2019 AMC 12A Problems 1

Problem

The area of a pizza with radius 4 is N percent larger than the area of a pizza with radius 3 inches. What is the integer closest to N?

$\textbf{(A) } 25 \qquad\textbf{(B) } 33 \qquad\textbf{(C) } 44 \qquad\textbf{(D) } 66 \qquad\textbf{(E) } 78$

Solution

The area of the larger pizza is 16π, while the area of the smaller pizza is 9π. Therefore, the larger pizza is $\frac{7\pi}{9\pi} \cdot 100\%$ bigger than the smaller pizza. $\frac{7\pi}{9\pi} \cdot 100\% = 77.777....$, which is closest to $\boxed{\textbf{(E) } 78}$.

2019 AMC 12A Problems/Problem 2

Problem

Suppose a is 150% of b. What percent of a is $3b$?

$\textbf{(A) } 50 \qquad \textbf{(B) } 66\frac{2}{3} \qquad \textbf{(C) } 150 \qquad \textbf{(D) } 200 \qquad \textbf{(E) } 450$

Solution 1

Since $a = 1.5b$, that means $b = \frac{a}{1.5}$. We multiply by 3 to get a $3b$ term, yielding $3b = 2a$, and $2a$ is $\boxed{\textbf{(D) } 200\%}$ of a.

Solution 2

Without loss of generality, let $b = 100$. Then, we have $a = 150$ and $3b = 300$. Thus, $\frac{3b}{a} = \frac{300}{150} = 2$, so $3b$ is 200% of a. Hence the answer is $\boxed{\textbf{(D) } 200\%}$.

2019 AMC 10A Problems/Problem 4

(Redirected from 2019 AMC 12A Problems/Problem 3)

The following problem is from both the 2019 AMC 10A #4 and 2019 AMC 12A #3, so both problems redirect to this page.

Problem

A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn$?$

$\textbf{(A) } 75 \qquad \textbf{(B) } 76 \qquad \textbf{(C) } 79 \qquad \textbf{(D) } 84 \qquad \textbf{(E) } 91$

Solution

By choosing the maximum number of balls while getting < 15 of each color, we could have chosen 14 red balls, 14 green balls, 14 yellow balls, 13 blue balls, 11 white balls, and 9 black balls, for a total of 75 balls. Picking one more ball guarantees that we will get 15 balls of a color -- either red, green, or yellow. Thus the answer is $75 + 1 = \boxed{\textbf{(B) } 76}$

2019 AMC 10A Problems/Problem 5

(Redirected from 2019 AMC 12A Problems/Problem 4)

The following problem is from both the 2019 AMC 10A #5 and 2019 AMC 12A #4, so both problems redirect to this page.

Problem

What is the greatest number of consecutive integers whose sum is $45?$

$\textbf{(A) } 9 \qquad \textbf{(B) } 25 \qquad \textbf{(C) } 45 \qquad \textbf{(D) } 90 \qquad \textbf{(E) } 120$

Solution 1

We might at first think that the answer would be 9, because $1 + 2 + 3 \dots + n = 45$ when $n = 9$. But note that the problem says that they can be integers, not necessarily positive. Observe also that every term in the sequence $-44, -43, \cdots, 44, 45$ cancels out except 45. Thus, the answer is, intuitively, $\boxed{\textbf{(D) } 90}$ integers.

Though impractical, a proof of maximality can proceed as follows: Let the desired sequence of consecutive integers be $a, a+1, \cdots, a+(N-1)$, where there are N terms, and we want to maximize N. Then the sum of the terms in this sequence is $aN + \frac{(N-1)(N)}{2} = 45$. Rearranging and factoring, this reduces to $N(2a+N-1) = 90$. Since N must divide 90, and we know that 90 is an attainable value of the sum, 90 must be the maximum.

Solution 2

To maximize the number of integers, we need to make the average of them as low as possible while still being positive. The average can be $\frac{1}{2}$ if the middle two numbers are 0 and 1, so the answer is $\frac{45}{\frac{1}{2}} = \boxed{\textbf{(D) } 90}$.

2019 AMC 10A Problems/Problem 7

(Redirected from 2019 AMC 12A Problems/Problem 5)

The following problem is from both the 2019 AMC 10A #7 and 2019 AMC 12A #5, so both problems redirect to this page.

Problem

Two lines with slopes $\frac{1}{2}$ and 2 intersect at $(2, 2)$. What is the area of the triangle enclosed by these two lines and the line $x + y = 10?$

$\textbf{(A) } 4 \qquad\textbf{(B) } 4\sqrt{2} \qquad\textbf{(C) } 6 \qquad\textbf{(D) } 8 \qquad\textbf{(E) } 6\sqrt{2}$

Solution 1

Let's first work out the slope-intercept form of all three lines: $(x,y)=(2,2)$ and $y=\frac{x}{2}+b$ implies $2=\frac{2}{2}+b=1+b$ so $b=1$, while $y=2x+c$ implies $2=2\cdot 2+c=4+c$ so $c=-2$. Also, $x+y=10$ implies $y=-x+10$. Thus the lines are $y=\frac{x}{2}+1, y=2x-2,$ and $y=-x+10$. Now we find the intersection points between each of the lines with $y=-x+10$, which are $(6,4)$ and $(4,6)$. Using the distance formula and then the Pythagorean Theorem, we see that we have an isosceles triangle with base $2\sqrt{2}$ and height $3\sqrt{2}$, whose area is $\boxed{\textbf{(C) } 6}$.

Solution 2

Like in Solution 1, we determine the coordinates of the three vertices of the triangle. Now, using the [Shoelace Theorem](), we can directly find that the area is $\boxed{\textbf{(C) } 6}$.

Solution 3

Like in the other solutions, solve the systems of equations to see that the triangle's two other vertices are at $(4, 6)$ and $(6, 4)$. Then apply Heron's

Formula: the semi-perimeter will be $s = \sqrt{2} + \sqrt{20}$, so the area reduces nicely to a difference of squares, making it $\boxed{\textbf{(C) } 6}$.

Solution 4

Like in the other solutions, we find, either using algebra or simply by drawing the lines on squared paper, that the three points of intersection are $(2,2)$, $(4,6)$, and $(6,4)$. We can now draw the bounding square with vertices $(2,2)$, $(2,6)$, $(6,6)$ and $(6,2)$, and deduce that the triangle's area is $16-4-2-4=\boxed{\textbf{(C) } 6}$.

Solution 5

Like in other solutions, we find that the three points of intersection are $(2,2)$, $(4,6)$, and $(6,4)$. Using graph paper, we can see that this triangle has 6 boundary lattice points and 4 interior lattice points. By Pick's Theorem, the area is $\frac{6}{2}+4-1=\boxed{\textbf{(C) } 6}$.

Solution 6

Like in other solutions, we find the three points of intersection. Label these $A(2,2)$, $B(4,6)$, and $C(6,4)$. By the Pythagorean Theorem, $AB=AC=2\sqrt{5}$ and $BC=2\sqrt{2}$. By the Law of Cosines,

$$\cos A = \frac{(2\sqrt{5})^2+(2\sqrt{5})^2-(2\sqrt{2})^2}{2\cdot 2\sqrt{5}\cdot 2\sqrt{5}} = \frac{20+20-8}{40} = \frac{32}{40} = \frac{4}{5}$$

Therefore, $\sin A = \sqrt{1-\cos^2 A} = \frac{3}{5}$, so the area is $\frac{1}{2}bc\sin A = \frac{1}{2}\cdot 2\sqrt{5}\cdot 2\sqrt{5}\cdot\frac{3}{5} = \boxed{\textbf{(C) } 6}$.

Solution 7

Like in other solutions, we find that the three points of intersection are $(2,2)$, $(4,6)$, and $(6,4)$. The area of the triangle is half the absolute value of the determinant of the matrix determined by these points. $\frac{1}{2}\left|\begin{vmatrix}2&2&1\\4&6&1\\6&4&1\end{vmatrix}\right| = \frac{1}{2}|-12| = \frac{1}{2}\cdot 12 = \boxed{\textbf{(C) } 6}$

Solution 8

Like in other solutions, we find the three points of intersection. Label these $A(2,2)$, $B(4,6)$, and $C(6,4)$. Then vectors $\overrightarrow{AB} = \langle 2,4\rangle$ and $\overrightarrow{AC} = \langle 4,2\rangle$. The area of the triangle is half the magnitude of the cross product of these two vectors. $\frac{1}{2}\left|\begin{vmatrix}i&j&k\\2&4&0\\4&2&0\end{vmatrix}\right| = \frac{1}{2}|-12k| = \frac{1}{2}\cdot 12 = \boxed{\textbf{(C) } 6}$

Solution 9

Like in other solutions, we find that the three points of intersection are $(2,2)$, $(4,6)$, and $(6,4)$. By the Pythagorean theorem, this is an isosceles triangle with base $2\sqrt{2}$ and equal length $2\sqrt{5}$. The area of an isosceles triangle with base b and equal length l is $\frac{b\sqrt{4l^2-b^2}}{4}$. Plugging in $b = 2\sqrt{2}$ and $l = 2\sqrt{5}$,

$$\frac{2\sqrt{2}\cdot\sqrt{80-8}}{4} = \frac{\sqrt{576}}{4} = \frac{24}{4} = \boxed{\textbf{(C) } 6}$$

Solution 10

Like in other solutions, we find the three points of intersection. Label these $A(2,2)$, $B(4,6)$, and $C(6,4)$. By the Pythagorean Theorem, $AB=AC=2\sqrt{5}$ and $BC=2\sqrt{2}$. By the Law of Cosines,

$$\cos A=\frac{(2\sqrt{5})^2+(2\sqrt{5})^2-(2\sqrt{2})^2}{2\cdot 2\sqrt{5}\cdot 2\sqrt{5}}=\frac{20+20-8}{40}=\frac{32}{40}=\frac{4}{5}$$

Therefore, $\sin A=\sqrt{1-\cos^2 A}=\frac{3}{5}$. By the extended Law of Sines, $2R=\frac{a}{\sin A}=\frac{2\sqrt{2}}{\frac{3}{5}}=\frac{10\sqrt{2}}{3}R=\frac{5\sqrt{2}}{3}$ Then the area is $\frac{abc}{4R}=\frac{2\sqrt{2}\cdot 2\sqrt{5}^2}{4\cdot\frac{5\sqrt{2}}{3}}=\boxed{\textbf{(C) } 6}$.

Solution 11

The area of a triangle formed by three lines, $a_1x+a_2y+a_3=0$ $b_1x+b_2y+b_3=0c_1x+c_2y+c_3=0$ is the absolute value of

$$\frac{1}{2}\cdot\frac{1}{(b_1c_2-b_2c_1)(a_1c_2-a_2c_1)(a_1b_2-a_2b_1)}\cdot\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}^2$$

Plugging in the three lines, $-x+2y-2=0-2x+y+2=0$ $x+y-10=0$ the area is the absolute value of

$$\frac{1}{2}\cdot\frac{1}{(-2-1)(-1-2)(-1+4)}\cdot\begin{vmatrix}-1 & 2 & -2\\ -2 & 1 & 2\\ 1 & 1 & -10\end{vmatrix}^2=\frac{1}{2}\cdot\frac{1}{27}\cdot 18^2=\boxed{\textbf{(C) } 6}$$

Source: Orrick, Michael L. “THE AREA OF A TRIANGLE FORMED BY THREE LINES.” Pi Mu Epsilon Journal, vol. 7, no. 5, 1981, pp. 294–298. JSTOR, www.jstor.org/stable/24336991.

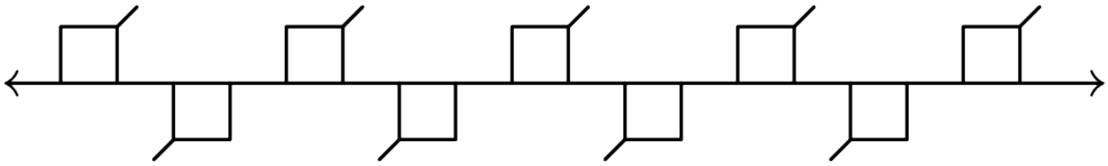
2019 AMC 10A Problems/Problem 8

(Redirected from 2019 AMC 12A Problems/Problem 6)

The following problem is from both the 2019 AMC 10A #8 and 2019 AMC 12A #6, so both problems redirect to this page.

Problem

The figure below shows line ℓ with a regular, infinite, recurring pattern of squares and line segments.



How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?

- some rotation around a point of line ℓ
- some translation in the direction parallel to line ℓ
- the reflection across line ℓ
- some reflection across a line perpendicular to line ℓ

$\textbf{(A) } 0 \qquad\textbf{(B) } 1 \qquad\textbf{(C) } 2 \qquad\textbf{(D) } 3 \qquad\textbf{(E) } 4$

Solution

Statement 1 is true. A 180° rotation about the point half way between an up-facing square and a down-facing square will yield the same figure.

Statement 2 is also true. A translation to the left or right will place the image onto itself when the figures above and below the line realign (the figure goes on infinitely in both directions).

Statement 3 is false. A reflection across line ℓ will change the up-facing squares to down-facing squares and vice versa.

Finally, statement 4 is also false because it will cause the diagonal lines extending from the squares to switch direction. Thus, only $\boxed{\textbf{(C) } 2}$ statements are true.

2019 AMC 10A Problems/Problem 12

(Redirected from 2019 AMC 12A Problems/Problem 7)

The following problem is from both the 2019 AMC 10A #12 and 2019 AMC 12A #7, so both problems redirect to this page.

Problem

Melanie computes the mean μ, the median M, and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, . . . , 12 28s, 11 29s, 11 30s, and 7 31s. Let d be the median of the modes. Which of the following statements is true?

$\textbf{(A) } \mu < d < M \qquad \textbf{(B) } M < d < \mu \qquad \textbf{(C) } d = M = \mu \qquad \textbf{(D) } d < M < \mu \qquad \textbf{(E) } d < \mu < M$

Solution 1

First of all, d obviously has to be smaller than M, since when calculating M, we must take into account the 29s, 30s, and 31s. So we can eliminate choices B and C. Since there are 365 total entries, the median, M, must be the 183rd one, at which point we note that $12 \cdot 15$ is 180, so 16 has to be the median (because 183 is between $12 \cdot 15 + 1 = 181$ and $12 \cdot 16 = 192$). Now, the mean, μ, must be smaller than 16, since there are many fewer 29s, 30s, and 31s. d is less than μ, because when calculating μ, we would include 29, 30, and 31. Thus the answer is $\boxed{\textbf{(E) } d < \mu < M}$.

Solution 2

As in Solution 1, we find that the median is 16. Then, looking at the modes $(1-28)$, we realize that even if we were to have 12 of each, their median would remain the same, being 14.5. As for the mean, we note that the mean of the first 28 is simply the same as the median of them, which is 14.5. Hence, since we in fact have 29's, 30's, and 31's, the mean has to be higher than 14.5. On the other hand, since there are fewer 29's, 30

's, and 31's than the rest of the numbers, the mean has to be lower than 16 (the median). By comparing these values, the answer is $\boxed{\textbf{(E)}}$.

Solution 3 (direct calculation)

We can solve this problem simply by carefully calculating each of the values, which turn out to be $M = 16$, $d = 14.5$, and $\mu \approx 15.7$. Thus the answer is $\boxed{\textbf{(E)}}$.

2019 AMC 10A Problems/Problem 14

(Redirected from 2019 AMC 12A Problems/Problem 8)

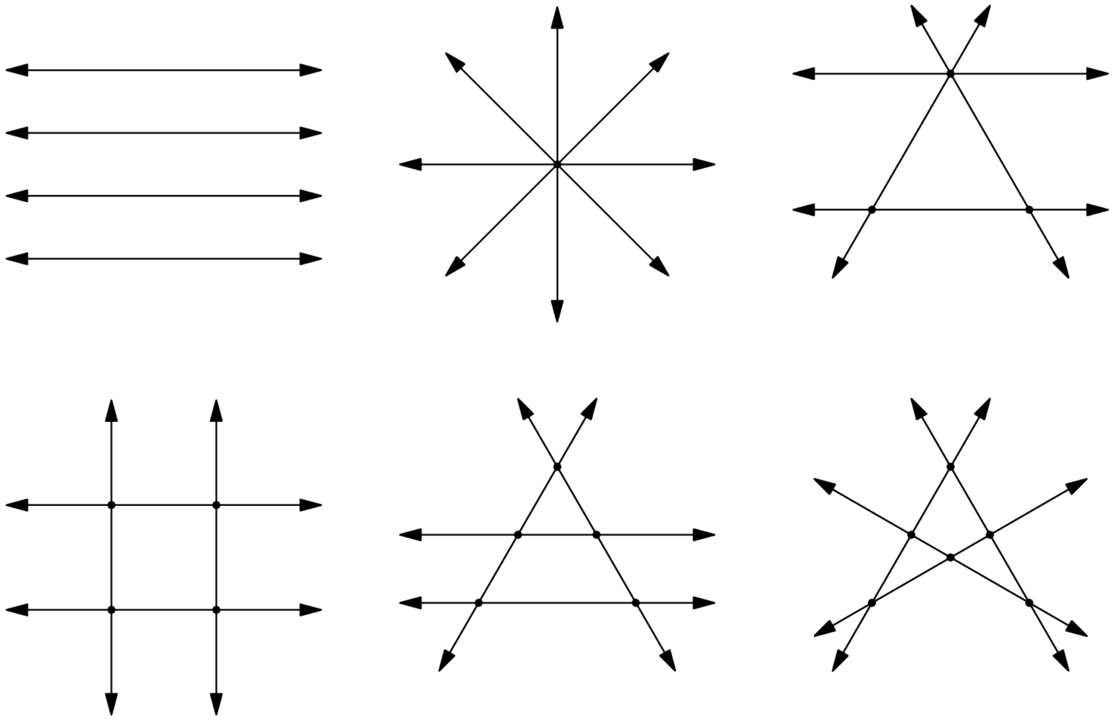
The following problem is from both the 2019 AMC 10A #14 and 2019 AMC 12A #8, so both problems redirect to this page.

For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

$\textbf{(A) } 14 \qquad \textbf{(B) } 16 \qquad \textbf{(C) } 18 \qquad \textbf{(D) } 19 \qquad \textbf{(E) } 21$

Solution

It is possible to obtain 0, 1, 3, 4, 5, and 6 points of intersection, as demonstrated in the following figures:



It is clear that the maximum number of possible intersections is $\binom{4}{2} = 6$, since each pair of lines can intersect at most once. We now prove that it is impossible to obtain two intersections.

We proceed by contradiction. Assume a configuration of four lines exists such that there exist only two intersection points. Let these intersection points be A and B. Consider two cases:

Case 1: No line passes through both A and B

Then, since an intersection is obtained by an intersection between at least two lines, two lines pass through each of A and B. Then, since there can be no additional intersections, no line that passes through A can intersect a line that passes through B, and so each line that passes through A must be parallel to every line that passes through B. Then the two lines passing through B are parallel to each other by transitivity of parallelism, so they coincide, contradiction.

Case 2: There is a line passing through A and B

Then there must be a line l_a passing through A, and a line l_b passing through B. These lines must be parallel. The fourth line l must pass through either A or B. Without loss of generality, suppose l passes through A.

Then since l and l_a cannot coincide, they cannot be parallel.

Then l and l_b cannot be parallel either, so they intersect, contradiction.

All possibilities have been exhausted, and thus we can conclude that two intersections is impossible. Our answer is given by the sum $0+1+3+4+5+6=\boxed{\textbf{(D) } 19}$.

2019 AMC 10A Problems/Problem 15

(Redirected from 2019 AMC 12A Problems/Problem 9)

The following problem is from both the 2019 AMC 10A #15 and 2019 AMC 12A #9, so both problems redirect to this page.

Problem

A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and $a_n=\frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$ for all $n \geq 3$ Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p+q?$

$\textbf{(A) } 2020 \qquad\textbf{(B) } 4039 \qquad\textbf{(C) } 6057 \qquad\textbf{(D) } 6061 \qquad\textbf{(E) } 8078$

Solution 1

Using the recursive formula, we find $a_3=\frac{3}{11}$, $a_4=\frac{3}{15}$, and so on. It appears that $a_n=\frac{3}{4n-1}$, for all n. Setting $n=2019$, we find $a_{2019}=\frac{3}{8075}$, so the answer is $\boxed{\textbf{(E) } 8078}$.

To prove this formula, we use induction. We are given that $a_1 = 1$ and $a_2 = \frac{3}{7}$, which satisfy our formula. Now assume the formula holds true for all $n \leq m$ for some positive integer m. By our assumption, $a_{m-1} = \frac{3}{4m-5}$ and $a_m = \frac{3}{4m-1}$. Using the recursive formula,

$$a_{m+1} = \frac{a_{m-1} \cdot a_m}{2a_{m-1} - a_m} = \frac{\frac{3}{4m-5} \cdot \frac{3}{4m-1}}{2 \cdot \frac{3}{4m-5} - \frac{3}{4m-1}} = \frac{\left(\frac{3}{4m-5} \cdot \frac{3}{4m-1}\right)(4m-5)(4m-1)}{\left(2 \cdot \frac{3}{4m-5} - \frac{3}{4m-1}\right)(4m-5)(4m-1)} = \frac{9}{6(4m-1) - 3(4m-5)} = \frac{3}{4(m+1)-1},$$

so our induction is complete.

Solution 2

Since we are interested in finding the sum of the numerator and the denominator, consider the sequence defined by $b_n = \frac{1}{a_n}$.

We have $\frac{1}{a_n} = \frac{2a_{n-2} - a_{n-1}}{a_{n-2} \cdot a_{n-1}} = \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}}$, so in other words,

$b_n = 2b_{n-1} - b_{n-2} = 3b_{n-2} - 2b_{n-3} = 4b_{n-3} - 3b_{n-4} = \dots$.

By recursively following this pattern, we can see that $b_n = (n-1) \cdot b_2 - (n-2) \cdot b_1$.

By plugging in 2019, we thus find $b_{2019} = 2018 \cdot \frac{7}{3} - 2017 = \frac{8075}{3}$. Since the numerator and the denominator are relatively prime, the answer is $\boxed{\textbf{(E) } 8078}$.

-eric2020

Solution 3

It seems reasonable to transform the equation into something else. Let $a_n = x$, $a_{n-1} = y$, and $a_{n-2} = z$. Therefore, we have $x = \frac{zy}{2z - y} 2xz - xy = zy 2xz = y(x+z)$

$y = \frac{2xz}{x+z}$ Thus, y is the harmonic mean of x and z. This implies a_n is a harmonic sequence or equivalently $b_n = \frac{1}{a_n}$ is arithmetic. Now, we have $b_1 = 1$, $b_2 = \frac{7}{3}$, $b_3 = \frac{11}{3}$, and so on. Since the common difference is $\frac{4}{3}$, we can express b_n explicitly as $b_n = \frac{4}{3}(n-1) + 1$.

This gives $b_{2019} = \frac{4}{3}(2019-1) + 1 = \frac{8075}{3}$ which implies $a_{2019} = \frac{3}{8075} = \frac{p}{q}$. $p + q = \boxed{\textbf{(E) } 8078}$ ~jakeg314

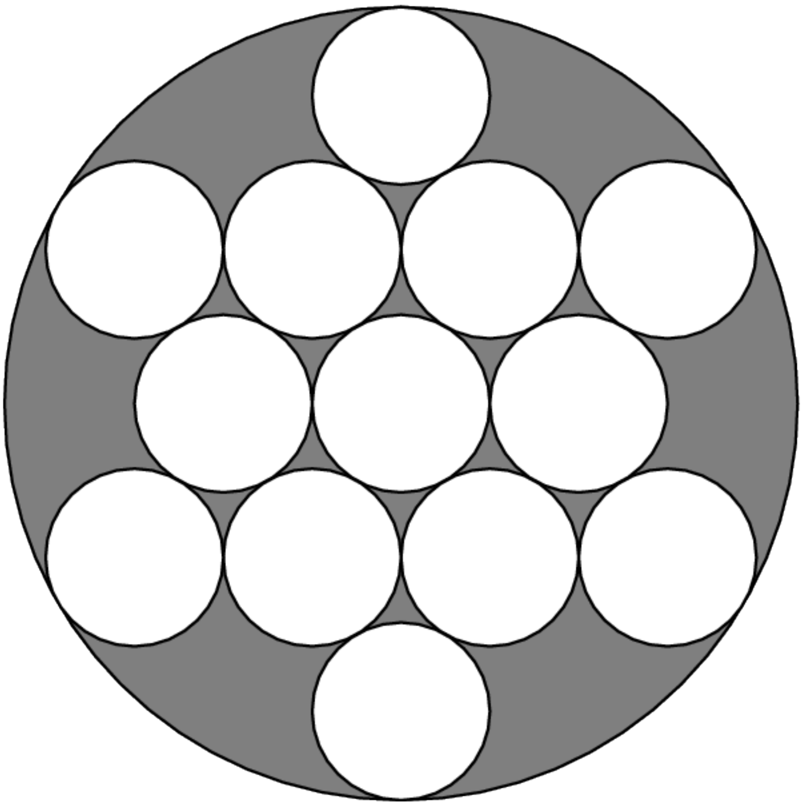
2019 AMC 10A Problems/Problem 16

(Redirected from 2019 AMC 12A Problems/Problem 10)

The following problem is from both the 2019 AMC 10A #16 and 2019 AMC 12A #10, so both problems redirect to this page.

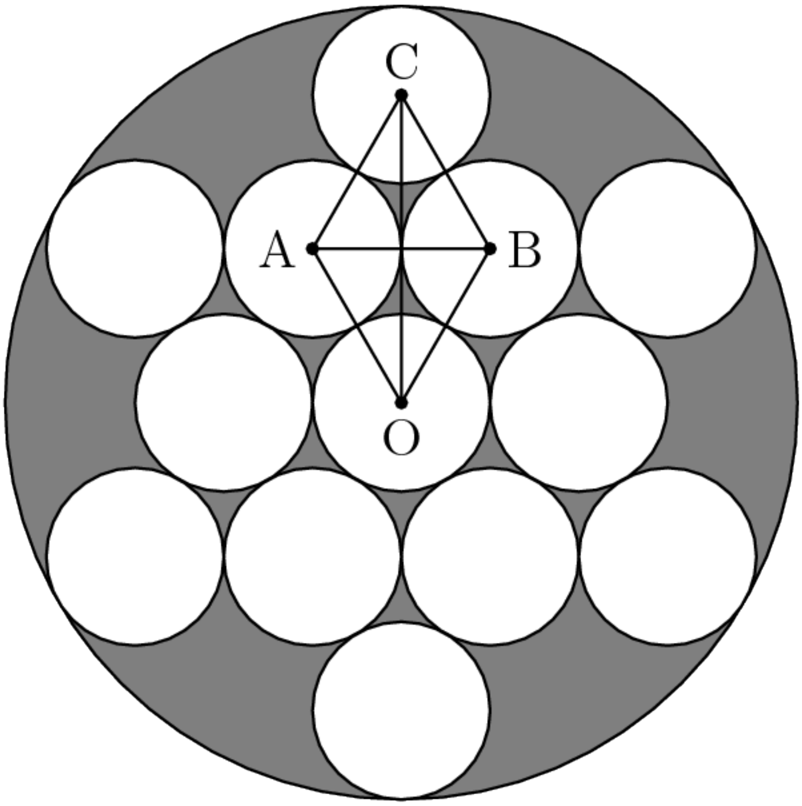
Problem

The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius $1?$



(A) $4\pi\sqrt{3}$ **(B)** 7π **(C)** $\pi\left(3\sqrt{3}+2\right)$ **(D)** $10\pi\left(\sqrt{3}-1\right)$ **(E)** $\pi\left(\sqrt{3}+6\right)$

Solution 1



In the diagram above, notice that triangle OAB and triangle ABC are congruent and equilateral with side length 2. We can see the radius of the larger circle is two times the altitude of OAB plus 1 (the distance from point C to the edge of the circle). Using $30^\circ - 60^\circ - 90^\circ$ triangles, we know the altitude is $\sqrt{3}$. Therefore, the radius of the larger circle is $2\sqrt{3}+1$.

The area of the larger circle is thus $\left(2\sqrt{3}+1\right)^2\pi = \left(13+4\sqrt{3}\right)\pi$, and the sum of the areas of the smaller circles is 13π, so the area of the dark region is $\left(13+4\sqrt{3}\right)\pi - 13\pi = \boxed{\textbf{(A) } 4\pi\sqrt{3}}$.

Solution 2

We can form an equilateral triangle with side length 6 from the centers of three of the unit circles tangent to the outer circle. The radius of the outer circle is the circumradius of the triangle plus 1. By using $R = \frac{abc}{4A}$ or $R = \frac{a}{2\sin A}$, we get the radius as $\frac{6}{\sqrt{3}} + 1$.

The shaded area is thus $\pi((\frac{6}{\sqrt{3}} + 1)^2 - 13) = \boxed{\textbf{(A) }4\pi\sqrt{3}}$.

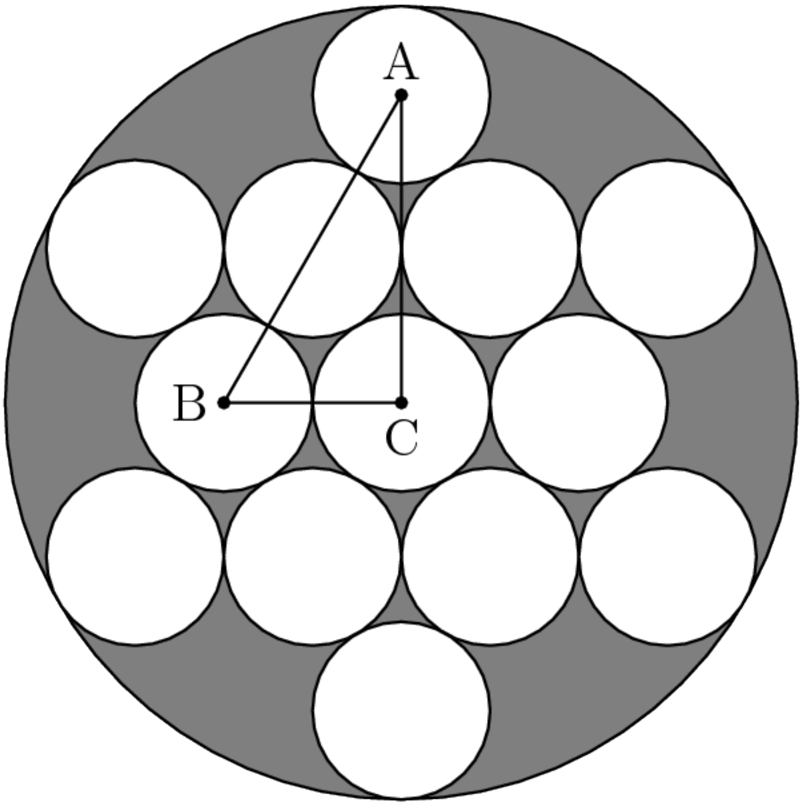
Solution 3

Like in Solution 2, we can form an equilateral triangle with side length 6 from the centers of three of the unit circles tangent to the outer circle. We can find the height of this triangle to be $3\sqrt{3}$. Then, we can form another equilateral triangle from the centers of the second and third circles in the third row and the center of the bottom circle with side length 2. The height of this triangle is clearly $\sqrt{3}$. Therefore the diameter of the large circle is $4\sqrt{3} + 2$ and the radius is $\frac{4\sqrt{3} + 2}{2} = 2\sqrt{3} + 1$. The area of the large circle is thus

$$\pi\left(2\sqrt{3} + 1\right)^2 = \pi \cdot \left(13 + 4\sqrt{3}\right) = 13\pi + 4\pi\sqrt{3}.$$

The total area of the 13 smaller circles is 13π, so the shaded area is $\left(13\pi + 4\pi\sqrt{3}\right) - 13\pi = \boxed{\textbf{(A) }4\pi\sqrt{3}}$.

Solution 4



In the diagram above, $AB = 4$ and $BC = 2$,

so $AC = \sqrt{4^2 - 2^2} = 2\sqrt{3}$. The larger circle's radius

is $AC + 1 = 2\sqrt{3} + 1$, so the larger circle's area

is $\pi\left(2\sqrt{3}+1\right)^2 = \pi\left(13+4\sqrt{3}\right) = 13\pi + 4\pi\sqrt{3}$.

Now, subtracting the combined area of the smaller circles

gives $13\pi + 4\pi\sqrt{3} - 13\pi = \boxed{\textbf{(A) } 4\pi\sqrt{3}}$.

2019 AMC 10A Problems/Problem 18

(Redirected from 2019 AMC 12A Problems/Problem 11)

The following problem is from both the 2019 AMC 10A #18 and 2019 AMC 12A #11, so both problems redirect to this page.

Problem

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323..._k$. What is k?

$\textbf{(A)}\ 13 \qquad \textbf{(B)}\ 14 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 16 \qquad \textbf{(E)}\ 17$

Solution 1

We can expand the fraction $0.\overline{23}_k$ as follows:

$0.\overline{23}_k = 2\cdot k^{-1} + 3 \cdot k^{-2} + 2 \cdot k^{-3} + 3 \cdot k^{-4} + ...$

Notice that this is equivalent to

$2(k^{-1} + k^{-3} + k^{-5} + ...) + 3 (k^{-2} + k^{-4} + k^{-6} + ...)$

By summing the geometric series and simplifying, we have $\frac{2k+3}{k^2-1} = \frac{7}{51}$. Solving this quadratic equation (or simply testing the answer choices) yields the answer $k = \boxed{\textbf{(D)}\ 16}$.

Solution 2

Let $a = 0.2323\dots_k$. Therefore, $k^2a=23.2323\dots_k$.

From this, we see that $k^2a-a=23_k$,

so $a = \frac{23_k}{k^2-1} = \frac{2k+3}{k^2-1} = \frac{7}{51}$.

Now, similar to in Solution 1, we can either test if $2k+3$ is a multiple of 7 with the answer choices, or actually solve the quadratic, so that the answer is $\boxed{\textbf{(D)}\ 16}$.

Solution 3 (bash)

We can simply plug in all the answer choices as values of k, and see which one works. After lengthy calculations, this eventually gives us $\boxed{\textbf{(D) } 16}$ as the answer.

Solution 4

Just as in Solution 1, we arrive at the equation $\frac{2k+3}{k^2-1} = \frac{7}{51}$.

We can now rewrite this as $\frac{2k+3}{(k-1)(k+1)} = \frac{7}{51} = \frac{7}{3 \cdot 17}$.

Notice that $2k+3 = 2(k+1)+1 = 2(k-1)+5$.

As 17 is a prime, we therefore must have that one of $k-1$and $k+1$ is divisible by 17. Now, checking each of the answer choices, this gives $\boxed{\textbf{(D) } 16}$.

2019 AMC 12A Problems/Problem 12

Problem

Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $(\log_2 \tfrac{x}{y})^2$?

$\textbf{(A) } \frac{25}{2} \qquad \textbf{(B) } 20 \qquad \textbf{(C) } \frac{45}{2} \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 32$

Solution 1

Let $\log_2 x = \log_y 16 = k$, so that $2^k = x$ and $y^k = 16 \implies y = 2^{\frac{4}{k}}$. Then we have $(2^k)(2^{\frac{4}{k}}) = 2^{k+\frac{4}{k}} = 2^6$.

We therefore have $k+\frac{4}{k}=6$, and deduce $k^2-6k+4=0$. The solutions to this are $k=3\pm\sqrt{5}$.

To solve the problem, we now find $(\log_2\frac{x}{y})^2=(\log_2 x-\log_2 y)^2=(k-\frac{4}{k})^2=(3\pm\sqrt{5}-\frac{4}{3\pm\sqrt{5}})^2=(3\pm\sqrt{5}-3\mp\sqrt{5})^2=(\pm 2\sqrt{5})^2=\boxed{20}$
.

Solution 2 (slightly simpler)

After obtaining $k+\frac{4}{k}=6$, notice that the required answer is $(k-\frac{4}{k})^2=k^2-8+\frac{16}{k^2}=\left(k^2+8+\frac{16}{k^2}\right)-16=\left(k+\frac{4}{k}\right)^2-16=6^2-16=20$
, as before.

Solution 3

From the given data, $\log_2(x)=\frac{1}{\log_{16}(y)}$, or $\log_2(x)=\frac{4}{\log_2(y)}$

We know that $xy=64$, so $x=\frac{64}{y}$.

Thus $\log_2(\frac{64}{y})=\frac{4}{\log_2(y)}$, so $6-\log_2(y)=\frac{4}{\log_2(y)}$,

so $6(\log_2(y))-(\log_2(y))^2=4$.

Solving for $\log_2(y)$, we obtain $\log_2(y)=3+\sqrt{5}$.

Easy resubstitution further gives $\log_2(x)=\frac{4}{3+\sqrt{5}}$. Simplifying, we obtain $\log_2(x)=3-\sqrt{5}$.

Looking back at the original problem, we have What is $(\log_2\frac{x}{y})^2$?

Deconstructing this expression using log rules, we get $(\log_2 x - \log_2 y)^2$.

Plugging in our known values, we get $((3-\sqrt{5})-(3+\sqrt{5}))^2$ or $(-2\sqrt{5})^2$.

Our answer is $\boxed{\textbf{(B) } 20}$.

Solution 4

Multiplying the first equation by $\log_2 y$, we obtain $\log_2 x \cdot \log_2 y = 4$.

From the second equation we have $\log_2 x + \log_2 y = \log_2(xy) = 6$.

Then,

$(\log_2 \frac{x}{y})^2 = (\log_2 x - \log_2 y)^2 = (\log_2 x + \log_2 y)^2 - 4\log_2 x \cdot \log_2 y = (6)^2 - 4(4) = 20 \Rightarrow \boxed{B}$

.

Solution 5

Let $A = \log_2 x$ and $B = \log_2 y$.

Writing the first given as $\log_2 x = \dfrac{\log_2 16}{\log_2 y}$ and the second as $\log_2 x + \log_2 y = \log_2 64$, we get $A \cdot B = 4$ and $A + B = 6$.

Solving for B we get $B = 3 \pm \sqrt{5}$.

Our goal is to find $(A-B)^2$. From the above, it is equal to $(6-2B) = \left(2\sqrt{5}\right)^2 = 20 \Rightarrow \boxed{B}$.

.

2019 AMC 12A Problems/Problem 13

Problem

How many ways are there to paint each of the integers $2, 3, \dots, 9$ either red, green, or blue so that each number has a different color from each of its proper divisors?

$\textbf{(A)}\ 144 \qquad\textbf{(B)}\ 216 \qquad\textbf{(C)}\ 256 \qquad\textbf{(D)}\ 384 \qquad\textbf{(E)}\ 432$

Solution 1

The 5 and 7 can be painted with no restrictions because the set of integers does not contain a multiple or proper factor of 5 or 7. There are 3 ways to paint each, giving us $\underline{9}$ ways to paint both. The 2 is the most restrictive number. There are $\underline{3}$ ways to paint 2, but without loss of generality, let it be painted red. 4 cannot be the same color as 2 or 8, so there are $\underline{2}$ ways to paint 4, which automatically determines the color for 8. 6 cannot be painted red, so there are $\underline{2}$ ways to paint 6, but WLOG, let it be painted blue. There are $\underline{2}$choices for the color for 3, which is either red or green in this case. Lastly, there are $\underline{2}$ ways to choose the color for 9.

$9\cdot3\cdot2\cdot2\cdot2\cdot2=\boxed{\textbf{(E)}\ 432}$.

Solution 2

We note that the primes can be colored any of the 3 colors since they don't have any proper divisors other than 1, which is not in the list. Furthermore, 6 is the only number in the list that has 2 distinct prime factors (namely, 2 and 3), so we do casework on 6.

Case 1: 2 and 3 are the same color

In this case, we have 3 primes to choose the color for (2, 5, and 7). Afterwards, 4, 6, and 9 have two possible colors, which will determine the color of 8. Thus, there are $3^3 \cdot 2^3 = 216$ possibilities here.

Case 2: 2 and 3 are different colors

In this case, we have 4 primes to color. Without loss of generality, we'll color the 2 first, then the 3. Then there are 3 color choices for $2, 5, 7$, and 2 color choices for 3. This will determine the color of 6 as well. After that, we only need to choose the color for 4 and 9, which each have 2 choices. Thus, there are $3^3 \cdot 2^3 = 216$ possibilities here as well.

Adding up gives $216 + 216 = \boxed{\textbf{(E) } 432}$.

Solution 3

$2, 4, 8$ require different colors each, so there are 6 ways to color these.

5 and 7 can be any color, so there are 3×3 ways to color these.

6 can have 2 colors once 2 is colored, and thus 3 also has 2 colors following 6, which leaves another 2 for 9.

All together: $6 \times 3 \times 3 \times 2 \times 2 \times 2 = 432 \Rightarrow \boxed{E}$.

2019 AMC 12A Problems/Problem 14

Problem

For a certain complex number c, the polynomial

$P(x) = (x^2 - 2x + 2)(x^2 - cx + 4)(x^2 - 4x + 8)$ has exactly 4 distinct roots. What is $|c|$?

$\textbf{(A) } 2 \qquad \textbf{(B) } \sqrt{6} \qquad \textbf{(C) } 2\sqrt{2} \qquad \textbf{(D) } 3 \qquad \textbf{(E) } \sqrt{10}$

Solution

The polynomial can be factored further broken down into

$P(x) = (x - [1 - i])(x - [1 + i])(x - [2 - 2i])(x - [2 + 2i])(x^2 - cx + 4)$

by using the quadratic formula on the quadratic factors. Since the first four roots are all distinct, the term $(x^2 - cx + 4)$ must be a product of any combination of two (not necessarily distinct) factors from the set: $(x - [1 - i]), (x - [1 + i]), (x - [2 - 2i])$, and $(x - [2 + 2i])$. We need the two factors to yield a constant term of 4 when multiplied together. The only combinations that work are $(x - [1 - i])$ and $(x - [2 + 2i])$, or $(x - [1 + i])$ and $(x - [2 - 2i])$. When multiplied together, the polynomial is either $(x^2 + [-3 + i]x + 4)$ or $(x^2 + [-3 - i]x + 4)$.

Therefore, $c = -3 \pm i$ and $|c| = \boxed{\textbf{(E) } \sqrt{10}}$.

2019 AMC 12A Problems/Problem 15

Problem

Positive real numbers a and b have the property that $\sqrt{\log a} + \sqrt{\log b} + \log \sqrt{a} + \log \sqrt{b} = 100$ and all four terms on the left are positive integers, where log denotes the base 10 logarithm. What is ab?

$\textbf{(A) } 10^{52} \qquad \textbf{(B) } 10^{100} \qquad \textbf{(C) } 10^{144} \qquad \textbf{(D) } 10^{164} \qquad \textbf{(E) } 10^{200}$

Solution 1

Since all four terms on the left are positive integers, from $\sqrt{\log a}$, we know that both $\log a$ has to be a perfect square and a has to be a power of ten. The same applies to b for the same reason. Setting a and $b$to 10^x and 10^y, where x and y are the perfect squares, $ab = 10^{x+y}$. By listing all the perfect squares up to 14^2 (as 15^2 is larger than the largest possible sum of x and y of 200 from answer choice E), two of those perfect squares must add up to one of the possible sums of x and y given from the answer choices (52, 100, 144, 164, or 200).

Only a few possible sums are seen: $16+36=52$, $36+64=100$, $64+100=164$, $100+100=200$, and $4+196=200$. By testing each of these (seeing whether $\sqrt{x}+\sqrt{b}+\frac{x}{2}+\frac{y}{2}=100$), only the pair $x=64$ and $y=100$ work. Therefore, a and b are 10^{64} and 10^{100}, and our answer is $\boxed{\textbf{(D) } 10^{164}}$.

Solution 2

Given that $\sqrt{\log a}$ and $\sqrt{\log b}$ are both integers, a and b must be in the form 10^{m^2} and 10^{n^2}, respectively for some positive integers m and n. Note that $\log \sqrt{a} = \frac{m^2}{2}$. By substituting for a and b, the equation becomes $m+n+\frac{m^2}{2}+\frac{n^2}{2}=100$. After multiplying the equation by 2 and completing the square with respect to m and n, the equation becomes $(m+1)^2+(n+1)^2=202$. Testing squares of positive integers that add to 202, 11^2+9^2 is the only option. Without loss of generality, let $m=10$ and $n=8$. Plugging in m and n to solve for a and b gives us $a=10^{100}$ and $b=10^{64}$.

Therefore, $ab = \boxed{\textbf{(D) } 10^{164}}$.

2019 AMC 10A Problems/Problem 20

(Redirected from 2019 AMC 12A Problems/Problem 16)

The following problem is from both the 2019 AMC 10A #20 and 2019 AMC 12A #16, so both problems redirect to this page.

Problem

The numbers $1, 2, \dots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

$\textbf{(A) }\frac{1}{21}\qquad\textbf{(B) }\frac{1}{14}\qquad\textbf{(C) }\frac{5}{63}\qquad\textbf{(D) }\frac{2}{21}\qquad\textbf{(E) }\frac{1}{7}$

Solution 1

Note that odd sums can only be formed by (e,e,o) or (o,o,o), so we focus on placing the evens: we need to have each even be with another even in each row/column. It can be seen that there are 9 ways to do this. There are then $5!$ ways to permute the odd numbers, and $4!$ ways to permute the even numbers, thus giving the answer as $\frac{5!\cdot 4!\cdot 9}{9!}=\boxed{\textbf{(B) }\frac{1}{14}}$.

Solution 2

By the Pigeonhole Principle, there must be at least one row with 2 or more odd numbers in it. Therefore, that row must contain 3 odd numbers in order to have an odd sum. The same thing can be done with the columns. Thus we simply have to choose one row and one column to be filled with odd numbers, so the number of valid odd/even configurations (without regard to which particular odd and even numbers are placed where) is $3\cdot 3=9$. The denominator will be $\binom{9}{4}$, the total number of ways we could choose which 4 of the 9 squares will contain an even number. Hence the answer is $\frac{9}{\binom{9}{4}}=\boxed{\textbf{(B) }\frac{1}{14}}$

2019 AMC 12A Problems/Problem 17

Problem

Let s_k denote the sum of the kth powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $s_0 = 3$, $s_1 = 5$, and $s_2 = 9$. Let a, b, and c be real numbers such that $s_{k+1} = a\, s_k + b\, s_{k-1} + c\, s_{k-2}$ for $k = 2, 3, \ldots.$ What is $a + b + c$?

$\textbf{(A)}\ -6 \qquad \textbf{(B)}\ 0 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 10 \qquad \textbf{(E)}\ 26$

Solution 1

Applying Newton's Sums (see this link), we have $s_{k+1} + (-5)s_k + (8)s_{k-1} + (-13)s_{k-2} = 0,$ so $s_{k+1} = 5s_k - 8s_{k-1} + 13s_{k-2},$ we get the answer as $5 + (-8) + 13 = 10$.

Solution 2

Let p, q, and r be the roots of the polynomial. Then,

$$p^3 - 5p^2 + 8p - 13 = 0$$

$$q^3 - 5q^2 + 8q - 13 = 0$$

$$r^3 - 5r^2 + 8r - 13 = 0$$

Adding these three equations, we get

$$(p^3 + q^3 + r^3) - 5(p^2 + q^2 + r^2) + 8(p + q + r) - 39 = 0$$

$$s_3 - 5s_2 + 8s_1 = 39$$

39 can be written as $13s_0$, giving

$$s_3 = 5s_2 - 8s_1 + 13s_0$$

We are given that $s_{k+1} = a\,s_k + b\,s_{k-1} + c\,s_{k-2}$ is satisfied for $k = 2, 3, \ldots$, meaning it must be satisfied when $k = 2$, giving us $s_3 = a\,s_2 + b\,s_1 + c\,s_0$.

Therefore, $a = 5, b = -8$, and $c = 13$ by matching coefficients.

$$5 - 8 + 13 = \boxed{\textbf{(D) } 10}.$$

Solution 3

Let p, q, and r be the roots of the polynomial. By Vieta's Formulae, we have

$$p + q + r = 5$$

$$pq + qr + rp = 8$$

$$pqr = 13.$$

We know $s_k = p^k + q^k + r^k$. Consider $(p + q + r)(s_k) = 5s_k$.

$$5s_k = [p^{k+1} + q^{k+1} + r^{k+1}] + p^k q + p^k r + pq^k + q^k r + pr^k + qr^k$$

Using $pqr = 13$ and $s_{k-2} = p^{k-2} + q^{k-2} + r^{k-2}$, we see $13s_{k-2} = p^{k-1}qr + pq^{k-1}r + pqr^{k-1}$.

We have

$$\begin{aligned} 5s_k + 13s_{k-2} &= s_{k+1} + (p^k q + p^k r + p^{k-1}qr) + (pq^k + pq^{k-1}r + q^k r) + (pqr^{k-1} + pr^k + qr^k) \\ &= s_{k+1} + p^{k-1}(pq + pr + qr) + q^{k-1}(pq + pr + qr) + r^{k-1}(pq + pr + qr) \\ &= s_{k+1} + (p^{k-1} + q^{k-1} + r^{k-1})(pq + pr + qr) \\ &= 5s_k + 13s_{k-2} = s_{k+1} + 8s_{k-1} \end{aligned}$$

Rearrange to get $s_{k+1} = 5s_k - 8s_{k-1} + 13s_{k-2}$

So, $a+b+c=5-8+13=\boxed{\textbf{(D) }10}$.

2019 AMC 10A Problems/Problem 21

(Redirected from 2019 AMC 12A Problems/Problem 18)

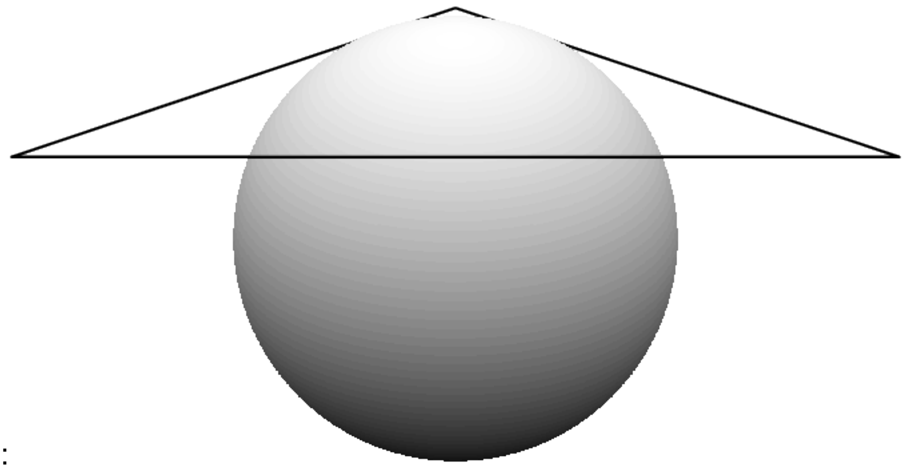
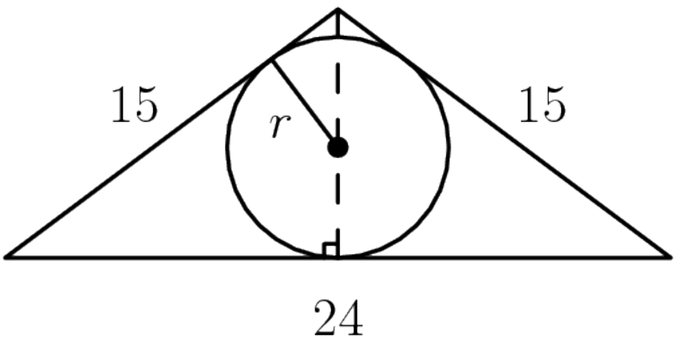
The following problem is from both the 2019 AMC 10A #21 and 2019 AMC 12A #18, so both problems redirect to this page.

Problem

A sphere with center O has radius 6. A triangle with sides of length $15, 15,$ and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle?

$\textbf{(A) }2\sqrt{3}\qquad \textbf{(B) }4\qquad \textbf{(C) }3\sqrt{2}\qquad \textbf{(D) }2\sqrt{5}\qquad \textbf{(E) }5$

Diagram

3D:  Plane through triangle: 

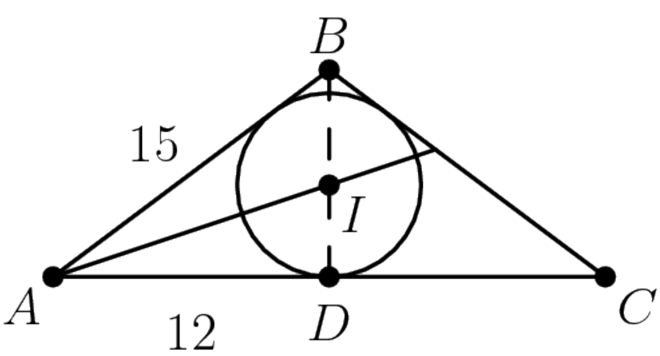
Solution 1

The triangle is placed on the sphere so that its three sides are tangent to the sphere. The cross-section of the sphere created by the plane of the triangle is also the incircle of the triangle. To find the inradius, use $\text{area} = \text{inradius} \cdot \text{semiperimeter}$. The area of the triangle can be found by drawing an altitude from the vertex between sides with length 15 to the midpoint of the side with length 24. The Pythagorean triple 9 - 12 - 15 allows us easily to determine that the base is 24 and the height is 9. The formula $\frac{\text{base} \cdot \text{height}}{2}$ can also be used to find the area of the triangle as 108, while the semiperimeter is simply $\frac{15+15+24}{2} = 27$. After plugging into the equation, we thus get $108 = \text{inradius} \cdot 27$, so the inradius is 4. Now, let the distance between O and the triangle be x. Choose a point on the incircle and denote it by A. The distance OA is 6, because it is just the radius of the sphere. The distance from point A to the center of the incircle is 4, because it is the radius of the incircle. By using the Pythagorean Theorem, we thus find $x = \sqrt{6^2 - 4^2} = \sqrt{20} = \boxed{\textbf{(D) } 2\sqrt{5}}$.

Solution 2

As in Solution 1, we note that by the Pythagorean Theorem, the height of the triangle is 9, and that the three sides of the triangle are tangent to the sphere, so the circle in the cross-section of the sphere is the incenter of the triangle.

Recall that the inradius is the intersection of the angle bisectors. To find the inradius of the incircle, we use the Angle Bisector



Theorem.

$$\begin{aligned} &\frac{AB}{BI} = \frac{AD}{DI} \\ \Rightarrow &\frac{15}{BI} = \frac{12}{DI} \\ \Rightarrow &\frac{BI}{5} = \frac{DI}{4} \end{aligned}$$

Since we know that $BI + DI$ (the height) is equal to 9, DI (the inradius) is 4. From here, the problem can be solved in the same way as in Solution 1. The answer is $\boxed{\textbf{(D) } 2\sqrt{5}}$.

2019 AMC 12A Problems/Problem 19

Problem

In $\triangle ABC$ with integer side lengths,

$$\cos A = \frac{11}{16}, \qquad \cos B = \frac{7}{8}, \qquad \text{and} \qquad \cos C = -\frac{1}{4}.$$

What is the least possible perimeter for $\triangle ABC$?

$\textbf{(A) } 9 \qquad \textbf{(B) } 12 \qquad \textbf{(C) } 23 \qquad \textbf{(D) } 27 \qquad \textbf{(E) } 44$

Solution 1

Notice that by the Law of Sines, $a : b : c = \sin A : \sin B : \sin C$, so let's flip all the cosines using $\sin^2 x + \cos^2 x = 1$ (sine is positive for $0^\circ \leq x \leq 180^\circ$, so we're good there).

$$\sin A = \frac{3\sqrt{15}}{16}, \qquad \sin B = \frac{\sqrt{15}}{8}, \qquad \text{and} \qquad \sin C = \frac{\sqrt{15}}{4}$$

These are in the ratio $3 : 2 : 4$, so our minimal triangle has side lengths 2, 3, and 4. $\boxed{\textbf{(A) } 9}$ is our answer.

Solution 2

$\angle ACB$ is obtuse since its cosine is negative, so we let the foot of the altitude from C to AB be H. Let $AH = 11x$, $AC = 16x$, $BH = 7y$, and $BC = 8y$. By the Pythagorean Theorem, $CH = \sqrt{256x^2 - 121x^2} = 3x\sqrt{15}$ and $CH = \sqrt{64y^2 - 49y^2} = y\sqrt{15}$. Thus, $y = 3x$. The sides of the triangle are then $16x$, $11x + 7(3x) = 32x$, and $24x$, so for some integers a, b, $16x = a$ and $24x = b$, where a and b are minimal. Hence, $\frac{a}{16} = \frac{b}{24}$, or $3a = 2b$. Thus the smallest possible positive integers a and b that satisfy this are $a = 2$ and $b = 3$, so $x = \frac{1}{8}$. The sides of the triangle are 2, 3, and 4, so $\boxed{\textbf{(A) } 9}$ is our answer.

2019 AMC 10A Problems/Problem 22

(Redirected from 2019 AMC 12A Problems/Problem 20)

The following problem is from both the 2019 AMC 10A #22 and 2019 AMC 12A #20, so both problems redirect to this page.

Problem

Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0, 1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x - y| > \frac{1}{2}$?

$\textbf{(A)}\ \frac{1}{3} \qquad \textbf{(B)}\ \frac{7}{16} \qquad \textbf{(C)}\ \frac{1}{2} \qquad \textbf{(D)}\ \frac{9}{16} \qquad \textbf{(E)}\ \frac{2}{3}$

Solution

There are several cases depending on what the first coin flip is when determining x and what the first coin flip is when determining y.

The four cases are:

Case 1: x is either 0 or 1, and y is either 0 or 1.

Case 2: x is either 0 or 1, and y is chosen from the interval $[0, 1]$.

Case 3: x is is chosen from the interval $[0, 1]$, and y is either 0 or 1.

Case 4: x is is chosen from the interval $[0, 1]$, and y is also chosen from the interval $[0, 1]$.

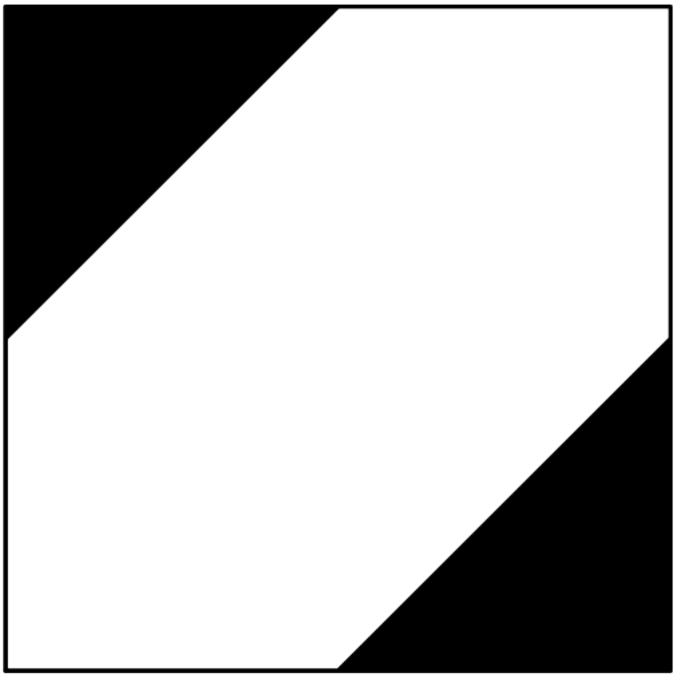
Each case has a $\frac{1}{4}$ chance of occurring (as it requires two coin flips).

For Case 1, we need x and y to be different. Therefore, the probability for success in Case 1 is $\frac{1}{2}$.

For Case 2, if x is 0, we need y to be in the interval $\left(\frac{1}{2}, 1\right]$. If x is 1, we need y to be in the interval $\left[0, \frac{1}{2}\right)$. Regardless of what x is, the probability for success for Case 2 is $\frac{1}{2}$.

By symmetry, Case 3 has the same success rate as Case 2.

For Case 4, we must use geometric probability because there are an infinite number of pairs (x, y) that can be selected, whether they satisfy the inequality or not. Graphing $|x - y| > \tfrac{1}{2}$ gives us the following picture where the shaded area is the set of all the points that fulfill the inequality:



The shaded area is $\frac{1}{4}$, which means the probability for success for case 4 is $\frac{1}{4}$ (since the total area of the bounding square, containing all possible pairs, is 1).

Adding up the success rates from each case, we get:

$$\left(\frac{1}{4}\right) \cdot \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}\right) = \boxed{\textbf{(B) } \frac{7}{16}}.$$

2019 AMC 12A Problems/Problem 21

Problem

Let $z = \frac{1+i}{\sqrt{2}}.$ What is

$$\left(z^{1^2}+z^{2^2}+z^{3^2}+\dots+z^{{12}^2}\right)\cdot\left(\frac{1}{z^{1^2}}+\frac{1}{z^{2^2}}+\frac{1}{z^{3^2}}+\dots+\frac{1}{z^{{12}^2}}\right)?$$

$\textbf{(A) } 18 \qquad \textbf{(B) } 72-36\sqrt2 \qquad \textbf{(C) } 36 \qquad \textbf{(D) } 72 \qquad \textbf{(E) } 72+36\sqrt2$

Solution 1

Note that $z = \operatorname{cis}(45^{\circ})$.

Also note that $z^{k} = z^{k + 8}$ for all positive integers k because of De Moivre's Theorem. Therefore, we want to look at the exponents of each term modulo 8.

$1^2, 5^2$, and 9^2 are all $1 \pmod{8}$

$2^2, 6^2$, and 10^2 are all $4 \pmod{8}$

$3^2, 7^2$, and 11^2 are all $1 \pmod{8}$

$4^2, 8^2$, and 12^2 are all $0 \pmod{8}$

Therefore,

$$z^{1^2} = z^{5^2} = z^{9^2} = \operatorname{cis}(45^{\circ})$$

$$z^{2^2} = z^{6^2} = z^{10^2} = \operatorname{cis}(180^{\circ}) = -1$$

$$z^{3^2} = z^{7^2} = z^{11^2} = \operatorname{cis}(45^{\circ})$$

$$z^{4^2} = z^{8^2} = z^{12^2} = \operatorname{cis}(0^{\circ}) = 1$$

The term thus $\left(z^{1^2}+z^{2^2}+z^{3^2}+\dots+z^{{12}^2}\right)$ simplifies to $6\operatorname{cis}(45^{\circ})$, while the term $\left(\frac{1}{z^{1^2}}+\frac{1}{z^{2^2}}+\frac{1}{z^{3^2}}+\dots+\frac{1}{z^{{12}^2}}\right)$ simplifies to $\frac{6}{\operatorname{cis}(45^{\circ})}$. Upon multiplication, the $\operatorname{cis}(45^{\circ})$ cancels out and leaves us with $\boxed{\textbf{(C) } 36}$.

Solution 2

It is well known that if $|z| = 1$ then $\bar{z} = \frac{1}{z}$. Therefore, we have that the desired expression is equal to $\left(z^1 + z^4 + z^9 + ... + z^{144}\right)\left(\bar{z}^1 + \bar{z}^4 + \bar{z}^9 + ... + \bar{z}^{144}\right)$

We know that $z = e^{\frac{i\pi}{4}}$ so $\bar{z} = e^{\frac{i7\pi}{4}}$. Then, by De Moivre's Theorem, we have $\left(e^{\frac{i\pi}{4}} + e^{i\pi} + ... + e^{2i\pi}\right)\left(e^{\frac{i7\pi}{4}} + e^{i7\pi} + ... + e^{2i\pi}\right)$

which can easily be computed as $\boxed{36}$.

See Also

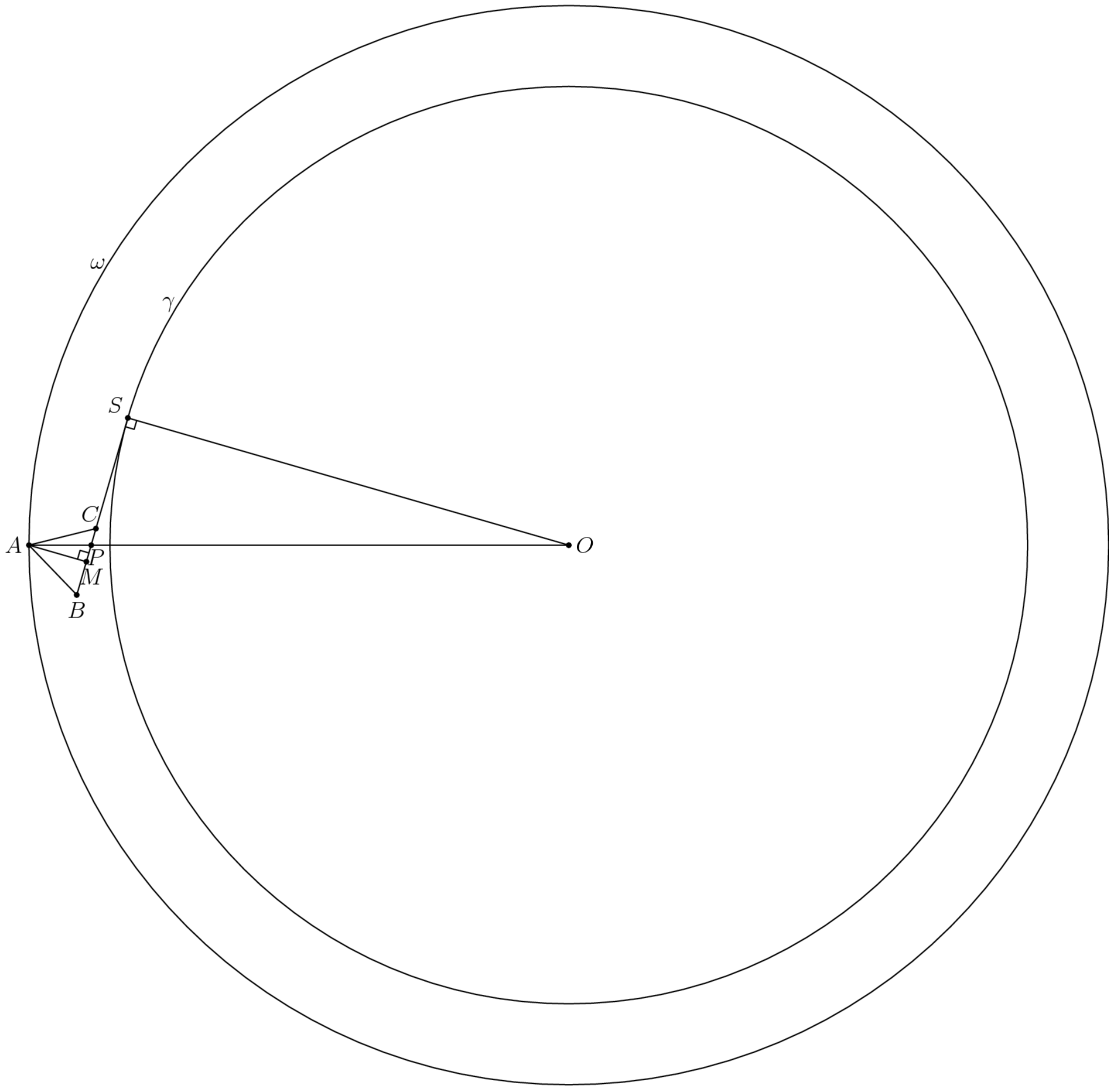
2019 AMC 12A Problems/Problem 22

Problem

Circles ω and γ, both centered at O, have radii 20 and 17, respectively. Equilateral triangle ABC, whose interior lies in the interior of ω but in the exterior of γ, has vertex A on ω, and the line containing side $\overline{BC}$ is tangent to γ. Segments $\overline{AO}$ and $\overline{BC}$ intersect at P, and $\dfrac{BP}{CP} = 3$. Then AB can be written in the form $\dfrac{m}{\sqrt{n}} - \dfrac{p}{\sqrt{q}}$ for positive integers m, n, p, q with $\gcd(m,n) = \gcd(p,q) = 1$. What is $m+n+p+q$?

$\textbf{(A) } 42 \qquad \textbf{(B) } 86 \qquad \textbf{(C) } 92 \qquad \textbf{(D) } 114 \qquad \textbf{(E) } 130$

Solution



Let S be the point of tangency between $\overline{BC}$ and γ, and M be the midpoint of $\overline{BC}$. Note that $AM \perp BS$and $OS \perp BS$. This implies that $\angle OAM \cong \angle AOS$, and $\angle AMP \cong \angle OSP$.
Thus, $\triangle PMA \sim \triangle PSO$.

If we let s be the side length of $\triangle ABC$, then it follows that $AM = \dfrac{\sqrt{3}}{2}s$ and $PM = \dfrac{s}{4}$. This implies that $AP = \dfrac{\sqrt{13}}{4}s$,

so $\dfrac{AM}{AP} = \dfrac{2\sqrt{3}}{\sqrt{13}}$.

Furthermore, $\dfrac{AM + SO}{AO} = \dfrac{AM}{AP}$ (because $\triangle PMA \sim \triangle PSO$

) so this gives us the equation $\frac{\frac{\sqrt{3}}{2}s+17}{20}=\frac{2\sqrt{3}}{\sqrt{13}}$ to solve for the side length s, or AB. Thus, $\frac{\sqrt{39}}{2}s+17\sqrt{13}=40\sqrt{3}$ $\frac{\sqrt{39}}{2}s=40\sqrt{3}-17\sqrt{13}s=\frac{80}{\sqrt{13}}-\frac{34}{\sqrt{3}}=AB$ The problem asks for $m+n+p+q=80+13+34+3=\boxed{\textbf{(E)}\ 130}$.

2019 AMC 12A Problems/Problem 23

Problem

Define binary operations $\diamondsuit$ and $\heartsuit$ by

$a \diamondsuit b = a^{\log_{7}(b)} \qquad \text{and} \qquad a \heartsuit b = a^{\frac{1}{\log_{7}(b)}}$ for all real numbers a and b for which these expressions are defined. The sequence (a_n) is defined recursively by $a_3 = 3\heartsuit 2$ and $a_n = (n\heartsuit (n-1)) \diamondsuit a_{n-1}$ for all integers $n \geq 4$. To the nearest integer, what is $\log_{7}(a_{2019})$?

$\textbf{(A) } 8 \qquad \textbf{(B) } 9 \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 11 \qquad \textbf{(E) } 12$

Solution 1

By definition, the recursion becomes $a_n = \left(n^{\frac{1}{\log_7(n-1)}}\right)^{\log_7(a_{n-1})}=n^{\frac{\log_7(a_{n-1})}{\log_7(n-1)}}$. By the change of base formula, this reduces to $a_n = n^{\log_{n-1}(a_{n-1})}$. Thus, we have $\log_n(a_n) = \log_{n-1}(a_{n-1})$. Thus, for each positive integer $m \geq 3$, the value of $\log_m(a_m)$ must be some constant value k.

We now compute k from a_3. It is given that $a_3 = 3 \heartsuit 2 = 3^{\frac{1}{\log_7(2)}}$, so $k = \log_3(a_3) = \log_3\left(3^{\frac{1}{\log_7(2)}}\right) = \dfrac{1}{\log_7(2)} = \log_2(7)$.

Now, we must have $\log_{2019}(a_{2019}) = k = \log_2(7)$. At this point, we simply switch some bases around. For those who are unfamiliar with logarithms, we can turn the logarithms into fractions which are less intimidating to work with.

$$\frac{\log a_{2019}}{\log 2019} = \frac{\log 7}{\log 2} \implies \frac{\log a_{2019}}{\log 7} = \frac{\log 2019}{\log 2} \implies \log_7(a_{2019}) = \log_2(2019)$$

We conclude that $\log_7(a_{2019}) = \log_2(2019) \approx \boxed{11}$, or choice $\boxed{\text{D}}$.

Solution 2

Using the recursive definition, $a_4 = (4 \heartsuit 3) \diamondsuit (3 \heartsuit 2)$ or $a_4 = (4^m)^n$ where $m = \dfrac{1}{\log_7(3)}$ and $n = \log_7(3^{\frac{1}{\log_7(2)}})$. Using logarithm rules, we can remove the exponent of the 3 so that $n = \dfrac{\log_7(3)}{\log_7(2)}$.

Therefore, $a_4 = 4^{\frac{1}{\log_7(2)}}$, which is $4 \heartsuit 2$.

We claim that $a_n = n \heartsuit 2$ for all $n \geq 3$. We can prove this through induction.

$$a_n = (n \heartsuit (n-1)) \diamondsuit ((n-1) \heartsuit 2)$$

This can be simplified as $a_n = ((n^{\log_{n-1}(7)}) \diamondsuit ((n-1)^{\log_2(7)}))$.

Applying the diamond operation, we can simplify $a_n = n^h$ where $h = \log_{n-1}(7) \cdot \log_7(n-1)^{\log_2(7)}$. By using logarithm rules to remove the exponent of $\log_7(n-1)$ and after cancelling, $h = \frac{1}{\log_7(2)}$.

Therefore, $a_n = n^{\frac{1}{\log_7(2)}} = n \heartsuit 2$ for all $n \geq 3$, completing the induction.

We have $a_{2019} = 2019^{\log_2(7)}$. Taking $\log_{2019}$ of both sides gives us $\log_{2019}(a_{2019}) = \log_2(7)$. Then, by changing to base 7 and after cancellation, we arrive at $\log_7(a_{2019}) = \log_2(2019)$.

Because $2^{11} = 2048$ and $2^{10} = 1024$, our answer is $\boxed{\textbf{(D) } 11}$.

2019 AMC 10A Problems/Problem 25

(Redirected from 2019 AMC 12A Problems/Problem 24)

The following problem is from both the 2019 AMC 10A #25 and 2019 AMC 12A #24, so both problems redirect to this page.

Problem

For how many integers n between 1 and 50, inclusive, is $\frac{(n^2-1)!}{(n!)^n}$ an integer? (Recall that $0! = 1$.)

$\textbf{(A) } 31 \qquad \textbf{(B) } 32 \qquad \textbf{(C) } 33 \qquad \textbf{(D) } 34 \qquad \textbf{(E) } 35$

Solution

The main insight is that

$$\frac{(n^2)!}{(n!)^{n+1}}$$

is always an integer. This is true because it is precisely the number of ways to split up n^2 objects into n unordered groups of size n. Thus,

$$\frac{(n^2-1)!}{(n!)^n} = \frac{(n^2)!}{(n!)^{n+1}} \cdot \frac{n!}{n^2}$$

is an integer if $n^2 \mid n!$, or in other words, if $n \mid (n-1)!$. This condition is false precisely when $n = 4$ or n is prime, by Wilson's Theorem. There are 15 primes between 1 and 50, inclusive, so there are 15 + 1 = 16 terms for which

$$\frac{(n^2-1)!}{(n!)^n}$$

is potentially not an integer. It can be easily verified that the above expression is not an integer for $n = 4$ as there are more factors of 2 in the denominator than the numerator. Similarly, it can be verified that the above expression is not an integer for any prime $n = p$, as there are more factors of p in the denominator than the numerator. Thus all 16 values of n make the expression not an integer and the answer is $50 - 16 = \boxed{\textbf{(D) } 34}$.

2019 AMC 12A Problems/Problem 25

Problem

Let $\triangle A_0B_0C_0$ be a triangle whose angle measures are exactly 59.999°, 60°, and 60.001°. For each positive integer n define A_n to be the foot of the altitude from A_{n-1} to line $B_{n-1}C_{n-1}$. Likewise, define B_n to be the foot of the altitude from B_{n-1} to line $A_{n-1}C_{n-1}$, and C_n to be the foot of the altitude from C_{n-1} to line $A_{n-1}B_{n-1}$. What is the least positive integer n for which $\triangle A_nB_nC_n$ is obtuse?

$\textbf{(A) } 10 \qquad \textbf{(B) } 11 \qquad \textbf{(C) } 13 \qquad \textbf{(D) } 14 \qquad \textbf{(E) } 15$

Solution

For all nonnegative integers n, let $\angle C_nA_nB_n = x_n$, $\angle A_nB_nC_n = y_n$, and $\angle B_nC_nA_n = z_n$.

Note that quadrilateral $A_0B_0A_1B_1$ is cyclic since $\angle A_0A_1B_0 = \angle A_0B_1B_0 = 90^\circ$; thus, $\angle A_0A_1B_1 = \angle A_0B_0B_1 = 90^\circ - x_0$. By a similar argument, $\angle A_0A_1C_1 = \angle A_0C_0C_1 = 90^\circ - x_0$. Thus, $x_1 = \angle A_0A_1B_1 + \angle A_0A_1C_1 = 180^\circ - 2x_0$. By a similar argument, $y_1 = 180^\circ - 2y_0$ and $z_1 = 180^\circ - 2z_0$.

Therefore, for any positive integer n, we have $x_n = 180^\circ - 2x_{n-1}$ (identical recurrence relations can be derived for y_n and z_n). To find an explicit form for this recurrence, we guess that the constant term is related exponentially to n (and the coefficient of x_0 is $(-2)^n$). Hence, we let $x_n = pq^n + r + (-2)^n x_0$. We will solve for p, q, and r by iterating the recurrence to obtain $x_1 = 180^\circ - 2x_0$, $x_2 = 4x_0 - 180^\circ$, and $x_3 = 540 - 8x_0$. Letting $n = 1, 2, 3$ respectively, we have

$$pq + r = 180 \qquad (1)$$

$$pq^2 + r = -180 \qquad (2)$$

$$pq^3 + r = 540 \qquad (3)$$

Subtracting (1) from (3), we have $pq(q^2-1)=360$, and subtracting (1) from (2) gives $pq(q-1)=-360$. Dividing these two equations gives $q+1=-1$, so $q=-2$. Substituting back, we get $p=-60$ and $r=60$.

We will now prove that for all positive integers n,

$x_n=-60(-2)^n+60+(-2)^n x_0=(-2)^n(x_0-60)+60$

via induction. Clearly the base case of $n=1$ holds, so it is left to prove that $x_{n+1}=(-2)^{n+1}(x_0-60)+60$ assuming our inductive hypothesis holds for n. Using the recurrence relation, we have

$$\begin{aligned} x_{n+1} &= 180-2x_n \\ &= 180-2((-2)^n(x_0-60)+60) \\ &= (-2)^{n+1}(x_0-60)+60 \end{aligned}$$

Our induction is complete, so for all positive integers n, $x_n=(-2)^n(x_0-60)+60$. Identical equalities hold for y_n and z_n.

The problem asks for the smallest n such that either x_n, y_n, or z_n is greater than 90°. WLOG, let $x_0=60^\circ$, $y_0=59.999^\circ$, and $z_0=60.001^\circ$. Thus, $x_n=60^\circ$ for all n, $y_n=-(-2)^n(0.001)+60$, and $z_n=(-2)^n(0.001)+60$. Solving for the smallest possible value of n in each sequence, we find that $n=15$ gives $y_n>90^\circ$. Therefore, the answer is $\boxed{\textbf{(E) }15}$.