2019 AMC 10B Problems/Problem 1

The following problem is from both the 2019 AMC 10B #1 and 2019 AMC 12B #1, so both problems redirect to this page.

Contents

Problem

Alicia had two containers. The first was $\frac{5}{6}$ full of water and the second was empty. She poured all the water from the first container into the second container, at which point the second container was $\frac{3}{4}$ full of water. What is the ratio of the volume of the first container to the volume of the second container?

$$\textbf{(A) } \frac{5}{8} \qquad \textbf{(B) } \frac{4}{5} \qquad \textbf{(C) } \frac{7}{8} \qquad \textbf{(D) } \frac{9}{10} \qquad \textbf{(E) } \frac{11}{12}$$

Solution 1

Let the first jar's volume be A and the second's be B. It is given that $\frac{3}{4}A = \frac{5}{6}B$.

We find that $\frac{A}{B} = \frac{\left(\frac{3}{4}\right)}{\left(\frac{5}{6}\right)} = \boxed{\textbf{(D) } \frac{9}{10}}$.

We already know that this is the ratio of the smaller to the larger volume because it is less than 1.

Solution 2

An alternate solution is to plug in some maximum volume for the first container - let's say 72, so there was a volume of 60 in the first container, and then the second container also has a volume of 60, so you get $60 \cdot \frac{4}{3} = 80$. Thus the answer is $\frac{72}{80} = \boxed{\textbf{(D) } \frac{9}{10}}$.

~IronicNinja

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 2

The following problem is from both the 2019 AMC 10B #2 and 2019 AMC 12B #2, so both problems redirect to this page.

Problem

Consider the statement, "If n is not prime, then $n-2$ is prime." Which of the following values of n is a counterexample to this statement?

$\textbf{(A) } 11 \qquad \textbf{(B) } 15 \qquad \textbf{(C) } 19 \qquad \textbf{(D) } 21 \qquad \textbf{(E) } 27$

Solution

Since a counterexample must be value of n which is not prime, n must be composite, so we eliminate A and C. Now we subtract 2 from the remaining answer choices, and we see that the only time $n-2$ is **not** prime is when $n = \boxed{\textbf{(E) } 27}$.

~IronicNinja

minor edit (the inclusion of not) by AlcBoy1729

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 3

Contents

Problem

In a high school with 500 students, 40% of the seniors play a musical instrument, while 30% of the non-seniors do not play a musical instrument. In all, 46.8% of the students do not play a musical instrument. How many non-seniors play a musical instrument?

$\textbf{(A) }66 \qquad\textbf{(B) }154 \qquad\textbf{(C) }186 \qquad\textbf{(D) }220 \qquad\textbf{(E) }266$

Solution 1

60% of seniors do not play a musical instrument. If we denote x as the number of seniors, then

$$\frac{3}{5}x + \frac{3}{10} \cdot (500 - x) = \frac{468}{1000} \cdot 500$$

$$\frac{3}{5}x + 150 - \frac{3}{10}x = 234$$

$$\frac{3}{10}x = 84$$

$$x = 84 \cdot \frac{10}{3} = 280$$

Thus there are $500 - x = 220$ non-seniors. Since 70% of the non-seniors play a musical instrument, $220 \cdot \frac{7}{10} = \boxed{\textbf{(B)}\ 154}$.

~IronicNinja

Solution 2

Let x be the number of seniors, and y be the number of non-seniors. Then

$$\frac{3}{5}x + \frac{3}{10}y = \frac{468}{1000} \cdot 500 = 234$$

Multiplying both sides by 10 gives us

$$6x + 3y = 2340$$

Also, $x + y = 500$ because there are 500 students in total.

Solving these system of equations give us $x = 280$, $y = 220$.

Since 70% of the non-seniors play a musical instrument, the answer is simply 70% of 220, which gives us $\boxed{\textbf{(B)}\ 154}$.

Solution 3 (using the answer choices)

We can clearly deduce that 70% of the non-seniors do play an instrument, but, since the total percentage of instrument players is 46.8%, the non-senior population is quite low. By intuition, we can therefore see that the answer is around B or C. Testing both of these gives us the answer $\boxed{\textbf{(B)}\ 154}$.

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 4

Contents

Problem

All lines with equation $ax + by = c$ such that a, b, c form an arithmetic progression pass through a common point. What are the coordinates of that point?

$\textbf{(A) } (-1,2) \qquad \textbf{(B) } (0,1) \qquad \textbf{(C) } (1,-2) \qquad \textbf{(D) } (1,0) \qquad \textbf{(E) } (1,2)$

Solution 1

If all lines satisfy the condition, then we can just plug in values for a, b, and c that form an arithmetic progression. Let's use $a = 1, b = 2, c = 3$, and $a = 1, b = 3, c = 5$. Then the two lines we get are:

$$x + 2y = 3$$

$$x + 3y = 5$$

Use elimination to deduce

$$y = 2$$

and plug this into one of the previous line equations. We get

$$x + 4 = 3 \Rightarrow x = -1$$

Thus the common point is $\boxed{\textbf{(A) } (-1,2)}$.

~IronicNinja

Solution 2

We know that a, b, and c form an arithmetic progression, so if the common difference is d, we can say $a, b, c = a, a+d, a+2d.$ Now we have $ax + (a+d)y = a + 2d$, and expanding gives $ax + ay + dy = a + 2d.$ Factoring gives

$a(x+y-1)+d(y-2)=0$. Since this must always be true (regardless of the values of x and y), we must have $x+y-1=0$ and $y-2=0$, so $x, y=-1, 2$, and the common point is $\boxed{\textbf{(A) } (-1, 2)}$.

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 5

Contents

Problem

Triangle ABC lies in the first quadrant. Points A, B, and C are reflected across the line $y = x$ to points A', B', and C', respectively. Assume that none of the vertices of the triangle lie on the line $y = x$. Which of the following statements is _not_ always true?

$\textbf{(A)}$ Triangle $A'B'C'$ lies in the first quadrant.

$\textbf{(B)}$ Triangles ABC and $A'B'C'$ have the same area.

$\textbf{(C)}$ The slope of line AA' is -1.

$\textbf{(D)}$ The slopes of lines AA' and CC' are the same.

$\textbf{(E)}$ Lines AB and $A'B'$ are perpendicular to each other.

Solution

Let's analyze all of the options separately.

$\textbf{(A)}$: Clearly $\textbf{(A)}$ is true, because a point in the first quadrant will have non-negative x- and y-coordinates, and so its reflection, with the coordinates swapped, will also have non-negative x- and y-coordinates.

$\textbf{(B)}$: The triangles have the same area, since $\triangle ABC$ and $\triangle A'B'C'$ are the same triangle (congruent). More formally, we can say that area is *invariant* under reflection.

$\mathbf{(C)}$: If point A has coordinates (p, q), then A' will have coordinates (q, p). The gradient is thus $\frac{p-q}{q-p} = -1$, so this is true. (We know $p \neq q$ since the question states that none of the points A, B, or C lies on the line $y = x$, so there is no risk of division by zero).

$\mathbf{(D)}$: Repeating the argument for $\mathbf{(C)}$, we see that both lines have slope -1, so this is also true.

$\mathbf{(E)}$: By process of elimination, this must now be the answer. Indeed, if point A has coordinates (p, q) and point B has coordinates (r, s), then A' and B' will, respectively, have coordinates (q, p) and (s, r). The product of the gradients of AB and $A'B'$ is $\frac{s-q}{r-p} \cdot \frac{r-p}{s-q} = 1 \neq -1$, so in fact these lines are **never** perpendicular to each other (using the "negative reciprocal" condition for perpendicularity).

Thus the answer is $\boxed{\mathbf{(E)}}$.

Counterexamples

If $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (7, 1)$, then the slope of AB, m_{AB}, is $\frac{1-3}{7-2} = -\frac{2}{5}$, while the slope of $A'B'$, $m_{A'B'}$, is $\frac{7-2}{1-3} = -\frac{5}{2}$. $m_{A'B'}$ is the **reciprocal** of m_{AB}, but it is not the negative reciprocal of m_{AB}. To generalize, let (x_1, y_1) denote the coordinates of point A, let (x_2, y_2) denote the coordinates of point B, let m_{AB} denote the slope of segment $\overline{AB}$, and let $m_{A'B'}$ denote the slope of segment $\overline{A'B'}$. Then, the coordinates of A' are (y_1, x_1), and of B' are (y_2, x_2). Then, $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$, and $m_{A'B'} = \frac{x_2 - x_1}{y_2 - y_1} = \frac{1}{m_{ab}}$. If $y_1 \neq y_2$ and $x_1 \neq x_2$, $\frac{1}{m_{AB}} \neq \frac{1}{m_{A'B'}} \Rightarrow m_{AB} \neq m_{A'B'}$, and in these cases, the condition is false.

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 6

The following problem is from both the 2019 AMC 10B #6 and 2019 AMC 12B #4, so both problems redirect to this page.

Contents

Problem

There is a positive integer n such that $(n+1)! + (n+2)! = n! \cdot 440$. What is the sum of the digits of n?

$\textbf{(A) }3\qquad\textbf{(B) }8\qquad\textbf{(C) }10\qquad\textbf{(D) }11\qquad\textbf{(E) }12$

Solution 1

$$\begin{aligned} & (n+1)n! + (n+2)(n+1)n! = 440 \cdot n! \\ \Rightarrow\ & n![n+1+(n+2)(n+1)] = 440 \cdot n! \\ \Rightarrow\ & n+1+n^2+3n+2 = 440 \\ \Rightarrow\ & n^2+4n-437 = 0 \end{aligned}$$

Solving by the quadratic formula, $n = \frac{-4 \pm \sqrt{16+437 \cdot 4}}{2} = \frac{-4 \pm 42}{2} = \frac{38}{2} = 19$ (since clearly $n \geq 0$). The answer is therefore $1+9 = \boxed{\textbf{(C) }10}$.

Solution 2

Dividing both sides by $n!$ gives

$$(n+1)+(n+2)(n+1) = 440 \Rightarrow n^2+4n-437 = 0 \Rightarrow (n-19)(n+23) = 0$$

Since n is non-negative, $n = 19$. The answer is $1+9 = \boxed{\textbf{(C) }10}$.

Solution 3

Dividing both sides by $n!$ as before gives $(n+1)+(n+1)(n+2)=440$. Now factor out $(n+1)$, giving $(n+1)(n+3)=440$. By considering the prime factorization of 440, a bit of experimentation gives us $n+1=20$ and $n+3=22$, so $n=19$, so the answer is $1+9=\boxed{\textbf{(C) } 10}$.

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 7

The following problem is from both the 2019 AMC 10B #7 and 2019 AMC 12B #5, so both problems redirect to this page.

Contents

Problem

Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

$\textbf{(A) } 18 \qquad \textbf{(B) } 21 \qquad \textbf{(C) } 24 \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 28$

Solution 1

If he has enough money to buy 12 pieces of red candy, 14 pieces of green candy, and 15 pieces of blue candy, then the smallest amount of money he could have is $\text{lcm}(12, 14, 15) = 420$ cents. Since a piece of purple candy costs 20 cents, the smallest possible value of n is $\frac{420}{20} = \boxed{\textbf{(B) } 21}$.

~IronicNinja

Solution 2

We simply need to find a value of $20n$ that is divisible by $12, 14$, and 15. Observe that $20 \cdot 18$ is divisible by 12 and 15, but not 14. $20 \cdot 21$ is divisible by $12, 14$, and 15, meaning that we have exact change (in this case, 420 cents) to buy each type of candy, so the minimum value of n is $\boxed{\textbf{(B) } 21}$.

Solution 3

We can notice that the number of purple candy times 20 has to be divisible by 7, because of the 14 green candies, and 3, because of the 12 red candies.
$7 * 3 = 21$, so the answer has to be $\boxed{\textbf{(B) }21}$.

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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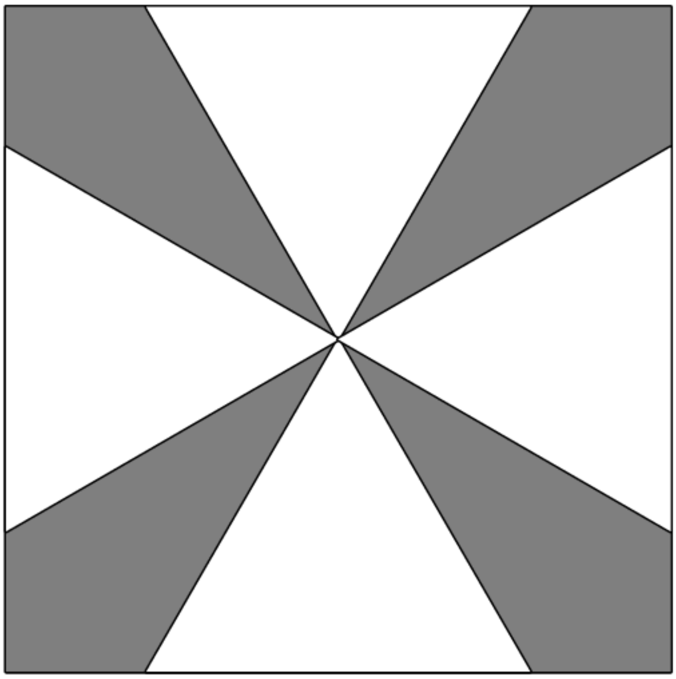
2019 AMC 10B Problems/Problem 8

Contents

Problem

The figure below shows a square and four equilateral triangles, with each triangle having a side lying on a side of the square, such that each triangle has side length 2 and the third vertices of the triangles meet at the center of the square. The region inside the square but outside the triangles is shaded. What is the area of the shaded region?



$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 12 - 4\sqrt{3} \qquad \textbf{(C)}\ 3\sqrt{3} \qquad \textbf{(D)}\ 4\sqrt{3} \qquad \textbf{(E)}\ 16 - 4\sqrt{3}$

Solution 1

We notice that the square can be split into 4 congruent smaller squares, with the altitude of the equilateral triangle being the side of this smaller square. Therefore, the area of each shaded part that resides within a square is the total area of the square subtracted from each triangle (which has already been split in half). #storm

When we split an equilateral triangle in half, we get two $30^\circ-60^\circ-90^\circ$ triangles. Therefore, the altitude, which is also the side length of one of the smaller squares, is $\sqrt{3}$. We can then compute the area of the two triangles as $2\cdot\frac{1\cdot\sqrt{3}}{2}=\sqrt{3}$.

The area of the each small squares is the square of the side length, i.e. $\left(\sqrt{3}\right)^2=3$. Therefore, the area of the shaded region in each of the four squares is $3-\sqrt{3}$.

Since there are 4 of these squares, we multiply this by 4 to get $4\left(3-\sqrt{3}\right)=\boxed{\textbf{(B) }12-4\sqrt{3}}$ as our answer.

Solution 2

We can see that the side length of the square is $2\sqrt{3}$ by considering the altitude of the equilateral triangle as in Solution 1. Using the Pythagorean Theorem, the diagonal of the square is thus $\sqrt{12+12}=\sqrt{24}=2\sqrt{6}$. Because of this, the height of one of the four shaded kites is $\sqrt{6}$. Now, we just need to find the length of that kite. By the Pythagorean Theorem again, this length is $\frac{2\sqrt{3}-2}{2}\times\sqrt{2}=\sqrt{3}-1=\sqrt{6}-\sqrt{2}$. Now using $\text{area}=\frac{1}{2}\cdot\text{length}\cdot\text{width}$, the area of one of the four kites is $2\sqrt{6}\times\left(\sqrt{6}-\sqrt{2}\right)=12-2\sqrt{12}=\boxed{\textbf{(B) }12-4\sqrt{3}}$.

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 9

Contents

Problem

The function f is defined by

$$f(x) = \lfloor|x|\rfloor - |\lfloor x \rfloor|$$

for all real numbers x, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to the real number r. What is the range of f?

$\textbf{(A) } \{-1, 0\} \qquad \textbf{(B) } \text{The set of nonpositive integers} \qquad \textbf{(C) } \{-1, 0, 1\} \qquad \textbf{(D) } \{0\} \qquad \textbf{(E) }$ T

Solution 1

There are four cases we need to consider here.

Case 1: x is a positive integer. Without loss of generality, assume $x=1$. Then $f(1) = 1 - 1 = 0$.

Case 2: x is a positive fraction. Without loss of generality, assume $x=\frac{1}{2}$. Then $f\left(\frac{1}{2}\right) = 0 - 0 = 0$.

Case 3: x is a negative integer. Without loss of generality, assume $x=-1$. Then $f(-1) = 1 - 1 = 0$.

Case 4: x is a negative fraction. Without loss of generality, assume $x=-\frac{1}{2}$. Then $f\left(-\frac{1}{2}\right) = 0 - 1 = -1$.

Thus the range of the function f is $\boxed{\textbf{(A) } \{-1, 0\}}$.

~IronicNinja, edited by someone else hehe

Solution 2

It is easily verified that when x is an integer, $f(x)$ is zero. We therefore need only to consider the case when x is not an integer.

When x is positive, $\lfloor x\rfloor \geq 0$, so

$$\begin{aligned} f(x) &= \lfloor |x| \rfloor - |\lfloor x \rfloor| \\ &= \lfloor x \rfloor - \lfloor x \rfloor \\ &= 0 \end{aligned}$$

When x is negative, let $x = -a-b$ be composed of integer part a and fractional part b (both ≥ 0):

$$\begin{aligned} f(x) &= \lfloor |-a-b| \rfloor - |\lfloor -a-b \rfloor| \\ &= \lfloor a+b \rfloor - |-a-1| \\ &= a-(a+1) = -1 \end{aligned}$$

Thus, the range of f is $\boxed{\textbf{(A) } \{-1, 0\}}$.

Note: One could solve the case of x as a negative non-integer in this way:

$$\begin{aligned} f(x) &= \lfloor |x| \rfloor - |\lfloor x \rfloor| \\ &= \lfloor -x \rfloor - |-\lfloor -x \rfloor - 1| \\ &= \lfloor -x \rfloor - (\lfloor -x \rfloor + 1) = -1 \end{aligned}$$

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 10

The following problem is from both the 2019 AMC 10B #10 and 2019 AMC 12B #6, so both problems redirect to this page.

Contents

Problem

In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units?

$\textbf{(A) } 0 \qquad\textbf{(B) } 2 \qquad\textbf{(C) } 4 \qquad\textbf{(D) } 8 \qquad\textbf{(E) } \text{infinitely many}$

Solution 1

Notice that whatever point we pick for C, AB will be the base of the triangle. Without loss of generality, let points A and B be $(0,0)$ and $(0,10)$, since for any other combination of points, we can just rotate the plane to make them $(0,0)$ and $(0,10)$ under a new coordinate system. When we pick point C, we have to make sure that its y-coordinate is ±20, because that's the only way the area of the triangle can be 100.

Now when the perimeter is minimized, by symmetry, we put C in the middle, at $(5, 20)$. We can easily see that AC and BC will both be $\sqrt{20^2+5^2} = \sqrt{425}$. The perimeter of this minimal triangle is $2\sqrt{425} + 10$, which is larger than 50. Since the minimum perimeter is greater than 50, there is no triangle that satisfies the condition, giving us $\boxed{\textbf{(A) } 0}$.

~IronicNinja

Solution 2

Without loss of generality, let AB be a horizontal segment of length 10. Now realize that C has to lie on one of the lines parallel to AB and vertically 20 units away from it. But $10+20+20$ is already 50, and this doesn't form a triangle. Otherwise, without loss of generality, $AC<20$. Dropping altitude CD, we have a right triangle ACD with hypotenuse $AC<20$ and leg $CD=20$, which is clearly impossible, again giving the answer as $\boxed{\textbf{(A) } 0}$.

Solution 3

Area = 100, perimeter = 50, semiperimeter $s = 50/2 = 25$, $z = AB = 10$, $x = AC$ and $y = 50 - 10 - x = 40 - x$.

Using the generic formula for triangle area using semiperimeter s and sides x, y, and z, area = $\sqrt{(s)(s-x)(s-y)(s-z)}$. (Heron's formula)

$$100 = \sqrt{(25)(25-10)(25-x)(25-(40-x))} = \sqrt{(375)(25-x)(x-15)}$$
.

Square both sides, divide by 375 and expand the polynomial to get $35x - x^2 - 375 = 80/3$.

$x^2 - 35x + (375 + 80/3) = 0$ and the discriminant is $((-35)^2 - 4 * 1 * 401.\overline{6}) < 0$ so no real solutions.

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 11

Contents

Problem

Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is $9:1$, and the ratio of blue to green marbles in Jar 2 is $8:1$. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

$\textbf{(A)}\ 5\qquad\textbf{(B)}\ 10\qquad\textbf{(C)}\ 25\qquad\textbf{(D)}\ 45\qquad\textbf{(E)}\ 50$

Solution

Call the number of marbles in each jar x (because the problem specifies that they each contain the same number). Thus, $\frac{x}{10}$ is the number of green marbles in Jar 1, and $\frac{x}{9}$ is the number of green marbles in Jar 2. Since $\frac{x}{9}+\frac{x}{10}=\frac{19x}{90}$, we have $\frac{19x}{90}=95$, so there are $x=450$ marbles in each jar.

Because $\frac{9x}{10}$ is the number of blue marbles in Jar 1, and $\frac{8x}{9}$ is the number of blue marbles in Jar 2, there are $\frac{9x}{10}-\frac{8x}{9}=\frac{x}{90}=5$ more marbles in Jar 1 than Jar 2. This means the answer is $\boxed{\textbf{(A)}\ 5}$.

Video Solution

https://youtu.be/mXvetCMMzpU

~IceMatrix

See Also

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2019 AMC 10B Problems/Problem 12

Contents

Problem

What is the greatest possible sum of the digits in the base-seven representation of a positive integer less than 2019?

$\textbf{(A) } 11 \qquad \textbf{(B) } 14 \qquad \textbf{(C) } 22 \qquad \textbf{(D) } 23 \qquad \textbf{(E) } 27$

Solution 1

Observe that $2019_{10} = 5613_7$. To maximize the sum of the digits, we want as many 6s as possible (since 6 is the highest value in base 7), and this will occur with either of the numbers or . Thus, the answer is $4 + 6 + 6 + 6 = 5 + 5 + 6 + 6 = \boxed{\textbf{(C) } 22}$.

~IronicNinja went through this test 100 times

Solution 2

Note that all base 7 numbers with 5 or more digits are in fact greater than 2019. Since the first answer that is possible using a 4 digit number is , we start with the smallest base 7 number that whose digits sum to , namely . But this is greater than , so we continue by trying , which is less than 2019. So the answer is .

LaTeX code fix by EthanYL

Video Solution

https://youtu.be/mXvetCMMzpU

~IceMatrix

See Also

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2019 AMC 10B Problems/Problem 13

The following problem is from both the 2019 AMC 10B #13 and 2019 AMC 12B #7, so both problems redirect to this page.

Contents

Problem

What is the sum of all real numbers x for which the median of the numbers $4, 6, 8, 17,$ and x is equal to the mean of those five numbers?

$$\textbf{(A) } -5 \qquad\textbf{(B) } 0 \qquad\textbf{(C) } 5 \qquad\textbf{(D) } \frac{15}{4} \qquad\textbf{(E) } \frac{35}{4}$$

Solution

The mean is $\frac{4+6+8+17+x}{5}=\frac{35+x}{5}$.

There are three possibilities for the median: it is either 6, 8, or x.

Let's start with 6.

$\frac{35+x}{5}=6$ has solution $x=-5$, and the sequence is $-5, 4, 6, 8, 17$, which does have median 6, so this is a valid solution.

Now let the median be 8.

$\frac{35+x}{5}=8$ gives $x=5$, so the sequence is $4, 5, 6, 8, 17$, which has median 6, so this is not valid.

Finally we let the median be x.

$\frac{35+x}{5}=x \implies 35+x=5x \implies x=\frac{35}{4}=8.75$, and the sequence is $4, 6, 8, 8.75, 17$, which has median 8. This case is therefore again not valid.

Hence the only possible value of x is $\boxed{\textbf{(A) } -5}$.

Video Solution

https://youtu.be/mXvetCMMzpU

~IceMatrix

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 14

Contents

Problem

The base-ten representation for $19!$ is $121,6T5,100,40M,832,H00$, where T, M, and H denote digits that are not given. What is $T+M+H$?

$\textbf{(A) }3 \qquad\textbf{(B) }8 \qquad\textbf{(C) }12 \qquad\textbf{(D) }14 \qquad\textbf{(E) }17$

Solution 1

We can figure out $H = 0$ by noticing that $19!$ will end with 3 zeroes, as there are three 5s in its prime factorization. Next, we use the fact that $19!$ is a multiple of both 11 and 9. Their divisibility rules (see Solution 2) tell us that $T + M \equiv 3 \pmod{9}$ and that $T - M \equiv 7 \pmod{11}$. By guess and checking, we see that $T = 4, M = 8$ is a valid solution. Therefore the answer is $4 + 8 + 0 = \boxed{\textbf{(C) }12}$.

Solution 2 (similar to Solution 1)

We know that $H = 0$, because $19!$ ends in three zeroes (see Solution 1). Furthermore, we know that 9 and 11 are both factors of $19!$. We can simply use the divisibility rules for 9 and 11 for this problem to find T and M. For $19!$ to be divisible by 9, the sum of digits must simply be divisible by 9. Summing the digits, we get that $T + M + 33$ must be divisible by 9. This leaves either $\textrm{A}$ or $\textrm{C}$ as our answer choice. Now we test for divisibility by 11. For a number to be divisible by 11, the alternating sum must be divisible by 11 (for example, with the number 2728, $2-7+2-8 = -11$, so 2728 is divisible by 11). Applying the alternating sum test to this problem, we see that $T-M-7$ must be divisible by 11. By inspection, we can see that this holds if $T=4$ and $M=8$. The sum is $8+4+0=\boxed{\textbf{(C) } 12}$.

Video Solution

https://youtu.be/mXvetCMMzpU

~IceMatrix

See Also

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2019 AMC 10B Problems/Problem 15

Contents

Problem

Right triangles T_1 and T_2 have areas of 1 and 2, respectively. A side of T_1 is congruent to a side of T_2 and a different side of T_1 is congruent to a different side of T_2. What is the square of the product of the lengths of the other (third) side of T_1 and T_2?

$\textbf{(A) }\frac{28}{3}\qquad\textbf{(B) }10\qquad\textbf{(C) }\frac{32}{3}\qquad\textbf{(D) }\frac{34}{3}\qquad\textbf{(E) }12$

Solution 1

First of all, let the two sides which are congruent be x and y, where $y > x$. The only way that the conditions of the problem can be satisfied is if x is the shorter leg of T_2 and the longer leg of T_1, and y is the longer leg of T_2 and the hypotenuse of T_1.

Notice that this means the value we are looking for is the square of $\sqrt{x^2+y^2}\cdot\sqrt{y^2-x^2} = \sqrt{y^4-x^4}$, which is just y^4-x^4.

The area conditions give us two equations: $\frac{xy}{2}=2$ and $\frac{x\sqrt{y^2-x^2}}{2}=1$.

This means that $y=\frac{4}{x}$ and that $\frac{4}{x^2}=y^2-x^2$.

Taking the second equation, we get $x^2y^2-x^4=4$, so since $xy=4$, $x^4=12$.

Since $y=\frac{4}{x}$, we get $y^4=\frac{256}{12}=\frac{64}{3}$.

The value we are looking for is just $y^4-x^4=\frac{64-36}{3}=\frac{28}{3}$ so the answer is $\boxed{\textbf{(A) }\frac{28}{3}}$.

Solution 2

Like in Solution 1, we have $\frac{xy}{2}=2$ and $\frac{x\sqrt{y^2-x^2}}{2}=1$.

Squaring both equations yields $x^2y^2 = 16$ and $x^2(y^2 - x^2) = 4$.

Let $a = x^2$ and $b = y^2$. Then $b = \frac{16}{a}$, and
$a\left(\frac{16}{a} - a\right) = 4 \implies 16 - a^2 = 4 \implies a = 2\sqrt{3}$, so $b = \frac{16}{2\sqrt{3}} = \frac{8\sqrt{3}}{3}$.

We are looking for the value of $y^4 - x^4 = b^2 - a^2$, so the answer is $\frac{64}{3} - 12 = \boxed{\textbf{(A)}\ \frac{28}{3}}$.

Solution 3

Firstly, let the right triangles be $\triangle ABC$ and $\triangle EDF$, with $\triangle ABC$ being the smaller triangle. As in Solution 1, let $\overline{AB} = \overline{EF} = x$ and $\overline{BC} = \overline{DF} = y$. Additionally, let $\overline{AC} = z$ and $\overline{DE} = w$.

We are given that $[ABC] = 1$ and $[EDF] = 2$, so using $\text{area} = \frac{bh}{2}$, we have $\frac{xy}{2} = 1$ and $\frac{xw}{2} = 2$. Dividing the two equations, we get $\frac{xy}{xw}$ = $\frac{y}{w} = 2$, so $y = 2w$.

Thus $\triangle EDF$ is a $30^\circ - 60^\circ - 90^\circ$ right triangle, meaning that $x = w\sqrt{3}$. Now by the Pythagorean Theorem in $\triangle ABC$,
$\left(w\sqrt{3}\right)^2 + (2w)^2 = z^2 \Rightarrow 3w^2 + 4w^2 = z^2 \Rightarrow 7w^2 = z^2 \Rightarrow w\sqrt{7} = z$.

The problem requires the square of the product of the third side lengths of each triangle, which is $(wz)^2$. By substitution, we see that wz = $(w)\left(w\sqrt{7}\right) = w^2\sqrt{7}$. We also know

$$\frac{xw}{2} = 1 \Rightarrow \frac{(w)\left(w\sqrt{3}\right)}{2} = 1 \Rightarrow (w)\left(w\sqrt{3}\right) = 2 \Rightarrow w^2\sqrt{3} = 2 \Rightarrow w^2 = \frac{2\sqrt{3}}{3}$$
.

Since we want $\left(w^2\sqrt{7}\right)^2$, multiplying both sides by $\sqrt{7}$ gets us $w^2\sqrt{7} = \frac{2\sqrt{21}}{3}$. Now squaring gives $\left(\frac{2\sqrt{21}}{3}\right)^2 = \frac{4 * 21}{9} = \boxed{\textbf{(A)}\ \frac{28}{3}}$.

Video Solution

https://youtu.be/mXvetCMMzpU

~IceMatrix

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 16

Contents

Problem

In $\triangle ABC$ with a right angle at C, point D lies in the interior of $\overline{AB}$ and point E lies in the interior of $\overline{BC}$ so that $AC=CD, DE=EB,$ and the ratio $AC:DE=4:3$. What is the ratio $AD:DB?$

$\textbf{(A) }2:3\qquad\textbf{(B) }2:\sqrt{5}\qquad\textbf{(C) }1:1\qquad\textbf{(D) }3:\sqrt{5}\qquad\textbf{(E) }3:2$

Solution 1

Without loss of generality, let $AC = CD = 4$ and $DE = EB = 3$. Let $\angle A = \alpha$ and $\angle B = \beta = 90^{\circ} - \alpha$. As $\triangle ACD$ and $\triangle DEB$ are isosceles, $\angle ADC = \alpha$ and $\angle BDE = \beta$. Then $\angle CDE = 180^{\circ} - \alpha - \beta = 90^{\circ}$, so $\triangle CDE$ is a $3-4-5$ triangle with $CE = 5$.

Then $CB = 5 + 3 = 8$, and $\triangle ABC$ is a $1-2-\sqrt{5}$ triangle.

In isosceles triangles $\triangle ACD$ and $\triangle DEB$, drop altitudes from C and E onto AB; denote the feet of these altitudes by P_C and P_E respectively. Then $\triangle ACP_C \sim \triangle ABC$ by AAA similarity, so we get that $AP_C = P_CD = \frac{4}{\sqrt{5}}$, and $AD = 2 \times \frac{4}{\sqrt{5}}$. Similarly we get $BD = 2 \times \frac{6}{\sqrt{5}}$, and $AD:DB = \boxed{\textbf{(A) }2:3}$.

Solution 2

Let $AC=CD=4x$, and $DE=EB=3x$. (For this solution, A is above C, and B is to the right of C). Also let $\angle A = t^{\circ}$, so $\angle ACD = (180-2t)^{\circ}$, which implies $\angle DCB = (2t-90)^{\circ}$. Similarly, $\angle B = (90-t)^{\circ}$, which implies $\angle BED = 2t^{\circ}$. This further implies that $\angle DEC = (180-2t)^{\circ}$.

Now we see that $\angle CDE = 180^{\circ} - \angle ECD - \angle DEC = 180^{\circ} - 2t^{\circ} + 90^{\circ} - 180^{\circ} + 2t^{\circ} = 90^{\circ}$. Thus $\triangle CDE$ is a right triangle, with side lengths of $3x$, $4x$, and $5x$ (by the Pythagorean Theorem, or simply the Pythagorean triple $3-4-5$). Therefore $AC = 4x$ (by definition), $BC = 5x+3x = 8x$, and $AB = 4\sqrt{5}x$. Hence $\cos{(2t^{\circ})} = 2\cos^2{t^{\circ}}-1$ (by the double angle formula), giving $2\left(\frac{1}{\sqrt{5}}\right)^2 - 1 = -\frac{3}{5}$.

By the Law of Cosines in $\triangle BED$, if $BD = d$, we have

$$d^2 = (3x)^2 + (3x)^2 - 2 \cdot \frac{-3}{5}(3x)(3x)$$
$$\Rightarrow d^2 = 18x^2 + \frac{54x^2}{5} = \frac{144x^2}{5}$$
$$\Rightarrow d = \frac{12x}{\sqrt{5}}$$

Now $AD = AB - BD = 4x\sqrt{5} - \frac{12x}{\sqrt{5}} = \frac{8x}{\sqrt{5}}$. Thus the answer is

$$\frac{\left(\frac{8x}{\sqrt{5}}\right)}{\left(\frac{12x}{\sqrt{5}}\right)} = \frac{8}{12} = \boxed{\textbf{(A) } 2:3}.$$

See Also

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2019 AMC 12B Problems/Problem 13

The following problem is from both the 2019 AMC 10B #17 and 2019 AMC 12B #13, so both problems redirect to this page.

Contents

Problem

A red ball and a green ball are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k = 1, 2, 3....$ What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

$$\textbf{(A) } \frac{1}{4} \qquad\textbf{(B) } \frac{2}{7} \qquad\textbf{(C) } \frac{1}{3} \qquad\textbf{(D) } \frac{3}{8} \qquad\textbf{(E) } \frac{3}{7}$$

Solution 1

By symmetry, the probability of the red ball landing in a higher-numbered bin is the same as the probability of the green ball landing in a higher-numbered bin. Clearly, the probability of both landing in the same bin is $\sum_{k=1}^{\infty} 2^{-k} \cdot 2^{-k} = \sum_{k=1}^{\infty} 2^{-2k} = \frac{1}{3}$ (by the geometric series sum formula). Therefore, since the other two probabilities have to both the same, they have to be $\frac{1-\frac{1}{3}}{2} = \boxed{\textbf{(C) } \frac{1}{3}}$.

Solution 2

Suppose the green ball goes in bin i, for some $i \geq 1$. The probability of this occurring is $\frac{1}{2^i}$. Given that this occurs, the probability that the red ball goes in a higher-numbered bin is $\frac{1}{2^{i+1}} + \frac{1}{2^{i+2}} + \ldots = \frac{1}{2^i}$ (by the geometric series sum formula). Thus the probability that the green ball goes in bin i, and the red ball goes in a bin greater than i, is $\left(\frac{1}{2^i}\right)^2 = \frac{1}{4^i}$. Summing from $i = 1$ to infinity, we get

$$\sum_{i=1}^{\infty} \frac{1}{4^i} = \boxed{\textbf{(C) } \frac{1}{3}}$$

where we again used the geometric series sum formula. (Alternatively, if this sum equals n, then by writing out the terms and multiplying both sides by 4, we see $4n = n + 1$, which gives $n = \frac{1}{3}$.)

Solution 3

For red ball in bin k, $\Pr(\text{Green Below Red}) = \sum_{i=1}^{k-1} 2^{-i}$ (GBR) and $\Pr(\text{Red in Bin k} = 2^{-k}$ (RB).

$$\Pr(\text{GBR}|\text{RB}) = \sum_{k=1}^{\infty} 2^{-k} \sum_{i=1}^{k-1} 2^{-i} = \sum_{k=1}^{\infty} 2^{-k} \cdot \frac{1}{2}\left(\frac{1-(1/2)^{k-1}}{1-1/2}\right)$$

$$\sum_{k=1}^{\infty} \frac{1}{2^{-k}} - 2\sum_{k=1}^{\infty} \frac{1}{(2^2)^{-k}} \implies 1 - 2/3 = \boxed{\textbf{D)} \frac{1}{3}}$$

Solution 4

The probability that the two balls will go into adjacent bins is $\frac{1}{2\times4}+\frac{1}{4\times8}+\frac{1}{8\times16}+...=\frac{1}{8}+\frac{1}{32}+\frac{1}{128}+\cdots=\frac{1}{6}$ by the geometric series sum formula. Similarly, the probability that the two balls will go into bins that have a distance of 2 from each other is $\frac{1}{2\times8}+\frac{1}{4\times16}+\frac{1}{8\times32}+\cdots=\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots=\frac{1}{12}$ (again recognizing a geometric series). We can see that each time we add a bin between the two balls, the probability halves. Thus, our answer is $\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\cdots$, which, by the geometric series sum formula, is $\boxed{\textbf{(C) } \frac{1}{3}}$.

Solution 5 (quick, conceptual)

Define a win as a ball appearing in higher numbered box.

Start from the first box.

There are 4 possible results in the box: Red, Green, Red and Green, or none, with an equal probability of $\frac{1}{4}$ for each. If none of the balls is in the first box, the game restarts at the second box with the same kind of probability distribution, so if p is the probability that Red wins, we can write $p=\frac{1}{4}+\frac{1}{4}p$: there is a $\frac{1}{4}$ probability that "Red" wins immediately, a 0 probability in the cases "Green" or "Red and Green", and in the "None" case (which occurs with $\frac{1}{4}$ probability), we then start again, giving the same probability p. Hence, solving the equation, we get $p=\boxed{\textbf{(C) } \frac{1}{3}}$.

Solution 6

Write out the infinite geometric series as $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$. To find the probablilty that red goes in a higher-numbered bin than green, we can simply remove all odd-index terms (i.e term 1, term 3, etc.), and then sum the remaining terms - this is in fact precisely equivalent to the method of Solution 2. Writing this out as another infinite geometric sequence, we are left with $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \cdots$. Summing, we get

$$\sum_{i=1}^{\infty} \frac{1}{4^i} = \boxed{\textbf{(C)}\ \frac{1}{3}}$$

Solution 7

This immediately seems like a geometric series problem, so fixing the green ball to fall into bin 1 gives a probability of $\frac{1}{2}(\frac{1}{2^2} + \frac{1}{2^3} + ...)$ for the red ball to fall into a higher bin. Fixing the green ball to fall into bin 2 gives a probability of $\frac{1}{2^2}(\frac{1}{2^3} + \frac{1}{2^4} + ...)$. Factoring out the denominator of the first fraction in each probability gives $\frac{1}{2^3}(1 + \frac{1}{2} + \frac{1}{2^2} + ..) + \frac{1}{2^5}(1 + \frac{1}{2} + \frac{1}{2^2} + ...) + ...$ so factoring out $(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...)$ results in the probability simplifying to $(\frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + ...)(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...)$ and using the formula $\frac{a}{1-r}$ to find both series, we obtain $(\frac{\frac{1}{2^3}}{1 - \frac{1}{4}})(\frac{1}{1 - \frac{1}{2}})$ which simplifies to $\boxed{\textbf{(C)}\ \frac{1}{3}}$ -- OGBooger

Video Solution

For those who want a video solution: https://youtu.be/VP7ltu-XEq8

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 18

Contents

Problem

Henry decides one morning to do a workout, and he walks $\frac{3}{4}$ of the way from his home to his gym. The gym is 2 kilometers away from Henry's home. At that point, he changes his mind and walks $\frac{3}{4}$ of the way from where he is back toward home. When he reaches that point, he changes his mind again and walks $\frac{3}{4}$ of the distance from there back toward the gym. If Henry keeps changing his mind when he has walked $\frac{3}{4}$ of the distance toward either the gym or home from the point where he last changed his mind, he will get very close to walking back and forth between a point A kilometers from home and a point B kilometers from home. What is $|A-B|$?

$\textbf{(A) } \frac{2}{3} \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 1\frac{1}{5} \qquad \textbf{(D) } 1\frac{1}{4} \qquad \textbf{(E) } 1\frac{1}{2}$

Solution 1

Let the two points that Henry walks in between be P and Q, with P being closer to home. As given in the problem statement, the distances of the points P and Q from his home are A and B respectively. By symmetry, the distance of point Q from the gym is the same as the distance from home to point P. Thus, $A = 2 - B$. In addition, when he walks from point Q to home, he walks $\dfrac{3}{4}$ of the distance, ending at point P. Therefore, we know that $B - A = \dfrac{3}{4}B$. By substituting, we get

$B - A = \frac{3}{4}(2 - A)$. Adding these equations now gives $2(B - A) = \frac{3}{4}(2 + B - A)$. Multiplying by 4, we get $8(B - A) = 6 + 3(B - A)$, so $B - A = \frac{6}{5} = \boxed{\textbf{(C) } 1\frac{1}{5}}$.

Solution 2 (not rigorous)

We assume that Henry is walking back and forth exactly between points P and Q, with P closer to Henry's home than Q. Denote Henry's home as a point H and the gym as a point G. Then $HP : PQ = 1 : 3$ and $PQ : QG = 3 : 1$, so $HP : PQ : QG = 1 : 3 : 1$. Therefore,

$$|A - B| = PQ = \frac{3}{1 + 3 + 1} \cdot 2 = \frac{6}{5} = \boxed{\textbf{(C) } 1\frac{1}{5}}.$$

Solution 3 (not rigorous; similar to 2)

Since Harry is very close to walking back and forth between two points, let us denote A closer to his house, and B closer to the gym. Then, let us denote the distance from A to B as x. If Harry was at B and walked $\frac{3}{4}$ of the way, he would end up at A, vice versa. Thus we can say that the distance from A to the gym is $\frac{1}{4}$ the distance from B to his house. That means it is $\frac{1}{3}x$. This also applies to the other side. Furthermore, we can say $\frac{1}{3}x$ + x + $\frac{1}{3}x$ = 2. We solve for x and get $x = \frac{6}{5}$. Therefore, the answer is $\boxed{\textbf{(C) } 1\frac{1}{5}}$.

~aryam

Video Solution

For those who want a video solution: https://youtu.be/45kdBy3htOg

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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2019 AMC 10B Problems/Problem 19

The following problem is from both the 2019 AMC 10B #19 and 2019 AMC 12B #14, so both problems redirect to this page.

Problem

Let S be the set of all positive integer divisors of $100,000.$ How many numbers are the product of two distinct elements of $S?$

$\textbf{(A) }98\qquad\textbf{(B) }100\qquad\textbf{(C) }117\qquad\textbf{(D) }119\qquad\textbf{(E) }121$

Solution

The prime factorization of $100,000$ is $2^5 \cdot 5^5$. Thus, we choose two numbers 2^a5^b and 2^c5^d where $0 \le a,b,c,d \le 5$ and $(a,b) \neq (c,d)$, whose product is $2^{a+c}5^{b+d}$, where $0 \le a+c \le 10$ and $0 \le b+d \le 10$.

Notice that this is analogous to choosing a divisor of $100,000^2 = 2^{10}5^{10}$, which has $(10+1)(10+1) = 121$ divisors. However, some of the divisors of $2^{10}5^{10}$ cannot be written as a product of two distinct divisors of $2^5 \cdot 5^5$, namely: $1 = 2^05^0$, $2^{10}5^{10}$, 2^{10}, and 5^{10}. The last two cannot be so written because the maximum factor of $100,000$ containing only 2s or 5s (and not both) is only 2^5 or 5^5. Since the factors chosen must be distinct, the last two numbers cannot be so written because they would require $2^5 \cdot 2^5$ or $5^5 \cdot 5^5$. The first two would require $1 \cdot 1$ and $2^55^5 \cdot 2^55^5$, respectively. This gives $121-4 = 117$ candidate numbers. It is not too hard to show that every number of the form 2^p5^q, where $0 \le p, q \le 10$, and p,q are not both 0 or 10, can be written as a product of two distinct elements in S. Hence the answer is $\boxed{\textbf{(C) }117}$.

See Also

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2019 AMC 10B Problems/Problem 20

The following problem is from both the 2019 AMC 10B #20 and 2019 AMC 12B #15, so both problems redirect to this page.

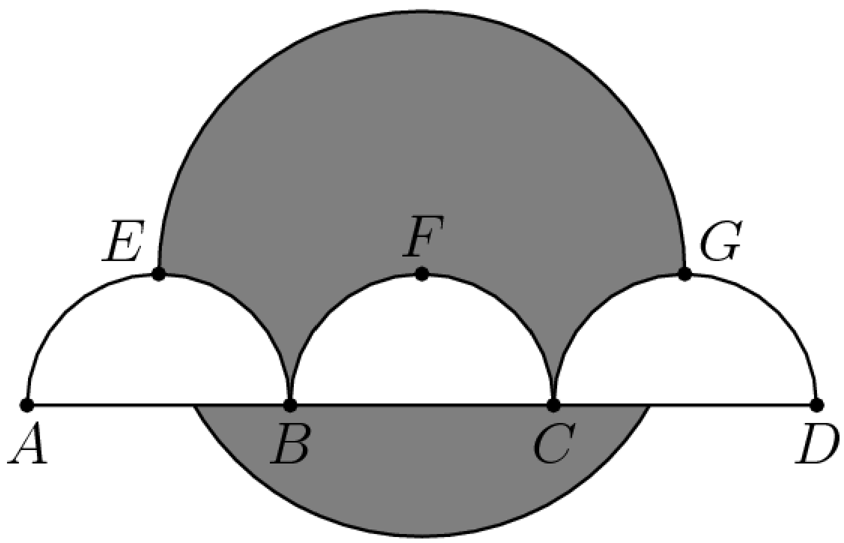
Contents

Problem

As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2.$ Three semicircles of radius $1,$ $\overarc{\rm AEB}, \overarc{\rm BFC},$ and $\overarc{\rm CGD},$ have their diameters on $\overline{AD},$ and are tangent to line EG at $E, F,$ and $G,$ respectively. A circle of radius 2 has its center on $F.$ The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form

$$\frac{a}{b} \cdot \pi - \sqrt{c} + d,$$

where $a, b, c,$ and d are positive integers and a and b are relatively prime. What is $a+b+c+d$?



$\textbf{(A) } 13 \qquad \textbf{(B) } 14 \qquad \textbf{(C) } 15 \qquad \textbf{(D) } 16 \qquad \textbf{(E) } 17$

Solution

Divide the circle into four parts: the top semicircle by connecting E, F, and G(A); the bottom sector (B), whose arc angle is 120° because the large circle's radius is 2 and the short length (the radius of the smaller semicircles) is 1, giving a $30^\circ - 60^\circ - 90^\circ$ triangle; the triangle formed by the radii of A and the chord (C); and the four parts which are the corners of a circle inscribed in a square (D). Then the area is $A+B-C+D$ (in $B-C$, we find the area of the bottom shaded region, and in D we find the area of the shaded region above the semicircles but below the diameter).

The area of A is $\frac{1}{2}\pi \cdot 2^2 = 2\pi$.

The area of B is $\frac{120^\circ}{360^\circ}\pi \cdot 2^2 = \frac{4\pi}{3}$.

For the area of C, the radius of 2, and the distance of 1 (the smaller semicircles' radius) to BC, creates two $30^\circ - 60^\circ - 90^\circ$ triangles, so C's area is $2 \cdot \frac{1}{2} \cdot 1 \cdot \sqrt{3} = \sqrt{3}$.

The area of D is $4 \cdot 1 - \frac{1}{4}\pi \cdot 2^2 = 4 - \pi$.

Hence, finding $A+B-C+D$, the desired area is $\frac{7\pi}{3} - \sqrt{3} + 4$, so the answer is $7+3+3+4=\boxed{\textbf{(E) } 17}$.

Video Solution

Video Solution from Youtube- https://www.youtube.com/watch?v=ZbWOZMfXtL8

See Also

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2019 AMC 10B Problems/Problem 21

Contents

Problem

Debra flips a fair coin repeatedly, keeping track of how many heads and how many tails she has seen in total, until she gets either two heads in a row or two tails in a row, at which point she stops flipping. What is the probability that she gets two heads in a row but she sees a second tail before she sees a second head?

$$\textbf{(A) } \frac{1}{36} \qquad \textbf{(B) } \frac{1}{24} \qquad \textbf{(C) } \frac{1}{18} \qquad \textbf{(D) } \frac{1}{12} \qquad \textbf{(E) } \frac{1}{6}$$

Solution 1

We first want to find out which sequences of coin flips satisfy the given condition. For Debra to see the second tail before the second head, her first flip can't be heads, as that would mean she would either end with double tails before seeing the second head, or would see two heads before she sees two tails. Therefore, her first flip must be tails. The shortest sequence of flips by which she can get two heads in a row and see the second tail before she sees the second head is $THTHH$, which has a probability of $\frac{1}{2^5} = \frac{1}{32}$. Furthermore, she can prolong her coin flipping by adding an extra TH, which itself has a probability of $\frac{1}{2^2} = \frac{1}{4}$. Since she can do this indefinitely, this gives an infinite geometric series, which means the answer (by the geometric series sum formula) is $\frac{\frac{1}{32}}{1-\frac{1}{4}} = \boxed{\textbf{(B) } \frac{1}{24}}$.

Solution 2 (Easier)

Note that the sequence must start in THT, which happens with $\frac{1}{8}$ probability. Now, let P be the probability that Debra will get two heads in a row after flipping THT. Either Debra flips two heads in a row immediately (probability $\frac{1}{4}$), or flips a head and then a tail and reverts back to the "original position" (probability $\frac{1}{4}P$). Therefore, $P = \frac{1}{4} + \frac{1}{4}P$, so $P = \frac{1}{3}$, so our final answer is $\frac{1}{8} \times \frac{1}{3} = \boxed{\textbf{(B) } \frac{1}{24}}$. - Stormersyle

Video Solution

https://www.youtube.com/watch?v=2f1zEvfUe9o

See Also

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2019 AMC 10B Problems/Problem 22

The following problem is from both the 2019 AMC 10B #22 and 2019 AMC 12B #19, so both problems redirect to this page.

Contents

Problem

Raashan, Sylvia, and Ted play the following game. Each starts with $\$1$. A bell rings every 15 seconds, at which time each of the players who currently have money simultaneously chooses one of the other two players independently and at random and gives $\$1$ to that player. What is the probability that after the bell has rung 2019 times, each player will have $\$1$? (For example, Raashan and Ted may each decide to give $\$1$ to Sylvia, and Sylvia may decide to give her her dollar to Ted, at which point Raashan will have $\$0$, Sylvia will have $\$2$, and Ted will have $\$1$, and that is the end of the first round of play. In the second round Rashaan has no money to give, but Sylvia and Ted might choose each other to give their $\$1$ to, and the holdings will be the same at the end of the second round.)

$\textbf{(A) } \frac{1}{7} \qquad\textbf{(B) } \frac{1}{4} \qquad\textbf{(C) } \frac{1}{3} \qquad\textbf{(D) } \frac{1}{2} \qquad\textbf{(E) } \frac{2}{3}$

Solution 1

On the first turn, each player starts off with $\$1$. Each turn after that, there are only two possibilities: either everyone stays at $\$1$, which we will write as $(1-1-1)$, or the distribution of money becomes $\$2-\$1-\$0$ in some order, which we write as $(2-1-0)$. We will consider these two states separately.

In the $(1-1-1)$ state, each person has two choices for whom to give their dollar to, meaning there are $2^3=8$ possible ways that the money can be rearranged. Note that there are only two ways that we can reach $(1-1-1)$ again:

1. Raashan gives his money to Sylvia, who gives her money to Ted, who gives his money to Raashan.

2. Raashan gives his money to Ted, who gives his money to Sylvia, who gives her money to Raashan.

Thus, the probability of staying in the $(1-1-1)$ state is $\frac{1}{4}$, while the probability of going to the $(2-1-0)$ state is $\frac{3}{4}$ (we can check that the 6 other possibilities lead to $(2-1-0)$)

In the $(2-1-0)$ state, we will label the person with $\$2$ as person A, the person with $\$1$ as person B, and the person with $\$0$ as person C. Person A has two options for whom to give money to, and person B has 2 options for whom to give money to, meaning there are total $2 \cdot 2 = 4$ ways the money can be redistributed. The only way that the distribution can return to $(1-1-1)$ is if A gives $\$1$ to B, and B gives $\$1$ to C. We check the other possibilities to find that they all lead back to $(2-1-0)$. Thus, the probability of going to the $(1-1-1)$ state is $\frac{1}{4}$, while the probability of staying in the $(2-1-0)$ state is $\frac{3}{4}$.

No matter which state we are in, the probability of going to the $(1-1-1)$ state is always $\frac{1}{4}$. This means that, after the bell rings 2018 times, regardless of what state the money distribution is in, there is a $\frac{1}{4}$ probability of going to the $(1-1-1)$ state after the 2019th bell ring. Thus, our answer is simply $\boxed{\textbf{(B) } \frac{1}{4}}$.

Solution 2 (Symmetry)

After the first ring, either nothing changes, or someone has $\$2$. No one can have $\$3$, since in that hypothetical round, that person would have to give away $\$1$.

Thus, the outcome is either $1-1-1$ or six symmetrical cases where one person gets $\$2$ (e.g. a $1-2-0$ or $2-1-0$ split). There are two ways for the three people to exchange dollars to get to the same $1-1-1$ result. As such, there are 8 overall possibilities (which make sense, since each person has 2 choices when giving away his or her dollar, resulting in 2^3 total possibilities). As such, from the $1-1-1$ case, there is a $1/4$ chance of returning to $1-1-1$.

Without loss of generality, take the $1-2-0$ case. Only 2 people can give money, so there are now 2^2 possible outcomes after the bell rings. It either decomposes back into $1-1-1$, remains unchanged, turns into $0-1-2$, or turns into $0-2-1$. As such, from the $1-1-1$ case, there is a $1/4$ chance of returning to $1-1-1$. Notice that this works for any of the 6 cases.

Since the starting state has a $1/4$ chance of remaining unchanged, and each of the different 6 symmetric states all also have a $1/4$ chance of reverting back to $1-1-1$, the chance of it being 1-1-1 after any state is always $\boxed{\textbf{(B) } \frac{1}{4}}$

Video Solution

https://youtu.be/XT440PjAFmQ

See Also

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2019 AMC 10B Problems/Problem 23

The following problem is from both the 2019 AMC 10B #23 and 2019 AMC 12B #20, so both problems redirect to this page.

Contents

Problem

Points $A(6, 13)$ and $B(12, 11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω ?

$\textbf{(A) }\frac{83\pi}{8}\qquad\textbf{(B) }\frac{21\pi}{2}\qquad\textbf{(C) }\frac{85\pi}{8}\qquad\textbf{(D) }\frac{43\pi}{4}\qquad\textbf{(E) }\frac{87\pi}{8}$

Solution 1

First, observe that the two tangent lines are of identical length. Therefore, supposing that the point of intersection is $(x, 0)$, the Pythagorean Theorem gives $\sqrt{(x-6)^2 + 13^2} = \sqrt{(x-12)^2 + 11^2}$. This simplifies to $x = 5$.

Further, notice (due to the right angles formed by a radius and its tangent line) that the quadrilateral (a kite) defined by the circle's center, A, B, and $(5, 0)$ is cyclic. Therefore, we can apply Ptolemy's Theorem to give $2\sqrt{170}x = d\sqrt{40}$, where x is the radius of the circle and d is the distance between the circle's center and $(5, 0)$.

Therefore, $d = \sqrt{17}x$. Using the Pythagorean Theorem on the triangle formed by the point $(5, 0)$, either one of A or B, and the circle's center, we find that $170 + x^2 = 17x^2$, so $x^2 = \frac{85}{8}$, and thus the answer is $\boxed{\textbf{(C) } \frac{85}{8}\pi}$.

Solution 2 (coordinate bash)

We firstly obtain $x = 5$ as in Solution 1. Label the point $(5, 0)$ as C. The midpoint M of segment AB is $(9, 12)$. Notice that the center of the circle must lie on the line passing through the points C and M. Thus, the center of the circle lies on the line $y = 3x - 15$.

Line AC is $y = 13x - 65$. Therefore, the slope of the line perpendicular to AC is $-\frac{1}{13}$, so its equation is $y = -\frac{x}{13} + \frac{175}{13}$.

But notice that this line must pass through $A(6, 13)$ and $(x, 3x - 15)$. Hence $3x - 15 = -\frac{x}{13} + \frac{175}{13} \Rightarrow x = \frac{37}{4}$. So the center of the circle is $\left(\frac{37}{4}, \frac{51}{4}\right)$.

Finally, the distance between the center, $\left(\frac{37}{4}, \frac{51}{4}\right)$, and point A is $\frac{\sqrt{170}}{4}$. Thus the area of the circle is $\boxed{\textbf{(C) } \frac{85}{8}\pi}$.

Solution 3

The midpoint of AB is $D(9, 12)$. Let the tangent lines at A and B intersect at $C(a, 0)$ on the x-axis. Then CD is the perpendicular bisector of AB. Let the center of the circle be O. Then $\triangle AOC$ is similar to $\triangle DAC$, so $\frac{OA}{AC} = \frac{AD}{DC}$.

The slope of AB is $\frac{13 - 11}{6 - 12} = \frac{-1}{3}$, so the slope of CD is 3. Hence, the equation of CD is $y - 12 = 3(x - 9) \Rightarrow y = 3x - 15$. Letting $y = 0$, we have $x = 5$, so $C = (5, 0)$.

Now, we compute $AC = \sqrt{(6-5)^2 + (13-0)^2} = \sqrt{170}$, $AD = \sqrt{(6-9)^2 + (13-12)^2} = \sqrt{10}$, and $DC = \sqrt{(9-5)^2 + (12-0)^2} = \sqrt{160}$.

Therefore $OA = \frac{AC \cdot AD}{DC} = \sqrt{\frac{85}{8}}$, and consequently, the area of the circle is $\pi \cdot OA^2 = \boxed{\textbf{(C) } \frac{85}{8}\pi}$.

Solution 4 (how fast can you multiply two-digit numbers?)

Let $(x, 0)$ be the intersection on the x-axis. By Power of a Point Theorem, $(x-6)^2 + 13^2 = (x-12)^2 + 11^2 \implies x = 5$. Then the equations are $13(x-6)+13 = y$ and $\frac{11}{7}(x-12)+11 = y$ for the tangent lines passing A and B respectively. Then the lines normal to them are $-\frac{1}{13}(x-6)+13 = y$ and $-\frac{7}{11}(x-12)+11 = y$. Thus,

$$-\frac{7}{11}(x-12)+11 = -\frac{1}{13}(x-6)+13$$

$$\frac{13 \cdot 7x - 11x}{13 \cdot 11} = \frac{84 \cdot 13 - 6 \cdot 11 - 2 \cdot 11 \cdot 13}{11 \cdot 13}$$

$$13 \cdot 7x - 11x = 84 \cdot 13 - 6 \cdot 11 - 2 \cdot 11 \cdot 13$$

After condensing, $x = \frac{37}{4}$. Then, the center of ω is $\left(\frac{37}{4}, \frac{51}{4}\right)$. Apply distance formula. WLOG, assume you use A. Then, the area of ω is

$$\sqrt{\frac{1^2}{4^2}+\frac{13^2}{4^2}}^2 \pi = \frac{170\pi}{16} \implies \boxed{\textbf{(C) } \frac{85}{8}\pi}$$

. ~ minor LaTeX edits by dolphin7

Solution 5 (power of a point)

Firstly, the point of intersection of the two tangent lines has an equal distance to points A and B due to power of a point theorem. This means we can easily find the point, which is $(5,0)$. Label this point X. $\triangle XAB$ is an isosceles triangle with lengths, $\sqrt{170}$, $\sqrt{170}$, and $2\sqrt{10}$. Label the midpoint of segment AB as M. The height of this triangle, or $\overline{XM}$, is $4\sqrt{10}$. Since $\overline{XM}$ bisects $\overline{AB}$, $\overleftrightarrow{XM}$ contains the diameter of circle ω. Let the two points on circle ω where $\overleftrightarrow{XM}$ intersects be P and Q with $\overline{XP}$ being the shorter of the two. Now let $\overline{MP}$ be x and $\overline{MQ}$ be y. By Power of a Point on $\overline{PQ}$ and $\overline{AB}$, $xy = (\sqrt{10})^2 = 10$. Applying Power of a Point again on $\overline{XQ}$ and $\overline{XA}$, $(4\sqrt{10}-x)(4\sqrt{10}+y) = (\sqrt{170})^2 = 170$. Expanding while using the fact that $xy = 10$, $y = x + \frac{\sqrt{10}}{2}$. Plugging this into ,

. Using the quadratic formula, ,

and since , . Since this is the diameter, the radius of circle ω is $\frac{\sqrt{170}}{4}$, and so the area of circle ω is

.

See Also

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2019 AMC 10B Problems/Problem 24

The following problem is from both the 2019 AMC 10B #24 and 2019 AMC 12B #22, so both problems redirect to this page.

Contents

Problem

Define a sequence recursively by $x_0 = 5$ and

$$x_{n+1} = \frac{x_n^2 + 5x_n + 4}{x_n + 6}$$

for all nonnegative integers n. Let m be the least positive integer such that

$$x_m \leq 4 + \frac{1}{2^{20}}.$$

In which of the following intervals does m lie?

$\textbf{(A) } [9,26] \qquad \textbf{(B) } [27,80] \qquad \textbf{(C) } [81,242] \qquad \textbf{(D) } [243,728] \qquad \textbf{(E) } [729,\infty)$

Solution 1

We first prove that $x_n > 4$ for all $n \geq 0$, by induction. Observe that

$$x_{n+1} - 4 = \frac{x_n^2 + 5x_n + 4 - 4(x_n + 6)}{x_n + 6} = \frac{(x_n - 4)(x_n + 5)}{x_n + 6}$$

so (since x_n is clearly positive for all n, from the initial definition), $x_{n+1} > 4$ if and only if $x_n > 4$.

We similarly prove that x_n is decreasing, since

$$x_{n+1} - x_n = \frac{x_n^2 + 5x_n + 4 - x_n(x_n + 6)}{x_n + 6} = \frac{4 - x_n}{x_n + 6} < 0$$

Now we need to estimate the value of $x_{n+1} - 4$, which we can do using the rearranged equation

$$x_{n+1} - 4 = (x_n - 4) \cdot \frac{x_n + 5}{x_n + 6}$$

Since x_n is decreasing, $\frac{x_n + 5}{x_n + 6}$ is clearly also decreasing, so we have

$$\frac{9}{10} < \frac{x_n+5}{x_n+6} \le \frac{10}{11}$$

and

$$\frac{9}{10}(x_n-4) < x_{n+1}-4 \le \frac{10}{11}(x_n-4)$$

This becomes

$$\left(\frac{9}{10}\right)^n = \left(\frac{9}{10}\right)^n (x_0-4) < x_n - 4 \le \left(\frac{10}{11}\right)^n (x_0-4) = \left(\frac{10}{11}\right)^n$$

The problem thus reduces to finding the least value of n such that

$$\left(\frac{9}{10}\right)^n < x_n - 4 \le \frac{1}{2^{20}} \text{ and } \left(\frac{10}{11}\right)^{n-1} > x_{n-1} - 4 > \frac{1}{2^{20}}$$

Taking logarithms, we get $n \ln \frac{9}{10} < -20 \ln 2$ and $(n-1) \ln \frac{10}{11} > -20 \ln 2$, i.e.

$$n > \frac{20 \ln 2}{\ln \frac{10}{9}} \text{ and } n - 1 < \frac{20 \ln 2}{\ln \frac{11}{10}}$$

As approximations, we can use $\ln \frac{10}{9} \approx \frac{1}{9}$, $\ln \frac{11}{10} \approx \frac{1}{10}$, and $\ln 2 \approx 0.7$. These allow us to estimate that

$$126 < n < 141$$

which gives the answer as $\boxed{\textbf{(C) } [81, 242]}$.

Solution 2

The condition where $x_m \le 4 + \frac{1}{2^{20}}$ gives the motivation to make a substitution to change the equilibrium from 4 to 0. We can substitute $x_n = y_n + 4$ to achieve that.

Now, we need to find the smallest value of m such that $y_m \le \frac{1}{2^{20}}$ given that $y_0 = 1$ and the recursion $y_{n+1} = \frac{y_n^2 + 9y_n}{y_n + 10}$.

Using wishful thinking, we can simplify the recursion as follows:

$$y_{n+1} = \frac{y_n^2 + 9y_n + y_n - y_n}{y_n + 10}$$

$$y_{n+1} = \frac{y_n(y_n + 10) - y_n}{y_n + 10}$$

$$y_{n+1} = y_n - \frac{y_n}{y_n + 10}$$

$$y_{n+1} = y_n \left(1 - \frac{1}{y_n + 10}\right)$$

The recursion looks like a geometric sequence with the ratio changing slightly after each term. Notice from the recursion that the y_n sequence is strictly decreasing, so all the terms after y_0 will be less than 1. Also, notice that all the terms in sequence will be positive. Both of these can be proven by induction.

With both of those observations in mind, $\frac{9}{10} < 1 - \frac{1}{y_n + 10} \leq \frac{10}{11}$. Combining this with the fact that the recursion resembles a geometric sequence, we conclude that $\left(\frac{9}{10}\right)^n < y_n \leq \left(\frac{10}{11}\right)^n$.

$\frac{9}{10}$ is approximately equal to $\frac{10}{11}$ and the ranges that the answer choices give us are generous, so we should use either $\frac{9}{10}$ or $\frac{10}{11}$ to find a rough estimate for m. $\left(\frac{9}{10}\right)^3$ is 0.729, while $\frac{1}{\sqrt{2}}$ is close to 0.7 because $(0.7)^2$ is 0.49, which is close to $\frac{1}{2}$.

Therefore, we can estimate that $2^{\frac{-1}{2}} < y_3$.

Putting both sides to the 40th power, we get $2^{-20} < (y_3)^{40}$

But $y_3 = (y_0)^3$, so $2^{-20} < (y_0)^{120}$ and therefore, $2^{-20} < y_{120}$.

This tells us that m is somewhere around 120, so our answer is $\boxed{\textbf{(C) } [81, 242]}$.

Solution 3

Since the choices are rather wide ranges, we can use approximation to make it easier. Notice that

$$x_{n+1} - x_n = \frac{4 - x_n}{x_n + 6}$$

And $x_0 = 5$, we know that x_n is a declining sequence, and as it get close to 4 its decline will slow, never falling below 4. So we'll use 4 to approximate x_n in the denominator so that we have a solvable difference equation:

$$x_{n+1} - x_n = \frac{4 - x_n}{10}$$

$$x_{n+1} = \frac{9}{10}x_n + \frac{2}{5}$$

Solve it with $x_0 = 5$, we have

$$x_n = 4 + (\frac{9}{10})^n$$

Now we wish to find n so that

$$(\frac{9}{10})^n \approx \frac{1}{2^{20}}$$

$$n \approx \frac{\log 2^{20}}{\log 10 - \log 9} \approx \frac{20 * 0.3}{0.05} = 120$$

Since 120 is safely within the range of [81,242], we have the answer. $\boxed{\textbf{(C) } [81, 242]}$.

-Mathdummy

Video Solution

Video Solution: https://www.youtube.com/watch?v=0k7bYOdiF6M

See Also

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2019 AMC 12B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolvin g.com/Forum/resources.php?c=182&cid=44&year=2019))	
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2019 AMC 10B Problems/Problem 25

The following problem is from both the 2019 AMC 10B #25 and 2019 AMC 12B #23, so both problems redirect to this page.

Contents

Problem

How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

$\textbf{(A) }55\qquad\textbf{(B) }60\qquad\textbf{(C) }65\qquad\textbf{(D) }70\qquad\textbf{(E) }75$

Solution 1 (recursion)

We can deduce, from the given restrictions, that any valid sequence of length n will start with a 0 followed by either 10 or 110. Thus we can define a recursive function $f(n) = f(n-3) + f(n-2)$, where $f(n)$ is the number of valid sequences of length n.

This is because for any valid sequence of length n, you can append either 10 or 110 and the resulting sequence will still satisfy the given conditions.

It is easy to find $f(5) = 1$ with the only possible sequence being 01010 and $f(6) = 2$ with the only two possible sequences being 011010 and 010110 by hand, and then by the recursive formula, we have $f(19) = \boxed{\textbf{(C) }65}$.

Solution 2 (casework)

After any particular 0, the next 0 in the sequence must appear exactly 2 or 3 positions down the line. In this case, we start at position 1 and end at position 19, i.e. we move a total of 18 positions down the line. Therefore, we must add a series of 2s and 3s to get 18. There are a number of ways to do this:

Case 1: nine 2s - there is only 1 way to arrange them.

Case 2: two 3s and six 2s - there are $\binom{8}{2} = 28$ ways to arrange them.

Case 3: four 3s and three 2s - there are $\binom{7}{4} = 35$ ways to arrange them.

Case 4: six 3s - there is only 1 way to arrange them.

Summing the four cases gives $1 + 28 + 35 + 1 = \boxed{\textbf{(C)}\ 65}$.

Video Solution

For those who want a video solution: https://youtu.be/VamT49PjmdI

See Also

2019 AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2019))	
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