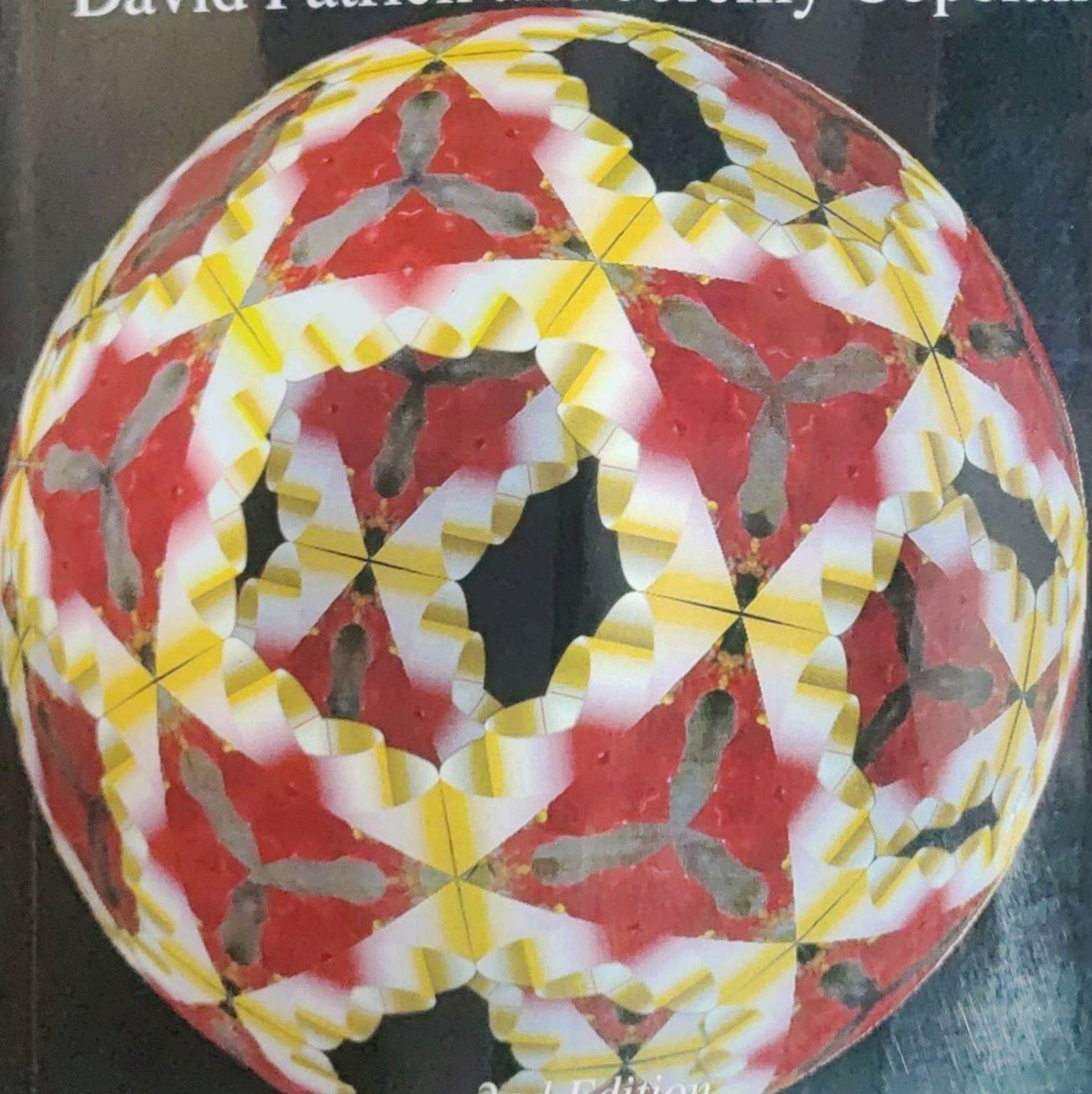
the Art of Problem Solving

Calculus
Solutions Manual

David Patrick and Jeremy Copeland



2nd Edition

Solutions Manual
Calculus
Second Edition

David Patrick and Jeremy Copeland
Art of Problem Solving

Published by: AoPS Incorporated
10865 Rancho Bernardo Rd Ste 100
San Diego, CA 92127
`books@artofproblemsolving.com`

ISBN-13: 978-1-934124-25-3

Visit the Art of Problem Solving website at `http://www.artofproblemsolving.com`

Scan this code with your mobile device to visit the Art of Problem Solving website, to view our other books, our free videos and interactive resources, our online community, and our online school.

Cover image designed by Vanessa Rusczyk using KaleidoTile software.

Printed in the United States of America.

Second Edition. Printed in 2020.

Preface

This book contains the full solution to every Exercise, Review Problem, and Challenge Problem in the text *Calculus*.

In most non-proof problems, the final answer is contained in a box, $\boxed{\text{like this}}$. However, we strongly recommend against just looking up the final answer and moving on to the next problem. Instead, even if you got the right answer, read the solution in this book. It might show you a different way of solving the problem that you might not have discovered.

Also, please keep in mind that the answer you get may not exactly match the format of the answer in this book. For example, the expressions

$$x^{\frac{3}{2}} \quad \text{and} \quad \sqrt{x^3}$$

or the quantities

$$\log 5 - 2\log 2 \quad \text{and} \quad \log \frac{5}{4}$$

are each exactly the same, even if at first glance they appear different.

If you believe that you have found an error in one of our solutions, please check the book's links page at

`http://www.artofproblemsolving.com/booklinks/calculus`

We will maintain on this page an errata list of all of the errors (in both the text and solutions manual) that get reported to us. If you find an error that is not yet in the errata list on the above webpage, please send us an email at `books@artofproblemsolving.com` to let us know.

If you don't understand a solution, or you think you have a better way of solving the problem, we invite you to come to our message board at

`www.artofproblemsolving.com`

and discuss it. Our message board is free to use and includes thousands of the world's most eager mathematical problem-solvers. (See more about our website on page 122.)

Contents

Preface iii

1 Sets and Functions 1

2 Limits and Continuity 15

3 The Derivative 23

4 Applications of the Derivative 33

5 Integration 48

6 Infinity 68

7 Series 80

8 Plane Curves 96

9 Differential Equations 108

CHAPTER 1

Sets and Functions

Exercises for Section 1.1

1.1.1 Suppose that $x \in A$. Then since $A \subseteq B$, we have $x \in B$, and since $A \subseteq C$, we have $x \in C$. Therefore, $x \in (B \cap C)$. Thus, every element of A is also an element of $B \cap C$, and we conclude that $A \subseteq (B \cap C)$.

The statement is not necessarily true if we have proper subsets. To construct a simple example, let $A = \{1\}$, $B = \{1,2\}$, and $C = \{1,3\}$. Then clearly $A \subset B$ and $A \subset C$, but $A = B \cap C = \{1\}$, so $A \not\subset (B \cap C)$.

1.1.2 First, we show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Suppose that $x \in A \cup (B \cap C)$. We have two cases:

Case 1: $x \in A$. Then $x \in (A \cup B)$ and $x \in (A \cup C)$, so $x \in (A \cup B) \cap (A \cup C)$ as desired.

Case 2: $x \in (B \cap C)$. Then $x \in B$ and $x \in C$. Therefore, $x \in (A \cup B)$ and $x \in (A \cup C)$, so $x \in (A \cup B) \cap (A \cup C)$ as desired.

We have shown that each element x in $A \cup (B \cap C)$ is an element of $(A \cup B) \cap (A \cup C)$, so $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now we will show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Suppose that $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. We now divide the possibilities for x into two cases:

Case 1: $x \in A$. Then $x \in A \cup (B \cap C)$ as desired.

Case 2: $x \notin A$. Then since $x \in A \cup B$, we have $x \in B$, and since $x \in A \cup C$, we have $x \in C$. Therefore, $x \in B \cap C$, and hence $x \in A \cup (B \cap C)$ as desired.

Therefore, we have shown that any element x in $(A \cup B) \cap (A \cup C)$ is an element of $A \cup (B \cap C)$, so $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Thus, $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

1.1.3 This is essentially a tautology. If $x \in A$, then $x \in A$, so $A \subseteq A$.

1.1.4

(a) Set difference is *not* commutative. For example, let $A = \{1,2\}$ and $B = \{1,3\}$. Then $A \setminus B = \{2\}$ whereas $B \setminus A = \{3\}$.

(b) Set difference is *not* associative. For example, let $A = \{1,2,3\}$, $B = \{1,2\}$, and $C = \{1\}$. Then

$$(A \setminus B) \setminus C = (\{1,2,3\} \setminus \{1,2\}) \setminus \{1\} = \{3\} \setminus \{1\} = \{3\},$$

whereas

$$A \setminus (B \setminus C) = \{1,2,3\} \setminus (\{1,2\} \setminus \{1\}) = \{1,2,3\} \setminus \{2\} = \{1,3\}.$$

(c) We claim that x is an element of each of these sets if and only if $x \in A$, $x \notin B$, and $x \notin C$. We prove this separately for each set:

$$\begin{aligned}
x \in A \setminus (B \cup C) &\Leftrightarrow x \in A \text{ and } x \notin (B \cup C)\\
&\Leftrightarrow x \in A \text{ and } ((x \notin B) \text{ and } (x \notin C))\\
&\Leftrightarrow x \in A \text{ and } x \notin B \text{ and } x \notin C.\\
x \in (A \setminus B) \cap (A \setminus C) &\Leftrightarrow (x \in (A \setminus B)) \text{ and } (x \in (A \setminus C))\\
&\Leftrightarrow (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C)\\
&\Leftrightarrow x \in A \text{ and } x \notin B \text{ and } x \notin C.\\
x \in (A \setminus B) \setminus C &\Leftrightarrow x \in (A \setminus B) \text{ and } x \notin C\\
&\Leftrightarrow (x \in A \text{ and } x \notin B) \text{ and } x \notin C\\
&\Leftrightarrow x \in A \text{ and } x \notin B \text{ and } x \notin C.
\end{aligned}$$

(d) $A \setminus \emptyset = A$, because there are no elements to exclude. On the other hand, $\emptyset \setminus A = \emptyset$, because $\emptyset \setminus A$ is a subset of $\emptyset$ by definition, and the only subset of the empty set is itself.

1.1.5 We show these sets are equal by showing that each is a subset of the other.

Step 1: First we show that $(A \cup B) \setminus (C \setminus A) \subseteq A \cup (B \setminus C)$. Let x be an element of the left hand side. If $x \in A$, then certainly $x \in A \cup (B \setminus C)$. If $x \notin A$, that means that $x \in B$ but not in $C \setminus A$. Since we have assumed already that $x \notin A$, x cannot be in C, so x must be in $B \setminus C$. This shows

$$(A \cup B) \setminus (C \setminus A) \subseteq A \cup (B \setminus C).$$

Step 2: Next we show that $A \cup (B \setminus C) \subseteq (A \cup B) \setminus (C \setminus A)$. If $x \in A$, then $x \in A \cup B$ and also $x \notin C \setminus A$, showing

$$A \subseteq (A \cup B) \setminus (C \setminus A).$$

Likewise, since $(C \setminus A)$ is a subset of C, we see that

$$B \setminus C \subseteq B \setminus (C \setminus A) \subseteq (A \cup B) \setminus (C \setminus A).$$

Combining these two we get that

$$A \cup (B \setminus C) \subseteq (A \cup B) \setminus (C \setminus A).$$

Exercises for Section 1.2

1.2.1 Suppose a and b are each greatest lower bounds for S. Then since a is a greatest lower bound and b is a lower bound, we have $b \le a$; but since b is a greatest lower bound and a is a lower bound, we have $a \le b$. Thus $a = b$, and hence the greatest lower bound is unique.

1.2.2 We must show that if $c, d \in (a, b)$ and $x \in \mathbb{R}$ is such that $c \le x \le d$, then $x \in (a, b)$. But we know that $a < c$ and $d < b$, so we have

$$a < c \le x \le d < b,$$

so in particular $a < x < b$, and thus $x \in (a, b)$.

Similarly, if $c, d \in [a, b]$ and $x \in \mathbb{R}$ is such that $c \le x \le d$, then

$$a \le c \le x \le d \le b,$$

so $a \le x \le b$, and hence $x \in [a, b]$.

1.2.3

(a) Let I, J be intervals, and suppose $a, b \in I \cap J$ and $x \in \mathbb{R}$ is such that $a \leq x \leq b$. Thus, since $a, b \in I$ and I is an interval, we have $x \in I$; similarly, since $a, b, \in J$ and J is an interval, we have $x \in J$. So $x \in I \cap J$, and thus, by definition, $I \cap J$ is an interval.

(b) There are many examples. For example, $S = (0,1) \cup (2,3)$ is not an interval, since $\frac{1}{2}, \frac{5}{2} \in S$, and $\frac{1}{2} \leq \frac{3}{2} \leq \frac{5}{2}$, but $\frac{3}{2} \notin S$.

1.2.4 First notice that since A and B are bounded, any upper bound for A is an upper bound for $A \cap B$. (The same holds for lower bounds.) Therefore $A \cap B$ is bounded and nonempty, so has a least upper bound.

Since $\sup A$ and $\sup B$ are both upper bounds for $A \cap B$, the smaller of the two must certainly be an upper bound for $A \cap B$. Therefore

$$\sup(A \cap B) \leq \min\{\sup A, \sup B\}.$$

We complete the proof by showing that this inequality must be an equality.

Assume that $\sup(A \cap B) < \min\{\sup A, \sup B\}$. Let x be any number such that $\sup(A \cap B) < x < \min\{\sup A, \sup B\}$ (for example, we can set x to be the average of $\sup(A \cap B)$ and $\min\{\sup A, \sup B\}$). Now, since $\sup(A \cap B) < x$, there is some $c \in A \cap B$ such that $c < x$. Notice also that, since $x < \sup A$, there is some $d \in A$ such that $x < d$ (otherwise x is an upper bound for A, contradicting the fact that x is less than the least upper bound). Thus, since c and d are in A and

$$c < x < d,$$

we get that $x \in A$ as well. Likewise there is some $d' \in B$ which is larger than x, so

$$c < x < d'$$

and c and d' are both in B, so $x \in B$. Combining these we see that $x \in A \cap B$. However we assumed that $\sup(A \cap B) < x$, which is a contradiction. Therefore

$$\sup(A \cap B) = \min\{\sup A, \sup B\}.$$

Exercises for Section 1.3

1.3.1 For $f(x)$ to be defined, we must have $x^2 - 7x + 12 > 0$. This factors as $(x-4)(x-3) > 0$, so we must have $x < 3$ or $x > 4$. Thus, the domain is $\boxed{(-\infty, 3) \cup (4, +\infty)}$. (The numerator is irrelevant to the domain, as it is defined for all $x \in \mathbb{R}$.)

1.3.2 The function is defined for all $x \in \mathbb{R}$, so $\boxed{\mathrm{Dom}(f) = \mathbb{R}}$. The $|2x - 3|$ term can be any nonnegative real number, so $\boxed{\mathrm{Rng}(f) = [5, +\infty)}$.

1.3.3 A function can have domain $\emptyset$, but then we cannot plug any value into it. So such a function must also have range $\emptyset$. A function can have range $\emptyset$ only if its domain is also $\emptyset$.

1.3.4 Let $y \in f(A)$. Then by definition $y = f(x)$ for some $x \in A$. But since $A \subseteq B$, we also have $x \in B$, so $y \in f(B)$. Since every element of $f(A)$ is also an element of $f(B)$, we conclude that $f(A) \subseteq f(B)$.

If the inclusion $A \subset B$ is proper, this does not necessarily mean that $f(A) \subset f(B)$. For a trivial example, consider the constant function $f(x) = 1$ for all $x \in \mathbb{R}$. Then $f(A) = \{1\}$ for all nonempty subsets $A \subseteq \mathbb{R}$, and hence $f(A) = f(B)$ for any nonempty subsets $A \subset B$.

1.3.5 The symbol $f^{-1}(y)$ denotes the unique element $x \in \mathrm{Dom}(f)$ such that $f(x) = y$. This element is unique since if $f(x') = y$,

$$x' = f^{-1}(f(x)) = x.$$

Likewise, $f^{-1}(\{y\})$ denotes the set of x such that $f(x) = y$. By above,

$$f^{-1}(\{y\}) = \{x\}.$$

Therefore $f^{-1}(y) \in f^{-1}(\{y\})$, and is the unique element thereof. We can write this as $\boxed{f^{-1}(\{y\}) = \{f^{-1}(y)\}}$.

Exercises for Section 1.4

1.4.1 If such a line exists, suppose it intersects the graph of f at points (c, b) and (d, b) with $c \neq d$. Then, since $f(c) = b$, we would require $f^{-1}(b) = c$, but since $f(d) = b$, we would require $f^{-1}(b) = d$. Since $f^{-1}(b)$ must be a unique value, we conclude that f^{-1} cannot exist, and thus f does not have an inverse.

1.4.2

(a) We see immediately that $\mathrm{Dom}(f) = \mathbb{R}$. To compute the range of f, notice that the graph of f is a parabola opening upward with roots 0 and 2, so its vertex is at $x = 1$, where $f(1) = -1$. Thus the range of f is $[-1, +\infty)$.

The function g is defined for $1 - x \geq 0$, or $1 \geq x$, so its domain is $(-\infty, 1]$. Since the square root can be any nonnegative real number, we have $\mathrm{Rng}(g) = [0, +\infty)$.

(b) We compute that

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{1-x}) = (\sqrt{1-x})^2 - 2\sqrt{1-x} = 1 - x - 2\sqrt{1-x}.$$

Since $\mathrm{Dom}(f) = \mathbb{R}$, the function $(f \circ g)$ is defined wherever g is defined, so $\mathrm{Dom}(f \circ g) = (-\infty, 1]$. Furthermore,

$$\mathrm{Rng}(f \circ g) = f(\mathrm{Rng}(g)) = f([0, +\infty)) = \mathrm{Rng}(f) = [-1, +\infty),$$

since 0 is to the left of the vertex of f.

Also,

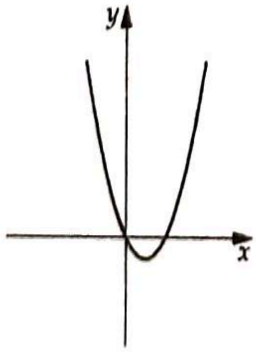
$$(g \circ f)(x) = g(f(x)) = g(x^2 - 2x) = \sqrt{1 - x^2 + 2x}.$$

For this to be defined, we must have $1 - x^2 + 2x \geq 0$, or $x^2 - 2x - 1 \leq 0$. This quadratic has roots $1 \pm \sqrt{2}$, so the domain of $g \circ f$ is $[1 - \sqrt{2}, 1 + \sqrt{2}]$. Also, the quadratic $-x^2 + 2x + 1$ has its maximum at its vertex $(1, 2)$, so the range of $g \circ f$ is $[0, \sqrt{2}]$.

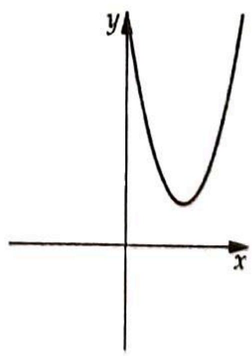
(c) As discussed in part (a), the graph of $f(x)$ is a parabola with vertex $(1, -1)$ and roots at $x = 0$ and $x = 2$; its graph is shown to the left below. The graph of $f(x-2) + 3$ shifts the graph 2 units to the right and 3 units up, so has vertex

$$(1 + 2, -1 + 3) = (3, 2).$$

Its graph is shown to the right below.

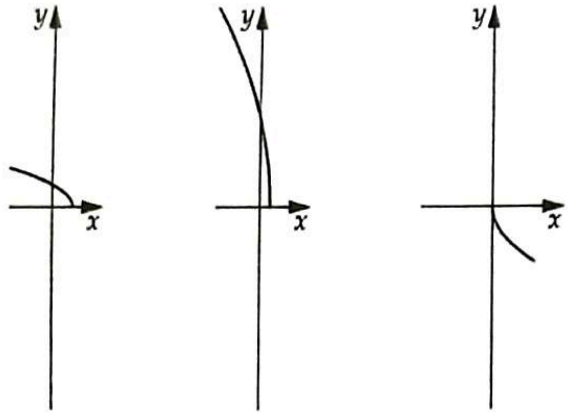


$f(x)$



$f(x-2) + 3$

(d) The graph of $g(x)$ is the top half of a leftward-opening parabola with vertex at $(1,0)$. Its graph is shown on the left below. The graph of $4g(2x)$ compacts by a factor of 2 in the horizontal direction and expands by a factor of 4 in the vertical direction; its graph is shown in the middle below. The graph of $-g(-3x+1)$ reflects over the y-axis, compacts horizontally by a factor of 3, shifts left by $\frac{1}{3}$ unit, and then reflects over the x-axis; its graph is shown to the right below.



$g(x)$ $4g(2x)$ $-g(-3x+1)$

1.4.3 We can define the domain of f as

$$\mathrm{Dom}(f) = \{a \in \mathbb{R} \mid \text{the line } x = a \text{ intersects } A\}.$$

Then, since each such line $x = a$ intersects A in a unique point (a, b), we then may define $f(a) = b$ for all $a \in \mathrm{Dom}(f)$. This defines our function f such that A is the graph $y = f(x)$.

Exercises for Section 1.5

1.5.1 Recall that $\sec = \dfrac{1}{\cos}$. Therefore the domain of secant is all x for which $\cos x \neq 0$:

$$\mathrm{Dom}(\sec) = \{x \mid \cos x \neq 0\} = \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\right\}.$$

Likewise, since $\csc = \dfrac{1}{\sin}$ and $\cot = \dfrac{\cos}{\sin}$, these functions are defined whenever sine is nonzero. Therefore

$$\mathrm{Dom}(\csc) = \mathrm{Dom}(\cot) = \{x \mid \sin(x) \neq 0\} = \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}.$$

Since sin and cos each have range $[-1, 1]$, we have

$$\mathrm{Rng}(\sec) = \mathrm{Rng}(\csc) = (-\infty, -1] \cup [1, +\infty).$$

Finally, cot has the same range as tan, so $\mathrm{Rng}(\cot) = \mathbb{R} = (-\infty, +\infty)$.

1.5.2

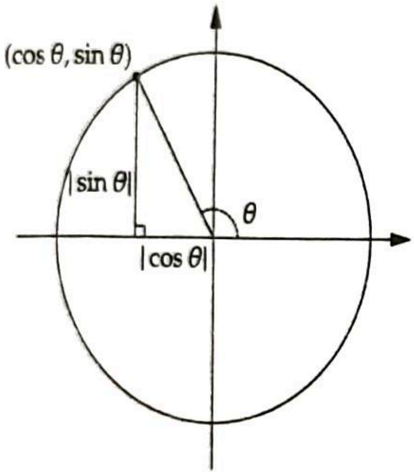
(a) Referring to the unit circle, $\sin(-\theta)$ is the y-coordinate of a point at angle $-\theta$. This point is the reflection through the x-axis of the point at angle θ. Reflecting through the x-axis changes the sign of the y-coordinate, so $\boxed{\sin(-\theta) = -\sin(\theta)}$.

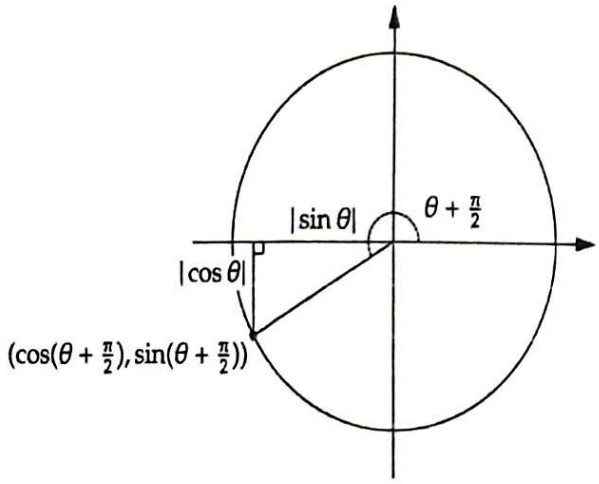
(b) As in part (a), $\cos(-\theta)$ is the x-coordinate of the point at angle $-\theta$, which is the reflection of the point at θ through the x-axis. Reflecting through the x-axis does not affect the x-coordinate, so $\boxed{\cos(-\theta) = \cos(\theta)}$.

1.5.3 If $\theta \neq (k+\frac{1}{2})\pi$, then $\tan\theta$ and $\tan(\theta+\pi)$ are both defined. Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, we see that $\tan\theta$ is the slope of the line through the origin O and the point $P = (\cos\theta, \sin\theta)$. In addition, $\theta+\pi$ is the angle of the point Q that is a rotation by angle π about O of the point P. This tells us that $\overleftrightarrow{OP} = \overleftrightarrow{OQ}$, and thus

$$\tan(\theta+\pi) = \text{slope}(\overleftrightarrow{OQ}) = \text{slope}(\overleftrightarrow{OP}) = \tan\theta.$$

1.5.4 The angle $\frac{\pi}{2}+\theta$ is the image of θ upon a counterclockwise rotation of $\frac{\pi}{2}$. If we draw the right triangle with one leg on the x-axis and hypotenuse the segment from the origin to the terminal point for each of θ and $\theta+\frac{\pi}{2}$, we get the following figures.





Notice that since $\frac{\pi}{2}$ is a right angle, this rotation gives congruent triangles. The terminal point of θ is $(\cos\theta, \sin\theta)$. By congruence, we can read the lengths of sides of the triangle associated to $\theta+\frac{\pi}{2}$; the x- and y-coordinates of θ become the y- and x-coordinates of $\theta+\frac{\pi}{2}$, up to sign:

$$\cos\left(\theta+\frac{\pi}{2}\right) = \pm\sin\theta,$$
$$\sin\left(\theta+\frac{\pi}{2}\right) = \pm\cos\theta.$$

Next we need to determine the signs of these values. If θ is in the first quadrant then $\theta+\frac{\pi}{2}$ is in the second quadrant. In the first quadrant, the sine and cosine are positive and in the second, cosine is negative. Therefore if θ is in the first quadrant, then

$$\boxed{\cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta, \qquad \sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta}.$$

Considering the cases where θ is in a different quadrant shows that the same equalities hold for all θ.

1.5.5

(a) $\cos\left(\frac{\pi}{2}-\theta\right) = \cos\left(\frac{\pi}{2}+(-\theta)\right) = -\sin(-\theta) = \boxed{\sin\theta}$.

(b) $\sin(\pi-\theta) = \sin\left(\frac{\pi}{2}+\left(\frac{\pi}{2}-\theta\right)\right) = \cos\left(\frac{\pi}{2}-\theta\right) = \boxed{\sin\theta}$.

(c) $\cos(\pi+\theta) = \cos\left(\frac{\pi}{2}+\left(\frac{\pi}{2}+\theta\right)\right) = -\sin\left(\frac{\pi}{2}+\theta\right) = \boxed{-\cos\theta}$.

1.5.6 First we prove that cosine has no period smaller than 2π. If t is a period for cosine, then $\cos t = \cos 0 = 1$. However, by inspecting the unit circle, we see that the values of t for which $\cos t = 1$ are those values where the point $(\cos t, \sin t)$ lies on the positive x-axis. This happens if and only if t is a multiple of 2π. Thus the smallest possible period for cosine is 2π.

Similarly for sine, we can consider the equation $\sin\left(\frac{\pi}{2}+t\right) = \sin\frac{\pi}{2} = 1$. The only values of t for which this equation holds are integer multiples of 2π, so there is no period of sine which is less than 2π.

However, we know $\tan(\theta+\pi) = \tan\theta$ for all $\theta \in \mathrm{Dom}(\tan)$, so π is a period of tangent. It must be the smallest possible period, since $\tan x \neq \tan y$ for any $-\frac{\pi}{2} < x < y < \frac{\pi}{2}$.

1.5.7

(a) $\cos^{-1} 0$ is the angle $\theta \in [0,\pi]$ such that $\cos\theta = 0$. Therefore $\cos^{-1} 0 = \boxed{\frac{\pi}{2}}$.

(b) $\sin^{-1}\frac{1}{2}$ is the angle $\theta \in [-\frac{\pi}{2},\frac{\pi}{2}]$ such that $\sin\theta = \frac{1}{2}$. Therefore $\sin^{-1}\frac{1}{2} = \boxed{\frac{\pi}{6}}$.

(c) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is the angle $\theta \in [0,\pi]$ such that $\cos\theta = -\frac{\sqrt{3}}{2}$. Therefore $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$.

(d) $\tan^{-1}(-1)$ is the angle $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$ such that $\tan\theta = -1$. Since $\frac{\sin(-\frac{\pi}{4})}{\cos(-\frac{\pi}{4})} = \frac{-1}{1} = -1$, we have $\tan^{-1}(-1) = \boxed{-\frac{\pi}{4}}$.

1.5.8 We note that the range of $\cos^{-1}\circ\cos$ is $[0,\pi]$, while the range of $\sin^{-1}\circ\sin$ is $[-\frac{\pi}{2},\frac{\pi}{2}]$. Thus, for any θ in $(\frac{\pi}{2},\pi]$, we have $\cos^{-1}(\cos\theta) = \theta$, but we cannot have $\sin^{-1}(\sin\theta)$ since $\theta \notin \mathrm{Rng}(\sin^{-1})$. For example,

$$\cos^{-1}(\cos\pi) = \cos^{-1}(-1) = \pi,$$
$$\sin^{-1}(\sin\pi) = \sin^{-1} 0 = 0 \neq \pi.$$

Exercises for Section 1.6

1.6.1 By the double-angle formula for cosine, we have

$$\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right) = 1 - 2\sin^2\left(\frac{\pi}{12}\right).$$

Since $\frac{\pi}{12}$ is in the first quadrant, we are looking for the positive square root of $\sin^2$. Thus

$$\boxed{\sin\frac{\pi}{12} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}}.$$

Similarly,

$$\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right) = 2\cos^2\left(\frac{\pi}{12}\right) - 1.$$

Thus, since $\cos\frac{\pi}{12}$ is positive,

$$\boxed{\cos\frac{\pi}{12} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}}.$$

Notice that $(\sqrt{6} \pm \sqrt{2})^2 = (8 \pm 2\sqrt{12}) = 4(2 \pm \sqrt{3})$, so we can also write

$$\boxed{\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}, \qquad \cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}}.$$

1.6.2 We notice that the double-angle identity

$$\cos(2x) = 2\cos^2 x - 1$$

is an expression containing only an angle and its double. Substituting $x = \theta/2$ we get

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}.$$

We take the square root to get:

$$\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}}.$$

The sign is determined by locating the quadrant of $\frac{\theta}{2}$. For example, if $-\pi < \theta < \pi$, then $\cos\frac{\theta}{2} > 0$.

An alternative double-angle identity for cosine is

$$\cos(2x) = 1 - 2\sin^2 x.$$

Again we substitute $x = \frac{\theta}{2}$ to find

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

or

$$\boxed{\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}},$$

where again the sign is determined by the quadrant of $\frac{\theta}{2}$.

1.6.3 We already know the angle-addition formulas for sine and cosine, so we rewrite tangent in terms of these and apply the angle addition identities:

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}\end{aligned}$$

To get these in terms of tangent, we then divide the numerator and denominator by $\cos\alpha\cos\beta$. Since cosine is nonzero anywhere tangent is defined, we will get an expression that is equivalent to the expression above whenever $\tan\alpha$ and $\tan\beta$ are defined.

$$\tan(\alpha+\beta) = \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \boxed{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}.$$

Reconsidering the steps above, we see that if the denominator is zero, we get that $\cos(\alpha+\beta) = 0$, so $\tan(\alpha+\beta)$ is undefined. Therefore the equation we found is true whenever $\tan\alpha$, $\tan\beta$, and $\tan(\alpha+\beta)$ are all defined.

1.6.4 The double-angle formula for tangent follows from the previous exercise:

$$\tan 2\theta = \tan(\theta+\theta) = \boxed{\frac{2\tan\theta}{1-\tan^2\theta}}.$$

To find $\tan\frac{\theta}{2}$ we recall the double-angle formulas for sine and cosine: $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 2\cos^2 x - 1$. Letting $x = \frac{\theta}{2}$ in these formulas and applying some algebra, we have:

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \boxed{\frac{\sin\theta}{\cos\theta+1}}.$$

This identity holds when $\cos(\theta) \neq -1$, which is exactly when $\tan\frac{\theta}{2}$ is defined.

Notice that we can use $\sin^2\theta = 1 - \cos^2\theta = (1-\cos\theta)(1+\cos\theta)$ to simplify the denominator, which gives an alternative version of the half-angle formula for tangent:

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{(\sin\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} = \frac{(\sin\theta)(1-\cos\theta)}{\sin^2\theta} = \boxed{\frac{1-\cos\theta}{\sin\theta}}.$$

In deriving this identity, we have assumed that $\cos\theta \neq \pm 1$, so the identity holds for those values. In fact the right hand side is undefined for both of these values, but the left hand side *is* defined (and zero) when $\cos\theta = 1$.

Exercises for Section 1.7

1.7.1

(a) $\log(e^5) = \boxed{5}$, since the logarithm and exponential are inverses.

(b) $\log(\sqrt{e}) = \log(e^{\frac{1}{2}}) = \boxed{\frac{1}{2}}$.

(c) $e^{2\log 3} = e^{\log(3^2)} = e^{\log 9} = \boxed{9}$.

(d) $(e^{-(\log 4)})^2 = (e^{\log(4^{-1})})^2 = (4^{-1})^2 = \boxed{\frac{1}{16}}$.

1.7.2

(a) Note that $\log_3(7)$ is the solution to the equation $3^y = 7$. If we take the natural logarithm of both sides we get $y\log 3 = \log 7$. We may divide by $\log 3$ to solve for y, giving $\boxed{\log_3 7 = \frac{\log 7}{\log 3}}$.

(b) We start with $a^{\log_a b} = b$, so taking the log of both sides gives

$$\log_a b \log a = \log b.$$

Dividing by $\log a \neq 0$ we get $\boxed{\log_a b = \frac{\log b}{\log a}}$.

1.7.3 We know that any function which passes the Horizontal Line Test has an inverse. Let b be any real number. We want to show that the graph of $f(x)$ intersects the graph of $y = b$ at most once. Assume that these graphs intersect (otherwise we're done). Then there is some $a \in \mathbb{R}$ with $f(a) = b$. Since f is strictly increasing, if $x < a$, then $f(x) < f(a)$ and if $x > a$, then $f(x) > f(a)$. These show that if $x \neq a$, then $f(x) \neq b$. Thus the graph of f intersects this line exactly once, as necessary, so the graph passes the Horizontal Line Test, and hence has an inverse.

Review Problems

1.31 If $A \subset B$ and $B \subset A$, then every element of A is an element of B, and every element of B is an element of A. But this means that $A = B$, which contradicts the assertion that the subsets are proper.

1.32

(a) If x is in exactly one of A and B, then x is in exactly one of B and A, so symmetric difference is commutative (or symmetric, as the name would imply).

(b) We can break elements into cases based on which sets they are in, but this is rather tedious. More simply, we observe that the set on each side of the associative law is defined as the elements in exactly one of A, B, or C together with elements in all three sets.

(c) $A \ominus \emptyset = A$ because if an element is in A it is not in the empty set, so it is in exactly one of A and $\emptyset$. There are no elements in $\emptyset$, so we do not have to consider any other cases. Therefore, this is $\boxed{A}$.

(d) The set $A \ominus B \ominus C \ominus D$ consists of all elements that are in an odd number of A, B, C, or D. That is, it is the set of elements in exactly one or exactly three of the sets. By the commutativity and associativity of symmetric difference, it is only necessary to check this for an element in one particular combination of membership in any given number of sets. (This is also why we don't need parentheses when we write $A \ominus B \ominus C \ominus D$.) For example, if x is in A but not in B, C, or D, then $x \notin C \ominus D$, so $x \notin B \ominus (C \ominus D)$, and hence $x \in A \ominus (B \ominus C \ominus D)$. Similar checks can be made for elements in exactly 0, 2, 3, or 4 of the sets.

1.33

(a) This is not an interval, since no elements between b and c are in the set.

(b) This is the interval $\boxed{\emptyset}$.

(c) This is the interval $\boxed{(a,d)}$.

(d) This is the interval $\boxed{(b,c)}$.

1.34 For $f(x)$ to be defined, we must have $2 - x - x^2 \geq 0$. This factors as $(2 + x)(1 - x) \geq 0$. This occurs when $-2 \leq x \leq 1$, so the domain of f is $\boxed{[-2,1]}$.

The quadratic has a maximum at its vertex, which is at $x = -\frac{1}{2}$, at which the quadratic equals $2 + \frac{1}{2} - \left(-\frac{1}{2}\right)^2 = \frac{9}{4}$. Since $\sqrt{\frac{9}{4}} = \frac{3}{2}$, the range of f is $\boxed{\left[0, \frac{3}{2}\right]}$.

1.35

(a) The domain is $\boxed{(-1,1)}$ and the range is $\boxed{(-3,5)}$.

(b) The domain is $\boxed{[0,4)}$. We cannot determine the range explicitly: the range of $f(\sqrt{x})$ is equal to the image $f([0,2))$, but we do not know precisely what subset of $(-3,5)$ this image is.

(c) If f^{-1} exists, then the domain of f^{-1} is $\boxed{(-3,5)}$ and its range is $\boxed{(-2,2)}$. (The domain and range of the inverse are the range and domain of the original function, respectively.)

1.36

(a) $\tan \frac{\pi}{4} = \boxed{1}$.

(b) $\cos \frac{7\pi}{4} = \cos \frac{\pi}{4} = \boxed{\dfrac{\sqrt{2}}{2}}$.

(c) $\sin \frac{5\pi}{3} = \sin\left(-\frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = \boxed{-\dfrac{\sqrt{3}}{2}}$.

(d) $\csc \frac{3\pi}{4} = \dfrac{1}{\sin \frac{3\pi}{4}} = \dfrac{1}{\frac{1}{\sqrt{2}}} = \boxed{\sqrt{2}}$.

(e) $\cot \frac{21\pi}{4} = \cot \frac{\pi}{4} = \boxed{1}$.

(f) $\sin\left(\pi \sin \frac{\pi}{6}\right) = \sin\left(\pi \cdot \frac{1}{2}\right) = \boxed{1}$.

(g) $\tan 21\pi = \tan 0 = \boxed{0}$.

(h) $\sec\left(-\frac{7\pi}{2}\right) = \sec\frac{\pi}{2} = \frac{1}{\cos\left(\frac{\pi}{2}\right)}$. Since cosine is zero at $\frac{\pi}{2}$, $\sec\left(-\frac{7\pi}{2}\right)$ is $\boxed{\text{undefined}}$.

1.37

(a) $\cos(\pi - \theta) = \cos\left(\frac{\pi}{2} + \left(\frac{\pi}{2} - \theta\right)\right) = -\sin\left(\frac{\pi}{2} - \theta\right) = -\cos(-\theta) = \boxed{-\cos\theta}$.

(b) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(-\theta) = \boxed{\cos\theta}$.

(c) $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\left(\frac{\pi}{2} + \left(\frac{3\pi}{2} + \theta\right)\right) = -\cos(2\pi + \theta) = \boxed{-\cos\theta}$.

1.38 Using the angle-addition formula,

$$\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \sin\frac{\pi}{6}\cos\frac{\pi}{4} + \cos\frac{\pi}{6}\sin\frac{\pi}{4} = \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}.$$

Alternatively, using the half-angle formula,

$$\sin\frac{5\pi}{12} = \sqrt{\frac{1-\cos\frac{5\pi}{6}}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}.$$

(You can verify that these two answers are in fact the same by squaring both.)

We can use either method above, or the fact that $\cos\frac{5\pi}{12} = \sqrt{1-\sin^2\frac{5\pi}{12}}$, to compute

$$\cos\frac{5\pi}{12} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}} = \boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}.$$

Finally,

$$\tan\frac{5\pi}{12} = \frac{\sin\frac{5\pi}{12}}{\cos\frac{5\pi}{12}} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{(\sqrt{6}+\sqrt{2})^2}{4} = \frac{8+2\sqrt{12}}{4} = \boxed{2+\sqrt{3}}.$$

1.39 Recall the identity $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$. Thus we're looking for the smallest positive solution to

$$\cos\left(\frac{\pi}{2} - 3\theta\right) = \cos 7\theta.$$

More generally, we know that $\cos\alpha = \cos\beta$ if and only if $\alpha \pm \beta$ is an integer multiple of 2π. Thus, our equation is satisfied if and only if $\frac{\pi}{2} - 10\theta = 2\pi k$ or $\frac{\pi}{2} + 4\theta = 2\pi k$, where k is an integer. Isolating θ we get the possible solutions

$$\theta = \frac{\pi}{20} - \frac{\pi}{5}k,$$
$$\theta = -\frac{\pi}{8} + \frac{\pi}{2}k,$$

where k is any integer. The smallest positive solution is thus $\theta = \boxed{\frac{\pi}{20}}$. Checking we see that $\sin\frac{3\pi}{20} = \cos\frac{7\pi}{20}$, since $\frac{3\pi}{20} + \frac{7\pi}{20} = \frac{\pi}{2}$. Furthermore, for smaller positive θ, sine is larger and cosine is smaller, so this must be the smallest positive solution.

1.40 We know that $\tan\frac{47\pi}{9} = \tan\left(\frac{47\pi}{9} - 5\pi\right) = \tan\frac{2\pi}{9}$. Thus, since $-\frac{\pi}{2} < \frac{2\pi}{9} < \frac{\pi}{2}$, we have

$$\arctan\left(\tan\frac{47\pi}{9}\right) = \arctan\left(\tan\frac{2\pi}{9}\right) = \boxed{\frac{2\pi}{9}}.$$

1.41 The range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, on which cosine is positive. Hence, letting $\theta = \sin^{-1} x$, we have

$$\begin{aligned}1 &= \cos^2\theta + \sin^2\theta \\ &= \cos^2(\sin^{-1} x) + \sin^2(\sin^{-1} x) \\ &= \cos^2(\sin^{-1} x) + x^2,\end{aligned}$$

and solving for the cosine term, we get $\cos(\sin^{-1} x) = \boxed{\sqrt{1-x^2}}$.

1.42

(a) Using logarithm properties, we have

$$10 = \log_5(25^{3x}) = 3x\log_5(25) = 3x(2) = 6x.$$

Thus $x = \boxed{\frac{5}{3}}$.

(b) This equation is the same as $\log_2(x^2 \cdot 3x) = 16$, so $3x^3 = 2^{16}$. Therefore $x = \sqrt[3]{\frac{2^{16}}{3}} = \boxed{32\sqrt[3]{\frac{2}{3}}}$.

(c) The equation is a quadratic in e^x. We factor:

$$(e^x - 4)(e^x + 1) = 0.$$

Since $e^x + 1$ is positive, we must have $e^x = 4$, so $x = \boxed{\log 4}$.

Challenge Problems

1.43

(a) We are given that if $x \in A$ or $x \in B$, then $x \in A$. Logically, this means that if $x \in B$, then $x \in A$. Therefore, $\boxed{B \subseteq A}$.

(b) We are given that if $x \in A$ and $x \in B$, then $x \in A$. This means that any element of A is also an element of B, so $\boxed{A \subseteq B}$.

1.44

(a) We have $x \in \text{Dom}(f)$ if and only if $cx + d \neq 0$, thus the domain is $\boxed{\mathbb{R} \setminus \left\{-\frac{d}{c}\right\}}$. (This assumes that $c \neq 0$; if $c = 0$, then the domain is $\mathbb{R}$).

If $y \in \text{Rng}(f)$, then letting $y = \frac{ax+b}{cx+d}$ and solving for x gives $x = \frac{b-dy}{cy-a}$. Thus we must have $cy - a \neq 0$, hence $y \neq \frac{a}{c}$. We verify that

$$f(x) = \frac{a(b-dy) + b(cy-a)}{c(b-dy) + d(cy-a)} = \frac{bcy - ady}{bc - ad} = y.$$

Thus $\text{Rng}(f) = \boxed{\mathbb{R} \setminus \left\{\frac{a}{c}\right\}}$.

(b) As determined in part (a), $f^{-1}(x) = \frac{b-dx}{cx-a}$. We have already checked that $f(f^{-1}(x)) = x$; we also have

$$f^{-1}(f(x)) = \frac{b(cx+d) - d(ax+b)}{c(ax+b) - a(cx+d)} = \frac{bcx - adx}{bc - ad} = x,$$

as desired.

(c) If $ad = bc$, then the domain of f is unchanged. However, we have

$$f(x) = \frac{ax+b}{cx+d} = \frac{ax+b}{cx+d} \cdot \frac{ad}{bc} = \frac{a}{c} \cdot \left(\frac{adx+bd}{bcx+bd}\right) = \frac{a}{c}$$

for all $x \in \text{Dom}(f)$; in other words, f is the constant function with range $\left\{\frac{a}{c}\right\}$, and f does not have an inverse.

1.45 We notice two solutions immediately: $\theta = 0$ and $\theta = \frac{\pi}{2}$. Furthermore, on the interval $(0, \pi/2)$, we know $\sin\theta$ and $\cos\theta$ are between 0 and 1, hence

$$\sin^5\theta < \sin^2\theta,$$
$$\cos^5\theta < \cos^2\theta.$$

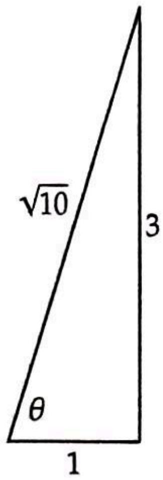
Combining these we get

$$\sin^5\theta + \cos^5\theta < \sin^2\theta + \cos^2\theta = 1,$$

so there are no other solutions. Therefore the only solutions are $\boxed{\theta \in \left\{0, \frac{\pi}{2}\right\}}$.

1.46 Since 3 is positive, arctan 3 is an acute angle, so we consider a right triangle with angle $\theta = \arctan 3$. We construct the right triangle with side opposite θ of length 3 and side adjacent to θ of length 1. The hypotenuse of this triangle has length $\sqrt{10}$. We can read from this triangle

$$\sin(\arctan 3) = \sin\theta = \boxed{\frac{3}{\sqrt{10}}}.$$



1.47 We know the double-angle formula $\sin 2x = 2\sin x\cos x$. It's not obvious how to get from $\cos x - \sin x$ to this value, but we notice that squaring $\cos x - \sin x$ gives us a $2\sin x\cos x$ term:

$$(\cos x - \sin x)^2 = \cos^2 x + \sin^2 x - 2\sin x\cos x.$$

Using $\sin^2 + \cos^2 = 1$ and the double-angle identity, we have.

$$(\cos x - \sin x)^2 = 1 - \sin(2x) = 1 - \frac{21}{25} = \frac{4}{25}.$$

Since we're given $\cos x > \sin x$, we can take the positive square root to find $\cos x - \sin x = \boxed{\frac{2}{5}}$.

1.48 Take the tangent and use the tangent angle-sum formula:

$$\tan\left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right) = \frac{\tan(\tan^{-1}\frac{1}{2}) + \tan(\tan^{-1}\frac{1}{3})}{1 - \tan(\tan^{-1}\frac{1}{2})\tan(\tan^{-1}\frac{1}{3})} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1.$$

Thus $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1} 1 = \boxed{\frac{\pi}{4}}$.

1.49

(a) We compute:

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{4}{4} = 1.\end{aligned}$$

(b) We can write

$$\sinh(x+y) = \frac{e^{x+y} - e^{-(x+y)}}{2}.$$

To try to get this in terms of sinh and cosh of just x and y, we experiment to see what happens when we multiply:

$$\sinh x \cosh y = \frac{(e^x - e^{-x})(e^y + e^{-y})}{4} = \frac{e^{x+y} - e^{-(x+y)} + e^{x-y} - e^{y-x}}{4}.$$

This is close, except for the e^{x-y} and e^{y-x} terms. We try swapping x and y:

$$\cosh x \sinh y = \frac{(e^x + e^{-x})(e^y - e^{-y})}{4} = \frac{e^{x+y} - e^{-(x+y)} - e^{x-y} + e^{y-x}}{4}.$$

Aha—we add these together and the unwanted terms will cancel:

$$\sinh x \cosh y + \cosh x \sinh y = \frac{2(e^{x+y} - e^{-(x+y)})}{4} = \sinh(x+y).$$

Similar calculations show that

$$\cosh(x+y) = \sinh x \sinh y + \cosh x \cosh y.$$

(c) We try to do the same thing as with the regular trig functions: think of cosh as the x-coordinate and sinh as the y-coordinate of some graph. Part (a) then gives us $x^2 - y^2 = 1$, whose graph is a hyperbola. (We restrict to the $x > 0$ half of the hyperbola since cosh is always positive.)

1.50 If $C \in C$, then C contains itself as a member, so by definition $C \notin C$, a contradiction. On the other hand, if $C \notin C$, then C does not contain itself as a member, so $C \in C$, again a contradiction. So we cannot have either $C \in C$ or $C \notin C$

However, the fundamental property of a "set" S is that for any object x, either $x \in S$ or $x \notin S$; that is, S is determined precisely by its elements. Since we cannot make this determination for the set C and the object C, we have a paradox.

The paradox is that such a set C cannot exist, so our notion of "set" is not broad enough. Mathematicians have developed various generalizations of set theory to deal with such complications.

CHAPTER 2

Limits and Continuity

Exercises for Section 2.1

2.1.1 The function from Problem 2.9 will do. $f(0) = 0$, but since the left and right hand limits at zero are different, there is no limit at 0.

2.1.2 Given ϵ, we need to find a δ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |x - a| < \epsilon.$$

Comparing these two, it is clear that choosing $\delta = \epsilon$ will do the trick. Thus $\lim\limits_{x \to a} x = a$.

2.1.3 Let $\epsilon > 0$ be given. By definition, we need to find a δ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |(cf)(x) - cL| < \epsilon.$$

If $c = 0$, then this last inequality is just $0 < \epsilon$, which is always true by definition. If $c \neq 0$, then $|(cf)(x) - cL| < \epsilon$ is the same as $|f(x) - L| < \frac{\epsilon}{|c|}$, and because we are given that $\lim\limits_{x \to a} f(x) = L$, there is some δ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \frac{\epsilon}{|c|} \quad \Rightarrow \quad |(cf)(x) - cL| < \epsilon.$$

Therefore, $\lim\limits_{x \to a}(cf)(x) = cL$.

2.1.4 We notice first that the greatest integer function is constant on the intervals $(n, n+1)$ between integers. Specifically if $1 < x < 2$, then $f(x) = 1$, and if $2 < x < 3$, then $f(x) = 2$.

Let $\epsilon > 0$ be given, and choose $\delta = 1$. Then

$$0 < 2 - x < \delta \quad \Rightarrow \quad 1 < x < 2 \quad \Rightarrow \quad f(x) = 1 \quad \Rightarrow \quad |f(x) - 1| = |1 - 1| = 0 < \epsilon.$$

Therefore $\lim\limits_{x \to 2^-} \lfloor x \rfloor = 1$.

Likewise, we check the limit from the right. Let $\epsilon > 0$ be given and let $\delta = 1$. Then

$$0 < x - 2 < \delta \quad \Rightarrow \quad 2 < x < 3 \quad \Rightarrow \quad f(x) = 2 \quad \Rightarrow \quad |f(x) - 2| = |2 - 2| = 0 < \epsilon.$$

Therefore $\lim\limits_{x \to 2^+} \lfloor x \rfloor = 2$.

2.1.5 Let's begin by trying to show that if f has a limit at a, then f has a limit from the left. Assume that $\lim\limits_{x \to a} f(x) = L$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We need to find δ' such that

$$0 < a - x < \delta' \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

However, notice that the δ from above works, since $0 < a - x < \delta'$ is *more strict* than $0 < |x - a| < \delta'$, meaning when the first is satisfied then the second is satisfied. So if $\delta' = \delta$, then

$$0 < a - x < \delta \quad \Rightarrow \quad 0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

This shows $\lim_{x \to a^-} f(x) = L$. Furthermore the same logic works to show that f has the same limit from the right:

$$0 < x - a < \delta \quad \Rightarrow \quad 0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon,$$

showing $\lim_{x \to a^+} f(x) = L$.

2.1.6 We begin by labeling $\lim_{x \to a} f(x) = F$, and $\lim_{x \to a} g(x) = G$. In general for problems involving limits, it is a good strategy to begin with the quantity we want to show is less than ϵ and work from there:

$$|f(x)g(x) - FG|.$$

What we want to do now is disentangle the appearances of the functions f and g. We apply a popular algebraic trick of adding and subtracting the function $f(x)G$ and applying the Triangle Inequality:

$$\begin{aligned} |f(x)g(x) - FG| &= |f(x)g(x) - f(x)G + f(x)G - FG| \\ &\leq |f(x)g(x) - f(x)G| + |f(x)G - FG| \\ &= |f(x)| \cdot |g(x) - G| + |f(x) - F| \cdot |G|. \end{aligned}$$

We applied the Triangle Inequality in the middle to separate the two quantities. This gave us a value which is an upper bound of the quantity we're interested in.

Notice that this tactic is not guaranteed to work: the bound we chose might have been poor. It may be possible that we can make the left-hand side arbitrarily small while the right-hand side could be ill-behaved. Fortunately what we've written looks very promising, since the quantities $|g(x) - G|$ and $|f(x) - F|$ are things we can make small by choosing δ small. We will try to force each of $|f(x)| \cdot |g(x) - G|$ and $|f(x) - F| \cdot |G|$ to be smaller than $\frac{\epsilon}{2}$.

First consider the quantity

$$|f(x) - F| \cdot |G|.$$

If $|G| = 0$, then this is always smaller than $\frac{\epsilon}{2}$. We notice that

$$|f(x) - F| \cdot |G| \leq |f(x) - F| \cdot (|G| + 1).$$

Now since $\lim_{x \to a} f(x) = F$, for any given ϵ, we can choose δ_1 such that

$$0 < |x - a| < \delta_1 \quad \Rightarrow \quad |f(x) - F| < \frac{\epsilon}{2(|G| + 1)}$$

from which it follows that

$$|f(x) - F| \cdot |G| \leq \frac{\epsilon}{2(|G| + 1)} \cdot |G| \leq \frac{\epsilon}{2}.$$

Next we try to bound

$$|f(x)| \cdot |g(x) - G|$$

by $\frac{\epsilon}{2}$. This is more difficult since the factor $|f(x)|$ varies based on x. However we notice that when x is near a, the value $f(x)$ should be near F. Specifically another application of the Triangle Inequality gives

$$|f(x)| = |f(x) - F + F| \leq |f(x) - F| + |F|.$$

From here we can see, because $\lim_{x \to a} f(x) = F$, that there must be some δ_2 such that

$$0 < |x - a| < \delta_2 \quad \Rightarrow \quad |f(x) - F| \leq 1.$$

Therefore

$$0 < |x - a| < \delta_2 \quad \Rightarrow \quad |f(x)| \le |F| + |f(x) - F| \le |F| + 1.$$

Therefore $|f(x)| \cdot |g(x) - G| \le (|F| + 1)|g(x) - G|$, and we are in the same situation as before since we are faced with bounding $|g(x) - G|(|F| + 1)$. Because $\lim_{x \to a} g(x) = G$, we can choose δ_3 such that

$$0 < |x - a| < \delta_3 \quad \Rightarrow \quad |g(x) - G| < \frac{\epsilon}{2(|F| + 1)}.$$

Putting all of this together, we can now construct the proof:

Proof. Let $\epsilon > 0$ be given. Choose δ_1, δ_2, and δ_3 such that

$$\begin{aligned} 0 < |x - a| < \delta_1 \quad &\Rightarrow \quad |f(x) - F| < \frac{\epsilon}{2(|G| + 1)} \\ 0 < |x - a| < \delta_2 \quad &\Rightarrow \quad |f(x) - F| \le 1 \\ 0 < |x - a| < \delta_3 \quad &\Rightarrow \quad |g(x) - G| < \frac{\epsilon}{2(|F| + 1)}. \end{aligned}$$

The Triangle Inequality gives

$$|f(x) - F| \le 1 \quad \Rightarrow \quad |f(x)| \le |F| + 1.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then if $0 < |x - a| < \delta$,

$$\begin{aligned} |f(x)g(x) - FG| &= |f(x)g(x) - f(x)G + f(x)G - FG| \\ &\le |f(x)g(x) - f(x)G| + |f(x)G - FG| \\ &= |f(x)| \cdot |g(x) - G| + |f(x) - F| \cdot |G| \\ &\le (|F| + 1) \cdot \frac{\epsilon}{2(|F| + 1)} + \frac{\epsilon}{2(|G| + 1)} \cdot |G| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

2.1.7 We want to prove that if $f(x) \le g(x) \le h(x)$ for all x and the limits of f and h at a are both L, then the limit of g at a must also be L. We start by considering the quantity we want to bound, namely

$$|g(x) - L|.$$

The one fact we need to use is that $f \le g \le h$. We see that this comes in handy if we subtract L from each term:

$$f(x) - L \le g(x) - L \le h(x) - L.$$

Observe now that if $g(x) \ge L$, then

$$0 \le g(x) - L \le h(x) - L = |h(x) - L|.$$

Also, if $g(x) \le L$, then

$$0 \le L - g(x) \le L - f(x) = |f(x) - L|.$$

In either case, we get that

$$|g(x) - L| \le \max\{|h(x) - L|, |f(x) - L|\}.$$

From here we can construct our proof.

Proof. Let $\epsilon > 0$ and let $\delta_f, \delta_h > 0$ be chosen such that

$$\begin{aligned} 0 < |x - a| < \delta_f \quad &\Rightarrow \quad |f(x) - L| < \epsilon \\ 0 < |x - a| < \delta_h \quad &\Rightarrow \quad |h(x) - L| < \epsilon. \end{aligned}$$

Now we let $\delta = \min\{\delta_f, \delta_h\}$. From this it follows that if $0 < |x - a| < \delta$, then

$$|g(x) - L| \le \max\{|h(x) - L|, |f(x) - L|\} < \epsilon.$$

Exercises for Section 2.2

2.2.1 For every $\epsilon > 0$ we want to find $\delta > 0$ for which

$$|x-a| < \delta \quad \Rightarrow \quad \big||x| - |a|\big| < \epsilon.$$

By the Triangle Inequality we know that

$$\big||x| - |a|\big| \leq |x-a|,$$

so we see that $\delta = \epsilon$ will work.

2.2.2 Let

$$f(x) = \begin{cases} 1 & \text{if } x = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

We see that at every point a, we have $\lim_{x \to a} f(x) = 0$, since the restriction of f to any interval not containing 0 or 2 is constant, and the limits at 0 and 2 are the same as those of the function $g(x) = 0$, so are also zero. However $f(0) = f(2) \neq 0$, so f is not continuous at these two points, but is continuous everywhere else.

2.2.3 If f and g are continuous at a, then

$$\lim_{x \to a} (fg)(x) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = f(a)g(a) = (fg)(a),$$

showing fg is continuous at a.

The function f/g is only defined at a if $g(a) \neq 0$. Thus f/g cannot be continuous at a unless $g(a) \neq 0$. Assume now that $g(a) \neq 0$. We still should confirm that it is possible to compute the limit of f/g at a. However, since g is continuous at a, there is some δ such that

$$|x-a| < \delta \quad \Rightarrow \quad |g(x) - g(a)| < \frac{|g(a)|}{2}.$$

applying the Triangle Inequality, we see

$$|g(x)| \geq |g(a)| - |g(x) - g(a)| > \frac{|g(a)|}{2}.$$

This shows that on a sufficiently small interval containing a, the function f/g is defined (and thus we may take limits). Therefore if $g(a) \neq 0$,

$$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} = \frac{f}{g}(a).$$

2.2.4 If f is continuous on $[a, b]$, then so is $-f = (-1)f$. By the Extreme Value Theorem, $-f$ attains a maximum, $-f(c)$, so

$$-f(c) \geq -f(x) \text{ for all } x \in [a, b].$$

Multiplying by -1, we see

$$f(c) \leq f(x) \text{ for all } x \in [a, b],$$

so $f(c)$ is a minimum for f on $[a, b]$.

2.2.5 We claim that f is constant. We prove this by contradiction. Assume that there are points $c < d$ in $[a, b]$ such that $f(c) \neq f(d)$. Then by the Intermediate Value Theorem, every irrational number between $f(c)$ and $f(d)$ will also be in the range of f, which is a contradiction of the fact that the image f lies in $\mathbb{Q}$.

Notice that there is a loose end here; we used the fact that between any two rational numbers there is an irrational number. This is not too difficult to show. Recall that we showed $\sqrt{2}$ is irrational, which implies $\frac{1}{\sqrt{2}}$ is irrational and between 0 and 1. From this it follows that

$$f(c) + \frac{f(d) - f(c)}{\sqrt{2}}$$

is irrational and between $f(c)$ and $f(d)$, provided $f(c) \neq f(d)$.

2.2.6

(a) We know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, and that $\lim_{x\to 0} x = 0$. For $x \neq 0$,

$$\sin x = \frac{\sin x}{x} \cdot x.$$

Therefore,

$$\lim_{x\to 0} \sin(x) = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} x\right) = 1 \cdot 0 = 0 = \sin 0.$$

Therefore sine is continuous at zero.

(b) Since sine is continuous at 0, so must be $\cos^2 x = 1 - \sin^2 x$. Unfortunately we haven't shown that the square root of a continuous function is continuous. However notice that for $x \in (-\pi/2, \pi/2)$,

$$\cos^2 x \leq \cos x \leq 1,$$

so we can apply the Squeeze Theorem to show that $\lim_{x\to 0} \cos x = 1$. (We could also invoke the inequality $|\cos x - 1| \leq |\cos^2 x - 1|$, and use the fact that $\cos^2 x$ converges to 1.)

(c) These are direct computation:

$$\begin{aligned}
\lim_{h\to 0} \sin(x+h) &= \lim_{h\to 0}(\sin x \cos h + \cos x \sin h) \\
&= \sin x \left(\lim_{h\to 0} \cos h\right) + \cos x \left(\lim_{h\to 0} \sin h\right) \\
&= \sin x \cdot 1 + \cos x \cdot 0 \\
&= \sin x.
\end{aligned}$$

The proof for cosine is similar, or we could observe that since $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, cosine must also be continuous.

2.2.7

(a) Multiplication of a polynomial by a nonzero constant does not change its zeros, so we may assume that the leading coefficient of the polynomial is 1 (that the polynomial is "monic"). Thus consider the polynomial

$$p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n.$$

Intuitively what we want to use is that p goes to $+\infty$ at $+\infty$ and goes to $-\infty$ at $-\infty$, and then apply the Intermediate Value Theorem.

First observe that

$$\lim_{x\to\infty} \frac{p(x)}{x^n} = \lim_{x\to\infty} 1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_n}{x^n} = 1.$$

Since $1 > 0$, there must exist some b such that $\frac{p(b)}{b^n} > 0$, so $p(b) > 0$. Likewise,

$$\lim_{x\to-\infty} \frac{p(x)}{x^n} = \lim_{x\to-\infty} 1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_n}{x^n} = 1,$$

so there is some $a < 0$ such that $\frac{p(a)}{a^n} > 0$. Since n is odd and $a < 0$, this tells us $p(a) < 0$. Now we know that $p(a) < 0 < p(b)$ and that all polynomials are continuous. Therefore, by the Intermediate Value Theorem, there is some $c \in [a, b]$ such that $p(c) = 0$.

(b) If a polynomial is even it needn't have any (real) roots. Consider, for example $p(x) = x^2 + 1$.

Review Problems

2.15 We want to be able to show that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x-2| < \delta \quad \Rightarrow \quad |x^3 - 8| < \epsilon.$$

For $\delta > 0$, if $0 < |x-2| < \delta$, then $2 - \delta < x < 2 + \delta$, and we have

$$(2-\delta)^3 < x^3 < (2+\delta)^3.$$

Expanding gives

$$8 - 12\delta + 6\delta^2 - \delta^3 < x^3 < 8 + 12\delta + 6\delta^2 + \delta^3,$$

thus

$$-12\delta + 6\delta^2 - \delta^3 < x^3 - 8 < 12\delta + 6\delta^2 + \delta^3.$$

But note that if $\delta < 1$, then $\delta^3 < \delta^2 < \delta$, and

$$12\delta + 6\delta^2 + \delta^3 < 19\delta,$$
$$-12\delta + 6\delta^2 - \delta^3 > -19\delta,$$

so we have $|x^3 - 8| < 19\delta$. Thus, choosing $\delta = \min\left\{1, \frac{\epsilon}{19}\right\}$ will ensure that

$$0 < |x-2| < \delta \quad \Rightarrow \quad |x^3 - 8| < 19\delta \le \epsilon.$$

2.16

(a) If $x \neq 1$, then $\dfrac{x^3-1}{x-1} = \dfrac{(x-1)(x^2+x+1)}{x-1} = x^2+x+1$ so $\lim\limits_{x\to1} \dfrac{x^3-1}{x-1} = \lim\limits_{x\to1}(x^2+x+1) = 1^2+1+1 = \boxed{3}$. This is valid because x^2+x+1 is a polynomial and thus continuous everywhere.

(b) If $x \neq 0$, then $\dfrac{(x+2)^2-4}{x} = \dfrac{x^2+4x}{x} = x+4$. Therefore, $\lim\limits_{x\to0} \dfrac{(x+2)^2-4}{x} = \lim\limits_{x\to0}(x+4) = \boxed{4}$.

(c) Factor the denominator as $(1+\sqrt{x})(1-\sqrt{x})$. Then we have $\lim\limits_{x\to1} \dfrac{1}{1+\sqrt{x}} = \boxed{\dfrac{1}{2}}$.

(d) This is just $0^2 = \boxed{0}$, since $\sqrt{x^2} = |x|$ is continuous.

(e) From the right, $x - \lfloor x \rfloor$ approaches $\boxed{0}$ at each integer.

(f) From the left, $x - \lfloor x \rfloor$ approaches $\boxed{1}$.

2.17 Here is an example:

$$f(x) = \begin{cases} 1/x \text{ if } x < 0 \\ 0 \text{ if } x \ge 0 \end{cases}$$

In this example, the limit from the left side does not exist, as $\frac{1}{x}$ is unbounded. On the right side, the limit is obviously 0. So $\lim\limits_{x\to0^+} f(x)$ exists, but $\lim\limits_{x\to0^-} f(x)$ does not exist, as desired.

2.18 If f is continuous on $\mathbb{R}$ and takes on both positive and negative values, then f must hit zero. Take $a < b$ such that $f(a)$ and $f(b)$ have different signs. Since f is continuous, by the Intermediate Value Theorem $f([a,b])$ contains all values between $f(a)$ and $f(b)$. Since one of these is positive and the other negative, $0 \in f([a,b]) \subseteq f(\mathbb{R})$.

2.19 Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \{-1,0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(x)$ is 0 except for $x \in \{-1,0,1\}$, we have $\lim\limits_{x\to a} f(x) = 0$ for any $a \in \mathbb{R}$. Therefore f is continuous on $\mathbb{R} \setminus \{-1,0,1\}$.

2.20

(a) Let $\epsilon > 0$ be given. There exists δ such that

$$0 < |x| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

This means that

$$0 < |cx| < \delta \quad \Rightarrow \quad |f(cx) - L| < \epsilon,$$

so

$$0 < |x| < \frac{\delta}{|c|} \quad \Rightarrow \quad |f(cx) - L| < \epsilon.$$

Therefore, $\lim_{x\to 0} f(cx) = L$.

(b) By part (a), and the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, we have $\lim_{x\to 0} \frac{\sin ax}{ax} = 1$. Therefore,

$$\lim_{x\to 0} \frac{\sin ax}{x} = \lim_{x\to 0} a\left(\frac{\sin ax}{ax}\right) = a\left(\lim_{x\to 0} \frac{\sin ax}{ax}\right) = a.$$

(c) Write

$$\frac{\sin ax}{\sin bx} = \frac{\sin ax}{x} \cdot \frac{x}{\sin bx}.$$

The limits of these two terms are a and $\frac{1}{b}$, respectively, so

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}.$$

2.21 f is continuous if and only if $\lim_{x\to a} f(x) = f(a)$ for all $a \in \mathrm{Dom}(f)$. This means that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon,$$

which matches the condition in the problem except for the "$0 <$" portion at the far left. However, if $0 = |x - a|$, then $x = a$ and $f(x) = f(a)$, so $|f(x) - f(a)| = 0 < \epsilon$. Thus, we have

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

Challenge Problems

2.22 We just use the limit definition with $\delta = \epsilon$: if $\epsilon > 0$ is given, then setting $\delta = \epsilon$ gives

$$0 < |x| < \delta \quad \Rightarrow \quad |f(x)| \le |x| < \delta = \epsilon.$$

So we have $\lim_{x\to 0} f(x) = 0$. Further, $|f(0)| \le 0$, so $f(0) = 0$. Since $\lim_{x\to 0} f(x) = f(0)$, we conclude that f is continuous at 0. (Alternatively, we can use the Squeeze Theorem on $f(x)$ and squeeze it between $-|x|$ and $|x|$.)

2.23 To avoid confusion, we write $\mathring{\sin} x$ to denote sine in degrees. We convert this to the sine function in radians.

$$f(x) = \frac{\mathring{\sin} x}{x} = \frac{\sin(\frac{\pi x}{180})}{x} = \frac{\pi}{180} \cdot \frac{\sin(\frac{\pi x}{180})}{\frac{\pi x}{180}}$$

Since we know that $\frac{\sin t}{t} \to 1$ as $t \to 0$, we get

$$\lim_{x\to 0} f(x) = \frac{\pi}{180} \lim_{x\to 0} \frac{\sin(\frac{\pi x}{180})}{\frac{\pi x}{180}} = \frac{\pi}{180}.$$

(Here we have applied the results from 2.20.)

2.24

(a) Let $\epsilon > 0$ be given. Since $\lim_{x\to 0} f(x) = L$, there exists some $\delta' > 0$ such that

$$|x - 0| < \delta' \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Let $\delta' = \delta^3$. Then

$$|x - 0| < \delta \quad \Rightarrow \quad |x^3 - 0| < \delta^3 = \delta' \quad \Rightarrow \quad |f(x^3) - L| < \epsilon.$$

Therefore, $\lim_{x\to 0} f(x^3) = L$.

(b) Let

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Then f has no limit at 0, but $f(x^2) = 0$ for all x, so $\lim_{x\to 0} f(x^2) = 0$.

2.25

(a) By definition, $f(I)$ is an interval if and only if $u, v \in f(I)$ and $u < w < v$ implies $w \in f(I)$. This follows immediately from the Intermediate Value Theorem.

(b) We know that $f(I)$ is an interval by part (a). By the Boundedness Lemma, f is bounded and the least upper bound and greatest lower bound are each elements of $f(I)$. Therefore $f(I)$ is a closed interval.

(c) Let $f(x) = \lfloor x \rfloor$. The set $I = [\frac{1}{2}, \frac{3}{2}]$ is a closed interval, but $f(I) = \{0, 1\}$ is not an interval.

2.26 If $f(0) = 0$ or $f(1) = 1$, then we're done. Otherwise, let $g(x) = f(x) - x$. This is still continuous. Since $f(0) > 0$, we have $g(0) > 0$. Since $f(1) < 1$, we have $g(1) < 0$. So by the Intermediate Value Theorem, there is some a such that $g(a) = 0$, meaning that $f(a) = a$.

It is not true on the open interval $(0, 1)$. For example, let $f(x) = x^2$. If we restrict the domain to $(0, 1)$, then the range is also $(0, 1)$, but there is no fixed point.

2.27 This is false. Let

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

and let $f(x) = 0$. Then $(g \circ f)(x) = 1$ and

$$\begin{aligned} \lim_{x\to 0} f(x) &= 0, \\ \lim_{x\to 0} g(x) &= 0, \\ \lim_{x\to 0} (g \circ f)(x) &= 1. \end{aligned}$$

2.28 We claim that $\lim_{x\to a} f(x) = 0$ for all $a \in \mathbb{R}$.

To prove this, let $\epsilon > 0$ be given. Then $\epsilon > \frac{1}{n}$ for some positive integer n. Pick δ small enough so that no integer multiple of $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-1}, \frac{1}{n}$, other than possibly a itself, is within δ of a. (We can always do this since there are only finitely many of these multiples in any bounded interval.) Then $f(x)$ takes values less than $\frac{1}{n}$ when $x \neq a$ is within δ of a, and hence

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x)| < \frac{1}{n} < \epsilon.$$

Therefore, $\lim_{x\to a} f(x) = 0$.

Thus, f is continuous exactly at the points at which $f(x) = 0$, and these are exactly the irrational numbers.

CHAPTER 3

The Derivative

Exercises for Section 3.2

3.2.1

(a) $f'(x) = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = \boxed{0}$.

(b) $f'(x) = \lim_{h\to 0} \frac{2(x+h)-2x}{h} = \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = \boxed{2}$.

(c) $f'(x) = \lim_{h\to 0} \frac{\left[(x+h)^2 - 3(x+h)\right] - \left[x^2 - 3x\right]}{h} = \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} = \lim_{h\to 0}(2x - 3 + h) = \boxed{2x-3}$.

(d) $f'(x) = \lim_{h\to 0} \frac{(x-1+h)^3 - (x-1)^3}{h} = \lim_{h\to 0} \frac{3(x-1)^2h + 3(x-1)h^2 + h^3}{h} = \lim_{h\to 0}(3(x-1)^2 + 3(x-1)h + h^2) = \boxed{3(x-1)^2}$.

(e) $f'(x) = \lim_{h\to 0} \frac{\frac{1}{2(x+h)} - \frac{1}{2x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{2hx(x+h)} = \lim_{h\to 0} \frac{-1}{2x(x+h)} = \boxed{-\frac{1}{2x^2}}$.

3.2.2 Substituting into the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{((x+h)^4 + 3) - (x^4+3)}{h} \\ &= \lim_{h\to 0} \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 3) - (x^4+3)}{h} \\ &= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h\to 0}(4x^3 + 6x^2h + 4xh^2 + h^3) \\ &= 4x^3. \end{aligned}$$

So the slope of the tangent line at $(1,4)$ is $4(1)^3 = 4$. Therefore, the tangent line is given by the equation $\boxed{y = 4(x-1) + 4 = 4x}$.

3.2.3 By the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^2/2 - x^2/2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{2h} \\ &= \lim_{h\to 0}\left(x + \frac{h}{2}\right) \\ &= x. \end{aligned}$$

The tangent line has slope $f'(2) = 2$, and so the line has equation $\boxed{y = 2(x-2) + 2 = 2x - 2}$.

3.2.4 An easy example is $f(x) = |x| + |x-1|$. This function is continuous on all of $\mathbb{R}$, but its graph has "sharp corners" at $x = 0$ and $x = 1$. We may also note that $f'(x) = -2$ if $x \in (-\infty, 0)$, $f'(x) = 0$ if $x \in (0,1)$, and $f'(x) = 2$ if $x \in (1, +\infty)$, so f is differentiable on all of $\mathbb{R} \setminus \{0, 1\}$.

3.2.5 First, we observe that for any function g,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} g(-x),$$

provided either limit is defined. This is because replacing x by $-x$ does not change anything in the δ-ϵ definition of the limit as $x \to 0$.

Therefore, we can replace h with $-h$ in the original limit definition of derivative, and we get:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}.$$

So the "new" definition of derivative presented in the problem is equivalent to the original definition.

Exercises for Section 3.3

3.3.1 We simply use the limit definition of derivative:

$$\begin{aligned}(cf)'(x) &= \lim_{h\to 0} \frac{(cf)(x+h) - (cf)(x)}{h} \\ &= \lim_{h\to 0} \frac{c(f(x+h) - f(x))}{h} \\ &= c \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= cf'(x).\end{aligned}$$

3.3.2

(a) $\boxed{12x^2 + 3 + 4x^{-3}}$

(b) By the Binomial Theorem we have

$$\begin{aligned}\left((3x^2+2)^5\right)' &= \left(1(3^5x^{10})(2^0) + 5(3^4x^8)(2^1) + 10(3^3x^6)(2^2) + 10(3^2x^4)(2^3) + 5(3^1x^2)(2^4) + 1(3^0x^0)(2^5)\right)' \\ &= 10 \cdot 1(3^5x^9)(2^0) + 8 \cdot 5(3^4x^7)(2^1) + 6 \cdot 10(3^3x^5)(2^2) + 4 \cdot 10(3^2x^3)(2^3) + 2 \cdot 5(3^1x^1)(2^4) \\ &= 30x\left(1(3^4x^8)(2^0) + 4(3^3x^6)(2^1) + 6(3^2x^4)(2^2) + 4(3^1x^2)(2^3) + 1(3^0x^0)(2^4)\right) \\ &= \boxed{30x(3x^2+2)^4}.\end{aligned}$$

(c) By the Product Rule,

$$\frac{d}{dx}(x^2+1)\sin x = (x^2+1)'(\sin x) + (x^2+1)(\sin x)' = \boxed{2x\sin x + (x^2+1)\cos x}.$$

(d) We start with the double angle formula $\cos 2x = \cos^2 x - \sin^2 x = (\cos x - \sin x)(\cos x + \sin x)$. Then

$$\begin{aligned}\frac{d}{dx}\cos 2x &= (-\sin x - \cos x)(\cos x + \sin x) + (\cos x - \sin x)(-\sin x + \cos x) \\ &= -4\sin x\cos x \\ &= \boxed{-2\sin 2x}.\end{aligned}$$

(e) We write $e^{2x} = e^x e^x$ and apply the Product Rule:

$$(e^{2x})' = (e^x)'e^x + e^x(e^x)' = e^x e^x + e^x e^x = \boxed{2e^{2x}}.$$

(f) By the Product Rule,

$$\frac{d}{dx}(5\log x)(1+\tan x) = (5\log x)'(1+\tan x) + (5\log x)(1+\tan x)' = \boxed{\frac{5}{x}(1+\tan x) + 5\log x\sec^2 x}.$$

(g) Using $\log(x^3) = 3\log x$, we get that the derivative is $\boxed{\frac{3}{x}}$.

(h) We apply the Product Rule twice:

$$\begin{aligned}(x^2e^x\sec x)' &= (x^2)'(e^x\sec x) + (x^2)(e^x\sec x)' \\ &= (x^2)'(e^x\sec x) + (x^2)(e^x)'(\sec x) + (x^2)(e^x)(\sec x)' \\ &= 2xe^x\sec x + x^2e^x\sec x + x^2e^x\sec x\tan x \\ &= \boxed{xe^x\sec x\,(2 + x + x\tan x)}.\end{aligned}$$

3.3.3

(a) By the limit definition:

$$\begin{aligned}\left(\frac{1}{g}\right)'(x) &= \lim_{h\to 0}\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} \\ &= \lim_{h\to 0}\frac{g(x) - g(x+h)}{g(x)g(x+h)h} \\ &= \lim_{h\to 0}\frac{g(x)-g(x+h)}{h}\cdot\lim_{h\to 0}\frac{1}{g(x)g(x+h)} \\ &= -g'(x)\cdot\frac{1}{(g(x))^2} = -\frac{g'(x)}{(g(x))^2}.\end{aligned}$$

Since we assumed that $g(x) \neq 0$ and g is differentiable at x, we know that g must be continuous at x, so for sufficiently small h, we have $g(x+h) \neq 0$. Thus we can take the limits above without worrying about dividing by zero.

(b) Use part (a) and apply the Product Rule. We get:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= f(x)\left(\frac{1}{g}\right)'(x) + f'(x)\left(\frac{1}{g}\right)(x) \\ &= -\frac{f(x)g'(x)}{(g(x))^2} + \frac{f'(x)}{g(x)} \\ &= \frac{-f(x)g'(x) + f'(x)g(x)}{(g(x))^2}.\end{aligned}$$

3.3.4 We apply the limit definition:

$$\begin{aligned}\frac{d}{dx}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h) - \cos(x)}{h} \\ &= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} \\ &= (\cos x)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) - (\sin x)\left(\lim_{h\to 0}\frac{\sin h}{h}\right) \\ &= (\cos x)(0) - (\sin x)(1) = -\sin x.\end{aligned}$$

3.3.5 Using the Quotient Rule:

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\frac{1}{\cos x} = -\frac{-\sin x}{\cos^2 x} = \boxed{\sec x \tan x},$$
$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\frac{1}{\sin x} = -\frac{\cos x}{\sin^2 x} = \boxed{-\csc x \cot x},$$
$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = \boxed{-\csc^2 x}.$$

Exercises for Section 3.4

3.4.1

(a) $\frac{d}{dx}(3x^4 + x)^5 = 5(3x^4 + x)^4 \cdot \frac{d}{dx}(3x^4 + x) = 5(3x^4 + x)^4(12x^3 + 1) = \boxed{5(12x^3 + 1)(3x^4 + x)^4}$.

(b) Since $\frac{d}{du}\sqrt{u} = \frac{1}{2\sqrt{u}}$, we have

$$\frac{d}{dx}\sqrt{x^2 - 1} = \frac{1}{2\sqrt{x^2 - 1}} \cdot \frac{d}{dx}(x^2 - 1) = \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x = \boxed{\frac{x}{\sqrt{x^2 - 1}}}.$$

(c) Since $\frac{d}{du}\log u = \frac{1}{u}$, we have

$$\frac{d}{dx}\log(x^3 + \cos x) = \frac{1}{x^3 + \cos x} \cdot \frac{d}{dx}(x^3 + \cos x) = \frac{1}{x^3 + \cos x} \cdot (3x^2 - \sin x) = \boxed{\frac{3x^2 - \sin x}{x^3 + \cos x}}.$$

(d) $(\sin\sqrt{x})' = (\cos\sqrt{x}) \cdot (\sqrt{x})' = \boxed{\frac{\cos\sqrt{x}}{2\sqrt{x}}}$.

(e) $\frac{d}{dx}e^{-2x^2} = e^{-2x^2} \cdot \frac{d}{dx}(-2x^2) = \boxed{-4xe^{-2x^2}}$.

(f) Since $\frac{d}{du}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}$, we have

$$\frac{d}{d\theta}\sin^{-1}\theta^2 = \frac{1}{\sqrt{1 - (\theta^2)^2}} \cdot \frac{d}{d\theta}(\theta^2) = \boxed{\frac{2\theta}{\sqrt{1 - \theta^4}}}.$$

(g) We apply the Chain Rule twice:

$$\begin{aligned}
\frac{d}{dx}\sqrt{1 + (x^2 + 1)^3} &= \frac{1}{2\sqrt{1 + (x^2 + 1)^3}} \cdot \frac{d}{dx}((x^2 + 1)^3) \\
&= \frac{1}{2\sqrt{1 + (x^2 + 1)^3}} \cdot 3(x^2 + 1)^2 \cdot \frac{d}{dx}(x^2 + 1) \\
&= \frac{1}{2\sqrt{1 + (x^2 + 1)^3}} \cdot 3(x^2 + 1)^2 \cdot 2x \\
&= \boxed{\frac{3x(x^2 + 1)^2}{\sqrt{1 + (x^2 + 1)^3}}}.
\end{aligned}$$

(h) $\left(e^{e^x}\right)' = e^{e^x} \cdot (e^x)' = e^x e^{e^x} = \boxed{e^{x+e^x}}$.

3.4.2 Letting $f(x) = e^x$, and noting $f'(x) = e^x$, we apply the Inverse Function Rule:

$$(\log x)' = (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\log x}} = \frac{1}{x}.$$

3.4.3

(a) Let $f(x) = x^n$ on the domain $[0, +\infty)$, and note that $f^{-1}(x) = x^{\frac{1}{n}}$. Then by the Inverse Function Rule, using $f'(x) = nx^{n-1}$, we have

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{n(x^{\frac{1}{n}})^{n-1}} = \frac{1}{n}x^{-\frac{n-1}{n}} = \frac{1}{n}x^{(\frac{1}{n}-1)}.$$

(b) Let $q = \frac{m}{n}$ where m is an integer and n is a positive integer. Then by part (a) and the Chain Rule,

$$\frac{d}{dx}x^q = \frac{d}{dx}\left(x^{\frac{1}{n}}\right)^m = m\left(x^{\frac{1}{n}}\right)^{m-1} \cdot \frac{d}{dx}x^{\frac{1}{n}} = mx^{\frac{m-1}{n}} \cdot \frac{1}{n}x^{(\frac{1}{n}-1)} = \frac{m}{n}x^{\left(\frac{(m-1)+1-n}{n}\right)} = \frac{m}{n}x^{(\frac{m}{n}-1)} = qx^{q-1}.$$

3.4.4 Letting $f(x) = \tan x$, and noting $f'(x) = \sec^2 x$, we apply the Inverse Function Rule:

$$(\tan^{-1}(x))' = \frac{1}{\sec^2(\tan^{-1} x)}.$$

But for any θ we have $\tan^2\theta + 1 = \sec^2\theta$, so

$$\sec^2(\tan^{-1} x) = \tan^2(\tan^{-1} x) + 1 = x^2 + 1,$$

and thus

$$(\tan^{-1}(x))' = \boxed{\frac{1}{x^2+1}}.$$

Both sides are defined for all real x.

3.4.5 Suppose that the leading term of $f(x)$ is cx^n, where n is a positive integer. Then the leading term of $f'(x)$ is ncx^{n-1}, and the leading term of $f''(x)$ is $n(n-1)cx^{n-2}$. So we must have

$$cx^n = (ncx^{n-1})(n(n-1)cx^{n-2}) = n^2(n-1)c^2x^{2n-3}.$$

Since these must be equal, we have $n = 2n - 3$, hence $n = 3$. Now, comparing the coefficients, we have $c = 18c^2$, so since c is nonzero, we have $c = \boxed{\frac{1}{18}}$. (Note that the polynomial $f(x) = \frac{x^3}{18}$ satisfies the condition.)

Exercises for Section 3.5

3.5.1 The conclusion is that f must be constant. To prove this, let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$. Then by the Mean Value Theorem, there is some $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But we know that $f'(c) = 0$ for all $c \in (x_1, x_2)$, so we conclude that $f(x_2) - f(x_1) = 0$; that is, $f(x_1) = f(x_2)$ for all $x_1, x_2 \in [a, b]$, and thus f is constant on $[a, b]$.

3.5.2 We note that $(f-g)' = f' - g' = 0$, so by the previous exercise, $f-g$ is a constant function.

3.5.3 Let the roots of f be $r_1 < r_2 < \cdots < r_n$. Since f is a polynomial, and thus differentiable, then by Rolle's Theorem, there must be some $s_1 \in (r_1, r_2)$ such that $f'(s_1) = 0$. Similarly, for any $1 \le k < n$, we can apply Rolle's Theorem to find $s_k \in (r_k, r_{k+1})$ such that $f'(s_k) = 0$. Thus, $\{s_1, s_2, \ldots, s_{n-1}\}$ are the $n-1$ distinct roots of f'.

3.5.4 By the Mean Value Theorem, there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

But $f'(c) > 0$ by assumption, thus

$$\frac{f(b)-f(a)}{b-a} > 0.$$

Since $a < b$, the denominator is positive, so the numerator must be positive as well, and thus $f(a) < f(b)$.

Exercises for Section 3.6

3.6.1 Implicit differentiation gives $2x - 2yy' = 0$, so $y' = \frac{x}{y}$. Substituting $x = 2$ and $y = \sqrt{3}$ gives $y' = \boxed{\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}}$.

3.6.2 Implicit differentiation gives

$$3y^2y' - 3x(y^2)' - (3x)'y^2 + 2x - xy' - x'y = 3y^2y' - 6xyy' - 3y^2 + 2x - xy' - y = 0.$$

Plugging in $x = -1$ and $y = 1$ gives

$$3y' + 6y' - 3 - 2 + y' - 1 = 0,$$

so $10y' = 6$ and thus $y' = \frac{3}{5}$. Thus, the equation of the line is $y - 1 = \frac{3}{5}(x+1)$, or $\boxed{y = \frac{3}{5}x + \frac{8}{5}}$.

3.6.3 Implicit differentiation gives

$$2x + y' = \frac{1}{y^2-1}\cdot\frac{d}{dx}(y^2-1) = \frac{1}{y^2-1}\cdot 2yy' = \frac{2yy'}{y^2-1}.$$

Solving for y' gives

$$y' = \frac{2x}{\frac{2y}{y^2-1} - 1} = \boxed{\frac{2x(y^2-1)}{-y^2+2y+1}}.$$

3.6.4 Implicitly differentiating the equation (using the Product Rule on both sides) gives

$$x\cos(x+y)(1+y') + \sin(x+y) = -y\sin(x-y)(1-y') + y'\cos(x-y).$$

Substituting $x = 0$ and $y = \frac{\pi}{2}$ gives

$$\sin\frac{\pi}{2} = -\frac{\pi}{2}\sin\left(-\frac{\pi}{2}\right)(1-y') + y'\cos\left(-\frac{\pi}{2}\right).$$

Evaluating the trig functions gives

$$1 = -\frac{\pi}{2}(-1)(1-y') + y'\cdot 0 = \frac{\pi}{2}(1-y'),$$

hence $y' = 1 - \frac{2}{\pi}$. Thus the slope of the tangent line is $\boxed{1 - \frac{2}{\pi}}$.

3.6.5 We take the equation $\sin^2\theta + \cos^2\theta = 1$ and differentiate with respect to θ. This gives

$$2\sin\theta \cdot \frac{d}{d\theta}\sin\theta + 2\cos\theta \cdot \frac{d}{d\theta}\cos\theta = 0.$$

Using $\frac{d}{d\theta}\sin\theta = \cos\theta$, we have

$$\sin\theta\cos\theta = -\cos\theta\left(\frac{d}{d\theta}\cos\theta\right),$$

from which we conclude that $\frac{d}{d\theta}\cos\theta = -\sin\theta$ everywhere that $\cos\theta \neq 0$. But these are isolated points, so by our assumption of continuity, we conclude that $\frac{d}{d\theta}\cos\theta = -\sin\theta$ everywhere.

Review Problems

3.26

(a) By the Chain Rule,

$$\frac{d}{dx}(2x+3)^{-3} = (-3)(2x+3)^{-4}(2) = \boxed{\frac{-6}{(2x+3)^4}}.$$

(b) By the Chain Rule,

$$\frac{d}{dx}\sqrt{x^3+e^x} = \frac{1}{2}(x^3+e^x)^{-\frac{1}{2}}(x^3+e^x)' = \frac{1}{2}(x^3+e^x)^{-\frac{1}{2}}(3x^2+e^x) = \boxed{\frac{3x^2+e^x}{2\sqrt{x^3+e^x}}}.$$

(c) Using the Chain Rule and then the Product Rule,

$$\begin{aligned}\frac{d}{dx}\log(\sin x\cos x) &= \frac{\frac{d}{dx}(\sin x\cos x)}{\sin x\cos x}\\ &= \frac{(\sin x)(-\sin x) + (\cos x)(\cos x)}{\sin x\cos x}\\ &= \boxed{\frac{\cos^2 x - \sin^2 x}{\sin x\cos x}}.\end{aligned}$$

If you are observant, you may notice that the numerator is $\cos 2x$ and the denominator is $\frac{1}{2}\sin 2x$, so the whole thing is $\boxed{2\cot 2x}$.

(d) Again, we use the Chain Rule repeatedly:

$$\begin{aligned}\frac{d}{dx}(1+(x+e^{x/2})^{1/2})^{1/3} &= (1/3)(1+(x+e^{x/2})^{1/2})^{-2/3}(1+(x+e^{x/2})^{1/2})'\\ &= (1/3)(1+(x+e^{x/2})^{1/2})^{-2/3}(1/2)(x+e^{x/2})^{-1/2}(x+e^{x/2})'\\ &= (1/6)(1+(x+e^{x/2})^{1/2})^{-2/3}(x+e^{x/2})^{-1/2}(1+e^{x/2}/2)\\ &= \boxed{\frac{2+e^{x/2}}{12(1+\sqrt{x+e^{x/2}})^{2/3}\sqrt{x+e^{x/2}}}}.\end{aligned}$$

(e) We use the Product Rule, and then the Chain Rule:

$$\frac{d}{dx}(axe^{-bx}) = (ax)'e^{-bx} + (ax)(e^{-bx})' = ae^{-bx} - abxe^{-bx} = \boxed{ae^{-bx}(1-bx)}.$$

(f) This requires the Chain Rule:

$$\frac{d}{dx}e^{-x^2} = e^{-x^2}(-x^2)' = \boxed{(-2x)e^{-x^2}}.$$

(g) Using the derivative of $\sin^{-1}$ and the Chain Rule, we get

$$\frac{d}{dx}\sin^{-1}(x^2) = \frac{1}{\sqrt{1-(x^2)^2}}\cdot(x^2)' = \boxed{\frac{2x}{\sqrt{1-x^4}}}.$$

(h) By the Chain Rule,

$$\frac{d}{dx}\sin^3 x = 3(\sin^2 x)(\sin x)' = \boxed{3\sin^2 x\cos x}.$$

3.27 First we find the slope of the tangent line, which is just the derivative $y' = 3(x-1)^2$ at $x = 3$. Thus we have $y'(3) = 3(3-1)^2 = 12$. Therefore, the line is $\boxed{y = 12(x-3)+10 = 12x-26}$.

3.28 We solve $2 = x^2 - 6x + 10$ to get $x^2 - 6x + 8 = (x-4)(x-2) = 0$, so since $x \geq 3$ we must have $x = 4$. Then, noting that $f'(x) = 2x - 6$, we have, by the Inverse Function Theorem,

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(4)} = \frac{1}{2(4)-6} = \boxed{\frac{1}{2}}.$$

3.29 Taking the implicit derivative with respect to x of both sides, we get

$$2yy' = 3x^2 - 3.$$

Plugging in $x = 2$ and $y = \sqrt{3}$ gives $2\sqrt{3}y' = 9$, so $y' = \frac{3\sqrt{3}}{2}$. Thus the tangent line is the graph of $\boxed{y = \frac{3\sqrt{3}}{2}(x-2)+\sqrt{3}}$.

3.30 We add $-f(a) + f(a) = 0$ to the numerator of the definition of $f^{\#}(a)$, and we obtain:

$$\begin{aligned}
f^{\#}(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)+f(a)-f(a-h)}{2h}\\
&= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{2h}+\frac{f(a)-f(a-h)}{2h}\right)\\
&= \lim_{h\to 0}\left(\frac{1}{2}\frac{f(a+h)-f(a)}{h}+\frac{1}{2}\frac{f(a-h)-f(a)}{-h}\right)\\
&= \frac{1}{2}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\frac{1}{2}\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}\\
&= \frac{1}{2}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\frac{1}{2}\lim_{-h\to 0}\frac{f(a+(-h))-f(a)}{(-h)}\\
&= \frac{1}{2}f'(a)+\frac{1}{2}f'(a) = f'(a).
\end{aligned}$$

It follows that $f^{\#}(a) = f'(a)$ for all a where $f'(a)$ exists. Notice that it is possible that $f^{\#}(a)$ may exist while $f'(a)$ does not! (The reader is encouraged to find an example of a function f where this occurs.)

3.31

(a) We apply the product rule twice:

$$(fgh)' = f'(gh) + f(gh)' = f'gh + fg'h + fgh'.$$

(b) We conjecture that

$$(f_1 f_2 \cdots f_k)' = (f_1' f_2 \cdots f_k) + (f_1 f_2' \cdots f_k) + (f_1 f_2 \cdots f_k'),$$

where there are n summands and each summand is a product that contains the derivative of exactly one of the f_j.

We can prove this by induction on the number of terms. Clearly the base case holds: $f' = f'$. For the inductive step, assume that the result holds for a product of $k-1$ terms. Then we compute:

$$\begin{aligned}(f_1 f_2 \cdots f_k)' &= f_1'(f_2 \cdots f_k) + f_1(f_2 \cdots f_k)' \\ &= f_1'(f_2 \cdots f_k) + f_1\left((f_2' \cdots f_k) + \cdots + (f_2 \cdots f_k')\right) \\ &= (f_1' f_2 \cdots f_k) + (f_1 f_2' \cdots f_k) + (f_1 f_2 \cdots f_k').\end{aligned}$$

3.32 Since we are trying to find an example where the Mean Value Theorem fails, we know that f must be non-differentiable at some point in (a, b). So we look for an example that is not differentiable. The simplest example is $|x|$. Indeed, if $f(x) = |x|$, then on the interval $[-1, 1]$ there is no point $c \in (-1, 1)$ such that $f'(c) = \frac{|1|-|-1|}{1-(-1)} = 0$.

3.33 The derivative is

$$f'(x) = 1 + 2x + 3x^2 + \cdots + 100x^{99}.$$

So $f'(1) = 1 + 2 + \cdots + 100 = \frac{(100)(101)}{2} = \boxed{5050}$.

Challenge Problems

3.34 Using $a^x = e^{x \log a}$, we see

$$\begin{aligned}\frac{d}{dx}(a^x) &= \frac{d}{dx} e^{x \log a} \\ &= (\log a) e^{x \log a} \\ &= \boxed{a^x \log a}.\end{aligned}$$

Using $\log_a x = \frac{\log x}{\log a}$, we get

$$\frac{d}{dx} \log_a x = \boxed{\frac{1}{x \log a}}.$$

3.35 We have $f'(x) = 3x^2 + a$, so the slopes of the tangent lines at a and b are $3a^2 + a$ and $3b^2 + a$. If these lines are parallel, then these slopes must be equal, so $3a^2 + a = 3b^2 + a$, or $a^2 = b^2$. Since $a \neq b$, this means that $a = -b$, so $a + b = 0$. Therefore, $f(1) = 1 + a + b = \boxed{1}$.

3.36 Assume that $f(-x) = f(x)$. Then taking the derivatives of both sides we get $-f'(-x) = f'(x)$, or $f'(-x) = -f'(x)$. Therefore if f is even then f' is odd.

Likewise, taking the derivative of $f(-x) = -f(x)$, we get $f'(-x) = f'(x)$. Therefore if f is odd then f' is even.

3.37 Define $g(x) = f(x) - f(2x)$. We are given that $g'(1) = 5$ and $g'(2) = 7$. Then note that

$$h(x) = f(x) - f(4x) = (f(x) - f(2x)) + (f(2x) - f(4x)) = g(x) + g(2x).$$

So $h'(x) = g'(x) + 2g'(2x)$, and hence $h'(1) = 5 + 2(7) = \boxed{19}$.

3.38 We prove this by induction. The 0^{th} derivative of a nonzero constant function (a degree 0 polynomial) is a nonzero constant. For the inductive step, assume that the result is true for a degree $n-1$ polynomial (where n is a positive integer). Now, if p is a degree n polynomial, then

$$p^{(n)} = (p')^{(n-1)}.$$

Since p is degree n, the polynomial p' must be degree $n-1$ (notice that if the leading coefficient of p is $a \neq 0$, then the leading coefficient of p' is $na \neq 0$). Applying the inductive assumption, we then have $p^{(n)} = (p')^{(n-1)} = c \neq 0$, so we are done.

3.39

(a) Using the Product Rule, we have

$$f'(x) = m(x-r)^{m-1}h(x) + (x-r)^m h'(x) = (x-r)^{m-1}(mh(x) + (x-r)h'(x)).$$

Letting $h_1(x) = mh(x) + (x-r)h'(x)$, we see that we can write $f'(x) = (x-r)^{m-1}h_1(x)$, and further $h_1(r) = mh(r) + (r-r)h'(r) = mh(r) \neq 0$. Thus, f' has root r with multiplicity $r-1$.

(b) By repeated application of part (a), we have that $f^{(k)}$ has root r with multiplicity $m-k$. Thus, for $1 \le k < m$, we have $f^{(k)}(r) = 0$.

3.40

(a) By the Product Rule,

$$(fg)' = f'g + fg'.$$

We simply apply the Product Rule again:

$$(f'g + fg')' = (f'g)' + (fg')' = f''g' + f'g' + f'g' + fg'' = f''g + 2f'g' + fg''.$$

(b) We prove this by induction. We have the base case for $n = 0$: $fg = fg$. Now assume that the result is true for some nonnegative integer n. Then

$$\begin{aligned}(fg)^{(n+1)} = ((fg)^{(n)})' &= \frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n-k)} \\ &= \sum_{k=0}^{n}\binom{n}{k}\left(f^{(k+1)}g^{(n-k)} + f^{(k)}g^{(n-k+1)}\right) \\ &= \left[\sum_{k=0}^{n}\binom{n}{k}f^{(k+1)}g^{(n-k)}\right] + \left[\sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n-k+1)}\right]\end{aligned}$$

Replacing k with $k-1$ in the first sum gives

$$\left[\sum_{k=1}^{n+1}\binom{n}{k-1}f^{(k)}g^{(n-(k-1))}\right] + \left[\sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{((n+1)-k)}\right].$$

Noting that $\binom{n}{-1} = \binom{n}{n+1} = 0$, we see that both of these sums can be taken from 0 to $n+1$:

$$\sum_{k=0}^{n+1}\left(\binom{n}{k-1} + \binom{n}{k}\right)f^{(k)}g^{((n+1)-k)}.$$

Finally we apply Pascal's identity, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ to get

$$(fg)^{(n+1)} = \sum_{k=0}^{n+1}\binom{n+1}{k}f^{(k)}g^{((n+1)-k)},$$

completing the proof.

CHAPTER 4

Applications of the Derivative

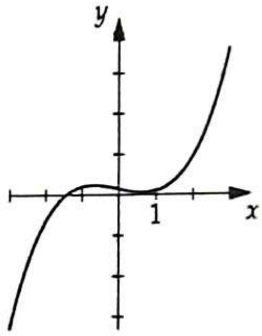
Exercises for Section 4.1

4.1.1 In each of these parts, let $f(x)$ be the function listed.

(a) We calculate $f'(x) = 9x^2 - 4$ and $f''(x) = 18x$.

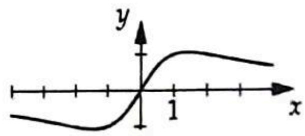
We know that f is increasing when f' is positive; this occurs when $9x^2 > 4$ or $|x| > 2/3$. Thus f is increasing on $(-\infty, -2/3) \cup (2/3, \infty)$. Similarly, f is decreasing when f' is negative, which occurs when $|x| < 2/3$, or $x \in (-2/3, 2/3)$.

Finally, f is concave up whenever $f''(x) > 0$, which occurs when $18x > 0$. Thus f is concave up on $(0, \infty)$. Similarly, f is concave down when $f''(x) < 0$, which occurs on $(-\infty, 0)$. Lastly, $f''(x) = 0$ where $x = 0$, and 0 is the only inflection point.



(b) We calculate

$$f'(x) = \frac{1(x^2+2) - x(2x)}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2}.$$



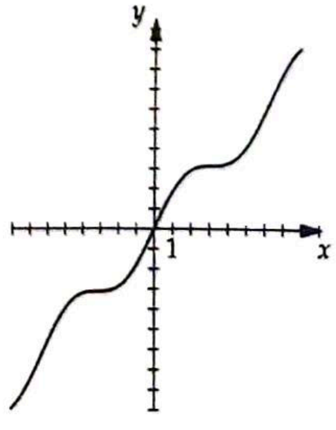
Then

$$f''(x) = \frac{-2x(x^2+2)^2 - (2-x^2)2(2x)(x^2+2)}{(x^2+2)^4} = \frac{-2x(x^2+2) - (2-x^2)(4x)}{(x^2+2)^3} = \frac{2x^3 - 12x}{(x^2+2)^3}.$$

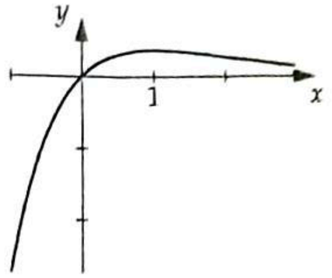
In both f' and f'', the denominator is always positive, and thus does not affect the sign.

Therefore, f is increasing whenever $2 - x^2 > 0$, or on $(-\sqrt{2}, \sqrt{2})$, and f is decreasing whenever $2 - x^2 < 0$, or on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$. Likewise, $2x^3 - 12x = 2x(x^2 - 6)$ is positive on $(-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$, so f is concave up on these intervals, and $2x(x^2 - 6)$ is negative on $(-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$, so f is concave down here. The points of inflection are $-\sqrt{6}$, 0, and $\sqrt{6}$.

(c) First we compute $f'(x) = 1 + \cos x$ and $f''(x) = -\sin x$. Since $\cos x \ge -1$, we have $f'(x) \ge 0$ for all x, and thus f is increasing everywhere. On the other hand, f'' alternates signs: in particular, f is concave down on $(2k\pi, (2k+1)\pi)$ for every integer k, and concave up on $((2k-1)\pi, 2k\pi)$ for every integer k. All integer multiples of π are inflection points.



(d) We calculate



$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x},$$
$$f''(x) = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}.$$

Since e^{-x} is positive for all x, we see that f' has the same sign as $1-x$ for all x, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. Also, f'' has the same sign as $x-2$ for all x, so f is concave up on $(2, \infty)$ and concave down on $(-\infty, 2)$, and $x = 2$ is the only inflection point.

4.1.2 f' is: positive at C, D (the function is strictly increasing at these points); negative at A, J (the function is strictly decreasing at these points); zero at B, E, G, I (the function "levels off" at these points); and undefined at F, H (the function has a "sharp corner" at these points).

f'' is: positive at A, B (the function is concave up at these points); negative at D, E, J (the function is concave down at these points); zero at C, G, I (C and I are inflection points, and the function is constant at G); and undefined at F, H (the function has a "sharp corner" at these points).

4.1.3 Suppose f is strictly increasing. Then for any y, we cannot have two values x_0, x_1 with $x_0 < x_1$ such that $f(x_0) = f(x_1) = y$, since the strictly increasing property of f implies $f(x_0) < f(x_1)$. Thus each value of y is the image under f of at most one point, and hence f has an inverse. Essentially the same argument holds if f is strictly decreasing.

A non-strictly monotonic function does not necessarily have an inverse; for example, any constant function with domain $\mathbb{R}$ is monotonic but not invertible.

4.1.4 This solution is mainly a matter of reversing all of the inequalities from the comparable result in the text for concave up functions.

Suppose that $b > a$ is a point in I. We need to show that $f(b)$ is less than the y-coordinate of the point on the tangent line with x-coordinate b. This point lies on the line through $(a, f(a))$ with slope $f'(a)$; this line has equation $y - f(a) = f'(a)(x-a)$ in point-slope form. So the point on this tangent line with x-coordinate b has y-coordinate

$$f(a) + f'(a)(b-a).$$

Thus, in order to show that the graph lies above the tangent line when $x = b$, we have to show that

$$f(b) < f(a) + f'(a)(b-a).$$

Let's rewrite this expression with all the terms on the same side:

$$\frac{f(b) - f(a)}{b-a} - f'(a) < 0.$$

By the Mean Value Theorem, there is some c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Now we just have to prove that $f'(c) - f'(a) < 0$, or $f'(c) < f'(a)$. But since $b > a$, we also have $c > a$, and since $f'' < 0$, we know that f' is strictly decreasing, so $f'(c) < f'(a)$.

If $b < a$, then the argument is essentially the same, except that dividing by $b - a$ will give the inequality

$$\frac{f(b) - f(a)}{b-a} - f'(a) > 0.$$

Then, applying the Mean Value Theorem gives $f'(c) - f'(a) > 0$ for some $b < c < a$; again, since $f'' < 0$, we know that f' is decreasing, so $f'(c) > f'(a)$, and we're done.

4.1.5 Let $g(x)$ be the linear function whose graph is the secant line from $(a, f(a))$ to $(b, f(b))$. We can write

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

However, it is more convenient to write g more symmetrically as

$$g(x) = \frac{b - x}{b - a}f(a) + \frac{x - a}{b - a}f(b).$$

(To quickly see that this must be the same function, note that g is linear with $g(a) = f(a)$ and $g(b) = f(b)$.) We must show that $g(c) > f(c)$ for all $c \in (a, b)$.

For any $c \in (a, b)$, we can compute:

$$\begin{aligned} g(c) - f(c) &= \frac{b - c}{b - a}f(a) + \frac{c - a}{b - a}f(b) - \left(\frac{b - c}{b - a} + \frac{c - a}{b - a}\right)f(c) \\ &= \frac{b - c}{b - a}(f(a) - f(c)) + \frac{c - a}{b - a}(f(b) - f(c)). \end{aligned}$$

By the Mean Value Theorem, there is some $\xi_1 \in (a, c)$ such that $f'(\xi_1) = \dfrac{f(a) - f(c)}{a - c}$, and there is some $\xi_2 \in (c, b)$ such that $f'(\xi_2) = \dfrac{f(b) - f(c)}{b - c}$. Thus, we have

$$\begin{aligned} g(c) - f(c) &= \frac{b - c}{b - a}(a - c)f'(\xi_1) + \frac{c - a}{b - a}(b - c)f'(\xi_2) \\ &= \frac{(c - a)(b - c)}{b - a}\left(f'(\xi_2) - f'(\xi_1)\right). \end{aligned}$$

But the quantity $\dfrac{(c - a)(b - c)}{b - a} > 0$ since $a < c < b$. Also, since f' is increasing (because $f'' > 0$) and $\xi_1 < \xi_2$, we have $f'(\xi_2) - f'(\xi_1) > 0$. Therefore, $g(c) - f(c) > 0$, so $g(c) > f(c)$ and hence the secant line between $(a, f(a))$ and $(b, f(b))$ lies above the graph of $f(x)$ on (a, b).

To prove the corresponding statement for concave down functions, we note that if f is concave down, then $-f$ is concave up, so the secant lines of $-f$ lie above the graph of $-f$. Therefore the secant lines of f lie below the graph of f.

Exercises for Section 4.2

4.2.1

(a) f is differentiable on $[0, 2]$, so the extremal points must occur at the endpoints 0 and 2 or where the derivative is equal to zero. The derivative is $f'(x) = 4x - 1$, which is zero at $x = \frac{1}{4}$. We evaluate at each of these points:

$$\begin{aligned} &f(0) = 3, \\ &f\left(\frac{1}{4}\right) = 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} + 3 = \frac{23}{8}, \\ &f(2) = 2(2)^2 - 2 + 3 = 9. \end{aligned}$$

Therefore, the maximum on $[0, 2]$ is $\boxed{9}$, and the minimum is $\boxed{\frac{23}{8}}$.

(b) The function f is differentiable on $(1,3]$, but grows without bound as x approaches 1 from the positive side. Therefore, f has no maximum, and its minimum occurs either at 3 or where its derivative is zero. By the quotient rule,

$$f'(x) = \frac{2x(x-1)-x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2}.$$

This is zero at $x = 0$ and $x = 2$, but only 2 is within $(1,3]$. Therefore, the minimum is either $f(2) = 2^2/(2-1) = 4$ or $f(3) = 3^2/(3-1) = 9/2$. Since $4 < 9/2$, the minimum is $\boxed{4}$ at $x = 2$.

(c) This function is differentiable on $[-1,1]$ with derivative $12x^2-16x$. This derivative has only one root in $[-1,1]$, namely $x = 0$. We check f at the critical points and the endpoints: $f(-1) = -11$, $f(0) = 1$, and $f(1) = -3$. Therefore the minimum is $\boxed{-11}$ and the maximum is $\boxed{1}$.

(d) This function is differentiable on $[0,2\pi]$ with derivative $\cos x + 1$, which is always nonnegative. Therefore the function is increasing, and thus achieves its extrema at the endpoints of the interval. The minimum is $f(0) = \boxed{0}$ and the maximum is $f(2\pi) = \boxed{2\pi}$.

4.2.2 If $f(x)$ has a local maximum at $x = -2$, we must have $f'(-2) = 0$. Calculating the derivative gives $f'(x) = 1 + \frac{k}{x^2}$, so $f'(-2) = 1 + \frac{k}{4} = 0$, making $\boxed{k = -4}$. We must check that this is a maximum, and the Second Derivative Test will do the job. We have $f'(x) = 1 - \frac{4}{x^2}$, so $f''(-2) = 8/(-2)^3 = -1 < 0$, thus f has a local maximum at $x = -2$.

4.2.3 We can write our revenue function as a piecewise-continuous function:

$$R(n) = \begin{cases} 1000n & \text{if } n < 100 \\ n(1000 - 5(n-100)) & \text{if } n \geq 100 \end{cases}$$

The cost function is just $C(n) = 40000 + 200n$. Clearly for each of the first 100 people, the marginal revenue is \$1000 but the marginal cost is only \$200, so they will want at least 100 people. So we can restrict ourselves to $n \geq 100$, and thus simply write $R(n) = n(1500 - 5n)$. Then the profit is

$$P(n) = n(1500 - 5n) - (40000 + 200n) = -5n^2 + 1300n - 40000.$$

We don't need calculus to maximize this! This is a downward-opening parabola whose vertex is at $n = 1300/10 = 130$. So $\boxed{130}$ people maximizes the profit.

4.2.4 We might expect the derivative to be a degree 4 polynomial, since it should have two regular roots plus a double-root (which is the critical point that is not an extremum). A simple example of such a function is $x^2(x-1)(x+1) = x^4 - x^2$. This function is the derivative of $\frac{1}{5}x^5 - \frac{1}{3}x^3$, but we can multiply by 15 to eliminate the fractions. Hence, our possible function is $\boxed{f(x) = 3x^5 - 5x^3}$.

We check that this polynomial satisfies the conditions that we want. As discussed above, the first derivative is $15(x^4-x^2) = 15x^2(x-1)(x+1)$, with roots of $0, 1$, and -1. The second derivative is $f''(x) = 15(4x^3-2x) = 30x(2x^2-1)$. Evaluating at the critical points we find $f''(0) = 0$, $f''(1) = 30$ and $f''(-1) = -30$. Therefore, f has a minimum at 1 and a maximum at -1. At 0, the Second Derivative Test is inconclusive, so we must demonstrate that 0 is not a local maximum or minimum by other means. On the interval $(-1,1)$, we have $15x^2(x^2-1) \leq 0$. Therefore the derivative is nonpositive on $(-1,1)$, and the function is decreasing on this interval. Therefore 0 is not a local extremum.

4.2.5 Suppose that the soup can has a base radius of r and a height of h, both in cm. The base has area πr^2, so the entire can has volume $\pi r^2 h = 200$ (in cm^3). The top and bottom of the can have combined area $2\pi r^2$, and the lateral surface area is equal to the circumference of the circle times the height, or $2\pi rh$, for a total surface area of $2\pi r(r+h)$ cm^2. This costs 0.2 cents per square centimeter, so our total cost is $0.4\pi r(r+h)$ cents.

Solving the volume formula for $h = 200/r^2\pi$ and plugging into this expression yields a cost of

$$c(r) = 0.4\pi r\left(r + \frac{200}{r^2\pi}\right) = 0.4\left(\pi r^2 + \frac{200}{r}\right).$$

This is continuous and differentiable on $(0, \infty)$, and as r goes towards either endpoint, the cost grows without bound, so its minimum must occur at a point where the derivative is equal to zero. The derivative of cost with respect to radius is

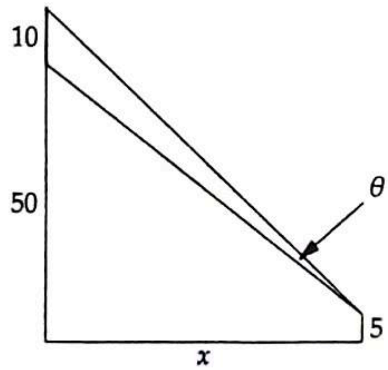
$$\frac{dc}{dr} = 0.4\left(2\pi r - \frac{200}{r^2}\right) = 0.8\left(\frac{\pi r^3 - 100}{r^2}\right),$$

which is zero when $\pi r^3 = 100$ or $r = \sqrt[3]{100/\pi}$. So the cost is

$$0.4\left(\pi r^2 + \frac{200}{r}\right) = 0.4\left(\frac{\pi r^3 + 200}{r}\right) = 0.4\left(\frac{300}{\sqrt[3]{100/\pi}}\right) = 0.4\left(\frac{300\sqrt[3]{10\pi}}{10}\right) = \boxed{12\sqrt[3]{10\pi} \approx 37.86 \text{ cents}}.$$

4.2.6 Let x denote the length of the side of the grazing area that is perpendicular to the wall. Thus the side that is parallel to the wall has length $40 - 2x$ (since we use $2x$ meters on fencing on the two perpendicular sides), and hence the area is $f(x) = x(40 - 2x) = 40x - 2x^2$. The domain is $x \in [0, 20]$, and we see that the area at $x = 0$ or $x = 20$ is 0, so the critical point of the function must be the global maximum. We compute $f'(x) = 40 - 4x$, so $x = 10$ is the critical point. This rectangle has area $(10)(20) = \boxed{200}$ square meters.

4.2.7 Consider the diagram at right. Based on right triangles drawn from eye level to the building, we have



$$\theta = \tan^{-1}\frac{55}{x} - \tan^{-1}\frac{45}{x}.$$

Knowing the derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$ and applying the Chain Rule, we get

$$\theta'(x) = \frac{-55/x^2}{1 + 55^2/x^2} - \frac{-45/x^2}{1 + 45^2/x^2} = \frac{45}{x^2 + 45^2} - \frac{55}{x^2 + 55^2}.$$

We solve $\theta'(x) = 0$ to find the critical point(s). We get

$$0 = \frac{45}{x^2 + 45^2} - \frac{55}{x^2 + 55^2},$$

so $45(x^2 + 55^2) = 55(x^2 + 45^2)$, or $10x^2 = 45 \cdot 55^2 - 55 \cdot 45^2 = 10 \cdot 45 \cdot 55$. This gives $x = \sqrt{2475} = 15\sqrt{11}$ as the only critical point. To verify that this critical point is a local maximum, write $\theta'(x)$ with a common denominator:

$$\theta'(x) = \frac{-10x^2 + (45 \cdot 55^2 - 55 \cdot 45^2)}{(x^2 + 45^2)(x^2 + 55^2)}.$$

We already determined that $x = 15\sqrt{11}$ is the value that makes the numerator of $\theta'(x)$ equal to 0. Since the denominator is always positive, it is clear that $\theta'(x) > 0$ for $0 < x < 15\sqrt{11}$, and $\theta'(x) < 0$ for $x > 15\sqrt{11}$. Thus, $\theta(x)$ is increasing on $[0, 15\sqrt{11})$ and is decreasing on $(15\sqrt{11}, \infty)$, and hence $x = 15\sqrt{11}$ is the global maximum. Thus, the optimal viewing distance is $\boxed{15\sqrt{11} \approx 49.75 \text{ feet}}$ away from the building.

Exercises for Section 4.3

4.3.1 The formula for the height of the rock is given by $h = -16t^2 + v_0 t + h_0$, where v_0 is the initial velocity in feet/second and h_0 is the initial height in feet. Plugging the given data into this equation gives $h = -16t^2 - 16t + 480$; note that the initial velocity is negative, since the rock is thrown downward. Setting $h = 0$, we can simplify:

$$-16t^2 - 16t + 480 = 0 \implies t^2 + t - 30 = 0 \implies (t - 5)(t + 6) = 0.$$

Since we begin at $t = 0$, the only positive solution is $\boxed{t = 5}$ seconds after it is thrown. The speed then is $v = h' = -32t - 16 = -32(5) - 16 = \boxed{-176}$ feet per second, or 176 feet per second downwards (assuming no air resistance of course).

4.3.2 When the cannonball is shot, it has a vertical velocity component of $30 \cdot \sin(30°) = 15$ meters per second. Plugging this and an initial height of 0 into our equation for motion under the force of gravity, we get $y = -4.9t^2+15t$. Besides $t = 0$, the other solution to $y = 0$ is $t = 15/4.9 = 150/49 \approx 3.06$ seconds later.

The horizontal component is easier to analyze. Initially, it is $30\cos(30°) \approx 25.98$ meters per second. Assuming no air resistance, it remains this for its entire flight. Thus, 3.06 seconds later, it has moved a total of $25.98 \cdot 3.06 \approx \boxed{79.5}$ meters horizontally.

4.3.3

(a) Speed is the derivative of position, so we have $\boxed{a'(t) \geq b'(t)}$.

(b) We can't just write $b'(t) = L(x)$, since one is a function of x and the other is a function of t. The correct equation is $\boxed{b'(t) = L(b(t))}$.

(c) This means that car A is always 1 hour behind car B.

(d) This means that car A is always traveling at the speed limit of the point where it was one hour ago.

Exercises for Section 4.4

4.4.1

(a) We consider the function $f(x) = x^{\frac{1}{4}}$, approximated at $x = 81$. The derivative of $x^{\frac{1}{4}}$ is $\frac{1}{4}x^{-\frac{3}{4}}$, which at 81 is $\frac{1}{4}(81^{-\frac{3}{4}}) = \frac{1}{108}$. Therefore,

$$\sqrt[4]{80} \approx \sqrt[4]{81} + \frac{1}{108}(80 - 81) = 3 - \frac{1}{108} = \boxed{\frac{323}{108}}.$$

(b) We convert 62° to radians: $62° = \frac{62\pi}{180} = \frac{\pi}{3} + \frac{\pi}{90}$. We approximate $\cos x$ at $x = \frac{\pi}{3}$ to get

$$\cos\left(\frac{\pi}{3} + \frac{\pi}{90}\right) \approx \cos\frac{\pi}{3} + \frac{\pi}{90}\left(-\sin\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}\pi}{180} = \boxed{\frac{90 - \sqrt{3}\pi}{180}} \approx 0.4698.$$

(c) We approximate $\log_2 x$ at $x = 16$. Write $\log_2 x = \frac{\log x}{\log 2}$, so that the derivative of $\log_2 x$ is $\frac{1}{x\log 2}$. Therefore,

$$\log_2 17 \approx \log_2 16 + \frac{1}{16\log 2} = \boxed{4 + \frac{1}{16\log 2}} \approx 4.0902.$$

4.4.2 If the actual side length is s, the volume is s^3. We know that s is close to 30 cm, so we can approximate the side length near $s = 30$. The derivative of the volume at $s = 30$ is $3s^2 = 3(30)^2 = 2700\text{ cm}^2$, so the linear approximation gives an error of $0.2 \cdot 2700 = \boxed{540\text{ cm}^3}$.

4.4.3 The volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius. We need to find the difference between a sphere with radius 500 mm and one with radius 502 mm. So we use the tangent line approximation of the formula for volume at $r = 500$. The derivative is $4\pi r^2$, so the added volume (paint) is approximately

$$4\pi(500)^2 \cdot 2 = \boxed{2000000\pi\text{ mm}^3 = 2000\pi\text{ cm}^3 = 2\pi\text{ liters}}.$$

4.4.4 The volume of such a cylinder is πh^3. If the error in the measurement is E, then the measured volume is approximately $\pi h^3 + 3\pi h^2 E$. The error in this measurement is $3\pi h^2 E$, making the percent error $\frac{3\pi h^2 E}{\pi h^3} = \frac{3E}{h}$. Therefore, if we require $\frac{3|E|}{h} < \frac{2}{100}$, then we have $\frac{|E|}{h} < \frac{2}{300}$, so we can tolerate a $\boxed{\frac{2}{3}\%}$ error in the measurement of the height.

4.4.5 We know that $f'(x) = nx^{n-1}$. Therefore, by the tangent line approximation,

$$f(x+\epsilon) \approx f(x) + \epsilon f'(x) = x^n + nx^{n-1}\epsilon.$$

But the Binomial Theorem says that

$$f(x+\epsilon) = (x+\epsilon)^n = x^n + nx^{n-1}\epsilon + \cdots + \epsilon^n.$$

We see that the tangent line approximation is just the first two terms of the Binomial Theorem, ignoring the other terms. When ϵ is small, this is not unreasonable; each of those terms includes ϵ^m for some $m \geq 2$.

Exercises for Section 4.5

4.5.1 The number $\sqrt[4]{2}$ is the positive root of the function $x^4 - 2$, whose derivative is $4x^3$. One possible starting point is $x = 1$. Then the tangent line approximation for $x^4 - 2$ is $4(x-1) - 1 = 4x - 5$, which hits the x-axis at 1.25. The tangent line approximation at $x = 1.25$ is $x^4 - 2 \approx (4 \cdot 1.25^3)(x - 1.25) + (1.25^4 - 2)$, which is zero when

$$x = 1.25 + \frac{2 - 1.25^4}{4 \cdot 1.25^3} = 1.25 + \frac{512 - 625}{2000} = 1.25 - .0565 = 1.1935.$$

A calculator gives $\sqrt[4]{2} \approx 1.1892$, so this achieves 3 digits of accuracy.

4.5.2 The step in Newton's method for $f(x) = x^{\frac{1}{3}}$ is

$$r_{n+1} = r_n - \frac{r_n^{1/3}}{\frac{1}{3}r_n^{-2/3}} = -2r_n.$$

Starting with $r_0 = 1$, the sequence we find is $1, -2, 4, -8, 16 \ldots$. Of course we know the only root is at $x = 0$.

The issue that we run into is that the slope grows (in magnitude) without bound as x approaches 0. This is forcing the linear approximation to be bad regardless of how close we get to 0. When we draw the tangent line at r_j, it overshoots the root and intersects the x-axis at a point which is on the other side of the y-axis and further from the root than our original point. It seems intuitively clear that if the slope of f between r_n and r_{n+1} remains relatively constant, then the Newton's method approximation should be good. This example gives some credence to that conjecture since it is an example of Newton's method's failing at a point near which the derivative is unbounded.

Exercises for Section 4.6

4.6.1 Let's put everything in kilometers first, and assign some variables. Let the horizontal distance from the camera to the spot on the ground directly below the plane be x, time be t, and the angle the camera makes with the ground be θ. Then trigonometry yields $\tan\theta = 10/x$, or $x = 10\cot\theta$ (in km). We differentiate to get

$$\frac{dx}{dt} = -10\csc^2\theta\frac{d\theta}{dt}.$$

We know that $\frac{dx}{dt} = -0.2$ km/s, where the sign is negative since the plane is approaching the camera. We are considering the moment when $\sin\theta = \frac{10}{15} = \frac{2}{3}$, so $\csc^2\theta = \frac{1}{\sin^2\theta} = \frac{9}{4}$. Therefore,

$$\frac{d\theta}{dt} = \frac{-.2}{-10\cdot\frac{9}{4}} = \boxed{\frac{2}{225}\frac{\text{rad}}{\text{sec}}}.$$

4.6.2 If pressure is P and volume V, then Boyle's Law for this gas states that $PV = 200\text{ cm}^3\cdot\text{atm}$. We differentiate both sides of this equation, yielding

$$P\cdot\frac{dV}{dt} + V\cdot\frac{dP}{dt} = 0.$$

We know that at the time in question, $P = 1 + 10\cdot .1 = 2$ atm and V therefore is 100 cm^3. Solving our equation,

$$\frac{dV}{dt} = -\frac{V}{P}\cdot\frac{dP}{dt} = -\frac{100\text{ cm}^3}{2\text{ atm}}\cdot(0.1)\text{ atm/min} = -5\text{ cm}^3/\text{min}.$$

Thus the volume is $\boxed{\text{decreasing at a rate of } 5\text{ cm}^3/\text{min}}$.

4.6.3 Let V be the volume of the snowball and x be its diameter. Using the formula for the volume of a sphere, we can write $V = \frac{4}{3}\pi\left(\frac{x}{2}\right)^3 = \frac{\pi}{6}x^3$. Then, differentiating both sides with respect to t, we get

$$\frac{dV}{dt} = \frac{\pi}{2}x^2\frac{dx}{dt}.$$

To find $\frac{dx}{dt}$, we just need $\frac{dV}{dt}$ and x. Since volume is proportional to the cube of diameter, at half its volume the snowball will have a diameter of $\frac{10\text{cm}}{\sqrt[3]{2}} = 5\sqrt[3]{4}$ cm. Thus,

$$\frac{dx}{dt} = \frac{dV}{dt}\cdot\frac{2}{\pi x^2} = (0.5\text{ cm}^3/\text{sec})\cdot\frac{2}{\pi 50\sqrt[3]{2}\text{ cm}^2} = \boxed{\frac{\sqrt[3]{4}}{100\pi}\frac{\text{cm}}{\text{sec}}}.$$

4.6.4 We are given that $\frac{dV}{dt}$ is proportional to surface area, which is proportional to r^2 (where r is the radius), so $\frac{dV}{dt} = kr^2$ for some constant k. We assume for now that the meteorite is spherical, so $V = \frac{4\pi}{3}r^3$. Upon differentiating by t, we get $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. Eliminating $\frac{dV}{dt}$ from this system we deduce that $\frac{dr}{dt} = \frac{k}{4\pi}$ is constant, so the meteorite's radius is decreasing $\boxed{\text{at a constant rate}}$.

Notice that we needn't have assumed that the meteorite was spherical, but only that the surface area was always proportional to $V^{2/3}$. Letting r denote the square root of surface area allows us to recycle the above computation to prove this more general case.

4.6.5 The position of the first car after t seconds is $4t^2$ north of the starting point, and the position of the second car after t seconds is $\frac{5}{2}t^2$ east of the starting point. So the distance between them is

$$\sqrt{(4t^2)^2 + \left(\frac{5}{2}t^2\right)^2} = t^2\sqrt{16 + \frac{25}{4}} = \frac{t^2\sqrt{89}}{2}.$$

The rate of change of this distance is $t\sqrt{89}$, so at $t = 6$ the distance between the cars is increasing by $\boxed{6\sqrt{89}}$ meters per second.

4.6.6 We let $t = 0$ represent 12:00 and measure time in minutes. Also, because we are working with a clock, it makes sense to measure angles clockwise from the 12:00 position. The minute hand sweeps an angle of 2π every 60 minutes, so its angular velocity is $\mu = \frac{2\pi}{60} = \frac{\pi}{30}$, and thus the position of the minute hand at time t is $\mu t = \frac{\pi}{30}t$ radians clockwise from the 12:00 position. Similarly, the hour hand sweeps an angle of 2π every $60\cdot 12 = 720$

minutes, so its angular velocity is $\eta = \frac{2\pi}{720} = \frac{\pi}{360}$, and thus the position of the hour hand at time t is $\eta t = \frac{\pi}{360}t$ radians clockwise from the 12:00 position. The angle between the two hands as a function of t is $\mu t - \eta t = (\mu - \eta)t$.

We are concerned with the time $t = 120$, representing 2:00. By the Law of Cosines, the distance x between the tips of the two hands satisfies

$$x^2 = 6^2 + 10^2 - 2 \cdot 10 \cdot 6\cos((\mu - \eta)t) = 136 - 120\cos((\mu - \eta)t).$$

Differentiating gives

$$2x\frac{dx}{dt} = 120(\mu - \eta)\sin((\mu - \eta)t).$$

Now we consider $t = 120$. The distance satisfies

$$x^2 = 136 - 120\cos\left(\left(\frac{\pi}{30} - \frac{\pi}{360}\right)(120)\right) = 136 - 120\cos\left(4\pi - \frac{\pi}{3}\right) = 136 - 120\cos\left(-\frac{\pi}{3}\right) = 76,$$

so $x = \sqrt{76} = 2\sqrt{19}$ is the distance between the tips of the two hands at $t = 120$.

We also have

$$x\frac{dx}{dt} = 60\left(\frac{\pi}{30} - \frac{\pi}{360}\right)\sin\left(\left(\frac{\pi}{30} - \frac{\pi}{360}\right)(120)\right) = \frac{11\pi}{6}\sin\left(4\pi - \frac{\pi}{3}\right) = -\frac{11\pi\sqrt{3}}{12}.$$

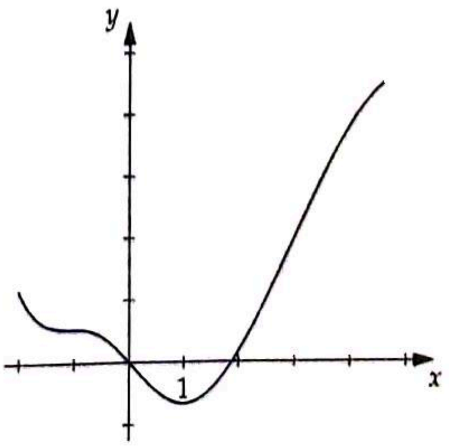
Therefore

$$\frac{dx}{dt} = \frac{-\frac{11\pi\sqrt{3}}{12}}{x} = \frac{-\frac{11\pi\sqrt{3}}{12}}{2\sqrt{19}} = -\frac{11\pi\sqrt{3}}{24\sqrt{19}}.$$

Thus, the distance between the tips of the hands is decreasing at a rate of $\boxed{\frac{11\pi\sqrt{3}}{24\sqrt{19}} \approx 0.572 \text{ cm/min}}$.

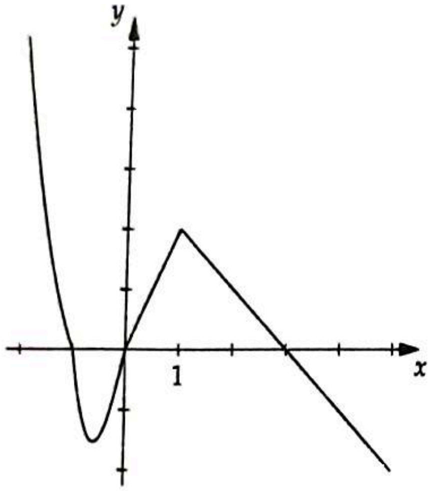
Review Problems

4.34 We see that f is decreasing for $x < 1$ (since $f'(x) < 0$ on this region) and increasing for $x > 1$. We also notice that f is concave up for $x < -1$ and $x \in (0, 3)$, since the derivative of f is increasing on these regions; f is concave down elsewhere. Also, we see that the graph of f is steepest near $x = 3$ (where f' is relatively large), and f has critical points at $x = -1$, $x = 1$, and $x = 5$, where the graph of f should have horizontal tangent lines. Using all of this information, we can sketch a graph of f, shown below.



Note that this graph can be translated vertically up or down as much as we wish. We have arbitrarily chosen to make $f(0) = 0$.

To graph f'', we graph the slopes of the tangent lines to the graph of f'. As noted earlier, f'' is positive on $x < -1$ and $x \in (0,3)$ and negative otherwise. We also note that $f''(x) = 0$ for $x \in \{-1,0,3\}$, as these are the critical points of f'. Below is one possible graph: note that f'' need not be itself differentiable (as evidenced by the sharp corners in the graph of f'').



4.35 Note that f' has no real roots, so $f' > 0$ for all x. Thus f is always strictly increasing, and we know that any strictly increasing function has an inverse. Another way to see this is that if we have $a < b$ with $f(a) = f(b)$, then by Rolle's Theorem some c between a and b must have $f'(c) = 0$, but this cannot happen since $f'(c) > 0$ for all c.

4.36

(a) Suppose $f(x) = ax^2 + bx + c$ with $a \neq 0$. Then $f'(x) = 2ax + b$, and $f''(x) = 2a \neq 0$. Thus, since f'' is continuous and $f'' \neq 0$, we conclude that f has no inflection points.

(b) Suppose $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$. Then $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$. We see that $f''(x) = 0$ if and only if $x = -\frac{b}{3a}$. Since f'' is positive to one side of $x = -\frac{b}{3a}$ and negative to the other side (which side is which depends on the sign of a), we see that $x = -\frac{b}{3a}$ is the (unique) inflection point of f.

(c) If f is of degree n, then $f(x) = a_n x^n + \cdots + a_0$, and $f''(x) = n(n-1)a_n x^{n-2} + \cdots$ is of degree $n-2$. Thus, we can say that f has at most $n-2$ inflection points. It is possible that f may have fewer than $n-2$ inflection points, if f'' has fewer than $n-2$ real roots.

4.37 f is differentiable with derivative $f'(x) = 3x^2 + p$. If $p \geq 0$, then f is increasing everywhere, since $3x^2 \geq 0$, and therefore has no local maxima or minima. Otherwise, f is increasing when $|x| > \sqrt{-\frac{p}{3}}$ and decreasing when $x \in \left(-\sqrt{-\frac{p}{3}}, \sqrt{-\frac{p}{3}}\right)$. Furthermore, the critical points are $\pm\sqrt{-\frac{p}{3}}$, and by the First Derivative Test, we see that $x = -\sqrt{-\frac{p}{3}}$ gives a local maximum, and $x = \sqrt{-\frac{p}{3}}$ gives a local minimum.

Also, note that $f''(x) = 6x$ is positive for $x > 0$ and negative for $x < 0$, so f is concave up on $(0,\infty)$, concave down on $(-\infty, 0)$ and has an inflection point at $x = 0$. This also confirms to us, via the Second Derivative Test, that in the case where $p < 0$, we have that $x = \sqrt{-\frac{p}{3}}$ gives a local minimum and $x = -\sqrt{-\frac{p}{3}}$ gives a local maximum.

4.38 We first note that $f(0) = f(1) = 0$, and $f(x) > 0$ for all $x \in (0,1)$. So f must have a local maximum on $(0,1)$. Continuing, we can calculate

$$f'(x) = mx^{m-1}(1-x)^n - nx^m(1-x)^{n-1} = x^{m-1}(1-x)^{n-1}(m(1-x) - nx) = x^{m-1}(1-x)^{n-1}(m - (m+n)x).$$

We see that $f'(x) = 0$ with $x \in (0,1)$ if and only if $x = \frac{m}{m+n}$. So f increases to a maximum at $x = \frac{m}{m+n}$, then decreases back to 0.

4.39 Suppose she sells x blivets at \$$(2500 - x)$ apiece. Her overall revenue is \$$(2500 - x)x$ = \$$(2500x - x^2)$. Her cost includes the factory and the blivets themselves, totaling \$2500 + \$$900x$ = \$$(2500 + 900x)$. Subtracting this from the revenue yields a profit of \$$(-x^2 + 1600x - 2500)$ per day. This has a maximum either at $x = 0$ (the lower bound of x) or where its derivative is equal to zero, since this is a downward opening parabola. But the derivative of the profit is \$$(-2x + 1600)$ per blivet, which is zero when $x = 800$ blivets. If she makes 800 blivets, she gets a total profit of

$$\$(-800^2 + 1600 \cdot 800 - 2500) = \$(-640000 + 1280000 - 2500) = \$637500,$$

which is definitely more than the −\$2500 she would make at $x = 0$, and thus is the maximum. Therefore, to maximize her profit, Tina should price the blivets at \$(2500 − 800) = $\boxed{\$1700}$ and make exactly 800 of them per day.

4.40 The area of such a sector is given by $\frac{1}{2}\theta r^2 = 300$, so $\theta = 600/r^2$. The perimeter consists of two radii and a portion of the circumference, with total length $L = 2r + \theta r = 2r + \frac{600}{r}$. The restriction that $0 < \theta < 2\pi$ corresponds to the restriction $r > \sqrt{\frac{600}{2\pi}}$. Note that L grows arbitrarily large as $r \to \infty$, so we look for critical points on $r \in \left(\sqrt{\frac{300}{\pi}}, \infty\right)$ where $L'(r) = 0$. We have $L'(r) = 2 - \frac{600}{r^2}$, which is zero where $\boxed{r = \sqrt{300} \text{ meters and } \theta = 2 \text{ radians}}$. This is a minimum since the second derivative $\frac{1200}{r^3}$ is positive.

4.41 We need to estimate how much the formula GM/r^2 changes when r goes from 6400 km to 6410 km. Since we are comparing with the initial value, we can ignore the constants G and M. We can approximate using a tangent line approximation at $r = 6400$ km. Since the derivative of $\frac{1}{r^2}$ is $-\frac{2}{r^3}$, at $r = 6400 + \Delta r$, we get a difference of

$$\frac{\left(-\frac{2}{6400^3}\right)\Delta r}{\frac{1}{6400^2}} = -\frac{\Delta r}{3200} = -\frac{\Delta r}{32}\%.$$

Using $\Delta r = 10$, this gives $\boxed{-\frac{10}{32}\% = -0.3125\%}$.

4.42 Her height above the ground is $h = 30 + 30\sin\theta$, where θ is the angle that her car forms with the center of the wheel (where, as usual, 0 means that she is horizontal to the center). Differentiating gives $\frac{dh}{dt} = 30\cos\theta\frac{d\theta}{dt}$. We know that $\frac{d\theta}{dt} = 8\pi$ rad/min. When she is ascending through 40 meters, we have $h = 40$, so $\sin\theta = \frac{1}{3}$, and thus $\cos\theta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$. This gives

$$\frac{dh}{dt} = 30 \cdot \frac{2\sqrt{2}}{3} \cdot 8\pi = \boxed{160\sqrt{2}\pi \text{ (in meters/minute)}},$$

or perhaps more meaningful would be $\boxed{\frac{8\sqrt{2}}{3}\pi \text{ (in meters/second)}}$.

4.43

(a) We use the given data to get $200 = 70 + 230e^{-10k}$, so $e^{-10k} = 13/23$, and hence $k = -(0.1)\log(13/23) \approx 0.057$. To find when the pie is 150 degrees, we must solve $150 = 70 + 230e^{-kt}$ for t, so $e^{-kt} = 8/23$ and

$$t = -\frac{1}{k}\log(8/23) = \boxed{10\frac{\log(8/23)}{\log(13/23)} \approx 18.51 \text{ minutes}}.$$

(b) To find the rate of cooling, we differentiate to get $\frac{dT}{dt} = -230ke^{-kt}$, and plugging in $k \approx 0.057$ and $t \approx 18.51$, we get

$$\frac{dT}{dt} \approx -230(0.057)e^{-(0.057)(18.51)} \approx -4.56,$$

so the pie is cooling at $\boxed{\text{approximately 4.56 degrees per minute}}$.

4.44

(a) We can differentiate to get $\frac{df}{dt} = k\frac{dg}{dt}$. So, the rates of change of f and g are also directly proportional, with the same constant of proportionality.

(b) We can differentiate to get $g\frac{df}{dt} + f\frac{dg}{dt} = 0$, or $\frac{df/dt}{dg/dt} = -\frac{f}{g}$. This implies that the ratio of the rates of change of f and g is equal to the negative of the ratio of f and g.

Challenge Problems

4.45 Define $h(x) = f(x) - g(x)$. Note that $h(a) = 0$ and $h'(x) > 0$ for all $x > a$. Then h is strictly increasing for $x > a$, and in particular $h(x) > 0$ for all $x > a$. Thus $f(x) > g(x)$ for all $x > a$.

The reason it's called the Racetrack Theorem: think of f and g as representing the positions of two cars on a racetrack. If they start at the same point at time a, and car f is always going faster than car g after time a, then car f will always be ahead of car g after time a.

4.46 We know that $(f^2)' = 2ff'$, so x is a critical point of f^2 if and only if either $(f^2)' = 0$, in which case either $f(x) = 0$ or $f'(x) = 0$, or $(f^2)'$ is undefined, which can only happen if f' is undefined. Therefore, the critical points of f^2 are precisely the critical points of f together with the roots of f.

4.47 a must be a critical point, so since f is differentiable we must have $f'(a) = 0$. This gives us the system of equations

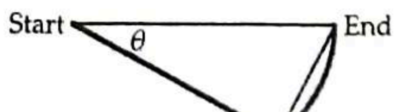
$$a = f(a) = a^4 - a^3 + 1,$$
$$0 = f'(a) = 4a^3 - 3a^2 - a.$$

The latter equation is $a(4a + 1)(a - 1) = 0$, so its roots are 0, 1, and $-\frac{1}{4}$. Inspection shows that only $a = 1$ satisfies the top equation, so we use the Second Derivative Test to check if this is a minimum. We have $f''(a) = 12a^2 - 6a - 2$, so $f''(1) = 4 > 0$, and indeed $\boxed{a = 1}$ is a local minimum.

4.48

(a) We may assume that Sam rows along a chord to some point on the lakeshore, and then walks directly to the end point. (He may in fact walk first and then row, but by the symmetry of the lake it doesn't matter. There is no advantage to mixing up shorter segments of rowing and walking: again by the symmetry, we can rearrange the segments so that all the rowing segments come first, and then it is clear that rowing in a straight line to the rowing endpoint is faster than rowing in multiple segments.)

Suppose that Sam rows at an angle θ below the diameter from his start point to his destination, as shown in the picture at right. The distance that he rows is $\cos\theta$ (it is the side adjacent to the angle with measure θ in a right triangle with hypotenuse 1), and the length that he walks is θ (an arc of measure 2θ of a circle of radius $\frac{1}{2}$). So the total amount of time that Sam takes is



$$f(\theta) = \frac{1}{4}\cos\theta + \frac{1}{6}\theta.$$

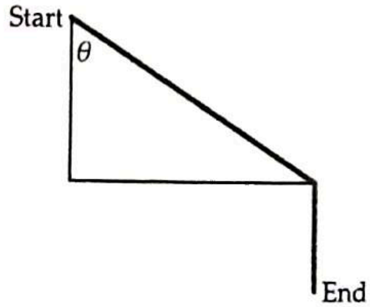
Note that the domain is $\theta \in \left[0, \frac{\pi}{2}\right]$, where $\theta = 0$ corresponds to rowing the entire distance, and $\theta = \frac{\pi}{2}$ corresponds to walking the entire distance. We then compute

$$f'(\theta) = -\frac{1}{4}\sin\theta + \frac{1}{6},$$

so a critical point is where $\sin\theta = \frac{2}{3}$, or $\theta = \sin^{-1}\frac{2}{3}$. However, using the Second Derivative Test, we see that $f''(\theta) = -\frac{1}{4}\cos\theta < 0$, so any critical point $\theta \in \left(0, \frac{\pi}{2}\right)$ must be a local maximum. We want to minimize the time, so we conclude that the minimum on $\left[0, \frac{\pi}{2}\right]$ must be at one of the endpoints. Rowing the entire distance (where $\theta = 0$) takes $f(0) = \frac{1}{4}$ hours, whereas walking the entire distance takes $f\left(\frac{\pi}{2}\right) = \frac{\pi}{12} > \frac{3}{12} = \frac{1}{4}$ hours, so Sam is best off rowing the whole way, which takes $\boxed{\frac{1}{4} \text{ hours}}$.

(b) We can assume that Sam rows in a straight line from his starting point to a point on the opposite side of the river, and then walks the rest of the way. (He might, in fact, walk on the first side, then row across, and then walk on the other side, but this doesn't alter his total time than if he had just done the rowing first.)

Suppose that Sam rows at an angle θ to the side that he starts on, as shown in the diagram. The distance that he rows is $\csc\theta$ (it is hypotenuse of a right triangle where the side opposite θ has length 1), and the length that he walks is $1 - \cot\theta$. So the total amount of time that Sam takes is



$$f(\theta) = \frac{1}{4}\csc\theta + \frac{1}{6}(1 - \cot\theta).$$

Note that the domain is $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, where $\theta = \frac{\pi}{4}$ corresponds to rowing the entire distance, and $\theta = \frac{\pi}{2}$ corresponds to rowing directly across, then walking the entire shoreline. We then compute

$$f'(\theta) = -\frac{1}{4}\csc\theta\cot\theta + \frac{1}{6}\csc^2\theta = \csc\theta\left(-\frac{1}{4}\cot\theta + \frac{1}{6}\csc\theta\right),$$

so, since $\csc\theta \neq 0$, a critical point is where $\cos\theta = \frac{\cot\theta}{\csc\theta} = \frac{2}{3}$, or $\theta = \cos^{-1}\frac{2}{3}$. Using the Second Derivative Test, we see that for $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$,

$$\begin{aligned}
f''(\theta) &= \csc\theta\left(\frac{1}{4}\csc^2\theta - \frac{1}{6}\cot\theta\csc\theta\right) - (\cot\theta\csc\theta)\left(-\frac{1}{4}\cot\theta + \frac{1}{6}\csc\theta\right) \\
&= \csc\theta\left(\frac{1}{4}\csc^2\theta - \frac{1}{3}\csc\theta\cot\theta + \frac{1}{4}\cot^2\theta\right) \\
&= \frac{1}{4}\csc\theta\left((\csc\theta - \cot\theta)^2 + \frac{2}{3}\csc\theta\cot\theta\right) \\
&> 0,
\end{aligned}$$

so any critical point $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ must be the minimum. Thus, the total time is minimized by choosing $\theta = \cos^{-1}\frac{2}{3}$, and the time necessary is

$$f\left(\cos^{-1}\frac{2}{3}\right) = \frac{1}{4}\csc\left(\cos^{-1}\frac{2}{3}\right) + \frac{1}{6}\left(1 - \cot\left(\cos^{-1}\frac{2}{3}\right)\right) = \frac{1}{4}\left(\frac{3}{\sqrt{5}}\right) + \frac{1}{6}\left(1 - \frac{2}{\sqrt{5}}\right) = \boxed{\frac{1}{6} + \frac{\sqrt{5}}{12} \approx 0.353 \text{ hours}}.$$

4.49 If we invest \$$m$ at r% interest, then the amount that we have after t years is $(\$m)\left(1 + \frac{r}{100}\right)^t$. So we are looking for the value of t such that

$$\left(1 + \frac{r}{100}\right)^t = 2.$$

Thus $t = \log_{(1+\frac{r}{100})} 2 = \dfrac{\log 2}{\log(1 + \frac{r}{100})}$.

We can approximate $\log(1 + \epsilon)$ by tangent line approximation: let $f(x) = \log x$, so $f'(x) = \frac{1}{x}$. Then

$$\log(1 + \epsilon) \approx \log(1) + \epsilon = \epsilon.$$

Thus we have

$$t = \frac{\log 2}{\log(1 + \frac{r}{100})} \approx \frac{100\log 2}{r}.$$

Finally, we can approximate $100 \log 2 \approx 69.3$, which is close enough to 72 to not matter too much if r is small. (Accountants and other investment advisors prefer 72 since it's divisible by lots of small numbers: 1, 2, 3, 4, 6, 8, ... Look up "Rule of 72" on the internet to see some examples.)

4.50 Let A, B, X, and θ_i be as in the diagram. Let point A be distance a from the x-axis, point B be distance b from the x-axis, and the horizontal distance between the two be d. Let the distance from the point on the interface nearest A and X be x, where $0 \le x \le d$. By Pythagorean Theorem, $AX = \sqrt{a^2 + x^2}$ and $XB = \sqrt{b^2 + (d-x)^2}$. Time is distance divided by rate, so the total time of travel is

$$f(x) = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c_2}.$$

Setting $f'(x) = 0$, we have

$$0 = f'(x) = \frac{x}{c_1 \sqrt{a^2 + x^2}} - \frac{d - x}{c_2 \sqrt{b^2 + (d-x)^2}}.$$

Now we note that the first fraction in this equation is equal to $\frac{\sin\theta_1}{c_1}$ and the second is $\frac{\sin\theta_2}{c_2}$. Therefore, we must have

$$\boxed{c_1 \sin\theta_2 = c_2 \sin\theta_1}.$$

We also note that $f'(x) < 0$ for small x (specifically, to the left of the critical point) and $f'(x) > 0$ for large x (specifically, to the right of the critical point), so by the First Derivative Test, the above critical point does indeed give a global minimum.

4.51 We start with

$$E(z) = f(z) - f(a) - f'(a)(z - a).$$

Divide by $z - a$ to get

$$\frac{|E(z)|}{z - a} = \left|\frac{f(z) - f(a)}{z - a} - f'(a)\right|.$$

By the Mean Value Theorem, there is some $a < c < z$ such that $f'(c) = \dfrac{f(z) - f(a)}{z - a}$, hence

$$\frac{|E(z)|}{z - a} = |f'(c) - f'(a)|.$$

Note that $(c - a) < (z - a)$, so dividing again gives

$$\frac{|E(z)|}{(z - a)^2} = \frac{|f'(c) - f'(a)|}{z - a} < \frac{|f'(c) - f'(a)|}{c - a},$$

and applying the Mean Value Theorem (to f') means that there is some $a < \xi < c$ such that $f''(\xi) = \dfrac{f'(c) - f'(a)}{c - a}$. Hence

$$\frac{|E(z)|}{(z - a)^2} < |f''(\xi)|,$$

and thus $|E(z)| < |f''(\xi)|(z - a)^2 \le M(z - a)^2$.

4.52 Suppose noon is h hours after it started snowing. We're told that the rate of change of the position of the plow (which we can denote as s) is inversely proportional to the amount of snow, which itself it proportional to the time since it started snowing. So we have $\frac{ds}{dt} = \frac{k}{t}$ for some constant k. This means that $s = k \log t + C$ for some constants k and C.

The given information about the position of the plow tells us that

$$k \log(h + 1) + C = 2 + k \log h + C$$

and

$$k\log(h+2)+C=3+k\log h+C.$$

So $k(\log(h+1)-\log(h))=2$ and $k(\log(h+2)-\log(h))=3$, meaning that

$$3\log\left(\frac{h+1}{h}\right)=2\log\left(\frac{h+2}{h}\right).$$

Taking the exponential of both sides gives

$$\left(\frac{h+1}{h}\right)^3=\left(\frac{h+2}{h}\right)^2,$$

so $(h+1)^3h^2=(h+2)^2h^3$, and thus $(h+1)^3=(h+2)^2h$, giving $h^2+h-1=0$. Thus

$$h=\frac{-1\pm\sqrt{5}}{2},$$

and only the positive solution makes sense. So $h=(\sqrt{5}-1)/2\approx 0.618$, so it started snowing 0.618 hours before noon, or at $\boxed{\text{about 11:23 a.m.}}$.

CHAPTER 5

Integration

Exercises for Section 5.1

5.1.1

(a) The region is a rectangle (actually a square) of width 4 and height 4, so its area is 16. Thus $\int_{-1}^{3} 4\,dx = \boxed{16}$.

(b) The region is a trapezoid of width $6-2=4$. The left end has height $3(2)-1=5$, and the right end has height $3(6)-1=17$. Thus, the area is $\frac{1}{2}(4)(17+5)=44$, and hence $\int_2^6 (3x-1)\,dx = \boxed{44}$.

(c) This is the same region as Problem 5.6 of the text, which had area $\frac{7}{3}$, except that it is 1 unit higher. This extra unit adds $(1)(1)=1$ unit of area. Thus, the area is $\frac{10}{3}$, and $\int_1^2 (x^2+1)\,dx = \boxed{\frac{10}{3}}$.

(d) Notice that the region from -2 is 0 is exactly the same as the region from 0 to 2, except it is under the x-axis instead of over the x-axis. Thus, the (negative) area over $[-2,0]$ will exactly cancel with the (positive) area over $[0,2]$, and hence $\int_{-2}^{2} x^3\,dx = \boxed{0}$.

(e) This is essentially the same as part (d): the negative area over $\left[-3,\frac{1}{2}\right]$ will cancel with the positive area over $\left[\frac{1}{2},4\right]$, and thus $\int_{-3}^{4}(2x-1)\,dx = \boxed{0}$.

(f) This region is the sum of three rectangles: a rectangle of height 0 over $[0,1]$, a rectangle of height 1 over $[1,2]$, and a rectangle of height 2 over $[2,3]$. Thus, the total area is $0+1+2=3$, and hence $\int_0^3 \lfloor x\rfloor\,dx = \boxed{3}$.

5.1.2 The region in question has base $[a,a]$, which has length 0. Thus, no matter what f is, the area is 0. Hence $\int_a^a f = \boxed{0}$ for any f.

5.1.3 Let $\mathcal{P}$ be given by $a = x_0 < x_1 < \cdots < x_n = b$. As in the text, for all $0 \le i < n$ let

$$h_i = \inf\{f(x) \mid x \in [x_i, x_{i+1}]\} \quad \text{and} \quad H_i = \sup\{f(x) \mid x \in [x_i, x_{i+1}]\}.$$

We see that $h_i \le H_i$ for all i, and thus

$$l(f,\mathcal{P}) = \sum_{i=0}^{n-1} h_i(x_{i+1}-x_i) \le \sum_{i=0}^{n-1} H_i(x_{i+1}-x_i) = u(f,\mathcal{P}).$$

Exercises for Section 5.2

5.2.1

(a) An antiderivative of x^4 is $\frac{1}{5}x^5$. Thus, $\displaystyle\int_{-1}^{2} x^4\,dx = \frac{1}{5}x^5\Big|_{-1}^{2} = \frac{1}{5}(2^5-(-1)^5) = \boxed{\frac{33}{5}}$.

(b) e^x is its own antiderivative, so $\displaystyle\int_0^1 e^x\,dx = e^x\Big|_0^1 = e^1 - e^0 = \boxed{e-1}$.

(c) An antiderivative of $\sec^2\theta$ is $\tan\theta$, so $\int_{\pi/6}^{\pi/4} \sec^2\theta\, d\theta = \tan\theta\Big|_{\pi/6}^{\pi/4} = \tan\frac{\pi}{4} - \tan\frac{\pi}{6} = \boxed{1 - \frac{\sqrt{3}}{3}}$.

(d) An antiderivative of $2x$ is x^2, hence $\int_{-4}^{-2} 2x\,dx = x^2\Big|_{-4}^{-2} = (-2)^2 - (-4)^2 = \boxed{-12}$. (One could instead note that this integral equals the negative area of a trapezoid with bases 8 and 4 and height 2, so it is $-(2)(6) = -12$.)

5.2.2 We use the Fundamental Theorem of Calculus:

$$\int_a^b f = F(b) - F(a) = (F(c) - F(a)) + (F(b) - F(c)) = \int_a^c f + \int_c^b f.$$

5.2.3 Let F be an antiderivative of f such that $F(0) = 0$. (For instance, we could define $F(x) = \int_0^x f(t)\,dt$ using the Fundamental Theorem of Calculus.) Then

$$\frac{d}{dx}F(-x) = \frac{dF}{dx}(-x)\cdot\frac{d}{dx}(-x) = f(-x)\cdot(-1) = -f(-x) = f(x).$$

That is, the function $G(x) = F(-x)$ is also an antiderivative of f. But $F(0) = G(0) = 0$, so since any two antiderivatives must differ by a constant, we have $F(x) = G(x) = F(-x)$; that is, F is an even function. To finish, we compute

$$\int_{-a}^{a} f(x)\,dx = F(a) - F(-a) = F(a) - F(a) = 0.$$

5.2.4 Let $G(x) = \int_0^x g(t)\,dt$, so that by the Fundamental Theorem of Calculus, $G'(x) = g(x)$. Then by definition,

$$\int_0^{f(x)} g(t)\,dt = G(f(x)),$$

and taking the derivative of both sides and applying the Chain Rule gives

$$\frac{d}{dx}\int_0^{f(x)} g(t)\,dt = \frac{d}{dx}G(f(x)) = G'(f(x))f'(x) = g(f(x))f'(x),$$

as desired.

Exercises for Section 5.3

5.3.1

(a) This is just a polynomial, which we know how to integrate:

$$\int (x^3 - x + 2)\,dx = \boxed{\frac{x^4}{4} - \frac{x^2}{2} + 2x + C}.$$

(b) If we write $\frac{2}{x^4}$ as $2x^{-4}$, we can integrate this using our usual rules:

$$\int \left(3x^5 - x - \frac{2}{x^4}\right) dx = \boxed{\frac{x^6}{2} - \frac{x^2}{2} + \frac{2x^{-3}}{3} + C}.$$

(c) We know that $\int \sin x\, dx = -\cos x$ and $\int \cos x\, dx = \sin x$. Combining these,

$$\int (2\sin x - 3\cos x)\, dx = \boxed{-2\cos x - 3\sin x + C}.$$

(d) First, we distribute the x, so $\frac{x^3+x^2+x+1}{x} = x^2 + x + 1 + x^{-1}$. Then we integrate:

$$\int (x^2 + x + 1 + x^{-1})\, dx = \boxed{\frac{x^3}{3} + \frac{x^2}{2} + x + \log|x| + C}.$$

(e) This requires a substitution of $u = -2x$. Then $du = -2\, dx$, so

$$\int e^{-2x}\, dx = -\frac{1}{2}\int e^u\, du = -\frac{1}{2}e^u + C = \boxed{-\frac{1}{2}e^{-2x} + C}.$$

(f) This requires the substitution $u = x^2$. Then $du = 2x\, dx$, so

$$\int x\sin(x^2)\, dx = \frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u + C = \boxed{-\frac{1}{2}\cos x^2 + C}.$$

(g) Noticing that the derivative of sine is cosine, we make the substitution $u = \sin\theta$. Then $du = \cos\theta\, d\theta$ so

$$\int \sin^4(\theta)\cos(\theta)\, d\theta = \int u^4\, du = \frac{1}{5}u^5 + C = \boxed{\frac{1}{5}\sin^5\theta + C}.$$

(h) We make the substitution $u = 2x - 3$. Then $du = 2\, dx$ so

$$\int \frac{1}{2x-3}\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\log|u| + C = \boxed{\frac{1}{2}\log|2x-3| + C}.$$

(i) Since $\frac{1}{\cos^2 x} = \sec^2 x = \frac{d}{dx}\tan x$, we make the substitution $u = \tan x$. Then

$$\int \frac{1}{\cos^2 x\sqrt{\tan x}}\, dx = \int u^{-\frac{1}{2}}\, du = 2\sqrt{u} + C = \boxed{2\sqrt{\tan x} + C}.$$

(j) We integrate by parts. Let $u = x$ and $v = \frac{1}{5}e^{5x}$, so $du = dx$ and $dv = e^{5x}\, dx$. Then

$$\int xe^{5x}\, dx = \int u\, dv = (uv) - \int v\, du = \frac{x}{5}e^{5x} - \int \frac{1}{5}e^{5x}\, dx = \frac{x}{5}e^{5x} - \frac{1}{25}e^{5x} = \boxed{\left(\frac{1}{5}x - \frac{1}{25}\right)e^{5x} + C}.$$

(k) We integrate by parts, with $u = 2x^2$ and $v = \frac{1}{3}e^{3x}$, so $du = 4x\, dx$ and $dv = e^{3x}\, dx$. Then

$$\int 2x^2e^{3x}\, dx = \int u\, dv = (uv) - \int v\, du = \frac{2}{3}x^2e^{3x} - \int \frac{4}{3}xe^{3x}\, dx.$$

To calculate this last integral, we use integration by parts again, using $u = x$ and $v = \frac{4}{9}e^{3x}$, so that $du = dx$ and $dv = \frac{4}{3}e^{3x}\, dx$. We have

$$\int \frac{4}{3}xe^{3x}\, dx = \int u\, dv = uv - \int v\, du = \frac{4}{9}xe^{3x} - \int \frac{4}{9}e^{3x}\, dx = \frac{4}{9}xe^{3x} - \frac{4}{27}e^{3x} = \left(\frac{4}{9}x - \frac{4}{27}\right)e^{3x}.$$

Substituting this for the integral above yields

$$\int 2x^2e^{3x}\, dx = \frac{2}{3}x^2e^{3x} - \left(\frac{4}{9}x - \frac{4}{27}\right)e^{3x} = \boxed{\left(\frac{2}{3}x^2 - \frac{4}{9}x + \frac{4}{27}\right)e^{3x} + C}$$

(l) We're going to integrate by parts, but we need to set it up correctly: Let $u = \log x$ and $v = \frac{1}{2}x^2$. Then $du = \frac{1}{x}\,dx$ and $dv = x\,dx$, so we get

$$\int x \log x\,dx = \int u\,dv = uv - \int v\,du = \frac{1}{2}x^2 \log x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 = \boxed{\frac{1}{4}x^2(2\log x - 1) + C}.$$

(m) We make the substitution $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$, and

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4\tan^2\theta + 4} 2\sec^2\theta\,d\theta = \int \frac{2\sec^2\theta}{4\sec^2\theta}\,d\theta = \int \frac{1}{2}\,d\theta = \frac{\theta}{2}.$$

Since we know that $\theta = \tan^{-1}\left(\frac{x}{2}\right)$, we get

$$\int \frac{1}{x^2+4} = \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}.$$

(n) Since $x^2 + x + 1$ has no real roots, we cannot use partial fractions. Instead, we complete the square:

$$\frac{1}{x^2+x+1}\,dx = \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx.$$

We make things clearer by substituting $y = x + \frac{1}{2}$. Then $dy = dx$ and we have

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{y^2+\frac{3}{4}}\,dy.$$

Now we substitute $y = \frac{\sqrt{3}}{2}\tan\theta$. This gives $dy = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$, so

$$\int \frac{1}{y^2+\frac{3}{4}}\,dy = \int \frac{1}{\frac{3}{4}\tan^2\theta + \frac{3}{4}} \cdot \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta = \frac{4}{3}\int \frac{1}{\sec^2\theta} \cdot \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta = \frac{2}{\sqrt{3}}\int d\theta = \frac{2\theta}{\sqrt{3}}.$$

Reversing the substitutions, we have

$$\theta = \tan^{-1}\left(\frac{2y}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}\right),$$

so our final answer is $\boxed{\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C}$.

(o) $x^2 - 4x + 3$ factors as $(x-3)(x-1)$, so we write the partial fraction decomposition:

$$\frac{1}{x^2-4x+3} = \frac{1}{2}\left(\frac{1}{x-3} - \frac{1}{x-1}\right).$$

Thus,

$$\int \frac{1}{x^2-4x+3}\,dx = \int \frac{1}{2}\left(\frac{1}{x-3} - \frac{1}{x-1}\right) = \frac{1}{2}(\log|x-3| - \log|x-1|) + C = \boxed{\log\sqrt{\left|\frac{x-3}{x-1}\right|} + C}.$$

(p) $x^2 - 4x - 5$ factors as $(x-5)(x+1)$. We use the partial fraction decomposition

$$\frac{x+3}{x^2-4x-5} = \frac{1}{3}\left(\frac{4}{x-5} - \frac{1}{x+1}\right).$$

Integrating, we get

$$\int \frac{x+3}{x^2-4x-5}\,dx = \int \frac{1}{3}\left(\frac{4}{x-5} - \frac{1}{x+1}\right)dx = \boxed{\frac{1}{3}(4\log|x-5| - \log|x+1|) + C}.$$

(q) The denominator factors as $x^3-x = x(x^2-1) = x(x+1)(x-1)$. Thus, we have the partial fraction decomposition

$$\frac{x^2-2}{x^3-x} = \frac{2}{x} - \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{x-1}.$$

Hence

$$\int \frac{x^2-2}{x^3-x}\,dx = \int\left(\frac{2}{x} - \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{x-1}\right)dx = 2\log|x| - \frac{1}{2}\log|x+1| - \frac{1}{2}\log|x-1| + C = \boxed{\log\left(\frac{x^2}{\sqrt{|x^2-1|}}\right) + C}.$$

(r) Notice that $\frac{1}{1+e^x} = 1 - \frac{e^x}{1+e^x}$. Taking the integral of both sides yields

$$\int \frac{1}{1+e^x}\,dx = \int 1\,dx - \int \frac{e^x}{1+e^x}\,dx = \boxed{x - \log(1+e^x) + C}.$$

(s) Let $u = \sqrt{x}$, so that $x = u^2$. Then $dx = 2u\,du$, so

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{2u}{1+u}\,du = \int\left(2 - \frac{2}{1+u}\right)du = 2u - 2\log(1+u) + C = \boxed{2\sqrt{x} - 2\log(1+\sqrt{x}) + C}.$$

(Since $\sqrt{x}$ is always positive, we don't need an absolute value on the log.)

(t) Let $u = \sqrt[3]{x}$, so that $x = u^3$ and $dx = 3u^2\,du$. Thus

$$\int \frac{1}{1+\sqrt[3]{x}}\,dx = \int \frac{3u^2}{1+u}\,du = \int\left(3u - 3 + \frac{3}{1+u}\right)du = \frac{3}{2}u^2 - 3u + 3\log|1+u| + C = \boxed{\frac{3}{2}x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 3\log|1+x^{\frac{1}{3}}| + C}.$$

(u) Let $u = \sqrt{2+\sqrt{x}}$. Then $x = (u^2-2)^2$, so $dx = 4u(u^2-2)\,du$ and

$$\int \sqrt{2+\sqrt{x}}\,dx = \int 4u^2(u^2-2)\,du = \frac{4}{5}u^5 - \frac{8}{3}u^3 + C = \boxed{\frac{4}{5}(2+\sqrt{x})^{\frac{5}{2}} - \frac{8}{3}(2+\sqrt{x})^{\frac{3}{2}} + C}.$$

5.3.2 Let $u = f(x)$ so that $du = f'(x)\,dx$. Then the integral becomes

$$\int_a^b f'(x)f(x)\,dx = \int_{f(a)}^{f(b)} u\,du = \left.\frac{u^2}{2}\right|_{f(a)}^{f(b)} = \boxed{\frac{(f(b))^2 - (f(a))^2}{2}}.$$

5.3.3

(a) Use integration by parts with $u = \sin^{n-1}x$, $du = (n-1)\sin^{n-2}x\cos x\,dx$ and $dv = \sin x\,dx$, $v = -\cos x$. Then the integral becomes

$$\int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx.$$

But using $\cos^2 x = 1 - \sin^2 x$, this becomes

$$\int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int(\sin^{n-2}x - \sin^n x)\,dx.$$

Adding $(n-1)\int \sin^n\,dx$ to both sides gives

$$n\int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx,$$

and dividing by n gives

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

Applying this repeatedly can produce a formula for any positive integer n. For example:

$$\begin{aligned}\int \sin^6 x\,dx &= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\int \sin^4 x\,dx\\ &= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx\right)\\ &= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{5}{8}\left(-\frac{1}{2}\sin x\cos x + \frac{1}{2}\int dx\right)\\ &= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x + C.\end{aligned}$$

Essentially the same computation gives the recursive formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx,$$

which can be applied repeatedly to compute $\int \cos^n x\,dx$ for any positive integer n.

(b) If either m or n is odd, we can use the identity $\sin^2 x + \cos^2 x = 1$ and make a direct substitution. For example:

$$\begin{aligned}\int \sin^5 x\cos^8 x\,dx &= \int \sin^4 x\cos^8 x\sin x\,dx\\ &= \int (1-\cos^2 x)^2\cos^8 x\sin x\,dx\\ &= -\int (1-u^2)^2 u^8\,du \qquad (\text{where } u=\cos x, du = -\sin x\,dx)\\ &= -\int (u^8 - 2u^{10} + u^{12})\,du\\ &= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13}\\ &= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C.\end{aligned}$$

If both m and n are even, then we use the substitutions

$$\sin^2 x = \frac{1-\cos(2x)}{2}, \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

to reduce the powers by a factor of 2, then either repeat or use the "m or n odd" technique described above.

5.3.4 We use the partial fraction decomposition

$$\frac{1}{x(x-1)^2} = \frac{1}{x} + \frac{-x+2}{(x-1)^2}.$$

We further decompose $\frac{-x+2}{(x-1)^2} = \frac{-1}{x-1} + \frac{1}{(x-1)^2}$, so that

$$\begin{aligned}\int_2^3 \frac{1}{x(x-1)^2}\,dx &= \int_2^3 \left(\frac{1}{x} + \frac{-1}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \left(\log|x| - \log|x-1| - \frac{1}{x-1}\right)\Big|_2^3 \\ &= \left(\log 3 - \log 2 - \frac{1}{2}\right) - (\log 2 - \log 1 - 1) \\ &= \log 3 - 2\log 2 + \frac{1}{2} = \boxed{\log\frac{3}{4} + \frac{1}{2}}.\end{aligned}$$

5.3.5 Let $u = \sin^{-1} x$, so $x = \sin u$ and $dx = \cos u\, du$. Then

$$\int \sin^{-1} x\, dx = \int u \cos u\, du.$$

Now we apply integration by parts:

$$\int u \cos u\, du = u \sin u - \int \sin u\, du = u \sin u + \cos u + C.$$

Converting back into x gives our answer:

$$\int \sin^{-1} x\, dx = \boxed{x \sin^{-1}(x) + \sqrt{1-x^2} + C}.$$

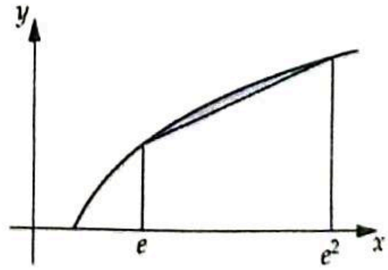
Exercises for Section 5.4

5.4.1

(a) These two curves intersect at $(0,0)$ and $(1,1)$. On $[0,1]$, we have $x^2 > x^3$, so we integrate $x^2 - x^3$ from 0 to 1:

$$\int_0^1 (x^2 - x^3)\, dx = \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}.$$

(b) The area in question is the area under $y = \log x$ on the interval $[e, e^2]$, minus the area of the trapezoid bordered by $x = e$, $x = e^2$, the line segment from $(e, 1)$ to $(e^2, 2)$, and the x-axis. The area under the curve is



$$\int_e^{e^2} \log x\, dx = (x \log x - x)\Big|_e^{e^2} = (e^2 \log e^2 - e^2) - (e \log e - e) = e^2.$$

The area of the trapezoid is the average of the lengths of the parallel sides ($\frac{1}{2}(1+2)$) times the distance between the parallel sides ($e^2 - e$), thus its area is $\frac{3}{2}(e^2 - e)$. Therefore, the area of the desired region is

$$e^2 - \frac{3}{2}(e^2 - e) = \boxed{\frac{3}{2}e - \frac{1}{2}e^2}.$$

(c) Note that $(a, a^2) = (a, \cos a)$ is the first intersection point of $y = x^2$ and $y = \cos x$, and we have $\cos x > x^2$ for $x < a$. Therefore, our desired integral is

$$\int_0^a (\cos x - x^2)\,dx = \left(\sin x - \frac{1}{3}x^3\right)\Bigg|_0^a = \sin a - \frac{1}{3}a^3.$$

But $\sin a = \sqrt{1-\cos^2 a} = \sqrt{1-a^4}$, so the answer is $\boxed{\sqrt{1-a^4} - \frac{1}{3}a^3}$.

5.4.2

(a) A cross-section is a circle with radius x^3, so has area $\pi(x^3)^2 = \pi x^6$. Thus the volume is

$$\pi \int_0^2 x^6\,dx = \frac{\pi}{7}x^7\Bigg|_0^2 = \boxed{\frac{128\pi}{7}}.$$

(b) The solid consists of cylindrical shells of surface area $2\pi x f(x) = 2\pi x^4$. Thus the volume is

$$2\pi \int_0^2 x^4\,dx = \frac{2}{5}\pi x^5\Bigg|_0^2 = \boxed{\frac{64\pi}{5}}.$$

(c) A cross-section is a circle with radius $\cos x$, so has area $\pi \cos^2 x$. Thus the volume is

$$\pi \int_0^{\pi/2} \cos^2 x\,dx.$$

We know that $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$, so

$$\pi \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{2}\int_0^{\pi/2}(1+\cos(2x))\,dx = \frac{\pi}{2}\left(x + \frac{1}{2}\sin(2x)\right)\Bigg|_0^{\pi/2} = \frac{\pi}{2}\left(\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - \left(\frac{1}{2}\sin 0\right)\right) = \boxed{\frac{\pi^2}{4}}.$$

(d) The part of the solid above the x-axis is equal to the area under $f(x) = \sqrt{x^2-1}$ rotated about the y-axis. The solid consists of cylindrical shells of surface area $2\pi x f(x) = 2\pi x\sqrt{x^2-1}$. Thus, when we multiply by 2 to get the total volume both above and below the x-axis, we get a total volume of

$$4\pi \int_2^3 x\sqrt{x^2-1}\,dx.$$

Substituting $u = x^2 - 1$ and $du = 2x\,dx$ yields

$$4\pi \int_2^3 x\sqrt{x^2-1}\,dx = 2\pi \int_3^8 \sqrt{u}\,du = \frac{4}{3}\pi\, u^{\frac{3}{2}}\Bigg|_3^8 = \boxed{\pi\left(\frac{64}{3}\sqrt{2} - 4\sqrt{3}\right)}.$$

(e) A cross-section perpendicular to the x-axis is a circle with radius $y = b\sqrt{1-\frac{x^2}{a^2}}$. Integrating from $-a$ to a, we get a volume of

$$\int_{-a}^a \pi b^2\left(1 - \frac{x^2}{a^2}\right)dx = \frac{\pi b^2}{a^2}\int_{-a}^a (a^2 - x^2)\,dx = \frac{2\pi b^2}{a^2}\int_0^a (a^2-x^2)\,dx,$$

where the last step follows because $a^2 - x^2$ is symmetric with respect to the y-axis. This integral gives the volume as

$$\frac{2\pi b^2}{a^2}\left(a^2 x - \frac{1}{3}x^3\right)\Bigg|_0^a = \boxed{\frac{4\pi}{3}b^2 a}.$$

5.4.3

(a) First we find $f'(x) = \frac{x^4}{4} - \frac{1}{x^4}$. Then $1 + (f'(x))^2 = 1 + \frac{x^8}{16} - \frac{1}{2} + \frac{1}{x^8} = \frac{x^8}{16} + \frac{1}{2} + \frac{1}{x^8} = \left(\frac{x^4}{4} + \frac{1}{x^4}\right)^2$. Thus the length is given by the definite integral

$$\int_2^4 \left(\frac{x^4}{4} + \frac{1}{x^4}\right) dx = \left(\frac{x^5}{20} - \frac{1}{3x^3}\right)\Bigg|_2^4 = \left(\frac{1024}{20} - \frac{1}{192}\right) - \left(\frac{32}{20} - \frac{1}{24}\right) = \frac{248}{5} + \frac{7}{192} = \boxed{\frac{47651}{960}}.$$

(b) We compute $1 + (f'(x))^2 = 1 + \left(\frac{9}{2}\sqrt{x}\right)^2 = 1 + \frac{81x}{4}$. Thus the length is

$$\int_0^4 \sqrt{1 + \frac{81x}{4}}\, dx = \frac{4}{81} \cdot \frac{2}{3}\left(1 + \frac{81x}{4}\right)^{\frac{3}{2}}\Bigg|_0^4 = \boxed{\frac{8}{243}(82^{3/2} - 1)}.$$

5.4.4 We can construct the torus by rotating, about the y-axis, the interior of a circle of radius b centered at $(a, 0)$. This circle is given by the equation

$$(x - a)^2 + y^2 = b^2.$$

Hence,

$$y = \sqrt{b^2 - (x - a)^2}$$

gives the upper half of a cross-section of the torus. A cylindrical shell thus has surface area $4\pi x\sqrt{b^2 - (x - a)^2}$ (note the factor of 4 rather than 2 because we need the region both above and below the x-axis.) Thus, the volume is

$$4\pi \int_{a-b}^{a+b} x\sqrt{b^2 - (x - a)^2}\, dx.$$

To compute this integral, substitute $u = x - a$, giving

$$4\pi \int_{-b}^{b} (u + a)\sqrt{b^2 - u^2}\, du.$$

Note that $u\sqrt{b^2 - u^2}$ is an odd function, so $\int_{-b}^{b} u\sqrt{b^2 - u^2}\, du = 0$. This just leaves

$$4a\pi \int_{-b}^{b} \sqrt{b^2 - u^2}\, du.$$

The integral is just the area of a semicircle of radius b, so it is $\frac{1}{2}\pi b^2$. Hence the volume of the torus is $\boxed{2\pi^2 ab^2}$.

5.4.5

(a) This is essentially just the Intermediate Value Theorem. We know that $\frac{1}{b-a}\int_a^b f(x)\, dx$ is the average value of f, so this must lie between the maximum and minimum values of f on $[a, b]$. So by the Intermediate Value Theorem, there is some c such that $f(c)$ equals the average value of f.

(b) Let M be the maximum value of f on $[a, b]$ and let m be the minimum value of f on $[a, b]$ (since f is continuous and $[a, b]$ is closed, these values must exist). Then since g is never negative, we have the inequality

$$m \int_a^b g(x)\, dx \le \int_a^b f(x)g(x)\, dx \le M \int_a^b g(x)\, dx,$$

and hence $\int_a^b f(x)g(x) = \xi \int_a^b g(x)\, dx$ for some $m \le \xi \le M$. But then, by the Intermediate Value Theorem, we must have $c \in [a, b]$ such that $f(c) = \xi$, and the result follows.

(c) A simple example is $f(x) = g(x) = x$ on the interval $[-1, 1]$. Then

$$\int_{-1}^{1} f(x)g(x)\,dx = \int_{-1}^{1} x^2\,dx = \frac{2}{3},$$

but since $\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} x\,dx = 0$, there cannot be any value c such that $c\int_{-1}^{1} g(x)\,dx = \frac{2}{3}$.

Exercises for Section 5.5

5.5.1 Let $f(x) = \frac{\sin x}{x}$. We list the relevant values of f below:

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
$f(x)$	0.8415	0.8020	0.7592	0.7134	0.6650	0.6145	0.5623	0.5088	0.4546

We now compute the various estimates:

Method	Computation	Result
Left-side rectangles	$\frac{1}{4}(f(1) + f(1.25) + f(1.5) + f(1.75))$	0.7070
Right-side rectangles	$\frac{1}{4}(f(1.25) + f(1.5) + f(1.75) + f(2))$	0.6103
Trapezoid Rule	$\frac{1}{8}(f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2))$	0.6586
Midpoint rectangles	$\frac{1}{4}(f(1.125) + f(1.375) + f(1.625) + f(1.875))$	0.6597
Simpson's Rule	$\frac{1}{24}(f(1) + 4f(1.125) + 2f(1.25) + 4f(1.375) + 2f(1.5)$ $+4f(1.625) + 2f(1.75) + 4f(1.875) + f(2))$	0.6593

A computer algebra system computes $\int_1^2 \frac{\sin x}{x}\,dx = 0.65933\ldots$, so the Simpson's Rule estimate is accurate to 4 decimal places.

Review Problems

5.54

(a) We can manipulate the fraction before integrating. Multiplying numerator and denominator by x^2 yields $\frac{x^2(x^2-1)}{1-x^2} = -x^2$. Therefore, the integral is just

$$\int_2^4 (-x^2)\,dx = -\frac{x^3}{3}\bigg|_2^4 = -\frac{4^3}{3} + \frac{2^3}{3} = \boxed{-\frac{56}{3}}.$$

(b) Write as

$$\int_0^8 2x^{\frac{2}{3}}\,dx = 2\,\frac{x^{\frac{5}{3}}}{\frac{5}{3}}\bigg|_0^8 = \frac{6}{5}\cdot 8^{\frac{5}{3}} = \boxed{\frac{192}{5}}.$$

(c) Seeing $x^3 + 1$ within that complicated expression with an x^2 term from its derivative outside, we make the substitution $u = x^3 + 1$. Then $du = 3x^2\,dx$, and $x \in [-1, 1]$ becomes $u \in [0, 2]$, so

$$\int_{-1}^{1} x^2\sqrt[3]{x^3+1}\,dx = \frac{1}{3}\int_0^2 u^{\frac{1}{3}}\,du = \frac{1}{3}\cdot\frac{3}{4}u^{\frac{4}{3}}\bigg|_0^2 = \frac{1}{4}\sqrt[3]{16} = \boxed{\frac{1}{2}\sqrt[3]{2}}.$$

(d) We break this into easier chunks by long division:

$$\frac{x^2}{x-3} = x + 3 + \frac{9}{x-3}.$$

Then

$$\int \frac{x^2}{x-3}\,dx = \int \left(x + 3 + \frac{9}{x-3}\right) dx = \boxed{\frac{1}{2}x^2 + 3x + 9\log|x-3| + C}.$$

(e) We write $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and realize that the derivative of $\cos\theta$ is $-\sin\theta$. Then we substitute $u = \cos\theta$ and get

$$\int_{\pi/6}^{\pi/3} \tan\theta\, d\theta = -\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{u}\,du = -\log|u|\Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} = -\log\frac{1}{2} + \log\frac{\sqrt{3}}{2} = \boxed{\log\sqrt{3} = \frac{1}{2}\log 3}.$$

(f) We can write this integral as $\int_0^2 xe^{-x}\,dx$ and then proceed by integration by parts, with $u = x$ and $v = -e^{-x}$, so $du = dx$ and $dv = e^{-x}\,dx$. Then

$$\int_0^2 xe^{-x}\,dx = -xe^{-x}\Big|_0^2 - \int_0^2 -e^{-x}\,dx = (-2e^{-2}) - e^{-x}\Big|_0^2 = -2e^{-2} - (e^{-2} - 1) = \boxed{1 - 3e^{-2} = 1 - \frac{3}{e^2}}.$$

(g) Seeing an x^2 makes us think to integrate by parts twice. First we let $u = x^2$ and $v = -\cos x$, so $du = 2x\,dx$ and $dv = \sin x\,dx$. Then

$$\int x^2 \sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx.$$

To find $\int 2x\cos x\,dx$, we again integrate by parts, this time with $u = 2x$ and $v = \sin x$, so $du = 2\,dx$ and $dv = \cos x\,dx$. Therefore,

$$\int 2x\cos x\,dx = 2x\sin x - \int 2\sin x\,dx = 2x\sin x + 2\cos x,$$

and

$$\int x^2\sin x\,dx = \boxed{-x^2\cos x + 2x\sin x + 2\cos x + C} = (2 - x^2)\cos x + 2x\sin x + C.$$

(h) We substitute $x = \sin\theta$, so $dx = \cos\theta\,d\theta$ (using $x = \cos\theta$ gives the same final result). We take $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so we know that $\cos\theta > 0$ for all θ. Then we get

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2\theta}\cos\theta\,d\theta = \int \cos^2\theta\,d\theta.$$

But $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, so we have

$$\int \cos^2\theta\,d\theta = \frac{1}{2}\int (1 + \cos 2\theta)\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C.$$

Substituting $\theta = \sin^{-1} x$, and using $\sin(2\theta) = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$, we conclude that

$$\int \sqrt{1-x^2}\,dx = \boxed{\frac{1}{2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right) + C}.$$

(i) We use the partial fraction decomposition

$$\frac{x+1}{x^2-4} = \frac{\frac{1}{4}}{x+2} + \frac{\frac{3}{4}}{x-2}.$$

Therefore,

$$\int_3^5 \frac{x+1}{x^2-4}\,dx = \frac{1}{4}\int_3^5 \frac{dx}{x+2} + \frac{3}{4}\int_3^5 \frac{dx}{x-2}$$
$$= \frac{1}{4}\log|x+2|\Big|_3^5 + \frac{3}{4}\log|x-2|\Big|_3^5$$
$$= \frac{1}{4}(\log 7 - \log 5) + \frac{3}{4}(\log 3 - \log 1)$$
$$= \frac{1}{4}\log\left(\frac{7\cdot 3^3}{5}\right) = \boxed{\frac{1}{4}\log\frac{189}{5}}.$$

(j) First we factor the denominator as $(x^2+1)^2$. Seeing x^2+1 as a factor of x^3+x, we can separate the integral to

$$\int \frac{x^3+x+2}{x^4+2x^2+1}\,dx = \int \frac{x^3+x}{(x^2+1)^2}\,dx + \int \frac{2}{(x^2+1)^2}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{2}{(x^2+1)^2}\,dx.$$

We evaluate the first integral easily by letting $u = x^2+1$, so $du = 2x\,dx$ and

$$\int \frac{x}{x^2+1}\,dx = \int \frac{du}{2u} = \frac{1}{2}\log|u| = \frac{1}{2}\log(x^2+1) + C.$$

To evaluate the second integral, we can substitute $x = \tan\theta$, so $dx = \sec^2\theta\,d\theta$ and $\frac{2dx}{(x^2+1)^2} = \frac{2\sec^2\theta\,d\theta}{\sec^4\theta} = 2\cos^2\theta\,d\theta$. Now we use the double angle formula for cosine: $2\cos^2\theta = 1+\cos 2\theta$. Thus

$$\int \frac{2}{(x^2+1)^2}\,dx = \int (1+\cos 2\theta)\,d\theta = \theta + \frac{1}{2}\sin 2\theta = \tan^{-1}x + \frac{1}{2}\sin(2\tan^{-1}x) + C.$$

Note also that $\sin 2t = 2\sin t\cos t = 2\tan t\cos^2 t$, and $\cos^2(\tan^{-1}x) = \frac{1}{1+x^2}$, thus $\frac{1}{2}\sin(2\tan^{-1}x) = \frac{x}{1+x^2}$. Hence,

$$\int \frac{2}{(x^2+1)^2}\,dx = \tan^{-1}x + \frac{x}{1+x^2} + C.$$

Therefore, the entire integral is $\boxed{\frac{1}{2}\log(x^2+1) + \tan^{-1}x + \frac{x}{1+x^2} + C}$.

(k) Let $u = \sqrt{e^x+1}$. Then $u^2 = e^x+1$, so $x = \log(u^2-1)$ and $dx = \frac{2u}{u^2-1}\,du$. Making this substitution, our integral becomes

$$\int \frac{2u}{u(u^2-1)}\,du = \int \frac{2}{u^2-1}\,du.$$

But we can use partial fractions to write this as

$$\int \frac{2}{u^2-1}\,du = \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = \log|u-1| - \log|u+1|,$$

so the answer is $\boxed{\log\left|\sqrt{e^x+1}-1\right| - \log\left|\sqrt{e^x+1}+1\right| + C}$. We can also write this as

$$\log\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right| + C = -2\coth^{-1}\left(\sqrt{e^x+1}\right) + C,$$

where coth is the hyperbolic cotangent function.

5.55

(a) Use the substitution $u = x + c$. Note $du = dx$. Then using $x = u - c$, we get

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(u-c)\,du.$$

(b) Use the substitution $u = cx$. Note $du = c\,dx$, and $x = \frac{u}{c}$, so we have

$$\int_a^b f(x)\,dx = \frac{1}{c}\int_{ca}^{cb} f\left(\frac{u}{c}\right)du.$$

5.56 Do integration by parts with $u = x$, $du = dx$, and $v = -\frac{1}{2}e^{-x^2}$, $dv = xe^{-x^2}\,dx$. Then

$$\int x^2e^{-x^2}\,dx = -\frac{1}{2}xe^{-x^2} + \frac{1}{2}\int e^{-x^2}\,dx = \boxed{-\frac{1}{2}xe^{-x^2} + \frac{1}{2}f(x)}.$$

5.57 If $a = b = 0$, then the function is 0, and the integral is just the constant C. If $a = 0$ and $b \neq 0$, then the integral is

$$\int \sin(bx)\,dx = -\frac{1}{b}\cos(bx) + C.$$

If a is nonzero, then integration by parts with $u = \sin bx$, $du = b\cos(bx)\,dx$ and $v = \frac{1}{a}e^{ax}$, $dv = e^{ax}\,dx$ yields

$$\int e^{ax}\sin(bx)\,dx = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\int e^{ax}\cos(bx)\,dx.$$

We then integrate $e^{ax}\cos(bx)$, again by parts:

$$-\frac{b}{a}\int e^{ax}\cos(bx)\,dx = -\frac{b}{a^2}e^{ax}\cos(bx) - \frac{b^2}{a^2}\int e^{ax}\sin(bx)\,dx.$$

The original integral appears again on the right side of the above expression, so solving for it we have the equation

$$\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\sin(bx)\,dx = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a^2}e^{ax}\cos(bx).$$

Multiply by a^2 to get rid of the fractions:

$$(a^2 + b^2)\int e^{ax}\sin(bx)\,dx = ae^{ax}\sin(bx) - be^{ax}\cos(bx).$$

Finally, $a^2 + b^2$ is nonzero, so we can divide by it, and

$$\int e^{ax}\sin(bx)\,dx = \boxed{\frac{ae^{ax}\sin(bx) - be^{ax}\cos(bx)}{a^2 + b^2} + C}.$$

5.58 We can make the substitution $u = -x$ and apply the evenness of f to get

$$\int_{-2}^{-1} f(x)\,dx = \int_1^2 f(x)\,dx.$$

Then

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_1^2 f(x)\,dx = \int_{-1}^{1} f(x)\,dx + 2\int_1^2 f(x)\,dx,$$

(c)

$$\int_1^2 f(x)\,dx = \frac{1}{2}\left(\int_{-2}^2 f(x)\,dx - \int_{-1}^1 f(x)\,dx\right) = \frac{1}{2}(12-8) = \boxed{2}.$$

5.59 Let (b,c) be the rightmost intersection point. Then the condition is that

$$\int_0^b (2x - 3x^3 - c)\,dx = 0.$$

This integral is

$$\left(x^2 - \frac{3}{4}x^4 - cx\right)\Big|_0^b = b^2 - \frac{3}{4}b^4 - bc,$$

so we need $b - \frac{3}{4}b^3 - c = 0$, or $4b - 3b^3 - 4c = 0$. But $c = 2b - 3b^3$, since (b,c) is on the graph of $y = 2x - 3x^3$, so we have

$$4b - 3b^3 - 4(2b - 3b^3) = 9b^3 - 4b = b(9b^2 - 4) = b(3b-2)(3b+2) = 0.$$

We clearly want the positive solution, so $b = \frac{2}{3}$ and $c = 2\left(\frac{2}{3}\right) - 3\left(\frac{2}{3}\right)^3 = \frac{4}{3} - \frac{8}{9} = \boxed{\frac{4}{9}}$.

5.60 Let F be an antiderivative of f. Then by the Fundamental Theorem of Calculus,

$$\int_{u(x)}^{v(x)} f(t)\,dt = F(v(x)) - F(u(x)).$$

Differentiating both sides and using the Chain Rule gives

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = F'(v(x))v'(x) - F'(u(x))u'(x) = \boxed{f(v(x))v'(x) - f(u(x))u'(x)}.$$

5.61 The derivative of $\sin x$ is $\cos x$, and we can use $[0, 2\pi]$ as the period. Therefore, the integral we want is $\boxed{\int_0^{2\pi} \sqrt{1+\cos^2 x}\,dx}$.

5.62

(a) Each cross-section is a circle with radius $x^2 - x$, so has area $\pi(x^2-x)^2 = \pi(x^4 - 2x^3 + x^2)$. Thus the volume is

$$\pi\int_1^2 (x^4 - 2x^3 + x^2)\,dx = \pi\left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3\right)\Big|_1^2 = \pi\left(\left(\frac{32}{5} - 8 + \frac{8}{3}\right) - \left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right)\right) = \boxed{\frac{31\pi}{30}}.$$

(b) The solid consists of cylinders of height $\cos x$ and radius $2\pi x$, which has area $2\pi x\cos x$. Thus, the volume is (where we use integration by parts):

$$\begin{aligned}
2\pi\int_0^{\pi/2} x\cos x\,dx &= 2\pi\left(x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx\right)\\
&= 2\pi\,(x\sin x + \cos x)\Big|_0^{\pi/2}\\
&= 2\pi\left(\frac{\pi}{2} - 1\right) = \boxed{\pi^2 - 2\pi}
\end{aligned}$$

(c) A cross-section that is x units from the top is a circle of radius $2 + \frac{x}{2}$, and thus has area $\pi\left(2 + \frac{x}{2}\right)^2$. Thus, the volume is the integral

$$\int_0^6 \pi\left(2+\frac{x}{2}\right)^2 dx = \pi\int_0^6 \left(4 + 2x + \frac{1}{4}x^2\right) dx = \pi\left(4x + x^2 + \frac{1}{12}x^3\right)\Big|_0^6 = \pi(24+36+18) = \boxed{78\pi}.$$

This also follows from the formula $\frac{1}{3}\pi r^2 h$ for the volume of a circular cone with radius r and height h. Our truncated cone is the difference of a cone with $(r,h) = (5,10)$ and a cone with $(r,h) = (2,4)$, and thus the volume is

$$\frac{1}{3}\pi(5)^2(10) - \frac{1}{3}\pi(2)^2(4) = \frac{1}{3}\pi(250-16) = 78\pi.$$

5.63 Consider cross-sections of the form $x = a$ from $a = 0$ to $a = 1$. Each cross-section is the interior of a circle of the form $y^2 + z^2 = 4 - a^2$, so it has area $\pi(4 - a^2)$. Integrating from $a = 0$ to $a = 1$, we conclude that the volume equals

$$\pi\int_0^1 (4-a^2)\, da = \left(4a - \frac{a^3}{3}\right)\Big|_0^1 = \pi\left(4 - \frac{1}{3}\right) = \boxed{\frac{11\pi}{3}}.$$

5.64 We can slice up the surface area into a sum of areas of cylindrical strips. The area of the strip at $x = x_i$ is the product of its circumference with its slant height. The circumference of each strip is $2\pi f(x_i)$, and the slant height is the arc length of the portion of the curve corresponding to the strip, which is approximated by $\sqrt{1+(f'(x_i))^2}\,(x_{i+1}-x_i)$. Thus the strip has area $2\pi f(x_i)\sqrt{1+(f'(x_i))^2}\,(x_{i+1}-x_i)$. Hence, the total area is approximated by the Riemann sum

$$\sum_{i=0}^{n-1} 2\pi f(x_i)\sqrt{1+(f'(x_i))^2}\,(x_{i+1}-x_i).$$

As the widths of the strips approach 0, the surface area is given by the definite integral

$$\boxed{2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx}.$$

To test this, let $y = \sqrt{r^2 - x^2}$. The result of rotating this graph about the x-axis on the interval $[-r, r]$ is a sphere of radius r. We compute

$$f'(x) = -\frac{x}{\sqrt{r^2-x^2}},$$

so $1 + (f'(x))^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2-x^2}$. Thus the surface area is

$$2\pi\int_{-r}^{r} f(x)\sqrt{1+(f'(x))^2}\, dx = 2\pi\int_{-r}^{r}\left(\sqrt{r^2-x^2}\cdot\sqrt{\frac{r^2}{r^2-x^2}}\right) dx = 2\pi\int_{-r}^{r} r\, dx = 2\pi r(2r) = 4\pi r^2,$$

as expected.

5.65 The average value is $\frac{1}{2a}\int_{-a}^{a} f(x)\, dx$. Let $f(x) = px^3 + qx^2 + rx + s$, so we have

$$\frac{1}{2a}\int_{-a}^{a}(px^3+qx^2+rx+s)\,dx = \frac{1}{2a}\left(\frac{p}{4}x^4 + \frac{q}{3}x^3 + \frac{r}{2}x^2 + sx\right)\Big|_{-a}^{a} = \frac{1}{2a}\left(\frac{2}{3}qa^3 + 2as\right) = \frac{1}{3}qa^2 + s.$$

On the other hand,

$$\frac{1}{2}\left(f\left(\frac{a}{\sqrt3}\right) + f\left(-\frac{a}{\sqrt3}\right)\right) = \frac{1}{2}\left(\left(\frac{p}{3\sqrt3}a^3 + \frac{q}{3}a^2 + \frac{r}{\sqrt3}a + s\right) + \left(-\frac{p}{3\sqrt3}a^3 + \frac{q}{3}a^2 - \frac{r}{\sqrt3}a + s\right)\right) = \frac{1}{3}qa^2 + s.$$

(Note that in both computations, the cubic and linear terms cancel out.)

Challenge Problems

5.66

(a) It may help to compute $\frac{d}{dx}(\log x)^2 = \frac{2}{x}(\log x)$, and notice that this is just the first term of the integrand divided by x. Furthermore, we notice that

$$\frac{d}{dx}x(\log x)^2 = (\log x)^2 + x\left(\frac{2}{x}(\log x)\right) = (\log x)^2 + 2\log x,$$

which is exactly the function we are integrating. Therefore,

$$\int (2\log x + (\log x)^2)\,dx = \int \frac{d}{dx}(x(\log x)^2)\,dx = \boxed{x(\log x)^2 + C}.$$

Note that this is essentially just a rearranged integration by parts:

$$\int (u\,dv + v\,du) = uv,$$

where in our example $u = x$ and $v = (\log x)^2$.

(b) The denominator factors as $x(x^3+1)$, and we have the convenient decomposition

$$\frac{2x^3-1}{x^4+x} = \frac{3x^2}{x^3+1} - \frac{1}{x}.$$

Thus,

$$\int \frac{2x^3-1}{x^4+x}\,dx = \int \frac{3x^2}{x^3+1}\,dx - \int \frac{1}{x}\,dx = \log|x^3+1| - \log|x| + C = \boxed{\log\left|\frac{x^3+1}{x}\right| + C}.$$

(c) Let $u = \sqrt[3]{x}$, so that $x = u^3$ and $dx = 3u^2\,du$. Our integral then becomes

$$\int \sin\sqrt[3]{x}\,dx = \int 3u^2 \sin u\,du.$$

This can now be integrated by parts (twice) to get

$$\begin{aligned}\int 3u^2\sin u\,du &= -3u^2\cos u + \int 6u\cos u\,du \\ &= -3u^2\cos u + 6u\sin u - \int 6\sin u\,du \\ &= -3u^2\cos u + 6u\sin u + 6\cos u + C.\end{aligned}$$

Reversing the substitution gives an answer of

$$\boxed{(-3\sqrt[3]{x^2}+6)\cos\sqrt[3]{x} + 6\sqrt[3]{x}\sin\sqrt[3]{x} + C}.$$

(d) Since we want to deal nicely with both a square root and a cube root of x, we try the substitution $x = u^6$. Then $dx = 6u^5\,du$, and the integral becomes

$$\int \frac{x^{-\frac{1}{2}}}{1+x^{\frac{1}{3}}}\,dx = \int \frac{u^{-3}}{1+u^2}(6u^5)\,du = \int \frac{6u^2}{u^2+1}\,du.$$

We now split the integral as

$$\int \frac{6u^2}{u^2+1}\,du = 6\int\left(1 - \frac{1}{u^2+1}\right)du = 6(u - \tan^{-1}u) + C.$$

Reversing the substitution gives the answer $\boxed{6(x^{\frac{1}{6}} - \tan^{-1}x^{\frac{1}{6}}) + C}$.

5.67

(a) Do long division:

$$\frac{x^4(1-x)^4}{1+x^2} = \frac{x^8 - 4x^7 + 6x^6 - 4x^5 + x^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}.$$

Integrating gives

$$\left(\frac{1}{7}x^7 - \frac{2}{3}x^6 + x^5 - \frac{4}{3}x^3 + 4x - 4\tan^{-1}(x)\right)\Bigg|_0^1 = \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\cdot\frac{\pi}{4} = \boxed{\frac{22}{7} - \pi}.$$

(b) Note that doing the substitution $u = \frac{\pi}{2} - x$ gives

$$\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx = \int_0^{\pi/2} \frac{\cos^3 u}{\sin^3 u + \cos^3 u}\,du.$$

Adding these together gives

$$2\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx = \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x}\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2},$$

and thus $\displaystyle\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx = \boxed{\frac{\pi}{4}}$.

(c) Integrate by parts with $u = (1-x)^7$, $du = -7(1-x)^6\,dx$ and $v = \frac{1}{201}x^{201}$, $dv = x^{200}\,dx$. We get

$$\binom{207}{7}\int_0^1 x^{200}(1-x)^7\,dx = \frac{1}{201}\binom{207}{7}x^{201}(1-x)^7\Bigg|_0^1 + \frac{7}{201}\binom{207}{7}\int_0^1 x^{201}(1-x)^6\,dx.$$

Note that the first term on the right side is 0, and that $\frac{7}{201}\binom{207}{7} = \binom{207}{6}$. So we have

$$\binom{207}{7}\int_0^1 x^{200}(1-x)^7\,dx = \binom{207}{6}\int_0^1 x^{201}(1-x)^6\,dx.$$

We can repeat the process six more times to get

$$\binom{207}{7}\int_0^1 x^{200}(1-x)^7\,dx = \binom{207}{0}\int_0^1 x^{207}\,dx = \boxed{\frac{1}{208}}.$$

(d) We'd like to make the substitution $u = x^5 + 3x^2 + x$, but for that we need the substitution $du = (5x^4 + 6x + 1)\,dx$. So we write

$$9x + 4 = (5x^4 + 15x + 5) - (5x^4 + 6x + 1).$$

So our integral becomes

$$\int_1^2 \frac{5x^4 + 15x + 5}{x^5 + 3x^2 + x}\,dx - \int_1^2 \frac{5x^4 + 6x + 1}{x^5 + 3x^2 + x}\,dx.$$

This is

$$\int_1^2 \frac{5}{x}\,dx - \int_5^{46} \frac{1}{u}\,du.$$

So the integral is $5\log 2 - \log 46 + \log 5$, which you can further simplify if you like to $\boxed{\log\frac{80}{23}}$.

5.68 If $a = b$, then

$$\int \frac{dx}{(x-a)(x-b)} = \int \frac{dx}{(x-a)^2} = -\frac{1}{x-a} + C.$$

If $a \neq b$, then we use partial fractions. We claim that

$$\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right).$$

This can be verified by multiplying by $(x-a)(x-b)$, yielding 1 on the left and $\frac{1}{a-b}((x-b)-(x-a)) = \frac{a-b}{a-b} = 1$ on the right. Then we integrate:

$$\int \frac{dx}{(x-a)(x-b)} = \int \frac{dx}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right) = \frac{1}{a-b}(\log|x-a| - \log|x-b|) + C = \boxed{\frac{1}{a-b}\log\left|\frac{x-a}{x-b}\right| + C}.$$

5.69 We compute $f'(x) = \dfrac{\cos x}{\sin x} = \cot x$, and $\sqrt{1+(f'(x))^2} = \sqrt{1+\cot^2 x} = \csc x$ for $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Thus the length is given by

$$\int_{\pi/4}^{\pi/2} \csc x\, dx.$$

To compute this integral, we consider two expressions that we know have a csc term in their derivative:

$$\frac{d}{dx}\cot x = -\csc^2 x \quad \text{and} \quad \frac{d}{dx}\csc x = -\cot x \csc x.$$

In particular, we notice that

$$\frac{d}{dx}(\cot x + \csc x) = -(\csc^2 x + \cot x \csc x) = -\csc x(\cot x + \csc x).$$

Thus, if $f(x) = \cot x + \csc x$, then we have

$$\csc x = -\frac{f'(x)}{f(x)} = -\frac{d}{dx}\log|f(x)|,$$

and hence $-\log|f(x)|$ is an antiderivative of $\csc x$. To finish, we compute

$$\int_{\pi/4}^{\pi/2} \csc x\, dx = (-\log|\cot x + \csc x|)\Big|_{\pi/4}^{\pi/2} = -\log|0+1| + \log|1+\sqrt{2}| = \boxed{\log(1+\sqrt{2})}.$$

5.70 The integral we want is

$$\int_0^2 \sqrt{1+(\cosh' x)^2}\, dx.$$

We note that $\cosh' x = \sinh x$, where $\sinh x = \dfrac{e^x - e^{-x}}{2}$. So our arc length is

$$\int_0^2 \sqrt{1+\sinh^2 x}\, dx.$$

It is easy to check that $1 + \sinh^2 x = \cosh^2 x$:

$$1 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1 + \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4} = \left(\frac{e^x + e^{-x}}{2}\right)^2,$$

so the integral is

$$\int_0^2 \cosh x\,dx = \sinh x\Big|_0^2 = \sinh 2 - \sinh 0 = \boxed{\sinh 2 = \frac{e^2 - e^{-2}}{2}}.$$

5.71

(a) Substitute $u = -\frac{t}{\sqrt{2}}$, so that $du = -\frac{dt}{\sqrt{2}}$. Then

$$\frac{1}{2\sqrt{\pi}}\int_{-x}^{x} e^{-\frac{t^2}{2}}\,dt = \frac{1}{\sqrt{2}\sqrt{\pi}}\int_{-x/\sqrt{2}}^{x/\sqrt{2}} e^{-u^2}\,du = \frac{2}{\sqrt{2}\sqrt{\pi}}\int_0^{x/\sqrt{2}} e^{-u^2}\,du = \boxed{\frac{1}{\sqrt{2}}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}.$$

(b) Note $\int_0^{x^2} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x^2)$. So taking the derivative, we get

$$\frac{\sqrt{\pi}}{2}\frac{d}{dx}\operatorname{erf}(x^2) = \frac{\sqrt{\pi}}{2}2x\operatorname{erf}'(x^2) = \boxed{2xe^{-x^4}}.$$

(c) We use the product rule:

$$\frac{d}{dx}(\sqrt{x}\operatorname{erf}(x)) = \sqrt{x}\operatorname{erf}'(x) + \frac{\operatorname{erf}(x)}{2\sqrt{x}} = \boxed{\frac{2}{\sqrt{\pi}}\sqrt{x}e^{-x^2} + \frac{\operatorname{erf}(x)}{2\sqrt{x}}}.$$

(d) Integrate by parts: let $u = \operatorname{erf}(x)$, so $du = \left(\frac{2}{\sqrt{\pi}}\right)e^{-x^2}\,dx$, and let $v = x$, so $dv = dx$. Then

$$\int \operatorname{erf}(x)\,dx = x\operatorname{erf}(x) - \frac{2}{\sqrt{\pi}}\int xe^{-x^2}\,dx = \boxed{x\operatorname{erf}(x) + \frac{1}{\sqrt{\pi}}e^{-x^2} + C}.$$

5.72 Assume that f is monotonically increasing (the proof where f is monotonically decreasing is essentially the same). Let $\mathcal{P}_n$ be the partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$ into n pieces of width $\frac{b-a}{n}$. Then, noting that $f(x_0) \le f(x_1) \le \cdots \le f(x_n)$, we have

$$l(f,\mathcal{P}_n) = \sum_{i=0}^{n-1}\frac{b-a}{n}\cdot f(x_i) = \frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)$$

and

$$u(f,\mathcal{P}_n) = \sum_{i=0}^{n-1}\frac{b-a}{n}\cdot f(x_{i+1}) = \frac{b-a}{n}\sum_{i=0}^{n-1} f(x_{i+1}).$$

When we compute the difference $u(f,\mathcal{P}_n) - l(f,\mathcal{P}_n)$, we note that all of the terms cancel, except for the last term of the upper sum and the first term of the lower sum; that is,

$$u(f,\mathcal{P}_n) - l(f,\mathcal{P}_n) = \frac{b-a}{n}(f(x_n) - f(x_0)) = \frac{b-a}{n}(f(b) - f(a)).$$

But as n gets large, the above quantity approaches 0 (since $(b-a)(f(b)-f(a))$ is constant). Thus, the difference between the lower sums and the upper sums approaches 0 for our choice $\mathcal{P}_n$ of partitions, and hence $\int_a^b f$ exists.

5.73 We have

$$\int_0^1 f(x)(\alpha^2 - 2\alpha x + x^2)\,dx = \alpha^2 - 2\alpha^2 + \alpha^2 = 0,$$

so

$$\int_0^1 f(x)(a-x)^2\,dx = 0.$$

But the function $f(x)(a-x)^2$ is positive for all $x \neq a$, so its integral must be positive too. Hence there are $\boxed{\text{no such functions}}$.

5.74 Let one cylinder have the x-axis as its axis, and have equation $y^2 + z^2 = 1$. Let the other cylinder have the y-axis as its axis, and have equation $x^2 + z^2 = 1$. So for a fixed z, the cross-section is bounded by $y = \pm\sqrt{1-z^2}$ and $x = \pm\sqrt{1-z^2}$. In other words, the cross-section is a square with side length $2\sqrt{1-z^2}$. So the cross-section has area $4(1-z^2)$, and thus the volume is

$$\int_{-1}^{1} 4(1-z^2)\,dz = 4\left(z - \frac{1}{3}z^3\right)\Big|_{-1}^{1} = 8\left(1-\frac{1}{3}\right) = \boxed{\frac{16}{3}}.$$

(The perhaps surprising part about this is that there is no π in the answer!)

5.75 If we let 4-dimensional space be represented by 4-tuples (x, y, z, w), then the 4-dimensional unit sphere is the set of points satisfying the equation

$$x^2 + y^2 + z^2 + w^2 = 1.$$

As x varies from -1 to 1, the cross-section of the 4-dimensional sphere is given by the set of triples (y, z, w) such that $y^2 + z^2 + w^2 = 1 - x^2$; that is, it is a 3-dimensional sphere of radius $\sqrt{1-x^2}$. This cross-section thus has volume $\frac{4}{3}\pi(\sqrt{1-x^2})^3$. Thus, to compute the volume of the 4-dimensional sphere, we sum the 3-dimensional cross-sectional volumes as x varies from -1 to 1, giving the definite integral

$$\frac{4}{3}\pi\int_{-1}^{1}(\sqrt{1-x^2})^3\,dx.$$

To compute this integral, make the trig substitution $x = \sin\theta$, so that $dx = \cos\theta\,d\theta$, and the integral becomes

$$\frac{4}{3}\pi\int_{-1}^{1}(\sqrt{1-x^2})^3\,dx = \frac{4}{3}\pi\int_{-\pi/2}^{\pi/2}(\sqrt{1-\sin^2\theta})^3(\cos\theta)\,d\theta = \frac{4}{3}\pi\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta.$$

Since cos is an even function, this quantity equals

$$\frac{8}{3}\pi\int_0^{\pi/2}\cos^4\theta\,d\theta.$$

Using the formula $\cos^2 t = \frac{1}{2}(1+\cos 2t)$, we have

$$\cos^4\theta = \left(\frac{1}{2}(1+\cos 2\theta)\right)^2 = \frac{1}{4}(1+2\cos 2\theta+\cos^2 2\theta) = \frac{1}{4}\left(1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right) = \frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta,$$

so we get

$$\frac{8}{3}\pi\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{8}{3}\pi\int_0^{\pi/2}\left(\frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta\right)d\theta = \frac{8}{3}\pi\left(\frac{3}{16}\pi+0+0\right) = \boxed{\frac{1}{2}\pi^2}.$$

CHAPTER 6

Infinity

Exercises for Section 6.1

6.1.1

(a) The limit as $x \to \infty$ of a rational function with equal degree in the numerator and denominator is the ratio of the leading coefficients, which in this problem is $\boxed{\frac{5}{2}}$. We can also compute this explicitly by dividing both numerator and denominator by x^3, which yields

$$\lim_{x\to\infty} \frac{5x^3 - x^2 + 3x - 2}{2x^3 + 3x^2 - x + 6} = \lim_{x\to\infty} \frac{5 - \frac{1}{x} + \frac{3}{x^2} - \frac{2}{x^3}}{2 + \frac{3}{x} - \frac{1}{x^2} + \frac{6}{x^3}} = \frac{5+0+0+0}{2+0+0+0} = \frac{5}{2}.$$

(b) Since the degree of the denominator is greater than the degree of the numerator, the limit is $\boxed{0}$. We can also compute this explicitly:

$$\lim_{x\to\infty} \frac{2x^4 + x^2 - 3}{-3x^5 - x^3 + 2x - 7} = \lim_{x\to\infty} \frac{\frac{2}{x} + \frac{1}{x^3} - \frac{3}{x^5}}{-3 - \frac{1}{x^2} + \frac{2}{x^4} - \frac{7}{x^5}} = \frac{0}{-3} = 0.$$

(c) As with a rational function, we divide by the highest power of x:

$$\lim_{x\to\infty} \frac{x^2 + \sin x}{2x^2 + \sin x} = \lim_{x\to\infty} \frac{1 + \frac{\sin x}{x^2}}{2 + \frac{\sin x}{x^2}}.$$

As x gets large, $\left|\frac{\sin x}{x^2}\right| \le \frac{1}{x^2}$, so by the Squeeze Theorem,

$$0 = \lim_{x\to\infty}\left(-\frac{1}{x^2}\right) \le \lim_{x\to\infty} \frac{\sin x}{x^2} \le \lim_{x\to\infty} \frac{1}{x^2} = 0,$$

so $\lim\limits_{x\to\infty} \frac{\sin x}{x^2} = 0$, and thus

$$\lim_{x\to\infty} \frac{x^2 + \sin x}{2x^2 + \sin x} = \lim_{x\to\infty} \frac{1 + \frac{\sin x}{x^2}}{2 + \frac{\sin x}{x^2}} = \frac{1+0}{2+0} = \boxed{\frac{1}{2}}.$$

6.1.2

(a) Let $\epsilon > 0$ be given. We know that there exist numbers N_f and N_g such that

$$x > N_f \quad \Rightarrow \quad |f(x) - L| < \frac{\epsilon}{2}$$
$$x > N_g \quad \Rightarrow \quad |g(x) - M| < \frac{\epsilon}{2}.$$

Let N be the larger of these two numbers. Then by the Triangle Inequality,

$$x > N \quad \Rightarrow \quad \left|(f+g)(x) - (L+M)\right| \le |f(x) - L| + |g(x) - M| < \epsilon,$$

so $\lim_{x\to\infty}(f+g) = L + M$.

(b) If $c = 0$, then the statement is clearly true. Assume $c \neq 0$ and let $\epsilon > 0$ be given. We know that there exists some N such that

$$x > N \quad \Rightarrow \quad |f(x) - L| < \frac{\epsilon}{|c|}.$$

For this N,

$$x > N \quad \Rightarrow \quad |(cf)(x) - cL| = |c| \cdot |f(x) - L| < |c|\frac{\epsilon}{|c|} = \epsilon,$$

and hence $\lim_{x\to\infty}(cf) = cL$.

6.1.3 First, suppose that $\lim_{x\to-\infty} f(x) = L$, and let $\epsilon > 0$ be given. We know that we can choose $N < 0$ such that

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Let $\delta = -\frac{1}{N}$. Then

$$-\delta < z < 0 \quad \Rightarrow \quad \frac{1}{z} < N \quad \Rightarrow \quad \left|f\left(\frac{1}{z}\right) - L\right| < \epsilon.$$

But this last statement is exactly the definition of $\lim_{z\to0^-} f\left(\frac{1}{z}\right) = L$.

Conversely, suppose that $\lim_{z\to0^-} f\left(\frac{1}{z}\right) = L$, and let $\epsilon > 0$ be given. We know that we can choose $\delta > 0$ such that

$$-\delta < z < 0 \quad \Rightarrow \quad \left|f\left(\frac{1}{z}\right) - L\right| < \epsilon.$$

Let $N = -\frac{1}{\delta}$. Then

$$x < N \quad \Rightarrow \quad -\delta < \frac{1}{x} < 0 \quad \Rightarrow \quad \left|f\left(\frac{1}{\frac{1}{x}}\right) - L\right| < \epsilon \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

This is the definition of $\lim_{x\to-\infty} f(x) = L$.

6.1.4 One possibility is that the limit of the derivative might not exist. For example, let $f(x) = \frac{1}{x}\sin x^2$. Note that for all $x \in \mathbb{R}$,

$$-\frac{1}{x} \le f(x) \le \frac{1}{x},$$

so by the Squeeze Theorem, $\lim_{x\to\infty} f(x) = 0$. We have

$$f'(x) = -\frac{1}{x^2}\sin x^2 + 2\cos x^2,$$

and as $x \to \infty$, the $-\frac{1}{x^2}\sin x^2$ term of $f'(x)$ approaches 0, but the $2\cos x^2$ term oscillates between -2 and 2, so $\lim_{x\to\infty} f'(x)$ does not exist.

On the contrary, if $\lim_{x\to\infty} f'(x)$ does exist, we expect that it is 0. This makes sense geometrically: the function f has a horizontal asymptote at $x = c$, so its slope approaches the slope of the line $x = c$, which is 0. To prove this, assume that the limit exists but is nonzero. For simplicity, assume that $\lim_{x\to\infty} f'(x) = L > 0$. (If the limit is less than 0, we can consider the function $-f$, and the proof is the same).

By definition, for any $\epsilon > 0$, there exists some N such that

$$x > N \quad \Rightarrow \quad |f'(x) - L| < \epsilon.$$

Choose $\epsilon = \frac{L}{2}$. Then

$$x > N \quad \Rightarrow \quad |f'(x) - L| < \frac{L}{2} \quad \Rightarrow \quad \frac{L}{2} < f'(x) < \frac{3L}{2}.$$

Specifically, for sufficiently large x, $f'(x) > \frac{L}{2}$.

Now let $x > N$. We apply the Mean Value Theorem. For every $x > N$, there exists some $\xi \in (N, x)$ such that

$$f(x) = f(N) + (x - N)f'(\xi) > f(N) + x\frac{L}{2} - \frac{3NL}{2}.$$

Thus

$$c = \lim_{x\to\infty} f(x) \geq \left(f(N) - \frac{3NL}{2}\right) + \frac{L}{2}\lim_{x\to\infty} x,$$

but the quantity on the right is unbounded, giving a contradiction.

In summary, $\lim_{x\to\infty} f'(x)$ is $\boxed{\text{0 or does not exist}}$.

Exercises for Section 6.2

6.2.1

(a) Let f be a real-valued function and $a \in \mathbb{R}$. We say $\lim_{x\to a} f(x) = -\infty$ if, for all N, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad f(x) < N.$$

(b) Let f be a real-valued function. We say $\lim_{x\to\infty} f(x) = \infty$ if, for all N, there exists M such that

$$x > M \quad \Rightarrow \quad f(x) > N.$$

6.2.2

(a) We factor the denominator:

$$\lim_{x\to2^+} \frac{1}{x^2 - 3x + 2} = \lim_{x\to2^+} \frac{1}{(x-2)(x-1)} = \left(\lim_{x\to2^+} \frac{1}{x-2}\right)\left(\lim_{x\to2^+} \frac{1}{x-1}\right) = \lim_{x\to2^+} \frac{1}{x-2}.$$

Since $x - 2 > 0$ as x approaches 2 from the right, the limit is $\boxed{+\infty}$.

(b) Since $x - 1$ is a factor of both the numerator and denominator,

$$\lim_{x\to1^-} \frac{x-1}{x^3 - x^2 + x - 1} = \lim_{x\to1^-} \frac{x-1}{(x-1)(x^2+1)} = \lim_{x\to1^-} \frac{1}{x^2+1} = \boxed{\frac{1}{2}}.$$

(c) Since tan increases without bound on $\left(0, \frac{\pi}{2}\right)$, we suspect that $\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$. We can prove this using the definition of limit. Let $N > 0$ be given and let $\delta = \frac{\pi}{2} - \arctan N$. Note that $\delta \in \left(0, \frac{\pi}{2}\right)$, and hence

$$0 < \frac{\pi}{2} - x < \delta \quad \Rightarrow \quad \tan\left(\frac{\pi}{2} - x\right) < \tan\delta \quad \Rightarrow \quad \tan x > \tan\left(\frac{\pi}{2} - \delta\right) = N.$$

This proves

$$\lim_{x\to\frac{\pi}{2}^-} \tan x = \boxed{+\infty}.$$

6.2.3 One example is $f(x) = \cos x$. Note that $\cos 2n\pi x = 1$ and $\cos(2n+1)\pi x = -1$ for all positive integers n. Thus, cos does not increase nor decrease without bound, so the limit cannot be $\pm\infty$. Also, if $\lim\limits_{x\to\infty} \cos x = a$ for some $a \in \mathbb{R}$, then we must have $|1 - a| < \epsilon$ and $|-1 - a| < \epsilon$ for all $\epsilon > 0$, and clearly this cannot be the case (for instance, choosing $\epsilon = \frac{1}{2}$ and applying the Triangle Inequality gives $2 < 1$, which is not true).

Informally, what is happening is that $\cos x$ oscillates between 1 and -1 infinitely often, so cannot approach a single value as x gets large.

Exercises for Section 6.3

6.3.1

(a) L'Hôpital's Rule gives

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \boxed{\frac{1}{2}}.$$

(b) L'Hôpital's Rule gives

$$\lim_{x\to \pi} \frac{\sin 3x}{\sin 4x} = \lim_{x\to \pi} \frac{3\cos 3x}{4\cos 4x} = \boxed{-\frac{3}{4}}.$$

(c) L'Hôpital's Rule gives

$$\lim_{x\to 0} \frac{xe^x}{1-e^x} = \lim_{x\to 0} \frac{xe^x + e^x}{-e^x} = \lim_{x\to 0} \frac{x+1}{-1} = \boxed{-1}.$$

(The last step is simply canceling the e^x terms.)

(d) Since $\log x$ grows (negatively) without bound as $x \to 0^+$, and $x^4 - 1$ is negative and bounded near 0, the limit is $\boxed{\infty}$.

6.3.2 Note that $\lim\limits_{x\to\infty} |f(x)| = \infty$ for any non-constant polynomial f, and in particular $f(x) \neq 0$ for sufficiently large x. Thus, for sufficiently large x,

$$0 \le \left|\frac{\sin x}{f(x)}\right| \le \frac{1}{|f(x)|}.$$

Thus, by the Squeeze Theorem, since $\lim\limits_{x\to\infty} \frac{1}{|f(x)|} = 0$, we have $\lim\limits_{x\to\infty} \left|\frac{\sin x}{f(x)}\right| = 0$, and thus $\lim\limits_{x\to\infty} \frac{\sin x}{f(x)} = 0$. Therefore, $f(x)$ dominates $\sin x$.

The exact same proof works for cosine: just replace "sin" with "cos" in the above argument.

6.3.3 Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where $n \ge 1$ and $a_n \neq 0$. Then

$$\lim_{x\to\infty} \frac{\sqrt{x}}{f(x)} = \lim_{x\to\infty} \frac{\sqrt{x}}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0} = \lim_{x\to\infty} \frac{1}{a_n x^{n-\frac{1}{2}} + a_n x^{n-\frac{3}{2}} + \cdots + a_0 x^{-\frac{1}{2}}} = 0,$$

since $n - \frac{1}{2} > 0$ forces the denominator to grow without bound.

6.3.4 We note that as $h \to 0$, the numerator and denominator both approach 0. Therefore l'Hôpital's Rule is applicable.

Treating x as a constant, we have $\frac{d}{dh}(f(x+h) - f(x)) = f'(x+h)$ and $\frac{d}{dh}(h) = 1$. Taking the limits as $h \to 0$, we get

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f'(x+h)}{1} = f'(x),$$

assuming that f' is continuous. This just recaptures our original definition of the derivative.

6.3.5 If we substitute $u = 1/x$, then

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \lim_{u\to0^-} \frac{f(1/u)}{g(1/u)}.$$

But $\lim_{u\to0^-} f(1/u) = \lim_{x\to-\infty} f(x) = 0$, and similarly $\lim_{u\to0^-} g(1/u) = \lim_{x\to-\infty} g(x) = 0$. So l'Hôpital's Rule applies, and we have

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \lim_{u\to0^-} \frac{f(1/u)}{g(1/u)} = \lim_{u\to0^-} \frac{-\frac{1}{u^2}\cdot f'(1/u)}{-\frac{1}{u^2}\cdot g'(1/u)} = \lim_{u\to0^-} \frac{f'(1/u)}{g'(1/u)}.$$

Undoing the transformation gives

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \lim_{u\to0^-} \frac{f(1/u)}{g(1/u)} = \lim_{u\to0^-} \frac{f'(1/u)}{g'(1/u)} = \lim_{x\to-\infty} \frac{f'(x)}{g'(x)},$$

as desired.

Exercises for Section 6.4

6.4.1

(a) We can write $x\log x = \frac{\log x}{\frac{1}{x}}$, and note that both numerator and denominator of this fraction approach $\pm\infty$ as $x \to 0^+$. Thus, by l'Hôpital's Rule,

$$\lim_{x\to0^+} x\log x = \lim_{x\to0^+} \frac{\log x}{\frac{1}{x}} = \lim_{x\to0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to0^+} -x = \boxed{0}.$$

(b) We can let $z = \frac{1}{x}$, and we compute:

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{z\to0^+} \frac{\sin z}{z} = \boxed{1}.$$

(c) This limit is just

$$\exp\left(\lim_{x\to1} \log\left(x^{\frac{1}{\sin(1-x)}}\right)\right).$$

We can write expression inside the limit as $\frac{x\log x}{\sin(1-x)}$. As $x \to 1$, both numerator and denominator approach 0, so we can try l'Hôpital's Rule:

$$\lim_{x\to1} \frac{x\log x}{\sin(1-x)} = \lim_{x\to1} \frac{\log x + 1}{-\cos(1-x)} = \frac{\log 1 + 1}{-\cos 0} = -1.$$

Exponentiating yields

$$\lim_{x\to1} x^{\left(\frac{1}{\sin(1-x)}\right)} = e^{-1} = \boxed{\frac{1}{e}}.$$

6.4.2 Since we have an exponential expression, we try taking a logarithm:

$$\log \lim_{x\to0^+} (1+x)^{\frac{1}{x}} = \lim_{x\to0^+} \log(1+x)^{\frac{1}{x}} = \lim_{x\to0^+} \frac{\log(1+x)}{x}.$$

Now we note that the numerator and denominator of this last fraction both approach 0 as $x \to 0^+$, so we use l'Hôpital's Rule:

$$\lim_{x\to0^+} \frac{\log(1+x)}{x} = \lim_{x\to0^+} \frac{\frac{1}{1+x}}{1} = \lim_{x\to0^+} \frac{1}{1+x} = 1.$$

Thus, exponentiating gives the original limit as $e^1 = \boxed{e}$.

6.4.3 As in the previous problem, we take the natural logarithm:

$$\log \lim_{x\to 0^+} (1+kx)^{\frac{1}{x}} = \lim_{x\to 0^+} \log(1+kx)^{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\log(1+kx)}{x}.$$

As $x \to 0^+$, this fraction approaches $\frac{\log(1)}{0} = \frac{0}{0}$, so we can use l'Hôpital's Rule:

$$\lim_{x\to 0^+} \frac{\log(1+kx)}{x} = \lim_{x\to 0^+} \frac{\frac{k}{1+kx}}{1} = \lim_{x\to 0^+} \frac{k}{1+kx} = k.$$

Therefore, the original limit is $\boxed{e^k}$.

6.4.4 We write

$$f - g = \frac{\frac{1}{g} - \frac{1}{f}}{\frac{1}{fg}}.$$

This is now a $\frac{0}{0}$ indeterminate form at ∞, so we can apply l'Hôpital's Rule:

$$\lim_{x\to\infty}(f(x) - g(x)) = \lim_{x\to\infty} \frac{-\frac{g'(x)}{(g(x))^2} + \frac{f'(x)}{(f(x))^2}}{-\frac{f'(x)g(x)+f(x)g'(x)}{(f(x)g(x))^2}} = \lim_{x\to\infty} \frac{f^2(x)g'(x) - g^2(x)f'(x)}{f(x)g'(x) + g(x)f'(x)}.$$

Of course, there's no guarantee that this limit will exist, but at least it gives us something to try!

Exercises for Section 6.5

6.5.1

(a) We set this up as a limit: $\displaystyle\lim_{b\to\infty} \int_3^b \frac{1}{(2x-1)^2}\,dx$. Evaluating the integral, we get

$$\int_3^b \frac{1}{(2x-1)^2}\,dx = \left(\frac{-1/2}{2x-1}\right)\Bigg|_3^b = -\frac{1}{2}\left(\frac{1}{2b-1} - \frac{1}{2\cdot 3 - 1}\right).$$

As $b \to \infty$, the fraction $\frac{1}{2b-1}$ approaches zero, giving

$$\int_3^\infty \frac{1}{(2x-1)^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{2}\left(\frac{1}{2b-1} - \frac{1}{2\cdot 3-1}\right)\right) = -\frac{1}{2}\left(-\frac{1}{5}\right) = \boxed{\frac{1}{10}}.$$

(b) This integral is $\displaystyle\lim_{b\to\infty} \int_2^b \frac{1}{x(\log x)^2}\,dx$. We compute the antiderivative using the substitution $u = \log x$ and $du = \frac{1}{x}\,dx$:

$$\int \frac{1}{x(\log x)^2}\,dx = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\log x}.$$

Therefore, the improper integral is

$$\int_2^\infty \frac{1}{x(\log x)^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{\log x}\right)\Bigg|_2^b = \lim_{b\to\infty}\left(-\frac{1}{\log b} + \frac{1}{\log 2}\right) = \boxed{\frac{1}{\log 2}},$$

since $\log b \to \infty$ when $b \to \infty$.

(c) As usual, we write this improper integral as a limit: $\lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx$. We first find an antiderivative, substituting $u=-x^2$ and $du=-2x\,dx$:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{e^u}{2} = -\frac{e^{-x^2}}{2}.$$

Now we evaluate the improper integral:

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\left(-\frac{e^{-x^2}}{2}\right)\Bigg|_0^b = \lim_{b\to\infty}\frac{e^0-e^{-b^2}}{2} = \boxed{\frac{1}{2}}.$$

(d) This function is not defined at $x=2$, so this is an improper integral and we must write it as a limit: $\lim_{b\to 2^-}\int_0^b \frac{1}{4-x^2}\,dx$. To evaluate this integral, we use partial fractions:

$$\frac{1}{4-x^2} = \frac{1}{(2+x)(2-x)} = \frac{1}{4}\left(\frac{1}{2-x}+\frac{1}{2+x}\right).$$

Thus,

$$\int\frac{1}{4-x^2}\,dx = \frac{1}{4}\int\left(\frac{1}{2-x}+\frac{1}{2+x}\right)dx = \frac{1}{4}(\log|2+x|-\log|2-x|) = \frac{1}{4}\log\left|\frac{2+x}{2-x}\right|.$$

Thus our improper integral is

$$\int_0^2\frac{1}{4-x^2}\,dx = \lim_{b\to 2^-}\frac{1}{4}\log\left|\frac{2+x}{2-x}\right|\Bigg|_0^b = \lim_{b\to 2^-}\frac{1}{4}\left(\log\left|\frac{2+b}{2-b}\right|-\log 1\right) = \frac{1}{4}\lim_{b\to 2^-}\log\left|\frac{2+b}{2-b}\right|.$$

But as $b\to 2^-$, the denominator of the logarithm goes to 0, so the log grows without bound. Hence, the integral $\boxed{\text{diverges}}$.

6.5.2

(a) We compute:

$$\int_0^\infty\frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b\frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\tan^{-1}x\Big|_0^b = \lim_{b\to\infty}\tan^{-1}b = \boxed{\frac{\pi}{2}}.$$

(b) Because the function is even, we have

$$\int_{-\infty}^\infty\frac{1}{1+x^2}\,dx = 2\int_0^\infty\frac{1}{1+x^2}\,dx = 2\left(\frac{\pi}{2}\right) = \boxed{\pi}.$$

6.5.3 We use integration by parts:

$$\begin{aligned}\int_0^\infty x^2e^{-x}\,dx &= -x^2e^{-x}\Big|_0^\infty + \int_0^\infty 2xe^{-x}\,dx\\ &= 0+2\int_0^\infty xe^{-x}\,dx\\ &= -2xe^{-x}\Big|_0^\infty + 2\int_0^\infty e^{-x}\,dx\\ &= 0-2e^{-x}\Big|_0^\infty\\ &= \boxed{2}.\end{aligned}$$

6.5.4 For simplicity, suppose that $c < d$ (the proof for $c > d$ is essentially the same). Note that

$$\int_c^b f = \lim_{x \to b^-} \int_c^x f = \lim_{x \to b^-} \left(\int_c^d f + \int_d^x f \right) = \int_c^d f + \lim_{x \to b^-} \int_d^x f.$$

Since the improper integral on the left side converges (by assumption), and the integral $\int_c^d f$ is defined, we have that

$$\int_d^b f = \lim_{x \to b^-} \int_d^x f = \int_c^b f - \int_c^d f.$$

A similar argument shows that

$$\int_a^d f = \int_a^c f + \int_c^d f,$$

and adding gives

$$\int_a^d f + \int_d^b f = \left(\int_c^b f - \int_c^d f \right) + \left(\int_a^c f + \int_c^d f \right) = \int_a^c f + \int_c^b f.$$

6.5.5

(a) If this integral converges, then both $\int_{-\infty}^0 f(x)\,dx$ and $\int_0^\infty f(x)\,dx$ converge. Thus

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx = \lim_{a \to \infty} \int_{-a}^0 f(x)\,dx + \lim_{a \to \infty} \int_0^a f(x)\,dx = \lim_{a \to \infty} \int_{-a}^a f(x)\,dx.$$

(b) We can take an odd function such as $f(x) = x$. Then

$$\int_{-a}^a x\,dx = \frac{1}{2}x^2 \Big|_{-a}^a = \frac{1}{2}(a^2 - (-a)^2) = 0.$$

Therefore, $\lim_{a \to \infty} \int_{-a}^a f(x)\,dx = 0$ converges. On the other hand,

$$\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^0 x\,dx + \int_0^\infty x\,dx,$$

and neither of the improper integrals on the right side converges, so the double-improper integral on the left side diverges. Thus, it is possible for $\lim_{a \to \infty} \int_{-a}^a f(x)\,dx$ to converge, but $\int_{-\infty}^{\infty} f(x)\,dx$ to diverge, as desired.

Review Problems

6.23

(a) At $x = 1$, both numerator and denominator are 0, so l'Hôpital's Rule gives us

$$\lim_{x \to 1} \frac{x^2 - 1}{\log x} = \lim_{x \to 1} \frac{2x}{\frac{1}{x}} = \boxed{2}.$$

(b) At $x = 0$, both numerator and denominator are 0, so l'Hôpital's Rule gives us

$$\lim_{x \to 0} \frac{\cos^2 x - 1}{x^2} = \lim_{x \to 0} \frac{-2 \sin x \cos x}{2x} = \lim_{x \to 0} \left(-\frac{\sin x \cos x}{x} \right).$$

This still evaluates to $\frac{0}{0}$, so we apply l'Hôpital's Rule again:

$$\lim_{x\to 0}\left(-\frac{\sin x\cos x}{x}\right) = \lim_{x\to 0}\left(-\frac{-\sin^2 x+\cos^2 x}{1}\right) = \boxed{-1}.$$

(c) At $x = 0$, both numerator and denominator are 0, so applying l'Hôpital's Rule, and then performing some simplification, gives

$$\lim_{x\to 0}\frac{10x^2-\frac{1}{2}x^3}{e^{4x^2}-1} = \lim_{x\to 0}\frac{20x-\frac{3}{2}x^2}{8xe^{4x^2}} = \lim_{x\to 0}\frac{x(20-\frac{3}{2}x)}{8xe^{4x^2}} = \lim_{x\to 0}\frac{20-\frac{3}{2}x}{8e^{4x^2}} = \frac{20}{8} = \boxed{\frac{5}{2}}.$$

6.24 By l'Hôpital's Rule, we have

$$\lim_{t\to 0}\frac{\sin at}{\sin bt} = \lim_{t\to 0}\frac{a\cos at}{b\cos bt} = \boxed{\frac{a}{b}}.$$

Rather than also using l'Hôptial's Rule on the ratio of tangents, we write it as

$$\lim_{t\to 0}\frac{\tan at}{\tan bt} = \lim_{t\to 0}\left(\frac{\sin at}{\sin bt}\cdot\frac{\cos bt}{\cos at}\right) = \frac{a}{b}\cdot 1 = \boxed{\frac{a}{b}}.$$

6.25

(a) We evaluate

$$\int_1^\infty e^{-2x}\,dx = \lim_{b\to\infty}\int_1^b e^{-2x}\,dx = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-2x}\Big|_1^b\right) = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-2b}+\frac{1}{2}e^{-2}\right) = \frac{1}{2}e^{-2} = \boxed{\frac{1}{2e^2}}.$$

It's also OK to write this without explicitly writing the limit:

$$\int_1^\infty e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_1^\infty = 0+\frac{1}{2}e^{-2} = \frac{1}{2e^2}.$$

(b) This is improper at the $x = 0$ end. We have

$$\int_0^2 \frac{1}{x^3}\,dx = -\frac{1}{2x^2}\Big|_0^2.$$

But $\lim_{a\to 0^+}\frac{1}{2x^2} = \infty$, so the improper integral $\boxed{\text{diverges}}$.

(c) Complete the square of the denominator as $x^2+2x+2 = (x+1)^2+1$. Then

$$\int_{-\infty}^\infty \frac{1}{x^2+2x+2}\,dx = \int_{-\infty}^\infty \frac{1}{(x+1)^2+1}.$$

Since this integral is evaluated over all of $\mathbb{R}$, we can do the change of variable $u = x+1$, and the integral is still evaluated over all of $\mathbb{R}$. Thus we have

$$\int_{-\infty}^\infty \frac{1}{x^2+2x+2}\,dx = \int_{-\infty}^\infty \frac{1}{u^2+1}\,du = \tan^{-1}u\Big|_{-\infty}^\infty = \boxed{\pi}.$$

6.26 This is an improper integral. Make the substitution $x = u^6$. Then the limits of integration are the same, and $dx = 6u^5\,du$. Therefore, the integral becomes

$$\int_0^1 \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = \int_0^1 \frac{6u^5}{u^3+u^2}\,du = 6\int_0^1 \frac{u^3}{u+1}\,du.$$

This is no longer improper. We can evaluate this integral by rewriting the function as

$$\frac{u^3}{u+1} = \frac{(u^3+1)-1}{u+1} = u^2 - u + 1 - \frac{1}{u+1},$$

hence we have

$$\int_0^1 \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = 6\int_0^1 \left(u^2 - u + 1 - \frac{1}{u+1}\right) du = \left(2u^3 - 3u^2 + 6u - 6\log|u+1|\right)\Big|_0^1 = \boxed{5 - 6\log 2}.$$

6.27 We notice that $\sqrt{x^2+x}$ approaches x as $x \to \infty$, so this "$\infty - \infty$" form may look like it approaches 0, but this is incorrect. Instead, we eliminate the ambiguous term by multiplying and dividing by its conjugate:

$$\sqrt{x^2+x} - x = \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} = \frac{x}{\sqrt{x^2+x}+x}.$$

The limit of the last term is

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \boxed{\frac{1}{2}}.$$

6.28

(a) If $h_1 \in o(f)$ and $h_2 \in o(g)$, then

$$\lim_{x\to\infty} \frac{h_1(x)}{f(x)} = \lim_{x\to\infty} \frac{h_2(x)}{g(x)} = 0.$$

Multiply these limits together to get

$$\lim_{x\to\infty} \frac{(h_1h_2)(x)}{(fg)(x)} = 0,$$

so $h_1h_2 \in o(fg)$.

(b) If $h_1 \in o(f)$ and $h_2 \in o(h_1)$, then

$$\lim_{x\to\infty} \frac{h_1(x)}{f(x)} = \lim_{x\to\infty} \frac{h_2(x)}{h_1(x)} = 0.$$

Multiply these limits together to get

$$\lim_{x\to\infty} \frac{h_2(x)}{f(x)} = 0,$$

so $h_2 \in o(f)$.

(c) We need $\lim\limits_{x\to\infty} \dfrac{f(x)}{e^x} = 0$. But if $f(x)$ is degree n, we can simply apply l'Hôpital's Rule n times: the degree of f will decrease by 1 on each application, until we eventually are reduced to

$$\lim_{x\to\infty} \frac{f(x)}{e^x} = \lim_{x\to\infty} \frac{f^{(n)}(x)}{e^x},$$

where $f^{(n)}$ denotes the n^{th} derivative of f. But $f^{(n)}$ is a constant, so the limit is 0, and $f \in o(\exp)$.

6.29 We just compute:

$$\int_a^b \sin mx\,dx = \frac{1}{m}(-\cos mx)\Big|_a^b = \frac{1}{m}(\cos ma - \cos mb).$$

The difference of the cosine terms is bounded (between -2 and 2), but $\frac{1}{m}$ approaches 0 as m grows large. So the limit of the integral is 0.

Challenge Problems

6.30

(a) By definition,

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_{x=0}^{\infty} = \boxed{1}.$$

(b) We start with

$$\Gamma(z+1) = \int_0^\infty x^z e^{-x}\,dx.$$

We use integration by parts with $u = x^z$, $du = zx^{z-1}\,dx$ and $v = -e^{-x}$, $dv = e^{-x}\,dx$. This gives:

$$\begin{aligned}\Gamma(z+1) &= -x^z e^{-x}\Big|_0^\infty + \int_0^\infty zx^{z-1}e^{-x}\,dx \\ &= \left(\lim_{x\to\infty} -x^z e^{-x}\right) + 0 + z\int_0^\infty x^{z-1}e^{-x}\,dx \\ &= z\Gamma(z),\end{aligned}$$

noting that the limit in the second line above is 0 because e^x dominates x^z as $x \to \infty$ (or you could prove this using l'Hôpital's Rule repeatedly). Note also that part (b) holds for any $z > 0$, even if z is not an integer.

(c) We have

$$\begin{aligned}\Gamma(2) &= 1\Gamma(1) = 1, \\ \Gamma(3) &= 2\Gamma(2) = 2, \\ \Gamma(4) &= 3\Gamma(3) = 6,\end{aligned}$$

and so on, and we see (and can easily prove by induction if we so desire) that $\boxed{\Gamma(n) = (n-1)!}$.

6.31 Let $z(t) = (ax^t + by^t)^{\frac{1}{t}}$. Taking the log gives

$$\log z(t) = \frac{\log(ax^t + by^t)}{t}.$$

At $t = 0$ this is a $\frac{0}{0}$ indeterminate form (since $\log(a+b) = \log(1) = 0$), so l'Hôpital's Rule gives us

$$\lim_{t\to 0^+} \log z(t) = \lim_{t\to 0^+} \frac{\log(ax^t + by^t)}{t} = \lim_{t\to 0^+} \frac{\frac{1}{ax^t+by^t}\cdot\frac{d}{dt}(ax^t + by^t)}{1} = \lim_{t\to 0^+} \frac{d}{dt}(ax^t + by^t).$$

Recall that $\frac{d}{dt}c^t = c^t \log c$, where c is a constant, so the above limit is

$$\lim_{t\to 0}(ax^t(\log x) + by^t(\log y)) = a\log x + b\log y = \log(x^a y^b).$$

Hence, upon exponentiation, we get that the original limit is $\boxed{x^a y^b}$.

6.32 First, note that since e^{-t^2} is an even function, we have

$$\int_{-\infty}^\infty e^{-t^2}\,dt = 2\int_0^\infty e^{-t^2}\,dt = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

If $b < 0$, then $e^{-\frac{(x-a)^2}{b}} \to \infty$ as $x \to \infty$, so the improper integral diverges. If $b > 0$, we make the substitution $u = \frac{x-a}{\sqrt{b}}$, so $du = \frac{1}{\sqrt{b}}\,dx$, and the limits of integration are the same (since u is linear in x and increasing as x is increasing). Therefore, our integral becomes

$$\sqrt{b}\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{b}\cdot\sqrt{\pi} = \boxed{\sqrt{b\pi}}.$$

6.33 Since f is continuous, there is a maximum value M of $|f(x)|$ on $[0,1]$. Therefore,

$$\left|\int_x^1 \frac{f(t)}{t}\,dt\right| \le M\int_x^1 \frac{1}{t}\,dt.$$

So we compute $\lim\limits_{x\to 0^+} x\displaystyle\int_x^1 \frac{1}{t}\,dt$. The integral goes to infinity as x approaches 0, so we rewrite it as:

$$\lim_{x\to 0^+} \frac{\int_x^1 \frac{1}{t}\,dt}{\frac{1}{x}},$$

and since this is an $\frac{\infty}{\infty}$ indeterminate form, we apply l'Hôptial's Rule:

$$\lim_{x\to 0^+} \frac{-\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} x = 0.$$

Therefore,

$$\lim_{x\to 0^+}\left|x\int_x^1 \frac{f(t)}{t}\,dt\right| \le \lim_{x\to 0^+} Mx\int_x^1 \frac{1}{t}\,dt = 0,$$

and thus the limit is $\boxed{0}$.

6.34 We'd like to take advantage of some symmetry, so we perform the change of variable $u = x-1$. Then $du = dx$ and we have

$$\int_1^{\infty} \frac{dx}{e^{x+1}+e^{3-x}} = \int_0^{\infty} \frac{du}{e^{u+2}+e^{2-u}}.$$

Now we can factor e^2 out of the denominator:

$$\frac{1}{e^2}\int_0^{\infty} \frac{du}{e^u+e^{-u}}\,du.$$

We next try to get rid of the negative power of e in the denominator by multiplying numerator and denominator by e^u:

$$\frac{1}{e^2}\int_0^{\infty} \frac{e^u}{(e^u)^2+1}\,du.$$

Now the "obvious" substitution is $v = e^u$, $dv = e^u\,du$, so we have

$$\frac{1}{e^2}\int_1^{\infty} \frac{dv}{v^2+1} = \frac{1}{e^2}\arctan v\Big|_1^{\infty}.$$

This gives our answer:

$$\frac{1}{e^2}(\arctan\infty - \arctan 1) = \frac{1}{e^2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \boxed{\frac{\pi}{4e^2}}.$$

(Of course, "arctan ∞" is an ugly shorthand for $\lim\limits_{x\to\infty} \arctan x$.)

CHAPTER 7

Series

Exercises for Section 7.1

7.1.1

(a) Just as we did with limits of rational functions, $a_n = \dfrac{1-\frac{1}{n^3}}{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}} \to \boxed{1}$ as $n \to \infty$.

(b) Letting $f(x) = \dfrac{3x^2+2e^x}{x-e^x}$, we apply l'Hôpital's Rule:

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2+2e^x}{x-e^x} = \lim_{x\to\infty} \frac{6x+2e^x}{1-e^x} = \lim_{x\to\infty} \frac{6+2e^x}{-e^x} = \lim_{x\to\infty} \frac{2e^x}{-e^x} = \boxed{-2}.$$

(c) Since $|\sin n| \le 1$ for all n, we have

$$\left|\frac{n\sin n}{n^2+1}\right| \le \frac{n}{n^2+1} \le \frac{n}{n^2} = \frac{1}{n}.$$

But $\frac{1}{n}$ goes to 0 as $n \to \infty$, so $\dfrac{n \sin n}{n^2+1}$ must also go to $\boxed{0}$.

(d) We can manipulate this via difference of squares:

$$\sqrt[4]{n^2+1} - \sqrt{n+1} = \left(\sqrt[4]{n^2+1} - \sqrt{n+1}\right) \cdot \frac{\sqrt[4]{n^2+1}+\sqrt{n+1}}{\sqrt[4]{n^2+1}+\sqrt{n+1}} = \frac{\sqrt{n^2+1}-(n+1)}{\sqrt[4]{n^2+1}+\sqrt{n+1}}.$$

This is still a bit unpleasant to work with, so we use difference of squares again on the numerator:

$$\frac{\sqrt{n^2+1}-(n+1)}{\sqrt[4]{n^2+1}+\sqrt{n+1}} = \frac{\sqrt{n^2+1}-(n+1)}{\sqrt[4]{n^2+1}+\sqrt{n+1}} \cdot \frac{\sqrt{n^2+1}+(n+1)}{\sqrt{n^2+1}+(n+1)} = \frac{-2n}{(\sqrt[4]{n^2+1}+\sqrt{n+1})(\sqrt{n^2+1}+(n+1))}.$$

Notice that $\sqrt{n^2+1}+(n+1) > n+n = 2n$, so

$$|\sqrt[4]{n^2+1} - \sqrt{n+1}| = \frac{2n}{(\sqrt[4]{n^2+1}+\sqrt{n+1})(\sqrt{n^2+1}+(n+1))} < \frac{1}{\sqrt[4]{n^2+1}+\sqrt{n+1}}.$$

As n gets large, the denominator on the right gets arbitrarily large, so this fraction goes to zero. But this means that $\sqrt[4]{n^2+1} - \sqrt{n+1} \to \boxed{0}$.

(e) Note that $\log a_n = \frac{\log(n^2+1)}{n}$. Then for all $n > 1$,

$$0 < \log a_n = \frac{\log(n^2+1)}{n} < \frac{\log n^3}{n} = \frac{3\log n}{n}.$$

But $\frac{\log n}{n} \to 0$ as $n \to \infty$, so $\log a_n \to 0$ as $n \to \infty$. Therefore, $a_n \to e^0 = \boxed{1}$.

7.1.2 Suppose the original sequence has common ratio r. Let a and b be two consecutive terms in the original sequence; thus we have $b = ra$. These terms become ka and kb in the new sequence, and we have $kb = k(ra) = r(ka)$ in the new sequence. Therefore, the new sequence is also geometric with constant ratio r.

7.1.3

(a) If x, y, and z are consecutive terms in a geometric sequence with constant ratio r, then $y = rx$ and $z = r^2x$ for some real number r. Then $y^2 = r^2x^2$ and $xz = r^2x^2$, so $y^2 = xz$. Conversely, if $y^2 = xz$, then dividing by zy gives us $\frac{z}{y} = \frac{y}{z}$, which means that x, y, z are consecutive terms of a geometric sequence (with common ratio $\frac{y}{x} = \frac{z}{y}$).

(b) In general, it is not true that if x, y, and z are consecutive terms of a geometric sequence, then $y = \sqrt{xz}$. For example, take $x = 1$, $y = -2$, and $z = 4$. Then x, y, and z are in consecutive terms of a geometric sequence with common ratio -2, but $\sqrt{xz} = 2 \neq y$.

7.1.4 Suppose $\{a_n\}$ converges to L. Choose any $\epsilon > 0$; then by definition of convergence there exists some N such that $L + \epsilon$ is an upper bound for $\{a_n \mid n > N\}$. However, the set $\{a_n \mid n \leq N\}$ is finite, and thus also has an upper bound, specifically $\sup\{a_n \mid n \leq N\}$. The larger of these two upper bounds is an upper bound for the sequence. Essentially the same argument shows that the sequence has a lower bound: the smaller of $L - \epsilon$ and $\inf\{a_n \mid n \leq N\}$ is a lower bound.

Exercises for Section 7.2

7.2.1

(a) If $m = 0$ then $\{c_n\}$ is simply the constant sequence 0, and the series clearly converges to $0 = 0A = mA$.

If $m \neq 0$, then let $\epsilon > 0$ be given. We must find N such that for any $n > N$,

$$\epsilon > \left|\sum_{i=1}^{n} c_i - mA\right| = \left|\sum_{i=1}^{n} ma_i - mA\right| = m\left|\sum_{i=1}^{n} a_i - A\right|.$$

But by the convergence of $\sum_{i=1}^{\infty} a_i$, we know that we can find an N such that for all $n > N$,

$$\frac{\epsilon}{m} > \left|\sum_{i=1}^{n} a_i - A\right|,$$

so that N will satisfy the above inequality, and we are done.

(b) Let $\epsilon > 0$ be given. We must find N such that for any $n > N$,

$$\epsilon > \left|\sum_{i=1}^{n} d_i - (A + B)\right|.$$

But by the Triangle Inequality,

$$\left|\sum_{i=1}^{n} d_i - (A + B)\right| = \left|\sum_{i=1}^{n} a_i - A + \sum_{i=1}^{n} b_i - B\right| \leq \left|\sum_{i=1}^{n} a_i - A\right| + \left|\sum_{i=1}^{n} b_i - B\right|.$$

By assumption, since $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ converge to A and B, respectively, we can find N sufficiently large so that both $\left|\sum_{i=1}^{n} a_i - A\right|$ and $\left|\sum_{i=1}^{n} b_i - B\right|$ are less than $\frac{\epsilon}{2}$. Thus, $\left|\sum_{i=1}^{n} d_i - (A + B)\right|$ is less than ϵ, as needed, and we are done.

7.2.2

(a) The first term is $\frac{1}{6}$ and the common ratio is $\frac{1}{6}$, so the sum of the series is $\dfrac{\frac{1}{6}}{1-\frac{1}{6}} = \boxed{\dfrac{1}{5}}$.

(b) The first term is 192 and the common ratio is $\frac{144}{192} = \frac{3}{4}$, so the sum of the series is $\dfrac{192}{1-\frac{3}{4}} = \boxed{768}$.

(c) The first term is 2 and the common ratio is $-\frac{1}{\sqrt{2}}$, so the sum of the series is

$$\frac{2}{1-\left(-\frac{1}{\sqrt{2}}\right)} = \frac{2}{1+\frac{1}{\sqrt{2}}} = \frac{2}{1+\frac{1}{\sqrt{2}}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{1+\sqrt{2}} = \frac{2\sqrt{2}(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})} = \boxed{4-2\sqrt{2}}.$$

7.2.3

(a) Note that for any k,

$$\sum_{i=1}^{k} a_i = (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_k - b_{k+1}) = b_1 - b_{k+1}.$$

Thus

$$\sum_{i=1}^{\infty} a_i = \lim_{k\to\infty}(b_1 - b_{k+1}) = b_1.$$

(b) Write each term as $\dfrac{1}{n^2+n} = \dfrac{1}{n} - \dfrac{1}{n+1}$. Then by part (a), we have $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+n} = \boxed{1}$.

(c) In order to use telescoping, we would like to write

$$\frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)} = \frac{c_n}{3^n-2^n} - \frac{c_{n+1}}{3^{n+1}-2^{n+1}}$$

for some sequence $\{c_n\}$. A little experimentation with small values of n will lead to the guess that $c_n = 2^n$. Indeed, we verify that

$$\frac{2^n}{3^n-2^n} - \frac{2^{n+1}}{3^{n+1}-2^{n+1}} = \frac{2^n3^{n+1} - 2^{2n+1} - 3^n2^{n+1} + 2^{2n+1}}{(3^{n+1}-2^{n+1})(3^n-2^n)} = \frac{3(6^n)-2(6^n)}{(3^{n+1}-2^{n+1})(3^n-2^n)} = \frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)}.$$

So the series is telescoping, and since

$$\lim_{n\to\infty}\frac{2^n}{3^n-2^n} = \lim_{n\to\infty}\frac{1}{\left(\frac{3}{2}\right)^n - 1} = 0,$$

we conclude that

$$\sum_{n=1}^{\infty}\frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)} = \sum_{n=1}^{\infty}\left(\frac{2^n}{3^n-2^n} - \frac{2^{n+1}}{3^{n+1}-2^{n+1}}\right) = \frac{2^1}{3^1-2^1} = \boxed{2}.$$

7.2.4 The given series can be expressed as the sum of two infinite geometric series:

$$\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+\cdots = \left(\frac{1}{7}+\frac{1}{7^3}+\cdots\right)+\left(\frac{2}{7^2}+\frac{2}{7^4}+\cdots\right) = \frac{\frac{1}{7}}{1-\left(\frac{1}{7}\right)^2} + \frac{\frac{2}{7^2}}{1-\left(\frac{1}{7}\right)^2} = \frac{7}{48}+\frac{1}{24} = \boxed{\frac{3}{16}}.$$

Exercises for Section 7.3

7.3.1

(a) We use the Limit Comparison Test with the series $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2+3n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{n^2+3n+1} = 1.$$

Therefore, since $\sum \frac{1}{n}$ diverges, so does the given series.

(b) We use the Limit Comparison Test with the series $\sum \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{\frac{2}{n^2-n+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^2}{n^2-n+1} = 2.$$

Therefore, since $\sum \frac{1}{n^2}$ converges, so does the given series. However, it is not readily determinable what the given series converges to.

(Note (a) and (b) are examples of the general rule for series whose entries are given by a rational function: such a series converges if and only if the degree of the denominator is at least 2 greater than the degree of the numerator.)

(c) This is just the sum of a geometric series with ratio $\frac{2}{5}$ and a geometric series with ratio $\frac{3}{5}$. So it converges and sums to

$$\frac{2}{5}\cdot\frac{1}{1-\frac{2}{5}} + \frac{3}{5}\cdot\frac{1}{1-\frac{3}{5}} = \frac{2}{5}\cdot\frac{5}{3} + \frac{3}{5}\cdot\frac{5}{2} = \frac{2}{3} + \frac{3}{2} = \boxed{\frac{13}{6}}.$$

(d) Use the Ratio Test:

$$\lim_{n\to\infty} \frac{\frac{(n+1)!^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}.$$

Since this is less than 1, the series converges.

(e) We use the Limit Comparison Test with the harmonic series $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{t\to 0^+} \frac{\sin t}{t} = 1.$$

Since the harmonic series diverges, so does the given series.

7.3.2 It is clear that

$$\sum_{n=1}^{\infty}(-a_n) = -\sum_{n=1}^{\infty} a_n.$$

We suppose $\{a_n\}$ and $\{b_n\}$ are two sequences such that $0 \geq a_n \geq b_n$ for all n. Then $0 \leq (-a_n) \leq (-b_n)$, so applying the nonnegative Series Convergence Test, we get:

- If $\sum b_n$ converges, then $\sum a_n$ converges too.
- If $\sum a_n$ diverges, then $\sum b_n$ diverges too.

7.3.3 We can do both parts at once.

Delete the first term of the series. Then a_2 determines the height $f(2)$ of a box between $x = 1$ and $x = 2$ that lies strictly under the curve, and a_3 determines the height $f(3)$ of a box between $x = 2$ and $x = 3$ that lies strictly under the curve, and so on. So $a_2, a_3, \ldots$ are the heights of a Riemann sum (actually a lower Darboux sum) from $x = 1$ to ∞ that lies strictly below the curve. Therefore,

$$\sum_{n=2}^{\infty} a_n < \int_1^{\infty} f(x)\,dx,$$

and hence if the integral converges, then so does the series.

Similarly, a_1 determines the height $f(1)$ of a box between $x = 1$ and $x = 2$ that lies strictly above the curve, and a_2 determines the height $f(2)$ of a box between $x = 2$ and $x = 3$ that lies strictly above the curve, and so on. So $a_1, a_2, \ldots$ are the heights of an upper Darboux sum from $x = 1$ to ∞ that lies strictly above the curve. Therefore,

$$\int_1^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n.$$

7.3.4

(a) Suppose $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$. Then for any $\epsilon > 0$, we can choose N such that

$$n > N \quad \Rightarrow \quad \left|\frac{a_n}{b_n} - c\right| < c.$$

Let $\epsilon = c$ and choose a corresponding N so that

$$n > N \quad \Rightarrow \quad \left|\frac{a_n}{b_n} - c\right| < c \quad \Rightarrow \quad 0 < \frac{a_n}{b_n} < 2c,$$

and in particular $a_n < 2cb_n$ for all $n > N$. Thus, if $\sum b_n$ converges, then $\sum 2cb_n$ converges, and by the Series Comparison Test, $\sum a_n$ converges.

Furthermore, if $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$, then $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{c}$, and we can use the same argument to show that if $\sum a_n$ converges, then $\sum b_n$ converges.

Thus $\sum a_n$ converges if and only if $\sum b_n$ converges, completing the proof.

(b) Let $\epsilon = 1$ and choose N such that

$$n > N \quad \Rightarrow \quad \left|\frac{a_n}{b_n} - 0\right| < 1 \quad \Rightarrow \quad a_n \leq b_n.$$

Then by the Series Comparison Test, if $\sum b_n$ converges, then $\sum a_n$ converges.

(c) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$, then $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = 0$. Hence, applying part (b), we conclude that if $\sum a_n$ converges, then $\sum b_n$ converges. (Equivalently, if $\sum b_n$ diverges, then $\sum a_n$ diverges.)

7.3.5 Since this looks close to the harmonic series, we try the Limit Comparison Test with $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{1}{n^{(n+1)/n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^{(n+1)/n}} = \lim_{n\to\infty} \frac{1}{n^{(1/n)}}.$$

We can evaluate $\lim\limits_{n\to\infty} n^{(1/n)}$ using l'Hôpital's Rule. Let $y = n^{(1/n)}$. Then $\log y = \frac{1}{n}\log n = \frac{\log n}{n}$. So

$$\lim_{n\to\infty} \log y = \lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

hence $\lim\limits_{n\to\infty} y = e^0 = 1$. Thus the result of the Limit Comparison test is 1. Hence, since $\sum \frac{1}{n}$ diverges, so does our given series.

7.3.6 Suppose $r < 1$. Let s be any number such that $r < s < 1$. Then, by definition, there exists some N such that $a_n < s^n$ for all $n > N$. Thus, after some initial terms, the series $\sum a_n$ is less than the convergent geometric series $\sum s^n$, so it converges.

Conversely, suppose $r > 1$. Let s be any number such that $r > s > 1$. Then, by definition, there exists some N such that $a_n > s^n$ for all $n > N$. Thus, after some initial terms, the series $\sum a_n$ is greater than the divergent geometric series $\sum s^n$, so it diverges.

Finally, we know that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, so for any p-series $\sum a_n = \sum \frac{1}{n^p}$, we see that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n}}\right)^p = 1.$$

But we know that the p-series diverges for $0 \le p \le 1$ and converges for $p > 1$, so the Root Test is inconclusive if $r = 1$.

Exercises for Section 7.4

7.4.1 All of the series, except for part (e), are alternating.

(a) This is a geometric series with ratio $-\frac{1}{3}$. It converges to $\dfrac{-\frac{1}{3}}{1-\left(-\frac{1}{3}\right)} = \boxed{-\frac{1}{4}}$. The absolute value of the terms of the series give a geometric series with ratio $\frac{1}{3}$, which converges, so the original series is absolutely convergent.

(b) The absolute value of the k^{th} term is $\dfrac{1}{k^2}$. This is strictly decreasing and converges to 0, so the series converges. (In fact, the series converges to $-\frac{\pi^2}{12}$, but this is difficult to show at present.) The absolute value of the terms of the series give the p-series where $p = 2$, which converges, so the original series is absolutely convergent.

(c) The absolute value of the k^{th} term is $\dfrac{1}{\sqrt{k}}$. This is strictly decreasing and converges to 0, so the series converges. The absolute value of the terms of the series give the p-series where $p = \frac{1}{2}$, which diverges, so the original alternating series is conditionally convergent.

(d) The absolute value of the k^{th} term is $\dfrac{\log k}{k}$. This is strictly decreasing (for $k > 1$) and converges to 0, so the series converges. Taking the absolute value of the terms of the series gives the series $\sum \frac{\log k}{k}$, which is strictly greater than the divergent harmonic series (for $k > 1$), so it diverges; that is, the original alternating series is conditionally convergent.

(e) This series is not alternating, since $(-1)^{2k} = (-1^2)^k = 1^k = 1$ for all k. Thus the series is just the p-series where $p = \frac{3}{2}$, and hence is convergent. It is also trivially absolutely convergent since taking the absolute value of the terms of the series gives the same series, since all the terms are positive.

(f) Since k and k^2 have the same parity (both are even or both are odd), we have $(-1)^{k^2} = (-1)^k$, so this series is alternating. The absolute value of the k^{th} term is $\dfrac{k^2}{k^3+1}$, which is strictly decreasing, so the series converges by the Alternating Series Test. However, by the Limit Comparison Test, the absolute value series behaves in the same way as the harmonic series, so it diverges. Thus the original series is conditionally convergent.

7.4.2 If $\sum a_n$ absolutely converges, we must have $|a_n| \to 0$ as $n \to \infty$. Thus, by definition, for some $N > 0$, we have $|a_n| < 1$ for all $n > N$, which means that $0 \le a_n^2 < |a_n| < 1$ for all $n > N$. Thus, by the Series Comparison Test, since $\sum |a_n|$ converges, so does $\sum a_n^2$.

Exercises for Section 7.5

7.5.1 We are approximating $f(x) = x^{\frac{1}{2}}$ using the Taylor polynomial at $x = 1$. The relevant derivatives are:

$$\begin{aligned} f(1) &= 1, \\ f'(x) &= \frac{1}{2}x^{-\frac{1}{2}}, \quad f'(1) = \frac{1}{2}, \\ f''(x) &= -\frac{1}{4}x^{-\frac{3}{2}}, \quad f''(1) = -\frac{1}{4}, \\ f'''(x) &= \frac{3}{8}x^{-\frac{5}{2}}, \quad f'''(1) = \frac{3}{8}. \end{aligned}$$

So the cubic Taylor polynomial is

$$p(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

(Don't forget to divide by $n!$ in the $(x-1)^n$ term!) Hence, our approximation is

$$p(0.9) = 1 + \frac{1}{2}(-0.1) - \frac{1}{8}(-0.1)^2 + \frac{1}{16}(-0.1)^3 = \boxed{0.9486875}.$$

To determine the error bound, we compute

$$f^{(4)}(x) = -\frac{15}{16}x^{-\frac{7}{2}}.$$

On $[0.9, 1]$, the value of $|f^{(4)}(x)|$ is maximized at $x = 0.9$, giving

$$|f^{(4)}(0.9)| = \frac{15}{16(0.9)^{\frac{7}{2}}} = \frac{625\sqrt{5}}{729\sqrt{2}} \approx 1.3556\ldots,$$

and hence the absolute value of the error is at most

$$\left|\frac{f^{(4)}(0.9)}{4!}(-0.1)^4\right| \approx 0.000005648\ldots.$$

Indeed, a calculator will say $\sqrt{0.9} = 0.948683298\ldots$, so our cubic polynomial estimate is accurate to about 0.000005.

7.5.2 We can write the cubic Taylor polynomial of $f(x) = (1+x)^p$ about $x = 0$. The relevant derivatives are:

$$\begin{aligned} f(0) &= 1, \\ f'(x) &= p(1+x)^{p-1}, \quad f'(0) = p, \\ f''(x) &= p(p-1)(1+x)^{p-2}, \quad f''(0) = p(p-1), \\ f'''(x) &= p(p-1)(p-2)(1+x)^{p-3}, \quad f'''(0) = p(p-1)(p-2). \end{aligned}$$

So the polynomial is

$$p(x) = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3,$$

and hence

$$(1+\epsilon)^{\frac{1}{n}} \approx \boxed{1 + \frac{\epsilon}{n} + \frac{\frac{1}{n}(\frac{1}{n}-1)}{2}\epsilon^2 + \frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)}{6}\epsilon^3}.$$

7.5.3 If $f(x) = e^x$, then all derivatives are also equal to e^x, so $f^{(n)}(0) = 1$ for all n. Therefore, the Taylor polynomial is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

where we can carry the polynomial to as many terms as we need. For 6 decimal places, we note that $(0.1)^5 = 0.00001$ and $5! = 120$, so the quintic term when $x = 0.1$ will be approximately $0.00000008\ldots$ and should not affect the first 6 decimal places. So we can use a quartic polynomial, and we get

$$e^{0.1} \approx 1 + 0.1 + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!},$$

which is $\boxed{1.105171}$ (to 6 decimal places). (And indeed a calculator will confirm that this is accurate to 6 decimal places.)

7.5.4 Note that $p'(x) = 2ax + b$ and $p''(x) = 2a$.

(a) We have $p(0) = c$, $p'(0) = b$, and $p''(0) = 2a$. Thus the Taylor polynomial is

$$c + bx + \frac{2a}{2}x^2 = \boxed{ax^2 + bx + c}.$$

Of course, this is exactly the same as the original quadratic $p(x)$.

(b) We have $p(1) = a + b + c$, $p'(1) = 2a + b$, and $p''(1) = 2a$, thus the Taylor polynomial is

$$a + b + c + (2a + b)(x - 1) + \frac{2a}{2}(x - 1)^2 = a + b + c + 2ax + bx - 2a - b + ax^2 - 2ax + a = \boxed{ax^2 + bx + c}.$$

Again, this is just $p(x)$.

(c) We have $p(r) = ar^2 + br + c$, $p'(r) = 2ar + b$, and $p''(r) = 2a$, thus the Taylor polynomial is

$$ar^2 + br + c + (2ar + b)(x - r) + a(x - r)^2.$$

Expanding gives

$$ar^2 + br + c + 2arx - 2ar^2 + bx - br + ax^2 - 2arx + ar^2 = \boxed{ax^2 + bx + c},$$

which is just $p(x)$, as before. In fact, note that we never used the fact that r is a root, so we will get $p(x)$ as our quadratic Taylor polynomial at any value of x.

Exercises for Section 7.6

7.6.1

(a) By putting $-x$ into the Taylor polynomial for e^x, we compute

$$\boxed{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots}.$$

This converges for all real x by the Ratio Test.

(b) Replace every x with x^2 in the Taylor polynomial for $\sin x$. This gives

$$\boxed{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots}.$$

This converges for all real x by the Ratio Test.

(c) Replace every x with x^2 in the answer from (a) and then multiply by x. The result is

$$\boxed{x - x^3 + \frac{x^5}{2!} - \frac{x^7}{3!} + \cdots}.$$

This converges for all real x by the Ratio Test.

(d) Let $f(x) = (x+1)^{\frac{1}{2}}$. Then

$$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}},$$
$$f''(x) = -\frac{1}{4}(x+1)^{-\frac{3}{2}},$$
$$f'''(x) = \frac{3}{8}(x+1)^{-\frac{5}{2}},$$

and thus $f'(2) = \frac{1}{2\sqrt{3}}$, $f''(2) = -\frac{1}{12\sqrt{3}}$, $f'''(2) = \frac{1}{24\sqrt{3}}$, and hence the Taylor series is

$$\boxed{\sqrt{3} + \frac{1}{2\sqrt{3}}(x-2) - \frac{1}{24\sqrt{3}}(x-2)^2 + \frac{1}{144\sqrt{3}}(x-2)^3 + \cdots}.$$

Let c_n be the coefficient of $(x-2)^n$ in this power series. Note that

$$|c_n| = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{2n-3}{2}}{n!}\cdot 3^{-\frac{2n-1}{2}},$$

so that

$$\left|\frac{c_n}{c_{n+1}}\right| = \frac{3(n+1)}{\frac{2n-1}{2}} = \frac{6n+6}{2n-1}.$$

As $n \to \infty$, this ratio approaches 3, and thus the radius of convergence is 3, so the series converges for $x \in (-1, 5)$. (The series may also converge at $x = -1$ and/or $x = 5$, but it is difficult to tell.)

(e) This is the sum of a geometric series with first term 1 and common ratio x^2, so the Taylor series is

$$\boxed{1 + x^2 + x^4 + x^6 + \cdots}.$$

This converges for all $|x| < 1$.

7.6.2 We have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Differentiating term-by-term gives

$$-x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots,$$

which is the Taylor series of $-\sin x$.

7.6.3

(a) The derivatives of f are

$$f(x) = e^x \sin x,$$
$$f'(x) = e^x(\sin x + \cos x),$$
$$f''(x) = e^x(2\cos x),$$
$$f'''(x) = e^x(2\cos x - 2\sin x),$$
$$f^{(4)}(x) = e^x(-4\sin x),$$
$$f^{(5)}(x) = e^x(-4\sin x - 4\cos x).$$

Setting $x = 0$ makes these $0, 1, 2, 2, 0, -4$ respectively. Using the formula for Taylor series, we get

$$e^x \sin x = x + \frac{2\cdot x^2}{2!} + \frac{2\cdot x^3}{3!} - \frac{4\cdot x^5}{5!} + \cdots$$
$$= \boxed{x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots}.$$

(b) The Taylor series for e^x and $\sin x$ are

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots, \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
&= 0 + x + 0x^2 - \frac{x^3}{3!} + 0x^4 + \cdots.
\end{aligned}$$

Multiplying these together for the first four terms,

$$e^x \sin x = x + x^2 + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{1}{3!} - \frac{1}{3!}\right)x^4 + \left(\frac{1}{5!} - \frac{1}{3!2!} + \frac{1}{4!}\right)x^5 + \cdots.$$

This simplifies to $\boxed{x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots}$, as computed in (a).

7.6.4 Recall that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Using this to rewrite the series gives

$$\frac{x^2}{1} - \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^3}{3} + \frac{x^4}{3} - \frac{x^4}{4} - \frac{x^5}{4} + \cdots = -x + \frac{x(1+x)}{1} - \frac{x^2(1+x)}{2} + \frac{x^3(1+x)}{3} - \frac{x^4(1+x)}{4} + \frac{x^5(1+x)}{5} - \cdots.$$

Recall that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

and that this converges when $|x| < 1$. Comparing this to our given series shows that the given series is

$$-x + (1+x)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) = \boxed{-x + (1+x)\log(1+x)}.$$

A simpler solution is to differentiate the original series, giving

$$f'(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

which we recognize as the power series of $\log(1+x)$. So our original series is an antiderivative of $\log(1+x)$, which is $(x+1)\log(x+1) - x + C$, and plugging in $x = 0$ verifies that $C = 0$.

Note that the sum of the absolute value of the terms of the power series,

$$\frac{|x|^2}{2} + \frac{|x|^3}{3\cdot 2} + \frac{|x|^4}{4\cdot 3} + \frac{|x|^5}{5\cdot 4} + \cdots,$$

converges for $|x| < 1$ by the Ratio Test. So the original series is absolutely convergent and there is no problem with reordering terms here.

Review Problems

7.36 Let $b_n = \log_2 a_n$ for all n. Then $b_1 = \frac{1}{2}$ and $b_n = \frac{1+b_{n-1}}{2}$ for all n. We claim that $b_n = 1 - \frac{1}{2^n}$ for all n, which we can show by induction. For the base case, $b_1 = 1 - \frac{1}{2} = 1 - \frac{1}{2^1}$ as desired, and for the inductive step, if $b_{n-1} = 1 - \frac{1}{2^{n-1}}$, then

$$b_n = \frac{1}{2}(1 + b_{n-1}) = \frac{1}{2}\left(1 + 1 - \frac{1}{2^{n-1}}\right) = 1 - \frac{1}{2^n},$$

as desired. Therefore, we have $b_n \to 1$, so by the continuity of $f(x) = \log_2 x$, we conclude that $a_n \to 2^1 = \boxed{2}$.

7.37

(a) This series has positive terms for $k > 1$, and we can compare it to $\sum \frac{1}{k^2}$ using the Limit Comparison Test. We have

$$\lim_{k\to\infty} \frac{\frac{2k-1}{3k^3+1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{2k^3-k^2}{3k^3+1} = \frac{2}{3}.$$

Since this is nonzero, we conclude that since the series $\sum \frac{1}{k^2}$ converges, so does our given series.

(b) This is the sum of two geometric series, one with ratio $\frac{3}{7}$ and one with ratio $\frac{4}{7}$. Thus the series converges to

$$\frac{\frac{3}{7}}{1-\frac{3}{7}} + \frac{\frac{4}{7}}{1-\frac{4}{7}} = \frac{3}{4} + \frac{4}{3} = \frac{25}{12}.$$

(c) This is an alternating series. The absolute value of the k^{th} term is $\frac{2^k}{3^k-1}$, which is decreasing for $k > 1$ and converges to 0, so the series converges conditionally. Furthermore, via the Limit Comparison Test with the geometric series with ratio $\frac{2}{3}$, we see that the series is absolutely convergent.

(d) We apply the Ratio Test:

$$\lim_{k\to\infty} \frac{\frac{(k+1)^2 3^{k+1}}{(k+1)!}}{\frac{k^2 3^k}{k!}} = \lim_{k\to\infty} \frac{3(k+1)^2}{k^2(k+1)} = 0.$$

Thus, the series converges.

(e) This is a p-series with $p = e > 1$, so it converges.

7.38

(a) We plug in $-x^2$ to the Taylor series of e^x:

$$e^{-x} = \boxed{1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots}.$$

(b) We plug in x^3 to the Taylor series of $\cos x$:

$$\cos x^3 = \boxed{1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots}.$$

(c) The Taylor series for e^{x^2} is

$$1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots.$$

We get the Taylor series for $e^{\sin x}$ by plugging in the power series for $\sin x$ into the Taylor series of e^x:

$$1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) + \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^2}{2!} + \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^3}{3!} + \cdots.$$

We can simplify this and collect terms:

$$1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right) + \left(\frac{x^2}{2} - \frac{x^4}{6} + \cdots\right) + \left(\frac{x^3}{6} - \frac{x^5}{12} + \cdots\right)$$

which gives

$$1 + x + \frac{x^2}{2} - \frac{x^4}{6} + \cdots.$$

(The higher degree terms won't matter as we only need to compute the first four terms.) Now we multiply e^{x^2} by $e^{\sin x}$:

$$e^{x^2+\sin x} = \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots\right)\left(1 + x + \frac{x^2}{2} - \frac{x^4}{6} + \cdots\right) = \boxed{1 + x + \frac{3}{2}x^2 + x^3 + \cdots}.$$

You could also solve this by taking the various derivatives of $e^{x^2+\sin x}$.

(d) We can start with the Taylor series of $\dfrac{1}{\sqrt{1+y}} = (1+y)^{-\frac{1}{2}}$:

$$\begin{aligned}(1+y)^{-\frac{1}{2}} &= 1 - \frac{1}{2}y + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}y^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}y^3 + \cdots \\ &= 1 - \frac{1}{2}y + \frac{3}{8}y^2 - \frac{5}{16}y^3 + \cdots,\end{aligned}$$

and then substitute $y = -4x^2$:

$$\frac{1}{\sqrt{1-4x^2}} = \boxed{1 + 2x^2 + 6x^4 + 20x^6 + \cdots}.$$

As a check, we can plug in $x = \frac{1}{4}$ and see if it's close:

$$\frac{1}{\sqrt{1-4\left(\frac{1}{4}\right)^2}} = \frac{1}{\sqrt{\frac{3}{4}}} = \sqrt{\frac{4}{3}} \approx 1.1547$$

$$1 + 2\left(\frac{1}{4}\right)^2 + 6\left(\frac{1}{4}\right)^4 + 20\left(\frac{1}{4}\right)^6 \approx 1 + 0.125 + 0.0234 + 0.0049 \approx 1.1533,$$

so that's pretty close. (The next term is $70x^8$, which adds 0.0011 to our approximation, giving 1.1544, and now it's very close.)

7.39 We note that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Substituting $x = 2$ gives

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!},$$

thus

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{n!} = \boxed{\frac{1}{2}e^2}.$$

7.40 The power series expansion for $\sin x^2$ is

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots.$$

Integrating, this becomes

$$g(x) = C + \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots,$$

for some C. By the Fundamental Theorem of Calculus it suffices to find $g(1) - g(0)$. When we do this, C will cancel, and

$$g(1) - g(0) = \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \cdots.$$

The fourth term is far less than 0.001, so we take the first three to get $\boxed{0.310}$.

7.41 Suppose $p > q$. Then we have

$$\lim_{n\to\infty} \sqrt[n]{p^n + q^n} = \lim_{n\to\infty} p\sqrt[n]{1 + \left(\frac{q}{p}\right)^n} = p \lim_{n\to\infty} \sqrt[n]{1 + \left(\frac{q}{p}\right)^n} = p,$$

since $\lim_{n\to\infty} \sqrt[n]{1+c^n} = 1$ if $0 \le c < 1$. Similarly, if $p < q$, then the limit is q. If $p = q$, then we have

$$\lim_{n\to\infty} \sqrt[n]{2p^n} = p \lim_{n\to\infty} \sqrt[n]{2} = p.$$

This can all be summarized as

$$\lim_{n\to\infty} \sqrt[n]{p^n + q^n} = \boxed{\max\{p,q\}},$$

which makes intuitive sense, as the larger term "dominates" the smaller term.

7.42 We can use the Ratio Test. We compute the limit of $\frac{a_{n+1}}{a_n}$, where a_n is the n^{th} term of the series:

$$L = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(c(n+1))^{n+1}}}{\frac{n!}{(cn)^n}} = \lim_{n\to\infty} \frac{(n+1)(n^n)}{c(n+1)^{n+1}} = \lim_{n\to\infty} \frac{1}{c}\left(\frac{n}{n+1}\right)^n.$$

But the last term of this limit is

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{1}{\frac{n+1}{n}}\right)^n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e}.$$

Thus we get $L = \frac{1}{ce}$. The series converges if $L < 1$, which occurs if $c > \frac{1}{e}$. It diverges if $L > 1$, which occurs if $c < \frac{1}{e}$.

7.43 We note that $f(x)$ is a geometric series with ratio $\frac{x}{2}$, so

$$f(x) = \frac{1}{1 - \frac{x}{2}} = \frac{2}{2-x},$$

with convergence if $-2 < x < 2$. The interval $[0,1]$ is within the radius of convergence, thus

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{2}{2-x}\,dx = -2\log(2-x)\Big|_0^1 = -2(\log 1 - \log 2) = 2\log 2.$$

Finally,

$$\sqrt{e^{\int_0^1 f(x)\,dx}} = \sqrt{e^{2\log 2}} = e^{\log 2} = \boxed{2}.$$

Challenge Problems

7.44 Let $b_k = \sum_{i=2^{k-1}}^{2^k-1} a_i$ for all $k \ge 1$. Note that $b_1 = a_1 > 0$, and the given inequality implies that $b_k \ge b_{k-1}$ for all $k > 1$. For example,

$$b_3 = a_4 + a_5 + a_6 + a_7 \le (a_8 + a_9) + (a_{10} + a_{11}) + (a_{12} + a_{13}) + (a_{14} + a_{15}) = b_4.$$

Thus the b_k's do not converge to 0 but rather increase, so $\sum_{k=1}^{\infty} b_k$ diverges. But $\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} b_k$, so the series $\sum_{n=1}^{\infty} a_n$ diverges as well.

7.45

(a) Let $f(x) = \frac{1}{x}$. Then since $f(x)$ is decreasing, the Riemann sum

$$f(1) + f(2) + \cdots + f(n) = \sum_{i=1}^{n} \frac{1}{i}$$

is strictly larger than the definite integral $\int_1^{n+1} f(x)\,dx = \log(n+1) > \log n$. Thus all the terms of $\{a_n\}$ are positive.

For the other inequality, we must prove that

$$0 < a_n - a_{n+1} = \log(n+1) - \log(n) - \frac{1}{n+1}.$$

But

$$\log(n+1) - \log(n) = \int_n^{n+1} \frac{1}{x}\,dx > \int_n^{n+1} \frac{1}{n+1}\,dx = \frac{1}{n+1},$$

so the inequality holds.

(b) Since $\{a_n\}$ is monotone decreasing and bounded below (by 0), it must converge, and in fact it converges to $\inf\{a_n\}$.

7.46 This looks more like a Riemann sum if we factor out $\frac{1}{n}$:

$$a_n = \frac{1}{n}\left(\frac{1}{\sqrt{1-\left(\frac{0}{n}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{1}{n}\right)^2}} + \cdots + \frac{1}{\sqrt{1-\left(\frac{n-1}{n}\right)^2}}\right) = \frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\sqrt{1-\left(\frac{i}{n}\right)^2}}.$$

Now it *is* a Riemann sum! Taking the limit as $n \to \infty$ gives

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx.$$

(The terms of the Riemann sum are the areas of boxes of width $\frac{1}{n}$ and height $f\left(\frac{i}{n}\right)$, where $f(x) = \frac{1}{\sqrt{1-x^2}}$.) So to evaluate the limit, we just need to compute this integral. Fortunately, this is easy:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x\Big|_0^1 = \sin^{-1} 1 - \sin^{-1} 0 = \boxed{\frac{\pi}{2}}.$$

7.47

(a) The sequence $\{a_n\}$ is clearly nondecreasing, and so $a_n = a_{n-1} + a_{n-2} \le 2a_{n-1}$. Now we apply the Ratio Test to $f(x)$:

$$\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| \le |2x|.$$

So if $|x| < \frac{1}{2}$, the series converges, as desired.

(b) Note that

$$\begin{array}{rcccccccccccc} f(x) = & 1 & + & x & + & 2x^2 & + & 3x^3 & + & 5x^4 & + & 8x^5 & + \cdots \\ xf(x) = & & & x & + & x^2 & + & 2x^3 & + & 3x^4 & + & 5x^5 & + \cdots \\ x^2 f(x) = & & & & & x^2 & + & x^3 & + & 2x^4 & + & 3x^5 & + \cdots \end{array}$$

If we subtract the second and third equations from the first, all the terms with x to a positive power will vanish by the recursive formula $a_n = a_{n-1} + a_{n-2}$. Therefore $(1 - x - x^2)f(x) = 1$ and $f(x) = \frac{1}{1-x-x^2}$. Since we showed the series converges for $|x| < \frac{1}{2}$ in (a), the series is identically equal to $\frac{1}{1-x-x^2}$ for $|x| < \frac{1}{2}$, as desired.

7.48 For a nonnegative integer k and real p, we say

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k!}.$$

Then the Binomial Theorem expression makes sense as the Taylor series of $(1+x)^p$ about $x = 0$. Notice that if $f(x) = (1+x)^p$, then $f^{(k)}(x) = p(p-1)\cdots(p-k+1)(1+x)^{p-k}$. Putting in $x = 0$ and dividing by $k!$ gives $\binom{p}{k}$, as desired.

By using the Ratio Test, we can verify that if p is not a nonnegative integer, then the expression converges only when $|x| < 1$. Note that $a_k = |p(p-1)\cdots(p-k+1)|$ is an unbounded sequence for any p that is not a nonnegative integer.

7.49 Let's first try it for $n = 2$. In this case, $C(-t-1)$ is the coefficient of x^2 in $(1+x)^{-t-1}$, which is $\frac{(-t-1)(-t-2)}{2} = \frac{(t+1)(t+2)}{2}$. So the integral is

$$\int_0^1 \left(\frac{(t+1)(t+2)}{2}\left(\frac{1}{t+1} + \frac{1}{t+2}\right)\right) dt.$$

This easily simplifies:

$$\frac{1}{2}\int_0^1 ((t+2) + (t+1))\, dt.$$

We can evaluate this directly very easily, but the "trick" here is to rewrite it as

$$\frac{1}{2}\int_0^1 \frac{d}{dt}((t+2)(t+1))\, dt = \frac{1}{2}\left.(t+2)(t+1)\right|_0^1 = \frac{1}{2}(6-2) = 2.$$

Let's now see the general case. We compute

$$C(-t-1) = \frac{(-t-1)(-t-2)(-t-3)\cdots(-t-n)}{n!} = (-1)^n\frac{(t+1)(t+2)\cdots(t+n)}{n!}.$$

So the integral becomes

$$\int_0^1 (-1)^n\frac{(t+1)(t+2)\cdots(t+n)}{n!}\left(\frac{1}{t+1} + \cdots + \frac{1}{t+n}\right) dt.$$

Rewrite the integrand as

$$\frac{(-1)^n}{n!}\int_0^1 \frac{d}{dt}((t+1)(t+2)\cdots(t+n))\, dt.$$

Evaluating this, we get

$$\frac{(-1)^n}{n!}\left.((t+1)(t+2)\cdots(t+n))\right|_0^1 = \frac{(-1)^n}{n!}((n+1)! - n!) = (-1)^n((n+1)-1) = (-1)^n n.$$

7.50

(a) We proceed by induction. Note that the base case $n = 0$ works with $p_0 = 1$.

For the inductive step, let n be a positive integer and assume that $f^{(n-1)}$ is of the specified form. For nonzero x, we have

$$\begin{aligned} f^{(n)}(x) = \frac{d}{dx}f^{(n-1)}(x) &= \frac{d}{dx}\left(p_{n-1}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}\right) \\ &= -\frac{1}{x^2}p'_{n-1}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}} + p_{n-1}\left(\frac{1}{x}\right)\frac{2}{x^3}e^{-\frac{1}{x^2}} \\ &= \left(-\frac{1}{x^2}p'_{n-1}\left(\frac{1}{x}\right) + \frac{2}{x^3}p_{n-1}\left(\frac{1}{x}\right)\right)e^{-\frac{1}{x^2}}. \end{aligned}$$

We define p_n by $p_n(y) = -y^2p'_{n-1}(y) + 2y^3p_{n-1}(y)$, and we have

$$p_n\left(\frac{1}{x}\right) = -\frac{1}{x^2}p'_{n-1}\left(\frac{1}{x}\right) + \frac{2}{x^3}p_{n-1}\left(\frac{1}{x}\right),$$

giving the result.

At $x = 0$, we must apply the definition of derivative:

$$f^{(n)}(0) = \lim_{h\to 0}\frac{f^{(n-1)}(h) - f^{(n-1)}(0)}{h} = \lim_{h\to 0}\frac{1}{h}p_{n-1}\left(\frac{1}{h}\right)e^{-\frac{1}{h^2}}.$$

This looks like a good place for the substitution $t = \frac{1}{h}$. However in this case we need to consider both the limit at ∞ and at $-\infty$.

$$\lim_{t\to\infty} tp_{n-1}(t)e^{-t^2} = \lim_{t\to\infty}\frac{tp_{n-1}(t)}{e^{t^2}} = 0,$$

where the limit is zero because exponentials dominate polynomials. Likewise the limit at $-\infty$ is zero, so $\lim_{h\to 0}\frac{1}{h}p_{n-1}\left(\frac{1}{h}\right)f(h) = 0$ for any polynomial p. Thus $f^{(n)}(0) = 0$, as required.

(b) The Taylor series for f is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{0}{n!}x^n = 0.$$

This Taylor series converges for all x; however, since $f(x)$ is always nonzero when $x \neq 0$, the Taylor series converges to the wrong value *for every* $x \neq 0$.

CHAPTER 8

Plane Curves

Exercises for Section 8.1

8.1.1 Note: because parameterizations are not unique, there are other possible answers.

(a) $u(t) = 1 + 2\cos t,\ v(t) = 1 + 2\sin t$.

(b) $u(t) = t\cos(\pi t/2),\ v(t) = t\sin(\pi t/2)$.

(c) $u(t) = 2\cos t + 2,\ v(t) = \sin t - 3$.

8.1.2 The position of the center of the circle is given by $(4\pi t, 2)$. Relative to the center of the circle, the point moves in a circle of radius 1, with y-coordinate $-\cos 2\pi t$ and x-coordinate $-\sin 2\pi t$. Then the position of the point with translation and rotation of the circle accounted for is

$$(u(t), v(t)) = (4\pi t - \sin 2\pi t, 2 - \cos 2\pi t).$$

To determine the speed, we need to compute $S(t) = \sqrt{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2}$. We find that

$$\left(\frac{du}{dt}, \frac{dv}{dt}\right) = (4\pi - 2\pi\cos 2\pi t, 2\pi\sin 2\pi t).$$

Then

$$\begin{aligned} S(t) &= 2\pi\sqrt{4 - 4\cos 2\pi t + \cos^2 2\pi t + \sin^2 2\pi t}, \\ &= \boxed{2\pi\sqrt{5 - 4\cos 2\pi t}}. \end{aligned}$$

Notice that for no t is this quantity 0. So the cycloid has no "cusps" as found on the cycloid traced by a point on the circumference of the circle.

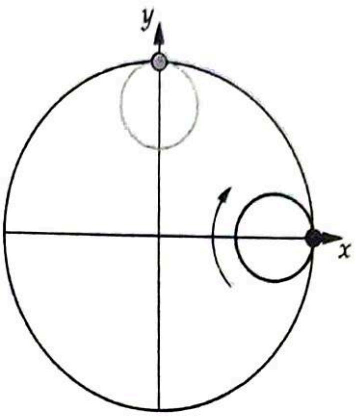
8.1.3

(a) Taking the derivatives of each component gives $(-\sin t, 1 + \cos t)$. The slope of the tangent line is then $\frac{1+\cos t}{-\sin t}$. Plugging in $t = \frac{\pi}{2}$ gives a slope of $\frac{1+0}{-1} = \boxed{-1}$.

(b) If $u(t) = \cos t$ and $v(t) = t + \sin t$, then the length for $t \in [0, \pi]$ equals the integral $\int_0^\pi \sqrt{(u'(t))^2 + (v'(t))^2}\,dt$. Plugging in $u'(t), v'(t)$ from computations in (a) gives that the length is

$$\int_0^\pi \sqrt{\sin^2 t + \cos^2 t + 2\cos t + 1}\,dt = \int_0^\pi 2\sqrt{\frac{1+\cos t}{2}}\,dt = \int_0^\pi 2\left|\cos\frac{t}{2}\right|dt = 4\sin\frac{\pi}{2} - 4\sin 0 = \boxed{4}.$$

8.1.4 We'll assume that the small circle takes 2π units of time to return to its original position, in order to attempt to match the parameterization of the astroid given in the text. There are two parts to the motion: the movement of the center of the small circle with respect to the origin, and the movement of the point in question about the center of the small circle.



The center of the small circle is always distance $\frac{3}{4}$ from the center of the large circle. Thus, the path traced by the center of the small circle is a circle of radius $\frac{3}{4}$ centered at $(0,0)$, and since the small circle takes 2π units of time to return to its original position, the path is parameterized by $\left(\frac{3}{4}\cos t, \frac{3}{4}\sin t\right)$.

The circumference of the small circle is $\frac{1}{4}$ that of the larger circle, so the point on the small circle will again touch the big circle after the small circle has moved $\frac{1}{4}$ of the way around the circle, as in the picture to the right. During the time it has taken the small circle to reach this new position, the point must have rotated $\frac{3}{4}$ of the way around the small circle, as shown in the picture. Thus, the small circle makes $4\cdot\frac{3}{4} = 3$ rotations during the 2π units of time that it takes for it to return to its original position. Therefore, noting that the small circle rotates in the clockwise direction, we have the parametric equations $\left(\frac{1}{4}\cos 3t, -\frac{1}{4}\sin 3t\right)$ for the movement of the point around the small circle.

Adding these two components gives the overall parameterization

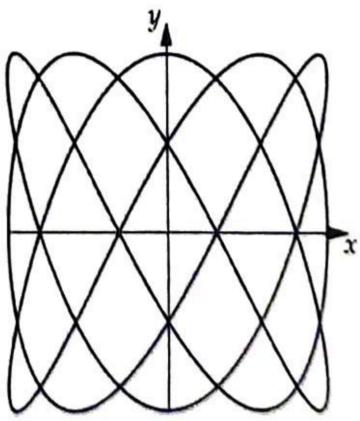
$$\boxed{\left(\frac{3}{4}\cos t + \frac{1}{4}\cos 3t, \frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right)}.$$

Using the triple angle formulas

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$

we see that the above parameterization is equal to $(\cos^3 t, \sin^3 t)$, which is the parameterization of the astroid as presented in the text.

8.1.5 Because of the trig functions, the curve lies entirely within $-1 \le x \le 1$ and $-1 \le y \le 1$, and repeats every 2π. Note that the curve starts at $(1,0)$ at $t = 0$.



Examining just the x-coordinate, we have 3 periods of the function $\cos 3t$ for $t \in [0, 2\pi]$, so the curve attains $x = 1$ three times (counting the start at $t = 0$ and end at $t = 2\pi$ as just one time), attains $x = -1$ three times, and crosses the y-axis six times. Similarly, examining just the y-coordinate, we have 5 periods of the function $\sin 5t$ for $t \in [0, 2\pi]$, so the curve attains $y = 1$ five times, attains $y = -1$ five times, and crosses the x-axis ten times (counting $t = 0$ and $t = 2\pi$ as one combined "crossing").

A sketch of the curve is shown at right. This curve is known as a *Lissajous curve*. More generally, we can consider the curve given by the parameterization $(\cos at, \sin bt)$, where a and b are nonzero constants. Note that $a = b$ gives a circle. If a and b are relatively prime, then the curve will attain $x = 1$ or $x = -1$ a total of a times each, and will attain $y = 1$ or $y = -1$ a total of b times each.

Exercises for Section 8.2

8.2.1

(a) Note that $r = 0$ when $0 = a\cos\theta + b\sin\theta$, or $-\frac{a}{b} = \tan\theta$. So the origin of the plane is on the graph.

If $r \neq 0$, then multiply by r to get $r^2 = ar\cos\theta + br\sin\theta$. Then, converting to rectangular coordinates, we have $x^2 + y^2 = ax + by$, which can be rearranged (after completing the square in both x and y) as

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}.$$

The graph of this is a circle centered at the rectangular point $\left(\frac{a}{2}, \frac{b}{2}\right)$ with radius $\frac{\sqrt{a^2+b^2}}{2}$. (Note that this graph includes the origin.)

For the slope of the tangent line at $\theta = 0$, it is easiest to implicitly differentiate the rectangular equation, noting that $\theta = 0$ gives the rectangular point $(a, 0)$. Differentiating gives

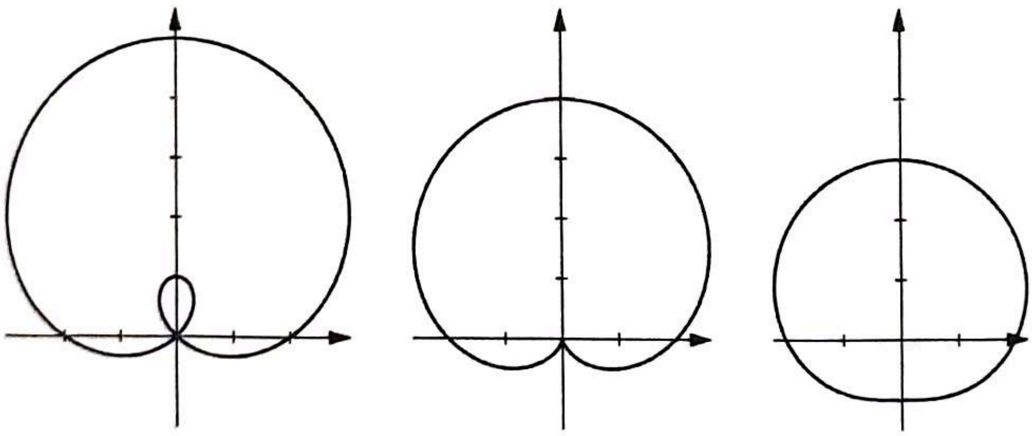
$$2\left(x - \frac{a}{2}\right) + 2\left(y - \frac{b}{2}\right) \cdot \frac{dy}{dx} = 0.$$

Plugging in $x = a$ and $y = 0$ gives $a - b\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = \boxed{\frac{a}{b}}$.

(b) Note that, as θ increases from 0 to 2π, the graph passes through the following points (in polar coordinates):

$$(a, 0),\ \left(a + b, \frac{\pi}{2}\right),\ (a, \pi),\ \left(a - b, \frac{3\pi}{2}\right),\ (a, 2\pi).$$

The nature of the graph will vary depending on whether $a - b$ is positive, zero, or negative. We sketch three graphs below, all using $a = 2$. The first graph uses $b = 3$, so r is negative for some values of θ. The second uses $b = 2$ and is a cardioid. The third uses $b = 1$, so r is positive throughout. The curves are collectively known as **limaçons**.



$r = 2 + 3\sin\theta$ $\quad$ $r = 2 + 2\sin\theta$ $\quad$ $r = 2 + \sin\theta$

For the slope of the tangent line, we note that $x = r\cos\theta = (a + b\sin\theta)(\cos\theta)$, so

$$\frac{dx}{d\theta} = (a + b\sin\theta)(-\sin\theta) + (b\cos\theta)(\cos\theta) = -a\sin\theta + b(\cos^2\theta - \sin^2\theta) = -a\sin\theta + b\cos 2\theta.$$

Similarly, $y = r\sin\theta = (a + b\sin\theta)(\sin\theta)$, so

$$\frac{dy}{d\theta} = (a + b\sin\theta)(\cos\theta) + (b\cos\theta)(\sin\theta) = a\cos\theta + 2b\sin\theta\cos\theta = a\cos\theta + b\sin 2\theta.$$

Thus,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta + b\sin 2\theta}{-a\sin\theta + b\cos 2\theta},$$

and at $\theta = \frac{\pi}{2}$, this slope equals

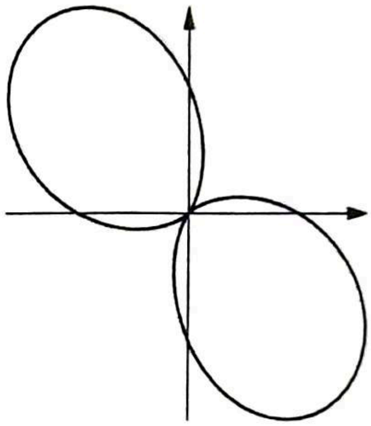
$$\frac{a\cos\frac{\pi}{2} + b\sin\pi}{-a\sin\frac{\pi}{2} + b\cos\pi} = \frac{0}{-(a+b)} = \boxed{0}.$$

Note that this answer is consistent with our pictures above: all of the graphs cross the positive y-axis at a point at which the graph has a tangent line of slope 0; that is, the tangent line at the positive y-intercept is horizontal.

(c) The curve passes through the following points (r, θ) in polar coordinates:

$$(1,0), \left(0, \frac{\pi}{4}\right), \left(1, \frac{\pi}{2}\right), \left(2, \frac{3\pi}{4}\right), (1, \pi).$$

Also note that if (r, θ) is on the curve, then so is $(r, \theta + \pi)$, thus the curve is symmetric under rotation by π (about the origin).



The graph of the curve is shown at right. To graph the correct behavior of the curve at the origin, it is useful to know the slope of the tangent line at the origin, which occurs when $\theta = \frac{\pi}{4}$.

We have $x = r\cos\theta = (1 - \sin 2\theta)(\cos\theta)$, so

$$\frac{dx}{d\theta} = (1 - \sin 2\theta)(-\sin\theta) + (-2\cos 2\theta)(\cos\theta),$$

and we have $y = r\sin\theta = (1 - \sin 2\theta)(\sin\theta)$, so

$$\frac{dy}{d\theta} = (1 - \sin 2\theta)(\cos\theta) + (-2\cos 2\theta)(\sin\theta).$$

Thus, the slope of the tangent line at the point corresponding to some value of θ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1 - \sin 2\theta)(\cos\theta) + (-2\cos 2\theta)(\sin\theta)}{(1 - \sin 2\theta)(-\sin\theta) + (-2\cos 2\theta)(\cos\theta)}.$$

However, at $\theta = \frac{\pi}{4}$, we have $\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0$, so the above expression for $\frac{dy}{dx}$ is undefined, and instead we must compute

$$\lim_{\theta \to \frac{\pi}{4}} \frac{dy/d\theta}{dx/d\theta},$$

which we can compute using l'Hôpital's Rule. We compute

$$\begin{aligned} \frac{d}{d\theta}\frac{dx}{d\theta} &= (1 - \sin 2\theta)(-\cos\theta) + (-2\cos 2\theta)(-\sin\theta) + (-2\cos 2\theta)(-\sin\theta) + (4\sin 2\theta)(\cos\theta) \\ &= (1 - \sin 2\theta)(-\cos\theta) + 4\cos 2\theta\sin\theta + 4\sin 2\theta\cos\theta, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{d\theta}\frac{dy}{d\theta} &= (1 - \sin 2\theta)(-\sin\theta) + (-2\cos 2\theta)(\cos\theta) + (-2\cos 2\theta)(\cos\theta) + (4\sin 2\theta)(\sin\theta) \\ &= (1 - \sin 2\theta)(-\sin\theta) - 4\cos 2\theta\cos\theta + 4\sin 2\theta\sin\theta. \end{aligned}$$

Then

$$\lim_{\theta \to \frac{\pi}{4}} \frac{dy/d\theta}{dx/d\theta} = \lim_{\theta \to \frac{\pi}{4}} \frac{d^2y/d\theta^2}{d^2x/d\theta^2} = \frac{4\sin\frac{\pi}{2}\cos\frac{\pi}{4}}{4\sin\frac{\pi}{2}\sin\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1,$$

so the slope of the tangent line at the origin is $\boxed{1}$.

8.2.2 Using the fact that (r_0, θ_0) has rectangular coordinates $(r_0\cos\theta_0, r_0\sin\theta_0)$, we can use the formula in the text to get:

$$r = r_0\left(\frac{\sin\theta_0 - m\cos\theta_0}{\sin\theta - m\cos\theta}\right).$$

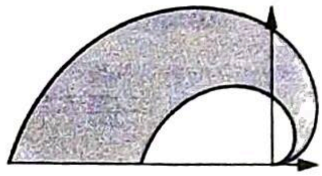
This can be more naturally expressed by letting $\psi = \arctan m$. Then our equation becomes

$$r = r_0\left(\frac{\cos\psi\sin\theta_0 - \sin\psi\cos\theta_0}{\cos\psi\sin\theta - \sin\psi\cos\theta}\right),$$

and then applying the angle-subtraction formula for sine, we have

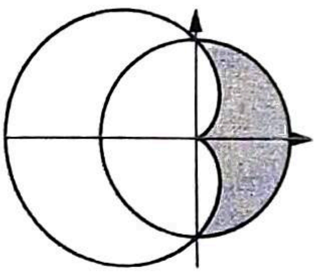
$$r = r_0 \cdot \frac{\sin(\theta_0 - \psi)}{\sin(\theta - \psi)}.$$

Exercises for Section 8.3



8.3.1 The graphs and the desired region are shown at right. The area inside the outer spiral is $\frac{1}{2}\int_0^{\pi}(2\theta)^2\,d\theta$. The area inside the inner spiral is $\frac{1}{2}\int_0^{\pi}\theta^2\,d\theta$. Thus, the desired area is the difference:

$$\frac{1}{2}\int_0^{\pi}3\theta^2\,d\theta = \frac{1}{2}\theta^3\Big|_0^{\pi} = \boxed{\frac{1}{2}\pi^3}.$$

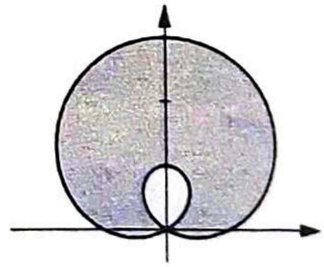


8.3.2 The graphs of the cardioid and the circle are shown at right, with the desired region shaded. The cardioid crosses $r = 1$ at $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$. Thus we want

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2}((1)^2 - (1-\cos\theta)^2)\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}(2\cos\theta - \cos^2\theta)\,d\theta.$$

Since the regions from $-\frac{\pi}{2}$ to 0 and from 0 to $\frac{\pi}{2}$ have equal area, this area is just

$$\int_0^{\pi/2}(2\cos\theta - \cos^2\theta)\,d\theta = \int_0^{\pi/2}\left(2\cos\theta - \frac{1}{2}(1+\cos 2\theta)\right)d\theta = \left(2\sin\theta - \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right)\Big|_0^{\pi/2} = \boxed{2 - \frac{\pi}{4}}.$$



8.3.3 The outer boundary of the desired region is the part of the limaçon where $r \geq 0$, which occurs when $2\sin\theta \geq -1$, or $\sin\theta \geq -\frac{1}{2}$. This occurs for $-\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}$. Thus, the entire area inside the limaçon—including both the shaded and unshaded regions—is given by

$$\frac{1}{2}\int_{-\pi/6}^{7\pi/6}(1+2\sin\theta)^2\,d\theta.$$

To exclude the unshaded region, we subtract its area, which is the region from $7\pi/6$ to $11\pi/6$, giving the area of the shaded region as

$$\frac{1}{2}\int_{-\pi/6}^{7\pi/6}(1+2\sin\theta)^2\,d\theta - \frac{1}{2}\int_{7\pi/6}^{11\pi/6}(1+2\sin\theta)^2\,d\theta.$$

Noting that the combined intervals of the integrals is $[-\pi/6, 11\pi/6]$, which is a full period for sine, gives us the idea to use a clever trick to make the computation a bit simpler. We note that for any function f and $a < c < b$,

$$\int_a^c f - \int_c^b f = \int_a^c f + \int_c^b f - 2\int_c^b f = \int_a^b f - 2\int_c^b f.$$

Using this makes our area equal to

$$\frac{1}{2}\int_{-\pi/6}^{11\pi/6}(1+2\sin\theta)^2\,d\theta - \int_{7\pi/6}^{11\pi/6}(1+2\sin\theta)^2\,d\theta.$$

The advantage of this is that, by the periodicity of the function $1+2\sin\theta$, the first integral is the same as the integral from 0 to 2π. Thus our area is

$$\frac{1}{2}\int_{0}^{2\pi}(1+2\sin\theta)^2\,d\theta - \int_{7\pi/6}^{11\pi/6}(1+2\sin\theta)^2\,d\theta.$$

We now note that

$$(1+2\sin\theta)^2 = 1+4\sin\theta+4\sin^2\theta = 1+4\sin\theta+(2-2\cos 2\theta) = 3+4\sin\theta-2\cos 2\theta,$$

which has antiderivative $3\theta - 4\cos\theta - \sin 2\theta$. Thus, our area evaluates to

$$\frac{1}{2}\left(3\theta-4\cos\theta-\sin 2\theta\right)\Big|_0^{2\pi} - \left(3\theta-4\cos\theta-\sin 2\theta\right)\Big|_{7\pi/6}^{11\pi/6}.$$

The first term is just 3π. In the second term, we have

$$\begin{aligned}\left(3\theta-4\cos\theta-\sin 2\theta\right)\Big|_{7\pi/6}^{11\pi/6} &= 3\left(\frac{11\pi}{6}-\frac{7\pi}{6}\right)-4\left(\cos\frac{11\pi}{6}-\cos\frac{7\pi}{6}\right)-\left(\sin\frac{11\pi}{3}-\sin\frac{7\pi}{3}\right)\\ &= 2\pi-4\sqrt{3}-(-\sqrt{3}) = 2\pi-3\sqrt{3}.\end{aligned}$$

Thus, the area is $3\pi-(2\pi-3\sqrt{3}) = \boxed{\pi+3\sqrt{3}}$.

Review Problems

8.21

(a) Constant speed 1 means that we make a revolution in time 4π, so the parameterization is $\boxed{\left(2\cos\frac{t}{2}, 2\sin\frac{t}{2}\right)}$.

(b) The horizontal movement of the cannonball is with constant velocity $100\cos(\pi/3) = 50$, so the horizontal position at time t is $50t$. The vertical movement is $100t\sin(\pi/3)+\frac{1}{2}gt^2 = 50\sqrt{3}t-4.9t^2$. So the parameterization is

$$\boxed{(50t, 50\sqrt{3}t-4.9t^2)}.$$

(c) The wheel itself is moving forward at a rate of 0.6π meters per second (as it travels one circumference each second). So the center of the wheel is given by the parameterization $(0.6\pi t, 0.3)$.

The point on the edge, relative to the center of the wheel, is given by the parameterization

$$(-0.3\sin 2\pi t, -0.3\cos 2\pi t).$$

(Note how this parameterization takes into account the clockwise motion, the starting position of $(0,-0.3)$, and the fact that the period of rotation is 1 second.)

Thus, the overall position of the point is the sum of these two parameterizations:

$$\boxed{(0.6\pi t-0.3\sin 2\pi t, 0.3-0.3\cos 2\pi t)}.$$

(d) The parameterization will be of the form $(2\cos f(t), -2\sin f(t))$ for some increasing function f. The speed at time t is

$$\begin{aligned}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} &= \sqrt{(-2f'(t)\sin f(t))^2+(-2f'(t)\cos f(t))^2}\\ &= 2f'(t)\sqrt{\sin^2 f(t)+\cos^2 f(t)}\\ &= 2f'(t).\end{aligned}$$

So $2f'(t) = \sqrt{t}$, hence $f(t) = \frac{1}{3}\sqrt{t^3}$. (Note $f(0) = 0$ as required to get the proper starting point.) Thus the parameterization is

$$\boxed{\left(2\cos\left(\frac{1}{3}\sqrt{t^3}\right), -2\sin\left(\frac{1}{3}\sqrt{t^3}\right)\right)}.$$

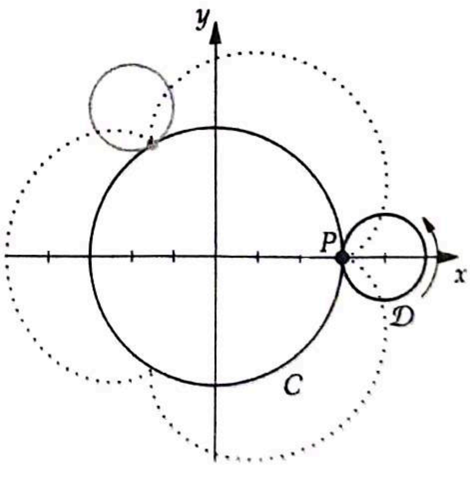
8.22 The point of tangency of the string moves around the circle at 2π radians per second. First, we compute the position of the point of tangency of the string with the bobbin. Because this is simply a revolution around a circle of radius 10, the parameterization of the point of tangency is $(10\cos 2\pi t, 10\sin 2\pi t)$.

Now we compute the position of the end of the string relative to the point of tangency with the bobbin. After t seconds have passed, the point of tangency has moved a length of $20\pi t$ around the bobbin. Thus, a length of $20\pi t$ of string has been unwound. The direction that the string points is tangent to the bobbin. By computing the derivatives of the components in $(10\cos 2\pi t, 10\sin 2\pi t)$, we realize that the direction the string points in is the direction of $(\sin 2\pi t, -\cos 2\pi t)$. Thus, the position of the end of the string relative to the point of tangency with the bobbin is parameterized by $(20\pi t\sin 2\pi t, -20\pi t\cos 2\pi t)$.

Adding these two together gives the parameterization

$$\boxed{(10\cos 2\pi t + 20\pi t\sin 2\pi t, 10\sin 2\pi t - 20\pi t\cos 2\pi t)}.$$

8.23 The starting position is shown in black in the picture to the right. There are two components to the motion of P: the rotation of $\mathcal{D}$ around C, and the rotation of P around $\mathcal{D}$.



The center of $\mathcal{D}$ rotates around C along a path that is a circle of radius 4 centered at $(0,0)$. Since it starts at $(4,0)$ and takes 1 unit of time to make 1 counterclockwise rotation, the parameterization of the motion of the center of $\mathcal{D}$ is given by $(4\cos 2\pi t, 4\sin 2\pi t)$.

The point P rotates counterclockwise around $\mathcal{D}$. Looking at the picture, since the circumference of C is 3 times that of $\mathcal{D}$, we see that P will again be tangent to C after $\mathcal{D}$ has moved $\frac{1}{3}$ of the way around C, as shown by the grey circle in the upper-left of the picture. During the time it takes for $\mathcal{D}$ to get from its starting position (in black) to this new position (in grey), we see that P has rotated a total of $\frac{4}{3}$ of the way around $\mathcal{D}$. Thus P makes $\frac{4}{3}$ rotations around $\mathcal{D}$ in $\frac{1}{3}$ second, and hence makes 1 rotation in $\frac{1}{4}$ second, or 4 rotations in 1 second. Thus, the parameterization of P relative to the center of $\mathcal{D}$ is given by $(-\cos 8\pi t, -\sin 8\pi t)$. (The minus signs in this parameterization are due to the fact that P starts at $(-1,0)$ relative to the center of $\mathcal{D}$.)

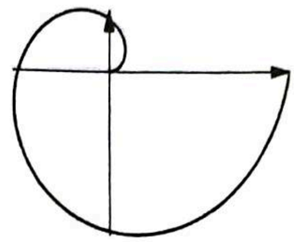
Adding the two parameterizations, we get

$$\boxed{(4\cos 2\pi t - \cos 8\pi t, 4\sin 2\pi t - \sin 8\pi t)}$$

as a parameterization of the epicycloid. The epicycloid is shown by the dotted line in the picture.

8.24

(a) Shown at right is the graph for θ from 0 to 2π.



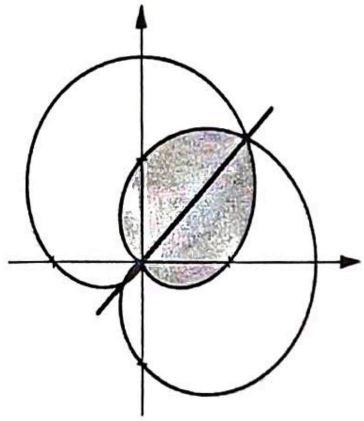
(b) Note that $x = r\cos\theta = \theta\cos\theta$, so $\frac{dx}{d\theta} = -\theta\sin\theta + \cos\theta$. Similarly, $y = r\sin\theta = \theta\sin\theta$, so $\frac{dy}{d\theta} = \theta\cos\theta + \sin\theta$. Thus,

$$\frac{dy}{dx} = \frac{\theta\cos\theta + \sin\theta}{-\theta\sin\theta + \cos\theta}.$$

(c) We must integrate between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Thus we have

$$\frac{1}{2}\int_{\pi/2}^{3\pi/2} \theta^2\,d\theta = \frac{1}{6}\,\theta^3\Big|_{\pi/2}^{3\pi/2} = \frac{1}{6}\left(\frac{27\pi^3}{8} - \frac{\pi^3}{8}\right) = \boxed{\frac{13\pi^3}{24}}.$$

8.25 We see that the picture is symmetric about the line $x = y$, as shown. We can also see this algebraically: reflecting across $x = y$ is the same as replacing θ with $\frac{\pi}{2} - \theta$, and using the identity $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ establishes that the two cardioids are reflections of each other.



Thus, the area is just twice the area of shaded region below the line $y = x$. This half of the overall region is the region inside the cardioid $r = 1 + \sin\theta$ from $\theta = -\frac{3\pi}{4}$ to $\theta = \frac{\pi}{4}$. Hence, the total area is

$$\begin{aligned}
2\cdot\frac{1}{2}\int_{-3\pi/4}^{\pi/4}(1+\sin\theta)^2\,d\theta &= \int_{-3\pi/4}^{\pi/4}(1 + 2\sin\theta + \sin^2\theta)\,d\theta \\
&= \int_{-3\pi/4}^{\pi/4}\left(1 + 2\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right)d\theta \\
&= \int_{-3\pi/4}^{\pi/4}\left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta \\
&= \left(\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right)\Bigg|_{-3\pi/4}^{\pi/4} \\
&= \frac{3}{2}\pi - 2\left(\cos\frac{\pi}{4} - \cos\left(-\frac{3\pi}{4}\right)\right) - \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin\left(-\frac{3\pi}{2}\right)\right) \\
&= \boxed{\frac{3}{2}\pi - 2\sqrt{2}}.
\end{aligned}$$

8.26 Note that r is nonzero, so we may multiply the equation by r to get

$$Ar\cos\theta + Br\sin\theta + C = 0.$$

This converts to rectangular form to get $Ax + By + C = 0$, which is the equation for a line. The slope is $-\frac{A}{B}$ and the y-intercept is $\left(0, -\frac{C}{B}\right)$. If $A = 0$ then the line is the horizontal line $y = -\frac{C}{B}$, and if $B = 0$ then the line is the vertical line $x = -\frac{C}{A}$. If both A and B are zero then the graph is empty if C is nonzero, and if they're all zero then the equation is $0 = 0$, which is content-free.

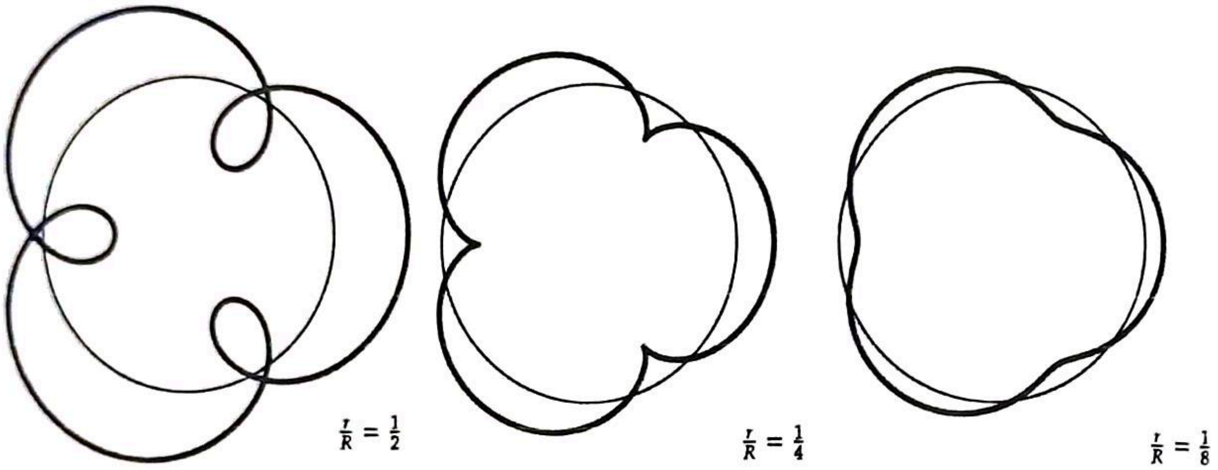
Challenge Problems

8.27

(a) The planet moves with parameterization $(R\cos 2\pi t, R\sin 2\pi t)$. The moon moves around the planet with parameterization $(r\cos 8\pi t, r\sin 8\pi t)$ relative to the center of the planet. (Note that as t goes from 0 to 1, the planet revolves once around the star, and the moon revolves 4 times around the planet.) Adding these, we get parametric equations for the moon relative to the star:

$$\boxed{(R\cos 2\pi t + r\cos 8\pi t, R\sin 2\pi t + r\sin 8\pi t)}.$$

(b) There are different behaviors depending on the ratio $\frac{r}{R}$. Below, we plot three different values:



The thin circle is the path of the planet, and the bold curve is the path of the moon. Note that if $\frac{r}{R} > \frac{1}{4}$, the path of the moon loops on itself, and if $\frac{r}{R} < \frac{1}{4}$, then the moon stays close to the path of the planet and doesn't loop. If $\frac{r}{R} = \frac{1}{4}$, then we get sharp corners, as we'll see in part (c).

(c) A full stop occurs when $\frac{dx}{dt} = \frac{dy}{dt} = 0$. We compute:

$$\frac{dx}{dt} = -2\pi R\sin 2\pi t - 8\pi r\sin 8\pi t,$$
$$\frac{dy}{dt} = 2\pi R\cos 2\pi t + 8\pi r\cos 8\pi t.$$

Setting these equal to 0 gives

$$0 = R\sin 2\pi t + 4r\sin 8\pi t,$$
$$0 = R\cos 2\pi t + 4r\cos 8\pi t.$$

One way to solve this system is to square both equations and add them:

$$0 = R^2 + (4r)^2 + 8Rr(\sin 2\pi t\sin 8\pi t + \cos 2\pi t\cos 8\pi t).$$

The trig expression in the parentheses is the cosine angle subtraction expansion of $\cos(8\pi t - 2\pi t)$, so the equation simplifies to

$$0 = R^2 + 16r^2 + 8Rr\cos 6\pi t,$$

hence

$$\cos 6\pi t = -\frac{R^2 + 16r^2}{8Rr} = -\frac{R}{8r} - \frac{2r}{R}.$$

Let $c = \frac{r}{R} < 1$, and we have

$$\cos 6\pi t = -\left(2c + \frac{1}{8c}\right).$$

This will have a solution if and only if the right side is between -1 and 1, so we must have

$$2c + \frac{1}{8c} \le 1.$$

Multiplying by c and rearranging gives $2c^2 - c + \frac{1}{8} \le 0$, or $\frac{1}{8}(4c-1)^2 \le 0$. Clearly this occurs if and only if $c = \frac{1}{4}$.

In summary: if $c = \frac{r}{R} = \frac{1}{4}$, then the planet comes to a full stop when $\cos 6\pi t = -\left(2c + \frac{1}{8c}\right) = -1$; this happens 3 times a year (at $t = \frac{1}{6}$, $t = \frac{1}{2}$, and $t = \frac{5}{6}$) and these times correspond to the sharp corners in the graph for the case $\frac{r}{R} = \frac{1}{4}$ shown above.

8.28 We have $r + \epsilon r \cos\theta = \ell$, so $\sqrt{x^2+y^2} + \epsilon x = \ell$. Thus

$$x^2 + y^2 = (\ell - \epsilon x)^2 = \ell^2 - 2\epsilon x + \epsilon^2 x^2.$$

If $\epsilon > 1$, this can be written in the form

$$y^2 - a(x-b)^2 = c^2$$

for some positive constant a and constants b and c. So the graph is a $\boxed{\text{hyperbola}}$. More specifically, it is a hyperbola with one focus at $(0,0)$.

8.29 To show that both parameterizations give the same curve, start with

$$C = \{(u(t), v(t)) \mid t \in [a,b]\}$$

and the "new" curve

$$C_{new} = \{(u(f(\tau)), v(f(\tau))) \mid \tau \in [c,d]\}.$$

(We use a different variable for C_{new} for clarity.) We wish to show that $C = C_{new}$. If $(u(t), v(t)) \in C$, then $t \in [a,b]$, so $t = f(\tau)$ for some $\tau \in [c,d]$, and hence $(u(t), v(t)) = (u(f(\tau)), v(f(\tau))) \in C_{new}$. Thus $C \subseteq C_{new}$. Conversely, if $(u(f(\tau)), v(f(\tau))) \in C_{new}$ for some $\tau \in [c,d]$, then $t = f(\tau) \in [a,b]$, and hence $(u(f(\tau)), v(f(\tau))) = (u(t), v(t)) \in C$. Thus $C_{new} \subseteq C$, and therefore $C_{new} = C$. So both parameterizations give the same curve.

The length of the curve C computed using the new parameterization is

$$\int_c^d \sqrt{\left(\frac{d}{d\tau}(u(f(\tau)))\right)^2 + \left(\frac{d}{d\tau}(v(f(\tau)))\right)^2}\, d\tau.$$

By the Chain Rule, this is equal to

$$\int_c^d \sqrt{(u'(f(\tau))f'(\tau))^2 + (v'(f(\tau))f'(\tau))^2}\, d\tau.$$

Since $f'(\tau) \ge 0$, we can pull $(f'(\tau))^2$ outside of the square root, giving

$$\int_c^d f'(\tau)\sqrt{(u'(f(\tau)))^2 + (v'(f(\tau)))^2}\, d\tau.$$

Now with the substitution $t = f(\tau)$ and $dt = f'(\tau)d\tau$, we have

$$\int_a^b \sqrt{(u'(t))^2 + (v'(t))^2}\, dt,$$

which equals the length of C computed using the original parameterization.

8.30

(a) If n is odd, we have n petals, but if n is even, we have $2n$ petals. That is because $\cos n\theta$ will have n periods from $\theta = 0$ to $\theta = 2\pi$, but in the even n case, each period will produce 2 petals (one where $r \geq 0$ and one where $r \leq 0$), whereas in the odd n case, the $r \leq 0$ petal will exactly overlap with an $r \geq 0$ petal from a different period. This is because $\cos(n(\theta + \pi)) = \cos n\theta$ when n is even, but $\cos(n(\theta + \pi)) = -\cos n\theta$ when n is odd.

(b) The area of the first petal is twice the area inside the curve from $\theta = 0$ to $\theta = \frac{\pi}{2n}$. So the integral that gives the area we want is

$$2 \cdot \frac{1}{2} \int_0^{\pi/2n} \cos^2 n\theta \, d\theta.$$

We then compute:

$$\begin{aligned}
\int_0^{\pi/2n} \cos^2 n\theta \, d\theta &= \frac{1}{2} \int_0^{\pi/2n} (1 + \cos 2n\theta) \, d\theta \\
&= \frac{\pi}{4n} + \frac{1}{2} \int_0^{\pi/2n} \cos 2n\theta \, d\theta \\
&= \frac{\pi}{4n} + \frac{1}{4n} \sin 2n\theta \Big|_0^{\pi/2n} \\
&= \frac{\pi}{4n} + \frac{1}{4n} (\sin \pi - \sin 0) \\
&= \boxed{\frac{\pi}{4n}}.
\end{aligned}$$

(c) Note that we have $y = r \sin\theta = \cos 2\theta \sin\theta$. We need to maximize this for the domain $\theta \in \left[0, \frac{\pi}{4}\right]$. This is easier to work with if we write the whole thing in terms of sine:

$$y = \cos 2\theta \sin\theta = (1 - 2\sin^2\theta)\sin\theta = \sin\theta - 2\sin^3\theta.$$

Now we can maximize by differentiating and finding the critical points:

$$\frac{dy}{d\theta} = \cos\theta - 6\sin^2\theta\cos\theta.$$

Since $\cos\theta \neq 0$ in the interval that we are considering, we see that the critical point occurs where $\sin^2\theta = \frac{1}{6}$, or $\sin\theta = \frac{1}{\sqrt{6}}$. Note that this critical point must be a local maximum, as the endpoints of the interval give $y = 0$ as local minimums. So our width is

$$2y = 2\left(\frac{1}{\sqrt{6}} - 2\frac{1}{6\sqrt{6}}\right) = \boxed{\frac{2\sqrt{6}}{9}}.$$

(d) We can do similar to part (c): we wish to maximize $y = \cos n\theta \sin\theta$ on the interval $\left[0, \frac{\pi}{2n}\right]$. We differentiate

$$\frac{dy}{d\theta} = \cos n\theta \cos\theta - n \sin n\theta \sin\theta.$$

We set this equal to 0 and solve for θ to get the value of θ that maximizes y. So the value of θ that we want is the solution to

$$0 = \cos n\theta \cos\theta - n \sin n\theta \sin\theta$$

for $\theta \in \left[0, \frac{\pi}{2n}\right]$. We can make this a little bit simpler by using trigonometric product-to-sum identities:

$$\begin{aligned}
0 &= \cos n\theta \cos\theta - n \sin n\theta \sin\theta \\
&= \frac{1}{2}(\cos(n+1)\theta + \cos(n-1)\theta) - \frac{n}{2}(\cos(n-1)\theta - \cos(n+1)\theta) \\
&= \frac{n+1}{2}\cos(n+1)\theta - \frac{n-1}{2}\cos(n-1)\theta.
\end{aligned}$$

Thus, we want the value of $\theta \in \left[0, \frac{\pi}{2n}\right]$ such that

$$(n-1)\cos(n-1)\theta = (n+1)\cos(n+1)\theta.$$

In part (c), where $n = 2$, this amounts to solving $\cos\theta = 3\cos 3\theta$, and you can verify that $\theta = \sin^{-1}\left(\frac{1}{\sqrt{6}}\right)$ is indeed the solution. However, for $n > 2$, this equation is difficult to solve. The $n = 3$ case is a bit tractable: we have $2\cos 2\theta = 4\cos 4\theta$. Letting $z = \cos 2\theta$ and using the cosine double-angle formula, this is equivalent to $2z = 4(2z^2 - 1)$, which has positive root $z = \frac{1+\sqrt{33}}{8}$. Thus,

$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right),$$

and we could use this value of θ to get the width $2y = 2\cos 3\theta \sin\theta$, but don't try this last step at home. If $n > 3$, then the computation becomes virtually impossible without a computer algebra system.

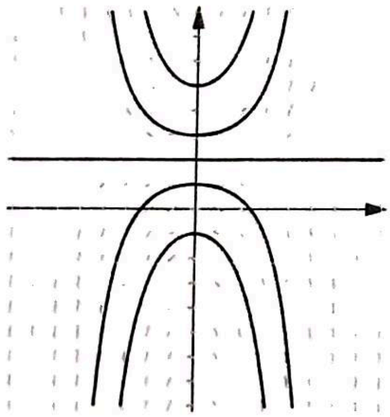
CHAPTER 9

Differential Equations

Exercises for Section 9.1

9.1.1

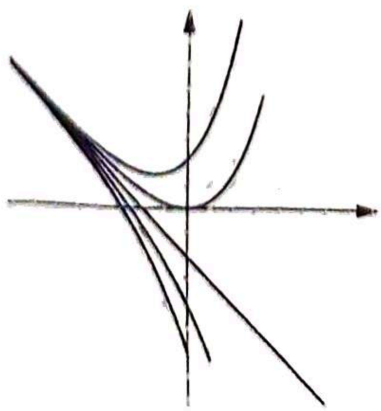
(a) Using separation of variables, we write $\frac{dy}{y-1} = x\,dx$. Integrating both sides gives $\log|y-1| = \frac{x^2}{2} + C$, or $y = \boxed{ce^{\frac{1}{2}x^2} + 1}$ for some constant c.



(b) If we ignore the x term, we get the equation $y' = y$, which we know has the solution $y = ce^x$ for a real constant c. Now we need to add some extra terms to this so that y' has an extra x. We can guess that these extra terms will have the form $a_1x + a_0$ for some real a_0, a_1. Let $y = ce^x + a_1x + a_0$, then substituting into $y' = y + x$ gives

$$ce^x + a_1 = ce^x + (a_1 + 1)x + a_0.$$

Equating corresponding coefficients gives $a_1 + 1 = 0$ and $a_0 = a_1$, so $a_0 = a_1 = -1$, and thus $\boxed{y = ce^x - x - 1}$ is our solution.



(c) Make the substitution $y = vx$, so $y' = v + v'x$. Then the equation becomes

$$v + v'x = \frac{1+v}{1-v},$$

giving

$$v'x = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v}.$$

Now using separation of variables, this becomes

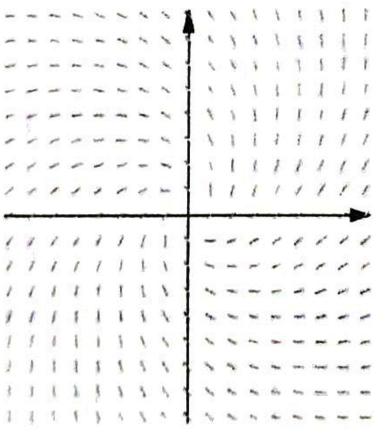
$$\frac{1-v}{1+v^2}\,dv = \frac{dx}{x}.$$

This can be solved, but it's fairly ugly, and then we have to undo the substitution via $v = \frac{y}{x}$. It turns out (and you can check this yourself if you like) that the solution curves are implicitly given by

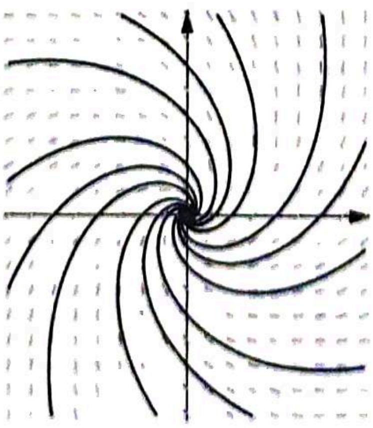
$$2\tan^{-1}\left(\frac{y}{x}\right) = \log(x^2 + y^2) + C,$$

where C is a constant. Also, this equation does not really give us a good idea of what the solution curves look like.

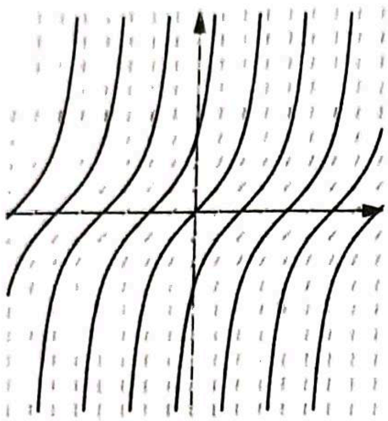
Rather than explicitly solve the differential equation, we can just draw the slope field and sketch some curves. First, the slope field:



The solutions look like they spiral towards $(0,0)$. We can draw some possible solution curves, resulting in the picture below.



(d) Using separation of variables, we write $\frac{dy}{1+y^2} = dx$. Integrating both sides gives $\arctan y = x + C$. Thus $\boxed{y = \tan(x + C)}$ is the general solution.



9.1.2

(a) We use separation of variables. Rewrite the equation as $\frac{y'}{y} = x$, then $\frac{dy}{y} = x\,dx$. Integrating both sides gives $\log|y| = \frac{x^2}{2} + C$ for some constant C. Thus $\boxed{y = ce^{x^2/2}}$ for some real c. We can verify that any solution of this form works.

(b) We use separation of variables. Rewrite the equation as $y'e^{-y} = x$. Then $e^{-y}\,dy = x\,dx$. Integrating both sides, we get $-e^{-y} = \frac{x^2}{2} + C$. We have $y = 0$ at $x = 0$, so putting these in gives $-1 = C$. So $e^{-y} = 1 - \frac{x^2}{2}$. Thus $\boxed{y = \log\left(\frac{1}{1-\frac{x^2}{2}}\right) = -\log\left(1 - \frac{1}{2}x^2\right)}$. Note that as x approaches $\sqrt{2}$ or $-\sqrt{2}$, the value of y goes to infinity, so the function is only defined for $x \in (-\sqrt{2}, \sqrt{2})$.

(c) Rewrite the equation as $\frac{y'}{y^2} = \sin x$. Then $\frac{dy}{y^2} = \sin x\,dx$ and $-\frac{1}{y} = -\cos x + C$ for a real constant C. Thus $\boxed{y = \frac{1}{\cos x - C}}$ is the general solution. We can verify that any solution of this form works.

9.1.3 We solve this by antidifferentiation. First, let $v = y'$. Then $v' = 2$ with $v(0) = 0$. The general solution is $v = 2t + C_1$ for some constant C_1. Putting in $t = 0, v = 0$ gives $C_1 = 0$, so the unique solution is $v = 2t$. Now we need to solve $y' = 2t$ with $y(0) = 1$. Again, taking the antiderivative gives $y = t^2 + C_2$. Putting in $t = 0, y = 1$ gives $C_2 = 1$, so the unique solution is $\boxed{y = t^2 + 1}$.

9.1.4 We separate variables and integrate:

$$\int \frac{dy}{2y - y^2} = \int dx.$$

The left side decomposes into partial fractions:

$$\frac{1}{2}\left(\int \frac{dy}{y} + \int \frac{dy}{2-y}\right) = \int dx.$$

Integrating gives

$$\frac{1}{2}(\log|y| - \log|2-y|) = x + C,$$

and simplifying gives

$$\log\left|\frac{y}{2-y}\right| = 2x + C'.$$

Taking the exponential gives

$$\frac{y}{2-y} = ce^{2x},$$

where c could be negative. Plugging in $y(0) = 1$ gives $\frac{1}{2-1} = ce^0 = c$, so $c = 1$. Hence

$$\frac{y}{2-y} = e^{2x},$$

and solving for y gives $\boxed{y = \frac{2}{1+e^{-2x}}}$.

9.1.5 Let y denote the difference in temperature between the milk and the room. Then the milk's temperature is governed by the differential equation $y' = ky$, which has solution $y = Ce^{kt}$. We are given that $y(0) = -18$ and $y(1) = -15$. Putting in $t = 0, y = -18$ gives $C = -18$, so $y = -18e^{kt}$. Putting in $t = 1, y = -15$ gives $-15 = -18e^k$. Thus $k = \log\left(\frac{5}{6}\right)$. Putting this in and simplifying gives $y = -18e^{(\log 5/6)t} = -18(1.2)^{-t}$. We want to know what t is when $y = -10$. Using logarithms, we can determine that this is $\boxed{t = \log_{1.2} 1.8 \approx 3.22 \text{ minutes}}$ after the milk is removed from the refrigerator.

9.1.6 We make the substitution $y = vx$, so that $y' = v + v'x$. Then we have $v + v'x = f(v)$, which we can write as $v'x = f(v) - v$. Now we separate the variables as

$$\frac{dv}{f(v) - v} = \frac{dx}{x}.$$

Finally, we can antidifferentiate both sides and solve.

Exercises for Section 9.2

9.2.1

(a) The characteristic equation $r^2 - 4r + 3 = (r-3)(r-1) = 0$ has roots 1 and 3, so the general solution is of the form

$$y = c_1e^x + c_2e^{3x}.$$

Plugging in $y(0) = 1$ gives $1 = c_1 + c_2$ and plugging in $y'(0) = 2$ gives $c_1 + 3c_2 = 2$. Thus $c_1 = c_2 = \frac{1}{2}$, and the solution is

$$\boxed{y = \frac{1}{2}e^x + \frac{1}{2}e^{3x}}.$$

(b) The characteristic equation $r^2 + 6r + 9 = (r+3)^2 = 0$ has a double root -3, so the general solution is of the form

$$y = c_1e^{-3x} + c_2xe^{-3x}.$$

Plugging in $y(0) = 0$ gives $c_1 = 0$. Then $y' = c_2(e^{-3x} - 3xe^{-ex})$, so $y'(0) = 1$ gives $c_2 = 1$. Thus the solution is $\boxed{y = xe^{-3x}}$.

(c) The characteristic equation $r^2 - 4r + 13 = 0$ has roots $2 \pm 3i$, so the general solution is of the form

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x).$$

Plugging in $y(0) = 2$ gives $c_1 = 2$. Differentiating gives

$$y' = e^{2x}((2c_1 + 3c_2)\cos 3x + (-3c_1 + 2c_2)\sin 3x),$$

so plugging in $y'(0) = -1$ gives $2c_1 + 3c_2 = -1$, hence $c_2 = -\frac{5}{3}$. Thus the solution is

$$\boxed{y = e^{2x}\left(2\cos 3x - \frac{5}{3}\sin 3x\right)}.$$

9.2.2 Let $y = e^{rx}\cos sx$. We then compute:

$$\begin{aligned}
y' &= re^{rx}\cos sx - se^{rx}\sin sx = e^{rx}(r\cos sx - s\sin sx),\\
y'' &= re^{rx}(r\cos sx - s\sin sx) + e^{rx}(-rs\sin sx - s^2\cos sx) = e^{rx}((r^2 - s^2)\cos sx - 2rs\sin sx).
\end{aligned}$$

This gives

$$y'' + ay' + by = e^{rx}(((r^2 - s^2) + ar + b)(\cos sx) + (-2rs - as)(\sin sx)).$$

But we know that $2r = -a$ and $r^2 + s^2 = b$ (because $r \pm si$ are roots of $\lambda^2 + a\lambda + b$), so both coefficients of the trig functions in the above expression are 0, and hence $y'' + ay' + by = 0$, as desired.

9.2.3 Let $y = e^{rx}(c_1x + c_2)$. We compute:

$$\begin{aligned}
y' &= re^{rx}(c_1x + c_2) + e^{rx}(c_1) = e^{rx}(rc_1x + c_1 + rc_2),\\
y'' &= re^{rx}(rc_1x + c_1 + rc_2) + e^{rx}(rc_1) = e^{rx}(r^2c_1x + 2rc_1 + r^2c_2).
\end{aligned}$$

This gives

$$y'' + ay' + by = e^{rx}((r^2 + ar + b)c_1x + (2rc_1 + r^2c_2 + a(c_1 + rc_2) + bc_2)).$$

Since r is a root of the characteristic polynomial, we know $r^2 + ar + b = 0$, so it suffices to show that

$$2rc_1 + r^2c_2 + a(c_1 + rc_2) + bc_2 = 0.$$

But we also know that $a = -2r$ and $b = r^2$, since r is a double root of the characteristic polynomial, so substituting these gives

$$2rc_1 + r^2c_2 + a(c_1 + rc_2) + bc_2 = 2rc_1 + r^2c_2 - 2r(c_1 + rc_2) + r^2c_2 = 0,$$

as desired.

9.2.4 First we solve the homogeneous equation $y'' + 2y' + 2y = 0$. The characteristic equation $r^2 + 2r + 2$ has roots $r = -1 \pm i$, so its general solution is

$$y = e^{-x}(c_1\cos x + c_2\sin x).$$

Next, we look for a solution to the original equation of the form $y = se^{-3x}$ for some constant s. Noting that $y' = -3se^{-3x}$ and $y'' = 9se^{-3x}$, we must have

$$e^{-3x} = y'' + 2y' + 2y = 9se^{-3x} + 2(-3se^{-3x}) + 2se^{-3x} = 5se^{-3x},$$

so $s = \frac{1}{5}$. Thus, the general solution is

$$\boxed{y = e^{-x}(c_1\cos x + c_2\sin x) + \frac{1}{5}e^{-3x}}.$$

Review Problems

9.19 Separate as $\dfrac{dy}{1+y^2} = dx$, so integration gives $\tan^{-1} y = x + C$. Plugging in $x = y = 0$ gives $C = 0$, so we have $\boxed{y = \tan x}$.

9.20 Expand as $(x^2 + 1)y' = x^2y^2$, and separate variables to give $\dfrac{dy}{y^2} = \dfrac{x^2}{x^2+1}\,dx$. Integrating both sides gives

$$-\frac{1}{y} = \int \frac{x^2}{x^2+1}\,dx = \int\left(1 - \frac{1}{x^2+1}\right)dx = x - \tan^{-1}x + C,$$

hence $y = -\frac{1}{x - \tan^{-1} x + C}$. Plugging in $y(1) = 2$ gives

$$2 = -\frac{1}{1 - \frac{\pi}{4} + C},$$

so $1 - \frac{\pi}{4} + C = -\frac{1}{2}$, hence $C = -\frac{3}{2} + \frac{\pi}{4}$. Thus, the solution is

$$\boxed{y = -\frac{1}{x - \tan^{-1} x - \frac{3}{2} + \frac{\pi}{4}}}.$$

This can be made to look slightly nicer by multiplying numerator and denominator by −4:

$$y = \frac{4}{6 - \pi - 4x + 4\tan^{-1} x}.$$

9.21 Separate variables to give $\frac{dy}{3y - y^2} = dt$. The integral of the left side is

$$\int \frac{dy}{3y - y^2} = \frac{1}{3}\int \left(\frac{1}{y} + \frac{1}{3-y}\right) dy = \frac{1}{3}(\log|y| - \log|3 - y|) = \frac{1}{3}\log\left|\frac{y}{3-y}\right| = \frac{1}{3}\log\left|\frac{3}{3-y} - 1\right|.$$

Thus we have

$$\frac{1}{3}\log\left|\frac{3}{3-y} - 1\right| = t + C,$$

and exponentiating gives

$$\left|\frac{3}{3-y} - 1\right| = ce^{3t}.$$

Thus,

$$\frac{3}{3-y} = ce^{3t} + 1,$$

and solving for y yields

$$y = 3 - \frac{3}{ce^{3t} + 1}.$$

Plugging in $t = 0$ gives $5 = 3 - \frac{3}{c+1}$, so $c = -\frac{5}{2}$, and the solution is

$$\boxed{y = 3 - \frac{3}{1 - \frac{5}{2}e^{3t}} = 3 + \frac{6}{5e^{3t} - 2}}.$$

As $t \to \infty$, the fraction $\frac{6}{2-5e^{3t}}$ approaches 0, so $\boxed{\lim_{t\to\infty} y(t) = 3}$.

9.22 Let $v = y'$, so the equation becomes $2v' + v^2 = -1$. This separates to $-2\frac{dv}{1+v^2} = dx$, and hence we integrate to get $-2\arctan v = x + C$, or $v = \tan\left(-\frac{x+C}{2}\right)$. Therefore

$$y = \int v\,dx = \boxed{-2\log\left|\sec\left(-\frac{x+C}{2}\right)\right| + D}.$$

We could write this a bit more simply as

$$y = 2\log\left|\left(\sec\left(-\frac{x+C}{2}\right)\right)^{-1}\right| + D = 2\log\left|\cos\left(-\frac{x+C}{2}\right)\right| + D = 2\log\left|\cos\left(\frac{x}{2} + C_0\right)\right| + D.$$

9.23 Let $z = y'$, so that $z' = z$. This is the basic exponential growth equation, so $y' = z = ce^x$. Integrating again gives $y = ce^x + d$. Plugging in the initial conditions gives the equations $1 = c + d$ and $2 = ce + d$, so $c(e-1) = 1$, hence $c = \frac{1}{e-1}$ and $d = 1 - \frac{1}{e-1}$. Thus the solution is $\boxed{y = 1 + \frac{e^x - 1}{e - 1}}$.

9.24 First, we find the explicit solution using separation of variables. We can write $\frac{dy}{y^2} = -0.5\,dt$. Integrating both sides gives $-\frac{1}{y} = -0.5t + C$ for a constant C. Therefore $y = \frac{1}{0.5t - C}$. We use the initial condition to find C. Putting in $t = 0$ and $y = 10$ gives $10 = \frac{1}{0-C}$, so $C = -0.1$. Thus $y = \frac{1}{0.5t+0.1}$. Putting in $y = 1$ and solving for t gives us that $t = 1.8$, so the object slows to 1 m/sec after $\boxed{\text{1.8 seconds}}$.

9.25

(a) A rumor spreads when a person who knows the rumor comes into contact with a person that doesn't know the rumor, and gossip ensues. Thus, the rate of spread of the rumor is proportional to the quantity

(population that knows the rumor)(population that doesn't know the rumor),

which is $y(1-y)$. Letting k be the constant of proportionality, we get $y' = ky(1-y)$.

(b) Separating variables and integrating gives

$$\int \frac{dy}{y(1-y)} = \int k\,dt = kt + C.$$

To evaluate the integral on the left side, we use the partial fraction decomposition

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y},$$

which then gives

$$\log|y| - \log|1-y| = kt + C.$$

The left side of the above is $\log\left|\frac{y}{1-y}\right|$, so exponentiating gives

$$\frac{y}{1-y} = c_0 e^{kt},$$

which is more useful when we take reciprocals, giving $\frac{1}{y} - 1 = ce^{-kt}$ (where $c = \frac{1}{c_0}$). Thus, solving for y, we have

$$\boxed{y = \frac{1}{1 + ce^{-kt}}}$$

for some positive constant c.

(c) We measure time in days starting from noon on Sunday. At $t = 0$, we have $0.1 = y = \frac{1}{1+c}$, so $c = 9$ and our equation is

$$y = \frac{1}{1 + 9e^{-kt}}.$$

At $t = 1$, we have $0.2 = y = \frac{1}{1+9e^{-k}}$, so $9e^{-k} = 4$, hence $k = -\log(4/9) \approx 0.8109$. To find when $y = 0.9$, we solve

$$0.9 = \frac{1}{1 + 9e^{\log(4/9)t}},$$

giving $e^{\log(4/9)t} = 1/81$, or $\log(4/9)t = \log(1/81)$, and finally

$$t = \frac{\log(1/81)}{\log(4/9)} \approx 5.419 \text{ days}.$$

Noting that 0.419 days is about 10 hours and 3 minutes, we conclude that 90% of the population will know the rumor at approximately $\boxed{\text{10:03 PM on Friday}}$.

9.26 The key observation is to note that the left side is the **derivative of a product**—the equation can we rewritten as

$$((\sin x)y)' = \tan x.$$

Therefore we can just integrate:

$$(\sin x)y = \int \tan x\, dx.$$

To integrate the right side, you can look it up, or write it as

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

and substitute $u = \cos x$, $du = -\sin x\, dx$. This makes our integral

$$\int \frac{\sin x}{\cos x}\, dx = -\int \frac{du}{u} = -\log|u| + C = -\log|\cos x| + C.$$

We usually write this as $\log|\sec x| + C$. Thus, we have that

$$(\sin x)y = \log|\sec x| + C,$$

and dividing by $\sin x$ solves for y: $\boxed{y = (\csc x)(\log|\sec x| + C)}$.

Challenge Problems

9.27 If the wrong "product rule" works, then we must have $f'g' = (fg)' = f'g + fg'$. Plugging in $f = e^{x^2}$ and $f' = 2xe^{x^2}$ gives

$$2xe^{x^2}g' = 2xe^{x^2}g + e^{x^2}g'.$$

Dividing by e^{x^2} leaves

$$2xg' = 2xg + g'.$$

This can be separated as

$$\frac{dg}{g} = \frac{2x}{2x-1}\, dx,$$

so

$$\log|g| = \int \frac{2x}{2x-1}\, dx = \int \left(1 + \frac{1}{2x-1}\right) dx = x + \frac{1}{2}\log|2x-1| + C.$$

Exponentiating both sides gives $\boxed{g(x) = ce^x\sqrt{2x-1}}$ on any interval with $x > \frac{1}{2}$.

9.28

(a) Population growth is proportional to the number of couples in the population (since it takes 2 people to reproduce), so this is the ay^2 term in the growth rate. Death, however, is proportional to the number of people, so this is the $-by$ term in the growth rate.

(b) Separation of variables gives

$$\int \frac{dy}{ay^2 - by} = \int dt.$$

Factoring out $\frac{1}{a}$ lets us use partial fractions:

$$\int \frac{dy}{ay^2 - by} = \frac{1}{a}\int \left(\frac{-a/b}{y} + \frac{a/b}{y - \frac{b}{a}}\right) dy.$$

This is

$$\frac{1}{b}\left(-\log|y| + \log\left|y - \frac{b}{a}\right|\right) = \frac{1}{b}\log\left|\frac{y - \frac{b}{a}}{y}\right| = \frac{1}{b}\log\left|1 - \frac{b}{ay}\right|.$$

Thus, we have the equation

$$\frac{1}{b}\log\left|1 - \frac{b}{ay}\right| = t + C,$$

hence exponentiating gives

$$1 - \frac{b}{ay} = ce^{bt},$$

and solving gives

$$\boxed{y = \frac{b}{a(1 - ce^{bt})}}.$$

(c) Plugging in $t = 0$ and $y = m$ gives $m = \dfrac{b}{a(1-c)}$, so $1 - c = \dfrac{b}{am}$, and hence $c = 1 - \dfrac{b}{am}$. Thus our equation becomes

$$y = \frac{b}{a(1 - ce^{bt})} = \frac{b}{a(1 - (1 - \frac{b}{am})e^{bt})}.$$

If $m > b/a$, then $(b/am) < 1$, and $0 < c < 1$. But note that some value of t (specifically, the value of t for which $e^{bt} = 1/c$) makes the denominator 0. So $y \to \infty$ in a *finite* amount of time (leading us to believe that this is a bad model).

If $m = b/a$, then $c = 0$, so we have a constant population $y = b/a$.

If $m < b/a$, then $c < 0$, and the population approaches 0 as $t \to \infty$. (People are not being born fast enough to make up for the people who are dying.)

This analysis leads us to believe that the differential equation is too simplistic to model population: the model only really "works" when the population is already exactly at the equilibrium size.

9.29

(a) We take our original equation $y' - 2y = e^{-x}$ and multiply through by e^{-2x}, giving:

$$e^{-2x}y' - 2e^{-2x}y = e^{-3x}.$$

It seems like we've made things more complicated by doing this, but in fact we've actually made things simpler! The left side is now simply $(e^{-2x}y)'$, and our differential equation has become

$$(e^{-2x}y)' = e^{-3x}.$$

Now we can integrate both sides with respect to x, giving:

$$e^{-2x}y = -\frac{1}{3}e^{-3x} + C.$$

To finish, we multiply by e^{2x} to solve for y, and the solution is

$$\boxed{y = -\frac{1}{3}e^{-x} + Ce^{2x}}.$$

(b) We can generalize the example from part (a) to develop a method to solve first-order linear differential equations. We start with

$$y' + p(x)y = q(x).$$

We want to multiply by some function $h(x)$ so that the left side becomes $(h(x)y)'$. Therefore, $h(x)$ must satisfy

$$h(x)(y' + p(x)y) = (h(x)y)' = h(x)y' + h'(x)y.$$

Comparing the y terms of the above equation, we see that this requires $h'(x) = h(x)p(x)$. This new equation for h is a separable differential equation: in particular, $\frac{h'(x)}{h(x)} = p(x)$. Integrating gives $\log|h(x)| = \int p(x)\,dx$, and then exponentiation gives

$$\boxed{h(x) = e^{\int p(x)\,dx}}.$$

(c) Where $h(x)$ is as in part (b) above, we have the equation

$$(yh)' = y'h + yh' = y'h + yhp = h(y' + py) = hq,$$

so integrating gives $yh = \int hq$, and hence the solution is

$$y = \frac{\int hq}{h} = \boxed{\frac{\int \left(e^{\int p(x)\,dx} q(x)\right) dx}{e^{\int p(x)\,dx}}}.$$

9.30

(a) We differentiate g:

$$g'(x) = \frac{d}{dx}\left((f(x))^2 + (f'(x))^2\right) = 2f(x)f'(x) + 2f'(x)f''(x) = 2f'(x)(f(x) + f''(x)) = 0.$$

So $g' = 0$, hence g is constant.

(b) We note that g is constant and $g(0) = (f(0))^2 + (f'(0))^2 = 0 + 0 = 0$. Hence $g = 0$. But the only way that the sum of two squares can be 0 is if each square is itself 0; in particular, we must have $f = 0$.

(c) We trivially observe that $f(x) = f'(0)\sin x + f(0)\cos x$ is a solution. If h is another solution with $h(0) = f(0)$ and $h'(0) = f'(0)$, then $f - h = 0$ by part (b), hence $h = f$. Thus, the functions shown are the only solutions.

9.31 Differentiate the given $f'(x) = f(1 - x)$ to get

$$f''(x) = -f'(1 - x) = -f(1 - (1 - x)) = -f(x).$$

Thus $f(x) = a\cos(x) + b\sin(x)$ for some constants a and b. Plugging in $f(0) = 1$ gives $a = 1$, so the function is $f(x) = \cos(x) + b\sin(x)$. To finish, we must solve for b, since $f(1) = f'(0) = -\sin(0) + b\cos(0) = b$.

However, we can compare

$$f'(x) = -\sin(x) + b\cos(x)$$

and

$$f(1 - x) = \cos(1 - x) + b\sin(1 - x).$$

Rewriting the latter in terms of $\cos(x)$ and $\sin(x)$ gives

$$\begin{aligned} f(1 - x) &= \cos 1 \cos x + \sin 1 \sin x + b \sin 1 \cos x - b \cos 1 \sin x \\ &= (\cos 1 + b \sin 1)\cos x + (\sin 1 - b\cos 1)\sin x. \end{aligned}$$

Thus, since $f'(x) = f(1 - x)$, we have the system

$$\begin{aligned} b &= \cos 1 + b\sin 1, \\ -1 &= \sin 1 - b\cos 1. \end{aligned}$$

These have the common solution $\boxed{f(1) = b = \frac{\cos 1}{1 - \sin 1} = \frac{1 + \sin 1}{\cos 1}}$. Note this can also be written as $b = \sec 1 + \tan 1$.

For case (1), we have

$$\begin{aligned}\int_{x-y}^{x+y} (2/c)\sinh(ct)\,dt &= \frac{2}{c^2}(\cosh(cx+cy)-\cosh(cx-cy))\\ &= \frac{4}{c^2}\sinh(cx)\sinh(cy) = f(x)f(y).\end{aligned}$$

So the solutions are:

$$\boxed{0,\ 2x,\ \frac{2}{c}\sin(cx),\ \frac{2}{c}\sinh(cx)},$$

where $c \neq 0$ is any constant.

www.artofproblemsolving.com

The Art of Problem Solving (AoPS) is:

- **Books**

  For over 26 years, the classic *Art of Problem Solving* books have been used by students as a resource for the American Mathematics Competitions and other national and local math events.

  Every school should have this in their math library.
  – Paul Zeitz, past coach of the U.S. International Mathematical Olympiad team

  The Art of Problem Solving Introduction and Intermediate texts, together with our *Prealgebra*, *Precalculus*, and *Calculus* texts, form a complete curriculum for outstanding math students in grades 6-12.

  The new book [Introduction to Counting & Probability] is great. I have started to use it in my classes on a regular basis. I can see the improvement in my kids over just a short period.
  – Jeff Boyd, 4-time MATHCOUNTS National Competition winning coach

- **Classes**

  The Art of Problem Solving offers online classes on topics such as number theory, counting, geometry, algebra, and more at beginning, intermediate, and Olympiad levels.

  All the children were very engaged. It's the best use of technology I have ever seen.
  – Mary Fay-Zenk, coach of National Champion California MATHCOUNTS teams

- **Forum**

  As of August 2020, the Art of Problem Solving Forum has over 510,000 members who have posted over 11,300,000 messages on our discussion board. Members can also participate in any of our free "Math Jams."

  I'd just like to thank the coordinators of this site for taking the time to set it up... I think this is a great site, and I bet just about anyone else here would say the same...
  – AoPS Community Member

- **Resources**

  We have links to summer programs, book resources, problem sources, national and local competitions, scholarship listings, a math wiki, and a LaTeX tutorial.

  I'd like to commend you on your wonderful site. It's informative, welcoming, and supportive of the math community. I wish it had been around when I was growing up.
  – AoPS Community Member

- **...and more!**

Membership is FREE! Come join the Art of Problem Solving community today!

Art of Problem Solving
Calculus

Calculus is part of the acclaimed Art of Problem Solving curriculum designed to challenge high-performing middle and high school students. **Calculus** covers all topics in a single-variable calculus course, including limits, continuity, differentiation, integration, power series, plane curves, and elementary differential equations. The text is written to challenge students at a much deeper level than a traditional high school or first-year college calculus course.

Calculus includes hundreds of problems, ranging from routine exercises to extremely challenging problems drawn from high school and undergraduate-level mathematics competitions. Many of the problems have full, detailed solutions in the text, and the remainder have full solutions in the accompanying solutions manual.

David Patrick is an instructor and curriculum developer at Art of Problem Solving. He is the co-author of Art of Problem Solving's *Prealgebra* and author of *Introduction to Counting & Probability* and *Intermediate Counting & Probability*. He had the sole perfect score in North America on the 1988 American High School Mathematics Examination, was a USA Mathematical Olympiad winner, and was a top-10 finisher on the William Lowell Putnam Mathematics Competition. Dr. Patrick is a graduate of Carnegie Mellon University and has a Ph.D. in Mathematics from MIT. He taught mathematics at the University of Washington before joining Art of Problem Solving in 2004.

Jeremy Copeland is the School Director for Art of Problem Solving's online school. Dr. Copeland is a graduate of Reed College and has a Ph.D. in Mathematics from the University of Chicago. He taught mathematics at MIT and was a fellow at the Clay Mathematics Institute before joining Art of Problem Solving in 2009.

ISBN 978-1-934124-25-3

90000 >

9 781934 124253