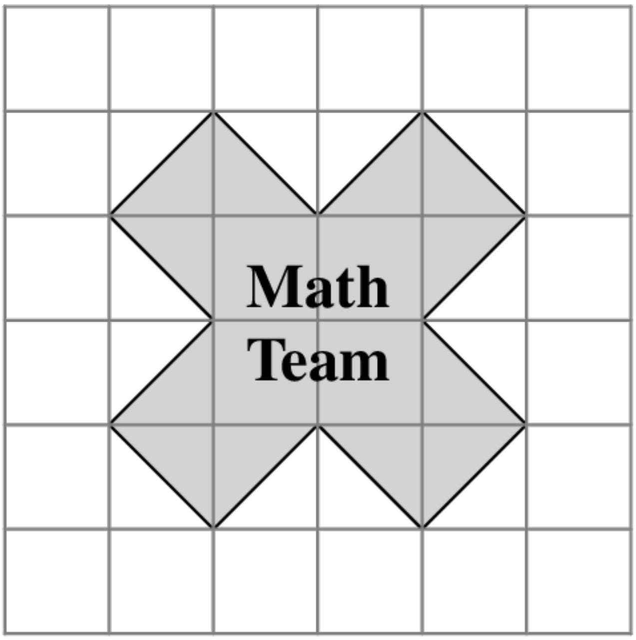
2022 AMC 8 Solutions

Problem1

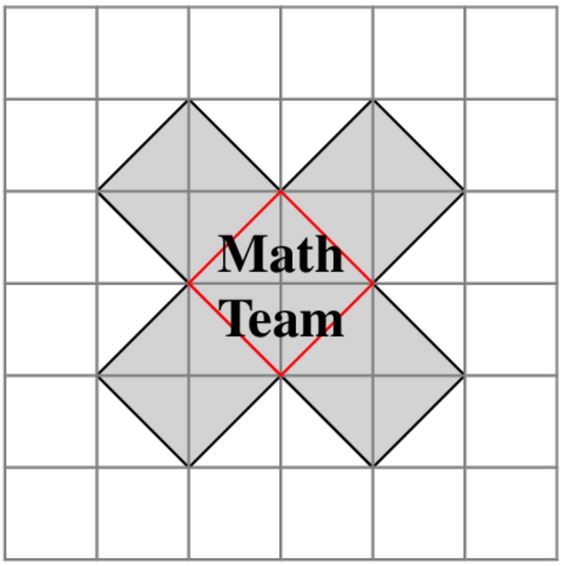
The Math Team designed a logo shaped like a multiplication symbol, shown below on a grid of 1-inch squares. What is the area of the logo in square inches?



$\textbf{(A) } 10 \qquad \textbf{(B) } 12 \qquad \textbf{(C) } 13 \qquad \textbf{(D) } 14 \qquad \textbf{(E) } 15$

Solution 1

Draw the following four lines as shown:



We see these lines split the figure into five squares with side length $\sqrt{2}$. Thus, the area is $5 \cdot \left(\sqrt{2}\right)^2 = 5 \cdot 2 = \boxed{\textbf{(A) } 10}$.

Solution 2

We can apply Pick's Theorem: There are 5 lattice points in the interior and 12 lattice points on the boundary of the figure. As a result, the area is $5 + \frac{12}{2} - 1 = \boxed{\textbf{(A) } 10}$.

Solution 3

Notice that the area of the figure is equal to the area of the 4×4 square subtracted by the 12 triangles that are half the area of each square, which is 1. The total area of the triangles not in the figure is $12 \cdot \frac{1}{2} = 6$, so the answer is $16 - 6 = \boxed{\textbf{(A) } 10}$.

Problem2

Consider these two operations:$$a \blacklozenge b = a^2 - b^2$$ $$a \bigstar b = (a-b)^2$$ What is the value of $(5 \blacklozenge 3) \bigstar 6?$

$\textbf{(A) } -20 \qquad \textbf{(B) } 4 \qquad \textbf{(C) } 16 \qquad \textbf{(D) } 100 \qquad \textbf{(E) } 220$

Solution

We have
$$\begin{aligned}(5 \blacklozenge 3) \bigstar 6 &= \left(5^2 - 3^2\right) \bigstar 6 \\ &= 16 \bigstar 6 \\ &= (16-6)^2 \\ &= \boxed{\textbf{(D) } 100}.\end{aligned}$$

Problem3

When three positive integers a, b, and c are multiplied together, their product is 100. Suppose $a < b < c$. In how many ways can the numbers be chosen?

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 3 \qquad \textbf{(E) } 4$

Solution

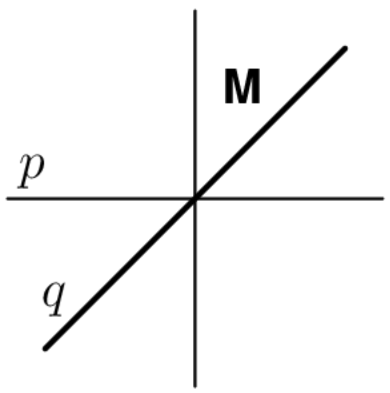
The positive divisors of 100 are $1, 2, 4, 5, 10, 20, 25, 50, 100$. It is clear that $10 \le c \le 50,$ so we apply casework to c :

- If $c = 10,$ then $(a, b, c) = (2, 5, 10).$
- If $c = 20,$ then $(a, b, c) = (1, 5, 20).$
- If $c = 25,$ then $(a, b, c) = (1, 4, 25).$
- If $c = 50,$ then $(a, b, c) = (1, 2, 50).$

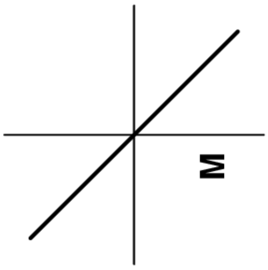
Together, the numbers $a, b,$ and c can be chosen in $\boxed{\textbf{(E) } 4}$ ways

Problem4

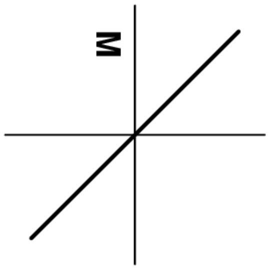
The letter **M** in the figure below is first reflected over the line q and then reflected over the line p. What is the resulting image?



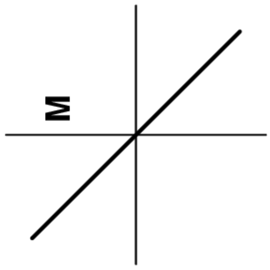
(A)



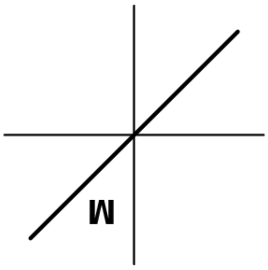
(B)



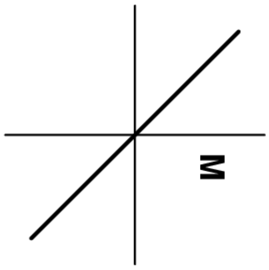
(C)



(D)

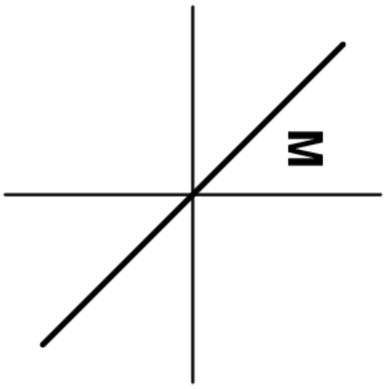


(E)

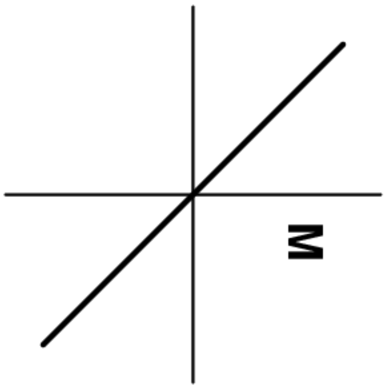


Solution

When **M** is first reflected over the line q, we obtain the following diagram:



When **M** is then reflected over the line p, we obtain the following diagram:



Therefore, the answer is $\boxed{\textbf{(E)}}$.

Problem5

Anna and Bella are celebrating their birthdays together. Five years ago, when Bella turned 6 years old, she received a newborn kitten as a birthday present. Today the sum of the ages of the two children and the kitten is 30 years. How many years older than Bella is Anna?

$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 4 \qquad \textbf{(E) } 5$

Solution

Five years ago, Bella was 6 years old, and the kitten was 0 years old.

Today, Bella is 11 years old, and the kitten is 5 years old. It follows that Anna is $30-11-5=14$ years old.

Therefore, Anna is $14-11=\boxed{\textbf{(C) } 3}$ years older than Bella.

Problem6

Three positive integers are equally spaced on a number line. The middle number is $15,$ and the largest number is 4 times the smallest number. What is the smallest of these three numbers?

$\textbf{(A) } 4 \qquad \textbf{(B) } 5 \qquad \textbf{(C) } 6 \qquad \textbf{(D) } 7 \qquad \textbf{(E) } 8$

Solution

Let the smallest number be $x.$ It follows that the largest number is $4x.$

Since $x,15,$ and $4x$ are equally spaced on a number line, we have
$$\begin{aligned} 4x-15 &= 15-x \\ 5x &= 30 \\ x &= \boxed{\textbf{(C) } 6}. \end{aligned}$$

Problem7

When the World Wide Web first became popular in the 1990s, download speeds reached a maximum of about 56 kilobits per second. Approximately how many minutes would the download of a 4.2-megabyte song have taken at that speed? (Note that there are 8000 kilobits in a megabyte.)

$\textbf{(A) } 0.6 \qquad \textbf{(B) } 10 \qquad \textbf{(C) } 1800 \qquad \textbf{(D) } 7200 \qquad \textbf{(E) } 36000$

Solution

Notice that the number of kilobits in this song is $4.2 \cdot 8000 = 8 \cdot 7 \cdot 6 \cdot 100.$

We must divide this by 56 in order to find out how many seconds this song would take to download: $\frac{\cancel{8} \cdot \cancel{7} \cdot 6 \cdot 100}{\cancel{56}} = 600.$

Finally, we divide this number by 60 because this is the number of ***seconds*** to get the answer $\frac{600}{60} = \boxed{\textbf{(B) } 10}.$

Problem8

What is the value of $\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdots \frac{18}{20} \cdot \frac{19}{21} \cdot \frac{20}{22}?$

$\textbf{(A) } \frac{1}{462} \qquad \textbf{(B) } \frac{1}{231} \qquad \textbf{(C) } \frac{1}{132} \qquad \textbf{(D) } \frac{2}{213} \qquad \textbf{(E) } \frac{1}{22}$

Solution 1

Note that common factors (from 3 to $20,$ inclusive) of the numerator and the denominator cancel. Therefore, the original expression becomes

$$\frac{1}{\cancel{3}} \cdot \frac{2}{\cancel{4}} \cdot \frac{\cancel{3}}{\cancel{5}} \cdots \frac{\cancel{18}}{\cancel{20}} \cdot \frac{\cancel{19}}{21} \cdot \frac{\cancel{20}}{22} = \frac{1 \cdot 2}{21 \cdot 22} = \frac{1}{21 \cdot 11} = \boxed{\textbf{(B) } \frac{1}{231}}.$$

Solution 2

The original expression becomes

$$\frac{20!}{22!/2!}=\frac{20!\cdot 2!}{22!}=\frac{20!\cdot 2}{20!\cdot 21\cdot 22}=\frac{2}{21\cdot 22}=\frac{1}{21\cdot 11}=\boxed{\textbf{(B)}\ \frac{1}{231}}.$$

Problem9

A cup of boiling water (212°F) is placed to cool in a room whose temperature remains constant at 68°F. Suppose the difference between the water temperature and the room temperature is halved every 5 minutes. What is the water temperature, in degrees Fahrenheit, after 15 minutes?

$\textbf{(A)}\ 77 \qquad \textbf{(B)}\ 86 \qquad \textbf{(C)}\ 92 \qquad \textbf{(D)}\ 98 \qquad \textbf{(E)}\ 104$

Solution

Initially, the difference between the water temperature and the room temperature is $212-68=144$ degrees Fahrenheit.

After 5 minutes, the difference between the temperatures is $144\div2=72$ degrees Fahrenheit.

After 10 minutes, the difference between the temperatures is $72\div2=36$ degrees Fahrenheit.

After 15 minutes, the difference between the temperatures is $36\div2=18$ degrees Fahrenheit. At this point, the water temperature is $68+18=\boxed{\textbf{(B)}\ 86}$ degrees Fahrenheit.

Remark

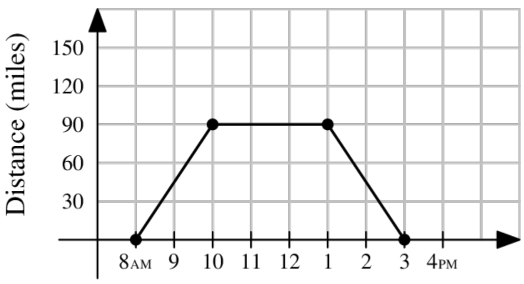
Alternatively, we can condense the solution above into the following equation:

$$68 + (212 - 68) \cdot \left(\frac{1}{2}\right)^{\frac{15}{5}} = 86.$$

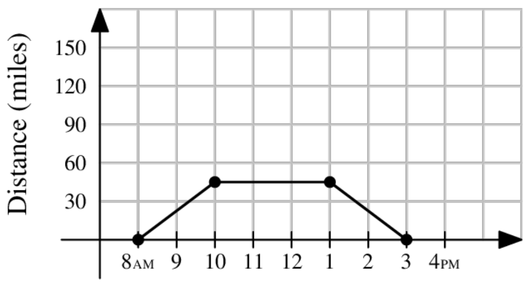
Problem10

One sunny day, Ling decided to take a hike in the mountains. She left her house at 8 AM, drove at a constant speed of 45 miles per hour, and arrived at the hiking trail at 10 AM. After hiking for 3 hours, Ling drove home at a constant speed of 60 miles per hour. Which of the following graphs best illustrates the distance between Ling's car and her house over the course of her trip?

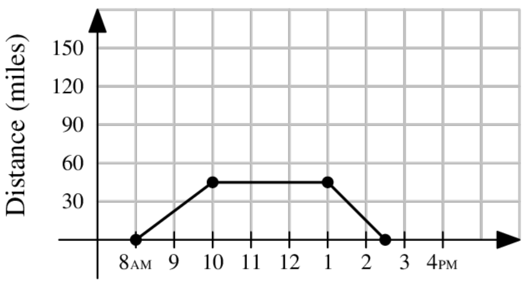
(A)



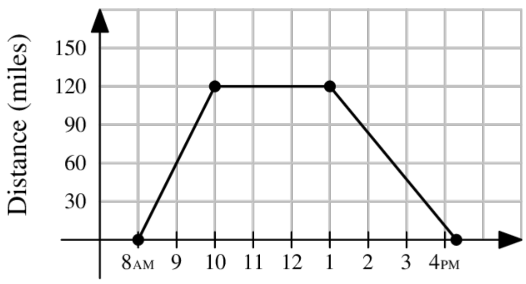
(B)



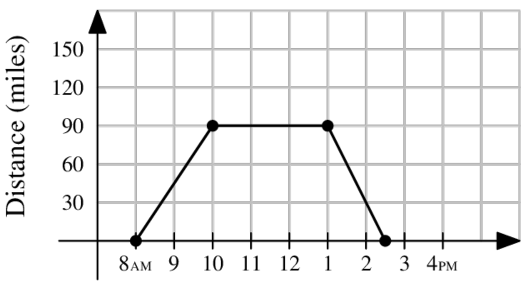
(C)



(D)



(E)



Solution 1 (Analysis)

Note that:

- **At $8\text{ AM},$ Ling's car was 0 miles from her house.**
- From 8 AM to $10\text{ AM},$ Ling drove to the hiking trail at a constant speed of 45 miles per hour.

  It follows that at $10\text{ AM},$ Ling's car was $45\cdot2=90$ miles from her house.
- From 10 AM to $1\text{ PM},$ Ling did not move her car.

  It follows that at $1\text{ PM},$ Ling's car was still 90 miles from her house.
- From $1\text{ PM},$ Ling drove home at a constant speed of 60 miles per hour. So, she arrived home $90\div60=1.5$ hour later.

  It follows that at $2:30\text{ PM},$ Ling's car was 0 miles from her house.

Therefore, the answer is $\boxed{\textbf{(E)}}$.

Solution 2 (Elimination)

Ling's trip took 2 hours, thus she traveled for $2\times45=90$ miles.

Choices $\textbf{(B)}$, $\textbf{(C)}$, and $\textbf{(D)}$ are eliminated. Ling drove 45 miles per hour (mph) to the mountains, and 60 mph back to her house, so the rightmost slope must be steeper than the leftmost one. Choice $\textbf{(A)}$ is eliminated. This leaves us with $\boxed{\textbf{(E)}}$.

Problem11

Henry the donkey has a very long piece of pasta. He takes a number of bites of pasta, each time eating 3inches of pasta from the middle of one piece. In the end, he has 10 pieces of pasta whose total length is 17 inches. How long, in inches, was the piece of pasta he started with?

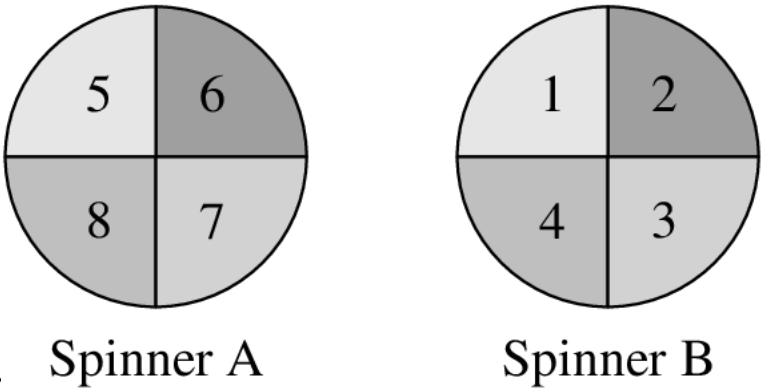
$\textbf{(A) } 34 \qquad \textbf{(B) } 38 \qquad \textbf{(C) } 41 \qquad \textbf{(D) } 44 \qquad \textbf{(E) } 47$

Solution

If there are 10 pieces of pasta, Henry took $10-1=9$ bites. Each of these 9 bites took 3 inches of pasta out, and thus his bites in total took away $9 \cdot 3 = 27$ inches of pasta. Thus, the original piece of pasta was $27+17=\boxed{\textbf{(D) } 44}$ inches long.

Problem12

The arrows on the two spinners shown below are spun. Let the number N equal 10 times the number on Spinner A, added to the number on Spinner B. What is the probability that N is a perfect square



number?

$\textbf{(A) } \frac{1}{16} \qquad \textbf{(B) } \frac{1}{8} \qquad \textbf{(C) } \frac{1}{4} \qquad \textbf{(D) } \frac{3}{8} \qquad \textbf{(E) } \frac{1}{2}$

Solution

First, we realize that there are a total of 16 possibilities. Now, we list all of them that can be spun. This includes 64 and 81. Then, our answer is $\frac{2}{16} = \boxed{\textbf{(B) } \frac{1}{8}}$.

Problem13

How many positive integers can fill the blank in the sentence below?

“One positive integer is _____ more than twice another, and the sum of the two numbers is 28.”

$\textbf{(A) } 6 \qquad \textbf{(B) } 7 \qquad \textbf{(C) } 8 \qquad \textbf{(D) } 9 \qquad \textbf{(E) } 10$

Solution

Let m and n be positive integers such that $m > n$ and $m + n = 28.$ It follows that $m = 2n + d$ for some positive integer $d.$ We wish to find the number of possible values for $d.$

By substitution, we have $(2n + d) + n = 28,$ from which $d = 28 - 3n.$ Note that $n = 1, 2, 3, \ldots, 9$ each generate a positive integer for $d,$ so there are $\boxed{\textbf{(D) } 9}$ possible values for $d.$

Problem14

In how many ways can the letters in $\textbf{BEEKEEPER}$ be rearranged so that two or more $\textbf{E}$s do not appear together?

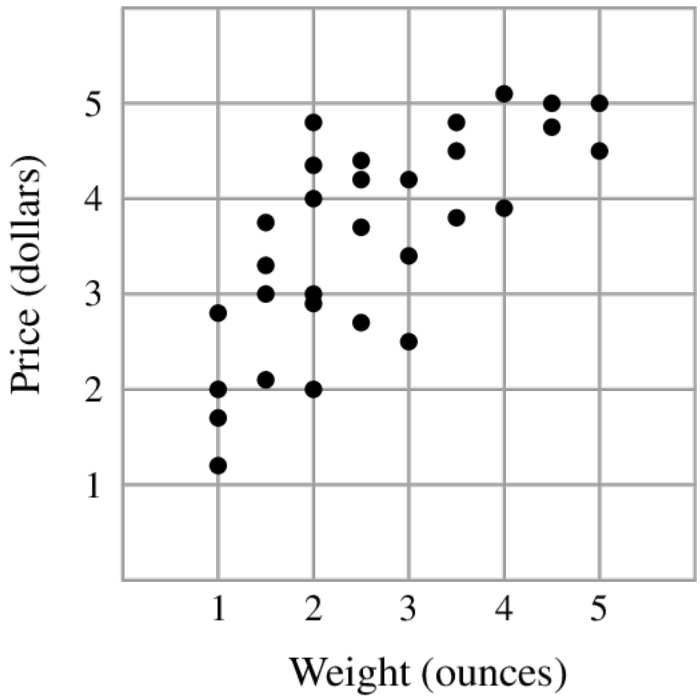
$\textbf{(A) } 1 \qquad \textbf{(B) } 4 \qquad \textbf{(C) } 12 \qquad \textbf{(D) } 24 \qquad \textbf{(E) } 120$

Solution

All valid arrangements of the letters must be of the form $\mathbf{E\_\_E\_\_E\_\_E\_\_E}$. The problem is equivalent to counting the arrangements of $\mathbf{B}, \mathbf{K}, \mathbf{P}$, and $\mathbf{R}$ into the four blanks, in which there are $4! = \boxed{\textbf{(D) } 24}$ ways.

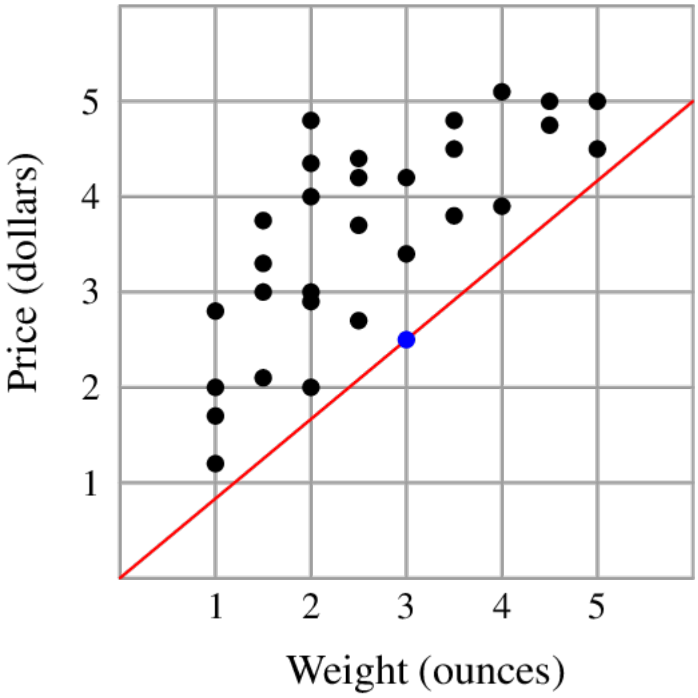
Problem15

Laszlo went online to shop for black pepper and found thirty different black pepper options varying in weight and price, shown in the scatter plot below. In ounces, what is the weight of the pepper that offers the lowest price per ounce?



$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 4 \qquad \textbf{(E) } 5$

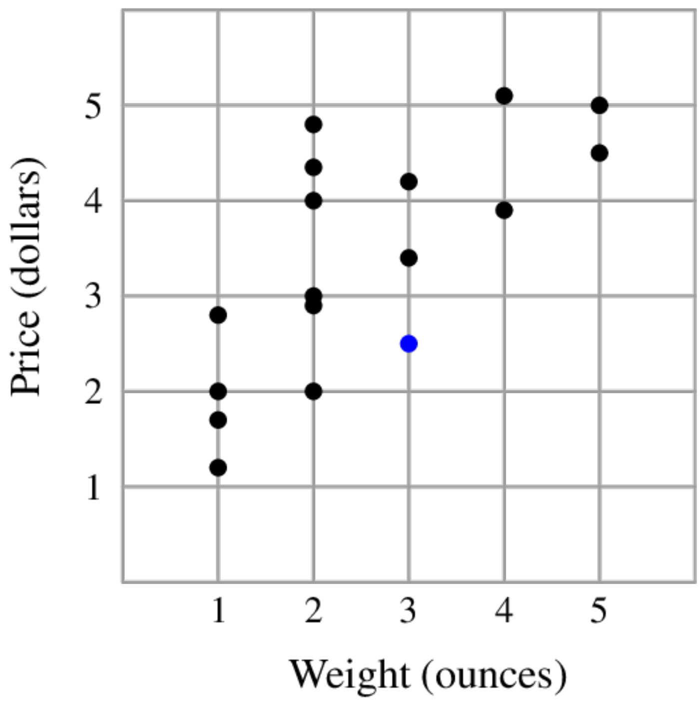
Solution



We are looking for a black point, that when connected to the origin, yields the lowest slope. The slope represents the price per ounce. It is clearly the blue point, which can be found visually. Also, it is the only one with a price per once significantly less than 1. Finally, we see that the blue point is over the category with a weight of $\boxed{\textbf{(C) } 3}$ ounces.

Solution 2 (Elimination)

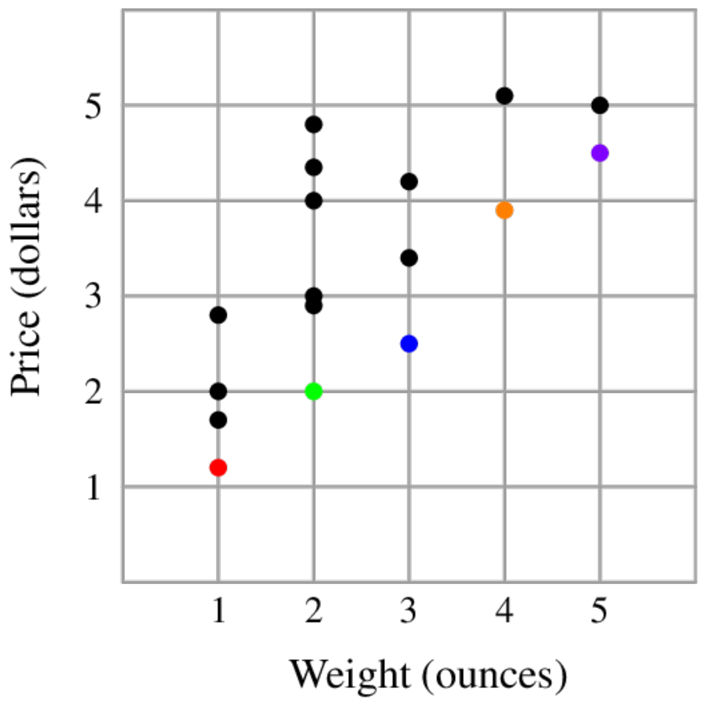
By the answer choices, we can disregard the points that do not have integer weights. As a result, we obtain the following diagram:



We then proceed in the same way that we had done in Solution 1. Following the steps, we figure out the blue dot that yields the lowest slope, along with passing the origin. We then can look at the x-axis(in this situation, the weight) and figure out it has $\boxed{\textbf{(C) } 3}$ ounces.

Solution 3 (Elimination)

We can find the lowest point in each line (1, 2, 3, 4, or 5) and find the price per pound. (Note that we don't need to find the points higher than the points below since we are finding the lowest price per pound.)



The red dot has a price per pound of something that is larger than 1. The green dot has a price per pound of 1. The blue dot has a price per pound of something like $\frac{2.5}{3}$. The orange dot has a price per pound that is less than 1, but close to it. The purple dot has a price per pound of something like $\frac{4.5}{5}$. We see that choices $\textbf{(A)}$, $\textbf{(B)}$,and $\textbf{(D)}$ are eliminated. Also, $\frac{4.5}{5} > \frac{2.5}{3}$ thus the answer is $\boxed{\textbf{(C) }3}$.

Problem16

Four numbers are written in a row. The average of the first two is 21, the average of the middle two is 26,and the average of the last two is 30. What is the average of the first and last of the numbers?

$\textbf{(A) }24 \qquad \textbf{(B) }25 \qquad \textbf{(C) }26 \qquad \textbf{(D) }27 \qquad \textbf{(E) }28$

Solution 1 (Arithmetic)

Note that the sum of the first two numbers is $21\cdot2=42,$ the sum of the middle two numbers is $26\cdot2=52,$ and the sum of the last two numbers is $30\cdot2=60.$

It follows that the sum of the four numbers is $42+60=102,$ so the sum of the first and last numbers is $102-52=50.$ Therefore, the average of the first and last numbers is $50\div2=\boxed{\textbf{(B) } 25}.$

Solution 2 (Algebra)

Let $a,b,c,$ and d be the four numbers in that order. We are given that

$$\begin{align*} \frac{a+b}{2}&=21, &(1) \\ \frac{b+c}{2}&=26, &(2) \\ \frac{c+d}{2}&=30, &(3) \end{align*}$$

and we wish to find $\frac{a+d}{2}.$

We add (1) and $(3),$ then subtract (2) from the result: $\frac{a+d}{2}=21+30-26=\boxed{\textbf{(B) } 25}.$ ~MRENTHUSIASM

Solution 3 (Assumption)

We can just assume some of the numbers. For example, let the first two numbers both be $21.$ It follows that the third number is $31,$ and the fourth number is 29. Therefore, the average of the first and last numbers is $\dfrac{21+29}{2}=\dfrac{50}{2}=\boxed{\textbf{(B) } 25}$.

We can check this with other sequences, such as $20, 22, 30, 30,$ where the average of the first and last numbers is still 25.

Problem17

If n is an even positive integer, the *double factorial* notation $n!!$ represents the product of all the even integers from 2 to n. For example, $8!!=2\cdot4\cdot6\cdot8$. What is the units digit of the following sum? $2!!+4!!+6!!+\cdots+2018!!+2020!!+2022!!$

$\textbf{(A) } 0 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 4 \qquad \textbf{(D) } 6 \qquad \textbf{(E) } 8$

Solution

Notice that once $n>8,$ the units digit of $n!!$ will be 0 because there will be a factor of 10. Thus, we only need to calculate the units digit of $2!!+4!!+6!!+8!!=2+8+48+48\cdot8$. We only care about units digits, so we have $2+8+8+8\cdot8,$ which has the same units digit as $2+8+8+4$. The answer is $\boxed{\textbf{(B) } 2}$.

Problem18

The midpoints of the four sides of a rectangle are $(-3,0), (2,0), (5,4),$ and $(0,4)$. What is the area of the rectangle?

$\textbf{(A) } 20 \qquad \textbf{(B) } 25 \qquad \textbf{(C) } 40 \qquad \textbf{(D) } 50 \qquad \textbf{(E) } 80$

Solution 1

The midpoints of the four sides of every rectangle are the vertices of a rhombus whose area is half the area of the rectangle.

Let $A = (-3,0), B = (2,0), C = (5,4),$ and $D = (0,4).$

Note that $A, B, C,$ and D are the vertices of a rhombus whose diagonals have lengths $AC = 4\sqrt{5}$ and $BD = 2\sqrt{5}.$ It follows that the area of rhombus $ABCD$ is $\frac{4\sqrt{5} \cdot 2\sqrt{5}}{2} = 20,$ so the area of the rectangle is $20 \cdot 2 = \boxed{\textbf{(C) } 40}.$

Solution 2

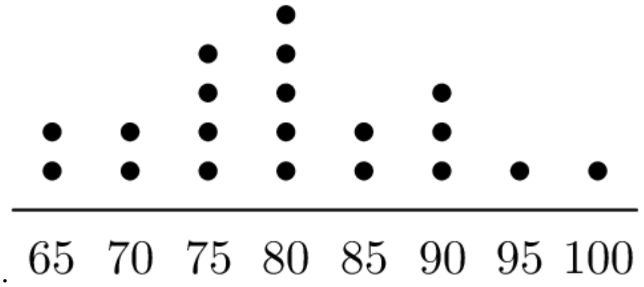
If a rectangle has area $K,$ then the area of the quadrilateral formed by its midpoints is $\frac{K}{2}.$

Define points $A, B, C,$ and D as Solution 1 does.

Since $A, B, C,$ and D are the midpoints of the rectangle, the rectangle's area is $2[ABCD].$ Now, note that $ABCD$ is a parallelogram since $AB = CD$ and $\overline{AB} \parallel \overline{CD}.$ As the parallelogram's height from D to $\overline{AB}$ is 4 and $AB = 5,$ its area is $4 \cdot 5 = 20.$ Therefore, the area of the rectangle is $20 \cdot 2 = \boxed{\textbf{(C) } 40}.$

Problem19

Mr. Ramos gave a test to his class of 20 students. The dot plot below shows the distribution of test scores.



Later Mr. Ramos discovered that there was a scoring error on one of the questions. He regraded the tests, awarding some of the students 5 extra points, which increased the median test score to 85. What is the minimum number of students who received extra points?

(Note that the *median* test score equals the average of the 2 scores in the middle if the 20 test scores are arranged in increasing order.)

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 6$

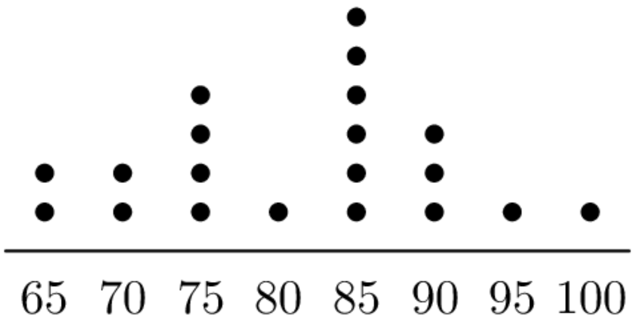
Solution

Before Mr. Ramos added scores, the median was $\frac{80+80}{2}=80$. There are two cases now:

Case #1: The middle two scores are 80 and 90. To do this, we firstly suppose that the two students who got 85 are awarded the extra 5 points. We then realize that this case will have a lot of students who receive the extra points, therefore we reject this case.

Case #2: The middle two scores are both 85. To do this, we simply need to suppose that some of the students who got 80 are awarded the extra 5 points. Note that there are 8 students who got 75 or less. Therefore there must be only 1 student who got 80 so that the middle two scores are both 85. Therefore our answer is $\boxed{\textbf{(C)}\ 4}$.

A diagram to visualize better what I explained here:



Problem20

The grid below is to be filled with integers in such a way that the sum of the numbers in each row and the sum of the numbers in each column are the same. Four numbers are missing. The number x in the lower left corner is larger than the other three missing numbers. What is the smallest possible value of x?

-2	9	5
		-1
x		8

$\textbf{(A) } -1 \qquad \textbf{(B) } 5 \qquad \textbf{(C) } 6 \qquad \textbf{(D) } 8 \qquad \textbf{(E) } 9$

Solution 1

The sum of the numbers in each row is 12. Consider the second row. In order for the sum of the numbers in this row to equal 12, the two shaded numbers must add up to 13:

-2	9	5
		-1
x		8

If two numbers add up to 13, one of them must be at least 7: If both shaded numbers are no more than 6, their sum can be at most 12. Therefore, for x to be larger than the three missing numbers, x must be at least 8. We can construct a working scenario where $x = 8$:

-2	9	5
6	7	-1
8	-4	8

So, our answer is $\boxed{\textbf{(D) } 8}$.

Solution 2

The sum of the numbers in each row is $-2+9+5=12,$ and the sum of the numbers in each column is $5+(-1)+8=12.$

Let y be the number in the lower middle. It follows that $x+y+8=12,$ or $x+y=4.$

We express the other two missing numbers in terms of x and $y,$ as shown below:

-2	9	5
$y+10$	$x-1$	-1
x	y	8

We have $x>x-1, x>y+10,$ and $x>y.$ Note that the first inequality is true for all values of $x.$ We only need to solve the second inequality so that the third inequality is true for all values of $x.$ By substitution, we get $x>(4-x)+10,$ from which $x>7.$

Therefore, the smallest possible value of x is $\boxed{\textbf{(D) } 8}$.

Problem21

Steph scored 15 baskets out of 20 attempts in the first half of a game, and 10 baskets out of 10 attempts in the second half. Candace took 12 attempts in the first half and 18 attempts in the second. In each half, Steph scored a higher percentage of baskets than Candace. Surprisingly they ended with the same overall percentage of baskets scored. How many more baskets did Candace score in the second half than in the first?

	First Half	Second Half
Steph	$\frac{15}{20}$	$\frac{10}{10}$
Candace	$\frac{\square}{12}$	$\frac{\square}{18}$

$\textbf{(A) } 7 \qquad \textbf{(B) } 8 \qquad \textbf{(C) } 9 \qquad \textbf{(D) } 10 \qquad \textbf{(E) } 11$

Solution

Let x be the number of shots that Candace made in the first half, and let y be the number of shots Candace made in the second half. Since Candace and Steph took the same number of attempts, with an equal percentage of baskets scored, we have $x+y=10+15=25.$ In addition, we have the following inequalities: $\frac{x}{12}<\frac{15}{20} \implies x<9,$ and $\frac{y}{18}<\frac{10}{10} \implies y<18.$ Pairing this up with $x+y=25$ we see the ***only*** possible solution is $(x,y)=(8,17),$ for an answer of $17-8=\boxed{\textbf{(C) } 9}$.

Problem22

A bus takes 2 minutes to drive from one stop to the next, and waits 1 minute at each stop to let passengers board. Zia takes 5 minutes to walk from one bus stop to the next. As Zia reaches a bus stop, if the bus is at the previous stop or has already left the previous stop, then she will wait for the bus. Otherwise she will start walking toward the next stop. Suppose the bus and Zia start at the same time toward the library, with the bus 3 stops behind. After how many minutes will Zia board the bus?

$\textbf{(A)}\ 17 \qquad \textbf{(B)}\ 19 \qquad \textbf{(C)}\ 20 \qquad \textbf{(D)}\ 21 \qquad \textbf{(E)}\ 23$

Solution

Initially, suppose that the bus is at Stop 0 (starting point) and Zia is at Stop $3.$

We construct the following table of 5-minute intervals:

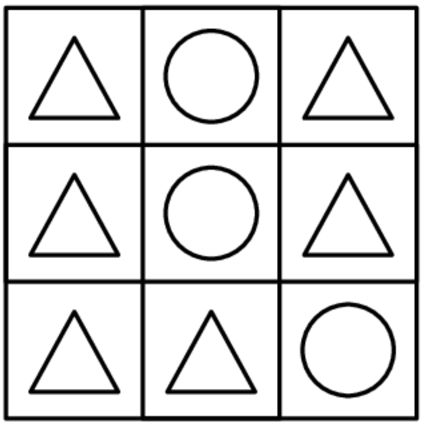
Time	Bus's Location	Zia's Location
5 Minutes	Stop 2 (Waiting)	Stop 4
10 Minutes	Midpoint of Stops 3 and 4	Stop 5
15 Minutes	Stop 5 (Leaving)	Stop 6

Note that Zia will wait for the bus after 15 minutes, and the bus will arrive 2 minutes later.

Therefore, the answer is $15+2=\boxed{\textbf{(A)}\ 17}.$

Problem23

A $\triangle$ or $\bigcirc$ is placed in each of the nine squares in a 3-by-3 grid. Shown below is a sample configuration with three $\triangle$s in a line.



How many configurations will have three $\triangle$s in a line and three $\bigcirc$s in a line?

$\textbf{(A) } 39 \qquad \textbf{(B) } 42 \qquad \textbf{(C) } 78 \qquad \textbf{(D) } 84 \qquad \textbf{(E) } 96$

Solution 1

Notice that diagonals and a vertical-horizontal pair can never work, so the only possibilities are if all lines are vertical or if all lines are horizontal. These are essentially the same, so we'll count up how many work with all lines of shapes vertical, and then multiply by 2 at the end.

We take casework:

Case 1: 3 lines In this case, the lines would need to be 2 of one shape and 1 of another, so there are $\frac{3!}{2}=3$ ways to arrange the lines and 2 ways to pick which shape has only one line. In total, this is $3\cdot2=6.$

Case 2: 2 lines In this case, the lines would be one line of triangles, one line of circles, and the last one can be anything that includes both shapes. There are $3!=6$ ways to arrange the lines and $2^3-2=6$ ways to choose the last line. In total, this is $6\cdot6=36.$

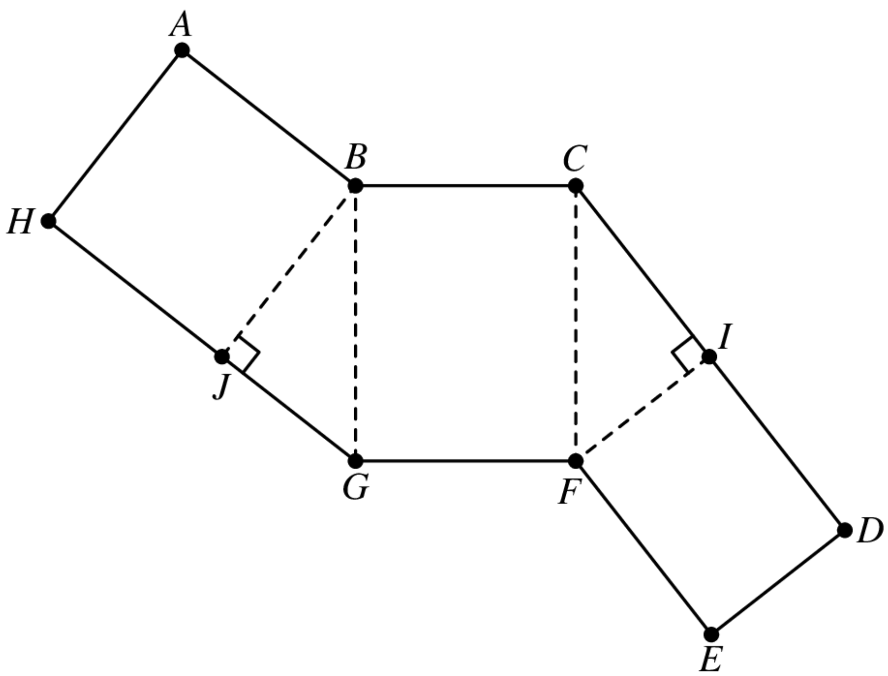
Finally, we add and multiply: $2(36+6)=2(42)=\boxed{\textbf{(D) } 84}$.

Solution 2

We will only consider columns, but at the end our answer should be multiplied by 2. There are 3 ways to choose a column for $\bigcirc$ and 2 ways to choose a column for $\triangle$. The third column can be filled in $2^3 = 8$ways. Therefore, we have $3 \cdot 2 \cdot 8 = 48$ ways. However, we overcounted the cases with 2 complete columns of with one symbol and 1 complete columns with another symbol. This happens in $2 \cdot 3 = 6$ways. $48 - 6 = 42$. However, we have to remember to double our answer giving us $\boxed{\textbf{(D) } 84}$.

Problem24

The figure below shows a polygon $ABCDEFGH$, consisting of rectangles and right triangles. When cut out and folded on the dotted lines, the polygon forms a triangular prism. Suppose that $AH = EF = 8$ and $GH = 14$. What is the volume of the prism?



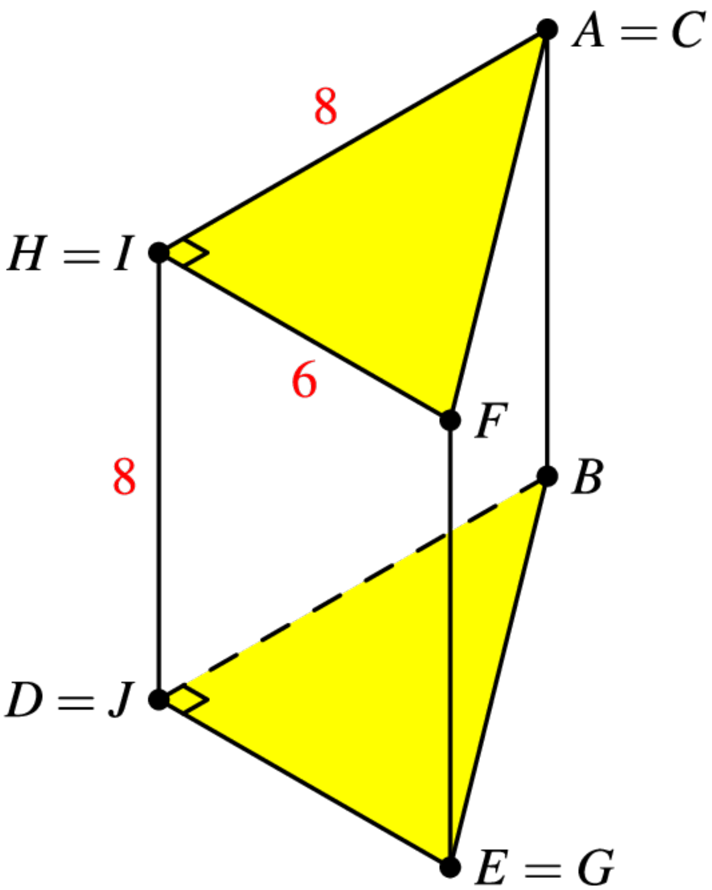
$\textbf{(A) } 112 \qquad \textbf{(B) } 128 \qquad \textbf{(C) } 192 \qquad \textbf{(D) } 240 \qquad \textbf{(E) } 288$

Solution

While imagining the folding, $\overline{AB}$ goes on $\overline{BC}$, $\overline{AH}$ goes on $\overline{CI}$, and $\overline{EF}$ goes on $\overline{FG}$. So, $BJ = CI = 8$ and $FG = BC = 8$. Also, $\overline{HJ}$ becomes an edge parallel to $\overline{FG}$, so that means $HJ = 8$.

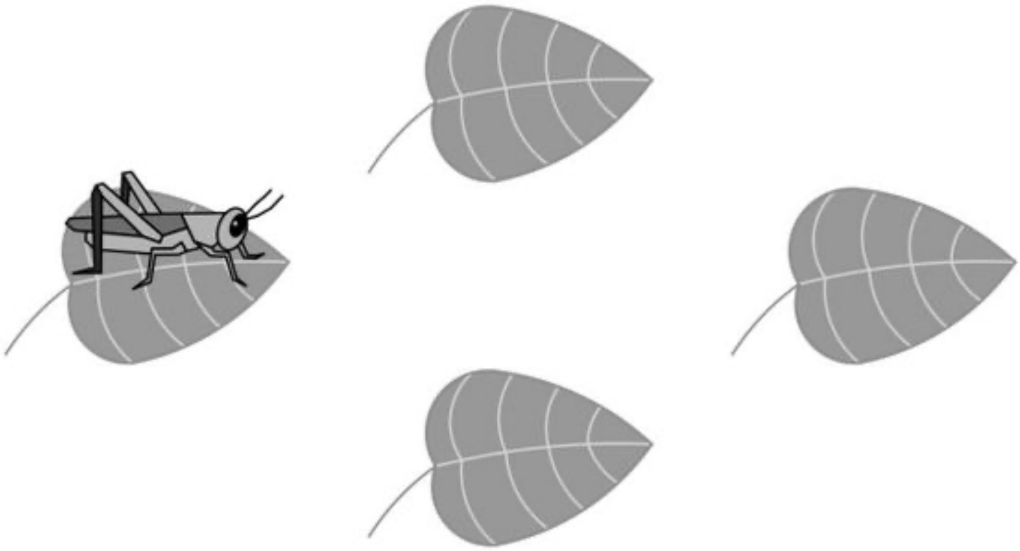
Since $GH = 14$, then $JG = 14 - 8 = 6$. So, the area of $\triangle BJG$ is $\dfrac{8 \cdot 6}{2} = 24$. If we let $\triangle BJG$ be the base, then the height is $FG = 8$. So, the volume is $24 \cdot 8 = \boxed{\textbf{(C) } 192}$.

After folding polygon $ABCDEFGH$ on the dotted lines, we obtain the following triangular prism:



Problem25

A cricket randomly hops between 4 leaves, on each turn hopping to one of the other 3 leaves with equal probability. After 4 hops what is the probability that the cricket has returned to the leaf where it started?



$\textbf{(A)}\ \frac{2}{9} \qquad \textbf{(B)}\ \frac{19}{80} \qquad \textbf{(C)}\ \frac{20}{81} \qquad \textbf{(D)}\ \frac{1}{4} \qquad \textbf{(E)}\ \frac{7}{27}$

Solution 1 (Casework)

Let A denote the leaf where the cricket starts and B denote one of the other 3 leaves. Note that:

- If the cricket is at $A,$ then the probability that it hops to B next is $1.$
- If the cricket is at $B,$ then the probability that it hops to A next is $\frac{1}{3}.$
- If the cricket is at $B,$ then the probability that it hops to B next is $\frac{2}{3}.$

We apply casework to the possible paths of the cricket:

1. $A \rightarrow B \rightarrow A \rightarrow B \rightarrow A$

The probability for this case is $1 \cdot \frac{1}{3} \cdot 1 \cdot \frac{1}{3} = \frac{1}{9}$.

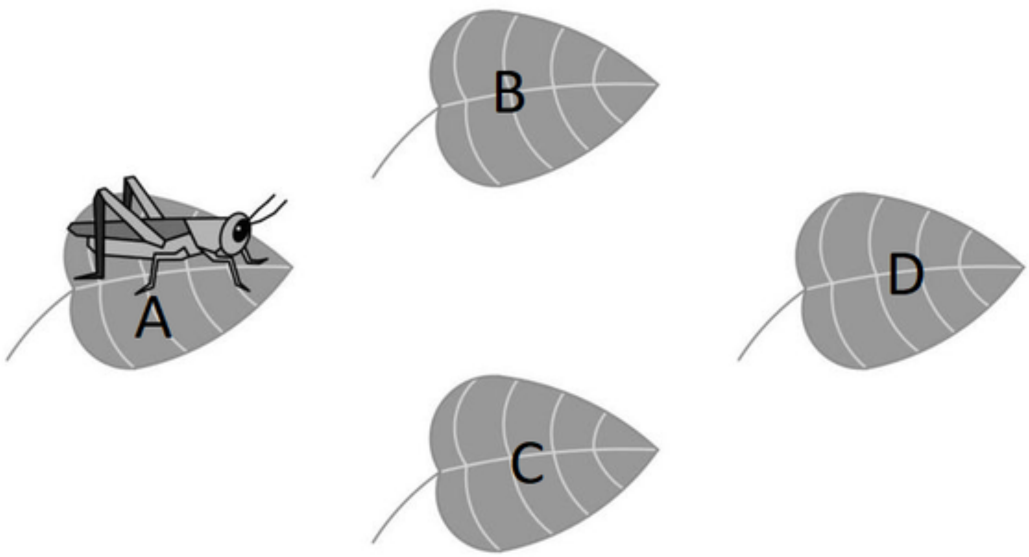
2. $A \rightarrow B \rightarrow B \rightarrow B \rightarrow A$

The probability for this case is $1 \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{27}$.

Together, the probability that the cricket returns to A after 4 hops is $\frac{1}{9} + \frac{4}{27} = \boxed{\textbf{(E) } \frac{7}{27}}$.

Solution 2 (Casework)

We can label the leaves as shown below:



Carefully counting cases, we see that there are 7 ways for the cricket to return to leaf A after four hops if its first hop was to leaf B:

1. $A \rightarrow B \rightarrow A \rightarrow B \rightarrow A$
2. $A \rightarrow B \rightarrow A \rightarrow C \rightarrow A$
3. $A \rightarrow B \rightarrow A \rightarrow D \rightarrow A$
4. $A \rightarrow B \rightarrow C \rightarrow B \rightarrow A$
5. $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$
6. $A \rightarrow B \rightarrow D \rightarrow B \rightarrow A$

7. $A \to B \to D \to C \to A$

By symmetry, we know that there are 7 ways if the cricket's first hop was to leaf C, and there are 7 ways if the cricket's first hop was to leaf D. So, there are 21 ways in total for the cricket to return to leaf A after four hops.

Since there are $3^4 = 81$ possible ways altogether for the cricket to hop to any other leaf four times, the answer is $\frac{21}{81} = \boxed{\textbf{(E)}\ \frac{7}{27}}$.

Solution 3 (Recursion)

Denote P_n to be the probability that the cricket would return back to the first point after n hops. Then, we get the recursive formula $P_n = \frac{1}{3}(1 - P_{n-1})$ because if the leaf is not on the target leaf, then there is a $\frac{1}{3}$ probability that it will make it back.

With this formula and the fact that $P_1 = 0$ (After one hop, the cricket can never be back to the target leaf.), we have $P_2 = \frac{1}{3}, P_3 = \frac{2}{9}, P_4 = \frac{7}{27},$ so our answer is $\boxed{\textbf{(E)}\ \frac{7}{27}}$.

Solution 4 (Dynamic Programming)

Let A denote the leaf cricket starts at, and B, C, D be the other leaves, similar to Solution 2.

Let $A(n)$ be the probability the cricket lands on A after n hops, $B(n)$ be the probability the cricket lands on B after crawling n hops, and etc.

Note that $A(1)=0$ and $B(1)=C(1)=D(1)=\frac{1}{3}$. For $n\geq 2$, the probability that the cricket land on each leaf after n hops is $\frac{1}{3}$ the sum of the probability the cricket land on other leaves after $n-1$ hops. So, we have

$$\begin{aligned}
A(n)&=\frac{1}{3}\cdot[B(n-1)+C(n-1)+D(n-1)],\\
B(n)&=\frac{1}{3}\cdot[A(n-1)+C(n-1)+D(n-1)],\\
C(n)&=\frac{1}{3}\cdot[A(n-1)+B(n-1)+D(n-1)],\\
D(n)&=\frac{1}{3}\cdot[A(n-1)+B(n-1)+C(n-1)].
\end{aligned}$$

It follows that $A(n)=B(n-1)=C(n-1)=D(n-1).$

We construct the following table:

n	$A(n)$	$B(n)$	$C(n)$	$D(n)$
1	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
2	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$
3	$\frac{2}{9}$	$\frac{7}{27}$	$\frac{7}{27}$	$\frac{7}{27}$
4	$\frac{7}{27}$	$\frac{20}{81}$	$\frac{20}{81}$	$\frac{20}{81}$

Therefore, the answer is $A(4)=\boxed{\textbf{(E) }\frac{7}{27}}$.

Solution 5 (Generating Function)

Assign the leaves to $0, 1, 2,$ and 3 modulo $4,$ and let 0 be the starting leaf. We then use generating functions with relation to the change of leaves. For example, from 3 to 1 would be a change of $2,$ and from 1 to 2 would be a change of $1.$ This generating function is equal to $(x + x^2 + x^3)^4.$ It is clear that we want the coefficients in the form of $x^{4n},$ where n is a positive integer. One application of roots of unity filter gives us a successful case count of $\frac{81+1+1+1}{4} = 21.$

Therefore, the answer is $\frac{21}{3^4} = \boxed{\textbf{(E) } \frac{7}{27}}.$