2019 AMC 10B Problems/Problem 1

(Redirected from 2019 AMC 12B Problems/Problem 1)

The following problem is from both the 2019 AMC 10B #1 and 2019 AMC 12B #1, so both problems redirect to this page.

Problem

Alicia had two containers. The first was $\frac{5}{6}$ full of water and the second was empty. She poured all the water from the first container into the second container, at which point the second container was $\frac{3}{4}$ full of water. What is the ratio of the volume of the first container to the volume of the second container?

$\textbf{(A) } \frac{5}{8} \qquad \textbf{(B) } \frac{4}{5} \qquad \textbf{(C) } \frac{7}{8} \qquad \textbf{(D) } \frac{9}{10} \qquad \textbf{(E) } \frac{11}{12}$

Solution 1

Let the first jar's volume be A and the second's be B. It is given that $\frac{3}{4}A = \frac{5}{6}B$. We find that $\frac{B}{A} = \frac{\left(\frac{3}{4}\right)}{\left(\frac{5}{6}\right)} = \boxed{\textbf{(D) } \frac{9}{10}}$.

We already know that this is the ratio of the smaller to the larger volume because it is less than 1.

Solution 2

We can set up a ratio to solve this problem. If x is the volume of the first container, and y is the volume of the second container, then: $\frac{5}{6}x = \frac{3}{4}y$

Cross-multiplying allows us to get $\frac{x}{y} = \frac{3}{4} \cdot \frac{6}{5} = \frac{18}{20} = \frac{9}{10}$. Thus the ratio of the volume of the first container to the second container is $\boxed{\textbf{(D) } \frac{9}{10}}$.

2019 AMC 10B Problems/Problem 2

(Redirected from 2019 AMC 12B Problems/Problem 2)

The following problem is from both the 2019 AMC 10B #2 and 2019 AMC 12B #2, so both problems redirect to this page.

Problem

Consider the statement, "If n is not prime, then $n-2$ is prime." Which of the following values of n is a counterexample to this statement?

$\textbf{(A) } 11 \qquad \textbf{(B) } 15 \qquad \textbf{(C) } 19 \qquad \textbf{(D) } 21 \qquad \textbf{(E) } 27$

Solution

Since a counterexample must be value of n which is not prime, n must be composite, so we eliminate $\text{A}$and C. Now we subtract 2 from the remaining answer choices, and we see that the only time $n-2$ is **not**prime is when $n = \boxed{\textbf{(E) } 27}$.

2019 AMC 12B Problems/Problem 3

Problem

Which of the following rigid transformations (isometries) maps the line segment $\overline{AB}$ onto the line segment $\overline{A'B'}$ so that the image of $A(-2, 1)$ is $A'(2, -1)$ and the image of $B(-1, 4)$ is $B'(1, -4)$?

$\textbf{(A)}$ reflection in the y-axis

$\textbf{(B)}$ counterclockwise rotation around the origin by 90°

$\textbf{(C)}$ translation by 3 units to the right and 5 units down

$\textbf{(D)}$ reflection in the x-axis

$\textbf{(E)}$ clockwise rotation about the origin by 180°

Solution

We can simply graph the points, or use coordinate geometry, to realize that both A' and B' are, respectively, obtained by rotating A and B by 180° about the origin. Hence the rotation by 180° about the origin maps the line segment $\overline{AB}$ to the line segment $\overline{A'B'}$, so the answer is $\boxed{(\text{E})}$.

2019 AMC 10B Problems/Problem 6

(Redirected from 2019 AMC 12B Problems/Problem 4)

> *The following problem is from both the 2019 AMC 10B #6 and 2019 AMC 12B #4, so both problems redirect to this page.*

Problem

There is a real n such that $(n+1)! + (n+2)! = n! \cdot 440$. What is the sum of the digits of n?

$\textbf{(A) } 3 \qquad \textbf{(B) } 8 \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 11 \qquad \textbf{(E) } 12$

Solution 1

$$\begin{aligned} &(n+1)n! + (n+2)(n+1)n! = 440 \cdot n! \\ \Rightarrow\ &n![n+1+(n+2)(n+1)] = 440 \cdot n! \\ \Rightarrow\ &n+1+n^2+3n+2 = 440 \\ \Rightarrow\ &n^2+4n-437 = 0 \end{aligned}$$

Solving by the quadratic formula,

$n = \frac{-4 \pm \sqrt{16 + 437 \cdot 4}}{2} = \frac{-4 \pm 42}{2} = \frac{38}{2} = 19$ (since clearly $n \geq 0$). The answer is therefore $1 + 9 = \boxed{\textbf{(C) } 10}$.

~IronicNinja

Solution 2

Dividing both sides by $n!$ gives $(n+1)+(n+2)(n+1) = 440 \Rightarrow n^2+4n-437 = 0 \Rightarrow (n-19)(n+23) = 0.$ Since n is non-negative, $n = 19$. The answer is $1 + 9 = \boxed{\textbf{(C) } 10}$.

Solution 3

Dividing both sides by $n!$ as before gives $(n + 1) + (n + 1)(n + 2) = 440$. Now factor out $(n + 1)$, giving $(n + 1)(n + 3) = 440$. By considering the prime factorization of 440, a bit of experimentation gives us $n + 1 = 20$ and $n + 3 = 22$, so $n = 19$, so the answer is $1 + 9 = \boxed{\textbf{(C) } 10}$.

2019 AMC 10B Problems/Problem 7

(Redirected from 2019 AMC 12B Problems/Problem 5)

The following problem is from both the 2019 AMC 10B #7 and 2019 AMC 12B #5, so both problems redirect to this page.

Problem

Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

$\textbf{(A) } 18 \qquad \textbf{(B) } 21 \qquad \textbf{(C) } 24 \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 28$

Solution 1

If he has enough money to buy 12 pieces of red candy, 14 pieces of green candy, and 15 pieces of blue candy, then the smallest amount of money he could have is $\text{lcm}(12,14,15)=420$ cents. Since a piece of purple candy costs 20 cents, the smallest possible value of n is $\frac{420}{20}=\boxed{\textbf{(B) } 21}$.

~IronicNinja

Solution 2

We simply need to find a value of $20n$ that is divisible by 12, 14, and 15. Observe that $20 \cdot 18$ is divisible by 12 and 15, but not 14. $20 \cdot 21$ is divisible by 12, 14, and 15, meaning that we have exact change (in this case, 420 cents) to buy each type of candy, so the minimum value of n is $\boxed{\textbf{(B) } 21}$.

2019 AMC 10B Problems/Problem 10

(Redirected from 2019 AMC 12B Problems/Problem 6)

The following problem is from both the 2019 AMC 10B #10 and 2019 AMC 12B #6, so both problems redirect to this page.

Problem

In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units?

$\textbf{(A) } 0 \qquad\textbf{(B) } 2 \qquad\textbf{(C) } 4 \qquad\textbf{(D) } 8 \qquad\textbf{(E) } \text{infinitely many}$

Solution 1

Notice that whatever point we pick for C, AB will be the base of the triangle. Without loss of generality, let points A and B be $(0,0)$ and $(0,10)$, since for any other combination of points, we can just rotate the plane to make them $(0,0)$ and $(0,10)$ under a new coordinate system. When we pick point C, we have to make sure that its y-coordinate is ± 20, because that's the only way the area of the triangle can be 100.

Now when the perimeter is minimized, by symmetry, we put C in the middle, at $(5,20)$. We can easily see that AC and BC will both be $\sqrt{20^2+5^2}=\sqrt{425}$. The perimeter of this minimal triangle is $2\sqrt{425}+10$, which is larger than 50. Since the minimum perimeter is greater than 50, there is no triangle that satisfies the condition, giving us $\boxed{\textbf{(A) }0}$.

~IronicNinja

Solution 2

Without loss of generality, let AB be a horizontal segment of length 10. Now realize that C has to lie on one of the lines parallel to AB and vertically 20 units away from it. But $10+20+20$ is already 50, and this doesn't form a triangle. Otherwise, without loss of generality, $AC<20$. Dropping altitude CD, we have a right triangle ACD with hypotenuse $AC<20$ and leg $CD=20$, which is clearly impossible, again giving the answer as $\boxed{\textbf{(A) }0}$.

2019 AMC 10B Problems/Problem 13

(Redirected from 2019 AMC 12B Problems/Problem 7)

The following problem is from both the 2019 AMC 10B #13 and 2019 AMC 12B #7, so both problems redirect to this page.

Problem

What is the sum of all real numbers x for which the median of the numbers $4,6,8,17,$ and x is equal to the mean of those five numbers?

$\textbf{(A) } -5 \qquad\textbf{(B) } 0 \qquad\textbf{(C) } 5 \qquad\textbf{(D) } \frac{15}{4} \qquad\textbf{(E) } \frac{35}{4}$

Solution

The mean is $\dfrac{4+6+8+17+x}{5}=\dfrac{35+x}{5}$.

There are three possibilities for the median: it is either 6, 8, or x.

Let's start with 6.

$\dfrac{35+x}{5}=6$ has solution $x=-5$, and the sequence is $-5, 4, 6, 8, 17$, which does have median 6, so this is a valid solution.

Now let the median be 8.

$\dfrac{35+x}{5}=8$ gives $x=5$, so the sequence is $4, 5, 6, 8, 17$, which has median 6, so this is not valid.

Finally we let the median be x.

$$\frac{35+x}{5}=x \implies 35+x=5x \implies x=\frac{35}{4}=8.75$$

, and the sequence is $4, 6, 8, 8.75, 17$, which has median 8. This case is therefore again not valid.

Hence the only possible value of x is $\boxed{\textbf{(A) } -5}$.

2019 AMC 12B Problems/Problem 8

Problem

Let $f(x)=x^2(1-x)^2$. What is the value of the sum

$$f\left(\frac{1}{2019}\right)-f\left(\frac{2}{2019}\right)+f\left(\frac{3}{2019}\right)-f\left(\frac{4}{2019}\right)+\cdots+f\left(\frac{2017}{2019}\right)-f\left(\frac{2018}{2019}\right)?$$

$\textbf{(A) }0\qquad\textbf{(B) }\frac{1}{2019^4}\qquad\textbf{(C) }\frac{2018^2}{2019^4}\qquad\textbf{(D) }\frac{2020^2}{2019^4}\qquad\textbf{(E) }1$

Solution

First, note that $f(x) = f(1-x)$. We can see this since

$$f(x) = x^2(1-x)^2 = (1-x)^2x^2 = (1-x)^2\left(1-(1-x)\right)^2 = f(1-x)$$

Using this result, we regroup the terms accordingly:

$$\left(f\left(\frac{1}{2019}\right)-f\left(\frac{2018}{2019}\right)\right)+\left(f\left(\frac{2}{2019}\right)-f\left(\frac{2017}{2019}\right)\right)+\cdots+\left(f\left(\frac{1009}{2019}\right)-f\left(\frac{1010}{2019}\right)\right)$$
$$=\left(f\left(\frac{1}{2019}\right)-f\left(\frac{1}{2019}\right)\right)+\left(f\left(\frac{2}{2019}\right)-f\left(\frac{2}{2019}\right)\right)+\cdots+\left(f\left(\frac{1009}{2019}\right)-f\left(\frac{1009}{2019}\right)\right)$$

Now it is clear that all the terms will cancel out (the series telescopes), so the answer is $\boxed{\textbf{(A) } 0}$.

2019 AMC 12B Problems/Problem 9

Problem

For how many integral values of x can a triangle of positive area be formed having side lengths $\log_2 x, \log_4 x, 3$?

$\textbf{(A) } 57 \qquad \textbf{(B) } 59 \qquad \textbf{(C) } 61 \qquad \textbf{(D) } 62 \qquad \textbf{(E) } 63$

Solution

For these lengths to form a triangle of positive area, the Triangle Inequality tells us that we need $\log_2 x + \log_4 x > 3$, $\log_2 x + 3 > \log_4 x$, and $\log_4 x + 3 > \log_2 x$. The second inequality is redundant, as it's always less restrictive than the last inequality.

Let's raise the first inequality to the power of 4. This gives

$$4^{\log_2 x} \cdot 4^{\log_4 x} > 64 \Rightarrow \left(2^2\right)^{\log_2 x} \cdot x > 64 \Rightarrow x^2 \cdot x > 64$$

. Thus, $x > 4$.

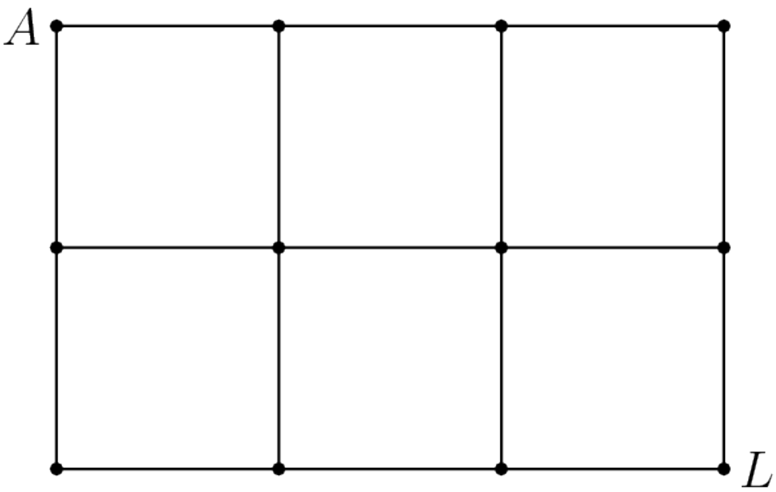
Doing the same for the second inequality gives $4^{\log_4 x} \cdot 64 > 4^{\log_2 x} \Rightarrow 64x > x^2 \Rightarrow x < 64$ (where we are allowed to divide both sides by x since x must be positive in order for the logarithms given in the problem statement to even have real values).

Combining our results, x is an integer strictly between 4 and 64, so the number of possible values of x is $64-4-1=\boxed{\textbf{(B) } 59}$.

2019 AMC 12B Problems/Problem 10

Problem

The figure below is a map showing 12 cities and 17 roads connecting certain pairs of cities. Paula wishes to travel along exactly 13 of those roads, starting at city A and ending at city L, without traveling along any portion of a road more than once. (Paula is allowed to visit a city more than



once.)

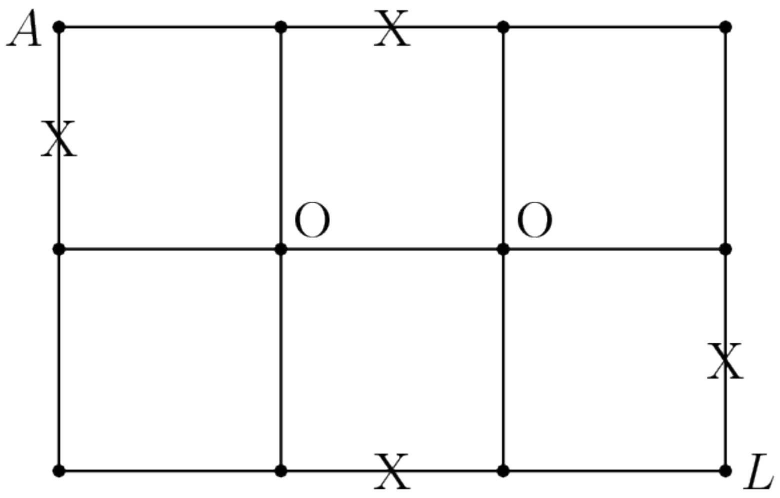
How many different routes can Paula take?

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 3 \qquad \textbf{(E) } 4$

Solution 1 (graph theory)

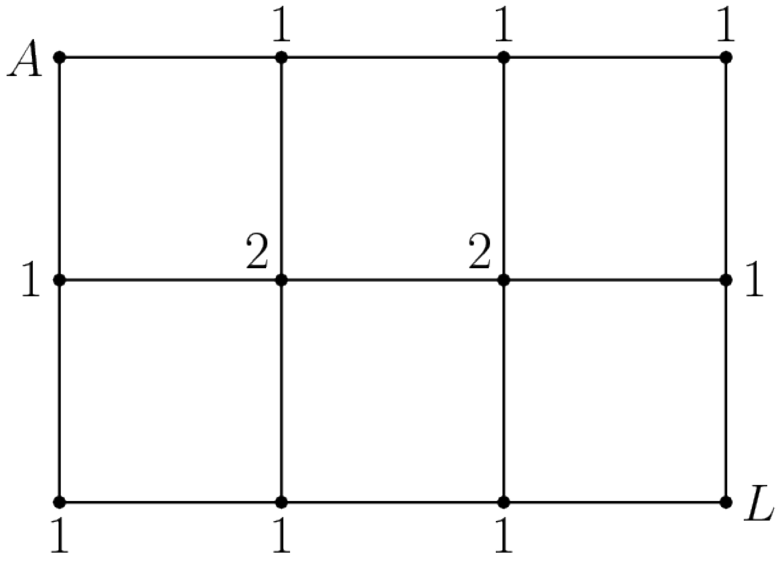
Note that of the 12 cities, 6 of them (2 on the top, 2 on the bottom, and 1 on each side) have 3 edges coming into/out of them (i.e., in graph theory terms, they have degree 3). Therefore, at least 1 edge connecting to each of these cities cannot be used. Additionally, the same applies to the start and end points, since we don't want to return to them. Thus there are $6+2=8$ vertices that we know have 1unused edge, and we have $17-13=4$ unused edges to work with (since there are 17 edges in total, and we must use exactly 13 of them). It is not hard to find that there is only one configuration satisfying these conditions:

Note: Xs represent unused edges.



Observe that at each of the 2 cities marked with an O on a path, there are 2 possibilities: we can either continue straight and cross back over the path later, or we can make a left turn, then turn right when we approach the junction again. This gives us a total of $2 \cdot 2 = \boxed{\textbf{(E) } 4}$ valid paths.

Solution 2 (longer graph theory)

Let the bottom-left vertex be $(0, 0)$, and let each of the edges have length 1, so that all of the vertices are at lattice points. Firstly, notice that for any vertex V on the graph (other than A or L), we can visit it at most $M(V) = \left\lfloor \frac{\deg(V)}{2} \right\rfloor$ times, where $\deg(V)$ is, as usual, the degree of V (i.e. the number of edges coming into/out of V). This is because to visit any of these vertices, we would have to enter and exit it by two different edges, in order to avoid using any portion of a road more than once. (Those who know some graph theory will recognise this a well-known principle.) We will label each vertex with this number.  Additionally, notice that if we visit n vertices (not necessarily distinct, i.e. counting a vertex which is visited twice as two vertices) along our path, we must traverse $n-1$ edges (this is quite easy to prove). Thus, if we want to traverse 13 edges in total, we have to visit 14 vertices. We must visit A and L, leaving 12vertices from the rest of the graph to visit.

If we sum the maximum numbers of visits to each vertex, we find that this is exactly equal to the 12 found above. This means that we have to visit each vertex $M(V)$ times, and must traverse $2 \cdot M(V)$ edges connected to each vertex. Specifically, we must traverse all 4 of the edges connected to the two central vertices at $(1,1)$ and $(1,2)$, as well as both edges connected to the 2 corner vertices (excluding A and L), and any 2 edges connected to the other vertices along the outside edge of the rectangle.

With this information, we can now proceed by dividing into cases.

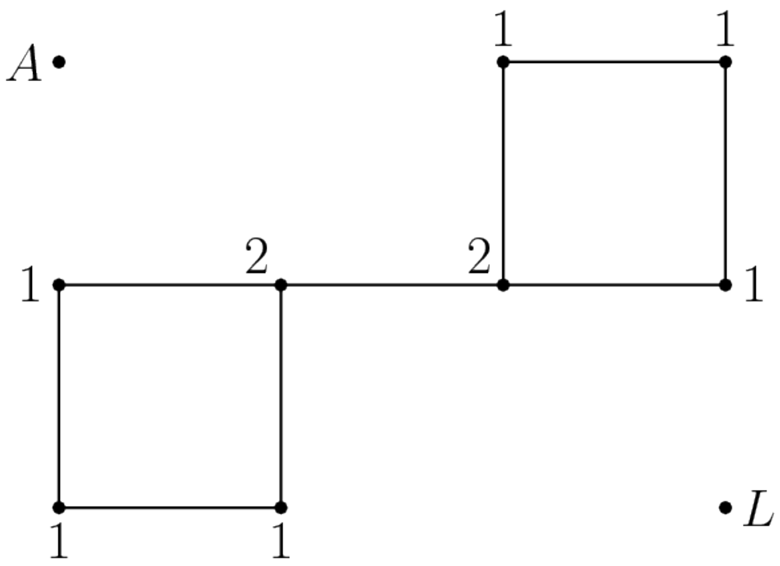
Case 1: We first move down from A.

In this case, we see that we must immediately move right to $(1,1)$ in order to traverse the edge from $(0,1)$to $(1,1)$, since we can never revisit $(0,1)$.

However, by the same logic, we must immediately move to $(0,0)$. This is a contradiction, so there are no possible paths in this case.

Case 2: We first move right from A.

Similar to the last case, we see that we must immediately move to $(1,1)$. By symmetry, we conclude that our last two moves must be $(2,1) \to (2,0) \to (3,0)$. With this information, we have reduced the problem to traveling from $(1,1)$ to $(2,1)$ with the same constraints as before. We redraw the graph, removing the edges we have already used and updating our labels (note that $(1,1)$ and $(2,1)$ are still labeled with 2 since we will pass through them twice, at the start and the end). Then, we remove the vertices with label 0 and the edges we know we can never traverse, giving:



Now, it is clear that we must complete a cycle in the lower left square, return to $(1,1)$, go to $(2,1)$, and complete a cycle in the top right square, returning to $(2,1)$. There are two ways to traverse each cycle (clockwise and anti-clockwise), giving us a total of $2 \cdot 2 = \boxed{\textbf{(E) } 4}$ paths of length 13 from A to L.

Solution 3 (Condensed version of Solution 1)

Ovserve that only the two central vertices can be visited twice. Since the path is of length 13, we need to repeat a vertex. Caseworking on each vertice, we can find there are two paths that go through each central vertex twice, for an answer of $\boxed{4}$.

2019 AMC 12B Problems/Problem 11

Problem

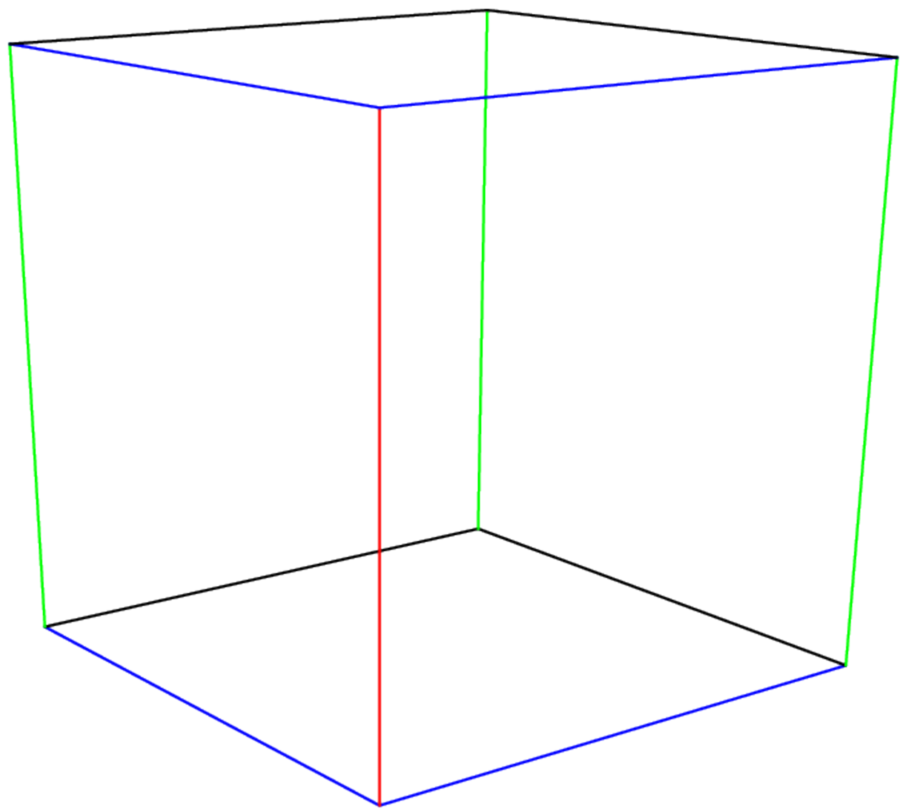
How many unordered pairs of edges of a given cube determine a plane?

$\textbf{(A) } 12 \qquad \textbf{(B) } 28 \qquad \textbf{(C) } 36 \qquad \textbf{(D) } 42 \qquad \textbf{(E) } 66$

Solution 1

Without loss of generality, choose one of the 12 edges of the cube to be among the two selected. We now calculate the probability that a randomly-selected second edge makes the pair satisfy the condition in the problem statement.

For two lines in space to determine a common plane, they must either intersect or be parallel (in other words, they cannot be skew lines). If all 12 line segments are extended to lines, the first (arbitrarily chosen) edge's line intersects 4 lines and is parallel to another 3. Thus $4+3=7$ of the $12-1=11$ remaining line segments (which could be chosen for the second edge) give a pair of lines determining a common plane. To see this, observe that in the diagram below, the red edge is parallel to the 3 green edges and intersects with the 4 blue edges.



This means that the probability that a randomly-selected pair of edges determine a plane is $\frac{7}{11}$, and we calculate that there are $\binom{12}{2}=66$ total pairs of edges that could be chosen (without the restriction). Thus the answer is $\frac{7}{11} \cdot 66 = \boxed{\textbf{(D) } 42}$.

Solution 2

As in Solution 1, we observe that the two edges must either be parallel or intersect. Clearly the edges will intersect if and only if they are part of the same face. We can thus divide into two cases:

Case 1: The two edges are part of the same face. There are 6 faces, and $\binom{4}{2} = 6$ ways to choose 2 of the 4 edges of the square, giving a total of $6 \cdot 6 = 36$ possibilities.

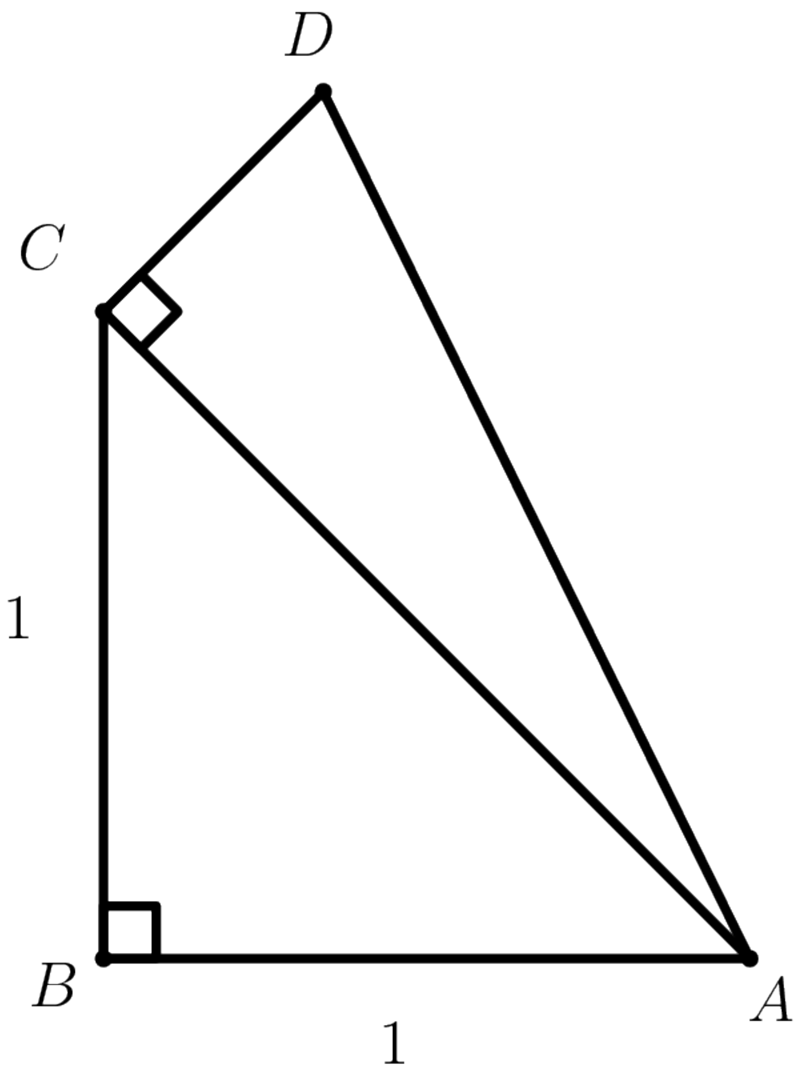
Case 2: The two edges are parallel and not part of the same face. Observe that each of the 12 edges is parallel to exactly 1 edge that is not part of its face. The edges can thus be paired up, giving $\frac{12}{2} = 6$ possibilities for this case.

Adding the two cases, the answer is hence $36 + 6 = \boxed{\textbf{(D) } 42}$.

2019 AMC 12B Problems/Problem 12

Problem

Right triangle ACD with right angle at C is constructed outwards on the hypotenuse $\overline{AC}$ of isosceles right triangle ABC with leg length 1, as shown, so that the two triangles have equal perimeters. What is $\sin(2\angle BAD)$?



$\textbf{(A)}\ \frac{1}{3} \qquad \textbf{(B)}\ \frac{\sqrt{2}}{2} \qquad \textbf{(C)}\ \frac{3}{4} \qquad \textbf{(D)}\ \frac{7}{9} \qquad \textbf{(E)}\ \frac{\sqrt{3}}{2}$

Solution 1

Firstly, note by the Pythagorean Theorem in $\triangle ABC$ that $AC = \sqrt{2}$. Now, the equal perimeter condition means that $BC + BA = 2 = CD + DA$, since side AC is common to both triangles and thus can be discounted. This relationship, in combination with the Pythagorean Theorem in $\triangle ACD$, gives $AC^2 + CD^2 = \left(\sqrt{2}\right)^2 + (2 - DA)^2 = DA^2$.

Hence $2+4-4DA+DA^2=DA^2$, so $DA=\frac{3}{2}$, and thus $CD=\frac{1}{2}$.

Next,

since $\angle BAC=45^\circ$, $\sin(\angle BAC)=\cos(\angle BAC)=\frac{1}{\sqrt{2}}$.

Using the lengths found above, $\sin(\angle CAD)=\frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{2}\right)}=\frac{1}{3}$,

and $\cos(\angle CAD)=\frac{\sqrt{2}}{\left(\frac{3}{2}\right)}=\frac{2\sqrt{2}}{3}$.

Thus, by the addition formulae for $\sin$ and $\cos$, we have

$$\begin{aligned}\sin(\angle BAD)&=\sin(\angle BAC+\angle CAD)\\&=\sin(\angle BAC)\cos(\angle CAD)+\cos(\angle BAC)\sin(\angle CAD)\\&=\frac{1}{\sqrt{2}}\cdot\frac{2\sqrt{2}}{3}+\frac{1}{\sqrt{2}}\cdot\frac{1}{3}\\&=\frac{2\sqrt{2}+1}{3\sqrt{2}}\end{aligned}$$

and

$$\begin{aligned}\cos(\angle BAD)&=\cos(\angle BAC+\angle CAD)\\&=\cos(\angle BAC)\cos(\angle CAD)-\sin(\angle BAC)\sin(\angle CAD)\\&=\frac{1}{\sqrt{2}}\cdot\frac{2\sqrt{2}}{3}-\frac{1}{\sqrt{2}}\cdot\frac{1}{3}\\&=\frac{2\sqrt{2}-1}{3\sqrt{2}}\end{aligned}$$

Hence, by the double angle formula for $\sin$,

$$\sin(2\angle BAD)=2\sin(\angle BAD)\cos(\angle BAD)=\frac{2(8-1)}{18}=\boxed{\textbf{(D) }\frac{7}{9}}$$

.

Solution 2 (coordinate geometry)

We use the Pythagorean Theorem, as in Solution 1, to find $AD = \frac{3}{2}$ and $CD = \frac{1}{2}$. Now notice that the angle between CD and the vertical (i.e. the y-axis) is 45° – to see this, drop a perpendicular from D to BA which meets BA at E, and use the fact that the angle sum of quadrilateral $CBED$ must be 360°. Anyway, this implies that the line CD has slope 1, so since C is the point $(0, 1)$ and the length of CD is $\frac{1}{2}$, D has coordinates $\left(0 + \frac{\left(\frac{1}{2}\right)}{\sqrt{2}}, 1 + \frac{\left(\frac{1}{2}\right)}{\sqrt{2}}\right) = \left(\frac{1}{2\sqrt{2}}, 1 + \frac{1}{2\sqrt{2}}\right)$.

Thus we have the lengths $DE = 1 + \frac{1}{2\sqrt{2}}$ (it is just the y-coordinate) and $AE = 1 - \frac{1}{2\sqrt{2}}$. By simple trigonometry in $\triangle DAE$, we now find

$$\sin\left(\angle BAD\right) = \frac{\left(1 + \frac{1}{2\sqrt{2}}\right)}{\left(\frac{3}{2}\right)} = \frac{\left(2 + \frac{1}{\sqrt{2}}\right)}{3} = \frac{2\sqrt{2} + 1}{3\sqrt{2}}$$

and

$$\cos\left(\angle BAD\right) = \frac{\left(1 - \frac{1}{2\sqrt{2}}\right)}{\left(\frac{3}{2}\right)} = \frac{\left(2 - \frac{1}{\sqrt{2}}\right)}{3} = \frac{2\sqrt{2} - 1}{3\sqrt{2}}$$

just as before. We can then use the double angle formula (as in Solution 1) to deduce $\sin\left(2\angle BAD\right) = \boxed{\textbf{(D) } \frac{7}{9}}$.

Solution 3 (easier finish to Solution 1)

Again, use Pythagorean Theorem to find that $AD = \frac{3}{2}$ and $CD = \frac{1}{2}$.

Let $\angle DAC = \theta$. Note that we want $\sin{(90+2\theta)} = \cos{2\theta}$ which is easy to compute:

$$\cos{\theta} = \frac{2\sqrt{2}}{3} \implies \cos{2\theta} = 2(\frac{8}{9}) - 1 = \boxed{\textbf{(D)} \ \frac{7}{9}}$$

2019 AMC 12B Problems/Problem 13

The following problem is from both the [2019 AMC 10B #17] and [2019 AMC 12B #13], so both problems redirect to this page.

Problem

A red ball and a green ball are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k = 1,2,3....$ What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

$$\textbf{(A) } \frac{1}{4} \qquad\textbf{(B) } \frac{2}{7} \qquad\textbf{(C) } \frac{1}{3} \qquad\textbf{(D) } \frac{3}{8} \qquad\textbf{(E) } \frac{3}{7}$$

Solution 1

By symmetry, the probability of the red ball landing in a higher-numbered bin is the same as the probability of the green ball landing in a higher-numbered bin. Clearly, the probability of both landing in the same bin is $\sum_{k=1}^{\infty} 2^{-k} \cdot 2^{-k} = \sum_{k=1}^{\infty} 2^{-2k} = \frac{1}{3}$ (by the geometric series sum formula). Therefore the other two probabilities have to both be $\frac{1-\frac{1}{3}}{2} = \boxed{\textbf{(C) } \frac{1}{3}}$.

Solution 2

Suppose the green ball goes in bin i, for some $i \geq 1$. The probability of this occurring is $\frac{1}{2^i}$. Given that this occurs, the probability that the red ball goes in a higher-numbered bin is $\frac{1}{2^{i+1}} + \frac{1}{2^{i+2}} + \ldots = \frac{1}{2^i}$ (by the geometric series sum formula). Thus the probability that the green ball goes in bin i, and the red ball goes in a bin greater than i, is $\left(\frac{1}{2^i}\right)^2 = \frac{1}{4^i}$.

Summing from $i = 1$ to infinity, we get

$\sum_{i=1}^{\infty} \frac{1}{4^i} = \boxed{\textbf{(C) } \frac{1}{3}}$ where we again used the geometric series sum formula. (Alternatively, if this sum equals n, then by writing out the terms and multiplying both sides by 4, we see $4n = n + 1$, which gives $n = \frac{1}{3}$.)

Solution 3

The probability that the two balls will go into adjacent bins is

$$\frac{1}{2 \times 4} + \frac{1}{4 \times 8} + \frac{1}{8 \times 16} + ... = \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots = \frac{1}{6}$$

by the geometric series sum formula. Similarly, the probability that the two balls will go into bins that have a distance of 2 from each other is

$$\frac{1}{2 \times 8} + \frac{1}{4 \times 16} + \frac{1}{8 \times 32} + \cdots = \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1}{12}$$

(again recognizing a geometric series). We can see that each time we add a bin between the two balls, the probability halves. Thus, our answer is $\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\cdots$, which, by the geometric series sum formula, is $\boxed{\textbf{(C) } \frac{1}{3}}$.

-fidgetboss_4000

Solution 4 (quick, conceptual)

Define a win as a ball appearing in higher numbered box.

Start from the first box.

There are 4 possible results in the box: Red, Green, Red and Green, or none, with an equal probability of $\frac{1}{4}$for each. If none of the balls is in the first box, the game restarts at the second box with the same kind of probability distribution, so if p is the probability that Red wins, we can write $p=\frac{1}{4}+\frac{1}{4}p$: there is a $\frac{1}{4}$probability that "Red" wins immediately, a 0 probability in the cases "Green" or "Red and Green", and in the "None" case (which occurs with $\frac{1}{4}$ probability), we then start again, giving the same probability p. Hence, solving the equation, we get $p=\boxed{\textbf{(C) } \frac{1}{3}}$.

2019 AMC 10B Problems/Problem 19

(Redirected from 2019 AMC 12B Problems/Problem 14)

The following problem is from both the 2019 AMC 10B #19 and 2019 AMC 12B #14, so both problems redirect to this page.

Problem

Let S be the set of all positive integer divisors of $100,000.$ How many numbers are the product of two distinct elements of $S?$

$\textbf{(A) } 98 \qquad \textbf{(B) } 100 \qquad \textbf{(C) } 117 \qquad \textbf{(D) } 119 \qquad \textbf{(E) } 121$

Solution

The prime factorization of $100,000$ is $2^5 \cdot 5^5$. Thus, we choose two numbers 2^a5^b and 2^c5^d where $0 \le a,b,c,d \le 5$ and $(a,b) \neq (c,d)$, whose product is $2^{a+c}5^{b+d}$, where $0 \le a+c \le 10$ and $0 \le b+d \le 10$.

Notice that this is analogous to choosing a divisor of $100,000^2 = 2^{10}5^{10}$, which has $(10+1)(10+1) = 121$ divisors. However, some of the divisors of $2^{10}5^{10}$ cannot be written as a product of two distinct divisors of $2^5 \cdot 5^5$, namely: $1 = 2^05^0$, $2^{10}5^{10}$, 2^{10}, and 5^{10}. The last two cannot be so written because the maximum factor of $100,000$ containing only 2s or 5s (and not both) is only 2^5 or 5^5. Since the factors chosen must be distinct, the last two numbers cannot be so written because they would require $2^5 \cdot 2^5$ or $5^5 \cdot 5^5$. This gives $121 - 4 = 117$ candidate numbers. It is not too hard to show that every number of the form 2^p5^q, where $0 \le p, q \le 10$, and p,q are not both 0 or 10, can be written as a product of two distinct elements in S. Hence the answer is $\boxed{\textbf{(C) } 117}$.

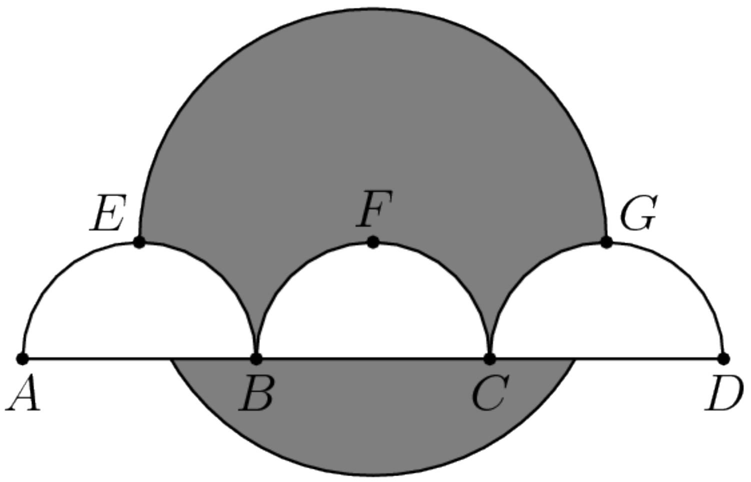
2019 AMC 10B Problems/Problem 20

(Redirected from 2019 AMC 12B Problems/Problem 15)

The following problem is from both the 2019 AMC 10B #20 and 2019 AMC 12B #15, so both problems redirect to this page.

Problem

As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, $\overset{\frown}{\mathrm{AEB}}$, $\overset{\frown}{\mathrm{BFC}}$, and $\overset{\frown}{\mathrm{CGD}}$, have their diameters on $\overline{AD}$, and are tangent to line EG at E, F, and G, respectively. A circle of radius 2 has its center on F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form $\frac{a}{b} \cdot \pi - \sqrt{c} + d$, where a, b, c, and d are positive integers and a and b are relatively prime. What is $a + b + c + d$?



$\textbf{(A) } 13 \qquad \textbf{(B) } 14 \qquad \textbf{(C) } 15 \qquad \textbf{(D) } 16 \qquad \textbf{(E) } 17$

Solution

Divide the circle into four parts: the top semicircle (A); the bottom sector (B), whose arc angle is 120° because the large circle's radius is 2 and the short length (the radius of the smaller semicircles) is 1, giving a $30^\circ - 60^\circ - 90^\circ$ triangle; the triangle formed by the radii of A and the chord (C), and the four parts which are the corners of a circle inscribed in a square (D). Then the area is $A + B - C + D$ (in $B - C$, we find the area of the shaded region above the semicircles but below the diameter, and in D we find the area of the bottom shaded region).

The area of A is $\frac{1}{2}\pi \cdot 2^2 = 2\pi$.

The area of B is $\frac{120^\circ}{360^\circ}\pi \cdot 2^2 = \frac{4\pi}{3}$.

For the area of C, the radius of 2, and the distance of 1 (the smaller semicircles' radius) to BC, creates two $30^\circ - 60^\circ - 90^\circ$ triangles, so C's area is $2 \cdot \frac{1}{2} \cdot 1 \cdot \sqrt{3} = \sqrt{3}$.

The area of D is $4 \cdot 1 - \frac{1}{4}\pi \cdot 2^2 = 4 - \pi$.

Hence, finding $A + B - C + D$, the desired area is $\frac{7\pi}{3} - \sqrt{3} + 4$, so the answer is $7 + 3 + 3 + 4 = \boxed{\textbf{(E) } 17}$

2019 AMC 12B Problems/Problem 16

Problem

There are lily pads in a row numbered 0 to 11, in that order. There are predators on lily pads 3 and 6, and a morsel of food on lily pad 10. Fiona the frog starts on pad 0, and from any given lily pad, has a $\frac{1}{2}$ chance to hop to the next pad, and an equal chance to jump 2 pads. What is the probability that Fiona reaches pad 10without landing on either pad 3 or pad 6?

$\textbf{(A) } \frac{15}{256} \qquad \textbf{(B) } \frac{1}{16} \qquad \textbf{(C) } \frac{15}{128} \qquad \textbf{(D) } \frac{1}{8} \qquad \textbf{(E) } \frac{1}{4}$

Solution 1

Firstly, notice that if Fiona jumps over the predator on pad 3, she must on pad 4. Similarly, she must land on 7 if she makes it past 6. Thus, we can split the problem into 3 smaller sub-problems, separately finding the probability Fiona skips 3, the probability she skips 6 (starting at 4) and the probability she *doesn't* skip 10 (starting at 7). Notice that by symmetry, the last of these three sub-problems is the complement of the first sub-problem, so the probability will be $1 - \text{the probability obtained in the first sub-problem}$.

In the analysis below, we call the larger jump a 2-jump, and the smaller a 1-jump.

For the first sub-problem, consider Fiona's options. She can either go 1-jump, 1-jump, 2-jump, with probability $\frac{1}{8}$, or she can go 2-jump, 2-jump, with probability $\frac{1}{4}$. These are the only two options, so they together make the answer $\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$. We now also know the answer to the last sub-problem is $1-\frac{3}{8}=\frac{5}{8}$.

For the second sub-problem, Fiona *must* go 1-jump, 2-jump, with probability $\frac{1}{4}$, since any other option would result in her death to a predator.

Thus, since the three sub-problems are independent, the final answer is $\frac{3}{8} \cdot \frac{1}{4} \cdot \frac{5}{8} = \boxed{\textbf{(A) } \frac{15}{256}}$.

Solution 2

Observe that since Fiona can only jump at most 2 places per move, and still wishes to avoid pads 3 and 6, she must also land on numbers 2, 4, 5, and 7.

There are two ways to reach lily pad 2, namely 1-jump, 1-jump, with probability $\frac{1}{4}$, or just a 2-jump, with probability $\frac{1}{2}$. The total is thus $\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$. Fiona must now make a 2-jump to lily pad 4, again with probability $\frac{1}{2}$, giving $\frac{3}{4}\cdot\frac{1}{2}=\frac{3}{8}$.

Similarly, Fiona must now make a 1-jump to reach lily pad 5, again with probability $\frac{1}{2}$, giving $\frac{3}{8}\cdot 12=\frac{3}{16}$. Then she must make a 2-jump to reach lily pad 7, with probability $\frac{1}{2}$, yielding $\frac{3}{16}\cdot\frac{1}{2}=\frac{3}{32}$.

Finally, to reach lily pad 10, Fiona has a few options - she can make 3 consecutive 1-jumps, with probability $\frac{1}{8}$, or 1-jump, 2-jump, with probability $\frac{1}{4}$, or 2-jump, 1-jump, again with probability $\frac{1}{4}$. The final answer is thus $\frac{3}{32}\cdot\left(\frac{1}{8}+\frac{1}{4}+\frac{1}{4}\right)=\frac{3}{32}\cdot\frac{5}{8}=\boxed{\textbf{(A) } \frac{15}{256}}$.

Solution 3 (recursion)

Let p_n be the probability of landing on lily pad n. Observe that if there are no restrictions, we would have $p_n=\frac{1}{2}\cdot p_{n-1}+\frac{1}{2}\cdot p_{n-2}$

This is because, given that Fiona is at lily pad $n-2$, there is a $\frac{1}{2}$ probability that she will make a 2-jump to reach lily pad n, and the same applies for a 1-jump to reach lily pad $n-1$. We will now compute the values of p_n recursively, but we will skip over 3 and 6. That is, we will not consider any jumps from lily pads 3 or $6$when considering the probabilities. We obtain the following chart, where an X represents an unused/uncomputed value:

1	$1/2$	$3/4$	X	$3/8$	$3/16$	X	$3/32$	$3/64$	$9/128$	$15/256$	X
0	1	2	3	4	5	6	7	8	9	10	11

Hence the answer is $\boxed{\textbf{(A) }\frac{15}{256}}$.

Note: If we let p_n be the probability of surviving if the frog is on lily pad n, using p_{10} = 1, we can solve backwards and obtain the following chart:

X	1	1/2	3/4	5/8	X	5/16	5/32	X	5/64	5/128	15/256
11	10	9	8	7	6	5	4	3	2	1	0

2019 AMC 12B Problems/Problem 17

Problem

How many nonzero complex numbers z have the property that $0, z,$ and $z^3,$ when represented by points in the complex plane, are the three distinct vertices of an equilateral triangle?

$\textbf{(A) }0\qquad\textbf{(B) }1\qquad\textbf{(C) }2\qquad\textbf{(D) }4\qquad\textbf{(E) }\text{infinitely many}$

Solution 1

Convert z and z^3 into modulus-argument (polar) form, giving $z = r\text{cis}(\theta)$ for some r and θ. Thus, by De Moivre's Theorem, $z^3 = r^3\text{cis}(3\theta)$. Since the distance from 0 to z is r, and the triangle is equilateral, the distance from 0 to z^3 must also be r, so $r^3 = r$, giving $r = 1$. (We know $r \neq 0$ since the problem statement specifies that z must be nonzero.)

Now, to get from z to z^3, which should be a rotation of 120° if the triangle is equilateral, we multiply by $z^2 = r^2\text{cis}(2\theta)$, again using De Moivre's Theorem. Thus we require $2\theta = \pm\frac{\pi}{3} + 2\pi k$ (where k can be any integer). If $0 < \theta < \frac{\pi}{2}$, we must have $\theta = \frac{\pi}{6}$, while if $\frac{\pi}{2} \leq \theta < \pi$, we must have $\theta = \frac{5\pi}{6}$. Hence there are 2 values that work for $0 < \theta < \pi$.

By symmetry, the interval $\pi \leq \theta < 2\pi$ will also give 2solutions. The answer is thus $2+2=\boxed{\textbf{(D) } 4}$.

Note: Here's a graph showing how z and z^3 move as θ increases: https://www.desmos.com/calculator/xtnpzoqkgs.

Solution 2

For the triangle to be equilateral, the vector from z to z^3, i.e $z^3 - z$, must be a 60° rotation of the vector from 0 to z, i.e. just z. Thus we must have

$$\frac{(z^3 - z)}{(z-0)} = \text{cis}(\pi/3) \text{ or } \text{cis}(5\pi/3)$$

Simplifying gives $z^2 - 1 = \text{cis}(\pi/3) \text{ or } z^2 - 1 = \text{cis}(5\pi/3)$

so$z^2 = 1 + \text{cis}(\pi/3) \text{ or } z^2 = 1 + \text{cis}(5\pi/3)$

Since any nonzero complex number will have two square roots, each equation gives two solutions. Thus, as before, the total number of possible values of z is $\boxed{\textbf{(D) } 4}$.

2019 AMC 12B Problems/Problem 18

Problem

Square pyramid $ABCDE$ has base $ABCD$, which measures 3 cm on a side, and altitude AEperpendicular to the base, which measures 6 cm. Point P lies on BE, one third of the way from B to E; point Q lies on DE, one third of the way from D to E; and point R lies on CE, two thirds of the way from C to E. What is the area, in square centimeters, of $\triangle PQR$?

$\textbf{(A)}\ \frac{3\sqrt{2}}{2}\qquad\textbf{(B)}\ \frac{3\sqrt{3}}{2}\qquad\textbf{(C)}\ 2\sqrt{2}\qquad\textbf{(D)}\ 2\sqrt{3}\qquad\textbf{(E)}\ 3\sqrt{2}$

Solution 1 (coordinate bash)

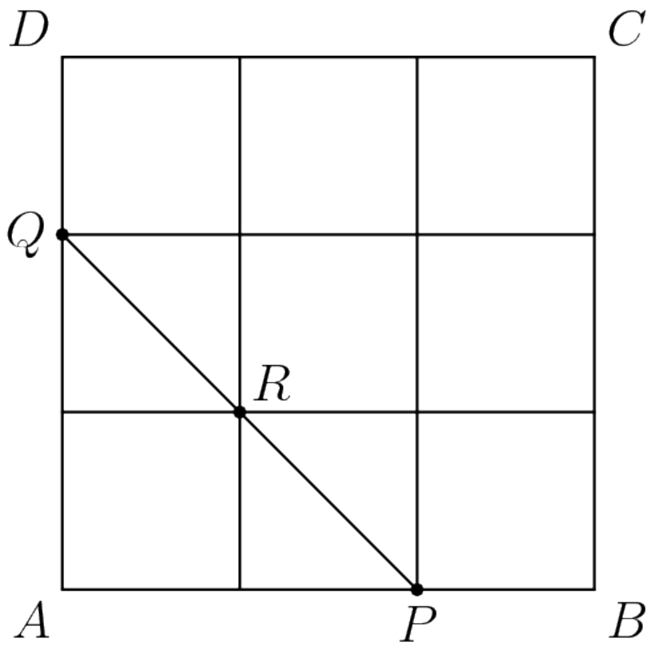
Using the given data, we can label the points $A(0,0,0), B(3,0,0), C(3,3,0), D(0,3,0)$, and $E(0,0,6)$. We can also find the points $P=B+\frac{1}{3}\overrightarrow{BE}=(3,0,0)+\frac{1}{3}(-3,0,6)=(3,0,0)+(-1,0,2)=(2,0,2)$. Similarly, $Q=(0,2,2)$ and $R=(1,1,4)$.

Using the distance formula, $PQ=\sqrt{(-2)^2+2^2+0^2}=2\sqrt{2}$, $PR=\sqrt{(-1)^2+1^2+2^2}=\sqrt{6}$, and $QR=\sqrt{1^2+(-1)^2+2^2}=\sqrt{6}$. Using Heron's formula, or by dropping an altitude from P to find the height, we can then find that the area of $\triangle PQR$ is $\boxed{\textbf{(C)}\ 2\sqrt{2}}$.

Note: After finding the coordinates of P, Q, and R, we can alternatively find the vectors $\overrightarrow{PQ}=[-2,2,0]$ and $\overrightarrow{PR}=[-1,1,2]$, then apply the formula $\text{area}=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|$. In this case, the cross product equals $[4,4,0]$, which has magnitude $4\sqrt{2}$, giving the area as $2\sqrt{2}$ like before.

Solution 2

As in Solution 1, let $A=(0,0,0), B=(3,0,0), C=(3,3,0), D=(0,3,0),$ and $E=(0,0,6)$, and calculate the coordinates of P, Q, and R as $P=(2,0,2), Q=(0,2,2), R=(1,1,4)$. Now notice that the plane determined by $\triangle PQR$ is perpendicular to the plane determined by $ABCD$. To see this, consider the *bird's-eye view*, looking down upon P, Q, and R projected onto $ABCD$:



Additionally, we know that PQ is parallel to the plane determined by $ABCD$, since P and Q have the same z-coordinate. Hence the height of $\triangle PQR$ is equal to the z-coordinate of R minus the z-coordinate of P, giving $4-2=2$. By the distance formula, $\overline{PQ}=2\sqrt{2}$, so the area of $\triangle PQR$ is $\frac{1}{2}\cdot 2\sqrt{2}\cdot 2=\boxed{\textbf{(C) }2\sqrt{2}}$.

Solution 3 (geometry)

By the Pythagorean Theorem, we can calculate $EB=ED=3\sqrt{5}, EC=3\sqrt{6}, ER=\sqrt{6},$ and

$EP = EQ = 2\sqrt{5}$. Now by the Law of Cosines in $\triangle EBC$, we have $\cos\left(\angle EBC\right) = \dfrac{EB^2 + EC^2 - BC^2}{2 \cdot EB \cdot EC} = \dfrac{\sqrt{30}}{6}$.

Similarly, by the Law of Cosines in $\triangle EPR$, we have

$PR^2 = ER^2 + EP^2 - 2 \cdot ER \cdot EP \cdot \cos\left(\angle EBC\right) = 6$, so $PR = \sqrt{6}$. Observe that $\triangle ERP \cong \triangle ERQ$ (by *side-angle-side*), so $QR = PR = \sqrt{6}$.

Next, notice that PQ is parallel to DB, and therefore $\triangle EQP$ is similiar to $\triangle EDB$. Thus we have $\dfrac{QP}{DB} = \dfrac{EP}{EB} = \dfrac{2}{3}$. Since $DB = 3\sqrt{2}$, this gives $PQ = 2\sqrt{2}$.

Now we have the three side lengths of isosceles $\triangle PQR$: $PR = QR = \sqrt{6}$, $PQ = 2\sqrt{2}$. Letting the midpoint of PQ be S, RS is the perpendicular bisector of PQ, and so can be used as a height of $\triangle PQR$ (taking PQ as the base). Using the Pythagorean Theorem again, we have $RS = \sqrt{PR^2 - PS^2} = 2$, so the area of $\triangle PQR$ is $\dfrac{1}{2} \cdot PQ \cdot RS = \dfrac{1}{2} \cdot 2\sqrt{2} \cdot 2 = \boxed{\textbf{(C) } 2\sqrt{2}}$.

2019 AMC 10B Problems/Problem 22

(Redirected from 2019 AMC 12B Problems/Problem 19)

The following problem is from both the 2019 AMC 10B #22 and 2019 AMC 12B #19, so both problems redirect to this page.

Problem

Raashan, Sylvia, and Ted play the following game. Each starts with $\$1$. A bell rings every 15 seconds, at which time each of the players who currently have money simultaneously chooses one of the other two players independently and at random and gives $\$1$ to that player. What is the probability that after the bell has rung 2019 times, each player will have $\$1$? (For example, Raashan and Ted may each decide to give $\$1$to Sylvia, and Sylvia may decide to give her her dollar to Ted, at which point Raashan will have $\$0$, Sylvia will have $\$2$, and Ted will have $\$1$, and that is the end of the first round of play. In the second round Rashaan has no money to give, but Sylvia and Ted might choose each other to give their $\$1$ to, and the holdings will be the same at the end of the second round.)

$\textbf{(A) } \frac{1}{7} \qquad\textbf{(B) } \frac{1}{4} \qquad\textbf{(C) } \frac{1}{3} \qquad\textbf{(D) } \frac{1}{2} \qquad\textbf{(E) } \frac{2}{3}$

Solution

On the first turn, each player starts off with $\$1$. Each turn after that, there are only two possibilities: either everyone stays at $\$1$, which we will write as $(1-1-1)$, or the distribution of money becomes $\$2-\$1-\$0$ in some order, which we write as $(2-1-0)$. We will consider these two states separately.

In the $(1-1-1)$ state, each person has two choices for whom to give their dollar to, meaning there are $2^3=8$ possible ways that the money can be rearranged. Note that there are only two ways that we can reach $(1-1-1)$ again:

1. Raashan gives his money to Sylvia, who gives her money to Ted, who gives his money to Raashan.

2. Raashan gives his money to Ted, who gives his money to Sylvia, who gives her money to Raashan.

Thus, the probability of staying in the $(1-1-1)$ state is $\frac{1}{4}$, while the probability of going to the $(2-1-0)$ state is $\frac{3}{4}$ (we can check that the 6 other possibilities lead to $(2-1-0)$)

In the $(2-1-0)$ state, we will label the person with $\$2$ as person A, the person with $\$1$ as person B, and the person with $\$0$ as person C. Person A has two options for whom to give money to, and person B has 2 options for whom to give money to, meaning there are total $2 \cdot 2 = 4$ ways the money can be redistributed. The only way that the distribution can return to $(1-1-1)$ is if A gives $\$1$ to B, and B gives $\$1$ to C. We check the other possibilities to find that they all lead back to $(2-1-0)$. Thus, the probability of going to the $(1-1-1)$ state is $\frac{1}{4}$, while the probability of staying in the $(2-1-0)$ state is $\frac{3}{4}$.

No matter which state we are in, the probability of going to the $(1-1-1)$ state is always $\frac{1}{4}$. This means that, after the bell rings 2018 times, regardless of what state the money distribution is in, there is a $\frac{1}{4}$ probability of going to the $(1-1-1)$ state after the 2019th bell ring.

Thus, our answer is simply $\boxed{\textbf{(B) } \frac{1}{4}}$.

2019 AMC 10B Problems/Problem 23

(Redirected from 2019 AMC 12B Problems/Problem 20)

The following problem is from both the 2019 AMC 10B #23 and 2019 AMC 12B #20, so both problems redirect to this page.

Problem

Points $A(6,13)$ and $B(12,11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and Bintersect at a point on the x-axis. What is the area of ω?

$\textbf{(A) }\frac{83\pi}{8}\qquad\textbf{(B) }\frac{21\pi}{2}\qquad\textbf{(C) }\frac{85\pi}{8}\qquad\textbf{(D) }\frac{43\pi}{4}\qquad\textbf{(E) }\frac{87\pi}{8}$

Solution 1

First, observe that the two tangent lines are of identical length. Therefore, supposing that the point of intersection is $(x,0)$, the Pythagorean Theorem gives $x=5$.

Further, notice (due to the right angles formed by a radius and its tangent line) that the quadrilateral (a kite) defined by the circle's center, A, B, and $(5,0)$ is cyclic. Therefore, we can apply Ptolemy's Theorem to give $2\sqrt{170}x=d\sqrt{40}$, where d is the distance between the circle's center and $(5,0)$. Therefore, $d=\sqrt{17}x$. Using the Pythagorean Theorem on the triangle formed by the point $(5,0)$, either one of A or B, and the circle's center, we find that $170+x^2=17x^2$, so $x^2=\frac{85}{8}$, and thus the answer is $\boxed{\textbf{(C) }\frac{85}{8}\pi}$.

Solution 2

We firstly obtain $x = 5$ as in Solution 1. Label the point $(5, 0)$ as C. The midpoint M of segment AB is $(9, 12)$. Notice that the center of the circle must lie on the line passing through the points C and M. Thus, the center of the circle lies on the line $y = 3x - 15$.

Line AC is $y = 13x - 65$. Therefore, the slope of the line perpendicular to AC is $-\frac{1}{13}$, so its equation is $y = -\frac{x}{13} + \frac{175}{13}$.

But notice that this line must pass through $A(6, 13)$ and $(x, 3x - 15)$.

Hence $3x - 15 = -\frac{x}{13} + \frac{175}{13} \Rightarrow x = \frac{37}{4}$. So the center of the circle is $\left(\frac{37}{4}, \frac{51}{4}\right)$.

Finally, the distance between the center, $\left(\frac{37}{4}, \frac{51}{4}\right)$, and point A is $\frac{\sqrt{170}}{4}$. Thus the area of the circle is $\boxed{\textbf{(C)}\ \frac{85}{8}\pi}$.

Solution 3

The midpoint of AB is $D(9, 12)$. Let the tangent lines at A and B intersect at $C(a, 0)$ on the x-axis. Then CD is the perpendicular bisector of AB. Let the center of the circle be O.

Then $\triangle AOC$ is similar to $\triangle DAC$, so $\frac{OA}{AC} = \frac{AD}{DC}$. The slope of AB is $\frac{13 - 11}{6 - 12} = \frac{-1}{3}$, so the slope of CD is 3. Hence, the equation of CD is $y-12=3(x-9) \Rightarrow y=3x-15$.

Letting $y=0$, we have $x=5$, so $C=(5,0)$.

Now, we

compute $AC=\sqrt{(6-5)^2+(13-0)^2}=\sqrt{170}$,

$AD=\sqrt{(6-9)^2+(13-12)^2}=\sqrt{10}$,

and $DC=\sqrt{(9-5)^2+(12-0)^2}=\sqrt{160}$.

Therefore $OA=\frac{AC\cdot AD}{DC}=\sqrt{\frac{85}{8}}$, and consequently, the area

of the circle is $\pi\cdot OA^2=\boxed{\textbf{(C) }\frac{85}{8}\pi}$.

2019 AMC 12B Problems/Problem 21

Problem

How many quadratic polynomials with real coefficients are there such that the set of roots equals the set of coefficients? (For clarification: If the polynomial is $ax^2+bx+c, a\neq 0,$ and the roots are r and s, then the requirement is that $\{a,b,c\}=\{r,s\}$.)

$\textbf{(A) } 3 \qquad\textbf{(B) } 4 \qquad\textbf{(C) } 5 \qquad\textbf{(D) } 6 \qquad\textbf{(E) } \text{infinitely many}$

Solution

Firstly, if $r = s$, then $a = b = c$, so the equation becomes $ax^2 + ax + a = 0 \Rightarrow x^2 + x + 1 = 0$, which has no real roots.

Hence there are three cases we need to consider:

Case 1: $a = b = r$ and $c = s \neq r$: The equation becomes $ax^2 + ax + c = 0$, and by Vieta's Formulas, we have $a + c = -1$ and $ac = \frac{c}{a}$. This second equation becomes $(a^2 - 1)c = 0$. Hence one possibility is $c = 0$, in which case $a = -1$, giving the equation $-x^2 - x = 0$, which has roots 0 and -1. This gives one valid solution. On the other hand, if $c \neq 0$, then $a^2 - 1 = 0$, so $a = \pm 1$. If $a = 1$, we have $c = -2$, and the equation is $x^2 + x - 2 = 0$, which clearly works, giving a second valid solution. If $a = -1$, then we have $c = 0$, which has already been considered, so this possibility gives no further valid solutions.

Case 2: $a = c = r$, $b = s \neq r$: The equation becomes $ax^2 + bx + a = 0$, so by Vieta's Formulas, we have $a + b = -\frac{b}{a}$ and $ab = 1$. These equations reduce to $a^3 + a + 1 = 0$. By sketching a graph of this function, we see that there is exactly one real root. (Alternatively, note that as it is of odd degree, there is at least one real root, and by differentiation, it has no stationary points, so there is at most one real root. Combining these gives exactly one real root.) This gives a third valid solution.

Case 3: $a = r$, $b = c = s \neq r$: The equation becomes $ax^2 + bx + b = 0$, so by Vieta's Formulas, we have $a+b=-\frac{b}{a}$ and $ab=\frac{b}{a}$. Observe that $b\neq 0$, as if it were 0, the equation would just have one real root, 0, so this would not give a valid solution. Thus, taking the second equation and dividing both sides by b, we deduce have $a=\frac{1}{a}$, so $a=\pm 1$. If $a=1$, we have $1+b=-b$, giving $b=-\frac{1}{2}$, so the equation is $x^2-\frac{1}{2}x-\frac{1}{2}=0$, which is a fourth valid solution. If $a=-1$, we have $1+b=b$, which is a contradiction, so this case gives no further valid solutions.

Hence the total number of valid solutions is $\boxed{\textbf{(B) }4}$.

2019 AMC 10B Problems/Problem 24

(Redirected from 2019 AMC 12B Problems/Problem 22)

The following problem is from both the 2019 AMC 10B #24 and 2019 AMC 12B #22, so both problems redirect to this page.

Problem

Define a sequence recursively by $x_0=5$ and $x_{n+1}=\frac{x_n^2+5x_n+4}{x_n+6}$ for all nonnegative integers n. Let m be the least positive integer such that $x_m\leq 4+\frac{1}{2^{20}}.$ In which of the following intervals does m lie?

$\textbf{(A) } [9,26] \qquad\textbf{(B) } [27,80] \qquad\textbf{(C) } [81,242]\qquad\textbf{(D) } [243,728] \qquad\textbf{(E) } [729,\infty)$

Solution 1

We first prove that $x_n>4$ for all $n\geq 0$, by induction. Observe that

$$x_{n+1} - 4 = \frac{x_n^2 + 5x_n + 4 - 4(x_n + 6)}{x_n + 6} = \frac{(x_n - 4)(x_n + 5)}{x_n + 6}$$

so (since x_n is clearly positive for all n, from the initial definition), $x_{n+1} > 4$ if and only if $x_n > 4$.

We similarly prove that x_n is decreasing, since

$$x_{n+1} - x_n = \frac{x_n^2 + 5x_n + 4 - x_n(x_n + 6)}{x_n + 6} = \frac{4 - x_n}{x_n + 6} < 0$$

Now we need to estimate the value of $x_{n+1} - 4$, which we can do using the rearranged equation $x_{n+1} - 4 = (x_n - 4) \cdot \frac{x_n + 5}{x_n + 6}$

Since x_n is decreasing, $\frac{x_n + 5}{x_n + 6}$ is clearly also decreasing, so we have $\frac{9}{10} < \frac{x_n + 5}{x_n + 6} \leq \frac{10}{11}$

and $\frac{9}{10}(x_n - 4) < x_{n+1} - 4 \leq \frac{10}{11}(x_n - 4)$

This becomes

$$\left(\frac{9}{10}\right)^n = \left(\frac{9}{10}\right)^n (x_0 - 4) < x_n - 4 \leq \left(\frac{10}{11}\right)^n (x_0 - 4) = \left(\frac{10}{11}\right)^n$$

The problem thus reduces to finding the least value of n such that

$$\left(\frac{9}{10}\right)^n < x_n - 4 \leq \frac{1}{2^{20}} \text{ and } \left(\frac{10}{11}\right)^{n-1} > x_{n-1} - 4 > \frac{1}{2^{20}}$$

Taking logarithms, we get $n \ln \frac{9}{10} < -20 \ln 2$ and $(n-1) \ln \frac{10}{11} > -20 \ln 2$, i.e.

$$n > \frac{20 \ln 2}{\ln \frac{10}{9}} \text{ and } n - 1 < \frac{20 \ln 2}{\ln \frac{11}{10}}$$

As approximations, we can use $\ln\frac{10}{9}\approx\frac{1}{9}$, $\ln\frac{11}{10}\approx\frac{1}{10}$, and $\ln 2\approx 0.7$. These allow us to estimate that $126<n<141$ which gives the answer as $\boxed{\textbf{(C) } [81,242]}$.

Solution 2

The condition where $x_m\leq 4+\frac{1}{2^{20}}$ gives the motivation to make a substitution to change the equilibrium from 4 to 0. We can substitute $x_n=y_n+4$ to achieve that.

Now, we need to find the smallest value of m such that $y_m\leq\frac{1}{2^{20}}$ given that $y_0=1$ and the recursion $y_{n+1}=\frac{y_n^2+9y_n}{y_n+10}$.

Using wishful thinking, we can simplify the recursion as follows:

$$y_{n+1}=\frac{y_n^2+9y_n+y_n-y_n}{y_n+10}$$

$$y_{n+1}=\frac{y_n(y_n+10)-y_n}{y_n+10}$$

$$y_{n+1}=y_n-\frac{y_n}{y_n+10}$$

$$y_{n+1}=y_n(1-\frac{1}{y_n+10})$$

The recursion looks like a geometric sequence with the ratio changing slightly after each term. Notice from the recursion that the y_n sequence is strictly decreasing, so all the terms after y_0 will be less than 1. Also, notice that all the terms in sequence will be positive. Both of these can be proven by induction.

With both of those observations in mind, $\frac{9}{10} < 1 - \frac{1}{y_n + 10} \leq \frac{10}{11}$. Combining this with the fact that the recursion resembles a geometric sequence, we conclude that $(\frac{9}{10})^n < y_n \leq (\frac{10}{11})^n$.

$\frac{9}{10}$ is approximately equal to $\frac{10}{11}$ and the ranges that the answer choices give us are generous, so we should use either $\frac{9}{10}$ or $\frac{10}{11}$ to find a rough estimate for m. $(\frac{9}{10})^3$ is 0.729, while $\frac{1}{\sqrt{2}}$ is close to 0.7 because $(0.7)^2$ is 0.49, which is close to $\frac{1}{2}$.

Therefore, we can estimate that $2^{\frac{-1}{2}} < y_3$.

Putting both sides to the 40th power, we get $2^{-20} < (y_3)^{40}$

But $y_3 = (y_0)^3$, so $2^{-20} < (y_0)^{120}$ and therefore, $2^{-20} < y_{120}$.

This tells us that m is somewhere around 120, so our answer is $\boxed{\textbf{(C) } [81,242]}$.

See Also

2019 AMC 10B Problems/Problem 25

(Redirected from 2019 AMC 12B Problems/Problem 23)

The following problem is from both the 2019 AMC 10B #25 and 2019 AMC 12B #23, so both problems redirect to this page.

Problem

How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

$\textbf{(A) }55\qquad\textbf{(B) }60\qquad\textbf{(C) }65\qquad\textbf{(D) }70\qquad\textbf{(E) }75$

Solution 1 (recursion)

We can deduce, from the given restrictions, that any valid sequence of length n will start with a 0 followed by either 10 or 110. Thus we can define a recursive function $f(n) = f(n-3) + f(n-2)$,

where $f(n)$is the number of valid sequences of length n.

This is because for any valid sequence of length n, you can append either 10 or 110 and the resulting sequence will still satisfy the given conditions.

It is easy to find $f(5) = 1$ and $f(6) = 2$ by hand, and then by the recursive formula, we have $f(19) = \boxed{\textbf{(C) }65}$.

Solution 2 (casework)

After any particular 0, the next 0 in the sequence must appear exactly 2 or 3 positions down the line. In this case, we start at position 1 and end at position 19, i.e. we move a total of 18 positions down the line. Therefore, we must add a series of 2s and 3s to get 18. There are a number of ways to do this:

Case 1: nine 2s - there is only 1 way to arrange them.

Case 2: two 3s and six 2s - there are $\binom{8}{2} = 28$ ways to arrange them.

Case 3: four 3s and three 2s - there are $\binom{7}{3} = 35$ ways to arrange them.

Case 4: six 3s - there is only 1 way to arrange them.

Summing the four cases gives $1+28+35+1=\boxed{\textbf{(C) } 65}$.

2019 AMC 12B Problems/Problem 24

Problem

Let $\omega=-\tfrac{1}{2}+\tfrac{1}{2}i\sqrt{3}.$ Let S denote all points in the complex plane of the form $a+b\omega+c\omega^2$, where $0\leq a \leq 1, 0\leq b\leq 1$, and $0\leq c\leq 1.$ What is the area of S?

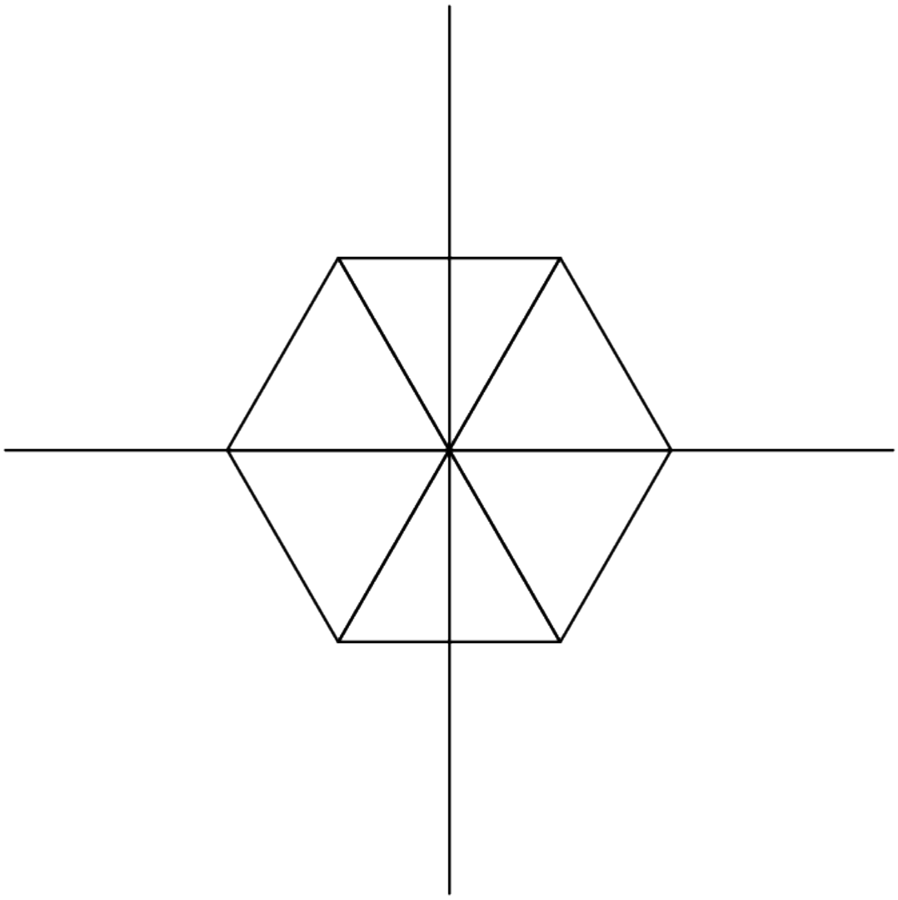
$\textbf{(A) } \frac{1}{2}\sqrt3 \qquad\textbf{(B) } \frac{3}{4}\sqrt3 \qquad\textbf{(C) } \frac{3}{2}\sqrt3 \qquad\textbf{(D) } \frac{1}{2}\pi\sqrt3 \qquad\textbf{(E) } \pi$

Solution 1

Notice that $\omega = e^{\frac{2i\pi}{3}}$, which is one of the cube roots of unity. We wish to find the span of $(a+b\omega+c\omega^2)$for reals $0 \le a, b, c \le 1$. Observe also that if $a, b, c > 0$, then replacing a, b, and c by $a - \min(a, b, c), b - \min(a, b, c)$, and $c - \min(a, b, c)$ leaves the value of $a + b\omega + c\omega^2$ unchanged.

Therefore, assume that at least one of a, b, c is equal to 0. If exactly one of them is 0, we can form an equilateral triangle of side length 1 using the remaining terms. A similar argument works if exactly two of them are 0. In total, we get $3+\binom{3}{2}=6$ equilateral triangles, whose total area is $6\cdot\frac{\sqrt{3}}{4}=\boxed{\textbf{(C) } \frac{3}{2}\sqrt{3}}$.

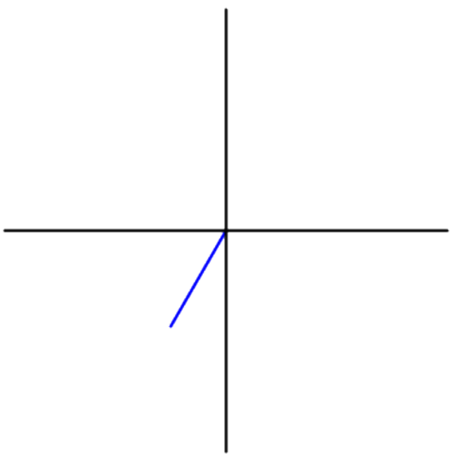
Note: A diagram of the six equilateral triangles is shown



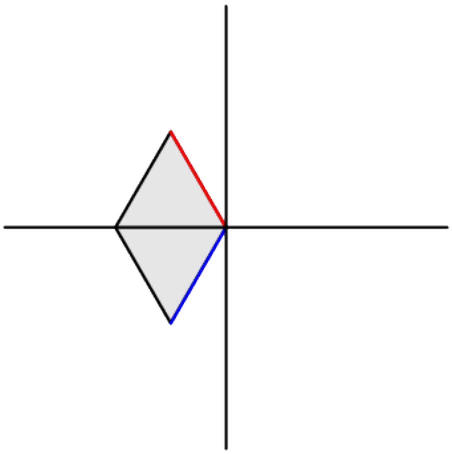
below.

Solution 2

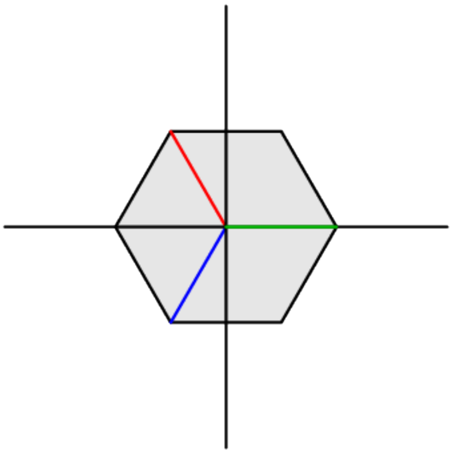
We can add on each term one at a time. Firstly, the possible values of $c\omega^2 = c\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$ lie on the following line:



For each point on the line, we can add $b\omega = b\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$. This means that we can extend the area to



by "moving" the blue line along the red line. Finally, we can add a to every point, giving



by "moving" the previous area along the green line. This leaves us with a regular hexagon with side length 1, so, as in Solution 1, the total area is $\boxed{\textbf{(C) } \frac{3}{2}\sqrt{3}}$.

2019 AMC 12B Problems/Problem 25

Problem

Let $ABCD$ be a convex quadrilateral with $BC = 2$ and $CD = 6.$ Suppose that the centroids of $\triangle ABC, \triangle BCD,$ and $\triangle ACD$ form the vertices of an equilateral triangle. What is the maximum possible value of $ABCD$?

$\textbf{(A)}\ 27 \qquad \textbf{(B)}\ 16\sqrt3 \qquad \textbf{(C)}\ 12+10\sqrt3 \qquad \textbf{(D)}\ 9+12\sqrt3 \qquad \textbf{(E)}\ 30$

Solution 1 (vectors)

Place an origin at A, and assign position vectors of $B = \vec{p}$ and $D = \vec{q}$.

Since AB is not parallel to AD, vectors $\vec{p}$ and $\vec{q}$ are linearly independent, so we can write $C = m\vec{p} + n\vec{q}$ for some constants m and n. Now, recall that the centroid of a triangle $\triangle XYZ$ has position vector $\frac{1}{3}\left(\vec{x} + \vec{y} + \vec{z}\right)$.

Thus the centroid of $\triangle ABC$ is $g_1 = \frac{1}{3}(m+1)\vec{p} + \frac{1}{3}n\vec{q}$; the centroid of $\triangle BCD$ is $g_2 = \frac{1}{3}(m+1)\vec{p} + \frac{1}{3}(n+1)\vec{q}$; and the centroid of $\triangle ACD$ is $g_3 = \frac{1}{3}m\vec{p} + \frac{1}{3}(n+1)\vec{q}$.

Hence $\overrightarrow{G_1G_2} = \frac{1}{3}\vec{q}$, $\overrightarrow{G_2G_3} = -\frac{1}{3}\vec{p}$, and $\overrightarrow{G_3G_1} = \frac{1}{3}\vec{p} - \frac{1}{3}\vec{q}$.

For $\triangle G_1G_2G_3$ to be equilateral, we need $\left|\overrightarrow{G_1G_2}\right| = \left|\overrightarrow{G_2G_3}\right| \Rightarrow |\vec{p}| = |\vec{q}| \Rightarrow AB = AD$.

Further, $\left|\overrightarrow{G_1G_2}\right| = \left|\overrightarrow{G_1G_3}\right| \Rightarrow |\vec{p}| = |\vec{p} - \vec{q}| = BD$. Hence we have $AB = AD = BD$, so $\triangle ABD$ is equilateral.

Now let the side length of $\triangle ABD$ be k, and let $\angle BCD = \theta$. By the Law of Cosines in $\triangle BCD$, we have $k^2 = 2^2 + 6^2 - 2 \cdot 2 \cdot 6 \cdot \cos\theta = 40 - 24\cos\theta$.

Since $\triangle ABD$ is equilateral, its area is $\frac{\sqrt{3}}{4}k^2 = 10\sqrt{3} - 6\sqrt{3}\cos\theta$, while the area of $\triangle BCD$ is $\frac{1}{2}\cdot 2\cdot 6\cdot \sin\theta = 6\sin\theta$. Thus the total area of $ABCD$ is

$$10\sqrt{3} + 6\left(\sin\theta - \sqrt{3}\cos\theta\right) = 10\sqrt{3} + 12\sin\left(\theta - 60^\circ\right)$$

, where in the last step we used the subtraction formula for $\sin$. Observe that $\sin\left(\theta - 60^\circ\right)$ has maximum value 1 when e.g. $\theta = 150^\circ$, which is a valid configuration, so the maximum area is $10\sqrt{3} + 12(1) = \boxed{\textbf{(C)}\ 12 + 10\sqrt{3}}$.

Solution 2

Let G_1, G_2, G_3 be the centroids of ABC, BCD, and CDA respectively, and let M be the midpoint of BC. A, G_1, and M are collinear due to well-known properties of the centroid. Likewise, D, G_2, and M are collinear as well. Because (as is also well-known) $AG_1 = 3AM$ and $DG_2 = 3DM$, we have $\triangle MG_1G_2 \sim \triangle MAD$. This implies that AD is parallel to G_1G_2, and in terms of lengths, $AD = 3G_1G_2$.

We can apply the same argument to the pair of triangles $\triangle BCD$ and $\triangle ACD$, concluding that AB is parallel to G_2G_3 and $AB = 3G_2G_3$. Because $3G_1G_2 = 3G_2G_3$ (due to the triangle being equilateral), $AB = AD$, and the pair of parallel lines preserve the 60° angle, meaning $\angle BAD = 60^\circ$. Therefore $\triangle BAD$ is equilateral.

At this point, we can finish as in Solution 1, or, to avoid using trigonometry, we can continue as follows:

Let $BD = 2x$, where $2 < x < 4$ due to the Triangle Inequality in $\triangle BCD$. By breaking the quadrilateral into $\triangle ABD$ and $\triangle BCD$, we can create an expression for the area of $ABCD$. We use the formula for the area of an equilateral triangle given its side length to find the area of $\triangle ABD$ and Heron's formula to find the area of $\triangle BCD$.

After simplifying,

$$[ABCD] = x^2\sqrt{3} + \sqrt{36 - (x^2 - 10)^2}$$

Substituting $k = x^2 - 10$, the expression becomes

$$[ABCD] = k\sqrt{3} + \sqrt{36 - k^2} + 10\sqrt{3}$$

We can ignore the $10\sqrt{3}$ for now and focus on $k\sqrt{3} + \sqrt{36 - k^2}$.

By the Cauchy-Schwarz Inequality,

$$\left(k\sqrt{3} + \sqrt{36 - k^2}\right)^2 \leq \left(\left(\sqrt{3}\right)^2 + 1^2\right)\left((k)^2 + \left(\sqrt{36 - k^2}\right)^2\right)$$

The RHS simplifies to 12^2, meaning the maximum value of $k\sqrt{3} + \sqrt{36 - k^2}$ is 12.

Thus the maximum possible area of $ABCD$ is $\boxed{\textbf{(C)}\ 12 + 10\sqrt{3}}$.