2003 AMC 12A Problems/Problem 1

The following problem is from both the 2003 AMC 12A #1 and 2003 AMC 10A #1, so both problems redirect to this page.

Contents

Problem

What is the difference between the sum of the first 2003 even counting numbers and the sum of the first 2003 odd counting numbers?

$\mathrm{(A) \ } 0\qquad \mathrm{(B) \ } 1\qquad \mathrm{(C) \ } 2\qquad \mathrm{(D) \ } 2003\qquad \mathrm{(E) \ } 4006$

Solution

Solution 1

The first 2003 even counting numbers are $2,4,6,...,4006$.

The first 2003 odd counting numbers are $1,3,5,...,4005$.

Thus, the problem is asking for the value of $(2+4+6+...+4006)-(1+3+5+...+4005)$.

$(2+4+6+...+4006)-(1+3+5+...+4005) = (2-1)+(4-3)+(6-5)+...+(4006-4005)$

$= 1+1+1+...+1 = \boxed{\mathrm{(D)}\ 2003}$

Solution 2

Using the sum of an arithmetic progression formula, we can write this as $\frac{2003}{2}(2 + 4006) - \frac{2003}{2}(1 + 4005) = \frac{2003}{2} \cdot 2 = \boxed{\mathrm{(D)}\ 2003}$.

Solution 3

The formula for the sum of the first n even numbers, is $S_E = n^2 + n$, (E standing for even).

Sum of first n odd numbers, is $S_O = n^2$, (O standing for odd).

Knowing this, plug 2003 for n,

$S_E - S_O = (2003^2 + 2003) - (2003^2) = 2003 \Rightarrow \boxed{\mathrm{(D)}\ 2003}$.

See also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by First Question	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A Problems/Problem 2

The following problem is from both the 2003 AMC 12A #2 and 2003 AMC 10A #2, so both problems redirect to this page.

Problem

Members of the Rockham Soccer League buy socks and T-shirts. Socks cost \$4 per pair and each T-shirt costs \$5 more than a pair of socks. Each member needs one pair of socks and a shirt for home games and another pair of socks and a shirt for away games. If the total cost is \$2366, how many members are in the League?

$\mathrm{(A) \ } 77\qquad \mathrm{(B) \ } 91\qquad \mathrm{(C) \ } 143\qquad \mathrm{(D) \ } 182\qquad \mathrm{(E) \ } 286$

Solution

Since T-shirts cost 5 dollars more than a pair of socks, T-shirts cost $5+4=9$ dollars.

Since each member needs 2 pairs of socks and 2 T-shirts, the total cost for 1 member is $2(4+9)=26$ dollars.

Since 2366 dollars was the cost for the club, and 26 was the cost per member, the number of members in the League is $2366\div 26=\boxed{\mathrm{(B)}\ 91}$.

See Also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2003 AMC 12A Problems/Problem 3

The following problem is from both the 2003 AMC 12A #3 and 2003 AMC 10A #3, so both problems redirect to this page.

Problem

A solid box is 15 cm by 10 cm by 8 cm. A new solid is formed by removing a cube 3 cm on a side from each corner of this box. What percent of the original volume is removed?

$\mathrm{(A)}\ 4.5\% \qquad \mathrm{(B)}\ 9\% \qquad \mathrm{(C)}\ 12\% \qquad \mathrm{(D)}\ 18\% \qquad \mathrm{(E)}\ 24\%$

Solution

The volume of the original box is $15\cdot10\cdot8=1200$

The volume of each cube that is removed is $3\cdot3\cdot3=27$

Since there are 8 corners on the box, 8 cubes are removed.

So the total volume removed is $8\cdot27=216$.

Therefore, the desired percentage is $\frac{216}{1200}\cdot100 = \boxed{\mathrm{(D)}\ 18\%}$

See Also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2003 AMC 12A Problems/Problem 4

The following problem is from both the 2003 AMC 12A #4 and 2003 AMC 10A #4, so both problems redirect to this page.

Contents

Problem

It takes Mary 30 minutes to walk uphill 1 km from her home to school, but it takes her only 10 minutes to walk from school to her home along the same route. What is her average speed, in km/hr, for the round trip?

$\mathrm{(A)}\ 3\qquad\mathrm{(B)}\ 3.125\qquad\mathrm{(C)}\ 3.5\qquad\mathrm{(D)}\ 4\qquad\mathrm{(E)}\ 4.5$

Solution

Solution 1

Since she walked 1 km to school and 1 km back home, her total distance is $1+1=2$ km.

Since she spent 30 minutes walking to school and 10 minutes walking back home, her total time is $30+10=40$ minutes = $\frac{40}{60}=\frac{2}{3}$ hours.

Therefore her average speed in km/hr is $\frac{2}{\frac{2}{3}}=\boxed{\mathrm{(A)}\ 3}$.

Solution 2

The average speed of two speeds that travel the same distance is the harmonic mean of the speeds, or $\frac{2}{\frac{1}{x}+\frac{1}{y}}=\frac{2xy}{x+y}$ (for speeds x and y). Mary's speed going to school is $2\,\mathrm{km/hr}$, and her speed coming back is $6\,\mathrm{km/hr}$. Plugging the numbers in, we get that the average speed is $\frac{2\times6\times2}{6+2}=\frac{24}{8}=\boxed{\mathrm{(A)}\ 3}$.

See Also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A Problems/Problem 5

The following problem is from both the 2003 AMC 12A #5 and 2003 AMC 10A #11, so both problems redirect to this page.

Problem

The sum of the two 5-digit numbers $AMC10$ and $AMC12$ is 123422. What is $A+M+C$?

$\mathrm{(A) \ } 10\qquad \mathrm{(B) \ } 11\qquad \mathrm{(C) \ } 12\qquad \mathrm{(D) \ } 13\qquad \mathrm{(E) \ } 14$

Solution

$AMC10+AMC12=123422$

$AMC00+AMC00=123400$

$AMC+AMC=1234$

$2\cdot AMC=1234$

$AMC=\frac{1234}{2}=617$

Since A, M, and C are digits, $A=6$, $M=1$, $C=7$.

Therefore, $A+M+C = 6+1+7 = \boxed{\mathrm{(E)}\ 14}$.

See Also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 6

The following problem is from both the 2003 AMC 12A #6 and 2003 AMC 10A #6, so both problems redirect to this page.

Problem

Define $x\heartsuit y$ to be $|x-y|$ for all real numbers x and y. Which of the following statements is not true?

$\mathrm{(A)}\ x\heartsuit y = y\heartsuit x$ for all x and y

$\mathrm{(B)}\ 2(x\heartsuit y) = (2x)\heartsuit(2y)$ for all x and y

$\mathrm{(C)}\ x\heartsuit 0 = x$ for all x

$\mathrm{(D)}\ x\heartsuit x = 0$ for all x

$\mathrm{(E)}\ x\heartsuit y > 0$ if $x \neq y$

Solution

Examining statement C:

$x\heartsuit 0 = |x-0| = |x|$

$|x| \neq x$ when $x<0$, but statement C says that it does for all x.

Therefore the statement that is not true is $\boxed{\mathrm{(C)}\ x\heartsuit 0 = x\ \text{for all}\ x}$

See Also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 7

The following problem is from both the 2003 AMC 12A #7 and 2003 AMC 10A #7, so both problems redirect to this page.

Problem

How many non-congruent triangles with perimeter 7 have integer side lengths?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution

By the triangle inequality, no side may have a length greater than the semiperimeter, which is $\frac{1}{2}\cdot7=3.5$.

Since all sides must be integers, the largest possible length of a side is 3. Therefore, all such triangles must have all sides of length 1, 2, or 3. Since $2+2+2=6<7$, at least one side must have a length of 3. Thus, the remaining two sides have a combined length of $7-3=4$. So, the remaining sides must be either 3 and 1 or 2 and 2. Therefore, the number of triangles is $\boxed{\textbf{(B)}\ 2}$.

See Also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2003 AMC 12A Problems/Problem 8

The following problem is from both the 2003 AMC 12A #8 and 2003 AMC 10A #8, so both problems redirect to this page.

Contents

Problem

What is the probability that a randomly drawn positive factor of 60 is less than 7?

$$\mathrm{(A)}\ \frac{1}{10}\qquad\mathrm{(B)}\ \frac{1}{6}\qquad\mathrm{(C)}\ \frac{1}{4}\qquad\mathrm{(D)}\ \frac{1}{3}\qquad\mathrm{(E)}\ \frac{1}{2}$$

Solution

Solution 1

For a positive number n which is not a perfect square, exactly half of the positive factors will be less than $\sqrt{n}$.

Since 60 is not a perfect square, half of the positive factors of 60 will be less than $\sqrt{60}\approx 7.746$.

Clearly, there are no positive factors of 60 between 7 and $\sqrt{60}$.

Therefore half of the positive factors will be less than 7.

So the answer is $\boxed{\mathrm{(E)}\ \frac{1}{2}}$.

Solution 2

Testing all numbers less than 7, numbers $1, 2, 3, 4, 5$, and 6 divide 60. The prime factorization of 60 is $2^2\cdot 3\cdot 5$. Using the formula for the number of divisors, the total number of divisors of 60 is $(3)(2)(2)=12$. Therefore, our desired probability is $\frac{6}{12}=\boxed{\mathrm{(E)}\ \frac{1}{2}}$

See Also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by Problem 7	Followed by Problem 9
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A Problems/Problem 9

Problem

A set S of points in the xy-plane is symmetric about the origin, both coordinate axes, and the line $y = x$. If $(2, 3)$ is in S, what is the smallest number of points in S?

$\mathrm{(A) \ } 1\qquad \mathrm{(B) \ } 2\qquad \mathrm{(C) \ } 4\qquad \mathrm{(D) \ } 8\qquad \mathrm{(E) \ } 16$

Solution

If $(2, 3)$ is in S, then $(3, 2)$ is also, and quickly we see that every point of the form $(\pm 2, \pm 3)$ or $(\pm 3, \pm 2)$ must be in S. Now note that these 8 points satisfy all of the symmetry conditions. Thus the answer is $\boxed{\mathrm{(D)}\ 8}$.

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2003_AMC_12A_Problems/Problem_9&oldid=66718"

Category: Introductory Algebra Problems

2003 AMC 12A Problems/Problem 10

Problem

Al, Bert, and Carl are the winners of a school drawing for a pile of Halloween candy, which they are to divide in a ratio of $3:2:1$, respectively. Due to some confusion they come at different times to claim their prizes, and each assumes he is the first to arrive. If each takes what he believes to be the correct share of candy, what fraction of the candy goes unclaimed?

$\mathrm{(A) \ } \frac{1}{18}\qquad \mathrm{(B) \ } \frac{1}{6}\qquad \mathrm{(C) \ } \frac{2}{9}\qquad \mathrm{(D) \ } \frac{5}{18}\qquad \mathrm{(E) \ } \frac{5}{12}$

Solution

Because the ratios are $3:2:1$, Al, Bert, and Carl believe that they need to take $1/2$, $1/3$, and $1/6$ of the pile when they each arrive, respectively. After each person comes, $1/2$, $2/3$, and $5/6$ of the pile's size (just before each came) remains. The pile starts at 1, and at the end $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{5}{6} \cdot 1 = \frac{5}{18}$ of the original pile goes unclaimed. (Note that because of the properties of multiplication, it does not matter what order the three come in.) Hence the answer is $\boxed{\mathrm{(D)}\ \frac{5}{18}}$.

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2003 AMC 12A Problems/Problem 11

Problem 11

A square and an equilateral triangle have the same perimeter. Let A be the area of the circle circumscribed about the square and B the area of the circle circumscribed around the triangle. Find A/B.

$$\text{(A) } \frac{9}{16} \qquad \text{(B) } \frac{3}{4} \qquad \text{(C) } \frac{27}{32} \qquad \text{(D) } \frac{3\sqrt{6}}{8} \qquad \text{(E) } 1$$

Solution

Suppose that the common perimeter is P. Then, the side lengths of the square and triangle, respectively, are $\frac{P}{4}$ and $\frac{P}{3}$ The circle circumscribed about the square has a diameter equal to the diagonal of the square, which is $\frac{P\sqrt{2}}{4}$ Therefore, the radius is $\frac{P\sqrt{2}}{8}$ and the area of the circle is

$$\pi \cdot \left(\frac{P\sqrt{2}}{8}\right)^2 = \pi \cdot \frac{2P^2}{64} = \boxed{\frac{P^2\pi}{32} = A}$$

Now consider the circle circumscribed around the equilateral triangle. Due to symmetry, the circle must share a center with the equilateral triangle. The radius of the circle is simply the distance from the center of the triangle to a vertex. This distance is $\frac{2}{3}$ of an altitude. By $30-60-90$ right triangle properties, the altitude is $\frac{\sqrt{3}}{2} \cdot s$ where s is the side. So, the radius is $\frac{2}{3} \cdot \frac{\sqrt{3}}{2} \cdot \frac{P}{3} = \frac{P\sqrt{3}}{9}$ The area of the circle is $\pi \cdot \left(\frac{P\sqrt{3}}{9}\right)^2 = \pi \cdot \frac{3P^2}{81} = \boxed{\frac{P^2\pi}{27} = B}$ So,

$$\frac{A}{B} = \frac{\frac{P^2\pi}{32}}{\frac{P^2\pi}{27}} = \frac{P^2\pi}{32} \cdot \frac{27}{P^2\pi} = \boxed{\frac{27}{32} \implies \text{(C) } \frac{27}{32}}$$

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 12

The following problem is from both the 2003 AMC 12A #12 and 2003 AMC 10A #24, so both problems redirect to this page.

Problem

Sally has five red cards numbered 1 through 5 and four blue cards numbered 3 through 6. She stacks the cards so that the colors alternate and so that the number on each red card divides evenly into the number on each neighboring blue card. What is the sum of the numbers on the middle three cards?

$\mathrm{(A) \ } 8\qquad \mathrm{(B) \ } 9\qquad \mathrm{(C) \ } 10\qquad \mathrm{(D) \ } 11\qquad \mathrm{(E) \ } 12$

Solution

Let R_i and B_j designate the red card numbered i and the blue card numbered j, respectively.

B_5 is the only blue card that R_5 evenly divides, so R_5 must be at one end of the stack and B_5 must be the card next to it.

R_1 is the only other red card that evenly divides B_5, so R_1 must be the other card next to B_5.

B_4 is the only blue card that R_4 evenly divides, so R_4 must be at one end of the stack and B_4 must be the card next to it.

R_2 is the only other red card that evenly divides B_4, so R_2 must be the other card next to B_4.

R_2 doesn't evenly divide B_3, so B_3 must be next to R_1, B_6 must be next to R_2, and R_3 must be in the middle.

This yields the following arrangement from top to bottom: $\{R_5,B_5,R_1,B_3,R_3,B_6,R_2,B_4,R_4\}$

Therefore, the sum of the numbers on the middle three cards is $3+3+6=\boxed{\mathrm{(E)}\ 12}$.

See Also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by Problem 23	Followed by Problem 25
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

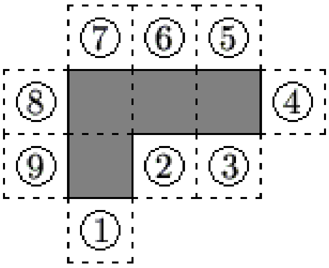
2003 AMC 12A Problems/Problem 13

The following problem is from both the 2003 AMC 12A #13 and 2003 AMC 10A #10, so both problems redirect to this page.

Contents

Problem

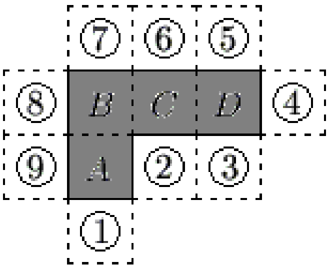
The polygon enclosed by the solid lines in the figure consists of 4 congruent squares joined edge-to-edge. One more congruent square is attached to an edge at one of the nine positions indicated. How many of the nine resulting polygons can be folded to form a cube with one face missing?



$\mathrm{(A)}\ 2\qquad\mathrm{(B)}\ 3\qquad\mathrm{(C)}\ 4\qquad\mathrm{(D)}\ 5\qquad\mathrm{(E)}\ 6$

Solution

Solution 1



Let the squares be labeled A, B, C, and D.

When the polygon is folded, the "right" edge of square A becomes adjacent to the "bottom edge" of square C, and the "bottom" edge of square A becomes adjacent to the "bottom" edge of square D.

So, any "new" square that is attatched to those edges will prevent the polygon from becoming a cube with one face missing.

Therefore, squares 1, 2, and 3 will prevent the polygon from becoming a cube with one face missing.

Squares 4, 5, 6, 7, 8, and 9 will allow the polygon to become a cube with one face missing when folded.

Thus the answer is $\boxed{\mathrm{(E)}\ 6}$.

Solution 2

Another way to think of it is that a cube missing one face has 5 of its 6 faces. Since the shape has 4 faces already, we need another face. The only way to add another face is if the added square does not overlap any of the others. $1, 2,$ and 3 overlap, while squares 4 to 9 do not. The answer is $\boxed{\mathrm{(E)}\ 6}$

See Also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 12	Followed by Problem 14
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

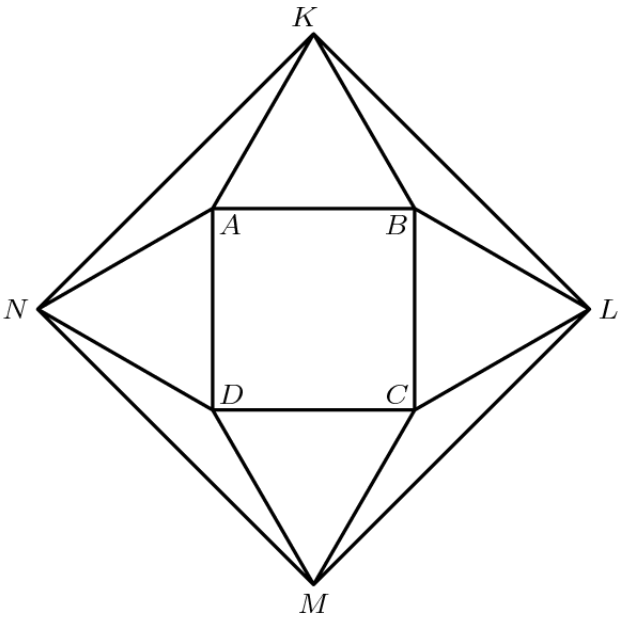
Category: Introductory Geometry Problems

2003 AMC 12A Problems/Problem 14

Contents

Problem

Points $K, L, M,$ and N lie in the plane of the square $ABCD$ such that AKB, BLC, CMD, and DNA are equilateral triangles. If $ABCD$ has an area of 16, find the area of $KLMN$.



$\mathrm{(A) \ } 32 \qquad \mathrm{(B) \ } 16+16\sqrt{3} \qquad \mathrm{(C) \ } 48 \qquad \mathrm{(D) \ } 32+16\sqrt{3} \qquad \mathrm{(E) \ } 64$

Solution

Solution 1

Since the area of square ABCD is 16, the side length must be 4. Thus, the side length of triangle AKB is 4, and the height of AKB, and thus DMC, is $2\sqrt{3}$.

The diagonal of the square KNMC will then be $4 + 4\sqrt{3}$. From here there are 2 ways to proceed:

First: Since the diagonal is $4 + 4\sqrt{3}$, the side length is $\frac{4+4\sqrt{3}}{\sqrt{2}}$, and the area is thus $\frac{16+48+32\sqrt{3}}{2} = \boxed{\mathrm{(D)}\ 32+16\sqrt{3}}$.

Solution 2

Since a square is a rhombus, the area of the square is $\frac{d_1 d_2}{2}$, where d_1 and d_2 are the diagonals of the rhombus. Since the diagonal is $4+4\sqrt{3}$, the area is $\frac{(4+4\sqrt{3})^2}{2} = \boxed{\mathrm{(D)}\ 32+16\sqrt{3}}$.

See Also

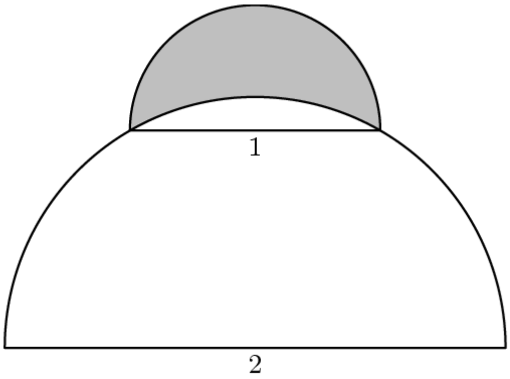
2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 15

The following problem is from both the 2003 AMC 12A #15 and 2003 AMC 10A #19, so both problems redirect to this page.

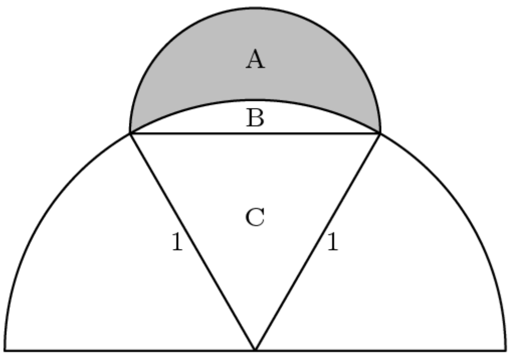
Problem

A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a lune. Determine the area of this lune.



$\mathrm{(A)}\ \frac{1}{6}\pi-\frac{\sqrt{3}}{4} \qquad \mathrm{(B)}\ \frac{\sqrt{3}}{4}-\frac{1}{12}\pi \qquad \mathrm{(C)}\ \frac{\sqrt{3}}{4}-\frac{1}{24}\pi \qquad \mathrm{(D)}\ \frac{\sqrt{3}}{4}+\frac{1}{24}\pi \qquad \mathrm{(E)}\ \frac{\sqrt{3}}{4}+\frac{1}{12}\pi$

Solution



Let $[X]$ denote the area of region X in the figure above.

The shaded area $[A]$ is equal to the area of the smaller semicircle $[A+B]$ minus the area of a sector of the larger circle $[B+C]$ plus the area of a triangle formed by two radii of the larger semicircle and the diameter of the smaller semicircle $[C]$.

The area of the smaller semicircle is $[A+B] = \frac{1}{2}\pi\cdot(\frac{1}{2})^{2} = \frac{1}{8}\pi$.

Since the radius of the larger semicircle is equal to the diameter of the smaller semicircle, the triangle is an equilateral triangle and the sector measures 60°.

The area of the 60° sector of the larger semicircle is $[B+C] = \frac{60}{360}\pi\cdot(\frac{2}{2})^{2} = \frac{1}{6}\pi$.

The area of the triangle is $[C] = \frac{1^{2}\sqrt{3}}{4} = \frac{\sqrt{3}}{4}$.

So the shaded area is

$$[A] = [A + B] - [B + C] + [C] = \left(\frac{1}{8}\pi\right) - \left(\frac{1}{6}\pi\right) + \left(\frac{\sqrt{3}}{4}\right) = \boxed{(\text{C})\ \frac{\sqrt{3}}{4} - \frac{1}{24}\pi}.$$

See Also

2003 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2003))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Categories: Introductory Geometry Problems | Area Problems

2003 AMC 12A Problems/Problem 16

Contents

Problem

A point P is chosen at random in the interior of equilateral triangle ABC. What is the probability that $\triangle ABP$ has a greater area than each of $\triangle ACP$ and $\triangle BCP$?

$\textbf{(A)}\ \frac{1}{6} \qquad \textbf{(B)}\ \frac{1}{4} \qquad \textbf{(C)}\ \frac{1}{3} \qquad \textbf{(D)}\ \frac{1}{2} \qquad \textbf{(E)}\ \frac{2}{3}$

Solution

Solution 1

After we pick point P, we realize that ABC is symmetric for this purpose, and so the probability that ACP is the greatest area, or ABP or BCP, are all the same. Since they add to 1, the probability that ACP has the greatest area is $\boxed{\mathrm{(C)}\ \frac{1}{3}}$

Solution 2

We will use geometric probability. Let us take point P, and draw the perpendiculars to BC, CA, and AB, and call the feet of these perpendiculars D, E, and F respectively. The area of triangle ACP is simply $\frac{1}{2} * AC * PF$. Similarly we can find the area of triangles BCP and ABP. If we add these up and realize that it equals the area of the entire triangle, we see that no matter where we choose P, PD + PE + PF = the height of the triangle. Setting the area of triangle ABP greater than ACP and BCP, we want PF to be the largest of PF, PD, and PE. We then realize that PF = PD = PE when P is the incenter of ABC. Let us call the incenter of the triangle Q. If we want PF to be the largest of the three, by testing points we realize that P must be in the interior of quadrilateral QDCE. So our probability (using geometric probability) is the area of QDCE divided by the area of ABC. We will now show that the three quadrilaterals, QDCE, QEAF, and QFBD are congruent. As the definition of point Q yields, QF = QD = QE. Since ABC is equilateral, Q is also the circumcenter of ABC, so QA = QB = QC. By the Pythagorean Theorem, BD = DC = CE = EA = AF = FB. Also, angles BDQ, BFQ, CEQ, CDQ, AFQ, and AEQ are all equal to 90 degrees. Angles DBF, FAE, ECD are all equal to 60 degrees, so it is now clear that QDCE, QEAF, QFBD are all congruent. Summing up these areas gives us the area of ABC. QDCE contributes to a third of that area so $\frac{[QDCE]}{[ABC]} = \frac{1}{3}$ (C).

See Also

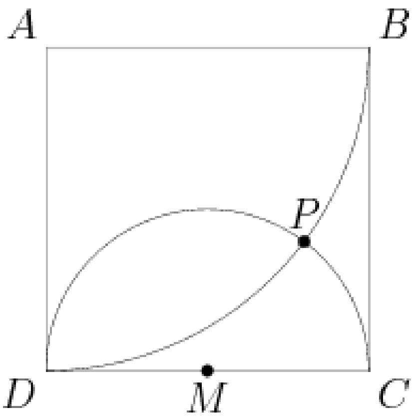
2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 15	Followed by Problem 17
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 17

Contents

Problem

Square $ABCD$ has sides of length 4, and M is the midpoint of $\overline{CD}$. A circle with radius 2 and center M intersects a circle with radius 4 and center A at points P and D. What is the distance from P to $\overline{AD}$?



$\mathrm{(A) \ } 3 \qquad \mathrm{(B) \ } \frac{16}{5} \qquad \mathrm{(C) \ } \frac{13}{4} \qquad \mathrm{(D) \ } 2\sqrt{3} \qquad \mathrm{(E) \ } \frac{7}{2}$

Solution 1

Let D be the origin. A is the point $(0,4)$ and M is the point $(2,0)$. We are given the radius of the quarter circle and semicircle as 4 and 2, respectively, so their equations, respectively, are:

$$x^2 + (y-4)^2 = 4^2$$

$$(x-2)^2 + y^2 = 2^2$$

Subtract the second equation from the first:

$$x^2 + (y - 4)^2 - (x - 2)^2 - y^2 = 12$$

$$4x - 8y + 12 = 12$$

$$x = 2y.$$

Then substitute:

$$(2y)^2 + (y-4)^2 = 16$$

$$4y^2 + y^2 - 8y + 16 = 16$$

$$5y^2 - 8y = 0$$

$$y(5y - 8) = 0.$$

Thus $y = 0$ and $y = \frac{8}{5}$ making $x = 0$ and $x = \frac{16}{5}$.

The first value of 0 is obviously referring to the x-coordinate of the point where the circles intersect at the origin, D, so the second value must be referring to the x coordinate of P. Since $\overline{AD}$ is the y-axis, the distance to it from P is the same as the x-value of the coordinate of P, so the distance from P to $\overline{AD}$ is $\frac{16}{5} \Rightarrow B$

Solution 2

Note that P is merely a reflection of D over AM. Call the intersection of AM and DP X. Drop perpendiculars from X and P to AD, and denote their respective points of intersection by J and K. We then have $\triangle DXJ \sim \triangle DPK$, with a scale factor of 2. Thus, we can find XJ and double it to get our answer. With some analytical geometry, we find that $XJ = \frac{8}{5}$, implying that $PK = \frac{16}{5}$.

Solution 3

As in Solution 2, draw in DP and AM and denote their intersection point X. Next, drop a perpendicular from P to AD and denote the foot as Z. $AP \cong AD$ as they are both radii and similarly $DM \cong MP$ so $APMD$ is a kite and $DX \perp XM$ by a well-known theorem.

Pythagorean theorem gives us $AM = 2\sqrt{5}$. Clearly $\triangle XMD \sim \triangle XDA \sim \triangle DMA \sim \triangle ZDP$ by angle-angle and $\triangle XMD \cong \triangle XMP$ by Hypotenuse Leg. Manipulating similar triangles gives us $PZ = \frac{16}{5}$

Solution 4

Using the double-angle formula for sine, what we need to find is

$$AP \cdot \sin(DAP) = AP \cdot 2\sin(DAM)\cos(DAM) = 4 \cdot 2 \cdot \frac{2}{\sqrt{20}} \cdot \frac{4}{\sqrt{20}} = \frac{16}{5}.$$

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 18

Contents

Problem

Let n be a 5-digit number, and let q and r be the quotient and the remainder, respectively, when n is divided by 100. For how many values of n is $q+r$ divisible by 11?

$\mathrm{(A) \ } 8180\qquad \mathrm{(B) \ } 8181\qquad \mathrm{(C) \ } 8182\qquad \mathrm{(D) \ } 9000\qquad \mathrm{(E) \ } 9090$

Solution 1

When a 5-digit number is divided by 100, the first 3 digits become the quotient, q, and the last 2 digits become the remainder, r.

Therefore, q can be any integer from 100 to 999 inclusive, and r can be any integer from 0 to 99 inclusive.

For each of the $9 \cdot 10 \cdot 10=900$ possible values of q, there are at least $\lfloor \frac{100}{11} \rfloor = 9$ possible values of r such that $q+r \equiv 0\pmod{11}$.

Since there is 1 "extra" possible value of r that is congruent to $0\pmod{11}$, each of the $\lfloor \frac{900}{11} \rfloor = 81$ values of q that are congruent to $0\pmod{11}$ have 1 more possible value of r such that $q+r \equiv 0\pmod{11}$.

Therefore, the number of possible values of n such that $q+r \equiv 0\pmod{11}$ is $900\cdot9+81\cdot1=8181 \Rightarrow \boxed{(B)}$.

Solution 2

Notice that $q+r \equiv 0 \pmod{11} \Rightarrow 100q+r \equiv 0 \pmod{11}$. This means that any number whose quotient and remainder sum is divisible by 11 must also be divisible by 11. Therefore, there are $\frac{99990-10010}{11}+1=8181$ possible values. The answer is $\boxed{(B)}$.

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 19

Problem

A parabola with equation $y = ax^2 + bx + c$ is reflected about the x-axis. The parabola and its reflection are translated horizontally five units in opposite directions to become the graphs of $y = f(x)$ and $y = g(x)$, respectively. Which of the following describes the graph of $y = (f + g)(x)$?

$\textbf{(A)}$ a parabola tangent to the x-axis
$\textbf{(B)}$ a parabola not tangent to the x-axis $\qquad \textbf{(C)}$ a horizontal line
$\textbf{(D)}$ a non-horizontal line $\qquad \textbf{(E)}$ the graph of a cubic function

Solution

If we take the parabola $ax^2 + bx + c$ and reflect it over the x - axis, we have the parabola $-ax^2 - bx - c$. Without loss of generality, let us say that the parabola is translated 5 units to the left, and the reflection to the right. Then:

$$f(x) = a(x+5)^2 + b(x+5) + c = ax^2 + (10a+b)x + 25a + 5b + c$$
$$g(x) = -a(x-5)^2 - b(x-5) - c = -ax^2 + 10ax - bx - 25a + 5b - c$$

Adding them up produces:

$$(f+g)(x) = ax^2 + (10a+b)x + 25a + 5b + c - ax^2 + 10ax - bx - 25a + 5b - c = 20ax + 10b$$

This is a line with slope $20a$. Since a cannot be 0 (because $ax^2 + bx + c$ would be a line) we end up with $\boxed{\textbf{(D)}\text{ a non-horizontal line}}$

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 20

Problem

How many 15-letter arrangements of 5 A's, 5 B's, and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters, and no C's in the last 5 letters?

$$\text{(A)}\ \sum_{k=0}^{5}\binom{5}{k}^{3}\qquad\text{(B)}\ 3^5\cdot 2^5\qquad\text{(C)}\ 2^{15}\qquad\text{(D)}\ \frac{15!}{(5!)^3}\qquad\text{(E)}\ 3^{15}$$

Solution

The answer is $\boxed{\text{(A)}}$.

Note that the first five letters must be B's or C's, the next five letters must be C's or A's, and the last five letters must be A's or B's. If there are k B's in the first five letters, then there must be $5-k$ C's in the first five letters, so there must be k C's and $5-k$ A's in the next five letters, and k A's and $5-k$ B's in the last five letters. Therefore the number of each letter in each group of five is determined completely by the number of B's in the first 5 letters, and the number of ways to arrange these 15 letters with this restriction is $\binom{5}{k}^3$ (since there are $\binom{5}{k}$ ways to arrange k B's and $5-k$ C's).

Therefore the answer is $\sum_{k=0}^{5}\binom{5}{k}^{3}$.

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2003_AMC_12A_Problems/Problem_20&oldid=53940"

Category: Introductory Combinatorics Problems

2003 AMC 12A Problems/Problem 21

Problem

The graph of the polynomial

$$P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

has five distinct x-intercepts, one of which is at $(0,0)$. Which of the following coefficients cannot be zero?

$\text{(A)}\ a \qquad \text{(B)}\ b \qquad \text{(C)}\ c \qquad \text{(D)}\ d \qquad \text{(E)}\ e$

Contents

Solution

Solution 1

Let the roots be $r_1 = 0, r_2, r_3, r_4, r_5$. According to Vieta's formulas, we have $d = r_1r_2r_3r_4 + r_1r_2r_3r_5 + r_1r_2r_4r_5 + r_1r_3r_4r_5 + r_2r_3r_4r_5$. The first four terms contain $r_1 = 0$ and are therefore zero, thus $d = r_2r_3r_4r_5$. This is a product of four non-zero numbers, therefore d must be non-zero $\Longrightarrow \mathrm{(D)}$.

Solution 2

Clearly, since $(0,0)$ is an intercept, e must be 0. But if d was 0, x^2 would divide the polynomial, which means it would have a double root at 0, which is impossible, since all five roots are distinct.

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 20	Followed by Problem 22
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 22

Problem

Objects A and B move simultaneously in the coordinate plane via a sequence of steps, each of length one. Object A starts at $(0,0)$ and each of its steps is either right or up, both equally likely. Object B starts at $(5,7)$ and each of its steps is either to the left or down, both equally likely. Which of the following is closest to the probability that the objects meet?

$\mathrm{(A)}\ 0.10 \qquad \mathrm{(B)}\ 0.15 \qquad \mathrm{(C)}\ 0.20 \qquad \mathrm{(D)}\ 0.25 \qquad \mathrm{(E)}\ 0.30$

Solution

If A and B meet, their paths connect $(0,0)$ and $(5,7)$. There are $\binom{12}{5}=792$ such paths. Since the path is 12 units long, they must meet after each travels 6 units, so the probability is $\frac{792}{2^{6}\cdot 2^{6}} \approx 0.20 \Rightarrow \boxed{C}$

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 21	Followed by Problem 23
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 23

Problem

How many perfect squares are divisors of the product $1! \cdot 2! \cdot 3! \cdot \ldots \cdot 9!$?

$\textbf{(A)}\ 504 \qquad \textbf{(B)}\ 672 \qquad \textbf{(C)}\ 864 \qquad \textbf{(D)}\ 936 \qquad \textbf{(E)}\ 1008$

Solution

We want to find the number of perfect square factors in the product of all the factorials of numbers from $1-9$. We can write this out and take out the factorials, and then find a prime factorization of the entire product. We can also find this prime factorization by finding the number of times each factor is repeated in each factorial. This comes out to be equal to $2^{30} * 3^{13} * 5^5 * 7^3$. To find the amount of perfect square factors, we realize that each exponent in the prime factorization must be even: $2^{15} * 3^6 * 5^2 * 7^1$. To find the total number of possibilities, we add 1 to each exponent and multiply them all together. This gives us $16 * 7 * 3 * 2 = 672 \Rightarrow \boxed{\text{(B)}}$.

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Intermediate Algebra Problems

2003 AMC 12A Problems/Problem 24

Problem

If $a \geq b > 1$, what is the largest possible value of $\log_a(a/b) + \log_b(b/a)$?

$\text{(A)}\ -2 \qquad \text{(B)}\ 0 \qquad \text{(C)}\ 2 \qquad \text{(D)}\ 3 \qquad \text{(E)}\ 4$

Solution

Using logarithmic rules, we see that

$$\log_a a - \log_a b + \log_b b - \log_b a = 2 - (\log_a b + \log_b a)$$

$$= 2 - (\log_a b + \frac{1}{\log_a b})$$

Since a and b are both positive, using AM-GM gives that the term in parentheses must be at least 2, so the largest possible values is $2 - 2 = \boxed{0}$.

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 23	Followed by [1] (http://www.artofproblemsolving.com/Wiki/index.php/2003_AMC_12A_Problems/Problem_25)
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2003 AMC 12A Problems/Problem 25

Problem

Let $f(x) = \sqrt{ax^2 + bx}$. For how many real values of a is there at least one positive value of b for which the domain of f and the range of f are the same set?

$\mathrm{(A) \ 0 } \qquad \mathrm{(B) \ 1 } \qquad \mathrm{(C) \ 2 } \qquad \mathrm{(D) \ 3 } \qquad \mathrm{(E) \ \text{infinitely many} }$

Solution

The function $f(x) = \sqrt{x(ax+b)}$ has a codomain of all non-negative numbers, or $0 \le f(x)$. Since the domain and the range of f are the same, it follows that the domain of f also satisfies $0 \le x$.

The function has two zeroes at $x = 0, \frac{-b}{a}$, which must be part of the domain. Since the domain and the range are the same set, it follows that $\frac{-b}{a}$ is in the codomain of f, or $0 \le \frac{-b}{a}$. This implies that one (but not both) of a, b is non-positive. If a is positive, then $\lim_{x \rightarrow -\infty} ax^2 + bx \ge 0$, which implies that a negative number falls in the domain of $f(x)$, contradiction. Thus a must be non-positive, b is non-negative, and the domain of the function occurs when $x(ax+b) > 0$, or

$$0 \le x \le \frac{-b}{a}.$$

Completing the square, $f(x) = \sqrt{a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a}} \le \sqrt{\frac{-b^2}{4a}}$ by the Trivial Inequality (remember that $a \le 0$). Since f is continuous and assumes this maximal value at $x = \frac{-b}{2a}$, it follows that the range of f is

$$0 \le f(x) \le \sqrt{\frac{-b^2}{4a}}.$$

As the domain and the range are the same, we have that $\frac{-b}{a} = \sqrt{\frac{-b^2}{4a}} = \frac{b}{2\sqrt{-a}} \Longrightarrow a(a+4) = 0$ (we can divide through by b since it is given that b is positive). Hence $a = 0, -4$, which both we can verify work, and the answer is $\mathrm{(C)}$.

See Also

2003 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2003))	
Preceded by Problem 24	Followed by Last question
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	