2015 AMC 10A Problems/Problem 1

The following problem is from both the 2015 AMC 12A #1 and 2015 AMC 10A #1, so both problems redirect to this page.

Problem

What is the value of $(2^0 - 1 + 5^2 - 0)^{-1} \times 5?$

$\textbf{(A)}\ -125 \qquad \textbf{(B)}\ -120 \qquad \textbf{(C)}\ \frac{1}{5} \qquad \textbf{(D)}\ \frac{5}{24} \qquad \textbf{(E)}\ 25$

Solution

$$(2^0 - 1 + 5^2 - 0)^{-1} \times 5 = (1 - 1 + 25 - 0)^{-1} \times 5 = 25^{-1} \times 5 = \frac{1}{25} \times 5 = \boxed{\textbf{(C)}\ \frac{1}{5}}.$$

See Also

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2015 AMC 12A Problems/Problem 2

Problem

Two of the three sides of a triangle are 20 and 15. Which of the following numbers is not a possible perimeter of the triangle?

$\textbf{(A)}\ 52\qquad\textbf{(B)}\ 57\qquad\textbf{(C)}\ 62\qquad\textbf{(D)}\ 67\qquad\textbf{(E)}\ 72$

Solution

Letting x be the third side, then by the triangle inequality, $20-15<x<20+15$, or $5<x<35$. Therefore the perimeter must be greater than 40 but less than 70. 72 is not in this range, so $\boxed{\textbf{(E)}\ 72}$ is our answer.

See Also

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2015 AMC 10A Problems/Problem 5

The following problem is from both the 2015 AMC 12A #3 and 2015 AMC 10A #5, so both problems redirect to this page.

Contents

Problem

Mr. Patrick teaches math to 15 students. He was grading tests and found that when he graded everyone's test except Payton's, the average grade for the class was 80. After he graded Payton's test, the test average became 81. What was Payton's score on the test?

$\textbf{(A)}\ 81\qquad\textbf{(B)}\ 85\qquad\textbf{(C)}\ 91\qquad\textbf{(D)}\ 94\qquad\textbf{(E)}\ 95$

Solution

If the average of the first 14 peoples' scores was 80, then the sum of all of their tests is $14*80 = 1120$. When Payton's score was added, the sum of all of the scores became $15*81=1215$. So, Payton's score must be $1215-1120 = \boxed{\textbf{(E) }95}$

Alternate Solution

The average of a set of numbers is the value we get if we evenly distribute the total across all entries. So assume that the first 14 students each scored 80. If Payton also scored an 80, the average would still be 80. In order to increase the overall average to 81, we need to add one more point to all of the scores, including Payton's. This means we need to add a total of 15 more points, so Payton needs $80+15 = \boxed{\textbf{(E) }95}$

See also

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2015 AMC 10A Problems/Problem 6

The following problem is from both the 2015 AMC 12A #4 and 2015 AMC 10A #6, so both problems redirect to this page.

Problem

The sum of two positive numbers is 5 times their difference. What is the ratio of the larger number to the smaller number?

$\textbf{(A)}\ \frac{5}{4}\qquad\textbf{(B)}\ \frac{3}{2}\qquad\textbf{(C)}\ \frac{9}{5}\qquad\textbf{(D)}\ 2 \qquad\textbf{(E)}\ \frac{5}{2}$

Solution

Let a be the bigger number and b be the smaller.

$a + b = 5(a - b)$.

Solving gives $\frac{a}{b} = \frac{3}{2}$, so the answer is $\boxed{\textbf{(B)}\ \frac{3}{2}}$.

See Also

2015 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2015))	
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2015 AMC 12A Problems/Problem 5

Problem

Amelia needs to estimate the quantity $\frac{a}{b}-c$, where $a, b,$ and c are large positive integers. She rounds each of the integers so that the calculation will be easier to do mentally. In which of these situations will her answer necessarily be greater than the exact value of $\frac{a}{b}-c$?

$\textbf{(A)}$ She rounds all three numbers up.
$\textbf{(B)}$ She rounds a and b up, and she rounds $c$down.
$\textbf{(C)}$ She rounds a and c up, and she rounds $b$down.
$\textbf{(D)}$ She rounds a up, and she rounds b and $c$down.
$\textbf{(E)}$ She rounds c up, and she rounds a and $b$down.

Solution

To maximize our estimate, we want to maximize $\frac{a}{b}$ and minimize c, because both terms are positive values. Therefore we round c down. To maximize $\frac{a}{b}$, round a up and b down. $\Rightarrow \boxed{\textbf{(D)}}$

See Also

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2015 AMC 10A Problems/Problem 8

The following problem is from both the 2015 AMC 12A #6 and 2015 AMC 10A #8, so both problems redirect to this page.

Problem

Two years ago Pete was three times as old as his cousin Claire. Two years before that, Pete was four times as old as Claire. In how many years will the ratio of their ages be 2 : 1 ?

$\textbf{(A)}\ 2 \qquad\textbf{(B)}\ 4 \qquad\textbf{(C)}\ 5 \qquad\textbf{(D)}\ 6 \qquad\textbf{(E)}\ 8$

Solution

This problem can be converted to a system of equations. Let p be Pete's current age and c be Claire's current age.

The first statement can be written as $p-2=3(c-2)$. The second statement can be written as $p-4=4(c-4)$

To solve the system of equations:

$p=3c-4$

$p=4c-12$

$3c-4=4c-12$

$c=8$

$p=20$

Let x be the number of years until Pete is twice as old as Claire.

$20+x=2(8+x)$

$20+x=16+2x$

$x=4$

The answer is $\boxed{\textbf{(B)}\ 4}$.

See Also

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2015 AMC 10A Problems/Problem 9

The following problem is from both the 2015 AMC 12A #7 and 2015 AMC 10A #9, so both problems redirect to this page.

Problem

Two right circular cylinders have the same volume. The radius of the second cylinder is 10% more than the radius of the first. What is the relationship between the heights of the two cylinders?

$\textbf{(A)}$ The second height is 10% less than the first.
$\textbf{(B)}$ The first height is 10% more than the second.
$\textbf{(C)}$ The second height is 21% less than the first.
$\textbf{(D)}$ The first height is 21% more than the second.
$\textbf{(E)}$ The second height is 80% of the first.

Solution

Let the radius of the first cylinder be r_1 and the radius of the second cylinder be r_2. Also, let the height of the first cylinder be h_1 and the height of the second cylinder be h_2. We are told

$$r_2 = \frac{11r_1}{10}$$

$$\pi r_1^2 h_1 = \pi r_2^2 h_2$$

Substituting the first equation into the second and dividing both sides by π, we get

$$r_1^2 h_1 = \frac{121r_1^2}{100}h_2 \implies h_1 = \frac{121h_2}{100}.$$

Therefore, $\boxed{\textbf{(D)}\ \text{The first height is } 21\% \text{ more than the second.}}$

See Also

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2015 AMC 10A Problems/Problem 11

The following problem is from both the 2015 AMC 12A #8 and 2015 AMC 10A #11, so both problems redirect to this page.

Problem 11

The ratio of the length to the width of a rectangle is $4:3$. If the rectangle has diagonal of length d, then the area may be expressed as kd^2 for some constant k. What is k?

$\textbf{(A)}\ \frac{2}{7}\qquad\textbf{(B)}\ \frac{3}{7}\qquad\textbf{(C)}\ \frac{12}{25}\qquad\textbf{(D)}\ \frac{16}{25}\qquad\textbf{(E)}\ \frac{3}{4}$

Solution

Let the rectangle have length $4x$ and width $3x$. Then by $3-4-5$ triangles (or the Pythagorean Theorem), we have $d = 5x$, and so $x = \frac{d}{5}$. Hence, the area of the rectangle is $3x \cdot 4x = 12x^2 = \frac{12d^2}{25}$, so the answer is $\boxed{\textbf{(C)}\ \frac{12}{25}}$

See also

2015 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2015))	
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Category: Introductory Geometry Problems

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2015 AMC 12A Problems/Problem 9

Contents

Problem

A box contains 2 red marbles, 2 green marbles, and 2 yellow marbles. Carol takes 2 marbles from the box at random; then Claudia takes 2 of the remaining marbles at random; and then Cheryl takes the last 2 marbles. What is the probability that Cheryl gets 2 marbles of the same color?

$\textbf{(A)}\ \frac{1}{10} \qquad\textbf{(B)}\ \frac{1}{6} \qquad\textbf{(C)}\ \frac{1}{5} \qquad\textbf{(D)}\ \frac{1}{3} \qquad\textbf{(E)}\ \frac{1}{2}$

Solution 1

If Cheryl gets two marbles of the same color, then Claudia and Carol must take all four marbles of the two other colors. The probability of this happening, given that Cheryl has two marbles of a certian color is $\frac{4}{6} * \frac{3}{5} * \frac{2}{4} * \frac{1}{3} = \frac{1}{15}$. Since there are three different colors, our final probability is $3 * \frac{1}{15} = \frac{1}{5}\ \textbf{(C)}$.

Solution 2

The order of the girls' drawing the balls really does not matter. Thus, we can let Cheryl draw first, so after she draws one ball, the other must be of the same color. Thus, the answer is $\frac{1}{5}\ \textbf{(C)}$.

Solution 3

The total number of ways they can draw is $\binom{6}{2}\binom{4}{2}\binom{2}{2}$. Let Cheryl draw first and since there are three colors, there are $\binom{3}{1}$ ways she can get 2 marbles of the same color. The other two pick two each, which leads to $\binom{4}{2}$ and $\binom{2}{2}$, respectively. $\frac{\binom{3}{1}\binom{4}{2}\binom{2}{2}}{\binom{6}{2}\binom{4}{2}\binom{2}{2}} = \frac{1}{5}\ \textbf{(C)}$

See Also

2015 AMC 12A Problems/Problem 10

Problem

Integers x and y with $x > y > 0$ satisfy $x + y + xy = 80$. What is x?

$\textbf{(A)}\ 8 \qquad \textbf{(B)}\ 10 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 18 \qquad \textbf{(E)}\ 26$

Solution

Use SFFT to get $(x+1)(y+1) = 81$. The terms $(x+1)$ and $(y+1)$ must be factors of 81, which include $1, 3, 9, 27, 81$. Because $x > y$, $x + 1$ is equal to 27 or 81. But if $x + 1 = 81$, then $y = 0$ and so $x = \boxed{\textbf{(E)}\ 26}$.

See Also

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2015 AMC 12A Problems/Problem 11

Problem

On a sheet of paper, Isabella draws a circle of radius 2, a circle of radius 3, and all possible lines simultaneously tangent to both circles. Isabella notices that she has drawn exactly $k \geq 0$ lines. How many different values of k are possible?

$\textbf{(A)}\ 2 \qquad\textbf{(B)}\ 3 \qquad\textbf{(C)}\ 4 \qquad\textbf{(D)}\ 5 \qquad\textbf{(E)}\ 6$

Solution

Isabella can get 0 lines if the circles are concentric, 1 if internally tangent, 2 if overlapping, 3 if externally tangent, and 4 if non-overlapping and not externally tangent. There are $\boxed{\textbf{(D)}\ 5}$ values of k.

See Also

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2015 AMC 12A Problems/Problem 12

Problem

The parabolas $y = ax^2 - 2$ and $y = 4 - bx^2$ intersect the coordinate axes in exactly four points, and these four points are the vertices of a kite of area 12. What is $a + b$?

$\textbf{(A)}\ 1\qquad\textbf{(B)}\ 1.5\qquad\textbf{(C)}\ 2\qquad\textbf{(D)}\ 2.5\qquad\textbf{(E)}\ 3$

Solution

Clearly, the parabolas must intersect the x-axis at the same two points. Their distance multiplied by $4 - (-2)$ (the distance between the y-intercepts), all divided by 2 is equal to 12, the area of the kite (half the product of the diagonals). That distance is thus 4, and so the x-intercepts are $(2, 0), (-2, 0).$ Then $0 = 4a - 2 \rightarrow a = 0.5$, and $0 = 4 - 4b \rightarrow b = 1.$ Then $a + b = 1.5$, or $\textbf{(B)}$.

See Also

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2015 AMC 12A Problems/Problem 13

Problem

A league with 12 teams holds a round-robin tournament, with each team playing every other team exactly once. Games either end with one team victorious or else end in a draw. A team scores 2 points for every game it wins and 1 point for every game it draws. Which of the following is NOT a true statement about the list of 12 scores?

$\textbf{(A)}$ There must be an even number of odd scores.
$\textbf{(B)}$ There must be an even number of even scores.
$\textbf{(C)}$ There cannot be two scores of 0.
$\textbf{(D)}$ The sum of the scores must be at least 100.
$\textbf{(E)}$ The highest score must be at least 12.

Solution

We can eliminate answer choices $\textbf{(A)}$ and $\textbf{(B)}$ because there are an even number of scores, so if one is false, the other must be false too. Answer choice $\textbf{(C)}$ must be true since every team plays every other team, so it is impossible for two teams to lose every game. Answer choice $\textbf{(D)}$ must be true since each game gives out a total of two points, and there are $\frac{11 \times 12}{2} = 66$ games, for a total of 132 points. Answer choice $\textbf{(E)}$ is false (and thus our answer). If everyone draws each of their 11 games, then every team will tie for first place with 11 points each.

See Also

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2015 AMC 12A Problems/Problem 14

Problem

What is the value of a for which $\frac{1}{\log_2 a}+\frac{1}{\log_3 a}+\frac{1}{\log_4 a}=1$?

$\textbf{(A)}\ 9\qquad\textbf{(B)}\ 12\qquad\textbf{(C)}\ 18\qquad\textbf{(D)}\ 24\qquad\textbf{(E)}\ 36$

Solution

We use the change of base formula to show that

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}.$$

Thus, our equation becomes

$$\log_a 2 + \log_a 3 + \log_a 4 = 1,$$

which becomes after combining:

$$\log_a 24 = 1.$$

Hence $a=24$, and the answer is $\textbf{(D)}.$

See Also

2015 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2015))	
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2015 AMC 12A Problems/Problem 15

Contents

Problem

What is the minimum number of digits to the right of the decimal point needed to express the fraction $\frac{123456789}{2^{26}\cdot 5^4}$ as a decimal?

$\textbf{(A)}\ 4\qquad\textbf{(B)}\ 22\qquad\textbf{(C)}\ 26\qquad\textbf{(D)}\ 30\qquad\textbf{(E)}\ 104$

Solution

We can rewrite the fraction as $\frac{123456789}{2^{22}\cdot 10^4} = \frac{12345.6789}{2^{22}}$. Since the last digit of the numerator is odd, a 5 is added to the right if the numerator is divided by 2, and this will continuously happen because 5, itself, is odd. Indeed, this happens twenty-two times since we divide by 2 twenty-two times, so we will need 22 more digits. Hence, the answer is $4 + 22 = 26\ \textbf{(C)}$.

Alternate Solution

Note that 123456789 is not a multiple of 2 or 5, and therefore shares no factors with the original denominator. Multiple the numerator and denominator of the fraction by 5^{22} to give $\frac{5^{22}\cdot 123456789}{10^{26}}$. This fraction will require 26 divisions by ten to write as a decimal, and since the original fraction is less than 1 all of the digits will be to the right of the decimal point. Answer: $\textbf{(C)}$

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2015 AMC 10A Problems/Problem 21

The following problem is from both the 2015 AMC 12A #16 and 2015 AMC 10A #21, so both problems redirect to this page.

Contents

Problem

Tetrahedron $ABCD$ has $AB=5$, $AC=3$, $BC=4$, $BD=4$, $AD=3$, and $CD=\tfrac{12}{5}\sqrt2$. What is the volume of the tetrahedron?

$\textbf{(A) }3\sqrt2\qquad\textbf{(B) }2\sqrt5\qquad\textbf{(C) }\dfrac{24}5\qquad\textbf{(D) }3\sqrt3\qquad\textbf{(E) }\dfrac{24}5\sqrt2$

Solution 1

Let the midpoint of CD be E. We have $CE = \dfrac{6}{5}\sqrt{2}$, and so by the Pythagorean Theorem $AE = \dfrac{\sqrt{153}}{5}$ and $BE = \dfrac{\sqrt{328}}{5}$. Because the altitude from A of tetrahedron $ABCD$ passes touches plane BCD on BE, it is also an altitude of triangle ABE. The area A of triangle ABE is, by Heron's Formula, given by

$$16A^2 = 2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4 = -(a^2+b^2-c^2)^2+4a^2b^2.$$

Substituting $a=AE, b=BE, c=5$ and performing huge (but manageable) computations yield $A^2=18$, so $A = 3\sqrt{2}$. Thus, if h is the length of the altitude from A of the tetrahedron, $BE \cdot h = 2A = 6\sqrt{2}$. Our answer is thus

$$V = \frac{1}{3} Bh = \frac{1}{3} h \cdot BE \cdot \frac{6\sqrt{2}}{5} = \frac{24}{5},$$

and so our answer is $\boxed{\textbf{(C) } \dfrac{24}{5}}$

Solution 2

Drop altitudes of triangle ABC and triangle ABD down from C and D, respectively. Both will hit the same point; let this point be T. Because both triangle ABC and triangle ABD are 3-4-5 triangles, $CT = DT = \dfrac{3 \cdot 4}{5} = \dfrac{12}{5}$. Because $CT^2 + DT^2 = 2 \left(\dfrac{12}{5}\right)^2 = \left(\dfrac{12}{5}\sqrt{2}\right)^2 = CD^2$, it follows that the CTD is a right triangle, meaning that $\angle CTD = 90^{\circ}$, and it follows that planes ABC and ABD are perpendicular to each other. Now, we can treat ABC as the base of the tetrahedron and TD as the height. Thus, the desired volume is

$$V=\frac{1}{3}Bh=\frac{1}{3}\cdot[ABC]\cdot TD=\frac{1}{3}\cdot 6\cdot\frac{12}{5}=\frac{24}{5}$$

which is answer $\boxed{\textbf{(C) } \frac{24}{5}}$

See Also

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Category: Introductory Geometry Problems

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2015 AMC 10A Problems/Problem 22

The following problem is from both the 2015 AMC 12A #17 and 2015 AMC 10A #22, so both problems redirect to this page.

Contents

Problem

Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

$$\textbf{(A)}\dfrac{47}{256}\qquad\textbf{(B)}\dfrac{3}{16}\qquad\textbf{(C)}\dfrac{49}{256}\qquad\textbf{(D)}\dfrac{25}{128}\qquad\textbf{(E)}\dfrac{51}{256}$$

Solution

Solution 1

We will count how many valid standing arrangements there are (counting rotations as distinct), and divide by $2^8 = 256$ at the end. We casework on how many people are standing.

Case $1: 0$ people are standing. This yields 1 arrangement.

Case $2: 1$ person is standing. This yields 8 arrangements.

Case $3: 2$ people are standing. This yields $\dbinom{8}{2} - 8 = 20$ arrangements, because the two people cannot be next to each other.

Case $4: 4$ people are standing. Then the people must be arranged in stand-sit-stand-sit-stand-sit-stand-sit fashion, yielding 2 possible arrangements.

More difficult is:

Case $5: 3$ people are standing. First, choose the location of the first person standing (8 choices). Next, choose 2 of the remaining people in the remaining 5 legal seats to stand, amounting to 6 arrangements considering that these two people cannot stand next to each other. However, we have to divide by $3,$ because there are 3 ways to choose the first person given any three. This yields $\dfrac{8 \cdot 6}{3} = 16$ arrangements for Case $5.$

Alternate Case 5: Use complementary counting. Total number of ways to choose 3 people from 8 which is $\binom{8}{3}$. Sub-case 1: three people are next to each other which is $\binom{8}{1}$. Sub-case 2: two people are next to each other and the third person is not $\binom{8}{1}\binom{4}{1}$. This yields $\binom{8}{3}-\binom{8}{1}-\binom{8}{1}\binom{4}{1}=16$

Summing gives $1+8+20+2+16=47,$ and so our probability is $\boxed{\textbf{(A) } \frac{47}{256}}$.

Solution 2

We will count how many valid standing arrangements there are counting rotations as distinct and divide by 256 at the end. Line up all 8 people linearly. In order for no two people standing to be adjacent, we will place a sitting person to the right of each standing person. In effect, each standing person requires 2 spaces and the standing people are separated by sitting people. We just need to determine the number of combinations of pairs and singles and the problem becomes very similar to pirates and gold aka stars and bars aka ball and urn.

If there are 4 standing, there are $\binom{4}{4}=1$ ways to place them. For $3,$ there are $\binom{3+2}{3}=10$ ways. etc. Summing, we get
$\binom{4}{4}+\binom{5}{3}+\binom{6}{2}+\binom{7}{1}+\binom{8}{0}=1+10+15+7+1=34$ ways.

Now we consider that the far right person can be standing as well, so we have
$\binom{3}{3}+\binom{4}{2}+\binom{5}{1}+\binom{6}{0}=1+6+5+1=13$ ways

Together we have $34+13=47$, and so our probability is $\boxed{\textbf{(A) } \frac{47}{256}}$.

Solution 3

We will count how many valid standing arrangements there are (counting rotations as distinct), and divide by $2^8=256$ at the end. If we suppose for the moment that the people are in a line, and decide from left to right whether they sit or stand. If the leftmost person sits, we have the same number of arrangements as if there were only 7 people. If they stand, we count the arrangements with 6 instead because the person second from the left must sit. We notice that this is the Fibonacci sequence, where with 1 person there are two ways and with 2 people there are three ways. Carrying out the Fibonacci recursion until we get to 8 people, we find there are 55 standing arrangements. Some of these were illegal however, since both the first and last people stood. In these cases, both the leftmost and rightmost two people are fixed, leaving us to subtract the number of ways for 4 people to stand in a line, which is 8 from our sequence. Therefore our probability is $\frac{55-8}{256}=\boxed{\textbf{(A) } \frac{47}{256}}$

See Also

2015 AMC 12A Problems/Problem 18

Contents

Problem

The zeros of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

$\textbf{(A)}\ 7\qquad\textbf{(B)}\ 8\qquad\textbf{(C)}\ 16\qquad\textbf{(D)}\ 17\qquad\textbf{(E)}\ 18$

Solution 1

The problem asks us to find the sum of every integer value of a such that the roots of $x^2 - ax + 2a = 0$ are both integers.

The quadratic formula gives the roots of the quadratic equation: $x=\frac{a\pm\sqrt{a^2-8a}}{2}$

As long as the numerator is an even integer, the roots are both integers. But first of all, the radical term in the numerator needs to be an integer; that is, the discriminant $a^2 - 8a$ equals k^2, for some nonnegative integer k.

$a^2-8a=k^2$

$a(a-8)=k^2$

$((a-4)+4)((a-4)-4)=k^2$

$(a-4)^2-4^2=k^2$

$(a-4)^2=k^2+4^2$

From this last equation, we are given a hint of the Pythagorean theorem. Thus, $(k,4,|a-4|)$ must be a Pythagorean triple unless $k=0$.

In the case $k=0$, the equation simplifies to $|a-4|=4$. From this equation, we have $a=0,8$. For both $a=0$ and $a=8$, $\frac{a\pm\sqrt{a^2-8a}}{2}$ yields two integers, so these values satisfy the constraints from the original problem statement. (Note: the two zero roots count as "two integers.")

If k is a positive integer, then only one Pythagorean triple could match the triple $(k,4,|a-4|)$ because the only Pythagorean triple with a 4 as one of the values is the classic $(3,4,5)$ triple. Here, $k=3$ and $|a-4|=5$. Hence, $a=-1,9$. Again, $\frac{a\pm\sqrt{a^2-8a}}{2}$ yields two integers for both $a=-1$ and $a=9$, so these two values also satisfy the original constraints.

There are a total of four possible values for a: $-1,0,8,$ and 9. Hence, the sum of all of the possible values of a is $\boxed{\textbf{(C)}\ 16}$.

Solution 2

Let m and n be the roots of $x^2 - ax + 2a$

By Vieta's Formulas, $n + m = a$ and $mn = 2a$

Substituting gets us $n + m = \frac{mn}{2}$

$2n - mn + 2m = 0$

Using Simon's Favorite Factoring Trick:

$n(2 - m) + 2m = 0$

$-n(2 - m) - 2m = 0$

$-n(2 - m) - 2m + 4 = 4$

$(2 - n)(2 - m) = 4$

This means that the values for (m, n) are $(0, 0), (4, 4), (3, 6), (1, -2)$ giving us a values of $-1, 0, 8,$ and 9. Adding these up gets $\boxed{\textbf{(C)}\ 16}$.

Solution 3

The quadratic formula gives

$$x = \frac{a \pm \sqrt{a(a-8)}}{2}$$

. For x to be an integer, it is necessary (and sufficient!) that $a(a-8)$ to be a perfect square. So we have $a(a-8) = b^2$; this is a quadratic in itself and the quadratic formula gives

$$a = 4 \pm \sqrt{16 + b^2}$$

We want $16 + b^2$ to be a perfect square. From smartly trying small values of b, we find $b = 0, b = 3$ as solutions, which correspond to $a = -1, 0, 8, 9$. These are the only ones; if we want to make sure then we must hand check up to $b = 8$. Indeed, for $b \geq 9$ we have that the differences between consecutive squares are greater than 16 so we can't have $b^2 + 16$ be a perfect square. So summing our values for a we find 16 (C) as the answer.

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2015 AMC 10A Problems/Problem 24

The following problem is from both the 2015 AMC 12A #19 and 2015 AMC 10A #24, so both problems redirect to this page.

Contents

Problem

For some positive integers p, there is a quadrilateral $ABCD$ with positive integer side lengths, perimeter p, right angles at B and C, $AB=2$, and $CD=AD$. How many different values of $p<2015$ are possible?

$\textbf{(A) }30\qquad\textbf{(B) }31\qquad\textbf{(C) }61\qquad\textbf{(D) }62\qquad\textbf{(E) }63$

Solution 1

Let $BC = x$ and $CD = AD = y$ be positive integers. Drop a perpendicular from A to CD to show that, using the Pythagorean Theorem, that

$$x^2 + (y - 2)^2 = y^2.$$

Simplifying yields $x^2 - 4y + 4 = 0$, so $x^2 = 4(y - 1)$. Thus, y is one more than a perfect square.

The perimeter $p = 2 + x + 2y = 2y + 2\sqrt{y - 1} + 2$ must be less than 2015. Simple calculations demonstrate that $y = 31^2 + 1 = 962$ is valid, but $y = 32^2 + 1 = 1025$ is not. On the lower side, $y = 1$ does not work (because $x > 0$), but $y = 1^2 + 1$ does work. Hence, there are 31 valid y (all y such that $y = n^2 + 1$ for $1 \le n \le 31$), and so our answer is $\boxed{\textbf{(B) }31}$

Solution 2

If $AD=CD=x$, then $BC=\sqrt{x^2-(2-x)^2}=2\sqrt{x-1}$. Thus, $p=2x+2\sqrt{x-1}+2<2015$ and thus $2x+2\sqrt{x-1}<2013$. Since $\sqrt{x-1}$ must be an integer, we note that $x-1$ is a perfect square and by simple computations it is seen that $x=31^2+1$ works but $x=32^2+1$ doesn't. Therefore $x=1$ to 31 work, giving an answer of $\boxed{\textbf{(B) }31}$.

See Also

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2015 AMC 12A Problems/Problem 20

Contents

Problem

Isosceles triangles T and T' are not congruent but have the same area and the same perimeter. The sides of T have lengths 5, 5, and 8, while those of T' have lengths a, a, and b. Which of the following numbers is closest to b?

$\textbf{(A) }3\qquad\textbf{(B) }4\qquad\textbf{(C) }5\qquad\textbf{(D) }6\qquad\textbf{(E) }8$

Solution

Solution 1

The area of T is $\dfrac{1}{2} \cdot 8 \cdot 3 = 12$ and the perimeter is 18.

The area of T' is $\dfrac{1}{2}b\sqrt{a^2 - (\dfrac{b}{2})^2}$ and the perimeter is $2a + b$.

Thus $2a + b = 18$, so $2a = 18 - b$.

Thus $12 = \dfrac{1}{2}b\sqrt{a^2 - (\dfrac{b}{2})^2}$, so $48 = b\sqrt{4a^2 - b^2} = b\sqrt{(18 - b)^2 - b^2} = b\sqrt{324 - 36b}$.

We square and divide 36 from both sides to obtain $64 = b^2(9 - b)$, so $b^3 - 9b^2 + 64 = 0$. This factors as $(b - 8)(b^2 - b - 8) = 0$. Because clearly $b \neq 8$ but $b > 0$, we have $b = \dfrac{1 + \sqrt{33}}{2} < \dfrac{1 + 6}{2} = 3.5$. The answer is $\textbf{(A)}$.

Solution 2

Triangle T, being isosceles, has an area of $\dfrac{1}{2}(8)\sqrt{5^2 - 4^2} = 12$ and a perimeter of $5 + 5 + 8 = 18$. Triangle T' similarly has an area of $\dfrac{1}{2}(b)\left(\sqrt{a^2 - \dfrac{b^2}{4}}\right) = 12$ and $2a + b = 18$.

Now we apply our computational fortitude.

$$\frac{1}{2}(b)\left(\sqrt{a^2 - \frac{b^2}{4}}\right) = 12$$

$$(b)\left(\sqrt{a^2-\frac{b^2}{4}}\right)=24$$

$$(b)\sqrt{4a^2-b^2}=48$$

$$b^2(4a^2-b^2)=48^2$$

$$b^2(2a+b)(2a-b)=48^2$$

Plug in $2a+b=18$ to obtain

$$18b^2(2a-b)=48^2$$

$$b^2(2a-b)=128$$

Plug in $2a=18-b$ to obtain

$$b^2(18-2b)=128$$

$$2b^3-18b^2+128=0$$

$$b^3-9b^2+64=0$$

We know that $b=8$ is a valid solution by T. Factoring out $b-8$, we obtain

$$(b-8)(b^2-b-8)=0 \Rightarrow b^2-b-8=0$$

Utilizing the quadratic formula gives

$$b=\frac{1\pm\sqrt{33}}{2}$$

We clearly must pick the positive solution. Note that $5<\sqrt{33}<6$, and so $3<\frac{1+\sqrt{33}}{2}<\frac{7}{2}$, which clearly gives an answer of $\boxed{\text{A}}$, as desired.

Solution 3

Triangle T has perimeter $5+5+8=18$ so $18=2a+b$.

Using Heron's, we get $\sqrt{(9)(4)^2(1)}=\sqrt{(\frac{2a+b}{2})(\frac{b}{2})^2(\frac{2a-b}{2})}$.

We know that $2a+b=18$ from above so we plug that in, and we also know that then $2a-b=18-2b$.

$12=3\frac{b}{2}\sqrt{9-b}$

$64=9b^2-b^3$

We plug in 3 for b in the LHS, and we get 54 which is too low. We plug in 4 for b in the LHS, and we get 80 which is too high. We now know that b is some number between 3 and 4.

If $b\geq 3.5$, then we would round up to 4, but if $b<3.5$, then we would round down to 3. So let us plug in 3.5 for b.

We get 67.375 which is too high, so we know that $b < 3.5$.

The answer is 3. $\textbf{(A)}$

See Also

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2015 AMC 12A Problems/Problem 21

Problem

A circle of radius r passes through both foci of, and exactly four points on, the ellipse with equation $x^2 + 16y^2 = 16$. The set of all possible values of r is an interval $[a, b)$. What is $a + b$?

$\textbf{(A)}\ 5\sqrt{2}+4 \qquad \textbf{(B)}\ \sqrt{17}+7 \qquad \textbf{(C)}\ 6\sqrt{2}+3 \qquad \textbf{(D)}\ \sqrt{15}+8 \qquad \textbf{(E)}\ 12$

Solution

We can graph the ellipse by seeing that the center is $(0, 0)$ and finding the ellipse's intercepts. The points where the ellipse intersects the coordinate axes are $(0, 1)$, $(0, -1)$, $(4, 0)$, and $(-4, 0)$. Recall that the two foci lie on the major axis of the ellipse and are a distance of c away from the center of the ellipse, where $c^2 = a^2 - b^2$, with a being half the length of the major (longer) axis and b being half the minor (shorter) axis of the ellipse. We have that $c^2 = 4^2 - 1^2 \implies c^2 = 15 \implies c = \pm\sqrt{15}$. Hence, the coordinates of both of our foci are $(\sqrt{15}, 0)$ and $(-\sqrt{15}, 0)$. In order for a circle to pass through both of these foci, we must have that the center of this circle lies on the y-axis.

The minimum possible value of r belongs to the circle whose diameter's endpoints are the foci of this ellipse, so $a = \sqrt{15}$. The value for b is achieved when the circle passes through the foci and only three points on the ellipse, which is possible when the circle touches $(0, 1)$ or $(0, -1)$. Which point we use does not change what value of b is attained, so we use $(0, -1)$. Here, we must find the point $(0, y)$ such that the distance from $(0, y)$ to both foci and $(0, -1)$ is the same. Now, we have the two following equations.

$$(\sqrt{15})^2 + (y)^2 = b^2$$

$$y + 1 = b \implies y = b - 1$$

Substituting for y, we have that

$$15 + (b-1)^2 = b^2 \implies -2b + 16 = 0.$$

Solving the above simply yields that $b = 8$, so our answer is $a + b = \sqrt{15} + 8\ \textbf{(D)}$.

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2015 AMC 12A Problems/Problem 22

Problem

For each positive integer n, let $S(n)$ be the number of sequences of length n consisting solely of the letters A and B, with no more than three As in a row and no more than three Bs in a row. What is the remainder when $S(2015)$ is divided by 12?

$\textbf{(A)}\ 0\qquad\textbf{(B)}\ 4\qquad\textbf{(C)}\ 6\qquad\textbf{(D)}\ 8\qquad\textbf{(E)}\ 10$

Solution

One method of approach is to find a recurrence for $S(n)$.

Let us define $A(n)$ as the number of sequences of length n ending with an A, and $B(n)$ as the number of sequences of length n ending in B. Note that $A(n) = B(n)$ and $S(n) = A(n) + B(n)$, so $S(n) = 2A(n)$.

For a sequence of length n ending in A, it must be a string of As appended onto a sequence ending in B of length $n-1, n-2, \text{or } n-3$. So we have the recurrence:

$$A(n) = B(n{-}1){+}B(n{-}2){+}B(n{-}3) = A(n{-}1){+}A(n{-}2){+}A(n{-}3)$$

We can thus begin calculating values of $A(n)$. We see that the sequence goes (starting from $A(0) = 1$): $1, 1, 2, 4, 7, 13, 24...$

A problem arises though: the values of $A(n)$ increase at an exponential rate. Notice however, that we need only find $S(2015) \bmod 12$. In fact, we can abuse the fact that $S(n) = 2A(n)$ and only find $A(2015) \bmod 6$. Going one step further, we need only find $A(2015) \bmod 2$ and $A(2015) \bmod 3$ to find $A(2015) \bmod 6$.

Here are the values of $A(n) \bmod 2$, starting with $A(0)$:

$$1, 1, 0, 0, 1, 1, 0, 0...$$

Since the period is 4 and $2015 \equiv 3 \bmod 4$, $A(2015) \equiv 0 \bmod 2$.

Similarly, here are the values of $A(n) \bmod 3$, starting with $A(0)$:

$$1, 1, 2, 1, 1, 1, 0, 2, 0, 2, 1, 0, 0, 1, 1, 2...$$

Since the period is 13 and $2015 \equiv 0 \bmod 13$, $A(2015) \equiv 1 \bmod 3$.

Knowing that $A(2015) \equiv 0 \bmod 2$ and $A(2015) \equiv 1 \bmod 3$, we see that $A(2015) \equiv 4 \bmod 6$, and $S(2015) \equiv 8 \bmod 12$. Hence, the answer is $\textbf{(D)}$.

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2015 AMC 12A Problems/Problem 23

Problem

Let S be a square of side length 1. Two points are chosen independently at random on the sides of S. The probability that the straight-line distance between the points is at least $\frac{1}{2}$ is $\frac{a-b\pi}{c}$, where $a, b,$ and c are positive integers and $\gcd(a,b,c)=1$. What is $a+b+c$?

$\textbf{(A) }59\qquad\textbf{(B) }60\qquad\textbf{(C) }61\qquad\textbf{(D) }62\qquad\textbf{(E) }63$

Solution

Divide the boundary of the square into halves, thereby forming 8 segments. Without loss of generality, let the first point A be in the bottom-left segment. Then, it is easy to see that any point in the 5 segments not bordering the bottom-left segment will be distance at least $\frac{1}{2}$ apart from A. Now, consider choosing the second point on the bottom-right segment. The probability for it to be distance at least 0.5 apart from A is $\frac{0+1}{2}=\frac{1}{2}$ because of linearity of the given probability. (Alternatively, one can set up a coordinate system and use geometric probability.)

If the second point B is on the left-bottom segment, then if A is distance x away from the left-bottom vertex, then B must be at least $\frac{1}{2}-\sqrt{0.25-x^2}$ away from that same vertex. Thus, using an averaging argument we find that the probability in this case is

$$\frac{1}{\frac{1}{2}^2}\int_0^{\frac{1}{2}}\frac{1}{2}-\sqrt{0.25-x^2}dx=4\left(\frac{1}{4}-\frac{\pi}{16}\right)=1-\frac{\pi}{4}.$$

(Alternatively, one can equate the problem to finding all valid (x,y) with $0<x,y<\frac{1}{2}$ such that $x^2+y^2\geq\frac{1}{4}$, i.e. (x, y) is outside the unit circle with radius 0.5.)

Thus, averaging the probabilities gives

$$P=\frac{1}{8}\left(5+\frac{1}{2}+1-\frac{\pi}{4}\right)=\frac{1}{32}(26-\pi).$$

Our answer is $\textbf{(A)}$.

See Also

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2015 AMC 12A Problems/Problem 24

Problem

Rational numbers a and b are chosen at random among all rational numbers in the interval $[0, 2)$ that can be written as fractions $\dfrac{n}{d}$ where n and d are integers with $1 \le d \le 5$. What is the probability that

$$(\cos(a\pi) + i\sin(b\pi))^4$$

is a real number?

$\textbf{(A)}\ \frac{3}{5} \qquad \textbf{(B)}\ \frac{4}{25} \qquad \textbf{(C)}\ \frac{41}{200} \qquad \textbf{(D)}\ \frac{6}{25} \qquad \textbf{(E)}\ \frac{13}{50}$

Solution

Let $\cos(a\pi) = x$ and $\sin(b\pi) = y$. Consider the binomial expansion of the expression:

$$x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4.$$

We notice that the only terms with i are the second and the fourth terms. Thus for the expression to be a real number, either $\cos(a\pi)$ or $\sin(b\pi)$ must be 0, or the second term and the fourth term cancel each other out (because in the fourth term, you have $i^2 = -1$).

Case 1: Either $\cos(a\pi)$ or $\sin(b\pi)$ is 0.

The two a's satisfying this are $\frac{1}{2}$ and $\frac{3}{2}$, and the two b's satisfying this are 0 and 1. Because a and b can both be expressed as fractions with a denominator less than or equal to 5, there are a total of 20 possible values for a and b:

$$0, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{3},$$

$$\frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{1}{4}, \frac{3}{4},$$

$$\frac{5}{4}, \frac{7}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5},$$

$$\frac{4}{5}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \text{and} \frac{9}{5}.$$

Calculating the total number of sets of (a, b) results in $20 \cdot 20 = 400$ sets. Calculating the total number of invalid sets (sets where a doesn't equal $\frac{1}{2}$ or $\frac{3}{2}$ and b doesn't equal 0 or 1), resulting in $(20 - 2) \cdot (20 - 2) = 324$.

Thus the number of valid sets is 76.

Case 2: The two terms cancel.

We then have:

$$\cos^3(a\pi) \cdot \sin(b\pi) = \cos(a\pi) \cdot \sin^3(b\pi).$$

So:

$$\cos^2(a\pi) = \sin^2(b\pi),$$

which means for a given value of $\cos(a\pi)$ or $\sin(b\pi)$, there are 4 valid values(one in each quadrant).

When either $\cos(a\pi)$ or $\sin(b\pi)$ are equal to 1, however, there are only two corresponding values. We don't count the sets where either $\cos(a\pi)$ or $\sin(b\pi)$ equals 0, for we would get repeated sets. We also exclude values where the denominator is an odd number, for we cannot find any corresponding values(for example, if a is $\frac{1}{5}$, then b must be $\frac{3}{10}$, which we don't have). Thus the total number of sets for this case is $4 \cdot 4 + 2 \cdot 2 = 20$.

Thus, our final answer is $\dfrac{(20+76)}{400} = \dfrac{6}{25}$, which is $\boxed{\textbf{(D)}}$.

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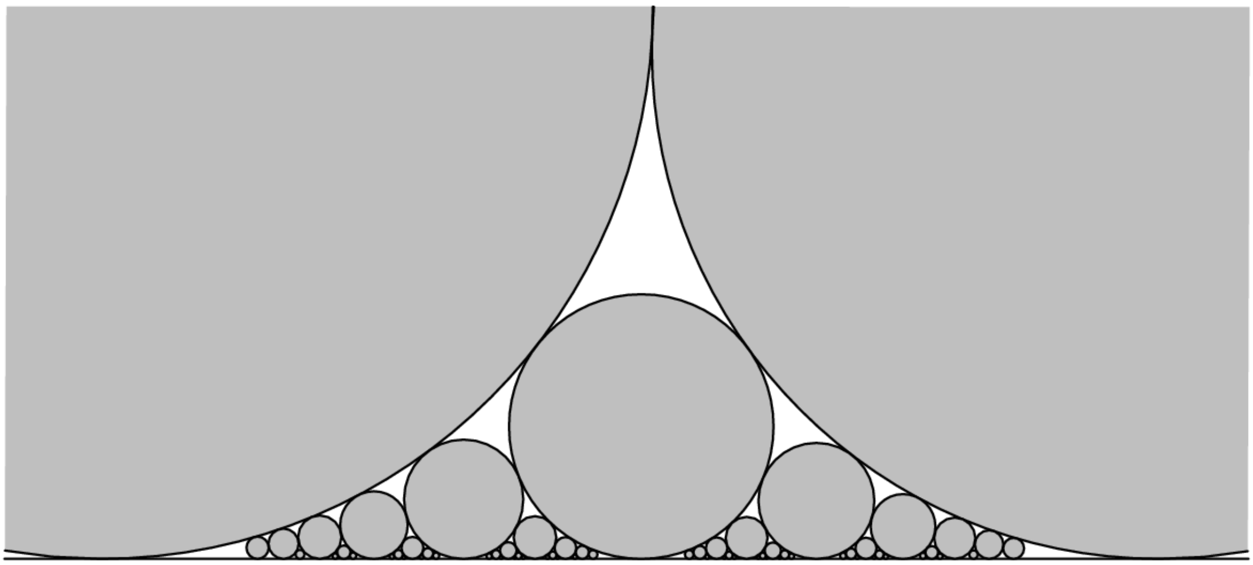
2015 AMC 12A Problems/Problem 25

Problem

A collection of circles in the upper half-plane, all tangent to the x-axis, is constructed in layers as follows. Layer L_0 consists of two circles of radii 70^2 and 73^2 that are externally tangent. For $k \geq 1$, the circles in $\bigcup_{j=0}^{k-1} L_j$ are ordered according to their points of tangency with the x-axis. For every pair of consecutive circles in this order, a new circle is constructed externally tangent to each of the two circles in the pair. Layer L_k consists of the 2^{k-1} circles constructed in this way. Let $S = \bigcup_{j=0}^{6} L_j$, and for every circle C denote by $r(C)$ its radius. What is

$$\sum_{C \in S} \frac{1}{\sqrt{r(C)}}?$$



$\textbf{(A)}\ \frac{286}{35} \qquad \textbf{(B)}\ \frac{583}{70} \qquad \textbf{(C)}\ \frac{715}{73} \qquad \textbf{(D)}\ \frac{143}{14} \qquad \textbf{(E)}\ \frac{1573}{146}$

Solution

Let us start with the two circles in L_0 and the circle in L_1. Let the larger circle in L_0 be named circle X with radius x and the smaller be named circle Y with radius y. Also let the single circle in L_1 be named circle Z with radius z. Draw radii x, y, and z perpendicular to the x-axis. Drop altitudes a and b from the center of Z to these radii x and y, respectively, and drop altitude c from the center of Y to radius x perpendicular to the x-axis. Connect the centers of circles x, y, and z with their radii, and utilize the Pythagorean Theorem. We attain the following equations.

$$(x-z)^2 + a^2 = (x+z)^2 \implies a^2 = 4xz$$

$$(y-z)^2 + b^2 = (y+z)^2 \implies b^2 = 4yz$$

$$(x-y)^2 + c^2 = (x+y)^2 \implies c^2 = 4xy$$

We see that $a = 2\sqrt{xz}$, $b = 2\sqrt{yz}$, and $c = 2\sqrt{xy}$. Since $a + b = c$, we have that $2\sqrt{xz} + 2\sqrt{yz} = 2\sqrt{xy}$. Divide this equation by $2\sqrt{xyz}$, and this equation becomes the well-known relation of Descartes's Circle Theorem $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{z}}$. We can apply this relationship recursively with the circles in layers $L_2, L_3, \cdots, L_6$.

Here, let $S(n)$ denote the sum of the reciprocals of the square roots of all circles in layer n. The notation in the problem asks us to find the sum of the reciprocals of the square roots of the radii in each circle in this collection, which is $\sum_{n=0}^{6} S(n)$. We already have that $S(0) = S(1) = \frac{1}{\sqrt{z}} = \frac{1}{73} + \frac{1}{70}$. Then, $S(2) = 2S(1) + S(0) = 3S(0)$. Additionally, $S(3) = 2S(2) + 2S(1) + S(0) = 9S(0)$, and $S(4) = 2S(3) + 2S(2) + 2S(1) + S(0) = 27S(0)$. Now, we notice that $S(n+1) = 3S(n)$ because $S(n+1) = 2S(n) + 2S(n-1) + \cdots + 2S(1) + S(0)$, which is a power of 3. Hence, our desired sum is $(1 + 1 + 3 + 9 + 27 + 81 + 243)(S(0)) = 365\left(\frac{1}{73} + \frac{1}{70}\right)$. This simplifies to $365\left(\frac{143}{73(70)}\right) = \frac{143}{14}\ \textbf{(D)}$.

Note that the circles in this question are known as Ford circles.

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