2010 AMC 12A Problems/Problem 1

Problem

What is $(20-(2010-201))+(2010-(201-20))$?

$\textbf{(A)}\ -4020 \qquad \textbf{(B)}\ 0 \qquad \textbf{(C)}\ 40 \qquad \textbf{(D)}\ 401 \qquad \textbf{(E)}\ 4020$

Solution

$$20-2010+201+2010-201+20 = 20+20 = \boxed{\textbf{(C)}\ 40}.$$

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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Category: Introductory Algebra Problems

2010 AMC 12A Problems/Problem 2

Problem

A ferry boat shuttles tourists to an island every hour starting at 10 AM until its last trip, which starts at 3 PM. One day the boat captain notes that on the 10 AM trip there were 100 tourists on the ferry boat, and that on each successive trip, the number of tourists was 1 fewer than on the previous trip. How many tourists did the ferry take to the island that day?

$\textbf{(A)}\ 585 \qquad \textbf{(B)}\ 594 \qquad \textbf{(C)}\ 672 \qquad \textbf{(D)}\ 679 \qquad \textbf{(E)}\ 694$

Solution

It is easy to see that the ferry boat takes 6 trips total. The total number of people taken to the island is

$$\begin{aligned}&100+99+98+97+96+95\\&=6(100)-(1+2+3+4+5)\\&=600-15\\&=\boxed{585\ \textbf{(A)}}\end{aligned}$$

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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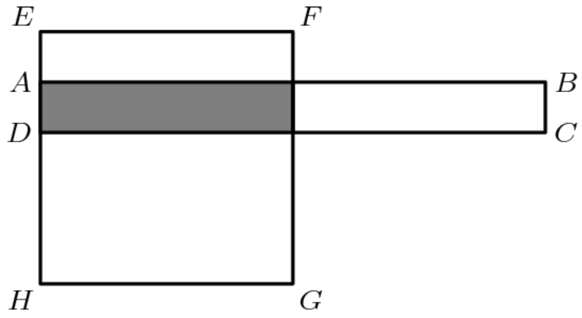
Category: Introductory Algebra Problems

2010 AMC 12A Problems/Problem 3

Problem

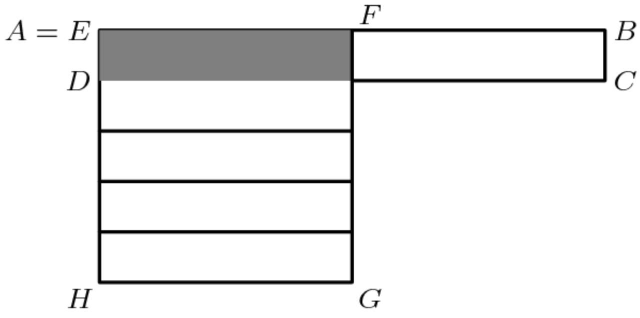
Rectangle $ABCD$, pictured below, shares 50% of its area with square $EFGH$. Square $EFGH$ shares 20% of its area with rectangle $ABCD$. What is $\frac{AB}{AD}$?



$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 10$

Solution

If we shift A to coincide with E, and add new horizontal lines to divide $EFGH$ into five equal parts:



This helps us to see that $AD=a/5$ and $AB=2a$, where $a=EF$. Hence $\frac{AB}{AD}=\frac{2a}{a/5}=10$.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12A Problems/Problem 4

Problem

If $x < 0$, then which of the following must be positive?

$\textbf{(A)}\ \frac{x}{|x|} \qquad \textbf{(B)}\ -x^2 \qquad \textbf{(C)}\ -2^x \qquad \textbf{(D)}\ -x^{-1} \qquad \textbf{(E)}\ \sqrt[3]{x}$

Solution

x is negative, so we can just place a negative value into each expression and find the one that is positive. Suppose we use -1.

$$\textbf{(A)} \Rightarrow \frac{-1}{|-1|} = -1$$

$$\textbf{(B)} \Rightarrow -(-1)^2 = -1$$

$$\textbf{(C)} \Rightarrow -2^{(-1)} = -\frac{1}{2}$$

$$\textbf{(D)} \Rightarrow -(-1)^{(-1)} = 1$$

$$\textbf{(E)} \Rightarrow \sqrt[3]{-1} = -1$$

Obviously only $\boxed{\textbf{(D)}}$ is positive.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2010 AMC 12A Problems/Problem 5

Problem

Halfway through a 100-shot archery tournament, Chelsea leads by 50 points. For each shot a bullseye scores 10 points, with other possible scores being 8, 4, 2, and 0 points. Chelsea always scores at least 4 points on each shot. If Chelsea's next n shots are bullseyes she will be guaranteed victory. What is the minimum value for n?

$\textbf{(A)}\ 38 \qquad \textbf{(B)}\ 40 \qquad \textbf{(C)}\ 42 \qquad \textbf{(D)}\ 44 \qquad \textbf{(E)}\ 46$

Solution

Let k be the number of points Chelsea currently has. In order to guarantee victory, we must consider the possibility that the opponent scores the maximum amount of points by getting only bullseyes.

$$k + 10n + 4(50-n) > (k-50) + 50\cdot 10$$
$$6n > 250$$

The lowest integer value that satisfies the inequality is $\boxed{42\ \textbf{(C)}}$.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2010 AMC 12A Problems/Problem 6

Problem

A palindrome, such as 83438, is a number that remains the same when its digits are reversed. The numbers x and $x+32$ are three-digit and four-digit palindromes, respectively. What is the sum of the digits of x ?

$\textbf{(A)}\ 20 \qquad \textbf{(B)}\ 21 \qquad \textbf{(C)}\ 22 \qquad \textbf{(D)}\ 23 \qquad \textbf{(E)}\ 24$

Solution

x is at most 999, so $x+32$ is at most 1031. The minimum value of $x+32$ is 1000. However, the only palindrome between 1000 and 1032 is 1001, which means that $x+32$ must be 1001.

It follows that x is 969, so the sum of the digits is $\boxed{\textbf{(E)}\ 24}$.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2010 AMC 12A Problems/Problem 7

Problem

Logan is constructing a scaled model of his town. The city's water tower stands 40 meters high, and the top portion is a sphere that holds 100,000 liters of water. Logan's miniature water tower holds 0.1 liters. How tall, in meters, should Logan make his tower?

$\textbf{(A)}\ 0.04 \qquad \textbf{(B)}\ \frac{0.4}{\pi} \qquad \textbf{(C)}\ 0.4 \qquad \textbf{(D)}\ \frac{4}{\pi} \qquad \textbf{(E)}\ 4$

Solution

The water tower holds $\frac{100000}{0.1} = 1000000$ times more water than Logan's miniature. Therefore, Logan should make his tower $\sqrt[3]{1000000} = 100$ times shorter than the actual tower. This is $\frac{40}{100} = \boxed{0.4}$ meters high, or choice $\textbf{(C)}$.

Also, the fact that $1\text{ L} = 1\text{ cm}^3$ doesn't matter since only the ratios are important.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2010))	
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All AMC 10 Problems and Solutions	

Categories: Introductory Geometry Problems | 3D Geometry Problems

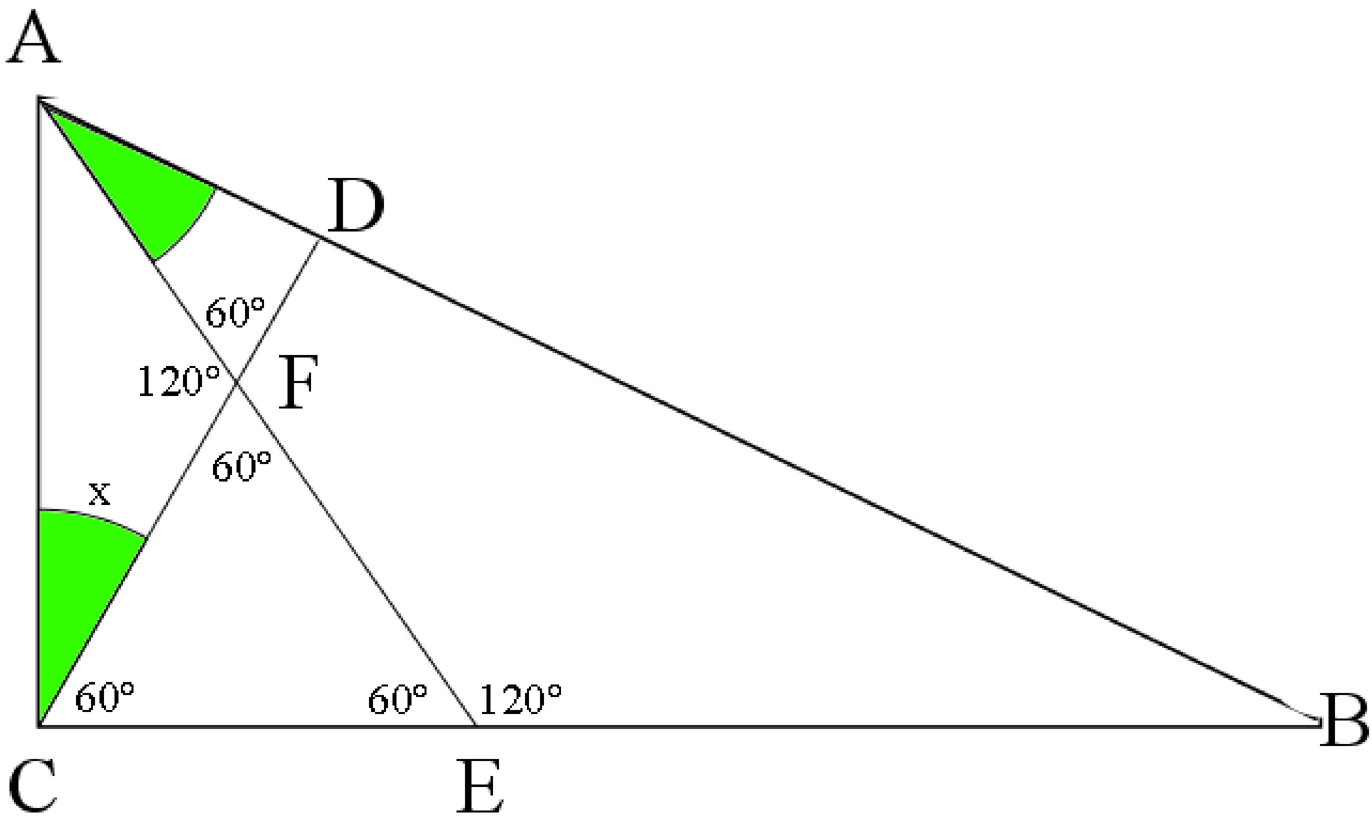
2010 AMC 12A Problems/Problem 8

Problem

Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?

$\textbf{(A)}\ 60^\circ \qquad \textbf{(B)}\ 75^\circ \qquad \textbf{(C)}\ 90^\circ \qquad \textbf{(D)}\ 105^\circ \qquad \textbf{(E)}\ 120^\circ$

Solution



Let $\angle BAE = \angle ACD = x$.

$$\begin{aligned}
\angle BCD &= \angle AEC = 60^\circ \\
\angle EAC + \angle FCA + \angle ECF + \angle AEC &= \angle EAC + x + 60^\circ + 60^\circ = 180^\circ \\
\angle EAC &= 60^\circ - x \\
\angle BAC &= \angle EAC + \angle BAE = 60^\circ - x + x = 60^\circ
\end{aligned}$$

Since $\frac{AC}{AB} = \frac{1}{2}$, triangle ABC is a $30-60-90$ triangle, so $\angle BCA = \boxed{90^\circ\ \textbf{(C)}}$.

See also

2010 AMC 12A Problems/Problem 9

Contents

- 1 Problem
- 2 Solution
  - 2.1 Solution 1
  - 2.2 Solution 2
  - 2.3 Solution 3
  - 2.4 Solution 4
- 3 See also

Problem

A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume, in cubic inches, of the remaining solid?

$\textbf{(A)}\ 7 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 10 \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 15$

Solution

Solution 1

Imagine making the cuts one at a time. The first cut removes a box $2\times 2\times 3$. The second cut removes two boxes, each of dimensions $2\times 2\times 0.5$, and the third cut does the same as the second cut, on the last two faces. Hence the total volume of all cuts is $12 + 4 + 4 = 20$.

Therefore the volume of the rest of the cube is $3^3 - 20 = 27 - 20 = \boxed{7\ \textbf{(A)}}$.

Solution 2

We can use Principle of Inclusion-Exclusion (PIE) to find the final volume of the cube.

There are 3 "cuts" through the cube that go from one end to the other. Each of these "cuts" has $2 \times 2 \times 3 = 12$ cubic inches. However, we can not just sum their volumes, as the central $2\times 2\times 2$ cube is included in each of these three cuts. To get the correct result, we can take the sum of the volumes of the three cuts, and subtract the volume of the central cube twice.

Hence the total volume of the cuts is $3(2 \times 2 \times 3) - 2(2\times 2\times 2) = 36 - 16 = 20$.

Therefore the volume of the rest of the cube is $3^3 - 20 = 27 - 20 = \boxed{7\ \textbf{(A)}}$.

Solution 3

We can visualize the final figure and see a cubic frame. We can find the volume of the figure by adding up the volumes of the edges and corners.

Each edge can be seen as a $2\times 0.5\times 0.5$ box, and each corner can be seen as a $0.5\times 0.5\times 0.5$ box.

$$12 \cdot \frac{1}{2} + 8 \cdot \frac{1}{8} = 6 + 1 = \boxed{7\ \textbf{(A)}}.$$

Solution 4

First, you can find the volume, which is 27. Now, imagine there are three prisms of dimensions $2 \times 2 \times 3$. Now subtract the prism volumes from 27. We have -9. From here we add two times 2^3, because we over-removed. This is $16-9=\boxed{7\,\textbf{(A)}}$.

Note: Isn't this the same thing as solution 2? thanks.-rulai

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2010 AMC 12A Problems/Problem 10

Problem

The first four terms of an arithmetic sequence are p, 9, $3p-q$, and $3p+q$. What is the 2010^{th} term of this sequence?

$\textbf{(A)}\ 8041 \qquad \textbf{(B)}\ 8043 \qquad \textbf{(C)}\ 8045 \qquad \textbf{(D)}\ 8047 \qquad \textbf{(E)}\ 8049$

Solution

$3p-q$ and $3p+q$ are consecutive terms, so the common difference is $(3p+q)-(3p-q) = 2q$.

$$\begin{aligned} p+2q &= 9 \\ 9+2q &= 3p-q \\ q &= 2 \\ p &= 5 \end{aligned}$$

The common difference is 4. The first term is 5 and the 2010^{th} term is

$$5+4(2009) = \boxed{8041\ \textbf{(A)}}$$

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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Category: Introductory Algebra Problems

2010 AMC 12A Problems/Problem 11

Problem

The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?

$\textbf{(A)}\ \frac{7}{15} \qquad \textbf{(B)}\ \frac{7}{8} \qquad \textbf{(C)}\ \frac{8}{7} \qquad \textbf{(D)}\ \frac{15}{8} \qquad \textbf{(E)}\ \frac{15}{7}$

Solution

This problem is quickly solved with knowledge of the laws of exponents and logarithms.

$$\begin{align*} 7^{x+7} &= 8^x \\ 7^x * 7^7 &= 8^x \\ \left(\frac{8}{7}\right)^x &= 7^7 \\ x &= \log_{8/7} 7^7 \end{align*}$$

Since we are looking for the base of the logarithm, our answer is $\boxed{\textbf{(C)}\ \frac{8}{7}}$.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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Category: Introductory Number Theory Problems

2010 AMC 12A Problems/Problem 12

Problem

In a magical swamp there are two species of talking amphibians: toads, whose statements are always true, and frogs, whose statements are always false. Four amphibians, Brian, Chris, LeRoy, and Mike live together in this swamp, and they make the following statements.

Brian: "Mike and I are different species."

Chris: "LeRoy is a frog."

LeRoy: "Chris is a frog."

Mike: "Of the four of us, at least two are toads."

How many of these amphibians are frogs?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 4$

Solution

Start with Brian. If he is a toad, he tells the truth, hence Mike is a frog. If Brian is a frog, he lies, hence Mike is a frog, too. Thus Mike must be a frog.

As Mike is a frog, his statement is false, hence there is at most one toad.

As there is at most one toad, at least one of Chris and LeRoy is a frog. But then the other one tells the truth, and therefore is a toad.

Hence we must have one toad and $\boxed{\textbf{(D)}\ 3}$ frogs.

See also

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Category: Introductory Algebra Problems

2010 AMC 12A Problems/Problem 13

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- 2 Solution 1
- 3 Solution 2
- 4 See also

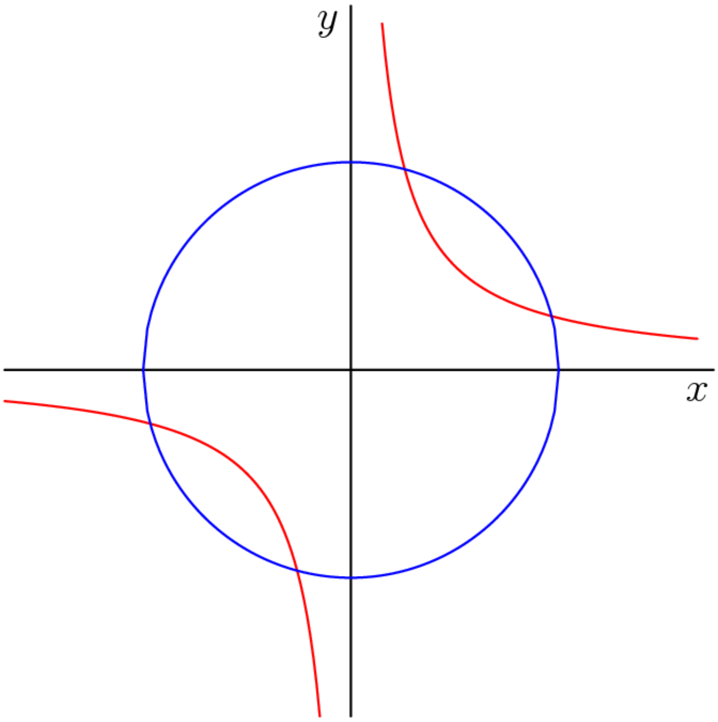
Problem

For how many integer values of k do the graphs of $x^2+y^2=k^2$ and $xy=k$ not intersect?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 8$

Solution 1

The image below shows the two curves for $k=4$. The blue curve is $x^2+y^2=k^2$, which is clearly a circle with radius k, and the red curve is a part of the curve $xy=k$.



In the special case $k=0$ the blue curve is just the point $(0,0)$, and as $0\cdot 0=0$, this point is on the red curve as well, hence they intersect.

The case $k<0$ is symmetric to $k>0$: the blue curve remains the same and the red curve is flipped according to the x axis. Hence we just need to focus on $k>0$.

Clearly, on the red curve there will always be points arbitrarily far from the origin: for example, as x approaches 0, y approaches ∞. Hence the red curve intersects the blue one if and only if it contains a point whose distance from the origin is at most k.

At this point we can guess that on the red curve the point where $x=y$ is always closest to the origin, and skip the rest of this solution.

For an exact solution, fix k and consider any point (x,y) on the red curve. Its distance from the origin is $\sqrt{x^2+(k/x)^2}$. To minimize this distance, it is enough to minimize $x^2+(k/x)^2$. By the Arithmetic Mean-Geometric Mean Inequality we get that this value is at least $2k$, and that equality holds whenever $x^2=(k/x)^2$, i.e., $x=\pm\sqrt{k}$.

Now recall that the red curve intersects the blue one if and only if its closest point is at most k from the origin. We just computed that the distance between the origin and the closest point on the red curve is $\sqrt{2k}$. Therefore, we want to find all positive integers k such that $\sqrt{2k}>k$.

Clearly the only such integer is $k=1$, hence the two curves are only disjoint for $k=1$ and $k=-1$. This is a total of $\boxed{2\ \textbf{(C)}}$ values.

Solution 2

From the graph shown above, we see that there is a specific point closest to the center of the circle. Using some logic, we realize that as long as said furthest point is not inside or on the graph of the circle. This should be enough to conclude that the hyperbola does not intersect the circle.

Therefore, for each value of k, we only need to check said value to determine intersection. Let said point, closest to the circle have coordinates $(x,k/x)$ derived from the equation. Then, all coordinates that satisfy $\sqrt{x^2+(k/x)^2}\le k$ intersect the circle. Squaring, we find $x^2+(k/x)^2\le k^2$. After multiplying through by x^2 and rearranging, we find $x^4-x^2k^2+k^2\le 0$. We see this is a quadratic in x^2 and consider taking the determinant, which tells us that solutions are real when, after factoring: $k^2(k^2-4)\le 0$ We plot this inequality on the number line to find it is satisfied for all values except: $(-1,0,1)$. We then eliminate 0 because it is extraneous as both $xy=0$ and $x^2+y^2=0$ are points which coincide. Therefore, there are a total of $\boxed{2\ \textbf{(C)}}$ values.

See also

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All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2010 AMC 12A Problems/Problem 14

Problem

Nondegenerate $\triangle ABC$ has integer side lengths, $\overline{BD}$ is an angle bisector, $AD = 3$, and $DC = 8$. What is the smallest possible value of the perimeter?

$\textbf{(A)}\ 30 \qquad \textbf{(B)}\ 33 \qquad \textbf{(C)}\ 35 \qquad \textbf{(D)}\ 36 \qquad \textbf{(E)}\ 37$

Solution

By the Angle Bisector Theorem, we know that $\frac{AB}{3} = \frac{BC}{8}$. If we use the lowest possible integer values for AB and BC (the measures of AD and DC, respectively), then $AB + BC = AD + DC = AC$, contradicting the Triangle Inequality. If we use the next lowest values ($AB = 6$ and $BC = 16$), the Triangle Inequality is satisfied. Therefore, our answer is $6 + 16 + 3 + 8 = \boxed{33}$, or choice $\textbf{(B)}$.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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Category: Introductory Geometry Problems

2010 AMC 12A Problems/Problem 15

Problem

A coin is altered so that the probability that it lands on heads is less than $\frac{1}{2}$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $\frac{1}{6}$. What is the probability that the coin lands on heads?

$\textbf{(A)}\ \frac{\sqrt{15}-3}{6} \qquad \textbf{(B)}\ \frac{6-\sqrt{6\sqrt{6}+2}}{12} \qquad \textbf{(C)}\ \frac{\sqrt{2}-1}{2} \qquad \textbf{(D)}\ \frac{3-\sqrt{3}}{6} \qquad \textbf{(E)}\ \frac{\sqrt{3}-1}{2}$

Solution

Let x be the probability of flipping heads. It follows that the probability of flipping tails is $1-x$.

The probability of flipping 2 heads and 2 tails is equal to the number of ways to flip it times the product of the probability of flipping each coin.

$$\begin{aligned}
{4\choose2}x^2(1-x)^2 &= \frac{1}{6}\\
6x^2(1-x)^2 &= \frac{1}{6}\\
x^2(1-x)^2 &= \frac{1}{36}\\
x(1-x) &= \pm\frac{1}{6}
\end{aligned}$$

As for the desired probability x both x and $1-x$ are nonnegative, we only need to consider the positive root, hence

$$\begin{aligned}
x(1-x) &= \frac{1}{6}\\
6x^2-6x+1 &= 0
\end{aligned}$$

Applying the quadratic formula we get that the roots of this equation are $\frac{3\pm\sqrt{3}}{6}$. As the probability of heads is less than $\frac{1}{2}$, we get that the answer is $\boxed{\textbf{(D)}\ \frac{3-\sqrt{3}}{6}}$.

See also

2010 AMC 12A Problems/Problem 16

Problem

Bernardo randomly picks 3 distinct numbers from the set $\{1,2,3,4,5,6,7,8,9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1,2,3,4,5,6,7,8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

$\textbf{(A)}\ \frac{47}{72} \qquad \textbf{(B)}\ \frac{37}{56} \qquad \textbf{(C)}\ \frac{2}{3} \qquad \textbf{(D)}\ \frac{49}{72} \qquad \textbf{(E)}\ \frac{39}{56}$

Solution

We can solve this by breaking the problem down into 2 cases and adding up the probabilities.

Case 1: Bernardo picks 9. If Bernardo picks a 9 then it is guaranteed that his number will be larger than Silvia's. The probability that he will pick a 9 is $\frac{\frac{8\cdot7}{2}}{\frac{9\cdot8\cdot7}{3\cdot2\cdot1}} = \frac{1}{3}$.

Case 2: Bernardo does not pick 9. Since the chance of Bernardo picking 9 is $\frac{1}{3}$, the probability of not picking 9 is $\frac{2}{3}$.

If Bernardo does not pick 9, then he can pick any number from 1 to 8. Since Bernardo is picking from the same set of numbers as Silvia, the probability that Bernardo's number is larger is equal to the probability that Silvia's number is larger.

Ignoring the 9 for now, the probability that they will pick the same number is the number of ways to pick Bernardo's 3 numbers divided by the number of ways to pick any 3 numbers.

We get this probability to be $\frac{3!}{8\cdot 7\cdot 6} = \frac{1}{56}$

Probability of Bernardo's number being greater is

$$\frac{1-\frac{1}{56}}{2} = \frac{55}{112}$$

Factoring the fact that Bernardo could've picked a 9 but didn't:

$$\frac{2}{3}\cdot\frac{55}{112} = \frac{55}{168}$$

Adding up the two cases we get $\frac{1}{3}+\frac{55}{168} = \boxed{\frac{37}{56}\ \textbf{(B)}}$

See also

2010 AMC 12A Problems/Problem 17

Problem

Equiangular hexagon $ABCDEF$ has side lengths $AB = CD = EF = 1$ and $BC = DE = FA = r$. The area of $\triangle ACE$ is 70% of the area of the hexagon. What is the sum of all possible values of r?

$\textbf{(A)}\ \frac{4\sqrt{3}}{3} \qquad \textbf{(B)} \frac{10}{3} \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ \frac{17}{4} \qquad \textbf{(E)}\ 6$

Solution

It is clear that $\triangle ACE$ is an equilateral triangle. From the Law of Cosines, we get that $AC^2 = r^2+1^2-2r\cos{\frac{2\pi}{3}} = r^2+r+1$. Therefore, the area of $\triangle ACE$ is $\frac{\sqrt{3}}{4}(r^2+r+1)$.

If we extend BC, DE and FA so that FA and BC meet at X, BC and DE meet at Y, and DE and FA meet at Z, we find that hexagon $ABCDEF$ is formed by taking equilateral triangle XYZ of side length $r+2$ and removing three equilateral triangles, ABX, CDY and EFZ, of side length 1. The area of $ABCDEF$ is therefore

$\frac{\sqrt{3}}{4}(r+2)^2-\frac{3\sqrt{3}}{4} = \frac{\sqrt{3}}{4}(r^2+4r+1)$.

Based on the initial conditions,

$$\frac{\sqrt{3}}{4}(r^2+r+1) = \frac{7}{10}\left(\frac{\sqrt{3}}{4}\right)(r^2+4r+1)$$

Simplifying this gives us $r^2-6r+1 = 0$. By Vieta's Formulas we know that the sum of the possible value of r is $\boxed{\textbf{(E)}\ 6}$.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12A Problems/Problem 18

Problem

A 16-step path is to go from $(-4,-4)$ to $(4,4)$ with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \le x \le 2$, $-2 \le y \le 2$ at each step?

$\textbf{(A)}\ 92 \qquad \textbf{(B)}\ 144 \qquad \textbf{(C)}\ 1568 \qquad \textbf{(D)}\ 1698 \qquad \textbf{(E)}\ 12,800$

Solution

Each path must go through either the second or the fourth quadrant. Each path that goes through the second quadrant must pass through exactly one of the points $(-4,4)$, $(-3,3)$, and $(-2,2)$.

There is 1 path of the first kind, $\binom{8}{1}^2 = 64$ paths of the second kind, and $\binom{8}{2}^2 = 28^2 = 784$ paths of the third type. Each path that goes through the fourth quadrant must pass through exactly one of the points $(4,-4)$, $(3,-3)$, and $(2,-2)$. Again, there are 1 paths of the first kind, $\binom{8}{1}^2 = 64$ paths of the second kind, and $\binom{8}{2}^2 = 28^2 = 784$ paths of the third type.

Hence the total number of paths is $2(1+64+784) = \boxed{1698}$.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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Category: Introductory Combinatorics Problems

2010 AMC 12A Problems/Problem 19

Problem

Each of 2010 boxes in a line contains a single red marble, and for $1 \leq k \leq 2010$, the box in the kth position also contains k white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let $P(n)$ be the probability that Isabella stops after drawing exactly n marbles. What is the smallest value of n for which $P(n) < \frac{1}{2010}$?

$\textbf{(A)}\ 45 \qquad \textbf{(B)}\ 63 \qquad \textbf{(C)}\ 64 \qquad \textbf{(D)}\ 201 \qquad \textbf{(E)}\ 1005$

Solution

The probability of drawing a white marble from box k is $\frac{k}{k+1}$, and the probability of drawing a red marble from box k is $\frac{1}{k+1}$.

To stop after drawing n marbles, we must draw a white marble from boxes $1, 2, \ldots, n-1,$ and draw a red marble from box $n.$ Thus,

$$P(n) = \left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n}\right) \cdot \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

So, we must have $\frac{1}{n(n+1)} < \frac{1}{2010}$ or $n(n+1) > 2010.$

Since $n(n+1)$ increases as n increases, we can simply test values of n; after some trial and error, we get that the minimum value of n is $\boxed{\textbf{(A)}\ 45}$, since $45(46) = 2070$ but $44(45) = 1980.$

See also

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Category: Introductory Combinatorics Problems

2010 AMC 12A Problems/Problem 20

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- 1 Problem
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Problem

Arithmetic sequences (a_n) and (b_n) have integer terms with $a_1 = b_1 = 1 < a_2 \le b_2$ and $a_n b_n = 2010$ for some n. What is the largest possible value of n?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 288 \qquad \textbf{(E)}\ 2009$

Solution

Solution 1

Since (a_n) and (b_n) have integer terms with $a_1 = b_1 = 1$, we can write the terms of each sequence as

$$(a_n) \Rightarrow \{1, x+1, 2x+1, 3x+1, ...\}$$
$$(b_n) \Rightarrow \{1, y+1, 2y+1, 3y+1, ...\}$$

where x and y $(x \le y)$ are the common differences of each, respectively.

Since

$$a_n = (n-1)x + 1$$
$$b_n = (n-1)y + 1$$

it is easy to see that

$a_n \equiv b_n \equiv 1 \mod (n-1)$.

Hence, we have to find the largest n such that $\frac{a_n - 1}{n-1}$ and $\frac{b_n - 1}{n-1}$ are both integers.

The prime factorization of 2010 is $2 \cdot 3 \cdot 5 \cdot 67$. We list out all the possible pairs that have a product of 2010

$$(2, 1005), (3, 670), (5, 402), (6, 335), (10, 201), (15, 134), (30, 67)$$

and soon find that the largest $n-1$ value is 7 for the pair $(15, 134)$, and so the largest n value is $\boxed{8\ \textbf{(C)}}$.

Solution 2

As above, let $a_n = (n-1)x+1$ and $b_n = (n-1)y+1$ for some $1 \leq x \leq y$.

Now we get $2010 = a_n b_n = (n-1)^2 xy + (n-1)(x+y) + 1$, hence $2009 = (n-1)((n-1)xy + x + y)$. Therefore $n-1$ divides $2009 = 7^2 \cdot 41$. And as the second term is greater than the first one, we only have to consider the options $n-1 \in \{1, 7, 41\}$.

For $n = 42$ we easily see that for $x = y = 1$ the right side is less than 49 and for any other (x, y) it is way too large.

For $n = 8$ we are looking for (x, y) such that $7xy + x + y = 2009/7 = 7 \cdot 41$. Note that $x + y$ must be divisible by 7. We can start looking for the solution by trying the possible values for $x + y$, and we easily discover that for $x + y = 21$ we get $xy + 3 = 41$, which has a suitable solution $(x, y) = (2, 19)$.

Hence $n = 8$ is the largest possible n. (There is no need to check $n = 2$ anymore.)

See also

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Category: Introductory Algebra Problems

2010 AMC 12A Problems/Problem 21

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- 3 Solution 2
- 4 See also

Problem

The graph of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ lies above the line $y = bx + c$ except at three values of x, where the graph and the line intersect. What is the largest of these values?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 8$

Solution 1

The x values in which $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ intersect at $y = bx + c$ are the same as the zeros of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2 - bx - c$.

Since there are 3 zeros and the function is never negative, all 3 zeros must be double roots because the function's degree is 6.

Suppose we let p, q, and r be the roots of this function, and let $x^3 - ux^2 + vx - w$ be the cubic polynomial with roots p, q, and r.

$$\begin{align*}(x-p)(x-q)(x-r) &= x^3 - ux^2 + vx - w\\ (x-p)^2(x-q)^2(x-r)^2 &= x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2 - bx - c = 0\\ \sqrt{x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2 - bx - c} &= x^3 - ux^2 + vx - w = 0\end{align*}$$

In order to find $\sqrt{x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2 - bx - c}$ we must first expand out the terms of $(x^3 - ux^2 + vx - w)^2$.

$$(x^3 - ux^2 + vx - w)^2$$
$$= x^6 - 2ux^5 + (u^2 + 2v)x^4 - (2uv + 2w)x^3 + (2uw + v^2)x^2 - 2vwx + w^2$$

[Quick note: Since we don't know a, b, and c, we really don't even need the last 3 terms of the expansion.]

$$\begin{align*}2u &= 10\\ u^2 + 2v &= 29\\ 2uv + 2w &= 4\\ u &= 5\\ v &= 2\\ w &= -8\\ \sqrt{x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2 - bx - c} &= x^3 - 5x^2 + 2x + 8\end{align*}$$

All that's left is to find the largest root of $x^3 - 5x^2 + 2x + 8$.

$$x^3 - 5x^2 + 2x + 8 = (x-4)(x-2)(x+1)$$

$$\boxed{\textbf{(A)}\ 4}$$

Solution 2

The x values in which $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ intersect at $y = bx + c$ are the same as the zeros of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2 - bx - c$. We also know that this graph has 3 places tangent to the x-axis, which means that each root has to have a multiplicity of 2. Let the function be $(x-p)^2(x-q)^2(x-r)^2$.

Applying Vieta's formulas, we get $2p + 2q + 2r = 10$ or $p + q + r = 5$. Applying it again, we get, after simplification, $p^2 + q^2 + r^2 + 4pq + 4pr + 4qr = 29$.

Notice that squaring the first equation yields $p^2 + q^2 + r^2 + 2pq + 2qr + 2pr = 25$, which is similar to the second equation.

Subtracting this from the second equation, we get $2pq + 2pr + 2qr = 4$. Now that we have to $pq + pr + qr$ term, we can manpulate the equations to yield the sum of squares. $2(p^2 + q^2 + r^2 + 2pq + 2qr + 2pr) - 2pq - 2pr - 2qr = 25 * 2 - 4$ or $2p^2 + 2q^2 + 2r^2 + 2pq + 2qr + 2pr = 46$. We finally reach $(p+q)^2 + (q+r)^2 + (p+r)^2 = 46$.

Since the answer choices are integers, we can guess and check squares to get $\{(p+q)^2, (q+r)^2, (p+r)^2\} = \{1, 9, 36\}$ in some order. We can check that this works by adding then and seeing $2p + 2q + 2r = 10$. We just need to take the lowest value in the set, square root it, and subtract the resulting value from 5 to get $\boxed{\textbf{(A)}\ 4}$.

Alternative method:

After reaching $p + q + r = 5$ and $pq + qr + rp = 2$, we can algebraically derive pqr.

Applying Vieta's formulas on the x^3 term yields $2p^2q + 2pq^2 + 2q^2r + 2qr^2 + 2r^2p + 2rp^2 + 8pqr = 4$.

Notice that $(p+q+r)(pq+qr+rp) = p^2q + pq^2 + q^2r + qr^2 + r^2p + rp^2 + 3pqr$, so

$$2p^2q + 2pq^2 + 2q^2r + 2qr^2 + 2r^2p + 2rp^2 + 6pqr = 2(p+q+r)(pq+qr+rp) = 20.$$

Subtracting this from $2p^2q + 2pq^2 + 2q^2r + 2qr^2 + 2r^2p + 2rp^2 + 8pqr = 4$ yields $2pq = -16$, so $pqr = -8$, which means that p, q, and r are the roots of the cubic $x^3 - 5x^2 + 2x + 8$, and it is not hard to find that these roots are -1, 2, and 4. The largest of these values is $\boxed{\textbf{(A)}\ 4}$.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12A Problems/Problem 22

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  - 2.4 Solution 3
- 3 See also

Problem

What is the minimum value of $f(x) = |x-1| + |2x-1| + |3x-1| + \cdots + |119x - 1|$?

$\textbf{(A)}\ 49 \qquad \textbf{(B)}\ 50 \qquad \textbf{(C)}\ 51 \qquad \textbf{(D)}\ 52 \qquad \textbf{(E)}\ 53$

Solution

Solution 1

If we graph each term separately, we will notice that all of the zeros occur at $\frac{1}{m}$, where m is any integer from 1 to 119, inclusive: $|mx-1|=0 \implies mx=1 \implies x=1/m$.

The minimum value of $f(x)$ occurs where the absolute value of the sum of the slopes is at a minimum ≥ 0, since it is easy to see that the value will be increasing on either side. That means the minimum must happen at some $\frac{1}{m}$.

The sum of the slope at $x = \frac{1}{m}$ is

$$\begin{aligned}&\sum_{i=m+1}^{119} i - \sum_{i=1}^{m} i \\ &= \sum_{i=1}^{119} i - 2\sum_{i=1}^{m} i \\ &= -m^2 - m + 7140\end{aligned}$$

Now we want to minimize $-m^2-m+7140$. The zeros occur at -85 and 84, which means the slope is 0 where $m = 84, 85$.

We can now verify that both $x = \frac{1}{84}$ and $x = \frac{1}{85}$ yield $\boxed{49\ \textbf{(A)}}$.

You can also think of the slopes playing 'tug of war', where the slope of each absolute function upon passing its x-intercept is negated, positively tugging on the remaining negative slopes.

The sum of the slopes is $1+2+3+4\ldots119 = \sum_{m=1}^{119} m = \frac{119 \cdot 120}{2} = 60 \cdot 119 = 7140$

So we need to find the least integer a such that

$$1+2+3+\ldots a = \sum_{n=1}^{a} n = \frac{a(a+1)}{2} \ge \frac{7140}{2} = 3570:$$

$$a(a+1) \geq 7140 \implies a^2+a-7140 \geq 0 \to a = 84 \text{ exactly!}$$

This "exactly" means that the slope is ZERO between the whole interval $x \in \left(\frac{1}{85}, \frac{1}{84}\right)$. We can explicitly evaluate both to check that they are both equal to the desired minimum value of $f(x)$:

$$\frac{84+83+\cdots+2+1+1+2+\cdots+33+34}{85} = \frac{84(85)/2+34(35)/2}{85} = \frac{85(14+84)/2}{85} = 49$$

$$\frac{83+82+\cdots+2+1+1+2+\cdots+34+35}{84} = \frac{83(84)/2+35(36)/2}{84} = \frac{84(15+83)/2}{84} = 49$$

Thus the minimum value of $f(x)$ is $\boxed{\textbf{(A) } 49}$.

Solution 2

Rewrite the given expression as follows:

$$1|x-1| + 2\left|x-\frac{1}{2}\right| + \cdots + 119\left|x-\frac{1}{119}\right|$$

Imagine the real line. For each $n \in \{1,\ldots,119\}$ imagine that there are n boys standing at the coordinate $\frac{1}{n}$. We now need to place a donut on the real line in such a way that the sum of its distances from all the boys is minimal, and we need to compute this sum.

Note that there are $B = 1+2+\cdots+119 = 119 \cdot 60 = 7140$ boys in total. Let's label them from 1 (the only boy placed at 1) to B (the last boy placed at $\frac{1}{119}$.

Clearly, the minimum sum is achieved if the donut's coordinate is the median of the boys' coordinates. To prove this, place the donut at the median coordinate. If you now move it in any direction by any amount d, there will be $B/2$ boys such that it moves d away from this boy. For each of the remaining boys, it moves at most d closer, hence the total sum of distances does not decrease.

Hence the optimal solution is to place the donut at the median coordinate. Or, more precisely, as B is even, we can place it anywhere on the segment formed by boy $B/2$ and boy $(B/2)+1$: by extending the previous argument, anywhere on this segment the sum of distances is the same.

By trial and error, or by solving the quadratic equation $z(z+1)/2 = 7140/2$ we get that boy number $B/2$ is the last boy placed at $\frac{1}{84}$ and the next boy is the one placed at $\frac{1}{85}$. Hence the given expression is minimized for any $x \in \left[\frac{1}{85}, \frac{1}{84}\right]$.

Common part of both solutions

To find the minimum, we want to balance the expression so that it is neither top nor bottom heavy.

$$\frac{119(120)}{2(2)} = \frac{7140}{2} = 3570 = \frac{84(85)}{2} = \frac{119(120)}{2} - \frac{84(85)}{2}.$$

Now that we know that the sum of the first 84 x's is equivalent to the sum of x's 85 to 119, we can plug either $\frac{1}{84}$ or $\frac{1}{85}$ to find the minimum.

Note that the terms $x-1$ to $83x-1$ are negative, and the terms $85x-1$ to $119x-1$ are positive. Hence we get:

$$\begin{aligned}
& |x-1|+|2x-1|+\cdots+|83x-1| \\
= & (1-x)+(1-2x)+\cdots+(1-83x) \\
= & 83-x(1+2+\cdots+83) \\
= & 83-\frac{1}{84}\cdot\frac{83\cdot 84}{2} \\
= & 83-\frac{83}{2} \\
= & \frac{83}{2}
\end{aligned}$$

and

$$\begin{aligned}
& |85x-1|+|86x-1|+\cdots+|119x-1| \\
= & (85x-1)+(86x-1)+\cdots+(119x-1) \\
= & x(85+86+\cdots+119)-(119-84) \\
= & \frac{1}{84}\cdot\frac{84\cdot 85}{2}-35 \\
= & \frac{85}{2}-35 \\
= & \frac{15}{2}
\end{aligned}$$

Hence the total sum of distances is $\frac{83}{2}+\frac{15}{2}=49$.

Solution 3

Since the minimum exists, we want all the xs to cancel out. Thus, we want to find some n such that

$$1+2+3+...+n=(n+1)+(n+2)+(n+3)+...+119$$

$$\frac{n(n+1)}{2}=\frac{119\cdot 120}{2}-\frac{n(n+1)}{2}$$

$$n^2+n-7140=0$$

$$n=84$$

Then, $x=\frac{1}{n}=\frac{1}{84}$. The answer(expression's value) is then $84*1+(119-85+1)*(-1)$, which becomes $84-35=\boxed{49}$.

See also

2010 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12A Problems/Problem 23

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Problem

The number obtained from the last two nonzero digits of $90!$ is equal to n. What is n?

$\textbf{(A)}\ 12 \qquad \textbf{(B)}\ 32 \qquad \textbf{(C)}\ 48 \qquad \textbf{(D)}\ 52 \qquad \textbf{(E)}\ 68$

Hints and Method of Attack

Let P be the result of dividing $90!$ by tens such that P is not divisible by 10. We want to consider $P \mod 100$. But because 100 is not prime, and because P is obviously divisible by 4 (if in doubt, look at the answer choices), we only need to consider $P \mod 25$.

However, 25 is a very particular number. $1*2*3*4 \equiv -1 \mod 25$, and so is $6*7*8*9$. How can we group terms to take advantage of this fact?

There might be a problem when you cancel out the 10s from $90!$. One method is to cancel out a factor of 2 from an existing number along with a factor of 5. But this might prove cumbersome, as the grouping method will not be as effective. Instead, take advantage of inverses in modular arithmetic. Just leave the negative powers of 2 in a "storage base," and take care of the other terms first. Then, use Fermat's Little Theorem to solve for the power of 2.

Solution

We will use the fact that for any integer n,

$$\begin{aligned}(5n+1)(5n+2)(5n+3)(5n+4) &= [(5n+4)(5n+1)][(5n+2)(5n+3)]\\ &= (25n^2+25n+4)(25n^2+25n+6) \equiv 4\cdot 6\\ &= 24 \pmod{25} \equiv -1 \pmod{25}.\end{aligned}$$

First, we find that the number of factors of 10 in $90!$ is equal to $\left\lfloor \frac{90}{5}\right\rfloor + \left\lfloor\frac{90}{25}\right\rfloor = 18+3=21$. Let $N=\frac{90!}{10^{21}}$. The n we want is therefore the last two digits of N, or $N \pmod{100}$. If instead we find $N \pmod{25}$, we know that $N \pmod{100}$, what we are looking for, could be $N \pmod{25}$, $N \pmod{25}+25$, $N \pmod{25}+50$, or $N \pmod{25}+75$. Only one of these numbers will be a multiple of four, and whichever one that is will be the answer, because $N \pmod{100}$ has to be a multiple of 4.

If we divide N by 5^{21} by taking out all the factors of 5 in N, we can write N as $\frac{M}{2^{21}}$ where

$$M = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 2 \cdots 89 \cdot 18,$$

where every multiple of 5 is replaced by the number with all its factors of 5 removed. Specifically, every number in the form $5n$ is replaced by n, and every number in the form $25n$ is replaced by n.

The number M can be grouped as follows:

$$\begin{aligned}M= &(1 \cdot 2 \cdot 3 \cdot 4)(6 \cdot 7 \cdot 8 \cdot 9)\cdots(86 \cdot 87 \cdot 88 \cdot 89)\\ &\cdot (1 \cdot 2 \cdot 3 \cdot 4)(6 \cdot 7 \cdot 8 \cdot 9)\cdots(16 \cdot 17 \cdot 18)\\ &\cdot (1 \cdot 2 \cdot 3).\end{aligned}$$

Where the first line is composed of the numbers in $90!$ that aren't multiples of five, the second line is the multiples of five and not 25 after they have been divided by five, and the third line is multiples of 25 after they have been divided by 25.

Using the identity at the beginning of the solution, we can reduce M to

$$\begin{aligned}M &\equiv (-1)^{18} \cdot (-1)^3(16 \cdot 17 \cdot 18) \cdot (1 \cdot 2 \cdot 3)\\ &= 1 \cdot -21 \cdot 6\\ &= -1 \pmod{25} = 24 \pmod{25}.\end{aligned}$$

Using the fact that $2^{10}=1024 \equiv -1 \pmod{25}$ (or simply the fact that $2^{21}=2097152$ if you have your powers of 2 memorized), we can deduce that $2^{21} \equiv 2 \pmod{25}$. Therefore $N = \frac{M}{2^{21}} \equiv \frac{24}{2} \pmod{25} = 12 \pmod{25}$.

Finally, combining with the fact that $N \equiv 0 \pmod 4$ yields $n = \boxed{\textbf{(A)}\ 12}$.

Solution 2

Let P be $90!$ after we truncate its zeros. Notice that $90!$ has exactly (floored) $\lfloor\frac{90}{5}\rfloor + \lfloor\frac{90}{25}\rfloor = 21$ factors of 5; thus,

$$P = 2^{-21} * 5^{-21} * 90!.$$

We shall consider P modulo 4 and 25, to determine its residue modulo 100. It is easy to prove that P is divisible by 4 (consider the number of 2s dividing $90!$ minus the number of 5s dividing $90!$), and so we only need to consider P modulo 25.

Now, notice that for integers a, n we have

$$(5n+a)(5n-a) \equiv -a^2 \mod 25.$$

Thus, for integral a:

$$(10a+1)(10a+2)(10a+3)(10a+4)(10a+6)(10a+7)(10a+8)(10a+9) \equiv (-1)(-4)(-9)(-16) \equiv 576 \equiv 1 \mod 25$$

Using this process, we can essentially remove all the numbers which had not formerly been a multiple of 5 in $90!$ from consideration.

Now, we consider the remnants of the 5, 10, 15, 20, ..., 90 not yet eliminated. The 10, 20, 30, ..., 90 becomes 1, 2, 3, 4, 1, 6, 7, 8, 9, whose product is 1 mod 25. Also, the 5, 5, 15, 25, ..., 85 becomes 1, 1, 3, 1, 7, 9, 11, 13, 3, 17 and 2^{-12}. We deduce that from multiplying out the 1, 1, 3, 1, 7, ..., 17 is equivalent to 2 modulo 25, and so we need to compute 2^{-11}. But this is simply by Fermat's Little Theorem $2^9 = 512 \equiv 12 \mod 25$. Because 12 is also a multiple of 4, we can utilize the Chinese Remainder Theorem to show that $P = 12 \mod 100$ and so the answer is $\boxed{12}$.

See also

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Category: Intermediate Number Theory Problems

2010 AMC 12A Problems/Problem 24

Problem

Let $f(x) = \log_{10}\left(\sin(\pi x) \cdot \sin(2\pi x) \cdot \sin(3\pi x) \cdots \sin(8\pi x)\right)$. The intersection of the domain of $f(x)$ with the interval $[0,1]$ is a union of n disjoint open intervals. What is n?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 12 \qquad \textbf{(C)}\ 18 \qquad \textbf{(D)}\ 22 \qquad \textbf{(E)}\ 36$

Solution

The question asks for the number of disjoint open intervals, which means we need to find the number of disjoint intervals such that the function is defined within them.

We note that since all of the $\sin$ factors are inside a logarithm, the function is undefined where the inside of the logarithm is equal to or less than 0.

First, let us find the number of zeros of the inside of the logarithm.

$$\begin{aligned}
\sin(\pi x) \cdot \sin(2\pi x) \cdot \sin(3\pi x) \cdots \sin(8\pi x) &= 0 \\
\sin(\pi x) &= 0 \\
x &= 0, 1 \\
\sin(2\pi x) &= 0 \\
x &= 0, \frac{1}{2}, 1 \\
\sin(3\pi x) &= 0 \\
x &= 0, \frac{1}{3}, \frac{2}{3}, 1 \\
\sin(4\pi x) &= 0 \\
x &= 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1 \\
&\cdots
\end{aligned}$$

After counting up the number of zeros for each factor and eliminating the excess cases we get 23 zeros and 22 intervals.

In order to find which intervals are negative, we must first realize that at every zero of each factor, the sign changes. We also have to be careful, as some zeros are doubled, or even tripled, quadrupled, etc.

The first interval $(0, \frac{1}{8})$ is obviously positive. This means the next interval $(\frac{1}{8}, \frac{1}{7})$ is negative. Continuing the pattern and accounting for doubled roots (which do not flip sign), we realize that there are 5 negative intervals from 0 to $\frac{1}{2}$. Since the function is symmetric, we know that there are also 5 negative intervals from $\frac{1}{2}$ to 1.

And so, the total number of disjoint open intervals is $22 - 2 \cdot 5 = \boxed{12\ \textbf{(B)}}$

See also

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Problem

Two quadrilaterals are considered the same if one can be obtained from the other by a rotation and a translation. How many different convex cyclic quadrilaterals are there with integer sides and perimeter equal to 32?

$\textbf{(A)}\ 560 \qquad \textbf{(B)}\ 564 \qquad \textbf{(C)}\ 568 \qquad \textbf{(D)}\ 1498 \qquad \textbf{(E)}\ 2255$

Solution 1

It should first be noted that given any quadrilateral of fixed side lengths, the angles can be manipulated so that the quadrilateral becomes cyclic.

Denote a, b, c, and d as the integer side lengths of the quadrilateral. Without loss of generality, let $a \ge b \ge c \ge d$.

Since $a+b+c+d=32$, the Triangle Inequality implies that $a \le 15$.

We will now split into 5 cases.

Case 1: $a=b=c=d$ (4 side lengths are equal)

Clearly there is only 1 way to select the side lengths $(8,8,8,8)$, and no matter how the sides are rearranged only 1 unique quadrilateral can be formed.

Case 2: $a=b=c>d$ or $a>b=c=d$ (3 side lengths are equal)

If 3 side lengths are equal, then each of those side lengths can only be integers from 6 to 10 except for 8 (because that is counted in the first case). Obviously there is still only 1 unique quadrilateral that can be formed from one set of side lengths, resulting in a total of 4 quadrilaterals.

Case 3: $a=b>c=d$ (2 pairs of side lengths are equal)

a and b can be any integer from 9 to 15, and likewise c and d can be any integer from 1 to 7. However, a single set of side lengths can form 2 different cyclic quadrilaterals (a rectangle and a kite), so the total number of quadrilaterals for this case is $7 \cdot 2 = 14$.

Case 4: $a=b>c>d$ or $a>b=c>d$ or $a>b>c=d$ (2 side lengths are equal)

If the 2 equal side lengths are each 1, then the other 2 sides must each be 15, which we have already counted in an earlier case. If the equal side lengths are each 2, there is 1 possible set of side lengths. Likewise, for side lengths of 3 there are 2 sets. Continuing this pattern, we find a total of $1+2+3+4+4+5+7+5+4+4+3+2+1 = 45$ sets of side lengths. (Be VERY careful when adding up the total for this case!) For each set of side lengths, there are 3 possible quadrilaterals that can be formed, so the total number of quadrilaterals for this case is $3 \cdot 45 = 135$.

Case 5: $a>b>c>d$ (no side lengths are equal) Using the same counting principles starting from $a=15$ and eventually reaching $a=9$, we find that the total number of possible side lengths is 69. There are $4!$ ways to arrange the 4 side lengths, but there is only 1 unique quadrilateral for 4 rotations, so the number of quadrilaterals for each set of side lengths is $\frac{4!}{4} = 6$. The total number of quadrilaterals is $6 \cdot 69 = 414$.

And so, the total number of quadrilaterals that can be made is $414 + 135 + 14 + 4 + 1 = \boxed{568 \textbf{ (C)}}$.

Solution 2

As with solution 1 we would like to note that given any quadrilateral we can change its angles to make a cyclic one.

Let $a \geq b \geq c \geq d$ be the sides of the quadrilateral.

There are $\binom{31}{3}$ ways to partition 32. However, some of these will not be quadrilaterals since they would have one side bigger than the sum of the other three. This occurs when $a \geq 16$. For $a = 16$, $b + c + d = 16$. There are $\binom{15}{2}$ ways to partition 16. Since a could be any of the four sides, we have counted $4\binom{15}{2}$ degenerate quadrilaterals. Similarly, there are $4\binom{14}{2}$, $4\binom{13}{2} \cdots 4\binom{2}{2}$ for other values of a. Thus, there are $\binom{31}{3} - 4\left(\binom{15}{2} + \binom{14}{2} + \cdots + \binom{2}{2}\right) = \binom{31}{3} - 4\binom{16}{3} = 2255$ non-degenerate partitions of 32 by the hockey stick theorem. However, for $a \neq b \neq c \neq d$ or $a = b \neq c \neq d$, each quadrilateral is counted 4 times, 1 for each rotation. Also, for $a = b \neq c = d$, each quadrilateral is counted twice. Since there is 1 quadrilateral for which $a = b = c = d$, and 7 for which $a = b \neq c = d$, there are $2255 - 1 - 2 \cdot 7 = 2240$ quads for which $a \neq b \neq c \neq d$ or $a = b \neq c \neq d$. Thus there are $1 + 7 + \frac{2240}{4} = \boxed{568} = \boxed{\textbf{(C)}}$ total quadrilaterals.

See also

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