2016 AMC 10A Problems/Problem 1

Contents

Problem

What is the value of $\frac{11!-10!}{9!}$?

$\textbf{(A)}\ 99 \qquad\textbf{(B)}\ 100 \qquad\textbf{(C)}\ 110 \qquad\textbf{(D)}\ 121 \qquad\textbf{(E)}\ 132$

Solution 1

Factoring out $10!$ from the numerator and cancelling out $9!$ from the numerator and the denominator, we have

$$\frac{11!-10!}{9!} = \frac{11\cdot 10! - 1\cdot 10!}{9!} = \frac{(10!)\cdot(11-1)}{9!} = 10{\cdot}10 = \boxed{\textbf{(B)}\ 100}.$$

Solution 2

We can use subtraction of fractions to get

$$\frac{11!-10!}{9!} = \frac{11!}{9!} - \frac{10!}{9!} = 110 - 10 = \boxed{\textbf{(B)}\ 100}.$$

Solution 3

Factoring out $9!$ gives $\frac{11!-10!}{9!} = \frac{9!(11\cdot 10 - 10)}{9!} = 110 - 10 = \boxed{\textbf{(B)}\ 100}$.

See Also

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2016 AMC 10A Problems/Problem 2

Contents

Problem

For what value of x does $10^x \cdot 100^{2x} = 1000^5$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution 1

We can rewrite $10^x \cdot 100^{2x} = 1000^5$ as $10^{5x} = 10^{15}$:

$$\begin{split} 10^x \cdot 100^{2x} &= 10^x \cdot (10^2)^{2x} \\ 10^x \cdot 10^{4x} &= (10^3)^5 \\ 10^{5x} &= 10^{15} \end{split}$$

Since the bases are equal, we can set the exponents equal, giving us $5x = 15$. Solving the equation gives us $x = \boxed{\textbf{(C)}\ 3}$.

Solution 2

We can rewrite this expression as $\log(10^x \cdot 100^{2x}) = \log(1000^5)$, which can be simplified to $\log(10^x \cdot 10^{4x}) = 5\log(1000)$, and that can be further simplified to $\log(10^{5x}) = 5\log(10^3)$. This leads to $5x = 15$. Solving this linear equation yields $x = \boxed{\textbf{(C)}\ 3}$.

See Also

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2016 AMC 10A Problems/Problem 4

Problem

The remainder can be defined for all real numbers x and y with $y \neq 0$ by

$$\text{rem}(x, y) = x - y\left\lfloor \frac{x}{y} \right\rfloor$$

where $\left\lfloor \tfrac{x}{y} \right\rfloor$ denotes the greatest integer less than or equal to $\tfrac{x}{y}$. What is the value of $\text{rem}(\tfrac{3}{8}, -\tfrac{2}{5})$?

$$\textbf{(A) } -\frac{3}{8} \qquad \textbf{(B) } -\frac{1}{40} \qquad \textbf{(C) } 0 \qquad \textbf{(D) } \frac{3}{8} \qquad \textbf{(E) } \frac{31}{40}$$

Solution

The value, by definition, is

$$\begin{aligned}
\text{rem}\left(\frac{3}{8}, -\frac{2}{5}\right) &= \frac{3}{8} - \left(-\frac{2}{5}\right)\left\lfloor \frac{\frac{3}{8}}{-\frac{2}{5}} \right\rfloor \\
&= \frac{3}{8} - \left(-\frac{2}{5}\right)\left\lfloor \frac{3}{8} \times \frac{-5}{2} \right\rfloor \\
&= \frac{3}{8} - \left(-\frac{2}{5}\right)\left\lfloor \frac{-15}{16} \right\rfloor \\
&= \frac{3}{8} - \left(-\frac{2}{5}\right)(-1) \\
&= \frac{3}{8} - \frac{2}{5} \\
&= \boxed{\textbf{(B) } -\frac{1}{40}}.
\end{aligned}$$

See Also

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2016 AMC 10A Problems/Problem 7

Contents

Problem

The mean, median, and mode of the 7 data values $60, 100, x, 40, 50, 200, 90$ are all equal to x. What is the value of x?

$\textbf{(A)}\ 50 \qquad\textbf{(B)}\ 60 \qquad\textbf{(C)}\ 75 \qquad\textbf{(D)}\ 90 \qquad\textbf{(E)}\ 100$

Solution

Since x is the mean,

$$\begin{aligned} x &= \frac{60+100+x+40+50+200+90}{7} \\ &= \frac{540+x}{7}. \end{aligned}$$

Therefore, $7x=540+x$, so $x=\boxed{\textbf{(D) }90}$.

Check

Order the list: $\{40,50,60,90,100,200\}$. x must be either 60 or 90 because it is both the median and the mode of the set. Thus 90 is correct.

See Also

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2016 AMC 12A Problems/Problem 5

Problem

Goldbach's conjecture states that every even integer greater than 2 can be written as the sum of two prime numbers (for example, $2016 = 13 + 2003$). So far, no one has been able to prove that the conjecture is true, and no one has found a counterexample to show that the conjecture is false. What would a counterexample consist of?

$\textbf{(A)}$ an odd integer greater than 2 that can be written as the sum of two prime numbers
$\textbf{(B)}$ an odd integer greater than 2 that cannot be written as the sum of two prime numbers
$\textbf{(C)}$ an even integer greater than 2 that can be written as the sum of two numbers that are not prime
$\textbf{(D)}$ an even integer greater than 2 that can be written as the sum of two prime numbers
$\textbf{(E)}$ an even integer greater than 2 that cannot be written as the sum of two prime numbers

Solution

In this case, a counterexample is a number that would prove Goldbach's conjecture false. The conjecture asserts what can be done with even integers greater than 2. Therefore the solution is

$$\textbf{(E)}\text{ an even integer greater than 2 that cannot be written as the sum of two prime numbers.}$$

See Also

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2016 AMC 10A Problems/Problem 9

Problem

A triangular array of 2016 coins has 1 coin in the first row, 2 coins in the second row, 3 coins in the third row, and so on up to N coins in the Nth row. What is the sum of the digits of N?

$\textbf{(A)}\ 6\qquad\textbf{(B)}\ 7\qquad\textbf{(C)}\ 8\qquad\textbf{(D)}\ 9\qquad\textbf{(E)}\ 10$

Solution

We are trying to find the value of N such that

$$1+2+3\cdots+(N-1)+N=\frac{N(N+1)}{2}=2016.$$

Noticing that $\frac{63\cdot 64}{2}=2016,$ we have $N=63,$ so our answer is $\boxed{\textbf{(D)}\ 9}$.

Notice that we were attempting to solve $\frac{N(N+1)}{2} = 2016 \Rightarrow N(N+1) = 2016\cdot 2 = 4032.$ Approximating $N(N+1) \approx N^2$, we were looking for a square that is close to, but less than, $4032.$ Since $64^2=4096$, we see that $N = 63$ is a likely candidate. Multiplying $63\cdot 64$ confirms that our assumption is correct.

See Also

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2016 AMC 12A Problems/Problem 7

Contents

Problem

Which of these describes the graph of $x^2(x+y+1)=y^2(x+y+1)$?

$\textbf{(A)}$ two parallel lines
$\textbf{(B)}$ two intersecting lines
$\textbf{(C)}$ three lines that all pass through a common point
$\textbf{(D)}$ three lines that do not all pass through a comment point
$\textbf{(E)}$ a line and a parabola

Solution 1

The equation $x^2(x+y+1)=y^2(x+y+1)$ tells us $x^2=y^2$ or $x+y+1=0$. $x^2=y^2$ generates two lines $y=x$ and $y=-x$. $x+y+1=0$ is another straight line. The only intersection of $y=x$ and $y=-x$ is $(0,0)$, which is not on $x+y+1=0$. Therefore, the graph is three lines that do not have a common intersection, or

$$\boxed{\textbf{(D)}\text{ three lines that do not all pass through a common point}}$$

Solution 2

If $x+y+1\neq 0,$ then dividing both sides of the equation by $x+y+1$ gives us $x^2=y^2.$ Rearranging and factoring, we get $x^2-y^2=(x+y)(x-y)=0.$ If $x+y+1=0,$ then the equation is satisfied. Thus either $x+y=0,$ $x-y=0,$ or $x+y+1=0.$ These equations can be rearranged into the lines $y=-x,$ $y=x,$ and $y=-x-1,$ respectively. Since these three lines are distinct, the answer is $\boxed{\textbf{(D)}\text{ three lines that do not all pass through a common point}}$.

Solution 3

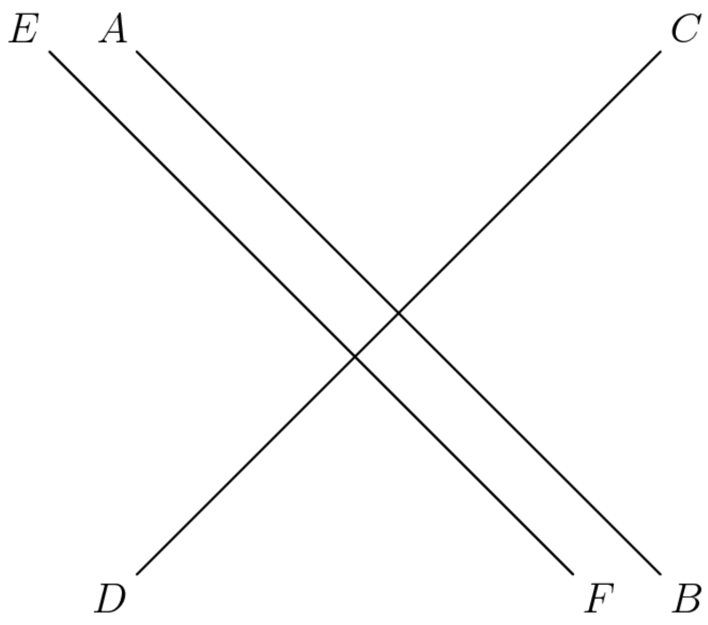
Subtract $y^2(x+y+1)$ on both sides of the equation to get $x^2(x+y+1)-y^2(x+y+1)=0$. Factoring $x+y+1$ gives us $(x+y+1)(x^2-y^2)=(x+y+1)(x+y)(x-y)=0,$ so either $x+y+1=0,$ $x+y=0,$ or $x-y=0.$ Continue on with the second half of solution 2.

Diagram:

$AB: y=x$

$CD: y=-x$

$EF: x+y+1=0$



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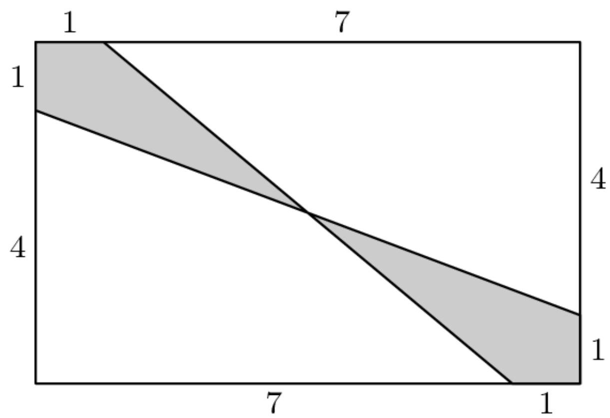
2016 AMC 10A Problems/Problem 11

Contents

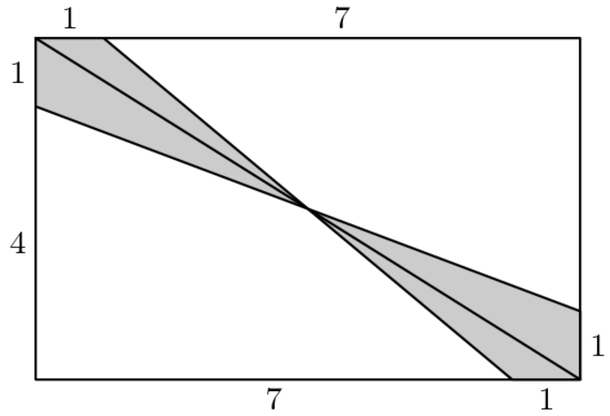
Problem

What is the area of the shaded region of the given 8×5 rectangle?



$\textbf{(A)}\ 4\dfrac{3}{5} \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 5\dfrac{1}{4} \qquad \textbf{(D)}\ 6\dfrac{1}{2} \qquad \textbf{(E)}\ 8$

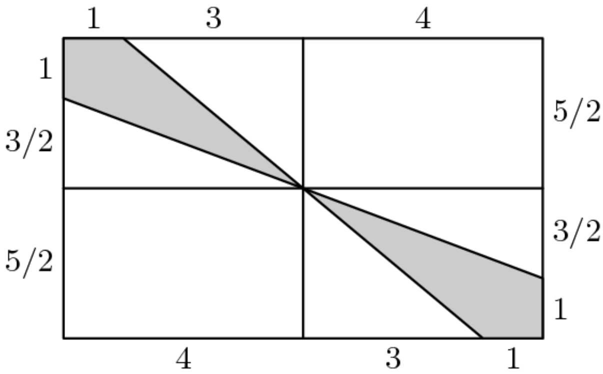
Solution 1



The bases of these triangles are all 1, and their heights are 4, $\frac{5}{2}$, 4, and $\frac{5}{2}$. Thus, their areas are 2, $\frac{5}{4}$, 2, and $\frac{5}{4}$, which add to the area of the shaded region, which is $\boxed{6\frac{1}{2}}$.

Solution 2

Find the area of the unshaded area by calculating the area of the triangles and rectangles outside of the shaded region. We can do this by splitting up the unshaded areas into various triangles and rectangles as shown.



Notice that the two added lines bisect each of the 4 sides of the large rectangle.

Subtracting the unshaded area from the total area gives us $40 - 33\frac{1}{2} = \boxed{6\frac{1}{2}}$, so the correct answer is $\boxed{\textbf{(D)}}$.

See Also

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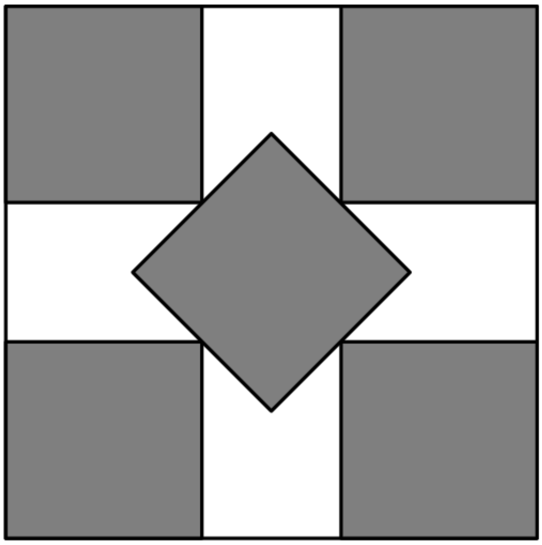
2016 AMC 12A Problems/Problem 9

Problem 9

The five small shaded squares inside this unit square are congruent and have disjoint interiors. The midpoint of each side of the middle square coincides with one of the vertices of the other four small squares as shown. The common side length is $\frac{a-\sqrt{2}}{b}$, where a and b are positive integers. What is $a+b$?



$\textbf{(A)}\ 7 \qquad\textbf{(B)}\ 8 \qquad\textbf{(C)}\ 9 \qquad\textbf{(D)}\ 10 \qquad\textbf{(E)}\ 11$

Solution

Let s be the side length of the small squares.

The diagonal of the big square can be written in two ways: $\sqrt{2}$ and $s\sqrt{2}+s+s\sqrt{2}$.

Solving for s, we get $s=\dfrac{4-\sqrt{2}}{7}$, so our answer is $4+7 \Rightarrow \boxed{\textbf{(E)} 11}$

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2016 AMC 10A Problems/Problem 13

Contents

Problem

Five friends sat in a movie theater in a row containing 5 seats, numbered 1 to 5 from left to right. (The directions "left" and "right" are from the point of view of the people as they sit in the seats.) During the movie Ada went to the lobby to get some popcorn. When she returned, she found that Bea had moved two seats to the right, Ceci had moved one seat to the left, and Dee and Edie had switched seats, leaving an end seat for Ada. In which seat had Ada been sitting before she got up?

$\textbf{(A) }1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 4\qquad \textbf{(E) } 5$

Solution 1

Bash: we see that the following configuration works.

Bea - Ada - Ceci - Dee - Edie

After moving, it becomes

Ada - Ceci - Bea - Edie - Dee.

Thus, Ada was in seat $\boxed{2}$.

Solution 2

Process of elimination of possible configurations.

Let's say that Ada=A, Bea=B, Ceci=C, Dee=D, and Edie=E.

Since B moved more to the right than C did left, this implies that B was in a LEFT end seat originally:

$$B, -, C \rightarrow -, C, B$$

This is affirmed because $DE \rightarrow ED$, which there is no new seats uncovered. So A, B, C are restricted to the same $1, 2, 3$ seats. Thus, it must be $B, A, C \rightarrow A, C, B$, and more specifically:

$$B, A, C, D, E \rightarrow A, C, B, E, D$$

So A, Ada, was originally in seat $\boxed{\textbf{(B) } 2}$.

Solution 3

The seats are numbered 1 through 5, so let each letter (A, B, C, D, E) correspond to a number. Let a move to the left be subtraction and a move to the right be addition.

We know that $1 + 2 + 3 + 4 + 5 = A + B + C + D + E = 15$. After everyone moves around, however, our equation looks like $(A + x) + B + 2 + C - 1 + D + E = 15$ because D and E switched seats, B moved two to the right, and C moved 1 to the left.

For this equation to be true, x has to be -1, meaning A moves 1 left from her original seat. Since A is now sitting in a corner seat, the only possible option for the original placement of A is in seat number $\boxed{\textbf{(B) } 2}$.

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2016 AMC 12A Problems/Problem 11

Contents

Problem

Each of the 100 students in a certain summer camp can either sing, dance, or act. Some students have more than one talent, but no student has all three talents. There are 42 students who cannot sing, 65 students who cannot dance, and 29 students who cannot act. How many students have two of these talents?

$\textbf{(A)}\ 16\qquad\textbf{(B)}\ 25\qquad\textbf{(C)}\ 36\qquad\textbf{(D)}\ 49\qquad\textbf{(E)}\ 64$

Solution

Let a be the number of students that can only sing, b can only dance, and c can only act.

Let ab be the number of students that can sing and dance, ac can sing and act, and bc can dance and act.

From the information given in the problem, $a + ab + b = 29, b + bc + c = 42,$ and $a + ac + c = 65$.

Adding these equations together, we get $2(a + b + c) + ab + bc + ac = 136$.

Since there are a total of 100 students, $a + b + c + ab + bc + ac = 100$.

Subtracting these equations, we get $a + b + c = 36$.

Our answer is $ab + bc + ac = 100 - (a + b + c) = 100 - 36 = \boxed{\textbf{(E)}\ 64}$

Solution 2

An easier way to solve the problem: Since 42 students cannot sing, there are $100-42=58$ students who can.

Similarly 65 students cannot dance, there are $100-65=35$ students who can.

And 29 students cannot act, there are $100-29=71$ students who can.

Therefore, there are $58+35+71=164$ students in all ignoring the overlaps between 2 of 3 talent categories. There are no students who have all 3 talents, nor those who have none (0), so only 1 or 2 talents are viable.

Thus, there are $164-100=\boxed{\textbf{(E)}\ 64}$ students who have 2 of 3 talents.

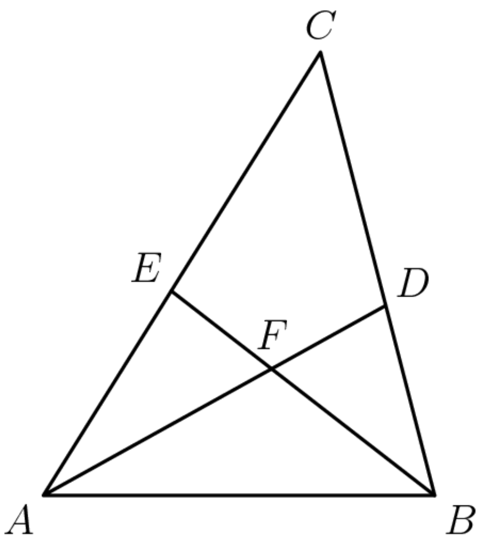
See Also

2016 AMC 12A Problems/Problem 12

Contents

Problem 12

In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio AF : FD?



$\textbf{(A)}\ 3:2\qquad\textbf{(B)}\ 5:3\qquad\textbf{(C)}\ 2:1\qquad\textbf{(D)}\ 7:3\qquad\textbf{(E)}\ 5:2$

Solution 1

Applying the angle bisector theorem to $\triangle ABC$ with $\angle CAB$ being bisected by AD, we have

$$\frac{CD}{AC} = \frac{BD}{AB}.$$

Thus, we have

$$\frac{CD}{8} = \frac{BD}{6},$$

and cross multiplying and dividing by 2 gives us

$$3 \cdot CD = 4 \cdot BD.$$

Since $CD + BD = BC = 7$, we can substitute $CD = 7 - BD$ into the former equation. Therefore, we get $3(7 - BD) = 4BD$, so $BD = 3$.

Apply the angle bisector theorem again to $\triangle ABD$ with $\angle ABC$ being bisected. This gives us

$$\frac{AB}{AF} = \frac{BD}{FD},$$

and since $AB = 6$ and $BD = 3$, we have

$$\frac{6}{AF} = \frac{3}{FD}.$$

Cross multiplying and dividing by 3 gives us

$$AF = 2 \cdot FD,$$

and dividing by FD gives us

$$\frac{AF}{FD} = \frac{2}{1}.$$

Therefore,

$$AF : FD = \frac{AF}{FD} = \frac{2}{1} = \boxed{\textbf{(C) } 2 : 1}.$$

Solution 2

By the angle bisector theorem, $\frac{AB}{AE} = \frac{CB}{CE}$

$\frac{6}{AE} = \frac{7}{8 - AE}$ so $AE = \frac{48}{13}$

Similarly, $CD = 4$.

Now, we use mass points. Assign point C a mass of 1.

$mC \cdot CD = mB \cdot DB$, so $mB = \frac{4}{3}$

Similarly, A will have a mass of $\frac{7}{6}$

$mD = mC + mB = 1 + \frac{4}{3} = \frac{7}{3}$

So $\frac{AF}{AD} = \frac{mD}{mA} = \boxed{\textbf{(C) } 2 : 1}$

Solution 3

Denote $[\triangle ABC]$ as the area of triangle ABC and let r be the inradius. Also, as above, use the angle bisector theorem to find that $BD = 3$. There are two ways to continue from here:

1. Note that F is the incenter. Then, $\frac{AF}{FD} = \frac{[\triangle AFB]}{[\triangle BFD]} = \frac{AB * \frac{r}{2}}{BD * \frac{r}{2}} = \frac{AB}{BD} = \boxed{\textbf{(C)}\; 2 : 1}$

2. Apply the angle bisector theorem on $\triangle ABD$ to get $\frac{AF}{FD} = \frac{AB}{BD} = \frac{6}{3} = \boxed{\textbf{(C)}\; 2 : 1}$

Solution 4

INDUCTIVE REASONING IS STILL REASONING

Tear a piece of your scrap paper and mark the length of AF on it with your pencil. Do the same for FD. Clearly AF is twice FD. Thus $\frac{AF}{FD} = \boxed{\textbf{(C)}\; 2 : 1}$

See Also

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2016 AMC 10A Problems/Problem 17

Contents

Problem

Let N be a positive multiple of 5. One red ball and N green balls are arranged in a line in random order. Let $P(N)$ be the probability that at least $\tfrac{3}{5}$ of the green balls are on the same side of the red ball. Observe that $P(5)=1$ and that $P(N)$ approaches $\tfrac{4}{5}$ as N grows large. What is the sum of the digits of the least value of N such that $P(N) < \tfrac{321}{400}$?

$\textbf{(A) } 12 \qquad \textbf{(B) } 14 \qquad \textbf{(C) } 16 \qquad \textbf{(D) } 18 \qquad \textbf{(E) } 20$

Solution

Let $n = \dfrac{N}{5}$. Then, consider 5 blocks of n green balls in a line, along with the red ball. Shuffling the line is equivalent to choosing one of the $N + 1$ positions between the green balls to insert the red ball. Less than $\dfrac{3}{5}$ of the green balls will be on the same side of the red ball if the red ball is inserted in the middle block of n balls, and there are $n - 1$ positions where this happens. Thus, $P(N) = 1 - \dfrac{n - 1}{N + 1} = \dfrac{4n + 2}{5n + 1}$, so

$$P(N) = \frac{4n+2}{5n+1} < \frac{321}{400}.$$

Multiplying both sides of the inequality by $400(5n+1)$, we have

$$400(4n+2) < 321(5n+1),$$

and by the distributive property,

$$1600n + 800 < 1605n + 321.$$

Subtracting $1600n + 321$ on both sides of the inequality gives us

$$479 < 5n.$$

Therefore, $N = 5n > 479$, so the least possible value of N is 480. The sum of the digits of 480 is $\boxed{\textbf{(A) } 12}$.

Pattern Solution

Let $N=5$, $P(N)=1$ $(Given)$

Let $N = 10, \; P(N) = \frac{10}{11}$

Let $N = 15, \; P(N) = \frac{14}{16}$

Notice that the fraction can be written as $1 - \frac{\frac{N}{5} - 1}{N + 1}$

Now it's quite simple to write the inequality as $1 - \frac{\frac{N}{5} - 1}{N + 1} < \frac{321}{400}$

We can subtract 1 on both sides to obtain $-\frac{\frac{N}{5} - 1}{N + 1} < -\frac{79}{400}$

Dividing both sides by -1, we derive $\frac{\frac{N}{5} - 1}{N + 1} > \frac{79}{400}$. (Switch the inequality sign when dividing by -1)

We then cross multiply to get $80N - 400 > 79N + 79$

Finally we get $N > 479$

To achieve $N = 480$

So the sum of the digits of N = $\boxed{\textbf{(A) } 12}$

- Solution by $AOPS12142015$

See Also

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2016 AMC 10A Problems/Problem 18

Contents

Problem

Each vertex of a cube is to be labeled with an integer 1 through 8, with each integer being used once, in such a way that the sum of the four numbers on the vertices of a face is the same for each face. Arrangements that can be obtained from each other through rotations of the cube are considered to be the same. How many different arrangements are possible?

$\textbf{(A)}\ 1\qquad\textbf{(B)}\ 3\qquad\textbf{(C)}\ 6\qquad\textbf{(D)}\ 12\qquad\textbf{(E)}\ 24$

Solution 1

First of all, the adjacent faces have same sum $(18$, because $1+2+3+4+5+6+7+8=36$, $36/2=18)$, consider the $opposite\ sides$ (the two sides which are parallel but not in same face of the cube) they must have same sum value too. Now think about the extreme condition 1 and 8 , if they are not sharing the same side, which means they would become end points of $opposite\ sides$, we should have $1+X=8+Y$, but no solution for $[2,7]$, contradiction.

Now we know 1 and 8 must share the same side, which sum is 9, the $opposite\ side$ also must have sum of 9 , same thing for the other two parallel sides.

Now we have 4 parallel sides $1-8,2-7,3-6,4-5$. thinking about 4 end points number need to have sum of 18. it is easy to notice only $1-7-6-4$ vs $8-2-3-5$ would work.

So if we fix one direction $1-8($or $8-1)$ all other 3 parallel sides must lay in one particular direction. $(1-8,7-2,6-3,4-5)$ or $(8-1,2-7,3-6,5-4)$

Now, the problem is same as the problem to arrange 4 points in a $2-D$ square. which is $4!/4=\boxed{\textbf{(C)}\ 6.}$

Solution 2

Again, all faces sum to $18.$ If x,y,z are the vertices next to one, then the remaining vertices are $17-x-y,17-y-z,17-x-z,x+y+z-16.$ Now it remains to test possibilities. Note that we must have $x+y+z>17.$ WLOG let $x<y<z.$

$3,7,8:$ Does not work. $4,6,8:$ Works. $5,6,7:$ Does not work. $5,6,8:$ Works. $5,7,8:$ Does not work. $6,7,8:$ Works.

So our answer is $3\cdot 2=\boxed{\textbf{(C)}\ 6.}$

Solution 3

We know the sum of each face is $18.$ If we look at an edge of the cube whose numbers sum to x, it must be possible to achieve the sum $18-x$ in two distinct ways, looking at the two faces which contain the edge. If 8 and 6 were on the same face, it is possible to achieve the desired sum only with the numbers 1 and 3 since the values must be distinct. Similarly, if 8 and 7 were on the same face, the only way to get the sum is with 1 and 2. This means that 6 and 7 are not on the same edge as 8, or in other words they are diagonally across from it on the same face, or on the other end of the cube.

Now we look at three cases, each yielding two solutions which are reflections of each other:

1) 6 and 7 are diagonally opposite 8 on the same face. 2) 6 is diagonally across the cube from 8, while 7 is diagonally across from 8 on the same face. 3) 7 is diagonally across the cube from 8, while 6 is diagonally across from 8 on the same face.

This means the answer is $3 \cdot 2 = \boxed{\textbf{(C) } 6.}$

See Also

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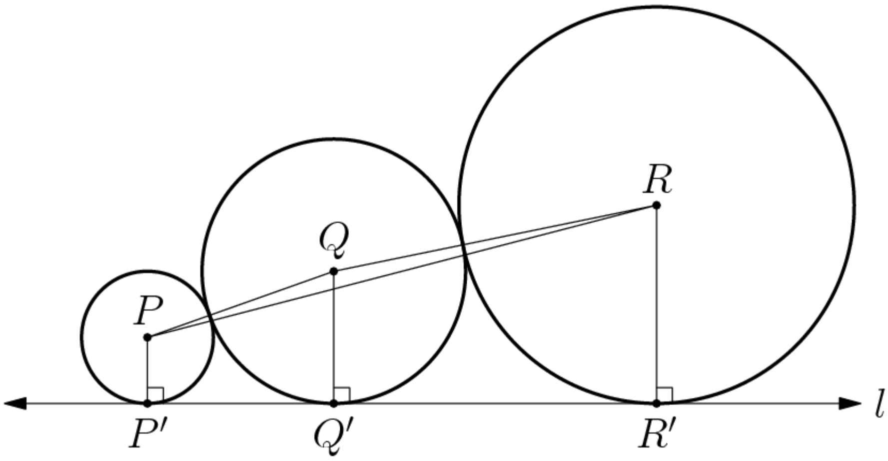
2016 AMC 10A Problems/Problem 21

Circles with centers P, Q and R, having radii $1, 2$ and 3, respectively, lie on the same side of line l and are tangent to l at P', Q' and R', respectively, with Q' between P' and R'. The circle with center Q is externally tangent to each of the other two circles. What is the area of triangle PQR?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ \sqrt{\frac{2}{3}} \qquad \textbf{(C)}\ 1 \qquad \textbf{(D)}\ \sqrt{6}-\sqrt{2} \qquad \textbf{(E)}\ \sqrt{\frac{3}{2}}$

Solution



Notice that we can find $[P'PQRR']$ in two different ways: $[P'PQQ'] + [Q'QRR']$ and $[PQR] + [P'PRR']$, so $[P'PQQ'] + [Q'QRR'] = [PQR] + [P'PRR']$

$P'Q' = \sqrt{PQ^2 - (QQ' - PP')^2} = \sqrt{9 - 1} = \sqrt{8} = 2\sqrt{2}$. Additionally, $Q'R' = \sqrt{QR^2 - (RR' - QQ')^2} = \sqrt{5^2 - 1^2} = \sqrt{24} = 2\sqrt{6}$. Therefore, $[P'PQQ'] = \dfrac{P'P + Q'Q}{2} * 2\sqrt{2} = \dfrac{1+2}{2} * 2\sqrt{2} = 3\sqrt{2}$. Similarly, $[Q'QRR'] = 5\sqrt{6}$. We can calculate $[P'PRR']$ easily because $P'R' = P'Q' + Q'R' = 2\sqrt{2} + 2\sqrt{6}$. $[P'PRR'] = 4\sqrt{2} + 4\sqrt{6}$.

Plugging into first equation, the two sums of areas, $3\sqrt{2} + 5\sqrt{6} = 4\sqrt{2} + 4\sqrt{6} + [PQR]$.

$[PQR] = \sqrt{6} - \sqrt{2} \rightarrow \boxed{\text{D}}$.

Solution 2

Use the Shoelace Theorem.

Let the center of the first circle of radius 1 be at (0, 1).

Draw the trapezoid $PQQ'P'$ and using the Pythagorean Theorem, we get that $P'Q' = 2\sqrt{2}$ so the center of the second circle of radius 2 is at $(2\sqrt{2}, 2)$.

Draw the trapezoid $QRR'Q'$ and using the Pythagorean Theorem, we get that $Q'R' = 2\sqrt{6}$ so the center of the third circle of radius 3 is at $(2\sqrt{2} + 2\sqrt{6}, 3)$.

Now, we may use the Shoelace Theorem!

$(0, 1)$

$(2\sqrt{2}, 2)$

$(2\sqrt{2}+2\sqrt{6}, 3)$

$$\frac{1}{2}|(2\sqrt{2}+4\sqrt{2}+4\sqrt{6})-(6\sqrt{2}+2\sqrt{2}+2\sqrt{6})|$$

$$=\sqrt{6}-\sqrt{2}\ \boxed{\text{D}}.$$

See Also

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2016 AMC 12A Problems/Problem 16

Contents

Problem 16

The graphs of $y = \log_3 x, y = \log_x 3, y = \log_{\frac{1}{3}} x$, and $y = \log_x \dfrac{1}{3}$ are plotted on the same set of axes. How many points in the plane with positive x-coordinates lie on two or more of the graphs?

$\textbf{(A)}\ 2\qquad\textbf{(B)}\ 3\qquad\textbf{(C)}\ 4\qquad\textbf{(D)}\ 5\qquad\textbf{(E)}\ 6$

Solution

Setting the first two equations equal to each other, $\log_3 x = \log_x 3$.

Solving this, we get $(3,1)$ and $\left(\frac{1}{3}, -1\right)$.

Similarly with the last two equations, we get $(3,-1)$ and $\left(\frac{1}{3}, 1\right)$.

Now, by setting the first and third equations equal to each other, we get $(1,0)$.

Pairing the first and fourth or second and third equations won't work because then $\log x \le 0$.

Pairing the second and fourth equations will yield $x = 1$, but since you can't divide by $\log 1 = 0$, it doesn't work.

After trying all pairs, we have a total of 5 solutions $\rightarrow \boxed{\textbf{(D)} 5}$

Solution 2

Note that $\log_b a = \log_c a / \log_c b$.

Then $\log_b a = \log_a a / \log_a b = 1 / \log_a b$

$\log_{\frac{1}{a}} b = \log_a \frac{1}{a} / \log_a b = -1 / \log_a b$

$\log_{\frac{1}{b}} a = -\log_a b$

Therefore, the system of equations can be simplified to:

$y = t$

$y = -t$

$y = \dfrac{1}{t}$

$$y = -\frac{1}{t}$$

where $t = \log_3 x$. Note that all values of t correspond to exactly one positive x value, so all (t, y) intersections will correspond to exactly one (x, y) intersection in the positive-x area.

Graphing this system of easy-to-graph functions will generate a total of 5 solutions $\rightarrow \boxed{\textbf{(D)} 5}$

See Also

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2016 AMC 12A Problems/Problem 17

Problem 17

Let $ABCD$ be a square. Let E, F, G and H be the centers, respectively, of equilateral triangles with bases $\overline{AB}, \overline{BC}, \overline{CD},$ and $\overline{DA},$ each exterior to the square. What is the ratio of the area of square $EFGH$ to the area of square $ABCD$?

$$\textbf{(A)}\ 1\qquad\textbf{(B)}\ \frac{2+\sqrt{3}}{3}\qquad\textbf{(C)}\ \sqrt{2}\qquad\textbf{(D)}\ \frac{\sqrt{2}+\sqrt{3}}{2}\qquad\textbf{(E)}\ \sqrt{3}$$

Solution

The center of an equilateral triangle is its centroid, where the three medians meet.

The distance along the median from the centroid to the base is one third the length of the median.

Let the side length of the square be 1. The height of $\triangle E$ is $\frac{\sqrt{3}}{2}$, so the distance from E to the midpoint of $\overline{AB}$ is $\frac{\sqrt{3}}{2} \cdot \frac{1}{3} = \frac{\sqrt{3}}{6}$

$EG = 2 \cdot \frac{\sqrt{3}}{6}$ (from above) $+1$ (side length of the square).

Since EG is the diagonal of square $EFGH$, $\frac{[EFGH]}{ABCD} = \frac{\frac{EG^2}{2}}{1^2} = \boxed{\textbf{(B)}\ \frac{2+\sqrt{3}}{3}}$

See Also

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2016 AMC 10A Problems/Problem 22

Contents

Problem

For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

$\textbf{(A) }110\qquad\textbf{(B) }191\qquad\textbf{(C) }261\qquad\textbf{(D) }325\qquad\textbf{(E) }425$

Solution 1

Since the prime factorization of 110 is $2 \cdot 5 \cdot 11$, we have that the number is equal to $2 \cdot 5 \cdot 11 \cdot n^3$. This has $2 \cdot 2 \cdot 2=8$ factors when $n=1$. This needs a multiple of 11 factors, which we can achieve by setting $n=2^3$, so we have $2^{10} \cdot 5 \cdot 11$ has 44 factors. To achieve the desired 110 factors, we need the number of factors to also be divisible by 5, so we can set $n=2^3 \cdot 5$, so $2^{10} \cdot 5^4 \cdot 11$ has 110 factors. Therefore, $n=2^3 \cdot 5$. In order to find the number of factors of $81n^4$, we raise this to the fourth power and multiply it by 81, and find the factors of that number. We have $3^4 \cdot 2^{12} \cdot 5^4$, and this has $5 \cdot 13 \cdot 5=\boxed{\textbf{(D) }325}$ factors.

Solution 2

$110n^3$ clearly has at least three distinct prime factors, namely 2, 5, and 11.

Furthermore, since the number of factors of $p_1^{n_1}\cdots p_k^{n_k}$ is $(n_1+1)\cdots(n_k+1)$ when the p's are distinct primes, we see that there can be at most three distinct prime factors for a number with 110 factors.

We conclude that $110n^3$ has only the three prime factors 2, 5, and 11 and that the multiplicities are 1, 4, and 10 in some order. I.e., there are six different possible values of n all of the form $n=p_1\cdot p_2^3$.

$81n^4$ thus has prime factorization $81n^4=3^4\cdot p_1^4\cdot p_2^{12}$ and a factor count of $5\cdot 5\cdot 13=\boxed{\textbf{(D) }325}$

See Also

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2016 AMC 12A Problems/Problem 19

Problem

Jerry starts at 0 on the real number line. He tosses a fair coin 8 times. When he gets heads, he moves 1 unit in the positive direction; when he gets tails, he moves 1 unit in the negative direction. The probability that he reaches 4 at some time during this process $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a + b$? (For example, he succeeds if his sequence of tosses is $HTHHHHHH$.)

$\textbf{(A)}\ 69\qquad\textbf{(B)}\ 151\qquad\textbf{(C)}\ 257\qquad\textbf{(D)}\ 293\qquad\textbf{(E)}\ 313$

Solution

For 6 to 8 heads, we are guaranteed to hit 4 heads, so the sum here is $\binom{8}{2} + \binom{8}{1} + \binom{8}{0} = 28 + 8 + 1 = 37$.

For 4 heads, you have to hit the 4 heads at the start so there's only one way, 1.

For 5 heads, we either start off with 4 heads, which gives us $\textbf{4C1} = 4$ ways to arrange the other flips, or we start off with five heads and one tail, which has 6 ways minus the 2 overlapping cases, $\textbf{HHHHHTTT}$ and $\textbf{HHHHTHTT}$. Total ways: 8.

Then we sum to get 46. There are a total of $2^8 = 256$ possible sequences of 8 coin flips, so the probability is $\frac{46}{256} = \frac{23}{128}$. Summing, we get $23 + 128 = \boxed{\textbf{(B)}\ 151}$.

See Also

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2016 AMC 10A Problems/Problem 23

Contents

Problem

A binary operation $\diamondsuit$ has the properties that $a\,\diamondsuit\,(b\,\diamondsuit\,c) = (a\,\diamondsuit\,b)\cdot c$ and that $a\,\diamondsuit\,a = 1$ for all nonzero real numbers $a, b,$ and c. (Here $\cdot$ represents multiplication). The solution to the equation $2016\,\diamondsuit\,(6\,\diamondsuit\,x) = 100$ can be written as $\tfrac{p}{q}$, where p and q are relatively prime positive integers. What is $p+q?$

$\textbf{(A) }109\qquad\textbf{(B) }201\qquad\textbf{(C) }301\qquad\textbf{(D) }3049\qquad\textbf{(E) }33,601$

Solution

Solution 1

We see that $a \diamond a = 1$, and think of division. Testing, we see that the first condition $a \diamond (b \diamond c) = (a \diamond b) \cdot c$ is satisfied, because $\frac{a}{\frac{b}{c}} = \frac{a}{b} \cdot c$. Therefore, division is the operation $\diamond$. Solving the equation,

$$\frac{2016}{\frac{6}{x}} = \frac{2016}{6} \cdot x = 336x = 100 \implies x = \frac{100}{336} = \frac{25}{84},$$

so the answer is $25 + 84 = \boxed{\textbf{(A) }109.}$

Solution 2

We can manipulate the given identities to arrive at a conclusion about the binary operator $\diamondsuit$. Substituting $b = c$ into the second identity yields

$$(a\,\diamondsuit\,b){\cdot}b = a\,\diamondsuit\,(b\,\diamondsuit\,b) = a\,\diamondsuit\,1 = a\,\diamondsuit\,(a\,\diamondsuit\,a) = (a\,\diamondsuit\,a){\cdot}a = a.$$

Hence, $(a\,\diamondsuit\,b)\cdot b = a,$ or, dividing both sides of the equation by $b,$ $(a\,\diamondsuit\,b) = \frac{a}{b}.$

Hence, the given equation becomes $\frac{2016}{\frac{6}{x}} = 100$. Solving yields $x = \frac{100}{336} = \frac{25}{84},$ so the answer is $25 + 84 = \boxed{\textbf{(A) }109.}$

Solution 3

One way to eliminate the $\diamondsuit$ in this equation is to make $a = b$ so that $a\,\diamondsuit\,(b\,\diamondsuit\,c) = c$. In this case, we can make $b = 2016$.

$$2016 \diamondsuit (6 \diamondsuit x) = 100 \implies (2016 \diamondsuit 6) \cdot x = 100$$

By multiplying both sides by $\frac{6}{x}$, we get:

$$(2016 \diamondsuit 6) \cdot 6 = \frac{600}{x} \implies 2016 \diamondsuit (6 \diamondsuit 6) = \frac{600}{x}$$

Because $6 \diamondsuit 6 = 2016 \diamondsuit 2016 = 1$:

$$2016 \diamondsuit (2016 \diamondsuit 2016) = \frac{600}{x} \implies (2016 \diamondsuit 2016) \cdot 2016 = \frac{600}{x} \implies 2016 = \frac{600}{x}$$

Therefore, $x = \frac{600}{2016} = \frac{25}{84}$, so the answer is $25 + 84 = \boxed{\textbf{(A) } 109.}$

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2016 AMC 10A Problems/Problem 24

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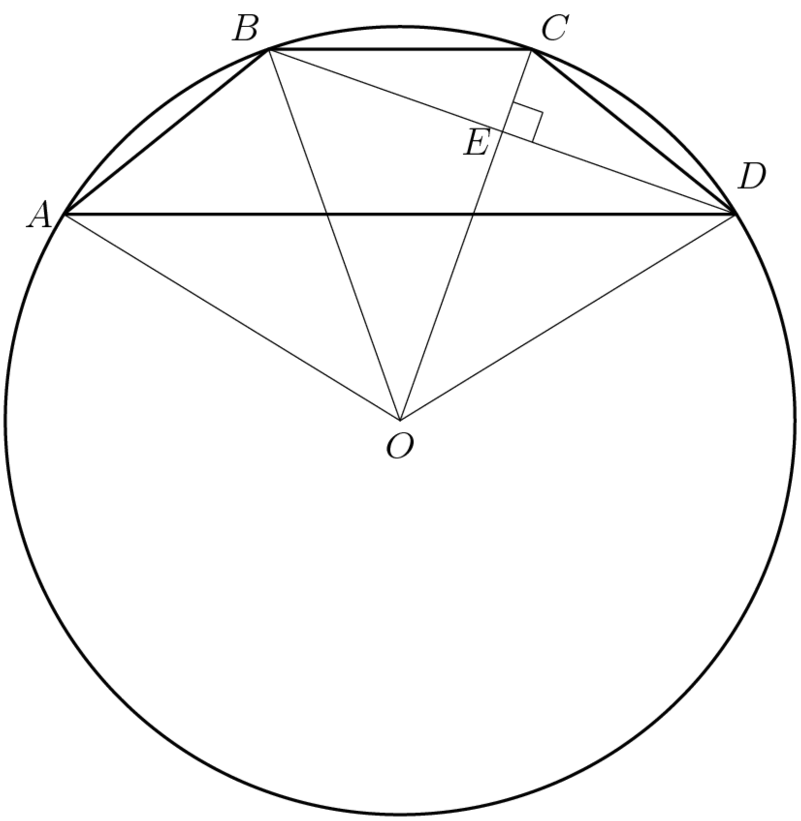
Problem

A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

$\textbf{(A) }200\qquad \textbf{(B) }200\sqrt{2}\qquad\textbf{(C) }200\sqrt{3}\qquad\textbf{(D) }300\sqrt{2}\qquad\textbf{(E) } 500$

Solution 1 (Algebra)

To save us from getting big numbers with lots of zeros behind them, let's divide all side lengths by 200 for now, then multiply it back at the end of our solution.



Construct quadrilateral $ABCD$ on the circle with AD being the missing side (Notice that since the side length is less than the radius, it will be very small on the top of the circle). Now, draw the radii from center O to $A, B, C,$ and D. Let the intersection of BD and OC be point E. Notice that BD and OC are perpendicular because $BCDO$ is a kite.

We set lengths $BE = ED$ equal to x. By the Pythagorean Theorem,

$$\sqrt{1^2 - x^2} + \sqrt{(\sqrt{2})^2 - x^2} = \sqrt{2}$$

We solve for x:

$$1 - x^2 + 2 - x^2 + 2\sqrt{(1 - x^2)(2 - x^2)} = 2$$

$$2\sqrt{(1 - x^2)(2 - x^2)} = 2x^2 - 1$$

$$4(1 - x^2)(2 - x^2) = (2x^2 - 1)^2$$

$$8 - 12x^2 + 4x^4 = 4x^4 - 4x^2 + 1$$

$$8x^2 = 7$$

$$x = \frac{\sqrt{14}}{4}$$

By Ptolemy's Theorem,

$$AB \cdot CD + BC \cdot AD = AC \cdot BD = BD^2 = (2 \cdot BE)^2$$

Substituting values,

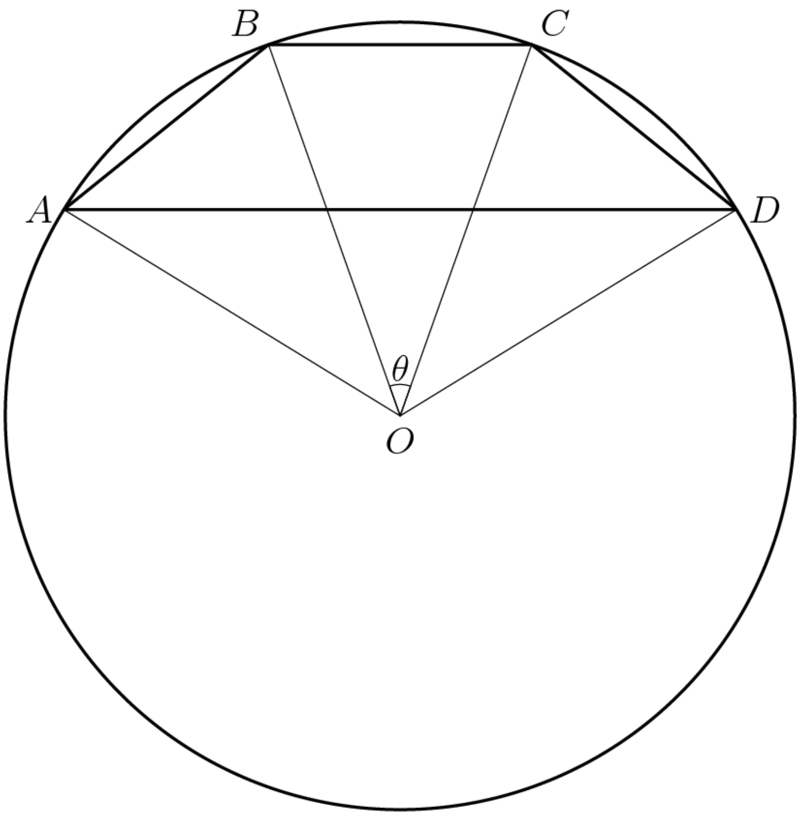
$$1^2 + 1 \cdot AD = 4\left(\frac{\sqrt{14}}{4}\right)^2$$

$$1 + AD = \frac{7}{2}$$

$$AD = \frac{5}{2}$$

Finally, we multiply back the 200 that we divided by at the beginning of the problem to get $AD = \boxed{500(E)}$.

Solution 2 (Trigonometry Bash)



Construct quadrilateral $ABCD$ on the circle with AD being the missing side (Notice that since the side length is less than the radius, it will be very small on the top of the circle). Now, draw the radii from center O to $A, B, C,$ and D. Apply law of cosines on ΔBOC; let $\theta = \angle BOC$. We get the following equation:

$$(BC)^2 = (OB)^2 + (OC)^2 - 2 \cdot OB \cdot OC \cdot \cos\theta$$

Substituting the values in, we get

$$(200)^2 = 2 \cdot (200)^2 + 2 \cdot (200)^2 - 2 \cdot 2 \cdot (200)^2 \cdot \cos\theta$$

Canceling out, we get

$$\cos\theta = \frac{3}{4}$$

Because $\angle AOB$, $\angle BOC$, and $\angle COD$ are congruent, $\angle AOD = 3\theta$. To find the remaining side (AD), we simply have to apply the law of cosines to ΔAOD . Now, to find $\cos 3\theta$, we can derive a formula that only uses $\cos\theta$:

$$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \cdot (2\sin\theta\cos\theta)$$

$$\cos 3\theta = \cos\theta(\cos 2\theta - 2\sin^2\theta) = \cos\theta(2\cos^2\theta - 1 + 2\cos^2\theta)$$

$$\Rightarrow \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

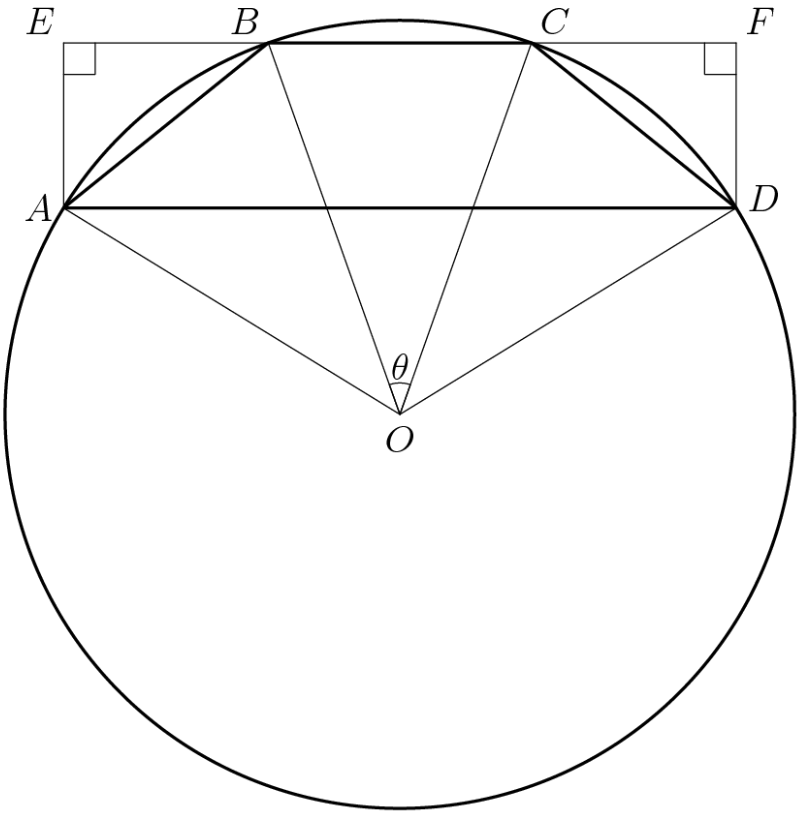
Plugging in $\cos\theta = \frac{3}{4}$, we get $\cos 3\theta = -\frac{9}{16}$. Now, applying law of cosines on triangle OAD, we get

$$(AD)^2 = 2 \cdot (200)^2 + 2 \cdot (200)^2 + 2 \cdot 200\sqrt{2} \cdot 200\sqrt{2} \cdot \frac{9}{16}$$

$$\Rightarrow 2 \cdot (200)^2 \cdot (1 + 1 + \frac{9}{8}) = (200)^2 \cdot \frac{25}{4}$$

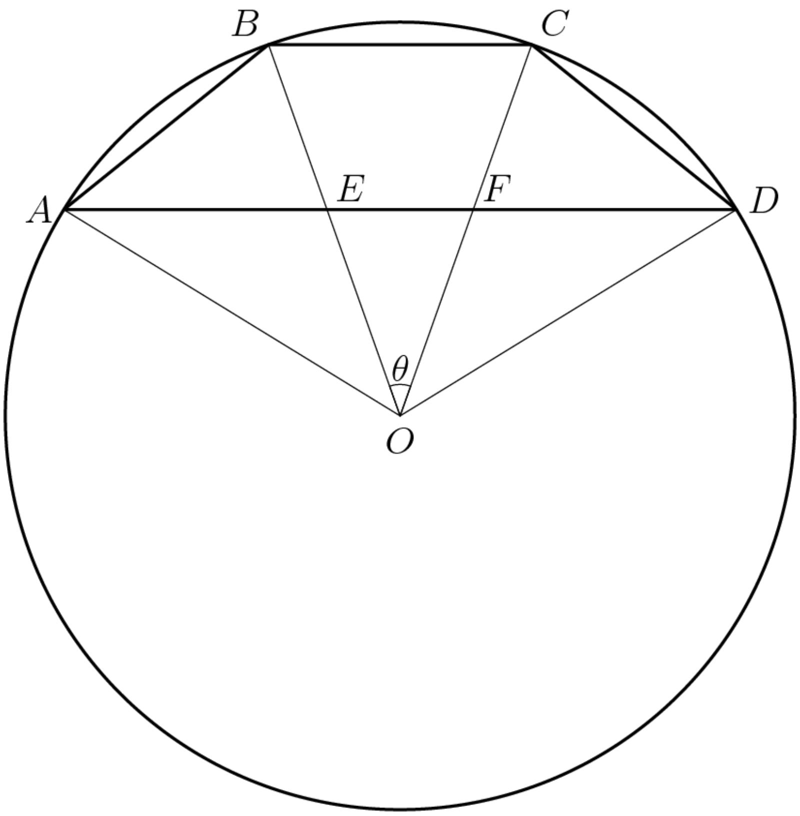
$$AD = 200 \cdot \frac{5}{2} = \boxed{500}$$

Solution 3 (Easier trig)



Construct quadrilateral $ABCD$ on the circle O with AD being the missing side. Then, drop perpendiculars from A and D to (extended) line BC, and let these points be E and F, respectively. Also, let $\theta = \angle BOC$. From Law of Cosines on $\triangle BOC$, we have $\cos\theta = \frac{3}{4}$. Now, since $\triangle BOC$ is isosceles with $OB = OC$, we have that $\angle BCO = \angle CBO = 90 - \frac{\theta}{2}$. By SSS congruence, we have that $\triangle OBC \cong \triangle OCD$, so we have that $\angle OCD = \angle BCO = 90 - \frac{\theta}{2}$, so $\angle DCF = \theta$. Thus, we have $\frac{FC}{DC} = \cos\theta = \frac{3}{4}$, so $FC = 150$. Similarly, $BE = 150$, and $AD = 150 + 200 + 150 = \boxed{500}$.

Solution 4 (Just Geometry)



Label AD intercept OB at E and OC at F.

$$\overset{\frown}{\mathrm{AB}} = \overset{\frown}{\mathrm{BC}} = \overset{\frown}{\mathrm{CD}} = \theta$$

$$\angle BAD = \frac{1}{2} \cdot \overset{\frown}{\mathrm{BCD}} = \theta = \angle AOB$$

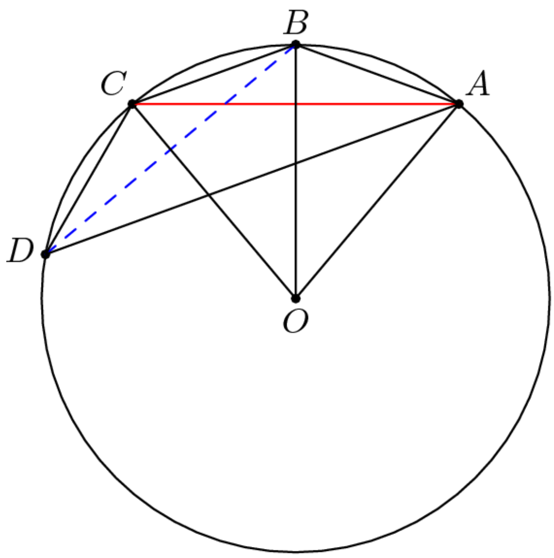
From there, $\triangle OAB \sim \triangle ABE$, thus:

$$\frac{OA}{AB} = \frac{AB}{BE} = \frac{OB}{AE}$$

$OA = OB$ because they are both radii of $\odot O$. Since $\frac{OA}{AB} = \frac{OB}{AE}$, we have that $AB = AE$. Similarly, $CD = DF$.

$OE = 100\sqrt{2} = \frac{OB}{2}$ and $EF = \frac{BC}{2} = 100$, so
$AD = AE + EF + FD = 200 + 100 + 200 = \boxed{\textbf{(E) } 500}$

Solution 5 (Ptolemy's Theorem)



Let $s = 200$. Let O be the center of the circle. Then AC is twice the altitude of $\triangle OBC$. Since $\triangle OBC$ is isosceles we can compute its area to be $s^2\sqrt{7}/4$, hence $CA = 2\frac{2\cdot s^2\sqrt{7}/4}{s\sqrt{2}} = s\sqrt{7/2}$.

Now by Ptolemy's Theorem we have $CA^2 = s^2 + AD \cdot s \implies AD = (7/2 - 1)s$. This gives us:

$$\boxed{\textbf{(E) } 500.}$$

Solution 6 (Trigonometry)

Since all three sides equal 200, they subtend three equal angles from the center. The right triangle between the center of the circle, a vertex, and the midpoint between two vertices has side lengths $100, 100\sqrt{7}, 200\sqrt{2}$ by the Pythagorean Theorem. Thus, the sine of half of the subtended angle is $\dfrac{100}{200\sqrt{2}} = \dfrac{\sqrt{2}}{4}$. Similarly, the cosine is $\dfrac{100\sqrt{7}}{200\sqrt{2}} = \dfrac{\sqrt{14}}{4}$. Since there are three sides, and since $\sin\theta = \sin(180-\theta)$, we seek to find $2r\sin 3\theta$.

First, $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\left(\dfrac{\sqrt{2}}{4}\right)\left(\dfrac{\sqrt{14}}{4}\right) = \dfrac{2\sqrt{2}\sqrt{14}}{16} = \dfrac{\sqrt{7}}{4}$ and $\cos 2\theta = \dfrac{3}{4}$ by Pythagorean.

$$\sin 3\theta = \sin(2\theta+\theta) = \sin 2\theta\cos\theta + \sin\theta\cos 2\theta = \frac{\sqrt{7}}{4}\left(\frac{\sqrt{14}}{4}\right) + \frac{\sqrt{2}}{4}\left(\frac{3}{4}\right) = \frac{7\sqrt{2}+3\sqrt{2}}{16} = \frac{5\sqrt{2}}{8}$$

$$2r\sin 3\theta = 2\left(200\sqrt{2}\right)\left(\frac{5\sqrt{2}}{8}\right) = 400\sqrt{2}\left(\frac{5\sqrt{2}}{8}\right) = \frac{800\cdot 5}{8} = \boxed{\textbf{(E) } 500}$$

Solution 7 (Area)

For simplicity, scale everything down by a factor of 100. Let the inscribed trapezoid be $ABCD$, where $AB = BC = CD = 2$ and DA is the missing side length. Let $DA = 2x$. If M and N are the midpoints of BC and AD, respectively, the height of the trapezoid is $OM - ON$. By the pythagorean theorem, $OM = \sqrt{OB^2 - BM^2} = \sqrt{7}$ and $ON = \sqrt{OA^2 - AN^2} = \sqrt{8 - x^2}$. Thus the height of the trapezoid is $\sqrt{7} - \sqrt{8-x^2}$, so the area is $\dfrac{(2+2x)(\sqrt{7}-\sqrt{8-x^2})}{2} = (x+1)(\sqrt{7}-\sqrt{8-x^2})$. By Brahmagupta's formula, the area is $\sqrt{(x+1)(x+1)(x+1)(3-x)}$. Setting these two equal, we get $(x+1)(\sqrt{7}-\sqrt{8-x^2}) = \sqrt{(x+1)(x+1)(x+1)(3-x)}$. Dividing both sides by $x+1$ and then squaring, we get $7 - 2(\sqrt{7})(\sqrt{8-x^2}) + 8 - x^2 = (x+1)(3-x)$. Expanding the right hand side and canceling the x^2 terms gives us $15-2(\sqrt{7})(\sqrt{8-x^2})=2x+3$. Rearranging and dividing by two, we get $(\sqrt{7})(\sqrt{8-x^2})=6-x$. Squaring both sides, we get $56-7x^2=x^2-12x+36$. Rearranging, we get $8x^2-12x-20=0$. Dividing by 4 we get $2x^2-3x-5=0$. Factoring we get, $(2x-5)(x+1)=0$, and since x cannot be negative, we get $x=2.5$. Since $DA=2x$, $DA=5$. Scaling up by 100, we get $\boxed{\textbf{(E) } 500}$.

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2016 AMC 10A Problems/Problem 25

Contents

Problem

How many ordered triples (x, y, z) of positive integers satisfy $\operatorname{lcm}(x, y) = 72, \operatorname{lcm}(x, z) = 600$ and $\operatorname{lcm}(y, z) = 900$?

$\textbf{(A)}\ 15\qquad\textbf{(B)}\ 16\qquad\textbf{(C)}\ 24\qquad\textbf{(D)}\ 27\qquad\textbf{(E)}\ 64$

Solution

Solution 1

We prime factorize $72, 600,$ and 900. The prime factorizations are $2^3 \times 3^2$, $2^3 \times 3 \times 5^2$ and $2^2 \times 3^2 \times 5^2$, respectively. Let $x = 2^a \times 3^b \times 5^c$, $y = 2^d \times 3^e \times 5^f$ and $z = 2^g \times 3^h \times 5^i$. We know that

$$\max(a, d) = 3$$

$$\max(b, e) = 2$$

$$\max(a, g) = 3$$

$$\max(b, h) = 1$$

$$\max(c, i) = 2$$

$$\max(d, g) = 2$$

$$\max(e, h) = 2$$

and $c = f = 0$ since $\operatorname{lcm}(x, y)$ isn't a multiple of 5. Since $\max(d, g) = 2$ we know that $a = 3$. We also know that since $\max(b, h) = 1$ that $e = 2$. So now some equations have become useless to us...let's take them out.

$$\max(b, h) = 1$$

$$\max(d, g) = 2$$

are the only two important ones left. We do casework on each now. If $\max(b, h) = 1$ then $(b, h) = (1, 0), (0, 1)$ or $(1, 1)$. Similarly if $\max(d, g) = 2$ then $(d, g) = (2, 0), (2, 1), (2, 2), (1, 2), (0, 2)$. Thus our answer is $5 \times 3 = \boxed{15 (\text{A})}$.

Solution 2

It is well known that if the $\operatorname{lcm}(a,b)=c$ and c can be written as $p_1^a p_2^b p_3^c\ldots$, then the highest power of all prime numbers $p_1,p_2,p_3\ldots$ must divide into either a and/or b. Or else a lower $c_0=p_1^{a-\epsilon}p_2^{b-\epsilon}p_3^{c-\epsilon}\ldots$ is the lcm.

Start from x:$\operatorname{lcm}(x,y)=72$ so $8\mid x$ or $9\mid x$ or both. But $9\nmid x$ because $\operatorname{lcm}(x,z)=600$ and $9\nmid 600$. So $x=8,24$.

y can be $9,18,36$ in both cases of x but NOT 72 because $\operatorname{lcm} y,z=900$ and $72\nmid 900$.

So there are six sets of x,y and we will list all possible values of z based on those.

$25\mid z$ because z must source all powers of 5. $z\in\{25,50,75,100,150,300\}$. $z\neq\{200,225\}$ because of lcm restrictions.

By different sourcing of powers of 2 and 3,

$$(8,9): z=300$$

$$(8,18): z=300$$

$$(8,36): z=75,150,300$$

$$(24,9): z=100,300$$

$$(24,18): z=100,300$$

$$(24,36): z=25,50,75,100,150,300$$

$z=100$ is "enabled" by x sourcing the power of 3. $z=75,150$ is uncovered by y sourcing all powers of 2. And $z=25,50$ is uncovered by x and y both at full power capacity.

Counting the cases, $1+1+3+2+2+6=\boxed{\textbf{(A) } 15}$.

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2016 AMC 12A Problems/Problem 23

Contents

Problem

Three numbers in the interval $[0,1]$ are chosen independently and at random. What is the probability that the chosen numbers are the side lengths of a triangle with positive area?

$\textbf{(A)}\ \frac{1}{6}\qquad\textbf{(B)}\ \frac{1}{3}\qquad\textbf{(C)}\ \frac{1}{2}\qquad\textbf{(D)}\ \frac{2}{3}\qquad\textbf{(E)}\ \frac{5}{6}$

Solution

Solution 1: Super WLOG

WLOG assume a is the largest. Scale the triangle to $1, b/a, c/a$ or $1, x, y$ Then the solution is $\boxed{\textbf{(C)}\ 1/2}$ (Insert graph with square of side length 1 and the line $x+y=1$ that cuts it in half)

Solution 2: Conditional Probability

WLOG, let the largest of the three numbers drawn be $a>0$. Then the other two numbers are drawn uniformly and independently from the interval $[0,a]$. The probability that their sum is greater than a is $\boxed{\textbf{(C)}\ 1/2.}$

Solution 3: Calculus

When $a>b$, consider two cases:

1) $0<a<\frac{1}{2}$, then $\int_0^{\frac{1}{2}}\int_0^a 2b\,\mathrm{d}b\,\mathrm{d}a=\frac{1}{24}$

2) $\frac{1}{2}<a<1$, then $\int_{\frac{1}{2}}^1\left(\int_0^{1-a} 2b\,\mathrm{d}b+\int_{1-a}^a 1+b-a\,db\right)\mathrm{d}a=\frac{5}{24}$

$a<b$ is the same. Thus the answer is $\frac{1}{2}$.

Solution 4: Geometry

The probability of this occurring is the volume of the corresponding region within a $1\times1\times1$ cube, where each point (x,y,z) corresponds to a choice of values for each of x, y, and z. The region where, WLOG, side z is too long, $z\geq x+y$, is a pyramid with a base of area $\frac{1}{2}$ and height 1, so its volume is

$\dfrac{\frac{1}{2} \cdot 1}{3} = \dfrac{1}{6}$. Accounting for the corresponding cases in x and y multiplies our answer by 3, so we have excluded a total volume of $\dfrac{1}{2}$ from the space of possible probabilities. Subtracting this from 1 leaves us with a final answer of $\dfrac{1}{2}$.

Solution 5: More Calculus

The probability of this occurring is the volume of the corresponding region within a $1 \times 1 \times 1$ cube, where each point (x, y, z) corresponds to a choice of values for each of $x, y,$ and z. We take a horizontal cross section of the cube, essentially picking a value for z. The area where the triangle inequality will not hold is when $x + y < z$, which has area $\dfrac{z^2}{2}$ or when $x + z < y$ or $y + z < x$, which have an area of $\dfrac{(1-z)^2}{2} + \dfrac{(1-z)^2}{2} = (1-z)^2$. Integrating this expression from 0 to 1 in the form

$$\int_0^1 \frac{z^2}{2} + (1-z)^2 dz = \frac{z^3}{2} - z^2 + z \Big|_0^1 = \frac{1}{2} - 1 + 1 = \frac{1}{2}$$

Solution 6: Geometry in 2-D

WLOG assume that z is the largest number and hence the largest side. Then $x, y \leq z$. We can set up a square that is z by z in the xy plane. We are wanting all the points within this square that satisfy $x + y > z$. This happens to be a line dividing the square into 2 equal regions. Thus the answer is $\dfrac{1}{2}$.

[][] diagram for this problem goes here (z by z square)

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2016 AMC 12A Problems/Problem 24

Contents

Problem

There is a smallest positive real number a such that there exists a positive real number b such that all the roots of the polynomial x^3-ax^2+bx-a are real. In fact, for this value of a the value of b is unique. What is this value of b?

$\textbf{(A)}\ 8\qquad\textbf{(B)}\ 9\qquad\textbf{(C)}\ 10\qquad\textbf{(D)}\ 11\qquad\textbf{(E)}\ 12$

Solution

Solution 1

The acceleration must be zero at the x-intercept; this intercept must be an inflection point for the minimum a value. Derive $f(x)$ so that the acceleration $f''(x)=0$:
$x^3-ax^2+bx-a \rightarrow 3x^2-2ax+b \rightarrow 6x-2a \rightarrow x=\dfrac{a}{3}$ for the inflection point/root.
Furthermore, the slope of the function must be zero - maximum - at the intercept, thus having a triple root at $x=a/3$ (if the slope is greater than zero, there will be two complex roots and we do not want that).

The function with the minimum a:

$$f(x)=\left(x-\frac{a}{3}\right)^3$$

$$x^3-ax^2+\left(\frac{a^2}{3}\right)x-\frac{a^3}{27}$$

Since this is equal to the original equation x^3-ax^2+bx-a,

$$\frac{a^3}{27}=a \rightarrow a^2=27 \rightarrow a=3\sqrt{3}$$

$$b=\frac{a^2}{3}=\frac{27}{3}=\boxed{\textbf{(B) }9}$$

The actual function: $f(x)=x^3-\left(3\sqrt{3}\right)x^2+9x-3\sqrt{3}$

$f(x)=0 \rightarrow x=\sqrt{3}$ triple root. "Complete the cube."

Solution 2

Note that since both a and b are positive, all 3 roots of the polynomial are positive as well.

Let the roots of the polynomial be r, s, t. By Vieta's $a = r + s + t$ and $a = rst$.

Since r, s, t are positive we can apply AM-GM to get $\dfrac{r+s+t}{3} \geq \sqrt[3]{rst} \rightarrow \dfrac{a}{3} \geq \sqrt[3]{a}$. Cubing both sides and then dividing by a (since a is positive we can divide by a and not change the sign of the inequality) yields $\dfrac{a^2}{27} \geq 1 \rightarrow a \geq 3\sqrt{3}$.

Thus, the smallest possible value of a is $3\sqrt{3}$ which is achieved when all the roots are equal to $\sqrt{3}$. For this value of a, we can use Vieta's to get $b = \boxed{\textbf{(B) } 9}$.

Solution 3

All three roots are identical. Therefore, comparing coefficients, the root of this cubic function is $\sqrt{3}$. Using Vieta's, the coefficient we desire is the sum of the pairwise products of the roots. Because our root is unique, the answer is simply $b = \boxed{\textbf{(B) } 9}$.

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2016 AMC 12A Problems/Problem 25

Problem

Let k be a positive integer. Bernardo and Silvia take turns writing and erasing numbers on a blackboard as follows: Bernardo starts by writing the smallest perfect square with k+1 digits. Every time Bernardo writes a number, Silvia erases the last k digits of it. Bernardo then writes the next perfect square, Silvia erases the last k digits of it, and this process continues until the last two numbers that remain on the board differ by at least 2. Let f(k) be the smallest positive integer not written on the board. For example, if k = 1, then the numbers that Bernardo writes are $16, 25, 36, 49, 64$, and the numbers showing on the board after Silvia erases are 1, 2, 3, 4, and 6, and thus f(1) = 5. What is the sum of the digits of $f(2)+f(4)+f(6)+\cdots+f(2016)$?

$\textbf{(A)}\ 7986\qquad\textbf{(B)}\ 8002\qquad\textbf{(C)}\ 8030\qquad\textbf{(D)}\ 8048\qquad\textbf{(E)}\ 8064$

Solution

Consider $f(2)$. The numbers left on the blackboard will show the hundreds place at the end. In order for the hundreds place to differ by 2, the difference between two perfect squares needs to be at least 100. Calculus $\left(\frac{\mathrm{d}}{\mathrm{d}x}x^2=2x\right)$ and a bit of thinking says this first happens at $x \geq 100/2=50$. The perfect squares from here go: $2500, 2601, 2704, 2809\ldots$. Note that the ones and tens also make the perfect squares, $1^2, 2^2, 3^2\ldots$. After the ones and tens make 100, the hundreds place will go up by 2, thus reaching our goal. Since $10^2=100$, the last perfect square to be written will be $(50+10)^2=60^2=3600$. The missing number is one less than the number of hundreds $(k=2)$ of 3600, or 35.

Now consider f(4). Instead of the difference between two squares needing to be 100, the difference must now be 10000. This first happens at $x \geq 5000$. After this point, similarly, $\sqrt{10000}=100$ more numbers are needed to make the 10^4 th's place go up by 2. This will take place at $(5000+100)^2=5100^2=26010000$. Removing the last four digits (the zeros) and subtracting one yields 2600 for the skipped value.

In general, each new value of $f(k+2)$ will add two digits to the "5" and one digit to the "1". This means that the last number Bernardo writes for $k=6$ is $(500000+1000)^2$, the last for $k=8$ will be $(50000000+10000)^2$, and so on until $k=2016$. Removing the last k digits as Silvia does will be the same as removing $k/2$ trailing zeroes on the number to be squared. This means that the last number on the board for $k=6$ is 5001^2, $k=8$ is 50001^2, and so on. So the first missing number is $5001^2-1, 50001^2-1$ etc. The squaring will make a "25" with two more digits than the last number, a "10" with one more digit, and a "1". The missing number is one less than that, so the "1" will be subtracted from $f(k)$. In other words, $f(k)=25\cdot10^{k-2}+1\cdot10^{k/2}$.

Therefore:

$$f(2)=35=25+10$$

$$f(4)=2600=2500+100$$

$$f(6)=251000=250000+1000$$

$$f(8)=25010000=25000000+10000$$

And so on. The sum $f(2)+f(4)+f(6)+\cdots+f(2016)$ is:

$2.52525252525\ldots2525 \cdot 10^{2015}$ + $1.11111\ldots110 \cdot 10^{1008}$, with 2016 repetitions each of "25" and "1". There is no carrying in this addition. Therefore each $f(k)$ adds $2+5+1=8$ to the sum of the digits. Since $2n = 2016$, $n = 1008$, and $8n = 8064$, or $\boxed{\textbf{(E)}\ 8064}$.

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