2011 AMC 12B Problems/Problem 1

Problem

What is $\frac{2+4+6}{1+3+5} - \frac{1+3+5}{2+4+6}?$

$\textbf{(A)}\ -1 \qquad \textbf{(B)}\ \frac{5}{36} \qquad \textbf{(C)}\ \frac{7}{12} \qquad \textbf{(D)}\ \frac{147}{60} \qquad \textbf{(E)}\ \frac{43}{3}$

Solution

Add up the numbers in each fraction to get $\frac{12}{9} - \frac{9}{12}$, which equals $\frac{4}{3} - \frac{3}{4}$. Doing the subtraction yields $\boxed{\frac{7}{12}\ \textbf{(C)}}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by First Problem	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 2

Problem

Josanna's test scores to date are $90, 80, 70, 60,$ and $85.$ Her goal is to raise her test average at least 3 points with her next test. What is the minimum test score she would need to accomplish this goal?

$\textbf{(A)}\ 80 \qquad\textbf{(B)}\ 82 \qquad\textbf{(C)}\ 85 \qquad\textbf{(D)}\ 90 \qquad\textbf{(E)}\ 95$

Solution

Take the average of her current test scores, which is

$$\frac{90+80+70+60+85}{5} = \frac{385}{5} = 77$$

This means that she wants her test average after the sixth test to be $80.$ Let x be the score that Josanna receives on her sixth test. Thus, our equation is

$$\frac{90+80+70+60+85+x}{6} = 80$$

$$385 + x = 480$$

$$x = \boxed{95 \textbf{(E)}}$$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2011 AMC 12B Problems/Problem 3

Problem

LeRoy and Bernardo went on a week-long trip together and agreed to share the costs equally. Over the week, each of them paid for various joint expenses such as gasoline and car rental. At the end of the trip it turned out that LeRoy had paid A dollars and Bernardo had paid B dollars, where $A < B.$ How many dollars must LeRoy give to Bernardo so that they share the costs equally?

$\textbf{(A)}\ \frac{A+B}{2} \qquad \textbf{(B)}\ \frac{A-B}{2} \qquad \textbf{(C)}\ \frac{B-A}{2} \qquad \textbf{(D)}\ B-A \qquad \textbf{(E)}\ A+B$

Solution

The total amount of money that was spent during the trip was

$$A+B$$

So each person should pay

$$\frac{A+B}{2}$$

if they were to share the costs equally. Because LeRoy has already paid A dollars of his part, he still has to pay

$$\frac{A+B}{2} - A =$$

$$= \boxed{\frac{B-A}{2} \textbf{(C)}}$$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 4

Problem

In multiplying two positive integers a and b, Ron reversed the digits of the two-digit number a. His erroneous product was $161.$ What is the correct value of the product of a and b?

$\textbf{(A)}\ 116 \qquad \textbf{(B)}\ 161 \qquad \textbf{(C)}\ 204 \qquad \textbf{(D)}\ 214 \qquad \textbf{(E)}\ 224$

Solution

Taking the prime factorization of 161 reveals that it is equal to $23*7.$ Therefore, the only ways to represent 161 as a product of two positive integers is $161*1$ and $23*7.$ Because neither 161 nor 1 is a two-digit number, we know that a and b are 23 and $7.$ Because 23 is a two-digit number, we know that a, with its two digits reversed, gives $23.$ Therefore, $a=32$ and $b=7.$ Multiplying our two correct values of a and b yields

$$a*b=32*7=$$

$$=\boxed{224 \textbf{(E)}}$$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 5

Problem

Let N be the second smallest positive integer that is divisible by every positive integer less than 7. What is the sum of the digits of N?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 9$

Solution

N must be divisible by every positive integer less than 7, or $1, 2, 3, 4, 5,$ and 6. Each number that is divisible by each of these is is a multiple of their least common multiple. $LCM(1, 2, 3, 4, 5, 6) = 60$, so each number divisible by these is a multiple of 60. The smallest multiple of 60 is clearly 60, so the second smallest multiple of 60 is $2 \times 60 = 120$. Therefore, the sum of the digits of N is $1 + 2 + 0 = \boxed{3\ \textbf{(A)}}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 6

Problem

Two tangents to a circle are drawn from a point A. The points of contact B and C divide the circle into arcs with lengths in the ratio $2:3$. What is the degree measure of $\angle BAC$?

$\textbf{(A)}\ 24 \qquad \textbf{(B)}\ 30 \qquad \textbf{(C)}\ 36 \qquad \textbf{(D)}\ 48 \qquad \textbf{(E)}\ 60$

Solution

In order to solve this problem, use of the tangent-tangent intersection theorem (Angle of intersection between two tangents dividing a circle into arc length A and arc length B = 1/2 (Arc A° - Arc B°).

In order to utilize this theorem, the degree measures of the arcs must be found. First, set A (Arc length A) equal to 3d, and B (Arc length B) equal to 2d.

Setting 3d+2d = 360° will find d = 72° , and so therefore Arc length A in degrees will equal 216° and arc length B will equal 144° .

Finally, simply plug the two arc lengths into the tangent-tangent intersection theorem, and the answer:

1/2 (216° -144°) = 1/2 (72°) $= \boxed{36 \textbf{(C)}}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 7

Problem

Let x and y be two-digit positive integers with mean 60. What is the maximum value of the ratio $\frac{x}{y}$?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ \frac{33}{7} \qquad \textbf{(C)}\ \frac{39}{7} \qquad \textbf{(D)}\ 9 \qquad \textbf{(E)}\ \frac{99}{10}$

Solution

If x and y have a mean of 60, then $\frac{x+y}{2}=60$ and $x+y=120$. To maximize $\frac{x}{y}$, we need to maximize x and minimize y. Since they are both two-digit positive integers, the maximum of x is 99 which gives $y=21$. y cannot be decreased because doing so would increase x, so this gives the maximum value of $\frac{x}{y}$, which is $\frac{99}{21}=\boxed{\frac{33}{7}\ \textbf{(B)}}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 8

Problem

Keiko walks once around a track at exactly the same constant speed every day. The sides of the track are straight, and the ends are semicircles. The track has width 6 meters, and it takes her 36 seconds longer to walk around the outside edge of the track than around the inside edge. What is Keiko's speed in meters per second?

$\textbf{(A)}\ \frac{\pi}{3} \qquad \textbf{(B)}\ \frac{2\pi}{3} \qquad \textbf{(C)}\ \pi \qquad \textbf{(D)}\ \frac{4\pi}{3} \qquad \textbf{(E)}\ \frac{5\pi}{3}$

Solution

To find Keiko's speed, all we need to find is the difference between the distance around the inside edge of the track and the distance around the outside edge of the track, and divide it by the difference in time it takes her for each distance. We are given the difference in time, so all we need to find is the difference between the distances.

The track is divided into lengths and curves. The lengths of the track will exhibit no difference in distance between the inside and outside edges, so we only need to concern ourselves with the curves.

The curves of the track are semicircles, but since there are two of them, we can consider both of the at the same time by treating them as a single circle. We need to find the difference in the circumferences of the inside and outside edges of the circle.

The formula for the circumference of a circle is $C = 2 * \pi * r$ where r is the radius of the circle.

Let's define the circumference of the inside circle as C_1 and the circumference of the outside circle as C_2.

If the radius of the inside circle (r_1) is n, then given the thickness of the track is 6 meters, the radius of the outside circle (r_2) is $n + 6$.

Using this, the difference in the circumferences is:

$$C_2 - C_1 = 2 * \pi * (r_2 - r_1) = 2 * \pi * (n + 6 - n) = 2 * \pi * 6 = 12\pi$$

12π is the difference between the inside and outside lengths of the track. Divided by the time differential, we get:

$$12\pi \div 36 = \boxed{\textbf{(A)}\ \frac{\pi}{3}}$$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 7	Followed by Problem 9
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 9

Problem

Two real numbers are selected independently and at random from the interval $[-20, 10]$. What is the probability that the product of those numbers is greater than zero?

$\textbf{(A)}\ \frac{1}{9} \qquad \textbf{(B)}\ \frac{1}{3} \qquad \textbf{(C)}\ \frac{4}{9} \qquad \textbf{(D)}\ \frac{5}{9} \qquad \textbf{(E)}\ \frac{2}{3}$

Solution

For the product to be greater than zero, we must have either both numbers negative or both positive.

Both numbers are negative with a $\frac{2}{3} * \frac{2}{3} = \frac{4}{9}$ chance.

Both numbers are positive with a $\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$ chance.

Therefore, the total probability is $\frac{4}{9} + \frac{1}{9} = \frac{5}{9}$ and we are done. $\boxed{D}$

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 10

Problem

Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

$\textbf{(A)}\ 15 \qquad \textbf{(B)}\ 30 \qquad \textbf{(C)}\ 45 \qquad \textbf{(D)}\ 60 \qquad \textbf{(E)}\ 75$

Solution

Since $AB \parallel CD$, $\angle AMD = \angle CDM$ hence $CM = CD = 6$. Therefore $\angle BMC = 30^\circ$. Therefore $\angle AMD = \boxed{\textbf{(E)}\ 75^\circ}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 11

Problem

A frog located at (x,y), with both x and y integers, makes successive jumps of length 5 and always lands on points with integer coordinates. Suppose that the frog starts at $(0,0)$ and ends at $(1,0)$. What is the smallest possible number of jumps the frog makes?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 6$

Solution

Since the frog always jumps in length 5 and lands on a lattice point, the sum of its coordinates must change either by 5 (by jumping parallel to the x- or y-axis), or by 3 or 4 (based off the 3-4-5 right triangle).

Because either 1, 5, or 7 is always the change of the sum of the coordinates, the sum of the coordinates will always change from odd to even or vice versa. Thus, it is impossible for the frog to go from $(0,0)$ to $(1,0)$ in an even number of moves. Therefore, the frog cannot reach $(1,0)$ in two moves.

However, a path is possible in 3 moves: from $(0,0)$ to $(3,4)$ to $(6,0)$ to $(1,0)$.

Thus, the answer is $=\boxed{3 \textbf{(B)}}$.

See also

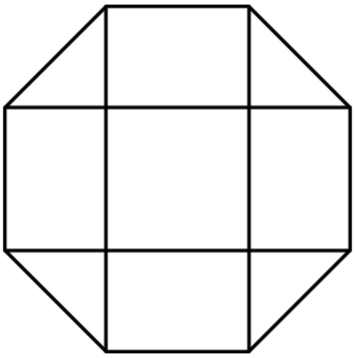
2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2011 AMC 12B Problems/Problem 12

Problem

A dart board is a regular octagon divided into regions as shown below. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. What is the probability that the dart lands within the center square?



$\textbf{(A)}\ \frac{\sqrt{2}-1}{2} \qquad \textbf{(B)}\ \frac{1}{4} \qquad \textbf{(C)}\ \frac{2-\sqrt{2}}{2} \qquad \textbf{(D)}\ \frac{\sqrt{2}}{4} \qquad \textbf{(E)}\ 2-\sqrt{2}$

Solution

Let's assume that the side length of the octagon is x. The area of the center square is just x^2. The triangles are all $45-45-90$ triangles, with a side length ratio of $1:1:\sqrt{2}$. The area of each of the 4 identical triangles is $\left(\frac{x}{\sqrt{2}}\right)^2\times\frac{1}{2}=\frac{x^2}{4}$, so the total area of all of the triangles is also x^2. Now, we must find the area of all of the 4 identical rectangles. One of the side lengths is x and the other side length is $\frac{x}{\sqrt{2}}=\frac{x\sqrt{2}}{2}$, so the area of all of the rectangles is $2x^2\sqrt{2}$. The ratio of the area of the square to the area of the octagon is $\frac{x^2}{2x^2+2x^2\sqrt{2}}$. Cancelling x^2 from the fraction, the ratio becomes $\frac{1}{2\sqrt2+2}$. Multiplying the numerator and the denominator each by $2\sqrt{2}-2$ will cancel out the radical, so the fraction is now $\frac{1}{2\sqrt2+2}\times\frac{2\sqrt2-2}{2\sqrt2-2}=\frac{2\sqrt2-2}{4}=\boxed{\mathrm{(A)}\ \frac{\sqrt{2}-1}{2}}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 13

Problem

Brian writes down four integers $w > x > y > z$ whose sum is 44. The pairwise positive differences of these numbers are $1, 3, 4, 5, 6$ and 9. What is the sum of the possible values of w?

$\textbf{(A)}\ 16 \qquad \textbf{(B)}\ 31 \qquad \textbf{(C)}\ 48 \qquad \textbf{(D)}\ 62 \qquad \textbf{(E)}\ 93$

Solution

Assume that $y - z = a, x - y = b, w - x = c$. $w - z$ results in the greatest pairwise difference, and thus it is 9. This means $a + b + c = 9$. a, b, c must be in the set $1, 3, 4, 5, 6$. The only way for 3 numbers in the set to add up to 9 is if they are $1, 3, 5$. $a + b$, and $b + c$ then must be the remaining two numbers which are 4 and 6. The ordering of (a, b, c) must be either $(3, 1, 5)$ or $(5, 1, 3)$.

Case 1 $(a, b, c) = (3, 1, 5)$ $x = w - 5$ $y = w - 5 - 1$ $x = w - 5 - 1 - 3$
$w + x + y + z = 4w - 20 = 44$ $w = 16$

Case 2 $(a, b, c) = (5, 1, 3)$ $x = w - 3$ $y = w - 3 - 1$ $x = w - 3 - 1 - 5$
$w + x + y + z = 4w - 16 = 44$ $w = 15$

The sum of the two w's is $15 + 16 = 31$ $\boxed{B}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 12	Followed by Problem 14
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 14

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See also

Problem

A segment through the focus F of a parabola with vertex V is perpendicular to $\overline{FV}$ and intersects the parabola in points A and B. What is $\cos\left(\angle AVB\right)$?

$\textbf{(A)}\ -\frac{3\sqrt{5}}{7}\qquad\textbf{(B)}\ -\frac{2\sqrt{5}}{5}\qquad\textbf{(C)}\ -\frac{4}{5}\qquad\textbf{(D)}\ -\frac{3}{5}\qquad\textbf{(E)}\ -\frac{1}{2}$

Solution 1

Name the directrix of the parabola l. Define $d(X,k)$ to be the distance between a point X and a line k.

Now we remember the geometric definition of a parabola: given any line l (called the directrix) and any point F (called the focus), the parabola corresponding to the given directrix and focus is the locus of the points that are equidistant from F and l. Therefore $FV = d(V,l)$. Let this distance be d. Now note that $d(F,l) = 2d$, so $d(A,l) = d(B,l) = 2d$. Therefore $AF = BF = 2d$. We now use the Pythagorean Theorem on triangle AFV; $AV = \sqrt{AF^2 + FV^2} = d\sqrt{5}$. Similarly, $BV = d\sqrt{5}$. We now use the Law of Cosines:

$$AB^2 = AV^2 + VB^2 - 2AV\cdot VB\cos\angle AVB \Rightarrow 16d^2 = 10d^2 - 10d^2\cos\angle AVB$$

$$\Rightarrow \cos\angle AVB = -\frac{3}{5}$$

This shows that the answer is $\boxed{\textbf{(D)}}$.

Solution 2

WLOG we can assume that the parabola is $y = x^2$. Therefore $V = (0,0)$ and $F = (0,\frac{1}{4})$. Also $A = (-\frac{1}{2},\frac{1}{4})$ and $B = (\frac{1}{2},\frac{1}{4})$.

$AB = 1$ and $AV = VB = \sqrt{(\frac{1}{2})^2 + (\frac{1}{4})^2} = \frac{\sqrt{5}}{4}$ by the pythagorean theorem.

Now using the law of cosines on $\triangle AVB$ we have:

$$AB^2 = 2AV^2 - 2AV\cos(\angle AVB) = 2AV^2(1 - \cos(\angle AVB))$$

$$1 = \frac{5}{8}\cos(\angle AVB)$$

and $\cos(\angle AVB) = -\frac{3}{5}\boxed{\textbf{(D)}}$.

(solution by mihirb)

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 15

Problem 15

How many positive two-digits integers are factors of $2^{24}-1$?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 10 \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 14$

Solution

From repeated application of difference of squares:

$$2^{24}-1=(2^{12}+1)(2^6+1)(2^3+1)(2^3-1)$$

$$2^{24}-1=(2^{12}+1)*65*9*7$$

$$2^{24}-1=(2^{12}+1)*5*13*3^2*7$$

Applying sum of cubes:

$$2^{12}+1=(2^4+1)(2^8-2^4+1)$$

$$2^{12}+1=17*241$$

A quick check shows 241 is prime. Thus, the only factors to be concerned about are $3^2*5*7*13*17$, since multiplying by 241 will make any factor too large.

Multiply 17 by 3 or 5 will give a two digit factor; 17 itself will also work. The next smallest factor, 7, gives a three digit number. Thus, there are 3 factors which are multiples of 17.

Multiply 13 by $3, 5$ or 7 will also give a two digit factor, as well as 13 itself. Higher numbers will not work, giving an additional 4 factors.

Multiply 7 by $3, 5,$ or 3^2 for a two digit factor. There are no mare factors to check, as all factors which include 13 are already counted. Thus, there are an additional 3 factors.

Multiply 5 by 3 or 3^2 for a two digit factor. All higher factors have been counted already, so there are 2 more factors.

Thus, the total number of factors is $3+4+3+2=\boxed{12\ \textbf{(D)}}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 10B Problems/Problem 20

The following problem is from both the 2011 AMC 12B #16 and 2011 AMC 10B #20, so both problems redirect to this page.

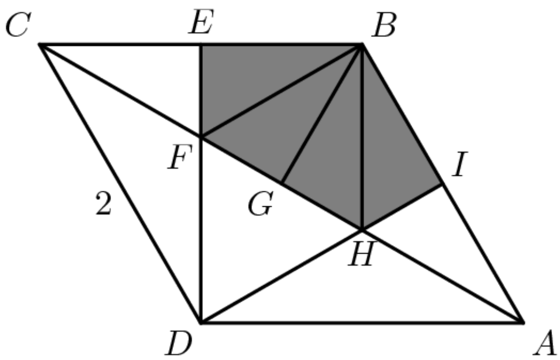
Problem

Rhombus $ABCD$ has side length 2 and $\angle B = 120^\circ$. Region R consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of R?

$\textbf{(A)}\ \frac{\sqrt{3}}{3} \qquad\textbf{(B)}\ \frac{\sqrt{3}}{2} \qquad\textbf{(C)}\ \frac{2\sqrt{3}}{3} \qquad\textbf{(D)}\ 1+\frac{\sqrt{3}}{3} \qquad\textbf{(E)}\ 2$

Solution

Suppose that P is a point in the rhombus $ABCD$ and let ℓ_{BC} be the perpendicular bisector of $\overline{BC}$. Then $PB < PC$ if and only if P is on the same side of ℓ_{BC} as B. The line ℓ_{BC} divides the plane into two half-planes; let S_{BC} be the half-plane containing B. Let us define similarly ℓ_{BD}, S_{BD} and ℓ_{BA}, S_{BA}. Then R is equal to $ABCD \cap S_{BC} \cap S_{BD} \cap S_{BA}$. The region turns out to be an irregular pentagon. We can make it easier to find the area of this region by dividing it into four triangles:



Since $\triangle BCD$ and $\triangle BAD$ are equilateral, ℓ_{BC} contains D, ℓ_{BD} contains A and C, and ℓ_{BA} contains D. Then $\triangle BEF \cong \triangle BGF \cong \triangle BGH \cong \triangle BIH$ with $BE = 1$ and $EF = \frac{1}{\sqrt{3}}$ so $[BEF] = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{3}$. Multiply this by 4 and it turns out that the pentagon has area $\boxed{(C) \frac{2\sqrt{3}}{3}}$.

See Also

2011 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2011))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2011 AMC 12B Problems/Problem 17

Problem

Let $f(x)=10^{10x}, g(x)=\log_{10}\left(\frac{x}{10}\right), h_1(x)=g(f(x))$, and $h_n(x)=h_1(h_{n-1}(x))$ for integers $n\geq 2$. What is the sum of the digits of $h_{2011}(1)$?

$\textbf{(A)}\ 16081 \qquad \textbf{(B)}\ 16089 \qquad \textbf{(C)}\ 18089 \qquad \textbf{(D)}\ 18098 \qquad \textbf{(E)}\ 18099$

Solution

$g(x)=\log_{10}\left(\frac{x}{10}\right)$ $=\log_{10}\left(x\right)$ - 1

$h_{1}(x)=g(f(x))=g(10^{10x})=\log_{10}\left(10^{10x}\right)-1=10x-1$

Proof by induction that $h_{n}(x)=10^n x-(1+10+10^2+...+10^{n-1})$:

For $\text{n}=1$, $h_{1}(x)=10x-1$

Assume $h_{n}(x)=10^n x-(1+10+10^2+...+10^{n-1})$ is true for n:

$$h_{n+1}(x)=h_{1}(h_{n}(x))=10h_{n}(x)-1=10\left(10^n x-(1+10+10^2+...+10^{n-1})\right)-1$$
$$=10^{n+1}x-(10+10^2+...+10^{n})-1$$
$$=10^{n+1}x-(1+10+10^2+...+10^{(n+1)-1})$$

Therefore, if it is true for n, then it is true for n+1; since it is also true for n = 1, it is true for all positive integers n.

$h_{2011}(1)=10^{2011}\times 1-(1+10+10^2+...+10^{2010})$, which is the 2011-digit number 8888...8889

The sum of the digits is 8 times 2010 plus 9, or $\boxed{16089 \textbf{(B)}}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 18

Problem

A pyramid has a square base with side of length 1 and has lateral faces that are equilateral triangles. A cube is placed within the pyramid so that one face is on the base of the pyramid and its opposite face has all its edges on the lateral faces of the pyramid. What is the volume of this cube?

$\textbf{(A)}\ 5\sqrt{2} - 7 \qquad \textbf{(B)}\ 7 - 4\sqrt{3} \qquad \textbf{(C)}\ \frac{2\sqrt{2}}{27} \qquad \textbf{(D)}\ \frac{\sqrt{2}}{9} \qquad \textbf{(E)}\ \frac{\sqrt{3}}{9}$

Solution

We can use the Pythagorean Theorem to split one of the triangular faces into two 30-60-90 triangles with side lengths $\frac{1}{2}, 1$ and $\frac{\sqrt{3}}{2}$.

Next, take a cross-section of the pyramid, forming a triangle with the top of the triangle and the midpoints of two opposite sides of the square base.

This triangle is isosceles with a base of 1 and two sides of length $\frac{\sqrt{3}}{2}$.

The height of this triangle will equal the height of the pyramid. To find this height, split the triangle into two right triangles, with sides $\frac{1}{2}, \frac{\sqrt{2}}{2}$ and $\frac{\sqrt{3}}{2}$.

The cube, touching all four triangular faces, will form a similar pyramid which sits on top of the cube. If the cube has side length x, the pyramid has side length $\frac{x\sqrt{2}}{2}$.

Thus, the height of the cube plus the height of the smaller pyramid equals the height of the larger pyramid.

$x + \frac{x\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$.

$x\left(1 + \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$

$x\left(2 + \sqrt{2}\right) = \sqrt{2}$

$x = \frac{\sqrt{2}}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{2\sqrt{2}-2}{4-2} = \sqrt{2} - 1 =$ side length of cube.

$\left(\sqrt{2}-1\right)^3 = (\sqrt{2})^3 + 3(\sqrt{2})^2(-1) + 3(\sqrt{2})(-1)^2 + (-1)^3 = 2\sqrt{2} - 6 + 3\sqrt{2} - 1 = \textbf{(A)} 5\sqrt{2} - 7$

See Also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 19

Problem

A lattice point in an xy-coordinate system is any point (x, y) where both x and y are integers. The graph of $y = mx + 2$ passes through no lattice point with $0 < x \leq 100$ for all m such that $\frac{1}{2} < m < a$. What is the maximum possible value of a?

$\textbf{(A)}\ \frac{51}{101} \qquad \textbf{(B)}\ \frac{50}{99} \qquad \textbf{(C)}\ \frac{51}{100} \qquad \textbf{(D)}\ \frac{52}{101} \qquad \textbf{(E)}\ \frac{13}{25}$

Solution

It is very easy to see that the $+2$ in the graph does not impact whether it passes through lattice.

We need to make sure that m cannot be in the form of $\frac{a}{b}$ for $1 \leq b \leq 100$, otherwise the graph $y = mx$ passes through lattice point at $x = b$. We only need to worry about $\frac{a}{b}$ very close to $\frac{1}{2}$, $\frac{m+1}{2m+1}$, $\frac{m+1}{2m}$ will be the only case we need to worry about and we want the minimum of those, clearly for $1 \leq b \leq 100$, the smallest is $\frac{50}{99}$, so answer is $\boxed{\frac{50}{99}\textbf{(B)}}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 20

Contents

- 1 Problem
- 2 Solutions
  - 2.1 Solution 1
  - 2.2 Solution 2
  - 2.3 Solution 3
- 3 See also

Problem

Triangle ABC has $AB = 13, BC = 14$, and $AC = 15$. The points D, E, and F are the midpoints of $\overline{AB}, \overline{BC}$, and $\overline{AC}$ respectively. Let $X \neq E$ be the intersection of the circumcircles of $\triangle BDE$ and $\triangle CEF$. What is $XA + XB + XC$?

$\textbf{(A)}\ 24 \qquad \textbf{(B)}\ 14\sqrt{3} \qquad \textbf{(C)}\ \frac{195}{8} \qquad \textbf{(D)}\ \frac{129\sqrt{7}}{14} \qquad \textbf{(E)}\ \frac{69\sqrt{2}}{4}$

Solutions

Solution 1

Answer: (C)

Let us also consider the circumcircle of $\triangle ADF$.

Note that if we draw the perpendicular bisector of each side, we will have the circumcenter of $\triangle ABC$ which is P, Also, since $m\angle ADP = m\angle AFP = 90^\circ$. $ADPF$ is cyclic, similarly, $BDPE$ and $CEPF$ are also cyclic. With this, we know that the circumcircles of $\triangle ADF$, $\triangle BDE$ and $\triangle CEF$ all intersect at P, so P is X.

The question now becomes calculate the sum of distance from each vertices to the circumcenter.

We can calculate the distances with coordinate geometry. (Note that $XA = XB = XC$ because X is the circumcenter.)

Let $A = (5, 12)$, $B = (0, 0)$, $C = (14, 0)$, $X = (x_0, y_0)$

Then X is on the line $x = 7$ and also the line with slope $-\frac{5}{12}$ and passes through $(2.5, 6)$.

$$y_0 = 6 - \frac{45}{24} = \frac{33}{8}$$

So $X = (7, \frac{33}{8})$

and $XA + XB + XC = 3XB = 3\sqrt{7^2 + \left(\frac{33}{8}\right)^2} = 3 \times \frac{65}{8} = \frac{195}{8}$

Solution 2

Consider an additional circumcircle on $\triangle ADF$. After drawing the diagram, it is noticed that each triangle has side values: 7, $\frac{15}{2}$, $\frac{13}{2}$. Thus they are congruent, and their respective circumcircles are. By inspection, we see that XA, XB, and XC are the circumdiameters, and so they are congruent. Therefore, the solution can be found by calculating one of these circumdiameters and multiplying it by a factor of 3. We can find the circumradius quite easily with the formula $\sqrt{(s)(s-a)(s-b)(s-c)} = \frac{abc}{4R}$, s.t. $s = \frac{a+b+c}{2}$ and R is the circumradius. Since $s = \frac{21}{2}$:

$$\sqrt{(\frac{21}{2})(4)(3)(\frac{7}{2})} = \frac{\frac{15}{2} \cdot \frac{13}{2} \cdot 7}{4R}$$

After a few algebraic manipulations:

$$\Rightarrow R = \frac{65}{16} \Rightarrow D = 2R = \frac{65}{8} \Rightarrow 3D = \boxed{\frac{195}{8}}.$$

Solution 3

Let O be the circumcenter of $\triangle ABC$, and h_A denote the length of the altitude from A. Note that a homothety centered at B with ratio $\frac{1}{2}$ takes the circumcircle of $\triangle BAC$ to the circumcircle of $\triangle BDE$. It also takes the point diametrically opposite B on the circumcircle of $\triangle BAC$ to O. Therefore, O lies on the circumcircle of $\triangle BDE$. Similarly, it lies on the circumcircle of $\triangle CEF$. By Pythagorean triples, $h_A = 12$. Finally, our answer is

$$3R = 3 \cdot \frac{abc}{4\{ABC\}} = 3 \cdot \frac{abc}{2ah_A} = 3 \cdot \frac{bc}{2h_A} = \boxed{\frac{195}{8}.}$$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 21

Contents

- 1 Problem
- 2 Solution
- 3 Sidenote
- 4 See also

Problem

The arithmetic mean of two distinct positive integers x and y is a two-digit integer. The geometric mean of x and y is obtained by reversing the digits of the arithmetic mean. What is $|x-y|$?

$\textbf{(A)}\ 24 \qquad \textbf{(B)}\ 48 \qquad \textbf{(C)}\ 54 \qquad \textbf{(D)}\ 66 \qquad \textbf{(E)}\ 70$

Solution

Answer: (D)

$\frac{x+y}{2}=10a+b$ for some $1\le a\le 9, 0\le b\le 9$.

$\sqrt{xy}=10b+a$

$100a^2+20ab+b^2=\frac{x^2+2xy+y^2}{4}$

$xy=100b^2+20ab+a^2$

$\frac{x^2+2xy+y^2}{4}-xy=\frac{x^2-2xy+y^2}{4}=\left(\frac{x-y}{2}\right)^2=99a^2-99b^2=99(a^2-b^2)$

$|x-y|=2\sqrt{99(a^2-b^2)}$

Note that in order for x-y to be integer, (a^2-b^2) has to be $11n$ for some perfect square n. Since a is at most 9, $n=1$ or 4

If $n=1$, $|x-y|=66$, if $n=4$, $|x-y|=132$. In AMC, we are done. Otherwise, we need to show that $a^2-b^2=44$ is impossible.

$(a-b)(a+b)=44$ -> $a-b=1$, or 2 or 4 and $a+b=44$, 22, 11 respectively. And since $a+b\le 18$, $a+b=11$, $a-b=4$, but there is no integer solution for a, b.

In addition: Note that $11n$ with $n=1$ may be obtained with $a=6$ and $b=5$ as $a^2-b^2=36-25=11$.

Sidenote

It is easy to see that $(a,b)=(6,5)$ is the only solution. This yields $(x,y)=(98,32)$. Their arithmetic mean is 65 and their geometric mean is 56.

See also

2011 AMC 12B Problems/Problem 22

Problem

Let T_1 be a triangle with sides 2011, 2012, and 2013. For $n \geq 1$, if $T_n = \Delta ABC$ and D, E, and F are the points of tangency of the incircle of ΔABC to the sides AB, BC, and AC, respectively, then T_{n+1} is a triangle with side lengths AD, BE, and CF, if it exists. What is the perimeter of the last triangle in the sequence (T_n)?

$\textbf{(A)}\ \frac{1509}{8} \qquad \textbf{(B)}\ \frac{1509}{32} \qquad \textbf{(C)}\ \frac{1509}{64} \qquad \textbf{(D)}\ \frac{1509}{128} \qquad \textbf{(E)}\ \frac{1509}{256}$

Solution

Answer: (D)

Let $AB = c$, $BC = a$, and $AC = b$

Then $AD = AF$, $BE = BD$ and $CF = CE$

Then $a = BE + CF$, $b = AD + CF$, $c = AD + BE$

Hence:

$$AD = AF = \frac{b+c-a}{2}$$
$$BE = BD = \frac{a+c-b}{2}$$
$$CF = CE = \frac{a+b-c}{2}$$

Note that $a+1=b$ and $a-1=c$ for $n=1$, I claim that it is true for all n, assume for induction that it is true for some n, then

$$AD = AF = \frac{a}{2}$$
$$BE = BD = \frac{a-2}{2} = AD - 1$$
$$CF = CE = \frac{a+2}{2} = AD + 1$$

Furthermore, the average for the sides is decreased by a factor of 2 each time.

So T_n is a triangle with side length $\frac{2012}{2^{n-1}} - 1$, $\frac{2012}{2^{n-1}}$, $\frac{2012}{2^{n-1}} + 1$

and the perimeter of such T_n is $\frac{(3)(2012)}{2^{n-1}}$

Now we need to find when T_n fails the triangle inequality. So we need to find the last n such that $\frac{2012}{2^{n-1}} > 2$

$$\frac{2012}{2^{n-1}} > 2$$
$$2012 > 2^n$$
$$n \leq 10$$

For $n = 10$, perimeter is $\frac{(3)(2012)}{2^9} = \frac{1509}{2^7} = \frac{1509}{128}$

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 21	Followed by Problem 23
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 23

Problem

A bug travels in the coordinate plane, moving only along the lines that are parallel to the x-axis or y-axis. Let $A = (-3, 2)$ and $B = (3, -2)$. Consider all possible paths of the bug from A to B of length at most 20. How many points with integer coordinates lie on at least one of these paths?

$\textbf{(A)}\ 161 \qquad \textbf{(B)}\ 185 \qquad \textbf{(C)}\ 195 \qquad \textbf{(D)}\ 227 \qquad \textbf{(E)}\ 255$

Solution

Answer: (C)

If a point (x, y) satisfy the property that $|x-3|+|y+2|+|x+3|+|y-2| \le 20$, then it is in the desirable range because $|x-3|+|y+2|$ is the shortest path from (x, y) to B, and $|x+3|+|y-2|$ is the shortest path from (x, y) to A

If $-3 \le x \le 3$, then $-7 \le y \le 7$ satisfy the property. there are $15 \times 7 = 105$ lattice points here.

else let $3 < x \le 8$ (and for $-8 \le x < -3$ it is symmetrical,
$-7 + (x-3) \le y \le 7 - (x-3)$,

$-4 + x \le y \le 4 - x$

So for $x = 4$, there are 13 lattice points,

for $x = 5$, there are 11 lattice points,

etc.

For $x = 8$, there are 5 lattice points.

Hence, there are a total of $105 + 2(13 + 11 + 9 + 7 + 5) = \boxed{195}$ lattice points.

See also

2011 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12B Problems/Problem 24

Problem

Let $P(z)=z^8+\left(4\sqrt{3}+6\right)z^4-\left(4\sqrt{3}+7\right)$. What is the minimum perimeter among all the 8-sided polygons in the complex plane whose vertices are precisely the zeros of $P(z)$?

$\textbf{(A)}\ 4\sqrt{3}+4 \qquad \textbf{(B)}\ 8\sqrt{2} \qquad \textbf{(C)}\ 3\sqrt{2}+3\sqrt{6}$

$\textbf{(D)}\ 4\sqrt{2}+4\sqrt{3} \qquad \textbf{(E)}\ 4\sqrt{3}+6$

Solution

Answer: (B)

First of all, we need to find all z such that $P(z)=0$

$$P(z)=\left(z^4-1\right)\left(z^4+\left(4\sqrt{3}+7\right)\right)$$

So $z^4=1$ or $z=e^{i\frac{n\pi}{2}}$

or $z^4=-\left(4\sqrt{3}+7\right)$

$$z^2=\pm i\sqrt{4\sqrt{3}+7}=e^{i\frac{(2n+1)\pi}{2}}\left(\sqrt{3}+2\right)$$

$$z=e^{i\frac{(2n+1)\pi}{4}}\sqrt{\sqrt{3}+2}=e^{i\frac{(2n+1)\pi}{4}}\left(\frac{\sqrt{3}}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)$$

Now we have a solution at $\frac{n\pi}{4}$ if we look at them in polar coordinate, further more, the 8-gon is symmetric (it is an regular octagon) . So we only need to find the side length of one and multiply by 8.

So answer $=8\times$ distance from 1 to $\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)(1+i)$

Side length $=\sqrt{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)^2}=\sqrt{2\left(\frac{3}{4}+\frac{1}{4}\right)}=\sqrt{2}$

Hence, answer is $8\sqrt{2}$.

Easier method: Use the law of cosines. We make a the distance. Now, since the angle does not change the distance from the origin, we can just use the distance.

$a^2=(\frac{\sqrt{3}}{\sqrt{2}}+\frac{1}{\sqrt{2}})^2+1^2-2\times\frac{\sqrt{3}}{\sqrt{2}}+\frac{1}{\sqrt{2}}\times 1\times\cos\frac{\pi}{4}$, which simplifies to $a^2=2+\sqrt{3}+1-1-\sqrt{3}$, or $a^2=2$, or $a=\sqrt{2}$. Multiply the answer by 8 to get $\boxed{(B) 8\sqrt{2}}$

See also

2011 AMC 12B Problems/Problem 25

Contents

- 1 Problem
- 2 Solution
  - 2.1 Solution 1
  - 2.2 Proof:
  - 2.3 Solution 2
- 3 See also

Problem

For every m and k integers with k odd, denote by $\left[\frac{m}{k}\right]$ the integer closest to $\frac{m}{k}$. For every odd integer k, let $P(k)$ be the probability that

$$\left[\frac{n}{k}\right]+\left[\frac{100-n}{k}\right]=\left[\frac{100}{k}\right]$$

for an integer n randomly chosen from the interval $1\leq n\leq 99!$. What is the minimum possible value of $P(k)$ over the odd integers k in the interval $1\leq k\leq 99$?

$\textbf{(A)}\ \frac{1}{2}\qquad\textbf{(B)}\ \frac{50}{99}\qquad\textbf{(C)}\ \frac{44}{87}\qquad\textbf{(D)}\ \frac{34}{67}\qquad\textbf{(E)}\ \frac{7}{13}$

Solution

Solution 1

Answer: $(D)\frac{34}{67}$

First of all, you have to realize that

if $\left[\frac{n}{k}\right]+\left[\frac{100-n}{k}\right]=\left[\frac{100}{k}\right]$

then $\left[\frac{n-k}{k}\right]+\left[\frac{100-(n-k)}{k}\right]=\left[\frac{100}{k}\right]$

So, we can consider what happen in $1\leq n\leq k$ and it will repeat. Also since range of n is 1 to $99!$, it is always a multiple of k. So we can just consider $P(k)$ for $1\leq n\leq k$.

Let $\text{fpart}(x)$ be the fractional part function

This is an AMC exam, so use the given choices wisely. With the given choices, and the previous explanation, we only need to consider $k=99$, 87, 67, 13. $1\leq n\leq k$

For $k>\frac{200}{3}$, $\left[\frac{100}{k}\right]=1$. 3 of the k that should consider lands in here.

For $n < \frac{k}{2}$, $\left[\frac{n}{k}\right] = 0$, then we need $\left[\frac{100-n}{k}\right] = 1$

else for $\frac{k}{2} < n < k$, $\left[\frac{n}{k}\right] = 1$, then we need $\left[\frac{100-n}{k}\right] = 0$

For $n < \frac{k}{2}$, $\left[\frac{100-n}{k}\right] = \left[\frac{100}{k} - \frac{n}{k}\right] = 1$

So, for the condition to be true, $100 - n > \frac{k}{2}$. ($k > \frac{200}{3}$, no worry for the rounding to be > 1)

$100 > k > \frac{k}{2} + n$, so this is always true.

For $\frac{k}{2} < n < k$, $\left[\frac{100-n}{k}\right] = 0$, so we want $100 - n < \frac{k}{2}$, or $100 < \frac{k}{2} + n$

$100 < \frac{k}{2} + n < \frac{3k}{2}$

For k = 67, $67 > n > 100 - \frac{67}{2} = 66.5$

For k = 69, $69 > n > 100 - \frac{69}{2} = 67.5$

etc.

We can clearly see that for this case, $k = 67$ has the minimum $P(k)$, which is $\frac{34}{67}$. Also, $\frac{7}{13} > \frac{34}{67}$.

So for AMC purpose, answer is $\boxed{\textbf{(D)}\ \frac{34}{67}}$.

Proof:

Notice that for these integers $99, 87, 67$:

$0 \to 49, 50, 51 \to 98$

$100 \to 51, 50, 49 \to 2$

$P = \frac{98}{99}$

$0 \to 43, 44 \to 56, 57 \to 86$

$87 \to 57, 56 \to 44, 43 \to 14$

$P = \frac{74}{87}$

$0 \to 33, 34 \to 66$

$100 \to 67, 66 \to 34$

$P = \frac{34}{67}$

That the probability is $\frac{2k-100}{k} = 2 - \frac{100}{k}$. Even for $k = 13$, $P(13) = \frac{9}{13} = \frac{100}{13} - 7$. And $P(11) = \frac{10}{11} = 10 - \frac{100}{11}$.

Perhaps the probability for a given k is $\left\lceil \frac{100}{k} \right\rceil - \frac{100}{k}$ if $\left[\frac{100}{k} \right] = \left\lfloor \frac{100}{k} \right\rfloor$ and $\frac{100}{k} - \left\lfloor \frac{100}{k} \right\rfloor$ if $\left[\frac{100}{k} \right] = \left\lceil \frac{100}{k} \right\rceil$.

So $P > \frac{1}{2}$ and $P_{\min} = \frac{k_{\min}+1}{2k_{\min}} = \frac{101}{201}$. Because $201 = 3 \cdot 67 \mid 99!$!

Now, let's say we are not given any answer, we need to consider $k < \frac{200}{3}$.

I claim that $P(k) \geq \frac{1}{2} + \frac{1}{2k}$

If $\left[\frac{100}{k} \right]$ got round down, then $1 \leq n \leq \frac{k}{2}$ all satisfy the condition along with $n = k$

because if $\operatorname{fpart}\left(\frac{100}{k}\right) < \frac{1}{2}$ and $\operatorname{fpart}\left(\frac{n}{k}\right) < \frac{1}{2}$, so must $\operatorname{fpart}\left(\frac{100-n}{k}\right) < \frac{1}{2}$

and for $n = k$, it is the same as $n = 0$.

, which makes

$P(k) \geq \frac{1}{2} + \frac{1}{2k}$.

If $\left[\frac{100}{k} \right]$ got round up, then $\frac{k}{2} \leq n \leq k$ all satisfy the condition along with $n = 1$

because if $\operatorname{fpart}\left(\frac{100}{k}\right) > \frac{1}{2}$ and $\operatorname{fpart}\left(\frac{n}{k}\right) > \frac{1}{2}$

Case 1) $\operatorname{fpart}\left(\frac{100-n}{k}\right) < \frac{1}{2}$

-> $\operatorname{fpart}\left(\frac{100}{k}\right) = \operatorname{fpart}\left(\frac{n}{k}\right) + \operatorname{fpart}\left(\frac{100-n}{k}\right)$

Case 2)

$$\text{fpart}\left(\frac{100-n}{k}\right) > \frac{1}{2}$$

$$\text{-> fpart}\left(\frac{100}{k}\right) + 1 = \text{fpart}\left(\frac{n}{k}\right) + \text{fpart}\left(\frac{100-n}{k}\right)$$

and for $n = 1$, since k is odd, $\left[\frac{99}{k}\right] \neq \left[\frac{100}{k}\right]$

-> $99.5 = k(p + .5)$ -> $199 = k(2p + 1)$, and 199 is prime so $k = 1$ or $k = 199$, which is not in this set

, which makes

$$P(k) \geq \frac{1}{2} + \frac{1}{2k}.$$

Now the only case without rounding, $k = 1$. It must be true.

Solution 2

It suffices to consider $0 \leq n < k - 1$. Now for each of these n, let $f(n) = [\frac{n}{k}], g(n) = [\frac{100}{k}] - [\frac{100-n}{k}]$. If we let $k = 67$, then the following graphs result for f and g.

$f:$

$g:$

Our probability is the number of $0 \leq i < k$ such that $f(i) + g(i) = 0$ over k. Of course, this always holds for $i = 0$. If we let k vary, then the graph of f is always very similar to what it looks like above (groups of $\frac{k+1}{2}, \frac{k-1}{2}$ dots). However, the graph of g can vary greatly. In the above diagram, $g(i) = 0$ for all i, while it is possible for $g(i) = -1$ for all $i \neq 0$. In order to minimize the number of i which satisfy $f(i) + g(i) = 0$, we either want $g(i) = 0$ for $0 \leq i < k$, or $g(i) = -1$ for $1 \leq i < k$. This way, we see that at least half of the numbers from 1 to $k - 1$ satisfy the given equation. So, our desired probability is at least $\frac{k+1}{2k}$. As shown by the diagram above, the probability is $\frac{34}{67}$ for $k = 67$. Clearly no better solutions can exist when $k < 67$. On the other hand, for $k > 67$ 87 and 99 do not yield better probabilities. Therefore, our answer is $\boxed{67.}$

See also