2018 AMC 12A Problems/Problem 1

Contents

Problem

A large urn contains 100 balls, of which 36% are red and the rest are blue. How many of the blue balls must be removed so that the percentage of red balls in the urn will be 72%? (No red balls are to be removed.)

$\textbf{(A)}\ 28 \qquad\textbf{(B)}\ 32 \qquad\textbf{(C)}\ 36 \qquad\textbf{(D)}\ 50 \qquad\textbf{(E)}\ 64$

Solution 1

There are 36 red balls; for these red balls to comprise 72% of the urn, there must be only 14 blue balls. Since there are currently 64 blue balls, this means we must remove $50 = \boxed{\textbf{(D)}.}$

Solution 2

There are 36 red balls and 64 blue balls. For the percentage of the red balls to double from 36% to 72% of the urn, half of the total number of balls must be removed. Therefore, the number of blue balls that need to be removed is $50 = \boxed{\textbf{(D)}}$

See Also

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2018 AMC 12A Problems/Problem 2

Contents

Problem

While exploring a cave, Carl comes across a collection of 5-pound rocks worth $\$14$ each, 4-pound rocks worth $\$11$ each, and 1-pound rocks worth $\$2$ each. There are at least 20 of each size. He can carry at most 18 pounds. What is the maximum value, in dollars, of the rocks he can carry out of the cave?

$\textbf{(A) } 48 \qquad \textbf{(B) } 49 \qquad \textbf{(C) } 50 \qquad \textbf{(D) } 51 \qquad \textbf{(E) } 52$

Solution

The answer is just $3 \cdot 18 = 54$ minus the minimum number of rocks we need to make 18 pounds, or

$$54 - 4 = \boxed{\textbf{(C)} 50.}$$

Solution 2

The ratio of dollar per pound is greatest for the 5 pound rock, then the 4 pound, lastly the 1 pound. So we should take two 5 pound rocks and two 4 pound rocks.

Total weight:

$$2 \cdot 14 + 2 \cdot 11 = \boxed{\textbf{(C)} 50.}$$

~steakfails

See Also

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2018 AMC 10A Problems/Problem 4

Contents

Problem

How many ways can a student schedule 3 mathematics courses -- algebra, geometry, and number theory -- in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)

$\textbf{(A) }3\qquad\textbf{(B) }6\qquad\textbf{(C) }12\qquad\textbf{(D) }18\qquad\textbf{(E) }24$

Solution 1

We must place the classes into the periods such that no two classes are in the same period or in consecutive periods.

Ignoring distinguishability, we can thus list out the ways that three periods can be chosen for the classes when periods cannot be consecutive:

Periods $1, 3, 5$

Periods $1, 3, 6$

Periods $1, 4, 6$

Periods $2, 4, 6$

There are 4 ways to place 3 nondistinguishable classes into 6 periods such that no two classes are in consecutive periods. For each of these ways, there are $3! = 6$ orderings of the classes among themselves.

Therefore, there are $4 \cdot 6 = \boxed{\textbf{(E) } 24}$ ways to choose the classes.

-Versailles15625

Solution 2

Realize that the number of ways of placing, regardless of order, the 3 mathematics courses in a 6-period day so that no two are consecutive is the same as the number of ways of placing 3 mathematics courses in a sequence of 4 periods regardless of order and whether or not they are consecutive.

To see that there is a one to one correlation, note that for every way of placing 3 mathematics courses in 4 total periods (as above) one can add a non-mathematics course between each pair (2 total) of consecutively occurring mathematics courses (not necessarily back to back) to ensure there will be no two consecutive mathematics courses in the resulting 6-period day. For example, where M denotes a math course and O denotes a non-math course:
$MOMM \rightarrow MOOMOM$

For each 6-period sequence consisting of Ms and Os, we have $3!$ orderings of the 3 distinct mathematics courses.

So, our answer is $\binom{4}{3}(3!) = \boxed{\textbf{(E) } 24}$

- Gregwwl

Solution 3

Counting what we don't want is another slick way to solve this problem. Use PIE to count two cases: 1. Two classes consecutive, 2. Three classes consecutive.

Case 1: Consider two consecutive periods as a "block" of which there are 5 places to put in(1,2; 2,3; 3,4; 4,5; 5,6). Then we simply need to place two classes within the block, $3 \cdot 2$. Finally we have 4 periods remaining to place the final math class. Thus there are $5 \cdot 3 \cdot 2 \cdot 4$ ways to place two consecutive math classes with disregard to the third.

Case 2: Now consider three consecutive periods as a "block" of which there are now 4 places to put in(1,2,3; 2,3,4; 3,4,5; 4,5,6). We now need to arrange the math classes in the block, $3 \cdot 2 \cdot 1$. Thus there are $4 \cdot 3 \cdot 2 \cdot 1$ ways to place all three consecutive math classes.

By PIE we subtract Case 1 by Case 2 in order to not overcount: $120 - 24$. Then we subtract that answer from the total ways to place the classes with no restrictions: $(6 \cdot 5 \cdot 4) - 96 = \boxed{\textbf{(E) } 24}$

-LitJamal

See Also

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Category: Introductory Combinatorics Problems

2018 AMC 10A Problems/Problem 5

Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, "We are at least 6 miles away," Bob replied, "We are at most 5 miles away." Charlie then remarked, "Actually the nearest town is at most 4 miles away." It turned out that none of the three statements were true. Let d be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of d?

$\textbf{(A) } (0,4) \qquad \textbf{(B) } (4,5) \qquad \textbf{(C) } (4,6) \qquad \textbf{(D) } (5,6) \qquad \textbf{(E) } (5,\infty)$

Solution

From Alice and Bob, we know that $5 < d < 6.$ From Charlie, we know that $4 < d.$ We take the union of these two intervals to yield $\boxed{\textbf{(D) } (5,6)}$, because the nearest town is between 5 and 6 miles away.

See Also

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Category: Introductory Algebra Problems

2018 AMC 12A Problems/Problem 5

Problem

What is the sum of all possible values of k for which the polynomials $x^2 - 3x + 2$ and $x^2 - 5x + k$ have a root in common?

$\textbf{(A) }3 \qquad\textbf{(B) }4 \qquad\textbf{(C) }5 \qquad\textbf{(D) }6 \qquad\textbf{(E) }10$

Solution

We factor $x^2 - 3x + 2$ into $(x-1)(x-2)$. Thus, either 1 or 2 is a root of $x^2 - 5x + k$. If 1 is a root, then $1^2 - 5 \cdot 1 + k = 0$, so $k = 4$. If 2 is a root, then $2^2 - 5 \cdot 2 + k = 0$, so $k = 6$. The sum of all possible values of k is $\boxed{\textbf{(E)} 10}$.

See Also

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2018 AMC 12A Problems/Problem 6

Contents

Problem

For positive integers m and n such that $m + 10 < n + 1$, both the mean and the median of the set $\{m, m + 4, m + 10, n + 1, n + 2, 2n\}$ are equal to n. What is $m + n$?

$\textbf{(A)} 20 \qquad \textbf{(B)} 21 \qquad \textbf{(C)} 22 \qquad \textbf{(D)} 23 \qquad \textbf{(E)} 24$

Solution 1

The mean and median are

$$\frac{3m+4n+17}{6} = \frac{m+n+11}{2} = n,$$

so $3m + 17 = 2n$ and $m + 11 = n$. Solving this gives $(m, n) = (5, 16)$ for $m + n = \boxed{21}$. (trumpeter)

Solution 2

This is an alternate solution if you don't want to solve using algebra. First, notice that the median n is the average of $m + 10$ and $n + 1$. Therefore, $n = m + 11$, so the answer is $m + n = 2m + 11$, which must be odd. This leaves two remaining options: $(B) 21$ and $(D) 23$. Notice that if the answer is (B), then m is odd, while m is even if the answer is (D). Since the average of the set is an integer n, the sum of the terms must be even. $4 + 10 + 1 + 2 + 2n$ is odd by definition, so we know that $3m + 2n$ must also be odd, thus with a few simple calculations m is odd. Because all other answers have been eliminated, (B) is the only possibility left. Therefore, $m + n = \boxed{21}$. $\blacksquare$ --anna0kear

See Also

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2018 AMC 10A Problems/Problem 7

For how many (not necessarily positive) integer values of n is the value of $4000 \cdot \left(\tfrac{2}{5}\right)^n$ an integer?

$\textbf{(A) }3 \qquad \textbf{(B) }4 \qquad \textbf{(C) }6 \qquad \textbf{(D) }8 \qquad \textbf{(E) }9$

Solution

The prime factorization of 4000 is $2^5 \cdot 5^3$. Therefore, the maximum number for n is 3, and the minimum number for n is -5. Then we must find the range from -5 to 3, which is $3-(-5)+1=8+1=\boxed{\textbf{(E) }9}$.

See Also

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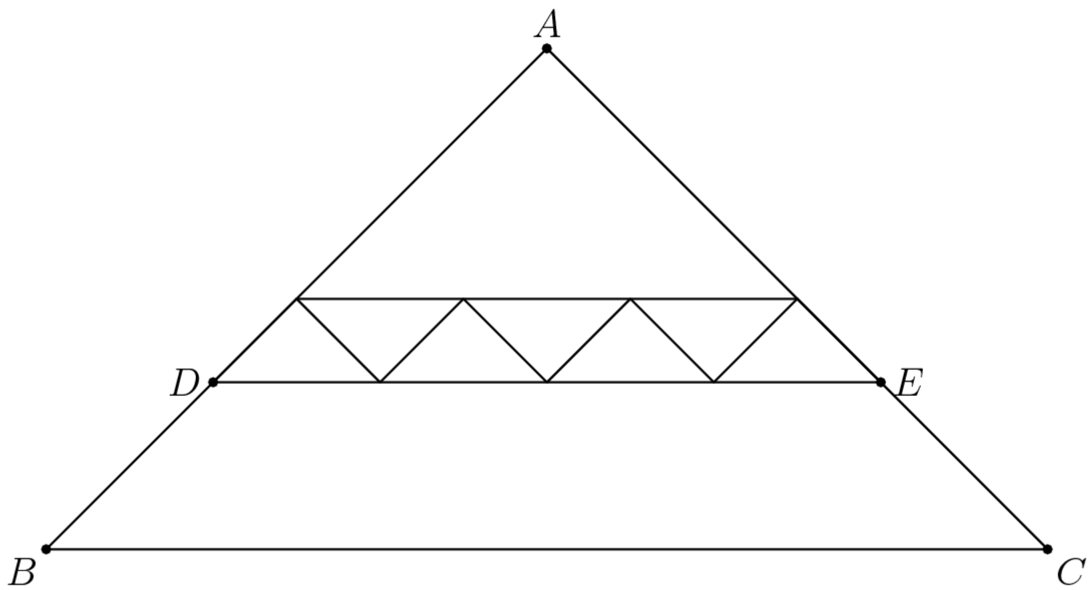
Category: Introductory Number Theory Problems

2018 AMC 10A Problems/Problem 9

Contents

Problem

All of the triangles in the diagram below are similar to isosceles triangle ABC, in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?



$\textbf{(A) } 16 \qquad \textbf{(B) } 18 \qquad \textbf{(C) } 20 \qquad \textbf{(D) } 22 \qquad \textbf{(E) } 24$

Solution 1

Let x be the area of ADE. Note that x is comprised of the 7 small isosceles triangles and a triangle similar to ADE with side length ratio $3:4$ (so an area ratio of $9:16$). Thus, we have

$$x = 7 + \frac{9}{16}x$$

This gives $x = 16$, so the area of $DBCE = 40 - x = \boxed{24}$.

Solution 2

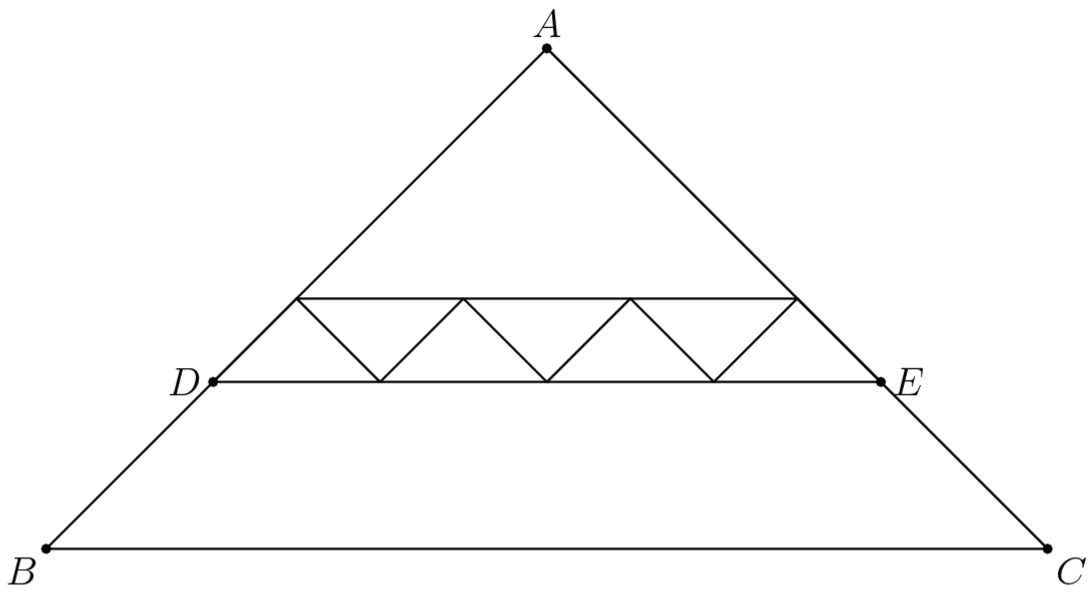
Let the base length of the small triangle be x. Then, there is a triangle ADE encompassing the 7 small triangles and sharing the top angle with a base length of $4x$. Because the area is proportional to the square of the side, let the base BC be $\sqrt{40}x$. Then triangle ADE has an area of 16. So the area is $40 - 16 = \boxed{24}$.

Solution 3

Notice $[DBCE] = [ABC] - [ADE]$. Let the base of the small triangles of area 1 be x, then the base length of $\Delta ADE = 4x$. Notice, $(\frac{DE}{BC})^2 = \frac{1}{40} \implies \frac{x}{BC} = \frac{1}{\sqrt{40}}$, then

$4x = \frac{4BC}{\sqrt{40}} \implies [ADE] = (\frac{4}{\sqrt{40}})^2 \cdot [ABC] = \frac{2}{5}[ABC]$ Thus,

$[DBCE] = [ABC] - [ADE] = [ABC](1 - \frac{2}{5}) = \frac{3}{5} \cdot 40 = \boxed{24}$



Solution by ktong

Solution 4

The area of ADE is 16 times the area of the small triangle, as they are similar and their side ratio is $4:1$. Therefore the area of the trapezoid is $40 - 16 = \boxed{24}$.

Solution 5

You can see that we can create a "stack" of 5 triangles congruent to the 7 small triangles shown here, arranged in a row above those 7, whose total area would be 5. Similarly, we can create another row of 3, and finally 1 more at the top, as follows. We know this cumulative area will be $7+5+3+1=16$, so to find the area of such trapezoid $BCED$, we just take $40-16=\boxed{24}$, like so. ■ --anna0kear

Solution 6

The combined area of the small triangles is 7, and from the fact that each small triangle has an area of 1, we can deduce that the larger triangle above has an area of 9 (as the sides of the triangles are in a proportion of $\frac{1}{3}$, so will their areas have a proportion that is the square of the proportion of their sides, or $\frac{1}{9}$). Thus, the combined area of the top triangle and the trapezoid immediately below is $7+9=16$. The area of trapezoid $BCED$ is thus the area of triangle $ABC - 16 = \boxed{24}$. --lepetitmoulin

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Category: Introductory Geometry Problems

2018 AMC 12A Problems/Problem 9

Contents

Problem

Which of the following describes the largest subset of values of y within the closed interval $[0, \pi]$ for which

$$\sin(x+y) \leq \sin(x) + \sin(y)$$

for every x between 0 and π, inclusive?

$$\textbf{(A) } y=0 \qquad \textbf{(B) } 0\leq y\leq \frac{\pi}{4} \qquad \textbf{(C) } 0\leq y\leq \frac{\pi}{2} \qquad \textbf{(D) } 0\leq y\leq \frac{3\pi}{4} \qquad \textbf{(E) } 0\leq y\leq \pi$$

Solution 1

On the interval $[0, \pi]$ sine is nonnegative; thus $\sin(x+y) = \sin x \cos y + \sin y \cos x \leq \sin x + \sin y$ for all $x, y \in [0, \pi]$. The answer is $\boxed{\textbf{(E) } 0\leq y\leq \pi}$. (CantonMathGuy)

Solution 2

Expanding,

$$\cos y \sin x + \cos x \sin y \leq \sin x + \sin y$$

Let $\sin x = a \geq 0$, $\sin y = b \geq 0$. We have that

$$(\cos y)a + (\cos x)b \leq a + b$$

Comparing coefficients of a and b gives a clear solution: both $\cos y$ and $\cos x$ are less than or equal to one, so the coefficients of a and b on the left are less than on the right. Since $a, b \geq 0$, that means that this equality is always satisfied over this interval, or $\boxed{\textbf{(E) } 0\leq y\leq \pi}$.

Solution 3

If we plug in π, we can see that $\sin(x+\pi) \leq \sin(x)$. Note that since $\sin(x)$ is always nonnegative, $\sin(x+\pi)$ is always nonpositive. So, the inequality holds true when $y = \pi$. The only interval that contains π in the answer choices is $\boxed{\textbf{(E) } 0\leq y\leq \pi}$.

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2018 AMC 10A Problems/Problem 12

How many ordered pairs of real numbers (x, y) satisfy the following system of equations?

$$x + 3y = 3$$

$$\big||x| - |y|\big| = 1$$

$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 4 \qquad \textbf{(E) } 8$

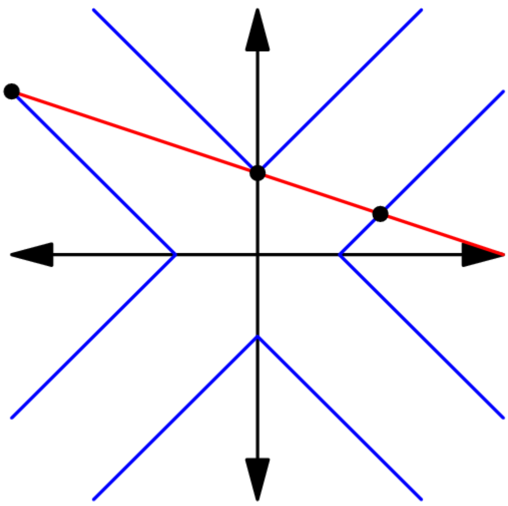
Contents

Solutions

Solution 1

We can solve this by graphing the equations. The second equation looks challenging to graph, but start by graphing it in the first quadrant only (which is easy since the inner absolute value signs can be ignored), then simply reflect that graph into the other quadrants.

The graph looks something like this:



Now, it becomes clear that there are $\boxed{\textbf{(C) } 3}$ intersection points. (pinetree1) BOI

Solution 2

$x + 3y = 3$ can be rewritten to $x = 3 - 3y$. Substituting $3 - 3y$ for x in the second equation will give $||3 - 3y| - y| = 1$. Splitting this question into casework for the ranges of y will give us the total number of solutions.

$\textbf{Case 1:}\, y > 1$: $3 - 3y$ will be negative so $|3 - 3y| = 3y - 3$. $|3y - 3 - y| = |2y - 3| = 1$

Subcase 1: $y > \dfrac{3}{2}$

$2y-3$ is positive so $2y-3=1$ and $y=2$ and $x=3-3(2)=-3$

$$\text{Subcase 2: } 1<y<\frac{3}{2}$$

$2y-3$ is negative so $|2y-3|=3-2y=1$. $2y=2$ and so there are no solutions (y can't equal to 1)

$\textbf{Case 2:}\, y=1$: It is fairly clear that $x=0$.

$\textbf{Case 3:}\, y<1$: $3-3y$ will be positive so $|3-3y-y|=|3-4y|=1$

$$\text{Subcase 1: } y>\frac{4}{3}$$

$3-4y$ will be negative so $4y-3=1 \rightarrow 4y=4$. There are no solutions (again, y can't equal to 1)

$$\text{Subcase 2: } y<\frac{4}{3}$$

$3-4y$ will be positive so $3-4y=1 \rightarrow 4y=2$. $y=\frac{1}{2}$ and $x=\frac{3}{2}$. Thus, the solutions are: $(-3,2),(0,1),\left(\frac{3}{2},\frac{1}{2}\right)$, and the answer is $\boxed{\textbf{(C) } 3}$. $\LaTeX$ edit by pretzel, very minor $\LaTeX$ edits by Bryanli, very very minor $\LaTeX$ edit by ssb02

Solution 3

Note that $||x|-|y||$ can take on either of four values: $x+y$, $x-y$, $-x+y$, $-x-y$. Solving the equations (by elimination, either adding the two equations or subtracting), we obtain the three solutions: $(0,1)$, $(-3,2)$, $(1.5,0.5)$ so the answer is $\boxed{\textbf{(C) } 3}$. One of those equations overlap into $(0,1)$ so there's only 3 solutions.

~trumpeter, ccx09 ~minor edit, XxHalo711

Solution 4

Just as in solution 2, we derive the equation $||3-3y|-|y||=1$. If we remove the absolute values, the equation collapses into four different possible values. $3-2y$, $3-4y$, $2y-3$, and $4y-3$, each equal to either 1 or -1. Remember that if $P-Q=a$, then $Q-P=-a$. Because we have already taken 1 and -1 into account, we can eliminate one of the conjugates of each pair, namely $3-2y$ and $2y-3$, and $3-4y$ and $4y-3$. Find the values of y when $3-2y=1$, $3-2y=-1$, $3-4y=1$ and $3-4y=-1$. We see that $3-2y=1$ and $3-4y=-1$ give us the same value for y, so the answer is $\boxed{\textbf{(C) } 3}$

~Zeric Hang

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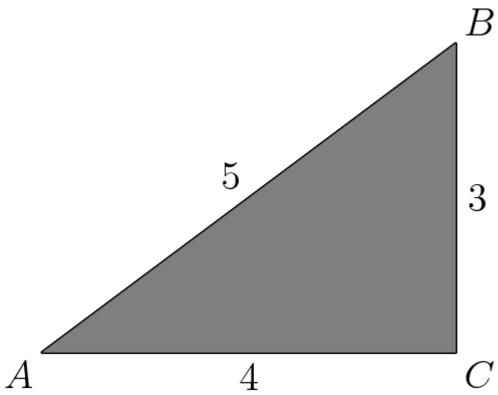
Category: Intermediate Algebra Problems

2018 AMC 10A Problems/Problem 13

Contents

Problem

A paper triangle with sides of lengths 3,4, and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease?



$\textbf{(A) } 1+\frac{1}{2}\sqrt{2} \qquad \textbf{(B) } \sqrt{3} \qquad \textbf{(C) } \frac{7}{4} \qquad \textbf{(D) } \frac{15}{8} \qquad \textbf{(E) } 2$

Solution 1

First, we need to realize that the crease line is just the perpendicular bisector of side AB, the hypotenuse of right triangle $\triangle ABC$. Call the midpoint of AB point D. Draw this line and call the intersection point with AC as E. Now, $\triangle ACB$ is similar to $\triangle ADE$ by AA similarity. Setting up the ratios, we find that

$$\frac{BC}{AC}=\frac{DE}{AD} \Rightarrow \frac{3}{4}=\frac{DE}{\frac{5}{2}} \Rightarrow DE=\frac{15}{8}.$$

Thus, our answer is $\boxed{\textbf{D) } \frac{15}{8}}$.

~Nivek

Note

In general, whenever we are asked to make a crease, think about that crease as a line of reflection over which the diagram is reflected. This is why the crease must be the perpendicular bisector of AB, because A must be reflected onto B. (by pulusona)

Solution 2

Use the ruler and graph paper you brought to quickly draw a 3-4-5 triangle of any scale (don't trust the diagram in the booklet). Very carefully fold the acute vertices together and make a crease. Measure the crease with the ruler. If you were reasonably careful, you should see that it measures somewhat more than $\frac{7}{4}$ units and somewhat less than 2 units.

The only answer choice in range is $\boxed{\textbf{D) } \frac{15}{8}}$.

This is pretty much a cop-out, but it's allowed in the rules technically.

See Also

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Category: Introductory Geometry Problems

2018 AMC 10A Problems/Problem 17

Contents

Problem

Let S be a set of 6 integers taken from $\{1, 2, \dots, 12\}$ with the property that if a and b are elements of S with $a < b$, then b is not a multiple of a. What is the least possible value of an element in S?

$\textbf{(A)}\ 2\qquad\textbf{(B)}\ 3\qquad\textbf{(C)}\ 4\qquad\textbf{(D)}\ 5\qquad\textbf{(E)}\ 7$

Solution

If we start with 1, we can include nothing else, so that won't work.

If we start with 2, we would have to include every odd number except 1 to fill out the set, but then 3 and 9 would violate the rule, so that won't work.

Experimentation with 3 shows it's likewise impossible. You can include 7, 11, and either 5 or 10 (which are always safe). But after adding either 4 or 8 we have nowhere else to go.

Finally, starting with 4, we find that the sequence $4, 5, 6, 7, 9, 11$ works, giving us $\boxed{\textbf{(C)}\ 4}$. (Random_Guy)

Solution 2

We know that all the odd numbers (except 1) can be used.

$$3, 5, 7, 9, 11$$

Now we have 7 to choose from for the last number (out of $1, 2, 4, 6, 8, 10, 12$). We can eliminate 1, 2, 10, and 12, and we have $4, 6, 8$ to choose from. But wait, 9 is a multiple of 3! Now we have to take out either 3 or 9 from the list. If we take out 9, none of the numbers would work, but if we take out 3, we get:

$$4, 5, 6, 7, 9, 11$$

So the least number is 4, so the answer is $\boxed{\textbf{(C)}\ 4}$.

-Baolan

See Also

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2018 AMC 10A Problems/Problem 18

Contents

Problem

How many nonnegative integers can be written in the form

$$a_7\cdot3^7+a_6\cdot3^6+a_5\cdot3^5+a_4\cdot3^4+a_3\cdot3^3+a_2\cdot3^2+a_1\cdot3^1+a_0\cdot3^0,$$

where $a_i\in\{-1,0,1\}$ for $0\le i \le 7$?

$\textbf{(A) } 512 \qquad \textbf{(B) } 729 \qquad \textbf{(C) } 1094 \qquad \textbf{(D) } 3281 \qquad \textbf{(E) } 59,048$

Solution 1

This looks like balanced ternary, in which all the integers with absolute values less than $\frac{3^n}{2}$ are represented in n digits. There are 8 digits. Plugging in 8 into the formula for the balanced ternary gives a maximum bound of $|x|=3280.5$, which means there are 3280 positive integers, 0, and 3280 negative integers. Since we want all nonnegative integers, there are $3280+1=\boxed{3281}$ integers or $\boxed{D}$.

Solution 2

Note that all numbers formed from this sum are either positive, negative or zero. The number of positive numbers formed by this sum is equal to the number of negative numbers formed by this sum, because of symmetry. There is only one way to achieve a sum of zero, if all $a_i = 0$. The total number of ways to pick a_i from $i = 1, 2, 3, ...7$ is $3^8 = 6561$. $\frac{6561-1}{2} = 3280$ gives the number of possible negative integers. The question asks for the number of nonnegative integers, so subtracting from the total gives $6561-3280 = \boxed{3281}$. (RegularHexagon)

Solution 3

Note that the number of total possibilities (ignoring the conditions set by the problem) is $3^8 = 6561$. So, E is clearly unrealistic.

Note that if a_7 is 1, then it's impossible for

$$a_7\cdot3^7+a_6\cdot3^6+a_5\cdot3^5+a_4\cdot3^4+a_3\cdot3^3+a_2\cdot3^2+a_1\cdot3^1+a_0\cdot3^0,$$

to be negative. Therefore, if a_7 is 1, there are $3^7 = 2187$ possibilities. (We also must convince ourselves that these 2187 different sets of coefficients must necessarily yield 2187 different integer results.)

As A, B, and C are all less than 2187, the answer must be $\boxed{(D) 3281}$

Solution 4

Note that we can do some simple casework: If $a_7 = 1$, then we can choose anything for the other 7 variables, so this give us 3^7. If $a_7 = 0$ and $a_6 = 1$, then we can choose anything for the other 6 variables, giving us 3^6. If $a_7=0$, $a_6=0$, and $a_5=1$, then we have 3^5. Continuing in this vein, we have $3^7+3^6+\cdots+3^1+3^0$ ways to choose the variables' values, except we have to add 1 because we haven't counted the case where all variables are 0. So our total sum is $\boxed{(D) 3281}$. Note that we have counted all possibilities, because the largest positive positive power of 3 must be greater than or equal to the largest negative positive power of 3, for the number to be nonnegative.

Solution 5

The key is to realize that this question is basically taking place in $a \in \{0, 1, 2\}$ if each value of a was increased by 1, essentially making it into base 3. Then the range would be from $0 \cdot 3^7 + 0 \cdot 3^6 + 0 \cdot 3^5 + 0 \cdot 3^4 + 0 \cdot 3^3 + 0 \cdot 3^2 + 0 \cdot 3^1 + 0 \cdot 3^0 = 0$ to $2 \cdot 3^7 + 2 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^4 + 2 \cdot 3^3 + 2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0 = 3^8 - 1 = 6561 - 1 = 6560$, yielding 6561 different values. Since the distribution for all $a_i \in \{-1, 0, 1\}$ the question originally gave is symmetrical, we retain the 3280 positive integers and one 0 but discard the 3280 negative integers. Thus, we are left with the answer, $\boxed{\textbf{(D)} 3281}$. $\blacksquare$ --anna0kear

Solution 6

First, set $a_i = 0$ for all $i \geq 1$. The range would be the integers for which $[-1, 1]$. If $a_i = 0$ for all $i \geq 2$, our set expands to include all integers such that $-4 \leq \mathbb{Z} \leq 4$. Similarly, when $i \geq 3$ we get $-13 \leq \mathbb{Z} \leq 13$, and when $i \geq 4$ the range is $-40 \leq \mathbb{Z} \leq 40$. The pattern continues until we reach $i = 7$, where $-3280 \leq \mathbb{Z} \leq 3280$. Because we are only looking for positive integers, we filter out all $\mathbb{Z} < 0$, leaving us with all integers between $0 \leq \mathbb{Z} \leq 3280$, inclusive. The answer becomes $\boxed{(D) 3281}$. $\blacksquare$ --anna0kear

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Category: Intermediate Number Theory Problems

2018 AMC 12A Problems/Problem 14

Contents

Problem

The solutions to the equation $\log_{3x} 4 = \log_{2x} 8$, where x is a positive real number other than $\frac{1}{3}$ or $\frac{1}{2}$, can be written as $\frac{p}{q}$ where p and q are relatively prime positive integers. What is $p + q$?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 13 \qquad \textbf{(C)}\ 17 \qquad \textbf{(D)}\ 31 \qquad \textbf{(E)}\ 35$

Solution 1

Base switch to log 2 and you have $\frac{\log_2 4}{\log_2 3x} = \frac{\log_2 8}{\log_2 2x}$.

$$\frac{2}{\log_2 3x} = \frac{3}{\log_2 2x}$$

$$2 * \log_2 2x = 3 * \log_2 3x$$

Then $\log_2 (2x)^2 = \log_2 (3x)^3$. so $4x^2 = 27x^3$ and we have $x = \frac{4}{27}$ leading to $\boxed{\textbf{(D)} 31}$ (jeremylu)

Solution 2

If you multiply both sides by $\log_2(3x)$

then it should come out to $\log_2(3x)$ * $\log_{3x}(4)$ = $\log_2 3x$ * $\log_{2x}(8)$

that then becomes $\log_2(4)$ * $\log_{3x}(3x)$ = $\log_2(8)$ * $\log_{2x}(3x)$

which simplifies to 2*1 = 3$\log_{2x}(3x)$

so now $\frac{2}{3}$ = $\log_{2x}(3x)$ putting in exponent form gets

$(2x)^2$ = $(3x)^3$

so 4x^2 = 27x^3

dividing yields x = 4/27 and

4+27 = $\boxed{\textbf{(D)} 31}$

```
- Pikachu13307
```

Solution 3

We can convert both 4 and 8 into 2^2 and 2^3, respectively, giving:

$$2\log_{3x}(2) = 3\log_{2x}(2)$$

Converting the bases of the right side, we get $\log_{2x} 2 = \frac{\ln 2}{\ln(2x)}$

$$\frac{2}{3} * \log_{3x}(2) = \frac{\ln 2}{\ln(2x)}$$

$$2^{\frac{2}{3}} = (3x)^{\frac{\ln 2}{\ln(2x)}}$$

$$\frac{2}{3} * \ln 2 = \frac{\ln 2}{\ln(2x)} * \ln(3x)$$

Dividing both sides by $\ln 2$, we get

$$\frac{2}{3} = \frac{\ln(3x)}{\ln(2x)}$$

Which simplifies to

$$2\ln(2x) = 3\ln(3x)$$

Log expansion allows us to see that

$2\ln 2 + 2\ln(x) = 3\ln 3 + 3\ln(x)$, which then simplifies to

$$\ln(x) = 2\ln 2 - 3\ln 3$$

Thus,

$$x = e^{2\ln 2 - 3\ln 3} = \frac{e^{2\ln 2}}{e^{3\ln 3}}$$

And

$$x = \frac{2^2}{3^3} = \frac{4}{27} = \boxed{\textbf{(D)} 31}$$

-lepetitmoulin

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2018 AMC 10A Problems/Problem 20

A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called $symmetric$ if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

$\textbf{(A)} \text{ 510} \qquad \textbf{(B)} \text{ 1022} \qquad \textbf{(C)} \text{ 8190} \qquad \textbf{(D)} \text{ 8192} \qquad \textbf{(E)} \text{ 65,534}$

Solution 1

Draw a 7×7 square.

K	J	H	G	H	J	K
J	F	E	D	E	F	J
H	E	C	B	C	E	H
G	D	B	A	B	D	G
H	E	C	B	C	E	H
J	F	E	D	E	F	J
K	J	H	G	H	J	K

Start from the center and label all protruding cells symmetrically.

More specifically, since there are 4 given lines of symmetry (2 diagonals, 1 vertical, 1 horizontal) and they split the plot into 8 equivalent sections, we can take just one-eighth and study it in particular. Each of these sections has 10 distinct sub-squares, whether partially or in full. So since each can be colored either white or black, we choose $2^{10} = 1024$ but then subtract the 2 cases where all are white or all are black. That leaves us with $\boxed{(B)}, 1022.$ ■

There are only ten squares we get to actually choose, and two independent choices for each, for a total of $2^{10} = 1024$ codes. Two codes must be subtracted (due to the rule that there must be at least one square of each color) for an answer of $\boxed{\textbf{(B)} \text{ } 1022}$.

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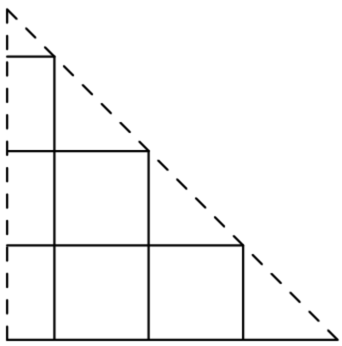
~Nosysnow
```

Note that this problem is very similar to the 1996 AIME Problem 7.

Solution 2



Imagine folding the scanning code along its lines of symmetry. There will be 10 regions which you have control over coloring. Since we must subtract off 2 cases for the all-black and all-white cases, the answer is $2^{10} - 2 = \boxed{\textbf{(B)} \text{ 1022.}}$

-EatingStuff

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Category: Intermediate Combinatorics Problems

2018 AMC 10A Problems/Problem 21

Contents

Problem

Which of the following describes the set of values of a for which the curves $x^2+y^2=a^2$ and $y=x^2-a$ in the real xy-plane intersect at exactly 3 points?

$\textbf{(A) }a=\frac14 \qquad \textbf{(B) }\frac14 < a < \frac12 \qquad \textbf{(C) }a>\frac14 \qquad \textbf{(D) }a=\frac12 \qquad \textbf{(E) }a>\frac12$

Solution 1

Substituting $y=x^2-a$ into $x^2+y^2=a^2$, we get

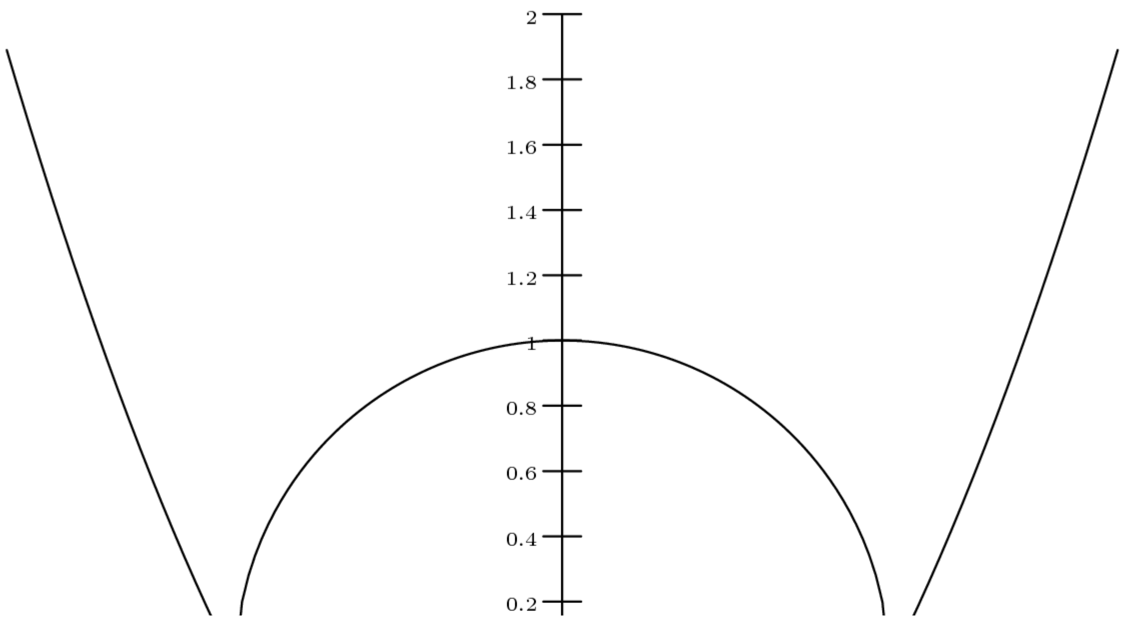
$$x^2+(x^2-a)^2=a^2 \implies x^2+x^4-2ax^2=0 \implies x^2(x^2-(2a-1))=0$$

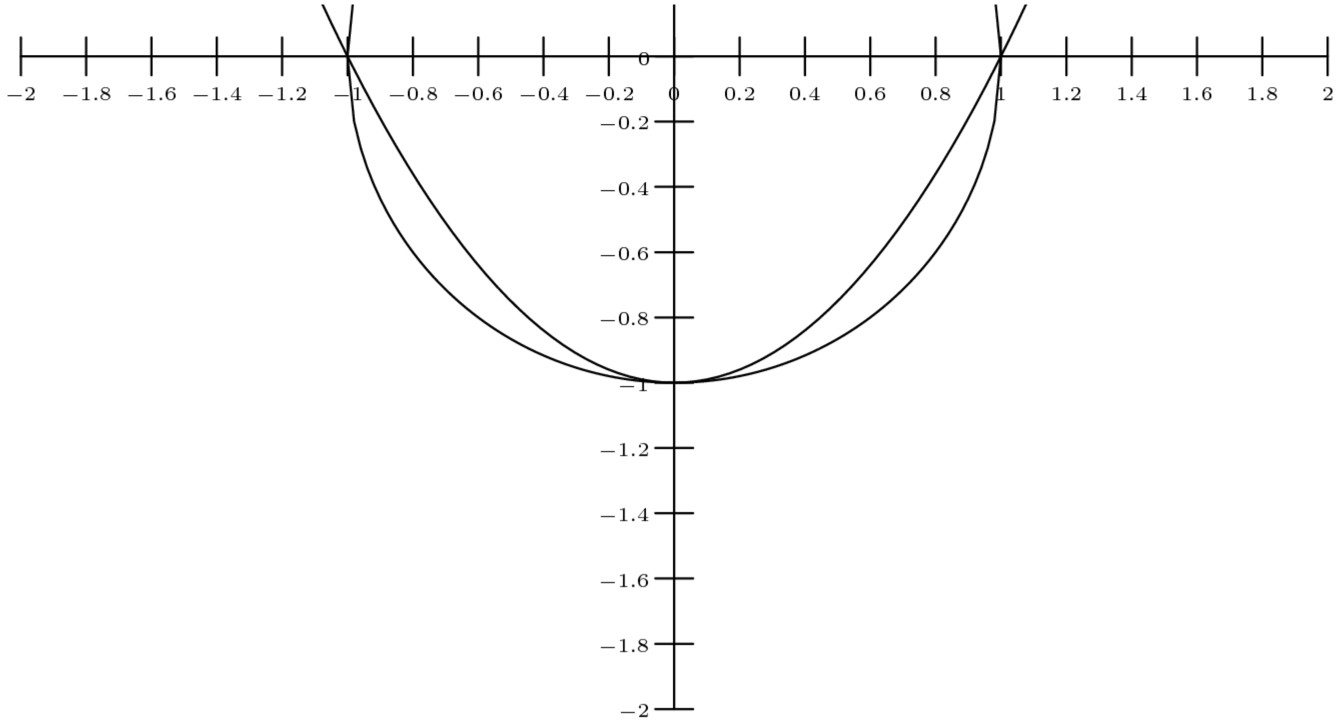
Since this is a quartic, there are 4 total roots (counting multiplicity). We see that $x=0$ always at least one intersection at $(0,-a)$ (and is in fact a double root).

The other two intersection points have x coordinates $\pm\sqrt{2a-1}$. We must have $2a-1>0$, otherwise we are in the case where the parabola lies entirely above the circle (tangent to it at the point $(0,a)$). This only results in a single intersection point in the real coordinate plane. Thus, we see $\boxed{\textbf{(E) }a>\frac12}$.

(projecteulerlover)

Solution 2





Looking at a graph, it is obvious that the two curves intersect at (0, -a). We also see that if the parabola goes 'in' the circle. Then, by going out of it (as it will), it will intersect five times, an impossibility. Thus we only look for cases where the parabola becomes externally tangent to the circle. We have $x^2 - a = -\sqrt{a^2 - x^2}$. Squaring both sides and solving yields $x^4 - (2a-1)x^2 = 0$. Since x = 0 is already accounted for, we only need to find 1 solution for $x^2 = 2a - 1$, where the right hand side portion is obviously increasing. Since a = 1/2 begets x = 0 (an overcount), we have $\boxed{\textbf{(E) } a > \frac{1}{2}}$ is the right answer.

Solution by JohnHankock

Solution 3

This describes a unit parabola, with a circle centered on the axis of symmetry and tangent to the vertex. As the curvature of the unit parabola at the vertex is 2, the radius of the circle that matches it has a radius of $\frac{1}{2}$. This circle is tangent to an infinitesimally close pair of points, one on each side. Therefore, it is tangent to only 1 point. When a larger circle is used, it is tangent to 3 points because the points on either side are now separated from the vertex. Therefore, $\boxed{\textbf{(E) } a > \frac{1}{2}}$ is correct.

$QED\blacksquare$

Solution 4

Notice, the equations are of that of a circle of radius a centered at the origin and a parabola translated down by a units. They always intersect at the point $(0, a)$, and they have symmetry across the y-axis, thus, for them to intersect at exactly 3 points, it suffices to find the y solution.

First, rewrite the second equation to $y = x^2 - a \implies x^2 = y + a$ And substitute into the first equation: $y + a + y^2 = a^2$ Since we're only interested in seeing the interval in which a can exist, we find the discriminant: $1 - 4a + 4a^2$. This value must not be less than 0 (It is the square root part of the quadratic formula). To find when it is 0, we find the roots:

$$4a^2 - 4a + 1 = 0 \implies a = \frac{4 \pm \sqrt{16-16}}{8} = \frac{1}{2}$$

Since $\lim_{a\to\infty}(4a^2-4a+1)=\infty$, our range is $\boxed{\textbf{(E) } a > \frac{1}{2}}$.

Solution by ktong

Solution 5 (Cheating with Answer Choices)

Simply plug in $a=0,\frac{1}{2},\frac{1}{4},1$ and solve the systems. (This shouldn't take too long.) And then realize that only $a=1$ yields three real solutions for x, so we are done and the answer is $\boxed{\textbf{(E) } a > \frac{1}{2}}$.

~ ccx09

Solution 6 (Calculus Needed)

In order to solve for the values of a, we need to just count multiplicities of the roots when the equations are set equal to each other: in other words, take the derivative. We know that $\sqrt{x^2-a^2}=x^2-a$. Now, we take square of both sides, and rearrange to obtain $x^4-(2a+1)x^2+2a^2=0$. Now, we make take the second derivative of the equation to obtain $6x^2-(4a+2)=0$. Now, we must take discriminant. Since we need the roots of that equation to be real and not repetitive (otherwise they would not intersect each other at three points), the discriminant must be greater than zero. Thus,

$b^2-4ac>0\to 0-4(6)(4a+2)>0\to a>\frac{1}{2}$ The answer is $\boxed{\textbf{(E) } a > \frac{1}{2}}$ and we are done.

~awesome1st

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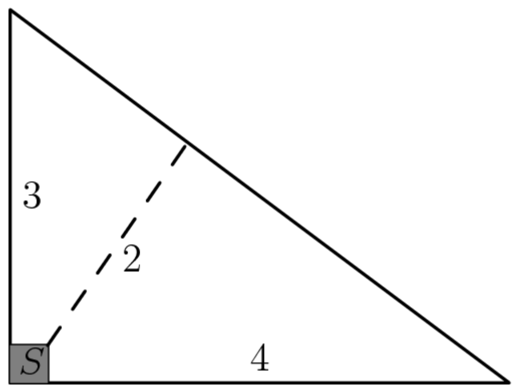
Category: Introductory Algebra Problems

2018 AMC 10A Problems/Problem 23

Contents

Problem

Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



$\textbf{(A) } \frac{25}{27} \qquad \textbf{(B) } \frac{26}{27} \qquad \textbf{(C) } \frac{73}{75} \qquad \textbf{(D) } \frac{145}{147} \qquad \textbf{(E) } \frac{74}{75}$

Solution 1

Let the square have side length x. Connect the upper-right vertex of square S with the two vertices of the triangle's hypotenuse. This divides the triangle in several regions whose areas must add up to the area of the whole triangle, which is 6.

Square S has area x^2, and the two thin triangle regions have area $\dfrac{x(3-x)}{2}$ and $\dfrac{x(4-x)}{2}$. The final triangular region with the hypotenuse as its base and height 2 has area 5. Thus, we have

$$x^2+\frac{x(3-x)}{2}+\frac{x(4-x)}{2}+5=6$$

Solving gives $x=\dfrac{2}{7}$. The area of S is $\dfrac{4}{49}$ and the desired ratio is $\dfrac{6-\frac{4}{49}}{6}=\boxed{\frac{145}{147}}$.

Alternatively, once you get $x=\dfrac{2}{7}$, you can avoid computation by noticing that there is a denominator of 7, so the answer must have a factor of 7 in the denominator, which only $\boxed{\frac{145}{147}}$ does.

Solution 2

Let the square have side length s. If we were to extend the sides of the square further into the triangle until they intersect on point on the hypotenuse, we'd have a similar right triangle formed between the hypotenuse and the two new lines, and 2 smaller similar triangles that share a side of length 2. Using the side-to-side ratios of these triangles, we can find that the length of the big similar triangle is $\frac{5}{3}(2) = \frac{10}{3}$. Now, let's extend this big similar right triangle to the left until it hits the side of length 3. Now, the length is $\frac{10}{3} + s$, and using the ratios of the side lengths, the height is $\frac{3}{4}(\frac{10}{3} + s) = \frac{5}{2} + \frac{3s}{4}$. Looking at the diagram, if we add the height of this triangle to the side length of the square, we'd get 3, so

$$\frac{5}{2}+\frac{3s}{4}+s = \frac{5}{2}+\frac{7s}{4} = 3\frac{7s}{4} = \frac{1}{2}s = \frac{2}{7} \implies \text{area of square is } (\frac{2}{7})^2 = \frac{4}{49}$$

Now comes the easy part: finding the ratio of the areas:
$\frac{3\cdot 4\cdot \frac{1}{2} - \frac{4}{49}}{3\cdot 4\cdot \frac{1}{2}} = \frac{6-\frac{4}{49}}{6} = \frac{294-4}{294} = \frac{290}{294} = \boxed{\frac{145}{147}}$.

Solution by ktong

Solution 3

We use coordinate geometry. Let the right angle be at $(0, 0)$ and the hypotenuse be the line $3x + 4y = 12$ for $0 \le x \le 3$. Denote the position of S as (s, s), and by the point to line distance formula, we know that

$$\frac{|3s + 4s - 12|}{5} = 2$$

$$\Rightarrow |7s - 12| = 10$$

Obviously $s < \frac{22}{7}$, so $s = \frac{2}{7}$, and from here the rest of the solution follows to get $\boxed{\frac{145}{147}}$.

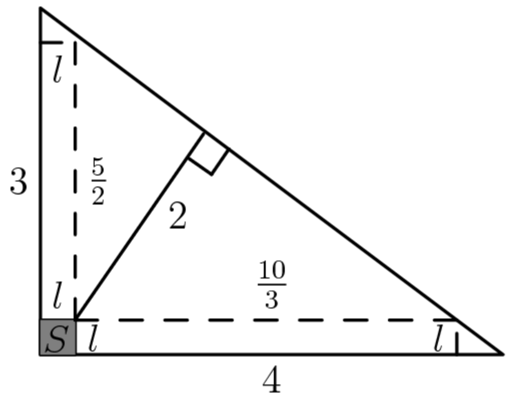
Solution 4

Let the side length of the square be x. First off, let us make a similar triangle with the segment of length 2 and the top-right corner of S. Therefore, the longest side of the smaller triangle must be $2 \cdot \frac{5}{4} = \frac{5}{2}$. We then do operations with that side in terms of x. We subtract x from the bottom, and $\frac{3x}{4}$ from the top. That gives us the equation of $3 - \frac{7x}{4} = \frac{5}{2}$. Solving,

$$12 - 7x = 10 \implies x = \frac{2}{7}.$$

Thus, $x^2 = \frac{4}{49}$, so the fraction of the triangle (area 6) covered by the square is $\frac{2}{147}$. The answer is then $\boxed{\frac{145}{147}}$.

Solution 5



On the diagram above, find two smaller triangles similar to the large one with side lengths 3, 4, and 5; consequently, the segments with length $\frac{5}{2}$ and $\frac{10}{3}$.

Find an expression for l: using the hypotenuse, we can see that $\frac{3}{2}+\frac{8}{3}+\frac{5}{4}l+\frac{5}{3}l=5$. Simplifying, $\frac{35}{12}l=\frac{5}{6}$, or $l=2/7$.

A different calculation would yield $l+\frac{3}{4}l+\frac{5}{2}=3$, so $\frac{7}{4}l=\frac{1}{2}$. In other words, $l=\frac{2}{7}$, while to check, $l+\frac{4}{3}l+\frac{10}{3}=4$. As such, $\frac{7}{3}l=\frac{2}{3}$, and $l=\frac{2}{7}$.

Finally, we get $A(\square S)=l^2=\frac{4}{49}$, to finish. As a proportion of the triangle with area 6, the answer would be $1-\frac{4}{49\cdot 6}=1-\frac{2}{147}=\frac{145}{147}$, so $\boxed{D}$ is correct. $\blacksquare$ --anna0kear

Solution 6

Let s be the side length of the square. The area of the triangle is 6. Connect the inside corner of the square to the three corners. Then, the area of the triangle is also $5+\frac{3}{2}s+2s=5+\frac{7}{2}s$. Solving gives $s=\frac{2}{7}$. That makes the answer $\frac{6-\left(\frac{2}{7}\right)^2}{6}=\frac{145}{147}$. $\boxed{\textbf{D.}}$

\

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2018 AMC 10A Problems/Problem 24

Contents

Problem

Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

$\textbf{(A) }60 \qquad \textbf{(B) }65 \qquad \textbf{(C) }70 \qquad \textbf{(D) }75 \qquad \textbf{(E) }80 \qquad$

Solution 1

Let $BC = a$, $BG = x$, $GC = y$, and the length of the perpendicular to BC through A be h. By angle bisector theorem, we have that

$$\frac{50}{x} = \frac{10}{y},$$

where $y = -x + a$. Therefore substituting we have that $BG = \frac{5a}{6}$. By similar triangles, we have that $DF = \frac{5a}{12}$, and the height of this trapezoid is $\frac{h}{2}$. Then, we have that $\frac{ah}{2} = 120$. We wish to compute $\frac{5a}{8} \cdot \frac{h}{2}$, and we have that it is $\boxed{75}$ by substituting.

(rachanamadhu)

I may have read this solution incorrectly, but it seems to me that the author mistakenly assumed that the angle bisector is a perpendicular bisector, which is false since the triangle is not isosceles. -bobert1

Solution 2

$\overline{DE}$ is midway from A to $\overline{BC}$, and $DE = \frac{BC}{2}$. Therefore, $\triangle ADE$ is a quarter of the area of $\triangle ABC$, which is 30. Subsequently, we can compute the area of quadrilateral $BDEC$ to be $120 - 30 = 90$. Using the angle bisector theorem in the same fashion as the previous problem, we get that $\overline{BG}$ is 5 times the length of $\overline{GC}$. We want the larger piece, as described by the problem. Because the heights are identical, one area is 5 times the other, and $\frac{5}{6} \cdot 90 = \boxed{75}$.

Solution 3

The area of $\triangle ABG$ to the area of $\triangle ACG$ is $5:1$ by Law of Sines. So the area of $\triangle ABG$ is 100. Since $\overline{DE}$ is the midsegment of $\triangle ABC$, so $\overline{DF}$ is the midsegment of $\triangle ABG$. So the area of $\triangle ACG$ to the area of $\triangle ABG$ is $1:4$, so the area of $\triangle ACG$ is 25, by similar triangles. Therefore the area of quad $FDBG$ is $100 - 25 = \boxed{75}$ (steakfails)

Solution 4

The area of quadrilateral $FDBG$ is the area of $\triangle ABG$ minus the area of $\triangle ADF$. Notice, $\overline{DE}||\overline{BC}$, so $\triangle ABG \sim \triangle ADF$, and since $\overline{AD} : \overline{AB} = 1 : 2$, the area of $\triangle ADF : \triangle ABG = (1:2)^2 = 1:4$. Given that the area of $\triangle ABC$ is 120, using $\frac{bh}{2}$ on side AB yields $\frac{50h}{2} = 120 \implies h = \frac{240}{50} = \frac{24}{5}$. Using the Angle Bisector Theorem, $\overline{BG} : \overline{BC} = 50 : (10+50) = 5 : 6$, so the height of $\triangle ABG : \triangle ACB = 5 : 6$. Therefore our answer is

$$[FDBG] = [ABG] - [ADF] = [ABG](1 - \frac{1}{4}) = \frac{3}{4} \cdot \frac{bh}{2} = \frac{3}{8} \cdot 50 \cdot \frac{5}{6} \cdot \frac{24}{5} = \frac{3}{8} \cdot 200 = \boxed{75}$$

-Solution by ktong

See Also

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Category: Intermediate Geometry Problems

2018 AMC 12A Problems/Problem 19

Contents

Problem

Let A be the set of positive integers that have no prime factors other than 2, 3, or 5. The infinite sum

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{12}+\frac{1}{15}+\frac{1}{16}+\frac{1}{18}+\frac{1}{20}+\cdots$$

of the reciprocals of the elements of A can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

$\textbf{(A) } 16 \qquad \textbf{(B) } 17 \qquad \textbf{(C) } 19 \qquad \textbf{(D) } 23 \qquad \textbf{(E) } 36$

Solution

It's just

$$\sum_{a\ge 0}\frac{1}{2^a}\sum_{b\ge 0}\frac{1}{3^b}\sum_{c\ge 0}\frac{1}{5^c}=2\cdot\frac{3}{2}\cdot\frac{5}{4}=\frac{15}{4}\Rightarrow\textbf{(C)}.$$

since this represents all the numbers in the denominator.

(ayushk)

Solution 2

Separate into 7 separate infinite series's so we can calculate each and find the original sum. The first infinite sequence shall be all the reciprocals of the powers of 2, the second shall be reciprocals of the powers of 3, and the third is reciprocals of the powers of 5. We can easily calculate these to be $1, 1/2, 1/4$ respectively. The fourth infinite series shall be all real numbers in the form $1/(2^a3^b)$, where a and b are greater than or equal to 1. The fifth is all real numbers in the form $1/(2^a5^b)$, where a and b are greater than or equal to 1. The sixth is all real numbers in the form $1/(3^a5^b)$, where a and b are greater than or equal to 1. The seventh infinite series is all real numbers in the form $1/(2^a3^b5^c)$, where a and b and c are greater than or equal to 1. Let us denote the first sequence as a_1, the second as a_2, etc. We know $a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/4$, let us find a_4. factoring out $1/6$ from the terms in this subsequence, we would get $a_4 = 1/6(1 + a_1 + a_2 + a_4)$. Knowing a_1 and a_2, we can substitute and solve for a_4, and we get $1/2$. If we do the similar procedures for the fifth and sixth sequences, we can solve for them too, and we get after solving them $1/4$ and $1/8$. Finally, for the seventh sequence, we see $a_7 = 1/30(a_8)$, where a_8 is the infinite series the problem is asking us to solve for. The sum of all seven subsequences will equal the one we are looking for, so solving, we get $1 + 1/2 + 1/4 + 1/2 + 1/4 + 1/8 + 1/30(a_8) = a_8$, but when we separated the sequence into its parts, we ignored the $1/1$, so adding in the 1, we get $1 + 1 + 1/2 + 1/4 + 1/2 + 1/4 + 1/8 + 1/30(a_8) = a_8$, which when we solve for, we get $29/8 = 29/30(a_8)$, $1/8 = 1/30(a_8)$, $30/8 = (a_8)$, $15/4 = (a_8)$. So our answer is 15/4, but we are asked to add the numerator and denominator, which sums up to 19, which is the answer.

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2018 AMC 12A Problems/Problem 20

Contents

Problem

Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the midpoint of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, the length CI can be written as $\frac{a-\sqrt{b}}{c}$, where a, b, and c are positive integers and b is not divisible by the square of any prime. What is the value of $a+b+c$?

$\textbf{(A) }9 \qquad \textbf{(B) }10 \qquad \textbf{(C) }11 \qquad \textbf{(D) }12 \qquad \textbf{(E) }13$

Solution 1

Observe that $\triangle EMI$ is isosceles right (M is the midpoint of diameter arc EI), so $MI = 2, MC = \frac{3}{\sqrt{2}}$. With $\angle MCI = 45^\circ$, we can use Law of Cosines to determine that $CI = \frac{3 \pm \sqrt{7}}{2}$. The same calculations hold for BE also, and since $CI < BE$, we deduce that CI is the smaller root, giving the answer of $\boxed{12}$. (trumpeter)

Solution 2 (Using Ptolemy)

We first claim that $\triangle EMI$ is isosceles and right.

Proof: Construct $\overline{MF} \perp \overline{AB}$ and $\overline{MG} \perp \overline{AC}$. Since $\overline{AM}$ bisects $\angle BAC$, one can deduce that $MF = MG$. Then by AAS it is clear that $MI = ME$ and therefore $\triangle EMI$ is isosceles. Since quadrilateral $AIME$ is cyclic, one can deduce that $\angle EMI = 90^\circ$. Q.E.D.

Since the area of $\triangle EMI$ is 2, we can find that $MI = ME = 2, EI = 2\sqrt{2}$

Since M is the mid-point of $\overline{BC}$, it is clear that $AM = \frac{3\sqrt{2}}{2}$.

Now let $AE = a$ and $AI = b$. By Ptolemy's Theorem, in cyclic quadrilateral $AIME$, we have $2a + 2b = 6$. By Pythagorean Theorem, we have $a^2 + b^2 = 8$. One can solve the simultaneous system and find $b = \frac{3+\sqrt{7}}{2}$. Then by deducting the length of $\overline{AI}$ from 3 we get $CI = \frac{3-\sqrt{7}}{2}$, giving the answer of $\boxed{12}$. (Surefire2019)

Solution 3 (More Elementary)

Like above, notice that $\triangle EMI$ is isosceles and right, which means that $\frac{ME \cdot MI}{2} = 2$, so $MI^2 = 4$ and $MI = 2$. Then construct $\overline{MF} \perp \overline{AB}$ and $\overline{MG} \perp \overline{AC}$ as well as $\overline{MI}$. It's clear that $MG^2 + GI^2 = MI^2$ by Pythagorean, so knowing that $MG = \frac{AB}{2} = \frac{3}{2}$ allows one to solve to get $GI = \frac{\sqrt{7}}{2}$. By just looking at the diagram, $CI = AC - MF - GI = \frac{3-\sqrt{7}}{2}$. The answer is thus $3+7+2=12$.

Solution 4 (Coordinate Geometry)

Let A lie on $(0,0)$, E on $(0,y)$, I on $(x,0)$, and M on $(\frac{3}{2},\frac{3}{2})$. Since $AIME$ is cyclic, $\angle EMI$ (which is opposite of another right angle) must be a right angle; therefore, $\vec{ME}\cdot\vec{MI} =< \frac{-3}{2}, y-\frac{3}{2} > \cdot < x-\frac{3}{2}, -\frac{3}{2} >= 0$. Compute the dot product to arrive at the relation $y = 3 - x$. We can set up another equation involving the area of $\triangle EMI$ using the Shoelace Theorem. This is $2 = (\frac{1}{2})[(\frac{3}{2})(y-\frac{3}{2}) + (x)(-y) + (x+\frac{3}{2})(\frac{3}{2})]$. Multiplying, substituting $3-x$ for y, and simplifying, we get $x^2 - 3x + \frac{1}{2} = 0$. Thus, $(x,y) = (\frac{3\pm\sqrt{7}}{2}, \frac{3\mp\sqrt{7}}{2})$. But $AI > AE$, meaning $x = AI = \frac{3+\sqrt{7}}{2} \rightarrow CI = 3 - \frac{3+\sqrt{7}}{2} = \frac{3-\sqrt{7}}{2}$, and the final answer is $3+7+2=\boxed{12}$.

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2018 AMC 12A Problems/Problem 21

Contents

Problem

Which of the following polynomials has the greatest real root?
$\textbf{(A) } x^{19}+2018x^{11}+1 \qquad \textbf{(B) } x^{17}+2018x^{11}+1 \qquad \textbf{(C) } x^{19}+2018x^{13}+1 \qquad \textbf{(D) } x^{17}+2018x^{13}+1 \qquad \textbf{(E) } 2019x+2018$

Solution 1

We can see that our real solution has to lie in the open interval $(-1, 0)$. From there, note that $x^a < x^b$ if a, b are odd positive integers so $a < b$, so hence it can only either be B or E(as all of the other polynomials will be larger than the polynomial B). Finally, we can see that plugging in the root of $2019x + 2018$ into B gives a negative, and so the answer is $\boxed{\text{B}}$. (cpma213)

Solution 2 (Calculus version of solution 1)

Note that $a(-1) = b(-1) = c(-1) = d(-1) < 0$ and $a(0) = b(0) = c(0) = d(0) > 0$. Calculating the definite integral for each function on the interval $[-1, 0]$, we see that $B(x)|_{-1}^{0}$ gives the most negative value. To maximize our real root, we want to maximize the area of the curve under the x-axis, which means we want our integral to be as negative as possible and thus the answer is $\boxed{\text{B}}$.

Solution 3 (Alternate Calculus Version)

Newton's Method is used to approximate the zero x_1 of any real valued function given an estimation for the root x_0: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. After looking at all the options, $x_0 = -1$ gives a reasonable estimate. For options A to D, $f(-1) = -1$ and the estimation becomes $x_1 = -1 + \frac{1}{f'(-1)}$. Thus we need to minimize the derivative, giving us B. Now after comparing B and E through Newton's method, we see that B has the higher root, so the answer is $\boxed{\text{B}}$. (Qcumber)

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2018 AMC 12A Problems/Problem 22

Contents

Problem

The solutions to the equations $z^2 = 4 + 4\sqrt{15}i$ and $z^2 = 2 + 2\sqrt{3}i$, where $i = \sqrt{-1}$, form the vertices of a parallelogram in the complex plane. The area of this parallelogram can be written in the form $p\sqrt{q} - r\sqrt{s}$, where p, q, r, and s are positive integers and neither q nor s is divisible by the square of any prime number. What is $p + q + r + s$?

$\textbf{(A)}20 \qquad \textbf{(B)}21 \qquad \textbf{(C)}22 \qquad \textbf{(D)}23 \qquad \textbf{(E)}24$

Solution 1

The roots are $\pm\left(\sqrt{10} + i\sqrt{6}\right), \pm\left(\sqrt{3} + i\right)$ (easily derivable by using DeMoivre and half-angle). From there, shoelace on $(0,0), \left(\sqrt{10}, \sqrt{6}\right), \left(\sqrt{3}, 1\right)$ and multiplying by 4 gives the area of $6\sqrt{2} - 2\sqrt{10}$, so the answer is $\boxed{20}$. (trumpeter)

Solution 2 (No DeMoivre's)

Write z as $a + bi$. For the first equation,

$$(a + bi)^2 = 4 + 4\sqrt{15}i$$

$$a^2 + 2abi - b^2 = 4 + 4\sqrt{15}i$$

Setting the real parts equal and imaginary parts equal, we have:

$$a^2 - b^2 = 4$$

$$ab = 2\sqrt{15}$$

Squaring the second equation gives $a^2b^2 = 60$. We now need two numbers that have a difference of 4 and a product of 60. By inspection, 10 and 6 work, so $a^2 = 10$ and $b^2 = 6$. Since ab is positive, a and b must have the same sign. Thus we have two solutions for (a, b):

$$(-\sqrt{10}, -\sqrt{6})$$

$$(\sqrt{10}, \sqrt{6})$$

Repeating the process for the second equation, we have two solutions:

$$(-\sqrt{3}, -1)$$

$$(\sqrt{3}, 1)$$

In a clockwise direction, the points are $(-\sqrt{10}, -\sqrt{6}), (-\sqrt{3}, -1), (\sqrt{10}, \sqrt{6}), (\sqrt{3}, 1)$. Now we can use the shoelace theorem. The area is $6\sqrt{2} - 2\sqrt{10}$, so the answer is $\boxed{20}$.

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2018 AMC 12A Problems/Problem 23

Contents

Problem

In $\triangle PAT, \angle P = 36^\circ, \angle A = 56^\circ$, and $PA = 10.$ Points U and G lie on sides $\overline{TP}$ and $\overline{TA}$, respectively, so that $PU = AG = 1.$ Let M and N be the midpoints of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines MN and PA?

$\textbf{(A)}76 \qquad \textbf{(B)}77 \qquad \textbf{(C)}78 \qquad \textbf{(D)}79 \qquad \textbf{(E)}80$

Solution

Let P be the origin, and PA lie on the x axis.

We can find $U = (\cos(36), \sin(36))$ and $G = (10 - \cos(56), \sin(56))$

Then, we have $M = (5, 0)$ and $N = \left(\dfrac{10 + \cos(36) - \cos(56)}{2}, \dfrac{\sin(36) + \sin(56)}{2}\right)$

Notice that the tangent of our desired points is the the absolute difference between the y coordinates of the two points divided by the absolute difference between the x coordinates of the two points.

This evaluates to

$$\frac{\sin(36) + \sin(56)}{\cos(36) - \cos(56)}$$

Now, using sum to product identities, we have this equal to

$$\frac{2\sin(46)\cos(10)}{-2\sin(46)\sin(-10)} = \frac{\sin(80)}{\cos(80)} = \tan(80)$$

so the answer is $\boxed{\textbf{(E)}.}$ (lifeisgood03)

Note: Though this solution is excellent, setting $M = (0,0)$ makes life a tad bit easier ~ MathleteMA

Solution 2 (Overkill)

Note that X, the midpoint of major arc PA on (PAT) is the Miquel Point of $PUAG$ (Because $PU = AG$). Then, since $1 = \dfrac{UN}{NG} = \dfrac{PM}{MA}$, this spiral similarity carries M to N. Thus, we have $\triangle XMN \sim \triangle XAG$, so $\angle XMN = \angle XAG$.

But, we have

$\angle XAG = \angle PAG = \angle PAX = 56 - \dfrac{180 - \angle PXA}{2} = 56 - \dfrac{180 - \angle T}{2} = 56 - \dfrac{\angle A + \angle P}{2} = 56 - \dfrac{56 + 36}{2} = 56 - 46 = 10$; thus $\angle XMN = 10$.

Then, as X is the midpoint of the major arc, it lies on the perpendicular bisector of PA, so $\angle XMA = 90$. Since we want the acute angle, we have $\angle NMA = \angle XMA - \angle XMN = 90 - 10 = 80$, so the answer is $\boxed{\textbf{(E)}}$.

(stronto)

Solution 3 (Nice, I Think?)

Let the bisector of $\angle ATP$ intersect PA at $X.$ We have $\angle ATX = \angle PTX = 44^\circ$, so $\angle TXA = 80^\circ.$ We claim that MN is parallel to this angle bisector, meaning that the acute angle formed by MN and PA is 80°, meaning that the answer is $\boxed{\textbf{(E)}}$.

To prove this, let $N(x)$ be the midpoint of $U(x)G(x)$, where $U(x)$ and $G(x)$ are the points on PT and AT, respectively, such that $PU = AG = x.$ (The points given in this problem correspond to $x = 1$, but the idea we're getting at is that x will ultimately not matter.) Since $U(x)$ and $G(x)$ vary linearly with x, the locus of all points $N(x)$ must be a line. Notice that $N(0) = M$, so M lies on this line. Let $N(x_0)$ be the intersection of this line with PT (we know that this line will intersect PT and not AT because $PT > AT$). Notice that $G(x_0) = T.$

Let $AT = a, TP = b, PT = c.$ Then $AG(x_0) = PU(x_0) = AT = a$ and $PG(x_0) = PT = b.$ Thus, $PN(x_0) = \dfrac{a+b}{2}$. By the Angle Bisector Theorem, $\dfrac{PX}{AX} = \dfrac{PT}{AT} = \dfrac{b}{a}$, so $PX = \dfrac{bc}{a+b}$. Since M is the midpoint of AP, we also have $PM = \dfrac{c}{2}$. Notice that:

$$\frac{PM}{PX} = \frac{\frac{c}{2}}{\frac{bc}{a+b}} = \frac{a+b}{2b}$$

$$\frac{PN(x_0)}{PT} = \frac{\frac{a+b}{2}}{b} = \frac{a+b}{2b}$$

Since $\frac{PN(x_0)}{PT} = \frac{PM}{PX}$, the line containing all points $N(x)$ must be parallel to TX. This concludes the proof.

The critical insight to finding this solution is that the length 1 probably shouldn't matter because a length ratio of $1:5$ or $1:10$ (as in the problem) is exceedingly unlikely to generate nice angles. This realization then motivates the idea of looking at all points similar to N, which then leads to looking at the most convenient such point (in this case, the one that lies on PT).

(sujaykazi) Shoutout to Richard Yi and Mark Kong for working with me to discover the necessary insights to this problem!

Solution 4

Let the mid-point of $\overline{AT}$ be B and the mid-point of $\overline{GT}$ be C. Since $\overline{BC} = \overline{CG} - \overline{BG}$ and $\overline{CG} = \overline{AB} - \frac{1}{2}$, we can conclude that $\overline{BC} = \frac{1}{2}$. Similarly, we can conclude that $\overline{BM} - \overline{CN} = \frac{1}{2}$. Construct $ND//BC$ and intersects $\overline{BM}$ at D, which gives $\overline{MD} = \overline{DN} = \frac{1}{2}$. Since $\angle ABD = \angle BDN, \overline{MD} = \overline{DN}$, we can find the value of $\angle DMN$, which is equal to $\frac{1}{2}T = 44^\circ$. Since $BM//PT$, which means $\angle DMN + \angle MNP + \angle P = 180^\circ$, we can infer that $\angle MNP = 100^\circ$. As we are required to give the acute angle formed, the final answer would be 80°, which is $\boxed{\textbf{(E)}}$. (Surefire2019)

Solution 5 (Simplest, I think)

Link PN, extend PN to Q so that $QN = PN$. Then link QG and QA.

$\because M, N$ is the middle point of AP and QD

$\therefore MN$ is the middle line of $\triangle PAQ$

$\therefore \angle QAP = \angle NMP$

Notice that $\triangle PDN \cong \triangle QGN$

As a result, $QG = AG = DP = 1, \angle AQG = \angle QAG, \angle GQN = \angle NPD$

Also, $\angle GQN + \angle QPA = \angle QPD + \angle QPA = \angle DPA = 36^\circ$

As a result, $2\angle QAG = 180^\circ - 56^\circ - 36^\circ = 88^\circ$

Therefore, $\angle QAP = \angle QAG + \angle TAP = 56^\circ + 44^\circ = 100^\circ$

As a result, the answer of this problem is $\boxed{100^\circ}$ or $\boxed{80^\circ}$

Look into the 5 choices, the answer is $\boxed{E}$

~Solution by $BladeRunnerAUG$ (Frank FYC)

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2018 AMC 12A Problems/Problem 24

Contents

Problem

Alice, Bob, and Carol play a game in which each of them chooses a real number between 0 and 1. The winner of the game is the one whose number is between the numbers chosen by the other two players. Alice announces that she will choose her number uniformly at random from all the numbers between 0 and 1, and Bob announces that he will choose his number uniformly at random from all the numbers between $\frac{1}{2}$ and $\frac{2}{3}$. Armed with this information, what number should Carol choose to maximize her chance of winning?

$$\textbf{(A) }\frac{1}{2}\qquad \textbf{(B) }\frac{13}{24} \qquad \textbf{(C) }\frac{7}{12} \qquad \textbf{(D) }\frac{5}{8} \qquad \textbf{(E) }\frac{2}{3}$$

Solution 1

Plug in all the answer choices to get $\boxed{\textbf{(B)}}$.

Solution 2

Let the value we want be x. The probability that Alice's number is less than Carol's number and Bob's number is greater than Carol's number is $x(\dfrac{2}{3}-x)$. Similarly, the probability that Bob's number is less than Carol's number and Alice's number is greater than Carol's number is $(x-\dfrac{1}{2})(1-x)$. Adding these together, the probability that Carol wins given a certain number x is $-2x^2+\dfrac{13}{6}x-\dfrac{1}{2}$. Using calculus or the fact that the extremum of a parabola occurs at $\dfrac{-b}{2a}$, the maximum value occurs at $x=\dfrac{13}{24}$, which is $\boxed{\textbf{(B)}}$.

Solution 3

The expected value of Alice's number is $\dfrac{1}{2}$ and the expected value of Bob's number is $\dfrac{7}{12}$. To maximize her chance of winning, Carol would choose number exactly in between the two expected values, giving: $\dfrac{6+7}{12*2}=\dfrac{13}{24}$. This is $\boxed{\textbf{(B)}}$. (Random_Guy)

EDIT: I believe this method is incorrect. Assume Bob can only choose $\dfrac{7}{12}$ but Alice chooses from the same range as before. The answer, using the above method, remains $\dfrac{13}{24}$ which is clearly wrong in this case. Correct me if I misunderstood the solution. (turnip123)

EDIT2: It would make sense for the answer to remain the same, given that Bob's expected value stays the same. Why should the answer change in your scenario? (KenV)

EDIT3: I realized my mistake. The method works in all situations where the expected value falls within both of their range. In my case, Carol's number would be less than Bob's number if Carol chooses any number from the range [0, 7/12). She would then want to maximize the chances of picking a number greater than Alice, which is achieved by picking the largest number possible from the range [0, 7/12), which is not 13/24.

Solution 4

Let’s call Alice’s number a, Bob’s number b, and Carol’s number c. Then, in order to maximize her chance of choosing a number that is in between a and b, she should choose c = (a+b)/2.

We need to find the average value of (a+b)/2 over the region [0, 1] x[1/2, 2/3] in the a-b plane.

We can set up a double integral with bounds 0 to 1 for the outer integral and 1/2 to 2/3 for the inner integral with an integrand of (a+b)/2. We need to divide our answer by 1/6, the area of the region of interest. This should yield 13/24, B.

Solution 5

Have Carol pick a point x on the real line. The probability that Alice's number is below is thus x, and the probability that Alice's number is above is $1-x$. Now, Carol must pick a point between $\frac{1}{2}$ and $\frac{2}{3}$, exclusive. If she does not, then it is impossible for Carol's point to be between both Alice and Bob's point. Now what is the probability that Bob's point is below Carol's? To make it simple, scale Bob's number to the normal number line by subtracting $\frac{1}{2}$ and multiplying by 6. So thus the chance Bob's number is below is $6(x-\frac{1}{2})$ and above is $1-6(x-\frac{1}{2})$.

Now we want to maximixe the probability of (Alice number below x) AND (Bob number above x) + (Alice number above x) AND (Bob number below x). This gives the formula $x(1-6(x-\frac{1}{2}))+(1-x)6(x-\frac{1}{2})=-3+13x-12x^2$. Then proceed using the process of completing the square or averaging the roots to get that the maximum of this is $\frac{13}{24}$.

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2018 AMC 10A Problems/Problem 25

Contents

Problem

For a positive integer n and nonzero digits a, b, and c, let A_n be the n-digit integer each of whose digits is equal to a; let B_n be the n-digit integer each of whose digits is equal to b, and let C_n be the $2n$-digit (not n-digit) integer each of whose digits is equal to c. What is the greatest possible value of $a + b + c$ for which there are at least two values of n such that $C_n - B_n = A_n^2$?

$\textbf{(A) } 12 \qquad \textbf{(B) } 14 \qquad \textbf{(C) } 16 \qquad \textbf{(D) } 18 \qquad \textbf{(E) } 20$

Solution 1

Observe $A_n = a(1 + 10 + \dots + 10^{n-1}) = a \cdot \tfrac{10^n - 1}{9}$; similarly $B_n = b \cdot \tfrac{10^n - 1}{9}$ and $C_n = c \cdot \tfrac{10^{2n} - 1}{9}$. The relation $C_n - B_n = A_n^2$ rewrites as

$$c \cdot \frac{10^{2n} - 1}{9} - b \cdot \frac{10^n - 1}{9} = a^2 \cdot \left(\frac{10^n - 1}{9}\right)^2.$$

Since $n > 0$, $10^n > 1$ and we may cancel out a factor of $\tfrac{10^n - 1}{9}$ to obtain

$$c \cdot (10^n + 1) - b = a^2 \cdot \frac{10^n - 1}{9}.$$

This is a linear equation in 10^n. Thus, if two distinct values of n satisfy it, then all values of n will. Matching coefficients, we need

$$c = \frac{a^2}{9} \quad \text{and} \quad c - b = -\frac{a^2}{9} \implies b = \frac{2a^2}{9}.$$

To maximize $a + b + c = a + \tfrac{a^2}{3}$, we need to maximize a. Since b and c must be integers, a must be a multiple of 3. If $a = 9$ then b exceeds 9. However, if $a = 6$ then $b = 8$ and $c = 4$ for an answer of $\boxed{\textbf{(D) } 18}$.

(CantonMathGuy)

Solution 2 (quicker?)

Immediately start trying $n = 1$ and $n = 2$. These give the system of equations $11c - b = a^2$ and $1111c - 11b = (11a)^2$ (which simplifies to $101c - b = 11a^2$). These imply that $a^2 = 9c$, so the possible (a, c) pairs are $(9, 9)$, $(6, 4)$, and $(3, 1)$. The first puts b out of range but the second makes $b = 8$. We now know the answer is at least $6 + 8 + 4 = 18$.

We now only need to know whether $a + b + c = 20$ might work for any larger n. We will always get equations like $100001c - b = 11111a^2$ where the c coefficient is very close to being nine times the a coefficient. Since the b term will be quite insignificant, we know that once again a^2 must equal $9c$, and thus $a = 9, c = 9$ is our only hope to reach 20. Substituting and dividing through by 9, we will have something like $100001 - b/9 = 99999$. No matter what n really was, b is out of range (and certainly isn't 2 as we would have needed).

The answer then is $\boxed{\textbf{(D) } 18}$.

Solution 3

Notice that $(0.\overline{3})^2 = 0.\overline{1}$ and $(0.\overline{6})^2 = 0.\overline{4}$. Setting $a = 3$ and $c = 1$, we see $b = 2$ works for all possible values of n. Similarly, if $a = 6$ and $c = 4$, then $b = 8$ works for all possible values of n. The second solution yields a greater sum of $\boxed{\textbf{(D)}\ 18}$.

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