2006 AMC 12A Problems/Problem 1

The following problem is from both the 2006 AMC 12A #1 and 2006 AMC 10A #1, so both problems redirect to this page.

Problem

Sandwiches at Joe's Fast Food cost $\$3$ each and sodas cost $\$2$ each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas?

$\mathrm{(A)}\ 31\qquad\mathrm{(B)}\ 32\qquad\mathrm{(C)}\ 33\qquad\mathrm{(D)}\ 34\qquad\mathrm{(E)}\ 35$

Solution

The 5 sandwiches cost $5\cdot 3=15$ dollars. The 8 sodas cost $8\cdot 2=16$ dollars. In total, the purchase costs $15+16=31$ dollars. The answer is $\mathrm{(A)}$.

See also

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
Preceded by First Question	Followed by Problem 2
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2006 AMC 12A Problems/Problem 2

The following problem is from both the 2006 AMC 12A #2 and 2006 AMC 10A #2, so both problems redirect to this page.

Problem

Define $x \otimes y = x^3 - y$. What is $h \otimes (h \otimes h)$?

$\mathrm{(A)}\ -h\qquad\mathrm{(B)}\ 0\qquad\mathrm{(C)}\ h\qquad\mathrm{(D)}\ 2h\qquad\mathrm{(E)}\ h^3$

Solution

By the definition of $\otimes$, we have $h \otimes h = h^3 - h$. Then $h \otimes (h \otimes h) = h \otimes (h^3 - h) = h^3 - (h^3 - h) = h$. The answer is $\mathrm{(C)}$.

See also

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
Preceded by Problem 1	Followed by Problem 3
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2006 AMC 12A Problems/Problem 3

The following problem is from both the 2006 AMC 12A #3 and 2006 AMC 10A #3, so both problems redirect to this page.

Problem

The ratio of Mary's age to Alice's age is $3:5$. Alice is 30 years old. How old is Mary?

$\mathrm{(A)}\ 15\qquad \mathrm{(B)}\ 18\qquad \mathrm{(C)}\ 20\qquad \mathrm{(D)}\ 24\qquad \mathrm{(E)}\ 50$

Solution

Let m be Mary's age. Then $\frac{m}{30}=\frac{3}{5}$. Solving for m, we obtain $m=18$. The answer is $\mathrm{(B)}$.

See also

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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Preceded by Problem 2	Followed by Problem 4
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2006 AMC 12A Problems/Problem 4

The following problem is from both the 2006 AMC 12A #4 and 2008 AMC 10A #4, so both problems redirect to this page.

Problem

A digital watch displays hours and minutes with AM and PM. What is the largest possible sum of the digits in the display?

$\mathrm{(A)}\ 17\qquad\mathrm{(B)}\ 19\qquad\mathrm{(C)}\ 21\qquad\mathrm{(D)}\ 22\qquad\mathrm{(E)}\ 23$

Solution

From the greedy algorithm, we have 9 in the hours section and 59 in the minutes section.
$9+5+9=23 \Rightarrow \mathrm{(E)}$

See also

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
Preceded by Problem 3	Followed by Problem 5
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Preceded by Problem 3	Followed by Problem 5
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Category: Introductory Number Theory Problems

2006 AMC 12A Problems/Problem 5

The following problem is from both the 2006 AMC 12A #5 and 2006 AMC 10A #5, so both problems redirect to this page.

Problem

Doug and Dave shared a pizza with 8 equally-sized slices. Doug wanted a plain pizza, but Dave wanted anchovies on half the pizza. The cost of a plain pizza was 8 dollars, and there was an additional cost of 2 dollars for putting anchovies on one half. Dave ate all the slices of anchovy pizza and one plain slice. Doug ate the remainder. Each paid for what he had eaten. How many more dollars did Dave pay than Doug?

$\mathrm{(A) \ } 1\qquad \mathrm{(B) \ } 2\qquad \mathrm{(C) \ } 3\qquad \mathrm{(D) \ } 4\qquad \mathrm{(E) \ } 5$

Solution

Dave and Doug paid $8+2=10$ dollars in total. Doug paid for three slices of plain pizza, which cost $\frac{3}{8}\cdot 8=3$. Dave paid $10-3=7$ dollars. Dave paid $7-3=4$ more dollars than Doug. The answer is $\mathrm{(D)}$.

See also

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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Preceded by Problem 4	Followed by Problem 6
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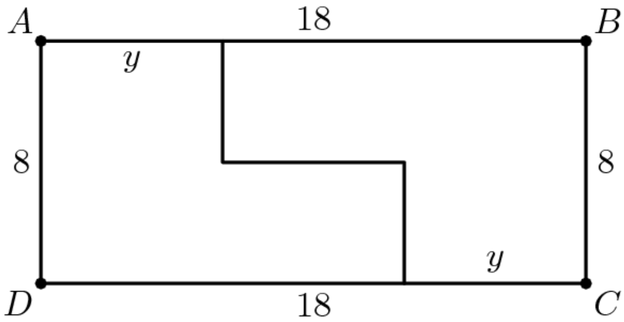
Category: Introductory Algebra Problems

2006 AMC 12A Problems/Problem 6

The following problem is from both the 2006 AMC 12A #6 and 2006 AMC 10A #7, so both problems redirect to this page.

Problem

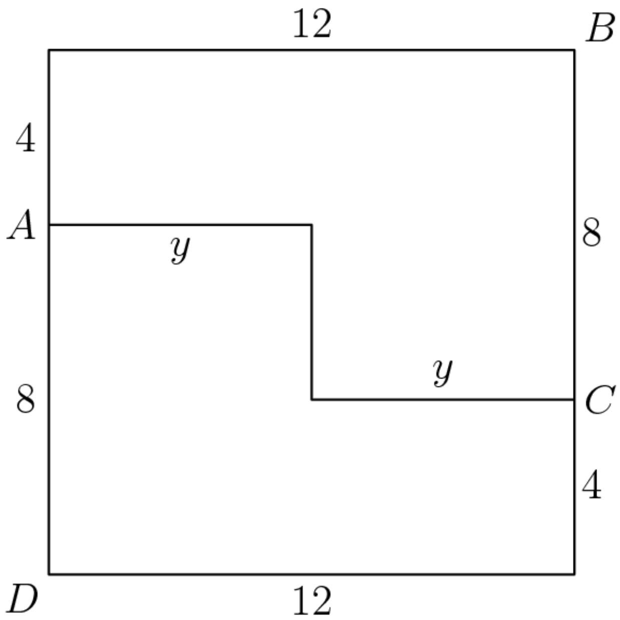
The 8×18 rectangle $ABCD$ is cut into two congruent hexagons, as shown, in such a way that the two hexagons can be repositioned without overlap to form a square. What is y?



$\textbf{(A) } 6 \qquad \textbf{(B) } 7 \qquad \textbf{(C) } 8 \qquad \textbf{(D) } 9 \qquad \textbf{(E) } 10$

Solution

Since the two hexagons are going to be repositioned to form a square without overlap, the area will remain the same. The rectangle's area is $18 \cdot 8 = 144$. This means the square will have four sides of length 12. The only way to do this is shown below.



As you can see from the diagram, the line segment denoted as y is half the length of the side of the square, which leads to $y = \frac{12}{2} = 6 \Longrightarrow \text{(A)}$.

See also

2006 AMC 12A Problems/Problem 7

Problem

Mary is 20% older than Sally, and Sally is 40% younger than Danielle. The sum of their ages is 23.2 years. How old will Mary be on her next birthday?

$\mathrm{(A) \ } 7 \qquad \mathrm{(B) \ } 8 \qquad \mathrm{(C) \ } 9 \qquad \mathrm{(D) \ } 10 \qquad \mathrm{(E) \ } 11$

Solution

Let m be Mary's age, let s be Sally's age, and let d be Danielle's age. We have $s = .6d$, and $m = 1.2s = 1.2(.6d) = .72d$. The sum of their ages is $m + s + d = .72d + .6d + d = 2.32d$. Therefore, $2.32d = 23.2$, and $d = 10$. Then $m = .72(10) = 7.2$. Mary will be 8 on her next birthday. The answer is $\mathrm{(B)}$.

See also

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
Preceded by Problem 6	Followed by Problem 8
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2006 AMC 12A Problems/Problem 8

> The following problem is from both the 2006 AMC 12A #8 and 2008 AMC 10A #9, so both problems redirect to this page.

Problem

How many sets of two or more consecutive positive integers have a sum of 15?

$\mathrm{(A) \ } 1\qquad \mathrm{(B) \ } 2\qquad \mathrm{(C) \ } 3\qquad \mathrm{(D) \ } 4\qquad \mathrm{(E) \ } 5$

Solution

Notice that if the consecutive positive integers have a sum of 15, then their average (which could be a fraction) must be a divisor of 15. If the number of integers in the list is odd, then the average must be either 1, 3, or 5, and 1 is clearly not possible. The other two possibilities both work:

- $1 + 2 + 3 + 4 + 5 = 15$
- $4 + 5 + 6 = 15$

If the number of integers in the list is even, then the average will have a $\frac{1}{2}$. The only possibility is $\frac{15}{2}$, from which we get:

- $15 = 7 + 8$

Thus, the correct answer is 3, answer choice $\mathrm{(C)}$.

See also

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12A Problems/Problem 9

Problem

Oscar buys 13 pencils and 3 erasers for 1.00. A pencil costs more than an eraser, and both items cost a whole number of cents. What is the total cost, in cents, of one pencil and one eraser?

$\mathrm{(A)}\ 10\qquad\mathrm{(B)}\ 12\qquad\mathrm{(C)}\ 15\qquad\mathrm{(D)}\ 18\qquad\mathrm{(E)}\ 20$

Solution

Let the price of a pencil be p and an eraser e. Then $13p + 3e = 100$ with $p > e > 0$. Since p and e are positive integers, we must have $e \geq 1$ and $p \geq 2$.

Considering the equation $13p + 3e = 100$ modulo 3 (that is, comparing the remainders when both sides are divided by 3) we have $p + 0e \equiv 1 \pmod{3}$ so p leaves a remainder of 1 on division by 3.

Since $p \geq 2$, possible values for p are 4, 7, 10 ....

Since 13 pencils cost less than 100 cents, $13p < 100$. $13 \times 10 = 130$ is too high, so p must be 4 or 7.

If $p = 4$ then $13p = 52$ and so $3e = 48$ giving $e = 16$. This contradicts the pencil being more expensive. The only remaining value for p is 7; then the 13 pencils cost $7 \times 13 = 91$ cents and so the 3 erasers together cost 9 cents and each eraser costs $\frac{9}{3} = 3$ cents.

Thus one pencil plus one eraser cost $7 + 3 = 10$ cents, which is answer choice $\mathrm{(A)}$.

See also

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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Category: Introductory Number Theory Problems

2006 AMC 12A Problems/Problem 10

The following problem is from both the 2006 AMC 12A #10 and 2006 AMC 10A #10, so both problems redirect to this page.

Problem

For how many real values of x is $\sqrt{120-\sqrt{x}}$ an integer?

$\mathrm{(A)}\ 3\qquad \mathrm{(B)}\ 6\qquad \mathrm{(C)}\ 9\qquad \mathrm{(D)}\ 10\qquad \mathrm{(E)}\ 11$

Solution

For $\sqrt{120-\sqrt{x}}$ to be an integer, $120-\sqrt{x}$ must be a perfect square.

Since $\sqrt{x}$ can't be negative, $120-\sqrt{x} \leq 120$.

The perfect squares that are less than or equal to 120 are $\{0,1,4,9,16,25,36,49,64,81,100\}$, so there are 11 values for $120-\sqrt{x}$.

Since every value of $120-\sqrt{x}$ gives one and only one possible value for x, the number of values of x is $11 \Rightarrow E$.

See also

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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Category: Introductory Algebra Problems

2006 AMC 12A Problems/Problem 11

The following problem is from both the 2006 AMC 12A #11 and 2008 AMC 10A #11, so both problems redirect to this page.

Problem

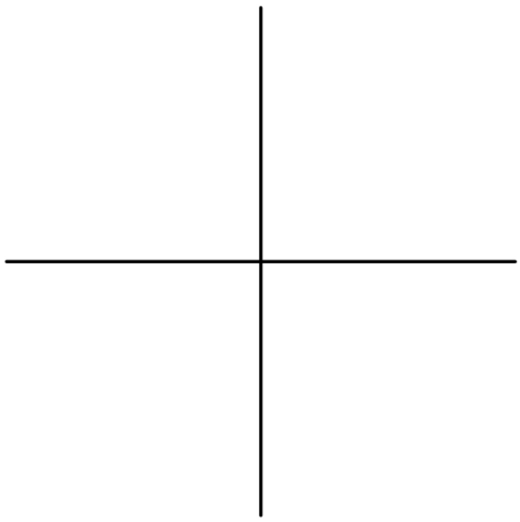
Which of the following describes the graph of the equation $(x+y)^2 = x^2 + y^2$?

$\mathrm{(A)}\ \text{the empty set}\qquad\mathrm{(B)}\ \text{one point}\qquad\mathrm{(C)}\ \text{two lines}\qquad\mathrm{(D)}\ \text{a circle}\qquad\mathrm{(E)}\ \text{the entire plane}$

Solution

$$\begin{aligned}(x+y)^2 &= x^2+y^2\\ x^2+2xy+y^2 &= x^2+y^2\\ 2xy &= 0\end{aligned}$$

Either $x=0$ or $y=0$. The union of them is 2 lines, and thus the answer is $\mathrm{(C)}$.



See also

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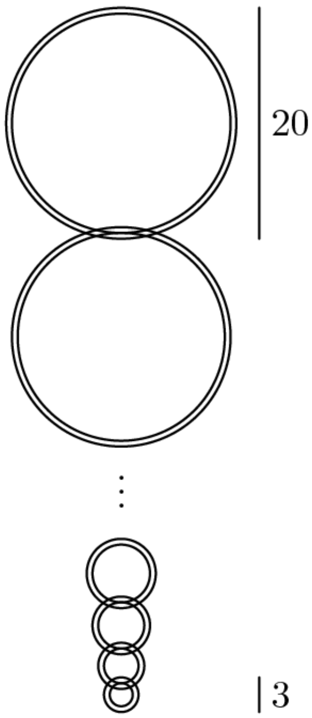
2006 AMC 12A Problems/Problem 12

The following problem is from both the 2004 AMC 12A #12 and 2004 AMC 10A #14, so both problems redirect to this page.

Contents

Problem

A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the outer rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?



$\textbf{(A) } 171 \qquad \textbf{(B) } 173 \qquad \textbf{(C) } 182 \qquad \textbf{(D) } 188 \qquad \textbf{(E) } 210$

Solutions

Solution 1

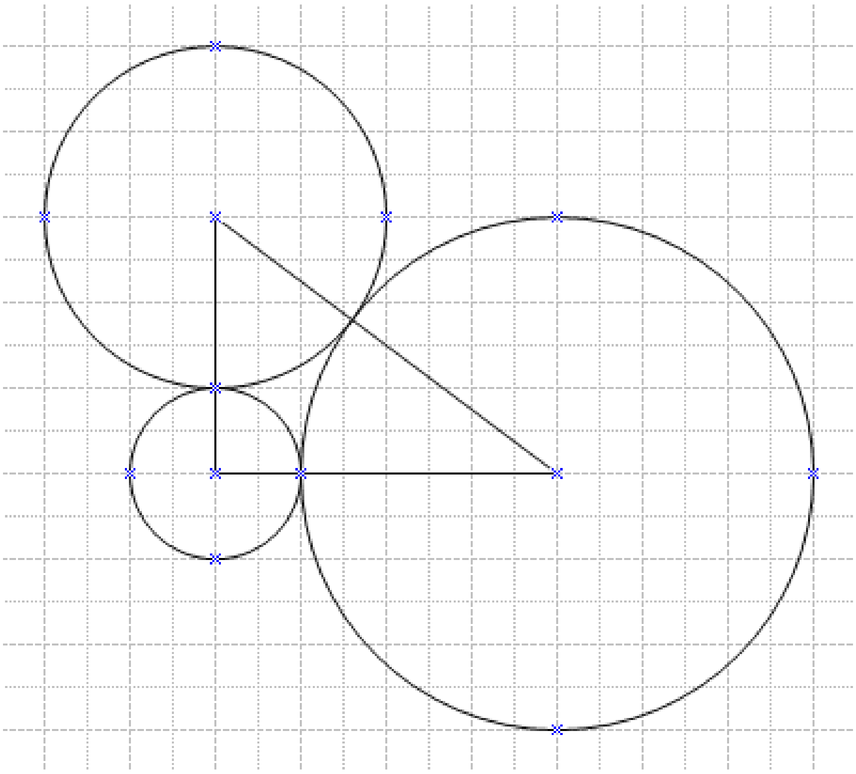
The inside diameters of the rings are the positive integers from 1 to 18. The total distance needed is the sum of these values plus 2 for the top of the first ring and the bottom of the last ring. Using the formula for the sum of an arithmetic series, the answer is $\frac{18 \cdot 19}{2} + 2 = 173 \Rightarrow \text{(B)}$.

Solution 2

Alternatively, the sum of the consecutive integers from 3 to 20 is $\frac{1}{2}(18)(3+20) = 207$. However, the 17 intersections between the rings must be subtracted, and we also get $207 - 2(17) = 173$.

2006 AMC 12A Problems/Problem 13

Problem



The vertices of a $3-4-5$ right triangle are the centers of three mutually externally tangent circles, as shown. What is the sum of the areas of the three circles?

$\mathrm{(A)}\ 12\pi\qquad\mathrm{(B)}\ \frac{25\pi}{2}\qquad\mathrm{(C)}\ 13\pi\qquad\mathrm{(D)}\ \frac{27\pi}{2}\qquad\mathrm{(E)}\ 14\pi$

Solution

Let the radius of the smallest circle be a. We find that the radius of the largest circle is $4-a$ and the radius of the second largest circle is $3-a$. Thus, $4-a+3-a=5 \iff a=1$. The radii of the other circles are 3 and 2. The sum of their areas is $\pi+9\pi+4\pi=14\pi \iff \mathrm{(E)}$

See also

- 2006 AMC 12A Problems

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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2006 AMC 12A Problems/Problem 14

The following problem is from both the 2006 AMC 12A #14 and 2006 AMC 10A #22, so both problems redirect to this page.

Contents

Problem

Two farmers agree that pigs are worth 300 dollars and that goats are worth 210 dollars. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a 390 dollar debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

$\mathrm{(A) \ } 5\qquad \mathrm{(B) \ } 10\qquad \mathrm{(C) \ } 30\qquad \mathrm{(D) \ } 90\qquad \mathrm{(E) \ } 210$

Solutions

Solution 1

The problem can be restated as an equation of the form $300p + 210g = x$, where p is the number of pigs, g is the number of goats, and x is the positive debt. The problem asks us to find the lowest x possible. p and g must be integers, which makes the equation a Diophantine equation. The Euclidean algorithm tells us that there are integer solutions to the Diophantine equation $am + bn = c$, where c is the greatest common divisor of a and b, and no solutions for any smaller c. Therefore, the answer is the greatest common divisor of 300 and 210, which is 30, $\mathrm{(C)}$

Solution 2

Alternatively, note that $300p + 210g = 30(10p + 7g)$ is divisible by 30 no matter what p and g are, so our answer must be divisible by 30. In addition, three goats minus two pigs give us $630 - 600 = 30$ exactly. Since our theoretical best can be achieved, it must really be the best, and the answer is $\mathrm{(C)}$. debt that can be resolved.

Solution 3

Let us simplify this problem. Dividing by 30, we get a pig to be: $\frac{300}{30} = 10$, and a goat to be $\frac{210}{30} = 7$. It becomes evident that if you exchange 5 pigs for 7 goats, we get the smallest positive difference - $5 \cdot 10 - 7 \cdot 7 = 50 - 49 = 1$. Since we originally divided by 30, we need to multiply again, thus getting the answer: $1 \cdot 30 = \mathrm{(C)} 30$

See also

2006 AMC 12A Problems/Problem 15

Contents

Problem

Suppose $\cos x = 0$ and $\cos(x+z) = 1/2$. What is the smallest possible positive value of z?

$\text{(A) } \frac{\pi}{6} \qquad \text{(B) } \frac{\pi}{3} \qquad \text{(C) } \frac{\pi}{2} \qquad \text{(D) } \frac{5\pi}{6} \quad \text{(E) } \frac{7\pi}{6}$

Solutions

Solution 1

- For $\cos x = 0$, x must be in the form of $\frac{\pi}{2} + \pi n$, where n denotes any integer.
- For $\cos(x+z) = 1/2$, $x + z = \frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n$.

The smallest possible value of z will be that of $\frac{5\pi}{3} - \frac{3\pi}{2} = \frac{\pi}{6} \Rightarrow \text{(A)}$.

See also

- 2006 AMC 12A Problems

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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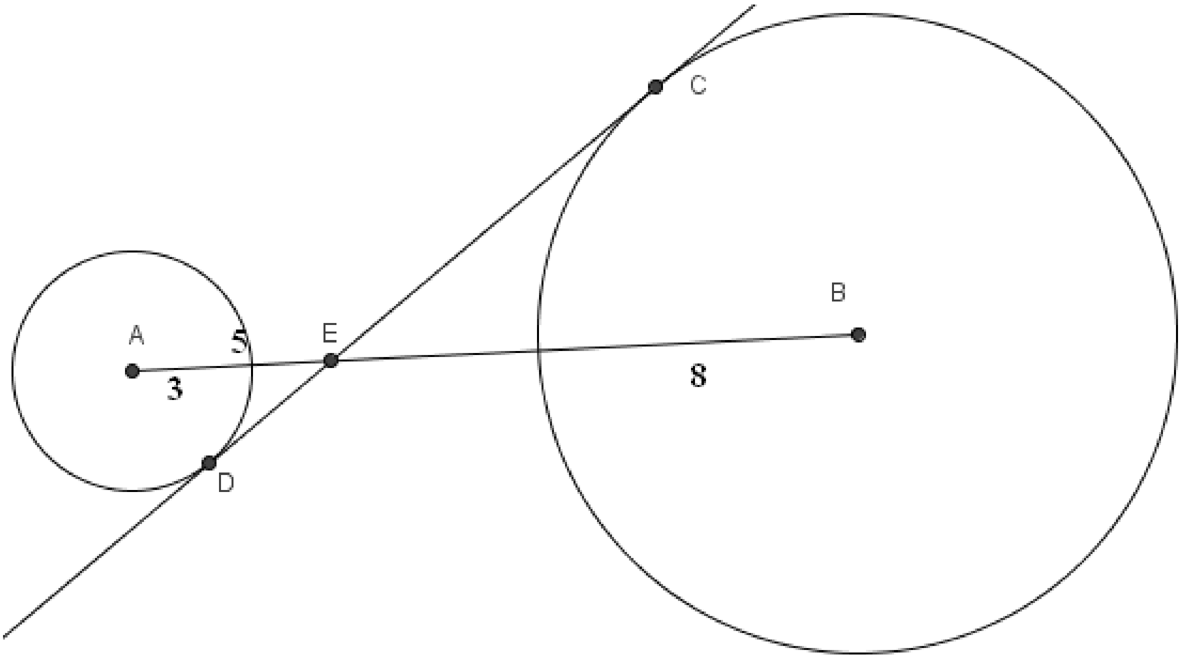
2006 AMC 12A Problems/Problem 16

The following problem is from both the 2006 AMC 12A #16 and 2006 AMC 10A #23, so both problems redirect to this page.

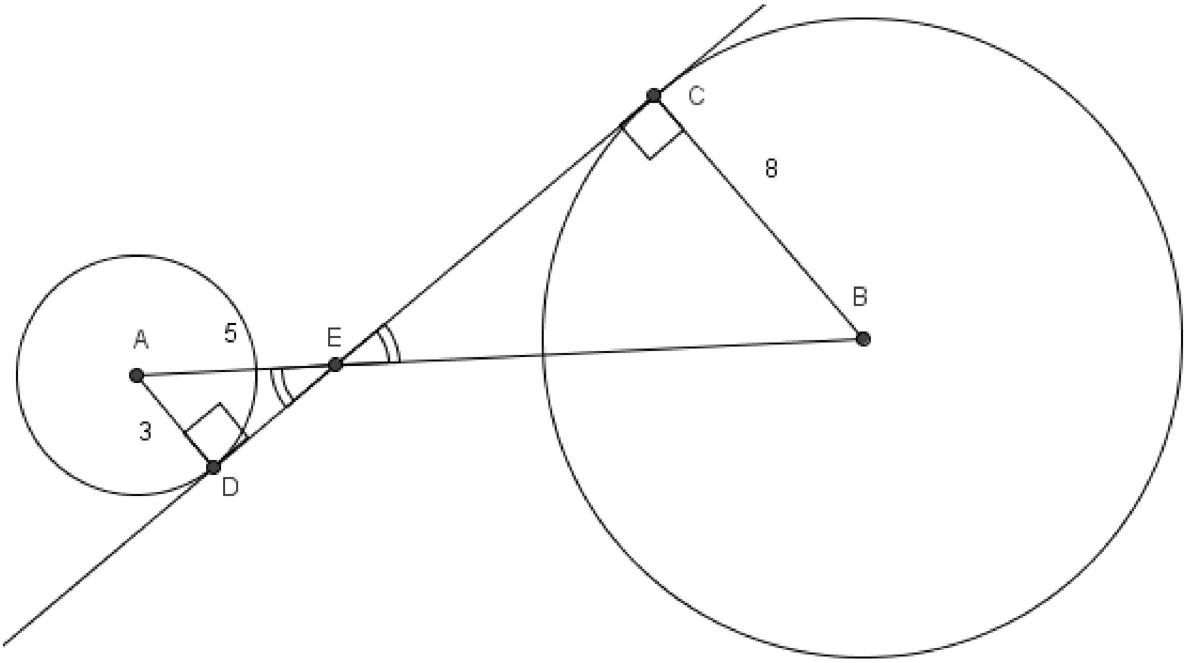
Problem

Circles with centers A and B have radii 3 and 8, respectively. A common internal tangent intersects the circles at C and D, respectively. Lines AB and CD intersect at E, and $AE = 5$. What is CD?

$\text{(A)}\ 13 \qquad \text{(B)}\ \frac{44}{3} \qquad \text{(C)}\ \sqrt{221} \qquad \text{(D)}\ \sqrt{255} \qquad \text{(E)}\ \frac{55}{3}$



Solution



$\angle AED$ and $\angle BEC$ are vertical angles so they are congruent, as are angles $\angle ADE$ and $\angle BCE$ (both are right angles because the radius and tangent line at a point on a circle are always perpendicular). Thus, $\triangle ACE \sim \triangle BDE$.

By the Pythagorean Theorem, line segment $DE = 4$. The sides are proportional, so $\frac{DE}{AD} = \frac{CE}{BC} \Rightarrow \frac{4}{3} = \frac{CE}{8}$. This makes $CE = \frac{32}{3}$ and $CD = CE + DE = 4 + \frac{32}{3} = \frac{44}{3} \Longrightarrow \mathrm{B}$.

See also

2006 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2006))	
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Category: Introductory Geometry Problems

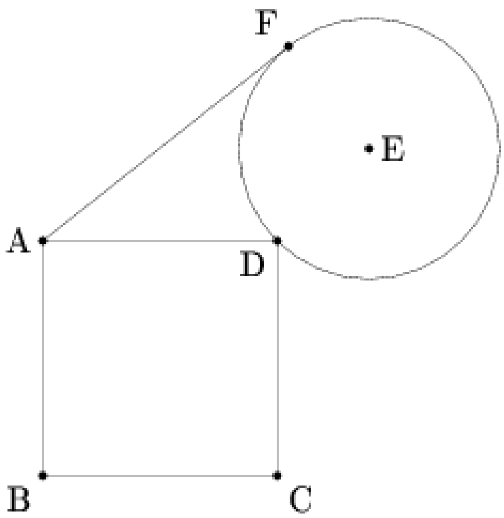
2006 AMC 12A Problems/Problem 17

Contents

Problem

Square $ABCD$ has side length s, a circle centered at E has radius r, and r and s are both rational. The circle passes through D, and D lies on $\overline{BE}$. Point F lies on the circle, on the same side of $\overline{BE}$ as A. Segment AF is tangent to the circle, and $AF=\sqrt{9+5\sqrt{2}}$. What is r/s?

$\mathrm{(A) \ } \frac{1}{2}\qquad \mathrm{(B) \ } \frac{5}{9}\qquad \mathrm{(C) \ } \frac{3}{5}\qquad \mathrm{(D) \ } \frac{5}{3}\qquad \mathrm{(E) \ } \frac{9}{5}$



Solution

Solution 1

One possibility is to use the coordinate plane, setting B at the origin. Point A will be $(0,s)$ and E will be $\left(s+\frac{r}{\sqrt{2}},\ s+\frac{r}{\sqrt{2}}\right)$ since B, D, and E are collinear and contain a diagonal of $ABCD$. The Pythagorean theorem results in

$$AF^2+EF^2=AE^2$$

$$r^2+\left(\sqrt{9+5\sqrt{2}}\right)^2=\left(\left(s+\frac{r}{\sqrt{2}}\right)-0\right)^2+\left(\left(s+\frac{r}{\sqrt{2}}\right)-s\right)^2$$

$$r^2 + 9 + 5\sqrt{2} = s^2 + rs\sqrt{2} + \frac{r^2}{2} + \frac{r^2}{2}$$

$$9 + 5\sqrt{2} = s^2 + rs\sqrt{2}$$

This implies that $rs = 5$ and $s^2 = 9$; dividing gives us $\frac{r}{s} = \frac{5}{9} \Rightarrow B$.

Solution 2

First note that angle $\angle AFE$ is right since $\overline{AF}$ is tangent to the circle. Using the Pythagorean Theorem on $\triangle AFE$, then, we see

$$AE^2 = 9 + 5\sqrt{2} + r^2.$$

But it can also be seen that $\angle BDA = 45^\circ$. Therefore, since D lies on $\overline{BE}$, $\angle ADE = 135^\circ$. Using the Law of Cosines on $\triangle ADE$, we see

$$AE^2 = s^2 + r^2 - 2sr\cos(135^\circ)$$

$$AE^2 = s^2 + r^2 - 2sr\left(-\frac{1}{\sqrt{2}}\right)$$

$$AE^2 = s^2 + r^2 + \sqrt{2}sr$$

$$9 + 5\sqrt{2} + r^2 = s^2 + r^2 + \sqrt{2}sr$$

$$9 + 5\sqrt{2} = s^2 + \sqrt{2}sr.$$

Thus, since r and s are rational, $s^2 = 9$ and $sr = 5$. So $s = 3$, $r = \frac{5}{3}$, and $\frac{r}{s} = \frac{5}{9}$.

See also

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2006 AMC 12A Problems/Problem 18

Problem

The function f has the property that for each real number x in its domain, $1/x$ is also in its domain and

$$f(x) + f\left(\frac{1}{x}\right) = x$$

What is the largest set of real numbers that can be in the domain of f?

$(\mathrm{A})\ \{x|x \neq 0\} \qquad (\mathrm{B})\ \{x|x < 0\}$

$(\mathrm{C})\ \{x|x > 0\} (\mathrm{D})\ \{x|x \neq -1 \text{ and x} \neq 0 \text{ and x} \neq 1\}$

$(\mathrm{E})\ \{-1, 1\}$

Solution

Quickly verifying by plugging in values verifies that -1 and 1 are in the domain.

$$f(x) + f\left(\frac{1}{x}\right) = x$$

Plugging in $\frac{1}{x}$ into the function:

$$f\left(\frac{1}{x}\right) + f\left(\frac{1}{\frac{1}{x}}\right) = \frac{1}{x}$$

$$f\left(\frac{1}{x}\right) + f(x) = \frac{1}{x}$$

Since $f(x) + f\left(\frac{1}{x}\right)$ cannot have two values:

$$x = \frac{1}{x}$$

$$x^2 = 1$$

$$x = \pm 1$$

Therefore, the largest set of real numbers that can be in the domain of f is $\{-1, 1\} \Rightarrow E$

See also

- 2006 AMC 12A Problems

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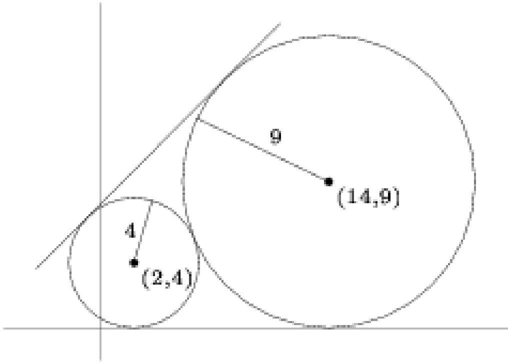
2006 AMC 12A Problems/Problem 19

Problem

Circles with centers $(2,4)$ and $(14,9)$ have radii 4 and 9, respectively. The equation of a common external tangent to the circles can be written in the form $y=mx+b$ with $m>0$. What is b?



$\mathrm{(A)}\ \frac{908}{119}\qquad\mathrm{(B)}\ \frac{909}{119}\qquad\mathrm{(C)}\ \frac{130}{17}\qquad\mathrm{(D)}\ \frac{911}{119}\qquad\mathrm{(E)}\ \frac{912}{119}$

Solution

Let L_1 be the line that goes through $(2,4)$ and $(14,9)$, and let L_2 be the line $y=mx+b$. If we let θ be the measure of the acute angle formed by L_1 and the x-axis, then $\tan\theta=\frac{5}{12}$. L_1 clearly bisects the angle formed by L_2 and the x-axis, so $m=\tan{2\theta}=\frac{2\tan\theta}{1-\tan^2{\theta}}=\frac{120}{119}$. We also know that L_1 and L_2 intersect at a point on the x-axis. The equation of L_1 is $y=\frac{5}{12}x+\frac{19}{6}$, so the coordinate of this point is $\left(-\frac{38}{5},0\right)$. Hence the equation of L_2 is $y=\frac{120}{119}x+\frac{912}{119}$, so $b=\frac{912}{119}$, and our answer choice is $\boxed{\mathrm{E}}$.

See also

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2006 AMC 12A Problems/Problem 20

The following problem is from both the 2006 AMC 12A #20 and 2006 AMC 10A #25, so both problems redirect to this page.

Contents

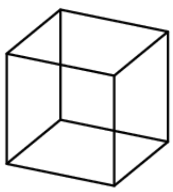
Problem

A bug starts at one vertex of a cube and moves along the edges of the cube according to the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug will have visited every vertex exactly once?

$$\textbf{(A) } \frac{1}{2187}\qquad \textbf{(B) } \frac{1}{729}\qquad \textbf{(C) } \frac{2}{243}\qquad \textbf{(D) } \frac{1}{81}\qquad \textbf{(E) } \frac{5}{243}$$

Solutions

Solution 1



Let us count the good paths. The bug starts at an arbitrary vertex, moves to a neighboring vertex (3 ways), and then to a new neighbor (2 more ways). So, without loss of generality, let the cube have vertices $ABCDEFGH$ such that $ABCD$ and $EFGH$ are two opposite faces with A above E and B above F. The bug starts at A and moves first to B, then to C.

From this point, there are two cases.

Case 1: the bug moves to D. From D, there is only one good move available, to H. From H, there are two ways to finish the trip, either by going $H \to G \to F \to E$ or $H \to E \to F \to G$. So there are 2 good paths in this case.

Case 2: the bug moves to G. Case 2a: the bug moves $G \to H$. In this case, there are 0 good paths because it will not be possible to visit both D and E without double-visiting some vertex. Case 2b: the bug moves $G \to F$. There is a unique good path in this case, $F \to E \to H \to D$.

Thus, all told we have 3 good paths after the first two moves, for a total of $3 \cdot 3 \cdot 2 = 18$ good paths. There were $3^7 = 2187$ possible paths the bug could have taken, so the probability a random path is good is the ratio of good paths to total paths, $\frac{18}{2187} = \frac{2}{243} \Rightarrow \boxed{(C)}$.

Solution 2 (using the answer choices)

As in Solution 1, the bug can move from its arbitrary starting vertex to a neighboring vertex in 3 ways. After this, the bug can move to a new neighbor in 2 ways (it cannot return to the first vertex). The total number of paths (as stated above) is 3^7 or 2187. Therefore, the probability of the bug following a good path is equal to $\frac{6x}{2187}$ for some positive integer x. The only answer choice which can be expressed in this form is $\frac{2}{243} \Rightarrow \boxed{(C)}$.

See also

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Category: Introductory Combinatorics Problems

2006 AMC 12A Problems/Problem 21

Problem

Let

$S_1 = \{(x, y) | \log_{10}(1 + x^2 + y^2) \le 1 + \log_{10}(x + y)\}$

and

$S_2 = \{(x, y) | \log_{10}(2 + x^2 + y^2) \le 2 + \log_{10}(x + y)\}$.

What is the ratio of the area of S_2 to the area of S_1?

$\text{(A) } 98 \qquad \text{(B) } 99 \qquad \text{(C) } 100 \qquad \text{(D) } 101 \qquad \text{(E) } 102$

Solution

Looking at the constraints of S_1:

$\log_{10}(1 + x^2 + y^2) \le 1 + \log_{10}(x + y)$

$\log_{10}(1 + x^2 + y^2) \le \log_{10} 10 + \log_{10}(x + y)$

$\log_{10}(1 + x^2 + y^2) \le \log_{10}(10x + 10y)$

$1 + x^2 + y^2 \le 10x + 10y$

$x^2 - 10x + y^2 - 10y \le -1$

$x^2 - 10x + 25 + y^2 - 10y + 25 \le 49$

$(x - 5)^2 + (y - 5)^2 \le (7)^2$

S_1 is a circle with a radius of 7. So, the area of S_1 is 49π.

Looking at the constraints of S_2:

$\log_{10}(2 + x^2 + y^2) \le 2 + \log_{10}(x + y)$

$\log_{10}(2 + x^2 + y^2) \le \log_{10} 100 + \log_{10}(x + y)$

$\log_{10}(2 + x^2 + y^2) \le \log_{10}(100x + 100y)$

$2 + x^2 + y^2 \le 100x + 100y$

$x^2 - 100x + y^2 - 100y \le -2$

$x^2 - 100x + 2500 + y^2 - 100y + 2500 \le 4998$

$(x - 50)^2 + (y - 50)^2 \le (7\sqrt{102})^2$

S_2 is a circle with a radius of $7\sqrt{102}$. So, the area of S_2 is 4998π.

So the desired ratio is $\frac{4998\pi}{49\pi} = 102 \Rightarrow \boxed{E}$.

See also

2006 AMC 12A Problems/Problem 22

Problem

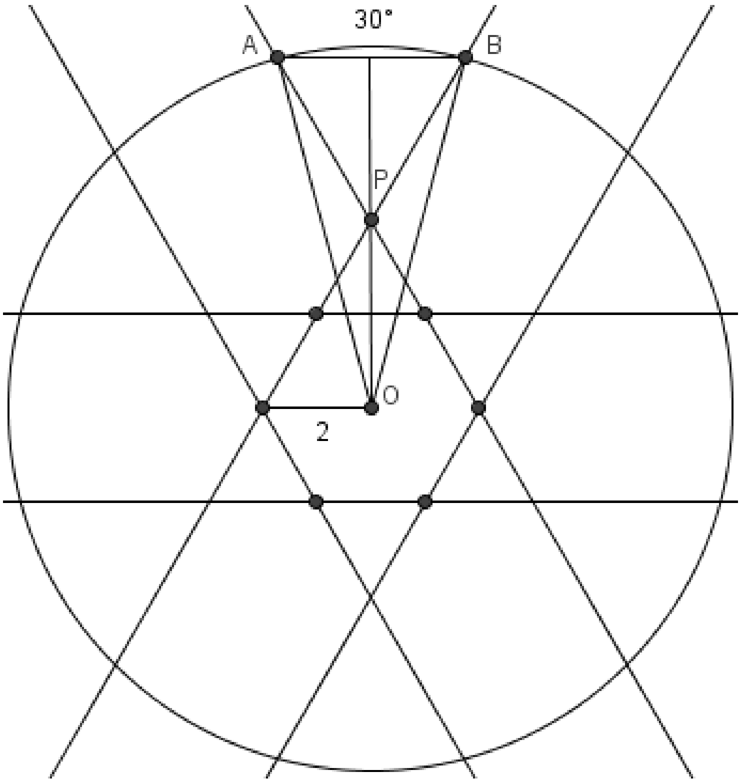
A circle of radius r is concentric with and outside a regular hexagon of side length 2. The probability that three entire sides of hexagon are visible from a randomly chosen point on the circle is $1/2$. What is r?

$\mathrm{(A) \ } 2\sqrt{2}+2\sqrt{3}\qquad \mathrm{(B) \ } 3\sqrt{3}+\sqrt{2}\qquad \mathrm{(C) \ } 2\sqrt{6}+\sqrt{3}\qquad \mathrm{(D) \ } 3\sqrt{2}+\sqrt{6}$

$\mathrm{(E) \ } 6\sqrt{2}-\sqrt{3}$

Solution

Project any two non-adjacent and non-opposite sides of the hexagon to the circle; the arc between the two points formed is the location where all three sides of the hexagon can be fully viewed. Since there are six such pairs of sides, there are six arcs. The probability of choosing a point is $1/2$, or if the total arc degree measures add up to $\frac{1}{2}\cdot 360^\circ = 180^\circ$. Each arc must equal $\frac{180}{6}=30^\circ$.



Call the center O, and the two endpoints of the arc A and B, so $\angle AOB = 30^\circ$. Let P be the intersections of the projections of the sides of the hexagon corresponding to $\overline{AB}$. Notice that $\triangle APO$ is an isosceles triangle: $\angle AOP = 15^\circ$ and $\angle OAP = OAB - 60 = \frac{180-30}{2} - 60 = 15^\circ$. Since OA is a radius and OP can be found in terms of a side of the hexagon, we are almost done.

If we draw the altitude of APO from P, then we get a right triangle. Using simple trigonometry,

$$\cos 15 = \frac{\frac{r}{2}}{2\sqrt{3}} = \frac{r}{4\sqrt{3}}.$$

Since $\cos 15 = \cos(45-30) = \frac{\sqrt{6}+\sqrt{2}}{4}$, we get

$$r = \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)\cdot 4\sqrt{3} = 3\sqrt{2}+\sqrt{6} \Rightarrow \mathrm{(D)}.$$

2006 AMC 12A Problems/Problem 23

Problem

Given a finite sequence $S = (a_1, a_2, \ldots, a_n)$ of n real numbers, let $A(S)$ be the sequence

$$\left(\frac{a_1 + a_2}{2}, \frac{a_2 + a_3}{2}, \ldots, \frac{a_{n-1} + a_n}{2}\right)$$

of $n-1$ real numbers. Define $A^1(S) = A(S)$ and, for each integer m, $2 \le m \le n-1$, define $A^m(S) = A(A^{m-1}(S))$. Suppose $x > 0$, and let $S = (1, x, x^2, \ldots, x^{100})$. If $A^{100}(S) = (1/2^{50})$, then what is x?

$$\text{(A) } 1 - \frac{\sqrt{2}}{2} \qquad \text{(B) } \sqrt{2} - 1 \qquad \text{(C) } \frac{1}{2} \qquad \text{(D) } 2 - \sqrt{2} \qquad \text{(E) } \frac{\sqrt{2}}{2}$$

Solution

$$A^1(S) = \left(\frac{1+x}{2}, \frac{x+x^2}{2}, \ldots, \frac{x^{99}+x^{100}}{2}\right)$$
$$A^2(S) = \left(\frac{1+2x+x^2}{2^2}, \frac{x+2x^2+x^3}{2^2}, \ldots, \frac{x^{98}+2x^{99}+x^{100}}{2^2}\right)$$
$$\Rightarrow A^2(S) = \left(\frac{(x+1)^2}{2^2}, \frac{x(x+1)^2}{2^2}, \ldots, \frac{x^{98}(x+1)^2}{2^2}\right)$$

In general, $A^n(S) = \left(\frac{(x+1)^n}{2^n}, \frac{x(x+1)^n}{2^n}, \ldots, \frac{x^{100-n}(x+1)^n}{2^n}\right)$ such that $A^n(s)$ has $101 - n$ terms. Specifically, $A^{100}(S) = \frac{(x+1)^{100}}{2^{100}}$ To find x, we need only solve the equation $\frac{(x+1)^{100}}{2^{100}} = \frac{1}{2^{50}}$. Algebra yields $x = \sqrt{2} - 1$.

See also

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2006 AMC 12A Problems/Problem 24

Contents

Problem

The expression

$(x+y+z)^{2006}+(x-y-z)^{2006}$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

$\mathrm{(A)}\ 6018\qquad\mathrm{(B)}\ 671,676\qquad\mathrm{(C)}\ 1,007,514\qquad\mathrm{(D)}\ 1,008,016\qquad\mathrm{(E)}\ 2,015,028$

Solution 1

By the Multinomial Theorem, the summands can be written as

$$\sum_{a+b+c=2006}\frac{2006!}{a!b!c!}x^ay^bz^c$$

and

$$\sum_{a+b+c=2006}\frac{2006!}{a!b!c!}x^a(-y)^b(-z)^c,$$

respectively. Since the coefficients of like terms are the same in each expression, each like term either cancel one another out or the coefficient doubles. In each expansion there are:

$$\binom{2006+2}{2}=2015028$$

terms without cancellation. For any term in the second expansion to be negative, the parity of the exponents of y and z must be opposite. Now we find a pattern:

if the exponent of y is 1, the exponent of z can be all even integers up to 2004, so there are 1003 terms.

if the exponent of y is 3, the exponent of z can go up to 2002, so there are 1002 terms.

$\vdots$

if the exponent of y is 2005, then z can only be 0, so there is 1 term.

If we add them up, we get $\frac{1003\cdot 1004}{2}$ terms. However, we can switch the exponents of y and z and these terms will still have a negative sign. So there are a total of $1003\cdot 1004$ negative terms.

By subtracting this number from 2015028, we obtain $\boxed{D}$ or 1008016 as our answer.

Solution 2

Alternatively, we can use a generating function to solve this problem. The goal is to find the generating function for the number of unique terms in the simplified expression (in terms of k). In other words, we want to find $f(x)$ where the coefficient of x^k equals the number of unique terms in $(x+y+z)^k+(x-y-z)^k$.

First, we note that all unique terms in the expression have the form, $Cx^ay^bz^c$, where $a+b+c=k$ and C is some constant. Therefore, the generating function for the MAXIMUM number of unique terms possible in the simplified expression of $(x+y+z)^k+(x-y-z)^k$ is

$$(1+x+x^2+x^3\cdots)^3=\frac{1}{(1-x)^3}$$

Secondly, we note that a certain number of terms of the form, $Cx^ay^bz^c$, do not appear in the simplified version of our expression because those terms cancel. Specifically, we observe that terms cancel when $1\equiv b+c \pmod 2$ because every unique term is of the form:

$$\binom{k}{a,b,c}x^ay^bz^c+(-1)^{b+c}\binom{k}{a,b,c}x^ay^bz^c$$

for all possible a,b,c.

Since the generating function for the maximum number of unique terms is already known, it is logical that we want to find the generating function for the number of terms that cancel, also in terms of k. With some thought, we see that this desired generating function is the following:

$$2(x+x^3+x^5\cdots)(1+x^2+x^4\cdots)(1+x+x^2+x^3\cdots)=\frac{2x}{(1-x)^3(1+x)^2}$$

Now, we want to subtract the latter from the former in order to get the generating function for the number of unique terms in $(x+y+z)^k+(x-y-z)^k$, our initial goal:

$$\frac{1}{(1-x)^3}-\frac{2x}{(1-x)^3(1+x)^2}=\frac{x^2+1}{(1-x)^3(1+x)^2}$$

which equals

$$(x^2+1)(1+x+x^2\cdots)^3(1-x+x^2-x^3\cdots)^2$$

The coefficient of x^{2006} of the above expression equals

$$\sum_{a=0}^{2006}\binom{2+a}{2}\binom{1+2006-a}{1}(-1)^a+\sum_{a=0}^{2004}\binom{2+a}{2}\binom{1+2004-a}{1}(-1)^a$$

Evaluating the expression, we get 1008016, as expected.

Solution 3

Define P such that $P = y + z$. Then the expression in the problem becomes: $(x+P)^{2006} + (x-P)^{2006}$.

Expanding this using binomial theorem gives $(x^n + Px^{n-1} + ... + P^{n-1}x + P^n) + (x^n - Px^{n-1} + ... - P^{n-1}x + P^n)$, where $n = 2006$ (we may omit the coefficients, as we are seeking for the number of terms, not the terms themselves).

Simplifying gives: $2(x^n + x^{n-2}P^2 + ... + x^2P^{n-2} + P^n)$. Note that two terms that come out of different powers of P cannot combine and simplify, as their exponent of x will differ. Therefore, we simply add the number of terms produced from each addend. By the Binomial Theorem, $P^k = (y+z)^k$ will have $k+1$ terms, so the answer is $1 + 3 + 5 + ... + 2007 = 1004^2 = 1,008,016 \implies \boxed{D}$.

See also

- 2006 AMC 12A Problems

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2006 AMC 12A Problems/Problem 25

Contents

Problem

How many non- empty subsets S of $\{1,2,3,\ldots,15\}$ have the following two properties?

(1) No two consecutive integers belong to S.

(2) If S contains k elements, then S contains no number less than k.

$\mathrm{(A) \ } 277\qquad \mathrm{(B) \ } 311\qquad \mathrm{(C) \ } 376\qquad \mathrm{(D) \ } 377\qquad \mathrm{(E) \ } 405$

Solution 1

This question can be solved fairly directly by casework and pattern-finding. We give a somewhat more general attack, based on the solution to the following problem:

How many ways are there to choose k elements from an ordered n element set without choosing two consecutive members?

You want to choose k numbers out of n with no consecutive numbers. For each configuration, we can subtract $i-1$ from the i-th element in your subset. This converts your configuration into a configuration with k elements where the largest possible element is $n-k+1$, with no restriction on consectutive numbers. Since this process is easily reversible, we have a bijection. Without consideration of the second condition, we have: ${15 \choose 1} + {14 \choose 2} + {13 \choose 3} + ... + {9 \choose 7} + {8 \choose 8}$

Now we examine the second condition. It simply states that no element in our original configuration (and hence also the modified configuration, since we don't move the smallest element) can be less than k, which translates to subtracting k from the "top" of each binomial coefficient. Now we have, after we cancel all the terms ${n \choose k}$ where $n < k$, ${15 \choose 1} + {13 \choose 2} + {11 \choose 3} + {9 \choose 4} + {7 \choose 5}= 15 + 78 + 165 + 126 + 21 = \boxed{405} \Longrightarrow \mathrm{(E)}$

Solution 2

We have the same setup as in the previous solution.

Note that if $n < 2k-1$, the answer will be 0. Otherwise, the k elements we choose define $k+1$ boxes (which divide the nonconsecutive numbers) into which we can drop the $n-k$ remaining elements, with the caveat that each of the middle $k-1$ boxes must have at least one element (since the numbers are nonconsecutive). This is equivalent to dropping $n-2k+1$ elements into $k+1$ boxes, where each box is allowed to be empty. And this is equivalent to arranging $n-k+1$ objects, k of which are dividers, which we can do in $F(n,k) = {n-k+1 \choose k}$ ways.

Now, looking at our original question, we see that the thing we want to calculate is just

$$F(15,1) + F(14,2) + F(13,3) + F(12,4) + F(11,5) = {15 \choose 1} + {13 \choose 2} + {11 \choose 3} + {9 \choose 4} + {7 \choose 5} = 15 + 78 + 165 + 126 + 21 = 405 \Longrightarrow (\mathrm{I}$$

See also

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