2012 AMC 12B Problems/Problem 1

The following problem is from both the 2012 AMC 12B #1 and 2012 AMC 10B #1, so both problems redirect to this page.

Contents

Problem

Each third-grade classroom at Pearl Creek Elementary has 18 students and 2 pet rabbits. How many more students than rabbits are there in all 4 of the third-grade classrooms?

$\textbf{(A)}\ 48\qquad\textbf{(B)}\ 56\qquad\textbf{(C)}\ 64\qquad\textbf{(D)}\ 72\qquad\textbf{(E)}\ 80$

Solution

Solution 1

Multiplying 18 and 2 by 4 we get 72 students and 8 rabbits. We then subtract: $72-8=\boxed{\textbf{(C)}\ 64}$.

Solution 2

In each class, there are $18-2=16$ more students than rabbits. So for all classrooms, the difference between students and rabbits is $16\times 4=\boxed{\textbf{(C)}\ 64}$

See Also

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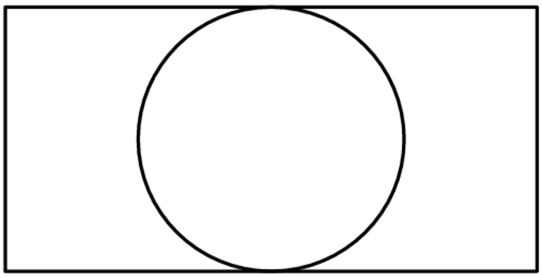
2012 AMC 12B Problems/Problem 2

Problem

A circle of radius 5 is inscribed in a rectangle as shown. The ratio of the length of the rectangle to its width is 2:1. What is the area of the rectangle?



$\textbf{(A)}\ 50\qquad\textbf{(B)}\ 100\qquad\textbf{(C)}\ 125\qquad\textbf{(D)}\ 150\qquad\textbf{(E)}\ 200$

Solution

If the radius is 5, then the width is 10, hence the length is 20. $10\times 20 = \boxed{\textbf{(E)}\ 200}$.

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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2012 AMC 12B Problems/Problem 3

Problem

For a science project, Sammy observed a chipmunk and squirrel stashing acorns in holes. The chipmunk hid 3 acorns in each of the holes it dug. The squirrel hid 4 acorns in each of the holes it dug. They each hid the same number of acorns, although the squirrel needed 4 fewer holes. How many acorns did the chipmunk hide?

$\textbf{(A)}\ 30\qquad\textbf{(B)}\ 36\qquad\textbf{(C)}\ 42\qquad\textbf{(D)}\ 48\qquad\textbf{(E)}\ 54$

Solution

If x is the number of holes that the chipmunk dug, then $3x = 4(x-4)$, so $3x = 4x - 16$, and $x = 16$. The number of acorns hidden by the chipmunk is equal to $3x = \boxed{\textbf{(D)}\ 48}$

See Also

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2012 AMC 12B Problems/Problem 4

Problem

Suppose that the euro is worth 1.3 dollars. If Diana has 500 dollars and Etienne has 400 euros, by what percent is the value of Etienne's money greater that the value of Diana's money?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 6.5 \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 13$

Solution

The ratio $\frac{400\text{ euros}}{500\text{ dollars}}$ can be simplified using conversion factors:

$$\frac{400\text{ euros}}{500\text{ dollars}} \cdot \frac{1.3\text{ dollars}}{1\text{ euro}} = \frac{520}{500} = 1.04$$

which means the money is greater by $\boxed{\textbf{(B)}\ 4}$ percent.

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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2012 AMC 12B Problems/Problem 5

Contents

Problem

Two integers have a sum of 26. when two more integers are added to the first two, the sum is 41. Finally, when two more integers are added to the sum of the previous 4 integers, the sum is 57. What is the minimum number of even integers among the 6 integers?

$\textbf{(A)}\ 1\qquad\textbf{(B)}\ 2\qquad\textbf{(C)}\ 3\qquad\textbf{(D)}\ 4\qquad\textbf{(E)}\ 5$

Solution

So, x+y=26, x could equal 15, and y could equal 11, so no even integers required here. 41-26=15. a+b=15, a could equal 9 and b could equal 6, so one even integer is required here. 57-41=16. m+n=16, m could equal 9 and n could equal 7, so no even integers required here, meaning only 1 even integer is required; A.

Solution 2

Just worded and formatted a little differently than above.

The first two integers sum up to 26. Since 26 is even, in order to minimize the number of even integers, we make both of the first two odd.

The second two integers sum up to $41-26=15$. Since 15 is odd, we must have at least one even integer in these next two.

Finally, $57-41=16$, and once again, 16 is an even number so both of these integers can be odd.

Therefore, we have a total of one even integer and our answer is $\boxed{\text{(A)}}$.

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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2012 AMC 12B Problems/Problem 6

Problem

In order to estimate the value of $x - y$ where x and y are real numbers with $x > y > 0$, Xiaoli rounded x up by a small amount, rounded y down by the same amount, and then subtracted her rounded values. Which of the following statements is necessarily correct?

(A) Her estimate is larger than $x - y$.

(B) Her estimate is smaller than $x - y$.

(C) Her estimate equals $x - y$.

(D) Her estimate equals $y - x$.

(E) Her estimate is 0.

Solution

The original expression $x - y$ now becomes $(x + k) - (y - k) = (x - y) + 2k > x - y$, where k is a positive constant, hence the answer is (A).

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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2012 AMC 12B Problems/Problem 7

Problem

Small lights are hung on a string 6 inches apart in the order red, red, green, green, green, red, red, green, green, green, and so on continuing this pattern of 2 red lights followed by 3 green lights. How many feet separate the 3rd red light and the 21st red light?

Note: 1 foot is equal to 12 inches.

$\textbf{(A)}\ 18\qquad\textbf{(B)}\ 18.5\qquad\textbf{(C)}\ 20\qquad\textbf{(D)}\ 20.5\qquad\textbf{(E)}\ 22.5$

Solution

We know the repeating section is made of 2 red lights and 3 green lights. The 3rd red light would appear in the 2nd section of this pattern, and the 21st red light would appear in the 11th section. There would then be a total of 44 lights in between the 3rd and 21st red light, translating to 45 6-inch gaps. Since it wants the answer in feet, so the answer is $\frac{45*6}{12} \rightarrow \boxed{\textbf{(E)}\ 22.5}$

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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2012 AMC 12B Problems/Problem 8

Contents

Problem 8

A dessert chef prepares the dessert for every day of a week starting with Sunday. The dessert each day is either cake, pie, ice cream, or pudding. The same dessert may not be served two days in a row. There must be cake on Friday because of a birthday. How many different dessert menus for the week are possible?

$\textbf{(A)}\ 729\qquad\textbf{(B)}\ 972\qquad\textbf{(C)}\ 1024\qquad\textbf{(D)}\ 2187\qquad\textbf{(E)}\ 2304$

Solution

We can count the number of possible foods for each day and then multiply to enumerate the number of combinations.

On Friday, we have one possibility: cake.

On Saturday, we have three possibilities: pie, ice cream, or pudding. This is the end of the week.

On Thursday, we have three possibilities: pie, ice cream, or pudding. We can't have cake because we have to have cake the following day, which is the Friday with the birthday party.

On Wednesday, we have three possibilities: cake, plus the two things that were not eaten on Thursday.

Similarly, on Tuesday, we have three possibilities: the three things that were not eaten on Wednesday.

Likewise on Monday: three possibilities, the three things that were not eaten on Tuesday.

On Sunday, it is tempting to think there are four possibilities, but remember that cake must be served on Friday. This serves to limit the number of foods we can eat on Sunday, with the result being that there are three possibilities: The three things that were not eaten on Monday.

So the number of menus is $3\cdot3\cdot3\cdot3\cdot3\cdot1\cdot3=729.$ The answer is $\boxed{A}$.

Solution 2

We can perform casework as an understandable means of getting the answer. We can organize our counting based on the food that was served on Wednesday, because whether cake is or is not served on Wednesday seems to significantly affect the number of ways the chef can make said foods for that week.

Case 1: Cake is served on Wednesday. Here, we have three choices for food on Thursday and Saturday since cake must be served on Friday, and none of these choices are cake, which was served Wednesday. Likewise, we have three choices (pie, ice cream, and pudding) for the food served on Tuesday and thus three choices for those served on Monday and Sunday, with these three choices being whatever was not served on Tuesday and Monday, respectively. Hence, for this case, there are $3\cdot3\cdot3\cdot3\cdot3=243$ possibilities.

Case 2: Cake is not served on Wednesday. Obviously, this means that pie, ice cream, and pudding are our only choices for Wednesday's food. Since cake must be served on Friday, only ice cream, pudding, and cake can be served on Thursday. However, since one of those foods was chosen for Wednesday, we only have two possibilities for Thursday's food. Like our first case, we have three possibilities for the food served on Tuesday, Monday, and Sunday: whatever was not served on Wednesday, Tuesday, and Monday, respectively. $3\cdot3\cdot3\cdot3\cdot2\cdot3=486$ possibilities thus exist for this case.

Adding the number of possibilities together yields that $243+486=729$ is the total number of menus, making our answer $\boxed{A}$.

2012 AMC 12B Problems/Problem 9

Problem

It takes Clea 60 seconds to walk down an escalator when it is not moving, and 24 seconds when it is moving. How seconds would it take Clea to ride the escalator down when she is not walking?

$\textbf{(A)}\ 36\qquad\textbf{(B)}\ 40\qquad\textbf{(C)}\ 42\qquad\textbf{(D)}\ 48\qquad\textbf{(E)}\ 52$

Solution

She walks at a rate of x units per second to travel a distance y. As $vt = d$, we find $60x = y$ and $24*(x+k) = y$, where k is the speed of the escalator. Setting the two equations equal to each other, $60x = 24x + 24k$, which means that $k = 1.5x$. Now we divide 60 by 1.5 because you add the speed of the escalator but remove the walking, leaving the final answer that it takes to ride the escalator alone as $\boxed{\textbf{(B)}\ 40}$

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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2012 AMC 12B Problems/Problem 10

Problem

What is the area of the polygon whose vertices are the points of intersection of the curves $x^2+y^2=25$ and $(x-4)^2+9y^2=81?$

$\textbf{(A)}\ 24\qquad\textbf{(B)}\ 27\qquad\textbf{(C)}\ 36\qquad\textbf{(D)}\ 37.5\qquad\textbf{(E)}\ 42$

Solution

The first curve is a circle with radius 5 centered at the origin, and the second curve is an ellipse with center $(4,0)$ and end points of $(-5,0)$ and $(13,0)$. Finding points of intersection, we get $(-5,0)$, $(4,3)$, and $(4,-3)$, forming a triangle with height of 9 and base of 6. So the area of this triangle is $9\cdot 6\cdot 0.5=27$ $\textbf{(B)}$.

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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Category: Introductory Geometry Problems

2012 AMC 12B Problems/Problem 11

Problem

In the equation below, A and B are consecutive positive integers, and A, B, and $A+B$ represent number bases:

$$132_A+43_B=69_{A+B}.$$

What is $A+B$?

$\textbf{(A)}\ 9\qquad\textbf{(B)}\ 11\qquad\textbf{(C)}\ 13\qquad\textbf{(D)}\ 15\qquad\textbf{(E)}\ 17$

Solution

Change the equation to base 10:

$$A^2 + 3A +2 + 4B +3= 6A + 6B + 9$$

$$A^2 - 3A - 2B - 4 = 0$$

Either $B = A+1$ or $B = A-1$, so either $A^2 - 5A - 6, B = A+1$ or $A^2 - 5A - 2, B = A-1$. The second case has no integer roots, and the first can be re-expressed as $(A-6)(A+1) = 0, B = A + 1$. Since A must be positive, $A = 6, B = 7$ and $A+B = 13$ (C).

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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Category: Introductory Number Theory Problems

2012 AMC 12B Problems/Problem 12

Contents

Problem

How many sequences of zeros and ones of length 20 have all the zeros consecutive, or all the ones consecutive, or both?

$\textbf{(A)}\ 190\qquad\textbf{(B)}\ 192\qquad\textbf{(C)}\ 211\qquad\textbf{(D)}\ 380\qquad\textbf{(E)}\ 382$

Solution

Solution 1

There are $\binom{20}{2}$ selections; however, we count these twice, therefore

$2*\binom{20}{2} = 380.$ The wording of the question implies D not E.

MAA decided to accept both D and E, however.

Solution 2

Consider the 20 term sequence of 0's and 1's. Keeping all other terms 1, a sequence of $k>0$ consecutive 0's can be placed in $21-k$ locations. That is, there are 20 strings with 1 zero, 19 strings with 2 consecutive zeros, 18 strings with 3 consecutive zeros, ..., 1 string with 20 consecutive zeros. Hence there are $20+19+\cdots+1=\binom{21}{2}$ strings with consecutive zeros. The same argument shows there are $\binom{21}{2}$ strings with consecutive 1's. This yields $2\binom{21}{2}$ strings in all. However, we have counted twice those strings in which all the 1's and all the 0's are consecutive. These are the cases $01111...$, $00111...$, $000111...$, ..., $000...0001$ (of which there are 19) as well as the cases $10000...$, $11000...$, $111000...$, ..., $111...110$ (of which there are 19 as well). This yields $2\binom{21}{2}-2\cdot 19 = 382$ so that the answer is $\boxed{\textbf{E}}$.

See Also

2012 AMC 12B Problems/Problem 13

Contents

Problem

Two parabolas have equations $y = x^2 + ax + b$ and $y = x^2 + cx + d$, where $a, b, c,$ and d are integers, each chosen independently by rolling a fair six-sided die. What is the probability that the parabolas will have a least one point in common?

Solution

Solution 1

Set the two equations equal to each other: $x^2 + ax + b = x^2 + cx + d$. Now remove the x squared and get x's on one side: $ax - cx = d - b$. Now factor x: $x(a - c) = d - b$. If a cannot equal c, then there is always a solution, but if $a = c$, a 1 in 6 chance, leaving a 1080 out 1296, always having at least one point in common. And if $a = c$, then the only way for that to work, is if $d = b$, a 1 in 36 chance, however, this can occur 6 ways, so a 1 in 6 chance of this happening. So adding one sixth to $\frac{1080}{1296}$, we get the simplified fraction of $\frac{31}{36}$; answer (D).

Solution 2

Proceed as above to obtain $x(a - c) = d - b$. The probability that the parabolas have at least 1 point in common is 1 minus the probability that they do not intersect. The equation $x(a - c) = d - b$ has no solution if and only if $a = c$ and $d \neq b$. The probability that $a = c$ is $\frac{1}{6}$ while the probability that $d \neq b$ is $\frac{5}{6}$. Thus we have $1 - \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{31}{36}$ for the probability that the parabolas intersect.

See Also

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2012 AMC 12B Problems/Problem 14

Contents

Problem

Bernardo and Silvia play the following game. An integer between 0 and 999 inclusive is selected and given to Bernardo. Whenever Bernardo receives a number, he doubles it and passes the result to Silvia. Whenever Silvia receives a number, she adds 50 to it and passes the result to Bernardo. The winner is the last person who produces a number less than 1000. Let N be the smallest initial number that results in a win for Bernardo. What is the sum of the digits of N?

$\textbf{(A)}\ 7\qquad\textbf{(B)}\ 8\qquad\textbf{(C)}\ 9\qquad\textbf{(D)}\ 10\qquad\textbf{(E)}\ 11$

Solution

Solution 1

The last number that Bernado says has to be between 950 and 999. Note that $1\rightarrow 2\rightarrow 52\rightarrow 104\rightarrow 154\rightarrow 308\rightarrow 358\rightarrow 716\rightarrow 766$ contains 4 doubling actions. Thus, we have $x \rightarrow 2x \rightarrow 2x+50 \rightarrow 4x+100 \rightarrow 4x+150 \rightarrow 8x+300 \rightarrow 8x+350 \rightarrow 16x+700$.

Thus, $950<16x+700<1000$. Then, $16x>250 \implies x \geq 16$. If $x=16$, we have $16x+700=956$. Working backwards from 956,

$$956 \rightarrow 478 \rightarrow 428 \rightarrow 214 \rightarrow 164 \rightarrow 82 \rightarrow 32 \rightarrow 16.$$

So the starting number is 16, and our answer is $1+6=\boxed{7}$, which is A.

Solution 2

Work backwards. The last number Bernardo produces must be in the range $[950,999]$. That means that before this, Silvia must produce a number in the range $[475,499]$. Before this, Bernardo must produce a number in the range $[425,449]$. Before this, Silvia must produce a number in the range $[213,224]$. Before this, Bernardo must produce a number in the range $[163,174]$. Before this, Silvia must produce a number in the range $[82,87]$. Before this, Bernardo must produce a number in the range $[32,37]$. Before this, Silvia must produce a number in the range $[16,18]$. Silvia could not have added 50 to any number before this to obtain a number in the range $[16,18]$, hence the minimum N is 16 with the sum of digits being $\boxed{\textbf{(A)}\ 7}$.

See Also

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All AMC 10 Problems and Solutions	

2012 AMC 12B Problems/Problem 15

Problem

Jesse cuts a circular disk of radius 12, along 2 radii to form 2 sectors, one with a central angle of 120. He makes two circular cones using each sector to form the lateral surface of each cone. What is the ratio of the volume of the smaller cone to the larger cone?

$\textbf{(A)}\ \frac{1}{8} \qquad \textbf{(B)}\ \frac{1}{4} \qquad \textbf{(C)}\ \frac{\sqrt{10}}{10} \qquad \textbf{(D)}\ \frac{\sqrt{5}}{6} \qquad \textbf{(E)}\ \frac{\sqrt{5}}{5}$

Solution

If the original radius is 12, then the circumference is 24π; since arcs are defined by the central angles, the smaller arc, a 120 degree angle, is half the size of the larger sector. so the smaller arc is 8π, and the larger is 16π. Those two arc lengths become the two circumferences of the new cones; so the radius of the smaller cone is 4 and the larger cone is 8. Using the Pythagorean theorem, the height of the larger cone is $4 \cdot \sqrt{5}$ and the smaller cone is $8 \cdot \sqrt{2}$, and now for volume just square the radii and multiply by $\frac{1}{3}$ of the height to get the volume of each cone: $128 \cdot \sqrt{2}$ and $256 \cdot \sqrt{5}$ [both multiplied by three as ratio come out the same. now divide the volumes by each other to get the final ratio of $\boxed{\textbf{(C)}\ \frac{\sqrt{10}}{10}}$

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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Category: Introductory Geometry Problems

2012 AMC 12B Problems/Problem 16

The following problem is from both the 2012 AMC 12B #16 and 2012 AMC 10B #24, so both problems redirect to this page.

Contents

Problem

Amy, Beth, and Jo listen to four different songs and discuss which ones they like. No song is liked by all three. Furthermore, for each of the three pairs of the girls, there is at least one song liked by those two girls but disliked by the third. In how many different ways is this possible?

$\textbf{(A)}\ 108\qquad\textbf{(B)}\ 132\qquad\textbf{(C)}\ 671\qquad\textbf{(D)}\ 846\qquad\textbf{(E)}\ 1105$

Solutions

Solution 1

Let the ordered triple (a,b,c) denote that a songs are liked by Amy and Beth, b songs by Beth and Jo, and c songs by Jo and Amy. We claim that the only possible triples are $(1,1,1), (2,1,1), (1,2,1) (1,1,2)$.

To show this, observe these are all valid conditions. Second, note that none of a,b,c can be bigger than 3. Suppose otherwise, that $a = 3$. Without loss of generality, say that Amy and Beth like songs 1, 2, and 3. Then because there is at least one song liked by each pair of girls, we require either b or c to be at least 1. In fact, we require either b or c to equal 1, otherwise there will be a song liked by all three. Suppose $b = 1$. Then we must have $c=0$ since no song is liked by all three girls, a contradiction.

Case 1:How many ways are there for (a,b,c) to equal $(1,1,1)$? There are 4 choices for which song is liked by Amy and Beth, 3 choices for which song is liked by Beth and Jo, and 2 choices for which song is liked by Jo and Amy. The fourth song can be liked by only one of the girls, or none of the girls, for a total of 4 choices. So $(a,b,c)=(1,1,1)$ in $4\cdot3\cdot2\cdot4 = 96$ ways.

Case 2:To find the number of ways for $(a,b,c) = (2,1,1)$, observe there are $\binom{4}{2} = 6$ choices of songs for the first pair of girls. There remain 2 choices of songs for the next pair (who only like one song). The last song is given to the last pair of girls. But observe that we let any three pairs of the girls like two songs, so we multiply by 3. In this case there are $6\cdot2\cdot3=36$ ways for the girls to like the songs.

That gives a total of $96 + 36 = \boxed{132}$ ways for the girls to like the song, so the answer is $\textbf{(B)}$.

Solution 2

We begin by noticing that there are four ways to assign a song liked by both Amy and Beth, three ways to assign a song liked by both Amy and Jo (because Jo may not like the song liked by both Amy and Beth), and two ways to assign a song liked by both Beth and Jo (because both Beth and Jo may not like the song liked by the previous pairs. Additionally, there are $2 \cdot 2 \cdot 2 = 8$ ways to assign song preferences for the fourth song. Multiplying, we obtain an answer of $4 \cdot 3 \cdot 2 \cdot 8 = 192$. However, in doing so, we have committed an egregious error. We have in fact over counted the cases in which the fourth song is liked by two girls but not the third.

We proceed again by ignoring the cases in which the fourth song is liked by two girls but not the third. There are $4 \cdot 3 \cdot 2 \cdot 5 = 120$ of these cases. However, in doing so, we have committed yet another egregious error. The cases in which the fourth song is liked by two girls but not the third have not been accounted for!

In performing our past two calculations, we have, however, established that the answer has a lower bound of 120 and an upper bound of 192. As $\textbf{(B)}$ is the only answer within these bounds, we conclude that the answer must be $\textbf{(B)}$.

Solution 3: A Different Way of Looking at Solution 1

Let AB, BJ, and AJ denote a song that is liked by Amy and Beth (but not Jo), Beth and Jo (but not Amy), and Amy and Jo (but not Beth), respectively. Similarly, let $A, B, J,$ and N denote a song that is liked by only Amy, only Beth, only Jo, and none of them, respectively. Since we know that there is at least 1 AB, BJ, and AJ, they must be 3 songs out of the 4 that Amy, Beth, and Jo listened to. The fourth song can be of any type N, A, B, J, AB, BJ, and AJ (there is no ABJ because no song is liked by all three, as stated in the problem.) Therefore, we must find the number of ways to rearrange AB, BJ, AJ, and a song from the set $\{N, A, B, J, AB, BJ, AJ\}$.

Case 1: Fourth song = N, A, B, J

Note that in Case 1, all four of the choices for the fourth song are different from the first three songs.

Number of ways to rearrange = $(4!)$ rearrangements for each choice $* 4$ choices = 96.

Case 2: Fourth song = AB, BJ, AJ

Note that in Case 2, all three of the choices for the fourth song repeat somewhere in the first three songs.

Number of ways to rearrange = $(4!/2!)$ rearrangements for each choice $* 3$ choices = 36.

$96 + 36 = \boxed{\textbf{(B)}\ 132}$.

See Also

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2012 AMC 12B Problems/Problem 17

Contents

Problem

Square $PQRS$ lies in the first quadrant. Points $(3,0), (5,0), (7,0),$ and $(13,0)$ lie on lines $SP, RQ, PQ,$ and SR, respectively. What is the sum of the coordinates of the center of the square $PQRS$?

$\textbf{(A)}\ 6 \qquad \textbf{(B)}\ 6.2 \qquad \textbf{(C)}\ 6.4 \qquad \textbf{(D)}\ 6.6 \qquad \textbf{(E)}\ 6.8$

Solutions

Solution 1

An image is supposed to go here. You can help us out by creating one and editing it in (https://artofproblemsolving.com/wiki/index.php?title=2012_AMC_12B_Problems/Problem_17&action=edit). Thanks.

Let the four points be labeled P_1, P_2, P_3, and P_4, respectively. Let the lines that go through each point be labeled L_1, L_2, L_3, and L_4, respectively. Since L_1 and L_2 go through SP and RQ, respectively, and SP and RQ are opposite sides of the square, we can say that L_1 and L_2 are parallel with slope m. Similarly, L_3 and L_4 have slope $-\frac{1}{m}$. Also, note that since square $PQRS$ lies in the first quadrant, L_1 and L_2 must have a positive slope. Using the point-slope form, we can now find the equations of all four lines: $L_1: y = m(x-3)$, $L_2: y = m(x-5)$, $L_3: y = -\frac{1}{m}(x-7)$, $L_4: y = -\frac{1}{m}(x-13)$.

Since $PQRS$ is a square, it follows that Δx between points P and Q is equal to Δy between points Q and R. Our approach will be to find Δx and Δy in terms of m and equate the two to solve for m. L_1 and L_3 intersect at point P. Setting the equations for L_1 and L_3 equal to each other and solving for x, we find that they intersect at $x = \frac{3m^2+7}{m^2+1}$. L_2 and L_3 intersect at point Q. Intersecting the two equations, the x-coordinate of point Q is found to be $x = \frac{5m^2+7}{m^2+1}$. Subtracting the two, we get $\Delta x = \frac{2m^2}{m^2+1}$. Substituting the x-coordinate for point Q found above into the equation for L_2, we find that the y-coordinate of point Q is $y = \frac{2m}{m^2+1}$. L_2 and L_4 intersect at point R. Intersecting the two equations, the y-coordinate of point R is found to be $y = \frac{8m}{m^2+1}$. Subtracting the two, we get

$\Delta y = \dfrac{6m}{m^2+1}$. Equating Δx and Δy, we get $2m^2 = 6m$ which gives us $m = 3$. Finally, note that the line which goes though the midpoint of P_1 and P_2 with slope 3 and the line which goes through the midpoint of P_3 and P_4 with slope $-\dfrac{1}{3}$ must intersect at at the center of the square. The equation of the line going through $(4,0)$ is given by $y = 3(x-4)$ and the equation of the line going through $(10,0)$ is $y = -\dfrac{1}{3}(x-10)$. Equating the two, we find that they intersect at $(4.6, 1.8)$. Adding the x and y-coordinates, we get 6.4. Thus, answer choice $\boxed{\textbf{(C)}}$ is correct.

Solution 2

Note that the center of the square lies along a line that has an $x-$intercept of $\dfrac{3+5}{2} = 4$, and also along another line with $x-$intercept $\dfrac{7+13}{2} = 10$. Since these 2 lines are parallel to the sides of the square, they are perpendicular (since the sides of a square are). Let m be the slope of the first line. Then $-\dfrac{1}{m}$ is the slope of the second line. We may use the point-slope form for the equation of a line to write $l_1 : y = m(x-4)$ and $l_2 : y = -\dfrac{1}{m}(x-10)$. We easily calculate the intersection of these lines using substitution or elimination to obtain $\left(\dfrac{4m^2+10}{m^2+1}, \dfrac{6m}{m^2+1}\right)$ as the center or the square. Let θ denote the (acute) angle formed by l_1 and the $x-$axis. Note that $\tan\theta = m$. Let s denote the side length of the square. Then $\sin\theta = s/2$. On the other hand the acute angle formed by l_2 and the $x-$axis is $90-\theta$ so that $\cos\theta = \sin(90-\theta) = s/6$. Then $m = \tan\theta = 3$. Substituting into $\left(\dfrac{4m^2+10}{m^2+1}, \dfrac{6m}{m^2+1}\right)$ we obtain $\left(\dfrac{23}{5}, \dfrac{9}{5}\right)$ so that the sum of the coordinates is $\dfrac{32}{5} = 6.4$. Hence the answer is $\boxed{\text{C}}$.

Solution 3 (Fast)

Suppose

$$SP : y = m(x-3)$$

$$RQ : y = m(x-5)$$

$$PQ : -my = x-7$$

$$SR : -my = x-13$$

where $m > 0$.

Recall that the distance between two parallel lines $Ax+By+C=0$ and $Ax+By+C_1=0$ is $|C-C_1|/\sqrt{A^2+B^2}$, we have distance between SP and RQ equals to $2m/\sqrt{1+m^2}$, and the distance between PQ and SR equals to $6/\sqrt{1+m^2}$. Equating them, we get $m=3$.

Then, the center of the square is just the intersection between the following two "mid" lines:

$$L_1 : y = 3(x-4)$$

$$L_2 : -3y = x - 10$$

The solution is $(4.6, 1.8)$, so we get the answer $4.6 + 1.8 = 6.4$. $\boxed{\text{C}}$.

See Also

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2012 AMC 12B Problems/Problem 18

Contents

Problem 18

Let $(a_1,a_2,\dots,a_{10})$ be a list of the first 10 positive integers such that for each $2\le i\le10$ either a_i+1 or a_i-1 or both appear somewhere before a_i in the list. How many such lists are there?

$\textbf{(A)}\ 120\qquad\textbf{(B)}\ 512\qquad\textbf{(C)}\ 1024\qquad\textbf{(D)}\ 181,440\qquad\textbf{(E)}\ 362,880$

Solution 1

Let $1\leq k\leq 10$. Assume that $a_1=k$. If $k<10$, the first number appear after k that is greater than k must be $k+1$, otherwise if it is any number x larger than $k+1$, there will be neither $x-1$ nor $x+1$ appearing before x. Similarly, one can conclude that if $k+1<10$, the first number appear after $k+1$ that is larger than $k+1$ must be $k+2$, and so forth.

On the other hand, if $k>1$, the first number appear after k that is less than k must be $k-1$, and then $k-2$, and so forth.

To count the number of possibilities when $a_1=k$ is given, we set up 9 spots after k, and assign $k-1$ of them to the numbers less than k and the rest to the numbers greater than k. The number of ways in doing so is 9 choose $k-1$.

Therefore, when summing up the cases from $k=1$ to 10, we get

$$\binom{9}{0}+\binom{9}{1}+\cdots+\binom{9}{9}=2^9=512......\boxed{\text{B}}$$

Solution 2 (Noticing Stuff)

If there is only 1 number, the number of ways to order would be 1. If there are 2 numbers, the number of ways to order would be 2. If there are 3 numbers, by listing out, the number of ways is 4. We can then make a conjecture that the problem is simply powers of 2.

$2^{10-1}=512.$

See Also

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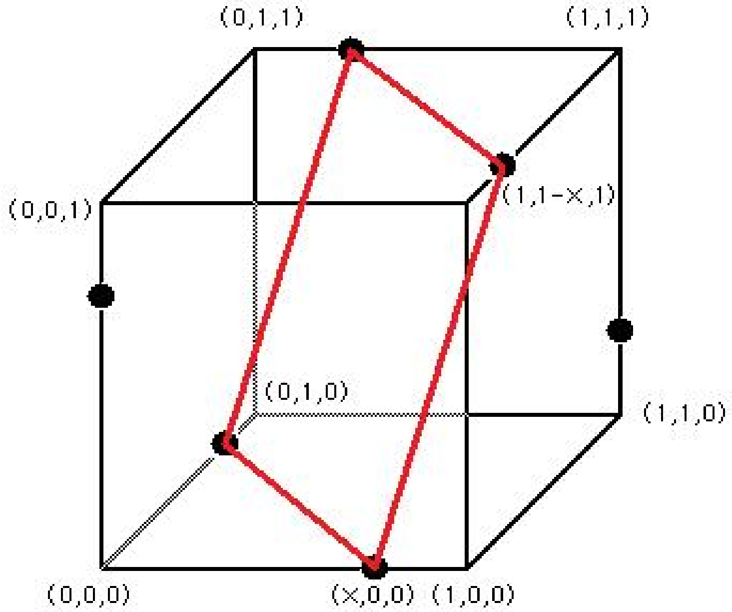
2012 AMC 12B Problems/Problem 19

Problem 19

A unit cube has vertices $P_1, P_2, P_3, P_4, P_1', P_2', P_3',$ and P_4'. Vertices P_2, P_3, and P_4 are adjacent to P_1, and for $1 \le i \le 4,$ vertices P_i and P_i' are opposite to each other. A regular octahedron has one vertex in each of the segments P_1P_2, P_1P_3, P_1P_4, $P_1'P_2'$, $P_1'P_3'$, and $P_1'P_4'$. What is the octahedron's side length?

$\textbf{(A)}\ \frac{3\sqrt{2}}{4}\qquad\textbf{(B)}\ \frac{7\sqrt{6}}{16}\qquad\textbf{(C)}\ \frac{\sqrt{5}}{2}\qquad\textbf{(D)}\ \frac{2\sqrt{3}}{3}\qquad\textbf{(E)}\ \frac{\sqrt{6}}{2}$

Solution



Observe the diagram above. Each dot represents one of the six vertices of the regular octahedron. Three dots have been placed exactly x units from the (0,0,0) corner of the unit cube. The other three dots have been placed exactly x units from the (1,1,1) corner of the unit cube. A red square has been drawn connecting four of the dots to provide perspective regarding the shape of the octahedron. Observe that the three dots that are near (0,0,0) are each (x)($\sqrt{2}$) from each other. The same is true for the three dots that are near (1,1,1). There is a unique x for which the rectangle drawn in red becomes a square. This will occur when the distance from (x,0,0) to (1,1-x, 1) is (x)($\sqrt{2}$).

Using the distance formula we find the distance between the two points to be: $\sqrt{(1-x)^2+(1-x)^2+1}$ = $\sqrt{2x^2-4x+3}$. Equating this to (x)($\sqrt{2}$) and squaring both sides, we have the equation:

$2x^2$ - $4x+3$ = $2x^2$

$-4x+3=0$

x = $\frac{3}{4}$.

Since the length of each side is (x)($\sqrt{2}$), we have a final result of $\frac{3\sqrt{2}}{4}$. Thus, Answer choice A is correct.

(If someone can draw a better diagram with the points labeled P1,P2, etc., I would appreciate it).

--Jm314 14:55, 26 February 2012 (EST)

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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2012 AMC 12B Problems/Problem 20

Problem 20

A trapezoid has side lengths 3, 5, 7, and 11. The sums of all the possible areas of the trapezoid can be written in the form of $r_1\sqrt{n_1}+r_2\sqrt{n_2}+r_3$, where r_1, r_2, and r_3 are rational numbers and n_1 and n_2 are positive integers not divisible by the square of any prime. What is the greatest integer less than or equal to $r_1+r_2+r_3+n_1+n_2$?

$\textbf{(A)}\ 57\qquad\textbf{(B)}\ 59\qquad\textbf{(C)}\ 61\qquad\textbf{(D)}\ 63\qquad\textbf{(E)}\ 65$

Solution

Name the trapezoid $ABCD$, where AB is parallel to CD, $AB<CD$, and $AD<BC$. Draw a line through B parallel to AD, crossing the side CD at E. Then $BE=AD$, $EC=DC-DE=DC-AB$. One needs to guarantee that $BE+EC>BC$, so there are only three possible trapezoids:

$$AB=3, BC=7, CD=11, DA=5, CE=8$$

$$AB=5, BC=7, CD=11, DA=3, CE=6$$

$$AB=7, BC=5, CD=11, DA=3, CE=4$$

In the first case, $\cos(\angle BCD)=(8^2+7^2-5^2)/(2\cdot 7\cdot 8)=11/14$, so $\sin(\angle BCD)=\sqrt{1-121/196}=5\sqrt{3}/14$. Therefore the area of this trapezoid is $\frac{1}{2}(3+11)\cdot 7\cdot 5\sqrt{3}/14=\frac{35}{2}\sqrt{3}$.

In the second case, $\cos(\angle BCD)=(6^2+7^2-3^2)/(2\cdot 6\cdot 7)=19/21$, so $\sin(\angle BCD)=\sqrt{1-361/441}=4\sqrt{5}/21$. Therefore the area of this trapezoid is $\frac{1}{2}(5+11)\cdot 7\cdot 4\sqrt{5}/21=\frac{32}{3}\sqrt{5}$.

In the third case, $\angle BCD=90^\circ$, therefore the area of this trapezoid is $\frac{1}{2}(7+11)\cdot 3=27$.

So $r_1+r_2+r_3+n_1+n_2=17.5+10.666...+27+3+5$, which is rounded down to $63...\boxed{D}$.

See Also

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2012 AMC 12B Problems/Problem 21

Problem 21

Square $AXYZ$ is inscribed in equiangular hexagon $ABCDEF$ with X on $\overline{BC}$, Y on $\overline{DE}$, and Z on $\overline{EF}$. Suppose that $AB=40$, and $EF=41(\sqrt{3}-1)$. What is the side-length of the square?

$\textbf{(A)}\ 29\sqrt{3} \qquad \textbf{(B)}\ \frac{21}{2}\sqrt{2}+\frac{41}{2}\sqrt{3} \qquad \textbf{(C)}\ 20\sqrt{3}+16 \qquad \textbf{(D)}\ 20\sqrt{2}+13\sqrt{3} \qquad \textbf{(E)}\ 21\sqrt{6}$

Solution (Long)

Extend AF and YE so that they meet at G. Then $\angle FEG=\angle GFE=60^\circ$, so $\angle FGE=60^\circ$ and therefore AB is parallel to YE. Also, since AX is parallel and equal to YZ, we get $\angle BAX=\angle ZYE$, hence $\triangle ABX$ is congruent to $\triangle YEZ$. We now get $YE=AB=40$.

Let $a_1=EY=40$, $a_2=AF$, and $a_3=EF$.

Drop a perpendicular line from A to the line of EF that meets line EF at K, and a perpendicular line from Y to the line of EF that meets EF at L, then $\triangle AKZ$ is congruent to $\triangle ZLY$ since $\angle YZL$ is complementary to $\angle KZA$. Then we have the following equations:

$$\frac{\sqrt{3}}{2}a_2=AK=ZL=ZE+\frac{1}{2}a_1$$

$$\frac{\sqrt{3}}{2}a_1=YL=ZK=ZF+\frac{1}{2}a_2$$

The sum of these two yields that

$$\frac{\sqrt{3}}{2}(a_1+a_2)=\frac{1}{2}(a_1+a_2)+ZE+ZF=\frac{1}{2}(a_1+a_2)+EF$$

$$\frac{\sqrt{3}-1}{2}(a_1+a_2)=41(\sqrt{3}-1)$$

$$a_1+a_2=82$$

$$a_2=82-40=42.$$

So, we can now use the law of cosines in $\triangle AGY$:

$$2AZ^2=AY^2=AG^2+YG^2-2AG\cdot YG\cdot\cos 60^\circ$$

$$=(a_2+a_3)^2+(a_1+a_3)^2-(a_2+a_3)(a_1+a_3)$$

$$=(41\sqrt{3}+1)^2+(41\sqrt{3}-1)^2-(41\sqrt{3}+1)(41\sqrt{3}-1)$$

$$=6\cdot 41^2+2-3\cdot 41^2+1=3(41^2+1)=3\cdot 1682$$

$$AZ^2=3\cdot 841=3\cdot 29^2$$

Therefore $AZ = 29\sqrt{3}...\boxed{\text{A}}$

See Also

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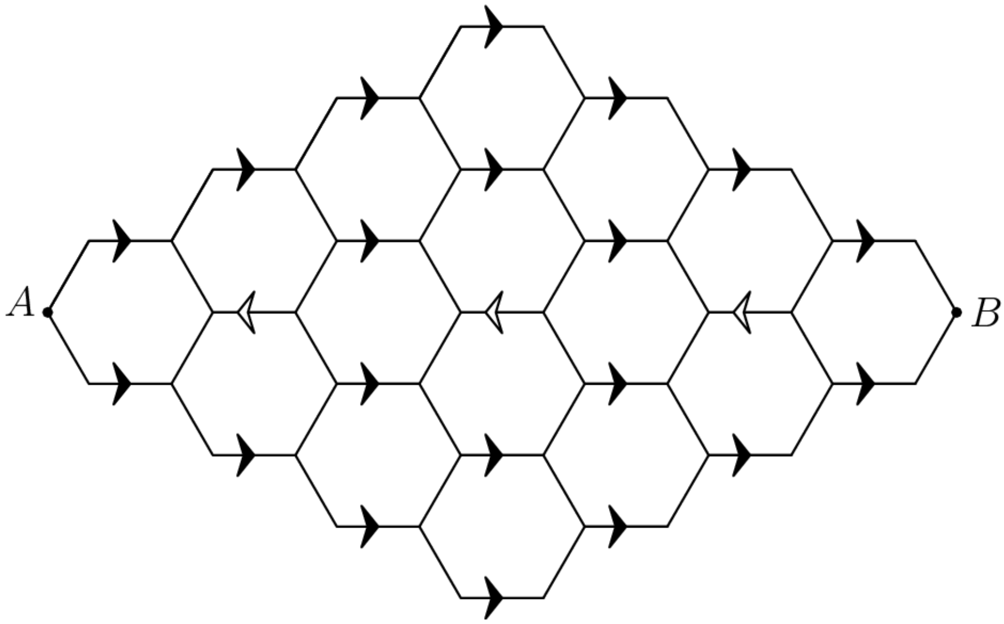
2012 AMC 10B Problems/Problem 25

The following problem is from both the 2012 AMC 12B #22 and 2012 AMC 10B #25, so both problems redirect to this page.

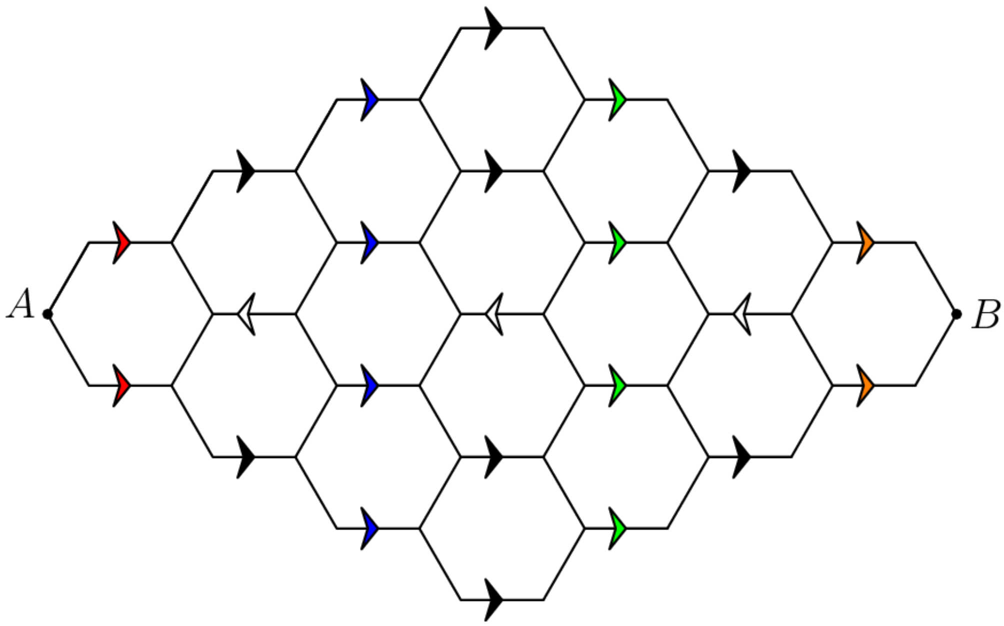
Problem 25

A bug travels from A to B along the segments in the hexagonal lattice pictured below. The segments marked with an arrow can be traveled only in the direction of the arrow, and the bug never travels the same segment more than once. How many different paths are there?



$\textbf{(A)}\ 2112 \qquad \textbf{(B)}\ 2304 \qquad \textbf{(C)}\ 2368 \qquad \textbf{(D)}\ 2384 \qquad \textbf{(E)}\ 2400$

Solution 1

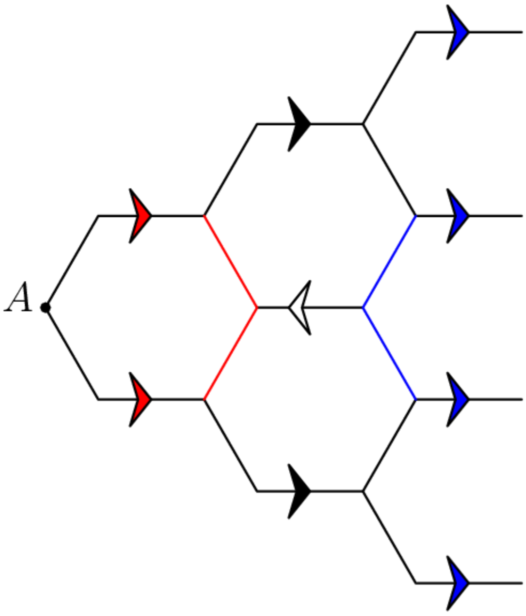


There is 1 way to get to any of the red arrows. From the first (top) red arrow, there are 2 ways to get to each of the first and the second (top 2) blue arrows; from the second (bottom) red arrow, there are 3 ways to get to each of the first and the second blue arrows. So there are in total 5 ways to get to each of the blue arrows.

From each of the first and second blue arrows, there are respectively 4 ways to get to each of the first and the second green arrows; from each of the third and the fourth blue arrows, there are respectively 8 ways to get to each of the first and the second green arrows. Therefore there are in total $5\cdot(4+4+8+8)=120$ ways to get to each of the green arrows.

Finally, from each of the first and second green arrows, there are respectively 2 ways to get to the first orange arrow; from each of the third and the fourth green arrows, there are 3 ways to get to the first orange arrow. Therefore there are $120\cdot(2+2+3+3)=1200$ ways to get to each of the orange arrows, hence 2400 ways to get to the point B. $\boxed{\textbf{(E)}\ 2400}$

Solution 2 (using the answer choices)



For every blue arrow, there are $2\cdot2=4$ ways to reach it without using the reverse arrow since the bug can choose any of 2 red arrows to pass through and 2 black arrows to pass through. If the bug passes through the white arrow, the red arrow that the bug travels through must be the closest to the first black arrow. Otherwise, the bug will have to travel through both red segments, which is impossible because now there is no path to take after the bug emerges from the reverse arrow. Similarly, with the blue segments, the second black arrow taken must be the one that is closest to the blue arrow that was taken. Also, it is trivial that the two black arrows taken must be different. Therefore, if the reverse arrow is taken, the blue arrow taken determines the entire path and there is 1 path for every arrow. Since the bug cannot return once it takes a blue arrow, the answer must be divisible by 5. $\boxed{\textbf{(E)}\ 2400}$ is the only answer that is.

2012 AMC 12B Problems/Problem 23

Contents

Problem 23

Consider all polynomials of a complex variable, $P(z) = 4z^4 + az^3 + bz^2 + cz + d$, where $a, b, c,$ and d are integers, $0 \le d \le c \le b \le a \le 4$, and the polynomial has a zero z_0 with $|z_0| = 1.$ What is the sum of all values $P(1)$ over all the polynomials with these properties?

$\textbf{(A)}\ 84 \qquad \textbf{(B)}\ 92 \qquad \textbf{(C)}\ 100 \qquad \textbf{(D)}\ 108 \qquad \textbf{(E)}\ 120$

Solution (doubtful)

Since z_0 is a root of P, and P has integer coefficients, z_0 must be algebraic. Since z_0 is algebraic and lies on the unit circle, z_0 must be a root of unity (Comment: this is not true. See this link: [1] (http://math.stackexchange.com/questions/4323/are-all-algebraic-integers-with-absolute-value-1-roots-of-unity)). Since P has degree 4, it seems reasonable (and we will assume this only temporarily) that z_0 must be a 2nd, 3rd, or 4th root of unity. These are among the set $\{\pm1, \pm i, (-1 \pm i\sqrt{3})/2\}$. Since complex roots of polynomials come in conjugate pairs, we have that P has one (or more) of the following factors: $z+1$, $z-1$, z^2+1, or z^2+z+1. If $z=1$ then $a+b+c+d+4=0$; a contradiction since a, b, c, d are non-negative. On the other hand, suppose $z=-1$. Then $(a+c)-(b+d)=4$. This implies $a+b=8,7,6,5,4$ while $b+d=4,3,2,1,0$ correspondingly. After listing cases, the only such valid a, b, c, d are $4,4,4,0$, $4,3,3,0$, $4,2,2,0$, $4,1,1,0$, and $4,0,0,0$.

Now suppose $z=i$. Then $4=(a-c)i+(b-d)$ whereupon $a=c$ and $b-d=4$. But then $a=b=c$ and $d=a-4$. This gives only the cases a, b, c, d equals $4,4,4,0$, which we have already counted in a previous case.

Suppose $z=-i$. Then $4=i(c-a)+(b-d)$ so that $a=c$ and $b=4+d$. This only gives rise to a, b, c, d equal $4,4,4,0$ which we have previously counted.

Finally suppose z^2+z+1 divides P. Using polynomial division ((or that $z^3=1$ to make the same deductions) we ultimately obtain that $b=4+c$. This can only happen if a, b, c, d is $4,4,0,0$.

Hence we've the polynomials

$$4x^4+4x^3+4x^2+4x$$

$$4x^4+4x^3+3x^2+3x$$

$$4x^4+4x^3+2x^2+2x$$

$$4x^4+4x^3+x^2+x$$

$$4x^4+4x^3$$

$$4x^4+4x^3+4x^2$$

However, by inspection $4x^4+4x^3+4x^2+4x+4$ has roots on the unit circle, because $x^4+x^3+x^2+x+1=(x^5-1)/(x-1)$ which brings the sum to 92 (choice B). Note that this polynomial has a 5th root of unity as a root. We will show that we were \textit{almost} correct in our initial assumption; that is that z_0 is at most a 5th root of unity, and that the last polynomial we obtained is the last polynomial with the given properties. Suppose that z_0 in an nth root of unity where $n>5$, and z_0 is not a 3rd or 4th root of unity. (Note that 1st and 2nd roots of unity are themselves 4th roots of unity). If n is prime, then \textit{every} nth root of unity except 1 must satisfy our polynomial, but since $n>5$ and the degree of our polynomial is 4, this is impossible. Suppose n is composite. If it has a prime factor p greater than 5 then again every pth root of unity must satisfy our polynomial and we arrive at the same contradiction. Therefore suppose n is divisible only by 2,3,or 5. Since by hypothesis z_0 is not a 2nd or 3rd root of unity, z_0 must be a 5th root of unity. Since 5 is prime, every 5th root of unity except 1 must satisfy our polynomial. That is, the other 4 complex 5th roots of unity must satisfy $P(z_0)=0$. But $(x^5-1)/(x-1)$ has exactly all 5th roots of unity excluding 1, and $(x^5-1)/(x-1)=x^4+x^3+x^2+x+1$. Thus this must divide P which implies $P(x)=4(x^4+x^3+x^2+x+1)$. This completes the proof.

Solution

First, assume that $z_0\in\mathbb{R}$, so $z_0=1$ or -1. 1 does not work because $P(1)\geq 4$. Assume that $z_0=-1$. Then $0=P(-1)=4-a+b-c+d$, we have $4+b+d=a+c\leq 4+b$, so $d=0$. Also, $a=4$ has to be true since $4+b=a+c\leq a+b$. Now $4+b=4+c$ gives $b=c$, therefore the only possible choices for (a,b,c,d) are $(4,t,t,0)$. In these cases, $P(-1)=4-4+t-t+0=0$. The sum of $P(-1)$ over these cases is $\sum_{t=0}^{4}(4+4+t+t)=40+20=60.$

Second, assume that $z_0\in\mathbb{C}\backslash\mathbb{R}$, so $z_0=x_0+iy_0$ for some real x_0,y_0, $|x_0|<1$. By conjugate roots theorem we have that $P(z_0)=P(z_0^*)=0$, therefore $(z-z_0)(z-z_0^*)=(z^2-2x_0*z+1)$ is a factor of $P(z)$, and we may assume that

$$P(z)=(z^2-2x_0z+1)(4z^2+pz+d)$$

for some real p. Expanding this polynomial and comparing the coefficients, we have the following equations:

$$p-8x_0=a$$

$$d+4-2px_0=b$$

$$p-2dx_0=c$$

From the first and the third we may deduce that $2x_0=\dfrac{a-c}{d-4}$ and that $p=\dfrac{da-4c}{d-4}$, if $d\neq 4$ (we will consider $d=4$ by the end). Let $k=2px_0=\dfrac{(a-c)(da-4c)}{(4-d)^2}$. From the second equation, we know that $k=d+4-b$ is non-negative.

Consider the following cases:

Case 1: $a=c$. Then $k=0$, $b=d+4$, so $a=b=c=4$, $d=0$. However, this has already been found (i.e. the form of $(4,t,t,0)$).

Case 2: $a>c\geq 0$. Then since $k\geq 0$, we have $da-4c\geq 0$. However, $da\leq 4c$, therefore $da-4c=0$. This is true only when $d=c$. Also, we get $k=0$ again. In this case, $b=d+4$, so $a=b=4$, $c=d=0$, $x_0=-1/2$. $P(z)$ has a root $z_0=e^{i2\pi/3}$. $P(1)=12$.

Last case: $d=4$. We have $a=b=c=d=4$ and that $P(z)$ has a root $z_0=e^{i2\pi/5}$. $P(1)=20$
.

Therefore the desired sum is $60+12+20=92...\boxed{\text{B}}$.

See Also

2012 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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2012 AMC 12B Problems/Problem 24

Problem

Define the function f_1 on the positive integers by setting $f_1(1)=1$ and if $n=p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ is the prime factorization of $n>1$, then

$$f_1(n)=(p_1+1)^{e_1-1}(p_2+1)^{e_2-1}\cdots(p_k+1)^{e_k-1}.$$

For every $m\ge 2$, let $f_m(n)=f_1(f_{m-1}(n))$. For how many N in the range $1\le N\le 400$ is the sequence $(f_1(N),f_2(N),f_3(N),\dots)$ unbounded?

Note: A sequence of positive numbers is unbounded if for every integer B, there is a member of the sequence greater than B.

$\textbf{(A) }15\qquad\textbf{(B) }16\qquad\textbf{(C) }17\qquad\textbf{(D) }18\qquad\textbf{(E) }19$

Solution

First of all, notice that for any odd prime p, the largest prime that divides $p+1$ is no larger than $\frac{p+1}{2}$, therefore eventually the factorization of $f_k(N)$ does not contain any prime larger than 3. Also, note that $f_2(2^m)=f_1(3^{m-1})=2^{2m-4}$, when $m=4$ it stays the same but when $m\ge 5$ it grows indefinitely. Therefore any number N that is divisible by 2^5 or any number N such that $f_k(N)$ is divisible by 2^5 makes the sequence $(f_1(N),f_2(N),f_3(N),\dots)$ unbounded. There are 12 multiples of 2^5 within 400. $2^4 5^2=400$ also works: $f_2(2^4 5^2)=f_1(3^4\cdot 2)=2^6$.

Now let's look at the other cases. Any first power of prime in a prime factorization will not contribute the unboundedness because $f_1(p^1)=(p+1)^0=1$. At least a square of prime is to contribute. So we test primes that are less than $\sqrt{400}=20$:

$f_1(3^4)=4^3=2^6$ works, therefore any number ≤ 400 that are divisible by 3^4 works: there are 4 of them.

$3^3\cdot Q^2$ could also work if Q^2 satisfies $2\mid f_1(Q^2)$, but $3^3\cdot 5^2>400$.

$f_1(5^3)=6^2=2^2 3^2$ does not work.

$f_1(7^3)=8^2=2^6$ works. There are no other multiples of 7^3 within 400.

$7^2\cdot Q^2$ could also work if $4\mid f_1(Q^2)$, but $7^2\cdot 3^2>400$ already.

For number that are only divisible by $p=11,13,17,19$, they don't work because none of these primes are such that $p+1$ could be a multiple of 2^5 nor a multiple of 3^4.

In conclusion, there are $12+1+4+1=18$ number of N's ... $\boxed{D}$.

See Also

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2012 AMC 12B Problems/Problem 25

Problem 25

Let $S = \{(x, y) : x \in \{0, 1, 2, 3, 4\}, y \in \{0, 1, 2, 3, 4, 5\}, \text{ and } (x, y) \neq (0, 0)\}$. Let T be the set of all right triangles whose vertices are in S. For every right triangle $t = \triangle ABC$ with vertices A, B, and C in counter-clockwise order and right angle at A, let $f(t) = \tan(\angle CBA)$. What is

$$\prod_{t \in T} f(t)?$$

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ \frac{625}{144} \qquad \textbf{(C)}\ \frac{125}{24} \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ \frac{625}{24}$

Solution

Consider reflections. For any right triangle ABC with the right labeling described in the problem, any reflection $A'B'C'$ labeled that way will give us $\tan CBA \cdot \tan C'B'A' = 1$. First we consider the reflection about the line $y = 2.5$. Only those triangles $\subseteq T$ that have one vertex at $(0, 5)$ do not reflect to a traingle $\subseteq T$. Within those triangles, consider a reflection about the line $y = 5 - x$. Then only those triangles $\subseteq T$ that have one vertex on the line $y = 0$ do not reflect to a triangle $\subseteq T$. So we only need to look at right triangles that have vertices $(0, 5), (*, 0), (*, *)$. There are three cases:

Case 1: $A = (0, 5)$. Then $B = (*, 0)$ is impossible.

Case 2: $B = (0, 5)$. Then we look for $A = (x, y)$ such that $\angle BAC = 90^\circ$ and that $C = (*, 0)$. They are: $(A = (x, 5), C = (x, 0))$, $(A = (3, 2), C = (1, 0))$ and $(A = (4, 1), C = (3, 0))$. The product of their values of $\tan \angle CBA$ is $\frac{5}{1} \cdot \frac{5}{2} \cdot \frac{5}{3} \cdot \frac{5}{4} \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{625}{144}$.

Case 3: $C = (0, 5)$. Then $A = (*, 0)$ is impossible.

Therefore $\boxed{\textbf{(B)}\ \frac{625}{144}}$ is the answer.

See Also

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