2020 AMC 8 Problems

Problem1

Luka is making lemonade to sell at a school fundraiser. His recipe requires 4 times as much water as sugar and twice as much sugar as lemon juice. He uses 3 cups of lemon juice. How many cups of water does he need?

$\textbf{(A) }6 \qquad \textbf{(B) }8 \qquad \textbf{(C) }12 \qquad \textbf{(D) }18 \qquad \textbf{(E) }24$

Solution 1

Luka will need $3\cdot 2=6$ cups of sugar, and thus $6\cdot 4=24$ cups of water.

The answer is $\boxed{\textbf{(E) }24}$.

Solution 2

We have that

$\text{lemonade} : \text{water} : \text{lemon juice} = 4\cdot 2 : 2 : 1 = 8 : 2 : 1$,

so Luka needs $3\cdot 8=\boxed{\textbf{(E) }24}$ cups.

Problem2

Four friends do yardwork for their neighbors over the weekend, earning $\$15, \$20, \$25,$ and $\$40,$ respectively. They decide to split their earnings equally among themselves. In total how much will the friend who earned $\$40$ give to the others?

$\textbf{(A) }\$5 \qquad \textbf{(B) }\$10 \qquad \textbf{(C) }\$15 \qquad \textbf{(D) }\$20 \qquad \textbf{(E) }\$25$

Solution

The friends earn $\$\left(15+20+25+40\right)=\$100$ in total. Since they decided to split their earnings equally, it follows that each person will get $\$\left(\frac{100}{4}\right) = \$25$. Since the friend who earned $\$40$ will need to leave with $\$25$, he will have to give $\$\left(40-25\right) = \boxed{\textbf{(C) }\$15}$ to the others.

Problem3

Carrie has a rectangular garden that measures 6 feet by 8 feet. She plants the entire garden with strawberry plants. Carrie is able to plant 4 strawberry plants per square foot, and she harvests an average of 10 strawberries per plant. How many strawberries can she expect to harvest?

$\textbf{(A) }560 \qquad \textbf{(B) }960 \qquad \textbf{(C) }1120 \qquad \textbf{(D) }1920 \qquad \textbf{(E) }3840$

Solution 1

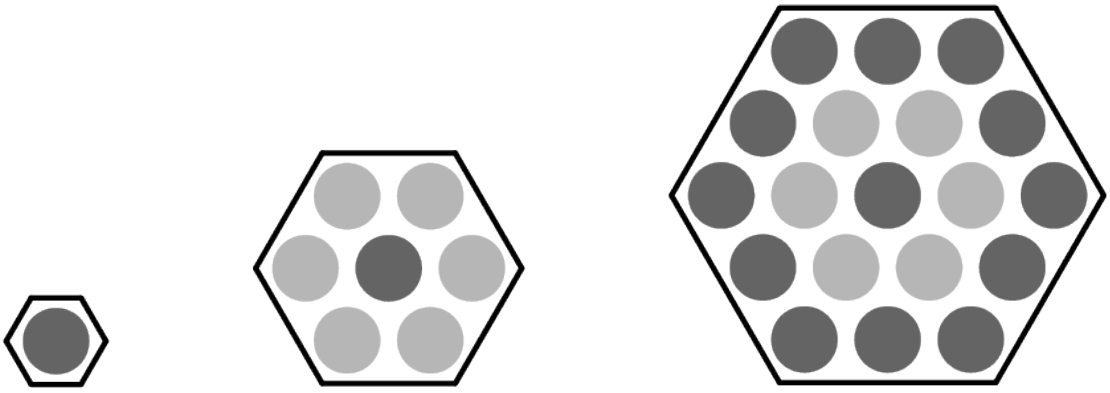
The area of the garden is $6 \cdot 8 = 48$ square feet. Since Carrie plants 4 strawberry plants per square foot, there are a total of $48 \cdot 4 = 192$ strawberry plants, each of which produces 10 strawberries on average. Accordingly, she can expect to harvest $192 \cdot 10 = \boxed{\textbf{(D) }1920}$ strawberries.

Solution 2

Looking at the units of each quantity, we observe that the answer will be the product of the number of square feet, the number of plants per square foot, and the number of strawberries per plant. This gives $6 \cdot 8 \cdot 4 \cdot 10 = \boxed{\textbf{(D) }1920}$.

Problem4

Three hexagons of increasing size are shown below. Suppose the dot pattern continues so that each successive hexagon contains one more band of dots. How many dots are in the next hexagon?



$\textbf{(A) }35 \qquad\textbf{(B) }37 \qquad\textbf{(C) }39 \qquad\textbf{(D) }43 \qquad\textbf{(E) }49$

Solution 1

Looking at the rows of each hexagon, we see that the first hexagon has 1 dot, the second has $2+3+2$dots and the third has $3+4+5+4+3$ dots, and given the way the hexagons are constructed, it is clear that this pattern continues. Hence the fourth hexagon has $4+5+6+7+6+5+4=\boxed{\textbf{(B) }37}$ dots.

Solution 2

The first hexagon has 1 dot, the second hexagon has $1+6$ dots, the third hexagon $1+6+12$ dots, and so on. The pattern continues since to go from hexagon n to hexagon $(n+1)$, we add a new ring of hexagons around the outside of the existing ones, with each side of the ring having side length $(n+1)$. Thus the number of hexagons added is $6(n+1)-6=6n$ (we subtract 6 as each of the corner hexagons in the ring is counted as part of two sides), confirming the pattern. We therefore predict that that the fourth hexagon has $1+6+12+18=\boxed{\textbf{(B) }37}$ dots.

Solution 3 (variant of Solution 2)

Let the number of dots in the first hexagon be $h_0 = 1$. By the same argument as in Solution 2, we have $h_n = h_{n-1} + 6n$ for $n > 0$. Using this, we find that $h_1 = 7$, $h_2 = 19$, and $h_3 = \boxed{\textbf{(B) } 37}$.

Problem5

Three fourths of a pitcher is filled with pineapple juice. The pitcher is emptied by pouring an equal amount of juice into each of 5 cups. What percent of the total capacity of the pitcher did each cup receive?

$\textbf{(A) } 5 \qquad \textbf{(B) } 10 \qquad \textbf{(C) } 15 \qquad \textbf{(D) } 20 \qquad \textbf{(E) } 25$

Solution 1

Each cup is filled with $\frac{3}{4} \cdot \frac{1}{5} = \frac{3}{20}$ of the amount of juice in the pitcher, so the percentage is $\frac{3}{20} \cdot 100 = \boxed{\textbf{(C) } 15}$.

Solution 2

The pitcher is $\frac{3}{4}$ full, i.e. 75% full. Therefore each cup receives $\frac{75}{5} = \boxed{\textbf{(C) } 15}$ percent of the total capacity.

Solution 3

Assume that the pitcher has a total capacity of 100 ounces. Since it is filled three fourths with pineapple juice, it contains 75 ounces of pineapple juice, which means that each cup will contain $\frac{75}{5} = 15$ ounces of pineapple juice. Since the total capacity of the pitcher was 100 ounces, it follows that each cup received 15% of the total capacity of the pitcher, yielding $\boxed{\textbf{(C)}\ 15}$ as the answer.

Problem6

Aaron, Darren, Karen, Maren, and Sharon rode on a small train that has five cars that seat one person each. Maren sat in the last car. Aaron sat directly behind Sharon. Darren sat in one of the cars in front of Aaron. At least one person sat between Karen and Darren. Who sat in the middle car?

$\textbf{(A)}\ \text{Aaron} \qquad \textbf{(B)}\ \text{Darren} \qquad \textbf{(C)}\ \text{Karen} \qquad \textbf{(D)}\ \text{Maren} \qquad \textbf{(E)}\ \text{Sharon}$

Solution

Write the order of the cars as $\square\square\square\square\square$, where the left end of the row represents the back of the train and the right end represents the front. Call the people A, D, K, M, and S respectively. The first condition gives $M\square\square\square\square$, so we try $MAS\square\square$, $M\square AS\square$, and $M\square\square AS$. In the first case, as D sat in front of A, we must have $MASDK$ or $MASKD$, both of which do not comply with the last condition. In the second case, we obtain $MKASD$, which works, while the third case is obviously impossible, since it results in there being no way for D to sit in front of A. It follows that, with the only possible arrangement being $MKASD$, the person sitting in the middle car is $\boxed{\textbf{(A)}\ \text{Aaron}}$.

Problem7

How many integers between 2020 and 2400 have four distinct digits arranged in increasing order? (For example, 2347 is one integer.)

$\textbf{(A)}\ 9 \qquad \textbf{(B)}\ 10 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 21 \qquad \textbf{(E)}\ 28$

Solution 1

Firstly, observe that the second digit of such a number cannot be 1 or 2, because the digits must be distinct and increasing. The second digit also cannot be 4 as the number must be less than 2400, so it must be 3. It remains to choose the latter two digits, which must be 2 distinct digits from $\{4,5,6,7,8,9\}$. That can be done in $\binom{6}{2} = \frac{6 \cdot 5}{2 \cdot 1} = 15$ ways; there is then only 1 way to order the digits, namely in increasing order. This means the answer is $\boxed{\textbf{(C) } 15}$.

Solution 2 (without using the "choose" function)

As in Solution 1, we find that the first two digits must be 23, and the third digit must be at least 4. If it is 4, then there are 5 choices for the last digit, namely 5, 6, 7, 8, or 9. Similarly, if the third digit is 5, there are 4choices for the last digit, namely 6, 7, 8, and 9; if 6, there are 3 choices; if 7, there are 2 choices; and if 8, there is 1 choice. It follows that the total number of such integers is $5+4+3+2+1=\boxed{\textbf{(C) } 15}$.

Problem8

Ricardo has 2020 coins, some of which are pennies (1-cent coins) and the rest of which are nickels (5-cent coins). He has at least one penny and at least one nickel. What is the difference in cents between the greatest possible and least possible amounts of money that Ricardo can have?

$\textbf{(A) } 8062 \qquad \textbf{(B) } 8068 \qquad \textbf{(C) } 8072 \qquad \textbf{(D) } 8076 \qquad \textbf{(E) } 8082$

Solution 1

Clearly, the amount of money Ricardo has will be maximized when he has the maximum number of nickels. Since he must have at least one penny, the greatest number of nickels he can have is 2019, giving a total of $(2019 \cdot 5 + 1)$ cents. Analogously, the amount of money he has will be least when he has the greatest number of pennies; as he must have at least one nickel, the greatest number of pennies he can have is also 2019, giving a total of $(2019 \cdot 1 + 5)$ cents. Hence the required difference is

$(2019 \cdot 5 + 1) - (2019 \cdot 1 + 5) = 2019 \cdot 4 - 4 = 4 \cdot 2018 = \boxed{\textbf{(C) } 8072}$

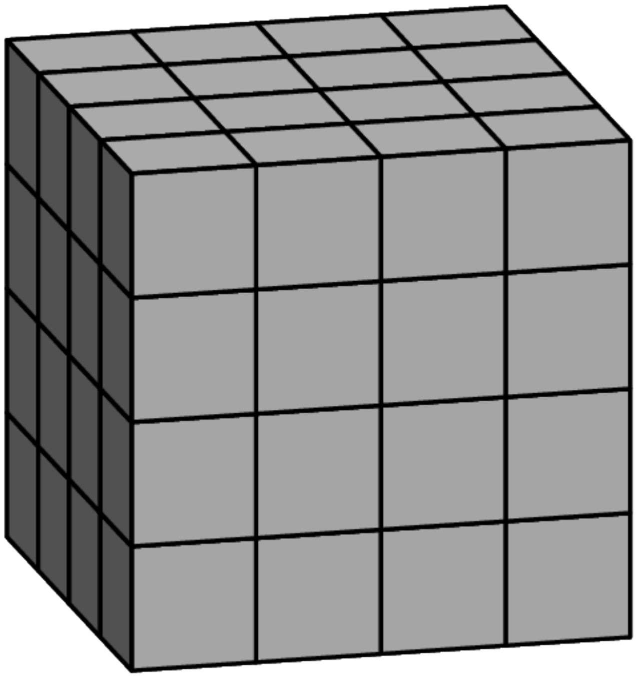
Solution 2

Suppose Ricardo has p pennies, so then he has $(2020 - p)$ nickels. In order to have at least one penny and at least one nickel, we require $p \geq 1$ and $2020 - p \geq 1$, i.e. $1 \leq p \leq 2019$. The number of cents he has is $p + 5(2020 - p) = 10100 - 4p$, so the maximum is $10100 - 4 \cdot 1$ and the minimum is $10100 - 4 \cdot 2019$, and the difference is therefore

$(10100 - 4 \cdot 1) - (10100 - 4 \cdot 2019) = 4 \cdot 2019 - 4 = 4 \cdot 2018 = \boxed{\textbf{(C) } 8072}$

Problem9

Akash's birthday cake is in the form of a $4 \times 4 \times 4$ inch cube. The cake has icing on the top and the four side faces, and no icing on the bottom. Suppose the cake is cut into 64 smaller cubes, each measuring $1 \times 1 \times 1$ inch, as shown below. How many of the small pieces will have icing on exactly two sides?



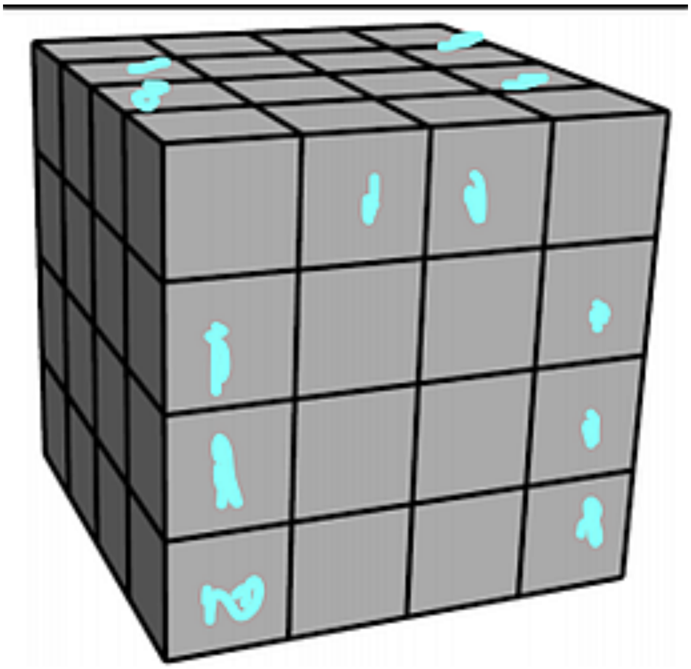
$\textbf{(A)}\ 12 \qquad \textbf{(B)}\ 16 \qquad \textbf{(C)}\ 18 \qquad \textbf{(D)}\ 20 \qquad \textbf{(E)}\ 24$

Solution 1

Notice that, for a small cube which does not form part of the bottom face, it will have exactly 2 faces with icing on them only if it is one of the 2 center cubes of an edge of the larger cube. There are $12-4=8$such edges (as we exclude the 4 edges of the bottom face), so this case yields $2\cdot 8=16$ small cubes. As for the bottom face, we can see that only the 4 corner cubes have exactly 2 faces with icing, so the total is $16+4=\boxed{\textbf{(D)}\ 20}$.

Solution 2

The following diagram shows 12 of the small cubes having exactly 2 faces with icing on them; that is all of them except for those on the hidden face directly opposite to the front face.



But the hidden face is an exact copy of the front face, so the answer is $12+8=\boxed{\textbf{(D)}\ 20}$

Problem10

Zara has a collection of 4 marbles: an Aggie, a Bumblebee, a Steelie, and a Tiger. She wants to display them in a row on a shelf, but does not want to put the Steelie and the Tiger next to one another. In how many ways can she do this?

$\textbf{(A) }6\qquad\textbf{(B) }8\qquad\textbf{(C) }12\qquad\textbf{(D) }18\qquad\textbf{(E) }24$

Solution 1

By the Georgeooga-Harryooga Theorem there are $\frac{(4-2)!(4-2+1)!}{(4-2\cdot 2+1)!}=\boxed{\textbf{(C)}\ 12}$ way to arrange the marbles.

Solution by RedFireTruck

Solution 2

We can arrange our marbles like so $\square A\square B\square$.

To arrange the A and B we have $2! = 2$ ways.

To place the S and T in the blanks we have ${}_3P_2 = 6$ ways.

By fundamental counting principle our final answer is $2 \cdot 6 = \boxed{\textbf{(C) } 12}$

Solution by RedFireTruck

Solution 3

Let the Aggie, Bumblebee, Steelie, and Tiger, be referred to by $A, B, S,$ and T, respectively. If we ignore the constraint that S and T cannot be next to each other, we get a total of $4! = 24$ ways to arrange the 4 marbles. We now simply have to subtract out the number of ways that S and T can be next to each other. If we place S and T next to each other in that order, then there are three places that we can place them, namely in the first two slots, in the second two slots, or in the last two slots (i.e. $ST\square\square, \square ST\square, \square\square ST$). However, we could also have placed S and T in the opposite order (i.e. $TS\square\square, \square TS\square, \square\square TS$). Thus there are 6 ways of placing S and T directly next to each other. Next, notice that for each of these placements, we have two open slots for placing A and B. Specifically, we can place A in the first open slot and B in the second open slot or switch their order and place B in the first open slot and A in the second open slot. This gives us a total of $6 \times 2 = 12$ ways to place S and T next to each other. Subtracting this from the total number of arrangements gives us $24 - 12 = 12$ total arrangements $\implies \boxed{\textbf{(C) } 12}$.

We can also solve this problem directly by looking at the number of ways that we can place S and T such that they are not directly next to each other. Observe that there are three ways to place S and T (in that order) into the four slots so they are not next to each other (i.e. $S\square T\square, \square S\square T, S\square\square T$). However, we could also have placed S and T in the opposite order (i.e. $T\square S\square, \square T\square S, T\square\square S$). Thus there are 6 ways of

placing S and T so that they are not next to each other. Next, notice that for each of these placements, we have two open slots for placing A and B. Specifically, we can place A in the first open slot and B in the second open slot or switch their order and place B in the first open slot and A in the second open slot. This gives us a total of $6 \times 2 = 12$ ways to place S and T such that they are not next to each other $\implies \boxed{\textbf{(C) } 12}$.
~junaidmansuri

Solution 4

Let's try complementary counting. There $4!$ ways to arrange the 4 marbles. However, there are $2 \cdot 3!$arrangements where Steelie and Tiger are next to each other. (Think about permutations of the element ST, A, and B or TS, A, and B). Thus, $4! - 2 \cdot 3! = \boxed{12 \textbf{(C)}}$

Solution 5

We use complementary counting: we will count the numbers of ways where Steelie and Tiger are together and subtract that from the total count. Treat the Steelie and the Tiger as a "super marble." There are $2!$ways to arrange Steelie and Tiger within this "super marble." Then there are $3!$ ways to arrange the "super marble" and Zara's two other marbles in a row. Since there are $4!$ ways to arrange the marbles without any restrictions, the answer is given by $4! - 2! \cdot 3! = \textbf{(C) } 12$

-franzliszt

Solution 6

We will use the following

$\textbf{Georgeooga-Harryooga Theorem:}$ The Georgeooga-Harryooga Theorem states that if you have $a$distinguishable objects and b of them cannot be together, then there are $\dfrac{(a-b)!(a-b+1)!}{b!}$ ways to arrange the objects.

Proof. *(Created by AoPS user RedFireTruck)*

Let our group of a objects be represented like so $1, 2, 3, ..., a-1, a$. Let the last b objects be the ones we can't have together.

Then we can organize our objects like so $\square 1 \square 2 \square 3 \square ... \square a-b-1 \square a-b \square$.

We have $(a-b)!$ ways to arrange the objects in that list.

Now we have $a-b+1$ blanks and b other objects so we have ${}_{a-b+1}P_b = \dfrac{(a-b+1)!}{(a-2b+1)!}$ ways to arrange the objects we can't put together.

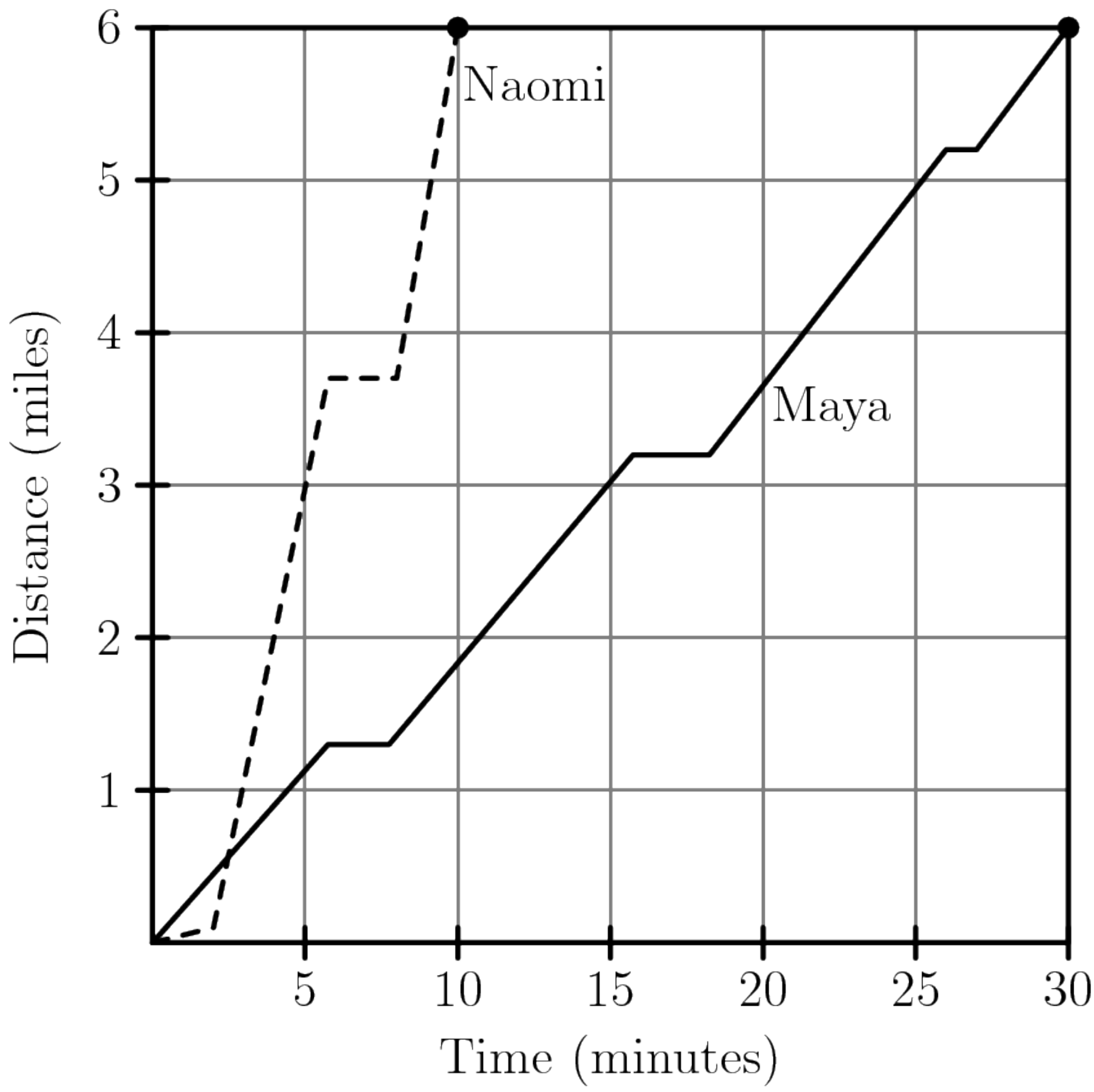
By fundamental counting principal our answer is $\dfrac{(a-b)!(a-b+1)!}{(a-2b+1)!}$.

Proof by RedFireTruck

Back to the problem. By the Georgeooga-Harryooga Theorem, our answer is $\dfrac{(4-2)!(4-2+1)!}{(4-2\cdot 2+1)!} = \textbf{(C)}\ 12$

Problem11

After school, Maya and Naomi headed to the beach, 6 miles away. Maya decided to bike while Naomi took a bus. The graph below shows their journeys, indicating the time and distance traveled. What was the difference, in miles per hour, between Naomi's and Maya's average speeds?



$\textbf{(A) } 6 \qquad \textbf{(B) } 12 \qquad \textbf{(C) } 18 \qquad \textbf{(D) } 20 \qquad \textbf{(E) } 24$

Solution 1

Naomi travels 6 miles in a time of 10 minutes, which is equivalent to $\dfrac{1}{6}$ of an hour. Since $\text{speed} = \dfrac{\text{distance}}{\text{time}}$, her speed is $\dfrac{6}{\left(\frac{1}{6}\right)} = 36$ mph. By a similar calculation, Maya's speed is 12 mph, so the answer is $36-12=\boxed{\textbf{(E) } 24}$.

Solution 2 (variant of Solution 1)

Naomi's speed of 6 miles in 10 minutes is equivalent to $6 \cdot 6 = 36$ miles per hour, while Maya's speed of 6miles in 30 minutes (i.e. half an hour) is equivalent to $6 \cdot 2 = 12$ miles per hour. The difference is consequently $36 - 12 = \boxed{\textbf{(E) } 24}$.

Problem12

For a positive integer n, the factorial notation $n!$ represents the product of the integers from n to 1. (For example, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.) What value of N satisfies the following equation? $5! \cdot 9! = 12 \cdot N!$

$\textbf{(A) } 10 \qquad \textbf{(B) } 11 \qquad \textbf{(C) } 12 \qquad \textbf{(D) } 13 \qquad \textbf{(E) } 14$

Solution 1

We have $5! = 2 \cdot 3 \cdot 4 \cdot 5$, and $2 \cdot 5 \cdot 9! = 10 \cdot 9! = 10!$. Therefore the equation becomes $3 \cdot 4 \cdot 10! = 12 \cdot N!$, and so $12 \cdot 10! = 12 \cdot N!$. Cancelling the 12s, it is clear that $N = \boxed{\textbf{(A) } 10}$.

Solution 2 (variant of Solution 1)

Since $5! = 120$, we obtain $120 \cdot 9! = 12 \cdot N!$, which becomes $12 \cdot 10 \cdot 9! = 12 \cdot N!$and thus $12 \cdot 10! = 12 \cdot N!$. We therefore deduce $N = \boxed{\textbf{(A) } 10}$.

Solution 3 (using answer choices)

We can see that the answers $\textbf{(B)}$ to $\textbf{(E)}$ contain a factor of 11, but there is no such factor of 11 in $5! \cdot 9!$. Therefore, the answer must be $\boxed{\textbf{(A) } 10}$.

Problem13

Jamal has a drawer containing 6 green socks, 18 purple socks, and 12 orange socks. After adding more purple socks, Jamal noticed that there is now a 60% chance that a sock randomly selected from the drawer is purple. How many purple socks did Jamal add?

$\textbf{(A) } 6 \qquad \textbf{(B) } 9 \qquad \textbf{(C) } 12 \qquad \textbf{(D) } 18 \qquad \textbf{(E) } 24$

Solution 1

After Jamal adds x purple socks, he has $(18+x)$ purple socks and $6+18+12+x=(36+x)$ total socks. This means the probability of drawing a purple sock is $\frac{18+x}{36+x}$, so we obtain $\frac{18+x}{36+x}=\frac{3}{5}$. Since $\frac{18+9}{36+9}=\frac{27}{45}=\frac{3}{5}$, the answer is $\boxed{\textbf{(B) } 9}$.

Solution 2 (variant of Solution 1)

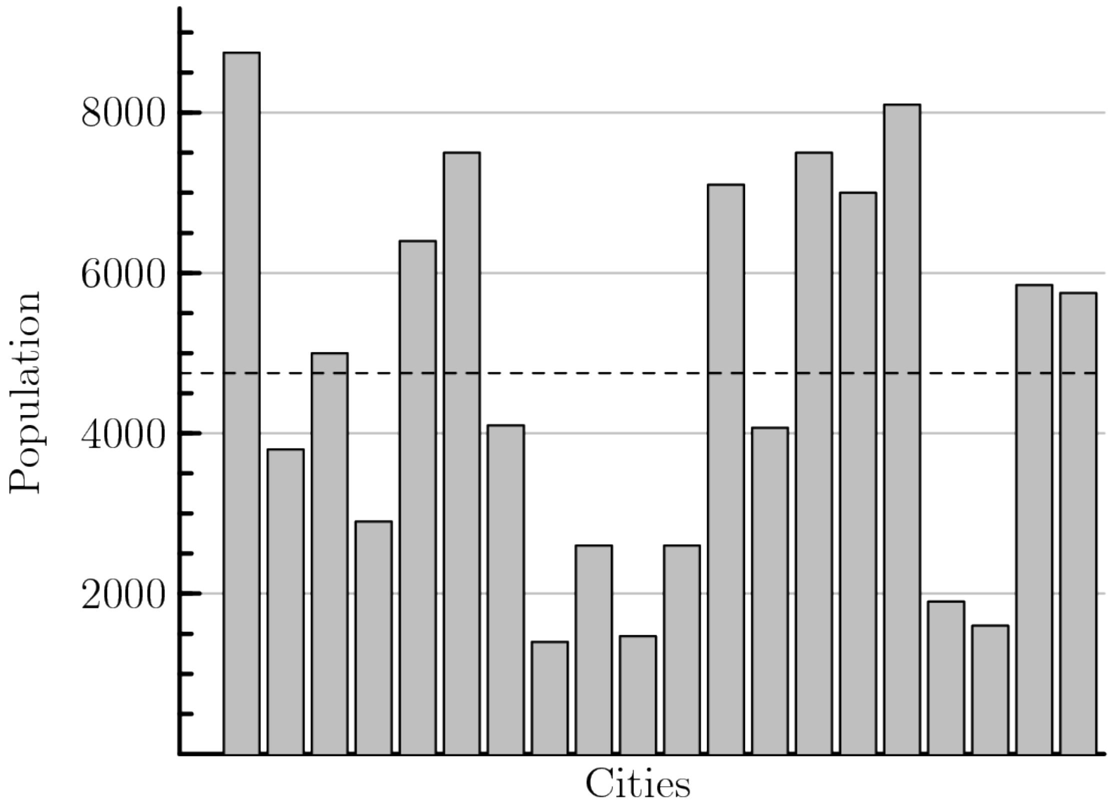
As in Solution 1, we have the equation $\frac{18+x}{36+x}=\frac{3}{5}$. Cross-multiplying yields $90+5x=108+3x \Rightarrow 2x=18 \Rightarrow x=9$. Thus, Jamal added $\boxed{\textbf{(B) } 9}$ purple socks.

Solution 3 (non-algebraic)

6 green socks and 12 orange socks together should be $100\% - 60\% = 40\%$ of the new total number of socks, so that new total must be $\dfrac{6+12}{0.4} = 45$. It follows that $45 - 6 - 18 - 12 = \boxed{\textbf{(B) } 9}$ purple socks were added.

Problem14

There are 20 cities in the County of Newton. Their populations are shown in the bar chart below. The average population of all the cities is indicated by the horizontal dashed line. Which of the following is closest to the total population of all 20 cities?



$\textbf{(A) } 65{,}000 \qquad \textbf{(B) } 75{,}000 \qquad \textbf{(C) } 85{,}000 \qquad \textbf{(D) } 95{,}000 \qquad \textbf{(E) } 105{,}000$

Solution 1

We can see that the dotted line is halfway between $4{,}500$ and $5{,}000$, so is at $4{,}750$. As this is the average population of all 20 cities, the total population is simply $4{,}750 \cdot 20 = \boxed{\textbf{(D) } 95{,}000}$.

Solution 2 (estimation)

The dashed line, which represents the average population of each city, is slightly below $5{,}000$. Since there are 20 cities, the answer is slightly less than $20 \cdot 5{,}000 \approx 100{,}000$, which is closest to

Problem15

Suppose 15% of x equals 20% of y. What percentage of x is y?

$\textbf{(A) }5 \qquad \textbf{(B) }35 \qquad \textbf{(C) }75 \qquad \textbf{(D) }133\frac{1}{3} \qquad \textbf{(E) }300$

Solution 1

Since $20\% = \frac{1}{5}$, multiplying the given condition by 5 shows that y is $15 \cdot 5 = \boxed{\textbf{(C) }75}$ percent of x.

Solution 2

Letting $x = 100$ (without loss of generality), the condition becomes $0.15 \cdot 100 = 0.2 \cdot y \Rightarrow 15 = \frac{y}{5} \Rightarrow y = 75$. Clearly, it follows that y is 75% of x, so the answer is $\boxed{\textbf{(C) }75}$.

Solution 3

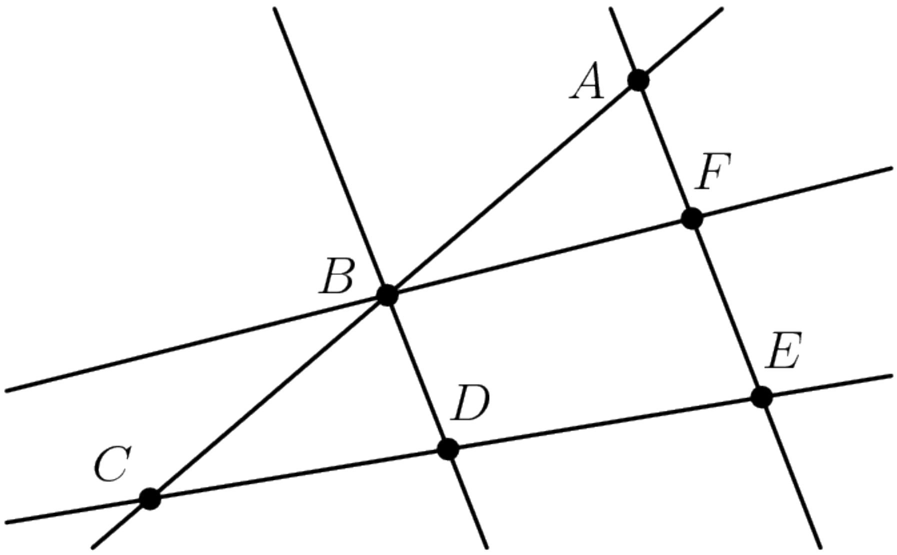
We have $15\% = \frac{3}{20}$ and $20\% = \frac{1}{5}$, so $\frac{3}{20}x = \frac{1}{5}y$. Solving for y, we multiply by 5 to give $y = \frac{15}{20}x = \frac{3}{4}x$, so the answer is $\boxed{\textbf{(C) }75}$.

Solution 4

We are given $0.15x = 0.20y$, so we may assume without loss of generality that $x = 20$ and $y = 15$. This means $\frac{y}{x} = \frac{15}{20} = \frac{75}{100}$, and thus answer is $\boxed{\textbf{(C) } 75}$.

Problem 16

Each of the points $A, B, C, D, E,$ and F in the figure below represents a different digit from 1 to 6. Each of the five lines shown passes through some of these points. The digits along each line are added to produce five sums, one for each line. The total of the five sums is $47.$ What is the digit represented by B?



$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 4 \qquad \textbf{(E) } 5$

Solution

We can form the following expressions for the sum along each line: $\begin{cases} A+B+C \\ A+E+F \\ C+D+E \\ B+D \\ B+F \end{cases}$ Adding these together, we must have $2A+3B+2C+2D+2E+2F=47$,

i.e. $2(A+B+C+D+E+F)+B=47$.

Since A,B,C,D,E,F are unique integers between 1 and 6, we obtain $A+B+C+D+E+F=1+2+3+4+5+6=21$ (where the order doesn't matter as addition is commutative), so our equation simplifies to $42+B=47$. This means $B=\boxed{\textbf{(E) }5}$.

Problem17

How many positive integer factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely $1,2,3,4,6,$ and 12.)

$\textbf{(A) }6 \qquad \textbf{(B) }7 \qquad \textbf{(C) }8 \qquad \textbf{(D) }9 \qquad \textbf{(E) }10$

Solution 1

Since $2020=2^2\cdot5\cdot101$, we can simply list its factors: $1,2,4,5,10,20,101,202,404,505,1010,2020.$

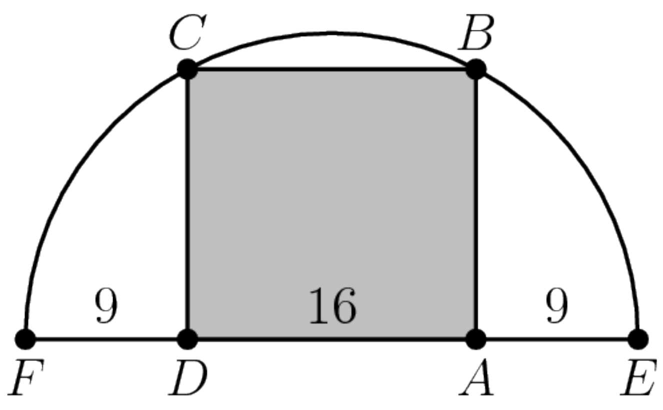
There are 12 of these; only $1,2,4,5,101$ (i.e. 5 of them) have at most 3 factors, so the remaining $12-5=\boxed{\textbf{(B) }7}$ factors have more than 3 factors.

Solution 2

As in Solution 1, we prime factorize 2020 as $2^2 \cdot 5 \cdot 101$, and we recall the standard formula that the number of positive factors of an integer is found by adding 1 to each exponent in its prime factorization, and then multiplying these. Thus 2020 has $(2+1)(1+1)(1+1) = 12$ factors. The only number which has one factor is 1. For a number to have exactly two factors, it must be prime, and the only prime factors of 2020 are 2, 5, and 101. For a number to have three factors, it must be a square of a prime (this follows from the standard formula mentioned above), and from the prime factorization, the only square of a prime that is a factor of 2020 is 4. Thus, there are 5 factors of 2020 which themselves have 1, 2, or 3 factors (namely 1, 2, 4, 5, and 101), so the number of factors of 2020 that have more than 3 factors is $12 - 5 = \boxed{\textbf{(B) } 7}$.

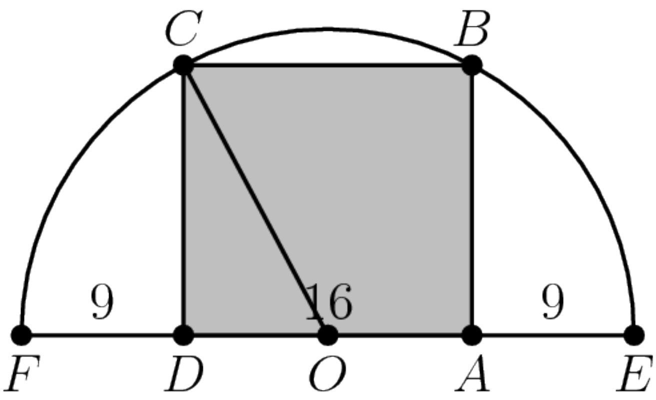
Problem18

Rectangle $ABCD$ is inscribed in a semicircle with diameter $\overline{FE}$, as shown in the figure. Let $DA = 16$, and let $FD = AE = 9$. What is the area of $ABCD$?



$\textbf{(A) } 240 \qquad \textbf{(B) } 248 \qquad \textbf{(C) } 256 \qquad \textbf{(D) } 264 \qquad \textbf{(E) } 272$

Solution 1



Let O be the center of the semicircle. The diameter of the semicircle is $9+16+9=34$, so $OC=17$. By symmetry, O is in fact the midpoint of DA, so $OD=OA=\frac{16}{2}=$. By the Pythagorean theorem in right-angled triangle ODC (or OBA), we have that CD (or AB) is $\sqrt{17^2-8^2}=15$. Accordingly, the area of $ABCD$ is $16\cdot 15=\boxed{\textbf{(A) }240}$.

Solution 2 (coordinate geometry)

Let the midpoint of segment FE be the origin. Evidently, point $D=(-8,0)$ and $A=(8,0)$. Since points C and B share x-coordinates with D and A respectively, it suffices to find the y-coordinate of B (which will be the height of the rectangle) and multiply this by DA (which we know is 16). The radius of the semicircle is $\frac{9+16+9}{2}=17$, so the whole circle has equation $x^2+y^2=289$; as already stated, Bhas the same x-coordinate as A, i.e. 8, so substituting this into the equation shows that $y=\pm 15$. Since $y>0$ at B, the y-coordinate of B is 15. Therefore, the answer is $16\cdot 15=\boxed{\textbf{(A) }240}$.

(Note that the synthetic solution (Solution 1) is definitely faster and more elegant. However, this is the solution that you should use if you can't see any other easier strategy.)

Solution 3

We can use a result from the Art of Problem Solving *Introduction to Algebra* book: for a semicircle with diameter $(1+n)$, such that the 1 part is on one side and the n part is on the other side, the height from the end of the 1 side (or the start of the n side) is $\sqrt{n}$. To use this, we scale the figure down by 9; then the height is $\sqrt{1+\frac{16}{9}}=\sqrt{\frac{16+9}{9}}=\sqrt{\frac{25}{9}}=\frac{5}{3}$. Now, scaling back up by 9, the height DC is $9\cdot\frac{5}{3}=15$. The answer is $15\cdot 16=\boxed{\textbf{(A) }240}$.

Problem19

A number is called flippy if its digits alternate between two distinct digits. For example, 2020 and $37373$are flippy, but 3883 and 123123 are not. How many five-digit flippy numbers are divisible by $15?$

$\textbf{(A) }3 \qquad \textbf{(B) }4 \qquad \textbf{(C) }5 \qquad \textbf{(D) }6 \qquad \textbf{(E) }8$

Solution 1

A number is divisible by 15 precisely if it is divisible by 3 and 5. The latter means the last digit must be either 5 or 0, and the former means the sum of the digits must be divisible by 3. If the last digit is 0, the first digit would be 0 (because the digits alternate), which is not possible. Hence the last digit must be 5, and the number is of the form $5\square5\square5$. If the unknown digit is x, we deduce $5+x+5+x+5\equiv 0 \pmod{3} \Rightarrow 2x\equiv 0 \pmod{3}$.

We know 2^{-1} exists modulo 3 because 2 is relatively prime to 3, so we conclude that x (i.e. the second and fourth digit of the number) must be a multiple of 3. It can be 0, 3, 6, or 9, so there are $\boxed{\textbf{(B) } 4}$ options: 50505, 53535, 56565, and 59595.

Solution 2 (variant of Solution 1)

As in Solution 1, we find that such numbers must start with 5 and alternate with 5 (i.e. must be of the form $5\square 5\square 5$), where the two digits between the 5s need to be the same. Call that digit x. For the number to be divisible by 3, the sum of the digits must be divisible by 3; since the sum of the three 5s is 15, which is already a multiple of 3, it must also be the case that $x+x=2x$ is a multiple of 3. Thus, the problem reduces to finding the number of digits from 0 to 9 for which $2x$ is a multiple of 3. This leads to $x=0$, 3, 6, or 9, so there are $\boxed{\textbf{(B) } 4}$ possible numbers (namely 50505, 53535, 56565, and 59595).

Problem20

A scientist walking through a forest recorded as integers the heights of 5 trees standing in a row. She observed that each tree was either twice as tall or half as tall as the one to its right. Unfortunately some of her data was lost when rain fell on her notebook. Her notes are shown below, with blanks indicating the missing numbers. Based on her observations, the scientist was able to reconstruct the lost data. What was the average height of the trees, in meters?

Tree 1	___ meters
Tree 2	11 meters
Tree 3	___ meters
Tree 4	___ meters
Tree 5	___ meters
Average height	___ .2 meters

$\textbf{(A) } 22.2 \qquad \textbf{(B) } 24.2 \qquad \textbf{(C) } 33.2 \qquad \textbf{(D) } 35.2 \qquad \textbf{(E) } 37.2$

Solution 1

We will show that 22, 11, 22, 44, and 22 meters are the heights of the trees from left to right. We are given that all tree heights are integers, so since Tree 2 has height 11 meters, we can deduce that Trees 1 and 3 both have a height of 22 meters. There are now three possible cases for the heights of Trees 4 and 5 (in order for them to be integers), namely heights of 11 and 22, 44 and 88, or 44 and 22. Checking each of these, in the first case, the average is 17.6 meters, which doesn't end in $.2$ as the problem requires. Therefore, we consider the other cases. With 44 and 88, the average is 37.4 meters, which again does not end in $.2$, but with 44 and 22, the average is 24.2 meters, which does. Consequently, the answer is $\boxed{\textbf{(B) } 24.2}$.

Solution 2

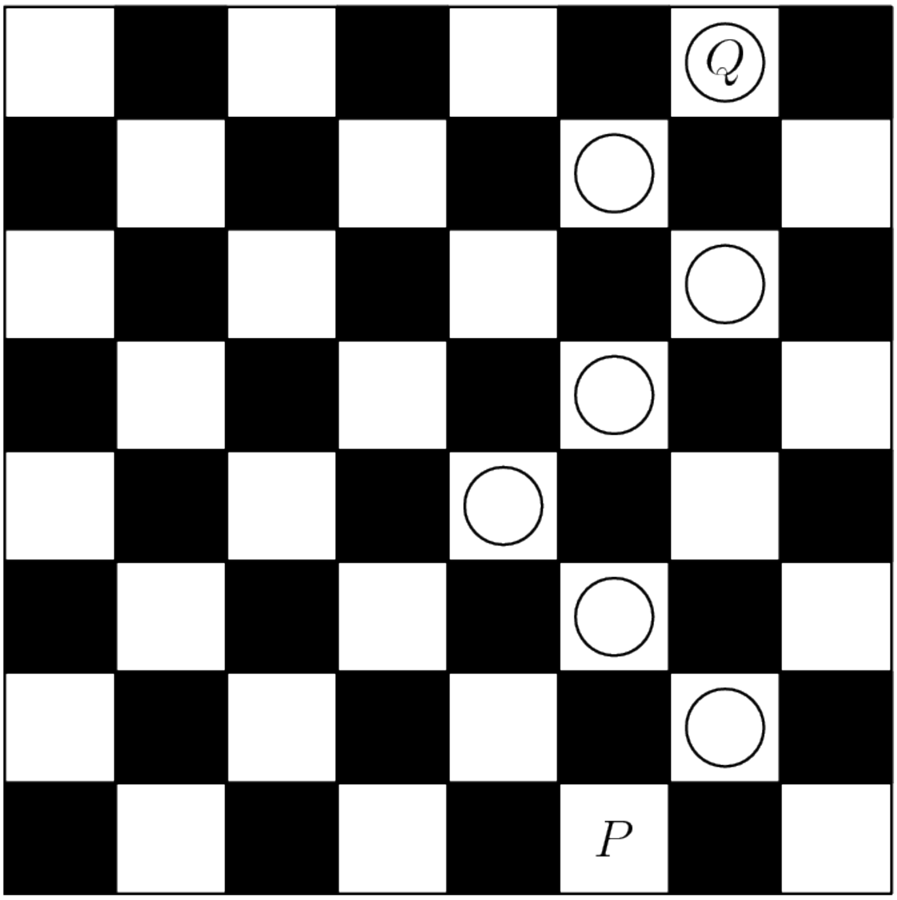
As in Solution 1, we shall show that the heights of the trees are 22, 11, 22, 44, and 22 meters. Let S be the sum of the heights, so that the average height will be $\frac{S}{5}$ meters. We note that $0.2 = \frac{1}{5}$, so in order for $\frac{S}{5}$ to end in $.2$, S must be one more than a multiple of 5. Moreover, as all the heights are integers, the heights of Tree 1 and Tree 3 are both 22 meters. At this point, our table looks as follows:

Tree 1	22 meters
Tree 2	11 meters
Tree 3	22 meters
Tree 4	___ meters
Tree 5	___ meters
Average height	___ .2 meters

If Tree 4 now has a height of 11, then Tree 5 would need to have height 22, but in that case S would equal 88, which is not 1 more than a multiple of 5. So we instead take Tree 4 to have height 44. Then the sum of the heights of the first 4 trees is $22+11+22+44=99$, so using a height of 22 for Tree 5 gives $S=121$, which is 1 more than a multiple of 5 (whereas 88 gives $S=187$, which is not). Thus the average height of the trees is $\frac{121}{5}=\boxed{\textbf{(B) } 24.2}$ meters.

Problem21

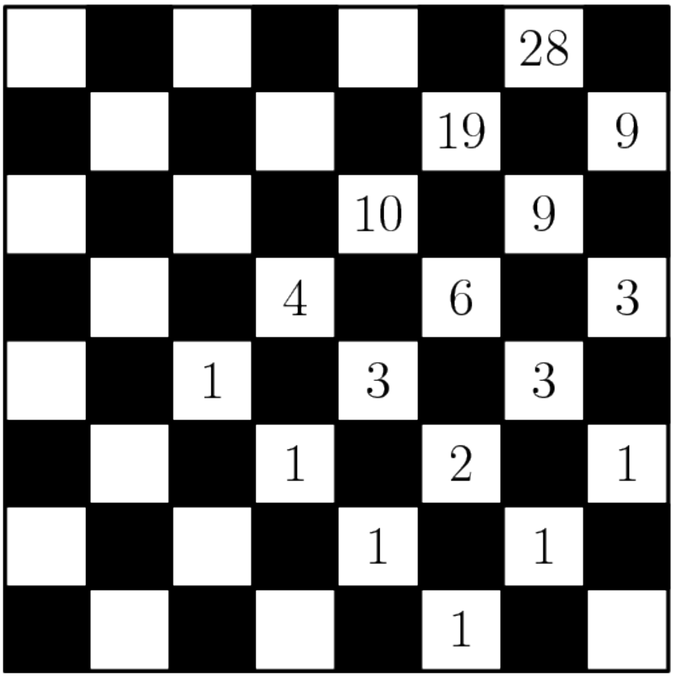
A game board consists of 64 squares that alternate in color between black and white. The figure below shows square P in the bottom row and square Q in the top row. A marker is placed at $P.$ A step consists of moving the marker onto one of the adjoining white squares in the row above. How many 7-step paths are there from P to $Q?$ (The figure shows a sample path.)



$\textbf{(A) } 28 \qquad \textbf{(B) } 30 \qquad \textbf{(C) } 32 \qquad \textbf{(D) } 33 \qquad \textbf{(E) } 35$

Solution 1

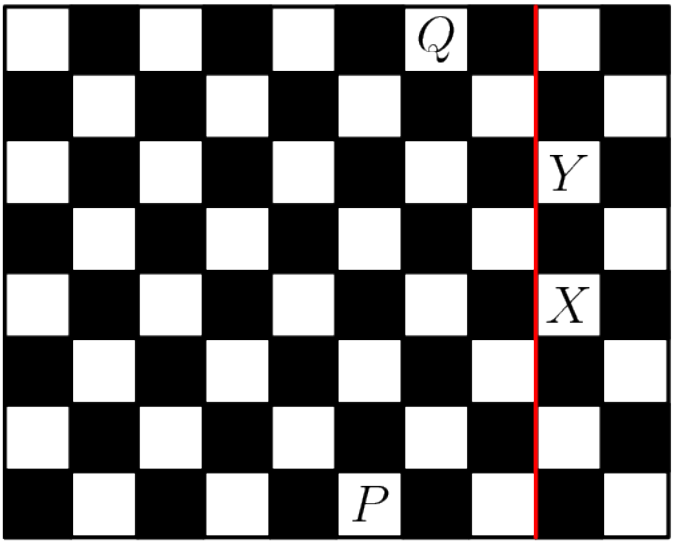
Notice that, in order to step onto any particular white square, the marker must have come from one of the 1 or 2 white squares immediately beneath it (since the marker can only move on white squares). This means that the number of ways to move from P to that square is the sum of the numbers of ways to move from P to each of the white squares immediately beneath it. To solve the problem, we can accordingly construct the following diagram, where each number in a square is calculated as the sum of the numbers on the white squares immediately beneath that square (and thus will represent the number of ways to remove from P to that square, as already stated).



The answer is therefore $\boxed{\textbf{(A) } 28}$.

Solution 2

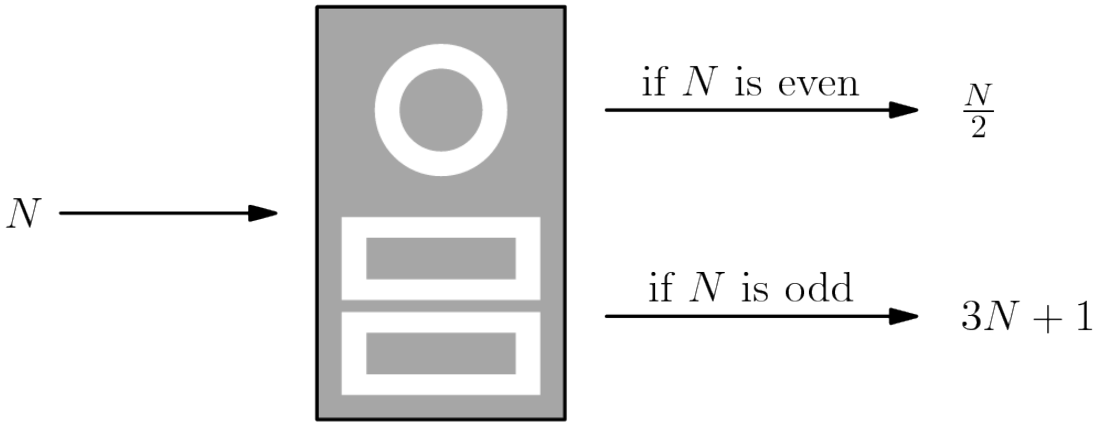
Suppose we "extend" the chessboard infinitely with 2 additional columns to the right, as shown below. The red line shows the right-hand edge of the original board.



The total number of paths from P to Q, including invalid paths which cross over the red line, is then the number of paths which make 4 steps up-and-right and 3 steps up-and-left, which is $\binom{4+3}{3} = \binom{7}{3} = 35$. We need to subtract the number of invalid paths, i.e. the number of paths that pass through X or Y. To get to X, the marker has to make 3 up-and-right steps, after which it can proceed to Q with 3 steps up-and-left and 1 step up-and-right. Thus, the number of paths from P to Q that pass through X is $1 \cdot \binom{3+1}{3} = 4$. Similarly, the number of paths that pass through Y is $\binom{4+1}{1} \cdot 1 = 5$. However, we have now double-counted the invalid paths which pass through both X and Y; from the diagram, it is clear that there are only 2 of these (as the marker can get from X to Y by a step up-and-left and a step-up-and-right in either order). Hence the number of invalid paths is $4+5-2=7$, and the number of valid paths from P to Q is $35-7=\boxed{\textbf{(A) } 28}$.

Problem22

When a positive integer N is fed into a machine, the output is a number calculated according to the rule shown below.



For example, starting with an input of $N=7,$ the machine will output $3\cdot 7+1=22.$ Then if the output is repeatedly inserted into the machine five more times, the final output is $26.$

$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26$ When the same 6-step process is applied to a different starting value of $N,$ the final output is $1.$ What is the sum of all such integers $N?$

$N \to \_\_\_ \to \_\_\_ \to \_\_\_ \to \_\_\_ \to \_\_\_ \to 1$

$\textbf{(A) }73 \qquad \textbf{(B) }74 \qquad \textbf{(C) }75 \qquad \textbf{(D) }82 \qquad \textbf{(E) }83$

Solution 1

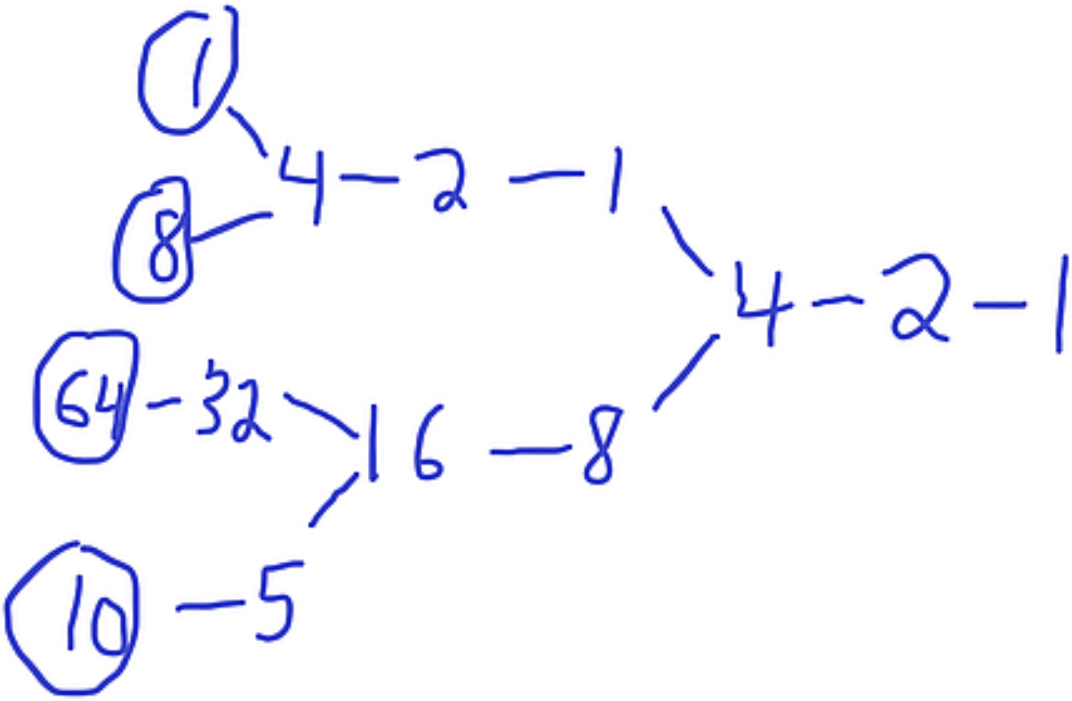
We start with the final output of 1 and work backwards, taking care to consider all possible inputs that could have resulted in any particular output. This produces the following set of possibilities at each stage:

$\{1\}\leftarrow\{2\}\leftarrow\{4\}\leftarrow\{1,8\}\leftarrow\{2,16\}\leftarrow\{4,5,32\}\leftarrow\{1,8,10,64\}$

where, for example, 2 must come from 4 (as there is no integer n satisfying $3n+1=2$), but 16 could come from 32 or 5 (as $\frac{32}{2}=3\cdot 5+1=16$, and 32 is even while 5 is odd). By construction, the last set in this sequence contains all the numbers which will lead to the number 1 at the end of the 6-step process, and their sum is $1+8+10+64=\boxed{\textbf{(E) }83}$.

Solution 2 (variant of Solution 1)

As in Solution 1, we work backwards from 1, this time showing the possible cases in a tree diagram:



The possible numbers are those at the "leaves" of the tree (the ends of the various branches), which are 1, 8, 64, and 10. Thus the answer is $1+8+64+10=\boxed{\textbf{(E) }83}$.

Solution 3 (algebraic)

We begin by finding the inverse of the function that the machine uses. Call the input I and the output O. If I is even, $O=\frac{I}{2}$, and if I is odd, $O=3I+1$. We can therefore see that $I=2O$ when I is even and $I=\frac{O-1}{3}$ when I is odd. Therefore, starting with 1, if I is even, $I=2$, and if I is odd, $I=0$, but the latter is not valid since 0 is not actually odd. This means that the 2nd-to-last number in the sequence has to be 2. Now, substituting 2 into the inverse formulae, if I is even, $I=4$ (which is indeed even), and if I is odd, $I=\frac{1}{3}$, which is not an integer. This means the 3rd-to-last number in the sequence has to be 4. Substituting in 4, if I is even, $I=8$, but if I is odd, $I=1$. Both of these are valid solutions, so the 4th-to-last number can be either 1 or 8. If it is 1, then by the argument we have just made, the 5th-to-last number has to be 2, the 6th-to-last number has to be 4, and the 7th-to-last number, which is the first number, must be either 1 or 8. In this way, we have ultimately found two solutions: $N=1$ and $N=8$.

On the other hand, if the 4th-to-last number is 8, substituting 8 into the inverse formulae shows that the 5th-to-last number is either 16 or $\frac{7}{3}$, but the latter is not an integer. Substituting 16 shows that if I is even, $I=32$, but if I is odd, $I=5$, and both of these are valid. If the 6th-to-last number is 32, then the first number must be 64, since $\frac{31}{3}$ is not an integer; if the 6th-to-last number is 5, then the first number has to be 10, as $\frac{4}{3}$ is not an integer. This means that, in total, there are 4 solutions for N, specifically, 1, 8, 10, and 64, which sum to $\boxed{\textbf{(E) } 83}$.

Problem23

Five different awards are to be given to three students. Each student will receive at least one award. In how many different ways can the awards be distributed?

$\textbf{(A) } 120 \qquad \textbf{(B) } 150 \qquad \textbf{(C) } 180 \qquad \textbf{(D) } 210 \qquad \textbf{(E) } 240$

Solution 1 (Complementary Counting)

Without the restriction that each student receives at least one award, we could simply take each of the 5awards and choose one of the 3 students to give it to, so that there would be $3^5=243$ ways to distribute the awards. We now need to subtract the cases where at least one student doesn't receive an award. If a student doesn't receive an award, there are 3 choices for which student that is, then $2^5 = 32$ ways of choosing a student to receive each of the awards, for a total of $3 \cdot 32 = 96$. However, if 2 students both don't receive an award, then such a case would be counted twice among our 96, so we need to add back in these cases. Of course, 2 students both not receiving an award is equivalent to only 1 student receiving all 5 awards, so there are simply 3 choices for which student that would be. It follows that the total number of ways of distributing the awards is $243 - 96 + 3 = \boxed{\textbf{(B) } 150}$.

Solution 2

Firstly, observe that it is not possible for a single student to receive 4 or 5 awards because this would mean that one of the other students receives no awards. Thus, each student must receive either 1, 2, or 3awards. If a student receives 3 awards, then the other two students must each receive 1 award; if a student receives 2 awards, then another student must also receive 2 awards and the remaining student must receive 1 award. We consider each of these two cases in turn. If a student receives three awards, there are 3 ways to choose which student this is, and $\binom{5}{3}$ ways to give that student 3 out of the 5 awards. Next, there are 2 students left and 2 awards to give out, with each student getting one award. There are clearly just 2 ways to distribute these two awards out, giving $3 \cdot \binom{5}{3} \cdot 2 = 60$ ways to distribute the awards in this case.

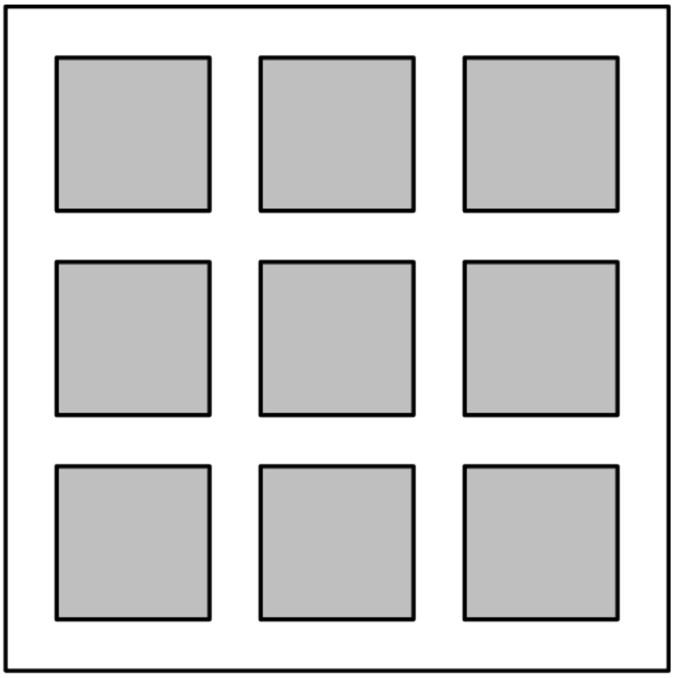
In the other case, a student receives 2 awards. We first have to choose which of the two students we will select to give two awards each to. There are $\binom{3}{2}$ ways to do this, after which there are $\binom{5}{2}$ ways to give the first student his two awards, leaving 3 awards yet to distribute. There are then $\binom{3}{2}$ ways to give the second student his 2 awards. Finally, there is only 1 student and 1 award left, so there is only 1 way to distribute this award. This results in $\binom{3}{2} \cdot \binom{5}{2} \cdot \binom{3}{2} \cdot 1 = 90$ ways to distribute the awards in this case. Adding the results of these two cases, we get $60 + 90 = \boxed{\textbf{(B) } 150}$.

Solution 3 (variation of Solution 2)

If each student must receive at least one award, then, as in Solution 2, we deduce that the only possible ways to split up the 5 awards are $3, 1, 1$ and $2, 2, 1$ (i.e. one student gets three awards and the others get one each, or two students each get two awards and the other student is left with the last one). In the first case, there are 3 choices for which student gets 3 awards, and $\binom{5}{3} = 10$ choices for which 3 awards they get. We are then left with 2 awards, and there are exactly 2 choices depending on which remaining student gets which. This yields a total for this case of $3 \cdot 10 \cdot 2 = 60$. For the second case, there are similarly 3 choices for which student gets only 1 award, and 5 choices for which award he gets. There are then 4 remaining awards, from which we choose 2 to give to one student and 2 to give to the other, which can be done in $\binom{4}{2} = 6$ ways (and we can say that e.g. the 2 chosen this way go to the first remaining student and the other 2 go to the second remaining student, which counts all possibilities). This means the total for the second case is $3 \cdot 5 \cdot 6 = 90$, and the answer is $60 + 90 = \boxed{\textbf{(B) } 150}$.

Problem24

A large square region is paved with n^2 gray square tiles, each measuring s inches on a side. A border $d$inches wide surrounds each tile. The figure below shows the case for $n = 3$. When $n = 24$, the 576 gray tiles cover 64% of the area of the large square region. What is the ratio $\frac{d}{s}$ for this larger value of n?



$\textbf{(A) }\frac{6}{25} \qquad \textbf{(B) }\frac{1}{4} \qquad \textbf{(C) }\frac{9}{25} \qquad \textbf{(D) }\frac{7}{16} \qquad \textbf{(E) }\frac{9}{16}$

Solution 1

The area of the shaded region is $(24s)^2$. To find the area of the large square, we note that there is a d-inch border between each of the 23 pairs of consecutive squares, as well as from between first/last squares and the large square, for a total of $23+2 = 25$ times the length of the border, i.e. $25d$. Adding this to the total length of the consecutive squares, which is $24s$, the side length of the large square is $(24s+25d)$, yielding the equation $\frac{(24s)^2}{(24s+25d)^2}=\frac{64}{100}$. Taking the square root of both sides (and using the fact that lengths are non-negative) gives $\frac{24s}{24s+25d}=\frac{8}{10} = \frac{4}{5}$, and cross-multiplying now gives

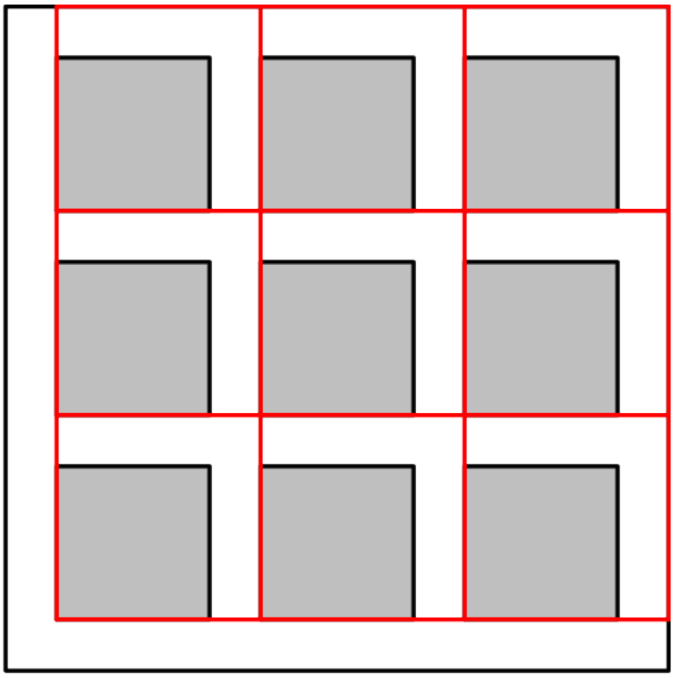
$$120s = 96s + 100d \Rightarrow 24s = 100d \Rightarrow \frac{d}{s} = \frac{24}{100} = \boxed{\textbf{(A) }\frac{6}{25}}$$

.

Solution 2

Without loss of generality, we may let $s = 1$ (since d will be determined by the scale of s, and we are only interested in the ratio $\frac{d}{s}$). Then, as the total area of the 576 gray tiles is simply 576, the large square has area $\frac{576}{0.64} = 900$, making the side of the large square $\sqrt{900} = 30$. As in Solution 1, the the side length of the large square consists of the total length of the gray tiles and 25 lots of the border, so the length of the border is $d = \frac{30 - 24}{25} = \frac{6}{25}$. Since $\frac{d}{s} = d$ if $s = 1$, the answer is $\boxed{\textbf{(A) }\frac{6}{25}}$.

Solution 3 (using answer choices)

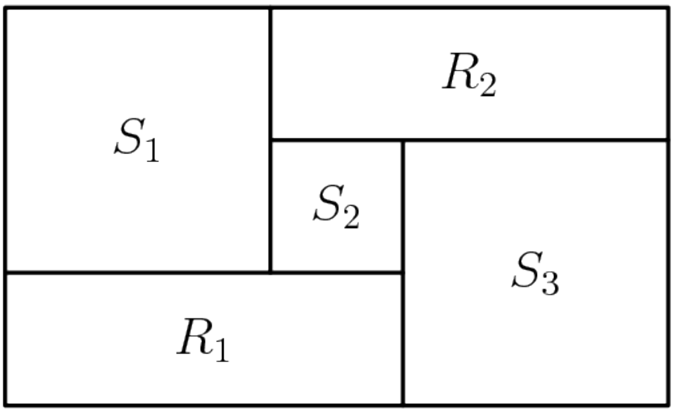
As in Solution 2, we let $s = 1$ without loss of generality. For sufficiently large n, we can approximate the percentage of the area covered by the gray tiles by subdividing most of the region into congruent squares, as shown:  Each red square has side length $(1+d)$, so by solving $\dfrac{1^2}{(1+d)^2}=\dfrac{64}{100}\iff\dfrac{1}{1+d}=\dfrac{4}{5}$, we obtain $d=\dfrac{1}{4}$.

The actual fraction of the total area covered by the gray tiles will be slightly less than $\dfrac{1}{(1+d)^2}$, which implies $\dfrac{1}{(1+d)^2}>\dfrac{64}{100}\iff\dfrac{1}{1+d}>\dfrac{4}{5}\iff d<\dfrac{1}{4}$.

Hence d (and thus $\dfrac{d}{s}$, since we are assuming $s=1$) is less than $\dfrac{1}{4}$, and the only choice that satisfies this is $\boxed{\textbf{(A) } \dfrac{6}{25}}$.

Problem25

Rectangles R_1 and R_2, and squares S_1, S_2, and S_3, shown below, combine to form a rectangle that is 3322 units wide and 2020 units high. What is the side length of S_2 in units?



$\textbf{(A) } 651 \qquad \textbf{(B) } 655 \qquad \textbf{(C) } 656 \qquad \textbf{(D) } 662 \qquad \textbf{(E) } 666$

Solution 1

Let the side length of each square S_k be s_k. Then, from the diagram, we can line up the top horizontal lengths of S_1, S_2, and S_3 to cover the top side of the large rectangle, so $s_1 + s_2 + s_3 = 3322$. Similarly, the short side of R_2 will be $s_1 - s_2$, and lining this up with the left side of S_3 to cover the vertical side of the large rectangle gives $s_1 - s_2 + s_3 = 2020$. We subtract the second equation from the first to obtain $2s_2 = 1302$, and thus $s_2 = \boxed{\textbf{(A) } 651}$.

Solution 2

Assuming that the problem is well-posed, it should be true in the particular case where $S_1 \cong S_3$ and $R_1 \cong R_2$. Let the sum of the side lengths of S_1 and S_2 be x, and let the length of rectangle R_2 be y. Under our assumption, we then have the system $\begin{cases} x + y = 3322 \\ x - y = 2020 \end{cases}$ which we solve to find that $y = \boxed{\textbf{(A) } 651}$.

Solution 3 (fast)

Since the sum of the side lengths of each pair of squares/rectangles has a sum of 3322 or 2020, and a difference of S_2, we see that the answer is $\frac{3322-2020}{2}=\frac{1302}{2}=\boxed{\textbf{(A) } 651}$.