2007 AMC 12A Problems/Problem 1

The following problem is from both the 2007 AMC 12A #1 and 2007 AMC 10A #1, so both problems redirect to this page.

Problem

One ticket to a show costs $\$20$ at full price. Susan buys 4 tickets using a coupon that gives her a 25% discount. Pam buys 5 tickets using a coupon that gives her a 30% discount. How many more dollars does Pam pay than Susan?

$\mathrm{(A)}\ 2\qquad\mathrm{(B)}\ 5\qquad\mathrm{(C)}\ 10\qquad\mathrm{(D)}\ 15\qquad\mathrm{(E)}\ 20$

Solution 1

P = the amount Pam spent S = the amount Susan spent

- $P = 5\cdot(20\cdot.7) = 70$
- $S = 4\cdot(20\cdot.75) = 60$

Pam pays 10 more dollars than Susan $\Rightarrow \boxed{\mathrm{C}}$

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by First question	Followed by Problem 2
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Preceded by First question	Followed by Problem 2
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Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 2

Problem

An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50 cm. It is filled with water to a height of 40 cm. A brick with a rectangular base that measures 40 cm by 20 cm and a height of 10 cm is placed in the aquarium. By how many centimeters does the water rise?

$\mathrm{(A)}\ 0.5\qquad \mathrm{(B)}\ 1\qquad \mathrm{(C)}\ 1.5\qquad \mathrm{(D)}\ 2\qquad \mathrm{(E)}\ 2.5$

Solution

The brick has volume $8000 cm^3$. The base of the aquarium has area $4000 cm^2$. For every inch the water rises, the volume increases by $4000 cm^3$; therefore, when the volume increases by $8000 cm^3$, the water level rises $2 cm \Rightarrow \boxed{\text{D}}$

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 1	Followed by Problem 3
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Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 3

Problem

The larger of two consecutive odd integers is three times the smaller. What is their sum?

$\mathrm{(A)}\ 4 \qquad \mathrm{(B)}\ 8 \qquad \mathrm{(C)}\ 12 \qquad \mathrm{(D)}\ 16 \qquad \mathrm{(E)}\ 20$

Solution

Solution 1 Let n be the smaller term. Then $n+2=3n \Longrightarrow 2n=2 \Longrightarrow n=1$

- Thus, the answer is $1+(1+2)=4 \mathrm{(A)}$

Solution 2

- By trial and error, 1 and 3 work. 1+3=4.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 2	Followed by Problem 4
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Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 4

Problem

Kate rode her bicycle for 30 minutes at a speed of 16 mph, then walked for 90 minutes at a speed of 4 mph. What was her overall average speed in miles per hour?

$\mathrm{(A)}\ 7\qquad \mathrm{(B)}\ 9\qquad \mathrm{(C)}\ 10\qquad \mathrm{(D)}\ 12\qquad \mathrm{(E)}\ 14$

Solution

$$16\cdot\frac{30}{60}+4\cdot\frac{90}{60}=14$$

$$\frac{14}{2}=7\Rightarrow\boxed{A}$$

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 5

The following problem is from both the 2007 AMC 12A #5 and 2007 AMC 10A #7, so both problems redirect to this page.

Problem

Last year Mr. Jon Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. He paid a total of $\$10500$ for both taxes. How many dollars was his inheritance?

$\mathrm{(A)}\ 30000 \qquad \mathrm{(B)}\ 32500 \qquad \mathrm{(C)}\ 35000 \qquad \mathrm{(D)}\ 37500 \qquad \mathrm{(E)}\ 40000$

Solution

After paying his taxes, he has $0.8*0.9=0.72$ of the inheritance left. Since 10500 is 0.28 of the inheritance, the whole inheritance is $\frac{10500}{0.28} = 37500\ \mathrm{(D)}$.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
Preceded by Problem 4	Followed by Problem 6
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Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 6

The following problem is from both the 2007 AMC 12A #6 and 2007 AMC 10A #8, so both problems redirect to this page.

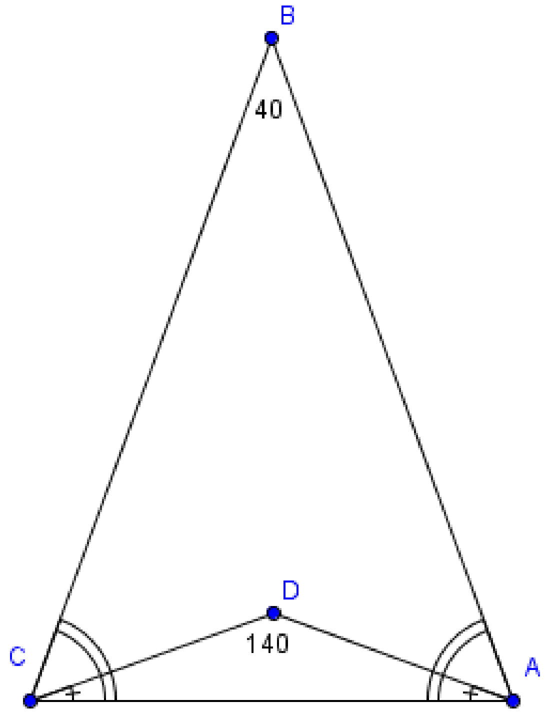
Contents

Problem

Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside triangle ABC, angle ABC measures 40 degrees, and angle ADC measures 140 degrees. What is the degree measure of angle BAD?

$\mathrm{(A)}\ 20 \qquad \mathrm{(B)}\ 30 \qquad \mathrm{(C)}\ 40 \qquad \mathrm{(D)}\ 50 \qquad \mathrm{(E)}\ 60$

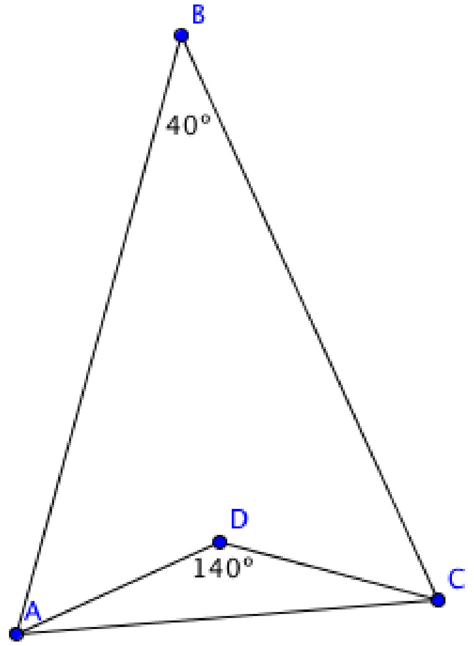
Solution 1



We angle chase, and find out that:

- $DAC = \frac{180-140}{2} = 20$
- $BAC = \frac{180-40}{2} = 70$
- $BAD = BAC - DAC = 50\ \mathrm{(D)}$

Solution 2



Since triangle ABC is isosceles we know that angle $\angle BAC = \angle BCA$.

Also since triangle ADC is isosceles we know that $\angle DAC = \angle DCA$.

This implies that $\angle BAD = \angle BCD$.

Then the sum of the angles in quadrilateral $ABCD$ is $40 + 220 + 2\angle BAD = 360$.

Solving the equation we get $\angle BAD = 50$.

Therefore the answer is (D).

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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2007 AMC 12A Problems/Problem 7

Problem

Let a, b, c, d, and e be five consecutive terms in an arithmetic sequence, and suppose that $a+b+c+d+e=30$. Which of a, b, c, d, or e can be found?

$\text{(A)}\ a \qquad \text{(B)}\ b \qquad \text{(C)}\ c \qquad \text{(D)}\ d \qquad \text{(E)}\ e$

Solution

Let f be the common difference between the terms.

- $a = c - 2f$
- $b = c - f$
- $c = c$
- $d = c + f$
- $e = c + 2f$

$a+b+c+d+e = 5c = 30$, so $c = 6$. But we can't find any more variables, because we don't know what f is. So the answer is C.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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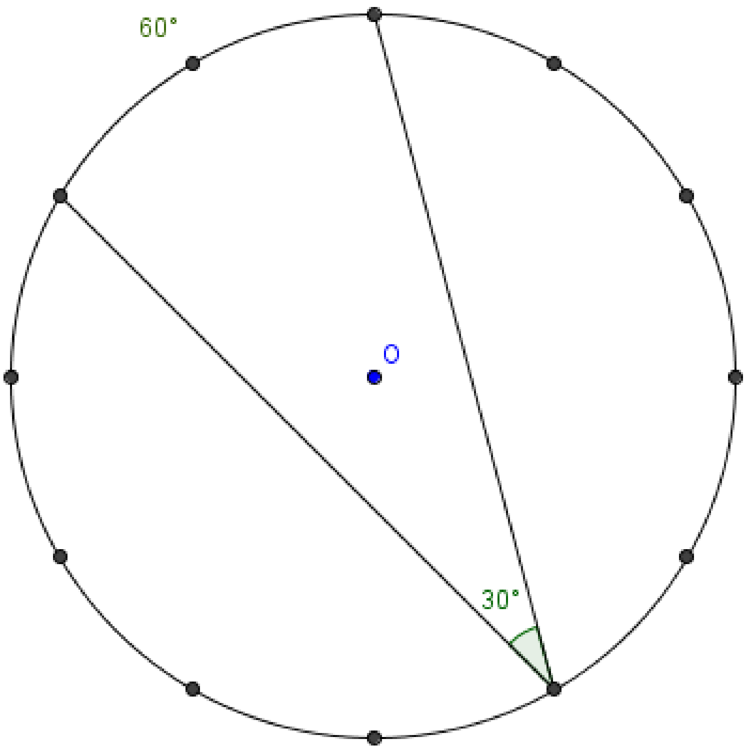
2007 AMC 12A Problems/Problem 8

Problem

A star-polygon is drawn on a clock face by drawing a chord from each number to the fifth number counted clockwise from that number. That is, chords are drawn from 12 to 5, from 5 to 10, from 10 to 3, and so on, ending back at 12. What is the degree measure of the angle at each vertex in the star polygon?

$\text{(A)}\ 20 \qquad \text{(B)}\ 24 \qquad \text{(C)}\ 30 \qquad \text{(D)}\ 36 \qquad \text{(E)}\ 60$

Solution



We look at the angle between 12, 5, and 10. It subtends $\frac{1}{6}$ of the circle, or 60 degrees (or you can see that the arc is $\frac{2}{3}$ of the right angle). Thus, the angle at each vertex is an inscribed angle subtending 60 degrees, making the answer $\frac{1}{2}60 = 30^\circ \Longrightarrow \mathrm{(C)}$

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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2007 AMC 12A Problems/Problem 9

The following problem is from both the 2007 AMC 12A #9 and 2007 AMC 10A #13, so both problems redirect to this page.

Problem

Yan is somewhere between his home and the stadium. To get to the stadium he can walk directly to the stadium, or else he can walk home and then ride his bicycle to the stadium. He rides 7 times as fast as he walks, and both choices require the same amount of time. What is the ratio of Yan's distance from his home to his distance from the stadium?

$\mathrm{(A)}\ \frac{2}{3}\qquad\mathrm{(B)}\ \frac{3}{4}\qquad\mathrm{(C)}\ \frac{4}{5}\qquad\mathrm{(D)}\ \frac{5}{6}\qquad\mathrm{(E)}\ \frac{7}{8}$

Solution

Let the distance from Yan's initial position to the stadium be a and the distance from Yan's initial position to home be b. We are trying to find b/a, and we have the following identity given by the problem:

$$a = b + \frac{a+b}{7}$$
$$\frac{6a}{7} = \frac{8b}{7} \Longrightarrow 6a = 8b$$

Thus $b/a = 6/8 = 3/4$ and the answer is $\mathrm{(B)}\ \frac{3}{4}$

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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2007 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2007))	
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2007 AMC 12A Problems/Problem 10

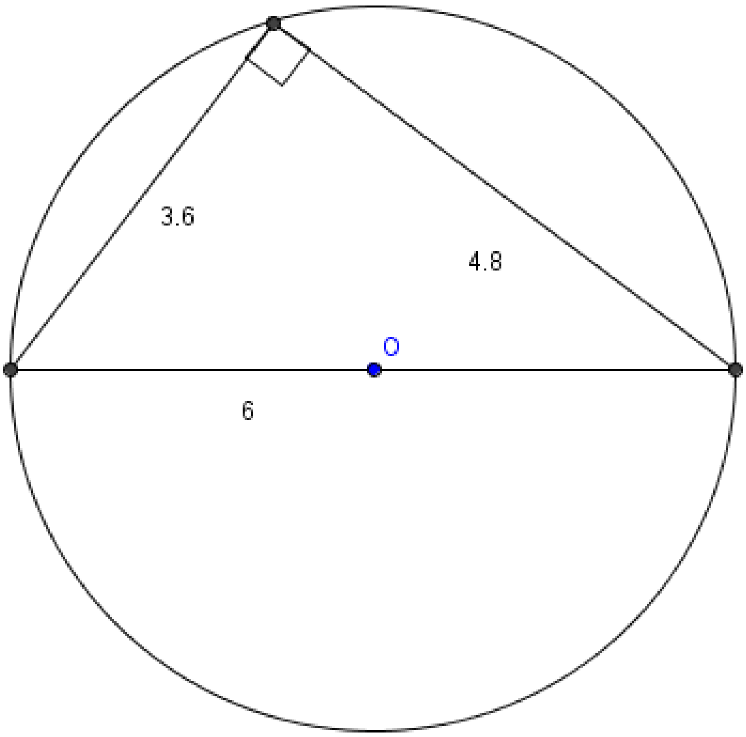
> The following problem is from both the 2007 AMC 12A #10 and 2007 AMC 10A #14, so both problems redirect to this page.

Problem

A triangle with side lengths in the ratio $3:4:5$ is inscribed in a circle with radius 3. What is the area of the triangle?

$\mathrm{(A)}\ 8.64\qquad\mathrm{(B)}\ 12\qquad\mathrm{(C)}\ 5\pi\qquad\mathrm{(D)}\ 17.28\qquad\mathrm{(E)}\ 18$

Solution



Since 3-4-5 is a Pythagorean triple, the triangle is a right triangle. Since the hypotenuse is a diameter of the circumcircle, the hypotenuse is $2r = 6$. Then the other legs are $\frac{24}{5} = 4.8$ and $\frac{18}{5} = 3.6$. The area is $\frac{4.8 \cdot 3.6}{2} = 8.64\ \mathrm{(A)}$

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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2007 AMC 12A Problems/Problem 11

The following problem is from both the 2007 AMC 12A #11 and 2007 AMC 10A #22, so both problems redirect to this page.

Problem

A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with the terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime factor that always divides S?

$\mathrm{(A)}\ 3\qquad \mathrm{(B)}\ 7\qquad \mathrm{(C)}\ 13\qquad \mathrm{(D)}\ 37\qquad \mathrm{(E)}\ 43$

Solution

A given digit appears as the hundreds digit, the tens digit, and the units digit of a term the same number of times. Let k be the sum of the units digits in all the terms. Then $S=111k=3*37k$, so S must be divisible by $37\ \mathrm{(D)}$. To see that it need not be divisible by any larger prime, the sequence $123, 231, 312$ gives $S=666=2\cdot 3^2\cdot 37$.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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2007 AMC 12A Problems/Problem 12

The following problem is from both the 2007 AMC 12A #12 and 2007 AMC 10A #16, so both problems redirect to this page.

Problem

Integers $a, b, c,$ and d, not necessarily distinct, are chosen independently and at random from 0 to 2007, inclusive. What is the probability that $ad - bc$ is even?

$\mathrm{(A)}\ \frac{3}{8} \qquad \mathrm{(B)}\ \frac{7}{16} \qquad \mathrm{(C)}\ \frac{1}{2} \qquad \mathrm{(D)}\ \frac{9}{16} \qquad \mathrm{(E)}\ \frac{5}{8}$

Solution

The only times when $ad - bc$ is even is when ad and bc are of the same parity. The chance of ad being odd is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, so it has a $\frac{3}{4}$ probability of being even. Therefore, the probability that $ad - bc$ will be even is $\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{5}{8}\ \mathrm{(E)}$.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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Category: Introductory Combinatorics Problems

2007 AMC 12A Problems/Problem 13

Problem

A piece of cheese is located at $(12, 10)$ in a coordinate plane. A mouse is at $(4, -2)$ and is running up the line $y = -5x + 18$. At the point (a, b) the mouse starts getting farther from the cheese rather than closer to it. What is $a + b$?

$\mathrm{(A)}\ 6 \qquad \mathrm{(B)}\ 10 \qquad \mathrm{(C)}\ 14 \qquad \mathrm{(D)}\ 18 \qquad \mathrm{(E)}\ 22$

Solution

We are trying to find the foot of a perpendicular from $(12, 10)$ to $y = -5x + 18$. Then the slope of the line that passes through the cheese and (a, b) is the negative reciprocal of the slope of the line, or $\frac{1}{5}$. Therefore, the line is $y = \frac{1}{5}x + \frac{38}{5}$. The point where $y = -5x + 18$ and $y = \frac{1}{5}x + \frac{38}{5}$ intersect is $(2, 8)$, and $2 + 8 = 10\ (B)$.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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All AMC 12 Problems and Solutions	

Category: Introductory Geometry Problems

2007 AMC 12A Problems/Problem 14

Problem

Let a, b, c, d, and e be distinct integers such that

$$(6-a)(6-b)(6-c)(6-d)(6-e)=45$$

What is $a+b+c+d+e$?

$\mathrm{(A)}\ 5\qquad \mathrm{(B)}\ 17\qquad \mathrm{(C)}\ 25\qquad \mathrm{(D)}\ 27\qquad \mathrm{(E)}\ 30$

Solution

If 45 is expressed as a product of five distinct integer factors, the absolute value of the product of any four it as least $|(-3)(-1)(1)(3)| = 9$, so no factor can have an absolute value greater than 5. Thus the factors of the given expression are five of the integers $\pm 3, \pm 1, \pm 5$. The product of all six of these is $-225 = (-5)(45)$, so the factors are -3, -1, 1, 3, and 5. The corresponding values of a, b, c, d, and e are 9, 7, 5, 3, and 1, and their sum is 25 (C).

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 15

Problems

The set $\{3, 6, 9, 10\}$ is augmented by a fifth element n, not equal to any of the other four. The median of the resulting set is equal to its mean. What is the sum of all possible values of n?

$\mathrm{(A)}\ 7\qquad \mathrm{(B)}\ 9\qquad \mathrm{(C)}\ 19\qquad \mathrm{(D)}\ 24\qquad \mathrm{(E)}\ 26$

Solution

The median must either be $6, 9,$ or n. Casework:

- Median is 6: Then $n \le 6$ and $\frac{3+6+9+10+n}{5} = 6 \Longrightarrow n = 2$.
- Median is 9: Then $n \ge 9$ and $\frac{3+6+9+10+n}{5} = 9 \Longrightarrow n = 17$.
- Median is n: Then $6 < n < 9$ and $\frac{3+6+9+10+n}{5} = n \Longrightarrow n = 7$.

All three cases are valid, so our solution is $2 + 7 + 17 = 26 \Longrightarrow \mathrm{(E)}$.

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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All AMC 12 Problems and Solutions	

Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 16

Problems

How many three-digit numbers are composed of three distinct digits such that one digit is the average of the other two?

$\textbf{(A)}\ 96 \qquad \textbf{(B)}\ 104 \qquad \textbf{(C)}\ 112 \qquad \textbf{(D)}\ 120 \qquad \textbf{(E)}\ 256$

Solution

We can find the number of increasing arithmetic sequences of length 3 possible from 0 to 9, and then find all the possible permutations of these sequences.

Common difference	Sequences possible	Number of sequences
1	$012, \ldots, 789$	8
2	$024, \ldots, 579$	6
3	$036, \ldots, 369$	4
4	$048, \ldots, 159$	2

This gives us a total of $2+4+6+8=20$ sequences. There are $3! = 6$ to permute these, for a total of 120.

However, we note that the conditions of the problem require three-digit numbers, and hence our numbers cannot start with zero. There are $2! \cdot 4 = 8$ numbers which start with zero, so our answer is $120 - 8 = 112 \Longrightarrow \mathrm{(C)}$.

See also

similar problem

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 17

Problem

Suppose that $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$. What is $\cos(a-b)$?

$\textbf{(A)}\ \sqrt{\frac{5}{3}}-1 \qquad \textbf{(B)}\ \frac{1}{3} \qquad \textbf{(C)}\ \frac{1}{2} \qquad \textbf{(D)}\ \frac{2}{3} \qquad \textbf{(E)}\ 1$

Solution

We can make use the of the trigonometric Pythagorean identities: square both equations and add them up:

$$\begin{aligned}
\sin^2 a + \sin^2 b + 2\sin a \sin b + \cos^2 a + \cos^2 b + 2\cos a \cos b &= \frac{5}{3} + 1 \\
2 + 2\sin a \sin b + 2\cos a \cos b &= \frac{8}{3} \\
2(\cos a \cos b + \sin a \sin b) &= \frac{2}{3}
\end{aligned}$$

This is just the cosine difference identity, which simplifies to $\cos(a-b) = \frac{1}{3} \Longrightarrow \mathrm{(B)}$

See also

2007 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2007))	
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Category: Introductory Trigonometry Problems

2007 AMC 12A Problems/Problem 18

Contents

Problem

The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2+i) = 0$. What is $a+b+c+d$?

$\mathrm{(A)}\ 0 \qquad \mathrm{(B)}\ 1 \qquad \mathrm{(C)}\ 4 \qquad \mathrm{(D)}\ 9 \qquad \mathrm{(E)}\ 16$

Solution 1

A fourth degree polynomial has four roots. Since the coefficients are real(meaning that complex roots come in conjugate pairs), the remaining two roots must be the complex conjugates of the two given roots, namely $2-i, -2i$. Now we work backwards for the polynomial:

$$(x-(2+i))(x-(2-i))(x-2i)(x+2i) = 0$$

$$(x^2 - 4x + 5)(x^2 + 4) = 0$$

$$x^4 - 4x^3 + 9x^2 - 16x + 20 = 0$$

Thus our answer is $-4+9-16+20 = 9\ \mathrm{(D)}$.

Solution 2

Just like in Solution 1 we realize that the roots come in conjugate pairs. Which means the roots are $2i, i+2, -2i, 2-i$ So our polynomial is

(1) $f(x) = (x-2i)(x+2i)(x-i-2)(x-2+i)$

Looking at the equation of the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$. We see that $a+b+c+d = f(1) - 1$

If we plug in 1 into equation (1) we get $f(1) = (1-2i)(1+2i)(-1-i)(-1+i)$.

Now if we multiply a complex number by its conjugate we get the sum of the squares of its real and imaginary parts. Using this property on the above we multiply and get $f(1) = (1-2i)(1+2i)(-1-i)(-1+i) = (1^2+2^2)(1^2+1^2) = 10$ So the answer is $f(1) - 1 = 10 - 1 = 9$. $\boxed{\text{D}}$

See also

2007 AMC 12A Problems/Problem 19

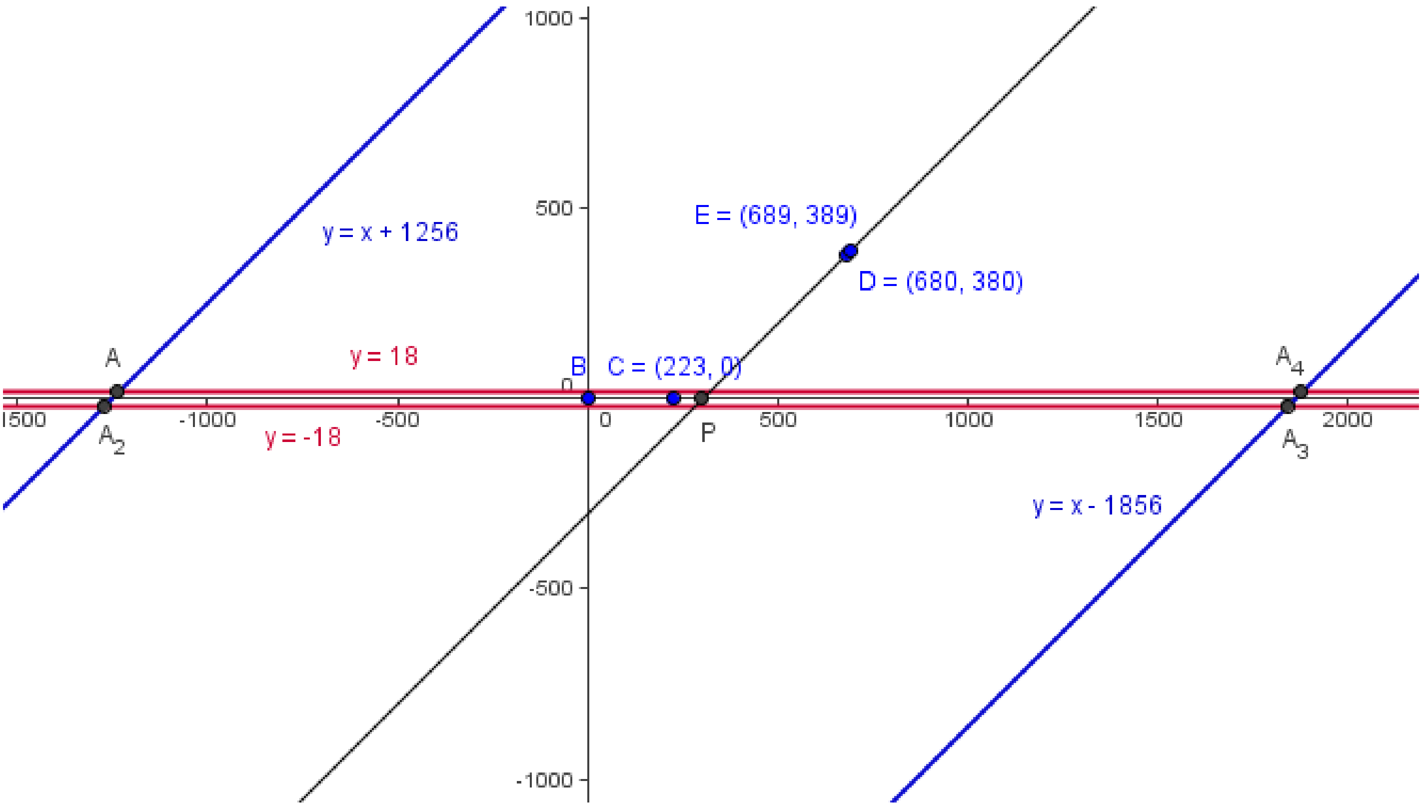
Problem

Triangles ABC and ADE have areas 2007 and $7002,$ respectively, with $B=(0,0),$ $C=(223,0),$ $D=(680,380),$ and $E=(689,389).$ What is the sum of all possible x-coordinates of A?

$\mathrm{(A)}\ 282 \qquad \mathrm{(B)}\ 300 \qquad \mathrm{(C)}\ 600 \qquad \mathrm{(D)}\ 900 \qquad \mathrm{(E)}\ 1200$

Contents

Solution



Solution 1

From $k = [ABC] = \frac{1}{2}bh$, we have that the height of $\triangle ABC$ is $h = \frac{2k}{b} = \frac{2007 \cdot 2}{223} = 18.$ Thus A lies on the lines $y = \pm 18 \quad (1).$

$DE = 9\sqrt{2}$ using 45-45-90 triangles, so in $\triangle ADE$ we have that $h = \frac{2 \cdot 7002}{9\sqrt{2}} = 778\sqrt{2}.$ The slope of DE is $1,$ so the equation of the line is $y = x + b \Longrightarrow b = (380) - (680) = -300 \Longrightarrow y = x - 300.$ The point A lies on one of two parallel lines that are $778\sqrt{2}$ units away from $\overline{DE}.$ Now take an arbitrary point on the line $\overline{DE}$ and draw the perpendicular to one of the parallel lines; then draw a line straight down from the same arbitrary

point. These form a 45-45-90 $\triangle$, so the straight line down has a length of $778\sqrt{2} \cdot \sqrt{2} = 1556$. Now we note that the y-intercept of the parallel lines is either 1556 units above or below the y-intercept of line $\overline{DE}$; hence the equation of the parallel lines is
$y = x - 300 \pm 1556 \Longrightarrow x = y + 300 \pm 1556 \quad (2)$.

We just need to find the intersections of these two lines and sum up the values of the x-coordinates. Substituting the (1) into (2), we get $x = \pm 18 + 300 \pm 1556 = 4(300) = 1200 \Longrightarrow \mathrm{(E)}$.

Solution 2

We are finding the intersection of two pairs of parallel lines, which will form a parallelogram. The centroid of this parallelogram is just the intersection of $\overline{BC}$ and $\overline{DE}$, which can easily be calculated to be $(300, 0)$. Now the sum of the x-coordinates is just $4(300) = 1200$.

See also

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Category: Introductory Geometry Problems

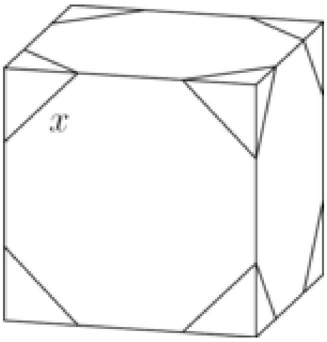
2007 AMC 12A Problems/Problem 20

Problem

Corners are sliced off a unit cube so that the six faces each become regular octagons. What is the total volume of the removed tetrahedra?

$\mathrm{(A)}\ \frac{5\sqrt{2}-7}{3}\qquad \mathrm{(B)}\ \frac{10-7\sqrt{2}}{3}\qquad \mathrm{(C)}\ \frac{3-2\sqrt{2}}{3}\qquad \mathrm{(D)}\ \frac{8\sqrt{2}-11}{3}\qquad \mathrm{(E)}\ \frac{6-4\sqrt{2}}{3}$

Solution



Since the sides of a regular polygon are equal in length, we can call each side x. Examine one edge of the unit cube: each contains two slanted diagonal edges of an octagon and one straight edge. The diagonal edges form $45-45-90\triangle$ right triangles, making the distance on the edge of the cube $\frac{x}{\sqrt{2}}$. Thus,

$2\cdot\frac{x}{\sqrt{2}}+x=1$, and $x=\frac{1}{\sqrt{2}+1}\cdot\left(\frac{\sqrt{2}-1}{\sqrt{2}-1}\right)=\sqrt{2}-1$.

Each of the cut off corners is a pyramid, whose volume can be calculated by $V=\frac{1}{3}Bh$. Use the base as one of the three congruent isosceles triangles, with the height being one of the edges of the pyramid that sits on the edges of the cube. The height is $\frac{x}{\sqrt{2}}=1-\frac{1}{\sqrt{2}}$. The base is a $45-45-90\triangle$ with leg of length $1-\frac{1}{\sqrt{2}}$, making its area $\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)^2=\frac{3-2\sqrt{2}}{4}$. Plugging this in, we get that the area of one of the tetrahedra is $\frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)\left(\frac{3-2\sqrt{2}}{4}\right)=\frac{10-7\sqrt{2}}{24}$. Since there are 8 removed corners, we get an answer of $\frac{10-7\sqrt{2}}{3}\Longrightarrow \mathrm{B}$

See also

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2007 AMC 12A Problems/Problem 21

Problem

The sum of the zeros, the product of the zeros, and the sum of the coefficients of the function $f(x) = ax^2 + bx + c$ are equal. Their common value must also be which of the following?

$$\begin{array}{l} \text{(A) the coefficient of } x^2 \quad \text{(B) the coefficient of } x \\ \text{(C) the y-intercept of the graph of } y = f(x) \\ \text{(D) one of the x-intercepts of the graph of } y = f(x) \\ \text{(E) the mean of the x-intercepts of the graph of } y = f(x) \end{array}$$

Solution

By Vieta's formulas, the sum of the roots of a quadratic equation is $\frac{-b}{a}$, the product of the zeros is $\frac{c}{a}$, and the sum of the coefficients is $a+b+c$. Setting equal the first two tells us that $\frac{-b}{a} = \frac{c}{a} \Rightarrow b = -c$. Thus, $a+b+c = a+b-b = a$, so the common value is also equal to the coefficient of $x^2 \Longrightarrow \mathrm{A}$.

To disprove the others, note that:

- B: then $b = \frac{-b}{a}$, which is not necessarily true.
- C: the y-intercept is c, so $c = \frac{c}{a}$, not necessarily true.
- D: an x-intercept of the graph is a root of the polynomial, but this excludes the other root.
- E: the mean of the x-intercepts will be the sum of the roots of the quadratic divided by 2.

See also

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Category: Introductory Algebra Problems

2007 AMC 12A Problems/Problem 22

The following problem is from both the 2007 AMC 12A #22 and 2007 AMC 10A #25, so both problems redirect to this page.

Problem

For each positive integer n, let $S(n)$ denote the sum of the digits of n. For how many values of n is $n + S(n) + S(S(n)) = 2007?$

$\mathrm{(A)}\ 1 \qquad \mathrm{(B)}\ 2 \qquad \mathrm{(C)}\ 3 \qquad \mathrm{(D)}\ 4 \qquad \mathrm{(E)}\ 5$

Contents

Solution

Solution 1

For the sake of notation let $T(n) = n + S(n) + S(S(n))$. Obviously $n<2007$. Then the maximum value of $S(n) + S(S(n))$ is when $n = 1999$, and the sum becomes $28 + 10 = 38$. So the minimum bound is 1969. We do casework upon the tens digit:

Case 1: $196u \Longrightarrow u = 9$. Easy to directly disprove.

Case 2: $197u$. $S(n) = 1 + 9 + 7 + u = 17 + u$, and $S(S(n)) = 8 + u$ if $u \le 2$ and $S(S(n)) = 2 + (u-3) = u - 1$ otherwise.

> Subcase a: $T(n) = 1970 + u + 17 + u + 8 + u = 1995 + 3u = 2007 \Longrightarrow u = 4$. This exceeds our bounds, so no solution here.
>
> Subcase b: $T(n) = 1970 + u + 17 + u + u - 1 = 1986 + 3u = 2007 \Longrightarrow u = 7$. First solution.

Case 3: $198u$. $S(n) = 18 + u$, and $S(S(n)) = 9 + u$ if $u \le 1$ and $2 + (u-2) = u$ otherwise.

> Subcase a: $T(n) = 1980 + u + 18 + u + 9 + u = 2007 + 3u = 2007 \Longrightarrow u = 0$. Second solution.
>
> Subcase b: $T(n) = 1980 + u + 18 + u + u = 1998 + 3u = 2007 \Longrightarrow u = 3$. Third solution.

Case 4: $199u$. But $S(n) > 19$, and $n + S(n)$ clearly sum to > 2007.

Case 5: $200u$. So $S(n) = 2 + u$ and $S(S(n)) = 2 + u$ (recall that $n < 2007$), and $2000 + u + 2 + u + 2 + u = 2004 + 3u = 2007 \Longrightarrow u = 1$. Fourth solution.

In total we have $4 \mathrm{(D)}$ solutions, which are $1977, 1980, 1983,$ and 2001.

Solution 2

Clearly, $n > 1950$. We can break this into three cases:

Case 1: $n \geq 2000$

Inspection gives $n = 2001$.

Case 2: $n < 2000$, $n = 19xy$(not to be confused with $19 * x * y$), $x + y < 10$

If you set up an equation, it reduces to

$$4x + y = 32$$

which has as its only solution satisfying the constraints $x = 8,\ y = 0$.

Case 3: $n < 2000$, $n = 19xy$, $x + y \geq 10$

This reduces to

$4x + y = 35$. The only two solutions satisfying the constraints for this equation are $x = 7$, $y = 7$ and $x = 8,\ y = 3$.

The solutions are thus $1977, 1980, 1983, 2001$ and the answer is $\text{(D)}\ 4$.

Solution 3

As in Solution 1, we note that $S(n) \leq 28$ and $S(S(n)) \leq 10$.
Obviously, $n \equiv S(n) \equiv S(S(n)) \pmod 9$.
As $2007 \equiv 0 \pmod 9$, this means that $n \bmod 9 \in \{0, 3, 6\}$, or equivalently that $n \equiv S(n) \equiv S(S(n)) \equiv 0 \pmod 3$.

Thus $S(S(n)) \in \{3, 6, 9\}$. For each possible $S(S(n))$ we get three possible $S(n)$.
(E. g., if $S(S(n)) = 6$, then $S(n) = x$ is a number such that $x \leq 28$ and $S(x) = 6$, therefore $S(n) \in \{6, 15, 24\}$.)

For each of these nine possibilities we compute $n_?$ as $2007 - S(n) - S(S(n))$ and check whether $S(n_?) = S(n)$.
We'll find out that out of the 9 cases, in 4 the value $n_?$ has the correct sum of digits.
This happens for $n_? \in \{1977, 1980, 1983, 2001\}$.

Solution 4 (This is not allowed)

This is mainly for fun. We can create a python program to do this for us:

```
def calculateSumOfDigits(number):
    number  = str(number)
    digits = []
    for i in range(len(number)):
        digits.append(int(number[i]))
    return sum(digits)
for i in range(2007):
```

```
    if(i+calculateSumOfDigits(i)+calculateSumOfDigits(calculateSumOfDigits(i)) == 2007):
```

```
        print(i)
```

See also

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Category: Introductory Number Theory Problems

2007 AMC 12A Problems/Problem 23

Problem

Square $ABCD$ has area $36,$ and $\overline{AB}$ is parallel to the x-axis. Vertices $A,$ B, and C are on the graphs of $y = \log_a x,$ $y = 2\log_a x,$ and $y = 3\log_a x,$ respectively. What is $a?$

$\mathrm{(A)}\ \sqrt[6]{3}\qquad \mathrm{(B)}\ \sqrt{3}\qquad \mathrm{(C)}\ \sqrt[3]{6}\qquad \mathrm{(D)}\ \sqrt{6}\qquad \mathrm{(E)}\ 6$

Solution

Let x be the x-coordinate of B and C, and x_2 be the x-coordinate of A and y be the y-coordinate of A and B. Then $2\log_a x = y \Longrightarrow a^{y/2} = x$ and $\log_a x_2 = y \Longrightarrow x_2 = a^y = \left(a^{y/2}\right)^2 = x^2$. Since the distance between A and B is 6, we have $x^2 - x - 6 = 0$, yielding $x = -2, 3$.

However, we can discard the negative root (all three logarithmic equations are underneath the line $y = 3$ and above $y = 0$ when x is negative, hence we can't squeeze in a square of side 6). Thus $x = 3$.

Substituting back, $3\log_a x - 2\log_a x = 6 \Longrightarrow a^6 = x$, so $a = \sqrt[6]{3}\ \mathrm{(A)}$.

See also

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Categories: Introductory Algebra Problems | Introductory Geometry Problems

2007 AMC 12A Problems/Problem 24

Contents

Problem

For each integer $n > 1$, let $F(n)$ be the number of solutions to the equation $\sin x = \sin (nx)$ on the interval $[0, \pi]$. What is $\sum_{n=2}^{2007} F(n)$?

$\text{(A)}\ 2014524 \qquad \text{(B)}\ 2015028 \qquad \text{(C)}\ 2015033 \qquad \text{(D)}\ 2016532 \qquad \text{(E)}\ 2017033$

Solution

Solution 1

$F(2) = 3$

By looking at various graphs, we obtain that, for most of the graphs

$F(n) = n + 1$

However, when $n \equiv 1 \pmod 4$, the middle apex of the sine curve touches the sine curve at the top only one time (instead of two), so we get here $F(n) = n$.

$3 + 4 + 5 + 5 + 7 + 8 + 9 + 9 + \cdots + 2008$
$= (1 + 2 + 3 + 4 + 5 + \cdots + 2008) - 3 - 501 = \frac{(2008)(2009)}{2} - 504 = 2016532\ \text{(D)}$

Solution 2

$$\sin nx - \sin x = 2\left(\cos \frac{n+1}{2}x\right)\left(\sin \frac{n-1}{2}x\right)$$

So $\sin nx = \sin x$ if and only if $\cos \frac{n+1}{2}x = 0$ or $\sin \frac{n-1}{2}x = 0$.

The first occurs whenever $\frac{n+1}{2}x = (j+1/2)\pi$, or $x = \frac{(2j+1)\pi}{n+1}$ for some nonnegative integer j. Since $x \le \pi$, $j \le n/2$. So there are $1 + \lfloor n/2 \rfloor$ solutions in this case.

The second occurs whenever $\frac{n-1}{2}x = k\pi$, or $x = \frac{2k\pi}{n-1}$ for some nonnegative integer k. Here $k \le \frac{n-1}{2}$ so that there are $\left\lfloor \frac{n+1}{2} \right\rfloor$ solutions here.

However, we overcount intersections. These occur whenever

$$\frac{2j+1}{n+1} = \frac{2k}{n-1}$$

$$k = \frac{(2j+1)(n-1)}{2(n+1)}$$

which is equivalent to $2(n+1)$ dividing $(2j+1)(n-1)$. If n is even, then $(2j+1)(n-1)$ is odd, so this never happens. If $n \equiv 3 \pmod 4$, then there won't be intersections either, since a multiple of 8 can't divide a number which is not even a multiple of 4.

This leaves $n \equiv 1 \pmod 4$. In this case, the divisibility becomes $\frac{n+1}{2}$ dividing $(2j+1)\frac{n-1}{4}$. Since $\frac{n+1}{2}$ and $\frac{n-1}{4}$ are relatively prime (subtracting twice the second number from the first gives 1), $\frac{n+1}{2}$ must divide $2j+1$. Since $j \leq \frac{n-1}{2}$, $2j+1 \leq n < 2 \cdot \frac{n+1}{2}$. Then there is only one intersection, namely when $j = \frac{n-1}{4}$.

Therefore we find $F(n)$ is equal to $1 + \lfloor n/2 \rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor = n+1$, unless $n \equiv 1 \pmod 4$, in which case it is one less, or n. The problem may then be finished as in Solution 1.

See also

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Category: Introductory Trigonometry Problems

2007 AMC 12A Problems/Problem 25

Problem

Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1,2,3,\ldots,12\}$, including the empty set, are spacy?

$$\mathrm{(A)}\ 121 \qquad \mathrm{(B)}\ 123 \qquad \mathrm{(C)}\ 125 \qquad \mathrm{(D)}\ 127 \qquad \mathrm{(E)}\ 129$$

Contents

Solution

Solution 1

Let S_n denote the number of spacy subsets of $\{1,2,...n\}$. We have $S_0=1, S_1=2, S_2=3$.

The spacy subsets of S_{n+1} can be divided into two groups:

- $A =$ those not containing $n+1$. Clearly $|A|=S_n$.
- $B =$ those containing $n+1$. We have $|B|=S_{n-2}$, since removing $n+1$ from any set in B produces a spacy set with all elements at most equal to $n-2$, and each such spacy set can be constructed from exactly one spacy set in B.

Hence,

$$S_{n+1}=S_n+S_{n-2}$$

From this recursion, we find that

$S(0)$	$S(1)$	$S(2)$	$S(3)$	$S(4)$	$S(5)$	$S(6)$	$S(7)$	$S(8)$	$S(9)$	$S(10)$	$S(11)$	$S(12)$
1	2	3	4	6	9	13	19	28	41	60	88	129

Solution 2

Since each of the elements of the subsets must be spaced at least two apart, a divider counting argument can be used.

From the set $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ we choose at most four numbers. Let those numbers be represented by balls. Between each of the balls there are at least two dividers. So for example, o | | o | | o | | o | | represents $1,4,7,10$.

For subsets of size k there must be $2(k-1)$ dividers between the balls, leaving $12-k-2(k-1)=12-3k+2$ dividers to be be placed in $k+1$ spots between the balls. The number of way this can be done is $\binom{(12-3k+2)+(k+1)-1}{k}=\binom{12-2k+2}{k}$.

Therefore, the number of spacy subsets is $\binom{6}{4}+\binom{8}{3}+\binom{10}{2}+\binom{12}{1}+\binom{14}{0}=129$.

Solution 3

A shifting argument is also possible, and is similiar in spirit to Solution 2. Clearly we can have at most 4 elements. Given any arrangment, we subract $2i-2$ from the $i-th$ element in our subset, when the elements are arranged in increasing order. This creates a bijection with the number of size k subsets of the set of the first $14-2k$ positive integers. For instance, the arrangment o | | o | | o | | | o | corresponds to the arrangment o o o | o |. Notice that there is no longer any restriction on consectutive numbers. Therefore, we can easily plug in the possible integers 0, 1, 2, 3, 4, 5 for k:

$$\binom{14}{0}+\binom{12}{1}+\binom{10}{2}+\binom{8}{3}+\binom{6}{4}=\boxed{129}$$

In general, the number of subsets of a set with n element and with no k consecutive numbers is $\sum_{i=0}^{\lfloor\frac{n}{k}\rfloor}\binom{n-(k-1)(i-1)}{i}$.

See also

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