2018 AMC 10B Problems/Problem 1

Contents

Problem

Kate bakes a 20-inch by 18-inch pan of cornbread. The cornbread is cut into pieces that measure 2 inches by 2 inches. How many pieces of cornbread does the pan contain?

$\textbf{(A) } 90 \qquad \textbf{(B) } 100 \qquad \textbf{(C) } 180 \qquad \textbf{(D) } 200 \qquad \textbf{(E) } 360$

Solution 1

The area of the pan is $20 \cdot 18 = 360$. Since the area of each piece is 4, there are $\frac{360}{4} = 90$ pieces. Thus, the answer is $\boxed{A}$.

Solution 2

By dividing each of the dimensions by 2, we get a 10×9 grid which makes 90 pieces. Thus, the answer is $\boxed{A}$.

See Also

2018 AMC 10B (**Problems** • **Answer Key** • Resources)	
Preceded by **First Problem**	Followed by **Problem 2**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **First Problem**	Followed by **Problem 2**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 10B Problems/Problem 2

Problem

Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?

$\textbf{(A) } 64 \qquad \textbf{(B) } 65 \qquad \textbf{(C) } 66 \qquad \textbf{(D) } 67 \qquad \textbf{(E) } 68$

Solution

Let Sam drive at exactly 60 mph in the first half hour, 65 mph in the second half hour, and x mph in the third half hour.

Due to $rt = d$, and that 30 min is half an hour, he covered $60 \cdot \frac{1}{2} = 30$ miles in the first 30 mins.

SImilarly, he covered $\frac{65}{2}$ miles in the 2nd half hour period.

The problem states that Sam drove 96 miles in 90 min, so that means that he must have covered $96 - \left(30 + \frac{65}{2}\right) = 33\frac{1}{2}$ miles in the third half hour period.

$rt = d$, so $x \cdot \frac{1}{2} = 33\frac{1}{2}$.

Therefore, Sam was driving $\boxed{\textbf{(D) } 67}$ miles per hour in the third half hour.

See Also

2018 AMC 10B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 1**	Followed by **Problem 3**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 1**	Followed by **Problem 3**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 12B Problems/Problem 3

Contents

Problem

A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the x-intercepts of these two lines?

$\textbf{(A) } 5 \qquad \textbf{(B) } 10 \qquad \textbf{(C) } 20 \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 50$

Solution 1

Using the slope-intercept form, we get the equations $y-30 = 6(x-40)$ and $y-30 = 2(x-40)$. Simplifying, we get $6x-y=210$ and $2x-y=50$. Letting $y=0$ in both equations and solving for x gives the x-intercepts: $x=35$ and $x=25$, respectively. Thus the distance between them is $35-25=10 \Rightarrow \boxed{\textbf{(B) } 10}$

Solution 2

In order for the line with slope 2 to travel "up" 30 units (from $y=0$), it must have traveled $30/2=15$ units to the right. Thus, the x-intercept is at $x=40-15=25$. As for the line with slope 6, in order for it to travel "up" 30 units it must have traveled $30/6=5$ units to the right. Thus its x-intercept is at $x=40-5=35$. Then the distance between them is $35-25=10 \Rightarrow \boxed{\textbf{(B) } 10}$

See Also

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 2**	Followed by **Problem 4**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2018 AMC 12B Problems/Problem 4

Problem

A circle has a chord of length 10, and the distance from the center of the circle to the chord is 5. What is the area of the circle?

$\textbf{(A)}\ 25\pi \qquad \textbf{(B)}\ 50\pi \qquad \textbf{(C)}\ 75\pi \qquad \textbf{(D)}\ 100\pi \qquad \textbf{(E)}\ 125\pi$

Solution

The shortest segment that connects the center of the circle to a chord is the perpendicular bisector of the chord. Applying the Pythagorean theorem, we find that

$$r^2 = 5^2 + 5^2 = 50$$

The area of a circle is πr^2, so the answer is $\boxed{\text{(B)} 50\pi}$

See Also

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 3**	Followed by **Problem 5**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 12B Problems/Problem 5

Contents

Problem

How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

$\textbf{(A)} 128 \qquad \textbf{(B)} 192 \qquad \textbf{(C)} 224 \qquad \textbf{(D)} 240 \qquad \textbf{(E)} 256$

Solution 1

Since an element of a subset is either in or out, the total number of subsets of the 8 element set is $2^8 = 256$. However, since we are only concerned about the subsets with at least 1 prime in it, we can use complementary counting to count the subsets without a prime and subtract that from the total. Because there are 4 non-primes, there are $2^8 - 2^4 = 240$ subsets with at least 1 prime so the answer is $\Rightarrow \boxed{\textbf{(D)} 240}$

Solution 2

We can construct our subset by choosing which primes are included and which composites are included. There are $2^4 - 1$ ways to select the primes (total subsets minus the empty set) and 2^4 ways to select the composites. Thus, there are $15 * 16$ ways to choose a subset of the eight numbers, or $\boxed{\textbf{(D)} 240}$.

See Also

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 4**	Followed by **Problem 6**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Combinatorics Problems

2018 AMC 12B Problems/Problem 6

Problem

Suppose S cans of soda can be purchased from a vending machine for Q quarters. Which of the following expressions describes the number of cans of soda that can be purchased for D dollars, where 1 dollar is worth 4 quarters?

$\textbf{(A)}\ \frac{4DQ}{S} \qquad \textbf{(B)}\ \frac{4DS}{Q} \qquad \textbf{(C)}\ \frac{4Q}{DS} \qquad \textbf{(D)}\ \frac{DQ}{4S} \qquad \textbf{(E)}\ \frac{DS}{4Q}$

Solution 1

The unit price for a can of soda (in quarters) is $\frac{S}{Q}$. Thus, the number of cans which can be bought for D dollars ($4D$ quarters) is $\boxed{\textbf{(B)}\ \frac{4DS}{Q}}$

See Also

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 5**	Followed by **Problem 7**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 12B Problems/Problem 7

Contents

Problem

What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27?$$

$\textbf{(A) } 3 \qquad \textbf{(B) } 3\log_{7}23 \qquad \textbf{(C) } 6 \qquad \textbf{(D) } 9 \qquad \textbf{(E) } 10$

Solution 1

Change of base makes this $\frac{\prod_{i=3}^{13} \log (2i+1)}{\prod_{i=1}^{11}\log (2i+1)} = \frac{\log 25 \log 27}{\log 3 \log 5} = \log_3 27 \cdot \log_5 25 = \boxed{6}$

Solution 2

Using the chain rule for logarithms ($\log_a b \cdot \log_b c = \log_a c$), we get $\log_3 7 \cdot \log_5 9 \cdots \log_{23} 27 = (\log_3 7 \cdot \log_7 11 \cdots \log_{23} 27) \cdot (\log_5 9 \cdot \log_9 13 \cdots \log_{21} 25) = \log_3 27 \cdot \log_5 25 = 3 \cdot 2 = 6$.

See Also

2018 AMC 12B (Problems • Answer Key • Resources)	
Preceded by **Problem 6**	Followed by **Problem 8**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2018 AMC 12B Problems/Problem 8

Problem

Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

$\textbf{(A)} 25 \qquad \textbf{(B)} 32 \qquad \textbf{(C)} 50 \qquad \textbf{(D)} 63 \qquad \textbf{(E)} 75$

Solution

Draw the Median connecting C to the center O of the circle. Note that the centroid is $\frac{1}{3}$ of the distance from O to C.

Thus, as C traces a circle of radius 12, the Centroid will trace a circle of radius $\frac{12}{3} = 4$.

The area of this circle is $\pi \cdot 4^2 = 16\pi \approx \boxed{50}$.

See Also

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 7**	Followed by **Problem 9**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 12B Problems/Problem 9

Contents

Problem

What is

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j)?$$

$\textbf{(A) }100,100 \qquad \textbf{(B) }500,500\qquad \textbf{(C) }505,000 \qquad \textbf{(D) }1,001,000 \qquad \textbf{(E) }1,010,000$

Solution 1

We can start by writing out the first couple of terms:

$$(1+1)+(1+2)+(1+3)+\cdots+(1+100)$$

$$(2+1)+(2+2)+(2+3)+\cdots+(2+100)$$

$$(3+1)+(3+2)+(3+3)+\cdots+(3+100)$$

$$\vdots$$

$$(100+1)+(100+2)+(100+3)+\cdots+(100+100)$$

Looking at the second terms in the parentheses, we can see that $1+2+3+\cdots+100$ occurs 100 times. It goes horizontally and exists 100 times vertically. Looking at the first terms in the parentheses, we can see that $1+2+3+\cdots+100$ occurs 100 times. It goes vertically and exists 100 times horizontally.

Thus, we have:

$$2\left(\frac{100\cdot101}{2}\cdot100\right).$$

This gives us:

$$\boxed{\textbf{(E) }1010000}.$$

Solution 2

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j)=\sum_{i=1}^{100}(100i+5050)=100\cdot5050+5050\cdot100=\boxed{1,010,000}$$

Solution 3

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j)=\sum_{i=1}^{100}\sum_{i=1}^{100}2i=(100)*(5050*2)=\boxed{1,010,000}$$

See Also

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 8**	Followed by **Problem 10**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 10B Problems/Problem 14

A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

$\textbf{(A)}\ 202\qquad\textbf{(B)}\ 223\qquad\textbf{(C)}\ 224\qquad\textbf{(D)}\ 225\qquad\textbf{(E)}\ 234$

Solution

To minimize the number of values, we want to maximize the number of times they appear. So, we could have 223 numbers appear 9 times, 1 number appear once, and the mode appear 10 times, giving us a total of $223+1+1$ = $\boxed{\textbf{(D)}\ 225}$

See Also

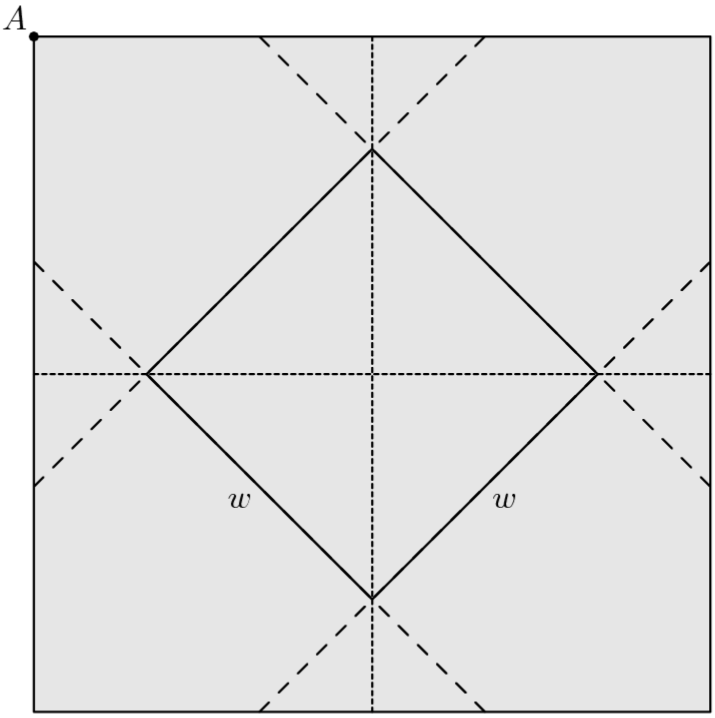
2018 AMC 10B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 13**	Followed by **Problem 15**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

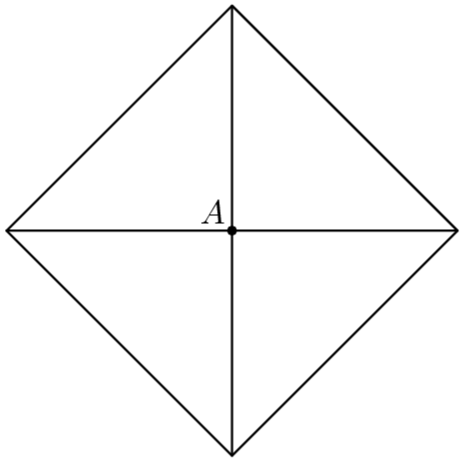
2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 9**	Followed by **Problem 11**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Combinatorics Problems

2018 AMC 10B Problems/Problem 15

A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h. What is the area of the sheet of wrapping paper?





$\textbf{(A) } 2(w+h)^2 \qquad \textbf{(B) } \frac{(w+h)^2}{2} \qquad \textbf{(C) } 2w^2+4wh \qquad \textbf{(D) } 2w^2 \qquad \textbf{(E) } w^2h$

Solution 1

Consider one-quarter of the image (the wrapping paper is divided up into 4 congruent squares). The length of each dotted line is h. The area of the rectangle that is w by h is wh. The combined figure of the two triangles with base h is a square with h as its diagonal. Using the Pythagorean Theorem, each side of this square is $\sqrt{\frac{h^2}{2}}$. Thus, the area is the side length squared which is $\frac{h^2}{2}$. Similarly, the combined figure of the two triangles with base w is a square with area $\frac{w^2}{2}$. Adding all of these together, we get $\frac{w^2}{2}+\frac{h^2}{2}+wh$. Since we have four of these areas in the entire wrapping paper, we multiply this by 4, getting $4(\frac{w^2}{2}+\frac{h^2}{2}+wh)=2(w^2+h^2+2wh)=\boxed{\textbf{(A) } 2(w+h)^2}$.

Solution 2

The sheet of paper is made out of the surface area of the box plus the sum of the four triangles. The surface area is $2w^2+2wh+2wh$ which equals $2w^2+4wh$.The four triangles each have a height and a base of h, so they each have an area of $\frac{h^2}{2}$. There are four of them, so multiplied by four is $2h^2$. Together, paper's area is $2w^2+4wh+2h^2$. This can be factored and written as $\boxed{\textbf{(A) } 2(w+h)^2}$.

See Also

2018 AMC 10B (Problems • Answer Key • Resources)	
Preceded by **Problem 14**	Followed by **Problem 16**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12B (Problems • Answer Key • Resources)	
Preceded by **Problem 10**	Followed by **Problem 12**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2018 AMC 12B Problems/Problem 12

Problem

Side $\overline{AB}$ of $\triangle ABC$ has length 10. The bisector of angle A meets $\overline{BC}$ at D, and $CD = 3$. The set of all possible values of AC is an open interval (m,n). What is $m+n$?

$$\textbf{(A) }16 \qquad \textbf{(B) }17 \qquad \textbf{(C) }18 \qquad \textbf{(D) }19 \qquad \textbf{(E) }20$$

Solution

Let $BD = x$. Then by Angle Bisector Theorem, we have $AC = 30/x$. Now, by the triangle inequality, we have three inequalities.

- $10 + x + 3 > AC$, so $13 + x > 30/x$. Solve this to find that $x > 2$, so $AC < 15$.
- $AC + 10 > x + 3$, so $30/x > x - 7$. Solve this to find that $x < 10$, so $AC > 3$.
- The third inequality can be disregarded, because $30/x > 7 - x$ has no real roots.

Then our interval is simply $(3, 15)$ to get $18 \boxed{C}$.

See Also

2018 AMC 12B (Problems • Answer Key • Resources)	
Preceded by **Problem 11**	Followed by **Problem 13**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

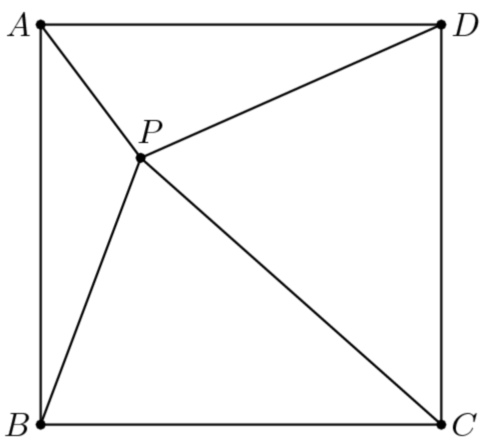
Category: Introductory Geometry Problems

2018 AMC 12B Problems/Problem 13

Contents

Problem

Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ are the vertices of a convex quadrilateral. What is the area of that quadrilateral?



$\textbf{(A) }100\sqrt{2}\qquad\textbf{(B) }100\sqrt{3}\qquad\textbf{(C) }200\qquad\textbf{(D) }200\sqrt{2}\qquad\textbf{(E) }200\sqrt{3}$

Solution 1 (Drawing an Accurate Diagram)

We can draw an accurate diagram by using centimeters and scaling everything down by a factor of 2. The centroid is the intersection of the three medians in a triangle.

After connecting the 4 centroids, we see that the quadrilateral looks like a square with side length of 7. However, we scaled everything down by a factor of 2, so the length is 14. The area of a square is s^2, so the area is:

$$\boxed{\textbf{(C) }200}.$$

Solution 2

The centroid of a triangle is $\frac{2}{3}$ of the way from a vertex to the midpoint of the opposing side. Thus, the length of any diagonal of this quadrilateral is 20. The diagonals are also parallel to sides of the square, so they are perpendicular to each other, and so the area of the quadrilateral is $\frac{20\cdot20}{2} = 200$, $\boxed{(C)}$.

Solution 3

The midpoints of the sides of the square form another square, with side length $15\sqrt{2}$ and area 450. Dilating the corners of this square through point P by a factor of $3:2$ results in the desired quadrilateral (also a square). The area of this new square is $\frac{2^2}{3^2}$ of the area of the original dilated square. Thus, the answer is $\frac{4}{9}*450=\boxed{C}$

Solution 4

We put the diagram on a coordinate plane. The coordinates of the square are $(0,0),(30,0),(30,30),(0,30)$ and the coordinates of point P are (x,y). By using the centroid formula, we find that the coordinates of the centroids are $(\frac{x}{3},10+\frac{y}{3}),(10+\frac{x}{3},\frac{y}{3}),(20+\frac{x}{3},10+\frac{y}{3})$, and $(10+\frac{x}{3},20+\frac{y}{3})$. Shifting the coordinates down by $(\frac{x}{3},\frac{y}{3})$does not change its area, and we ultimately get that the area is equal to the area covered by $(0,10),(10,0),(20,10),(10,20)$ which has an area of $\boxed{(C)200.}$

See Also

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 12**	Followed by **Problem 14**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Intermediate Geometry Problems

2018 AMC 10B Problems/Problem 19

Contents

Problem

Joey and Chloe and their daughter Zoe all have the same birthday. Joey is 1 year older than Chloe, and Zoe is exactly 1 year old today. Today is the first of the 9 birthdays on which Chloe's age will be an integral multiple of Zoe's age. What will be the sum of the two digits of Joey's age the next time his age is a multiple of Zoe's age?

$\textbf{(A) }7 \qquad \textbf{(B) }8 \qquad \textbf{(C) }9 \qquad \textbf{(D) }10 \qquad \textbf{(E) }11$

Solution 1

Let Joey's age be j, Chloe's age be c, and we know that Zoe's age is 1.

We know that there must be 9 values $k \in \mathbb{Z}$ such that $c+k=a(1+k)$ where a is an integer.

Therefore, $c-1+(1+k)=a(1+k)$ and $c-1=(1+k)(a-1)$. Therefore, we know that, as there are 9 solutions for k, there must be 9 solutions for $c-1$. We know that this must be a perfect square. Testing perfect squares, we see that $c-1=36$, so $c=37$. Therefore, $j=38$. Now, since $j-1=37$, by similar logic, $37=(1+k)(a-1)$, so $k=36$ and Joey will be $38+36=74$ and the sum of the digits is $\boxed{\text{(E) } 11}$

Solution 2

Here's a different way of saying the above solution:

If a number is a multiple of both Chloe's age and Zoe's age, then it is a multiple of their difference. Since the difference between their ages does not change, then that means the difference between their ages has 9 factors. Therefore, the difference between Chloe and Zoe's age is 36, so Chloe is 37, and Joey is 38. The common factor that will divide both of their ages is 37, so Joey will be 74. 7 + 4 = $\boxed{\text{(E) } 11}$

Solution 3

Similar approach to above, just explained less concisely and more in terms of the problem (less algebra-y)

Let $C+n$ denote Chloe's age, $J+n$ denote Joey's age, and $Z+n$ denote Zoe's age, where n is the number of years from now. We are told that $C+n$ is a multiple of $Z+n$ exactly nine times. Because $Z+n$ is 1 at $n=0$ and will increase until greater than $C-Z$, it will hit every natural number less than $C-Z$, including every factor of $C-Z$. For $C+n$ to be an integral multiple of $Z+n$, the difference $C-Z$ must also be a multiple of Z, which happens iff Z is a factor of $C-Z$. Therefore, $C-Z$ has nine factors. The smallest number that has nine positive factors is $2^2 3^2=36$ (we want it to be small so that Joey will not have reached three digits of age before his age is a multiple of Zoe's). We also know $Z=1$ and $J=C+1$. Thus,

$$C-Z=36$$

$$J-Z=37$$

By our above logic, the next time $J-Z$ is a multiple of $Z+n$ will occur when $Z+n$ is a factor of $J-Z$. Because 37 is prime, the next time this happens is at $Z+n=37$, when $J+n=74$. $7+4=\boxed{\textbf{(E)} 11}$

See Also

Category: Introductory Number Theory Problems

2018 AMC 12B Problems/Problem 15

Contents

Problem

How many odd positive 3-digit integers are divisible by 3 but do not contain the digit 3?

$\textbf{(A) } 96 \qquad \textbf{(B) } 97 \qquad \textbf{(C) } 98 \qquad \textbf{(D) } 102 \qquad \textbf{(E) } 120$

Solution 1 (For Dummies)

Analyze that the three-digit integers divisible by 3 start from 102. In the 200's, it starts from 201. In the 300's, it starts from 300. We see that the units digits is $0, 1,$ and 2.

Write out the 1- and 2-digit multiples of 3 starting from $0, 1,$ and 2. Count up the ones that meet the conditions. Then, add up and multiply by 3, since there are three sets of three from 1 to 9. Then, subtract the amount that started from 0, since the 300's all contain the digit 3.

We get:

$$3(12+12+12)-12.$$

This gives us:

$$\boxed{\textbf{(A) } 96}.$$

Solution 2

There are 4 choices for the last digit ($1, 5, 7, 9$), and 8 choices for the first digit (exclude 0). We know what the second digit mod 3 is, so there are 3 choices for it (pick from one of the sets $\{0, 6, 9\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$). The answer is $4 \cdot 8 \cdot 3 = \boxed{96}$ (Plasma_Vortex)

Solution 3

Consider the number of 2-digit numbers that do not contain the digit 3, which is $90-18=72$. For any of these 2-digit numbers, we can append $1, 5, 7,$ or 9 to reach a desirable 3-digit number. However, $1 \equiv 7 \equiv 1 \,(mod\, 3)$, and thus we need to count any 2-digit number $\equiv 2 \,(mod\, 3)$ twice. There are $(98-11)/3+1=30$ total such numbers that have remainder 2, but 6 of them $(23, 32, 35, 38, 53, 83)$ contain 3, so the number we want is $30-6=24$. Therefore, the final answer is $72+24=\boxed{96}$.

See Also

2018 AMC 12B (Problems • Answer Key • Resources)	
Preceded by **Problem 14**	Followed by **Problem 16**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 12B Problems/Problem 16

Problem

The solutions to the equation $(z+6)^8=81$ are connected in the complex plane to form a convex regular polygon, three of whose vertices are labeled $A, B,$ and C. What is the least possible area of $\triangle ABC?$

$\textbf{(A) } \frac{1}{6}\sqrt{6} \qquad \textbf{(B) } \frac{3}{2}\sqrt{2}-\frac{3}{2} \qquad \textbf{(C) } 2\sqrt3-3\sqrt2 \qquad \textbf{(D) } \frac{1}{2}\sqrt{2} \qquad \textbf{(E) } \sqrt 3-1$

Solution

The answer is the same if we consider $z^8=81.$ Now we just need to find the area of the triangle bounded by $\sqrt{3}i, \sqrt{3},$ and $\frac{\sqrt{3}}{\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{2}}i.$ This is just $\boxed{\textbf{B.}}$

See Also

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 15**	Followed by **Problem 17**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Categories: Intermediate Algebra Problems | Intermediate Geometry Problems

2018 AMC 12B Problems/Problem 17

Contents

Problem

Let p and q be positive integers such that

$$\frac{5}{9} < \frac{p}{q} < \frac{4}{7}$$

and q is as small as possible. What is $q - p$?

$\textbf{(A)}\ 7 \qquad \textbf{(B)}\ 11 \qquad \textbf{(C)}\ 13 \qquad \textbf{(D)}\ 17 \qquad \textbf{(E)}\ 19$

Solution 1

We claim that, between any two fractions a/b and c/d, if $bc - ad = 1$, the fraction with smallest denominator between them is $\frac{a+c}{b+d}$. To prove this, we see that

$$\frac{1}{bd} = \frac{c}{d} - \frac{a}{b} = \left(\frac{c}{d} - \frac{p}{q}\right) + \left(\frac{p}{q} - \frac{a}{b}\right) \geq \frac{1}{dq} + \frac{1}{bq},$$

which reduces to $q \geq b + d$. We can easily find that $p = a + c$, giving an answer of $\boxed{\textbf{(A)}\ 7}$.

Solution 2 (requires justification)

Assume that the difference $\frac{p}{q} - \frac{5}{9}$ results in a fraction of the form $\frac{1}{9q}$. Then,

$9p - 5q = 1$

Also assume that the difference $\frac{4}{7} - \frac{p}{q}$ results in a fraction of the form $\frac{1}{7q}$. Then,

$4q - 7p = 1$

Solving the system of equations yields $q = 16$ and $p = 9$. Therefore, the answer is $\boxed{\textbf{(A)}\ 7}$

Solution 3

Cross-multiply the inequality to get

$$35q < 63p < 36q.$$

Then,

$$0 < 63p - 35q < q,$$

$$0 < 7(9p - 5q) < q.$$

Since p, q are integers, $9p - 5q$ is an integer. To minimize q, start from $9p - 5q = 1$, which gives $p = \frac{5q+1}{9}$. This limits q to be greater than 7, so test values of q starting from $q = 8$. However, $q = 8$ to $q = 14$ do not give integer values of p.

Once $q > 14$, it is possible for $9p - 5q$ to be equal to 2, so p could also be equal to $\frac{5q+2}{9}$. The next value, $q = 15$, is not a solution, but $q = 16$ gives $p = \frac{5 \cdot 16 + 1}{9} = 9$. Thus, the smallest possible value of q is 16, and the answer is $16 - 9 = \boxed{\textbf{(A) } 7}$.

Solution 4

Graph the regions $y > \frac{5}{9}x$ and $y < \frac{4}{7}x$. Note that the lattice point $(16, 9)$ is the smallest magnitude one which appears within the region bounded by the two graphs. Thus, our fraction is $\frac{9}{16}$ and the answer is $16 - 9 = \boxed{\textbf{(A) } 7}$.

Remark: This also gives an intuitive geometric proof of the mediant using vectors.

Solution 5 (Using answer choices to prove mediant)

As the other solutions do, the mediant $= \frac{9}{16}$ is between the two fractions, with a difference of $\boxed{\textbf{(A) } 7}$. Suppose that the answer was not A, then the answer must be B or C as otherwise p would be negative. Then, the possible fractions with lower denominator would be $\frac{k-11}{k}$ for $k = 12, 13, 14, 15$ and $\frac{k-13}{k}$ for $k = 14, 15$, which are clearly not anywhere close to $\frac{4}{7} \approx 0.6$

Solution 6

Inverting the given inequality we get

$$\frac{7}{4} < \frac{q}{p} < \frac{9}{5}$$

which simplifies to

$$35p < 20q < 36p$$

We can now substitute $q = p + k$. Note we need to find k.

$$35p < 20p + 20k < 36p$$

which simplifies to

$$15p < 20k < 16p$$

Cleary p is greater than k. We will now substitute $p = k + x$ to get

$$15k + 15x < 20k < 16k + 16x$$

The inequality $15k+15x<20k$ simplifies to $3x<k$. The inequality $20k<16k+16x$ simplifies to $k<4x$. Combining the two we get

$$3x<k<4x$$

Since x and k are integers, the smallest values of x and k that satisfy the above equation are 2 and 7 respectively. Substituting these back in, we arrive with an answer of $\boxed{\textbf{(A) } 7}$.

Solution 7

Start with $\frac{5}{9}$. Repeat the following process until you arrive at the answer: if the fraction is less than or equal to $\frac{5}{9}$, add 1 to the numerator; otherwise, if it is greater than or equal to $\frac{4}{7}$, add one 1 to the denominator. We have:

$$\frac{5}{9},\frac{6}{9},\frac{6}{10},\frac{6}{11},\frac{7}{11},\frac{7}{12},\frac{7}{13},\frac{8}{13},\frac{8}{14},\frac{8}{15},\frac{9}{15},\frac{9}{16}$$

$16-9=\boxed{\textbf{(A) } 7}$.

Solution 8

Because q and p are positive integers with $p<q$, we can let $q=p+k$ where $k\in\mathbb{Z}$. Now, the problem condition reduces to

$\frac{5}{9}<\frac{p}{p+k}<\frac{4}{7}$

Our first inequality is $\frac{5}{9}<\frac{p}{p+k}$ which gives us $5p+5k<9p \implies \frac{5}{4}k<p$.

Our second inequality is $\frac{p}{p+k}<\frac{4}{7}$ which gives us $7p<4p+4k \implies p<\frac{4}{3}k$.

Hence, $\frac{5}{4}k<p<\frac{4}{3}k \implies 15k<12p<16k$.

It is clear that we are aiming to find the least positive integer value of k such that there is at least one value of p that satisfies the inequality.

Now, simple casework through the answer choices of the problem reveals that $q-p=p+k-p=k \implies k\geq\boxed{7}$.

See Also

2018 AMC 12B (Problems • Answer Key • Resources)	
Preceded by **Problem 16**	Followed by **Problem 18**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 10B Problems/Problem 20

Contents

Problem

A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

$\textbf{(A)}\ 2016 \qquad \textbf{(B)}\ 2017 \qquad \textbf{(C)}\ 2018 \qquad \textbf{(D)}\ 2019 \qquad \textbf{(E)}\ 2020$

Solution 1

$f\left(n\right) = f\left(n-1\right) - f\left(n-2\right) + n$

$= \left(f\left(n-2\right) - f\left(n-3\right) + n - 1\right) - f\left(n-2\right) + n$

$= 2n - 1 - f\left(n-3\right)$

$= 2n - 1 - \left(2\left(n-3\right) - 1 - f\left(n-6\right)\right)$

$= f\left(n-6\right) + 6$

Thus, $f\left(2018\right) = 2016 + f\left(2\right) = 2017.$ $\boxed{B}$

Solution 2 (A Bit Bashy)

Start out by listing some terms of the sequence.

$$f(1) = 1$$

$$f(2) = 1$$

$$f(3) = 3$$

$$f(4) = 6$$

$$f(5) = 8$$

$$f(6) = 8$$

$$f(7) = 7$$

$$f(8) = 7$$

$$f(9) = 9$$

$$f(10) = 12$$

$$f(11) = 14$$

$$f(12) = 14$$

$$f(13) = 13$$

$$f(14) = 13$$

$$f(15) = 15$$

$$.....$$

Notice that $f(n) = n$ whenever n is an odd multiple of 3, and the pattern of numbers that follow will always be +3, +2, +0, -1, +0. The closest odd multiple of 3 to 2018 is 2013, so we have

$$f(2013) = 2013$$

$$f(2014) = 2016$$

$$f(2015) = 2018$$

$$f(2016) = 2018$$

$$f(2017) = 2017$$

$$f(2018) = \boxed{(B) 2017}.$$

Solution 3(Bashy Pattern Finding)

Writing out the first few values, we get: $1, 1, 3, 6, 8, 8, 7, 7, 9, 12, 14, 14, 13, 13, 15, 18, 20, 20, 19, 19...$. Examining, we see that every number x where $x \equiv 1 \pmod{6}$ has $f(x) = x$, $f(x+1) = f(x) = x$, and $f(x-1) = f(x-2) = x+1$. The greatest number that's 1 (mod 6) and less 2018 is 2017, so we have $f(2017) = f(2018) = 2017. \boxed{B}$

See Also

2018 AMC 10B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 19**	Followed by **Problem 21**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 17**	Followed by **Problem 19**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 10B Problems/Problem 21

Contents

Problem

Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1, 2, \ldots, \dfrac{n}{2}, n$. At some moment Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323?

$\textbf{(A) } 324 \qquad \textbf{(B) } 330 \qquad \textbf{(C) } 340 \qquad \textbf{(D) } 361 \qquad \textbf{(E) } 646$

Solution 1

Prime factorizing 323 gives you $17 \cdot 19$. The desired answer needs to be a multiple of 17 or 19, because if it is not a multiple of 17 or 19, the LCM, or the least possible value for n, will not be more than 4 digits. Looking at the answer choices, $\boxed{\text{(C) } 340}$ is the smallest number divisible by 17 or 19. Checking, we can see that n would be 6460.

Solution 2

Let the next largest divisor be k. Suppose $\gcd(k, 323) = 1$. Then, as $323|n, k|n$, therefore, $323 \cdot k|n.$ However, because $k > 323$, $323k > 323 \cdot 324 > 9999$. Therefore, $\gcd(k, 323) > 1$. Note that $323 = 17 \cdot 19$. Therefore, the smallest the gcd can be is 17 and our answer is $323 + 17 = \boxed{\text{(C) } 340}$.

See Also

2018 AMC 10B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 20**	Followed by **Problem 22**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 18**	Followed by **Problem 20**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Intermediate Number Theory Problems

2018 AMC 10B Problems/Problem 24

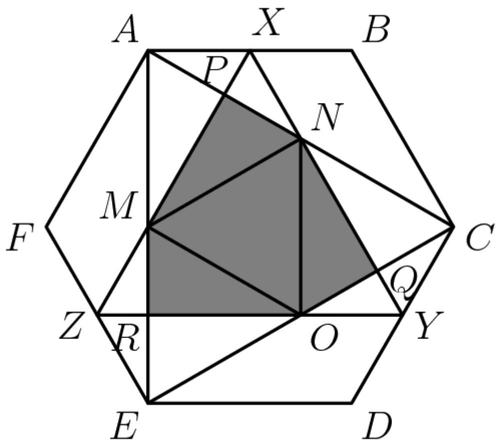
Contents

Problem

Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y, and Z the midpoints of sides $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$, respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

$\textbf{(A)}\frac{3}{8}\sqrt{3} \qquad \textbf{(B)}\frac{7}{16}\sqrt{3} \qquad \textbf{(C)}\frac{15}{32}\sqrt{3} \qquad \textbf{(D)}\frac{1}{2}\sqrt{3} \qquad \textbf{(E)}\frac{9}{16}\sqrt{3}$

Solution 1



The desired area (hexagon $MPNQOR$) consists of an equilateral triangle ($\triangle MNO$) and three right triangles ($\triangle MPN$, $\triangle NQO$, and $\triangle ORM$).

Notice that $\overline{AD}$ (not shown) and $\overline{BC}$ are parallel. $\overline{XY}$ divides transversals $\overline{AB}$ and $\overline{CD}$ into a $1:1$ ratio. Thus, it must also divide transversal $\overline{AC}$ and transversal $\overline{CO}$ into a $1:1$ ratio. By symmetry, the same applies for $\overline{CE}$ and $\overline{EA}$ as well as $\overline{EM}$ and $\overline{AN}$

In $\triangle ACE$, we see that $\frac{[MNO]}{[ACE]}=\frac{1}{4}$ and $\frac{[MPN]}{[ACE]}=\frac{1}{8}$. Our desired area becomes

$$(\frac{1}{4}+3\cdot\frac{1}{8})\cdot\frac{(\sqrt{3})^2\cdot\sqrt{3}}{4}=\frac{15}{32}\sqrt{3}=\boxed{C}$$

Solution 2

Now, if we look at the figure, we can see that the complement of the hexagon we are trying to find is composed of 3 isosceles trapezoids ($AXFZ$, $XBCY$, and $ZYED$), and 3 right triangles, with one right angle on each of X, Y, and Z. Finding the trapezoid's area, we know that one base of each trapezoid is just the side length of the hexagon, which is 1, and the other base is 3/2 (It is halfway in between the side and the longest diagonal, which has length 2) with a height of $\frac{\sqrt{3}}{4}$ (by using the Pythagorean theorem and the fact that it is an isosceles trapezoid) to give each trapezoid having an area of $\frac{5\sqrt{3}}{16}$ for a total area of $\frac{15\sqrt{3}}{16}$. (Alternatively, we could have calculated the area of hexagon $ABCDEF$ and subtracted the area of $\triangle XYZ$, which, as we showed before, had a side length of 3/2). Now, we need to find the area of each of the small triangles, which, if we look at the triangle that has a vertex on X, is similar to the triangle with a base of $YC = 1/2.$ Using similar triangles we calculate the base to be 1/4 and the height to be $\frac{\sqrt{3}}{4}$ giving us an area of $\frac{\sqrt{3}}{32}$ per triangle, and a total area of $3\frac{\sqrt{3}}{32}$. Adding the two areas together, we get $\frac{15\sqrt{3}}{16}+\frac{3\sqrt{3}}{32}=\frac{33\sqrt{3}}{32}$. Finding the total area, we get $6\cdot 1^2\cdot\frac{\sqrt{3}}{4}=\frac{3\sqrt{3}}{2}$. Taking the complement, we get $\frac{3\sqrt{3}}{2}-\frac{33\sqrt{3}}{32}=\frac{15\sqrt{3}}{32}=(C)\frac{15}{32}\sqrt{3}$

Solution 3 (Trig)

Notice, the area of the convex hexagon formed through the intersection of the 2 triangles can be found by finding the area of the triangle formed by the midpoints of the sides and subtracting the smaller triangles that are formed by the region inside this triangle but outside the other triangle. First, let's find the area of the area of the triangle formed by the midpoint of the sides. Notice, this is an equilateral triangle, thus all we need is to find the length of its side. To do this, we look at the isosceles trapezoid outside this triangle but inside the outer hexagon. Since the interior angle of a regular hexagon is 120° and the trapezoid is isosceles, we know that the angle opposite is 60°, and thus the side length of this triangle is $1+2(\frac{1}{2}\cos(60^{\circ})=1+\frac{1}{2}=\frac{3}{2}$. So the area of this triangle is $\frac{\sqrt{3}}{4}s^2=\frac{9\sqrt{3}}{16}$ Now let's find the area of the smaller triangles. Notice, triangle ACE cuts off smaller isosceles triangles from the outer hexagon. The base of these isosceles triangles is perpendicular to the base of the isosceles trapezoid mentioned before, thus we can use trigonometric ratios to find the base and height of these smaller triangles, which are all congruent due to the rotational symmetry of a regular hexagon. The area is then $\frac{1}{2}(\frac{1}{2})\cos(60^{\circ}))(\frac{1}{2}\sin(60^{\circ}))=\frac{\sqrt{3}}{32}$ and the sum of the areas is $3\cdot\frac{\sqrt{3}}{32}=\frac{3\sqrt{3}}{32}$ Therefore, the area of the convex hexagon is

$$\frac{9\sqrt{3}}{16}-\frac{3\sqrt{3}}{32}=\frac{18\sqrt{3}}{32}-\frac{3\sqrt{3}}{32}=\boxed{\frac{15\sqrt{3}}{32}}\implies\boxed{C}$$

See Also

2018 AMC 10B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 23**	Followed by **Problem 25**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 19**	Followed by **Problem 21**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 12B Problems/Problem 21

Contents

Problem

In $\triangle ABC$ with side lengths $AB = 13$, $AC = 12$, and $BC = 5$, let O and I denote the circumcenter and incenter, respectively. A circle with center M is tangent to the legs AC and BC and to the circumcircle of $\triangle ABC$. What is the area of $\triangle MOI$?

$\textbf{(A)}\ 5/2\qquad\textbf{(B)}\ 11/4\qquad\textbf{(C)}\ 3\qquad\textbf{(D)}\ 13/4\qquad\textbf{(E)}\ 7/2$

Solution 1

Let the triangle have coordinates $(0,0),(5,0),(0,12).$ Then the coordinates of the incenter and circumcenter are $(2,2)$ and $(2.5,6),$ respectively. If we let $M=(x,x),$ then x satisfies

$$\sqrt{(2.5-x)^2+(6-x)^2}+x=6.5$$

$$2.5^2-5x+x^2+6^2-12x+x^2=6.5^2-13x+x^2$$

$$x^2=(5+12-13)x$$

$$x\neq 0\implies x=4.$$

Now the area of our triangle can be calculated with the Shoelace Theorem. The answer turns out to be $\boxed{\textbf{E.}}$

Solution 2

Notice that we can let $M = C$. If $O = (0,0)$, then $C = \left(6, -\frac{5}{2}\right)$ and $I = \left(4, -\frac{1}{2}\right)$. Using shoelace formula, we get $[COI] = \frac{7}{2}$. $\boxed{\textbf{E.}}$

See Also

2018 AMC 12B (Problems • Answer Key • Resources)	
Preceded by **Problem 20**	Followed by **Problem 22**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2018 AMC 12B Problems/Problem 22

Contents

Problem

Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many such polynomials satisfy $P(-1) = -9$?

$\textbf{(A) } 110 \qquad \textbf{(B) } 143 \qquad \textbf{(C) } 165 \qquad \textbf{(D) } 220 \qquad \textbf{(E) } 286$

Solution

Suppose our polynomial is equal to

$$ax^3+bx^2+cx+d$$

Then we are given that

$$-9=b+d-a-c.$$

If we let $b=-9-b', d=-9-d'$ then we have

$$-9=a+c+b'+d'.$$

The number of solutions to this equation is simply $\binom{12}{3}=220$ by stars and bars, so our answer is $\boxed{\textbf{D.}}$

Solution 2

Suppose our polynomial is equal to

$$ax^3+bx^2+cx+d$$

Then we are given that

$$9=b+d-a-c.$$

Then the polynomials

$$cx^3+bx^2+ax+d$$

,

$$ax^3+dx^2+cx+b$$

,

$$cx^3+dx^2+ax+b$$

also have

$$b+d-a-c=-9$$

when

$$x=-1$$

. So the number of solutions must be divisible by 4. So the answer must be $\boxed{\textbf{D.}}$

See Also

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 21**	Followed by **Problem 23**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Categories: Intermediate Combinatorics Problems | Intermediate Algebra Problems

2018 AMC 12B Problems/Problem 23

Problem

Ajay is standing at point A near Pontianak, Indonesia, 0° latitude and 110° E longitude. Billy is standing at point B near Big Baldy Mountain, Idaho, USA, 45° N latitude and 115° W longitude. Assume that Earth is a perfect sphere with center C. What is the degree measure of $\angle ACB$?

$$\textbf{(A) }105 \qquad \textbf{(B) }112\frac{1}{2} \qquad \textbf{(C) }120 \qquad \textbf{(D) }135 \qquad \textbf{(E) }150$$

Solution

Suppose that Earth is a unit sphere with center $(0,0,0)$. We can let

$$A=(1,0,0), B=\left(-\frac{1}{2},\frac{1}{2},\frac{\sqrt{2}}{2}\right).$$

The angle θ between these two vectors satisfies $\cos\theta=A\cdot B=-\frac{1}{2}$, yielding $\theta=120^\circ$, or $\boxed{\textbf{C.}}$

See Also

2018 AMC 12B (**Problems** • **Answer Key** • Resources)	
Preceded by **Problem 22**	Followed by **Problem 24**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

Category: Intermediate Geometry Problems

2018 AMC 10B Problems/Problem 25

Contents

Problem

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many real numbers x satisfy the equation $x^2 + 10,000\lfloor x \rfloor = 10,000x$?

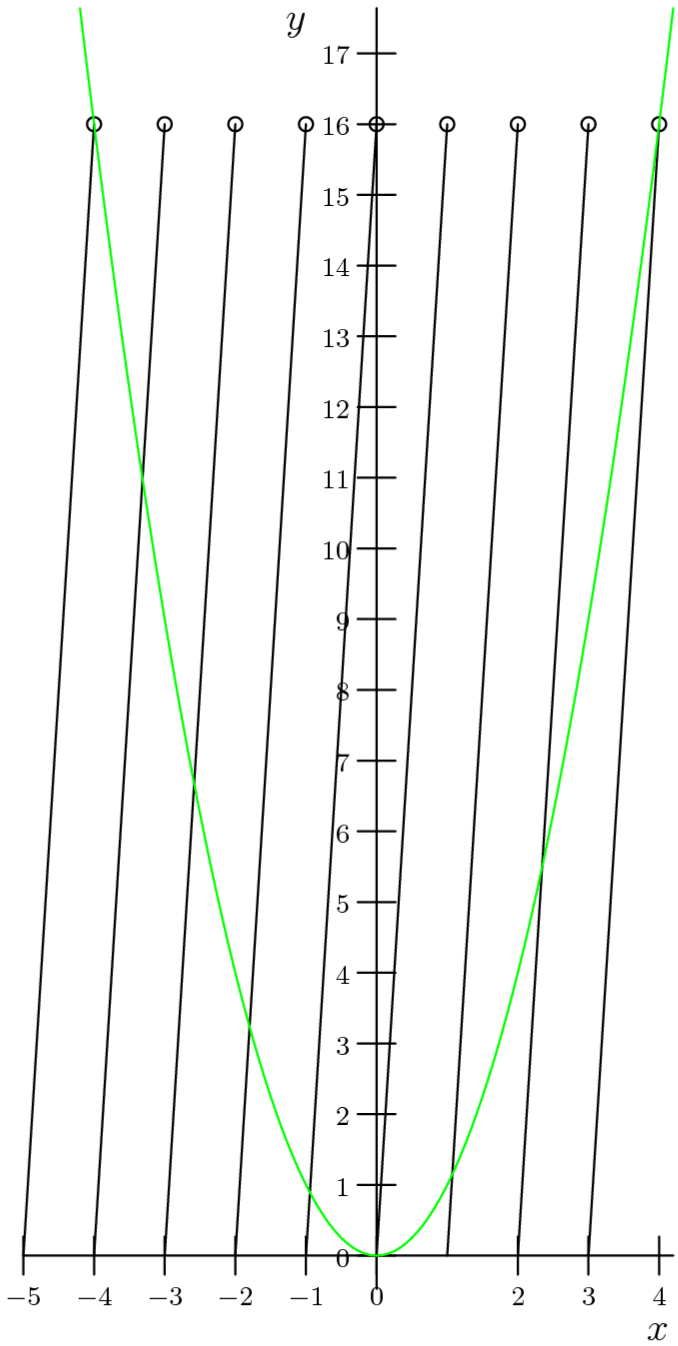
$\textbf{(A) } 197 \qquad \textbf{(B) } 198 \qquad \textbf{(C) } 199 \qquad \textbf{(D) } 200 \qquad \textbf{(E) } 201$

Solution 1

This rewrites itself to $x^2 = 10,000\{x\}$.

Graphing $y = 10,000\{x\}$ and $y = x^2$ we see that the former is a set of line segments with slope $10,000$ from 0 to 1 with a hole at $x = 1$, then 1 to 2 with a hole at $x = 2$ etc.

Here is a graph of $y = x^2$ and $y = 16\{x\}$ for visualization.



Now notice that when $x = \pm 100$ then graph has a hole at $(\pm 100, 10,000)$ which the equation $y = x^2$ passes through and then continues upwards. Thus our set of possible solutions is bounded by $(-100, 100)$. We can see that $y = x^2$ intersects each of the lines once and there are $99 - (-99) + 1 = 199$ lines for an answer of $\boxed{\text{(C) } 199}$.

Solution 2

Same as the first solution, $x^2 = 10,000\{x\}$.

We can write x as $\lfloor x \rfloor + \{x\}$. Expanding everything, we get a quadratic in x in terms of $\lfloor x \rfloor$:
$\{x\}^2 + (2\lfloor x \rfloor - 10,000)\{x\} + \lfloor x \rfloor^2 = 0$

We use the quadratic formula to solve for {x}:
$$\{x\} = \frac{-2\lfloor x \rfloor + 10,000 \pm \sqrt{(-2\lfloor x \rfloor + 10,000^2 - 4\lfloor x \rfloor^2)}}{2}$$

Since $0 \leq \{x\} < 1$, we get an inequality which we can then solve. After simplifying a lot, we get that $\lfloor x \rfloor^2 + 2\lfloor x \rfloor - 9999 < 0$.

Solving over the integers, $-101 < \lfloor x \rfloor < 99$, and since $\lfloor x \rfloor$ is an integer, there are $\boxed{\text{(C) } 199}$ solutions. Each value of $\lfloor x \rfloor$ should correspond to one value of x, so we are done.

Solution 3

Let $x = a + k$ where a is the integer part of x and k is the fractional part of x. We can then rewrite the problem below:

$$(a+k)^2 + 10000a = 10000(a+k)$$

From here, we get

$$(a+k)^2 + 10000a = 10000a + 10000k$$

Solving for $a + k = x$

$$(a+k)^2 = 10000k$$

$$x = a + k = \pm 100\sqrt{k}$$

Because $0 \leq k < 1$, we know that $a + k$ cannot be less than or equal to -100 nor greater than or equal to 100. Therefore:

$$-99 \leq x \leq 99$$

There are 199 elements in this range, so the answer is $\boxed{\textbf{(C) } 199}$.

Solution 4

Notice the given equation is equivilent to $(\lfloor x \rfloor + \{x\})^2 = 10,000\{x\}$

Now we now that $\{x\} < 1$ so plugging in 1 for $\{x\}$ we can find the upper and lower bounds for the values.

$$(\lfloor x \rfloor + 1)^2 = 10,000(1)$$

$$(\lfloor x \rfloor + 1) = \pm 100$$

$$\lfloor x \rfloor = 99, -101$$

And just like Solution 2, we see that $-101 < \lfloor x \rfloor < 99$, and since $\lfloor x \rfloor$ is an integer, there are $\boxed{\text{(C) } 199}$ solutions. Each value of $\lfloor x \rfloor$ should correspond to one value of x, so we are done.

Solution 5

First, we can let $\{x\} = b$, $\lfloor x \rfloor = a$. We know that $a + b = x$ by definition. We can rearrange the equation to obtain

$$x^2 = 10^4(x-a).$$

By taking square root on both sides, we obtain $x = \pm 100\sqrt{b}$ (because $x - a = b$). We know since b is the fractional part of x, it must be that $0 \leq b < 1$. Thus, x may take any value in the interval $-100 < x < 100$. Hence, we know that there are $\boxed{\text{(C) } 199}$ potential values for $\lfloor x \rfloor$ in that range and we are done.

~awesome1st

See Also

2018 AMC 10B (Problems • Answer Key • Resources)	
Preceded by **Problem 24**	Followed by **Last Problem**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2018 AMC 12B (Problems • Answer Key • Resources)	
Preceded by **Problem 23**	Followed by **Problem 25**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

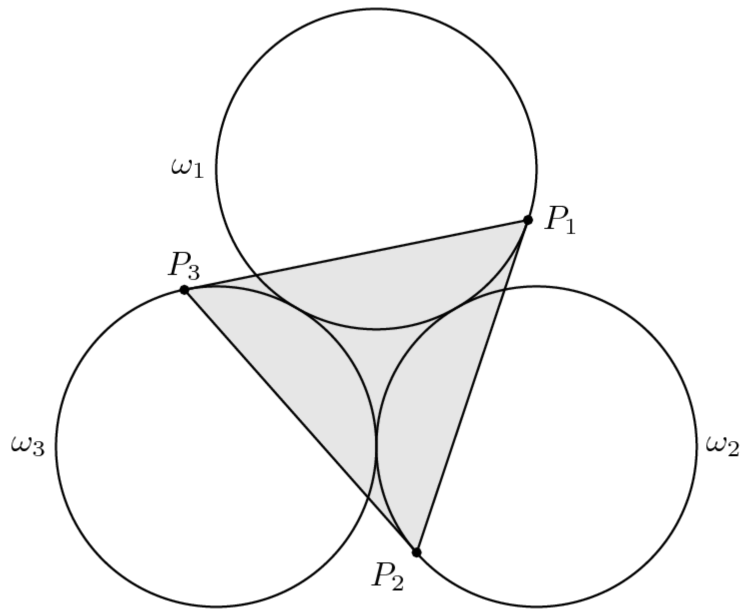
Category: Intermediate Algebra Problems

2018 AMC 12B Problems/Problem 25

Contents

Problem

Circles ω_1, ω_2, and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points P_1, P_2, and P_3 lie on ω_1, ω_2, and ω_3 respectively such that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i for each $i = 1, 2, 3$, where $P_4 = P_1$. See the figure below. The area of $\triangle P_1P_2P_3$ can be written in the form $\sqrt{a} + \sqrt{b}$ for positive integers a and b. What is $a + b$?



$\textbf{(A) } 546 \qquad \textbf{(B) } 548 \qquad \textbf{(C) } 550 \qquad \textbf{(D) } 552 \qquad \textbf{(E) } 554$

Solution 1

Let O_i be the center of circle ω_i for $i = 1, 2, 3$, and let K be the intersection of lines O_1P_1 and O_2P_2. Because $\angle P_1P_2P_3 = 60^\circ$, it follows that $\triangle P_2KP_1$ is a $30 - 60 - 90^\circ$ triangle. Let $d = P_1K$; then $P_2K = 2d$ and $P_1P_2 = \sqrt{3}d$. The Law of Cosines in $\triangle O_1KO_2$ gives

$$8^2 = (d+4)^2 + (2d-4)^2 - 2(d+4)(2d-4)\cos 60^\circ,$$

which simplifies to $3d^2 - 12d - 16 = 0$. The positive solution is $d = 2 + \frac{2}{3}\sqrt{21}$. Then $P_1P_2 = \sqrt{3}d = 2\sqrt{3} + 2\sqrt{7}$, and the required area is

$$\frac{\sqrt{3}}{4} \cdot \left(2\sqrt{3} + 2\sqrt{7}\right)^2 = 10\sqrt{3} + 6\sqrt{7} = \sqrt{300} + \sqrt{252}.$$

The requested sum is $300 + 252 = \boxed{552}$.

Solution 2

Let O_1 and O_2 be the centers of ω_1 and ω_2 respectively and draw O_1O_2, O_1P_1, and O_2P_2. Note than $\angle OP_1P_2$ and $\angle O_2P_2P_3$ are both right. Furthermore, since $\triangle P_1P_2P_3$ is equilateral, $m\angle P_1P_2P_3 = 60^\circ$ and $m\angle O_2P_2P_1 = 30^\circ$. Mark M as the base of the altitude from O_2 to P_1P_2. By special right triangles, $O_2M = 2$

and $P_2M = 2\sqrt{3}$. since $O_1O_2 = 8$ and $O_1P_1 = 4$, we can find find $P_1M = \sqrt{8^2 - (4+2)^2} = 2\sqrt{7}$. Thus, $P_1P_2 = P_1M + P_2M = 2\sqrt{3} + 2\sqrt{7}$. This makes $[P_1P_2P_3] = \dfrac{\left(2\sqrt{3}+2\sqrt{7}\right)^2\sqrt{3}}{4} = 10\sqrt{3} + 6\sqrt{7} = \sqrt{252} + \sqrt{300}$. This makes the answer $252 + 300 = 552$. $\boxed{\textbf{D.}}$

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