1. What is the value of

$$2^{1+2+3} - \left(2^1 + 2^2 + 2^3\right)?$$

A. 0 B. 50 C. 52 D. 54 E. 57

Answer: B

Solution: We evaluate the given expression to get that

$$2^{1+2+3} - \left(2^1 + 2^2 + 2^3\right) = 2^6 - \left(2^1 + 2^2 + 2^3\right) = 64 - 2 - 4 - 8 = 50 \Longrightarrow [B]$$

2. Under what conditions is true, where a and b are real numbers?

$$\sqrt{a^2 + b^2} = a + b$$

A. It is never true

B. It is true if and only if $ab = 0$

C. It is true if and only if $a + b \leq 0$

D. It is true if and only if $ab = 0$ and $a + b \leq 0$

E. It is always true

Answer: D

Solution: Square both sides to get $a^2 + b^2 = a^2 + 2ab + b^2$. Then, $0 = 2ab \rightarrow ab = 0$. Also, it is clear that both sides of the equation must be nonnegative. The answer is (**D**)

3. The sum of two natural numbers is 17,402. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

A. 10,272 B. 11,700 C. 13,362 D. 14,238 E. 15,426

Answer: D

Solution: Since the ones place of a multiple of 10 is 0 , this implies the other integer has to end with a 2 since both integers sum up to a number that ends with a 2 . Thus, the ones place of the difference has to be $10 - 2 = 8$, and the only answer choice that ends with an 8 is $[D]14238$

4. Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that
I. all of his happy snakes can add
II. none of his purple snakes can subtract, and
III. all of his snakes that can’ t subtract also can’ t add.
Which of these conclusions can be drawn about Tom’ s snakes?

A. Purple snakes can add.

B. Purple snakes are happy.

C. Snakes that can add are purple.

D. Happy snakes are not purple.

E. Happy snakes can’ t subtract.

Answer: D

Solution: We know that purple snakes cannot subtract, thus they cannot add either. Since happy snakes must be able to add, the purple snakes cannot be happy. Therefore, we know that the happy snakes are not purple and the answer is $(\mathbf{D})$

5. When a student multiplied the number 66 by the repeating decimal,
 1. $ababab\cdots = 1 \cdot ab$
 Where a and b are digits, he did not notice the notation and just multiplied 66 times
 $1.ab$.
 Later he found that his answer is 0.5 less than the correct answer. What is the 2-digit integer ab?

A. 15 B. 30 C. 45 D. 60 E. 75

Answer: E

Solution: It is known that $0 \cdot \overline{ab} = \frac{ab}{99}$ and $0.ab = \frac{ab}{100}$. Let $\overline{ab} = x$. We have that $66\left(1 + \frac{x}{100}\right) + 0.5 = 66\left(1 + \frac{x}{99}\right)$. Solving gives that $100x - 75 = 99x$ so $x = (\mathrm{E})75$

6. A deck of cards has only red cards and black cards. The probability of a randomly chosen card being red is 1/3. When 4 black cards are added to the deck, the probability of choosing red becomes 1/4. How many cards were in the deck originally?

A. 6 B. 9 C. 12 D. 15 E. 18

Answer: C

Solution: If the probability of choosing a red card is $\frac{1}{3}$, the red and black cards are in ratio 1: 2 . This means at the beginning there are x red cards and $2x$ black cards. After 4 black cards are added, there are $2x + 4$ black cards. This time, the probability of choosing a red card is $\frac{1}{4}$ so the ratio of red to black cards is 1: 3 . This means in the new deck the number of black cards is also $3x$ for the same x red cards. So, $3x = 2x + 4$ and $x = 4$ meaning there are 4 red cards in the deck at the start and $2(4) = 8$ black cards. So the answer is $8 + 4 = 12 = (\mathbf{C})$

7. What is the least possible value of

$$(xy - 1)^2 + (x + y)^2$$

for real numbers x and y?

A. 0 B. $\frac{1}{4}$ C. $\frac{1}{2}$ D. 1 E. 2

Answer: D

Solution: Expanding, we get that the expression is $x^2+2xy+y^2+x^2y^2-2xy+1$ or $x^2+y^2+x^2y^2+1$. By the trivial inequality(all squares are nonnegative) the minimum value for this is (D)1, which can be achieved at $x=y=0$.

8. A sequence of numbers is defined by $D_0=0, D_1=0, D_2=1$ and $D_n=D_{n-1}+D_{n-3}$ for $n\geq 3$. What are the parities (evenness or oddness) of the triple of numbers $(D_{2021}, D_{2022}, D_{2013})$ where E denotes even and O denotes odd.

A. (O, E, O) B. (E, E, O) C. (E, O, E) D. (O, O, E) E. (O, O, O)

Answer: C

Solution: Making a small chart, we have

D0	D1	D2	D3	D4	D5	D6	D7	D8	D9
0	0	1	1	1	2	3	4	6	9
E	E	O	O	O	E	O	E	E	O

This starts repeating every 7 terms, so $D_{2021}=D_5=E, D_{2022}=D_6=O$, and $D_{2023}=D_7=E$. Thus, the answer is $(\mathbf{C})(E,O,E)$

9. Which of the following is equivalent to

$$(2+3)\left(2^2+3^2\right)\left(2^4+3^4\right)\left(2^8+3^8\right)\left(2^{16}+3^{16}\right)\left(2^{32}+3^{32}\right)\left(2^{64}+3^{64}\right)$$

A. $3^{127}+2^{127}$

B. $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$

C. $3^{128}-2^{128}$

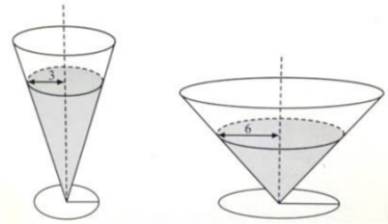
D. $3^{128}+2^{128}$

E. 5^{127}

Answer: C

Solution: All you need to do is multiply the entire equation by $(3-2)$. Then all the terms will easily simplify by difference of squares and you will get $3^{128}-2^{128}$ or $[C]$ as your final answer. Notice you don't need to worry about $3-2$ because that's equal to 1.

10. Two right circular cones with vertices facing down as shown in the fgure below contain the same amount of liquid. The radii of the top of the liquid surfaces are 3cm and 6cm. Into each cone is dropped a spherical marble of radius 1cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the liquid level in the wide cone?



A. 1:1 B. 47:43 C. 2:1 D. 40:13 E. 4:1

Answer: E

Solution: The heights of the cones are not given, so suppose the heights are very large (i.e. tending towards infinity) in order to approximate the cones as cylinders with base radii 3 and 6 and infinitely large height. Then the base area of the wide cylinder is 4 times that of the narrow cylinder. Since we are dropping a ball of the same volume into each cylinder, the water level in the narrow cone/cylinder should rise (E) 4 times as much.

11. A laser is placed at the point (3,5) . The laser beam travels in a straight line. Larry wants the beam to hit and bounce off the y -axis, then hit and bounce off the x -axis, the hit the point (7,5) . What is the total distance the beam will travel along this path?

A. $2\sqrt{10}$ B. $5\sqrt{2}$ C. $10\sqrt{2}$ D. $15\sqrt{2}$ E. $10\sqrt{5}$

Answer: C

Solution: Every time the laser bounces off a wall, instead we can imagine it going straight by reflecting it about the wall. Thus, the laser starts at (3,5) and ends at (-7,-5) , so the path's length is $\sqrt{10^2+10^2} =$ (C) $10\sqrt{2}$.

12. All the roots of polynomial

$$z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$$

are positive integers, possibly repeated. What is the value of B?

A. -88 B. -80 C. -64 D. -41 E. -40

Answer: A

Solution: By Vieta's formulae, the sum of the 6 roots is 10 and the product of the 6 roots is 16. By inspection, we see the roots are $1, 1, 2, 2, 2,$ and 2, so the function is $(z-1)^2(z-2)^4 = \left(z^2-2z+1\right)\left(z^4-8z^3+24z^2-32z+16\right)$. Therefore, $B = -32 - 48 - 8 = (A) - 88$

13. Of the following complex number z, which one has the property that z^5 has the greatest real part?

A. -2 B. $-\sqrt{3}+i$ C. $-\sqrt{2}+\sqrt{2}i$ D. $-1+\sqrt{3}i$ E. $2i$

Answer: B

Solution: First, $(B) = 2\,\text{cis}(150), (C) = 2\text{cis}(135), (D) = 2\text{cis}(120)$ Taking the real part of the 5 th power of each we have:

$(A) : (-2)^5 = -32$

$(B) : 32\cos(650) = 32\cos(30) = 16\sqrt{3}$

$(C) : 32\cos(675) = 32\cos(-45) = 16\sqrt{2}$

$(D) : 32\cos(600) = 32\cos(240)$ which is negative

$(\mathbf{E}) : (2i)^5$ which is zero

Thus, the answer is $[B]$

14. What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

A. 21 B. $100\log_5 3$ C. $200\log_3 5$ D. 2,200 E. 21,000

Answer: E

Solution: This equals

$$\left(\sum_{k=1}^{20} k\log_5(3)\right)\left(\sum_{k=1}^{100} \log_9(25)\right) = \frac{20 \cdot 21}{2} \cdot \log_5(3) \cdot 100\log_3(5) = \text{ (E) } 21000$$

15. A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the numbers of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of groups that could be selected. What is the remainder when N is divided by 100?

A. 47 B. 48 C. 83 D. 95 E. 96

Answer: D

Solution: We know the choose function and we know the pair multiplication MN so we do the multiplications and additions. $\binom{6}{0}\left(\binom{8}{4}+\binom{8}{8}\right)+\binom{6}{1}\left(\binom{8}{1}+\binom{8}{5}\right)+\binom{6}{2}\left(\binom{8}{2}+\binom{8}{6}\right)+\binom{6}{3}\left(\binom{8}{3}+\binom{8}{7}\right)+\binom{6}{4}\left(\binom{8}{0}+\binom{8}{4}+\binom{8}{8}\right)+\binom{6}{5}\left(\binom{8}{1}+\binom{8}{5}\right)+\binom{6}{6}\left(\binom{8}{2}+\binom{8}{6}\right) = 4095 \equiv [(D)95] \pmod{100}$

16. In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

A. 100.5 B. 134 C. 142 D. 150.5 E. 167

Answer: C

Solution: The x th number of this sequence is $\left[\frac{-1\pm\sqrt{1+8x}}{2}\right]$ via the quadratic formula. We can see that if we halve x we end up getting $\left[\frac{-1\pm\sqrt{1+4x}}{2}\right]$. This is approximately the number divided by $\sqrt{2}\cdot\frac{200}{\sqrt{2}} = 141.4$ and since 142 looks like the only number close to it, it is answer $(C)142$

17. Trapezoid $ABCD$ has $AB||CD, BC = CD = 43$, and $AD \perp BD$. Let O be the intersection of the diagonals AC and BD, and let P be the midpoint of BD Given that $OP = 11$, the length AD can be written in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. What is $m+n$?

A. 65 B. 132 C. 157 D. 194 E. 215

Answer: D

Solution: Angle chasing reveals that $\triangle BPC \sim \triangle BDA$, therefore

$2 = \frac{BD}{BP} = \frac{AB}{BC} = \frac{AB}{43}$

$AB = 86$

Additional angle chasing shows that $\triangle ABO \sim \triangle CDO$, therefore

$2 = \frac{AB}{CD} = \frac{BP}{PD} = \frac{\frac{BD}{2}+11}{\frac{BD}{2}-11}$

$$BD = 66$$

Since $\triangle ADB$ is right, the Pythagorean theorem implies that

$AD = \sqrt{86^2 - 66^2}$

$AD = 4\sqrt{190}$

$4\sqrt{190} \Longrightarrow 4 + 190 = \text{(D) } 194$

18. Let f be function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Suppose the f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$.

A. $\frac{17}{32}$ B. $\frac{11}{16}$ C. $\frac{7}{9}$ D. $\frac{7}{6}$ E. $\frac{25}{11}$

Answer: E

Solution: Looking through the solutions we can see that $f\left(\frac{25}{11}\right)$ can be expressed as $f\left(\frac{25}{11}\cdot 11\right) = f(11)+f\left(\frac{25}{11}\right)$ so using the prime numbers to piece together what we have we can get $10 = 11 + f\left(\frac{25}{11}\right)$, so $f\left(\frac{25}{11}\right) = -1$ or E

19. How many solutions does the equation sin

$$\frac{\pi}{2}\cos(x) = \cos\left(\frac{\pi}{2}\sin x\right)$$

have in the closed interval $[0, \pi]$

A. 0 B. 1 C. 2 D. 3 E. 4

Answer: C

Solution: $\sin\left(\frac{\pi}{2}\cos x\right) = \cos\left(\frac{\pi}{2}\sin x\right)$

The ranges of $\frac{\pi}{2}\sin x$ and $\frac{\pi}{2}\cos x$ are both $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is included in the range of arcsin, so we c

$\frac{\pi}{2}\cos x = \arcsin\left(\cos\left(\frac{\pi}{2}\sin x\right)\right)$

$\frac{\pi}{2}\cos x = \frac{\pi}{2} - \frac{\pi}{2}\sin x$

$\cos x = 1 - \sin x$

$\cos x + \sin x = 1$

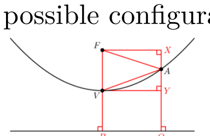
This only happens at $x = 0, \frac{\pi}{2}$ on the interval $[0, \pi]$, because one of sin and cos must be 1 and the other 0 . Therefore, the answer is $[C]2$

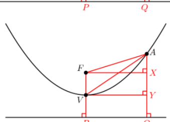
20. Suppose that on a parabola with vertex V and focus F there exists a point A such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length FV?

A. 13 B. $\frac{40}{3}$ C. $\frac{41}{3}$ D. 14 E. $\frac{43}{3}$

Answer: B

Solution: Let ℓ be the directrix of $\mathcal{P}$; recall that $\mathcal{P}$ is the set of points T such that the distance from T to ℓ is equal to TF. Let P and Q be the orthogonal projections of F and A onto ℓ, and further let X and Y be the orthogonal projections of F and V onto AQ. Because $AF < AV$, there are two possible configurations which may arise, and they are shown below.





Set $d = FV$, which by the definition of a parabola also equals VP. Then as $AQ = AF = 20$, we have $AY = 20 - d$ and $AX = |20 - 2d|$. Since $FXYV$ is a rectangle, $FX = VY$, so by Pythagorean Theorem on triangles AFX and AVY,

$21^2 - (20 - d)^2 = AV^2 - AY^2 = VY^2$

$= FX^2 = AF^2 - AX^2 = 20^2 - (20 - 2d)^2$

This equation simplifies to $3d^2 - 40d + 41 = 0$, which has solutions $d = \frac{20 \pm \sqrt{277}}{3}$. Both values of d work

- the smaller solution with the right configuration and the larger solution with the left configuration
- and so the requested answer is $\frac{40}{3}$

21. The five solutions to the equation $(z-1)\left(z^2+2z+4\right)\left(z^2+4z+6\right)$ may be written in the form $x_k + iy_k$ for $1 \le k \le 5$, where x_k and y_k are real. Let ϵ be the unique ellipse that passes through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and (x_5, y_5). The eccentricity of ϵ can be written as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. What is $m+n$? (Recall that the eccentricity of an ellipse ϵ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of ϵ, and $2c$ is the distance between its two foci.

A. 7 B. 9 C. 11 D. 13 E. 15

Answer: A

Solution: The solutions to this equation are $z=1, z=-1 \pm i\sqrt{3}$, and $z=-2 \pm i\sqrt{2}$. Consider the five points $(1,0), (-1, \pm\sqrt{3})$, and $(-2, \pm\sqrt{2})$; these are the five points which lie on $\mathcal{E}$. Note that since these five points are symmetric about the x -axis, so must $\mathcal{E}$. Now let $r := b/a$ denote the ratio of the length of the minor axis of $\mathcal{E}$ to the length of its major axis. Remark that if we perform a transformation of the plane which scales every x -coordinate by a factor of $r, \mathcal{E}$ is sent to a circle $\mathcal{E}'$. Thus, the problem is equivalent to finding the value of r such that $(r,0), (-r, \pm\sqrt{3})$, and $(-2r, \pm\sqrt{2})$ all lie on a common circle; equivalently, it suffices to determine the value of r such that the circumcenter of the triangle formed by the points $P_1 = (r, 0), P_2 = (-r, \sqrt{3})$, and $P_3 = (-2r, \sqrt{2})$ lies on the x -axis.

Recall that the circumcenter of a triangle ABC is the intersection point of the perpendicular bisectors of its three sides. The equations of the perpendicular bisectors of the segments $\overline{P_1P_2}$ and $\overline{P_1P_3}$ are $y = \frac{\sqrt{3}}{2} + \frac{2r}{\sqrt{3}}x$ and $y = \frac{\sqrt{2}}{2} + \frac{3r}{\sqrt{2}}\left(x + \frac{r}{2}\right)$ respectively. These two lines have different slopes for $r \neq 0$, so indeed they will intersect at some point (x_0, y_0); we want $y_0 = 0$. Plugging $y=0$ into the first equation yields $x = -\frac{3}{44}$, and so plugging $y=0$ into the second equation and simplifying yields $-\frac{1}{3r} = x + \frac{r}{2} = -\frac{3}{4r} + \frac{r}{2}$ Solving yields $r = \sqrt{\frac{5}{6}}$.

Finally, recall that the lengths a, b, and c (where c is the distance between the foci of $\mathcal{E}$) satisfy $c = \sqrt{a^2 - b^2}$. Thus the eccentricity of $\mathcal{E}$ is $\frac{c}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2} = \sqrt{\frac{1}{6}}$ and the requested answer is 7 (A)

22. Suppose that the root of the polynomial $x^3 + ax^2 + bx + c$ are $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$ where angles are in radians. What is abc?

A. $-\frac{3}{49}$ B. $-\frac{1}{28}$ C. $-\frac{\sqrt[3]{7}}{64}$ D. $\frac{1}{32}$ E. $\frac{3}{28}$

Answer: D

Solution: Note sum of roots of unity equal zero, sum of real parts equal zero, and $\operatorname{Re}\omega^m = \operatorname{Re}\omega^{-m}$, thus $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = 1/2(0 - \cos 0) = -1/2$ which means $A = \frac{1}{2}$. By product to sum, $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7}\cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\cos\frac{6\pi}{7} = \frac{1}{2}\left(2\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{10\pi}{7}\right)$ $= \frac{1}{2}\left(2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7}\right) = -1/2$, so $B = -\frac{1}{2}$ By product to sum, $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7} = \frac{1}{2}\left(\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7}\right)\cos\frac{6\pi}{7} = \frac{1}{4}\left(\cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right) + \frac{1}{4}\left(1 + \cos\frac{12\pi}{7}\right) = \frac{1}{4}\left(1 + \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right) = 1/8$, so $C = -1/8$ $ABC = \frac{1}{32}$

23. Frieda the frog begins a sequence of hops on a 33 grid of squares, moving one square on each hop and choosing at random the direction of each hop up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example If Frieda begins in the center square and makes two hops "up", the frst hop would place her In the top row middle square, and the second hop would cause Frieda to Jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping If she lands on a comer square. What Is the probability that she reaches a corner square on one of the four hops?

   A. $\frac{9}{16}$ B. $\frac{5}{8}$ C. $\frac{3}{4}$ D. $\frac{25}{32}$ E. $\frac{13}{16}$

   Answer: D

   Solution: We can draw a state diagram with three states: center, edge, and corner. Denote center by **M**, edge by **E**, and corner by **C**. There are a few ways Frieda can reach a corner in four or less moves: EC, EEC, EEEC, EMEC. Then, calculating the probabilities of each of these cases happening, we have $1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \cdot 1 \cdot \frac{1}{2} = \frac{25}{32}$, so the answer is D.

24. Semicircle Γ has diameter AB of length 14. Circle Ω lies tangent to AB at a point P and intersects Γ at Q and R. If $QR = 3\sqrt{3}$, and $\angle QPR = 60°$, then the area of $\triangle PQR$ is $\frac{a\sqrt{b}}{c}$ where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. What is $a + b + c$?

   A. 110 B. 114 C. 118 D. 122 E. 126

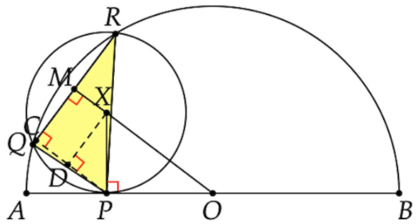
   Answer: D

   Solution: Let $O = \Gamma$ be the center of the semicircle, $X = \Omega$ be the center of the circle, and M be the midpoint of $\overline{QR}$. By the Perpendicular Chord Theorem Converse, we have $\overline{XM} \perp \overline{QR}$ and $\overline{OM} \perp \overline{QR}$. Together, points O, X, and M must be collinear. Applying the Extended Law of Sines on $\triangle PQR$, we have

$$XP = \frac{QR}{2 \cdot \sin \angle QPR} = \frac{3\sqrt{3}}{2 \cdot \frac{\sqrt{3}}{2}} = 3$$

   in which the radius of $\odot\Omega$ is 3.

   By the SAS Congruence, we have $\triangle QXM \cong \triangle RXM$, both of which are $30° - 60° - 90°$ triangles. By the side-length ratios, $RM = \frac{3\sqrt{3}}{2}, RX = 3$, and $MX = \frac{3}{2}$. By the Pythagorean Theorem in $\triangle ORM$, we get $OM = \frac{13}{2}$ and $OX = OM - XM = 5$. By the Pythagorean Theorem on $\triangle OXP$, we obtain $OP = 4$.



   As shown above, we construct an altitude $\overline{PC}$ of $\triangle PQR$. Since $\overline{PC} \perp \overline{RQ}$ and $\overline{OM} \perp \overline{RQ}$, we know that $\overline{PC} \| \overline{OM}$. We construct D on $\overline{PC}$ such that $\overline{XD} \perp \overline{PC}$. Clearly, $MXDC$ is a rectangle. Since $\angle XPD = \angle OXP$ by alternate interior angles, we have $\triangle XPD \sim \triangle OXP$ by the AA Similarity, with ratio of similitude $\frac{XP}{OX} = \frac{3}{5}$. Therefore, we get that $PD = \frac{9}{5}$ and $PC = PD + DC = PD + MX = \frac{9}{5} + \frac{3}{2} = \frac{33}{10}$ The area of $\triangle PQR$ is

$$\frac{1}{2}(RQ)(PC) = \frac{1}{2}(3\sqrt{3})\left(\frac{33}{10}\right) = \frac{99\sqrt{3}}{20}$$

and the answer is $99 + 3 + 20 =$ (D) 122.

25. Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1, d(2) = 2$, and $d(12) = 6$. (This function is known as the divisor function.) Let $f(n) = \frac{d(n)}{\sqrt[3]{n}}$.
There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$.
what is the sum of the digits of N?

A. 5 B. 6 C. 7 D. 8 E. 9

Answer: E

Solution: Using the answer choices to our advantage, we can show that N must be divisible by 9 without explicitly computing N, by exploiting the following fact: Claim: If n is not divisible by 3, then $f(9n) > f(3n) > f(n)$. Proof: Since $d(\cdot)$ is a multiplicative function, we have $d(3n) = d(3)d(n) = 2d(n)$ and $d(9n) = 3d(n)$. Then

$$f(3n) = \frac{2d(n)}{\sqrt[3]{3n}} \approx 1.38 f(n)$$
$$f(9n) = \frac{3d(n)}{\sqrt[3]{9n}} \approx 1.44 f(n)$$

Note that the values $\frac{2}{\sqrt[3]{3}}$ and $\frac{3}{\sqrt[3]{9}}$ do not have to be explicitly computed; we only need the fact that $\frac{3}{\sqrt[3]{9}} > \frac{2}{\sqrt[3]{3}} > 1$ which is easy to show by hand The above claim automatically implies N is a multiple of 9: if N was not divisible by 9 , then $f(9N) > f(N)$ which is a contradiction, and if N was divisible by 3 and not 9 , then $f(3N) > f(N) > f\left(\frac{N}{3}\right)$, also a contradiction. Then the sum of digits of N must be a multiple of 9 , so only choice $[E]$9 works.