2000 AMC 12 Problems/Problem 1

The following problem is from both the 2000 AMC 12 #1 and 2000 AMC 10 #1, so both problems redirect to this page.

Problem

In the year 2001, the United States will host the International Mathematical Olympiad. Let $I, M,$ and O be distinct positive integers such that the product $I \cdot M \cdot O = 2001$. What is the largest possible value of the sum $I + M + O$?

$\text{(A)}\ 23 \qquad \text{(B)}\ 55 \qquad \text{(C)}\ 99 \qquad \text{(D)}\ 111 \qquad \text{(E)}\ 671$

Solution

The sum is the highest if two factors are the lowest.

So, $1 \cdot 3 \cdot 667 = 2001$ and $1 + 3 + 667 = 671 \Longrightarrow \boxed{\text{(E)}}$.

See Also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
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Categories: Introductory Algebra Problems | Introductory Number Theory Problems

2000 AMC 12 Problems/Problem 2

The following problem is from both the 2000 AMC 12 #2 and 2000 AMC 10 #2, so both problems redirect to this page.

Problem

$2000(2000^{2000}) =$

$\text{(A)}\ 2000^{2001} \qquad \text{(B)}\ 4000^{2000} \qquad \text{(C)}\ 2000^{4000} \qquad \text{(D)}\ 4,000,000^{2000} \qquad \text{(E)}\ 2000^{4,000,000}$

Solution

$2000(2000^{2000}) = (2000^1)(2000^{2000}) = 2000^{2001} \Rightarrow \boxed{A}$

See also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
Preceded by Problem 1	Followed by Problem 3
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2000 AMC 10 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2000))	
Preceded by Problem 1	Followed by Problem 3
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All AMC 10 Problems and Solutions	

Category: Introductory Algebra Problems

2000 AMC 12 Problems/Problem 3

The following problem is from both the 2000 AMC 12 #3 and 2000 AMC 10 #3, so both problems redirect to this page.

Problem

Each day, Jenny ate 20% of the jellybeans that were in her jar at the beginning of that day. At the end of the second day, 32 remained. How many jellybeans were in the jar originally?

$\textbf{(A)}\ 40\qquad\textbf{(B)}\ 50\qquad\textbf{(C)}\ 55\qquad\textbf{(D)}\ 60\qquad\textbf{(E)}\ 75$

Solution

Since Jenny eats 20% of her jelly beans per day, $80\% = \frac{4}{5}$ of her jelly beans remain after one day.

Let x be the number of jelly beans in the jar originally.

$$\frac{4}{5}\cdot\frac{4}{5}\cdot x = 32$$

$$\frac{16}{25}\cdot x = 32$$

$$x = \frac{25}{16}\cdot 32 = 50 \Rightarrow \boxed{B}$$

See also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
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2000 AMC 10 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2000))	
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All AMC 10 Problems and Solutions	

2000 AMC 12 Problems/Problem 4

The following problem is from both the 2000 AMC 12 #4 and 2000 AMC 10 #6, so both problems redirect to this page.

Problem

The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ starts with two 1s, and each term afterwards is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?

$\text{(A) } 0 \qquad \text{(B) } 4 \qquad \text{(C) } 6 \qquad \text{(D) } 7 \qquad \text{(E) } 9$

Solution

Note that any digits other than the units digit will not affect the answer. So to make computation quicker, we can just look at the Fibonacci sequence in $\bmod 10$:

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, \ldots.$$

The last digit to appear in the units position of a number in the Fibonacci sequence is $6 \Longrightarrow \boxed{\text{C}}$.

See also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
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Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2000_AMC_12_Problems/Problem_4&oldid=53553"

Category: Introductory Combinatorics Problems

2000 AMC 12 Problems/Problem 5

The following problem is from both the 2000 AMC 12 #5 and 2000 AMC 10 #9, so both problems redirect to this page.

Problem

If $|x-2|=p$, where $x<2$, then $x-p=$

$\text{(A)} \ -2 \qquad \text{(B)} \ 2 \qquad \text{(C)} \ 2-2p \qquad \text{(D)} \ 2p-2 \qquad \text{(E)} \ |2p-2|$

Solution

When $x<2, x-2$ is negative so $|x-2|=2-x=p$ and $x=2-p$.

Thus $x-p=(2-p)-p=2-2p \Longrightarrow \text{(C)}$.

See also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
Preceded by Problem 4	Followed by Problem 6
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2000 AMC 10 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2000))	
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Category: Introductory Algebra Problems

2000 AMC 12 Problems/Problem 6

The following problem is from both the 2000 AMC 12 #6 and 2000 AMC 10 #11, so both problems redirect to this page.

Contents

Problem

Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

$\textbf{(A)}\ 21 \qquad \textbf{(B)}\ 60 \qquad \textbf{(C)}\ 119 \qquad \textbf{(D)}\ 180 \qquad \textbf{(E)}\ 231$

Solution 1

All prime numbers between 4 and 18 have an odd product and an even sum. Any odd number minus an even number is an odd number, so we can eliminate B and D. Since the highest two prime numbers we can pick are 13 and 17, the highest number we can make is $(13)(17)-(13+17) = 221 - 30 = 191$. Thus, we can eliminate E. Similarly, the two lowest prime numbers we can pick are 5 and 7, so the lowest number we can make is $(5)(7)-(5+7) = 23$. Therefore, A cannot be an answer. So, the answer must be (C).

Solution 2

Let the two primes be p and q. We wish to obtain the value of $pq-(p+q)$, or $pq-p-q$. Using Simon's Favorite Factoring Trick, we can rewrite this expression as $(1-p)(1-q)-1$ or $(p-1)(q-1)-1$. Noticing that $(13-1)(11-1)-1 = 120-1 = 119$, we see that the answer is (C).

See also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
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2000 AMC 12 Problems/Problem 7

Problem

How many positive integers b have the property that $\log_b 729$ is a positive integer?

$\mathrm{(A) \ 0 } \qquad \mathrm{(B) \ 1 } \qquad \mathrm{(C) \ 2 } \qquad \mathrm{(D) \ 3 } \qquad \mathrm{(E) \ 4 }$

Solution

If $\log_b 729 = n$, then $b^n = 729$. Since $729 = 3^6$, b must be 3 to some factor of 6. Thus, there are four (3, 9, 27, 729) possible values of $b \Longrightarrow \mathrm{E}$.

See also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
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All AMC 12 Problems and Solutions	

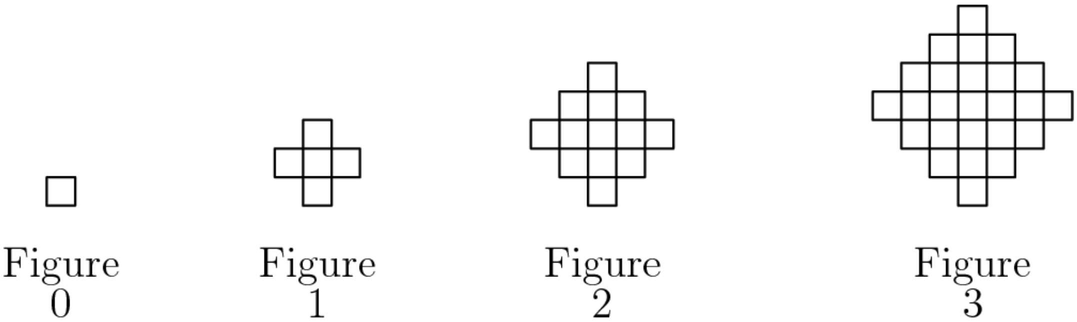
2000 AMC 12 Problems/Problem 8

The following problem is from both the 2000 AMC 12 #8 and 2000 AMC 10 #12, so both problems redirect to this page.

Contents

Problem

Figures 0, 1, 2, and 3 consist of 1, 5, 13, and 25 nonoverlapping unit squares, respectively. If the pattern were continued, how many nonoverlapping unit squares would there be in figure 100?



$\text{(A)}\ 10401 \qquad\text{(B)}\ 19801 \qquad\text{(C)}\ 20201 \qquad\text{(D)}\ 39801 \qquad\text{(E)}\ 40801$

Solution

Solution 1

We have a recursion:

$$A_n = A_{n-1} + 4n.$$

I.E. we add increasing multiples of 4 each time we go up a figure.

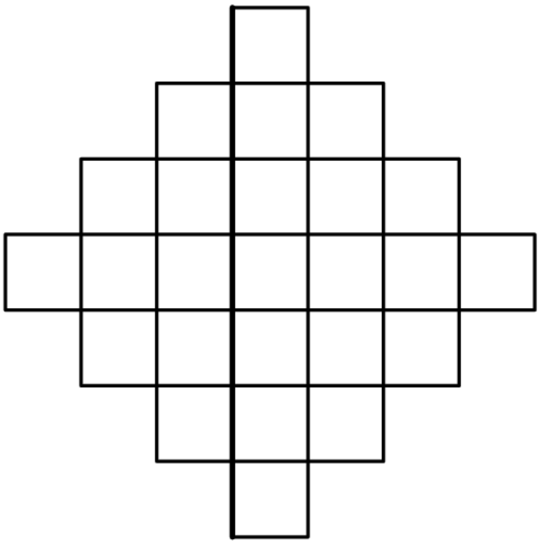
So, to go from Figure 0 to 100, we add

$$4 \cdot 1 + 4 \cdot 2 + ... + 4 \cdot 99 + 4 \cdot 100 = 4 \cdot 5050 = 20200.$$

We then add 20200 to the number of squares in Figure 0 to get 20201, which is choice $\boxed{\text{C}}$

Solution 2

We can divide up figure n to get the sum of the sum of the first $n+1$ odd numbers and the sum of the first n odd numbers. If you do not see this, here is the example for $n=3$:



The sum of the first n odd numbers is n^2, so for figure n, there are $(n+1)^2+n^2$ unit squares. We plug in $n=100$ to get 20201, which is choice $\boxed{\text{C}}$

Solution 3

Using the recursion from solution 1, we see that the first differences of $4, 8, 12, ...$ form an arithmetic progression, and consequently that the second differences are constant and all equal to 4. Thus, the original sequence can be generated from a quadratic function.

If $f(n) = an^2 + bn + c$, and $f(0) = 1$, $f(1) = 5$, and $f(2) = 13$, we get a system of three equations in three variables:

$f(0)=0$ gives $c = 1$

$f(1)=5$ gives $a+b+c=5$

$f(2)=13$ gives $4a+2b+c=13$

Plugging in $c=1$ into the last two equations gives

$a+b=4$

$4a+2b=12$

Dividing the second equation by 2 gives the system:

$a+b=4$

$2a+b=6$

Subtracting the first equation from the second gives $a=2$, and hence $b=2$. Thus, our quadratic function is:

$f(n) = 2n^2 + 2n + 1$

Calculating the answer to our problem, $f(100) = 20000 + 200 + 1 = 20201$, which is choice $\boxed{\text{C}}$

See Also

2000 AMC 12 Problems/Problem 9

The following problem is from both the 2000 AMC 12 #9 and 2000 AMC 10 #14, so both problems redirect to this page.

Contents

Problem

Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were $71, 76, 80, 82,$ and 91. What was the last score Mrs. Walters entered?

$\text{(A)}\ 71 \qquad \text{(B)}\ 76 \qquad \text{(C)}\ 80 \qquad \text{(D)}\ 82 \qquad \text{(E)}\ 91$

Solution

Solution 1

The first number is divisible by 1.

The sum of the first two numbers is even.

The sum of the first three numbers is divisible by 3.

The sum of the first four numbers is divisible by 4.

The sum of the first five numbers is 400.

Since 400 is divisible by 4, the last score must also be divisible by 4. Therefore, the last score is either 76 or 80.

Case 1: 76 is the last number entered.

Since $400 \equiv 76 \equiv 1 \pmod{3}$, the fourth number must be divisible by 3, but none of the scores are divisible by 3.

Case 2: 80 is the last number entered.

Since $80 \equiv 2 \pmod{3}$, the fourth number must be $2 \pmod{3}$. That number is 71 and only 71. The next number must be 91, since the sum of the first two numbers is even. So the only arrangement of the scores $76, 82, 91, 71, 80 \Rightarrow \text{(C)}$

Solution 2

We know the first sum of the first three numbers must be divisible by 3, so we write out all 5 numbers $\pmod{3}$, which gives $2, 1, 2, 1, 1$, respectively. Clearly the only way to get a number divisible by 3 by adding three of these is by adding the three ones. So those must go first. Now we have an odd sum, and since the next average must be divisible by 4, 71 must be next. That leaves 80 for last, so the answer is C.

See also

2000 AMC 12 Problems/Problem 10

Problem

The point $P = (1,2,3)$ is reflected in the xy-plane, then its image Q is rotated by 180° about the x-axis to produce R, and finally, R is translated by 5 units in the positive-y direction to produce S. What are the coordinates of S?

$\text{(A)}\ (1,7,-3) \qquad \text{(B)}\ (-1,7,-3) \qquad \text{(C)}\ (-1,-2,8) \qquad \text{(D)}\ (-1,3,3) \qquad \text{(E)}\ (1,3,3)$

Solution

Step 1: Reflect in the xy plane. Replace z with its additive inverse: $(1,2,-3)$

Step 2: Rotate around x-axis 180 degrees. Replace y and z with their respective additive inverses. $(1,-2,3)$

Step 3: Translate 5 units in positive-y direction. Replace y with y+5. $(1,3,3) \Rightarrow \text{(E)}$

See Also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
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Categories: Introductory Geometry Problems | 3D Geometry Problems

2000 AMC 12 Problems/Problem 11

> The following problem is from both the 2000 AMC 12 #11 and 2000 AMC 10 #15, so both problems redirect to this page.

Problem

Two non-zero real numbers, a and b, satisfy $ab = a - b$. Which of the following is a possible value of $\frac{a}{b} + \frac{b}{a} - ab$?

$\text{(A)}\ -2 \qquad \text{(B)}\ \frac{-1}{2} \qquad \text{(C)}\ \frac{1}{3} \qquad \text{(D)}\ \frac{1}{2} \qquad \text{(E)}\ 2$

Solution

$$\frac{a}{b} + \frac{b}{a} - ab = \frac{a^2+b^2}{ab} - (a-b) = \frac{a^2+b^2}{a-b} - \frac{(a-b)^2}{(a-b)} = \frac{2ab}{a-b} = 2 \Rightarrow \text{(E)}.$$

Alternatively, we could test simple values, like $(a,b) = \left(1, \frac{1}{2}\right)$, which would yield $\frac{a}{b} + \frac{b}{a} - ab = 2$.

Another way is to solve the equation for b, giving $b = \frac{a}{a+1}$; then substituting this into the expression and simplifying gives the answer of 2.

See also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
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2000 AMC 10 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2000))	
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2000 AMC 12 Problems/Problem 12

Problem

Let $A, M,$ and C be nonnegative integers such that $A+M+C=12$. What is the maximum value of $A\cdot M\cdot C+A\cdot M+M\cdot C+A\cdot C$?

$\text{(A)}\ 62 \qquad\text{(B)}\ 72 \qquad\text{(C)}\ 92 \qquad\text{(D)}\ 102 \qquad\text{(E)}\ 112$

Solution

It is not hard to see that

$$(A+1)(M+1)(C+1) =$$

$$AMC+AM+AC+MC+A+M+C+1$$

Since $A+M+C=12$, we can rewrite this as

$$(A+1)(M+1)(C+1) =$$

$$AMC+AM+AC+MC+13$$

So we wish to maximize

$$(A+1)(M+1)(C+1)-13$$

Which is largest when all the factors are equal (consequence of AM-GM). Since $A+M+C=12$, we set $A=B=C=4$ Which gives us

$$(4+1)(4+1)(4+1)-13=112$$

so the answer is $\boxed{\text{E}}$.

See also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2000 AMC 12 Problems/Problem 13

The following problem is from both the 2000 AMC 12 #13 and 2000 AMC 10 #22, so both problems redirect to this page.

Contents

Problem

One morning each member of Angela's family drank an 8-ounce mixture of coffee with milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Angela drank a quarter of the total amount of milk and a sixth of the total amount of coffee. How many people are in the family?

$\text{(A)}\ 3 \qquad \text{(B)}\ 4 \qquad \text{(C)}\ 5 \qquad \text{(D)}\ 6 \qquad \text{(E)}\ 7$

Solution

Solution 1:

Let c be the total amount of coffee, m of milk, and p the number of people in the family. Then each person drinks the same total amount of coffee and milk (8 ounces), so

$$\left(\frac{c}{6}+\frac{m}{4}\right)p = c+m$$

Regrouping, we get $2c(6-p)=3m(p-4)$. Since both c,m are positive, it follows that $6-p$ and $p-4$ are also positive, which is only possible when $p=5\ \text{(C)}$.

Solution 2 (less rigorous):

One could notice that (since there are only two components to the mixture) Angela must have more than her "fair share" of milk and less then her "fair share" of coffee in order to ensure that everyone has 8 ounces. The "fair share" is $1/p$. So,

$$\frac{1}{6}<\frac{1}{p}<\frac{1}{4}$$

Which requires that p be $p=5\ \text{(C)}$, since p is a whole number.

Solution 3:

Again, let c, m, and p be the total amount of coffee, total amount of milk, and number of people in the family, respectively. Then the total amount that is drunk is $8p$, and also $c+m$. Thus, $c+m=8p$, so $m=8p-c$ and $c=8p-m$.

We also know that the amount Angela drank, which is $\frac{c}{6}+\frac{m}{4}$, is equal to 8 ounces, thus $\frac{c}{6}+\frac{m}{4}=8.$ Rearranging gives

$$24p-c=96.$$

Now notice that $c > 0$ (by the problem statement). In addition, $m > 0$, so $c = 8p - m < 8p$. Therefore, $0 < c < 8p$, and so $24p > 24p - c > 16p$. We know that $24p - c = 96$, so

$$24p > 96 > 16p.$$

From the leftmost inequality, we get $p > 4$, and from the rightmost inequality, we get $p < 6$. The only possible value of p is $p = 5\ (\mathrm{C})$.

Sidenote

If we now solve for c and m, we find that $m = 16$ and $c = 24$. Thus in total the family drank 16 ounces of milk and 24 ounces of coffee. Angela drank exactly 4 ounces of milk and 4 ounces of coffee.

See also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
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Category: Introductory Number Theory Problems

2000 AMC 12 Problems/Problem 14

The following problem is from both the 2000 AMC 12 #14 and 2000 AMC 10 #23, so both problems redirect to this page.

Problem

When the mean, median, and mode of the list

$$10, 2, 5, 2, 4, 2, x$$

are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

$\text{(A)}\ 3 \qquad \text{(B)}\ 6 \qquad \text{(C)}\ 9 \qquad \text{(D)}\ 17 \qquad \text{(E)}\ 20$

Solution

- The mean is $\frac{10+2+5+2+4+2+x}{7} = \frac{25+x}{7}$.
- Arranged in increasing order, the list is $2, 2, 2, 4, 5, 10$, so the median is either $2, 4$ or x depending upon the value of x.
- The mode is 2, since it appears three times.

We apply casework upon the median:

- If the median is 2 $(x \le 2)$, then the arithmetic progression must be non-constant, which is fine.
- If the median is 4 $(x \ge 4)$, then the mean can either be $0, 3, 6$ to form an arithmetic progression. Solving for x yields $-25, -4, 17$ respectively, of which only 17 works.
- If the median is x $(2 \le x \le 4)$, then the mean can either be $1, 5/2, 4$ to form an arithmetic progression. Solving for x yields $-18, -7.5, 3$ respectively, of which only 3 works.

The answer is $3 + 17 = 20\ \text{(E)}$.

SOLUTION DOESNT MAKE SENSE 2,2,4,5,10 IS NON CONSTANT

See also

2000 AMC 12 (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2000))	
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2000 AMC 12 Problems/Problem 15

The following problem is from both the 2000 AMC 12 #15 and 2000 AMC 10 #24, so both problems redirect to this page.

Problem

Let f be a function for which $f(x/3) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

$\text{(A)}\ -1/3 \qquad \text{(B)}\ -1/9 \qquad \text{(C)}\ 0 \qquad \text{(D)}\ 5/9 \qquad \text{(E)}\ 5/3$

Solution

Let $y = \frac{x}{3}$; then $f(y) = (3y)^2 + 3y + 1 = 9y^2 + 3y + 1$. Thus $f(3z) - 7 = 81z^2 + 9z - 6 = 3(9z-2)(3z+1) = 0$, and $z = -\frac{1}{3}, \frac{2}{9}$. These sum up to $\boxed{-\frac{1}{9} \text{(B)}}$. (We can also use Vieta's formulas to find the sum more quickly.)

Alternative solution: Set $f(\frac{x}{3}) = x^2 + x + 1 = 7$ to get $x^2 + x - 6 = 0$. From either finding the roots or using Vieta's formulas, we find the sum of these roots to be -1. Each root of this equation is 9 times greater than a corresponding root of $f(3z) = 7$ (because $\frac{x}{3} = 3z$ gives $x = 9z$), thus the sum of the roots in the equation $f(3z) = 7$ is $-\frac{1}{9}$.

Alternate Solution 2:

Since we have $f(x/3)$, $f(3z)$ occurs at $x = 9z$. Thus, $f(9z/3) = f(3z) = (9z)^2 + 9z + 1$. We set this equal to 7:

$81z^2 + 9z + 1 = 7 \Longrightarrow 81z^2 + 9z - 6 = 0$. For any quadratic $ax^2 + bx + c = 0$, the sum of the roots is $-\frac{b}{a}$. Thus, the sum of the roots of this equation is $-\frac{9}{81} = \boxed{-\frac{1}{9}} \Longrightarrow \boxed{\text{(B)}}$

See also

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2000 AMC 12 Problems/Problem 16

Problem

A checkerboard of 13 rows and 17 columns has a number written in each square, beginning in the upper left corner, so that the first row is numbered $1, 2, \ldots, 17$, the second row $18, 19, \ldots, 34$, and so on down the board. If the board is renumbered so that the left column, top to bottom, is $1, 2, \ldots, 13,$, the second column $14, 15, \ldots, 26$ and so on across the board, some squares have the same numbers in both numbering systems. Find the sum of the numbers in these squares (under either system).

$\text {(A)}\ 222 \qquad \text {(B)}\ 333\qquad \text {(C)}\ 444 \qquad \text {(D)}\ 555 \qquad \text {(E)}\ 666$

Solution

Index the rows with $i = 1, 2, 3, ..., 13$ Index the columns with $j = 1, 2, 3, ..., 17$

For the first row number the cells $1, 2, 3, ..., 17$ For the second, $18, 19, ..., 34$ and so on

So the number in row = i and column = j is $f(i, j) = 17(i-1) + j = 17i + j - 17$

Similarly, numbering the same cells columnwise we find the number in row = i and column = j is $g(i, j) = i + 13j - 13$

So we need to solve

$$f(i, j) = g(i, j) 17i + j - 17 = i + 13j - 13 16i = 4 + 12j 4i = 1 + 3j i = (1 + 3j)/4$$

We get
$(i, j) = (1, 1), f(i, j) = g(i, j) = 1 (i, j) = (4, 5), f(i, j) = g(i, j) = 56 (i, j) = (7, 9), f(i, j) = g(i, j) = 111 (i, j) = (10, 13), f(i, j) = g(i, j) = 166 (i, j) = (13, 17), f(i, j) = g(i, j) = 221$

$\boxed{D}$ 555

See also

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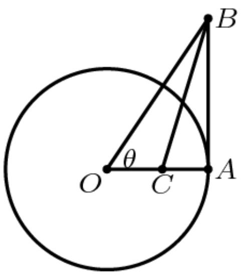
2000 AMC 12 Problems/Problem 17

Contents

Problem

A circle centered at O has radius 1 and contains the point A. The segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on $\overline{OA}$ and $\overline{BC}$ bisects $\angle ABO$, then $OC =$



$\text{(A)}\ \sec^2 \theta - \tan \theta \qquad \text{(B)}\ \frac 12 \qquad \text{(C)}\ \frac{\cos^2 \theta}{1 + \sin \theta}\qquad \text{(D)}\ \frac{1}{1+\sin\theta} \qquad \text{(E)}\ \frac{\sin \theta}{\cos^2 \theta}$

Solution

Since $\overline{AB}$ is tangent to the circle, $\triangle OAB$ is a right triangle. This means that $OA = 1$, $AB = \tan \theta$ and $OB = \sec \theta$. By the Angle Bisector Theorem,

$$\frac{OB}{OC} = \frac{AB}{AC} \Longrightarrow AC \sec \theta = OC \tan \theta$$

We multiply both sides by $\cos \theta$ to simplify the trigonometric functions,

$$AC = OC \sin \theta$$

Since $AC + OC = 1$, $1 - OC = OC \sin \theta \Longrightarrow OC = \dfrac{1}{1+\sin \theta}$. Therefore, the answer is $\boxed{\textbf{(D)} \dfrac{1}{1+\sin \theta}}$.

Alternatively, one could notice that OC approaches the value 1/2 as theta gets close to 90 degrees. The only choice that is consistent with this is (D).

Solution (with minimal trig)

Let's assign a value to θ so we don't have to use trig functions to solve. 60 is a good value for θ, because then we have a $30-60-90 \triangle$ -- $\angle BAC = 90$ because AB is tangent to Circle O.

Using our special right triangle, since $AO = 1$, $OB = 2$, and $AB = \sqrt{3}$.

Let $OC = x$. Then $CA = 1-x$. since BC bisects $\angle ABO$, we can use the angle bisector theorem:

$$\frac{2}{x} = \frac{\sqrt{3}}{1-x}$$
$$2 - 2x = \sqrt{3}x$$
$$2 = (\sqrt{3} + 2)x$$
$$x = \frac{2}{\sqrt{3} + 2}.$$

Now, we only have to use a bit of trig to guess and check: the only trig facts we need to know to finish the problem is:

$$sin\theta = \frac{Opposite}{Hypotenuse}$$
$$cos\theta = \frac{Adjacent}{Hypotenuse}$$
$$tan\theta = \frac{Opposite}{Adjacent}.$$

With a bit of guess and check, we get that the answer is $\boxed{D}$.

See also

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2000 AMC 12 Problems/Problem 18

The following problem is from both the 2000 AMC 12 #18 and 2000 AMC 10 #25, so both problems redirect to this page.

Problem

In year N, the 300^{th} day of the year is a Tuesday. In year $N+1$, the 200^{th} day is also a Tuesday. On what day of the week did the 100^{th} day of year $N-1$ occur?

$\text{(A)}\ \text{Thursday} \qquad \text{(B)}\ \text{Friday}\qquad \text{(C)}\ \text{Saturday}\qquad \text{(D)}\ \text{Sunday}\qquad \text{(E)}\ \text{Monday}$

Solution

There are either $65 + 200 = 265$ or $66 + 200 = 266$ days between the first two dates depending upon whether or not year $N+1$ is a leap year. Since 7 divides into 266, then it is possible for both dates to be Tuesday; hence year $N+1$ is a leap year and N is not a leap year. There are $265 + 300 = 565$ days between the date in years $N, N-1$, which leaves a remainder of 5 upon division by 7. Since we are subtracting days, we count 5 days before Tuesday, which gives us $\text{Thursday} \text{(A)}$.

See also

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Category: Introductory Number Theory Problems

2000 AMC 12 Problems/Problem 19

Contents

Problem

In triangle ABC, $AB = 13$, $BC = 14$, $AC = 15$. Let D denote the midpoint of $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle BAC. Which of the following is closest to the area of the triangle ADE?

$\text{(A)}\ 2 \qquad \text{(B)}\ 2.5 \qquad \text{(C)}\ 3 \qquad \text{(D)}\ 3.5 \qquad \text{(E)}\ 4$

Solution 1

The answer is exactly 3, choice (C). We can find the area of triangle ADE by using the simple formula $\frac{bh}{2}$. Dropping an altitude from A, we see that it has length 12 (we can split the large triangle into a $9-12-15$ and a $5-12-13$ triangle). Then we can apply the Angle Bisector theorem on triangle ABC to solve for BE. Solving $\frac{13}{BE} = \frac{15}{14-BE}$, we get that $BE = \frac{13}{2}$. D is the midpoint of BC so $BD = 7$. Thus we get the base of triangle $ADE, DE,$ to be $\frac{1}{2}$ units long. Applying the formula $\frac{bh}{2}$, we get $\frac{12*\frac{1}{2}}{2} = 3$.

Solution 2

The area of ADE is $\frac{DE\cdot h}{2}=\frac{DE}{BC}\cdot\frac{BC\cdot h}{2}=\frac{DE}{BC}[ABC]$ where h is the height of triangle ABC. Using Angle Bisector Theorem, we find $\frac{13}{BE} = \frac{15}{14-BE}$, which we solve to get $BE = \frac{13}{2}$. D is the midpoint of BC so $BD = 7$. Thus we get the base of triangle $ADE, DE,$ to be $\frac{1}{2}$ units long. We can now use Heron's Formula on ABC.

$$s=\frac{AB+BC+AC}{2}=21$$

$$[ABC]=\sqrt{(s)(s-AB)(s-BC)(s-AC)}=\sqrt{(21)(8)(7)(6)}=84$$

$$\frac{DE}{BC}[ABC]=\frac{\frac{1}{2}}{14}\cdot 84=3$$

Therefore, the answer is C.

See also

2000 AMC 12 Problems/Problem 20

Problem

If $x, y,$ and z are positive numbers satisfying

$$x+1/y=4,\qquad y+1/z=1,\qquad \text{and}\qquad z+1/x=7/3$$

Then what is the value of xyz ?

$\text{(A)}\ 2/3 \qquad \text{(B)}\ 1 \qquad \text{(C)}\ 4/3 \qquad \text{(D)}\ 2 \qquad \text{(E)}\ 7/3$

Contents

Solution

Solution 1

Multiplying all three expressions together,

$$\begin{aligned} \left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right) &= xyz+x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{xyz} \\ (4)(1)\left(\frac{7}{3}\right) &= 4+1+\frac{7}{3}+xyz+\frac{1}{xyz} \\ 2 &= xyz+\frac{1}{xyz} \\ 0 &= (xyz-1)^2 \end{aligned}$$

Thus $xyz = 1 \Rightarrow B$

Solution 2

We have a system of three equations and three variables, so we can apply repeated substitution.

$$4=x+\frac{1}{y}=x+\frac{1}{1-\frac{1}{z}}=x+\frac{1}{1-\frac{1}{7/3-1/x}}=x+\frac{7x-3}{4x-3}$$

Multiplying out the denominator and simplification yields

$4(4x-3)=x(4x-3)+7x-3 \Longrightarrow (2x-3)^2=0$, so $x=\frac{3}{2}$. Substituting leads to $y=\frac{2}{5}, z=\frac{5}{3}$, and the product of these three variables is 1.

See also

2000 AMC 12 Problems/Problem 21

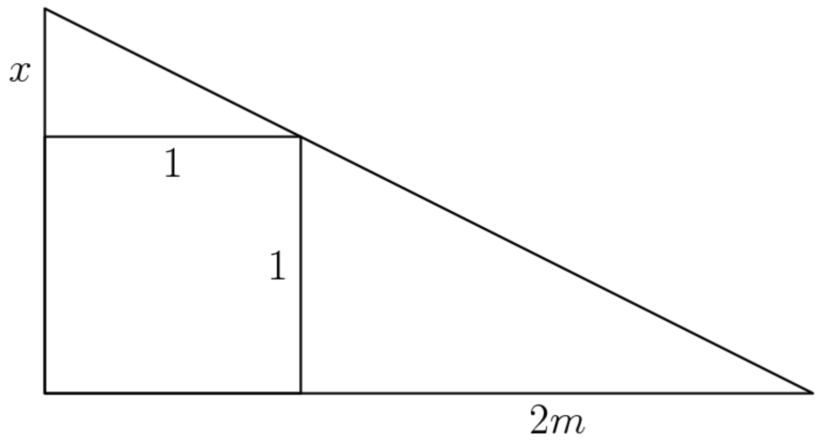
The following problem is from both the 2000 AMC 12 #21 and 2000 AMC 10 #19, so both problems redirect to this page.

Problem

Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is m times the area of the square. The ratio of the area of the other small right triangle to the area of the square is

$$\text{(A)}\ \frac{1}{2m+1} \qquad \text{(B)}\ m \qquad \text{(C)}\ 1-m \qquad \text{(D)}\ \frac{1}{4m} \qquad \text{(E)}\ \frac{1}{8m^2}$$

Solution



WLOG, let a side of the square be 1. Simple angle chasing shows that the two right triangles are similar. Thus the ratio of the sides of the triangles are the same. Since $A = \frac{1}{2}bh = \frac{h}{2}$, the height of the triangle with area m is $2m$. Therefore $\frac{2m}{1} = \frac{1}{x}$ where x is the base of the other triangle. $x = \frac{1}{2m}$, and the area of that triangle is $\frac{1}{2} \cdot 1 \cdot \frac{1}{2m} = \frac{1}{4m}$ (D).

See also

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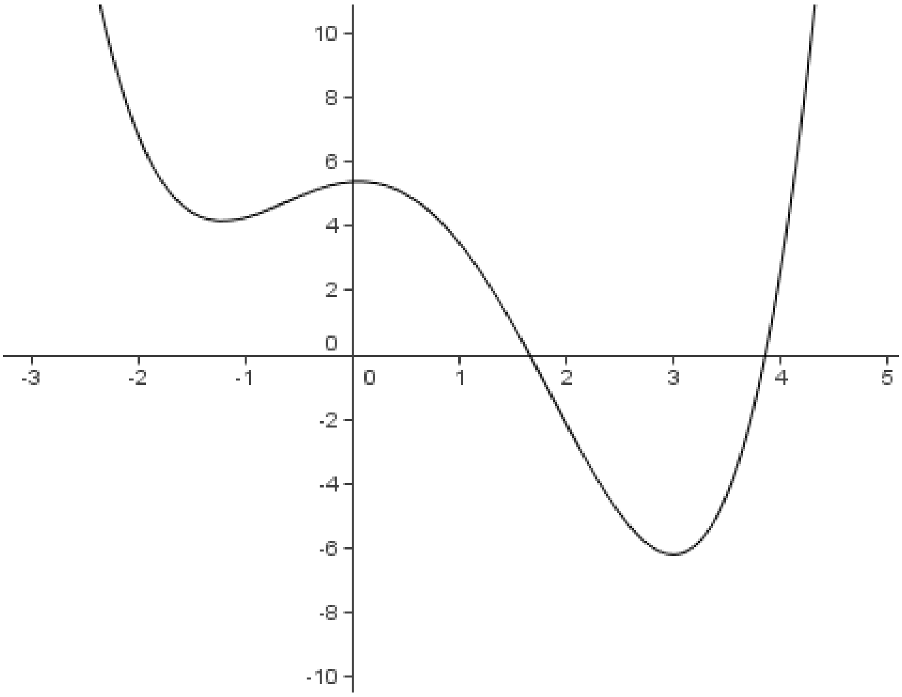
2000 AMC 12 Problems/Problem 22

Problem

The graph below shows a portion of the curve defined by the quartic polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$. Which of the following is the smallest?

(A) $P(-1)$
(B) The product of the zeros of P
(C) The product of the non-real zeros of P
(D) The sum of the coefficients of P
(E) The sum of the real zeros of P



Solution

Note that there are 3 maxima/minima. Hence we know that the rest of the graph is greater than 10. We approximate each of the above expressions:

1. According to the graph, $P(-1) > 4$
2. The product of the roots is d by Vieta's formulas. Also, $d = P(0) > 5$ according to the graph.
3. The product of the real roots is > 5, and the total product is < 6 (from above), so the product of the non-real roots is $< \frac{6}{5}$.
4. The sum of the coefficients is $P(1) > 1.5$
5. The sum of the real roots is > 5.

Clearly (C) is the smallest.

See also

2000 AMC 12 Problems/Problem 23

Problem

Professor Gamble buys a lottery ticket, which requires that he pick six different integers from 1 through 46, inclusive. He chooses his numbers so that the sum of the base-ten logarithms of his six numbers is an integer. It so happens that the integers on the winning ticket have the same property— the sum of the base-ten logarithms is an integer. What is the probability that Professor Gamble holds the winning ticket?

$\text{(A)}\ 1/5 \qquad \text{(B)}\ 1/4 \qquad \text{(C)}\ 1/3 \qquad \text{(D)}\ 1/2 \qquad \text{(E)}\ 1$

Solution

The product of the numbers have to be a power of 10 in order to have an integer base ten logarithm. Thus all of the numbers must be in the form 2^m5^n. Listing out such numbers from 1 to 46, we find $1,2,4,5,8,10,16,20,25,32,40$ are the only such numbers. Immediately it should be noticed that there are a larger number of powers of 2 than of 5. Since a number in the form of 10^k must have the same number of 2s and 5s in its factorization, we require larger powers of 5 than we do of 2. To see this, for each number subtract the power of 5 from the power of 2. This yields $0,1,2,-1,3,0,4,1,-2,5,2,$ and indeed the only non-positive terms are $0,0,-1,-2$. Since there are only two zeros, the largest number that Professor Gamble could have picked would be 2.

Thus Gamble picks numbers which fit $-2+-1+0+0+1+2$, with the first four having already been determined to be $\{25,5,1,10\}$. The choices for the 1 include $\{2,20\}$ and the choices for the 2 include $\{4,40\}$. Together these give four possible tickets, which makes Professor Gamble’ s probability $1/4\ \mathrm{(B)}$.

See also

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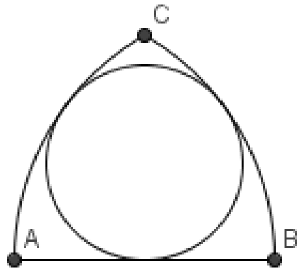
2000 AMC 12 Problems/Problem 24

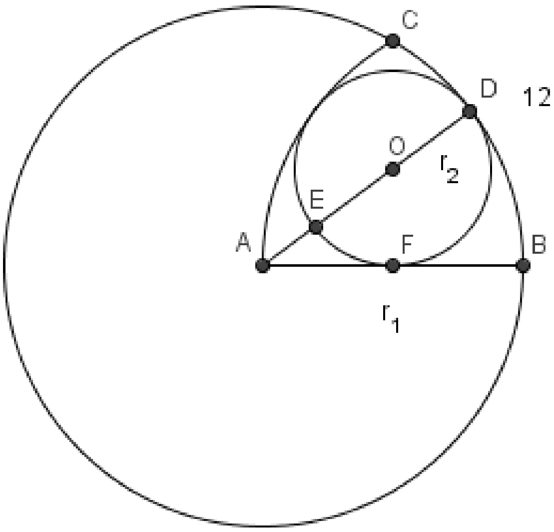
Problem

If circular arcs AC and BC have centers at B and A, respectively, then there exists a circle tangent to both $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$, and to $\overline{AB}$. If the length of $\overset{\frown}{BC}$ is 12, then the circumference of the circle is



$\text{(A)}\ 24 \qquad \text{(B)}\ 25 \qquad \text{(C)}\ 26 \qquad \text{(D)}\ 27 \qquad \text{(E)}\ 28$

Solution



Since AB, BC, AC are all radii, it follows that $\triangle ABC$ is an equilateral triangle.

Draw the circle with center A and radius $\overline{AB}$. Then let D be the point of tangency of the two circles, and E be the intersection of the smaller circle and $\overline{AD}$. Let F be the intersection of the smaller circle and $\overline{AB}$. Also define the radii $r_1 = AB, r_2 = \frac{DE}{2}$ (note that DE is a diameter of the smaller circle, as D is the point of tangency of both circles, the radii of a circle is perpendicular to the tangent, hence the two centers of the circle are collinear with each other and D).

By the Power of a Point Theorem,

$$AF^2 = AE \cdot AD \Longrightarrow \left(\frac{r_1}{2}\right)^2 = (AD - 2r_2) \cdot AD.$$

Since $AD = r_1$, then $\frac{r_1^2}{4} = r_1(r_1 - 2r_2) \Longrightarrow r_2 = \frac{3r_1}{8}$. Since ABC is equilateral, $\angle BAC = 60^{\circ}$, and so $\overset{\frown}{BC} = 12 = \frac{60}{360}2\pi r_1 \Longrightarrow r_1 = \frac{36}{\pi}$. Thus $r_2 = \frac{27}{2\pi}$ and the circumference of the circle is $27\ \mathrm{(D)}$.

(Alternatively, the Pythagorean Theorem can also be used to find r_2 in terms of r_1. Notice that since AB is tangent to circle O, $\overline{OF}$ is perpendicular to $\overline{AF}$. Therefore,

$$AF^2 + OF^2 = AO^2$$

$$(\frac{r_1}{2})^2 + r_2^2 = (r_1 - r_2)^2$$

After simplification, $r_2 = \frac{3r_1}{8}$.)

See also

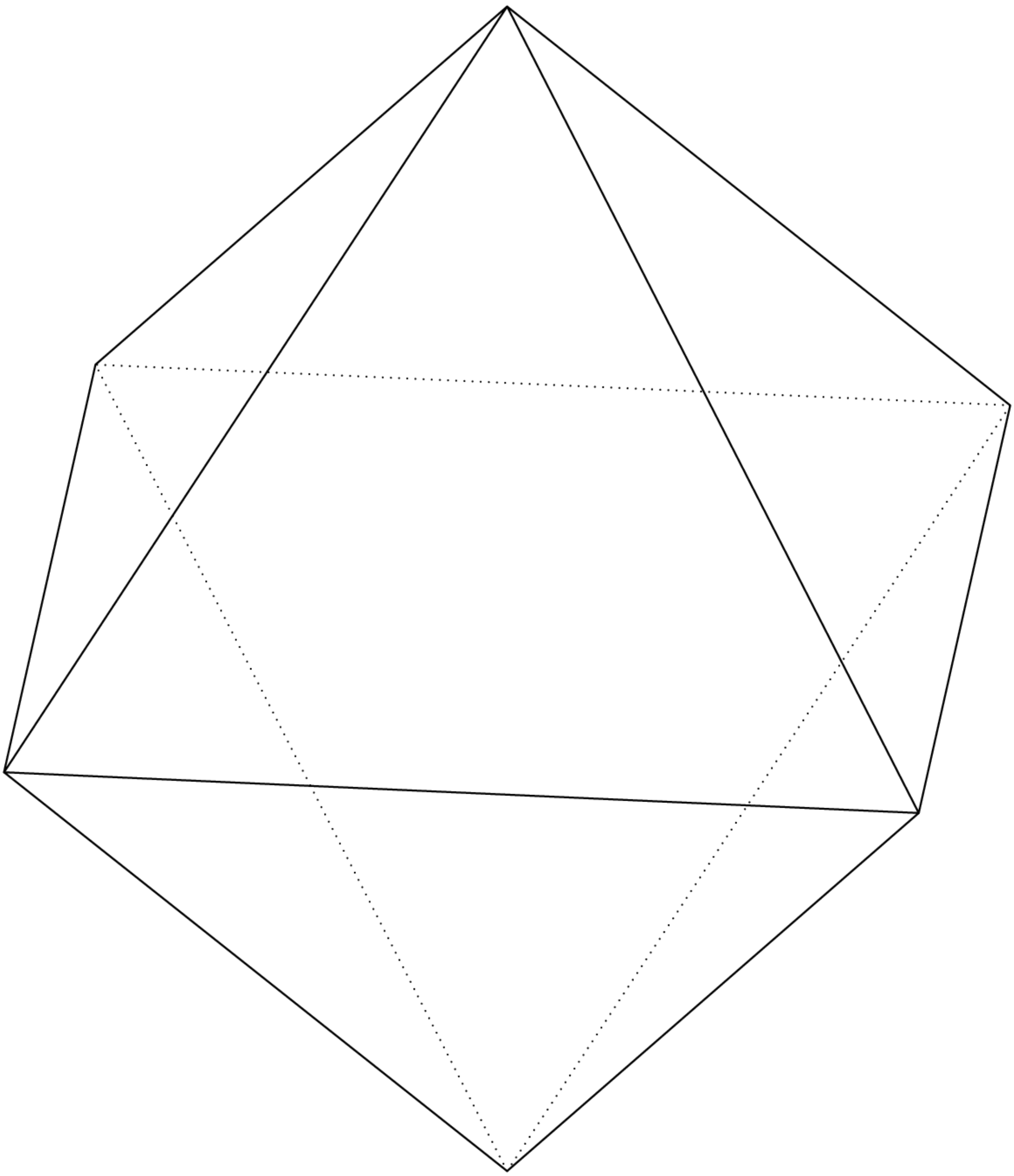
2000 AMC 12 Problems/Problem 25

Contents

Problem

Eight congruent equilateral triangles, each of a different color, are used to construct a regular octahedron. How many distinguishable ways are there to construct the octahedron? (Two colored octahedrons are distinguishable if neither can be rotated to look just like the other.)

$\text{(A)}\ 210 \qquad \text{(B)}\ 560 \qquad \text{(C)}\ 840 \qquad \text{(D)}\ 1260 \qquad \text{(E)}\ 1680$

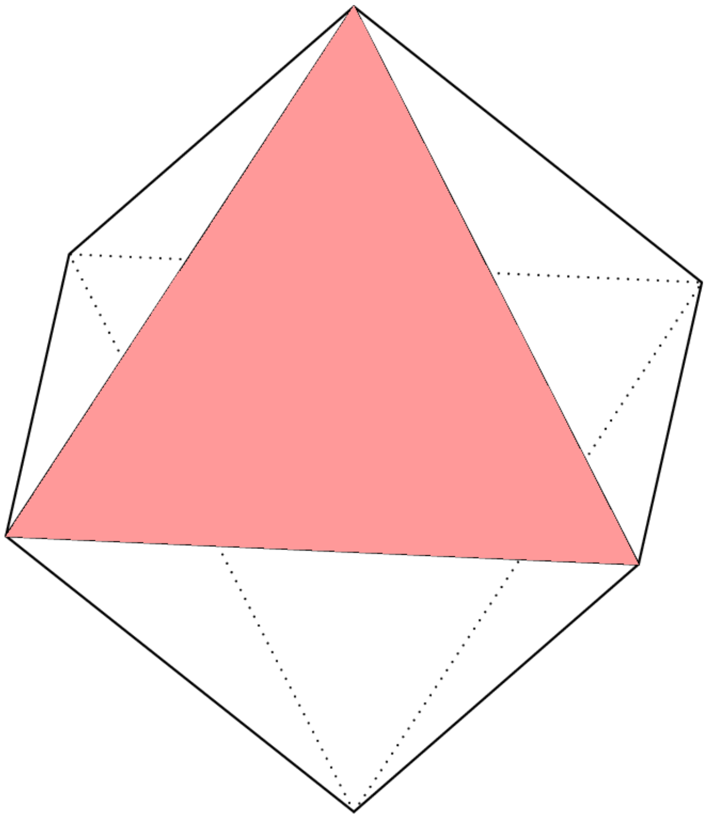


Solution 1

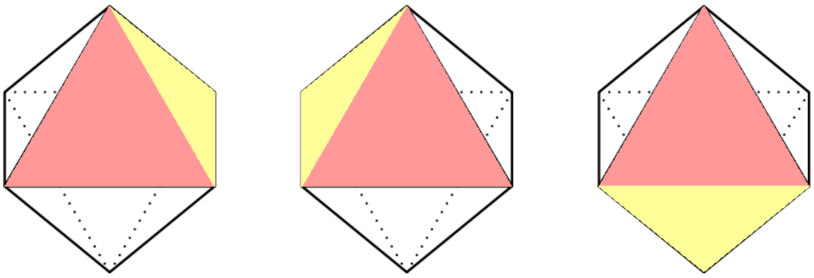
We consider the dual of the octahedron, the cube; a cube can be inscribed in an octahedron with each of its vertices at a face of the octahedron. So the problem is equivalent to finding the number of ways to color the vertices of a cube.

Select any vertex and call it A; there are 8 color choices for this vertex, but this vertex can be rotated to any of 8 locations. After fixing A, we pick another vertex B adjacent to A. There are seven color choices for B, but there are only three locations to which B can be rotated to (since there are three edges from A). The remaining six vertices can be colored in any way and their locations are now fixed. Thus the total number of ways is $\frac{8}{8} \cdot \frac{7}{3} \cdot 6! = 1680 \Rightarrow \text{(E)}$.

Though the cube may be easier to think about, the octahedron can be directly considered. Since the octahedron is indistinguishable by rotations, without loss of generality fix a face to be red.



There are $7!$ ways to arrange the remaining seven colors, but there still are three possible rotations about the fixed face, so the answer is $7!/3 = 1680$.



Solution 2

There are 8! ways to place eight colors on a fixed octahedron. An octahedron has six vertices, of which one can face the top, and for any vertex that faces the top, there are four different triangles around that vertex that can be facing you. Thus there are 6*4 = 24 ways to orient an octahedron, and $8!/24 = 1680 \Rightarrow \text{(E)}$

See also

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