2022 AMC 10B Solutions

Problem1

Define $x \diamondsuit y$ to be $|x-y|$ for all real numbers x and y. What is the value of $(1 \diamondsuit (2 \diamondsuit 3)) - ((1 \diamondsuit 2) \diamondsuit 3)$?

$\textbf{(A)}\ -2 \qquad \textbf{(B)}\ -1 \qquad \textbf{(C)}\ 0 \qquad \textbf{(D)}\ 1 \qquad \textbf{(E)}\ 2$

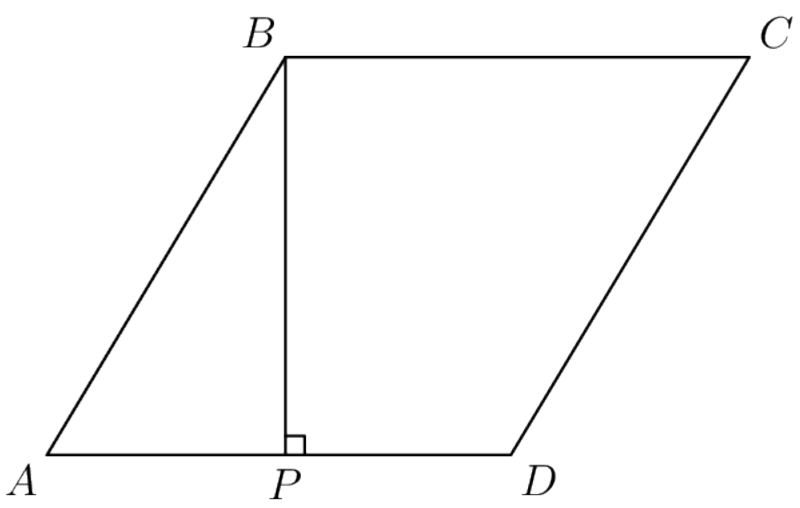
Solution

We have

$$\begin{aligned}(1 \diamondsuit (2 \diamondsuit 3)) - ((1 \diamondsuit 2) \diamondsuit 3) &= |1-|2-3|| - ||1-2|-3| \\ &= |1-1| - |1-3| \\ &= 0-2 \\ &= \boxed{\textbf{(A)}\ -2}.\end{aligned}$$

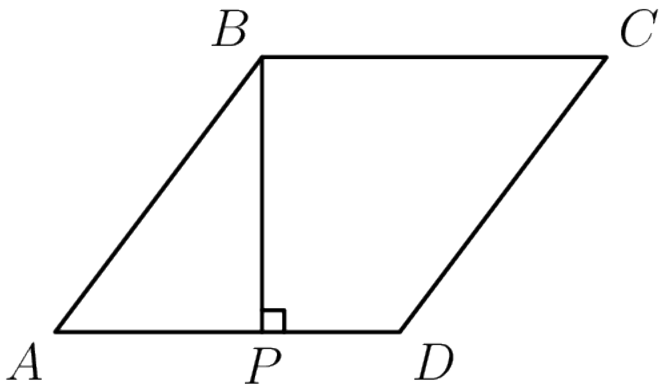
Problem2

In rhombus $ABCD$, point P lies on segment $\overline{AD}$ so that $\overline{BP} \perp \overline{AD}$, $AP = 3$, and $PD = 2$. What is the area of $ABCD$? (Note: The figure is not drawn to scale.)



$\textbf{(A) } 3\sqrt{5} \qquad \textbf{(B) } 10 \qquad \textbf{(C) } 6\sqrt{5} \qquad \textbf{(D) } 20 \qquad \textbf{(E) } 25$

Solution 1



(Figure redrawn to scale.)

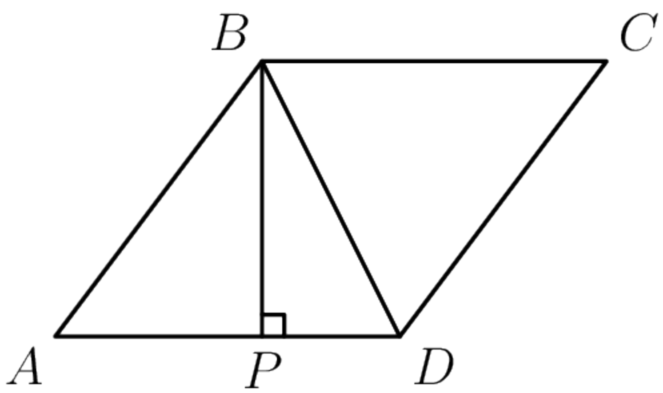
$AD = AP + PD = 3 + 2 = 5$.

$ABCD$ is a rhombus, so $AB = AD = 5$.

$\triangle APB$ is a 3-4-5 right triangle, so $BP = 4$.

The area of the rhombus $= bh = (AD)(BP) = 5 \cdot 4 = \boxed{\textbf{(D) } 20}$.

Solution 2 (The Area Of A Triangle)



The diagram is from as Solution 1, but a line is constructed at BD.

When it comes to the sides of a rhombus, their opposite sides are congruent and parallel. This means that $\angle ABD \cong \angle BDC$, by the Alternate Interior Angles Theorem.

By SAS Congruence, we get $\triangle ABD \cong \triangle BDC$.

Since $AP = 3$ and $AB = 5$, we know that $BP = 4$ because $\triangle APB$ is a 3-4-5 right triangle, as stated in Solution 1.

The area of $\triangle ABD$ would be 10, since the area of the triangle is $\frac{bh}{2}$.

Since we know that $\triangle ABD \cong \triangle BDC$ and that $ABCD = \triangle ABD + \triangle BDC$, so we can double the area of $\triangle ADB$ to get $10 \cdot 2 = \boxed{\textbf{(D) } 20} \blacksquare$.

Problem3

How many three-digit positive integers have an odd number of even digits?

$\textbf{(A) } 150 \qquad \textbf{(B) } 250 \qquad \textbf{(C) } 350 \qquad \textbf{(D) } 450 \qquad \textbf{(E) } 550$

Solution

There are only 2 ways for an odd number of even digits: 1 even digit or all even digits.

Case 1: 1 even digit

There are $5 \cdot 5 = 25$ ways to choose the odd digits, 5 ways for the even digit, and 3 ways to order the even digit. So, $25 \cdot 5 \cdot 3 = 375$. However, there are $5 \cdot 5 = 25$ ways that the hundred's digit is 0 and we must subtract this from 375, leaving us with 350 ways.

Case 2: all even digits

There are $5 \cdot 5 \cdot 5 = 125$ ways to choose the even digits,

and $5 \cdot 5 = 25$ ways where the hundred's digit is 0.

So, $125 - 25 = 100$.

Adding up the cases, the answer is $100 + 350 = \boxed{\textbf{(D) } 450}$.

Solution 2 (Bijection)

We will show that the answer is 450 by proving a bijection between the three digit integers that have an even number of even digits and the three digit integers that have an odd number of even digits. For every even number with an odd number of even digits, we increment the number's last digit by 1, unless the last digit is 9, in which case it becomes 0. It is very easy to show that every number with an even number of even digits is mapped to every number with an odd number of even digits, and vice versa. Thus, the answer is half the number of three digit numbers, or $\boxed{\text{D. } 450}$

Problem4

A donkey suffers an attack of hiccups and the first hiccup happens at $4:00$ one afternoon. Suppose that the donkey hiccups regularly every 5 seconds. At what time does the donkey's 700th hiccup occur?

$\textbf{(A) }15 \text{ seconds after } 4:58$

$\textbf{(B) }20 \text{ seconds after } 4:58$

$\textbf{(C) }25 \text{ seconds after } 4:58$

$\textbf{(D) }30 \text{ seconds after } 4:58$

$\textbf{(E) }35 \text{ seconds after } 4:58$

Solution 1

Since the donkey hiccupped the 1st hiccup at $4:00$, he hiccupped for $5 \cdot (700-1) = 3495$ seconds, which is 58 minutes and 15 seconds, so the answer is $\boxed{\textbf{(A) } 15 \text{ seconds after } 4:58}$.

Solution 2 (Faster)

We see that the minute has already been determined. The donkey hiccups once every 5 seconds, or 12 times a minute. $700 \equiv 4$ (mod 12), so the 700th hiccup happened on the same second as the 4th, which occurred on the $5(4-1) = 15$th second. $\boxed{\textbf{(A) } 15 \text{ seconds after } 4:58}$.

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Bogus Solution

Obviously, the donkey will have its 700th hiccup $700 \cdot 5 = 3500$ seconds after the moment it started. This is $4{:}00 + 3500 \text{ seconds} = \boxed{\textbf{(B) } 20 \text{ seconds after } 4{:}58}$.

This is not correct, though. Hiccup number 1 occurred at $4{:}00$, so actually the time of hiccup n is $4{:}00 + 5(n-1) \text{ seconds}$.

Problem5

What is the value of $$\frac{\left(1+\frac13\right)\left(1+\frac15\right)\left(1+\frac17\right)}{\sqrt{\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)}}?$$

$\textbf{(A) } \sqrt3 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } \sqrt{15} \qquad \textbf{(D) } 4 \qquad \textbf{(E) } \sqrt{105}$

Solution 1 (Difference of Squares)

We apply the difference of squares to the denominator, and then regroup factors:

$$\begin{aligned}\frac{\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)}{\sqrt{\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)}} &= \frac{\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)}{\sqrt{\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)}\cdot\sqrt{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)}} \\ &= \frac{\sqrt{\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)}}{\sqrt{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)}} \\ &= \frac{\sqrt{\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}}}{\sqrt{\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}}} \\ &= \frac{\sqrt{4\cdot6\cdot8}}{\sqrt{2\cdot4\cdot6}} \\ &= \frac{\sqrt{8}}{\sqrt{2}} \\ &= \boxed{\textbf{(B) } 2}.\end{aligned}$$

Solution 2 (Brute Force)

Since these numbers are fairly small, we can use brute force as follows:

$$\frac{(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})}{\sqrt{(1-\frac{1}{3^2})(1-\frac{1}{5^2})(1-\frac{1}{7^2})}} = \frac{\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}}{\sqrt{\frac{8}{9}\cdot\frac{24}{25}\cdot\frac{48}{49}}} = \frac{\frac{4\cdot6\cdot8}{3\cdot5\cdot7}}{\sqrt{\frac{(2^3)(2^3\cdot3^1)(2^4\cdot3^1)}{(3^2)(5^2)(7^2)}}} = \frac{\frac{64}{35}}{\frac{96}{105}} = \frac{64}{35}\cdot\frac{105}{96} = \boxed{\textbf{(B) } 2}.$$

~not_slay

Solution 3 (Brute Force)

This solution starts off exactly as the one above. We simplify to get:

$$\frac{(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})}{\sqrt{(1-\frac{1}{3^2})(1-\frac{1}{5^2})(1-\frac{1}{7^2})}} = \frac{\frac{4\cdot6\cdot8}{3\cdot5\cdot7}}{\sqrt{\frac{(2^3)(2^3\cdot3^1)(2^4\cdot3^1)}{(3^2)(5^2)(7^2)}}}$$

But now, we can get a nice simplification as shown:

$$\frac{\frac{4\cdot6\cdot8}{3\cdot5\cdot7}}{\sqrt{\frac{(2^3)(2^3\cdot3^1)(2^4\cdot3^1)}{(3^2)(5^2)(7^2)}}} = \frac{\frac{4\cdot6\cdot8}{3\cdot5\cdot7}}{\sqrt{\frac{2^{10}\cdot3^2}{3^2\cdot5^2\cdot7^2}}} = \frac{\frac{4\cdot6\cdot8}{3\cdot5\cdot7}}{\frac{2^5\cdot3}{3\cdot5\cdot7}} = \frac{4\cdot6\cdot8}{3\cdot5\cdot7}\cdot\frac{3\cdot5\cdot7}{2^5\cdot3} = \frac{2^6\cdot3}{2^5\cdot3} = \boxed{\textbf{(B) } 2}.$$

Problem6

How many of the first ten numbers of the sequence $121, 11211, 1112111, \ldots$ are prime numbers?

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 3 \qquad \textbf{(E) } 4$

Solution 1 (Generalization)

The nth term of this sequence is

$$\sum_{k=n}^{2n}10^k + \sum_{k=0}^{n}10^k = 10^n\sum_{k=0}^{n}10^k + \sum_{k=0}^{n}10^k = \left(10^n+1\right)\sum_{k=0}^{n}10^k.$$

It follows that the terms are

$$\begin{align*} 121 &= 11\cdot11, \\ 11211 &= 101\cdot111, \\ 1112111 &= 1001\cdot1111, \\ &\vdots \end{align*}$$

Therefore, there are $\boxed{\textbf{(A) } 0}$ prime numbers in this sequence.

Solution 2 (Educated Guesses)

Note that it's obvious that 121 is divisible by 11 and 11211 is divisible by 3; therefore, since this an AMC 10 problem 6, we may safely assume that we do not need to check two-digit prime divisibility or use obscure theorems. So, the answer is $\boxed{\textbf{(A) } 0}$.

Problem7

For how many values of the constant k will the polynomial $x^{2}+kx+36$ have two distinct integer roots?

$\textbf{(A) }6 \qquad \textbf{(B) }8 \qquad \textbf{(C) }9 \qquad \textbf{(D) }14 \qquad \textbf{(E) }16$

Solution 1

Let p and q be the roots of $x^2+kx+36.$ By Vieta's Formulas, we have $p+q=-k$ and $pq=36.$

This shows that p and q must be distinct factors of $36.$ The possibilities of $\{p,q\}$ are $\pm\{1,36\},\pm\{2,18\},\pm\{3,12\},\pm\{4,9\}.$ Each unordered pair gives a unique value of $k.$ Therefore, there are $\boxed{\textbf{(B) } 8}$ values of k, namely $\pm37,\pm20,\pm15,\pm13.$

Solution 2

Note that k must be an integer. By the quadratic formula, $x=\dfrac{-k\pm\sqrt{k^2-144}}{2}.$ Since 144 is a multiple of 4, k and k^2-144 have the same parity, so x is an integer if and only if k^2-144 is a perfect square.

Let $k^2-144=n^2.$ Then, $(k+n)(k-n)=144.$ Since k is an integer and 144 is even, $k+n$ and $k-n$ must both be even. Assuming that k is positive, we get 5 possible values of $k+n$, namely $2,4,8,6,12$, which will give distinct positive values of k, but $k+n=12$ gives $k+n=k-n$ and $n=0$, giving 2 identical integer roots. Therefore, there are 4 distinct positive values of $k.$ Multiplying that by 2 to take the negative values into account, we get $4\cdot2=\boxed{\textbf{(B) } 8}$ values of $k.$

Solution 3 (Pythagorean Triples)

Proceed similar to Solution 2 and deduce that the discriminant of $x^2 + kx + 36$ must be a perfect square greater than 0 to satisfy all given conditions. Seeing something like $k^2 - 144$ might remind us of a right triangle where k is the hypotenuse, and 12 is a leg. There are four ways we could have this: a $9 - 12 - 15$ triangle, a $12 - 16 - 20$ triangle, a $5 - 12 - 13$ triangle, and a $12 - 35 - 37$ triangle. Multiply by two to account for negative k values (since k is being squared) and our answer is $\boxed{\textbf{(B) } 8}$.

Problem8

Consider the following 100 sets of 10 elements each:

$$\begin{aligned}&\{1, 2, 3, \ldots, 10\},\\ &\{11, 12, 13, \ldots, 20\},\\ &\{21, 22, 23, \ldots, 30\},\\ &\vdots\\ &\{991, 992, 993, \ldots, 1000\}.\end{aligned}$$

How many of these sets contain exactly two multiples of 7?

$\textbf{(A)}\ 40 \qquad \textbf{(B)}\ 42 \qquad \textbf{(C)}\ 43 \qquad \textbf{(D)}\ 49 \qquad \textbf{(E)}\ 50$

Solution 1

We apply casework to this problem. The only sets that contain two multiples of seven are those for which:

1. The multiples of 7 are $1 \pmod{10}$ and $8 \pmod{10}$. That is, the first and eighth elements of such sets are multiples of 7.

The first element is $1+10k$ for some integer $0 \leq k \leq 99.$ It is a multiple of 7 when $k = 2, 9, 16, \ldots, 93.$

2. The multiples of 7 are $2 \pmod{10}$ and $9 \pmod{10}.$ That is, the second and ninth elements of such sets are multiples of $7.$

   The second element is $2+10k$ for some integer $0 \leq k \leq 99.$ It is a multiple of 7 when $k = 4, 11, 18, \ldots, 95.$

3. The multiples of 7 are $3 \pmod{10}$ and $0 \pmod{10}.$ That is, the third and tenth elements of such sets are multiples of $7.$

   The third element is $3+10k$ for some integer $0 \leq k \leq 99.$ It is a multiple of 7 when $k = 6, 13, 20, \ldots, 97.$

Each case has $\left\lfloor\frac{100}{7}\right\rfloor = 14$ sets. Therefore, the answer is $14 \cdot 3 = \boxed{\textbf{(B)}\ 42}.$

Solution 2

Each set contains exactly 1 or 2 multiples of 7.

There are $\frac{1000}{10} = 100$ total sets and $\left\lfloor\frac{1000}{7}\right\rfloor = 142$ multiples of 7.

Thus, there are $142 - 100 = \boxed{\textbf{(B)}\ 42}$ sets with 2 multiples of 7.

Solution 3

$$\{1,2,3,\ldots,10\},$$
$$\{11,12,13,\ldots,20\},$$
$$\{21,22,23,\ldots,30\},$$
$$\vdots$$

We find a pattern. $\{991,992,993,\ldots,1000\}$. Through quick listing $7,14,21,28,35,42,49,56,63,70,77,84,91,98$, we can figure out that the first set has 1 multiple of 7. The second set has 1 multiple of 7. The third set has 2 multiples of 7. The fourth set has 1 multiple of 7. The fifth set has 2 multiples of 7. The sixth set has 1 multiple of 7. The seventh set has 2 multiples of 7. The eighth set has 1 multiple of 7. The ninth set has 1 multiples of 7. The tenth set has 2 multiples of 7. We see that the pattern for the number of multiples per set goes: $1,1,2,1,2,1,2,1,1,2.$ We can reasonably conclude that the pattern $1,1,2,1,2,1,2$ repeats every 7 times. So, for every 7 sets, there are three multiples of 7. We calculate $\left\lfloor\frac{100}{7}\right\rfloor$ and multiply that by 3(We disregard the remainder of 2 since it doesn't add any extra sets with 2 multiples of 7.). We get $14\cdot 3=\boxed{\textbf{(B) }42}$.

Problem9

The sum $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{2021}{2022!}$ can be expressed as $a-\frac{1}{b!}$, where a and b are positive integers. What is $a+b$?

$\textbf{(A) }2020\qquad\textbf{(B) }2021\qquad\textbf{(C) }2022\qquad\textbf{(D) }2023\qquad\textbf{(E) }2024$

Solution 1

Note that $\frac{n}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$, and therefore this sum is a telescoping sum, which is equivalent to $1 - \frac{1}{2022!}$. Our answer is $1 + 2022 = \boxed{\textbf{(D) } 2023}$.

Solution 2

We have $\left(\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{2021}{2022!}\right) + \frac{1}{2022!} = \left(\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{2020}{2021!}\right) + \frac{1}{2021!}$ from canceling a 2022 from $\frac{2021+1}{2022!}$. This sum clearly telescopes, thus we end up with $\left(\frac{1}{2!} + \frac{2}{3!}\right) + \frac{1}{3!} = \frac{2}{2!} = 1$. Thus the original equation is equal to $1 - \frac{1}{2022}$, and $1 + 2022 = 2023$. $\boxed{\textbf{(D) } 2023}$.

Solution 3 (Induction)

By looking for a pattern, we see that $\frac{1}{2!} = 1 - \frac{1}{2!}$ and $\frac{1}{2!} + \frac{2}{3!} = \frac{5}{6} = 1 - \frac{1}{3!}$, so we can conclude by engineer's induction that the sum in the problem is equal to $1 - \frac{1}{2022!}$, for an answer of $\boxed{\textbf{(D) } 2023}$. This can be proven with actual induction as well; we have already established 2base cases, so now assume that $\frac{1}{2!} + \frac{2}{3!} + \cdots \frac{n-1}{n!} = 1 - \frac{1}{n!}$ for $n = k$. For $n = k + 1$ we get

$\frac{1}{2!} + \frac{2}{3!} + \cdots \frac{n-1}{n!} + \frac{n}{(n+1)!} = 1 - \frac{1}{n!} + \frac{n}{(n+1)!} = 1 - \frac{n+1}{(n+1)!} + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

, completing the proof.

Solution 4

Let $x = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{2022!}.$

Note that

$$\begin{aligned}\left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{2021}{2022!}\right) + \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{2022!}\right) &= \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \cdots + \frac{2022}{2022!} \\ \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{2021}{2022!}\right) + x &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{2021!} \\ \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{2021}{2022!}\right) + x &= x + 1 - \frac{1}{2022!} \\ \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{2021}{2022!}\right) &= 1 - \frac{1}{2022!}.\end{aligned}$$

Therefore, the answer is $1 + 2022 = \boxed{\textbf{(D) } 2023}.$

Solution 5 (Combinatorics)

Suppose you are picking a permutation of 2022 distinct elements. Suppose that the correct order of the permutation is $x_1, x_2, x_3, \ldots, x_{2021}, x_{2022}$We want to find the probability of picking the permutation in the wrong order.

Suppose that we have picked everything to the correct order except our last 2 elements. That is we have$x_1, x_2, x_3, \ldots, x_{2019}, x_{2020}$We want pick the next element such that it does not equal to x_{2021} . There are 1 ways to choose that, so we add $\frac{1}{2!}$ to the probability.

Suppose that we have picked everything to the correct order except our last 3 elements. That is we have$x_1, x_2, x_3, \ldots, x_{2018}, x_{2019}$We want pick the next element such that it does not equal to x_{2020}. There are 2 ways to choose that, so we add $\frac{2}{3!}$ to the probability.

This series ends up being the probability of making a permutation in the wrong order and that is of course $1-\frac{1}{2022!}$

Solution 6 (Desperate)

Because the fractions get smaller, it is obvious that the answer is less than 1, so we can safely assume that $a=1$ (this can also be guessed by intuition using similar math problems). Looking at the answer choices, $2018<b<2024$. Because the last term consists of $2022!$ (and the year is 2022) we can guess that $b=2022$. Adding them yields $1+2022=\boxed{\textbf{(D) } 2023}$.

Problem10

Camila writes down five positive integers. The unique mode of these integers is 2 greater than their median, and the median is 2 greater than their arithmetic mean. What is the least possible value for the mode?

$\textbf{(A) } 5 \qquad \textbf{(B) } 7 \qquad \textbf{(C) } 9 \qquad \textbf{(D) } 11 \qquad \textbf{(E) } 13$

Solution 1

Let M be the median. It follows that the two largest integers are $M+2.$

Let a and b be the two smallest integers such that $a<b.$ The sorted list is $a,b,M,M+2,M+2.$ Since the median is 2 greater than their arithmetic mean, we have $\frac{a+b+M+(M+2)+(M+2)}{5}+2=M,$ or $a+b+14=2M.$ Note that $a+b$ must be even. We minimize this sum so that the arithmetic mean, the median, and the unique mode are

minimized. Let $a = 1$ and $b = 3$, from

which $M = 9$ and $M + 2 = \boxed{\textbf{(D) } 11}$.

Solution 2

We can also easily test all the answer choices.

For answer choice $\textbf{(A)}$, the mode is 5, the median is 3, and the arithmetic mean is 1. However, we can quickly see this doesn't work, as there are five integers, and they can't have an arithmetic mean of 1 while having a mode of 5.

Trying answer choice $\textbf{(B)}$, the mode is 7, the median is 5, and the arithmetic mean is 3. From the arithmetic mean, we know that all the numbers have to sum to 15. We know three of the numbers: __, __, $5, 7, 7$. This exceeds the sum of 15.

Now we try answer choice $\textbf{(C)}$. The mode is 9, the median is 7, and the arithmetic mean is 5. From the arithmetic mean, we know that the list sums to 25. Three of the numbers are __, __, $7, 9, 9$, which is exactly 25. However, our list needs positive integers, so this won't work.

Since we were really close on answer choice $\textbf{(C)}$, we can intuitively feel that the answer is probably going to be $\textbf{(D)}$. We can confirm this by creating a list that satisfies the problem and choose $\textbf{(D)}: 1, 3, 9, 11, 11.$

So, our answer is $\boxed{\textbf{(D) } 11}$.

Problem11

All the high schools in a large school district are involved in a fundraiser selling T-shirts. Which of the choices below is logically equivalent to the statement "No school bigger than Euclid HS sold more T-shirts than Euclid HS"?

$\textbf{(A)}$ All schools smaller than Euclid HS sold fewer T-shirts than Euclid HS.

$\textbf{(B)}$ No school that sold more T-shirts than Euclid HS is bigger than Euclid HS.

$\textbf{(C)}$ All schools bigger than Euclid HS sold fewer T-shirts than Euclid HS.

$\textbf{(D)}$ All schools that sold fewer T-shirts than Euclid HS are smaller than Euclid HS.

$\textbf{(E)}$ All schools smaller than Euclid HS sold more T-shirts than Euclid HS.

Solution 1

Let B denote a school that is bigger than Euclid HS, and M denote a school that sold more T-shirts than Euclid HS.

It follows that $\neg B$ denotes a school that is **not bigger than** Euclid HS,

and $\neg M$ denotes a school that **did not sell more** T-shirts than Euclid HS.

Converting everything to conditional statements (if-then form), the given statement becomes $B \implies \neg M$.Its contrapositive is $M \implies \neg B$, which is $\boxed{\textbf{(B)}}$.

Note that "not bigger than" does not mean "smaller than", and "not selling more" does not mean "selling less". There is an equality case. Therefore, none of the other answer choices is equivalent to $B \implies \neg M.$

Solution 2 (Elimination)

Suppose we have five schools: Euclid HS with 50 students and 10 T-shirts sold, school A with 51 students and 10 T-shirts sold, school B with 49students and 10 T-shirts sold, school C with 49 students and 9 T-shirts sold, and school D with 51 students and 9 T-shirts sold (This configuration is legal.). Then, school B rules out $\textbf{(A)}$, school A rules out $\textbf{(C)}$, school D rules out $\textbf{(D)}$, and school C rules out $\textbf{(E)}$, leaving us with $\boxed{\textbf{(B)}}$ as the correct answer.

Problem12

A pair of fair 6-sided dice is rolled n times. What is the least value of n such that the probability that the sum of the numbers face up on a roll equals 7 at least once is greater than $\frac{1}{2}$?

$\textbf{(A) } 2 \qquad \textbf{(B) } 3 \qquad \textbf{(C) } 4 \qquad \textbf{(D) } 5 \qquad \textbf{(E) } 6$

Solution

Rolling a pair of fair 6-sided dice, the probability of getting a sum of 7 is $\frac{1}{6}:$ Regardless what the first die shows, the second die has exactly one outcome to make the sum $7.$ We consider the complement: The probability of not getting a sum of 7 is $1-\frac{1}{6}=\frac{5}{6}.$ Rolling the pair of dice $n$times, the probability of getting a sum of 7 at least once is $1-\left(\frac{5}{6}\right)^n.$

Therefore, we have $1-\left(\frac{5}{6}\right)^n>\frac{1}{2},$ or $\left(\frac{5}{6}\right)^n<\frac{1}{2}.$

Since $\left(\frac{5}{6}\right)^4<\frac{1}{2}<\left(\frac{5}{6}\right)^3,$ the least integer n satisfying the inequality is $\boxed{\textbf{(C) } 4}.$

Solution 2 (99% Accurate Guesswork)

Let's try the answer choices. We can quickly find that when we roll 3 dice, either the first and second sum to 7, the first and third sum to 7, or the second and third sum to 7. There are 6 ways for the first and second dice to sum to 7, 6 ways for the first and third to sum to 7, and 6 ways for the second and third dice to sum to 7. However, we overcounted (but not by much) so we can assume that the answer is $\boxed{\textbf{(C) } 4}$

Problem13

The positive difference between a pair of primes is equal to 2, and the positive difference between the cubes of the two primes is 31106. What is the sum of the digits of the least prime that is greater than those two primes?

$\textbf{(A) } 8 \qquad \textbf{(B) } 10 \qquad \textbf{(C) } 11 \qquad \textbf{(D) } 13 \qquad \textbf{(E) } 16$

Solution 1

Let the two primes be a and b. We would have $a-b=2$ and $a^3-b^3=31106$. Using difference of cubes, we would have $(a-b)(a^2+ab+b^2)=31106$. Since we know $a-b$ is equal to 2, $(a-b)(a^2+ab+b^2)$ would become $2(a^2+ab+b^2)=31106$. Simplifying more, we would get $a^2+ab+b^2=15553$.

Now let's introduce another variable. Instead of using a and b, we can express the primes as $x+2$ and x where a is $x+2$ and b is x.

Plugging x and $x+2$ in, we would have $(x+2)^2+x(x+2)+x^2$. When we expand the parenthesis, it would become $x^2+4x+4+x^2+2x+x^2$. Then we combine like terms to get $3x^2+6x+4$ which equals 15553. Then we subtract 4 from both sides to get $3x^2+6x=15549$. Since all three numbers are divisible by 3, we can divide by 3 to get $x^2+2x=5183$.

Notice how if we had 1 to both sides, the left side would become a perfect square trinomial: $x^2+2x+1=5184$ which is $(x+1)^2=5184$. Since 2 is too small to be a valid number, the two primes must be odd, therefore $x+1$ is the number in the middle of them. Conveniently enough, $5184=72^2$ so the two numbers are 71 and 73. The next prime number is 79, and $7+9=16$ so the answer is $\boxed{\textbf{(E) } 16}$.

Solution 2

Let the two primes be a and b, with a being the smaller prime. We have $a-b=2$, and $a^3-b^3=31106$. Using difference of cubes, we obtain $a^2+ab+b^2=15553$. Now, we use the equation $a-b=2$ to obtain $a^2-2ab+b^2=4$.
Hence,

$$a^2+ab+b^2-(a^2-2ab+b^2)=3ab=15553-4=15549$$

$ab=5183$.Because we have $b=a+2$, $ab=(a+1)^2-(1)^2$.
Thus, $(a+1)^2=5183+1=5184$, so $a+1=72$. This implies $a=71$, $b=73$, and thus the next biggest prime is 79, so our answer is $7+9=\boxed{\textbf{(E) } 16}$

Solution 3 (Estimation)

Let the two primes be p and q such that $p-q=2$ and $p^3-q^3=31106$

By the difference of cubes formula, $p^3-q^3=(p-q)(p^2+pq+q^2)$

Plugging in $p-q=2$ and $p^3-q^3=31106$,

$31106=2(p^2+pq+q^2)$

Through the givens, we can see that $p\approx q$.

Thus,

$$31106=2(p^2+pq+q^2)\approx 6p^2$$
$$p^2\approx \frac{31106}{6}\approx 5200$$
$$p\approx\sqrt{5200}\approx 72$$

Checking prime pairs near 72, we find that $p=73, q=71$

The least prime greater than these two primes is 79 $\implies \boxed{\textbf{(E)}\ 16}$

Problem14

Suppose that S is a subset of $\{1,2,3,\cdots,25\}$ such that the sum of any two (not necessarily distinct) elements of S is never an element of S.What is the maximum number of elements S may contain?

Solution 1 (Pigeonhole Principle)

Denote by M the largest number in S. We categorize numbers $\{1,2,\ldots,M-1\}$ (except $\frac{M}{2}$ if M is even) into $\left\lfloor \frac{M-1}{2} \right\rfloor$ groups, such that the ith group contains two numbers i and $M-i$.

Recall that $M \in S$ and the sum of two numbers in S cannot be equal to M, and the sum of numbers in each group above is equal to S. Thus, each of the above $\left\lfloor \frac{M-1}{2} \right\rfloor$ groups can have at most one number in S.

Therefore,
$$\begin{aligned} |S| &\leq 1 + \left\lfloor \frac{M-1}{2} \right\rfloor \\ &\leq 1 + \left\lfloor \frac{25}{2} \right\rfloor \\ &= 13. \end{aligned}$$

Next, we construct an instance of S with $|S| = 13$.

Let $S = \{13, 14, \ldots, 25\}$. Thus, this set is feasible. Therefore, the most number of elements in S is $\boxed{\textbf{(B) } 13}$.

Solution 2

We know that two odd numbers sum to an even number, so we can easily say that odd numbers $1-25$ can be included in the list, making for 13 elements. But, how do we know we can't include even numbers for a higher element value? Well, to get a higher element value than 13, odd numbers as well as even numbers would have to be included in the list (since there are only 12 even numbers from $1-25$, and many of those even numbers are the sum of even numbers). However, for every even value we add to our odd list, we have to take away an odd number because there are either two odd numbers that sum to that even value, or that even value and another odd number will sum to an odd number later in the list. So, $\boxed{\textbf{(B) } 13}$ elements is the highest we can go.

Problem15

Let S_n be the sum of the first n term of an arithmetic sequence that has a common difference of 2. The quotient $\frac{S_{3n}}{S_n}$ does not depend on n. What is S_{20}?

$\textbf{(A) } 340 \qquad \textbf{(B) } 360 \qquad \textbf{(C) } 380 \qquad \textbf{(D) } 400 \qquad \textbf{(E) } 420$

Solution 1

Suppose that the first number of the arithmetic sequence is a. We will try to compute the value of S_n. First, note that the sum of an arithmetic sequence is equal to the number of terms multiplied by the median of the sequence. The median of this sequence is equal to $a+n-1$. Thus, the value of S_n is $n(a+n-1)=n^2+n(a-1)$.

Then, $\frac{S_{3n}}{S_n}=\frac{9n^2+3n(a-1)}{n^2+n(a-1)}=9-\frac{6n(a-1)}{n^2+n(a-1)}$. Of course, for this value to be constant, $6n(a-1)$ must be 0 for all values of n, and thus $a=1$. Finally, we have $S_{20}=20^2=\boxed{\textbf{(D) } 400}$.

Solution 2

Let's say that our sequence is $a, a+2, a+4, a+6, a+8, a+10, \ldots$. Then, since the value of n doesn't matter in the quotient $\frac{S_{3n}}{S_n}$, we can say that $\frac{S_3}{S_1} = \frac{S_6}{S_2}$.

Simplifying, we get $\frac{3a+6}{a} = \frac{6a+30}{2a+2}$, from

which $\frac{a+2}{a} = \frac{a+5}{a+1}$. Solving for a, we get that $a = 1$.

Now, we proceed similar to Solution 1 and get

that $S_{20} = 20^2 = \boxed{\textbf{(D) } 400}$.

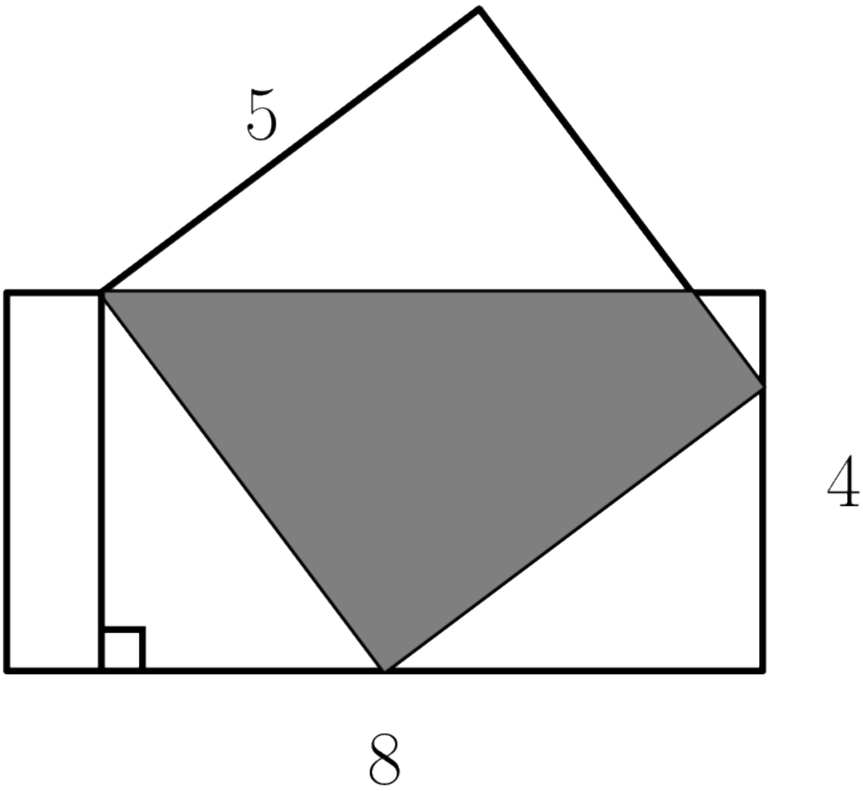
Solution 3 (Quick Insight)

Recall that the sum of the first n odd numbers is n^2.

Since $\frac{S_{3n}}{S_n} = \frac{9n^2}{n^2} = 9$, we have $S_n = 20^2 = \boxed{\textbf{(D) } 400}$.

Problem16

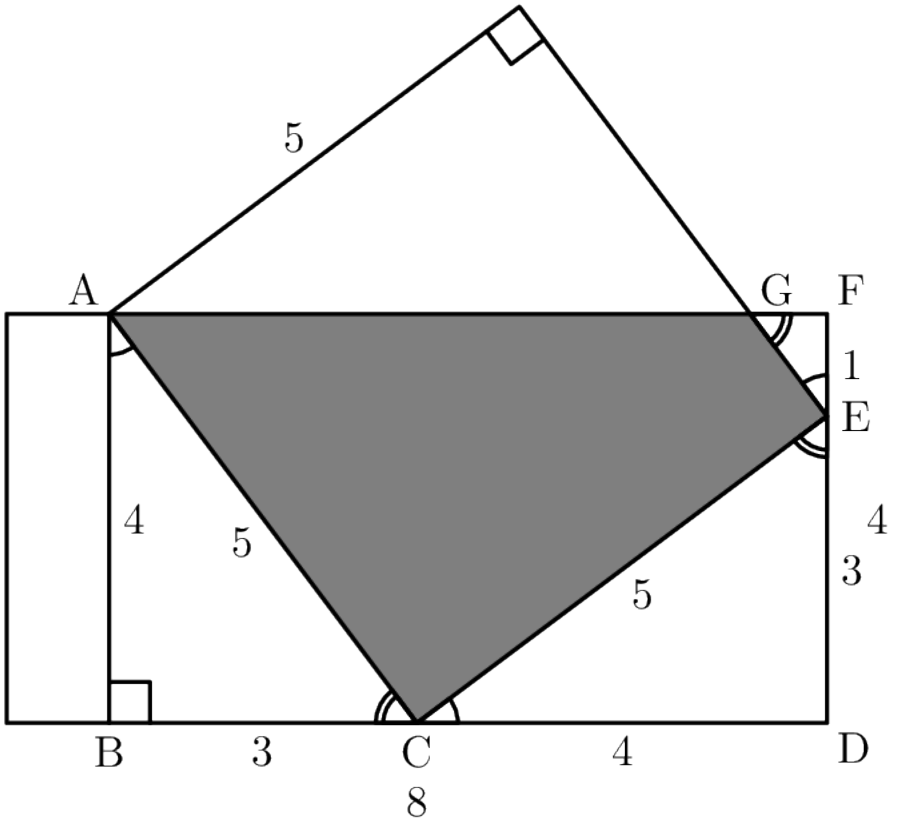
The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?



$\textbf{(A)}\ 15\frac{1}{8} \qquad \textbf{(B)}\ 15\frac{3}{8} \qquad \textbf{(C)}\ 15\frac{1}{2} \qquad \textbf{(D)}\ 15\frac{5}{8} \qquad \textbf{(E)}\ 15\frac{7}{8}$

Solution 1

Let us label the points on the diagram.



By doing some angle chasing using the fact that $\angle ACE$ and $\angle CEG$ are right angles, we find that $\angle BAC \cong \angle DCE \cong \angle FEG$.

Similarly, $\angle ACB \cong \angle CED \cong \angle EGF$.

Therefore, $\triangle ABC \sim \triangle CDE \sim \triangle EFG$.

As we are given a rectangle and a square, $AB = 4$ and $AC = 5$.

Therefore, $\triangle ABC$ is a 3-4-5 right triangle and $BC = 3$.

CE is also 5. So, using the similar triangles, $CD = 4$ and $DE = 3$.

$EF = DF - DE = 4 - 3 = 1$. Using the similar triangles again, EF is $\frac{1}{4}$ of the corresponding AB. So,

$$\begin{aligned}[\triangle EFG] &= \left(\frac{1}{4}\right)^2 \cdot [\triangle ABC] \\ &= \frac{1}{16} \cdot 6 \\ &= \frac{3}{8}.\end{aligned}$$

Finally, we have

$$\begin{aligned}[ACEG] &= [ABDF] - [\triangle ABC] - [\triangle CDE] - [\triangle EFG] \\ &= 7 \cdot 4 - \frac{1}{2} \cdot 3 \cdot 4 - \frac{1}{2} \cdot 3 \cdot 4 - \frac{3}{8} \\ &= 28 - 6 - 6 - \frac{3}{8} \\ &= \boxed{\textbf{(D)}\ 15\frac{5}{8}}.\end{aligned}$$

Solution 2 (Clever)

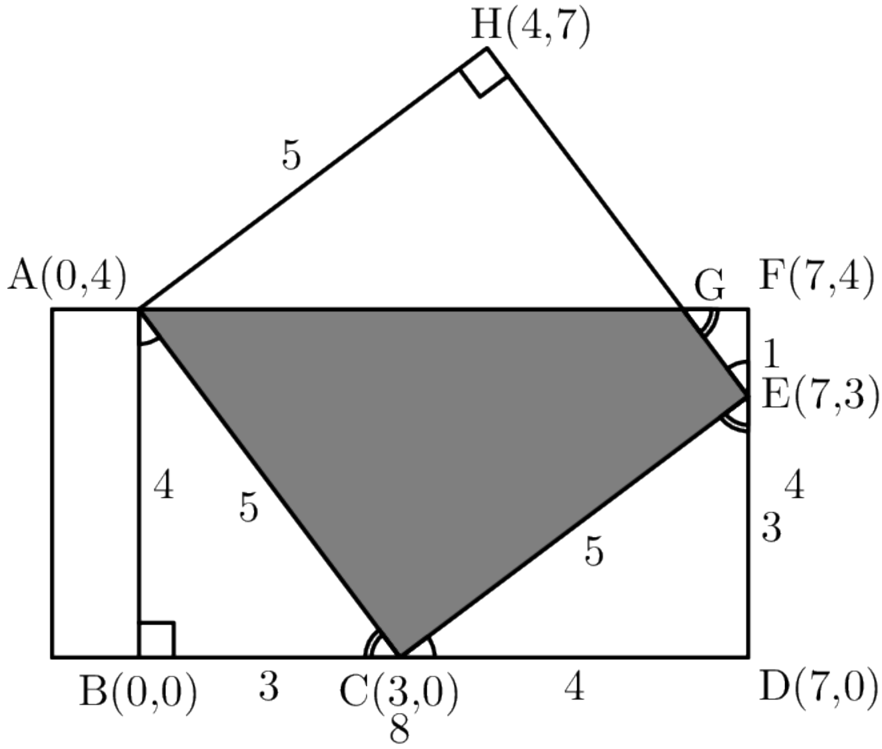
(Refer to the diagram above) Proceed the same way as solution 1 until you get all of the side lengths. Then, it is clear that due to the answer choices, we only need to find the fractional part of the shaded area. The area of the whole rectangle is integral, as is the area of $\triangle ABC$, $\triangle CDE$, and the rectangle to the far left of the diagram. The area of EFG is $\frac{3}{8}$ and thus the fractional part of the answer is $\frac{5}{8}$. Our answer is $\boxed{\textbf{(D)}\ 15\frac{5}{8}}$

Solution 3 (Area of trapezoid)

Proceed similar to solution 1 and use similar triangles to find side length of GE. Then use area of a trapezoid to solve for the area of ACEG.

Solution 4 (Coordinate Geometry)

Same diagram as solution 1, but added point H, which is (4,7). I also renamed all the points to form coordinates using B as the



origin.

In order to find the area, point G's coordinates must be found. Notice how EH and AG intercept at point G. This means that we need to find the equations for EH and AG and make a system of linear equations.

Using the slope formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, we get the slope for EH, which means $m = \dfrac{3-7}{7-4}$ = $-\dfrac{4}{3}$

Then, by using point-slope form. $y - y_1 = m(x - x_1)$. We can say that the equation for EH is $y - 7 = -\dfrac{4}{3}(x - 4)$ or in this case, $y = -\dfrac{4}{3}x + 12\dfrac{1}{3}$.

And it is easy to figure out that the equation for AG is $y = 4$.

The best way to solve the system of linear equations is to substitute the y for the 4 in equation EH. $4=-\frac{4}{3}x+12\frac{1}{3}$, so $x=6\frac{1}{4}$ and $y=4$ This would mean $G\left(6\frac{1}{4},4\right)$.

Since we have our G coordinate, we can continue with solution 3, with the area of the trapezoid -> $\left(\frac{(EG+AC)}{2}\right)(CE)$ where $EG=\frac{5}{4}$ (using distance formula for E to G), $AC=5$, and $CE=5$.

By substitution, we get this, $\left(\frac{\frac{5}{4}+5}{2}\right)(5)=\boxed{\textbf{(D) } 15\frac{5}{8}}$.

Problem17

One of the following numbers is not divisible by any prime number less than $10.$ Which is it?

$\textbf{(A) } 2^{606}-1 \qquad \textbf{(B) } 2^{606}+1 \qquad \textbf{(C) } 2^{607}-1 \qquad \textbf{(D) } 2^{607}+1 \qquad \textbf{(E) } 2^{607}+3^{607}$

Solution 1 (Modular Arithmetic)

For $\textbf{(A)}$ modulo $3,$
$$\begin{aligned} 2^{606}-1 &\equiv (-1)^{606}-1 \\ &\equiv 1-1 \\ &\equiv 0. \end{aligned}$$
Thus, $2^{606}-1$ is divisible by $3.$

For $\textbf{(B)}$ modulo $5,$

$$\begin{aligned}2^{606}+1 &\equiv 2^{\operatorname{Rem}(606,\phi(5))}+1 \\ &\equiv 2^{\operatorname{Rem}(606,4)}+1 \\ &\equiv 2^2+1 \\ &\equiv 0.\end{aligned}$$

Thus, $2^{606}+1$ is divisible by $5.$

For $\textbf{(D)}$ modulo $3,$

$$\begin{aligned}2^{607}+1 &\equiv (-1)^{607}+1 \\ &\equiv -1+1 \\ &\equiv 0.\end{aligned}$$

Thus, $2^{607}+1$ is divisible by $3.$

For $\textbf{(E)}$ modulo $5,$

$$\begin{aligned}2^{607}+3^{607} &\equiv 2^{607}+(-2)^{607} \\ &\equiv 2^{607}-2^{607} \\ &\equiv 0.\end{aligned}$$

Thus, $2^{607}+3^{607}$ is divisible by $5.$

Therefore, the answer is $\boxed{\textbf{(C) } 2^{607}-1}.$

Solution 2 (Factoring)

We have

$$\begin{aligned} 2^{606}-1 &= 4^{303}-1 = (4-1)(4^{302}+4^{301}+4^{300}+\cdots+4^0), \\ 2^{606}+1 &= 4^{303}+1 = (4+1)(4^{302}-4^{301}+4^{300}-\cdots+4^0), \\ 2^{607}+1 &= (2+1)(2^{606}-2^{605}+2^{604}-\cdots+2^0), \\ 2^{607}+3^{607} &= (2+3)(2^{606}\cdot 3^0-2^{605}\cdot 3^1+2^{604}\cdot 3^2-\cdots+2^0\cdot 3^{606}). \end{aligned}$$

We conclude that $\textbf{(A)}$ is divisible by 3, $\textbf{(B)}$ is divisible by 5, $\textbf{(D)}$ is divisible by 3, and $\textbf{(E)}$ is divisible by 5.

Since all of the other choices have been eliminated, we are left with $\boxed{\textbf{(C) } 2^{607}-1}$.

Solution 3 (Elimination)

Mersenne Primes are primes of the form $2^n - 1$, where n is prime. Using the process of elimination, we can eliminate every option except for $\textbf{(A)}$ and $\textbf{(C)}$.

Clearly, 606 isn't prime, so the answer must be $\boxed{\textbf{(C)}\ 2^{607}-1}$.

Problem18

Consider systems of three linear equations with unknowns x, y, and z,

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

where each of the coefficients is either 0 or 1 and the system has a solution other than $x=y=z=0$. For example, one such system is

$$\{1x+1y+0z=0, 0x+1y+1z=0, 0x+0y+0z=0\}$$

with a nonzero solution of $\{x,y,z\}=\{1,-1,1\}$. How many such systems of equations are there? (The equations in a system need not be distinct, and two systems containing the same equations in a different order are considered different.)

$\textbf{(A)}\ 302 \qquad \textbf{(B)}\ 338 \qquad \textbf{(C)}\ 340 \qquad \textbf{(D)}\ 343 \qquad \textbf{(E)}\ 344$

Solution 1 (Casework)

Let $M_1 = \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}, M_2 = \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix}$, and $M_3 = \begin{bmatrix} a_3 & b_3 & c_3 \end{bmatrix}.$

We wish to count the ordered triples (M_1, M_2, M_3) of row matrices. We perform casework:

1. $M_1 = M_2 = M_3.$

There are $2^3 = 8$ options for M_1. Once M_1 is chosen, M_2 and M_3 are uniquely determined.

In this case, we have $\boldsymbol{8}$ ordered triples $\boldsymbol{(M_1, M_2, M_3)}$.

2. Exactly two of M_1, M_2, and M_3 are equal.

   For $M_1 = M_2 \neq M_3$, there are $2^3 = 8$ options for M_1. Once M_1 is chosen, M_2 is uniquely determined, and M_3 has $2^3 - 1 = 7$ options. So, there are $8 \cdot 7 = 56$ ordered triples (M_1, M_2, M_3).
   Similarly, for each of $M_1 = M_3 \neq M_2$ and $M_2 = M_3 \neq M_1$, there are 56 ordered triples (M_1, M_2, M_3).

   In this case, we have $\boldsymbol{56 \cdot 3 = 168}$ ordered triples $\boldsymbol{(M_1, M_2, M_3)}$.

3. All of M_1, M_2, and M_3 are different.

   There are two subcases:

   A. Exactly one of M_1, M_2, and M_3 is $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

      For $M_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, there are $2^3 - 1 = 7$ options for M_2 and $2^3 - 2 = 6$ options for M_3. So, there are $7 \cdot 6 = 42$ ordered triples (M_1, M_2, M_3).

Similarly, for each of $M_2 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and $M_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, there are 42 ordered triples (M_1, M_2, M_3).

In this subcase, we have $\mathbf{42 \cdot 3 = 126}$ ordered triples $(\boldsymbol{M_1, M_2, M_3})$.

B. The sum of two of M_1, M_2, and M_3 is equal to the third matrix.

For $M_1 + M_2 = M_3$:

- If $M_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, then
$$M_2 \in \left\{ \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}.$$

  More generally, if M_1 consists of one 1 and two 0's, then M_2 has 3 options, and M_3 is uniquely determined. So, there are $\binom{3}{1} \cdot 3 = 9$ ordered triples (M_1, M_2, M_3).

- If $M_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, then
$$M_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

  More generally, if M_1 consists of two 1's and one 0, then M_2 and M_3 are uniquely determined. So, there are $\binom{3}{2} = 3$ ordered triples $(M_1, M_2, M_3).$

There are $9 + 3 = 12$ ordered triples $(M_1, M_2, M_3).$

Similarly, for each of $M_1 + M_3 = M_2$ and $M_2 + M_3 = M_1,$ there are 12 ordered triples $(M_1, M_2, M_3).$

In this subcase, we have $\mathbf{12 \cdot 3 = 36}$ ordered triples $\boldsymbol{(M_1, M_2, M_3)}.$

Together, the answer is $8 + 168 + 126 + 36 = \boxed{\textbf{(B) } 338}.$

Solution 2 (Complementary Counting)

We will use complementary counting and do casework on the equations.

There are 8 possible equations:

Equation 1: $0 = 0$

Equation 2: $x = 0$

Equation 3: $y = 0$

Equation 4: $z = 0$

Equation 5: $x + y = 0$

Equation 6: $x + z = 0$

Equation 7: $y + z = 0$

Equation 8: $x + y + z = 0$

We will continue to refer to the equations by their number on this list.

$8^3 = 512$ total systems. Note that no two equations by themselves can force $x = y = z = 0$. Therefore no system with Equation 1 or with repeated equations can force $x = y = z = 0$.

$\textbf{Case 1:}$ Equation 8 ($x + y + z = 0$) is present.

Case 1^a: Equation 8, and two equations from $\{5, 6, 7\}$.

There are $\binom{3}{2} = 3$ ways to choose two equations from $\{5, 6, 7\}$ and $3! = 6$ ways to arrange each case. The number of options that force $x = y = z = 0$ is $3 \cdot 3! = 18$.

Case 1^b: Equation 8, one equation from $\{5, 6, 7\}$, and one equation from $\{2, 3, 4\}$.

There are $\binom{3}{1} = 3$ ways to choose one equation from $\{5, 6, 7\}$. WLOG let us choose Equation 7. Given $x + y + z = 0$ and $y + z = 0$, we conclude that $x = 0$. The third equation can be either $y = 0$ or $z = 0$. There are $3!$ ways to arrange each case. The number of options that force $x = y = z = 0$ is $3 \cdot 2 \cdot 3! = 36$.

Case 1^c: Equation 8, and two equations from $\{2, 3, 4\}$.

There are $\binom{3}{2} = 3$ ways to choose two equations from $\{2, 3, 4\}$ and $3! = 6$ ways to arrange each case. Each of these cases forces $x = y = z = 0$. $3 \cdot 3! = 18$ total options.

$\textbf{Case 2:}$ Equation 8 is $\textbf{not}$ present, at least one equation from $\{5, 6, 7\}$ is present.

Case 2^a: Equations $\{5, 6, 7\}$ are all present.

There are $3!$ ways to arrange the three equations. 6 options.

Case 2^b: Two equations from $\{5, 6, 7\}$ are present. One equation from $\{2, 3, 4\}$ is present.

There are $\binom{3}{2}$ ways to choose two equations from $\{5, 6, 7\}$. WLOG let Equations 5 and 6 be in our system: $x + y = 0$ and $x + z = 0$. Any equation from $\{2, 3, 4\}$ will force $x = y = z = 0$. There are $3!$ ways to arrange the equations. The number of options that force $x = y = z = 0$ is $\binom{3}{2} \cdot \binom{3}{1} \cdot 3! = 54$.

Case 2^c: One equation from $\{5, 6, 7\}$ is present. Two equations from $\{2, 3, 4\}$ are present.

There are $\binom{3}{1}$ ways to choose one equation from $\{5, 6, 7\}$. WLOG let Equation 5 ($x + y = 0$) be present. One of the two equations from $\{2,3,4\}$ must be Equation 4, $z=0$, since it is the only equation that restricts z. The last equation can be either 2 or 3. There are $3!$ ways to arrange the equations. The number of options that force $x=y=z=0$ is $\binom{3}{1}\cdot\binom{2}{1}\cdot 3! = 36$.

$\textbf{Case 3:}$ Only equations $\{2,3,4\}$ are present.

There are $3!$ ways to arrange the three equations. 6 options.

We add up the cases: $18+36+18+6+54+36+6=174$ total systems force $x=y=z=0$. Thus $512-174=\boxed{\textbf{(B) } 338}$ do not.

Solution 3 (Complementary Counting, Linear Dependence, Vector Analysis)

Denote vector $\overrightarrow{i}=(i_1,i_2,i_3)^T$ for $i\in\{a,b,c\}$. Thus, we need to count how many vector tuples $\left(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}\right)$ are linearly dependent.

We do complementary counting.

First, the total number of vector tuples $\left(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}\right)$ is $\left(2^3\right)^3=512$.

Second, we count how many many vector tuples $\left(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}\right)$ are linearly independent.

To meet this condition, no vector can be a zero vector $\overrightarrow{0}=(0,0,0)^T$.

Next, we do the casework analysis.

Case 1^c: Three vectors are all on axes.

In this case, the number of $\left(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right)$ is $3!$.

Case 2^c: Two vectors are on axes and the third vector is not.

We construct such an instance in the following steps.

Step 1: We determine which two vectors lie on axes.

The number of ways is 3.

Step 2: For two vectors selected in Step 1, we determine which two axes they lie on.

The number of ways is $3 \cdot 2$.

Step 3: For the third unselected vector, we determine its value.

To make three vectors linear independent, the third vector cannot be on the plane formed by the first two vectors. So the number of ways is 3.

Following from the rule of product, the number of $\left(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right)$ in this case is $3 \cdot 3 \cdot 2 \cdot 3$.

Case 3^c: One vector is on an axis and the other two are not.

We construct such an instance in the following steps.

Step 1: We determine which vector lies on an axis.

The number of ways is 3.

Step 2: For the selected vector, we determine which axis it lies on.

The number of ways is 3.

Step 3: We determine the values of the two unselected vectors.

First, to be linearly independent, these two vectors are distinct. Second, to be linearly independent, we cannot have one vector $(1, 1, 1)$ and another one that is a diagonal vector on the plane that is perpendicular to the first selected vector.

Thus, the number or ways in this step is $4 \cdot 3 - 2 = 10$.

Following from the rule of product, the number of $\left(\vec{a}, \vec{b}, \vec{c}\right)$ in this case is $3 \cdot 3 \cdot 10$.

Case $(4.4)^c$: No vector is on any axis.

In this case, any three distinct vectors are linearly independent. So the number of $\left(\vec{a}, \vec{b}, \vec{c}\right)$ in this case is $4 \cdot 3 \cdot 2$.

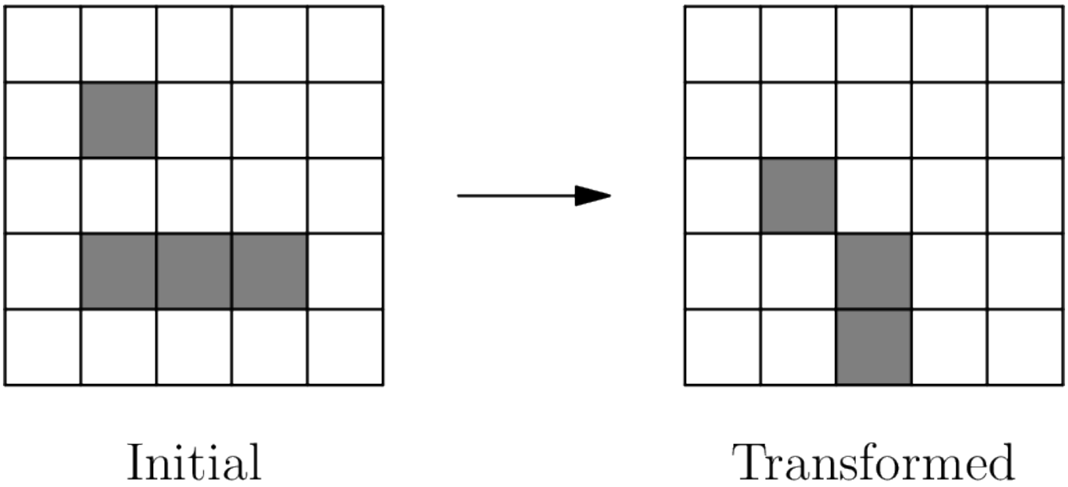
Putting all cases together, the number of vector tuples $\left(\vec{a}, \vec{b}, \vec{c}\right)$ that are linearly independent
is
$8^3 - 3! - 3 \cdot 3 \cdot 2 \cdot 3 - 3 \cdot 3 \cdot 10 - 4 \cdot 3 \cdot 2 = \boxed{\textbf{(B) } 338}$.

Problem19

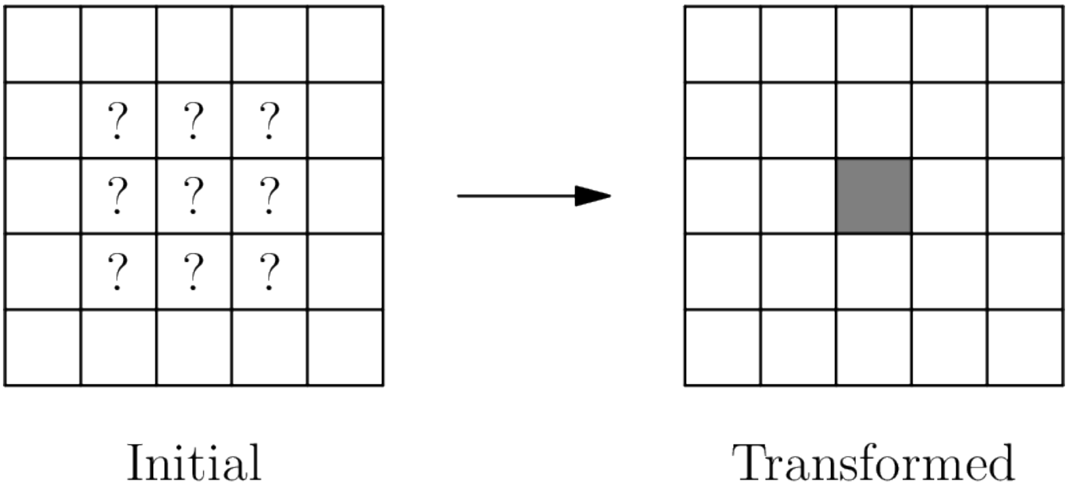
Each square in a 5×5 grid is either filled or empty, and has up to eight adjacent neighboring squares, where neighboring squares share either a side or a corner. The grid is transformed by the following rules:

- Any filled square with two or three filled neighbors remains filled.
- Any empty square with exactly three filled neighbors becomes a filled square.
- All other squares remain empty or become empty.

A sample transformation is shown in the figure below.



Suppose the 5×5 grid has a border of empty squares surrounding a 3×3 subgrid. How many initial configurations will lead to a transformed grid consisting of a single filled square in the center after a single transformation? (Rotations and reflections of the same configuration are considered different.)



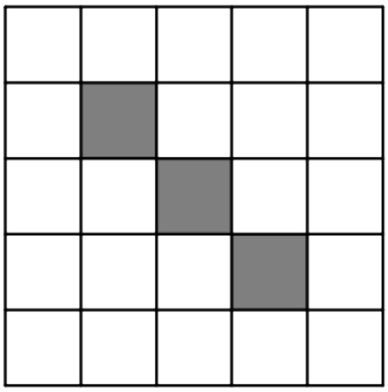
$\textbf{(A) } 14 \qquad \textbf{(B) } 18 \qquad \textbf{(C) } 22 \qquad \textbf{(D) } 26 \qquad \textbf{(E) } 30$

Solution

There are two cases for the initial configuration:

1. The center square is filled.

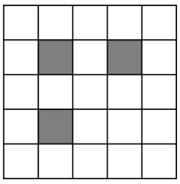
   Exactly two of the eight adjacent neighboring squares of the center are filled. Clearly, the only possibility is that the squares along one diagonal are filled, as shown below:



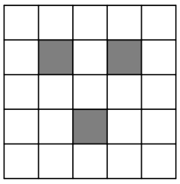
$2\ \text{Configurations}$ In this case, there are 2 possible initial configurations. All rotations and reflections are considered.

2. The center square is empty.

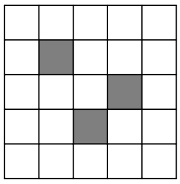
   Exactly three of the eight adjacent neighboring squares of the center are filled. The possibilities are shown below:



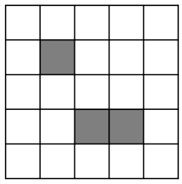
4 Configurations



4 Configurations



4 Configurations



8 Configurations

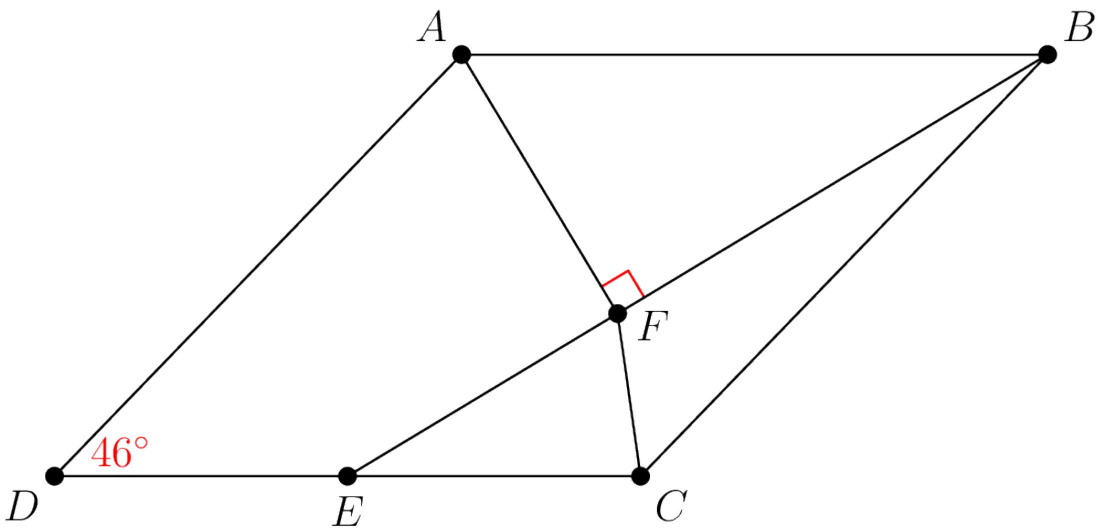
   In this case, there are $4+4+4+8=20$ possible initial configurations. All rotations and reflections are considered.

Together, the answer is $2+20=\boxed{\textbf{(C) } 22}$.

Problem20

Let $ABCD$ be a rhombus with $\angle ADC = 46^\circ$. Let E be the midpoint of $\overline{CD}$, and let F be the point on $\overline{BE}$ such that $\overline{AF}$ is perpendicular to $\overline{BE}$. What is the degree measure of $\angle BFC$?

Diagram



Solution 1 (Law of Sines and Law of Cosines)

Without loss of generality, we assume the length of each side of $ABCD$ is 2.

Because E is the midpoint of CD, $CE = 1$.

Because $ABCD$ is a rhombus, $\angle BCE = 180^\circ - \angle D$.

In $\triangle BCE$, following from the law of sines, $\frac{CE}{\sin \angle FBC} = \frac{BC}{\sin \angle BEC}$.

We have $\angle BCE = 180^\circ - \angle FBC - \angle BCE = 46^\circ - \angle FBC$.

Hence, $\frac{1}{\sin \angle FBC} = \frac{2}{\sin \left(46^\circ - \angle FBC \right)}$.

By solving this equation, we get $\tan \angle FBC = \frac{\sin 46^\circ}{2 + \cos 46^\circ}$.

Because $AF \perp BF$,

$$
\begin{aligned}
BF & = AB \cos \angle ABF \\
& = 2 \cos \left(46^\circ - \angle FBC \right) .
\end{aligned}
$$

In $\triangle BFC$, following from the law of sines, $\frac{BF}{\sin \angle BCF} = \frac{BC}{\sin \angle BFC}$.

Because $\angle BCF = 180^\circ - \angle BFC - \angle FBC$, the equation above can be converted as $\frac{BF}{\sin\left(\angle BFC + \angle FBC\right)} = \frac{BC}{\sin \angle BFC}$.

Therefore,

$$\begin{aligned}
\tan \angle BFC &= \frac{\sin \angle FBC}{\cos\left(46^\circ - \angle FBC\right) - \cos \angle FBC} \\
&= \frac{1}{\sin 46^\circ - \left(1 - \cos 46^\circ\right) \cot \angle FBC} \\
&= \frac{\sin 46^\circ}{\cos 46^\circ - 1} \\
&= - \frac{\sin 134^\circ}{1 + \cos 134^\circ} \\
&= - \tan \frac{134^\circ}{2} \\
&= - \tan 67^\circ \\
&= \tan\left(180^\circ - 67^\circ\right) \\
&= \tan 113^\circ .
\end{aligned}$$

Therefore, $\angle BFC = \boxed{\textbf{(D) } 113}$.

Solution 2

Extend segments $\overline{AD}$ and $\overline{BE}$ until they meet at point G.

Because $\overline{AB} \parallel \overline{ED}$, we have $\angle ABG = \angle DEG$ and $\angle GDE = \angle GAB$, so $\triangle ABG \sim \triangle DEG$ by AA.

Because $ABCD$ is a rhombus, $AB = CD = 2DE$, so $AG = 2GD$, meaning that D is a midpoint of segment $\overline{AG}$.

Now, $\overline{AF} \perp \overline{BE}$, so $\triangle GFA$ is right and median $FD = AD$.

So now, because $ABCD$ is a rhombus, $FD = AD = CD$. This means that there exists a circle from D with radius AD that passes through F, A, and C.

AG is a diameter of this circle because $\angle AFG = 90^\circ$. This means that $\angle GFC = \angle GAC = \frac{1}{2}\angle GDC$, so $\angle GFC = \frac{1}{2}(180^\circ - 46^\circ) = 67^\circ$, which means that $\angle BFC = \boxed{\textbf{(D) } 113}$

Solution 3

Let $\overline{AC}$ meet $\overline{BD}$ at O, then $AOFB$ is cyclic and $\angle FBO = \angle FAO$.

Also, $AC \cdot BO = [ABCD] = 2 \cdot [ABE] = AF \cdot BE$, so $\frac{AF}{BO} = \frac{AC}{BE}$, thus $\triangle AFC \sim \triangle BOE$ by SAS, and $\angle OEB = \angle ACF$, then $\angle CFE = \angle EOC = \angle DAC = 67^\circ$,

and $\angle BFC = \boxed{\textbf{(D) } 113}$

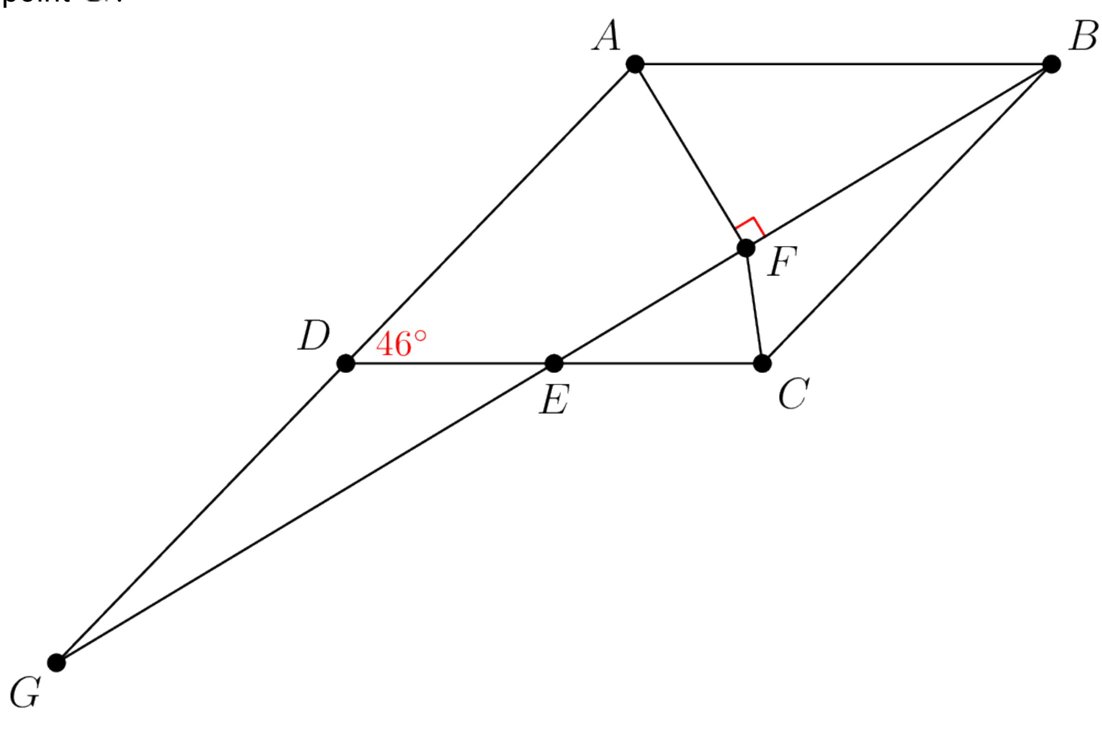
Solution 4

Observe that all answer choices are close to $112.5 = 90 + \frac{45}{2}$. A quick solve shows that having $\angle D = 90^\circ$ yields $\angle BFC = 135^\circ = 90 + \frac{90}{2}$, meaning that $\angle BFC$ increases with $\angle D$.

Substituting, $\angle BFC = 90 + \frac{46}{2} = \boxed{\textbf{(D) } 113}$

Solution 5 (Similarity and Circle Geometry)

Let's make a diagram, but extend AD and BE to point G.

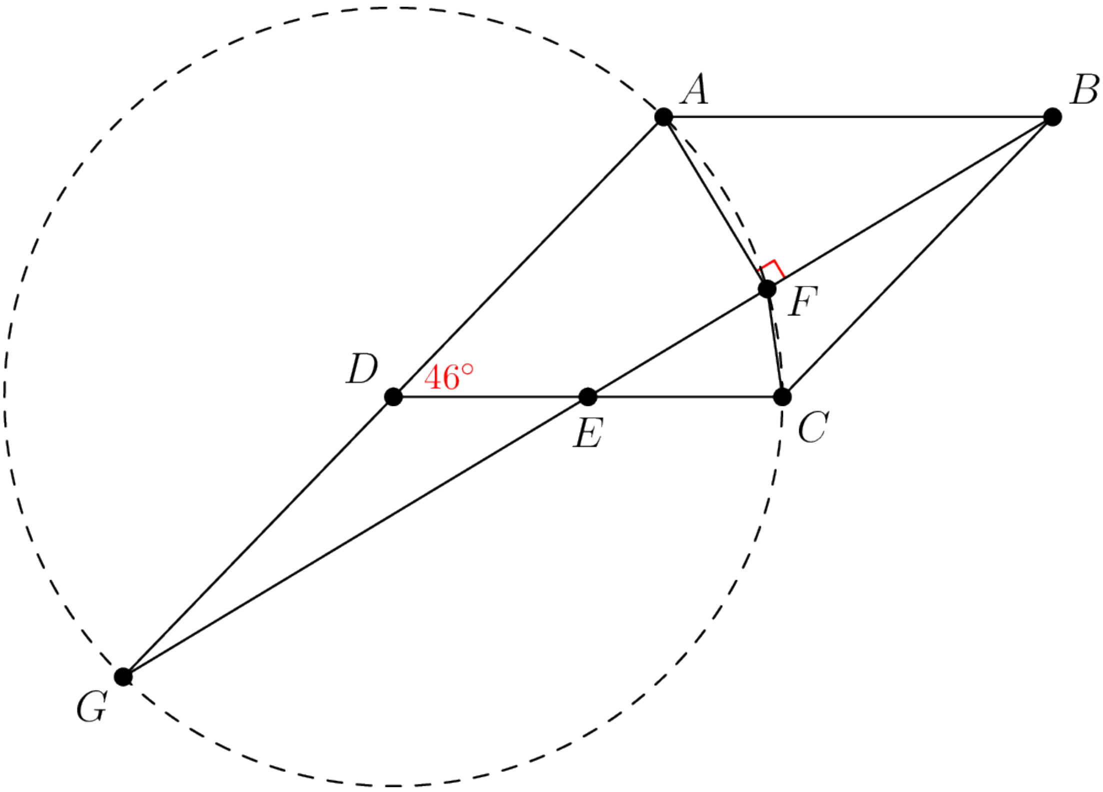


We know that $AB = AD = 2$ and $CE = DE = 1$.

By AA Similarity, $\triangle ABG \sim \triangle DEG$ with a ratio of $2:1$. This implies that $2AD = AG$ and $AD \cong DG$,

so $AG = 2AD = 2 \cdot 2 = 4$. That is, D is the midpoint of AG.

Now, let's redraw our previous diagram, but construct a circle with radius AD or 2 centered at D and by extending CD to point H, which is on the
circle.



Notice how F and C are on the circle and that $\angle CFE$ intercepts with $\overset{\frown}{CG}$.

Let's call $\angle CFE = \theta$.

Note that $\angle CDG$ also intercepts $\overset{\frown}{CG}$, So $\angle CDG = 2\angle CFE$.

Let $\angle CDG = 2\theta$. Notice how $\angle CDG$ and $\angle ADC$ are

$$2\theta = 180 - \angle ADC$$
$$2\theta = 180 - 46$$
$$2\theta = 134$$

supplementary to each other. We conclude that $\theta = 67.$

Since $\angle BFC = 180 - \theta$, we

have $\angle BFC = 180 - 67 = \boxed{\textbf{(D) } 113}$.

Problem21

Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial $x^2 + x + 1$, the remainder is $x + 2$, and when $P(x)$ is divided by the polynomial $x^2 + 1$, the remainder is $2x + 1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

Solution 1 (Experimentation)

Given that all the answer choices and coefficients are integers, we hope that $P(x)$ has positive integer coefficients.

Throughout this solution, we will express all polynomials in base x.
E.g. $x^2 + x + 1 = 111_x$.

We are given: $111a + 12 = 101b + 21 = P(x)$. We add 111 and 101 to each side and balance respectively: $111(a-1) + 123 = 101(b-1) + 122 = P(x).$ We make the units digits equal: $111(a-1) + 123 = 101(b-2) + 223 = P(x)$. We now notice that: $111(a-11)+1233=101(b-12)+1233=P(x).$

Therefore $a=11_x=x+1$, $b=12_x=x+2$,

and $P(x)=1233_x=x^3+2x^2+3x+3$. 3 is the minimal degree of $P(x)$ since there is no way to influence the x's digit in $101b+21$ when b is an integer. The desired sum is $1^2+2^2+3^2+3^2=\boxed{\textbf{(E) } 23}$

Solution 2

Let $P(x)=Q(x)(x^2+x+1)+x+2$,

then $P(x)=Q(x)(x^2+1)+xQ(x)+x+2$,

therefore $xQ(x)+x+2\equiv 2x+1 \pmod{x^2+1}$,

or $xQ(x)\equiv x-1 \pmod{x^2+1}$. Clearly the minimum is when $Q(x)=x+1$, and expanding gives $P(x)=x^3+2x^2+3x+3$. Summing the squares of coefficients gives $\boxed{\textbf{(E) } 23}$

Solution 3

Let $P(x)=(x^2+x+1)Q_1(x)+x+2$,

then $P(x)=(x^2+1)Q_1(x)+xQ_1(x)+x+2$

Also $P(x)=(x^2+1)Q_2(x)+2x+1$

We infer that $Q_1(x)$ and $Q_2(x)$ have same degree, we can assume $Q_1(x) = x + a$, and $Q_2(x) = x + b$, since $P(x)$ has least degree. If this cannot work, we will try quadratic, etc.

Then we get: $(x^2+1)(Q_1(x) - Q_2(x)) + xQ_1(x) - x + 1 = 0$

The constant term gives us: $(Q_1(x) - Q_2(x)) + 1 = 0$

So $Q_1(x) - Q_2(x) = -1$

Substituting this in gives: $-(x^2+1) + xQ_1(x) - x + 1 = 0$

Solving this equation, we get $Q_1(x) = x + 1$

Plugging this into our original equation we get $P(x) = x^3 + 2x^2 + 3x + 3$

Verify this works with $P(x) = (x^2+1)Q_2(x) + 2x + 1$

Therefore the answer is $1^2 + 2^2 + 3^2 + 3^2 = \boxed{\textbf{(E) } 23}$

Solution 4 (Undetermined Coefficients)

Notice that we cannot have the quotients equal to some constants, since the same constant will yield different constant terms for $P(x)$ (which is bad) and different constants will yield different first coefficients (also bad). Thus, we try setting the quotients equal to linear terms (for minimizing degree).

Let $P(x) = (x^2+x+1)(ax+b) + (x+2)$ and $P(x) = (x^2+1)(ax+c) + (2x+1)$. The quotients have the same x coefficient, since $P(x)$ must have the same x^3 coefficient in both cases. Expanding, we get $P(x) = ax^3 + (a+b)x^2 + (a+b+1)x + (b+2)$ and $P(x) = ax^3 + cx^2 + (a+2)x + (c+1)$.

Equating coefficients, we get $b+2 = c+1$, $a+b+1 = a+2$, and $a+b=c$. From the second equation, we get $b=1$, then substituting into the first, $c=2$. Finally, from $a+b=c$, we have $a=1$.

Now, $P(x) = (x^2+x+1)(ax+b) + (x+2) = (x^2+x+1)(x+1) + (x+2) = x^3+2x^2+3x+3$ and our answer is $1^2+2^2+3^2+3^2 = \boxed{\textbf{(E) } 23}$.

Solution 5: Quick (But not quicker than 2)

We construct the following equations in terms of $P(x)$ and the information given by the problem:$\textbf{(1)}\ \ P(x) = (x^2+x+1)\cdot Q(x) + x + 2$ $\textbf{(2)}\ \ P(x) = (x^2+1)\cdot R(x) + 2x + 1$ Upon inspection, $Q(x)$ and $R(x)$ cannot be constant, so the smallest possible degree of $P(x)$ is 3, and both $Q(x)$ and $R(x)$ are linear.

Let $Q(x) = x - q$ and $R(x) = x - r$. We know there will be values for q and r that make the below equation hold, so we can assume that $P(x)$ has a leading coefficient of 1.

Substituting these values in, and setting $\textbf{(1)}$ and $\textbf{(2)}$ equal to each other,

$$(x^2+x+1)(x-q) + x + 2 = (x^2+1)(x-r) + 2x + 1.$$

We plug in $x=0$, yielding $r+1=q$. Substituting this value into the above equation,

$$(x^2+x+1)(x-r-1) + x + 2 = (x^2+1)(x-r) + 2x + 1.$$

Letting $x = 1,$ we conclude that $r = -2,$ so $R(x) = x + 2.$ Therefore,

$$P(x) = (x^2 + 1)(x + 2) + 2x + 1 = x^3 + 2x^3 + 3x + 3.$$

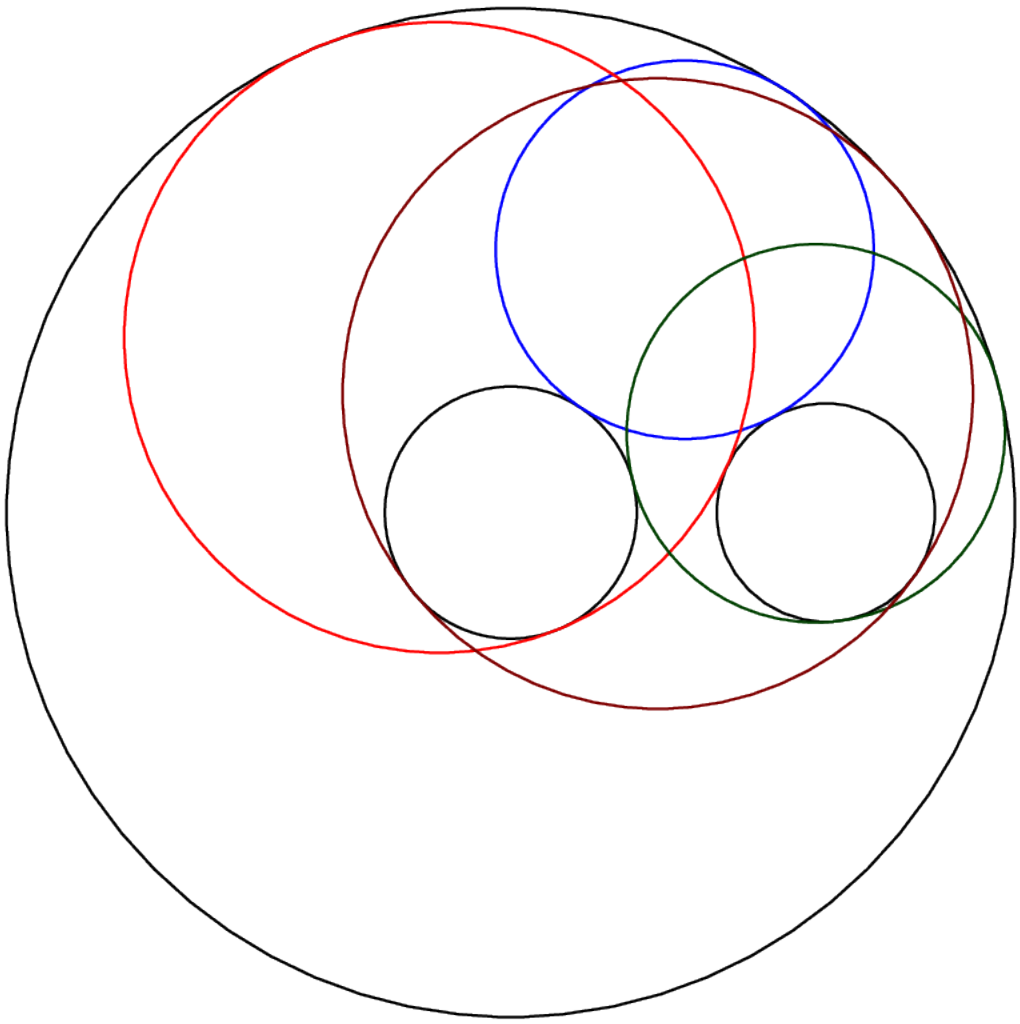
The requested sum is $1^2 + 2^2 + 3^2 + 3^2 = \boxed{\textbf{(E) } 23}$

Problem22

Let S be the set of circles in the coordinate plane that are tangent to each of the three circles with equations $x^2 + y^2 = 4$, $x^2 + y^2 = 64$, and $(x-5)^2 + y^2 = 3$. What is the sum of the areas of all circles in S?

$\textbf{(A) } 48\pi \qquad \textbf{(B) } 68\pi \qquad \textbf{(C) } 96\pi \qquad \textbf{(D) } 102\pi \qquad \textbf{(E) } 136\pi$

Solution 1



The circles match up as follows: Case 1 is brown, Case 2 is blue, Case 3 is green, and Case 4 is red. Let $x^2 + y^2 = 64$ be circle O, $x^2 + y^2 = 4$ be circle P, and $(x-5)^2 + y^2 = 3$ be circle Q. All the circles in S are internally tangent to circle O. There are four cases with two circles belonging to each:

$*$ P and Q are internally tangent to S.

$*$ P and Q are externally tangent to S.

$*$ P is externally and Circle Q is internally tangent to S.

$*$ P is internally and Circle Q is externally tangent to S.

Consider Cases 1 and 4 together. Since circles O and P have the same center, the line connecting the center of S and the center of O will pass through the tangency point of both S and O and the tangency point of S and P. This line will be the diameter of S and have length $r_P + r_O = 10$. Therefore the radius of S in these cases is 5.

Consider Cases 2 and 3 together. Similarly to Cases 1 and 4, the line connecting the center of S to the center of O will pass through the tangency points. This time, however, the diameter of S will have length $r_P - r_O = 6$. Therefore, the radius of S in these cases is 3.

The set of circles S consists of 8 circles - 4 of which have radius 5 and 4 of which have radius 3. The total area of all circles in S is $4(5^2\pi + 3^2\pi) = 136\pi \Rightarrow \boxed{\textbf{(E)}}$.

Solution 2

We denote by C_1 the circle that has the equation $x^2 + y^2 = 4$. We denote by C_2 the circle that has the equation $x^2 + y^2 = 64$. We denote by C_3 the circle that has the equation $(x-5)^2 + y^2 = 3$.

We denote by C_0 a circle that is tangent to C_1, C_2 and C_3. We denote by (u, v) the coordinates of circle C_0, and r the radius of this circle.

From the graphs of circles C_1, C_2, C_3, we observe that if C_0 is tangent to all of them, then C_0 must be internally tangent to C_2. We have

$$u^2 + v^2 = (8-r)^2 . \qquad (1)$$

We do the following casework analysis in terms of the whether C_0 is externally tangent to C_1 and C_3.

Case 1: C_0 is externally tangent to C_1 and C_3.

We have $u^2 + v^2 = (r+2)^2 \qquad (2)$

and $(u-5)^2 + v^2 = \left(r+\sqrt{3}\right)^2. \qquad (3)$

Taking $(2)-(1)$, we get $r+2 = 8-r$. Thus, $r=3$. We can further compute (omitted here) that there exist feasible (u,v) with this given r.

Case 2: C_1 is internally tangent to C_0 and C_3 is externally tangent to C_0.

We have $u^2 + v^2 = (r-2)^2 \qquad (2)$

and $(u-5)^2 + v^2 = \left(r+\sqrt{3}\right)^2. \qquad (3)$

Taking $(2)-(1)$, we get $r-2 = 8-r$. Thus, $r=5$. We can further compute (omitted here) that there exist feasible (u,v) with this given r.

Case 3: C_1 is externally tangent to C_0 and C_3 is internally tangent to C_0.

We have $u^2 + v^2 = (r+2)^2 \qquad (2)$

and $(u-5)^2 + v^2 = \left(r-\sqrt{3}\right)^2. \qquad (3)$

Taking $(2)-(1)$, we get $r+2 = 8-r$. Thus, $r=3$. We can further compute (omitted here) that there exist feasible (u,v) with this given r.

Case 4: C_1 is internally tangent to C_0 and C_3 is internally tangent to C_0.

We have $u^2 + v^2 = (r-2)^2 \qquad (2)$

and $(u-5)^2 + v^2 = \left(r-\sqrt{3}\right)^2. \qquad (3)$

Taking $(2)-(1)$, we get $r-2=8-r$. Thus, $r=5$. We can further compute (omitted here) that there exist feasible (u,v) with this given r.

Because the graph is symmetric with the x-axis, and for each case above, the solution of v is not 0. Hence, in each case, there are two congruent circles whose centers are symmetric through the x-axis.

Therefore, the sum of the areas of all the circles in S is $2\left(3^2\pi+5^2\pi+3^2\pi+5^2\pi\right)=\boxed{\textbf{(E) } 136\pi}$.

Problem23

Ant Amelia starts on the number line at 0 and crawls in the following manner. For $n=1,2,3,$ Amelia chooses a time duration t_n and an increment x_n independently and uniformly at random from the interval $(0,1)$. During the nth step of the process, Amelia moves x_n units in the positive direction, using up t_n minutes. If the total elapsed time has exceeded 1 minute during the nth step, she stops at the end of that step; otherwise, she continues with the next step, taking at most 3 steps in all. What is the probability that Amelia's position when she stops will be greater than 1?

$\textbf{(A) } \frac{1}{3} \qquad \textbf{(B) } \frac{1}{2} \qquad \textbf{(C) } \frac{2}{3} \qquad \textbf{(D) } \frac{3}{4} \qquad \textbf{(E) } \frac{5}{6}$

Solution 1

We use the following lemma to solve this problem.

Let $y_1, y_2, \cdots, y_n$ be independent random variables that are uniformly distributed on $(0,1)$. Then for $n=2$, $\mathbb{P}\left(y_1+y_2\leq 1\right)=\frac{1}{2}$.

For $n=3$,

$$\mathbb{P}\left(y_1+y_2+y_3\leq 1\right)=\frac{1}{6}. \text{ (Check remark for proof)}$$

Now, we solve this problem.

We denote by τ the last step Amelia moves. Thus, $\tau \in \{2, 3\}$. We have

$$\begin{aligned} P\left(\sum_{n=1}^{\tau} x_n > 1\right) & = P\left(x_1 + x_2 > 1 | t_1 + t_2 > 1\right) P\left(t_1 + t_2 > 1\right) \\ & \quad + P\left(x_1 + x_2 + x_3 > 1 | t_1 + t_2 \leq 1\right) P\left(t_1 + t_2 \leq 1\right) \\ & = P\left(x_1 + x_2 > 1\right) P\left(t_1 + t_2 > 1\right) + P\left(x_1 + x_2 + x_3 > 1\right) P\left(t_1 + t_2 \leq 1\right) \\ & = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{6}\right)\frac{1}{2} \\ & = \boxed{\textbf{(C) } \frac{2}{3}}, \end{aligned}$$

where the second equation follows from the property that $\{x_n\}$ and $\{t_n\}$ are independent sequences, the third equality follows from the lemma above.

Solution 2 (Clever)

There are two cases: Amelia takes two steps or three steps.

The former case has a probability of $\frac{1}{2}$, as stated above, and thus the latter also has a probability of $\frac{1}{2}$.

The probability that Amelia passes 1 after two steps is also $\frac{1}{2}$, as it is symmetric to the probability above.

Thus, if the probability that Amelia passes 1 after three steps is x, our total probability is $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot x$. We know that $0 < x < 1$, and it is relatively obvious that $x > \frac{1}{2}$ (because the probability that $x > \frac{3}{2}$ is $\frac{1}{2}$). This means that our total probability is between $\frac{1}{2}$ and $\frac{3}{4}$, non-inclusive, so the only answer choice that fits is $\boxed{\textbf{(C) } \frac{2}{3}}$

Solution 3

Obviously the chance of Amelia stopping after only 1 step is 0.

When Amelia takes 2 steps, then the sum of the time taken during the steps is greater than 1 minute. Let the time taken be x and y respectively, then we need $x+y>1$ for $0<x<1, 0<y<1$, which has a chance of $\frac{1}{2}$. Let the lengths of steps be a and b respectively, then we need $a+b>1$ for $0<a<1, 0<b<1$, which has a chance of $\frac{1}{2}$. Thus the total chance for this case is $\frac{1}{4}$.

When Amelia takes 3 steps, then by complementary counting the chance of taking 3 steps is $1-\frac{1}{2}=\frac{1}{2}$. Let the lengths of steps be a, b and c respectively, then we need $a+b+c>1$ for $0<a<1, 0<b<1, 0<c<1$, which has a chance of $\frac{5}{6}$ (Check remark for proof). Thus the total chance for this case is $\frac{5}{12}$.

Thus the answer is $\frac{1}{4}+\frac{5}{12}=\frac{2}{3}$.

Problem24

Consider functions f that satisfy $|f(x)-f(y)|\leq\frac12|x-y|$ for all real numbers x and y. Of all such functions that also satisfy the equation $f(300)=f(900)$, what is the greatest possible value of $f(f(800))-f(f(400))$?

$\textbf{(A)}\ 25 \qquad \textbf{(B)}\ 50 \qquad \textbf{(C)}\ 100 \qquad \textbf{(D)}\ 150 \qquad \textbf{(E)}\ 200$

Solution 1 (Absolute Values and Inequalities)

By definition, we have

$$\begin{aligned} |f(f(800))-f(f(400))| &\leq \frac12|f(800)-f(400)| && (\bigstar) \\ &\leq \frac12\left|\frac12|800-400|\right| \\ &= 100, \end{aligned}$$

from which we eliminate answer choices $\textbf{(D)}$ and $\textbf{(E)}$.

Note that
$$\begin{aligned} |f(800)-f(300)| &\leq 250, \\ |f(800)-f(900)| &\leq 50, \\ |f(400)-f(300)| &\leq 50, \\ |f(400)-f(900)| &\leq 250. \end{aligned}$$

Let $a=f(300)=f(900).$ Together, we conclude that
$$\begin{aligned} |f(800)-a| &\leq 50, \\ |f(400)-a| &\leq 50. \end{aligned}$$
We

rewrite $(\bigstar)$ as

$$
\begin{aligned}
|f(f(800)) - f(f(400))| &\leq \frac{1}{2}|f(800) - f(400)| \\
&= \frac{1}{2}|(f(800) - a) - (f(400) - a)| \\
&\leq \frac{1}{2}|50 - (-50)| \\
&= \boxed{\textbf{(B) } 50}.
\end{aligned}
$$

Solution 2 (Lipschitz Condition)

Denote $f(900) - f(600) = a$.

Because $f(300) = f(900)$, $f(300) - f(600) = a$.

Following from the Lipschitz condition given in this problem, $|a| \leq 150$ and

$$f(800) - f(600) \leq \min\left\{a + 50, 100\right\}$$

and $f(400) - f(600) \geq \max\left\{a - 50, -100\right\}$.

Thus,

$$
\begin{aligned}
f(800) - f(400) &\leq \min\left\{a + 50, 100\right\} - \max\left\{a - 50, -100\right\} \\
&= 100 + \min\left\{a, 50\right\} - \max\left\{a, -50\right\} \\
&= 100 + \begin{cases} a + 50 & \text{if } a \leq -50 \\ 0 & \text{if } -50 < a < 50 \\ -a + 50 & \text{if } a \geq 50 \end{cases}.
\end{aligned}
$$

Thus, $f(800) - f(400)$ is maximized at $a = 0$, $f(800) - f(600) = 50$, $f(400) - f(600) = -50$, with the maximal value 100.

By symmetry, following from an analogous argument, we can show that $f(800) - f(400)$ is minimized at $a=0$, $f(800)-f(600)=-50$,

$f(400)-f(600)=50$, with the minimal value -100.

Following from the Lipschitz condition,

$$\begin{aligned} f(f(800))-f(f(400)) &\leq \frac{1}{2}\left|f(800)-f(400)\right| \\ &\leq 50. \end{aligned}$$

We have already construct instances in which the second inequality above is augmented to an equality.

Now, we construct an instance in which the first inequality above is augmented to an equality.

Consider the following piecewise-linear function:

$$f(x)=\begin{cases} \frac{1}{2}(x-300) & \text{if } x\leq 300 \\ -\frac{1}{2}(x-300) & \text{if } 300<x\leq 400 \\ \frac{1}{2}(x-600) & \text{if } 400<x\leq 800 \\ -\frac{1}{2}(x-900) & \text{if } x>800 \end{cases}.$$

Therefore, the maximum value of $f(f(800))-f(f(400))$ is $\boxed{\textbf{(B) } 50}$.

Solution 3 (Educated Guess)

Divide both sides by $|x-y|$ to get $\frac{|f(x)-f(y)|}{|x-y|}\leq\frac{1}{2}$. This means that when we take any two points on f, the absolute value of the slope between the two points is at most $\frac{1}{2}$.

Let $f(300)=f(900)=c$, and since we want to find the maximum value of $|f(800)-f(400)|$, we can take the most extreme case and draw a line with slope $\frac{-1}{2}$ down from $f(300)$ to $f(400)$ and a line with slope $\frac{-1}{2}$ up from $f(900)$ to $f(800)$.

Then $f(400) = c - 50$ and $f(800) = c + 50$,

so $|f(800) - f(400)| = |c + 50 - (c - 50)| = 100$, and this is attainable because the slope of the line connecting $f(400)$ and $f(800)$ still has absolute value less than $\frac{1}{2}$.

Therefore,

$|f(f(800)) - f(f(400))| \leq \frac{1}{2}|f(800) - f(400)| = \frac{1}{2}(100) = \boxed{\textbf{(B) } 50}$

.

Problem25

Let $x_0, x_1, x_2, \dots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k$$

Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \geq 1$. What is the value of the sum $x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022}$

$\textbf{(A) } 6 \qquad \textbf{(B) } 7 \qquad \textbf{(C) } 12 \qquad \textbf{(D) } 14 \qquad \textbf{(E) } 15$

Solution

First, notice that

$$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022} = \frac{S_{2023} - S_{2019}}{2^{2019}}$$

Then since S_n is the modular inverse of 7 in $\mathbb{Z}_{2^n}$, we can perform the Euclidean algorithm to find it for $n = 2019, 2023$.

Starting with 2019, $7S_{2019} \equiv 1 \pmod{2^{2019}}$

$7S_{2019} = 2^{2019}k + 1$ Now, take both sides $\bmod 7$

$0 \equiv 2^{2019}k + 1 \pmod 7$ Using Fermat's Little Theorem, $2^{2019} = (2^{336})^6 \cdot 2^3 \equiv 2^3 \equiv 1 \pmod 7$

Thus,

$$0 \equiv k + 1 \pmod 7 \implies k \equiv 6 \pmod 7 \implies k = 7j + 6$$

Therefore,

$$7S_{2019} = 2^{2019}(7j + 6) + 1 \implies S_{2019} = \frac{2^{2019}(7j+6)+1}{7}$$

We may repeat this same calculation with S_{2023} to yield $S_{2023} = \dfrac{2^{2023}(7h+3)+1}{7}$ Now, we notice that S_n is basically an integer expressed in binary form with n bits. This gives rise to a simple inequality, $0 \leqslant S_n \leqslant 2^n$ Since the maximum possible number that can be generated with n bits is $\underbrace{11111\ldots1_2}_{n} = \sum_{k=0}^{n-1} 2^k = 2^n - 1 \leqslant 2^n$

Looking at our calculations for S_{2019} and S_{2023}, we see that the only valid integers that satisfy that constraint are $j = h = 0$

$$\frac{S_{2023} - S_{2019}}{2^{2019}} = \frac{\frac{2^{2023}\cdot 3+1}{7} - \frac{2^{2019}\cdot 6+1}{7}}{2^{2019}} = \frac{2^4 \cdot 3 - 6}{7} = \boxed{\textbf{(A) } 6}$$

Solution 2 (Base-2 Analysis)

We solve this problem with base 2. To avoid any confusion, for a base-2 number, we index the kth rightmost digit as digit $k - 1$.

We have $S_n = (x_{n-1} x_{n-2} \cdots x_1 x_0)_2$.

In the base-2 representation, $7 S_n \equiv 1 \pmod{2^n}$ is equivalent to

$$(x_{n-1} x_{n-2} \cdots x_1 x_0 000)_2 - (x_{n-1} x_{n-2} \cdots x_1 x_0)_2 - (1)_2 = \left(\cdots \underbrace{00 \cdots 0}_{n \text{ digits}} \right)_2 .$$

In the rest of the analysis, to lighten notation, we ease the base-2 subscription from all numbers. The equation above can be reformulated as:

\begin{table} \begin{tabular}{ccccccccc}

```
    & ... & 0 & ... & 0 & 0 & 0 & 0 & 0 \\
    &   &   &   &   &   &   &   & 1 \\
    +&  & x_{n-1} & ... & x_4 & x_3 & x_2 & x_1 & x_0 \\
\hline
    & x_{n-1} x_{n-2} x_{n-3} & x_{n-4} & ... & x_1 & x_0 & 0 & 0 & 0\\
```

\end{tabular} \end{table}

Therefore, $x_0 = x_1 = x_2 = 1$, $x_3 = 0$, and

for $k \geq 4$, $x_k = x_{k-3}$.

Therefore,

$$\begin{aligned} x_{2019} + 2 x_{2020} + 4 x_{2021} + 8 x_{2022} &= x_3 + 2 x_1 + 4 x_2 + 8 x_3 \\ &= \boxed{\textbf{(A) } 6}. \end{aligned}$$

Solution 3

As in Solution 1, we note that

$$x_{2019} + 2 x_{2020} + 4 x_{2021} + 8 x_{2022} = \frac{S_{2023} - S_{2019}}{2^{2019}} .$$

We also know that $7 S_{2023} \equiv 1 \pmod{2^{2023}}$ and $7 S_{2019} \equiv 1 \pmod{2^{2019}}$, this implies:

$$\textbf{(1)}\ \ 7 S_{2023} = 2^{2023} \cdot x + 1 \textbf{(2)}\ \ 7 S_{2019} = 2^{2019} \cdot y + 1.$$

Dividing by 7, we can isolate the previous sums:

$$\textbf{(3)}\ \ S_{2023} = \frac{2^{2023}\cdot x+1}{7}\textbf{(4)}\ \ S_{2019} = \frac{2^{2019}\cdot y+1}{7}.$$

The maximum value of S_n occurs when every x_i is equal to 1. Even when this happens, the value of S_n is less than 2^n. Therefore, we can construct the following inequalities:

$$\textbf{(3)}\ \ S_{2023} = \frac{2^{2023}\cdot x+1}{7} < 2^{2023}$$

$$\textbf{(4)}\ \ S_{2019} = \frac{2^{2019}\cdot y+1}{7} < 2^{2019}.$$

From these two equations, we can deduce that both x and y are less than 7.

Reducing $\mathbf{1}$ and $\mathbf{2}$ $\pmod 7,$ we see that

$$2^{2023}\cdot x \equiv 6 \pmod 7 \text{and} 2^{2019}\cdot y \equiv 6 \pmod 7.$$

The powers of 2 repeat every $3, \pmod 7.$

Therefore, $2^{2023} \equiv 2 \pmod 7$ and $2^{2019} \equiv 1 \pmod 7.$ Substituing this back into the above equations,

$$2x \equiv 6 \pmod 7 \text{and} y \equiv 6 \pmod 7.$$

Since x and y are integers less than 7, the only values of x and y are 3 and 6 respectively.

The requested sum is

$$\frac{S_{2023}-S_{2019}}{2^{2019}} = \frac{\frac{2^{2023}\cdot x+1}{7} - \frac{2^{2019}\cdot y+1}{7}}{2^{2019}}$$

$$= \frac{1}{2^{2019}}\left(\frac{2^{2023}\cdot 3+1}{7} - \left(\frac{2^{2019}\cdot 6+1}{7}\right)\right)$$

$$= \frac{3\cdot 2^4 - 6}{7} = \boxed{\textbf{(A) }6}.$$