2016 AMC 12B Problems/Problem 1

Problem

What is the value of $\frac{2a^{-1}+\frac{a^{-1}}{2}}{a}$ when $a=\frac{1}{2}$?

$\textbf{(A)}\ 1\qquad\textbf{(B)}\ 2\qquad\textbf{(C)}\ \frac{5}{2}\qquad\textbf{(D)}\ 10\qquad\textbf{(E)}\ 20$

Solution

By: Dragonfly

We find that a^{-1} is the same as 2, since a number to the power of -1 is just the reciprocal of that number. We then get the equation to be

$$\frac{2\times2+\frac{2}{2}}{\frac{1}{2}}$$

We can then simplify the equation to get $\boxed{\textbf{(D)}\ 10}$

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
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All AMC 12 Problems and Solutions	

2016 AMC 12B Problems/Problem 2

Problem

The harmonic mean of two numbers can be calculated as twice their product divided by their sum. The harmonic mean of 1 and 2016 is closest to which integer?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 45 \qquad \textbf{(C)}\ 504 \qquad \textbf{(D)}\ 1008 \qquad \textbf{(E)}\ 2015$

Solution

By: dragonfly

Since the harmonic mean is 2 times their product divided by their sum, we get the equation

$$\frac{2\times1\times2016}{1+2016}$$

which is then

$$\frac{4032}{2017}$$

which is finally closest to $\boxed{\textbf{(A)}\ 2}$.

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
Preceded by Problem 1	Followed by Problem 3
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All AMC 12 Problems and Solutions	

2016 AMC 12B Problems/Problem 3

Problem

Let $x=-2016$. What is the value of $\Bigg\vert\Big\vert |x|-x\Big\vert-|x|\Bigg\vert-x$?

Solution

By: dragonfly

First of all, lets plug in all of the x's into the equation.

$$\Bigg\vert\Big\vert |-2016|-(-2016)\Big\vert-|-2016|\Bigg\vert-(-2016)$$

Then we simplify to get

$$\Bigg\vert\Big\vert 2016+2016\Big\vert-2016\Bigg\vert+2016$$

which simplifies into

$$\Bigg\vert 2016\Bigg\vert+2016$$

and finally we get $\boxed{\textbf{(D)}\ 4032}$

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
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2016 AMC 12B Problems/Problem 4

Problem

The ratio of the measures of two acute angles is $5:4$, and the complement of one of these two angles is twice as large as the complement of the other. What is the sum of the degree measures of the two angles?

$\textbf{(A)}\ 75\qquad\textbf{(B)}\ 90\qquad\textbf{(C)}\ 135\qquad\textbf{(D)}\ 150\qquad\textbf{(E)}\ 270$

Solution

By: dragonfly

We set up equations to find each angle. The larger angle will be represented as x and the larger angle will we represented as y, in degrees. This implies that

$$4x=5y$$

and

$$2\times(90-x)=90-y$$

since the larger the original angle, the smaller the complement.

We then find that $x=75$ and $y=60$, and their sum is $\boxed{\textbf{(C)}\ 135}$

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
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2016 AMC 12B Problems/Problem 5

Problem

The War of 1812 started with a declaration of war on Thursday, June 18, 1812. The peace treaty to end the war was signed 919 days later, on December 24, 1814. On what day of the week was the treaty signed?

$\textbf{(A)}\ \text{Friday} \qquad \textbf{(B)}\ \text{Saturday} \qquad \textbf{(C)}\ \text{Sunday} \qquad \textbf{(D)}\ \text{Monday} \qquad \textbf{(E)}\ \text{Tuesday}$

Solution

By: dragonfly

To find what day of the week it is in 919 days, we have to divide 919 by 7 to see the remainder, and then add the remainder to the current day. We get that $\frac{919}{7}$ has a remainder of 2, so we increase the current day by 2 to get $\boxed{\textbf{(B)}\text{Saturday}}$

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
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2016 AMC 12B Problems/Problem 6

Problem

All three vertices of $\triangle ABC$ lie on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 64. What is the length of BC?

$\textbf{(A)}\ 4\qquad\textbf{(B)}\ 6\qquad\textbf{(C)}\ 8\qquad\textbf{(D)}\ 10\qquad\textbf{(E)}\ 16$

Solution

By: Albert471

Plotting points B and C on the graph shows that they are at $\left(-x,x^2\right)$ and $\left(x,x^2\right)$, which is isosceles. By setting up the triangle area formula you get: $64=\frac{1}{2}*2x*x^2 = 64=x^3$ Making x=4, and the length of BC is $2x$, so the answer is $\boxed{\textbf{(C)}8}$.

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
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All AMC 12 Problems and Solutions	

2016 AMC 12B Problems/Problem 7

Problem

Josh writes the numbers $1, 2, 3, \dots, 99, 100$. He marks out 1, skips the next number (2), marks out 3, and continues skipping and marking out the next number to the end of the list. Then he goes back to the start of his list, marks out the first remaining number (2), skips the next number (4), marks out 6, skips 8, marks out 10, and so on to the end. Josh continues in this manner until only one number remains. What is that number?

$\textbf{(A)}\ 13 \qquad \textbf{(B)}\ 32 \qquad \textbf{(C)}\ 56 \qquad \textbf{(D)}\ 64 \qquad \textbf{(E)}\ 96$

Solution

By Albert471

Following the pattern, you are crossing out...

Time 1: Every non-multiple of 2

Time 2: Every non-multiple of 4

Time 3: Every non-multiple of 8

Following this pattern, you are left with every multiple of 64 which is only $\boxed{\textbf{(D)} 64}$.

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
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2016 AMC 12B Problems/Problem 8

Contents

Problem

A thin piece of wood of uniform density in the shape of an equilateral triangle with side length 3 inches weighs 12 ounces. A second piece of the same type of wood, with the same thickness, also in the shape of an equilateral triangle, has side length of 5 inches. Which of the following is closest to the weight, in ounces, of the second piece?

$\textbf{(A)}\ 14.0\qquad\textbf{(B)}\ 16.0\qquad\textbf{(C)}\ 20.0\qquad\textbf{(D)}\ 33.3\qquad\textbf{(E)}\ 55.6$

Solution

By: dragonfly

We can solve this problem by using similar triangles, since two equilateral triangles are always similar. We can then use

$$\left(\frac{3}{5}\right)^2=\frac{12}{x}.$$

We can then solve the equation to get $x=\frac{100}{3}$ which is closest to $\boxed{\textbf{(D)}\ 33.3}$

Solution 2

Another approach to this problem, very similar to the previous one but perhaps explained more thoroughly, is to use proportions. First, since the thickness and density are the same, we can set up a proportion based on the principle that $d=\frac{m}{V}$, thus $dV=m.$

However, since density and thickness are the same and $A\propto s^2$ (recognizing that the area of an equilateral triangle is $\frac{(s)^2\sqrt{3}}{4}$), we can say that $m\propto s^2.$

Then, by increasing s by a factor of $\frac{5}{3}$, s^2 is increased by a factor of $\frac{25}{9}$, thus $m=12*\frac{25}{9}$ or $\boxed{\textbf{(D)}\ 33.3}$.

See Also

2016 AMC 12B Problems/Problem 9

Problem

Carl decided to fence in his rectangular garden. He bought 20 fence posts, placed one on each of the four corners, and spaced out the rest evenly along the edges of the garden, leaving exactly 4 yards between neighboring posts. The longer side of his garden, including the corners, has twice as many posts as the shorter side, including the corners. What is the area, in square yards, of Carl's garden?

$\textbf{(A)}\ 256 \qquad \textbf{(B)}\ 336 \qquad \textbf{(C)}\ 384 \qquad \textbf{(D)}\ 448 \qquad \textbf{(E)}\ 512$

Solution

By Albert471

To start, use algebra to determine the number of posts on each side. You have (the long sides count for 2 because there are twice as many) $6x = 20 + 4$ (each corner is double counted so you must add 4) Making the shorter end have 4, and the longer end have 8. $((8-1)*4)*((4-1)*4) = 28*12 = 336$. Therefore, the answer is $\boxed{\textbf{(B)}\ 336}$

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
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2016 AMC 12B Problems/Problem 10

Contents

Problem

A quadrilateral has vertices $P(a,b)$, $Q(b,a)$, $R(-a,-b)$, and $S(-b,-a)$, where a and b are integers with $a>b>0$. The area of $PQRS$ is 16. What is $a+b$?

$\textbf{(A)}\ 4 \qquad\textbf{(B)}\ 5 \qquad\textbf{(C)}\ 6 \qquad\textbf{(D)}\ 12 \qquad\textbf{(E)}\ 13$

Solution

By distance formula we have $(a-b)^2+(b-a)^2*2*(a+b)^2=256$. SImplifying we get $(a-b)(a+b)=8$. Thus $a+b$ and $a-b$ have to be a factor of 8. The only way for them to be factors of 8 and remain integers is if $a+b=4$ and $a-b=2$. So the answer is $\boxed{\textbf{(A)}\ 4}$

Solution by I_Dont_Do_Math

Solution 2

Solution by e_power_pi_times_i

By the Shoelace Theorem, the area of the quadrilateral is $2a^2-2b^2$, so $a^2-b^2=8$. Since a and b are integers, $a=3$ and $b=1$, so $a+b=\boxed{\textbf{(A)}\ 4}$.

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
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2016 AMC 12B Problems/Problem 11

Problem

How many squares whose sides are parallel to the axes and whose vertices have coordinates that are integers lie entirely within the region bounded by the line $y=\pi x$, the line $y=-0.1$ and the line $x=5.1?$

$\textbf{(A)}\ 30 \qquad \textbf{(B)}\ 41 \qquad \textbf{(C)}\ 45 \qquad \textbf{(D)}\ 50 \qquad \textbf{(E)}\ 57$

Solution

Solution by e_power_pi_times_i

(Note: diagram is needed)

If we draw a picture showing the triangle, we see that it would be easier to count the squares vertically and not horizontally. The upper bound is 16 squares $(y = 5.1*\pi)$, and the limit for the x-value is 5 squares. First we count the $1*1$ squares. In the back row, there are 12 squares with length 1 because $y = 4*\pi$ generates squares from $(4,0)$ to $(4,4\pi)$, and continuing on we have 9, 6, and 3 for x-values for 1, 2, and 3 in the equation $y = \pi x$. So there are $12+9+6+3 = 30$ squares with length 1 in the figure. For $2*2$ squares, each square takes up 2 un left and 2 un up. Squares can also overlap. For $2*2$ squares, the back row stretches from $(3,0)$ to $(3,3\pi)$, so there are 8 squares with length 2 in a 2 by 9 box. Repeating the process, the next row stretches from $(2,0)$ to $(2,2\pi)$, so there are 5 squares. Continuing and adding up in the end, there are $8+5+2 = 15$ squares with length 2 in the figure. Squares with length 3 in the back row start at $(2,0)$ and end at $(2,2\pi)$, so there are 4 such squares in the back row. As the front row starts at $(1,0)$ and ends at $(1,\pi)$ there are $4+1 = 5$ squares with length 3. As squares with length 4 would not fit in the triangle, the answer is $30+15+5$ which is $\boxed{\textbf{D) }50}$.

See Also

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2016 AMC 12B Problems/Problem 12

Contents

Problem

All the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ are written in a 3×3 array of squares, one number in each square, in such a way that if two numbers are consecutive then they occupy squares that share an edge. The numbers in the four corners add up to 18. What is the number in the center?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 7 \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 9$

Solution

Solution by Mlux: Draw a 3×3 matrix. Notice that no adjacent numbers could be in the corners since two consecutive numbers must share an edge. Now find 4 nonconsecutive numbers that add up to 18. Trying $1 + 3 + 5 + 9 = 18$ works. Place each odd number in the corner in a clockwise order. Then fill in the spaces. There has to be a 2 in between the 1 and 3. There is a 4 between 3 and 5. The final grid should similar to this.

$1, 2, 3$
$8, 7, 4$
$9, 6, 5$

$\textbf{(C)} 7$ is in the middle.

Solution 2

If we color the square like a chessboard, since the numbers altrenate between even and odd, and there are five odd numbers and four even numbers, the odd numbers must be in the corners/center, while the even numbers must be on the edges. Since the odd numbers add up to 25, and the numbers in the corners add up to 18, the number in the center must be 25-18=7

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
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2016 AMC 12B Problems/Problem 13

Contents

Problem

Alice and Bob live 10 miles apart. One day Alice looks due north from her house and sees an airplane. At the same time Bob looks due west from his house and sees the same airplane. The angle of elevation of the airplane is 30° from Alice's position and 60° from Bob's position. Which of the following is closest to the airplane's altitude, in miles?

$\textbf{(A)}\ 3.5 \qquad\textbf{(B)}\ 4 \qquad\textbf{(C)}\ 4.5 \qquad\textbf{(D)}\ 5 \qquad\textbf{(E)}\ 5.5$

Solution

Let's set the altitude = z, distance from Alice to airplane's ground position (point right below airplane)=y and distance from Bob to airplane's ground position=x

From Alice's point of view, $\tan(\theta) = \frac{z}{y}$. $\tan 30 = \frac{\sin 30}{\cos 30} = \frac{1}{\sqrt{3}}$. So, $y = z*\sqrt{3}$

From Bob's point of view, $\tan(\theta) = \frac{z}{x}$. $\tan 60 = \frac{\sin 60}{\cos 60} = \sqrt{3}$. So, $x = \frac{z}{\sqrt{3}}$

We know that x^2 + y^2 = 10^2

Solving the equation (by plugging in x and y), we get z=$\sqrt{30}$ = about 5.5.

So, answer is $E)5.5$

solution by sudeepnarala

Solution 2

Non-trig solution by e_power_pi_times_i

Set the distance from Alice's and Bob's position to the point directly below the airplane to be x and y, respectively. From the Pythagorean Theorem, $x^2+y^2=100$. As both are $30-60-90$ triangles, the altitude of the airplane can be expressed as $\frac{x\sqrt{3}}{3}$ or $y\sqrt{3}$. Solving the equation $\frac{x\sqrt{3}}{3} = y\sqrt{3}$, we get $x = 3y$. Plugging this into the equation $x^2+y^2=100$, we get $10y^2 = 100$, or $y = \sqrt{10}$ (y cannot be negative), so the altitude is $\sqrt{3*10} = \sqrt{30}$, which is closest to $\boxed{\textbf{E)}\ 5.5}$

See Also

2016 AMC 12B Problems/Problem 14

Contents

Problem

The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of $S?$

$\textbf{(A)}\ \frac{1+\sqrt{5}}{2} \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ \sqrt{5} \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 4$

Solution

The second term in a geometric series is $a_2 = a \cdot r$, where r is the common ratio for the series and a is the first term of the series. So we know that $a\cdot r = 1$ and we wish to find the minimum value of the infinite sum of the series. We know that: $S_\infty = \frac{a}{1-r}$ and substituting in $a=\frac{1}{r}$, we get that $S_\infty = \frac{\frac{1}{r}}{1-r} = \frac{1}{r(1-r)} = \frac{1}{r}+\frac{1}{1-r}$. From here, you can either use calculus or AM-GM.

Calculus: Let $f(x) = \frac{1}{x-x^2} = (x-x^2)^{-1}$, then $f'(x) = -(x-x^2)^{-2}\cdot (1-2x)$. Since $f(0)$ and $f(1)$ are undefined $x \neq 0, 1$. This means that we only need to find where the derivative equals 0, meaning $1-2x = 0 \Rightarrow x = \frac{1}{2}$. So $r = \frac{1}{2}$, meaning that $S_\infty = \frac{1}{\frac{1}{2} - (\frac{1}{2})^2} = \frac{1}{\frac{1}{2}-\frac{1}{4}} = \frac{1}{\frac{1}{4}} = 4$

AM-GM For 2 positive real numbers a and b, $\frac{a+b}{2} \geq \sqrt{ab}$. Let $a = \frac{1}{r}$ and $b = \frac{1}{1-r}$. Then: $\frac{\frac{1}{r}+\frac{1}{1-r}}{2} \geq \sqrt{\frac{1}{r}\cdot\frac{1}{1-r}} = \sqrt{\frac{1}{r}+\frac{1}{1-r}}$. This implies that $\frac{S_\infty}{2} \geq \sqrt{S_\infty}$. or $S_\infty^2 \geq 4\cdot S_\infty$. Rearranging : $(S_\infty-2)^2 \geq 4 \Rightarrow S_\infty - 2 \geq 2 \Rightarrow S_\infty \geq 4$. Thus, the smallest value is $S_\infty = 4$.

Solution 2

A simple approach is to initially recognize that $S_\infty = \frac{a}{1-r}$ and $a = \frac{1}{r}$. We know that $|r| \leq 1$, since the series must converge. We can start by observing the greatest answer choice, 4. Therefore, $r \not\leq \frac{1}{3}$, because that would make $\frac{1}{r} \geq 3$, which would make the series exceed 4. In order to minimize both the initial term and the rest of the series, we can recognize that $r = \frac{1}{2}$ is the opitimal ratio, thus the answer is $\boxed{\textbf{(E)}\ 4}$.

See Also

2016 AMC 12B Problems/Problem 15

Problem

All the numbers $2, 3, 4, 5, 6, 7$ are assigned to the six faces of a cube, one number to each face. For each of the eight vertices of the cube, a product of three numbers is computed, where the three numbers are the numbers assigned to the three faces that include that vertex. What is the greatest possible value of the sum of these eight products?

$\textbf{(A)}\ 312 \qquad \textbf{(B)}\ 343 \qquad \textbf{(C)}\ 625 \qquad \textbf{(D)}\ 729 \qquad \textbf{(E)}\ 1680$

Solution

First assign each face the letters a, b, c, d, e, f. The sum of the product of the faces is $abc + acd + ade + aeb + fbc + fcd + fde + feb$. We can factor this into $(a+f)(b+c)(d+e)$ which is the product of the sum of each pair of opposite faces. In order to maximize $(a+f)(b+c)(d+e)$ we use the numbers $(7+2)(6+3)(5+4)$ or $\boxed{\textbf{(D)}\ 729}$.

See Also

2016 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2016))	
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2016 AMC 12B Problems/Problem 16

Contents

Problem

In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

$\textbf{(A)}\ 1\qquad\textbf{(B)}\ 3\qquad\textbf{(C)}\ 5\qquad\textbf{(D)}\ 6\qquad\textbf{(E)}\ 7$

Solution

We proceed with this problem by considering two cases, when: 1) There are an odd number of consecutive numbers, 2) There are an even number of consecutive numbers.

For the first case, we can cleverly choose the convenient form of our sequence to be

$$a-n,\cdots,a-1,a,a+1,\cdots,a+n$$

because then our sum will just be $(2n+1)a$. We now have

$$(2n+1)a=345$$

and a will have a solution when $\dfrac{345}{2n+1}$ is an integer, namely when $2n+1$ is a divisor of 345. We check that

$$2n+1=3,5,15,23$$

work, and no more, because $2n+1=1$ does not satisfy the requirements of two or more consecutive integers, and when $2n+1$ equals the next biggest factor, 69, there must be negative integers in the sequence. Our solutions are $\{114,115,116\},\{67,\cdots,71\},\{16,\cdots,30\},\{4,\cdots,26\}$.

For the even cases, we choose our sequence to be of the form:

$$a-(n-1),\cdots,a,a+1,\cdots,a+n$$

so the sum is $\dfrac{(2n)(2a+1)}{2}=n(2a+1)$. In this case, we find our solutions to be $\{172,173\},\{55,\cdots,60\},\{30,\cdots,39\}$.

We have found all 7 solutions and our answer is $\boxed{\textbf{(E)}\ 7}$.

Solution 2

The sum from a to b where a and b are integers and $a>b$ is

$S=\dfrac{(a-b+1)(a+b)}{2}$

$$345 = \frac{(a-b+1)(a+b)}{2}$$

$$2 \cdot 3 \cdot 5 \cdot 23 = (a-b+1)(a+b)$$

Let $c = a - b + 1$ and $d = a + b$

$$2 \cdot 3 \cdot 5 \cdot 23 = c \cdot d$$

If we factor 690 into all of its factor groups $(\text{exg } (10, 69) \text{ or } (15, 46))$ we will have several ordered pairs (c, d) where $c < d$

The number of possible values for c is half the number of factors of 690 which is $\frac{1}{2} \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 8$

However, we have one extraneous case of $(1, 690)$ because here, $a = b$ and we have the sum of one consecutive number which is not allowed by the question.

Thus the answer is $8 - 1 = 7$

$\boxed{\textbf{(E) } 7}$.

See Also

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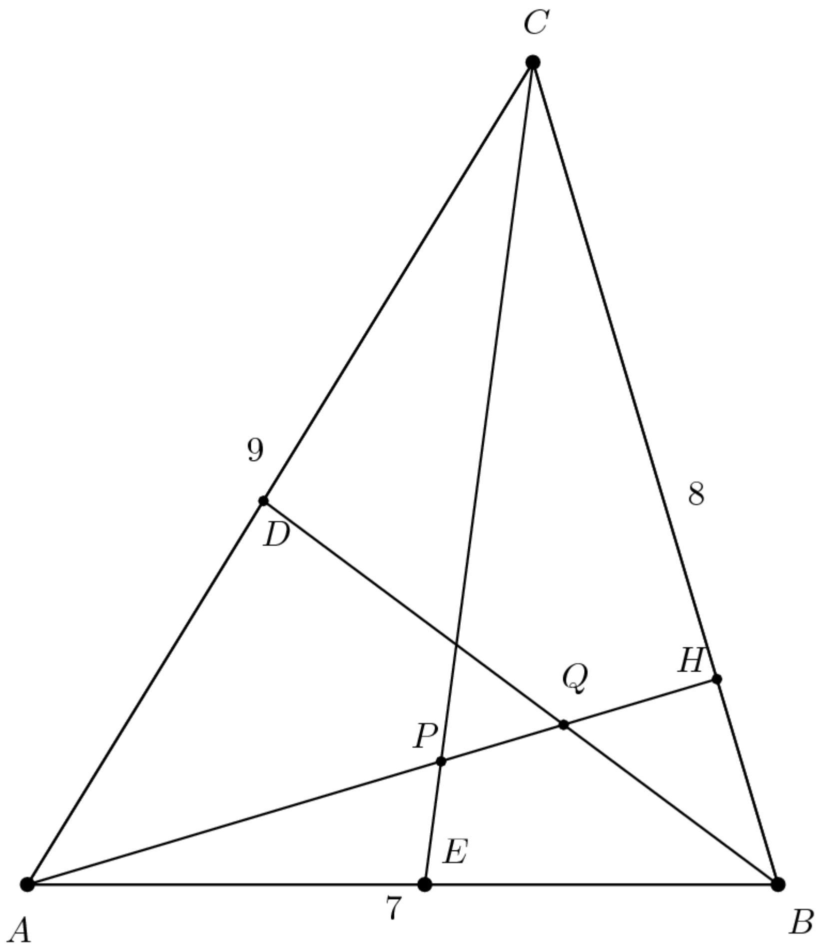
2016 AMC 12B Problems/Problem 17

Problem

In $\triangle ABC$ shown in the figure, $AB = 7$, $BC = 8$, $CA = 9$, and $\overline{AH}$ is an altitude. Points D and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $\overline{BD}$ and $\overline{CE}$ are angle bisectors, intersecting $\overline{AH}$ at Q and P, respectively. What is PQ?



$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ \frac{5}{8}\sqrt{3} \qquad \textbf{(C)}\ \frac{4}{5}\sqrt{2} \qquad \textbf{(D)}\ \frac{8}{15}\sqrt{5} \qquad \textbf{(E)}\ \frac{6}{5}$

Solution

Get the area of the triangle by heron's formula:

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(12)(3)(4)(5)} = 12\sqrt{5}$$

Use the area to find the height AH with known base BC:

$$Area = 12\sqrt{5} = \frac{1}{2}bh = \frac{1}{2}(8)(AH)$$

$$AH = 3\sqrt{5}$$

$$BH = \sqrt{AB^2 - AH^2} = \sqrt{7^2 - (3\sqrt{5})^2} = 2$$

$$CH = BC - BH = 8 - 2 = 6$$

Apply angle bisector theorem on triangle ACH and triangle ABH, we get $AP : PH = 9 : 6$ and $AQ : QH = 7 : 2$, respectively. From now, you can simply use the answer choices because only choice $\mathbf{D}$ has $\sqrt{5}$ in it and we know that $AH = 3\sqrt{5}$ the segments on it all have integral lengths so that $\sqrt{5}$ will remain there. However, by scaling up the length ratio: $AH : AP : PH = 45 : 27 : 18$ and $AQ : QH = 45 : 35 : 10$. we get $AH : PQ = 45 : (18 - 10) = 45 : 8$.

$$PQ = 3\sqrt{5} * \frac{8}{45} = \boxed{\textbf{(D)}\, \frac{8}{15}\sqrt{5}}$$

See Also

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2016 AMC 12B Problems/Problem 18

Problem

What is the area of the region enclosed by the graph of the equation $x^2+y^2=|x|+|y|$?

$\textbf{(A)}\ \pi+\sqrt{2} \qquad \textbf{(B)}\ \pi+2 \qquad \textbf{(C)}\ \pi+2\sqrt{2} \qquad \textbf{(D)}\ 2\pi+\sqrt{2} \qquad \textbf{(E)}\ 2\pi+2\sqrt{2}$

Solution

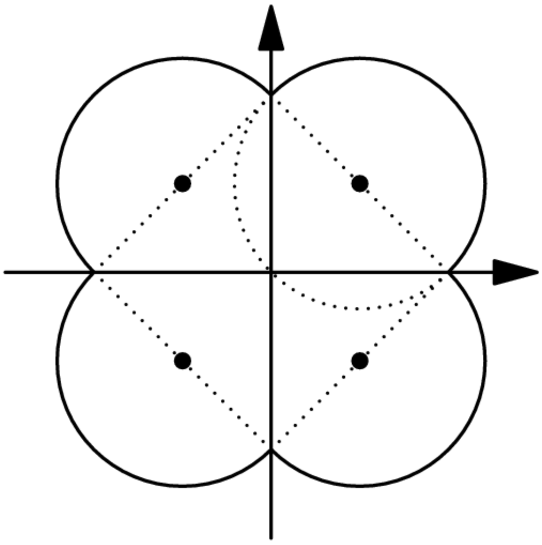
Consider the case when $x > 0$, $y > 0$.

$$x^2+y^2=x+y$$

$$(x-\frac{1}{2})^2+(y-\frac{1}{2})^2=\frac{1}{2}$$

Notice the circle intersect the axes at points $(0,1)$ and $(1,0)$. Find the area of this circle in the first quadrant. The area is made of a semi-circle with radius of $\frac{\sqrt{2}}{2}$ and a triangle:

$$A=\frac{\pi}{4}+\frac{1}{2}$$



Because of symmetry, the area is the same in all four quadrants. The answer is $\boxed{\textbf{(B)}\ \pi+2}$

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2016 AMC 12B Problems/Problem 19

Contents

Problem

Tom, Dick, and Harry are playing a game. Starting at the same time, each of them flips a fair coin repeatedly until he gets his first head, at which point he stops. What is the probability that all three flip their coins the same number of times?

$\textbf{(A)}\ \frac{1}{8} \qquad \textbf{(B)}\ \frac{1}{7} \qquad \textbf{(C)}\ \frac{1}{6} \qquad \textbf{(D)}\ \frac{1}{4} \qquad \textbf{(E)}\ \frac{1}{3}$

Solution 1

By: dragonfly

We can solve this problem by listing it as an infinite geometric equation. We get that to have the same amount of tosses, they have a $\frac{1}{8}$ chance of getting all heads. Then the next probability is all of them getting tails and then on the second try, they all get heads. The probability of that happening is $\left(\frac{1}{8}\right)^2$ .We then get the geometric equation

$$x=\frac{1}{8}+\left(\frac{1}{8}\right)^2+\left(\frac{1}{8}\right)^3...$$

And then we find that x equals to $\boxed{\textbf{(B)}\ \frac{1}{7}}$ because of the formula of the sum for an infinite series,

$$\frac{\frac{1}{8}}{1-\frac{1}{8}}=\frac{1}{8}*\frac{8}{7}=\frac{1}{7}.$$

Solution 2

Call it a "win" if the boys all flip their coins the same number of times, and the probability that they win is P. The probability that they win on their first flip is $\frac{1}{8}$. If they don't win on their first flip, that means they all flipped tails (which also happens with probability $\frac{1}{8}$) and that their chances of winning have returned to what they were at the beginning. This covers all possible sequences of winning flips. So we have

$$P=\frac{1}{8}+\frac{1}{8}P$$

Solving for P gives $\boxed{\textbf{(B)}\ \frac{1}{7}}$.

2016 AMC 12B Problems/Problem 20

Problem

A set of teams held a round-robin tournament in which every team played every other team exactly once. Every team won 10 games and lost 10 games; there were no ties. How many sets of three teams $\{A, B, C\}$ were there in which A beat B, B beat C, and C beat A?

$\textbf{(A)}\ 385 \qquad \textbf{(B)}\ 665 \qquad \textbf{(C)}\ 945 \qquad \textbf{(D)}\ 1140 \qquad \textbf{(E)}\ 1330$

Solution

We use complementary counting. Firstly, because each team played 20 other teams, there are 21 teams total. All sets that do not have A beat B, B beat C, and C beat A have one team that beats both the other teams. Thus we must count the number of sets of three teams such that one team beats the two other teams and subtract that number from the total number of ways to choose three teams.

There are 21 ways to choose the team that beat the two other teams, and $\binom{10}{2} = 45$ to choose two teams that the first team both beat. This is $21 * 45 = 945$ sets. There are $\binom{21}{3} = 1330$ sets of three teams total. Subtracting, we obtain $1330 - 945 = \boxed{A) 385}$ as our answer.

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2016 AMC 12B Problems/Problem 21

Let $ABCD$ be a unit square. Let Q_1 be the midpoint of $\overline{CD}$. For $i = 1, 2, \dots,$ let P_i be the intersection of $\overline{AQ_i}$ and $\overline{BD}$, and let Q_{i+1} be the foot of the perpendicular from P_i to $\overline{CD}$. What is

$$\sum_{i=1}^{\infty} \text{Area of } \triangle DQ_iP_i \,?$$

$\textbf{(A)}\ \frac{1}{6} \qquad \textbf{(B)}\ \frac{1}{4} \qquad \textbf{(C)}\ \frac{1}{3} \qquad \textbf{(D)}\ \frac{1}{2} \qquad \textbf{(E)}\ 1$

Solution

(By Qwertazertl)

We are tasked with finding the sum of the areas of every $\triangle DQ_iP_i$ where i is a positive integer. We can start by finding the area of the first triangle, $\triangle DQ_1P_1$. This is equal to $\frac{1}{2} \cdot DQ_1 \cdot P_1Q_2$. Notice that since triangle $\triangle DQ_1P_1$ is similar to triangle $\triangle ABP_1$ in a 1 : 2 ratio, P_1Q_2 must equal $\frac{1}{3}$ (since we are dealing with a unit square whose side lengths are 1). DQ_1 is of course equal to $\frac{1}{2}$ as it is the mid-point of CD. Thus, the area of the first triangle is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3}$.

The second triangle has a base DQ_2 equal to that of P_1Q_2 (see that $\triangle DQ_2P_1 \sim \triangle DCB$) and using the same similar triangle logic as with the first triangle, we find the area to be $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4}$. If we continue and test the next few triangles, we will find that the sum of all $\triangle DQ_iP_i$ is equal to

$$\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n(n+1)}$$

or

$$\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n}-\frac{1}{n+1}$$

This is known as a telescoping series because we can see that every term after the first $\frac{1}{n}$ is going to cancel out. Thus, the the summation is equal to $\frac{1}{2}$ and after multiplying by the half out in front, we find that the answer is $\boxed{\textbf{(B)}\ \frac{1}{4}}$.

See Also

2016 AMC 12B Problems/Problem 22

Problem

For a certain positive integer n less than 1000, the decimal equivalent of $\frac{1}{n}$ is $0.\overline{abcdef}$, a repeating decimal of period of 6, and the decimal equivalent of $\frac{1}{n+6}$ is $0.\overline{wxyz}$, a repeating decimal of period 4. In which interval does n lie?

$\textbf{(A)}\ [1,200]\qquad\textbf{(B)}\ [201,400]\qquad\textbf{(C)}\ [401,600]\qquad\textbf{(D)}\ [601,800]\qquad\textbf{(E)}\ [801,999]$

Solution

Solution by e_power_pi_times_i

If $\frac{1}{n} = 0.\overline{abcdef}$, n must be a factor of 999999. Also, by the same procedure, $n+6$ must be a factor of 9999. Checking through all the factors of 999999 and 9999 that are less than 1000, we see that $n = 297$ is a solution, so the answer is $\boxed{\textbf{(B)}}$.

Note: $n = 27$ is also a solution, which invalidates this method. However, we need to examine all factors of 999999 that are not factors of 99999, 999, or 99, or 9. Additionally, we need $n+6$ to be a factor of 9999 but not 999, 99, or 9. Indeed, 297 satisfies these requirements.

For anyone who wants more information about repeating decimals, visit: https://en.wikipedia.org/wiki/Repeating_decimal

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2016 AMC 12B Problems/Problem 23

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Problem

What is the volume of the region in three-dimensional space defined by the inequalities $|x|+|y|+|z|\le1$ and $|x|+|y|+|z-1|\le1$

$\textbf{(A)}\ \frac{1}{6}\qquad \textbf{(B)}\ \frac{1}{4}\qquad \textbf{(C)}\ \frac{1}{3}\qquad \textbf{(D)}\ \frac{1}{2}\qquad \textbf{(E)}\ 1$

Solution 1 (Non Calculus)

The first inequality refers to the interior of a regular octahedron with top and bottom vertices $(0, 0, 1),\ (0, 0, -1)$. Its volume is $8\cdot\tfrac{1}{6} = \tfrac{4}{3}$. The second inequality describes an identical shape, shifted $+1$ upwards along the Z axis. The intersection will be a similar octahedron, linearly scaled down by half. Thus the volume of the intersection is one-eighth of the volume of the first octahedron, giving an answer of $\textbf{(A)}\ \tfrac{1}{6}$.

Solution 2 (Calculus)

Let $z \rightarrow z-1/2$, then we can transform the two inequalities to $|x|+|y|+|z-1/2|\le 1$ and $|x|+|y|+|z+1/2|\le 1$. Then it's clear that $-1/2\le z\le 1/2$, consider $0\le z\le 1/2$, $|x|+|y|\le 1/2-z$, then since the area of $|x|+|y|\le k$ is $2k^2$, the volume is $\displaystyle\int_0^{1/2} 2k^2\,dk = \frac{1}{12}$. By symmetry, the case when $\dfrac{-1}{2}\le z\le 0$ is the same. Thus the answer is $\dfrac{1}{6}$.

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2016 AMC 12B Problems/Problem 24

Problem

There are exactly $77,000$ ordered quadruplets (a, b, c, d) such that $\gcd(a, b, c, d) = 77$ and $\operatorname{lcm}(a, b, c, d) = n$. What is the smallest possible value for n?

$\textbf{(A)}\ 13,860 \qquad \textbf{(B)}\ 20,790 \qquad \textbf{(C)}\ 21,560 \qquad \textbf{(D)}\ 27,720 \qquad \textbf{(E)}\ 41,580$

Solution

Let $A = a \div 77,\ B = b \div 77$, etc., so that $\gcd(A, B, C, D) = 1$. Then for each prime power p^k in the prime factorization of $N = n \div 77$, at least one of the prime factorizations of (A, B, C, D) has p^k, at least one has p^0, and all must have p^m with $0 \le m \le k$.

Let $f(k)$ be the number of ordered quadruplets of integers (m_1, m_2, m_3, m_4) such that $0 \le m_i \le k$ for all i, the largest is k, and the smallest is 0. Then for the prime factorization $N = 2^{k_2}3^{k_3}5^{k_5}\dots$ we must have $77000 = f(k_2)f(k_3)f(k_5)\dots$ So let's take a look at the function $f(k)$ by counting the quadruplets we just mentioned..

There are 14 quadruplets which consist only of 0 and k. Then there are $24(k-1)$ quadruplets which include three different values, and $12(k-1)^2$ with four. Thus $f(k) = 14 + 12(2k - 2 + (k-1)^2) = 14 + 12(k^2 - 1)$ and the first few values from $k = 1$ onwards are

$$14, 50, 110, 194, 302, 434, 590, 770, \dots$$

Straight away we notice that $14 \cdot 50 \cdot 110 = 77000$, so the prime factorization of N can use the exponents $1, 2, 3$. To make it as small as possible, assign the larger exponents to smaller primes. The result is $N = 2^3 3^2 5^1 = 360$, so $n = 360 \cdot 77 = 27720$ which is answer $\textbf{(D)}$.

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2016 AMC 12B Problems/Problem 25

Contents

Problem

The sequence (a_n) is defined recursively by $a_0 = 1$, $a_1 = \sqrt[19]{2}$, and $a_n = a_{n-1}a_{n-2}^2$ for $n \geq 2$. What is the smallest positive integer k such that the product $a_1a_2\cdots a_k$ is an integer?

$\textbf{(A)}\ 17\qquad\textbf{(B)}\ 18\qquad\textbf{(C)}\ 19\qquad\textbf{(D)}\ 20\qquad\textbf{(E)}\ 21$

Solution 1

Let $b_i = 19\text{log}_2a_i$. Then $b_0 = 0, b_1 = 1,$ and $b_n = b_{n-1} + 2b_{n-2}$ for all $n \geq 2$. The characteristic polynomial of this linear recurrence is $x^2 - x - 2 = 0$, which has roots 2 and -1.

Therefore, $b_n = k_12^n + k_2(-1)^n$ for constants to be determined k_1, k_2. Using the fact that $b_0 = 0, b_1 = 1,$ we can solve a pair of linear equations for k_1, k_2:

$k_1 + k_2 = 0$ $2k_1 - k_2 = 1$.

Thus $k_1 = \frac{1}{3}$, $k_2 = -\frac{1}{3}$, and $b_n = \frac{2^n - (-1)^n}{3}$.

Now, $a_1a_2\cdots a_k = 2^{\frac{(b_1 + b_2 + \cdots + b_k)}{19}}$, so we are looking for the least value of k so that

$b_1 + b_2 + \cdots + b_k \equiv 0 \pmod{19}$.

Note that we can multiply all b_i by three for convenience, as the b_i are always integers, and it does not affect divisibility by 19.

Now, for all even k the sum (adjusted by a factor of three) is $2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 2$. The smallest k for which this is a multiple of 19 is $k = 18$ by Fermat's Little Theorem, as it is seen with further testing that 2 is a primitive root $\pmod{19}$.

Now, assume k is odd. Then the sum (again adjusted by a factor of three) is $2^1 + 2^2 + \cdots + 2^k + 1 = 2^{k+1} - 1$. The smallest k for which this is a multiple of 19 is $k = 17$, by the same reasons. Thus, the minimal value of k is $\textbf{(A)}\ 17$.

Solution 2

Since the product $a_1a_2\cdots a_k$ is an integer, the sum of the logarithms $\log_2 a_k$ must be an integer. Multiply all of these logarithms by 19, so that the sum must be a multiple of 19. We take these vales modulo 19 to save calculation time. Using the recursion $a_n = a_{n-1}a_{n-2}^2$:

$$a_0 = 0, a_1 = 1 \ldots \implies 0, 1, 1, 3, 5, 11, 2, 5, 9, 0, 18, 18, 16, 14, 8, 17, 14, 10, 0 \ldots$$

Listing the numbers out is expedited if you notice $a_{n+1} = 2a_n + (-1)^n \pmod{19}$. Notice that $a_k + a_{k+9} \equiv 0 \bmod 19$. The cycle repeats every $9 + 9 = 18$ terms. Since $a_0 = 0$ and $a_{18} = 0$, we only need the first 17 terms to sum up to a multiple of 19: $\boxed{\textbf{(A)}\ 17}$. (NOTE: This solution proves 17 is the upper bound, but since 17 is the lowest answer choice, it is correct. To rigorously prove it, you will have to add up the mods listed until you get $0 \pmod{19}$.

$$a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17} = 2^{87381/19} = 2^{4599} \approx 2.735 \cdot 10^{1384}$$

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