2008 AMC 12A Problems/Problem 1

The following problem is from both the 2008 AMC 12A #1 and 2008 AMC 10A #1, so both problems redirect to this page.

Problem

A bakery owner turns on his doughnut machine at 8:30 AM. At 11:10 AM the machine has completed one third of the day's job. At what time will the doughnut machine complete the job?

$\mathrm{(A)}\ \text{1:50 PM}\qquad\mathrm{(B)}\ \text{3:00 PM}\qquad\mathrm{(C)}\ \text{3:30 PM}\qquad\mathrm{(D)}\ \text{4:30 PM}\qquad\mathrm{(E)}\ \text{5:50 PM}$

Solution

The machine completes one-third of the job in $\text{11:10} - \text{8:30} = \text{2:40}$ hours. Thus, the entire job is completed in $3\cdot(\text{2:40}) = \text{8:00}$ hours.

Since the machine was started at 8:30 AM, the job will be finished 8 hours later, at 4:30 PM. The answer is $\mathrm{(D)}$.

See Also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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2008 AMC 12A Problems/Problem 2

Contents

Problem

What is the reciprocal of $\frac{1}{2}+\frac{2}{3}$?

$\mathrm{(A)}\ \frac{6}{7}\qquad\mathrm{(B)}\ \frac{7}{6}\qquad\mathrm{(C)}\ \frac{5}{3}\qquad\mathrm{(D)}\ 3\qquad\mathrm{(E)}\ \frac{7}{2}$

Solution

Solution 1

Here's a cheapshot: Obviously, $\frac{1}{2}+\frac{2}{3}$ is greater than 1. Therefore, its reciprocal is less than 1, and the answer must be $\boxed{\frac{6}{7}}$.

Solution 2

$$\left(\frac{1}{2}+\frac{2}{3}\right)^{-1}=\left(\frac{3}{6}+\frac{4}{6}\right)^{-1}=\left(\frac{7}{6}\right)^{-1}=\frac{6}{7}\Rightarrow A.$$

See Also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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2008 AMC 12A Problems/Problem 3

The following problem is from both the 2008 AMC 12A #3 and 2008 AMC 10A #4, so both problems redirect to this page.

Problem

Suppose that $\frac{2}{3}$ of 10 bananas are worth as much as 8 oranges. How many oranges are worth as much as $\frac{1}{2}$ of 5 bananas?

$\mathrm{(A)}\ 2\qquad\mathrm{(B)}\ \frac{5}{2}\qquad\mathrm{(C)}\ 3\qquad\mathrm{(D)}\ \frac{7}{2}\qquad\mathrm{(E)}\ 4$

Solution

If $\frac{2}{3}\cdot 10\text{ bananas} = 8\text{ oranges}$, then

$\frac{1}{2}\cdot 5\text{ bananas} = \left(\frac{1}{2}\cdot 5\text{ bananas}\right)\cdot\left(\frac{8\text{ oranges}}{\frac{2}{3}\cdot 10\text{ bananas}}\right) = 3\text{ oranges} \Longrightarrow \mathrm{(C)}.$

See Also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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2008 AMC 12A Problems/Problem 4

> The following problem is from both the 2008 AMC 12A #4 and 2008 AMC 10A #5, so both problems redirect to this page.

Problem

Which of the following is equal to the product

$$\frac{8}{4}\cdot\frac{12}{8}\cdot\frac{16}{12}\cdot\cdots\cdot\frac{4n+4}{4n}\cdot\cdots\cdot\frac{2008}{2004}?$$

$\textbf{(A)}\ 251\qquad\textbf{(B)}\ 502\qquad\textbf{(C)}\ 1004\qquad\textbf{(D)}\ 2008\qquad\textbf{(E)}\ 4016$

Contents

Solution

Solution 1

$\frac{8}{4}\cdot\frac{12}{8}\cdot\frac{16}{12}\cdots\frac{4n+4}{4n}\cdots\frac{2008}{2004}=\frac{1}{4}\cdot\left(\frac{8}{8}\cdot\frac{12}{12}\cdots\frac{4n}{4n}\cdots\frac{2004}{2004}\right)\cdot 2008=\frac{2008}{4}=502\Rightarrow B.$

Solution 2

Notice that everything cancels out except for 2008 in the numerator and 4 in the denominator.

Thus, the product is $\frac{2008}{4}=502$, and the answer is $\textbf{(B)}$.

See Also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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2008 AMC 12A Problems/Problem 5

The following problem is from both the 2008 AMC 12A #5 and 2008 AMC 10A #9, so both problems redirect to this page.

Problem

Suppose that

$$\frac{2x}{3} - \frac{x}{6}$$

is an integer. Which of the following statements must be true about x?

$\mathrm{(A)}$ It is negative.
$\mathrm{(B)}$ It is even, but not necessarily a multiple of 3.
$\mathrm{(C)}$ It is a multiple of 3, but not necessarily even.
$\mathrm{(D)}$ It is a multiple of 6, but not necessarily a multiple of 12.
$\mathrm{(E)}$ It is a multiple of 12.

Solution

$$\frac{2x}{3} - \frac{x}{6} \implies \frac{4x}{6} - \frac{x}{6} \implies \frac{3x}{6} \implies \frac{x}{2}$$

For $\frac{x}{2}$ to be an integer, x must be even, but not necessarily divisible by 3. Thus, the answer is $\mathrm{(B)}$.

See also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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2008 AMC 12A Problems/Problem 6

The following problem is from both the 2008 AMC 12A #6 and 2004 AMC 10A #8, so both problems redirect to this page.

Contents

Problem

Heather compares the price of a new computer at two different stores. Store A offers 15% off the sticker price followed by a $\$90$ rebate, and store B offers 25% off the same sticker price with no rebate. Heather saves $\$15$ by buying the computer at store A instead of store B. What is the sticker price of the computer, in dollars?

$\mathrm{(A)}\ 750\qquad\mathrm{(B)}\ 900\qquad\mathrm{(C)}\ 1000\qquad\mathrm{(D)}\ 1050\qquad\mathrm{(E)}\ 1500$

Solution

Solution 1

Let the sticker price be x.

The price of the computer is $0.85x-90$ at store A, and $0.75x$ at store B.

Heather saves $\$15$ at store A, so $0.85x-90+15=0.75x$.

Solving, we find $x=750$, and the thus answer is $\mathrm{(A)}$.

Solution 2

The $\$90$ in store A is $\$15$ better than the additional 10% off at store B.

Thus the 10% off is equal to $\$90$ - $\$15=\$75$, and therefore the sticker price is $\$750$.

See Also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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2008 AMC 12A Problems/Problem 7

The following problem is from both the 2008 AMC 12A #7 and 2008 AMC 10A #11, so both problems redirect to this page.

Problem

While Steve and LeRoy are fishing 1 mile from shore, their boat springs a leak, and water comes in at a constant rate of 10 gallons per minute. The boat will sink if it takes in more than 30 gallons of water. Steve starts rowing toward the shore at a constant rate of 4 miles per hour while LeRoy bails water out of the boat. What is the slowest rate, in gallons per minute, at which LeRoy can bail if they are to reach the shore without sinking?

$\mathrm{(A)}\ 2\qquad\mathrm{(B)}\ 4\qquad\mathrm{(C)}\ 6\qquad\mathrm{(D)}\ 8\qquad\mathrm{(E)}\ 10$

Solution

It will take $\frac{1}{4}$ of an hour or 15 minutes to get to shore.

Since only 30 gallons of water can enter the boat, only $\frac{30}{15}=2$ net gallons can enter the boat per minute.

Since 10 gallons of water enter the boat each minute, LeRoy must bail $10-2=8$ gallons per minute $\Rightarrow \mathrm{(D)}$.

See Also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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Category: Introductory Algebra Problems

2008 AMC 12A Problems/Problem 8

Problem

What is the volume of a cube whose surface area is twice that of a cube with volume 1?

$\mathrm{(A)}\ \sqrt{2}\qquad\mathrm{(B)}\ 2\qquad\mathrm{(C)}\ 2\sqrt{2}\qquad\mathrm{(D)}\ 4\qquad\mathrm{(E)}\ 8$

Solution

A cube with volume 1 has a side of length $\sqrt[3]{1}=1$ and thus a surface area of $6 \cdot 1^2=6$.

A cube whose surface area is $6\cdot2=12$ has a side of length $\sqrt{\frac{12}{6}}=\sqrt{2}$ and a volume of $(\sqrt{2})^3=2\sqrt{2}\Rightarrow\mathrm{(C)}$.

See Also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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Categories: Introductory Geometry Problems | 3D Geometry Problems

2008 AMC 12A Problems/Problem 9

The following problem is from both the 2008 AMC 12A #9 and 2008 AMC 10A #14, so both problems redirect to this page.

Problem

Older television screens have an aspect ratio of $4:3$. That is, the ratio of the width to the height is $4:3$. The aspect ratio of many movies is not $4:3$, so they are sometimes shown on a television screen by "letterboxing" - darkening strips of equal height at the top and bottom of the screen, as shown. Suppose a movie has an aspect ratio of $2:1$ and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 2.25 \qquad \textbf{(C)}\ 2.5 \qquad \textbf{(D)}\ 2.7 \qquad \textbf{(E)}\ 3$

Solution

Let the width and height of the screen be $4x$ and $3x$ respectively, and let the width and height of the movie be $2y$ and y respectively.

By the Pythagorean Theorem, the diagonal is $\sqrt{(3x)^2+(4x)^2}=5x=27$. So $x=\frac{27}{5}$.

Since the movie and the screen have the same width, $2y=4x\Rightarrow y=2x$.

Thus, the height of each strip is $\frac{3x-y}{2}=\frac{3x-2x}{2}=\frac{x}{2}=\frac{27}{10}=2.7 \Longrightarrow \textbf{(D)}$.

See Also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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2008 AMC 12A Problems/Problem 10

The following problem is from both the 2008 AMC 12A #10 and 2008 AMC 10A #13, so both problems redirect to this page.

Contents

Problem

Doug can paint a room in 5 hours. Dave can paint the same room in 7 hours. Doug and Dave paint the room together and take a one-hour break for lunch. Let t be the total time, in hours, required for them to complete the job working together, including lunch. Which of the following equations is satisfied by t?

$\text{(A)}\ \left(\frac{1}{5}+\frac{1}{7}\right)\left(t+1\right)=1\qquad\text{(B)}\ \left(\frac{1}{5}+\frac{1}{7}\right)t+1=1\qquad\text{(C)}\ \left(\frac{1}{5}+\frac{1}{7}\right)t=1\\ \text{(D)}\ \left(\frac{1}{5}+\frac{1}{7}\right)\left(t-1\right)=1\qquad\text{(E)}\ \left(5+7\right)t=1$

Solution

Solution 1

Doug can paint $\frac{1}{5}$ of a room per hour, Dave can paint $\frac{1}{7}$ of a room in an hour, and the time they spend working together is $t-1$.

Since rate times time gives output, $\left(\frac{1}{5}+\frac{1}{7}\right)\left(t-1\right)=1 \Rightarrow \text{(D)}$

Solution 2

If one person does a job in a hours and another person does a job in b hours, the time it takes to do the job together is $\frac{ab}{a+b}$ hours.

Since Doug paints a room in 5 hours and Dave paints a room in 7 hours, they both paint in $\frac{5*7}{5+7} = \frac{35}{12}$ hours. They also take 1 hour for lunch, so the total time $t = \frac{35}{12} + 1$ hours.

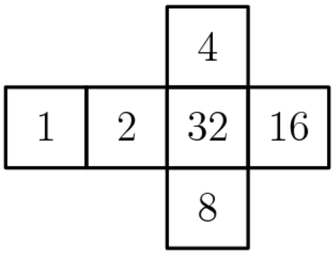
Looking at the answer choices, (D) is the only one satisfied by $t = \frac{35}{12} + 1$.

See Also

2008 AMC 12A Problems/Problem 11

Problem

Three cubes are each formed from the pattern shown. They are then stacked on a table one on top of another so that the 13 visible numbers have the greatest possible sum. What is that sum?



$\textbf{(A)}\ 154 \qquad \textbf{(B)}\ 159 \qquad \textbf{(C)}\ 164 \qquad \textbf{(D)}\ 167 \qquad \textbf{(E)}\ 189$

Solution

To maximize the sum of the 13 faces that are showing, we can minimize the sum of the numbers of the 5 faces that are not showing.

The bottom 2 cubes each have a pair of opposite faces that are covered up. When the cube is folded, $(1,32)$; $(2,16)$; and $(4,8)$ are opposite pairs. Clearly $4+8=12$ has the smallest sum.

The top cube has 1 number that is not showing. The smallest number on a face is 1.

So, the minimum sum of the 5 unexposed faces is $2\cdot12+1=25$. Since the sum of the numbers on all the cubes is $3(32+16+8+4+2+1)=189$, the maximum possible sum of 13 visible numbers is $189-25=164 \Rightarrow C$.

See Also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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2008 AMC 12A Problems/Problem 12

Problem

A function f has domain $[0, 2]$ and range $[0, 1]$. (The notation $[a, b]$ denotes $\{x : a \leq x \leq b\}$.) What are the domain and range, respectively, of the function g defined by $g(x) = 1 - f(x + 1)$?

$\mathrm{(A)}\ [-1, 1], [-1, 0] \qquad \mathrm{(B)}\ [-1, 1], [0, 1] \qquad \textbf{(C)}\ [0, 2], [-1, 0] \qquad \mathrm{(D)}\ [1, 3], [-1, 0] \qquad \mathrm{(E)}\ [1, 3], [0, 1$

Solution

$g(x)$ is defined if $f(x+1)$ is defined. Thus the domain is all $x | x + 1 \in [0, 2] \rightarrow x \in [-1, 1]$.

Since $f(x + 1) \in [0, 1]$, $-f(x + 1) \in [-1, 0]$. Thus $g(x) = 1 - f(x + 1) \in [0, 1]$ is the range of $g(x)$.

Thus the answer is $[-1, 1], [0, 1] \Rightarrow B$.

See Also

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Category: Introductory Algebra Problems

2008 AMC 12A Problems/Problem 13

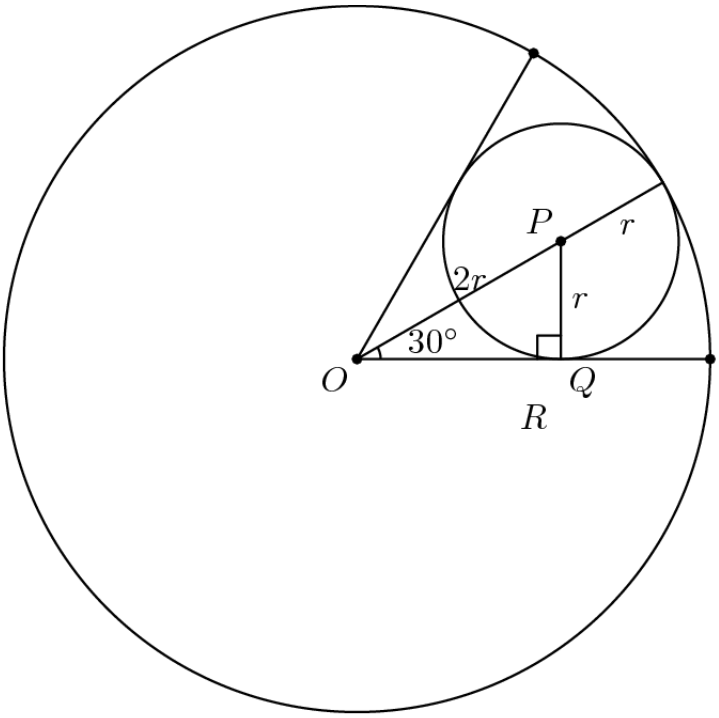
The following problem is from both the 2008 AMC 12A #13 and 2008 AMC 10A #16, so both problems redirect to this page.

Problem

Points A and B lie on a circle centered at O, and $\angle AOB = 60^\circ$. A second circle is internally tangent to the first and tangent to both $\overline{OA}$ and $\overline{OB}$. What is the ratio of the area of the smaller circle to that of the larger circle?

$\mathrm{(A)}\ \frac{1}{16} \qquad \mathrm{(B)}\ \frac{1}{9} \qquad \mathrm{(C)}\ \frac{1}{8} \qquad \mathrm{(D)}\ \frac{1}{6} \qquad \mathrm{(E)}\ \frac{1}{4}$

Solution



Let P be the center of the small circle with radius r, and let Q be the point where the small circle is tangent to OA. Also, let C be the point where the small circle is tangent to the big circle with radius R.

Then PQO is a right triangle, and a $30-60-90$ triangle at that. Therefore, $OP=2PQ$.

Since $OP = OC - PC = OC - r = R - r$, we have $R - r = 2PQ$, or $R - r = 2r$, or $\frac{1}{3} = \frac{r}{R}$.

Then the ratio of areas will be $\frac{1}{3}$ squared, or $\frac{1}{9} \Rightarrow \boxed{\text{B}}$.

See also

2008 AMC 12A Problems/Problem 14

Problem

What is the area of the region defined by the inequality $|3x-18|+|2y+7|\le 3$?

$\mathrm{(A)}\ 3\qquad\mathrm{(B)}\ \frac{7}{2}\qquad\mathrm{(C)}\ 4\qquad\mathrm{(D)}\ \frac{9}{2}\qquad\mathrm{(E)}\ 5$

Solution

Area is invariant under translation, so after translating left 6 and up $7/2$ units, we have the inequality

$$|3x|+|2y|\le 3$$

which forms a diamond centered at the origin and vertices at $(\pm 1, 0), (0, \pm 1.5)$. Thus the diagonals are of length 2 and 3. Using the formula $A = \frac{1}{2} d_1 d_2$, the answer is $\frac{1}{2}(2)(3) = 3 \Rightarrow \mathrm{(A)}$.

See also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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2008 AMC 12A Problems/Problem 15

The following problem is from both the 2008 AMC 12A #15 and 2008 AMC 10A #24, so both problems redirect to this page.

Problem

Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

$\mathrm{(A)}\ 0\qquad\mathrm{(B)}\ 2\qquad\mathrm{(C)}\ 4\qquad\mathrm{(D)}\ 6\qquad\mathrm{(E)}\ 8$

Solution

$k \equiv 2008^2 + 2^{2008} \equiv 8^2 + 2^4 \equiv 4+6 \equiv 0 \pmod{10}$.

So, $k^2 \equiv 0 \pmod{10}$. Since $k = 2008^2+2^{2008}$ is a multiple of four and the units digit of powers of two repeat in cycles of four, $2^k \equiv 2^4 \equiv 6 \pmod{10}$.

Therefore, $k^2+2^k \equiv 0+6 \equiv 6 \pmod{10}$. So the units digit is $6 \Rightarrow D$.

See Also

2008 AMC 12A (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2008))	
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2008 AMC 12A Problems/Problem 16

Problem

The numbers $\log(a^3b^7)$, $\log(a^5b^{12})$, and $\log(a^8b^{15})$ are the first three terms of an arithmetic sequence, and the 12^{th} term of the sequence is $\log b^n$. What is n?

$\mathrm{(A)}\ 40\qquad\mathrm{(B)}\ 56\qquad\mathrm{(C)}\ 76\qquad\mathrm{(D)}\ 112\qquad\mathrm{(E)}\ 143$

Contents

Solution 1

Let $A = \log(a)$ and $B = \log(b)$.

The first three terms of the arithmetic sequence are $3A + 7B$, $5A + 12B$, and $8A + 15B$, and the 12^{th} term is nB.

Thus, $2(5A + 12B) = (3A + 7B) + (8A + 15B) \Rightarrow A = 2B$.

Since the first three terms in the sequence are $13B$, $22B$, and $31B$, the kth term is $(9k + 4)B$.

Thus the 12^{th} term is $(9 \cdot 12 + 4)B = 112B = nB \Rightarrow n = 112 \Rightarrow \boxed{D}$.

Solution 2

If $\log(a^3b^7)$, $\log(a^5b^{12})$, and $\log(a^8b^{15})$ are in arithmetic progression, then a^3b^7, a^5b^{12}, and a^8b^{15} are in geometric progression. Therefore,

$$a^2b^5 = a^3b^3 \Rightarrow a = b^2$$

Therefore, $a^3b^7 = b^{13}$, $a^5b^{12} = b^{22}$, therefore the 12th term in the sequence is $b^{13+9*11} = b^{112} \Rightarrow \boxed{D}$

Solution 3

If a, b, and c are in a arithmetic progression then $b = \frac{a+c}{2}$ which means
$\log(a^5b^{12}) = \frac{\log(a^3b^7) + \log(a^8b^{15})}{2} = \frac{\log(a^{11}b^{22})}{2}$ therefore
$2\log(a^5b^{12}) = \log(a^{10}b^{24}) = \log(a^{11}b^{22}) \Rightarrow a = b^2$
This means that the Kth term of the series would be $\log(b^{13+9(k-1)})$
The 12th term would be $\log(b^{112}) \Rightarrow n = 112 \Rightarrow D$

See Also

2008 AMC 12A Problems/Problem 17

Contents

Problem

Let $a_1, a_2, \ldots$ be a sequence determined by the rule $a_n = a_{n-1}/2$ if a_{n-1} is even and $a_n = 3a_{n-1} + 1$ if a_{n-1} is odd. For how many positive integers $a_1 \le 2008$ is it true that a_1 is less than each of a_2, a_3, and a_4?

$\mathrm{(A)}\ 250\qquad\mathrm{(B)}\ 251\qquad\mathrm{(C)}\ 501\qquad\mathrm{(D)}\ 502\qquad\mathrm{(E)}1004$

Solution

All positive integers can be expressed as $4n$, $4n+1$, $4n+2$, or $4n+3$, where n is a nonnegative integer.

- If $a_1 = 4n$, then $a_2 = \frac{4n}{2} = 2n < a_1$.
- If $a_1 = 4n+1$, then $a_2 = 3(4n+1)+1 = 12n+4$, $a_3 = \frac{12n+4}{2} = 6n+2$, and $a_4 = \frac{6n+2}{2} = 3n+1 < a_1$.
- If $a_1 = 4n+2$, then $a_2 = 2n+1 < a_1$.
- If $a_1 = 4n+3$, then $a_2 = 3(4n+3)+1 = 12n+10$, $a_3 = \frac{12n+10}{2} = 6n+5$, and $a_4 = 3(6n+5)+1 = 18n+16$.

Since $12n+10, 6n+5, 18n+16 > 4n+3$, every positive integer $a_1 = 4n+3$ will satisfy $a_1 < a_2, a_3, a_4$.

Since one fourth of the positive integers $a_1 \le 2008$ can be expressed as $4n+3$, where n is a nonnegative integer, the answer is $\frac{1}{4} \cdot 2008 = 502 \Rightarrow D$.

Alternate Solution

After checking the first few a_n such as 1, 2 through 7, we can see that the only a_1 that satisfy the conditions are odd numbers that when tripled and added 1 to, are double an odd number. For example, for $a_n = 3$, we notice the sequence yields 10, 5, and 16, a valid sequence.

So we can set up an equation, $3x+1 = 2(2k-1)$ where x is equal to a_1. Rearranging the equation yields $(3x+3)/4 = k$. Experimenting yields that every 4th x after 3 creates an integer, and thus satisfies the sequence condition. So the number of valid solutions is equal to $2008/4 = 502 \Rightarrow D$.

See Also

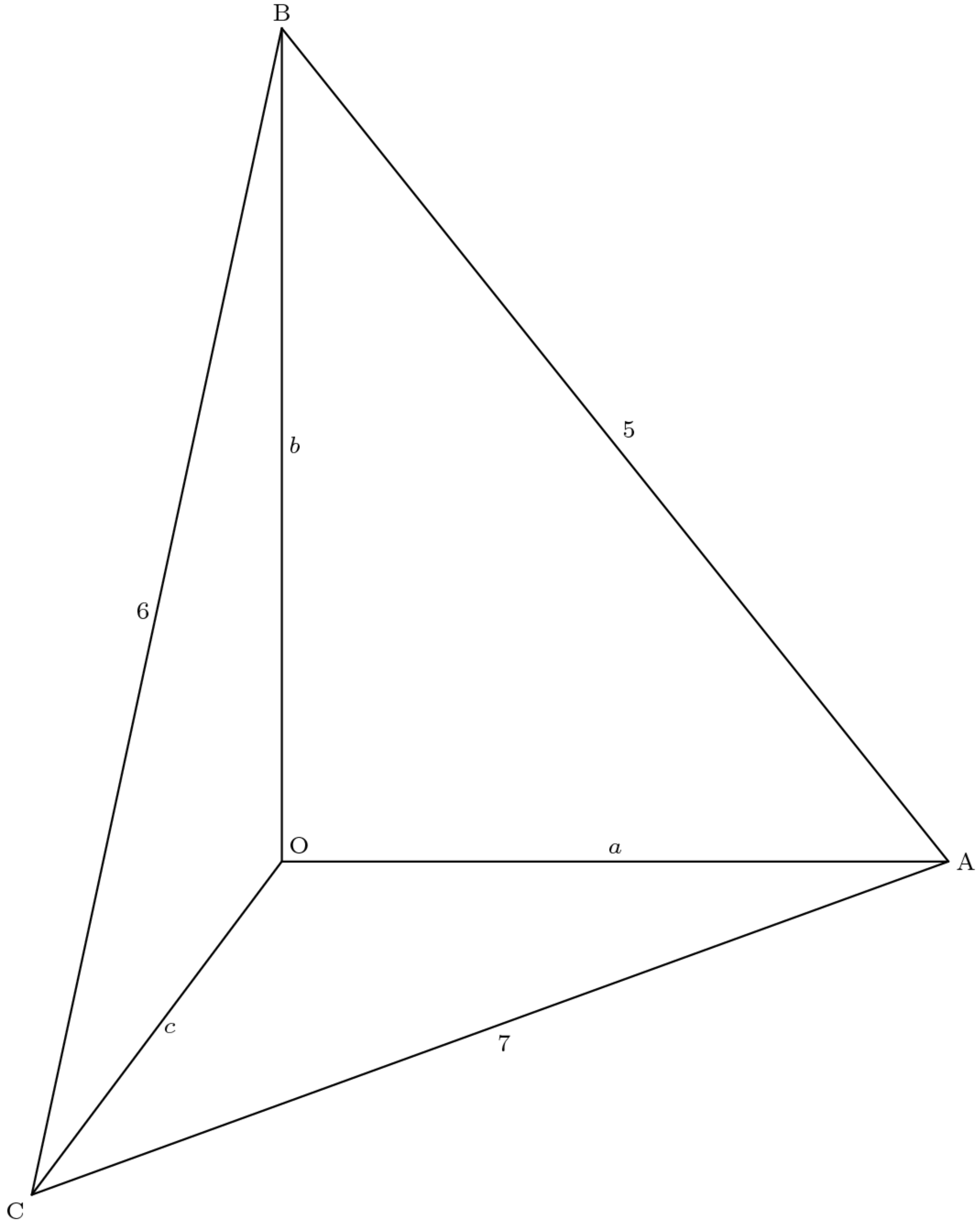
2008 AMC 12A Problems/Problem 18

Problem

Triangle ABC, with sides of length 5, 6, and 7, has one vertex on the positive x-axis, one on the positive y-axis, and one on the positive z-axis. Let O be the origin. What is the volume of tetrahedron $OABC$?

$\mathrm{(A)}\ \sqrt{85}\qquad\mathrm{(B)}\ \sqrt{90}\qquad\mathrm{(C)}\ \sqrt{95}\qquad\mathrm{(D)}\ 10\qquad\mathrm{(E)}\ \sqrt{105}$

Solution



Without loss of generality, let A be on the x axis, B be on the y axis, and C be on the z axis, and let AB, BC, CA have respective lengths of 5, 6, and 7. Let a, b, c denote the lengths of segments $OA, OB, OC,$ respectively. Then by the Pythagorean Theorem,

$$a^2 + b^2 = 5^2,$$
$$b^2 + c^2 = 6^2,$$
$$c^2 + a^2 = 7^2,$$

so $a^2 = (5^2 + 7^2 - 6^2)/2 = 19$; similarly, $b^2 = 6$ and $c^2 = 30$. Since OA, OB, and OC are mutually perpendicular, the tetrahedron's volume is

$$abc/6$$

because we can consider the tetrahedron to be a right triangular pyramid.

$$abc/6 = \sqrt{a^2b^2c^2}/6 = \frac{\sqrt{19 \cdot 6 \cdot 30}}{6} = \sqrt{95},$$

which is answer choice $\boxed{\text{C}}$.

See also

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Category: Introductory Geometry Problems

2008 AMC 12A Problems/Problem 19

Contents

Problem

In the expansion of

$$\left(1+x+x^2+\cdots+x^{27}\right)\left(1+x+x^2+\cdots+x^{14}\right)^2,$$

what is the coefficient of x^{28}?

$\mathrm{(A)}\ 195\qquad\mathrm{(B)}\ 196\qquad\mathrm{(C)}\ 224\qquad\mathrm{(D)}\ 378\qquad\mathrm{(E)}\ 405$

Solution 1

Let $A=\left(1+x+x^2+\cdots+x^{14}\right)$ and $B=\left(1+x+x^2+\cdots+x^{27}\right)$. We are expanding $A\cdot A\cdot B$.

Since there are 15 terms in A, there are $15^2=225$ ways to choose one term from each A. The product of the selected terms is x^n for some integer n between 0 and 28 inclusive. For each $n\neq 0$, there is one and only one x^{28-n} in B. For example, if I choose x^2 from A , then there is exactly one power of x in B that I can choose; in this case, it would be x^{24}. Since there is only one way to choose one term from each A to get a product of x^0, there are $225-1=224$ ways to choose one term from each A and one term from B to get a product of x^{28}. Thus the coefficient of the x^{28} term is $224\Rightarrow\boxed{C}$.

Solution 2

Let $P(x)=\left(1+x+x^2+\cdots+x^{14}\right)^2=a_0+a_1x+a_2x^2+\cdots+a_{28}x^{28}$. Then the x^{28} term from the product in question $\left(1+x+x^2+\cdots+x^{27}\right)\left(a_0+a_1x+a_2x^2+\cdots+a_{28}x^{28}\right)$ is

$$1a_{28}x^{28}+xa_{27}x^{27}+x^2a_{26}x^{26}+\cdots+x^{27}a_1x=(a_1+a_2+\cdots a_{28})\,x^{28}$$

So we are trying to find the sum of the coefficients of $P(x)$ minus a_0. Since the constant term a_0 in $P(x)$ (when expanded) is 1, and the sum of the coefficients of $P(x)$ is $P(1)$, we find the answer to be
$P(1)-a_0=\left(1+1+1^2+\cdots1^{14}\right)^2-1=15^2-1=224\Rightarrow\boxed{C}$.

Solution 3

We expand $(1+x+x^2+x^3+\cdots+x^{14})^2$ to
$(1+x+x^2+x^3+\cdots+x^{14})*(1+x+x^2+x^3+\cdots+x^{14})$ and use FOIL to multiply. It expands out to:

$$\begin{aligned}&1+x+x^2+x^3+x^4+\cdots+x^{14}+\\&\quad x+x^2+x^3+x^4+\cdots+x^{14}+x^{15}+\\&\qquad x^2+x^3+x^4+\cdots+x^{14}+x^{15}+x^{16}+\cdots\end{aligned}$$

It becomes apparent that

$$(1+x+x^2+x^3+\cdots+x^{14})^2=1+2x+3x^2+4x^3+\cdots+15x^{14}+14x^{15}+13x^{16}+\cdots+x^{28}$$
.

Now we have to find the coefficient of x^{28} in the product:

$(1+2x+3x^2+4x^3+\cdots+15x^{14}+14x^{15}+13x^{16}+\cdots+x^{28})\cdot(1+x+x^2+x^3+\cdots+x^{27})$
.

We quickly see that the we get x^{28} terms from $x^{27}\cdot 2x$, $x^{26}\cdot 3x^2$, $x^{25}\cdot 4x^3$, ... $15x^{14}\cdot x^{14}$, ... $x^{28}\cdot 1$. The coefficient of x^{28} is just the sum of the coefficients of all these terms.
$1+2+3+4+\cdots+15+14+13+\cdots+4+3+2=224$, so the answer is $\boxed{C}$.

Solution 4

Rewrite the product as $\frac{(x^{28}-1)(x^{15}-1)(x^{15}-1)}{(x-1)^3}$. It is known that

$$\frac{1}{(1-x)^n}=\binom{n-1}{n-1}+\binom{n}{n-1}x+\binom{n+1}{n-1}x^2+\binom{n+2}{n-1}x^3+\cdots+\binom{n-1+k}{n-1}x^k+\cdots.$$

Thus, our product becomes

$$-\left(x^{28}-1\right)\left(x^{15}-1\right)\left(x^{15}-1\right)\left(\binom{2}{2}+\binom{3}{2}x+\binom{4}{2}x^2+\cdots\right).$$

$$=-\left(x^{28}-1\right)\left(x^{15}-1\right)\left(x^{15}-1\right)\left(1+3x+6x^2+\cdots\right).$$

We determine the x^{28} coefficient by doing casework on the first three terms in our product. We can obtain an x^{28} term by choosing x^{28} in the first term, -1 in the second and third terms, and 1 in the fourth term. We can get two x^{28} terms by choosing x^{15} in either the second or third term, -1 in the first term, -1 in the second or third term from which x^{15} has not been chosen, and the $\binom{15}{2}x^{13}$ in the fourth term. We get $\binom{15}{2}*2=210$ x^{28} terms this way. (We multiply by 2 because the x^{15} term could have been chosen from the second term or the third term). Lastly, we can get an x^{28} term by choosing -1 in the first three terms and a $\binom{30}{2}x^{28}$ from the fourth term. We have a total of $1+210-435=-224$ for the x^{28} coefficient, but we recall that we have a negative sign in front of our product, so we obtain an answer of $224\Rightarrow\boxed{(C)}$.

See Also

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Category: Introductory Algebra Problems

2008 AMC 12A Problems/Problem 20

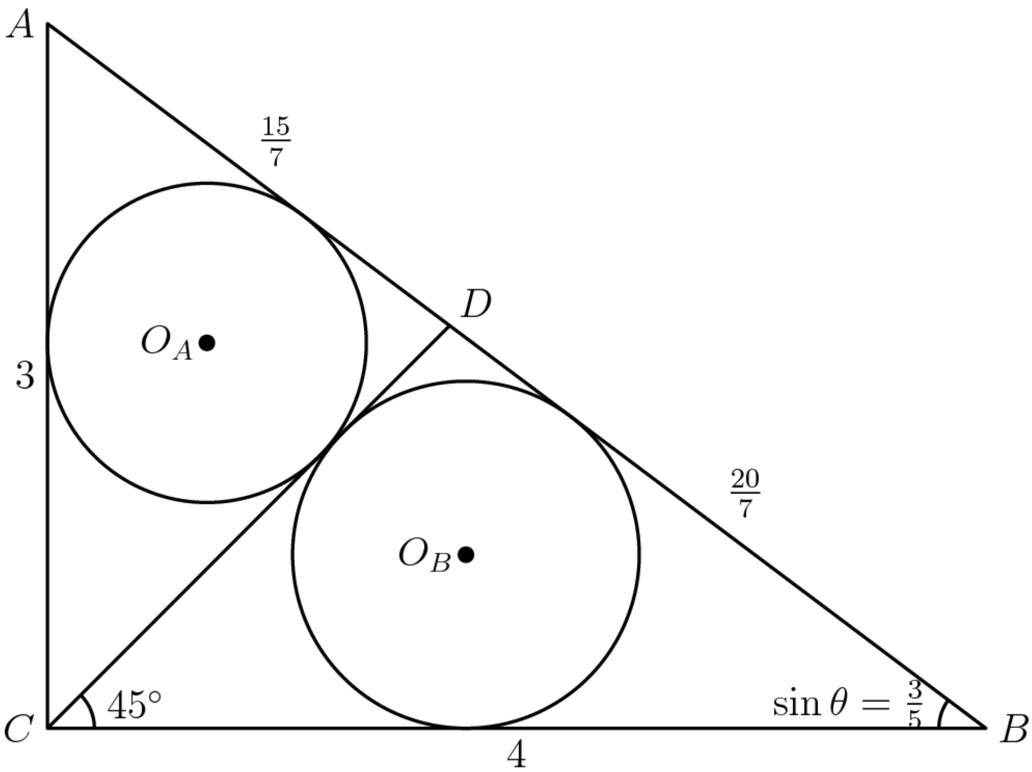
Contents

Problem

Triangle ABC has $AC = 3$, $BC = 4$, and $AB = 5$. Point D is on $\overline{AB}$, and $\overline{CD}$ bisects the right angle. The inscribed circles of $\triangle ADC$ and $\triangle BCD$ have radii r_a and r_b, respectively. What is r_a/r_b?

$\mathrm{(A)}\ \frac{1}{28}\left(10-\sqrt{2}\right) \qquad \mathrm{(B)}\ \frac{3}{56}\left(10-\sqrt{2}\right) \qquad \mathrm{(C)}\ \frac{1}{14}\left(10-\sqrt{2}\right) \qquad \mathrm{(D)}\ \frac{5}{56}\left(10-\sqrt{2}\right)$
$\mathrm{(E)}\ \frac{3}{28}\left(10-\sqrt{2}\right)$

Solution 1



By the Angle Bisector Theorem,

$$\frac{BD}{4} = \frac{5-BD}{3} \implies BD = \frac{20}{7}$$

By Law of Sines on $\triangle BCD$,

$$\frac{BD}{\sin 45^\circ} = \frac{CD}{\sin \angle B} \implies \frac{20/7}{\sqrt{2}/2} = \frac{CD}{3/5} \implies CD = \frac{12\sqrt{2}}{7}$$

Since the area of a triangle satisfies $[\triangle] = rs$, where $r =$ the inradius and $s =$ the semiperimeter, we have

$$\frac{r_A}{r_B} = \frac{[ACD] \cdot s_B}{[BCD] \cdot s_A}$$

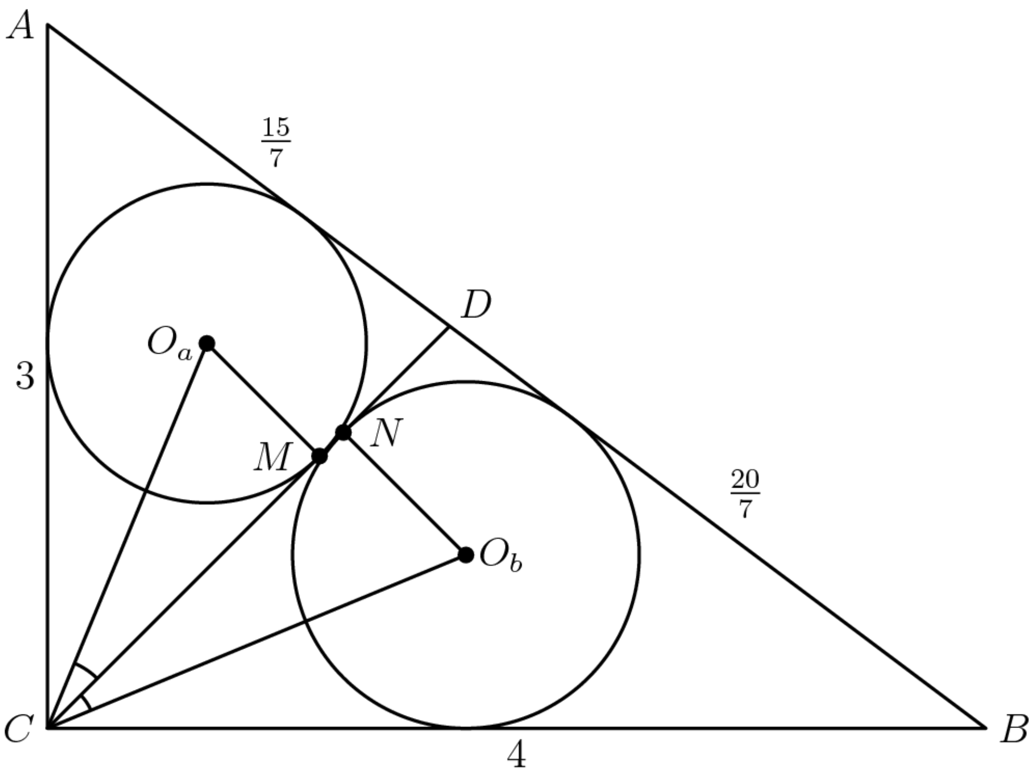
Since $\triangle ACD$ and $\triangle BCD$ share the altitude (to $\overline{AB}$), their areas are the ratio of their bases, or

$$\frac{[ACD]}{[BCD]} = \frac{AD}{BD} = \frac{3}{4}$$

The semiperimeters are $s_A = \left(3 + \frac{15}{7} + \frac{12\sqrt{2}}{7}\right) / 2 = \frac{18 + 6\sqrt{2}}{7}$ and $s_B = \frac{24 + 6\sqrt{2}}{7}$. Thus,

$$\begin{aligned} \frac{r_A}{r_B} &= \frac{[ACD] \cdot s_B}{[BCD] \cdot s_A} = \frac{3}{4} \cdot \frac{(24 + 6\sqrt{2})/7}{(18 + 6\sqrt{2})/7} \\ &= \frac{3(4 + \sqrt{2})}{4(3 + \sqrt{2})} \cdot \left(\frac{3 - \sqrt{2}}{3 - \sqrt{2}}\right) = \frac{3}{28}(10 - \sqrt{2}) \Rightarrow \text{(E)} \qquad \blacksquare \end{aligned}$$

Solution 2



We start by finding the length of AD and BD as in solution 1. Using the angle bisector theorem, we see that $AD = \frac{15}{7}$ and $BD = \frac{20}{7}$. Using Stewart's Theorem gives us the equation $5d^2 + \frac{1500}{49} = \frac{240}{7} + \frac{180}{7}$, where d is the length of CD. Solving gives us $d = \frac{12\sqrt{2}}{7}$, so $CD = \frac{12\sqrt{2}}{7}$.

Call the incenters of triangles ACD and BCD O_a and O_b respectively. Since O_a is an incenter, it lies on the angle bisector of $\angle ACD$. Similarly, O_b lies on the angle bisector of $\angle BCD$. Call the point on CD tangent to O_a M, and the point tangent to O_b N. Since $\triangle CO_aM$ and $\triangle CO_bN$ are right, and $\angle O_aCM = \angle O_bCN$, $\triangle CO_aM \sim \triangle CO_bN$. Then, $\frac{r_a}{r_b} = \frac{CM}{CN}$.

We now use common tangents to find the length of CM and CN. Let $CM = m$, and the length of the other tangents be n and p. Since common tangents are equal, we can write that $m + n = \frac{12\sqrt{2}}{7}$, $n + p = \frac{15}{7}$ and $m + p = 3$. Solving gives us that $CM = m = \frac{6\sqrt{2}+3}{7}$. Similarly, $CN = \frac{6\sqrt{2}+4}{7}$.

We see now that $\frac{r_a}{r_b} = \frac{\frac{6\sqrt{2}+3}{7}}{\frac{6\sqrt{2}+4}{7}} = \frac{6\sqrt{2}+3}{6\sqrt{2}+4} = \frac{60-6\sqrt{2}}{56} = \frac{3}{28}(10-\sqrt{2}) \Rightarrow \boxed{E}$

(Thanks to above solution writer for the framework of my diagram)

See Also

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2008 AMC 12A Problems/Problem 21

Problem

A permutation $(a_1, a_2, a_3, a_4, a_5)$ of $(1, 2, 3, 4, 5)$ is heavy-tailed if $a_1 + a_2 < a_4 + a_5$. What is the number of heavy-tailed permutations?

$\mathrm{(A)}\ 36\qquad\mathrm{(B)}\ 40\qquad\textbf{(C)}\ 44\qquad\mathrm{(D)}\ 48\qquad\mathrm{(E)}\ 52$

Solution

There are $5! = 120$ total permutations.

For every permutation $(a_1, a_2, a_3, a_4, a_5)$ such that $a_1 + a_2 < a_4 + a_5$, there is exactly one permutation such that $a_1 + a_2 > a_4 + a_5$. Thus it suffices to count the permutations such that $a_1 + a_2 = a_4 + a_5$.

$1 + 4 = 2 + 3$, $1 + 5 = 2 + 4$, and $2 + 5 = 3 + 4$ are the only combinations of numbers that can satisfy $a_1 + a_2 = a_4 + a_5$.

There are 3 combinations of numbers, 2 possibilities of which side of the equation is $a_1 + a_2$ and which side is $a_4 + a_5$, and $2^2 = 4$ possibilities for rearranging a_1, a_2 and a_4, a_5. Thus, there are $3 \cdot 2 \cdot 4 = 24$ permutations such that $a_1 + a_2 = a_4 + a_5$.

Thus, the number of heavy-tailed permutations is $\frac{120 - 24}{2} = 48 \Rightarrow D.$

See also

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Category: Introductory Combinatorics Problems

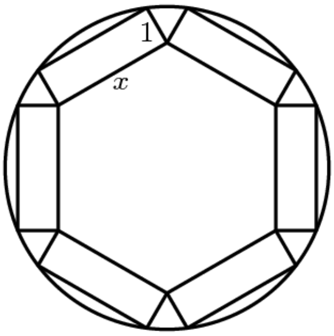
2008 AMC 12A Problems/Problem 22

The following problem is from both the 2008 AMC 12A #22 and 2008 AMC 10A #25, so both problems redirect to this page.

Contents

Problem

A round table has radius 4. Six rectangular place mats are placed on the table. Each place mat has width 1 and length x as shown. They are positioned so that each mat has two corners on the edge of the table, these two corners being end points of the same side of length x. Further, the mats are positioned so that the inner corners each touch an inner corner of an adjacent mat. What is x?

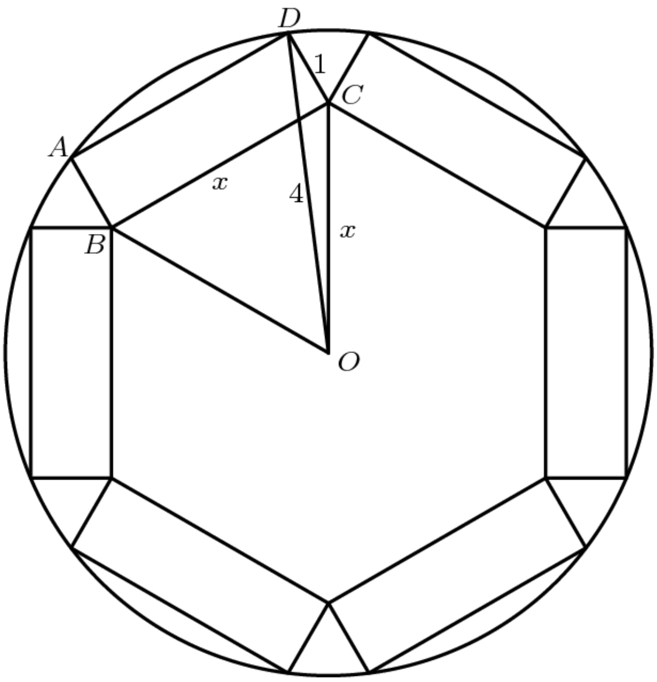


$\mathrm{(A)}\ 2\sqrt{5}-\sqrt{3}\qquad \mathrm{(B)}\ 3\qquad \mathrm{(C)}\ \frac{3\sqrt{7}-\sqrt{3}}{2}\qquad \mathrm{(D)}\ 2\sqrt{3}\qquad \mathrm{(E)}\ \frac{5+2\sqrt{3}}{2}$

Solution

Solution 1 (trigonometry)

Let one of the mats be $ABCD$, and the center be O as shown:



Since there are 6 mats, ΔBOC is equilateral. So, $BC = CO = x$. Also, $\angle OCD = \angle OCB + \angle BCD = 60^\circ + 90^\circ = 150^\circ$.

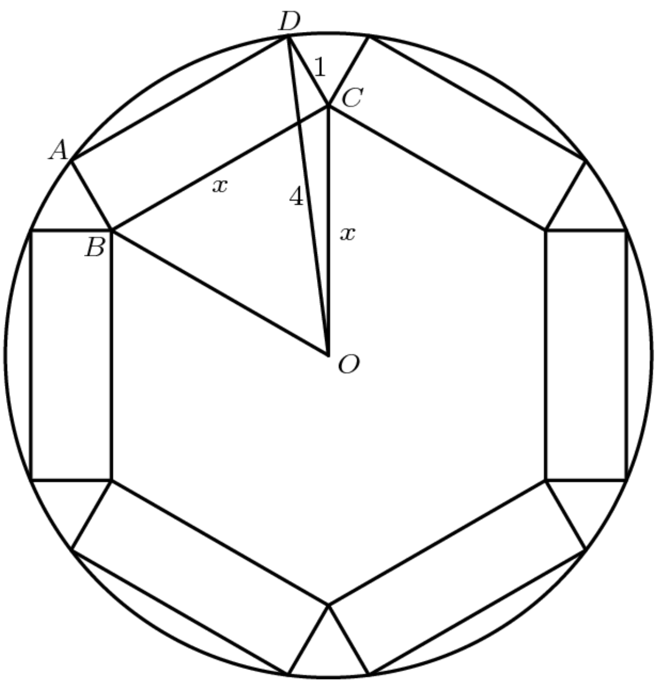
By the Law of Cosines:

$$4^2 = 1^2 + x^2 - 2 \cdot 1 \cdot x \cdot \cos(150^\circ) \Rightarrow x^2 + x\sqrt{3} - 15 = 0 \Rightarrow x = \frac{-\sqrt{3} \pm 3\sqrt{7}}{2}.$$

Since x must be positive, $x = \dfrac{3\sqrt{7} - \sqrt{3}}{2} \Rightarrow C$.

Solution 2 (without trigonometry)

Draw OD and OC as in the diagram. Draw the altitude from O to BC and call the intersection E



As proved in the first solution, $\angle OCD = 150^\circ$. That makes ΔOCE a $30-60-90$ triangle, so $OE = \frac{x}{2}$ and $CE = \frac{x\sqrt{3}}{2}$

Since ΔOED is a right triangle, $\left(\frac{x}{2}\right)^2 + \left(\frac{x\sqrt{3}}{2}+1\right)^2 = 4^2 \Rightarrow x^2 + x\sqrt{3} - 15 = 0$

Solving for x gives $x = \frac{3\sqrt{7}-\sqrt{3}}{2} \Rightarrow C$

See Also

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Categories: Introductory Geometry Problems | Introductory Trigonometry Problems

2008 AMC 12A Problems/Problem 23

Problem

The solutions of the equation $z^4 + 4z^3i - 6z^2 - 4zi - i = 0$ are the vertices of a convex polygon in the complex plane. What is the area of the polygon?

$\mathrm{(A)}\ 2^{\frac{5}{8}} \qquad \mathrm{(B)}\ 2^{\frac{3}{4}} \qquad \mathrm{(C)}\ 2 \qquad \mathrm{(D)}\ 2^{\frac{5}{4}} \qquad \mathrm{(E)}\ 2^{\frac{3}{2}}$

Solution

Looking at the coefficients, we are immediately reminded of the binomial expansion of $(x+1)^4$.

Modifying this slightly, we can write the given equation as:

$$(z+i)^4 = 1 + i = 2^{\frac{1}{2}} \cdot \text{cis}\,\frac{\pi}{4}$$

We can apply a translation of $-i$ and a rotation of $-\frac{\pi}{4}$ (both operations preserve area) to simplify the problem:

$$z^4 = 2^{\frac{1}{2}}$$

Because the roots of this equation are created by rotating $\frac{\pi}{2}$ radians successively about the origin, the quadrilateral is a square.

We know that half the diagonal length of the square is $\left(2^{\frac{1}{2}}\right)^{\frac{1}{4}} = 2^{\frac{1}{8}}$

Therefore, the area of the square is $\frac{\left(2 \cdot 2^{\frac{1}{8}}\right)^2}{2} = \frac{2^{\frac{9}{4}}}{2} = 2^{\frac{5}{4}} \Rightarrow D.$

See Also

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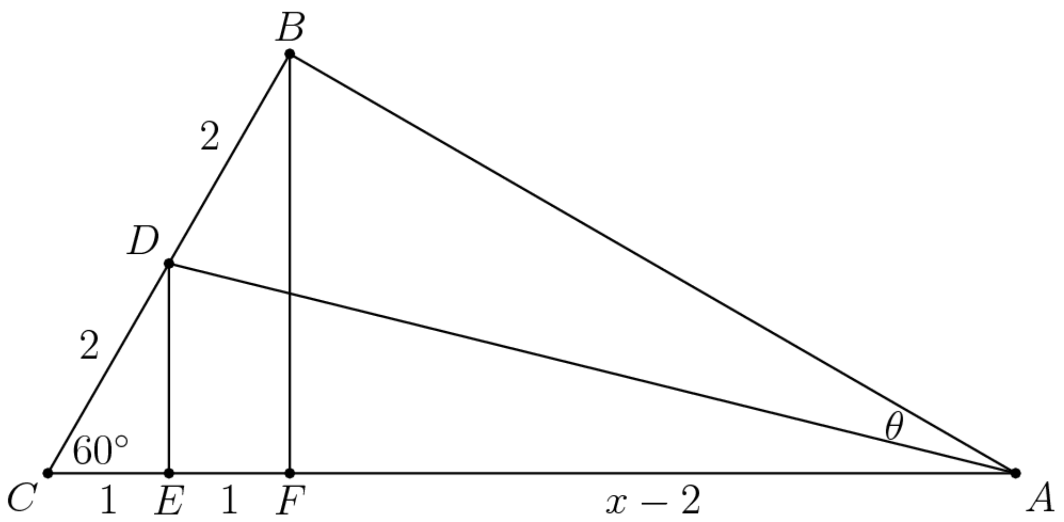
2008 AMC 12A Problems/Problem 24

Problem

Triangle ABC has $\angle C = 60^{\circ}$ and $BC = 4$. Point D is the midpoint of BC. What is the largest possible value of $\tan \angle BAD$?

$\mathrm{(A)}\ \frac{\sqrt{3}}{6} \qquad \mathrm{(B)}\ \frac{\sqrt{3}}{3} \qquad \mathrm{(C)}\ \frac{\sqrt{3}}{2\sqrt{2}} \qquad \mathrm{(D)}\ \frac{\sqrt{3}}{4\sqrt{2}-3} \qquad \mathrm{(E)}\ 1$

Solution



Let $x = CA$. Then $\tan\theta = \tan(\angle BAF - \angle DAE)$, and since $\tan\angle BAF = \frac{2\sqrt{3}}{x-2}$ and $\tan\angle DAE = \frac{\sqrt{3}}{x-1}$, we have

$$\tan\theta = \frac{\frac{2\sqrt{3}}{x-2} - \frac{\sqrt{3}}{x-1}}{1 + \frac{2\sqrt{3}}{x-2}\cdot\frac{\sqrt{3}}{x-1}} = \frac{x\sqrt{3}}{x^2-3x+8}$$

With calculus, taking the derivative and setting equal to zero will give the maximum value of $\tan\theta$. Otherwise, we can apply AM-GM:

$$\frac{x^2 - 3x + 8}{x} = \left(x + \frac{8}{x}\right) - 3 \geq 2\sqrt{x \cdot \frac{8}{x}} - 3 = 4\sqrt{2} - 3$$

$$\frac{x}{x^2 - 3x + 8} \leq \frac{1}{4\sqrt{2}-3}$$

$$\frac{x\sqrt{3}}{x^2 - 3x + 8} = \tan\theta \leq \frac{\sqrt{3}}{4\sqrt{2}-3}$$

Thus, the maximum is at $\frac{\sqrt{3}}{4\sqrt{2}-3} \Rightarrow \mathrm{(D)}$.

2008 AMC 12A Problems/Problem 25

Problem

A sequence (a_1, b_1), (a_2, b_2), (a_3, b_3), $\ldots$ of points in the coordinate plane satisfies

$(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n)$ for $n = 1, 2, 3, \ldots$.

Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?

$\mathrm{(A)}\ -\frac{1}{2^{97}} \qquad \mathrm{(B)}\ -\frac{1}{2^{99}} \qquad \mathrm{(C)}\ 0 \qquad \mathrm{(D)}\ \frac{1}{2^{98}} \qquad \mathrm{(E)}\ \frac{1}{2^{96}}$

Solution

This sequence can also be expressed using matrix multiplication as follows:

$$\left[\begin{array}{c} a_{n+1} \\ b_{n+1} \end{array}\right] = \left[\begin{array}{cc} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{array}\right] \left[\begin{array}{c} a_n \\ b_n \end{array}\right] = 2\left[\begin{array}{cc} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{array}\right] \left[\begin{array}{c} a_n \\ b_n \end{array}\right].$$

Thus, (a_{n+1}, b_{n+1}) is formed by rotating (a_n, b_n) counter-clockwise about the origin by 30° and dilating the point's position with respect to the origin by a factor of 2.

So, starting with (a_{100}, b_{100}) and performing the above operations 99 times in reverse yields (a_1, b_1).

Rotating $(2, 4)$ clockwise by $99 \cdot 30^\circ \equiv 90^\circ$ yields $(4, -2)$. A dilation by a factor of $\frac{1}{2^{99}}$ yields the point $(a_1, b_1) = \left(\frac{4}{2^{99}}, -\frac{2}{2^{99}}\right) = \left(\frac{1}{2^{97}}, -\frac{1}{2^{98}}\right)$.

Therefore, $a_1 + b_1 = \frac{1}{2^{97}} - \frac{1}{2^{98}} = \frac{1}{2^{98}} \Rightarrow D$.

Shortcut: no answer has 3 in the denominator. So the point cannot have orientation $(2, 4)$ or $(-2, -4)$. Also there are no negative answers. Any other non-multiple of 90° rotation of $30n^\circ$ would result in the need of radicals. So either it has orientation $(4, -2)$ or $(-4, 2)$. Both answers add up to 2. Thus, $2/2^{99} = \boxed{\textbf{(D)}\ \frac{1}{2^{98}}}$.

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