2011 AMC 12A Problems/Problem 1

Problem

A cell phone plan costs 20 dollars each month, plus 5 cents per text message sent, plus 10 cents for each minute used over 30 hours. In January Michelle sent 100 text messages and talked for 30.5 hours. How much did she have to pay?

$\textbf{(A)}\ 24.00 \qquad \textbf{(B)}\ 24.50 \qquad \textbf{(C)}\ 25.50 \qquad \textbf{(D)}\ 28.00 \qquad \textbf{(E)}\ 30.00$

Solution

The base price of Michelle's cell phone plan is 20 dollars. If she sent 100 text messages and it costs 5 cents per text, then she must have spent 500 cents for texting, or 5 dollars. She talked for 30.5 hours, but $30.5 - 30$ will give us the amount of time that she has to pay an additional amount for. $30.5 - 30 = .5$ hours $= 30$ minutes. Since the price for phone calls is 10 cents per minute, the additional amount Michelle has to pay for phone calls is $30 * 10 = 300$ cents, or 3 dollars. Adding $20 + 5 + 3$ dollars $= 28$ dollars $= \boxed{\textbf{D}}$.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by First Problem	Followed by Problem 2
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

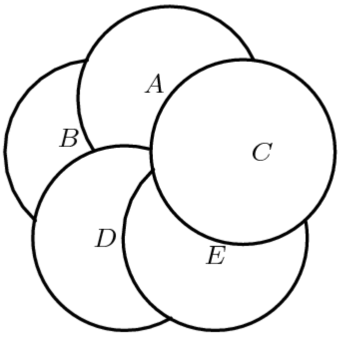
Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_1&oldid=53823"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 2

Problem

There are 5 coins placed flat on a table according to the figure. What is the order of the coins from top to bottom?



$\textbf{(A)}\ (C, A, E, D, B) \qquad \textbf{(B)}\ (C, A, D, E, B) \qquad \textbf{(C)}\ (C, D, E, A, B) \qquad \textbf{(D)}\ (C, E, A, D, B)$
$\textbf{(E)}\ (C, E, D, A, B)$

Solution

By careful inspection and common sense, the answer is $\textbf{(E)}$.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 1	Followed by Problem 3
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_2&oldid=79060"

Category: Introductory Combinatorics Problems

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 3

Problem

A small bottle of shampoo can hold 35 milliliters of shampoo, whereas a large bottle can hold 500 milliliters of shampoo. Jasmine wants to buy the minimum number of small bottles necessary to completely fill a large bottle. How many bottles must she buy?

$\textbf{(A)}\ 11 \qquad \textbf{(B)}\ 12 \qquad \textbf{(C)}\ 13 \qquad \textbf{(D)}\ 14 \qquad \textbf{(E)}\ 15$

Solution

To find how many small bottles we need, we can simply divide 500 by 35. This simplifies to $\frac{100}{7} = 14\frac{2}{7}$. Since the answer must be an integer greater than 14, we have to round up to 15 bottles, or $\boxed{\textbf{E}}$

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 2	Followed by Problem 4
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_3&oldid=53825"

2011 AMC 12A Problems/Problem 4

Problem

At an elementary school, the students in third grade, fourth grade, and fifth grade run an average of 12, 15, and 10 minutes per day, respectively. There are twice as many third graders as fourth graders, and twice as many fourth graders as fifth graders. What is the average number of minutes run per day by these students?

$\textbf{(A)}\ 12 \qquad \textbf{(B)}\ \frac{37}{3} \qquad \textbf{(C)}\ \frac{88}{7} \qquad \textbf{(D)}\ 13 \qquad \textbf{(E)}\ 14$

Solution

Let us say that there are f fifth graders. According to the given information, there must be $2f$ fourth graders and $4f$ third graders. The average time run by each student is equal to the total amount of time run divided by the number of students. This gives us $\frac{12\cdot 4f + 15\cdot 2f + 10\cdot f}{4f+2f+f} = \frac{88f}{7f} = \frac{88}{7} \Rightarrow \boxed{C}$

If you want to simplify the problem even more, just imagine/assume that only 1 fifth grader existed. Then you can simply get rid of the variables.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 3	Followed by Problem 5
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_4&oldid=79062"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 5

Problem

Last summer 30% of the birds living on Town Lake were geese, 25% were swans, 10% were herons, and 35% were ducks. What percent of the birds that were not swans were geese?

$\textbf{(A)}\ 20 \qquad \textbf{(B)}\ 30 \qquad \textbf{(C)}\ 40 \qquad \textbf{(D)}\ 50 \qquad \textbf{(E)}\ 60$

Solution

To simplify the problem, let us say that there were a total of 100 birds. The number of birds that are not swans is 75. The number of geese is 30. Therefore the percentage is just $\frac{30}{75} \times 100 = 40 \Rightarrow \boxed{C}$

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 4	Followed by Problem 6
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_5&oldid=69969"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 6

Problem

The players on a basketball team made some three-point shots, some two-point shots, and some one-point free throws. They scored as many points with two-point shots as with three-point shots. Their number of successful free throws was one more than their number of successful two-point shots. The team's total score was 61 points. How many free throws did they make?

$\textbf{(A)}\ 13 \qquad \textbf{(B)}\ 14 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 16 \qquad \textbf{(E)}\ 17$

Solution

For the points made from two-point shots and from three-point shots to be equal, the numbers of made shots are in a $3:2$ ratio. Therefore, assume they made $3x$ and $2x$ two- and three- point shots, respectively, and thus $3x+1$ free throws. The total number of points is

$$2 \times (3x) + 3 \times (2x) + 1 \times (3x+1) = 15x + 1$$

Set that equal to 61, we get $x = 4$, and therefore the number of free throws they made $3 \times 4 + 1 = 13 \Rightarrow \boxed{A}$

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 5	Followed by Problem 7
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_6&oldid=53828"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 7

Problem

A majority of the 30 students in Ms. Demeanor's class bought pencils at the school bookstore. Each of these students bought the same number of pencils, and this number was greater than 1. The cost of a pencil in cents was greater than the number of pencils each student bought, and the total cost of all the pencils was 17.71. What was the cost of a pencil in cents?

$\textbf{(A)}\ 7 \qquad \textbf{(B)}\ 11 \qquad \textbf{(C)}\ 17 \qquad \textbf{(D)}\ 23 \qquad \textbf{(E)}\ 77$

Solution

The total cost of the pencils can be found by $(\text{students} * \text{pencils purchased by each} * \text{price of each pencil})$. As the cost is 17.71 dollars, or 1771 cents, the cost of the pencils must divide the total cost. Scanning the answer choices, they all unfortunately divide into 1771, so we have to start from the beginning (although this does give us some indication that we are on the right track, because MAA doesn't want to make things too easy for us).

Therefore, since 1771 is the product of three sets of values, we can begin with prime factorization, since it gives some insight into the values: $7, 11, 23$. Since neither (C) nor (E) are any of these factors, they can be eliminated immediately, leaving (A), (B), and (D).

Beginning with $(A) 7$, we see that the number of pencils purchased by each student must be either 11 or 23. However, the problem states that the price of each pencil must exceed the number of pencils purchased, so we can eliminate this.

Continuing with $(B) 11$, we can conclude that the only case that fulfills the restrictions are that there are 23 students who each purchased 7 such pencils, so the answer is $\boxed{B}$. We can apply the same logic to (E) as we applied to (A), if one wants to make doubly sure.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 6	Followed by Problem 8
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_7&oldid=75937"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 8

Contents

Problem

In the eight term sequence A, B, C, D, E, F, G, H, the value of C is 5 and the sum of any three consecutive terms is 30. What is $A+H$?

$\textbf{(A)}\ 17 \qquad \textbf{(B)}\ 18 \qquad \textbf{(C)}\ 25 \qquad \textbf{(D)}\ 26 \qquad \textbf{(E)}\ 43$

Solution

Solution 1

Let $A=x$. Then from $A+B+C=30$, we find that $B=25-x$. From $B+C+D=30$, we then get that $D=x$. Continuing this pattern, we find $E=25-x$, $F=5$, $G=x$, and finally $H=25-x$. So $A+H=x+25-x=25 \rightarrow \boxed{\textbf{C}}$

Solution 2

Given that the sum of 3 consecutive terms is 30, we have
$(A+B+C)+(C+D+E)+(F+G+H)=90$ and
$(B+C+D)+(E+F+G)=60$

It follows that $A+B+C+D+E+F+G+H=85$ because $C=5$.

Subtracting, we have that $A+H=25 \rightarrow \boxed{\textbf{C}}$.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 7	Followed by Problem 9
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?

2011 AMC 12A Problems/Problem 9

Problem

At a twins and triplets convention, there were 9 sets of twins and 6 sets of triplets, all from different families. Each twin shook hands with all the twins except his/her siblings and with half the triplets. Each triplet shook hands with all the triplets except his/her siblings and with half the twins. How many handshakes took place?

$\textbf{(A)}\ 324 \qquad \textbf{(B)}\ 441 \qquad \textbf{(C)}\ 630 \qquad \textbf{(D)}\ 648 \qquad \textbf{(E)}\ 882$

Solution

There are 18 total twins and 18 total triplets. Each of the twins shakes hands with the 16 twins not in their family and 9 of the triplets, a total of 25 people. Each of the triplets shakes hands with the 15 triplets not in their family and 9 of the twins, for a total of 24 people. Dividing by two to accommodate the fact that each handshake was counted twice, we get a total of

$$\frac{1}{2} \times 18 \times (25 + 24) = 9 \times 49 = 441 \rightarrow \boxed{\textbf{B}}$$

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 8	Followed by Problem 10
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_9&oldid=53831"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 10

Problem

A pair of standard 6-sided dice is rolled once. The sum of the numbers rolled determines the diameter of a circle. What is the probability that the numerical value of the area of the circle is less than the numerical value of the circle's circumference?

$\textbf{(A)}\ \frac{1}{36} \qquad \textbf{(B)}\ \frac{1}{12} \qquad \textbf{(C)}\ \frac{1}{6} \qquad \textbf{(D)}\ \frac{1}{4} \qquad \textbf{(E)}\ \frac{5}{18}$

Solution

For the circumference to be greater than the area, we must have $\pi d > \pi \left(\frac{d}{2}\right)^2$, or $d < 4$. Now since d is determined by a sum of two dice, the only possibilities for d are thus 2 and 3. In order for two dice to sum to 2, they most both show a value of 1. The probability of this happening is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. In order for two dice to sum to 3, one must show a 1 and the other must show a 2. Since this can happen in two ways, the probability of this event occurring is $2 \times \frac{1}{6} \times \frac{1}{6} = \frac{2}{36}$. The sum of these two probabilities now gives the final answer: $\frac{1}{36} + \frac{2}{36} = \frac{3}{36} = \frac{1}{12} \rightarrow \boxed{\textbf{B}}$

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 9	Followed by Problem 11
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_10&oldid=53832"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 11

Contents

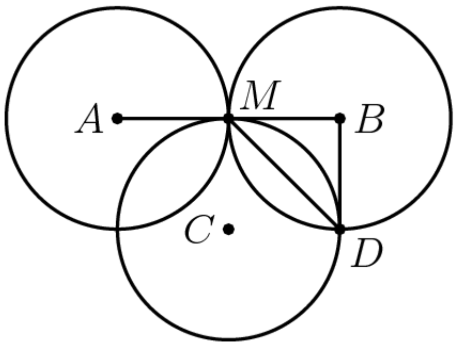
Problem

Circles $A, B,$ and C each have radius 1. Circles A and B share one point of tangency. Circle C has a point of tangency with the midpoint of $\overline{AB}.$ What is the area inside circle C but outside circle A and circle $B?$

$\textbf{(A)}\ 3 - \frac{\pi}{2} \qquad \textbf{(B)}\ \frac{\pi}{2} \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ \frac{3\pi}{4} \qquad \textbf{(E)}\ 1+\frac{\pi}{2}$

Solution

Solution 1



The requested area is the area of C minus the area shared between circles A, B and C.

Let M be the midpoint of $\overline{AB}$ and D be the other intersection of circles C and B.

Then area shared between C, A and B is 4 of the regions between arc $\widehat{MD}$ and line $\overline{MD}$, which is (considering the arc on circle B) a quarter of the circle B minus $\triangle MDB$:

$$\frac{\pi r^2}{4} - \frac{bh}{2}$$

$$b = h = r = 1$$

(We can assume this because $\angle DBM$ is 90 degrees, since $CDBM$ is a square, due the application of the tangent chord theorem at point M)

So the area of the small region is

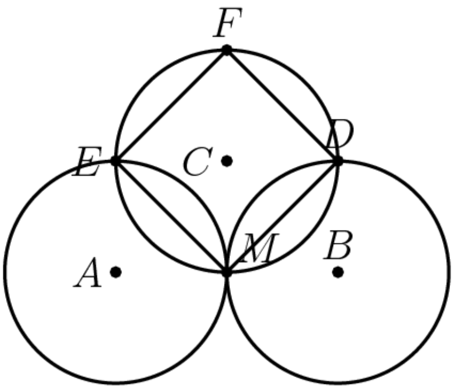
$$\frac{\pi}{4} - \frac{1}{2}$$

The requested area is area of circle C minus 4 of this area:

$$\pi 1^2 - 4(\frac{\pi}{4} - \frac{1}{2}) = \pi - \pi + 2 = 2$$

$\boxed{\textbf{C}}$.

Solution 2



We can move the area above the part of the circle above the segment EF down, and similarly for the other side. Then, we have a square, whose diagonal is 2, so the area is then just $\left(\frac{2}{\sqrt{2}}\right)^2 = 2$.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 10	Followed by Problem 12
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_11&oldid=69970"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 12

Contents

Problem

A power boat and a raft both left dock A on a river and headed downstream. The raft drifted at the speed of the river current. The power boat maintained a constant speed with respect to the river. The power boat reached dock B downriver, then immediately turned and traveled back upriver. It eventually met the raft on the river 9 hours after leaving dock A. How many hours did it take the power boat to go from A to B?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 3.5 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 4.5 \qquad \textbf{(E)}\ 5$

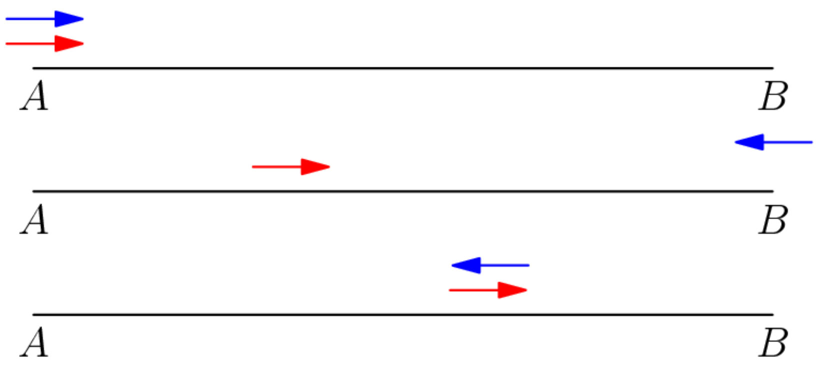
Solution

Solution 1

Since the speed of the river is not specified, the outcome of the problem must be independent of this speed. We may thus trivially assume that the river has a speed of 0. In this case, when the powerboat travels from A to B, the raft remains at A. Thus the trip from A to B takes the same time as the trip from B to the raft. Since these times are equal and sum to 9 hours, the trip from A to B must take half this time, or 4.5 hours. The answer is thus $\boxed{\textbf{D}}$.

Solution 2

What's important in this problem is to consider everything in terms of the power boat and the raft, since that is how the problem is given to us. Think of the blue arrow as the power boat and the red arrow as the raft in the following three diagrams, which represent different time intervals of the problem.



Thinking about the distance covered as their distances with respect to each other, they are 0 distance apart in the first diagram when they haven't started to move yet, some distance d apart in the second diagram when the power boat reaches B, and again 0 distance apart in the third diagram when they meet. Therefore, with respect to each other, the boat and the raft cover a distance of d on the way there, and again cover a distance of d on when drawing closer. This makes sense, because from the 1st diagram to the second, the raft moves in the same direction as the boat, while from the 2nd to the 3rd, the boat and raft move in opposite directions.

Let b denote the speed of the power boat (only the power boat, not factoring in current) and r denote the speed of the raft, which, as given by the problem, is also equal to the speed of the current. Thus, from A to B, the boat travels at a velocity of $b+r$, and on the way back, travels at a velocity of

$-(b-r)=r-b$, since the current aids the boat on the way there, and goes against the boat on the way back. With respect to the raft then, the boat's velocity from A to B becomes $(r+b)-r=b$, and on the way back it becomes $(r-b)-r=-b$. Since the boat's velocities with respect to the raft are exact opposites, b and $-b$, we therefore know that the boat and raft travel apart from each other at the same rate that they travel toward each other.

From this, we have that the boat travels a distance d at rate b with respect to the raft both on the way to B and on the way back. Thus, using $\frac{distance}{speed}=time$, we have $\frac{2d}{b}=9 \text{ hours}$, and to see how long it took to travel half the distance, we have $\frac{d}{b}=4.5 \text{ hours} \implies \boxed{\textbf{D}}$

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 11	Followed by Problem 13
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_12&oldid=75931"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 13

Problem

Triangle ABC has side-lengths $AB = 12$, $BC = 24$, and $AC = 18$. The line through the incenter of $\triangle ABC$ parallel to $\overline{BC}$ intersects $\overline{AB}$ at M and $\overline{AC}$ at N. What is the perimeter of $\triangle AMN$?

$\textbf{(A)}\ 27 \qquad \textbf{(B)}\ 30 \qquad \textbf{(C)}\ 33 \qquad \textbf{(D)}\ 36 \qquad \textbf{(E)}\ 42$

Solution

Let O be the incenter. Because $MO \parallel BC$ and BO is the angle bisector, we have

$$\angle MBO = \angle CBO = \angle MOB = \frac{1}{2}\angle MBC$$

It then follows due to alternate interior angles and base angles of isosceles triangles that $MO = MB$. Similarly, $NO = NC$. The perimeter of $\triangle AMN$ then becomes

$$\begin{aligned} AM + MN + NA &= AM + MO + NO + NA \\ &= AM + MB + NC + NA \\ &= AB + AC \\ &= 30 \rightarrow \boxed{(B)} \end{aligned}$$

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 12	Followed by Problem 14
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_13&oldid=68874"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 14

Problem

Suppose a and b are single-digit positive integers chosen independently and at random. What is the probability that the point (a, b) lies above the parabola $y = ax^2 - bx$?

$\textbf{(A)}\ \frac{11}{81} \qquad \textbf{(B)}\ \frac{13}{81} \qquad \textbf{(C)}\ \frac{5}{27} \qquad \textbf{(D)}\ \frac{17}{81} \qquad \textbf{(E)}\ \frac{19}{81}$

Solution

If (a, b) lies above the parabola, then b must be greater than $y(a)$. We thus get the inequality $b > a^3 - ba$. Solving this for b gives us $b > \frac{a^3}{a+1}$. Now note that $\frac{a^3}{a+1}$ constantly increases when a is positive. Then since this expression is greater than 9 when $a = 4$, we can deduce that a must be less than 4 in order for the inequality to hold, since otherwise b would be greater than 9 and not a single-digit integer. The only possibilities for a are thus 1, 2, and 3.

For $a = 1$, we get $b > \frac{1}{2}$ for our inequality, and thus b can equal any integer from 1 to 9.

For $a = 2$, we get $b > \frac{8}{3}$ for our inequality, and thus b can equal any integer from 3 to 9.

For $a = 3$, we get $b > \frac{27}{4}$ for our inequality, and thus b can equal any integer from 7 to 9.

Finally, if we total up all the possibilities we see there are 19 points that satisfy the condition, out of $9 \times 9 = 81$ total points. The probability of picking a point that lies above the parabola is thus $\frac{19}{81} \rightarrow \boxed{\textbf{E}}$

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 13	Followed by Problem 15
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_14&oldid=53836"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 15

Problem

The circular base of a hemisphere of radius 2 rests on the base of a square pyramid of height 6. The hemisphere is tangent to the other four faces of the pyramid. What is the edge-length of the base of the pyramid?

$\textbf{(A)}\ 3\sqrt{2} \qquad \textbf{(B)}\ \frac{13}{3} \qquad \textbf{(C)}\ 4\sqrt{2} \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ \frac{13}{2}$

Solution

Let $ABCDE$ be the pyramid with $ABCD$ as the square base. Let O and M be the center of square $ABCD$ and the midpoint of side AB respectively. Lastly, let the hemisphere be tangent to the triangular face ABE at P.

Notice that $\triangle EOM$ has a right angle at O. Since the hemisphere is tangent to the triangular face ABE at P, $\angle EPO$ is also 90°. Hence $\triangle EOM$ is similar to $\triangle EPO$.

$$\frac{OM}{2} = \frac{6}{EP}$$

$$OM = \frac{6}{EP} \times 2$$

$$OM = \frac{6}{\sqrt{6^2 - 2^2}} \times 2 = \frac{3\sqrt{2}}{2}$$

The length of the square base is thus $2 \times \frac{3\sqrt{2}}{2} = 3\sqrt{2} \rightarrow \boxed{\textbf{A}}$

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 14	Followed by Problem 16
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_15&oldid=53837"

Categories: Introductory Geometry Problems | 3D Geometry Problems

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 16

Problem

Each vertex of convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

$\textbf{(A)}\ 2520 \qquad \textbf{(B)}\ 2880 \qquad \textbf{(C)}\ 3120 \qquad \textbf{(D)}\ 3250 \qquad \textbf{(E)}\ 3750$

Solution

We can do some casework when working our way around the pentagon from A to E. At each stage, there will be a makeshift diagram.

1.) For A, we can choose any of the 6 colors.

```
        A : 6
```

2.) For B, we can either have the same color as A, or any of the other 5 colors. We do this because each vertex of the pentagon is affected by the 2 opposite vertices, and D will be affected by both A and B.

```
      A : 6
B:1         B:5
```

3.) For C, we cannot have the same color as A. Also, we can have the same color as B (E will be affected), or any of the other 4 colors. Because C can't be the same as A, it can't be the same as B if B is the same as A, so it can be any of the 5 other colors.

```
      A : 6
B:1         B:5
C:5     C:4   C:1
```

4.) D is affected by A and B. If they are the same, then D can be any of the other 5 colors. If they are different, then D can be any of the (6-2)=4 colors.

```
      A : 6
B:1         B:5
C:5     C:4   C:1
D:5     D:4   D:4
```

5.) E is affected by B and C. If they are the same, then E can be any of the other 5 colors. If they are different, then E can be any of the (6-2)=4 colors.

```
      A : 6
B:1         B:5
C:5     C:4   C:1
D:5     D:4   D:4
E:4     E:4   E:5
```

6.) Now, we can multiply these three paths and add them:
$$(6 \times 1 \times 5 \times 5 \times 4) + (6 \times 5 \times 4 \times 4 \times 4) + (6 \times 5 \times 1 \times 4 \times 5) = 600 + 1920 + 600 = 3120$$

7.) Our answer is C!

See also

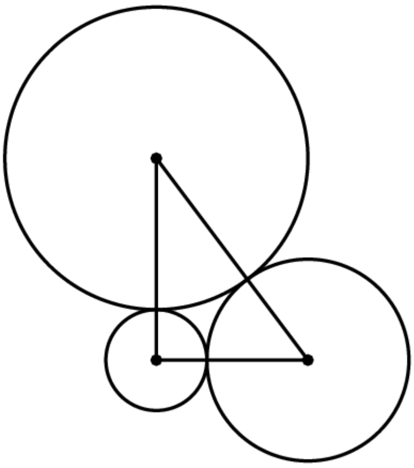
2011 AMC 12A Problems/Problem 17

Problem

Circles with radii 1, 2, and 3 are mutually externally tangent. What is the area of the triangle determined by the points of tangency?

$\textbf{(A)}\ \frac{3}{5} \qquad \textbf{(B)}\ \frac{4}{5} \qquad \textbf{(C)}\ 1 \qquad \textbf{(D)}\ \frac{6}{5} \qquad \textbf{(E)}\ \frac{4}{3}$

Solution



The centers of these circles form a 3-4-5 triangle, which has an area equal to 6.

The areas of the three triangles determined by the center and the two points of tangency of each circle are, by Law of Sines,

$$\frac{1}{2} \cdot 1 \cdot 1 \cdot 1 = \frac{1}{2}$$

$$\frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{4}{5} = \frac{8}{5}$$

$$\frac{1}{2} \cdot 3 \cdot 3 \cdot \frac{3}{5} = \frac{27}{10}$$

which add up to 4.8. The area we're looking for is the large 3-4-5 triangle minus the three smaller triangles, or $6 - 4.8 = 1.2 = \frac{6}{5} \rightarrow \boxed{(D)}$.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 16	Followed by Problem 18
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12A Problems/Problem 18

Contents

Problem

Suppose that $|x+y|+|x-y|=2$. What is the maximum possible value of x^2-6x+y^2?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 7 \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 9$

Solution 1

Plugging in some values, we see that the graph of the equation $|x+y|+|x-y| = 2$ is a square bounded by $x= \pm 1$ and $y = \pm 1$.

Notice that $x^2 - 6x + y^2 = (x-3)^2 + y^2 - 9$ means the square of the distance from a point (x,y) to point $(3,0)$ minus 9. To maximize that value, we need to choose the point in the feasible region farthest from point $(3,0)$, which is $(-1, \pm 1)$. Either one, when substituting into the function, yields $8 \rightarrow \boxed{(D)}$.

Solution 2

Since the equation $|x+y|+|x-y|=2$ is dealing with absolute values, the following could be deduced: $(x+y)+(x+y)=2, (x+y)-(x-y)=2$, $-(x+y)+(x-y)=2$, and $-(x+y)-(x-y)=2$. Simplifying would give $x=1$, $y=1$, $y=-1$, and $x=-1$. In x^2-6x+y^2, it does not matter whether x or y is -1 or 1. To maximize $-6x$, though, x would have to be -1. Therefore, when $x=-1$ and $y=-1$ or $y=1$, the equation evaluates to $\boxed{(D)}$ 8.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 17	Followed by Problem 19
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_18&oldid=72893"

2011 AMC 12A Problems/Problem 19

Problem

At a competition with N players, the number of players given elite status is equal to $2^{1+\lfloor \log_2 (N-1) \rfloor} - N$. Suppose that 19 players are given elite status. What is the sum of the two smallest possible values of N?

$\textbf{(A)}\ 38 \qquad \textbf{(B)}\ 90 \qquad \textbf{(C)}\ 154 \qquad \textbf{(D)}\ 406 \qquad \textbf{(E)}\ 1024$

Solution

We start with $2^{1+\lfloor \log_2 (N-1) \rfloor} - N = 19$. After rearranging, we get $\lfloor \log_2 (N-1) \rfloor = \log_2 \left(\frac{N+19}{2}\right)$.

Since $\lfloor \log_2 (N-1) \rfloor$ is a positive integer, $\frac{N+19}{2}$ must be in the form of 2^m for some positive integer m. From this fact, we get $N = 2^{m+1} - 19$.

If we now check integer values of N that satisfy this condition, starting from $N = 19$, we quickly see that the first values that work for N are $2^6 - 19$ and $2^7 - 19$, giving values of 5 and 6 for m, respectively. Adding up these two values for N, we get $45 + 109 = 154 \rightarrow \boxed{\textbf{C}}$

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 18	Followed by Problem 20
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_19&oldid=53841"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 20

Problem

Let $f(x) = ax^2 + bx + c$, where a, b, and c are integers. Suppose that $f(1) = 0$, $50 < f(7) < 60$, $70 < f(8) < 80$, $5000k < f(100) < 5000(k+1)$ for some integer k. What is k?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution

From $f(1) = 0$, we know that $a + b + c = 0$.

From the first inequality, we get $50 < 49a + 7b + c < 60$. Subtracting $a + b + c = 0$ from this gives us $50 < 48a + 6b < 60$, and thus $\frac{25}{3} < 8a + b < 10$. Since $8a + b$ must be an integer, it follows that $8a + b = 9$.

Similarly, from the second inequality, we get $70 < 64a + 8b + c < 80$. Again subtracting $a + b + c = 0$ from this gives us $70 < 63a + 7b < 80$, or $10 < 9a + b < \frac{80}{7}$. It follows from this that $9a + b = 11$.

We now have a system of three equations: $a + b + c = 0$, $8a + b = 9$, and $9a + b = 11$. Solving gives us $(a, b, c) = (2, -7, 5)$ and from this we find that $f(100) = 2(100)^2 - 7(100) + 5 = 19305$

Since $15000 < 19305 < 20000 \to 5000(3) < 19305 < 5000(4)$, we find that $k = 3 \to \boxed{\textbf{(C)}}$.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 19	Followed by Problem 21
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_20&oldid=53842"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 21

Problem

Let $f_1(x)=\sqrt{1-x}$, and for integers $n \geq 2$, let $f_n(x)=f_{n-1}(\sqrt{n^2 - x})$. If N is the largest value of n for which the domain of f_n is nonempty, the domain of f_N is $[c]$. What is $N+c$?

$\textbf{(A)}\ -226 \qquad \textbf{(B)}\ -144 \qquad \textbf{(C)}\ -20 \qquad \textbf{(D)}\ 20 \qquad \textbf{(E)}\ 144$

Solution

The domain of $f_{1}(x)=\sqrt{1-x}$ is defined when $x\leq1$.

$$f_{2}(x)=f_{1}\left(\sqrt{4-x}\right)=\sqrt{1-\sqrt{4-x}}$$

Applying the domain of $f_{1}(x)$ and the fact that square roots must be positive, we get $0\leq\sqrt{4-x}\leq1$. Simplifying, the domain of $f_{2}(x)$ becomes $3\leq x\leq4$.

Repeat this process for $f_{3}(x)=\sqrt{1-\sqrt{4-\sqrt{9-x}}}$ to get a domain of $-7\leq x\leq0$.

For $f_{4}(x)$, since square roots are positive, we can exclude the negative values of the previous domain to arrive at $\sqrt{16-x}=0$ as the domain of $f_{4}(x)$. We now arrive at a domain with a single number that defines x, however, since we are looking for the largest value for n for which the domain of f_{n} is nonempty, we must continue until we arrive at a domain that is empty. We continue with $f_{5}(x)$ to get a domain of $\sqrt{25-x}=16$. Solve for x to get $x=-231$. Since square roots cannot be negative, this is the last nonempty domain. We add to get $5-231=\boxed{\textbf{(A)}\ -226}$.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 20	Followed by Problem 22
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_21&oldid=57138"

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 22

Problem

Let R be a square region and $n \geq 4$ an integer. A point X in the interior of R is called n-ray partitional if there are n rays emanating from X that divide R into n triangles of equal area. How many points are 100-ray partitional but not 60-ray partitional?

$\textbf{(A)}\ 1500 \qquad \textbf{(B)}\ 1560 \qquad \textbf{(C)}\ 2320 \qquad \textbf{(D)}\ 2480 \qquad \textbf{(E)}\ 2500$

Solution

There must be four rays emanating from X that intersect the four corners of the square region. Depending on the location of X, the number of rays distributed among these four triangular sectors will vary. We start by finding the corner-most point that is 100-ray partitional (let this point be the bottom-left-most point).

We first draw the four rays that intersect the vertices. At this point, the triangular sectors with bases as the sides of the square that the point is closest to both do not have rays dividing their areas. Therefore, their heights are equivalent since their areas are equal. The remaining 96 rays are divided among the other two triangular sectors, each sector with 48 rays, thus dividing these two sectors into 49 triangles of equal areas.

Let the distance from this corner point to the closest side be a and the side of the square be s. From this, we get the equation $\frac{a\times s}{2}=\frac{(s-a)\times s}{2}\times\frac1{49}$. Solve for a to get $a=\frac s{50}$. Therefore, point X is $\frac1{50}$ of the side length away from the two sides it is closest to. By moving X $\frac s{50}$ to the right, we also move one ray from the right sector to the left sector, which determines another 100-ray partitional point. We can continue moving X right and up to derive the set of points that are 100-ray partitional.

In the end, we get a square grid of points each $\frac s{50}$ apart from one another. Since this grid ranges from a distance of $\frac s{50}$ from one side to $\frac{49s}{50}$ from the same side, we have a 49×49 grid, a total of 2401 100-ray partitional points. To find the overlap from the 60-ray partitional, we must find the distance from the corner-most 60-ray partitional point to the sides closest to it. Since the 100-ray partitional points form a 49×49 grid, each point $\frac s{50}$ apart from each other, we can deduce that the 60-ray partitional points form a 29×29 grid, each point $\frac s{30}$ apart from each other. To find the overlap points, we must find the common divisors of 30 and 50 which are $1, 2, 5,$ and 10. Therefore, the overlapping points will form grids with points s, $\frac s{2}$, $\frac s{5}$, and $\frac s{10}$ away from each other respectively. Since the grid with points $\frac s{10}$ away from each other includes the other points, we can disregard the other grids. The total overlapping set of points is a 9×9 grid, which has 81 points. Subtract 81 from 2401 to get $2401-81=\boxed{\textbf{(C)}\ 2320}$.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 21	Followed by Problem 23
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2011 AMC 12A Problems/Problem 23

Contents

Problem

Let $f(z) = \frac{z+a}{z+b}$ and $g(z) = f(f(z))$, where a and b are complex numbers. Suppose that $|a| = 1$ and $g(g(z)) = z$ for all z for which $g(g(z))$ is defined. What is the difference between the largest and smallest possible values of $|b|$?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ \sqrt{2}-1 \qquad \textbf{(C)}\ \sqrt{3}-1 \qquad \textbf{(D)}\ 1 \qquad \textbf{(E)}\ 2$

Solution

By algebraic manipulations, we obtain

$$h(z) = g(g(z)) = f(f(f(f(z)))) = \frac{Pz+Q}{Rz+S}$$

where

$$P = (a+1)^2 + a(b+1)^2$$

$$Q = a(b+1)(b^2+2a+1)$$

$$R = (b+1)(b^2+2a+1)$$

$$S = a(b+1)^2 + (a+b^2)^2$$

In order for $h(z) = z$, we must have $R=0$, $Q=0$, and $P=S$.

$R=0$ implies $b=-1$ or $b^2+2a+1=0$.

$Q=0$ implies $a=0$, $b=-1$, or $b^2+2a+1=0$.

$P=S$ implies $b=\pm1$ or $b^2+2a+1=0$.

Since $|a| = 1 \neq 0$, in order to satisfy all 3 conditions we must have either $b=\pm1$ or $b^2+2a+1=0$. In the first case $|b|=1$.

For the latter case note that

$$|b^2+1| = |-2a| = 2$$

$$2 = |b^2+1| \leq |b^2| + 1$$

and hence,

$$1 \leq |b|^2 \Rightarrow 1 \leq |b|$$

. On the other hand,

$$2 = |b^2 + 1| \geq |b^2| - 1$$

so,

$$|b^2| \leq 3 \Rightarrow 0 \leq |b| \leq \sqrt{3}$$

. Thus $1 \leq |b| \leq \sqrt{3}$. Hence the maximum value for $|b|$ is $\sqrt{3}$ while the minimum is 1 (which can be achieved for instance when $|a| = 1, |b| = \sqrt{3}$ or $|a| = 1, |b| = 1$ respectively). Therefore the answer is $\boxed{\textbf{(C)}\ \sqrt{3} - 1}$.

Shortcut

We only need Q in $f^4(z) = g^2(z) = \frac{Pz + Q}{Rz + S}$.

Set $Q = 0$: $a(b+1)\left(b^2 + 2a + 1\right) = 0$. Since $|a| = 1$, either $b + 1 = 0$ or $b^2 + 2a + 1 = 0$.

$b + 1 = 0 \rightarrow b = -1$ so $|b| = 1$.

$b^2 + 2a + 1 = 0 \rightarrow b^2 = -1 - 2a$. This is a circle in the complex plane centered at $(-1, 0)$ with radius 2 since $|a| = 1$. The maximum distance from the origin is 3 at $(-3, 0)$ and similarly the minimum distance is 1 at $(1, 0)$. So $1 \leq |b^2| \leq 3 \rightarrow 1 \leq |b| \leq \sqrt{3}$.

Both solutions give the same lower bound, 1. So the range is $\sqrt{3} - 1 = \boxed{\textbf{(C)}\ \sqrt{3} - 1}$.

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 22	Followed by Problem 24
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "http://artofproblemsolving.com/wiki/index.php?title=2011_AMC_12A_Problems/Problem_23&oldid=76002"

Category: Intermediate Algebra Problems

Copyright © 2016 Art of Problem Solving

2011 AMC 12A Problems/Problem 24

Contents

Problem

Consider all quadrilaterals $ABCD$ such that $AB=14$, $BC=9$, $CD=7$, and $DA=12$. What is the radius of the largest possible circle that fits inside or on the boundary of such a quadrilateral?

$\textbf{(A)}\ \sqrt{15} \qquad \textbf{(B)}\ \sqrt{21} \qquad \textbf{(C)}\ 2\sqrt{6} \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 2\sqrt{7}$

Solution

Solution 1

Note as above that ABCD must be cyclic to obtain the circle with maximal radius. Let E, F, G, and H be the points on AB, BC, CD, and DA respectively where the circle is tangent. Let $\theta=\angle BAD$ and $\alpha=\angle ADC$. Since the quadrilateral is cyclic, $\angle ABC=180^{\circ}-\alpha$ and $\angle BCD=180^{\circ}-\theta$. Let the circle have center O and radius r. Note that OHD, OGC, OFB, and OEA are right angles.

Hence $FOG=\theta$, $GOH=180^{\circ}-\alpha$, $EOH=180^{\circ}-\theta$, and $FOE=\alpha$.

Therefore, $AEOH\sim OFCG$ and $EBFO\sim HOGD$.

Let $x=CG$. Then $CF=x$, $BF=BE=9-x$, $GD=DH=7-x$, and $AH=AE=x+5$. Using $AEOH\sim OFCG$ and $EBFO\sim HOGD$ we have $r/(x+5)=x/r$, and $(9-x)/r=r/(7-x)$. By equating the value of r^2 from each, $x(x+5)=(7-x)(9-x)$. Solving we obtain $x=3$ so that $\boxed{\textbf{(C)}\ 2\sqrt{6}}$.

Solution 2

To maximize the radius of the circle, we also need to maximize its area. To maximize the area of the circle, the quadrilateral must be tangential (have an incircle). In a tangential quadrilateral, the sum of opposite sides is equal to the semiperimeter of the quadrilateral. $14+7=12+9$, so this particular quadrilateral has an incircle. By definition, given 4 side lengths, a cyclic quadrilateral has the maximum area of any quadrilateral with those side lengths. Therefore, to maximize the area of the quadrilateral and thus the incircle, we assume that this quadrilateral is cyclic.

For cyclic quadrilaterals, Brahmagupta's formula gives the area as $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where s is the semiperimeter and $a,b,c,$ and d are the side lengths. Breaking it up into 4 triangles, we see the area of a tangential quadrilateral is also equal to $r*s$. Equate these two equations. Substituting s, the semiperimeter, and A, the area and solving for r, we get $\boxed{\textbf{(C)}\ 2\sqrt{6}}$.

See also

2011 AMC 12A Problems/Problem 25

Problem

Triangle ABC has $\angle BAC = 60^\circ$, $\angle CBA \le 90^\circ$, $BC = 1$, and $AC \ge AB$. Let H, I, and O be the orthocenter, incenter, and circumcenter of $\triangle ABC$, respectively. Assume that the area of pentagon $BCOIH$ is the maximum possible. What is $\angle CBA$?

$\textbf{(A)}\ 60^\circ \qquad \textbf{(B)}\ 72^\circ \qquad \textbf{(C)}\ 75^\circ \qquad \textbf{(D)}\ 80^\circ \qquad \textbf{(E)}\ 90^\circ$

Solution

Let $\angle CAB = A$, $\angle ABC = B$, $\angle BCA = C$ for convenience.

It's well-known that $\angle BOC = 2A$, $\angle BIC = 90 + \frac{A}{2}$, and $\angle BHC = 180 - A$ (verifiable by angle chasing). Then, as $A = 60$, it follows that $\angle BOC = \angle BIC = \angle BHC = 120$ and consequently pentagon $BCOIH$ is cyclic. Observe that $BC = 1$ is fixed, whence the circumcircle of cyclic pentagon $BCOIH$ is also fixed. Similarly, as $OB = OC$(both are radii), it follows that O and also $[BCO]$ is fixed. Since $[BCOIH] = [BCO] + [BOIH]$ is maximal, it suffices to maximize $[BOIH]$.

Verify that $\angle IBC = \frac{B}{2}$, $\angle HBC = 90 - C$ by angle chasing; it follows that $\angle IBH = \angle HBC - \angle IBC = 90 - C - \frac{B}{2} = \frac{A}{2} - \frac{C}{2} = 30 - \frac{C}{2}$ since $A + B + C = 180 \implies \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = 90$ by Triangle Angle Sum. Similarly, $\angle OBC = (180 - 120)/2 = 30$ (isosceles base angles are equal), whence

$$\angle IBO = \angle IBC - \angle OBC = \frac{B}{2} - 30 = 60 - \frac{A}{2} - \frac{C}{2} = 30 - \frac{C}{2}$$

Since $\angle IBH = \angle IBO$, $IH = IO$ by Inscribed Angles.

There are two ways to proceed.

Letting O' and R be the circumcenter and circumradius, respectively, of cyclic pentagon $BCOIH$, the most straightforward is to write $[BOIH] = [OO'I] + [IO'H] + [HO'B] - [BO'O]$, whence

$$[BOIH] = \frac{1}{2}R^2(\sin(60 - C) + \sin(60 - C) + \sin(2C - 60) - \sin(60))$$

and, using the fact that R is fixed, maximize $2\sin(60 - C) + \sin(2C - 60)$ with Jensen's Inequality.

A more elegant way is shown below.

Lemma: $[BOIH]$ is maximized only if $HB = HI$.

Proof by contradiction: Suppose $[BOIH]$ is maximized when $HB \neq HI$. Let H' be the midpoint of minor arc BI be and I' the midpoint of minor arc $H'O$. Then $[BOIH'] = [IBO] + [IBH'] > [IBO] + [IBH] = [BOIH]$ since the altitude from H' to BI is greater than that from H to BI; similarly $[BH'I'O] > [BOIH'] > [BOIH]$. Taking H', I' to be the new orthocenter, incenter, respectively, this contradicts the maximality of $[BOIH]$, so our claim follows. $\blacksquare$

With our lemma($HB = HI$) and $IH = IO$ from above, along with the fact that inscribed angles that intersect the same length chords are equal,

$$\angle ABC = 2\angle IBC = 2(\angle OBC + \angle OBI) = 2((30 - \frac{C}{2}) + (30 + \frac{C}{2} + \frac{1}{3}\angle OCB)) = 2(30 + \frac{1}{3}\angle OCB) = 80 \implies \boxed{(D)}$$

-Solution by thecmd999

See also

2011 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2011))	
Preceded by Problem 24	Followed by Last Problem
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	