Keys and Solutions

Chapter VI

Algebra
Key and Solutions

Problems within the text are ordered with the last three digits reversed. This way, if you are looking for a solution you will not accidently see the answer to the next problem. Problems marked with a do not include a written solution, only the answer.

1.001 $(x-5)(x-4)$
1.010 6hrs
1.011 $(x^2+9)(x+3)\cdot(x-3)$
1.020 -1
1.021 $(8x+1)(2x-5)$
1.030 7.5
1.031 $6\sqrt{3}$
1.040 45
1.041 $4n^{-8}$ or $4/n^8$
1.050 6
1.051 10
1.061 1,000,000
1.070 $7a-8$
1.071 1
1.080 $a^2-4a-21$
1.081 $2\sqrt{34}$
1.090 34cm^2
1.091 1/18
1.100 $x=5$
1.101 $(x-28)(x+3)$
1.110 15 problems
1.111 $(x+2)(x-2)\cdot(x+1)(x-1)$
1.120 16/15
1.121 $4\sqrt{3}$
1.130 4
1.131 46/3cm
1.140 65
1.141 1/4
1.150 18
1.151 97
1.161 159
1.170 $3x-3$
1.171 $\sqrt{14}-3$
1.180 $y^2-22y+121$
1.181 140
1.190 21
1.191 677
1.200 $x=-5/3$
1.201 $(3x+1)^2$
1.210 15 minutes
1.211 $(a-3b)^2$
1.220 $2x-3y=5$
1.221 5
1.230 401 years
1.231 36ft
1.240 10
1.241 $x=625$
1.250 17/21
1.251 16
1.261 55/9
1.270 $7a-4b$
1.271 25
1.280 $4x^2-9$
1.281 600 miles.
1.290 7
1.291 143 minutes.
1.300 $x=5$
1.301 $(7x-9)^2$
1.310 9
1.311 $5(c-3)(c-2)$
1.320 $-1/2$
1.321 (see solutions)
1.330 340lbs
1.331 24ft
1.340 37.5%
1.341 $4\sqrt[3]{9}$ cm
1.350 $56\pi\ \text{cm}^2$
1.351 11
1.361 33
1.370 $6a^2-17a+12$
1.371 $(3+\sqrt{13})/2$
1.380 $6x^2-13x-5$
1.381 9
1.390 12cm
1.391 168 feet
1.400 $x=4$
1.401 $(11x+1)(11x-1)$
1.410 -36

1.420 $b=-1$

1.421 $x=13$

1.430 10sec

1.431 1.5in

1.440 \$1.00

1.441 $5\sqrt{2}-1$

1.450 30

1.451 62

1.461 4

1.470 $8x^2+23x-3$

1.471 $(a^{n+1}-1)/(a-1)$

1.480 $4a^2+4ab+b^2$

1.481 7/3

1.490 2,000

1.491 22.5

1.500 $x=15$

1.501 $(5x+3)(5x-3)$

1.510 48cm^2

1.520 108 units^2

1.521 4.5cm

1.530 8,160

1.531 27min

1.540 $162°$

1.541 $x=11$

1.550 25.2 mph

1.551 119

1.561 35

1.570 $8a^2-3a+14ab-6b-4b^2$

1.571 $(1+\sqrt{7})/2$

1.580 $a^3+9a^2-4a-36$

1.581 5

1.600 $x=7$

1.601 $8xy(x-3)$

1.610 30 miles

1.620 $b=4$

1.621 16

1.630 \$6.00

1.631 7/9

1.640 74.5

1.641 64

1.650 $y=-2x+9$

1.651 8

1.661 19

1.671 144

1.681 12

1.701 $2(x-3)(x-4)$

1.710 2cm

1.720 $9x-4y=-72$

1.721 6/7

1.730 \$4,500

1.731 1

1.741 72cm^3

1.750 6

1.751 92

1.761 1,023/1,024

1.771 $\sqrt{2}$

1.781 3:40pm

1.801 $4(x+3)(x-3)$

1.810 15min

1.820 10

1.821 $3\cdot 5^2\cdot 11\cdot 31\cdot 41$

1.841 $9+5\sqrt{3}$

1.851 3

1.861 2/3

1.881 72 min

1.900 6hrs 45min

1.901 $7(a-2)^2$

1.910 1hr 15min

1.920 10

1.921 $38+17\sqrt{5}$

1.941 400

1.981 $5\sqrt{3}$

1.010 Intuitively, there are $\frac{15}{9}=\frac{5}{3}$ as many beavers doing twice the work. It will take $\frac{3}{5}\cdot 2=\frac{6}{5}$ the time, or 6 hours. Alternatively, using $w=rt(b)$ to solve for the rate we have $1=5(9)r$ so $r=\frac{1}{45}$. Plug this in and solve for t in the equation $2=\frac{1}{45}(15)t$ and we get $t=$ **6** hours.

1.011 $x^4-81=(x^2+9)(x^2-9)=(x^2+9)(x+3)(x-3)$.

1.020 Using the slope formula: $m=\frac{y_2-y_1}{x_2-x_1}=\frac{0-9}{0-(-9)}=-1$

1.021 For $16x^2-38x-5$, we multiply *ac* to get $-80x^2$ and look for factors which have a sum of $-38x$. Use $-40x$ and $2x$ to rewrite the expression then factor by grouping:

$$16x^2-40x+2x-5$$
$$=8x(2x-5)+1(2x-5)$$
$$=(8x+1)(2x-5)$$

1.030 Solve the system of equations by elimination:

$$\begin{array}{r} a+b=20 \\ +\ a-b=5 \\ \hline 2a \quad\ =25 \end{array}$$

, which makes $a=12.5$. Substituting this back into either equation we get $b=$ **7.5** (which is the smaller of the two numbers).

1.031 We have the equations: $a+b=12$ and $ab=9$. Solve the first equation for a: $a=12-b$ and substitute this into our second equation to get $(12-b)b=9$ which gives us the quadratic: $b^2-12b+9=0$. It is not factorable, so we substitute values into the quadratic formula and solve:

$$x=\frac{12\pm\sqrt{144-4(9)}}{2}=\frac{12\pm\sqrt{108}}{2}=\frac{12\pm6\sqrt{3}}{2}=6\pm3\sqrt{3},$$

giving us two roots whose difference is $(6+3\sqrt{3})-(6-3\sqrt{3})=6\sqrt{3}$.

1.040 The ratio of sugar to cookies must remain the same, so we write a proportion:

$\frac{2\frac{2}{3}}{24}=\frac{5}{c}$ gives us $\frac{8}{3}c=5(24)$, so $c=$ **45** cookies.

1.041 $\left[2(n^2)^{-2}\right]^2=\left[2(n^{-4})\right]^2=4n^{-8}$ or $\frac{4}{n^8}$.

1.050 Intuitively, it takes 6 lumberjacks 7 minutes, or 42 "lumberjack minutes" to saw through 8 trees. If there are 7 lumberjacks, it will take them 42/7 = **6** minutes to saw through 8 trees.

1.051 $\frac{1+3+5+7+9+11+13+15+17+19}{10}=\mathbf{10}.$

The mean of the first *n* positive odd integers is *n*.

1.061 The average of the first 1,000 odd integers is 1,000, the same as the median (see **p.53**). The first term is 1 and the last term is 1 + 999(2) = 1,999, so the median is (1 + 1,999)/2 = 1,000. There are 1,000 integers, so their sum is $1{,}000^2$ = **1,000,000**. The sum of the first *n* odd integers for any value *n* is n^2.

1.070 $3a-4(2-a)=3a-8+4a=7a-8$

1.071 Let $x = 0.\bar{9}$, then $10x = 9.\bar{9}$ and $10x - x = 9.\bar{9} - 0.\bar{9}$, making $9x = 9$ and $x = $ **1**.

1.081 We have the equations: $a + b = 16$ and $ab = 30$. Solve the first equation for a: $a = 16 - b$ and substitute this into our second equation to get $(16 - b)b = 30$ which gives us the quadratic: $b^2 - 16b + 30 = 0$. It is not factorable, so we substitute values into the quadratic formula and solve:

$$x = \frac{16 \pm \sqrt{256 - 4(30)}}{2} = \frac{16 \pm \sqrt{136}}{2} = \frac{16 \pm 2\sqrt{34}}{2} = 8 \pm \sqrt{34},$$

giving us two roots whose difference is $(8 + \sqrt{34}) - (8 - \sqrt{34}) = 2\sqrt{34}$.

1.090 We have $s^2 = 50$ and we are looking for $(s + 4)(s - 4)$, which is a difference of squares equal to $s^2 - 16$. We know the value of $s^2 = 50$, so $s^2 - 16 = 50 - 16 =$ **34cm²**.

1.091 We use s to represent the amount of sugar and j to represent the amount of juice used in the original recipe. When the sugar is doubled, we have:

$\dfrac{2s}{2s + l} = \dfrac{10}{100}$ (the ratio of the sugar to the total drink is 10%). Solving, we get $20s = 2s + l$, which makes $18s = l$, or $18 = \dfrac{l}{s}$ and $\dfrac{s}{l} = \dfrac{1}{18}$.

1.110 We use $w = rt(s)$ to find the rate: $9 = 30(4)r$ gives us $r = \dfrac{3}{40}$. Plug this in with the new values to solve for the work (problems) done by 5 students in 40 minutes: $w = \dfrac{3}{40} \cdot 40 \cdot 5 =$ **15** problems.

1.111 The equation $x^4 - 5x^2 + 4$ factors into $(x^2 - 4)(x^2 - 1)$. Each factor is a difference of squares and can be factored further:
$(x^2 - 4)(x^2 - 1) = (x+2)(x-2)(x+1)(x-1)$.

1.120 The intercepts of $3x - 5y = 8$ occur when $x = 0$ and when $y = 0$. When $x = 0$ we have $-5y = 8$, so the y-intercept occurs at $y = -\frac{8}{5}$. When $y = 0$ we have $3x = 8$, so the x-intercept occurs at $x = \frac{8}{3}$. The sum of the intercepts is: $-\frac{8}{5} + \frac{8}{3} = \frac{16}{15}$.

1.121 If we label the sides of the rectangle a and b, we have perimeter $2(a+b) = 16$ so $a + b = 8$, and the area of the rectangle is $ab = 8$. By the Pythagorean theorem, the diagonal length we are looking for will be $\sqrt{a^2 + b^2}$. We do not have to solve for a and b separately once we recognize that:
$a^2 + b^2 = (a+b)^2 - 2ab = (8)^2 - 2(8) = 48$, which makes the diagonal length $\sqrt{48} = 4\sqrt{3}$.

1.130 This one we can do without writing a system in our heads using the 'cheat' method: If she had 18 dimes, this would be \$1.80, which is \$0.60 short of the correct amount. Each time we replace a dime with a quarter, we add \$0.15, so we need to replace 4 dimes with **4** quarters.

1.131 If we call the width w, the length is $3w + 1$ and the area equation becomes $w(3w+1) = 10$, or
$3w^2 + w - 10 = 0$. This can be factored into
$(3w-5)(w+2) = 0$ which gives us $w = \frac{5}{3}$ (-2 does not make sense as the side length of a rectangle).
The perimeter is therefore:
$2w + 2(3w+1) = 8w + 2 = 8 \cdot \frac{5}{3} + 2 = \frac{46}{3}$ cm.

1.140 We can write the following equations based on the information given, where c and g represent the number of cherry and grape candies after your friend eats some of them:

$\frac{c+1}{2g}=\frac{3}{5}$ and $\frac{c}{g}=\frac{7}{6}$. These equations become:

$5c+5=6g$ and $6c=7g$. Solve this system of equations to get $c = 35$ and $g = 30$, so there are **65** candies left in the bag (there were 96 to begin with).

1.141 $\sqrt{\frac{13}{56}}\cdot\sqrt{\frac{7}{26}}=\sqrt{\frac{13\cdot 7}{56\cdot 26}}=\sqrt{\frac{1\cdot 1}{8\cdot 2}}=\sqrt{\frac{1}{16}}=\frac{1}{4}$

1.150 We are given that $\frac{p}{a}=\frac{120}{100}=\frac{6}{5}$ and $\frac{b}{a}=\frac{4}{3}$. We want whole numbers for p, a, and b in $p{:}a{:}b$, so we set a to 15 (the LCM of 3 and 15), which gives us $\frac{p}{a}=\frac{18}{15}$ and $\frac{b}{a}=\frac{20}{15}$. Therefore $p{:}a{:}b$ = 18:15:20, and the smallest possible number of pears is **18**.

1.151 The sum of his first 7 test scores is 7(88) = 616, and to have an average of 90 for 9 tests requires a sum of 9(90) = 810. This requires 810 – 616 = 194 more points in 2 tests, or an average of 194/2 = **97**. Alternatively, we need to add 2 points to an average which is spread over 9 tests. This requires a score 9(2)/2 = 9 points higher than his current average: $88+9=97$.

1.161 The common difference is found by $\frac{24-18}{9-5}=\frac{6}{4}=\frac{3}{2}$. We must add this common difference 90 times to get from the 9th to the 99th term: $24+90\cdot\frac{3}{2}=\mathbf{159}$.

1.170 $8x-(3+5x)=8x-3-5x=3x-3$.

1.171 For $x=\cfrac{5}{6+\cfrac{5}{6+\cfrac{5}{6+\ldots}}}$ we have $x=\frac{5}{6+x}$ or

$x(6+x)=5$. Solve the quadratic $x^2+6x-5=0$ to get $x=-3\pm\sqrt{14}$. Only the positive solution makes sense: $x=\sqrt{14}-3$.

1.181 We will express joogs and zoogs in terms of 15 moogs. 15 moogs = 9 zoogs and 15 moogs = 35 joogs, so 9 zoogs = 35 joogs. 36 zoogs would therefore be worth **140** joogs.

1.190 For a and b we have $a + b = 5$ and $ab = 2$, and we are looking for $a^2 + b^2$.

$a^2+b^2=(a+b)^2-2ab$, so $(5)^2-2(2)=\mathbf{21}$.

1.191 First we note that a sum of squares a^2+b^2 can be written as $(a+b)^2-2ab$. In this case, we get a difference of squares:

$3^{12}+2^2=(3^6+2)^2-2(2)(3^6)=(3^6+2)^2-2^2 3^6$.

Consider $a=3^6+2$ and $b=2\cdot 3^3$,

$(3^6+2)^2-2^2 3^6=a^2-b^2=(a+b)(a-b)$.

$a=731$ and $b=54$, so we have

$(731+54)(731-54)=785\cdot 677=5\cdot 157\cdot 677$.

677 is the largest prime factor. This is a hard problem, don't worry too much if you missed it.

1.210 Working together for 12 minutes, Jeremy and Michael can solve 8 cubes (3 by Jeremy who take 4 minutes per cube and 5 by Michael). This is 1:30 per cube while working together. It would therefore take them **15** minutes to solve 10 cubes.

1.211 This is a perfect square. $a^2-6ab+9b^2=(a-3b)^2$.

1.220 First, find the slope: $\dfrac{-5-(-1)}{-5-1}=\dfrac{-4}{-6}=\dfrac{2}{3}$.

We will use point-slope form (there are many other ways which work well). I used the first point.

$y+1=\dfrac{2}{3}(x-1)$ Multiply the equation by 3 to get

$3y+3=2x-2$, so $2x-3y=5$.

1.221 We are given $a^2-b^2=80$, and $a+b=16$ and asked for $a-b$: $a^2-b^2=(a+b)(a-b)=(16)(a-b)=80$, so

$a-b=\dfrac{80}{16}=\mathbf{5}$.

1.230 Label the age (in 2005) of the first painting a and the age of the second painting b. The statements can be written algebraically: $2005-a=2(2005-b)$ and $b=3a$. Substituting $3a$ for b in the first equation, we get $2005-a=2(2005-3a)$. Solving, we get $a=$ **401** years. (The ages are 401 and 1,203, the dates are 1604 and 802).

1.231 Labeled as shown, we can write the equations: $4x+2y=48$ and $xy=72$. Solve the first equation for y to get $y=24-2x$, which we substitute into the second equation to get $x(24-2x)=72$:

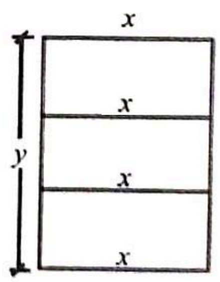
$24x-2x^2=72$

$12x-x^2=36$

$0=x^2-12x+36$

$0=(x-6)^2$

$x=6$



This make y = 12 and the perimeter of the enclosure is **36** feet.

1.240 Starting with 100 pounds of plums, there are 92 pounds of water (92%) and 8 pounds of pulp. After drying, the prunes still contain 8 pounds of pulp. If a prune is 20% water, then 8 pounds of pulp represents 80% of the mass, so the prunes weigh **10** pounds.

1.241 $\sqrt[6]{x} = \sqrt[3]{5^2}$ means the same thing as $x^{\frac{1}{6}} = 5^{\frac{2}{3}}$, so

$\left(x^{\frac{1}{6}}\right)^6 = \left(5^{\frac{2}{3}}\right)^6$ gives us $x = 5^4 = \mathbf{625}$.

1.250 5% more than a is b means $1.05a = b$, and b is 15% less than c can be written $b = 0.85c$. This leads us

to $1.05a = 0.85c$, or $21a = 17c$, so $\dfrac{a}{c} = \dfrac{17}{21}$.

1.251 Start with 7 integers labeled in order *a*, *b*, *c*, *d*, *e*, *f*, and *g*. We are given the median: $d = 8$. The distinct mode of 9 means that *e* and *f* must be 9 (or *e*, *f*, and *g* could all be 9). The mean gives us the sum: 49. To maximize the range we want *a*, *b*, and *c* to be as small as possible, allowing *f* to be as large as possible while keeping the sum equal to 49. Make *a*, *b*, and *c* equal to 1, 2, and 3. This makes the set (whose sum is 49) 1, 2, 3, 8, 9, 9, and *f*.

$1+2+3+8+9+9+f = 49$, so $f = 17$ and the range is $17 - 1 = \mathbf{16}$.

1.261 Call the first three terms *a*, *b*, and *c*, and the next three terms *d*, *e*, and *f*. We note that *b* is the average of the first three terms: 20/3, and *e* is the average of the next three terms: 25/3. The common difference can be found using *b* and *e* (the 2nd and 5th terms):

$\dfrac{25/3 - 20/3}{5-2} = \dfrac{5/3}{3} = \dfrac{5}{9}$. Subtract this from *b* to find

the first term *a*: $\dfrac{20}{3} - \dfrac{5}{9} = \dfrac{55}{9}$.

1.270 $(3a-b)-(3b-4a)=3a-b-3b+4a=7a-4b$.

1.271 Forget the first 20 momentarily and just let $x=\sqrt{20+\sqrt{20+\sqrt{20...}}}$, then $x^2=20+x$, so $x^2-x-20=0$. This factors into $(x-5)(x+4)=0$, so x = 5 (it is assumed that we are looking for the positive square root). Add the 20 from the beginning of the expression and we get **25**.

1.281 Use $d = rt$ to write two equations: $d=3(r+25)$ and $d=4(r-25)$ which gives us $3(r+25)=4(r-25)$. Solving for r we get r = 175 (mph). Plug this into either of the first two equations to get d = **600** miles.

1.290 Note that $ab + bc = b(a + c)$, so $7 + 5 = b(4)$. Therefore, $b = 3$ and $a + b + c = (a + c) + b = 4 + 3 =$ **7**.

1.291 We can use the harmonic mean of 11 and 13 minutes to find the time it takes for Rohan and Lewis to both chop 6 pounds of carrots (12 pounds total, working together). We then multiply that value by 12 (because they are chopping 144 pounds):

$$\frac{2(11)(13)}{11+13}\cdot 12=\frac{2(11)(13)}{24}\cdot 12=\mathbf{143}\text{ minutes.}$$

1.310 Start with the equation $j=3m-3$. Plug-in 33 for j and we solve to get m = 12. A nice thing to know is that when James is twice Molly's age, she will be the age he was when she was born. The difference in their ages gives us the age James was when Molly was born: 21. When James is twice Molly's age he will be 42, which is **9** years from now.

1.311 I generally re-arrange these first: $30-25c+5c^2=$ $5c^2-25c+30=5(c^2-5c+6)=5(c-3)(c-2)$.

1.320 The graph of $4x-2y=3$ has a slope of $\frac{-4}{-2}=2$, and the perpendicular slope is $-\frac{1}{2}$.

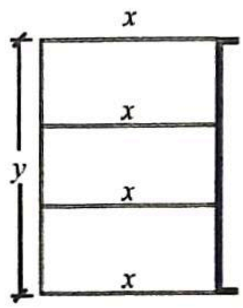
1.321 $256x^8 - 1 = (16x^4 + 1)(16x^4 - 1)$
$$= (16x^4 + 1)(4x^2 + 1)(4x^2 - 1)$$
$$= (16x^4 + 1)(4x^2 + 1)(2x + 1)(2x - 1)$$

1.330 Take the four equations and add them together:

$$\begin{array}{r} a+b+c=240 \\ b+c+d=250 \\ a+c+d=260 \\ +\ a+b+d=270 \\ \hline 3a+3b+3c+3d=1{,}020 \end{array}$$

Divide both sides of the equation by 3 to get $a + b + c + d =$ **340** pounds.

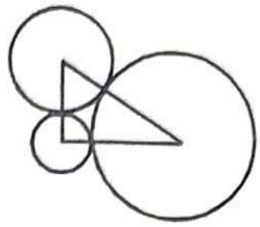
1.331 Labeled as shown, we can write the equations: $4x + y = 48$ and $xy = 144$. Solve the first equation for y to get $y = 48 - 4x$, which we substitute into the second equation to get $x(48 - 4x) = 144$, which gives us $48x - 4x^2 = 144$, or $4x^2 - 48x + 144 = 0$. Simplify this and factor: $x^2 - 12x + 36 = 0$, so $(x-6)^2 = 0$. The solution $x = 6$ makes $y =$ **24 feet.**



1.340 The prices are currently 80% of their original value, so we look for a value x where $0.8x = 0.5$. This gives us $x = 5/8 = 0.625$ or 62.5% of the already marked-down prices, which is an additional **37.5%** off.

1.341 If we can triple the volume by multiplying the edge length by $\sqrt[3]{3}$, then to get one-third the volume, we will need to divide the edge length by $\sqrt[3]{3}$: $\frac{12}{\sqrt[3]{3}}$. To rationalize the denominator, we multiply by a cube root that will make $\sqrt[3]{3}$ a perfect cube:

$$\frac{12}{\sqrt[3]{3}} \cdot \frac{\sqrt[3]{9}}{\sqrt[3]{9}} = \frac{12\sqrt[3]{9}}{\sqrt[3]{27}} = \frac{12\sqrt[3]{9}}{3} = 4\sqrt[3]{9}\text{ cm (about 8.3cm).}$$



1.350 Call the three radii a, b, and c. The short leg of the triangle is therefore $a + b$, the other leg $a + c$ and the hypotenuse $b + c$. This gives us the system of three equations:

$a+b=6$, $a+c=8$, and $b+c=10$.

Subtract the second from the first to get $b-c=-2$. Add this to the third equation to get $2b = 8$, so $b = 4$. Substitute this value into the other equations to find $a = 2$ and $c = 6$. The combined area of the three circles is therefore $4\pi + 16\pi + 36\pi = 56\pi$ units^2.

1.351 The current set {3, 4, 5, 5, 8} has a mean, median, and mode of 5. Of the four integers added to the set, at least three must be 6's to make the mode 6. The mean of the new set must be 6 as well, which means the sum of the integers in the set must be 54. The 7 integers we know already have a sum of 3 + 4 + 5 + 5 + 6 + 6 + 6 + 8 = 43. This means that the fourth (and largest) number added to the set must be $54 - 43 =$ **11**. The new set: {3, 4, 5, 5, 6, 6, 6, 8, 11}.

1.361 The arithmetic sequence is easy enough to just count up to... 16, 24, 32, 40, <u>48</u> is the 5th term. The common ratio in the geometric sequence is 3/2, so the 5th term will be $16\left(\frac{3}{2}\right)^4 = \underline{81}$. (Perhaps we could have also counted to this one). $81 - 48 =$ **33**.

1.371 For $x = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \ldots}}}$ we have $x = 3 + \frac{1}{x}$ or

$x - 3 = \frac{1}{x}$, so $x^2 - 3x = 1$ or $x^2 - 3x - 1 = 0$. The

positive solution to this quadratic is $x = \frac{3 + \sqrt{13}}{2}$.

1.381 We want to add n such that the mean of $\{6, 11, 13, 7, 14, n\}$ is $n + 1$:

$$\frac{6+11+13+7+14+n}{6} = n+1, \text{ so } 51+n = 6n+6.$$

Solving we get $n =$ **9**.

1.390 Given the area, we have $(x-3)(x+3) = 216$ or $x^2 - 9 = 216$, so $x = \sqrt{225} = 15$ and the short side is $15 - 3 =$ **12cm** long.

1.391 We are looking for the sum of the terms: 24 + 18 + 18 + 13.5 + 13.5 + ... which resembles an infinite geometric series (except that all but the first term is repeated). Remove that first term (for now) and solve for 36 + 27 + 20.25 ... which is an infinite geometric series with common ratio 3/4 and first term 36. Use the formula (p. 56) to get the total distance (less the initial drop): $d = \dfrac{36}{1-0.75} = 144\,\text{ft}$, then add back the 24-foot initial drop to get **168** feet.

1.410 Even integers have a positive difference of 2, so four consecutive even integers can be labeled n, $(n + 2)$, $(n + 4)$, and $(n + 6)$. This allows us to write the equation: $n + (n+2) + (n+4) + (n+6) = 3n$. Solving, we get $4n + 12 = 3n$, so $n = -12$, and the sum of the integers is $3(-12) =$ **-36**.

1.420 Given the point $(b,-3)$, we can substitute these two values into the equation $y = 2x + b$, giving us $-3 = 2(b) + b$. Solve for b to get $b =$ **-1**.

1.421 In the equation $\dfrac{2x^2 - 5x - 12}{2x+3} = 9$ we can factor the numerator to get $\dfrac{(2x+3)(x-4)}{2x+3} = 9$ which simplifies to $x - 4 = 9$, so $x =$ **13**.

1.430 We will call Kenny's running speed in flights per minute k and the speed of the escalator (also in flights per minute) e. When Kenny runs up the down escalator, we have $k - e = 2$ (the two here represents 2 flights of stairs per minute because it takes 30 seconds). When Kenny runs up the up escalator we have $k + e = 10$. Adding these two equations we get $2k = 12$, so Kenny's speed alone is 12/2 = 6 flights per minute, or **10 seconds** per flight of stairs.

1.431 If the width of the photograph is 4.25, then the width of the frame is $4.25 + 2x$ and the height is $7.75 + 2x$. We are given the area of the border: 45. This gives us the equation solved below:

$$\begin{aligned}(4.25 + 2x)(7.75 + 2x) - (4.25)(7.75) &= 45\\ 32.9375 + 8.5x + 15.5x + 4x^2 - 32.9375 &= 45\\ 4x^2 + 24x &= 45\\ 4x^2 + 24x - 45 &= 0\\ (2x - 3)(2x + 15) &= 0\end{aligned}$$

We use the positive solution: x = **1.5 inches**.

1.440 Between them they have 100 ounces, so each gets 100/3 ounces. Harold has 120/3 ounces, so he gives Greg 20/3 ounces, while Jake has 180/3 ounces and must give Greg 80/3 ounces. Harold contributed 1/5 of the water that Greg drank, so he should get **$1** (which is1/5 of the money Greg gave them).

1.441 Rationalizing each denominator is key here:

$$\frac{1}{\sqrt{1}+\sqrt{2}} \cdot \frac{\sqrt{1}-\sqrt{2}}{\sqrt{1}-\sqrt{2}} = \frac{\sqrt{1}-\sqrt{2}}{-1} = \frac{\sqrt{2}-\sqrt{1}}{1}$$

and we are left with the sum of:

$$\frac{\sqrt{2}-\sqrt{1}}{1} + \frac{\sqrt{3}-\sqrt{2}}{1} + \frac{\sqrt{4}-\sqrt{3}}{1} + \ldots + \frac{\sqrt{50}-\sqrt{49}}{1}$$

$$= -\sqrt{1} + \left(\sqrt{2}-\sqrt{2}\right) + \left(\sqrt{3}-\sqrt{3}\right) + \ldots + \left(\sqrt{49}-\sqrt{49}\right) + \sqrt{50}$$

$$= \sqrt{50} - \sqrt{1} = 5\sqrt{2} - 1.$$

1.450 Using m to represent my current age and f to represent my father's current age, the first equation is:

$$m = \frac{1}{2}f - 6, \text{ or } 2m + 12 = f.$$

In six years, m becomes $(m + 6)$ and f becomes $(f + 6)$ and the second equation is:

$$m + 6 = \frac{1}{3}(f + 6) + 6, \text{ or } 3m - 6 = f.$$

Both equations were solved for f to allow us to substitute and solve $2m + 12 = 3m - 6$ to get $m = 18$, which makes $f = 48$. If my father is 30 years older than I am, he was **30** when I was born.

1.451 Intuitively, if a tenth score of 80 increases a quiz average by 2 points, we can consider taking 20 of these points and distributing 2 each to every score. This means the old average must have been 60 and the new average is therefore 62. For a more algebraic approach, call the old average a. This gives us the equation:

$\frac{9a + 80}{10} = a + 2$, so $9a + 80 = 10a + 20$, making his old average $a = 60$ and his new average is $60 + 2 = \mathbf{62}$.

1.461 First note that $2^0 = 1$ and $2^{10} = 1{,}024$. We can use a common ratio of 2^{10}, 2^5, 2^2, or 2^1 for a total of **4** geometric sequences:

1, 1,024.
1, 32, 1,024.
1, 4, 16, 64, 256, 1,024.
1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1,024.

1.471 Consider the sum $a^0 + a^1 + a^2 + a^3 ... + a^n$:
if $x = a^0 + a^1 + a^2 + a^3 ... + a^n$,
then $ax = a^1 + a^2 + a^3 + a^4 ... + a^{n+1}$, and
$ax - x = a^{n+1} - a^0$. Factor out the x on the left:
$x(a-1) = a^{n+1} - 1$ and divide by $(a-1)$ to get:

$$x = \frac{a^{n+1} - 1}{a - 1}.$$

1.481 If $3x + \frac{3}{x} = 5$, then $\left(3x + \frac{3}{x}\right)^2 = 25$, which gives us:

$9x^2 + 18 + \frac{9}{x^2} = 25$, so $9x^2 + \frac{9}{x^2} = 7$. Divide both

sides of this equation by 3 to get $3x^2 + \frac{3}{x^2} = \frac{7}{3}$.

1.490 For the sum to be as small as possible, we want the numbers to be as close together as possible. Note that $999{,}996 = 1{,}000{,}000 - 4 = (1{,}000 + 2)(1{,}000 - 2)$, so $1{,}002 \cdot 998 = 999{,}996$ and the smallest possible sum is 1,002 + 998 = **2,000.**

1.491 We can find a^2, b^2, and c^2 separately using the following:

$$a^2 = \frac{ab \cdot ac}{bc} = \frac{40}{5} = 8$$

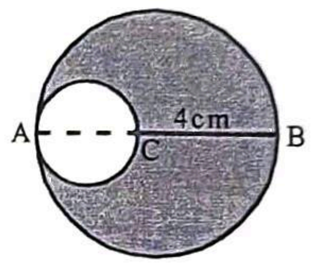
$$b^2 = \frac{ab \cdot bc}{ac} = \frac{20}{10} = 2$$

$$c^2 = \frac{bc \cdot ac}{ab} = \frac{50}{4} = 12.5 \text{ so } a^2 + b^2 + c^2 = \mathbf{22.5}$$

1.510 If we label the sides of the rectangle x and $3x$, the perimeter is $x + 3x + x + 3x = 8x = 32$, so $x = 4$. The sides are therefore 4 and 12, which gives an area of **48cm²**.

1.520 If we find the x and y-intercepts of the graph, we can use these as the height and base of the triangle bounded by both axis. The y-intercept is 12 or (0, 12) if you prefer. We find the x-intercept by setting y to zero and solving for x, which gives us 18 for the x-intercept. The area of the triangle is therefore: $A = \frac{1}{2} \cdot 12 \cdot 18 = \mathbf{108\ units^2}$.

1.521 Call the small radius r and the large radius R. $\pi(R^2 - r^2) = 9\pi$, so $R^2 - r^2 = 9$ which makes $(R+r)(R-r) = 9$. We notice that $(R-r) = 2$ (half the difference in the diameters), so $2(R+r) = 9$. This gives us the sum of the radii: $R + r = \mathbf{4.5}$.



1.530 We are given enough information to write three equations: $a + b = 37$, $b + c = 41$, and $a + c = 44$. Subtract the first from the second to get $c - a = 4$. Add this to $a + c = 44$ and we get $2c = 48$, so $c = 24$. Plug-in to get $a = 20$ and $b = 17$. $abc = \mathbf{8{,}160}$.

1.531 Using $d=rt$, we have distance 5.4 miles, rate x mph, and time (in minutes) $x + 15$. Convert the time in minutes to hours by dividing by 60 and we have:

$$5.4 = x\left(\frac{x+15}{60}\right).$$

This leads us the quadratic $x^2 + 15x - 324 = 0$ and factors into $(x-12)(x+27) = 0$. The positive solution $x = 12$ means that he rode 12mph and took $12 + 15 = \mathbf{27}$ minutes to get to school.

1.540 The sum of the interior angles in a pentagon is 540° (**p.180**). Labeling the angles $2x$, $3x$, $4x$, $5x$, and $6x$ we have $2x+3x+4x+5x+6x=540$, so $20x = 540$ and $x = 27$. This makes the largest angle $6(27)$ = **162°**.

1.541

$$2^x = \left(4^{\frac{1}{3}} \cdot 16^{\frac{3}{4}}\right)^3$$

$$2^x = \left((2^2)^{\frac{1}{3}} \cdot (2^4)^{\frac{3}{4}}\right)^3$$

$$2^x = \left(2^{\frac{2}{3}} \cdot 2^3\right)^3$$

$$2^x = \left(2^{\frac{11}{3}}\right)^3 = 2^{11} \text{ so } x = \mathbf{11}.$$

1.550 Let's put the formula for harmonic mean to work here, using r to represent the rate in mph that Franklin must ride on the return trip:

$$\frac{2 \cdot 18 \cdot r}{18+r} = 21, \text{ so } 36r = 378 + 21r.$$

Solving for r we get **25.2 mph.**

1.551 This is a case where guess-and-check can save some time if we recognize that the integers can be 101, 103, 105, 107... until the 10th integer is 119. A more rigorous approach probably looks something like this: Call the integers a, b, c, d, ...i, j.

$a + b = 2(102) = 204$

$a + b + c = 3(103) = 309$, so $c = 105$.

$a + b + c + d = 4(104) = 416$, so $d = 107$.

... until

$a + b + c + d + ... + i = 9(109) = 981$, and

$a + b + c + d + ... + j = 10(110) = 1{,}100$, so j = **119**.

1.561 Her 12th birthday was exactly halfway between her 7th and 17th birthdays, so we only need to find the geometric mean of 5 and 245: $\sqrt{5 \cdot 245}$ = **35**.

1.571 Let $x = 1+\cfrac{3}{2+\cfrac{3}{1+\cfrac{3}{2+\cfrac{3}{1+\ldots}}}}$ Then: $x = 1+\cfrac{3}{2+\cfrac{3}{x}}$

$$= 1+\frac{3}{\left(\frac{2x+3}{x}\right)} = 1+\frac{3x}{2x+3}, \text{ so } x-1 = \frac{3x}{2x+3}, \text{ and}$$

$2x^2 + x - 3 = 3x$. Subtract $3x$ to get the quadratic

$2x^2 - 2x - 3 = 0$, which has solutions: $x = \dfrac{1 \pm \sqrt{7}}{2}$.

Only the positive solution makes sense, so:

$$x = \frac{1+\sqrt{7}}{2}.$$

1.581 To begin, write an equation to model the question:

$$\frac{a-b}{\frac{1}{b}-\frac{1}{a}} = 5 \text{ means that } \frac{a-b}{\left(\frac{a-b}{ab}\right)} = 5, \text{ so}$$

$a-b \cdot \dfrac{ab}{a-b} = 5$ giving us $ab = \mathbf{5}$.

1.610 We use $d = rt$ to write two equations, but we must be careful with the units. Anil's speed is given in miles per hour, so 5 minutes is converted to 1/12 hour.

On the highways: $d = 45t$

On the back roads: $d - 3 = 36\left(t + \dfrac{1}{12}\right)$

Convert the second equation into $d = 36t + 6$, which means that $45t = 36t + 6$ (because $d = d$). Solving for the time in hours gives us $t = \dfrac{2}{3}$ hr, which is equal to 40 minutes. Averaging 45mph on the highway route for 40 minutes, Anil will travel **30 miles**. This is really a system of equations as discussed in chapter 1.3.

1.620 Remember that the slope of a standard form linear equation is $-A/B$. The slope of $2x-5y=7$ is 2/5. We are looking for a value b for which $10x+by=7$ has a slope of $-5/2$:

$\dfrac{-10}{b}=\dfrac{-5}{2}$, so b = **4**.

1.621 If the sides of the rectangle are a and b, we are given that $ab=9$ and $\sqrt{a^2+b^2}=\sqrt{46}$. This gives us $a^2+b^2=46$. We are looking for $2(a+b)$. Notice that $(a+b)^2=a^2+b^2+2ab$, so $(a+b)^2=46+2(9)$ = 64. If $(a+b)^2 = 64$ then $a+b=8$ ($a+b$ cannot be negative for the side lengths of a rectangle). The perimeter is $2(a+b)$ = **16cm.**

1.630 This is more a puzzle than an algebra problem, but it does involve the concept of substitution. Consider the following method of purchasing the letters used to spell TWELVE: Buy the words TWO and ELEVEN, then sell back the letters you don't need: O, N, and E (which happen to spell ONE!). \$3 + \$5 minus \$2 = **\$6** to buy TWELVE.

1.631 The sum of the roots can be found by $\dfrac{-b}{a}=\dfrac{7}{9}$.

1.640 If her time decreases by 2% six times, her new time will be $(0.98)^6$ times what her original time was. This means that if x is her original time, then

$x(0.98)^6=66$, so $x=\dfrac{66}{(0.98)^6}\approx$ **74.5** seconds.

1.641 We need the smallest integer for which $n^{\frac{1}{3}}$ and $n^{\frac{1}{4}}$ are both integers, so we look for the smallest possible value of a^{12}, in which case $\left(a^{12}\right)^{\frac{1}{3}} = a^4$ and $\left(a^{12}\right)^{\frac{1}{4}} = a^3$. If we use $a = 2$ we have $\left(2^{12}\right)^{\frac{1}{3}} = 16$, $\left(2^{12}\right)^{\frac{1}{4}} = 8$, and $\sqrt{2^{12}} = 2^6 = \mathbf{64}$.

1.650 The line which passes through the two points of intersection is the perpendicular bisector of the segment which connects the centers of the circles (this is easiest to see by connecting all four points to make a rhombus, in which the diagonals are perpendicular bisectors, see **p.183**). The slope between the centers of the circles is 1/2, so the perpendicular slope is -2. The midpoint is (3,3). (Average the x and y coordinates to get the midpoint.) The equation of the line which passes through (3,3) with a slope of -2 is $y = -2x + 9$.

1.651 Intuitively, the difference between 19 and 35 points is 16 points. If these points are evenly distributed among the games she has played there will be a 2-point difference in her average score, meaning the 16 points would be evenly distributed between **8** games. Algebraically, call t the total number of points she has scored so far this season and g the total number of games in a season. If we add 19 to her total we get an average of 18 points per game means: $t + 19 = 18g$, while adding 35 to her total makes her average 20 points per game: $t + 35 = 20g$. Solving this system of equations also gives us g = **8 games**.

1.661 After some quick guessing-and-checking, we see that the common ratio must certainly be less than 2 (otherwise, the long side is always greater than the sum of the two shorter sides, as in 1-2-4). Using a common ratio of 3/2 allows us to use 4-6-9 as the lengths of the sides, in which case the perimeter of the triangle is 4 + 6 + 9 = **19** units.

1.671 Let $x = 2\sqrt{3\sqrt{2\sqrt{3\sqrt{2...}}}}$, then $x^2 = 4\cdot 3\sqrt{x} = 12\sqrt{x}$ and $x^4 = 144x$, making $x^3 = 144$, so $x = \sqrt[3]{144}$. **144** is our answer.

1.681 Look at this in reverse, we are trying to find $a+\sqrt{b}$ for which $(a+\sqrt{b})^2$ is equal to $32+10\sqrt{7}$. When we square $a+\sqrt{b}$ we get $a^2+2a\sqrt{b}+b$, so a^2+b must equal 32 and $2a\sqrt{b}$ must equal $10\sqrt{7}$, making $a = 5$ and $b = 7$. $a + b =$ **12**.

1.701 $2x^2-14x+24 = 2(x^2-7x+12) = 2(x-3)(x-4)$

1.710 For the hexagon we have the sum of the sides:
$n+(n+1)+(n+2)+(n+3)+(n+4)+(n+5) = 45$, or $6n + 15 = 45$ which gives us $n = 5$.
For the pentagon we have
$n+(n+1)+(n+2)+(n+3)+(n+4) = 45$, or $5n + 10 = 45$ which gives us $n = 7$. The difference in the lengths of the shortest sides is **2cm.**

1.720 The easiest way to do this is to convert the equation into standard form and then switch the coefficients of x and y.

$y = \frac{4}{9}x - 8$	Multiply the equation by 9.
$9y = 4x - 72$	Subtract $9y$ and add 72.
$4x - 9y = 72$	The y-intercept is $\frac{72}{-9} = -8$ $(0,-8)$.
	The x-intercept is $\frac{72}{4} = 18$ $(18,0)$.

To switch intercepts, we can just switch the a and b values of the equation (the coefficients of x and y): $-9x+4y = 72$. Multiply by -1 to get $9x-4y=-72$.

1.721 We are given $a+b=6$ and $a^2+b^2=22$. We are asked to find $\frac{1}{a}+\frac{1}{b}$, which is equal to $\frac{a+b}{ab}$, so all we need is ab. Because $a+b=6$, $(a+b)^2=36$, and $(a+b)^2=a^2+b^2+2ab$, if we subtract $a^2+b^2=22$ from $(a+b)^2=36$ we are left with $2ab=14$, so $ab=7$ and $\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}=\frac{6}{7}$.

1.730 We can do the math in our heads: \$8,000 loses 50% to become \$4,000. \$4,000 gains 50% back to get to \$6,000. Down 50% again gives us \$3,000, and finally back up again by 50% gets us to **\$4,500**. We can do this with any starting value by multiplying by 0.5 and 1.5 repeatedly. The investment loses 25% every 2 days at this rate.

1.731 First, complete the distribution on $(j-1)(1-k)$ to get $j+k-1-jk=(j+k)-jk-1$, so we only need the sum and product of the roots (not the actual roots themselves). The sum of the roots of $2x^2-5x+1=0$ is $\frac{-b}{a}=\frac{5}{2}$ and the product is $\frac{c}{a}=\frac{1}{2}$, so $(j-1)(1-k)=(j+k)-jk-1=\frac{5}{2}-\frac{1}{2}-1=\mathbf{1}$.

1.741 Call the edge lengths of the prism a, b, and c. We are given $ab = 12$, $ac = 16$, and $bc = 27$ and asked to find abc. If we multiply all three of our equations together we get $a^2b^2c^2=12\cdot16\cdot27$. To get abc, we take the square root of both sides of the equation. $abc=\sqrt{12\cdot16\cdot27}=\sqrt{4\cdot16\cdot81}=2\cdot4\cdot9=\mathbf{72cm^3}$.

1.750 Every spider (s) has 7 more "legs than bugs" (for every spider, there are 8 legs but only one bug, hence, 7 more legs than bugs), while every beetle (b) has 5 more "legs than bugs". This gives us the equation $7s+5b=162$. We are also told that $b=s+18$. Substitute and solve $7s+5(s+18)=162$, so $12s+90=162$. This gives us $s =$ **6 spiders**.

1.751 A 95 on his next three tests will increase his average to 93, which will be out of 9 tests. $93(9) = 837$ points on 9 tests. Subtract $95(3) = 285$, leaving 552 points on his first 6 tests, for an average of $552/6 =$ **92**.

1.761 Let $x = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{1{,}024}$, then

$$2x = 1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{512}$$ and subtracting the two:

$$2x - x = 1 - \frac{1}{1{,}024} = \frac{1{,}023}{1{,}024}.$$

1.771 Let $x = 1 + \cfrac{1}{1 + \cfrac{2}{1 + \cfrac{1}{1 + ...}}}$, then $x = 1 + \cfrac{1}{1 + \cfrac{2}{x}}$, so

$$x = 1 + \frac{1}{\frac{x+2}{x}} = 1 + \frac{x}{x+2} = \frac{x+2}{x+2} + \frac{x}{x+2} = \frac{2x+2}{x+2}$$

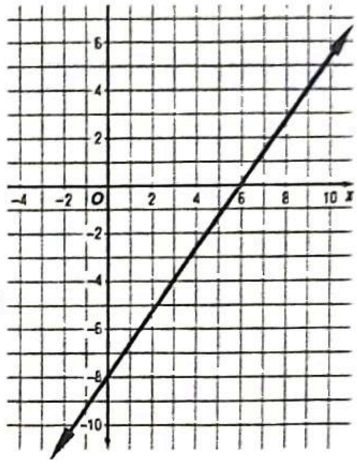
Multiplying both sides by x, we get $x(x+2) = 2x + 2$, or $x^2 + 2x = 2x + 2$. This makes $x^2 = 2$, so $x = \sqrt{2}$. (only the positive solution makes sense here).

1.781 Take this one step at a time. Tejas paints 1/3 of the fence in 2 hours, so he paints at a rate of 1/6 fence per hour. Working together, Tejas and Jason finish $\frac{4}{5} - \frac{1}{3} = \frac{7}{15}$ of the fence, of which $\frac{1}{6}$ was completed by Tejas, meaning that on his own, Jason can complete $\frac{7}{15} - \frac{1}{6} = \frac{3}{10}$ on his own in one hour. Only $\frac{1}{5}$ of the fence is left to be painted, which will take Jason $\frac{1/5}{3/10} = \frac{2}{3}$ of an hour or 40 minutes. He will finish painting at **3:40pm**.

1.801 $4x^2 - 36 = 4(x^2 - 9) = 4(x+3)(x-3)$

1.810 We will use the harmonic mean to find his average speed on the way there: $\dfrac{2 \cdot 4 \cdot 6}{4+6} = \dfrac{24}{5}$ miles per hour for 25 minutes (5/12 hour) means that the distance is $\dfrac{24}{5} \cdot \dfrac{5}{12} = 2$ miles. Running 2 miles at 8mph will take 2/8 hour = **15 minutes.**

1.820 Find the intercepts of the graph: The y-intercept is at $(0,-8)$, and the x-intercept is at $(-6,0)$. We can use the distance formula (**p.201**) or simply recognize that we have a right triangle with legs 6 and 8 and hypotenuse $\sqrt{6^2+8^2}$ = **10 units.**



1.821 $2^{20} - 1 = (2^{10}+1)(2^{10}-1) = (2^{10}+1)(2^5+1)(2^5-1)$
$= (1{,}025)(33)(31) = (5 \cdot 5 \cdot 41)(3 \cdot 11)(31)$
$= \mathbf{3 \cdot 5^2 \cdot 11 \cdot 31 \cdot 41}$

1.841 By the Pythagorean theorem, we have: $x^2 + (x+1)^2 = (x+3)^2$, or $2x^2 + 2x + 1 = x^2 + 6x + 9$, which simplifies to $x^2 - 4x - 8 = 0$. Using the quadratic formula to solve, we get positive solution $x = 2 + 2\sqrt{3}$, which is the short leg of the triangle. This makes $3 + 2\sqrt{3}$ the length of the longer leg, and the area of the triangle is:

$$\frac{1}{2}(2+2\sqrt{3})(3+2\sqrt{3}) = \frac{1}{2}(18+10\sqrt{3}) = 9+5\sqrt{3}.$$

1.851 The middle number must be 7, and the sum of the three integers must be 18 (meaning the sum of the first and last integers must be 11). We cannot use 5 and 6 (6 would be the median) or 4 and 7 (7 would be the mode), so we only have {3, 7, 8}, {2, 7, 9}, and {1, 7, 10}. There are **3** distinct sets.

1.861 The sum of a geometric series is given by the formula $\frac{a}{1-r}$. The common ratio is $\frac{2}{5}$, as is the first term of the series. This gives us $\frac{2/5}{1-2/5}=\frac{2/5}{3/5}=\frac{2}{3}$.

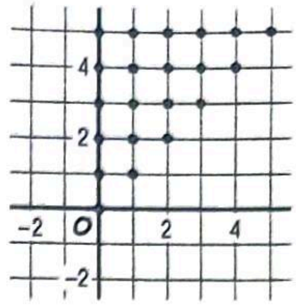
1.881 We can see that if we wanted to record for 6 hours and then play the 6-hour tape back, we would need 3 battery packs to record and 2 to play back for a total of 5 battery packs. If 5 battery packs will allow 6 hours of filming, then 1 battery pack will allow 6/5 = 1.2 hours or **72 minutes**. Alternatively, using the harmonic mean of 2 and 3, we see that a battery will last $\frac{2\cdot 2\cdot 3}{2+3}=\frac{12}{5}$ hours if half is used for taping and half is used for playback, so we can record for half that length of time (6/5 hour = 72 minutes).

1.900 If 3 men take 9 hours to paint 4 rooms, it would take 1 man 3 times as long to paint 4 rooms: 27 hours. This means it will take 27/4 = 6.75 hours or **6 hours and 45 minutes** for one man to paint one room.

1.901 $7a^2-28a+28=7(a^2-4a+4)=7(a-2)^2$

1.910 If Tobey mows 1 lawn in 5/6 hour, then he mows at a rate of 6/5 lawns per hour. This means that in 30 minutes, Tobey will mow 6/10 or 3/5 of the lawn, meaning Nick must mow 2/5 lawn in a half-hour or 4/5 lawn per hour. At this rate, it will take him 5/4 hours or **1 hour 15 minutes** to mow a whole lawn.

1.920 Begin by mapping the possible coordinate pairs where $0 \le x \le y \le 5$. The equation passes through the origin and one of the 21 points shown. We are looking for all distinct slopes $\frac{y}{x}$.



This now becomes simply a matter of careful counting and avoiding duplicates. Starting with $y = 5$ we have $\frac{5}{5}$, $\frac{5}{4}$, $\frac{5}{3}$, $\frac{5}{2}$, and $\frac{5}{1}$. $\frac{5}{0}$ cannot be used as a slope m in the equation given. When $y = 4$ we have three new slopes: $\frac{4}{3}$, $\frac{4}{2}$, and $\frac{4}{1}$. For $y = 3$ we have $\frac{3}{2}$ and $\frac{3}{1}$, and for $y = 2$ and $y = 1$ we have no new slopes. This gives us a total of **10** distinct slopes.

1.921 $(a+\sqrt{b})^2 = a^2 + 2a\sqrt{b} + b = 9 + 4\sqrt{5}$, so b must be 5 (the value under the radical) and a must be 2 (because $2a\sqrt{5} = 4\sqrt{5}$. To find $(a+\sqrt{b})^3$ we just multiply $(2+\sqrt{5})$ by $(9+4\sqrt{5})$:

$$(2+\sqrt{5})(9+4\sqrt{5}) = 18 + 8\sqrt{5} + 9\sqrt{5} + 20$$
$$= 38 + 17\sqrt{5}.$$

1.941 We look for a way to get 5^n in the expression 25^{2n+1}:

$$25^{2n+1} = (5^2)^{2n+1} = (5^2)^{2n} \cdot 5^2 = 5^{4n} \cdot 5^2 = (5^n)^4 \cdot 5^2.$$

Now we can substitute 2 for 5^n to get $(2)^4 \cdot 5^2$ = **400**.

1.981 If $a + b = 15$, then $(a+b)^2 = 225$, which is also equal to $a^2 + 2ab + b^2$. Subtract $a^2 + b^2 = 150$ from $a^2 + 2ab + b^2 = 225$ to get $2ab = 75$. Now we have $a+b=15$, so $a=15-b$, and we plug-this into $2ab=75$ to get $2(15-b)b=75$ which leads to the quadratic $2b^2-30b+75=0$. The solutions are $\frac{15\pm5\sqrt{3}}{2}$, and their positive difference is:

$$\frac{15+5\sqrt{3}}{2}-\frac{15-5\sqrt{3}}{2}=5\sqrt{3}.$$

Counting
Key and Solutions

Problems within the text are ordered with the last three digits reversed. This way, if you are looking for a solution you will not accidently see the answer to the next problem. Problems marked with a ▒ do not include a written solution, only the answer.

2.001	6
2.010	46
2.011	15
2.020	12
2.021	81
2.030	435
2.031	336
2.040	480
2.041	60
2.051	126
2.060	18,432
2.061	1,820
2.070	25
2.071	1,024
2.080	720
2.081	2,940
2.090	151,200
2.091	96
2.100	99
2.101	120
2.110	10
2.111	15
2.120	3
2.121	816
2.130	120
2.131	20,160
2.140	128
2.141	455
2.150	465
2.151	55
2.161	66
2.170	12
2.171	32
2.180	20,160
2.181	1,270
2.191	34
2.200	49
2.201	24
2.210	200
2.211	0
2.220	33
2.221	480
2.230	210
2.231	325
2.240	256
2.241	28ft
2.250	99
2.261	2,700
2.270	1
2.271	16
2.280	120
2.281	278,208
2.291	360
2.300	190
2.301	48
2.310	27
2.311	1,140
2.320	27
2.321	27
2.330	408
2.331	35
2.340	1,024
2.341	80,640
2.350	56
2.361	56
2.371	6
2.380	120
2.381	20
2.400	3,075
2.401	5,040
2.410	21
2.411	56
2.420	428
2.421	386
2.430	90
2.431	34,650
2.440	512
2.441	10
2.450	17
2.461	60
2.471	382

2.480	336	**2.611**	380	**2.771**	16
2.481	5,400	**2.620**	3	**2.780**	360
2.500	16	**2.621**	100	**2.781**	40
2.501	360	**2.630**	61	**2.800**	900
2.511	100	**2.631**	54	**2.801**	90
2.520	31	**2.640**	5	**2.811**	243
2.521	55	**2.650**	17,576	**2.840**	200
2.530	8	**2.671**	340	**2.850**	18
2.531	2,520	**2.680**	24	**2.880**	180
2.540	1,536	**2.681**	148	**2.881**	252
2.541	256	**2.700**	19	**2.900**	25
2.550	666	**2.701**	117	**2.901**	60,480
2.561	5,376	**2.711**	196	**2.911**	1,020
2.571	386	**2.720**	$n(n-1)/2$	**2.920**	10
2.580	2,520	**2.721**	120	**2.930**	216
2.581	10	**2.731**	1,072	**2.950**	120
2.600	18	**2.740**	10	**2.980**	840
2.601	1,440	**2.750**	768	**2.981**	10,079

2.001 Consider three tiles: $\boxed{\text{F}}\boxed{\text{OO}}\boxed{\text{T}}$. There are $3! =$ **6** ways to arrange them, and each arrangement will include a double-O.

2.010 $100 = 10^2$ and $10{,}000 = 100^2$, so we are really just looking for the number of even integers between 10 and 100 inclusive. There are $100 - 10 + 1 = 91$ integers. The first and last are even, so there are 46 even (and 45 odd). This gives us **46** even perfect squares.

2.020 22 spotted dogs plus 30 short-haired dogs makes 52 dogs (but there are only 40). This means that 12 dogs must have been counted twice, so at least **12** have short hair and spots.

2.021 There are $99 - 10 + 1 = 90$ two-digit integers. 11, 22, 33, 44, ... 88, and 99 (9 integers) use the same digit twice, so there are $90 - 9 =$ **81** two-digit integers that use two different digits.

2.030 Each point must be connected to 29 other points, so it would appear to take $29 \cdot 30 = 870$ lines, however, connecting point B to A is the same as connecting A to B, so we divide by 2 to eliminate the duplicates and are left with **435** lines.

2.031 The gold can be awarded to 8 different dogs, after which 7 dogs can win silver and 6 can win bronze: $8 \cdot 7 \cdot 6 =$ **336** ways.

2.040 3 styles times 8 waist sizes times 10 inseam lengths times 2 colors = **480** different pairs.

2.041 There are 5 choices for the hundreds digit (1, 3, 5, 7, or 9) then 4 for the tens digit and 3 for the ones digit for a total of $5 \cdot 4 \cdot 3 =$ **60** whole numbers.

2.051 Consider placing 5 dividers to separate 4 items (each can either be fries, apple pie, onion rings, chicken nuggets, hamburger, or cheeseburger). Using | to represent dividers and o to represent items, one such arrangement of dividers and items would be:

|oo||o||o

The diagram above represents 0 fries, 2 pies, 0 onion rings, 1 chicken nuggets, 0 hamburgers, and 1 cheeseburger. There are $_9C_4 =$ **126** ways to arrange the four o's between the five dividers (which separate the items into six categories).

2.060 There are $3 \cdot 6 = 18$ ways to choose a lettuce and a dressing. Each of the toppings can either be on the salad or not on the salad, so each topping represents two choices. This gives $2^{10} = 1{,}024$ ways to add toppings. Multiply by the 18 salad/dressing combinations to get **18,432** possible salads.

2.061 Consider 12 bills and 4 dividers (which separate the bills into five numbered briefcases). There are $_{16}C_4$ = **1,820** ways to arrange 4 dividers among 12 bills. (There are 16 objects to arrange, so we can place the 4 dividers in $_{16}C_4$ ways).

2.070 $\frac{25!}{24!} = \frac{25 \cdot \cancel{24} \cdot \cancel{23} \cdot \ldots \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{24} \cdot \cancel{23} \cdot \ldots \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = \mathbf{25}.$

2.071 $\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \ldots + \binom{10}{0}$ is the sum of the entries in the tenth row of Pascal's triangle: $2^{10} = \mathbf{1{,}024}$.

2.080 $P(10,3) = 10 \cdot 9 \cdot 8 = \mathbf{720}$.

2.081 The 6 slices can be divided among the 4 students who are not vegetarians. Use 3 dividers and 6 slices which gives us ${}_9C_3 = 84$ ways to divide the pepperoni slices. The 3 slices of cheese pizza can be divided among all 5 students. Using 4 dividers and 3 slices we get ${}_7C_3 = 35$ ways to distribute the cheese slices. This gives us a total of $35 \cdot 84 =$ **2,940** ways to distribute all 9 slices.

2.090 BOOKKEEPER has two O's, two K's, and three E's. There are 10! ways to arrange all ten letters, but many of these arrangements create the same word. The O's can be arranged 2 ways without changing the word, the K's can be arranged 2 ways, and the E's can be arranged $3! = 6$ ways. We divide the 10! arrangements by $(2 \cdot 2 \cdot 6)$ to eliminate duplicates and get **151,200**.

2.091 The three couples can be arranged $3! = 6$ ways on the sofa from left to right, after which each couple can be arranged 2 different ways for a total of $2^3 = 8$ ways. After the couples are seated, there are two places for grandma to be seated (between the first two couples or between the 2nd and 3rd couple). $6 \cdot 8 \cdot 2 =$ **96** seating arrangements.

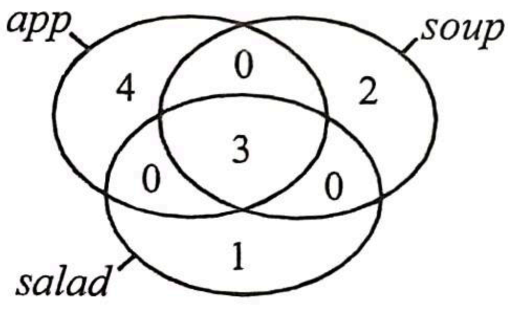
2.100 From 500 to 600 exclusive there are $600 - 500 - 1 =$ **99** integers.

2.101 Consider arrangements of the tiles: $\boxed{R}\boxed{EE}\boxed{C}\boxed{I}\boxed{P}$. There are $5! =$ **120** ways to arrange the tiles, each creating an arrangement which includes a double-E.

2.110 Around a 30-foot circle, you can set **10** place-settings if they are placed every 3 feet (exactly as you would expect, this one is not "tricky".)

2.111 This problem (paired with the one before it) was intended to help you make the discovery that $C(6,2) = C(6,4)$. Choosing 2 of 6 items can be done the same number of ways as choosing 4 of 6 (there will always be 2 left over). $C(6,4) = C(6,2) =$ **15**.

2.120 We can solve this problem with or without a Venn diagram. The number in the overlapping region will be counted three times (two extra times). When we add the number of appetizers, soups, and salads, we get 16. This is 6 more than the number of people. Because no person ordered just two items, **3** items must have been counted two extra times.



2.121 There are $_7C_4 = 35$ ways to choose four spaces, and only one of these arrangements does not have any spaces next to each other (leaving 34 combinations). Each of these arrangements has $4!$ ways to assign coworkers to the spaces, so we have $34(4!) =$ **816** ways to assign the four spaces.

2.130 Each of the 16 teams plays 15 games. To avoid counting games twice (the Cavaliers playing the Hokies is the same as the Hokies playing the Cavaliers) we divide by 2. $(16 \cdot 15)/2 =$ **120** games.

2.131 PRACTICE has 8 letters with two C's. $8!/2 =$ **20,160**.

2.140 There are two possible outcomes for each of 7 flips for a total of $2^7 =$ **128** possible outcomes. Note that HHHTTTT is considered different from TTTTHHH.

2.141 Choosing 3 from a group of 15, there are 15 choices for the first selection, 14 next, and then 13 for the final selection on the team. Picking Al, Bill, and Carly is the same as choosing these three players in any of $3! = 6$ arrangements.

$$_{15}C_3 = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = \mathbf{455} \text{ possible teams.}$$

2.150 The sum of $1 + 2 + \ldots + 30 = \dfrac{30(31)}{2} =$ **465**.

2.151 Three of the roses have already been selected because we know that there must be at least one of each color. This leaves 9 roses to choose from three colors. Place 2 dividers among 9 roses to divide them into red, white, and pink groups. There are $_{11}C_2 =$ **55** ways to place 2 dividers among 9 roses. (There are 11 objects to arrange, 2 are dividers).

2.161 We place two dividers among 10 points to divide them into groups representing speed, handling, and acceleration. There are $_{12}C_2 =$ **66** ways to arrange 2 dividers among 10 points (12 items, 2 are dividers).

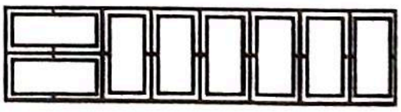
2.170 $\dfrac{5!}{10} = \dfrac{\cancel{5} \cdot 4 \cdot 3 \cdot \cancel{2}}{\cancel{5} \cdot \cancel{2}} = \mathbf{12}$

2.171 We look to the odd entries in the 6th row of Pascal's triangle: 1 <u>6</u> 15 <u>20</u> 15 <u>6</u> 1 which represent $_6C_1$, $_6C_3$, and $_6C_5$. The sum of these entries is **32**. Note for future reference that the sum of the odd entries in row n is equal to the sum of the even entries, or half the total: 2^{n-1}.

2.180 $_8P_6 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 =$ **20,160**.

2.181 After the 1 in 1,000,000 there are 6 digits. At first it seems that we can simply place 4 fives $_6C_4 = 15$ ways and then choose any one of 10 digits for each of the remaining two places $10^2 = 100$ ways for a total of 1,500 ways. However, we over-count when there are more than 4 fives. We must count three cases. When there are exactly four 5's there are $_6C_4 = 15$ times $9^2 = 81$ ways to fill the remaining two places (with something other than a 5) = 1,215 numbers. With exactly five 5's there are $_6C_5 = 6$ ways to place the fives and 9 ways to fill the remaining place = 54 numbers. There is only one way to have six 5's. This gives us a total of 1,215 + 54 + 1 = **1,270** numbers.

2.191 There is 1 way that we can arrange all 8 dominoes vertically. If we place dominoes horizontally, they must be placed in pairs (treat the pair as one unit). With one horizontal pair there are 6 vertical dominoes and we have $_7C_1 = 7$ ways to arrange them. With 2 horizontal pairs, there are 4 remaining vertical dominoes which makes $_6C_2 = 15$ ways to arrange them. With 3 horizontal pairs, there are 2 remaining vertical dominoes which makes $_5C_3 = 10$ ways to arrange them. Finally, four horizontal pairs leaves zero vertical dominoes and only 1 way to arrange them. 1 + 7 + 15 + 10 + 1 = **34** ways to arrange the 8 dominoes. Alternatively, we can tie this to the Fibonacci sequence. There



is clearly only one way to tile a 1 by 2 board. On a 2 by 2 board, the dominoes can be placed either vertically or horizontally for a total of 2 ways. If we go to a 3 by 2 board, we can place the first domino vertically leaving a 2 by 2 board (giving us two ways to tile it), or we can place two dominoes horizontally leaving a 1 by 2 board with one way to tile it.

This gives us $1 + 2 = 3$ ways to tile a 2 by 3 board. Moving up to a 4 by 2 board, the first tile can be vertical (leaving a 3 by 2 board with 3 ways to tile it) or horizontal (leaving a 2 by 2 board with 2 possible tilings). This gives $2 + 3 = 5$ possible arrangements. Continue the pattern and you can see that the number of ways to tile an 8 by 2 board is equal to the number of ways to tile a 7 by 2 plus the number of ways to tile a 6 by 2. The pattern is the Fibonacci sequence and we are looking for the 8th term if we start: 1, 2, 3, 5, 8, 13, 21, **34** ...

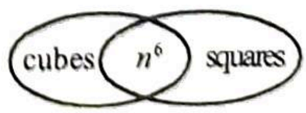
2.200 Of the 99 integers between 500 and 600, the first and last are odd, meaning there is one more odd than even. There are therefore 50 odds and **49** evens.

2.201 If we place a T at the beginning of the word, we are left with 4, 3, 2, and 1 letter to place 2nd through 5th. There are $4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{24}$ letter arrangements.

2.210 The problem asks for integers, and consecutive negative integers have a positive product so we cannot forget to count them. We can start with $-100(-99)$ and get all the way up to $99 \cdot 100$. The first integer in each pair can be anywhere from -100 to 99 (inclusive). $99-(-100)+1= \mathbf{200}$ pairs of consecutive integers whose product is less than 10,000.

2.211 $C(85, 25)$ is equal to $C(85, 60)$, which makes their difference **0**. When you choose 25 items from a set of 85, the leftover items form a set of 60. (See **2.111**)

2.220 There are 27 perfect squares from 1^2 to 27^2. There are 9 perfect cubes from 1^3 to 9^3, however, some of these are also perfect squares and will be counted twice. Any integer raised to the 6th power is both a perfect square and a perfect cube. 1^6, 2^6, and 3^6 are each counted twice. This gives us $27+9-3 = \mathbf{33}$ squares and cubes from 1 to 729.



2.221 It is easier to count the ways in which six friends can line up with Alice and David standing together, and subtract them from the total number of ways the students can stand in line without restrictions. There are $6! = 720$ ways that 6 students can line up. If we pair Alice and David together (Alice immediately to the left of David), there are $5! = 120$ ways to arrange them. David and Alice can switch places (Alice immediately to the right of David), doubling this number to 240. $720 - 240 = \mathbf{480}$ ways.

2.230 $20 + 19 + 18 + \ldots + 1 = \dfrac{20(21)}{2} = \mathbf{210}$ cans.

2.231 26 students each complete a project with each of 25 other students, however, Dhruv working with Calvin is the same as Calvin working with Dhruv. We must divide by 2 to avoid over-counting:

$$\frac{26(25)}{2} = \mathbf{325} \text{ projects.}$$

2.240 Because digits may be repeated, there are 4 choices for each digit which gives us $4^4 = \mathbf{256}$ four-digit numbers.

2.241 Each side includes three sides of 4-inch posts (1 foot) and two 3-foot spaces between them (6 feet). The length of each side is 7 feet, so the perimeter is $4 \cdot 7 = \mathbf{28}$ feet.

2.250 Watch-out for overcounting! 99 people each shake the hand of 2 other people, but if we count each of these 198 handshakes, we have counted the handshakes twice. Divide by 2 to get **99**. Consider each person shaking the hand of the person on his/her right in turn. The person whose hand they are shaking is shaking the hand of the person to his/her left.

2.261 Consider each set of bills separately. For the \$1 bills, use 2 dividers to separate 4 bills. There are ${}_6C_2 = 15$ ways to do this. Use 2 dividers to separate the three \$5 bills in ${}_5C_2 = 10$ ways. The two \$20 bills can be distributed ${}_4C_2 = 6$ ways and the \$100 bill can be given to any one of the three (3 ways). This gives us $15 \cdot 10 \cdot 6 \cdot 3 =$ **2,700** ways to distribute the money.

2.270 $\dfrac{20 \cdot 19!}{20!} = \dfrac{20 \cdot 19 \cdot 18 \cdot \ldots \cdot 2 \cdot 1}{20 \cdot 19 \cdot 18 \cdot \ldots \cdot 2 \cdot 1} = \mathbf{1}$

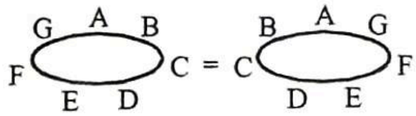
2.271 $\binom{9}{5}$ and $\binom{9}{6}$ are entries on the 9th row of Pascal's triangle. The sum of these two entries is the entry on row 10 which falls between the two: $\binom{10}{6}$.

$10 + 6 =$ **16**. The rule: $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

2.280 The first digit can be any one of the 6 numbered balls, after which there are 5 left and then 4 remaining for the last draw, for a total of $6 \cdot 5 \cdot 4 =$ **120** different 3-digit numbers.

2.281 We are filling the blanks in A _ _ - _ _ _ where the first blank must be filled with one of 24 remaining letters (not A or O), the second blank will be filled with one of 23 letters which remain. The first number can be chosen from among 9 (no 0), the second will be one of the remaining 8, and one of the 7 digits that have not been used will fill the fifth blank. $24 \cdot 23 \cdot 9 \cdot 8 \cdot 7 =$ **278,208** standard license plates.

2.291 Call the charms A, B, C, D, E, F, and G. Placed in a row, there are 7! ways to order the charms. On a bracelet, however, ABCDEFG is the same as BCDEFGA because the charms can be rotated around the bracelet. We can rotate each arrangement so that charm A is always first. There are then 6! ways to arrange the charms after A, however, the bracelet can also be flipped over, so that the arrangement ABCDEFG is the same as AGFEDCB (see below). We divide 6! by 2 to get **360** distinct arrangements of charms on the bracelet.



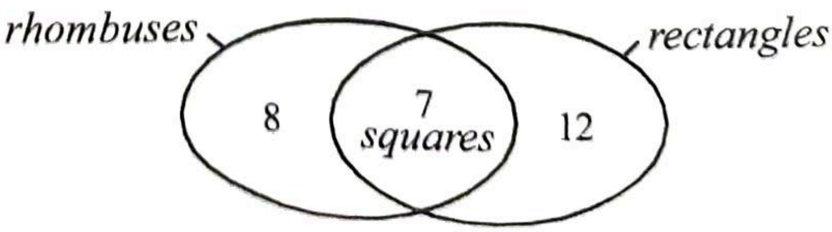
2.300 If the first number is 130, you will add five once to get the 2nd number, twice to get the 3rd number, and so on until you add five 12 times to get the 13th number. $130 + 12(5) = \mathbf{190}$.

2.301 There are 2 choices for the first letter (either E or I), after which there are 4 letters available to be the 2nd letter, then 3, 2, and 1 left to finish the word. This gives a total of $2(4!) = 2 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 48$ possible arrangements.

2.310 There are 33 multiples of 3 ($3 \cdot 1$ through $3 \cdot 33$). Multiples of 3 that are multiples of 5 are multiples of 15. Six multiples of 15 ($15 \cdot 1$ through $15 \cdot 6$) must be excluded to give us **27** whole numbers.

2.311 $_{20}C_3 = \mathbf{1{,}140}$.

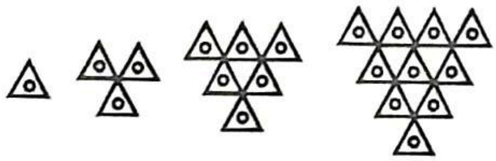
2.320 A Venn diagram helps us organize. The area in the diagram where rectangles and rhombuses overlap represents squares. Fill this region in first then complete the diagram. All of these shapes are parallelograms. There are **27** parallelograms.



2.321 It is much easier to count the integers less than 50 which are divisible by 3 or 5. There are 16 numbers less than 50 divisible by 3 ($3 \cdot 1$ through $3 \cdot 16$) and 9 which are divisible by 5 ($5 \cdot 1$ through $5 \cdot 9$), however, we have counted multiples of 15 twice: 15, 30, and 45, so we subtract these from the total. This makes $16+9-3=22$ numbers less than 50 that are divisible by 3 or 5. There are 49 integers less than 50, so there are $49-22=$ **27** positive integers less than 50 which are not divisible by 3 or 5.

2.330 If we place a "bowling pin" at the center of the triangles as shown, we can see that this is just a bowling pin counting problem. There are 3 toothpicks for each "bowling pin" for a total of:

$\frac{16(17)}{2} \cdot 3 = \mathbf{408}$ toothpicks.



2.331 The mouse must move up 3 times and right 4 times. There are ${}_7C_3$ = **35** ways to arrange three "up" moves among 7 total moves.

2.340 There are 2 choices for each of 10 digits, so we have 2^{10} = **1,024** numbers.

2.341 Whether we start with 27 or 28, there are 7 digits left. There are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 40{,}320$ ways to fill these digits without replacement. 2 choices for the first pair of digits followed by 40,320 possibilities for the remaining digits gives a total of **80,640** NC zip codes that do not repeat digits.

2.350 For the sum of two numbers to be odd, one must be odd and the other must be even. Each of the 8 odd integers must be connected to each of 7 even integers. $8 \cdot 7$ = **56** lines. Note that there is no need to correct for over-counting in this problem as each case was only counted once.

2.361 There are too many cases to count quickly with casework. Consider placing three dividers among 9 dots, for example, the diagram below would represent rolling a 2, then a 4, then a 2, then a 1 ($2 + 4 + 2 + 1 = 9$).

●●|●●●●|●●|●

We cannot place a divider at the front or back, and dividers cannot be adjacent. This gives us 8 slots to fill with 3 dividers or $_8C_3 = \mathbf{56}$ ways. None of these includes a number greater than 6, so we do not need to discount any impossible scenarios (below we see a representation of rolling 1, 1, 1, 6).

●|●|●|●●●●●●

2.371 We look for a row in Pascal's triangle in which the sum of the first three entries (0, 1, and 2 heads) is 22. This occurs on row 6, so $n = \mathbf{6}$.

$$\binom{6}{0}+\binom{6}{1}+\binom{6}{2}=1+6+15=22.$$

2.380 5 books can be placed 1st, then 4, 3, 2, and 1. $5! = \mathbf{120}$.

2.381 No two of the numbers rolled can be the same. There are $6C3 = 20$ ways to select three numbers (from the six possible) and there is one way to order each of these from least to greatest for white/green/red. This leaves us with just **20** ways.

2.400 The check numbering can be considered an inclusive counting problem. We look for n where $3{,}474 - n + 1 = 400$. Solving, we get $n = \mathbf{3{,}075}$.

2.401 This is the same as asking for the number of arrangements of the letters in the word BEGINNIG (removing one N to be placed at the front). This leaves 8 letters with two N's, I's, and G's so we have:

$$\frac{8!}{2\cdot 2\cdot 2} = \mathbf{5{,}040} \text{ arrangements.}$$

2.410 There are several ways to look at this problem, but drawing each post seems unreasonable. If we place a post at each vertex we are left with 114 posts. 114/6 = 19 additional posts to be placed on each side for a total of **21** on each side when you include the post at each vertex. If we divide 120 by 6 to get 20, we miss the fact that 6 posts are each counted on two sides. Include the missed post to get 21.

2.411 $8C3$ = **56** leadership teams.

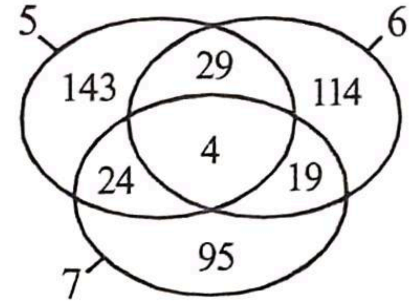
2.420 The trick is to avoid over-counting. The floor value of a number, indicated by $\lfloor x \rfloor$ represents the greatest integer value of x and basically means round down, for example: $\lfloor 9.8 \rfloor = 9$ and $\lfloor 73/11 \rfloor = 6$. There are $\left\lfloor \frac{1{,}000}{5} \right\rfloor = 200$ multiples of 5, $\left\lfloor \frac{1{,}000}{6} \right\rfloor = 166$ multiples of 6, and $\left\lfloor \frac{1{,}000}{7} \right\rfloor = 142$ multiples of 7. If we add these three values to get 508, we will double-count multiples of two of these integers and triple-count multiples of all three. We must subtract multiples of both 5 and 6 (which are multiples of 30): $\left\lfloor \frac{1{,}000}{30} \right\rfloor = 33$, multiples of 5 and 7: $\left\lfloor \frac{1{,}000}{35} \right\rfloor = 28$, and multiples of 6 and 7: $\left\lfloor \frac{1{,}000}{42} \right\rfloor = 23$. By subtracting these we eliminate duplicates, but we have actually removed too many numbers. The numbers which are multiples of 5, 6, *and* 7 have been removed 3 times (we wanted to remove them twice because they were originally triple-counted). We must add back $\left\lfloor \frac{1{,}000}{5 \cdot 6 \cdot 7} \right\rfloor = 4$ integers. (continued)

Summarizing, we have:

$$\left\lfloor\frac{1{,}000}{5}\right\rfloor+\left\lfloor\frac{1{,}000}{6}\right\rfloor+\left\lfloor\frac{1{,}000}{7}\right\rfloor-\left\lfloor\frac{1{,}000}{30}\right\rfloor-\left\lfloor\frac{1{,}000}{35}\right\rfloor-\left\lfloor\frac{1{,}000}{42}\right\rfloor$$

$$+\left\lfloor\frac{1{,}000}{210}\right\rfloor=200+166+142-33-28-23+4=\mathbf{428}$$

integer multiples of 5, 6, and/or 7 from 1 to 1,000.



2.421 There are ${}_{10}C_5 = 252$ ways to flip the same number of heads as tails (choose any 5 of the 10 to be heads), and $2^{10} = 1{,}024$ ways to flip a coin 10 times. Half of the rest of the times there will be more heads than tails. $(1{,}024-252)/2=\mathbf{386}$ ways. Pascal's triangle can be used for an alternate method (see **p.97**).

2.430 Each vertex can be connected to every other vertex by a diagonal except that it cannot be connected to itself or the two adjacent vertices. This means that each of n vertices on a polygon can be connected by a diagonal to $n-3$ vertices. For a 15-gon, each vertex will be connected to 12 others by a diagonal. This gives $15 \cdot 12 = 180$ diagonals, but each has been counted twice because diagonals AB and BA are the same. Divide by 2 to get **90** diagonals in a 15-gon.

2.431 MISSISSIPPI has 11 letters with four I's, four S's, and two P's: $\dfrac{11!}{4!\cdot 4!\cdot 2!} = \mathbf{34{,}650}$ arrangements.

2.440 This is different from the previous problem in that the first digit must be a 1 (for example: 0,010,101,011 is not a 10-digit number). There are $2^9 = \mathbf{512}$ ways to select the other 9 digits.

2.441 Every 6-unit path from the top left corner to the bottom right corner can be covered with three dominoes. Starting from the top we must find a path which goes right twice and down three times. There are ${}_5C_2$ = **10** ways to arrange 2 moves right among 3 down.

2.450 22 isosceles + 25 right triangles = 47 triangles, but there are only 30. We must have counted **17** triangles twice. These triangles are isosceles *and* right.

2.461 The first man must have at least CA on his chest, and the last man must have at least RS. This leaves VALIE for the middle man. You can find all 10 ways to assign VALIE with careful casework or consider placing 2 dividers, each in one of these 6 places: _V_A_L_I_E_. There are ${}_6C_2$ = 15 ways, but we cannot have |V|ALIE or V|A|LIE or any of the 5 arrangements which leave just one letter for the middle man. This gives us 10 ways to divide the letters and there are 3! = 6 ways to choose who stands 1st, 2nd, and 3rd for a total of $10 \cdot 6 =$ **60** ways.

2.471 We look to row 9 of Pascal's triangle to find values for ${}_9C_0 + {}_9C_1 + {}_9C_2 + {}_9C_3 + {}_9C_4 + {}_9C_5 = 1 + 9 + 36 + 84 + 126 + 126 =$ **382** combinations of toppings.

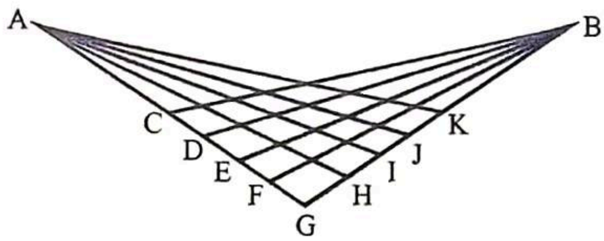
2.480 8 students can be chosen as president leaving 7 for vice president and 6 to be chosen as secretary for a total of ${}_8P_3 = 8 \cdot 7 \cdot 6 =$ **336** leadership teams.

2.481 ${}_{10}C_2 \cdot {}_{10}C_3 = 45 \cdot 120 =$ **5,400** choices.

2.500 Besides just counting them on your fingers, this can be solved by counting the number of problems from 10 to 40 inclusive: $40 - 10 + 1 = 31$. You start and end on an even problem so there are more evens than odds: **16** evens and 15 odds.

2.501 There are 6! = 720 arrangements of the 6 digits. In these 720 arrangements, each can be paired with its palindrome (the same number written in reverse order, like 235,146 and 641,532). Only one of the two will have the 1 to the left of the 2, so we need to divide 720 by 2 to get **360** arrangements.

2.511 Every triangle has either point A or point B as one vertex. Using B as one vertex, we need two more vertices to make a triangle. There are 5 points on segment AK that can be used as vertices of a triangle with vertex B. We can choose 2 of these in ${}_5C_2 = 10$ ways. The same can be done on AJ, AI, AH, and AG for a total of 50 triangles. Using A we have a symmetrical situation in which 50 triangles have one vertex at A for a total of **100** triangles.



2.520 Using a common denominator, we have 45/60 students who like sweet tea, 36/60 like unsweet tea, and 10/60 like neither. This gives us a total of 91/60, which means that 31/60 of students must like both sweet and unsweet tea. The fewest number of students who could have been surveyed is 60, so at least **31** students must like both.

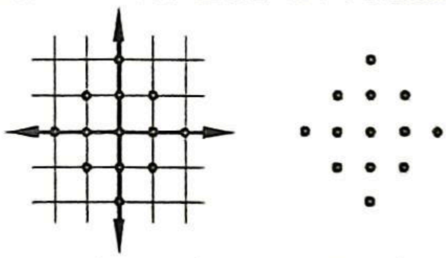
2.521 There are $8^2 = 64$ possible rolls. It is easier to count rolls which produce a prime number than a composite. A prime product will occur only when one of the dice shows a 1 and the other shows a prime: (1, 2) (1, 3) (1, 5) and (1, 7). These four can all be reversed for a total of 8 rolls which have a product that is prime. Finally (and easy to forget), rolling two ones produces a product that is neither prime nor composite, so we have $64 - 8 - 1 = \mathbf{55}$ rolls.

2.530 If n people shake $n-2$ hands (they do not shake their spouses hand or their own hand) for a total of 112 handshakes, then $\frac{n(n-2)}{2} = 112$. You can solve the quadratic or guess-check to get $n = 16$, so **8** couples attended.

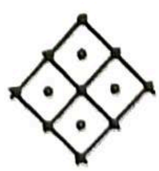
2.531 There are $7! = 5{,}040$ ways to arrange 1st through 7th place. In exactly half of these arrangements, Raj finishes ahead of Thomas (there are an equal number of arrangements with Thomas ahead as with Raj ahead). $5{,}040/2 =$ **2,520** ways.

2.540 It is a common mistake to assume $3 \cdot 9 = 27$ pizzas, however, each of the 9 toppings creates two possibilities: it can be included or not included on a pizza. Think of this as filling out a pizza order by putting checks beside the toppings you want. Each of 9 boxes will be checked or not checked for a total of $2^9 = 512$ possible topping combinations (including no toppings at all). Now we multiply by the three choices of size. $3 \cdot 512 =$ **1,536** pizza choices.

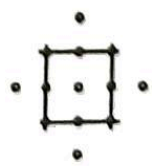
2.541 There are 13 lattice points which satisfy $|x| + |y| \leq 0$ shown below and then removed from the coordinate plane for clarity:



It is necessary to choose three of these 13 points to make a triangle and at first we might guess that $_{13}C_3 = 286$ is the answer, however, not every set of 3 points will form a triangle, so we must subtract. There are two rows of 5 points (on each axis). $2(_5C_3) = 20$ sets of points on these two rows do not form triangles. We can also count 10 rows of 3 which will not form a triangle (and have not been counted already, shown below): $286 - 20 - 10 =$ **256** ways.



(6)



(4)

2.550 There are 499 even integers ($2 \cdot 1$ through $2 \cdot 499$), and 33 multiples of 3 ($3 \cdot 1$ through $3 \cdot 333$). To avoid over-counting, we must subtract from this total any number which is divisible by both 2 and 3 (all multiples of 6). There are 166 multiples of 6 ($6 \cdot 1$ through $6 \cdot 166$) giving us $499 + 333 - 166 =$ **666** positive integers. It seems obvious after the fact that 2 out of 3 positive integers are divisible by 2, 3, or both: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, ...

2.561 There are $2^8 = 256$ ways to assign 8 arrows either left or right. We must then place gaps between arrows to separate them into words. Consider the set of 8 arrows: >>>>>>>>. There are $_7C_2 = 21$ ways to choose two spaces in this (or any other) arrangement, giving us $256 \cdot 21 =$ **5,376** code phrases.

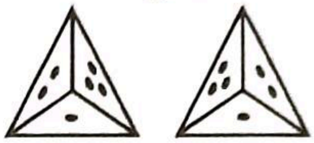
2.571 We can have $_{10}C_0$, $_{10}C_1$, $_{10}C_2$, $_{10}C_3$ or $_{10}C_4$ combinations of teammates depending on how many we want to choose. Look to the 10th row of Pascal's triangle to get these values quickly: $1 + 10 + 45 + 120 + 210$ = **386** possible combinations of teammates.

2.580 There are 7 choices for your top pick, then 6, 5, 4, and 3 choices: $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 =$ **2,520** rankings.

2.581 This is tricky casework. The following list has all of the ways that the dots can be placed on 4 faces:

(6, 0, 0, 0)	(5, 1, 0, 0)	
(4, 2, 0, 0)	(4, 1, 1, 0)	
(3, 3, 0, 0)	(3, 2, 1, 0)	(3, 1, 1, 1)
(2, 2, 2, 0)	(2, 2, 1, 1)	

Next, it requires some mental gymnastics to visualize which (if any) of the above can have the dots placed in multiple ways so that the die cannot be rotated to appear the same. For example, it should be clear that only one die can have 6 dots on one face. No matter what face the dots are on, we can place that face down and it will appear like any of the others. The only dot pattern where arrangement matters is the (3, 2, 1, 0) pattern. Consider the face with 0 dots placed on a table. The 3, 2, 1 can be placed in either way shown, and no rotation will make these dice appear the same. This gives us two arrangements for (3, 2, 1, 0) and one of each for the other 8 for a total of **10** distinct dice.

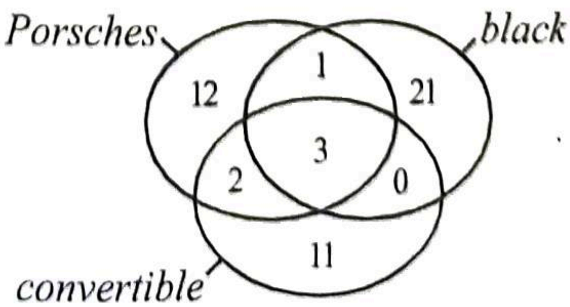


2.600 There are $30-10+1=21$ jerseys. If there are 3 left over, 18 jerseys were handed out to **18** players.

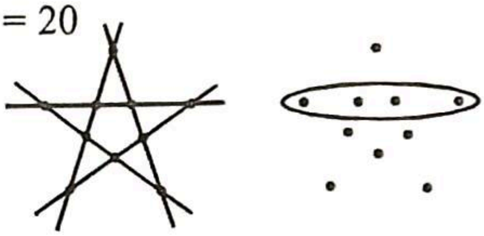
2.601 Consider Molly and Katie as one person and arrange the seven students as if they were 6. There are 6! = 720 ways to do this. Additionally, Katie and Molly can stand in two orders (either KM or MK), doubling the number of arrangements to **1,440**.

2.611 There are $_6C_3 = 20$ ways to walk 3 blocks north and three blocks west (consider the number of arrangements of NNNWWW). On the way back, he cannot take the same path, so there are 19 paths to choose for the walk back. This gives us $20 \cdot 19 =$ **380** different ways to walk to the deli and back.

2.620 The dealer has 50 cars, but when we add the number of Porches, black cars, and convertibles we get 18 + 25 + 16 = 59 cars. This means that there were 9 over-counts (some twice, some three times). There are 3 black convertibles, 4 black Porches, and 5 convertible Porches. If we subtract these 12 over-counts from 59 we get 47, so we need to add back 3 (because the black convertible Porches would have been subtracted too many times). There are **3** black convertible Porches. The Venn diagram below shows the distribution.

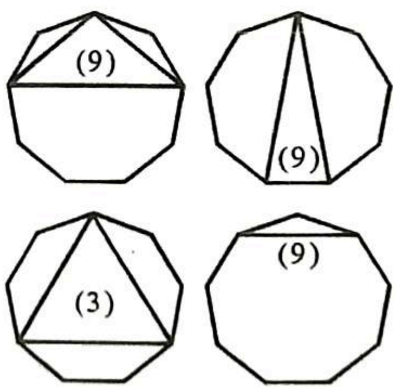


2.621 When we draw five lines with each intersecting the other four, we get $(5 \cdot 4)/2 = 10$ points of intersection (this is a handshake problem). The simplest way to do this is to draw a star. Connecting three of these points to create a triangle requires that we select 3 of the 10 points of intersection ${}_{10}C_3 = 120$ ways, however, there are 5 sets of 4 collinear points (selecting three of these will not make a triangle). We must subtract $5({}_4C_3) = 20$ sets of points which will not form a triangle, leaving us with **100** triangles.

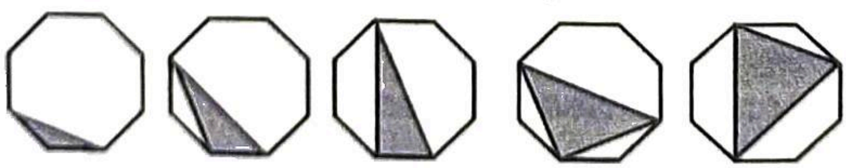


2.630 $1 + 2 + 3 + \ldots + 59 + 60 = \dfrac{60(61)}{2} = 30(61)$, which gives us a prime factorization of $2 \cdot 3 \cdot 5 \cdot \mathbf{61}$.

2.631 There are ${}_9C_3 = 84$ ways to connect the vertices of a nonagon to form a triangle. I find that it is easier to count the isosceles and equilateral triangles: there are 3 types of isosceles triangles (that are not equilateral) for each of the nine vertices of the nonagon. There are also 3 equilateral triangles. This leaves $84 - 3(9) - 3 =$ **54** scalene triangles.



2.640 Somewhat surprisingly, there are only **5**:

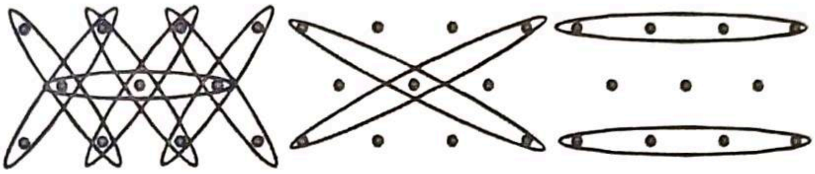


2.650 There are 26 letters and repeats are allowed. $26^3 =$ **17,576** codes.

2.671 We are looking for ${}_{10}C_3 + {}_{10}C_6 + {}_{10}C_9$. These are the 4th, 7th, and 10th entries on the 10th row of Pascal's triangle: $120 + 210 + 10 =$ **340** ways.

2.680 4 letters can be 1st, then 3, 2, and 1. $4! =$ **24** arrangements.

2.681 There are $_{11}C_3 = 165$ ways to choose 3 of the 11 points. When three points are collinear they do not form a triangle. Careful counting reveals 9 rows of 3 points and 2 rows of 4.



Each row of 4 points has $_4C_3 = 4$ sets of 3 points which will not form a triangle. This makes a total of $9 + 4 + 4 = 17$ sets of 3 points that will not form a triangle. Subtract 17 from 165 to get **148** triangles.

2.700 One slice gives us two pieces, 2 gives us 3, etc. until **19** slices make 20 pieces.

2.701 Put RED on a single tile, then we can arrange the five tiles $5! = 120$ ways. $\boxed{\text{RED}}$ $\boxed{\text{O}}\boxed{\text{R}}\boxed{\text{D}}\boxed{\text{E}}$, however, we over-count three situations where the word RED appears twice: REDREDO, REDORED, and OREDRED (for example, $\boxed{\text{RED}}$ $\boxed{\text{R}}\boxed{\text{E}}\boxed{\text{D}}\boxed{\text{O}}$ is the same as $\boxed{\text{R}}\boxed{\text{E}}\boxed{\text{D}}$ $\boxed{\text{RED}}\boxed{\text{O}}$). Subtract these three over-counts to get **117** ways.

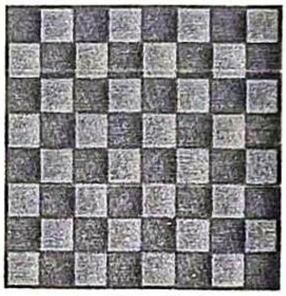
2.711 There are $_{12}C_3 = 220$ ways to choose three of the twelve points. There are 6 rows of 4 points. If three points are chosen which are on the same line, they cannot be connected to form a triangle. For each row of 4 dots there are $_4C_3 = 4$ sets of points which will not form a triangle. This makes a total of $4 \cdot 6 =$ 24 sets which will not form a triangle. $220 - 24 =$ **196** triangles.

2.720 If there are n people, each person will shake $n-1$ hands (every one but his own). We divide by 2 to avoid counting each handshake twice.

The formula is therefore $\dfrac{n(n-1)}{2}$.

2.721 To count the total number of ways to get from A to B, we must go up 4 times and right 6 times. This gives us ${}_{10}C_4 = 210$ ways to get from A to B. There are ${}_4C_2 = 6$ ways to get from A to X and ${}_6C_2 = 15$ ways to get from X to B. This makes $6 \cdot 15 = 90$ ways to get from A to B through X. Subtract this from 210 to get **120** ways which do not pass through X.

2.731 Of course it is easier to count the rectangles with 1 or fewer red squares. Each of the 64 single squares and the 112 double squares  (7 per row, with 8 horizontal and 8 vertical rows) has one or fewer red squares. Finally, there are three triple squares  per row which have only one red square (48 total). Subtract these from 1,296 to get **1,072** rectangles with more than one red square.



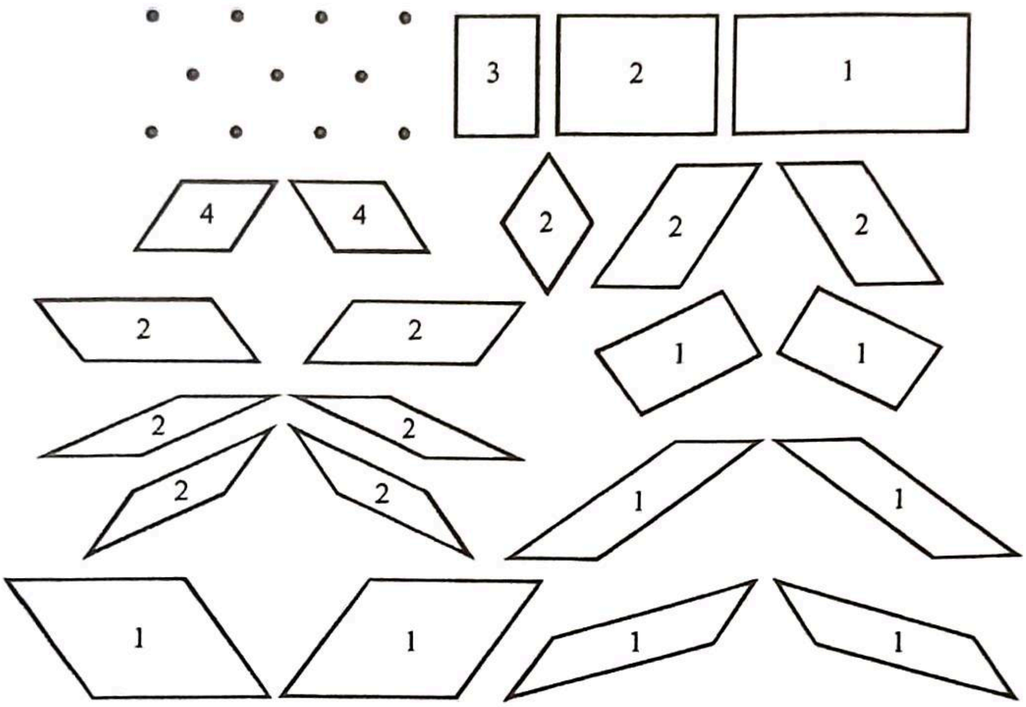
2.740 Call the faces of the cube: top, bottom, left, right, front, and back. There is 1 cube whose faces are all white. There is 1 cube with 1 black face. There are 2 cubes with 2 black faces: one with the top and bottom black (or any two opposite faces), and one with the top and left black (or any two faces which share an edge). There are 2 cubes with 3 black and 3 white faces: one with top/bottom/front and one with top/left/front. There are 2 cubes with 4 black faces (this is the same as having 2 white faces), 1 with 5 black faces, and 1 with 6 black faces: $1 + 1 + 2 + 2 + 2 + 1 + 1 =$ **10** distinct cubes.

2.750 The first digit must be a 1, 2, or 3 (not a 0). Each of the next 4 digits can be a 0, 1, 2, or 3. This gives us $3 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 3 \cdot 4^4 =$ **768** integers.

2.771 The number of ways is given by the entries in Pascal's triangle. Each time you move down a row, there are 2 choices. We go down 4 rows so we have $2^4 =$ **16** ways to spell COUNT.

2.780 LADDER has 6 letters with two D's: $\frac{6!}{2!}=\mathbf{360}$ arrangements.

2.781 This is a massive exercise in casework, and somewhat difficult to show clearly. Below are all the parallelograms with the number of each within (**40** total).



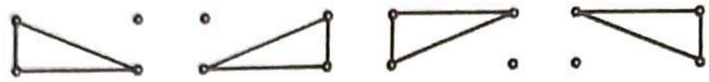
If you find another, please write me on my web site.

2.800 From 100 to 999 there are $999-100+1=\mathbf{900}$ 3-digit whole numbers.

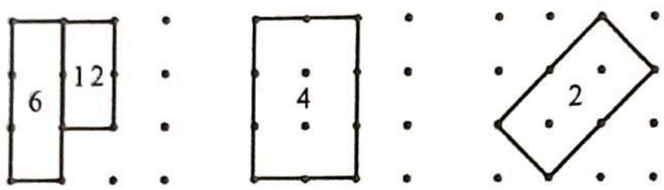
2.801 Fortunately, there are two D's, two E's and two R's. For a word beginning with D and ending with D, we have to arrange 5 more letters, two of which are E's and two of which are R's: $\frac{5!}{2!\cdot 2!}=30$ words which begin and end with D. There are an equal number which begin and end with E and with R for a total of $3\cdot 30=\mathbf{90}$ arrangements.

2.811 Every possible pentagon uses one of the three parallel lines on each side. This gives us three choices for each of 5 sides. $3^5=\mathbf{243}$ pentagons.

2.840 First, note that for any four points which form a rectangle, there 4 right triangles:

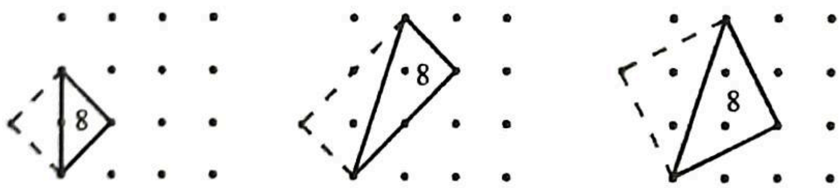


Now we just need to count rectangles. We have already counted the squares (there are 20 in the example on the same page). Now we look for rectangles that are not squares:



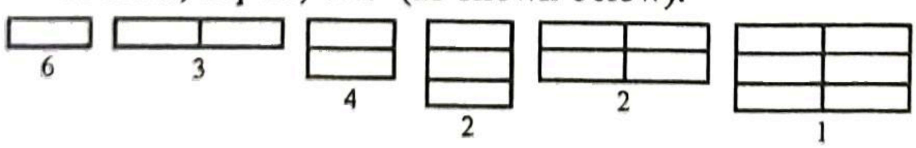
This gives us 44 rectangles (including squares) with four right triangles each, for a total of 176.

Unfortunately, we missed some triangles that are not part of any of these rectangles (in fact, I missed all of the triangles below in the first printing of this book, there are 8 of each type).



This brings our total to **200**.

2.850 My strategy on these problems is to count singles, doubles, triples, etc. (as shown below).



There are a total of **18** rectangles.

2.880 ICICLE has 6 letters with two I's and two C's:

$\frac{6!}{2!\cdot 2!} = \mathbf{180}$ letter arrangements.

2.881 This is a sticks-and-stones problem in disguise. Think of this as distributing "points" to each place value. A number less than 1,000,000 can have a maximum of 6 digits: ones, tens, hundreds, thousands, ten-thousands and hundred-thousands. Use 5 dividers to separate 5 "points" to be distributed among six place values. For example, in the diagram below, points are distributed: 0, 1, 1, 2, 0, 1. This represents the number 11,201.

|•|•|••||•

There are ${}_{10}C_5$ ways to arrange 5 dividers and 5 "points". ${}_{10}C_5 = 252$ numbers.

2.900 There are 7 whole numbers from 0 through 6. Between each of the 6 pairs of consecutive whole numbers there are 3 marks, making 18 additional marks. This gives us a total of **25** marks.

2.901 First, place the principal in one of the three shady spots. This leaves 8 spaces for 6 teachers. There are ${}_8P_6 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 20{,}160$ ways for the remaining teachers to park. Multiply this by the three places the principal can park and we get **60,480** ways.

2.911 With 20 vertices, there are ${}_{20}C_3 = 1{,}140$ ways to choose three points as vertices of a triangle. There are ${}_5C_3 = 10$ of these triangles on each of the 12 pentagonal faces, for a total of 120 triangles on the faces of the dodecahedron. Then, from any vertex, you can go in three different directions along the edges to another vertex. We can connect these three vertices to form an equilateral triangle where each side is a face diagonal. There are 20 of these (one per vertex). So, there are a total of 1,140−120−20=**1,000** triangles.

2.920 Each of the 5 points must be connected by a line to 4 others. We divide by 2 to avoid counting each line twice. $(5 \cdot 4)/2 =$ **10** lines.

2.930 Each roll has 6 possible outcomes for a total of 6^3 = **216** possible outcomes.

2.950 Each of the 16 points must be connected by a line to 15 others. We divide by 2 to avoid counting each line twice. $(16 \cdot 15)/2 =$ **120** lines.

2.980 COOKBOOK has 8 letters with four O's and two K's:

$$\frac{8!}{4! \cdot 2!} = \mathbf{840} \text{ arrangements.}$$

2.981 MISSPELL has 8 letters with two S's and two L's.

$\dfrac{8!}{2! \cdot 2!} = 10{,}080$ arrangements, but one of these is the correct spelling, so there are **10,079** misspellings.

3.010 The loss of a fish ... [illegible]

A [illegible] ability [illegible]

3.020 Let O represent [illegible] and X represent a [illegible] 2/5,

so the probability of a miss is 3/5. The [illegible]

$\frac{2}{5} \cdot \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{3}{5}$ and there are $\binom{5}{2} = \frac{5 \cdot 4}{2}$ ways to

arrange the missed shots (for example, OOXXX and XOXOX). Multiplying gives

$\frac{5 \cdot 4}{2} \cdot [illegible] = \frac{[illegible]}{625}$

3.030 There are [illegible] = 84 ways to choose 3 digits from [illegible] of [illegible] 84 ways to [illegible] 7 [illegible] five [illegible] (starting with [illegible] and ending [illegible] 9). [illegible]

3.040 There are [illegible] = 1,287 ways to choose 5 cards [illegible] set. [illegible] we have KQJ in the set, we must choose 2 of the remaining 10 cards to complete [illegible]. There are 10C2 = [illegible] ways to complete [illegible] making the probability [illegible] 5/143

3.050 It is easier to count the number of ways to roll [illegible] less. [illegible] a one way [illegible] a 2, [illegible] ways [illegible] (1,1,2) [illegible] (1,3) [illegible] and (3,1) [illegible] 26 possibilities, leaving [illegible] roll [illegible] a [illegible] 5/6

3.060 [illegible]

10 9 8 10

13 13 13 [illegible]

Probability
Key and Solutions

Problems within the text are ordered with the last three digits reversed. This way, if you are looking for a solution you will not accidently see the answer to the next problem.

3.010 1/2	**3.260** 19/27	**3.510** 5/36
3.020 144/625	**3.270** 2/5	**3.520** 14/33
3.030 1/12	**3.280** 15	**3.530** 5/32
3.040 5/143	**3.290** 12/47	**3.550** 27/28
3.050 5/6	**3.300** 3/4	**3.560** 234/425
3.060 30/91	**3.310** 1/100	**3.570** $0.133...
3.070 1/4	**3.320** 80/243	**3.580** $1
3.080 6/5 points	**3.330** 5/84	**3.590** 60.7%
3.090 7.75	**3.340** 5/14	**3.600** 1/102
3.100 1/4	**3.350** 3/4	**3.610** 1/2
3.110 1/12	**3.360** 3/7	**3.620** 5/1,296
3.120 35/66	**3.370** 49/81	**3.650** 3/4
3.130 3/35	**3.380** 20	**3.660** 12/35
3.140 9/44	**3.390** 25/648	**3.680** 84
3.150 3/4	**3.400** 11/221	**3.690** 2π units^2
3.160 6/35	**3.410** 5/324	**3.700** 1/52
3.170 1/2	**3.420** 29%	**3.710** 1/720
3.180 $ 239	**3.430** 3/5	**3.750** 11/12
3.190 6/11	**3.440** 34/95	**3.760** 1/2
3.200 1/8	**3.450** 19/49	**3.790** 2/35
3.210 125/729	**3.460** 65.7%	**3.800** 1/26
3.220 20/27	**3.470** 5/8	**3.810** 3/8
3.230 2/27	**3.480** 11.25	**3.890** 24.2%
3.240 2/11	**3.490** 6	**3.900** 2/51
3.250 3/4	**3.500** 44/4,165	**3.990** 3/8

3.010 The toss of a fair coin is independent of prior events. A *fair* coin will always land on heads with a probability of **1/2** regardless of previous outcomes.

3.020 Let O represent a made free-throw and X represent a miss. The probability of making a free throw is 2/5, so the probability of a miss is 3/5. The probability *P*(OOOXX) of making the first three and missing the last two is:

$\frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{3}{5}$ and there are $\binom{5}{2} = \frac{5 \cdot 4}{2}$ ways to

arrange the missed shots (for example, OOXOX and XOOOX). Multiplying gives us:

$$\frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{5 \cdot 4}{2} = \frac{144}{625}.$$

3.030 There are $_9C_3 = 84$ ways to choose 3 digits from a set of 9. Of these 84 ways, there are 7 sets of consecutive integers (starting with 1-2-3 and ending with 7-8-9). 7/84 = **1/12.**

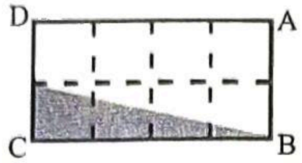
3.040 There are $_{13}C_5 = 1{,}287$ ways to choose 5 cards from a set of 13. If we have KQJ in the set, we must choose 2 of the remaining 10 cards to complete the set. There are $_{10}C_2 = 45$ ways to complete the set, making the probability 45/1,287 = **5/143**.

3.050 It is easier to count the number of ways to roll a 4 or less. There is one way (1,1) to roll a 2, there are 2 ways to roll a 3 (1,2) and (2,1), and 3 ways to roll a 4 (1,3), (2,2), and (3,1) for a total of 6 out of the $6^2 =$ 36 possible rolls, leaving 30 ways to roll greater than a 4. 30/36 = **5/6**.

3.060 Selecting blocks without replacement gives us:

$$\frac{10}{14} \cdot \frac{9}{13} \cdot \frac{8}{12} = \frac{30}{91}.$$

3.070 If point X is selected within the shaded area, the height of triangle ABX will be greater than 4 times the height of triangle BCX, making its area greater than twice that of triangle BCX (its base is half that of triangle BCX). The shaded area is **1/4** the total area.



3.080 Guessing randomly on 5 questions, you could expect to be correct once, earning 6 points. This gives each guess an expected value of **6/5** or **1.2** points.

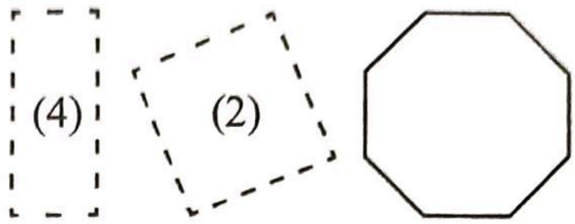
3.090 We will do digit sums and expected value for the months and years separately to simplify things. The expected value for the digit sum of a month is (1 + 2 + 3 + ... + 9) + (1 + 0) + (1 +1) + (1 + 2) = 51. Because each month occurs the same number of times we just divide this by 12 to get 4.25. The expected value for the digit sum of a year between 10 and 15 is [(1 + 0) + (1 + 1) + (1 + 2) + (1 + 3) + (1 + 4) + (1 + 5)]/6 = 21/6 = 3.5. The expected sum of the digits is therefore 4.25 + 3.5 = **7.75.**

3.100 There are 13 hearts in a standard deck of 52 cards, so the probability of drawing a heart is 13/52 = **1/4.**

3.110 The probability of rolling a 5 on the red die is 1/6 and the probability of an even number on the green die is 1/2, making the probability of a 5 on red and even on green: $\frac{1}{6}\cdot\frac{1}{2}=\frac{1}{12}$.

3.120 There are $_{12}C_2$ = 66 ways to select 2 representatives. With 7 boys and 5 girls there are 7 choices for the boy and 5 for the girl, or 7 · 5 = 35 different pairs, making the probability 35/66. Alternatively, we find the probability of selecting boy then a girl and add it to the probability of selecting a girl then a boy:
P(Boy, Girl) = (7/12)(5/11) = 35/132 and
P(Girl, Boy) = (5/12)(7/11) = 35/132, so *P*(Boy, Girl *or* Girl, Boy) = 35/132 + 35/132 = **35/66.**

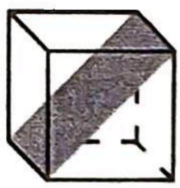
3.130 There are $_8C_4 = 70$ ways to select 4 of the 8 vertices. Only 6 of these combinations form rectangles (shown), making the probability 6/70 or **3/35**.



3.140 There are $_{12}C_3 = 220$ ways to select three students from a group of 12. If Adam is selected, there are 10 students left (excluding Billy) and we need to choose two to pair with Adam. $_{10}C_2 = 45$ pairs can be selected, meaning there are 45 groups of 3 students which include Adam but not Billy out of the 220 possible groups. 45/220 = **9/44**.

3.150 There are 4! = 24 ways to arrange the digits. To be divisible by 4, the integer formed must end in either 12, 24, or 32. Two of each of these makes 6 numbers divisible by 4: 3,412; 4,312; 1,324; 3,124; 1,432; and 4,132. This means there are 18 out of the 24 that are not divisible by 4 or **3/4**.

3.160 There are $_8C_4 = 70$ ways to select 4 of the 8 vertices of a cube. For all four points to be on the same plane, they can all be on the same face (6 ways), or on one of 6 diagonal planes slicing through the cube (one is shown). 12/70 = **6/35**.



3.170 The larger shaded ring has an area of $(6^2 - 5^2)\pi = 11\pi$ and the smaller ring has an area of $(4^2 - 3^2)\pi = 7\pi$, for a combined area of 18π. The entire circle has an area of 36π, making the probability of landing in a shaded area **1/2**.

3.180 \$0 +\$2 + \$10 + \$50 + \$100 + \$250 + \$1,000 = \$1,912. \$1912/8 = **\$239**.

3.190 Think of the game playing out in rounds. Each round consists of each player taking a roll (of course, Ben may lose before his roll in any given round). Alice has a 1/6 chance of rolling a 6 to win the game on the first roll of each round. The probability that Ben will win a round requires that Alice not roll a 6 (and Ben must roll a 6). This will occur with probability (5/6)(1/6) = 5/36. If no one wins a round, a new round begins and this continues until someone wins. Alice will win 1 of 6 (or 6 of 36) rounds, while Ben will win 5 of 36 rounds. The ratio of Alice's wins to losses (odds) is therefore 6:5, making the probability of Alice winning: **6/11** (with Ben winning 5 of 11 games).

3.200 With replacement, the probability that each selected card will be red is 1/2. $(1/2)^3$ = **1/8.**

3.210 The probability of drawing an odd digit from the digits 1 through 9 is 5/9. We multiply this three times:

$$\left(\frac{5}{9}\right)^3 = \frac{125}{729}.$$

3.220 The probability of flipping tails on a single toss is 2/3. The probability of flipping three tails in a row is $(2/3)^3$ = 8/27. There are three ways to flip two tails and a heads (TTH, THT, and HTT). The probability of each is $(1/3)(2/3)^2$ = 4/27. Multiply this by three to get 12/27. The probability of flipping more tails than heads is the combined probability of flipping two or three tails on three tosses. 8/27 + 12/27 = **20/27.**

3.230 The total number of ways colors can be chosen for the flag is $3^4 = 81$ (four stripes can each be one of three colors). Now look at the ways that we can choose colors where no two same-colored stripes share an edge. Begin by choosing any of the 3 colors for the vertical stripe. Once we choose a color for this stripe, none of the other stripes can be the same color. We can pick 1of the 2 remaining colors for the middle horizontal stripe. The remaining stripes must both be the remaining color. This gives us only 3! = 6 ways that no two same-colored stripes can share an edge. 6/81 = **2/27**.

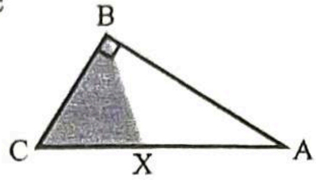
3.240 There are many ways to approach this problem: First, the probability of two girls being in any one group is:

$$3\left(\frac{2}{12}\cdot\frac{1}{11}\cdot\frac{10}{10}\right)=\frac{1}{22}$$ (the three is there because the group may be selected GGB, GBG, or BGG). There are four groups, making the probability 2/11. Alternatively, there are $_{12}C_3 = 220$ ways to form a group and 10 ways to form a group with the two girls (2 girls plus 1 of the 10 remaining boys) for a probability of 1/22 that any one of the groups has both girls. Once again, 4 groups quadruples this probability to give us 2/11. Finally (and perhaps easiest), consider the probability that one of the girls has already been assigned to a group. 2 of the 11 available spaces are in her group, making the probability **2/11** that the second girl is placed with the first.

3.250 There are $2^3 = 8$ ways to flip 2 coins. Only HHH and TTT do not show at least one tails or one heads, so there are 6 of 8 possibilities which show at least one of each, making the probability **3/4**.

3.260 The toast will land buttered-side down 2/3 of the time. At least one piece will land buttered-side up as long as all three pieces do not land buttered-side down. The probability that all three will land buttered-side down is $(2/3)^3 = 8/27$. Subtract this from 1 to get **19/27**.

3.270 Consider the triangle in the orientation shown. Triangles CXB and AXB have the same height (the perpendicular from B to the hypotenuse of triangle ABC). The ratio of their bases CX:AX = 2:3, so the ratio of their areas is $2h$:$3h$ or 2:3, making the shaded area 2/5 of the total area and the probability of a random point inside the triangle being selected within the shaded region is **2/5**.



3.280 The sum of the first nine digits is $\frac{9(10)}{2} = 45$, making the average of any row or column 45/3 = 15. The sum of each diagonal is 15 as well, so the expected value is **15**.

3.290 The surface area of the prism is $2(3 \cdot 4) + 2(3 \cdot 5) + 2(4 \cdot 5) = 94\,\text{cm}^2$, and the small faces have a combined area of $2(3 \cdot 4) = 24\,\text{cm}^2$. The probability of a randomly selected point being on one of the two smaller faces is therefore 24/94 = **12/47**.

3.300 Your first selection can be any card from the deck. The probability that the second card will be from a different suit is 39/52 = **3/4**.

3.310 There are 10^3 = 1,000 possible combinations of three digits for the last 3 digits of a phone number (000 through 999), and only 10 have the same three digits, so the probability is 10/1,000 or 1/100. Alternatively, the first digit doesn't matter. The probability of each of the next digits being the same as the first is 1/10, which gives us $(1/10)^2$ = **1/100**.

3.320 A student will select a chocolate chip cookie with a probability of 2/3. Using C for chocolate chip and O for oatmeal, the probability that the first three students will choose chocolate chip followed by two students selecting oatmeal is:

$P(\text{CCCOO}) = \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 = \frac{8}{243}$, but there are

${}_5C_2 = 10$ ways to arrange CCCOO, making the probability **80/243**.

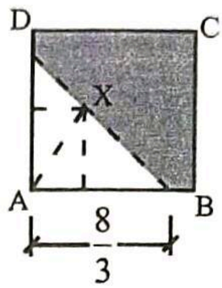
3.330 There are ${}_9C_3 = 84$ combinations of 3 numbers. Only two common factors greater than 1 are possible: 2 and 3. There are 4 even numbers, and ${}_4C_3 = 4$ possible sets of 3 even numbers. There is one set of three numbers which have 3 as a common factor $\{3,6,9\}$. This means that there are 5 out of 84 combinations of 3 numbers which share a common factor greater than 1: **5/84** is the probability of selecting one of these sets.

3.340 For a set of three numbers to have a product that is odd, all of the numbers must be odd. The question is therefore: What is the probability that one of the sets contains only odd numbers? There are ${}_9C_3 = 84$ combinations of 3 numbers and ${}_5C_3 = 10$ combinations of 3 odd numbers. The probability of any one set containing only odd numbers is 10/84 or 5/42, and there are three sets, which makes the probability 3(5/42) = 15/42 = **5/14**.

3.350 It is easiest to calculate the probability that the product is odd, which requires that both rolls be odd. The probability of rolling an odd number is 1/2, so the probability of rolling two odds in a row is $(1/2)^2 = 1/4$. Subtract this from 1 to get the probability that the product of the two rolls is even: **3/4**.

3.360 There are 8 vertices of a cube, which means there are $_8C_3 = 56$ ways to connect three of the vertices to form a triangle. There are $_4C_3 = 4$ possible triangles on each of 6 faces for a total of 24 out of 56 triangles: $24/56 =$ **3/7**.

3.370 Consider the area of quadrilateral ABXD as the sum of the areas of triangles ADX and ABX. Each has a base length 3, so the sum of the heights of the triangles must be 8/3 for the combined area to be greater than 4 (X must be anywhere within the shaded area). The area of the unshaded triangle is $(1/2)(8/3)^2 = 32/9$ and the area of the entire figure is 81/9, making the shaded area 49/9. The probability that point X is within the shaded area is therefore



$$\frac{49/9}{81/9} = \mathbf{49/81}.$$

3.380 The expected value is the average value of the bills in the bag. Let x represent \$1 bills and y represent \$5 bills: $x + 5y$ is the total value and $x + y$ is the number of bills:

$\frac{x+5y}{x+y} = 1.2$, so $x+5y = 1.2x+1.2y$. We will solve for the ratio x/y: $3.8y = 0.2x$ gives us $\frac{3.8}{0.2} = \frac{x}{y} = \frac{19}{1}$,

so there must be a minimum of **20** bills in the bag (one \$5 bill and nineteen \$1 bills).

3.390 There are $6^5 = 7{,}776$ ways to roll a set of 5 dice. We will use XXXOO to represent the full house, where X represents one number rolled and O represents a different number. There are $_5C_2 = 10$ arrangements of XXXOO, and we can choose from 6 numbers to use for X and 5 are left for O. This gives us $10 \cdot 6 \cdot 5 = 300$ of the 7,776 ways, or a probability of **25/648**.

3.400 There are 3 face cards in each suit for a total of 12 face cards in a deck of 52 cards. The probability that the top card is a face card is 12/52, and if the top card is a face card, the probability that the second card is a face card is 11/51. The probability that both are face cards is:

$$\frac{12}{52}\cdot\frac{11}{51}=\frac{11}{221}.$$

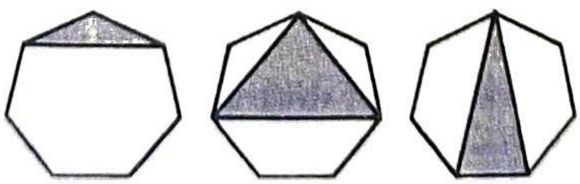
3.410 There are $6^6 = 46{,}656$ possible rolls. The first roll can be any of the first 6 digits, after which there are only 5 available rolls, then 4, 3, 2, and 1, meaning there are 6! ways to roll a 1-6 in any order:

$$\frac{6!}{6^6}=\frac{\cancel{6}\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}}{\cancel{6}\cdot 6\cdot 6\cdot \underset{3}{\cancel{6}}\cdot \underset{3}{\cancel{6}}\cdot \cancel{6}}=\frac{5}{324}$$

3.420 The probability of rain expressed as a fraction is 2/5 (and the probability that it will not rain is 3/5), so the probability of three rainy days in a week is

$\left(\frac{2}{5}\right)^3\cdot\left(\frac{3}{5}\right)^4=\frac{647}{78{,}125}$. We must multiply this by the number of ways to arrange these three rainy days in a 7-day week: $_7C_3 = 35$, so $35\cdot\frac{647}{78{,}125}\approx$ **29%**.

3.430 There are $_7C_3 = 35$ ways to connect three of the vertices to form a triangle. We can use any vertex of the heptagon as the apex angle for three different isosceles triangles for a total of 21 possible isosceles triangles (out of 35). This makes the probability **3/5**.



3.440 There are ${}_{20}C_3 = 1{,}140$ sets of 3 students who can be chosen from a group of 20. We are looking to form a group with 1 vegetarian and 2 non-vegetarians. There are ${}_{3}C_1 = 3$ ways to choose the vegetarian and ${}_{17}C_2 = 136$ ways to choose the non-vegetarians, for a total of $3 \cdot 136 = 408$ ways to form a group with one vegetarian and two non-vegetarians, which gives us a probability of $408/1{,}140 = 34/95$.
Alternatively, The probability of selecting vegetarian, non-vegetarian, non-vegetarian in that order is:

$\frac{3}{20} \cdot \frac{17}{19} \cdot \frac{16}{18} = \frac{34}{285}$, and we can arrange these picks three different ways (the veg. can be 1st, 2nd, or 3rd),

which gives us: $3 \cdot \frac{34}{285} = \frac{34}{95}$.

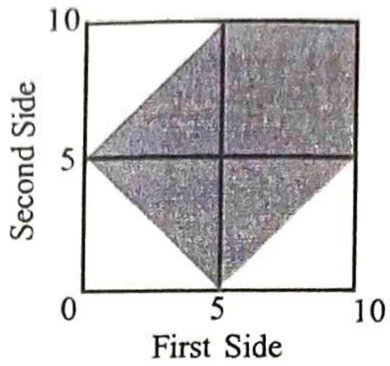
3.450 We will find the probability that no two dates fall on the same day of the week and subtract this from 1. The first date selected can be on any one of the seven days. The second date must fall on one of the six remaining days, and the third date must fall on one of the remaining 5 days for a probability of:

$\frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} = \frac{30}{49}$, making the probability that at least two of the dates fall on the same day of the week

$1 - \frac{30}{49} = \frac{19}{49}$.

3.460 There is a 70% chance that it will not snow on each of the next three days. The probability that it will not snow at all is $(0.7)^3 = 0.343$ or 34.3%. This leaves a probability that it will snow at least once:
$1 - 0.343 = 0.657 = \mathbf{65.7\%}$.

3.470 The remaining sides of the triangle must have a sum that is greater than 5 and a difference that is less than 5 as represented by the shaded area in the graph below. The shaded area is **5/8** the total area.



3.480 To solve this problem, we will solve for each of the four digits on a digital clock separately (2:34 will be considered 02:34). The digit farthest to the left will be a 1 for 3 out of 12 hours, for an expected digit value of 0.25. The next (hours) digit will be 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2 for equal amounts of time for an expected value of 48/12 = 4. The next digit will be 0, 1, 2, 3, 4, and 5 with equal frequency for an expected value of 2.5. Finally, the minutes digit will be 0 through 9 with equal frequency for an expected value of 4.5. The expected value for the sum of the digits is the sum of the expected values for each digit: $0.25 + 4 + 2.5 + 4.5 = \mathbf{11.25}$.

3.490 For the product of two integers to be odd, both integers must be odd. If there are three odd integers, then there are $_3C_2 = 3$ combinations of two odd integers, and this is 1/5 the total number of combinations. There must be 15 total combinations. $_6C_2 = 15$ so there are **6** integers in the set altogether (3 odd and 3 even).

3.500 The first card can be of any suit. The next three cards must be of the same suit as the first, so we have:

$$\frac{52}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}\cdot\frac{10}{49}=\frac{44}{4{,}165}$$

3.510 The modified die will have faces with 2, 3, 4, 5, 6, and 7 dots. There will still be 36 possible rolls with a pair of the modified dice. Of those 36 rolls, 5 will show an 8: (2,6), (3,5), (4,4), (5,3), (6,2), making the probability of rolling an 8: **5/36.**

3.520 The probability of selecting two blue socks then two black socks is found by:

$\frac{7}{12}\cdot\frac{6}{11}\cdot\frac{5}{10}\cdot\frac{4}{9}=\frac{7}{99}$, and there are ${}_4C_2 = 6$ ways to arrange the order of two black and two blue socks. Multiplying, we get a probability of:

$$6\cdot\frac{7}{99}=\frac{14}{33}.$$

Alternatively, there are ${}_{12}C_4 = 495$ ways to choose 4 socks from a set of 12. There are ${}_7C_2 = 21$ ways to choose two of the blue socks and ${}_5C_2 = 10$ ways to choose two black socks, which makes $21 \cdot 10 = 210$ ways to choose 2 blue and two black socks from the drawer. 210/495 = **14/33.**

3.530 Call the six faces of the cube: top, bottom, left, right, front, and back. With six faces and two choices of color for each face, there are $2^6 = 64$ ways to paint the faces of the cube. Of these 64 ways, there is one way to paint every face white, there are six ways to paint one face black, and three ways to paint two faces black so that no two black faces share an edge (front/back, left/right, or top/bottom). This makes 10 out of 64 ways, or a probability of **5/32.**

3.550 It is much simpler to calculate the probability that all three are the same color. There are three blocks of each color, so the probability is the same for each:

$\frac{3}{9}\cdot\frac{2}{8}\cdot\frac{1}{7}=\frac{1}{84}$. Multiply this by 3 because there are three colors to get 1/28. The probability that no two blocks will be different colors is 1/28, so the probability that at least two will be different colored is **27/28**.

3.560 In any suit, there are ${}_{13}C_2 = 78$ ways to select two cards, and there are 39 cards that can complete the set, making $78 \cdot 39 = 3{,}042$ ways to create a set with two cards from a given suit and one card from outside that suit. There are four suits, giving us $3{,}042 \cdot 4 = 12{,}168$ ways. There are ${}_{52}C_3 = 22{,}100$ ways to choose three cards from a set of 52. Simplifying 12,168/22,100 give us **234/425**. Alternatively, consider the probability of choosing two cards of a given suit followed by one from a different suit:

$\frac{13}{52} \cdot \frac{12}{51} \cdot \frac{39}{50} = \frac{39}{850}$. There are three ways to order the selection of the suits (two hearts and a "not heart" can be selected HHX, HXH, or XHH). We must also multiply by the 4 choices for the suit that has two cards. This gives us:

$3 \cdot 4 \cdot \frac{39}{850} = \frac{468}{850} = \frac{234}{425}$. With either method, this is an easy problem to mess up.

3.570 Let's start with a scenario using 64 pennies to see if we can find a pattern. If you insert 64 pennies, you can expect to get back 16 each of pennies, nickels, dimes, and quarters, for a total of 16($0.40) + 16 pennies to reinsert. When you reinsert the 16 pennies, you can expect to get back 4 of each coin for a total of 4($0.40) + 4. Finally, inserting the four pennies you can expect to get back $0.40 + a penny. We see that the expected value of 64 pennies is 16($0.40) + 4($0.40) + 1($0.40) + 1 penny.

Now, let's use this pattern to solve for *n* pennies: For every *n* pennies inserted, we can expect to get:

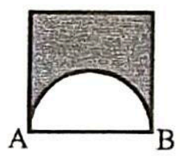
$$\frac{n}{4}(0.40)+\frac{n}{16}(0.40)+\frac{n}{64}(0.40)+\ldots$$

$$=n(0.10)+\frac{n}{4}(0.10)+\frac{n}{16}(0.10)+\ldots$$

$=0.10n\left(1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}\ldots\right)$. The infinite series in parenthesis is equal to $\frac{4}{3}$ (see **p.56**), so we have $0.10n\left(\frac{4}{3}\right)=0.1\bar{3}n$. The expected value of each penny is \$0.40/3 = **\$0.1$\bar{3}$**.

3.580 There are 12 blocks, so there are $_{12}C_2$ = 66 ways to draw two blocks. Assume you draw all 66 possible pairs. There are $_4C_2$ = 6 ways to draw two red and $_5C_2$ = 10 ways to draw two blue blocks (earning \$48 total) and there are $_3C_2$ = 3 ways to draw two white blocks (earning \$18). In 66 draws, you could expect to win \$66 for an expected value of **\$1**.

3.590 If point X is selected on the semicircle, angle AXB will be a right angle (see inscribed angles **p.185-186**). If X is chosen utside of the semicircle the angle will be acute. Inside the circle, the angle will be obtuse. Use 4cm for the side of the square. The area of the semicircle is 2π cm^2 and the area of the square is 16cm^2. The shaded area is therefore $(16-2\pi)$ cm^2 and the probability that point X will be within the shaded area is:



$$\frac{16-2\pi}{16}\approx\mathbf{60.7\%}.$$

3.600 There are 2 red aces in a 52-card deck. If the top card is a red ace, 13 of the remaining 51 cards are spades: $\frac{2}{52}\cdot\frac{13}{51}=\frac{1}{102}$.

3.610 The units digit of each integer can be odd or even, with equal probability. For the sum of the three units digits to be even, we can have either 2 odd digits and 1 even digit: {O,O,E}, {O,E,O}, or {E,O,O}, or we can have 3 even digits: {E,E,E}. There are $2^3 = 8$ possibilities (think of this as counting coin flips, where each flip determines whether the units digit is odd or even), which makes the probability that the sum is even 4/8 = **1/2**.

3.620 There are 6^6 possible outcomes for six rolls of a standard die. Consider OOOOOX as a representation of five of the same digit and one different digit. There are 6 ways to arrange OOOOOX, six numbers to select for the O, and five numbers to select for the X. This gives us: $\frac{6\cdot 6\cdot 5}{6^6}=\frac{5}{6^4}=\frac{5}{1{,}296}$.

3.650 We will look for the probability that A and B end up in the same pile. There are $_9C_3 = 84$ ways to choose 3 of 9 letters for a pile. There are 7 possible sets that include A, B, and one of the remaining 7 letters. This makes the probability that any pile of 3 letters will include both A and B 7/84 = 1/12. There are 3 piles, so the probability that 1 of 3 piles will include both A and B is 3(1/12) = 3/12 or 1/4. The probability that A and B will be in different piles is therefore $1-1/4=$ 3/4. Alternatively, place the A first:

A _ _   _ _ _   _ _ _   _ _ _

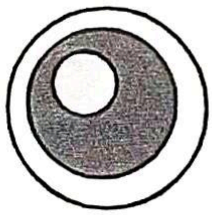
This leaves 8 places for the B. 2 of those places are in the same pile as the A, while 6 are in one of the other piles. 6/8 = **3/4**.

3.660 There are 7C3 = 35 possible combinations. We are looking for combinations whose sum is prime. Be methodical. The smallest possible sum is 9 and the largest possible sum is 21. Primes between 9 and 21 are 11, 13, 17, and 19. Look for ways to get each:
11 = 2+4+5 = 2+3+6 (2 ways),
13 = 2+5+6 = 2+4+7 = 2+3+8 = 3+4+6 (4 ways),
17 = 2+7+8 = 3+6+8 = 4+6+7 = 4+5+8 (4 ways),
19 = 4+7+8 = 5+6+8 (2 ways).
This gives us a total of 12 out of 35 sums that are prime, or a probability of **12/35**.

3.680 Call the number of \$1 and \$2 tokens x. The total value of all the tokens is $x + 2x + 18(5) = 3x + 90$. The total number of tokens is $2x + 18$, therefore:

$\frac{3x+90}{2x+18} = 2.25$, so $3x + 90 = 4.5x + 40.5$. Solving for x we get $x = 33$. There are 33 + 33 + 18 = **84** tokens in the bag.

3.690 Call the smallest radius a, the medium radius b, and the largest radius c. The shaded area must equal half the entire area:



$\pi(b^2 - a^2) = \frac{\pi c^2}{2}$, or $b^2 - a^2 = \frac{c^2}{2}$.

This simplifies the guess-and-check process, as we look for a difference of squares that is equal to half a perfect square. When $a = 1$, $b = 3$, and $c = 4$: $3^2 - 1^2 = 4^2/2$. The radii can be 1, 3, and 4. However, the problem states that we only need integral diameters. We can use diameters $a = 1$, $b = 3$, $c = 4$ and maintain the proportions of the figure. The shaded area then becomes half the area of the largest circle: $\frac{\pi(2)^2}{2} = 2\pi$ units².

3.700 There are two cases to consider: the first card can be an ace of spades, or an ace from another suit. If the ace is the ace of spades, the probability is:

$$\frac{1}{52}\cdot\frac{12}{51}=\frac{12}{52\cdot 51}.$$

If the top card is the ace of another suit we have:

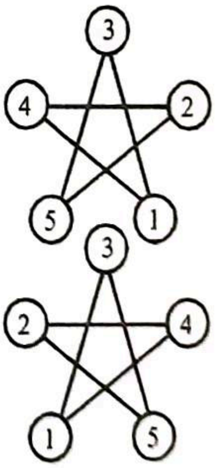
$$\frac{3}{52}\cdot\frac{13}{51}=\frac{39}{52\cdot 51}.$$

Added together (now you see why I didn't multiply the $52\cdot 51$):

$$\frac{51}{52\cdot 51}=\frac{1}{52}.$$

3.710 There are $6! = 720$ ways to arrange 6 students in order, and only one of these ways seats the students from oldest to youngest, so the probability is **1/720**.

3.750 Start by placing the 3. Any arrangement can be rotated so that the three is located on top. When we place the three on top, this leaves 4 blanks to fill with 4 integers for $4! = 24$ possible ways to complete the diagram. To avoid consecutive integers, we can have a 1 and a 5 connected to the three as shown in two ways with the 2 and the 4 forced (depending on the placement of the 1 and the 5). This gives us just 2 ways to avoid consecutive integers (shown), so there are 22/24 = **11/12** arrangements with at least one connected pair of consecutive integers.



3.760 Consider the question, "What is the probability that there are more tails showing than heads?". The answer to both questions must be the same, and it is impossible to flip an equal number of heads and tails with 27 flips. The answer must therefore be **1/2**.

3.790 There are $_8C_2 = 28$ ways to choose the first pair of vertices, and $_6C_2 = 15$ ways to select the second pair of vertices, and because we can switch the order of the selection (the first pair can be selected second), we find that there are $(28 \cdot 15)/2 = 210$ ways to connect two pairs of two vertices. Alternatively, there are $_8C_4 = 70$ ways to choose four vertices (call these four A, B, C, and D) after which we can connect A to any of the three other vertices (then connecting the remaining pair) to get $70 \cdot 3 = 210$. Intersecting vertices can either intersect in the center of the cube or at the center of one of the six faces. There are 6 pairs which have their intersection on a face. There are 4 diagonals which pass through the center of the cube. We can choose any two of these for $_4C_2 = 6$ possible combinations of diagonals which intersect at the center of the cube, bringing the total to 12 out of 210 = **2/35**.

3.800 There are two cases to consider: the ace can be an ace of spades, or an ace from another suit. We multiply by 2 because the ace can be the top card or the second card. If the ace is the ace of spades, the probability is:

$$2 \cdot \frac{1}{52} \cdot \frac{12}{51} = \frac{24}{52 \cdot 51}.$$

If the ace is of another suit:

$$2 \cdot \frac{3}{52} \cdot \frac{13}{51} = \frac{78}{52 \cdot 51}.$$

Added together: $\dfrac{102}{52 \cdot 51} = \dfrac{2}{52} = \dfrac{1}{26}$.

3.810 The product will be positive if the two spins are either both positive or both negative.

$$P(\text{both positive}) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

$P(\text{both negative}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

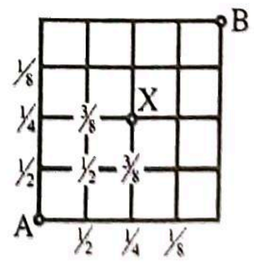
Add these together to get a probability of **3/8**.

3.890 If there is a 75% chance that it will erupt within the next 5 years, then there is a 25% chance that it will not erupt. This means that if n is the probability (expressed as a decimal) that it will not erupt during each of the five years, then $n^5 = 0.25$. $\sqrt[5]{0.25} \approx 0.758$ gives us a probability of about 75.8% each year that it will not erupt, so the probability of eruption during each of the five years is about **24.2%**.

3.900 There are 13 pairs of consecutive cards in each suit (Ace-2 through King-Ace) for a total of 52 pairs of consecutive cards. There are ${}_{52}C_2 = (52 \cdot 51)/2$ combinations that could be the top two cards of a shuffled deck: $\frac{52}{(52 \cdot 51)/2} = \frac{52}{26 \cdot 51} = \frac{2}{51}$.

Alternatively, for any top card, 2 of the remaining 51 cards in the deck are consecutive. **2/51.**

3.990 Do not be fooled into solving this by counting the number of paths from A to B which pass through X (36) and dividing by the total number of paths from A to B (70). Coin flips create the probabilities shown, as each intersection offers two equally likely choices. There are two points from which Jennifer can walk to X, each has a 3/8 probability of being reached with Jennifer's coin flip method. Half of the time from each point she will travel to X:



$$\frac{1}{2}\left(\frac{3}{8} + \frac{3}{8}\right) = \frac{3}{8}.$$

Number Theory
Key and Solutions

Problems within the text are ordered with the last three digits reversed. This way, if you are looking for a solution you will not accidently see the answer to the next problem.

4.001 abundant
4.002 2
4.010 907,654 & 14,256
4.011 GCF: 15, LCM: 180
4.012 $0.\overline{41}$
4.020 8
4.021 6
4.022 5
4.030 $41 \cdot 7$, $17 \cdot 23$, prime
4.031 42
4.032 4
4.040 72: 12 180: 18
210: 16 112: 10
4.041 31
4.042 199
4.050 20
4.051 16
4.052 15
4.060 280
4.061 1
4.070 12, 6, 96, 96, 6^{th}
4.071 $5{,}000{,}200_8$
4.080 $3^6 \cdot 5^3$
4.081 250_8
4.090 50: 93
405: 726
210: 576
4.091 83,195,146
4.100 2, 3, 4, 6, 8
4.101 deficient
4.102 0
4.110 2
4.111 GCF: 16, LCM: 560
4.112 $0.\overline{02}$
4.120 4
4.121 4
4.122 1
4.130 7
4.131 60
4.132 i
4.140 12
4.141 720
4.142 3
4.150 9
4.151 37
4.152 1/2
4.160 315
4.170 9, 196, 14, 14, 9^{th}
4.171 16
4.180 a^b
4.181 $436{,}000_8$
4.190 8
4.191 4
4.200 2, 3, 4, 5, 6, 9, 10
4.201 deficient
4.202 0
4.210 23,232
4.211 GCF: 7, LCM: 420
4.212 $0.0\overline{71}$
4.220 4
4.221 Wed.
4.222 7/90

4.231 19,344
4.232 6
4.240 48
4.241 19
4.242 11,232
4.250 12
4.251 4
4.252 37/150
4.260 720
4.271 a. 83 b. 5
c. 511
4.280 4
4.281 666_8
4.290 a. 127 b. 255
c. 511 d. $2^{31}-1$
4.291 68
4.301 abundant
4.310 99,990
4.311 105
4.312 $0.\overline{06}$
4.320 3,312
4.321 60
4.322 21
4.331 3,600
4.332 26
4.340 9
4.341 409
4.342 6
4.350 9
4.351 1,067
4.352 8,748
4.371 215
4.380 9
4.381 $1{,}110_6$
4.390 7^3 or 343
4.391 5
4.401 perfect
4.410 8
4.411 36
4.412 $0.\overline{045}$

4.420 18
4.421 16
4.422 $0.\overline{471}$
4.431 97
4.432 248
4.440 7
4.441 8
4.442 2
4.450 13
4.451 32
4.452 0
4.471 5
4.480 27/32
4.481 $1{,}332_4$
4.490 12.4
4.491 6
4.500 2,232
4.510 4
4.511 64
4.512 $0.\overline{054}$
4.520 5,744
4.521 19
4.522 3
4.530 440: $2^3 \cdot 5 \cdot 11$
432: $2^4 \cdot 3^3$
209: $11 \cdot 19$
4.531 6^8
4.532 < 17 times.
4.540 2^{30}
4.542 1,023
4.550 7
4.551 2209
4.552 23 students
4.571 $2{,}102_4$
4.581 111_8
4.590 48
4.591 4
4.600 10,008
4.610 90,002
4.620 1,998

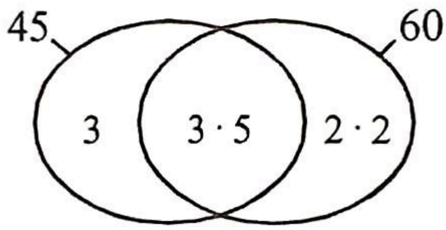
4.622 999
4.630 441: $3^2 \cdot 7^2$
256: 2^8
576: $2^6 \cdot 3^2$
(squares)
4.631 10,800
4.632 407
4.640 60, 72, 84, 90, 96.
4.642 2
4.650 10
4.651 27
4.652 94
4.671 $1{,}000{,}000{,}000{,}000_3$
4.681 25_6
4.690 1
4.720 9,876,513,240
4.722 714,285
4.730 25^2
4.731 416 or 7^{416}
4.742 6
4.750 24
4.751 7^{57}
4.752 44
4.771 0.56
4.781 $2{,}400_6$
4.790 992
4.820 3
4.830 12^2
4.831 16
4.842 13/90
4.850 10
4.851 11
4.881 101_2
4.920 24,442
4.930 270^2
4.931 33
4.942 1,327,104
4.950 180
4.951 42
4.981 300_9

4.001 $40 = 2^3 \cdot 5$. Its factor sum is $(1+2+4+8)(1+5) = 90$, which is greater than 2(40), making 40 **abundant**.

4.002 The units digit of 1976^{61} is 6 (any number ending in 6 raised to a power ends in 6). The units digit of 2007^{61} is 7 (the digits cycle in sets of four 7-9-3-1, and 60 is divisible by 4, so 2007^{60} ends in a 1 and 2007^{61} ends in a 7). The units digit of $(1976^{61})(2007^{61})$ will be the same as the units digit of $6 \cdot 7$: **2**.

4.010 3,951 is not divisible by 11. **907,654** is divisible by 11. Sum alternating digits and subtract:
$(9+7+5)-(0+6+4)=21-10=11$.
14,256 is also divisible by 11:
$(1+2+6)-(4+5)=0$

4.011 The GCF is the product of the factors in the overlapping region, while the LCM is the product of all the factors. The **GCF is 15** and the **LCM is 180**.



4.020 For 45,2_8 to be divisible by 3, the sum of the digits 4 + 5 + 2 + _ + 8 must be divisible by 3. This narrows our choices to 2, 5, and 8. For 45,2_8 to be divisible by eight, 2_8 must be divisible by 8. Of 2, 5, and 8, only the **8** makes 2_8 divisible by 8.

4.021 We find the LCM of 24, 30, and 100 is 600, so you must buy **6** packs of napkins to have the same number of cups, plates, and napkins.

4.022 $\dfrac{1}{256}=\dfrac{1}{2^8}=\left(\dfrac{1}{2}\right)^8=(0.5)^8$, which must end in a **5** (true for all $1/2^n$ where n is a positive integer).

4.031 $\frac{12!}{11!}+\frac{10!}{9!}+\frac{8!}{7!}+\frac{6!}{5!}+\frac{4!}{3!}+\frac{2!}{1!}$ simplifies to $12 + 10 + 8 + 6 + 4 + 2 =$ **42**.

4.032 The digits in $0.\overline{02439}$ repeat in blocks of 5, so the 1,896,250th digit is a 9. The third digit after every 9 will be a 4, so the 1,896,253rd digit is a **4**.

4.041 The number of zeros at the end of a number is determined by the number of times 10 is a factor, so we are looking to pair up as many 2's and 5's in the prime factorization as possible. There are many more 2's than 5's, so we will concentrate on counting 5's. Every number that is divisible by 5 will contribute at least one 5. From $1(5) = 5$ to $25(5) = 125$ there are 25 multiples of 5. Additionally, multiples of 25 contribute two 5's. There are 5 multiples of 25. This gives us an additional five 5's. Finally, 125 contributes three 5's (we have already counted it twice, so we add just one more 5). This gives us a total of $25 + 5 + 1 = 31$ fives. This means that 125! has 5^{31} in its prime factorization. Pairing each five with a 2, we see that 10^{31} divides 125!, and 125! therefore ends in **31** zeros.

4.042 199 is prime. The product of its integer factors is **199**.

4.050 $480 = 2^5 \cdot 3^1 \cdot 5^1$, so it has $(5+1)(1+1)(1+1) = 24$ factors. There are $(1+1)(1+1) = 4$ odd factors, so **20** of its factors are even.

4.051 To be divisible by 99, the integer $4\underline{a},\underline{b}13$ must be divisible by both 11 and 9. The sum of its digits must be divisible by 9 and $4 + 1 + 3 = 8$, so $a + b$ must be either 1, 10, or 18. To be divisible by 11, $(4+b+3)-(a+1)$ must be either 0, or 11. This eliminates $a + b = 1$ and $a + b = 18$, so $a + b = 10$. Some quick guess-and-check gives us $a = 8$ and $b = 2$, so $ab =$ **16**, or solve the system of equations (p.16): $a + b = 10$ and $(4+b+3)-(a+1)=0$ to get $a = 8$ and $b = 2$. The system when $(4+b+3)-(a+1)=11$ gives us 2.5 and 7.5 for a and b.

4.052 For $1/n$ to be represented by a terminating decimal, the prime factorization of n must be of the form $2^a \cdot 5^b$. Be careful counting combinations where $2^a \cdot 5^b \leq 100$: Powers of 2: 2^0 through 2^6. Powers of 5: 5^1 and 5^2. 5 times a power of 2: 10, 20, 40, 80. 25 times a power of 2: 50, 100.
We have $n = 1, 2, 4, 8, 16, 32, 64, 5, 25, 10, 20, 40, 80, 50$, and 100 for a total of **15** values.

4.060 The prime factorization can have a 2 or a 3, but not both. It makes sense to use only the 2. We quickly arrive at $2^3 \cdot 5^1 \cdot 7^1 =$ **280**, which has $4 \cdot 2 \cdot 2 = 16$ factors and is the smallest such integer not divisible by 6. Other candidates: $2^7 \cdot 5^1 = 640$, $2^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 = 770$, and $2^3 \cdot 5^3 = 1{,}000$

4.061 For a fraction $n/496$ in simplest terms to have a numerator of 1, n must be a factor of 496. To find the sum of all such fractions, the numerator will be the factor sum of 496 (**4.790**), not including 496. 496 is a perfect number whose factor sum is 992. If we exclude 496 from the sum, we get 496. Therefore, the sum of all fractions $n/496$ which can be simplified so that the numerator is 1 is $496/496 =$ **1**.

4.071 The place values in base 8 are $\underline{8^6}$, $\underline{8^5}\,\underline{8^4}\,\underline{8^3}$, $\underline{8^2}\,\underline{8^1}\,\underline{8^0}$, so representing $5(8^6) + 2(8^2)$ is easy: $\mathbf{5{,}000{,}200_8}$.

4.080 The prime factorization of 45 is $3^2 \cdot 5$, which means 45 has $3 \cdot 2 = 6$ factors (3 pairs) and each pair of factors has a product of 45 ($3^2 \cdot 5$), so the product of these factors is $(3^2 \cdot 5)^3 = \mathbf{3^6 \cdot 5^3}$.

4.081 In any positive integer base, when we multiply by 10, we add a zero. $25_8 \cdot 10_8 = \mathbf{250_8}$.

4.090 $50 = 2 \cdot 5^2$ has a factor sum $(1 + 2)(1 + 5 + 25) =$ **93**. $405 = 3^4 \cdot 5$ has a factor sum $(1 + 3 + 9 + 27 + 81)(1 + 5) =$ **726**. $210 = 2 \cdot 3 \cdot 5 \cdot 7$ has a factor sum $(1 + 2)(1 + 3)(1 + 5)(1 + 7) =$ **576**.

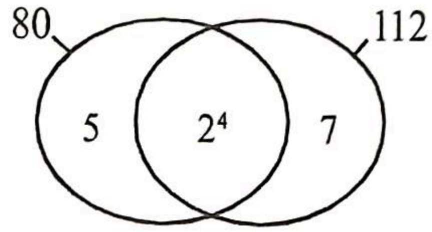
4.091 $9{,}089^2$ ends in a 1 ($9^2 = 81$) and 765^2 ends in a 5, so their sum ends in a $1 + 5 = 6$. Given the choices, **83,195,146** is the only one that makes sense.

4.101 A prime number has only two factors: 1 and itself. The factor sum of a prime number n is therefore $n + 1$. All primes are therefore **deficient**.

4.102 The units digits of the powers of 3 cycle in sets of 4: 3 - 9 - 7 - 1 - 3 - 9 - 7 - 1 ... so to find the units digit for the sum of four consecutive powers of 3, we add $3 + 9 + 7 + 1 = 20$. The units digit is **0**.

4.110 Using the divisibility rule for 11, note that for $89\underline{x}43$ to be divisible by 11, $8 + x + 3$ must equal $9 + 4$ (or the difference of the sums must be divisible by 11, which is not possible in this case). Therefore, $x =$ **2**.

4.111 GCF: **16**
LCM: **560**



4.120 We will use a casework approach. There are no 1-digit integers that work. There are two 2-digit integers: 24 and 42. 3-digit integers 222 and 444 both work, giving us a total of **4** multiples of 3.

4.121 The LCM of 90 and 78 seconds gives us the amount of time it takes for Janice and Kiera to finish a lap at the same time. The LCM is 1,170 seconds, which is equal to 19.5 minutes. They will need to run for another 19.5 minutes to cross the finish line in a whole number of minutes (39 minutes). In 39 minutes (2,340 seconds), Kiera completes 2,340/90 = 26 laps, while Janice completes 2,340/78 = 30 laps, or **4** more laps than Kiera. We could also have found the LCM of 60 (seconds), 78, and 90 to get the number of seconds it would take Janice and Kiera to cross the finish line at the same time in a whole number of minutes.

4.122 $\frac{5}{33}=\frac{15}{99}=0.\overline{15}$. The odd digits are 1's and the even digits are 5's, so the 59th digit is a **1**.

4.130 The 7 other primes are 11, 17, 71, 37, 73, 79, and 97.

4.131 $7!=7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1=7\cdot2\cdot3\cdot5\cdot2\cdot2\cdot3\cdot2=$ $2^4\cdot3^2\cdot5\cdot7$, which has $5\cdot3\cdot2\cdot2=\mathbf{60}$ factors.

4.132 P-R-I-Y-A-N-K-A has 8 letters. $955\equiv3(\text{mod}8)$, so she completes her name and writes three more letters. The third letter in her name is **I**.

4.140 An integer with six factors has its prime factorization in the form a^5 or a^2b. $2^2\cdot3=\mathbf{12}$ is the smallest such number.

4.141 The prime factorization of 10! is $2^8\cdot3^4\cdot5^2\cdot7$. Notice that $(2^4\cdot3^2\cdot5)^2$ is a factor of 10!, so $2^4\cdot3^2\cdot5=\mathbf{720}$ is the largest n where n^2 is a factor of (10!).

4.142 The prime factorization of 900 is $2^2\cdot3^2\cdot5^2$. We can remove a 2, a 3, or a 5 and leave a number that has 18 factors, so there are **3** factors of 900 which have exactly 18 factors (180, 300, and 450).

4.150 The prime factorization of 900 is $2^2\cdot3^2\cdot5^2$. Odd factors cannot include any 2's, so 900 has $3\cdot3=\mathbf{9}$ odd factors.

4.151 When we look for prime factors of 1,517, we check factors up to $\sqrt{1{,}517}$ (about 39). 37 would typically be the last prime factor we check and we find that $1{,}517=\mathbf{37}\cdot41$.

4.152 In base-5, the place values after the decimal point are $5^{-1}, 5^{-2}, 5^{-3}$, etc.

Let $0.\overline{2}_5=x=\frac{2}{5}+\frac{2}{25}+\frac{2}{125}\ldots$ (see **p.56**)

Then $5x=2+\frac{2}{5}+\frac{2}{25}+\frac{2}{125}\ldots$, so $5x-x=2$, $x=\frac{1}{2}$.

4.160 We are looking for an integer with a prime factorization in the form $a^2 \cdot b \cdot c$, or $a^3 \cdot b^2$, or perhaps $a^5 \cdot b$ using prime factors greater than 2. $3^2 \cdot 5 \cdot 7 = 315$, $3^3 \cdot 5^2 = 675$, and $3^5 \cdot 5 = 1{,}215$ so **315** is the smallest odd integer with 12 factors.

4.171 The place values in base 2 are $\underline{2^4}\,\underline{2^3}$, $\underline{2^2}\,\underline{2^1}\,\underline{2^0}$, so the 1 in 10,000 is in the 2^4 place and represents **16.**

4.180 If a has $2b$ factors, it has b pairs of factors, each of which has a product a. The product of the factors of a is therefore $\boldsymbol{a^b}$.

4.190 $6 = 2 \cdot 3$: $(1 + 2)(1 + 3) = 12$
8 $= 2^3$: $(1 + 2 + 4 + 8) = 15$
$9 = 3^2$: $(1 + 3 + 9) = 13$

4.191 Looking only at units digits:
$(346^2 + 364^2)^2 = (...6 + ...6)^2 = (...2)^2 = ...\mathbf{4}$

4.201 Let x be the sum of the factors of 2^n: $(1 + 2 + ...\ 2^n)$:

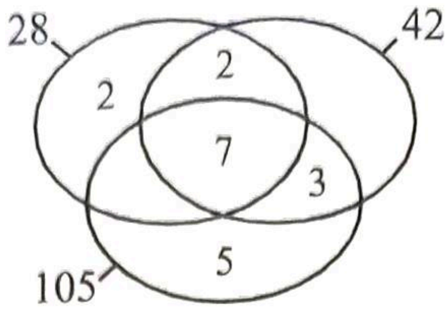
$$\begin{aligned} x &= 1+2+4+...+2^n \\ 2x &= 2+4+...+2^n+2^{n+1} \\ 2x-x &= 2^{n+1}-1 \\ x &= 2(2^n)-1 \end{aligned}$$

We see that the factors sum is always one less than twice the power of 2, which makes all powers of 2 **deficient** (just barely).

4.202 Units digits of the powers of 9 cycle 1-9-1-9 ... , so consecutive powers of 9 will end in a 1 and a 9, meaning the sum of consecutive powers of 9 will always end in a zero: $...1 + ...9 = ...\mathbf{0}$.

4.210 The sum of the 2nd and 4th digits must equal the sum of the 1st, 3rd, and 5th digits, so we must use three 2's and two 3's (placed in the tens and thousands place) to make **23,232**.

4.211 GCF: **7**
LCM: **420**



4.220 Integers divisible by both 8 and 9 are divisible by 72. There are **4** multiples of 72 less than 360.

4.221 The number of days it takes for Ken and Larry to get their hair cut again on the same day is the LCM of 20 and 26, which is 260. 7 divides 259 (37 times), so the 259th day is Tuesday again, and day 260 is a **Wednesday**.

4.222 As fractions, $0.\overline{07}=\frac{7}{99}=\frac{70}{990}$ and $0.0\overline{07}=\frac{7}{990}$.

To convert: $x=0.0\overline{07}$, $10x=0.\overline{07}=\frac{7}{99}$, so

$x=\frac{7}{990}$. Added together we get $\frac{77}{990}=\frac{7}{90}$.

4.231 $7!=7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1=2^4\cdot3^2\cdot5\cdot7$, so the factor sum is $(1+2+4+8+16)(1+3+9)(1+5)(1+7)$, or $(31)(13)(6)(8)=\mathbf{19{,}344}$.

4.232 If we call Katy student 1, then Katy $\equiv1(\text{mod}8)$, and Meera wants to be $\equiv1(\text{mod}8)$. $350\equiv6(\text{mod}8)$, and counting backwards we get $345\equiv1(\text{mod}8)$, which is **6 places** to Katy's right (there are 5 students between Meera and Katy).

4.240 An integer with ten factors will have a prime factorization of the form a^9 or $a^4\cdot b$, so $2^4\cdot3=\mathbf{48}$ is the smallest integer with exactly 10 factors.

4.241 $17!+18!=17!(1+18)=19\cdot17!$, making **19** the largest prime factor of $17!+18!$.

4.242 The sum of the five digits must be 9, and the last digit must be a 2 (a number divisible by 8 must be even). We want to place as many leading ones as possible to keep the integer as small as possible, while keeping the sum of the digits equal to 9 and using at least one 3: this leads us to **11,232**, which is divisible by both 8 and 9.

4.250 $440 = 2^3 \cdot 5 \cdot 11$, so 440 has $4 \cdot 2 \cdot 2 = 16$ factors. Odd factors cannot have any 2's. If we exclude the 2's there are $2 \cdot 2 = 4$ odd factors, meaning **12** of the factors of 440 are even.

4.251 $12{,}321 = 111^2 = 3^2 \cdot 37^2$. Perfect square factors have only even exponents (including zero): $3^0 \cdot 37^0 = 1$, $3^2 \cdot 37^0 = 9$, $3^0 \cdot 37^2 = 1{,}369$, and $3^2 \cdot 37^2 = 12{,}321$ for a total of **4** perfect square factors. (See **p.368**).

4.252 $0.\bar{2} = \dfrac{2}{9} = \dfrac{200}{900}$, $0.0\bar{2} = \dfrac{2}{9} \cdot \dfrac{1}{10} = \dfrac{2}{90} = \dfrac{20}{900}$, and

$$0.00\bar{2} = \frac{2}{9} \cdot \frac{1}{100} = \frac{2}{900}, \text{ so}$$

$$0.\bar{2} + 0.0\bar{2} + 0.00\bar{2} = \frac{200 + 20 + 2}{900} = \frac{222}{900} = \frac{37}{150}.$$

4.260 For a to have 5 factors, its prime factorization must be in the form x^4 for some prime integer x, and for b to have 6 factors its prime factorization must be in the form y^2z (or y^5) for some prime integers y and z. The product ab must therefore have prime factorization in the form $x^4 \cdot y^5$ or $x^4 \cdot y^2 \cdot z$. To minimize this product we use $2^4 \cdot 3^2 \cdot 5 = \mathbf{720}$ ($2^5 \cdot 3^4 = 2{,}592$).

4.271
a. $215_6 = 2(6^2) + 1(6^1) + 5(6^0) = \mathbf{83}$
b. $101_2 = 1(2^2) + 0(2^1) + 1(2^0) = \mathbf{5}$
c. $777_8 = 7(8^2) + 7(8^1) + 7(8^0) = \mathbf{511}$

4.280 $30 = 2 \cdot 3 \cdot 5$, so 30 has 8 factors or 4 pairs of factors with each pair having a product of 30, so we have the product $30^4 = (2 \cdot 3 \cdot 5)^4 = 2^4 \cdot 3^4 \cdot 5^4$. $x = \mathbf{4}$

4.290 a. $64 = 2^6$, which has a factor sum:
$(1+2+4+8+16+32+64) = \mathbf{127} = 2^7 - 1$

b. $128 = 2^7$, which has a factor sum:
$(1+2+4+8+16+32+64+128) = \mathbf{255} = 2^8 - 1$

c. $256 = 2^8$, which has a factor sum:
$(1+2+4+8+16+32+64+128+256) = \mathbf{511}$.

d. 2^{30} has a factor sum:
$(2^0 + 2^1 + 2^2 + 2^3 + ... + 2^{30}) = \mathbf{2^{31} - 1}$.

The sum of the factors of 2^n is $2^{n+1} - 1$. (See **4.201**).

4.291 The units digit of $\sqrt{4{,}624}$ must be a 2 or an 8. 62 is too small, so we assume $\sqrt{4{,}624}$ must be **68**.

4.301 A multiple of 6 can be expressed $6n$. $6n$ has divisors 1, n, $2n$, $3n$, and $6n$ (and quite possibly many more, but these are all factors). The sum of these factors is $12n + 1$, so the least possible factor sum of any multiple of 6 is $12n + 1$ (one greater than twice the number). Every integer multiple of 6 is **abundant**. For the same reason, you can tell if any integer is abundant simply by knowing if any one of its factors is abundant (or perfect). For example, $20n$ has divisors 1, n, $2n$, $4n$, $5n$, $10n$, and $20n$ for a factor sum that is at least $42n + 1$.

4.310 The largest 5-digit integer is 99,999. We want the sum of alternating digits to be equal, so we change the units digit to a 0 to get **99,990**. This must be the largest 5-digit multiple of 11 because if we add 11 to 99,990 we get a 6-digit integer.

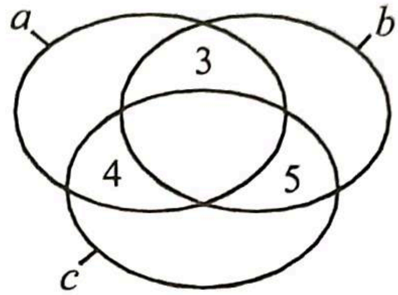
4.311 The product of two integers is equal to the product of their GCF and LCM, so we divide their product by their GCF to get their LCM: 315/3 = **105**.

4.312 $\frac{2}{33} = \frac{6}{99} = 0.\overline{06}$.

4.320 The digit sum must be at least 9 for the integer to be divisible by 9, so we need at least four digits: 1, 2, 3, and 3. The last digit must be a 2 for divisibility by 8. We try 1,332 and 3,132, but neither is divisible by 8 so **3,312** is the smallest integer divisible by 8 and 9 which uses each of the digits 1, 2, and 3 at least once.

4.321 We solve with a Venn diagram:

LCM(a,b,c) = **60**.

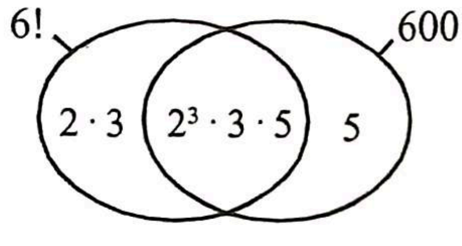


4.322 For the decimal expansion of n/168 to terminate, the denominator must have only 2's and 5's in its prime factorization. $168 = 2\cdot2\cdot2\cdot3\cdot7$, so we must have a numerator of $3 \cdot 7 =$ **21** to cancel the $3 \cdot 7$ in the denominator. ($21/168 = 1/8 = 0.125$).

4.331 $6! = 6\cdot5\cdot4\cdot3\cdot2 = 2^4\cdot3^2\cdot5$

$600 = 2^3\cdot3\cdot5^2$

LCM = **3,600**



4.332 $(1+2+3+\ldots+48)$ is equal to $\dfrac{(48)(49)}{2} = 24(49)$.

$49 \equiv -1(\text{mod}50)$, so $24(49) \equiv -24(\text{mod}50)$, and $-24(\text{mod}50) \equiv 26(\text{mod}50)$. We are looking for the value of 26^{49} in modulo 50. $26^1 \equiv 26(\text{mod}50)$ and $26^2 = 676 \equiv 26(\text{mod } 50)$, so $26^{49} \equiv 26(\text{mod } 50)$ (every power of 26 ends in 76). $(1+2+\ldots+48)^{49}$ leaves a remainder of **26** when divided by 50.

4.340 Integers which have an odd number of factors are all perfect squares. There are **9** perfect squares less than 100.

4.341 For $n!$ to end in exactly 100 zeros, it must have exactly one-hundred 5's in its prime factorization (when paired with a 2, these 5's contribute a factor of 10, adding a zero to the end of the number). We first look for multiples of 5: 5! has one 5, 10! has two, etc. 500! has 100 multiples of 5, however, multiples of 25 contributes two 5's and multiples of 125 contribute three 5's. 500! includes 20 multiples of 25 and 4 multiples of 125, so 500! ends in $100 + 40 + 4 = 144$ zeros. Some quick guess-and-check working backwards and we find that 400! ends in $80 + 16 + 3 = 99$ zeros, so 405! ends in $81 + 16 + 3 = 100$ zeros, but we can go as high as 409! without adding another 5, so n = **409** is our answer.

4.342 *Method 1*: Look for a pattern: 2^0 leaves a remainder of 1, $2^0 + 2^1$ leaves a remainder of 3, $2^0 + 2^1 + 2^2$ leaves a remainder of 7, and the pattern of remainders continues: 1 - 3 - 7 - 6 - 4 - 0 - 1 - 3 - 7 - 6 - 4 - 0 ... in a repeating block of 6 digits. 99 leaves a remainder of 3 when divided by 6, so the remainder will be the same as for $2^0 + 2^1 + 2^2 + 2^3$, which is **6**.

Method 2: $(2^0 + 2^1 + 2^2 \ldots + 2^{99}) = 2^{100} - 1$, so we first look for the value of 2^{100} in modulo 9.

$2^1 \equiv 2 \pmod 9$, $2^2 \equiv 4 \pmod 9$, $2^3 \equiv 8 \pmod 9$,

$2^4 \equiv 7 \pmod 9$, $2^5 \equiv 5 \pmod 9$, $2^6 \equiv 1 \pmod 9$, and the cycle repeats in a block of 6: 2 - 4 - 8 - 7 - 5 - 1 etc. Note that these remainders are easy to find by doubling the previous term and then dividing by 9 where necessary. 100 leaves a remainder of 4 when divided by 6, so 2^{100} leaves the same remainder as 2^4 when divided by 9 (7). Subtract 1 (we were looking for $2^{100} - 1$) and we get a remainder of **6**.

4.350 $1{,}296 = 2^4 \cdot 3^4$. To make a perfect square, we can use any even power of 2 with any even power of 3. There are three of each (2^0, 2^2, and 2^4 along with 3^0, 3^2, and 3^4), giving us $3 \cdot 3 = \mathbf{9}$ ways to create perfect square factors (1, 4, 9, 16, 36, 81, 144, 324, and 1,296).

4.351 For a multiple of 11 to have 10 factors, its prime factorization must be in the form 11^9, $11 \cdot a^4$, or $a \cdot 11^4$. We can rule-out 11^9 and $a \cdot 11^4$, as both are larger than 999, so we look for values of a which make $11 \cdot a^4 < 1{,}000$. Only two values work for a: 2 and 3. $11 \cdot 2^4 = 176$ and $11 \cdot 3^4 = 891$, so we have $176 + 891 = \mathbf{1{,}067}$.

4.352 There are an enormous number of sets of positive integers whose sum is 25, so there must be a method. Lets look for a pattern. For 6:
$1 + 5 = 6$ and $1 \cdot 5 = 5$,
$2 + 2 + 2 = 6$ and $2 \cdot 2 \cdot 2 = 8$,
$3 + 3 = 6$ and $3 \cdot 3 = 9$.
It appears that we are better off using 3's (not 2's) if we want to maximize the product for a given sum. Lets compare 3's to 5's:
$3 + 3 + 3 + 3 + 3 = 15$ and $3^5 = 243$,
while $5 + 5 + 5 = 15$ and $5^3 = 125$.
3's are better than 5's when trying to achieve a given sum while maximizing a product for a set of integers. 3's clearly do better than the larger primes like 7 and 11. We therefore look to use as many 3's as possible, with any remainder used in 2's. If the remainder is 1 then we are better off using one less three so that we leave a remainder of $4 = 2 \cdot 2$, which exceeds $1 \cdot 3$. For a sum of 25 we can use seven 3's and two 2's: $3(7) + 2(2) = 25$, $3^7 \cdot 2^2 = \mathbf{8{,}748}$.

4.371 The largest 3-digit number in base 6 is one less than 1000_6, which is $6^3 = 216$ (in base 10), so we have $555_6 = \mathbf{215}_{10}$.

4.380 9 has 3 factors: 1, 3, and 9, so their product is 3^3.

4.381 When we add we must carry a 1, because there is no digit 6 in base-6. $3_6 + 3_6 = 10_6$. and $3_6 + 3_6 + 1_6 = 11_6$.

$$\begin{array}{r} \overset{1\,1}{3}33_6 \\ +\ 333_6 \\ \hline 1{,}110_6 \end{array}$$

4.390 Guess-and-check gives us $(1 + 7 + 7^2 + 7^3) = (1 + 7 + 49 + 343) = 400$, so our answer is **$7^3$ or 343**.

4.391 $(1 + 2 + \ldots + 30) = \dfrac{30 \cdot 31}{2} = 15 \cdot 31$, which has a units digit of 5. The units digit remains a **5** when squared.

4.401 Because $8{,}128 = 2^6 \cdot 127$ (which is $64 \cdot 127$), its factor sum is $(1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6)(1 + 127) = (2^7 - 1)(128) = (127 \cdot 128)$, which is twice 8,128 (as we can see from the prime factorization of 8,128). 8,128 is therefore **perfect**.

4.410 For a three-digit multiple of 11 to end in a 2, either the tens digit must exceed the hundreds digit by 2 (so that the sum of the units digit and the hundreds digit is equal to the tens digit), or the hundreds digit must exceed the tens digit by 9 (so that the sum of the units digit and the hundreds digit is greater than the tens digit by 11). This gives us the following set of three-digit integers: 132, 242, 352, 462, 572, 682, 792, and 902 for a total of **8** such multiples of 11.

4.411 The product of the GCF and the LCM for a pair of integers is equal to the product of the integers, so $18(180) = 90x$. Solving for x gives us $x = \mathbf{36}$.

4.412 $\dfrac{5}{111} = \dfrac{45}{999} = 0.\overline{045}$.

4.420 The sum of the digits must be divisible by 9, so we try 9 2's, but there cannot be an odd number of 2's (the sum of alternating sets of digits must be equal) so we must use a minimum of **18** twos.

4.421 We must divide the chips, pecans, and M&Ms into an equal number of cookies, so we look for the GCF of

80, 112, and 128, which is **16**. (Each giant cookie gets 5 chocolate chips, 7 pecans, and 6 M&Ms).

4.422 $\frac{1}{37} = \frac{3}{111} = \frac{27}{999}$, $\frac{1}{9} = \frac{111}{999}$, and $\frac{1}{3} = \frac{333}{999}$, so

$$\frac{1}{37} + \frac{1}{9} + \frac{1}{3} = \frac{27}{999} + \frac{111}{999} + \frac{333}{999} = \frac{471}{999} = 0.\overline{471}.$$

4.431 $100! = 100 \cdot 99 \cdot 98 \cdot \mathbf{97} \cdot 96 \cdot 95 \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. There is no way to create a larger prime factor by multiplication, so 97 is the greatest prime factor of 100!.

4.432 We are looking for n such that $31n \equiv 1 \pmod{13}$, but 31 is congruent to 5 in modulo 13. This simplifies our expression to $5n \equiv 1 \pmod{13}$. We see that this works when $n = 8$, which gives us $40 \equiv 1 \pmod{13}$. The eighth multiple of 31 will therefore leave a remainder of 1 when divided by 13: $31 \cdot 8 = \mathbf{248}$.

4.440 Both a and b must be the square of a prime number (only the square of a prime number will have exactly three factors). We organize our work, looking for products $x^2y^2 < 1{,}000$ where $x < y$, and both x and y are prime. The following **7** pairs (x, y) are the only ones which satisfy the conditions:
(2, 3) (2, 5) (2, 7) (2, 11) (2, 13) (3, 5) and (3, 7).

4.441 The number of minutes in February is $60 \cdot 24 \cdot 28$. We work our way up starting at $2 \cdot 3 \cdot 4$ ... until we have the same factors at $8! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$. $n = \mathbf{8}$.

4.442 First, we find the smallest integer that is divisible by 2, 3, 4, 5, 6, 8, 9, 10, and 11: $2^3 \cdot 3^2 \cdot 5 \cdot 11 = 3{,}960$. Any multiple of 3,960 is divisible by 2, 3, 4, 5, 6, 8, 9, 10, and 11. There are **2** multiples of 3,960 less than 10,000: 3,960 and 7,920.

4.450 A number with 7 factors must have a prime factorization n^6, so $(n^6)^2 = n^{12}$ will have **13** factors.

4.451 $720 = 2^4 \cdot 3^2 \cdot 5$ has $5 \cdot 3 \cdot 2 = 30$ factors. The smallest integer with 32 factors is: $2^3 \cdot 3 \cdot 5 \cdot 7 = 840$. The smallest integer with 36 factors is $2^2 \cdot 3^2 \cdot 5 \cdot 7 =$ 1,260. This should be enough to convince us that **32** factors is the maximum for any 3-digit integer.

4.452 10! (whether the 10! is in base 9 or base 10) is divisible by 9 and therefore must end in a **0** in base-9.

4.471 242 is one less than 3^5, which is $100{,}000_3$, so it will take **5** digits to represent 242 in base-3: $22{,}222_3$.

4.480 $54 = 2 \cdot 3^3$ and has 8 factors (4 pairs). Each pair can be replaced with a 54 for a product of $54^4 = 2^4 \cdot 3^{12}$. $36 = 2^2 \cdot 3^2$ and has 9 factors, each of which can be replaced with a 6 for a product of $6^9 = 2^9 \cdot 3^9$:

$$\frac{2^4 \cdot 3^{12}}{2^9 \cdot 3^9} = \frac{3^3}{2^5} = \frac{27}{32}.$$

4.481 When we add we must carry a 1, because there is no digit 6 in base-4. $3_4 + 3_4 = 12_4$. and $3_4 + 3_4 + 1_4 = 13_4$.

$$\begin{array}{r} \scriptstyle 1\,1 \\ 333_4 \\ +\,333_4 \\ \hline 1{,}332_4 \end{array}$$

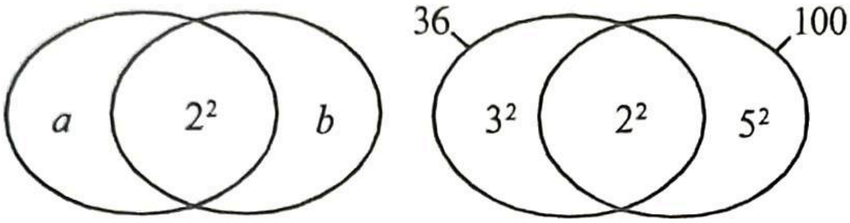
4.490 To find the average of the factors of 48, we find the sum and divide it by the number of factors. $48 = 2^4 \cdot 3$, which has 10 factors and a factor sum of $(1 + 2 + 4 + 8 + 16)(1 + 3) = 124$. The average factor of 48 is $124/10 =$ **12.4**.

4.491 An integer that is divisible by 2 but not 5 must end in a 2, 4, 6, or 8. $(n^2)^2 = n^4$, and 2^4, 4^4, 6^4, and 8^4 all end in a 6, so the units digit of any integer $(n^2)^2$ where n is divisible by 2 but not 5 will always end in a **6**.

4.500 It must end in a 2 and have a digit sum that is divisible by 3. We must use at least one 3, which means we need three 2's to satisfy the digit sum. Minimize this integer by placing the 3 in the tens place: **2,232**.

4.510 For 1,234,567: $1 + 3 + 5 + 7 = 16$ and $2 + 4 + 6 = 12$. If the units digit of 1,234,567 were a 3, the number would be divisible by 11. 1,234,563 is divisible by 11, so 1,234,567 leaves a remainder of **4** when divided by 11.

4.511 I am a big fan of the Venn diagram with these for organizing my thoughts. If their GCF is 4 and their product is 3,600, then their LCM is 3,600/4 = 900. We are looking for two perfect squares a and b in the diagram which are mutually prime and have a product of 900/4 = 225. $225 = 3^2 \cdot 5^2$, so we can use $a = 9$ and $b = 25$ or $a = 1$ and $b = 225$. To minimize the positive difference we use 9 and 25, so $9 \cdot 4 = 36$ and $25 \cdot 4 = 100$. Their positive difference is **64**.



4.512 $\dfrac{2}{37} = \dfrac{6}{111} = \dfrac{54}{999} = 0.\overline{054}$

4.520 We recognize immediately that the units digit must be a 2 or a 4. Look at the last three digits of a number to determine divisibility by 8. Check only those combinations which end in 2 or 4: ~~132~~, ~~142~~, 312, ~~342~~, ~~412~~, 432, ~~124~~, ~~134~~, ~~214~~, ~~234~~, ~~314~~, ~~324~~. This gives us two multiples of 8: 4,312 and 1,432. Their sum is **5,744**.

4.521 The slope of the line is 162/126, which simplifies to 9/7. Any x-coordinate which is a multiple of 7 on this line segment will have a corresponding y-coordinate that is an integer multiple of 9: (0,0), (7,9), (14, 18) ... (126, 162). There are $126/7 + 1 = 19$ multiples of 7 from 0 to 126 (remember to include 0) and 19 corresponding multiples of 9 for a total of **19** lattice points.

4.522 $\dfrac{110}{111} = \dfrac{990}{999} = 0.\overline{990}$. (**3** digits)

4.531 We are looking for multiples of 3 (there are 6 in 18!) and multiples of $3^2 = 9$ (there are 2). This gives us 8 threes in the prime factorization of 18! which, when paired with 8 twos, gives us 8 sixes or $\mathbf{6^8}$.

4.532 We are trying to advance the clock ahead by one minute. If we push the > button 9 times, we increase the time on the clock by 3 minutes: $9 \cdot 7 = 63 \equiv 3 \pmod{60}$. If we made 5 "laps" using this method (by pushing > 45 times) we would end up 15 minutes ahead of where we started: $5 \cdot 3 = 15 \equiv 1 \pmod{7}$. Pushing > 2 less times would put us one minute ahead at 6:04 (in 43 button presses).

However, if we push < 8 times, we end up 4 minutes ahead: $8(-7) = -56 \equiv 4 \pmod{60}$. Taking two "laps" in this manner by pressing < 16 times, we are left 8 minutes ahead: $2 \cdot 4 = 8 \equiv 1 \pmod{7}$. One more press of < and we are at 6:04, one minute ahead of where we started in a total of **17** presses of < .

4.540 To have 31 factors, the prime factorization of a positive integer must be in the form a^{30}. To minimize this value we use $a = 2$ to get $\mathbf{2^{30}}$.

4.542 $33{,}333_4 = 100{,}000_4 - 1 = 4^5 - 1 = \mathbf{1{,}023}$.

4.550 The prime factorization of an integer with exactly 3 factors is of the form a^2, so we are asked to find primes which have a square that is 3-digits long. Beginning with $11^2 = 121$ and ending with $31^2 = 961$, we have 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, and 31^2 for a total of **7**.

4.551 $1{,}849 = 43^2$. We are looking for the square of the next prime, $47^2 =$ **2,209**.

4.552 We are looking to find a value x where $38x \equiv -1 \pmod{35}$. Because $38 \equiv 3 \pmod{35}$, we simplify our search to $3x \equiv -1 \pmod{35}$. Note that any number which is congruent to $-1 \pmod{70}$ is also congruent to $-1 \pmod{35}$, which helps us see that $x = 23$ is a solution. **23 students** in 38 groups = 874 students (25 rows of 35 = 875 seats).

4.571 $222_8 = 2(8^2) + 2(8) + 2 = 2(64) + 16 + 2$ which can be easily converted to base-4: $= 2(4^3) + 1(4^2) + 2$, which is **$2{,}102_4$**.

4.581 No need for borrowing, this one is easy.

$$\begin{array}{r} 567_8 \\ -\ 456_8 \\ \hline 111_8 \end{array}$$

4.590 Look for abundant numbers that are less than half of 100, this is really an educated guess-and-check, choosing numbers which have a lot of factors. Nothing below 36 comes close, so we begin there:
$36 = 2^2 \cdot 3^2$. Factor sum: $(1 + 2 + 4)(1 + 3 + 9) = 91$.
$40 = 2^3 \cdot 3$. Factor sum: $(1 + 2 + 4 + 8)(1 + 5) = 90$.
$42 = 2 \cdot 3 \cdot 7$. Factor sum $(1 + 2)(1 + 3)(1 + 7) = 96$.
$44 = 2^2 \cdot 11$. Factor sum $(1 + 2 + 4)(1 + 11) = 84$.
$45 = 3^2 \cdot 5$. Factor sum $(1 + 3 + 9)(1 + 5) = 78$.
48 $= 2^4 \cdot 3$. Factor sum $(1 + 2 ... + 16)(1 + 3) = 124$.
Somewhat surprisingly, 48 is the smallest.

4.591 In order for there to be a remainder, the units digits of the consecutive integers can only be 1, 2, 3, and 4 or 6, 7, 8, and 9 (consider what happens when you include a units digit of 0 or 5). In either case, the units digit of the product of the integers will be a 4. This means that when the integer is divided by 10 the remainder is **4**.

4.600 **10,008** has a digit sum of 9 and is divisible by 8.

4.610 If we use a 2 for the units digit, the sums of alternating digits have a difference of $(9+0+2)-(0+0)=11$, so **90,002** is divisible by 11.

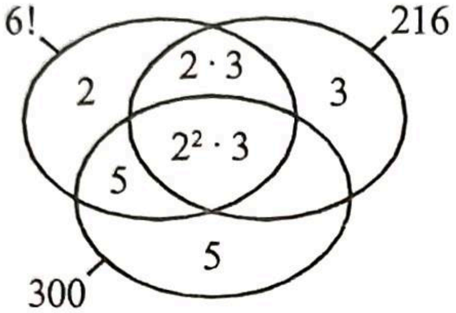
4.620 All three digits must be even for each of ABC, CAB, and BCA to be divisible by 6. The sum of the digits cannot be 9 or 27, so we look for three even digits whose sum is 18. The only three distinct digits which work are 4, 6, and 8. We can use 468 + 684 + 846 = **1,998** (note that 486, 648, and 864 yield the same sum).

4.622 Call the larger three-digit integer x and the smaller y. This gives us $1{,}000x + y = 6(1{,}000y + x)$. Simplify this to $1{,}000x + y = 6{,}000y + 6x$ or $994x = 5{,}999y$ which simplifies to $142x = 857y$. The obvious three-digit solutions to this equation are $x = 857$ and $y = 142$. $142 + 857 =$ **999**.

Alternatively, we recall that $\frac{1}{7} = 0.\overline{142857}$, and $\frac{6}{7} = 0.\overline{857142}$, giving us 142,857 and 857,142.

4.631 $6! = 2^4 \cdot 3^2 \cdot 5$
$216 = 2^3 \cdot 3^3$
$300 = 2^2 \cdot 3 \cdot 5^2$

LCM $=$ **10,800**



4.632 We begin by finding an integer x where $x \equiv 1 (\text{mod}\, 14)$ and $x \equiv 2 (\text{mod}\, 15)$. Looking for the smallest positive integer which leaves a remainder of 1 when divided by 14 and 2 when divided by 15 can be difficult, but notice that the difference between 14 and 15 is equal to the difference in the modular residues 1 and 2. This leads us to look for the first negative solution $x = -13$. Add to this $14 \cdot 15 = 210$ to maintain modular congruence and we get 197 trees. Unfortunately, the problem states that Craig has more than 200 trees so we must add another 210 to get **407** trees.

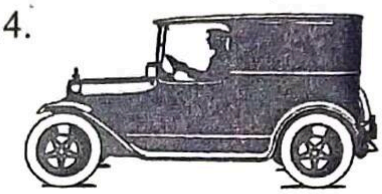
4.640 $2^2 \cdot 3 \cdot 5 = \mathbf{60}$, $2^3 \cdot 3^2 = \mathbf{72}$, $2^2 \cdot 3 \cdot 7 = \mathbf{84}$, $2 \cdot 3^2 \cdot 5 = \mathbf{90}$, and $2^5 \cdot 3 = \mathbf{96}$.

4.642 We are looking for the remainder when 500 is divided by 6, which is **2** (the 498th person will count off 6, with the last two people numbered 1 and 2).

4.650 The only multiple of 7 having 5 factors is 7^4, so $n = 7^4$ and $3n$ is $3 \cdot 7^4$, which has **10** factors.

4.651 $900/n$ is a positive integer only when n is a factor of 900. $900 = 2^2 \cdot 3^2 \cdot 5^2$ and has 27 factors. There are **27** values of n which make $900/n$ a positive integer.

4.652 The tires have circumference 27π inches and 30π inches. The LCM (270π) inches gives us the distance after which the tires will both reach their original position in the rotation. The stars will be positioned as shown in the drawing 5 times (every 54π inches) during this period. 1/4 mile = 15,840 inches. $15{,}840/(54\pi) = 93.4$. If the wheels begin in the position shown, this makes **94** times.



4.671 $9^6 = (3^2)^6 = 3^{12} = \mathbf{1{,}000{,}000{,}000{,}000_3}$.

4.681 When we borrow in base 10 we "borrow" 10, but in base 6 we "borrow" 6.

$$\begin{array}{r} {\scriptstyle 1\ 6\ 9} \\ \cancel{2}\cancel{1}\cancel{3}_6 \\ -\ 144_6 \\ \hline \mathbf{25_6} \end{array}$$

4.690 The two values are both equal to $(1 + 2)(1 + 3)(1 + 5)(1 + 7)(1 + 11)$. Their quotient is **1**.

4.720 I like to start these problems with the largest number possible (9,876,543,210) and modify it as necessary. Looking at the number, it is already divisible by 9 and 10, but not by 8 and 11. For 11, the digit sum is 45, so the sum of alternating digits cannot be equal. The difference must be 11 with alternating digit sums of 17 and 28. 9,876,543,210 has alternating digit sums of 25 and 20, so we must increase the digit sum of $9 + 7 + 5 + 3 + 1$ to 28. Trading the 1 for a 4 is the best way to do this $(9 + 7 + 5 + 3 + 4 = 28)$. Leave the largest five digits in order on the left and the zero on the right. Look for arrangements of the underlined digits which make the number divisible by 8 while maintaining the alternating digit sums. $9{,}876{,}5\underline{24{,}13}0$ is not divisible by 8, neither is $9{,}876{,}5\underline{23{,}14}0$, but **9,876,513,240** is.

4.722 We have a single misplaced digit which we will call x and a 5-digit number y. In the correct order, we have $10y + x$, but in the incorrect order written by Michael we have $100{,}000x + y$. The incorrect integer is 5 times the correct integer, giving us $5(10y + x) = 100{,}000x + y$ which simplifies to $49y = 99{,}995x$. Dividing by 7 we get $7y = 14{,}285x$. We can use $x = 7$ and $y = 14{,}285$ which makes Michael's number **714,285**. Alternatively (as in problem 4.622) the problem suggests a cyclic number found in the repeating block of digits in sevenths:

$$\frac{1}{7} = 0.\overline{142857} \text{ and } \frac{5}{7} = 0.\overline{714285}.$$

4.731 For this problem I will introduce a function called the floor value. The floor value of a number, indicated by $\lfloor x \rfloor$ represents the greatest integer value of x and basically means round down. For example: $\lfloor 9.8 \rfloor = 9$ and $\lfloor 73/11 \rfloor = 6$. To find the power of 7 in 2,500!

we add $\left\lfloor \frac{2{,}500}{7} \right\rfloor + \left\lfloor \frac{2{,}500}{7^2} \right\rfloor + \left\lfloor \frac{2{,}500}{7^3} \right\rfloor + \left\lfloor \frac{2{,}500}{7^4} \right\rfloor$.

$\left\lfloor \frac{2{,}500}{7} \right\rfloor = 357$, which means there are 357 multiples of 7 in 2,500! Each contributes a 7 to the prime factorization of 2,500! Continuing, there are 51 multiples of 7^2, and each contributes an additional 7. There are 7 multiples of 7^3 and 1 multiple of 7^4. This gives us a total of $357 + 51 + 7 + 1 = 416$ sevens in the prime factorization of 2,500!, making 416 ($\mathbf{7^{416}}$) the power of 7 in 2,500!.

4.742 The units digits in the powers of 2 repeat: 2-4-8-6-2-4-8-6..., and 222/4 leaves a remainder of 2. This means that 222^{222} must end in a 4. Multiplying this result by 9 would give us a units digit **6**.

4.750 For a multiple of 6 to have 9 factors, it must be a perfect square, making our integer $2^2 \cdot 3^2$. Multiplying by 10 we get $2^3 \cdot 3^2 \cdot 5$, which has **24** factors.

4.751 We use the floor function again as explained in **4.731**:

$$\left\lfloor \frac{343}{7} \right\rfloor + \left\lfloor \frac{343}{7^2} \right\rfloor + \left\lfloor \frac{343}{7^3} \right\rfloor = 49 + 7 + 1 = \mathbf{57, \text{ or } 7^{57}}.$$

4.752 For any pair of consecutive integers, one must be even and therefore divisible by 2, so we look for the least positive integer whose prime factorization is of the form $2^2 \cdot a$ or $a^2 \cdot 2$ for which an adjacent integer has 6 factors. $2^2 \cdot 11 =$ **44** and $3^2 \cdot 5 = 45$ are the first pair of consecutive integers which have exactly 6 factors each. Pairs like (75,76) and (98,99) work as well.

4.771 0.24_5 is $2(5^{-1}) + 4(5^{-2}) = \frac{2}{5} + \frac{4}{25} = \frac{14}{25} = \mathbf{0.56}$.

4.781 Begin with the subtraction shown on the right. To multiply by 100_6, just add two zeros to get $\mathbf{2{,}400_6}$.

$$\begin{array}{r} {}^{6\;9} \\ 113_6 \\ -\ 45_6 \\ \hline 24_6 \end{array}$$

4.790 $496 = 16 \cdot 31 = 2^4 \cdot 31$ which has a factor sum $(1 + 2 + 4 + 8 + 16)(1 + 31) = (31)(32) =$ **992**. 496 is a perfect number (**p.147**), a number p whose factor sum is $2p$. There are relatively few known perfect numbers, and all known perfect numbers are of the form: $p = 2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime (called a Mersenne prime). In order for $2^n - 1$ to be prime, n must be prime. The largest known primes are all Mersenne primes.

4.820 456,564,465,645 has a digit sum of $4(4+5+6) = 60$, so it is divisible by 3. It is not even, so it is not divisible by 6, but three less than 456,564,465,645 is divisible by 6, so dividing by 6 leaves a remainder of **3**.

4.831 We are looking for a number n for which $n!$ includes fifteen 2's in its prime factorization. This one is small enough that we can get the answer quickly simply by counting up by 2's and adding the powers contributed by each: 2, $4(2^2)$, 6, $8(2^3)$, 10, $12(3 \cdot 2^2)$, 14, $16(2^4)$. We are up to fifteen 2's, which means that 2^{15} divides **16**! (but will not divide 15!).

4.842 If $x = 0.1\bar{4}$, then $10x = 1.\bar{4} = 1\frac{4}{9} = \frac{13}{9}$. Dividing by 10 we get $x = \frac{13}{90}$.

4.850 Perfect squares have three factors only when they are the square of a prime. There are 19 perfect squares less than 400. Excluding 1 and all perfect squares of primes: 2^2, 3^2, 5^2, 7^2, 11^2, 13^2, 17^2, and 19^2 leaves us with $19 - 9 = \mathbf{10}$ perfect squares. (We could have just as easily counted composites).

4.851 $1{,}024 = 2^{10}$, so we are looking for $n!$ where there are fewer than 10 twos in the prime factorization. $2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12$ is divisible by 2^{10}, so n must be less than 12. **11**! is not divisible by 1,024 (but 12! is).

4.881 This one is done easily by converting to base 10, but it is good practice with the concept of borrowing for subtraction:

$$\begin{array}{r} {\scriptstyle 0\;1\;1\;1\;2} \\ \cancel{1}\cancel{0}\cancel{0}\cancel{0}\cancel{0}_2 \\ -\ 1011_2 \\ \hline 101_2 \end{array}$$

4.920 The sum of alternating digits must be equal (we cannot create a difference of 11 with two 5's and two 6's). The four ways this can be done are: 5,665; 5,566; 6,556; 6,655. Their sum is **24,442**.

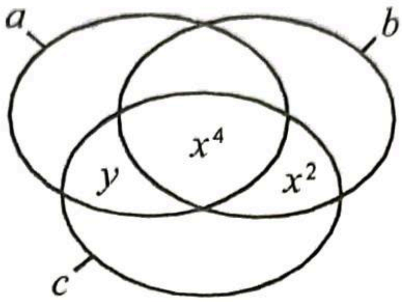
4.931 We are looking for a number n for which $n!$ includes fifteen 3's in its prime factorization. This one is small enough that we can get the answer quickly simply by counting up by threes and adding the number of 3's

contributed by each until we get to fifteen 3's:
3, 6, 9 (3^2), 12, 15, 18($2 \cdot 3^2$), 21, 24, 27(3^3), 30, 33.
Counting the 3's above gives us 15. This means that 3^{15} divides **33**! (but will not divide 32!).

4.942 A Venn diagram helps organize:

$a = x^4 \cdot y$
$b = x^6$
$c = x^6 \cdot y$

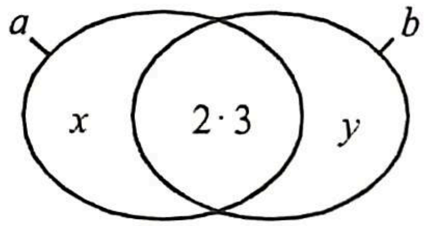
$abc = x^{16} \cdot y^2$



To minimize the product, we use primes $x = 2$ and $y = 3$ to get $x^{16} \cdot y^2 = 2^{16} \cdot 3^2 =$ **589,824**.

4.950 An integer with 18 factors will have a prime factorization of the form a^{17}, $a^8 \cdot b$, $a^5 \cdot b^2$, or $a^2 \cdot b^2 \cdot c$. It is safe to quickly rule out the first two options, and we try $2^5 \cdot 3^2 = 288$ against $2^2 \cdot 3^2 \cdot 5 = 180$. **180** is the smallest integer which has 18 factors.

4.951 Another quick Venn diagram gives us $xy = 12$, and we know that x and y must be mutually prime (otherwise the GCF would be greater than 6). Neither can be 1 (because a and b are greater than 6). We use $x = 4$ and $y = 3$ to give us $a = 24$ and $b = 18$, whose sum is **42**.



4.981 $444_9 = 4(81) + 4(9) + 4$. $11111_3 = 81 + 27 + 9 + 3 + 1$. Subtracting, we see that $4(81) - 81 = 3(81)$, $4(9) - (27 + 9) = 0$, and $4 - (3 + 1) = 0$, so we are left with just $3(81)$ or $3(9^2) =$ **300_9**.

Geometry
Key and Solutions

Problems within the text are ordered with the last three digits reversed. This way, if you are looking for a solution you will not accidently see the answer to the next problem. Problems marked with a do not include a written solution, only the answer.

5.001 $9\pi\sqrt{3}\text{ cm}^3$
5.010 21
5.011 $(576\pi+216\sqrt{3}) \approx 2{,}183.7\text{ in}^3$
5.020 22.5°
5.021 $x=24/5$
5.030 150°
5.031 13ft
5.040 $16\pi+8$
5.050 13mi
5.051 9/16
5.060 18in
5.070 7.5cm
5.071 $\pi\text{ cm}^2$
5.080 2/3 cm
5.090 32cm^2
5.100 42°
5.101 16in
5.110 3
5.111 2% decrease
5.120 26cm
5.121 $x=7$
5.130 100°
5.131 5cm
5.140 12π
5.141 $4\sqrt{10}$
5.150 $2\sqrt{3}$
5.151 12,600
5.160 1/3
5.161 5
5.170 425π
5.171 $\sqrt{5}$ cm.
5.180 29units^2
5.190 84cm^2
5.200 33°
5.201 9/2 cm
5.211 22
5.220 80cm
5.221 $x=48/9$
5.230 35°
5.231 $\sqrt{3}$
5.240 8π
5.241 $5\sqrt{35}$ cm
5.250 41cm
5.251 80 lbs
5.260 $2\sqrt{3}$
5.261 $3\sqrt{3}-\pi$
5.270 12cm
5.271 156cm^2
5.280 576cm^2
5.290 $100\pi-200$
5.300 156°
5.301 14ft 7in
5.311 $500\pi/3\text{in}^3$

5.320 18cm

5.321 $x = 16/3$

5.330 95°

5.331 $75\sqrt{7}$

5.340 $3\pi + 18$

5.341 $21\sqrt{5} + 47$

5.350 35.5cm

5.351 1/4 lb

5.360 $(1+\sqrt{2})$ ft

5.361 15.9

5.371 294

5.380 $(14\pi + 12\sqrt{3})$ cm

5.390 4π

5.400 18°

5.401 $3\sqrt{3}$ cm.

5.411 1/3

5.420 5°

5.430 112.5°

5.431 54in

5.440 13π

5.450 16ft

5.451 (640/9) cm³

5.460 $4\sqrt{3}$ cm

5.461 12

5.471 $\pi(2-\sqrt{3})$ units²

5.480 9in²

5.490 $4\sqrt{2} + 8$ cm²

5.500 132°

5.501 18π in³

5.511 $72\sqrt{2}$ cm³

5.520 22

5.530 40°

5.531 450/17

5.540 Proof, see solutions.

5.550 $\sqrt{a^2 + b^2 + c^2}$

5.551 3000 cm³

5.560 $\sqrt{3}/2$

5.561 5.5cm

5.571 38cm

5.580 $27\sqrt{3}$

5.590 1:3

5.600 720cm

5.601 84

5.611 $36\sqrt{3}$ in³

5.620 180cm

5.630 35°

5.650 7.4cm

5.651 990 lbs

5.660 3π cm²

5.661 $18\sqrt{2}$

5.671 $3\pi/2$ cm²

5.690 240cm²

5.700 60°

5.720 9cm

5.730 81°

5.760 $x^2/4$

5.761 $\sqrt{5}$ cm

5.790 1cm²

5.800 7

5.820 60cm²

5.830 70°

5.860 $3\sqrt{10}$

5.890 3.49cm²

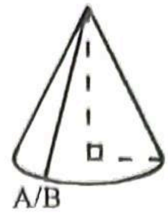
5.900 5cm

5.930 50°

5.960 $9\sqrt{2}$

5.990 24/5 cm

5.001 The radius of the semicircle becomes the slant height: 6cm. The arc length of the semicircle becomes the circumference of the cone: 6π cm. This makes the diameter of the cone 6cm, and the radius 3cm. Use the Pythagorean theorem to find the height of the cone: $3\sqrt{3}$ cm. The volume is:

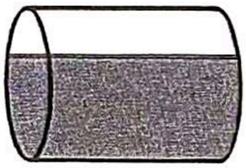


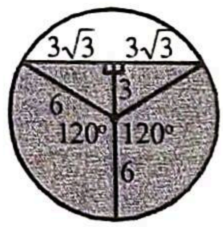
$$\frac{\pi \cdot 3^2 \cdot 3\sqrt{3}}{3} = 9\pi\sqrt{3}\,\text{cm}^3.$$

5.010 Use the diagram at right. The same strategy can be used for any polygon and the maximum number is $(n-2)(n-3)/2$ (similar to **2.430**), where n is the number of sides/vertices. For $n = 9$ we get **21** diagonals. With this method, we essentially create a polygon with $n-1$ vertices. Count its diagonals and add one (for the missing "top" of the diagram).

(sides connect below)

5.011 Find the area of the base of the cylinder. The base can be divided into a circle sector and two 30-60-90 triangles. The sector is 2/3 of the circle, which we add to the combined area of the triangles to get the area of the base:



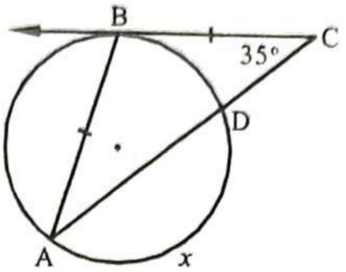


$$\frac{2}{3}\pi \cdot 36 + 3(3\sqrt{3}) = 24\pi + 9\sqrt{3}$$

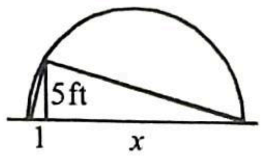
Multiply by 24 inches to get the volume of water in the cylinder: $24(24\pi + 9\sqrt{3}) = 576\pi + 216\sqrt{3}$.

5.020 ABHC is an isosceles trapezoid with angle BCH = 45°, and triangle ABC is isosceles with base angle BCA equal to 22.5°, so angle ACH = BCH - BCA = **22.5°**. Alternatively, if we circumscribe a circle about the octagon, we see that ACH is an inscribed angle (p.185) whose measure is half of the intercepted arc, which is 1/8 of 360°, making ACH 1/16 of 360° = **22.5°**.

5.030 Triangle ABC is isosceles, so inscribed angle A equals 35°, making minor arc BD = 70°. Angle CBA is 110°, making major arc BDA = 220°. Arc DA is the difference between arcs ADB and BD: 220° - 70° = **150°**.



5.031 Adding the lines shown, we see that the 5 foot height is the altitude of a right triangle, creating two smaller similar right triangles in which 1/5 = 5/x, so x = 25 and the diameter of the semicircle is 26. This makes the height at the center (equal to the radius) **13ft.**

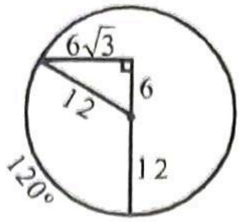


5.040 The small semicircle has a radius of 6 and arc length 6π, the larger semicircle has arc length 10π. The two short segments are equal to the difference of the radii (4cm). Added together, we get $16\pi + 8$.

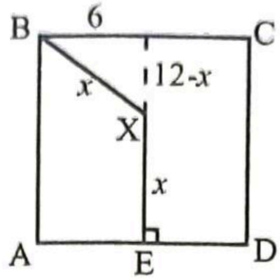
5.050 The total distance south is 12 miles, and the total distance west is 5 miles. Connect his starting location to his finish to form the hypotenuse of a 5-12-13 right triangle. Parker is **13 miles** from where he started.

5.051 The circles are inscribed within similar triangles. The ratio of their radii will be 15:20 or 3:4, so the ratio of their areas will be 3^2 to 4^2 or 9/16.

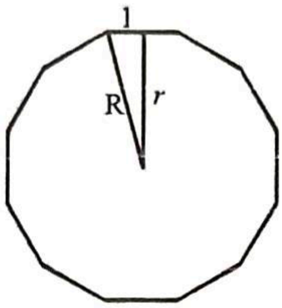
5.060 The 120 degree rotation carries the gum to a height of **18** inches as shown, using the 30-60-90 right triangle drawn.



5.070 With the diagram labeled as shown: $6^2 + (12 - x)^2 = x^2$. Solving for x gives us $180 = 24x$, or $x =$ **7.5cm.**



5.071 This solution actually works for any polygon of any edge length. Label the radius of the inscribed circle r, and the radius of the Circum-scribed circle R. By the Pythagorean theorem:



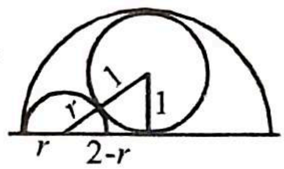
$R^2 - r^2 = 1^2$, so

$\pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi$ **cm².**

For any polygon of side length s, the difference in area of the inscribed and circumscribed circles is:

$$\pi\left(\frac{s}{2}\right)^2 = \frac{\pi s^2}{4}$$

5.080 Using the Pythagorean theorem to solve for r in the diagram as labeled gives us: $1^2 + (2 - r)^2 = (r + 1)^2$. Solving for r gives us r = **2/3cm.**



5.090 Rotate the inner square inside the circle until the vertices are at the midpoints of the sides of the larger square. Drawing the diagonals of the inner square should make it clear that the inner square is half the area of the outer square, or **32cm².**

5.100 At 96°, angle A cannot be a base angle of an isosceles triangle. Angles B and C are base angles, each measuring **42°**.

5.101 The volume of the spherical cannonball is equal to the volume of the cylinder of displaced water.

$$\frac{4}{3}\pi(4)^3 = \pi r^2 \cdot \frac{1}{3}.$$

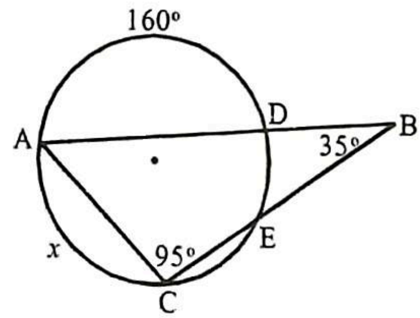
Solving, we get $4^4 = r^2$ which gives us $r =$ **16in.**

5.110 Only triangles of side length 2-3-4, 2-4-5, and 3-4-5 are scalene with a perimeter less than or equal to 12. Do you see why 1-2-3 or 2-4-6 are impossible? Try to draw one with a ruler. This gives us **3**.

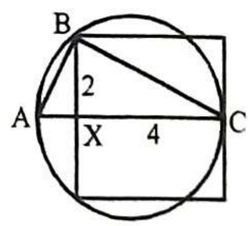
5.111 Increasing the radius by 40% increases the area of the base by $(1.4)^2$ or 1.96 times the original. Multiply by half the height and we get 0.98 times the original volume, or a **2% decrease** in volume.

5.120 Triangles ABC and BCD share two side lengths (AB = CD and BC = BC), so the difference in their perimeters (4) is the difference between the lengths of BD and AC, making BD + 4 = AC. We know that AC is twice BD, so BD = 4 and AC = 8, making AB + BC = BC + CD = 13. The perimeter of the parallelogram is twice that: **26cm.**

5.130 Angle A is $180^\circ - 95^\circ - 35^\circ = 50^\circ$, making arc DC = 100°. Arc AC is $360^\circ - 160^\circ - 100^\circ =$ **100°**.

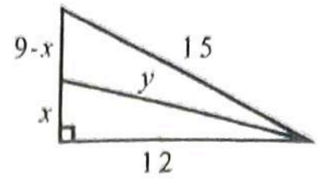


5.131 Adding AB and BC to the diagram, we get right triangle ABC with altitude BX and similar right triangles AXB and BXC, with AX/BX = BX/XC, so AX = 1 and the diameter of the circle is **5cm.**



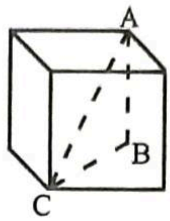
5.140 There are 3/4 each of 4 circles of diameter 4cm.

$$\frac{3}{4}\cdot 4 \cdot 4\pi = 12\pi \text{ cm}.$$



5.141 From the diagram, using angle bisector theorem:

$\dfrac{15}{9-x} = \dfrac{12}{x}$ gives us $x = 4$, and the Pythagorean theorem gives us $12^2 + 4^2 = y^2$. Solve for y to get $y = 4\sqrt{10}$.

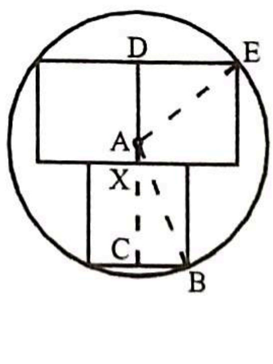


5.150 If the edge length is 2cm, then BC is the diagonal of a square of side length 2cm. BC = $2\sqrt{2}$. AC is the hypotenuse of a triangle whose legs are 2 and $2\sqrt{2}$ cm. $2^2 + (2\sqrt{2})^2 = (AC)^2$. AC = $\sqrt{12} = 2\sqrt{3}$ cm.

5.151 The side lengths are ten times the original, which will make the area 10^2 times greater, or **12,600 units^2.**

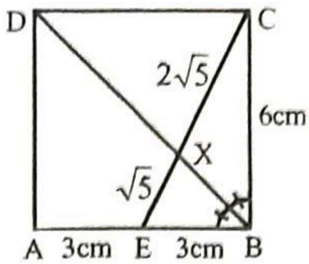
5.160 If CZ = BY = AX = 1, then BZ = CX = AY = 2, and XZ = ZY = XY = $\sqrt{3}$, making the ratio of the sides $\sqrt{3}/3$, and the ratio of the areas $(\sqrt{3}/3)^2$ = **1/3.**

5.161 In the case where angle B = 90°, AB would be 15 (long leg of an 8-15-17 Pythagorean triple) and length x would be $\sqrt{226}$, slightly bigger than 15. As angle B becomes more obtuse, x decreases until angle B approaches 180°. Length x must be greater than 10cm because triangle ACD has sides 7 and 17. Therefore, $\sqrt{226} > x > 10$, which means x may be 11, 12, 13, 14, or 15cm for a total of **5** integral lengths.



5.170 Label the length AX = x and the circle radius r. AC = $16 + x$, and AD = $16 - x$.
In ADE: $16^2 + (16-x)^2 = r^2$.
In ACB: $8^2 + (16+x)^2 = r^2$.
Substitution yields
$16^2 + (16-x)^2 = 8^2 + (16+x)^2$.
Solve for x to get $x = 3$. AD = 13, and DE = 16, so AE = $\sqrt{13^2 + 16^2} = \sqrt{425}$. The area of the circle is $\pi\sqrt{425}^2 = 425\pi$ cm^2.

5.171 We find EC = $3\sqrt{5}$ using the Pythagorean theorem, and angle bisector theorem (angle ABD = CBD = 45°) gives us EX:CX = 1:2, so EX = $\sqrt{5}$ cm.



5.180 The distance between opposite vertices is $\sqrt{(4-1)^2+(5+2)^2} = \sqrt{58}$. We divide by $\sqrt{2}$ to get the side length: $\sqrt{29}$. Square this length to get the area: **29 units²**.

5.190 The horizontal line between $(-7,2)$ and $(14,2)$ can be used as the base of length 21. The height to the point $(-1,10)$ is 8 units. Half the base times the height equals **84cm²**. Alternatively, use the distance formula to get side lengths 21, 10, and 17. Use these values in Heron's formula to get 84cm². Many triangle area problems on the plane are best solved by drawing a rectangle around the triangle and subtracting the area outside the triangle from the rectangle (not necessary here).

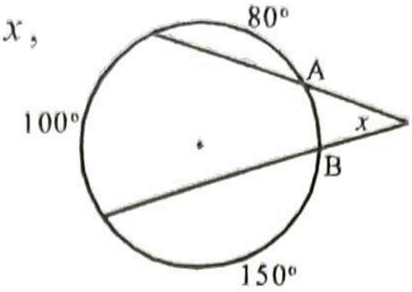
5.200 Solve the equation $x + (2x) + (2x + 15) = 180$ for x to get x = **33°**.

5.201 The surface area of a sphere is $4\pi r^2$. A hemisphere has half of the curved surface plus the area of the base: $2\pi r^2 + \pi r^2 = 3\pi r^2$. The volume of a hemisphere is $\frac{2}{3}\pi r^3$. Solve the equation $3\pi r^2 = \frac{2}{3}\pi r^3$ for r to get $\boldsymbol{r}$ **= 9/2**.

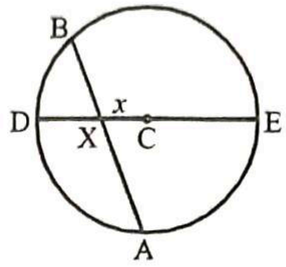
5.211 To use $f + v = e + 2$, we must find the number of edges. There are 20 kites, for a total of 80 sides, with each side shared by 2 kites. This gives us a total of 80/2 = 40 edges. 20 + v = 40 + 2. $\boldsymbol{v}$ **= 22.**

5.220 If the interior angle measure is 171 degrees, the exterior angles measure 9 degrees and there are 360/9 = 40 sides, each of length 2cm for a perimeter of **80cm.**

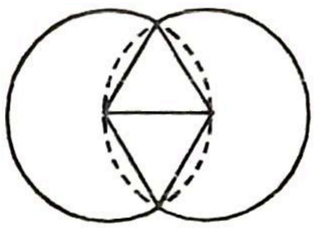
5.230 Arc AB = 30. $(100-30)/2 = x$, so x = 35°.



5.231 If we label length CX = x and the radius of the circle is 3, then EX = 3 + x and DX = 3 − x. (BX)(AX) = (DX)(EX), which gives us: $2 \cdot 3 = (3+x)(3-x)$. $6 = 9 - x^2$, $x^2 = 3$, so $x = \sqrt{3}$.

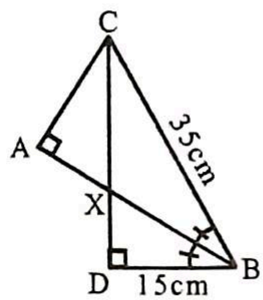


5.240 If we connect the centers of the circles to the points of intersection, we create two equilateral triangles (sides are radii of congruent circles), which leaves 240° for each major arc, or 2/3 of each circle of radius 3cm and circumference 6π:



$$\frac{2}{3} \cdot 2 \cdot 6\pi = 8\pi \text{ cm.}$$

5.241 Using the Pythagorean theorem, we get CD = $10\sqrt{10}$, and angle bisector theorem gives us the ratio of DX to CX is 15:35, making DX = $3\sqrt{10}$ and CX = $7\sqrt{10}$. Using the Pythagorean theorem again we find BX = $3\sqrt{35}$. Triangles ACX and DBX are similar, so AX/CX = DX/BX:



$$\frac{AX}{7\sqrt{10}} = \frac{3\sqrt{10}}{3\sqrt{35}} \text{ gives us AX} = \frac{210}{3\sqrt{35}} = \frac{70\sqrt{35}}{35} = 2\sqrt{35}.$$

AX + BX = AB = $5\sqrt{35}$.

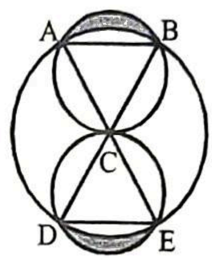
5.250 Unfold the left side of the box to see that we are solving for the diagonal length of a 9 by 40 rectangle. The Pythagorean theorem gives us **41cm.**

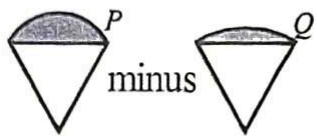
5.251 The replica is 1/7 the height of the original, so its volume/mass is $(1/7)^3$ of the original, or 1/343. If the original weighs 27,440 pounds: 27,440/343 = **80lbs.**

5.260 The diagonals are $2\sqrt{2}$ cm. The altitude of an equilateral triangle creates two 30-60-90 right triangles with short leg $\sqrt{2}$ (half the side length) and long leg $\sqrt{2}\cdot\sqrt{3}=\sqrt{6}$. The area is:

$$\frac{1}{2}\cdot 2\sqrt{2}\cdot\sqrt{6}=\sqrt{12}=2\sqrt{3}\text{ cm}^2.$$

5.261 Look first for a diagram that will help you solve the problem. To find each shaded area, use the diagram below.





To find the area of shaded figure *P*, find the area of the smaller circle. The radius of the smaller circle is $\sqrt{3}$ (p.199), so the area of the circle is 3π. Subtract the area of the triangle: $9\sqrt{3}/4$ and then divide by 3 (or multiply by 1/3) to get shaded area *P*:

$$\frac{1}{3}\left(3\pi-\frac{9\sqrt{3}}{4}\right)=\pi-\frac{9\sqrt{3}}{12}$$

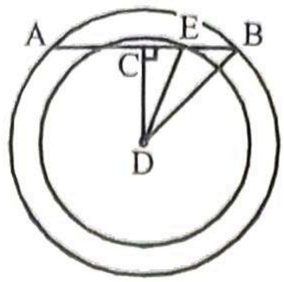
To find the area of shaded figure *Q*, subtract the area of the triangle from the area of the sector, which is 1/6 the area of the larger circle:

$$\frac{1}{6}\cdot 9\pi-\frac{9\sqrt{3}}{4}=\frac{3\pi}{2}-\frac{9\sqrt{3}}{4}$$

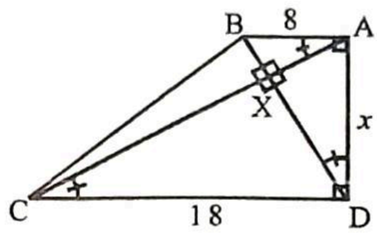
Subtract to get the area of one lune.

$$\pi-\frac{9\sqrt{3}}{12}-\left(\frac{3\pi}{2}-\frac{9\sqrt{3}}{4}\right)=-\frac{\pi}{2}+\frac{18\sqrt{3}}{12}=\frac{3\sqrt{3}}{2}-\frac{\pi}{2}$$

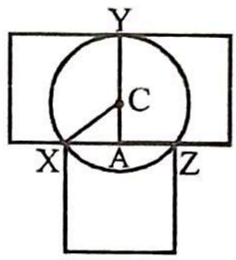
Double this to get the combined area of the lunes: $3\sqrt{3}-\pi$ cm².



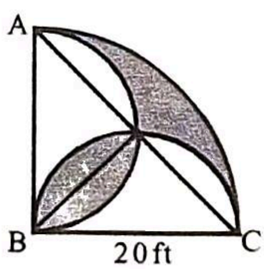
5.270 CD is a leg of right triangles CDE and CDB. Label CE = x and EB = $2x$ (because AB is trisected at the points of intersection, CE is half the length of EB).
$7^2 - x^2 = (CD)^2$ and $9^2 - (3x)^2 = (CD)^2$, so
$7^2 - x^2 = 9^2 - (3x)^2$. Solving for x we get $x = 2$, so AB = **12cm**.



5.271 Triangles BAD and ADC are similar, so: $\dfrac{8}{x} = \dfrac{x}{18}$. $x^2 = 144$ and $x = 12$, so the area of the figure is $\dfrac{12(8+18)}{2} = $ **156cm²**.



5.280 Label AX = x, then the side length of the square is $2x$, and CA = $2x - 15$ (the side of the square minus the radius CY=15). CX is 15. In right triangle CAX: $x^2 + (2x-15)^2 = 15^2$. Solving for x we get $x = 12$. The area of each square is $(2x)^2$ = **576cm²**.



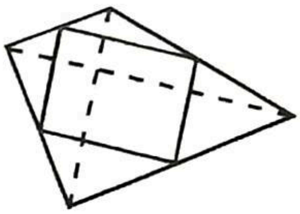
5.290 With the lines drawn as shown, it becomes clear that the two circle segments which make the "football" shape could be used to fill the white space outside the isosceles right triangle. We only need to subtract the area of triangle ABC from the quarter-circle: $100\pi - 200$

5.300 The exterior angle measure is easiest to find: 360/15 = 24, so the interior angle is its supplement: **156°**.

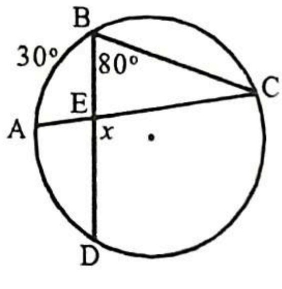
5.301 Find the total volume of water, then find the depth when half is in each container. The combined volume is $24\pi(4)^2 + 24\pi(3)^2 = 600\pi$ cm³. The half in the 4ft cylinder fills it to a depth of $300\pi / 16\pi$ = 18.75 feet, or 18ft 9in. The half in the 3ft cylinder fills it to a depth of $300\pi / 9\pi = 33.\overline{3}$ ft or 33ft 4in. The difference is **14ft 7in.**

5.311 Draw a diagonal from one point of contact where the inscribed cylinder touches the sphere through the center of the sphere to a point of contact on the opposite side (this is a diameter of the sphere). This diameter is the hypotenuse of a right triangle with legs 6in and 8in, making the diameter 10in and the radius 5in.
The volume is $\frac{4}{3}\pi \cdot 5^3 = \frac{500\pi}{3}$ in³.

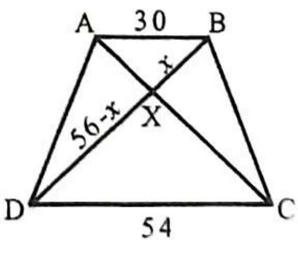
5.320 Connecting the midpoints of any quadrilateral creates a parallelogram in which all four sides are midsegments of triangles whose bases are the diagonals of the original quadrilateral. The sides are therefore half the length of the diagonals: 4 + 4 + 5 + 5 = **18cm.**

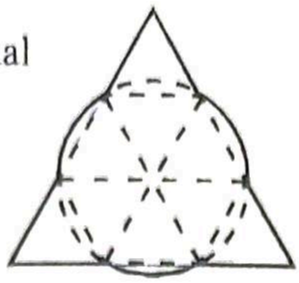


5.330 Angle BCA is inscribed in an arc measuring 30°, so its measure is 15°. The measure x is an exterior angle of triangle BEC equal to the sum of angles B and C. $\boldsymbol{x}$ **= 95°**.



5.331 In similar triangles ABX and CDX, $30/x = 54/(56\text{-}x)$. Solve for x using cross-products to get $x = 20$. Find the height of triangle ABX using the Pythagorean theorem: $h = 5\sqrt{7}$ and the area is $15 \cdot 5\sqrt{7} = 75\sqrt{7}$.

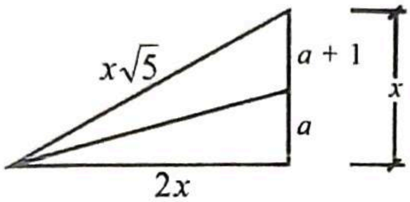




5.340 The side length of the hexagon is equal to the radius of the circle (3cm). This makes the diameter of the circle 6cm and the circumference 6π. Half the circumference is on the perimeter of the figure (3π). Add this to the six smaller segments (3cm each) to get $3\pi + 18$ cm.

5.341 The lengths of the sides are in the ratio $x : 2x : x\sqrt{5}$. We will label the parts of the short leg a and $a + 1$:

$$\frac{a}{2x} = \frac{a+1}{x\sqrt{5}}.$$



$$ax\sqrt{5} = 2ax + 2x$$

$$ax\sqrt{5} - 2ax = 2x$$

$$a(\sqrt{5} - 2) = 2$$

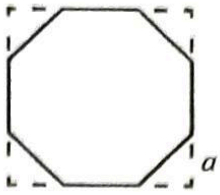
$$a = \frac{2}{\sqrt{5} - 2}$$

Rationalizing the denominator gives us: $a = 2\sqrt{5} + 4$. $x = a + (a+1) = 4\sqrt{5} + 9$, $2x = 8\sqrt{5} + 18$, and $x\sqrt{5} = 20 + 9\sqrt{5}$. The perimeter is the sum of these three sides: $21\sqrt{5} + 47$.

5.350 We look to "unfold" the box in a way that will give us the rectangle with the shortest diagonal. There are three ways in this case. Problem **5.250** gave us a 9 by 40 rectangle, using the top and the right sides give us a 10 by 39 rectangle, and using the top and the front gives us a 19 by 30 rectangle. To minimize the diagonal, we look for the rectangle whose sides are closest in length. The shortest distance is the diagonal of an 19cm by 30cm rectangle, **≈35.5 cm.**

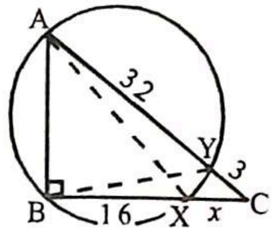
5.351 The larger hemisphere has a surface area 4 times the original. If the areas are in the ratio 1:4, then the radii are in the ratio 1:2, and the volumes are in the ratio 1:8. The smaller hemisphere must have 1/8 the volume and mass of the larger one, or **1/4 pound.**

5.360 If we extend the sides as shown, we create 45-45-90 triangles on each corner, each with a hypotenuse of 1. This makes the length $a = \sqrt{2}/2$. The height of the octagon is $1 + 2a$: $(1+\sqrt{2})$ ft.



5.361 The half-full cone is similar to and half the volume of the whole cone. If the ratio of the volumes is 1:2, then the ratio of the heights is $1:\sqrt[3]{2}$. If the height of the large cone is 20cm, we divide by $\sqrt[3]{2}$ to get the height of the small cone: $\approx$**15.9cm.**

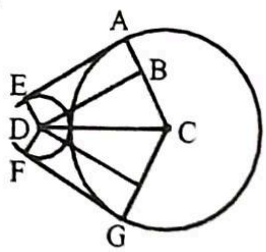
5.371 If we add AX and BY, we get similar triangles CXA and CYB. This gives us the proportion:



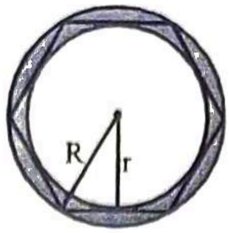
$$\frac{3}{16+x} = \frac{x}{35}, \text{ or } 105 = 16x + x^2.$$

Subtracting the 105 and factoring yields $0 = (x+21)(x-5)$, so $x = 5$ (the positive solution). Plugging this in, we see that triangle ABC has hypotenuse 35 and leg 21. This means that AB = 28 (not to scale), and the area of triangle ABC = **294** square units. Alternatively, we could add XY and solve using similar right triangles CYX and CBA.

5.380 Add segment BD parallel to AE. From the given information, we see that DE = 3, AB = 3, BC = 9 − 3 = 6, and CD = 12. This makes triangle CBD a 30-60-90 right triangle in which BD $=6\sqrt{3}=$ AE = FG. The total length of the band includes arcs EF, which is 120° (1/3 the smaller circumference), and major arc AG, which is 240° (2/3 of the larger circumference):



$$2\cdot 6\sqrt{3} + \frac{1}{3}(6\pi) + \frac{2}{3}(18\pi) = (14\pi + 12\sqrt{3})\text{ cm.}$$



5.390 From the diagram we have 30-60-90 triangle with short leg of length 2 (half the side of the hexagon). This makes $r = 2\sqrt{3}$ and $R = 4$, where r and R are the radii of the inscribed and circumscribed circles. The area of the annulus is $\pi(4)^2 - \pi(2\sqrt{3})^2 = 4\pi$ cm². Alternatively, we are looking for the difference between the radii squared and by the Pythagorean theorem, we see: $R^2 - r^2 = 2^2$, so $\pi(R^2 - r^2) = 4\pi$. (See **5.071**).

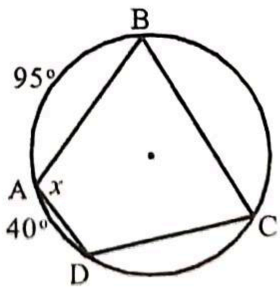
5.400 Triangle ACB is isosceles with obtuse vertex angle B = 144°, making its base angles A and C **18°**.

5.401 The surface area of the cone is $\pi(3)^2 + \pi(3)s$ (s is the slant height of the cone). The surface area of the hemisphere is $3\pi(3)^2$. Set these equal to get $9\pi + 3\pi s = 27\pi$, so $s = 6$. The slant height (6) is the hypotenuse of the 30-60-90 right triangle whose short leg is the radius (3cm). The height is the long leg: $3\sqrt{3}$ cm.

5.411 If we label the side length of a cube *s*, the long diagonal of a cube is $s\sqrt{3}$, making the radius of the circumscribed sphere $s\sqrt{3}/2$. The radius of the inscribed sphere is *s*/2. The ratio of the surface area of the small sphere to the large sphere is:

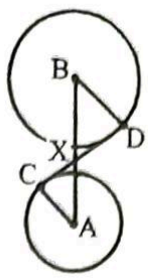
$$\frac{4\pi\left(\frac{s}{2}\right)^2}{4\pi\left(\frac{s\sqrt{3}}{2}\right)^2} = \frac{s^2}{(s\sqrt{3})^2} = \frac{s^2}{3s^2} = \frac{1}{3}$$

5.420 Connecting two adjacent points to any other vertex will create a **5°** angle (see **5.620**).



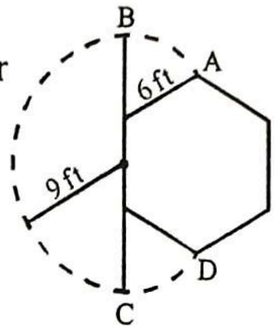
5.430 Arc BCD is 360° minus arcs AB and AD, or 225°. Angle measure x is half the intercepted arc: **112.5°**.

5.431 In similar right triangles BDX and ACX, BD = 44, and AC = 28. Label segment AX = x and BX = $90-x$. This gives us the proportion:

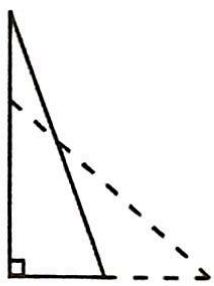


$\frac{28}{x}=\frac{44}{90-x}$, and cross products give us $2{,}520-28x=44x$. Solving for x we get $x=35$, so AX = 35 and BX = 55. Substituting and applying the Pythagorean theorem, we get CX = 21 and DX = 33. CX + DX = CD = **54 inches.**

5.440 Semicircle BC has an arc length of 9π. After the dog passes point B or C, three feet of the leash are on the side of the shed and the radius is limited to 6 feet. The arc measure of AB and CD is 60°. Their sum is 120°, which is 1/3 of a circle of diameter 12. Therefore, arc lengths AB + CD = $(1/3)12\pi=4\pi$ and the combined length of the dashed path is $4\pi+9\pi=13\pi$ feet.



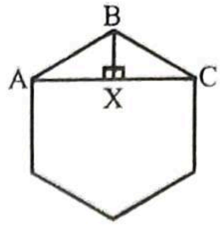
5.450 When we begin to climb, the triangle formed between the floor, the ladder, and the wall has lengths 14ft, 48ft, and 50ft (recognizing Pythagorean triples is key here) and when the ladder slips, the 48 foot height becomes 40 feet. The ladder (hypotenuse) remains 50, so the base of the ladder must now be 30 feet from the wall (3:4:5 right triangle), so the base has slipped $30-14=$ **16 feet.**



5.451 The ratio of their altitudes is 3:4, so the ratio of their volumes is $3^3:4^3=27:64$. Set up a proportion:

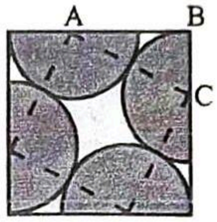
$\frac{27}{64}=\frac{30}{x}$ gives us x = **(640/9)cm³**, about 71.1cm³.

5.460 If the perimeter is 24cm, the side length is 4cm. Adding BX perpendicular to AC creates two 30-60-90 right triangles in which BX = 2 and AX = CX = $2\sqrt{3}$, therefore AC = $4\sqrt{3}$ cm.



5.461 20 triangles have a total of 60 sides, but each side (edge) is shared by two triangles, so an icosahedron has 30 edges. $f + v = e + 2$, gives us $20 + v = 32$, and solving we get $v = 12$, so there are **12** vertices.

5.471 We only need to find the radius (r) of the congruent semicircles. If we add the lines shown, we notice that triangle ABC is a right triangle with BC = r and AC = $2r$ which means that triangle ABC is a 30-60-90 triangle with AB = $r\sqrt{3}$. Segment AB is also equal to the side length of the square (1) minus the radius: AB = $1-r$, so $r\sqrt{3} = 1-r$. Add r to both sides: $r\sqrt{3}+r=1$. Factor out the r: $r(1+\sqrt{3})=1$. Divide and rationalize the denominator:

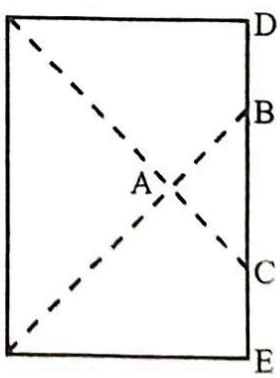


$$r = \frac{1}{1+\sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{1-\sqrt{3}}{-2} = \frac{\sqrt{3}-1}{2}$$

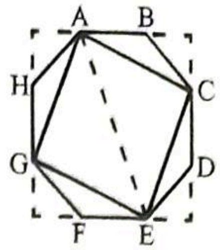
The combined area of the four semicircles is $2\pi r^2$:

$$2\pi\left(\frac{\sqrt{3}-1}{2}\right)^2 = \pi(2-\sqrt{3})\,\text{units}^2.$$

5.480 Because DC = BE = 8.5 inches and DE = 11 inches, DB = CE = 2.5 inches. This means that BC = 6 inches. BC is the hypotenuse of a 45-45-90 right triangle, so AB = $3\sqrt{2}$ and the area of triangle ABC = $(3\sqrt{2})^2/2 = \mathbf{9in^2}$.



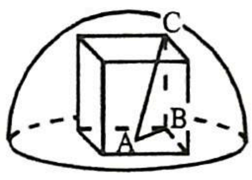
5.490 To find the area of the square, we will begin by finding the length of AE (the diagonal length of the square) using the Pythagorean theorem with right triangle AFE. We have learned to find the height (AF) of an octagon (**5.360**). $AF = 2+2\sqrt{2}$. FE = 2. By the Pythagorean theorem: $2^2+(2+2\sqrt{2})^2=(AE)^2$, so $8\sqrt{2}+16=(AE)^2$. $(AE)^2 = (AC)^2 + (CE)^2$, therefore $(AE)^2 = 2(AC)^2$. We are trying to find the area of the square $(AC)^2$:



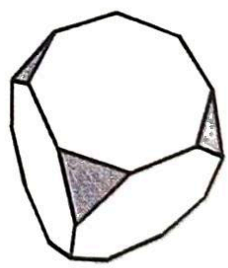
$$(AC)^2=\frac{(AE)^2}{2}=\frac{8\sqrt{2}+16}{2}=4\sqrt{2}+8\,\text{cm}^2.$$

5.500 We will label the angles $x-28$, x, $x+28$, and $x+56$. The sum of the angles is 360°, so $4x + 56 = 360°$. Solving for x, we get $x = 76$. The largest angle measure is $x + 56 = 76 + 56 =$ **132°**.

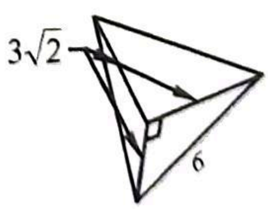
5.501 The radius of the hemisphere is equal to the distance from the midpoint of the base of the cube A to the vertex of the cube C. AC is the hypotenuse of a right triangle whose short leg AB is half the diagonal length of a square of side length 2. Therefore $AB = \sqrt{2}$, BC = 2, and $(\sqrt{2})^2+2^2=(AC)^2$. $AC = \sqrt{6}$ and the surface area of the hemisphere is $3\pi(\sqrt{6})^2=18\pi\,\text{in}^2$.



5.511 Eight triangular pyramids are removed. The base of each is an isosceles right triangle of area $(3\sqrt{2})^2/2 =$ 9cm² and whose height is $3\sqrt{2}$ cm. The combined volume of eight of these pyramids is $8(Bh/3)$:

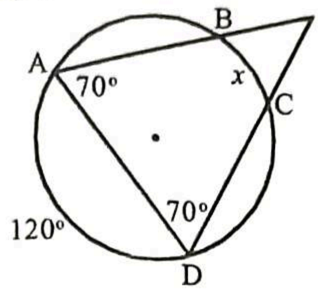


$$8\cdot\frac{9(3\sqrt{2})}{3}=72\sqrt{2}\,\text{cm}^3$$

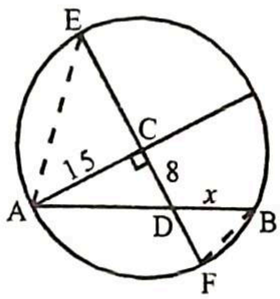


5.520 From number theory: $360/n$ equals the exterior angle measure, so for the exterior angle measure to be integral, n must be a factor of 360. 360 has 24 factors, but we cannot use 1 or 2 (a polygon cannot have 1 or 2 sides). $24 - 2 =$ **22** polygons.

5.530 Arcs AC and BD are twice the inscribed angles A and D, making AC = BD = 140°. If we add arcs AB + BC + CD we get 360° - 120° = 240°. When we add arcs AC and BD we get 280°, which is equal to AB + BC + BC + CD. If AB + BC + CD = 280° and AB + BC + BC + CD = 240° then arc measure BC = $\boldsymbol{x}$ = **40°**.



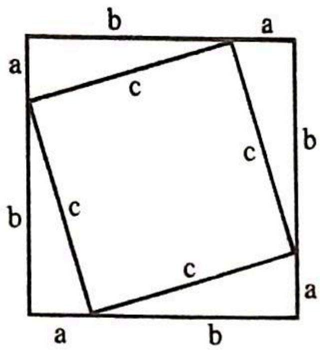
5.531 Segment AD = 17 by the Pythagorean theorem. EC = AC = 15 (both are radii). DF = 15 − 8 = 7. In similar triangles ADE and FDB, AD/DF = DE/BD: $\frac{17}{7} = \frac{23}{x}$, so $x = \frac{121}{17}$.



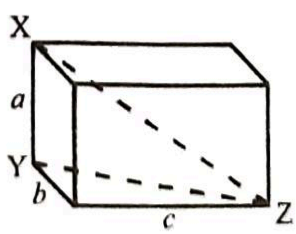
$$AB = AD + DB = 17 + \frac{161}{17} = \frac{289}{17} + \frac{161}{17} = \frac{450}{17}.$$

5.540 The area of the square is $(a+b)^2$, and can also be found by adding the center square to the four triangles: $c^2 + 2ab$. Therefore:

$$(a+b)^2 = c^2 + 2ab$$
$$a^2 + 2ab + b^2 = c^2 + 2ab$$
$$a^2 + b^2 = c^2$$

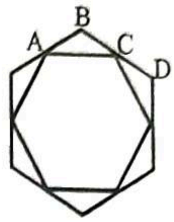


5.550 The length YZ= $\sqrt{b^2 + c^2}$, so XZ is the hypotenuse of right triangle XYZ. For diagonal XZ, $(XZ)^2 = a^2 + (\sqrt{b^2 + c^2})^2$ so XZ = = $\sqrt{a^2 + b^2 + c^2}$.

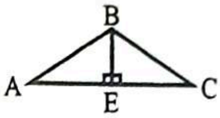


5.551 To increase the surface area 100 times, the edge length must be increased by a factor of 10, meaning the volume will be increased by a factor of 1,000. The new cube will have 1,000 times the volume of the original, or **3,000cm³**.

5.560 We are looking for the ratio AC:BD. Triangle ABC is isosceles with angle ABC = 120°. If we drop an altitude from B to AC, we bisect angle ABC creating two 30-60-90 triangles.



Assign a length of 2 to BC, and we see that CE = $\sqrt{3}$ (you could also call length BC = $2x$ and CE = $x\sqrt{3}$, but we are only looking for the ratio, so it is fine to assign a number that is easy to work with). Double both values to get the lengths of the sides of the hexagons.



AC = 2CE = $2\sqrt{3}$ and BD = 2BC = 4 so

$$\frac{AC}{BD}=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}.$$

5.561 Call the edge lengths a, b, and c. We are given that $ab = 6$, $bc = 9$, and $ac = 12$. We are looking for the diagonal length, given by $\sqrt{a^2+b^2+c^2}$ (**5.550**).

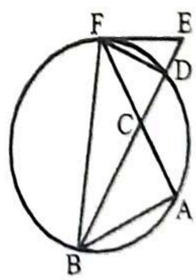
$$\frac{ab\cdot ac}{bc}=a^2=8,\ \frac{ab\cdot bc}{ac}=b^2=4.5,\ \frac{ac\cdot bc}{ab}=c^2=18.$$

The diagonal is therefore $\sqrt{8+4.5+18}\approx$ **5.5cm.**

5.571 If we add segments FD and FB we get similar triangles FDE and EFB (angles FED = BEF and EFD = EBF). Therefore, DE/EF = FE/EB:

$\frac{8}{20}=\frac{20}{EB}$, so EB = 50. EC = 20 so CB = 30.

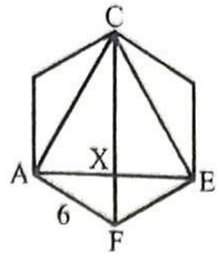
Triangles FDC and BAC are similar:



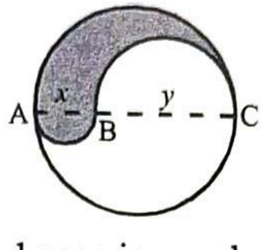
$\frac{FC}{BC}=\frac{DC}{AC}$. $\frac{20}{30}=\frac{12}{AC}$. AC = 18.

AF = AC + FC = **38cm.**

5.580 In the diagram, we see that triangles AXF and AXC are 30-60-90 right triangles. AF = 6, making AX = $3\sqrt{3}$ and XC = 9 which makes the area of triangle ACE $= 9 \cdot 3\sqrt{3} = 27\sqrt{3}$.



5.590 We will do this the hard way to prove that the solution works for all cases. Labeling the semicircles by their diameters, the shaded area is equal to semicircle AC minus semicircle BC plus semicircle AB. The unshaded area is equal to semicircle AC minus semicircle AB plus semicircle BC. The areas of the semicircles are directly proportional to their radii squared, or even their diameters squared, so we will use x^2 to represent the area of semicircle AB, y^2 to represent the area of semicircle BC, and $(x + y)^2$ to represent the area of semicircle AC. The ratio AB to BC is x:y, and we are looking for the ratio of the shaded area to the unshaded area:



$$\frac{\textit{shaded}}{\textit{unshaded}} = \frac{(x+y)^2 - y^2 + x^2}{(x+y)^2 - x^2 + y^2} = \frac{x^2 + 2xy + y^2 - y^2 + x^2}{x^2 + 2xy + y^2 - x^2 + y^2} =$$

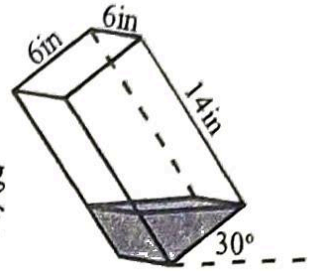
$$\frac{2x^2 + 2xy}{2y^2 + 2xy} = \frac{2x(x+y)}{2y(x+y)} = \frac{x}{y}.$$

Of course, we did not need to do all of this. It would have been easier to assign convenient values, but this allows us to see that for any ratio x:y in the given diagram, the ratio of the areas will be x:y.
If AB:BC = 1:3 then the ratio of the shaded to unshaded areas is also **1:3**.

5.600 If its interior angles measure 179°, its exterior angles are 1°, so it has 360/1 = 360 sides, each 2cm in length for a perimeter of **720cm**.

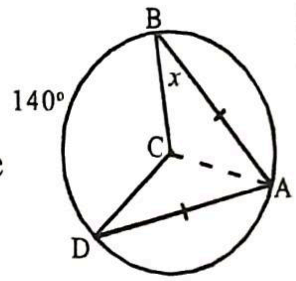
5.601 There are 38 total faces. We can solve for the number of edges with some common sense: 6 squares have 24 sides and 32 triangles have a total of 96 sides for a total of 120 sides, each shared by two faces. This gives us 60 edges. $f + v = e + 2$ can be used to give us $v = 24$. $v + e = 24 + 60 =$ **84**.

5.611 The triangle facing us is a 30-60-90 triangle (the water line will be parallel to the floor) with a long leg length 6in. This gives us a short leg of $2\sqrt{3}$ inches, making the area of the triangle $6\sqrt{3}$. Using the triangle as the base of a prism, we see that the prism's height would be 6in. The volume (Bh) is therefore $6(6\sqrt{3}) = 36\sqrt{3}$ in³.



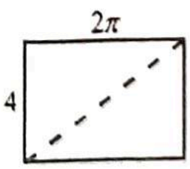
5.620 Triangle ACB is isosceles, so angle ABC = 170°. The exterior angles are 10°, and 360/10 = 36 sides of length 5cm makes the perimeter **180cm.**

5.630 Arcs AB and AD must be equal because they are intercepted by congruent chords, making each 110°. Angles CBA and CAB are base angles of isosceles triangle ABC and therefore equal, with angle BCA = 110°, both base angles must be 35°. $\boldsymbol{x = 35°}$.



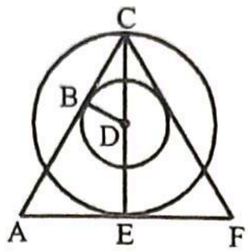
5.650 If we unroll the lateral surface of the cylinder, the path of the ant is the diagonal of a rectangle that is 4cm tall and half the circumference of the circle in width (2π cm). We use the Pythagorean theorem to find the diagonal length:

$$d = \sqrt{4^2 + (2\pi)^2} \approx \mathbf{7.4cm}.$$



5.651 The ratio of the diameters of the snowballs is 2:3:4, so the ratio of their masses is $2^3:3^3:4^3 = 8:27:64$. The mass of the smallest snowball is 80 pounds. Using our ratio, we get 270lbs for the medium and 640lbs for the large, for a total weight of **990lbs**.

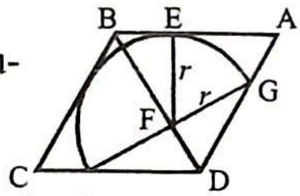
5.660 Segment AE = 4cm. Both AEC and BDC are 30-60-90 right triangles, so EC = $4\sqrt{3}$, CD = $2\sqrt{3}$, and BD = $\sqrt{3}$. The small circle therefore has an area of $\pi(\sqrt{3})^2 = 3\pi$ cm².



5.661 Call the edge lengths a, b, and c. We are given that $ab = 6$, $bc = 9$, and $ac = 12$ and we are looking for abc. $\sqrt{(ab)(bc)(ac)} = \sqrt{a^2b^2c^2} = abc$:

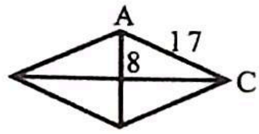
$$abc = \sqrt{(6)(9)(12)} = 18\sqrt{2}$$

5.671 The largest semicircle is inscribed as shown, with its diameter perpendicular to diagonal BD. Triangles BEF and DFG are congruent (both are 30-60-90 triangles with medium leg length r). This means that the ratio of BF to DF is 2:1, and because BD = 3, BF = 2 and FD =1. This makes



$r = \sqrt{3}$, and the area of semicircle is $\dfrac{\pi(\sqrt{3})^2}{2} = \dfrac{3\pi}{2}$.

5.690 The diagonals of a rhombus are perpendicular and bisect each other, and the area is half the product of the diagonals. AC is the hypotenuse of a right triangle of short leg 8. We use the Pythagorean theorem to find long leg 15cm, making the long diagonal 30cm and the area

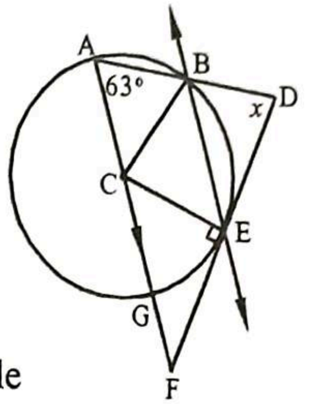


$\dfrac{30 \cdot 16}{2} = \mathbf{240cm^2}$.

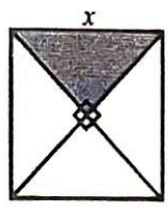
5.700 Ten 150° interior angles means there are ten 30° exterior angles for a total of 300°. For the exterior angle sum to equal 360°, the remaining exterior angle must measure **60°**, which is the largest.

5.720 If the perimeter of rhombus PQRS = 18cm, the side length is 18/4 = 4.5cm. The side of the rhombus forms the midsegment of triangle WXY, where WY is the base. WY is twice the midsegment = **9cm.**

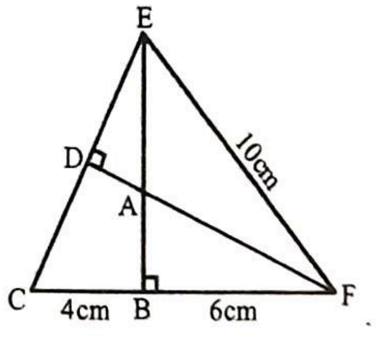
5.730 Triangle CAB is isosceles (CA and CB are radii) with base angles A = B = 63°. Angle ACB is 54°, making arc AB 54° as well. Because parallel lines intercept congruent arcs, arc EG = 54°, angle ECG = 54°, and angle EFC = 36°. In triangle ADF, angle A = 63°, F = 36°, and the measure of angle D = $\boldsymbol{x}$ = **81°**.



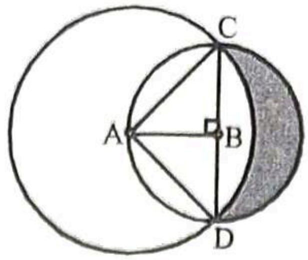
5.760 There are many ways to solve this. The diagram to the right should be convincing. The area of an isosceles right triangle with hypotenuse length x is 1/4 the area of a square of edge length x, or $\boldsymbol{x^2/4}$.

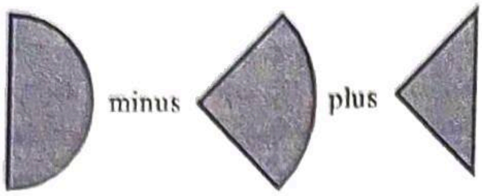


5.761 There are a lot of similar triangles in this figure. Mark congruent angles and convince yourself that triangles ABF, ADE, CBE, CDF, and EDF are all similar. We can use the Pythagorean theorem in triangle EBF to find EB = 8cm, so the ratio of the short leg to the long leg in each of the similar triangles is 1:2. Using the Pythagorean theorem with triangle CBE we get CE= $4\sqrt{5}$, DE= $2\sqrt{5}$, and AD is DE/2 = $\sqrt{5}$ cm.



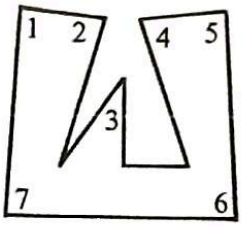
5.790 In the diagram, AB = BC = 1cm and AC = $\sqrt{2}$ by the Pythagorean theorem. We will find the area by:





$\frac{\pi}{2} \quad - \frac{\pi}{2} \quad +1 \quad = 1\text{cm}^2$.

5.800 The sum of the interior angles in a decagon is 180(8) = 1,440. To maximize the number of acute angles, we maximize the measure of the obtuse interior angles. The largest interior angle possible is just a hair smaller than 360°. With two (almost) 360 degree angles, the remaining 8 angles would have an average measure of slightly more than 90°, so we must have at least three obtuse interior angles and a maximum of **7** acute interior angles. One example is shown.

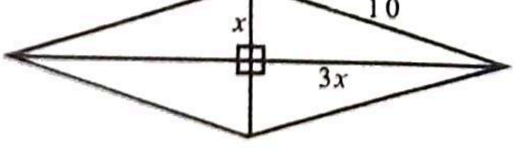


5.820 The diagonals of the rhombus intersect at right angles, so the sides of the rhombus form the hypotenuse of a right triangle of sides that can be labeled x and $3x$. This gives us:

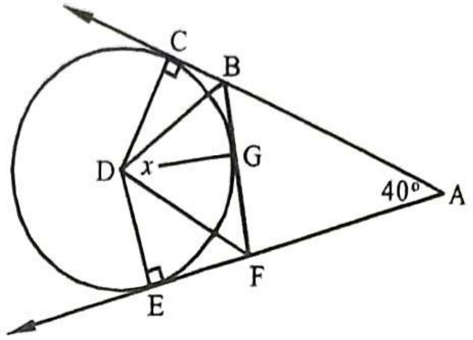
$x^2 + (3x)^2 = 10^2$ or $10x^2 = 100$, which means

$x = \sqrt{10}$. The area of a rhombus (it is just four right triangles) is half the product of the diagonals:

$\frac{2\sqrt{10} \cdot 6\sqrt{10}}{2} = 60$. The area is **60cm²**.

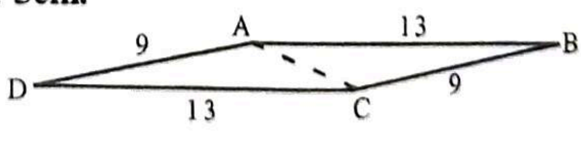


5.830 Quadrilateral ACDE is a kite with right angles ACD and DEA, which makes angles A and CDE supplementary, so angle CDE = 140°. Notice kites CBGD and DGFE as well, with diagonals DB and DF bisecting the vertex angles. This makes the measure x of angle BDF half the measure of CDE. $x = 70°$.

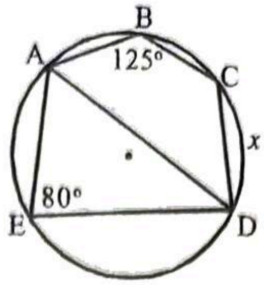


5.890 There are only a few triangles with integral side lengths a, b, and c that have perimeter = 15:
$(a, b, c) = (1, 7, 7)$ $(2, 6, 7)$, $(3, 6, 6)$, $(3, 5, 7)$, $(4, 5, 6)$ and $(4, 4, 7)$. It is easy to use Heron's formula here, because each has the same semiperimeter. We are looking to minimize the product
$(s-a)(s-b)(s-c)$, where $s = 7.5$. For each set above we get:
$(1, 7, 7)$: $(7.5-1)(7.5-7)(7.5-7) = 1.625$
$(2, 6, 7)$: $(7.5-2)(7.5-6)(7.5-7) = 4.125$
$(3, 6, 6)$: $(7.5-3)(7.5-6)(7.5-6) = 10.125$
$(3, 5, 7)$: $(7.5-3)(7.5-5)(7.5-7) = 5.625$
$(4, 5, 6)$: $(7.5-4)(7.5-5)(7.5-6) = 13.125$
$(4, 4, 7)$: $(7.5-4)(7.5-4)(7.5-7) = 6.125$
The $(1, 7, 7)$ triangle is the smallest, now complete Heron's formula to find the area:
$\sqrt{7.5(1.625)} \approx$ **3.49cm²**.

5.900 The diagonal length must be greater than the difference between side lengths, making the shortest possible integer length **5cm.**



5.930 Angle B is inscribed in major arc ADC, which measures 250°. This makes minor arc AC = 110°. Angle E is inscribed in minor arc AD which measures 160°. The measure of arc CD = 160° − 110° = **50°**.



5.990 The area of the rhombus is half the product of its diagonals, or 24cm². The diagonals bisect each other at right angles, creating 3-4-5 right triangles where 5cm is the side length of the rhombus. The area of a rhombus can also be found by multiplying its edge length by its altitude (a) which gives us $5a = 24$, so the altitude is **24/5 cm**.

Appendix

Chapter VII

In general, there are very few things that I recommend you just sit down and memorize. After solving a lot of difficult problems, you tend to build a mathematical vocabulary of items that are frequently useful. For example, most students who do well in mathematical competitions would not need to do any math to find the missing hypotenuse of a right triangle with legs of length 12 and 35 or the square root of 1,024.

Included in this chapter are formulas and values that are good to know. Much of this should be memorized, but don't spend too much time cramming numbers and formulas in your head. Refer to this section when you are not sure about a number cube or a formula and eventually you will start to remember them without ever having done much drill work.

It doesn't hurt to purposefully commit some of this material to memory. My memory is dreadful, and I have had to learn some patterns and tricks that help me remember most of the material in this section. Look for patterns that help you find ways to do the same.

Some of the material here is not covered in the text, but may be useful as you continue with practice from other sources.

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Squares:	Cubes:	Powers of 2:
		$2^0 = 1$
$11^2 = 121$	$1^3 = 1$	$2^1 = 2$
$12^2 = 144$	$2^3 = 8$	$2^2 = 4$
$13^2 = 169$	$3^3 = 27$	$2^3 = 8$
$14^2 = 196$	$4^3 = 64$	$2^4 = 16$
$15^2 = 225$	$5^3 = 125$	$2^5 = 32$
$16^2 = 256$	$6^3 = 216$	$2^6 = 64$
$17^2 = 289$	$7^3 = 343$	$2^7 = 128$
$18^2 = 324$	$8^3 = 512$	$2^8 = 256$
$19^2 = 361$	$9^3 = 729$	$2^9 = 512$
$20^2 = 400$	$10^3 = 1{,}000$	$2^{10} = 1{,}024$
$21^2 = 441$	$11^3 = 1{,}331$	

Squaring a Number Ending in 5 (or .5):

The number will always end in 25. Find the leading digits by multiplying the first digit(s) by the next integer:

$25^2 = (2 \cdot 3)\,25 = 625$
$35^2 = (3 \cdot 4)\,25 = 1{,}225$
$45^2 = (4 \cdot 5)\,25 = 2{,}025$
$55^2 = (5 \cdot 6)\,25 = 3{,}025$
$65^2 = (6 \cdot 7)\,25 = 4{,}225$
$75^2 = (7 \cdot 8)\,25 = 5{,}625$
$85^2 = (8 \cdot 9)\,25 = 7{,}225$
$95^2 = (9 \cdot 10)\,25 = 9{,}025$
$105^2 = (10 \cdot 11)\,25 = 11{,}025$
...

These come up frequently:
$0.5^2 = 0.25$
$1.5^2 = 2.25$
$2.5^2 = 6.25$
$3.5^2 = 12.25$
...

First 50 Primes: (There are 25 primes less than 100).

2	3	5	7	11	13	17	19	23	29	31
37	41	43	47	53	59	61	67	71	73	79
83	89	97	101	103	107	109	113	127	131	137
139	149	151	157	163	167	173	179	181	191	
193	197	199	211	223	227	229 ...				

Don't be fooled by these prime-looking numbers:

91 $(7 \cdot 13)$ **119:** $(7 \cdot 17)$ **133:** $(7 \cdot 19)$ **143** $(11 \cdot 13)$
161: $(7 \cdot 23)$ **187:** $(11 \cdot 17)$ **203:** $(7 \cdot 29)$ **217:** $(7 \cdot 31)$

Pythagorean Theorem: For a right triangle with legs a and b with hypotenuse c: $a^2 + b^2 = c^2$

Pythagorean Triples:

Primitive triples and other common triples where $a \leq 20$.

***a* is odd:**	***a* is even:**
note: $a^2 = b + c$	$(6, 8, 10)$
$(3, 4, 5)$	$(8, 15, 17)$: $8^2 = 2(15 + 17)$
$(5, 12, 13)$	$(10, 24, 26)$
$(7, 24, 25)$	$(12, 35, 37)$: $12^2 = 2(35 + 37)$
$(9, 40, 41)$	$(14, 48, 50)$
$(11, 60, 61)$	$(16, 63, 65)$* : $16^2 = 2(63 + 65)$
$(13, 84, 85)$	$(18, 80, 82)$
$(15, 112, 113)$	$(20, 21, 29)$**

* Also (16, 30, 34) which is just twice the (8, 15, 17) triple.

** Largest common primitive triple and does not fit easily into the patterns shown.

Less common primitive triples for $a < 40$:

$(28, 45, 53)$ $(33, 56, 65)$ $(36, 77, 85)$ $(39, 80, 89)$

Pascal's Triangle:

1 *row 0*

1 1 *row 1*

1 2 1 *row 2*

1 3 3 1 *row 3*

1 4 6 4 1 *row 4*

1 5 10 10 5 1 *row 5*

1 6 15 20 15 6 1 *row 6*

1 7 21 35 35 21 7 1 *row 7*

1 8 28 56 70 56 28 8 1 *row 8*

1 9 36 84 126 126 84 36 9 1 *row 9*

1 10 45 120 210 252 210 120 45 10 1 *row 10*

1 11 55 165 330 462 462 330 165 55 11 1 *row 11*

The 7th row is relatively easy to memorize. If you memorize the 7th row, the rows beneath are easy to construct.

Triangular numbers follow the third diagonal (1, 3, 6, 10...).

The sum of the numbers in the n^{th} row is 2^n.

Commonly Used Fractions:

Know all of these without having to think about them.
Obviously, this is not a comprehensive list. You should know your tenths, hundredths, twentieths, etc. as well.

$\frac{1}{8} = 0.125$ $\frac{1}{4} = 0.25$ $\frac{1}{3} = 0.\bar{3}$

$\frac{3}{8} = 0.375$ $\frac{1}{2} = 0.5$ $\frac{5}{8} = 0.625$

$\frac{2}{3} = 0.\bar{6}$ $\frac{7}{8} = 0.875$ $\frac{3}{4} = 0.75$

Sixths and Twelfths:

$\frac{1}{12} = 0.08\bar{3}$ $\frac{1}{6} = 0.1\bar{6}$ $\frac{5}{12} = 0.41\bar{6}$

$\frac{7}{12} = 0.58\bar{3}$ $\frac{5}{6} = 0.8\bar{3}$ $\frac{11}{12} = 0.91\bar{6}$

Sevenths: Remember 1/7 and the rest come easy. The order of the digits remains the same.

$\frac{1}{7} = 0.\overline{142857}$ $\frac{2}{7} = 0.\overline{285714}$ $\frac{3}{7} = 0.\overline{428571}$

$\frac{4}{7} = 0.\overline{571428}$ $\frac{5}{7} = 0.\overline{714285}$ $\frac{6}{7} = 0.\overline{857142}$

Ninths, 99ths, etc.:

$\frac{1}{9} = 0.\bar{1}$ $\frac{2}{9} = 0.\bar{2}$

$\frac{13}{99} = 0.\overline{13}$ $\frac{7}{99} = 0.\overline{07}$

$\frac{101}{999} = 0.\overline{101}$ $\frac{20}{999} = 0.\overline{020}$ etc.

Area Formulas:

Square: $A = s^2$

Parallelogram: $A = bh$

Rhombus or Kite: $A = \dfrac{ab}{2}$ (where a and b are the diagonals, this works for any quadrilateral with perpendicular diagonals).

Trapezoid: $A = \dfrac{h(b_1 + b_2)}{2}$ (where b_1 and b_2 are the parallel sides).

Circle: $A = \pi r^2$ Circumference: $C = 2\pi r$

Triangle: $A = \dfrac{bh}{2}$

$A = rs$ (where r is the inradius and s is the semiperimeter).

$A = \dfrac{abc}{4R}$ (a, b, and c are the sides and R is the circumradius).

$A = \dfrac{1}{2}ab\sin c$ (useful trig. formula not used in this text).

Heron's Formula: Used to find the area of a triangle with side lengths a, b, and c where s is the semiperimeter (half the perimeter):

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Volume:

Prism or Cylinder: $V = Bh$ Sphere: $V = \dfrac{4}{3}\pi r^3$

Pyramid or Cone: $V = \dfrac{Bh}{3}$

Surface Area:

Cylinder: $A = \pi r^2 + \pi rh$ Sphere: $V = 4\pi r^2$

Cone: $A = \pi r^2 + \pi rs$

Faces, Edges, and Vertices: $f + v = e + 2$

Ones!

$1^2 = 1$
$11^2 = 121$
$111^2 = 12{,}321 = (3 \cdot 37)^2$
$1{,}111^2 = 1{,}234{,}321 = (11 \cdot 101)^2$
$11{,}111^2 = 123{,}454{,}321 = (41 \cdot 271)^2$
$111{,}111^2 = 12{,}345{,}654{,}321 = (3 \cdot \underbrace{\boxed{7 \cdot 11 \cdot 13}}_{1{,}001} \cdot 37)^2$

...until the pattern breaks down after 12,345,678,987,654,321.

1,001 is boxed above because any six-digit number which repeats the same three digit is divisible by 1,001. Example: $317{,}317 = 1{,}001 \cdot 317 = 7 \cdot 11 \cdot 13 \cdot 317$

Clocks

Every minute, the minute hand on a standard clock travels 6 degrees and the hour hand travels 0.5 degrees. Perhaps more importantly, the angle between them changes by 5.5 degrees. This comes up often enough that it helps to have this memorized.

Permutations: $nPr = P(n,r) = \dfrac{n!}{(n-r)!}$

Combinations: $nCr = C(n,r) = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Linear Equations:

Slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Point-Slope Form: $y - y_1 = m(x - x_1)$

Standard Form: $Ax + By = C$

Slope in Standard Form: $-\dfrac{A}{B}$

Slope-Intercept Form: $y = mx + b$

Quadratics: $y = ax^2 + bx + c$

Quadratic Formula: Used to find the roots (zeros) of a quadratic equation in the form $0 = ax^2 + bx + c$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Vertex: The vertex is a maximum when $a < 0$ and a minimum when $a > 0$. The x-coordinate of the vertex is called the axis of symmetry and is found at:

$$x = \frac{-b}{2a}.$$

Sum of the Roots/Product of the Roots:

The sum of the roots of a quadratic in the form:

$0 = ax^2 + bx + c$ is equal to $\frac{-b}{a}$ or $-\frac{b}{a}$.

The product of the roots of a quadratic in the form:

$0 = ax^2 + bx + c$ is equal to $\frac{c}{a}$.

Special Factorizations:

Sum of Squares: $a^2 + b^2 = (a+b)^2 - 2ab$

Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$

Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Sum of the First n Cubes:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$$

Harmonic Mean: Used to find an average rate of a and b:

$$H = \frac{2ab}{a+b}.$$

$$\text{For } \{a_1, a_2, a_3, \ldots a_n\}\text{: } H = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_n}}.$$

Arithmetic Mean:

$$\text{For } \{a_1, a_2, a_3, \ldots a_n\}\text{: } A = \frac{a_1+a_2+a_3+\ldots+a_n}{n}.$$

Sum of an Arithmetic Series: The sum of an arithmetic series with terms a_1 through a_n:

$$a_1+a_2+a_3+\ldots+a_n = n\cdot\frac{a_1+a_n}{2}.$$

Geometric Mean: Of two numbers, a and b: $\sqrt{ab}$.

$$\text{For } \{a_1, a_2, a_3, \ldots a_n\}\text{: } G = \sqrt[n]{a_1\cdot a_2\cdot a_3\cdot\ldots\cdot a_n}$$

Sum of a Geometric Series: The sum of a geometric series with first term a and common ratio r:

$a+ar+ar^2+ar^3\ldots$ where $|r|<1$ is found by:

$$S = \frac{a}{1-r}.$$

Triangular Numbers: The n^{th} triangular number T_n is:

$$T_n = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

Sum of the first *n* Perfect Squares:

$$1^2+2^2+3^2+\ldots n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Jason Batterson is a curriculum developer at ArtofProblemSolving.com. He formerly taught math and ran the competition program at Ligon Middle School in Raleigh, NC. The school won state MATHCOUNTS championships in 2008, 2009, and 2010 and has consistently been the top middle school in the state for both the AMC8 and AMC10 competitions. In 2008 he was a recipient of the Edyth May Sliffe Award from the Mathematical Association of America for distinguished teaching in mathematics.

Patricia frequently misspells the word MISSPELL. How many ways are there to misspell the word MISSPELL if all of the correct letters are used, but are placed in the wrong order? (page 101)

How many of the factors of 900 have exactly 18 factors? (page 172)

What is the area of the largest semicircle which can be inscribed within rhombus ABCD of edge length 3cm if the measure of angle A is 60 degrees? (page 227)

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