1. How many integer values of x satisfy $|x| < 3\pi$?

   A. 9 B. 10 C. 18 D. 19 E. 20

   Answer: D

   Solution: Since 3π is about 9.42 , we multiply 9 by 2 and add 1 to get (D) 19

2. At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 pairs. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both students wearing yellow shirts?

   A. 23 B. 32 C. 37 D. 41 E. 64

   Answer: B

   Solution: There are 46 students paired with a blue partner. The other 11 students wearing blue shirts must each be paired with a partner wearing a shirt of the opposite color. There are 64 students remaining. Therefore the requested number of pairs is $\frac{64}{2} =$ (B) 32

3. Suppose

$$2 + \frac{1}{1 + \frac{1}{2 + \frac{2}{3+x}}} = \frac{144}{53}$$

   What is the value of x?

   A. $\frac{3}{4}$ B. $\frac{7}{8}$ C. $\frac{14}{15}$ D. $\frac{37}{38}$ E. $\frac{52}{53}$

   Answer: A

   Solution: Subtracting 2 from both sides and taking reciprocals gives $1 + \frac{1}{2+\frac{2}{3+x}} = \frac{53}{38}$. Subtracting 1 from both sides and taking reciprocals again gives $2 + \frac{2}{3+x} = \frac{38}{15}$. Subtracting 2 from both sides and taking reciprocals for the final time gives $\frac{x+3}{2} = \frac{15}{8}$ or $x = \frac{3}{4} \Rightarrow$ (A)

4. Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon class's mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the scores of all the students?

   A. 74 B. 75 C. 76 D. 77 E. 78

   Answer: C

   Solution: Assume there are 3 students in the morning class and 4 in the afternoon class. Then the average is $\frac{3 \cdot 84 + 4 \cdot 70}{7} =$ (C) 76

5. The point $P(a, b)$ in the xy-plane is first rotated counterclockwise by 90° around the point $(1, 5)$ and then reflected about the line $y = -x$. The image of P after these two transformations is at $(-6, 3)$. What is $b - a$?

A. 1 B. 3 C. 5 D. 7 E. 9

Answer: D

Solution: The final image of P is (-6,3) . We know the reflection rule for reflecting over $y = -x$ is $(x, y) - - > (-y, -x)$. So before the reflection and after rotation the point is (-3,6) . By definition of rotation, the slope between (-3,6) and (1,5) must be perpendicular to the slope between (a, b) and (1,5) . The first slope is $\frac{5-6}{1-(-3)} = \frac{-1}{4}$. This means the slope of P and (1,5) is 4 Rotations also preserve distance to the center of rotation, and since we only "travelled" up and down by the slope once to get from (3,-6) to (1,5) it follows we shall only use the slope once to travel from (1,5) to P. Therefore point P is located at $(1 + 1, 5 + 4) = (2, 9)$. The answer is $9 - 2 = 7 = \mathbf{(D)}$

6. An inverted cone with base radius 12cm and height 18cm is full of water. The water is poured into a tall cylinder whose horizontal base has radius of 24cm. What is the height in centimeters of the water in the cylinder?

A. 1.5 B. 3 C. 4 D. 4.5 E. 6

Answer: A

Solution: The water completely fills up the cone. For now, assume the radius of both cone and cylinder are the same. Then the cone has $\frac{1}{3}$ of the volume of the cylinder, and so the height is divided by 3 . Then, from the problem statement, the radius is doubled, meaning the area of the base is quadrupled (since $2^2 = 4$). Therefore, the height is divided by 3 and divided by 4, which is $18 \div 3 \div 4 = 1.5 = (A)$

7. Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

A. 1 : 16 B. 1 : 15 C. 1 : 14 D. 1 : 8 E. 1 : 3

Answer: C

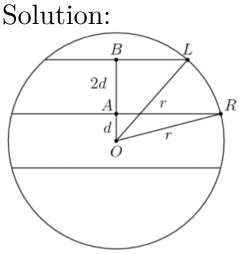
Solution: Prime factorize N to get $N = 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2$. For each odd divisor n of N, there exist even divisors $2n, 4n, 8n$ of N, therefore the ratio is $1 : (2 + 4 + 8) \to \mathbf{(C)}$

8. Three equally spaced parallel lines intersect a circle, creating three chords of lengths 38, 38, and 34. What is the distance between two adjacent parallel lines?

A. $5\frac{1}{2}$ B. 6 C. $6\frac{1}{2}$ D. 7 E. $7\frac{1}{2}$

Answer: B

Solution:



Since two parallel chords have the same length (38), they must be equidistant from the center of

the circle. Let the perpendicular distance of each chord from the center of the circle be d. Thus, the distance from the center of the circle to the chord of length 34 is $2d + d = 3d$ and the distance between each of the chords is just $2d$. Let the radius of the circle be r. Drawing radii to the points where the lines intersect the circle, we create two different right triangles:
One with base $\frac{38}{2} = 19$, height d, and hypotenuse $r(\triangle RAO$ on the diagram) Another with base $\frac{34}{2} = 17$, height $3d$, and hypotenuse $r(\triangle LBO$ on the diagram) By the Pythagorean theorem, we can create the following system of equations:

$19^2 + d^2 = r^2$

$17^2 + (3d)^2 = r^2$

Solving, we find $d = 3$, so $2d =$ (B) 6

9. What is the value of $\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}$?

   A. 0 B. 1 C. $\frac{5}{4}$ D. 2 E. $\log_2 5$

   Answer: D

   Solution: $\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}$ Note that $\log_{40} 2 = \frac{1}{\log_2 40}$, and similarly $\log_{20} 2 = \frac{1}{\log_2 20}$ $= \log_2 80 \cdot \log_2 40 - \log_2$ $= (\log_2 4 + \log_2 20)(\log_2 2 + \log_2 20) -$ $= (2 + \log_2 20)(1 + \log_2 20) - ($

   Expanding $2 + 2\log_2 20 + \log_2 20 + (\log_2 20)^2 - 3\log_2 20 - (\log_2 20)^2$ All the log terms cancel, so the answer is $2 \Longrightarrow (D)$

10. Two distinct numbers are selected from the set $\{1, 2, 3, 4, \ldots, 36, 37\}$ so that the sum of the remaining 35 numbers is the product of these two numbers. What is the difference of these two numbers?

    A. 5 B. 7 C. 8 D. 9 E. 10

    Answer: E

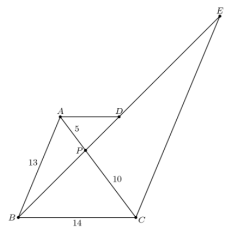
    Solution: The sum of the first 37 integers is given by $n(n + 1)/2$, so $37(37 + 1)/2 = 703$. Therefore, $703 - x - y = xy$ Rearranging, $xy + x + y = 703$ $(x + 1)(y + 1) = 704$ Looking at the possible divisors of $704 = 2^6 * 11, 22$ and 32 are within the constraints of $0 < x \leq y \leq 37$ so we try those: $(x + 1)(y + 1) = 22 * 32$ $x + 1 = 22, y + 1 = 32$ $x = 21, y = 31$ Therefore, the difference $y - x = 31 - 21 = 10$, choice E.

11. Triangle ABC has $AB = 13, BC = 14$ and $AC = 15$. Let P be the point on $\overline{AC}$ such that $PC = 10$. There are exactly two points D and E on line BP such that quadrilaterals $ABCD$ and $ABCE$ are trapezoids. What is the distance DE?

    A. $\frac{42}{5}$ B. $6\sqrt{2}$ C. $\frac{84}{5}$ D. $12\sqrt{2}$ E. 18

    Answer: D

    Solution:



Using Stewart's Theorem we find $BP = 8\sqrt{2}$. From the similar triangles $BPA \sim DPC$ and $BPC \sim EPA$ we have

$DP = BP \cdot \frac{PC}{PA} = 2BP$

$EP = BP \cdot \frac{PA}{PC} = \frac{1}{2}BP$

So $DE = \frac{3}{2}BP =$ (D) $12\sqrt{2}$

12. Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer in S is also removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S. What is the average value of all the integers in the set S?

A. 36.2 B. 36.4 C. 36.6 D. 36.8 E. 37

Answer: D

Solution: We should plug in 36.2 and assume everything is true except the 35 part. We then calculate that part and end up with 35.75 . We also see with the formulas we used with the plug in that when you increase by 0.2 the 35.75 part decreases by 0.25 . The answer is then (D)36.8. You can work backwards because it is multiple choice and you don't have to do critical thinking.

13. How many values of θ in the interval $0 < \theta \leq 2\pi$ satisfy $1 - 3\sin\theta + 5\cos 3\theta = 0$?

A. 2 B. 4 C. 5 D. 6 E. 8

Answer: D

Solution: First, move terms to get $1 + 5\cos 3x = 3\sin x$. After graphing, we find that there are 6 solutions (two in each period of $5\cos 3x$).

14. Let $ABCD$ be a rectangle and let $\overline{DM}$ be a segment perpendicular to the plane of $ABCD$. Suppose that $\overline{DM}$ has integer length, and the lengths of $\overline{MA}, \overline{MC}$, and $\overline{MB}$ are consecutive odd positive integers (in this order). What is the volume of pyramid $MABCD$?

A. $24\sqrt{5}$ B. 60 C. $28\sqrt{5}$ D. 66 E. $8\sqrt{70}$

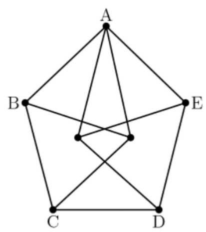
Answer: A

Solution: This question is just about pythagorean theorem

$$\begin{aligned}a^2+(a+2)^2-b^2&=(a+4)^2\\2a^2+4a+4-b^2&=a^2+8a+16\\a^2-4a+4-b^2&=16\\(a-2+b)(a-2-b)&=16\\a=3,b&=7\end{aligned}$$

With these calculation, we find out answer to be (A) $24\sqrt{5}$

15. The figure is constructed from 11 line segments, each of which has length 2 . The area of pentagon $ABCDE$ can be written is $\sqrt{m}+\sqrt{n}$, where m and n are positive integers. What is $m+n$?



A. 20 B. 21 C. 22 D. 23 E. 24

Answer: D

Solution: Let M be the midpoint of CD. Noting that AED and ABC are $120-30-30$ triangles because of the equilateral triangles, $AM=\sqrt{AD^2-MD^2}=\sqrt{12-1}=\sqrt{11}\Longrightarrow[ACD]=\sqrt{11}$. Also, $[AED]=2\cdot2\cdot\frac{1}{2}\cdot\sin120^\circ=\sqrt{3}$ and so $[ABCDE]=[ACD]+2[AED]=\sqrt{11}+2\sqrt{3}=\sqrt{11}+\sqrt{12}\Longrightarrow$ (D) 23

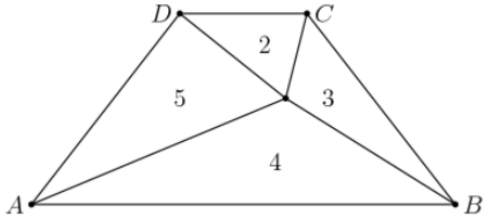
16. Let $g(x)$ be a polynomial with leading coefficient 1, whose three roots are the reciprocals of the three roots of $f(x)=x^3+ax^2+bx+c$ where $1<a<b<c$. What is $g(1)$ in terms of a,b, and c?

A. $\frac{1+a+b+c}{c}$ B. $1+a+b+c$ C. $\frac{1+a+b+c}{c^2}$ D. $\frac{a+b+c}{c^2}$ E. $\frac{1+a+b+c}{a+b+c}$

Answer: A

Solution: Note that $f(1/x)$ has the same roots as $g(x)$, if it is multiplied by some monomial so that the leading term is x^3 they will be equal. We have $f(1/x)=\frac{1}{x^3}+\frac{a}{x^2}+\frac{b}{x}+c$ so we can see that $g(x)=\frac{x^3}{c}f(1/x)$ Therefore $g(1)=\frac{1}{c}f(1)=$ (A) $\frac{1+a+b+c}{c}$

17. Let $ABCD$ be an isoceles trapezoid having parallel bases $\overline{AB}$ and $\overline{CD}$ with $AB>CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are $2,3,4$, and 5 starting with the triangle with base $\overline{CD}$ and moving clockwise as shown in the diagram below. What is the ratio $\frac{AB}{CD}$?



A. 3 B. $2+\sqrt{2}$ C. $1+\sqrt{6}$ D. $2\sqrt{3}$ E. $3\sqrt{2}$

Answer: B

Solution: Let b_1 be the bottom base, b_2 be the top base, h_1 be the height of the bottom triangle, h_2 be the height of the top triangle. Thus, $b_1h_1 = 8, b_2h_2 = 4, (b_1+b_2)(h_1+h_2) = 28$, so $b_1h_2 + b_2h_1 = 16$. Let $b_2 = 1, h_2 = 4$, so we get $b_1h_1 = 8, 4b_1 + h_1 = 16$. This gives us a quadratic in b_1, ie. $4b_1^2 + 8 = 16b_1$, so $b_1 = 2+\sqrt{2}$

18. Let z be a complex number satisfying $12|z|^2 = 2|z+2|^2 + \left|z^2+1\right|^2 + 31$. What is the value of $z + \frac{6}{z}$?

A. -2 B. -1 C. $\frac{1}{2}$ D. 1 E. 4

Answer: A

Solution: The answer being in the form $z + \frac{6}{z}$ means that there are two solutions, some complex number and its complex conjugate.

$a + bi = \frac{6}{a-bi}$

$a^2 + b^2 = 6$

We should then be able to test out some ordered pairs of (a, b). After testing it out, we get the ordered pairs of $(-1, \sqrt{5})$ and its conjugate $(-1, -\sqrt{5})$. Plugging this into answer format gives us $(A) - 2$

19. Two fair dice, each with at least 6 faces are rolled. On each face of each dice is printed a distinct integer from 1 to the number of faces on that die, inclusive. The probability of rolling a sum if 7 is $\frac{3}{4}$ of the probability of rolling a sum of 10, and the probability of rolling a sum of 12 is $\frac{1}{12}$ What is the least possible number of faces on the two dice combined?

A. 16 B. 17 C. 18 D. 19 E. 20

Answer: B

Solution: Suppose the dice have a and b faces, and WLOG $a \geq b$. Since each die has at least 6 faces, there will always be 6 ways to sum to 7 . As a result, there must be $\frac{4}{3} \cdot 6 = 8$ ways to sum to 10 . There are at most nine distinct ways to get a sum of 10 , which are possible whenever $a, b \geq 9$. To achieve exactly eight ways, b must have 8 faces, and $a \geq 9$. Let n be the number of ways to obtain a sum of 12 , then $\frac{n}{8a} = \frac{1}{12} \Longrightarrow n = \frac{2}{3}a$. Since $b = 8, n \leq 8 \Longrightarrow a \leq 12$. In addition to $3 / a$, we only have to test $a = 9, 12$, of which both work. Taking the smaller one, our answer becomes $a + b = 9 + 8 = \mathbf{(B) 17}$

20. Let $Q(z)$ and $R(z)$ be the unique polynomials such that $z^{2021} + 1 = \left(z^2 + z + 1\right) Q(z) + R(z)$ and the degree of R is less than 2. What is $R(z)$?

A. $-z$ B. -1 C. 2021 D. $z+1$ E. $2z+1$

Answer: A

Solution: Note that $z^3 - 1 \equiv 0 \quad \left(\bmod z^2 + z + 1\right)$ so if $F(z)$ is the remainder when dividing by $z^3 - 1$, $F(z) \equiv R(z) \quad \left(\bmod z^2 + z + 1\right)$ Now, $z^{2021} + 1 = \left(z^3 - 1\right)\left(z^{2018} + z^{2015} + \cdots + z^2\right) + z^2 + 1$ So $F(z) = z^2 + 1$, and $R(z) \equiv F(z) \equiv -z \quad \left(\bmod z^2 + z + 1\right)$ The answer is **(A)** $-z$

21. Let S be the sum of all positive real numbers x for which

$$x^{2^{\sqrt{2}}} = \sqrt{2}^{2^x}$$

Which of the following statements is true?

A. $S < \sqrt{2}$ B. $S = \sqrt{2}$ C. $\sqrt{2} < S < 2$ D. $2 \leq S < 6$ E. $S \geq 6$

Answer: D

Solution: $x^{2^{\sqrt{2}}} = \sqrt{2}^{2^x}$ $2^{\sqrt{2}} \log x = 2^x \log \sqrt{2}$ (At this point we see by inspection that $x = \sqrt{2}$ is a solution.) $\sqrt{2} \log 2 + \log \log x = x \log 2 + \log \log \sqrt{2}$ $\sqrt{2} + \log_2 \log_2 x = x + \log_2 \log_2 \sqrt{2} = x - 1$ $\log_2 \log_2 x = x - 1 - \sqrt{2}$ RHS is a line. LHS is a concave curve that looks like a logarithm and has x intercept at (2,0) . There are at most two solutions, one of which is $\sqrt{2}$. But note that at $x = 2$, we have $\log_2 \log_2(2) = 0 > 2 - 1 - \sqrt{2}$, meaning that the log log curve is above the line, so it must intersect the line again at a point $x > 2$. Now we check $x = 4$ and see that $\log_2 \log_2(4) = 1 < 4 - 1 - \sqrt{2}$, which means at $x = 4$ the line is already above the log log curve. Thus, the second solution lies in the interval (2,4) . The answer is $2 \leq S < 6$

22. Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: $(3, 2), (2, 1, 2), (4), (4, 1), (2, 2)$, or $(1, 1, 2)$.

Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

A. $(6, 1, 1)$ B. $(6, 2, 1)$ C. $(6, 2, 2)$ D. $(6, 3, 1)$ E. $(6, 3, 2)$

Answer: B

Solution: (6,1,1) can be turned into (2,2,1,1) by Arjun, which is symmetric, so Beth will lose. (6,3,1) can be turned into (3,1,3,1) by Arjun, which is symmetric, so Beth will lose. (6,2,2) can be turned into (2,2,2,2) by Arjun, which is symmetric, so Beth will lose. (6,3,2) can be turned into (3,2,3,2) by Arjun, which is symmetric, so Beth will lose. That leaves (6,2,1) or **(B)**

23. Three balls are randomly and independantly tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin i is 2^{-i} for $i = 1, 2, 3, \ldots$ More than one ball is

allowed in each bin. The probability that the balls end up evenly spaced in distinct bins is $\frac{p}{q}$, where p and q are relatively prime positive integers. (For example, the balls are evenly spaced if they are tossed into bins $3, 17$, and 10.) What is $p + q$?

A. 55 B. 56 C. 57 D. 58 E. 59

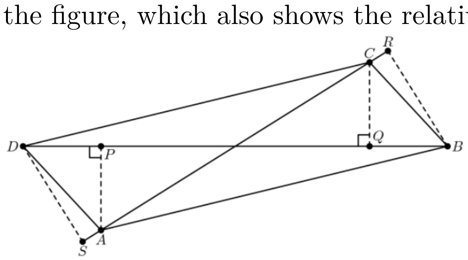
Answer: A

Solution: "Evenly spaced" just means the bins form an arithmetic sequence. Suppose the middle bin in the sequence is x. There are $x - 1$ different possibilities for the first bin, and these two bins uniquely determine the final bin. Now, the probability that these 3 bins are chosen is $6 \cdot 2^{-3x} = 6 \cdot \frac{1}{8^x}$, so the probability x is the middle bin is $6 \cdot \frac{x-1}{8^x}$. Then, we want the sum

$$\begin{aligned} 6\sum_{x=2}^{\infty} \frac{x-1}{8^x} &= \frac{6}{8}\left[\frac{1}{8} + \frac{2}{8^2} + \frac{3}{8^3}\cdots\right] \\ &= \frac{3}{4}\left[\left(\frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3}\right) + \left(\frac{1}{8^2} + \frac{1}{8^3} + \frac{1}{8^4}\right) + \cdots\right] \\ &= \frac{3}{4}\left[\frac{1}{7} \cdot \left(1 + \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3}\right)\right] \\ &= \frac{3}{4} \cdot \frac{8}{49} \\ &= \frac{6}{49} \end{aligned}$$

The answer is $6 + 49 = (A)55$

24. Let $ABCD$ be a parallelogram with area 15. Points P and Q are the projections of A and C, respectively, onto the line BD; and points R and S are the projections of B and D, respectively, onto the line AC. See the figure, which also shows the relative locations of these points.



Suppose $PQ = 6$ and $RS = 8$, and let d denote the length of $\overline{BD}$, the longer diagonal of $ABCD$. Then d^2 can be written in the form $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

A. 81 B. 89 C. 97 D. 105 E. 113

Answer: A

Solution: Let X be the intersection of diagonals AC and BD. By symmetry $[\triangle XCB] = \frac{15}{4}, XQ = 3$ and $XR = 4$, so now we have reduced all of the conditions one quadrant. Let $CQ = x.XC = \sqrt{x^2 + 9}, RB = \frac{4x}{3}$ by similar triangles and using the area condition we get $\frac{4}{3} \cdot x \cdot \sqrt{x^2 + 9} = \frac{15}{2}$. Note that it suffices to find $OB = \frac{4}{3}\sqrt{x^2+9}$ because we can double and square it to get d^2. Solving for $a = x^2$ in the above equation, and then using $d^2 = \frac{64}{9}\left(x^2+9\right) = 8\sqrt{41}+32 \Rightarrow 81$

25. Let S be the set of lattice points in the coordinate plane, both of whose coordinates are integers between 1 and 30 , inclusive. Exactly 300 points in S lie on or below a line with equation $y = mx$. The possible values of m lie in an interval of length $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a+b$?

A. 31 B. 47 C. 62 D. 72 E. 85

Answer: E

Solution: First, we find a numerical representation for the number of lattice points in S that are under the line $y = mx$. For any value of x, the highest lattice point under $y = mx$ is $\lfloor mx \rfloor$. Because every lattice point from $(x, 1)$ to $(x, \lfloor mx \rfloor)$ is under the line, the total number of lattice points under the line is $\sum_{x=1}^{30}(\lfloor mx \rfloor)$. Now, we proceed by finding lower and upper bounds for m. To find the lower bound, we start with an approximation. If 300 lattice points are below the line, then around $\frac{1}{3}$ of the area formed by S is under the line. By using the formula for a triangle's area, we find that when $x = 30, y \approx 20$. Solving for m assuming that (30,20) is a point on the line, we get $m = \frac{2}{3}$. Plugging in m to $\sum_{x=1}^{30}(\lfloor mx \rfloor)$, we get:

$$\sum_{x=1}^{30}\left(\left\lfloor \frac{2}{3}x \right\rfloor\right) = 0+1+2+2+3+\cdots+18+18+19+20$$

We have a repeat every 3 values (every time $y = \frac{2}{3}x$ goes through a lattice point). Thus, we can use arithmetic sequences to calculate the value above:

$$\begin{aligned}\sum_{x=1}^{30}\left(\left\lfloor \tfrac{2}{3}x \right\rfloor\right) &= 0+1+2+2+3+\cdots+18+18+19+20\\ &= \tfrac{20(21)}{2}+2+4+6+\cdots+18\\ &= 210+\tfrac{20}{2}\cdot 9\\ &= 300\end{aligned}$$

This means that $\frac{2}{3}$ is a possible value of m. Furthermore, it is the lower bound for m. This is because $y = \frac{2}{3}x$ goes through many points (such as (21,14)). If m was lower, $y = \frac{2}{3}x$ would no longer go through some of these points, and there would be less than 300 lattice points under it. Now, we find an upper bound for m. Imagine increasing m slowly and rotating the line $y = mx$, starting from the lower bound of $m = \frac{2}{3}$. The upper bound for m occurs when $y = mx$ intersects a lattice point again.

In other words, we are looking for the first $m > \frac{2}{3}$ that is expressible as a ratio of positive integers $\frac{p}{q}$ with $q \le 30$. For each $q = 1, \ldots, 30$, the smallest multiple of $\frac{1}{q}$ which exceeds $\frac{2}{3}$ is $1, \frac{2}{2}, \frac{3}{3}, \frac{3}{4}, \frac{4}{5}, \cdots, \frac{19}{27}, \frac{19}{28}, \frac{20}{29}, \frac{21}{30}$ respectively, and the smallest of these is $\frac{19}{28}$. Note: start listing $\frac{1}{q}$ from $\frac{21}{30}$ and observe that they get further and further away from $\frac{2}{3}$. Alternatively, see the method of finding upper bounds in the multiples of solution 2 The lower bound is $\frac{2}{3}$ and the upper bound is

$\frac{19}{28}$. Their difference is $\frac{1}{84}$, so the answer is $1 + 84 = 85$