2005 AMC 12A Problems/Problem 1

Problem

Two is 10% of x and 20% of y. What is $x - y$?

$\text{(A) } 1 \qquad \text{(B) } 2 \qquad \text{(C) } 5 \qquad \text{(D) } 10 \qquad \text{(E) } 20$

Solution

$$2 = \frac{1}{10}x \Longrightarrow x = 20, \quad 2 = \frac{1}{5}y \Longrightarrow y = 10, \quad x - y = 20 - 10 = 10 \text{(D)}.$$

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
Preceded by First question	Followed by Problem 2
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2005 AMC 12A Problems/Problem 2

The following problem is from both the 2005 AMC 12A #2 and 2005 AMC 10A #3, so both problems redirect to this page.

Problem

The equations $2x + 7 = 3$ and $bx - 10 = -2$ have the same solution. What is the value of b?

$\mathrm{(A)}\ -8 \qquad \mathrm{(B)}\ -4 \qquad \mathrm{(C)}\ 2 \qquad \mathrm{(D)}\ 4 \qquad \mathrm{(E)}\ 8$

Solution

$$2x + 7 = 3 \Longrightarrow x = -2, \quad -2b - 10 = -2 \Longrightarrow -2b = 8 \Longrightarrow b = -4\ \mathrm{(B)}$$

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
Preceded by Problem 1	Followed by Problem 3
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2005 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2005))	
Preceded by Problem 2	Followed by Problem 4
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All AMC 10 Problems and Solutions	

Category: Introductory Algebra Problems

2005 AMC 12A Problems/Problem 3

Problem

A rectangle with diagonal length x is twice as long as it is wide. What is the area of the rectangle?

$\text{(A) } \frac{1}{4}x^2 \qquad \text{(B) } \frac{2}{5}x^2 \qquad \text{(C) } \frac{1}{2}x^2 \qquad \text{(D) } x^2 \qquad \text{(E) } \frac{3}{2}x^2$

Solution

Let w be the width, so the length is $2w$. By the Pythagorean Theorem, $w^2 + 4w^2 = x^2 \Longrightarrow \frac{x}{\sqrt{5}} = w$. The area of the rectangle is $(w)(2w) = 2w^2 = 2\left(\frac{x}{\sqrt{5}}\right)^2 = \frac{2}{5}x^2 \text{ (B)}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources)

Preceded by Problem 2	Followed by Problem 4

1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25

All AMC 12 Problems and Solutions

Category:

- Introductory Geometry Problems

2005 AMC 12A Problems/Problem 4

Problem

A store normally sells windows at <dollar/>100 each. This week the store is offering one free window for each purchase of four. Dave needs seven windows and Doug needs eight windows. How much will they save if they purchase the windows together rather than separately?

$\mathrm{(A)}\ 100\qquad \mathrm{(B)}\ 200\qquad \mathrm{(C)}\ 300\qquad \mathrm{(D)}\ 400\qquad \mathrm{(E)}\ 500$

Solution

For n windows, the store offers a discount of $100 \cdot \left\lfloor \frac{n}{5} \right\rfloor$ (floor function). Dave receives a discount of $100 \cdot \left\lfloor \frac{7}{5} \right\rfloor = 100$ and Doug receives a discount of $100 \cdot \left\lfloor \frac{8}{5} \right\rfloor = 100$. These amount to 200 dollars in discounts. Together, they receive a discount of $100 \cdot \left\lfloor \frac{15}{5} \right\rfloor = 300$, so they save $300-200=100\ \mathrm{(A)}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
Preceded by Problem 3	Followed by Problem 5
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2005 AMC 12A Problems/Problem 5

Problem

The average (mean) of 20 numbers is 30, and the average of 30 other numbers is 20. What is the average of all 50 numbers?

$\mathrm{(A)}\ 23\qquad\mathrm{(B)}\ 24\qquad\mathrm{(C)}\ 25\qquad\mathrm{(D)}\ 10\qquad\mathrm{(E)}\ 27$

Solution

The sum of the first 20 numbers is $20 \cdot 30$ and the sum of the other 30 numbers is $30 \cdot 20$. Hence the overall average is $\frac{20 \cdot 30 + 30 \cdot 20}{50} = 24\ \mathrm{(B)}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
Preceded by Problem 4	Followed by Problem 6
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2005 AMC 12A Problems/Problem 6

Problem

Josh and Mike live 13 miles apart. Yesterday, Josh started to ride his bicycle toward Mike's house. A little later Mike started to ride his bicycle toward Josh's house. When they met, Josh had ridden for twice the length of time as Mike and at four-fifths of Mike's rate. How many miles had Mike ridden when they met?

$\mathrm{(A)}\ 4\qquad \mathrm{(B)}\ 5\qquad \mathrm{(C)}\ 6\qquad \mathrm{(D)}\ 7\qquad \mathrm{(E)}\ 8$

Solution

Let D_J, D_M be the distances traveled by Josh and Mike, respectively, and let r, t be the time and rate of Mike. Using $d = rt$, we have that $D_M = rt$ and $D_J = \left(\frac{4}{5}r\right)(2t) = \frac{8}{5}rt$. Then

$$13 = D_M + D_J = rt + \frac{8}{5}rt = \frac{13}{5}rt \Longrightarrow rt = D_M = 5\ \mathrm{(B)}.$$

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
Preceded by Problem 5	Followed by Problem 7
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

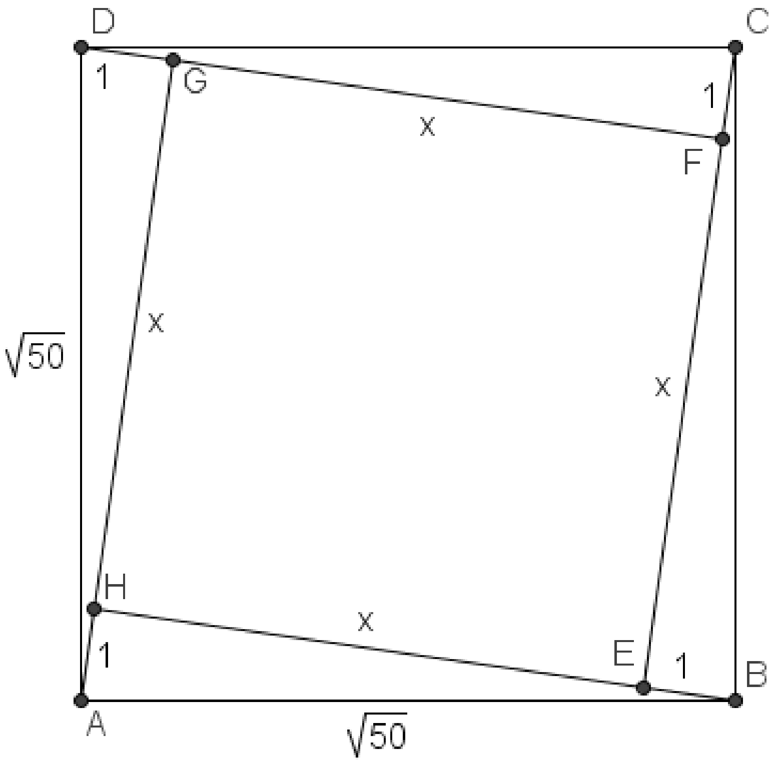
2005 AMC 12A Problems/Problem 7

Problem

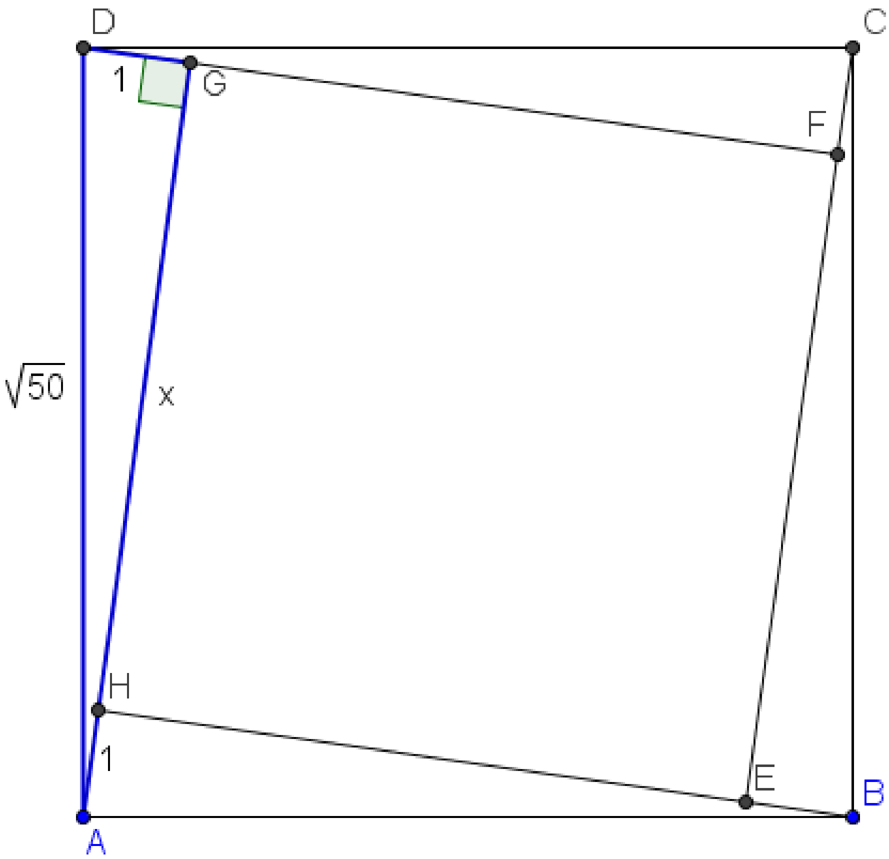
Square $EFGH$ is inside the square $ABCD$ so that each side of $EFGH$ can be extended to pass through a vertex of $ABCD$. Square $ABCD$ has side length $\sqrt{50}$ and $BE = 1$. What is the area of the inner square $EFGH$?

$\mathrm{(A)}\ 25 \qquad \mathrm{(B)}\ 32 \qquad \mathrm{(C)}\ 36 \qquad \mathrm{(D)}\ 40 \qquad \mathrm{(E)}\ 42$

Solution



Arguable the hardest part of this question is to visualize the diagram. Since each side of $EFGH$ can be extended to pass through a vertex of $ABCD$, we realize that $EFGH$ must be tilted in such a fashion. Let a side of $EFGH$ be x.



Notice the right triangle (in blue) with legs $1, x+1$ and hypotenuse $\sqrt{50}$. By the Pythagorean Theorem, we have $1^2 + (x+1)^2 = (\sqrt{50})^2 \implies (x+1)^2 = 49 \implies x = 6$. Thus, $[EFGH] = x^2 = 36\ (\mathrm{C})$

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
Preceded by Problem 6	Followed by Problem 8
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All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2005 AMC 12A Problems/Problem 8

Problem

Let A, M, and C be digits with

$$(100A + 10M + C)(A + M + C) = 2005$$

What is A?

$\mathrm{(A)}\ 1 \qquad \mathrm{(B)}\ 2 \qquad \mathrm{(C)}\ 3 \qquad \mathrm{(D)}\ 4 \qquad \mathrm{(E)}\ 5$

Solution

Clearly the two quantities are both integers, so we check the prime factorization of $2005 = 5 \cdot 401$. It is easy to see now that $(A, M, C) = (4, 0, 1)$ works, so the answer is $\mathrm{(D)}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
Preceded by Problem 7	Followed by Problem 9
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2005 AMC 12A Problems/Problem 9

Contents

Problem

There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of these values of a?

$\mathrm{(A)}\ -16 \qquad \mathrm{(B)}\ -8 \qquad \mathrm{(C)}\ 0 \qquad \mathrm{(D)}\ 8 \qquad \mathrm{(E)}\ 20$

Solution

Solution 1

A quadratic equation always has two roots, unless it has a double root. That means we can write the quadratic as a square, and the coefficients 4 and 9 suggest this. Completing the square, $0 = (2x \pm 3)^2 = 4x^2 \pm 12x + 9$, so $\pm 12 = a + 8 \Longrightarrow a = 4, -20$. The sum of these is $-20 + 4 = -16 \Rightarrow \mathrm{(A)}$.

Solution 2

Another method would be to use the quadratic formula, since our x^2 coefficient is given as 4, the x coefficient is $a+8$ and the constant term is 9. Hence, $x = \frac{-(a+8) \pm \sqrt{(a+8)^2 - 4(4)(9)}}{2(4)}$

Because we want only a single solution for x, the determinant must equal 0. Therefore, we can write $(a+8)^2 - 144 = 0$ which factors to $a^2 + 16a - 80 = 0$; using Vieta's formulas we see that the sum of the solutions for a is the opposite of the coefficient of a, or $-16 \Rightarrow \mathrm{(A)}$.

Solution 3

Using the discriminant, the result must equal 0. $D = b^2 - 4ac = (a+8)^2 - 4(4)(9)$ $= a^2 + 16a + 64 - 144 = a^2 + 16a - 80 = 0 \Rightarrow (a+20)(a-4) = 0$ Therefore, $a = -20$ or $a = 4$, giving a sum of $-16 \Rightarrow \mathrm{(A)}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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2005 AMC 12A Problems/Problem 10

Problem

A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

$\textbf{(A)}\ 3\qquad\textbf{(B)}\ 4\qquad\textbf{(C)}\ 5\qquad\textbf{(D)}\ 6\qquad\textbf{(E)}\ 7$

Solution

There are $6n^3$ sides total on the unit cubes, and $6n^2$ are painted red.

$$\frac{6n^2}{6n^3} = \frac{1}{4} \Rightarrow n = 4 \rightarrow \mathrm{B}$$

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
Preceded by Problem 9	Followed by Problem 11
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2005 AMC 12A Problems/Problem 11

Problem

How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

$\mathrm{(A)}\ 41\qquad\mathrm{(B)}\ 42\qquad\mathrm{(C)}\ 43\qquad\mathrm{(D)}\ 44\qquad\mathrm{(E)}\ 45$

Contents

Solution

Solution 1

Let the digits be A, B, C so that $B = \frac{A+C}{2}$. In order for this to be an integer, A and C have to have the same parity. There are 9 possibilities for A, and 5 for C. B depends on the value of both A and C and is unique for each (A,C). Thus our answer is $9 \cdot 5 \cdot 1 = 45 \implies E$.

Solution 2

Thus, the three digits form an arithmetic sequence.

- If the numbers are all the same, then there are 9 possible three-digit numbers.
- If the numbers are different, then we count the number of strictly increasing arithmetic sequences between 0 and 10 and multiply by 2 for the decreasing ones:

Common difference	Sequences possible	Number of sequences
1	$012, \ldots, 789$	8
2	$024, \ldots, 579$	6
3	$036, \ldots, 369$	4
4	$048, \ldots, 159$	2

This gives us $2(8+6+4+2) = 40$. However, the question asks for three-digit numbers, so we have to ignore the four sequences starting with 0. Thus our answer is $40 + 9 - 4 = 45 \Longrightarrow \mathrm{(E)}$.

See also

- similar problem

2005 AMC 12A Problems/Problem 12

Problem

A line passes through $A\ (1,1)$ and $B\ (100,1000)$. How many other points with integer coordinates are on the line and strictly between A and B?

$\mathrm{(A)}\ 0\qquad\mathrm{(B)}\ 2\qquad\mathrm{(C)}\ 3\qquad\mathrm{(D)}\ 8\qquad\mathrm{(E)}\ 9$

Solution

For convenience' s sake, we can transform A to the origin and B to $(99,999)$ (this does not change the problem). The line AB has the equation $y = \frac{999}{99}x = \frac{111}{11}x$. The coordinates are integers if $11|x$, so the values of x are $11, 22 \ldots 88$, with a total of $8\ \mathrm{(D)}$ coordinates.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
Preceded by Problem 11	Followed by Problem 13
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All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2005 AMC 12A Problems/Problem 13

Problem

The regular 5-point star $ABCDE$ is drawn and in each vertex, there is a number. Each $A, B, C, D,$ and E are chosen such that all 5 of them came from set $\{3, 5, 6, 7, 9\}$. Each letter is a different number (so one possible way is $A = 3, B = 5, C = 6, D = 7, E = 9$). Let AB be the sum of the numbers on A and B, and so forth. If $AB, BC, CD, DE,$ and EA form an arithmetic sequence (not necessarily in increasing order), find the value of CD.

$\mathrm{(A)}\ 9 \qquad \mathrm{(B)}\ 10 \qquad \mathrm{(C)}\ 11 \qquad \mathrm{(D)}\ 12 \qquad \mathrm{(E)}\ 13$

Solution

$AB + BC + CD + DE + EA = 2(A + B + C + D + E)$. The sum $A + B + C + D + E$ will always be $3 + 5 + 6 + 7 + 9 = 30$, so the arithmetic sequence has a sum of $2 \cdot 30 = 60$. Since CD is the middle term, it must be the average of the five numbers, of $\frac{60}{5} = 12 \Longrightarrow \mathrm{(D)}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2005 AMC 12A Problems/Problem 14

Problem

On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of dots?

$\mathrm{(A)}\ \frac{5}{11} \qquad \mathrm{(B)}\ \frac{10}{21} \qquad \mathrm{(C)}\ \frac{1}{2} \qquad \mathrm{(D)}\ \frac{11}{21} \qquad \mathrm{(E)}\ \frac{6}{11}$

Solution

There are $1 + 2 + 3 + 4 + 5 + 6 = 21$ dots total. Casework:

- The dot is removed from an even face. There is a $\frac{2 + 4 + 6}{21} = \frac{4}{7}$ chance of this happening. Then there are 4 odd faces, giving us a probability of $\frac{4}{7} \cdot \frac{4}{6} = \frac{8}{21}$.
- The dot is removed from an odd face. There is a $\frac{1 + 3 + 5}{21} = \frac{3}{7}$ chance of this happening. Then there are 2 odd faces, giving us a probability of $\frac{3}{7} \cdot \frac{2}{6} = \frac{1}{7}$.

Thus the answer is $\frac{8}{21} + \frac{1}{7} = \frac{11}{21} \Longrightarrow \mathrm{(D)}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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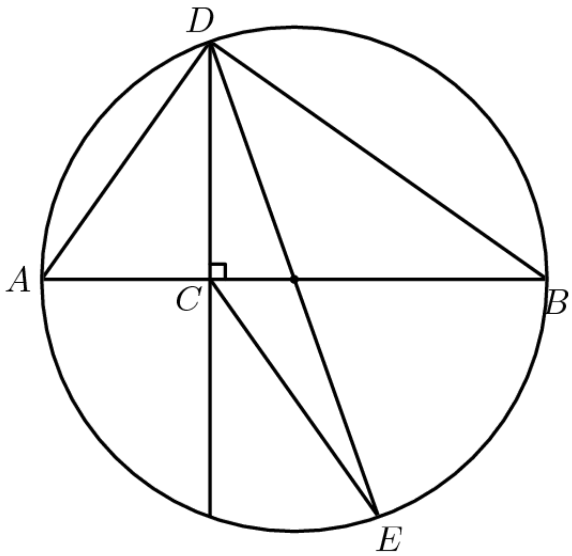
Category: Introductory Combinatorics Problems

2005 AMC 12A Problems/Problem 15

Contents

Problem

Let $\overline{AB}$ be a diameter of a circle and C be a point on $\overline{AB}$ with $2 \cdot AC = BC$. Let D and E be points on the circle such that $\overline{DC} \perp \overline{AB}$ and $\overline{DE}$ is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?



$(\mathrm{A})\ \frac{1}{6} \qquad (\mathrm{B})\ \frac{1}{4} \qquad (\mathrm{C})\ \frac{1}{3} \qquad (\mathrm{D})\ \frac{1}{2} \qquad (\mathrm{E})\ \frac{2}{3}$

Solution

Solution 1

Notice that the bases of both triangles are diameters of the circle. Hence the ratio of the areas is just the ratio of the heights of the triangles, or $\frac{CD}{CF}$ (F is the foot of the perpendicular from C to DE).

Call the radius r. Then $AC = \frac{1}{3}(2r) = \frac{2}{3}r$, $CO = \frac{1}{3}r$. Using the Pythagorean Theorem in $\triangle OCD$, we get $\frac{1}{3}r^2 + CD^2 = r^2 \Longrightarrow CD = \frac{2\sqrt{2}}{3}r$.

Now we have to find CF. Notice $\triangle OCD \sim \triangle OFC$, so we can write the proportion:

$$\frac{OF}{OC} = \frac{OC}{OD}$$

$$\frac{OF}{\frac{1}{3}r} = \frac{\frac{1}{3}r}{r}$$
$$OF = \frac{1}{9}r$$

By the Pythagorean Theorem in $\triangle OFC$, we have
$\left(\frac{1}{9}r\right)^2 + CF^2 = \left(\frac{1}{3}r\right)^2 \implies CF = \sqrt{\frac{8}{81}r^2} = \frac{2\sqrt{2}}{9}r.$

Our answer is $\frac{CD}{CF} = \frac{\frac{2\sqrt{2}}{3}r}{\frac{2\sqrt{2}}{9}r} = \frac{1}{3} \implies \text{(C)}.$

Solution 2

Let the center of the circle be O.

Note that $2 \cdot AC = BC \Rightarrow 3 \cdot AC = AB$.

O is midpoint of $AB \Rightarrow \frac{3}{2}AC = AO \Rightarrow CO = \frac{1}{3}AO \Rightarrow CO = \frac{1}{6}AB.$

O is midpoint of $DE \Rightarrow$ Area of $\triangle DCE = 2\cdot$ Area of $\triangle DCO = 2 \cdot (\frac{1}{6}\cdot$ Area of $\triangle ABD) = \frac{1}{3}\cdot$ Area of $\triangle ABD \implies \text{(C)}.$

Solution 3

Let r be the radius of the circle. Note that $AC + BC = 2r$ so $AC = \frac{2}{3}r$.

By Power of a Point Theorem, $CD^2 = AC \cdot BC = 2 \cdot AC^2$, and thus $CD = \sqrt{2} \cdot AC = \frac{2\sqrt{2}}{3}r$

Then the area of $\triangle ABD$ is $\frac{1}{2}AB \cdot CD = \frac{2\sqrt{2}}{3}r^2$. Similarly, the area of $\triangle DCE$ is $\frac{1}{2}(r - AC) \cdot 2 \cdot CD = \frac{2\sqrt{2}}{9}r^2$, so the desired ratio is $\frac{\frac{2\sqrt{2}}{9}r^2}{\frac{2\sqrt{2}}{3}r^2} = \frac{1}{3} \implies \text{(C)}$

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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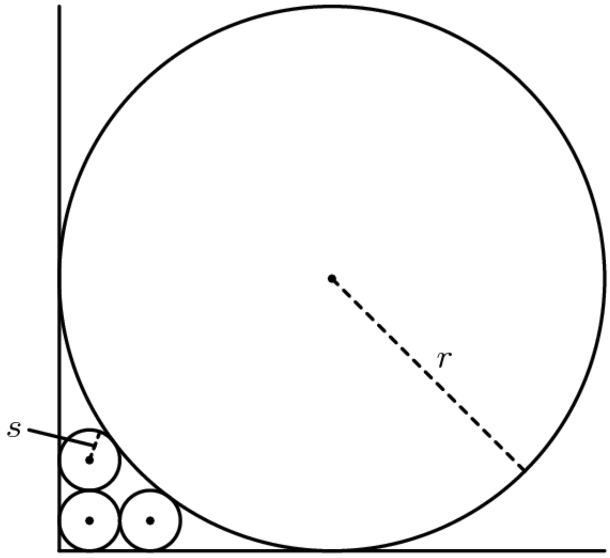
2005 AMC 12A Problems/Problem 16

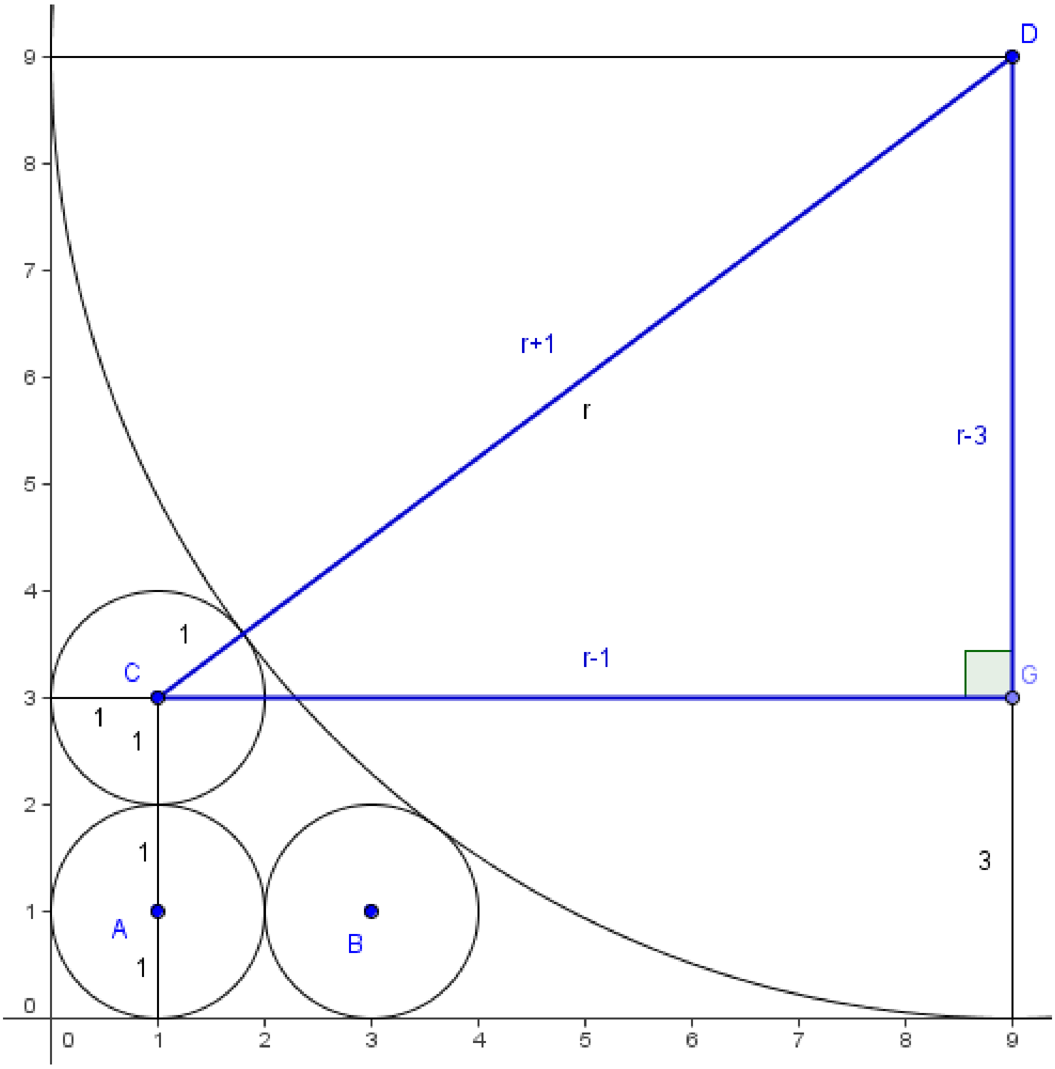
Problem

Three circles of radius s are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?



$\mathrm{(A)}\ 5 \qquad \mathrm{(B)}\ 6 \qquad \mathrm{(C)}\ 8 \qquad \mathrm{(D)}\ 9 \qquad \mathrm{(E)}\ 10$

Solution



Without loss of generality, let $s = 1$. Draw the segment between the center of the third circle and the large circle; this has length $r + 1$. We then draw the radius of the large circle that is perpendicular to the x-axis, and draw the perpendicular from this radius to the center of the third circle. This gives us a right triangle with legs $r - 3, r - 1$ and hypotenuse $r + 1$. The Pythagorean Theorem yields:

$$(r-3)^2 + (r-1)^2 = (r+1)^2$$
$$r^2 - 10r + 9 = 0$$
$$r = 1, 9$$

Quite obviously $r > s = 1$, so $r = 9$ and $\frac{r}{s} = \frac{9}{1} = 9 \Longrightarrow \mathrm{(D)}$.

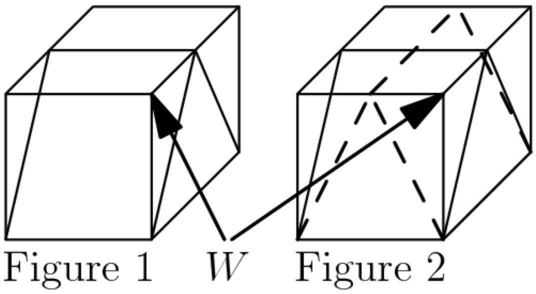
See also

2005 AMC 12A Problems/Problem 17

Problem

A unit cube is cut twice to form three triangular prisms, two of which are congruent, as shown in Figure 1. The cube is then cut in the same manner along the dashed lines shown in Figure 2. This creates nine pieces. What is the volume of the piece that contains vertex W?

$\textbf{(A)}\ \frac{1}{12} \qquad \textbf{(B)}\ \frac{1}{9} \qquad \textbf{(C)}\ \frac{1}{8} \qquad \textbf{(D)}\ \frac{1}{6} \qquad \textbf{(E)}\ \frac{1}{4}$



Solution

It is a pyramid, so $\frac{1}{3}\cdot\left(\frac{1}{4}\right)\cdot(1)=\frac{1}{12}\Rightarrow\boxed{\textbf{(A)}}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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2005 AMC 12A Problems/Problem 18

Problem

Call a number prime-looking if it is composite but not divisible by $2, 3,$ or $5.$ The three smallest prime-looking numbers are $49, 77$, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

$\mathrm{(A)}\ 100 \qquad \mathrm{(B)}\ 102 \qquad \mathrm{(C)}\ 104 \qquad \mathrm{(D)}\ 106 \qquad \mathrm{(E)}\ 108$

Solution

The given states that there are 168 prime numbers less than 1000, which is a fact we must somehow utilize. Since there seems to be no easy way to directly calculate the number of "prime-looking" numbers, we can apply complementary counting. We can split the numbers from 1 to 1000 into several groups: $\{1\}$, $\{\mathrm{numbers\ divisible\ by\ 2 = S_2}\}$, $\{\mathrm{numbers\ divisible\ by\ 3 = S_3}\}$, $\{\mathrm{numbers\ divisible\ by\ 5 = S_5}\}$, $\{\mathrm{primes\ not\ including\ 2,3,5}\}$, $\{\mathrm{prime-looking}\}$. Hence, the number of prime-looking numbers is $1000 - 165 - 1 - |S_2 \cup S_3 \cup S_5|$ (note that $2, 3, 5$ are primes).

We can calculate $S_2 \cup S_3 \cup S_5$ using the Principle of Inclusion-Exclusion: (the values of $|S_2| \ldots$ and their intersections can be found quite easily)

$$\begin{aligned}|S_2 \cup S_3 \cup S_5| &= |S_2| + |S_3| + |S_5| - |S_2 \cap S_3| - |S_3 \cap S_5| - |S_2 \cap S_5| + |S_2 \cap S_3 \cap S_5| \\ &= 500 + 333 + 200 - 166 - 66 - 100 + 33 = 734\end{aligned}$$

Substituting, we find that our answer is $1000 - 165 - 1 - 734 = 100 \Longrightarrow \mathrm{(A)}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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2005 AMC 12A Problems/Problem 19

Contents

Problem

A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. If the odometer now reads 002005, how many miles has the car actually traveled?

$\mathrm{(A)}\ 1404 \qquad \mathrm{(B)}\ 1462 \qquad \mathrm{(C)}\ 1604 \qquad \mathrm{(D)}\ 1605 \qquad \mathrm{(E)}\ 1804$

Solutions

Solution 1

We find the number of numbers with a 4 and subtract from 2005. Quick counting tells us that there are 200 numbers with a 4 in the hundreds place, 200 numbers with a 4 in the tens place, and 201 numbers with a 4 in the units place (counting 2004). Now we apply the Principle of Inclusion-Exclusion. There are 20 numbers with a 4 in the hundreds and in the tens, and 20 for both the other two intersections. The intersection of all three sets is just 2. So we get:

$$2005-(200+200+201-20-20-20+2) = 1462 \Longrightarrow \mathrm{(B)}$$

Solution 2

Alternatively, consider that counting without the number 4 is almost equivalent to counting in base 9; only, in base 9, the number 9 is not counted. Since 4 is skipped, the symbol 5 represents 4 miles of travel, and we have traveled 2004_9 miles. By basic conversion,

$$2005_9=9^3(2)+9^0(5)=729(2)+1(5)=1458+5=1463.$$

$$1463-1=\boxed{1462}$$

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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2005 AMC 12A Problems/Problem 20

Problem

For each x in $[0,1]$, define

$$\begin{array}{lll} f(x) & = 2x, & \text{if } 0 \le x \le \frac{1}{2}; \\ f(x) & = 2-2x, & \text{if } \frac{1}{2} < x \le 1. \end{array}$$

Let $f^{[2]}(x) = f(f(x))$, and $f^{[n+1]}(x) = f^{[n]}(f(x))$ for each integer $n \ge 2$. For how many values of x in $[0,1]$ is $f^{[2005]}(x) = \frac{1}{2}$?

$$\text{(A) } 0 \qquad \text{(B) } 2005 \qquad \text{(C) } 4010 \qquad \text{(D) } 2005^2 \qquad \text{(E) } 2^{2005}$$

Solution

For the two functions $f(x) = 2x, 0 \le x \le \frac{1}{2}$ and $f(x) = 2-2x, \frac{1}{2} \le x \le 1$, as long as $f(x)$ is between 0 and 1, x will be in the right domain, so we don't need to worry about the domain of x.

Also, every time we change $f(x)$, the expression for the final answer in terms of x will be in a different form(although they'll all satisfy the final equation), so we get a different starting value of x. Every time we have two choices for $f(x)$ and altogether we have to choose 2005 times. Thus, $2^{2005} \Rightarrow \boxed{E}$.

Note: the values of x that satisfy $f^{[n]}(x) = \frac{1}{2}$ are $\frac{1}{2^{n+1}}, \frac{3}{2^{n+1}}, \frac{5}{2^{n+1}}, \ldots, \frac{2^{n+1}-1}{2^{n+1}}$.

See Also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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Category: Introductory Algebra Problems

2005 AMC 12A Problems/Problem 21

Problem

How many ordered triples of integers (a,b,c), with $a \ge 2$, $b\ge 1$, and $c \ge 0$, satisfy both $\log_a b = c^{2005}$ and $a + b + c = 2005$?

$\mathrm{(A)}\ 0 \qquad \mathrm{(B)}\ 1 \qquad \mathrm{(C)}\ 2 \qquad \mathrm{(D)}\ 3 \qquad \mathrm{(E)}\ 4$

Solution

$a^{c^{2005}} = b$

Casework upon c:

- $c = 0$: Then $a^0 = b \Longrightarrow b = 1$. Thus we get $(2004,1,0)$.
- $c = 1$: Then $a^1 = b \Longrightarrow a = b$. Thus we get $(1002,1002,1)$.
- $c \ge 2$: Then the exponent of a becomes huge, and since $a \ge 2$ there is no way we can satisfy the second condition. Hence we have two ordered triples $\mathrm{(C)}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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2005 AMC 12A Problems/Problem 22

Problem

A rectangular box P is inscribed in a sphere of radius r. The surface area of P is 384, and the sum of the lengths of its 12 edges is 112. What is r?

$\mathrm{(A)}\ 8\qquad \mathrm{(B)}\ 10\qquad \mathrm{(C)}\ 12\qquad \mathrm{(D)}\ 14\qquad \mathrm{(E)}\ 16$

Solution

The box P has dimensions a, b, and c. Therefore,

$$2ab+2ac+2bc=384$$

$$4a+4b+4c=112 \Longrightarrow a+b+c=28$$

Now we make a formula for r. Since the diameter of the sphere is the space diagonal of the box,

$$r=\frac{\sqrt{a^2+b^2+c^2}}{2}$$

We square $a+b+c$:

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc=a^2+b^2+c^2+384=784$$

We get that

$$r=\frac{\sqrt{a^2+b^2+c^2}}{2}=\boxed{\mathrm{(B)}\ 10}$$

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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Categories: Intermediate Geometry Problems | 3D Geometry Problems

2005 AMC 12A Problems/Problem 23

Problem

Two distinct numbers a and b are chosen randomly from the set $\{2, 2^2, 2^3, ..., 2^{25}\}$. What is the probability that $\log_a b$ is an integer?

$\mathrm{(A)}\ \frac{2}{25}\qquad \mathrm{(B)}\ \frac{31}{300}\qquad \mathrm{(C)}\ \frac{13}{100}\qquad \mathrm{(D)}\ \frac{7}{50}\qquad \mathrm{(E)}\ \frac{1}{2}$

Solution

Let $\log_a b = z$, so $a^z = b$. Define $a = 2^x$, $b = 2^y$; then $(2^x)^z = 2^{xz} = 2^y$, so $x|y$. Here we can just make a table and count the number of values of y per value of x. The largest possible value of x is 12, and we get $\sum_{x=1}^{12} \lfloor \frac{25}{x} - 1 \rfloor = 24 + 11 + 7 + 5 + 4 + 3 + 2 + 2 + 1 + 1 + 1 + 1 = 62$.

The total number of ways to pick two distinct numbers is $\frac{25!}{(25-2)!} = 25 \cdot 24 = 600$, so we get a probability of $\frac{62}{600} = \boxed{\mathrm{(B)}\ \frac{31}{300}}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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Category: Introductory Combinatorics Problems

2005 AMC 12A Problems/Problem 24

Problem

Let $P(x) = (x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x)) = P(x) * R(x)$?

$\mathrm{(A) \ } 19 \qquad \mathrm{(B) \ } 22 \qquad \mathrm{(C) \ } 24 \qquad \mathrm{(D) \ } 27 \qquad \mathrm{(E) \ } 32$

Solution

Since $R(x)$ has degree three, then $P(x) \cdot R(x)$ has degree six. Thus, $P(Q(x))$ has degree six, so $Q(x)$ must have degree two, since $P(x)$ has degree three.

$$\begin{array}{l} P(Q(1)) = (Q(1)-1)(Q(1)-2)(Q(1)-3) = P(1) \cdot R(1) = 0, \\ P(Q(2)) = (Q(2)-1)(Q(2)-2)(Q(2)-3) = P(2) \cdot R(2) = 0, \\ P(Q(3)) = (Q(3)-1)(Q(3)-2)(Q(3)-3) = P(3) \cdot R(3) = 0. \end{array}$$

Hence, we conclude $Q(1)$, $Q(2)$, and $Q(3)$ must each be 1, 2, or 3. Since a quadratic is uniquely determined by three points, there can be $3*3*3 = 27$ different quadratics $Q(x)$ after each of the values of $Q(1)$, $Q(2)$, and $Q(3)$ are chosen.

However, we have included $Q(x)$ which are not quadratics. Namely,

$$\begin{array}{c} Q(1)=Q(2)=Q(3)=1 \Rightarrow Q(x)=1, \\ Q(1)=Q(2)=Q(3)=2 \Rightarrow Q(x)=2, \\ Q(1)=Q(2)=Q(3)=3 \Rightarrow Q(x)=3, \\ Q(1)=1, Q(2)=2, Q(3)=3 \Rightarrow Q(x)=x, \\ Q(1)=3, Q(2)=2, Q(3)=1 \Rightarrow Q(x)=4-x. \end{array}$$

Clearly, we could not have included any other constant functions. For any linear function, we have $2 \cdot Q(2) = Q(1) + Q(3)$ because $2(2) = 1 + 3$. So we have not included any other linear functions. Therefore, the desired answer is $27 - 5 = \boxed{\textbf{(B) }22}$.

See also

2005 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2005))	
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2005 AMC 12A Problems/Problem 25

Problem

Let S be the set of all points with coordinates (x, y, z), where x, y, and z are each chosen from the set $\{0, 1, 2\}$. How many equilateral triangles all have their vertices in S?

$(A)\ 72 \qquad (B)\ 76 \qquad (C)\ 80 \qquad (D)\ 84 \qquad (E)\ 88$

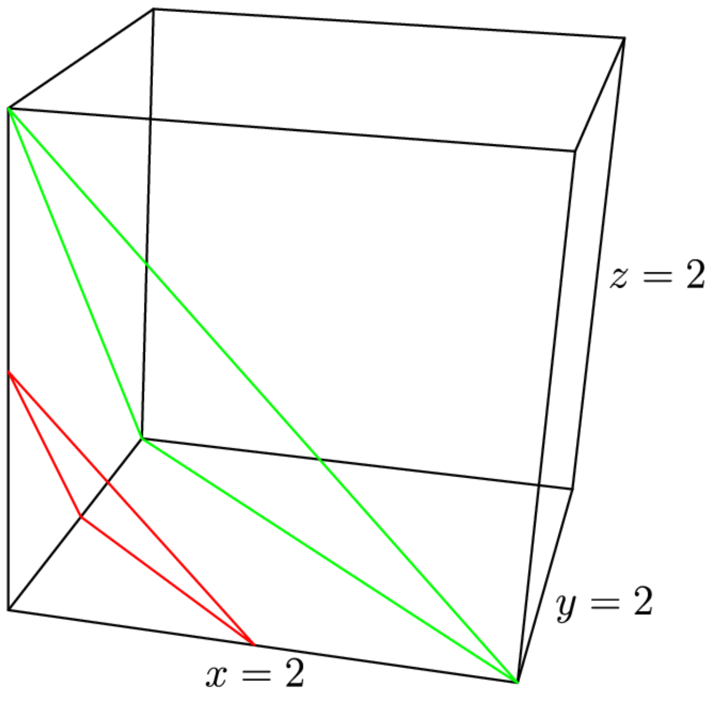
Contents

Solution

Solution 1 (non-rigorous)

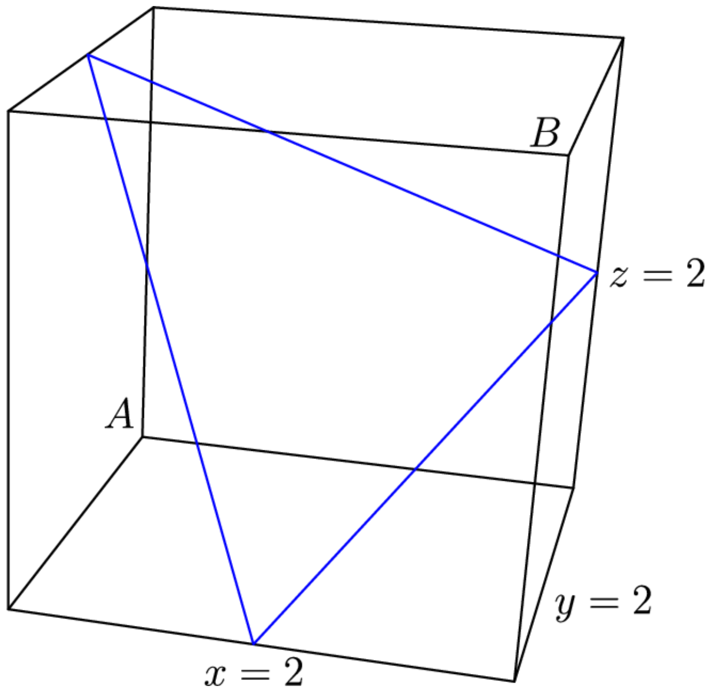
For this solution, we will just find as many solutions as possible, until it becomes intuitive that there are no more size of triangles left.

First, try to make three of its vertices form an equilateral triangle. This we find is possible by taking any vertex, and connecting the three adjacent vertices into a triangle. This triangle will have a side length of $\sqrt{2}$; a quick further examination of this cube will show us that this is the only possible side length (red triangle in diagram). Each of these triangles is determined by one vertex of the cube, so in one cube we have 8 equilateral triangles. We have 8 unit cubes, and then the entire cube (green triangle), giving us 9 cubes and $9 \cdot 8 = 72$ equilateral triangles.



NOTE: Connecting the centers of the faces will actually give an octahedron, not a cube, because it only has 6 vertices.

Now, we look for any additional equilateral triangles. Connecting the midpoints of three non-adjacent, non-parallel edges also gives us more equilateral triangles (blue triangle). Notice that picking these three edges leaves two vertices alone (labelled A and B), and that picking any two opposite vertices determine two equilateral triangles. Hence there are $\frac{8 \cdot 2}{2} = 8$ of these equilateral triangles, for a total of $\boxed{\textbf{(C)}\ 80}$.



Solution 2 (rigorous)

The three-dimensional distance formula shows that the lengths of the equilateral triangle must be $\sqrt{d_x^2 + d_y^2 + d_z^2}, 0 \le d_x, d_y, d_z \le 2$, which yields the possible edge lengths of

$$\begin{array}{ccc}
\sqrt{0^2+0^2+1^2} = \sqrt{1}, & \sqrt{0^2+1^2+1^2} = \sqrt{2}, & \sqrt{1^2+1^2+1^2} = \sqrt{3}, \\
\sqrt{0^2+0^2+2^2} = \sqrt{4}, & \sqrt{0^2+1^2+2^2} = \sqrt{5}, & \sqrt{1^2+1^2+2^2} = \sqrt{6}, \\
\sqrt{0^2+2^2+2^2} = \sqrt{8}, & \sqrt{1^2+2^2+2^2} = \sqrt{9}, & \sqrt{2^2+2^2+2^2} = \sqrt{12}
\end{array}$$

Some casework shows that $\sqrt{2}, \ \sqrt{6}, \ \sqrt{8}$ are the only lengths that work, from which we can use the same counting argument as above.

See also

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