2012 AMC 10A Problems/Problem 3

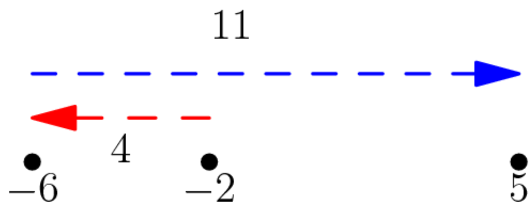
The following problem is from both the 2012 AMC 12A #1 and 2012 AMC 10A #3, so both problems redirect to this page.

Problem

A bug crawls along a number line, starting at -2. It crawls to -6, then turns around and crawls to 5. How many units does the bug crawl altogether?

$\textbf{(A)}\ 9\qquad\textbf{(B)}\ 11\qquad\textbf{(C)}\ 13\qquad\textbf{(D)}\ 14\qquad\textbf{(E)}\ 15$

Solution



Crawling from -2 to -6 takes it a distance of 4 units. Crawling from -6 to 5 takes it a distance of 11 units. Add 4 and 11 to get $\boxed{\textbf{(E)}\ 15}$

See Also

2012 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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2012 AMC 10A Problems/Problem 1

The following problem is from both the 2012 AMC 12A #2 and 2012 AMC 10A #1, so both problems redirect to this page.

Contents

Problem

Cagney can frost a cupcake every 20 seconds and Lacey can frost a cupcake every 30 seconds. Working together, how many cupcakes can they frost in 5 minutes?

$\textbf{(A)}\ 10\qquad\textbf{(B)}\ 15\qquad\textbf{(C)}\ 20\qquad\textbf{(D)}\ 25\qquad\textbf{(E)}\ 30$

Solution 1

Cagney can frost one in 20 seconds, and Lacey can frost one in 30 seconds. Working together, they can frost one in $\frac{20\cdot30}{20+30} = \frac{600}{50} = 12$ seconds. In 300 seconds (5 minutes), they can frost $\boxed{\textbf{(D)}\ 25}$ cupcakes.

Solution 2

In 300 seconds (5 minutes), Cagney will frost $\frac{300}{20} = 15$ cupcakes, and Lacey will frost $\frac{300}{30} = 10$ cupcakes. Therefore, working together they will frost $15 + 10 = \boxed{\textbf{(D)}\ 25}$ cupcakes.

Solution 3

Since Cagney frosts 3 cupcakes a minute, and Lacey frosts 2 cupcakes a minute, they together frost $3+2=5$ cupcakes a minute. Therefore, in 5 minutes, they frost $5\times5=25 \Rightarrow \boxed{\textbf{(D)}}$

See Also

2012 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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2012 AMC 12A Problems/Problem 3

Problem

A box 2 centimeters high, 3 centimeters wide, and 5 centimeters long can hold 40 grams of clay. A second box with twice the height, three times the width, and the same length as the first box can hold n grams of clay. What is n?

$\textbf{(A)}\ 120 \qquad \textbf{(B)}\ 160 \qquad \textbf{(C)}\ 200 \qquad \textbf{(D)}\ 240 \qquad \textbf{(E)}\ 280$

Solution

The first box has volume $2 \times 3 \times 5 = 30 \text{ cm}^3$, and the second has volume $(2 \times 2) \times (3 \times 3) \times (5) = 180 \text{ cm}^3$. The second has a volume that is 6 times greater, so it holds $6 \times 40 = \boxed{\textbf{(D)}\ 240}$ grams.

See Also

2012 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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2012 AMC 10A Problems/Problem 7

The following problem is from both the 2012 AMC 12A #4 and 2012 AMC 10A #7, so both problems redirect to this page.

Contents

Problem

In a bag of marbles, $\frac{3}{5}$ of the marbles are blue and the rest are red. If the number of red marbles is doubled and the number of blue marbles stays the same, what fraction of the marbles will be red?

$\textbf{(A)}\ \frac{2}{5}\qquad\textbf{(B)}\ \frac{3}{7}\qquad\textbf{(C)}\ \frac{4}{7}\qquad\textbf{(D)}\ \frac{3}{5}\qquad\textbf{(E)}\ \frac{4}{5}$

Solution 1

Assume that there are 5 total marbles in the bag. The actual number does not matter, since all we care about is the ratios, and the only operation performed on the marbles in the bag is doubling.

There are 3 blue marbles in the bag and 2 red marbles. If you double the amount of red marbles, there will still be 3 blue marbles but now there will be 4 red marbles. Thus, the answer is $\boxed{\textbf{(C)}\ \frac{4}{7}}$.

Solution 2

Let us say that there are x marbles in the bag. Therefore, $\frac{3x}{5x}$ are blue, and $\frac{2x}{5x}$ are red. When the red marbles are doubled, we now have $\frac{2*2x}{5x+2x}=\frac{4x}{7x}=\frac{4}{7}\Rightarrow\boxed{\textbf{(C)}}$

See Also

2012 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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2012 AMC 12A Problems/Problem 5

Problem

A fruit salad consists of blueberries, raspberries, grapes, and cherries. The fruit salad has a total of 280 pieces of fruit. There are twice as many raspberries as blueberries, three times as many grapes as cherries, and four times as many cherries as raspberries. How many cherries are there in the fruit salad?

$\textbf{(A)}\ 8\qquad\textbf{(B)}\ 16\qquad\textbf{(C)}\ 25\qquad\textbf{(D)}\ 64\qquad\textbf{(E)}\ 96$

Solution

So let the number of blueberries be $b,$ the number of raspberries be $r,$ the number of grapes be $g,$ and finally the number of cherries be $c.$

Observe that since there are 280 pieces of fruit,

$$b+r+g+c=280.$$

Since there are twice as many raspberries as blueberries,

$$2b=r.$$

The fact that there are three times as many grapes as cherries implies,

$$3c=g.$$

Because there are four times as many cherries as raspberries, we deduce the following:

$$4r=c.$$

Note that we are looking for $c.$ So, we try to rewrite all of the other variables in terms of $c.$ The third equation gives us the value of g in terms of c already. We divide the fourth equation by 4 to get that $r=\frac{c}{4}.$ Finally, substituting this value of r into the first equation provides us with the equation $b=\frac{c}{8}$ and substituting yields:

$$\frac{c}{4}+\frac{c}{8}+3c+c=280$$

Multiply this equation by 8 to get:

$$2c+c+24c+8c=8\cdot 280,$$

$$35c=8\cdot 280,$$

$$c=64.$$

$$\boxed{D}$$

See Also

2012 AMC 10A Problems/Problem 8

The following problem is from both the 2012 AMC 12A #6 and 2012 AMC 10A #8, so both problems redirect to this page.

Problem

The sums of three whole numbers taken in pairs are 12, 17, and 19. What is the middle number?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 8$

Solution

Let the three numbers be equal to a, b, and c. We can now write three equations:

$a + b = 12$

$b + c = 17$

$a + c = 19$

Adding these equations together, we get that

$2(a + b + c) = 48$ and

$a + b + c = 24$

Substituting the original equations into this one, we find

$c + 12 = 24$

$a + 17 = 24$

$b + 19 = 24$

Therefore, our numbers are 12, 7, and 5. The middle number is $\boxed{\textbf{(D)}\ 7}$

See Also

2012 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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2012 AMC 10A Problems/Problem 10

The following problem is from both the 2012 AMC 12A #7 and 2012 AMC 10A #10, so both problems redirect to this page.

Problem

Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?

$\textbf{(A)}\ 5\qquad\textbf{(B)}\ 6\qquad\textbf{(C)}\ 8\qquad\textbf{(D)}\ 10\qquad\textbf{(E)}\ 12$

Solution

If we let a be the smallest sector angle and r be the difference between consecutive sector angles, then we have the angles $a, a+r, a+2r, \cdots .a+11r$. Use the formula for the sum of an arithmetic sequence and set it equal to 360, the number of degrees in a circle.

$$\begin{align*} \frac{a+a+11r}{2}\cdot 12 &= 360\\ 2a+11r &= 60\\ a &= \frac{60-11r}{2} \end{align*}$$

All sector angles are integers so r must be a multiple of 2. Plug in even integers for r starting from 2 to minimize $a.$ We find this value to be 4 and the minimum value of a to be $\frac{60-11(4)}{2} = \boxed{\textbf{(C)}\ 8}$

See Also

2012 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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2012 AMC 10A Problems/Problem 13

The following problem is from both the 2012 AMC 12A #8 and 2012 AMC 10A #13, so both problems redirect to this page.

Contents

Problem

An iterative average of the numbers 1, 2, 3, 4, and 5 is computed the following way. Arrange the five numbers in some order. Find the mean of the first two numbers, then find the mean of that with the third number, then the mean of that with the fourth number, and finally the mean of that with the fifth number. What is the difference between the largest and smallest possible values that can be obtained using this procedure?

$\textbf{(A)}\ \frac{31}{16}\qquad\textbf{(B)}\ 2\qquad\textbf{(C)}\ \frac{17}{8}\qquad\textbf{(D)}\ 3\qquad\textbf{(E)}\ \frac{65}{16}$

Solution 1

The minimum and maximum can be achieved with the orders $5,4,3,2,1$ and $1,2,3,4,5$.

$$5,4,3,2,1 \Rightarrow \frac{9}{2},3,2,1 \Rightarrow \frac{15}{4},2,1 \Rightarrow \frac{23}{8},1 \Rightarrow \frac{31}{16}$$

$$1,2,3,4,5 \Rightarrow \frac{3}{2},3,4,5 \Rightarrow \frac{9}{4},4,5 \Rightarrow \frac{25}{8},5 \Rightarrow \frac{65}{16}$$

The difference between the two is $\frac{65}{16}-\frac{31}{16}=\frac{34}{16}=\boxed{\textbf{(C)}\ \frac{17}{8}}$.

Solution 2

The iterative average of any 5 integers a,b,c,d,e can be thought of as:

$$((((\frac{a+b}{2})+c)/2+d)/2+e)/2$$

Expanding this, we see that this fraction is equal to:

$$\frac{a+b+2c+4d+8e}{16}$$

Plugging in $1,2,3,4,5$ for a,b,c,d,e, we see that in order to maximize the fraction,

$$a=1,b=2,c=3,d=4,e=5,$$

and in order to minimize the fraction,

$$a=5,b=4,c=3,d=2,e=1.$$

After plugging in these values and finding the positive difference of the two fractions, we arrive with $\frac{34}{16} \Rightarrow \frac{17}{8}$, which is our answer of $\boxed{\textbf{(C)}}$

See Also

2012 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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2012 AMC 10A Problems/Problem 12

The following problem is from both the 2012 AMC 12A #9 and 2012 AMC 10A #12, so both problems redirect to this page.

Problem

A year is a leap year if and only if the year number is divisible by 400 (such as 2000) or is divisible by 4 but not 100 (such as 2012). The 200th anniversary of the birth of novelist Charles Dickens was celebrated on February 7, 2012, a Tuesday. On what day of the week was Dickens born?

$\textbf{(A)}\ \text{Friday}\qquad\textbf{(B)}\ \text{Saturday}\qquad\textbf{(C)}\ \text{Sunday}\qquad\textbf{(D)}\ \text{Monday}\qquad\textbf{(E)}\ \text{Tuesday}$

Solution

Each year we go back is one day back, because $365 = 1 \pmod 7$. Each leap year we go back is two days back, since $366 = 2 \pmod 7$. A leap year is usually every four years, so 200 years would have $\frac{200}{4}$ = 50 leap years, but the problem points out that 1900 does not count as a leap year.

This would mean a total of 151 regular years and 49 leap years, so $1(151) + 2(49)$ = 249 days back. Since $249 = 4 \pmod 7$, four days back from Tuesday would be $\boxed{\textbf{(A)}\ \text{Friday}}$.

See Also

2012 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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2012 AMC 12A Problems/Problem 10

Contents

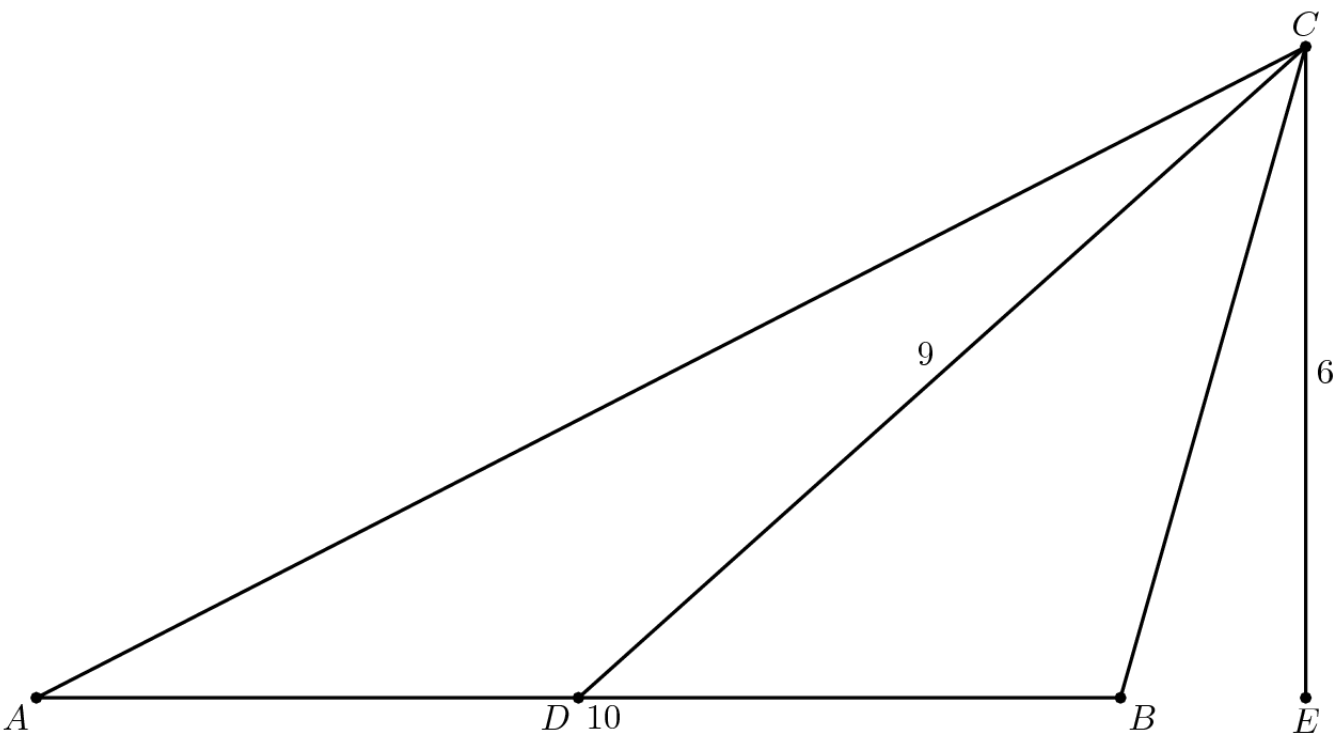
Problem

A triangle has area 30, one side of length 10, and the median to that side of length 9. Let θ be the acute angle formed by that side and the median. What is $\sin{\theta}$?

$\textbf{(A)}\ \frac{3}{10}\qquad\textbf{(B)}\ \frac{1}{3}\qquad\textbf{(C)}\ \frac{9}{20}\qquad\textbf{(D)}\ \frac{2}{3}\qquad\textbf{(E)}\ \frac{9}{10}$

Solution

Solution 1



AB is the side of length 10, and CD is the median of length 9. The altitude of C to AB is 6 because the 0.5(altitude)(base)=Area of the triangle. θ is $\angle CDE$. To find $\sin{\theta}$, just use opposite over hypotenuse with the right triangle $\triangle DCE$. This is equal to $\frac{6}{9}=\boxed{\textbf{(D)}\ \frac{2}{3}}$.

Solution 2

It is a well known fact that a median divides the area of a triangle into two smaller triangles of equal area. Therefore, the area of $\triangle BCD = 15$ in the above figure. Expressing the area in terms of $\sin\theta$, $\frac{1}{2} \cdot 5 \cdot 9 \cdot \sin\theta = 15$. Solving for $\sin\theta$ gives $\sin\theta = \frac{2}{3}$. $\boxed{D}$.

Solution 3

The area of a triangle with sides a, b and angle between them θ is $\frac{1}{2}ab\sin\theta$. Therefore, $30 = \frac{1}{2}(9 \cdot 5)\sin\theta + \frac{1}{2}(9 \cdot 5)\sin\left(180^\circ - \theta\right)$, as two angles along the same line must be supplementary. This simplifies to

$$\sin\theta + \sin\left(180^\circ - \theta\right) = \frac{4}{3} = \sin\theta + \sin 180^\circ \cos\theta - \cos 180^\circ \sin\theta.$$

$$2\sin\theta = \frac{4}{3} \rightarrow \sin\theta = \frac{2}{3}. \boxed{D}$$

See Also

2012 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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Category: Introductory Geometry Problems

2012 AMC 12A Problems/Problem 11

Problem

Alex, Mel, and Chelsea play a game that has 6 rounds. In each round there is a single winner, and the outcomes of the rounds are independent. For each round the probability that Alex wins is $\frac{1}{2}$, and Mel is twice as likely to win as Chelsea. What is the probability that Alex wins three rounds, Mel wins two rounds, and Chelsea wins one round?

$\textbf{(A)}\ \frac{5}{72}\qquad\textbf{(B)}\ \frac{5}{36}\qquad\textbf{(C)}\ \frac{1}{6}\qquad\textbf{(D)}\ \frac{1}{3}\qquad\textbf{(E)}\ 1$

Solution

If m is the probability Mel wins and c is the probability Chelsea wins, $m = 2c$ and $m + c = \frac{1}{2}$. From this we get $m = \frac{1}{3}$ and $c = \frac{1}{6}$. For Alex to win three, Mel to win two, and Chelsea to win one, in that order, is $\frac{1}{2^3 \cdot 3^2 \cdot 6} = \frac{1}{432}$. Multiply this by the number of permutations (orders they can win) which is $\frac{6!}{3!2!1!} = 60$.

$$\frac{1}{432} \cdot 60 = \frac{60}{432} = \boxed{\textbf{(B)}\ \frac{5}{36}}$$

See Also

2012 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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2012 AMC 12A Problems/Problem 12

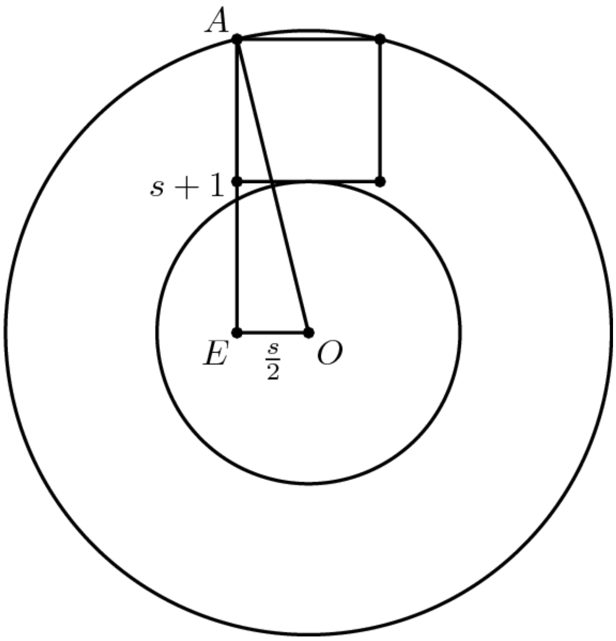
Contents

Problem

A square region $ABCD$ is externally tangent to the circle with equation $x^2+y^2=1$ at the point $(0,1)$ on the side CD. Vertices A and B are on the circle with equation $x^2+y^2=4$. What is the side length of this square?

$\textbf{(A)}\ \frac{\sqrt{10}+5}{10} \qquad \textbf{(B)}\ \frac{2\sqrt{5}}{5} \qquad \textbf{(C)}\ \frac{2\sqrt{2}}{3} \qquad \textbf{(D)}\ \frac{2\sqrt{19}-4}{5} \qquad \textbf{(E)}\ \frac{9-\sqrt{17}}{5}$

Solution 1



The circles have radii of 1 and 2. Draw the triangle shown in the figure above and write expressions in terms of s (length of the side of the square) for the sides of the triangle. Because AO is the radius of the larger circle, which is equal to 2, we can write the Pythagorean Theorem.

$$\left(\frac{s}{2} \right) ^2 + (s+1)^2 = 2^2$$
$$\frac14 s^2 + s^2 + 2s + 1 = 4$$
$$\frac54 s^2 +2s - 3 = 0$$
$$5s^2 + 8s - 12 = 0$$

Use the quadratic formula.

$$s = \frac{-8+\sqrt{8^2-4(5)(-12)}}{10} = \frac{-8+\sqrt{304}}{10} = \frac{-8+4\sqrt{19}}{10} = \boxed{\textbf{(D)}\ \frac{2\sqrt{19}-4}{5}}$$

Solution 2

Using the diagram above, we look at the top-right vertex of the square. Let us call this point (x, y). Then, we that since the square is symmetrical over the y-axis, that the y value is equal to $2x + 1$, since we can multiply the x value(which is half of s) by two to get s, and we add one since the square lies one unit above the origin. Now, all we must do is find the intersection of the larger circle, $x^2 + y^2 = 4$, and the line $y = 2x + 1$. Substituting the second equation into the first, we get:

$5x^2 + 4x - 3 = 0$

Using the quadratic formula, we arrive with $x = \frac{-4 \pm 2\sqrt{19}}{10}$. However, recall that the x value is only one half of the side length. Multiplying this value by 2, then, and using only the positive root(since the top right vertex of the square has a positive x value), we get:

$\frac{-4+2\sqrt{19}}{5} \Rightarrow \boxed{\textbf{(D)}}$

See Also

2012 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
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Category: Introductory Geometry Problems

2012 AMC 10A Problems/Problem 19

> The following problem is from both the 2012 AMC 12A #13 and 2012 AMC 10A #19, so both problems redirect to this page.

Problem 19

Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 P.M. How long, in minutes, was each day's lunch break?

$\textbf{(A)}\ 30\qquad\textbf{(B)}\ 36\qquad\textbf{(C)}\ 42\qquad\textbf{(D)}\ 48\qquad\textbf{(E)}\ 60$

Solution

Let Paula work at a rate of p, the two helpers work at a combined rate of h, and the time it takes to eat lunch be L, where p and h are in house/hours and L is in hours. Then the labor on Monday, Tuesday, and Wednesday can be represented by the three following equations:

$$(8-L)(p+h)=.50$$

$$(6.2-L)h=.24$$

$$(11.2-L)p=.26$$

With three equations and three variables, we need to find the value of L. Adding the second and third equations together gives us $6.2h+11.2p-L(p+h)=.50$. Subtracting the first equation from this new one gives us $-1.8h+3.2p=0$, so we get $h=\frac{16}{9}p$. Plugging into the second equation:

$$(6.2-L)\frac{16}{9}p=.24$$

$$(6.2-L)p=\frac{27}{200}$$

We can then subtract this from the third equation:

$$5p=.26-\frac{27}{200}$$

$$p=\frac{1}{40}$$

Plugging p into our third equation gives:

$$L=\frac{4}{5}$$

Converting L from hours to minutes gives us $L = 48$ minutes, which is $\boxed{\textbf{(D)}\ 48}$.

See Also

2012 AMC 10A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2012))	
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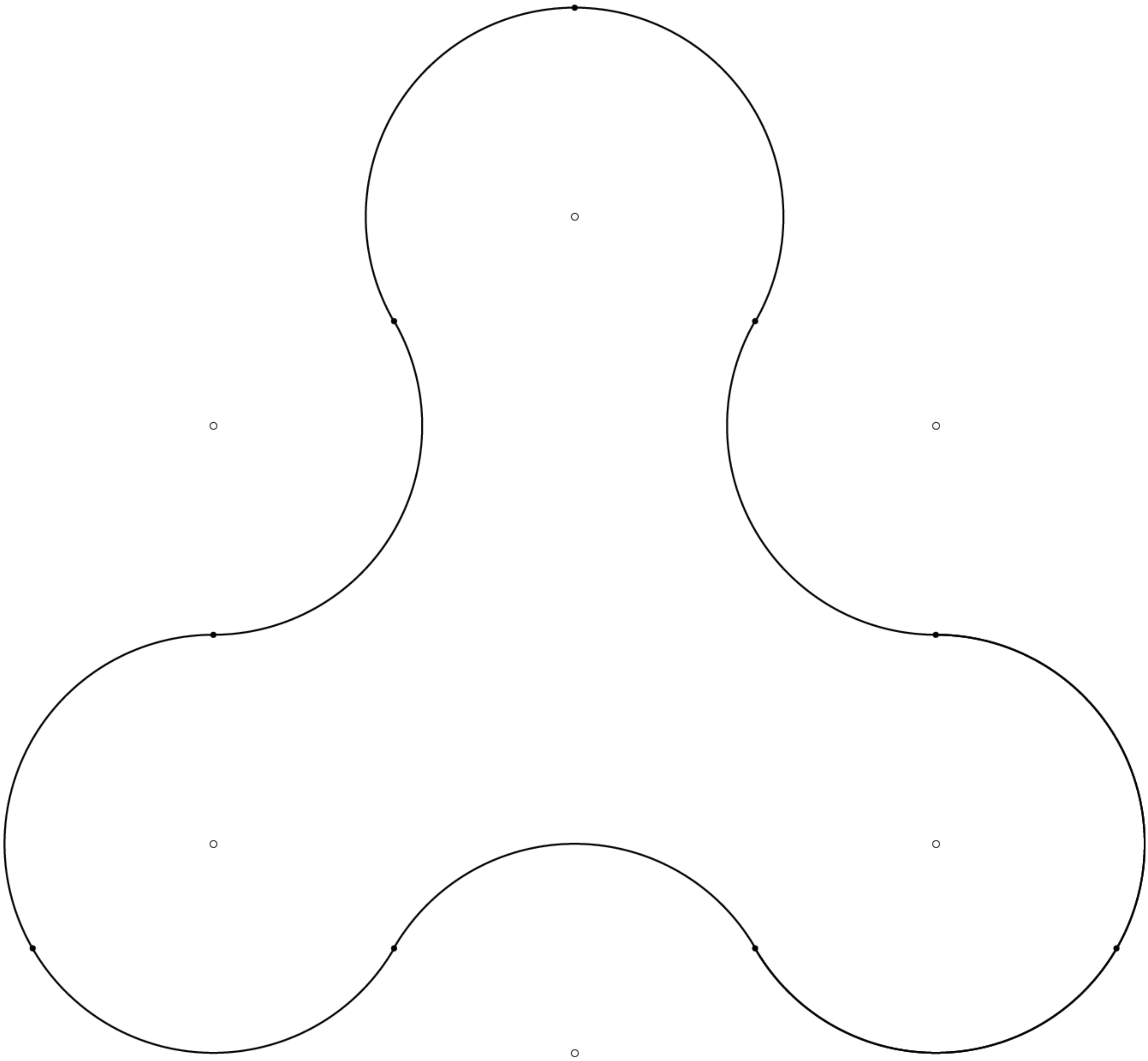
Category: Introductory Algebra Problems

2012 AMC 10A Problems/Problem 18

The following problem is from both the 2012 AMC 12A #14 and 2012 AMC 10A #18, so both problems redirect to this page.

Problem 18

The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve?



$\textbf{(A)}\ 2\pi+6\qquad\textbf{(B)}\ 2\pi+4\sqrt{3}\qquad\textbf{(C)}\ 3\pi+4\qquad\textbf{(D)}\ 2\pi+3\sqrt{3}+2\qquad\textbf{(E)}\ \pi+6\sqrt{3}$

Solution

Draw the hexagon between the centers of the circles, and compute its area $(6)(0.5)(2\sqrt{3}) = 6\sqrt{3}$. Then add the areas of the three sectors outside the hexagon (2π) and subtract the areas of the three sectors inside the hexagon but outside the figure(π) to get the area enclosed in the curved figure $(\pi + 6\sqrt{3})$, which is $\boxed{\textbf{(E)}\ \pi + 6\sqrt{3}}$.

See Also

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Categories: Introductory Geometry Problems | Area Problems

2012 AMC 10A Problems/Problem 20

The following problem is from both the 2012 AMC 12A #15 and 2012 AMC 10A #20, so both problems redirect to this page.

Problem

A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability the grid is now entirely black?

$\textbf{(A)}\ \frac{49}{512}\qquad\textbf{(B)}\ \frac{7}{64}\qquad\textbf{(C)}\ \frac{121}{1024}\qquad\textbf{(D)}\ \frac{81}{512}\qquad\textbf{(E)}\ \frac{9}{32}$

Solution

First, there is only one way for the middle square to be black because it is not affected by the rotation. Then we can consider the corners and edges separately. Let's first just consider the number of ways we can color the corners. There is 1 case with all black squares. There are four cases with one white square and all 4 work. There are six cases with two white squares, but only the 2 with the white squares diagonal from each other work. There are no cases with three white squares or four white squares. Then the total number of ways to color the corners is $1+4+2=7$. In essence, the edges work the same way, so there are also 7 ways to color them. The number of ways to fit the conditions over the number of ways to color the squares is

$$\frac{7\times 7}{2^9} = \boxed{\textbf{(A)}\ \frac{49}{512}}$$

See Also

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2012 AMC 12A Problems/Problem 16

Contents

Problem

Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ \sqrt{26} \qquad \textbf{(C)}\ 3\sqrt{3} \qquad \textbf{(D)}\ 2\sqrt{7} \qquad \textbf{(E)}\ \sqrt{30}$

Solution

Solution 1

Let r denote the radius of circle C_1. Note that quadrilateral $ZYOX$ is cyclic. By Ptolemy's Theorem, we have $11XY = 13r + 7r$ and $XY = 20r/11$. Let t be the measure of angle YOX. Since $YO = OX = r$, the law of cosines on triangle YOX gives us $\cos t = -79/121$. Again since $ZYOX$ is cyclic, the measure of angle $YZX = 180 - t$. We apply the law of cosines to triangle ZYX so that $XY^2 = 7^2 + 13^2 - 2(7)(13)\cos(180 - t)$. Since $\cos(180 - t) = -\cos t = 79/121$ we obtain $XY^2 = 12000/121$. But$XY^2 = 400r^2/121$ so that $r = \sqrt{30}$. $\boxed{E}$.

Solution 2

Let us call the r the radius of circle C_1, and R the radius of C_2. Consider $\triangle OZX$ and $\triangle OZY$. Both of these triangles have the same circumcircle (C_2). From the Extended Law of Sines, we see that $\frac{r}{\sin\angle OZY} = \frac{r}{\sin\angle OZX} = 2R$. Therefore, $\angle OZY \cong \angle OZX$. We will now apply the Law of Cosines to $\triangle OZX$ and $\triangle OZY$ and get the equations

$$r^2 = 13^2 + 11^2 - 2\cdot 13\cdot 11\cdot\cos\angle OZX,$$

$$r^2 = 11^2 + 7^2 - 2\cdot 11\cdot 7\cdot\cos\angle OZY,$$

respectively. Because $\angle OZY \cong \angle OZX$, this is a system of two equations and two variables. Solving for r gives $r = \sqrt{30}$. $\boxed{E}$.

Solution 3

Let r denote the radius of circle C_1. Note that quadrilateral $ZYOX$ is cyclic. By Ptolemy's Theorem, we have $11XY = 13r + 7r$ and $XY = 20r/11$. Consider isosceles triangle XOY. Pulling an altitude to XY from O, we obtain $\cos(\angle OXY) = \frac{10}{11}$. Since quadrilateral $ZYOX$ is cyclic, we have $\angle OXY = \angle OZY$, so $\cos(\angle OXY) = \cos(\angle OZY)$. Applying the Law of Cosines to triangle OZY, we obtain $\frac{10}{11} = \frac{7^2+11^2-r^2}{2(7)(11)}$. Solving gives $r = \sqrt{30}$. $\boxed{E}$.

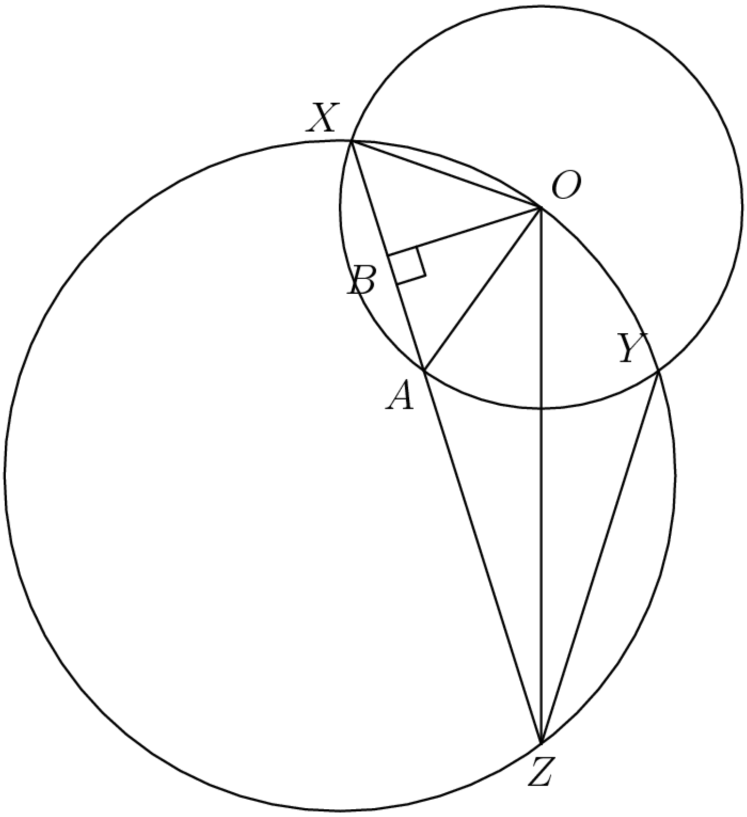
-Solution by thecmd999

Solution 4

Let $P = XY \cap OZ$. Consider an inversion about $C_1 \implies C_2 \to XY, Z \to P$. So, $OP \cdot OZ = r^2 \implies OP = r^2/11 \implies PZ = \frac{121-r^2}{11}$. Using $\triangle YPZ \sim OXZ \implies r = \sqrt{30} \implies \boxed{E}$.

-Solution by IDMasterz

Solution 5



Notice that $\angle YZO = \angle XZO$ as they subtend arcs of the same length. Let A be the point of intersection of C_2 and XZ. We now have $AZ = YZ = 7$ and $XA = 6$. Furthermore, notice that $\triangle XAO$ is isosceles, thus the altitude from O to XA bisects XZ at point B above. By the Pythagorean Theorem,

$$\begin{aligned} BZ^2 + BO^2 &= OZ^2 \\ (BA+AZ)^2 + OA^2 - BA^2 &= 11^2 \\ (3+7)^2 + r^2 - 3^2 &= 121 \\ r^2 &= 30 \end{aligned}$$

Thus, $r = \sqrt{30} \implies \boxed{\textbf{E}}$

Solution 6

Use the diagram above. Consider the power of point Z with respect to Circle O, we have $13 \cdot 7 = 11^2 - r^2$, which gives $r = \boxed{\sqrt{30}}$.

See Also

Category: Introductory Geometry Problems

2012 AMC 12A Problems/Problem 17

Problem

Let S be a subset of $\{1,2,3,\ldots,30\}$ with the property that no pair of distinct elements in S has a sum divisible by 5. What is the largest possible size of S?

$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 13 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 16 \qquad \textbf{(E)}\ 18$

Solution

Of the integers from 1 to 30, there are six each of $0,1,2,3,4 \pmod 5$. We can create several rules to follow for the elements in subset S. No element can be $1 \pmod 5$ if there is an element that is $4 \pmod 5$. No element can be $2 \pmod 5$ if there is an element that is $3 \pmod 5$. Thus we can pick 6 elements from either $1 \pmod 5$ or $4 \pmod 5$ and 6 elements from either $2 \pmod 5$ or $3 \pmod 5$ for a total of $6+6=12$ elements. Considering $0 \pmod 5$, there can be one element that is so because it will only be divisible by 5 if paired with another element that is $0 \pmod 5$. The final answer is $\boxed{\textbf{(B)}\ 13}$.

See Also

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2012 AMC 12A Problems/Problem 18

Problem

Triangle ABC has $AB = 27$, $AC = 26$, and $BC = 25$. Let I denote the intersection of the internal angle bisectors of $\triangle ABC$. What is BI?

$\textbf{(A)}\ 15 \qquad \textbf{(B)}\ 5+\sqrt{26}+3\sqrt{3} \qquad \textbf{(C)}\ 3\sqrt{26} \qquad \textbf{(D)}\ \frac{2}{3}\sqrt{546} \qquad \textbf{(E)}\ 9\sqrt{3}$

Solution

Inscribe circle C of radius r inside triangle ABC so that it meets AB at Q, BC at R, and AC at S. Note that angle bisectors of triangle ABC are concurrent at the center O(also I) of circle C. Let $x = QB$, $y = RC$ and $z = AS$. Note that $BR = x$, $SC = y$ and $AQ = z$. Hence $x + z = 27$, $x + y = 25$, and $z + y = 26$. Subtracting the last 2 equations we have $x - z = -1$ and adding this to the first equation we have $x = 13$.

By Heron's formula for the area of a triangle we have that the area of triangle ABC is $\sqrt{39(14)(13)(12)}$. On the other hand the area is given by $(1/2)25r + (1/2)26r + (1/2)27r$. Then $39r = \sqrt{39(14)(13)(12)}$ so that $r^2 = 56$.

Since the radius of circle O is perpendicular to BC at R, we have by the pythagorean theorem $BO^2 = BI^2 = r^2 + x^2 = 56 + 169 = 225$ so that $BI = 15$.

See Also

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Category: Introductory Geometry Problems

2012 AMC 10A Problems/Problem 23

The following problem is from both the 2012 AMC 12A #19 and 2012 AMC 10A #23, so both problems redirect to this page.

Problem

Adam, Benin, Chiang, Deshawn, Esther, and Fiona have internet accounts. Some, but not all, of them are internet friends with each other, and none of them has an internet friend outside this group. Each of them has the same number of internet friends. In how many different ways can this happen?

$\textbf{(A)}\ 60\qquad\textbf{(B)}\ 170\qquad\textbf{(C)}\ 290\qquad\textbf{(D)}\ 320\qquad\textbf{(E)}\ 660$

Solution

Note that if n is the number of friends each person has, then n can be any integer from 1 to 4, inclusive.

Also note that the cases of $n=1$ and $n=4$ are the same, since a map showing a solution for $n=1$ can correspond one-to-one with a map of a solution for $n=4$ by simply making every pair of friends non-friends and vice versa. The same can be said of configurations with $n=2$ when compared to configurations of $n=3$. Thus, we have two cases to examine, $n=1$ and $n=2$, and we count each of these combinations twice.

For $n=1$, if everyone has exactly one friend, that means there must be 3 pairs of friends, with no other interconnections. The first person has 5 choices for a friend. There are 4 people left. The next person has 3 choices for a friend. There are two people left, and these remaining two must be friends. Thus, there are 15 configurations with $n=1$.

For $n=2$, there are two possibilities. The group of 6 can be split into two groups of 3, with each group creating a friendship triangle. The first person has $\binom{5}{2} = 10$ ways to pick two friends from the other five, while the other three are forced together. Thus, there are 10 triangular configurations.

However, the group can also form a friendship hexagon, with each person sitting on a vertex, and each side representing the two friends that person has. The first person may be seated anywhere on the hexagon Without loss of generality. This person has $\binom{5}{2} = 10$ choices for the two friends on the adjoining vertices.

Each of the three remaining people can be seated "across" from one of the original three people, forming a different configuration. Thus, there are $10 \cdot 3! = 60$ hexagonal configurations, and in total 70 configurations for $n=2$.

As stated before, $n=3$ has 70 configurations, and $n=4$ has 15 configurations. This gives a total of $(70+15)\cdot 2 = 170$ configurations, which is option $\boxed{\textbf{(B)}\ 170}$.

See Also

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2012 AMC 12A Problems/Problem 20

Contents

Problem

Consider the polynomial

$$P(x)=\prod_{k=0}^{10}(x^{2^k}+2^k)=(x+1)(x^2+2)(x^4+4)\cdots(x^{1024}+1024)$$

The coefficient of x^{2012} is equal to 2^a. What is a?

$$\textbf{(A)}\ 5\qquad\textbf{(B)}\ 6\qquad\textbf{(C)}\ 7\qquad\textbf{(D)}\ 10\qquad\textbf{(E)}\ 24$$

Solution 1

Every term in the expansion of the product is formed by taking one term from each factor and multiplying them all together. Therefore, we pick a power of x or a power of 2 from each factor.

Every number, including 2012, has a unique representation by the sum of powers of two, and that representation can be found by converting a number to its binary form. $2012=11111011100_2$, meaning $2012=1024+512+256+128+64+16+8+4$.

Thus, the x^{2012} term was made by multiplying x^{1024} from the $(x^{1024}+1024)$ factor, x^{512} from the $(x^{512}+512)$ factor, and so on. The only numbers not used are 32, 2, and 1.

Thus, from the $(x^{32}+32),(x^2+2),(x+1)$ factors, 32, 2, and 1 were chosen as opposed to x^{32},x^2, and x.

Thus, the coefficient of the x^{2012} term is $32\times 2\times 1=64=2^6$. So the answer is $6\rightarrow\boxed{B}$.

Solution 2

The degree of $P(x)$ is $1024+512+256+\cdots+1=2047$. We want to find the coefficient of x^{2012}, so we need to omit powers of 2 that add up to $2047-2012=35$. Because 35 is odd, we know that one of these must be 2^0. Then, we can test all cases (there are very few of them) and we find that only $2^0+2^1+2^5$ works. From here, we know that the answer is $2^0\cdot 2^1\cdot 2^5=2^6$. Therefore, the answer is $\boxed{(B)\,6.}$

See Also

2012 AMC 10A Problems/Problem 24

The following problem is from both the 2012 AMC 12A #21 and 2012 AMC 10A #24, so both problems redirect to this page.

Problem

Let a, b, and c be positive integers with $a \ge b \ge c$ such that $a^2 - b^2 - c^2 + ab = 2011$ and $a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc = -1997$.

What is a?

$\textbf{(A)}\ 249 \qquad \textbf{(B)}\ 250 \qquad \textbf{(C)}\ 251 \qquad \textbf{(D)}\ 252 \qquad \textbf{(E)}\ 253$

Solution

Add the two equations.

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc = 14.$$

Now, this can be rearranged:

$$(a^2 - 2ab + b^2) + (a^2 - 2ac + c^2) + (b^2 - 2bc + c^2) = 14$$

and factored:

$$(a-b)^2 + (a-c)^2 + (b-c)^2 = 14$$

a, b, and c are all integers, so the three terms on the left side of the equation must all be perfect squares. Recognize that $14 = 9 + 4 + 1$.

$(a-c)^2 = 9 \rightarrow a - c = 3$, since $a - c$ is the biggest difference. It is impossible to determine by inspection whether $a - b = 2$ or 1, or whether $b - c = 1$ or 2.

We want to solve for a, so take the two cases and solve them each for an expression in terms of a. Our two cases are $(a, b, c) = (a, a-1, a-3)$ or $(a, a-2, a-3)$. Plug these values into one of the original equations to see if we can get an integer for a.

$a^2 - (a-1)^2 - (a-3)^2 + a(a-1) = 2011$, after some algebra, simplifies to $7a = 2021$. 2021 is not divisible by 7, so a is not an integer.

The other case gives $a^2 - (a-2)^2 - (a-3)^2 + a(a-2) = 2011$, which simplifies to $8a = 2024$. Thus, $a = 253$ and the answer is $\boxed{\textbf{(E)}\ 253}$.

See Also

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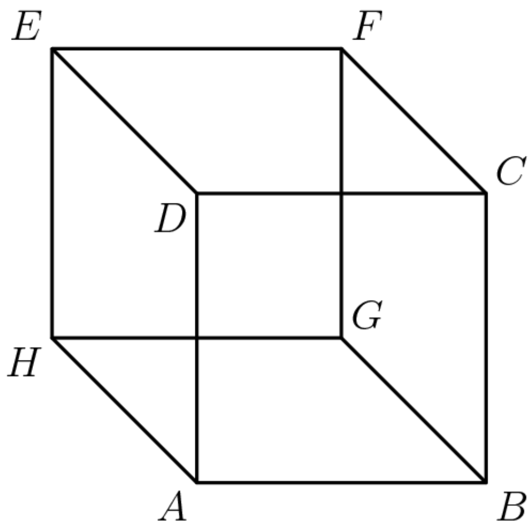
2012 AMC 12A Problems/Problem 22

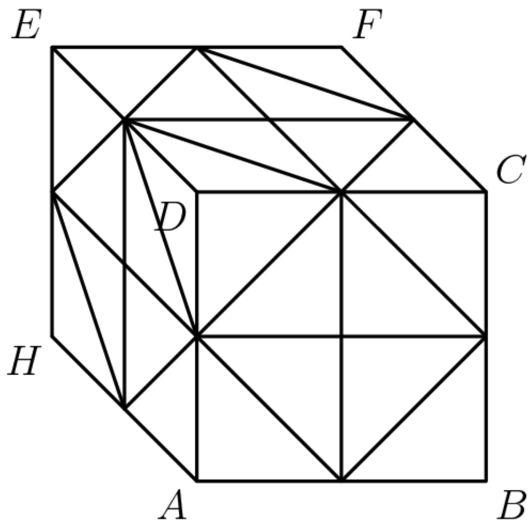
Problem

Distinct planes $p_1, p_2, ..., p_k$ intersect the interior of a cube Q. Let S be the union of the faces of Q and let $P = \bigcup_{j=1}^{k} p_j$. The intersection of P and S consists of the union of all segments joining the midpoints of every pair of edges belonging to the same face of Q. What is the difference between the maximum and minimum possible values of k?

$\textbf{(A)}\ 8\qquad\textbf{(B)}\ 12\qquad\textbf{(C)}\ 20\qquad\textbf{(D)}\ 23\qquad\textbf{(E)}\ 24$

Solution





We need two different kinds of planes that only intersect Q at the mentioned segments (we call them traces in this solution). These will be all the possible p_j's.

First, there are two kinds of segments joining the midpoints of every pair of edges belonging to the same face of Q: long traces are those connecting the midpoint of opposite sides of the same face of Q, and short traces are those connecting the midpoint of adjacent sides of the same face of Q.

Suppose p_j contains a short trace t_1 of a face of Q. Then it must also contain some trace t_2 of an adjacent face of Q, where t_2 share a common endpoint with t_1. So, there are three possibilities for t_2, each of which determines a plane p_j containing both t_1 and t_2.

Case 1: t_2 makes an acute angle with t_1. In this case, $p_j \cap Q$ is an equilateral triangle made by three short traces. There are 8 of them, corresponding to the 8 vertices.

Case 2: t_2 is a long trace. $p \cap Q$ is a rectangle. Each pair of parallel faces of Q contributes 4 of these rectangles so there are 12 such rectangles.

Case 3: t_2 is the short trace other than the one described in case 1, i.e. t_2 makes an obtuse angle with t_1. It is possible to prove that $p \cap Q$ is a regular hexagon (See note #1 for a proof) and there are 4 of them.

Case 4: p_j contains no short traces. This can only make $p_j \cap Q$ be a square enclosed by long traces. There are 3 such squares.

In total, there are $8 + 12 + 4 + 3 = 27$ possible planes in P. So the maximum of k is 27.

On the other hand, the most economic way to generate these long and short traces is to take all the planes in case 3 and case 4. Overall, they intersect at each trace exactly once (there is a quick way to prove this. See note #2 below.) and also covered all the $6 \times 4 + 4 \times 3 = 36$ traces. So the minimum of k is 7. The answer to this problem is then $27 - 7 = 20 \ldots \boxed{C}$.

Note 1: Indeed, let $t_1 = AB$ where $B = t_1 \cap t_2$, and C be the other endpoint of t_2 that is not B. Draw a line through A parallel to t_1. This line passes through the center O of the cube and therefore we see that the reflection of A, B, C, denoted by A', B', C', respectively, lie on the same plane containing A, B, C. Thus $p_j \cap Q$ is the regular hexagon $ABCA'B'C'$. To count the number of these hexagons, just notice that each short trace uniquely determine a hexagon (by drawing the plane through this trace and the center), and that each face has 4 short traces. Therefore, there are 4 such hexagons.

Note 2: The quick way to prove the fact that none of the planes described in case 3 and case 4 share the same trace is as follows: each of these plane contains the center and therefore the intersection of each pair of them is a line through the center, which obviously does not contain any traces.

See Also

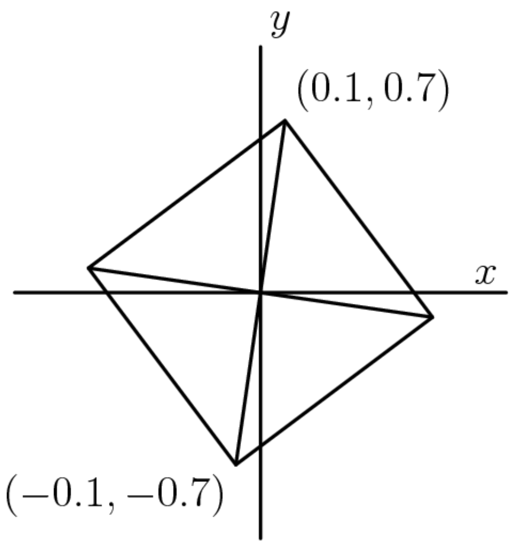
2012 AMC 12A Problems/Problem 23

Problem

Let S be the square one of whose diagonals has endpoints $(0.1, 0.7)$ and $(-0.1, -0.7)$. A point $v = (x, y)$ is chosen uniformly at random over all pairs of real numbers x and y such that $0 \le x \le 2012$ and $0 \le y \le 2012$. Let $T(v)$ be a translated copy of S centered at v. What is the probability that the square region determined by $T(v)$ contains exactly two points with integer coefficients in its interior?

$\textbf{(A)}\ 0.125 \qquad \textbf{(B)}\ 0.14 \qquad \textbf{(C)}\ 0.16 \qquad \textbf{(D)}\ 0.25 \qquad \textbf{(E)}\ 0.32$

Solution



The unit square's diagonal has a length of $\sqrt{0.2^2 + 1.4^2} = \sqrt{2}$. Because S square is not parallel to the axis, the two points must be adjacent.

Now consider the unit square U with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$. Let us first consider only two vertices, $(0,0)$ and $(1,0)$. We want to find the area of the region within U that the point $v = (x, y)$ will create the translation of S, $T(v)$ such that it covers both $(0,0)$ and $(1,0)$. By symmetry, there will be three equal regions that cover the other pairs of adjacent vertices.

For $T(v)$ to contain the point $(0,0)$, v must be inside square S. Similarly, for $T(v)$ to contain the point $(1,0)$, v must be inside a translated square S with center at $(1,0)$, which we will call S'. Therefore, the area we seek is Area$(U \cap S \cap S')$.

To calculate the area, we notice that Area$(U \cap S \cap S') = \frac{1}{2} \cdot$ Area$(S \cap S')$ by symmetry. Let $S_1 = (0.1, 0.7), S_2 = (0.7, -0.1), S_1' = (1.1, 0.7), S_2' = (0.3, 0.1)$. Let $M = (0.7, 0.4)$ be the midpoint of $S_1'S_2'$, and $N = (0.7, 0.7)$ along the line S_1S_1'. Let I be the intersection of S and S' within U, and J be the intersection of S and S' outside U. Therefore, the area we seek is $\frac{1}{2} \cdot$ Area $(S \cap S') = \frac{1}{2}[IS_2'JS_2]$. Because S_2, M, N all have x coordinate 0.7, they are collinear. Noting that the side length of S and S' is 1 (as shown above), we also see that $S_2M = MS_1' = 0.5$, so

$\triangle S_1'NM \cong \triangle S_2IM$. If follows that $IS_2 = NS_1' = 1.1 - 0.7 = 0.4$ and $IS'2 = MS_2' - MI = MS_2' - MN = 0.5 - 0.3 = 0.2$. Therefore, the area is $\frac{1}{2}$· Area $(S \cap S') = \frac{1}{2}[IS_2'JS_2] = \frac{1}{2} \cdot 0.2 \cdot 0.4 = 0.04$.

Because there are three other regions in the unit square U that we need to count, the total area of v within U such that $T(v)$ contains two adjacent lattice points is $0.04 \cdot 4 = 0.16$.

By periodicity, this probability is the same for all $0 \leq x \leq 2012$ and $0 \leq y \leq 2012$. Therefore, the answer is $\boxed{\textbf{(C)}\ 0.16}$.

See Also

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2012 AMC 12A Problems/Problem 24

Problem

Let $\{a_k\}_{k=1}^{2011}$ be the sequence of real numbers defined by $a_1 = 0.201$, $a_2 = (0.2011)^{a_1}$, $a_3 = (0.20101)^{a_2}$, $a_4 = (0.201011)^{a_3}$, and in general,

$$a_k = \begin{cases} (0.\underbrace{20101\cdots 0101}_{k+2 \text{ digits}})^{a_{k-1}} & \text{if } k \text{ is odd,} \\ (0.\underbrace{20101\cdots 01011}_{k+2 \text{ digits}})^{a_{k-1}} & \text{if } k \text{ is even.} \end{cases}$$

Rearranging the numbers in the sequence $\{a_k\}_{k=1}^{2011}$ in decreasing order produces a new sequence $\{b_k\}_{k=1}^{2011}$. What is the sum of all integers k, $1 \le k \le 2011$, such that $a_k = b_k$?

$\textbf{(A)}\ 671 \qquad \textbf{(B)}\ 1006 \qquad \textbf{(C)}\ 1341 \qquad \textbf{(D)}\ 2011 \qquad \textbf{(E)}\ 2012$

Solution

First, we must understand two important functions: $f(x) = b^x$ for $0 < b < 1$(decreasing exponential function), and $g(x) = x^k$ for $k > 0$(increasing exponential function for positive x). $f(x)$ is used to establish inequalities when we change the exponent and keep the base constant. $g(x)$ is used to establish inequalities when we change the base and keep the exponent constant.

We will now examine the first few terms.

Comparing a_1 and a_2, $0 < a_1 = (0.201)^1 < (0.201)^{a_1} < (0.2011)^{a_1} = a_2 < 1 \Rightarrow 0 < a_1 < a_2 < 1$.

Therefore, $0 < a_1 < a_2 < 1$.

Comparing a_2 and a_3,
$0 < a_3 = (0.20101)^{a_2} < (0.20101)^{a_1} < (0.2011)^{a_1} = a_2 < 1 \Rightarrow 0 < a_3 < a_2 < 1$.

Comparing a_1 and a_3, $0 < a_1 = (0.201)^1 < (0.201)^{a_2} < (0.20101)^{a_2} = a_3 < 1 \Rightarrow 0 < a_1 < a_3 < 1$.

Therefore, $0 < a_1 < a_3 < a_2 < 1$.

Comparing a_3 and a_4,
$0 < a_3 = (0.20101)^{a_2} < (0.20101)^{a_3} < (0.201011)^{a_3} = a_4 < 1 \Rightarrow 0 < a_3 < a_4 < 1$.

Comparing a_2 and a_4,
$0 < a_4 = (0.201011)^{a_3} < (0.201011)^{a_1} < (0.2011)^{a_1} = a_2 < 1 \Rightarrow 0 < a_4 < a_2 < 1$.

Therefore, $0 < a_1 < a_3 < a_4 < a_2 < 1$.

Continuing in this manner, it is easy to see a pattern(see Note 1).

Therefore, the only k when $a_k = b_k$ is when $2(k - 1006) = 2011 - k$. Solving gives $\boxed{\textbf{(C)} 1341}$.

Note 1: We claim that $0 < a_1 < a_3 < ... < a_{2011} < a_{2010} < ... < a_4 < a_2 < 1$.

We can use induction to prove this statement. (not necessary for AMC):

Base Case: We have already shown the base case above, where $0 < a_1 < a_2 < 1$.

Inductive Step:

Rearranging in decreasing order gives
$1 > b_1 = a_2 > b_2 = a_4 > ... > b_{1005} = a_{2010} > b_{1006} = a_{2011} > ... > b_{2010} = a_3 > b_{2011} = a_1 > 0$.

See Also

2012 AMC 12A Problems/Problem 25

Problem

Let $f(x) = |2\{x\} - 1|$ where $\{x\}$ denotes the fractional part of x. The number n is the smallest positive integer such that the equation

$$nf(xf(x)) = x$$

has at least 2012 real solutions. What is n? Note: the fractional part of x is a real number $y = \{x\}$ such that $0 \le y < 1$ and $x - y$ is an integer.

$\textbf{(A)}\ 30 \qquad \textbf{(B)}\ 31 \qquad \textbf{(C)}\ 32 \qquad \textbf{(D)}\ 62 \qquad \textbf{(E)}\ 64$

Solution

Our goal is to determine how many times intersects the line $y = x$. We begin by analyzing the behavior of $\{x\}$. It increases linearly with a slope of one, then when it reaches the next integer, it repeats itself. We can deduce that the function is like a sawtooth wave, with a period of one. We then analyze the function $f(x) = |2\{x\} - 1|$. The slope of the teeth is multiplied by 2 to get 2, and the function is moved one unit downward. The function can then be described as starting at -1, moving upward with a slope of 2 to get to 1, and then repeating itself, still with a period of 1. The absolute value of the function is then taken. This results in all the negative segments becoming flipped in the Y direction. The positive slope starting at -1 of the function ranging from u to $u.5$, where u is any arbitrary integer, is now a negative slope starting at positive 1. The function now looks like the letter V repeated within every square in the first row. It is now that we address the goal of this, which is to determine how many times the function intersects the line $y = x$. Since there are two line segments per box, the function has two chances to intersect the line $y = x$ for every integer. If the height of the function is higher than $y = x$ for every integer on an interval, then every chance within that interval intersects the line. Returning to analyzing the function, we note that it is multiplied by x, and then fed into $f(x)$. Since $f(x)$ is a periodic function, we can model it as multiplying the function's frequency by x. This gives us $2x$ chances for every integer, which is then multiplied by 2 once more to get $4x$ chances for every integer. The amplitude of this function is initially 1, and then it is multiplied by n, to give an amplitude of n. The function intersects the line $y = x$ for every chance in the interval of $0 \le x \le n$, since the function is n units high. The function ceases to intersect $y = x$ when $n < x$, since the height of the function is lower than $y = x$. The number of times the function intersects $y = x$ is then therefore equal to $\int_0^n 4x \,\mathrm{d}x$. It is easy to see that this is equal to $2n^2$. The problem then simplifies to the algebraic expression $2n^2 = 2012$, which simplifies to, $n^2 = 1006$, and then to $n = \sqrt{1006}$, which rounds up to 32. $\boxed{\text{C}}$.

See Also

2012 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2012))	
Preceded by Problem 24	Followed by Last Problem
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	