2020 AMC 12B Solution

Problem1

What is the value in simplest form of the following expression?$\sqrt{1}+\sqrt{1+3}+\sqrt{1+3+5}+\sqrt{1+3+5+7}$

$\textbf{(A) } 5 \qquad \textbf{(B) } 4+\sqrt{7}+\sqrt{10} \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 15 \qquad \textbf{(E) } 4+3\sqrt{3}+2\sqrt{5}+\sqrt{7}$

Solution

$\sqrt{1}+\sqrt{1+3}+\sqrt{1+3+5}+\sqrt{1+3+5+7}$ =

$\sqrt{1}+\sqrt{4}+\sqrt{9}+\sqrt{16} \ = 1+2+3+4 = \boxed{\textbf{(C) } 10}$

Note: This comes from the fact that the sum of the first n odds is n^2.

Problem2

What is the value of the following expression?$\frac{100^2-7^2}{70^2-11^2} \cdot \frac{(70-11)(70+11)}{(100-7)(100+7)}$

$\textbf{(A) } 1 \qquad \textbf{(B) } \frac{9951}{9950} \qquad \textbf{(C) } \frac{4780}{4779} \qquad \textbf{(D) } \frac{108}{107} \qquad \textbf{(E) } \frac{81}{80}$

Solution

Using difference of squares to factor the left term, we get$\frac{100^2-7^2}{70^2-11^2} \cdot \frac{(70-11)(70+11)}{(100-7)(100+7)} =$ $\frac{(100-7)(100+7)}{(70-11)(70+11)} \cdot \frac{(70-11)(70+11)}{(100-7)(100+7)}$Cancelling all the terms, we get $\boxed{\textbf{(A) } 1}$ as the answer.

Problem 3

The ratio of w to x is $4:3$, the ratio of y to z is $3:2$, and the ratio of z to x is $1:6$. What is the ratio of $w$to y?

$\textbf{(A) } 4:3 \qquad \textbf{(B) } 3:2 \qquad \textbf{(C) } 8:3 \qquad \textbf{(D) } 4:1 \qquad \textbf{(E) } 16:3$

Solution 1

WLOG, let $w=4$ and $x=3$.

Since the ratio of z to x is $1:6$, we can substitute in the value of x to get $\frac{z}{3}=\frac{1}{6} \implies z=\frac{1}{2}$.

The ratio of y to z is $3:2$, so $\frac{y}{\frac{1}{2}}=\frac{3}{2} \implies y=\frac{3}{4}$.

The ratio of w to y is then $\frac{4}{\frac{3}{4}}=\frac{16}{3}$ so our answer is $\boxed{\textbf{(E) } 16:3}$ ~quacker88

Solution 2

We need to somehow link all three of the ratios together. We can start by connecting the last two ratios together by multiplying the last ratio by two.

$z:x=1:6=2:12$, and since $y:z=3:2$, we can link them together to get $y:z:x=3:2:12$.

Finally, since $x:w=3:4=12:16$, we can link this again to get: $y:z:x:w=3:2:12:16$,

so $w:y=\boxed{\textbf{(E) } 16:3}$ ~quacker88

Problem4

The acute angles of a right triangle are a° and b°, where $a>b$ and both a and b are prime numbers. What is the least possible value of b?

$\textbf{(A) } 2 \qquad \textbf{(B) } 3 \qquad \textbf{(C) } 5 \qquad \textbf{(D) } 7 \qquad \textbf{(E) } 11$

Solution

Since the three angles of a triangle add up to 180° and one of the angles is 90° because it's a right triangle, $a^{\circ}+b^{\circ}=90^{\circ}$.

The greatest prime number less than 90 is 89. If $a=89^{\circ}$, then $b=90^{\circ}-89^{\circ}=1^{\circ}$, which is not prime.

The next greatest prime number less than 90 is 83. If $a=83^{\circ}$, then $b=7^{\circ}$, which IS prime, so we have our answer $\boxed{\textbf{(D)}\ 7}$ ~quacker88

Solution 2

Looking at the answer choices, only 7 and 11 are coprime to 90. Testing 7, the smaller angle, makes the other angle 83 which is prime, therefore our answer is $\boxed{\textbf{(D)}\ 7}$

Problem5

Teams A and B are playing in a basketball league where each game results in a win for one team and a loss for the other team. Team A has won $\tfrac{2}{3}$ of its games and team B has won $\tfrac{5}{8}$ of its games. Also, team B has won 7 more games and lost 7 more games than team $A.$ How many games has team A played?

$\textbf{(A)}\ 21 \qquad \textbf{(B)}\ 27 \qquad \textbf{(C)}\ 42 \qquad \textbf{(D)}\ 48 \qquad \textbf{(E)}\ 63$

Solution

First, let us assign some variables. Let

$$A_w=2x, A_l=x, A_g=3x,$$

$$B_w=5y, B_l=3y, B_g=8y,$$

where X_w denotes number of games won, X_l denotes number of games lost, and X_g denotes total games played for $X\in\{A, B\}$. Using the given information, we can set up the following two equations:

$$B_w = A_w + 7 \implies 5y = 2x + 7,$$

$$B_l = A_l + 7 \implies 3y = x + 7.$$

We can solve through substitution, as the second equation can be written as $x = 3y - 7$, and plugging this into the first equation gives $5y = 6y - 7 \implies y = 7$, which means $x = 3(7) - 7 = 14$. Finally, we want the total number of games team A has played, which is $A_g = 3(14) = \boxed{\textbf{(C)}\ 42}$.

~Argonauts16

Solution 2

Using the information from the problem, we can note that team A has lost $\frac{1}{3}$ of their matches. Using the answer choices, we can find the following list of possible win-lose scenarios for A, represented in the form (w, l) for convenience:

$$A \implies (14, 7)B \implies (18, 9)C \implies (28, 14)$$

$$D \implies (32, 16)E \implies (42, 21)$$

Thus, we have 5 matching B scenarios, simply adding 7 to w and l. We can then test each of the five Bscenarios for $\frac{w}{w+l} = \frac{5}{8}$ and find that $(35, 21)$ fits this description. Then working backwards and subtracting 7 from w and l gives us the point $(28, 14)$, making the answer $\boxed{\textbf{C}}$.

Problem 6

For all integers $n \geq 9$, the value of $\frac{(n+2)! - (n+1)!}{n!}$ is always which of the following?

$\textbf{(A) } \text{a multiple of } 4 \qquad \textbf{(B) } \text{a multiple of } 10 \qquad \textbf{(C) } \text{a prime number}$
$\textbf{(D) } \text{a perfect square} \qquad \textbf{(E) } \text{a perfect cube}$

Solution

We first expand the expression:
$$\frac{(n+2)!-(n+1)!}{n!} = \frac{(n+2)(n+1)n!-(n+1)n!}{n!}$$

We can now divide out a common factor of $n!$ from each term of this expression:

$$(n+2)(n+1)-(n+1)$$

Factoring out $(n+1)$, we get $(n+1)(n+2-1) = (n+1)^2$

which proves that the answer is $\boxed{\textbf{(D) } \text{a perfect square}}$.

Solution 2

Factor out an $n!$ to get: $\frac{(n+2)!-(n+1)!}{n!} = (n+2)(n+1)-(n+1)$ Now, without loss of generality, test values of n until only one answer choice is left valid:

$n=1 \implies (3)(2)-(2)=4$, knocking out $\textbf{B}$, $\textbf{C}$, and $\textbf{E}$.

$n=2 \implies (4)(3)-(3)=9$, knocking out $\textbf{A}$.

This leaves $\boxed{\textbf{(D) } \text{a perfect square}}$ as the only answer choice left.

With further testing it becomes clear that for all n, $(n+2)(n+1)-(n+1)=(n+1)^2$, proved in Solution 1.

Problem7

Two nonhorizontal, non vertical lines in the xy-coordinate plane intersect to form a 45° angle. One line has slope equal to 6 times the slope of the other line. What is the greatest possible value of the product of the slopes of the two lines?

$\textbf{(A)}\ \frac{1}{6} \qquad \textbf{(B)}\ \frac{2}{3} \qquad \textbf{(C)}\ \frac{3}{2} \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 6$

Solution

Let one of the lines have equation $y = ax$. Let θ be the angle that line makes with the x-axis, so $\tan(\theta) = a$. The other line will have a slope of $\tan(45^{\circ} + \theta) = \frac{\tan(45^{\circ}) + \tan(\theta)}{1 - \tan(45^{\circ})\tan(\theta)} = \frac{1+a}{1-a}$. Since the slope of one line is 6 times the other, and a is the smaller slope,

$6a = \frac{1+a}{1-a} \implies 6a - 6a^2 = 1 + a \implies 6a^2 - 5a + 1 = 0 \implies a = \frac{1}{2}, \frac{1}{3}$. If $a = \frac{1}{2}$, the other line will have slope $\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$. If $a = \frac{1}{3}$, the other line will have slope $\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$. The first case gives the bigger product of $\frac{3}{2}$, so our answer is $\boxed{\textbf{(C)}\ \frac{3}{2}}$.

~JHawk0224

Solution 2 (bash)

Place on coordinate plane. Lines are $y=mx, y=6mx$. The intersection point at the origin. Goes through $(0,0), (1,m), (1,6m), (1,0)$. So by law of sines, $\frac{5m}{\sin 45^{\circ}} = \frac{\sqrt{1+m^2}}{1/(\sqrt{1+36m^2})}$, lettin $a=m^2$ we want $6a$. Simplifying gives $50a = (1+a)(1+36a)$, so

$36a^2 - 13a + 1 = 0 \implies 36(a-1/4)(a-1/9) = 0,$ so max $a = 1/4,$ and $6a = 3/2$ $\boxed{(C)}$.

Law of sines on the green triangle with the red angle (45 deg) and blue angle, where sine blue angle is $1/(\sqrt{1+36m^2})$ from right triangle w vertices $(0,0), (1,0), (1,6m).$

~ccx09

Solution 3 (complex)

Let the intersection point is the origin. Let (a,b) be a point on the line of lesser slope. The mutliplication of $a+bi$ by cis 45.
$$(a+bi)(\frac{1}{\sqrt{2}} + i * \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}((a-b)+(a+b)*i)$$
and therefore $(a-b, a+b)$ lies on the line of greater slope.

Then, the rotation of (a,b) by 45 degrees gives a line of slope $\frac{a+b}{a-b}$.

We get the equation
$$\frac{6b}{a} = \frac{a+b}{a-b} \implies a^2 - 5ab + 6b^2 = (a-3b)(a-2b) = 0$$
and this gives our answer.

~jeffisepic

Solution 4 (matrix transformation)

Multiply by the rotation transformation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where $\theta = 45^\circ.$

Solution 5 (Cheating)

Let the smaller slope be m, then the larger slope is $6m$. Since we want the greatest product we begin checking each answer choice, starting with (E).

$$6m^2 = 6.$$

$$m^2 = 1.$$

This gives $m = 1$ and $6m = 6$. Checking with a protractor we see that this does not form a 45 degree angle.

Using this same method for the other answer choices, we eventually find that the answer is $\boxed{\textbf{(C)}\ \frac{3}{2}}$ since our slopes are $\frac{1}{2}$ and 3 which forms a perfect 45 degree angle.

Problem8

How many ordered pairs of integers (x, y) satisfy the equation $x^{2020} + y^2 = 2y?$

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ \text{infinitely many}$

Solution

Rearranging the terms and and completing the square for y yields the result $x^{2020} + (y-1)^2 = 1$. Then, notice that x can only be 0, 1 and -1 because any value of x^{2020} that is greater than 1 will cause the term $(y-1)^2$ to be less than 0, which is impossible as y must be real. Therefore, plugging in the above values for x gives the ordered pairs $(0, 0)$, $(1, 1)$, $(-1, 1)$, and $(0, 2)$ gives a total of $\boxed{\textbf{(D)}\ 4}$ ordered pairs.

Solution 2

Bringing all of the terms to the LHS, we see a quadratic equation $y^2-2y+x^{2020}=0$ in terms of y. Applying the quadratic formula, we get $y=\frac{2\pm\sqrt{4-4\cdot1\cdot x^{2020}}}{2}=\frac{2\pm\sqrt{4(1-x^{2020})}}{2}$. In order for y to be real, which it must be given the stipulation that we are seeking integral answers, we know that the discriminant, $4(1-x^{2020})$ must be nonnegative. Therefore, $4(1-x^{2020})\geq 0 \implies x^{2020}\leq 1$. Here, we see that we must split the inequality into a compound, resulting in $-1\leq x\leq 1$.

The only integers that satisfy this are $x\in\{-1,0,1\}$. Plugging these values back into the quadratic equation, we see that $x=\{-1,1\}$ both produce a discriminant of 0, meaning that there is only 1 solution for y. If $x=\{0\}$, then the discriminant is nonzero, therefore resulting in two solutions for y.

Thus, the answer is $2\cdot1+1\cdot2=\boxed{\textbf{(D) } 4}$.

~Tiblis

Solution 3, x first

Set it up as a quadratic in terms of y: $y^2-2y+x^{2020}=0$ Then the discriminant is $\Delta=4-4x^{2020}$ This will clearly only yield real solutions when $x^{2020}\leq 1$, because it is always positive. Then $x=-1,0,1$. Checking each one: -1 and 1 are the same when raised to the 2020th power: $y^2-2y+1=(y-1)^2=0$ This has only has solutions 1, so $(\pm1,1)$ are solutions. Next, if $x=0$: $y^2-2y=0$ Which has 2 solutions, so $(0,2)$ and $(0,0)$

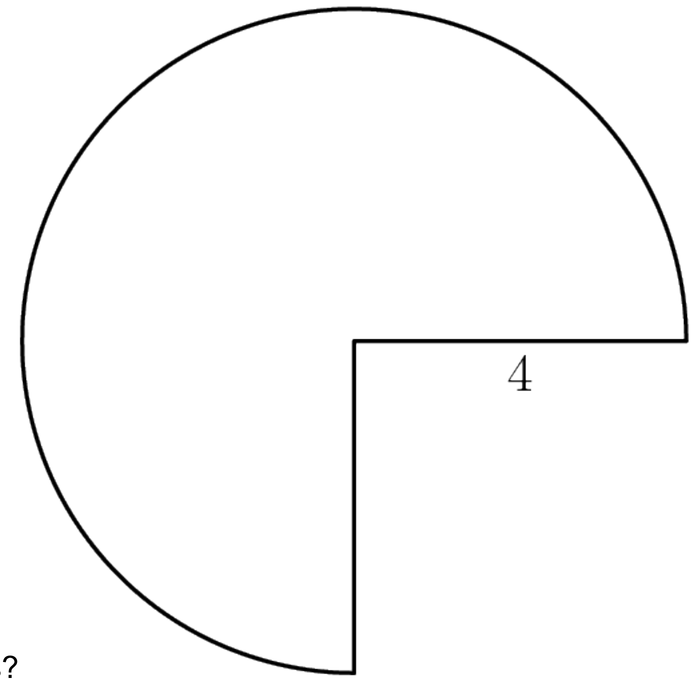
These are the only 4 solutions, so $\boxed{\textbf{(D) } 4}$

Solution 4, y first

Move the y^2 term to the other side to get $x^{2020} = 2y - y^2 = y(2-y)$. Because $x^{2020} \geq 0$ for all x, then $y(2-y) \geq 0 \Rightarrow y = 0, 1, 2$. If $y = 0$ or $y = 2$, the right side is 0 and therefore $x = 0$. When $y = 1$, the right side become 1, therefore $x = 1, -1$. Our solutions are $(0,2)$, $(0,0)$, $(1,1)$, $(-1,1)$. There are 4 solutions, so the answer is $\boxed{\textbf{(D) } 4}$ - wwt7535

Problem 9

A three-quarter sector of a circle of radius 4 inches together with its interior can be rolled up to form the lateral surface area of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic



inches?

$\textbf{(A) } 3\pi\sqrt{5} \qquad \textbf{(B) } 4\pi\sqrt{3} \qquad \textbf{(C) } 3\pi\sqrt{7} \qquad \textbf{(D) } 6\pi\sqrt{3} \qquad \textbf{(E) } 6\pi\sqrt{7}$

Solution

Notice that when the cone is created, the radius of the circle will become the slant height of the cone and the intact circumference of the circle will become the circumference of the base of the cone.

We can calculate that the intact circumference of the circle is $8\pi \cdot \frac{3}{4} = 6\pi$. Since that is also equal to the circumference of the cone, the radius of the cone is 3. We also have that the slant height of the cone is 4. Therefore, we use the Pythagorean Theorem to calculate that the height of the cone is $\sqrt{4^2 - 3^2} = \sqrt{7}$. The volume of the cone is $\frac{1}{3} \cdot \pi \cdot 3^2 \cdot \sqrt{7} = \boxed{\textbf{(C)}\ 3\pi\sqrt{7}}$ -PCChess

Solution 2 (Last Resort/Cheap)

Using a ruler, measure a circle of radius 4 and cut out the circle and then the quarter missing. Then, fold it into a cone and measure the diameter to be 6 cm $\implies r = 3$. You can form a right triangle with sides 3, 4, and then through the Pythagorean theorem the height h is found to be $h^2 = 4^2 - 3^2 \implies h = \sqrt{7}$. The volume of a cone is $\frac{1}{3}\pi r^2 h$. Plugging in we find $V = 3\pi\sqrt{7} \implies \boxed{\textbf{(C)}}$

Problem10

In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

$\textbf{(A) } \frac{\sqrt{5}}{12} \qquad \textbf{(B) } \frac{\sqrt{5}}{10} \qquad \textbf{(C) } \frac{\sqrt{5}}{9} \qquad \textbf{(D) } \frac{\sqrt{5}}{8} \qquad \textbf{(E) } \frac{2\sqrt{5}}{15}$

Solution 1 (Angle Chasing/Trig)

Let O be the center of the circle and the point of tangency between ω and $\overline{AD}$ be represented by K. We know that $\overline{AK} = \overline{KD} = \overline{DM} = \frac{1}{2}$. Consider the right triangle $\triangle ADM$. Let $\measuredangle AMD = \theta$.

Since ω is tangent to $\overline{DC}$ at M, $\measuredangle PMO = 90 - \theta$. Now, consider $\triangle POM$. This triangle is iscoceles because $\overline{PO}$ and $\overline{OM}$ are both radii of ω. Therefore, $\measuredangle POM = 180 - 2(90 - \theta) = 2\theta$.

We can now use Law of Cosines on $\angle POM$ to find the length of PM and subtract it from the length of AM to find AP.

Since $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = \frac{2}{\sqrt{5}}$, the double angle formula tells us that $\cos 2\theta = -\frac{3}{5}$. We have $PM^2 = \frac{1}{2} - \frac{1}{2}\cos 2\theta \implies PM = \frac{2\sqrt{5}}{5}$ By Pythagorean theorem, we find that $AM = \frac{\sqrt{5}}{2} \implies \boxed{\textbf{(B) } \frac{\sqrt{5}}{10}}$

~awesome1st

Solution 2(Coordinate Bash)

Place circle ω in the Cartesian plane such that the center lies on the origin. Then we can easily find the equation for ω as $x^2 + y^2 = \frac{1}{4}$, because it is not translated and the radius is $\frac{1}{2}$.

We have $A = \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $M = \left(0, -\frac{1}{2}\right)$. The slope of the line passing through these two points is $\frac{\frac{1}{2} + \frac{1}{2}}{-\frac{1}{2} - 0} = \frac{1}{-\frac{1}{2}} = -2$, and the y-

intercept is simply M. This gives us the line passing through both points as $y=-2x-\frac{1}{2}.$

We substitute this into the equation for the circle to get $x^2+\left(-2x-\frac{1}{2}\right)^2=\frac{1}{4}$, or $x^2+4x^2+2x+\frac{1}{4}=\frac{1}{4}.$

Simplifying gives $x(5x+2)=0$. The roots of this quadratic are $x=0$ and $x=-\frac{2}{5}$, but if $x=0$ we get point M, so we only want $x=-\frac{2}{5}.$

We plug this back into the linear equation to find $y=\frac{3}{10}$, and so $P=\left(-\frac{2}{5},\frac{3}{10}\right)$. Finally, we use distance formula on A and P to get

$$AP=\sqrt{\left(-\frac{5}{10}+\frac{4}{10}\right)^2+\left(\frac{5}{10}-\frac{3}{10}\right)^2}=\sqrt{\frac{1}{100}+\frac{4}{100}}=\boxed{\textbf{(B)}\frac{\sqrt{5}}{10}}$$

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~Argonauts16

Solution 3(Power of a Point)

Let circle ω intersect $\overline{AB}$ at point N. By Power of a Point, we have $AN^2=AP\cdot AM$. We know $AN=\frac{1}{2}$ because N is the midpoint of $\overline{AB}$, and we can easily find AM by the Pythagorean Theorem, which gives us $AM=\sqrt{1^2+\left(\frac{1}{2}\right)^2}=\frac{\sqrt{5}}{2}$. Our equation is now $\frac{1}{4} = AP \cdot \frac{\sqrt{5}}{2}$, or $AP = \frac{2}{4\sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{2 \cdot 5}$, thus our answer is $\boxed{\textbf{(B)}\ \frac{\sqrt{5}}{10}}$.

~Argonauts16

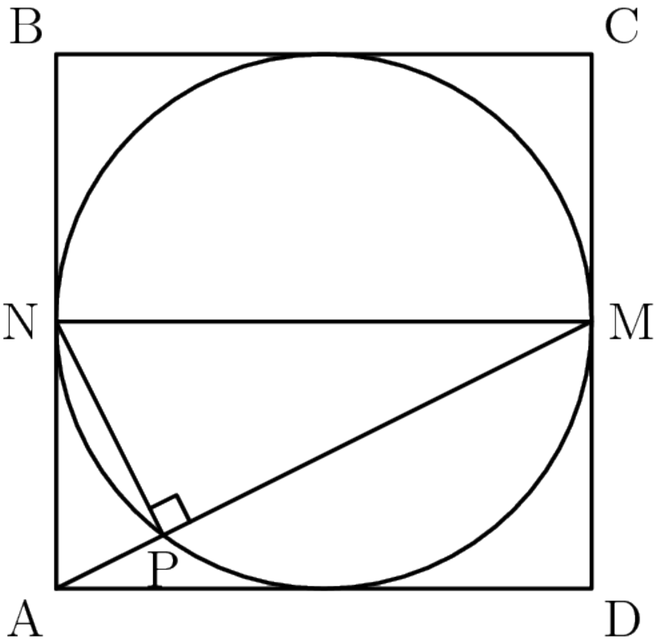
Solution 4

Take O as the center and draw segment ON perpendicular to $AM, ON \cap AM = N$, link OM. Then we have $OM \parallel AD$. So $\angle DAM = \angle OMA$. Since $AD = 2AM = 2OM = 1$, we have $\cos \angle DAM = \cos \angle OMP = \frac{2}{\sqrt{5}}$. As a result, $NM = OM \cos \angle OMP = \frac{1}{2} \cdot \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}}$. Thus $PM = 2NM = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$. Since $AM = \frac{\sqrt{5}}{2}$, we have $AP = AM - PM = \frac{\sqrt{5}}{10}$. Put $\boxed{B}$.

~FANYUCHEN20020715

Solution 5 (Similar Triangles)

Call the midpoint of $\overline{AB}$ point N. Draw in $\overline{NM}$ and $\overline{NP}$. Note that $\angle NPM = 90^\circ$ due to Thales's Theorem.

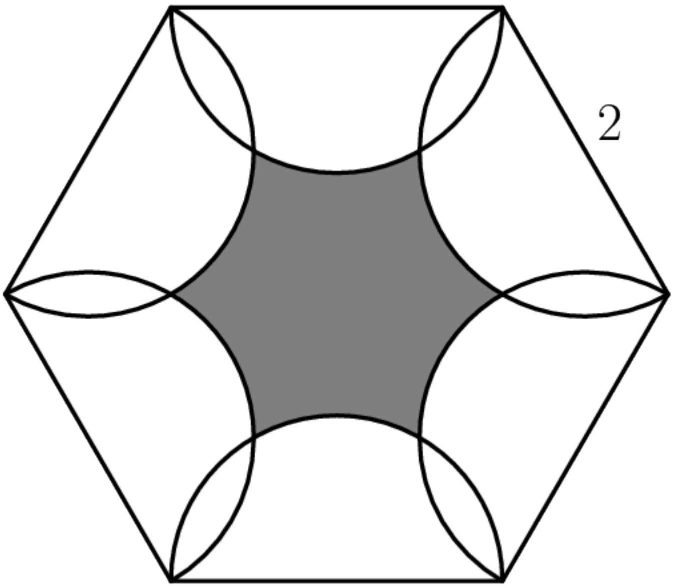


Using the Pythagorean theorem, $AM = \frac{\sqrt{5}}{2}$. Now we just need to find AP using similar triangles.

$$\triangle APN \sim \triangle ANM \Rightarrow \frac{AP}{AN} = \frac{AN}{AM} \Rightarrow \frac{AP}{\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{\sqrt{5}}{2}} \Rightarrow AP = \boxed{\textbf{(B)}\ \frac{\sqrt{5}}{10}}$$

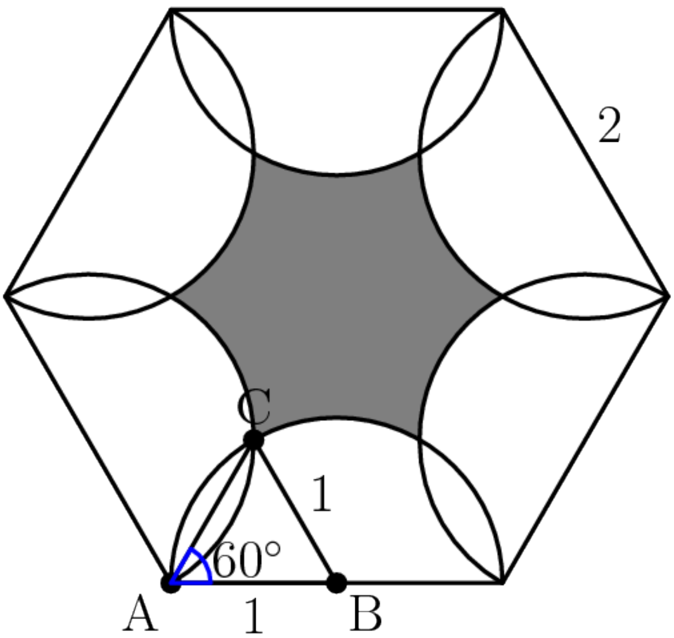
Problem11

As shown in the figure below, six semicircles lie in the interior of a regular hexagon with side length 2 so that the diameters of the semicircles coincide with the sides of the hexagon. What is the area of the shaded region — inside the hexagon but outside all of the semicircles?

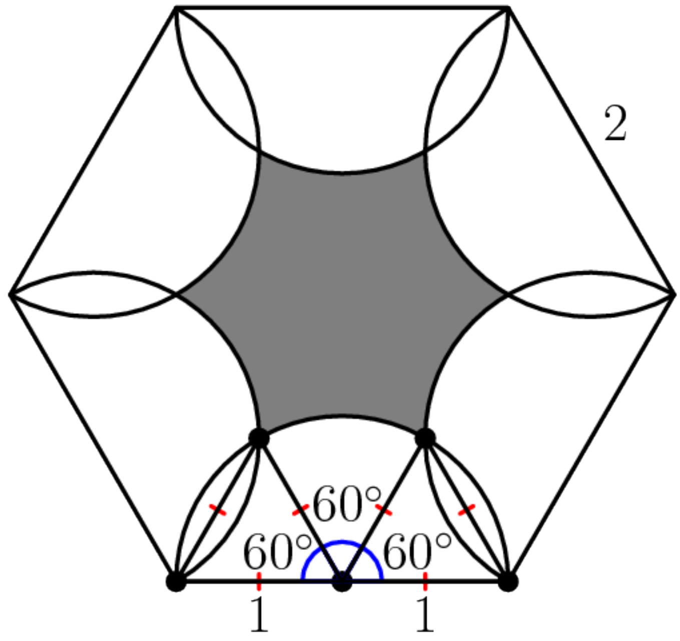


(A) $6\sqrt{3}-3\pi$ **(B)** $\frac{9\sqrt{3}}{2}-2\pi$ **(C)** $\frac{3\sqrt{3}}{2}-\frac{\pi}{3}$ **(D)** $3\sqrt{3}-\pi$ **(E)** $\frac{9\sqrt{3}}{2}-\pi$

Solution 1



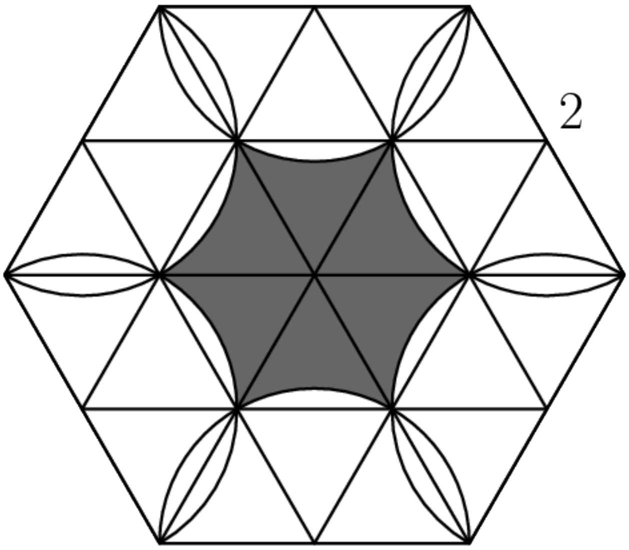
Let point A be a vertex of the regular hexagon, let point B be the midpoint of the line connecting point A and a neighboring vertex, and let point C be the second intersection of the two semicircles that pass through point A. Then, $BC = 1$, since B is the center of the semicircle with radius 1 that C lies on, $AB = 1$, since B is the center of the semicircle with radius 1 that A lies on, and $\angle BAC = 60^\circ$, as a regular hexagon has angles of 120∘, and $\angle BAC$ is half of any angle in this hexagon. Now, using the sine law, $\frac{1}{\sin \angle ACB} = \frac{1}{\sin 60^\circ}$, so $\angle ACB = 60^\circ$. Since the angles in a triangle sum to 180○, $\angle ABC$ is also 60○. Therefore, $\triangle ABC$ is an equilateral triangle with side lengths of 1.



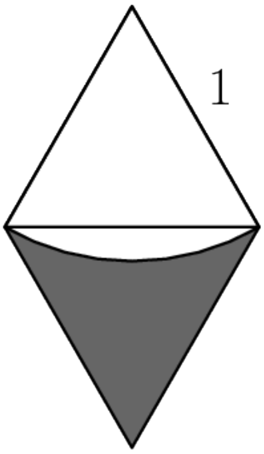
Since the area of a regular hexagon can be found with the formula $\frac{3\sqrt{3}s^2}{2}$, where S is the side length of the hexagon, the area of this hexagon is $\frac{3\sqrt{3}(2^2)}{2} = 6\sqrt{3}$. Since the area of an equilateral triangle can be found with the formula $\frac{\sqrt{3}}{4}s^2$, where S is the side length of the equilateral triangle, the area of an equilateral triangle with side lengths of 1 is $\frac{\sqrt{3}}{4}(1^2) = \frac{\sqrt{3}}{4}$.

Since the area of a circle can be found with the formula πr^2, the area of a sixth of a circle with radius 1 is $\frac{\pi(1^2)}{6} = \frac{\pi}{6}$. In each sixth of the hexagon, there are two equilateral triangles colored white, each with an area of $\frac{\sqrt{3}}{4}$, and one sixth of a circle with radius 1 colored white, with an area of $\frac{\pi}{6}$. The rest of the sixth is colored gray. Therefore, the total area that is colored white in each sixth of the hexagon is $2(\frac{\sqrt{3}}{4})+\frac{\pi}{6}$, which equals $\frac{\sqrt{3}}{2}+\frac{\pi}{6}$, and the total area colored white is $6(\frac{\sqrt{3}}{2}+\frac{\pi}{6})$, which equals $3\sqrt{3}+\pi$. Since the area colored gray equals the total area of the hexagon minus the area colored white, the area colored gray is $6\sqrt{3}-(3\sqrt{3}+\pi)$, which equals $\boxed{\textbf{(D) } 3\sqrt{3}-\pi}$.

Solution 2

First, subdivide the hexagon into 24 equilateral triangles with side length 1: 

Now note that the entire shaded

region is just 6 times this part:  The entire rhombus is just 2 equilatrial triangles with side lengths of 1, so it has an area of: $2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

The arc that is not included has an area of: $\frac{1}{6} \cdot \pi \cdot 1^2 = \frac{\pi}{6}$ Hence, the area of the shaded region in that section is $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$ For a final area of: $6\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = 3\sqrt{3} - \pi \Rightarrow \boxed{\text{(D)}}$

Problem12

Let $\overline{AB}$ be a diameter in a circle of radius $5\sqrt{2}$. Let $\overline{CD}$ be a chord in the circle that intersects $\overline{AB}$ at a point E such that $BE = 2\sqrt{5}$ and $\angle AEC = 45^\circ$. What is $CE^2 + DE^2$?

$\textbf{(A)}\ 96 \qquad \textbf{(B)}\ 98 \qquad \textbf{(C)}\ 44\sqrt{5} \qquad \textbf{(D)}\ 70\sqrt{2} \qquad \textbf{(E)}\ 100$

Solution 1

Let O be the center of the circle, and X be the midpoint of $\overline{CD}$.
Let $CX = a$ and $EX = b$. This implies that $DE = a - b$.
Since $CE = CX + EX = a + b$, we now want to

find $(a+b)^2+(a-b)^2=2(a^2+b^2)$. Since $\angle CXO$ is a right angle, by Pythagorean theorem $a^2+b^2=CX^2+OX^2=(5\sqrt{2})^2=50$. Thus, our answer is $2(50)=\boxed{\textbf{(E) } 100}$.

~JHawk0224

Solution 2 (Power of a Point)

Let O be the center of the circle, and X be the midpoint of CD. Draw triangle OCD, and median OX. Because $OC=OD$, OCD is isosceles, so OX is also an altitude of OCD. $OD=5\sqrt{2}-2\sqrt{5}$, and because angle OEC is 45 degrees and triangle OXE is right, $OX=EX=\dfrac{5\sqrt{2}-2\sqrt{5}}{\sqrt{2}}=5-\sqrt{10}$. Because triangle OXC is right, $CX=\sqrt{(5\sqrt{2})^2-(5-\sqrt{10})^2}=\sqrt{15+10\sqrt{10}}$.

Thus, $CD=2\sqrt{15+10\sqrt{10}}$. We are looking for CE^2 + DE^2 which is also $(CE+DE)^2-2\cdot CE\cdot DE$. Because

$CE+DE=CD=2\sqrt{15+10\sqrt{10}}, (CE+CD)^2=4(15+10\sqrt{10})=60+40\sqrt{10}$

. By power of a point,

$$CE\cdot DE=AE\cdot BE=2\sqrt{5}\cdot(10\sqrt{2}-2\sqrt{5})=20\sqrt{10}-20$$

so $2\cdot CE\cdot DE=40\sqrt{10}-40$.

Finally,

$$CE^2+DE^2=60+40\sqrt{10}-(40\sqrt{10}-40)=\boxed{(E)100}$$

.

~CT17

Solution 3 (Law of Cosines)

Let O be the center of the circle. Notice how $OC = OD = r$, where r is the radius of the circle. By applying the law of cosines on triangle OCE,

$r^2 = CE^2 + OE^2 - 2(CE)(OE)\cos 45 = CE^2 + OE^2 - (CE)(OE)\sqrt{2}$

. Similarly, by applying the law of cosines on triangle ODE,
$r^2 = DE^2 + OE^2 - 2(DE)(OE)\cos 135 = DE^2 + OE^2 + (DE)(OE)\sqrt{2}$
. By subtracting these two equations, we get $CE^2 - DE^2 - (CE)(OE)\sqrt{2} - (DE)(OE)\sqrt{2} = 0$

. We can rearrange it to get

$CE^2 - DE^2 = (CE)(OE)\sqrt{2} + (DE)(OE)\sqrt{2} = (CE + DE)(OE\sqrt{2})$

. Because both CE and DE are both positive, we can safely divide both sides by $(CE + DE)$ to obtain $CE - DE = OE\sqrt{2}$.

Because $OE = OB - BE = 5\sqrt{2} - 2\sqrt{5}$,
$(CE - DE)^2 = CE^2 + DE^2 - 2(CE)(DE) = (OE\sqrt{2})^2 = 2(5\sqrt{2} - 2\sqrt{5})^2 = 140 - 40\sqrt{10}$
. Through power of a point, we can find out that $(CE)(DE) = 20\sqrt{10} - 20$,

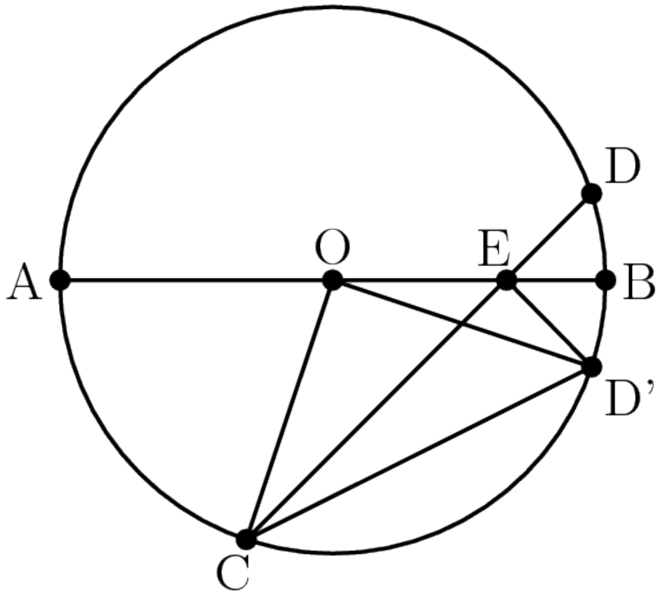
so

$CE^2 + DE^2 = (CE - DE)^2 + 2(CE)(DE) = (140 - 40\sqrt{10}) + 2(20\sqrt{10} - 20) = \boxed{\textbf{(E)}\ 100}$

.

~Math_Wiz_3.14

Solution 4 (Reflections)



Let O be the center of the circle. By reflecting D across the line AB to produce D', we have that $\angle BED' = 45$. Since $\angle AEC = 45$, $\angle CED' = 90$. Since $DE = ED'$, by the Pythagorean Theorem, our desired solution is just CD'^2. Looking next to circle arcs, we know that $\angle AEC = \dfrac{\overset{\frown}{\mathrm{AC}} + \overset{\frown}{\mathrm{BD}}}{2} = 45$, so $\overset{\frown}{\mathrm{AC}} + \overset{\frown}{\mathrm{BD}} = 90$. Since $\overset{\frown}{\mathrm{BD'}} = \overset{\frown}{\mathrm{BD}}$, and $\overset{\frown}{\mathrm{AC}} + \overset{\frown}{\mathrm{BD'}} + \overset{\frown}{\mathrm{CD'}} = 180$, $\overset{\frown}{\mathrm{CD'}} = 90$. Thus, $\angle COD' = 90$. Since $OC = OD' = 5\sqrt{2}$, by the Pythagorean Theorem, the desired $CD'^2 = \boxed{\textbf{(E) } 100}$.

Problem13

Which of the following is the value of $\sqrt{\log_2 6 + \log_3 6}$?

$\textbf{(A) } 1 \qquad \textbf{(B) } \sqrt{\log_5 6} \qquad \textbf{(C) } 2 \qquad \textbf{(D) } \sqrt{\log_2 3} + \sqrt{\log_3 2} \qquad \textbf{(E) } \sqrt{\log_2 6} + \sqrt{\log_3 6}$

Solution 1 (Logic)

Using the knowledge of the powers of 2 and 3, we know that $\log_2 6$ is greater than 2.5 and $\log_3 6$ is greater than 1.5. So that means $\sqrt{\log_2 6 + \log_3 6} > 2$.

Since $\boxed{\textbf{(D) } \sqrt{\log_2 3} + \sqrt{\log_3 2}}$ is the only option greater than 2, it's the answer. ~Baolan

Solution 2

$\sqrt{\log_2 6 + \log_3 6} = \sqrt{\log_2 2 + \log_2 3 + \log_3 2 + \log_3 3} = \sqrt{2 + \log_2 3 + \log_3 2}$

. If we call $\log_2 3 = x$, then we have

$\sqrt{2 + x + \frac{1}{x}} = \sqrt{x} + \frac{1}{\sqrt{x}} = \sqrt{\log_2 3} + \frac{1}{\sqrt{\log_3 2}} = \sqrt{\log_2 3} + \sqrt{\log_3 2}$

. So our answer is $\boxed{\textbf{(D)}}$

Problem14

Bela and Jenn play the following game on the closed interval $[0, n]$ of the real number line, where n is a fixed integer greater than 4. They take turns playing, with Bela going first. At his first turn, Bela chooses any real number in the interval $[0, n]$. Thereafter, the player whose turn it is chooses a real number that is more than one unit away from all numbers previously chosen by either player. A player unable to choose such a number loses. Using optimal strategy, which player will win the game?

$\textbf{(A)}\ \text{Bela will always win.} \qquad \textbf{(B)}\ \text{Jenn will always win.} \qquad \textbf{(C)}\ \text{Bela will win if and only if }n\text{ is odd.}$
$\textbf{(D)}\ \text{Jenn will win if and only if }n\text{ is odd.} \qquad \textbf{(E)}\ \text{Jenn will win if and only if }n>8.$

Solution

Notice that to use the optimal strategy to win the game, Bela must select the middle number in the range $[0, n]$ and then mirror whatever number Jenn selects. Therefore, if Jenn can select a number within the range, so can Bela.

Jenn will always be the first person to run out of a number to choose, so the answer is $\boxed{\textbf{(A)}\ \text{Bela will always win}}$.

Solution 2 (Guessing)

First of all, realize that the value of n should have no effect on the strategy at all. This is because they can choose real numbers, not integers, so even if n is odd, for example, they can still go halfway. Similarly, there is no reason the strategy would change when $n > 8$.

So we are left with (A) and (B). From here it is best to try out random numbers and try to find the strategy that will let Bela win, but if you can't find it, realize that it is more likely the answer is $\boxed{\textbf{(A)}\ \text{Bela will always win}}$ since Bela has the first move and thus has more control.

Problem15

There are 10 people standing equally spaced around a circle. Each person knows exactly 3 of the other 9people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the 10 people to split up into 5 pairs so that the members of each pair know each other?

$\textbf{(A)}\ 11 \qquad \textbf{(B)}\ 12 \qquad \textbf{(C)}\ 13 \qquad \textbf{(D)}\ 14 \qquad \textbf{(E)}\ 15$

Solution

Let us use casework on the number of diagonals.

Case 1: 0 diagonals There are 2 ways: either 1 pairs with 2, 3 pairs with 4, and so on or 10 pairs with 1, 2pairs with 3, etc.

Case 2: 1 diagonal There are 5 possible diagonals to draw (everyone else pairs with the person next to them.

Note that there cannot be 2 diagonals.

Case 3: 3 diagonals

Note that there cannot be a case with 4 diagonals because then there would have to be 5 diagonals for the two remaining people, thus a contradiction.

Case 4: 5 diagonals There is 1 way to do this.

Thus, in total there are $2+5+5+1=\boxed{13}$ possible ways.

Problem16

An urn contains one red ball and one blue ball. A box of extra red and blue balls lie nearby. George performs the following operation four times: he draws a ball from the urn at random and then takes a ball of the same color from the box and returns those two matching balls to the urn. After the four iterations the urn contains six balls. What is the probability that the urn contains three balls of each color?

$\textbf{(A) } \frac{1}{6} \qquad \textbf{(B) } \frac{1}{5} \qquad \textbf{(C) } \frac{1}{4} \qquad \textbf{(D) } \frac{1}{3} \qquad \textbf{(E) } \frac{1}{2}$

Solution

Let R denote that George selects a red ball and B that he selects a blue one. Now, in order to get 3 balls of each color, he needs 2 more of both R and B.

There are 6 cases: $RRBB, RBRB, RBBR, BBRR, BRBR, BRRB$ (we can confirm that there are only 6 since $\binom{4}{2} = 6$). However we can clump $RRBB + BBRR$, $RBRB + BRBR$, and $RBBR + BRRB$ together since they are equivalent by symmetry.

$\textbf{CASE 1:}$ $RRBB$ and $BBRR$

Let's find the probability that he picks the balls in the order of $RRBB$.

The probability that the first ball he picks is red is $\frac{1}{2}$.

Now there are 2 reds and 1 blue in the urn. The probability that he picks red again is now $\frac{2}{3}$.

There are 3 reds and 1 blue now. The probability that he picks a blue is $\frac{1}{4}$.

Finally, there are 3 reds and 2 blues. The probability that he picks a blue is $\frac{2}{5}$.

So the probability that the $RRBB$ case happens is $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{2}{5} = \frac{1}{30}$. However, since the $BBRR$case is the exact same by symmetry, case 1 has a probability of $\frac{1}{30} \cdot 2 = \frac{1}{15}$ chance of happening.

$\textbf{CASE 2:}$ $RBRB$ and $BRBR$

Let's find the probability that he picks the balls in the order of $RBRB$.

The probability that the first ball he picks is red is $\frac{1}{2}$.

Now there are 2 reds and 1 blue in the urn. The probability that he picks blue is $\frac{1}{3}$.

There are 2 reds and 2 blues now. The probability that he picks a red is $\frac{1}{2}$.

Finally, there are 3 reds and 2 blues. The probability that he picks a blue is $\frac{2}{5}$.

So the probability that the $RBRB$ case happens is $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{30}$. However, since the $BRBR$case is the exact same by symmetry, case 2 has a probability of $\frac{1}{30} \cdot 2 = \frac{1}{15}$ chance of happening.

$\textbf{CASE 3:}$ $RBBR$ and $BRRB$

Let's find the probability that he picks the balls in the order of $RBBR$.

The probability that the first ball he picks is red is $\frac{1}{2}$.

Now there are 2 reds and 1 blue in the urn. The probability that he picks blue is $\frac{1}{3}$.

There are 2 reds and 2 blues now. The probability that he picks a blue is $\frac{1}{2}$.

Finally, there are 2 reds and 3 blues. The probability that he picks a red is $\frac{2}{5}$.

So the probability that the $RBBR$ case happens is $\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{5}=\frac{1}{30}$. However, since the $BRBR$case is the exact same by symmetry, case 3 has a probability of $\frac{1}{30}\cdot 2=\frac{1}{15}$ chance of happening.

Adding up the cases, we have $\frac{1}{15}+\frac{1}{15}+\frac{1}{15}=\boxed{\textbf{(B) } \frac{1}{5}}$ ~quacker88

Solution 2

We know that we need to find the probability of adding 2 red and 2 blue balls in some order. There are 6 ways to do this, since there are $\binom{4}{2}=6$ ways to arrange $RRBB$ in some order. We will show that the probability for each of these 6 ways is the same.

We first note that the denominators should be counted by the same number. This number is $2\cdot 3\cdot 4\cdot 5=120$. This is because 2, 3, 4, and 5 represent how many choices there are for the four steps. No matter what the $k-th$ step involves $k+1$ numbers to choose from.

The numerators are the number of successful operations. No matter the order, the first time a red is added will come from 1 choice and the second time will come from 2 choices, since that is how many reds are in the urn originally. The same goes for the blue ones. The numerator must equal $(1 \cdot 2)^2$.

Therefore, the probability for each of the orderings of $RRBB$ is $\frac{4}{120} = \frac{1}{30}$. There are 6 of these, so the total probability is $\boxed{\textbf{(B) }\frac{1}{5}}$.

Solution 3

First, notice that when George chooses a ball he just adds another ball of the same color. On George's first move, he either chooses the red or the blue with a $\frac{1}{2}$ chance each. We can assume he chooses Red(chance $\frac{1}{2}$), and then multiply the final answer by two for symmetry. Now, there are two red balls and one blue ball in the urn. Then, he can either choose another Red(chance $\frac{2}{3}$), in which case he must choose two blues to get three of each, with probability $\frac{1}{4} \cdot \frac{2}{5} = \frac{1}{10}$ or a blue for two blue and two red in the urn, with chance $\frac{1}{3}$. If he chooses blue next, he can either choose a red then a blue, or a blue then a red. Each of these has a $\frac{1}{2} \cdot \frac{2}{5}$ for total of $2 \cdot \frac{1}{5} = \frac{2}{5}$. The total probability that he ends up with three red and three blue is $2 \cdot \frac{1}{2}(\frac{2}{3} \cdot \frac{1}{10} + \frac{1}{3} \cdot \frac{2}{5}) = \frac{1}{15} + \frac{2}{15} = \boxed{\textbf{(B) }\frac{1}{5}}$. ~aop2014

Solution 4

Let the probability that the urn ends up with more red balls be denoted $P(R)$. Since this is equal to the probability there are more blue balls, the probability there are equal amounts is $1-2P(R)$. $P(R)=$ the probability no more blues are chosen plus the probability only 1 more blue is chosen. The first case, $P(\text{no more blues})=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}=\frac{1}{5}$.

The second case, $P(1\text{ more blue})=4\cdot\frac{1\cdot1\cdot2\cdot3}{2\cdot3\cdot4\cdot5}=\frac{1}{5}$. Thus, the answer is $1-2\left(\frac{1}{5}+\frac{1}{5}\right)=1-\frac{4}{5}=\boxed{\textbf{(B) } \frac{1}{5}}$.

~JHawk0224

Solution 5

By conditional probability after 4 rounds we have 5 cases: RRRBBB, RRRRBB, RRBBBB, RRRRRB and RBBBBB. Thus the probability is $\frac{1}{5}$. Put $\boxed{B}$.

~FANYUCHEN20020715

Edited by Kinglogic

Solution 6

Here X stands for R or B, and Y for the remaining color. After 3 rounds one can either have a 4+1 configuration (XXXXY), or 3+2 configuration (XXXYY). The probability of getting to XXXYYY from XXXYY is $\frac{2}{5}$. Observe that the probability of arriving to 4+1 configuration is $\frac{2}{3}\cdot\frac{3}{4}=\frac{1}{2}$ ($\frac{2}{3}$ to get from XXY to XXXY, $\frac{3}{4}$ to get from XXXY to XXXXY). Thus the probability of arriving to 3+2 configuration is also $\frac{1}{2}$, and the answer is $\frac{1}{2}\cdot\frac{2}{5}=\boxed{\textbf{(B) } \frac{1}{5}}$.

Solution 7

We can try to use dynamic programming to solve this problem. (Informatics Olympiad hahaha)

We let $dp[i][j]$ be the probability that we end up with i red balls and j blue balls. Notice that there are only two ways that we can end up with i red balls and j blue balls: one is by fetching a red ball from the urn when we have $i-1$ red balls and j blue balls and the other is by fetching a blue ball from the urn when we have i red balls and $j-1$ blue balls.

Then we have

$$dp[i][j] = \frac{i-1}{i-1+j}dp[i-1][j] + \frac{j-1}{i-1+j}dp[i][j-1]$$

Then we start can with $dp[1][1] = 1$ and try to compute $dp[3][3]$.

$i \setminus j$	1	2	3
1	1	$1/2$	$1/3$
2	$1/2$	$1/3$	$1/4$
3	$1/3$	$1/4$	$1/5$

The answer is $\boxed{\textbf{(B) } \frac{1}{5}}$.

Problem17

How many polynomials of the form $x^5 + ax^4 + bx^3 + cx^2 + dx + 2020$, where a, b, c, and d are real numbers, have the property that whenever r is a root, so is $\frac{-1+i\sqrt{3}}{2} \cdot r$? (Note that $i=\sqrt{-1}$)

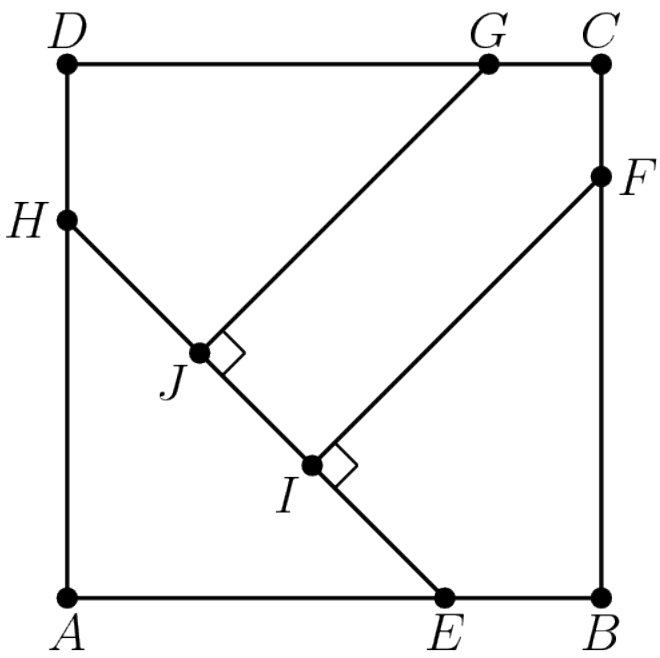
$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 3 \qquad \textbf{(E) } 4$

Solution

Let $P(x)=x^5+ax^4+bx^3+cx^2+dx+2020$. We first notice that $\frac{-1+i\sqrt{3}}{2}=e^{i\frac{2\pi}{3}}$, so in order r to be a root of P, $re^{i\frac{2\pi}{3}}$ must also be a root of P, meaning that 3 of the roots of P must be r, $re^{i\frac{2\pi}{3}}$, $re^{i\frac{4\pi}{3}}$. However, since P is degree 5, there must be two additional roots. Let one of these roots be w, if w is a root, then $we^{i\frac{2\pi}{3}}$ and $we^{i\frac{4\pi}{3}}$ must also be roots. However, P is a fifth degree polynomial, and can therefore only have 5 roots. This implies that w is either r, $re^{i\frac{2\pi}{3}}$, or $re^{i\frac{4\pi}{3}}$. Thus we know that the polynomial P can be written in the form $(x-r)^m(x-re^{i\frac{2\pi}{3}})^n(x-re^{i\frac{4\pi}{3}})^p$. Moreover, by Vieta's, we know that there is only one possible value for the magnitude of r as $||r||^5=2020$, meaning that the amount of possible polynomials P is equivalent to the possible sets (m,n,p). In order for the coefficients of the polynomial to all be real, $n=p$ due to $re^{i\frac{2\pi}{3}}$ and $re^{i\frac{4\pi}{3}}$ being conjugates and since $m+n+p=5$, (as the polynomial is 5th degree) we have two possible solutions for (m,n,p) which are $(1,2,2)$ and $(3,1,1)$yielding two possible polynomials. The answer is thus $\boxed{\textbf{(C) } 2}$.

Problem18

In square $ABCD$, points E and H lie on $\overline{AB}$ and $\overline{DA}$, respectively, so that $AE=AH.$ Points F and Glie on $\overline{BC}$ and $\overline{CD}$, respectively, and points I and J lie on $\overline{EH}$ so that $\overline{FI}\perp\overline{EH}$ and $\overline{GJ}\perp\overline{EH}$. See the figure below. Triangle AEH, quadrilateral $BFIE$, quadrilateral $DHJG$, and pentagon $FCGJI$each has area $1.$ What



is FI^2?

$\textbf{(A) } \frac{7}{3} \qquad \textbf{(B) } 8-4\sqrt2 \qquad \textbf{(C) } 1+\sqrt2 \qquad \textbf{(D) } \frac{7}{4}\sqrt2 \qquad \textbf{(E) } 2\sqrt2$

Solution

Since the total area is 4, the side length of square $ABCD$ is 2. We see that since triangle HAE is a right isosceles triangle with area 1, we can determine sides HA and AE both to be $\sqrt{2}$. Now, consider extending FB and IE until they intersect. Let the point of intersection be K. We note that EBK is also a right isosceles triangle with side $2-\sqrt{2}$ and find it's area to be $3-2\sqrt{2}$. Now, we notice that FIK is also a right isosceles triangle and find it's area to be $\frac{1}{2}FI^2$. This is also equal to $1+3-2\sqrt{2}$ or $4-2\sqrt{2}$. Since we are looking for FI^2, we want two times this. That gives $\boxed{\textbf{(B) } 8-4\sqrt{2}}$.~TLiu

Solution 2

Since this is a geometry problem involving sides, and we know that HE is 2, we can use our ruler and find the ratio between FI and HE. Measuring(on the booklet), we get that HE is about 1.8 inches and FI is

about 1.4 inches. Thus, we can then multiply the length of HE by the ratio of $\frac{1.4}{1.8}$, of which we then get $FI = \frac{14}{9}$. We take the square of that and get $\frac{196}{81}$, and the closest answer to that is $\boxed{\textbf{(B)}\ 8-4\sqrt{2}}$. ~Celloboy
(Note that this is just a strategy I happened to use that worked. Do not press your luck with this strategy, for it was a lucky guess)

Solution 3

Draw the auxiliary line AC. Denote by M the point it intersects with HE, and by N the point it intersects with GF. Last, denote by x the segment FN, and by y the segment FI. We will find two equations for $x$and y, and then solve for y^2.

Since the overall area of $ABCD$ is $4 \implies AB = 2$, and $AC = 2\sqrt{2}$. In addition, the area of $\triangle AME = \frac{1}{2} \implies AM = 1$.

The two equations for x and y are then:

- Length of AC: $1 + y + x = 2\sqrt{2} \implies x = (2\sqrt{2} - 1) - y$
- Area of CMIF: $\frac{1}{2}x^2 + xy = \frac{1}{2} \implies x(x + 2y) = 1$.

Substituting the first into the second, yields $\left[\left(2\sqrt{2} - 1\right) - y\right] \cdot \left[\left(2\sqrt{2} - 1\right) + y\right] = 1$

Solving for y^2 gives $\boxed{\textbf{(B)}\ 8-4\sqrt{2}}$ ~DrB

Solution 4

Plot a point F' such that F' and I are collinear and extend line FB to point B' such that $FIB'F'$ forms a square. Extend line AE to meet line $F'B'$ and point E' is the intersection of the two. The area of this square is equivalent to FI^2. We see that the area of square $ABCD$ is 4, meaning each side is of length 2. The area of the pentagon $EIFF'E'$ is 2.

Length $AE = \sqrt{2}$, thus $EB = 2 - \sqrt{2}$. Triangle $EB'E'$ is isosceles, and the area of this triangle is $\frac{1}{2}(4-2\sqrt{2})(2-\sqrt{2}) = 6 - 4\sqrt{2}$. Adding these two areas, we get $2 + 6 - 4\sqrt{2} = 8 - 4\sqrt{2} \rightarrow \boxed{\text{(B)}}$. --OGBooger

Solution 5 (HARD Calculation)

We can easily observe that the area of square $ABCD$ is 4 and its side length is 2 since all four regions that build up the square has area 1. Extend FI and let the intersection with AB be K. Connect AC, and let the intersection of AC and HE be L. Notice that since the area of triangle AEH is 1 and $AE = AH$, $AE = AH = \sqrt{2}$,

therefore $BE = HD = 2 - \sqrt{2}$. Let $CG = CF = m$, then $BF = DG = 2 - m$. Also notice that $KB = 2 - m$, thus $KE = KB - BE = 2 - m - (2 - \sqrt{2}) = \sqrt{2} - m$. Now use the condition that the area of quadrilateral $BFIE$ is 1, we can set up the following equation: $\frac{1}{2}(2-m)^2 - \frac{1}{4}(\sqrt{2}-m)^2 = 1$ We solve the equation and yield $m = \frac{8 - 2\sqrt{2} - \sqrt{64 - 32\sqrt{2}}}{2}$. Now notice that

$$FI = AC - AL = 2\sqrt{2} - 1 - \frac{\sqrt{2}}{2} * \frac{8 - 2\sqrt{2} - \sqrt{64 - 32\sqrt{2}}}{2}$$

$$= 2\sqrt{2} - 1 - \frac{8\sqrt{2} - 4 - \sqrt{128 - 64\sqrt{2}}}{4}$$

$$= \frac{\sqrt{128 - 64\sqrt{2}}}{4}.$$

Hence $FI^2 = \frac{128 - 64\sqrt{2}}{16} = 8 - 4\sqrt{2}$. -HarryW

Problem19

Square $ABCD$ in the coordinate plane has vertices at the points $A(1,1), B(-1,1), C(-1,-1)$, and $D(1,-1)$. Consider the following four transformations: L, a rotation of 90° counterclockwise around the origin; R, a rotation of 90° clockwise around the origin; H, a reflection across the x-axis; and V, a reflection across the y-axis.

Each of these transformations maps the squares onto itself, but the positions of the labeled vertices will change. For example, applying R and then V would send the vertex A at $(1,1)$ to $(-1,-1)$ and would send the vertex B at $(-1,1)$ to itself. How many sequences of 20 transformations chosen from $\{L, R, H, V\}$ will send all of the labeled vertices back to their original positions? (For example, R, R, V, H is one sequence of 4 transformations that will send the vertices back to their original positions.)

$\textbf{(A)}\ 2^{37} \qquad \textbf{(B)}\ 3 \cdot 2^{36} \qquad \textbf{(C)}\ 2^{38} \qquad \textbf{(D)}\ 3 \cdot 2^{37} \qquad \textbf{(E)}\ 2^{39}$

Solution

Let (+) denote counterclockwise/starting orientation and (-) denote clockwise orientation. Let 1,2,3, and 4 denote which quadrant A is in.

Realize that from any odd quadrant and any orientation, the 4 transformations result in some permutation of $(2+, 2-, 4+, 4-)$.

The same goes that from any even quadrant and any orientation, the 4 transformations result in some permutation of $(1+, 1-, 3+, 3-)$.

We start our first 19 moves by doing whatever we want, 4 choices each time. Since 19 is odd, we must end up on an even quadrant.

As said above, we know that exactly one of the four transformations will give us $(1+)$, and we must use that transformation.

Thus $4^{19} = \boxed{(C) 2^{38}}$

Solution 2

Hopefully, someone will think of a better one, but here is an indirect answer, use only if you are really desperate. 20 moves can be made, and each move have 4 choices, so a total of $4^{20} = 2^{40}$ moves. First, after the 20 moves, Point A can only be in first quadrant $(1, 1)$ or third quadrant $(-1, -1)$. Only the one in the first quadrant works, so divide by 2. Now, C must be in the opposite quadrant as A. B can be either in the second ($(-1, 1)$) or fourth quadrant ($(1, -1)$) , but we want it to be in the second quadrant, so divide by 2 again. Now as A and B satisfy the conditions, C and D will also be at their original spot. $\frac{2^{40}}{2 \cdot 2} = 2^{38}$. The answer is $\boxed{C}$ ~Kinglogic

Solution 3

The total number of sequence is $4^{20} = 2^{40}$.

Note that there can only be even number of reflections since they result in the same anti-clockwise orientation of the verices A, B, C, D. Therefore, the probability of having the same anti-clockwise orientation with the original arrangement after the transformation is $\frac{1}{2}$.

Next, even number of reflections mean that there must be even number of rotations since their sum is even. Even rotations result only in the original position or 180° rotation of it.

Since rotation R and rotation L cancels each other out, the difference between the numbers of them define the final position. The probability of the transformation returning the vertices to the orginal position given that there are even number of rotations is equivalent to the probability that

$$|n(R)-n(L)| \equiv 0 \pmod 4$$ when

$$|n(H)-n(V)| \equiv 0 \pmod 4$$

or

$$|n(R)-n(L)| \equiv 2 \pmod 4$$ when

$$|n(H)-n(V)| \equiv 2 \pmod 4$$

which is again, $\frac{1}{2}$.

Therefore, $2^{40} \cdot \frac{1}{2} \cdot \frac{1}{2} = \boxed{\textbf{(C) } 2^{38}}$ ~joshuamh111

Solution 4

Notice that any pair of two of these transformations either swaps the x and y-coordinates, negates the x and y-coordinates, swaps and negates the x and y-coordinates, or leaves the original unchanged. Furthermore, notice that for each of these results, if we apply another pair of transformations, one of these results will happen again, and with equal probability. Therefore, no matter what state after we apply the first 19 pairs of transformations, there is a $\frac{1}{4}$ chance the last pair of transformations will return the figure to its original position. Therefore, the answer is $\frac{4^{20}}{4} = 4^{19} = \boxed{\textbf{(C) } 2^{38}}$

Problem20

Two different cubes of the same size are to be painted, with the color of each face being chosen independently and at random to be either black or white. What is the probability that after they are painted, the cubes can be rotated to be identical in appearance?

$\textbf{(A)}\ \frac{9}{64} \qquad \textbf{(B)}\ \frac{289}{2048} \qquad \textbf{(C)}\ \frac{73}{512} \qquad \textbf{(D)}\ \frac{147}{1024} \qquad \textbf{(E)}\ \frac{589}{4096}$

Solution

Define two ways of painting to be in the same $class$ if one can be rotated to form the other.

We can count the number of ways of painting for each specific $class$.

Case 1: Red-blue color distribution is 0-6 (out of 6 total faces)

Trivially $1^2 = 1$ way to paint the cubes.

Case 2: Red-blue color distribution is 1-5

Trivially all $\binom{6}{5} = 6$ ways belong to the same $class$, so 6^2 ways to paint the cubes.

Case 3: Red-blue color distribution is 2-4

There are two $classes$ for this case: the $class$ where the two red faces are touching and the other $class$where the two red faces are on opposite faces. There are 3 members of the latter $class$ since there are 3unordered pairs of 2 opposite faces of a cube. Thus, there are $\binom{6}{4} - 3 = 12$ members of the former $class$. Thus, $12^2 + 3^2$ ways to paint the cubes for this case.

Case 4: Red-blue color distribution is 3-3

By simple intuition, there are also two $classes$ for this case, the $class$ where the three red faces meet at a single vertex, and the other class where the three red faces are in a "straight line" along the edges of the cube. Note that since there are 8 vertices in a cube, there are 8 members of the former class and $\binom{6}{3} - 8 = 12$ members of the latter class. Thus, $12^2 + 8^2$ ways to paint the cubes for this case.

Note that by symmetry (since we are literally only switching the colors), the number of ways to paint the cubes for Red-blue color distributions 4-2, 5-1, and 6-0 is 2-4, 1-5, and 0-6 (respectively).

Thus, our total answer is $\frac{2(6^2+1^2+12^2+3^2)+12^2+8^2}{2^{12}} = \frac{588}{4096} = \boxed{\textbf{(D) } \frac{147}{1024}}$.

Problem21

How many positive integers n satisfy $$\dfrac{n+1000}{70} = \lfloor \sqrt{n} \rfloor?$$(Recall that $\lfloor x\rfloor$ is the greatest integer not exceeding x.)

$\textbf{(A) } 2 \qquad\textbf{(B) } 4 \qquad\textbf{(C) } 6 \qquad\textbf{(D) } 30 \qquad\textbf{(E) } 32$

Solution

First notice that the graphs of $(x+1000)/70$ and $\sqrt{n}$ intersect at 2 points. Then, notice that $(n+1000)/70$ must be an integer. This means that n is congruent to $50 \pmod{70}$.

For the first intersection, testing the first few values of n (adding 70 to n each time and noticing the left side increases by 1 each time) yields $n=20$ and $n=21$. timating from the graph can narrow down the other cases, being $n=47$, d $n=50$. This results in a total of 6 cases, for an answer of $\boxed{\textbf{(C) } 6}$.

~DrJoyo

Solution 2 (Graphing)

One intuitive approach to the question is graphing. Obviously, you should know what the graph of the square root function is, and if any function is floored (meaning it is taken to the greatest integer less than a value), a stair-like figure should appear. The other function is simply a line with a slope of $1/70$. If you precisely draw out the two regions of the graph where the derivative of the square function nears the derivative of the linear function, you can now deduce that 3 values of intersection lay closer to the left side of the stair, and 3 values lay closer to the right side of the stair.

With meticulous graphing, you can realize that the answer is $\boxed{\textbf{(C) } 6}$.

A in-depth graph with intersection points is linked below.https://www.desmos.com/calculator/e5wk9adbuk

Solution 3

- Not a reliable or in-depth solution (for the guess and check students)

We can first consider the equation without a floor function:

$$\frac{n+1000}{70} = \sqrt{n}$$

Multiplying both sides by 70 and then squaring:

$$n^2 + 2000n + 1000000 = 4900n$$

Moving all terms to the left:

$$n^2 - 2900n + 1000000 = 0$$

Now we can use wishful thinking to determine the factors:

$$(n-400)(n-2500) = 0$$

This means that for $n = 400$ and $n = 2500$, the equation will hold without the floor function.

Now we can simply check the multiples of 70 around 400 and 2500 in the original equation:

For $n = 330$, left hand side $= 19$ but $18^2 < 330 < 19^2$ so right hand side $= 18$

For $n = 400$, left hand side $= 20$ and right hand side $= 20$

For $n = 470$, left hand side $= 21$ and right hand side $= 21$

For $n = 540$, left hand side $= 22$ but $540 > 23^2$ so right hand side $= 23$

Now we move to $n = 2500$

For $n = 2430$, left hand side $= 49$ and $49^2 < 2430 < 50^2$ so right hand side $= 49$

For $n = 2360$, left hand side $= 48$ and $48^2 < 2360 < 49^2$ so right hand side $= 48$

For $n = 2290$, left hand side $= 47$ and $47^2 < 2360 < 48^2$ so right hand side $= 47$

For $n = 2220$, left hand side $= 46$ but $47^2 < 2220$ so right hand side $= 47$

For $n = 2500$, left hand side $= 50$ and right hand side $= 50$

For $n = 2570$, left hand side $= 51$ but $2570 < 51^2$ so right hand side $= 50$

Therefore we have 6 total solutions, $n = 400, 470, 2290, 2360, 2430, 2500 = \boxed{\textbf{(C) } 6}$

Solution 4

This is my first solution here, so please forgive me for any errors.

We are given that $\dfrac{n+1000}{70} = \lfloor \sqrt{n} \rfloor$

$\lfloor \sqrt{n} \rfloor$ must be an integer, which means that $n + 1000$ is divisible by 70.

As $1000 \equiv 20 \pmod{70}$, this means that $n \equiv 50 \pmod{70}$, so we can write $n = 70k + 50$ for $k \in \mathbb{Z}$.

Therefore,
$$\frac{n+1000}{70}=\frac{70k+1050}{70}=k+15=\lfloor\sqrt{70k+50}\rfloor$$

Also, we can say that $\sqrt{70k+50}-1\leq k+15$ and $k+15\leq\sqrt{70k+50}$

Squaring the second inequality, we get $k^2+30k+225\leq 70k+50 \implies k^2-40k+175\leq 0 \implies (k-5)(k-35)\leq 0 \implies 5\leq k\leq 35$.

Similarly solving the first inequality gives us $k\leq 19-\sqrt{155}$ or $k\geq 19+\sqrt{155}$

$\sqrt{155}$ is slightly larger than 12, so instead, we can say $k\leq 6$ or $k\geq 32$.

Combining this with $5\leq k\leq 35$, we get $k=5,6,32,33,34,35$ are all solutions for k that give a valid solution for n, meaning that our answer is $\boxed{\textbf{(C)}6}$.

-Solution By Qqqwerw

Solution 5

We start with the given equation $\frac{n+1000}{70}=\lfloor\sqrt{n}\rfloor$ From there, we can start with the general inequality that $\lfloor\sqrt{n}\rfloor\leq\sqrt{n}<\lfloor\sqrt{n}\rfloor+1$. This means that $\frac{n+1000}{70}\leq\sqrt{n}<\frac{n+1070}{70}$ Solving each inequality separately gives us two inequalities:

$$n-70\sqrt{n}+1000\leq 0\rightarrow(\sqrt{n}-50)(\sqrt{n}-20)\leq 0\rightarrow 20\leq\sqrt{n}\leq 50$$

$$n-70\sqrt{n}+1070>0\rightarrow\sqrt{n}<35-\sqrt{155},\sqrt{n}>35+\sqrt{155}$$

Simplifying and approximating decimals yields 2 solutions for one inequality and 4 for the other. Hence $2+4=\boxed{\textbf{(C)} 6}$.

Problem 22

What is the maximum value of $\frac{(2^t-3t)t}{4^t}$ for real values of t?

$\textbf{(A)}\ \frac{1}{16} \qquad \textbf{(B)}\ \frac{1}{15} \qquad \textbf{(C)}\ \frac{1}{12} \qquad \textbf{(D)}\ \frac{1}{10} \qquad \textbf{(E)}\ \frac{1}{9}$

Solution 1

Set $u=t2^{-t}$. Then the expression in the problem can be written as

$$R=-3t^2 4^{-t}+t2^{-t}=-3u^2+u=-3(u-1/6)^2+\frac{1}{12}\leq\frac{1}{12}.$$

It is easy to see that $u=\frac{1}{6}$ is attained for some value of t between $t=0$ and $t=1$, thus the maximal value of R is $\textbf{(C)}\ \frac{1}{12}$.

Solution 2 (Calculus Needed)

We want to maximize $f(t)=\frac{(2^t-3t)t}{4^t}=\frac{t\cdot 2^t-3t^2}{4^t}$. We can use the first derivative test. Use quotient rule to get the following:

$\frac{(2^t+t\cdot\ln 2\cdot 2^t-6t)4^t-(t\cdot 2^t-3t^2)4^t\cdot 2\ln 2}{(4^t)^2}=0 \implies 2^t+t\cdot\ln 2\cdot 2^t-6t=(t\cdot 2^t-3t^2)2\ln 2$

$$\implies 2^t+t\cdot\ln 2\cdot 2^t-6t=2t\ln 2\cdot 2^t-6t^2\ln 2$$

$$\implies 2^t(1-t\cdot\ln 2)=6t(1-t\cdot\ln 2) \implies 2^t=6t$$

Therefore, we plug this back into the original equation to get $\boxed{\textbf{(C)}\frac{1}{12}}$

~awesome1st

Solution 3

First, substitute $2^t=x(\log_2 x=t)$ so that $\frac{(2^t-3t)t}{4^t}=\frac{x\log_2 x-3(\log_2 x)^2}{x^2}$

Notice that $\frac{x\log_2 x-3(\log_2 x)^2}{x^2}=\frac{\log_2 x}{x}-3\Big(\frac{\log_2 x}{x}\Big)^2.$

When seen as a function, $\frac{\log_2 x}{x}-3\Big(\frac{\log_2 x}{x}\Big)^2$ is a synthesis function that has $\frac{\log_2 x}{x}$ as its inner function.

If we substitute $\frac{\log_2 x}{x}=p$, the given function becomes a quadratic function that has a maximum value of $\frac{1}{12}$ when $p=\frac{1}{6}$.

Now we need to check if $\frac{\log_2 x}{x}$ can have the value of $\frac{1}{6}$ in the range of real numbers.

In the range of (positive) real numbers, function $\frac{\log_2 x}{x}$ is a continuous function whose value gets infinitely smaller as x gets closer to 0 (as log_2x also diverges toward negative infinity in the same condition).

When $x=2$, $\frac{\log_2 x}{x}=\frac{1}{2}$, which is larger than $\frac{1}{6}$.

Therefore, we can assume that $\frac{\log_2 x}{x}$ equals to $\frac{1}{6}$ when x is somewhere between 1 and 2 (at least), which means that the maximum value of $\frac{(2^t-3t)t}{4^t}$ is $\boxed{\textbf{(C)}\ \frac{1}{12}}$.

Solution 4 (definitely legit)

We see 2, 3, 4 in the expression. Guess $2\times 3\times 4$ is factor of the answer $\implies \boxed{C}$.

Problem 23

How many integers $n \geq 2$ are there such that whenever $z_1, z_2, \ldots, z_n$ are complex numbers such that

$$|z_1| = |z_2| = \ldots = |z_n| = 1 \text{ and } z_1 + z_2 + \ldots + z_n = 0,$$

then the numbers $z_1, z_2, \ldots, z_n$ are equally spaced on the unit circle in the complex plane?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution

Solution

For $n=2$, we see that if $z_1+z_2=0$, then $z_1=-z_2$, so they are evenly spaced along the unit circle.

For $n=3$, WLOG, we can set $z_1=1$. Notice that now $\Re(z_2+z_3)=-1$ and $\Im\{z_2\}=-\Im\{z_3\}$. This forces z_2 and z_3 to be equal to $e^{i\frac{2\pi}{3}}$ and $e^{-i\frac{2\pi}{3}}$, meaning that all three are equally spaced along the unit circle.

We can now show that we can construct complex numbers when $n \geq 4$ that do not satisfy the conditions in the problem.
Suppose that the condition in the problem holds for some $n = k$. We can now add two points z_{k+1} and z_{k+2} anywhere on the unit circle such that $z_{k+1} = -z_{k+2}$, which will break the condition. Now that we have shown that $n = 2$ and $n = 3$ works, by this construction, any $n \geq 4$ does not work, making the answer $\boxed{(B) 2}$.

Problem24

Let $D(n)$ denote the number of ways of writing the positive integer n as a product $n = f_1 \cdot f_2 \cdots f_k$, where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

$\textbf{(A) }112 \qquad \textbf{(B) }128 \qquad \textbf{(C) }144 \qquad \textbf{(D) }172 \qquad \textbf{(E) }184$

Solution

Note that $96 = 2^5 \cdot 3$. Since there are at most six not necessarily distinct factors > 1 multiplying to 96, we have six cases: $k = 1, 2, ..., 6.$ Now we look at each of the six cases.

$k = 1$: We see that there is 1 way, merely 96.
$k = 2$: This way, we have the 3 in one slot and 2 in another, and symmetry. The four other 2's leave us with 5 ways and symmetry doubles us so we have 10.

$k=3$: We have $3,2,2$ as our baseline. We need to multiply by 2 in 3 places, and see that we can split the remaining three powers of 2 in a manner that is $3-0-0$, $2-1-0$ or $1-1-1$.

A $3-0-0$ split has $6+3=9$ ways of happening ($24-2-2$ and symmetry; $2-3-16$ and symmetry), a $2-1-0$ split has $6\cdot 3=18$ ways of happening (due to all being distinct) and a $1-1-1$ split has 3 ways of happening ($6-4-4$ and symmetry) so in this case we have $9+18+3=30$ ways.

$k=4$: We have $3,2,2,2$ as our baseline, and for the two other 2's, we have a $2-0-0-0$ or $1-1-0-0$ split. The former grants us $4+12=16$ ways ($12-2-2-2$ and symmetry and $3-8-2-2$ and symmetry) and the latter grants us also $12+12=24$ ways ($6-4-2-2$ and symmetry and $3-4-4-2$ and symmetry) for a total of $16+24=40$ ways.

$k=5$: We have $3,2,2,2,2$ as our baseline and one place to put the last two: on another two or on the three. On the three gives us 5 ways due to symmetry and on another two gives us $5\cdot 4=20$ ways due to symmetry. Thus, we have $5+20=25$ ways.

$k=6$: We have $3,2,2,2,2,2$ and symmetry and no more twos to multiply, so by symmetry, we have 6ways.

Thus, adding, we have $1+10+30+40+25+6=\boxed{\textbf{(A) } 112}$.

~kevinmathz

Solution 2

As before, note that $96=2^5\cdot 3$, and we need to consider 6 different cases, one for each possible value of k, the number of factors in our factorization. However, instead of looking at each individually, find a general form for the number of possible factorizations with k factors. First, the factorization needs to contain one factor that is itself a multiple of 3, and there are k to choose from, and the rest must contain at least one factor of 2. Next, consider the remaining $6-n$ factors of 2 left to assign to the k factors. Using stars and bars, the number of ways to do this is $\binom{(6-k)+k-1}{6-k} = \binom{5}{6-k}$ This makes $k\binom{5}{6-k}$ possibilities for each k.

To obtain the total number of factorizations, add all possible values for k:

$$\sum_{k=1}^{6} k\binom{5}{6-k} = 1+10+30+40+25+6 = \boxed{\textbf{(A) } 112}$$

.

Solution 3

Begin by examining f_1. f_1 can take on any value that is a factor of 96 except 1.

For each choice of f_1, the resulting $f_2 \dots f_k$ must have a product of $96/f_1$.

This means the number of ways the rest $f_a, 1 < a <= k$ can be written by the scheme stated in the problem for each f_1 is equal to $D(96/f_1)$, since the product of $f_2 \cdot f_3 \dots \cdot f_k = x$ is counted as one valid product if and only if $f_1 \cdot x = 96$, the product x has the properties that factors are greater than 1, and differently ordered products are counted separately.

For example, say the first factor is 2. Then, the remaining numbers must multiply to 48, so the number of ways the product can be written beginning with 2 is $D(48)$. To add up all the number of solutions for every possible starting factor, $D(96/f_1)$ must be calculated and summed for all possible f_1, except 96 and 1, since a single 1 is not counted according to the problem statement. The 96 however, is counted, but only results in 1 possibility, the first and only factor being 96. This means

$D(96) = D(48) + D(32) + D(24) + D(16) + D(12) + D(8) + D(6) + D(4) + D(3) + D(2) + 1$
.

Instead of calculating D for the larger factors first, reduce $D(48)$, $D(32)$, and $D(24)$ into sums of $D(m)$where $m <= 16$ to ease calculation.

Following the recursive definition $D(n) = ($sums of $D(c)) + 1$where c takes on every divisor of n except for 1 and itself, the sum simplifies to
$D(96) = (D(24) + D(16) + D(12) + D(8) + D(6) + D(4) + D(3) + D(2) + 1)$
+
$(D(16) + D(8) + D(4) + D(2) + 1) + D(24) + D(16) + D(12) + D(8) + D(6) + D(4) + D(3) + D(2) + 1.$

$$D(24) = D(12) + D(8) + D(6) + D(4) + D(3) + D(2) + 1$$
, so the sum further simplifies to
$D(96) = 3D(16) + 4D(12) + 5D(8) + 4D(6) + 5D(4) + 4D(3) + 5D(2) + 5$
, after combining terms. From quick casework,
$D(16) = 8, D(12) = 8, D(8) = 4, D(6) = 3, D(4) = 2, D(3) = 1$
and $D(2) = 1$. Substituting these values into the expression above,

$$D(96) = 3 \cdot 8 + 4 \cdot 8 + 5 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 4 \cdot 1 + 5 \cdot 1 + 5 = \boxed{\textbf{(A) } 112}$$
.

~monmath a.k.a Fmirza

Solution 4

Note that $96 = 3 \cdot 2^5$, and that D of a perfect power of a prime is relatively easy to calculate. Also note that you can find $D(96)$ from $D(32)$ by simply totaling the number of ways there are to insert a 3 into a set of numbers that multiply to 32.

First, calculate $D(32)$. Since $32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, all you have to do was find the number of ways to divide the 2's into groups, such that each group has at least one 2. By stars and bars, this results in 1 way with five terms, 4 ways with four terms, 6 ways with three terms, 4 ways with two terms, and 1 way with one term. (The total, 16, is not needed for the remaining calculations.)

Then, to get $D(96)$, in each possible $D(32)$ sequence, insert a 3 somewhere in it, either by placing it somewhere next to the original numbers (in one of $n+1$ ways, where n is the number of terms in the $D(32)$ sequence), or by multiplying one of the numbers by 3 (in one of n ways). There are $2+1=3$ways to do this with one term, $3+2=5$ with two, 7 with three, 9 with four, and 11 with five.

The resulting number of possible sequences is $3 \cdot 1 + 5 \cdot 4 + 7 \cdot 6 + 9 \cdot 4 + 11 \cdot 1 = \boxed{\textbf{(A) } 112}$.

Problem 25

For each real number a with $0 \le a \le 1$, let numbers x and y be chosen independently at random from the intervals $[0, a]$ and $[0, 1]$, respectively, and let $P(a)$ be the probability that

$\sin^2{(\pi x)} + \sin^2{(\pi y)} > 1$What is the maximum value of $P(a)$?

$\textbf{(A)}\ \frac{7}{12} \qquad \textbf{(B)}\ 2 - \sqrt{2} \qquad \textbf{(C)}\ \frac{1+\sqrt{2}}{4} \qquad \textbf{(D)}\ \frac{\sqrt{5}-1}{2} \qquad \textbf{(E)}\ \frac{5}{8}$

Solution

Let's start first by manipulating the given inequality.

$$\sin^2{(\pi x)} + \sin^2{(\pi y)} > 1$$

$$\sin^2{(\pi x)} > 1 - \sin^2{(\pi y)} = \cos^2{(\pi y)}$$

Let's consider the boundary cases: $\sin^2{(\pi x)} = \cos^2{(\pi y)}$ and $\sin^2{(\pi x)} = -\cos^2{(\pi y)}$

$$\sin{(\pi x)} = \cos{(\pi y)} = \sin{(\frac{\pi}{2} \pm \pi y)}$$

Solving, we get $y=\frac{1}{2}-x$ and $y=x-\frac{1}{2}$. Solving the second case gives us $y=x+\frac{1}{2}$ and $y=\frac{3}{2}-x$. If we graph these equations in $[0,1]\times[0,1]$, we get a rhombus shape. Testing points in each section tells us that the inside of the rhombus satisfies the inequality in the problem statement.

From the region graph, notice that in order to maximize $P(a)$, $a\geq\frac{1}{2}$. We can solve the rest with geometric probability.

When $a\geq\frac{1}{2}, P(a)$ consists of a triangle with area $\frac{1}{4}$ and a trapezoid with bases 1 and $2-2a$ and height $a-\frac{1}{2}$. Finally, to calculate $P(a)$, we divide this area by a, so $P(a)=\frac{1}{a}\left(\frac{1}{4}+\frac{(a-\frac{1}{2})(3-2a)}{2}\right)$

After expanding out, we get $P(a)=\frac{-4a^2+8a-2}{4a}=2-a-\frac{1}{2a}$. In order to maximize this expression, we must minimize $a+\frac{1}{2a}$.

By AM-GM, $a+\frac{1}{2a}\geq 2\sqrt{\frac{a}{2a}}=\sqrt{2}$, which we can achieve by setting $a=\frac{\sqrt{2}}{2}$.

Therefore, the maximum value of $P(a)$ is $P\left(\frac{\sqrt{2}}{2}\right)=\boxed{\textbf{(B)} 2-\sqrt{2}}$