange in $\binom{9}{3}=\boxed{\textbf{(E) } 84}$ ways. (RegularHexagon)

Solution 3

Add possibilities. There are 3 ways to sum to 10, listed below.

$$4,1,1,1,1,1,1: 7$$

$$3,2,1,1,1,1,1: 42$$

$$2,2,2,1,1,1,1: 35.$$

Add up the possibilities: $35+42+7=\boxed{\textbf{(E) } 84}$.

Thus we have repeated Solution 1 exactly, but with less explanation.

~kevinmathz

Solution 4 (overkill)

We can use generating functions, where $(x+x^2+...+x^6)$ is the function for each die. We want to find the coefficient of x^{10} in $(x+x^2+...+x^6)^7$, which is the coefficient of x^3 in $\left(\frac{1-x^7}{1-x}\right)^7$. This evaluates to

$$\binom{-7}{3}\cdot(-1)^3=\boxed{\textbf{(E) } 84}$$

-wannabecharmander

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 10**	Followed by **Problem 12**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Probability Problems

2018 AMC 10A Problems/Problem 12

How many ordered pairs of real numbers (x, y) satisfy the following system of equations?

$$x + 3y = 3$$

$$\big||x| - |y|\big| = 1$$

$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 4 \qquad \textbf{(E) } 8$

Contents

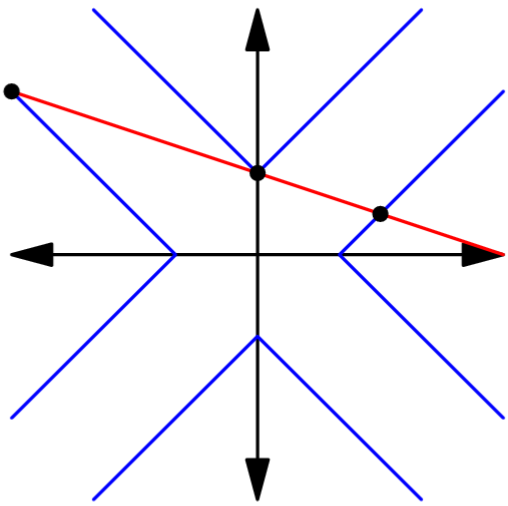
- 1 Solutions
  - 1.1 Solution 1
  - 1.2 Solution 2
  - 1.3 Solution 3
  - 1.4 Solution 4
- 2 See Also

Solutions

Solution 1

We can solve this by graphing the equations. The second equation looks challenging to graph, but start by graphing it in the first quadrant only (which is easy since the inner absolute value signs can be ignored), then simply reflect that graph into the other quadrants.

The graph looks something like this:



Now, it becomes clear that there are $\boxed{\textbf{(C) } 3}$ intersection points. (pinetree1) BOI

Solution 2

$x + 3y = 3$ can be rewritten to $x = 3 - 3y$. Substituting $3 - 3y$ for x in the second equation will give $||3 - 3y| - y| = 1$. Splitting this question into casework for the ranges of y will give us the total number of solutions.

$\textbf{Case 1:}$ $y > 1$: $3 - 3y$ will be negative so $|3 - 3y| = 3y - 3$. $|3y - 3 - y| = |2y - 3| = 1$

Subcase 1: $y > \dfrac{3}{2}$

$2y-3$ is positive so $2y-3=1$ and $y=2$ and $x=3-3(2)=-3$

$$\text{Subcase 2: } 1<y<\frac{3}{2}$$

$2y-3$ is negative so $|2y-3|=3-2y=1$. $2y=2$ and so there are no solutions (y can't equal to 1)

$\textbf{Case 2:}$ $y=1$: It is fairly clear that $x=0$.

$\textbf{Case 3:}$ $y<1$: $3-3y$ will be positive so $|3-3y-y|=|3-4y|=1$

$$\text{Subcase 1: } y>\frac{4}{3}$$

$3-4y$ will be negative so $4y-3=1 \rightarrow 4y=4$. There are no solutions (again, y can't equal to 1)

$$\text{Subcase 2: } y<\frac{4}{3}$$

$3-4y$ will be positive so $3-4y=1 \rightarrow 4y=2$. $y=\frac{1}{2}$ and $x=\frac{3}{2}$. Thus, the solutions are: $(-3,2),(0,1),\left(\frac{3}{2},\frac{1}{2}\right)$, and the answer is $\boxed{\textbf{(C) } 3}$. $\LaTeX$ edit by pretzel, very minor $\LaTeX$ edits by Bryanli, very very minor $\LaTeX$ edit by ssb02

Solution 3

Note that $||x|-|y||$ can take on either of four values: $x+y$, $x-y$, $-x+y$, $-x-y$. Solving the equations (by elimination, either adding the two equations or subtracting), we obtain the three solutions: $(0,1)$, $(-3,2)$, $(1.5,0.5)$ so the answer is $\boxed{\textbf{(C) } 3}$. One of those equations overlap into $(0,1)$ so there's only 3 solutions.

~trumpeter, ccx09 ~minor edit, XxHalo711

Solution 4

Just as in solution 2, we derive the equation $||3-3y|-|y||=1$. If we remove the absolute values, the equation collapses into four different possible values. $3-2y$, $3-4y$, $2y-3$, and $4y-3$, each equal to either 1 or -1. Remember that if $P-Q=a$, then $Q-P=-a$. Because we have already taken 1 and -1 into account, we can eliminate one of the conjugates of each pair, namely $3-2y$ and $2y-3$, and $3-4y$ and $4y-3$. Find the values of y when $3-2y=1$, $3-2y=-1$, $3-4y=1$ and $3-4y=-1$. We see that $3-2y=1$ and $3-4y=-1$ give us the same value for y, so the answer is $\boxed{\textbf{(C) } 3}$

~Zeric Hang

See Also

2018 AMC 10A (Problems • Answer Key • Resources)	
Preceded by **Problem 11**	Followed by **Problem 13**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 9**	Followed by **Problem 11**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Intermediate Algebra Problems

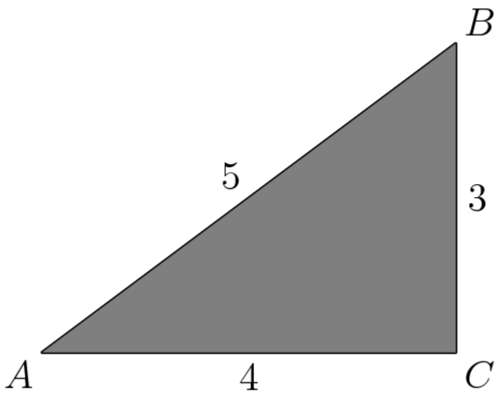
2018 AMC 10A Problems/Problem 13

Contents

- 1 Problem
- 2 Solution 1
  - 2.1 Note
- 3 Solution 2
- 4 See Also

Problem

A paper triangle with sides of lengths 3,4, and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease?



$\textbf{(A) } 1+\frac12 \sqrt2 \qquad \textbf{(B) } \sqrt3 \qquad \textbf{(C) } \frac74 \qquad \textbf{(D) } \frac{15}{8} \qquad \textbf{(E) } 2$

Solution 1

First, we need to realize that the crease line is just the perpendicular bisector of side AB, the hypotenuse of right triangle $\triangle ABC$. Call the midpoint of AB point D. Draw this line and call the intersection point with AC as E. Now, $\triangle ACB$ is similar to $\triangle ADE$ by AA similarity. Setting up the ratios, we find that

$$\frac{BC}{AC}=\frac{DE}{AD} \Rightarrow \frac{3}{4}=\frac{DE}{\frac{5}{2}} \Rightarrow DE=\frac{15}{8}.$$

Thus, our answer is $\boxed{\textbf{D) } \frac{15}{8}}$.

~Nivek

Note

In general, whenever we are asked to make a crease, think about that crease as a line of reflection over which the diagram is reflected. This is why the crease must be the perpendicular bisector of AB, because A must be reflected onto B. (by pulusona)

Solution 2

Use the ruler and graph paper you brought to quickly draw a 3-4-5 triangle of any scale (don't trust the diagram in the booklet). Very carefully fold the acute vertices together and make a crease. Measure the crease with the ruler. If you were reasonably careful, you should see that it measures somewhat more than $\frac{7}{4}$ units and somewhat less than 2 units.

The only answer choice in range is $\boxed{\textbf{D)}\ \frac{15}{8}}$.

This is pretty much a cop-out, but it's allowed in the rules technically.

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 12**	Followed by **Problem 14**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 10**	Followed by **Problem 12**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2018 AMC 10A Problems/Problem 14

What is the greatest integer less than or equal to

$$\frac{3^{100}+2^{100}}{3^{96}+2^{96}}?$$

$\textbf{(A) }80\qquad\textbf{(B) }81 \qquad\textbf{(C) }96 \qquad\textbf{(D) }97 \qquad\textbf{(E) }625\qquad$

Contents

- 1 Solution
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3
- 5 Solution 4
- 6 Solution 5 (eyeball it)
- 7 Solution 6 (Using the answer choices)
- 8 Solution 7 (The Slick Solution)
- 9 Solution 8
- 10 Solution 9
- 11 See Also

Solution

Solution 1

Let's set this value equal to x. We can write

$$\frac{3^{100}+2^{100}}{3^{96}+2^{96}}=x.$$

Multiplying by $3^{96}+2^{96}$ on both sides, we get

$$3^{100}+2^{100}=x(3^{96}+2^{96}).$$

Now let's take a look at the answer choices. We notice that 81, choice B, can be written as 3^4. Plugging this into out equation above, we get

$$3^{100}+2^{100} \stackrel{?}{=} 3^4(3^{96}+2^{96}) \Rightarrow 3^{100}+2^{100} \stackrel{?}{=} 3^{100}+3^4\cdot 2^{96}.$$

The right side is larger than the left side because

$$2^{100} \leq 2^{96}\cdot 3^4.$$

This means that our original value, x, must be less than 81. The only answer that is less than 81 is 80 so our answer is $\boxed{A}$.

~Nivek

Solution 2

$$\frac{3^{100}+2^{100}}{3^{96}+2^{96}}=\frac{2^{96}(\frac{3^{100}}{2^{96}})+2^{96}(2^4)}{2^{96}(\frac{3}{2})^{96}+2^{96}(1)}=\frac{\frac{3^{100}}{2^{96}}+2^4}{(\frac{3}{2})^{96}+1}=\frac{\frac{3^{100}}{2^{100}}*2^4+2^4}{(\frac{3}{2})^{96}+1}=\frac{2^4(\frac{3^{100}}{2^{100}}+1)}{(\frac{3}{2})^{96}+1}.$$

We can ignore the 1's on the end because they won't really affect the fraction. So, the answer is very very very close but less than the new fraction.

$$\frac{2^4(\frac{3^{100}}{2^{100}}+1)}{(\frac{3}{2})^{96}+1} < \frac{2^4(\frac{3^{100}}{2^{100}})}{(\frac{3}{2})^{96}}$$

$$\frac{2^4(\frac{3^{100}}{2^{100}})}{(\frac{3}{2})^{96}} = \frac{3^4}{2^4} * 2^4 = 3^4 = 81$$

So, our final answer is very close but not quite 81, and therefore the greatest integer less than the number is $\boxed{(A) 80}$

Solution 3

Let $x = 3^{96}$ and $y = 2^{96}$. Then our fraction can be written as $\frac{81x+16y}{x+y} = \frac{16x+16y}{x+y} + \frac{65x}{x+y} = 16 + \frac{65x}{x+y}$. Notice that $\frac{65x}{x+y} < \frac{65x}{x} = 65$. So , $16 + \frac{65x}{x+y} < 16 + 65 = 81$. And our only answer choice less than 81 is $\boxed{(A) 80}$ (RegularHexagon)

Solution 4

Let $x = \frac{3^{100}+2^{100}}{3^{96}+2^{96}}$. Multiply both sides by $(3^{96}+2^{96})$, and expand. Rearranging the terms, we get $3^{96}(3^4 - x) + 2^{96}(2^4 - x) = 0$. The left side is strictly decreasing, and it is negative when $x = 81$. This means that the answer must be less than 81; therefore the answer is $\boxed{(A)}$.

Solution 5 (eyeball it)

A faster solution. Recognize that for exponents of this size 3^n will be enormously greater than 2^n, so the terms involving 2 will actually have very little effect on the quotient. Now we know the answer will be very close to 81.

Notice that the terms being added on to the top and bottom are in the ratio $\frac{1}{16}$ with each other, so they must pull the ratio down from 81 very slightly. (In the same way that a new test score lower than your current cumulative grade always must pull that grade downward.) Answer: $\boxed{(A)}$.

Solution 6 (Using the answer choices)

We can compare the given value to each of our answer choices. We already know that it is greater than 80 because otherwise there would have been a smaller answer, so we move onto 81. We get:

$$\frac{3^{100}+2^{100}}{3^{96}+2^{96}} \ ? \ 3^4$$

Cross multiply to get:

$$3^{100}+2^{100} \ ? \ 3^{100}+(2^{96})(3^4)$$

Cancel out 3^{100} and divide by 2^{96} to get $2^4 \ ? \ 3^4$. We know that $2^4 < 3^4$, which means the expression is less than 81 so the answer is $\boxed{(A)}$.

Solution 7 (The Slick Solution)

Notice how $\frac{3^{100}+2^{100}}{3^{96}+2^{96}}$ can be rewritten as $\frac{81(3^{96})+16(2^{96})}{3^{96}+2^{96}} = \frac{81(3^{96})+81(2^{96})}{3^{96}+2^{96}} - \frac{65(2^{96})}{3^{96}+2^{96}} = 81 - \frac{65(2^{96})}{3^{96}+2^{96}}$. Note that $\frac{65(2^{96})}{3^{96}+2^{96}} < 1$, so the greatest integer less than or equal to $\frac{3^{100}+2^{100}}{3^{96}+2^{96}}$ is 80 or $\boxed{(\mathbf{A})}$ ~blitzkrieg21

Solution 8

For positive a, b, c, d, if $\frac{a}{b} < \frac{c}{d}$ then $\frac{c+a}{d+b} < \frac{c}{d}$. Let $a = 2^{100}, b = 2^{96}, c = 3^{100}, d = 3^{96}$. Then $\frac{c}{d} = 3^4$. So answer is less than 81, which leaves only one choice, 80.

- Note that the algebra here is synonymous to the explanation given in Solution 5. This is the algebraic reason to the logic of if you get a test score with a lower percentage than your average (no matter how many points/percentage of your total grade it was worth), it will pull your overall grade down.

~ ccx09

Solution 9

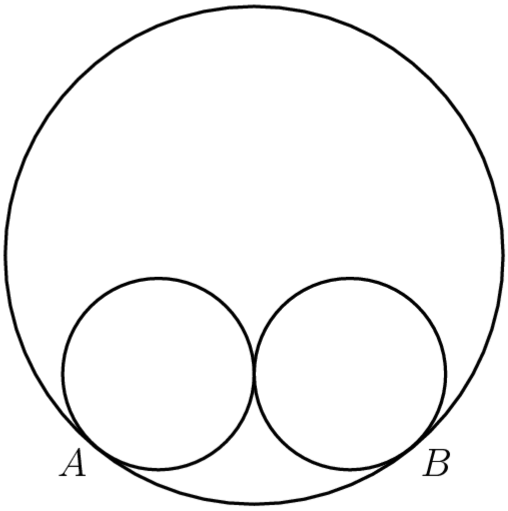
Try long division, and notice putting $3^4 = 81$ as denominator is too big and putting $3^4 - 1 = 80$ is too small. So we know that the answer is between 80 and 81, yielding 80 as our answer.

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 13**	Followed by **Problem 15**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

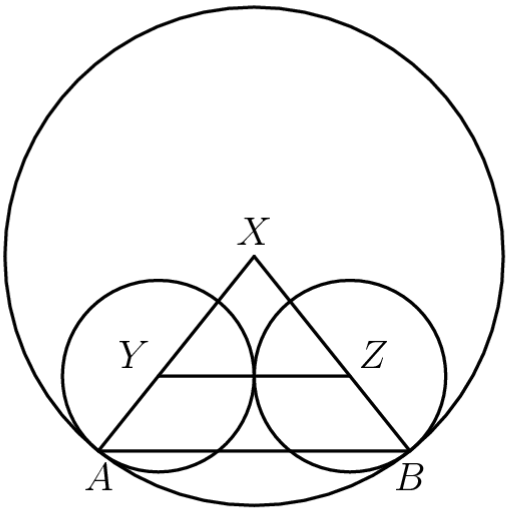
2018 AMC 10A Problems/Problem 15

Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\tfrac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?



$\textbf{(A) } 21 \qquad \textbf{(B) } 29 \qquad \textbf{(C) } 58 \qquad \textbf{(D) } 69 \qquad \textbf{(E) } 93$

Solution 1

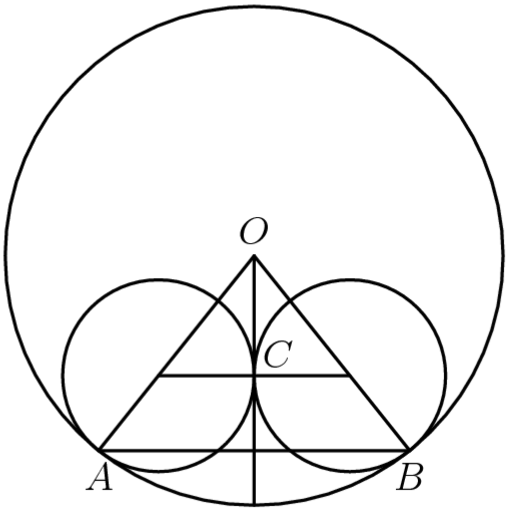


Let the center of the surrounding circle be X. The circle that is tangent at point A will have point Y as the center. Similarly, the circle that is tangent at point B will have point Z as the center. Connect AB, YZ, XA, and XB. Now observe that $\triangle XYZ$ is similar to $\triangle XAB$. Writing out the ratios, we get

$$\frac{XY}{XA} = \frac{YZ}{AB} \Rightarrow \frac{13-5}{13} = \frac{5+5}{AB} \Rightarrow \frac{8}{13} = \frac{10}{AB} \Rightarrow AB = \frac{65}{4}.$$

Therefore, our answer is $65+4=\boxed{\textbf{D) } 69}$.

Solution 2



Let the center of the large circle be O. Let the common tangent of the two smaller circles be C. Draw the two radii of the large circle, $\overline{OA}$ and $\overline{OB}$ and the two radii of the smaller circles to point C. Draw ray $\overrightarrow{OC}$ and $\overline{AB}$. This sets us up with similar triangles, which we can solve. The length of $\overline{OC}$ is equal to $\sqrt{39}$ by Pythagorean Theorem, the length of the hypotenuse is 8, and the other leg is 5. Using similar triangles, OB is 13, and therefore half of AB is $\frac{65}{8}$. Doubling gives $\frac{65}{4}$, which results in $65 + 4 = \boxed{\textbf{D) } 69}$. $QED\square$

(minor edits by elements2015)

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 14**	Followed by **Problem 16**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

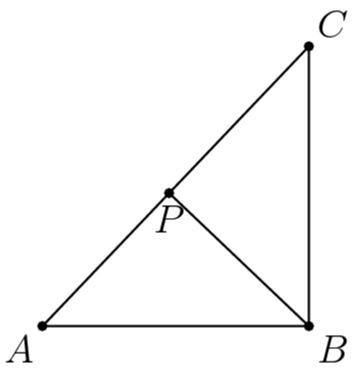
Category: Introductory Geometry Problems

2018 AMC 10A Problems/Problem 16

Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?

$\textbf{(A) }5 \qquad \textbf{(B) }8 \qquad \textbf{(C) }12 \qquad \textbf{(D) }13 \qquad \textbf{(E) }15$

Solution



As the problem has no diagram, we draw a diagram. The hypotenuse has length 29. Let P be the foot of the altitude from B to AC. Note that BP is the shortest possible length of any segment. Writing the area of the triangle in two ways, we can solve for $BP = \frac{20 \cdot 21}{29}$, which is between 14 and 15.

Let the line segment be BX, with X on AC. As you move X along the hypotenuse from A to P, the length of BX strictly decreases, hitting all the integer values from $20, 19, \dots 15$ (IVT). Similarly, moving X from P to C hits all the integer values from $15, 16, \dots, 21$. This is a total of $\boxed{(D) 13}$ line segments. (asymptote diagram added by elements2015)

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 15**	Followed by **Problem 17**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

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Category: Introductory Geometry Problems

2018 AMC 10A Problems/Problem 17

Contents

- 1 Problem
- 2 Solution
- 3 Solution 2
- 4 See Also

Problem

Let S be a set of 6 integers taken from $\{1, 2, \ldots, 12\}$ with the property that if a and b are elements of S with $a < b$, then b is not a multiple of a. What is the least possible value of an element in S?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 7$

Solution

If we start with 1, we can include nothing else, so that won't work.

If we start with 2, we would have to include every odd number except 1 to fill out the set, but then 3 and 9 would violate the rule, so that won't work.

Experimentation with 3 shows it's likewise impossible. You can include 7, 11, and either 5 or 10 (which are always safe). But after adding either 4 or 8 we have nowhere else to go.

Finally, starting with 4, we find that the sequence $4, 5, 6, 7, 9, 11$ works, giving us $\boxed{\textbf{(C)}\ 4}$. (Random_Guy)

Solution 2

We know that all the odd numbers (except 1) can be used.

$$3, 5, 7, 9, 11$$

Now we have 7 to choose from for the last number (out of $1, 2, 4, 6, 8, 10, 12$). We can eliminate 1, 2, 10, and 12, and we have $4, 6, 8$ to choose from. But wait, 9 is a multiple of 3! Now we have to take out either 3 or 9 from the list. If we take out 9, none of the numbers would work, but if we take out 3, we get:

$$4, 5, 6, 7, 9, 11$$

So the least number is 4, so the answer is $\boxed{\textbf{(C)}\ 4}$.

-Baolan

See Also

2018 AMC 10A (Problems • Answer Key • Resources)	
Preceded by **Problem 16**	Followed by **Problem 18**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12A (Problems • Answer Key • Resources)	
Preceded by **Problem 11**	Followed by **Problem 13**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 10A Problems/Problem 18

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3
- 5 Solution 4
- 6 Solution 5
- 7 Solution 6
- 8 See Also

Problem

How many nonnegative integers can be written in the form

$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0,$$

where $a_i \in \{-1, 0, 1\}$ for $0 \le i \le 7$?

$\textbf{(A)}\ 512 \qquad \textbf{(B)}\ 729 \qquad \textbf{(C)}\ 1094 \qquad \textbf{(D)}\ 3281 \qquad \textbf{(E)}\ 59,048$

Solution 1

This looks like balanced ternary, in which all the integers with absolute values less than $\frac{3^n}{2}$ are represented in n digits. There are 8 digits. Plugging in 8 into the formula for the balanced ternary gives a maximum bound of $|x| = 3280.5$, which means there are 3280 positive integers, 0, and 3280 negative integers. Since we want all nonnegative integers, there are $3280 + 1 = \boxed{3281}$ integers or $\boxed{D}$.

Solution 2

Note that all numbers formed from this sum are either positive, negative or zero. The number of positive numbers formed by this sum is equal to the number of negative numbers formed by this sum, because of symmetry. There is only one way to achieve a sum of zero, if all $a_i = 0$. The total number of ways to pick a_i from $i = 1, 2, 3, ...7$ is $3^8 = 6561$. $\frac{6561 - 1}{2} = 3280$ gives the number of possible negative integers. The question asks for the number of nonnegative integers, so subtracting from the total gives $6561 - 3280 = \boxed{3281}$. (RegularHexagon)

Solution 3

Note that the number of total possibilities (ignoring the conditions set by the problem) is $3^8 = 6561$. So, E is clearly unrealistic.

Note that if a_7 is 1, then it's impossible for

$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0,$$

to be negative. Therefore, if a_7 is 1, there are $3^7 = 2187$ possibilities. (We also must convince ourselves that these 2187 different sets of coefficients must necessarily yield 2187 different integer results.)

As A, B, and C are all less than 2187, the answer must be $\boxed{(D) 3281}$

Solution 4

Note that we can do some simple casework: If $a_7 = 1$, then we can choose anything for the other 7 variables, so this give us 3^7. If $a_7 = 0$ and $a_6 = 1$, then we can choose anything for the other 6 variables, giving us 3^6. If $a_7 = 0$, $a_6 = 0$, and $a_5 = 1$, then we have 3^5. Continuing in this vein, we have $3^7 + 3^6 + \cdots + 3^1 + 3^0$ ways to choose the variables' values, except we have to add 1 because we haven't counted the case where all variables are 0. So our total sum is $\boxed{(D)3281}$. Note that we have counted all possibilities, because the largest positive positive power of 3 must be greater than or equal to the largest negative positive power of 3, for the number to be nonnegative.

Solution 5

The key is to realize that this question is basically taking place in $a \in \{0, 1, 2\}$ if each value of a was increased by 1, essentially making it into base 3. Then the range would be from $0 \cdot 3^7 + 0 \cdot 3^6 + 0 \cdot 3^5 + 0 \cdot 3^4 + 0 \cdot 3^3 + 0 \cdot 3^2 + 0 \cdot 3^1 + 0 \cdot 3^0 = 0$ to $2 \cdot 3^7 + 2 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^4 + 2 \cdot 3^3 + 2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0 = 3^8 - 1 = 6561 - 1 = 6560$, yielding 6561 different values. Since the distribution for all $a_i \in \{-1, 0, 1\}$ the question originally gave is symmetrical, we retain the 3280 positive integers and one 0 but discard the 3280 negative integers. Thus, we are left with the answer, $\boxed{\textbf{(D)} 3281}$. $\blacksquare$ --anna0kear

Solution 6

First, set $a_i = 0$ for all $i \geq 1$. The range would be the integers for which $[-1, 1]$. If $a_i = 0$ for all $i \geq 2$, our set expands to include all integers such that $-4 \leq \mathbb{Z} \leq 4$. Similarly, when $i \geq 3$ we get $-13 \leq \mathbb{Z} \leq 13$, and when $i \geq 4$ the range is $-40 \leq \mathbb{Z} \leq 40$. The pattern continues until we reach $i = 7$, where $-3280 \leq \mathbb{Z} \leq 3280$. Because we are only looking for positive integers, we filter out all $\mathbb{Z} < 0$, leaving us with all integers between $0 \leq \mathbb{Z} \leq 3280$, inclusive. The answer becomes $\boxed{(D)3281}$. $\blacksquare$ --anna0kear

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 17**	Followed by **Problem 19**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 12**	Followed by **Problem 14**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Intermediate Number Theory Problems

2018 AMC 10A Problems/Problem 19

A number m is randomly selected from the set $\{11, 13, 15, 17, 19\}$, and a number n is randomly selected from $\{1999, 2000, 2001, \ldots, 2018\}$. What is the probability that m^n has a units digit of 1?

$\textbf{(A)}\ \frac{1}{5} \qquad \textbf{(B)}\ \frac{1}{4} \qquad \textbf{(C)}\ \frac{3}{10} \qquad \textbf{(D)}\ \frac{7}{20} \qquad \textbf{(E)}\ \frac{2}{5}$

Solution 1

Since we only care about the unit digit, our set $\{11, 13, 15, 17, 19\}$ can be turned into $\{1, 3, 5, 7, 9\}$. Call this set A and call $\{1999, 2000, 2001, \cdots, 2018\}$ set B. Let's do casework on the element of A that we choose. Since $1 \cdot 1 = 1$, any number from B can be paired with 1 to make 1^n have a units digit of 1. Therefore, the probability of this case happening is $\frac{1}{5}$ since there is a $\frac{1}{5}$ chance that the number 1 is selected from A. Let us consider the case where the number 3 is selected from A. Let's look at the unit digit when we repeatedly multiply the number 3 by itself:

$$3 \cdot 3 = 9$$

$$9 \cdot 3 = 7$$

$$7 \cdot 3 = 1$$

$$1 \cdot 3 = 3$$

We see that the unit digit of 3^x, for some integer x, will only be 1 when x is a multiple of 4. Now, let's count how many numbers in B are divisible by 4. This can be done by simply listing:

$$2000, 2004, 2008, 2012, 2016.$$

There are 5 numbers in B divisible by 4 out of the $2018 - 1999 + 1 = 20$ total numbers. Therefore, the probability that 3 is picked from A and a number divisible by 4 is picked from B is $\frac{1}{5} \cdot \frac{5}{20} = \frac{1}{20}$. Similarly, we can look at the repeating units digit for 7:

$$7 \cdot 7 = 9$$

$$9 \cdot 7 = 3$$

$$3 \cdot 7 = 1$$

$$1 \cdot 7 = 7$$

We see that the unit digit of 7^y, for some integer y, will only be 1 when y is a multiple of 4. This is exactly the same conditions as our last case with 3 so the probability of this case is also $\frac{1}{20}$. Since $5 \cdot 5 = 25$ and 25 ends in 5, the units digit of 5^w, for some integer, w will always be 5. Thus, the probability in this case is 0. The last case we need to consider is when the number 9 is chosen from A. This happens with probability $\frac{1}{5}$. We list out the repeating units digit for 9 as we have done for 3 and 7:

$$9 \cdot 9 = 1$$

$$1 \cdot 9 = 9$$

We see that the units digit of 9^z, for some integer z, is 1 only when z is an even number. From the 20 numbers in B, we see that exactly half of them are even. The probability in this case is $\frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$. Finally, we can add all of our probabilities together to get

$$\frac{1}{5} + \frac{1}{20} + \frac{1}{20} + \frac{1}{10} = \boxed{\frac{2}{5}}.$$

~Nivek

Solution 2

Since only the units digit is relevant, we can turn the first set into $\{1, 3, 5, 7, 9\}$. Note that $x^4 \equiv 1 \mod 10$ for all odd digits x, except for 5. Looking at the second set, we see that it is a set of all integers between 1999 and 2018. There are 20 members of this set, which means that, $\mod 4$, this set has 5 values which correspond to $\{0, 1, 2, 3\}$, making the probability equal for all of them. Next, check the values for which it is equal to $1 \mod 10$. There are $4 + 1 + 0 + 1 + 2 = 8$ values for which it is equal to 1, remembering that $5^{4n} \equiv 1 \mod 10$ only if $n = 0$, which it is not. There are 20 values in total, and simplifying $\frac{8}{20}$ gives us $\boxed{\frac{2}{5}}$ or $\boxed{E}$.

$QED \blacksquare$

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 18**	Followed by **Problem 20**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Categories: Introductory Number Theory Problems | Introductory Probability Problems

2018 AMC 10A Problems/Problem 20

A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called $symmetric$ if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

$\textbf{(A)}\ 510 \qquad \textbf{(B)}\ 1022 \qquad \textbf{(C)}\ 8190 \qquad \textbf{(D)}\ 8192 \qquad \textbf{(E)}\ 65{,}534$

Solution 1

Draw a 7×7 square.

K	J	H	G	H	J	K
J	F	E	D	E	F	J
H	E	C	B	C	E	H
G	D	B	A	B	D	G
H	E	C	B	C	E	H
J	F	E	D	E	F	J
K	J	H	G	H	J	K

Start from the center and label all protruding cells symmetrically.

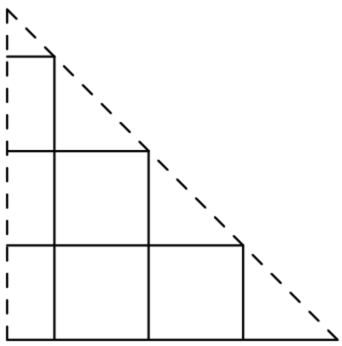
More specifically, since there are 4 given lines of symmetry (2 diagonals, 1 vertical, 1 horizontal) and they split the plot into 8 equivalent sections, we can take just one-eighth and study it in particular. Each of these sections has 10 distinct sub-squares, whether partially or in full. So since each can be colored either white or black, we choose $2^{10} = 1024$ but then subtract the 2 cases where all are white or all are black. That leaves us with $\boxed{(B)}, 1022.$ ■

There are only ten squares we get to actually choose, and two independent choices for each, for a total of $2^{10} = 1024$ codes. Two codes must be subtracted (due to the rule that there must be at least one square of each color) for an answer of $\boxed{\textbf{(B)} \text{ } 1022}$.

```
~Nosysnow
```

Note that this problem is very similar to the 1996 AIME Problem 7.

Solution 2



Imagine folding the scanning code along its lines of symmetry. There will be 10 regions which you have control over coloring. Since we must subtract off 2 cases for the all-black and all-white cases, the answer is
$2^{10} - 2 = \boxed{\textbf{(B)} \text{ } 1022.}$

-EatingStuff

See Also

2018 AMC 10A (Problems • Answer Key • Resources)	
Preceded by **Problem 19**	Followed by **Problem 21**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12A (Problems • Answer Key • Resources)	
Preceded by **Problem 14**	Followed by **Problem 16**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Intermediate Combinatorics Problems

2018 AMC 10A Problems/Problem 21

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3
- 5 Solution 4
- 6 Solution 5 (Cheating with Answer Choices)
- 7 Solution 6 (Calculus Needed)
- 8 See Also

Problem

Which of the following describes the set of values of a for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real xy-plane intersect at exactly 3 points?

$\textbf{(A) }a=\frac14 \qquad \textbf{(B) }\frac14 < a < \frac12 \qquad \textbf{(C) }a>\frac14 \qquad \textbf{(D) }a=\frac12 \qquad \textbf{(E) }a>\frac12$

Solution 1

Substituting $y = x^2 - a$ into $x^2 + y^2 = a^2$, we get

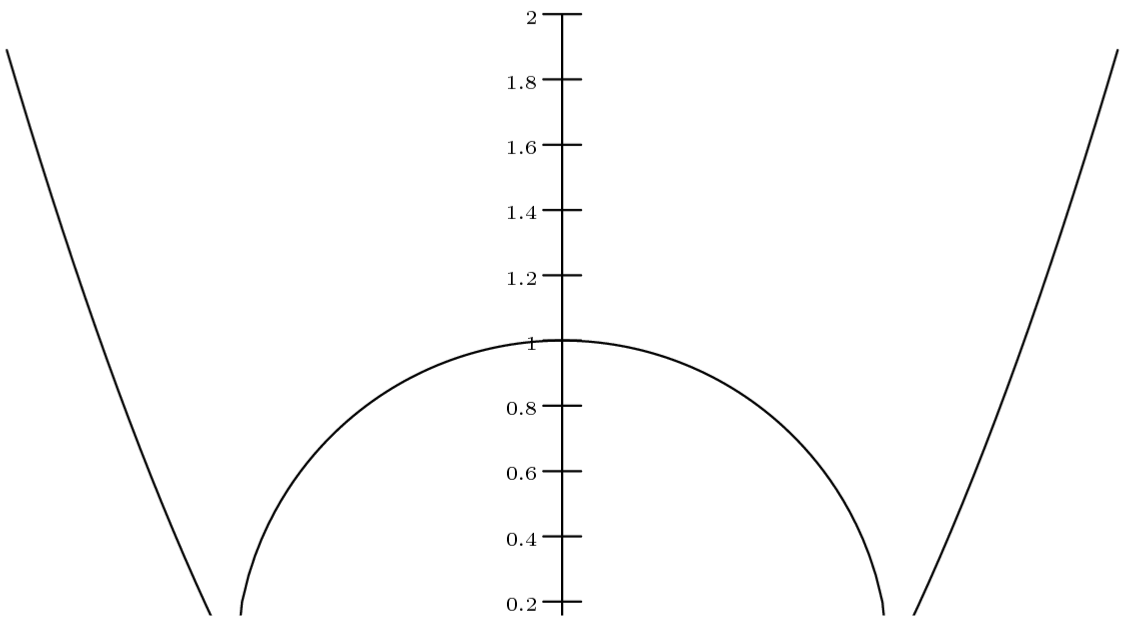
$$x^2+(x^2-a)^2=a^2 \implies x^2+x^4-2ax^2=0 \implies x^2(x^2-(2a-1))=0$$

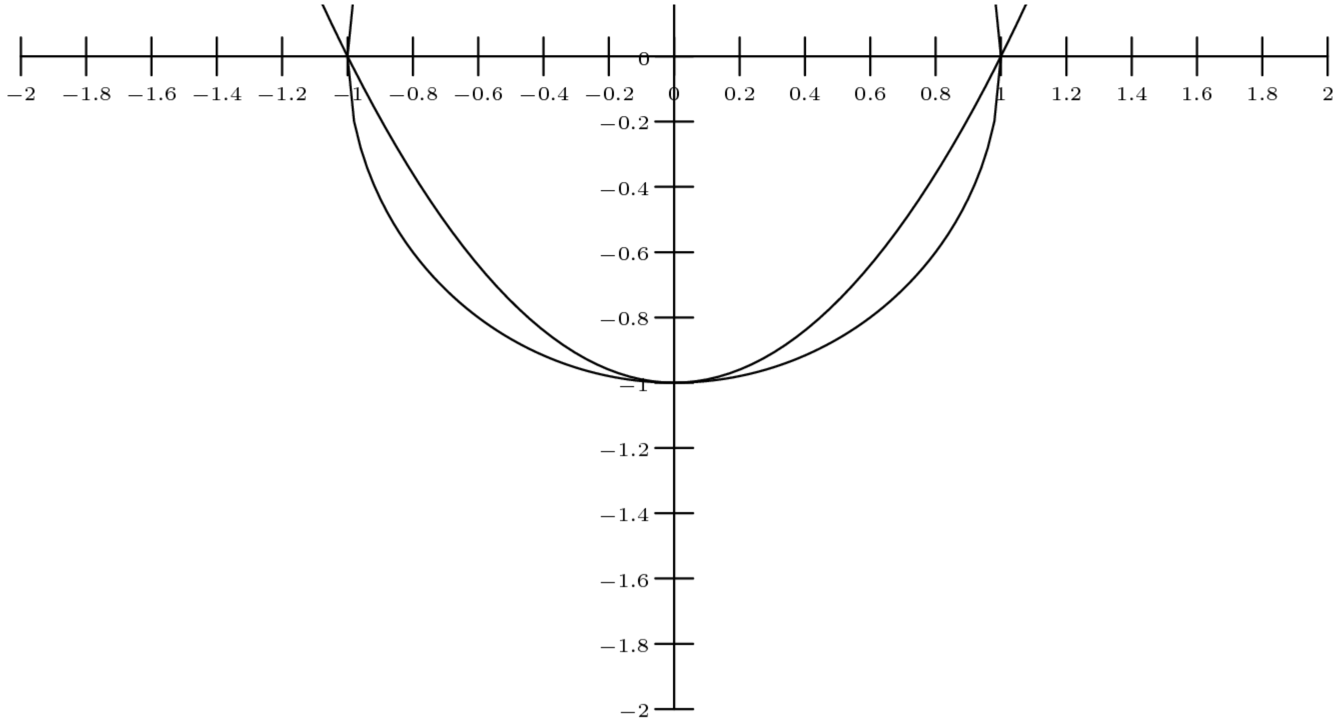
Since this is a quartic, there are 4 total roots (counting multiplicity). We see that $x = 0$ always at least one intersection at $(0, -a)$ (and is in fact a double root).

The other two intersection points have x coordinates $\pm\sqrt{2a-1}$. We must have $2a - 1 > 0$, otherwise we are in the case where the parabola lies entirely above the circle (tangent to it at the point $(0, a)$). This only results in a single intersection point in the real coordinate plane. Thus, we see $\boxed{\textbf{(E) }a>\frac12}$.

(projecteulerlover)

Solution 2





Looking at a graph, it is obvious that the two curves intersect at (0, -a). We also see that if the parabola goes 'in' the circle. Then, by going out of it (as it will), it will intersect five times, an impossibility. Thus we only look for cases where the parabola becomes externally tangent to the circle. We have $x^2 - a = -\sqrt{a^2 - x^2}$. Squaring both sides and solving yields $x^4 - (2a-1)x^2 = 0$. Since x = 0 is already accounted for, we only need to find 1 solution for $x^2 = 2a - 1$, where the right hand side portion is obviously increasing. Since a = 1/2 begets x = 0 (an overcount), we have $\boxed{\textbf{(E) }a > \frac{1}{2}}$ is the right answer.

Solution by JohnHankock

Solution 3

This describes a unit parabola, with a circle centered on the axis of symmetry and tangent to the vertex. As the curvature of the unit parabola at the vertex is 2, the radius of the circle that matches it has a radius of $\frac{1}{2}$. This circle is tangent to an infinitesimally close pair of points, one on each side. Therefore, it is tangent to only 1 point. When a larger circle is used, it is tangent to 3 points because the points on either side are now separated from the vertex. Therefore, $\boxed{\textbf{(E) }a > \frac{1}{2}}$ is correct.

$QED\blacksquare$

Solution 4

Notice, the equations are of that of a circle of radius a centered at the origin and a parabola translated down by a units. They always intersect at the point $(0, a)$, and they have symmetry across the y-axis, thus, for them to intersect at exactly 3 points, it suffices to find the y solution.

First, rewrite the second equation to $y = x^2 - a \implies x^2 = y + a$ And substitute into the first equation: $y + a + y^2 = a^2$ Since we're only interested in seeing the interval in which a can exist, we find the discriminant: $1 - 4a + 4a^2$. This value must not be less than 0 (It is the square root part of the quadratic formula). To find when it is 0, we find the roots:

$$4a^2 - 4a + 1 = 0 \implies a = \frac{4 \pm \sqrt{16-16}}{8} = \frac{1}{2}$$

Since $\lim_{a \to \infty} (4a^2 - 4a + 1) = \infty$, our range is $\boxed{\textbf{(E) } a > \frac{1}{2}}$.

Solution by ktong

Solution 5 (Cheating with Answer Choices)

Simply plug in $a = 0, \frac{1}{2}, \frac{1}{4}, 1$ and solve the systems. (This shouldn't take too long.) And then realize that only $a = 1$ yields three real solutions for x, so we are done and the answer is $\boxed{\textbf{(E) } a > \frac{1}{2}}$.

~ ccx09

Solution 6 (Calculus Needed)

In order to solve for the values of a, we need to just count multiplicities of the roots when the equations are set equal to each other: in other words, take the derivative. We know that $\sqrt{x^2 - a^2} = x^2 - a$. Now, we take square of both sides, and rearrange to obtain $x^4 - (2a+1)x^2 + 2a^2 = 0$. Now, we make take the second derivative of the equation to obtain $6x^2 - (4a+2) = 0$. Now, we must take discriminant. Since we need the roots of that equation to be real and not repetitive (otherwise they would not intersect each other at three points), the discriminant must be greater than zero. Thus,

$b^2 - 4ac > 0 \to 0 - 4(6)(4a+2) > 0 \to a > \frac{1}{2}$ The answer is $\boxed{\textbf{(E) } a > \frac{1}{2}}$ and we are done.

~awesome1st

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 20**	Followed by **Problem 22**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 15**	Followed by **Problem 17**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2018 AMC 10A Problems/Problem 22

Let $a, b, c,$ and d be positive integers such that $\gcd(a, b) = 24$, $\gcd(b, c) = 36$, $\gcd(c, d) = 54$, and $70 < \gcd(d, a) < 100$. Which of the following must be a divisor of a?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 7 \qquad \textbf{(C)}\ 11 \qquad \textbf{(D)}\ 13 \qquad \textbf{(E)}\ 17$

Contents

- 1 Solution 1
- 2 Solution 1.1
- 3 Solution 2 (Better notation)
- 4 See Also

Solution 1

We can say that a and b 'have' $2^3 * 3$, that b and c have $2^2 * 3^2$, and that c and d have $3^3 * 2$. Combining 1 and 2 yields b has (at a minimum) $2^3 * 3^2$, and thus a has $2^3 * 3$ (and no more powers of 3 because otherwise $gcd(a,b)$ would be different). In addition, c has $3^3 * 2^2$, and thus d has $3^3 * 2$ (similar to a, we see that d cannot have any other powers of 2). We now assume the simplest scenario, where $a = 2^3 * 3$ and $d = 3^3 * 2$. According to this base case, we have $gcd(a, d) = 2 * 3 = 6$. We want an extra factor between the two such that this number is between 70 and 100, and this new factor cannot be divisible by 2 or 3. Checking through, we see that $6 * 13$ is the only one that works. Therefore the answer is $\boxed{\textbf{(D)}\ 13}$

Solution by JohnHankock

Solution 1.1

Elaborating on to what Solution 1 stated, we are not able to add any extra factor of 2 or 3 to $gcd(a,d)$ because doing so would later the gcd of (a,b) and (c,d). This is why:

The $gcd(a,b)$ is $2^3 * 3$ and the gcd of (c,d) is $2 * 3^3$. However, the gcd of $(b,c) = 2^2 * 3^2$ (meaning both are divisible by 36). Therefore, a is only divisible by 3^1 (and no higher power of 3), while d is divisible by only 2^1 (and no higher power of 2).

Thus, the gcd of (a,d) can be expressed in the form $2 * 3 * k$ for which k is a number not divisible by 2 or 3. The only answer choice that satisfies this (and the other condition) is $\boxed{\textbf{(D)}\ 13}$.

Solution 2 (Better notation)

First off, note that 24, 36, and 54 are all of the form $2^x \times 3^y$. The prime factorizations are $2^3 \times 3^1$, $2^2 \times 3^2$ and $2^1 \times 3^3$, respectively. Now, let a_2 and a_3 be the number of times 2 and 3 go into a,respectively. Define b_2, b_3, c_2, and c_3 similiarly. Now, translate the lcms into the following:

$$\min(a_2, b_2) = 3$$

$$\min(a_3, b_3) = 1$$

$$\min(b_2, c_2) = 2$$

$$\min(b_3, c_3) = 2$$

$$\min(a_2, c_2) = 1$$

$$\min(a_3, c_3) = 3$$

.

(Unfinished) ~Rowechen Zhong

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 21**	Followed by **Problem 23**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Intermediate Number Theory Problems

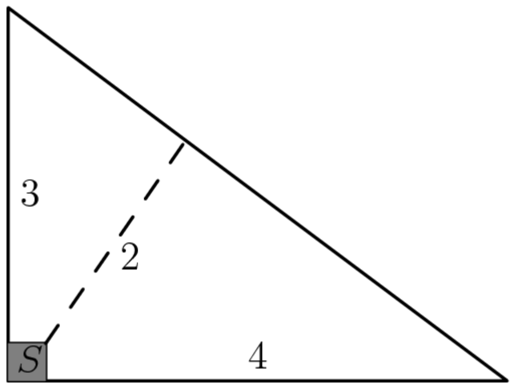
2018 AMC 10A Problems/Problem 23

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3
- 5 Solution 4
- 6 Solution 5
- 7 Solution 6
- 8 See Also

Problem

Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



$\textbf{(A) } \frac{25}{27} \qquad \textbf{(B) } \frac{26}{27} \qquad \textbf{(C) } \frac{73}{75} \qquad \textbf{(D) } \frac{145}{147} \qquad \textbf{(E) } \frac{74}{75}$

Solution 1

Let the square have side length x. Connect the upper-right vertex of square S with the two vertices of the triangle's hypotenuse. This divides the triangle in several regions whose areas must add up to the area of the whole triangle, which is 6.

Square S has area x^2, and the two thin triangle regions have area $\frac{x(3-x)}{2}$ and $\frac{x(4-x)}{2}$. The final triangular region with the hypotenuse as its base and height 2 has area 5. Thus, we have

$$x^2+\frac{x(3-x)}{2}+\frac{x(4-x)}{2}+5=6$$

Solving gives $x=\frac{2}{7}$. The area of S is $\frac{4}{49}$ and the desired ratio is $\frac{6-\frac{4}{49}}{6}=\boxed{\frac{145}{147}}$.

Alternatively, once you get $x=\frac{2}{7}$, you can avoid computation by noticing that there is a denominator of 7, so the answer must have a factor of 7 in the denominator, which only $\boxed{\frac{145}{147}}$ does.

Solution 2

Let the square have side length s. If we were to extend the sides of the square further into the triangle until they intersect on point on the hypotenuse, we'd have a similar right triangle formed between the hypotenuse and the two new lines, and 2 smaller similar triangles that share a side of length 2. Using the side-to-side ratios of these triangles, we can find that the length of the big similar triangle is $\frac{5}{3}(2)=\frac{10}{3}$. Now, let's extend this big similar right triangle to the left until it hits the side of length 3. Now, the length is $\frac{10}{3}+s$, and using the ratios of the side lengths, the height is $\frac{3}{4}(\frac{10}{3}+s)=\frac{5}{2}+\frac{3s}{4}$. Looking at the diagram, if we add the height of this triangle to the side length of the square, we'd get 3, so

$$\frac{5}{2}+\frac{3s}{4}+s=\frac{5}{2}+\frac{7s}{4}=3\frac{7s}{4}=\frac{1}{2}s=\frac{2}{7}\implies \text{area of square is }(\frac{2}{7})^2=\frac{4}{49}$$

Now comes the easy part: finding the ratio of the areas: $\frac{3\cdot 4\cdot\frac{1}{2}-\frac{4}{49}}{3\cdot 4\cdot\frac{1}{2}}=\frac{6-\frac{4}{49}}{6}=\frac{294-4}{294}=\frac{290}{294}=\boxed{\frac{145}{147}}$.

Solution by ktong

Solution 3

We use coordinate geometry. Let the right angle be at $(0,0)$ and the hypotenuse be the line $3x+4y=12$ for $0\le x\le 3$. Denote the position of S as (s,s), and by the point to line distance formula, we know that

$$\frac{|3s+4s-12|}{5}=2$$

$$\Rightarrow |7s-12|=10$$

Obviously $s<\frac{22}{7}$, so $s=\frac{2}{7}$, and from here the rest of the solution follows to get $\boxed{\frac{145}{147}}$.

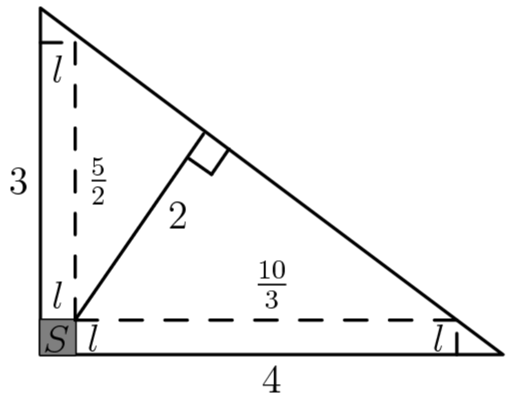
Solution 4

Let the side length of the square be x. First off, let us make a similar triangle with the segment of length 2 and the top-right corner of S. Therefore, the longest side of the smaller triangle must be $2\cdot\frac{5}{4}=\frac{5}{2}$. We then do operations with that side in terms of x. We subtract x from the bottom, and $\frac{3x}{4}$ from the top. That gives us the equation of $3-\frac{7x}{4}=\frac{5}{2}$. Solving,

$$12-7x=10\implies x=\frac{2}{7}.$$

Thus, $x^2=\frac{4}{49}$, so the fraction of the triangle (area 6) covered by the square is $\frac{2}{147}$. The answer is then $\boxed{\frac{145}{147}}$.

Solution 5



On the diagram above, find two smaller triangles similar to the large one with side lengths 3, 4, and 5; consequently, the segments with length $\frac{5}{2}$ and $\frac{10}{3}$.

Find an expression for l: using the hypotenuse, we can see that $\frac{3}{2}+\frac{8}{3}+\frac{5}{4}l+\frac{5}{3}l=5$. Simplifying, $\frac{35}{12}l=\frac{5}{6}$, or $l=2/7$.

A different calculation would yield $l+\frac{3}{4}l+\frac{5}{2}=3$, so $\frac{7}{4}l=\frac{1}{2}$. In other words, $l=\frac{2}{7}$, while to check, $l+\frac{4}{3}l+\frac{10}{3}=4$. As such, $\frac{7}{3}l=\frac{2}{3}$, and $l=\frac{2}{7}$.

Finally, we get $A(\square S)=l^2=\frac{4}{49}$, to finish. As a proportion of the triangle with area 6, the answer would be $1-\frac{4}{49\cdot 6}=1-\frac{2}{147}=\frac{145}{147}$, so $\boxed{D}$ is correct. $\blacksquare$ --anna0kear

Solution 6

Let s be the side length of the square. The area of the triangle is 6. Connect the inside corner of the square to the three corners. Then, the area of the triangle is also $5+\frac{3}{2}s+2s=5+\frac{7}{2}s$. Solving gives $s=\frac{2}{7}$. That makes the answer $\frac{6-\left(\frac{2}{7}\right)^2}{6}=\frac{145}{147}$. $\boxed{\textbf{D.}}$

\

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 22**	Followed by **Problem 24**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 16**	Followed by **Problem 18**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 10A Problems/Problem 24

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Solution 3
- 5 Solution 4
- 6 See Also

Problem

Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

$\textbf{(A) }60 \qquad \textbf{(B) }65 \qquad \textbf{(C) }70 \qquad \textbf{(D) }75 \qquad \textbf{(E) }80$

Solution 1

Let $BC = a$, $BG = x$, $GC = y$, and the length of the perpendicular to BC through A be h. By angle bisector theorem, we have that

$$\frac{50}{x} = \frac{10}{y},$$

where $y = -x + a$. Therefore substituting we have that $BG = \frac{5a}{6}$. By similar triangles, we have that $DF = \frac{5a}{12}$, and the height of this trapezoid is $\frac{h}{2}$. Then, we have that $\frac{ah}{2} = 120$. We wish to compute $\frac{5a}{8} \cdot \frac{h}{2}$, and we have that it is $\boxed{75}$ by substituting.

(rachanamadhu)

I may have read this solution incorrectly, but it seems to me that the author mistakenly assumed that the angle bisector is a perpendicular bisector, which is false since the triangle is not isosceles. -bobert1

Solution 2

$\overline{DE}$ is midway from A to $\overline{BC}$, and $DE = \frac{BC}{2}$. Therefore, $\triangle ADE$ is a quarter of the area of $\triangle ABC$, which is 30. Subsequently, we can compute the area of quadrilateral $BDEC$ to be $120 - 30 = 90$. Using the angle bisector theorem in the same fashion as the previous problem, we get that $\overline{BG}$ is 5 times the length of $\overline{GC}$. We want the larger piece, as described by the problem. Because the heights are identical, one area is 5 times the other, and $\frac{5}{6} \cdot 90 = \boxed{75}$.

Solution 3

The area of $\triangle ABG$ to the area of $\triangle ACG$ is $5:1$ by Law of Sines. So the area of $\triangle ABG$ is 100. Since $\overline{DE}$ is the midsegment of $\triangle ABC$, so $\overline{DF}$ is the midsegment of $\triangle ABG$. So the area of $\triangle ACG$ to the area of $\triangle ABG$ is $1:4$, so the area of $\triangle ACG$ is 25, by similar triangles. Therefore the area of quad $FDBG$ is $100 - 25 = \boxed{75}$ (steakfails)

Solution 4

The area of quadrilateral $FDBG$ is the area of $\triangle ABG$ minus the area of $\triangle ADF$. Notice, $\overline{DE}||\overline{BC}$, so $\triangle ABG \sim \triangle ADF$, and since $\overline{AD} : \overline{AB} = 1 : 2$, the area of $\triangle ADF : \triangle ABG = (1:2)^2 = 1 : 4$. Given that the area of $\triangle ABC$ is 120, using $\frac{bh}{2}$ on side AB yields $\frac{50h}{2} = 120 \implies h = \frac{240}{50} = \frac{24}{5}$. Using the Angle Bisector Theorem, $\overline{BG} : \overline{BC} = 50 : (10 + 50) = 5 : 6$, so the height of $\triangle ABG : \triangle ACB = 5 : 6$. Therefore our answer is

$$[FDBG] = [ABG] - [ADF] = [ABG](1 - \frac{1}{4}) = \frac{3}{4} \cdot \frac{bh}{2} = \frac{3}{8} \cdot 50 \cdot \frac{5}{6} \cdot \frac{24}{5} = \frac{3}{8} \cdot 200 = \boxed{75}$$

-Solution by ktong

See Also

2018 AMC 10A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 23**	Followed by **Problem 25**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12A (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 17**	Followed by **Problem 19**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Intermediate Geometry Problems

2018 AMC 10A Problems/Problem 25

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2 (quicker?)
- 4 Solution 3
- 5 See Also

Problem

For a positive integer n and nonzero digits a, b, and c, let A_n be the n-digit integer each of whose digits is equal to a; let B_n be the n-digit integer each of whose digits is equal to b, and let C_n be the $2n$-digit (not n-digit) integer each of whose digits is equal to c. What is the greatest possible value of $a+b+c$ for which there are at least two values of n such that $C_n - B_n = A_n^2$?

$\textbf{(A) } 12 \qquad \textbf{(B) } 14 \qquad \textbf{(C) } 16 \qquad \textbf{(D) } 18 \qquad \textbf{(E) } 20$

Solution 1

Observe $A_n = a(1 + 10 + \dots + 10^{n - 1}) = a \cdot \tfrac{10^n - 1}{9}$; similarly $B_n = b \cdot \tfrac{10^n - 1}{9}$ and $C_n = c \cdot \tfrac{10^{2n} - 1}{9}$. The relation $C_n - B_n = A_n^2$ rewrites as

$$c \cdot \frac{10^{2n} - 1}{9} - b \cdot \frac{10^n - 1}{9} = a^2 \cdot \left(\frac{10^n - 1}{9}\right)^2.$$

Since $n > 0$, $10^n > 1$ and we may cancel out a factor of $\tfrac{10^n - 1}{9}$ to obtain

$$c \cdot (10^n + 1) - b = a^2 \cdot \frac{10^n - 1}{9}.$$

This is a linear equation in 10^n. Thus, if two distinct values of n satisfy it, then all values of n will. Matching coefficients, we need

$$c = \frac{a^2}{9} \quad \text{and} \quad c - b = -\frac{a^2}{9} \implies b = \frac{2a^2}{9}.$$

To maximize $a + b + c = a + \tfrac{a^2}{3}$, we need to maximize a. Since b and c must be integers, a must be a multiple of 3. If $a = 9$ then b exceeds 9. However, if $a = 6$ then $b = 8$ and $c = 4$ for an answer of $\boxed{\textbf{(D) } 18}$.

(CantonMathGuy)

Solution 2 (quicker?)

Immediately start trying $n=1$ and $n=2$. These give the system of equations $11c-b=a^2$ and $1111c-11b=(11a)^2$ (which simplifies to $101c-b=11a^2$). These imply that $a^2=9c$, so the possible (a,c) pairs are $(9,9)$, $(6,4)$, and $(3,1)$. The first puts b out of range but the second makes $b=8$. We now know the answer is at least $6+8+4=18$.

We now only need to know whether $a+b+c=20$ might work for any larger n. We will always get equations like $100001c-b=11111a^2$ where the c coefficient is very close to being nine times the a coefficient. Since the b term will be quite insignificant, we know that once again a^2 must equal $9c$, and thus $a=9, c=9$ is our only hope to reach 20. Substituting and dividing through by 9, we will have something like $100001-b/9=99999$. No matter what n really was, b is out of range (and certainly isn't 2 as we would have needed).

The answer then is $\boxed{\textbf{(D) } 18}$.

Solution 3

Notice that $(0.\overline{3})^2 = 0.\overline{1}$ and $(0.\overline{6})^2 = 0.\overline{4}$. Setting $a = 3$ and $c = 1$, we see $b = 2$ works for all possible values of n. Similarly, if $a = 6$ and $c = 4$, then $b = 8$ works for all possible values of n. The second solution yields a greater sum of $\boxed{\textbf{(D) } 18}$.

See Also

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Preceded by **Problem 24**	Followed by **Last Problem**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

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Preceded by **Problem 24**	Followed by **Last Problem**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
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