2014 AMC 10A Problems/Problem 1

Problem

What is $10 \cdot \left(\tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{10}\right)^{-1}?$

$\textbf{(A)}\ 3 \qquad\textbf{(B)}\ 8 \qquad\textbf{(C)}\ \frac{25}{2} \qquad\textbf{(D)}\ \frac{170}{3} \qquad\textbf{(E)}\ 170$

Solution

We have

$$10 \cdot \left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10}\right)^{-1}$$

Making the denominators equal gives

$$\implies 10 \cdot \left(\frac{5}{10} + \frac{2}{10} + \frac{1}{10}\right)^{-1}$$

$$\implies 10 \cdot \left(\frac{5+2+1}{10}\right)^{-1}$$

$$\implies 10 \cdot \left(\frac{8}{10}\right)^{-1}$$

$$\implies 10 \cdot \left(\frac{4}{5}\right)^{-1}$$

$$\implies 10 \cdot \frac{5}{4}$$

$$\implies \frac{50}{4}$$

Finally, simplifying gives

$$\implies \boxed{\textbf{(C)}\ \frac{25}{2}}$$

See Also

2014 AMC 12A Problems/Problem 2

Problem

At the theater children get in for half price. The price for 5 adult tickets and 4 child tickets is 24.50. How much would 8 adult tickets and 6 child tickets cost?

$\textbf{(A) }35\qquad\textbf{(B) }38.50\qquad\textbf{(C) }40\qquad\textbf{(D) }42\qquad\textbf{(E) }42.50$

Solution

Suppose x is the price of an adult ticket. The price of a child ticket would be $\frac{x}{2}$.

$$\begin{aligned} 5x+4(x/2)=7x &= 24.50 \\ x &= 3.50 \end{aligned}$$

Plug in for 8 adult tickets and 6 child tickets.

$$\begin{aligned} 8x+6(x/2) &= 8(3.50)+3(3.50) \\ &= \boxed{\textbf{(B) }38.50} \end{aligned}$$

See Also

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2014 AMC 10A Problems/Problem 4

The following problem is from both the 2014 AMC 12A #3 and 2014 AMC 10A #4, so both problems redirect to this page.

Contents

Problem

Walking down Jane Street, Ralph passed four houses in a row, each painted a different color. He passed the orange house before the red house, and he passed the blue house before the yellow house. The blue house was not next to the yellow house. How many orderings of the colored houses are possible?

$\textbf{(A)}\ 2\qquad\textbf{(B)}\ 3\qquad\textbf{(C)}\ 4\qquad\textbf{(D)}\ 5\qquad\textbf{(E)}\ 6$

Solution 1

Attack this problem with very simple casework. The only possible locations for the yellow house (Y) is the 3rd house and the last house.

Case 1: Y is the 3rd house.

The only possible arrangement is $B-O-Y-R$

Case 2: Y is the last house.

There are two possible ways:

$B-O-R-Y$ and

$O-B-R-Y$ so our answer is $\boxed{\textbf{(B)} 3}$

Solution 2

There are 24 possible arrangements of the houses. The number of ways with the blue house next to the yellow house is $3!\cdot 2! = 12$, as we can consider the arrangements of O, (RB), and Y. Thus there are $24-12$ arrangements with the blue and yellow houses non-adjacent.

Exactly half of these have the orange house before the red house by symmetry, and exactly half of those have the blue house before the yellow house (also by symmetry), so our answer is $12\cdot\frac{1}{2}\cdot\frac{1}{2}=3$.

See Also

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2014 AMC 10A Problems/Problem 6

The following problem is from both the 2014 AMC 12A #4 and 2014 AMC 10A #6, so both problems redirect to this page.

Contents

Problem

Suppose that a cows give b gallons of milk in c days. At this rate, how many gallons of milk will d cows give in e days?

$\textbf{(A)}\ \frac{bde}{ac}\qquad\textbf{(B)}\ \frac{ac}{bde}\qquad\textbf{(C)}\ \frac{abde}{c}\qquad\textbf{(D)}\ \frac{bcde}{a}\qquad\textbf{(E)}\ \frac{abc}{de}$

Solution 1

We need to multiply b by $\frac{d}{a}$ for the new cows and $\frac{e}{c}$ for the new time, so the answer is $b\cdot \frac{d}{a}\cdot \frac{e}{c}=\frac{bde}{ac}$, or $\boxed{\textbf{(A)}}$.

Solution 2

We plug in $a=2$, $b=3$, $c=4$, $d=5$, and $e=6$. Hence the question becomes "2 cows give 3 gallons of milk in 4 days. How many gallons of milk do 5 cows give in 6 days?"

If 2 cows give 3 gallons of milk in 4 days, then 2 cows give $\frac{3}{4}$ gallons of milk in 1 day, so 1 cow gives $\frac{3}{4\cdot 2}$ gallons in 1 day. This means that 5 cows give $\frac{5\cdot 3}{4\cdot 2}$ gallons of milk in 1 day. Finally, we see that 5 cows give $\frac{5\cdot 3\cdot 6}{4\cdot 2}$ gallons of milk in 6 days. Substituting our values for the variables, this becomes $\frac{dbe}{ac}$, which is $\boxed{\textbf{(A)}\ \frac{bde}{ac}}$.

Solution 3

We see that the the number of cows is inversely proportional to the number of days and directly proportional to the gallons of milk. So our constant is $\frac{ac}{b}$.

Let g be the answer to the question. We have

$$\frac{de}{g}=\frac{ac}{b}\implies gac=bde\implies g=\frac{bde}{ac}\implies \boxed{\textbf{(A)}\ \frac{bde}{ac}}$$

Solution 4

The problem specifics "rate," so it would be wise to first find the rate at which cows produce milk. We can find the rate of work/production by finding the gallons produced by a single cow in a single day. To do this, we divide the amount produced by the number of cows and number of days

$$\implies \text{rate} = \frac{b}{ac}$$

Now that we have the gallons produced by a single cow in a single day, we simply multiply that rate by the number of cows and the number of days

$$\implies \frac{bde}{ac} \implies \boxed{\textbf{(A)}}$$

See Also

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Category: Introductory Algebra Problems

2014 AMC 10A Problems/Problem 5

The following problem is from both the 2014 AMC 12A #5 and 2014 AMC 10A #5, so both problems redirect to this page.

Problem

On an algebra quiz, 10% of the students scored 70 points, 35% scored 80 points, 30% scored 90 points, and the rest scored 100 points. What is the difference between the mean and median score of the students' scores on this quiz?

$\textbf{(A)}\ 1\qquad\textbf{(B)}\ 2\qquad\textbf{(C)}\ 3\qquad\textbf{(D)}\ 4\qquad\textbf{(E)}\ 5$

Solution

Without loss of generality, let there be 20 students(the least whole number possible) who took the test. We have 2 students score 70 points, 7 students score 80 points, 6 students score 90 points and 5 students score 100 points.

The median can be obtained by eliminating members from each group. The median is 90 points.

The mean is equal to the total number of points divided by the number of people, which gives 87

Thus, the difference between the median and the mean is equal to $90-87=\boxed{\textbf{(C)}\ 3}$

See Also

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Category: Introductory Algebra Problems

2014 AMC 12A Problems/Problem 6

Contents

Problem

The difference between a two-digit number and the number obtained by reversing its digits is 5 times the sum of the digits of either number. What is the sum of the two digit number and its reverse?

$\textbf{(A) }44\qquad\textbf{(B) }55\qquad\textbf{(C) }77\qquad\textbf{(D) }99\qquad\textbf{(E) }110$

Solution 1

Let the two digits be a and b. Then, $5a + 5b = 10a + b - 10b - a = 9a - 9b$, or $2a = 7b$. This yields $a = 7$ and $b = 2$ because $a, b < 10$. Then, $72 + 27 = \boxed{\textbf{(D) }99}$.

Solution 2 (Meta)

We start like above. Let the two digits be a and b. Therefore, $5(a + b) = 10a + b - 10b - a = 9(a - b)$. Since we are looking for $10a + b + 10b + a = 11(a + b)$ and we know that $a + b$ must be a multiple of 9, the only answer choice that works is $\boxed{\textbf{(D) }99}$.

See Also

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2014 AMC 12A Problems/Problem 7

Problem 7

The first three terms of a geometric progression are $\sqrt{3}$, $\sqrt[3]{3}$, and $\sqrt[6]{3}$. What is the fourth term?

$\textbf{(A) }1\qquad\textbf{(B) }\sqrt[7]{3}\qquad\textbf{(C) }\sqrt[8]{3}\qquad\textbf{(D) }\sqrt[9]{3}\qquad\textbf{(E) }\sqrt[10]{3}$

Solution

The terms are $\sqrt{3}$, $\sqrt[3]{3}$, and $\sqrt[6]{3}$, which are equivalent to $3^{\frac{3}{6}}$, $3^{\frac{2}{6}}$, and $3^{\frac{1}{6}}$. So the next term will be $3^{\frac{0}{6}} = 1$, so the answer is $\boxed{\textbf{(A)}}$.

See Also

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2014 AMC 10A Problems/Problem 11

The following problem is from both the 2014 AMC 12A #8 and 2014 AMC 10A #11, so both problems redirect to this page.

Problem

A customer who intends to purchase an appliance has three coupons, only one of which may be used:

Coupon 1: 10% off the listed price if the listed price is at least $\$50$

Coupon 2: $\$20$ off the listed price if the listed price is at least $\$100$

Coupon 3: 18% off the amount by which the listed price exceeds $\$100$

For which of the following listed prices will coupon 1 offer a greater price reduction than either coupon 2 or coupon 3?

$\textbf{(A)}\ \$179.95 \qquad \textbf{(B)}\ \$199.95 \qquad \textbf{(C)}\ \$219.95 \qquad \textbf{(D)}\ \$239.95 \qquad \textbf{(E)}\ \$259.95$

Solution 1

Let the listed price be x. Since all the answer choices are above $\$100$, we can assume $x > 100$. Thus the discounts after the coupons are used will be as follows:

Coupon 1: $x \times 10\% = .1x$

Coupon 2: 20

Coupon 3: $18\% \times (x-100) = .18x - 18$

For coupon 1 to give a greater price reduction than the other coupons, we must have $.1x > 20 \implies x > 200$ and $.1x > .18x - 18 \implies .08x < 18 \implies x < 225$.

From the first inequality, the listed price must be greater than $\$200$, so answer choices $\textbf{(A)}$ and $\textbf{(B)}$ are eliminated.

From the second inequality, the listed price must be less than $\$225$, so answer choices $\textbf{(D)}$ and $\textbf{(E)}$ are eliminated.

The only answer choice that remains is $\boxed{\textbf{(C)}\ \$219.95}$.

See Also

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2014 AMC 10A Problems/Problem 10

The following problem is from both the 2014 AMC 12A #9 and 2014 AMC 10A #10, so both problems redirect to this page.

Contents

Problem

Five positive consecutive integers starting with a have average b. What is the average of 5 consecutive integers that start with b?

$\textbf{(A)}\ a+3\qquad\textbf{(B)}\ a+4\qquad\textbf{(C)}\ a+5\qquad\textbf{(D)}\ a+6\qquad\textbf{(E)}\ a+7$

Solution 1

Let $a=1$. Our list is $\{1,2,3,4,5\}$ with an average of $15\div 5=3$. Our next set starting with 3 is $\{3,4,5,6,7\}$. Our average is $25\div 5=5$.

Therefore, we notice that $5=1+4$ which means that the answer is $\boxed{\textbf{(B)}\ a+4}$.

Solution 2

We are given that

$$b=\frac{a+a+1+a+2+a+3+a+4}{5}$$

$$\implies b=a+2$$

We are asked to find the average of the 5 consecutive integers starting from b in terms of a. By substitution, this is

$$\frac{a+2+a+3+a+4+a+5+a+6}{5}=a+4$$

Thus, the answer is $\boxed{\textbf{(B)}\ a+4}$

See Also

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2014 AMC 12A Problems/Problem 10

Contents

Problem

Three congruent isosceles triangles are constructed with their bases on the sides of an equilateral triangle of side length 1. The sum of the areas of the three isosceles triangles is the same as the area of the equilateral triangle. What is the length of one of the two congruent sides of one of the isosceles triangles?

$\textbf{(A)}\ \dfrac{\sqrt3}4 \qquad \textbf{(B)}\ \dfrac{\sqrt3}3 \qquad \textbf{(C)}\ \dfrac23 \qquad \textbf{(D)}\ \dfrac{\sqrt2}2 \qquad \textbf{(E)}\ \dfrac{\sqrt3}2$

Solution 1

Reflect each of the triangles over its respective side. Then since the areas of the triangles total to the area of the equilateral triangle, it can be seen that the triangles fill up the equilateral one and the vertices of these triangles concur at the circumcenter of the equilateral triangle. Hence the desired answer is just its circumradius, or $\boxed{\dfrac{\sqrt3}{3}\textbf{ (B)}}$.

(Solution by djmathman)

Solution 2

Since the total area of each congruent isosceles triangle is the same, the area of each is $\dfrac{1}{3}$ the total area of the equilateral triangle of side length 1, or $\dfrac{1}{3}$ x $\dfrac{\sqrt{3}}{4}$. Likewise, the area of each can be defined as $\dfrac{bh}{2}$ with base b equaling 1, meaning that $\dfrac{h}{2}$ = $\dfrac{1}{3}$ x $\dfrac{\sqrt{3}}{4}$, or h = $\dfrac{\sqrt{3}}{6}$. A side length of the isosceles triangle is the hypotenuse with legs $\dfrac{b}{2}$ and h. Using the Pythagorean Theorem, the side length is $\sqrt{(\dfrac{1}{2})^2+(\dfrac{\sqrt{3}}{6})^2}$, or $\boxed{\dfrac{\sqrt3}{3}\textbf{ (B)}}$.

(Solution by johnstucky)

See Also

2014 AMC 10A Problems/Problem 15

The following problem is from both the 2014 AMC 12A #11 and 2014 AMC 10A #15, so both problems redirect to this page.

Contents

Problem

David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

$\textbf{(A) }140\qquad\textbf{(B) }175\qquad\textbf{(C) }210\qquad\textbf{(D) }245\qquad\textbf{(E) }280$

Solution 1 (Algebra)

Note that he drives at 50 miles per hour after the first hour and continues doing so until he arrives.

Let d be the distance still needed to travel after the first 1 hour. We have that $\frac{d}{50} + 1.5 = \frac{d}{35}$, where the 1.5 comes from 1 hour late decreased to 0.5 hours early.

Simplifying gives $7d + 525 = 10d$, or $d = 175$.

Now, we must add an extra 35 miles traveled in the first hour, giving a total of $\boxed{\textbf{(C) }210}$ miles.

Solution 2 (Answer Choices)

Instead of spending time thinking about how one can set up an equation to solve the problem, one can simply start checking the answer choices. Quickly checking, we know that neither choice $\textbf{(A)}$ or choice $\textbf{(B)}$ work, but $\textbf{(C)}$ does. We can verify as follows. After 1 hour at 35 mph, David has 175 miles left. This then takes him 3.5 hours at 50 mph. But $210/35 = 6 \text{ hours}$. Since $1 + 3.5 = 4.5 \text{ hours}$ is 1.5 hours less than 6, our answer is $\boxed{\textbf{(C) }210}$.

See Also

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2014 AMC 12A Problems/Problem 12

Contents

Problem

Two circles intersect at points A and B. The minor arcs AB measure 30° on one circle and 60° on the other circle. What is the ratio of the area of the larger circle to the area of the smaller circle?

$\textbf{(A) } 2 \qquad \textbf{(B) } 1+\sqrt{3} \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 2+\sqrt{3} \qquad \textbf{(E) } 4$

Solution 1

Let the radius of the larger and smaller circles be x and y, respectively. Also, let their centers be O_1 and O_2, respectively. Then the ratio we need to find is

$$\frac{\pi x^2}{\pi y^2} = \frac{x^2}{y^2}$$

Draw the radii from the centers of the circles to A and B. We can easily conclude that the 30° belongs to the larger circle, and the 60 degree arc belongs to the smaller circle. Therefore, $m\angle AO_1B = 30^\circ$ and $m\angle AO_2B = 60^\circ$. Note that ΔAO_2B is equilateral, so when chord AB is drawn, it has length y. Now, applying the Law of Cosines on ΔAO_1B:

$$y^2 = x^2 + x^2 - 2x^2 \cos 30 = 2x^2 - x^2\sqrt{3} = (2-\sqrt{3})x^2$$

$$\frac{x^2}{y^2} = \frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3} = \boxed{\textbf{(D)}}$$

(Solution by brandbest1)

Solution 2

Again, let the radius of the larger and smaller circles be x and y, respectively, and let the centers of these circles be O_1 and O_2, respectively. Let X bisect segment AB. Note that $\triangle AXO_1$ and $\triangle AXO_2$ are right triangles, with $\angle AO_1X = 15^\circ$ and $\angle AO_2X = 30^\circ$. We have $\sin 15 = \frac{AX}{x}$ and $\sin 30 = \frac{AX}{y}$ and $\frac{x}{y} = \frac{\sin 30}{\sin 15}$. Since the ratio of the area of the larger circle to that of the smaller circle is simply $\frac{\pi x^2}{\pi y^2} = \left(\frac{x}{y}\right)^2 = \left(\frac{\sin 30}{\sin 15}\right)^2$, we just need to find $\sin 30$ and $\sin 15$. We know $sin 30 = \frac{1}{2}$, and we can use the angle sum formula or half angle formula to compute $\sin 15 = \frac{\sqrt{6}-\sqrt{2}}{4}$. Plugging this into the previous expression, we get:

$$\left(\frac{x}{y}\right)^2 = \left(\frac{1/2}{\frac{\sqrt{6}-\sqrt{2}}{4}}\right)^2 = \left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 = 2+\sqrt{3} = \boxed{\textbf{(D)}}$$

(Solution by kevin38017)

Solution 3

Let the radius of the smaller and larger circle be r and R, respectively. We see that half the length of the chord is equal to $r\sin 30^\circ$, which is also equal to $R\sin 15^\circ$. Recall that $\sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$ and $\sin 30^\circ = \frac{1}{2}$. From this, we get $r = \frac{\sqrt{6}-\sqrt{2}}{2}R$, or $r^2 = \frac{8-2\sqrt{12}}{4}R^2 = \left(2-\sqrt{3}\right)R^2$, which is equivalent to $R^2 = \left(2+\sqrt{3}\right)r^2$.

(solution by soy_un_chemisto)

Solution 4

As in the previous solutions let the radius of the smaller and larger circles be r and R, respectively. Also, let their centers be O_1 and O_2, respectively. Now draw two congruent chords from points A and B to the end of the smaller circle, creating an isosceles triangle. Label that point X. Recalling the Inscribed Angle Theorem, we then see that $m\angle AXB = \frac{m\angle AO_1B}{2} = 30^\circ = m\angle AO_2B$. Based on this information, we can conclude that triangles AXB and AO_2B are congruent via ASA Congruence.

Next draw the height of AXB from X to AB. Note we've just created a right triangle with hypotenuse R, base $\frac{r}{2}$, and height $\frac{r\sqrt{3}}{2}+r$ Thus using the Pythagorean Theorem we can express R^2 in terms of r

$$R^2 = (\frac{r}{2})^2+(\frac{r\sqrt{3}}{2}+r)^2 = r^2+\frac{r^2}{4}+\frac{3r^2}{4}+(2)(\frac{r\sqrt{3}}{2})(r) = 2r^2+r^2\sqrt{3} = r^2(2+\sqrt{3})$$

We can now determine the ratio between the larger and smaller circles:

$$\frac{Area[O_2]}{Area[O_1]} = \frac{\pi R^2}{\pi r^2} = \frac{\pi r^2(2+\sqrt{3})}{\pi r^2} = \boxed{\textbf{(D)} 2+\sqrt{3}}$$

(Solution by derekxu)

See Also

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2014 AMC 12A Problems/Problem 13

Contents

Problem

A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

$\textbf{(A) }2100\qquad\textbf{(B) }2220\qquad\textbf{(C) }3000\qquad\textbf{(D) }3120\qquad\textbf{(E) }3125$

Solution 1

We can discern three cases.

Case 1: Each room houses one guest. In this case, we have 5 guests to choose for the first room, 4 for the second, ..., for a total of $5!=120$ assignments.

Case 2: Three rooms house one guest; one houses two. We have $\binom{5}{3}$ ways to choose the three rooms with 1 guest, and $\binom{2}{1}$ to choose the remaining one with 2. There are $5\cdot4\cdot3$ ways to place guests in the first three rooms, with the last two residing in the two-person room, for a total of $\binom{5}{3}\binom{2}{1}\cdot5\cdot4\cdot3=1200$ ways.

Case 3: Two rooms house two guests; one houses one. We have $\binom{5}{2}$ to choose the two rooms with two people, and $\binom{3}{1}$ to choose one remaining room for one person. Then there are 5 choices for the lonely person, and $\binom{4}{2}$ for the two in the first two-person room. The last two will stay in the other two-room, so there are $\binom{5}{2}\binom{3}{1}\cdot5\cdot\binom{4}{2}=900$ ways.

In total, there are $120+1200+900=2220$ assignments, or $\boxed{\textbf{(B)}}$.

(Solution by AwesomeToad16)

Solution 2

We can work in reverse by first determining the number of combinations in which there are more than 2 friends in at least one room. There are three cases:

Case 1: Three friends are in one room. Since there are 5 possible rooms in which this can occur, we are choosing three friends from the five, and the other two friends can each be in any of the four remaining rooms, there are $5 \cdot \binom{5}{3} \cdot 4 \cdot 4 = 800$ possibilities.

Case 2: Four friends are in one room. Again, there are 5 possible rooms, we are choosing four of the five friends, and the other one can be in any of the other four rooms, so there are $5 \cdot \binom{5}{4} \cdot 4 = 100$ possibilities.

Case 3: Five friends are in one room. There are 5 possible rooms in which this can occur, so there are 5 possibilities.

Since there are $5^5 = 3125$ possible combinations of the friends, the number fitting the given criteria is $3125 - (800 + 100 + 5) = \boxed{2220}$.

See Also

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2014 AMC 12A Problems/Problem 14

Contents

Problem

Let $a < b < c$ be three integers such that a, b, c is an arithmetic progression and a, c, b is a geometric progression. What is the smallest possible value of c?

$\textbf{(A)}\ -2 \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 6$

Solution 1

We have $b - a = c - b$, so $a = 2b - c$. Since a, c, b is geometric, $c^2 = ab = (2b - c)b \Rightarrow 2b^2 - bc - c^2 = (2b + c)(b - c) = 0$. Since $a < b < c$, we can't have $b = c$ and thus $c = -2b$. Then our arithmetic progression is $4b, b, -2b$. Since $4b < b < -2b$, $b < 0$. The smallest possible value of $c = -2b$ is $(-2)(-1) = 2$, or $\boxed{\textbf{(C)}}$.

(Solution by AwesomeToad)

Solution 2

Taking the definition of an arithmetic progression, there must be a common difference between the terms, giving us $(b - a) = (c - b)$. From this, we can obtain the expression $a = 2b - c$. Again, by taking the definition of a geometric progression, we can obtain the expression, $c = ar$ and $b = ar^2$, where r serves as a value for the ratio between two terms in the progression. By substituting b and c in the arithmetic progression expression with the obtained values from the geometric progression, we obtain the equation, $a = 2ar^2 - ar$ which can be simplified to $(r - 1)(2r + 1) = 0$ giving us $r = 1$ of $r = -1/2$. Thus, from the geometric progression, $a = a$, $b = 1/4a$ and $c = -1/2a$. Looking at the initial conditions of $a < b < c$ we can see that the lowest integer value that would satisfy the above expressions is if $a = -4$, thus making $c = 2$ or or $\boxed{\textbf{(C)}}$

(Solution by thatuser)

See Also

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2014 AMC 12A Problems/Problem 15

Contents

Problem

A five-digit palindrome is a positive integer with respective digits $abcba$, where a is non-zero. Let S be the sum of all five-digit palindromes. What is the sum of the digits of S?

$\textbf{(A)}\ 9 \qquad \textbf{(B)}\ 18 \qquad \textbf{(C)}\ 27 \qquad \textbf{(D)}\ 36 \qquad \textbf{(E)}\ 45$

Solution 1

For each digit $a=1,2,\ldots,9$ there are $10\cdot 10$ (ways of choosing b and c) palindromes. So the as contribute $(1+2+\cdots+9)(100)(10^4+1)$ to the sum. For each digit $b=0,1,2,\ldots,9$ there are $9\cdot 10$ (since $a\neq 0$) palindromes. So the bs contribute $(0+1+2+\cdots+9)(90)(10^3+10)$ to the sum. Similarly, for each $c=0,1,2,\ldots,9$ there are $9\cdot 10$ palindromes, so the c contributes $(0+1+2+\cdots+9)(90)(10^2)$ to the sum.

It just so happens that

$$(1+2+\cdots+9)(100)(10^4+1)+(1+2+\cdots+9)(90)(10^3+10)+(1+2+\cdots+9)(90)(10^2)=49500000$$

so the sum of the digits of the sum is $\boxed{\textbf{(B)}\ 18}$.

Solution 2

Notice that $10001+99999=110000.$ In fact, ordering the palindromes in ascending order, we find that the sum of the nth palindrome and the nth to last palindrome is $110000.$ We have $9*10*10$ palindromes, or 450 pairs of palindromes summing to $110000.$ Performing the multiplication gives 49500000, so the sum $\boxed{\textbf{(B)}\ 18}$.

Solution 3

As shown above, there are a total of 900 five-digit palindromes. We can calculate their sum by finding the expected value of a randomly selected palindrome satisfying the conditions given, then multiplying it by 900 to get our sum. The expected value for the ten-thousands and the units digit is $\frac{1+2+3+\cdots+9}{9}=5$, and the expected value for the thousands, hundreds, and tens digit is $\frac{0+1+2+\cdots+9}{10}=4.5$. Therefore our expected value is $5\times 10^4+4.5\times 10^3+4.5\times 10^2+4.5\times 10^1+5\times 10^0=55{,}000$. Since the question asks for the sum of the digits of the resulting sum, we do not need to keep the trailing zeros of either $55{,}000$ or 900. Thus we only need to calculate $55\times 9=495$, and the desired sum is $\boxed{\textbf{(B)}\ 18}$.

See Also

2014 AMC 10A Problems/Problem 20

The following problem is from both the 2014 AMC 12A #16 and 2014 AMC 10A #20, so both problems redirect to this page.

Problem

The product $(8)(888\dots8)$, where the second factor has k digits, is an integer whose digits have a sum of 1000. What is k?

$\textbf{(A)}\ 901\qquad\textbf{(B)}\ 911\qquad\textbf{(C)}\ 919\qquad\textbf{(D)}\ 991\qquad\textbf{(E)}\ 999$

Solution

We can list the first few numbers in the form $8*(8....8)$

$8*8=64$

$8*88=704$

$8*888=7104$

$8*8888=71104$

$8*88888=711104$

By now it's clear that the numbers will be in the form 7, $k-2$ 1s, and 04. We want to make the numbers sum to 1000, so $7+4+(k-2)=1000$. Solving, we get $k=991$, meaning the answer is $\boxed{(D)}$

See Also

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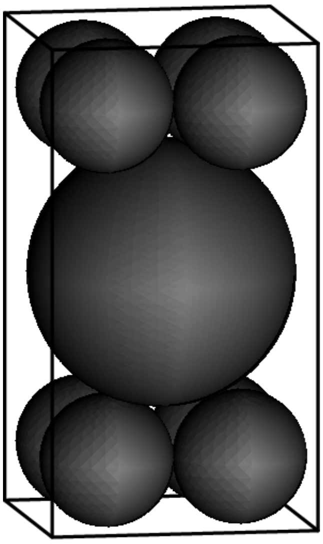
2014 AMC 12A Problems/Problem 17

Problem

A $4 \times 4 \times h$ rectangular box contains a sphere of radius 2 and eight smaller spheres of radius 1. The smaller spheres are each tangent to three sides of the box, and the larger sphere is tangent to each of the smaller spheres. What is h?



$\textbf{(A)}\ 2+2\sqrt{7} \qquad \textbf{(B)}\ 3+2\sqrt{5} \qquad \textbf{(C)}\ 4+2\sqrt{7} \qquad \textbf{(D)}\ 4\sqrt{5} \qquad \textbf{(E)}\ 4\sqrt{7}$

Solution

Let A be the point in the same plane as the centers of the top spheres equidistant from said centers. Let B be the analogous point for the bottom spheres, and let C be the midpoint of $\overline{AB}$ and the midpoint of the large sphere. Let D and E be the points at which line AB intersects the top of the box and the bottom, respectively.

Let O be the center of any of the top spheres (you choose!). We have $AO = 1 \cdot \sqrt{2}$, and $CO = 3$, so $AC = \sqrt{3^2 - \sqrt{2}^2} = \sqrt{7}$. Similarly, $BC = \sqrt{7}$. $\overline{AD}$ and $\overline{BE}$ are clearly equal to the radius of the small spheres, 1. Thus the total height is $AD + AC + BC + BE = 2 + 2\sqrt{7}$, or $\boxed{\textbf{(A)}}$.

(Solution by AwesomeToad)

See Also

2014 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
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2014 AMC 12A Problems/Problem 18

Contents

Problem

The domain of the function $f(x)=\log_{\frac12}(\log_4(\log_{\frac14}(\log_{16}(\log_{\frac1{16}}x))))$ is an interval of length $\tfrac mn$, where m and n are relatively prime positive integers. What is $m+n$?

$\textbf{(A) }19\qquad\textbf{(B) }31\qquad\textbf{(C) }271\qquad\textbf{(D) }319\qquad\textbf{(E) }511$

Solution 1

For simplicity, let $a=\log_{\frac{1}{16}}x, b=\log_{16}a, c=\log_{\frac14}b,$ and $d=\log_4 c.$

The domain of $\log_{\frac12}x$ is $x\in (0,\infty)$, so $d\in (0,\infty)$. Thus, $\log_4 c\in (0,\infty)\Rightarrow c\in (1,\infty)$.

Since $c=\log_{\frac14}b$ we have $b\in \left(0,\left(\frac14\right)^1\right)=\left(0,\frac14\right)$. Since $b=\log_{16}a$, we have

$a\in (16^0,16^{1/4})=(1,2)$. Finally, since $a=\log_{\frac1{16}}x$,

$x\in \left(\left(\frac1{16}\right)^2,\left(\frac1{16}\right)^1\right)=\left(\frac1{256},\frac1{16}\right)$.

The length of the x interval is $\frac1{16}-\frac1{256}=\frac{15}{256}$ and the answer is $\boxed{271\ \textbf{(C)}}$.

Solution 2

The domain of $f(x)$ is the range of the inverse function $f^{-1}(x)=\left(\frac{1}{16}\right)^{16^{\left(\frac14\right)^{4^{\left(\frac12\right)^x}}}}$. Now $f^{-1}(x)$ can be seen to be strictly decreasing, since $\left(\frac12\right)^x$ is decreasing, so $4^{\left(\frac12\right)^x}$ is decreasing, so $\left(\frac14\right)^{4^{\left(\frac12\right)^x}}$ is increasing, so $16^{\left(\frac14\right)^{4^{\left(\frac12\right)^x}}}$ is increasing, therefore $\left(\frac{1}{16}\right)^{16^{\left(\frac14\right)^{4^{\left(\frac12\right)^x}}}}$ is decreasing.

Therefore, the range of $f^{-1}(x)$ is the open interval $\left(\lim_{x\to\infty}f^{-1}(x), \lim_{x\to-\infty}f^{-1}(x)\right)$. We find:

$$\begin{aligned}
\lim_{x\to-\infty}\left(\frac{1}{16}\right)^{16^{\left(\frac{1}{4}\right)^{4^{\left(\frac{1}{2}\right)^{x}}}}} &= \lim_{a\to\infty}\left(\frac{1}{16}\right)^{16^{\left(\frac{1}{4}\right)^{4^{a}}}} \\
&= \lim_{b\to\infty}\left(\frac{1}{16}\right)^{16^{\left(\frac{1}{4}\right)^{b}}} \\
&= \left(\frac{1}{16}\right)^{16^{0}} \\
&= \frac{1}{16}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{1}{16}\right)^{16^{\left(\frac{1}{4}\right)^{4^{\left(\frac{1}{2}\right)^{x}}}}} &= \left(\frac{1}{16}\right)^{16^{\left(\frac{1}{4}\right)^{4^{0}}}} \\
&= \left(\frac{1}{16}\right)^{16^{\frac{1}{4}}} \\
&= \left(\frac{1}{16}\right)^{2} \\
&= \frac{1}{256}.
\end{aligned}$$

Hence the range of $f^{-1}(x)$ (which is then the domain of $f(x)$) is $\left(\frac{1}{256},\frac{1}{16}\right)$ and the answer is $\boxed{271\ \textbf{(C)}}$.

See Also

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2014 AMC 12A Problems/Problem 19

Problem

There are exactly N distinct rational numbers k such that $|k| < 200$ and

$$5x^2 + kx + 12 = 0$$

has at least one integer solution for x. What is N?

Solution

Factor the quadratic into

$$\left(5x + \frac{12}{n}\right)(x+n) = 0$$

where $-n$ is our integer solution. Then,

$$k = \frac{12}{n} + 5n,$$

which takes rational values between -200 and 200 when $|n| \le 39$, excluding $n = 0$. This leads to an answer of $2 \cdot 39 = \boxed{\textbf{(E)}\ 78}$.

See Also

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2014 AMC 12A Problems/Problem 20

Problem

In $\triangle BAC$, $\angle BAC = 40^\circ$, $AB = 10$, and $AC = 6$. Points D and E lie on $\overline{AB}$ and $\overline{AC}$ respectively. What is the minimum possible value of $BE + DE + CD$?

$\textbf{(A) }6\sqrt 3+3\qquad \textbf{(B) }\dfrac{27}2\qquad \textbf{(C) }8\sqrt 3\qquad \textbf{(D) }14\qquad \textbf{(E) }3\sqrt 3+9\qquad$

Solution

Let C_1 be the reflection of C across $\overline{AB}$, and let C_2 be the reflection of C_1 across $\overline{AC}$. Then it is well-known that the quantity $BE+DE+CD$ is minimized when it is equal to C_2B. (Proving this is a simple application of the triangle inequality; for an example of a simpler case, see Heron's Shortest Path Problem.) As A lies on both AB and AC, we have $C_2A=C_1A=CA=6$. Furthermore, $\angle CAC_1 = 2\angle CAB = 80^\circ$ by the nature of the reflection, so $\angle C_2AB = \angle C_2AC+\angle CAB = 80^\circ + 40^\circ = 120^\circ$. Therefore by the Law of Cosines

$$BC_2^2 = 6^2+10^2-2\cdot 6\cdot 10 \cos 120^\circ = 196 \implies BC_2 = \boxed{14 \textbf{ (D)}}.$$

See Also

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2014 AMC 12A Problems/Problem 21

Problem

For every real number x, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x, and let

$$f(x) = \lfloor x \rfloor (2014^{x - \lfloor x \rfloor} - 1).$$

The set of all numbers x such that $1 \leq x < 2014$ and $f(x) \leq 1$ is a union of disjoint intervals. What is the sum of the lengths of those intervals?

$\textbf{(A) }1\qquad \textbf{(B) }\dfrac{\log 2015}{\log 2014}\qquad \textbf{(C) }\dfrac{\log 2014}{\log 2013}\qquad \textbf{(D) }\dfrac{2014}{2013}\qquad \textbf{(E) }2014^{\frac{1}{2014}}$

Solution

Let $\lfloor x \rfloor = k$ for some integer $1 \leq k \leq 2013$. Then we can rewrite $f(x)$ as $k(2014^{x-k} - 1)$. In order for this to be less than or equal to 1, we need $2014^{x-k} - 1 \leq \dfrac{1}{k} \implies x \leq k + \log_{2014}\left(\dfrac{k+1}{k}\right)$. Combining this with the fact that $\lfloor x \rfloor = k$ gives that $x \in \left[k, k + \log_{2014}\left(\dfrac{k+1}{k}\right)\right]$, and so the length of the interval is $\log_{2014}\left(\dfrac{k+1}{k}\right)$. We want the sum of all possible intervals such that the inequality holds true; since all of these intervals must be disjoint, we can sum from $k = 1$ to $k = 2013$ to get that the desired sum is

$$\sum_{i=1}^{2013} \log_{2014}\left(\frac{i+1}{i}\right) = \log_{2014}\left(\prod_{i=1}^{2013} \frac{i+1}{i}\right) = \log_{2014}\left(\frac{2014}{1}\right) = \boxed{\textbf{1 (A)}}.$$

See Also

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2014 AMC 10A Problems/Problem 25

The following problem is from both the 2014 AMC 12A #22 and 2014 AMC 10A #25, so both problems redirect to this page.

Problem

The number 5^{867} is between 2^{2013} and 2^{2014}. How many pairs of integers (m,n) are there such that $1\leq m\leq 2012$ and

$$5^n<2^m<2^{m+2}<5^{n+1}?$$

$\textbf{(A) }278\qquad \textbf{(B) }279\qquad \textbf{(C) }280\qquad \textbf{(D) }281\qquad \textbf{(E) }282$

Solution

Between any two consecutive powers of 5 there are either 2 or 3 powers of 2 (because $2^2<5^1<2^3$). Consider the intervals $(5^0,5^1),(5^1,5^2),\dots (5^{866},5^{867})$. We want the number of intervals with 3 powers of 2.

From the given that $2^{2013}<5^{867}<2^{2014}$, we know that these 867 intervals together have 2013 powers of 2. Let x of them have 2 powers of 2 and y of them have 3 powers of 2. Thus we have the system

$$x+y=867$$

$$2x+3y=2013$$

from which we get $y=279$, so the answer is $\boxed{\textbf{(B)}}$.

See Also

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2014 AMC 12A Problems/Problem 23

Contents

Problem

The fraction

$$\frac{1}{99^2}=0.\overline{b_{n-1}b_{n-2}\ldots b_2b_1b_0},$$

where n is the length of the period of the repeating decimal expansion. What is the sum $b_0+b_1+\cdots+b_{n-1}$?

$\textbf{(A)}\ 874\qquad\textbf{(B)}\ 883\qquad\textbf{(C)}\ 887\qquad\textbf{(D)}\ 891\qquad\textbf{(E)}\ 892$

Solution 1

the fraction $\frac{1}{99}$ can be written as

$$\sum_{n=1}^{\infty} \frac{1}{10^{2n}}$$

. similarly the fraction $\frac{1}{99^2}$ can be written as $\sum_{m=1}^{\infty} \frac{1}{10^{2m}} \sum_{n=1}^{\infty} \frac{1}{10^{2n}}$ which is equivalent to

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{10^{2(m+n)}}$$

and we can see that for each $n+m=k$ there are $k-1$ (n,m) combinations so the above sum is equivalent to:

$$\sum_{k=2}^{\infty} \frac{k-1}{10^{2k}}$$

we note that the sequence starts repeating at $k=102$ yet consider

$$\sum_{k=99}^{101} \frac{k-1}{10^{2k}} = \frac{98}{10^{198}}+\frac{99}{10^{200}}+\frac{100}{10^{202}} = \frac{1}{10^{198}}(98+\frac{99}{100}+\frac{100}{10000}) = \frac{1}{10^{198}}(98+\frac{99}{100}+\frac{1}{100}) = \frac{1}{10^{198}}(98+\frac{100}{100}) = \frac{1}{10^{198}}(99$$

so the decimal will go from 1 to 99 skipping the number 98 and we can easily compute the sum of the digits from 0 to 99 to be

$$45 \cdot 10 \cdot 2 = 900$$

subtracting the sum of the digits of 98 which is 17 we get

$$900-17=883 \textbf{(B)}$$

Solution 2

$\frac{1}{99^2}$

$=\frac{1}{99} \cdot \frac{1}{99}$

$=\frac{0.\overline{01}}{99}$

$=0.\overline{00010203...9799}$

So, the answer is $0+0+0+1+0+2+0+3+...+9+7+9+9=2\cdot 10\cdot \frac{9\cdot 10}{2}-(9+8)$ or $\boxed{\textbf{(B)}\ 883}$.

There are two things to notice here. First, $\frac{1}{99}$ has a very simple and unique decimal expansion, as shown. Second, for $\frac{0.\overline{01}}{99}$ to itself produce a repeating decimal, 99 has to evenly divide a sufficiently extended number of the form $101010101..$. This number will have 99 ones (197 digits in total), as to be divisible by 9 and 11. The enormity of this number forces us to look for a pattern, and so we divide out as shown. Indeed, upon division (seeing how the remainders always end in "501" or "601" or, at last, "9801"), we find the repeating part

$.0001020304050607080910111213 15....9799$. If we wanted to further check our pattern, we could count the total number of digits in our quotient (not counting the first three): 195. Since $99 < 100$, multiplying by it will produce either 1 or 2 extra digits, so our quotient passes the test.

See Also

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2014 AMC 12A Problems/Problem 24

Contents

Problem

Let $f_0(x) = x + |x - 100| - |x + 100|$, and for $n \geq 1$, let $f_n(x) = |f_{n-1}(x)| - 1$. For how many values of x is $f_{100}(x) = 0$?

$\textbf{(A) }299\qquad\textbf{(B) }300\qquad\textbf{(C) }301\qquad\textbf{(D) }302\qquad\textbf{(E) }303$

Solution 1

1. Draw the graph of $f_0(x)$ by dividing the domain into three parts.

2. Look at the recursive rule. Take absolute of the previous function and down by 1 to get the next function.

3. Count the x intercepts of the each function and find the pattern.

The pattern turns out to be $3n+3$ solutions, for x interval:[1,99], the function gain only one extra solution after $f_{99}(x)$ because there is no summit on the graph any more, and the answer is thus $\textbf{(C) }301$. (Revised by Flamedragon & Jason,C)

Solution 2

First, notice that the recursion and the definition of $f_0(x)$ require that for all x such that $-100 \leq x \leq 100$, if $f_{100}(x) = 0$, then $f_0(x)$ is even. Now, we can do case work on x to find which values of x (such that $-100 \leq x \leq 100$) make $f_0(x)$ even. The answer comes out to be all the even values of x in the range $-100 \leq x \leq 100$, in the domain $-300 \leq x \leq 300$. So, the answer is $2 \cdot 150 + 1$ or $\boxed{\textbf{(C) }301}$.

See Also

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2014 AMC 12A Problems/Problem 25

Contents

Problem

The parabola P has focus $(0,0)$ and goes through the points $(4,3)$ and $(-4,-3)$. For how many points $(x,y)\in P$ with integer coordinates is it true that $|4x+3y|\leq 1000$?

$\textbf{(A) }38\qquad\textbf{(B) }40\qquad\textbf{(C) }42\qquad\textbf{(D) }44\qquad\textbf{(E) }46$

Solution

The parabola is symmetric through $y=-\frac{4}{3}x$, and the common distance is 5, so the directrix is the line through $(1,7)$ and $(-7,1)$, which is the line

$$3x-4y=-25.$$

Using the point-line distance formula, the parabola is the locus

$$x^2+y^2=\frac{\left\lvert 3x-4y+25\right\rvert^2}{3^2+4^2}$$

which rearranges to $(4x+3y)^2=25(6x-8y+25)$.

Let $m=4x+3y\in\mathbb{Z}$, $|m|\leq 1000$. Put $m=25k$ to obtain

$$25k^2=6x-8y+25$$

$$25k=4x+3y.$$

and accordingly we find by solving the system that $x=\frac{1}{2}(3k^2-3)+4k$ and $y=-2k^2+3k+2$.

One can show that the values of k that make (x,y) an integer pair are precisely odd integers k. For $|25k|\leq 1000$ this is $k=-39,-37,-35,\dots,39$, so 40 values work and the answer is $\boxed{\textbf{(B)}}$.

(Solution by v_Enhance)

Solution 2

Consider the rotation of axes such that the axes are the lines passing through the origin with slope $\frac{3}{4}$ and $-\frac{4}{3}$ for x-axis and y-axis, respectively, and let the point on the rotated axis be (x_1,y_1). We can check that $x=\frac{4}{5}x_1-\frac{3}{5}y_1$ and $y=\frac{3}{5}x_1+\frac{4}{5}y_1$ by the distance from a point to line formula

$\dfrac{ax_0+by_0+c}{\sqrt{a^2+b^2}}$ where the equation of the line is $ax_0+by_0+c=0$ and (x_0,y_0) is the point. We have the focus as $(0,0)$ and $(5,0)$ and $(-5,0)$ as points on the parabola(on the rotated axes). Therefore, the directrix is $y=\pm 5$, and it doesn't matter which one(due to the absolute value) so WLOG we choose $y_1=-5$. The vertex is the midpoint between the focus and the foot of the altitude from focus to directrix, so the vertex is $(0,-\frac{5}{2})$. Therefore, the equation is $y_1=\frac{x_1^2}{10}-\frac{5}{2}$, and from the equations above we have $|3x+4y|=5x_1$, so $|x_1|<200$. One can check with $4x+3y$ and $4y-3x$ that the only time x and y can both be integers is when x_1 and y_1 are both integer multiples of $\frac{1}{5}$. Therefore, the only time is when x_1 is an odd multiple of 5(otherwise y_1 is not a multiple of $\frac{1}{5}$, and this is obviously sufficient because y_1 is also a multiple of 5. The values that satisfy thus are $x_1=-195,-185,-175,...,195$, and there are $\boxed{(B)40}$ such numbers.

(Solution by Shaddoll)

See Also

2014 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2014))	
Preceded by Problem 24	Followed by Last Question
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All AMC 12 Problems and Solutions	

1) The line of symmetry is NOT y= -x but 4x + 3y = 0

2) In the expression for x, it is NOT 8 but 8k.

With these minor corrections, the solution still holds good.