2020 AMC 12A Solution

Problem1

Carlos took 70% of a whole pie. Maria took one third of the remainder. What portion of the whole pie was left?

$\textbf{(A)}\ 10\% \qquad \textbf{(B)}\ 15\% \qquad \textbf{(C)}\ 20\% \qquad \textbf{(D)}\ 30\% \qquad \textbf{(E)}\ 35\%$

Solution 1

If Carlos took 70% of the pie, $(100-70)=30\%$ must be remaining.

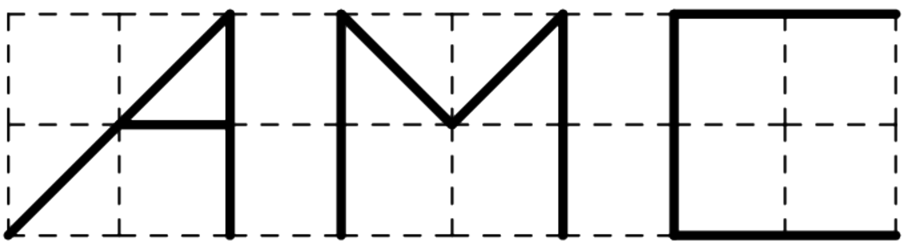
After Maria takes $\frac{1}{3}$ of the remaining 30%, $1-\frac{1}{3}=\frac{2}{3}$ is left.

Therefore:

$$30\% \cdot \frac{2}{3} = \boxed{\textbf{C)}\ 20\%}$$

Problem2

The acronym AMC is shown in the rectangular grid below with grid lines spaced 1 unit apart. In units, what is the sum of the lengths of the line segments that form the acronym AMC$?$



$\textbf{(A)}\ 17 \qquad \textbf{(B)}\ 15+2\sqrt{2} \qquad \textbf{(C)}\ 13+4\sqrt{2} \qquad \textbf{(D)}\ 11+6\sqrt{2} \qquad \textbf{(E)}\ 21$

Solution

Each of the straight line segments have length 1 and each of the slanted line segments have length $\sqrt{2}$(this can be deducted using $45-45-90$, pythag, trig, or just sense)

There area a total of 13 straight lines segments and 4 slanted line segments.

The sum is $\boxed{\textbf{C) } 13+4\sqrt{2}}$~quacker88

You could have also just counted 4 slanted line segments and realized that the only answer choice involving $4\sqrt{2}$ was $\boxed{\textbf{C) } 13+4\sqrt{2}}$.

Problem3

A driver travels for 2 hours at 60 miles per hour, during which her car gets 30 miles per gallon of gasoline. She is paid $\$0.50$ per mile, and her only expense is gasoline at $\$2.00$ per gallon. What is her net rate of pay, in dollars per hour, after this expense?

$\textbf{(A)}\ 20\qquad\textbf{(B)}\ 22\qquad\textbf{(C)}\ 24\qquad\textbf{(D)}\ 25\qquad\textbf{(E)}\ 26$

Solution

Since the driver travels 60 miles per hour and each hour she uses 2 gallons of gasoline, she spends $4 per hour on gas. If she gets $0.50 per mile, then she gets $30 per hour of driving. Subtracting the gas cost, her net rate of pay per hour is $\boxed{\textbf{(E)}\ 26}$.

Problem4

How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by $5?$

$\textbf{(A)}\ 80\qquad\textbf{(B)}\ 100\qquad\textbf{(C)}\ 125\qquad\textbf{(D)}\ 200\qquad\textbf{(E)}\ 500$

Solution

The ones digit, for all numbers divisible by 5, must be either 0 or 5. However, from the restriction in the problem, it must be even, giving us exactly one choice (0) for this digit. For the middle two digits, we may choose any even integer from $[0, 8]$, meaning that we have 5 total options. For the first digit, we follow similar intuition but realize that it cannot be 0, hence giving us 4 possibilities.

Therefore, using the multiplication rule, we get $4 \times 5 \times 5 \times 1 = \boxed{\textbf{(B) } 100}$. ~ciceronii

Problem5

The 25 integers from -10 to $14,$ inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

$\textbf{(A) } 2 \qquad \textbf{(B) } 5 \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 50$

Solution

Without loss of generality, consider the five rows in the square. Each row must have the same sum of numbers, meaning that the sum of all the numbers in the square divided by 5 is the total value per row. The sum of the 25 integers is $-10 + 9 + ... + 14 = 11 + 12 + 13 + 14 = 50$, and the common sum is $\frac{50}{5} = \boxed{\text{(C) } 10}$.

Solution 2

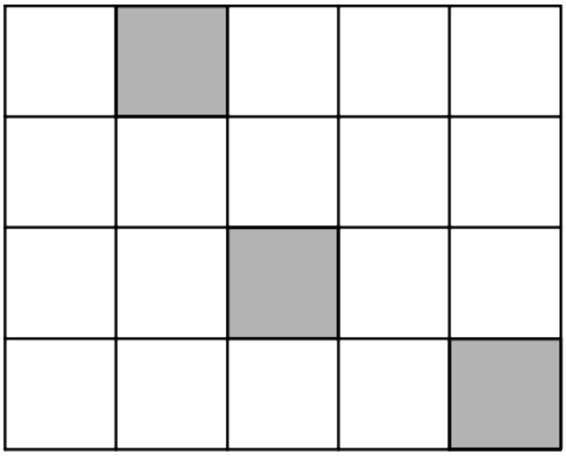
Take the sum of the middle 5 values of the set (they will turn out to be the mean of each row). We get $0+1+2+3+4 = \boxed{\textbf{(C) } 10}$ as our answer. ~Baolan

Solution 3

Taking the average of the first and last terms, -10 and 14, we have that the mean of the set is 2. There are 5 values in each row, column or diagonal, so the value of the common sum is $5 \cdot 2$, or $\boxed{\textbf{(C) } 10}$. ~Arctic_Bunny, edited by KINGLOGIC

Problem 6

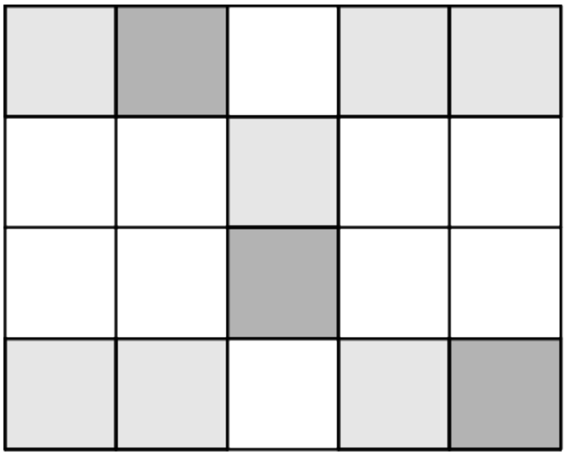
In the plane figure shown below, 3 of the unit squares have been shaded. What is the least number of additional unit squares that must be shaded so that the resulting figure has two lines of symmetry?



$\textbf{(A) }4 \qquad \textbf{(B) }5 \qquad \textbf{(C) }6 \qquad \textbf{(D) }7 \qquad \textbf{(E) }8$

Solution

The two lines of symmetry must be horizontally and vertically through the middle. We can then fill the boxes in like so:



where the light gray boxes are the ones we have filled. Counting these, we get $\boxed{\textbf{(D) }7}$ total boxes. ~ciceronii

Problem7

Seven cubes, whose volumes are 1, 8, 27, 64, 125, 216, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

$\textbf{(A) } 644 \qquad \textbf{(B) } 658 \qquad \textbf{(C) } 664 \qquad \textbf{(D) } 720 \qquad \textbf{(E) } 749$

Solution 1

The volume of each cube follows the pattern of n^3 ascending, for n is between 1 and 7.

We see that the total surface area can be comprised of three parts: the sides of the cubes, the tops of the cubes, and the bottom of the $7 \times 7 \times 7$ cube (which is just $7 \times 7 = 49$). The sides areas can be measured as the sum $4\sum_{n=0}^{7} n^2$, giving us 560. Structurally, if we examine the tower from the top, we see that it really just forms a 7×7 square of area 49. Therefore, we can say that the total surface area is $560+49+49=\boxed{\textbf{(B) } 658}$.
Alternatively, for the area of the tops, we could have found the sum $\sum_{n=0}^{6}((n+1)^2-n^2)$, giving us 49 as well.

~ciceronii

Solution 2

It can quickly be seen that the side lengths of the cubes are the integers from 1 to 7, inclusive.

First, we will calculate the total surface area of the cubes, ignoring overlap. This value
is

$$6(1^2+2^2+\cdots+7^2)=6\sum_{n=1}^{7} n^2=6\left(\frac{7(7+1)(2\cdot 7+1)}{6}\right)=7\cdot 8\cdot 15=840$$

. Then, we need to subtract out the overlapped parts of the cubes. Between each consecutive pair of cubes, one of the smaller cube's faces is completely covered, along with an equal area of one of the larger cube's faces. The total area of the overlapped parts of the cubes is thus equal to $2\sum_{n=1}^{6} n^2=182$. Subtracting the overlapped surface area from the total surface area, we get $840 - 182 = \boxed{\textbf{(B) }658}$. ~emerald_block

Solution 3 (a bit more tedious than others)

It can be seen that the side lengths of the cubes using cube roots are all integers from 1 to 7, inclusive.

Only the cubes with side length 1 and 7 have 5 faces in the surface area and the rest have 3. Also, since the

cubes are stacked, we have to find the difference between

each n^2 and $(n-1)^2$ side length as n ranges from 7 to

2.

We then come up with this:
$5(49)+13+4(36)+11+4(25)+9+4(16)+7+4(9)+5+4(4)+3+5(1)$
.

We then add all of this and get $\boxed{\textbf{(B) }658}$.

Problem 8

What is the median of the following list of 4040 numbers$?$

$$1, 2, 3, ..., 2020, 1^2, 2^2, 3^2, ..., 2020^2$$

$\textbf{(A)}\ 1974.5 \qquad \textbf{(B)}\ 1975.5 \qquad \textbf{(C)}\ 1976.5 \qquad \textbf{(D)}\ 1977.5 \qquad \textbf{(E)}\ 1978.5$

Solution 1

We can see that 44^2 is less than 2020. Therefore, there are 1976 of the 4040 numbers after 2020. Also, there are 2064 numbers that are under and equal to 2020. Since 44^2 is equal to 1936, it, with the other squares, will shift our median's placement up 44. We can find that the median of the whole set is 2020.5, and $2020.5-44$ gives us 1976.5. Our answer is $\boxed{\textbf{(C) }1976.5}$.

~aryam

Solution 2

As we are trying to find the median of a 4040-term set, we must find the average of the 2020th and 2021st terms.

Since $45^2 = 2025$ is slightly greater than 2020, we know that

the 44 perfect squares 1^2 through 44^2 are less than 2020, and the rest are greater. Thus, from the number 1 to the number 2020, there are $2020 + 44 = 2064$ terms. Since 44^2 is $44 + 45 = 89$ less

than $45^2 = 2025$ and 84 less than 2020, we will only need to consider the perfect square terms going down from the 2064th term, 2020, after going down 84 terms. Since the 2020th and 2021st terms are only 44 and 43 terms away from the 2064th term, we can simply subtract 44 from 2020 and 43 from 2020 to get the two terms, which are 1976 and 1977. Averaging the two, we get $\boxed{\textbf{(C) } 1976.5}$. ~emerald_block

Solution 3

We want to know the 2020th term and the 2021th term to get the median.

We know that $44^2 = 1936$

So numbers $1^2, 2^2, \ldots, 44^2$ are in between 1 to 1936.

So the sum of 44 and 1936 will result in 1980, which means that 1936 is the 1980th number.

Also, notice that $45^2 = 2025$, which is larger than 2021.

Then the 2020th term will be $1936 + 40 = 1976$, and similarly the 2021th term will be 1977.

Solving for the median of the two numbers, we get $\boxed{\textbf{(C) } 1976.5}$

Problem9

How many solutions does the equation $\tan(2x) = \cos(\frac{x}{2})$ have on the interval $[0, 2\pi]$?

$\textbf{(A) }1\qquad\textbf{(B) }2\qquad\textbf{(C) }3\qquad\textbf{(D) }4\qquad\textbf{(E) }5$

Solution

Draw a graph of tan$(2x)$ and cos$(\frac{x}{2})$

tan$(2x)$ has a period of $\frac{\pi}{2}$, asymptotes at $x = \frac{\pi}{4} + \frac{k\pi}{2}$, and zeroes at $\frac{k\pi}{2}$. It is positive from $(0,\frac{\pi}{4}) \cup (\frac{\pi}{2},\frac{3\pi}{4}) \cup (\pi,\frac{5\pi}{4}) \cup (\frac{3\pi}{2},\frac{7\pi}{4})$ and negative elsewhere.

cos$(\frac{x}{2})$ has a period of 4π and zeroes at π. It is positive from $[0,\pi)$ and negative elsewhere.

Drawing such a graph would get $\boxed{\textbf{E) }5}$ ~lopkiloinm

Problem10

There is a unique positive integer n such that $\log_2{(\log_{16}{n})} = \log_4{(\log_4{n})}$.What is the sum of the digits of $n?$

$\textbf{(A) } 4 \qquad \textbf{(B) } 7 \qquad \textbf{(C) } 8 \qquad \textbf{(D) } 11 \qquad \textbf{(E) } 13$

Solution

Any logarithm in the form $\log_{a^b} c = \frac{1}{b} \log_a c$. (this can be proved easily by using change of base formula to base a).

so $\log_2{(\log_{2^4}{n})} = \log_{2^2}{(\log_{2^2}{n})}$

becomes

$$\log_2(\frac{1}{4}\log_2 n) = \frac{1}{2}\log_2(\frac{1}{2}\log_2 n)$$

Using $\log$ property of addition, we can expand the parentheses into

$$\log_2{(\frac{1}{4})} + \log_2{(\log_2{n})} = \frac{1}{2}(\log_2{(\frac{1}{2})} + \log_2{(\log_2{n})})$$

Expanding the RHS and simplifying the logs without variables, we have

$$-2 + \log_2{(\log_2{n})} = -\frac{1}{2} + \frac{1}{2}(\log_2{(\log_2{n})})$$

Subtracting $\frac{1}{2}(\log_2{(\log_2{n})})$ from both sides and adding 2 to both sides gives us

$$\frac{1}{2}(\log_2{(\log_2{n})}) = \frac{3}{2}$$

Multiplying by 2, raising the logs to exponents of base 2 to get rid of the logs and simplifying gives us

$$(\log_2{(\log_2{n})}) = 3$$

$$2^{\log_2{(\log_2{n})}} = 2^3$$

$$\log_2{n} = 8$$

$$2^{\log_2{n}} = 2^8$$
$$n = 256$$

Adding the digits together, we have $2+5+6 = \boxed{\textbf{(E) }13}$ ~quacker88

Solution 2

We know that, as the answer is an integer, n must be some power of 16. Testing 16 yields $\log_2{(\log_{16}{16})} = \log_4{(\log_4{16})}$

$\log_2{1} = \log_4{2}$ $0 = \frac{1}{2}$ which does not work. We then try 256, giving us

$\log_2{(\log_{16}{256})} = \log_4{(\log_4{256})}$ $\log_2{2} = \log_4{4}$ $1 = 1$

which holds true. Thus, $n = 256$, so the answer is $2+5+6=\boxed{\textbf{(E) } 13}$.

(Don't use this technique unless you absolutely need to! Guess and check methods aren't helpful for learning math.)

~ciceronii

Solution 3-Change of Base

Using the change of base formula on the RHS of the initial equation yields $$\log_2{(\log_{16}{n})} = \frac{\log_2{(\log_4{n})}}{\log_2{4}}$$ This means we can multiply each side by 2 for $\log_2{(\log_{16}{n})}^2 = \log_2{(\log_4{n})}$ Canceling out the logs gives $(\log_{16}{n})^2 = \log_4{n}$ We use change of base on the RHS to see that $$(log_{16}n)^2 = \frac{log_{16}n}{\log_{16}{4}}$$ or $(log_{16}n)^2 = 2log_{16}n$ Substituting in $m = log_{16}n$ gives $m^2 = 2m$, so m is either 0 or 2. Since $m = 0$ yields no solution for n (since a log cannot be equal to 0), we get $2 = log_{16}n$, or $n = 16^2 = 256$, for a sum of $2+5+6=\boxed{\textbf{(E) } 13}$. ~aop2014

Solution 4

Suppose $\log_2(\log_{16} n) = k \implies \log_{16} n = 2^k \implies n = 16^{2^k}$. Similarly, we have $\log_4(\log_4 n) = k \implies \log_4 n = 4^k \implies n = 4^{4^k}$. Thus, we have $16^{2^k} = (4^2)^{2^k} = 4^{2^{k+1}}$ and $4^{4^k} = 4^{2^{2k}}$, so $k + 1 = 2k \implies k = 1$. Plugging this in to either one of the expressions for n gives 256, and the requested answer is $2 + 5 + 6 = \boxed{\textbf{(E) } 13}$.

Problem 11

A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0)$, $(0, 4)$, $(4, 4)$, and $(4, 0)$. What is the probability that the sequence of jumps ends on a vertical side of the square?

$\textbf{(A) } \frac{1}{2} \qquad \textbf{(B) } \frac{5}{8} \qquad \textbf{(C) } \frac{2}{3} \qquad \textbf{(D) } \frac{3}{4} \qquad \textbf{(E) } \frac{7}{8}$

Solution

Drawing out the square, it's easy to see that if the frog goes to the left, it will immediately hit a vertical end of the square. Therefore, the probability of this happening is $\frac{1}{4} * 1 = \frac{1}{4}$. If the frog goes to the right, it will be in the center of the square at $(2, 2)$, and by symmetry (since the frog is equidistant from all sides of the square), the chance it will hit a vertical side of a square is $\frac{1}{2}$. The probability of this happening is $\frac{1}{4} * \frac{1}{2} = \frac{1}{8}$.

If the frog goes either up or down, it will hit a line of symmetry along the corner it is closest to and furthest to, and again, is equidistant relating to the two closer sides and also equidistant relating the two further sides. The probability for it to hit a vertical wall is $\frac{1}{2}$. Because there's a $\frac{1}{2}$ chance of the frog going up and down, the total probability for this case is $\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$ and summing up all the cases, $\frac{1}{4} + \frac{1}{8} + \frac{1}{4} = \frac{5}{8} \implies \boxed{\textbf{(B)}\ \frac{5}{8}.}$

Solution 2

Let's say we have our four by four grid and we work this out by casework. A is where the frog is, while B and C are possible locations for his second jump, while O is everything else. If we land on a C, we have reached the vertical side. However, if we land on a B, we can see that there is an equal chance of reaching the horizontal or vertical side, since we are symmetrically between them. So we have the probability of landing on a C is 1/4, while B is 3/4. Since C means that we have "succeeded", while B means that we have a half chance, we compute $1 \cdot C + \frac{1}{2} \cdot B$.

$1 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{31}{44} + \frac{3}{8}$We get $\frac{5}{8}$, or BO O O O OO B O O O C A B O OO B O O OO O O O O-yeskay

Solution 3

If the frog is on one of the 2 diagonals, the chance of landing on vertical or horizontal each becomes $\frac{1}{2}$. Since it starts on $(1, 2)$, there is a $\frac{3}{4}$ chance (up, down, or right) it will reach a diagonal on the first jump and $\frac{1}{4}$ chance (left) it will reach the vertical side. The probablity of landing on a vertical is $\frac{1}{4}+\frac{3}{4}*\frac{1}{2}=\boxed{\textbf{(B)}\frac{5}{8}.}$ - Lingjun.

Solution 4 (Complete States)

Let $P_{(x,y)}$ denote the probability of the frog's sequence of jumps ends with it hitting a vertical edge when it is at (x,y). Note that $P_{(1,2)}=P_{(3,2)}$ by reflective symmetry over the line $x=2$.

Similarly, $P_{(1,1)}=P_{(1,3)}=P_{(3,1)}=P_{(3,3)}$, and $P_{(2,1)}=P_{(2,3)}$.

Now we create equations for the probabilities at each of these points/states by considering the probability of going either up, down, left, or right from that point: $P_{(1,2)}=\frac{1}{4}+\frac{1}{2}P_{(1,1)}+\frac{1}{4}P_{(2,2)}$

$$P_{(2,2)}=\frac{1}{2}P_{(1,2)}+\frac{1}{2}P_{(2,1)}$$

$$P_{(1,1)}=\frac{1}{4}+\frac{1}{4}P_{(1,2)}+\frac{1}{4}P_{(2,1)}$$

$P_{(2,1)}=\frac{1}{2}P_{(1,1)}+\frac{1}{4}P_{(2,2)}$ We have a system of 4 equations in 4 variables, so we can solve for each of these probabilities. Plugging the second equation into the fourth equation $P_{(2,1)}=\frac{1}{2}P_{(1,1)}+\frac{1}{4}\left(\frac{1}{2}P_{(1,2)}+\frac{1}{2}P_{(2,1)}\right)$ gives

$$P_{(2,1)}=\frac{8}{7}\left(\frac{1}{2}P_{(1,1)}+\frac{1}{8}P_{(1,2)}\right)=\frac{4}{7}P_{(1,1)}+\frac{1}{7}P_{(1,2)}$$

Plugging in the third equation into this $P_{(2,1)}=\frac{4}{7}\left(\frac{1}{4}+\frac{1}{4}P_{(1,2)}+\frac{1}{4}P_{(2,1)}\right)+\frac{1}{7}P_{(1,2)}$ gives

$P_{(2,1)}=\frac{7}{6}\left(\frac{1}{7}+\frac{2}{7}P_{(1,2)}\right)=\frac{1}{6}+\frac{1}{3}P_{(1,2)}\ (*)$ Next, plugging in the second and third equation into the first equation yields

$$P_{(1,2)} = \frac{1}{4} + \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}P_{(1,2)} + \frac{1}{4}P_{(2,1)}\right) + \frac{1}{4}\left(\frac{1}{2}P_{(1,2)} + \frac{1}{2}P_{(2,1)}\right)$$

$$P_{(1,2)} = \frac{3}{8} + \frac{1}{4}P_{(1,2)} + \frac{1}{4}P_{(2,1)}$$

Now plugging in (*) into this, we get $P_{(1,2)} = \frac{3}{8} + \frac{1}{4}P_{(1,2)} + \frac{1}{4}\left(\frac{1}{6} + \frac{1}{3}P_{(1,2)}\right)$

$$P_{(1,2)} = \frac{3}{2} \cdot \frac{5}{12} = \boxed{\textbf{(B) } \frac{5}{8}}$$

Problem12

Line l in the coordinate plane has equation $3x - 5y + 40 = 0$. This line is rotated 45° counterclockwise about the point $(20, 20)$ to obtain line k. What is the x-coordinate of the x-intercept of line k?

$\textbf{(A) } 10 \qquad \textbf{(B) } 15 \qquad \textbf{(C) } 20 \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 30$

Solution

The slope of the line is $\frac{3}{5}$. We must transform it by 45°.

45° creates an isosceles right triangle since the sum of the angles of the triangle must be 180° and one angle is 90° which means the last leg angle must also be 45°.

In the isosceles right triangle, the two legs are congruent. We can, therefore, construct an isosceles right triangle with a line of $\frac{3}{5}$ slope on graph paper. That line with $\frac{3}{5}$ slope starts at $(0, 0)$ and will go to $(5, 3)$, the vector $< 5, 3 >$.

Construct another line from $(0,0)$ to $(3,-5)$, the vector $<3,-5>$. This is $\perp$ and equal to the original line segment. The difference between the two vectors is $<2,8>$, which is the slope 4, and that is the slope of line k.

Furthermore, the equation $3x-5y+40=0$ passes straight through $(20,20)$ since $3(20)-5(20)+40=60-100+40=0$, which means that any rotations about $(20,20)$ would contain $(20,20)$. We can create a line of slope 4 through $(20,20)$. The x-intercept is therefore $20-\frac{20}{4}=\boxed{\textbf{(B) } 15.}$~lopkiloinm

Solution 2

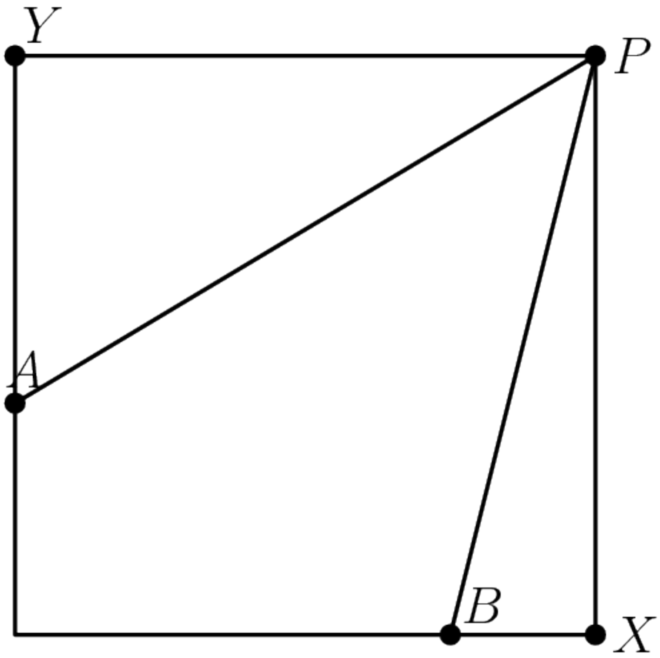
Since the slope of the line is $\frac{3}{5}$, and the angle we are rotating around is x, then $\tan x = \frac{3}{5}$

$$\tan(x+45^\circ) = \frac{\tan x + \tan(45^\circ)}{1 - \tan x * \tan(45^\circ)} = \frac{0.6+1}{1-0.6} = \frac{1.6}{0.4} = 4$$

Hence, the slope of the rotated line is 4. Since we know the line intersects the point $(20,20)$, then we know the line is $y=4x-60$. Set $y=0$ to find the x-intercept, and so $x=\boxed{15}$

~Solution by IronicNinja

Solution 3



Let P be $(20, 20)$ and X, Y be $(20, 0)$ and $(0, 20)$ respectively. Since the slope of the line is $3/5$ we know that $\tan \angle YPA = 3/5.$ Segments $\overline{PA}$ and $\overline{PB}$ represent the before and after of rotating l by 45 counterclockwise.
Thus, $\angle XPB = 45 - \angle YPA$ and $BX = 20 \tan \angle XPB = 20 \cdot \dfrac{1 - 3/5}{1 + 3/5} = 5$ by tangent addition formula. Since BX is 5 and the sidelength of the square is 20 the answer is $20 - 5 \implies \boxed{\textbf{B}}$.

Solution 4 (Cheap)

Using the protractor you brought, carefully graph the equation and rotate the given line 45° counter-clockwise about the point $(20, 20)$. Scaling everything down by a factor of 5 makes this process easier.

It should then become fairly obvious that the x intercept is $x = \boxed{15}$ (only use this as a last resort).

~Silverdragon

Solution 5 (Rotation Matrix)

First note that the given line goes through $(20, 20)$ with a slope of $\frac{3}{5}$. This means that $(25, 23)$ is on the line. Now consider translating the graph so that $(20, 20)$ goes to the origin, then $(25, 23)$ becomes $(5, 3)$. We now rotate the line 45° about the origin using a rotation matrix. This maps $(5, 3)$ to $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 4\sqrt{2} \end{bmatrix}$ The line through the origin and $(\sqrt{2}, 4\sqrt{2})$ has slope 4. Translating this line so that the origin is mapped to $(20, 20)$, we find that the equation for the new line is $4x - 60$, meaning that the x-intercept is $x = \frac{60}{4} = \boxed{\textbf{(B) } 15}$.

Problem13

There are integers $a, b,$ and $c,$ each greater than $1,$ such that

$$\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}} = \sqrt[36]{N^{25}}$$

for all $N > 1$. What is b?

$\textbf{(A) } 2 \qquad \textbf{(B) } 3 \qquad \textbf{(C) } 4 \qquad \textbf{(D) } 5 \qquad \textbf{(E) } 6$

Solution

$\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}}$ can be simplified to $N^{\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}}$.

The equation is then $N^{\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}} = N^{\frac{25}{36}}$ which implies that $\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} = \frac{25}{36}$.

a has to be 2 since $\frac{25}{36} > \frac{7}{12}$. $\frac{7}{12}$ is the result when $a, b,$ and c are $3, 2,$ and 2

b being 3 will make the fraction $\frac{2}{3}$ which is close to $\frac{25}{36}$.

Finally, with c being 6, the fraction becomes $\frac{25}{36}$. In this case $a, b,$ and c work, which means that b must equal $\boxed{\textbf{(B) }3.}$~lopkiloinm

Solution 2

As above, notice that you get $\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}=\frac{25}{36}$.

Now, combine the fractions to get $\frac{bc+c+1}{abc}=\frac{25}{36}$.

Assume that $bc+c+1=25$ and $abc=36$.

From the first equation we get $c(b+1)=24$. Note also that from the second equation, b and c must both be factors of 36.
After some casework we find that $c=6$ and $b=3$ works, with $a=2$. So our answer is $\boxed{\textbf{(B) }3.}$

~Silverdragon

Solution 3

Collapsed, $\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}} = \sqrt[abc]{N^{bc+c+1}}$. Comparing this to $\sqrt[36]{N^{25}}$, observe that $bc+c+1=25$and $abc=36$. The first can be rewritten as $c(b+1)=24$. Then, $b+1$ has to factor into 24 while 1 less than that also must factor into 36. The prime factorizations are as follows $36 = 2^2 3^2$ and $24 = 2^3 3$. Then, $b = \boxed{\textbf{B)} 3}$, as only 4 and 3 factor into 36 and 24 while being 1 apart.

Problem 14

Regular octagon $ABCDEFGH$ has area n. Let m be the area of quadrilateral $ACEG$. What is $\frac{m}{n}$?

$\textbf{(A) } \frac{\sqrt{2}}{4} \qquad \textbf{(B) } \frac{\sqrt{2}}{2} \qquad \textbf{(C) } \frac{3}{4} \qquad \textbf{(D) } \frac{3\sqrt{2}}{5} \qquad \textbf{(E) } \frac{2\sqrt{2}}{3}$

Solution 1

$ACEG$ is a square. WLOG $AB = 1$, then using Law of Cosines,

$AC^2 = [ACEG] = 1^2 + 1^2 - 2\cos 135 = 2 + \sqrt{2}$. The area of the octagon is just $[ACEG]$ plus the area of the four congruent (by symmetry) isosceles triangles, all an angle of 135 in between two sides of length 1.

Now, $\frac{m}{n} = \frac{2+\sqrt{2}}{2+\sqrt{2}+4\cdot\frac{1}{2}\sin 135} = \frac{2+\sqrt{2}}{2+2\sqrt{2}} = \frac{\sqrt{2}}{2}$. The answer is $\boxed{\textbf{(B)}}$.

Solution 2

Refer to the diagram. Call one of the side lengths of the square S. Since quadrilateral ACEG is a square, the area of the square would just be S^2, which we can find by applying Law of Cosines on one of the four triangles. Assume each of the sides of the octagon has length 1. Since each angle measures 135° in an octagon,

then $S^2 = 1^2 + 1^2 - 2 * \cos(135^\circ) = 2 + \sqrt{2}$

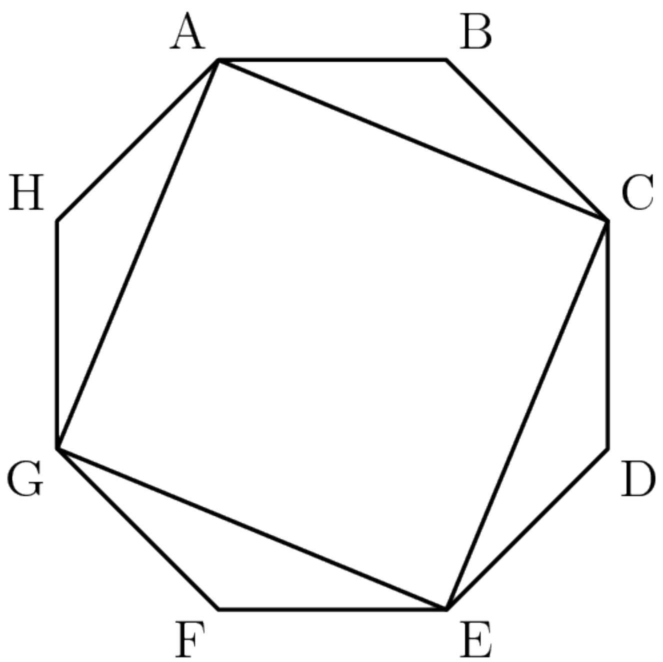
There are many ways to find the area of the octagon, but one way is to split the octagon into two trapezoids and one rectangle. We can easily compute AF to be $1+\sqrt{2}$ from splitting one of the sides into two $45-45-90$ triangles. So the area of the octagon is $2*\frac{1+\sqrt{2}}{2}+1+\sqrt{2} \Rightarrow 2+2\sqrt{2}$.

Hence, $\frac{m}{n}=\frac{2+\sqrt{2}}{2+2\sqrt{2}} \Rightarrow \frac{2+\sqrt{2}}{2+2\sqrt{2}} \cdot \frac{2-2\sqrt{2}}{2-2\sqrt{2}} \Rightarrow \frac{-2\sqrt{2}}{-4}$

$\Rightarrow \boxed{\frac{\sqrt{2}}{2}}$

~Solution by IronicNinja

Solution 3 (Deriving Formulas)



The first thing to notice is that $ACEG$ is a square. This is because, as $\triangle ABC \cong \triangle CDE \cong \triangle EFG \cong \triangle GHE$, they all have the same base, meaning that $AC = DE = EG = GA$. Hence, we have that it is a square. To determine the area of this square, we can determine the length of its diagonals.

In order to do this, we first determine the area of the octagon. Letting the side length be a, we can create a square of length s around it (see

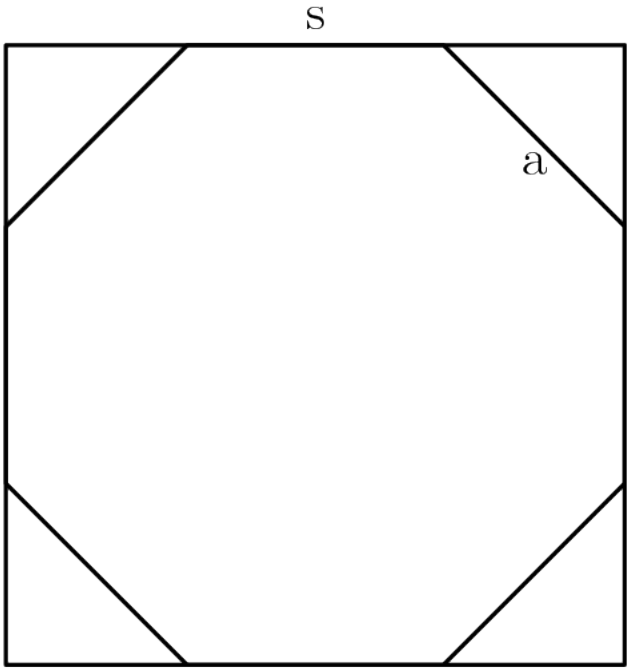


figure).

Creating a small square of side length a from the corners of this figure gives us an area of a^2. Thus, $s^2 - a^2 = n$ where n is the area of the octagon. We know from the Pythagorean Theorem that $s = a + \frac{a}{\sqrt{2}}$, meaning that $n = (a + \frac{a}{\sqrt{2}})^2 - a^2 = 2a^2(1+\sqrt{2})$.

Dividing this by 8 gives us the area of each triangular segment which makes up the octagon. Further dividing by 2 gives us the area of a smaller segment consisting of the right triangle with legs of the apothem and $\frac{a}{2}$. Using the area of a triangle as $\frac{1}{2}bh$, we can determine the length of apothem r from

$$\frac{2a^2(1+\sqrt{2})}{8 \times 2} = \frac{\frac{a}{2} \times r 4a^2(1+\sqrt{2})}{2 \qquad\quad 8} = ar$$

$$\frac{a(1+\sqrt{2})}{2} = r$$

.

From the apothem, we can once again use the Pythagorean Theorem, giving us the length of the circumradius R.

$$R^2 = (\frac{a(1+\sqrt{2})}{2})^2 + (\frac{a}{2})^2 R^2 = \frac{a^2(1+\sqrt{2})^2}{4} + \frac{a^2}{4}$$

$$R^2 = \frac{a^2(3+2\sqrt{2})}{4} + \frac{a^2}{4} R^2 = \frac{a^2(4+2\sqrt{2})}{4}$$

$$R = \sqrt{\frac{a^2(4+2\sqrt{2})}{4}} = \frac{a\sqrt{4+2\sqrt{2}}}{2}.$$

Doubling this gives us the diagonal of both the square and the octagon. From here, we can use the formula $A = \frac{1}{2}d^2$ for the area of the square:

$$A = \frac{(a\sqrt{4+2\sqrt{2}})^2}{2} A = \frac{a^2(4+2\sqrt{2})}{2} = m.$$

Thus we now only need to find the ratio $\frac{m}{n}$. This can be easily done through some algebra:

$$\frac{m}{n} = \frac{a^2(4+2\sqrt{2})}{2(2a^2(1+\sqrt{2})} \frac{m}{n} = \frac{4+2\sqrt{2}}{4+4\sqrt{2}} \frac{m}{n} = \frac{2+\sqrt{2}}{2+2\sqrt{2}}$$

Rationalizing the denominator by multiplying by the conjugate, we get $\frac{m}{n} = \boxed{\textbf{(B) } \frac{\sqrt{2}}{2}}$. ~ciceronii

- $\textbf{Note:}$ this can more easily be done if you know any of these formulas. This was an entire derivation of the area of an octagon, it's apothem, and it's circumradius, but it can be much simpler if you have any of these memorized.

Solution 3 (Elementary Geometry)

WLOG, let $AE = 1$. Let the intersection of AF and GD be point I. GIF is an isosceles right triangle ($m\angle HGF = 135^\circ$), so $IG = IF = \frac{1}{\sqrt{2}}$. The distance between each side and the center is then $IF + \frac{1}{2}GH = \frac{1}{2} + \frac{1}{\sqrt{2}}$. $ABCDEFGH$ is 8 triangles of base 1 and altitude $\frac{1}{2} + \frac{1}{\sqrt{2}}$, so its area is $8(\frac{1}{2})(\frac{1}{2} + \frac{1}{\sqrt{2}})$ or $2 + 2\sqrt{2}$Similarly, $ACEG$ is clearly a square of area GA^2. By the Pythagorean Theorem, $GA^2 = GI^2 + IA^2 = (\frac{1}{\sqrt{2}})^2 + (1 + \frac{1}{\sqrt{2}})^2$ or $2 + \sqrt{2}\frac{m}{n} = \frac{2 + \sqrt{2}}{2 + 2\sqrt{2}}$From some fast manipulations, see that it is $\boxed{\textbf{B)}\frac{\sqrt{2}}{2}}$

Problem15

In the complex plane, let A be the set of solutions to $z^3 - 8 = 0$ and let B be the set of solutions to $z^3 - 8z^2 - 8z + 64 = 0.$ What is the greatest distance between a point of A and a point of B?

$\textbf{(A) } 2\sqrt{3} \qquad \textbf{(B) } 6 \qquad \textbf{(C) } 9 \qquad \textbf{(D) } 2\sqrt{21} \qquad \textbf{(E) } 9 + \sqrt{3}$

Solution

Realize that $z^3 - 8 = 0$ will create an equilateral triangle on the complex plane with the first point at $2 + 0i$ and two other points with equal magnitude at $-1\pm i\sqrt{3}$.

Also, realize that $z^3 - 8z^2 - 8z + 64$ can be factored through grouping: $z^3 - 8z^2 - 8z + 64 = (z-8)(z^2-8)$.

$(z-8)(z^2-8)$ will create points at $8+0i$ and $\pm 2\sqrt{2} + 0i$.

Plotting the points and looking at the graph will make you realize that $-1 \pm i\sqrt{3}$ and $8+0i$ are the farthest apart and through Pythagorean Theorem, the answer is revealed to be $\sqrt{\sqrt{3}^2 + (8-(-1)^2} = \sqrt{84} = \boxed{\textbf{(D) } 2\sqrt{21}.}$ ~lopkiloinm

Problem16

A point is chosen at random within the square in the coordinate plane whose vertices are $(0,0), (2020,0), (2020,2020),$ and $(0,2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x,y) is a lattice point if x and y are both integers.) What is d to the nearest tenth$?$

$\textbf{(A) } 0.3 \qquad \textbf{(B) } 0.4 \qquad \textbf{(C) } 0.5 \qquad \textbf{(D) } 0.6 \qquad \textbf{(E) } 0.7$

Solution 1

Diagram

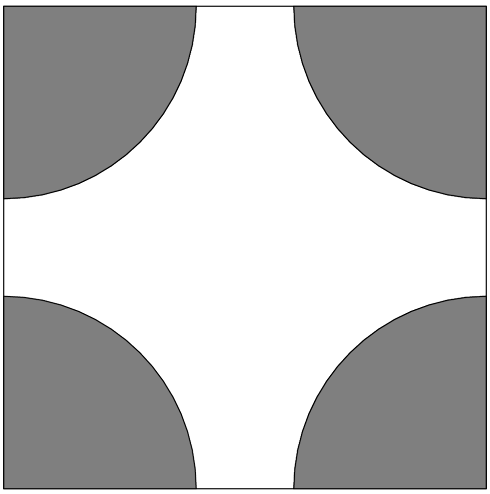


Diagram by MathandSki Using Asymptote

Note: The diagram represents each unit square of the given $2020 * 2020$ square.

Solution

We consider an individual one-by-one block.

If we draw a quarter of a circle from each corner (where the lattice points are located), each with radius d, the area covered by the circles should be 0.5. Because of this, and the fact that there are four circles, we write

$$4 * \frac{1}{4} * \pi d^2 = \frac{1}{2}$$

Solving for d, we obtain $d = \frac{1}{\sqrt{2\pi}}$, where with $\pi \approx 3$, we get $d = \frac{1}{\sqrt{6}}$, and from here, we simplify and see that $d \approx 0.4 \implies \boxed{\textbf{(B) } 0.4.}$ ~Crypthes

$\textbf{Note:}$ To be more rigorous, note that $d < 0.5$ since if $d \geq 0.5$ then clearly the probability is greater than $\frac{1}{2}$. This would make sure the above solution works, as if $d \geq 0.5$ there is overlap with the quartercircles. - $\textbf{Emathmaster}$

Solution 2

As in the previous solution, we obtain the equation $4 * \frac{1}{4} * \pi d^2 = \frac{1}{2}$, which simplifies to $\pi d^2 = \frac{1}{2} = 0.5$. Since π is slightly more than 3, d^2 is slightly less than $\frac{0.5}{3} = 0.1\overline{6}$. We notice that $0.1\overline{6}$ is slightly more than $0.4^2 = 0.16$, so d is roughly $\boxed{\textbf{(B) } 0.4}$. ~emerald_block

Solution 3 (Estimating)

As above, we find that we need to estimate $d = \frac{1}{\sqrt{2\pi}}$.

Note that we can approximate $2\pi \approx 6.28 \approx 6.25$ and so $\frac{1}{\sqrt{2\pi}} \approx \frac{1}{\sqrt{6.25}} = \frac{1}{2.5} = 0.4$.

And so our answer is $\boxed{\textbf{(B) } 0.4}$.

Problem 17

The vertices of a quadrilateral lie on the graph of $y = \ln x$, and the x-coordinates of these vertices are consecutive positive integers. The area of the quadrilateral is $\ln \frac{91}{90}$. What is the x-coordinate of the leftmost vertex?

$\textbf{(A) } 6 \qquad \textbf{(B) } 7 \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 12 \qquad \textbf{(E) } 13$

Solution 1

Let the coordinates of the quadrilateral be $(n, \ln(n)), (n+1, \ln(n+1)), (n+2, \ln(n+2)), (n+3, \ln(n+3))$. We have by shoelace's theorem, that the area is

$$\frac{\ln(n)(n+1)+\ln(n+1)(n+2)+\ln(n+2)(n+3)+n\ln(n+3)}{2}-\frac{\ln(n+1)\ln(n)+\ln(n+2)\ln(n+1)+\ln(n+3)(n+2)+\ln(n)(n+3)}{2}=$$

$$\frac{\ln\left(\frac{n^{n+1}(n+1)^{n+2}(n+2)^{n+3}(n+3)^n}{(n+1)^n(n+2)^{n+1}(n+3)^{n+2}n^{n+3}}\right)}{2} = \ln\left(\sqrt{\frac{(n+1)^2(n+3)^2}{n^2(n+2)^2}}\right) = \ln\left(\frac{(n+1)(n+2)}{n(n+3)}\right) = \ln\left(\frac{91}{90}\right).$$

We now that the numerator must have a factor of 13, so given the answer choices, n is either 12 or 11. If $n = 11$, the expression $\frac{(n+1)(n+2)}{n(n+3)}$ does not evaluate to $\frac{91}{90}$, but if $n = 12$, the expression evaluates to $\frac{91}{90}$. Hence, our answer is $\boxed{12}$.

~AopsUser101

Solution 2

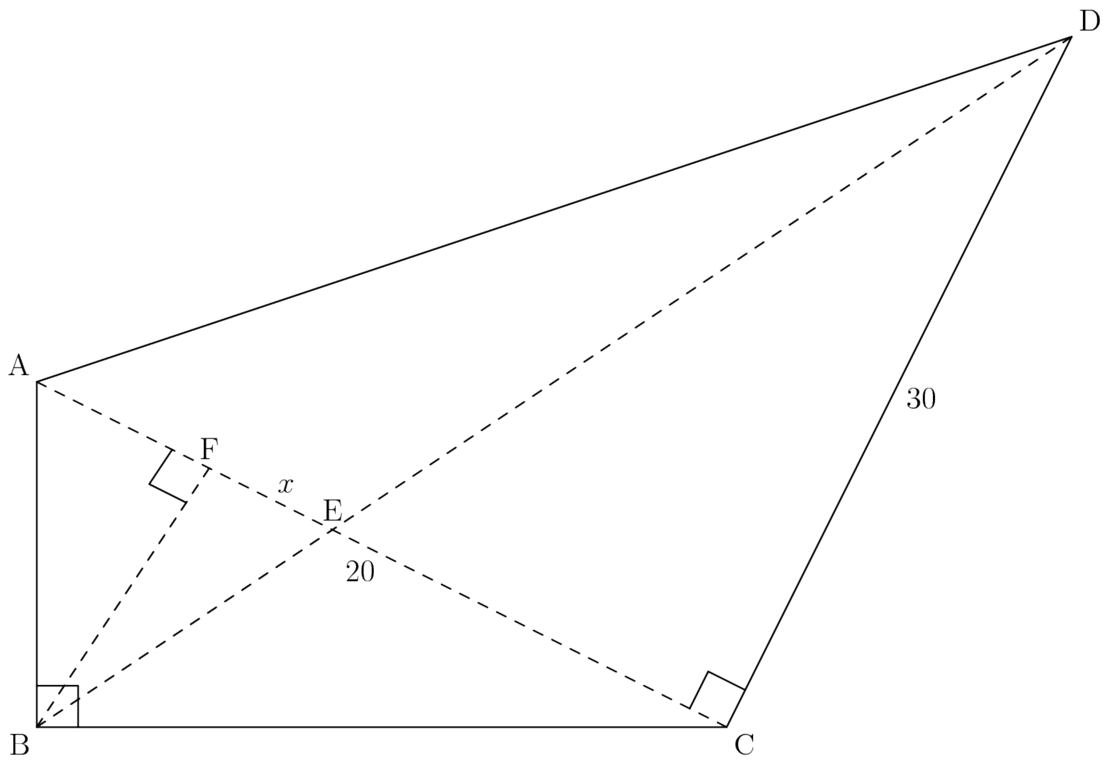
Like above, use the shoelace formula to find that the area of the triangle is equal to $\ln \frac{(n+1)(n+2)}{n(n+3)}$. Because the final area we are looking for is $\ln \frac{91}{90}$, the numerator factors into 13 and 7, which one of $n+1$ and $n+2$ has to be a multiple of 13 and the other has to be a multiple of 7. Clearly, the only choice for that is $\boxed{12}$

Problem18

Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ, AC = 20,$ and $CD = 30.$ Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point $E,$ and $AE = 5.$ What is the area of quadrilateral $ABCD?$

$\textbf{(A) } 330 \qquad \textbf{(B) } 340 \qquad \textbf{(C) } 350 \qquad \textbf{(D) } 360 \qquad \textbf{(E) } 370$

Solution 1 (Just Drop An Altitude)



It's crucial to draw a good diagram for this one.

Since $AC = 20$ and $CD = 30$, we get $[ACD] = 300$. Now we need to find $[ABC]$ to get the area of the whole quadrilateral. Drop an altitude from B to AC and call the point of intersection F. Let $FE = x$. Since $AE = 5$, then $AF = 5 - x$. By dropping this altitude, we can also see two similar triangles, BFE and DCE. Since EC is $20 - 5 = 15$, and $DC = 30$, we get that $BF = 2x$. Now, if we redraw another diagram just of ABC, we get that $(2x)^2 = (5 - x)(15 + x)$. Now expanding, simplifying, and dividing by the GCF, we get $x^2 + 2x - 15 = 0$. This factors to $(x + 5)(x - 3)$. Since lengths cannot be negative, $x = 3$.

Since $x = 3$, $[ABC] = 60$.

So

$$[ABCD] = [ACD] + [ABC] = 300 + 60 = \boxed{\textbf{(D) } 360}$$

(I'm very sorry if you're a visual learner but now you have a diagram by ciceronii)

~ Solution by Ultraman

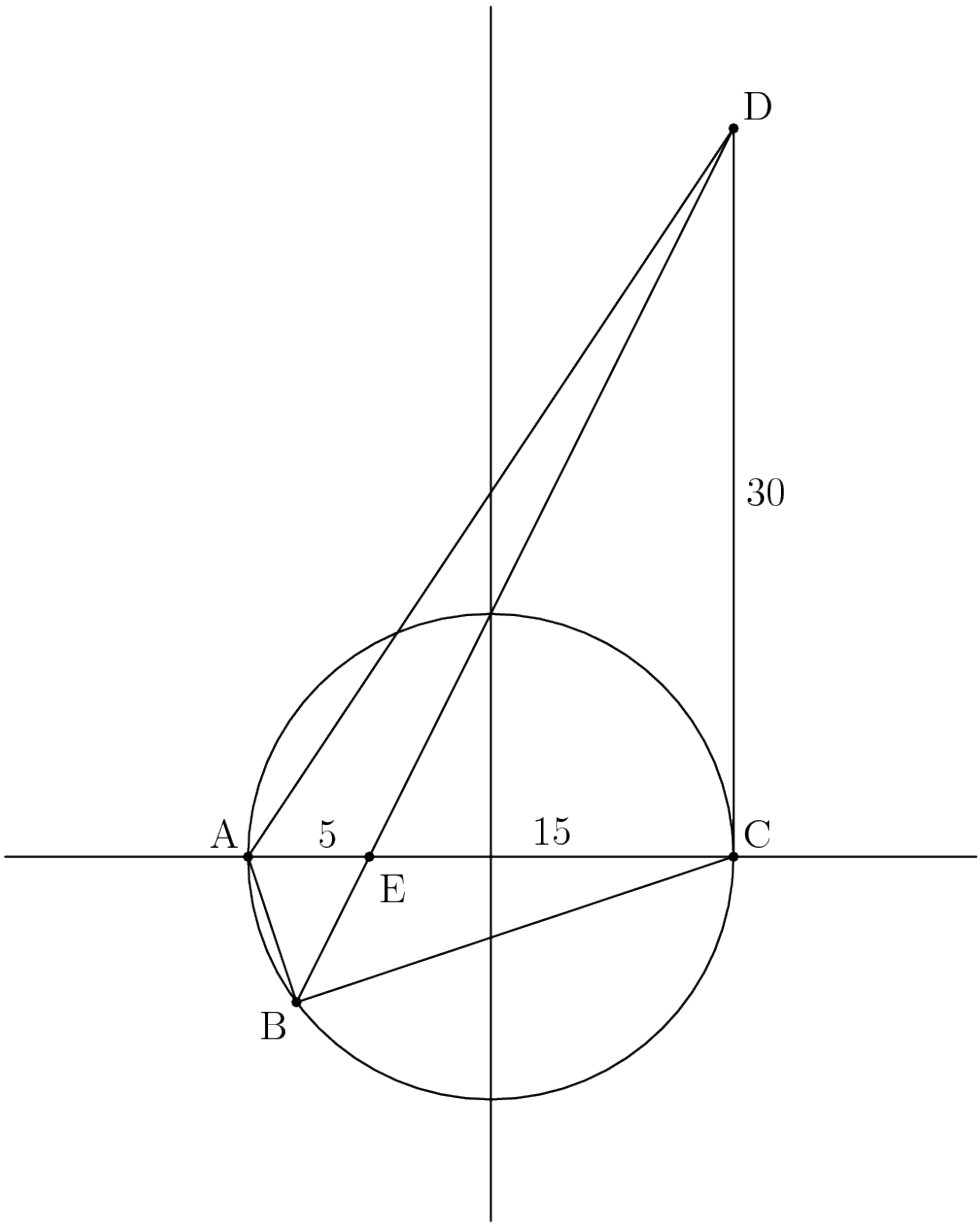
~ Diagram by ciceronii

Solution 2 (Pro Guessing Strats)

We know that the big triangle has area 300. Use the answer choices which would mean that the area of the little triangle is a multiple of 10. Thus the product of the legs is a multiple of 20. Guess that the legs are equal to $\sqrt{20a}$ and $\sqrt{20b}$, and because the hypotenuse is 20 we get $a + b = 20$. Testing small numbers, we get that when $a = 2$ and $b = 18$, ab is indeed a square. The area of the triangle is thus 60, so the answer is $\boxed{\textbf{(D) } 360}$.

~tigershark22 ~(edited by HappyHuman)

Solution 3 (coordinates)



Let the points be $A(-10,0)$, $B(x,y)$, $C(10,0)$, $D(10,30)$,and $E(-5,0)$, respectively. Since B lies on line DE, we know that $y=2x+10$. Furthermore, since $\angle ABC = 90^\circ$, B lies on the circle with diameter AC, so $x^2+y^2=100$. Solving for x and y with these equations, we get the solutions $(0,10)$and $(-8,-6)$. We immediately discard the $(0,10)$ solution as y should be negative. Thus, we conclude that

$$[ABCD]=[ACD]+[ABC]=\frac{20\cdot 30}{2}+\frac{20\cdot 6}{2}=\boxed{\textbf{(D) }360}$$

.

Solution 4 (Trigonometry)

Let $\angle C=\angle ACB$ and $\angle B=\angle CBE$. Using Law of Sines on $\triangle BCE$ we get $\frac{BE}{\sin C}=\frac{CE}{\sin B}=\frac{15}{\sin B}$ and LoS on $\triangle ABE$ yields $\frac{BE}{\sin(90-C)}=\frac{5}{\sin(90-B)}=\frac{BE}{\cos C}=\frac{5}{\cos B}$. Divide the two to get $\tan B=3\tan C$. Now, $\tan\angle CED=2=\tan\angle B+\angle C=\frac{4\tan C}{1-3\tan^2 C}$ and solve the quadratic, taking the positive solution (C is acute) to get $\tan C=\frac{1}{3}$. So if $AB=a$, then $BC=3a$ and $[ABC]=\frac{3a^2}{2}$. By Pythagorean Theorem, $10a^2=400\iff\frac{3a^2}{2}=60$ and the answer is $300+60\iff\boxed{\textbf{(D)}}$.

(This solution is incomplete, can someone complete it please-Lingjun) ok Latex edited by kc5170

We could use the famous m-n rule in trigonometry in triangle ABC with Point E [Unable to write it here.Could anybody write the expression] We will find that BD is angle bisector of triangle ABC(because we will get tan (x)=1) Therefore by converse of angle bisector theorem AB:BC = 1:3. By using phythagorean theorem we have values of AB and AC. AB.AC = 120. Adding area of ABC and ACD Answer••360

Problem19

There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \ldots < a_k$ such that $\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k}$. What is k?

$\textbf{(A) } 117 \qquad \textbf{(B) } 136 \qquad \textbf{(C) } 137 \qquad \textbf{(D) } 273 \qquad \textbf{(E) } 306$

Solution 1

First, substitute 2^{17} with a. Then, the given equation becomes $\frac{a^{17}+1}{a+1} = a^{16} - a^{15} + a^{14} ... - a^1 + a^0$. Now consider only $a^{16} - a^{15}$. This equals $a^{15}(a-1) = a^{15} * (2^{17}-1)$. Note that $2^{17}-1$ equals $2^{16} + 2^{15} + ... + 1$, since the sum of a geometric sequence is $\frac{a^n - 1}{a-1}$. Thus, we can see that $a^{16} - a^{15}$ forms the sum of 17 different powers of 2. Applying the same method to each of $a^{14} - a^{13}$, $a^{12} - a^{11}$, ... , $a^2 - a^1$, we can see that each of the pairs forms the sum of 17 different powers of 2. This gives us $17 * 8 = 136$. But we must count also the a^0 term. Thus, Our answer is $136 + 1 = \boxed{\textbf{(C) } 137}$.

~seanyoon777

Solution 2

(This is similar to solution 1) Let $x = 2^{17}$. Then, $2^{289} = x^{17}$. The LHS can be rewritten as

$\frac{x^{17}+1}{x+1} = x^{16} - x^{15} + \cdots + x^2 - x + 1 = (x-1)(x^{15} + x^{13} + \cdots + x^1) + 1$

. Plugging 2^{17}back in for x, we have $(2^{17}-1)(2^{15\cdot17} + 2^{13\cdot17} + \cdots + 2^{1\cdot17}) + 1 = (2^{16} + 2^{15} + \cdots + 2^0)(2^{15\cdot17} + 2^{13\cdot17} + \cdots + 2^{1\cdot17}) + 1$

. When expanded, this will have $17 \cdot 8 + 1 = 137$ terms. Therefore, our answer is $\boxed{\textbf{(C) } 137}$.

Solution 3 (Intuitive)

Multiply both sides by $2^{17}+1$ to get $2^{289}+1 = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k} + 2^{a_1+17} + 2^{a_2+17} + \ldots + 2^{a_k+17}$.

Notice that $a_1 = 0$, since there is a 1 on the LHS. However, now we have an extra term of 2^{18} on the right from 2^{a_1+17}. To cancel it, we let $a_2 = 18$. The two 2^{18}'s now combine into a term of 2^{19}, so we let $a_3 = 19$. And so on, until we get to $a_{18} = 34$. Now everything we don't want telescopes into 2^{35}. We already have that term since we let $a_2 = 18 \implies a_2 + 17 = 35$.

Everything from now on will automatically telescope to 2^{52}. So we let a_{19} be 52.

As you can see, we will have to add 17 a_n's at a time, then "wait" for the sum to automatically telescope for the next 17 numbers, etc, until we get to 2^{289}. We only need to add a_n's between odd multiples of 17 and even multiples. The largest even multiple of 17 below 289 is $17 \cdot 16$, so we will have to add a total of $17 \cdot 8$ a_n's. However, we must not forget we let $a_1 = 0$ at the beginning, so our answer is $17 \cdot 8 + 1 = \boxed{\textbf{(C) } 137}$.

Solution 4

Note that the expression is equal to something slightly lower than 2^{272}. Clearly, answer choices (D) and (E) make no sense because the lowest sum for 273 terms is $2^{273}-1$. (A) just makes no sense. (B) and (C) are 1 apart, but because the expression is odd, it will have to contain $2^0=1$, and because (C) is 1bigger, the answer is $\boxed{\textbf{(C) } 137}$.

~Lcz

Solution 5

In order to shorten expressions, $\#$ will represent 16 consecutive 0s when expressing numbers.

Think of the problem in binary. We have

$$\frac{1\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#1_2}{1\#1_2}$$

Note that

$$(2^{17}+1)(2^0+2^{34}+2^{68}+\cdots+2^{272}) = 2^0(2^{17}+1)+2^{34}(2^{17}+1)+2^{68}(2^{17}+1)+\cdots 2^{272}(2^{17}+1)$$
$$= 1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1_2$$

and

$$(2^{17}+1)(2^{17}+2^{51}+2^{85}+\cdots+2^{255}) = 2^{17}(2^{17}+1)+2^{51}(2^{17}+1)+2^{85}(2^{17}+1)+\cdots 2^{255}(2^{17}+1)$$
$$= 1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#0_2$$

Since

$$\begin{aligned} &1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1_2 \\ - \quad &1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#0_2 \\ = \; &1\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#1_2 \end{aligned}$$

this means that

$$(2^{17}+1)(2^0+2^{34}+2^{68}+\cdots+2^{272}) - (2^{17}+1)(2^{17}+2^{51}+2^{85}+\cdots+2^{255}) = 2^{289}$$

so

$$\frac{2^{289}+1}{2^{17}+1}=(2^0+2^{34}+2^{68}+\cdots+2^{272})-(2^{17}+2^{51}+2^{85}+\cdots+2^{255})$$

$$=2^0+(2^{34}-2^{17})+(2^{68}-2^{51})+\cdots+(2^{272}-2^{255})$$

Expressing each of the pairs of the form $2^{n+17}-2^n$ in binary, we have

$$\begin{array}{rr} & 100000000000000000\cdots0_2 \\ - & 10\cdots0_2 \\ = & 11111111111111110\cdots0_2 \end{array}$$

or

$$2^{n+17}-2^n=2^{n+16}+2^{n+15}+2^{n+14}+\cdots+2^n$$

This means that each pair has 17 terms of the form 2^n.

Since there are 8 of these pairs, there are a total of $8\cdot 17=136$ terms.

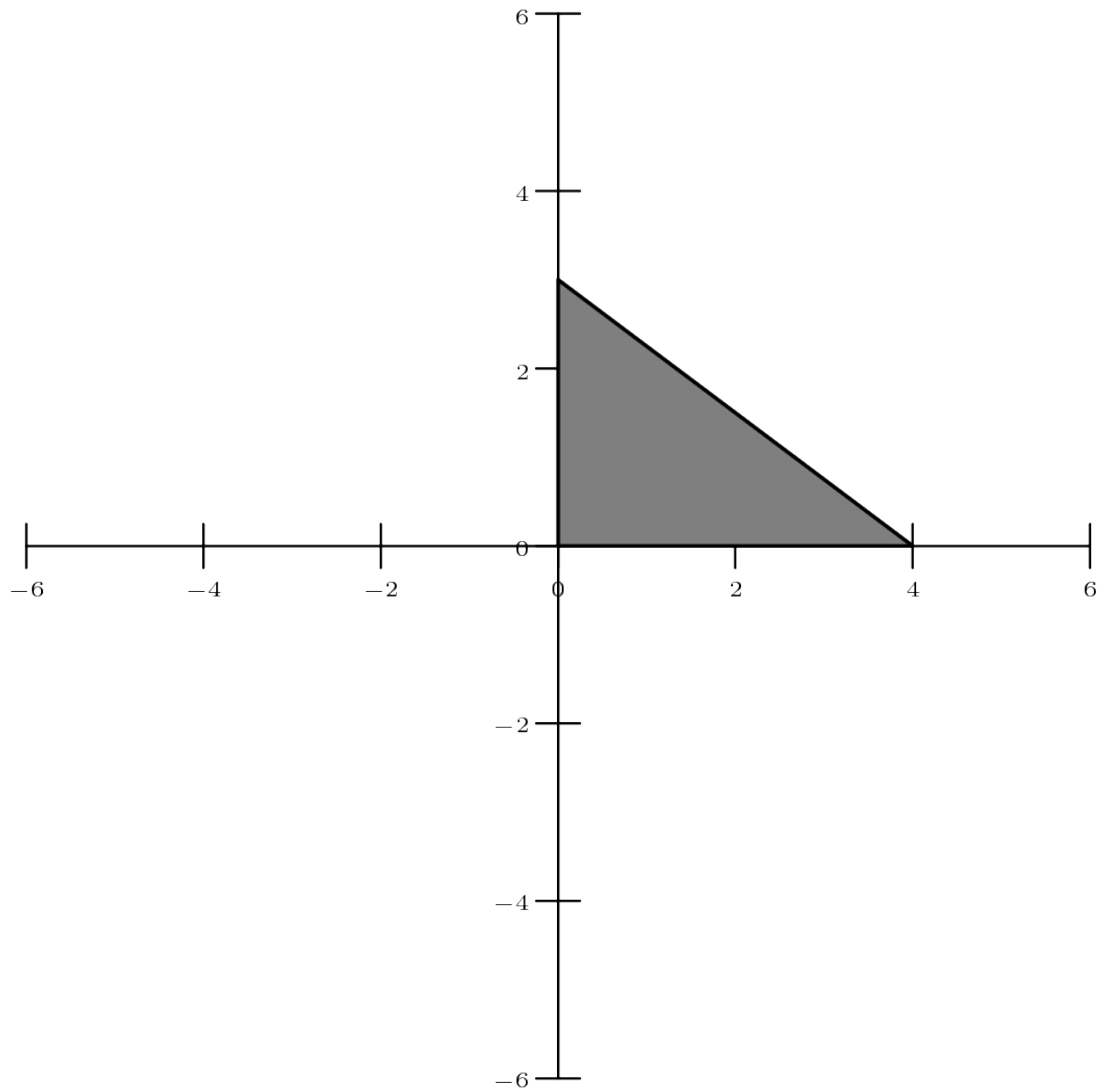
Accounting for the 2^0 term, which was not in the pair, we have a total of $136+1=\boxed{\textbf{(C) } 137}$ terms

Problem20

Let T be the triangle in the coordinate plane with vertices $(0,0)$, $(4,0)$, and $(0,3)$. Consider the following five isometries (rigid transformations) of the plane: rotations of 90°, 180°, and 270° counterclockwise around the origin, reflection across the x-axis, and reflection across the y-axis. How many of the 125sequences of three of these transformations (not necessarily distinct) will return T to its original position? (For example, a 180° rotation, followed by a reflection across the x-axis, followed by a reflection across the y-axis will return T to its original position, but a 90° rotation, followed by a reflection across the x-axis, followed by another reflection across the x-axis will not return T to its original position.)

$\textbf{(A) } 12 \qquad \textbf{(B) } 15 \qquad \textbf{(C) } 17 \qquad \textbf{(D) } 20 \qquad \textbf{(E) } 25$

Solution



First, any combination of motions we can make must reflect T an even number of times. This is because every time we reflect T, it changes orientation. Once T has been flipped once, no combination of rotations will put it back in place because it is the mirror image; however, flipping it again changes it back to the original orientation. Since we are only allowed 3 transformations and an even number of them must be reflections, we either reflect T 0 times or 2 times.

Case 1: 0 reflections on T

In this case, we must use 3 rotations to return T to its original position. Notice that our set of rotations, $\{90^\circ, 180^\circ, 270^\circ\}$, contains every multiple of 90° except for 0°. We can start with any two rotations a, b in $\{90^\circ, 180^\circ, 270^\circ\}$ and there must be exactly one $c \equiv -a - b \pmod{360^\circ}$ such that we can use the three rotations (a, b, c) which ensures that $a + b + c \equiv 0^\circ \pmod{360^\circ}$.

That way, the composition of rotations a, b, c yields a full rotation. For example, if $a = b = 90^\circ$, then $c \equiv -90^\circ - 90^\circ = -180^\circ \pmod{360^\circ}$, so $c = 180^\circ$ and the rotations $(90^\circ, 90^\circ, 180^\circ)$ yields a full rotation.

The only case in which this fails is when c would have to equal 0°. This happens when (a, b) is already a full rotation, namely, $(a, b) = (90^\circ, 270^\circ), (180^\circ, 180^\circ)$, or $(270^\circ, 90^\circ)$. However, we can simply subtract these three cases from the total. Selecting (a, b) from $\{90^\circ, 180^\circ, 270^\circ\}$ yields $3 \cdot 3 = 9$choices, and with 3 that fail, we are left with 6 combinations for case 1.

Case 2: 2 reflections on T

In this case, we first eliminate the possibility of having two of the same reflection. Since two reflections across the x-axis maps T back to itself, inserting a rotation before, between, or after these two reflections would change T's final location, meaning that any combination involving two reflections across the x-axis would not map T back to itself. The same applies to two reflections across the y-axis.

Therefore, we must use one reflection about the x-axis, one reflection about the y-axis, and one rotation. Since a reflection about the x-axis changes the sign of the y component, a reflection about the y-axis changes the sign of the x component, and a 180° rotation changes both signs, these three transformation composed (in any order) will suffice. It is therefore only a question of arranging the three, giving us $3! = 6$combinations for case 2.

Combining both cases we get $6 + 6 = \boxed{\textbf{(A)}\ 12}$

Solution 2(Rewording solution 1)

As in the previous solution, note that we must have either 0 or 2 reflections because of orientation since reflection changes orientation that is impossible to fix by rotation. We also know we can't have the same reflection twice, since that would give a net of no change and would require an identity rotation.

Suppose there are no reflections. Denote 90° as 1, 180° as 2, and 270° as 3, just for simplification purposes. We want a combination of 3 of these that will sum to either 4 or 8(0 and 12 is impossible since the minimum is 3 and the max is 9). 4 can be achieved with any permutation of $(1-1-2)$ and 8 can be achieved with any permutation of $(2-3-3)$. This case can be done in $3+3=6$ ways.

Suppose there are two reflections. As noted already, they must be different, and as a result will take the triangle to the opposite side of the origin if we don't do any rotation. We have 1 rotation left that we can do though, and the only one that will return to the original position is 2, which is 180° AKA reflection across origin. Therefore, since all 3 transformations are distinct. The three transformations can be applied anywhere since they are commutative(think quadrants). This gives 6 ways.

$$6+6=\boxed{(A)12}$$

Problem 21

How many positive integers n are there such that n is a multiple of 5, and the least common multiple of $5!$and n equals 5 times the greatest common divisor of $10!$ and $n?$

$\textbf{(A) } 12 \qquad \textbf{(B) } 24 \qquad \textbf{(C) } 36 \qquad \textbf{(D) } 48 \qquad \textbf{(E) } 72$

Solution

We set up the following equation as the problem states:

$$\operatorname{lcm}(5!, n) = 5\operatorname{gcd}(10!, n).$$

Breaking each number into its prime factorization, we see that the equation becomes

$$\operatorname{lcm}(2^3\cdot 3\cdot 5, n) = 5\operatorname{gcd}(2^8\cdot 3^4\cdot 5^2\cdot 7, n).$$

We can now determine the prime factorization of n. We know that its prime factors belong to the set $\{2, 3, 5, 7\}$, as no factor of $10!$ has 11 in its prime factorization, nor anything greater. Next, we must find exactly how many different possibilities exist for each.

There can be anywhere between 3 and 8 2's and 1 to 4 3's. However, since n is a multiple of 5, and we multiply the $\gcd$ by 5, there can only be 3 5's in n's prime factorization. Finally, there can either 0 or 1 7's.

Thus, we can multiply the total possibilities of n's factorization to determine the number of integers $n$which satisfy the equation, giving us $6 \times 4 \times 1 \times 2 = \boxed{\textbf{(D)}\ 48}$

Problem22

Let (a_n) and (b_n) be the sequences of real numbers such that$(2+i)^n = a_n + b_n i$for all integers $n \geq 0$, where $i = \sqrt{-1}$.

What is$\sum_{n=0}^{\infty} \frac{a_n b_n}{7^n}$?

$\textbf{(A)}\ \frac{3}{8} \qquad \textbf{(B)}\ \frac{7}{16} \qquad \textbf{(C)}\ \frac{1}{2} \qquad \textbf{(D)}\ \frac{9}{16} \qquad \textbf{(E)}\ \frac{4}{7}$

Solution 1

Square the given equality to yield$(3+4i)^n = (a_n + b_n i)^2 = (a_n^2 - b_n^2) + 2a_n b_n i,$ so $a_n b_n = \frac{1}{2}\operatorname{Im}((3+4i)^n)$ and

$$\sum_{n\geq 0} \frac{a_n b_n}{7^n} = \frac{1}{2}\operatorname{Im}\left(\sum_{n\geq 0} \frac{(3+4i)^n}{7^n}\right) = \frac{1}{2}\operatorname{Im}\left(\frac{1}{1-\frac{3+4i}{7}}\right) = \boxed{\frac{7}{16}}.$$

Solution 2 (DeMoivre's Formula)

Note that $(2+i) = \sqrt{5} \cdot \left(\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} i\right)$.

Let $\theta = \arctan(1/2)$, then, we know that $(2+i)=\sqrt{5}\cdot(\cos\theta+i\sin\theta)$,

so $(2+i)^n=(\cos(n\theta)+i\sin(n\theta))(\sqrt{5})^n$.

Therefore, $\sum_{n=0}^{\infty}\frac{a_nb_n}{7^n}=\sum_{n=0}^{\infty}\frac{\cos(n\theta)\sin(n\theta)(5)^n}{7^n}=$

$\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{5}{7}\right)^n\sin(2n\theta)=\frac{1}{2}\text{im}\left(\sum_{n=0}^{\infty}\left(\frac{5}{7}\right)^n e^{2i\theta n}\right)$.

Aha $\sum_{n=0}^{\infty}\left(\frac{5}{7}\right)^n e^{2i\theta n}$ is a geometric sequence that evaluates

to $\frac{1}{1-\frac{5}{7}e^{2\theta i}}$. We can quickly see

that $\sin(2\theta)=2\cdot\sin\theta\cdot\cos\theta=2\cdot\frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}=\frac{4}{5}$.

$\cos(2\theta)=\cos^2\theta-\sin^2\theta=\frac{4}{5}-\frac{1}{5}=\frac{3}{5}$.

Therefore, $\frac{1}{1-\frac{5}{7}e^{2\theta i}}=\frac{1}{1-\frac{5}{7}\left(\frac{3}{5}+\frac{4}{5}i\right)}=\frac{7}{8}+\frac{7}{8}i$. The

imaginary part is $\frac{7}{8}$, so our answer is $\frac{1}{2}\cdot\frac{7}{8}=\boxed{\frac{7}{16}}$.

Problem23

Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

$\textbf{(A) } \frac{7}{36} \qquad \textbf{(B) } \frac{5}{24} \qquad \textbf{(C) } \frac{2}{9} \qquad \textbf{(D) } \frac{17}{72} \qquad \textbf{(E) } \frac{1}{4}$

Solution 1

Consider the probability that rolling two dice gives a sum of s, where $s \leq 7$. There are $s-1$ pairs that satisfy this, namely $(1, s-1), (2, s-2), \ldots, (s-1, 1)$, out of $6^2 = 36$ possible pairs. The probability is $\frac{s-1}{36}$.

Therefore, if one die has a value of a and Jason rerolls the other two dice, then the probability of winning is $\frac{7-a-1}{36} = \frac{6-a}{36}$.

In order to maximize the probability of winning, a must be minimized. This means that if Jason rerolls two dice, he must choose the two dice with the maximum values.

Thus, we can let $a \leq b \leq c$ be the values of the three dice, which we will call A, B, and C respectively. Consider the case when $a + b < 7$. If $a + b + c = 7$, then we do not need to reroll any dice. Otherwise, if we reroll one die, we can roll dice C in the hope that we get the value that makes the sum of the three dice 7. This happens with probability $\frac{1}{6}$. If we reroll two dice, we will roll B and C, and the probability of winning is $\frac{6-a}{36}$, as stated above.

However, $\frac{1}{6} > \frac{6-a}{36}$, so rolling one die is always better than rolling two dice if $a + b < 7$.

Now consider the case where $a + b \geq 7$. Rerolling one die will not help us win since the sum of the three dice will always be greater than 7. If we reroll two dice, the probability of winning is, once again, $\frac{6-a}{36}$. To find the probability of winning if we reroll all three dice, we can let each dice have 1 dot and find the number of ways to distribute the remaining 4 dots. By Stars and Bars, there are $\binom{6}{2} = 15$ ways to do this, making the probability of winning $\frac{15}{6^3} = \frac{5}{72}$.

In order for rolling two dice to be more favorable than rolling three dice, $\frac{6-a}{36} > \frac{5}{72} \rightarrow a \leq 3$.

Thus, rerolling two dice is optimal if and only if $a \leq 3$ and $a+b \geq 7$. The possible triplets (a,b,c) that satisfy these conditions, and the number of ways they can be permuted,

are $(3,4,4) \rightarrow 3$ ways. $(3,4,5) \rightarrow 6$ ways. $(3,4,6) \rightarrow 6$ ways.

$(3,5,5) \rightarrow 3$ ways. $(3,5,6) \rightarrow 6$ ways. $(3,6,6) \rightarrow 3$ ways.

$(2,5,5) \rightarrow 3$ ways. $(2,5,6) \rightarrow 6$ ways. $(2,6,6) \rightarrow 3$ ways.

$(1,6,6) \rightarrow 3$ ways.

There are $3+6+6+3+6+3+3+6+3+3 = 42$ ways in which rerolling two dice is optimal, out of $6^3 = 216$ possibilities, Therefore, the probability that Jason will reroll two dice is $\frac{42}{216} = \boxed{\textbf{(A)}\ \frac{7}{36}}$

Solution 2

We count the numerator. Jason will pick up no dice if he already has a 7 as a sum. We need to assume he does not have a 7 to begin with. If Jason decides to pick up all the dice to re-roll, by Stars and Bars(or whatever...), there will be 2 bars and 4 stars(3 of them need to be guaranteed because a roll is at least 1) for a probability of $\frac{15}{216} = \frac{2.5}{36}$. If Jason picks up 2 dice and leaves a die showing k, he will need the other two to sum to $7-k$. This happens with probability $\frac{6-k}{36}$ for integers $1 \leq k \leq 6$. If the roll is not 7, Jason will pick up exactly one die to re-roll if there can remain two other dice with sum less than 7, since this will give him a $\frac{1}{6}$ chance which is a larger probability than all the cases unless he has a 7 to begin with. We have $\frac{1}{6} > \underline{\frac{5,4,3}{36}} > \frac{2.5}{36} > \frac{2,1,0}{36}$. We count the underlined part's frequency for the numerator without upsetting the probability greater than it. Let a be the roll we keep. We know a is at most 3 since 4 would cause Jason to pick up all the dice. When $a=1$, there are 3 choices for whether it is rolled 1st, 2nd, or 3rd, and in this case the other two rolls have to be at least 6(or he would have only picked up 1). This give $3 \cdot 1^2 = 3$ ways.

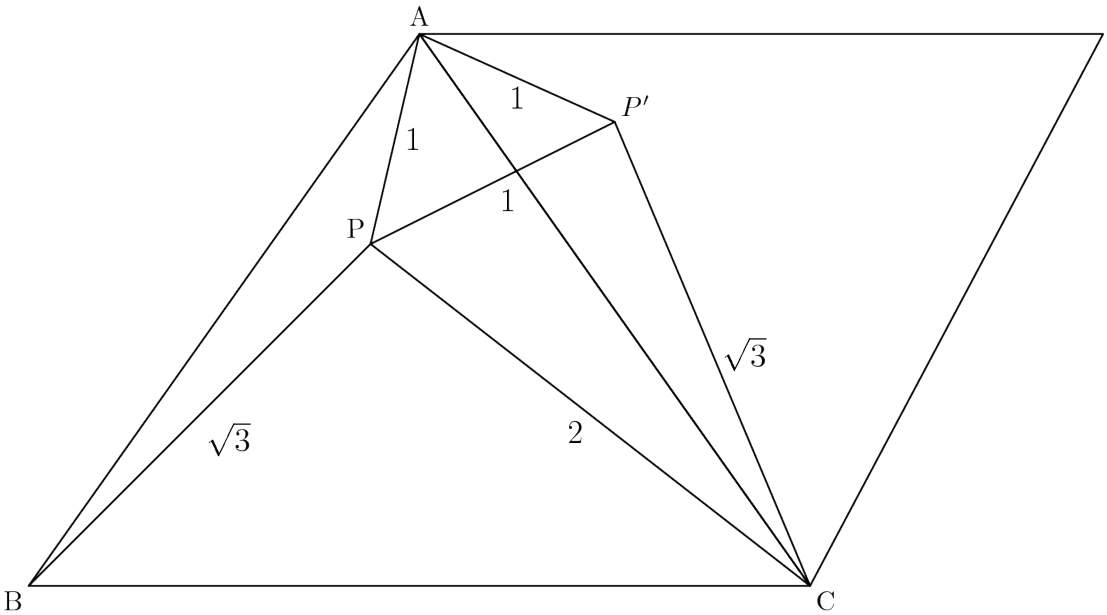
Similarly, $a=2$ gives $3 \cdot 2^2 = 12$ because the 2 can be rolled in 3 places and the other two rolls are at least 5. $a=3$ gives $3 \cdot 3^2 = 27$. Summing together gives the numerator of 42. The denominator is $6^3 = 216$, so we have $\frac{42}{216} = \boxed{(A)\frac{7}{36}}$

Problem 24

Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1$, $BP = \sqrt{3}$, and $CP = 2$. What is s?

$\textbf{(A) } 1+\sqrt{2} \qquad \textbf{(B) } \sqrt{7} \qquad \textbf{(C) } \frac{8}{3} \qquad \textbf{(D) } \sqrt{5+\sqrt{5}} \qquad \textbf{(E) } 2\sqrt{2}$

Solution



We begin by rotating $\triangle ABC$ by 60° about A, such that in $\triangle A'B'C'$, $B' = C$. We see that $\triangle APP'$ is equilateral with side length 1, meaning that $\angle APP' = 60^\circ$. We also see that $\triangle CPP'$ is a $30-60-90$ right triangle, meaning that $\angle CPP' = 60^\circ$. Thus, by adding the two together, we see that $\angle APC = 120^\circ$. We can now use the law of cosines as following:

$$s^2 = (AP)^2 + (CP)^2 - 2(AP)(CP)\cos\angle APC$$

$$s^2 = 1 + 4 - 2(1)(2)\cos 120^\circ s^2 = 5 - 4(-\frac{1}{2})$$

$$s = \sqrt{5+2}$$

giving us that $s = \boxed{\textbf{(B) } \sqrt{7}}$. ~ciceronii

Problem 25

The number $a = \frac{p}{q}$, where p and q are relatively prime positive integers, has the property that the sum of all real numbers x satisfying $\lfloor x \rfloor \cdot \{x\} = a \cdot x^2$ is 420, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. What is $p + q$?

$\textbf{(A) }245 \qquad \textbf{(B) }593 \qquad \textbf{(C) }929 \qquad \textbf{(D) }1331 \qquad \textbf{(E) }1332$

Solution 1

Let $1 < k < 2$ be the unique solution in this range. Note that ck is also a solution as long as $ck < c + 1$, hence all our solutions are $k, 2k, ..., bk$ for some b. This sum 420 must be between $\frac{b(b+1)}{2}$ and $\frac{(b+1)(b+2)}{2}$, which gives $b = 28$ and $k = \frac{420}{406} = \frac{30}{29}$. Plugging this back in gives $a = \frac{29 \cdot 1}{30^2} = \frac{29}{900} \implies \boxed{\textbf{C}}$.

Solution 2

First note that $\lfloor x \rfloor \cdot \{x\} < 0$ when $x < 0$ while $ax^2 \geq 0 \forall x \in \mathbb{R}$. Thus we only need to look at positive solutions ($x = 0$ doesn't affect the sum of the solutions). Next, we breakdown $\lfloor x \rfloor \cdot \{x\}$ down for each interval $[n, n+1)$, where n is a positive integer. Assume $\lfloor x \rfloor = n$, then $\{x\} = x - n$. This means that when $x \in [n, n+1)$, $\lfloor x \rfloor \cdot \{x\} = n(x-n) = nx - n^2$. Setting this equal to ax^2 gives

$$nx - n^2 = ax^2 \implies ax^2 - nx + n^2 = 0 \implies x = \frac{n \pm \sqrt{n^2 - 4an^2}}{2a}$$

We're looking at the solution with smaller x, which is $x = \frac{n - n\sqrt{1-4a}}{2a} = \frac{n}{2a}\left(1 - \sqrt{1-4a}\right)$. Note that if $\lfloor x \rfloor = n$ is the greatest n such that $\lfloor x \rfloor \cdot \{x\} = ax^2$ has a solution, the sum of all these solutions is slightly over $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, which is 406 when $n = 28$, just under 420. Checking this gives

$$\sum_{k=1}^{28} \frac{k}{2a}\left(1 - \sqrt{1-4a}\right) = \frac{1-\sqrt{1-4a}}{2a} \cdot 406 = 420$$

$$\frac{1-\sqrt{1-4a}}{2a} = \frac{420}{406} = \frac{30}{29} 29 - 29\sqrt{1-4a} = 60a$$

$$29\sqrt{1-4a} = 29 - 60a$$

$$29^2 - 4 \cdot 29^2 a = 29^2 + 3600a^2 - 120 \cdot 29a$$

$$3600a^2 = 116a^{a} = \frac{116}{3600} = \frac{29}{900} \implies \boxed{\textbf{(C) } 929}$$