2015 AMC 12B Problems/Problem 1

Problem

What is the value of $2-(-2)^{-2}$?

$\textbf{(A)}\ -2\qquad\textbf{(B)}\ \frac{1}{16}\qquad\textbf{(C)}\ \frac{7}{4}\qquad\textbf{(D)}\ \frac{9}{4}\qquad\textbf{(E)}\ 6$

Solution

$$2-(-2)^{-2}=2-\frac{1}{(-2)^2}=2-\frac{1}{4}=\frac{8}{4}-\frac{1}{4}=\boxed{\textbf{(C)}\ \frac{7}{4}}$$

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 2

Problem

Marie does three equally time-consuming tasks in a row without taking breaks. She begins the first task at 1:00 PM and finishes the second task at 2:40 PM. When does she finish the third task?

$\textbf{(A)}\text{ 3:10 PM}\qquad\textbf{(B)}\text{ 3:30 PM}\qquad\textbf{(C)}\text{ 4:00 PM}\qquad\textbf{(D)}\text{ 4:10 PM}\qquad\textbf{(E)}\text{ 4:30 PM}$

Solution

The first two tasks took $\text{2:40 PM} - \text{1:00 PM} = 100$ minutes. Thus, each task takes $100 \div 2 = 50$ minutes. So the third task finishes at $\text{2:40 PM} + 50$ minutes $= \boxed{\textbf{(B)}\text{ 3:30 PM}}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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All AMC 12 Problems and Solutions	

2015 AMC 12B Problems/Problem 3

Problem

Isaac has written down one integer two times and another integer three times. The sum of the five numbers is 100, and one of the numbers is 28. What is the other number?

$\textbf{(A)}\; 8 \qquad\textbf{(B)}\; 11 \qquad\textbf{(C)}\; 14 \qquad\textbf{(D)}\; 15 \qquad\textbf{(E)}\; 18$

Solution

Let a be the number written two times, and b the number written three times. Then $2a + 3b = 100$. Plugging in a = 28 doesn't yield an integer for b, so it must be that $b = 28$, and we get $2a + 84 = 100$. Solving for a, we obtain $a = \boxed{\textbf{(A)}\; 8}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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All AMC 12 Problems and Solutions	

2015 AMC 12B Problems/Problem 4

Problem

David, Hikmet, Jack, Marta, Rand, and Todd were in a 12-person race with 6 other people. Rand finished 6 places ahead of Hikmet. Marta finished 1 place behind Jack. David finished 2 places behind Hikmet. Jack finished 2 places behind Todd. Todd finished 1 place behind Rand. Marta finished in 6th place. Who finished in 8th place?

$\textbf{(A)}\ \text{David} \qquad\textbf{(B)}\ \text{Hikmet} \qquad\textbf{(C)}\ \text{Jack} \qquad\textbf{(D)}\ \text{Rand} \qquad\textbf{(E)}\ \text{Todd}$

Solution

Let $-$ denote any of the 6 racers not named. Then the correct order following all the logic looks like:

$$-, \text{Rand}, \text{Todd}, -, \text{Jack}, \text{Marta}, -, \text{Hikmet}, -, \text{David}, -, -$$

Clearly the 8th place runner is $\boxed{\textbf{(B)}\ \text{Hikmet}}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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All AMC 12 Problems and Solutions	

2015 AMC 12B Problems/Problem 5

Problem

The Tigers beat the Sharks 2 out of the 3 times they played. They then played N more times, and the Sharks ended up winning at least 95% of all the games played. What is the minimum possible value for N?

$\textbf{(A)}\ 35 \qquad \textbf{(B)}\ 37 \qquad \textbf{(C)}\ 39 \qquad \textbf{(D)}\ 41 \qquad \textbf{(E)}\ 43$

Solution

The ratio of the Shark's victories to games played is $\frac{1}{3}$. For N to be at its smallest, the Sharks must win all the subsequent games because $\frac{1}{3} < \frac{95}{100}$. Then we can write the equation

$$\frac{1+N}{3+N} = \frac{95}{100} = \frac{19}{20}$$

Cross-multiplying yields $20(1+N) = 19(3+N)$, and we find that $N = \boxed{\textbf{(B)}\ 37}$.

See Also

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2015 AMC 12B Problems/Problem 6

Problem

Back in 1930, Tillie had to memorize her multiplication facts from 0×0 to 12×12. The multiplication table she was given had rows and columns labeled with the factors, and the products formed the body of the table. To the nearest hundredth, what fraction of the numbers in the body of the table are odd?

$\textbf{(A)}\ 0.21 \qquad\textbf{(B)}\ 0.25 \qquad\textbf{(C)}\ 0.46 \qquad\textbf{(D)}\ 0.50 \qquad\textbf{(E)}\ 0.75$

Solution

There are a total of $(12+1)\times(12+1)=169$ products, and a product is odd if and only if both its factors are odd. There are 6 odd numbers between 0 and 12, namely $1,3,5,7,9,11,$ hence the number of odd products is $6\times 6=36$. Therefore the answer is $36/169 \doteq \boxed{\textbf{(A)}\ 0.21}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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All AMC 12 Problems and Solutions	

2015 AMC 12B Problems/Problem 7

Problem

A regular 15-gon has lines of symmetry, and the smallest positive angle for which it has rotational symmetry is R degrees. What is $L+R$?

$\textbf{(A)}\ 24 \qquad\textbf{(B)}\ 27 \qquad\textbf{(C)}\ 32 \qquad\textbf{(D)}\ 39 \qquad\textbf{(E)}\ 54$

Solution

From consideration of a smaller regular polygon with an odd number of sides (e.g. a pentagon), we see that the lines of symmetry go through a vertex of the polygon and bisect the opposite side. Hence $L = 15$, the number of sides / vertices. The smallest angle for a rotational symmetry transforms one side into an adjacent side, hence $R = 360^\circ/15 = 24^\circ$, the number of degrees between adjacent sides. Therefore the answer is $L+R = 15 + 24 = \boxed{\textbf{(D)}\ 39}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 8

Problem

What is the value of $(625^{\log_5 2015})^{\frac{1}{4}}$?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ \sqrt[4]{2015} \qquad \textbf{(C)}\ 625 \qquad \textbf{(D)}\ 2015 \qquad \textbf{(E)}\ \sqrt[4]{5^{2015}}$

Solution

$$(625^{\log_5 2015})^{\frac{1}{4}} = ((5^4)^{\log_5 2015})^{\frac{1}{4}} = (5^{4\cdot\log_5 2015})^{\frac{1}{4}} = (5^{\log_5 2015\cdot 4})^{\frac{1}{4}} = ((5^{\log_5 2015})^4)^{\frac{1}{4}} = (2015^4)^{\frac{1}{4}} = \boxed{\textbf{(D)}\ 2015}$$

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 9

Problem

Larry and Julius are playing a game, taking turns throwing a ball at a bottle sitting on a ledge. Larry throws first. The winner is the first person to knock the bottle off the ledge. At each turn the probability that a player knocks the bottle off the ledge is $\tfrac{1}{2}$, independently of what has happened before. What is the probability that Larry wins the game?

$\textbf{(A)}\; \dfrac{1}{2} \qquad \textbf{(B)}\; \dfrac{3}{5} \qquad \textbf{(C)}\; \dfrac{2}{3} \qquad \textbf{(D)}\; \dfrac{3}{4} \qquad \textbf{(E)}\; \dfrac{4}{5}$

Solution

If Larry wins, he either wins on the first move, or the third move, or the fifth move, etc. Let W represent "player wins", and L represent "player loses". Then the events corresponding to Larry winning are $W, LLW, LLLLW, LLLLLLW, \ldots$

Thus the probability of Larry winning is

$$\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}+\cdots$$

This is a geometric series with ratio $\dfrac{1}{2^2}=\dfrac{1}{4}$, hence the answer is $\dfrac{1}{2}\cdot\dfrac{1}{1-\frac{1}{4}}=\boxed{\textbf{(C)}\; \frac{2}{3}}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 10

Problem

How many noncongruent integer-sided triangles with positive area and perimeter less than 15 are neither equilateral, isosceles, nor right triangles?

$\textbf{(A)}\ 3 \qquad\textbf{(B)}\ 4 \qquad\textbf{(C)}\ 5 \qquad\textbf{(D)}\ 6 \qquad\textbf{(E)}\ 7$

Solution

Since we want non-congruent triangles that are neither isosceles nor equilateral, we can just list side lengths $(a, b, with $a<b<c$. Furthermore, "positive area" tells us that $c<a+b$ and the perimeter constraints means $a+b+c<15$.

There are no triangles when $a=1$ because then c must be less than $b+1$, implying that $b\geq c$, contrary to $b<c$.

When $a=2$, similar to above, c must be less than $b+2$, so this leaves the only possibility $c=b+1$. This gives 3 triangles $(2,3,4),(2,4,5),(2,5,6)$ within our perimeter constraint.

When $a=3$, c can be $b+1$ or $b+2$, which gives triangles $(3,4,5),(3,4,6),(3,5,6)$. Note that $(3,4,5)$ is a right triangle, so we get rid of it and we get only 2 triangles.

All in all, this gives us $3+2=\boxed{\textbf{(C)}\ 5}$ triangles.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 11

Problem

The line $12x + 5y = 60$ forms a triangle with the coordinate axes. What is the sum of the lengths of the altitudes of this triangle?

$\textbf{(A)}\ 20 \qquad \textbf{(B)}\ \frac{360}{17} \qquad \textbf{(C)}\ \frac{107}{5} \qquad \textbf{(D)}\ \frac{43}{2} \qquad \textbf{(E)}\ \frac{281}{13}$

Solution

Clearly the line and the coordinate axes form a right triangle. Since the x-intercept and y-intercept are 5 and 12 respectively, 5 and 12 are two sides of the triangle that are not the hypotenuse, and are thus two of the three heights. In order to find the third height, we can use different equations of the area of the triangle. Using the lengths we know, the area of the triangle is $\frac{1}{2} \times 5 \times 12 = 30$. We can use the hypotenuse as another base to find the third height. Using the distance formula, the length of the hypotenuse is $\sqrt{5^2 + 12^2} = 13$. Then $\frac{1}{2} \times 13 \times h = 30$, and so $h = \frac{60}{13}$. Therefore the sum of all the heights is $5 + 12 + \frac{60}{13} = \boxed{\textbf{(E)}\ \frac{281}{13}}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 12

Contents

Problem

Let a, b, and c be three distinct one-digit numbers. What is the maximum value of the sum of the roots of the equation $(x-a)(x-b)+(x-b)(x-c)=0$?

$\textbf{(A) }15\qquad\textbf{(B) }15.5\qquad\textbf{(C) }16\qquad\textbf{(D) }16.5\qquad\textbf{(E) }17$

Solution 1

The left-hand side of the equation can be factored as $(x-b)(x-a+x-c)=(x-b)(2x-(a+c))$, from which it follows that the roots of the equation are $x=b$, and $x=\frac{a+c}{2}$. The sum of the roots is therefore $b+\frac{a+c}{2}$, and the maximum is achieved by choosing $b=9$, and $\{a,c\}=\{7,8\}$. Therefore the answer is $9+\frac{7+8}{2}=9+7.5=\boxed{\textbf{(D) }16.5}$.

Solution 2

Expand the polynomial. We get $(x-a)(x-b)+(x-b)(x-c)=x^2-(a+b)x+ab+x^2-(b+c)+bc=2x^2-(a+2b+c)x+(ab+bc)$

Now, consider a general quadratic equation $ax^2+bx+c=0.$ The two solutions to this are

$$\frac{-b}{2a}+\frac{\sqrt{b^2-4ac}}{2a},\frac{-b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}.$$

The sum of these roots is

$$\frac{-b}{a}.$$

Therefore, reconsidering the polynomial of the problem, the sum of the roots is

$$\frac{a+2b+c}{2}.$$

Now, to maximize this, it is clear that $b=9.$ Also, we must have $a=8, b=7$ (or vice versa). The reason a,b have to equal these values instead of larger values is because each of a,b,c is distinct.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 13

Problem

Quadrilateral $ABCD$ is inscribed in a circle with $\angle BAC = 70^\circ, \angle ADB = 40^\circ, AD = 4,$ and $BC = 6$. What is AC?

$\textbf{(A)}\; 3+\sqrt{5} \qquad\textbf{(B)}\; 6 \qquad\textbf{(C)}\; \dfrac{9}{2}\sqrt{2} \qquad\textbf{(D)}\; 8-\sqrt{2} \qquad\textbf{(E)}\; 7$

Solution

$\angle ADB$ and $\angle ACB$ are both subtended by segment AB, hence $\angle ACB = \angle ADB = 40^\circ$. By considering $\triangle ABC$, it follows that $\angle ABC = 180^\circ - (70^\circ + 40^\circ) = 70^\circ$. Hence $\triangle ABC$ is isosceles, and $AC = BC = \boxed{\textbf{(B)}\; 6}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 14

Problem

A circle of radius 2 is centered at A. An equilateral triangle with side 4 has a vertex at A. What is the difference between the area of the region that lies inside the circle but outside the triangle and the area of the region that lies inside the triangle but outside the circle?

$\textbf{(A)}\ 8-\pi \qquad \textbf{(B)}\ \pi+2 \qquad \textbf{(C)}\ 2\pi-\frac{\sqrt{2}}{2} \qquad \textbf{(D)}\ 4(\pi-\sqrt{3}) \qquad \textbf{(E)}\ 2\pi-\frac{\sqrt{3}}{2}$

Solution

The area of the circle is $\pi \cdot 2^2 = 4\pi$, and the area of the triangle is $\frac{4^2 \cdot \sqrt{3}}{4} = 4\sqrt{3}$. The difference between the area of the region that lies inside the circle but outside the triangle and the area of the region that lies inside the triangle but outside the circle is the same as the difference between the area of the circle and the area of the triangle (because from both pieces we are subtracting the area of the two shapes' intersection), so the answer is $4\pi - 4\sqrt{3} = \boxed{\textbf{(D)}\ 4(\pi-\sqrt{3})}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 15

Contents

Problem

At Rachelle's school an A counts 4 points, a B 3 points, a C 2 points, and a D 1 point. Her GPA on the four classes she is taking is computed as the total sum of points divided by 4. She is certain that she will get As in both Mathematics and Science, and at least a C in each of English and History. She thinks she has a $\frac{1}{6}$ chance of getting an A in English, and a $\frac{1}{4}$ chance of getting a B. In History, she has a $\frac{1}{4}$ chance of getting an A, and a $\frac{1}{3}$ chance of getting a B, independently of what she gets in English. What is the probability that Rachelle will get a GPA of at least 3.5?

$\textbf{(A)}\; \frac{11}{72} \qquad\textbf{(B)}\; \frac{1}{6} \qquad\textbf{(C)}\; \frac{3}{16} \qquad\textbf{(D)}\; \frac{11}{24} \qquad\textbf{(E)}\; \frac{1}{2}$

Solution 1

The probability that Rachelle gets a C in English is $1-\frac{1}{6}-\frac{1}{4}=\frac{7}{12}$.

The probability that she gets a C in History is $1-\frac{1}{4}-\frac{1}{3}=\frac{5}{12}$.

We see that the sum of Rachelle's "point" scores must be 14 or more. We know that in Mathematics and Science we have a total point score of 8 (since she will get As in both), so we only need a sum of 6 in English and History. This can be achieved by getting two As, one A and one B, one A and one C, or two Bs. We can evaluate these cases.

The probability that she gets two As is $\frac{1}{6}\cdot\frac{1}{4}=\frac{1}{24}$.

The probability that she gets one A and one B is $\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{18}+\frac{1}{16}=\frac{8}{144}+\frac{9}{144}=\frac{17}{144}$.

The probability that she gets one A and one C is $\frac{1}{6}\cdot\frac{5}{12}+\frac{1}{4}\cdot\frac{7}{12}=\frac{5}{72}+\frac{7}{48}=\frac{31}{144}$.

The probability that she gets two Bs is $\frac{1}{4}\cdot\frac{1}{3}=\frac{1}{12}$.

Adding these, we get $\frac{1}{24}+\frac{17}{144}+\frac{31}{144}+\frac{1}{12}=\frac{66}{144}=\boxed{\textbf{(D)}\; \frac{11}{24}}$.

Solution 2

We can break it up into three mutually exclusive cases: A in english, at least a C in history; B in english and at least a B in history; C in english and an A in history. This gives

$$\frac{1}{6} \cdot 1 + \frac{1}{4} \cdot \left(\frac{1}{3} + \frac{1}{4}\right) + \left(1 - \frac{1}{6} - \frac{1}{4}\right) \cdot \frac{1}{4} = \boxed{\textbf{(D)}\ \frac{11}{24}}.$$

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 16

Problem

A regular hexagon with sides of length 6 has an isosceles triangle attached to each side. Each of these triangles has two sides of length 8. The isosceles triangles are folded to make a pyramid with the hexagon as the base of the pyramid. What is the volume of the pyramid?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 162 \qquad \textbf{(C)}\ 36\sqrt{21} \qquad \textbf{(D)}\ 18\sqrt{138} \qquad \textbf{(E)}\ 54\sqrt{21}$

Solution

The distance from a corner to the center is 6, and from the corner to the top of the pyramid is 8, so the height is $\sqrt{8^2-6^2}=\sqrt{64-36}=\sqrt{28}=2\sqrt{7}$.

The area of the hexagon is

$$\frac{3\sqrt{3}}{2}\cdot(\text{side})^2=\frac{3\sqrt{3}}{2}\cdot 6^2=54\sqrt{3}$$

Thus, The volume of the pyramid is

$$\frac{1}{3}\times\text{base}\times\text{height}=\frac{54\sqrt{3}\cdot 2\sqrt{7}}{3}=\boxed{\textbf{(C)}\ 36\sqrt{21}}$$

.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 17

Problem

An unfair coin lands on heads with a probability of $\frac{1}{4}$. When tossed $n > 1$ times, the probability of exactly two heads is the same as the probability of exactly three heads. What is the value of n?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 10 \qquad \textbf{(D)}\ 11 \qquad \textbf{(E)}\ 13$

Solution

When tossed n times, the probability of getting exactly 2 heads and the rest tails is

$$\binom{n}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{n-2}.$$

Similarly, the probability of getting exactly 3 heads is

$$\binom{n}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{n-3}.$$

Now set the two probabilities equal to each other and solve for n:

$$\binom{n}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{n-2}=\binom{n}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{n-3}$$

$$\frac{n(n-1)}{2!}\cdot\frac{3}{4}=\frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{4}$$

$$3=\frac{n-2}{3}$$

$$n-2=9$$

$$n=\boxed{\textbf{(D)}\ 11}$$

Note: the original problem did not specify $n > 1$, so $n = 1$ was a solution, but this was fixed in the Wiki problem text so that the answer would make sense. — @adihaya (talk) 15:23, 19 February 2016 (EST)

See Also

2015 AMC 12B Problems/Problem 18

Contents

Problem

For every composite positive integer n, define $r(n)$ to be the sum of the factors in the prime factorization of n. For example, $r(50) = 12$ because the prime factorization of 50 is 2×5^2, and $2 + 5 + 5 = 12$. What is the range of the function , $\{r(n) : n \text{ is a composite positive integer}\}$?

$\textbf{(A)}\;$ the set of positive integers
$\textbf{(B)}\;$ the set of composite positive integers
$\textbf{(C)}\;$ the set of even positive integers
$\textbf{(D)}\;$ the set of integers greater than 3
$\textbf{(E)}\;$ the set of integers greater than 4

Solution

Solution 1

This problem becomes simple once we recognize that the domain of the function is $\{4, 6, 8, 9, 10, 12, 14, 15, \dots\}$. By evaluating $r(4)$ to be 4, we can see that $\textbf{(E)}$ is incorrect. Evaluating $r(6)$ to be 5, we see that both $\textbf{(B)}$ and $\textbf{(C)}$ are incorrect. Since our domain consists of composite numbers, which, by definition, are a product of at least two positive primes, the minimum value of $r(n)$ is 4, so $\textbf{(A)}$ is incorrect. That leaves us with $\boxed{\textbf{(D)}\text{ the set of integers greater than 3}}$.

Solution 2

Think backwards. The range is the same as the numbers y that can be expressed as the sum of two or more prime positive integers.

The lowest number we can get is $y = 2 + 2 = 4$. For any number greater than 4, we can get to it by adding some amount of 2's and then possibly a 3 if that number is odd. For example, 23 can be obtained by adding 2 ten times and adding a 3; this corresponds to the argument $n = 2^{10} \times 3$. Thus our answer is $\boxed{\textbf{(D)}\text{ the set of integers greater than 3}}$.

See Also

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2015 AMC 12B Problems/Problem 19

Contents

Problem

In $\triangle ABC$, $\angle C = 90^\circ$ and $AB = 12$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. What is the perimeter of the triangle?

$\textbf{(A)}\ 12+9\sqrt{3}\qquad\textbf{(B)}\ 18+6\sqrt{3}\qquad\textbf{(C)}\ 12+12\sqrt{2}\qquad\textbf{(D)}\ 30\qquad\textbf{(E)}\ 32$

Solution 1

First, we should find the center and radius of this circle. We can find the center by drawing the perpendicular bisectors of WZ and XY and finding their intersection point. This point happens to be the midpoint of AB, the hypotenuse. Let this point be M. To find the radius, determine MY, where $MY^2 = MA^2 + AY^2$, $MA = \frac{12}{2} = 6$, and $AY = AB = 12$. Thus, the radius $= r = MY = 6\sqrt{5}$.

Next we let $AC = b$ and $BC = a$. Consider the right triangle ACB first. Using the pythagorean theorem, we find that $a^2 + b^2 = 12^2 = 144$. Next, we let M' to be the midpoint of WZ, and we consider right triangle $ZM'M$. By the pythagorean theorem, we have that $\left(\frac{b}{2}\right)^2 + \left(b+\frac{a}{2}\right)^2 = r^2 = 180$. Expanding this equation, we get that

$$\frac{1}{4}(a^2+b^2) + b^2 + ab = 180$$

$$\frac{144}{4} + b^2 + ab = 180$$

$$b^2 + ab = 144 = a^2 + b^2$$

$$ab = a^2$$

$$b = a$$

This means that ABC is a 45-45-90 triangle, so $a = b = \frac{12}{\sqrt{2}} = 6\sqrt{2}$. Thus the perimeter is $a + b + AB = 12\sqrt{2} + 12$ which is answer $\boxed{\textbf{(C)}\ 12+12\sqrt{2}}$. image needed

Solution 2

The center of the circle on which X, Y, Z, and W lie must be equidistant from each of these four points. Draw the perpendicular bisectors of $\overline{XY}$ and of $\overline{WZ}$. Note that the perpendicular bisector of $\overline{XY}$ is parallel to $\overline{BX}$ and passes through the midpoint of $\overline{AC}$. Therefore, the triangle that is formed by A, the midpoint of $\overline{AC}$, and the point at which this perpendicular bisector intersects $\overline{AB}$ must be similar to $\triangle ABC$, and the ratio of a side of the smaller triangle to a side of $\triangle ABC$ is 1:2. Consequently, the perpendicular bisector of $\overline{XY}$ passes through the midpoint of $\overline{AB}$. The perpendicular bisector of $\overline{WZ}$ must include the midpoint of $\overline{AB}$ as well. Since all points on a perpendicular bisector of any two points M and N are equidistant from M and N, the center of the circle must be the midpoint of $\overline{AB}$.

Now the distance between the midpoint of $\overline{AB}$ and Z, which is equal to the radius of this circle, is $\sqrt{12^2+6^2}=\sqrt{180}$. Let $a=AC$. Then the distance between the midpoint of $\overline{AB}$ and Y, also equal to the radius of the circle, is given by $\sqrt{\left(\frac{a}{2}\right)^2+\left(a+\frac{\sqrt{144-a^2}}{2}\right)^2}$ (the ratio of the similar triangles is involved here). Squaring these two expressions for the radius and equating the results, we have

$$\left(\frac{a}{2}\right)^2+\left(a+\frac{\sqrt{144-a^2}}{2}\right)^2=180$$

$$144-a^2=a\sqrt{144-a^2}$$

$$(144-a^2)^2=a^2(144-a^2)$$

Since a cannot be equal to 12, the length of the hypotenuse of the right triangle, we can divide by $(144-a^2)$, and arrive at $a=6\sqrt{2}$. The length of other leg of the triangle must be $\sqrt{144-72}=6\sqrt{2}$. Thus, the perimeter of the triangle is $12+2(6\sqrt{2})=\boxed{\textbf{(C) } 12+12\sqrt{2}}$.

Solution 3

In order to solve this problem, we can search for similar triangles. Begin by drawing triangle ABC and squares $ABXY$ and $ACWZ$. Draw segments $\overline{YZ}$ and $\overline{WX}$. Because we are given points X, Y, Z, and W lie on a circle, we can conclude that $WXYZ$ forms a cyclic quadrilateral. Take $\overline{AC}$ and extend it through a point P on $\overline{YZ}$. Now, we must do some angle chasing to prove that $\triangle WBX$ is similar to $\triangle YAZ$.

Let α denote the measure of $\angle ABC$. Following this, $\angle BAC$ measures $90-\alpha$. By our construction, $\overline{CAP}$ is a straight line, and we know $\angle YAB$ is a right angle. Therefore, $\angle PAY$ measures α. Also, $\angle CAZ$ is a right angle and thus, $\angle ZAP$ is a right angle. Sum $\angle ZAP$ and $\angle PAY$ to find $\angle ZAY$, which measures $90+\alpha$. We also know that $\angle WBY$ measures $90+\alpha$. Therefore, $\angle ZAY=\angle WBX$.

Let β denote the measure of $\angle AZY$. It follows that $\angle WZY$ measures $90+\beta^\circ$. Because $WXYZ$ is a cyclic quadrilateral, $\angle WZY+\angle YXW=180^\circ$. Therefore, $\angle YXW$ must measure $90-\beta$, and $\angle BXW$ must measure β. Therefore, $\angle AZY=\angle BXW$.

$\angle ZAY=\angle WBX$ and $\angle AZY=\angle BXW$, so $\triangle AZY\sim\triangle BXW$! Let $x=AC=WC$. By Pythagorean theorem, $BC=\sqrt{144-x^2}$. Now we have $WB=WC+BC=x+\sqrt{144-x^2}$, $BX=12$, $YA=12$, and $AZ=x$. We can set up an equation:

$$\frac{YA}{AZ} = \frac{WB}{BX}$$

$$\frac{12}{x} = \frac{x + \sqrt{144 - x^2}}{12}$$

$$144 = x^2 + x\sqrt{144 - x^2}$$

$$12^2 - x^2 = x\sqrt{144 - x^2}$$

$$12^4 - 2 * 12^2 * x^2 + x^4 = 144x^2 - x^4$$

$$2x^4 - 3(12^2)x^2 + 12^4 = 0$$

$$(2x^2 - 144)(x^2 - 144) = 0$$

Solving for x, we find that $x = 6\sqrt{2}$ or $x = 12$, which we omit. The perimeter of the triangle is $12 + x + \sqrt{144 - x^2}$. Plugging in $x = 6\sqrt{2}$, we get $\boxed{\textbf{(C)}\ 12 + 12\sqrt{2}}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 20

Problem

For every positive integer n, let $\text{mod}_5(n)$ be the remainder obtained when n is divided by 5. Define a function $f : \{0, 1, 2, 3, \dots\} \times \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$ recursively as follows:

$$f(i,j) = \begin{cases} \text{mod}_5(j+1) & \text{if } i = 0 \text{ and } 0 \le j \le 4, \\ f(i-1,1) & \text{if } i \ge 1 \text{ and } j = 0, \text{ and} \\ f(i-1, f(i,j-1)) & \text{if } i \ge 1 \text{ and } 1 \le j \le 4. \end{cases}$$

What is $f(2015,2)$?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 4$

Solution

Simply draw a table of values of $f(i,j)$ for the first few values of i:

$i \setminus j$	0	1	2	3	4
0	1	2	3	4	0
1	2	3	4	0	1
2	3	0	2	4	1
3	0	3	4	1	0
4	3	1	3	1	3
5	1	1	1	1	1

It is now clear that for $i \ge 5$, $f(i,j) = 1$ for all values $0 \le j \le 4$.

Thus, $f(2015,2) = \boxed{\textbf{(B)}\ 1}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 21

Contents

Problem

Cozy the Cat and Dash the Dog are going up a staircase with a certain number of steps. However, instead of walking up the steps one at a time, both Cozy and Dash jump. Cozy goes two steps up with each jump (though if necessary, he will just jump the last step). Dash goes five steps up with each jump (though if necessary, he will just jump the last steps if there are fewer than 5 steps left). Suppose that Dash takes 19 fewer jumps than Cozy to reach the top of the staircase. Let s denote the sum of all possible numbers of steps this staircase can have. What is the sum of the digits of s?

$\textbf{(A) }9\qquad\textbf{(B) }11\qquad\textbf{(C) }12\qquad\textbf{(D) }13\qquad\textbf{(E) }15$

Solution 1

We can translate this wordy problem into this simple equation:

$$\left\lceil \frac{s}{2} \right\rceil - 19 = \left\lceil \frac{s}{5} \right\rceil$$

We will proceed to solve this equation via casework.

Case 1: $\left\lceil \frac{s}{2} \right\rceil = \frac{s}{2}$

Our equation becomes $\frac{s}{2} - 19 = \frac{s}{5} + \frac{j}{5}$, where $j \in \{0,1,2,3,4\}$ Using the fact that s is an integer, we quickly find that $j = 1$ and $j = 4$ yield $s = 64$ and $s = 66$, respectively.

Case 2:

Our equation becomes $\frac{s}{2} + \frac{1}{2} - 19 = \frac{s}{5} + \frac{j}{5}$, where $j \in \{0,1,2,3,4\}$ Using the fact that s is an integer, we quickly find that $j = 2$ yields $s = 63$.

Summing up we get $63 + 64 + 66 = 193$. The sum of the digits is $\boxed{\textbf{(D) }13}$.

Solution 2

It can easily be seen that the problem can be expressed by the equation:

$$\left\lceil \frac{s}{2} \right\rceil - \left\lceil \frac{s}{5} \right\rceil = 19$$

However, because the ceiling function is difficult to work with, we can rewrite the previous equation as:

$$\frac{s+a}{2}-\frac{s+b}{5}=19$$

Where $a \in \{0,1\}$ and $b \in \{0,1,2,3,4\}$ Multiplying both sides by ten and simplifying, we get:

$$5s+5a-2s-2b=190$$

$$3s=190+2b-5a$$

$$s=63+\frac{1+2b-5a}{3}$$

Because s must be an integer, we need to find the values of a and b such that $2b-5a \equiv 2 \mod 3$. We solve using casework.

Case 1: $a=0$

If $a=0$, we have $2b \equiv 2 \mod 3$. We can easily see that $b=1$ or $b=4$, which when plugged into our original equation lead to $s=64$ and $s=66$ respectively.

Case 2: $a=1$

If $a=1$, we have $2b-5 \equiv 2 \mod 3$, which can be rewritten as $2b \equiv 1 \mod 3$. We can again easily see that $b=2$ is the only solution, which when plugged into our original equation lead to $s=63$.

Adding these together we get $64+66+63=193$. The sum of the digits is $\boxed{\textbf{(D)}\ 13}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 22

Contents

Problem

Six chairs are evenly spaced around a circular table. One person is seated in each chair. Each person gets up and sits down in a chair that is not the same chair and is not adjacent to the chair he or she originally occupied, so that again one person is seated in each chair. In how many ways can this be done?

$\textbf{(A)}\; 14 \qquad\textbf{(B)}\; 16 \qquad\textbf{(C)}\; 18 \qquad\textbf{(D)}\; 20 \qquad\textbf{(E)}\; 24$

Solution 1

Consider shifting every person over three seats (left or right) after each person has gotten up and sat back down again. Now, instead of each person being seated not in the same chair and not in an adjacent chair, each person will be seated either in the same chair or in an adjacent chair. The problem now becomes the number of ways in which six people can sit down in a chair that is either the same chair or an adjacent chair in a circle.

Consider the similar problem of n people sitting in a chair that is either the same chair or an adjacent chair in a row. Call the number of possibilities for this F_n. Then if the leftmost person stays put, the problem is reduced to a row of $n-1$ chairs, and if the leftmost person shifts one seat to the right, the new person sitting in the leftmost seat must be the person originally second from the left, reducing the problem to a row of $n-2$ chairs. Thus, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Clearly $F_1 = 1$ and $F_2 = 2$, so $F_3 = 3$, $F_4 = 5$, and $F_5 = 8$.

Now consider the six people in a circle and focus on one person. If that person stays put, the problem is reduced to a row of five chairs, for which there are $F_5 = 8$ possibilities. If that person moves one seat to the left, then the person who replaces him in his original seat will either be the person originally to the right of him, which will force everyone to simply shift over one seat to the left, or the person originally to the left of him, which reduces the problem to a row of four chairs, for which there are $F_4 = 5$ possibilities, giving $1+5=6$ possibilities in all. By symmetry, if that person moves one seat to the right, there are another 6 possibilities, so we have a total of $8+6+6=\boxed{\textbf{(D)}\; 20}$ possibilities.

Solution 2

Label the people sitting at the table $A, B, C, D, E, F,$ and assume that they are initially seated in the order $ABCDEF$. The possible new positions for $A, B, C, D, E,$ and F are respectively (a dash indicates a non-allowed position):

-	-	A	A	A	-
-	-	-	B	B	B
C	-	-	-	C	C
D	D	-	-	-	D
E	E	E	-	-	-
-	F	F	F	-	-

The permutations we are looking for should use one letter from each column, and there should not be any repeated letters:

$$\begin{array}{c} \text{CDEFAB} \\ \text{CEAFBD} \\ \text{CEFABD} \\ \text{CEFBAD} \\ \text{CFEABD} \\ \text{CFEBAD} \\ \text{DEAFBC} \\ \text{DEAFCB} \\ \text{DEFABC} \\ \text{DEFACB} \\ \text{DEFBAC} \\ \text{DFEABC} \\ \text{DFEACB} \\ \text{DFEBAC} \\ \text{EDAFBC} \\ \text{EDAFCB} \\ \text{EDFABC} \\ \text{EDFACB} \\ \text{EDFBAC} \\ \text{EFABCD} \end{array}$$

There are $\boxed{\textbf{(D) }20}$ such permutations.

Solution 3

We can represent each rearrangement as a permutation of the six elements $\{1,2,3,4,5,6\}$ in cycle notation. Note that any such permutation cannot have a 1-cycle, so the only possible types of permutations are 2,2,2-cycles, 4,2-cycles, 3,3-cycles, and 6-cycles. We deal with each case separately.

For 2,2,2-cycles, suppose that one of the 2-cycles switches the people across from each other, i.e. (14), (25), or (36). WLOG, we may assume it to be (14). Then we could either have both of the other 2-cycles be across from each other, giving the permutation $(14)(25)(36)$, or else neither of the other 2-cycles are across from each other, in which case the only possible permutation is $(14)(26)(35)$. This can happen for (25) and (36) as well. So since the first permutation is not counted twice, we find a total of $1+3=4$ permutations that are 2,2,2-cycles where at least one of the 2-cycles switches people diametrically opposite from each other. Otherwise, since the elements in a 2-cycle cannot differ by 1, 3, or 5 mod 6, they must differ by 2 or 4 mod 6, i.e. they must be of the same parity. But since we have three odd and three even elements, this is impossible. Hence there are exactly 4 such permutations that are 2,2,2-cycles.

For 4,2-cycles, we assume for the moment that 1 is part of the 2-cycle. Then the 2-cycle can be (13), (15), or (14). The first two are essentially the same by symmetry, and we must arrange the elements 2, 4, 5, 6 into a 4-cycle. However, 5 must have two neighbors that are not next to it, which is impossible, hence the first two cases yield no permutations. If the 2-cycle is (14), then we must arrange the elements 2, 3, 5, 6 into a 4-cycle. Then 2 must have the neighbors 5 and 6. We find that the 4-cycles (2536) and (2635) satisfy the desired properties, yielding the permutations $(14)(2536)$ and $(14)(2635)$. This can be done for the 2-cycles (25) and (36) as well, so we find a total of 6 such permutations that are 4,2-cycles.

For 3,3-cycles, note that if 1 neighbors 4, then the third element in the cycle will neighbor one of 1 and 4, so this is impossible. Therefore, the 3-cycle containing 1 must consist of the elements 1, 3, and 5. Therefore, we obtain the four 3,3-cycles $(135)(246)$, $(153)(246)$, $(135)(264)$, and $(153)(264)$.

For 6-cycles, note that the neighbors of 1 can be 3 and 4, 3 and 5, or 4 and 5. In the first case, we may assume that it looks like $(314\dots)$ -- the form $(413\dots)$ is also possible, but equivalent to this case. Then we must place the elements 2, 5, and 6. Note that 5 and 6 cannot go together, so 2 must go in between them. Also, 5 cannot neighbor 4, so we are left with one possibility, namely (314625), which has an analogous possibility (413526). In the second case, we assume that it looks like $(315\dots)$. Clearly, the 2 must go next to the 5, and the 6 must go last (to neighbor the 3), so the only possibility here is (315246), with the analogous possibility (513642). In the final case, we may assume that it looks like $(415\dots)$. Then the 2 and 3 cannot go together, so the 6 must go in between them. Therefore, the only possibility is (415362), with the analogous possibility (514263). We have covered all possibilities for 6-cycles, and we have found 6 of them.

Therefore, there are $4+6+4+6=\boxed{\textbf{(D)}\ 20}$ such permutations.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12B Problems/Problem 23

Problem

A rectangular box measures $a \times b \times c$, where a, b, and c are integers and $1 \leq a \leq b \leq c$. The volume and the surface area of the box are numerically equal. How many ordered triples (a, b, c) are possible?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 10 \qquad \textbf{(C)}\ 12 \qquad \textbf{(D)}\ 21 \qquad \textbf{(E)}\ 26$

Solution

The surface area is $2(ab+bc+ca)$, and the volume is abc, so equating the two yields

$$2(ab+bc+ca)=abc.$$

Divide both sides by $2abc$ to obtain

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}.$$

First consider the bound of the variable a. Since $\frac{1}{a}<\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$, we have $a>2$, or $a\geqslant 3$.

Also note that $c\geq b\geq a>0$, hence $\frac{1}{a}\geq\frac{1}{b}\geq\frac{1}{c}$. Thus, $\frac{1}{2}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{3}{a}$, so $a\leq 6$.

So we have $a=3,4,5$ or 6.

Before the casework, let's consider the possible range for b if $\frac{1}{b}+\frac{1}{c}=k>0$. From $\frac{1}{b}<k$, we have $b>\frac{1}{k}$. From $\frac{2}{b}\geq\frac{1}{b}+\frac{1}{c}=k$, we have $b\leq\frac{2}{k}$. Thus $\frac{1}{k}<b\leq\frac{2}{k}$.

When $a=3$, we get $\frac{1}{b}+\frac{1}{c}=\frac{1}{6}$, so $b=7,8,9,10,11,12$. We find the solutions $(a,b,c)=(3,7,42)$, $(3,8,24)$, $(3,9,18)$, $(3,10,15)$, $(3,12,12)$, for a total of 5 solutions.

When $a=4$, we get $\frac{1}{b}+\frac{1}{c}=\frac{1}{4}$, so $b=5,6,7,8$. We find the solutions $(a,b,c)=(4,5,20)$, $(4,6,12)$, $(4,8,8)$, for a total of 3 solutions.

When $a=5$, we get $\frac{1}{b}+\frac{1}{c}=\frac{3}{10}$, so $b=5,6$. The only solution in this case is $(a,b,c)=(5,5,10)$.

When $a=6$, b is forced to be 6, and thus $(a,b,c)=(6,6,6)$.

Thus, there are $5+3+1+1=\boxed{\textbf{(B)}\ 10}$ solutions.

See Also

2015 AMC 12B Problems/Problem 24

Problem

Four circles, no two of which are congruent, have centers at A, B, C, and D, and points P and Q lie on all four circles. The radius of circle A is $\frac{5}{8}$ times the radius of circle B, and the radius of circle C is $\frac{5}{8}$ times the radius of circle D. Furthermore, $AB = CD = 39$ and $PQ = 48$. Let R be the midpoint of $\overline{PQ}$. What is $AR+BR+CR+DR$?

$\textbf{(A)}\; 180 \qquad\textbf{(B)}\; 184 \qquad\textbf{(C)}\; 188 \qquad\textbf{(D)}\; 192\qquad\textbf{(E)}\; 196$

Solution

First, note that PQ lies on the radical axis of any of the pairs of circles. Suppose that O_1 and O_2 are the centers of two circles C_1 and C_2 that intersect exactly at P and Q, with O_1 and O_2 lying on the same side of PQ, and $O_1O_2 = 39$. Let $x = O_1R$, $y = O_2R$, and suppose that the radius of circle C_1 is r and the radius of circle C_2 is $\frac{5}{8}r$.

Then the power of point R with respect to C_1 is

$$(r+x)(r-x) = r^2 - x^2 = 24^2$$

and the power of point R with respect to C_2 is

$$\left(\frac{5}{8}r+y\right)\left(\frac{5}{8}r-y\right) = \frac{25}{64}r^2 - y^2 = 24^2.$$

Also, note that $x - y = 39$.

Subtract the above two equations to find that $\frac{39}{64}r^2 - x^2 + y^2 = 0$ or $39r^2 = 64(x^2 - y^2)$. As $x - y = 39$, we find that $r^2 = 64(x+y) = 64(2y+39)$. Plug this into an earlier equation to find that $25(2y+39) - y^2 = 24^2$. This is a quadratic equation with solutions $y = \frac{50 \pm 64}{2}$, and as y is a length, it is positive, hence $y = 57$, and $x = y + 39 = 96$. This is the only possibility if the two centers lie on the same same of their radical axis.

On the other hand, if they lie on opposite sides, then it is clear that there is only one possibility, and then it is clear that $O_1R + O_2R = O_1O_2 = 39$. Therefore, we obtain exactly four possible centers, and the sum of the desired lengths is $57 + 96 + 39 = \boxed{\textbf{(D)}\; 192}$.

See Also

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2015 AMC 12B Problems/Problem 25

Problem

A bee starts flying from point P_0. She flies 1 inch due east to point P_1. For $j \geq 1$, once the bee reaches point P_j, she turns 30° counterclockwise and then flies $j+1$ inches straight to point P_{j+1}. When the bee reaches P_{2015} she is exactly $a\sqrt{b}+c\sqrt{d}$ inches away from P_0, where a, b, c and d are positive integers and b and d are not divisible by the square of any prime. What is $a+b+c+d$?

$\textbf{(A)}\ 2016 \qquad\textbf{(B)}\ 2024 \qquad\textbf{(C)}\ 2032 \qquad\textbf{(D)}\ 2040 \qquad\textbf{(E)}\ 2048$

Solution

Let $x = e^{i\pi/6}$, a 30° counterclockwise rotation centered at the origin. Notice that P_k on the complex plane is:

$$1 + 2x + 3x^2 + \cdots + (k+1)x^k$$

We need to find the magnitude of P_{2015} on the complex plane. This is an arithmetic/geometric series.

$$\begin{aligned}
S &= 1 + 2x + 3x^2 + \cdots + 2015x^{2014} \\
xS &= x + 2x^2 + 3x^3 + \cdots + 2015x^{2015} \\
(1-x)S &= 1 + x + x^2 + \cdots + x^{2014} - 2015x^{2015} \\
S &= \frac{1-x^{2015}}{(1-x)^2} - \frac{2015x^{2015}}{1-x}
\end{aligned}$$

We want to find $|S|$. First, note that $x^{2015} = x^{11} = x^{-1}$ because $x^{12} = 1$. Therefore

$$S = \frac{1-\frac{1}{x}}{(1-x)^2} - \frac{2015}{x(1-x)} = -\frac{1}{x(1-x)} - \frac{2015}{x(1-x)} = -\frac{2016}{x(1-x)}.$$

Hence, since $|x| = 1$, we have $|S| = \dfrac{2016}{|1-x|}$.

Now we just have to find $|1-x|$. This can just be computed directly:

$$1 - x = 1 - \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$|1-x|^2 = \left(1 - \sqrt{3} + \frac{3}{4}\right) + \frac{1}{4} = 2 - \sqrt{3} = \left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2$$

$$|1-x| = \frac{\sqrt{6}-\sqrt{2}}{2}.$$

Therefore $|S| = 2016 \cdot \dfrac{2}{\sqrt{6}-\sqrt{2}} = 2016\left(\dfrac{\sqrt{6}+\sqrt{2}}{2}\right) = 1008\sqrt{2} + 1008\sqrt{6}.$

Thus the answer is $1008 + 1008 + 2 + 6 = \boxed{\textbf{(B) } 2024}$.

See Also

2015 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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