2018 AMC 10B Problems/Problem 1

Contents

Problem

Kate bakes a 20-inch by 18-inch pan of cornbread. The cornbread is cut into pieces that measure 2 inches by 2 inches. How many pieces of cornbread does the pan contain?

$\textbf{(A) } 90 \qquad \textbf{(B) } 100 \qquad \textbf{(C) } 180 \qquad \textbf{(D) } 200 \qquad \textbf{(E) } 360$

Solution 1

The area of the pan is $20 \cdot 18$ = 360. Since the area of each piece is 4, there are $\frac{360}{4} = 90$ pieces. Thus, the answer is $\boxed{A}$.

Solution 2

By dividing each of the dimensions by 2, we get a 10×9 grid which makes 90 pieces. Thus, the answer is $\boxed{A}$.

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2018 AMC 10B Problems/Problem 2

Problem

Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?

$\textbf{(A) } 64 \qquad \textbf{(B) } 65 \qquad \textbf{(C) } 66 \qquad \textbf{(D) } 67 \qquad \textbf{(E) } 68$

Solution

Let Sam drive at exactly 60 mph in the first half hour, 65 mph in the second half hour, and x mph in the third half hour.

Due to $rt = d$, and that 30 min is half an hour, he covered $60 \cdot \frac{1}{2} = 30$ miles in the first 30 mins.

SImilarly, he covered $\frac{65}{2}$ miles in the 2nd half hour period.

The problem states that Sam drove 96 miles in 90 min, so that means that he must have covered $96 - \left(30 + \frac{65}{2}\right) = 33\frac{1}{2}$ miles in the third half hour period.

$rt = d$, so $x \cdot \frac{1}{2} = 33\frac{1}{2}$.

Therefore, Sam was driving $\boxed{\textbf{(D) } 67}$ miles per hour in the third half hour.

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2018 AMC 10B Problems/Problem 3

Contents

Problem

In the expression $(\underline{\qquad}\times\underline{\qquad})+(\underline{\qquad}\times\underline{\qquad})$ each blank is to be filled in with one of the digits $1,2,3,$ or $4,$ with each digit being used once. How many different values can be obtained?

$\textbf{(A) }2 \qquad\textbf{(B) }3\qquad\textbf{(C) }4 \qquad\textbf{(D) }6 \qquad\textbf{(E) }24$

Solution

We have $\binom{4}{2}$ ways to choose the pairs, and we have 2 ways for the values to be switched so $\frac{6}{2} = \boxed{3.}$

Solution 2

We have four available numbers $(1, 2, 3, 4)$. Because different permutations do not matter because they are all addition and multiplication, if we put 1 on the first space, it is obvious there are $\boxed{3}$ possible outcomes $(2, 3, 4)$.

Solution 3

There are $4!$ ways to arrange the numbers and $2!2!2!$ overcounts per way due to commutativity. Therefore, the answer is $\frac{4!}{2!2!2!}=\boxed{3}$

See Also

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2018 AMC 10B Problems/Problem 4

Contents

Problem

A three-dimensional rectangular box with dimensions X, Y, and Z has faces whose surface areas are 24, 24, 48, 48, 72, and 72 square units. What is X + Y + Z?

$\textbf{(A) }18 \qquad \textbf{(B) }22 \qquad \textbf{(C) }24 \qquad \textbf{(D) }30 \qquad \textbf{(E) }36 \qquad$

Solution 1

Let X be the length of the shortest dimension and Z be the length of the longest dimension. Thus, $XY = 24$, $YZ = 72$, and $XZ = 48$. Divide the first two equations to get $\frac{Z}{X} = 3$. Then, multiply by the last equation to get $Z^2 = 144$, giving $Z = 12$. Following, $X = 4$ and $Y = 6$.

The final answer is $4+6+12 = 22$. $\boxed{B}$

Solution 2

Simply use guess and check to find that the dimensions are 4 by 6 by 12. Therefore, the answer is $4+6+12 = 22$. $\boxed{B}$

Solution 3

If you find the GCD of 24, 48, and 72 you get your first number, 12. After this, do $48 \div 12$ and $72 \div 12$ to get 4 and 6, the other 2 numbers. When you add up your 3 numbers, you get 22 which is $\boxed{B}$.

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2018 AMC 10B Problems/Problem 5

Contents

Problem

How many subsets of $\{2,3,4,5,6,7,8,9\}$ contain at least one prime number?

$\textbf{(A) } 128 \qquad \textbf{(B) } 192 \qquad \textbf{(C) } 224 \qquad \textbf{(D) } 240 \qquad \textbf{(E) } 256$

Solution 1

Consider finding the number of subsets that do not contain any primes. There are four primes in the set: 2, 3, 5, and 7. This means that the number of subsets without any primes is the number of subsets of $\{4,6,8,9\}$, which is just $2^4=16$. The number of subsets with at least one prime is the number of subsets minus the number of subsets without any primes. The number of subsets is $2^8=256$. Thus, the answer is $256-16=\boxed{\textbf{(D) } 240}$

Solution 2 (Using Answer Choices)

Well, there are 4 composite numbers, and you can list them in a 1 number format, a 2 number, 3 number, and a 4 number format. Now, we can use permutations

$\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}=15$. Using the answer choices, the only multiple of 15 is $\boxed{\textbf{(D) } 240}$

By: K6511

See Also

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2018 AMC 10B Problems/Problem 6

A box contains 5 chips, numbered 1, 2, 3, 4, and 5. Chips are drawn randomly one at a time without replacement until the sum of the values drawn exceeds 4. What is the probability that 3 draws are required?

$\textbf{(A)}\ \frac{1}{15} \qquad \textbf{(B)}\ \frac{1}{10} \qquad \textbf{(C)}\ \frac{1}{6} \qquad \textbf{(D)}\ \frac{1}{5} \qquad \textbf{(E)}\ \frac{1}{4}$

Solution 1

Notice that the only four ways such that no more than 2 draws are required are $1, 2$; $1, 3$; $2, 1$; and $3, 1$. Notice that each of those cases has a $\frac{1}{5} \cdot \frac{1}{4}$ chance, so the answer is $\frac{1}{5} \cdot \frac{1}{4} \cdot 4 = \frac{1}{5}$, or $\boxed{D}$.

Solution 2

Notice that only the first two draws are important, it doesn't matter what number we get third because no matter what combination of 3 numbers is picked, the sum will always be greater than 5. Also, note that it is necessary to draw a 1 in order to have 3 draws, otherwise 5 will be attainable in two or less draws. So the probability of getting a 1 is $\frac{1}{5}$. It is necessary to pull either a 2 or 3 on the next draw and the probability of that is $\frac{1}{2}$. But, the order of the draws can be switched so we get:

$\frac{1}{5} \cdot \frac{1}{2} \cdot 2 = \frac{1}{5}$, or $\boxed{D}$

By: Soccer_JAMS

See Also

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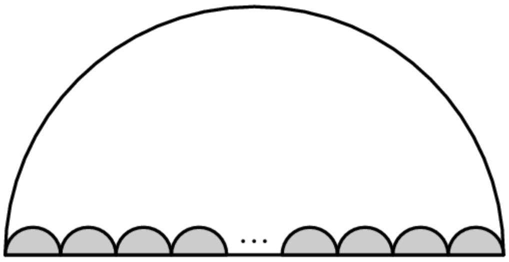
2018 AMC 10B Problems/Problem 7

In the figure below, N congruent semicircles lie on the diameter of a large semicircle, with their diameters covering the diameter of the large semicircle with no overlap. Let A be the combined area of the small semicircles and B be the area of the region inside the large semicircle but outside the semicircles. The ratio $A : B$ is $1 : 18$. What is N?



$\textbf{(A) } 16 \qquad \textbf{(B) } 17 \qquad \textbf{(C) } 18 \qquad \textbf{(D) } 19 \qquad \textbf{(E) } 36$

Contents

Solution 1 (Work using Answer Choices)

Use the answer choices and calculate them. The one that works is D

Solution 2 (More Algebraic Approach)

Let the number of semicircles be n and let the radius of each semicircle to be r. To find the total area of all of the small semicircles, we have $n \cdot \frac{\pi \cdot r^2}{2}$.

Next, we have to find the area of the larger semicircle. The radius of the large semicircle can be deduced to be $n \cdot r$. So, the area of the larger semicircle is $\frac{\pi \cdot n^2 \cdot r^2}{2}$.

Now that we have found the area of both A and B, we can find the ratio. $\frac{A}{B} = \frac{1}{18}$, so part-to-whole ratio is $1 : 19$ .When we divide the area of the small semicircles combined by the area of the larger semicircles, we get $\frac{1}{n}$. This is equal to $\frac{1}{19}$. By setting them equal, we find that $n = 19$. This is our answer, which corresponds to choice $\boxed{\text{(D)} 19}$.

Solution by: Archimedes15

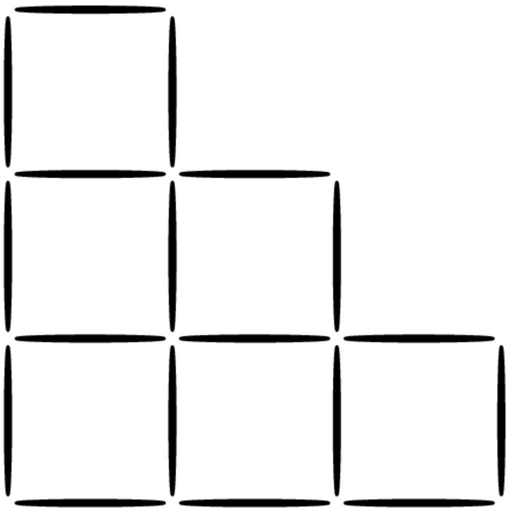
Solution 3

Each small semicircle is $\frac{1}{N^2}$ of the large semicircle. Since N small semicircles make $\frac{1}{19}$ of the large one, $\frac{N}{N^2} = \frac{1}{19}$. Solving this, we get $\boxed{19}$.

See Also

2018 AMC 10B Problems/Problem 8

Sara makes a staircase out of toothpicks as shown:



This is a 3-step staircase and uses 18 toothpicks. How many steps would be in a staircase that used 180 toothpicks?

$\textbf{(A) }10\qquad\textbf{(B) }11\qquad\textbf{(C) }12\qquad\textbf{(D) }24\qquad\textbf{(E) }30$

Contents

Solution

A staircase with n steps contains $4 + 6 + 8 + ... + 2n + 2$ toothpicks. This can be rewritten as $(n+1)(n+2) -2$.

So, $(n+1)(n+2) - 2 = 180$

So, $(n+1)(n+2) = 182.$

Inspection could tell us that $13 * 14 = 182$, so the answer is $13 - 1 = \boxed{(C) 12}$

Solution 2

Layer 1: 4 steps

Layer $1, 2$: 10 steps

Layer $1, 2, 3$: 18 steps

Layer $1, 2, 3, 4$: 28 steps

From inspection, we can see that with each increase in layer the difference in toothpicks between the current layer and the previous increases by 2. Using this pattern:

$4, 10, 18, 28, 40, 54, 70, 88, 108, 130, 154, 180$

From this we see that the solution is indeed $\boxed{(C) 12}$

By: Soccer_JAMS

Solution 3

We can find a function that gives us the number of toothpicks for every layer. Using finite difference, we know that the degree must be 2 and the leading coefficient is 1. The function is $f(n) = n^2 + 3n$ where n is the layer and $f(n)$ is the number of toothpicks.

We have to solve for n when $n^2 + 3n = 180 \Rightarrow n^2 + 3n - 180 = 0$. Factor to get $(n - 12)(n + 15)$. The roots are 12 and -15. Clearly -15 is impossible so the answer is $\boxed{(C) 12}$.

~Zeric Hang

See Also

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2018 AMC 10B Problems/Problem 9

The faces of each of 7 standard dice are labeled with the integers from 1 to 6. Let p be the probabilities that when all 7 dice are rolled, the sum of the numbers on the top faces is 10. What other sum occurs with the same probability as p?

$\textbf{(A)}\ 13 \qquad \textbf{(B)}\ 26 \qquad \textbf{(C)}\ 32 \qquad \textbf{(D)}\ 39 \qquad \textbf{(E)}\ 42$

Contents

Solution 1

It can be seen that the probability of rolling the smallest number possible is the same as the probability of rolling the largest number possible, the probability of rolling the second smallest number possible is the same as the probability of rolling the second largest number possible, and so on. This is because the number of ways to add a certain number of ones to an assortment of 7 ones is the same as the number of ways to take away a certain number of ones from an assortment of 7 6s.

So, we can match up the values to find the sum with the same probability as 10. We can start by noticing that 7 is the smallest possible roll and 42 is the largest possible role. The pairs with the same probability are as follows:

(7, 42), (8, 41), (9, 40), (10, 39), (11, 38)...

However, we need to find the number that matches up with 10. So, we can stop at (10, 39) and deduce that the sum with equal probability as 10 is 39. So, the correct answer is $\boxed{\textbf{(D)}\ 39}$, and we are done.

Written By: Archimedes15

Solution 2

Let's call the unknown value x. By symmetry, we realize that the difference between 10 and the minimum value of the rolls is equal to the difference between the maximum and x. So,

$$10-7=42-x$$

$x=39$ and our answer is $\boxed{\textbf{(D)}\ 39}$ By: Soccer_JAMS

Solution 3 (Simple Logic)

For the sums to have equal probability, the average sum of both sets of 7 dies has to be (6+1) x 7 = 49. Since having 10 is similar to not having 10, you just subtract 10 from the expected total sum. 49 - 10 = 39 so the answer is $\boxed{\textbf{(D)}\ 39}$

By: epicmonster

Solution 4

The expected value of the sums of the die rolls is $3.5*7=24.5$, and since the probabilities should be distributed symmetrically on both sides of 24.5, the answer is $24.5+24.5-10=39$, which is $\boxed{\textbf{(D)}\ 39}$.

By: dajeff

Note

Calculating the probability of getting a sum of 10 is also easy. There are 3 cases:

Case 1: $\{1,1,1,1,1,1,4\}$

$\frac{7!}{6!} = 7$ cases

Case 2: $\{1,1,1,1,1,2,3\}$

$\frac{7!}{5!} = 6*7 = 42$ cases

Case 3: $\{1,1,1,1,2,2,2\}$

$\frac{7!}{4!3!} = 5*7 = 35$ cases

The probability is $\frac{84}{6^7} = \frac{14}{6^6}$.

Calculating 6^6:

$6^6 = (6^3)^2 = 216^2 = 46656$

Therefore, the probability is $\frac{14}{46656} = \boxed{\frac{7}{23328}}$

~Zeric Hang

See Also

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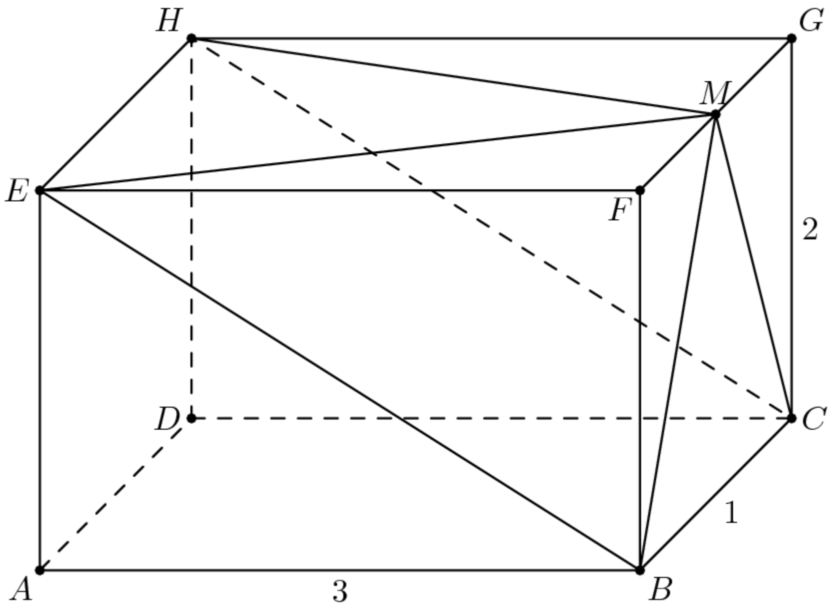
Contents

Problem

In the rectangular parallelpiped shown, AB = 3, BC = 1, and CG = 2. Point M is the midpoint of $\overline{FG}$. What is the volume of the rectangular pyramid with base $BCHE$ and apex M?



$\textbf{(A) } 1 \qquad \textbf{(B) } \frac{4}{3} \qquad \textbf{(C) } \frac{3}{2} \qquad \textbf{(D) } \frac{5}{3} \qquad \textbf{(E) } 2$

Solution 1

Consider the cross-sectional plane and label its area b. Note that the volume of the triangular prism that encloses the pyramid is $bh/2=3$, and we want the rectangular pyramid that shares the base and height with the triangular prism. The volume of the pyramid is $bh/3$, so the answer is $\boxed{2}$. (AOPS12142015)

Solution 2

We can start by finding the total volume of the parallelepiped. It is $2 \cdot 3 \cdot 1 = 6$, because a rectangular parallelepiped is a rectangular prism.

Next, we can consider the wedge-shaped section made when the plane $BCHE$ cuts the figure. We can find the volume of the triangular pyramid with base EFB and apex M. The area of EFB is $\frac{1}{2} \cdot 2 \cdot 3 = 3$. Since BC is given to be 1, we have that FM is $\frac{1}{2}$. Using the formula for the volume of a triangular pyramid, we have $V = \frac{1}{3} \cdot \frac{1}{2} \cdot 3 = \frac{1}{2}$. Also, since the triangular pyramid with base HGC and apex M has the exact same dimensions, it has volume $\frac{1}{2}$ as well.

The original wedge we considered in the last step has volume 3, because it is half of the volume of the parallelepiped. We can subtract out the parts we found to have $3 - \frac{1}{2} \cdot 2 = 2$. Thus, the volume of the figure we are trying to find is 2. This means that the correct answer choice is $\boxed{E}$.

Written by: Archimedes15

NOTE: For those who think that it isn't a rectangular prism, please read the problem. It says "rectangular parallelepiped." If a parallelepiped is such that all of the faces are rectangles, it is a rectangular prism.

Solution 3

If you look carefully, you will see that on the either side of the pyramid in question, there are two congruent tetrahedra. The volume of one is $\frac{1}{3}Bh$, with its base being half of one of the rectangular prism's faces and its height being half of one of the edges, so its volume is $\frac{1}{3}(3 \times 2/2 \times \frac{1}{2}) = \frac{1}{2}$. We can obtain the answer by subtracting twice this value from the diagonal half prism, or $(\frac{1}{2} \times 3 \times 2 \times 1) - (2 \times \frac{1}{2}) = \boxed{2}$

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Categories: Introductory Geometry Problems | 3D Geometry Problems

2018 AMC 10B Problems/Problem 11

Which of the following expressions is never a prime number when p is a prime number?

$\textbf{(A) } p^2+16 \qquad \textbf{(B) } p^2+24 \qquad \textbf{(C) } p^2+26 \qquad \textbf{(D) } p^2+46 \qquad \textbf{(E) } p^2+96$

Contents

Solution 1

Because squares of a non-multiple of 3 is always $1 \mod 3$, the only expression is always a multiple of 3 is $\boxed{\textbf{(C) } p^2+26}$. This is excluding when $p\equiv 0 \mod 3$, which only occurs when $p=3$, then $p^2+26=35$ which is still composite.

Solution 2 (Not Recommended Solution)

We proceed with guess and check:
$5^2+16=41 \qquad 7^2+24=73 \qquad 5^2+46=71 \qquad 19^2+96=457$. Clearly only $\boxed{\textbf{(C)}}$ is our only option left. (franchester)

Solution 3

Primes can only be 1 or $-1 \mod 6$. Therefore, the square of a prime can only be $1 \mod 6$. p^2+26 then must be $3 \mod 6$, so it is always divisible by 3. Therefore, the answer is $\boxed{\text{(C)}}$.

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2018 AMC 10B Problems/Problem 12

Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

$\textbf{(A)}\ 25 \qquad \textbf{(B)}\ 38 \qquad \textbf{(C)}\ 50 \qquad \textbf{(D)}\ 63 \qquad \textbf{(E)}\ 75$

Solution

Let $A=(-12,0), B=(12,0)$. Therefore, C lies on the circle with equation $x^2+y^2=144$. Let it have coordinates (x,y). Since we know the centroid of a triangle with vertices with coordinates of $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ is $\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$, the centroid of $\triangle ABC$ is $\left(\frac{x}{3},\frac{y}{3}\right)$. Because $x^2+y^2=144$, we know that $\left(\frac{x}{3}\right)^2+\left(\frac{y}{3}\right)^2=16$, so the curve is a circle centered at the origin. Therefore, its area is $16\pi\approx\boxed{\text{(C) } 50}$. -tdeng

Solution 2 (no coordinates)

We know the centroid of a triangle splits the medians into segments of ratio 2:1, and the median of the triangle that goes to the center of the circle is the radius (length 12), so the length from the centroid of the triangle to the center of the circle is always $\frac{1}{3}\cdot 12=4$. The area of a circle with radius 4 is 16π, or around $\boxed{\textbf{(C) } 50}$. -That_Crazy_Book_Nerd

See Also

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2018 AMC 10B Problems/Problem 13

Contents

Problem

How many of the first 2018 numbers in the sequence $101, 1001, 10001, 100001, \dots$ are divisible by 101?

$\textbf{(A) }253 \qquad \textbf{(B) }504 \qquad \textbf{(C) }505 \qquad \textbf{(D) }506 \qquad \textbf{(E) }1009$

Solution

Note that $10^{2k}+1$ for some odd k will suffice $\mod 101$. Each $2k \in \{2,4,6,\dots,2018\}$, so the answer is $\boxed{\textbf{(C) } 505}$ (AOPS12142015)

Solution 2

If we divide each number by 101, we see a pattern occuring in every 4 numbers. $101, 1000001, 10000000001, \dots$. We divide 2018 by 4 to get 504 with 2 left over. One divisible number will be in the 2 left over, so out answer is $\boxed{\textbf{(C) } 505}$.

Solution 3

Note that 909 is divisible by 101, and thus 9999 is too. We know that 101 is divisible and 1001 isn't so let us start from 10001. We subtract 9999 to get 2. Likewise from 100001 we subtract, but we instead subtract 9999 times 10 or 99990 to get 11. We do it again and multiply the 9's by 10 to get 101. Following the same knowledge, we can use mod 101 to finish the problem. The sequence will just be subtracting 1, multiplying by 10, then adding 1. Thus the sequence is $0, -9, -99(2), 11, 0, \dots$. Thus it repeats every four. Consider the sequence after the 1st term and we have 2017 numbers. Divide 2017 by four to get 504 remainder 1. Thus the answer is 504 plus the 1st term or $\boxed{\textbf{(C) } 505}$.

-googleghosh

See Also

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2018 AMC 10B Problems/Problem 14

A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

$\textbf{(A)}\ 202\qquad\textbf{(B)}\ 223\qquad\textbf{(C)}\ 224\qquad\textbf{(D)}\ 225\qquad\textbf{(E)}\ 234$

Solution

To minimize the number of values, we want to maximize the number of times they appear. So, we could have 223 numbers appear 9 times, 1 number appear once, and the mode appear 10 times, giving us a total of $223 + 1 + 1$ = $\boxed{\textbf{(D)}\ 225}$

See Also

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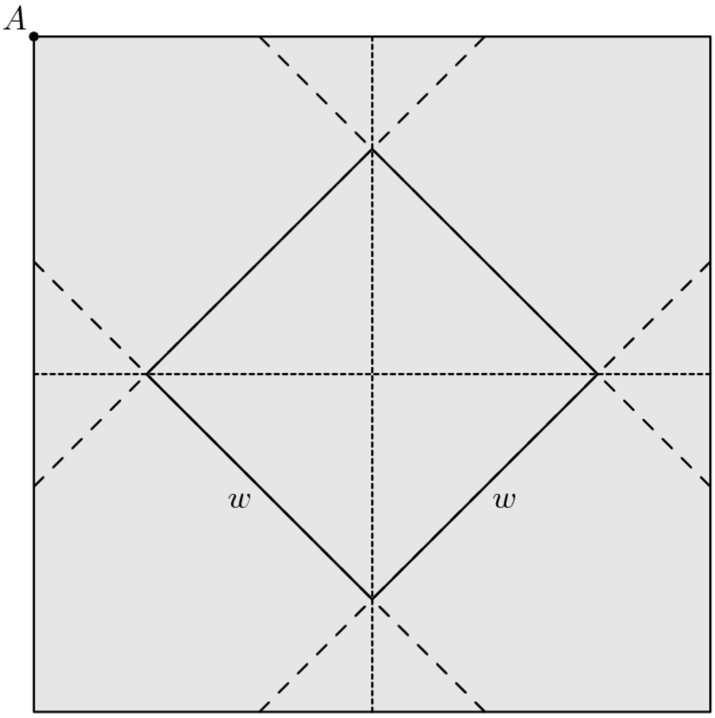
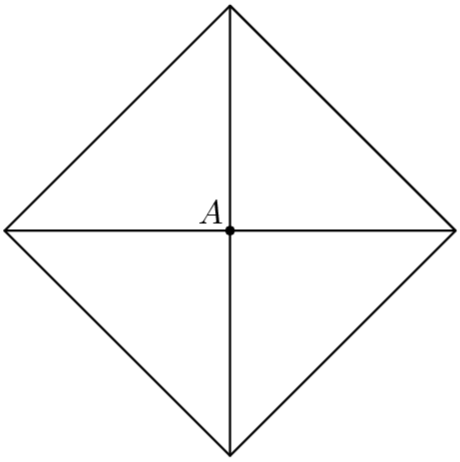
Category: Introductory Combinatorics Problems

2018 AMC 10B Problems/Problem 15

A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h. What is the area of the sheet of wrapping paper?





$\textbf{(A) } 2(w+h)^2 \qquad \textbf{(B) } \frac{(w+h)^2}{2} \qquad \textbf{(C) } 2w^2+4wh \qquad \textbf{(D) } 2w^2 \qquad \textbf{(E) } w^2h$

Solution 1

Consider one-quarter of the image (the wrapping paper is divided up into 4 congruent squares). The length of each dotted line is h. The area of the rectangle that is w by h is wh. The combined figure of the two triangles with base h is a square with h as its diagonal. Using the Pythagorean Theorem, each side of this square is $\sqrt{\frac{h^2}{2}}$. Thus, the area is the side length squared which is $\frac{h^2}{2}$. Similarly, the combined figure of the two triangles with base w is a square with area $\frac{w^2}{2}$. Adding all of these together, we get $\frac{w^2}{2} + \frac{h^2}{2} + wh$. Since we have four of these areas in the entire wrapping paper, we multiply this by 4, getting $4(\frac{w^2}{2}+\frac{h^2}{2}+wh) = 2(w^2+h^2+2wh) = \boxed{\textbf{(A) } 2(w+h)^2}$.

Solution 2

The sheet of paper is made out of the surface area of the box plus the sum of the four triangles. The surface area is $2w^2+2wh+2wh$ which equals $2w^2+4wh$.The four triangles each have a height and a base of h, so they each have an area of $\frac{h^2}{2}$. There are four of them, so multiplied by four is $2h^2$. Together, paper's area is $2w^2+4wh+2h^2$. This can be factored and written as $\boxed{\textbf{(A) } 2(w+h)^2}$.

See Also

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Category: Introductory Geometry Problems

2018 AMC 10B Problems/Problem 16

Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}.$$

What is the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 4$

Contents

Solution 1

One could simply list out all the residues to the third power $\mod 6$. (Edit: Euler's totient theorem is not a valid approach to showing that they are all congruent $\mod 6$. This is due to the fact that a_k need not be relatively prime to 6.)

Therefore the answer is congruent to $2018^{2018} \equiv 2^{2018} \pmod{6} = \boxed{(E) 4}$

Solution 2

(not very good one)

Note that
$$(a_1 + a_2 + \cdots + a_{2018})^3 = a_1^3 + a_2^3 + \cdots + a_{2018}^3 + 3a_1^2(a_1 + a_2 + \cdots + a_{2018} - a_1) + 3a_2^2(a_1 + a_2 + \cdots + a_{2018} - a_2) + \cdots +$$
$$3a_{2018}^2(a_1 + a_2 + \cdots + a_{2018} - a_{2018}) + 6\sum_{i \neq j \neq k}^{2018} a_i a_j a_k$$

Note that
$$a_1^3 + a_2^3 + \cdots + a_{2018}^3 + 3a_1^2(a_1 + a_2 + \cdots + a_{2018} - a_1) + 3a_2^2(a_1 + a_2 + \cdots + a_{2018} - a_2) + \cdots + 3a_{2018}^2(a_1 + a_2 + \cdots + a_{2018} - a_{2018}) +$$
$$6\sum_{i \neq j \neq k}^{2018} a_i a_j a_k \equiv a_1^3 + a_2^3 + \cdots + a_{2018}^3 + 3a_1^2(2018^{2018} - a_1) + 3a_2^2(2018^{2018} - a_2) + \cdots + 3a_{2018}^2(2018^{2018} - a_{2018}) \equiv -2(a_1^3 + a_2^3 + \cdots + a_{2018}^3$$
$$\pmod{6}$$

Therefore, $-2(a_1^3 + a_2^3 + \cdots + a_{2018}^3) \equiv (2018^{2018})^3 \equiv (2^{2018})^3 \equiv 4^3 \equiv 4 \pmod{6}$.

Thus, $a_1^3 + a_2^3 + \cdots + a_{2018}^3 \equiv 1 \pmod{3}$. However, since cubing preserves parity, and the sum of the individual terms is even, the some of the cubes is also even, and our answer is $\boxed{\text{(E) } 4}$

Solution 3

We first note that $1^3 + 2^3 + \ldots = (1 + 2 + \ldots)^2$. So what we are trying to find is what $(2018^{2018})^2 = (2018^{4036})$ mod 6. We start by noting that 2018 is congruent to 2 mod 6. So we are trying to find (2^{4036}) mod 6. Instead of trying to do this with some number theory skills, we could just look for a pattern. We start with small powers of 2 and see that 2^1 is 2 mod 6, 2^2 is 4 mod 6, 2^3 is 2 mod 6, 2^4 is 4 mod 6, and so on... So we see that since (2^{4036}) has an even power, it must be congruent to 4 mod 6, thus giving our answer $\boxed{\text{(E) } 4}$. You can prove this pattern using mods. But I thought this was easier.

-TheMagician

Solution 4 (Lazy solution)

Assume $a_1, a_2, \ldots a_{2017}$ are multiples of 6 and find $2018^{2018} \pmod{6}$ (which happens to be 4). Then ${a_1}^3 + \ldots + {a_{2018}}^3$ is congruent to $64 \pmod{6}$ or just 4.

-Patrick4President

Solution 5 (Fermat's Little Theorem)

First note that each $a_i^3 \equiv a_i \pmod{3}$ by Fermat's Little Theorem. This implies that $a_1^3 + \ldots + a_{2018}^3 \equiv a_1 + \ldots + a_{2018} \equiv 2^{2018} \equiv 1 \pmod{3}$. Also, all $a_i^2 \equiv a_i \pmod{2}$, hence $a_i^3 \equiv (a_i)(a_i^2) \equiv a_i^2 \equiv a_i \pmod{2}$ by Fermat's Little Theorem.Thus, $a_1^3 + \ldots a_{2018}^3 \equiv 2^{2018} \equiv 0 \pmod{2}$. Now set $x = a_1^3 + \ldots + a_{2018}^3$. Then, we have the congruences $x \equiv 0 \pmod{2}$ and $x \equiv 1 \pmod{3}$. By the Chinese Remainder Theorem, a solution must exist, and indeed solving the congruence we get that $x \equiv 4 \pmod{6}$. Thus, the answer is $\boxed{(E) 4}$

See Also

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2018 AMC 10B Problems/Problem 17

Contents

Problem

In rectangle $PQRS$, $PQ = 8$ and $QR = 6$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, points E and F lie on $\overline{RS}$, and points G and H lie on $\overline{SP}$ so that $AP = BQ < 4$ and the convex octagon $ABCDEFGH$ is equilateral. The length of a side of this octagon can be expressed in the form $k + m\sqrt{n}$, where k, m, and n are integers and n is not divisible by the square of any prime. What is $k + m + n$?

$\textbf{(A) } 1 \qquad \textbf{(B) } 7 \qquad \textbf{(C) } 21 \qquad \textbf{(D) } 92 \qquad \textbf{(E) } 106$

Solution 1

Let $AP = BQ = x$. Then $AB = 8 - 2x$.

Now notice that since $CD = 8 - 2x$ we have $QC = DR = x - 1$.

Thus by the Pythagorean Theorem we have $x^2 + (x-1)^2 = (8-2x)^2$ which becomes $2x^2 - 30x + 63 = 0 \implies x = \frac{15 - 3\sqrt{11}}{2}$.

Our answer is $8 - (15 - 3\sqrt{11}) = 3\sqrt{11} - 7 \implies \boxed{\text{(B) } 7}$. (Mudkipswims42)

Solution 2

Denote the length of the equilateral octagon as x. The length of $\overline{BQ}$ can be expressed as $\frac{8-x}{2}$. By Pythagoras, we find that:

$$\left(\frac{8-x}{2}\right)^2 + \overline{CQ}^2 = x^2 \implies \overline{CQ} = \sqrt{x^2 - \left(\frac{8-x}{2}\right)^2}$$

Since $\overline{CQ} = \overline{DR}$, we can say that $x + 2\sqrt{x^2 - \left(\frac{8-x}{2}\right)^2} = 6 \implies x = -7 \pm 3\sqrt{11}$. We can discard the negative solution, so $k + m + n = -7 + 3 + 11 = \boxed{\textbf{(B) } 7}$ ~ blitzkrieg21

See Also

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2018 AMC 10B Problems/Problem 18

Three young brother-sister pairs from different families need to take a trip in a van. These six children will occupy the second and third rows in the van, each of which has three seats. To avoid disruptions, siblings may not sit right next to each other in the same row, and no child may sit directly in front of his or her sibling. How many seating arrangements are possible for this trip?

$\textbf{(A) } 60 \qquad \textbf{(B) } 72 \qquad \textbf{(C) } 92 \qquad \textbf{(D) } 96 \qquad \textbf{(E) } 120$

Solution 1 (Casework)

We can begin to put this into cases. Let's call the pairs a, b and c, and assume that a member of pair a is sitting in the leftmost seat of the second row. We can have the following cases then.

Case 1: Second Row: a b c Third Row: b c a

Case 2: Second Row: a c b Third Row: c b a

Case 3: Second Row: a b c Third Row: c a b

Case 4: Second Row: a c b Third Row: b a c

For each of the four cases, we can flip the siblings, as they are distinct. So, each of the cases has $2 \cdot 2 \cdot 2 = 8$ possibilities. Since there are four cases, when pair a has someone in the leftmost seat of the second row, there are 32 ways to rearrange it. However, someone from either pair a, b, or c could be sitting in the leftmost seat of the second row. So, we have to multiply it by 3 to get our answer of $32 \cdot 3 = 96$. So, the correct answer is $\boxed{\textbf{(D) } 96}$.

Written By: Archimedes15

Solution 2

Call the siblings A_1, A_2, B_1, B_2, C_1, and C_2.

There are 6 choices for the child in the first seat, and it doesn't matter which one takes it, so suppose Without loss of generality that A_1 takes it (a $\circ$ is an empty seat):

$A_1 \circ \circ$
$\circ \ \circ \circ$

Then there are 4 choices for the second seat (B_1, B_2, C_1, or C_2). Again, it doesn't matter who takes the seat, so WLOG suppose it is B_1:

$A_1 B_1 \circ$
$\circ \ \circ \circ$

The last seat in the first row cannot be A_2 because it would be impossible to create a second row that satisfies the conditions. Therefore, it must be C_1 or C_2. Suppose WLOG that it is C_1. There are two ways to create a second row:

$A_1 B_1 C_1$
$B_2 C_2 A_2$

$A_1 B_1 C_1$
$C_2 A_2 B_2$

Therefore, there are $6 \cdot 4 \cdot 2 \cdot 2 = \boxed{\textbf{(D) } 96}$ possible seating arrangements.

Written by: R1ceming

See Also

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2018 AMC 10B Problems/Problem 19

Contents

Problem

Joey and Chloe and their daughter Zoe all have the same birthday. Joey is 1 year older than Chloe, and Zoe is exactly 1 year old today. Today is the first of the 9 birthdays on which Chloe's age will be an integral multiple of Zoe's age. What will be the sum of the two digits of Joey's age the next time his age is a multiple of Zoe's age?

$\textbf{(A) }7 \qquad \textbf{(B) }8 \qquad \textbf{(C) }9 \qquad \textbf{(D) }10 \qquad \textbf{(E) }11$

Solution 1

Let Joey's age be j, Chloe's age be c, and we know that Zoe's age is 1.

We know that there must be 9 values $k \in \mathbb{Z}$ such that $c+k=a(1+k)$ where a is an integer.

Therefore, $c-1+(1+k)=a(1+k)$ and $c-1=(1+k)(a-1)$. Therefore, we know that, as there are 9 solutions for k, there must be 9 solutions for $c-1$. We know that this must be a perfect square. Testing perfect squares, we see that $c-1=36$, so $c=37$. Therefore, $j=38$. Now, since $j-1=37$, by similar logic, $37=(1+k)(a-1)$, so $k=36$ and Joey will be $38+36=74$ and the sum of the digits is $\boxed{\text{(E) }11}$

Solution 2

Here's a different way of saying the above solution:

If a number is a multiple of both Chloe's age and Zoe's age, then it is a multiple of their difference. Since the difference between their ages does not change, then that means the difference between their ages has 9 factors. Therefore, the difference between Chloe and Zoe's age is 36, so Chloe is 37, and Joey is 38. The common factor that will divide both of their ages is 37, so Joey will be 74. 7 + 4 = $\boxed{\text{(E) }11}$

Solution 3

Similar approach to above, just explained less concisely and more in terms of the problem (less algebra-y)

Let $C+n$ denote Chloe's age, $J+n$ denote Joey's age, and $Z+n$ denote Zoe's age, where n is the number of years from now. We are told that $C+n$ is a multiple of $Z+n$ exactly nine times. Because $Z+n$ is 1 at $n=0$ and will increase until greater than $C-Z$, it will hit every natural number less than $C-Z$, including every factor of $C-Z$. For $C+n$ to be an integral multiple of $Z+n$, the difference $C-Z$ must also be a multiple of Z, which happens iff Z is a factor of $C-Z$. Therefore, $C-Z$ has nine factors. The smallest number that has nine positive factors is $2^2 3^2=36$ (we want it to be small so that Joey will not have reached three digits of age before his age is a multiple of Zoe's). We also know $Z=1$ and $J=C+1$. Thus,

$$C-Z=36$$

$$J-Z=37$$

By our above logic, the next time $J-Z$ is a multiple of $Z+n$ will occur when $Z+n$ is a factor of $J-Z$. Because 37 is prime, the next time this happens is at $Z+n=37$, when $J+n=74$. $7+4=\boxed{\textbf{(E)} 11}$

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Category: Introductory Number Theory Problems

2018 AMC 10B Problems/Problem 20

Contents

Problem

A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

$\textbf{(A)}\ 2016 \qquad \textbf{(B)}\ 2017 \qquad \textbf{(C)}\ 2018 \qquad \textbf{(D)}\ 2019 \qquad \textbf{(E)}\ 2020$

Solution 1

$f\left(n\right) = f\left(n-1\right) - f\left(n-2\right) + n$

$= \left(f\left(n-2\right) - f\left(n-3\right) + n - 1\right) - f\left(n-2\right) + n$

$= 2n - 1 - f\left(n-3\right)$

$= 2n - 1 - \left(2\left(n-3\right) - 1 - f\left(n-6\right)\right)$

$= f\left(n-6\right) + 6$

Thus, $f\left(2018\right) = 2016 + f\left(2\right) = 2017. \boxed{B}$

Solution 2 (A Bit Bashy)

Start out by listing some terms of the sequence.

$$f(1) = 1$$

$$f(2) = 1$$

$$f(3) = 3$$

$$f(4) = 6$$

$$f(5) = 8$$

$$f(6) = 8$$

$$f(7) = 7$$

$$f(8) = 7$$

$$f(9) = 9$$

$$f(10) = 12$$

$$f(11) = 14$$

$$f(12) = 14$$

$$f(13) = 13$$

$$f(14) = 13$$

$$f(15) = 15$$

$$.....$$

Notice that $f(n) = n$ whenever n is an odd multiple of 3, and the pattern of numbers that follow will always be +3, +2, +0, -1, +0. The closest odd multiple of 3 to 2018 is 2013, so we have

$$f(2013) = 2013$$

$$f(2014) = 2016$$

$$f(2015) = 2018$$

$$f(2016) = 2018$$

$$f(2017) = 2017$$

$$f(2018) = \boxed{(B) 2017}.$$

Solution 3(Bashy Pattern Finding)

Writing out the first few values, we get: $1, 1, 3, 6, 8, 8, 7, 7, 9, 12, 14, 14, 13, 13, 15, 18, 20, 20, 19, 19...$. Examining, we see that every number x where $x \equiv 1 \pmod{6}$ has $f(x) = x$, $f(x+1) = f(x) = x$, and $f(x-1) = f(x-2) = x+1$. The greatest number that's 1 (mod 6) and less 2018 is 2017, so we have $f(2017) = f(2018) = 2017. \boxed{B}$

See Also

2018 AMC 10B Problems/Problem 21

Contents

Problem

Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1, 2, ..., \dfrac{n}{2}, n$. At some moment Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323?

$\textbf{(A) } 324 \qquad \textbf{(B) } 330 \qquad \textbf{(C) } 340 \qquad \textbf{(D) } 361 \qquad \textbf{(E) } 646$

Solution 1

Prime factorizing 323 gives you $17 \cdot 19$. The desired answer needs to be a multiple of 17 or 19, because if it is not a multiple of 17 or 19, the LCM, or the least possible value for n, will not be more than 4 digits. Looking at the answer choices, $\boxed{\text{(C) } 340}$ is the smallest number divisible by 17 or 19. Checking, we can see that n would be 6460.

Solution 2

Let the next largest divisor be k. Suppose $\gcd(k, 323) = 1$. Then, as $323|n, k|n$, therefore, $323 \cdot k|n$. However, because $k > 323$, $323k > 323 \cdot 324 > 9999$. Therefore, $\gcd(k, 323) > 1$. Note that $323 = 17 \cdot 19$. Therefore, the smallest the gcd can be is 17 and our answer is $323 + 17 = \boxed{\text{(C) } 340}$.

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Category: Intermediate Number Theory Problems

2018 AMC 10B Problems/Problem 22

Real numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. Which of the following numbers is closest to the probability that $x, y,$ and 1 are the side lengths of an obtuse triangle?

$\textbf{(A) } 0.21 \qquad \textbf{(B) } 0.25 \qquad \textbf{(C) } 0.29 \qquad \textbf{(D) } 0.50 \qquad \textbf{(E) } 0.79$

Solution

The Pythagorean Inequality tells us that in an obtuse triangle, $a^2 + b^2 < c^2$. The triangle inequality tells us that $a + b > c$. So, we have two inequalities:

$$x^2 + y^2 < 1$$

$$x + y > 1$$

The first equation is $\frac{1}{4}$ of a circle with radius 1, and the second equation is a line from $(0, 1)$ to $(1, 0)$. So, the area is $\frac{\pi}{4} - \frac{1}{2}$ which is approximately 0.29.

Solution 2

Note that the obtuse angle in the triangle has to be opposite the side that is always length 1. This is because the largest angle is always opposite the largest side, and if 2 sides of the triangle were 1, the last side would have to be greater than 1 to make an obtuse triangle. Using this observation, we can set up a law of cosines where the angle is opposite 1:

$$1^2 = x^2 + y^2 - 2xy\cos(\theta)$$

where x and y are the sides that go from $[0, 1]$ and θ is the angle opposite the side of length 1.

By isolating $\cos(\theta)$, we get:

$$\frac{1 - x^2 - y^2}{-2xy} = \cos(\theta)$$

For θ to be obtuse, $\cos(\theta)$ must be negative. Therefore, $\frac{1 - x^2 - y^2}{-2xy}$ is negative. Since x and y must be positive, $-2xy$ must be negative, so we must make $1 - x^2 - y^2$ positive. From here, we can set up the inequality

$$x^2 + y^2 < 1$$

Additionally, to satisfy the definition of a triangle, we need:

$$x + y > 1$$

The solution should be the overlap between the two equations in the 1st quadrant.

By observing that $x^2 + y^2 < 1$ is the equation for a circle, the amount that is in the 1st quadrant is $\frac{\pi}{4}$. The line can also be seen as a chord that goes from $(0, 1)$ to $(1, 0)$. By cutting off the triangle of area $\frac{1}{2}$ that is not part of the overlap, we get $\frac{\pi}{4} - \frac{1}{2} \approx \boxed{0.29}$.

-allenle873

See Also

2018 AMC 10B Problems/Problem 23

How many ordered pairs (a, b) of positive integers satisfy the equation

$$a \cdot b + 63 = 20 \cdot \text{lcm}(a, b) + 12 \cdot \text{gcd}(a, b),$$

where $\text{gcd}(a, b)$ denotes the greatest common divisor of a and b, and $\text{lcm}(a, b)$ denotes their least common multiple?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 8$

Solution

Let $x =$lcm(a, b), and $y =$gcd(a, b). Therefore, $a \cdot b =$lcm$(a, b)\cdot$gcd$(a, b) = x \cdot y$. Thus, the equation becomes

$$x \cdot y + 63 = 20x + 12y$$

$$x \cdot y - 20x - 12y + 63 = 0$$

Using Simon's Favorite Factoring Trick, we rewrite this equation as

$$(x - 12)(y - 20) - 240 + 63 = 0$$

$$(x - 12)(y - 20) = 177$$

Since $177 = 3 \cdot 59$ and $x > y$, we have $x - 12 = 59$ and $y - 20 = 3$, or $x - 12 = 177$ and $y - 20 = 1$. This gives us the solutions $(71, 23)$ and $(189, 21)$. Obviously, the first pair does not work. Assume $a > b$. We must have $a = 21 \cdot 9$ and $b = 21$, and we could then have $a < b$, so there are $\boxed{2}$ solutions. (awesomeag)

Edited by IronicNinja~

See Also

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2018 AMC 10B Problems/Problem 24

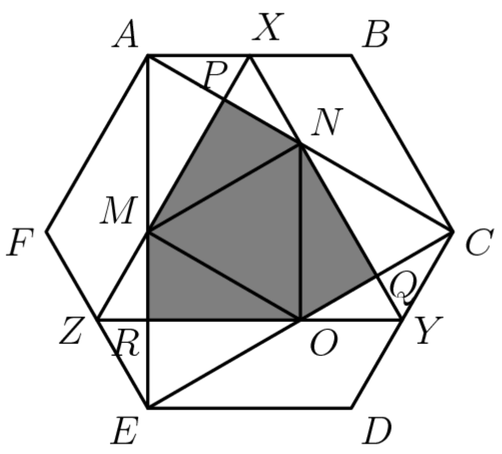
Contents

Problem

Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y, and Z the midpoints of sides $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$, respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

$\textbf{(A)} \frac{3}{8}\sqrt{3} \qquad \textbf{(B)} \frac{7}{16}\sqrt{3} \qquad \textbf{(C)} \frac{15}{32}\sqrt{3} \qquad \textbf{(D)} \frac{1}{2}\sqrt{3} \qquad \textbf{(E)} \frac{9}{16}\sqrt{3}$

Solution 1



The desired area (hexagon $MPNQOR$) consists of an equilateral triangle ($\triangle MNO$) and three right triangles ($\triangle MPN$, $\triangle NQO$, and $\triangle ORM$).

Notice that $\overline{AD}$ (not shown) and $\overline{BC}$ are parallel. $\overline{XY}$ divides transversals $\overline{AB}$ and $\overline{CD}$ into a $1:1$ ratio. Thus, it must also divide transversal $\overline{AC}$ and transversal $\overline{CO}$ into a $1:1$ ratio. By symmetry, the same applies for $\overline{CE}$ and $\overline{EA}$ as well as $\overline{EM}$ and $\overline{AN}$

In $\triangle ACE$, we see that $\frac{[MNO]}{[ACE]}=\frac{1}{4}$ and $\frac{[MPN]}{[ACE]}=\frac{1}{8}$. Our desired area becomes

$$(\frac{1}{4}+3\cdot\frac{1}{8})\cdot\frac{(\sqrt{3})^2\cdot\sqrt{3}}{4}=\frac{15}{32}\sqrt{3}=\boxed{C}$$

Solution 2

Now, if we look at the figure, we can see that the complement of the hexagon we are trying to find is composed of 3 isosceles trapezoids ($AXFZ$, $XBCY$, and $ZYED$), and 3 right triangles, with one right angle on each of X, Y, and Z. Finding the trapezoid's area, we know that one base of each trapezoid is just the side length of the hexagon, which is 1, and the other base is 3/2 (It is halfway in between the side and the longest diagonal, which has length 2) with a height of $\frac{\sqrt{3}}{4}$ (by using the Pythagorean theorem and the fact that it is an isosceles trapezoid) to give each trapezoid having an area of $\frac{5\sqrt{3}}{16}$ for a total area of $\frac{15\sqrt{3}}{16}$. (Alternatively, we could have calculated the area of hexagon $ABCDEF$ and subtracted the area of $\triangle XYZ$, which, as we showed before, had a side length of 3/2). Now, we need to find the area of each of the small triangles, which, if we look at the triangle that has a vertex on X, is similar to the triangle with a base of $YC = 1/2.$ Using similar triangles we calculate the base to be 1/4 and the height to be $\frac{\sqrt{3}}{4}$ giving us an area of $\frac{\sqrt{3}}{32}$ per triangle, and a total area of $3\frac{\sqrt{3}}{32}$. Adding the two areas together, we get $\frac{15\sqrt{3}}{16} + \frac{3\sqrt{3}}{32} = \frac{33\sqrt{3}}{32}$. Finding the total area, we get $6 \cdot 1^2 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$. Taking the complement, we get $\frac{3\sqrt{3}}{2} - \frac{33\sqrt{3}}{32} = \frac{15\sqrt{3}}{32} = (C)\frac{15}{32}\sqrt{3}$

Solution 3 (Trig)

Notice, the area of the convex hexagon formed through the intersection of the 2 triangles can be found by finding the area of the triangle formed by the midpoints of the sides and subtracting the smaller triangles that are formed by the region inside this triangle but outside the other triangle. First, let's find the area of the area of the triangle formed by the midpoint of the sides. Notice, this is an equilateral triangle, thus all we need is to find the length of its side. To do this, we look at the isosceles trapezoid outside this triangle but inside the outer hexagon. Since the interior angle of a regular hexagon is 120^{o} and the trapezoid is isosceles, we know that the angle opposite is 60^{o}, and thus the side length of this triangle is $1 + 2(\frac{1}{2}\cos(60^{\text{o}}) = 1 + \frac{1}{2} = \frac{3}{2}$. So the area of this triangle is $\frac{\sqrt{3}}{4}s^2 = \frac{9\sqrt{3}}{16}$ Now let's find the area of the smaller triangles. Notice, triangle ACE cuts off smaller isosceles triangles from the outer hexagon. The base of these isosceles triangles is perpendicular to the base of the isosceles trapezoid mentioned before, thus we can use trigonometric ratios to find the base and height of these smaller triangles, which are all congruent due to the rotational symmetry of a regular hexagon. The area is then $\frac{1}{2}(\frac{1}{2})\cos(60^{\text{o}}))(\frac{1}{2}\sin(60^{\text{o}})) = \frac{\sqrt{3}}{32}$ and the sum of the areas is $3 \cdot \frac{\sqrt{3}}{32} = \frac{3\sqrt{3}}{32}$ Therefore, the area of the convex hexagon is

$$\frac{9\sqrt{3}}{16} - \frac{3\sqrt{3}}{32} = \frac{18\sqrt{3}}{32} - \frac{3\sqrt{3}}{32} = \boxed{\frac{15\sqrt{3}}{32}} \implies \boxed{C}$$

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Contents

Problem

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many real numbers x satisfy the equation $x^2 + 10,000\lfloor x \rfloor = 10,000x$?

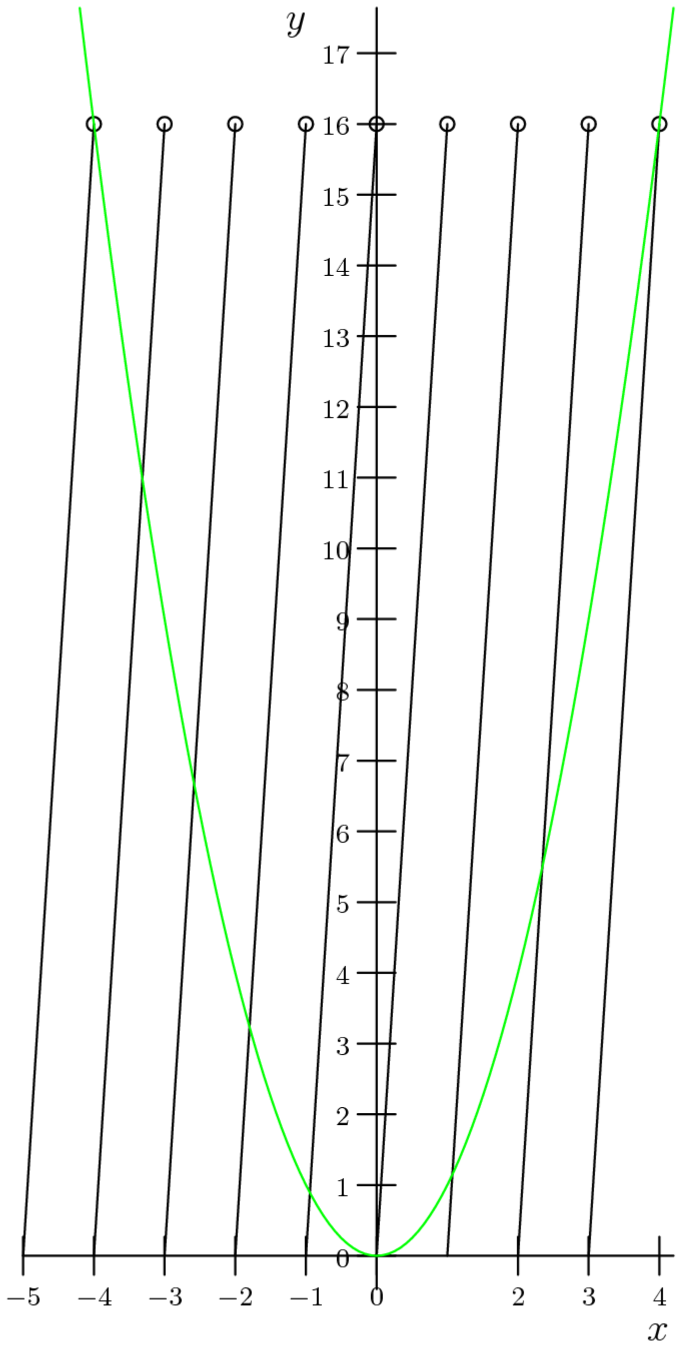
$\textbf{(A) } 197 \qquad \textbf{(B) } 198 \qquad \textbf{(C) } 199 \qquad \textbf{(D) } 200 \qquad \textbf{(E) } 201$

Solution 1

This rewrites itself to $x^2 = 10,000\{x\}$.

Graphing $y = 10,000\{x\}$ and $y = x^2$ we see that the former is a set of line segments with slope $10,000$ from 0 to 1 with a hole at $x = 1$, then 1 to 2 with a hole at $x = 2$ etc.

Here is a graph of $y = x^2$ and $y = 16\{x\}$ for visualization.



Now notice that when $x = \pm 100$ then graph has a hole at $(\pm 100, 10,000)$ which the equation $y = x^2$ passes through and then continues upwards. Thus our set of possible solutions is bounded by $(-100, 100)$. We can see that $y = x^2$ intersects each of the lines once and there are $99 - (-99) + 1 = 199$ lines for an answer of $\boxed{(\text{C})\ 199}$.

Solution 2

Same as the first solution, $x^2 = 10,000\{x\}$.

We can write x as $\lfloor x \rfloor + \{x\}$. Expanding everything, we get a quadratic in x in terms of $\lfloor x \rfloor$:

$$\{x\}^2 + (2\lfloor x \rfloor - 10,000)\{x\} + \lfloor x \rfloor^2 = 0$$

We use the quadratic formula to solve for {x}:

$$\{x\} = \frac{-2\lfloor x \rfloor + 10,000 \pm \sqrt{(-2\lfloor x \rfloor + 10,000^2 - 4\lfloor x \rfloor^2)}}{2}$$

Since $0 \le \{x\} < 1$, we get an inequality which we can then solve. After simplifying a lot, we get that $\lfloor x \rfloor^2 + 2\lfloor x \rfloor - 9999 < 0$.

Solving over the integers, $-101 < \lfloor x \rfloor < 99$, and since $\lfloor x \rfloor$ is an integer, there are $\boxed{\text{(C) } 199}$ solutions. Each value of $\lfloor x \rfloor$ should correspond to one value of x, so we are done.

Solution 3

Let $x = a + k$ where a is the integer part of x and k is the fractional part of x. We can then rewrite the problem below:

$$(a+k)^2 + 10000a = 10000(a+k)$$

From here, we get

$$(a+k)^2 + 10000a = 10000a + 10000k$$

Solving for $a + k = x$

$$(a+k)^2 = 10000k$$

$$x = a + k = \pm 100\sqrt{k}$$

Because $0 \le k < 1$, we know that $a + k$ cannot be less than or equal to -100 nor greater than or equal to 100. Therefore:

$$-99 \le x \le 99$$

There are 199 elements in this range, so the answer is $\boxed{\textbf{(C) } 199}$.

Solution 4

Notice the given equation is equivilent to $(\lfloor x \rfloor + \{x\})^2 = 10,000\{x\}$

Now we now that $\{x\} < 1$ so plugging in 1 for $\{x\}$ we can find the upper and lower bounds for the values.

$$(\lfloor x \rfloor + 1)^2 = 10,000(1)$$

$$(\lfloor x \rfloor + 1) = \pm 100$$

$$\lfloor x \rfloor = 99, -101$$

And just like Solution 2, we see that $-101 < \lfloor x \rfloor < 99$, and since $\lfloor x \rfloor$ is an integer, there are $\boxed{\text{(C) } 199}$ solutions. Each value of $\lfloor x \rfloor$ should correspond to one value of x, so we are done.

Solution 5

First, we can let $\{x\} = b$, $\lfloor x \rfloor = a$. We know that $a + b = x$ by definition. We can rearrange the equation to obtain

$$x^2 = 10^4(x - a).$$

By taking square root on both sides, we obtain $x = \pm 100\sqrt{b}$ (because $x - a = b$). We know since b is the fractional part of x, it must be that $0 \le b < 1$. Thus, x may take any value in the interval $-100 < x < 100$. Hence, we know that there are $\boxed{\text{(C) } 199}$ potential values for $\lfloor x \rfloor$ in that range and we are done.

~awesome1st

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