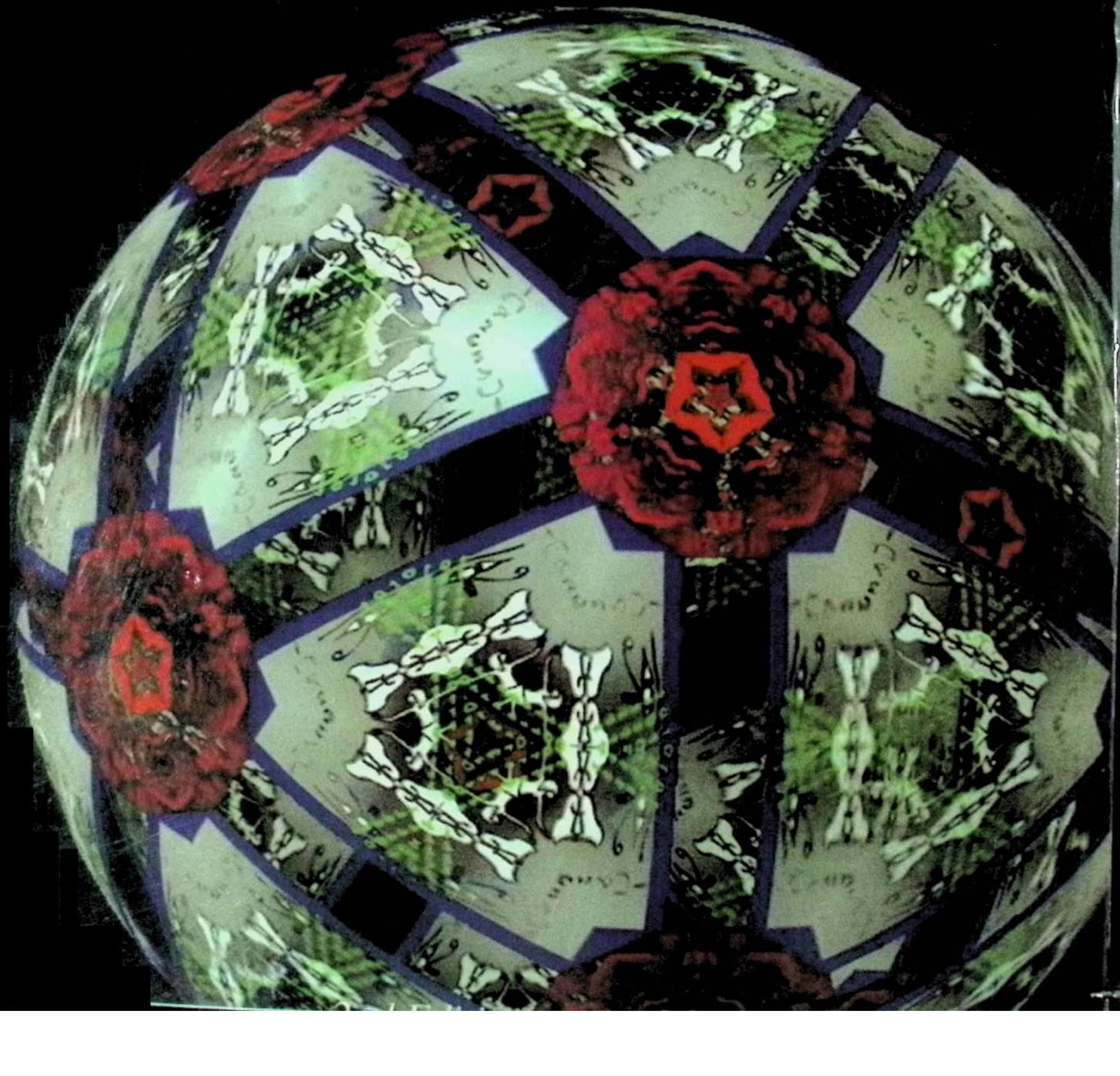
the Art of Problem Solving

Introduction to Number Theory *Solutions Manual*

Mathew Crawford



Solutions Manual

Introduction to Number Theory

2nd edition

Mathew Crawford
Art of Problem Solving

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10865 Rancho Bernardo Rd Ste 100
San Diego, CA 92127
(858) 675-4555
books@artofproblemsolving.com

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CHAPTER 1

Integers: The Basics

Exercises for Section 1.2

1.2.1 No. When we subtract a positive integer from a smaller one, the result is negative. For instance, $2 - 5 = -3$.

1.2.2 Yes. When we multiply a positive integer, such as 5, by another positive integer, such as 3, we are really just adding 5 together 3 times. The sum of several integers is always an integer. Adding positive integers means counting to the right side on the number line, so the result is always positive as well.

1.2.3 No. For instance, $1 \div 2 = \frac{1}{2}$ is not an integer.

1.2.4 Lists help us identify numbers with certain properties, so we list some perfect squares.

$$\begin{aligned} 10^2 &= 100 \\ 11^2 &= 121 \\ 12^2 &= 144 \\ 13^2 &= 169 \\ 14^2 &= 196 \\ 15^2 &= 225 \end{aligned}$$

All other perfect squares are either smaller or greater than these, so the perfect squares between 100 and 200 are $\boxed{121}$, $\boxed{144}$, $\boxed{169}$, and $\boxed{196}$.

Exercises for Section 1.3

1.3.1 No, because the quotient $22 \div 4 = 5.5$ is not an integer.

1.3.2 Yes, because the quotient $63 \div 9 = 7$ is an integer, which means $63 = 9 \cdot 7$.

1.3.3 Yes, because the quotient $-18 \div 6 = -3$ is an integer, which means $-18 = 6 \cdot -3$.

1.3.4 No, because the quotient $23 \div -13 = -1\frac{10}{13}$ is not an integer.

1.3.5 Multiples of 7 can be written in the form $7n$ for integer values of n. The 10 smallest positive multiples of 7 come from letting $n = 1, 2, \cdots, 10$. For instance, $7 \cdot 1 = \boxed{7}$ and $7 \cdot 2 = \boxed{14}$. We can also count by 7's to generate the rest of them: $\boxed{21}$, $\boxed{28}$, $\boxed{35}$, $\boxed{42}$, $\boxed{49}$, $\boxed{56}$, $\boxed{63}$, $\boxed{70}$.

1.3.6 We count to every tenth positive integer until we have included ten of them: $\boxed{10}$, $\boxed{20}$, $\boxed{30}$, $\boxed{40}$, $\boxed{50}$, $\boxed{60}$, $\boxed{70}$, $\boxed{80}$, $\boxed{90}$, $\boxed{100}$.

1.3.7 We will do this by counting every thirteenth positive integer until we have counted five of them. $\boxed{13}$, $\boxed{26}$, $\boxed{39}$, $\boxed{52}$, $\boxed{65}$.

1.3.8 We count backwards (going down from zero) every thirteenth negative integer until we have included five of them. $\boxed{-13}$, $\boxed{-26}$, $\boxed{-39}$, $\boxed{-52}$, $\boxed{-65}$.

1.3.9 First 10 positive multiples of 4: 4, 8, 12, 16, 20, 24, 28, 32, 36, 40.
First 10 positive multiples of 6: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60.
The smallest number in both these lists is $\boxed{12}$.

1.3.10 First 10 positive multiples of 7: 7, 14, 21, 28, 35, 42, 49, 56, 63, 70.
Dividing 4 into each of these multiples of 7, we find that $\boxed{28}$ is the smallest positive multiple of 7 that is also a multiple 4.

Exercises for Section 1.4

1.4.1

(a) $11 \div 6 = 1\frac{5}{6}$, which is not an integer, so 11 is not divisible by 6.

(b) $12 \div 6 = 2$, an integer, so $\boxed{12}$ is divisible by 6.

(c) $13 \div 6 = 2\frac{1}{6}$, which is not an integer, so 13 is not divisible by 6.

(d) $20 \div 6 = 3\frac{1}{3}$, which is not an integer, so 20 is not divisible by 6.

(e) $35 \div 6 = 5\frac{5}{6}$, which is not an integer, so 35 is not divisible by 6.

(f) $60 \div 6 = 10$, an integer, so $\boxed{60}$ is divisible by 6.

(g) $77 \div 6 = 12\frac{5}{6}$, which is not an integer, so 77 is not divisible by 6.

(h) $198 \div 6 = 33$, an integer, so $\boxed{198}$ is divisible by 6.

1.4.2

(a) $11 \div 11 = 1$, an integer, so $\boxed{11}$ is divisible by 11.

(b) $12 \div 11 = 1\frac{1}{11}$, which is not an integer, so 12 is not divisible by 11.

(c) $13 \div 11 = 1\frac{2}{11}$, which is not an integer, so 13 is not divisible by 11.

(d) $20 \div 11 = 1\frac{9}{11}$, which is not an integer, so 20 is not divisible by 11.

(e) $35 \div 11 = 3\frac{2}{11}$, which is not an integer, so 35 is not divisible by 11.

(f) $60 \div 11 = 5\frac{5}{11}$, which is not an integer, so 60 is not divisible by 11.

(g) $77 \div 11 = 7$, an integer, so $\boxed{77}$ is divisible by 11.

(h) $198 \div 11 = 18$, an integer, so $\boxed{198}$ is divisible by 11.

1.4.3 $5 \div 2 = 2\frac{1}{2}$, which is not an integer, so 5 is not divisible by 2.

1.4.4 $8 \div 4 = 2$, an integer, so 8 is divisible by 4.

1.4.5 $40 \div 8 = 5$, an integer, so 40 is divisible by 8.

1.4.6 $44 \div 8 = 5\frac{1}{2}$, which is not an integer, so 44 is not divisible by 8.

1.4.7 $60 \div 12 = 5$, an integer, so 60 is divisible by 12.

1.4.8 $100 \div 12 = 8\frac{1}{3}$, which is not an integer, so 100 is not divisible by 12.

1.4.9 $100 \div 10 = 10$, an integer, so 100 is divisible by 10.

Exercises for Section 1.5

1.5.1

(a) 1, 2, 4
(b) 1, 7
(c) 1, 2, 4, 8
(d) 1, 2, 5, 10
(e) 1, 13
(f) 1, 2, 7, 14
(g) 1, 3, 5, 15
(h) 1, 2, 4, 8, 16
(i) 1, 19
(j) 1, 2, 4, 5, 10, 20
(k) 1, 3, 7, 21
(l) 1, 2, 3, 4, 6, 8, 12, 24

Exercises for Section 1.6

1.6.1 There are $\frac{18}{n}$ apples in each bag, so n must be a divisor of 18. We are told that there are between 1 and 18 apples in each bag, so n could be $\boxed{2}$, $\boxed{3}$, $\boxed{6}$, or $\boxed{9}$.

1.6.2 We begin by translating the word problem into a number theory problem: "Which divisors of 120 are between 3 and 12 inclusive?" Dividing each of the integers from 3 to 12 into 120 we find that 3, 4, 5, 6, 8, 10, or 12 could be the number of teachers in each group, so there are $\boxed{7}$ possible group sizes.

Exercises for Section 1.7

1.7.1

(a) False. $18 \div 4 = 4\frac{1}{2}$, which is not an integer, so $4 \nmid 18$.

(b) False. $3 \div 9 = \frac{1}{3}$, which is not an integer, so $9 \nmid 3$.

(c) True. $49 \div 7 = 7$, which is an integer, so $7 \mid 49$.

(d) True. $12 \div 4 = 3$, which is an integer, so $4 \mid 12$.

(e) False. $9 \div 3 = 3$, which is an integer, so $3 \mid 9$.

(f) True. $14 \div 5 = 2\frac{4}{5}$, which is not an integer, so $5 \nmid 14$.

(g) True. $84 \div 3 = 28$, which is an integer, so $3 \mid 84$.

(h) True. $72 \div 12 = 6$, which is an integer, so $12 \mid 72$.

(i) True. $5 \div 5 = 1$, which is an integer, so $5 \mid 5$.

(j) False. $2 \div 18 = \frac{1}{9}$, which is not an integer, so $18 \nmid 2$.

(k) False. $84 \div 9 = 9\frac{1}{3}$, which is not an integer, so $9 \nmid 84$.

(l) True. $72 \div 4 = 18$, which is an integer, so $4 \mid 72$.

Review Problems

1.11

$$\begin{aligned} 14^2 &= 196 \\ 15^2 &= 225 \\ 16^2 &= 256 \\ 17^2 &= 289 \\ 18^2 &= 324 \end{aligned}$$

All other perfect squares are lower or higher than these, so $\boxed{225}$, $\boxed{256}$, and $\boxed{289}$ are the only perfect squares between 200 and 300.

1.12

(a) $84 \div 3 = 28$, an integer, so 84 is divisible by 3.

(b) $84 \div -14 = -6$, an integer, so 84 is divisible by -14.

(c) $193 \div 17 = 11\frac{6}{17}$, which is not an integer, so 193 is not divisible by 17.

(d) $1080 \div 18 = 60$, an integer, so 1080 is divisible by 18.

(e) $93 \div -2 = -46\frac{1}{2}$, which is not an integer, so 93 is not a multiple of -2.

(f) $93 \div 3 = 31$, which means $3 \cdot 31 = 93$, so 93 is a multiple of 3.

(g) $-140 \div 7 = -20$, which means $7 \cdot -20 = -140$, so -140 is a multiple of 7.

(h) $142 \div 7 = 20\frac{2}{7}$, which is not an integer, so 142 is not a multiple of 7.

(i) $30 \div 4 = 7\frac{1}{2}$, which is not an integer, so 30 is not a multiple of 4.

(j) $300 \div 4 = 75$, which means $4 \cdot 75 = 300$, so 300 is a multiple of 4.

1.13 We can count by 9's or list the values of $9n$ for $n = 1, 2, \ldots, 8$: 9, 18, 27, 36, 45, 54, 63, and 72.

1.14

(a) 1, 5, 25

(b) 1, 2, 13, 26

(c) 1, 3, 9, 27

(d) 1, 2, 4, 7, 14, 28

1.15

(a) False. $73 \div 3 = 24\frac{1}{3}$, which is not an integer, so $3 \nmid 73$.

(b) True. $273 \div 3 = 91$, which is an integer, so $3 \mid 273$.

(c) True. $182 \div 4 = 45\frac{1}{2}$, which is not an integer, so $4 \nmid 182$.

(d) False. $1182 \div 4 = 295\frac{1}{2}$, which is not an integer, so $4 \nmid 1182$.

(e) True. $1182 \div 6 = 197$, which is an integer, so $6 \mid 1182$.

(f) True. $2186 \div 14 = 156\frac{1}{7}$, which is not an integer, so $14 \nmid 2186$.

Challenge Problems

1.16 Let Scott's score be $81 + n$, where n is an integer between 0 and 6 inclusive. The average of all four scores is

$$\frac{94 + 91 + 95 + (81 + n)}{4} = \frac{361 + n}{4} = 90 + \frac{1 + n}{4}.$$

The average will be an integer when $4 \mid (n + 1)$. The value of $n + 1$ is between 1 and 7 inclusive, so it must be 4 (the only multiple of 4 in that range). Since $n + 1 = 4$, $n = 3$ and Scott's score is $81 + 3 = \boxed{84}$.

1.17

$$\begin{aligned} 10^3 &= 1000 \\ 11^3 &= 1331 \\ 12^3 &= 1728 \\ 13^3 &= 2197 \end{aligned}$$

All other perfect cubes are smaller or larger than these so the perfect cubes between 1000 and 2000 are $\boxed{1331}$ and $\boxed{1728}$.

1.18 We can narrow down the range by noting that

$$20^3 = 8000 < 10000 < 27000 = 30^3.$$

Since 10000 is much closer to 8000 than it is to 27000, we begin searching from 20^3 upwards:

$$\begin{aligned} 20^3 &= 8000 \\ 21^3 &= 9261 \\ 22^3 &= 10648 \end{aligned}$$

Since all perfect cubes are smaller or larger than these, the largest integer whose cube is less than 10000 is $\boxed{21}$.

1.19 First, we determine the largest possible exponent of a perfect power between 1 and 100. We do this by noting that $2^m < n^m$ for any positive integer $n > 2$ and any positive integer m. This means we can determine the largest possible power of a perfect power less than 100 by determining the largest power of 2 less than 100:

$$\begin{aligned} 2^1 &= 2 \\ 2^2 &= 4 \\ 2^3 &= 8 \\ 2^4 &= 16 \\ 2^5 &= 32 \\ 2^6 &= 64 \\ 2^7 &= 128 \end{aligned}$$

Now we note that $9^2 < 10^2 = 100$, so we don't need to check for powers of any integers larger than 9.

The perfect squares between 1 and 100 are $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, and $9^2 = 81$.

The perfect cubes between 1 and 100 are $2^3 = 8$, $3^3 = 27$, $4^3 = 64$. We know there are none smaller or larger because $1^3 = 1$ and $5^3 = 125 > 100$.

Any perfect fourth power is also a square: $n^4 = (n^2)^2$, so we've counted them already.

Since $1^5 = 1$ and $3^5 = 243 > 100$, we know that $2^5 = 32$ is the only perfect fifth power between 1 and 100.

Any perfect sixth power is also a square: $n^6 = (n^3)^2$, so we've counted them already.

We count 8 perfect squares, 3 perfect cubes, and 1 perfect fifth power, but one of the squares, 64, is also a cube, so there are $8 + 3 + 1 - 1 = \boxed{11}$ perfect powers between 1 and 100.

1.20 This problem is only harder than other divisor problems in that any mistake could throw our count off. The integers between 1 and 9 inclusive that are divisors of 24,516 are 1, 2, 3, 4, 6, and 9, so there are $\boxed{6}$ of them. If you counted any extra or didn't include any of these, go back and check your division.

1.21 Yes. $11111 \div 41 = 271$, an integer, so 11111 is divisible by 41.

1.22 Yes. $111111 \div 37 = 3003$, an integer, so 111111 is divisible by 37.

1.23 We check each of the smallest positive integers for divisibility of 5040 until we find a positive integer that is not a divisor of 5040.
$5040 \div 1 = 5040$, which is an integer, so 1 is a divisor of 5040.
$5040 \div 2 = 2520$, which is an integer, so 2 is a divisor of 5040.
$5040 \div 3 = 1680$, which is an integer, so 3 is a divisor of 5040.
$5040 \div 4 = 1260$, which is an integer, so 4 is a divisor of 5040.
$5040 \div 5 = 1008$, which is an integer, so 5 is a divisor of 5040.

$5040 \div 6 = 840$, which is an integer, so 6 is a divisor of 5040.
$5040 \div 7 = 720$, which is an integer, so 7 is a divisor of 5040.
$5040 \div 8 = 630$, which is an integer, so 8 is a divisor of 5040.
$5040 \div 9 = 560$, which is an integer, so 9 is a divisor of 5040.
$5040 \div 10 = 504$, which is an integer, so 10 is a divisor of 5040.
$5040 \div 11 = 458\frac{2}{11}$, which is not an integer, so $\boxed{11}$ is the smallest positive integer that is not a divisor of 5040.

1.24 The area of a circle in terms of its diameter is $\frac{d^2\pi}{4}$, so we are looking for the value of d such that

$$100 < \frac{d^2\pi}{4} < 120.$$

Multiplying everything in the inequality by 4, we get

$$400 < d^2\pi < 480.$$

Dividing everything by $\pi \approx 3.14$ and estimating, we get

$$127.39 < d^2 < 152.87.$$

Since $11^2 = 121$, $12^2 = 144$, and $13^2 = 169$, and all other perfect squares are larger or smaller than these, $d = \boxed{12}$.

CHAPTER 2

Primes and Composites

Exercises for Section 2.2

2.2.1 The prime numbers are (a) 19, (e) 23, and (k) 29.

The rest are composite as we can see by the fact that we can represent each as a product of integers between 1 and itself:

$$\begin{aligned} 20 &= 2\cdot 10 & 24 &= 2\cdot 12 & 27 &= 3\cdot 9\\ 21 &= 3\cdot 7 & 25 &= 5\cdot 5 & 28 &= 2\cdot 14\\ 22 &= 2\cdot 11 & 26 &= 2\cdot 13 & 30 &= 2\cdot 15 \end{aligned}$$

2.2.2 Any even positive integer is a multiple of 2. Since a prime cannot be a multiple of any integer between 1 and itself, the only even prime is 2. The answer is $\boxed{1}$.

2.2.3 Let m be the number of rows in the grid of students and let n be the number of columns. The total number of students is mn. If the only way to express mn as a product of positive integers is for one of the integers to be 1, then 1 and mn are the only divisors of mn, so mn is prime. The number of students in Jon's class is $\boxed{23}$, the only prime between 20 and 28.

2.2.4 Let c be a composite number, so $c = ab$ for positive integers a and b that are between 1 and c. Any positive multiple of c can be written as cn for some integer n. Note that since $c = ab$,

$$cn = (ab)n = a(bn),$$

showing that cn is the product of a and bn, two positive integers both greater than 1, meaning that cn is composite. So any positive multiple of a composite number c is itself composite.

Exercises for Section 2.3

2.3.1 The circled numbers are prime and the rest are composite:

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~
101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	~~115~~	~~116~~	~~117~~	~~118~~	~~119~~	~~120~~
~~121~~	~~122~~	~~123~~	~~124~~	~~125~~	~~126~~	127	~~128~~	~~129~~	~~130~~
131	~~132~~	~~133~~	~~134~~	~~135~~	~~136~~	137	~~138~~	139	~~140~~
~~141~~	~~142~~	~~143~~	~~144~~	~~145~~	~~146~~	~~147~~	~~148~~	149	~~150~~
151	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	157	~~158~~	~~159~~	~~160~~
~~161~~	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	~~169~~	~~170~~
~~171~~	~~172~~	173	~~174~~	~~175~~	~~176~~	~~177~~	~~178~~	179	~~180~~
181	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	~~187~~	~~188~~	~~189~~	~~190~~
191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199	~~200~~

Figure 2.1: The Sieve to 200

2.3.2 We can see above that $\boxed{173}$ is the largest prime less than 200 with no composite digits.

Exercises for Section 2.4

2.4.1 The primes are (a) 197, (d) 499, and (f) 773.

Each of (b), (c), and (e) are composite since we can express them as products of two smaller divisors: $297 = 3 \cdot 99$, $323 = 17 \cdot 19$, and $553 = 7 \cdot 79$.

2.4.2 A composite number is the product of two smaller positive integers. If a composite has no prime divisors less than 10, then the smallest that product can be is $11 \cdot 11 = \boxed{121}$.

Review Problems

2.6 The circled numbers are prime and the rest are composite:

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~
101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	~~115~~	~~116~~	~~117~~	~~118~~	~~119~~	~~120~~
~~121~~	~~122~~	~~123~~	~~124~~	~~125~~	~~126~~	127	~~128~~	~~129~~	~~130~~
131	~~132~~	~~133~~	~~134~~	~~135~~	~~136~~	137	~~138~~	139	~~140~~
~~141~~	~~142~~	~~143~~	~~144~~	~~145~~	~~146~~	~~147~~	~~148~~	149	~~150~~
151	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	157	~~158~~	~~159~~	~~160~~
~~161~~	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	~~169~~	~~170~~
~~171~~	~~172~~	173	~~174~~	~~175~~	~~176~~	~~177~~	~~178~~	179	~~180~~
181	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	~~187~~	~~188~~	~~189~~	~~190~~
191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199	~~200~~
~~201~~	~~202~~	~~203~~	~~204~~	~~205~~	~~206~~	~~207~~	~~208~~	~~209~~	~~210~~
211	~~212~~	~~213~~	~~214~~	~~215~~	~~216~~	~~217~~	~~218~~	~~219~~	~~220~~
~~221~~	~~222~~	223	~~224~~	~~225~~	~~226~~	227	~~228~~	229	~~230~~
~~231~~	~~232~~	233	~~234~~	~~235~~	~~236~~	~~237~~	~~238~~	239	~~240~~
241	~~242~~	~~243~~	~~244~~	~~245~~	~~246~~	~~247~~	~~248~~	~~249~~	~~250~~
251	~~252~~	~~253~~	~~254~~	~~255~~	~~256~~	257	~~258~~	~~259~~	~~260~~
~~261~~	~~262~~	263	~~264~~	~~265~~	~~266~~	~~267~~	~~268~~	269	~~270~~
271	~~272~~	~~273~~	~~274~~	~~275~~	~~276~~	277	~~278~~	~~279~~	~~280~~
281	~~282~~	283	~~284~~	~~285~~	~~286~~	~~287~~	~~288~~	~~289~~	~~290~~
~~291~~	~~292~~	293	~~294~~	~~295~~	~~296~~	~~297~~	~~298~~	~~299~~	~~300~~

Figure 2.2: The Sieve to 300

2.7 In the Sieve of Eratosthenes, we cross out every positive multiple of 3 other than 3 itself (which is prime). So, there is only $\boxed{1}$ prime multiple of 3.

2.8 Since $2 \mid 10$, we know that $2 \mid 10n$ for any integer n. This means any multiple of 10 has 2 as a divisor and is therefore composite. There are $\boxed{0}$ prime multiples of 10.

2.9

(a) Since $17^2 < 313 < 18^2$, we check to see if 313 is divisible by any prime up to 17. It is not divisible by any of them, so 313 is $\boxed{\text{prime}}$.

(b) Since $18^2 < 343 < 19^2$, we check to see if 343 is divisible by any prime up to 17. Since $343 = 7^3$, 343 is $\boxed{\text{composite}}$.

(c) Since $19^2 < 391 < 20^2$, we check to see if 391 is divisible by any prime up to 19. Since $391 = 17 \cdot 23$, 391 is $\boxed{\text{composite}}$.

(d) Since $20^2 < 427 < 21^2$, we check to see if 427 is divisible by any prime up to 19 (the largest prime less than 20). Since $427 = 7 \cdot 61$, 427 is $\boxed{\text{composite}}$.

(e) Since $23^2 < 569 < 24^2$, we check to see if 569 is divisible by any prime up to 23. It is not divisible by any of them, so 569 is $\boxed{\text{prime}}$.

(f) Since $29^2 < 853 < 30^2$, we check to see if 853 is divisible by any prime up to 29. It is not divisible by any of them, so 853 is $\boxed{\text{prime}}$.

2.10 Let n be the number of students at class. Betty noticed that n has no divisors between 1 and itself, so n is prime. Wilma noticed that $n+2$ is prime. This means we are looking for the smaller of two primes that differ by 2 that are between 30 and 50. The primes in that range are 31, 37, 41, 43, and 47. Since 41 and 43 differ by 2, $n = \boxed{41}$.

2.11 A two-digit number whose digits are prime is made up of the digits 2, 3, 5, or 7. We check the possible numbers from the largest in this list and work backwards until we find a prime number:
77 is divisible by 7.
75 is divisible by 3.
$\boxed{73}$ is prime, so it is the largest two-digit prime number whose digits are both prime.

2.12 A perfect cube is the product of three integers: $n \cdot n \cdot n = n^3$. Unless $n < 2$, the perfect cube has a divisor between 1 and n^3 (namely n) and is therefore composite. When $n = 1$, $n^3 = 1$, which is also not prime, so there are $\boxed{0}$ prime cubes.

Challenge Problems

2.13 The sum of the six smallest primes is $2 + 3 + 5 + 7 + 11 + 13 = 41$. The seventh is 17 and the remainder when 41 is divided by 17 is $\boxed{7}$.

2.14 A number with digits xy has value $10x + y$, while the number obtained while reversing its digits, yx, has value $10y + x$. Setting the sum of these two equal to 110, we get

$$10x + y + 10y + x = 11(x + y) = 110.$$

Dividing by 11 we see that the sum of the digits is 10: $x + y = 10$. For xy and yx to be two digit prime numbers, both x and y must be odd. So x and y can be 9 and 1, 7 and 3, or 5 and 5. 19 is prime, but 91 is divisible by 7 and is therefore composite. 55 is divisible by 5, so it is also composite. Both 73 and 37 are prime, and the larger of this pair is $\boxed{73}$.

2.15 Since 5^{23} and 7^{17} are both odd, their sum is even and therefore divisible by 2. There are no smaller primes than $\boxed{2}$, so it is the smallest prime divisor of the sum.

2.16 We must find the largest prime c such that $a + b + c = 25$ where a and b are also (different) primes. One method is to check the largest possible values of c working downwards.

The greatest prime less than 25 is 23. If $c = 23$, $a + b = 2$, and no two prime numbers sum to 2.

If $c = 19$, $a + b = 6$. The only two primes that sum to 6 are 3 and 3, but a and b cannot be the same.

If $c = 17$, $a + b = 8$, and when $a = 3$ and $b = 5$, a and b are both prime. So $\boxed{17}$ is the greatest number of pennies possible in any of the three piles.

2.17 All two-digit numbers whose digits sum to 9 are composite:

$$\begin{aligned} 18 &= 2\cdot 9 \\ 27 &= 3\cdot 9 \\ 36 &= 4\cdot 9 \\ 45 &= 5\cdot 9 \\ 54 &= 6\cdot 9 \\ 63 &= 7\cdot 9 \\ 72 &= 8\cdot 9 \\ 81 &= 9\cdot 9 \\ 90 &= 10\cdot 9 \end{aligned}$$

Since none of the 21 two-digit primes have a digit sum of 9, the answer is $\boxed{0}$.

2.18

$$9409 = 10000 - 2(3)(100) + 3^2 = (100-3)^2 = 97^2.$$

Since 9409 has a divisor between 1 and itself, it is composite, not prime.

2.19 So long as we can find 5 primes between 1000 and $37^2 = 1369$, we can check the integers after 1000 one by one to see which have no prime divisors less than 37. The *Sieve of Nygard* speeds up the process. Performing this check, we find that $\boxed{1009}$, $\boxed{1013}$, $\boxed{1019}$, $\boxed{1021}$, and $\boxed{1031}$ are the five smallest primes greater than 1000.

2.20 None of the even years after 2000 are prime, and 2001 is a multiple of 3. Now we check to see if 2003 is prime.

$$44^2 = 1936 < 2003 < 2025 = 45^2,$$

so we need only check to see if 2003 has prime divisors less than 45. It turns out that no prime less than 45 is a divisor of 2003, so $\boxed{2003}$ is the first year in the twenty-first century that is prime.

CHAPTER 3

Multiples and Divisors

Exercises for Section 3.2

3.2.1 We make lists of divisors and find the divisors common to each list:

Positive divisors of 12:	$1, 2, 3, 4, 6, 12$
Positive divisors of 18:	$1, 2, 3, 6, 9, 18$

The common divisors of 12 and 18 are $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{6}$.

3.2.2 We make lists of divisors and find the divisors common to each list:

Positive divisors of 14:	$1, 2, 7, 14$
Positive divisors of 42:	$1, 2, 3, 6, 7, 14, 21, 42$

The common divisors of 14 and 42 are $\boxed{1}$, $\boxed{2}$, $\boxed{7}$, and $\boxed{14}$.

3.2.3 We make lists of divisors and find the divisors common to each list:

Positive divisors of 20:	$1, 2, 4, 5, 10, 20$
Positive divisors of 35:	$1, 5, 7, 35$

The common divisors of 20 and 35 are $\boxed{1}$ and $\boxed{5}$.

3.2.4 Since 23 is prime, its only positive divisors are 1 and 23. Since $1 \mid 36$ and $23 \nmid 36$, the only positive common divisor of 23 and 36 is $\boxed{1}$.

3.2.5

(a) The number of eggs in each basket is a common divisor of 18 and 24 that is at least 4. The common divisors of 18 and 24 are 1, 2, 3, and 6, so there are $\boxed{6}$ eggs in each basket.

(b) The number of green baskets needed for 18 eggs is $\frac{18}{6} = \boxed{3}$.

(c) The number of blue baskets needed for 24 eggs is $\frac{24}{6} = \boxed{4}$.

3.2.6 Since 30 or 40 students can be broken up into groups of n students, n must be a common divisor of 30 and 40.

Positive divisors of 30:	$1, 2, 3, 5, 6, 10, 15, 30$
Positive divisors of 40:	$1, 2, 4, 5, 8, 10, 20, 40$

Since $n > 1$, the possible values of n are $\boxed{2}$, $\boxed{5}$, and $\boxed{10}$.

Exercises for Section 3.3

3.3.1 If you missed any of the following, go back and check your lists of common divisors for each pair of integers.

(a)	4	(d)	6	(g)	10
(b)	5	(e)	5	(h)	12
(c)	3	(f)	1	(i)	5

3.3.2 Since the number of pencils in a package must be a divisor of both 24 and 40, the largest possible number of pencils in a package is $\gcd(24, 40) = \boxed{8}$.

Exercises for Section 3.4

3.4.1

(a) 30

(b) 30, 60, 90, 120, 150

(c) 56

(d) 56, 112, 168, 224, 280

(e) 60

(f) 60, 120, 180, 240, 300

(g) 126

(h) 126, 252, 378, 504, 630

(i) 168

3.4.2 Let x be the smallest number of people that can be broken up into 15 groups of equal membership and into 48 groups of equal membership. This means x must be a multiple of both 15 and 48. The smallest such number is $\text{lcm}[15, 48] = 240$, so $x = \boxed{240}$.

Exercises for Section 3.5

3.5.1 An integer that leaves a remainder of 3 when divided by 5 can be written as $5n + 3$ for some nonnegative integer n. The largest permissible value of n will lead us to the largest value of $5n + 3$ less than 80, so we solve the inequality.

$$5n + 3 < 80.$$

Subtracting 3 from both sides gives $5n < 77$. Dividing both sides by 5, we have

$$n < 15\frac{2}{5},$$

so the largest permissible value of n is 15 and the largest integer less than 80 that leaves a remainder of 3 when divided by 5 is $5 \cdot 15 + 3 = \boxed{78}$.

3.5.2 The integer we seek leaves a remainder of 8 when divided by 13, so we can write it in the form $13n + 8$ for some nonnegative integer n. Since $13n + 8$ is a three-digit number, we need to find the largest value of n that satisfies the inequality chain:

$$100 \le 13n + 8 < 1000.$$

Really, we are only concerned with the right-hand part of the inequality, because there are clearly some three-digit numbers that leave remainders of 8 when divided by 13. Subtracting 8 from both sides of the inequality gives us

$$13n < 992.$$

Dividing by 13 and rounding down we find that $n \le 76$, so the largest three-digit number that leaves a remainder of 8 when divided by 13 is $13 \cdot 76 + 8 = \boxed{996}$.

3.5.3 Our goal is to count the two-digit integers in the form $8n + 2$ for integer values of n. We examine the inequality,

$$10 \le 8n + 2 < 100.$$

Subtracting 2 from all parts simplifies things:

$$8 \le 8n < 98.$$

Dividing everything by 8 isolates the possible values of n:

$$1 \le n < 12\frac{1}{4}.$$

Since n can be any integer from 1 to 12, there are $\boxed{12}$ two-digit integers in the form $8n + 2$ (that leave a remainder of 2 when divided by 8).

3.5.4 We want to count the three-digit integers in the form $9n + 4$ for integer values of n. We have the inequality,

$$100 \le 9n + 4 < 1000.$$

Subtracting 4 from all parts of the inequality gives $96 \le 9n < 996$. Dividing by 9 we see that $10\frac{2}{3} \le n < 110\frac{2}{3}$. So, there is a three-digit integer that leaves a remainder of 4 for every value of n from 11 to 110 inclusive, giving a total of $110 - 11 + 1 = \boxed{100}$.

3.5.5 Integers that leave a remainder of 1 when divided by 5 can be written in the form $5n+1$ for integer values of n. We must count the integers n for which

$$100 < 5n + 1 < 700.$$

Subtracting 1 from all parts of the inequality we get $99 < 5n < 699$. Dividing by 5 we have

$$19\frac{4}{5} < n < 139\frac{4}{5},$$

so $20 \le n \le 139$ and there are $139 - 20 + 1 = \boxed{120}$ such integers.

Some students might note that integers that leave a remainder of 1 when divided by 5 have units digits of 1 or 6, so there are 2 of them for every 10 consecutive integers. The integers from 100 to 699 can be divided into 60 groups of 10 consecutive integers:

$$\begin{array}{cccccccccc}
100 & 101 & 102 & 103 & 104 & 105 & 106 & 107 & 108 & 109 \\
110 & 111 & 112 & 113 & 114 & 115 & 116 & 117 & 118 & 119 \\
120 & 121 & 122 & 123 & 124 & 125 & 126 & 127 & 128 & 129 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
680 & 681 & 682 & 683 & 684 & 685 & 686 & 687 & 688 & 689 \\
690 & 691 & 692 & 693 & 694 & 695 & 696 & 697 & 698 & 699
\end{array}$$

Two integers in each of the 60 rows have units digits of 1 or 6 and all of those integers are greater than 100, so there are $2 \cdot 60 = 120$ of them. This kind of observation should become more automatic after later chapters in the text.

3.5.6 An integer that is 2 more than a multiple of 7 can be written in the form $7n + 2$ for some integer n. We need to count the values of n that satisfy

$$-120 < 7n + 2 < 120.$$

Subtracting 2 from all parts of the inequality gives us $-122 < 7n < 118$. Dividing everything by 7 yields $-17\frac{3}{7} < n < 16\frac{6}{7}$, so we are counting the integers from -17 to 16 inclusive. There are $16 - (-17) + 1 = \boxed{34}$ of them.

Exercises for Section 3.6

3.6.1

$$\begin{array}{rcrcccc}
\boxed{1} \cdot 42 & + & \boxed{4} \cdot 77 & = & 350 & = & 50 \cdot 7 \\
\boxed{3} \cdot 42 & + & \boxed{3} \cdot 77 & = & 357 & = & 51 \cdot 7 \\
\boxed{5} \cdot 42 & + & \boxed{2} \cdot 77 & = & 364 & = & 52 \cdot 7 \\
\boxed{7} \cdot 42 & + & \boxed{1} \cdot 77 & = & 371 & = & 53 \cdot 7 \\
\boxed{9} \cdot 42 & + & \boxed{0} \cdot 77 & = & 378 & = & 54 \cdot 7 \\
\boxed{0} \cdot 42 & + & \boxed{5} \cdot 77 & = & 385 & = & 55 \cdot 7 \\
\boxed{2} \cdot 42 & + & \boxed{4} \cdot 77 & = & 392 & = & 56 \cdot 7
\end{array}$$

3.6.2

$$\begin{array}{rcrcrcc}
\boxed{3} \cdot 36 & - & \boxed{5} \cdot 21 & = & 3 & = & 1 \cdot 3 \\
\boxed{6} \cdot 36 & - & \boxed{10} \cdot 21 & = & 6 & = & 2 \cdot 3 \\
\boxed{2} \cdot 36 & - & \boxed{3} \cdot 21 & = & 9 & = & 3 \cdot 3 \\
\boxed{5} \cdot 36 & - & \boxed{8} \cdot 21 & = & 12 & = & 4 \cdot 3 \\
\boxed{1} \cdot 36 & - & \boxed{1} \cdot 21 & = & 15 & = & 5 \cdot 3 \\
\boxed{4} \cdot 36 & - & \boxed{6} \cdot 21 & = & 18 & = & 6 \cdot 3
\end{array}$$

3.6.3 Using addition or subtraction, you can produce any multiple of 9 using just 45 and 27. Here are a few examples:

$$\begin{aligned} 45 + 27 &= 72 \\ 45 + 27 + 27 &= 99 \\ 45 + 45 + 27 &= 117 \\ 45 - 27 &= 18 \\ 27 - 45 &= -18 \end{aligned}$$

3.6.4 Using addition or subtraction, you can produce any multiple of 12 using just 96 and 60. Here are a few examples:

$$\begin{aligned} 96 + 60 &= 156 \\ 96 + 60 + 60 &= 216 \\ 96 + 96 + 60 &= 252 \\ 96 - 60 &= 36 \\ 60 - 96 &= -36 \end{aligned}$$

3.6.5 Three multiples of 4 can be written as $4a$, $4b$, and $4c$ for integers a, b, and c.

$$4a + 4b + 4c = 4(a + b + c).$$

Since $a + b + c$ is an integer, the sum of three multiples of 4 can be expressed as 4 times an integer. In other words, the sum is also a multiple of 4.

3.6.6 Four multiples of n can be written as an, bn, cn, and dn for integers a, b, c, and d.

$$an + bn + cn + dn = (a + b + c + d)n.$$

Since $a + b + c + d$ is an integer, the sum of four multiples of n can be expressed as n times an integer and is therefore a multiple of n.

Exercises for Section 3.7

3.7.1

(a) $\gcd(44, 128) = \gcd(44, 40) = \gcd(4, 40) = \gcd(4, 0) = \boxed{4}$.

(b) $\gcd(92, 529) = \gcd(92, 69) = \gcd(23, 69) = \gcd(23, 0) = \boxed{23}$.

(c) $\gcd(121, 748) = \gcd(121, 22) = \gcd(11, 22) = \gcd(11, 0) = \boxed{11}$.

(d) $\gcd(680, 4624) = \gcd(680, 544) = \gcd(136, 544) = \gcd(136, 0) = \boxed{136}$.

(e) $\gcd(2000, 14400) = \gcd(2000, 400) = \gcd(0, 400) = \boxed{400}$.

(f) $\gcd(3330, 39960) = \gcd(3330, 0) = \boxed{3330}$.

Review Problems

3.23

(a) $\gcd(14,24) = \gcd(14,10) = \gcd(4,10) = \gcd(4,2) = \gcd(0,2) = \boxed{2}$

(b) $\gcd(15,25) = \gcd(15,10) = \gcd(5,10) = \gcd(5,0) = \boxed{5}$

(c) $\gcd(42,96) = \gcd(42,12) = \gcd(6,12) = \gcd(6,0) = \boxed{6}$

(d) $\gcd(60,144) = \gcd(60,24) = \gcd(12,24) = \gcd(12,0) = \boxed{12}$

(e) $\gcd(129,311) = \gcd(129,53) = \gcd(3,53) = \gcd(3,2) = \gcd(1,2) = \gcd(1,0) = \boxed{1}$

(f) $\gcd(315,441) = \gcd(315,126) = \gcd(63,126) = \gcd(63,0) = \boxed{63}$

(g) $\gcd(1938,3306) = \gcd(1938,1368) = \gcd(570,1368)$
$= \gcd(570,228) = \gcd(114,228) = \gcd(114,0) = \boxed{114}$

(h) $\gcd(2001,25001) = \gcd(2001,989) = \gcd(23,989) = \gcd(23,0) = \boxed{23}$

3.24

(a) $1, 2, 3, 6$

(b) $1, 2$

(c) $1, 2, 5, 10$

(d) $1, 2, 3, 4, 6, 12$

3.25

(a) 42

(b) $42, 84, 126, 168, 210, 252$

(c) 420

(d) $420, 840, 1260, 1680, 2100, 2520$

(e) 126

(f) $126, 252, 378, 504, 630, 756$

(g) 30

3.26 It's probably easier to first find the LCM of two of the integers. The LCM of 12 and 18 is 36. Every common multiple of 12 and 18 will be a multiple of their LCM, so the LCM of all three integers must be a multiple of 36. The LCM of all three integers is the LCM of 36 and 30, which is $\boxed{180}$.

3.27 Let x be this smallest number of marbles that can be broken up into bags of 18 or bags of 42. This means x must be a multiple of both 18 and 42. The smallest such number is $\text{lcm}[18,42] = 126$, so $x = \boxed{126}$.

3.28 $\gcd(6432,132) = \gcd(96,132) = \gcd(96,36) = \gcd(24,36) = \gcd(24,12) = \gcd(0,12) = 12$. Decreasing this number by 8, the result is $12 - 8 = \boxed{4}$.

3.29 An integer that is divisible by 7 can be written in the form $7n$ for some integer n. Our goal is to count the number of integers such that

$$0 < 7n \le 100.$$

Dividing this inequality by 7, we get $0 < n \le 14\frac{2}{7}$, so n can be any integer from 1 to 14, corresponding to the $\boxed{14}$ positive multiples of 7 that do not exceed 100.

Challenge Problems

3.30 The number of cookies in each package must be a divisor of each of 252, 105, and 168. Since $\gcd(252, 105) = 21$, we check to see if 21 is also a divisor of 168. It is, so $\boxed{21}$ is the greatest possible number of cookies in each package.

3.31 There are many ways to create each multiple of 5 out of combinations of 25's and 15's, but here are simple ones for the 10 smallest positive multiples of 5:

$$\begin{array}{rcrcrcr}
2\cdot 25 & - & 3\cdot 15 & = & 5 & = & 1\cdot 5\\
1\cdot 25 & - & 1\cdot 15 & = & 10 & = & 2\cdot 5\\
0\cdot 25 & + & 1\cdot 15 & = & 15 & = & 3\cdot 5\\
2\cdot 25 & - & 2\cdot 15 & = & 20 & = & 4\cdot 5\\
1\cdot 25 & + & 0\cdot 15 & = & 25 & = & 5\cdot 5\\
0\cdot 25 & + & 2\cdot 15 & = & 30 & = & 6\cdot 5\\
2\cdot 25 & - & 1\cdot 15 & = & 35 & = & 7\cdot 5\\
1\cdot 25 & + & 1\cdot 15 & = & 40 & = & 8\cdot 5\\
0\cdot 25 & + & 3\cdot 15 & = & 45 & = & 9\cdot 5\\
2\cdot 25 & + & 0\cdot 15 & = & 50 & = & 10\cdot 5
\end{array}$$

As an extra challenge, see if you can find any patterns in the numbers above.

3.32 Of the 25 primes less than 100, only 1 is even. The other 24 are odd and their sum is therefore even. Adding the even prime, 2, the sum of all 25 primes is $\boxed{\text{even}}$.

If you aren't yet convinced, pair up the 24 odd primes and add each pair together. Each sum is even. Adding those sums and adding 2 to that is a sum of multiples of 2, which must itself be a multiple of 2.

3.33 There are 20 positive multiples of 5 less than 101. There are 14 positive multiples of 7 less than 101. However, there are 2 positive common multiples of 5 and 7 less than 101. This means there are 18 multiples of 5 that aren't multiples of 7 and 12 multiples of 7 that are not multiples of 5 for a total of $18 + 12 = \boxed{30}$.

Note that we could also add the numbers of multiples of 5 and 7 and subtract twice the number of common multiples (since they were counted in both as multiples of 5 and as multiples of 7):

$$20 + 14 - 2(2) = 30.$$

3.34 The key to solving this problem without using prime factorization (which we'll get to later in the book) is using variables to express every piece of information and relating all quantities in terms of multiplication by integers.

Let n be the total number of coins in the collections of Tom, Dick, and Harry. Since Tom owns $\frac{1}{3}n$ coins and Dick owns $\frac{1}{4}n$ coins, the number of coins Harry owns is the number remaining from the total:

$$n - \frac{n}{3} - \frac{n}{4} = \frac{5n}{12}.$$

Let m be the number of coins Harry owns. We know that m is an integer equal to $\frac{5n}{12}$, so

$$12m = 5n = a,$$

for some integer a. Since a is a multiple of both 5 and 12, a is a multiple of lcm[5, 12] = 60. This means we can write $a = 60b$ for some integer b:

$$12m = 5n = 60b.$$

Since $12m = 60b$, we have $m = 5b$, so the number of coins in Harry's collection is a multiple of 5. Since $5n = 60b$, we have $n = 12b$, so the total number of coins is a multiple of 12.

3.35 Using the kind of algebra we use in the Division Theorem, we look for a four-digit number a and a one-digit number b such that

$$a = b \cdot 432 + 2.$$

We want the smallest possible value of a, so we can test all the smallest possible values of b until we get to a four-digit number:

$$\begin{array}{lrcl} b = 1 & 434 & = & 1 \cdot 432 + 2 \\ b = 2 & 866 & = & 2 \cdot 432 + 2 \\ b = 3 & 1298 & = & 3 \cdot 432 + 2 \end{array}$$

The smallest possible four-digit value of a is $\boxed{1298}$.

3.36 There are equally many a's, b's, c's, and d's on the left-hand sides of the equations, so we add them to get a "balanced" equation:

$$3a + 3b + 3c + 3d = 5p.$$

The left-hand side of this equation is equal to $3(a + b + c + d)$, which is a multiple of 3. Since this multiple of 3 is equal to $5p$, we know that $3 \mid 5p$. This means $5p = 3x = y$ for some integers x and y. Since $3 \mid y$ and $5 \mid y$, y is a multiple of lcm[3, 5] = 15. This means $5p = 15z$ for some integer z. Dividing by 5, we get $p = 3z$. We see that $3 \mid p$, but p is prime, so it must be equal to $\boxed{3}$.

We didn't even have to solve the system of equations! However, note that from the equation we got from adding all four together,

$$a + b + c + d = 5.$$

We can plug in the value of p into each of the original equations and then subtract them one at a time from the sum of a, b, c, and d to find that $(a, b, c, d) = (2, -3, 1, 5)$.

CHAPTER 4

Prime Factorization

Exercises for Section 4.2

4.2.1

(a)	15	=	$\boxed{3^1 \cdot 5^1}$	(i)	68	=	$\boxed{2^2 \cdot 17^1}$
(b)	16	=	$\boxed{2^4}$	(j)	70	=	$\boxed{2^1 \cdot 5^1 \cdot 7^1}$
(c)	18	=	$\boxed{2^1 \cdot 3^2}$	(k)	90	=	$\boxed{2^1 \cdot 3^2 \cdot 5^1}$
(d)	21	=	$\boxed{3^1 \cdot 7^1}$	(l)	125	=	$\boxed{5^3}$
(e)	36	=	$\boxed{2^2 \cdot 3^2}$	(m)	240	=	$\boxed{2^4 \cdot 3^1 \cdot 5^1}$
(f)	40	=	$\boxed{2^3 \cdot 5^1}$	(n)	344	=	$\boxed{2^3 \cdot 43^1}$
(g)	45	=	$\boxed{3^2 \cdot 5^1}$	(o)	540	=	$\boxed{2^2 \cdot 3^3 \cdot 5^1}$
(h)	54	=	$\boxed{2^1 \cdot 3^3}$	(p)	999	=	$\boxed{3^3 \cdot 37^1}$

Exercises for Section 4.3

4.3.1

(a) $8 = 2^3$, $15 = 3^1 \cdot 5^1$, so $\text{lcm}[8, 15] = 2^3 \cdot 3^1 \cdot 5^1 = \boxed{120}$.

(b) $12 = 2^2 \cdot 3^1$, $42 = 2^1 \cdot 3^1 \cdot 7^1$, so $\text{lcm}[12, 42] = 2^2 \cdot 3^1 \cdot 7^1 = \boxed{84}$.

(c) $10 = 2^1 \cdot 5^1$, $2^3 \cdot 7^1 = 56$, so $\text{lcm}[10, 56] = 2^3 \cdot 5^1 \cdot 7^1 = \boxed{280}$.

(d) $18 = 2^1 \cdot 3^2$, $63 = 3^2 \cdot 7^1$, so $\text{lcm}[18, 63] = 2^1 \cdot 3^2 \cdot 7^1 = \boxed{126}$.

(e) $24 = 2^3 \cdot 3^1$, $90 = 2^1 \cdot 3^2 \cdot 5^1$, so $\text{lcm}[24, 90] = 2^3 \cdot 3^2 \cdot 5^1 = \boxed{360}$.

(f) $42 = 2^1 \cdot 3^1 \cdot 7^1$, $70 = 2^1 \cdot 5^1 \cdot 7^1$, so $\text{lcm}[42, 70] = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 = \boxed{210}$.

(g) $6 = 2^1 \cdot 3^1$, $8 = 2^3$, and $14 = 2^1 \cdot 7^1$, so $\text{lcm}[6, 8, 14] = 2^3 \cdot 3^1 \cdot 7^1 = \boxed{168}$.

(h) $9 = 3^2$, $12 = 2^2 \cdot 3^1$, and $16 = 2^4$, so $\text{lcm}[9, 12, 16] = 2^4 \cdot 3^2 = \boxed{144}$.

4.3.2 $18 = 2^1 \cdot 3^2$ and $30 = 2^1 \cdot 3^1 \cdot 5^1$, so $\text{lcm}[18,30] = 2^1 \cdot 3^2 \cdot 5^1 = 90$. Common multiples of 18 and 30 are multiples of 90, so they can be expressed in the form $90n$ for various integer values of n. The 8 smallest possible values come from plugging in $n = 1, 2, \ldots, 8$ to compute the values of $90n$:

$$\begin{aligned} 90 \cdot 1 &= \boxed{90} & 90 \cdot 5 &= \boxed{450} \\ 90 \cdot 2 &= \boxed{180} & 90 \cdot 6 &= \boxed{540} \\ 90 \cdot 3 &= \boxed{270} & 90 \cdot 7 &= \boxed{630} \\ 90 \cdot 4 &= \boxed{360} & 90 \cdot 8 &= \boxed{720} \end{aligned}$$

4.3.3 $14 = 2^1 \cdot 7^1$, $15 = 3^1 \cdot 5^1$, and $16 = 2^4$, so $\text{lcm}[14,15,16] = 2^4 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 1680$. Common multiples of 14, 15, and 16 are multiples of 1680. We compute the 6 smallest positive values of $1680n$:

$$\begin{aligned} 1680 \cdot 1 &= \boxed{1680} & 1680 \cdot 4 &= \boxed{6720} \\ 1680 \cdot 2 &= \boxed{3360} & 1680 \cdot 5 &= \boxed{8400} \\ 1680 \cdot 3 &= \boxed{5040} & 1680 \cdot 6 &= \boxed{10080} \end{aligned}$$

4.3.4 The LCM of 2, 3, 4, 5, 6, and 7 is $2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 420$. Common multiples of 2, 3, 4, 5, 6, and 7 are of the form $420n$ where n is an integer. The smallest positive integer n that makes $420n$ a four-digit number is 3, so the answer is $420 \cdot 3 = \boxed{1260}$.

Exercises for Section 4.4

4.4.1

(a) $12 = 2^2 \cdot 3^1$ and $20 = 2^2 \cdot 5^1$, so $\gcd(12, 20) = 2^2 = \boxed{4}$.

(b) $120 = 2^3 \cdot 3^1 \cdot 5^1$ and $96 = 2^5 \cdot 3^1$, so $\gcd(120, 96) = 2^3 \cdot 3^1 = \boxed{24}$.

(c) $84 = 2^2 \cdot 3^1 \cdot 7^1$ and $126 = 2^1 \cdot 3^2 \cdot 7^1$, so $\gcd(84, 126) = 2^1 \cdot 3^1 \cdot 7^1 = \boxed{42}$.

(d) $91 = 7^1 \cdot 13^1$ and $72 = 2^3 \cdot 3^2$, so $\gcd(91, 72) = \boxed{1}$.

(e) $140 = 2^2 \cdot 5^1 \cdot 7^1$ and $320 = 2^6 \cdot 5^1$, so $\gcd(140, 320) = 2^2 \cdot 5^1 = \boxed{20}$.

(f) $77 = 7^1 \cdot 11^1$ and $224 = 2^5 \cdot 7^1$, so $\gcd(77, 224) = 7^1 = \boxed{7}$.

4.4.2 The positive common divisors of two integers are the positive divisors of their GCD.

(a) $\gcd(30, 70) = 10$, and the positive divisors of 10 are $\boxed{1}$, $\boxed{2}$, $\boxed{5}$, and $\boxed{10}$.

(b) $\gcd(32, 92) = 4$, and the positive divisors of 4 are $\boxed{1}$, $\boxed{2}$, and $\boxed{4}$.

(c) $\gcd(126, 147) = 21$, and the positive divisors of 21 are $\boxed{1}$, $\boxed{3}$, $\boxed{7}$, and $\boxed{21}$.

(d) $\gcd(108, 360) = 36$, and the positive divisors of 36 are $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\boxed{6}$, $\boxed{9}$, $\boxed{12}$, $\boxed{18}$, and $\boxed{36}$.

Exercises for Section 4.5

4.5.1 Expressing each numerator and denominator as a prime factorization helps us reduce each fraction to lowest terms:

$$
\begin{array}{lcccccccc}
\text{(a)} & \dfrac{6}{10} & = & \dfrac{2^1 \cdot 3^1}{2^1 \cdot 5^1} & = & \dfrac{3^1}{5^1} & = & \boxed{\dfrac{3}{5}} \\[2ex]
\text{(b)} & \dfrac{39}{12} & = & \dfrac{3^1 \cdot 13^1}{2^2 \cdot 3^1} & = & \dfrac{13^1}{2^2} & = & \boxed{\dfrac{13}{4}} \\[2ex]
\text{(c)} & \dfrac{16}{-24} & = & -\dfrac{2^4}{2^3 \cdot 3^1} & = & -\dfrac{2^1}{3^1} & = & \boxed{-\dfrac{2}{3}} \\[2ex]
\text{(d)} & \dfrac{35}{63} & = & \dfrac{5^1 \cdot 7^1}{3^2 \cdot 7^1} & = & \dfrac{5^1}{3^2} & = & \boxed{\dfrac{5}{9}} \\[2ex]
\text{(e)} & -\dfrac{140}{50} & = & -\dfrac{2^2 \cdot 5^1 \cdot 7^1}{2^1 \cdot 5^2} & = & -\dfrac{2^1 \cdot 7^1}{5^1} & = & \boxed{-\dfrac{14}{5}} \\[2ex]
\text{(f)} & \dfrac{180}{40} & = & \dfrac{2^2 \cdot 3^2 \cdot 5^1}{2^3 \cdot 5^1} & = & \dfrac{3^2}{2^1} & = & \boxed{\dfrac{9}{2}} \\[2ex]
\text{(g)} & \dfrac{2040}{85} & = & \dfrac{2^3 \cdot 3^1 \cdot 5^1 \cdot 17^1}{5^1 \cdot 17^1} & = & 2^3 \cdot 3^1 & = & \boxed{24} \\[2ex]
\text{(h)} & \dfrac{-1008}{2520} & = & -\dfrac{2^4 \cdot 3^2 \cdot 7^1}{2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1} & = & -\dfrac{2^1}{5^1} & = & \boxed{-\dfrac{2}{5}}
\end{array}
$$

Exercises for Section 4.6

4.6.1 Since $6 = 2^1 \cdot 3^1$, positive multiples of 6 include at least one power of each of 2 and 3 in their prime factorizations. Since the powers of primes in the factorizations of squares are even, there are at least two powers of each of 2 and 3 in the prime factorizations of positive multiples of 6 that are perfect squares. This means that the numbers we want are multiples of $2^2 \cdot 3^2 = 36$ and can be written in the form $36n^2$ for positive integers n. The five smallest are

$$
\begin{aligned}
36 \cdot 1^2 &= \boxed{36} \\
36 \cdot 2^2 &= \boxed{144} \\
36 \cdot 3^2 &= \boxed{324} \\
36 \cdot 4^2 &= \boxed{576} \\
36 \cdot 5^2 &= \boxed{900}
\end{aligned}
$$

4.6.2 Let's take a look at the prime factorizations of a few perfect fifth powers:

$$\begin{aligned} 24^5 &= (2^3 \cdot 3^1)^5 &= 2^{15} \cdot 3^5 \\ 25^5 &= (5^2)^5 &= 5^{10} \\ 26^5 &= (2^1 \cdot 13^1)^5 &= 2^5 \cdot 13^5 \\ 27^5 &= (3^3)^5 &= 3^{15} \\ 28^5 &= (2^2 \cdot 7^1)^5 &= 2^{10} \cdot 7^5 \end{aligned}$$

Since the power of each prime gets multiplied by 5, the exponent of each prime in the prime factorization of a perfect fifth power is a multiple of 5.

4.6.3 The exponents of the primes in the prime factorization of an integer that is both a perfect square and a perfect cube must be both even and a multiple of 3. Therefore, these exponents must be multiples of lcm[2, 3] = 6. In other words, a perfect square that is a perfect cube is a perfect sixth power. The six smallest positive examples are

$$\begin{aligned} 1^6 &= \boxed{1} &= 1^2 &= 1^3 \\ 2^6 &= \boxed{64} &= 8^2 &= 4^3 \\ 3^6 &= \boxed{729} &= 27^2 &= 9^3 \\ 4^6 &= \boxed{4096} &= 64^2 &= 16^3 \\ 5^6 &= \boxed{15625} &= 125^2 &= 25^3 \\ 6^6 &= \boxed{46656} &= 216^2 &= 36^3 \end{aligned}$$

Notice that each can be expressed as the cube of a square *and* the square of a cube.

4.6.4 A multiple of $30 = 2^1 \cdot 3^1 \cdot 5^1$ that is a perfect cube must have at least 3 powers of each of 2, 3, and 5 in its prime factorization. This means we are looking for the four smallest positive multiples of $2^3 \cdot 3^3 \cdot 5^3 = 27000$ that are perfect cubes. These can be written in the form $27000n^3$ for positive integers n. The four smallest are

$$\begin{aligned} 27000 \cdot 1^3 &= \boxed{27000} \\ 27000 \cdot 2^3 &= \boxed{216000} \\ 27000 \cdot 3^3 &= \boxed{729000} \\ 27000 \cdot 4^3 &= \boxed{1728000} \end{aligned}$$

4.6.5 A perfect square that is a multiple of $20 = 2^2 \cdot 5^1$ must be a multiple of $2^2 \cdot 5^2 = 100$. A perfect cube that is a multiple of 20 must be a multiple of $2^3 \cdot 5^3 = 1000$. Our goal is thus to count the multiples of 20 from 100 to 1000 inclusive:

$$100 \le 20n \le 1000.$$

Dividing this entire inequality by 20 we get $5 \le n \le 50$, so there are $50 - 5 + 1 = \boxed{46}$ integers in Cameron's list.

Exercises for Section 4.7

4.7.1

$$\begin{array}{rrclclcl}
\text{(a)} & \gcd(50,90) & = & 10\cdot\gcd(5,9) & = & 10\cdot 1 & = & \boxed{10} \\
\text{(b)} & \text{lcm}[60,70] & = & 10\cdot\text{lcm}[6,7] & = & 10\cdot 42 & = & \boxed{420} \\
\text{(c)} & \gcd(750,600) & = & 50\cdot\gcd(15,12) & = & 50\cdot 3 & = & \boxed{150} \\
\text{(d)} & \text{lcm}[444,555] & = & 111\cdot\text{lcm}[4,5] & = & 111\cdot 20 & = & \boxed{2220}
\end{array}$$

In the following problems, remember that the product of two positive integers is equal to the product of their GCD and LCM.

4.7.2 $\gcd(100,120)\cdot \text{lcm}[100,120] = 100\cdot 120 = \boxed{12000}$.

4.7.3

$$40n = \gcd(n,40)\cdot\text{lcm}[n,40] = 10\cdot 280 = 2800.$$

Solving $40n = 2800$, we get $n = \boxed{70}$.

4.7.4 Let the unknown number be x.

$$18x = \gcd(18,x)\cdot\text{lcm}[18,x] = 6\cdot 450 = 2700.$$

Solving $18x = 2700$, we get $x = \boxed{150}$.

Review Problems

4.24

(a) $66 = \boxed{2^1\cdot 3^1\cdot 11^1}$

(b) $95 = \boxed{5^1\cdot 19^1}$

(c) $150 = \boxed{2^1\cdot 3^1\cdot 5^2}$

(d) $336 = \boxed{2^4\cdot 3^1\cdot 7^1}$

(e) $440 = \boxed{2^3\cdot 5^1\cdot 11^1}$

(f) $828 = \boxed{2^2\cdot 3^2\cdot 23^1}$

4.25

(a) $16 = 2^4$ and $44 = 2^2\cdot 11^1$, so $\gcd(16,44) = 2^2 = \boxed{4}$.

(b) $36 = 2^2\cdot 3^2$ and $132 = 2^2\cdot 3^1\cdot 11^1$, so $\gcd(36,132) = 2^2\cdot 3^1 = \boxed{12}$.

(c) $75 = 3^1\cdot 5^2$ and $360 = 2^3\cdot 3^2\cdot 5^1$, so $\gcd(75,360) = 3^1\cdot 5^1 = \boxed{15}$.

(d) $350 = 2^1\cdot 5^2\cdot 7^1$ and $588 = 2^2\cdot 3^1\cdot 7^2$, so $\gcd(350,588) = 2^1\cdot 7^1 = \boxed{14}$.

(e) $1200 = 2^4\cdot 3^1\cdot 5^2$ and $1620 = 2^2\cdot 3^4\cdot 5^1$, so $\gcd(1200,1620) = 2^2\cdot 3^1\cdot 5^1 = \boxed{60}$.

(f) $8 = 2^3$ and $21 = 3^1\cdot 7^1$, so $\text{lcm}[8,21] = 2^3\cdot 3^1\cdot 7^1 = \boxed{168}$.

(g) $36 = 2^2\cdot 3^2$ and $132 = 2^2\cdot 3^1\cdot 11^1$, so $\text{lcm}[36,132] = 2^2\cdot 3^2\cdot 11^1 = \boxed{396}$.

(h) $90 = 2^1\cdot 3^2\cdot 5^1$ and $192 = 2^6\cdot 3^1$, so $\text{lcm}[90,192] = 2^6\cdot 3^2\cdot 5^1 = \boxed{2880}$.

(i) $375 = 3^1 \cdot 5^3$ and $825 = 3^1 \cdot 5^2 \cdot 11^1$, so $\text{lcm}[375, 825] = 3^1 \cdot 5^3 \cdot 11^1 = \boxed{4125}$.

(j) $10 = 2^1 \cdot 5^1$, $18 = 2^1 \cdot 3^2$, and $48 = 2^4 \cdot 3^1$, so $\text{lcm}[10, 18, 48] = 2^4 \cdot 3^2 \cdot 5^1 = \boxed{720}$.

4.26 Common multiples of 54 and 99 are multiples of $\text{lcm}[54, 99] = 594$, so they can be expressed in the form $594n$. The 8 smallest positive common multiples of 54 and 99 are

$$\begin{aligned} 594 \cdot 1 &= \boxed{594} & 594 \cdot 5 &= \boxed{2970} \\ 594 \cdot 2 &= \boxed{1188} & 594 \cdot 6 &= \boxed{3564} \\ 594 \cdot 3 &= \boxed{1782} & 594 \cdot 7 &= \boxed{4158} \\ 594 \cdot 4 &= \boxed{2376} & 594 \cdot 8 &= \boxed{4752} \end{aligned}$$

4.27 We want to find the smallest four-digit number that is a multiple of $\text{lcm}[2, 3, 5, 7] = 210$, so we need to find the smallest value of n such that

$$210n \ge 1000.$$

Dividing this inequality by 210 we get $n \ge 4\frac{16}{21}$, so $n = 5$ gives us the smallest four-digit multiple of 210: $210 \cdot 5 = \boxed{1050}$.

4.28 The positive common divisors of 84 and 132 are the positive divisors of $\gcd(84, 132) = 12 = 2^2 \cdot 3^1$. These divisors are formed by multiplying the powers of 2 from 2^0 up to 2^2 by the powers of 3 from 3^0 up to 3^1:

$$\begin{aligned} 2^0 \cdot 3^0 &= \boxed{1} & 2^0 \cdot 3^1 &= \boxed{3} \\ 2^1 \cdot 3^0 &= \boxed{2} & 2^1 \cdot 3^1 &= \boxed{6} \\ 2^2 \cdot 3^0 &= \boxed{4} & 2^2 \cdot 3^1 &= \boxed{12} \end{aligned}$$

4.29 The positive common divisors of 162 and 540 are the positive divisors of $\gcd(162, 540) = 54 = 2^1 \cdot 3^3$. These divisors are formed by multiplying the powers of 2 from 2^0 up to 2^1 by the powers of 3 from 3^0 up to 3^3:

$$\begin{aligned} 2^0 \cdot 3^0 &= \boxed{1} & 2^1 \cdot 3^0 &= \boxed{2} \\ 2^0 \cdot 3^1 &= \boxed{3} & 2^1 \cdot 3^1 &= \boxed{6} \\ 2^0 \cdot 3^2 &= \boxed{9} & 2^1 \cdot 3^2 &= \boxed{18} \\ 2^0 \cdot 3^3 &= \boxed{27} & 2^1 \cdot 3^3 &= \boxed{54} \end{aligned}$$

4.30

(a) $$\frac{12}{15} = \frac{2^2 \cdot 3^1}{3^1 \cdot 5^1} = \frac{2^2}{5^1} = \boxed{\frac{4}{5}}$$

(b) $$-\frac{40}{88} = -\frac{2^3 \cdot 5^1}{2^3 \cdot 11^1} = -\frac{5^1}{11^1} = \boxed{-\frac{5}{11}}$$

(c) $$\frac{144}{9} = \frac{2^4 \cdot 3^2}{3^2} = 2^4 = \boxed{16}$$

(d) $-\frac{930}{156} = -\frac{2^1 \cdot 3^1 \cdot 5^1 \cdot 31^1}{2^2 \cdot 3^1 \cdot 13^1} = -\frac{5^1 \cdot 31^1}{2^1 \cdot 13^1} = \boxed{-\frac{155}{26}}$

(e) $\frac{646}{-57} = -\frac{2^1 \cdot 17^1 \cdot 19^1}{3^1 \cdot 19^1} = -\frac{2^1 \cdot 17^1}{3^1} = \boxed{-\frac{34}{3}}$

(f) $\frac{770}{100100} = \frac{2^1 \cdot 5^1 \cdot 7^1 \cdot 11^1}{2^2 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1} = \frac{1}{2^1 \cdot 5^1 \cdot 13^1} = \boxed{\frac{1}{130}}$

4.31

Positive multiples of $15 = 3^1 \cdot 5^1$ have at least 1 power of each of 3 and 5 in their prime factorizations. The prime factorizations of perfect squares have even powers of each prime, so we are looking for integers with at least 2 powers of both 3 and 5 in their prime factorizations. We want squares that are multiples of $3^2 \cdot 5^2 = 225$, which can be written as $225n^2$ for each positive integer n. The five smallest are shown at right.

$$\begin{aligned} 225 \cdot 1^2 &= \boxed{225} \\ 225 \cdot 2^2 &= \boxed{900} \\ 225 \cdot 3^2 &= \boxed{2025} \\ 225 \cdot 4^2 &= \boxed{3600} \\ 225 \cdot 5^2 &= \boxed{5625} \end{aligned}$$

4.32

Positive multiples of $49 = 7^2$ include at least 2 powers of 7 in their prime factorizations. The power of each exponent in the prime factorization of a cube is a multiple of 3, so we are looking for cubes that are positive multiples of $7^3 = 343$. We can write them in the form $343n^3$ for each positive integer n. The six smallest are shown at right.

$$\begin{aligned} 343 \cdot 1^3 &= \boxed{343} \\ 343 \cdot 2^3 &= \boxed{2744} \\ 343 \cdot 3^3 &= \boxed{9261} \\ 343 \cdot 4^3 &= \boxed{21952} \\ 343 \cdot 5^3 &= \boxed{42875} \\ 343 \cdot 6^3 &= \boxed{74088} \end{aligned}$$

4.33 Positive multiples of $6 = 2^1 \cdot 3^1$ that are perfect squares are multiples of $2^2 \cdot 3^2 = 36$ that can be written in the form $36n^2$ for each positive integer n. Positive multiples of 6 that are perfect cubes are multiples of $2^3 \cdot 3^3 = 216$ that can be written in the form $216n^3$ for each positive integer n.

When we look at the first four of each case, we can find the four smallest positive multiples of 6 that are either perfect squares or perfect cubes:

$$\begin{aligned} 36 \cdot 1^2 &= \boxed{36} & \qquad 216 \cdot 1^3 &= \boxed{216} \\ 36 \cdot 2^2 &= \boxed{144} & 216 \cdot 2^3 &= 1728 \\ 36 \cdot 3^2 &= \boxed{324} & 216 \cdot 3^3 &= 5832 \\ 36 \cdot 4^2 &= 576 & 216 \cdot 4^3 &= 13824 \end{aligned}$$

4.34 The product of two positive integers is equal to the product of their GCD and LCM, so

$$\gcd(18, 42) \cdot \text{lcm}[18, 42] = 18 \cdot 42 = \boxed{756}.$$

4.35 Since the product of two integers is equal to the product of their GCD and LCM, we have the equation

$$216n = 24 \cdot 864.$$

We could just solve this equation, but this is a case in which the prime factorizations of the integers make the arithmetic easier to see:

$$n \cdot (2^3 \cdot 3^3) = (2^3 \cdot 3^1) \cdot (2^5 \cdot 3^3),$$

Dividing everything by $2^3 \cdot 3^3$, we get

$$n = \frac{(2^3 \cdot 3^1) \cdot (2^5 \cdot 3^3)}{2^3 \cdot 3^3} = 3^1 \cdot 2^5 = \boxed{96}.$$

Challenge Problems

4.36

$$\begin{aligned} 2520 &= 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1 \\ 7200 &= 2^5 \cdot 3^2 \cdot 5^2 \\ \gcd(2520, 7200) &= 2^3 \cdot 3^2 \cdot 5^1 \quad = \boxed{360} \end{aligned}$$

4.37 $999999 = 999 \cdot 1001 = (3^3 \cdot 37^1) \cdot (7^1 \cdot 11^1 \cdot 13^1) = \boxed{3^3 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 37^1}.$

4.38

$$\begin{aligned} 12 &= 2^2 \cdot 3^1 \\ 15 &= 3^1 \cdot 5^1 \\ 20 &= 2^2 \cdot 5^1 \\ 420 &= 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 \end{aligned}$$

Since $420 = \text{lcm}[12, 15, 20, k]$, the prime factorization of 420 is made up of the largest powers of any primes in the prime factorizations of 12, 15, 20, and k. Since there are 2 powers of 2 in each of 12 and 20, 1 power of 3 in each of 12 and 15, and 1 power of 5 in each of 15 and 20, the only prime that must be in the prime factorization of k is a single power of 7. The smallest possible value of k is therefore $7^1 = \boxed{7}$.

4.39 First, none of the first five friends could have been incorrect. If one of them were, and their number was n, then friend $2n$ would also be incorrect, but the incorrect friends were consecutively numbered, so they would have to be friends 1 and 2 and friend 1 cannot possibly have been incorrect.

Additionally, if friend 6 were incorrect, then at least one of friends 2 and 3 were incorrect. If friend 10 were incorrect, then at least one of friends 2 and 5 were incorrect. But again, the incorrect friends must be consecutively numbered, so this is impossible.

The incorrect students were either students 7 and 8 or students 8 and 9. We can simply calculate the LCMs of the remaining numbers to see which is smallest, which must be our answer:

$$\begin{aligned} \text{lcm}[1, 2, 3, 4, 5, 6, 9, 10] &= 2^2 \cdot 3^2 \cdot 5^1 &&= \boxed{180} \\ \text{lcm}[1, 2, 3, 4, 5, 6, 7, 10] &= 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 &&= 420 \end{aligned}$$

4.40 Since 2, 3, and 5 divide $30N$ but not 7, they do not divide $30N + 7$. Similarly, 7 only divides $30N + 7$ if 7 divides $30N$, which means N must be a multiple of 7 for 7 to divide it. Since no number less than 11 divides $30N + 7$ while $N < 7$, we only need to check when $30N + 7 \geq 11^2$. When $N = 4$, $30N + 7 = 127$ is prime. When $N = 5$, $30N + 7 = 157$ is prime. However, when $N = \boxed{6}$, $30N + 7 = 187 = 11 \cdot 17$ is composite.

4.41 Since perfect squares that are perfect cubes have exponents in their prime factorizations that are both multiples of 2 and multiples of 3, all the exponents must be multiples of 6. Common multiples of 18 and 30 are multiples of $\text{lcm}[18, 30] = 90 = 2^1 \cdot 3^2 \cdot 5^1$. A multiple of 90 that is both a square and a cube must be in the form

$$2^6 \cdot 3^6 \cdot 5^6 \cdot n^6 = 729000000n^6$$

for some positive integer n. The five smallest are

$$\begin{aligned}
729000000 \cdot 1^6 &= \boxed{729000000} \\
729000000 \cdot 2^6 &= \boxed{46656000000} \\
729000000 \cdot 3^6 &= \boxed{531441000000} \\
729000000 \cdot 4^6 &= \boxed{2985984000000} \\
729000000 \cdot 5^6 &= \boxed{11390625000000}
\end{aligned}$$

When numbers get this large, its reasonable to simply leave them in exponential form: $\boxed{30^6}$, $\boxed{60^6}$, $\boxed{90^6}$, $\boxed{120^6}$, and $\boxed{150^6}$.

4.42 None of the integers is 1 because we could always remove it to reduce the sum while keeping the product the same. Now, note that the sum of two integers both greater than 1 is never greater than their product:

$$(x+1)(y+1) = xy + x + y + 1 = (xy - 1) + (x + 1) + (y + 1) \geq (x + 1) + (y + 1),$$

where the inequality holds for any pair of positive integers x and y since $xy - 1 \geq 0$. One example is that $7 + 10 < 70$. This tells us that we can generate a smaller sum by removing 70 from a particular product and replacing it with 7 and 10 ($70 \cdot 2 = 7 \cdot 10 \cdot 2$). Likewise, we can remove any composite from the product. In this way, the product of integers with the smallest sum must be the prime factorization of the integer itself. Since $140 = 2^2 \cdot 5^1 \cdot 7^1$, the answer is $2 + 2 + 5 + 7 = \boxed{16}$.

4.43 Any common multiple of 14, 26, and 66 must be a multiple of their LCM. No common divisor of these multiples can be larger than this LCM (since it must be a divisor of the LCM). Therefore, the GCD of the common multiples of 14, 26, and 66 is the LCM of 14, 26, and 66.

$$\begin{aligned}
14 &= 2^1 \cdot 7^1 \\
26 &= 2^1 \cdot 13^1 \\
66 &= 2^1 \cdot 3^1 \cdot 11^1 \\
\text{lcm}[14, 26, 66] &= 2^1 \cdot 3^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 = \boxed{6006}
\end{aligned}$$

4.44 In general, when a proper divisor d of an integer n is large, the value of $\frac{n}{d}$, which must itself be a proper divisor of n, is small. Since 1999, 2000, and 2001 are very close in value, we are really looking for the one that has the smallest divisor greater than 1. The smallest possible divisor greater than 1 is 2, and since $\frac{2000}{2} = 1000$ is a proper divisor of $\boxed{2000}$, we have our answer.

4.45

(a) $\text{lcm}[24, 90] = 2^3 \cdot 3^2 \cdot 5^1 = \boxed{360}$.

(b) $\boxed{\text{Yes}}$. Since $36 \mid 360$, $36a \mid 360a$ for any integer a, so $36 \mid 360a$ and every common multiple of 24 and 90 can be written as $360a$ some some integer a.

(c) $\boxed{\text{No}}$. The simplest counterexample is that 360 is not a multiple of 2160.

Every common multiple of two positive integers is a multiple of any divisor of their LCM. However, it isn't necessarily true that a common multiple of two integers is a multiple of their product.

4.46 For a positive integer n, we know that $8n = 2^3 \cdot n$ has 3 more powers of 2 in its prime factorization. This means that there is an even power of 2 in the prime factorization of one of n and $8n$ and an odd power of 2 in the other. Since all the exponents in the prime factorization of a perfect square must be even, at least one of n and $8n$ be must non-square.

4.47 The sum of the reciprocals is a fraction with denominator 60 in lowest terms. This means that 60 is a divisor of the LCM of the three consecutive integers.

Now, note that the sum of the reciprocals is greater than 3 times the reciprocal of the largest of the three integers. Also, the sum of the reciprocals is less than 3 times the reciprocal of the smallest of the three integers. In other words,

$$3 \cdot \frac{1}{n} > \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} > 3 \cdot \frac{1}{n+2}.$$

Now, note that

$$3 \cdot \frac{1}{3} > \frac{47}{60} > 3 \cdot \frac{1}{4}.$$

The largest of the integers is greater than 3. The smallest of the integers is less than 4. The only possibilities are $(2, 3, 4)$ and $(3, 4, 5)$. Since 60 is a multiple of 5, only the latter possibility works. Finally, we note that

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60},$$

and $3 + 4 + 5 = \boxed{12}$.

4.48 The number of stamps Jenna puts on each page must divide the number of stamps she puts into each book, so the largest possible number of stamps she puts on each page is $\gcd(840, 1008) = \boxed{168}$.

4.49 We use the formula for the sum of the n smallest positive integers, $\frac{n(n+1)}{2}$:

$$\frac{70 \cdot 71}{2} = 35 \cdot 71 = 5^1 \cdot 7^1 \cdot 71^1,$$

so $\boxed{71}$ is the largest prime divisor of the sum.

4.50 Let n be an integer that leaves a remainder of 1 when divided by each of 2, 3, ..., 8, and 9. That means $n - 1$ is a common multiple of $2, 3, \ldots, 8$, and 9 and therefore a multiple of

$$\text{lcm}[2,3,4,5,6,7,8,9] = 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1 = 2520.$$

This means $n - 1 = 2520m$ for some integer m, so $n = 2520m + 1$. Since we are looking for the next value of n greater than 1, we let $m = 1$ to get our answer: $n = 2520 \cdot 1 + 1 = \boxed{2521}$.

4.51

$$200n = 2^3 \cdot 5^2 \cdot n$$

This partial prime factorization of a positive multiple of 200 gets us started. All the exponents in the prime factorization of a perfect cube are multiples of 3, so n must add at least 1 power of 5 to the product. So, the smallest possible value of n is $\boxed{5}$, where the smallest positive integer that is both a multiple of 200 and a perfect cube is $200 \cdot 5 = 1000 = 10^3$.

4.52 Jayne erases every integer in the list that includes any prime greater than 2 in its prime factorization. This means the only integers left are the powers of 2.

$$\begin{aligned} 2^0 &= 1 \\ 2^1 &= 2 \\ 2^2 &= 4 \\ &\vdots \\ 2^9 &= 512 \\ 2^{10} &= 1024 \end{aligned}$$

Since $2^{11} = 2048 > 2000$, the powers of 2 in the list correspond to the exponents of the powers of 2 above. These are the integers from 0 to 10 inclusive, so there are $\boxed{11}$ integers left in the list when Jayne finishes.

4.53

$$abc = 72 = 2^3 \cdot 3^2,$$

where a, b, and c are positive divisors of 72. Note that many ordered triplets (a, b, c) have the same sum because they use the same divisors of 72:

$$2 + 4 + 9 = 4 + 9 + 2 = 4 + 2 + 9 = 15.$$

We can make life easier on ourselves by just looking for the situations where $a \le b \le c$, which will give us all possible sums.

We find all the possible sums $a + b + c$ through casework on the values of a (which must have a prime factorization with no more than 3 powers of 2 and 2 powers of 3 and no other primes) and by noting that $c = \frac{72}{ab}$. Here are the 12 cases showing $\boxed{11}$ different sums:

$$\begin{array}{llllll}
a = 1 & b = 1 & c = \dfrac{72}{1} = 72 & \Rightarrow & a+b+c = 74 \\[2ex]
a = 1 & b = 2 & c = \dfrac{72}{2} = 36 & \Rightarrow & a+b+c = 39 \\[2ex]
a = 1 & b = 3 & c = \dfrac{72}{3} = 24 & \Rightarrow & a+b+c = 28 \\[2ex]
a = 1 & b = 4 & c = \dfrac{72}{4} = 18 & \Rightarrow & a+b+c = 23 \\[2ex]
a = 1 & b = 6 & c = \dfrac{72}{6} = 12 & \Rightarrow & a+b+c = 19 \\[2ex]
a = 1 & b = 8 & c = \dfrac{72}{8} = 9 & \Rightarrow & a+b+c = 18 \\[2ex]
a = 2 & b = 2 & c = \dfrac{72}{4} = 18 & \Rightarrow & a+b+c = 22
\end{array}$$

$$\begin{array}{llllll}
a = 2 & b = 3 & c = \frac{72}{6} & = 12 & \Rightarrow & a+b+c = 17 \\
a = 2 & b = 4 & c = \frac{72}{8} & = 9 & \Rightarrow & a+b+c = 15 \\
a = 2 & b = 6 & c = \frac{72}{12} & = 6 & \Rightarrow & a+b+c = 14 \\
a = 3 & b = 3 & c = \frac{72}{9} & = 8 & \Rightarrow & a+b+c = 14 \\
a = 3 & b = 4 & c = \frac{72}{12} & = 6 & \Rightarrow & a+b+c = 13
\end{array}$$

4.54 Matthew and Alex count the same number at every common multiple of 6 and 4. Every common multpile of 6 and 4 is a multiple of $\text{lcm}[6,4] = 12$. Positive multiples of 12 can be written as $12n$ for positive integers n. These are no greater than 400:

$$12n \le 400 \qquad \text{so} \qquad n \le 33\frac{1}{3}.$$

There are $\boxed{33}$ numbers counted by both Alex and Matthew.

4.55 Let's evaluate the three pieces of information given by the problem:

(1) Since N is a divisor of $30 \cdot 72 = 2^4 \cdot 3^3 \cdot 5^1$, we know that N has no more than 4 powers of 2, no more than 3 powers of 3, and no more than 1 power of 5 in its prime factorization.

(2) Since 30 is a divisor of $72N$, we know that $\frac{72N}{30} = \frac{12N}{5}$ is an integer. This means N must have at least 1 power of 5 in its prime factorization.

(3) Since 72 is a divisor of $30N$, we know that $\frac{30N}{72} = \frac{5N}{12}$ is an integer. This means $12 \mid N$, so N has at least 2 powers of 2 and 1 power of 3 in its prime factorization.

Now, we are looking for the smallest possible value of N, and facts (2) and (3) together tell us that N is a multiple of $2^2 \cdot 3^1 \cdot 5^1 = 60$. Since $\boxed{60}$ fits all three facts, it is our answer.

4.56

$$\begin{aligned} 6 &= 2^1 \cdot 3^1 \\ 600 &= 2^3 \cdot 3^1 \cdot 5^2 \end{aligned}$$

Since these prime factorizations represent the larger and smaller of the powers of each prime in the prime factorizations of m and n, we know each of the following:

- One of m and n has 2^1 in its prime factorization; the other has 2^3 in its prime factorization.
- Both m and n have 3^1 in their prime factorizations.
- One of m and n does not have 5 in its prime factorization; the other has 5^2 in its prime factorization.

- There are no primes besides 2, 3, and 5 in the prime factorizations of m and n.

These answer parts (a) through (c). In order to answer part (d), we note that the first and third facts above can be combined in $\boxed{2}$ ways: either one of m or n contains both the larger powers of the primes 2 and 5, or both m and n contains the larger power of one of the primes. In fact, the possible pairs of integers that could be m and n are

$$\begin{array}{rclcrcl} 2^1 \cdot 3^1 & = & 6 & \text{with} & 2^3 \cdot 3^1 \cdot 5^2 & = & 600 \\ 2^3 \cdot 3^1 & = & 24 & \text{with} & 2^1 \cdot 3^1 \cdot 5^2 & = & 150 \end{array}$$

4.57 Since the numbers are not all different, at least two of them are the same, so we can let the numbers be x, x and y. Since $x^2y = n^2$ is a perfect square,

$$y = \frac{n^2}{x^2} = \left(\frac{n}{x}\right)^2,$$

so y must be a perfect square. The only perfect square from 2 to 8 is 4, so $y = 4$.

Now, we consider what the players know and what they can deduce:

- If a player sees a 4 and an x, then they know their number must be x.
- If a player sees two x's, then they know their number must be 4.

Since these are the only possibilities, all $\boxed{3}$ players can now deduce the numbers on their foreheads.

CHAPTER 5

Divisor Problems

Exercises for Section 5.2

5.2.1 Add 1 to each exponent in the prime factorization, and multiply when necessary:

$$\begin{array}{llcrcrcrcl}
\text{(a)} & 4 & = & 2^2 & \Rightarrow & t(4) & = & (2+1) & = & \boxed{3} \\
\text{(b)} & 6 & = & 2^1\cdot 3^1 & \Rightarrow & t(6) & = & (1+1)(1+1) & = & \boxed{4} \\
\text{(c)} & 12 & = & 2^2\cdot 3^1 & \Rightarrow & t(12) & = & (2+1)(1+1) & = & \boxed{6} \\
\text{(d)} & 15 & = & 3^1\cdot 5^1 & \Rightarrow & t(15) & = & (1+1)(1+1) & = & \boxed{4} \\
\text{(e)} & 25 & = & 5^2 & \Rightarrow & t(25) & = & (2+1) & = & \boxed{3} \\
\text{(f)} & 30 & = & 2^1\cdot 3^1\cdot 5^1 & \Rightarrow & t(30) & = & (1+1)(1+1)(1+1) & = & \boxed{8} \\
\text{(g)} & 60 & = & 2^2\cdot 3^1\cdot 5^1 & \Rightarrow & t(60) & = & (2+1)(1+1)(1+1) & = & \boxed{12} \\
\text{(h)} & 124 & = & 2^2\cdot 31^1 & \Rightarrow & t(124) & = & (2+1)(1+1) & = & \boxed{6} \\
\text{(i)} & 180 & = & 2^2\cdot 3^2\cdot 5^1 & \Rightarrow & t(180) & = & (2+1)(2+1)(1+1) & = & \boxed{18} \\
\text{(j)} & 280 & = & 2^3\cdot 5^1\cdot 7^1 & \Rightarrow & t(280) & = & (3+1)(1+1)(1+1) & = & \boxed{16} \\
\text{(k)} & 441 & = & 3^2\cdot 7^2 & \Rightarrow & t(441) & = & (2+1)(2+1) & = & \boxed{9} \\
\text{(l)} & 504 & = & 2^3\cdot 3^2\cdot 7^1 & \Rightarrow & t(504) & = & (3+1)(2+1)(1+1) & = & \boxed{24}
\end{array}$$

5.2.2

$$2002 = 2^1\cdot 7^1\cdot 11^1\cdot 13^1 \quad\Rightarrow\quad t(2002) = (1+1)(1+1)(1+1)(1+1) = \boxed{16}.$$

5.2.3 If the number of marbles in each box is n, then $mn = 600$, so m and n are both divisors of 600.

$$600 = 2^3\cdot 3^1\cdot 5^2 \quad\Rightarrow\quad t(600) = (3+1)(1+1)(2+1) = 24.$$

However, $m > 1$ and $n > 1$, so m can be neither 1 nor 600. This leaves $24 - 2 = \boxed{22}$ possible values for m.

Exercises for Section 5.3

5.3.1

$$252 = 2^2 \cdot 3^2 \cdot 7^1$$

An even number contains at least one power of 2 in its prime factorization. This means that an even divisor of 252 must be in the form $2^a \cdot 3^b \cdot 7^c$, where there are 2 choices for a (1 or 2), three are 3 choices for b (0, 1, or 2), and 2 choices for c (0 or 1). This means that $2 \cdot 3 \cdot 2 = \boxed{12}$ of the positive divisors of 252 are even. Now, see if you can find a complementary counting approach.

5.3.2

$$2160 = 2^4 \cdot 3^3 \cdot 5^1$$

A multiple of 3 has at least one power of 3 in its prime factorization, so we are counting divisors in the form $2^a \cdot 3^b \cdot 5^c$, where there are 5 choices for a (0-4), 3 choices for b (1-3), and 2 choices for c (0 or 1). Of the positive divisors of 2160, $5 \cdot 3 \cdot 2 = \boxed{30}$ of them are multiples of 3.

5.3.3 Since $xy = 144$, x can be any positive divisor of 144. Since $y = \dfrac{144}{x}$, there is exactly one positive integer y for each positive integer x. We can count the ordered pairs by counting the values of x, which are the divisors of 144:

$$144 = 2^4 \cdot 3^2 \qquad \Rightarrow \qquad t(144) = (4+1)(2+1) = \boxed{15}.$$

5.3.4 The common divisors of two integers are the divisors of their GCD.

$$t(\gcd(48, 156)) = t(12) = \boxed{6}.$$

5.3.5

$$840 = 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1 \qquad \Rightarrow \qquad t(840) = (3+1)(1+1)(1+1)(1+1) = 32.$$

Since 840 has 4 prime divisors (2, 3, 5, and 7), the proportion of prime divisors is $\dfrac{4}{32} = \boxed{\dfrac{1}{8}}$.

5.3.6 Since $t(n) = 11$ is prime and is the product of 1 more than each of the exponents in the prime factorization of n, there can be only one exponent, and therefore one prime in the prime factorization of n. This means $n = p^{10}$ for some odd prime number p, so

$$8n^3 = 2^3 \cdot p^{30} \qquad \Rightarrow \qquad t(8n^3) = (3+1)(30+1) = \boxed{124}.$$

5.3.7 Let $m = p_1^{e_1} \cdot p_2^{e_2} \cdots p_j^{e_j}$ and $n = q_1^{f_1} \cdot q_2^{f_2} \cdots q_k^{f_k}$ be prime factorizations of relatively prime m and n, so

$$\begin{aligned} t(m) &= (e_1+1)(e_2+1)\cdots(e_j+1) = 8 \\ t(n) &= (f_1+1)(f_2+1)\cdots(f_k+1) = 12 \end{aligned}$$

Since no two of the primes in these factorizations are the same,

$$mn = p_1^{e_1} \cdot p_2^{e_2} \cdots p_j^{e_j} \cdot q_1^{f_1} \cdot q_2^{f_2} \cdots q_k^{f_k}$$

is the prime factorization of their product and

$$t(mn) = (e_1 + 1)(e_2 + 1)\cdots(e_j + 1)(f_1 + 1)(f_2 + 1)\cdots(f_k + 1).$$

Substituting for the products of one more than each exponent, we get $t(mn) = 8 \cdot 12 = \boxed{96}$.

5.3.8

(a) The positive common divisors of 840 and 960 are the positive divisors of their GCD:

$$t(\gcd(840, 960)) = t(120) = \boxed{16}.$$

(b) The positive common divisors of 840 and 1200 are the positive divisors of their GCD:

$$t(\gcd(840, 1200)) = t(120) = \boxed{16}.$$

(c) The positive common divisors of 960 and 1200 are the positive divisors of their GCD:

$$t(\gcd(960, 1200)) = t(240) = \boxed{20}.$$

(d) The GCD of 840, 960, and 1200 is 120, so they have $t(120) = \boxed{16}$ common positive divisors.

(e) Any divisor of 120 is a divisor of all three integers, so the only way that a number is a divisor of exactly two of the three numbers is if it is a divisor of 240 but not 120. Since any divisor of 120 is a divisor of 240, we simply subtract the number of divisors of 120 from the number of divisors of 240 to get $20 - 16 = \boxed{4}$.

5.3.9 Let p and q be the prime divisors of n, so we can write $n = p^a \cdot q^b$ for positive integers a and b. This means $n^2 = p^{2a} \cdot q^{2b}$, so $t(n^2) = (2a + 1)(2b + 1) = 27$. Since $2a + 1$ and $2b + 1$ are both greater than 1 and are divisors of 27, we know they are 3 and 9 (in no particular order). This means that a and b are 1 and 4 (in no particular order), so

$$t(n) = (a + 1)(b + 1) = (1 + 1)(4 + 1) = \boxed{10}.$$

5.3.10 12 factors into $2 \cdot 2 \cdot 3$, $2 \cdot 6$, and $3 \cdot 4$, so any number with 12 divisors has a prime factorization in one of the forms $a^1b^1c^2, a^1b^5, a^2b^3$, or a^{11} (if there is only one prime divisor).

$$45000 = 2^3 \cdot 3^2 \cdot 5^4,$$

and we must match each of the above forms of divisors to one that corresponds to a divisor of 45000.

- The 3 divisors in the form $a^1b^1c^2$ are $2^1 \cdot 3^1 \cdot 5^2 = 150$, $2^1 \cdot 3^2 \cdot 5^1 = 90$, and $2^2 \cdot 3^1 \cdot 5^1 = 60$.
- There are no divisors of 45000 in the form a^1b^5.
- The 4 divisors in the form a^2b^3 are $2^3 \cdot 3^2 = 72$, $2^3 \cdot 5^2 = 200$, $2^2 \cdot 5^3 = 500$, and $3^2 \cdot 5^3 = 1125$.
- There are no divisors of 45000 in the form a^{11}.

There are $3 + 4 = \boxed{7}$ positive divisors of 45000 that have exactly 12 positive divisors of their own.

5.3.11 An integer has an odd number of divisors if and only if it is a perfect square. So n is a perfect square, so $n = m^2$ for some integer m. This means $36n = 36m^2 = (6m)^2$, so $36n$ is a perfect square as well, which means it has an odd number of divisors.

Exercises for Section 5.4

5.4.1 $78 = 2^1 \cdot 3^1 \cdot 13^1$, so $t(78) = (1+1)(1+1)(1+1) = 8$. The product of the divisors of 78 is $78^{8/2} = \boxed{78^4}$.

5.4.2 $120 = 2^3 \cdot 3^1 \cdot 5^1$, so $t(120) = (3+1)(1+1)(1+1) = 16$. The product of the divisors of 120 is $120^{16/2} = \boxed{120^8}$.

5.4.3 $144 = 2^4 \cdot 3^2$, so $t(144) = (4+1)(2+1) = 15$. The product of the divisors of 144 is $144^{15/2} = \boxed{12^{15}}$.

5.4.4 $3240 = 2^3 \cdot 3^4 \cdot 5^1$, so $t(3240) = (3+1)(4+1)(1+1) = 40$. The product of the divisors of 3240 is $3240^{40/2} = 3240^{20}$. However, this includes the divisor 3240 itself, so the product of the proper divisors of 3240 is $3240^{20-1} = \boxed{3240^{19}}$.

5.4.5 For an equal number of chickens to be held in n cages, 48 must be a multiple of n, which means n must be a divisor of 48. Our goal is to find the product of the positive divisors of $48 = 2^4 \cdot 3^1$. Since $t(48) = (4+1)(1+1) = 10$, the product of the possible values of n is $48^{10/2} = \boxed{48^5}$.

5.4.6 First, let's identify the even divisors of 180:

$$180 = 2^2 \cdot 3^2 \cdot 5^1,$$

so even divisors of 180 must be in the form $2^a \cdot 3^b \cdot 5^c$, where $1 \le a \le 2$, $0 \le b \le 2$, and $0 \le c \le 1$. Note that if we divide each of these divisors by 2, we get all the divisors of

$$2^1 \cdot 3^2 \cdot 5^1 = 90.$$

Since $t(90) = 12$, the product of the positive divisors of 90 is 90^6. Since each of the 12 even divisors of 180 is twice a divisor of 90, the product of the even divisors of 180 is $\boxed{2^{12} \cdot 90^6}$, which can also be expressed as $\boxed{360^6}$.

5.4.7

$$2400 = 2^5 \cdot 3^1 \cdot 5^2$$

Each of the divisors of 2400 that are multiples of 6 can be written in the form $6d$, where d is a divisor of

$$2^4 \cdot 5^2 = 400.$$

Since $t(400) = 15$, the product of the positive divisors of 400 is $400^{15/2} = 20^{15}$. Since the 15 divisors of 2400 that are multiples of 6 are each 6 times the divisors of 400, their product is $6^{15} \cdot 20^{15} = \boxed{120^{15}}$.

5.4.8

$$3200 = 2^7 \cdot 5^2$$

Perfect squares have prime factorizations with even exponents, and those that are divisors of 3200 are in the form $2^{2a} \cdot 5^{2b}$ where $0 \le a \le 3$ and $0 \le b \le 1$. In fact, the square roots of these square divisors are the divisors of $2^3 \cdot 5^1 = 40$. There are $t(40) = 8$ of these divisors and their product is 40^4. Since these are the square roots of the perfect square divisors we want, the product of the square divisors of 3200 is $(40^4)^2 = \boxed{40^8}$.

Review Problems

5.13

$$\begin{array}{lrcrcrcrcl}
\text{(a)} & 18 & = & 2^1\cdot 3^2 & \Rightarrow & t(18) & = & (1+1)(2+1) & = & \boxed{6} \\
\text{(b)} & 52 & = & 2^2\cdot 13^1 & \Rightarrow & t(52) & = & (2+1)(1+1) & = & \boxed{6} \\
\text{(c)} & 216 & = & 2^3\cdot 3^3 & \Rightarrow & t(216) & = & (3+1)(3+1) & = & \boxed{16} \\
\text{(d)} & 420 & = & 2^2\cdot 3^1\cdot 5^1\cdot 7^1 & \Rightarrow & t(420) & = & (2+1)(1+1)(1+1)(1+1) & = & \boxed{24} \\
\text{(e)} & 2520 & = & 2^3\cdot 3^2\cdot 5^1\cdot 7^1 & \Rightarrow & t(2520) & = & (3+1)(2+1)(1+1)(1+1) & = & \boxed{48} \\
\text{(f)} & 3750 & = & 2^1\cdot 3^1\cdot 5^4 & \Rightarrow & t(3750) & = & (1+1)(1+1)(4+1) & = & \boxed{20}
\end{array}$$

5.14

$$(30^4) = (2^1\cdot 3^1\cdot 5^1)^4 = 2^4\cdot 3^4\cdot 5^4$$

Since $t(30^4) = (4+1)^3 = 125$, taking out 1 and (30^4) leaves $125 - 2 = \boxed{123}$ positive divisors.

5.15 Timothy's quotient is an integer if and only if Joseph's number is a divisor of 1000. Our goal is to count the positive divisors of $1000 = 2^3\cdot 5^3$, so $t(1000) = (3+1)(3+1) = \boxed{16}$ is the number of integers that would make Timothy's quotient an integer.

5.16

$$3240 = 2^3\cdot 3^4\cdot 5^1$$

A positive divisor of 3240 is a multiple of 3 when it has a prime factorization in the form $2^a\cdot 3^b\cdot 5^c$ where $0\le a\le 3, 1\le b\le 4$, and $0\le c\le 1$. There are $4\cdot 4\cdot 2 = \boxed{32}$ choices for a, b, and c, giving the number of positive divisors of 3240 that are multiples of 3.

5.17 The perfect square divisors of 3240 are in the form $2^{2a}\cdot 3^{2b}\cdot 5^{2c}$, where $0\le a\le 1, 0\le b\le 2$, and $c = 0$. This gives $3\cdot 2\cdot 1 = \boxed{6}$ choices for the exponents to the primes that make up the prime factorizations of the perfect square divisors of 3240.

5.18

(a) $28 = 2^2\cdot 7^1$, so $t(28) = (2+1)(1+1) = 6$. The product is $28^{t(28)/2} = \boxed{28^3}$.

(b) $72 = 2^3\cdot 3^2$, so $t(72) = (3+1)(2+1) = 12$. The product is $72^{t(72)/2} = \boxed{72^6}$.

(c) $180 = 2^2\cdot 3^2\cdot 5^1$, so $t(180) = (2+1)(2+1)(1+1) = 18$. The product is $180^{t(180)/2} = \boxed{180^9}$.

(d) $9216 = 2^{10}\cdot 3^2$, so $t(9216) = (10+1)(2+1) = 33$. The product is $9216^{t(9216)/2} = 9216^{33/2} = \boxed{96^{33}}$.

5.19

$$240 = 2^4\cdot 3^1\cdot 5^1$$

The even divisors of 240 can be written as $2d$, where the values of d are the divisors of

$$2^3\cdot 3^1\cdot 5^1 = 120.$$

The product of these $t(120) = 16$ divisors of 120 is 120^8. Each of the 16 even divisors of 240 is twice a divisor of 120, so their product is $2^{16}\cdot 120^8 = \boxed{480^8}$.

Challenge Problems

5.20

$$999999 = 3^3 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 37^1,$$

so $t(999999) = (3+1)(1+1)(1+1)(1+1)(1+1) = \boxed{64}$.

5.21 Since m and n are relatively prime, the primes in their prime factorizations are entirely distinct. This means that the exponents in the prime factorization of mn are simply all those in the prime factorizations of m and n. Since we multiply 1 more than each of those exponents to count divisors, we have $t(mn) = t(m)t(n) = 12 \cdot 10 = 120$ positive divisors of mn. We multiply these divisors by noting that together they make $120/2 = 60$ pairs of divisors with products of mn. The product of all the divisors is the product of these pairs, which is $\boxed{(mn)^{60}}$.

5.22 For a positive integer n, $t(n)$ is a product of choices for exponents in the prime factorizations of the divisors of n. So, when $t(n) = p$ for some prime number p, there must be only one number in that product. This means there is only 1 exponent of $\boxed{1}$ prime in the prime factorization of n.

5.23 If $t(n) = 5$ for some integer n, then $n = p^4$ for some prime number p. In fact, we must be looking for the fourth powers of the three smallest primes. Their sum is

$$2^4 + 3^4 + 5^4 = 16 + 81 + 625 = \boxed{722}.$$

5.24

(a) $3200 = \boxed{2^7 \cdot 5^2}$

(b) $t(3200) = (7+1)(2+1) = \boxed{24}$

(c) There are only 7 choices for the power of 2 in the prime factorization of an even divisor of 3200, so there are $7 \cdot 3 = \boxed{21}$ even divisors.

(d) We are counting divisors in the form $2^{2a} \cdot 5^{2b}$ where $0 \le a \le 3$ and $0 \le b \le 1$, so there are $\boxed{8}$ divisors of 3200 that are perfect squares.

(e) We are looking for "square free" divisors of 3200, which must have prime factorizations in the form $2^a \cdot 5^b$ where neither a nor b is greater than 1. This means there are 2 choices for the values of each a and b, so 3200 has $2 \cdot 2 = \boxed{4}$ positive square free divisors.

5.25 Let $n = p^a \cdot q^b$ where p and q are prime. We are given that $t(n) = (a+1)(b+1) = 9$. Since both $a+1$ and $b+1$ are divisors of 9 greater than 1, they must both be 3, so $a = b = 2$.

$$n^2 = (p^2 \cdot q^2)^2 = p^4 \cdot q^4,$$

so $t(n^2) = (4+1)(4+1) = \boxed{25}$.

5.26 Since n can be written as the product of 2 factors in exactly 3 different ways excluding $n \cdot 1$, there are 4 ways in which n can be written as a product of divisors. This means that $t(n) = 8$ when n is not a perfect square and $t(n) = 7$ when n is a perfect square (where two of the factors whose product is n are the same).

When $t(n) = 7$, we know that $n = p^6$ is the prime factorization for some prime p. The smallest such value for n is $n = 2^6 = 64$.

When $t(n) = 8$, the prime factorization of n can have the forms p^7, $p^3 \cdot q^1$, and $p^1 \cdot q^1 \cdot r^1$. We can exclude the first possibility since it would only be larger than the case for $n = p^6$.

- We get the smallest possible value of $p^3 \cdot q^1$ for picking the smallest primes, where the smaller of the pair is taken to the third power, so $n = 2^3 \cdot 3^1 = 24$.
- We get the smallest possible value of $p^1 \cdot q^1 \cdot r^1$ by choosing the three smallest primes: $2^1 \cdot 3^1 \cdot 5^1 = 30$.

From these cases, we see that the smallest possible value of n is $\boxed{24}$ where

$$24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6.$$

5.27

$$14400 = 2^6 \cdot 3^2 \cdot 5^2$$

Since the only ways to break 8 down into a product of choices for exponents in a prime factorization are $(7+1)$, $(3+1)(1+1)$, and $(1+1)(1+1)(1+1)$, an integer with 8 positive divisors must have a prime factorization with one of the following forms: p^7, $p^3 \cdot q^1$, or $p^1 \cdot q^1 \cdot r^1$.

- Since the largest exponent of any prime in the prime factorization of 14400 is 6, there are no divisors of 14400 of the form p^7.
- Since 2 is the only prime in the prime factorization of 14400 with an exponent that is at least 3, the form $p^3 \cdot q^1$ has 2 possibilities: $2^3 \cdot 3^1 = 24$ and $2^3 \cdot 5^1 = 40$.
- There are only 3 primes in the prime factorization of 14400, so $p^1 \cdot q^1 \cdot r^1$ must be $2^1 \cdot 3^1 \cdot 5^1 = 30$.

There are $2 + 1 = \boxed{3}$ positive divisors of 14400 that themselves have 8 positive divisors each.

5.28

$$\gcd(7560, 8400) = 840 = 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1$$

The common divisors of 7560 and 8400 are the divisors of their GCD:

$$t(840) = (3+1)(1+1)(1+1)(1+1) = \boxed{32}.$$

5.29

(a) $t(p^{19}) = 20$ for any prime p, so $\boxed{1}$ prime could divide the integer.

(b) If $t(n) = 20$, then n has the largest possible number of prime divisors when 20 is the product of the largest possible number of integers greater than 1. This means breaking 20 down into a product of primes:

$$20 = 5 \cdot 2 \cdot 2 = (4+1)(1+1)(1+1),$$

so $n = p^4 \cdot q^1 \cdot r^1$ for some $\boxed{3}$ primes p, q, and r.

(c) First we examine the ways in which 20 can be broken down into a product of factors, each greater than 1:

$$20 = (4+1)(1+1)(1+1) = (4+1)(3+1) = (9+1)(1+1) = (19+1),$$

means that it $t(n) = 20$, then the prime factorization of n is one of the following:

$$p^4 \cdot q^1 \cdot r^1, \qquad p^4 \cdot q^3, \qquad p^9 \cdot q^1, \qquad p^{19}$$

- When $n = p^4 \cdot q^1 \cdot r^1$, we minimize n by choosing the three smallest primes, letting the smallest prime match the largest exponent: $n = 2^4 \cdot 3^1 \cdot 5^1 = 240$.
- When $n = p^4 \cdot q^3$, we minimize n by choosing the two smallest primes, letting the smallest prime match the larger exponent: $n = 2^4 \cdot 3^3 = 432$.
- When $n = p^9 \cdot q^1$, we minimize n by choosing the two smallest primes, letting the smallest prime match the larger exponent: $n = 2^9 \cdot 3^1 = 1536$.
- We minimize $n = p^{19}$ by letting p be the smallest prime: $n = 2^{19} = 524288$.

From these cases, we see that $\boxed{240}$ is the smallest positive integer with exactly 20 positive divisors.

(d) Without going into all the details, there are only a few cases that are really worthy of testing. One of the ways we see this is by noting that $240 < 256 = 2^8$, so any prime factorization with an exponent greater than 7 can be dismissed. We can make other limiting arguments.

- If $t(n) = 21$, then the minimum value of n is $2^6 \cdot 3^2 = 576$.
- If $t(n) = 24$, then the minimum value of n is $2^3 \cdot 3^2 \cdot 5^1 = 360$.
- If $t(n) = 25$, then the minimum value of n is $2^4 \cdot 3^4 = 1296$.
- If $t(n) = 30$, then the minimum value of n is $2^4 \cdot 3^2 \cdot 5^1 = 720$.

Other cases stand no chance, so the answer is $\boxed{\text{no}}$.

5.30

(a) $t(6n) = 9 = (8+1) = (2+1)(2+1)$ has either 1 prime factor or 2 prime factors, but since it has a factor of two and a factor of three (since it is a multiple of 6), it has $\boxed{2}$ prime divisors.

(b) Each prime has an exponent of $\boxed{2}$ because $t(6n) = (2+1)(2+1)$.

(c) $6n = 2^2 \cdot 3^2 = 36$, since it has a factor of two and a factor of three, two prime divisors, and each has an exponent of two. This means $n = \boxed{6}$.

5.31

$$12 = (11+1) = (5+1)(1+1) = (3+1)(2+1) = (2+1)(1+1)(1+1),$$

so we are looking for integers with prime factorizations in one of the four following forms:

$$p^{11}, \qquad p^5 \cdot q^1, \qquad p^3 \cdot q^2, \qquad p^2 \cdot q^1 \cdot r^1$$

- The smallest possible value for the first case is $p^{11} = 2^{11} = 2048$.
- The smallest possible value for the second case is $p^5 \cdot q^1 = 2^5 \cdot 3^1 = 96$. The next smallest is $2^5 \cdot 5^1 = 160 < 3^5 \cdot 2^1 = 486$.

- The smallest possible values for the third case are

$$\begin{aligned} 2^3 \cdot 3^2 &= 72 \\ 2^3 \cdot 5^2 &= 200 \\ 3^3 \cdot 2^2 &= 108 \end{aligned}$$

- The smallest possible values for the fourth case are

$$\begin{aligned} 2^2 \cdot 3^1 \cdot 5^1 &= 60 \\ 2^2 \cdot 3^1 \cdot 7^1 &= 84 \\ 2^2 \cdot 3^1 \cdot 11^1 &= 132 \\ 2^2 \cdot 5^1 \cdot 7^1 &= 140 \\ 3^2 \cdot 2^1 \cdot 5^1 &= 90 \\ 3^2 \cdot 2^1 \cdot 7^1 &= 126 \end{aligned}$$

Any larger primes from these cases produce integers greater than 100, so the sum of the positive integers less than 100 with exactly 12 positive divisors is

$$60 + 72 + 84 + 90 + 96 = \boxed{402}.$$

5.32

(a) The glitches that affect the cell numbered m are the ones that correspond to the divisors of m.

(b) For cell m to be unlocked in the morning, it needed to have had an odd number of glitches lock and relock it, which mean m has an odd number of divisors. The only integers that have odd numbers of divisors are perfect squares, so all the perfect square numbered cells are unlocked. Since $1^2 < 2^2 < 3^2 < \cdots < 44^2 = 1936 < 2000 < 2025 = 45^2$, there are 44 perfect squares up to 2000 corresponding to $\boxed{44}$ cells that are unlocked.

5.33 First, note that 6 perfect square divisors come from 6 choices for the possible even exponents of divisors. This means either there is 1 exponent (with 6 choices) or two exponents (with 2 and 3 choices).

Since we are looking for the smallest integer n with 6 perfect square divisors, we can simply test these cases with the smallest possible primes:

- When there is one prime with 6 even choices for the exponent, the smallest possible value is $2^{10} = 1024$.

- When there are two primes with 2 and 3 even choices for the exponents, the smallest possible value is $2^4 \cdot 3^2 = 144$.

The smaller of these cases is 144, and the sum of its perfect square divisors is

$$1^2 + 2^2 + 3^2 + 4^2 + 6^2 + 12^2 = 1 + 4 + 9 + 16 + 36 + 144 = \boxed{210}.$$

5.34

$$\frac{6480}{12} = 540 = 2^2 \cdot 3^3 \cdot 5^1$$

All the divisors of 6480 that are multiples of 12 can be written in the form $12d$ where d is any divisor of 540. The product of the $t(540) = 24$ positive divisors of 540 is 540^{12}. Multiplying 12 in for each divisor, we get the product of the 24 positive divisors of 6480 that are multiples of 12: $12^{24} \cdot 540^{12} = \boxed{77760^{12}}$.

5.35 The number of positive divisors of an integer is the product of the possible exponents of each prime in the integer's prime factorization, where each possible number of choices is greater than 1. Since $30 = 2^1 \cdot 3^1 \cdot 5^1$, we can express 30 as the product of at most 3 positive integers greater than 1. However, we must still confirm that this is possible.

Since the GCD of m and n has a prime number of positive divisors, the GCD must be in the form p^4 for some prime number p. Now we construct possible examples of integers m and n whose LCM has 30 positive divisors:

$$\begin{aligned} m &= p^4 \\ n &= p^4 \cdot q^2 \cdot r^1 \end{aligned}$$

Here, p, q, and r can be any distinct primes, and their LCM, $p^4 \cdot q^2 \cdot r^1$ has $(4+1)(2+1)(1+1) = 30$ positive divisors, so the answer is $\boxed{3}$ prime divisors.

5.36 We are looking for integers with 3 proper divisors, and therefore 4 total divisors. Such numbers either have 4 possible choices for the exponent to one prime in their prime factorizations, or 2 choices for each of 2 primes. Their prime factorizations must be of one of the two forms p^3 or $p^1 \cdot q^1$.

The largest proper divisor of p^3 is p^2, so we are looking for primes p such that $p^2 < 50$. Since $7^2 < 50 < 11^2$, there are 4 such primes: 2, 3, 5, and 7. Each of these corresponds to an integer p^3 with 3 proper divisors, all less than 50.

The largest proper divisor of $p^1 \cdot q^1$ is the larger of the two primes p and q. There are 15 primes less than 50. There are $\binom{15}{2} = \frac{15 \cdot 14}{2} = 105$ distinct pairs of primes that are both less than 50. Each pair corresponds to an integer $p^1 \cdot q^1$ with 3 proper divisors.

From these two cases, we see that there are $4 + 105 = \boxed{109}$ integers with exactly 3 proper divisors, each of which is less than 50.

CHAPTER 6 Special Numbers

Exercises for Section 6.2

6.2.1 $2^{13}-1 = 8191 < 10000$. Checking all the primes between 1 and 100 (since $\sqrt{8191} < \sqrt{10000} = 100$), we see that none of them is a divisor of 8191, so 8191 is $\boxed{\text{prime}}$.

6.2.2 For integers less than $11^2 = 121$, we only need to check the divisibility of 2, 3, 5 and 7 to look for primes. Going up from 103, we find that both $\boxed{107}$ and $\boxed{109}$ are prime.

We can also use the Sieve of Nygard to make things easier. Note that since $105 = 3 \cdot 5 \cdot 7$, there are no odd multiples of 3 between 105 and 111, no odd multiples of 5 between 105 and 115, and no odd multiples of 7 between 105 and 119.

Exercises for Section 6.3

6.3.1

(a) $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{5040}$

(b) $8! = 8 \cdot 7! = 8 \cdot 5040 = \boxed{40320}$

(c) $9! = 9 \cdot 8! = 9 \cdot 40320 = \boxed{362880}$

6.3.2 It's easier to first find the prime factorizations of the positive integers up to 10 and multiply them together than it is to compute 10! and then factor it.

$$\begin{aligned} 10! &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= (2^1 \cdot 5^1)(3^2)(2^3)(7^1)(2^1 \cdot 3^1)(5^1)(2^2)(3^1)(2^1)(1) \\ &= \boxed{2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1} \end{aligned}$$

6.3.3 $(3!)(2!)(m)$ can be simplified to $(6)(2)(m) = 12m$. Since $(3!)(2!)(m) = 720$, we have $12m = 720$, which means $m = \boxed{60}$.

We could also solve this problem by isolating m before we perform any calculations:

$$m = \frac{6!}{(3!)(2!)} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{(3!)(2!)} = \frac{6 \cdot 5 \cdot 4}{2!} = 6 \cdot 5 \cdot 2 = \boxed{60}.$$

6.3.4 Since $12! = 12 \cdot 11!$, we can examine the sum better by factoring $11!$ out of both parts:

$$11! + 12! = 11! + 12 \cdot 11! = 11!(1 + 12) = 11! \cdot 13.$$

Since no prime greater than 11 divides $11!$, $\boxed{13}$ is the largest prime factor of $11! + 12!$.

6.3.5 We use the prime factorizations of $6!$ and $(4!)^2$ to find their LCM (as we would with most pairs of integers):

$$\begin{aligned} 6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 &= 2^4 \cdot 3^2 \cdot 5^1 \\ (4!)^2 &= (4 \cdot 3 \cdot 2 \cdot 1)^2 &= 2^6 \cdot 3^2 \\ \text{lcm}[6!, (4!)^2] &= 2^6 \cdot 3^2 \cdot 5^1 &= \boxed{2880} \end{aligned}$$

6.3.6 Any prime number less than or equal to 20 divides $20!$. Since no prime greater than 20 can be a divisor of any of the 20 smallest positive integers, $20!$ has no prime divisors greater than 20. The prime factors of $20!$ are $\boxed{2}$, $\boxed{3}$, $\boxed{5}$, $\boxed{7}$, $\boxed{11}$, $\boxed{13}$, $\boxed{17}$, and $\boxed{19}$.

6.3.7

$$6! = 720 = 2^4 \cdot 3^2 \cdot 5^1.$$

Using this prime factorization, we find the number of positive divisors of $6!$:

$$t(6!) = (4 + 1)(2 + 1)(1 + 1) = \boxed{30}.$$

6.3.8 Using the prime factorization from the last problem to help,

$$7! = 7 \cdot 6! = 7(2^4 \cdot 3^2 \cdot 5^1) = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1.$$

The number of positive divisors of $7!$ is thus

$$(4 + 1)(2 + 1)(1 + 1)(1 + 1) = 60.$$

These 60 divisors make up 30 pairs with products of $7!$, so the product of all of them is

$$(7!)^{\frac{t(7!)}{2}} = (7!)^{30} = \boxed{5040^{30}}.$$

Exercises for Section 6.4

6.4.1

(a)

$$s(8) = 1 + 2 + 4 = 7 < 8,$$

so 8 is $\boxed{\text{deficient}}$.

(b)

$$s(16) = 1 + 2 + 4 + 8 = 15 < 16,$$

so 16 is $\boxed{\text{deficient}}$.

(c)

$$s(32) = 1 + 2 + 4 + 8 + 16 = 31 < 32,$$

so 32 is $\boxed{\text{deficient}}$.

In fact, all perfect powers of 2 are deficient. We know this because $s(2^{n+1}) - s(2^n) = 2^n = 2^{n+1} - 2^n$. That is, the amount that each side is increased from one power of 2 to the next is the same.

(d)

$$s(60) = 1 + 2 + 3 + 4 + 5 + 6 + 10 + 12 + 15 + 20 + 30 = 108 > 60,$$

so 60 is $\boxed{\text{abundant}}$.

Note that $15 + 20 + 30 = 65 > 60$, so we could declare that 60 is abundant after summing just a few of its divisors.

6.4.2 A prime has only one proper divisor, which is 1, so $s(p) = 1$ for any prime p. Since all primes are greater than 1, $s(p) = 1 < p$, so *all primes are deficient*. This means there are $\boxed{0}$ primes that are perfect numbers and $\boxed{0}$ that are abundant.

6.4.3 No primes are abundant numbers and none of the powers of 2 that are less than 30 are abundant. Also, 6 and 28 are perfect numbers. These facts cut down on the amount of work we need to do. We now simply test the rest:

$$\begin{array}{rcrcrcr}
s(9) & = & 1+3 & = & 4 & < & 9 \\
s(10) & = & 1+2+5 & = & 8 & < & 10 \\
s(12) & = & 1+2+3+4+6 & = & 16 & > & 12 \\
s(14) & = & 1+2+7 & = & 10 & < & 14 \\
s(15) & = & 1+3+5 & = & 9 & < & 15 \\
s(18) & = & 1+2+3+6+9 & = & 21 & > & 18 \\
s(20) & = & 1+2+4+5+10 & = & 22 & > & 20 \\
s(21) & = & 1+3+7 & = & 11 & < & 21 \\
s(22) & = & 1+2+11 & = & 14 & < & 22 \\
s(24) & = & 1+2+3+4+6+8+12 & = & 36 & > & 24 \\
s(25) & = & 1+5 & = & 6 & < & 25 \\
s(26) & = & 1+2+13 & = & 16 & < & 26 \\
s(27) & = & 1+3+9 & = & 13 & < & 27
\end{array}$$

The only abundant numbers less than 30 are 12, 18, 20, and 24, so there are $\boxed{4}$.

6.4.4 We could add up all the proper divisors of 4320, but we really only need to sum the largest ones to see that 4320 is $\boxed{\text{abundant}}$:

$$s(4320) > 2160 + 1440 + 720 = 4320.$$

6.4.5 A positive multiple of 6 that is greater than 6 can be written as $6n$ for some integer $n > 1$. Now, we sum a few of the divisors of $6n$ in order to learn what we can about $s(6n)$:

$$s(6n) \geq 3n + 2n + n + 1 = 6n + 1 > 6n.$$

Since $s(6n) > 6n$ for any $n > 1$, we know that all of the positive multiples of 6 that are greater than 6 are abundant numbers.

Exercises for Section 6.5

6.5.1 In a four-digit palindrome, the first digit is the same as the last, and the second digit is the same as the third digit. There are 9 options for the first/last digit (1 through 9 – the first digit cannot be 0), and there are 10 options for the second/third digit (0 through 9). This gives us $9 \cdot 10 = \boxed{90}$ four-digit palindromes.

6.5.2 In an even four-digit palindrome, the first/last digit must be an even number between 1 and 9, which means it is either 2, 4, 6, or 8 (4 options). The second/third digit can be anything digit from 0 and 9 inclusive (10 options). This gives us $4 \cdot 10 = \boxed{40}$ even four-digit palindromes.

6.5.3 There are a lot of ways to go about this one. Some students might initially find the answer by applying the Euclidean Algorithm to several pairs of four-digit palindromes.

If a four-digit palindrome has digits x and y in the form $xyyx$, then it can be expressed as

$$1000x + 100y + 10y + x = 1001x + 110y.$$

Since $\gcd(1001, 110) = 11$, we know that 11 is a divisor of both $1001x$ and $110y$ and is therefore a divisor of their sum, so $\boxed{11}$ divides every four-digit palindrome.

6.5.4 None of the one-digit palindromes (1 through 9) are abundant. Checking the two-digit palindromes, we find

$$s(66) = 1 + 2 + 3 + 6 + 11 + 22 + 33 = 78 > 66,$$

and $\boxed{66}$ is the smallest palindrome that is abundant.

6.5.5 We can use the distributive property of multiplication to multiply a three-digit palindrome aba (where a and b are digits) with 101:

$$101 \cdot aba = (100 + 1) \cdot aba = aba00 + aba = ab(2a)ba.$$

Here, the digits of the product are a, b, $2a$, b, and a, unless carrying occurs. In fact, this product is a palindrome unless carrying occurs, and that could only happen when $2a \geq 10$. Since we want the smallest such palindrome in which carrying occurs, we want the smallest possible value of a such that $2a \geq 10$ and the smallest possible value of b. This gives us $\boxed{505}$ as our answer and we see that $101 \cdot 505 = 51005$ is not a palindrome.

Review Problems

6.12 Since $17^2 = 289$, any composite number less than 289 must be divisible by a prime number less than $\sqrt{289} = 17$. The largest prime that is less than 17 is 13, so we use the Sieve of Eratosthenes on primes up to 13, and we see that $\boxed{227 \text{ and } 229}$ are prime, and that $\boxed{239 \text{ and } 241}$ are prime.

6.13 The only prime numbers that divide 30! are less than or equal to 30. So 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 are primes that divide 30!, and there are $\boxed{10}$ of these.

6.14 Dividing by $2^4 \cdot 3^2$ isolates n and makes computation simple:

$$n = \frac{6!}{2^4 \cdot 3^2} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2^4 \cdot 3^2} = \frac{2^4 \cdot 3^2 \cdot 5^1}{2^4 \cdot 3^2} = \boxed{5}.$$

6.15

(a) $9! = \boxed{362880}$

(b)

$$\begin{aligned} 9! &= 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= (3^2)(2^3)(7^1)(2^1 \cdot 3^1)(5^1)(2^2)(3^1)(2^1)(1) \\ &= \boxed{2^7 \cdot 3^4 \cdot 5^1 \cdot 7^1} \end{aligned}$$

(c) $t(9!) = (7+1)(4+1)(1+1)(1+1) = \boxed{160}$

(d) Since 9! has 80 pairs of divisors, each of which has a product of 9!, we can multiply all those pairwise products together to get the product of all the divisors of 9!:

$$(9!)^{t(9!)/2} = \boxed{(9!)^{80}}.$$

6.16

$$\begin{aligned} 7! &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 &&= 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1 \\ (5!)^2 &= (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^2 &&= 2^6 \cdot 3^2 \cdot 5^2 \\ \gcd(7!, (5!)^2) &= 2^4 \cdot 3^2 \cdot 5^1 &&= \boxed{720} \end{aligned}$$

6.17

$$\begin{aligned} 12! &= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= (2^2 \cdot 3^1)(11^1)(2^1 \cdot 5^1)(3^2)(2^3)(7^1)(2^1 \cdot 3^1)(5^1)(2^2)(3^1)(2^1)(1) \\ &= \boxed{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1} \end{aligned}$$

6.18

(a)

$$s(15) = 1 + 3 + 5 = 9 < 15,$$

so 15 is $\boxed{\text{not abundant}}$.

(b) We need only to sum the largest proper divisors of 30 to see that 30 is $\boxed{\text{abundant}}$:

$$s(30) > 15 + 10 + 6 = 31 > 30.$$

(c) We need only to sum the largest proper divisors of 96 to see that 96 is $\boxed{\text{abundant}}$:

$$s(96) > 48 + 32 + 24 = 104 > 96.$$

(d)

$$s(98) = 1 + 2 + 7 + 14 + 49 = 73 < 98,$$

so 98 is $\boxed{\text{not abundant}}$.

6.19 Every multiple of 3 less than or equal to 60 contributes at least one factor of 3 to 60!. Every multiple of $3^2 = 9$ that is less than or equal to 60 contributes one more additional factor of 3 to 60!. Every multiple of 27 contributes yet one more additional extra factor of 3 to 60!.

$$\frac{60}{3} = 20 \qquad \frac{20}{3} = 6\frac{2}{3} \qquad \frac{6}{3} = 2$$

Since 60! is a product of 20 multiples of 3, of which 6 are multiples of 3^2, and of which 2 are multiples of 3^3, there are a total of $20 + 6 + 2 = 28$ factors of 3 in 60!. This means $\boxed{3^{28}}$ is the largest power of 3 that divides 60!.

6.20 It's easiest to sum the proper divisors from greatest to least. As soon as the sum exceeds the integer itself, we know the number is abundant.

$$\begin{array}{rcrcrcr}
s(70) & = & 1+2+5+7+10+14+35 & = & 74 & > & 70 \\
s(71) & & & = & 1 & < & 71 \\
s(72) & > & 36+24+18 & = & 78 & > & 72 \\
s(73) & & & = & 1 & < & 73 \\
s(74) & = & 1+2+37 & = & 40 & < & 74 \\
s(75) & = & 1+3+5+15+25 & = & 44 & < & 75 \\
s(76) & = & 1+2+4+19+38 & = & 64 & < & 76 \\
s(77) & = & 1+7+11 & = & 19 & < & 77 \\
s(78) & > & 39+26+13+6 & = & 84 & > & 78 \\
s(79) & & & = & 1 & < & 79 \\
s(80) & > & 40+20+16+10 & = & 86 & > & 80
\end{array}$$

The abundant numbers between 70 and 80 inclusive are $\boxed{70}$, $\boxed{72}$, $\boxed{78}$, and $\boxed{80}$.

6.21 In an even three-digit palindrome, the first and the last digit are the same even nonzero digit (the palindrome cannot start with 0). There are 4 such digits. For each of these 4 digits, the middle digit can be any of the 10 digits from 0 to 9 inclusive. The total number of even, three-digit palindromes is $4 \cdot 10 = \boxed{40}$.

6.22 A five-digit palindrome has digits in the form *abcba*. Since the first digit cannot be 0, there are 9 choices for a. There are 10 choices for each of b and c. Each different choice of a, b, and c creates a distinct five-digit palindrome, so there are a total of $9 \cdot 10 \cdot 10 = \boxed{900}$ of them.

6.23 In Problem 6.5.4, we saw that 66 is the smallest abundant palindrome. In Problem 6.20, we saw that 77 is deficient, so we test 88:

$$s(88) = 44 + 22 + 11 + 8 + 4 + 2 + 1 = 92 > 88,$$

So $\boxed{88}$ is the second smallest abundant palindrome.

Challenge Problems

6.24 Because of the symmetric nature of the number 11, it is a divisor of many palindromes. Expanding powers of $x+y$ (where $x = 10$ and $y = 1$) helps us see why the first few powers of 11 are all palindromes:

$$\begin{array}{rcl} (x+y)^2 &=& x^2+2xy+y^2 \\ (x+y)^3 &=& x^3+3x^2y+3xy^2+y^3 \\ (x+y)^4 &=& x^4+4x^3y+6x^2y^2+4xy^3+y^4 \\ (x+y)^5 &=& x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5 \end{array} \qquad \begin{array}{rcr} 11^2 &=& 121 \\ 11^3 &=& 1331 \\ 11^4 &=& 14641 \\ 11^5 &=& 161051 \end{array}$$

We see that $\boxed{161051}$ is the smallest perfect power of 11 that is not a palindrome due to the carrying that must occur when some of the coefficients in the binomial expansion on the left reach 10.

6.25 First, we note that since n has exactly 4 positive divisors that its prime factorization is either in the form a^3 or $b^1 \cdot c^1$, where a, b, and c are all prime.

If $n = a^3$, then $a^3 \mid 100!$. Since we are looking for the largest prime a, we are looking for a such that there are three multiples of a from 1 to 100 inclusive. This means $3a \le 100$, so $a \le 33$. The largest possible value of a is 31, yielding $n = 31^3$ as the largest value of n for this case.

If $n = b^1 \cdot c^1$, we are looking for the largest different primes b and c less than 100. They are 89 and 97 (remember, $91 = 7 \cdot 13$ is composite). The largest possible value of n in this case is $89 \cdot 97$.

Comparing the values of n from the two cases we see that

$$31^3 = 29791 > 8633 = 89 \cdot 97,$$

so $\boxed{29791}$ is the largest integer with exactly 4 positive divisors that divides $100!$.

6.26 Since $100^{100} = (10^2)^{100} = 10^{200}$, we know that 100^{100} expanded has 201 digits: a 1 followed by 200 0's.

Now we count the number of 0's in which $100!$ terminates. There are more powers of 2 than 5 in the prime factorization of $100!$, so in order to count the powers of 10 (total number of terminal 0's), we count the powers of 5:

$$\frac{100}{5} = 20 \qquad \frac{20}{5} = 4$$

There are 20 multiples of 5 from 1 to 100 inclusive, of which 4 are multiples of 5^2, so there are $20+4 = 24$ terminating 0's in $100!$.

When we subtract $100^{100} - 100!$, we note that both integers are multiples of 10^{24}, while only the first is a multiple of 10^{25}. This means the difference is a multiple of 10^{24}, but will leave a positive remainder when divided by 10^{25}. This means the difference ends in exactly $\boxed{24}$ 0's.

6.27

$$3! \cdot 5! \cdot 7! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$$

Any positive divisor of this product must be in the form $2^a \cdot 3^b \cdot 5^c \cdot 7^d$ for nonnegative integers a, b, c, and d where $0 \le a \le 8$, $0 \le b \le 4$, $0 \le c \le 2$, and $0 \le d \le 1$. Since the exponents in the prime factorization of a perfect cube must be multiples of 3, there are 3 choices for the value of a (0, 3, and 6), there are 2 choices for the value of b (0 and 3), and 1 choice for each of the values of c and d (just 0). This gives the total number of cubes as $3 \cdot 2 \cdot 1 \cdot 1 = \boxed{6}$.

6.28 The only proper divisor of a prime number p is 1, so $s(p) = 1 < p$, meaning every prime is deficient.

6.29 Since $(n+1)! = (n+1)n!$ for any positive integer n, we can relate nearby factorials without a lot of computation:

$$\begin{aligned} 20! &= 20 \cdot 19! &= 20C \\ 21! &= 21 \cdot 20! &= 420C \\ 21! - 20! &= 420C - 20C &= \boxed{400C} \end{aligned}$$

6.30 Five-digit palindromes are in the form *abcba*, while six-digit palindromes are in the form *abccba*. In either case, there are 9 possibilities for a (1 through 9), and 10 possibilities for each b and c, for a total of $9 \cdot 10 \cdot 10 = 900$ five-digit palindromes and 900 six-digit palindromes. *Neither total is greater*.

6.31 First, note the formula for the sum of the first n positive integers:

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

We are looking for values of n up to 24 such that this quantity divides $n!$. This happens when we can reduce the following fraction to an integer:

$$\frac{n!}{\frac{n(n+1)}{2}} = \frac{2n!}{n(n+1)} = \frac{2n(n-1)!}{n(n+1)} = \frac{2(n-1)!}{n+1}.$$

We are looking for values of n such that $(n+1) \mid 2(n-1)!$. Let's reduce a few of the fractions to lowest terms to see what's going on. Obviously $n = 1$ works, so we'll go up from there:

$$\begin{aligned}
n = 2 &\Rightarrow \frac{2(n-1)!}{n+1} = \frac{2 \cdot 1!}{3} = \frac{2}{3} \\
n = 3 &\Rightarrow \frac{2(n-1)!}{n+1} = \frac{2 \cdot 2!}{4} = 1 \\
n = 4 &\Rightarrow \frac{2(n-1)!}{n+1} = \frac{2 \cdot 3!}{5} = \frac{12}{5} \\
n = 5 &\Rightarrow \frac{2(n-1)!}{n+1} = \frac{2 \cdot 4!}{6} = 8 \\
n = 6 &\Rightarrow \frac{2(n-1)!}{n+1} = \frac{2 \cdot 5!}{7} = \frac{240}{7} \\
n = 7 &\Rightarrow \frac{2(n-1)!}{n+1} = \frac{2 \cdot 6!}{8} = 180 \\
n = 8 &\Rightarrow \frac{2(n-1)!}{n+1} = \frac{2 \cdot 7!}{9} = 1120
\end{aligned}$$

The non-integer results occur when $(n+1)$ has a prime factorization that includes a power of a prime not in the product $2(n-1)!$. If $(n+1)$ is composite, it can be broken down into a product of two smaller positive integers, both of which are in the product of $2(n-1)!$. Fractions therefore occur when $(n+1)$ is prime, but the prime must be greater than 2, since $2 \mid 2(n-1)!$. The possible values of $(n+1)$ (where $1 \le n \le 24$) are 3, 5, 7, 11, 13, 17, 19, and 23, so there are 8 values of n for which $n!$ is not divisible by the sum of the positive integers up to n, leaving $24 - 8 = \boxed{16}$ values of n for which it is.

6.32 The only prime that is a multiple of 3 is 3. This means that any pair of twin primes other than 3 and 5 must be two integers that are 1 more and 1 less than a multiple of 3. If the smaller one is 1 more than a multiple of 3, then the larger one is a multiple of 3, which is impossible. This means that a pair of twin primes greater than 3 and 5 must be in the form $3n-1$ and $3n+1$ for some positive integer n. Their sum is $6n$, and $3 \mid 6n$, so the sum of any pair of twin primes other than 3 and 5 is a multiple of 3.

6.33 As we look for clues that relate the sum of two three-digit palindromes to a four-digit palindrome, we note that since the three-digit palindromes are both less than 1000, their sum is less than 2000. This means the four-digit palindrome is in the form $1aa1$ for some digit a. Now, we have a better view of the addition of two three-digit palindromes, bcb and ded:

$$\begin{array}{r} bcb \\ +\ ded \\ \hline 1aa1 \end{array}$$

Digits b and d are lead digits of palindromes, so both are greater than 0 and their sum is greater than 1. Their sum must also be no greater than 18. From the units digit of the sum of the two three-digit palindromes, we see that $b+d$ has a units digit of 1, so $b+d=11$. Now can now examine the sum a little more completely:

$$\begin{aligned} bcb + ded &= (100b + 10c + b) + (100d + 10e + d) \\ &= 101b + 10c + 101d + 10e \\ &= 101(b+d) + 10(c+e) \\ &= 101(11) + 10(c+e) \\ &= 1111 + 10(c+e) \end{aligned}$$

Since $0 \le c+e \le 18$, the value of $10(c+e)$ will only affect the two middle digits of the sum. Here are some possibilities:

$$\begin{array}{rclcrcl} c+e &=& 0 & \Rightarrow & 1111+10(c+e) &=& 1111 \\ c+e &=& 1 & \Rightarrow & 1111+10(c+e) &=& 1121 \\ c+e &=& 2 & \Rightarrow & 1111+10(c+e) &=& 1131 \\ & \vdots & & \Rightarrow & & & \vdots \\ c+e &=& 17 & \Rightarrow & 1111+10(c+e) &=& 1281 \\ c+e &=& 18 & \Rightarrow & 1111+10(c+e) &=& 1291 \end{array}$$

In fact, since 1111 is a palindrome, $c+e$ must be either 0 or 11 in order for the sum to be a palindrome. Since we want the sum to be largest, we let $c+e=11$, giving us the largest possible four-digit palindrome that is the sum of two three-digit palindromes:

$$1aa1 = 1111 + 10(c+e) = 1111 + 10 \cdot 11 = 1111 + 110 = \boxed{1221}.$$

Depending on our choices for b, c, d, and e, we can produce this sum in many different ways. Here are a few:

$$\begin{aligned} 232 + 989 &= 1221 \\ 242 + 979 &= 1221 \\ 525 + 696 &= 1221 \\ 727 + 494 &= 1221 \end{aligned}$$

6.34 A ten-digit palindrome has digits in the form $abcdeedcba$. There are 9 possibilities for a (1 through 9), since the ten-digit number cannot start with zero, and 10 possibilities each for b, c, d, and e (0 through 9). So, the total number of ten-digit palindromes is $9 \cdot 10^4 = \boxed{90000}$.

6.35 Since $8^n = (2^3)^n = 2^{3n}$, we can solve the problem by first finding the largest power of 2 that divides 100!:

$$\begin{aligned} \tfrac{100}{2} &= 50 & \tfrac{50}{2} &= 25 & \tfrac{25}{2} &= 12.5 \\ \tfrac{12}{2} &= 6 & \tfrac{6}{2} &= 3 & \tfrac{3}{2} &= 1.5 \end{aligned}$$

There are $50 + 25 + 12 + 6 + 3 + 1 = 97$ factors of 2 in 100!. The largest power of 8 corresponds to a power of 2 that is a multiple of 3, which 97 is not. Since $2^{96} = 8^{32}$ is the highest power of 8 that divides 100!, the answer is $n = \boxed{32}$.

6.36 $T(n)$ is the number of terminal zeros of $n!$, so $10^{T(n)} \mid n!$. This means $n!$ has at least $T(n)$ factors of 2 and $T(n)$ factors of 5. Since $n!$ always has at least as many factors of 2 than 5, $n!$ has exactly $T(n)$ factors of 5.

(a) If $T(n) > T(n-1)$, $n!$ must have at least one more factor of 5 than $(n-1)!$, which means n must be a multiple of 5.

(b) If $T(n) > T(n-1) + 1$, $n!$ must have at least two more factors of 5 than $(n-1)!$, which means n must be a multiple of 25.

(c) If $T(n) > T(n-1) + 2$, $n!$ must have at least three more factors of 5 than $(n-1)!$, which means n must be a multiple of 125.

(d) If there are no values of n such that $T(n) = m$ or $T(n) = m + 1$, then there is some value of n such that $T(n-1) < m$ and $T(n) > m + 1$. This means that $T(n) > T(n-1) + 2$. From (c) we know that n must be a multiple of 125. Let's take a look at the values of $T(n-1)$ and $T(n)$ where n is one of the smallest positive multiples of 125:

$$\begin{aligned} T(124) &= 28 & T(125) &= 31 \\ T(249) &= 59 & T(250) &= 62 \\ T(374) &= 90 & T(375) &= 93 \\ T(499) &= 121 & T(500) &= 124 \end{aligned}$$

The only positive integers m less than 100 for which neither m nor $m + 1$ are ever the value of $T(n)$ are $\boxed{29}$, $\boxed{60}$, and $\boxed{91}$.

6.37 We can rewrite the given product as the quotient of two factorials:

$$115 \times 116 \times 117 \times \cdots \times 201 = \frac{201!}{114!}.$$

The number of trailing 0's in the quotient is the number of factors of 5 in the quotient (since the number of factors of 2 is much larger). The number of factors of 5 in the quotient is the number of factors of 5 in 201! minus the number of factors of 5 in 114!.

$$\frac{200}{5} = 40 \qquad \frac{40}{5} = 8 \qquad \frac{8}{5} = 1.6$$

The number of factors of 5 in 201! is $40 + 8 + 1 = 49$.

$$\frac{114}{5} = 22.8 \qquad \frac{22}{5} = 4.4$$

The number of factors of 5 in 114! is $22 + 4 = 26$. This means the total number of trailing 0's in the original product is $49 - 26 = \boxed{23}$.

6.38 Let p^n be a power of a prime. The proper divisors of p^n are $1, p, p^2, \dots, p^{n-1}$, so

$$s(p^n) = p^{n-1} + \cdots + p + 1,$$

which is the sum of a geometric series:

$$s(p^n) = p^{n-1} + \cdots + p + 1 = \frac{p^n - 1}{p - 1} \leq p^n - 1 < p^n,$$

so p^n is deficient.

6.39

(a) Looking at the two smallest five-digit palindromes, 10001 and 10101, we see that

$$\gcd(10001, 10101) = \gcd(10001, 100) = \gcd(1, 100) = \gcd(1, 1) = 1,$$

so the smallest possible GCD of five digit palindromes is $\boxed{1}$.

(b) We could hone in on the answer in a similar way as we did in part (a), but we can use algebraic factorization to help us. Any six-digit palindrome is in the form *abccba*, which can be expressed as

$$100001a + 10010b + 1100c = 11(9091x + 910y + 100z).$$

So, all six-digit palindromes are multiples of 11, which means the GCD of any pair of them is at least 11. Checking the GCD of the smallest two six-digit palindromes,

$$\begin{aligned} \gcd(100001, 101101) &= \gcd(100001, 1100) \\ &= \gcd(1001, 1100) \\ &= \gcd(1001, 99) \\ &= \gcd(11, 99) \\ &= \gcd(11, 11) = 11, \end{aligned}$$

so we can be certain that the smallest possible GCD is $\boxed{11}$.

6.40

$$\frac{a!}{b!} = a \cdot (a-1) \cdots (b+1).$$

The number of integers in the product on the right is $a-b$, so we are really looking for the largest possible number of integers in the product $a \cdot (a-1) \cdots (b+1)$.

Every other integer is a multiple of 2, and every other even integer is a multiple of 4. That means that if $a \cdot (a-1) \cdots (b+1)$ is the product of four or more integers, it is the product of two even integers, one of which is a multiple of 4. The product would have to be a multiple of $2 \cdot 4 = 8$. This means that if the product is a multiple of 4, but not 8, there are at most 3 integers from $(b+1)$ to a inclusive. In fact, $\frac{5!}{2!} = 60$, which is a multiple of 4, but not 8, so $\boxed{3}$ is the largest possible value of $a-b$.

6.41 If n is perfect or abundant, then $s(n) \geq n$. Let $d_1, d_2, d_3, \ldots, d_k$ be the proper divisors of n, so

$$s(n) = d_1 + d_2 + \cdots + d_k \geq n.$$

Additionally, since $d_m \mid n$ for $1 \leq m \leq k$, we know that $ad_m \mid an$ for any positive integer $a > 1$. So, some of the proper divisors of an are a times the proper divisors of n: $ad_1, ad_2, ad_3, \ldots, ad_k$. Note that since $a > 1$, each of these proper divisors of an is greater than 1, so 1 is an additional proper divisor of an, thus

$$s(an) \geq 1 + ad_1 + ad_2 + \cdots + ad_k = 1 + a(d_1 + d_2 + \cdots + d_k) = 1 + as(n) > an,$$

so an is abundant, where an is any multiple of a perfect or abundant number n that is greater than n.

Note that this theorem can be useful when evaluating whether or not certain numbers are abundant. For instance, we can quickly determine that 102 is abundant by noting that one of its divisors, 6, is a perfect number.

6.42 Since $12^n = 2^{2n} \cdot 3^n$, we are looking for the largest value of n such that 2^{2n} and 3^n are divisors of 20!.

$$\frac{20}{2} = 10 \qquad \frac{10}{2} = 5 \qquad \frac{5}{2} = 2.5 \qquad \frac{2}{2} = 1$$

The largest power of 2 that divides 20! is $2^{(10+5+2+1)} = 2^{18}$.

$$\frac{20}{3} = 6\frac{2}{3} \qquad \frac{6}{3} = 2$$

The largest power of 3 that divides 20! is $3^{(6+2)} = 3^8$. Since there are 18 powers of 2 and 8 powers of 3 in 20!, we want the largest value of n such that $2n \leq 18$ and $n \leq 8$, so $\boxed{8}$ is the answer and 12^8 is the largest power of 12 that divides 20!.

6.43 It's easier to work with integers than fractions, so we multiply through by an integer that leaves us with only integers. In this case, multiplying by 7! does the trick:

$$5 \cdot 6! = (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3)a_2 + (7 \cdot 6 \cdot 5 \cdot 4)a_3 + (7 \cdot 6 \cdot 5)a_4 + (7 \cdot 6)a_5 + 7a_6 + a_7.$$

Looking for some clue as to how to use this equation, we note that the only part of the sum on the right side that is not a multiple of 7 is a_7. This means that a_7 must be the remainder when the left-hand side is divided by 7. Since $5 \cdot 6! = 3600$ and $3600 = 7 \cdot 514 + 2$, we know $a_7 = 2$.

Subtracting the value of a_7 out of the equation, we have

$$3598 = (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3)a_2 + (7 \cdot 6 \cdot 5 \cdot 4)a_3 + (7 \cdot 6 \cdot 5)a_4 + (7 \cdot 6)a_5 + 7a_6.$$

Now we divide both sides by 7 to get a simpler equation:

$$514 = (6 \cdot 5 \cdot 4 \cdot 3)a_2 + (6 \cdot 5 \cdot 4)a_3 + (6 \cdot 5)a_4 + 6a_5 + a_6.$$

Similarly as before we note that everything on the right-hand side of the equation is a multiple of 6 except a_6, so a_6 is equal to the remainder when 514 is divided by 6, which is 4.

Subtracting $a_6 = 4$ out of the equation, we have

$$510 = (6 \cdot 5 \cdot 4 \cdot 3)a_2 + (6 \cdot 5 \cdot 4)a_3 + (6 \cdot 5)a_4 + 6a_5.$$

Dividing everything by 6 we get

$$85 = (5 \cdot 4 \cdot 3)a_2 + (5 \cdot 4)a_3 + 5a_4 + a_5.$$

Now we see that a_5 is the remainder when 85 is divided by 5, so $a_5 = 0$.

Now we can get rid of a_5 and divide everything else by 5 to get

$$17 = (4 \cdot 3)a_2 + 4a_3 + a_4.$$

We see that $a_4 = 1$ is the remainder when 17 is divided by 4.

Subtracting $a_4 = 1$ from both sides of the equation and dividing by 4 we get $4 = 3a_2 + a_3$. At this point, using similar ideas as before, we see that $a_2 = a_3 = 1$, so the answer is

$$1 + 1 + 1 + 0 + 4 + 2 = \boxed{9}.$$

The two main keys behind this solution are first getting rid of the fractions to get a nicer look at the equation and then representing the coefficients in that new equation in terms of products.

CHAPTER 7

Algebra With Integers

Challenge Problems

7.10 Perfect squares greater than 0 have only even exponents in their prime factorizations. Integers that are divisible by 8 include 2 to *at least* the third power in their prime factorizations. But, since the exponents must all be even, squares that are divisible by $2^3 = 8$ must also be divisible by $2^4 = 16$.

Now, factoring 16 out of the prime factorizations of such squares, we can write them all in the form $16n^2$ where n is a positive integer. Now, we need to count the values of n (one for each perfect square that is divisible by 8) such that

$$0 \le 16n^2 < 500.$$

Dividing by 16 we have

$$0 \le n^2 < 31.25.$$

When we take the square root of all parts of this inequality, the upper boundary for n will be irrational, but since $5^2 < 31.25 < 6^2$, we can still write the boundaries for n easily as $0 \le n \le 5$. There are $\boxed{6}$ perfect squares less than 500 that are multiples of 8:

$$\begin{array}{lcl} n = 0 & \Rightarrow & 16n^2 = 0 \\ n = 1 & \Rightarrow & 16n^2 = 16 \\ n = 2 & \Rightarrow & 16n^2 = 64 \\ n = 3 & \Rightarrow & 16n^2 = 144 \\ n = 4 & \Rightarrow & 16n^2 = 256 \\ n = 5 & \Rightarrow & 16n^2 = 400 \end{array}$$

7.11 Let the number be AB where A is the tens digit and B is the units digit. We are given that

$$10A + B = 4(A + B).$$

Grouping like terms we have $6A = 3B$, so $2A = B$. We are looking for two-digit integers whose units digits are twice their tens digits. Those are $\boxed{12}$, $\boxed{24}$, $\boxed{36}$, and $\boxed{48}$.

7.12 We are trying to determine when $n^2 - 3n + 2$ is prime. Since primality is based on the existence of factors between 1 and a quantity, it seems natural to look for an algebraic factorization: $n^2 - 3n + 2 = (n-1)(n-2)$. A prime number has only one divisor greater than 1, so the smaller of the two factors $n - 1$ and $n - 2$ must be 1. Thus, $n - 2 = 1$ and $n = 3$ is the only solution and the answer is $\boxed{1}$.

7.13 Let the three two-digit integers we are looking for be ab, cd, and ef (where a, b, c, d, e, and f are the values of the digits). Since $3^2 + 4^2 + 5^2 = 50$, we have

$$\begin{aligned} a^2 + b^2 &= 50 \\ c^2 + d^2 &= 50 \\ e^2 + f^2 &= 50 \end{aligned}$$

Now we can simply check for perfect squares less than 50 and see which we can piece together to sum to 50. We find that $1^2 + 7^2 = 1 + 49 = 50$ and $5^2 + 5^2 = 25 + 25 = 50$. From these squares we construct the two-digit numbers 17, 55, and 71. Their sum is $17 + 55 + 71 = \boxed{143}$.

7.14 An integer that is a multiple of 15, 20, and 25 must be a multiple of their LCM, which is 300. We can express such integers in the form $300n$ for integer values of n. Our goal is to count the integers in this form such that

$$1000 \le 300n < 10000.$$

Dividing each part of the inequality by 300 isolates n:

$$\frac{10}{3} \le n < \frac{100}{3},$$

so $4 \le n \le 33$. There are $33 - 4 + 1 = 30$ integer values of n in this range, each of which corresponds to a four-digit multiple of each of 15, 20, and 25, so $\boxed{30}$ is the answer.

7.15 Let q, d, and n represent the numbers of quarters, dimes, and nickels in John's jar, respectively. We express d and n in terms of q to reduce the number of variables we have to work with:

$$\begin{aligned} d &= \frac{5}{8}q \\ n &= \frac{11}{6}d = \frac{11}{6}\left(\frac{5}{8}q\right) = \frac{55}{48}q \end{aligned}$$

The quantities q, $\frac{5}{8}q$, and $\frac{55}{48}q$ must all be integers, so q must be a multiple of the LCM of the denominators of all the fractions, which is 48. This means $q = 48n$ for some positive integer n. This means there are $48n$ quarters, $30n$ dimes, and $55n$ nickels and a total of $48n + 30n + 55n = 133n$ coins in John's jar. We are given that

$$500 < 133n < 600.$$

We could divide everything in the inequality by 133, but it's just as easy to hunt-and-check to find that $n = 4$ results in 532 coins in the jar. There are $4 \cdot 48 = 192$ quarters, $4 \cdot 30 = 120$ dimes, and $4 \cdot 55 = 220$ nickels, so the total amount of money in John's jar of coins is

$$192(\$0.25) + 120(\$0.10) + 220(\$0.05) = \$48.00 + \$12.00 + \$11.00 = \boxed{\$71.00}.$$

7.16 Let's take a look at what happens when we subtract the sum of the digits of a given two-digit number AB (where A and B are digits) from the number itself:

$$(10A + B) - (A + B) = 9A.$$

We are given that this difference has a units digit of 6. The value of B doesn't matter at all as it cancels out in subtraction. Now we check the possible values of A and see which produces a units digit of 6 when multiplied by 9:

$$\begin{aligned} 9(0) &= 0 \\ 9(1) &= 9 \\ 9(2) &= 18 \\ 9(3) &= 27 \\ 9(4) &= 36 \\ 9(5) &= 45 \\ 9(6) &= 54 \\ 9(7) &= 63 \\ 9(8) &= 72 \\ 9(9) &= 81 \end{aligned}$$

The tens digit must be 4 and any units digit will do, so there are $\boxed{10}$ two-digit integers with this property.

7.17 We are looking for values of n that are at least 3 (because a polygon must have at least 3 sides), such that

$$180 - \frac{360}{n}$$

is an integer. If this quantity is an integer, then 180 less than this quantity is an integer, so we are looking for values of n such that $\frac{360}{n}$ is an integer. In other words, we need to count the divisors of $360 = 2^3 \cdot 3^2 \cdot 5^1$. There are $(3+1)(2+1)(1+1) = 24$ of them, but two of them are less than 3, so there are $\boxed{22}$ regular polygons with angles of integer degree measures.

7.18 We know that for a and b and some positive integers m and n that

$$\begin{aligned} a+b &= m^2 \\ a-b &= n^2 \end{aligned}$$

Our goal is to find the smallest possible value of a, so it might help to find an expression for a. Adding the two equations together and dividing by 2 we get

$$a = \frac{m^2+n^2}{2}.$$

We can start plugging in the smallest possible (different) values for m and n, but it helps to note that m and n must either both be even or both be odd if the average of their squares is an integer. The smallest possible Benson rectangular number is $\frac{1^2+3^2}{2} = \frac{10}{2} = \boxed{5}$.

7.19 It's easier to work without fractions, so we multiply both sides of the equation by $6xy$ to get

$$3y + 2x = xy.$$

Now we reorganize the equation in order to apply Simon's Favorite Factoring Trick:

$$xy - 2x - 3y = 0 \Rightarrow xy - 2x - 3y + 6 = 6 \Rightarrow (x-3)(y-2) = 6.$$

Since $(x-3) \mid 6$, we get the possible values of x by solving for equations made by equating $x-3$ with various divisors of 6. Noting that $x > 1$, meaning $x - 3 > -2$, cuts down on the need to solve for all negative factors of 6 (as the right-hand side). Note that we solve for y after plugging back in each value of x:

$$\begin{array}{rclcrclcl}
x-3 & = & -2 & \Rightarrow & x & = & 1 & \Rightarrow & y=-1<0, \text{ no solution} \\
x-3 & = & -1 & \Rightarrow & x & = & 2 & \Rightarrow & y=-4<0, \text{ no solution} \\
x-3 & = & 1 & \Rightarrow & x & = & 4 & \Rightarrow & \boxed{(4,8)} \\
x-3 & = & 2 & \Rightarrow & x & = & 5 & \Rightarrow & \boxed{(5,5)} \\
x-3 & = & 3 & \Rightarrow & x & = & 6 & \Rightarrow & \boxed{(6,4)} \\
x-3 & = & 6 & \Rightarrow & x & = & 9 & \Rightarrow & \boxed{(9,3)}
\end{array}$$

Make sure you see why we could just as easily solved for y first.

7.20 For any decimal digit d, we have

$$dddddd = d \cdot 111111 = d \cdot 3^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 37^1.$$

The largest prime divisor common to all such integers is thus $\boxed{37}$.

7.21 It is natural for students who don't see the answer right away to explore a bit, so before we get into the solution, let's take a look at some examples:

$$\begin{array}{rclcl}
3 \cdot 5 & = & 15 & = & 4^2 - 1 \\
5 \cdot 7 & = & 35 & = & 6^2 - 1 \\
11 \cdot 13 & = & 143 & = & 12^2 - 1 \\
17 \cdot 19 & = & 323 & = & 18^2 - 1 \\
29 \cdot 31 & = & 899 & = & 30^2 - 1
\end{array}$$

We see that the squares involved are the squares of the integers right in between each pair of twin primes. If we write each prime from a pair of twin primes in terms of n, where n is the integer between them, then we have

$$(n-1)(n+1) + 1 = (n^2 - 1) + 1 = n^2,$$

which is exactly what we wanted to show.

7.22 We apply the Division Theorem to this problem. We are told that when x is divided by y that the quotient is u and the remainder is v:

$$x = yu + v.$$

We want to know the remainder when $x + 2uy$ is divided by y. We can rewrite $x + 2uy$ as

$$yu + v + 2uy = 3yu + v = y(3u) + v.$$

The new quotient when dividing by y is $3u$ and the remainder is once again $\boxed{v}$.

7.23 A perfect square that is a multiple of both 3 and 4 must have exponents of at least 2 for prime factors 2 and 3, so it must be a multiple of $2^2 \cdot 3^2 = 36$. These integers can be written in the form $36n^2$ for integers n. Since we are looking for three-digit integers, we have the inequality

$$100 \le 36n^2 < 1000.$$

Dividing through by 36 we have

$$\frac{25}{9} \le n^2 < \frac{250}{9} \qquad \text{so} \qquad 2 < n^2 < 28.$$

Taking the square roots of each part of this inequality we see that

$$1 < \sqrt{2} < 2 \le n \le 5 < \sqrt{28} < 6,$$

so there are $5 - 2 + 1 = 4$ possible values of n corresponding to $\boxed{4}$ three-digit perfect squares that are multiples of both 3 and 4.

7.24 The first thing we do is change the word problem into a math problem. Let A be the number of pigs the first farmer gives to the second and let B denote the number of goats the first farmer gives to the second. Let n be the total amount of debt settled in the transaction. Note that either A or B can be a negative integer. This would represent the first farmer receiving animals. Our equation is

$$300A + 210B = n.$$

Since $\gcd(300, 210) = 30$, we factor the left-hand side of the equation:

$$30(10A + 7B) = n.$$

Our goal is to find the smallest possible positive value of n. We see that n must be a multiple of 30, so we start by seeing if we can find (A, B) such that $30(10A + 7B) = 30$. Dividing by 30 we get $10A + 7B = 1$. After plugging in a few values we note that when $A = -2$ and $B = 3$ that $10(-2) + 7(3) = 1$. So, $\boxed{\$30}$ is in fact our answer.

7.25

We start with 100 tiles and remove 10, so 90 remain. Since $9^2 < 90 < 10^2$, we next remove 9 tiles, so 81 remain. As we continue, a nice pattern emerges.

# of Operations Performed	# of Tiles Remaining	
0	100	$= 10^2$
1	90	$= 9 \cdot 10$
2	81	$= 9^2$
3	72	$= 8 \cdot 9$
4	64	$= 8^2$
5	56	$= 7 \cdot 8$
6	49	$= 7^2$
		$\vdots$

We could trust the pattern and simply count the operations until we get down to $1 = 1^2$, but let's take a look at why the pattern works. For a positive integer m, the number of perfect squares $(1^2, 2^2, \ldots, m^2)$ between 1 and m^2 inclusive is m. Removing m tiles with squares leaves $m^2 - m$ tiles remaining.

Counting the perfect squares from 1 to $m^2 - m$ is a little more difficult, but we note that

$$(m-1)^2 < m(m-1) = m^2 - m < m^2.$$

There are $m - 1$ perfect squares up to $m^2 - m$. Subtracting $m - 1$ tiles leaves

$$m^2 - m - (m-1) = m^2 - 2m + 1 = (m-1)^2$$

tiles remaining.

Now we see that the number of tiles remaining alternates between perfect squares and the products of consecutive integers between those squares. Going from 10^2 tiles to 1^2 tile, we go down 9 perfect squares, requiring a total of $9 + 9 = \boxed{18}$ operations.

7.26 We are given several pieces of information to work with and we'll have to combine them. We are looking for even integers, so they must have positive powers of 2 in their prime factorizations. The numbers are also squares and cubes, so the exponents in their prime factorizations must be multiples of 2 and 3, and therefore 6. We are looking for integers in the form $2^6n^6 = 64n^6$ that have four digits:

$$1000 \le 64n^6 < 10000.$$

Dividing everything through by 64 we have

$$15.625 \le n^6 < 156.25.$$

It's easy to hunt for integers n that have sixth powers in this range: $1^6 = 1$, $2^6 = 64$, $3^6 = 729$. Larger values of n will not be in the range, so there is only $\boxed{1}$ even four-digit integer that is both a square and a cube, that integer being $64 \cdot 2^6 = 64 \cdot 64 = 4096$.

7.27 This one turns out to be less difficult than it initially looks. Since we are dealing with products of integers, we take a second look at the equation where the right-hand side is prime factored:

$$(100A + 10M + C)(A + M + C) = 5^1 \cdot 401^1.$$

The left-hand side is the product of a three-digit integer and the sum of its digits. The only possible digit sums are 1 and 5, but 1 doesn't work. The three-digit integer must be 401, so $A = \boxed{4}$.

7.28 We are given that $a^2 - b^2 = p$ for some positive integers a and b and some prime number p. We can factor the difference of squares:

$$(a+b)(a-b) = p.$$

Since p has no divisors between 1 and itself, $a - b$ must be equal to 1. This means that

$$(a+b)(a-b) = a + b = p,$$

so the sum of a and b is prime.

CHAPTER 8

Base Number Systems

Exercises for Section 8.2

8.2.1

(a) 258
(b) 50917
(c) 251
(d) 178

Exercises for Section 8.3

8.3.1

(a) 43_8
(b) 310_8
(c) 7601_8
(d) 500200_8

8.3.2

(a) 302_4
(b) 11231_4
(c) 313313_4
(d) 20101303_4

8.3.3

1_4	2_4	3_4	10_4
11_4	12_4	13_4	20_4
21_4	22_4	23_4	30_4
31_4	32_4	33_4	100_4

8.3.4

1_5	2_5	3_5	4_5	10_5
11_5	12_5	13_5	14_5	20_5
21_5	22_5	23_5	24_5	30_5
31_5	32_5	33_5	34_5	40_5
41_5	42_5	43_5	44_5	100_5
101_5	102_5	103_5	104_5	110_5
111_5	112_5	113_5	114_5	120_5
121_5				

8.3.5

1_6	2_6	3_6	4_6	5_6	10_6
11_6	12_6	13_6	14_6	15_6	20_6
21_6	22_6	23_6	24_6	25_6	30_6
31_6	32_6	33_6	34_6	35_6	40_6
41_6	42_6	43_6	44_6	45_6	50_6
51_6	52_6	53_6	54_6	55_6	100_6
101_6	102_6	103_6	104_6	105_6	110_6
111_6	112_6	113_6	114_6	115_6	120_6
121_6					

8.3.6

(a) 22_3	(d) 110_3	(g) 222_3
(b) 100_3	(e) 111_3	(h) 1000_3
(c) 101_3	(f) 112_3	(i) 1101_3

Exercises for Section 8.4

8.4.1

1_{13}	2_{13}	3_{13}	4_{13}	5_{13}	6_{13}	7_{13}	8_{13}	9_{13}	A_{13}	B_{13}	C_{13}	10_{13}
11_{13}	12_{13}	13_{13}	14_{13}	15_{13}	16_{13}	17_{13}	18_{13}	19_{13}	$1A_{13}$	$1B_{13}$	$1C_{13}$	20_{13}

8.4.2 The largest three-digit base-14 integer is 1 less than the smallest four-digit base-14 integer, which is

$$1000_{14} = 1 \cdot 14^3 = 2744.$$

Thus, the largest three-digit base-14 integer is $2744 - 1 = \boxed{2743}$.

Exercises for Section 8.5

8.5.1

(a) $10101_2 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 16 + 4 + 1 = \boxed{21}$.

(b) $10101_3 = 1 \cdot 3^4 + 0 \cdot 3^3 + 1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 = 81 + 9 + 1 = \boxed{91}$.

(c) $431_5 = 4 \cdot 5^2 + 3 \cdot 5^1 + 1 \cdot 5^0 = 100 + 15 + 1 = \boxed{116}$.

(d) $1312_6 = 1 \cdot 6^3 + 3 \cdot 6^2 + 1 \cdot 6^1 + 2 \cdot 6^0 = 216 + 108 + 6 + 2 = \boxed{332}$.

(e) $3206_7 = 3 \cdot 7^3 + 2 \cdot 7^2 + 0 \cdot 7^1 + 6 \cdot 7^0 = 1029 + 98 + 6 = \boxed{1133}$.

(f) $1A5_{13} = 1 \cdot 13^2 + 10 \cdot 13^1 + 5 \cdot 13^0 = 169 + 130 + 5 = \boxed{304}$.

8.5.2 In these solutions, we keep taking out the largest possible base digit bundles from 1407, converting it to each number base.

(a) $2^{10} < 1407 < 2^{11}$
$1407 = 1 \cdot 2^{10} + 383$
$1407 = 1 \cdot 2^{10} + 1 \cdot 2^8 + 127$
$1407 = 1 \cdot 2^{10} + 1 \cdot 2^8 + 1 \cdot 2^6 + 63$
$1407 = 1 \cdot 2^{10} + 1 \cdot 2^8 + 1 \cdot 2^6 + 1 \cdot 2^5 + 31$
$1407 = 1 \cdot 2^{10} + 1 \cdot 2^8 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 15$
$1407 = 1 \cdot 2^{10} + 1 \cdot 2^8 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 7$
$1407 = 1 \cdot 2^{10} + 1 \cdot 2^8 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 3$
$1407 = 1 \cdot 2^{10} + 1 \cdot 2^8 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1$
$1407 = 1 \cdot 2^{10} + 1 \cdot 2^8 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$
$1407 = \boxed{10101111111_2}$

(b) $3^6 < 1407 < 3^7$
$1407 = 1 \cdot 3^6 + 678$
$1407 = 1 \cdot 3^6 + 2 \cdot 3^5 + 192$
$1407 = 1 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^4 + 30$
$1407 = 1 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^4 + 1 \cdot 3^3 + 3$
$1407 = 1 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^4 + 1 \cdot 3^3 + 1 \cdot 3^1$
$1407 = \boxed{1221010_3}$

(c) $5^4 < 1407 < 5^5$
$1407 = 2 \cdot 5^4 + 157$
$1407 = 2 \cdot 5^4 + 1 \cdot 5^3 + 32$
$1407 = 2 \cdot 5^4 + 1 \cdot 5^3 + 1 \cdot 5^2 + 7$
$1407 = 2 \cdot 5^4 + 1 \cdot 5^3 + 1 \cdot 5^2 + 1 \cdot 5^1 + 2$
$1407 = 2 \cdot 5^4 + 1 \cdot 5^3 + 1 \cdot 5^2 + 1 \cdot 5^1 + 2 \cdot 5^0$
$1407 = \boxed{21112_5}$

(d) $6^4 < 1407 < 6^5$
$1407 = 1 \cdot 6^4 + 111$
$1407 = 1 \cdot 6^4 + 3 \cdot 6^2 + 3$
$1407 = 1 \cdot 6^4 + 3 \cdot 6^2 + 3 \cdot 6^0$
$1407 = \boxed{10303_6}$

(e) $11^3 < 1407 < 11^4$
$1407 = 1 \cdot 11^3 + 76$
$1407 = 1 \cdot 11^3 + 6 \cdot 11^1 + 10$
$1407 = 1 \cdot 11^3 + 6 \cdot 11^1 + 10 \cdot 11^0$
$1407 = \boxed{106A_{11}}$

(f) $37^2 < 1407 < 37^3$
$1407 = 1 \cdot 37^2 + 38$
$1407 = 1 \cdot 37^2 + 1 \cdot 37^1 + 1$
$1407 = 1 \cdot 37^2 + 1 \cdot 37^1 + 1 \cdot 37^0$
$1407 = \boxed{111_{37}}$

8.5.3 In each of the following solutions, we first convert each base number to decimal form and then into another number base.

(a) $23_9 = 2 \cdot 9^1 + 3 \cdot 9^0 = 21$
$2^4 < 21 < 2^5$
$21 = 1 \cdot 2^4 + 5$
$21 = 1 \cdot 2^4 + 1 \cdot 2^2 + 1$
$21 = 1 \cdot 2^4 + 1 \cdot 2^2 + 1 \cdot 2^0$
$21 = \boxed{10101_2}$

(b) $427_8 = 4 \cdot 8^2 + 2 \cdot 8^1 + 7 \cdot 8^0 = 279$
$5^3 < 279 < 5^4$
$279 = 2 \cdot 5^3 + 29$
$279 = 2 \cdot 5^3 + 1 \cdot 5^2 + 4$
$279 = 2 \cdot 5^3 + 1 \cdot 5^2 + 4 \cdot 5^0$
$279 = \boxed{2104_5}$

(c) $253_6 = 2 \cdot 6^2 + 5 \cdot 6^1 + 3 \cdot 6^0 = 105$
$7^2 < 105 < 7^3$
$105 = 2 \cdot 7^2 + 7$
$105 = 2 \cdot 7^2 + 1 \cdot 7^1$
$105 = \boxed{210_7}$

(d) $C10_{14} = 12 \cdot 14^2 + 1 \cdot 14^1 = 2366$
$4^5 < 2366 < 4^6$
$2366 = 2 \cdot 4^5 + 318$
$2366 = 2 \cdot 4^5 + 1 \cdot 4^4 + 62$
$2366 = 2 \cdot 4^5 + 1 \cdot 4^4 + 3 \cdot 4^2 + 14$
$2366 = 2 \cdot 4^5 + 1 \cdot 4^4 + 3 \cdot 4^2 + 3 \cdot 4^1 + 2$
$2366 = 2 \cdot 4^5 + 1 \cdot 4^4 + 3 \cdot 4^2 + 3 \cdot 4^1 + 2 \cdot 4^0$
$2366 = \boxed{210332_4}$

(e) $120221_3 = 430$
$19^2 < 430 < 19^3$
$430 = 1 \cdot 19^2 + 69$
$430 = 1 \cdot 19^2 + 3 \cdot 19^1 + 12$
$430 = 1 \cdot 19^2 + 3 \cdot 19^1 + 12 \cdot 19^0$
$430 = \boxed{13C_{19}}$

(f) $231_7 = 2 \cdot 7^2 + 3 \cdot 7^1 + 1 \cdot 7^0 = 120$
$8^2 < 120 < 8^3$
$120 = 1 \cdot 8^2 + 56$
$120 = 1 \cdot 8^2 + 7 \cdot 8^1$
$120 = \boxed{170_8}$

(g) $345_9 = 3 \cdot 9^2 + 4 \cdot 9^1 + 5 \cdot 9^0 = 284$
$3^5 < 284 < 3^6$
$284 = 1 \cdot 3^5 + 41$
$284 = 1 \cdot 3^5 + 1 \cdot 3^3 + 14$

$$284 = 1 \cdot 3^5 + 1 \cdot 3^3 + 1 \cdot 3^2 + 5$$
$$284 = 1 \cdot 3^5 + 1 \cdot 3^3 + 1 \cdot 3^2 + 1 \cdot 3^1 + 2$$
$$284 = 1 \cdot 3^5 + 1 \cdot 3^3 + 1 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0$$
$$284 = \boxed{101112_3}$$

8.5.4 For 47 to be expressed in two digits in base b, $47 < 100_b = 1 \cdot b^2$. So $\sqrt{47} < b$, which means the smallest positive integer b can be is $\boxed{7}$. We can check to be sure: $47 = 65_7 = 115_6$.

Exercises for Section 8.6

8.6.1 When we multiply a base integer n_b by 10_b, we multiply the entire number by b, adding 1 to the power of b in every digit bundle. The result is appending the digit 0 to the end of n_b. Likewise, a 0 gets added for each time we multiply a base-b integer by 10_b. So, for instance, multiplying by $10_b \cdot 10_b = 10_b^2$ added two 0's and multiplying by $1000_b = 10_b^3$ adds three 0's.

(a) $23_4 \cdot 10_4 = \boxed{230_4}$

(b) $10_5 \cdot 312_5 = \boxed{3120_5}$

(c) $10_8 \cdot 10_8 \cdot 31_8 = 10_8 \cdot 310_8 = \boxed{3100_8}$

(d) $10_7 \cdot 216_7 = \boxed{2160_7}$

(e) $10_{12} \cdot 3A9_{12} = \boxed{3A90_{12}}$

(f) $1000_2 \cdot 1011_2 = \boxed{1011000_2}$

8.6.2 The theme of this problem is converting directly between bases. A couple of these problem parts involve recognizing that we can change powers directly. The others can be solved by ordinary base number conversion techniques, but also point to algebraic relationships such as those involved in squaring binomials: $(x + y)^2 = x^2 + 2xy + y^2$.

(a) Convert directly between bases:

$$\begin{aligned}
623_9 &= 6 \cdot 9^2 + 2 \cdot 9^1 + 3 \cdot 9^0 \\
&= 6 \cdot (3^2)^2 + 2 \cdot (3^2)^1 + 3 \cdot (3^2)^0 \\
&= 6 \cdot 3^4 + 2 \cdot 3^2 + 3 \cdot 3^0 \\
&= (2 \cdot 3^1) \cdot 3^4 + 2 \cdot 3^2 + (1 \cdot 3^1) \cdot 3^0 \\
&= 2 \cdot 3^5 + 2 \cdot 3^2 + 1 \cdot 3^1 \\
&= \boxed{200210_3}
\end{aligned}$$

(b) Since $9 = 8 + 1$, we can use an understanding of squared binomials to solve this one:

$$\begin{aligned} 100_9 &= (10_9)^2 \\ &= (1 \cdot 9^1)^2 \\ &= 9^2 \\ &= (8+1)^2 \\ &= (1 \cdot 8^1 + 1 \cdot 8^0)^2 \\ &= 1 \cdot 8^2 + 2 \cdot 8^1 + 1 \cdot 8^0 \\ &= \boxed{121_8} \end{aligned}$$

(c) This time, let's convert directly between bases by grouping the new digits according to the old, keeping in mind that since $2^4 = 16$, each binary digit block must be four digits long:

$$\begin{aligned} D_{16} &= 1101_2 \\ C_{16} &= 1100_2 \\ 1_{16} &= 0001_2 \\ 7_{16} &= 0111_2 \end{aligned}$$

Now we arrange the binary digits according to the order of the digits in $DC17_{16}$ to get $\boxed{1101110000010111_2}$.

(d) We use a squared binomial to convert directly:

$$\begin{aligned} 100_{14} &= (10_{14})^2 \\ &= (1 \cdot 14^1)^2 \\ &= 14^2 \\ &= (13+1)^2 \\ &= (1 \cdot 13^1 + 1 \cdot 13^0)^2 \\ &= 1 \cdot 13^2 + 2 \cdot 13^1 + 1 \cdot 13^0 \\ &= \boxed{121_{13}} \end{aligned}$$

(e) We use a squared binomial to convert directly:

$$\begin{aligned} 100_{14} &= 14^2 \\ &= (12+2)^2 \\ &= (1 \cdot 12^1 + 2 \cdot 12^0)^2 \\ &= 1 \cdot 12^2 + 4 \cdot 12^1 + 4 \cdot 12^0 \\ &= \boxed{144_{12}} \end{aligned}$$

(f) We use a squared binomial to convert directly:

$$\begin{aligned} 100_{14} &= 14^2 \\ &= (11+3)^2 \\ &= (1 \cdot 11^1 + 3 \cdot 11^0)^2 \\ &= 1 \cdot 11^2 + 6 \cdot 11^1 + 9 \cdot 11^0 \\ &= \boxed{169_{11}} \end{aligned}$$

8.6.3 Refer to Chapter 6 to review the methods used in parts (a) and (b).

(a)

$$\frac{694}{2} = 347 \qquad \frac{347}{2} = 173.5 \qquad \frac{173}{2} = 86.5 \qquad \frac{86}{2} = 43$$
$$\frac{43}{2} = 21.5 \qquad \frac{21}{2} = 10.5 \qquad \frac{10}{2} = 5 \qquad \frac{5}{2} = 2.5 \qquad \frac{2}{2} = 1$$
$$347 + 173 + 86 + 43 + 21 + 10 + 5 + 2 + 1 = \boxed{688}$$

(b)

$$\frac{694}{7} = 99\frac{1}{7} \qquad \frac{99}{7} = 14\frac{1}{7} \qquad \frac{14}{7} = 2$$
$$99 + 14 + 2 = \boxed{115}$$

(c) The prime factorization of 694! includes 2^{688} and 7^{115}. When converted to base 14, 694! will have a terminal 0 for every power of 14 that divides it. Since $14 = 2 \cdot 7$, the number of powers of 14 comes from the total number of times we can pair a 2 and a 7 together in multiplication. There are far more 2's than 7's, so the number of factors of 14 is the same as the number of factors of 7. The number of terminal zeros is $\boxed{115}$.

8.6.4 Since we are only using the standard base-3 digits, the base-4 integers we are counting are the same as the positive base-3 integers less than 10000_3:

$$\begin{array}{cc} 1_4 & 1_3 \\ 2_4 & 2_3 \\ 10_4 & 10_3 \\ 11_4 & 11_3 \\ \vdots & \vdots \\ 2221_4 & 2221_3 \\ 2222_4 & 2222_3 \end{array}$$

Counting the positive base-3 integers is easier: we are counting all of them less than $10000_3 = 3^4 = 81$, so there are $81 - 1 = \boxed{80}$ positive integers that use only the digits 0, 1, and 2 in base 4.

8.6.5 The positive integers that can be written using the standard binary digits 0 and 1 when in base 5 correspond to the positive binary integers:

$$\begin{array}{cc} 1_5 & 1_2 \\ 10_5 & 10_2 \\ 11_5 & 11_2 \\ 100_5 & 100_2 \\ \vdots & \vdots \\ 11100_5 & 11100_2 \\ 11101_5 & 11101_2 \end{array}$$

The 29th smallest positive binary integer is $29 = 11101_2$. The 29th smallest positive base-5 integer written this way is

$$11101_5 = 1 \cdot 5^4 + 1 \cdot 5^3 + 1 \cdot 5^2 + 0 \cdot 5^1 + 1 \cdot 5^0 = 625 + 125 + 25 + 1 = \boxed{776}.$$

Review Problems

8.17

(a) 201_3

(b) 11021_3

(c) 10202122_3

8.18

$$\begin{array}{cccc} 1_4 & 2_4 & 3_4 & 10_4 \\ 11_4 & 12_4 & 13_4 & 20_4 \\ 21_4 & 22_4 & 23_4 & 30_4 \\ 31_4 & 32_4 & 33_4 & 100_4 \\ 101_4 & 102_4 & 103_4 & 110_4 \end{array}$$

8.19

(a) $76_8 = 7\cdot 8^1 + 6\cdot 8^0 = 56 + 6 = \boxed{62}$

(b) $101_9 = 1\cdot 9^2 + 1\cdot 9^0 = 81 + 1 = \boxed{82}$

(c) $213_5 = 2\cdot 5^2 + 1\cdot 5^1 + 3\cdot 5^0 = 50 + 5 + 3 = \boxed{58}$

(d) $311_9 = 3\cdot 9^2 + 1\cdot 9^1 + 1\cdot 9^0 = 243 + 9 + 1 = \boxed{253}$

(e) $514_{12} = 5\cdot 12^2 + 1\cdot 12^1 + 4\cdot 12^0 = 720 + 12 + 4 = \boxed{736}$

(f) $1001_2 = 1\cdot 2^3 + 1\cdot 2^0 = 8 + 1 = \boxed{9}$

(g) $3155_6 = 3\cdot 6^3 + 1\cdot 6^2 + 5\cdot 6^1 + 5\cdot 6^0 = 648 + 36 + 30 + 5 = \boxed{719}$

(h) $20211_3 = 2\cdot 3^4 + 2\cdot 3^2 + 1\cdot 3^1 + 1\cdot 3^0 = 162 + 18 + 3 + 1 = \boxed{184}$

(i) $A19B9_{13} = 10\cdot 13^4 + 1\cdot 13^3 + 9\cdot 13^2 + 11\cdot 13^1 + 9\cdot 13^0 = 285610 + 2197 + 1521 + 143 + 9 = \boxed{289480}$

8.20

$$\begin{array}{lrclclcll} \text{(a)} & 29 & = & 27+2 & = & 1\cdot 27 + 2\cdot 1 & = & 1\cdot 3^3 + 2\cdot 3^0 & = \boxed{1002_3} \\ \text{(b)} & 141 & = & 125+15+1 & = & 1\cdot 125 + 3\cdot 5 + 1\cdot 1 & = & 1\cdot 5^3 + 3\cdot 5^1 + 1\cdot 5^0 & = \boxed{1031_5} \\ \text{(c)} & 494 & = & 343+147+4 & = & 1\cdot 343 + 3\cdot 49 + 4\cdot 1 & = & 1\cdot 7^3 + 3\cdot 7^2 + 4\cdot 7^0 & = \boxed{1304_7} \\ \text{(d)} & 539 & = & 512+24+3 & = & 1\cdot 512 + 3\cdot 8 + 3\cdot 1 & = & 1\cdot 8^3 + 3\cdot 8^1 + 3\cdot 8^0 & = \boxed{1033_8} \end{array}$$

8.21

$$\begin{array}{lrclclcll} \text{(a)} & 7 & = & 4+2+1 & = & 1\cdot 4 + 1\cdot 2 + 1\cdot 1 & = & 1\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 & = \boxed{111_2} \\ \text{(b)} & 9 & = & 8+1 & = & 1\cdot 8 + 1\cdot 1 & = & 1\cdot 2^3 + 1\cdot 2^0 & = \boxed{1001_2} \\ \text{(c)} & 15 & = & 8+4+2+1 & = & 1\cdot 8 + 1\cdot 4 + 1\cdot 2 + 1\cdot 1 & = & 1\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 & = \boxed{1111_2} \end{array}$$

8.22

(a) We convert this one to decimal and then to base 5:

$$\begin{aligned} 211_3 &= 2 \cdot 3^2 + 1 \cdot 3^1 + 1 \cdot 3^0 \\ &= 18 + 3 + 1 \\ &= 22 \\ &= 20 + 2 \\ &= 4 \cdot 5^1 + 2 \cdot 5^0 \\ &= \boxed{42_5} \end{aligned}$$

(b) Since $2^2 = 4$, we can convert directly to base 4 by pairing digits together starting from the right side of our base-2 integer:

$$\begin{aligned} 01_2 &= 1_4 \\ 01_2 &= 1_4 \\ 10_2 &= 2_4 \\ 01_2 &= 1_4 \end{aligned}$$

Putting the base-4 digits together, we get $1011001_2 = \boxed{1121_4}$.

(c) We convert this one to decimal and then to base 9:

$$\begin{aligned} 4133_5 &= 4 \cdot 5^3 + 1 \cdot 5^2 + 3 \cdot 5^1 + 3 \cdot 5^0 \\ &= 500 + 25 + 15 + 3 \\ &= 543 \\ &= 486 + 54 + 3 \\ &= 6 \cdot 9^2 + 6 \cdot 9^1 + 3 \cdot 9^0 \\ &= \boxed{663_9} \end{aligned}$$

(d) Since $9 = 3^2$, we can convert directly to base 3 by expanding each base-9 digit into two base-3 digits:

$$\begin{aligned} 8_9 &= 22_3 \\ 1_9 &= 01_3 \\ 3_9 &= 10_3 \end{aligned}$$

Putting the base-3 digit pairs together, we get $813_9 = \boxed{220110_3}$.

(e) We convert this one to decimal and then to base 11:

$$\begin{aligned} 3166_7 &= 3 \cdot 7^3 + 1 \cdot 7^2 + 6 \cdot 7^1 + 6 \cdot 7^0 \\ &= 1029 + 49 + 42 + 6 \\ &= 1126 \\ &= 1089 + 33 + 4 \\ &= 9 \cdot 11^2 + 3 \cdot 11^1 + 4 \cdot 11^0 \\ &= \boxed{934_{11}} \end{aligned}$$

8.23 We are looking for the smallest base b such that $100_b \le 62 < 1000_b$, which is the same as saying that $b^2 \le 62 < b^3$. The smallest perfect cube greater than 62 is 64, so the smallest possible value of b is $\sqrt[3]{64} = \boxed{4}$.

Challenge Problems

8.24 The number of terminal zeros in which an integer ends in base 12 is the number of factors of 12 that divide that integer. Since $12 = 2^2 \cdot 3^1$, we must find the number of factors of both 2 and 3 in 10! in order to find the number of factors of 12.

$$\frac{10}{2} = 5 \qquad \frac{5}{2} = 2\frac{1}{2} \qquad \frac{2}{2} = 1$$

$$\frac{10}{3} = 3\frac{1}{3} \qquad \frac{3}{3} = 1$$

There are $5 + 2 + 1 = 8$ powers of 2 in the prime factorization of 10! and $3 + 1 = 4$ powers of 3. For every power of 12, we need 2 powers of 2 and 1 power of 3, so 12^4 is the highest power of 12 that divides 10!. This means that 10! ends in $\boxed{4}$ terminal zeros when written in base 12.

8.25 The goal is to count in base 3 using only binary digits. The 100^{th} smallest positive binary integer is $100 = 1100100_2$, so the 100^{th} smallest positive integer that can be written with only the binary digits is $1100100_3 = \boxed{981}$.

8.26 It's easier to count the integers that *don't* have a 7 digit than it is to count those that *do*, so we use complementary counting to solve this problem.

Since $512 = 8^3 = 1000_8$, our goal is to count the number of these positive integers that use only base-7 digits (0-6) when written in base 8. There are $8^3 = 512$ positive integers up to 1000_8 and there are $7^3 = 343$ positive integers up to 1000_7. This means that there are $512 - 343 = \boxed{169}$ positive integers up to 1000_8 that use a digit that is not used in base 7, meaning the digit 7.

8.27 The smallest positive integer that needs 4 digits when expressed in base b is $1000_b = b^3$. Therefore, the largest integer that can be expressed in base b using 3 digits is $b^3 - 1$.

$$\begin{array}{llcccc}
\text{(a)} & 222_3 & = & 3^3 - 1 & = & \boxed{26} \\
\text{(b)} & 333_4 & = & 4^3 - 1 & = & \boxed{63} \\
\text{(c)} & 888_9 & = & 9^3 - 1 & = & \boxed{728} \\
\text{(d)} & JJJ_{20} & = & 20^3 - 1 & = & \boxed{7999}
\end{array}$$

8.28 Since $15 = 3^1 \cdot 5^1$, the largest possible value of n for which $15^n \mid 942!$ is the largest possible value of n for which both $3^n \mid 942!$ and $5^n \mid 942!$. Since 942! has many more factors of 3 than it does 5, our answer will be the number of factors of 5 in 942!.

$$\frac{942}{5} = 188\frac{2}{5} \qquad \frac{188}{5} = 37\frac{3}{5} \qquad \frac{37}{5} = 7\frac{2}{5} \qquad \frac{7}{5} = 1\frac{2}{5}$$

There are $188 + 37 + 7 + 1 = 233$ factors of 5 in 942!, so the largest possible value of n is $\boxed{233}$.

8.29 The answer is $\boxed{\text{no}}$.

$$422_b = 4b^2 + 2b + 2 = 2(2b^2 + b + 1)$$

Since $b > 0$, $2b^2 + b + 1 > 0$, so 422_b is the product of two integers, 2 and $2b^2 + b + 1$, that are both greater than 1 and is therefore always composite, never prime.

8.30 Since $3^2 = 9$, we can convert directly from base 3 to base 9 by converting each pair of digits (starting with the rightmost digits) to base 9. Since $x = 12112211122211112222_3$, the leftmost digit in base 9 is the conversion of the leftmost digit pair from base 3, to base 9. Since $12_3 = 5_9$, the answer is $\boxed{5}$.

8.31 Since $23_b = 2b + 3$ and $b > 3$, 23_b can be any odd integer greater than $2(3) + 3 = 9$. We are looking for the next smallest odd perfect square, which is $5^2 = 25$. Since $2b + 3 = 25$, $b = \boxed{11}$ is our answer.

8.32 An eight-bit binary word is a nonnegative integer with less than 9 digits when expressed in binary. This means we are counting the integers from 0 to $2^8 - 1$ inclusive, of which there are $2^8 = \boxed{256}$. We could also view each digit as a "choice". There are 2 choices for each of 8 digits, so 2^8 eight-bit binary words are possible.

8.33 We can count 1 down from $32768 = 100000_8$ to get $\boxed{77777_8}$.

8.34 We are looking for integers n such that

$$\begin{array}{ccccccccc} 144 & = & 12^2 & \le & n & < & 12^3 & = & 1728 \\ 512 & = & 8^3 & \le & n & < & 8^4 & = & 4096 \end{array}$$

Since all these inequalities must be satisfied, $512 \le n < 1728$, and the number of possible values for n is $1728 - 512 = \boxed{1216}$.

8.35 It might at first be tempting to think that we can create a correspondence between the three digits we are using and the three digits used in base 3. However, this correspondence is made difficult by the fact that the smallest base-3 digit 0 cannot be the lead digit of a positive integer, whereas all three of 1, 3, and 5 can.

We can however, count the number of positive integers that use only the three digits 1, 3, and 5 in base 7. There are 3^n positive n-digit integers that use only 1, 3, and 5 in base 7. The number of such one-digit, two-digit, and three-digit integers is $3 + 9 + 27 = 39$. Since there are $3^4 = 81$ such four-digit integers, the integer we are looking for is a four-digit base-7 integer. In fact, $100 - 39 = 61$, so we are looking for the 61^{st} smallest four-digit base-7 integer written using only 1, 3, and 5.

There are $3^3 = 27$ base-7 integers that begin with 1 and use only 1, 3, and 5 for digits and there are 27 more that begin with 3. Now we have counted up through $39 + 27 + 27 = 93$ base-7 integers. The next one is 5111_7 and we simply finish counting up from there (at right).

$$\begin{array}{rcl} 94^{\text{th}} & = & 5111_7 \\ 95^{\text{th}} & = & 5113_7 \\ 96^{\text{th}} & = & 5115_7 \\ 97^{\text{th}} & = & 5131_7 \\ 98^{\text{th}} & = & 5133_7 \\ 99^{\text{th}} & = & 5135_7 \\ 100^{\text{th}} & = & 5151_7 \end{array}$$

We convert to decimal form to write our answer: $5151_7 = \boxed{1800}$.

8.36 Since $47_a = 74_b$, we know that $4a + 7 = 7b + 4$ where a and b are each at least 8 (since the largest digit of each base number is 7). Since $b \geq 8$, the smallest possible value of the integer represented as 47_a and 74_b is $7 \cdot 8 + 4 = 60$. However, $4a + 7 = 4(a + 1) + 3$, so we are looking for an integer that leaves a remainder of 3 when divided by 4. We try $b = 9$ and get $7b + 4 = 67$. Since $4a + 7 = 67$ has a solution $a = 15$, we have a pair of bases that work.

Since the value of a must go up as the value of b goes up in the equation $4a + 7 = 7b + 4$, and we found the smallest value of b for which there is a solution for a, we have found the smallest solutions for both a and b. The smallest possible value of their sum is $9 + 15 = \boxed{24}$.

8.37 $343 = 7^3 = 1000_7$, so the first 343 positive integers in base 7 are $1_7, 2_7, \ldots 1000_7$. Any number in this list that neither includes 4 or 5 only includes the digits 0, 1, 2, 3, and 6. If we replace 6 with 4, these have the same decimal expansions as the integers in base 5. Since there are $5^3 = 125$ positive integers less than or equal to 1000_5, there are 125 integers less than or equal to 1000_7 that contain no 4's or 5's in base 7, which means there are $343 - 125 = \boxed{218}$ integers that include a 4 or a 5.

8.38

We begin by numbering the digits with base 3 integers from left to right.

1_3	9
2_3	8
10_3	7
11_3	6
$\vdots$	$\vdots$
21102_3	0

The last digit corresponds to $21102_3 = 200$. Now, when we remove all but every third digit, we remove all of them except the ones that correspond to integers that end in 0 in base 3.

10_3	7
20_3	4
100_3	1
110_3	8
$\vdots$	$\vdots$
21100_3	2

Now we can renumber them counting up from 1 in base 3 by chopping off the last digit (which is 0) of each of the base-3 numbers that remained when we kept only every third digit.

1_3	7
2_3	4
10_3	1
11_3	8
$\vdots$	$\vdots$
2110_3	2

When we repeat the process of keeping only every third integer, we again keep only those that have a units digit of 0 in base 3. After we have repeated the process several times, we are left with only the two base-3 integers less than 200 that end with the most 0's, which are 10000_3 and 20000_3. These correspond to the 81^{st} and 162^{nd} digits in the original 200-digit number, which are 9 and 8 respectively, so the resulting two-digit number is $\boxed{98}$.

8.39 Let a, b, and c be digits such that $N = abc_7 = cba_9$. This gives us the equation

$$49a + 7b + c = 81c + 9b + a.$$

After cancelling like terms, we get $48a = 2b + 80c$. Dividing by 2, this equation simplifies to $24a = b + 40c$. Since $24a$ and $40c$ are both multiples of 8, we have $b = 24a - 40c = 8(3a - 5c)$ must also be a multiple of 8. However, b is a base-7 digit, so it must be $\boxed{0}$.

8.40 We can use binomial expansion to more easily convert each power of 10_{b+1} to each given base b:

$$\begin{aligned}(b+1)^2 &= b^2 + 2b + 1\\ (b+1)^3 &= b^3 + 3b^2 + 3b + 1\\ (b+1)^4 &= b^4 + 4b^3 + 6b^2 + 4b + 1\end{aligned}$$

$$\begin{array}{llllllllllll}
\text{(a)} & 100_7 & = & (10_7)^2 & = & 7^2 & = & (6+1)^2 & = & 6^2 + 2\cdot 6^1 + 1 & = & \boxed{121_6}\\
\text{(b)} & 100_8 & = & (10_8)^2 & = & 8^2 & = & (7+1)^2 & = & 7^2 + 2\cdot 7^1 + 1 & = & \boxed{121_7}\\
\text{(c)} & 100_9 & = & (10_9)^2 & = & 9^2 & = & (8+1)^2 & = & 8^2 + 2\cdot 8^1 + 1 & = & \boxed{121_8}\\
\text{(d)} & 1000_5 & = & (10_5)^3 & = & 5^3 & = & (4+1)^3 & = & 4^3 + 3\cdot 4^2 + 3\cdot 4^1 + 1 & = & \boxed{1331_4}\\
\text{(e)} & 1000_6 & = & (10_6)^3 & = & 6^3 & = & (5+1)^3 & = & 5^3 + 3\cdot 5^2 + 3\cdot 5^1 + 1 & = & \boxed{1331_5}\\
\text{(f)} & 1000_7 & = & (10_7)^3 & = & 7^3 & = & (6+1)^3 & = & 6^3 + 3\cdot 6^2 + 3\cdot 6^1 + 1 & = & \boxed{1331_6}\\
\text{(g)} & 10000_8 & = & (10_8)^4 & = & 8^4 & = & (7+1)^4 & = & 7^4 + 4\cdot 7^3 + 6\cdot 7^2 + 4\cdot 7^1 + 1 & = & \boxed{14641_7}\\
\text{(h)} & 10000_9 & = & (10_9)^4 & = & 9^4 & = & (8+1)^4 & = & 8^4 + 4\cdot 8^3 + 6\cdot 8^2 + 4\cdot 8^1 + 1 & = & \boxed{14641_8}
\end{array}$$

8.41 The integers that can only be written using 0, 1, 2, 3, 5, 6, 7, 8, 9 in base 10 correspond with the integers that are written with 0, 1, 2, 3, 4, 5, 6, 7, 8 in base 9, where 5 in base 10 corresponds with 4 in base 9, 6 in base 10 corresponds with 5 in base 9, etc. So 2005 on the odometer is the same as 2004_9, which is $\boxed{1462}$ miles in base 10.

8.42 Since the integer has 2 terminal zeros in base 3, it is a multiple of 3^2. Since it has a terminal zero in each base 4 and base 5, it is a multiple of each 4 and 5. This means the integer is a multiple is lcm[9, 4, 5] $= 2^2 \cdot 3^2 \cdot 5^1 = 180$. If the integer has a terminal zero in base b, then it is a multiple of b. Since the integer must have $t(180) = 18$ positive divisors (possibly others), there are $18 - 4 = \boxed{14}$ other bases (other than 1, 3, 4, and 5) in which the integer must have at least one terminal zero.

8.43 Since VYZ, VYX, and VVW are three consecutive base-5 integers, their units digits must be 3, 4, and 0 respectively because their middle digit changed between the second and third of these numbers. This means that $Z = 3$, $X = 4$, and $W = 0$ and the three consecutive integers are $VY3$, $VY4$, and $VV0$. Additionally, $V = Y + 1$, because when the middle digit went up, it couldn't have gone up from 4 to 0 because those digits are already used by other letters. This means $V = 2$ and $Y = 1$, so $XYZ_5 = 413_5 = \boxed{108}$.

8.44 Base-3 integers that don't use the digit 2 correspond to binary integers. We need to count all the integers up to 1992 that have a base-3 representation that looks like binary, so we start by converting 1992 to base 3: $1992 = 2201210_3$. The largest integer less than this that has only digits 0 and 1 is 1111111_3. This corresponds to the binary integer that is the last we need to count: $1111111_2 = 2^7 - 1 = \boxed{127}$.

CHAPTER 9

Base Number Arithmetic

Exercises for Section 9.2

9.2.1

(a)
$$\begin{array}{r} 1_6 \\ +\ 5_6 \\ \hline 10_6 \end{array}$$

(b)
$$\begin{array}{r} 15_9 \\ +\ 23_9 \\ \hline 38_9 \end{array}$$

(c)
$$\begin{array}{r} 4_6 \\ +\ 14_6 \\ \hline 22_6 \end{array}$$

(d)
$$\begin{array}{r} 54_7 \\ +\ 126_7 \\ \hline 213_7 \end{array}$$

(e)
$$\begin{array}{r} 2303_5 \\ +\ 131_5 \\ \hline 2434_5 \end{array}$$

(f)
$$\begin{array}{r} 39A_{12} \\ +\ 488_{12} \\ \hline 866_{12} \end{array}$$

(g)
$$\begin{array}{r} 6543_8 \\ +\ 7426_8 \\ \hline 16171_8 \end{array}$$

(h)
$$\begin{array}{r} 1211021_3 \\ +\ 2102_3 \\ \hline 1220200_3 \end{array}$$

(i)
$$\begin{array}{r} 122_9 \\ 13_9 \\ +\ 12_9 \\ \hline 147_9 \end{array}$$

(j)
$$\begin{array}{r} 11101_2 \\ 1100_2 \\ 101_2 \\ +\ 11_2 \\ \hline 110001_2 \end{array}$$

(k)
$$\begin{array}{r} 2121_3 \\ 1002_3 \\ +\ 211_3 \\ \hline 11111_3 \end{array}$$

(l)
$$\begin{array}{r} 12D94_{16} \\ +\ A1B2_{16} \\ \hline 1CF46_{16} \end{array}$$

Exercises for Section 9.3

9.3.1

(a) $\begin{array}{r} 11_7 \\ -\ 2_7 \\ \hline 6_7 \end{array}$

(e) $\begin{array}{r} 132_8 \\ -\ 75_8 \\ \hline 35_8 \end{array}$

(b) $\begin{array}{r} 58_9 \\ -\ 18_9 \\ \hline 40_9 \end{array}$

(f) $\begin{array}{r} 13A9_{12} \\ -\ 48B_{12} \\ \hline B1A_{12} \end{array}$

(c) $\begin{array}{r} 41_6 \\ -\ 14_6 \\ \hline 23_6 \end{array}$

(g) $\begin{array}{r} 3434_5 \\ -\ 1441_5 \\ \hline 1443_5 \end{array}$

(d) $\begin{array}{r} 126_7 \\ -\ 54_7 \\ \hline 42_7 \end{array}$

(h) $\begin{array}{r} 2102102_3 \\ -\ 1200212_3 \\ \hline 201120_3 \end{array}$

We use the rest of the exercises to display ordinary principles of arithmetic as they occur [also] in base number arithmetic. In some cases, there is no one right way to proceed, but you can judge for yourself which is easier, so long as you follow the order of operations and don't lose track of signs.

(i) $817_9 - 145_9 - 266_9 = 817_9 - (145_9 + 266_9) = 817_9 - 422_9 = \boxed{385_9}$.

(j) $1011_2 + 101_2 - 1100_2 + 1101_2 = (1011_2 + 101_2) + (-1100_2 + 1101_2) = 10000_2 + 1_2 = \boxed{10001_2}$.

(k) $A0A1_{11} - 3087_{11} - 2AA7_{11} = A0A1_{11} - (3087_{11} + 2AA7_{11}) = A0A1_{11} - 6083_{11} = \boxed{4019_{11}}$.

(l) In this last one, we manipulate the difference by chopping one of the numbers into simpler parts:

$$\begin{aligned} 3CD77_{16} - 19E8E_{16} &= 3CD77 - (1A000_{16} - 172_{16}) \\ &= 3CD77_{16} - 1A000_{16} + 172_{16} \\ &= (3CD77_{16} - 1A000_{16}) + 172_{16} \\ &= 22D77_{16} + 172_{16} \\ &= \boxed{22EE9_{16}} \end{aligned}$$

Exercises for Section 9.4

9.4.1

(a)
$$\begin{array}{r} 6_8 \\ \times\ 7_8 \\ \hline 52_8 \end{array}$$

(b)
$$\begin{array}{r} 201_3 \\ \times\ 12_3 \\ \hline 1102_3 \\ +\ 2010_3 \\ \hline 10112_3 \end{array}$$

(c)
$$\begin{array}{r} 76_8 \\ \times\ 57_8 \\ \hline 662_8 \\ +4660_8 \\ \hline 5542_8 \end{array}$$

(d)
$$\begin{array}{r} 3113_5 \\ \times\ 11_5 \\ \hline 3113_5 \\ +31130_5 \\ \hline 34243_5 \end{array}$$

(e)
$$\begin{array}{r} 3AA_{11} \\ \times\ 192_{11} \\ \hline 7A9_{11} \\ 32A20_{11} \\ +3AA00_{11} \\ \hline 73619_{11} \end{array}$$

(f)
$$\begin{array}{r} 4213_6 \\ \times\ 1215_6 \\ \hline 33513_6 \\ 42130_6 \\ 1243000_6 \\ +\ 4213000_6 \\ \hline 10020043_6 \end{array}$$

Exercises for Section 9.5

9.5.1 Remember that you can always check the results of division by multiplying the quotient by the divisor to ensure that it equals the dividend.

(a) $134_9 = 7_9 \cdot 17_9$. The quotient is $\boxed{17_9}$ and there is $\boxed{\text{no remainder}}$.

(b) $11111_2 = 101_2 \cdot 110_2 + 1_2$. The quotient is $\boxed{110_2}$ and the remainder is $\boxed{1_2}$.

(c) $1444_6 = 31_6 \cdot 32_6 + 12_6$. The quotient is $\boxed{32_6}$ and the remainder is $\boxed{12_6}$.

(d) $4516_8 = 43_8 \cdot 104_8 + 2_8$. The quotient is $\boxed{104_8}$ and the remainder is $\boxed{2_8}$.

(e) $81818_{11} = 81_{11} \cdot 1010_{11} + 8_{11}$. The quotient is $\boxed{1010_{11}}$ and the remainder is $\boxed{8_{11}}$.

(f) $9A71B_{16} = 3E9_{16} \cdot 277_{16} + 3CC_{16}$. The quotient is $\boxed{277_{16}}$ and the remainder is $\boxed{3CC_{16}}$.

9.5.2 $\boxed{\text{Yes,}}$ because $2246_8 \div 16_8 = 125_8$ is an integer.

9.5.3 $\boxed{\text{Yes,}}$ because $4554_7 \div 11_7 = 414_7$ is an integer.

9.5.4 A little creativity makes this problem much easier. In each of the following solutions, we break the base number into a sum of base digit bundles instead of using division.

(a)

$$1221_3 = 1\cdot 3^3 + 2\cdot 3^2 + 2\cdot 3^1 + 1\cdot 3^0 = 3(1\cdot 3^2 + 2\cdot 3^1 + 2\cdot 3^0) + 1$$

We factor 3 out of the first three digit bundles. However, the last digit bundles leaves us with a remainder when we divide the whole sum by 3. So, 1221_3 is not a multiple of 3.

(b)

$$334_5 = 3\cdot 5^2 + 3\cdot 5^1 + 4\cdot 5^0 = 3(1\cdot 5^2 + 1\cdot 5^1 + 1\cdot 5^0) + 1$$

We factor 3 out of the first two digit bundles, but the third leaves a remainder. The sum is not a multiple of 3, so 334_5 is not a multiple of 3.

(c)

$$4113_6 = 4\cdot 6^3 + 1\cdot 6^2 + 1\cdot 6^1 + 3\cdot 6^0$$

Since 6 is a multiple of 3, so is 6^n when n is a positive integer. Also, $3\cdot 6^0 = 3$ is a multiple of 3. So, 4113_6 is the sum of multiples of 3 and is therefore itself a multiple of 3.

(d)

$$7881_9 = 7\cdot 9^3 + 8\cdot 9^2 + 8\cdot 9^1 + 1\cdot 9^0 = 3^2(7\cdot 9^2 + 8\cdot 9^1 + 8\cdot 9^0) + 1$$

We factor $9 = 3^2$ out of the first three digit bundles. The sum of the first three digit bundles is a multiple of 3. However, the last digit bundles leaves a remainder when divided by 3, so the sum of the digit bundles leaves a remainder when divided by 3. So, 7881_9 is not a multiple of 3.

Review Problems

9.11

(a)

$$\begin{array}{r} 42_9 \\ +\ 12_9 \\ \hline 54_9 \end{array}$$

(b)

$$\begin{array}{r} 411_5 \\ +\quad 34_5 \\ \hline 1000_5 \end{array}$$

(c)

$$\begin{array}{r} 101110_2 \\ 11011_2 \\ +\quad 1001_2 \\ \hline 1010010_2 \end{array}$$

9.12

(a)

$$\begin{array}{r} 5144_6 \\ -\ 1023_6 \\ \hline 4121_6 \end{array}$$

(b)

$$\begin{array}{r} 713_{12} \\ -\quad A9_{12} \\ \hline 626_{12} \end{array}$$

9.13

(a)

$$\begin{array}{r} 54_7 \\ \times\ 21_7 \\ \hline 54_7 \\ +\ 141 \\ \hline 1464_7 \end{array}$$

(b)

$$\begin{array}{r} 2102_3 \\ \times\ 121_3 \\ \hline 2102_3 \\ 11211 \\ +\ 2102 \\ \hline 110211 2_3 \end{array}$$

9.14

(a) $205_6 \div 15_6 = \boxed{11_6}$

(b) $1510_8/52_8 = \boxed{24_8}$

9.15

(a) $117_9 = 5_9 \cdot 21_9 + 2_9$, so 117_9 is not a multiple of 5.

(b) $111101_2 = 101_2 \cdot 1100_2 + 1_2$, so 111101_2 is not a multiple of 5.

(c) $111101_4 = 11_4 \cdot 10100_4 + 1_4$, so 111101_4 is not a multiple of 5.

(d) $4105_6 = 5_6 \cdot 501_6$, with no remainder, so 4105_6 is a multiple of 5. Note that since $6 - 1 = 5$, you could also apply the method from Problem 9.10 (d) in the text.

(e)
$$A1BA = 10 \cdot 15^3 + 1 \cdot 15^2 + 11 \cdot 15^1 + 10 \cdot 15^0 = 15(10 \cdot 15^2 + 1 \cdot 15^1 + 11 \cdot 15^0) + 5(2 \cdot 15^0)$$
Since 15 is a multiple of 5, 15^n is a multiple of 5 for any positive integer n. This means that the first three digit bundles are multiples of 5. Since $10 \cdot 15^0 = 10$ is also a multiple of 5, $A1BA_{15}$ is a sum of multiples of 5 and is therefore itself a multiple of 5.

9.16 All of these problems can be solved by converting the integers to base 10 and using regular division or even using base number division. However, we look for methods that can be applied more easily once they are fully understood. Make sure you see why each method works!

(a) Since $12 = 3 \cdot 4$, a multiple of 12 must be a multiple of 4. However,
$$717_8 = 7 \cdot 8^2 + 1 \cdot 8^1 + 7 \cdot 8^0 = 8(7 \cdot 8^1 + 1 \cdot 8^0) + 7$$
is not a multiple of 4 because the first two digit bundles are multiples of 4 and the last is not. Therefore, 717_8 is not a multiple of 12 either.

(b)
$$2120213_3 = 2 \cdot 3^5 + 1 \cdot 3^4 + 2 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0$$
All of the base-3 digit bundles except the last are multiples of 3, so 212021_3 is not a multiple of 3. Since a multiple of 12 is a multiple of 3, 212021_3 is not a multiple of 12.

(c) In order for an integer to be a multiple of 12, it must be even. However,
$$14202_5 = 1 \cdot 5^4 + 4 \cdot 5^3 + 2 \cdot 5^2 + 0 \cdot 5^1 + 2 \cdot 5^0$$
is a sum of one odd integer and four even ones (including $0 \cdot 5^1 = 0$), so 14202_5 is odd and therefore not a multiple of 12.

(d) Similar to part (c), we note that
$$6234_7 = 6 \cdot 7^3 + 2 \cdot 7^2 + 3 \cdot 7^1 + 4 \cdot 7^0$$
is the sum of three even integers and one odd integer, so it is itself odd and not a multiple of 12.

(e) Using the method from Problem 9.10 (d), we find that a base-13 integer is a multiple of 12 only if the sum of its digits is a multiple of 12.
$$C_{13} + 1_{13} + 0_{13} + B_{13} = 12 + 1 + 0 + 11 = 24.$$
Since $12 \mid 24$, $C10B_{13}$ is a multiple of 12.

Challenge Problems

9.17 In the two rightmost columns of addition, there is no carrying, but in the third, there is, so $6_b + 1_b = 10_b$ and $b = \boxed{7}$.

9.18 This problem seems at first like a bigger headache than it really is. However, starting patiently by adding the binary numbers in the parentheses quickly makes the job easier:

$$\begin{aligned}(10101_2 + 1011_2) \cdot (110011_2 + 1101_2) \div (1000 + 100_2 + 10_2 + 1_2 + 1_2) &= 100000_2 \cdot 1000000_2 \div 10000_2 \\ &= 2^5 \cdot 2^6 \div 2^4 \\ &= 2^{5+6-4} \\ &= 2^7 \\ &= \boxed{10000000_2}\end{aligned}$$

9.19 When we rewrite the base numbers as sums of digit bundles, we get the equation

$$4 \cdot (b + 2) = b^2 + 3 \quad \Rightarrow \quad b^2 - 4b - 5 = 0.$$

Solving this quadratic equation, we get $b = 5$ and $b = -1$. But, since the base must be positive, $b = \boxed{5}$.

9.20 We know $b > 4$ since the digit 4 appears. For every $b > 4$, it's true that $11_b \cdot 313_b = 3443_b$. There are a number of ways in which we can see why this is true. Straightforward multiplication requires no carrying and long division reveals a similar result.

We can also take a look at the algebra of the base-b digit bundles:

$$(b + 1)(3b^2 + b + 3) = 3b^3 + 4b^2 + 4b + 3.$$

Since $b > 4$, no carrying is required, and the product is 3443_b. So $\boxed{\text{there is no base } b}$ for which 3443_b is prime.

9.21 Note that

$$1_2 + 10_2 + 100_2 + \cdots + 100000000_2 = 111111111_2 = 1000000000_2 - 1 = 2^9 - 1.$$

We can factor $2^9 - 1 = 8^3 - 1$ as a difference of cubes to make our task easier:

$$8^3 - 1 = (8 - 1)(8^2 + 8 + 1) = 7 \cdot 73.$$

Since $\boxed{73}$ is prime, it is the largest prime divisor of the sum.

9.22 There are a number of ways to approach this problem. Here, we focus on one possible solution that highlights algebraic methods and a bit of arithmetic creativity.

(a) Let $x = 11111111111111111_2 = 100000000000000000_2 - 1 = 2^{17} - 1$.

$$\begin{aligned}3x &= (2^2 - 1)(2^{17} - 1) \\ &= 2^{19} - 2^{17} - 2^2 + 1 \\ &= 10000000000000000000_2 - 100000000000000000_2 - 100_2 + 1_2 \\ &= 1011111111111111101_2\end{aligned}$$

The result is a $\boxed{19}$-digit binary number with $19 - 2 = \boxed{17}$ 1's. We could also have noted in the second to last line of arithmetic that we were subtracting an 18 digit binary number from a 20 digit binary number. The result is a 19 digit binary number starting with two 1's (and the rest 0's). From there, we subtract 100_2, cascading a carrying of digits from the first 1 to the third to last digit, before adding a 1. The result is a 19 digit binary integer with only two 0's (the second from each the right and the left).

9.23 We are given that $12_b \cdot 15_b \cdot 16_b = 3146_b$. We can express each integer in terms of base-b digit bundles, giving us a polynomial equation:

$$(b+2)(b+5)(b+6) = 3b^3 + b^2 + 4b + 6.$$

Expanding the left-hand side, we have

$$b^3 + 13b^2 + 52b + 60 = 3b^3 + b^2 + 4b + 6.$$

We can move all terms to the right-hand side and combine like terms:

$$0 = 2b^3 - 12b^2 - 48b - 54.$$

Dividing by 2, we now see that b is a solution to $b^3 - 6b^2 - 24b - 27$. Since the largest digit used in any of the numbers is 6, we know that $b \geq 7$. All integers are rational numbers and by the Rational Roots Theorem, we know that a rational solution to the polynomial is a (positive or negative) divisor of 27. In this case, the only possibilities are 9 and 27. As it turns out, $b = 9$ is a solution to the polynomial and thus our original base number product as well. The answer is $s = 12_9 + 15_9 + 16_9 = \boxed{44_9}$.

9.24 Let the digits of the integer from left to right be $d_1, d_2, \ldots, 7, 1, 7, 7, 7$. We can write the integer as a sum of base-12 digit bundles:

$$d_1 \cdot 12^{71776} + d_2 \cdot 12^{71775} + \cdots + 7 \cdot 12^4 + 1 \cdot 12^3 + 7 \cdot 12^2 + 7 \cdot 12^1 + 7 \cdot 12^0.$$

Since $3 \mid 12$, we know that $3 \mid 12^n$ for each positive integer n. Therefore, when we divide the large integer by 3, we can ignore all the base-12 digit bundles except the last one. Since $3 \nmid 7$, $\boxed{\text{the large integer } \textit{is not} \text{ a multiple of 3}}$. Thus Cleo need only use the units digit of the base-12 integer to determine whether or not it is a multiple of 3.

9.25 Berris must come up with a method to distinguish the three two-digit numbers. Base 100 to the rescue! Since x, y, and z are distinct base-100 digits and the base-100 number xyz_{100} is unique, Berris can let $A = 100^2$, $B = 100^1$, and $C = 100^0$ so that

$$Ax + By + Cz = x \cdot 100^2 + y \cdot 100^1 + z \cdot 100^0.$$

While the result is really a 6-digit decimal integer, it was base 100 that gave us the idea, and Berris can simply use the digits of the result to secure his freedom.

Harris announces, "$Ax + By + Cz = 659134$."

Berris replies, "Your numbers are 65, 91, and 34. And by the way...you smell funny."

"Curse you Berris! Curse you!" But Harris is a beaten villain, and he knows it.

9.26 Finding the sum of the three-digit base-5 palindromes directly seems tedious, so we look for a nicer method. The sum is equal to the total number of them multiplied by their average.

There are 4 choices for the first digit (and the last digit must be the same) and 5 ways to pick the middle digit, so there are 20 three-digit base-5 palindromes.

Finding their average is a little more difficult. We look for a helpful symmetry:

$$\begin{aligned}
101_5 + 444_5 &= 1100_5 \\
111_5 + 434_5 &= 1100_5 \\
121_5 + 424_5 &= 1100_5 \\
&\vdots \\
232_5 + 313_5 &= 1100_5 \\
242_5 + 303_5 &= 1100_5
\end{aligned}$$

The 20 three-digit palindromes make 10 pairs that sum to 1100_5. Aha! Now we don't even need the average, though looking for it pointed us in the right direction. We simply sum the ten pairs to get the total. Since $10 = 20_5$, this is equal to

$$20_5 \cdot 1100_5 = \boxed{22000_5}.$$

CHAPTER 10

Units Digits

Exercises for Section 10.2

10.2.1

(a) $4 + 3 = 7$, so the units digit of the sum is $\boxed{7}$.

(b) $4 + 8 = 12$, so the units digit of the sum is $\boxed{2}$.

(c) $3 + 9 = 12$, so the units digit of the sum is $\boxed{2}$.

(d) $6 + 7 + 8 + 9 = 30$, so the units digit of the sum is $\boxed{0}$.

(e) $7 - 4 = 3$, so the units digit of the difference is $\boxed{3}$.

(f) We need to carry, so instead of subtracting from 1, we subtract from 11. $11 - 3 = \boxed{8}$ is the units digit of the difference.

(g) $15 - 7 = \boxed{8}$ is the units digit of the difference.

(h) $17 - 2 - 9 = \boxed{6}$

(i) $7 + 2 - 5 + 2 - 8 = -2$. Since we need to carry from the tens digit, we add back 10 to get $\boxed{8}$ as the units digit.

(j) Each units digit gets added 10 times. $10(1 + 2 + \cdots + 9 + 0)$ is a multiple of 10 and has a units digit of $\boxed{0}$.

(k) $3 \cdot 1 = 3$, so the units digit of the product is $\boxed{3}$.

(l) $5 \cdot 3 = 15$, so the units digit of the product is $\boxed{5}$.

(m) $9 \cdot 9 + 1 \cdot 1 = 81 + 1 = 82$, so the units digit is $\boxed{2}$.

(n) $9^3 = 729$ and $7^5 = 16807$. Using the units digits of these results, $9 \cdot 7 = 63$, so the units digit is $\boxed{3}$.

(o) $7^3 = 343$, so $\boxed{3}$ is the units digit. Sometimes there is little or no reprieve from effort.

(p) We establish a pattern:

$$\begin{aligned} 7^1 &= 7 \\ 7^2 &= 49 \\ 7^3 &= 343 \\ 7^4 &= 2401 \\ 7^5 &= 16807 \end{aligned}$$

The pattern repeats every fourth multiple of 7, so 7^{12} has the same units digit as $7^0 = 1$, so its units digit is $\boxed{1}$.

(q) Since the units digits of $7 \cdot 17 \cdot 1977$ and 7^3 are the same, their difference has a units digit of $\boxed{0}$.

(r) The units digit of 18^6 is the same as in 8^6. There are several ways we could go about finding that units digit, but notice that $8^6 = 2^{18}$. It's easy to find the pattern of units digits for powers of 2:

$$\begin{aligned} 2^1 &= 2 \\ 2^2 &= 4 \\ 2^3 &= 8 \\ 2^4 &= 16 \\ 2^5 &= 32 \end{aligned}$$

The pattern repeats every fourth power of 2, and since $18 = 4 \cdot 4 + 2$, we conclude that 2^{18} has the same units digit as $2^2 = \boxed{4}$.

(s) 312^8 has the same units digit as 2^8, which, as we see above, is the same as the units digit of 2^4, which is $\boxed{6}$.

(t) Since the tens digits don't matter to us, the problem is the same as finding the units digit of $3^{19} \cdot 9^{13} = 3^{19} \cdot 3^{26} = 3^{45}$. Now we go hunting for the pattern of units digits of powers of 3:

$$\begin{aligned} 3^1 &= 3 \\ 3^2 &= 9 \\ 3^3 &= 27 \\ 3^4 &= 81 \\ 3^5 &= 243 \end{aligned}$$

The cycle of units digits is 4 long. Since $45 = 4 \cdot 11 + 1$, we know that 3^{45} has the same units digit as $3^1 = \boxed{3}$.

10.2.2 $(3^3)^5 = 3^{15}$. We see that $15 = 4 \cdot 3 + 3$. As in the problem above, the units digit of 3^{15} is the same as in $3^3 = 27$, so the units digit is $\boxed{7}$.

10.2.3 We begin by finding the units digit within each set of parentheses. We get

$$4 \cdot 6 \cdot 8 + 6 \cdot 4 \cdot 2 - 7^3.$$

Now we combine the units digit of each part to get $2 + 8 - 3 = \boxed{7}$.

10.2.4 Let Jeff's age be x, which means Tim's age is $3x$. Since Tim and Jeff's age have the same unit digit, the difference of their two ages ($3x - x = 2x$) has a units digit of 0. This means $10 \mid 2x$, so $5 \mid x$. Since $10 < x < 20$, Jeff's age is $x = \boxed{15}$.

10.2.5

$$\begin{aligned} 3 \cdot 1 &= 3 \\ 3 \cdot 3 &= 9 \\ 3 \cdot 5 &= 15 \\ 3 \cdot 7 &= 21 \\ 3 \cdot 9 &= 27 \end{aligned}$$

Let x be the number with units digit 3, and let y be the other positive integer. Since xy ends in a 1, y must end in a 7, so $x + y = 10$. The units digit of $x + y$ is $\boxed{0}$. Notice that we only needed to check odd values of y because the product was odd.

10.2.6 The largest of any of 5 consecutive odd integers is 8 more than the smallest. This is not enough to cause a repeat of any units digit. This means the 5 consecutive integers have different units digits. Since there are only 5 odd units digits, they must all be used in the product:

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = 945,$$

which has a units digit of 5, not 3. Either Gary incorrectly computed the product, or his 5 integers are not consecutive odd integers.

Exercises for Section 10.3

10.3.1

(a) Since $2_7 + 2_7 = 4_7$, the units digit is $\boxed{4}$.

(b) Since $3_{11} + 5_{11} = 8_{11}$, the units digit is $\boxed{8}$.

(c) Since $4_5 + 3_5 = 12_5$, the units digit is $\boxed{2}$.

(d) Since $13_6 - 5_6 = 4_6$, the units digit is $\boxed{4}$.

(e) Since $1_2 \cdot 1_2 = 1_2$, the units digit is $\boxed{1}$.

(f) Since $5_{18} \cdot 8_{18} = 24_{18}$, the units digit is $\boxed{4}$.

(g) Since $(4_8)^2 = 20_8$, the units digit is $\boxed{0}$.

(h) Since $(5_7)^3 = 236_7$, the units digit is $\boxed{6}$.

(i) We look for a pattern of units digits of powers of 7_9:

$$\begin{aligned} (7_9)^1 &= 7_9 \\ (7_9)^2 &= 54_9 \\ (7_9)^3 &= 421_9 \\ (7_9)^4 &= 3257_9 \end{aligned}$$

The units digits of powers of 7_9 repeat in cycles of 3. Since $313 = 3 \cdot 104 + 1$, we see that $(7_9)^{313}$ has the same units digit as $(7_9)^1 = 7_9$, which is $\boxed{7}$.

Note that the table we used allowed us to find units digits of powers of 7_9 without significant computation. We could simply multiply each given units digit by 7_9 to find the next units digit. For instance, 4 is the units digit of $(7_9)^2$, so we find the units digit of $(7_9)^3$ by multiplying $7_9 \cdot 4_9 = 31_9$.

10.3.2

(a) Since $2_3 \cdot 2_3 = 11_3$, the units digit is $\boxed{1}$.

(b) Since $2_4 \cdot 2_4 = 10_4$, the units digit is $\boxed{0}$.

(c) Since $2_5 \cdot 2_5 = 4_5$, the units digit is $\boxed{4}$.

(d) Since $2_6 \cdot 2_6 = 4_6$, the units digit is $\boxed{4}$.

Notice that for any integer $b > 4$, the units digit is 4. Make sure you see why.

10.3.3

$$\begin{aligned} 0_6 \cdot 1_6 &= 0_6 \\ 1_6 \cdot 2_6 &= 2_6 \\ 2_6 \cdot 3_6 &= 10_6 \\ 3_6 \cdot 4_6 &= 20_6 \\ 4_6 \cdot 5_6 &= 32_6 \\ 5_6 \cdot 0_6 &= 0_6 \end{aligned}$$

If we first express the integers in base 6, we can examine all possible products of the units digits of consecutive integers. Of these, there are two possible cases in which the product does not have a units digit of 6. In these cases we have $1_6 + 2_6 = 3_6$ and $4_6 + 5_6 = 13_6$ meaning that $\boxed{3}$ *must* be the base-6 units digit of their sum.

Exercises for Section 10.4

10.4.1 From the table in Problem 10.14, we see that an integer's square has a units digit of 4 only when the integer itself has a units digit of 2 or 8. Notice that $2 + 8 = 10$, the base we work in. Is this a coincidence?

10.4.2

$$\begin{aligned} 0 \cdot 8 &= 0 & 5 \cdot 8 &= 40 \\ 1 \cdot 8 &= 8 & 6 \cdot 8 &= 48 \\ 2 \cdot 8 &= 16 & 7 \cdot 8 &= 56 \\ 3 \cdot 8 &= 24 & 8 \cdot 8 &= 64 \\ 4 \cdot 8 &= 32 & 9 \cdot 8 &= 72 \end{aligned}$$

Checking all the possible units digits we see that if one of the integers has a units digit of 8 and their product has a units digit of 6, then the possible units digits of the other integer are $\boxed{2}$ and $\boxed{7}$.

10.4.3

$$\begin{aligned} (0_8)^2 &= 0_8 \\ (1_8)^2 &= 1_8 \\ (2_8)^2 &= 4_8 \\ (3_8)^2 &= 11_8 \\ (4_8)^2 &= 20_8 \\ (5_8)^2 &= 31_8 \\ (6_8)^2 &= 44_8 \\ (7_8)^2 &= 61_8 \end{aligned}$$

Squaring each possible base-8 units digit, we see that $\boxed{0}$, $\boxed{1}$, and $\boxed{4}$ are the only possible units digits of perfect squares expressed in base 8. Note that this means a perfect square must leave a remainder of 0, 1, or 4 when divided by 8.

10.4.4

(a) Since the units digits of $n!$ is always 0 when n is an integer greater than 4, we just sum the first few factorials to get our answer: $1 + 2 + 6 + 24 = 33$, so $\boxed{3}$ is the units digit.

(b) We can ignore $n!$ when n is 7 or higher, because each is a multiple of 7 and will therefore have a base-7 units digit of 0. There are several ways we can proceed from here. One is to just sum the factorials we want, then find its remainder when divided by 7, which will be the base-7 units digit:

$$1! + 2! + 3! + 4! + 5! + 6! = 1 + 2 + 6 + 24 + 120 + 720 = 873,$$

and $873 = 7 \cdot 124 + \boxed{5}$.

Review Problems

10.19

(a) $\boxed{1}$
(b) $\boxed{7}$
(c) $\boxed{9}$
(d) $\boxed{6}$
(e) $\boxed{8}$
(f) $\boxed{0}$
(g) $\boxed{7}$
(h) $\boxed{0}$
(i) $\boxed{5}$
(j) $\boxed{1}$
(k) $\boxed{8}$
(l) $\boxed{2}$

10.20 The units digit of powers of 2 and 3 both repeat in cycles four long. Since $1986 \div 4$ has a remainder of 2, the units digit of $3^{1986} - 2^{1986}$ is the same as $3^2 - 2^2 = \boxed{5}$.

10.21 Since Jenny's new system of arranging stamps includes 10 on each page, the number of the last page will be the units digit of her total number of stamps (in base 10). That units digit is the same as the units digit of $8 \cdot 2 \cdot 6 = 96$, which is $\boxed{6}$.

10.22 Since the product is positive, we need only check the possible odd units digits and we find that $\boxed{7}$ is the only one that works: $7 \cdot 7 = 49$.

10.23 First, we note that the product is odd, so neither of the primes is 2. Also, since the units digit of the product is neither 0 nor 5, the product is not a multiple of 5. Since all integers with units digits of 5 are multiples of 5, we don't need to worry about 5 as a possible units digit. We test the remaining cases. We need only test 10 cases since we can organize the products with the smaller units digit first:

$$\begin{array}{rcl@{\qquad}rcl}
1 \cdot 1 & = & 1 & 3 \cdot 7 & = & 21 \\
1 \cdot 3 & = & 3 & 3 \cdot 9 & = & 27 \\
1 \cdot 7 & = & 7 & 7 \cdot 7 & = & 49 \\
1 \cdot 9 & = & 9 & 7 \cdot 9 & = & 63 \\
3 \cdot 3 & = & 9 & 9 \cdot 9 & = & 81
\end{array}$$

The only possibilities are $1 + 3 = 4$ and $7 + 9 = 16$, which give units digits of $\boxed{4}$ and $\boxed{6}$.

10.24

(a)	$\boxed{4}$	(f)	$\boxed{3}$
(b)	$\boxed{2}$	(g)	$\boxed{6}$
(c)	$\boxed{2}$	(h)	$\boxed{1}$
(d)	$\boxed{0}$	(i)	$\boxed{1}$
(e)	$\boxed{3}$	(j)	$\boxed{1}$

10.25

(a) Since $2_4 \cdot 3_4 = 12_4$, the units digit is $\boxed{2}$.

(b) Since $2_5 \cdot 3_5 = 11_5$, the units digit is $\boxed{1}$.

(c) Since $2_6 \cdot 3_6 = 10_6$, the units digit is $\boxed{0}$.

(d) Since $2_{7171977} \cdot 3_{7171977} = 6_{7171977}$, the units digit is $\boxed{6}$.

10.26

$$\begin{aligned}(0_5)^2 &= 0_5\\(1_5)^2 &= 1_5\\(2_5)^2 &= 4_5\\(3_5)^2 &= 14_5\\(4_5)^2 &= 31_5\end{aligned}$$

Perfect squares can only have units digits of $\boxed{0}$, $\boxed{1}$, or $\boxed{4}$ in base 5.

10.27

Since $n^4 = (n^2)^2$, our goal is the find the possible units digits of a square of a square. We already know the only possible units digits of perfect squares: 0, 1, 4, 5, 6, and 9. We square each of these to find that $\boxed{0}$, $\boxed{1}$, $\boxed{5}$, and $\boxed{6}$ are the only possible units digits of a perfect fourth power.

$$\begin{aligned}0^2 &= 0\\1^2 &= 1\\4^2 &= 16\\5^2 &= 25\\6^2 &= 36\\9^2 &= 81\end{aligned}$$

10.28 First, we simplify the given factorial fraction:

$$\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1).$$

We now see that our task is to find the possible units digits of the product of two consecutive positive integers.

$$\begin{aligned}0\cdot 1 &= 0 & 5\cdot 6 &= 30\\1\cdot 2 &= 2 & 6\cdot 7 &= 42\\2\cdot 3 &= 6 & 7\cdot 8 &= 56\\3\cdot 4 &= 12 & 8\cdot 9 &= 72\\4\cdot 5 &= 20 & 9\cdot 10 &= 90\end{aligned}$$

The only possible units digits are $\boxed{0}$, $\boxed{2}$, and $\boxed{6}$. Do you see a reason why the units digits on each row above are the same?

10.29

We know that neither of the units digits is 0, so we test the other 5 cases of consecutive units digits in base 7 and find that the two integers must have units digits of 3 and 4. Since $3_7 + 4_7 = 10_7$, the units digit of their sum of $\boxed{0}$. Notice that the units digits of the products of the consecutive digits go $2\ldots6\ldots5\ldots6\ldots2$. Do you see a reason behind this "palindromic" pattern?

$$\begin{aligned} 1_7 \cdot 2_7 &= 2_7 \\ 2_7 \cdot 3_7 &= 6_7 \\ 3_7 \cdot 4_7 &= 15_7 \\ 4_7 \cdot 5_7 &= 26_7 \\ 5_7 \cdot 6_7 &= 42_7 \end{aligned}$$

Challenge Problems

10.30 Since $n^4 = (n^2)^2$, we are looking for the possible base-5 units digits of squares of squares. From Problem 10.26, we know that squares have base-5 units digits of 0, 1, or 4. We also know that the squares of such integers have base-5 units digits of 0, 1, or 1 respectively, so $\boxed{0}$ and $\boxed{1}$ are the only possible units digits of perfect fourth powers in base 5. Notice that this means perfect fourth powers leave remainders of only 0 or 1 when divided by 5.

10.31

(a) $\boxed{0}$ and $\boxed{1}$.

(b) $\boxed{0}$, $\boxed{1}$, $\boxed{3}$, and $\boxed{4}$.

(c) $\boxed{0}$, $\boxed{1}$, $\boxed{2}$, and $\boxed{4}$.

(d) $\boxed{0}$, $\boxed{1}$, $\boxed{4}$, and $\boxed{7}$.

10.32 The units digit of $(133^{13})^3$ is the same as the units digit of $(3^{13})^3 = 3^{39}$. As we've seen, the units digits of powers of three repeat in a cycle of 4. Since $39 = 4 \cdot 9 + 3$, the units digit of 3^{39} is the same as that of $3^3 = 27$, which is $\boxed{7}$.

10.33 The pages are consecutive integers and in Problem 10.28 we saw the possible units digits of products of consecutive integers. We sum the units digits of the consecutive integers in both cases where the product has a units digit of 6: $2 + 3 = 5$ and $7 + 8 = 15$ have the same units digit, $\boxed{5}$.

10.34 As we've seen already, the units digits of powers of 2 repeat in a cycle of 4. Since there are powers of 2 with units digit 6, we know that $100 \div 4 = \boxed{25}$ of the given powers of 2 have units digits of 6.

10.35 The units digit of mn is $1^6 = 1$. Searching for a units digit for n (which is clearly odd), we find that $7 \cdot 3 = 1$, so $\boxed{3}$ is the units digit of n.

10.36 First we note that the units digit of $m^2n = 39^{39}$ is the same as the units digit of 9^{39}. The units digits of powers of 9 repeat in a cycle of 2, so this is the same as the units digit of $9^1 = 9$. Since the units digit of n is 1, we know the units digit of m^2 is 9. Searching through the possibilities, we know that the units digit of m is either $\boxed{3}$ or $\boxed{7}$.

10.37

Looking for clues, we have a units digit of 2 in the product of the tens digit of the second factor and the units digit of the first. Since the units digit of the first is 8, we are looking for d such that $8d$ has a units digit of 2. The possibilities are $d = 4$ and $d = 9$, shown at right.

$$\begin{array}{r} \_\,8 \\ \times\, 4\,\_ \\ \hline 3\,\_\,\_ \\ \_\,1\,2\;\; \\ \hline \_\,\_\,\_\,\_ \end{array} \qquad \begin{array}{r} \_\,8 \\ \times\, 9\,\_ \\ \hline 3\,\_\,\_ \\ \_\,1\,2\;\; \\ \hline \_\,\_\,\_\,\_ \end{array}$$

Since the product is larger than 4000, our number sense tells us that it's 9, not 4, but in order to be certain, we continue to look for clues.

Looking at the product of the tens digit of the second factor and the whole first factor, we see that $4 \cdot 8 = 32$, but the product is 12 more than a multiple of 100. If we let the tens digit of the first factor be a, then $40a$ is 80 more than a multiple of 100, so either $a = 2$ or $a = 7$ and we can complete those parts of the product.

$$\begin{array}{r} 7\,8 \\ \times\, 4\,\_ \\ \hline 3\,\_\,\_ \\ 3\,1\,2\;\; \\ \hline \_\,\_\,\_\,\_ \end{array} \qquad \begin{array}{r} 2\,8 \\ \times\, 4\,\_ \\ \hline 3\,\_\,\_ \\ 1\,1\,2\;\; \\ \hline \_\,\_\,\_\,\_ \end{array}$$

Since $78 \cdot 50 = 3900 < 4000$, and $78 \cdot 50$ is greater than either possible product, we know that 4 was not the tens digit of the second factor.

Again, let a be the tens digit of the first factor. We know that $9(10a + 8)$ is 12 more than a multiple of 100, meaning $90a$ is 40 more than a multiple of 100. This means $9a$ has a units digit of 4. The only possible digit is $a = 6$. We complete as much of the product as we can now.

$$\begin{array}{r} 6\,8 \\ \times\, 9\,\_ \\ \hline 3\,\_\,\_ \\ 6\,1\,2\;\; \\ \hline \_\,\_\,\_\,\_ \end{array}$$

Finally, we know that 68 times the units digit of the second factor is at least 300 and less than 400.

$$4 < \frac{300}{68} < 5 < \frac{400}{68} < 6$$

The only possibility is 5, so the larger factor is the second factor, which is $\boxed{95}$.

10.38 An integer that is a multiple of both 4 and 6 is a multiple of lcm[4, 6] = 12. Our goal is to count the three-digit multiples of 12 with units digits of 2. For a positive integer n, $12n$ has a units digit of 2 when n has a units digit of 1 or 6. In other words, n is 1 more than a multiple of 5. We can rewrite n as $5m + 1$ for nonnegative integers m.

Since $12n = 12(5m + 1) = 60m + 12$, our goal is now to count integers m such that

$$100 \le 60m + 12 < 1000.$$

Subtracting 12 from each part of the inequality, we get

$$88 \le 60m < 988.$$

Dividing by 60 and noting that m is an integer, we find that $2 \le m \le 16$, so there are $16 - 1 = \boxed{15}$ values for the integer m that give us the three-digit multiples of 4 and 6 that have units digits of 2.

10.39 A group of 10 consecutive positive integers includes an integer that has each units digit exactly once. Their product has the same units digit of the product of their units digits, which is 0. Their sum has the same units digit as $0+1+2+\cdots+9=45$, which is 5. Subtracting $P-S$ we get the same units digit as $10-5=\boxed{5}$.

10.40

(a) The powers of 3 alternate between the 2 units digits 1 and 3 in base 4. Since 2006 leaves no remainder when divided by 2, 3^{2006} has the same units digit as $3^2=9=21_4$, which is $\boxed{1}$.

(b) The powers of 3 alternate among the 4 units digits 1, 3, 4, and 2 in base 5. Since 2006 leaves a remainder of 2 when divided by 4, 3^{2006} has the same units digit as $3^2=9=14_5$, which is $\boxed{4}$.

(c) The powers of 3 alternate among the 6 units digits 1, 3, 2, 6, 4, and 5 in base 7. Since 2006 leaves a remainder of 2 when divided by 6, 3^{2006} has the same units digit as $3^2=9=12_7$, which is $\boxed{2}$.

(d) The powers of 3 alternate between the 2 units digits 1 and 3 in base 8. Since 2006 leaves no remainder when divided by 2, 3^{2006} has the same units digit as $3^2=9=11_8$, which is $\boxed{1}$.

(e) All powers of 3 greater than 3^1 are multiples of 9, so they all have units digits of $\boxed{0}$ in base 9.

10.41 Cubing each units digit, we find that a cube has a units digit of 3 only when the units digit of its cube root has a units digit of 7, so $B=7$. Also, we note that

$$400^3=64000000<108531333<125000000=500^3.$$

This means $A=4$, so $A+B=\boxed{11}$.

10.42

(a) $2!=2=\boxed{2}_3$.

(b) $4!=24=4\boxed{4}_5$.

(c) $6!=720=24\boxed{6}_7$.

(d) Since 10! is a large number, we take a different approach, building the base-11 units digit by multiplying in each factor from 1 through 10, one at a time:

$$\begin{aligned}
2_{11}\cdot 1_{11} &= 2_{11}\\
3_{11}\cdot 2_{11} &= 6_{11}\\
4_{11}\cdot 6_{11} &= 22_{11}\\
5_{11}\cdot 2_{11} &= A_{11}\\
6_{11}\cdot A_{11} &= 55_{11}\\
7_{11}\cdot 5_{11} &= 32_{11}\\
8_{11}\cdot 2_{11} &= 15_{11}\\
9_{11}\cdot 5_{11} &= 41_{11}\\
A_{11}\cdot 1_{11} &= \boxed{A}_{11}
\end{aligned}$$

Notice that 3, 5, 7, and 11 are all prime. Do you notice a pattern in the answers?

10.43 Let M be Margaret's age and W be Wylia's age. We know that $M-2=W$, and that $W+M=(M-2)+M=2M-2$ has a units digit of 6, so $2M$ has a units digit of 8. The only possible units digits of M are 4 and 9. Since Wylia's age is even and she is two years younger than Margaret, M is even, so $M = \boxed{14}$.

10.44 Let N be the integer we seek. We know that N is 2 more than a multiple of each of 3, 4, 5, and 6, so $N - 2$ is a multiple of $\text{lcm}[3, 4, 5, 6] = 60$. Since $N > 2$, the number we seek is $N = \boxed{62}$.

10.45 The very first thing we notice is that $b \geq 4$ because we are using digits up to 3 already. Our sum is really a product of 7 and 32_b, so we know that $7 \cdot 2 = 14$ has a units digit of 2 in base b. This means $14 = nb + 2$ for some integer n and $nb = 12$ means that $b \mid 12$. The divisors of 12 greater than 3 are 4, 6, and 12, and all $\boxed{3}$ satisfy the problem statement.

10.46 Mersenne primes are primes of the form $2^p - 1$ for prime values of p. If a particular Mersenne prime has a units digit of 3, then 2^p has a units digit of 4. However, in the repeating cycle of units digits of powers of 2, only the powers of 2 in the form 2^{4n+2} for some nonnegative integer n have units digits of 4. Since $4n + 2 = 2(n + 1) = p$, $2 \mid p$, so $p = 2$. This proves there is only $\boxed{1}$ Mersenne prime with a units digit of 3, namely, $2^2 - 1 = 3$.

10.47

(a) $1599 = 6 \cdot 16^2 + 3 \cdot 16^1 + 15 \cdot 16^0 = \boxed{63F_{16}}$.

(b) Since $n^4 = (n^2)^2$, we are looking for the units digits of squares of squares in base 16. First, we find the possible units digits of squares:

$$\begin{array}{rclrcl}
(0_{16})^2 & = & 0_{16} & (8_{16})^2 & = & 40_{16} \\
(1_{16})^2 & = & 1_{16} & (9_{16})^2 & = & 51_{16} \\
(2_{16})^2 & = & 4_{16} & (A_{16})^2 & = & 64_{16} \\
(3_{16})^2 & = & 9_{16} & (B_{16})^2 & = & 79_{16} \\
(4_{16})^2 & = & 10_{16} & (C_{16})^2 & = & 90_{16} \\
(5_{16})^2 & = & 19_{16} & (D_{16})^2 & = & A9_{16} \\
(6_{16})^2 & = & 24_{16} & (E_{16})^2 & = & C4_{16} \\
(7_{16})^2 & = & 31_{16} & (F_{16})^2 & = & E1_{16}
\end{array}$$

The possible base-16 units digits of squares are 0, 1, 4, and 9. Squaring each of those, we find that the base-16 units digit of a perfect fourth power is either 0 or 1.

(c) We begin by rewriting the equation in base 16:

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 63F_{16}.$$

Only the units digits of the perfect fourth powers on the left-hand side of the equation contribute to the units digit of the sum, which is F. However, the sum of the units digits on the left is just the sum of up to fourteen 1's, which is at most E_{16}. This means that there are $\boxed{0}$ groups of 14 perfect fourth powers that add up to 1599.

Perhaps you can make this solution even simpler after learning modular arithmetic later in the book.

CHAPTER 11

Decimals and Fractions

Exercises for Section 11.2

11.2.1 When the denominator of a fraction expressed in lowest terms has a prime factorization in the form $2^a \cdot 5^b$, the number of digits past the decimal point needed to express that fraction is the greater of a and b. Each answer represents the exponent of the smallest power of 10 that is a multiple of each denominator.

(a) The denominator is $4 = 2^2 \cdot 5^0$, so this number needs $\boxed{2}$ digits to the right of the decimal.

(b) The denominator is $25 = 2^0 \cdot 5^2$, so this number needs $\boxed{2}$ digits to the right of the decimal.

(c) The denominator is $20 = 2^2 \cdot 5^1$, so this number needs $\boxed{2}$ digits to the right of the decimal.

(d) The denominator is $50 = 2^1 \cdot 5^2$, so this number needs $\boxed{2}$ digits to the right of the decimal.

(e) The denominator is $125 = 2^0 \cdot 5^3$, so this number needs $\boxed{3}$ digits to the right of the decimal.

(f) The denominator is $500 = 2^2 \cdot 5^3$, so this number needs $\boxed{3}$ digits to the right of the decimal.

(g) The denominator is $2000 = 2^4 \cdot 5^3$, so this number needs $\boxed{4}$ digits to the right of the decimal.

(h) The denominator is $1600 = 2^6 \cdot 5^2$, so this number needs $\boxed{6}$ digits to the right of the decimal.

11.2.2

(a) $$\frac{14}{25} = \frac{2^2 \cdot 14}{2^2 \cdot 5^2} = \frac{56}{100} = \boxed{0.56}$$

(b) $$\frac{83}{125} = \frac{2^3 \cdot 83}{2^3 \cdot 5^3} = \frac{664}{1000} = \boxed{0.664}$$

(c) $$\frac{3}{8} = \frac{5^3 \cdot 3}{2^3 \cdot 5^3} = \frac{375}{1000} = \boxed{0.375}$$

(d) $$\frac{91}{200} = \frac{5^1 \cdot 91}{2^3 \cdot 5^3} = \frac{455}{1000} = \boxed{0.455}$$

Exercises for Section 11.3

11.3.1 In some of the following solutions, we multiply in powers of 2 or 5 in order to pair all factors of 2 and 5 into powers of 10 in the denominator in order to make computation simpler. This method can save a great deal of valuable time.

(a) $\frac{1}{9} = \boxed{0.\overline{1}}$

(b) $\frac{1}{11} = \boxed{0.\overline{09}}$

(c) $\frac{1}{15} = \frac{2}{2} \cdot \frac{1}{15} = \frac{2}{30} = \frac{2}{3} \cdot \frac{1}{10} = (0.\overline{6})(0.1) = \boxed{0.0\overline{6}}$

(d) $\frac{4}{15} = \frac{2}{2} \cdot \frac{4}{15} = \frac{8}{30} = \frac{8}{3} \cdot \frac{1}{10} = (2.\overline{6})(0.1) = \boxed{0.2\overline{6}}$

(e) $\frac{1}{24} = \frac{5^3}{5^3} \cdot \frac{1}{24} = \frac{125}{3000} = \frac{125}{3} \cdot \frac{1}{1000} = (41.\overline{6})(0.001) = \boxed{0.041\overline{6}}$

(f) $\frac{29}{24} = \frac{5^3}{5^3} \cdot \frac{29}{24} = \frac{3625}{3000} = \frac{3625}{3} \cdot \frac{1}{1000} = (1208.\overline{3})(0.001) = \boxed{1.208\overline{3}}$

(g) $\frac{1}{90} = \frac{1}{9} \cdot \frac{1}{10} = (0.\overline{1})(0.1) = \boxed{0.0\overline{1}}$

(h) $\frac{223}{90} = \frac{223}{9} \cdot \frac{1}{10} = (24.\overline{7})(0.1) = \boxed{2.4\overline{7}}$

(i) $\frac{1}{18} = \frac{5^1}{5^1} \cdot \frac{1}{18} = \frac{5}{90} = \frac{5}{9} \cdot \frac{1}{10} = (0.\overline{5})(0.1) = \boxed{0.0\overline{5}}$

(j) $\frac{1}{27} = \boxed{0.\overline{037}}$

(k) $\frac{1}{108} = \frac{5^2}{5^2} \cdot \frac{1}{108} = \frac{25}{2700} = \frac{25}{27} \cdot \frac{1}{100} = (0.\overline{925})(0.01) = \boxed{0.00\overline{925}}$

(l) $\frac{1}{1080} = \frac{1}{108} \cdot \frac{1}{10} = (0.00\overline{925})(0.1) = \boxed{0.000\overline{925}}$

11.3.2 First, we find the repeating decimal expansion of 5/14:

$$\frac{5}{14} = \frac{5}{5} \cdot \frac{5}{14} = \frac{25}{70} = \frac{25}{7} \cdot \frac{1}{10} = (3.\overline{571428})(0.1) = 0.3\overline{571428}.$$

The 1314th digit after the decimal point is the 1313th digit in the 6-digit repeating block 5-7-1-4-2-8. Since $1313 \div 6$ leaves a remainder of 5, our answer is the 5th digit in the 6-digit block, which is $\boxed{2}$.

Exercises for Section 11.4

11.4.1

(a) Let $x = 0.\overline{5}$.

$$10x - x = 5.\overline{5} - 0.\overline{5} = 5 \Rightarrow x = \boxed{\frac{5}{9}}.$$

(b) Let $x = 0.00\overline{5}$.

$$10x - x = 0.0\overline{5} - 0.00\overline{5} = 0.05 \Rightarrow x = \frac{\frac{1}{20}}{9} = \boxed{\frac{1}{180}}.$$

Notice that we could also have multiplied the fraction from part (a) by 1/100.

(c) Let $x = 1.\overline{27}$.

$$100x - x = 127.\overline{27} - 1.\overline{27} = 126 \Rightarrow x = \frac{126}{99} = \boxed{\frac{14}{11}}.$$

(d) We use the result from part (c):

$$0.1\overline{27} = \frac{1.\overline{27}}{10} = \frac{\frac{14}{11}}{10} = \boxed{\frac{7}{55}}.$$

(e) Let $x = 4.\overline{054}$.

$$1000x - x = 4054.\overline{054} - 4.\overline{054} = 4050 \Rightarrow x = \frac{4050}{999} = \boxed{\frac{150}{37}}.$$

(f) We use the result from part (e):

$$0.\overline{405} = \frac{4.\overline{054}}{10} = \frac{\frac{150}{37}}{10} = \boxed{\frac{15}{37}}.$$

(g) Let $x = 0.\overline{76}$.

$$100x - x = 76.\overline{76} - 0.\overline{76} = 76 \Rightarrow x = \boxed{\frac{76}{99}}.$$

(h) We use the result from part (g):

$$0.2\overline{76} = 0.2 + 0.0\overline{76} = \frac{2}{10} + \frac{0.\overline{76}}{10} = \frac{2 + \frac{76}{99}}{10} = \frac{\frac{274}{99}}{10} = \frac{274}{990} = \boxed{\frac{137}{495}}.$$

Exercises for Section 11.5

11.5.1

(a)

$$\frac{19}{64} = \frac{1 \cdot 4^2 + 0 \cdot 4^1 + 3 \cdot 4^0}{4^3} = 1 \cdot 4^{-1} + 0 \cdot 4^{-2} + 3 \cdot 4^{-3} = \boxed{0.103_4}.$$

(b)

$$\frac{729}{1331} = \frac{6 \cdot 11^2 + 0 \cdot 11^1 + 3 \cdot 11^0}{11^3} = 6 \cdot 11^{-1} + 0 \cdot 11^{-2} + 3 \cdot 11^{-3} = \boxed{0.603_{11}}.$$

(c)

$$\left(\frac{6}{5}\right)^3 = \left(1 + \frac{1}{5}\right)^3 = 1 + 3\left(\frac{1}{5}\right) + 3\left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 = 1 \cdot 5^0 + 3 \cdot 5^{-1} + 3 \cdot 5^{-2} + 1 \cdot 5^{-3} = \boxed{1.331_5}.$$

(d)

$$\frac{1}{12}=\frac{1}{2^4}+\frac{1}{2^6}+\frac{1}{2^8}+\cdots=\boxed{0.00\overline{01}_2}.$$

(e)

$$\frac{1}{12}=\frac{2}{3^3}+\frac{2}{3^5}+\frac{2}{3^7}+\cdots=\boxed{0.0\overline{02}_3}.$$

(f)

$$\frac{1}{7}=\frac{1}{8}+\frac{1}{8^2}+\frac{1}{8^3}+\cdots=\boxed{0.\overline{1}_8}.$$

(g)

$$\frac{1}{3}=\frac{2}{7^1}+\frac{2}{7^2}+\frac{2}{7^3}+\cdots=\boxed{0.\overline{2}_7}.$$

(h)

$$\frac{1}{12}=\frac{10}{11^2}+\frac{10}{11^4}+\frac{10}{11^6}+\cdots=\boxed{0.\overline{0A}_{11}}.$$

11.5.2

(a)

$$0.10101_2=\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}=\frac{21}{32}=\boxed{0.65625}.$$

(b)

$$2.\overline{5}_7=2+\frac{5}{7^1}+\frac{5}{7^2}+\frac{5}{7^3}\cdots=2+\frac{\frac{5}{7}}{1-\frac{1}{7}}=2+\frac{5}{6}=\boxed{2.8\overline{3}}.$$

(c)

$$\begin{aligned}0.\overline{21} &= \frac{2}{3^1}+\frac{1}{3^2}+\frac{2}{3^3}+\frac{1}{3^4}+\cdots\\ &= \frac{2\cdot 3+1}{3^2}+\frac{2\cdot 3+1}{3^4}+\cdots\\ &= \frac{7}{3^2}+\frac{7}{3^4}+\frac{7}{3^6}+\cdots\\ &= \frac{\frac{7}{9}}{1-\frac{1}{9}}=\frac{7}{8}=\boxed{0.875}\end{aligned}$$

(d)

$$3.\overline{4}_{13}=3+\frac{4}{13^1}+\frac{4}{13^2}+\frac{4}{13^3}+\cdots=3+\frac{\frac{4}{13}}{1-\frac{1}{13}}=3+\frac{1}{3}=\boxed{3.\overline{3}}.$$

Review Problems

11.18

(a) $\boxed{1.75}$
(b) $\boxed{-0.325}$
(c) $\boxed{0.34}$
(d) $\boxed{17.42}$
(e) $\boxed{2.052}$
(f) $\boxed{8.9125}$
(g) $\boxed{0.03125}$
(h) $\boxed{6.\overline{6}}$
(i) $\boxed{-0.\overline{7}}$
(j) $\boxed{8.\overline{857142}}$
(k) $\boxed{0.0\overline{714285}}$
(l) $\boxed{1.91\overline{6}}$
(m) $\boxed{0.\overline{73}}$
(n) $\boxed{0.0\overline{89}}$
(o) $\boxed{3.\overline{115}}$

11.19 We don't need to convert a single fraction. We just count the ones with prime factorizations that include any prime other than 2 or 5: 3, 6, 7, and 9, so $\boxed{4}$ of them.

11.20 We expand the factorials and then convert to decimal form:

$$\frac{4!+3!}{3!+2!} = \frac{24+6}{6+2} = \frac{30}{8} = \frac{15}{4} = \boxed{3.75}.$$

11.21 First, we use long division or another method to find that $\frac{1}{37} = 0.\overline{027}$. We are looking for the 291st digit in the 3-digit repeating block 0-2-7. Since 291 is a multiple of 3, we want the last digit in the trio, which is $\boxed{7}$.

11.22

(a) $\boxed{\frac{21}{5}}$
(b) $\boxed{\frac{2}{25}}$
(c) $\boxed{-\frac{5}{4}}$
(d) $\boxed{\frac{364}{125}}$
(e) $\boxed{\frac{23}{3}}$
(f) $\boxed{\frac{4}{9}}$
(g) $\boxed{\frac{65}{11}}$
(h) $\boxed{\frac{13}{22}}$
(i) $\boxed{-\frac{24}{7}}$
(j) $\boxed{\frac{105}{37}}$
(k) $\boxed{\frac{21}{74}}$
(l) $\boxed{\frac{7}{55}}$

Challenge Problems

11.23

(a)

$$\frac{1}{25} = 1\cdot 5^{-2} = \boxed{0.01_5}.$$

(b)

$$\frac{130}{49} = 2 \cdot 7^0 + 4 \cdot 7^{-1} + 4 \cdot 7^{-2} = \boxed{2.44_7}.$$

(c) With some effort, we find a geometric series that gives us what we want:

$$(6 \cdot 8^{-2} + 3 \cdot 8^{-3} + 1 \cdot 8^{-4} + 4 \cdot 8^{-5})(1 + 8^{-4} + 8^{-8} + \cdots) = \boxed{0.0\overline{6314}_8}.$$

11.24

(a)

$$11.011_2 = 1 \cdot 2^1 + 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = \boxed{\frac{27}{8}}.$$

(b)

$$3.\bar{2}_4 = 3 \cdot 4^0 + \frac{2}{4^1} + \frac{2}{4^2} + \frac{2}{4^3} + \cdots = 3 + \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{11}{3} = \boxed{3.\bar{6}}.$$

(c)

$$-0.\overline{41}_6 = -\left(\frac{4}{6^1} + \frac{1}{6^2}\right)\left(1 + \frac{1}{6^2} + \frac{1}{6^4} + \frac{1}{6^6} + \cdots\right) = -\frac{25}{36} \cdot \frac{36}{35} = -\frac{5}{7} = \boxed{-0.\overline{714285}}.$$

11.25

(a) $\boxed{0.\overline{027}}$ (b) $\boxed{0.\overline{729}}$ (c) $\boxed{0.\overline{02439}}$

(d) $\boxed{0.\overline{0099}}$ (e) $\boxed{9.\overline{9009}}$ (f) $\boxed{0.\overline{0588235294117647}}$

(g) $\boxed{0.0\overline{1176470588235294}}$ (h) $\boxed{0.0\overline{105263157894736842}}$ (i) $\boxed{0.\overline{047619}}$

11.26 First, we find the repeating decimal expansion of our fraction: $\frac{1}{17} = 0.\overline{0588235294117647}$. We want the 4037th digit of a 16-digit block of digits. The remainder from $4037 \div 16$ is 5, so our answer is the 5th digit of the block, which is $\boxed{2}$.

11.27 The first term is 1/100 and the common ratio is 7/100, so

$$\frac{7^0}{100^1} + \frac{7^1}{100^2} + \frac{7^2}{100^3} + \frac{7^3}{100^4} \cdots = \frac{\frac{1}{100}}{1 - \frac{7}{100}} = \frac{\frac{1}{100}}{\frac{93}{100}} = \boxed{\frac{1}{93}}.$$

We add these fractions, noting that after the first few terms, the sum of the rest becomes negligible:

$$\frac{1}{93} = \boxed{0.010752\ldots}.$$

These are the first 6 digits of the repeating decimal block, but they are not the entire block.

11.28 During the long division process, there are only $p - 1$ remainders after division by p at each step (digit). Once a remainder repeats itself during long division, the entire long division process repeats from that point forward, so the cycle of digits can be at most $\boxed{p - 1}$ digits long.

11.29

(a) Each power of 10 adds exactly one digit, but note that $10^0 = 1$ has 1 digit, so 10^{30} has $\boxed{31}$ digits.

(b) We don't need to multiply out so many powers of 2. We note that $2^{10} = 1024$, which is very close to a power of 10. We use this result along with the algebraic expansion $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ to find the number of digits using an estimate of 2^{30}:

$$2^{30} = (10^3 + 24)^3 = 10^9 + 3(10^6)(24) + 3(10^3)(24^2) + 24^3.$$

The terms after the first are certainly much smaller than the first, so $10^9 < 2^{30} < 10^{10}$, meaning 2^{30} has $\boxed{10}$ digits.

(c) We want to multiply out 5^{30} even less than we wanted to multiply out 2^{30}, so we look for a helpful relationship between these large integers. Since $2^{30} \cdot 5^{30} = 10^{30}$, we note that

$$10^9 < 2^{30} < 10^{10} \quad \Rightarrow \quad \frac{10^{30}}{10^9} > \frac{10^{30}}{2^{30}} > \frac{10^{30}}{10^{10}}.$$

Simplifying all the fractions we find that

$$10^{21} > 5^{30} > 10^{20}.$$

5^{30} has the same number of digits as 10^{20}, which is $\boxed{21}$.

Notice the relationship between the answers: $31 = 10 + 21$. This relation is no accident. Understanding how it works can help you solve problems like this more quickly – or use this kind of calculation as a starting point for solving harder problems.

11.30 Creating a geometric series whose denominators are powers of 10 helps us see that computing the first few digits is just a matter of computing a few powers of 2:

$$\frac{1}{998} = \frac{1}{1000} + \frac{2^1}{1000^2} + \frac{2^2}{1000^3} + \cdots = \boxed{0.001002004008016032\ldots}.$$

Note that these are *not* the only digits in the repeating decimal block for $\frac{1}{998}$.

11.31

$$0.\overline{ab} = \frac{ab}{100^1} + \frac{ab}{100^2} + \frac{ab}{100^3} + \cdots = \frac{ab}{99}$$

In reduced form, the denominator is 99 divided by $\gcd(ab, 99)$, and must therefore be a divisor of 99. This divisor must be greater than 1 since $ab < 99$. Of the 6 positive divisors of 99, this leaves $6 - 1 = \boxed{5}$ possible denominators (3, 9, 11, 33, and 99).

11.32 If a number has a terminating decimal in base 8 then we can multiply it by $8^k = 2^{3k}$ for some positive integer k and the result will be a base-8 integer. That is, the result will be integral. This means we can express the number as a fraction whose denominator is a $\boxed{\text{pure power of 2}}$.

We generalize by noting that in base b, a number has a terminating decimal only when it can be multiplied by b^k to produce an integer. This means that m/n has a terminating decimal in base b when

$$\frac{m}{n} = \frac{a}{b^k}$$

for some integers a and k. Since $n = \frac{mb^k}{a}$, and $\gcd(m, n) = 1$, we know that $n \mid b^k$. In other words, the prime factorization of n contains only prime divisors in the prime factorization of b.

CHAPTER 12

Introduction to Modular Arithmetic

Exercises for Section 12.2

12.2.1

(a) 4	(e) 1	(i) 7
(b) 7	(f) 1	(j) 9
(c) 8	(g) 3	(k) 5
(d) 0	(h) 0	(l) 8

12.2.2

(a) $\frac{-18-6}{8} = \frac{-24}{8} = -3$ is an integer, so $-18 \equiv 6 \pmod 8$.

(b) $\frac{27-6}{8} = \frac{21}{8}$ is not an integer, so $27 \not\equiv 6 \pmod 8$.

(c) $\frac{54-6}{8} = \frac{48}{8} = 6$ is an integer, so $54 \equiv 6 \pmod 8$.

(d) $\frac{254-6}{8} = \frac{248}{8} = 31$ is an integer, so $254 \equiv 6 \pmod 8$.

(e) $\frac{754-6}{8} = \frac{748}{8} = \frac{187}{2}$ is not an integer, so $754 \not\equiv 6 \pmod 8$.

(f) $\frac{1036-6}{8} = \frac{1030}{8} = \frac{515}{4}$ is not an integer, so $1036 \not\equiv 6 \pmod 8$.

(g) $\frac{6310-6}{8} = \frac{6304}{8} = 788$ is an integer, so $6310 \equiv 6 \pmod 8$.

(h) $\frac{13254-6}{8} = \frac{13248}{8} = 1656$ is an integer, so $13254 \equiv 6 \pmod 8$.

12.2.3

(a) $\frac{-311-3}{11} = \frac{-314}{11}$ is not an integer, so $-311 \not\equiv 3 \pmod{11}$.

(b) $\frac{-8-3}{11} = \frac{-11}{11} = -1$ is an integer, so $-8 \equiv 3 \pmod{11}$.

(c) $\frac{8-3}{11} = \frac{5}{11}$ is not an integer, so $8 \not\equiv 3 \pmod{11}$.

(d) $\frac{33-3}{11} = \frac{30}{11}$ is not an integer, so $33 \not\equiv 3 \pmod{11}$.

(e) $\frac{410-3}{11} = \frac{407}{11} = 37$ is an integer, so $410 \equiv 3 \pmod{11}$.

(f) $\frac{2379-3}{11} = \frac{2376}{11} = 216$ is an integer, so $2379 \equiv 3 \pmod{11}$.

12.2.4 We are looking for multiples of 15. Make sure you see why.

(a) $\frac{-415}{15} = -\frac{83}{3}$ is not an integer, so $-415 \not\equiv 0 \pmod{15}$.

(b) $\frac{-75}{15} = -5$ is an integer, so $-75 \equiv 0 \pmod{15}$.

(c) $\frac{25}{15} = \frac{5}{3}$ is not an integer, so $25 \not\equiv 0 \pmod{15}$.

(d) $\frac{155}{15} = \frac{31}{3}$ is not an integer, so $155 \not\equiv 0 \pmod{15}$.

(e) $\frac{555}{15} = 37$ is an integer, so $555 \equiv 0 \pmod{15}$.

(f) $\frac{7275}{15} = 485$ is an integer, so $7275 \equiv 0 \pmod{15}$.

12.2.5

(a) $\frac{118-25}{13} = \frac{93}{13}$ is not an integer, so $118 \not\equiv 25 \pmod{13}$. $\boxed{\text{False.}}$

(b) $\frac{2401-147}{49} = \frac{2254}{49} = 46$ is an integer, so $2401 \equiv 147 \pmod{49}$. $\boxed{\text{True.}}$

(c) $\frac{183-291}{6} = \frac{-108}{6} = -18$ is an integer, so $183 \equiv 291 \pmod{6}$. $\boxed{\text{True.}}$

(d) $\frac{2701-14393}{8} = \frac{-11692}{8} = -\frac{2923}{2}$ is not an integer, so $2701 \not\equiv 14393 \pmod{8}$. $\boxed{\text{False.}}$

(e) $\frac{493-873}{10} = \frac{-380}{10} = -38$ is an integer, so $493 \equiv 873 \pmod{10}$. $\boxed{\text{True.}}$

(f) $\frac{4113-396}{9} = \frac{3717}{9} = 413$ is an integer, so $4113 \equiv 396 \pmod{9}$. $\boxed{\text{True.}}$

12.2.6 An integer congruent to 1 (mod 9) can be written in the form $9n+1$ for some integer n. We want to count the number of integers n such that

$$1 \le 9n + 1 \le 200.$$

Subtracting 1 from all parts of the inequality, we get $0 \le 9n \le 199$. Dividing by 9 we get $0 \le n \le 22\frac{1}{9}$. There are $22 - 0 + 1 = \boxed{23}$ values of n corresponding to positive integers from 1 to 200 inclusive that are congruent to 1 (mod 9).

Exercises for Section 12.3

12.3.1 The modulo-3 residues are the integers r such that $0 \le r < 3$, which are $\boxed{0}$, $\boxed{1}$, and $\boxed{2}$.

12.3.2

(a) $11 = 8 \cdot 1 + 3$.

(b) $23 = 8 \cdot 2 + 7$.

(c) $54 = 8 \cdot 6 + 6$.

(d) $99 = 8 \cdot 12 + 3$.

(e) $434 = 8 \cdot 54 + 2$.

(f) $812 = 8 \cdot 101 + 4$.

12.3.3

$$\begin{array}{llcllcrcl}
(a) & 71 & = & 23\cdot 3+2 & \Rightarrow & 71 & \equiv & \boxed{2} \pmod{3}\\
(b) & -14 & = & -2\cdot 8+2 & \Rightarrow & -14 & \equiv & \boxed{2} \pmod{8}\\
(c) & 14 & = & 1\cdot 8+6 & \Rightarrow & 14 & \equiv & \boxed{6} \pmod{8}\\
(d) & 194 & = & 17\cdot 11+7 & \Rightarrow & 194 & \equiv & \boxed{7} \pmod{11}\\
(e) & 3944 & = & 438\cdot 9+2 & \Rightarrow & 3944 & \equiv & \boxed{2} \pmod{9}\\
(f) & 471 & = & 22\cdot 21+9 & \Rightarrow & 471 & \equiv & \boxed{9} \pmod{21}
\end{array}$$

12.3.4

$$\begin{array}{llcllcrcl}
(a) & 11 & = & 1\cdot 6+5 & \Rightarrow & 11 & \equiv & \boxed{5} \pmod{6}\\
(b) & 23 & = & 3\cdot 6+5 & \Rightarrow & 23 & \equiv & \boxed{5} \pmod{6}\\
(c) & 37 & = & 6\cdot 6+1 & \Rightarrow & 37 & \equiv & \boxed{1} \pmod{6}\\
(d) & 54 & = & 9\cdot 6+0 & \Rightarrow & 54 & \equiv & \boxed{0} \pmod{6}\\
(e) & 99 & = & 16\cdot 6+3 & \Rightarrow & 99 & \equiv & \boxed{3} \pmod{6}\\
(f) & 219 & = & 36\cdot 6+3 & \Rightarrow & 219 & \equiv & \boxed{3} \pmod{6}\\
(g) & 434 & = & 72\cdot 6+2 & \Rightarrow & 434 & \equiv & \boxed{2} \pmod{6}\\
(h) & 812 & = & 135\cdot 6+2 & \Rightarrow & 812 & \equiv & \boxed{2} \pmod{6}\\
(i) & 1529 & = & 254\cdot 6+5 & \Rightarrow & 1529 & \equiv & \boxed{5} \pmod{6}
\end{array}$$

12.3.5 Since $a \equiv b \pmod{m}$, we know that $a - b = k_1 m$ for some integer k_1. Likewise, since $b \equiv c \pmod{m}$, we know that $b - c = k_2 m$ for some integer $k_2 m$. Now,

$$(a-b)+(b-c) = k_1 m + k_2 m \quad\Rightarrow\quad a - c = (k_1 + k_2)m.$$

Therefore, $a - c$ is a multiple of m and $a \equiv c \pmod{m}$.

Exercises for Section 12.4

12.4.1 The numbers on the right side of each modulus are residues for the modulus. Comparing the residues of the numbers on the left-hand side to the numbers on the right-hand side is an easy way to determine whether or not each statement of congruence is true.

(a) $\boxed{\text{True.}}$

(b) $\boxed{\text{True.}}$

(c) $\boxed{\text{False.}}$ $403 + 397 \equiv 3 + 5 \not\equiv 3 + 7 \pmod{8}$.

(d) $\boxed{\text{True.}}$

(e) $\boxed{\text{False.}}$ $134 + 453 - 217 \equiv 2 + 9 - 1 \not\equiv 2 + 3 - 1 \pmod{12}$.

(f) $\boxed{\text{True.}}$

12.4.2 Grouping residues helps make some series computations easier:

$$1+2+3+0+1+2+3+0+1+2+3+0 \equiv 3(1+2+3+0) \equiv 18 \equiv \boxed{2} \pmod{4}.$$

12.4.3 The marbles will be grouped into piles of 10. We might as well group the number of marbles each of Sally, Wei-Hwa, and Zoe brought into as many piles of 10 as possible before sorting out the rest. This means we only need to consider the modulo-10 residues of the numbers of marbles each of them brought:

$$\begin{aligned} 239 &\equiv 9 \pmod{10} \\ 174 &\equiv 4 \pmod{10} \\ 83 &\equiv 3 \pmod{10} \end{aligned}$$

Our goal is to find the modulo-10 residue of the total number of marbles. We find this by adding the residues above: $9+4+3=16 \equiv \boxed{6} \pmod{10}$. Since we were working modulo 10, this is the same as a units digit calculation.

12.4.4 The remainder when $a+b+c$ is divided by 13 is just the modulo-13 residue of $a+b+c$. In order to find that residue, we sum the modulo-13 residues of a, b, and c:

$$a+b+c \equiv 4+7+9 \equiv 20 \equiv \boxed{7} \pmod{13}.$$

12.4.5 We use residues of the numbers of each type of coin to determine the number of dimes and quarters leftover:

$$\begin{aligned} 83+129 &\equiv 3+9 \equiv 12 \pmod{40} \\ 159+266 &\equiv 9+16 \equiv 25 \pmod{50} \end{aligned}$$

The total value of the leftover quarters and dimes is

$$12(\$0.25)+25(\$0.10) = \$3.00+\$2.50 = \boxed{\$5.50}.$$

Exercises for Section 12.5

12.5.1

(a) $17 \cdot 18 \equiv 1 \cdot 2 \equiv \boxed{2} \pmod{4}$.

(b) $523 \cdot 421 \equiv 3 \cdot 1 \equiv \boxed{3} \pmod{4}$.

(c) $15^{15} \equiv (-1)^{15} \equiv -1 \equiv \boxed{3} \pmod{4}$.

(d) $121 \cdot 122 \cdot 123 \equiv 1 \cdot 2 \cdot 3 \equiv 6 \equiv \boxed{2} \pmod{4}$.

(e) $100! = 100 \cdot 99! \equiv 0 \cdot 99! \equiv \boxed{0} \pmod{4}$.

(f) $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \equiv 1 \cdot (-1) \cdot 1 \cdot (-1) \cdot 1 \equiv \boxed{1} \pmod{4}$.

12.5.2 Since the remainder of the product is just the modulo-11 residue, we multiply residues to get our answer:

$$514 \cdot 891 \equiv 8 \cdot 0 \equiv \boxed{0} \pmod{11}.$$

12.5.3 The product of the three integers is equivalent modulo 5 to the product of the modulo-5 residues of the three integers. We multiply these residues to find the remainder:

$$1 \cdot 2 \cdot 3 \equiv 6 \equiv \boxed{1} \pmod{5}.$$

12.5.4 Sometimes working with negative numbers is even easier than working with residues in modular arithmetic:

$$317 \cdot 5^{51} \equiv (-1) \cdot (-1)^{51} \equiv \boxed{1} \pmod{6}.$$

12.5.5 First, we compute the total amount of money in each of John's bags:

$$\$0.25 + \$0.10 + \$0.05 + \$0.01 = \$0.41.$$

The values of the denominations John receives other than pennies (dollars and dimes) are each worth a number of cents that is a multiple of 10. This means that the number of pennies he gets is the modulo-10 residue of the number of cents he has total. We use the modulo-10 residues of 73 and 41 to find that residue:

$$73 \cdot 41 \equiv 3 \cdot 1 \equiv \boxed{3} \pmod{10}.$$

12.5.6 We note that $24 \equiv -2 \pmod{13}$ and $15 \equiv 2 \pmod{13}$. We cleverly use these congruences to set up values that vanish in the arithmetic:

$$24^{50} - 15^{50} \equiv (-2)^{50} - 2^{50} \equiv 2^{50} - 2^{50} \equiv \boxed{0} \pmod{13}.$$

Exercises for Section 12.6

12.6.1 We use the first pure power of each integer [base] congruent to 1 modulo 5 to simplify computation:

(a) $2^4 = 16 \equiv 1 \pmod{5}$, so $2^8 = 2^{2\cdot 4} = (2^4)^2 = 16^2 \equiv 1^2 \equiv \boxed{1} \pmod{5}$.

(b) $3^4 = 81 \equiv 1 \pmod{5}$, so $3^{19} = 3^{4\cdot 4+3} = (3^4)^4 \cdot 3^3 \equiv 81^4 \cdot 27 \equiv 1^4 \cdot 2 \equiv \boxed{2} \pmod{5}$.

(c) $4^2 = 16 \equiv 1 \pmod{5}$, so $4^{55} = 4^{2\cdot 27+1} = (4^2)^{27} \cdot 4^1 \equiv 1^{27} \cdot 4 \equiv \boxed{4} \pmod{5}$.

(d) $7^4 \equiv 2^4 = 16 \equiv 1 \pmod{5}$, so $7^{17} = 7^{4\cdot 4+1} = (7^4)^4 \cdot 7^1 \equiv 1^4 \cdot 2 \equiv \boxed{2} \pmod{5}$.

(e) $19^2 \equiv (-1)^2 \equiv 1 \pmod{5}$, so $19^{77} = 19^{2\cdot 38+1} = (19^2)^{38} \cdot 19^1 \equiv 1^{38} \cdot 4 \equiv \boxed{4} \pmod{5}$.

(f) $2^4 = 16 \equiv 1 \pmod{5}$, so $14^{92} \cdot 17^{76} \equiv (-1)^{92} \cdot 2^{76} \equiv 2^{76} \equiv 2^{4\cdot 19} \equiv (2^4)^{19} \equiv 1^{19} \equiv \boxed{1} \pmod{5}$.

12.6.2

(a) We square each possible modulo-4 residue:

$$\begin{aligned} 0^2 &= 0 \equiv 0 \pmod{4} \\ 1^2 &= 1 \equiv 1 \pmod{4} \\ 2^2 &= 4 \equiv 0 \pmod{4} \\ 3^2 &= 9 \equiv 1 \pmod{4} \end{aligned}$$

The only possible modulo-4 residues of a perfect square are $\boxed{0}$ and $\boxed{1}$.

(b) Note that $10511 \equiv 3 \pmod{4}$. We are looking for squares a^2 and b^2 such that

$$a^2 + b^2 = 10511 \equiv 3 \pmod{4}.$$

But, since a perfect square must be congruent to 0 or 1 (mod 4), the sum of two perfect squares must be one of the following:

$$\begin{aligned} 0+0 &\equiv 0 \pmod 4 \\ 0+1 &\equiv 1 \pmod 4 \\ 1+1 &\equiv 2 \pmod 4 \end{aligned}$$

The sum of two squares cannot be congruent to 3 (mod 4), so there are no solutions to the equation.

12.6.3 For a positive integer k, the number $2k$ is even and

$$9^{2k} = 81^k \equiv 1^k \equiv 1 \pmod{10}.$$

This means that 9^{2k} has a units digit of 1 for any integer k. Since 8 is even, 8^7 is even. This means that 9^{8^7} can be written as 9^{2k} for some integer k. This means that 9^{8^7} has a units digit of $\boxed{1}$.

12.6.4 The tens and units digits of a positive integer are the same as in its modulo-100 residue. Note that $7^4 = 2401 \equiv 1 \pmod{100}$. This allows us to simplify the exponential modulo 100:

$$7^{2006} = 7^{501\cdot 4+2} = (7^4)^{501} \cdot 7^2 \equiv 1^{501} \cdot 49 \equiv 49 \pmod{100}.$$

So the tens digit is $\boxed{4}$ and the units digit is $\boxed{9}$. While exploring this problem, you should have noticed that there are only 4 sets of units/tens digits for a pure power of 7: 01, 07, 49, and 43.

12.6.5 First, we note that the integers 1 through 99 inclusive cycle through modulo-9 residues 11 times. This means that

$$1^2+2^2+3^2+\cdots+9^2 \equiv 11(1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+0^2) \pmod 9.$$

We could simply compute what's inside the parentheses, but we can use negative numbers to simplify calculations:

$$\begin{aligned} 11(1^2+2^2+\cdots+8^2+0^2) &\equiv 11[1^2+2^2+3^2+4^2+(-4)^2+(-3)^2+(-2)^2+(-1)^2] \\ &\equiv 11[1^2+2^2+3^2+4^2+4^2+3^2+2^2+1^2] \\ &\equiv 22(1^2+2^2+3^2+4^2) \\ &\equiv 22(1+4+9+16) \\ &\equiv 22(30) \\ &\equiv 4(3) \\ &\equiv 3 \pmod 9 \end{aligned}$$

Another solution involves use of the formula for the sum of the squares of the n smallest positive integers. See if you can find this solution on your own.

12.6.6 Since $a \equiv b \pmod{m_1}$, we know that $a - b = k_1m_1$ for some integer k_1. Additionally, $m_2 \mid m_1$, meaning that $m_1 = k_2m_2$ for some integer k_2. Thus,

$$a - b = k_1m_1 = k_1(k_2m_2) = k_1k_2m_2.$$

Since $a - b$ is a multiple of m_2, we know that $a \equiv b \pmod{m_2}$.

Review Problems

12.23

(a) $13 \equiv \boxed{1} \pmod 6$.

(b) $53 \equiv \boxed{5} \pmod 6$.

(c) $84 \equiv \boxed{0} \pmod 6$.

(d) $184 \equiv \boxed{4} \pmod 6$.

(e) $63 + 91 \equiv 3 + 1 \equiv \boxed{4} \pmod 6$.

(f) $141 - 78 \equiv 3 - 0 \equiv \boxed{3} \pmod 6$.

(g) $519 - 444 + 37 \equiv 3 - 0 + 1 \equiv \boxed{4} \pmod 6$.

(h) $12 - 11 + 10 - 9 + 8 - 7 = (12 - 11) + (10 - 9) + (8 - 7) = 1 + 1 + 1 \equiv \boxed{3} \pmod 6$.

(i) $43 \cdot 32 \equiv 1 \cdot 2 \equiv \boxed{2} \pmod 6$.

(j) $59 \cdot 159 \equiv 5 \cdot 3 \equiv 15 \equiv \boxed{3} \pmod 6$.

(k) $14^6 \equiv 2^6 \equiv 64 \equiv \boxed{4} \pmod 6$.

(l) $101^{99} \equiv (-1)^{99} \equiv -1 \equiv \boxed{5} \pmod 6$.

12.24

(a) $\frac{177-17}{8} = \frac{160}{8} = 20$ is an integer, so $177 \equiv 17 \pmod 8$. $\boxed{\text{True.}}$

(b) $\frac{871-713}{29} = \frac{158}{29}$ is not an integer, so $871 \not\equiv 713 \pmod{29}$. $\boxed{\text{False.}}$

(c) $\frac{1322-5294}{12} = \frac{-3972}{12} = -331$ is an integer, so $1322 \equiv 5294 \pmod{12}$. $\boxed{\text{True.}}$

(d) $\frac{5141-8353}{11} = \frac{-3212}{11} = -292$ is an integer, so $5141 \equiv 8353 \pmod{11}$. $\boxed{\text{True.}}$

(e) $\frac{13944-8919}{13} = \frac{5025}{13}$ is not an integer, so $13944 \not\equiv 8919 \pmod{13}$. $\boxed{\text{False.}}$

(f) $67 \cdot 73 \equiv 2 \cdot 3 \equiv 6 \not\equiv 1 \cdot 3 \pmod 5$. $\boxed{\text{False.}}$

(g) $17 \cdot 18 \cdot 19 \cdot 20 \equiv 1 \cdot 2 \cdot 3 \cdot 4 \equiv 4! \pmod 8$. $\boxed{\text{True.}}$

(h) $83^{144} \equiv (-2)^{144} \equiv 15^{144} \pmod{17}$. $\boxed{\text{True.}}$

12.25 Integers congruent to 5 (mod 7) are the ones that can be written as $7n + 5$ for some integer n. We are counting the values of n such that

$$1000 \le 7n + 5 \le 3000.$$

Subtracting 5 from each part of the inequality, we get

$$995 \le 7n \le 2995.$$

Dividing each part by 7, we get $142\,\frac{1}{7} \le n \le 427\,\frac{6}{7}$. There are $427 - 143 + 1 = \boxed{285}$ such values of n.

12.26 If $n \equiv -n \pmod{12}$, then $n - (-n) = 12k$ for some integer k.

$$2n = 12k \Rightarrow n = 6k.$$

Now we solve the inequality chain

$$40 \le 6k \le 80$$

for integers k. Dividing everything by 6, we get

$$6\frac{2}{3} \le k \le 13\frac{1}{3},$$

so $7 \le k \le 13$ and there are $13 - 7 + 1 = \boxed{7}$ integers k that correspond to the values of n such that $n \equiv -n \pmod{12}$.

12.27 Let the number of jelly beans in the individual jars be $a, b, c,$ and d. The modulo-12 residues of a, $b, c,$ and d are 3, 5, 7, and 11. So,

$$a + b + c + d \equiv 3 + 5 + 7 + 11 = 26 \equiv 2 \pmod{12}.$$

Combining the jars of jelly beans before packaging would have left out only $\boxed{2}$ jelly beans.

12.28

(a) $6 + 13 = 19 \equiv \boxed{5} \pmod{14}$.

(b) $6 - 13 \equiv -7 \equiv 7 \equiv 13 - 6 \pmod{14}$. Regardless of which way we subtract, the remainder is $\boxed{7}$.

(c) $6 \cdot 13 = 78 \equiv \boxed{8} \pmod{14}$.

(d) Since 7 is a divisor of 14, we can use the modulo-14 residue of the product to find the modulo-7 residue:

$$8 = 1 \cdot 7 + 1 \equiv \boxed{1} \pmod{7}.$$

12.29 Multiplying $a \equiv 19 \pmod{30}$ by 3, we get $3a \equiv 57 \equiv 27 \pmod{30}$. This means that

$$3a - 27 = 30k$$

for some integer k. Adding 20 to both sides of this equation and factoring, we get

$$3a - 7 = 30k + 20 = 10(3k + 2).$$

Since $3a - 7$ is a multiple of 10, $3a \equiv 7 \pmod{10}$.

12.30 It takes a while to find that $7^{10} \equiv 1 \pmod{11}$. We use it to break down the problem:

$$7^{255} = 7^{10 \cdot 25 + 5} = (7^{10})^{25} \cdot 7^5 \equiv 1^{25} \cdot 16807 \equiv 1 \cdot 10 \equiv \boxed{10} \pmod{11}.$$

12.31 Note that $9 \equiv 2 \pmod{7}$ and $5 \equiv -2 \pmod{7}$:

$$9^{42} - 5^{42} \equiv 2^{42} - (-2)^{42} \equiv 2^{42} - 2^{42} \equiv \boxed{0} \pmod{7}.$$

12.32 Squaring each of the modulo-6 residues, we find that perfect squares can only have modulo-6 residues of 0, 1, 3, and 4. As Susie already knows, this means that no perfect square leaves a remainder of 2 when divided by 6.

Challenge Problems

12.33 The last two digits of a positive integer are its modulo-100 residue.

(a) $99^{2005} \equiv (-1)^{2005} \equiv -1 \equiv \boxed{99} \pmod{100}$.

(b) $7^{603} \equiv 7^{4\cdot 150+3} \equiv (7^4)^{150} \cdot 7^3 \equiv 1^{150} \cdot 343 \equiv \boxed{43} \pmod{100}$.

12.34 One solution involves directly applying a formula for the sum of the first n positive integers (or for a more general arithmetic series). Our solution groups integers into pairs with constants sums (in the spirit of the derivation of formulas for arithmetic series). These pairings allow us to easily compute the modulo-200 residue:

$$\begin{aligned} 1+2+3+\cdots+200 &= (1+200)+(2+199)+(3+198)+\cdots+(100+101) \\ &= 100(201) \\ &\equiv 100(1) \\ &\equiv \boxed{100} \pmod{200} \end{aligned}$$

12.35 The difference between $617n$ and $943n$ is a multiple of 18, so

$$\frac{943n-617n}{18} = \frac{326n}{18} = \frac{163n}{9}$$

is an integer. This means n must be a multiple of 9 and the smallest possible value is $\boxed{9}$.

12.36 Let's take a look at the modulo-8 residues of the first few summands:

$$\begin{aligned} 17 &\equiv 1 \pmod 8 \\ 177 &\equiv 1 \pmod 8 \\ 1777 &\equiv 1 \pmod 8 \\ &\vdots \end{aligned}$$

It appears that all 20 summands are congruent to 1 (mod 8). We can be sure by considering the differences between consecutive terms:

$$\begin{array}{lcrcr} 177-17 &=& 160 &=& 20\cdot 8 \\ 1777-177 &=& 1600 &=& 200\cdot 8 \\ 17777-1777 &=& 16000 &=& 2000\cdot 8 \\ & & \vdots & & \vdots \end{array}$$

Now we see that

$$17+177+1777+\cdots+17777777777777777777 \equiv 1+1+1+\cdots+1 \equiv 20 \equiv \boxed{4} \pmod 8.$$

12.37 Note that $1^2 \equiv 2^2 \equiv 1 \pmod 3$. The only possible modulo-3 residue for a square that isn't a multiple of 3 is 1. Therefore, $a^2+b^2 \equiv 1+1 \equiv \boxed{2} \pmod 3$.

12.38 Zhenya begins by removing 1 stick from the pile so that 81 remain. From there, each time Ryun picks x sticks, Zhenya picks $5-x$ sticks. After each of Zhenya's moves, the total number of sticks left in the pile is 5 less than before. The number of sticks after Zhenya's moves is always congruent to 1 (mod 5). Following this strategy, Zhenya knows she can leave Ryun with the final stick and win the game.

12.39 From the units digit of n, we know that $n \equiv 10 \pmod{12}$. Since $6 \mid 12$, we also know that $n \equiv 10 \equiv 4 \pmod 6$. Squaring, we get $n^2 \equiv 4^2 \equiv 16 \equiv \boxed{4} \pmod 6$.

12.40 Since $69 \equiv 90 \equiv 125 \pmod N$, we know that both of the following are integers:

$$\frac{125-90}{N} = \frac{35}{N}$$

$$\frac{90-69}{N} = \frac{21}{N}$$

So, N is a divisor of $\gcd(21, 35) = 7$. Since $N > 1$, $N = 7$. So, $81 \equiv \boxed{4} \pmod 7$.

12.41 Let n be the positive integer in question.

(a) $n \equiv 4 \pmod 9$ implies that $n = 9x + 4 = 3(3x + 1) + 1$, which means $n \equiv 1 \pmod 3$.

(b) $n \equiv 5 \pmod 8$ implies that $n = 8x + 5 = 4(2x + 1) + 1$, which means $n \equiv 1 \pmod 4$.

(c) Since $n \equiv 1 \pmod 3$, $n \equiv 1 \pmod 4$, and $n \equiv 1 \pmod 5$, $n - 1$ is divisible by 3, 4, and 5, which means $n - 1$ must be divisible by $\text{lcm}[3, 4, 5] = 60$. Thus $n - 1 \equiv 0 \pmod{60}$, so $n \equiv 1 \pmod{60}$.

12.42 This problem looks rather daunting. Without a clear direction, we might as well work out the remainder when the first term is divided by 7. If we are lucky, we'll notice something along the way that will help us.

$$10^{10} \equiv 3^{10} \equiv 9^5 \equiv 2^5 \equiv 32 \equiv 4 \pmod 7.$$

Now we do the same with the second term using its relationship to the first term:

$$10^{100} \equiv (10^{10})^{10} \equiv 4^{10} \equiv (-3)^{10} \equiv 3^{10} \equiv 4 \pmod 7.$$

The remainder is the same. In fact, each term is the previous term to the 10^{th} power, and $4^{10} \equiv 4 \pmod 7$. So all the terms are congruent to 4 (mod 7). Now we compute the answer:

$$10^{10} + 10^{100} + \cdots + 10^{10000000000} \equiv 4 + 4 + \cdots + 4 \equiv 10(4) \equiv 40 \equiv \boxed{5} \pmod 7.$$

12.43 We prove more generally that for k people, the number of people who shook hands with an odd number of others must be even.

Let the number of handshakes made by each person be $n_1, n_2, n_3, \ldots, n_k$. Since each handshake involves two people, the sum of the individual handshake totals is always even:

$$n_1 + n_2 + n_3 + \cdots + n_k \equiv 0 \pmod 2.$$

Each handshake total $n_1, n_2, \ldots$ is congruent to either 0 or 1 (mod 2). The number congruent to 1 (mod 2) must be even in order for the sum to be 0 (mod 2).

12.44 The sum of two integers is a multiple of 7 if and only if the sum of their modulo-7 residues is a multiple of 7. In order to get a grip on how we can maximize a subset without such a pair whose sum is a multiple of 7, we divide the integers from 1 to 50 inclusive into modulo-7 residue classes:

	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50					

Residue classes make it easier to see how integers match up in pairs with sums that are multiples of 7. If we add a pair with residues 1 and 6, we get a multiple of 7. Likewise for pairs with residues of 2 and 5 or residues of 3 and 4. Adding a pair of integers with residues of 0 also results in a sum that is a multiple of 7.

Now we construct the largest possible subset. We can include at most 1 integer with a modulo 7 residue of 0. We can include all of the integers with residues of 1 or all of the integers with residues of 6. There are more integers with residues of 1, so we include all 8 of them. We include all 7 integers with residues of 2 and all 7 with residues of 3, leaving out those with residues of 4 or 5 (though we could just as easily reverse which residues we include). The final result is a subset S with $1+8+7+7=\boxed{23}$ integers.

12.45 The first thing we do is compute modulo-5 residues for small Fibonacci numbers. Our hope is to find something helpful:

$$
\begin{array}{lclclcll}
F_1 & = & 1 & & & \equiv & 1 & \pmod 5 \\
F_2 & = & 1 & & & \equiv & 1 & \pmod 5 \\
F_3 & = & F_1+F_2 & \equiv & 1+1 & \equiv & 2 & \pmod 5 \\
F_4 & = & F_2+F_3 & \equiv & 1+2 & \equiv & 3 & \pmod 5 \\
F_5 & = & F_3+F_4 & \equiv & 2+3 & \equiv & 0 & \pmod 5 \\
F_6 & = & F_4+F_5 & \equiv & 3+0 & \equiv & 3 & \pmod 5 \\
F_7 & = & F_5+F_6 & \equiv & 0+3 & \equiv & 3 & \pmod 5 \\
F_8 & = & F_6+F_7 & \equiv & 3+3 & \equiv & 1 & \pmod 5 \\
F_9 & = & F_7+F_8 & \equiv & 3+1 & \equiv & 4 & \pmod 5 \\
F_{10} & = & F_8+F_9 & \equiv & 1+4 & \equiv & 0 & \pmod 5 \\
F_{11} & = & F_9+F_{10} & \equiv & 4+0 & \equiv & 4 & \pmod 5 \\
F_{12} & = & F_{10}+F_{11} & \equiv & 0+4 & \equiv & 4 & \pmod 5 \\
F_{13} & = & F_{11}+F_{12} & \equiv & 4+4 & \equiv & 3 & \pmod 5 \\
F_{14} & = & F_{12}+F_{13} & \equiv & 4+3 & \equiv & 2 & \pmod 5 \\
F_{15} & = & F_{13}+F_{14} & \equiv & 3+2 & \equiv & 0 & \pmod 5
\end{array}
$$

$$
\begin{array}{lclclcll}
F_{16} & = & F_{14}+F_{15} & \equiv & 2+0 & \equiv & 2 & \pmod 5 \\
F_{17} & = & F_{15}+F_{16} & \equiv & 0+2 & \equiv & 2 & \pmod 5 \\
F_{18} & = & F_{16}+F_{17} & \equiv & 2+2 & \equiv & 4 & \pmod 5 \\
F_{19} & = & F_{17}+F_{18} & \equiv & 2+4 & \equiv & 1 & \pmod 5 \\
F_{20} & = & F_{18}+F_{19} & \equiv & 4+1 & \equiv & 0 & \pmod 5 \\
F_{21} & = & F_{19}+F_{20} & \equiv & 1+0 & \equiv & 1 & \pmod 5 \\
F_{22} & = & F_{20}+F_{21} & \equiv & 0+1 & \equiv & 1 & \pmod 5 \\
F_{23} & = & F_{21}+F_{22} & \equiv & 1+1 & \equiv & 2 & \pmod 5 \\
F_{24} & = & F_{22}+F_{23} & \equiv & 1+2 & \equiv & 3 & \pmod 5 \\
F_{25} & = & F_{23}+F_{24} & \equiv & 2+3 & \equiv & 0 & \pmod 5 \\
F_{26} & = & F_{24}+F_{25} & \equiv & 3+0 & \equiv & 3 & \pmod 5 \\
F_{27} & = & F_{25}+F_{26} & \equiv & 0+3 & \equiv & 3 & \pmod 5 \\
F_{28} & = & F_{26}+F_{27} & \equiv & 3+3 & \equiv & 1 & \pmod 5 \\
F_{29} & = & F_{27}+F_{28} & \equiv & 3+1 & \equiv & 4 & \pmod 5 \\
F_{30} & = & F_{28}+F_{29} & \equiv & 1+4 & \equiv & 0 & \pmod 5
\end{array}
$$

A lot of patterns emerge in these calculations. Most importantly, we see that after 20 numbers, a pattern of residues begins to repeat itself.

Each Fibonacci number (after the first two) is the sum of the previous two Fibonacci numbers, so its residue results from adding the residues of that previous pair. There are a limited number of possible pairs of modulo-5 residues, so the residues of a pair of consecutive Fibonacci numbers necessarily repeats:

$$
\begin{array}{lclclcll}
F_3 & = & F_1+F_2 & \equiv & 1+1 & \equiv & 2 & \pmod 5 \\
F_{23} & = & F_{21}+F_{22} & \equiv & 1+1 & \equiv & 2 & \pmod 5
\end{array}
$$

Once a pair of consecutive residues repeats, the following residues necessarily repeat as each is the sum of the previous two. Thus we know that $F_m = F_{m+20}$ for any positive integer m.

Now we relate F_{2006} to previous Fibonacci numbers:

$$F_{2006} \equiv F_{1986} \equiv F_{1966} \equiv \cdots \equiv F_6 \equiv \boxed{3} \pmod 5.$$

12.46 All primes greater than 2 are odd:

$$(2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1.$$

Since either k or $k+1$ is even, $4k(k+1)+1 \equiv 1 \pmod 8$, meaning the square of any odd integer is 1 more than a multiple of 8.

All primes greater than 3 are congruent to either 1 or 2 (mod 3) and $1^2 \equiv 2^2 \equiv 1 \pmod 3$, so the square of an integer that is not a multiple of 3 is 1 more than a multiple of 3.

Any prime $p > 3$ is neither a multiple of 2 nor of 3. This means p^2 is 1 more than a multiple of each of 3 and 8 and therefore a multiple of lcm[3, 8] = 24. Thus $24 \mid p^2 - 1$.

The only exceptions are $p = \boxed{2}$ where $2^2 - 1 = 3$ and $p = \boxed{3}$ where $3^2 - 1 = 8$.

12.47 The pigeonhole principle tells us that among any 51 integers, there are 2 with the same modulo-50 residue. Each of these integers can be written as $50n + r$ where n is an integer and r is the shared residue. Note that

$$(50n + r)^2 = 2500n^2 + 100nr + r^2 \equiv 100(25n^2 + nr) + r^2 \equiv r^2 \pmod{100}.$$

The value of n does not affect the modulo-100 residue of the square, so all integers congruent to r (mod 50) have squares with the same last two digits (that is, the same remainder when divided by 100).

12.48 When applying modular arithmetic to difficult number theory problems, the difficult part is often coming up with the right modulus to use. The number of possible residues of squares in modulos 3, 4, and 8 are limited, which is often helpful for solving problems. In this case, modulo 4 can help us.

The product of the n smallest primes is a multiple of 2, but not a multiple of 4. This means that

$$p_1p_2p_3\cdots p_n \equiv 2 \pmod 4 \quad\Rightarrow\quad p_1p_2p_3\cdots p_n + 1 \equiv 3 \pmod 4.$$

Since squares are congruent to either 0 or 1 (mod 4), the number $p_1p_2p_3\cdots p_n + 1$ is not a square.

CHAPTER 13

Divisibility Rules

Exercises for Section 13.2

13.2.1 Note that there are many ways to go about some or all of these problems other than the solutions below.

(a) Since the last two digits of 525 are $5^2 = 25$, we know that 5^2 is a divisor of 525. From there, $525 \div 25 = 21$ and we quickly finish off the prime factorization: $525 = \boxed{3^1 \cdot 5^2 \cdot 7^1}$.

(b) Since the last two digits of 1408 make a multiple of 4, we divide $1408 \div 4 = 352$. We make this observation several more times until we have $1408 = 4 \cdot 4 \cdot 4 \cdot 22 = \boxed{2^7 \cdot 11^1}$.

(c) We divide out factors of 2 until we're left with an odd number: $22572 = 2^2 \cdot 5643$. Since the sum of the digits of 5643 is 18, which is a multiple of 9, we factor out $9 = 3^2$ to get $22572 = 2^2 \cdot 3^2 \cdot 627$. The sum of the digits of 627 is 15, which is a multiple of 3, so we factor out one more multiple of 3: $22572 = 2^2 \cdot 3^3 \cdot 209$. The alternating digit sum of 209 is 11, so $11 \mid 209$. Finally, we have $22572 = \boxed{2^2 \cdot 3^3 \cdot 11^1 \cdot 19^1}$.

(d) The two trailing digits of 82200 tell us that $82200 = 822 \cdot 10^2 = 2^2 \cdot 5^2 \cdot 822$. Since 822 is even, we factor out another power of 2: $82200 = 2^3 \cdot 5^2 \cdot 411$. The sum of the digits of 411 is 6, which is a multiple of 3, so we can factor out a multiple of 3: $82200 = 2^3 \cdot 3^1 \cdot 5^2 \cdot 137$. After a little work we find that 137 is prime, so $82200 = \boxed{2^3 \cdot 3^1 \cdot 5^2 \cdot 137^1}$.

(e) First, $179010 = 17901 \cdot 10 = 2^1 \cdot 5^1 \cdot 17901$. The sum of the digits of 17901 is 18, which is a multiple of 9, so we can factor out 3^2: $179010 = 2^1 \cdot 3^2 \cdot 5^1 \cdot 1989$. The sum of the digits of 1989 is 18, so we can factor out two more powers of 3: $179010 = 2^1 \cdot 3^4 \cdot 5^1 \cdot 221$. Since $221 = 13 \cdot 17$, we have $179010 = \boxed{2^1 \cdot 3^4 \cdot 5^1 \cdot 13^1 \cdot 17^1}$.

(f) We start with $10485760 = 1048576 \cdot 10 = 2^1 \cdot 5^1 \cdot 1048576$. Now we divide out multiples of 2 until we are left with an odd number. It turns out that $1048576 = 2^{20}$, so $10485760 = \boxed{2^{21} \cdot 5^1}$.

13.2.2 The remainder when a nonnegative integer is divided by 8 is the same as when the three-digit number formed by its last 3 digits is divided by 8.

(a) $319 = 39 \cdot 8 + \boxed{7}$.

(b) $411 = 51 \cdot 8 + \boxed{3}$.

(c) $989 = 123 \cdot 8 + \boxed{5}$.

(d) $827 = 103 \cdot 8 + \boxed{3}$.

(e) $201 = 25 \cdot 8 + \boxed{1}$.

(f) First we note that $204001 \equiv 1 \pmod 8$. So,

$$204001^{34} \equiv 1^{34} \equiv 1 \pmod 8,$$

and the remainder is $\boxed{1}$.

13.2.3 The remainder when a nonnegative integer is divided by 9 is the same as its modulo-9 residue.

(a) $9319 \equiv 9 + 3 + 1 + 9 \equiv 22 \equiv 2 + 2 \equiv \boxed{4} \pmod 9$.

(b) $12411 \equiv 1 + 2 + 4 + 1 + 1 \equiv 9 \equiv \boxed{0} \pmod 9$.

(c) $65989 \equiv 6 + 5 + 9 + 8 + 9 \equiv 37 \equiv 3 + 7 \equiv 10 \equiv 1 + 0 \equiv \boxed{1} \pmod 9$.

(d) $91827 \equiv 9 + 1 + 8 + 2 + 7 \equiv 27 \equiv 2 + 7 \equiv 9 \equiv \boxed{0} \pmod 9$.

(e) $204201 \equiv 2 + 0 + 4 + 2 + 0 + 1 \equiv 9 \equiv \boxed{0} \pmod 9$.

(f) This one requires more work and this is by no means the only possible method:

$$\begin{aligned}
204001^{34} &\equiv (2+0+4+0+0+1)^{34} \\
&\equiv 7^{34} \\
&\equiv 7^{33} \cdot 7^{1} \\
&\equiv 343^{11} \cdot 7 \\
&\equiv (3+4+3)^{11} \cdot 7 \\
&\equiv 10^{11} \cdot 7 \\
&\equiv 1^{11} \cdot 7 \\
&\equiv 1 \cdot 7 \\
&\equiv \boxed{7} \pmod 9
\end{aligned}$$

13.2.4 Suppose that the least integer greater than 9000 that is a multiple of 11 is $9000 + n$. Using the alternating digit sum of 9000, we get

$$9000 + n \equiv 0 - 0 + 0 - 9 + n \equiv n - 9 \equiv 0 \pmod{11}.$$

This means $n \equiv 9 \pmod{11}$, and $n = 9$ gives us the least multiple of 11 greater than 9000: $9000+9 = \boxed{9009}$.

Exercises for Section 13.3

13.3.1 A nonnegative integer is a multiple of $25 = 5^2$ if and only if its last two digits form an integer that is a multiple of 25. This means $N5$ must be 00, 25, 50, or 75. However, the units digit of $8N5$ is 5, so either $N = \boxed{2}$ or $N = \boxed{7}$.

13.3.2 A nonnegative integer is a multiple of 9 if and only if the sum of its digits is a multiple of 9, so

$$5 + A + B + 3 = A + B + 8 \equiv 0 \pmod{9}.$$

This means $A + B \equiv 1 \pmod{9}$. Since $0 \le A + B \le 18$, either $A + B = 1$ or $A + B = 10$.

- When $A + B = 1$, we have 2 solutions: $(0, 1)$ and $(1, 0)$.
- When $A + B = 10$, we have 9 solutions: $(1, 9), (2, 8), (3, 7), \ldots, (9, 1)$.

In total, there are $2 + 9 = \boxed{11}$ ordered pairs (A, B) that make $5AB3$ a multiple of 9.

13.3.3 As in the last problem, we start with the sum of the digits:

$$5 + A + B + 4 = A + B + 9 \equiv A + B \equiv 0 \pmod{9}.$$

This gives us three possible cases:

- When $A + B = 0$, $(A, B) = (0, 0)$.
- When $A + B = 9$, there are 10 ordered pairs: $(0, 9), (1, 8), \ldots, (9, 0)$.
- When $A + B = 18$, $(A, B) = (9, 9)$.

In total, there are $1 + 10 + 1 = \boxed{12}$ ordered pairs (A, B) that make $5AB4$ a multiple of 9.

13.3.4 An integer is a multiple of 6 if and only if it is a multiple of both 2 and 3. In order for $5DDDD$ to be a multiple of 2, D must be even. In order for $5DDDD$ to be a multiple of 3, the sum of its digits must be a multiple of 3:

$$5 + D + D + D + D = 5 + 4D \equiv 2 + D \equiv 0 \pmod{3} \quad \Rightarrow \quad D \equiv 1 \pmod{3}.$$

The only even digit congruent to 1 (mod 3) is $\boxed{4}$.

13.3.5

(a) We are looking for (A, B) such that $10B + 8 \equiv 2B \equiv 0 \pmod{4}$. This is satisfied by every ordered pair (A, B) in which B is even. We will not list all 50 solutions.

(b) We are looking for (A, B) such that $4 + A + B + 8 \equiv A + B \equiv 0 \pmod{3}$. Since $0 \le A + B \le 18$, the cases are limited:

$A + B$	Ordered Pairs (A, B)
0	$(0, 0)$
3	$(0, 3), (1, 2), (2, 1), (3, 0)$
6	$(0, 6), (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 0)$
9	$(0, 9), (1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1), (9, 0)$
12	$(3, 9), (4, 8), (5, 7), (6, 6), (7, 5), (8, 4), (9, 3)$
15	$(6, 9), (7, 8), (8, 7), (9, 6)$
18	$(9, 9)$

(c) Since $\gcd(3,4) = 1$ and $\text{lcm}[3,4] = 12$, an integer is a multiple of 12 if and only if it is a multiple of both 3 and 4. This means that the ordered pairs (A, B) for which $\underline{4AB8}$ is a multiple of 12 are those that are solutions to both parts (a) and (b) (just those from part (b) in which B is even – there are 17 of them – make sure you can identify all 17).

13.3.6 Since $\underline{47D4}$ is 2 more than a multiple of 33, $\underline{47D4} - 2 = \underline{47D2}$ is a multiple of 33. An integer is a multiple of 33 if and only if it is a multiple of both 3 and 11.

Applying the divisibility rule for 3, we get

$$4 + 7 + D + 2 = D + 13 \equiv D + 1 \equiv 0 \pmod 3,$$

so $D \equiv 2 \pmod 3$.

Applying the divisibility rule for 11,

$$2 - D + 7 - 4 = 5 - D \equiv 0 \pmod{11},$$

so $D \equiv 5 \pmod{11}$. The only possible digit is $D = 5$. Since this satisfies $D \equiv 2 \pmod 3$, $D = \boxed{5}$ is the solution (and 4754 is 2 more than a multiple of 33).

13.3.7 There are a number of ways to go about this problem. However, since the last three digits of 1125 are 125, which equals 5^3, we might be able to use simple divisibility rules quickly. $1125 = 3^2 \cdot 5^3$. This prime factorization tells us that we need to apply the divisibility rules for both $3^2 = 9$ and $5^3 = 125$. We apply the latter first because it's more limiting: since the units digit of $\underline{52MN5}$ is 5, we know that $\underline{MN5}$ is one of 125, 375, 625, and 875. Now we test these cases to see if any of them make $\underline{52MN5}$ a multiple of 9:

$$\begin{aligned}
52125 &\equiv 5+2+1+2+5 \equiv 15 \not\equiv 0 \pmod 9 \\
52375 &\equiv 5+2+3+7+5 \equiv 22 \not\equiv 0 \pmod 9 \\
52625 &\equiv 5+2+6+2+5 \equiv 20 \not\equiv 0 \pmod 9 \\
52875 &\equiv 5+2+8+7+5 \equiv 27 \equiv 0 \pmod 9
\end{aligned}$$

Since only 52875 is a multiple of 9, $(M, N) = \boxed{(8,7)}$ is the only solution (and 52875 is a multiple of 1125).

Review Problems

13.11 An integer is a multiple of 9 if and only if the sum of its digits is a multiple of 9.

(a) $\boxed{\text{Yes.}}$ $8 + 7 + 1 + 2 = 18 \equiv 1 + 8 \equiv 9 \equiv 0 \pmod 9$, so 8712 is a multiple of 9.

(b) $\boxed{\text{No.}}$ $1 + 2 + 9 + 9 + 4 = 25 \equiv 2 + 5 \equiv 7 \pmod 9$, so 12994 is not a multiple of 9.

(c) $\boxed{\text{Yes.}}$ $5 + 2 + 5 + 1 + 5 = 18 \equiv 1 + 8 \equiv 9 \equiv 0 \pmod 9$, so 52515 is a multiple of 9.

(d) $\boxed{\text{Yes.}}$ $8 + 1 + 9 + 2 + 5 + 4 + 7 = 27 \equiv 2 + 7 \equiv 9 \equiv 0 \pmod 9$, so 8192547 is a multiple of 9.

13.12 An integer is a multiple of 11 if and only if the alternating sum of its digits is a multiple of 11.

(a) $\boxed{\text{Yes.}}$ $8 - 4 + 7 = 11 \equiv 0 \pmod{11}$, so 748 is a multiple of 11.

(b) $\boxed{\text{No.}}$ $7 - 5 + 5 - 8 = -1 \equiv 10 \pmod{11}$, so 8557 is not a multiple of 11.

(c) $\boxed{\text{Yes.}}$ $9 - 4 + 1 - 9 + 3 = 0 \equiv 0 \pmod{11}$, so 39149 is a multiple of 11.

(d) $\boxed{\text{No.}}$ $2 - 7 + 1 - 2 + 9 - 4 + 2 = 1 \equiv 1 \pmod{11}$, so 2492172 is not a multiple of 11.

13.13 An integer is a multiple of 8 if and only if its last three digits form a multiple of 8.

(a) $\boxed{\text{No.}}$ $1444 \equiv 444 \equiv 4 \pmod 8$, so 1444 is not a multiple of 8.

(b) $\boxed{\text{No.}}$ $83412 \equiv 412 \equiv 4 \pmod 8$, so 83412 is not a multiple of 8.

(c) $\boxed{\text{Yes.}}$ $971352 \equiv 352 \equiv 0 \pmod 8$, so 971352 is a multiple of 8.

(d) $\boxed{\text{No.}}$ $2222222220 \equiv 220 \equiv 4 \pmod 8$, so 2222222220 is not a multiple of 8.

13.14 Since $72 = 8 \cdot 9$ and $\gcd(8, 9) = 1$, we use the divisibility rules for 8 and 9 to find the digits A and B.

First, using the divisibility rule for 9,

$$\underline{60A5B} \equiv A + B + 11 \equiv A + B + 2 \equiv 0 \pmod 9.$$

This means that $\underline{60A5B}$ is a multiple of 9 if and only if $A+B \equiv 7 \pmod 9$, so either $A+B = 7$ or $A+B = 16$.

Now, using the divisibility rule for 8,

$$\underline{60A5B} \equiv \underline{A5B} \equiv 100A + 50 + B \equiv 4A + 2 + B \equiv 0 \pmod 8.$$

Since $4A$ is a multiple of 4, $2 + B$ must also be a multiple of 4. So, either $B = 2$ or $B = 6$. Also, since $B \le 6$, $A + B \le 15$, meaning $A + B$ is 7, not 16. The only remaining possibilities for (A, B) are $(5, 2)$ and $(1, 6)$. Plugging these in to $4A + 2 + B \equiv 0 \pmod 8$, we find that $\boxed{(5, 2)}$ is the only solution (and 60552 is a multiple of 72).

13.15 Since $45 = 5 \cdot 9$ and $\gcd(5, 9) = 1$, we use the divisibility rules for 5 and 9 to find the digits A and B.

First, by the divisibility rule for 5, either B is 0 or B is 5.

Now, by the divisibility rule for 9,

$$\underline{50A11B} \equiv A + B + 7 \equiv 0 \pmod 9,$$

so $A + B \equiv 2 \pmod 9$. Either $A + B = 2$ or $A + B = 11$.

If $A + B = 2$, then $B < 5$, so $(A, B) = \boxed{(2, 0)}$.

If $A + B = 11$, then $B > 0$, so $(A, B) = \boxed{(6, 5)}$.

Challenge Problems

13.16 We know the digits of the integer, so we think to apply any rule that comes to mind that uses all the digits, regardless of their order. Hence, we apply the divisibility rule for 9 and note that the nine-digit number is congruent to

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45 \equiv 0 \pmod 9.$$

Since the nine-digit number is multiple of 9, regardless of the order of its digits, the probability that it is prime is $\boxed{0}$.

13.17 An integer is a multiple of 4 if and only if its last two digits are a multiple of 4. Also, the units digit must be even in order for an integer to be even, so this gives us a good starting point for testing cases.

- If the units digit is 2, then $12 \equiv 32 \equiv 0 \pmod 4$, but $42 \equiv 2 \not\equiv 0 \pmod 4$, so if the last two digits are 12 or 32, then the integer is a multiple of 4.
- if the units digit is 4, then $24 \equiv 0 \pmod 4$, but $14 \equiv 34 \equiv 2 \not\equiv 0 \mod 4$, so if the last two digits are 24, then the integer is a multiple of 4.

From these possibilities, there are 6 such four-digit integers that are multiples of 4: 3412, 4312, 1432, 4132, 1324, and 3124. Since there are $4! = 24$ possible four-digit integers using the digits 1-4 each once, the probability that one of them is a multiple of 4 is $6/24 = \boxed{1/4}$.

13.18 A four-digit palindrome must be in the form $ABBA$ for some digits A and B. Such a palindrome is a multiple of 9 if and only if the sum of its digits is a multiple of 9. In other words, $2(A + B)$ must be a multiple of 9. Since 2 and 9 are relatively prime, $A + B$ must be a multiple of 9.

We could solve the problem in a number of ways from here, but note that $A + B$ is a multiple of 9 if and only if the two-digit integer AB is a multiple of 9. This makes the final count easy to find. From 18 to 99 inclusive, there are $\boxed{10}$ two-digit integers that are multiples of 9, corresponding to the palindromes 1881 to 9999 inclusive.

13.19 Since $99 = 9 \cdot 11$, and $\gcd(9, 11) = 1$, there is a temptation to try to build a divisibility rule by combining those for 9 and 11. However, noting that $100 = 10^2 \equiv 1 \pmod{99}$ gives us a nicer possibility.

Let's first examine the integer 133749 modulo 99 by dividing its digits up into consecutive pairs:

$$\begin{aligned}
133749 &= 13 \cdot (10^2)^2 + 37 \cdot (10^2)^1 + 49 \cdot (10^2)^0 \\
&\equiv 13 \cdot 1^2 + 37 \cdot 1^1 + 49 \cdot 1^0 \\
&\equiv 13 + 37 + 49 \\
&\equiv 99 \\
&\equiv 0 \pmod{99}
\end{aligned}$$

We find that 133749 is a multiple of 99 by breaking it up into two-digit integers.

Since $(10^2)^n = 100^n \equiv 1^n \equiv 1 \pmod{99}$, *we can always reduce integers modulo 99 to a sum of pairs of digits.* We must remember, however, to start from the rightmost digits when we pair digits for integers with odd numbers of digits. For instance, for the integer 3123491, we have

$$\begin{aligned}
3123491 &= 3 \cdot (10^2)^3 + 12 \cdot (10^2)^2 + 34 \cdot (10^2)^1 + 91 \cdot (10^2)^0 \\
&\equiv 3 + 12 + 34 + 91 \\
&\equiv 140 \\
&\equiv 41 \pmod{99}
\end{aligned}$$

So, 3123491 is not a multiple of 99. Just make sure not to group the digits starting from the left, where the units digit 1 would be the digit without a partner.

13.20 Let's get a grip on the arithmetic behind the divisibility rule. For any given integer n, we can rewrite n as $10A + B$, where B is the units digit of n. Now, let's be very precise about writing the meaning behind the divisibility rule mathematically. We are told that

$$10A + B \equiv 0 \pmod{p}$$

if and only if

$$A - 3B \equiv 0 \pmod{p},$$

for some prime p. Our goal is to find the value of p. There are a number of ways we could work with these congruences, but a common algebraic technique is to eliminate variables. We could eliminate either variable, but we'll multiply the first congruence by 3 and add it to the second:

$$3(10A + B) + A - 3B = 30A + 3B + A - 3B = 31A \equiv 0 \pmod{p}.$$

This new congruence must be true regardless of the value of A, so $p \mid 31$. Since 31 is prime, $p = 31$ and the given divisibility rule is a divisibility rule for 31.

13.21 There are lots of three-digit positive integers with digit sums of 13, and they aren't easily described in a useful way. So, we'll start with the fact that the integers we're counting are congruent to 1 (mod 4). We want integers whose last two digits are congruent to 1 (mod 4). Such integers end in $01, 05, 09, 13, 17, \ldots, 97$. Each of these $100/4 = 25$ pairs of digits can be associated with at most one hundreds digit such that the sum of all three digits is 13. Since the hundreds digit is between 1 and 9 inclusive, we want the pairs of last two digits to have a sum between 4 and 12 inclusive. We throw out $01, 21, 49, 69, 77, 85, 89$, and 97. This leaves $25 - 8 = \boxed{17}$ pairs of last two digits that can each be grouped with a hundreds digit to make a digit sum of 13.

13.22 We are looking for an integer that is a multiple of $\text{lcm}[7, 8, 9] = 504$. Let our six-digit number be $523ABC$ for some digits A, B, and C. First we note that $523ABC$ is a multiple of 504 if and only if $523ABC - 504000 = 19ABC$ is a multiple of 504. Now, we need to find an integer $19ABC$ that is a multiple of 504. We could apply divisibility rules to the problem, but we can also simply go hunting! What makes hunting easier is the fact that 19000/504 is approximately equal to $19000/500 = 38$. We note that $38 \cdot 504 = 19152$ and $39 \cdot 504 = 19656$. From these, we find that the six-digit numbers $\boxed{523152}$ and $\boxed{523656}$ are multiples of 504. The moral of the story is to keep an open mind as to when to use different methods to solve a problem.

13.23 Let's take a look at the three simplest examples:

$$\begin{aligned} \frac{1}{3^2} &= 0.\overline{1} \\ \frac{1}{3^3} &= 0.\overline{037} \\ \frac{1}{3^4} &= 0.\overline{012345679} \end{aligned}$$

Each repeating block of decimals is the result of dividing a previous repeating block by 3. This means means that the previous repeating block must be repeated enough times to make an integer that is a multiple of 3. For instance, $0.\overline{1} = 0.\overline{111}$. Dividing $111 \div 3 = 37$, we get $0.\overline{1} \div 3 = 0.\overline{037}$. Similarly, $37037037/3 = 12345679$. Each new repeating block is the result of dividing a multiple of 3 by 3 to get a non-multiple of 3. Thus we need to repeat each block 3 times before dividing by 3 again. This shows that if we continue our sequence of values of $\frac{1}{3^n}$ above, each will have 3 times as many digits in its repeating block, 3^{n-2}.

13.24 Let the base-b digits of n be (from right to left) $d_0, d_1, d_2, \ldots$. Since $b \equiv 1 \pmod{b-1}$, we have

$$\begin{aligned} n &= d_0 \cdot b^0 + d_1 \cdot b^1 + d_2 \cdot b^2 + \cdots \\ &\equiv d_0 \cdot 1^0 + d_1 \cdot 1^1 + d_2 \cdot 1^2 + \cdots \\ &\equiv d_0 + d_1 + d_2 + \cdots \pmod{b-1} \end{aligned}$$

Thus n is congruent to the sum of its base-b digits modulo $b-1$. This means n is a multiple of $b-1$ if and only if the sum of its base-b digits is a multiple of $b-1$. Note that when $b = 10$, we are working in the decimal system and we have the divisibility rule for 9.

13.25 First, we note that when $n = 2$, $n^3 - n = 6$. This means that $k \mid 6$. With this in mind, we check to see if $n^3 - n$ is always a multiple of 2 and 3. There are several possible ways to go about this, but we take a factoring approach:

$$n^3 - n = n(n^2 - 1) = n(n+1)(n-1) = (n-1)n(n+1).$$

Among any three consecutive integers, there is at least one even one. Also, the three consecutive integers must have all three modulo-3 residues (0, 1, and 2), meaning one of them is a multiple of 3. Since one of them is a multiple of 2 and one is a multiple of 3, their product must be a multiple of $2 \cdot 3 = 6$. This means that $k = \boxed{6}$ divides all values of $n^3 - n$.

13.26 Let A, B, and C be digits of the five-digit palindrome $ABCBA$ that is a multiple of 99. Thus

$$ABCBA = 100^2 \cdot A + 100 \cdot (10B + C) + (10B + A) \equiv A + 10B + C + 10B + A \equiv 2A + 20B + C \equiv 0 \pmod{99}.$$

Our goal is to minimize $ABCBA$, so our first priority is to minimize the value of A.

The expression $2A + 20B + C$ must be a multiple of 99, but the maximum value of the expression occurs when all of the digits are 9, in which case $2A + 20B + C = 207$. The only positive multiples of 99 less than 207 are 99 and 198. We have two cases:

$$\begin{aligned} 2A + 20B + C &= 99 \\ 2A + 20B + C &= 198 \end{aligned}$$

In both cases, the value of $20B$ dominates the value of the expression. In the first case, $B = 4$, because a lower value of B would require $A > 9$ or $C > 9$. A higher value of B would require a negative digit. In the second case, $B = 9$ for similar reasons. Now we are left with

$$\begin{aligned} 2A + C &= 19 \\ 2A + C &= 18 \end{aligned}$$

In both cases, the smallest possible value of A is 5. Now we want B to be as small as possible in order to minimize the value of $ABCBA$, so we throw out the second equation from where $B = 9$. This means $B = 4$. Since $2A + C = 19$, we have $C = 9$ and our 5-digit palindrome is $ABCBA = \boxed{54945}$.

13.27 Let $d_0, d_1, d_2, \ldots$ be the base-12 digits of n so that

$$n = 12^0 \cdot d_0 + 12^1 \cdot d_1 + 12^2 \cdot d_2 + \cdots + 12^m \cdot d_m + \cdots.$$

We group the digit bundles that are multiples of 12^m. Note that since $3 \mid 12$, $3^m \mid 12^m$.

$$\begin{aligned} n &= 12^0 \cdot d_0 + 12^1 \cdot d_1 + 12^2 \cdot d_2 + \cdots + (12^m \cdot d_m + \cdots) \\ &= 12^0 \cdot d_0 + 12^1 \cdot d_1 + 12^2 \cdot d_2 + \cdots + 12^m(12^0 \cdot d_m + 12^1 \cdot d_{m+1} + \cdots) \\ &\equiv 12^0 \cdot d_0 + 12^1 \cdot d_1 + 12^2 \cdot d_2 + \cdots + 12^{m-1} \cdot d_{m-1} \pmod{3^m} \end{aligned}$$

The last expression uses only the first m base-12 digits of n to determine the modulo-3^m residue of n.

13.28 The last four digits of an integer determine its modulo-2^4 residue. This motivates us to examine possible integers modulo 16. This seems reasonable since any power of 2 with at least 4 digits is a multiple of 16.

Let m be a nonnegative integer and let $n = 2^m$. If the last four digits of n are the same, then they are $dddd = 1111 \cdot d$ for some digit d. Since $1111 \equiv 7 \pmod{16}$,

$$n \equiv dddd \equiv 1111 \cdot d \equiv 7d \pmod{16}.$$

The only digit d that makes $7d$ a multiple of 16 is 0, but n is not a multiple of 10. Thus, n cannot be a pure power of 2, so there are no pure powers of 2 that end in four equal digits.

13.29 Since $15 = 3 \cdot 5$, and $\gcd(3, 5) = 1$, we use the divisibility rules for 3 and 5 to help us find n. Since $5 \mid n$, n has a units digit of either 0 or 5. However, we are told that all of n's digits are either 0 or 8, so the units digit of n is 0. Now we note that $3 \mid n$, so the sum of n's digits is a multiple of 3. However, the sum of n's digits is also the number of 8's that are digits of n. This smallest total of 8's is 3, where $3 \cdot 8 = 24$ is a multiple of 3. Finally, we want the smallest possible value of n, so we include no more digits and $n = 8880$ is the smallest positive multiple of 15 all of whose digits are 0 or 8. The answer to the problem is $n/15 = 8880/15 = \boxed{592}$.

13.30 We could solve all parts of this problem by dividing each of the four integers into longer strings of 1's until no remainder is left:

$$\begin{aligned} 1 &= 3 \cdot 0 + 1, \\ 11 &= 3 \cdot 3 + 2, \\ 111 &= 3 \cdot 37 + 0. \end{aligned}$$

However, some interesting observations can be made that provide a less gritty solution.

This problem involves dividing an integer with the same repeated digit into another integer with a different repeating digit. This fact allows us to reframe this problem in terms of repeating decimals. Consider $\frac{1}{9} = 0.\overline{1}$. This repeating decimal can be viewed as a repetition of any number of 1's:

$$\begin{aligned} \frac{1}{9} &= 0.\overline{1} \\ &= 0.\overline{11} \\ &= 0.\overline{111} \\ &\vdots \end{aligned}$$

Now, let m be an integer that is relatively prime to 10. When we divide $\frac{1}{9} \div m = \frac{1}{9m}$, we get another repeating decimal. The number of digits in the repeating decimal depends on the number of 1's needed to produce an integer that is divisible by m. When $m = 3$, we have

$$\frac{1}{9} \div 3 = 0.\overline{1} \div 3 = 0.\overline{111} \div 3 = 0.\overline{037}.$$

And $3 \cdot 37 = 111$.

In general, we can find the smallest number of 1's needed to produce an integer divisible by m by finding the number of digits in the repeating decimal expansion of $\frac{1}{9m}$. From here, we could simply compute the repeating decimals of $\frac{1}{9m}$ for each value of m. However, we note that it's easy to find the repeating decimal for each $\frac{1}{3m}$ which equals $\frac{1}{10^a-1}$ for some positive integer a. While this fact doesn't make a difference for part (a), it makes the other parts easier to compute. For part (a), we simply have

$$\frac{1}{9} \div 3 = 0.\overline{1} \div 3 = 0.\overline{111} \div 3 = 0.\overline{037}.$$

Notice that we rewrote the repeating block of 1/9 so that the number of 1's is a multiple of 3. This ensured that we had just enough of a repeating block to divide by 3 and get the repeating block for the result. The answer to (a) is $3 \cdot 37 = \boxed{111}$.

Similarly, we compute parts (b) and (c) by noting that we need to repeat the a-digit repeating decimal $\frac{1}{10^a-1}$ a total of 3 times in order to create a repeating block of digits that is divisible by 3.

$$\begin{array}{lllllll}
\text{(b)}\ 0.\overline{1} \div 33 & = \frac{1}{9} \cdot \frac{1}{33} & = \frac{1}{3} \cdot \frac{1}{99} & = \frac{1}{3}(0.\overline{01}) & = \frac{1}{3}(0.\overline{010101}) & = 0.\overline{003367} \\
\text{(c)}\ 0.\overline{1} \div 333 & = \frac{1}{9} \cdot \frac{1}{333} & = \frac{1}{3} \cdot \frac{1}{999} & = \frac{1}{3}(0.\overline{001}) & = \frac{1}{3}(0.\overline{001001001}) & = 0.\overline{000333667}
\end{array}$$

The answer to part (b) is $33 \cdot 3367 = \boxed{111111}$. The answer to part (c) is $333 \cdot 333667 = \boxed{111111111}$.

Similarly, we the answer to part (d) is $\boxed{\text{a 30-digit integer, all of whose digits are 1}}$. This comes from dividing the 10-digit repeating decimal for $\frac{1}{10^{10}-1}$ by 3.

CHAPTER 14

Linear Congruences

Exercises for Section 14.2

14.2.1

(a) Adding 5 to both sides of the congruence, we get $x \equiv 7 \pmod 3$. Since $7 \equiv 1 \pmod 3$, we can write the solutions as $\boxed{x \equiv 1 \pmod 3}$.

(b) Subtracting 223 from both sides of the congruence, we get

$$x \equiv -109 \equiv 3 \pmod 8,$$

so the solutions are $\boxed{x \equiv 3 \pmod 8}$.

(c) We look for a multiple of 5 that is 1 more than a multiple of 11:

$$5x \equiv 1 \equiv 12 \equiv 23 \equiv 34 \equiv 45 \pmod{11}.$$

Since 5 and 11 are relatively prime, 5 has a modulo 11 inverse. We multiply both sides of $5x \equiv 45 \pmod{11}$ by 5^{-1} to get

$$x \equiv 45 \cdot 5^{-1} \equiv 9 \cdot 5 \cdot 5^{-1} \equiv 9 \cdot 1 \equiv 9 \pmod{11}.$$

The solutions are $\boxed{x \equiv 9 \pmod{11}}$ and we note that $5^{-1} \equiv 9 \pmod{11}$.

(d) Subtracting 17 from both sides of the congruence, we get $2x \equiv -17 \equiv 1 \pmod 9$. Now we hunt for a multiple of 2 that is congruent to 1 (mod 9):

$$2x \equiv 1 \equiv 10 \pmod 9.$$

Since 2 and 9 are relatively prime, 2 has a modulo 9 inverse. We multiply both sides of $2x \equiv 10 \pmod 9$ by 2^{-1} to get

$$x \equiv 10 \cdot 2^{-1} \equiv 5 \cdot 2 \cdot 2^{-1} \equiv 5 \cdot 1 \equiv 5 \pmod 9,$$

so the solutions are $\boxed{x \equiv 5 \pmod 9}$.

14.2.2

$$\begin{array}{lcl}
1\cdot 1 \equiv 1 \pmod{11} & \Rightarrow & \boxed{1^{-1} \equiv 1 \pmod{11}} \\
2\cdot 6 \equiv 1 \pmod{11} & \Rightarrow & \boxed{2^{-1} \equiv 6 \pmod{11}} \\
3\cdot 4 \equiv 1 \pmod{11} & \Rightarrow & \boxed{3^{-1} \equiv 4 \pmod{11}} \\
4\cdot 3 \equiv 1 \pmod{11} & \Rightarrow & \boxed{4^{-1} \equiv 3 \pmod{11}} \\
5\cdot 9 \equiv 1 \pmod{11} & \Rightarrow & \boxed{5^{-1} \equiv 9 \pmod{11}} \\
6\cdot 2 \equiv 1 \pmod{11} & \Rightarrow & \boxed{6^{-1} \equiv 2 \pmod{11}} \\
7\cdot 8 \equiv 1 \pmod{11} & \Rightarrow & \boxed{7^{-1} \equiv 8 \pmod{11}} \\
8\cdot 7 \equiv 1 \pmod{11} & \Rightarrow & \boxed{8^{-1} \equiv 7 \pmod{11}} \\
9\cdot 5 \equiv 1 \pmod{11} & \Rightarrow & \boxed{9^{-1} \equiv 5 \pmod{11}} \\
10\cdot 10 \equiv 1 \pmod{11} & \Rightarrow & \boxed{10^{-1} \equiv 10 \pmod{11}}
\end{array}$$

14.2.3 An integer has an inverse in a modulus when it is relatively prime to the modulus. The modulo-15 residues that are relatively prime to 15 are $\boxed{1, 2, 4, 7, 8, 11, 13, 14}$.

14.2.4 First, we note that an integer cannot have an inverse at all unless the integer is relatively prime to the modulus. Now, let a be an integer that is relatively prime to a modulus m. Let k_1 and k_2 be distinct modulo-m residues. Consider the difference between ak_1 and ak_2:

$$ak_1 - ak_2 = a(k_1 - k_2).$$

Since k_1 and k_2 are distinct modulo-m residues, their difference is not a multiple of m. Since a is relatively prime to m, $a(k_1 - k_2)$ is not a multiple of m. This means that $ak_1 \not\equiv ak_2 \pmod{m}$. This means that the products of a with each modulo-m residue are incongruent modulo m, so only one of them can be congruent to 1 (mod m). Thus, a does not have more than one modulo m inverse.

Exercises for Section 14.3

14.3.1 Multiplying both sides of the congruence by 3^{-1}, we get

$$a \equiv 9 \cdot 3^{-1} \equiv 3 \cdot 3 \cdot 3^{-1} \equiv 3 \pmod{11},$$

so the solutions are $\boxed{a \equiv 3 \pmod{11}}$.

14.3.2 Adding 11 to both sides of the congruence, we get

$$5x \equiv 18 \pmod{8}.$$

Since $18 \equiv 2 \pmod{8}$, we look for a multiple of 5 that is congruent to 2 (mod 8):

$$5x \equiv 18 \equiv 10 \pmod{8}.$$

Multiplying both sides of the congruence by 5^{-1}, we get

$$x \equiv 10 \cdot 5^{-1} \equiv 2 \cdot 5 \cdot 5^{-1} \equiv 2 \pmod{8}.$$

The solutions are $\boxed{x \equiv 2 \pmod{8}}$.

14.3.3 Since 2 is a common divisor of 4, 8, and 10, the original congruence is equivalent to $2n \equiv 4 \pmod{5}$. Multiplying both sides of the congruence by 2^{-1}, we get

$$n \equiv 4 \cdot 2^{-1} \equiv 2 \cdot 2 \cdot 2^{-1} \equiv 2 \pmod{5}.$$

The solutions are $\boxed{n \equiv 2 \pmod{5}}$.

14.3.4 Subtracting 19 from both sides of the congruence and simplifying, we get

$$6a \equiv -12 \equiv 6 \pmod{18}.$$

Since 6 is a common divisor of 6, 6, and 18, the congruence $6a \equiv 6 \pmod{18}$ is equivalent to $\boxed{a \equiv 1 \pmod{3}}$.

Exercises for Section 14.4

14.4.1 From the congruences, we have

$$a = 8x + 3 = 9y + 5$$

for integers x and y. Looking at these equations modulo 8, we have

$$3 \equiv y + 5 \pmod{8} \Rightarrow y \equiv 6 \pmod{8}.$$

Thus $y = 8z + 6$ for some integer z and

$$a = 9y + 5 = 9(8z + 6) + 5 = 72z + 59.$$

The solutions are $\boxed{a \equiv 59 \pmod{72}}$.

14.4.2 We begin by simplifying both linear congruences:

$$\begin{aligned} 3N &\equiv 2 \pmod{4} &\Rightarrow\quad N &\equiv 2 \pmod{4} \\ 2N &\equiv 4 \pmod{7} &\Rightarrow\quad N &\equiv 2 \pmod{7} \end{aligned}$$

We could solve this simpler system of linear congruences by the usual means. However, we could also note that $N - 2$ is a multiple of both 4 and 7 and therefore of $\operatorname{lcm}[4,7] = 28$. This means that $\boxed{N \equiv 2 \pmod{28}}$.

14.4.3 We are looking for the largest $N < 400$ such that

$$N = 3a + 2 = 7b + 4$$

for integers a and b. Applying modulo 3 we get

$$2 \equiv b + 4 \Rightarrow b \equiv 1 \pmod{3}.$$

This means that $b = 3c + 1$ for some integer c. Thus

$$N = 7b + 4 = 7(3c + 1) + 4 = 21c + 11.$$

Now we apply the fact that $N < 400$:

$$21c + 11 < 400 \Rightarrow 21c < 389.$$

The largest possible value of c is 18. From this value we compute the largest N:

$$N = 21 \cdot 18 + 11 = 378 + 11 = \boxed{389}.$$

14.4.4 When an integer x leaves a remainder of 2 when divided by 4 and a remainder of 4 when divided by 5, we have

$$x = 4a + 2 = 5b + 4$$

for integers a and b. Looking at $4a + 2 = 5b + 4$ modulo 4, we see that $b \equiv 2 \pmod 4$. This means that $b = 4c + 2$ for some integer c and

$$x = 5b + 4 = 5(4c + 2) + 4 = 20c + 14.$$

In order to count the values of x such that $1 < x < 100$, we count the values of c such that

$$1 < 20c + 14 < 100.$$

Subtracting 14 from all parts of this inequality, we get

$$-13 < 20c < 86.$$

Dividing everything by 20 and rounding appropriately, we get $0 \le c \le 4$. There are $4 - 0 + 1 = 5$ possible values of c corresponding to $\boxed{5}$ possible values of x.

14.4.5 We begin by "solving" the last two congruences so that we can work with a simpler system of linear congruences:

$$\begin{aligned} x &\equiv 2 \pmod 3 \\ x &\equiv 4 \pmod 5 \\ x &\equiv 7 \pmod 8 \end{aligned}$$

Make sure you see that the last two congruences above are the solutions to the last two of the original three congruences.

Now, we could solve this system through ordinary means. However, we note that $x + 1$ is a multiple of 3, 5, and 8. This means that $x + 1$ is any multiple of $\text{lcm}[3, 5, 8] = 120$. Therefore, our solutions are $\boxed{x \equiv 119 \pmod{120}}$.

Review Problems

14.20

(a)

$$7x \equiv 1 \equiv 49 \pmod{12}.$$

Multiplying both sides of the congruence by 7^{-1}, we get $x \equiv \boxed{7} \pmod{12}$.

(b)

$$13x \equiv 1 \equiv 169 \pmod{14}.$$

Multiplying both sides of the congruence by 13^{-1}, we get $x \equiv \boxed{13} \pmod{14}$.

(c)

$$5x \equiv 1 \equiv 105 \pmod{26}.$$

Multiplying both sides of the congruence by 5^{-1}, we get $x \equiv \boxed{21} \pmod{26}$.

(d)

$$16x \equiv 1 \equiv 96 \pmod{19}.$$

Multiplying both sides of the congruence by 16^{-1}, we get $x \equiv \boxed{6} \pmod{19}$.

14.21 Modulo-20 residues with no inverses are the ones not relatively prime to 20. Those are $\boxed{0}$, $\boxed{2}$, $\boxed{4}$, $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{10}$, $\boxed{12}$, $\boxed{14}$, $\boxed{15}$, $\boxed{16}$, and $\boxed{18}$.

14.22 Since the product of $m-1$ with itself is congruent to 1 (mod m), $m-1$ is its own inverse:

$$(m-1)(m-1) \equiv (-1)(-1) \equiv 1 \pmod{m}.$$

14.23

(a) Since $3 \equiv 24 \pmod{7}$, we have $4x \equiv 24 \pmod{7}$. Multiplying both sides by 4^{-1}, we get $\boxed{x \equiv 6 \pmod{7}}$.

(b) Adding 13 to both sides, we get $6x \equiv 42 \pmod{72}$. Since 6 is a common divisor of 6, 42, and 72, this congruence is equivalent to $\boxed{x \equiv 7 \pmod{12}}$.

(c) Adding 473 to both sides, we get $26x \equiv 962 \pmod{20}$. This simplifies to $6x \equiv 2 \pmod{20}$. Since 2 is a common divisor of 6, 2, and 20, this congruence is the same as $3x \equiv 1 \pmod{10}$.

$$3x \equiv 1 \equiv 21 \pmod{10}.$$

Multiplying by 3^{-1}, we get $x \equiv 21 \cdot 3^{-1} \equiv \boxed{7 \pmod{10}}$.

14.24 We begin by solving each of the linear congruences in the system to get

$$\begin{aligned} x &\equiv 2 \pmod{3} \\ x &\equiv 5 \pmod{7} \end{aligned}$$

From this system of congruences, we know that

$$x = 3a + 2 = 7b + 5$$

for integers a and b. Looking at $3a + 2 = 7b + 5$ in modulo 3, we have

$$2 \equiv b + 5 \pmod{3} \Rightarrow b \equiv 0 \pmod{3}.$$

This means that $b = 3c$ for some integer c. Thus

$$x = 7b + 5 = 7(3c) + 5 = 21c + 5.$$

The solutions are $\boxed{x \equiv 5 \pmod{21}}$.

14.25 From the first two congruences, we know that

$$N = 3a + 2 = 4b + 1$$

for integers a and b. Looking at $3a + 2 = 4b + 1$ modulo 3, we get $2 \equiv b + 1 \pmod{3}$, so $b \equiv 1 \pmod{3}$. This means $b = 3c + 1$ for some integer c. Thus

$$N = 4b + 1 = 4(3c + 1) + 1 = 12c + 5.$$

The third congruence tells us that $N = 5d + 4$ for some integer d, so

$$12c + 5 = 5d + 4.$$

In modulo 5, this becomes $2c \equiv 4 \pmod{5}$. Solving this we get $c \equiv 2 \pmod{5}$. This means that $c = 5e + 2$ for some integer e. So,

$$N = 12c + 5 = 12(5e + 2) + 5 = 60e + 29.$$

Thus $\boxed{N \equiv 29 \pmod{60}}$.

14.26 We are looking for integers n such that $200 \le n \le 500$ and

$$n = 7a + 1 = 4b + 3$$

for some integers a and b. Looking at $7a + 1 = 4b + 3$ modulo 4, we get

$$3a + 1 \equiv 3 \pmod{4} \quad \Rightarrow \quad a \equiv 2 \pmod{4}.$$

This means that $a = 4c + 2$ for some integer c. So,

$$n = 7a + 1 = 7(4c + 2) + 1 = 28c + 15.$$

Now we solve $200 \le n \le 500$ by substituting for n:

$$200 \le 28c + 15 \le 500.$$

Subtracting 15, we have $185 \le 28c \le 485$. Dividing by 28 and rounding appropriately, we get

$$7 \le c \le 17.$$

There are $17 - 7 + 1 = \boxed{11}$ possible values of c corresponding to the integers n.

14.27 We are looking for the smallest positive integer x that satisfies the system of linear congruences

$$\begin{aligned} x &\equiv 5 \pmod{7} \\ x &\equiv 6 \pmod{11} \\ x &\equiv 4 \pmod{13} \end{aligned}$$

From the first two congruences, we know that

$$x = 7a + 5 = 11b + 6$$

for integers a and b. Looking at $7a + 5 = 11b + 6$ in modulo 7, we have

$$5 \equiv 4b + 6 \pmod 7 \Rightarrow 4b \equiv 6 \pmod 7.$$

Solving this last congruence, we get $b \equiv 5 \pmod 7$. Thus $b = 7c + 5$ for some integer c. This means that

$$x = 11b + 6 = 11(7c + 5) + 6 = 77c + 61.$$

Combining this information about x with the last of the linear congruences in our system, we see that

$$x = 77c + 61 = 13d + 4$$

where d is an integer. Looking at $77c + 61 = 13d + 4$ modulo 13, we have

$$12c + 9 \equiv 4 \pmod{13} \Rightarrow 12c \equiv 8 \pmod{13}.$$

Solving, we find that $c = 13e + 5$ for some integer e. Thus

$$x = 77c + 61 = 77(13e + 5) + 61 = 1001e + 446.$$

The smallest possible positive integer x corresponds to $e = 0$, so $x = \boxed{446}$ is the answer.

Challenge Problems

14.28 We need to identify integers congruent to 7 (mod 11) and 10 (mod 12). Call such an integer n. Then we know that

$$n = 11a + 7 = 12b + 10$$

for some integers a and b. We can apply equivalence modulo 12 to $11a + 7 = 12b + 10$ to get

$$-a + 7 \equiv 10 \pmod{12} \Rightarrow a \equiv -3 \equiv 9 \pmod{12}.$$

Therefore, $a = 12c + 9$ for some integer c, and thus

$$n = 11a + 7 = 11(12c + 9) + 7 = 132c + 106.$$

Taking this modulo 66 gives

$$n \equiv 106 \equiv \boxed{40} \pmod{66}.$$

14.29

$$101^2 \equiv 1 \pmod m \Rightarrow m \mid 101^2 - 1$$

Our goal is to count the divisors of $101^2 - 1$ that are greater than 101. In fact, it will be easier to count those that are at most 101 and subtract them from the total number of divisors of $101^2 - 1$.

$$101^2 - 1 = 101^2 - 1^2 = (101 + 1)(101 - 1) = 102 \cdot 100 = 2^3 \cdot 3^1 \cdot 5^2 \cdot 17^1.$$

The total number of positive divisors of $101^2 - 1 = 10200$ is $(3 + 1)(1 + 1)(2 + 1)(1 + 1) = 48$.

Now we must count the ones that are less than 101. Note that $101 \nmid 10200$. The 48 positive divisors of 10200 can be grouped into pairs whose products are 10200. If both divisors in a pair were greater than 101, their product would be at least $102^2 > 10200$, which is not the case. If both divisors in a pair were less than 101, their product would be at most $100^2 < 10200$, which is also not the case. This means exactly 1 of the divisors in each pair is less than 101. This means that the total number of possible values for the modulus m is $\frac{48}{2} = \boxed{24}$.

14.30 We want to identify mutual days of rest so that we can count them. Days of rest for Adam are in the form $4a$ for some positive integer a. Days of rest for Ben are in the form $10b-2$, $10b-1$, or $10b$ where b is a positive integer. We want days when Adam's and Ben's time off overlap. We want solutions, n, that satisfy any one of the three following systems of linear congruences:

$$\begin{aligned} n &\equiv 0 \pmod 4 \\ n &\equiv 0 \pmod{10} \end{aligned} \qquad \begin{aligned} n &\equiv 0 \pmod 4 \\ n &\equiv -1 \pmod{10} \end{aligned} \qquad \begin{aligned} n &\equiv 0 \pmod 4 \\ n &\equiv -2 \pmod{10} \end{aligned}$$

Solutions to the first system are $n \equiv 0 \pmod{20}$, there are no solutions to the middle system, and solutions to the last system are $n \equiv 8 \pmod{20}$. The first 1000 days go through $1000/20 = 50$ modulo 20 cycles. During each of these 50 cycles, Adam and Ben get 2 days off on the same day for a total of $50 \cdot 2 = \boxed{100}$ days off together.

14.31 Let A be the number of four-digit integers that leave a remainder of 2 when divided by 7. Let B be the number of four-digit integers that leave a remainder of 4 when divided by 5. Let C be the number of four-digit integers that both leave a remainder of 2 when divided by 7 and a remainder of 4 when divided by 5. Our goal is to compute $(A-C)+(B-C)$ (make sure you see why).

To calculate A, we note that we are looking for integers x such that

$$1000 \le 7x + 2 < 10000.$$

Subtracting 2, dividing by 7, and rounding appropriately, these inequalities become $143 \le x \le 1428$. So, $A = 1428 - 143 + 1 = 1286$.

To tabulate B, we note that we are looking for integers y such that

$$1000 \le 5y + 4 < 10000.$$

Subtracting 4, dividing by 5, and rounding appropriately, these inequalities become $200 \le y \le 1999$. So, $B = 1999 - 200 + 1 = 1800$.

In tabulating C, we first determine that we are looking for integers z such that

$$1000 \le 35z + 9 < 10000.$$

Make sure you see that these are the integers that have both remainders. Now, subtracting 9, dividing by 35, and rounding appropriately, our inequalities become $29 \le z \le 285$. So, $C = 285 - 29 + 1 = 257$.

Now we compute our answer:

$$(A-C)+(B-C) = (1286-257)+(1800-257) = 1029 + 1543 = \boxed{2572}.$$

14.32 The product of two modulo 120 units digits has a units digit of 1 if the units digits are modulo 120 inverses of each other. Only integers relatively prime to 120 have modulo 120 inverses. We count these integers using the principle of inclusion-exclusion:

$$120 - \frac{120}{2} - \frac{120}{3} - \frac{120}{5} + \frac{120}{6} + \frac{120}{10} + \frac{120}{15} - \frac{120}{30} = 32.$$

So, the first units digit has a 32/120 chance of having a modulo 120 inverse at all. When it does have an inverse, there is a 1/120 chance that the other units digit will be that inverse. These probabilities are independent, so the overall probability that the units digit of the product is 1 is

$$\frac{32}{120} \cdot \frac{1}{120} = \frac{32}{14400} = \boxed{\frac{1}{450}}.$$

14.33 A 50-digit prime is not a multiple of 2, 3, or 5. Thus, we know that one of the following two systems of linear congruences holds:

$$\begin{array}{rcl} x \not\equiv 0 \pmod 2 & \Rightarrow & x^2 \equiv 1 \pmod 8 \\ x \not\equiv 0 \pmod 3 & \Rightarrow & x^2 \equiv 1 \pmod 3 \\ x \not\equiv 0 \pmod 5 & \Rightarrow & x^2 \equiv 1 \pmod 5 \end{array}$$

or

$$\begin{array}{rcl} x \not\equiv 0 \pmod 2 & \Rightarrow & x^2 \equiv 1 \pmod 8 \\ x \not\equiv 0 \pmod 3 & \Rightarrow & x^2 \equiv 1 \pmod 3 \\ x \not\equiv 0 \pmod 5 & \Rightarrow & x^2 \equiv 4 \pmod 5 \end{array}$$

Solutions to the first system are $x^2 \equiv 1 \pmod{120}$, but we are told this is not the case. Solving the second system gives us the remainder we want: $x^2 \equiv \boxed{49} \pmod{120}$.

CHAPTER 15

Number Sense

Review Problems

15.20 We need to compute

$$\frac{1}{2}\cdot\frac{1}{10}\cdot\frac{1}{5}\cdot\frac{1}{2}\cdot 1{,}000{,}000.$$

Rather than multiplying all of this out sequentially, we notice that $\frac{1}{5}\cdot\frac{1}{2}=\frac{1}{10}$. Now we can group part of the multiplication to make things easier. Some students might even perform the calculation without paper:

$$\begin{aligned}\frac{1}{2}\cdot\frac{1}{10}\cdot\left(\frac{1}{5}\cdot\frac{1}{2}\right)\cdot 1{,}000{,}000 &= \frac{1}{2}\cdot\frac{1}{10}\cdot\frac{1}{10}\cdot 1{,}000{,}000\\ &= \frac{1}{2}\cdot 10{,}000\\ &= \boxed{5{,}000}\end{aligned}$$

15.21 We are told that the sum of two primes is equal to a third, so for prime numbers a, b, and c,

$$a+b=c.$$

There must be something about prime numbers that makes this information enough to solve the problem. Since all primes but 2 are odd, we can apply parity. The largest of these primes must be odd, so it is the sum of an even and an odd prime. The even prime is $\boxed{2}$, which is the smallest of all primes and is therefore our answer.

15.22 The smallest such integer is 2457. If our number were 4 or more times another number, then it would be at least $4\cdot 2457 = 9828$, which is too big. So our number is 2 or 3 times another number. Also note that the smaller number (the number we are multiplying 2 or 3 by) must begin with 2, because $2\cdot 4257 = 8514$ is too big.

If our number is 2 times another number, then since $2\cdot 4 = 8$ and $2\cdot 5 = 10$, our units digit of the smaller number must be 7. But we can check that neither $2\cdot 2457 = 4914$ nor $2\cdot 2547 = 5094$ work.

So, we're looking for two integers such that the larger is 3 times the smaller. This means that the larger integer is at least $3 \cdot 2457 = 7371$. The possibilities are 7425, 7452, 7524, and 7542. Dividing each by 3, we get 2475, 2484, 2508, and 2514. Only the first is a permutation of the digits 2, 4, 5, and 7, so our answer is $3 \cdot 2475 = \boxed{7425}$.

15.23 Since each of the two products includes 1212, we can factor to make things easier:

$$55 \cdot 1212 - 15 \cdot 1212 = (55 - 15) \cdot 1212 = 40 \cdot 1212 = \boxed{48480}.$$

15.24 All the numbers in this problem are powers of 2, so we can change everything into forms of powers of 2 in order to get the clearest look at the equation:

$$2^5 \cdot 2^9 \cdot 2^8 = 2^{2m}.$$

The left-hand side simplifies by summing the exponents, so $2^{22} = 2^{2m}$. This means $2m = 22$, so $m = \boxed{11}$.

15.25 We can start applying divisibility rules for each of the smallest primes in order, using our understanding of modular arithmetic to help.

$$11000 + 1100 + 11 \equiv 0 + 0 + 1 \equiv 1 \pmod 2,$$

so 2 is not a factor. Next,

$$11000 + 1100 + 11 \equiv 2 + 2 + 2 \equiv 6 \equiv 0 \pmod 3,$$

so $\boxed{3}$ is the smallest factor of $11000 + 1100 + 11$. Notice that we did not even need to compute the sum in order to apply these rules easily.

15.26 Since the product $8^{12} \cdot 25^8$ is a multiple of several powers of 10, it makes sense to reorganize the product in this way:

$$\begin{aligned} 8^{12} \cdot 25^8 &= 2^{36} \cdot 5^{16} \\ &= 2^{20} \cdot (2^{16} \cdot 5^{16}) \\ &= 2^{20} \cdot 10^{16} \end{aligned}$$

It might be a bit of a grind to calculate 2^{20}, but knowing your powers of 2 up to 2^{10} makes it easier:

$$2^{20} = (2^{10})^2 = 1{,}024^2 = (1{,}000 + 24)^2 = 1{,}000^2 + 2 \cdot 1{,}000 \cdot 24 + 24^2 = 1{,}000{,}000 + 48{,}000 + 576 = 1{,}048{,}576.$$

Even before we were done multiplying we could see that 2^{20} has 7 digits. When we multiply it by 10^{16}, we add 16 zeros on the end to give us a $7 + 16 = \boxed{23}$ digit number.

15.27 We are looking for the smallest positive integer n such that

$$1999n \equiv 2006 \pmod{10000}.$$

We take advantage of the fact that $1999 = 2000 - 1$:

$$2000n - n \equiv 2006 \pmod{10000}.$$

Now we see that viewing everything modulo 2000 can get us started quickly:

$$-n \equiv 6 \pmod{2000}.$$

We can rewrite n as $2000m - 6$ for some positive integer m. Plugging $n = 2000m - 6$ into $1999n \equiv 2006 \pmod{10000}$, we get

$$1999(2000m - 6) = 3998000m - 11994 \equiv 2006 \pmod{10000}.$$

Adding 11994 to both sides and subtracting out multiples of 10000, we get

$$8000m \equiv 4000 \pmod{10000}.$$

We can solve this linear congruence in a number of ways, but the quickest is to note that $3 \cdot 8 \equiv 4 \pmod{10}$, so $8000 \cdot 3 \equiv 4000 \pmod{10000}$, so $m = 3$ is the smallest possible value of m. This means that $2000 \cdot 3 - 6 = 5994$ is the smallest possible value of n. The smallest positive multiple of 1999 that ends in 2006 is

$$1999 \cdot 5994 = (2000 - 1)(6000 - 6) = 12000000 - 6000 - 12000 + 6 = \boxed{11982006}.$$

15.28 Once again, we apply parity to a sum of primes. Since the sum of the three primes is even, one of them must be the even prime, 2. This means the sum of the other two primes is 22. Our possible choices for the remaining prime numbers are 3 and 19, 5 and 17, and 11 and 11. There are six possible orderings of $(2, 3, 19)$, six possible orderings of $(2, 5, 17)$, and three possible orderings of $(2, 11, 11)$, for a total of $\boxed{15}$ ordered triples.

15.29 Solving an equation based on this problem would be tough, but we can make casework relatively easy by narrowing the range of integers whose cubes have four-digits. Since $10^3 = 1000$ and $22^3 = 10648$, there are only twelve 4-digit numbers which are cubes at all. We want the largest four-digit number equal to the cube of the sum of its digits, so we can work downwards from cubes of possible digit sums:

$$\begin{aligned} 21^3 &= 9261, & 9 + 2 + 6 + 1 = 18 \neq 21 \\ 20^3 &= 8000, & 8 + 0 + 0 + 0 = 8 \neq 20 \\ 19^3 &= 6859, & 6 + 8 + 5 + 9 = 28 \neq 19 \\ 18^3 &= 5832, & 5 + 8 + 3 + 2 = 18 \end{aligned}$$

Having checked all the largest four-digit cubes, we know that $\boxed{5832}$ is the largest four-digit number that is equal to the cube of the sum of its digit.

15.30 We see powers of 61 multiplied by $1, -3, 3$, and -1. These remind us of the coefficients of $(x - 1)^3$:

$$(x - 1)^3 = x^3 - 3x^2 + 3x - 1.$$

Plugging $x = 61$ into $(x - 1)^3$ gives us an easy way to calculate:

$$(61 - 1)^3 = 60^3 = \boxed{216000}.$$

15.31 Let's call the number we are looking for N. For each $m = 2, 3, \ldots, 10$ we have

$$N \equiv m - 1 \equiv -1 \pmod{m}.$$

Adding 1 to both sides of this congruence gives us $N + 1 \equiv 0 \pmod{m}$, so $N + 1$ is a multiple of each of $2, 3, \ldots, 10$. This means $N + 1$ is a multiple of the LCM of $2, 3, \ldots, 10$, which is 2520. We are looking for the smallest positive integer that is 1 less than a multiple of 2520, which is $\boxed{2519}$.

15.32 While it's usually easiest to find the GCD of a pair of integers using prime factorization, there is a nice relationship between the integers in this problem: $39654 = 39600 + 54 = 50 \cdot 792 + 54$. This allows an easy application of the Euclidean algorithm:

$$\gcd(39654, 792) = \gcd(792, 54) = \gcd(54, 36) = \gcd(36, 18) = \gcd(18, 18) = \boxed{18}.$$

15.33 If A and B were both odd, then $A + B$ would be the even prime 2. But 2 is less than the sum of any primes, so either A or B must be even and therefore 2. Since $A - B$ is prime, $A > B$, so $B = 2$.

The integers $A - 2$, A, and $A + 2$ are all prime and must be the only set of triplet primes, 3, 5, and 7 (there is only one set of triplet primes because in any set of three consecutive odd integers, one is a multiple of 3, meaning one of the primes must be 3). The sum of all four primes is $2 + 3 + 5 + 7 = \boxed{17}$.

15.34 Let a, b, and c be integers such that $0 < a < b < c$ and $a^2 + b^2 + c^2 = 125$. Since

$$125 = a^2 + b^2 + c^2 < 3c^2,$$

we know that $c^2 > \frac{125}{3} = 41.\overline{6}$. This tells us that the largest square is at least $7^2 = 49$. Since the largest square is at most $11^2 = 121$, we can check the five cases where $c = 7, 8, 9, 10$, or 11. When hunting for possible values of a and b, we can keep in mind that perfect squares are either congruent to 0 or 1 modulo 4 (0 for evens, 1 for odds). Since $125 \equiv 1 \pmod{4}$, exactly one of a, b, and c is odd.

$c = 7$	$a^2 + b^2 = 76$	no solutions
$c = 8$	$a^2 + b^2 = 61$	$(5, 6)$
$c = 9$	$a^2 + b^2 = 44$	no solutions
$c = 10$	$a^2 + b^2 = 25$	$(3, 4)$
$c = 11$	$a^2 + b^2 = 4$	no solutions

The only $\boxed{2}$ possible ways are $5^2 + 6^2 + 8^2$ and $3^2 + 4^2 + 10^2$.

15.35 We see powers of 3, so we change the integers to exponential form for easier computation:

$$\sqrt{(27)(243)} = \sqrt{3^3 \cdot 3^5} = \sqrt{3^8} = 3^4 = \boxed{81}.$$

15.36 We can use the relationship between 2^{2006}, 5^{2006}, and the fact that the decimal system is based on powers of 10. We note that $2^{2006} \cdot 5^{2006} = 10^{2006}$ is a 2007-digit number. We can write the powers of 2 and 5 in a way that makes the last step clear:

$$\begin{aligned} 2^{2006} &= a \cdot 10^m \\ 5^{2006} &= b \cdot 10^n \end{aligned}$$

Here m and n are positive integers and each a and b are between 1 and 10. The number of digits of 2^{2006} is $m + 1$ and the number of digits of 5^{2006} is $n + 1$. Let's look at what this means in terms of their product:

$$2^{2006} \cdot 5^{2006} = ab \cdot 10^{m+n} = 10^{2006}.$$

This means the product ab is a power of 10. Since a and b are each between 1 and 10, their product is between 1 and 100, so their product must be $10^1 = 10$, so

$$10 \cdot 10^{m+n} = 10^{2006},$$

meaning $m + n = 2005$, so the total number of digits in each of 2^{2006} and 5^{2006} is

$$(m+1) + (n+1) = (m+n) + 2 = 2005 + 2 = \boxed{2007}.$$

15.37 If n is one less than a multiple of each of $3, 5, 7, 9$, and 11, $n + 1$ must be a multiple of all of them, and therefore a multiple of their LCM. Since 3 is a divisor of 9, we can ignore it in the calculation. Since $5, 7, 9$, and 11 are all coprime in pairs, the LCM is

$$5 \cdot 7 \cdot 9 \cdot 11 = 5 \cdot 7 \cdot 99 = 35 \cdot 99 = 35 \cdot (100 - 1) = 3500 - 35 = 3465.$$

The smallest positive integer that is 1 less than a multiple of 3465 is $\boxed{3464}$.

15.38 We recall that any number ending in 2 or 4 must be divisible by 2, and any number ending in 5 must be divisible by 5. Since two prime numbers are formed, one of them must end in one of those three numbers. Since a prime can't be divisible by 2 unless it is equal to 2 or by 5 unless it is equal to 5, one of the primes must be either 2 or 5, and the other prime must end in 7. If one of the primes is 2, the other prime can either be 457 or 547. If one of the primes is 5, the other can either be 247 or 427. 247 is not prime, and the smallest of $2 \cdot 457$, $2 \cdot 547$, and $5 \cdot 427$ is $2 \cdot 457 = \boxed{914}$.

15.39 Factoring this difference of squares, we have

$$8008^2 - 7992^2 = (8008 + 7992)(8008 - 7992) = 16000 \cdot 16 = \boxed{256{,}000}.$$

15.40 There are not very many two-digit numbers that are 5 more than a multiple of 7, so we can just list them out:

$$12,\ 19,\ 26,\ 33,\ 40,\ 47,\ 54,\ 61,\ 68,\ 75,\ 82,\ 89,\ 96.$$

Of these, only 19, 82, and 96 have sums of digits which are multiples of 5, and their sum is $\boxed{197}$.

15.41 We can use what we know about the value of the square to narrow down the possibilities. We start with broad estimates: $300^2 = 90{,}000 < 152AB1 < 160{,}000 = 400^2$, so the first digit of the square root is 3. Indeed, $380^2 = 144{,}400 < 152AB1$, so the first two digits of the square root are either 38 or 39. Since the last digit of the number $152AB1$ is 1, its square root must have a last digit of either 1 or 9. Since $152AB1$ is close to midway between 144,400 and 160,000, the two possibilities we must check for its square root are 389 and 391. $389^2 = 151{,}321 < 152AB1$, but $391^2 = 152881$, which is of the required form, so the answer is $3 + 9 + 1 = \boxed{13}$.

15.42 Seeing all those powers of 5, we rewrite the whole expression in terms of 5:

$$\sqrt[3]{5^{24} + 5^{24} + 5^{25} + 5^{24}}.$$

Factoring out $\sqrt[3]{5^{24}} = 5^8$, we get

$$5^8\sqrt[3]{1 + 1 + 5 + 1} = 5^8\sqrt[3]{8} = 5^8 \cdot 2 = \boxed{781250}.$$

15.43 When an integer n is 1 more than a multiple of each 2 and 3, then $n-1$ is a multiple of both 2 and 3 and therefore a multiple of 6. We write $n = 6m+1$ for a nonnegative integer m. Since $0 < n < 100$,

$$0 < 6m + 1 < 100.$$

Subtracting 1 from both sides we get $-1 < 6m < 99$. Dividing by 6 we finally isolate m:

$$-\frac{1}{6} < m < 16\frac{1}{2}.$$

The possible nonnegative integers m are $0, 1, 2, \ldots, 16$, which each correspond to a value of n, so there are $\boxed{17}$ total.

15.44 Since the sum of the two prime numbers is 20, they can be expressed as $10 + x$ and $10 - x$ for some integer x. We can compute the product as a difference of squares:

$$(10 + x)(10 - x) = 100 - x^2.$$

The product is largest when x^2 is smallest. When $x = 0$, 1, and 2, at least one of $10 + x$ or $10 - x$ is composite. However, when $x = 3$, the primes are 7 and 13, giving a greatest possible product of $7 \cdot 13 = 100 - 3^2 = \boxed{91}$.

15.45 Notice that $(10)(6) = 2 \cdot (5)(6)$. The squares of two numbers and two times their product reminds us of the factorization $(x - y)^2 = x^2 - 2xy + y^2$. When $x = 5$ and $y = 6$, we get the expression we are trying to evaluate:

$$5^2 - (10)(6) + 6^2 = (5 - 6)^2 = \boxed{1}.$$

15.46 We recall that a number is a multiple of 5 if and only if its units digit is 0 or 5. Therefore, none of the given 2, 4, or 8 can be the last digit. We order 2, 4, and 8 from least to greatest to minimize the integer, so the smallest such number will be of the form $248x$. The smallest possibility is $\boxed{2480}$.

15.47 We factor $66666 = 3 \cdot 22222$. Then $A = 66666^4 = (3 \cdot 22222)^4 = 3^4 \cdot 22222^4 = 81B$. Dividing by B we get $A/B = \boxed{81}$.

15.48 Since $2^4 = 16$, we can use a difference of squares twice:

$$\begin{aligned} 5^8 - 16 &= 5^8 - 2^4 \\ &= (5^4 + 2^2)(5^4 - 2^2) \\ &= (5^4 + 2^2)(5^2 + 2)(5^2 - 2) \end{aligned}$$

The prime factorization we seek is the product of the prime factorizations of the three factors. $5^4 + 4 = 629$ factors as $17 \cdot 37$, $5^2 + 2 = 27$ factors as 3^3, and $5^2 - 2 = 23$ is prime. Hence the prime factorization of $5^8 - 16$ is $\boxed{3^3 \cdot 17 \cdot 23 \cdot 37}$. Students with excellent number sense might notice that $629 = 729 - 100 = 27^2 - 10^2 = (27 + 10)(27 - 10) = 37 \cdot 17$ makes the factorization even easier.

15.49 The temptation is to write out all of the first 25 primes, look at all of their remainders when divided by 4, and multiply, but there is a much simpler way. Since 2 is a prime, the product of the first 25 primes is going to be a multiple of 2, so its remainder when divided by 4 will be either 2 or 0. But this product is not a multiple of 4, since no other primes have a factor of 2, so the remainder cannot be 0. Therefore, the remainder is $\boxed{2}$.

15.50 Rather than exponentiating and subtracting, we factor 2^{19} out of each term, getting

$$2^{20} - 2^{19} = 2^{19}(2-1) = 2^{19},$$

so $x = \boxed{19}$.

15.51 Let $N > 1$ be an integer that, when divided by each of $2, 3, \ldots, 10$ and 11, leaves a remainder of 1. So, $N - 1$ is a multiple of each of $2, 3, \ldots, 10$ and 11 and is therefore a multiple of their LCM, 27720. We can express N in the parametric form $27720m + 1$. The two smallest such values of N occur when $m = 1$ and $m = 2$, so their difference is

$$(27720 \cdot 2 + 1) - (27720 \cdot 1 + 1) = \boxed{27720}.$$

15.52 The repeated digits give us a clue about a common factor. We write $8008 = 8 \cdot 1001$, and $14014 = 14 \cdot 1001$. Since, $\gcd(8, 14) = 2$,

$$\begin{aligned} \gcd(8008, 14014) &= 1001 \cdot \gcd(8, 14) \\ &= 1001 \cdot 2 \\ &= \boxed{2002} \end{aligned}$$

15.53 Any multiple of 9^3 can be written in the form $9^3 n$ for some integer n. Our goal is to count the number of integers in this form between 9^4 and 9^6. We divide the inequality

$$9^4 < 9^3 n < 9^6$$

by 9^3 in order to isolate n:

$$9 < n < 729.$$

It is now easy to count the possible values of n: $729 - 9 - 1 = \boxed{719}$.

15.54 Those coefficients look familiar (a row of Pascal's triangle!), and the fact that the powers of 53 decrease while those of 3 increase gives it away: the given expression is the expansion of $(53-3)^5$. Now the computation is relatively simple:

$$50^5 = \boxed{312{,}500{,}000}.$$

15.55 We can use both factorization and the trick for squaring integers with units digits of 5.

$$\begin{aligned} 35^4 - 25^4 &= (35^2 + 25^2)(35^2 - 25^2) \\ &= (35^2 + 25^2)(35 + 25)(35 - 25) \\ &= (1225 + 625)(60)(10) \\ &= (1850)(60)(10) \\ &= (3700)(30)(10) \\ &= (11100)(10)(10) \\ &= \boxed{1110000} \end{aligned}$$

Notice at the end that the products were regrouped in ways that allowed factors to be combined in the nicest ways possible. First and foremost, we organized multiples of 10. After that, we noticed the nice and common product $3 \cdot 37 = 111$.

15.56 Since each denominator is at least 1, we can determine the value of each variable in succession by subtracting out the largest positive integer less than or equal to the right-hand side of the equation.

$$a + \cfrac{1}{b + \cfrac{1}{c+\frac{1}{d}}} = \frac{931}{222},$$

so $a = 4$. Subtracting a from the left side and 4 from the right, we get

$$\cfrac{1}{b + \cfrac{1}{c+\frac{1}{d}}} = \frac{43}{222}.$$

We can get a similar (and simpler) equation to the original by taking the reciprocal of both sides:

$$b + \cfrac{1}{c + \frac{1}{d}} = \frac{222}{43}.$$

Repeating this process of subtracting out the integer part and taking the reciprocal, we find that $(a, b, c, d) = (4, 5, 6, 7)$.

15.57 There are a couple of nice approaches to this problem. We could start by factoring the difference of cubes, but since the numbers being used are constructed from such easy-to-multiply integers, we can work the problem algebraically.

$$\begin{aligned} 1005 &= 1000 + 5 \\ 995 &= 1000 - 5 \end{aligned}$$

Let $a = 1000$ and $b = 5$. Our goal is to compute

$$\begin{aligned} (a+b)^3 - (a-b)^3 &= (a^3 + 3a^2b + 3ab^2 + b^3) - (a^3 - 3a^2b + 3ab^2 - b^3) \\ &= 6a^2b + 2b^3 \end{aligned}$$

Plugging back in we get our answer:

$$6(1000^2)(5) + 2(5^3) = 30000000 + 250 = \boxed{30000250}.$$

15.58 We can cancel 2^{2000} from every part of the fraction for easy computation:

$$\frac{2^{2004} + 2^{2001}}{2^{2003} - 2^{2000}} = \frac{2^4 + 2}{2^3 - 1} = \boxed{\frac{18}{7}}.$$

15.59 Seeing the powers of 18 makes us think about a factorization of the cube of a sum. However, the "coefficients" don't match. We notice powers of 2 added – particularly the last term, 2^3. Indeed, we can compute this as the cube of a sum:

$$18^3 + 6 \cdot 18^2 + 12 \cdot 18 + 8 = 18^3 + 3 \cdot 18^2 \cdot 2^1 + 3 \cdot 18^1 \cdot 2^2 + 2^3 = (18+2)^3 = 20^3 = \boxed{8000}.$$

15.60 This will be easier to evaluate if we change everything to decimal form:

$$0.6 < \frac{a}{b} < 0.625.$$

Now we use trial and error to find the correct answer. Checking, we find that $b \le 12$ does not give a solution, but $\frac{8}{13} = 0.61538\ldots$ fits in the required range, so the answer is $b = \boxed{13}$.

15.61 The fraction $\dfrac{n-13}{5n+6}$ is reducible if and only if $\gcd(5n+6, n-13) > 1$. Using the Euclidean algorithm,

$$\begin{aligned} \gcd(5n+6, n-13) &= \gcd((5n+6) - 5(n-13), n-13) \\ &= \gcd(71, n-13) \end{aligned}$$

Since 71 is prime, the only way the GCD could be greater than 1 is for it to be 71. This means that $n - 13$ is the smallest positive multiple of 71, so $n - 13 = 71$, so $n = \boxed{84}$ is the answer.

15.62 The numbers are being listed such that an integer with a higher power of 3 in its prime factorization comes after an integer with a lower power of 3 in its prime factorization. This means the last number in the list will have the highest possible power of 3 in its prime factorization. We note that

$$729 = 3^6 < 2002 < 3^7 = 2187.$$

The last integers in the list will have 6 powers of 3 in their prime factorizations, namely, $3^6 = 729$, and $2 \cdot 3^6 = 1458$. The larger of the two will be last, which is $\boxed{1458}$.