2014 AMC 12B Problems/Problem 1

Problem

Leah has 13 coins, all of which are pennies and nickels. If she had one more nickel than she has now, then she would have the same number of pennies and nickels. In cents, how much are Leah's coins worth?

$\textbf{(A)}\ 33\qquad\textbf{(B)}\ 35\qquad\textbf{(C)}\ 37\qquad\textbf{(D)}\ 39\qquad\textbf{(E)}\ 41$

Solution

She has p pennies and n nickels, where $n+p=13$. If she had $n+1$ nickels then $n+1=p$, so $2n+1=13$ and $n=6$. So she has 6 nickels and 7 pennies, which clearly have a value of $\boxed{\textbf{(C)}\ 37}$ cents.

See also

<table>
<tr><td colspan="2">2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))</td></tr>
<tr><td>Preceded by
First Problem</td><td>Followed by
Problem 2</td></tr>
<tr><td colspan="2">1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25</td></tr>
<tr><td colspan="2">All AMC 12 Problems and Solutions</td></tr>
</table>

2014 AMC 12B Problems/Problem 2

Problem

Orvin went to the store with just enough money to buy 30 balloons. When he arrived he discovered that the store had a special sale on balloons: buy 1 balloon at the regular price and get a second at $\frac{1}{3}$ off the regular price. What is the greatest number of balloons Orvin could buy?

$\textbf{(A)}\ 33 \qquad \textbf{(B)}\ 34 \qquad \textbf{(C)}\ 36 \qquad \textbf{(D)}\ 38 \qquad \textbf{(E)}\ 39$

Solution

If every balloon costs n dollars, then Orvin has $30n$ dollars. For every balloon he buys for n dollars, he can buy another for $\frac{2n}{3}$ dollars. This means it costs him $\frac{5n}{3}$ dollars to buy a bundle of 2 balloons. With $30n$ dollars, he can buy $\frac{30n}{\frac{5n}{3}} = 18$ sets of two balloons, so the total number of balloons he can buy is $18 \times 2 \implies \boxed{\textbf{(C)}\ 36}$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 3

Problem

Randy drove the first third of his trip on a gravel road, the next 20 miles on pavement, and the remaining one-fifth on a dirt road. In miles, how long was Randy's trip?

$\textbf{(A)}\ 30\qquad\textbf{(B)}\ \frac{400}{11}\qquad\textbf{(C)}\ \frac{75}{2}\qquad\textbf{(D)}\ 40\qquad\textbf{(E)}\ \frac{300}{7}$

Solution

If the first and last legs of his trip account for $\frac{1}{3}$ and $\frac{1}{5}$ of his trip, then the middle leg accounts for $1 - \frac{1}{3} - \frac{1}{5} = \frac{7}{15}$ths of his trip. This is equal to 20 miles. Letting the length of the entire trip equal x, we have

$$\frac{7}{15}x = 20 \implies x = \boxed{\textbf{(E)}\ \frac{300}{7}}$$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 4

Problem

Susie pays for 4 muffins and 3 bananas. Calvin spends twice as much paying for 2 muffins and 16 bananas. A muffin is how many times as expensive as a banana?

$\textbf{(A)}\ \frac{3}{2}\qquad\textbf{(B)}\ \frac{5}{3}\qquad\textbf{(C)}\ \frac{7}{4}\qquad\textbf{(D)}\ 2\qquad\textbf{(E)}\ \frac{13}{4}$

Solution

Let m stand for the cost of a muffin, and let b stand for the value of a banana. We we need to find $\frac{m}{b}$, the ratio of the price of the muffins to that of the bananas. We have

$$2(4m+3b)=2m+16b$$

$$6m=10b$$

$$\frac{m}{b}=\boxed{\textbf{(B)}\ \frac{5}{3}}$$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 5

Problem

Doug constructs a square window using 8 equal-size panes of glass, as shown. The ratio of the height to width for each pane is $5:2$, and the borders around and between the panes are 2 inches wide. In inches, what is the side length of the square window?

$\textbf{(A)}\ 26\qquad\textbf{(B)}\ 28\qquad\textbf{(C)}\ 30\qquad\textbf{(D)}\ 32\qquad\textbf{(E)}\ 34$

Solution

Let the height of the panes equal $5x$, and let the width of the panes equal $2x$. Now notice that the total width of the borders equals 10, and the total height of the borders is 6. We have

$$10+4(2x)=6+2(5x)$$

$$x=2$$

Now, the total side length of the window equals

$$10+4(2x)=10+16=\boxed{\textbf{(A)}\ 26}$$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 6

Problem

Ed and Ann both have lemonade with their lunch. Ed orders the regular size. Ann gets the large lemonade, which is 50% more than the regular. After both consume $\frac{3}{4}$ of their drinks, Ann gives Ed a third of what she has left, and 2 additional ounces. When they finish their lemonades they realize that they both drank the same amount. How many ounces of lemonade did they drink together?

$\textbf{(A)}\ 30\qquad\textbf{(B)}\ 32\qquad\textbf{(C)}\ 36\qquad\textbf{(D)}\ 40\qquad\textbf{(E)}\ 50$

Solution

Let the size of Ed's drink equal x ounces, and let the size of Ann's drink equal $\frac{3}{2}x$ ounces. After both consume $\frac{3}{4}$ of their drinks, Ed and Ann have $\frac{x}{4}$ and $\frac{3x}{8}$ ounces of their drinks remaining. Ann gives away $\frac{x}{8}+2$ ounces to Ed.

In the end, Ed drank everything in his original lemonade plus what Ann gave him, and Ann drank everything in her original lemonade minus what she gave Ed. Thus we have

$$x+\frac{x}{8}+2=\frac{3x}{2}-\frac{x}{8}-2$$

$$x=16$$

The total amount the two of them drank is simply

$$x+\frac{3}{2}x=16+24=\boxed{\textbf{(D)}\ 40}$$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 7

Contents

Problem

For how many positive integers n is $\frac{n}{30-n}$ also a positive integer?

$\textbf{(A)}\ 4\qquad\textbf{(B)}\ 5\qquad\textbf{(C)}\ 6\qquad\textbf{(D)}\ 7\qquad\textbf{(E)}\ 8$

Solutions

Solution 1

We know that $n \leq 30$ or else $30-n$ will be negative, resulting in a negative fraction. We also know that $n \geq 15$ or else the fraction's denominator will exceed its numerator making the fraction unable to equal a positive integer value. Substituting all values n from 15 to 30 gives us integer values for $n = 15, 20, 24, 25, 27, 28, 29$. Counting them up, we have $\boxed{\textbf{(D)}\ 7}$ possible values for n.

Solution 2

Let $\frac{n}{30-n} = m$, where $m \in \mathbb{N}$. Solving for n, we find that $n = \frac{30m}{m+1}$. Because m and $m+1$ are relatively prime, $m+1|30$. Our answer is the number of proper divisors of $2^1 3^1 5^1$, which is $(1+1)(1+1)(1+1) - 1 = \boxed{\textbf{(D)}\ 7}$.

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 8

Problem

In the addition shown below A, B, C, and D are distinct digits. How many different values are possible for D?

$$\begin{array}{cccccc} & \mathrm{A} & \mathrm{B} & \mathrm{B} & \mathrm{C} & \mathrm{B} \\ + & \mathrm{B} & \mathrm{C} & \mathrm{A} & \mathrm{D} & \mathrm{A} \\ \hline & \mathrm{D} & \mathrm{B} & \mathrm{D} & \mathrm{D} & \mathrm{D} \end{array}$$

$\textbf{(A)}\ 2\qquad\textbf{(B)}\ 4\qquad\textbf{(C)}\ 7\qquad\textbf{(D)}\ 8\qquad\textbf{(E)}\ 9$

Solution

From the first column, we see $A+B<10$ because it yields a single digit answer. From the fourth column, we see that $C+D$ equals D and therefore $C=0$. We know that $A+B=D$. Therefore, the number of values D can take is equal to the number of possible sums less than 10 that can be formed by adding two distinct natural numbers. Letting $A=1$, and letting $B=2,3,4,5,6,7,8$, we have

$$D=3,4,5,6,7,8,9 \implies \boxed{\textbf{(C)}\ 7}$$

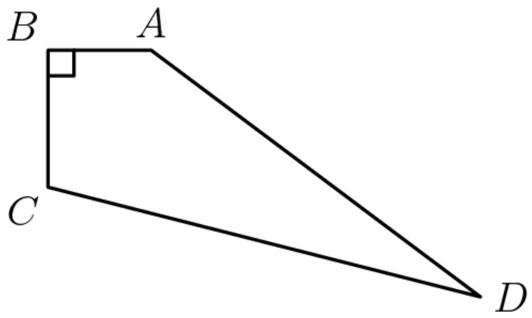
See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 7	Followed by Problem 9
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 9

Problem

Convex quadrilateral $ABCD$ has $AB = 3$, $BC = 4$, $CD = 13$, $AD = 12$, and $\angle ABC = 90^\circ$, as shown. What is the area of the quadrilateral?



$\textbf{(A)}\ 30 \qquad \textbf{(B)}\ 36 \qquad \textbf{(C)}\ 40 \qquad \textbf{(D)}\ 48 \qquad \textbf{(E)}\ 58.5$

Solution

Note that by the pythagorean theorem, $AC = 5$. Also note that $\angle CAD$ is a right angle because $\triangle CAD$ is a right triangle. The area of the quadrilateral is the sum of the areas of $\triangle ABC$ and $\triangle CAD$ which is equal to

$$\frac{3 \times 4}{2} + \frac{5 \times 12}{2} = 6 + 30 = \boxed{\textbf{(B)}\ 36}$$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2014 AMC 12B Problems/Problem 10

Contents

Problem

Danica drove her new car on a trip for a whole number of hours, averaging 55 miles per hour. At the beginning of the trip, abc miles was displayed on the odometer, where abc is a 3-digit number with $a \geq 1$ and $a+b+c \leq 7$. At the end of the trip, the odometer showed cba miles. What is $a^2+b^2+c^2?$.

$\textbf{(A)}\ 26\qquad\textbf{(B)}\ 27\qquad\textbf{(C)}\ 36\qquad\textbf{(D)}\ 37\qquad\textbf{(E)}\ 41$

Solution 1

We know that the number of miles she drove is divisible by 5, so a and c must either be the equal or differ by 5. We can quickly conclude that the former is impossible, so a and c must be 5 apart. Because we know that $c > a$ and $a + c \leq 7$ and $a \geq 1$, we find that the only possible values for a and c are 1 and 6, respectively. Because $a + b + c \leq 7$, $b = 0$. Therefore, we have

$$a^2 + b^2 + c^2 = 36 + 0 + 1 = \boxed{\textbf{(D) } 37}$$

Solution 2

Let the number of hours Danica drove be k. Then we know that $100a + 10b + c + 55k$ = $100c + 10b + a$. Simplifying, we have $99c - 99a = 55k$, or $9c - 9a = 5k$. Thus, k is divisible by 9. Because $55 * 18 = 990$, k must be 9, and therefore $c - a = 5$. Because $a + b + c \leq 7$ and $a \geq 1$, $a = 1$, $c = 6$ and $b = 0$, and our answer is $a^2 + b^2 + c^2 = 6^2 + 0^2 + 1^2 = 37$, or $\boxed{D}$.

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 11

Problem

A list of 11 positive integers has a mean of 10, a median of 9, and a unique mode of 8. What is the largest possible value of an integer in the list?

$\textbf{(A)}\ 24\qquad\textbf{(B)}\ 30\qquad\textbf{(C)}\ 31\qquad\textbf{(D)}\ 33\qquad\textbf{(E)}\ 35$

Solution

We start off with the fact that the median is 9, so we must have $a, b, c, d, e, 9, f, g, h, i, j$, listed in ascending order. Note that the integers do not have to be distinct.

Since the mode is 8, we have to have at least 2 occurrences of 8 in the list. If there are 2 occurrences of 8 in the list, we will have $a, b, c, 8, 8, 9, f, g, h, i, j$. In this case, since 8 is the unique mode, the rest of the integers have to be distinct. So we minimize a, b, c, f, g, h, i in order to maximize j. If we let the list be $1, 2, 3, 8, 8, 9, 10, 11, 12, 13, j$, then $j = 11 \times 10 - (1+2+3+8+8+9+10+11+12+13) = 33$.

Next, consider the case where there are 3 occurrences of 8 in the list. Now, we can have two occurrences of another integer in the list. We try $1, 1, 8, 8, 8, 9, 9, 10, 10, 11, j$. Following the same process as above, we get $j = 11 \times 10 - (1+1+8+8+8+9+9+10+10+11) = 35$. As this is the highest choice in the list, we know this is our answer. Therefore, the answer is $\boxed{\textbf{(E)}\ 35}$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 12

Problem

A set S consists of triangles whose sides have integer lengths less than 5, and no two elements of S are congruent or similar. What is the largest number of elements that S can have?

$\textbf{(A)}\ 8 \qquad\textbf{(B)}\ 9 \qquad\textbf{(C)}\ 10 \qquad\textbf{(D)}\ 11 \qquad\textbf{(E)}\ 12$

Solution

Define T to be the set of all integral triples (a, b, c) such that $a \ge b \ge c$, $b+c > a$, and $a, b, c < 5$. Now we enumerate the elements of T:

$(4, 4, 4)$

$(4, 4, 3)$

$(4, 4, 2)$

$(4, 4, 1)$

$(4, 3, 3)$

$(4, 3, 2)$

$(3, 3, 3)$

$(3, 3, 2)$

$(3, 3, 1)$

$(3, 2, 2)$

$(2, 2, 2)$

$(2, 2, 1)$

$(1, 1, 1)$

It should be clear that $|S|$ is simply $|T|$ minus the larger "duplicates" (e.g. $(2, 2, 2)$ is a larger duplicate of $(1, 1, 1)$). Since $|T|$ is 13 and the number of higher duplicates is 4, the answer is $13-4$ or $\boxed{\textbf{(B)}\ 9}$.

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 13

Problem

Real numbers a and b are chosen with $1 < a < b$ such that no triangle with positive area has side lengths $1, a,$ and b or $\frac{1}{b}, \frac{1}{a},$ and 1. What is the smallest possible value of b?

$\textbf{(A)}\ \frac{3+\sqrt{3}}{2} \qquad \textbf{(B)}\ \frac{5}{2} \qquad \textbf{(C)}\ \frac{3+\sqrt{5}}{2} \qquad \textbf{(D)}\ \frac{3+\sqrt{6}}{2} \qquad \textbf{(E)}\ 3$

Solution

Notice that $1 > \frac{1}{a} > \frac{1}{b}$. Using the triangle inequality, we find

$$a+1 > b \implies a > b-1$$

$$\frac{1}{a}+\frac{1}{b} > 1$$

In order for us the find the lowest possible value for b, we try to create two degenerate triangles where the sum of the smallest two sides equals the largest side. Thus we get

$$a = b-1$$

and

$$\frac{1}{a}+\frac{1}{b} = 1$$

Substituting, we get

$$\frac{1}{b-1}+\frac{1}{b} = \frac{b+b-1}{b(b-1)} = 1$$

$$\frac{2b-1}{b(b-1)} = 1$$

$$2b-1 = b^2-b$$

Solving for b using the quadratic equation, we get

$$b^2-3b+1 = 0 \implies b = \boxed{\textbf{(C)}\ \frac{3+\sqrt{5}}{2}}$$

See also

2014 AMC 12B Problems/Problem 14

Problem 14

A rectangular box has a total surface area of 94 square inches. The sum of the lengths of all its edges is 48 inches. What is the sum of the lengths in inches of all of its interior diagonals?

$\textbf{(A)}\ 8\sqrt{3}\qquad\textbf{(B)}\ 10\sqrt{2}\qquad\textbf{(C)}\ 16\sqrt{3}\qquad\textbf{(D)}\ 20\sqrt{2}\qquad\textbf{(E)}\ 40\sqrt{2}$

Solution

Let the side lengths of the rectangular box be x, y and z. From the information we get

$$4(x+y+z)=48 \Rightarrow x+y+z=12$$

$$2(xy+yz+xz)=94$$

The sum of all the lengths of the box's interior diagonals is

$$4\sqrt{x^2+y^2+z^2}$$

Squaring the first expression, we get:

$$144=x^2+y^2+z^2+94$$

Hence

$$x^2+y^2+z^2=50$$

$$4\sqrt{x^2+y^2+z^2}=\boxed{\textbf{(D)}\ 20\sqrt{2}}$$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 15

Problem

When $p = \sum_{k=1}^{6} k \ln k$, the number e^p is an integer. What is the largest power of 2 that is a factor of e^p?

$\textbf{(A)}\ 2^{12} \qquad \textbf{(B)}\ 2^{14} \qquad \textbf{(C)}\ 2^{16} \qquad \textbf{(D)}\ 2^{18} \qquad \textbf{(E)}\ 2^{20}$

Solution

Let's write out the sum. Our sum is equal to

$$1 \ln 1 + 2 \ln 2 + 3 \ln 3 + 4 \ln 4 + 5 \ln 5 + 6 \ln 6 =$$

$$\ln 1^1 + \ln 2^2 + \ln 3^3 + \ln 4^4 + \ln 5^5 + \ln 6^6 =$$

$$\ln \left(1^1 \times 2^2 \times 3^3 \times 4^4 \times 5^5 \times 6^6\right)$$

Raising e to the power of this quantity eliminates the natural logarithm, which leaves us with

$$e^p = 1^1 \times 2^2 \times 3^3 \times 4^4 \times 5^5 \times 6^6$$

This product has 2 powers of 2 in the 2^2 factor, $4 * 2 = 8$ powers of 2 in the 4^4 factor, and 6 powers of 2 in the 6^6 factor. This adds up to $2 + 8 + 6 = 16$ powers of two which divide into our quantity, so our answer is $\boxed{\textbf{(C)}\ 2^{16}}$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 16

Problem

Let P be a cubic polynomial with $P(0) = k$, $P(1) = 2k$, and $P(-1) = 3k$. What is $P(2) + P(-2)$?

$\textbf{(A)}\ 0\qquad\textbf{(B)}\ k\qquad\textbf{(C)}\ 6k\qquad\textbf{(D)}\ 7k\qquad\textbf{(E)}\ 14k$

Solution

Let $P(x) = Ax^3 + Bx^2 + Cx + D$. Plugging in 0 for x, we find $D = k$, and plugging in 1 and -1 for x, we obtain the following equations:

$$A + B + C + k = 2k$$

$$-A + B - C + k = 3k$$

Adding these two equations together, we get

$$2B = 3k$$

If we plug in 2 and -2 in for x, we find that

$$P(2)+P(-2) = 8A+4B+2C+k+(-8A+4B-2C+k) = 8B+2k$$

Multiplying the third equation by 4 and adding $2k$ gives us our desired result, so

$$P(2) + P(-2) = 12k + 2k = \boxed{\textbf{(E)}\ 14k}$$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 15	Followed by Problem 17
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 17

Contents

Problem

Let P be the parabola with equation $y = x^2$ and let $Q = (20, 14)$. There are real numbers r and s such that the line through Q with slope m does not intersect P if and only if $r < m < s$. What is $r + s$?

$\textbf{(A) } 1 \qquad \textbf{(B) } 26 \qquad \textbf{(C) } 40 \qquad \textbf{(D) } 52 \qquad \textbf{(E) } 80$

Solution (Algebra Based)

Let $y = m(x-20) + 14$. Equating them:

$x^2 = mx - 20m + 14$

$x^2 - mx + 20m - 14 = 0$

For there to be no solutions, the discriminant must be less than zero:

$m^2 - 4(20m - 14) < 0$

$m^2 - 80m + 56 < 0.$

So $m < 0$ for $r < m < s$ where r and s are the roots of $m^2 - 80m + 56 = 0$ and their sum by Vieta's formulas is $\boxed{\textbf{(E) } 80}$.

Solution (Calculus-based)

The line will begin to intercept the parabola when its slope equals that of the parabola at the point of tangency. Taking the derivative of the equation of the parabola, we get that the slope equals $2x$. Using the slope formula, we find that the slope of the tangent line to the parabola also equals $\frac{14 - x^2}{20 - x}$. Setting these two equal to each other, we get

$$2x = \frac{14 - x^2}{20 - x} \implies x^2 - 40x + 14 = 0$$

Solving for x, we get

$$x = 20 \pm \sqrt{386}$$

The sum of the two possible values for x where the line is tangent to the parabola is 40, and the sum of the slopes of these two tangent lines is equal to $2x$, or $\boxed{\textbf{(E) } 80}$.

See also

2014 AMC 12B Problems/Problem 18

The following problem is from both the 2014 AMC 12B #18 and 2014 AMC 10B #24, so both problems redirect to this page.

Problem

The numbers 1, 2, 3, 4, 5 are to be arranged in a circle. An arrangement is bad if it is not true that for every n from 1 to 15 one can find a subset of the numbers that appear consecutively on the circle that sum to n. Arrangements that differ only by a rotation or a reflection are considered the same. How many different bad arrangements are there?

$\textbf{(A)}\ 1 \qquad\textbf{(B)}\ 2 \qquad\textbf{(C)}\ 3 \qquad\textbf{(D)}\ 4 \qquad\textbf{(E)}\ 5$

Solution

We see that there are $5!$ total ways to arrange the numbers. However, we can always rotate these numbers so that, for example, the number 1 is always at the top of the circle. Thus, there are only $4!$ ways under rotation, which is not difficult to list out. We systematically list out all 24 cases.

Now, we must examine if they satisfy the conditions. We can see that by choosing one number at a time, we can always obtain subsets with sums 1, 2, 3, 4, and 5. By choosing the full circle, we can obtain 15. By choosing everything except for 1, 2, 3, 4, and 5, we can obtain subsets with sums of 10, 11, 12, 13, and 14.

This means that we now only need to check for 6, 7, 8, and 9. However, once we have found a set summing to 6, we can choose everything else and obtain a set summing to 9, and similarly for 7 and 8. Thus, we only need to check each case for whether or not we can obtain 6 or 7.

We find that there are only 4 arrangements that satisfy these conditions. However, each of these is a reflection of another. We divide by 2 for these reflections to obtain a final answer of $\boxed{\textbf{(B)}\ 2}$.

See Also

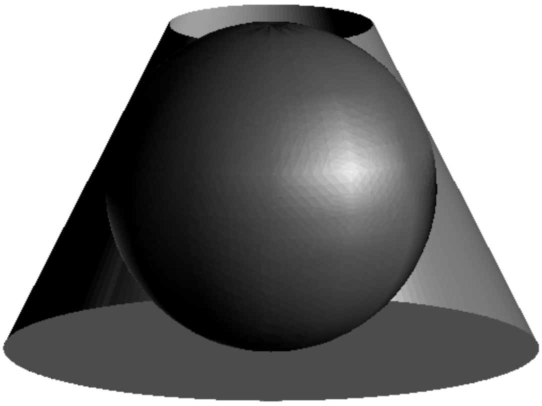
2014 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2014))	
Preceded by Problem 23	Followed by Problem 25
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 19

Problem

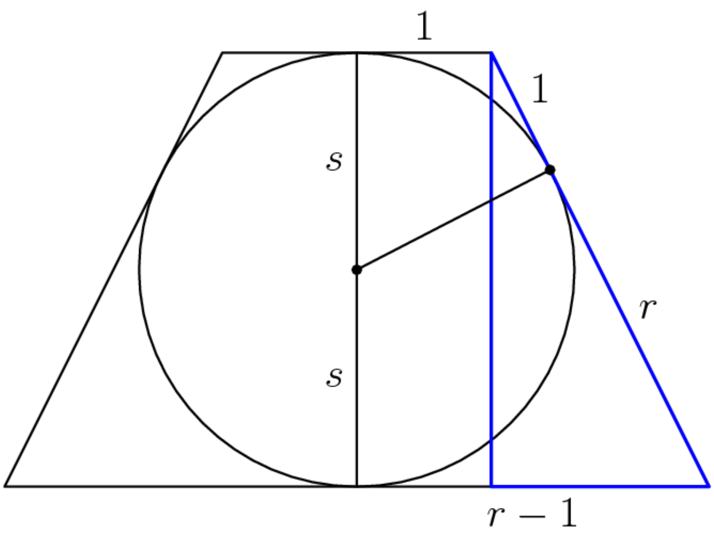
A sphere is inscribed in a truncated right circular cone as shown. The volume of the truncated cone is twice that of the sphere. What is the ratio of the radius of the bottom base of the truncated cone to the radius of the top base of the truncated cone?



$\text{(A) } \dfrac32 \quad \text{(B) } \dfrac{1+\sqrt5}2 \quad \text{(C) } \sqrt3 \quad \text{(D) } 2 \quad \text{(E) } \dfrac{3+\sqrt5}2$

Solution

First, we draw the vertical cross-section passing through the middle of the frustum. let the top base equal 2 and the bottom base to be equal to 2r



then using the Pythagorean theorem we have: $(r+1)^2 = (2s)^2 + (r-1)^2$ which is equivalent to: $r^2+2r+1 = 4s^2+r^2-2r+1$ subtracting r^2-2r+1 from both sides $4r = 4s^2$ solving for s we get:

$$s = \sqrt{r}$$

next we can find the area of the frustum and of the sphere and we know $V_{frustum} = 2V_{sphere}$ so we can solve for s using $V_{frustum} = \dfrac{\pi * h}{3}(R^2+r^2+Rr)$ we get:

$$V_{frustum} = \frac{\pi * 2\sqrt{r}}{3}(r^2+r+1)$$

using $V_{sphere} = \dfrac{4r^3\pi}{3}$ we get

$$V_{sphere} = \frac{4(\sqrt{r})^3\pi}{3}$$

so we have:

$$\frac{\pi * 2\sqrt{r}}{3}(r^2+r+1) = 2 * \frac{4(\sqrt{r})^3\pi}{3}$$

dividing by $\dfrac{2\pi * \sqrt{r}}{3}$ we get

$$r^2+r+1=4r$$

which is equivalent to

$$r^2-3r+1=0$$

$r = \dfrac{3 \pm \sqrt{(-3)^2-4*1*1}}{2*1}$ so

$$r = \frac{3+\sqrt{5}}{2} \longrightarrow \boxed{E}$$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 20

Problem

For how many positive integers x is $\log_{10}(x-40) + \log_{10}(60-x) < 2$?

$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 18 \qquad \textbf{(C)}\ 19 \qquad \textbf{(D)}\ 20 \qquad \textbf{(E)}\ \text{infinitely many}$

Solution

The domain of the LHS implies that

$$40 < x < 60$$

Begin from the left hand side

$$\log_{10}[(x-40)(60-x)] < 2$$

$$-x^2 + 100x - 2500 < 0$$

$$(x-50)^2 > 0$$

$$x \neq 50$$

Hence, we have integers from 41 to 49 and 51 to 59. There are $\boxed{\textbf{(B)} 18}$ integers.

See also

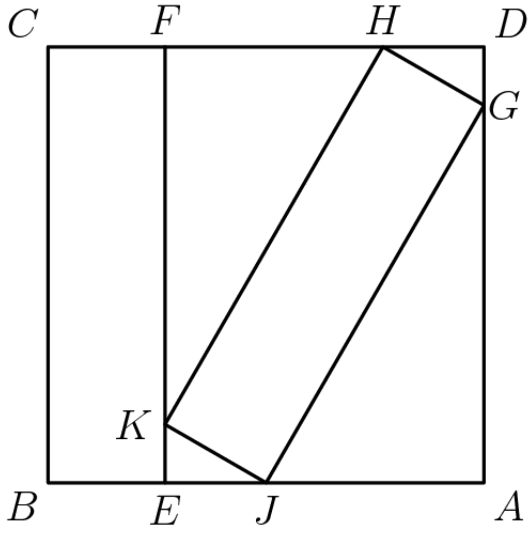
2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 21

Contents

Problem 21

In the figure, $ABCD$ is a square of side length 1. The rectangles $JKHG$ and $EBCF$ are congruent. What is BE?



$\textbf{(A) }\frac{1}{2}(\sqrt{6}-2)\qquad\textbf{(B) }\frac{1}{4}\qquad\textbf{(C) }2-\sqrt{3}\qquad\textbf{(D) }\frac{\sqrt{3}}{6}\qquad\textbf{(E) } 1-\frac{\sqrt{2}}{2}$

Solution 1

Draw the altitude from H to AB and call the foot L. Then $HL=1$. Consider HJ. It is the hypotenuse of both right triangles $\triangle HGJ$ and $\triangle HLJ$, and we know $JG=HL=1$, so we must have $\triangle HGJ\cong\triangle JLH$ by Hypotenuse-Leg congruence. From this congruence we have $LJ=HG=BE$.

Notice that all four triangles in this picture are similar. Also, we have $LA=HD=EJ$. So set $x=LJ=HG=BE$ and $y=LA=HD=EJ$. Now $BE+EJ+JL+LA=2(x+y)=1$. This means $x+y=\dfrac{1}{2}=BE+EJ=BJ$, so J is the midpoint of AB. So $\triangle AJG$, along with all other similar triangles in the picture, is a 30-60-90 triangle, and we have $AG=\sqrt{3}\cdot AJ=\sqrt{3}/2$ and subsequently $GD=\dfrac{2-\sqrt{3}}{2}=KE$. This means $EJ=\sqrt{3}\cdot KE=\dfrac{2\sqrt{3}-3}{2}$, which gives $BE=\dfrac{1}{2}-EJ=\dfrac{4-2\sqrt{3}}{2}=2-\sqrt{3}$, so the answer is $\textbf{(C)}$.

Solution 2

Let $BE = x$. Let $JA = y$. Because $\angle FKH = \angle EJK = \angle AGJ = \angle DHG$ and $\angle FHK = \angle EKJ = \angle AJG = \angle DGH$, $\triangle KEJ, \triangle JAG, \triangle GDH, \triangle HFK$ are all similar. Using proportions and the pythagorean theorem, we find

$$EK = xy$$

$$FK = \sqrt{1-y^2}$$

$$EJ = x\sqrt{1-y^2}$$

Because we know that $BE + EJ + AJ = EK + FK = 1$, we can set up a systems of equations

$$x + x\sqrt{1-y^2} + y = 1$$

$$xy + \sqrt{1-y^2} = 1$$

Solving for x in the second equation, we get

$$x = \frac{1-\sqrt{1-y^2}}{y}$$

Plugging this into the first equation, we get

$$\frac{1-\sqrt{1-y^2}}{y} + (\sqrt{1-y^2})\frac{1-\sqrt{1-y^2}}{y} + y = 1 \implies \frac{2y^2}{y} = 1 \implies y = \frac{1}{2}$$

Plugging into the previous equation with x, we get

$$x = 2\left(1-\sqrt{1-\frac{1}{4}}\right) = 2\left(\frac{2-\sqrt{3}}{2}\right) = \boxed{\textbf{(C) } 2-\sqrt{3}}$$

Solution 3

Let $BE = x$, $EK = a$, and $EJ = b$. Then $x^2 = a^2 + b^2$ and because $\triangle KEJ \cong \triangle GDH$ and $\triangle KEJ \sim \triangle JAG$, $\frac{GA}{1} = 1 - a = \frac{b}{x}$. Furthermore, the area of the four triangles and the two rectangles sums to 1:

$$1 = 2x + GA \cdot JA + ab$$

$$1 = 2x + (1-a)(1-(x+b)) + ab$$

$$1 = 2x + \frac{b}{x}(1-x-b) + \left(1-\frac{b}{x}\right)b$$

$$1 = 2x + \frac{b}{x} - b - \frac{b^2}{x} + b - \frac{b^2}{x}$$

$$x = 2x^2 + b - 2b^2$$

$$x-b=2(x-b)(x+b)$$

$$x+b=\frac{1}{2}$$

$$b=\frac{1}{2}-x$$

$$a=1-\frac{b}{x}=2-\frac{1}{2x}$$

By the Pythagorean theorem: $x^2=a^2+b^2$

$$x^2=\left(2-\frac{1}{2x}\right)^2+\left(\frac{1}{2}-x\right)^2$$

$$x^2=4-\frac{2}{x}+\frac{1}{4x^2}+\frac{1}{4}-x+x^2$$

$$0=\frac{1}{4x^2}-\frac{2}{x}+\frac{17}{4}-x$$

$$0=1-8x+17x^2-4x^3.$$

Then by the rational root theorem, this has roots $\frac{1}{4}$, $2-\sqrt{3}$, and $2+\sqrt{3}$. The first and last roots are extraneous because they imply $a=0$ and $x>1$, respectively, thus $x=\boxed{\textbf{(C) } 2-\sqrt{3}}$.

Solution 4

Let $\angle FKH$ = k and CF = a. It is shown that all four triangles in the picture are similar. From the square side lengths:

$$a+\sin(k)\cdot 1+\cos(k)\cdot a=1$$

$$\sin(k)a+\cos(k)=1$$

Solving for a we get:

$$a=\frac{1-\sin(k)}{\cos(k)+1}=\frac{1-\cos(k)}{\sin(k)}$$

$$(1-\sin(k))\cdot sin(k)=(1-\cos(k))\cdot(\cos(k)+1)$$

$$\sin(k)-\sin(k)^2=\cos(k)+1-\cos(k)^2-\cos(k)$$

$$\sin(k)-\sin(k)^2=\sin(k)^2$$

$$1-\sin(k)=\sin(k)$$

$$\sin(k)=\frac{1}{2},\cos(k)=\frac{\sqrt{3}}{2}$$

$$a=\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}=2-\sqrt{3}$$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 20	Followed by Problem 22
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2014 AMC 12B Problems/Problem 22

Problem

In a small pond there are eleven lily pads in a row labeled 0 through 10. A frog is sitting on pad 1. When the frog is on pad N, $0<N<10$, it will jump to pad $N-1$ with probability $\frac{N}{10}$ and to pad $N+1$ with probability $1-\frac{N}{10}$. Each jump is independent of the previous jumps. If the frog reaches pad 0 it will be eaten by a patiently waiting snake. If the frog reaches pad 10 it will exit the pond, never to return. What is the probability that the frog will escape without being eaten by the snake?

$\textbf{(A) }\frac{32}{79}\qquad \textbf{(B) }\frac{161}{384}\qquad \textbf{(C) }\frac{63}{146}\qquad \textbf{(D) }\frac{7}{16}\qquad \textbf{(E) }\frac{1}{2}$

Solution

A long, but straightforward bash:

Define $P(N)$ to be the probability that the frog survives starting from pad N.

Then note that by symmetry, $P(5)=1/2$, since the probabilities of the frog moving subsequently in either direction from pad 5 are equal.

We therefore seek to rewrite $P(1)$ in terms of $P(5)$, using the fact that

$$P(N)=\frac{N}{10}P(N-1)+\frac{10-N}{10}P(N+1)$$

as said in the problem.

Hence $P(1)=\frac{1}{10}P(0)+\frac{9}{10}P(2)=\frac{9}{10}P(2)$

$$\Rightarrow P(2)=\frac{10}{9}P(1)$$

Returning to our original equation:

$$P(1)=\frac{9}{10}P(2)=\frac{9}{10}\left(\frac{2}{10}P(1)+\frac{8}{10}P(3)\right)$$

$$=\frac{9}{50}P(1)+\frac{18}{25}P(3)\Rightarrow P(1)-\frac{9}{50}P(1)=\frac{18}{25}P(3)$$

$$\Rightarrow P(3)=\frac{41}{36}P(1)$$

Returning to our original equation:

$$P(1)=\frac{9}{50}P(1)+\frac{18}{25}\left(\frac{3}{10}P(2)+\frac{7}{10}P(4)\right)$$

$$=\frac{9}{50}P(1)+\frac{27}{125}P(2)+\frac{63}{125}P(4)$$

$$=\frac{9}{50}P(1)+\frac{27}{125}\left(\frac{10}{9}P(1)\right)+\frac{63}{125}\left(\frac{4}{10}P(3)+\frac{6}{10}P(5)\right)$$

Cleaing up the coefficients, we have:

$$=\frac{21}{50}P(1)+\frac{126}{625}P(3)+\frac{189}{625}P(5)$$

$$=\frac{21}{50}P(1)+\frac{126}{625}\left(\frac{41}{36}P(1)\right)+\frac{189}{625}\left(\frac{1}{2}\right)$$

Hence, $P(1)=\frac{525}{1250}P(1)+\frac{287}{1250}P(1)+\frac{189}{1250}$

$$\Rightarrow P(1)-\frac{812}{1250}P(1)=\frac{189}{1250}\Rightarrow P(1)=\frac{189}{438}$$

$$=\boxed{\frac{63}{146}\,(C)}$$

Or set $P(1)=a,P(2)=b,P(3)=c,P(4)=d,P(5)=e=1/2$:

$$a=0.1\emptyset+0.9b,b=0.2a+0.8c,c=0.3b+0.7d,d=0.4c+0.6e$$

$$10a=\emptyset+9b,10b=2a+8c,10c=3b+7d,10d=4c+6e$$

$$\implies b=\frac{10a-\emptyset}{9},c=\frac{5b-a}{4},d=\frac{10c-3b}{7},e=\frac{5d-2c}{3}=1/2$$

$$b=\frac{10a}{9}$$

$$c=\frac{5\left(\frac{10a}{9}\right)-a}{4}=\frac{\frac{50a}{9}-a}{9}=\frac{41a}{36}$$

$$d=\frac{10\left(\frac{41a}{36}\right)-3\left(\frac{30a}{9}\right)}{7}=\frac{\frac{205a}{18}-\frac{10a}{3}}{7}=\frac{145a}{126}$$

$$e = \frac{5\left(\frac{145a}{126}\right) - 2\left(\frac{41a}{36}\right)}{3} = \frac{\frac{725a}{126} - \frac{41a}{18}}{3} = \frac{73a}{63}$$

Since $e = \frac{1}{2}$, $\frac{73a}{63} = \frac{1}{2} \implies a = \boxed{\textbf{(C)}\ \frac{63}{146}}$.

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 21	Followed by Problem 23
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 23

Problem

The number 2017 is prime. Let $S=\sum_{k=0}^{62}\binom{2014}{k}$. What is the remainder when S is divided by 2017?

$\textbf{(A) }32\qquad\textbf{(B) }684\qquad\textbf{(C) }1024\qquad\textbf{(D) }1576\qquad\textbf{(E) }2016$

Solution

Note that $2014\equiv -3 \mod 2017$. We have for $k\ge1$

$$\dbinom{2014}{k}\equiv \frac{(-3)(-4)(-5)....(-2-k)}{k!}\mod 2017$$

$$\equiv (-1)^k\dbinom{k+2}{k} \mod 2017$$

$$\equiv (-1)^k\dbinom{k+2}{2} \mod 2017$$

Therefore

$$\sum \limits_{k=0}^{62} \dbinom{2014}{k}\equiv \sum \limits_{k=0}^{62}(-1)^k\dbinom{k+2}{2} \mod 2017$$

This is simply an alternating series of triangular numbers that goes like this: $1-3+6-10+15-21....$ After finding the first few sums of the series, it becomes apparent that

$$\sum \limits_{k=1}^{n}(-1)^k\dbinom{k+2}{2}\equiv -\left(\frac{n+1}{2} \right) \left(\frac{n+1}{2}+1 \right) \mod 2017 \textrm{ if n is odd}$$

and

$$\sum \limits_{k=1}^{n}(-1)^k\dbinom{k+2}{2}\equiv \left(\frac{n}{2}+1 \right)^2 \mod 2017 \textrm{ if n is even}$$

Obviously, 62 falls in the second category, so our desired value is

$$\left(\frac{62}{2}+1 \right)^2 = 32^2 = \boxed{\textbf{(C) }1024}$$

See also

2014 AMC 12B Problems/Problem 24

Problem

Let $ABCDE$ be a pentagon inscribed in a circle such that $AB = CD = 3$, $BC = DE = 10$, and $AE = 14$. The sum of the lengths of all diagonals of $ABCDE$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

$\textbf{(A) }129\qquad \textbf{(B) }247\qquad \textbf{(C) }353\qquad \textbf{(D) }391\qquad \textbf{(E) }421$

Solution

Let a denote the length of a diagonal opposite adjacent sides of length 14 and 3, b for sides 14 and 10, and c for sides 3 and 10. Using Ptolemy's Theorem on the five possible quadrilaterals in the configuration, we obtain:

$$\begin{align} c^2 &= 3a+100 \\ c^2 &= 10b+9 \\ ab &= 30+14c \\ ac &= 3c+140 \\ bc &= 10c+42 \end{align}$$

Using equations (1) and (2), we obtain:

$$a = \frac{c^2-100}{3}$$

and

$$b = \frac{c^2-9}{10}$$

Plugging into equation (4), we find that:

$$\begin{align*} \frac{c^2-100}{3}c &= 3c + 140\\ \frac{c^3-100c}{3} &= 3c + 140\\ c^3-100c &= 9c + 420\\ c^3-109c-420 &=0\\ (c-12)(c+7)(c+5)&=0 \end{align*}$$

Or similarly into equation (5) to check:

$$\frac{c^2-9}{10}c = 10c+42$$
$$\frac{c^3-9c}{10} = 10c+42$$
$$c^3-9c = 100c+420$$
$$c^3-109c-420 = 0$$
$$(c-12)(c+7)(c+5) = 0$$

c, being a length, must be positive, implying that $c=12$. In fact, this is reasonable, since $10+3 \approx 12$ in the pentagon with apparently obtuse angles. Plugging this back into equations (1) and (2) we find that $a=\frac{44}{3}$ and $b=\frac{135}{10}=\frac{27}{2}$.

We desire $3c+a+b=3\cdot 12+\frac{44}{3}+\frac{27}{2}=\frac{216+88+81}{6}=\frac{385}{6}$, so it follows that the answer is $385+6=\boxed{\textbf{(D) }391}$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 23	Followed by Problem 25
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2014 AMC 12B Problems/Problem 25

Problem

Find the sum of all the positive solutions of

$$2\cos 2x\left(\cos 2x-\cos\left(\frac{2014\pi^2}{x}\right)\right)=\cos 4x-1$$

$\textbf{(A)}\ \pi \qquad \textbf{(B)}\ 810\pi \qquad \textbf{(C)}\ 1008\pi \qquad \textbf{(D)}\ 1080\pi \qquad \textbf{(E)}\ 1800\pi$

Solution

Rewrite $\cos 4x - 1$ as $2\cos^2 2x - 2$. Now let $a = \cos 2x$, and let $b = \cos\left(\frac{2014\pi^2}{x}\right)$. We have

$$2a(a-b) = 2a^2 - 2$$

$$ab = 1$$

Notice that either $a = 1$ and $b = 1$ or $a = -1$ and $b = -1$. For the first case, $a = 1$ only when $x = k\pi$ and k is an integer. $b = 1$ when $\frac{2014\pi^2}{k\pi}$ is an even multiple of π, and since $2014 = 2 * 19 * 53$, $b = 1$ only when k is an odd divisor of 2014. This gives us these possible values for x:

$$x = \pi, 19\pi, 53\pi, 1007\pi$$

For the case where $a = -1$, $\cos 2x = -1$, so $x = \frac{m\pi}{2}$, where m is odd. $\frac{2014\pi^2}{\frac{m\pi}{2}}$ must also be an odd multiple of π in order for b to equal -1, so $\frac{4028}{m}$ must be odd. We can quickly see that dividing an even number by an odd number will never yield an odd number, so there are no possible values for m, and therefore no cases where $a = -1$ and $b = -1$. Therefore, the sum of all our possible values for x is

$$\pi + 19\pi + 53\pi + 1007\pi = \boxed{\textbf{(D)}\ 1080\pi}$$

See also

2014 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 24	Followed by Last Question
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	