2013 AMC 12B Problems/Problem 1

The following problem is from both the 2013 AMC 12B #1 and 2013 AMC 10B #3, so both problems redirect to this page.

Problem

On a particular January day, the high temperature in Lincoln, Nebraska, was 16 degrees higher than the low temperature, and the average of the high and low temperatures was 3°. In degrees, what was the low temperature in Lincoln that day?

$\textbf{(A)}\ -13 \qquad \textbf{(B)}\ -8 \qquad \textbf{(C)}\ -5 \qquad \textbf{(D)}\ -3 \qquad \textbf{(E)}\ 11$

Solution

Let L be the low temperature. The high temperature is $L+16$. The average is $\frac{L+(L+16)}{2}=3$. Solving for L, we get $L=\boxed{\textbf{(C)}\ -5}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
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2013 AMC 12B Problems/Problem 2

The following problem is from both the 2013 AMC 12B #2 and 2013 AMC 10B #2, so both problems redirect to this page.

Problem

Mr. Green measures his rectangular garden by walking two of the sides and finds that it is 15 steps by 20 steps. Each of Mr. Green's steps is 2 feet long. Mr. Green expects a half a pound of potatoes per square foot from his garden. How many pounds of potatoes does Mr. Green expect from his garden?

$\textbf{(A)}\ 600 \qquad \textbf{(B)}\ 800 \qquad \textbf{(C)}\ 1000 \qquad \textbf{(D)}\ 1200 \qquad \textbf{(E)}\ 1400$

Solution

Since each step is 2 feet, his garden is 30 by 40 feet. Thus, the area of $30(40) = 1200$ square feet. Since he is expecting $\dfrac{1}{2}$ of a pound per square foot, the total amount of potatoes expected is $1200 \times \dfrac{1}{2} = \boxed{\textbf{(A)}\ 600}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
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2013 AMC 12B Problems/Problem 3

The following problem is from both the 2013 AMC 12B #3 and 2013 AMC 10B #4, so both problems redirect to this page.

Problem

When counting from 3 to 201, 53 is the 51^{st} number counted. When counting backwards from 201 to 3, 53 is the n^{th} number counted. What is n?

$\textbf{(A)}\ 146 \qquad \textbf{(B)}\ 147 \qquad \textbf{(C)}\ 148 \qquad \textbf{(D)}\ 149 \qquad \textbf{(E)}\ 150$

Solution

Note that n is equal to the number of integers between 53 and 201, inclusive. Thus,
$n = 201 - 53 + 1 = \boxed{\textbf{(D)}\ 149}$

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
Preceded by Problem 2	Followed by Problem 4
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2013 AMC 12B Problems/Problem 4

The following problem is from both the 2013 AMC 12B #4 and 2013 AMC 10B #8, so both problems redirect to this page.

Problem

Ray's car averages 40 miles per gallon of gasoline, and Tom's car averages 10 miles per gallon of gasoline. Ray and Tom each drive the same number of miles. What is the cars' combined rate of miles per gallon of gasoline?
$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 16 \qquad \textbf{(C)}\ 25 \qquad \textbf{(D)}\ 30 \qquad \textbf{(E)}\ 40$

Solution

Let both Ray and Tom drive 40 miles. Ray's car would require $\frac{40}{40}=1$ gallon of gas and Tom's car would require $\frac{40}{10}=4$ gallons of gas. They would have driven a total of $40+40=80$ miles, on $1+4=5$ gallons of gas, for a combined rate of $\frac{80}{5}=\boxed{\textbf{(B)}\ 16}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
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2013 AMC 12B Problems/Problem 5

The following problem is from both the 2013 AMC 12B #5 and 2013 AMC 10B #6, so both problems redirect to this page.

Problem

The average age of 33 fifth-graders is 11. The average age of 55 of their parents is 33. What is the average age of all of these parents and fifth-graders?

$\textbf{(A)}\ 22 \qquad \textbf{(B)}\ 23.25 \qquad \textbf{(C)}\ 24.75 \qquad \textbf{(D)}\ 26.25 \qquad \textbf{(E)}\ 28$

Solution

The sum of the ages of the fifth graders is $33 * 11$, while the sum of the ages of the parents is $55 * 33$. Therefore, the total sum of all their ages must be 2178, and given $33 + 55 = 88$ people in total, their average age is $\frac{2178}{88} = \frac{99}{4} = \boxed{\textbf{(C)}\ 24.75}$.

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12B Problems/Problem 6

The following problem is from both the 2013 AMC 12B #6 and 2013 AMC 10B #11, so both problems redirect to this page.

Problem

Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 8$

Solution

If we complete the square after bringing the x and y terms to the other side, we get $(x-5)^2 + (y+3)^2 = 0$. Squares of real numbers are nonnegative, so we need both $(x-5)^2$ and $(y+3)^2$ to be 0, which only happens when $x = 5$ and $y = -3$. Therefore, $x + y = 5 + (-3) = \boxed{\textbf{(B)}\ 2}$.

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
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Categories: Introductory Algebra Problems | Algebraic Manipulations Problems

2013 AMC 10B Problems/Problem 7

Problem

Six points are equally spaced around a circle of radius 1. Three of these points are the vertices of a triangle that is neither equilateral nor isosceles. What is the area of this triangle?

$\textbf{(A)}\ \frac{\sqrt{3}}{3}\qquad\textbf{(B)}\ \frac{\sqrt{3}}{2}\qquad\textbf{(C)}\ 1\qquad\textbf{(D)}\ \sqrt{2}\qquad\textbf{(E)}\ 2$

Solution

If there are no two points on the circle that are adjacent, then the triangle would be equilateral. If the three points are all adjacent, it would be isosceles. Thus, the only possibility is two adjacent points and one point two away. Because one of the sides of this triangle is the diameter, the opposite angle is a right angle. Also, because the two adjacent angles are one sixth of the circle apart, the angle opposite them is thirty degrees. This is a $30-60-90$ triangle. If the original six points are connected, a regular hexagon is created. This hexagon consists of six equilateral triangles, so the radius is equal to one of its side lengths. The radius is 1, so the side opposite the thirty degree angle in the triangle is also 1. From rules with $30-60-90$ triangles, the area is $1 \cdot \sqrt{3}/2 = \boxed{\textbf{(B)}\ \frac{\sqrt{3}}{2}}$

See also

2013 AMC 10B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=43&year=2013))	
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Category: Introductory Geometry Problems

2013 AMC 12B Problems/Problem 8

Problem

Line l_1 has equation $3x - 2y = 1$ and goes through $A = (-1, -2)$. Line l_2 has equation $y = 1$ and meets line l_1 at point B. Line l_3 has positive slope, goes through point A, and meets l_2 at point C. The area of $\triangle ABC$ is 3. What is the slope of l_3?

$\textbf{(A)}\ \frac{2}{3} \qquad \textbf{(B)}\ \frac{3}{4} \qquad \textbf{(C)}\ 1 \qquad \textbf{(D)}\ \frac{4}{3} \qquad \textbf{(E)}\ \frac{3}{2}$

Solution

Line l_1 has the equation $y = 3x/2 - 1/2$ when rearranged. Substituting 1 for y, we find that line l_2 will meet this line at point $(1, 1)$, which is point B. We call $\overline{BC}$ the base and the altitude from A to the line connecting B and C, $y = -1$, the height. The altitude has length $|-2-1| = 3$, and the area of $\triangle ABC = 3$. Since $A = bh/2$, $b = 2$. Because l_3 has positive slope, it will meet l_2 to the right of B, and the point 2 to the right of B is $(3, 1)$. l_3 passes through $(-1, -2)$ and $(3, 1)$, and thus has slope $\frac{|1-(-2)|}{|3-(-1)|} = \boxed{\textbf{(B)}\ \frac{3}{4}}$.

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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Category: Introductory Geometry Problems

2013 AMC 12B Problems/Problem 9

Problem

What is the sum of the exponents of the prime factors of the square root of the largest perfect square that divides $12!$?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 7 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 10 \qquad \textbf{(E)}\ 12$

Solution

Looking at the prime numbers under 12, we see that there are $\left\lfloor \frac{12}{2} \right\rfloor + \left\lfloor \frac{12}{2^2} \right\rfloor + \left\lfloor \frac{12}{2^3} \right\rfloor = 6+3+1 = 10$ factors of 2, $\lfloor \frac{12}{3} \rfloor + \lfloor \frac{12}{3^2} \rfloor = 4+1 = 5$ factors of 3, and $\lfloor \frac{12}{5} \rfloor = 2$ factors of 5. All greater primes are represented once or not at all in $12!$, so they cannot be part of the square. Since we are looking for a perfect square, the exponents on its prime factors must be even, so we can only use 4 of the 5 factors of 3. The prime factorization of the square is therefore $2^{10} * 3^4 * 5^2$. To find the square root of this, we halve the exponents, leaving $2^5 * 3^2 * 5$. The sum of the exponents is $\boxed{\textbf{(C)}\ 8}$

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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Category: Introductory Number Theory Problems

2013 AMC 12B Problems/Problem 10

The following problem is from both the 2013 AMC 12B #10 and 2013 AMC 10B #17, so both problems redirect to this page.

Contents

Problem

Alex has 75 red tokens and 75 blue tokens. There is a booth where Alex can give two red tokens and receive in return a silver token and a blue token, and another booth where Alex can give three blue tokens and receive in return a silver token and a red token. Alex continues to exchange tokens until no more exchanges are possible. How many silver tokens will Alex have at the end?

$\textbf{(A)}\ 62 \qquad \textbf{(B)}\ 82 \qquad \textbf{(C)}\ 83 \qquad \textbf{(D)}\ 102 \qquad \textbf{(E)}\ 103$

Solution 1

We can approach this problem by assuming he goes to the red booth first. You start with 75R and 75B and at the end of the first booth, you will have 1R and 112B and 37S. We now move to the blue booth, and working through each booth until we have none left, we will end up with:1R, 2B and 103S. So, the answer is $\boxed{\textbf{(E)}103}$

Solution 2

Let x denote the number of visits to the first booth and y denote the number of visits to the second booth. Then we can describe the quantities of his red and blue coins as follows:

$$R(x,y) = -2x + y + 75$$

$$B(x,y) = x - 3y + 75$$

There are no legal exchanges when he has fewer than 2 red coins and fewer than 3 blue coins, namely when he has 1 red coin and 2 blue coins. We can then create a system of equations:

$$1 = -2x + y + 75$$

$$2 = x - 3y + 75$$

Solving yields $x = 59$ and $y = 44$. Since he gains one silver coin per visit to each booth, he has $x + y = 44 + 59 = \boxed{\textbf{(E)}103}$ silver coins in total.

See also

2013 AMC 12B Problems/Problem 11

Problem

Two bees start at the same spot and fly at the same rate in the following directions. Bee A travels 1 foot north, then 1 foot east, then 1 foot upwards, and then continues to repeat this pattern. Bee B travels 1 foot south, then 1 foot west, and then continues to repeat this pattern. In what directions are the bees traveling when they are exactly 10 feet away from each other?

$\textbf{(A)}\ A\ \text{east, } B\ \text{west}$
$\qquad \textbf{(B)}\ A\ \text{north, } B\ \text{south}$
$\qquad \textbf{(C)}\ A\ \text{north, } B\ \text{west}$
$\qquad \textbf{(D)}\ A\ \text{up, } B\ \text{south}$
$\qquad \textbf{(E)}\ A\ \text{up, } B\ \text{west}$

Solution

Let A and B begin at (0,0,0). In 6 steps, A will have done his route twice, ending up at (2,2,2), and B will have done his route three times, ending at (-3,-3,0). Their distance is $\sqrt{(2+3)^2+(2+3)^2+2^2}=\sqrt{54}<10$ We now move forward one step at a time until they are ten feet away: 7 steps: A moves north to (2,3,2), B moves south to (-3,-4,0), distance of $\sqrt{(2+3)^2+(3+4)^2+2^2}=\sqrt{78}<10$ 8 steps: A moves east to (3,3,2), B moves west to (-4,-4,0), distance of $\sqrt{(3+4)^2+(3+4)^2+2^2}=\sqrt{102}>10$

Thus they reach 10 feet away when A is moving east and B is moving west, between moves 7 and 8. Thus the answer is $\textbf{(A)}$

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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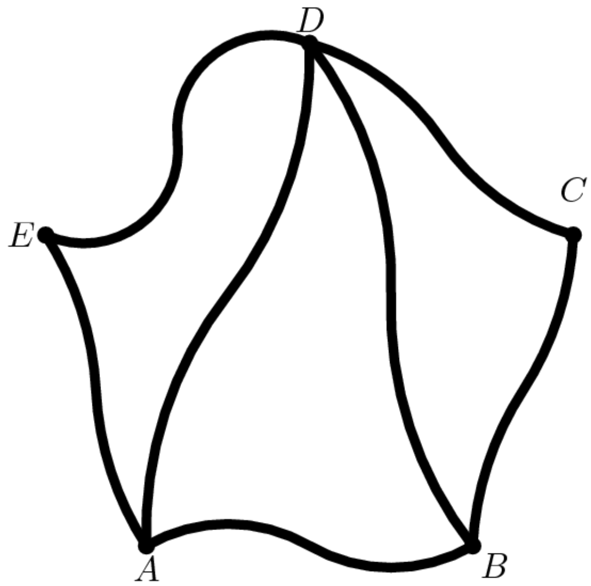
2013 AMC 12B Problems/Problem 12

Problem 12

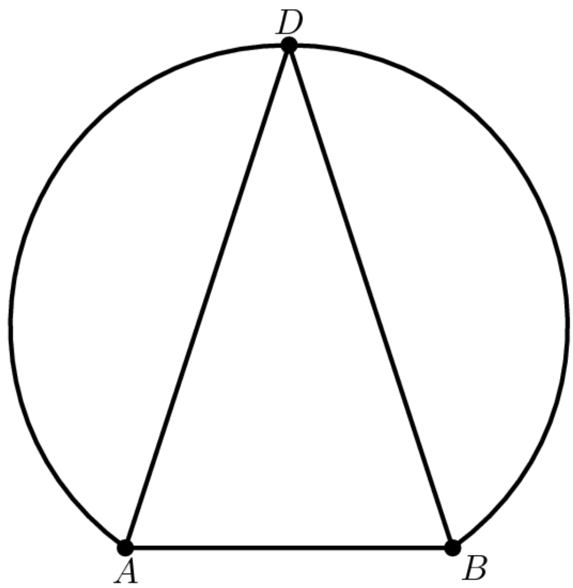
Cities A, B, C, D, and E are connected by roads $\widetilde{AB}$, $\widetilde{AD}$, $\widetilde{AE}$, $\widetilde{BC}$, $\widetilde{BD}$, $\widetilde{CD}$, and $\widetilde{DE}$. How many different routes are there from A to B that use each road exactly once? (Such a route will necessarily visit some cities more than once.)



$\textbf{(A)}\ 7 \qquad\textbf{(B)}\ 9 \qquad\textbf{(C)}\ 12 \qquad\textbf{(D)}\ 16 \qquad\textbf{(E)}\ 18$

Solution

Note that cities C and E can be removed when counting paths because if a path goes in to C or E, there is only one possible path to take out of cities C or E. So the diagram is as follows:



Now we proceed with casework. Remember that there are two ways to travel from A to D, D to A, B to D and D to B.:

Case 1 $A \Rightarrow D$: From D, if the path returns to A, then the next path must go to $B \Rightarrow D \Rightarrow B$. There are $2 \cdot 1 \cdot 2 = 4$ possibilities of the path $ADABDB$. If the path goes to D from B, then the path must continue with either $BDAB$ or $BADB$. There are $2 \cdot 2 \cdot 2 = 8$ possibilities. So, this case gives $4+8=12$ different possibilities.

Case 2 $A \Rightarrow B$: The path must continue with $BDADB$. There are $2 \cdot 2 = 4$ possibilities for this case.

Putting the two cases together gives $12+4=\boxed{\textbf{(D) }16}$

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12B Problems/Problem 13

Problem

The internal angles of quadrilateral $ABCD$ form an arithmetic progression. Triangles ABD and DCB are similar with $\angle DBA = \angle DCB$ and $\angle ADB = \angle CBD$. Moreover, the angles in each of these two triangles also form an arithemetic progression. In degrees, what is the largest possible sum of the two largest angles of $ABCD$?

$\textbf{(A)}\ 210 \qquad \textbf{(B)}\ 220 \qquad \textbf{(C)}\ 230 \qquad \textbf{(D)}\ 240 \qquad \textbf{(E)}\ 250$

Solution

Since the angles of Quadrilateral $ABCD$ form an arithmetic sequence, we can assign each angle with the value a, $a+d$, $a+2d$, and $a+3d$. Also, since these angles form an arithmetic progression, we can reason out that $(a)+(a+3d)=(a+d)+(a+2d)=180$.

For the sake of simplicity, lets rename the angles of each similar triangle. Lets call Angle DBA and Angle DCB Angle 1. Also we rename Angle ADB and Angle CBD Angle 2. Finally we rename Angles BAD and BDC Angle 3.

Now we can rename the four angles of Quadrilateral $ABCD$ as Angle 2, Angle $1+2$, Angle 3, and Angle $1+3$.

As for the similar triangles, whose Angles are equivalent, we can name them y, $y+b$, and $y+2b$. Therefore $y+y+b+y+2b=180$ and $y+b=60$. Because these 3 angles are each equal to one of the angles we named Angles 1, 2, and 3, we know that one of these three angles is equal to 60 degrees.

Now we we use trial and error to find out which of these 3 angles has a value of 60. If we substitute 60 degrees into Angle 1. This would cause the angle values of ABCD to be Angle 2, 60+Angle 2, Angle 3, and 60 + Angle 3. Since these four angles add up to 360, then Angle 2 + Angle 3 = 120. If we list them in increasing value, we get Angle 2, Angle 3, 60 + Angle 2, 60+Angle 3. Note that this is the only sequence that works because the common difference between each term cannot equal or exceed 45. So, this would give us the four angles 45, 75, 105, and 135. In this case, Angle 1, 2, and 3, the angles of both similar triangles, also form an arithmetic sequence with 45, 60, and 75, and the largest two angles of the quadrilateral add up to 240 which is an answer choice.

If we apply the same reasoning to Angles 2 and 3, we would get the sum of the highest two angles as 220, which works but is lower than 240. Therefore, $\boxed{\textbf{(D)}\ 240}$ is the correct answer.

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12B Problems/Problem 14

The following problem is from both the 2013 AMC 12B #14 and 2013 AMC 10B #21, so both problems redirect to this page.

Problem

Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. What is the smallest possible value of N ?

$\textbf{(A)}\ 55 \qquad \textbf{(B)}\ 89 \qquad \textbf{(C)}\ 104 \qquad \textbf{(D)}\ 144 \qquad \textbf{(E)}\ 273$

Solution

Let the first two terms of the first sequence be x_1 and x_2 and the first two of the second sequence be y_1 and y_2. Computing the seventh term, we see that $5x_1 + 8x_2 = 5y_1 + 8y_2$. Note that this means that x_1 and y_1 must have the same value modulo 8. To minimize, let one of them be 0; WLOG assume that $x_1 = 0$. Thus, the smallest possible value of y_1 is 8; since the sequences are non-decreasing $y_2 \geq 8$. To minimize, let $y_2 = 8$. Thus, $5y_1 + 8y_2 = 40 + 64 = \boxed{\textbf{(C)}\ 104}$.

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12B Problems/Problem 15

The following problem is from both the 2013 AMC 12B #15 and 2013 AMC 10B #20, so both problems redirect to this page.

Problem

The number 2013 is expressed in the form

$$2013 = \frac{a_1!a_2!...a_m!}{b_1!b_2!...b_n!},$$

where $a_1 \ge a_2 \ge \cdots \ge a_m$ and $b_1 \ge b_2 \ge \cdots \ge b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution

The prime factorization of 2013 is $61 \cdot 11 \cdot 3$. To have a factor of 61 in the numerator and to minimize a_1, a_1 must equal 61. Now we notice that there can be no prime p which is not a factor of 2013 such that $b_1 < p < 61$, because this prime will not be canceled out in the denominator, and will lead to an extra factor in the numerator. The highest p less than 61 is 59, so there must be a factor of 59 in the denominator. It follows that $b_1 = 59$ (to minimize b_1 as well), so the answer is $|61 - 59| = \boxed{\textbf{(B) } 2}$.

One possible way to express 2013 with $(a_1, b_1) = (61, 59)$ is

$$2013 = \frac{61! \cdot 19! \cdot 11!}{59! \cdot 20! \cdot 10!}.$$

See also

2013 AMC 12B (Problems • Answer Key • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=44&year=2013))	
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2013 AMC 12B Problems/Problem 16

Problem

Let $ABCDE$ be an equiangular convex pentagon of perimeter 1. The pairwise intersections of the lines that extend the sides of the pentagon determine a five-pointed star polygon. Let s be the perimeter of this star. What is the difference between the maximum and the minimum possible values of s.

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ \frac{1}{2} \qquad \textbf{(C)}\ \frac{\sqrt{5}-1}{2} \qquad \textbf{(D)}\ \frac{\sqrt{5}+1}{2} \qquad \textbf{(E)}\ \sqrt{5}$

Solution

The five pointed star can be thought of as five triangles sitting on the five sides of the pentagon. Because the pentagon is equiangular, each of its angles has measure $\frac{180^\circ(5-2)}{5} = 108^\circ$, and so the base angles of the aforementioned triangles (i.e., the angles adjacent to the pentagon) have measure $180^\circ - 108^\circ = 72^\circ$. The base angles are equal, so the triangles must be isosceles.

Let one of the sides of the pentagon have length x_1 (and the others x_2, x_3, x_4, x_5). Then, by trigonometry, the non-base sides of the triangle sitting on that side of the pentagon each has length $\frac{x_1}{2} \sec 72^\circ$, and so the two sides together have length $x_1 \sec 72^\circ$. To find the perimeter of the star, we sum up the lengths of the non-base sides for each of the five triangles to get $(x_1 + x_2 + x_3 + x_4 + x_5) \sec 72^\circ = (1) \sec 72^\circ = \sec 72^\circ$ (because the perimeter of the pentagon is 1). The perimeter of the star is constant, so the difference between the maximum and minimum perimeters is $\boxed{\textbf{(A)}\ 0}$.

See also

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Category: Introductory Geometry Problems

2013 AMC 12B Problems/Problem 17

Contents

Problem

Let $a, b,$ and c be real numbers such that

$$a+b+c=2, \text{ and}$$

$$a^2+b^2+c^2=12$$

What is the difference between the maximum and minimum possible values of c?

$\text{(A) } 2\qquad \text{(B) } \frac{10}{3}\qquad \text{(C) } 4 \qquad \text{(D) } \frac{16}{3}\qquad \text{(E) } \frac{20}{3}$

Solution 1

$a+b = 2-c$. Now, by Cauchy-Schwarz, we have that $(a^2+b^2) \ge \frac{(2-c)^2}{2}$. Therefore, we have that $\frac{(2-c)^2}{2}+c^2 \le 12$. We then find the roots of c that satisfy equality and find the difference of the roots. This gives the answer, $\boxed{\textbf{(D) }\frac{16}{3}}$.

Solution 2

This is similar to the first solution but is far more intuitive. From the given, we have

$$a+b=2-c$$

$$a^2+b^2=12-c^2$$

This immediately suggests use of the Cauchy-Schwarz inequality. By Cauchy, we have

$$2\left(a^2+b^2\right) \geq (a+b)^2$$

Substitution of the above results and some algebra yields

$$3c^2-4c-20 \leq 0$$

This quadratic inequality is easily solved, and it is seen that equality holds for $c=-2$ and $c=\frac{10}{3}$.

The difference between these two values is $\boxed{\textbf{(D) }\frac{16}{3}}$.

Solution 3

(no Cauchy-Schwarz)

From the first equation, we know that $c = 2 - a - b$. We substitute this into the second equation to find that

$$a^2 + b^2 + (2 - a - b)^2 = 12.$$

This simplifies to $2a^2 + 2b^2 - 4a - 4b + 2ab = 8$, which we can write as the quadratic $a^2 + (b-2)a + (b^2 - 2b - 4) = 0$. We wish to find real values for a and b that satisfy this equation. Therefore, the discriminant is nonnegative. Hence,

$$(b-2)^2 - 4(b^2 - 2b - 4) \geq 0,$$

or $-3b^2 + 4b + 20 \geq 0$. This factors as $-(3b - 10)(b + 2) \geq 0$. Therefore, $-2 \leq b \leq \frac{10}{3}$, and by symmetry this must be true for a and c as well.

Now $a = b = 2$ and $c = -2$ satisfy both equations, so we see that $c = -2$ must be the minimum possible value of c. Also, $c = \frac{10}{3}$ and $a = b = -\frac{2}{3}$ satisfy both equations, so we see that $c = \frac{10}{3}$ is the maximum possible value of c. The difference between these is $\frac{10}{3} - (-2) = \frac{16}{3}$, or $\boxed{\textbf{(D)}}$.

See also

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Category: Introductory Algebra Problems

2013 AMC 12B Problems/Problem 18

Problem

Barbara and Jenna play the following game, in which they take turns. A number of coins lie on a table. When it is Barbara’s turn, she must remove 2 or 4 coins, unless only one coin remains, in which case she loses her turn. What it is Jenna’s turn, she must remove 1 or 3 coins. A coin flip determines who goes first. Whoever removes the last coin wins the game. Assume both players use their best strategy. Who will win when the game starts with 2013 coins and when the game starts with 2014 coins?

$\textbf{(A)}$ Barbara will win with 2013 coins and Jenna will win with 2014 coins.

$\textbf{(B)}$ Jenna will win with 2013 coins, and whoever goes first will win with 2014 coins.

$\textbf{(C)}$ Barbara will win with 2013 coins, and whoever goes second will win with 2014 coins.

$\textbf{(D)}$ Jenna will win with 2013 coins, and Barbara will win with 2014 coins.

$\textbf{(E)}$ Whoever goes first will win with 2013 coins, and whoever goes second will win with 2014 coins.

Solution

We spit into 2 cases: 2013 coins, and 2014 coins.

$\textbf{2013 coins:}$ Notice that when there are 5 coins left, whoever moves first loses, as they must leave an amount of coins the other person can take. If Jenna goes first, she can take 3 coins. Then, whenever Barbara takes coins, Jenna will take the amount that makes the total coins taken in that round 5. (For instance, if Barbara takes 4 coins, Jenna will take 1). Eventually, since $2010 = 0 (\text{mod } 5)$ it will be Barbara’s move with 5 coins remaining, so she will lose. If Barbara goes first, on each round, Jenna will take the amount of coins that makes the total coins taken on that round 5. Since $2013 = 3 (\text{mod } 5)$, it will be Barbara’s move with 3 coins remaining, so she will have two take 2 coins, allowing Jenna to take the last coin. Therefore, Jenna will win with 2013 coins.

$\textbf{2014 coins:}$ If Jenna moves first, she will take 1 coin, leaving 2013 coins, and she wins as shown above. If Barbara moves first, she can take 4 coins, leaving 2010. After every move by Jenna, Barbara will then take the number of coins that makes the total taken in that round 5. Since $2010 = 0 (\text{mod } 5)$, it will be Jenna’s turn with 5 coins left, so Barbara will win. In this case, whoever moves first wins.

Based on this, the answer is $\boxed{\textbf{(B)}}$

See also

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2013 AMC 12B Problems/Problem 19

The following problem is from both the 2013 AMC 12B #19 and 2013 AMC 10B #23, so both problems redirect to this page.

Contents

Problem

In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

$\textbf{(A)}\ 18\qquad\textbf{(B)}\ 21\qquad\textbf{(C)}\ 24\qquad\textbf{(D)}\ 27\qquad\textbf{(E)}\ 30$

Solution 1

Since $\angle AFB = \angle ADB = 90^{\circ}$, quadrilateral $ABDF$ is cyclic. It follows that $\angle ADE = \angle ABF$. In addition, since $\angle AFB = \angle AED = 90$, triangles ABF and ADE are similar. It follows that $AF = (13)(\frac{4}{5}), BF = (13)(\frac{3}{5})$. By Ptolemy, we have $13DF + (5)(13)(\frac{4}{5}) = (12)(13)(\frac{3}{5})$. Cancelling 13, the rest is easy. We obtain $DF = \frac{16}{5} \implies 16 + 5 = 21 \implies \boxed{(B)}$

Solution 2

Using the similar triangles in triangle ADC gives $AE = \frac{48}{5}$ and $DE = \frac{36}{5}$. Quadrilateral $ABDF$ is cyclic, implying that $\angle B + \angle DFA$ = 180° . Therefore, $\angle B = \angle EFA$, and triangles AEF and ADB are similar. Solving the resulting proportion gives $EF = 4$. Therefore, $DF = ED - EF = \frac{16}{5}. \implies \boxed{(B)}$

See also

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2013 AMC 12B Problems/Problem 20

Problem

For $135^\circ < x < 180^\circ$, points $P=(\cos x, \cos^2 x), Q=(\cot x, \cot^2 x), R=(\sin x, \sin^2 x)$ and $S=(\tan x, \tan^2 x)$ are the vertices of a trapezoid. What is $\sin(2x)$?

$\textbf{(A)}\ 2-2\sqrt{2} \qquad \textbf{(B)}\ 3\sqrt{3}-6 \qquad \textbf{(C)}\ 3\sqrt{2}-5 \qquad \textbf{(D}\ -\frac{3}{4} \qquad \textbf{(E)}\ 1-\sqrt{3}$

Solution

Let f,g,h,j be $\sin, \cos, \tan, \cot$ (not respectively). Then we have four points $(f,f^2),(g,g^2),(h,h^2),(j,j^2)$, and a pair of lines each connecting two points must be parallel (as we are dealing with a trapezoid). WLOG, take the line connecting the first two points and the line connecting the last two points to be parallel, so that $\frac{g^2-f^2}{g-f}=\frac{j^2-h^2}{j-h}$, or $g+f=j+h$.

Now, we must find how to match up $\sin, \cos, \tan, \cot$ to f,g,h,j so that the above equation has a solution. On the interval $135^\circ < x < 180^\circ$, we have $\cot x < -1 < \cos x < 0 < \sin x$, and $\cot x < -1 < \tan x < 0 < \sin x$ so the sum of the largest and the smallest is equal to the sum of the other two, namely, $\sin x + \cot x = \cos x + \tan x$.

Now, we perform some algebraic manipulation to find $\sin(2x)$:

$$\sin x + \cot x = \cos x + \tan x$$
$$\sin x - \cos x = \tan x - \cot x = (\sin x - \cos x)(\sin x + \cos x)/(\sin x \cos x)$$
$$\sin x \cos x = \sin x + \cos x$$
$$(\sin x \cos x)^2 = (\sin x + \cos x)^2$$
$$(\sin x \cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x$$
$$(\sin x \cos x)^2 - 2\sin x \cos x - 1 = 0$$

Solve the quadratic to find $\sin x \cos x = \frac{2-2\sqrt{2}}{2}$, so that

$\sin(2x) = 2\sin x \cos x = \boxed{\textbf{(A)}\ 2-2\sqrt{2}}$.

See also

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2013 AMC 12B Problems/Problem 21

Problem

Consider the set of 30 parabolas defined as follows: all parabolas have as focus the point (0,0) and the directrix lines have the form $y = ax + b$ with a and b integers such that $a \in \{-2, -1, 0, 1, 2\}$ and $b \in \{-3, -2, -1, 1, 2, 3\}$. No three of these parabolas have a common point. How many points in the plane are on two of these parabolas?

$\textbf{(A) } 720 \qquad \textbf{(B) } 760 \qquad \textbf{(C) } 810 \qquad \textbf{(D) } 840 \qquad \textbf{(E) } 870$

Solution

Being on two parabolae means having the same distance from the common focus and both directrices. In particular, you have to be on an angle bisector of the directrices, and clearly on the same "side" of the directrices as the focus. So it's easy to see there are at most two solutions per pair of parabolae. Convexity and continuity imply exactly two solutions unless the directrices are parallel and on the same side of the focus.

So out of $2\binom{30}{2}$ possible intersection points, only $2*5*2*\binom{3}{2}$ fail to exist. This leaves $870-60=810=\boxed{\textbf{(C) } 810}$ solutions.

See also

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2013 AMC 12B Problems/Problem 22

Problem

Let $m > 1$ and $n > 1$ be integers. Suppose that the product of the solutions for x of the equation

$$8(\log_n x)(\log_m x) - 7\log_n x - 6\log_m x - 2013 = 0$$

is the smallest possible integer. What is $m + n$?

$\textbf{(A)}\ 12 \qquad \textbf{(B)}\ 20 \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 48 \qquad \textbf{(E)}\ 272$

Solution

Rearranging logs, the original equation becomes

$$\frac{8}{\log n \log m}(\log x)^2 - \left(\frac{7}{\log n} + \frac{6}{\log m}\right)\log x - 2013 = 0$$

By Vieta's Theorem, the sum of the possible values of $\log x$ is $\frac{\frac{7}{\log n} + \frac{6}{\log m}}{\frac{8}{\log n \log m}} = \frac{7\log m + 6\log n}{8} = \log \sqrt[8]{m^7n^6}$. But the sum of the possible values of $\log x$ is the logarithm of the product of the possible values of x. Thus the product of the possible values of x is equal to $\sqrt[8]{m^7n^6}$.

It remains to minimize the integer value of $\sqrt[8]{m^7n^6}$. Since $m, n > 1$, we can check that $m = 2^2$ and $n = 2^3$ work. Thus the answer is $4 + 8 = \boxed{\textbf{(A)}\ 12}$.

See also

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2013 AMC 10B Problems/Problem 25

The following problem is from both the 2013 AMC 12B #23 and 2013 AMC 10B #25, so both problems redirect to this page.

Problem

Bernardo chooses a three-digit positive integer N and writes both its base-5 and base-6 representations on a blackboard. Later LeRoy sees the two numbers Bernardo has written. Treating the two numbers as base-10 integers, he adds them to obtain an integer S. For example, if $N = 749$, Bernardo writes the numbers $10{,}444$ and $3{,}245$, and LeRoy obtains the sum $S = 13{,}689$. For how many choices of N are the two rightmost digits of S, in order, the same as those of $2N$?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 10 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 20 \qquad \textbf{(E)}\ 25$

Solution

First, we can examine the units digits of the number base 5 and base 6 and eliminate some possibilities.

Say that $N \equiv a \pmod{6}$

also that $N \equiv b \pmod{5}$

Substituting these equations into the question and setting the units digits of 2N and S equal to each other, it can be seen that $a = b$, and $b < 5$, (otherwise a and b always have different parities) so $N \equiv a \pmod{6}, N \equiv a \pmod{5}, \implies N = a \pmod{30}, 0 \le a \le 4$

Therefore, N can be written as $30x + y$ and $2N$ can be written as $60x + 2y$

Keep in mind that y can be one of five choices: $0, 1, 2, 3,$ or 4, ; Also, we have already found which digits of y will add up into the units digits of $2N$.

Now, examine the tens digit, x by using $\mod{25}$ and $\mod{36}$ to find the tens digit (units digits can be disregarded because $y = 0, 1, 2, 3, 4$ will always work) Then we see that $N = 30x + y$ and take it $\mod{25}$ and $\mod{36}$ to find the last two digits in the base 5 and 6 representation.

$$N \equiv 30x \pmod{36}$$

$$N \equiv 30x \equiv 5x \pmod{25}$$

Both of those must add up to

$$2N \equiv 60x \pmod{100}$$

$(33 \ge x \ge 4)$

Now, since $y = 0, 1, 2, 3, 4$ will always work if x works, then we can treat x as a units digit instead of a tens digit in the respective bases and decrease the mods so that x is now the units digit.

$$N \equiv 6x \equiv x \pmod{5}$$

$$N \equiv 5x \pmod{6}$$

$$2N \equiv 6x \pmod{10}$$

Say that $x = 5m + n$ (m is between 0-6, n is 0-4 because of constraints on x) Then

$$N \equiv 5m + n \pmod{5}$$

$$N \equiv 25m + 5n \pmod{6}$$

$$2N \equiv 30m + 6n \pmod{10}$$

and this simplifies to

$$N \equiv n \pmod{5}$$

$$N \equiv m + 5n \pmod{6}$$

$$2N \equiv 6n \pmod{10}$$

From inspection, when

$n = 0, m = 6$

$n = 1, m = 6$

$n = 2, m = 2$

$n = 3, m = 2$

$n = 4, m = 4$

This gives you 5 choices for x, and 5 choices for y, so the answer is $5 * 5 = \boxed{\textbf{(E) } 25}$

Shortcut

Notice that there are exactly $1000 - 100 = 900 = 5^2 \cdot 6^2$ possible values of n. This means, from $100 \leq n \leq 999$, every possible combination of 2 digits will happen exactly once. We know that $n = 900, 901, 902, 903, 904$ works because $900 \equiv \ldots 00_5 \equiv \ldots 00_6$.

We know for sure that the units digit will add perfectly every 30 added or subtracted, because $\text{lcm } 5, 6 = 30$. So we only have to care about cases of n every 30 subtracted. In each case, $2n$ subtracts 6/adds 4, n_5 subtracts 1 and n_6 adds 1 for the 10's digit.

$$\begin{array}{cccccccccccccccccccccccccccccccc} \textbf{5} & \color{red}0 & 4 & 3 & 2 & 1 & 0 & \color{red}4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & \color{red}3 & \color{red}2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & \color{red}1 \\ \textbf{6} & \color{red}0 & 1 & 2 & 3 & 4 & 5 & \color{red}0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & \color{red}5 & \color{red}0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & \color{red}5 \\ \textbf{10} & \color{red}0 & 4 & 8 & 2 & 6 & 0 & \color{red}4 & 8 & 2 & 6 & 0 & 4 & 8 & 2 & 6 & 0 & 4 & \color{red}8 & \color{red}2 & 6 & 0 & 4 & 8 & 2 & 6 & 0 & 4 & 8 & 2 & \color{red}6 \end{array}$$

As we can see, there are 5 cases, including the original, that work. These are highlighted in $\color{red}\text{red}$. So, thus, there are 5 possibilities for each case, and $5 \cdot 5 = \boxed{\textbf{(E) } 25}$.

== See also == Ciao

2013 AMC 12B Problems/Problem 24

Contents

Problem

Let ABC be a triangle where M is the midpoint of $\overline{AC}$, and $\overline{CN}$ is the angle bisector of $\angle ACB$ with N on $\overline{AB}$. Let X be the intersection of the median $\overline{BM}$ and the bisector $\overline{CN}$. In addition $\triangle BXN$ is equilateral with $AC = 2$. What is BN^2?

$\textbf{(A)}\ \frac{10-6\sqrt{2}}{7} \qquad \textbf{(B)}\ \frac{2}{9} \qquad \textbf{(C)}\ \frac{5\sqrt{2}-3\sqrt{3}}{8} \qquad \textbf{(D)}\ \frac{\sqrt{2}}{6} \qquad \textbf{(E)}\ \frac{3\sqrt{3}-4}{5}$

Solution

Let $BN = x$ and $NA = y$. From the conditions, let's deduct some convenient conditions that seems sufficient to solve the problem.

M is the midpoint of side AC.

This implies that $[ABX] = [CBX]$. Given that angle ABX is 60 degrees and angle BXC is 120 degrees, we can use the area formula to get

$$\frac{1}{2}(x+y)x\frac{\sqrt{3}}{2} = \frac{1}{2}x \cdot CX\frac{\sqrt{3}}{2}$$

So, $x + y = CX$ ..... (1)

CN is angle bisector.

In the triangle ABC, one has $BC/AC = x/y$, therefore $BC = 2x/y$..... (2)

Furthermore, triangle BCN is similar to triangle MCX, so $BC/CM = CN/CX$, therefore $BC = (CX + x)/CX = (2x + y)/(x + y)$.... (3)

By (2) and (3) and the subtraction law of ratios, we get

$$BC = 2x/y = (2x + y)/(y + x) = y/x$$

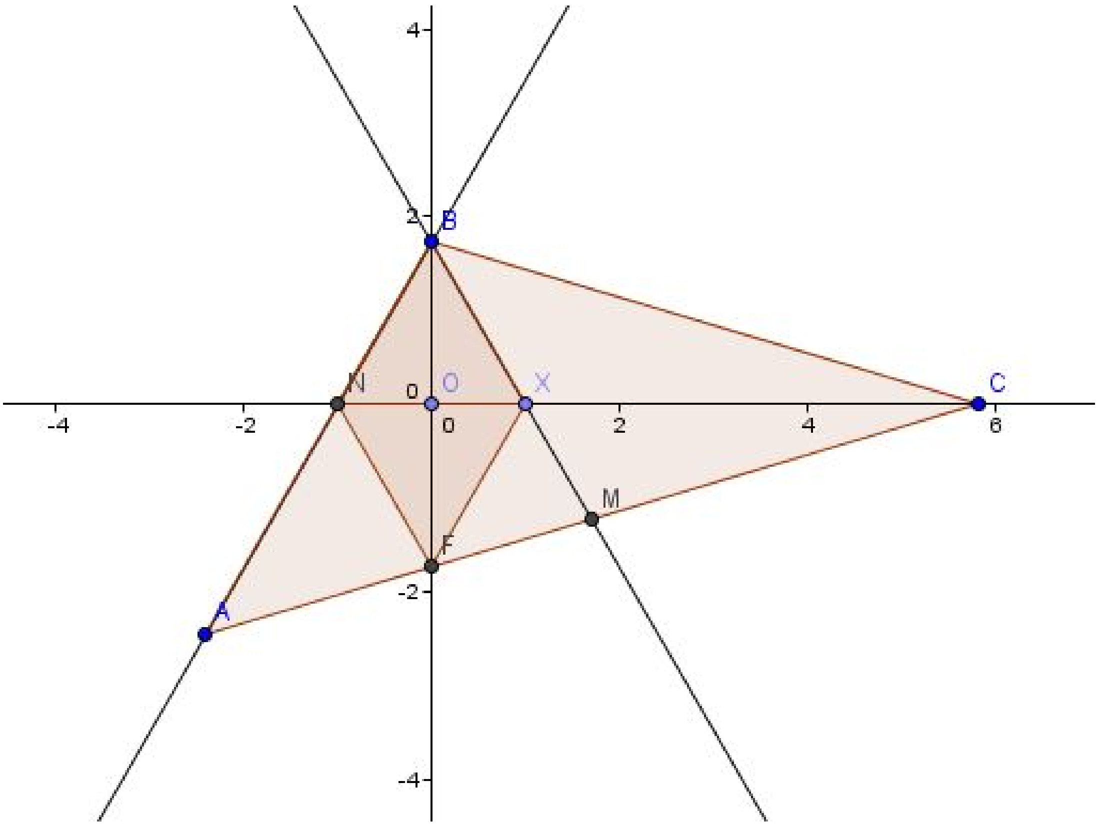
Therefore $2x^2 = y^2$, or $y = \sqrt{2}x$. So $BC = 2x/(\sqrt{2}x) = \sqrt{2}$.

Finally, using the law of cosine for triangle BCN, we get

$$2 = BC^2 = x^2 + (2x+y)^2 - x(2x+y) = 3x^2 + 3xy + y^2 = \left(5 + 3\sqrt{2}\right)x^2$$

$$x^2 = \frac{2}{5+3\sqrt{2}} = \boxed{\textbf{(A) } \frac{10-6\sqrt{2}}{7}}.$$

Solution 2 (Analytic)



Let us dilate triangle ABC so that the sides of equilateral triangle BXN are all equal to $2.$ The purpose of this is to ease the calculations we make in the problem. Given this, we aim to find the length of segment AM so that we can un-dilate triangle ABC by dividing each of its sides by AM. Doing so will make it so that $AM = 1$, as desired, and doing so will allow us to get the length of BN, whose square is our final answer.

Let O the foot of the altitude from B to $NX.$ On the coordinate plane, position O at $(0,0)$, and make NX lie on the x-axis. Since points N, X, and C, are collinear, C must also lie on the x-axis. Additionally, since $NX = 2$, $OB = \sqrt{3}$, meaning that we can position point B at $(0, \sqrt{3})$. Now, notice that line $\overline{AB}$ has the equation $y = \sqrt{3}x + \sqrt{3}$ and that line $\overline{BM}$ has the equation $y = -\sqrt{3}x + \sqrt{3}$ because angles BNX and BXN are both 60°. We can then position A at point $(n, \sqrt{3}(n+1))$ and C at point $(p, 0)$. Quickly note that, because CN is an angle bisector, AC must pass through the point $(0, -\sqrt{3})$.

We proceed to construct a system of equations. First observe that the midpoint M of AC must lie on BM, with the equation $y = -\sqrt{3}x + \sqrt{3}$. The coordinates of M are $\left(\frac{p+n}{2}, \frac{\sqrt{3}}{2}(n+1)\right)$, and we can plug in these coordinates into the equation of line BM, yielding that

$$\frac{\sqrt{3}}{2}(n+1) = -\sqrt{3}(\frac{p+n}{2})+\sqrt{3} \implies n+1 = -p-n+2 \implies p = -2n+1.$$

For our second equation, notice that line AC has equation $y = \frac{\sqrt{3}}{p}x - \sqrt{3}$. Midpoint M must also lie on this line, and we can substitute coordinates again to get

$$\frac{\sqrt{3}}{2}(n+1) = \frac{\sqrt{3}}{p}(\frac{p+n}{2}) - \sqrt{3} \implies n+1 = \frac{p+n}{p} - 2 \implies n+1 = \frac{n}{p} - 1$$

$$\implies p = \frac{n}{n+2}.$$

Setting both equations equal to each other and multiplying both sides by $(n+2)$, we have that $-2n^2 - 4n + n + 2 = n \implies -2n^2 - 4n + 2 = 0$, which in turn simplifies into $0 = n^2 + 2n - 1$ when dividing the entire equation by -2. Using the quadratic formula, we have that

$$n = \frac{-2 \pm \sqrt{4+4}}{2} = -1 - \sqrt{2}.$$

Here, we discard the positive root since A must lie to the left of the y-axis. Then, the coordinates of C are $(3 + 2\sqrt{2}, 0)$, and the coordinates of A are $(-1 - \sqrt{2}, -\sqrt{6})$. Seeing that segment AM has half the length of side AC, we have that the length of AM is

$$\frac{\sqrt{(3 + 2\sqrt{2} - (-1 - \sqrt{2}))^2 + (\sqrt{6})^2}}{2} = \frac{\sqrt{16 + 24\sqrt{2} + 18 + 6}}{2} = \sqrt{10 + 6\sqrt{2}}.$$

Now, we divide each side length of $\triangle ABC$ by AM, and from this, BN^2 will equal $\left(\frac{2}{\sqrt{10 + 6\sqrt{2}}}\right)^2 = \frac{2}{5 + 3\sqrt{2}} = \boxed{\textbf{(A)}\ \frac{10 - 6\sqrt{2}}{7}}.$

See also

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2013 AMC 12B Problems/Problem 25

Contents

Problem

Let G be the set of polynomials of the form

$$P(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_2z^2 + c_1z + 50,$$

where $c_1, c_2, \cdots, c_{n-1}$ are integers and $P(z)$ has distinct roots of the form $a + ib$ with a and b integers. How many polynomials are in G?

$\textbf{(A)}\ 288 \qquad \textbf{(B)}\ 528 \qquad \textbf{(C)}\ 576 \qquad \textbf{(D)}\ 992 \qquad \textbf{(E)}\ 1056$

Solution

Solution 1

If we factor into irreducible polynomials (in $\mathbb{Q}[x]$), each factor f_i has exponent 1 in the factorization and degree at most 2 (since the $a + bi$ with $b \neq 0$ come in conjugate pairs with product $a^2 + b^2$). Clearly we want the product of constant terms of these polynomials to equal 50; for $d \mid 50$, let $f(d)$ be the number of permitted f_i with constant term d. It's easy to compute $f(1) = 2$, $f(2) = 3$, $f(5) = 5$, $f(10) = 5$, $f(25) = 6$, $f(50) = 7$, and obviously $f(d) = 1$ for negative $d \mid 50$.

Note that by the distinctness condition, the only constant terms d that can be repeated are those with $d^2 \mid 50$ and $f(d) > 1$, i.e. $+1$ and $+5$. Also, the $+1$s don't affect the product, so we can simply count the number of polynomials with no constant terms of $+1$ and multiply by $2^{f(1)} = 4$ at the end.

We do casework on the (unique) even constant term $d \in \{\pm 2, \pm 10, \pm 50\}$ in our product. For convenience, let $F(d)$ be the number of ways to get a product of $50/d$ without using ± 1 (so only using $\pm 5, \pm 25$) and recall $f(-1) = 1$; then our final answer will be $2^{f(1)} \sum_{d \in \{2,10,50\}} (f(-d) + f(d))(F(-d) + F(d))$. It's easy to compute $F(-50) = 0$, $F(50) = 1$, $F(-10) = f(5) = 5$, $F(10) = f(-5) = 1$, $F(-2) = f(-25) + f(-5)f(5) = 6$, $F(2) = f(25) + \binom{f(5)}{2} = 16$, so we get

$$4[(1+3)(6+16)+(1+5)(1+5)+(1+7)(0+1)] = 4[132] = \boxed{\textbf{(B)}\ 528}$$

Solution 2

Disregard sign; we can tack on $x - 1$ if the product ends up being negative.

$1 : \pm i, -1$ (2) (1 is not included)

$2 : \pm 2, \pm 1 \pm i$ (4)

$5 : \pm 2 \pm i, \pm 1 \pm 2i, \pm 5$ (6)

$10 : \pm 3 \pm i, \pm 1 \pm 3i, \pm 10$ (6)

$25 : \pm 5, \pm 3 \pm 4i, \pm 4 \pm 3i, \pm 5i$ (7)

$50 : \pm 50, \pm 1 \pm 7i, \pm 7 \pm i, \pm 5 \pm 5i$ (8)

Our answer is $2^2 \left(4 \cdot \binom{6}{2} + 6 \cdot 6 + 4 \cdot 7 + 8 \right) = \boxed{528.}$

See also

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