2009 AMC 12A Problems/Problem 1

Problem

Kim's flight took off from Newark at 10:34 AM and landed in Miami at 1:18 PM. Both cities are in the same time zone. If her flight took h hours and m minutes, with $0 < m < 60$, what is $h + m$?

$\textbf{(A)}\ 46 \qquad \textbf{(B)}\ 47 \qquad \textbf{(C)}\ 50 \qquad \textbf{(D)}\ 53 \qquad \textbf{(E)}\ 54$

Solution

There is 1 hour and $60 - 34 = 26$ minutes between 10:34 AM and noon; and there is 1 hour and 18 minutes between noon and 1:18 PM. Hence the flight took 2 hours and $26 + 18 = 44$ minutes, and $h + m = 46\ \textbf{(A)}$.

See Also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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Category: Introductory Algebra Problems

2009 AMC 12A Problems/Problem 2

The following problem is from both the 2009 AMC 12A #2 and 2009 AMC 10A #3, so both problems redirect to this page.

Problem

Which of the following is equal to $1+\frac{1}{1+\frac{1}{1+1}}$?

$\textbf{(A)}\ \frac{5}{4} \qquad \textbf{(B)}\ \frac{3}{2} \qquad \textbf{(C)}\ \frac{5}{3} \qquad \textbf{(D)}\ 2 \qquad \textbf{(E)}\ 3$

Solution

We compute:

$$\begin{aligned} 1+\frac{1}{1+\frac{1}{1+1}} &= 1+\frac{1}{1+\frac{1}{1+1}} \\ &= 1+\frac{1}{1+\frac{1}{2}} \\ &= 1+\frac{1}{\frac{3}{2}} \\ &= 1+\frac{2}{3} \\ &= \frac{5}{3} \end{aligned}$$

This is choice $\boxed{C}$.

See Also

2009 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
Preceded by Problem 2	Followed by Problem 4
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2009 AMC 12A Problems/Problem 3

Contents

Problem

What number is one third of the way from $\frac{1}{4}$ to $\frac{3}{4}$?

$\textbf{(A)}\ \frac{1}{3} \qquad \textbf{(B)}\ \frac{5}{12} \qquad \textbf{(C)}\ \frac{1}{2} \qquad \textbf{(D)}\ \frac{7}{12} \qquad \textbf{(E)}\ \frac{2}{3}$

Solution

Solution 1

We can rewrite the two given fractions as $\frac{3}{12}$ and $\frac{9}{12}$. (We multiplied all numerators and denominators by 3.)

Now it is obvious that the interval between them is divided into three parts by the fractions $\boxed{\frac{5}{12}}$ and $\frac{7}{12}$.

Solution 2

The number we seek can be obtained as a weighted average of the two endpoints, where the closer one has weight 2 and the further one 1. We compute:

$$\frac{2\cdot\frac{1}{4}+1\cdot\frac{3}{4}}{3} = \frac{\frac{5}{4}}{3} = \boxed{\frac{5}{12}}$$

See Also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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2009 AMC 12A Problems/Problem 4

The following problem is from both the 2009 AMC 12A #4 and 2009 AMC 10A #2, so both problems redirect to this page.

Problem

Four coins are picked out of a piggy bank that contains a collection of pennies, nickels, dimes, and quarters. Which of the following could not be the total value of the four coins, in cents?

$\textbf{(A)}\ 15 \qquad \textbf{(B)}\ 25 \qquad \textbf{(C)}\ 35 \qquad \textbf{(D)}\ 45 \qquad \textbf{(E)}\ 55$

Solution

As all five options are divisible by 5, we may not use any pennies. (This is because a penny is the only coin that is not divisible by 5, and if we used between 1 and 4 pennies, the sum would not be divisible by 5.)

Hence the smallest coin we can use is a nickel, and thus the smallest amount we can get is $4\cdot 5 = 20$. Therefore the option that is not reachable is $\boxed{15} \Rightarrow (A)$.

We can verify that we can indeed get the other ones:

- $25 = 10 + 5 + 5 + 5$
- $35 = 10 + 10 + 10 + 5$
- $45 = 25 + 10 + 5 + 5$
- $55 = 25 + 10 + 10 + 10$

See Also

2009 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
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2009 AMC 12A Problems/Problem 5

The following problem is from both the 2009 AMC 12A #5 and 2009 AMC 10A #11, so both problems redirect to this page.

Problem

One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube?

$\textbf{(A)}\ 8 \qquad \textbf{(B)}\ 27 \qquad \textbf{(C)}\ 64 \qquad \textbf{(D)}\ 125 \qquad \textbf{(E)}\ 216$

Solution

Let the original cube have edge length a. Then its volume is a^3. The new box has dimensions $a-1$, a, and $a+1$, hence its volume is $(a-1)a(a+1) = a^3-a$. The difference between the two volumes is a. As we are given that the difference is 5, we have $a=5$, and the volume of the original cube was $5^3 = 125 \Rightarrow \boxed{\text{(D)}}$.

See Also

2009 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2009))	
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Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2009_AMC_12A_Problems/Problem_5&oldid=59750"

Categories: Introductory Geometry Problems | 3D Geometry Problems

2009 AMC 12A Problems/Problem 6

The following problem is from both the 2009 AMC 12A #6 and 2009 AMC 10A #13, so both problems redirect to this page.

Problem

Suppose that $P = 2^m$ and $Q = 3^n$. Which of the following is equal to 12^{mn} for every pair of integers (m, n)?

$\textbf{(A)}\ P^2Q \qquad \textbf{(B)}\ P^nQ^m \qquad \textbf{(C)}\ P^nQ^{2m} \qquad \textbf{(D)}\ P^{2m}Q^n \qquad \textbf{(E)}\ P^{2n}Q^m$

Solution

We have $12^{mn} = (2\cdot 2\cdot 3)^{mn} = 2^{2mn}\cdot 3^{mn} = (2^m)^{2n}\cdot (3^n)^m = \boxed{\textbf{E)} P^{2n}Q^m}$.

See Also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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2009 AMC 12A Problems/Problem 7

Problem

The first three terms of an arithmetic sequence are $2x-3$, $5x-11$, and $3x+1$ respectively. The n th term of the sequence is 2009. What is n?

$\textbf{(A)}\ 255 \qquad \textbf{(B)}\ 502 \qquad \textbf{(C)}\ 1004 \qquad \textbf{(D)}\ 1506 \qquad \textbf{(E)}\ 8037$

Solution

As this is an arithmetic sequence, the difference must be constant: $(5x-11)-(2x-3)=(3x+1)-(5x-11)$. This solves to $x=4$. The first three terms then are 5, 9, and 13. In general, the nth term is $1+4n$. Solving $1+4n=2009$, we get $n=\boxed{502}$.

See Also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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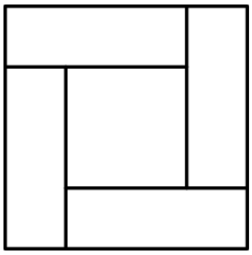
2009 AMC 12A Problems/Problem 8

The following problem is from both the 2009 AMC 12A #8 and 2009 AMC 10A #14, so both problems redirect to this page.

Problem

Four congruent rectangles are placed as shown. The area of the outer square is 4 times that of the inner square. What is the ratio of the length of the longer side of each rectangle to the length of its shorter side?



$\mathrm{(A)}\ 3 \qquad \mathrm{(B)}\ \sqrt{10} \qquad \mathrm{(C)}\ 2 + \sqrt{2} \qquad \mathrm{(D)}\ 2\sqrt{3} \qquad \mathrm{(E)}\ 4$

Solution

$\boxed{(A)}$ The area of the outer square is 4 times that of the inner square. Therefore the side of the outer square is $\sqrt{4} = 2$ times that of the inner square.

Then the shorter side of the rectangle is $1/4$ of the side of the outer square, and the longer side of the rectangle is $3/4$ of the side of the outer square, hence their ratio is $\boxed{3}$.

See Also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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2009 AMC 12A Problems/Problem 9

Contents

Problem

Suppose that $f(x+3)=3x^2+7x+4$ and $f(x)=ax^2+bx+c$. What is $a+b+c$?

$\textbf{(A)}\ -1 \qquad \textbf{(B)}\ 0 \qquad \textbf{(C)}\ 1 \qquad \textbf{(D)}\ 2 \qquad \textbf{(E)}\ 3$

Solution

Solution 1

As $f(x)=ax^2+bx+c$, we have $f(1)=a\cdot 1^2+b\cdot 1+c=a+b+c$.

To compute $f(1)$, set $x=-2$ in the first formula. We get $f(1)=f(-2+3)=3(-2)^2+7(-2)+4=12-14+4=\boxed{2}$.

Solution 2

Combining the two formulas, we know that $f(x+3)=a(x+3)^2+b(x+3)+c$.

We can rearrange the right hand side to $ax^2+(6a+b)x+(9a+3b+c)$.

Comparing coefficients we have $a=3$, $6a+b=7$, and $9a+3b+c=4$. From the second equation we get $b=-11$, and then from the third we get $c=10$. Hence $a+b+c=3-11+10=\boxed{2}$.

See Also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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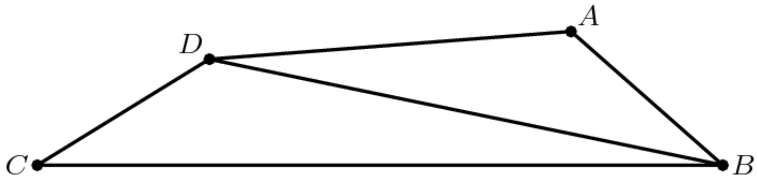
2009 AMC 12A Problems/Problem 10

The following problem is from both the 2009 AMC 12A #10 and 2009 AMC 10A #12, so both problems redirect to this page.

Problem

In quadrilateral $ABCD$, $AB = 5$, $BC = 17$, $CD = 5$, $DA = 9$, and BD is an integer. What is BD?



$\textbf{(A)}\ 11 \qquad \textbf{(B)}\ 12 \qquad \textbf{(C)}\ 13 \qquad \textbf{(D)}\ 14 \qquad \textbf{(E)}\ 15$

Solution

By the triangle inequality we have $BD < DA + AB = 9 + 5 = 14$, and also $BD + CD > BC$, hence $BD > BC - CD = 17 - 5 = 12$.

We got that $12 < BD < 14$, and as we know that BD is an integer, we must have $BD = \boxed{13}$.

See Also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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Category: Introductory Geometry Problems

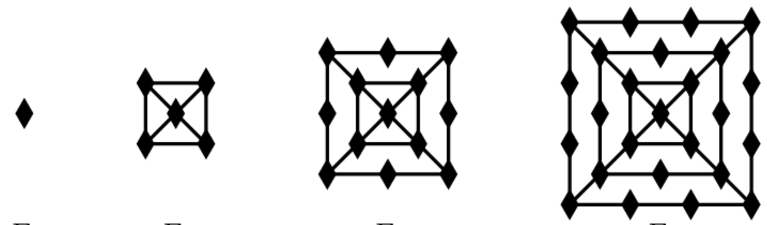
2009 AMC 12A Problems/Problem 11

The following problem is from both the 2009 AMC 12A #11 and 2009 AMC 10A #15, so both problems redirect to this page.

Contents

Problem

The figures F_1, F_2, F_3, and F_4 shown are the first in a sequence of figures. For $n \geq 3$, F_n is constructed from F_{n-1} by surrounding it with a square and placing one more diamond on each side of the new square than F_{n-1} had on each side of its outside square. For example, figure F_3 has 13 diamonds. How many diamonds are there in figure F_{20}?



$\textbf{(A)}\ 401 \qquad \textbf{(B)}\ 485 \qquad \textbf{(C)}\ 585 \qquad \textbf{(D)}\ 626 \qquad \textbf{(E)}\ 761$

Solution

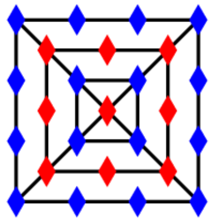
Solution 1

Split F_n into 4 congruent triangles by its diagonals (like in the pictures in the problem). This shows that the number of diamonds it contains is equal to 4 times the $(n-2)$th triangular number (i.e. the diamonds within the triangles or between the diagonals) and $4(n-1)+1$ (the diamonds on sides of the triangles or on the diagonals). The nth triangular number is $\frac{n(n+1)}{2}$. Putting this together for F_{20} this gives:

$$\frac{4(18)(19)}{2} + 4(19) + 1 = \boxed{761}$$

Solution 2

Color the diamond layers alternately blue and red, starting from the outside. You'll get the following pattern:



In the figure F_n, the blue diamonds form a $n \times n$ square, and the red diamonds form a $(n-1) \times (n-1)$ square. Hence the total number of diamonds in F_{20} is $20^2 + 19^2 = \boxed{761}$.

Solution 3

When constructing F_n from F_{n-1}, we add $4(n-1)$ new diamonds. Let d_n be the number of diamonds in F_n. We now know that $d_1 = 1$ and $\forall n > 1: \ d_n = d_{n-1} + 4(n-1)$.

Hence we get:

$$\begin{aligned} d_{20} &= d_{19} + 4 \cdot 19 \\ &= d_{18} + 4 \cdot 18 + 4 \cdot 19 \\ &= \cdots \\ &= 1 + 4(1 + 2 + \cdots + 18 + 19) \\ &= 1 + 4 \cdot \frac{19 \cdot 20}{2} \\ &= \boxed{761} \end{aligned}$$

See Also

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2009 AMC 12A Problems/Problem 12

Contents

Problem

How many positive integers less than 1000 are 6 times the sum of their digits?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 12$

Solution

Solution 1

The sum of the digits is at most $9+9+9=27$. Therefore the number is at most $6\cdot 27 = 162$. Out of the numbers 1 to 162 the one with the largest sum of digits is 99, and the sum is $9+9=18$. Hence the sum of digits will be at most 18.

Also, each number with this property is divisible by 6, therefore it is divisible by 3, and thus also its sum of digits is divisible by 3.

We only have six possibilities left for the sum of the digits: 3, 6, 9, 12, 15, and 18. These lead to the integers 18, 36, 54, 72, 90, and 108. But for 18 the sum of digits is $1+8=9$, which is not 3, therefore 18 is not a solution. Similarly we can throw away 36, 72, 90, and 108, and we are left with just $\boxed{1}$ solution: the number 54.

Solution 2

We can write each integer between 1 and 999 inclusive as $\overline{abc}=100a+10b+c$ where $a,b,c\in\{0,1,\dots,9\}$ and $a+b+c>0$. The sum of digits of this number is $a+b+c$, hence we get the equation $100a+10b+c = 6(a+b+c)$. This simplifies to $94a + 4b - 5c = 0$. Clearly for $a>0$ there are no solutions, hence $a=0$ and we get the equation $4b=5c$. This obviously has only one valid solution $(b,c)=(5,4)$, hence the only solution is the number 54.

Solution 3

The sum of the digits is at most $9+9+9=27$. Therefore the number is at most $6\cdot 27 = 162$. Since the number is 6 times the sum of its digits, it must be divisible by 6, therefore also by 3, therefore the sum of its digits must be divisible by 3. With this in mind we can conclude that the number must be divisible by 18, not just by 6. Since the number is divisible by 18, it is also divisible by 9, therefore the sum of its digits is divisible by 9, therefore the number is divisible by 54, which leaves us with 54, 108 and 162. Only 54 is 6 times its digits, hence the answer is $\boxed{1}$.

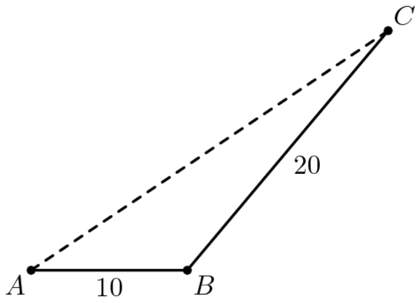
See Also

2009 AMC 12A Problems/Problem 13

Contents

Problem

A ship sails 10 miles in a straight line from A to B, turns through an angle between 45° and 60°, and then sails another 20 miles to C. Let AC be measured in miles. Which of the following intervals contains AC^2?



$\textbf{(A)}\ [400,500] \qquad \textbf{(B)}\ [500,600] \qquad \textbf{(C)}\ [600,700] \qquad \textbf{(D)}\ [700,800] \textbf{(E)}\ [800,900]$

Solution

Answering the question

To answer the question we are asked, it is enough to compute AC^2 for two different angles, preferably for both extremes (45 and 60 degrees). You can use the law of cosines to do so.

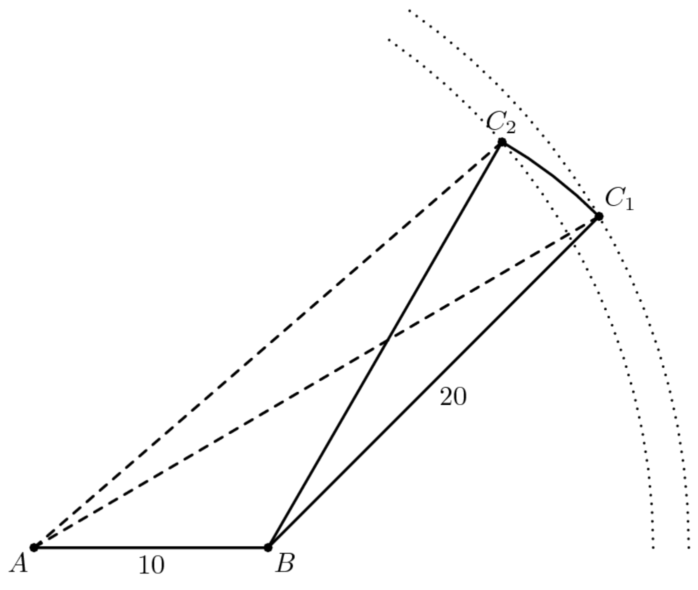
Alternately, it is enough to compute AC^2 for one of the extreme angles. In case it falls inside one of the given intervals, we are done. In case it falls on the boundary between two options, we also have to argue whether our AC^2 is the minimal or the maximal possible value of AC^2.

Below we show a complete solution in which we also show that all possible values of AC^2 do indeed lie in the given interval.

Complete solution

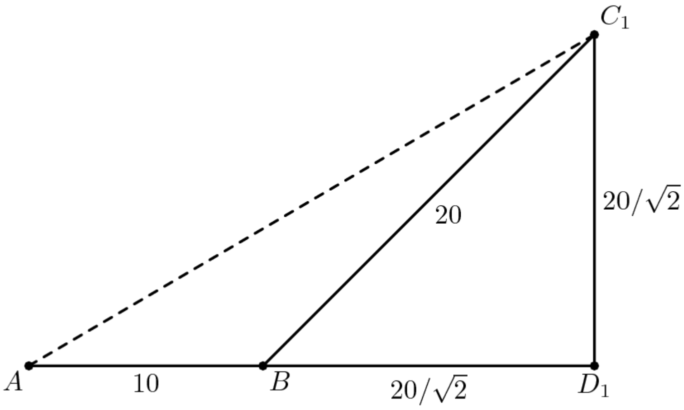
Let C_1 be the point the ship would reach if it turned 45°, and C_2 the point it would reach if it turned 60°. Obviously, C_1 is the furthest possible point from A, and C_2 is the closest possible point to A.

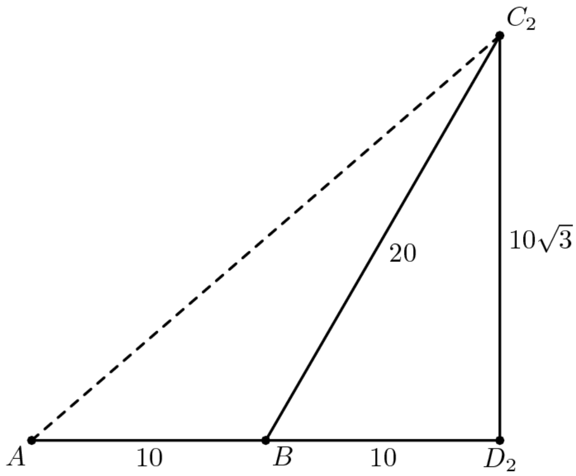
Hence the interval of possible values for AC^2 is $[AC_2^2, AC_1^2]$.



We can find AC_1^2 and AC_2^2 as follows:

Let D_1 and D_2 be the feet of the heights from C_1 and C_2 onto AB. The angles in the triangle BD_1C_1 are 45°, 45°, and 90°, hence $BD_1 = D_1C_1 = BC_1/\sqrt{2}$. Similarly, the angles in the triangle BD_2C_2 are 30°, 60°, and 90°, hence $BD_2 = BC_2/2$ and $D_2C_2 = BC_2\sqrt{3}/2$.





Hence we get:

$$AC_2^2 = AD_2^2 + D_2C_2^2 = 20^2 + (10\sqrt{3})^2 = 400 + 300 = 700$$

$$AC_1^2 = AD_1^2 + D_1C_1^2 = (10+20/\sqrt{2})^2 + (20\sqrt{2})^2 = 100+400/\sqrt{2}+200+200 = 500+200\sqrt{2} < 500+200\cdot 1.5 = 800$$

Therefore for any valid C the value AC^2 is surely in the interval $\boxed{\textbf{(D)} [700, 800]}$.

Alternate Solution

From the law of cosines, $500 - 400\cos 120^\circ < AC^2 < 500 - 400\cos 135^\circ \implies 700 < AC^2 < 500 + 200\sqrt{2}$. This is essentially the same solution as above. The answer is $\boxed{\textbf{(D)}}$.

See Also

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2009 AMC 12A Problems/Problem 14

Contents

Problem

A triangle has vertices $(0,0)$, $(1,1)$, and $(6m,0)$, and the line $y = mx$ divides the triangle into two triangles of equal area. What is the sum of all possible values of m?

$\textbf{A} -\frac{1}{3} \qquad \textbf{(B)} -\frac{1}{6} \qquad \textbf{(C)}\ \frac{1}{6} \qquad \textbf{(D)}\ \frac{1}{3} \qquad \textbf{(E)}\ \frac{1}{2}$

Solution

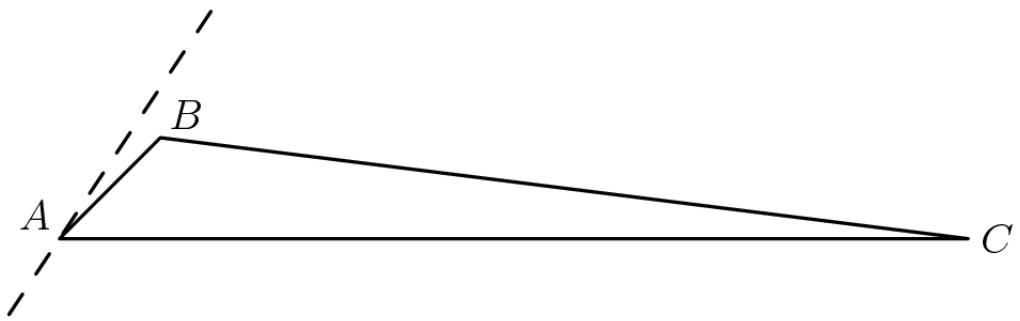
Let's label the three points as $A=(0,0)$, $B=(1,1)$, and $C=(6m,0)$.

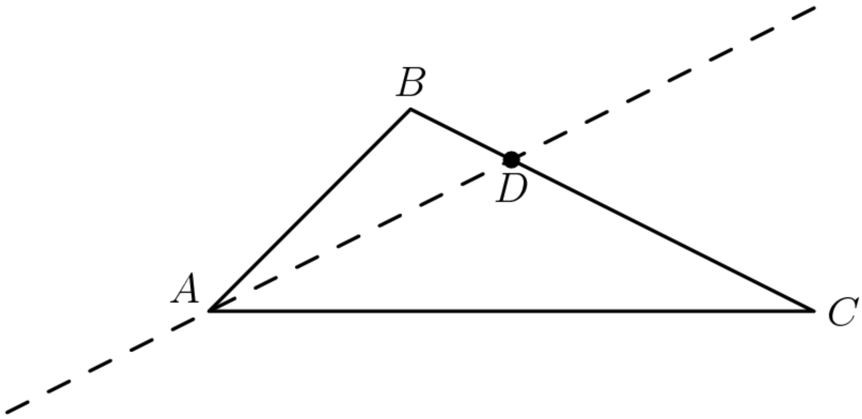
Clearly, whenever the line $y=mx$ intersects the inside of the triangle, it will intersect the side BC. Let D be the point of intersection.

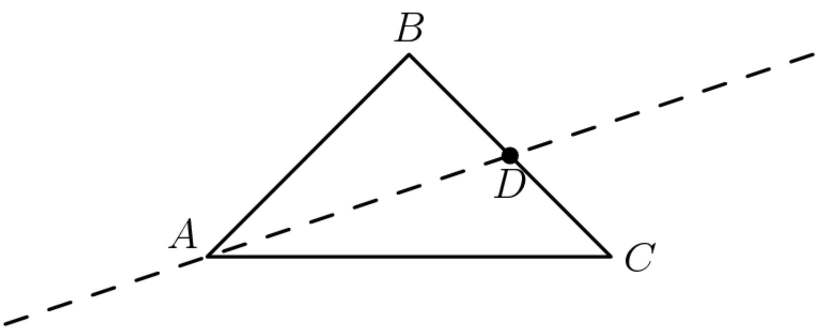
The triangles ABD and ACD have the same height, which is the distance between the point A and the line BC. Hence they have equal areas if and only if D is the midpoint of BC.

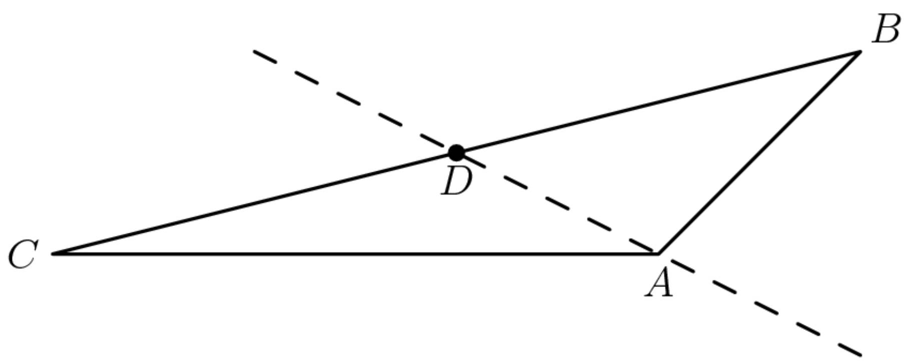
The midpoint of the segment BC has coordinates $\left(\frac{6m+1}{2}, \frac{1}{2} \right)$. This point lies on the line $y=mx$ if and only if $\frac{1}{2} = m \cdot \frac{6m+1}{2}$. This simplifies to $6m^2 + m - 1 = 0$. This is a quadratic equation with roots $m = \frac{1}{3}$ and $m = -\frac{1}{2}$. Both roots represent valid solutions, and their sum is $\frac{1}{3} - \frac{1}{2} = \boxed{-\frac{1}{6}}$.

For illustration, below are pictures of the situation for $m=1.5$, $m=0.5$, $m=1/3$, and $m=-1/2$.









Solution 2

The line must pass through the triangle's centroid, since the line divides the triangle in half. The coordinates of the centroid are found by averaging those of the vertices. The slope of the line from the origin through the centroid is thus $\frac{\frac{1}{3}}{\frac{1+6m}{3}}$, which is equal to m.

$$\frac{\frac{1}{3}}{\frac{1+6m}{3}} = m$$

$$\frac{1}{3} = m\left(\frac{1+6m}{3}\right)$$

$$1 = m(1+6m)$$

$$6m^2 + m - 1 = 0$$

Using Vieta's Formulas, the sum of the possible values of m is $\boxed{\textbf{(B)} \ -\frac{1}{6}}$

2009 AMC 12A Problems/Problem 15

Problem

For what value of n is $i + 2i^2 + 3i^3 + \cdots + ni^n = 48 + 49i$?

Note: here $i = \sqrt{-1}$.

$\textbf{(A)}\ 24 \qquad \textbf{(B)}\ 48 \qquad \textbf{(C)}\ 49 \qquad \textbf{(D)}\ 97 \qquad \textbf{(E)}\ 98$

Solution

Obviously, even powers of i are real and odd powers of i are imaginary. Hence the real part of the sum is $2i^2 + 4i^4 + 6i^6 + \dots$, and the imaginary part is $i + 3i^3 + 5i^5 + \cdots$.

Let's take a look at the real part first. We have $i^2 = -1$, hence the real part simplifies to $-2 + 4 - 6 + 8 - 10 + \cdots$. If there were an odd number of terms, we could pair them as follows: $-2 + (4 - 6) + (8 - 10) + \cdots$, hence the result would be negative. As we need the real part to be 48, we must have an even number of terms. If we have an even number of terms, we can pair them as $(-2 + 4) + (-6 + 8) + \cdots$. Each parenthesis is equal to 2, thus there are 24 of them, and the last value used is 96. This happens for $n = 96$ and $n = 97$. As $n = 96$ is not present as an option, we may conclude that the answer is $\boxed{97}$.

In a complete solution, we should now verify which of $n = 96$ and $n = 97$ will give us the correct imaginary part.

We can rewrite the imaginary part as follows: $i + 3i^3 + 5i^5 + \cdots = i(1 + 3i^2 + 5i^4 + \cdots) = i(1 - 3 + 5 - \cdots)$. We need to obtain $(1 - 3 + 5 - \cdots) = 49$. Once again we can repeat the same reasoning: If the number of terms were even, the left hand side would be negative, thus the number of terms is odd. The left hand side can then be rewritten as $1 + (-3 + 5) + (-7 + 9) + \cdots$. We need 24 parentheses, therefore the last value used is 97. This happens when $n = 97$ or $n = 98$, and we are done.

See Also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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2009 AMC 12A Problems/Problem 16

Problem

A circle with center C is tangent to the positive x and y-axes and externally tangent to the circle centered at $(3,0)$ with radius 1. What is the sum of all possible radii of the circle with center C?

$\textbf{(A) } 3 \qquad \textbf{(B) } 4 \qquad \textbf{(C) } 6 \qquad \textbf{(D) } 8 \qquad \textbf{(E) } 9$

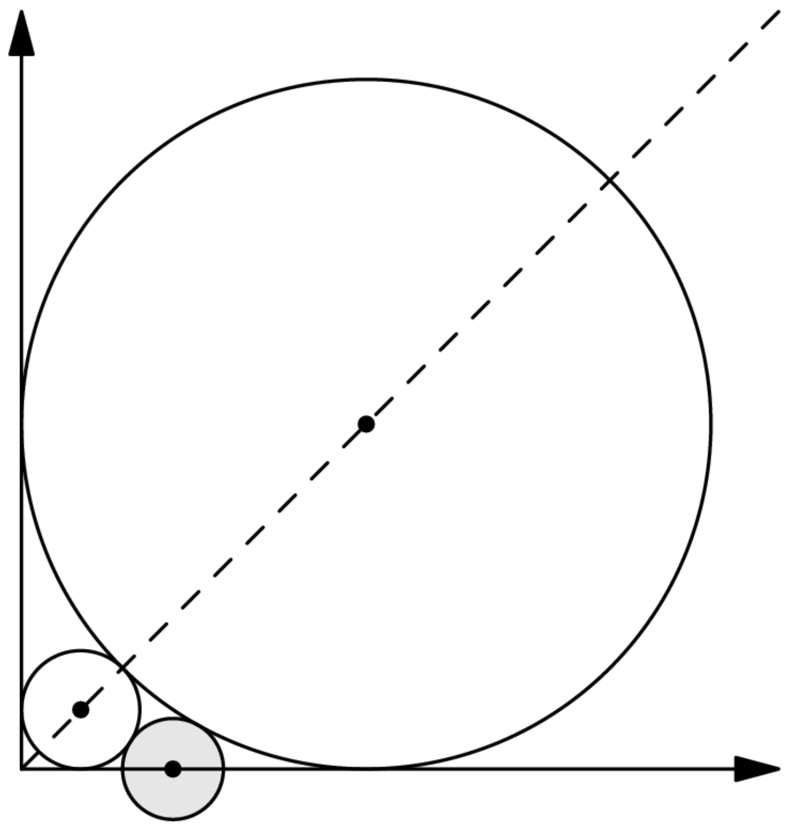
Solution

Let r be the radius of our circle. For it to be tangent to the positive x and y axes, we must have $C=(r,r)$. For the circle to be externally tangent to the circle centered at $(3,0)$ with radius 1, the distance between C and $(3,0)$ must be exactly $r+1$.

By the Pythagorean theorem the distance between (r,r) and $(3,0)$ is $\sqrt{(r-3)^2+r^2}$, hence we get the equation $(r-3)^2+r^2=(r+1)^2$.

Simplifying, we obtain $r^2-8r+8=0$. By Vieta's formulas the sum of the two roots of this equation is $\boxed{8}$.

(We should actually solve for r to verify that there are two distinct positive roots. In this case we get $r=4\pm 2\sqrt{2}$. This is generally a good rule of thumb, but is not necessary as all of the available answers are integers, and the equation obviously doesn't factor as integers.)



See Also

2009 AMC 12A Problems/Problem 17

Contents

Problem

Let $a + ar_1 + ar_1^2 + ar_1^3 + \cdots$ and $a + ar_2 + ar_2^2 + ar_2^3 + \cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is $r_1 + r_2$?

$\textbf{(A)}\ 0\qquad \textbf{(B)}\ \frac{1}{2}\qquad \textbf{(C)}\ 1\qquad \textbf{(D)}\ \frac{1+\sqrt{5}}{2}\qquad \textbf{(E)}\ 2$

Solution

Using the formula for the sum of a geometric series we get that the sums of the given two sequences are $\frac{a}{1-r_1}$ and $\frac{a}{1-r_2}$.

Hence we have $\frac{a}{1-r_1} = r_1$ and $\frac{a}{1-r_2} = r_2$. This can be rewritten as $r_1(1-r_1) = r_2(1-r_2) = a$.

As we are given that r_1 and r_2 are distinct, these must be precisely the two roots of the equation $x^2 - x + a = 0$.

Using Vieta's formulas we get that the sum of these two roots is $\boxed{1}$.

Alternate Solution

Using the formula for the sum of a geometric series we get that the sums of the given two sequences are $\frac{a}{1-r_1}$ and $\frac{a}{1-r_2}$.

Hence we have $\frac{a}{1-r_1} = r_1$ and $\frac{a}{1-r_2} = r_2$. This can be rewritten as $r_1(1-r_1) = r_2(1-r_2) = a$.

Which can be further rewritten as $r_1 - r_1^2 = r_2 - r_2^2$. Rearranging the equation we get $r_1 - r_2 = r_1^2 - r_2^2$. Expressing this as a difference of squares we get $r_1 - r_2 = (r_1 - r_2)(r_1 + r_2)$.

Dividing by like terms we finally get $r_1 + r_2 = \boxed{1}$ as desired.

Note: It is necessary to check that $r_1 - r_2 \neq 0$, as you cannot divide by zero. As the problem states that the series are different, $r_1 \neq r_2$, and so there is no division by zero error.

See Also

2009 AMC 12A Problems/Problem 18

The following problem is from both the 2009 AMC 12A #18 and 2009 AMC 10A #25, so both problems redirect to this page.

Contents

Problem

For $k > 0$, let $I_k = 10\ldots 064$, where there are k zeros between the 1 and the 6. Let $N(k)$ be the number of factors of 2 in the prime factorization of I_k. What is the maximum value of $N(k)$?

$\textbf{(A)}\ 6 \qquad \textbf{(B)}\ 7 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 9 \qquad \textbf{(E)}\ 10$

Solution

The number I_k can be written as $10^{k+2} + 64 = 5^{k+2}\cdot 2^{k+2} + 2^6$.

For $k\in\{1,2,3\}$ we have $I_k = 2^{k+2} \left(5^{k+2} + 2^{4-k} \right)$. The first value in the parentheses is odd, the second one is even, hence their sum is odd and we have $N(k)=k+2\leq 5$.

For $k>4$ we have $I_k=2^6 \left(5^{k+2}\cdot 2^{k-4} + 1 \right)$. For $k>4$ the value in the parentheses is odd, hence $N(k)=6$.

This leaves the case $k=4$. We have $I_4 = 2^6 \left(5^6 + 1 \right)$. The value $5^6 + 1$ is obviously even. And as $5\equiv 1 \pmod 4$, we have $5^6 \equiv 1 \pmod 4$, and therefore $5^6 + 1 \equiv 2 \pmod 4$. Hence the largest power of 2 that divides 5^6+1 is 2^1, and this gives us the desired maximum of the function N: $N(4) = \boxed{7}$.

Alternate Solution

Notice that 2 is a prime factor of an integer n if and only if n is even. Therefore, given any sufficiently high positive integral value of k, dividing I_k by 2^6 yields a terminal digit of zero, and dividing by 2 again leaves us with $2^7 \cdot a = I_k$ where a is an odd integer. Observe then that $\boxed{\textbf{(B)} 7}$ must be the maximum value for $N(k)$ because whatever value we choose for k, $N(k)$ must be less than or equal to 7.

EDIT: Isn't this solution incomplete because we need to show that $N(k) = 7$ can be reached?

See Also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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2009 AMC 12A Problems/Problem 19

Contents

Problem

Andrea inscribed a circle inside a regular pentagon, circumscribed a circle around the pentagon, and calculated the area of the region between the two circles. Bethany did the same with a regular heptagon (7 sides). The areas of the two regions were A and B, respectively. Each polygon had a side length of 2. Which of the following is true?

$$\textbf{(A)}\ A = \frac{25}{49}B \qquad \textbf{(B)}\ A = \frac{5}{7}B \qquad \textbf{(C)}\ A = B \qquad \textbf{(D)}\ A = \frac{7}{5}B \qquad \textbf{(E)}\ A = \frac{49}{25}B$$

Solution

In any regular polygon with side length 2, consider the isosceles triangle formed by the center of the polygon S and two consecutive vertices X and Y. We are given that $XY = 2$. Obviously $SX = SY = r$, where r is the radius of the circumcircle. Let T be the midpoint of XY. Then $XT = TY = 1$, and $TS = \rho$, where ρ is the radius of the incircle.

Applying the Pythagorean theorem on the triangle STX, we get that $\rho^2 + 1 = r^2$.

Then the area between the circumcircle and the incircle can be computed as $\pi r^2 - \pi\rho^2 = \pi r^2 - \pi(r^2 - 1) = \pi$.

Hence $A = \pi$, $B = \pi$, and therefore $\boxed{A = B}$.

Alternate Solution (Applying Basic Trig)

Similar to the first solution, consider the isosceles triangle formed by each polygon. If you drop an altitude to the side of each polygon, you get in both polygons a right triangle with base of 1. For both the pentagon and heptagon, the hypotenuse of these right triangles is the radii of the larger circles and the apothems (the altitude we dropped to the side of each polygon) are the radii of the smaller circles.

Label the apothem of the pentagon A_1 and the apothem of the heptagon A_2. Label the hypotenuse in the pentagon H_1 and the hypotenuse in the heptagon H_2.

Now, finding the angles in the triangles and applying trig functions to find these radii, we get the following:

A_1 = $\cot 36^\circ$

A_2 = $\cot \dfrac{180^\circ}{7}$

H_1 = $\csc 36^\circ$

H_2 = $\csc \dfrac{180^\circ}{7}$

Now, the areas in between the circles are:

Pentagon circles = $\pi((\csc 36^\circ)^2 - (\cot 36^\circ)^2)$

$$\text{Heptagon circles} = \pi((\csc\frac{180^\circ}{7})^2 - (\cot\frac{180^\circ}{7})^2)$$

Note the trig identity $(\cot\theta)^2 + 1 = (\csc\theta)^2$, from which we can easily get that $(\csc\theta)^2 - (\cot\theta)^2 = 1$

Thus, the area between the circles in both the heptagon and the pentagon are equivalent to π, and therefore the answer is $\boxed{\textbf{(C)} A = B}$.

See Also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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2009 AMC 12A Problems/Problem 20

The following problem is from both the 2009 AMC 12A #20 and 2009 AMC 10A #23, so both problems redirect to this page.

Problem

Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE?

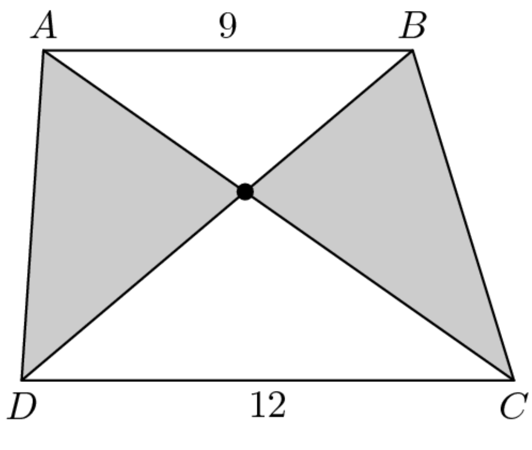
$\textbf{(A)}\ \frac{9}{2}\qquad\textbf{(B)}\ \frac{50}{11}\qquad\textbf{(C)}\ \frac{21}{4}\qquad\textbf{(D)}\ \frac{17}{3}\qquad\textbf{(E)}\ 6$

Contents

Solution

Solution 1

Let $[ABC]$ denote the area of triangle ABC. $[AED] = [BEC]$, so $[ABD] = [AED] + [AEB] = [BEC] + [AEB] = [ABC]$. Since triangles ABD and ABC share a base, they also have the same height and thus $\overline{AB}||\overline{CD}$ and $\triangle AEB \sim \triangle CED$ with a ratio of $3:4$. $AE = \frac{3}{7}AC = 6 \boxed{\textbf{(E)}}$.



Solution 2

Using the sine area formula on triangles AED and BEC, as $\angle AED = \angle BEC$, we see that

$$(AE)(ED) = (BE)(EC) \quad \Longrightarrow \quad \frac{AE}{EC} = \frac{BE}{ED}.$$

Since $\angle AEB = \angle DEC$, triangles AEB and DEC are similar. Their ratio is $\dfrac{AB}{CD} = \dfrac{3}{4}$. Since $AE + EC = 14$, we must have $EC = 8$, so $AE = 6$ $\textbf{(E)}$.

Solution 3 (which won't work when justification is required)

Consider an isosceles trapezoid with opposite bases of 9 and 12 and a diagonal of length 14. This trapezoid exists and satisfies all the conditions in the problem (the areas are congruent by symmetry). So, we can simply use this easier case to solve this problem, because the problem implies that the answer is invariant for all quadrilaterals satisfying the conditions.

Then, by similar triangles, the ratio of AE to EC is $3:4$, so $AE = \boxed{\textbf{(E)} 6}$.

See also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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Category: Introductory Geometry Problems

2009 AMC 12A Problems/Problem 21

Contents

Problem

Let $p(x) = x^3 + ax^2 + bx + c$, where a, b, and c are complex numbers. Suppose that

$$p(2009 + 9002\pi i) = p(2009) = p(9002) = 0$$

What is the number of nonreal zeros of $x^{12} + ax^8 + bx^4 + c$?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 10 \qquad \textbf{(E)}\ 12$

Solutions

Solution

From the three zeroes, we have $p(x) = (x - (2009 + 9002\pi i))(x - 2009)(x - 9002)$.

Then $p(x^4) = (x^4 - (2009 + 9002\pi i))(x^4 - 2009)(x^4 - 9002) = x^{12} + ax^8 + bx^4 + c$.

Let's do each factor case by case:

- $x^4 - (2009 + 9002\pi i) = 0$: Clearly, all the fourth roots are going to be complex.
- $x^4 - 2009 = 0$: The real roots are $\pm\sqrt[4]{2009}$, and there are two complex roots.
- $x^4 - 9002 = 0$: The real roots are $\pm\sqrt[4]{9002}$, and there are two complex roots.

So the answer is $4 + 2 + 2 = 8\ \textbf{(C)}$.

See also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
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2009 AMC 12A Problems/Problem 22

Problem

A regular octahedron has side length 1. A plane parallel to two of its opposite faces cuts the octahedron into the two congruent solids. The polygon formed by the intersection of the plane and the octahedron has area $\frac{a\sqrt{b}}{c}$, where a, b, and c are positive integers, a and c are relatively prime, and b is not divisible by the square of any prime. What is $a+b+c$?

$\textbf{(A)}\ 10\qquad \textbf{(B)}\ 11\qquad \textbf{(C)}\ 12\qquad \textbf{(D)}\ 13\qquad \textbf{(E)}\ 14$

Solution

Firstly, note that the intersection of the plane must be a hexagon. Consider the net of the octahedron. Notice that the hexagon becomes a line on the net. Also, notice that, given the parallel to the faces conditions, the line must be parallel to the sides of the net (precisely $\frac{1}{3}$ of them). Now, notice that, through symmetry, 2 of the hexagon's vertexes lie on the midpoint of the side of the "square" in the octahedron. In the net, the condition gives you that one of the intersections of the line with the net have to be on the midpoint of the side. However, if one is on the midpoint, because of the parallel conditions, all of the vertices are on the midpoint of a side. Thus, we have a regular hexagon with a side length of the midline of an equilateral triangle with side length 1, which is $\frac{1}{2}$. Thus, the answer is $\frac{3\sqrt{3}}{8}$, and $a+b+c = 14\ \textbf{(E)}$.

See also

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Categories: Introductory Geometry Problems | 3D Geometry Problems | Area Problems

2009 AMC 12A Problems/Problem 23

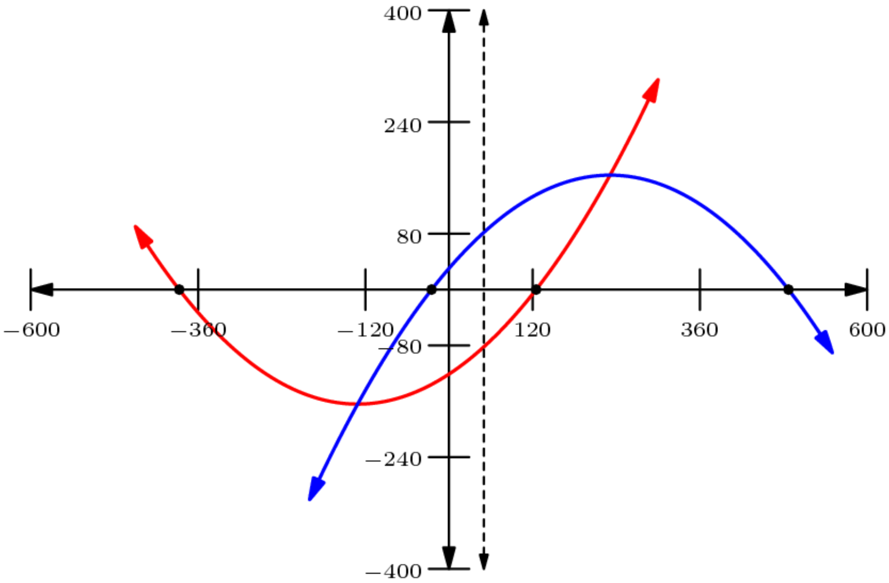
Contents

Problem

Functions f and g are quadratic, $g(x) = -f(100 - x)$, and the graph of g contains the vertex of the graph of f. The four x-intercepts on the two graphs have x-coordinates x_1, x_2, x_3, and x_4, in increasing order, and $x_3 - x_2 = 150$. The value of $x_4 - x_1$ is $m + n\sqrt{p}$, where m, n, and p are positive integers, and p is not divisible by the square of any prime. What is $m + n + p$?

$\textbf{(A)}\ 602 \qquad \textbf{(B)}\ 652 \qquad \textbf{(C)}\ 702 \qquad \textbf{(D)}\ 752 \qquad \textbf{(E)}\ 802$

Solution



The two quadratics are 180° rotations of each other about $(50, 0)$. Since we are only dealing with differences of roots, we can translate them to be symmetric about $(0, 0)$. Now $x_3 = -x_2 = 75$ and $x_4 = -x_1$. Say our translated versions of f and g are p and q, respectively, so that $p(x) = -q(-x)$. Let $x_3 = 75$ be a root of p and $x_2 = -75$ a root of q by symmetry. Note that since they each contain each other's vertex, x_1, x_2, x_3, and x_4 must be roots of alternating polynomials, so x_1 is a root of p and x_4 a root of q

$$p(x) = a(x - 75)(x - x_1)q(x) = -a(x + 75)(x + x_1)$$

The vertex of $p(x)$ is half the sum of its roots, or $\dfrac{75 + x_1}{2}$. We are told that the vertex of one quadratic lies on the other, so

$$\begin{aligned} p\left(\frac{75+x_1}{2}\right) &= a\left(\frac{75-x_1}{2}\right)\left(\frac{-75+x_1}{2}\right) \\ &= -\frac{a}{4}(x_1-75)^2 \\ -\frac{a}{4}(x_1-75)^2 &= q\left(\frac{75+x_1}{2}\right) \\ &= -a\left(\frac{x_1+225}{2}\right)\left(\frac{3x_1+75}{2}\right) \\ &= -\frac{a}{4}(x_1+225)(3x_1+75) \end{aligned}$$

Let $x_1 = 75u$ and divide through by 75^2, since this is a timed competition and it will drastically simplify computations. We know $u < -1$ and that $(u-1)^2 = (3u+1)(u+3)$, or

$$\begin{aligned} 0 &= (3u+1)(u+3) - (u-1)^2 \\ &= 3u^2 + 10u + 3 - (u^2 - 2u + 1) \\ &= 2u^2 + 12u + 2 \\ &= u^2 + 6u + 1 \end{aligned}$$

So $u = \dfrac{-6 \pm \sqrt{32}}{2} = -3 \pm 2\sqrt{2}$. Since $u < -1$, $u = -3 - 2\sqrt{2}$.

The answer is $r_4 - r_1 = (-r_1) - r_1 = -150u = 450 + 300\sqrt{2}$, and $450 + 300 + 2 = 752\ (\textbf{D})$.

Note

Actually it is not necessary to solve any quadratic equations, if one utilizes the two facts about the quadratic $ax^2 + bx + c$ ($a > 0$) that (i) the difference of the two quadratic roots equals to $\sqrt{\Delta}/a$, and (ii) that the minimum value of a quadratic equals to $-\Delta/4a$, where $\Delta = b^2 - 4ac$. Here is a possible adjustment to the solution:

Without loss of generality we may "shift" f, g, x_1, x_2, x_3, x_4 50 units to the left, then the differences of x_i remain the same, x_3 and x_2 are symmetrical about 0, so $x_2 = -75$, $x_3 = 75$. The relationship of f, g becomes $g(x) = -f(-x)$. So we may write:

$$f(x) = a(x-b)^2 + m$$

$$g(x) = -a(x+b)^2 - m$$

Again without loss of generality, we can assume $a > 0$ and $b > 0$ (Short argument is needed here instead of the lazy "wlog"). Also, the vertex of f is (b, m), so $m = g(b) = -4ab^2 - m$, or $m = -2ab^2 < 0$.

Since x_2, x_4 are roots of f, we have the following relationship of the roots:

$$x_2 + x_4 = 2b$$

$$x_4 - x_2 = \sqrt{\Delta}/a = \sqrt{-4am}/a = 2\sqrt{-m/a} = 2\sqrt{2b^2} = 2\sqrt{2}b = \sqrt{2}(x_2 + x_4)$$

So $(\sqrt{2} - 1)x_4 = 75(1 + \sqrt{2})$, or $x_4 = 75(1 + \sqrt{2})^2 = 75(3 + 2\sqrt{2})$.

Therefore $x_4 - x_1 = 2x_4 = 450 + 300\sqrt{2}$.

See also

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Category: Intermediate Algebra Problems

2009 AMC 12A Problems/Problem 24

Contents

Problem

The tower function of twos is defined recursively as follows: $T(1) = 2$ and $T(n+1) = 2^{T(n)}$ for $n \geq 1$. Let $A = (T(2009))^{T(2009)}$ and $B = (T(2009))^A$. What is the largest integer k such that

$$\underbrace{\log_2 \log_2 \log_2 \ldots \log_2 B}_{k \text{ times}}$$

is defined?

$\textbf{(A)}\ 2009 \qquad \textbf{(B)}\ 2010 \qquad \textbf{(C)}\ 2011 \qquad \textbf{(D)}\ 2012 \qquad \textbf{(E)}\ 2013$

Solution

We just look at the last three logarithms for the moment, and use the fact that $\log_2 T(k) = T(k-1)$. We wish to find:

$$\log_2 \log_2 \log_2 \left(T(2009)^{\left(T(2009)^{T(2009)}\right)}\right)$$

$$= \log_2(T(2009)\log_2(T(2009)\log_2 T(2009)))$$

$$= \log_2(T(2009)\log_2(T(2009)T(2008)))$$

$$= \log_2(T(2009)(T(2008) + T(2007)))$$

Now we realize that $T(n-1)$ is much smaller than $T(n)$. So we approximate this, remembering we have rounded down, as:

$$\log_2(T(2009)) = T(2008)$$

We have used 3 logarithms so far. Applying 2007 more to the left of our expression, we get $T(1) = 2$. Then we can apply the logarithm 2 more times, until we get to 0. So our answer is approximately $3 + 2007 + 2 = 2012$. But we rounded down, so that means that after 2012 logarithms we get a number slightly greater than 0, so we can apply logarithms one more time. We can be sure it is small enough so that the logarithm can only be applied 1 more time since $2012 + 1 = 2013$ is the largest answer choice. So the answer is $\textbf{(E)}$.

Alternative Solution

Let $L_k(x) = \log_2 \log_2 \dots \log_2(x)$ where there are k $\log_2$'s. $L_k(B)$ is defined iff $L_{k-1}(B) > 0$ iff $L_{k-2}(B) > 1$. Note $\log_2 T(k) = T(k-1)$, so $L_{k-2}(T(k-2)) = 1$. Thus, we seek the largest k such that $B > T(k-2)$. Now note that

$$T(2009)^{T(2009)^{T(2009)}} > 2^{2^{T(2009)}} = T(2011)$$

so $k = 2013$ satisfies the inequality. Since it is the largest choice, it is the answer.

See also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 23	Followed by Problem 25
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2009 AMC 12A Problems/Problem 25

Contents

Problem

The first two terms of a sequence are $a_1 = 1$ and $a_2 = \frac{1}{\sqrt{3}}$. For $n \ge 1$,

$$a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}}.$$

What is $|a_{2009}|$?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 2-\sqrt{3} \qquad \textbf{(C)}\ \frac{1}{\sqrt{3}} \qquad \textbf{(D)}\ 1 \qquad \textbf{(E)}\ 2+\sqrt{3}$

Solution

Consider another sequence $\{\theta_1, \theta_2, \theta_3 ...\}$ such that $a_n = \tan{\theta_n}$, and $0 \leq \theta_n < 180$.

The given recurrence becomes

$$\begin{aligned}
a_{n+2} &= \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}} \\
\tan{\theta_{n+2}} &= \frac{\tan{\theta_n} + \tan{\theta_{n+1}}}{1 - \tan{\theta_n}\tan{\theta_{n+1}}} \\
\tan{\theta_{n+2}} &= \tan(\theta_{n+1} + \theta_n)
\end{aligned}$$

It follows that $\theta_{n+2} \equiv \theta_{n+1} + \theta_n \pmod{180}$. Since $\theta_1 = 45, \theta_2 = 30$, all terms in the sequence $\{\theta_1, \theta_2, \theta_3 ...\}$ will be a multiple of 15.

Now consider another sequence $\{b_1, b_2, b_3 ...\}$ such that $b_n = \theta_n / 15$, and $0 \leq b_n < 12$. The sequence b_n satisfies $b_1 = 3, b_2 = 2, b_{n+2} \equiv b_{n+1} + b_n \pmod{12}$.

As the number of possible consecutive two terms is finite, we know that the sequence b_n is periodic. Write out the first few terms of the sequence until it starts to repeat.

$\{b_n\} = \{3, 2, 5, 7, 0, 7, 7, 2, 9, 11, 8, 7, 3, 10, 1, 11, 0, 11, 11, 10, 9, 7, 4, 11, 3, 2, 5, 7, ...\}$

Note that $b_{25} = b_1 = 3$ and $b_{26} = b_2 = 2$. Thus $\{b_n\}$ has a period of 24: $b_{n+24} = b_n$.

It follows that $b_{2009} = b_{17} = 0$ and $\theta_{2009} = 15 b_{2009} = 0$. Thus $a_{2009} = \tan{\theta_{2009}} = \tan{0} = 0$.

Our answer is $|a_{2009}| = \boxed{\textbf{(A)}\ 0}$.

Note

It is not actually difficult to list out the terms until it repeats. You will find that the period is 7 starting from term 2.

See also

2009 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2009))	
Preceded by Problem 24	Followed by Last question
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