2010 AMC 12B Problems/Problem 1

Problem

Makarla attended two meetings during her 9-hour work day. The first meeting took 45 minutes and the second meeting took twice as long. What percent of her work day was spent attending meetings?

$\textbf{(A)}\ 15 \qquad \textbf{(B)}\ 20 \qquad \textbf{(C)}\ 25 \qquad \textbf{(D)}\ 30 \qquad \textbf{(E)}\ 35$

Solution

The total number of minutes in here 9-hour work day is $9 \times 60 = 540.$ The total amount of time spend in meetings in minutes is $45 + 45 \times 2 = 135.$ The answer is then $\frac{135}{540} = \boxed{25\%}$ or $\boxed{(C)}$

See also

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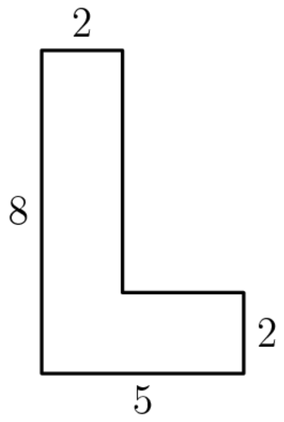
2010 AMC 12B Problems/Problem 2

Problem 2

A big L is formed as shown. What is its area?



$\textbf{(A)}\ 22 \qquad \textbf{(B)}\ 24 \qquad \textbf{(C)}\ 26 \qquad \textbf{(D)}\ 28 \qquad \textbf{(E)}\ 30$

Solution

We find the area of the big rectangle to be $8 \times 5 = 40$, and the area of the smaller rectangle to be $(8-2) \times (5-2) = 18$. The answer is then $40 - 18 = 22\ (A)$.

See also

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Categories: Introductory Geometry Problems | Area Problems

2010 AMC 12B Problems/Problem 3

Problem 3

A ticket to a school play cost x dollars, where x is a whole number. A group of 9$_{\text{th}}$ graders buys tickets costing a total of \$$48$, and a group of 10$_{\text{th}}$ graders buys tickets costing a total of \$$64$. How many values for x are possible?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution

We find the greatest common factor of 48 and 64 to be 16. The number of factors of 16 is 5 which is the answer (E).

See also

2010 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12B Problems/Problem 4

Problem 4

A month with 31 days has the same number of Mondays and Wednesdays.How many of the seven days of the week could be the first day of this month?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 6$

Solution

$31 \equiv 3 \pmod 7$ so the week cannot start with Saturday, Sunday, Tuesday or Wednesday as that would result in an unequal number of Mondays and Wednesdays. Therefore, Monday, Thursday, and Friday are valid so the answer is $\boxed{B}$.

See also

2010 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12B Problems/Problem 5

Problem 5

Lucky Larry's teacher asked him to substitute numbers for a, b, c, d, and e in the expression $a-(b-(c-(d+e)))$ and evaluate the result. Larry ignored the parenthese but added and subtracted correctly and obtained the correct result by coincidence. The number Larry substituted for a, b, c, and d were 1, 2, 3, and 4, respectively. What number did Larry substitute for e?

$\textbf{(A)}\ -5 \qquad \textbf{(B)}\ -3 \qquad \textbf{(C)}\ 0 \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 5$

Solution

We simply plug in the numbers

$$1-2-3-4+e=1-(2-(3-(4+e)))$$

$$-8+e=-2-e$$

$$2e=6$$

$$e=3\ (D)$$

See also

2010 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12B Problems/Problem 6

Problem 6

At the beginning of the school year, 50% of all students in Mr. Wells' math class answered "Yes" to the question "Do you love math", and 50% answered "No." At the end of the school year, 70% answered "Yes" and 30% answered "No." Altogether, $x\%$ of the students gave a different answer at the beginning and end of the school year. What is the difference between the maximum and the minimum possible values of x?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 20 \qquad \textbf{(C)}\ 40 \qquad \textbf{(D)}\ 60 \qquad \textbf{(E)}\ 80$

Solution

Clearly, the minimum possible value would be $70 - 50 = 20\%$. The maximum possible value would be $30 + 50 = 80\%$. The difference is $80 - 20 = \boxed{60}\ (D)$.

See also

2010 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12B Problems/Problem 7

Problem 7

Shelby drives her scooter at a speed of 30 miles per hour if it is not raining, and 20 miles per hour if it is raining. Today she drove in the sun in the morning and in the rain in the evening, for a total of 16 miles in 40 minutes. How many minutes did she drive in the rain?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 21 \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 27 \qquad \textbf{(E)}\ 30$

Solution

Let x be the time it is not raining, and y be the time it is raining, in hours.

We have the system: $30x + 20y = 16$ and $x + y = 2/3$

Solving gives $x = \frac{4}{15}$ and $y = \frac{2}{5}$

We want y in minutes, $\frac{2}{5} * 60 = 24 \Rightarrow C$

See also

2010 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12B Problems/Problem 8

Problem 8

Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37^{th} and 64^{th}, respectively. How many schools are in the city?

$\textbf{(A)}\ 22 \qquad \textbf{(B)}\ 23 \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 25 \qquad \textbf{(E)}\ 26$

Solution

There are x schools. This means that there are $3x$ people. Because no one's score was the same as another person's score, that means that there could only have been 1 median score. This implies that x is an odd number. x cannot be less than 23, because there wouldn't be a 64th place if there were. x cannot be greater than 23 either, because that would tie Andrea and Beth or Andrea's place would be worse than Beth's. Thus, the only possible answer is $23 \Rightarrow \boxed{B}$.

See also

2010 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12B Problems/Problem 9

Problem 9

Let n be the smallest positive integer such that n is divisible by 20, n^2 is a perfect cube, and n^3 is a perfect square. What is the number of digits of n?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 7$

Solution

We know that $n^2 = k^3$ and $n^3 = m^2$. Cubing and squaring the equalities respectively gives $n^6 = k^9 = m^4$. Let $a = n^6$. Now we know a must be a perfect 36-th power because $lcm(9, 4) = 36$, which means that n must be a perfect 6-th power. The smallest number whose sixth power is a multiple of 20 is 10, because the only prime factors of 20 are 2 and 5, and $10 = 2 \times 5$. Therefore our is equal to number $10^6 = 1000000$, with 7 digits $\Rightarrow \boxed{E}$.

See also

2010 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12B Problems/Problem 10

Problem 10

The average of the numbers $1, 2, 3, \cdots, 98, 99,$ and x is $100x$. What is x?

$\textbf{(A)}\ \frac{49}{101} \qquad \textbf{(B)}\ \frac{50}{101} \qquad \textbf{(C)}\ \frac{1}{2} \qquad \textbf{(D)}\ \frac{51}{101} \qquad \textbf{(E)}\ \frac{50}{99}$

Solution

We first sum the first 99 numbers: $\frac{99(100)}{2} = 99 \times 50 = 4,950$. Then, we know that the sum of the series is $4,950 + x$. Since the average is $100x$, and there are 100 terms, we also find the sum to equal $10,000x$. Setting equal - $10,000x = 4,\ 950 + x \Rightarrow 9,999x = 4,\ 950 \Rightarrow x = \frac{4,950}{9,999} \Rightarrow x = \frac{50}{101}$. Thus, the answer is $\boxed{\text{B}}$.

See also

2010 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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2010 AMC 12B Problems/Problem 11

Problem 11

A palindrome between 1000 and $10,000$ is chosen at random. What is the probability that it is divisible by 7?

$\textbf{(A)}\ \dfrac{1}{10} \qquad \textbf{(B)}\ \dfrac{1}{9} \qquad \textbf{(C)}\ \dfrac{1}{7} \qquad \textbf{(D)}\ \dfrac{1}{6} \qquad \textbf{(E)}\ \dfrac{1}{5}$

Solution

View the palindrome as some number with form (decimal representation): $a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$. But because the number is a palindrome, $a_3 = a_0, a_2 = a_1$. Recombining this yields $1001a_3 + 110a_2$. 1001 is divisible by 7, which means that as long as $a_2 = 0$, the palindrome will be divisible by 7. This yields 9 palindromes out of 90 $(9 \cdot 10)$ possibilities for palindromes. However, if $a_2 = 7$, then this gives another case in which the palindrome is divisible by 7. This adds another 9 palindromes to the list, bringing our total to $18/90 = \boxed{\frac{1}{5}} = \boxed{E}$

See also

2010 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2010))	
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Category: Introductory Combinatorics Problems

2010 AMC 12B Problems/Problem 12

Contents

Problem 12

For what value of x does

$$\log_{\sqrt{2}}\sqrt{x}+\log_{2}x+\log_{4}x^2+\log_{8}x^3+\log_{16}x^4=40?$$

$\textbf{(A) } 8 \qquad \textbf{(B) } 16 \qquad \textbf{(C) } 32 \qquad \textbf{(D) } 256 \qquad \textbf{(E) } 1024$

Solution 1

$$\log_{\sqrt{2}}\sqrt{x}+\log_{2}x+\log_{4}(x^2)+\log_{8}(x^3)+\log_{16}(x^4)=40$$

$$\frac{1}{2}\frac{\log_{2}x}{\log_{2}\sqrt{2}}+\log_{2}x+\frac{2\log_{2}x}{\log_{2}4}+\frac{3\log_{2}x}{\log_{2}8}+\frac{4\log_{2}x}{\log_{2}16}=40$$

$$\log_{2}x+\log_{2}x+\log_{2}x+\log_{2}x+\log_{2}x=40$$

$$5\log_{2}x=40$$

$$\log_{2}x=8$$

$$x=256 \ \ (D)$$

Solution 2

Using the fact that $\log_{x^n}y^n=\log_x y$, we see that the equation becomes $\log_2 x+\log_2 x+\log_2 x+\log_2 x+\log_2 x=40$. Thus, $5\log_2 x=40$ and $\log_2 x=8$, so $x=2^8=256$, or $\boxed{(D)}$.

See also

2010 AMC 12B Problems/Problem 13

Problem

In $\triangle ABC$, $\cos(2A-B)+\sin(A+B)=2$ and $AB=4$. What is BC?

$\textbf{(A)}\ \sqrt{2} \qquad \textbf{(B)}\ \sqrt{3} \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 2\sqrt{2} \qquad \textbf{(E)}\ 2\sqrt{3}$

Solution

We note that $-1\le \sin x\le 1$ and $-1\le \cos x\le 1$. Therefore, there is no other way to satisfy this equation other than making both $\cos(2A-B)=1$ and $\sin(A+B)=1$, since any other way would cause one of these values to become greater than 1, which contradicts our previous statement. From this we can easily conclude that $2A-B=0^\circ$ and $A+B=90^\circ$ and solving this system gives us $A=30^\circ$ and $B=60^\circ$. It is clear that $\triangle ABC$ is a $30^\circ-60^\circ-90^\circ$ triangle with $BC=2\implies (C)$.

See also

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Category: Introductory Geometry Problems

2010 AMC 12B Problems/Problem 14

Contents

Problem 14

Let a, b, c, d, and e be positive integers with $a+b+c+d+e=2010$ and let M be the largest of the sum $a+b$, $b+c$, $c+d$ and $d+e$. What is the smallest possible value of M?

$\textbf{(A)}\ 670 \qquad \textbf{(B)}\ 671 \qquad \textbf{(C)}\ 802 \qquad \textbf{(D)}\ 803 \qquad \textbf{(E)}\ 804$

Solution

We want to try make $a+b$, $b+c$, $c+d$, and $d+e$ as close as possible so that M, the maximum of these, if smallest.

Notice that $2010 = 670+670+670$. In order to express 2010 as a sum of 5 numbers, we must split up some of these numbers. There are two ways to do this (while keeping the sum of two numbers as close as possible): $2010 = 670+1+670+1+668$ or $2010 = 670+1+669+1+669$. We see that in both cases, the value of M is 671, so the answer is $671 \Rightarrow \boxed{B}$.

Solution 2

First, note that, simply by pigeonhole, at least one of a, b, c, d, e is greater than or equal to $\frac{2010}{5} = 402$, so none of C, D, or E can be the answer. Thus, the answer is A or B. We will show that A is unattainable, leaving us with B as the only possible answer.

Assume WLOG that $d+e$ is the largest sum. So $d+e=670$, meaning $a+b+c=2010-670=1340$. Because we let $d+e=M$, we must have $a+b \leq M = 670$ and $b+c \leq M = 670$. Adding these inequalities gives $a+2b+c \leq 1340$. But we just showed that $a+b+c=1340$, which means that $b=0$, a contradiction because we are told that all the variables are positive.

Therefore, the answer is $\boxed{B}$.

See also

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2010 AMC 12B Problems/Problem 15

Problem 15

For how many ordered triples (x, y, z) of nonnegative integers less than 20 are there exactly two distinct elements in the set $\{i^x, (1+i)^y, z\}$, where $i = \sqrt{-1}$?

$\textbf{(A)}\ 149 \qquad \textbf{(B)}\ 205 \qquad \textbf{(C)}\ 215 \qquad \textbf{(D)}\ 225 \qquad \textbf{(E)}\ 235$

Solution

We have either $i^x = (1+i)^y \neq z$, $i^x = z \neq (1+i)^y$, or $(1+i)^y = z \neq i^x$.

For $i^x = (1+i)^y$, this only occurs at 1. $(1+i)^y = 1$ has only one solution, namely, $y = 0$. $i^x = 1$ has five solutions between zero and nineteen, $x = 0, x = 4, x = 8, x = 12$, and $x = 16$. $z \neq 1$ has nineteen integer solutions between zero and nineteen. So for $i^x = (1+i)^y \neq z$, we have $5 \times 1 \times 19 = 95$ ordered triples.

For $i^x = z \neq (1+i)^y$, again this only occurs at 1. $(1+i)^y \neq 1$ has nineteen solutions, $i^x = 1$ has five solutions, and $z = 1$ has one solution, so again we have $5 \times 1 \times 19 = 95$ ordered triples.

For $(1+i)^y = z \neq i^x$, this occurs at 1 and 16. $(1+i)^y = 1$ and $z = 1$ both have one solution while $i^x \neq 1$ has fifteen solutions. $(1+i)^y = 16$ and $z = 16$ both have one solution, namely, $y = 8$ and $z = 16$, while $i^x \neq 16$ has twenty solutions. So we have $15 \times 1 \times 1 + 20 \times 1 \times 1 = 35$ ordered triples.

In total we have $95 + 95 + 35 = 225$ ordered triples $\Longrightarrow \boxed{D}$

See also

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2010 AMC 12B Problems/Problem 16

Problem 16

Positive integers a, b, and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \dots, 2010\}$. What is the probability that $abc + ab + a$ is divisible by 3?

$\textbf{(A)}\ \dfrac{1}{3} \qquad \textbf{(B)}\ \dfrac{29}{81} \qquad \textbf{(C)}\ \dfrac{31}{81} \qquad \textbf{(D)}\ \dfrac{11}{27} \qquad \textbf{(E)}\ \dfrac{13}{27}$

Solution

The value of 2010 is arbitrary other than it is divisible by 3, so the set $\{1, 2, 3, ..., 2010\}$ can be grouped into threes.

Obviously, if a is divisible by 3 (which has probability $\frac{1}{3}$) then the sum is divisible by 3. In the event that a is not divisible by 3 (which has probability $\frac{2}{3}$), then the sum is divisible by 3 if

$bc + b + 1 \equiv 0 \pmod 3$, which is the same as

$b(c+1) \equiv 2 \pmod 3$.

This only occurs when one of the factors b or $c+1$ is equivalent to $2 \pmod 3$ and the other is equivalent to $1 \pmod 3$. All four events $b \equiv 1 \pmod 3$, $c+1 \equiv 2 \pmod 3$, $b \equiv 2 \pmod 3$, and $c+1 \equiv 1 \pmod 3$ have a probability of $\frac{1}{3}$ because the set is grouped in threes.

In total the probability is $\frac{1}{3} + \frac{2}{3}(2(\frac{1}{3} \times \frac{1}{3})) = \frac{13}{27} \Rightarrow \boxed{E}$

See also

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2010 AMC 12B Problems/Problem 17

Contents

Problem

The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 24 \qquad \textbf{(C)}\ 36 \qquad \textbf{(D)}\ 42 \qquad \textbf{(E)}\ 60$

Solution 1

The first 4 numbers will form one of 3 tetris "shapes".

First, let's look at the numbers that form a 2x2 block, sometimes called tetris O:

1	2	
3	4	

1	3	
2	4	

Second, let's look at the numbers that form a vertical "L", sometimes called tetris J:

1	4	
2		
3		

1	3	
2		
4		

1	2	
3		
4		

Third, let's look at the numbers that form a horizontal "L", sometimes called tetris L:

1	2	3
4		

1	2	4
3		

1	3	4
2		

Now, the numbers 6-9 will form similar shapes (rotated by 180 degrees, and anchored in the lower-right corner of the 3x3 grid).

If you match up one tetris shape from the numbers 1-4 and one tetris shape from the numbers 6-9, there is only one place left for the number 5 to be placed.

So what shapes will physically fit in the 3x3 grid, together?

$$\begin{array}{lll} 1-4 \text{ shape} & 6-9 \text{ shape} & \text{number of pairings} \\ \quad O & \quad J & 2\times 3 = 6 \\ \quad O & \quad L & 2\times 3 = 6 \\ \quad J & \quad O & 3\times 2 = 6 \\ \quad J & \quad J & 3\times 3 = 9 \\ \quad L & \quad O & 3\times 2 = 6 \\ \quad L & \quad L & 3\times 3 = 9 \\ \quad O & \quad O & \qquad \text{They don't fit} \\ \quad J & \quad L & \qquad \text{They don't fit} \\ \quad L & \quad J & \qquad \text{They don't fit} \end{array}$$

The answer is $4\times 6 + 2\times 9 = \boxed{\text{(D) } 42}$.

Solution 2

This solution is trivial by the hook length theorem. The hooks look like this:

5	4	3
4	3	2
3	2	1

So, the answer is $\dfrac{9!}{5\cdot 4\cdot 3\cdot 4\cdot 3\cdot 2\cdot 3\cdot 2\cdot 1}$ = $\boxed{\text{(D) } 42}$

P.S. The hook length formula is a formula to calculate the number of standard Young tableaux of a Young diagram. Numberphile has an easy-to-understand video about it here: https://www.youtube.com/watch?v=vgZhrEs4tuk The full proof is quite complicated and is not given in the video, although the video hints at possible proofs.

See also

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2010 AMC 12B Problems/Problem 18

Contents

Problem

A frog makes 3 jumps, each exactly 1 meter long. The directions of the jumps are chosen independently at random. What is the probability that the frog's final position is no more than 1 meter from its starting position?

$\textbf{(A)}\ \dfrac{1}{6} \qquad \textbf{(B)}\ \dfrac{1}{5} \qquad \textbf{(C)}\ \dfrac{1}{4} \qquad \textbf{(D)}\ \dfrac{1}{3} \qquad \textbf{(E)}\ \dfrac{1}{2}$

Solution

Solution 1

We will let the moves be complex numbers a, b, and c, each of magnitude one. The frog starts on the origin. It is relatively easy to show that exactly one element in the set

$$\{|a+b+c|, |a+b-c|, |a-b+c|, |a-b-c|\}$$

has magnitude less than or equal to 1. Hence, the probability is $\boxed{(\text{C})\frac{1}{4}}$.

Solution 2

The first frog hop doesn't matter because no matter where the frog hops is lands on the border of the circle you want it to end in. The remaining places that the frog can jump to form a disk of radius 2 centered at the spot on which the frog first landed, and every point in the disk of radius 2 is equally likely to be reached in two jumps. The entirety of the circle you want the frog to land in is enclosed in this larger disk, so find the ratio of the two areas, which is $\boxed{(\text{C})\frac{1}{4}}$.

See also

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2010 AMC 12B Problems/Problem 19

Problem 19

A high school basketball game between the Raiders and Wildcats was tied at the end of the first quarter. The number of points scored by the Raiders in each of the four quarters formed an increasing geometric sequence, and the number of points scored by the Wildcats in each of the four quarters formed an increasing arithmetic sequence. At the end of the fourth quarter, the Raiders had won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

$\textbf{(A)}\ 30 \qquad \textbf{(B)}\ 31 \qquad \textbf{(C)}\ 32 \qquad \textbf{(D)}\ 33 \qquad \textbf{(E)}\ 34$

Solution

Let a, ar, ar^2, ar^3 be the quarterly scores for the Raiders. We know that the Raiders and Wildcats both scored the same number of points in the first quarter so let $a, a+d, a+2d, a+3d$ be the quarterly scores for the Wildcats. The sum of the Raiders scores is $a(1+r+r^2+r^3)$ and the sum of the Wildcats scores is $4a+6d$. Now we can narrow our search for the values of $a, d,$ and r. Because points are always measured in positive integers, we can conclude that a and d are positive integers. We can also conclude that r is a positive integer by writing down the equation:

$$a(1+r+r^2+r^3) = 4a+6d+1$$

Now we can start trying out some values of r. We try $r=2$, which gives

$$15a = 4a+6d+1$$

$$11a = 6d+1$$

We need the smallest multiple of 11 (to satisfy the <100 condition) that is $\equiv 1 \pmod{6}$. We see that this is 55, and therefore $a=5$ and $d=9$.

So the Raiders' first two scores were 5 and 10 and the Wildcats' first two scores were 5 and 14.

$$5+10+5+14 = 34 \longrightarrow \boxed{\textbf{(E)}}$$

See also

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2010 AMC 12B Problems/Problem 20

Problem

A geometric sequence (a_n) has $a_1 = \sin x$, $a_2 = \cos x$, and $a_3 = \tan x$ for some real number x. For what value of n does $a_n = 1 + \cos x$?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 8$

Solution

By the defintion of a geometric sequence, we have $\cos^2 x = \sin x \tan x$. Since $\tan x = \frac{\sin x}{\cos x}$, we can rewrite this as $\cos^3 x = \sin^2 x$.

The common ratio of the sequence is $\frac{\cos x}{\sin x}$, so we can write

$$a_1 = \sin x$$

$$a_2 = \cos x$$

$$a_3 = \frac{\cos^2 x}{\sin x}$$

$$a_4 = \frac{\cos^3 x}{\sin^2 x} = 1$$

$$a_5 = \frac{\cos x}{\sin x}$$

$$a_6 = \frac{\cos^2 x}{\sin^2 x}$$

$$a_7 = \frac{\cos^3 x}{\sin^3 x} = \frac{1}{\sin x}$$

$$a_8 = \frac{\cos x}{\sin^2 x} = \frac{1}{\cos^2 x}$$

Since $\cos^3 x = \sin^2 x = 1 - \cos^2 x$, we have $\cos^3 x + \cos^2 x = 1 \implies \cos^2 x(\cos x + 1) = 1 \implies \cos x + 1 = \frac{1}{\cos^2 x}$, which is a_8 , making our answer $8 \Rightarrow \boxed{E}$.

See also

2010 AMC 12B Problems/Problem 21

Problem 21

Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$$P(1) = P(3) = P(5) = P(7) = a, \text{ and}$$
$$P(2) = P(4) = P(6) = P(8) = -a.$$

What is the smallest possible value of a?

$\textbf{(A)}\ 105 \qquad \textbf{(B)}\ 315 \qquad \textbf{(C)}\ 945 \qquad \textbf{(D)}\ 7! \qquad \textbf{(E)}\ 8!$

Solution

There must be some polynomial $Q(x)$ such that $P(x) - a = (x-1)(x-3)(x-5)(x-7)Q(x)$

Then, plugging in values of $2, 4, 6, 8,$ we get

$$P(2) - a = (2-1)(2-3)(2-5)(2-7)Q(2) = -15Q(2) = -2a$$

$$P(4) - a = (4-1)(4-3)(4-5)(4-7)Q(4) = 9Q(4) = -2a$$

$$P(6) - a = (6-1)(6-3)(6-5)(6-7)Q(6) = -15Q(6) = -2a$$

$$P(8) - a = (8-1)(8-3)(8-5)(8-7)Q(8) = 105Q(8) = -2a$$

$-2a = -15Q(2) = 9Q(4) = -15Q(6) = 105Q(8).$ Thus, the least value of a must be the $\text{lcm}(15, 9, 15, 105)$. Solving, we receive 315, so our answer is $\boxed{\textbf{(B)}\ 315}$.

See also

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2010 AMC 12B Problems/Problem 22

Problem 22

Let $ABCD$ be a cyclic quadrilateral. The side lengths of $ABCD$ are distinct integers less than 15 such that $BC \cdot CD = AB \cdot DA$. What is the largest possible value of BD?

$\textbf{(A)}\ \sqrt{\dfrac{325}{2}} \qquad \textbf{(B)}\ \sqrt{185} \qquad \textbf{(C)}\ \sqrt{\dfrac{389}{2}} \qquad \textbf{(D)}\ \sqrt{\dfrac{425}{2}} \qquad \textbf{(E)}\ \sqrt{\dfrac{533}{2}}$

Solution

Let $AB = a$, $BC = b$, $CD = c$, and $AD = d$. We see that by the Law of Cosines on $\triangle ABD$, we have $BD^2 = a^2 + d^2 - 2ad\cos\angle BAD$. Also, $BD^2 = b^2 + c^2 - 2bc\cos\angle BCD$. Now, we know that $ad = bc$. Also, because $ABCD$ is a cyclic quadrilateral, we must have that $\angle BAD = 180 - \angle BCD$, so $\cos\angle BAD = -\cos\angle BCD$. Therefore, $2ad\cos\angle BAD = -2bc\cos\angle BCD$. Now, adding, we have $2BD^2 = a^2 + b^2 + c^2 + d^2$.

We now look at the equation $ad = bc$. Suppose that $a = 14$. Then, we must have either b or c equal 7. Suppose that $b = 7$. We let $d = 6$ and $c = 12$.

Now, $2BD^2 = 196 + 49 + 36 + 144 = 425$, so it is $\sqrt{\dfrac{425}{2}}$ or $\boxed{\textbf{(D)}}$.

See also

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2010 AMC 12B Problems/Problem 23

Problem 23

Monic quadratic polynomial $P(x)$ and $Q(x)$ have the property that $P(Q(x))$ has zeros at $x = -23, -21, -17,$ and -15, and $Q(P(x))$ has zeros at $x = -59, -57, -51$ and -49. What is the sum of the minimum values of $P(x)$ and $Q(x)$?

$\textbf{(A)}\ -100 \qquad \textbf{(B)}\ -82 \qquad \textbf{(C)}\ -73 \qquad \textbf{(D)}\ -64 \qquad \textbf{(E)}\ 0$

Solution

$P(x) = (x-a)^2 - b, Q(x) = (x-c)^2 - d$. Notice that $P(x)$ has roots $a \pm \sqrt{b}$, so that the roots of $P(Q(x))$ are the roots of $Q(x) = a + \sqrt{b}, a - \sqrt{b}$. For each individual equation, the sum of the roots will be $2c$ (symmetry or Vieta's). Thus, we have $4c = -23 - 21 - 17 - 15$, or $c = -19$. Doing something similar for $Q(P(x))$ gives us $a = -54$. We now have $P(x) = (x+54)^2 - b, Q(x) = (x+19)^2 - d$. Since Q is monic, the roots of $Q(x) = a + \sqrt{b}$ are "farther" from the axis of symmetry than the roots of $Q(x) = a - \sqrt{b}$. Thus, we have $Q(-23) = -54 + \sqrt{b}, Q(-21) = -54 - \sqrt{b}$, or $16 - d = -54 + \sqrt{b}, 4 - d = -54 - \sqrt{b}$. Adding these gives us $20 - 2d = -108$, or $d = 64$. Plugging this into $16 - d = -54 + \sqrt{b}$, we get $b = 36$. The minimum value of $P(x)$ is $-b$, and the minimum value of $Q(x)$ is $-d$. Thus, our answer is $-(b+d) = -100$, or answer $\boxed{\textbf{(A)}}$.

Alternate solution at: http://artofproblemsolving.com/community/c4h1256144_2010_amc_12b

See Also

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2010 AMC 12B Problems/Problem 24

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Problem 24

The set of real numbers x for which

$$\frac{1}{x-2009}+\frac{1}{x-2010}+\frac{1}{x-2011}\geq 1$$

is the union of intervals of the form $a<x\leq b$. What is the sum of the lengths of these intervals?

$\textbf{(A)}\ \frac{1003}{335} \qquad \textbf{(B)}\ \frac{1004}{335} \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ \frac{403}{134} \qquad \textbf{(E)}\ \frac{202}{67}$

Solution

Because the right side of the inequality is a horizontal line, the left side can be translated horizontally by any value and the intervals will remain the same. For simplicity of calculation, we will find the intervals where

$$\frac{1}{x+1}+\frac{1}{x}+\frac{1}{x-1}\geq 1$$

We shall say that $f(x)=\frac{1}{x+1}+\frac{1}{x}+\frac{1}{x-1}$. $f(x)$ has three vertical asymptotes at $x=\{-1,0,1\}$. As the sum of decreasing hyperbolas, the function is decreasing at all intervals. Values immediately to the left of each asymptote approach negative infinity, and values immediately to the right of each asymptote approach positive infinity. In addition, the function has a horizontal asymptote at $y=0$. The function intersects 1 at some point from $x=-1$ to $x=0$, and at some point from $x=0$ to $x=1$, and at some point to the right of $x=1$. The intervals where the function is greater than 1 are between the points where the function equals 1 and the vertical asymptotes.

If p, q, and r are values of x where $f(x)=1$, then the sum of the lengths of the intervals is $(p-(-1))+(q-0)+(r-1)=p+q+r$.

$$\frac{1}{x+1}+\frac{1}{x}+\frac{1}{x-1}=1$$

$$\implies x(x-1)+(x-1)(x+1)+x(x+1)=x(x-1)(x+1)$$

$$\implies x^3-3x^2-x+1=0$$

And now our job is simply to find the sum of the roots of x^3-3x^2-x+1. Using Vieta's formulas, we find this to be $3\Rightarrow\boxed{C}$.

NOTE': For the AMC, one may note that the transformed inequality should not yield solutions that involve big numbers like 67 or 134, and immediately choose C.

Solution 2

As in the first solution, note that the expression can be translated into

$$\frac{1}{x+1}+\frac{1}{x}+\frac{1}{x-1}\geq 1$$

without affecting the interval lengths.

This simplifies into

$$\frac{-x^3+3x^2+x-1}{(x)(x+1)(x-1)}\geq 0$$

and so

$$-x^3+3x^2+x-1\geq 0$$

. Each interval is $(-1,a),(0,b),(1,c)$, where a, b, and c are the roots of $-x^3+3x^2+x-1=0$ so the total length is $a+b+c$, which is the sum of the roots, or 3.

See also

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2010 AMC 12B Problems/Problem 25

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Problem 25

For every integer $n \geq 2$, let $\text{pow}(n)$ be the largest power of the largest prime that divides n. For example $\text{pow}(144) = \text{pow}(2^4 \cdot 3^2) = 3^2$. What is the largest integer m such that 2010^m divides

$$\prod_{n=2}^{5300} \text{pow}(n)?$$

$\textbf{(A)}\ 74 \qquad \textbf{(B)}\ 75 \qquad \textbf{(C)}\ 76 \qquad \textbf{(D)}\ 77 \qquad \textbf{(E)}\ 78$

Solution

Because 67 is the largest prime factor of 2010, it means that in the prime factorization of $\prod_{n=2}^{5300} \text{pow}(n)$, there'll be $p_1^{e1} \cdot p_2^{e2} \cdot ....67^x ...$ where x is the desired value we are looking for. Thus, to find this answer, we need to look for the number of times 67 is incorporated into the giant product.

All numbers $n = 67 \cdot x$, given $x = p_1^{e_1} \cdot p_2^{e^2} \cdot ... \cdot p_m^{e_m}$ such that for any integer x between 1 and m, prime p_x must be less than 67, contributes a 67 to the product. Considering $67 \cdot 79 < 5300 < 67 \cdot 80$, the possible values of x are $1, 2, ..., 70, 72, ...78$, since $x = 71, 79$ are primes that are greater than 67. However, $\text{pow}\left(67^2\right)$ contributes two 67s to the product, so we must count it twice. Therefore, the answer is $70 + 1 + 6 = \boxed{77} \Rightarrow \boxed{D}$.

Similar Solution

After finding the prime factorization of $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, divide 5300 by 67 and add 5300 divided by 67^2 in order to find the total number of multiples of 67 between 2 and 5300. $\lfloor \frac{5300}{67} \rfloor + \lfloor \frac{5300}{67^2} \rfloor = 80$ Since $71, 73$, and 79 are prime numbers greater than 67 and less than or equal to 80, subtract 3 from 80 to get the answer $80 - 3 = \boxed{77} \Rightarrow \boxed{D}$.

Need Discussion and Clarification

How do we know that we only have to check 67? There is no solid relationship between 67 being the largest prime factor in 2010 and 67 giving the smallest result of 77. Details in the Discussions page of this Article: http://artofproblemsolving.com/wiki/index.php?title=Talk:2010_AMC_12B_Problems/Problem_25

See Also