2019 AMC 8 Problems

Problem 1

Ike and Mike go into a sandwich shop with a total of $\$30.00$ to spend.

Sandwiches cost $\$4.50$ each and soft drinks cost $\$1.00$ each. Ike and Mike plan to buy as many sandwiches as they can and use the remaining money to buy soft drinks. Counting both soft drinks and sandwiches, how many items will they buy?

$\textbf{(A) }6\qquad\textbf{(B) }7\qquad\textbf{(C) }8\qquad\textbf{(D) }9\qquad\textbf{(E) }10$

Solution 1

We maximize the number of sandwiches Mike and Ike can buy by finding the lowest multiple of $\$4.50$ that is less than $\$30.$ This number is $6.$

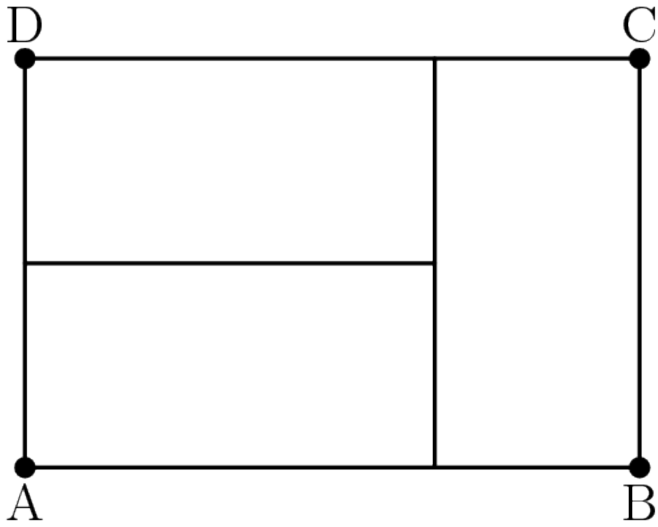
Therefore, they can buy 6 sandwiches for $\$4.50 \cdot 6 = \$27.$ They spend the remaining money on soft drinks, so they buy $30 - 27 = 3$ soft drinks.

Combining the items, Mike and Ike buy $6 + 3 = 9$ soft drinks.

The answer is $\boxed{\textbf{(D) }9}$.

Problem 2

Three identical rectangles are put together to form rectangle $ABCD$, as shown in the figure below. Given that the length of the shorter side of each of the smaller rectangles is 5 feet, what is the area in square feet of rectangle $ABCD$?



$\textbf{(A) } 45 \qquad \textbf{(B) } 75 \qquad \textbf{(C) } 100 \qquad \textbf{(D) } 125 \qquad \textbf{(E) } 150$

Solution 1

We know that the length of the shorter side of the 3 identical rectangles are all 5 so we can use that by seeing that the longer side of the right rectangle is the same as 2 of the shorter sides of the other 2 left rectangles. This means that $2 \cdot 5 \ = 10$ which is the longer side of the right rectangle, and because all the rectangles are congruent, we see that each of the rectangles have a longer side of 10 and a shorter side of 5. Now the bigger rectangle has a shorter length of 10(because the shorter side of the bigger rectangle is the bigger side of the shorter rectangle, which is 10) and so the bigger side of the bigger rectangle is the bigger side of the smaller rectangle + the smaller side of the smaller rectangle, which is $10 + 5 = 15$. Thus, the area is $15 \cdot 10 \ = 150$ for choice $\boxed{\textbf{(E) } 150}$ ~~Saksham27

Solution 2

Using the diagram we find that the larger side of the small rectangle is 2 times the length of the smaller side. Therefore the longer side is $5 \cdot 2 = 10$. So the area of the identical rectangles is $5 \cdot 10 = 50$. We have 3 identical rectangles that form the large rectangle. Therefore the area of the large rectangle is $50 \cdot 3 = \boxed{\textbf{(E) } 150}$. ~~fath2012

Solution 3

We see that if the short sides are 5, the long side has to be $5 \cdot 2 = 10$ because the long side is equal to the 2 short sides and because the rectangles are congruent. If that is to be, then the long side of the BIG rectangle(rectangle $ABCD$) is $10 + 5 = 15$ because long side + short side of the small rectangle is 15. The short side of rectangle $ABCD$ is 10 because it is the long side of the short rectangle. Multiplying $15$and 10 together gets us $15 \cdot 10$ which is $\boxed{\textbf{(E) } 150}$.
~~mathboy282

Problem 3

Which of the following is the correct order of the fractions $\frac{15}{11}, \frac{19}{15},$ and $\frac{17}{13},$ from least to greatest?

$\textbf{(A) } \frac{15}{11} < \frac{17}{13} < \frac{19}{15} \qquad \textbf{(B) } \frac{15}{11} < \frac{19}{15} < \frac{17}{13} \qquad \textbf{(C) } \frac{17}{13} < \frac{19}{15} < \frac{15}{11} \qquad \textbf{(D) } \frac{19}{15} < \frac{15}{11} < \frac{17}{13} \qquad \textbf{(E) } \frac{19}{15} < \frac{17}{13} < \frac{15}{11}$

Solution 1

Consider subtracting 1 from each of the fractions. Our new fractions would then be $\frac{4}{11}, \frac{4}{15},$ and $\frac{4}{13}$. Since $\frac{4}{15} < \frac{4}{13} < \frac{4}{11}$, it follows that the answer is $\boxed{\textbf{(E)} \frac{19}{15} < \frac{17}{13} < \frac{15}{11}}$

-will3145

Solution 2

We take a common denominator:
$\frac{15}{11}, \frac{19}{15}, \frac{17}{13} = \frac{15 \cdot 15 \cdot 13}{11 \cdot 15 \cdot 13}, \frac{19 \cdot 11 \cdot 13}{15 \cdot 11 \cdot 13}, \frac{17 \cdot 11 \cdot 15}{13 \cdot 11 \cdot 15} = \frac{2925}{2145}, \frac{2717}{2145}, \frac{2805}{2145}$.
Since $2717 < 2805 < 2925$ it follows that the answer is $\boxed{\textbf{(E)} \frac{19}{15} < \frac{17}{13} < \frac{15}{11}}$.

-xMidnightFirex

~ dolphin7 - I took your idea and made it an explanation.

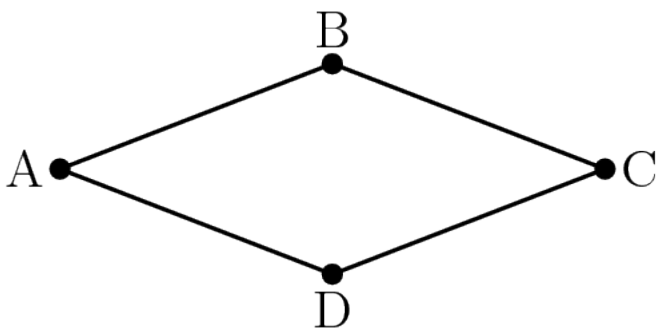
Solution 3

When $\frac{x}{y} > 1$ and $z > 0$, $\frac{x+z}{y+z} < \frac{x}{y}$. Hence, the answer is $\boxed{\textbf{(E)}\frac{19}{15}<\frac{17}{13}<\frac{15}{11}}$. ~ ryjs

This is also similar to Problem 20 on the AMC 2012.

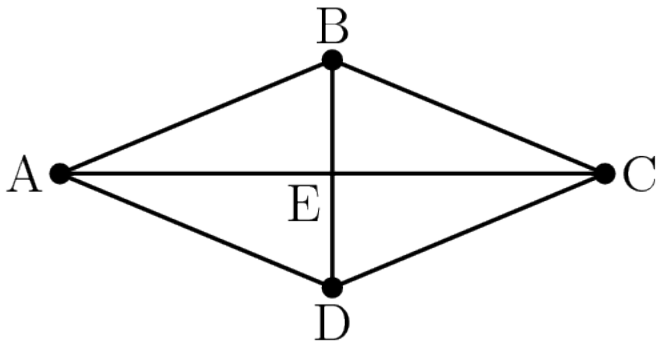
Problem 4

Quadrilateral $ABCD$ is a rhombus with perimeter 52 meters. The length of diagonal $\overline{AC}$ is 24 meters. What is the area in square meters of rhombus $ABCD$?



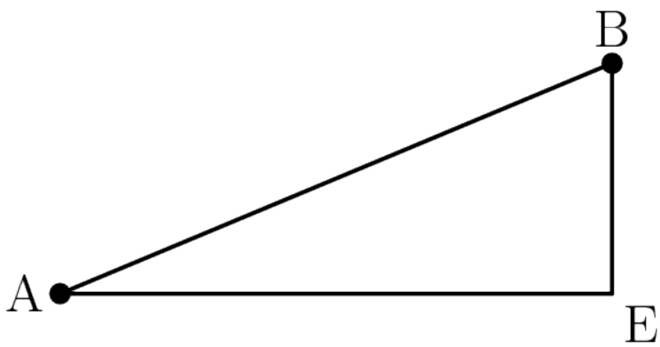
$\textbf{(A)}\ 60 \qquad \textbf{(B)}\ 90 \qquad \textbf{(C)}\ 105 \qquad \textbf{(D)}\ 120 \qquad \textbf{(E)}\ 144$

Solution 1



A rhombus has sides of equal length. Because the perimeter of the rhombus is 52, each side is $\frac{52}{4} = 13$. In a rhombus, diagonals are perpendicular and bisect each other, which means $\overline{AE}$ = 12 = $\overline{EC}$.

Consider one of the right triangles:

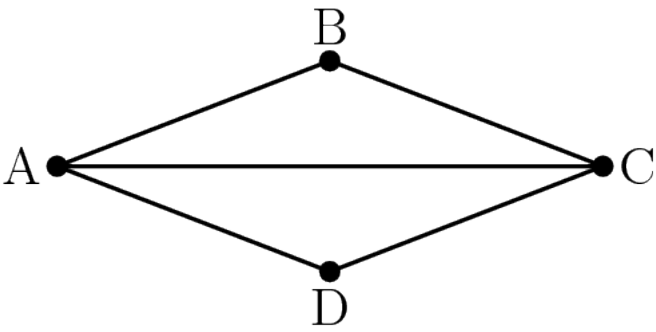


$\overline{AB}$ = 13, and $\overline{AE}$ = 12. Using Pythagorean theorem, we find that $\overline{BE}$ = 5.

Thus the values of the two diagonals are $\overline{AC}$ = 24 and $\overline{BD}$ = 10. The area of a rhombus is = $\frac{d_1 * d_2}{2}$ = $\frac{24 * 10}{2}$ = 120

$\boxed{\textbf{(D)}\ 120}$ ~phoenixfire

Solution 2 (Heron's)



Since a rhombus has sides of equal length, $AB = BC = CD = DA = \frac{52}{4} = 13$. In triangle ABC, $AB = BC = 13$ and $AC = 24$. Using Heron's formula, we have

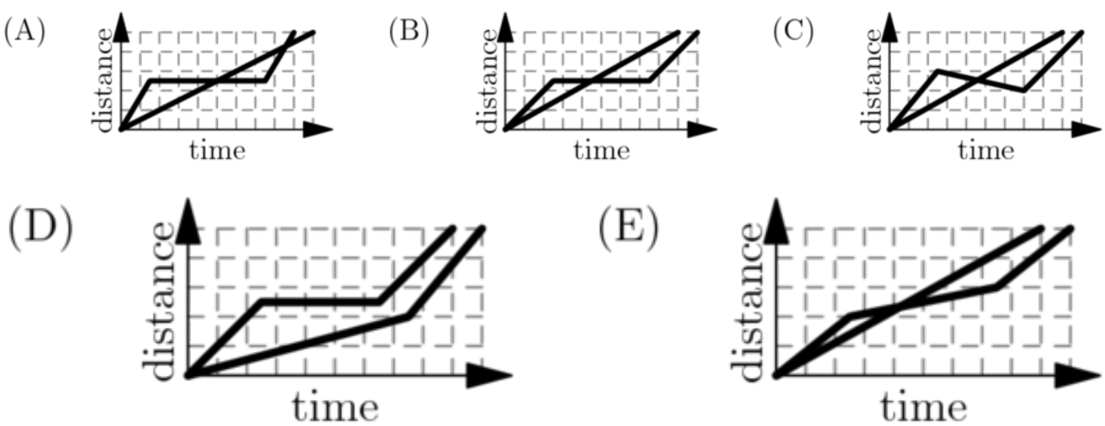
$[ABCD] = [ABC] + [ACD] = 2[ABC] = 2\sqrt{25 * 12 * 12 * 1}$

. Simplifying, we have $[ABC] = \sqrt{3600} = 60$ so

$[ABCD] = 2 * 60 = \boxed{\textbf{(D)} 120}$. ~~RWhite

Problem 5

A tortoise challenges a hare to a race. The hare eagerly agrees and quickly runs ahead, leaving the slow-moving tortoise behind. Confident that he will win, the hare stops to take a nap. Meanwhile, the tortoise walks at a slow steady pace for the entire race. The hare awakes and runs to the finish line, only to find the tortoise already there. Which of the following graphs matches the description of the race, showing the distance d traveled by the two animals over time t from start to finish?

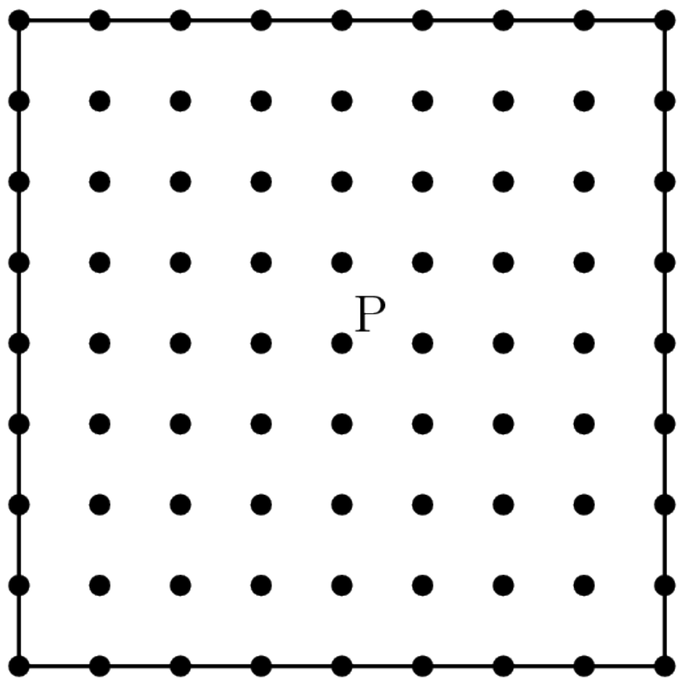


Solution 1 (Using the answer choices)

First, the tortoise walks at a constant rate, ruling out (D) Second, when the hare is resting, the distance will stay the same, ruling out (E) and (C). Third, the tortoise wins the race, meaning that the non-constant one should go off the graph last, ruling out (A). Therefore, the answer $\boxed{\textbf{(B)}}$ is the only one left.

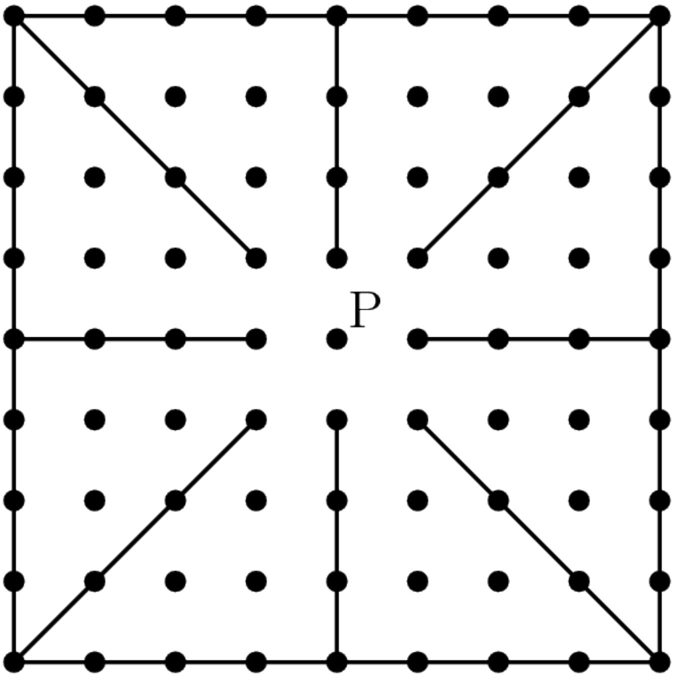
Problem 6

There are 81 grid points (uniformly spaced) in the square shown in the diagram below, including the points on the edges. Point P is in the center of the square. Given that point Q is randomly chosen among the other 80 points, what is the probability that the line PQ is a line of symmetry for the square?



$\textbf{(A)}\ \frac{1}{5}\qquad\textbf{(B)}\ \frac{1}{4}\qquad\textbf{(C)}\ \frac{2}{5}\qquad\textbf{(D)}\ \frac{9}{20}\qquad\textbf{(E)}\ \frac{1}{2}$

Solution 1



Lines of symmetry go through point P, and there are 8 directions the lines could go, and there are 4 dots at each direction. $\frac{4\times 8}{80} = \boxed{\textbf{(C)}\frac{2}{5}}$. ~heeeeeheeeeeee

Problem 7

Shauna takes five tests, each worth a maximum of 100 points. Her scores on the first three tests are 76, 94, and 87. In order to average 81 for all five tests, what is the lowest score she could earn on one of the other two tests?

$\textbf{(A)}\ 48 \qquad\textbf{(B)}\ 52 \qquad\textbf{(C)}\ 66 \qquad\textbf{(D)}\ 70 \qquad\textbf{(E)}\ 74$

Solution 1

We should notice that we can turn the information we are given into a linear equation and just solve for our set variables. I'll use the variables x and y for the scores on the last two tests. $\frac{76+94+87+x+y}{5} = 81,$ $\frac{257+x+y}{5} = 81.$ We can now cross multiply to get rid of the denominator. $257+x+y = 405, x+y = 148.$ Now that we have this equation, we will assign y as the lowest score of the two other tests, and so: $x=100$,$y=48$.Now we know that the lowest score on the two other tests is $\boxed{48}$.

~ aopsav

Solution 2

Right now, she scored $76, 94,$ and 87 points, with a total of 257 points. She wants her average to be $81$for her 5 tests so she needs to score 405 points in total. She needs to score a total of $(405-257)148$points in her 2 tests. So the minimum score she can get is when one of her 2 scores is 100. So the least possible score she can get is $\boxed{\textbf{(A)}\ 48}$. ~heeeeeeeheeeeee

Note: You can verify that $\boxed{48}$ is the right answer because it is the lowest answer out of the 5. Since it is possible to get 48, we are guaranteed that that is the right answer. ~~ gorefeebuddie

Solution 3

We can compare each of the scores with the average of 81: $76 \rightarrow -5$, $94 \rightarrow +13$, $87 \rightarrow +6$, $100 \rightarrow 19$;

So the last one has to be -33 (since all the differences have to sum to 0), which corresponds to $81-33=\boxed{48}$.

Problem 8

Gilda has a bag of marbles. She gives 20% of them to her friend Pedro. Then Gilda gives 10% of what is left to another friend, Ebony. Finally, Gilda gives 25% of what is now left in the bag to her brother Jimmy. What percentage of her original bag of marbles does Gilda have left for herself?

$$\textbf{(A)}\ 20\qquad\textbf{(B)}\ 33\frac{1}{3}\qquad\textbf{(C)}\ 38\qquad\textbf{(D)}\ 45\qquad\textbf{(E)}\ 54$$

Solution 1

After Gilda gives 20% of the marbles to Pedro, she has 80% of the marbles left. If she then gives 10% of what's left to Ebony, she has $(0.8 * 0.9)$ = 72% of what she had at the beginning. Finally, she gives 25% of what's left to her brother, so she has $(0.75 * 0.72) \boxed{\textbf{(E)}\ 54}$. of what she had in the beginning left.~heeeeeeeheeeeee

Solution 2

Suppose Gilda has 100 marbles.

Then she gives Pedro 20% of 100 = 20, she remains with 80 marbles.

Out of 80 marbles she gives 10% of 80 = 8 to Ebony.

Thus she remains with 72 marbles.

Then she gives 25% of 72 = 18 to Jimmy, finally leaving her with 54.

And $\frac{54}{100}$=54%=$\boxed{\textbf{(E)}\ 54}$

~phoenixfire

Solution 3 (Only if you have no time, do this method)

Since she gave away 20% and 10% of what is left and then another 25% of what is actually left, we can do 20+10+25 or 55%. But it is actually going to be a bit more than 55% because 10% of what is left is not 10% of the total amount. So the only option that is greater than 100% - 55% is $\boxed{\textbf{(E)}\ 54}$. ~~ gorefeebuddie

Problem 9

Alex and Felicia each have cats as pets. Alex buys cat food in cylindrical cans that are 6 cm in diameter and 12 cm high. Felicia buys cat food in cylindrical cans that are 12 cm in diameter and 6 cm high. What is the ratio of the volume one of Alex's cans to the volume one of Felicia's cans?

$\textbf{(A)}\ 1:4 \qquad \textbf{(B)}\ 1:2 \qquad \textbf{(C)}\ 1:1 \qquad \textbf{(D)}\ 2:1 \qquad \textbf{(E)}\ 4:1$

Solution 1

Using the formula for the volume of a cylinder, we get Alex, $\pi 108$, and Felicia, $\pi 216$. We can quickly notice that π cancels out on both sides, and that Alex's volume is $1/2$ of Felicia's leaving $1/2 = \boxed{1:2}$ as the answer.

~aopsav

Solution 2

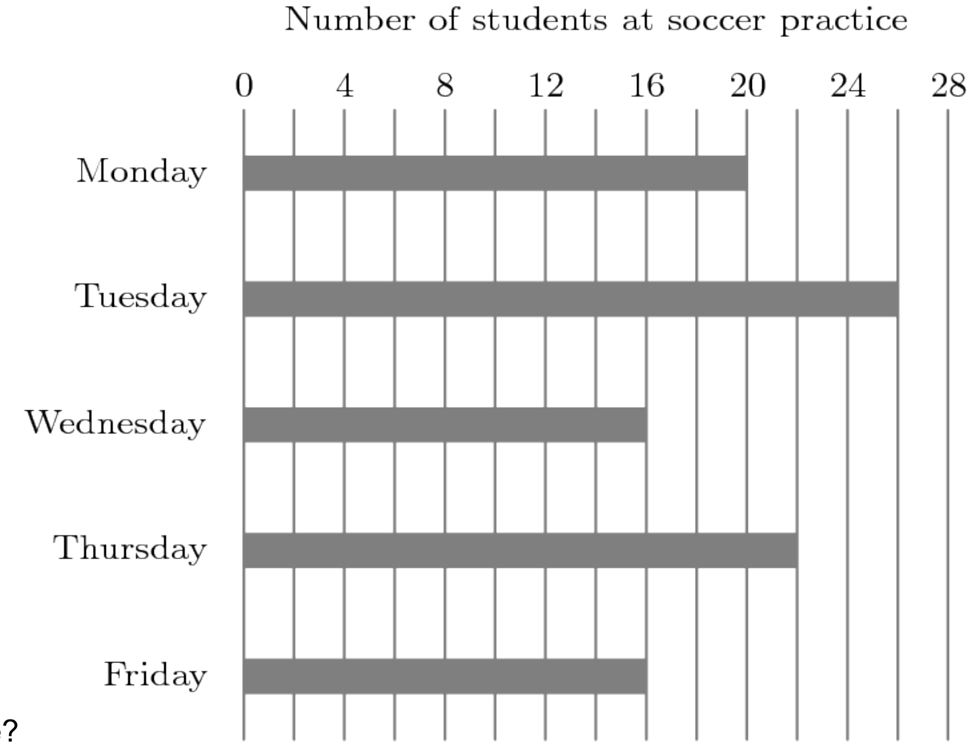
Using the formula for the volume of a cylinder, we get that the volume of Alex's can is $3^2 \cdot 12 \cdot \pi$, and that the volume of Felicia's can is $6^2 \cdot 6 \cdot \pi$. Now we divide the volume of Alex's can by the volume of Felicia's can, so we get $\frac{1}{2}$, which is $\boxed{\textbf{(B) }1:2}$ ~~SmileKat32

Solution 3

The ratio of the numbers is $1/2$. Looking closely at the formula $r^2 * h * \pi$, we see that the $r * h * \pi$ will cancel, meaning that the ratio of them will be $1(2)/2(2) = \boxed{\textbf{(B) }1:2}$

Problem 10

The diagram shows the number of students at soccer practice each weekday during last week. After computing the mean and median values, Coach discovers that there were actually 21 participants on Wednesday. Which of the following statements describes the change in the mean and median after the correction is



made?

(A) The mean increases by 1 and the median does not change.

(B) The mean increases by 1 and the median increases by 1.

(C) The mean increases by 1 and the median increases by 5.

(D) The mean increases by 5 and the median increases by 1.

(E) The mean increases by 5 and the median increases by 5.

Solution 1

On Monday, 20 people come. On Tuesday, 26 people come. On Wednesday, 16 people come. On Thursday, 22 people come. Finally, on Friday, 16 people come. $20+26+16+22+16=100$, so the mean is 20. The median is $(16,16,20,22,26) 20$. The coach figures out that actually 21 people come on Wednesday. The new mean is 21, while the new median is $(16,20,21,22,26) 21$. The median and mean both change, so the answer is $\boxed{\textbf{(B)}}$ Another way to compute the change in mean is to notice that the sum increased by 5 with the correction. So the average increased by $5/5 = 1$.

Problem 11

The eighth grade class at Lincoln Middle School has 93 students. Each student takes a math class or a foreign language class or both. There are 70 eighth graders taking a math class, and there are 54 eight graders taking a foreign language class. How many eighth graders take *only* a math class and *not* a foreign language class?

$\textbf{(A) }16\qquad\textbf{(B) }23\qquad\textbf{(C) }31\qquad\textbf{(D) }39\qquad\textbf{(E) }70$

Solution 1

Let x be the number of students taking both a math and a foreign language class.

By P-I-E, we get $70+54-x$ = 93.

Solving gives us $x=31$.

But we want the number of students taking only a math class.

Which is $70-31=39$.

$\boxed{\textbf{(D) }39}$

~phoenixfire

Solution 2

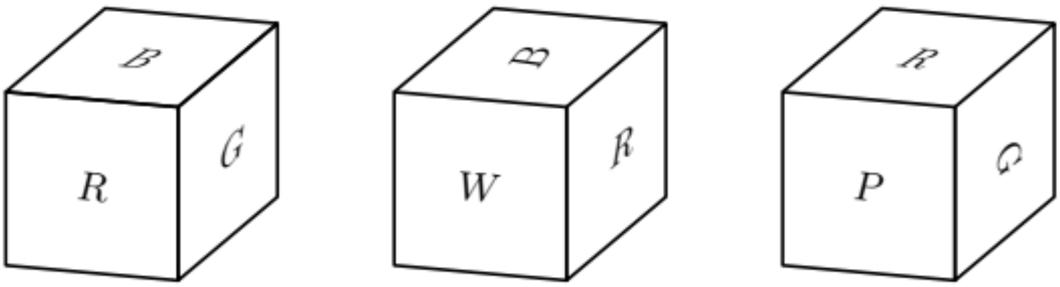
We have $70+54=124$ people taking classes. However we over-counted the number of people who take both classes. If we subtract the original amount of people who take classes we get that 31 people took the two classes. To find the amount of people who took only math class web subtract the people who didn't take only one math class, so we get $70-31=\boxed{\textbf{D }39}$ -fath2012

Solution 3

We're basically just finding the number of students not taking a foreign language since all the rest would be taking only math. $93-54=\boxed{\textbf{(D) } 39}$

Problem12

The faces of a cube are painted in six different colors: red (R), white (W), green (G), brown (B), aqua (A), and purple (P). Three views of the cube are shown below. What is the color of the face opposite the aqua face?



$\textbf{(A)}\ \text{red} \qquad \textbf{(B)}\ \text{white} \qquad \textbf{(C)}\ \text{green} \qquad \textbf{(D)}\ \text{brown} \qquad \textbf{(E)}\ \text{purple}$

Solution 1

B is on the top, and R is on the side, and G is on the right side. That means that (image 2) W is on the left side. From the third image, you know that P must be on the bottom since G is sideways. That leaves us with the back, so the back must be A. The front is opposite of the back, so the answer is $\boxed{\textbf{(A)}\ R}$.~heeeeeeeheeeee

Solution 2

Looking closely we can see that all faces are connected with R except for A. Thus the answer is $\boxed{\textbf{(A)}\ R}$.

Problem 13

A *palindrome* is a number that has the same value when read from left to right or from right to left. (For example 12321 is a palindrome.) Let N be the least three-digit integer which is not a palindrome but which is the sum of three distinct two-digit palindromes. What is the sum of the digits of N?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 6$

Solution

Note that the only positive 2-digit palindromes are multiples of 11, namely $11, 22, \ldots, 99$. Since N is the sum of 2-digit palindromes, N is necessarily a multiple of 11. The smallest 3-digit multiple of 11 which is not a palindrome is 110, so $N = 110$ is a candidate solution. We must check that 110 can be written as the sum of three distinct 2-digit palindromes; this suffices as $110 = 77 + 22 + 11$. Then $N = 110$, and the sum of the digits of N is $1 + 1 + 0 = \boxed{\textbf{(A) } 2}$.

Problem 14

Isabella has 6 coupons that can be redeemed for free ice cream cones at Pete's Sweet Treats. In order to make the coupons last, she decides that she will redeem one every 10 days until she has used them all. She knows that Pete's is closed on Sundays, but as she circles the 6 dates on her calendar, she realizes that no circled date falls on a Sunday. On what day of the week does Isabella redeem her first coupon?

$\textbf{(A) }\text{Monday}\qquad\textbf{(B) }\text{Tuesday}\qquad\textbf{(C) }\text{Wednesday}\qquad\textbf{(D) }\text{Thursday}\qquad\textbf{(E) }\text{Friday}$

Solution 1

Let $\text{Day } 1$ to $\text{Day } 2$ denote a day where one coupon is redeemed and the day when the second coupon is redeemed.

If she starts on a Monday she redeems her next coupon on Thursday.

Thursday to Sunday.

Thus $\textbf{(A)}\ \text{Monday}$ is incorrect.

If she starts on a Tuesday she redeems her next coupon on Friday.

Friday to Monday.

Monday to Thursday.

Thursday to Sunday.

Thus $\textbf{(B)}\ \text{Tuesday}$ is incorrect.

If she starts on a $Wednesday$ she redeems her next coupon on $Saturday$.

Saturday to Tuesday.

Tuesday to Friday.

Friday to Monday.

Monday to Thursday.

And on Thursday she redeems her last coupon.

No sunday occured thus $\boxed{\textbf{(C)}\ \text{Wednesday}}$ is correct.

Checking for the other options,

If she starts on a Thursday she redeems her next coupon on Sunday.

Thus $\textbf{(D)}\ \text{Thursday}$ is incorrect.

If she starts on a Friday she redeems her next coupon on Monday.

Monday to Thursday.

Thursday to Sunday.

Checking for the other options gave us negative results, thus the answer is $\boxed{\textbf{(C)}\ \text{Wednesday}}$.

~phoenixfire

Solution 2

Let

$$Sunday \equiv 0 \pmod{7}$$

$$Monday \equiv 1 \pmod{7}$$

$$Tuesday \equiv 2 \pmod{7}$$

$$Wednesday \equiv 3 \pmod{7}$$

$$Thursday \equiv 4 \pmod{7}$$

$$Friday \equiv 5 \pmod{7}$$

$$Saturday \equiv 6 \pmod{7}$$

$$10 \equiv 3 \pmod{7}$$

$$20 \equiv 6 \pmod{7}$$

$$30 \equiv 2 \pmod{7}$$

$$40 \equiv 5 \pmod{7}$$

$$50 \equiv 1 \pmod{7}$$

$$60 \equiv 4 \pmod{7}$$

Which clearly indicates if you start form a $x \equiv 3 \pmod 7$ you will not get a $y \equiv 0 \pmod 7$.

Any other starting value may lead to a $y \equiv 0 \pmod 7$.

Which means our answer is $\boxed{\textbf{(C)}\ Wednesday}$.

~phoenixfire

Solution 3

Like Solution 2, let the days of the week be numbers $\pmod 7$. 3 and 7 are coprime, so continuously adding 3 to a number $\pmod 7$ will cycle through all numbers from 0 to 6. If a string of 6 numbers in this cycle does not contain 0, then if you minus 3 from the first number of this cycle, it will always be 0. So, the answer is $\boxed{\textbf{(C)}\ Wednesday}$. ~~SmileKat32

Solution 4

Since Sunday is the only day that has not been counted yet. We can just add the 3 days as it will become $\boxed{\textbf{(C)}\ Wednesday}$. ~~ gorefeebuddie Note: This only works when 7 and 3 are relatively prime.

Problem 15

On a beach 50 people are wearing sunglasses and 35 people are wearing caps. Some people are wearing both sunglasses and caps. If one of the people wearing a cap is selected at random, the probability that this person is is also wearing sunglasses is $\frac{2}{5}$. If instead, someone wearing sunglasses is selected at random, what is the probability that this person is also wearing a cap?

$\textbf{(A)}\ \frac{14}{85} \qquad \textbf{(B)}\ \frac{7}{25} \qquad \textbf{(C)}\ \frac{2}{5} \qquad \textbf{(D)}\ \frac{4}{7} \qquad \textbf{(E)}\ \frac{7}{10}$

Solution 1

The number of people wearing caps and sunglasses is $\frac{2}{5}\cdot 35=14$. So then 14 people out of the 50 people wearing sunglasses also have caps. $\frac{14}{50}=\boxed{\textbf{(B)}\frac{7}{25}}$~heeeeeeheeeeee

Problem 16

Qiang drives 15 miles at an average speed of 30 miles per hour. How many additional miles will he have to drive at 55 miles per hour to average 50 miles per hour for the entire trip?

$\textbf{(A) }45\qquad\textbf{(B) }62\qquad\textbf{(C) }90\qquad\textbf{(D) }110\qquad\textbf{(E) }135$

Solution 1(answer options)

The only option that is easily divisible by 55 is 110. Which gives 2 hours of travel. And by the formula $\frac{15}{30}+\frac{110}{50}=\frac{5}{2}$

And Average Speed = $\frac{\text{Total Distance}}{\text{Total Time}}$

Thus $\frac{125}{50}=\frac{5}{2}$

Both are equal and thus our answer is $\boxed{\textbf{(D)}\ 110}$.

~phoenixfire

Solution 2

Note that the average speed is simply the total distance over the total time. Let the number of additional miles he has to drive be x. Therefore, the total distance is $15+x$ and the total time (in hours) is $\frac{15}{30}+\frac{x}{55}=\frac{1}{2}+\frac{x}{55}$. We can set up the following equation: $\dfrac{15+x}{\frac{1}{2}+\frac{x}{55}}=50.$ Simplifying the equation, we get $15+x=25+\dfrac{10x}{11}.$ Solving the equation yields $x=110,$ so our answer is $\boxed{\textbf{(D) }110}$.

~twinemma

Solution 3

If he travels 15 miles at a speed of 30 miles per hour, he travels for 30 min. Average rate is total distance over total time so $(15+d)/(0.5+t)=50$, where d is the distance left to travel and t is the time to travel that distance. solve for d to get $d=10+50t$. you also know that he has to travel 55 miles per hour for some time, so $d=55t$ plug that in for d to get $55t=10+50t$ and $t=2$ and since $d=55t, d=2\cdot 55=110$ the answer is $\boxed{\textbf{(D) }110}$.

Problem 17

What is the value of the product $\left(\frac{1\cdot3}{2\cdot2}\right)\left(\frac{2\cdot4}{3\cdot3}\right)\left(\frac{3\cdot5}{4\cdot4}\right)\cdots\left(\frac{97\cdot99}{98\cdot98}\right)\left(\frac{98\cdot100}{99\cdot99}\right)?$

$\textbf{(A) }\frac{1}{2}\qquad\textbf{(B) }\frac{50}{99}\qquad\textbf{(C) }\frac{9800}{9801}\qquad\textbf{(D) }\frac{100}{99}\qquad\textbf{(E) }50$

Solution 1

We rewrite: $\frac{1}{2}\cdot\left(\frac{3\cdot2}{2\cdot3}\right)\left(\frac{4\cdot3}{3\cdot4}\right)\cdots\left(\frac{99\cdot98}{98\cdot99}\right)\cdot\frac{100}{99}$

The middle terms cancel, leaving us with

$$\left(\frac{1 \cdot 100}{2 \cdot 99}\right) = \boxed{\textbf{(B)} \frac{50}{99}}$$

~phoenixfire

Solution 2

If you calculate the first few values of the equation, all of the values tend to $\frac{1}{2}$, but are not equal to it. The answer closest to $\frac{1}{2}$ but not equal to it is $\boxed{\textbf{(B)} \frac{50}{99}}$.~heeeeeeeheeeee

Solution 3

Rewriting the numerator and the denominator, we get $\frac{\frac{100!\cdot 98!}{2}}{(99!)^2}$. We can simplify by canceling 99! on both sides, leaving us with: $\frac{100 \cdot 98!}{2 \cdot 99!}$ We rewrite $99!$ as $99 \cdot 98!$ and cancel $98!$, which gets $\boxed{\frac{50}{99}}$. Answer B.

Problem 18

The faces of each of two fair dice are numbered 1, 2, 3, 5, 7, and 8. When the two dice are tossed, what is the probability that their sum will be an even number?

$\textbf{(A) } \frac{4}{9} \qquad \textbf{(B) } \frac{1}{2} \qquad \textbf{(C) } \frac{5}{9} \qquad \textbf{(D) } \frac{3}{5} \qquad \textbf{(E) } \frac{2}{3}$

Solution 1

The approach to this problem: There are two cases in which the sum can be an even number: both numbers are even and both numbers are odd. This results in only one case where the sum of the numbers are odd (one odd and one even in any order) . We can solve for how many ways the 2 numbers add up to an odd number and subtract the answer from 1.

How to solve the problem: The probability of getting an odd number first is $\frac{4}{6} = \frac{2}{3}$. In order to make the sum odd, we must select an even number next. The probability of getting an even number is $\frac{2}{6} = \frac{1}{3}$. Now we multiply the two fractions: $\frac{2}{3} \times \frac{1}{3} = 2/9$. However, this is not the answer because we could pick an even number first then an odd number. The equation is the same except backward and by the Communitive Property of Multiplication, the equations are it does not matter is the equation is backward or not. Thus we do $\frac{2}{9} \times 2 = \frac{4}{9}$. This is the probability of getting an odd number. In order to get the probability of getting an even number we do $1 - \frac{4}{9} = \boxed{(\textbf{C}) \frac{5}{9}}$

- ViratKohli2018 (VK18)

Solution 2

We have a 2 die with 2 evens and 4 odds on both dies. For the sum to be even, the rolls must consist of 2odds or 2 evens.

Ways to roll 2 odds (Case 1): The total number of ways to roll 2 odds is $4 * 4 = 16$, as there are 4 choices for the first odd on the first roll and 4 choices for the second odd on the second roll.

Ways to roll 2 evens (Case 2): Similarly, we have $2 * 2 = 4$ ways to roll 2 evens.

Totally, we have $6 * 6 = 36$ ways to roll 2 dies.

Therefore the answer is $\frac{16+4}{36} = \frac{20}{36} = \frac{5}{9}$, or $\boxed{\text{C}}$.

~A1337h4x0r

Solution 3 (Complementary Counting)

We count the ways to get an odd. If the sum is odd, then we must have an even and an odd. The probability of an even is $\frac{1}{3}$, and the probability of an odd is $\frac{2}{3}$. We have to multiply by $2!$ because the even and odd can be in any order. This gets us $\frac{4}{9}$, so the answer is $1-\frac{4}{9}=\frac{5}{9}=\boxed{\textbf{(C)}}$. - juliankuang

Solution 4

To get an even, you must get either 2 odds or 2 evens. The probability of getting 2 odds is $\frac{4}{6}*\frac{4}{6}$. The probability of getting 2 evens is $\frac{2}{6}*\frac{2}{6}$. If you add them together, you get $\frac{16}{36}+\frac{4}{36}$ = $\boxed{\textbf{(C)}\frac{5}{9}}$

Problem 19

In a tournament there are six teams that play each other twice. A team earns 3 points for a win, 1 point for a draw, and 0 points for a loss. After all the games have been played it turns out that the top three teams earned the same number of total points. What is the greatest possible number of total points for each of the top three teams?

$\textbf{(A) } 22 \qquad \textbf{(B) } 23 \qquad \textbf{(C) } 24 \qquad \textbf{(D) } 26 \qquad \textbf{(E) } 30$

Solution 1

After fully understanding the problem, we immediately know that the three top teams, say team A, team B, and team C, must beat the other three teams D, E, F. Therefore, A,B,C must each obtain $(3+3+3)=9$ points. However, they play against each team twice, for a total of 18 points against D, E, and F. For games between A, B, C, we have 2 cases. In both cases, there is an equality of points between A, B, and C.

Case 1: A team ties the two other teams. For a tie, we have 1 point, so we have $(1+1)*2=4$ points (they play twice). Therefore, this case brings a total of $4+18=22$ points.

Case 2: A team beats one team while losing to another. This gives equality, as each team wins once and loses once as well. For a win, we have 3 points, so a team gets $3\times 2=6$ points if they each win a game and lose a game. This case brings a total of $18+6=24$ points.

Therefore, we use Case 2 since it brings the greater amount of points, or $\boxed{24}$, so the answer is $\boxed{C}$.

~A1337h4x0r

Note that case 2 can be easily seen to be better as follows. Let x_A be the number of points A gets, x_B be the number of points B gets, and x_C be the number of points C gets. Since $x_A=x_B=x_C$, to maximize x_A, we can just maximize $x_A+x_B+x_C$. But in each match, if one team wins then the total sum increases by 3 points, whereas if they tie, the total sum increases by 2 points. So it is best if there are the fewest ties possible.

Solution 2

(1st match(3) + 2nd match(1)) * number of teams(6) = 24, $\boxed{C}$.

Explanation: So after reading the problem we see that there are 6 teams and each team versus each other twice. This means one of the two matches has to be a win, so 3 points so far. Now if we say that the team won again and make it 6 points, that would mean that team would be dominating the leader-board and the problem says that all the top 3 people have the same score. So that means the maximum amount of points we could get is 1 so that each team gets the same amount of matches won & drawn so that adds up to 4. 4 * the number of teams(6) = 24 so the answer is $\boxed{C}$

Problem 20

How many different real numbers x satisfy the equation $(x^2-5)^2=16?$

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 4 \qquad \textbf{(E) } 8$

Solution

We have that $(x^2-5)^2 = 16$ if and only if $x^2-5 = \pm 4$.

If $x^2-5 = 4$, then $x^2 = 9 \implies x = \pm 3$, giving 2 solutions.

If $x^2-5 = -4$, then $x^2 = 1 \implies x = \pm 1$, giving 2 more solutions. All four of these solutions work, so the answer is $\boxed{\textbf{(D)} 4}$. Further, the equation is a quartic in x, so by the Fundamental Theorem of Algebra, there can be at most four real solutions.

Problem 21

What is the area of the triangle formed by the lines $y = 5$, $y = 1 + x$, and $y = 1 - x$?

$\textbf{(A) } 4 \qquad \textbf{(B) } 8 \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 12 \qquad \textbf{(E) } 16$

Solution 1

You need to first find the coordinates where the graphs intersect. $y=5$, and $y=x+1$ intersect at $(4,5)$. $y=5$, and $y=1-x$ intersect at $(-4,5)$. $y=1-x$ and $y=1+x$ intersect at $(0,1)$. Using the Shoelace Theorem you get $\left(\frac{(20-4)-(-20+4)}{2}\right)$= $\frac{32}{2}$ =$\boxed{\textbf{(E) }16}$.~heeeeeeheeeee

Solution 2

Graphing the lines, we can see that the height of the triangle is 4, and the base is 8. Using the formula for the area of a triangle, we get $\frac{4 \cdot 8}{2}$ which is equal to $\boxed{\textbf{(E) } 16}$. ~SmileKat32

Problem 22

A store increased the original price of a shirt by a certain percent and then lowered the new price by the same amount. Given that the resulting price was 84% of the original price, by what percent was the price increased and decreased?

$\textbf{(A) } 16 \qquad \textbf{(B) } 20 \qquad \textbf{(C) } 28 \qquad \textbf{(D) } 36 \qquad \textbf{(E) } 40$

Solution 1

Suppose the fraction of discount is x. That means $(1-x)(1+x)=0.84$; so $1-x^2=0.84$, and $(x^2)=0.16$, obtaining $x=0.4$. Therefore, the price was increased and decreased by 40%, or $\boxed{\textbf{(E) } 40}$.

Solution 2(Answer options)

Let the price be 100 and then trying for each option leads to $\boxed{\textbf{(E) } 40}$.

-phoenixfire

Solution 3

Let x be the discount. We can also work in reverse such as $(84)$$(\frac{100}{100-x})$$(\frac{100}{100+x})$ = 100.

Thus 8400 = $(100+x)(100-x)$. Solving for x gives us $x=40, -40$. But x has to be positive. Thus x= 40.

Problem 23

After Euclid High School's last basketball game, it was determined that $\frac{1}{4}$ of the team's points were scored by Alexa and $\frac{2}{7}$ were scored by Brittany. Chelsea scored 15 points. None of the other 7 team members scored more than 2 points What was the total number of points scored by the other 7 team members?

$\textbf{(A) }10\qquad\textbf{(B) }11\qquad\textbf{(C) }12\qquad\textbf{(D) }13\qquad\textbf{(E) }14$

Solution 1

Since $\frac{\text{total points}}{4}$ and $\frac{2(\text{total points})}{7}$ are integers, we have $28|\text{total points}$. We see that the number of points scored by the other team members is less than or equal to 14 and greater than or equal to 0. We let the total number of points be t and the total number of points scored by the other team members, which means that

$$\frac{t}{4}+\frac{2t}{7}+15+x=t \qquad \implies \qquad 0\le\frac{13t}{28}-15=x\le 14$$

, which means $15\le\frac{13t}{28}\le 29$. The only value of t that satisfies all conditions listed is 56, so $x=\boxed{\textbf{(B)} 11}$. - juliankuang (lol im smart.)

Solution 2

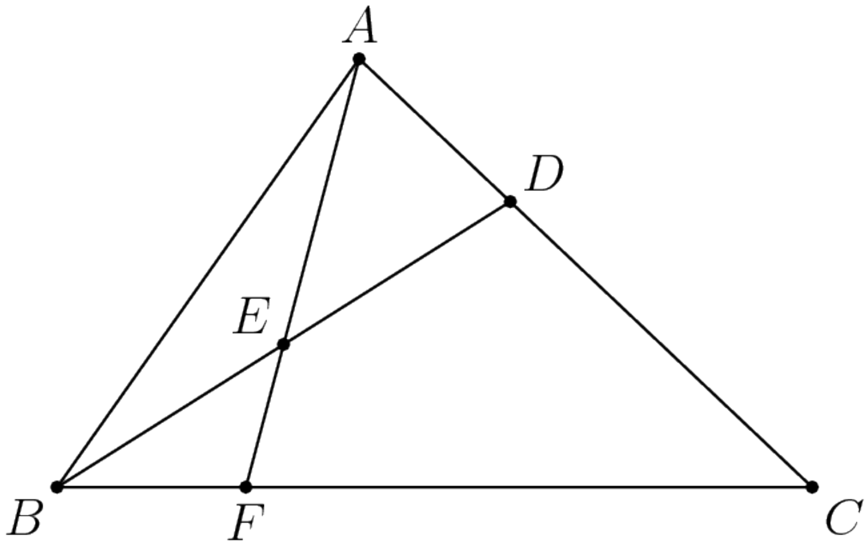
Starting from the above equation $\frac{t}{4}+\frac{2t}{7}+15+x=t$ where t is the total number of points scored and $x\leq 14$ is the number of points scored by the remaining 7 team members, we can simplify to obtain the Diophantine equation $x+15=\frac{13}{28}t$, or $28x+28\cdot 15=13t$. Since t is necessarily divisible by 28, let $t=28u$ where $u\geq 0$ and divide by 28 to obtain $x+15=13u$. Then it is easy to see $u=2$ ($t=56$) is the only candidate, giving $x=\boxed{\textbf{(B)}11}$. -scrabbler94

Solution 3

Fakesolve: We first start by setting the total number of points as 28, since $\text{lcm}(4,7)=28$. However, we see that this does not work since we surpass the number of points just with the information given ($28\cdot\frac{1}{4}+28\cdot\frac{2}{7}+15>28$). Next, we assume that the total number of points scored is 56. With this, we have that Alexa, Brittany, and Chelsea score: $56\cdot\frac{1}{4}+56\cdot\frac{2}{7}+15=45$, and thus, the other seven players would have scored a total of $56-45=\boxed{\textbf{(B)}11}$ (We see that this works since we could have 4 of them score 2 points, and the other 3 of them score 1 point) -aops5234

Problem 24

In triangle ABC, point D divides side $\overline{AC}$ so that $AD:DC=1:2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



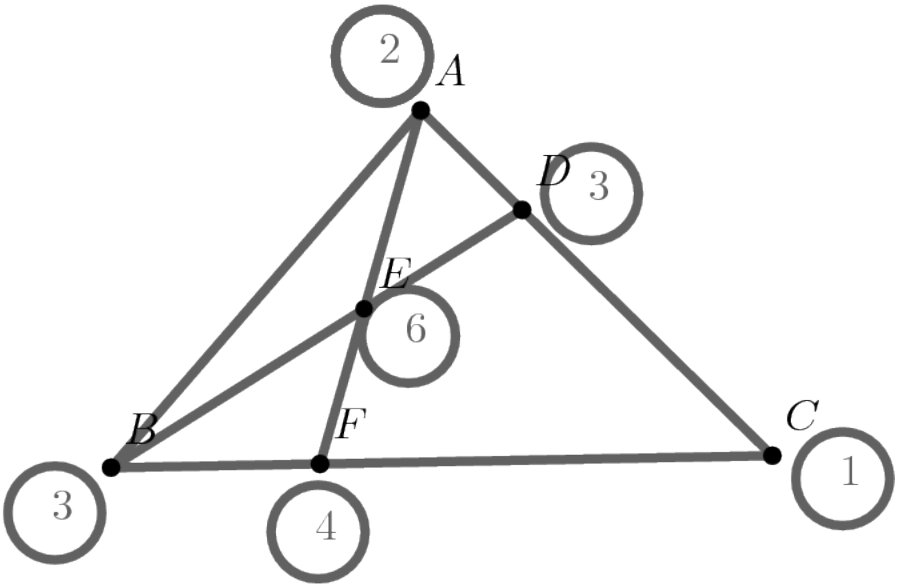
$\textbf{(A)}\ 24 \qquad \textbf{(B)}\ 30 \qquad \textbf{(C)}\ 32 \qquad \textbf{(D)}\ 36 \qquad \textbf{(E)}\ 40$

Solution 1

Draw X on $\overline{AF}$ such that XD is parallel to BC. That makes triangles BEF and EXD congruent since $BE = ED. FC = 3XD$ so $BC = 4BF$.

Since $AF = 3EF$ ($XE = EF$ and $AX = \frac{1}{3}AF$, so $XE = EF = \frac{1}{3}AF$), the altitude of triangle BEF is equal to $\frac{1}{3}$ of the altitude of ABC. The area of ABC is 360, so the area of $BEF = \frac{1}{3} \cdot \frac{1}{4} \cdot 360 = \boxed{\textbf{(B)}\ 30}$ ~heeeeeeeheeeee

Solution 2 (Mass Points)



First, when we see the problem, we see ratios, and we see that this triangle basically has no special properties (right, has medians, etc.) and this screams mass points at us.

First, we assign a mass of 2 to point A. We figure out that C has a mass of 1 since $2 \times 1 = 1 \times 2$. Then, by adding $1 + 2 = 3$, we get that point D has a mass of 3. By equality, point B has a mass of 3 also.

Now, we add $3 + 3 = 6$ for point E and $3 + 1 = 4$ for point F.

Now, BF is a common base for triangles ABF and EBF, so we figure out that the ratios of the areas is the ratios of the heights which is $\frac{AE}{EF} = 2 : 1$. So, EBF's area is one third the area of ABF, and we know the area of ABF is $\frac{1}{4}$ the area of ABC since they have the same heights but different bases.

So we get the area of EBF as $\frac{1}{3} \times \frac{1}{4} \times 360 = \boxed{\textbf{(B) } 30}$ -Brudder

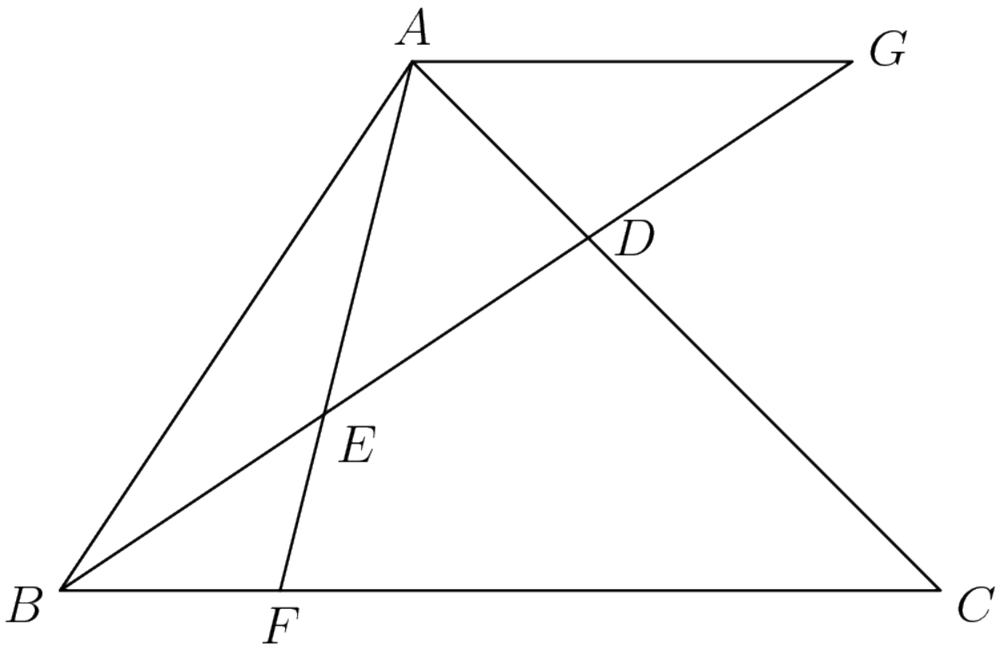
Note: We can also find the ratios of the areas using the reciprocal of the product of the mass points of EBF over the product of the mass points of ABC which is $\frac{2 \times 3 \times 1}{3 \times 6 \times 4} \times 360$ which also yields $\boxed{\textbf{(B) } 30}$ -Brudder

Solution 3

$\frac{BF}{FC}$ is equal to $\frac{\text{The area of triangle ABE}}{\text{The area of triangle ACE}}$. The area of triangle ABE is equal to 60 because it is equal to on half of the area of triangle ABD, which is equal to one third of the area of triangle ABC, which is 360. The area of triangle ACE is the sum of the areas of triangles AED and CED, which is respectively 60 and 120. So, $\frac{BF}{FC}$ is equal to $\frac{60}{180}$=$\frac{1}{3}$, so the area of triangle ABF is 90. That minus the area of triangle ABE is $\boxed{\textbf{(B) }30}$. ~~SmileKat32

Solution 4 (Similar Triangles)

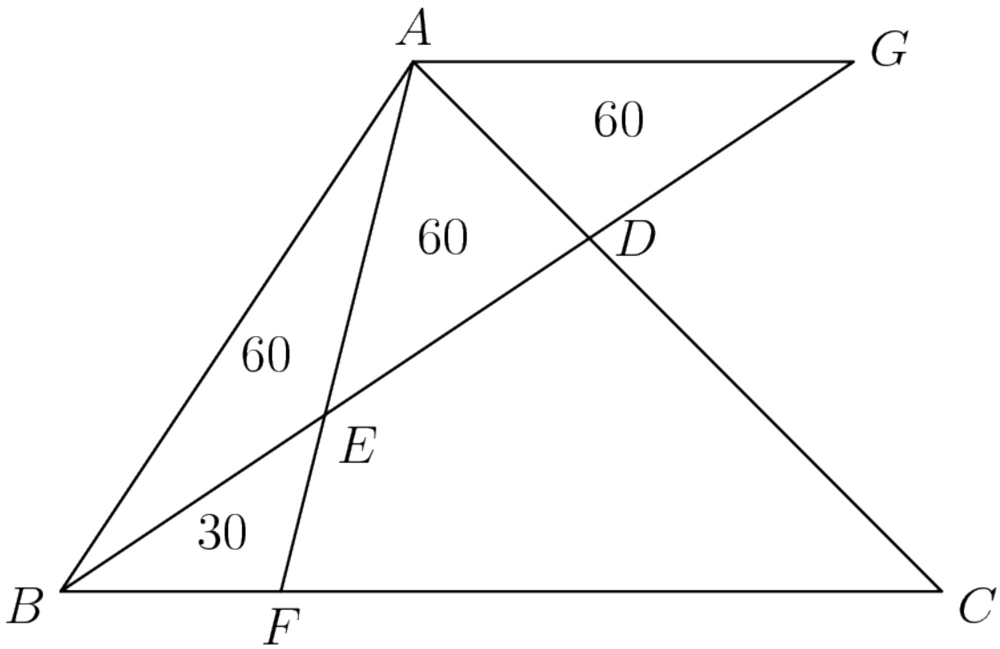
Extend $\overline{BD}$ to G such that $\overline{AG} \parallel \overline{BC}$ as



shown:

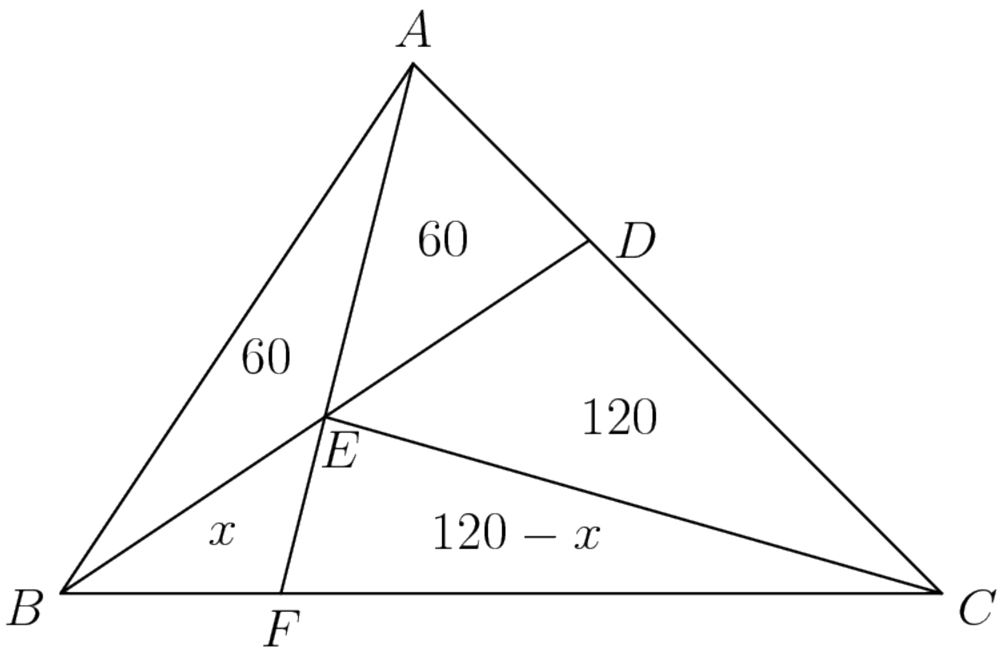
Then $\triangle ADG \sim \triangle CDB$ and $\triangle AEG \sim \triangle FEB$. Since $CD = 2AD$, triangle CDB has four times the area of triangle ADG. Since $[CDB] = 240$, we get $[ADG] = 60$.

Since $[AED]$ is also 60, we have $ED = DG$ because triangles AED and ADG have the same height and same areas and so their bases must be the congruent. Thus triangle AEG has twice the side lengths and therefore four times the area of triangle BEF, giving $[BEF] = (60+60)/4 = \boxed{\textbf{(B) } 30}$.



(Credit to MP8148 for the idea)

Solution 5 (Area Ratios)



As before we figure out the areas labeled in the diagram. Then we note that $\frac{EF}{AE}=\frac{x}{60}=\frac{120-x}{180}$. Solving gives $x=\boxed{\textbf{(B)}\ 30}$. (Credit to scrabbler94 for the idea)

Solution 6 (Coordinate Bashing)

Let ADB be a right triangle, and $BD=CD$

Let $A=(-2\sqrt{30},0)$

$B=(0,4\sqrt{30})$

$C=(4\sqrt{30},0)$

$D=(0,0)$

$E=(0,2\sqrt{30})$

$F=(\sqrt{30},3\sqrt{30})$

The line $\overleftrightarrow{AE}$ can be described with the equation $y=x-2\sqrt{30}$

The line $\overleftrightarrow{BC}$ can be described with $x+y=4\sqrt{30}$

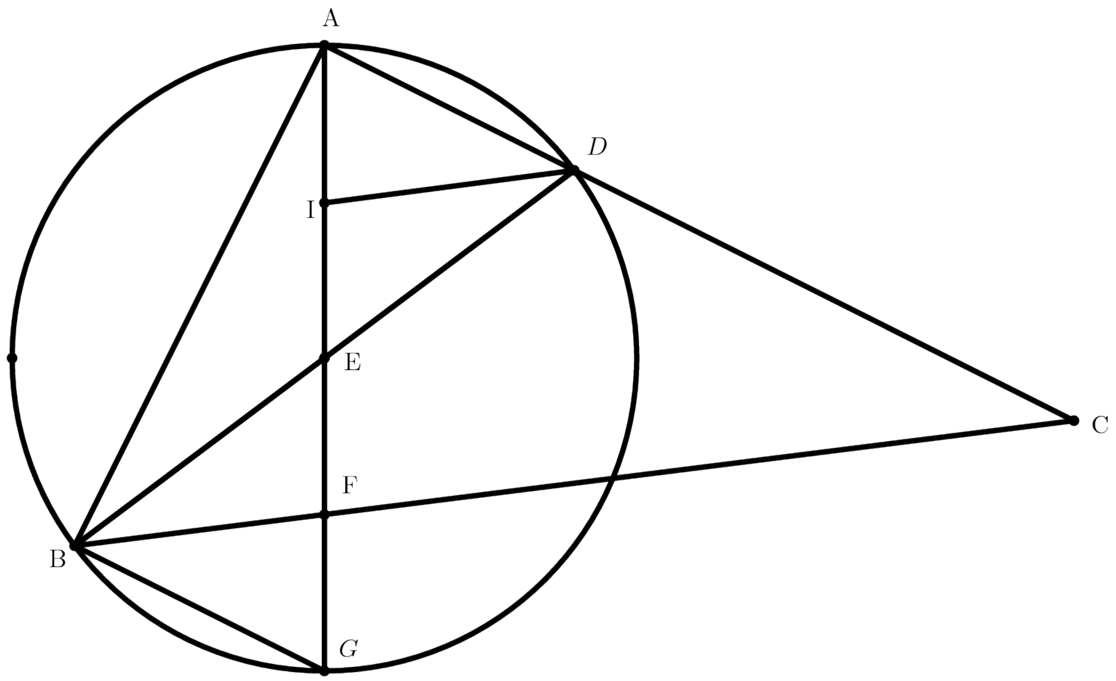
Solving, we get $x=3\sqrt{30}$ and $y=\sqrt{30}$

Now we can find $EF=BF=2\sqrt{15}$

$$[\triangle EBF]=\frac{(2\sqrt{15})^2}{2}=\boxed{\textbf{(B)}\ 30}\blacksquare$$

-Trex4days

Solution 7



Let $A[\Delta XYZ]$ = $Area\ of\ Triangle\ XYZ$

$A[\Delta ABD] : A[\Delta DBC] :: 1 : 2 :: 120 : 240$

$A[\Delta ABE] = A[\Delta AED] = 60$ (the median divides the area of the triangle into two equal parts)

Construction: Draw a circumcircle around ΔABD with BD as is diameter. Extend AF to G such that it meets the circle at G. Draw line BG.

$A[\Delta ABD] = A[\Delta ABG] = 120$ (Since $\Box ABGD$ is cyclic)

But $A[\Delta ABE]$ is common in both with an area of 60.

So, $A[\Delta AED] = A[\Delta BEG]$.

\therefore $A[\Delta AED] \cong A[\Delta BEG]$ (SAS Congruency Theorem).

In ΔAED, let DI be the median of ΔAED.

Which means $A[\Delta AID] = 30 = A[\Delta EID]$

Rotate ΔDEA to meet D at B and A at G. DE will fit exactly in BE (both are radii of the circle). From the above solutions, $\frac{AE}{EF} = 2:1$.

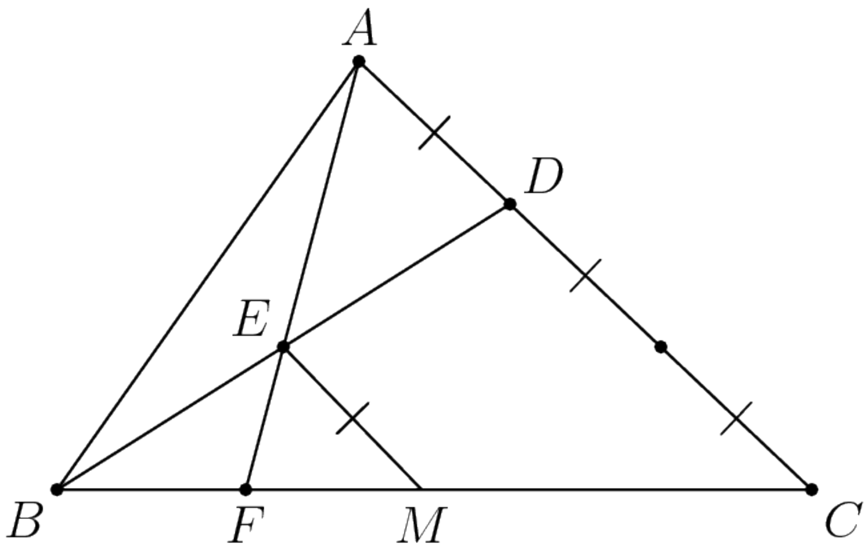
AE is a radius and EF is half of it implies EF = $\frac{radius}{2}$.

Which means $A[\Delta BEF] \cong A[\Delta DEI]$

Thus $A[\Delta BEF] = \boxed{\textbf{(B) } 30}$

~phoenixfire & flamewavelight

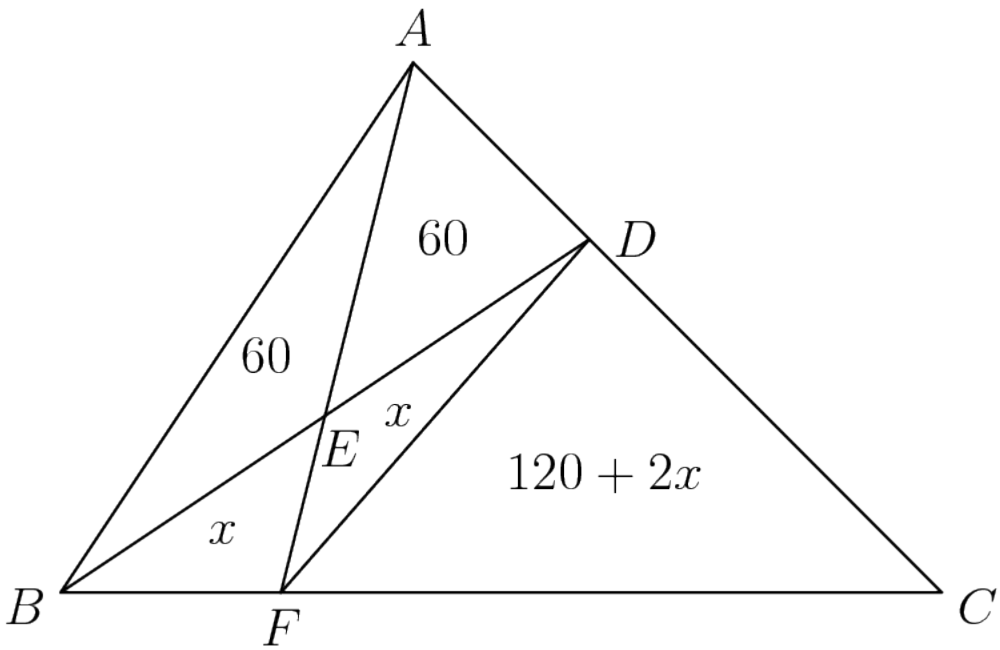
Solution 8



Using the ratio of $\overline{AD}$ and $\overline{CD}$, we find the area of $\triangle ADB$ is 120 and the area of $\triangle BDC$ is 240. Also using the fact that E is the midpoint of $\overline{BD}$, we know $\triangle ADE = \triangle ABE = 60$. Let M be a point such $\overline{EM}$ is parellel to $\overline{CD}$. We immediatley know that $\triangle BEM \sim BDC$ by 2. Using that we can conclude EM has ratio 1.

Using $\triangle EFM \sim \triangle AFC$, we get $EF : AE = 1 : 2$. Therefore using the fact that $\triangle EBF$ is in $\triangle ABF$, the area has ratio $\triangle BEF : \triangle ABE = 1 : 2$ and we know $\triangle ABE$ has area 60 so $\triangle BEF$ is $\boxed{\textbf{(B) }30}$. - fath2012

Solution 9

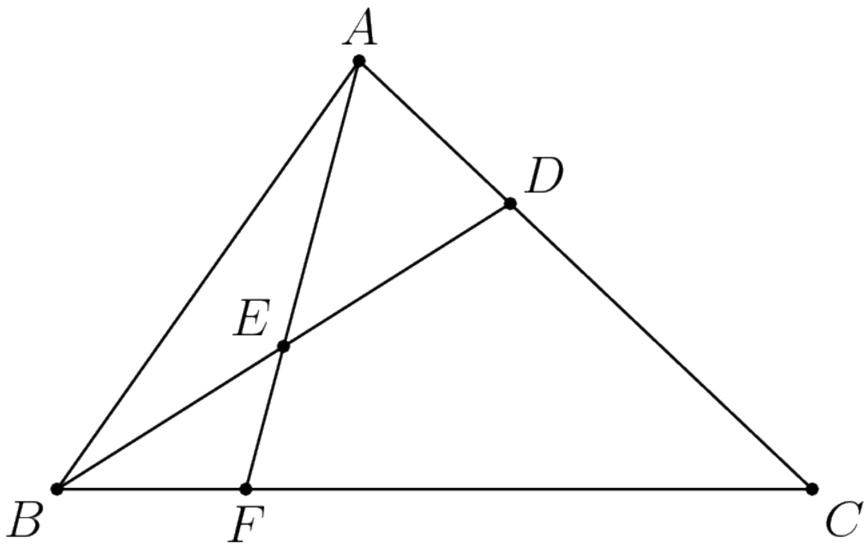


Labeling the areas in the diagram, we have:
$[DBC] = 240 = [BFE] + [FED] + [FDC] = x + x + 120 + 2x = 120 + 4x$

so $240 = 120 + 4x, 120 = 4x, 30 = x$.

So our answer is $\boxed{\textbf{(B) }30}$. ~~RWhite

Solution 10 (Menelaus's Theorem)



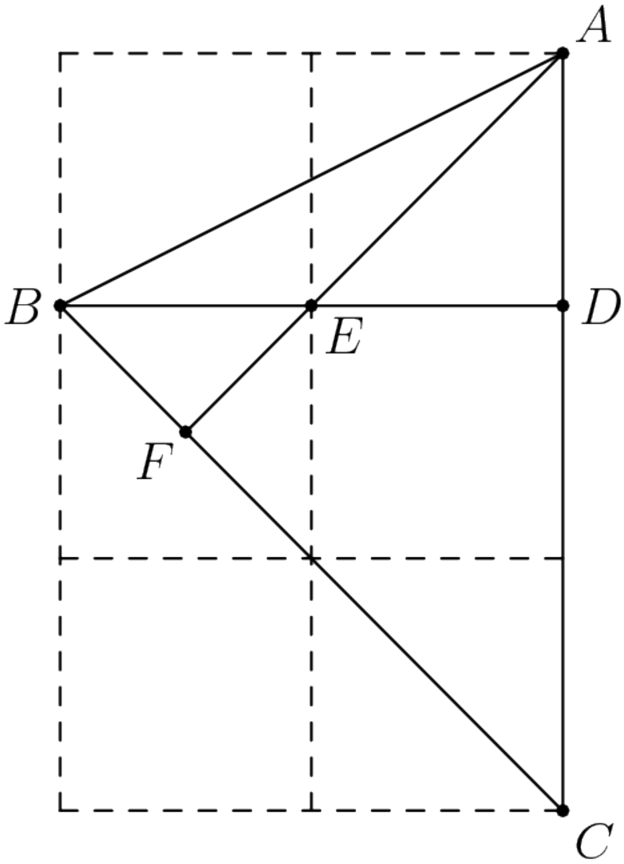
By Menelaus's Theorem on triangle BCD, we have

$$\frac{BF}{FC}\cdot\frac{CA}{DA}\cdot\frac{DE}{BE}=3\frac{BF}{FC}=1 \implies \frac{BF}{FC}=\frac{1}{3} \implies \frac{BF}{BC}=\frac{1}{4}.$$

Therefore,

$$[EBF]=\frac{BE}{BD}\cdot\frac{BF}{BC}\cdot[BCD]=\frac{1}{2}\cdot\frac{1}{4}\cdot\left(\frac{2}{3}\cdot[ABC]\right)=\boxed{\textbf{(B) } 30}.$$

Solution 11 (Graph Paper)



Note: If graph paper is unavailable, this solution can still be used by constructing a small grid on a sheet of blank paper.

As triangle ABC is loosely defined, we can arrange its points such that the diagram fits nicely on a coordinate plane. By doing so, we can construct it on graph paper and be able to visually determine the relative sizes of the triangles.

As point D splits line segment $\overline{AC}$ in a $1:2$ ratio, we draw $\overline{AC}$ as a vertical line segment 3 units long. Point D is thus 1 unit below point A and 2 units above point C. By definition, Point E splits line segment $\overline{BD}$ in a $1:1$ ratio, so we draw $\overline{BD}$ 2 units long directly left of D and draw E directly between B and D, 1 unit away from both.

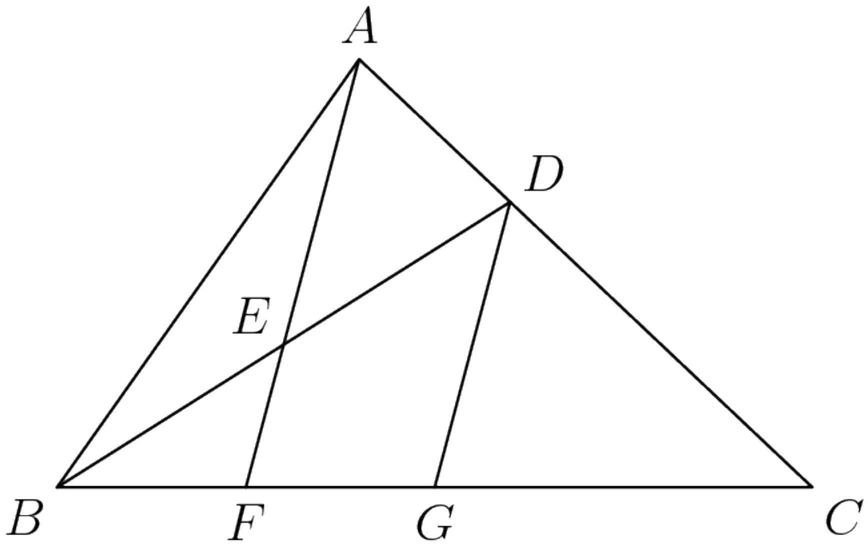
We then draw line segments $\overline{AB}$ and $\overline{BC}$. We can easily tell that triangle ABC occupies 3 square units of space. Constructing line AE and drawing F at the intersection of AE and BC, we can easily see that triangle EBF forms a right triangle occupying $\frac{1}{4}$ of a square unit of space.

The ratio of the areas of triangle EBF and triangle ABC is thus $\frac{1}{4} \div 3 = \frac{1}{12}$, and since the area of triangle ABC is 360, this means that the area of triangle EBF is $\frac{1}{12} \times 360 = \boxed{\textbf{(B) } 30}$.

~emerald_block

Additional note: There are many subtle variations of this triangle; this method is one of the more compact ones. ~i_equal_tan_90

Solution 12



We know that $AD = \frac{1}{3}AC$, so $[ABD] = \frac{1}{3}[ABC] = 120$. Using the same method, since $BE = \frac{1}{2}BD$, $[ABE] = \frac{1}{2}[ABD] = 60$.

Next, we draw G on $\overline{BC}$ such that $\overline{DG}$ is parallel to $\overline{AF}$ and create segment DG. We then observe that $\triangle AFC \sim \triangle DGC$, and since $AD : DC = 1 : 2$, $FG : GC$ is also equal to $1 : 2$. Similarly (no pun intended), $\triangle DBG \sim \triangle EBF$, and since $BE : ED = 1 : 1$, $BF : FG$ is also equal to $1 : 1$. Combining the information in these two ratios, we find that $BF : FG : GC = 1 : 1 : 2$, or equivalently, $BF = \frac{1}{4}BC$.

Thus, $[BFA] = \frac{1}{4}[BCA] = 90$. We already know that $[ABE] = 60$, so the area of $\triangle EBF$ is $[BFA] - [ABE] = \boxed{\textbf{(B) }30}$. ~i_equal_tan_90

Solution 13(fastest solution if you have no time)

The picture is misleading. Assume that the triangle ABC is right. Then find two factors of 720 that are the closest together so that the picture becomes easier in your mind. Quickly searching for squares near 720 to use difference of squares, we find 24 and 30 as our numbers. Then the coordinates of D are 10,16(note, A=0,0). E is then 5,8. Then the equation of the line AE is -16x/5+24=y. Plugging in y=0, we have x=15/2.Now notice that we have both the height and the base of EBF. Solving for the area, we have (8)(15/2)(1/2)=30.

Problem 25

Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

$\textbf{(A) }105\qquad\textbf{(B) }114\qquad\textbf{(C) }190\qquad\textbf{(D) }210\qquad\textbf{(E) }380$

Solution 1

We use stars and bars. The problem asks for the number of integer solutions (a,b,c) such that $a+b+c=24$ and $a,b,c\geq 2$. We can subtract 2 from a, b, c, so that we equivalently seek the number of non-negative integer solutions to $a'+b'+c'=18$. By stars and bars (using 18 stars and 2 bars), the number of solutions is $\binom{18+2}{2} = \binom{20}{2} = \boxed{\textbf{(C) }190}$.

Solution 2

Without loss of generality, let's assume that Alice has 2 apples. There are 19 ways to split the rest of the apples with Becky and Chris. If Alice has 3 apples, there are 18 ways to split the rest of the apples with Becky and Chris. If Alice has 4 apples, there are 17 ways to split the rest. So the total number of ways to split 24 apples between the three friends is equal to $19+18+17.........+1 = 20 \times \frac{19}{2} = \boxed{\textbf{(C) } 190}$

Solution 3

Let's assume that the three of them have x, y, z apples. Since each of them has to have at least 2 apples, we say that $a+2=x, b+2=y$ and $c+2=z$.

Thus, $a+b+c+6=24 \implies a+b+c=18$, and so by stars and bars, the number of solutions for this is

$$\binom{n+k-1}{k} \implies \binom{18+3-1}{3-1} \implies \binom{20}{2} = \boxed{\textbf{(C) } 190}$$