2006 AMC 12B Problems/Problem 1

Problem

What is $(-1)^1 + (-1)^2 + \cdots + (-1)^{2006}$?

$\textbf{(A)}\ -2006 \qquad \textbf{(B)}\ -1 \qquad \textbf{(C)}\ 0 \qquad \textbf{(D)}\ 1 \qquad \textbf{(E)}\ 2006$

Solution

$(-1)^n = 1$ if n is even and -1 if n is odd. So we have

$$-1+1-1+1-\cdots-1+1 = 0+0+\cdots+0+0 = 0 \Rightarrow \text{(C)}$$

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 2

Problem

For real numbers x and y, define $x\spadesuit y=(x+y)(x-y)$. What is $3\spadesuit(4\spadesuit 5)$?

$\mathrm{(A)}\ -72\qquad\mathrm{(B)}\ -27\qquad\mathrm{(C)}\ -24\qquad\mathrm{(D)}\ 24\qquad\mathrm{(E)}\ 72$

Solution

$4\spadesuit 5=-9$

$3\spadesuit -9=-72\Rightarrow\mathrm{(A)}$

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 3

Contents

Problem

A football game was played between two teams, the Cougars and the Panthers. The two teams scored a total of 34 points, and the Cougars won by a margin of 14 points. How many points did the Panthers score?

$\mathrm{(A)}\ 10\qquad \mathrm{(B)}\ 14\qquad \mathrm{(C)}\ 17\qquad \mathrm{(D)}\ 20\qquad \mathrm{(E)}\ 24$

Solution

Solution 1

If the Cougars won by a margin of 14 points, then the Panthers' score would be half of (34-14). That's 10 $\Rightarrow \boxed{\mathrm{(A)}}$.

Solution 2

Let the Panthers' score be x. The Cougars then scored $x+14$. Since the teams combined scored 34, we get

$$x+x+14=34$$
$$\rightarrow 2x+14=34$$
$$\rightarrow 2x=20$$
$$\rightarrow x=10 \quad ,$$

and the answer is $\boxed{\mathrm{(A)}}$.

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 4

Contents

Problem

Mary is about to pay for five items at the grocery store. The prices of the items are 7.99, 4.99, 2.99, 1.99, and 0.99. Mary will pay with a twenty-dollar bill. Which of the following is closest to the percentage of the 20.00 that she will receive in change?

$\text{(A) } 5 \qquad \text{(B) } 10 \qquad \text{(C) } 15 \qquad \text{(D) } 20 \qquad \text{(E) } 25$

Solution

The total price of the items is $(8 - .01) + (5 - .01) + (3 - .01) + (2 - .01) + (1 - .01) = 19 - .05 = 18.95$

$20 - 18.95 = 1.05$

$\frac{1.05}{20} = .0525 \Rightarrow (A)$

Alternative Solution

We can round the prices to 8, 5, 3, 2, and 1.

So $20 - (8 + 5 + 3 + 2 + 1) = 1$

We can make an equation: $20 * \frac{x}{100} = 1$

If we simplify the equation to "x", we get $x = 5$

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 5

Problem

John is walking east at a speed of 3 miles per hour, while Bob is also walking east, but at a speed of 5 miles per hour. If Bob is now 1 mile west of John, how many minutes will it take for Bob to catch up to John?

$\text{(A) } 30 \qquad \text{(B) } 50 \qquad \text{(C) } 60 \qquad \text{(D) } 90 \qquad \text{(E) } 120$

Solution

The speed that Bob is catching up to John is $5-3=2$ miles per hour. Since Bob is one mile behind John, it will take $\frac{1}{2} \Rightarrow \text{(A)}$ of an hour to catch up to John.

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 6

Problem

Francesca uses 100 grams of lemon juice, 100 grams of sugar, and 400 grams of water to make lemonade. There are 25 calories in 100 grams of lemon juice and 386 calories in 100 grams of sugar. Water contains no calories. How many calories are in 200 grams of her lemonade?

$\mathrm{(A)}\ 129\qquad \mathrm{(B)}\ 137\qquad \mathrm{(C)}\ 174\qquad \mathrm{(D)}\ 223\qquad \mathrm{(E)}\ 411$

Solution

Francesca makes a total of $100+100+400=600$ grams of lemonade, and in those 600 grams, there are 25 calories from the lemon juice and 386 calories from the sugar, for a total of $25+386=411$ calories per 600 grams. We want to know how many calories there are in $200=600/3$ grams, so we just divide 411 by 3 to get $137 \implies \boxed{\mathrm{(B)}}$.

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 7

Problem

Mr. and Mrs. Lopez have two children. When they get into their family car, two people sit in the front, and the other two sit in the back. Either Mr. Lopez or Mrs. Lopez must sit in the driver's seat. How many seating arrangements are possible?

$\text{(A) } 4 \qquad \text{(B) } 12 \qquad \text{(C) } 16 \qquad \text{(D) } 24 \qquad \text{(E) } 48$

Solution

First, we seat the children.

The first child can be seated in 3 spaces.

The second child can be seated in 2 spaces.

Now there are 2×1 ways to seat the adults.

$$3 \times 2 \times 2 = 12 \Rightarrow \text{(B)}$$

Alternative solution:

If there was no restriction, there would be 4!=24 ways to sit. However, only 2/4 of the people can sit in the driver's seat, so our answer is $\frac{2}{4} \cdot 24 = 12 \Rightarrow \text{(B)}$

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 8

Problem

The lines $x = \frac{1}{4}y + a$ and $y = \frac{1}{4}x + b$ intersect at the point $(1,2)$. What is $a + b$?

$\mathrm{(A)}\ 0 \qquad \mathrm{(B)}\ \frac{3}{4} \qquad \mathrm{(C)}\ 1 \qquad \mathrm{(D)}\ 2 \qquad \mathrm{(E)}\ \frac{9}{4}$

Solution

$4x - 4a = y$

$4x - 4a = \frac{1}{4}x + b$

$4 \cdot 1 - 4a = \frac{1}{4} \cdot 1 + b = 2$

$a = \frac{1}{2}$

$b = \frac{7}{4}$

$a + b = \frac{9}{4} \Rightarrow \mathrm{(E)}$

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 9

Contents

Problem

How many even three-digit integers have the property that their digits, read left to right, are in strictly increasing order?

$\text{(A) } 21 \qquad \text{(B) } 34 \qquad \text{(C) } 51 \qquad \text{(D) } 72 \qquad \text{(E) } 150$

Solution

Solution 1

Let's set the middle (tens) digit first. The middle digit can be anything from 2-7 (If it was 1 we would have the hundreds digit to be 0, if it was more than 8, the ones digit cannot be even).

If it was 2, there is 1 possibility for the hundreds digit, 3 for the ones digit. If it was 3, there are 2 possibilities for the hundreds digit, 3 for the ones digit. If it was 4, there are 3 possibilities for the hundreds digit, and 2 for the ones digit,

and so on.

So, the answer is $3(1+2)+2(3+4)+1(5+6)=\boxed{34}\Rightarrow B.$

Solution 2

The last digit is 4, 6, or 8.

If the last digit is x, the possibilities for the first two digits correspond to 2-element subsets of $\{1, 2, \dots, x-1\}$.

Thus the answer is $\binom{3}{2}+\binom{5}{2}+\binom{7}{2}=3+10+21=34.$

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 10

Problem

In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

$\mathrm{(A)}\ 43\qquad\mathrm{(B)}\ 44\qquad\mathrm{(C)}\ 45\qquad\mathrm{(D)}\ 46\qquad\mathrm{(E)}\ 47$

Solution

If the second size has length x, then the first side has length 3x, and we have the third side which has length 15. By the triangle inequality, we have:

$$x+15>3x\Rightarrow 2x<15\Rightarrow x<7.5$$

Now, since we want the greatest perimeter, we want the greatest integer x, and if $x<7.5$ then $x=7$. Then, the first side has length $3*7=21$, the second side has length 7, the third side has length 15, and so the perimeter is $21+7+15=43\Rightarrow\boxed{\mathrm{(A)}}$.

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 11

Problem

Joe and JoAnn each bought 12 ounces of coffee in a 16-ounce cup. Joe drank 2 ounces of his coffee and then added 2 ounces of cream. JoAnn added 2 ounces of cream, stirred the coffee well, and then drank 2 ounces. What is the resulting ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee?

$\text{(A)}\ \frac{6}{7} \qquad \text{(B)}\ \frac{13}{14} \qquad \text{(C)}\ 1 \qquad \text{(D)}\ \frac{14}{13} \qquad \text{(E)}\ \frac{7}{6}$

Solution

Joe has 2 ounces of cream, as stated in the problem.

JoAnn had 14 ounces of liquid, and drank $\frac{1}{7}$ of it. Therefore, she drank $\frac{1}{7}$ of her cream, leaving her $2 * \frac{6}{7}$.

$$\frac{2}{2 * \frac{6}{7}} = \frac{7}{6} \Rightarrow \boxed{\text{(E)}}$$

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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Category: Introductory Algebra Problems

2006 AMC 12B Problems/Problem 12

Problem

The parabola $y = ax^2 + bx + c$ has vertex (p, p) and y-intercept $(0, -p)$, where $p \neq 0$. What is b?

$\text{(A) } -p \qquad \text{(B) } 0 \qquad \text{(C) } 2 \qquad \text{(D) } 4 \qquad \text{(E) } p$

Solution

Substituting $(0, -p)$, we find that $y = -p = a(0)^2 + b(0) + c = c$, so our parabola is $y = ax^2 + bx - p$.

The x-coordinate of the vertex of a parabola is given by $x = p = \frac{-b}{2a} \Longleftrightarrow a = \frac{-b}{2p}$. Additionally, substituting (p, p), we find that

$y = p = a(p)^2 + b(p) - p \Longleftrightarrow ap^2 + (b-2)p = \left(\frac{-b}{2p}\right)p^2 + (b-2)p = p\left(\frac{b}{2} - 2\right) = 0$

. Since it is given that $p \neq 0$, then $\frac{b}{2} = 2 \Longrightarrow b = 4 \text{ (D)}$.

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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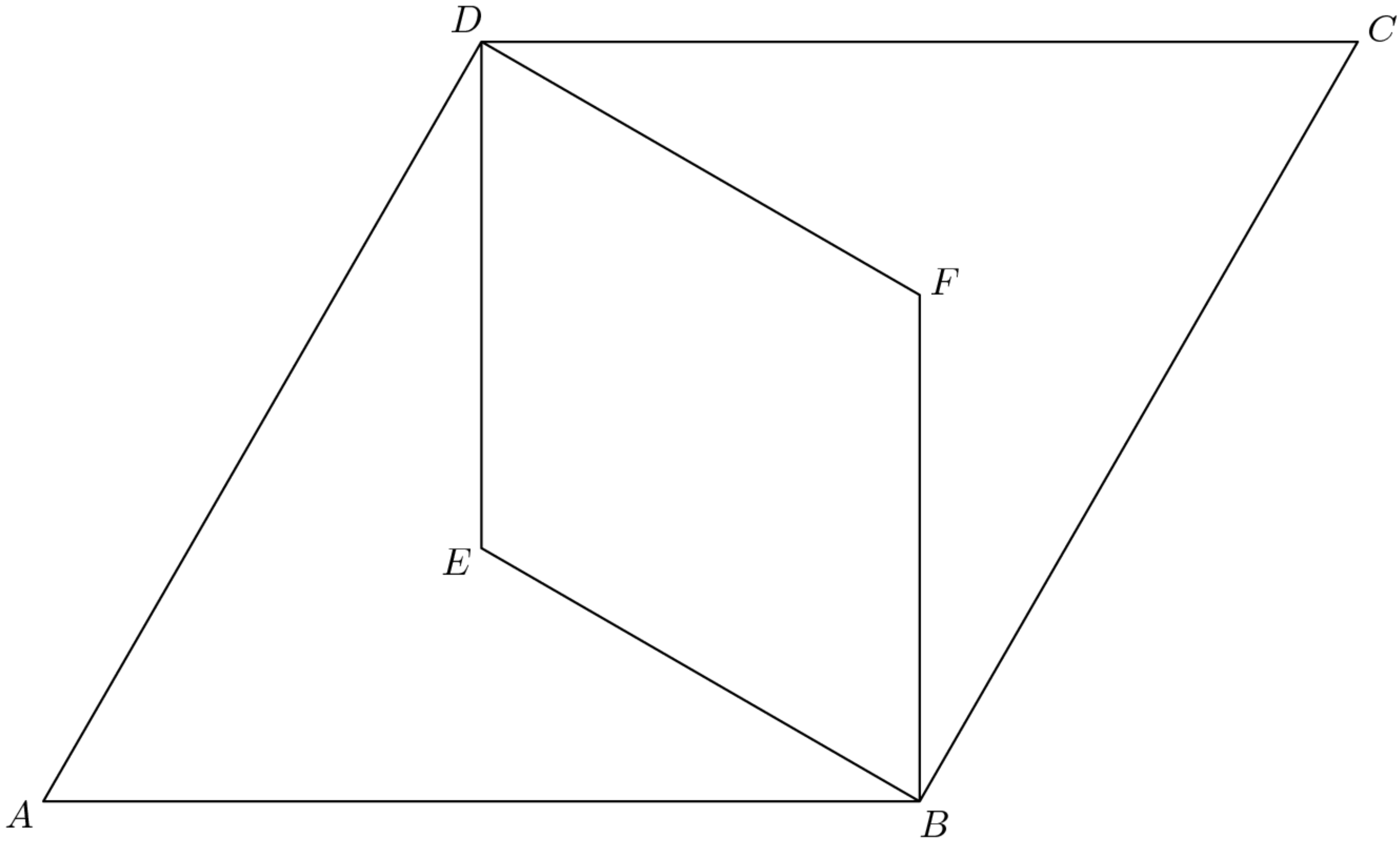
Category: Introductory Algebra Problems

2006 AMC 12B Problems/Problem 13

Problem

Rhombus $ABCD$ is similar to rhombus $BFDE$. The area of rhombus $ABCD$ is 24, and $\angle BAD = 60^\circ$. What is the area of rhombus $BFDE$?



$\mathrm{(A)}\ 6 \qquad \mathrm{(B)}\ 4\sqrt{3} \qquad \mathrm{(C)}\ 8 \qquad \mathrm{(D)}\ 9 \qquad \mathrm{(E)}\ 6\sqrt{3}$

Solution

Solution

The ratio of any length on $ABCD$ to a corresponding length on $BFDE^2$ is equal to the ratio of their areas. Since $\angle BAD = 60$, $\triangle ADB$ and $\triangle DBC$ are equilateral. DB, which is equal to AB, is the diagonal of rhombus $ABCD$. Therefore, $AC = \dfrac{DB(2)}{2\sqrt{3}} = \dfrac{DB}{\sqrt{3}}$. DB and AC are the longer diagonal of rhombuses $BEDF$ and $ABCD$, respectively. So the ratio of their areas is $(\dfrac{1}{\sqrt{3}})^2$ or $\dfrac{1}{3}$. One-third the area of $ABCD$ is equal to 8. So the answer is $\boxed{\text{C}}$.

See also

2006 AMC 12B Problems/Problem 14

Problem

Elmo makes N sandwiches for a fundraiser. For each sandwich he uses B globs of peanut butter at 4 cents per glob and J blobs of jam at 5 cents per glob. The cost of the peanut butter and jam to make all the sandwiches is 2.53. Assume that B, J and N are all positive integers with $N > 1$. What is the cost of the jam Elmo uses to make the sandwiches?

$\text{(A)}\ 1.05 \qquad \text{(B)}\ 1.25 \qquad \text{(C)}\ 1.45 \qquad \text{(D)}\ 1.65 \qquad \text{(E)}\ 1.85$

Solution

From the given, we know that

$253 = N(4B + 5J)$ (The numbers are in cents)

since $253 = 11 \cdot 23$, and since N is an integer, then $4B + 5J = 11$ or 23. It is easily deduced that 11 is impossible to make with B and J integers, so $N = 11$ and $4B + 5J = 23$. Then, it can be guessed and checked quite simply that if $B = 2$ and $J = 3$, then $4B + 5J = 4(2) + 5(3) = 23$. The problem asks for the total cost of jam, or $N(5J) = 11(15) = 165$ cents, or $1.65 \implies \text{(D)}$

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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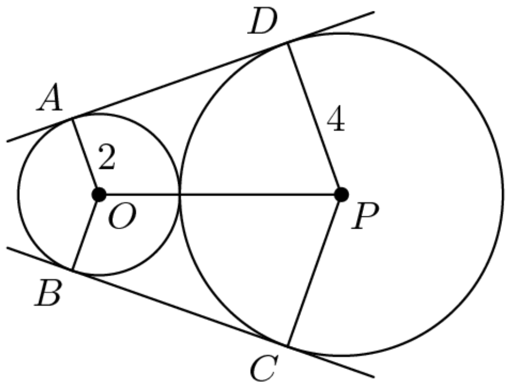
2006 AMC 12B Problems/Problem 15

Contents

Problem

Circles with centers O and P have radii 2 and 4, respectively, and are externally tangent. Points A and B are on the circle centered at O, and points C and D are on the circle centered at P, such that $\overline{AD}$ and $\overline{BC}$ are common external tangents to the circles. What is the area of hexagon $AOBCPD$?



$\textbf{(A)}\ 18\sqrt{3}\qquad\textbf{(B)}\ 24\sqrt{2}\qquad\textbf{(C)}\ 36\qquad\textbf{(D)}\ 24\sqrt{3}\qquad\textbf{(E)}\ 32\sqrt{2}$

Solution

Draw the altitude from O onto DP and call the point H. Because $\angle OAD$ and $\angle ADP$ are right angles due to being tangent to the circles, and the altitude creates $\angle OHD$ as a right angle. $ADHO$ is a rectangle with OH bisecting DP. The length OP is $4+2$ and HP has a length of 2, so by pythagorean's, OH is $\sqrt{32}$.

$2\cdot\sqrt{32}+\frac{1}{2}\cdot2\cdot\sqrt{32}=3\sqrt{32}=12\sqrt{2}$, which is half the area of the hexagon, so the area of the entire hexagon is $2\cdot12\sqrt{2}=\boxed{(B)}\qquad24\sqrt{2}$

Solution 2

$ADOP$ and $OPBC$ are congruent right trapezoids with legs 2 and 4 and with OP equal to 6. Draw an altitude from O to either DP or CP, creating a rectangle with width 2 and base x, and a right triangle with one leg 2, the hypotenuse 6, and the other x. Using the Pythagorean theorem, x is equal to $4\sqrt{2}$, and x is also equal to the height of the trapezoid. The area of the trapezoid is thus $\frac{1}{2}\cdot(4+2)\cdot4\sqrt{2}=12$, and the total area is two trapezoids, or $\boxed{24\sqrt{2}}$.

See also

2006 AMC 12B Problems/Problem 16

Problem

Regular hexagon $ABCDEF$ has vertices A and C at $(0, 0)$ and $(7, 1)$, respectively. What is its area?

$\text{(A) } 20\sqrt{3} \qquad \text{(B) } 22\sqrt{3} \qquad \text{(C) } 25\sqrt{3} \qquad \text{(D) } 27\sqrt{3} \qquad \text{(E) } 50$

Solution

To find the area of the regular hexagon, we only need to calculate the side length.

Drawing in points A, B, and C, and connecting A and C with an auxiliary line, we see two 30-60-90 triangles are formed.

Points A and C are a distance of $\sqrt{7^2+1^2} = \sqrt{50} = 5\sqrt{2}$ apart. Half of this distance is the length of the longer leg of the right triangles. Therefore, the side length of the hexagon is $\frac{5\sqrt{2}}{2} \cdot \frac{1}{\sqrt{3}} \cdot 2 = \frac{5\sqrt{6}}{3}$.

The apothem is thus $\frac{1}{2} \cdot \frac{5\sqrt{6}}{3} \cdot \sqrt{3} = \frac{5\sqrt{2}}{2}$, yielding an area of $\frac{1}{2} \cdot 10\sqrt{6} \cdot \frac{5\sqrt{2}}{2} = 25\sqrt{3}$.

See also

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2006 AMC 12B Problems/Problem 17

Problem

For a particular peculiar pair of dice, the probabilities of rolling 1, 2, 3, 4, 5 and 6 on each die are in the ratio $1:2:3:4:5:6$. What is the probability of rolling a total of 7 on the two dice?

$\mathrm{(A)}\ \frac{4}{63}\qquad \mathrm{(B)}\ \frac{1}{8}\qquad \mathrm{(C)}\ \frac{8}{63}\qquad \mathrm{(D)}\ \frac{1}{6}\qquad \mathrm{(E)}\ \frac{2}{7}$

Solution

The probability of getting an x on one of these dice is $\frac{x}{21}$.

The probability of getting 1 on the first and 6 on the second die is $\frac{1}{21}\cdot\frac{6}{21}$. Similarly we can express the probabilities for the other five ways how we can get a total 7. (Note that we only need the first three, the other three are symmetric.)

Summing these, the probability of getting a total 7 is:

$$2\cdot\left(\frac{1}{21}\cdot\frac{6}{21}+\frac{2}{21}\cdot\frac{5}{21}+\frac{3}{21}\cdot\frac{4}{21}\right)=\frac{56}{441}=\boxed{\frac{8}{63}}$$

See also 2016 AIME I Problems/Problem 2

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 18

Contents

Problem

An object in the plane moves from one lattice point to another. At each step, the object may move one unit to the right, one unit to the left, one unit up, or one unit down. If the object starts at the origin and takes a ten-step path, how many different points could be the final point?

$\mathrm{(A)}\ 120 \qquad \mathrm{(B)}\ 121 \qquad \mathrm{(C)}\ 221 \qquad \mathrm{(D)}\ 230 \qquad \mathrm{(E)}\ 231$

Solution

Let the starting point be $(0,0)$. After 10 steps we can only be in locations (x,y) where $|x|+|y|\leq 10$. Additionally, each step changes the parity of exactly one coordinate. Hence after 10 steps we can only be in locations (x,y) where $x+y$ is even. It can easily be shown that each location that satisfies these two conditions is indeed reachable.

Once we pick $x\in\{-10,\dots,10\}$, we have $11-|x|$ valid choices for y, giving a total of $\boxed{121}$ possible positions.

Solution 2

10 moves results in a lot of possible endpoints, so we try small cases first.

If the object only makes 1 move, it is obvious that there are only 4 possible points that the object can move to. If the object makes 2 moves, it can move to $(0,2)$, $(1,1)$, $(2,0)$, $(1,-1)$, $(0,-2)$, $(-1,-1)$, $(-2,0)$ as well as $(0,0)$, for a total of 9 moves. If the object makes 3 moves, it can end up at $(0,3)$, $(2,1)$, $(1,2)$, $(3,0)$, $(2,-1)$, $(1,-2)$, $(0,-3)$, $(-1,-2)$, $(-2,-1)$, $(0,-3)$ etc. for a total of 25.

At this point we can guess that for n moves, there are $(n+1)^2$ different endpoints. Thus, for 10 moves, there are $11^2=121$ endpoints, and the answer is $\boxed{B}$.

See also

2006 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12B Problems/Problem 19

Contents

Problem

Mr. Jones has eight children of different ages. On a family trip his oldest child, who is 9, spots a license plate with a 4-digit number in which each of two digits appears two times. "Look, daddy!" she exclaims. "That number is evenly divisible by the age of each of us kids!" "That's right," replies Mr. Jones, "and the last two digits just happen to be my age." Which of the following is not the age of one of Mr. Jones's children?

$\mathrm{(A) \ } 4\qquad \mathrm{(B) \ } 5\qquad \mathrm{(C) \ } 6\qquad \mathrm{(D) \ } 7\qquad \mathrm{(E) \ } 8$

Solution

First, The number of the plate is divisible by 9 and in the form of $aabb$, $abba$ or $abab$.

We can conclude straight away that $a+b=9$ using the 9 divisibility rule.

If $b=1$, the number is not divisible by 2 (unless it's 1818, which is not divisible by 4), which means there are no 2, 4, 6, or 8 year olds on the car, but that can't be true, as that would mean there are less than 8 kids on the car.

If $b=2$, then the only possible number is 7272. 7272 is divisible by 4, 6, and 8, but not by 5 and 7, so that doesn't work.

If $b=3$, then the only number is 6336, also not divisible by 5 or 7.

If $b=4$, the only number is 5544. It is divisible by 4, 6, 7, and 8.

Therefore, we conclude that the answer is $\mathrm{(B) \ } 5$

NOTE: Automatically, since there are 8 children and all of their ages are less than or equal to 9 and are different, the answer choices can be narrowed down to 5 or 8.

Alternate Solution

We know that the number of the plate is divisible by 9 and in the form $aabb$, $abba$, or $abab$ for distinct digits a,b.

Using the divisibility rule for 9, we can conclude that $a+b\equiv 0 \pmod{9}$.

We also know that the number of the plate is even, because you can only discard one number from the integers 1 through 9, inclusive (8 children, oldest is 9), and there's always going to be an even number left.

If one of the children was 5 years old, then the plate number would be divisible by both 5 and 2. Thus, the units digit must be 0. Then, the possible form of the plate number would be $aa00$, $0bb0$, or $a0a0$. The first case is not possible because 00 is not a possible age for the father.

We have the remaining forms $0bb0$ and $a0a0$.

Case 1: 0bb0: We know that $2b\equiv 0 \pmod{9}$, so b has to be either 0 or 9. b can't be 0 because a,b are distinct. So, $b=9$. However, the number 0990 is not divisible by both 4 and 7, so whichever number you discard, the other one will still be there. This creates a contradiction, so Case 1 cannot be true.

Case 2: a0a0: Similarly to Case 1, we can determine that a has to be 9. However, the number 9090 is not divisible by both 4 and 7. Using the same logic in case 1, we conclude that Case 2 cannot be true.

Since we have disproven all of our cases, we know that it is impossible for one of the children to be $\text{(B) } 5$ years old.

NOTE: This might look tedious, but it only took me around 30 seconds to do on paper.

See also

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2006 AMC 12B Problems/Problem 20

Problem

Let x be chosen at random from the interval $(0,1)$. What is the probability that $\lfloor\log_{10}4x\rfloor - \lfloor\log_{10}x\rfloor = 0$? Here $\lfloor x\rfloor$ denotes the greatest integer that is less than or equal to x.

$$\mathrm{(A)}\ \frac{1}{8} \qquad \mathrm{(B)}\ \frac{3}{20} \qquad \mathrm{(C)}\ \frac{1}{6} \qquad \mathrm{(D)}\ \frac{1}{5} \qquad \mathrm{(E)}\ \frac{1}{4}$$

Solution

Let k be an arbitrary integer. For which x do we have $\lfloor \log_{10} 4x\rfloor = \lfloor\log_{10} x\rfloor = k$?

The equation $\lfloor\log_{10} x\rfloor = k$ can be rewritten as $10^k \leq x < 10^{k+1}$. The second one gives us $10^k \leq 4x < 10^{k+1}$. Combining these, we get that both hold at the same time if and only if $10^k \leq x < \frac{10^{k+1}}{4}$.

Hence for each integer k we get an interval of values for which $\lfloor\log_{10}4x\rfloor - \lfloor\log_{10}x\rfloor = 0$. These intervals are obviously pairwise disjoint.

For any $k \geq 0$ the corresponding interval is disjoint with $(0,1)$, so it does not contribute to our answer. On the other hand, for any $k<0$ the entire interval is inside $(0,1)$. Hence our answer is the sum of the lengths of the intervals for $k<0$.

For a fixed k the length of the interval $\left[10^k, \frac{10^{k+1}}{4}\right)$ is $\frac{3}{2}\cdot 10^k$.

This means that our result is $\frac{3}{2}\left(10^{-1}+10^{-2}+\cdots\right) = \frac{3}{2}\cdot\frac{1}{9} = \boxed{\frac{1}{6}}$.

See also

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2006 AMC 12B Problems/Problem 22

Contents

Problem

Suppose a, b and c are positive integers with $a+b+c=2006$, and $a!b!c!=m\cdot 10^n$, where m and n are integers and m is not divisible by 10. What is the smallest possible value of n?

$\mathrm{(A)}\ 489 \qquad \mathrm{(B)}\ 492 \qquad \mathrm{(C)}\ 495 \qquad \mathrm{(D)}\ 498 \qquad \mathrm{(E)}\ 501$

Solution 1

The power of 10 for any factorial is given by the well-known algorithm

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n}{125} \right\rfloor + \cdots$$

It is rational to guess numbers right before powers of 5 because we won't have any extra numbers from higher powers of 5. As we list out the powers of 5, it is clear that $5^4=625$ is less than 2006 and $5^5=3125$ is greater. Therefore, set a and b to be 624. Thus, c is $2006-(624\cdot 2)=758$. Applying the algorithm, we see that our answer is $152+152+188=\boxed{492}$.

Solution 2

Clearly, the power of 2 that divides $n!$ is larger or equal than the power of 5 which divides it. Hence we are trying to minimize the power of 5 that will divide $a!b!c!$.

Consider $n!=1\cdot 2\cdot\dots\cdot n$. Each fifth term is divisible by 5, each 25-th one by 25, and so on. Hence the total power of 5 that divides n is $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \cdots$. (For any n only finitely many terms in the sum are non-zero.)

In our case we have $a<2006$, so the largest power of 5 that will be less than a is at most $5^4=625$. Therefore the power of 5 that divides $a!$ is equal to $\left\lfloor \frac{a}{5} \right\rfloor + \left\lfloor \frac{a}{25} \right\rfloor + \left\lfloor \frac{a}{125} \right\rfloor + \left\lfloor \frac{a}{625} \right\rfloor$. The same is true for b and c.

Intuition may now try to lure us to split 2006 into $a+b+c$ as evenly as possible, giving $a=b=669$ and $c=668$. However, this solution is not optimal.

To see how we can do better, let's rearrange the terms as follows:

$$\begin{aligned} result = \left\lfloor\frac{a}{5}\right\rfloor + \left\lfloor\frac{b}{5}\right\rfloor + \left\lfloor\frac{c}{5}\right\rfloor \\ + \left\lfloor\frac{a}{25}\right\rfloor + \left\lfloor\frac{b}{25}\right\rfloor + \left\lfloor\frac{c}{25}\right\rfloor \\ + \left\lfloor\frac{a}{125}\right\rfloor + \left\lfloor\frac{b}{125}\right\rfloor + \left\lfloor\frac{c}{125}\right\rfloor \\ + \left\lfloor\frac{a}{625}\right\rfloor + \left\lfloor\frac{b}{625}\right\rfloor + \left\lfloor\frac{c}{625}\right\rfloor \end{aligned}$$

The idea is that the rows of the above equation are roughly equal to $\left\lfloor\frac{n}{5}\right\rfloor$, $\left\lfloor\frac{n}{25}\right\rfloor$, etc.

More precisely, we can now notice that for any positive integers a, b, c, k we can write a, b, c in the form $a = a_0k + a_1$, $b = b_0k + b_1$, $c = c_0k + c_1$, where all a_i, b_i, c_i are integers and $0 \leq a_1, b_1, c_1 < k$.

It follows that

$$\left\lfloor\frac{a}{k}\right\rfloor + \left\lfloor\frac{b}{k}\right\rfloor + \left\lfloor\frac{c}{k}\right\rfloor = a_0 + b_0 + c_0$$

and

$$\left\lfloor\frac{a+b+c}{k}\right\rfloor = a_0 + b_0 + c_0 + \left\lfloor\frac{a_1+b_1+c_1}{k}\right\rfloor \leq a_0 + b_0 + c_0 + 2$$

Hence we get that for any positive integers a, b, c, k we have

$$\left\lfloor\frac{a}{k}\right\rfloor + \left\lfloor\frac{b}{k}\right\rfloor + \left\lfloor\frac{c}{k}\right\rfloor \quad \geq \quad \left\lfloor\frac{a+b+c}{k}\right\rfloor - 2$$

Therefore for any a, b, c the result is at least $\left\lfloor\frac{n}{5}\right\rfloor + \left\lfloor\frac{n}{25}\right\rfloor + \left\lfloor\frac{n}{125}\right\rfloor + \left\lfloor\frac{n}{625}\right\rfloor - 8 = 401 + 80 + 16 + 3 - 8 = 500 - 8 = 492$.

If we now show how to pick a, b, c so that we'll get the result 492, we will be done.

Consider the row with 625 in the denominator. We need to achieve sum 1 in this row, hence we need to make two of the numbers smaller than 625. Choosing $a = b = 624$ does this, and it will give us the largest possible remainders for a and b in the other three rows, so this is a pretty good candidate. We can compute $c = 2006 - a - b = 758$ and verify that this triple gives the desired result $\boxed{492}$.

See also

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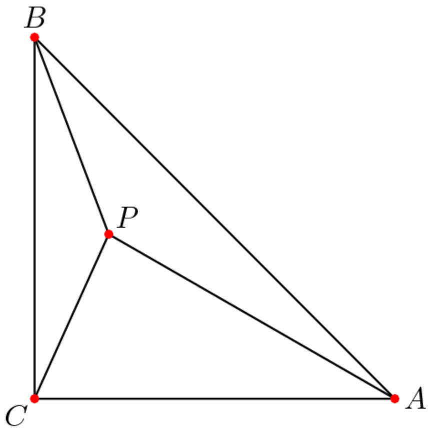
2006 AMC 12B Problems/Problem 23

Contents

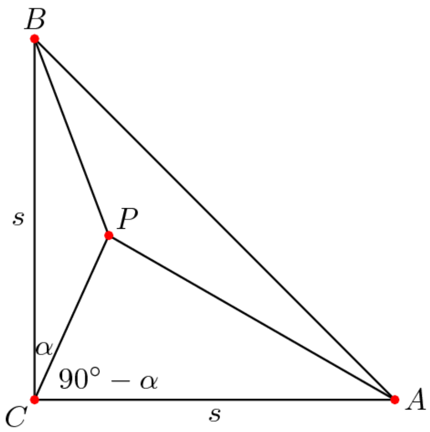
Problem

Isosceles $\triangle ABC$ has a right angle at C. Point P is inside $\triangle ABC$, such that $PA = 11$, $PB = 7$, and $PC = 6$. Legs $\overline{AC}$ and $\overline{BC}$ have length $s = \sqrt{a + b\sqrt{2}}$, where a and b are positive integers. What is $a + b$?



$\textbf{(A) } 85 \qquad \textbf{(B) } 91 \qquad \textbf{(C) } 108 \qquad \textbf{(D) } 121 \qquad \textbf{(E) } 127$

Solution



Using the Law of Cosines on $\triangle PBC$, we have:

$$PB^2 = BC^2 + PC^2 - 2 \cdot BC \cdot PC \cdot \cos(\alpha) \Rightarrow 49 = 36 + s^2 - 12s\cos(\alpha) \Rightarrow \cos(\alpha) = \frac{s^2 - 13}{12s}.$$

Using the Law of Cosines on $\triangle PAC$, we have:

$$PA^2 = AC^2 + PC^2 - 2 \cdot AC \cdot PC \cdot \cos(90^\circ - \alpha) \Rightarrow 121 = 36 + s^2 - 12s\sin(\alpha) \Rightarrow \sin(\alpha) = \frac{s^2 - 85}{12s}.$$

Now we use $\sin^2(\alpha) + \cos^2(\alpha) = 1$.

$$\begin{aligned}
\sin^2(\alpha)+\cos^2(\alpha)=1 &\Rightarrow \frac{s^4-26s^2+169}{144s^2}+\frac{s^4-170s^2+7225}{144s^2}=1\\
&\Rightarrow 2s^4-340s^2+7394=0\\
&\Rightarrow s^4-170s^2+3697=0\\
&\Rightarrow s^2=\frac{170\pm\sqrt{170^2-4\cdot 3697}}{2}\\
&\Rightarrow s^2=\frac{170\pm\sqrt{28900-14788}}{2}\\
&\Rightarrow s^2=\frac{170\pm\sqrt{14112}}{2}\\
&\Rightarrow s^2=\frac{170\pm\sqrt{2^5\cdot 3^2\cdot 7^2}}{2}\\
&\Rightarrow s^2=\frac{170\pm 84\sqrt{2}}{2}=85\pm 42\sqrt{2}
\end{aligned}$$

Note that we know that we want the solution with $s^2>85$ since we know that $\sin(\alpha)>0$. Thus, $a+b=85+42=\boxed{127}$.

Solution 2

Rotate triangle PAC 90 degrees counterclockwise about C so that the image of A rests on B. Now let the image of P be P'. Note that $P'C=6$, meaning triangle PCP' is right isosceles, and $\angle PP'C=45^\circ$. Then $PP'=6\sqrt{2}$. Now because $PB=7$ and $P'B=11$, we observe that $\angle P'PB=90^\circ$, by the Pythagorean Theorem on $P'PB$. Now we have that $\angle APC=\angle BP'C=\angle BP'P+\angle PP'C$. So we take the cosine of the second equality, using that fact that $\angle PP'C=45^\circ$, to get $\cos(BP'C)=\dfrac{6\sqrt{2}-7}{11\sqrt{2}}$. Finally, we use the fact that $\cos(BP'C)=\cos(CPA)$ and use the Law of Cosines on triangle CPA to arrive at the value of s^2.

Or notice that since $\angle P'PB=90^\circ$ and $\angle PP'C=45^\circ$, we have $\angle BPC=135^\circ$, and Law of Cosines on triangle BPC gives the value of s^2.

See also

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2006 AMC 12B Problems/Problem 24

Problem

Let S be the set of all point (x, y) in the coordinate plane such that $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq \frac{\pi}{2}$. What is the area of the subset of S for which

$$\sin^2 x - \sin x \sin y + \sin^2 y \leq \frac{3}{4}?$$

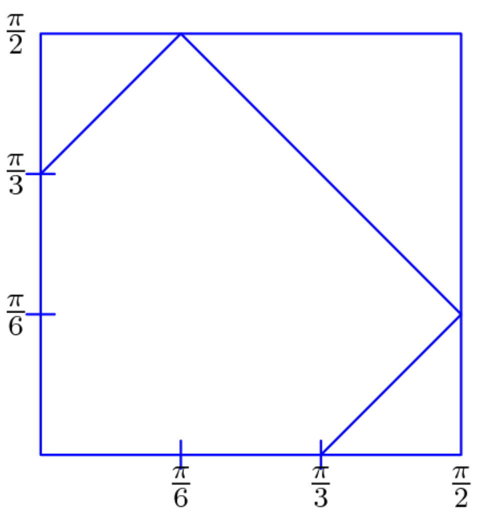
$\text{(A)}\ \frac{\pi^2}{9} \qquad \text{(B)}\ \frac{\pi^2}{8} \qquad \text{(C)}\ \frac{\pi^2}{6} \qquad \text{(D)}\ \frac{3\pi^2}{16} \qquad \text{(E)}\ \frac{2\pi^2}{9}$

Solution

We start out by solving the equality first.

$$\begin{aligned}
\sin^2 x - \sin x \sin y + \sin^2 y &= \frac{3}{4} \\
\sin x &= \frac{\sin y \pm \sqrt{\sin^2 y - 4(\sin^2 y - \frac{3}{4})}}{2} \\
&= \frac{\sin y \pm \sqrt{3 - 3\sin^2 y}}{2} \\
&= \frac{\sin y \pm \sqrt{3\cos^2 y}}{2} \\
&= \frac{1}{2}\sin y \pm \frac{\sqrt{3}}{2}\cos y \\
\sin x &= \sin(y \pm \frac{\pi}{3})
\end{aligned}$$

We end up with three lines that matter: $x = y + \frac{\pi}{3}$, $x = y - \frac{\pi}{3}$, and $x = \pi - (y + \frac{\pi}{3}) = \frac{2\pi}{3} - y$. We plot these lines below.



Note that by testing the point $(\pi/6, \pi/6)$, we can see that we want the area of the pentagon. We can calculate that by calculating the area of the square and then subtracting the area of the 3 triangles. (Note we could also do this by adding the areas of the isosceles triangle in the bottom left corner and the rectangle with the previous triangle's hypotenuse as the longer side.)

$$A = \left(\frac{\pi}{2}\right)^2 - 2 \cdot \frac{1}{2} \cdot \left(\frac{\pi}{6}\right)^2 - \frac{1}{2} \cdot \left(\frac{\pi}{3}\right)^2$$
$$= \pi^2 \left(\frac{1}{4} - \frac{1}{36} - \frac{1}{18}\right)$$
$$= \pi^2 \left(\frac{9-1-2}{36}\right) = \boxed{\text{(C) } \frac{\pi^2}{6}}$$

See also

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Category: Introductory Algebra Problems

2006 AMC 12B Problems/Problem 25

Problem

A sequence $a_1, a_2, \ldots$ of non-negative integers is defined by the rule $a_{n+2} = |a_{n+1} - a_n|$ for $n \geq 1$. If $a_1 = 999$, $a_2 < 999$ and $a_{2006} = 1$, how many different values of a_2 are possible?

$\text{(A)}\ 165 \qquad \text{(B)}\ 324 \qquad \text{(C)}\ 495 \qquad \text{(D)}\ 499 \qquad \text{(E)}\ 660$

Solution

We say the sequence (a_n) completes at i if i is the minimal positive integer such that $a_i = a_{i+1} = 1$. Otherwise, we say (a_n) does not complete.

Note that if $d = \gcd(999, a_2) \neq 1$, then $d | a_n$ for all $n \geq 1$, and d does not divide 1, so if $\gcd(999, a_2) \neq 1$, then (a_n) does not complete. (Also, a_{2006} cannot be 1 in this case since d does not divide 1, so we do not care about these a_2 at all.)

From now on, suppose $\gcd(999, a_2) = 1$.

We will now show that (a_n) completes at i for some $i \leq 2006$. We will do this with 3 lemmas.

Lemma: If $a_j \neq a_{j+1}$, and neither value is 0, then $\max(a_j, a_{j+1}) > \max(a_{j+2}, a_{j+3})$.

Proof: There are 2 cases to consider.

If $a_j > a_{j+1}$, then $a_{j+2} = a_j - a_{j+1}$, and $a_{j+3} = |a_j - 2a_{j+1}|$. So $a_j > a_{j+2}$ and $a_j > a_{j+3}$.

If $a_j < a_{j+1}$, then $a_{j+2} = a_{j+1} - a_j$, and $a_{j+3} = a_j$. So $a_{j+1} > a_{j+2}$ and $a_{j+1} > a_{j+3}$.

In both cases, $\max(a_j, a_{j+1}) > \max(a_{j+2}, a_{j+3})$, as desired.

Lemma: If $a_i = a_{i+1}$, then $a_i = 1$. Moreover, if instead we have $a_i = 0$ for some $i > 2$, then $a_{i-1} = a_{i-2} = 1$.

Proof: By the way (a_n) is constructed in the problem statement, having two equal consecutive terms $a_i = a_{i+1}$ implies that a_i divides every term in the sequence. So $a_i | 999$ and $a_i | a_2$, so $a_i | \gcd(999, a_2) = 1$, so $a_i = 1$. For the proof of the second result, note that if $a_i = 0$, then $a_{i-1} = a_{i-2}$, so by the first result we just proved, $a_{i-2} = a_{i-1} = 1$.

Lemma: (a_n) completes at i for some $i \leq 2000$.

Proof: Suppose (a_n) completed at some $i > 2000$ or not at all. Then by the second lemma and the fact that neither 999 nor a_2 are 0, none of the pairs $(a_1, a_2), \ldots, (a_{1999}, a_{2000})$ can have a 0 or be equal to $(1, 1)$. So the first lemma implies

$$\max(a_1, a_2) > \max(a_3, a_4) > \cdots > \max(a_{1999}, a_{2000}) > 0,$$

so $999 = \max(a_1, a_2) \geq 1000$, a contradiction. Hence (a_n) completes at i for some $i \leq 2000$.

Now we're ready to find exactly which values of a_2 we want to count.

Let's keep in mind that $2006 \equiv 2 \pmod 3$ and that $a_1 = 999$ is odd. We have two cases to consider.

Case 1: If a_2 is odd, then a_3 is even, so a_4 is odd, so a_5 is odd, so a_6 is even, and this pattern must repeat every three terms because of the recursive definition of (a_n), so the terms of (a_n) reduced modulo 2 are

$$1, 1, 0, 1, 1, 0, ...,$$

so a_{2006} is odd and hence 1 (since if (a_n) completes at i, then a_k must be 0 or 1 for all $k \geq i$).

Case 2: If a_2 is even, then a_3 is odd, so a_4 is odd, so a_5 is even, so a_6 is odd, and this pattern must repeat every three terms, so the terms of (a_n) reduced modulo 2 are

$$1, 0, 1, 1, 0, 1, ...,$$

so a_{2006} is even, and hence 0.

We have found that $a_{2006} = 1$ is true precisely when $\gcd(999, a_2) = 1$ and a_2 is odd. This tells us what we need to count.

There are $\phi(999) = 648$ numbers less than 999 and relatively prime to it (ϕ is the Euler totient function). We want to count how many of these are even. Note that

$$t \mapsto 999 - t$$

is a 1-1 correspondence between the odd and even numbers less than and relatively prime to 999. So our final answer is $648/2 = 324$, or $\boxed{\text{B}}$.

See also

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