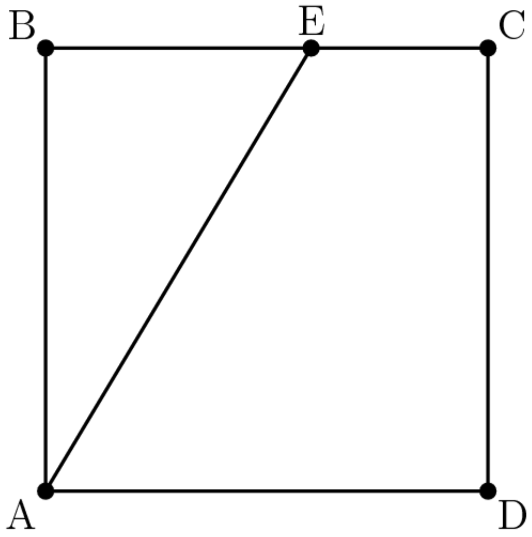
2013 AMC 12A Problems/Problem 1

Problem

Square $ABCD$ has side length 10. Point E is on $\overline{BC}$, and the area of $\triangle ABE$ is 40. What is BE?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 8$



Solution

We are given that the area of $\triangle ABE$ is 40, and that $AB = 10$.

The area of a triangle:

$$A = \frac{bh}{2}$$

Using AB as the height of $\triangle ABE$,

$$40 = \frac{10b}{2}$$

and solving for b,

$b = 8$, which is E

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12A Problems/Problem 2

Problem 2

A softball team played ten games, scoring $1, 2, 3, 4, 5, 6, 7, 8, 9$, and 10 runs. They lost by one run in exactly five games. In each of the other games, they scored twice as many runs as their opponent. How many total runs did their opponents score?

$\textbf{(A)}\ 35 \qquad \textbf{(B)}\ 40 \qquad \textbf{(C)}\ 45 \qquad \textbf{(D)}\ 50 \qquad \textbf{(E)}\ 55$

Solution

To score twice as many runs as their opponent, the softball team must have scored an even number.

Therefore we can deduce that when they scored an odd number of runs, they lost by one, and when they scored an even number of runs, they won by twice as much.

Therefore, the total runs by the opponent is $(2+4+6+8+10)+(1+2+3+4+5)=45$, which is C

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12A Problems/Problem 3

Problem

A flower bouquet contains pink roses, red roses, pink carnations, and red carnations. One third of the pink flowers are roses, three fourths of the red flowers are carnations, and six tenths of the flowers are pink. What percent of the flowers are carnations?

$\textbf{(A)}\ 15 \qquad \textbf{(B)}\ 30 \qquad \textbf{(C)}\ 40 \qquad \textbf{(D)}\ 60 \qquad \textbf{(E)}\ 70$

Solution

We are given that $\frac{6}{10} = \frac{3}{5}$ of the flowers are pink, so we know $\frac{2}{5}$ of the flowers are red.

Since $\frac{1}{3}$ of the pink flowers are roses, $\frac{2}{3}$ of the pink flowers are carnations.

We are given that $\frac{3}{4}$ of the red flowers are carnations.

The number of carnations are

$\frac{3}{5} * \frac{2}{3} + \frac{2}{5} * \frac{3}{4} = \frac{2}{5} + \frac{3}{10} = \frac{7}{10} = 70\%$, which is E

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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All AMC 12 Problems and Solutions	

2013 AMC 12A Problems/Problem 4

Problem

What is the value of

$$\frac{2^{2014}+2^{2012}}{2^{2014}-2^{2012}}?$$

$\textbf{(A)}\ -1\qquad\textbf{(B)}\ 1\qquad\textbf{(C)}\ \frac{5}{3}\qquad\textbf{(D)}\ 2013\qquad\textbf{(E)}\ 2^{4024}$

Solution

$$\frac{2^{2014}+2^{2012}}{2^{2014}-2^{2012}}$$

We can factor a 2^{2012} out of the numerator and denominator to obtain

$$\frac{2^{2012}*(2^2+1)}{2^{2012}*(2^2-1)}$$

The 2^{2012} cancels, so we get

$\frac{(2^2+1)}{(2^2-1)}=\frac{5}{3}$, which is C

See also

2013 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
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2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12A Problems/Problem 5

Problem

Tom, Dorothy, and Sammy went on a vacation and agreed to split the costs evenly. During their trip Tom paid $\$105$, Dorothy paid $\$125$, and Sammy paid $\$175$. In order to share the costs equally, Tom gave Sammy t dollars, and Dorothy gave Sammy d dollars. What is $t-d$?

$\textbf{(A)}\ 15 \qquad\textbf{(B)}\ 20 \qquad\textbf{(C)}\ 25 \qquad\textbf{(D)}\ 30 \qquad\textbf{(E)}\ 35$

Solution

Add up the amounts that Tom, Dorothy, and Sammy paid to get $\$405$, and divide by 3 to get $\$135$, the amount that each should have paid.

Tom, having paid $\$105$, owes Sammy $\$30$, and Dorothy, having paid $\$125$, owes Sammy $\$10$.

Thus, $t-d=30-10=20$, which is $\boxed{\textbf{(B)}}$

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12A Problems/Problem 6

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- 1 Problem
- 2 Solution
- 3 Alternative Solution
- 4 Additional note
- 5 See also

Problem

In a recent basketball game, Shenille attempted only three-point shots and two-point shots. She was successful on 20% of her three-point shots and 30% of her two-point shots. Shenille attempted 30 shots. How many points did she score?

$\textbf{(A)}\ 12 \qquad \textbf{(B)}\ 18 \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 30 \qquad \textbf{(E)}\ 36$

Solution

Let the number of 3-point shots attempted be x. Since she attempted 30 shots, the number of 2-point shots attempted must be $30 - x$.

Since she was successful on 20%, or $\frac{1}{5}$, of her 3-pointers, and 30%, or $\frac{3}{10}$, of her 2-pointers, then her score must be

$$\frac{1}{5} * 3x + \frac{3}{10} * 2(30 - x)$$

$$\frac{3}{5} * x + \frac{3}{5}(30 - x)$$

$$\frac{3}{5}(x + 30 - x)$$

$$\frac{3}{5} * 30$$

18, which is B

Alternative Solution

Since the problem doesn't specify the number of 3-point shots she attempted, it can be assumed that number doesn't matter, so let it be 0. Then, she must have made 30 2-point shots. So, her score must be:

$\frac{3}{10} * 30 * 2$, which is B.

Additional note

It is also reasonably easy to find all possibilities for the number of two-point and three-point shots she made. Just note that both numbers of successful throws have to be integers. For "30% of her two-point shots" to be an integer we need the number of two-point shots to be divisible by 10. This only leaves four possibilities for the number of two-point shots: 0, 10, 20, or 30. Each of them also works for the three-point shots, and as shown above, for each of them the total number of points scored is the same.

See also

2013 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
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2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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All AMC 12 Problems and Solutions	

2013 AMC 12A Problems/Problem 7

Problem

The sequence $S_1, S_2, S_3, \cdots, S_{10}$ has the property that every term beginning with the third is the sum of the previous two. That is,

$$S_n = S_{n-2} + S_{n-1} \text{ for } n \geq 3.$$

Suppose that $S_9 = 110$ and $S_7 = 42$. What is S_4?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 10 \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 16$

Solution

$S_9 = 110$, $S_7 = 42$

$S_8 = S_9 - S_7 = 110 - 42 = 68$

$S_6 = S_8 - S_7 = 68 - 42 = 26$

$S_5 = S_7 - S_6 = 42 - 26 = 16$

$S_4 = S_6 - S_5 = 26 - 16 = 10$

Therefore, the answer is $\boxed{\textbf{(C) }10}$

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12A Problems/Problem 8

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- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 See also

Problem

Given that x and y are distinct nonzero real numbers such that $x+\frac{2}{x}=y+\frac{2}{y}$, what is xy?

$\textbf{(A)}\ \frac{1}{4}\qquad\textbf{(B)}\ \frac{1}{2}\qquad\textbf{(C)}\ 1\qquad\textbf{(D)}\ 2\qquad\textbf{(E)}\ 4$

Solution 1

$x+\tfrac{2}{x}=y+\tfrac{2}{y}$

Since $x\neq y$, we may assume that $x=\dfrac{2}{y}$ and/or, equivalently, $y=\dfrac{2}{x}$.

Cross multiply in either equation, giving us $xy=2$.

Solution 2

$x+\tfrac{2}{x}=y+\tfrac{2}{y}$

$x-y+\dfrac{2}{x}-\dfrac{2}{y}=0$

$(x-y)+2(\dfrac{y-x}{xy})=0$

$(x-y)(1-\dfrac{2}{xy})=0$

Since $x\neq y$

$1=\dfrac{2}{xy}$

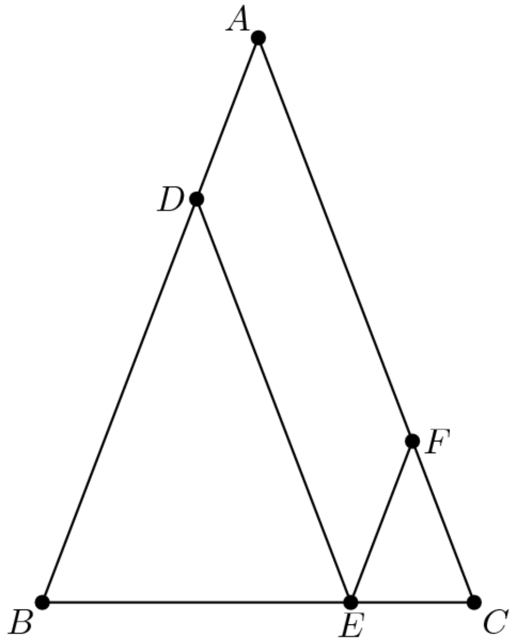
$xy=2$

See also

2013 AMC 12A Problems/Problem 9

Problem

In $\triangle ABC$, $AB = AC = 28$ and $BC = 20$. Points $D, E,$ and F are on sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, such that $\overline{DE}$ and $\overline{EF}$ are parallel to $\overline{AC}$ and $\overline{AB}$, respectively. What is the perimeter of parallelogram $ADEF$?



$\textbf{(A) }48\qquad\textbf{(B) }52\qquad\textbf{(C) }56\qquad\textbf{(D) }60\qquad\textbf{(E) }72$

Solution

Note that because $\overline{DE}$ and $\overline{EF}$ are parallel to the sides of $\triangle ABC$, the internal triangles $\triangle BDE$ and $\triangle EFC$ are similar to $\triangle ABC$, and are therefore also isosceles triangles.

It follows that $BD = DE$. Thus, $AD + DE = AD + DB = AB = 28$.

Since opposite sides of parallelograms are equal, the perimeter is $2 * (AD + DE) = 56$.

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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All AMC 12 Problems and Solutions	

2013 AMC 12A Problems/Problem 10

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- 5 See also

Problem

Let S be the set of positive integers n for which $\tfrac{1}{n}$ has the repeating decimal representation $0.\overline{ab} = 0.ababab\cdots,$ with a and b different digits. What is the sum of the elements of S?

$\textbf{(A)}\ 11 \qquad \textbf{(B)}\ 44 \qquad \textbf{(C)}\ 110 \qquad \textbf{(D)}\ 143 \qquad \textbf{(E)}\ 155$

Solution

Solution 1

Note that $\frac{1}{11} = 0.\overline{09}$.

Dividing by 3 gives $\frac{1}{33} = 0.\overline{03}$, and dividing by 9 gives $\frac{1}{99} = 0.\overline{01}$.

$S = \{11, 33, 99\}$

$11 + 33 + 99 = 143$

The answer must be at least 143, but cannot be 155 since no $n \leq 12$ other than 11 satisfies the conditions, so the answer is 143.

Solution 2

Let us begin by working with the condition $0.\overline{ab} = 0.ababab\cdots,$. Let $x = 0.ababab\cdots$. So, $100x - x = ab \Rightarrow x = \frac{ab}{99}$. In order for this fraction x to be in the form $\frac{1}{n}$, 99 must be a multiple of ab. Hence the possibilities of ab are $1, 3, 9, 11, 33, 99$. Checking each of these, $\frac{1}{99} = 0.\overline{01}$, $\frac{3}{99} = \frac{1}{33} = 0.\overline{03}$, $\frac{9}{99} = \frac{1}{11} = 0.\overline{09}$, $\frac{11}{99} = \frac{1}{9} = 0.\overline{1}$, $\frac{33}{99} = \frac{1}{3} = 0.\overline{3}$, and $\frac{99}{99} = 1$. So the only values of n that have distinct a and b are $11, 33,$ and 99. So,

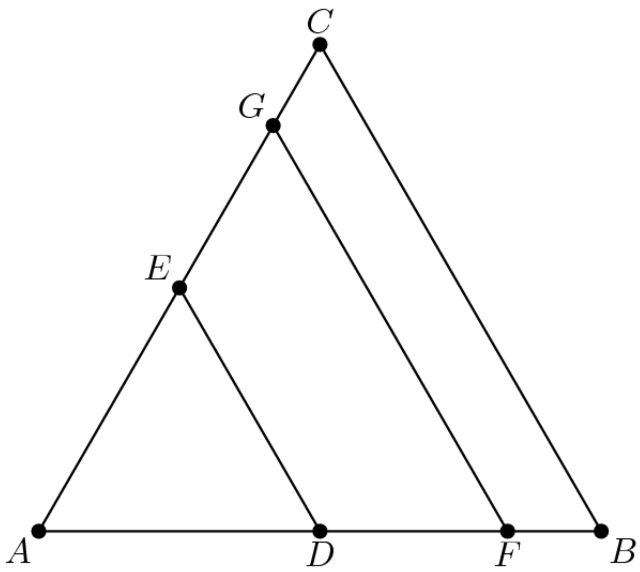
$11 + 33 + 99 = \boxed{\textbf{(D)} 143}$

See also

2013 AMC 12A Problems/Problem 11

Problem

Triangle ABC is equilateral with $AB = 1$. Points E and G are on $\overline{AC}$ and points D and F are on $\overline{AB}$ such that both $\overline{DE}$ and $\overline{FG}$ are parallel to $\overline{BC}$. Furthermore, triangle ADE and trapezoids $DFGE$ and $FBCG$ all have the same perimeter. What is $DE + FG$?



$\textbf{(A) }1\qquad\textbf{(B) }\frac{3}{2}\qquad\textbf{(C) }\frac{21}{13}\qquad\textbf{(D) }\frac{13}{8}\qquad\textbf{(E) }\frac{5}{3}$

Solution

Let $AD = x$, and $AG = y$. We want to find $DE + FG$, which is nothing but $x + y$.

Based on the fact that ADE, $DEFG$, and $BCFG$ have the same perimeters, we can say the following:

$$3x = x + 2(y - x) + y = y + 2(1 - y) + 1$$

Simplifying, we can find that

$$3x = 3y - x = 3 - y$$

Since $3 - y = 3x$, $y = 3 - 3x$.

After substitution, we find that $9 - 10x = 3x$, and x = $\frac{9}{13}$.

Again substituting, we find y = $\frac{12}{13}$.

Therefore, $x + y$ = $\frac{21}{13}$, which is C

See also

2013 AMC 12A Problems/Problem 12

Problem

The angles in a particular triangle are in arithmetic progression, and the side lengths are $4, 5, x$. The sum of the possible values of x equals $a + \sqrt{b} + \sqrt{c}$ where a, b, and c are positive integers. What is $a + b + c$?

$\textbf{(A)}\ 36 \qquad \textbf{(B)}\ 38 \qquad \textbf{(C)}\ 40 \qquad \textbf{(D)}\ 42 \qquad \textbf{(E)}\ 44$

Solution

Because the angles are in an arithmetic progression, and the angles add up to 180°, the second largest angle in the triangle must be 60°. Also, the side opposite of that angle must be the second longest because of the angle-side relationship. Any of the three sides, 4, 5, or x, could be the second longest side of the triangle.

The law of cosines can be applied to solve for x in all three cases.

When the second longest side is 5, we get that $5^2 = 4^2 + x^2 - 2(4)(x)cos60^\circ$, therefore $x^2 - 4x - 9 = 0$. By using the quadratic formula, $x = \frac{4 + \sqrt{16 + 36}}{2}$, therefore $x = 2 + \sqrt{13}$.

When the second longest side is x, we get that $x^2 = 5^2 + 4^2 - 40cos60^\circ$, therefore $x = \sqrt{21}$.

When the second longest side is 4, we get that $4^2 = 5^2 + x^2 - 2(5)(x)cos60^\circ$, therefore $x^2 - 5x + 9 = 0$. Using the quadratic formula, $x = \frac{5 + \sqrt{25 - 36}}{2}$. However, $\sqrt{-11}$ is not real, therefore the second longest side cannot equal 4.

Adding the two other possibilities gets $2 + \sqrt{13} + \sqrt{21}$, with $a = 2, b = 13$, and $c = 21$. $a + b + c = 36$, which is answer choice A.

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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Category: Introductory Geometry Problems

2013 AMC 12A Problems/Problem 13

Problem

Let points $A=(0,0)$, $B=(1,2)$, $C=(3,3)$, and $D=(4,0)$. Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through A. This line intersects $\overline{CD}$ at point $\left(\frac{p}{q},\frac{r}{s}\right)$, where these fractions are in lowest terms. What is $p+q+r+s$?

$\textbf{(A)}\ 54 \qquad \textbf{(B)}\ 58 \qquad \textbf{(C)}\ 62 \qquad \textbf{(D)}\ 70 \qquad \textbf{(E)}\ 75$

Solution

If you have graph paper, use Pick's Theorem to quickly and efficiently find the area of the quadrilateral. If not, just find the area by other methods.

Pick's Theorem states that

A = $I+\frac{B}{2}$ - 1, where I is the number of lattice points in the interior of the polygon, and B is the number of lattice points on the boundary of the polygon.

In this case,

A = $5+\frac{7}{2}$ - 1 = 7.5

so

$\frac{A}{2}$ = 3.75

The bottom half of the quadrilateral makes a triangle with base 4 and half the total area, so we can deduce that the height of the triangle must be $\frac{15}{8}$ in order for its area to be 3.75. This height is the y coordinate of our desired intersection point.

Note that segment CD lies on the line $y=-3x+12$. Substituting in $\frac{15}{8}$ for y, we can find that the x coordinate of our intersection point is $\frac{27}{8}$.

Therefore the point of intersection is ($\frac{27}{8}$, $\frac{15}{8}$), and our desired result is $27+8+15+8=58$, which is B.

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12A Problems/Problem 14

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- 2 Solution 1
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- 4 See also

Problem

The sequence

$$\log_{12}162,\ \log_{12}x,\ \log_{12}y,\ \log_{12}z,\ \log_{12}1250$$

is an arithmetic progression. What is x?

$\textbf{(A)}\ 125\sqrt{3} \qquad \textbf{(B)}\ 270 \qquad \textbf{(C)}\ 162\sqrt{5} \qquad \textbf{(D)}\ 434 \qquad \textbf{(E)}\ 225\sqrt{6}$

Solution 1

Since the sequence is arithmetic,

$\log_{12}162$ + $4d$ = $\log_{12}1250$, where d is the common difference.

Therefore,

$4d$ = $\log_{12}1250$ - $\log_{12}162$ = $\log_{12}(1250/162)$, and

d = $\frac{1}{4}$($\log_{12}(1250/162)$) = $\log_{12}(1250/162)^{1/4}$

Now that we found d, we just add it to the first term to find x:

$\log_{12}162$ + $\log_{12}(1250/162)^{1/4}$ = $\log_{12}((162)(1250/162)^{1/4})$

x = $(162)(1250/162)^{1/4}$ = $(162)(625/81)^{1/4}$ = $(162)(5/3)$ = 270, which is B

Solution 2

As the sequence $\log_{12}162,\ \log_{12}x,\ \log_{12}y,\ \log_{12}z,\ \log_{12}1250$ is an arithmetic progression, the sequence $162, x, y, z, 1250$ must be a geometric progression.

If we factor the two known terms we get $162=2\cdot 3^4$ and $1250=2\cdot 5^4$, thus the quotient is obviously $5/3$ and therefore $x=162\cdot(5/3)=270$.

See also

2013 AMC 12A Problems/Problem 15

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- 2 Solution 1
- 3 Solution 2
- 4 See also

Problem

Rabbits Peter and Pauline have three offspring—Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

$\textbf{(A)}\ 96 \qquad \textbf{(B)}\ 108 \qquad \textbf{(C)}\ 156 \qquad \textbf{(D)}\ 204 \qquad \textbf{(E)}\ 372$

Solution 1

There are two possibilities regarding the parents.

1) Both are in the same store. In this case, we can treat them both as a single bunny, and they can go in any of the 4 stores. The 3 baby bunnies can go in any of the remaining 3 stores. There are $4*3^3=108$ combinations.

2) The two are in different stores. In this case, one can go in any of the 4 stores, and the other can go in any of the 3 remaining stores. The 3 baby bunnies can each go in any of the remaining 2 stores. There are $4*3*2^3=96$ combinations.

Adding up, we get $108+96=204$ combinations.

Solution 2

We tackle the problem by sorting it by how many stores are involved in the transaction.

1) 2 stores are involved. There are $\binom{4}{2}=6$ ways to choose which of the stores are involved and 2 ways to choose which store recieves the parents. $6*2=12$ total arrangements.

2) 3 stores are involved. There are $\binom{4}{3}=4$ ways to choose which of the stores are involved. We then break the problem down to into two subsections - when the parents and grouped together or sold separately.

Separately: All children must be in one store. There are $3!$ ways to arrange this. 6 ways in total.

Together: Both parents are in one store and the 3 children are split between the other two. There are $\binom{3}{2}$ ways to split the children and $3!$ ways to choose to which store each group will be sold.

$3!*\binom{3}{2}=18.$

$(6+18)*4=96$ total arrangements.

3) All 4 stores are involved. We break down the problem as previously shown.

Separately: All children must be split between two stores. There are $\binom{3}{2} = 3$ ways to arrange this. We can then arrange which group is sold to which store in $4!$ ways. $4! * 3 = 72$.

Together: Both parents are in one store and the 3 children are each in another store. There are $4! = 24$ ways to arrange this.

$24 + 72 = 96$ total arrangements.

Final Answer: $12 + 96 + 96 = \boxed{\textbf{(D) } 204}$.

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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Category: Introductory Combinatorics Problems

2013 AMC 12A Problems/Problem 16

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- 2 Solution
  - 2.1 Solution 1
  - 2.2 Solution 2
  - 2.3 Solution 3
- 3 See also

Problem

A, B, C are three piles of rocks. The mean weight of the rocks in A is 40 pounds, the mean weight of the rocks in B is 50 pounds, the mean weight of the rocks in the combined piles A and B is 43 pounds, and the mean weight of the rocks in the combined piles A and C is 44 pounds. What is the greatest possible integer value for the mean in pounds of the rocks in the combined piles B and C?

$\textbf{(A)}\ 55 \qquad \textbf{(B)}\ 56 \qquad \textbf{(C)}\ 57 \qquad \textbf{(D)}\ 58 \qquad \textbf{(E)}\ 59$

Solution

Solution 1

Let pile A have A rocks, and so on.

The total weight of A and C can be expressed as $44(A+C)$.

To get the total weight of B and C, we add the weight of B and subtract the weight of A

$$44(A+C)+50B-40A=4A+44C+50B$$

Therefore, the mean of B and C is $\frac{4A+44C+50B}{B+C}$, which is simplified to $44+\frac{4A+6B}{B+C}$.

We now need to eliminate A in the numerator. Since we know that $40A+50B=43(A+B)$, we have $A=\frac{7}{3}B$

Substituting,

$$44+\frac{4(\frac{7}{3}B)+6B}{B+C}=44+\frac{46}{3}*\frac{B}{B+C}$$

$\frac{B}{B+C}<1$, so the maximum value occurs when $C=1$. Since $\frac{46}{3}$ must cancel to give an integer, and the only fraction that satisfies both conditions is $\frac{45}{46}$

Plugging in, we get

$$44+\frac{46}{3}*\frac{45}{46}=44+15=59$$

Solution 2

Suppose there are A, B, C rocks in the three piles, and that the mean of pile C is x, and that the mean of the combination of B and C is y. We are going to maximize y, subject to the following conditions:

$$40A + 50B = 43(A + B)$$

$$40A + xC = 44(A + C)$$

$$50B + xC = y(B + C)$$

which can be rearranged as:

$$7B = 3A$$

$$(x - 44)C = 4A$$

$$(x - y)C = (y - 50)B$$

Let us test $y = 59$ is possible. If so, it is already the answer. If not, there will be some contradiction. So the third equation becomes

$$(x - 59)C = 9B.$$

So $15C = (x - 44)C - (x - 59)C = 4A - 9B$, $45C = 4(3A) - 27B = 28B - 27B$, $105C = 28A - 9(7B) = A$, therefore,

$A = 105C, B = 45C, x = 4(105) + 44 = 464$, which gives us a consistent solution. Therefore $y = 59$ is the answer.

(Note: To further illustrate the idea, let us look at $y = 60$ and see what happens. We then get $7 \cdot 16C = 4A - 30A < 0$, which is a contradiction!)

Solution 3

Obtain the 3 equations as in solution 2.

$$7B = 3A$$

$$(x - 44)C = 4A$$

$$(x - y)C = (y - 50)B$$

Our goal is to try to isolate y into an inequality. The first equation gives $A = \frac{7}{3}B$, which we plug into the second equation to get

$$(x - 44)C = \frac{28}{3}B$$

To eliminate x, subtract equation 3 from equation 2:

$$(x-44)C-(x-y)C=\frac{28}{3}B-(y-50)B$$

$$(y-44)C=(\frac{178}{3}-y)B$$

In order for the coefficients to be positive,

$$44<y<\frac{178}{3}$$

Thus, the greatest integer value is $y=59$, choice (E).

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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Category: Introductory Algebra Problems

2013 AMC 12A Problems/Problem 17

Problem 17

A group of 12 pirates agree to divide a treasure chest of gold coins among themselves as follows. The k^{th} pirate to take a share takes $\frac{k}{12}$ of the coins that remain in the chest. The number of coins initially in the chest is the smallest number for which this arrangement will allow each pirate to receive a positive whole number of coins. How many coins does the 12^{th} pirate receive?

$\textbf{(A)}\ 720\qquad\textbf{(B)}\ 1296\qquad\textbf{(C)}\ 1728\qquad\textbf{(D)}\ 1925\qquad\textbf{(E)}\ 3850$

Solution

The first pirate takes $\frac{1}{12}$ of the x coins, leaving $\frac{11}{12}x$.

The second pirate takes $\frac{2}{12}$ of the remaining coins, leaving $\frac{10}{12}*\frac{11}{12}*x$.

Note that

$$12^{11} = (2^2*3)^{11} = 2^{22}*3^{11}$$

$$11! = 11*10*9*8*7*6*5*4*3*2$$

All the 2s and 3s cancel out of $11!$, leaving

$$11*5*7*5 = 1925$$

in the numerator.

We know there were just enough coins to cancel out the denominator in the fraction. So, at minimum, x is the numerator, leaving 1925 coins for the twelfth pirate.

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12A Problems/Problem 18

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Problem

Six spheres of radius 1 are positioned so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere?

$\textbf{(A)}\ \sqrt{2}\qquad\textbf{(B)}\ \frac{3}{2}\qquad\textbf{(C)}\ \frac{5}{3}\qquad\textbf{(D)}\ \sqrt{3}\qquad\textbf{(E)}\ 2$

Solution 1

Set up an isosceles triangle between the center of the 8th sphere and two opposite ends of the hexagon. Then set up another triangle between the point of tangency of the 7th and 8th spheres, and the points of tangency between the 7th sphere and 2 of the original spheres on opposite sides of the hexagon. Express each side length of the triangles in terms of r (the radius of sphere 8) and h (the height of the first triangle). You can then use Pythagorean Theorem to set up two equations for the two triangles, and find the values of h and r.

$(1+r)^2=2^2+h^2$

$(3\sqrt{2})^2=3^2+(h+r)^2$

$r=\boxed{\textbf{(B)}\ \frac{3}{2}}$

Solution 2

We have a regular hexagon with side lengths 2 and six spheres on each vertex with radius 1 that are internally tangent, therefore drawing radii going through all of them would create this regular hexagon.

There is a larger sphere which the 6 spheres are internally tangent to, with center in the center of the hexagon. To find the radius of the larger sphere we must first, either by prior knowledge or by deducing from the angle sum that the hexagon can be split into 6 equilateral triangles from it's vertices, that the radius is $2+1=3$

The 8th sphere is now, when thinking about it in 3D, sitting on top of the 6 spheres, which is the only possibility for it to tangent all the 6 small spheres externally and the larger sphere internally. The ring of the 6 small spheres is symmetrical and the 8th sphere will be resting with it's center aligned with the diameter of the large sphere.

We can therefore now create a triangle with the horizontal component 2, as it is from the vertex of the hexagon to the center of the hexagon. The vertical component is from the center of the large sphere to the center of the 8th sphere. This length equals 3, the radius of the large sphere, take away the radius of the 8th sphere, we can call it r, since the radius of the large sphere will include the diameter of the 8th sphere if we subtract radius we will reach the center. The last component is the hypotenuse of the right angled triangle. This consists of the radius of the small sphere - 1 - and the radius of the 8th sphere - r -.

We therefore now have a right angled triangle which when applied Pythagoras states $2^2 + (3-r)^2 = (1+r)^2$ Expanding brackets gives us $4 + 9 - 6r + r^2 = 1 + 2r + r^2$ here we can cancel out r^2 Isolating the r's $12 = 8r$ and then finally we have the answer: $r = \frac{12}{8} = \frac{3}{2}$

See also

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2013 AMC 12A Problems/Problem 19

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  - 2.1 Solution 1
  - 2.2 Solution 2
  - 2.3 Solution 3
- 3 See also

Problem

In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

$\textbf{(A)}\ 11 \qquad \textbf{(B)}\ 28 \qquad \textbf{(C)}\ 33 \qquad \textbf{(D)}\ 61 \qquad \textbf{(E)}\ 72$

Solution

Solution 1

Let $CX = x, BX = y$. Let the circle intersect AC at D and the diameter including AD intersect the circle again at E. Use power of a point on point C to the circle centered at A.

So $CX * CB = CD * CE => x(x+y) = (97-86)(97+86) => x(x+y) = 3 * 11 * 61$.

Obviously $x + y > x$ so we have three solution pairs for $(x, x+y) = (1, 2013), (3, 671), (11, 183), (33, 61)$. By the Triangle Inequality, only $x + y = 61$ yields a possible length of $BX + CX = BC$.

Therefore, the answer is D) 61.

Solution 2

Let $BX = q$, $CX = p$, and AC meet the circle at Y and Z, with Y on AC. Then $AZ = AY = 86$. Using the Power of a Point, we get that $p(p+q) = 11(183) = 11 * 3 * 61$. We know that $p + q > p$, and that $p > 13$ by the triangle inequality on $\triangle ACX$. Thus, we get that $BC = p + q = \boxed{\textbf{(D)}\ 61}$

Solution 3

Let x represent CX, and let y represent BX. Since the circle goes through B and X, $AB = AX = 86$. Then by Stewart's Theorem,

$$xy(x+y) + 86^2(x+y) = 97^2y + 86^2x.$$

$$x^2y + xy^2 + 86^2x + 86^2y = 97^2y + 86^2x$$

$$x^2 + xy + 86^2 = 97^2$$

(Since y cannot be equal to 0, dividing both sides of the equation by y is allowed.)

$$x(x+y) = (97+86)(97-86)$$

$$x(x+y) = 2013$$

The prime factors of 2013 are 3, 11, and 61. Obviously, $x < x + y$. In addition, by the Triangle Inequality, $BC < AB + AC$, so $x + y < 183$. Therefore, x must equal 33, and $x + y$ must equal $\boxed{\textbf{(D) } 61}$

See also

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2013 AMC 12A Problems/Problem 20

Problem 20

Let S be the set $\{1,2,3,...,19\}$. For $a,b \in S$, define $a \succ b$ to mean that either $0 < a-b \le 9$ or $b-a > 9$. How many ordered triples (x,y,z) of elements of S have the property that $x \succ y$, $y \succ z$, and $z \succ x$?

$\textbf{(A)}\ 810 \qquad \textbf{(B)}\ 855 \qquad \textbf{(C)}\ 900 \qquad \textbf{(D)}\ 950 \qquad \textbf{(E)}\ 988$

Solution

Imagine 19 numbers are just 19 persons sitting evenly around a circle C; each of them is facing to the center.

One may check that $x \succ y$ if and only if y is one of the 9 persons on the left of x, and $y \succ x$ if and only if y is one of the 9 persons on the right of x. Therefore, "$x \succ y$ and $y \succ z$ and $z \succ x$" implies that x, y, z cuts the circumference of C into three arcs, each of which has no more than 10 numbers sitting on it (inclusive).

We count the complement: where the cut generated by (x,y,z) has ONE arc that has more than 10 persons sitting on. Note that there can only be one such arc because there are only 19 persons in total.

Suppose the number of persons on the longest arc is $k > 10$. Then two places of x,y,z are just chosen from the two end-points of the arc, and there are $19-k$ possible places for the third person. Once the three places of x,y,z are chosen, there are three possible ways to put x,y,z into them clockwise. Also, note that for any $k > 10$, there are 19 ways to choose an arc of length k. Therefore the total number of ways (of the complement) is

$$\sum_{k=11}^{18} 3\cdot 19\cdot (19-k) = 3\cdot 19\cdot (1+\cdots+8) = 3\cdot 19\cdot 36$$

So the answer is

$$3\cdot \binom{19}{3} - 3\cdot 19\cdot 36 = 3\cdot 19\cdot (51-36) = 855$$

NOTE: this multiple choice problem can be done even faster -- after we realized the fact that each choice of the three places of x,y,z corresponds to 3 possible ways to put them in, and that each arc of length $k > 10$ has 19 equitable positions, it is evident that the answer should be divisible by $3\cdot 19$, which can only be 855 from the five choices.

See also

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2013 AMC 12A Problems/Problem 21

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Problem

Consider $A = \log(2013 + \log(2012 + \log(2011 + \log(\cdots + \log(3 + \log 2) \cdots))))$. Which of the following intervals contains A?

$\textbf{(A)}\ (\log 2016, \log 2017)$ $\textbf{(B)}\ (\log 2017, \log 2018)$ $\textbf{(C)}\ (\log 2018, \log 2019)$
$\textbf{(D)}\ (\log 2019, \log 2020)$ $\textbf{(E)}\ (\log 2020, \log 2021)$

Solution 1

Let $f(x) = \log(x + f(x-1))$ and $f(2) = log(2)$, and from the problem description, $A = f(2013)$

We can reason out an approximation, by ignoring the $f(x-1)$:

$$f_0(x) \approx \log x$$

And a better approximation, by plugging in our first approximation for $f(x-1)$ in our original definition for $f(x)$:

$$f_1(x) \approx \log(x + \log(x-1))$$

And an even better approximation:

$$f_2(x) \approx \log(x + \log(x - 1 + \log(x-2)))$$

Continuing this pattern, obviously, will eventually terminate at $f_{x-1}(x)$, in other words our original definition of $f(x)$.

However, at $x = 2013$, going further than $f_1(x)$ will not distinguish between our answer choices. $\log(2012 + \log(2011))$ is nearly indistinguishable from $\log(2012)$.

So we take $f_1(x)$ and plug in.

$$f(2013) \approx \log(2013 + \log 2012)$$

Since $1000 < 2012 < 10000$, we know $3 < log(2012) < 4$. This gives us our answer range:

$$\log 2016 < \log(2013 + \log 2012) < \log(2017)$$

$$(\log 2016, \log 2017)$$

Solution 2

Suppose $A = \log(x)$. Then $\log(2012 + \cdots) = x - 2013$. So if $x > 2017$, then $\log(2012 + log(2011 + \cdots)) > 4$. So $2012 + log(2011 + \cdots) > 10000$. Repeating, we then get $2011 + log(2010 + \cdots) > 10^{7988}$. This is clearly absurd (the RHS continues to grow more than exponentially for each iteration). So, x is not greater than 2017. So $A < \log(2017)$. But this leaves only one answer, so we are done.

See Also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12A Problems/Problem 22

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- 3 Solution 2
- 4 See also

Problem

A palindrome is a nonnegative integer number that reads the same forwards and backwards when written in base 10 with no leading zeros. A 6-digit palindrome n is chosen uniformly at random. What is the probability that $\frac{n}{11}$ is also a palindrome?

$\textbf{(A)}\ \frac{8}{25} \qquad \textbf{(B)}\ \frac{33}{100} \qquad \textbf{(C)}\ \frac{7}{20} \qquad \textbf{(D)}\ \frac{9}{25} \qquad \textbf{(E)}\ \frac{11}{30}$

Solution 1

Working backwards, we can multiply 5-digit palindromes $ABCBA$ by 11, giving a 6-digit palindrome:

$$A(A+B)(B+C)(B+C)(A+B)A$$

Note that if $A+B>10$ or $B+C>10$, then the symmetry will be broken by carried 1s

Simply count the combinations of (A,B,C) for which $A+B<10$ and $B+C<10$

$A=1$ implies 9 possible B (0 through 8), for each of which there are $10,9,8,7,6,5,4,3,2$ possible C, respectively. There are 54 valid palindromes when $A=1$

$A=2$ implies 8 possible B (0 through 7), for each of which there are $10,9,8,7,6,5,4,3$ possible C, respectively. There are 52 valid palindromes when $A=2$

Following this pattern, the total is

$$54+52+49+45+40+34+27+19+10=330$$

6-digit palindromes are of the form $XYZZYX$, and the first digit cannot be a zero, so there are $9*10*10=900$ combinations of (X,Y,Z)

So, the probability is $\frac{330}{900}=\frac{11}{30}$

Solution 2

Let the palindrome be the form in the previous solution which is $XYZZYX$. It doesn't matter what Z is because it only affects the middle digit. There are 90 ways to pick X and Y, and the only answer choice with denominator a factor of 90 is $\boxed{\textbf{(E)}\ \frac{11}{30}}$.

See also

2013 AMC 12A Problems/Problem 23

Problem

$ABCD$ is a square of side length $\sqrt{3}+1$. Point P is on $\overline{AC}$ such that $AP=\sqrt{2}$. The square region bounded by $ABCD$ is rotated 90° counterclockwise with center P, sweeping out a region whose area is $\frac{1}{c}(a\pi+b)$, where a, b, and c are positive integers and $\text{gcd}(a,b,c)=1$. What is $a+b+c$?

$\textbf{(A)}\ 15 \qquad \textbf{(B)}\ 17 \qquad \textbf{(C)}\ 19 \qquad \textbf{(D)}\ 21 \qquad \textbf{(E)}\ 23$

Solution

We first note that diagonal $\overline{AC}$ is of length $\sqrt{6}+\sqrt{2}$. It must be that $\overline{AP}$ divides the diagonal into two segments in the ratio $\sqrt{3}$ to 1. It is not difficult to visualize that when the square is rotated, the initial and final squares overlap in a rectangular region of dimensions 2 by $\sqrt{3}+1$. The area of the overall region (of the initial and final squares) is therefore twice the area of the original square minus the overlap, or $2(\sqrt{3}+1)^2-2(\sqrt{3}+1)=2(4+2\sqrt{3})-2\sqrt{3}-2=6+2\sqrt{3}$.

The area also includes 4 circular segments. Two are quarter-circles centered at P of radii $\sqrt{2}$ (the segment bounded by $\overline{PA}$ and $\overline{PA'}$) and $\sqrt{6}$ (that bounded by $\overline{PC}$ and $\overline{PC'}$). Assuming A is the bottom-left vertex and B is the bottom-right one, it is clear that the third segment is formed as B swings out to the right of the original square [recall that the square is rotated counterclockwise], while the fourth is formed when D overshoots the final square's left edge. To find these areas, consider the perpendicular from P to $\overline{BC}$. Call the point of intersection E. From the previous paragraph, it is clear that $PE=\sqrt{3}$ and $BE=1$. This means $PB=2$, and B swings back inside edge $\overline{BC}$ at a point 1 unit above E (since it left the edge 1 unit below). The triangle of the circular sector is therefore an equilateral triangle of side length 2, and so the angle of the segment is 60°. Imagining the process in reverse, it is clear that the situation is the same with point D.

The area of the segments can be found by subtracting the area of the triangle from that of the sector; it follows that the two quarter-segments have areas $\frac{1}{4}\pi(\sqrt{2})^2-\frac{1}{2}\sqrt{2}\sqrt{2}=\frac{\pi}{2}-1$ and $\frac{1}{4}\pi(\sqrt{6})^2-\frac{1}{2}\sqrt{6}\sqrt{6}=\frac{3\pi}{2}-3$. The other two segments both have area $\frac{1}{6}\pi(2)^2-\frac{(2)^2\sqrt{3}}{4}=\frac{2\pi}{3}-\sqrt{3}$.

The total area is therefore

$$(6+2\sqrt{3})+(\frac{\pi}{2}-1)+(\frac{3\pi}{2}-3)+2(\frac{2\pi}{3}-\sqrt{3})$$

$$=2+2\sqrt{3}+2\pi+\frac{4\pi}{3}-2\sqrt{3}$$

$$=\frac{10\pi}{3}+2$$

$$= \frac{1}{3}(10\pi + 6)$$

Since $a = 10$, $b = 6$, and $c = 3$, the answer is $a + b + c = 10 + 6 + 3 = \boxed{\textbf{(C) } 19}$.

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12A Problems/Problem 24

Problem

Three distinct segments are chosen at random among the segments whose end-points are the vertices of a regular 12-gon. What is the probability that the lengths of these three segments are the three side lengths of a triangle with positive area?

$\textbf{(A)}\ \frac{553}{715} \qquad \textbf{(B)}\ \frac{443}{572} \qquad \textbf{(C)}\ \frac{111}{143} \qquad \textbf{(D)}\ \frac{81}{104} \qquad \textbf{(E)}\ \frac{223}{286}$

Solution

Suppose p is the answer. We calculate $1-p$.

Assume that the circumradius of the 12-gon is 1, and the 6 different lengths are a_1, a_2, $\cdots$, a_6, in increasing order. Then

$$a_k = 2\sin(\frac{k\pi}{12}).$$

So $a_1 = (\sqrt{6}-\sqrt{2})/2 \approx 0.5$,

$a_2 = 1$,

$a_3 = \sqrt{2} \approx 1.4$,

$a_4 = \sqrt{3} \approx 1.7$,

$a_5 = (\sqrt{6}+\sqrt{2})/2 = a_1 + a_3$,

$a_6 = 2$.

Now, Consider the following inequalities:

- $a_1 + a_1 > a_2$: Since $6 > (1+\sqrt{2})^2$

- $a_1 + a_1 < a_3$.

- $a_1 + a_2$ is greater than a_3 but less than a_4.

- $a_1 + a_3$ is greater than a_4 but equal to a_5.

- $a_1 + a_4$ is greater than a_6.

- $a_2 + a_2 = 2 = a_6$. Then obviously any two segments with at least one them longer than a_2 have a sum greater than a_6.

Therefore, all triples (in increasing order) that can't be the side lengths of a triangle are the following. Note that x-y-z means (a_x, a_y, a_z):

```
1-1-3, 1-1-4, 1-1-5, 1-1-6,
1-2-4, 1-2-5, 1-2-6,
1-3-5, 1-3-6,
2-2-6
```

Note that there are 12 segments of each length of a_1, a_2, $\cdots$, a_5, respectively, and 6 segments of length a_6. There are 66 segments in total.

In the above list there are 3 triples of the type a-a-b without 6, 2 triples of a-a-6 where a is not 6, 3 triples of a-b-c without 6, and 2 triples of a-b-6 where a, b are not 6. So,

$$1-p=\frac{1}{66\cdot 65\cdot 64}(3\cdot 3\cdot 12\cdot 11\cdot 12+2\cdot 3\cdot 12\cdot 11\cdot 6+3\cdot 6\cdot 12^3+2\cdot 6\cdot 12^2\cdot 6)$$

$$=\frac{1}{66\cdot 65\cdot 64}(12^2(99+33)+12^3(18+6))=\frac{1}{66\cdot 65\cdot 64}(12^3\cdot 35)=\frac{63}{286}$$

So $p=223/286$.

See also

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2013 AMC 12A Problems/Problem 25

Problem

Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^2 + iz + 1$. How many complex numbers z are there such that $\text{Im}(z) > 0$ and both the real and the imaginary parts of $f(z)$ are integers with absolute value at most 10?

$\textbf{(A)}\ 399 \qquad \textbf{(B)}\ 401 \qquad \textbf{(C)}\ 413 \qquad \textbf{(D)}\ 431 \qquad \textbf{(E)}\ 441$

Solution

Suppose $f(z)=z^2+iz+1=c=a+bi$. We look for z with $\text{Im}(z)>0$ such that a,b are integers where $|a|, |b|\leq 10$.

First, use the quadratic formula:

$$z = \frac{1}{2} (-i \pm \sqrt{-1-4(1-c)}) = -\frac{i}{2} \pm \sqrt{ -\frac{5}{4} + c }$$

Generally, consider the imaginary part of a radical of a complex number: $\sqrt{u}$, where $u = v+wi = r e^{i\theta}$.

$$\Im (\sqrt{u}) = \Im(\pm \sqrt{r} e^{i\theta/2}) = \pm \sqrt{r} \sin(\theta/2) = \pm \sqrt{r}\sqrt{\frac{1-\cos\theta}{2}} = \pm \sqrt{\frac{r-v}{2}}.$$

Now let $u= -5/4 + c$, then $v = -5/4 + a$, $w=b$, $r=\sqrt{v^2 + w^2}$.

Note that $\Im(z)>0$ if and only if $\pm \sqrt{\frac{r-v}{2}}>\frac{1}{2}$. The latter is true only when we take the positive sign, and that $r-v > 1/2$,

or $v^2 + w^2 > (1/2 + v)^2 = 1/4 + v + v^2$, $w^2 > 1/4 + v$, or $b^2 > a-1$.

In other words, when $b^2 > a-1$, the equation $f(z)=a+bi$ has unique solution z in the region $\Im(z)>0$; and when $b^2 \leq a-1$ there is no solution. Therefore the number of desired solution z is the same as the number of ordered pairs (a,b) such that integers $|a|, |b| \leq 10$, and that $b^2 \geq a$.

When $a\leq 0$, there is no restriction on b so there are $11\cdot 21 = 231$ pairs;

when $a > 0$, there are $2(1+4+9+10+10+10+10+10+10+10)=2(84)=168$ pairs.

So there are $231+168=399$ in total.

See also

2013 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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