2007 AMC 12B Problems/Problem 1

The following problem is from both the 2007 AMC 12B #1 and 2007 AMC 10B #1, so both problems redirect to this page.

Problem

Isabella's house has 3 bedrooms. Each bedroom is 12 feet long, 10 feet wide, and 8 feet high. Isabella must paint the walls of all the bedrooms. Doorways and windows, which will not be painted, occupy 60 square feet in each bedroom. How many square feet of walls must be painted?

$\mathrm{(A)}\ 678 \qquad \mathrm{(B)}\ 768 \qquad \mathrm{(C)}\ 786 \qquad \mathrm{(D)}\ 867 \qquad \mathrm{(E)}\ 876$

Solution

There are four walls in each bedroom, since she can't paint floors or ceilings. So we calculate the number of square feet of wall there is in one bedroom:

$$(12*8)+(12*8)+(10*8)+(10*8)-60 = 160+192-60 = 292$$

We have three bedrooms, so she must paint

$$292*3 = 876 \Rightarrow \boxed{\mathrm{(E)}}$$

square feet of wall.

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by First question	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by First Problem	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

Category: Introductory Geometry Problems

2007 AMC 12B Problems/Problem 2

The following problem is from both the 2007 AMC 12B #2 and 2007 AMC 10B #3, so both problems redirect to this page.

Problem

A college student drove his compact car 120 miles home for the weekend and averaged 30 miles per gallon. On the return trip the student drove his parents' SUV and averaged only 20 miles per gallon. What was the average gas mileage, in miles per gallon, for the round trip?

$\textbf{(A) } 22 \qquad \textbf{(B) } 24 \qquad \textbf{(C) } 25 \qquad \textbf{(D) } 26 \qquad \textbf{(E) } 28$

Solution

The trip was 240 miles long and took $\dfrac{120}{30}+\dfrac{120}{20}=4+6=10$ gallons. Therefore, the average mileage was $\dfrac{240}{10}=\boxed{\text{(B) }24}$

See Also

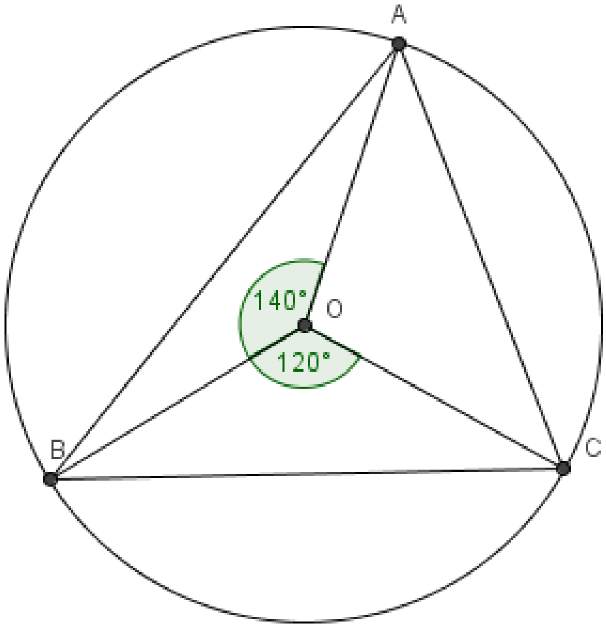
2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2007 AMC 12B Problems/Problem 3

Problem

The point O is the center of the circle circumscribed about triangle ABC, with $\angle BOC = 120^\circ$ and $\angle AOB = 140^\circ$, as shown. What is the degree measure of $\angle ABC$?



$(\mathrm{A})35 \qquad (\mathrm{B})40 \qquad (\mathrm{C})45 \qquad (\mathrm{D})50 \qquad (\mathrm{E})60$

Solution

$\angle AOC = 360 - 140 - 120 = 100 = 2\angle ABC$

$\angle ABC = 50 \Rightarrow \boxed{\mathrm{D}}$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 12B Problems/Problem 4

Problem

At Frank's Fruit Market, 3 bananas cost as much as 2 apples, and 6 apples cost as much as 4 oranges. How many oranges cost as much as 18 bananas?

$(\mathrm{A})6\ (\mathrm{B})8\ (\mathrm{C})9\ (\mathrm{D})12\ (\mathrm{E})18$

Solution

18 bananas cost the same as 12 apples, and 12 apples cost the same as 8 oranges, so 18 bananas cost the same as $8 \Rightarrow (\mathrm{B})$ oranges.

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 12B Problems/Problem 5

Problem

The 2007 AMC 12 contests will be scored by awarding 6 points for each correct response, 0 points for each incorrect response, and 1.5 points for each problem left unanswered. After looking over the 25 problems, Sarah has decided to attempt the first 22 and leave the last 3 unanswered. How many of the first 22 problems must she solve correctly in order to score at least 100 points?

$\mathrm{(A)}\ 13 \qquad \mathrm{(B)}\ 14 \qquad \mathrm{(C)}\ 15 \qquad \mathrm{(D)}\ 16 \qquad \mathrm{(E)} 17$

Solution

She must get at least $100-4.5=95.5$ points, and that can only be possible by answering at least $\lceil\frac{95.5}{6}\rceil=16 \Rightarrow \mathrm{(D)}$ questions correctly.

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

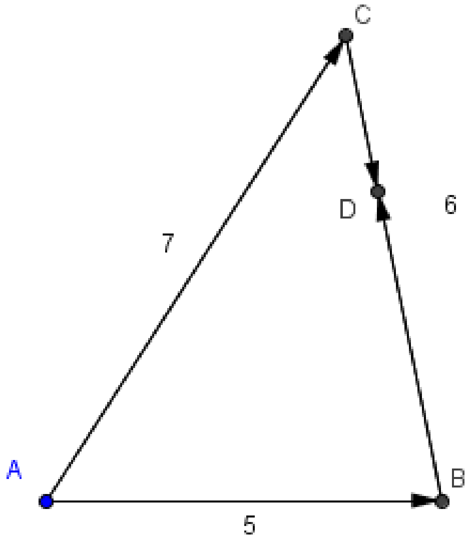
2007 AMC 12B Problems/Problem 6

Problem

Triangle ABC has side lengths $AB = 5$, $BC = 6$, and $AC = 7$. Two bugs start simultaneously from A and crawl along the sides of the triangle in opposite directions at the same speed. They meet at point D. What is BD?

$\mathrm{(A)}\ 1 \qquad \mathrm{(B)}\ 2 \qquad \mathrm{(C)}\ 3 \qquad \mathrm{(D)}\ 4 \qquad \mathrm{(E)}\ 5$

Solution



One bug goes to B. The path that he takes is $\frac{5+6+7}{2} = 9$ units long. The length of BD is $9 - AB = 9 - 5 = 4 \Rightarrow \mathrm{(D)}$

See also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

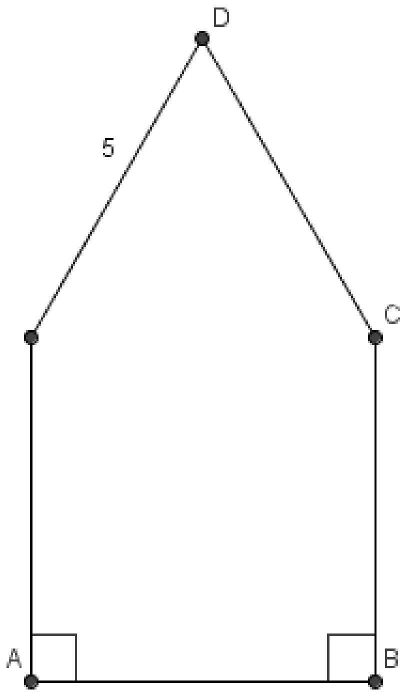
2007 AMC 12B Problems/Problem 7

Problem

All sides of the convex pentagon $ABCDE$ are of equal length, and $\angle A = \angle B = 90^\circ$. What is the degree measure of $\angle E$?

$\mathrm{(A)}\ 90 \qquad \mathrm{(B)}\ 108 \qquad \mathrm{(C)}\ 120 \qquad \mathrm{(D)}\ 144 \qquad \mathrm{(E)}\ 150$

Solution



Since A and B are right angles, and AE equals BC, $AECB$ is a square, and EC is 5. Since ED and CD are also 5, triangle CDE is equilateral. Angle E is therefore $90 + 60 = 150 \Rightarrow \mathrm{(E)}$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2007 AMC 10B Problems/Problem 12

> The following problem is from both the 2007 AMC 12B #8 and 2007 AMC 10B #12, so both problems redirect to this page.

Problem

Tom's age is T years, which is also the sum of the ages of his three children. His age N years ago was twice the sum of their ages then. What is T/N?

$\textbf{(A) } 2 \qquad\textbf{(B) } 3 \qquad\textbf{(C) } 4 \qquad\textbf{(D) } 5 \qquad\textbf{(E) } 6$

Solution

Tom's age N years ago was $T-N$. The ages of his three children N years ago was $T-3N$, since there are three people. If his age N years ago was twice the sum of the children's ages then,

$$\begin{aligned} T-N &= 2(T-3N) \\ T-N &= 2T-6N \\ T &= 5N \\ T/N &= \boxed{\text{(D) } 5} \end{aligned}$$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 7	Followed by Problem 9
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2007 AMC 12B Problems/Problem 9

A function f has the property that $f(3x-1)=x^2+x+1$ for all real numbers x. What is $f(5)$?

$(A) 7\ (B) 13\ (C) 31\ (D) 111\ (E) 211$

Solution

$3x-1=5 \implies x=2$

$f(3(2)-1)=2^2+2+1=7 \implies (A)$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10B Problems/Problem 14

The following problem is from both the 2007 AMC 12B #10 and 2007 AMC 10B #14, so both problems redirect to this page.

Contents

Problem

Some boys and girls are having a car wash to raise money for a class trip to China. Initially 40% of the group are girls. Shortly thereafter two girls leave and two boys arrive, and then 30% of the group are girls. How many girls were initially in the group?

$\mathrm{(A)}\ 4 \qquad\mathrm{(B)}\ 6 \qquad\mathrm{(C)}\ 8 \qquad\mathrm{(D)}\ 10 \qquad\mathrm{(E)}\ 12$

Solution

If we let p be the number of people initially in the group, the $0.4p$ is the number of girls. If two girls leave and two boys arrive, the number of people in the group is still p, but the number of girls is $0.4p-2$. Since only 30% of the group are girls,

$$\begin{align*}\frac{0.4p-2}{p} &= \frac{3}{10}\\ 4p-20 &= 3p\\ p &= 20\end{align*}$$

The number of girls is $0.4p = 0.4(20) = \boxed{(\mathrm{C})\ 8}$

Alternate Solution

There are the same number of total people before and after, but the number of girls has dropped by two and 10%. $\frac{2}{0.1} = 20$, and $40\% \cdot 20 = 8$, so the answer is (C).

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10B Problems/Problem 15

The following problem is from both the 2007 AMC 12B #11 and 2007 AMC 10B #15, so both problems redirect to this page.

Problem

The angles of quadrilateral $ABCD$ satisfy $\angle A = 2\angle B = 3\angle C = 4\angle D.$ What is the degree measure of $\angle A,$ rounded to the nearest whole number?

$\mathrm{(A)}\ 125 \qquad \mathrm{(B)}\ 144 \qquad \mathrm{(C)}\ 153 \qquad \mathrm{(D)}\ 173 \qquad \mathrm{(E)}\ 180$

Solution

The sum of the interior angles of any quadrilateral is $360^\circ.$

$$\begin{aligned} 360 &= \angle A + \angle B + \angle C + \angle D \\ &= \angle A + \frac{1}{2}A + \frac{1}{3}A + \frac{1}{4}A \\ &= \frac{12}{12}A + \frac{6}{12}A + \frac{4}{12}A + \frac{3}{12}A \\ &= \frac{25}{12}A \end{aligned}$$

$$\angle A = 360 \cdot \frac{12}{25} = 172.8 \approx \boxed{\mathrm{(D)}\ 173}$$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2007 AMC 10B Problems/Problem 16

The following problem is from both the 2007 AMC 10B #16 and 2007 AMC 12B #12, so both problems redirect to this page.

Problem

A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was $84.$ The juniors all received the same score, and the average score of the seniors was $83.$ What score did each of the juniors receive on the test?

$\textbf{(A)}\ 85 \qquad \textbf{(B)}\ 88 \qquad \textbf{(C)}\ 93 \qquad \textbf{(D)}\ 94 \qquad \textbf{(E)}\ 98$

Solution

We can assume there are 10 people in the class. Then there will be 1 junior and 9 seniors. The sum of everyone's scores is $10 \cdot 84 = 840.$ Since the average score of the seniors was $83,$ the sum of all the senior's scores is $9 \cdot 83 = 747.$ The only score that has not been added to that is the junior's score, which is $840 - 747 = \boxed{\mathrm{(C)}\ 93}$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 15	Followed by Problem 17
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2007 AMC 12B Problems/Problem 13

Problem 13

A traffic light runs repeatedly through the following cycle: green for 30 seconds, then yellow for 3 seconds, and then red for 30 seconds. Leah picks a random three-second time interval to watch the light. What is the probability that the color changes while she is watching?

$$\mathrm{(A)}\,\frac{1}{63}\qquad\mathrm{(B)}\,\frac{1}{21}\qquad\mathrm{(C)}\,\frac{1}{10}\qquad\mathrm{(D)}\,\frac{1}{7}\qquad\mathrm{(E)}\,\frac{1}{3}$$

Solution

The traffic light runs through a 63 second cycle.

Letting $t=0$ reference the moment it turns green, the light changes at three different times: $t=30$, $t=33$, and $t=63$

This means that the light will change if the beginning of Leah's interval lies in $[27,30]$, $[30,33]$ or $[60,63]$

This gives a total of 9 seconds out of 63

$$\frac{9}{63}=\frac{1}{7}\Rightarrow \mathrm{(D)}$$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 12	Followed by Problem 14
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Combinatorics Problems

2007 AMC 12B Problems/Problem 14

The following problem is from both the 2007 AMC 12B #14 and 2007 AMC 10B #17, so both problems redirect to this page.

Problem 14

Point P is inside equilateral $\triangle ABC$. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB?

$\mathrm{(A)}\ 4 \qquad \mathrm{(B)}\ 3\sqrt{3} \qquad \mathrm{(C)}\ 6 \qquad \mathrm{(D)}\ 4\sqrt{3} \qquad \mathrm{(E)}\ 9$

Solution

Drawing $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$, $\triangle ABC$ is split into three smaller triangles. The altitudes of these triangles are given in the problem as PQ, PR, and PS.

Summing the areas of each of these triangles and equating it to the area of the entire triangle, we get:

$$\frac{s(1)}{2} + \frac{s(2)}{2} + \frac{s(3)}{2} = \frac{s^2\sqrt{3}}{4}$$

where s is the length of a side

$$s = \boxed{\mathrm{(D)}\ 4\sqrt{3}}$$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2007 AMC 12B Problems/Problem 15

Contents

Problem 15

The geometric series $a + ar + ar^2 \ldots$ has a sum of 7, and the terms involving odd powers of r have a sum of 3. What is $a + r$?

$\textbf{(A)}\ \frac{4}{3} \qquad \textbf{(B)}\ \frac{12}{7} \qquad \textbf{(C)}\ \frac{3}{2} \qquad \textbf{(D)}\ \frac{7}{3} \qquad \textbf{(E)}\ \frac{5}{2}$

Solution

Solution 1

The sum of an infinite geometric series is given by $\frac{a}{1-r}$ where a is the first term and r is the common ratio.

In this series, $\frac{a}{1-r} = 7$

The series with odd powers of r is given as

$$ar + ar^3 + ar^5 \ldots$$

It's sum can be given by $\frac{ar}{1-r^2} = 3$

Doing a little algebra

$ar = 3(1-r)(1+r)$

$ar = 3\left(\frac{a}{7}\right)(1+r)$

$\frac{7}{3}r = 1 + r$

$r = \frac{3}{4}$

$a = 7(1-r) = \frac{7}{4}$

$a + r = \frac{5}{2} \Rightarrow \mathrm{(E)}$

Solution 2

The given series can be decomposed as follows:
$(a+ar+ar^2+\ldots)=(a+ar^2+ar^4+\ldots)+(ar+ar^3+ar^5+\ldots)$

Clearly $(a+ar^2+ar^4+\ldots)=(ar+ar^3+ar^5+\ldots)/r=3/r$. We obtain that $7=3/r+3$, hence $r=\frac{3}{4}$.

Then from $7=(a+ar+ar^2+\ldots)=\frac{a}{1-r}$ we get $a=\frac{7}{4}$, and thus $a+r=\frac{5}{2}$.

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 12B Problems/Problem 16

Contents

Problem 16

Each face of a regular tetrahedron is painted either red, white, or blue. Two colorings are considered indistinguishable if two congruent tetrahedra with those colorings can be rotated so that their appearances are identical. How many distinguishable colorings are possible?

$\mathrm{(A)}15\ \mathrm{(B)}18\ \mathrm{(C)}27\ \mathrm{(D)}54\ \mathrm{(E)}81$

Solution

A tetrahedron has 4 sides. The ratio of the number of faces with each color must be one of the following:

$4:0:0$, $3:1:0$, $2:2:0$, or $2:1:1$

The first ratio yields 3 appearances, one of each color.

The second ratio yields $3\cdot 2=6$ appearances, three choices for the first color, and two choices for the second.

The third ratio yields $\binom{3}{2}=3$ appearances since the two colors are interchangeable.

The fourth ratio yields 3 appearances. There are three choices for the first color, and since the second two colors are interchangeable, there is only one distinguishable pair that fits them.

The total is $3+6+3+3=15$ appearances $\Rightarrow \mathrm{(A)}$

Solution 2

Every colouring can be represented in the form (w,r,b), where w is the number of white faces, r is the number of red faces, and b is the number of blue faces. Every distinguishable colouring pattern can be represented like this in exactly one way, and every ordered whole number triple with a total sum of 4 represents exactly one colouring pattern (if two tetrahedra have rearranged colours on their faces, it is always possible to rotate one so that it matches the other).

Therefore, the number of colourings is equal to the number of ways 3 distinguishable nonnegative integers can add to 4. If you have 6 cockroaches in a row, this number is equal to the number of ways to pick two of the cockroaches to eat for dinner (because the remaining cockroaches in between are separated in to three sections with a non-negative number of cockroaches each), which is $\binom{6}{2}=15$

See Also

2007 AMC 12B Problems/Problem 17

Problem 17

If a is a nonzero integer and b is a positive number such that $ab^2 = \log_{10} b$, what is the median of the set $\{0, 1, a, b, 1/b\}$?

$$\text{(A)}0 \ \text{(B)}1 \ \text{(C)}a \ \text{(D)}b \ \text{(E)}\frac{1}{b}$$

Solution

Note that if a is positive, then, the equation will have no solutions for b. This becomes more obvious by noting that at $b = 1$, $ab^2 > \log_{10} b$. The LHS quadratic function will increase faster than the RHS logarithmic function, so they will never intersect.

This puts a as the smallest in the set since it must be negative.

Checking the new equation: $-b^2 = \log_{10} b$

Near $b = 0$, $-b^2 > \log_{10} b$ but at $b = 1$, $-b^2 < \log_{10} b$

This implies that the solution occurs somewhere in between: $0 < b < 1$

This also implies that $\frac{1}{b} > 1$

This makes our set (ordered) $\{a, 0, b, 1, 1/b\}$

The median is $b \Rightarrow \text{(D)}$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 12B Problems/Problem 18

Problem 18

Let a, b, and c be digits with $a \neq 0$. The three-digit integer abc lies one third of the way from the square of a positive integer to the square of the next larger integer. The integer acb lies two thirds of the way between the same two squares. What is $a+b+c$?

$\mathrm{(A)}\ 10\qquad \mathrm{(B)}\ 13\qquad \mathrm{(C)}\ 16\qquad \mathrm{(D)}\ 18\qquad \mathrm{(E)}\ 21$

Solution

The difference between acb and abc is given by

$$(100a+10c+b)-(100a+10b+c)=9(c-b)$$

The difference between the two squares is three times this amount or

$$27(c-b)$$

The difference between two consecutive squares is always an odd number, therefore $c-b$ is odd. We will show that $c-b$ must be 1. Otherwise we would be looking for two consecutive squares that are at least 81 apart. But already the equation $(x+1)^2-x^2=27\cdot 3$ solves to $x=40$, and 40^2 has more than three digits.

The consecutive squares with common difference 27 are $13^2=169$ and $14^2=196$. One third of the way between them is 178 and two thirds of the way is 187.

This gives $a=1$, $b=7$, $c=8$.

$$a+b+c=16\Rightarrow \mathrm{(C)}$$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

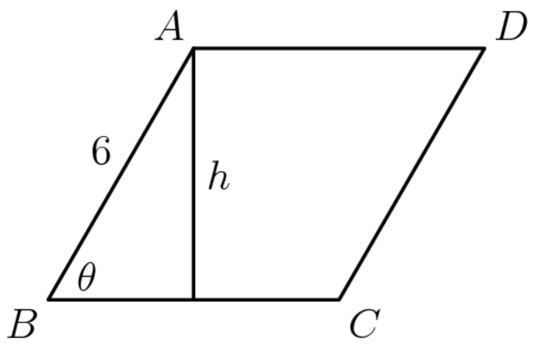
2007 AMC 12B Problems/Problem 19

Problem 19

Rhombus $ABCD$, with side length 6, is rolled to form a cylinder of volume 6 by taping $\overline{AB}$ to $\overline{DC}$. What is $\sin(\angle ABC)$?

$\mathrm{(A)}\ \frac{\pi}{9} \qquad \mathrm{(B)}\ \frac{1}{2} \qquad \mathrm{(C)}\ \frac{\pi}{6} \qquad \mathrm{(D)}\ \frac{\pi}{4} \qquad \mathrm{(E)}\ \frac{\sqrt{3}}{2}$

Solution



$$V_{Cylinder} = \pi r^2 h$$

Where $C = 2\pi r = 6$ and $h = 6\sin\theta$

$$r = \frac{3}{\pi}$$

$$V = \pi\left(\frac{3}{\pi}\right)^2 \cdot 6\sin\theta$$

$$6 = \frac{9}{\pi} \cdot 6\sin\theta$$

$$\sin\theta = \frac{\pi}{9} \Rightarrow \mathrm{(A)}$$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 12B Problems/Problem 20

Contents

Problem

The parallelogram bounded by the lines $y = ax + c$, $y = ax + d$, $y = bx + c$, and $y = bx + d$ has area 18. The parallelogram bounded by the lines $y = ax + c$, $y = ax - d$, $y = bx + c$, and $y = bx - d$ has area 72. Given that a, b, c, and d are positive integers, what is the smallest possible value of $a + b + c + d$?

$\mathrm{(A)}\ 13\qquad\mathrm{(B)}\ 14\qquad\mathrm{(C)}\ 15\qquad\mathrm{(D)}\ 16\qquad\mathrm{(E)}\ 17$

Solution

This article is a stub. Help us out by expanding it (https://artofproblemsolving.com/wiki/index.php?title=2007_AMC_12B_Problems/Problem_20&action=edit).

Plotting the parallelogram on the coordinate plane, the 4 corners are at $(0,c),(0,d),\left(\frac{d-c}{a-b},\frac{ad-bc}{a-b}\right),\left(\frac{c-d}{a-b},\frac{bc-ad}{a-b}\right)$. Because $72= 4 \cdot 18$, we have that $4(c-d)\left(\frac{c-d}{a-b}\right) = (c+d)\left(\frac{c+d}{a-b}\right)$ or that $2(c-d)=c+d$, which gives $c=3d$ (consider a homothety, or dilation, that carries the first parallelogram to the second parallelogram; because the area increases by $4\times$, it follows that the stretch along the diagonal, or the ratio of side lengths, is $2\times$). The area of triangular half of the parallelogram on the right side of the y-axis is given by $9 = \frac{1}{2} (c-d) \left(\frac{d-c}{a-b}\right)$, so substituting $c = 3d$:

$$\frac{1}{2} (c-d) \left(\frac{c-d}{a-b}\right) = 9 \quad \Longrightarrow \quad 2d^2 = 9(a-b)$$

Thus $3|d$, and we verify that $d = 3$, $a - b = 2 \Longrightarrow a = 3, b = 1$ will give us a minimum value for $a+b+c+d$. Then $a+b+c+d = 3 + 1 + 9 + 3 = \boxed{\mathrm{(D)}16}$.

Solution 2

This article is a stub. Help us out by expanding it (https://artofproblemsolving.com/wiki/index.php?title=2007_AMC_12B_Problems/Problem_20&action=edit).

The key to this solution is that area is invariant under translation. By suitably shifting the plane, the problem is mapped to the lines $c, d, (b-a)x + c, (b-a)x + d$ and $c, -d, (b-a)x + c, (b-a)x - d$. Now, the area of the parallelogram contained by is the former is equal to the area of a rectangle with sides $d-c$ and $\frac{d-c}{b-a}$, $\frac{(d-c)^2}{b-a} = 18$, and the area contained by the latter is $\frac{(c+d)^2}{b-a} = 72$. Thus, $d = 3c$ and $b - a$ must be even if the former quantity is to equal 18. $c^2 = 18(b-a)$ so c is a multiple of 3. Putting this all together, the minimal solution for $(a, b, c, d) = (3, 1, 3, 9)$, so the sum is $\boxed{\textbf{(D)} 16}$.

See also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 12B Problems/Problem 21

Problem 21

The first 2007 positive integers are each written in base 3. How many of these base-3 representations are palindromes? (A palindrome is a number that reads the same forward and backward.)

$\mathrm{(A)}\ 100 \qquad \mathrm{(B)}\ 101 \qquad \mathrm{(C)}\ 102 \qquad \mathrm{(D)}\ 103 \qquad \mathrm{(E)}\ 104$

Solution

$2007_{10} = 2202100_{3}$

All numbers of six or less digits in base 3 have been written.

The form of each palindrome is as follows

1 digit - a

2 digits - aa

3 digits - aba

4 digits - $abba$

5 digits - $abcba$

6 digits - $abccba$

Where a,b,c are base 3 digits

Since $a\neq 0$, this gives a total of $2+2+2\cdot3+2\cdot3+2\cdot3^2+2\cdot3^2 = 52$ palindromes so far.

7 digits - $abcdcba$, but not all of the numbers are less than 2202100_{3}

Case: $a=1$

All of these numbers are less than 2202100_{3} giving 3^3 more palindromes

Case: $a=2$, $b\neq 2$

All of these numbers are also small enough, giving $2\cdot3^2$ more palindromes

Case: $a=2$, $b=2$

It follows that $c=0$, since any other c would make the value too large. This leaves the number as $220d022_{3}$. Checking each value of d, all of the three are small enough, so that gives 3 more palindromes.

Summing our cases there are $52+3^3+2\cdot3^2+3 = 100 \Rightarrow \mathrm{(A)}$

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 20	Followed by Problem 22
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 12B Problems/Problem 22

Problem 22

Two particles move along the edges of equilateral $\triangle ABC$ in the direction

$$A \Rightarrow B \Rightarrow C \Rightarrow A,$$

starting simultaneously and moving at the same speed. One starts at A, and the other starts at the midpoint of $\overline{BC}$. The midpoint of the line segment joining the two particles traces out a path that encloses a region R. What is the ratio of the area of R to the area of $\triangle ABC$?

$\mathrm{(A)}\frac{1}{16} \qquad \mathrm{(B)}\frac{1}{12} \qquad \mathrm{(C)}\frac{1}{9} \qquad \mathrm{(D)}\frac{1}{6} \qquad \mathrm{(E)}\frac{1}{4}$

Solution

First, notice that each of the midpoints of AB, BC, and CA are on the locus. Suppose after some time the particles have each been displaced by a short distance x, to new positions A' and M' respectively. Consider $\triangle ABM$ and drop a perpendicular from M' to hit AB at Y. Then, $BA' = 1 - x$ and $BM' = 1/2 + x$. From here, we can use properties of a $30 - 60 - 90$ triangle to determine the lengths YA' and YM' as monomials in x. Thus, the locus of the midpoint will be linear between each of the three special points mentioned above. It follows that the locus consists of the only triangle with those three points as vertices. Comparing inradii between this "midpoint" triangle and the original triangle, the area contained by R must be $\mathrm{(A)}\frac{1}{16}$ of the total area.

See Also

2007 AMC 12B Problems/Problem 23

Contents

Problem 23

How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

$$\mathrm{(A)} 6\qquad \mathrm{(B)} 7\qquad \mathrm{(C)} 8\qquad \mathrm{(D)} 10\qquad \mathrm{(E)} 12$$

Solution

Let a and b be the two legs of the triangle.

We have $\frac{1}{2}ab = 3(a+b+c)$.

Then $ab=6\left(a+b+\sqrt{a^2+b^2}\right)$.

We can complete the square under the root, and we get, $ab=6\left(a+b+\sqrt{(a+b)^2-2ab}\right)$.

Let $ab=p$ and $a+b=s$, we have $p=6\left(s+\sqrt{s^2-2p}\right)$.

After rearranging, squaring both sides, and simplifying, we have $p=12s-72$.

Putting back a and b, and after factoring using $SFFT$, we've got $(a-12)(b-12)=72$.

Factoring 72, we get 6 pairs of a and b

$$(13,84),(14,48),(15,36),(16,30),(18,24),(20,21).$$

And this gives us 6 solutions $\Rightarrow \mathrm{(A)}$.

Alternatively, note that $72=2^3\cdot 3^2$. Then 72 has $(3+1)(2+1)=(4)(3)=12$ factors. However, half of these are repeats, so there we have $\frac{12}{2}=6$ solutions.

Solution #2

We will proceed by using the fact that $[ABC]=r\cdot s$, where r is the radius of the incircle and s is the semiperimeter $\left(s=\frac{p}{2}\right)$.

We are given $[ABC]=3p=6s\Rightarrow rs=6s\Rightarrow r=6$.

The incircle of ABC breaks the triangle's sides into segments such that $AB = x + y$, $BC = x + z$ and $AC = y + z$. Since ABC is a right triangle, one of x, y and z is equal to its radius, 6. Let's assume $z = 6$.

The side lengths then become $AB = x + y$, $BC = x + 6$ and $AC = y + 6$. Plugging into Pythagorean's theorem:

$$(x + y)^2 = (x + 6)^2 + (y + 6)^2$$

$$x^2 + 2xy + y^2 = x^2 + 12x + 36 + y^2 + 12y + 36$$

$$2xy - 12x - 12y = 72$$

$$xy - 6x - 6y = 36$$

$$(x - 6)(y - 6) - 36 = 36$$

$$(x - 6)(y - 6) = 72$$

We can factor 72 to arrive with 6 pairs of solutions: $(7, 78), (8, 42), (9, 30), (10, 24), (12, 18)$, and $(14, 15) \Rightarrow \text{(A)}$.

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 12B Problems/Problem 24

Contents

Problem 24

How many pairs of positive integers (a, b) are there such that $\gcd(a, b) = 1$ and

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer?

$$\mathrm{(A)}\, 4\quad \mathrm{(B)}\, 6\quad \mathrm{(C)}\, 9\quad \mathrm{(D)}\, 12\quad \mathrm{(E)}\, \text{infinitely many}$$

Solution

Combining the fraction, $\frac{9a^2 + 14b^2}{9ab}$ must be an integer.

Since the denominator contains a factor of 9, $9 | 9a^2 + 14b^2 \quad\implies\quad 9 | b^2 \quad\implies\quad 3 | b$

Since $b = 3n$ for some positive integer n, we can rewrite the fraction(divide by 9 on both top and bottom) as $\frac{a^2 + 14n^2}{3an}$

Since the denominator now contains a factor of n, we get $n | a^2 + 14n^2 \quad\implies\quad n | a^2$.

But since $1 = gcd(a, b) = gcd(a, 3n) = gcd(a, n)$, we must have $n = 1$, and thus $b = 3$.

For $b = 3$ the original fraction simplifies to $\frac{a^2 + 14}{3a}$.

For that to be an integer, a must be a factor of 14, and therefore we must have $a \in \{1, 2, 7, 14\}$. Each of these values does indeed yield an integer.

Thus there are four solutions: $(1, 3)$, $(2, 3)$, $(7, 3)$, $(14, 3)$ and the answer is $\mathrm{(A)}$

Solution 2

Let's assume that $\frac{a}{b} + \frac{14b}{9a} = m$ We get--

$$9a^2 - 9mab + 14b^2 = 0$$

Factoring this, we get 4 equations-

$$(3a - 2b)(3a - 7b) = 0$$

$$(3a - b)(3a - 14b) = 0$$

$(a-2b)(9a-7b)=0$

$(a-b)(9a-14b)=0$

(It's all negative, because if we had positive signs, a would be the opposite sign of b)

Now we look at these, and see that-

$3a=2b$

$3a=b$

$3a=7b$

$3a=14b$

$a=2b$

$9a=7b$

$a=b$

$9a=14b$

This gives us 8 solutions, but we note that the middle term needs to give you back $9m$.

For example, in the case

$(a-2b)(9a-7b)$, the middle term is $-25ab$, which is not equal by $-9m$ for whatever integer m.

Similar reason for the fourth equation. This elimnates the last four solutions out of the above eight listed, giving us 4 solutions total (A)

Solution 3

Let $u=\frac{a}{b}$. Then the given equation becomes $u+\frac{14}{9u}=\frac{9u^2+14}{9u}$.

Let's set this equal to some value, $k \Rightarrow \frac{9u^2+14}{9u}=k$.

Clearing the denominator and simplifying, we get a quadratic in terms of u:

$$9u^2-9ku+14=0 \Rightarrow u=\frac{9k \pm \sqrt{(9k)^2-504}}{18}$$

Since a and b are integers, u is a rational number. This means that $\sqrt{(9k)^2-504}$ is an integer.

Let $\sqrt{(9k)^2-504}=x$. Squaring and rearranging yields:

$(9k)^2-x^2=504$

$(9k+x)(9k-x)=504.$

In order for both x and a to be an integer, $9k+x$ and $9k-x$ must both be odd or even. (This is easily proven using modular arithmetic.) In the case of this problem, both must be even. Let $9k+x=2m$ and $9k-x=2n$.

Then:

$2m \cdot 2n=504$

$mn=126.$

Factoring 126, we get 6 pairs of numbers: $(1,126),(2,63),(3,42),(6,21),(7,18)$, and $(9,14)$.

Looking back at our equations for m and n, we can solve for $k=\frac{2m+2n}{18}=\frac{m+n}{9}$. Since k is an integer, there are only 2 pairs of (m,n) that work: $(3,42)$ and $(6,21)$. This means that there are 2 values of k such that u is an integer. But looking back at u in terms of k, we have $\pm$, meaning that there are 2 values of u for every k. Thus, the answer is $2\cdot 2=4 \Rightarrow \mathrm{(A)}$.

Solution 4

Rewriting the expression over a common denominator yields $\frac{9a^2+14b^2}{9ab}$. This expression must be equal to some integer m.

Thus, $\frac{9a^2+14b^2}{9ab}=m \to 9a^2+14b^2=9abm$. Taking this $\pmod{a}$ yields $14b^2\equiv 0 \pmod{a}$. Since $\gcd(a,b)=1$, $14\equiv 0 \pmod{a}$. This implies that $a|14$ so $a=1,2,7,14$.

We can then take $9a^2+14b^2=9abm \pmod{b}$ to get that $9\equiv 0 \pmod{b}$. Thus $b=1,3,9$.

However, taking $9a^2+14b^2=9abm \pmod{3}$, $b^2\equiv 0 \pmod{3}$ so b cannot equal 1.

Also, note that if $b=9$, $\frac{a}{b}+\frac{14b}{9a}=\frac{a}{9}+\frac{14}{a}$. Since $a|14$, $\frac{14}{a}$ will be an integer, but $\frac{a}{9}$ will not be an integer since none of the possible values of a are multiples of 9. Thus, b cannot equal 9.

Thus, the only possible values of b is 3, and a can be 1, 2, 7, or 14. This yields 4 possible solutions, so the answer is $\mathrm{(A)}$.

See Also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 23	Followed by Problem 25
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2007 AMC 12B Problems/Problem 25

Problem

Points A, B, C, D and E are located in 3-dimensional space with $AB = BC = CD = DE = EA = 2$ and $\angle ABC = \angle CDE = \angle DEA = 90^o$. The plane of $\triangle ABC$ is parallel to $\overline{DE}$. What is the area of $\triangle BDE$?

$(\mathrm{A})\sqrt{2} \qquad (\mathrm{B})\sqrt{3} \qquad (\mathrm{C})2 \qquad (\mathrm{D})\sqrt{5} \qquad (\mathrm{E})\sqrt{6}$

Solution

Let $A = (0,0,0)$, and $B = (2,0,0)$. Since $EA = 2$, we could let $C = (2,0,2)$, $D = (2,2,2)$, and $E = (2,2,0)$. Now to get back to A we need another vertex $F = (0,2,0)$. Now if we look at this configuration as if it was two dimensions, we would see a square missing a side if we don't draw FA. Now we can bend these three sides into an equilateral triangle, and the coordinates change: $A = (0,0,0)$, $B = (2,0,0)$, $C = (2,0,2)$, $D = (1,\sqrt{3},2)$, and $E = (1,\sqrt{3},0)$. Checking for all the requirements, they are all satisfied. Now we find the area of triangle BDE. It is a $2-2-2\sqrt{2}$ triangle, which is an isosceles right triangle. Thus the area of it is $\frac{2\cdot 2}{2} = 2 \Rightarrow (\mathrm{C})$.

See also

2007 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 24	Followed by Last Problem
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Categories: Introductory Geometry Problems | Area Problems