CAPE1150

UNIVERSITY OF LEEDS

# Engineering Mathematics

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University of Leeds

January 2025

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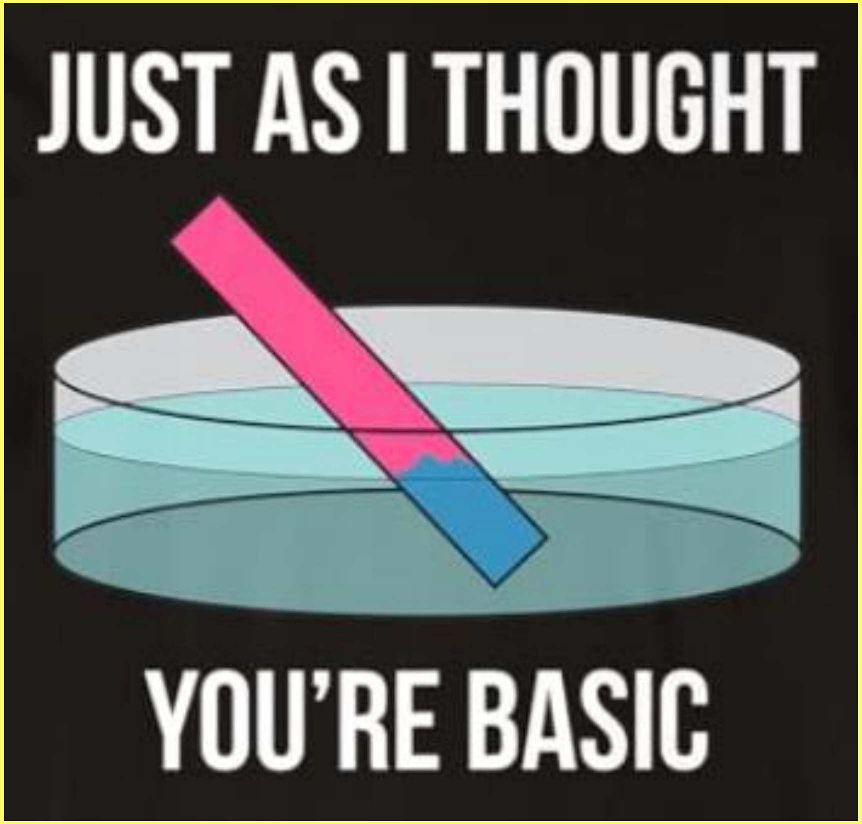
# Integration Methods: Outline of Lecture 2

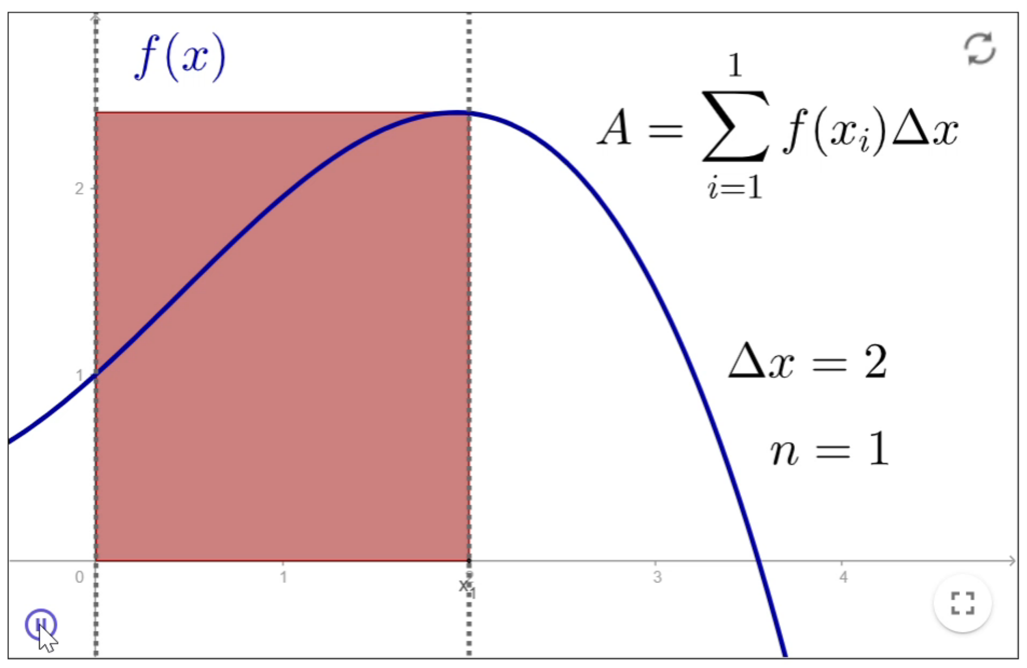
- Integration: Recap of the basics
- Integration by standard results
- Integration using trigonometric identities
- Reverse chain rule (inspection)
- Integration by substitution
- Integration by parts
- Integration using partial fractions




Integration: Recap of the Basics

# Integration: A Sum of an Infinite number of Infinitesimal Strips

- An area can be approximated by splitting it into a number $n$ of rectangles, each with width $\Delta x$ and height $f(x)$.
- The area of each rectangle at position $x_i$ along the $x$-axis would be $A_i = f(x_i) \times \Delta x$
- The total shaded area would then be the sum of these areas, that is $\sum_{i=1}^{n} f(x_i)\Delta x$
- As the number of rectangles, n, increases, and width $\Delta x$ decreases, the approximation of the area becomes more accurate.




- Calculus is able to take the limit such that as the number of rectangles $n \to \infty$, and $\Delta x \to dx$ (rectangles become infinitesimally thin).
- The sum now becomes an integral: $\sum_{i=1}^{n} f(x_i)\Delta x \ \to \int_0^2 f(x)\, dx$.
- The "approximation" is now no longer an approximation and the integration gives the **exact** area!

Integration can also be used to find volumes of very complex shapes!

# Definite Integrals: Some Important Results/Shortcuts

$$\int_a^b 0\, dx = [C]_a^b = C - C = 0$$

As $C$ does not depend on $x$, the bounds have no effect

$$\int_a^b dx = \int_a^b 1\, dx = [x]_a^b = b - a$$

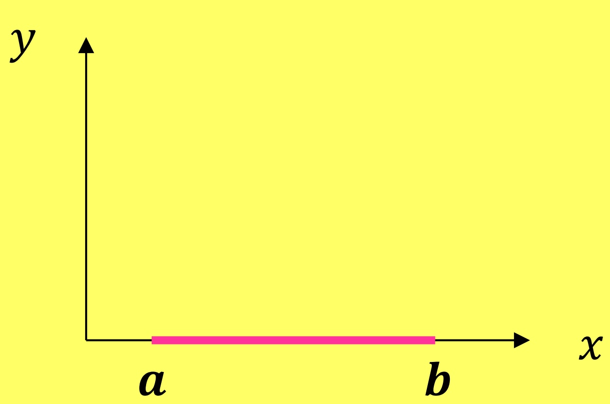
The definite integral of 1 with respect to any variable is just the difference between the bounds

E.g. $\int_{-2}^{3} dx = [x]_{-2}^{3} = 3 - -2 = 5$

**Line integral (1D)**

$$\int_a^b 1\, dx = b - a$$

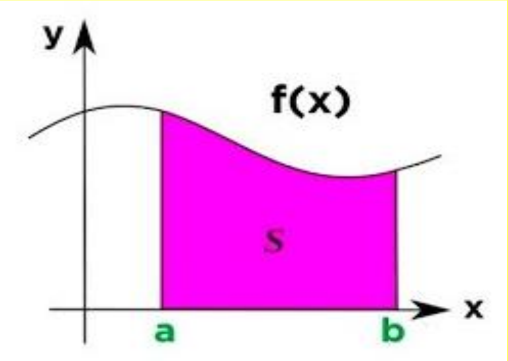
Gives the **length** of the line between $x = a$ and $x = b$




**Area Integral**

$$\int_a^b f(x)\, dx$$

Gives the **area** under curve $f(x)$ between $x = a$ and $x = b$




("Under" means between the curve and the $x$-axis)

# Standard Derivatives/Integrals (Reference Page)

## Standard Derivatives

| $y = f(x)$ | $\frac{dy}{dx} = f'(x)$ |
|---|---|
| $k$, constant | $0$ |
| $x^n$, any constant $n$ | $nx^{n-1}$ |
| $e^x$ | $e^x$ |
| $\ln x = \log_e x$ | $\frac{1}{x}$ |
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x = \frac{\sin x}{\cos x}$ | $\sec^2 x$ |
| $\operatorname{cosec} x = \frac{1}{\sin x}$ | $-\operatorname{cosec} x \cot x$ |
| $\sec x = \frac{1}{\cos x}$ | $\sec x \tan x$ |
| $\cot x = \frac{\cos x}{\sin x}$ | $-\operatorname{cosec}^2 x$ |
| $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1} x$ | $\frac{-1}{\sqrt{1-x^2}}$ |
| $\tan^{-1} x$ | $\frac{1}{1+x^2}$ |
| $\cosh x$ | $\sinh x$ |
| $\sinh x$ | $\cosh x$ |
| $\tanh x$ | $\operatorname{sech}^2 x$ |
| $\operatorname{sech} x$ | $-\operatorname{sech} x \tanh x$ |
| $\operatorname{cosech} x$ | $-\operatorname{cosech} x \coth x$ |
| $\coth x$ | $-\operatorname{cosech}^2 x$ |
| $\cosh^{-1} x$ | $\frac{1}{\sqrt{x^2-1}}$ |
| $\sinh^{-1} x$ | $\frac{1}{\sqrt{x^2+1}}$ |
| $\tanh^{-1} x$ | $\frac{1}{1-x^2}$ |

## Standard Integrals

| $f(x)$ | $\int f(x)\,dx = F(x) + c$ | |
|---|---|---|
| $k$, constant | $kx + c$ | |
| $x^n$, $(n \neq -1)$ | $\frac{x^{n+1}}{n+1} + c$ | |
| $x^{-1} = \frac{1}{x}$ | $\begin{cases} \ln x + c & x > 0 \\ \ln(-x) + c & x < 0 \end{cases}$ | |
| $e^x$ | $e^x + c$ | |
| $\cos x$ | $\sin x + c$ | |
| $\sin x$ | $-\cos x + c$ | |
| $\tan x$ | $\ln(\sec x) + c$ | $-\frac{\pi}{2} < x < \frac{\pi}{2}$ |
| $\sec x$ | $\ln(\sec x + \tan x) + c$ | $-\frac{\pi}{2} < x < \frac{\pi}{2}$ |
| $\operatorname{cosec} x$ | $\ln(\operatorname{cosec} x - \cot x) + c$ | $0 < x < \pi$ |
| $\cot x$ | $\ln(\sin x) + c$ | $0 < x < \pi$ |
| $\cosh x$ | $\sinh x + c$ | |
| $\sinh x$ | $\cosh x + c$ | |
| $\tanh x$ | $\ln \cosh x + c$ | |
| $\coth x$ | $\ln \sinh x + c$ | $x > 0$ |
| $\frac{1}{x^2+a^2}$ | $\frac{1}{a} \tan^{-1} \frac{x}{a} + c$ | $a > 0$ |
| $\frac{1}{x^2-a^2}$ | $\frac{1}{2a} \ln \frac{x-a}{x+a} + c$ | $\lvert x \rvert > a > 0$ |
| $\frac{1}{a^2-x^2}$ | $\frac{1}{2a} \ln \frac{a+x}{a-x} + c$ | $\lvert x \rvert < a$ |
| $\frac{1}{\sqrt{x^2+a^2}}$ | $\sinh^{-1} \frac{x}{a} + c$ | $a > 0$ |
| $\frac{1}{\sqrt{x^2-a^2}}$ | $\cosh^{-1} \frac{x}{a} + c$ | $x \geqslant a > 0$ |
| $\frac{1}{\sqrt{x^2+k}}$ | $\ln(x + \sqrt{x^2+k}) + c$ | |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1} \frac{x}{a} + c$ | $-a \leqslant x \leqslant a$ |
| $f(ax+b)$ | $\frac{1}{a} F(ax+b) + c$ | $a \neq 0$ |
| e.g. $\cos(2x-3)$ | $\frac{1}{2} \sin(2x-3) + c$ | |

**Inspection/Reverse Chain Rule**

These are “well known” results to use as needed

# Integrating Standard Functions

There's certain results you should be able to integrate straight off, by just thinking about the opposite of differentiation.

| $y$ | $\int y\, dx$ |
|---|---|
| $x^n$ | $\frac{1}{n+1}x^{n+1}+C$ |
| $e^x$ | $e^x+C$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert+C$ |
| $\cos x$ | $\sin x+C$ |
| $\sin x$ | $-\cos x+C$ |
| $\sec^2 x$ | $\tan x+C$ |
| $cosec\, x\cot x$ | $-cosec\, x+C$ |
| $cosec^2 x$ | $-\cot x+C$ |
| $\sec x\tan x$ | $\sec x+C$ |

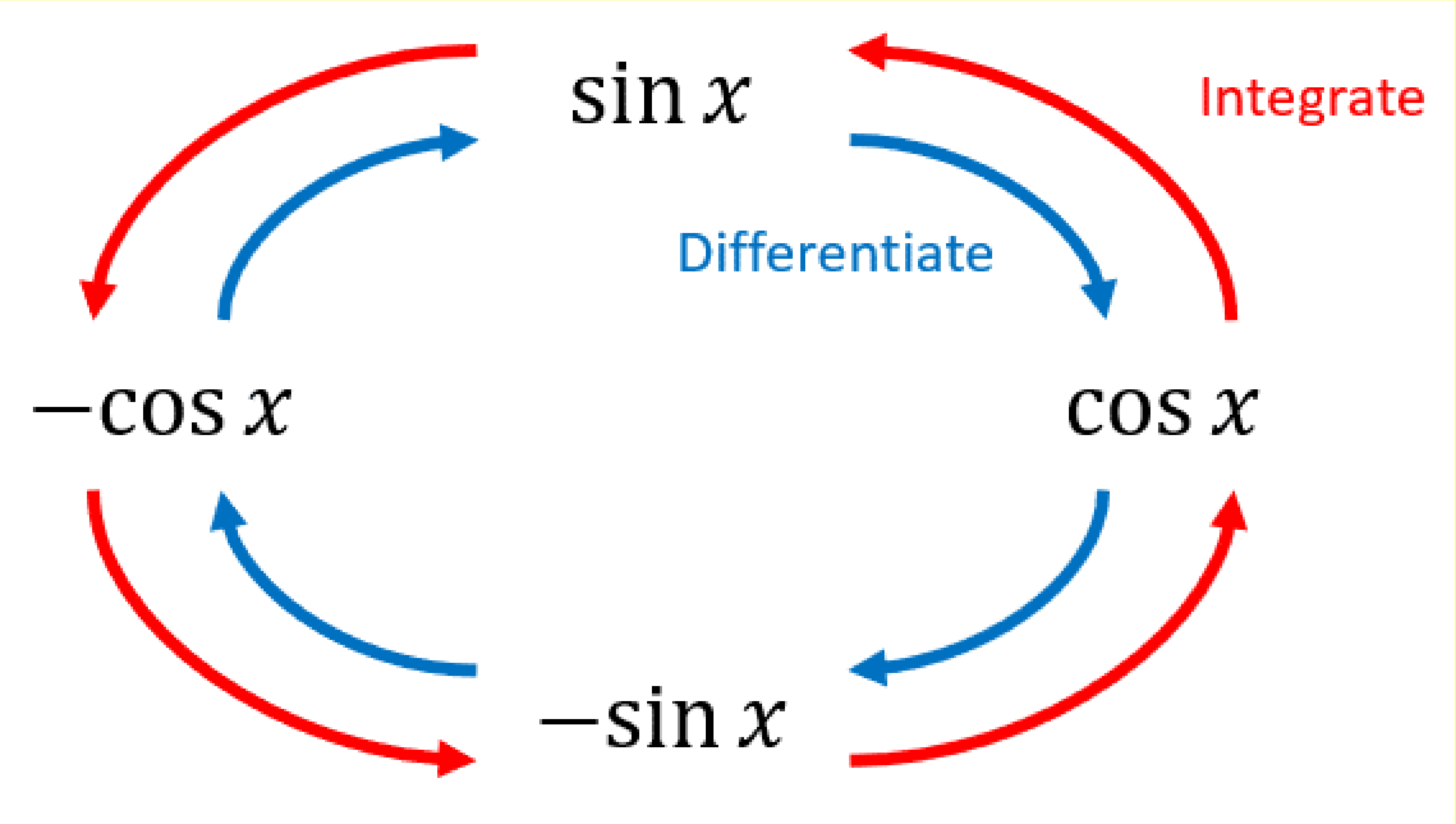
The $\lvert x\rvert$ rules out problems when $x$ is negative (when $\ln x$ is not defined)

Remember my memorisation trick of picturing sin above cos from so that 'going down' is differentiating and 'going up' is integrating, and we change the sign if the wrong way round.

You should remember this one.

# Trigonometric functions

They actually form a cycle (it is worth learning this):



# A bit more on logs

You may remember that:

$$\frac{d}{dx}(\ln ax) = \frac{1}{x}$$

This is because by chain rule

$$\frac{d}{dx}(\ln ax) = \frac{1}{ax} \times a$$

so actually, $\frac{dy}{dx} = \frac{1}{x} \Rightarrow y = \ln(ax) + c$

but using log laws: $\ln(ax) = \ln x + \ln a$

So $y = \ln x + \ln a + c$

But since $\ln a$ is constant, we can just say

$$y = \ln x + k$$

where $k = \ln a + c$ is an arbitrary constant.

# More general cases (working backwards)

| $y = f(x)$ | $\dfrac{dy}{dx} = f'(x)$ |
|---|---|
| $a$ (constant) | 0 |
| $x^n$ | $nx^{n-1}$ |
| $e^{ax}$ | $ae^{ax}$ |
| $\ln(ax)$ | $\dfrac{1}{x}$ |
| $\sin(ax)$ | $a\cos(ax)$ |
| $\cos(ax)$ | $-a\sin(ax)$ |
| $\tan(ax)$ | $a\sec^2(ax)$ |
| $\operatorname{cosec} ax$ | $-a \operatorname{cosec} ax \cot ax$ |
| $\sec ax$ | $a \sec ax \tan ax$ |
| $\cot ax$ | $-a \operatorname{cosec}^2 ax$ |

| $\boldsymbol{y}$ | $\boldsymbol{\int y\, dx}$ |
|---|---|
| $x^n$ | $\dfrac{1}{n+1}x^{n+1} + C$ |
| $e^{ax}$ | $\dfrac{1}{a}e^{ax} + C$ |
| $\dfrac{1}{x}$ | $\ln\lvert x\rvert + C$ |
| $\sin ax$ | $-\dfrac{1}{a}\cos ax + C$ |
| $\cos x$ | $\dfrac{1}{a}\sin ax + C$ |
| $\sec^2 ax$ | $\dfrac{1}{a}\tan ax + C$ |
| $cosec\ ax \cot ax$ | $-\dfrac{1}{a} cosec\ ax + C$ |
| $\sec ax \tan ax$ | $\dfrac{1}{a}\sec ax + C$ |
| $cosec^2 ax$ | $-\dfrac{1}{a}\cot ax + C$ |

We can "rearrange" the derivatives to find the integrals, this amounts to reverse chain rule…

For example, $\frac{d}{dx}\tan ax = a\sec^2 ax \quad \Rightarrow \sec^2 ax = \dfrac{1}{a}\dfrac{d}{dx}\tan ax \quad \Rightarrow \int \sec^2 ax = \dfrac{1}{a}\tan ax + C$

# Diagnostic Question

$$\int 2\cos x + \frac{3}{x} - \sqrt{x}\, dx =$$

**Y**

$$2\sin x + 3\ln|x| - \frac{3}{2}x^{\frac{3}{2}} + C$$

**C**

$$-2\sin x - \frac{3}{x^2} - \frac{1}{2}x^{-\frac{1}{2}} + C$$

**M**

$$-2\sin x + 3\ln|x| - \frac{2}{3}x^{\frac{3}{2}} + C$$

**A**

$$2\sin x + 3\ln|x| - \frac{2}{3}x^{\frac{3}{2}} + C$$

# Diagnostic Question

Given that $\int_a^{3a} \frac{2x+1}{x}\,dx = \ln 12$, find the exact value of $a$

**Y** $a = 0.347$

**M** $a = \frac{1}{4}\ln 4$

**C** $a = \frac{1}{2}\ln 12$

**A** $a = \frac{1}{4}\ln 9$

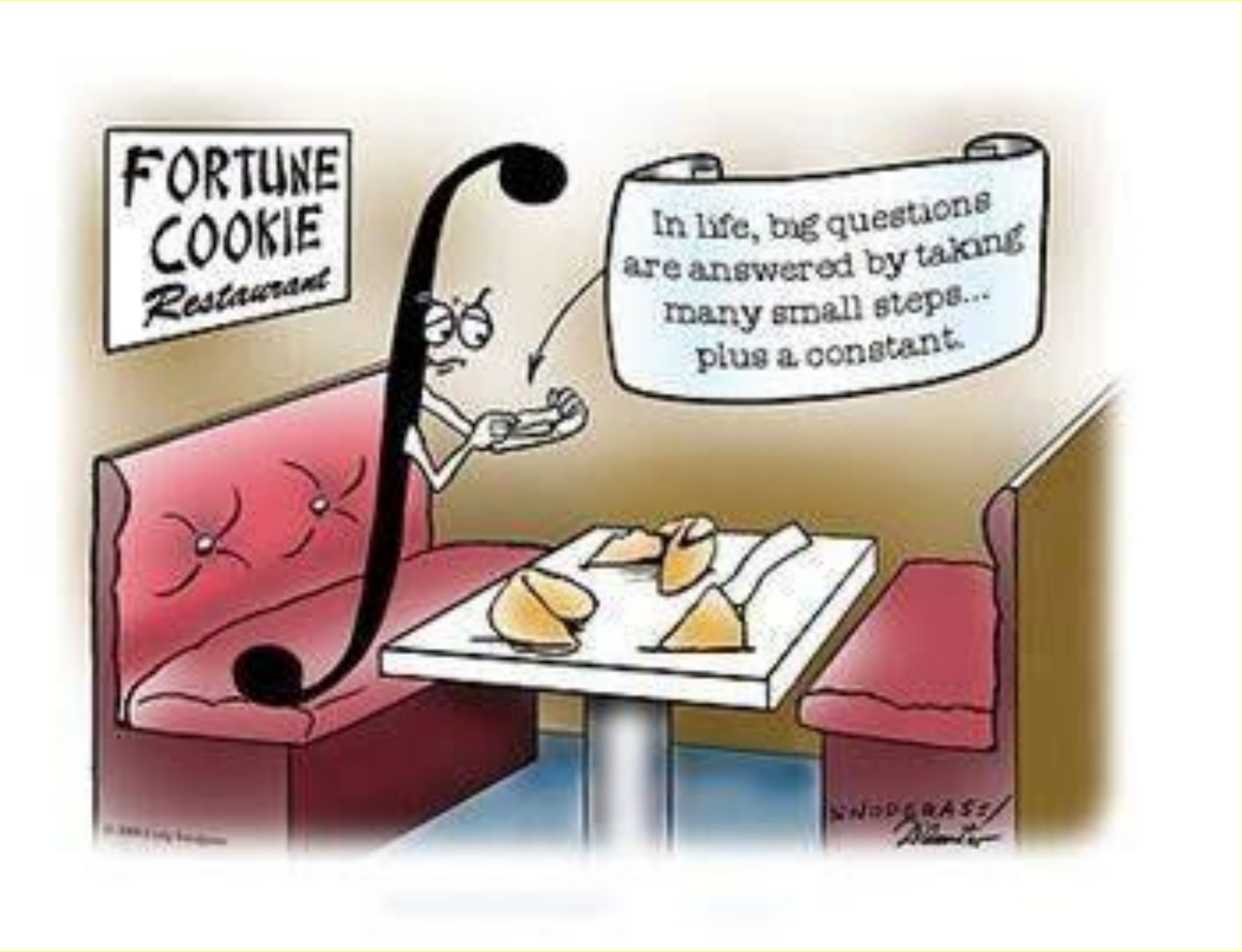
**Important Notes:**
We can simplify:

$$\frac{x+1}{x} \equiv \frac{x}{x} + \frac{1}{x} \equiv 1 + \frac{1}{x}$$

However it is **NOT** true that:

$$\frac{x}{x+1} \equiv \frac{x}{x} + \frac{x}{1}$$

In my experience students often don't spot when they can split up a fraction to then integrate.




## Integration by Standard Result

# Integrating $f(ax+b)$

$$\frac{d}{dx}(\sin(3x+1)) = 3\cos(3x+1)$$

Therefore:

$$\int \cos(3x+1)\ dx = \boldsymbol{\frac{1}{3}\sin(3x+1)+C}$$

For any expression where inner function is $ax+b$, integrate as before and $\div\, a$.

$$\int f'(ax+b)dx = \frac{1}{a}f(ax+b)+C$$

Quick Examples:

$$\int e^{3x}\ dx = \frac{1}{3}e^{3x}+C$$

$$\int \frac{1}{5x+2}dx = \frac{1}{5}\ln|5x+2|+C$$

$$\int 2\sec^2(3x-2)\ dx = 2\tan(3x-2)/3+C$$

$$= \frac{2}{3}\tan(3x-2)+C$$

$$\int (10x+11)^{12} = \frac{(10x+11)^{13}}{13}/10+C$$

$$= \frac{1}{130}(10x+11)^{13}+C$$

**Tip: For $\int (ax+b)^n\ dx$, ensure you divide by the $(n+1)$ <u>and</u> the $a$**

# Diagnostic Question

$$\int e^{3x+1}\,dx =$$

**Y**

$$3e^{3x+1} + C$$

**C**

$$\frac{1}{3}e^{3x+1} + C$$

**M**

$$\frac{1}{3x+2}e^{3x+2} + C$$

**A**

$$e^{\frac{3x^2}{2}+x} + C$$

# Diagnostic Question

$$\int \frac{1}{1-2x}\,dx =$$

**Y**

$$-\frac{1}{2}\ln|1-2x|+C$$

**C**

$$\ln|1-2x|+C$$

**M**

$$-2\ln|1-2x|+C$$

**A**

$$\frac{1}{2}\ln|1-2x|+C$$

# Diagnostic Question

$$\int \frac{1}{3(4x-2)^2}\,dx =$$

**Y**

$$\frac{1}{3}\ln(4x-2)^2$$

**C**

$$\frac{1}{12}\ln(4x-2)^2 + C$$

**M**

$$-\frac{1}{12}(4x-2)^{-1} + C$$

**A**

$$\frac{1}{12}(4x-2)^{-1} + C$$

# Diagnostic Question

$$\int \sin(1-5x)\,dx =$$

**Y**

$$-\frac{1}{5}\cos(1-5x)+C$$

**C**

$$\frac{1}{5}\cos(1-5x)+C$$

**M**

$$-\cos(1-5x)+C$$

**A**

$$5\cos(1-5x)+C$$

# Diagnostic Question

$$\int \sec^2 3x \, dx =$$

**Y**

$\tan 3x + C$

**C**

$\frac{1}{3} \tan 3x + C$

**M**

$3 \tan 3x + C$

**A**

$\frac{1}{3} \sec^3 3x + C$

# Diagnostic Question

$$\int \sec(3x)\tan(3x)\,dx =$$

**Y**

$$\frac{1}{3}\sec(3x) + C$$

**C**

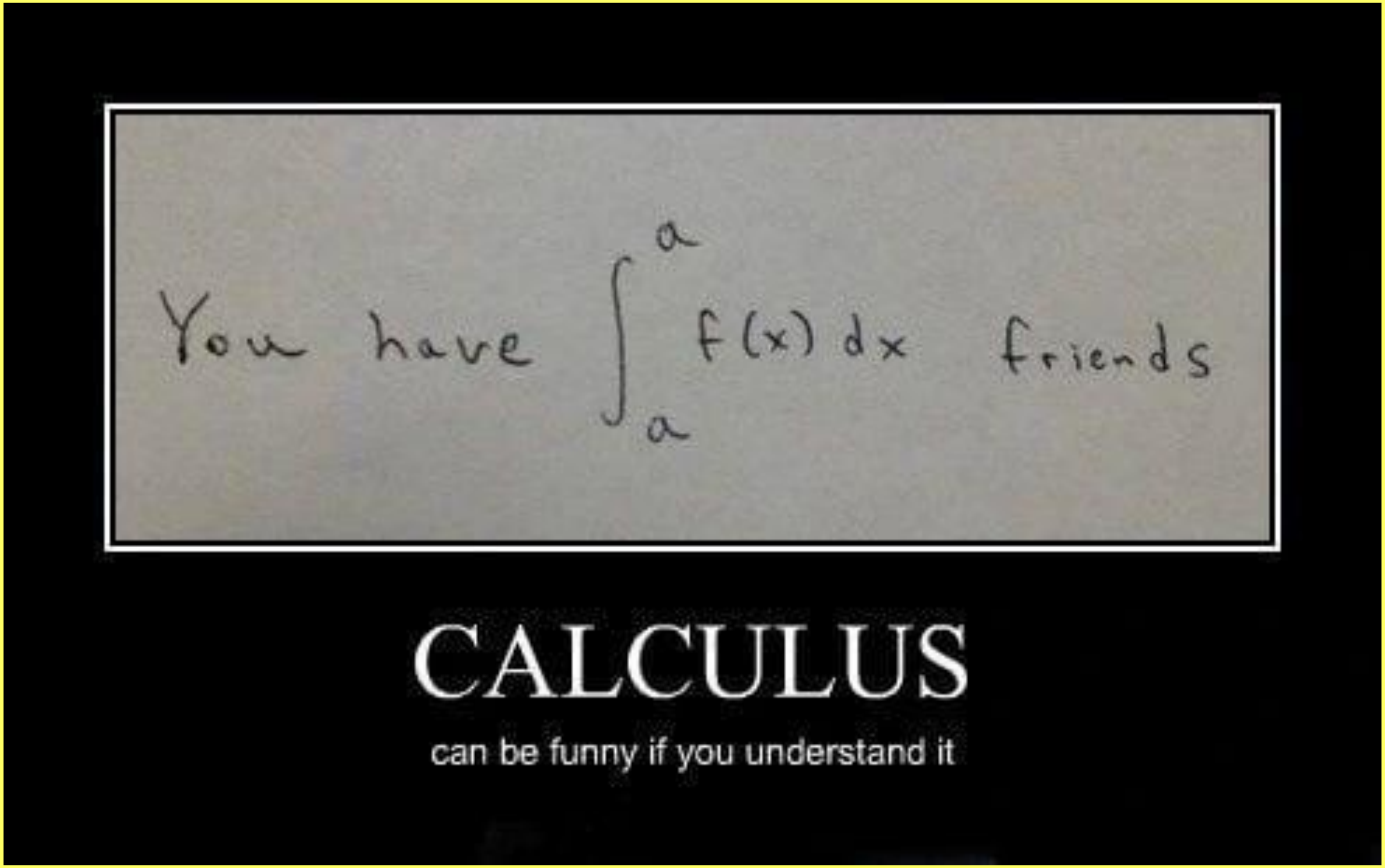
$$\frac{1}{3}\sec(3x) + C$$

**M**

$$\frac{1}{9}\sec(3x) + C$$

**A**

$$3\sec(3x) + C$$




## Integrating using Trig Identities

# Integrating using Trig Identities

Some expressions, such as $\sin^2 x$ and $\sin x \cos x$ can't be integrated directly, but we can use our trig identities to replace it with an expression we can easily integrate.

**E.g. 1** Find $\int \sin^2 x \;\; dx$

Do you know a trig identity involving $\sin^2 x$?

$$\cos 2x = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\int \frac{1}{2} - \frac{1}{2}\cos 2x \;\; dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

**E.g. 2** Find $\int \tan^2 x \;\; dx$

Recall that $1 + \tan^2 x \equiv \sec^2 x$

$$\int \tan^2 x \;\; dx = \int \sec^2 x - 1 \;\; dx$$

$$= \tan x - x + C$$

**E.g. 3** Find $\int \sin 3x \cos 3x \;\; dx$

$$\sin 6x \equiv 2\sin 3x \cos 3x$$

$$\sin 3x \cos 3x = \frac{1}{2}\sin 6x$$

$$\int \sin 3x \cos 3x \;\; dx = -\frac{1}{12}\cos 6x + C$$

**E.g. 4** Find $\int (\sec x + \tan x)^2 \; dx$

$$\int (\sec x + \tan x)^2 \; dx$$

$$= \int \sec^2 x + 2\sec x \tan x + \tan^2 x \;\; dx$$

$$= \tan x + 2\sec x + \tan x - x + C$$

From E.g. 2

$$= 2\tan x - 2\sec x - x + C$$

# Diagnostic Question

Find $\int \cos^2 x \; dx$

**Y**

$$\frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

**C**

$$\frac{1}{2}x - \frac{1}{2}\cos 2x + C$$

**M**

$$\frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

**A**

$$\frac{1}{2}x + \frac{1}{2}\sin 2x + C$$


**Reverse Chain Rule/Inspection**

# Reverse Chain Rule (Shortcut for Substitution)

There's certain more complicated expressions which look like the result of having applied the chain rule. I call this process '**consider then scale**':

1. **Consider some expression that will differentiate to something similar to it.**
2. **Differentiate, and adjust for any scale difference.**

$$\int x(x^2+5)^3\,dx$$

The first $x$ looks like it arose from differentiating the $x^2$ inside the brackets.

Consider $y=(x^2+5)^4$

$$\frac{dy}{dx}=4(x^2+5)^3\times 2x$$
$$=8x(x^2+5)^3$$

$$\therefore \int x(x^2+5)^3\,dx$$
$$=\frac{1}{8}(x^2+5)^4+C$$

$$\int \cos x \sin^2 x\;dx$$

The $\cos x$ probably arose from differentiating the $sin$.

Consider $y=\sin^3 x$

$$\frac{dy}{dx}=3\sin^2 x\cos x$$

$$\therefore \int \cos x\sin^2 x\;dx$$
$$=\frac{1}{3}\sin^3 x+C$$

$$\int \frac{2x}{x^2+1}\,dx$$

The $2x$ probably arose from differentiating the $x^2$.

Consider $y=\ln|x^2+1|$

$$\frac{dy}{dx}=\frac{2x}{x^2+1}$$

$$\therefore \int \frac{2x}{x^2+1}\,dx=\ln|x^2+1|+C$$

Note: All these could be found by using substitution, so if unsure, use substitution!.

# Reverse Chain Rule

**Integration by Inspection/Reverse Chain Rule**: Use common sense to **consider some expression** that would differentiate to the expression given. Then **scale** appropriately.

Common patterns:

$$\int k\frac{f'(x)}{f(x)}dx \rightarrow \quad Try\ \ln|f(x)|$$

In words: "If the bottom of a fraction differentiates to give the top (forgetting scaling), try ln of the bottom".

$$\int kf'(x)[f(x)]^n \rightarrow \quad Try\ [f(x)]^{n+1}$$

**E.g. 1**

$$\int \frac{x^2}{x^3+1}\,dx$$

Consider $y = \ln|x^3+1|$

$$\frac{dy}{dx} = \frac{3x^2}{x^3+1}$$

Therefore:

$$\int \frac{x^2}{x^3+1}\,dx$$

$$= \frac{1}{3}\ln|x^3+1| + C$$

**E.g. 2**

$$\int x\, e^{x^2+1}\,dx$$

Consider $y = e^{x^2+1}$

$$\frac{dy}{dx} = 2x\, e^{x^2+1}$$

Therefore:

$$\int x\, e^{x^2+1}dx$$

$$= \frac{1}{2}e^{x^2+1} + C$$

**E.g. 3**

$$\int \frac{x}{(x^2+5)^3}dx$$

**Tip:** If there's as power around the whole denominator, DON'T use $ln$: re-express the expression as a product. e.g. $x(x^2+5)^{-3}$

$$= \int x(x^2+5)^{-3}dx$$

$$= -\frac{1}{4}(x^2+5)^{-2} + C$$

# $\sin^n x \cos x$ vs $\sec^n x \tan x$

Notice when we differentiate $\sin^5 x$, then power decreases:

$$\boldsymbol{\frac{d}{dx}(\sin^5 x) = \frac{d}{dx}\left((\sin x)^5\right) = 5\sin^4 x \cos x}$$

However, when we differentiate $\sec^5 x$:

$$\boldsymbol{\frac{d}{dx}\left((\sec x)^5\right) = 5\sec^4 x \times \sec x \tan x = 5\sec^5 x \tan x}$$

Notice that the power of $sec$ didn't go down. Keep this in mind when integrating.

**For Example**

$$\int \sec^4 x \tan x \; dx$$

Consider $y = \sec^4 x$

$$\frac{dy}{dx} = 4\sec^3 x \times \sec x \tan x = 4\sec^4 x \tan x$$

$$\therefore \int \sec^4 x \tan x \; dx = \frac{1}{4}\sec^4 x + C$$

# Diagnostic Question

$$\int \cos x \; e^{\sin x}\, dx =$$

**Y**

$e^{-\cos x} + C$

**C**

$-e^{-\cos x} + C$

**M**

$e^{\sin x} + C$

**A**

$\cot x \, e^{\sin x} + C$

# Diagnostic Question

$$\int x^2 \, e^{x^3+1} \, dx =$$

**Y**

$$e^{x^3+1} + C$$

**C**

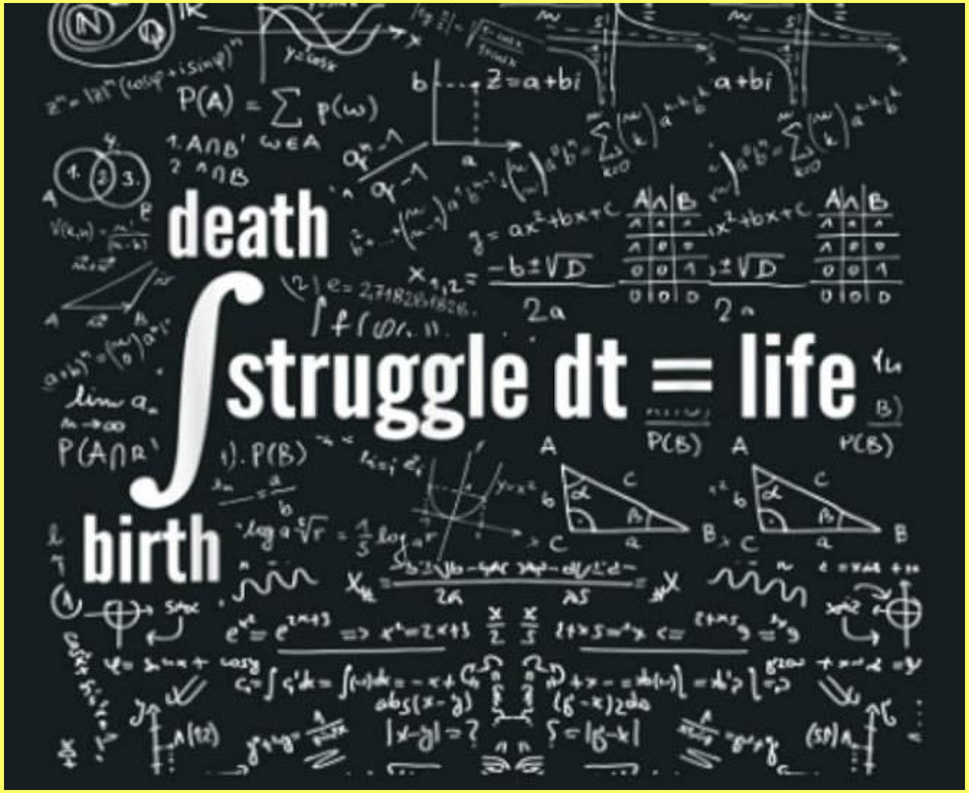
$$\frac{1}{3} e^{x^3+1} + C$$

**M**

$$\frac{e^{x^3+1}}{3x^2} + C$$

**A**

$$3e^{x^3+1} + C$$




# Integration by Substitution

# Integration by Substitution

For some integrations involving a complicated expression, we can make a substitution to turn it into an equivalent integration that is simpler. We wouldn't be able to use 'reverse chain rule' on the following:

**E.g. 1** Use the substitution $u = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

The aim is to completely remove any reference to $x$, and replace it with $u$. We'll have to work out $x$ and $dx$ so that we can replace them.

**STEP 1:** Using substitution, work out $x$ and $dx$ (or variant)

$$u = 2x + 5 \rightarrow \frac{du}{dx} = 2 \quad \rightarrow \quad dx = \frac{1}{2}du$$

$$x = \frac{u-5}{2}$$

**Tip**: Be careful about ensuring you reciprocate when rearranging for $dx$.

**STEP 2:** Substitute these into expression.

$$\int x\sqrt{2x+5}\ dx = \int \frac{u-5}{2}\sqrt{u}\frac{1}{2}du = \frac{1}{4}\int \sqrt{u}(u-5)\ du$$

$$= \frac{1}{4}\int u^{\frac{3}{2}} - 5u^{\frac{1}{2}}$$

**Tip**: If you have a constant factor, factor it out of the integral.

**STEP 3:** Integrate simplified expression.

$$= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{10}{3}u^{\frac{3}{2}}\right) + C$$

**STEP 4:** Write answer in terms of $x$.

$$= \frac{(2x+5)^{\frac{5}{2}}}{10} - \frac{5(2x+5)^{\frac{3}{2}}}{6} + C$$

# Integration by Substitution (Technicalities)

When integrating by substitution, when we change the $dx$ to another variable, what we're really doing is using the chain rule!

Informally, you can think of the $du$'s cancelling to leave the $dx$.

$$\int \dots dx \quad \rightarrow \quad \int \dots \frac{dx}{du} du$$

So, for the previous example, technically:

$$u = 2x + 5 \qquad \frac{du}{dx} = 2 \quad \rightarrow \quad \frac{dx}{du} du = \frac{1}{2} du$$

But for the sake of ease, you can just "rearrange for $dx$"

# How can we tell what substitution to use?

**There's no hard and fast rule, but it's often helpful to replace to replace expressions inside roots, powers or the denominator of a fraction.**

| Integral: | Sensible substitution: |
| --- | --- |
| $\int \cos x \sqrt{1+\sin x}\, dx$ | $\boldsymbol{u = 1 + \sin x}$ |
| $\int 6x\, e^{x^2}\, dx$ | $\boldsymbol{u = x^2}$ |
| $\int \frac{xe^x}{1+x}\, dx$ | $\boldsymbol{u = 1 + x}$ |
| $\int e^{\frac{1-x}{1+x}}\, dx$ | $\boldsymbol{u = \frac{1-x}{1+x}}$ |

A good indicator to use substitution is when an integral is of the form:
$\int f'(x) g(f(x))$
i.e. A composite function is multiplied by the derivative of its inner function.
That way the derivative product will cancel with the substitution.

# Another Example

**E.g. 2**

Find $\int x^3 e^{3x^4}\ dx$

**STEP 1:** Using substitution, work out $x$ and $dx$ (or variant)

$$u = 3x^4 \qquad \frac{du}{dx} = 12x^3 \quad \rightarrow \quad dx = \frac{du}{12x^3}$$

**STEP 2:** Substitute these into expression.

$$\therefore \int x^3 e^{3x^4}\ dx = \int \cancel{x^3} e^u \frac{du}{12\cancel{x^3}} = \frac{1}{12}\int e^u\ du$$

**STEP 3:** Integrate simplified expression.

$$= \frac{1}{12}e^u + C$$

**STEP 4:** Write answer in terms of $x$.

$$= \frac{1}{12}e^{3x^4} + C$$

# A "standard trick"

**E.g. 3**

Find $\int \frac{x}{x+1}\,dx$

As it is, we cannot split this fraction to integrate directly

**STEP 1:** Using substitution, work out $x$ and $dx$ (or variant)

$$u = x + 1 \qquad \frac{du}{dx} = 1 \quad \rightarrow \quad du = dx$$

$$x = u - 1$$

**STEP 2:** Substitute these into expression.

$$\therefore \int \frac{x}{x+1}\,dx = \int \frac{u-1}{u}\,du = \int 1 - \frac{1}{u}\,du$$

**STEP 3:** Integrate simplified expression.

$$= u - \ln u + C$$

**STEP 4:** Write answer in terms of $x$.

$$= x + 1 - \ln(x+1) + C$$

$$= x - \ln(x+1) + C'$$

**Where $C' = C + 1$** (you should state this to be fully correct)

# Another Example

**E.g. 4**

Find $\int \cos x \sin x\,(1+\sin x)^3\ dx$

**STEP 1:** Using substitution, work out $x$ and $dx$ (or variant)

$$u = 1 + \sin x \qquad \frac{du}{dx} = \cos x \quad \rightarrow \quad dx = \frac{du}{\cos x}$$

$$\sin x = u - 1$$

**STEP 2:** Substitute these into expression.

$$\therefore \int \cos x \sin x\,(1+\sin x)^3\ dx = \int \cancel{\cos x}\,(u-1)u^3\,\frac{du}{\cancel{\cos x}}$$

$$= \int (u-1)u^3\ du \quad = \int u^4 - u^3\ du$$

**STEP 3:** Integrate simplified expression.

$$= \frac{1}{5}u^5 - \frac{1}{4}u^4 + C$$

**STEP 4:** Write answer in terms of $x$.

$$= \frac{1}{5}(\sin x + 1)^5 - \frac{1}{4}(\sin x + 1)^4 + C$$

# Using substitutions involving implicit differentiation

When a root is involved, it makes things tidier if we use $u^2 = \cdots$

**E.g. 5**

Find $\int x\sqrt{2x+5}\ dx$

**STEP 1:** Using substitution, work out $x$ and $dx$ (or variant)

$$u^2 = 2x + 5 \qquad 2u\frac{du}{dx} = 2 \quad \rightarrow \quad dx = u\,du$$

$$x = \frac{u^2 - 5}{2}$$

(implicit differentiation/chain rule)

**STEP 2:** Substitute these into expression.

$$\int x\sqrt{2x+5}\ dx = \int \frac{u^2-5}{2} u \times u\,du = \int \frac{1}{2}u^4 - \frac{5}{2}u^2\,du$$

**STEP 3:** Integrate simplified expression.

$$= \frac{1}{10}u^5 - \frac{5}{6}u^3 + C$$

**STEP 4:** Write answer in terms of $x$.

$$= \frac{(2x+5)^{\frac{5}{2}}}{10} - \frac{5(2x+5)^{\frac{3}{2}}}{6} + C$$

This was slightly less tedious than when we used $u = 2x + 5$, as we didn't have fractional powers to deal with.

# Using Substitution to Prove Standard Integrals

**E.g. 6**

$$\int \tan x \; dx$$

Rewrite $$\int \tan x \; dx = \int \frac{\sin x}{\cos x} dx$$

Let $u = \cos x$ $$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$$

$$\int \frac{\sin x}{\cos x} dx = -\int \frac{\cancel{\sin x}}{u} \frac{du}{\cancel{\sin x}} = -\int \frac{1}{u} du$$

$$= -\ln u + C \qquad = -\ln|\cos x| + C$$

The final result can be rewritten using the law of logarithm powers:

$$-\ln|\cos x| + C = \ln(\cos x)^{-1} + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$

$$\boxed{\int \tan(x) \; dx = \ln|\sec(x)| + C}$$

# Substitution Method: Definite Integration

- We work in exactly the same way BUT we must also **substitute for the limits**, since they are values of $x$ and we are changing the variable to $u$.

- Once you have rewritten the integral in terms of $u$, the limits must also be in terms of $u$ or the expression is incorrect.

- A definite integral gives a value so it is not necessary to substitute back to get an equation in $x$.

# Definite Integration (Substitution)

**E.g. 7**

$$\int_0^1 \frac{x}{(1+x^2)^3}\,dx$$

Let $u = 1 + x^2$

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

**Now because we've changed from $x$ to $u$, we have to work out what values of $u$ would have given those limits for $x$:**

Limits: $x = 0 \Rightarrow u = 1 + x^2 = 1$

$x = 1 \Rightarrow u = 1 + x^2 = 2$

$$\therefore \int_0^1 \frac{x}{(1+x^2)^3}\,dx = \int_1^2 \frac{\cancel{x}}{u^3}\frac{du}{2\cancel{x}} = \frac{1}{2}\int_1^2 u^{-3}\,du$$

$$= \frac{1}{2}\left[\frac{u^{-2}}{-2}\right]_1^2 = \left[-\frac{1}{4u^2}\right]_1^2 = \left[-\frac{1}{16}\right] - \left[-\frac{1}{4}\right] = \frac{3}{16}$$

# Definite Integration (Substitution)

**E.g. 8**

$$\int_0^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin x}\ dx$$

Use substitution: $\boldsymbol{u = 1 + \sin x}$

$$\frac{du}{dx} = \cos x \quad \rightarrow \quad dx = \frac{du}{\cos x} \qquad\qquad \sin x = u - 1$$

**Limits:**

When $x = \frac{\pi}{2}$, $\quad u = 2$

When $x = 0$, $\quad u = 1$

$$\int_0^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin x}\ dx = \int_1^2 u^{\frac{1}{2}}\, du = \left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^2 = \frac{2}{3}\left(2\sqrt{2} - 1\right)$$

# Diagnostic Question

$$\int \frac{x^3}{x^4 - 1} dx =$$

**Y**

$\ln|x^4 - 1| + C$

**C**

$4\ln|x^4 - 1| + C$

**M**

$\dfrac{1}{4x^3}\ln|x^4 - 1| + C$

**A**

$\dfrac{1}{4}\ln|x^4 - 1| + C$

# Diagnostic Question

$$\int \frac{x}{(x^2+5)^3}\,dx =$$

**Y**

$$\frac{1}{2}\ln(x^2+5)^3 + C$$

**C**

$$\frac{-1}{2(x^2+5)^2} + C$$

**M**

$$\frac{-1}{8(x^2+5)^4} + C$$

**A**

$$\frac{-1}{4(x^2+5)^2} + C$$

# Diagnostic Question

$$\int \frac{\cos x}{\sin x + 2}\,dx =$$

**Y**

$\ln|\sin x + 2| + C$

**C**

$\cos x \ln|\sin x + 2| + C$

**M**

$\ln|-\cos x + 2x| + C$

**A**

$\sin x \ln|\sin x + 2| + C$

# Inverse Trigonometric Functions

- The following results can be proved using trigonometric substitutions:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

# Using Substitution to Prove Standard Integrals

**E.g. 9**

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

Let $x = \sin u$ $\qquad \frac{dx}{du} = \cos u \Rightarrow dx = \cos u\, du$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2 u}}\cos u\, du$$

$$\sin^2 u + \cos^2 u = 1 \Rightarrow 1 - \sin^2 u = \cos^2 u$$

$$\therefore \int \frac{1}{\sqrt{1-\sin^2 u}}\cos u\, du = \int \frac{1}{\cancel{\sqrt{\cos^2 u}}}\cancel{\cos u}\, du$$

$$= \int 1\, du = u + C$$

Substituting back for $u$ gives: $\quad u = \sin^{-1} x \qquad \therefore \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

Using $x = a\sin u$ we can get a more general result:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$


**Integration by Parts**

# Proof of Formula

Start with the product rule for differentiation:

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Integrating each term gives:

$$\int \frac{d(uv)}{dx}dx = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

Simplifying the LHS:

$$uv = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

After rearranging

$$\boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

This is the formula for <u>Integration by Parts</u>

Here $u$ and $v$ must both be functions of $x$

(a constant still counts as a function of $x$)

# Integration by Parts

**E.g. 1** Find $\int x\cos x \; dx$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}\,dx$$

**STEP 1**: Decide which thing will be $u$ (and which $\frac{dv}{dx}$).

$u = x \qquad \frac{dv}{dx} = \cos x$

You're about to differentiate $u$ and integrate $\frac{dv}{dx}$, so the idea is to pick them so differentiating $u$ makes it 'simpler', and $\frac{dv}{dx}$ can be integrated easily.
$u$ will always be the $x^n$ term **UNLESS** one term is $\ln x$.

**STEP 2**: Find $\frac{du}{dx}$ and $v$.

$\frac{du}{dx} = 1 \qquad v = \sin x$

**STEP 3**: Use the formula.

I just remember it as "$uv$ minus the integral of the two new things multiplied together"

$$\int x\cos x \; dx = x\sin x - \int 1\sin x \; dx$$

$$= x\sin x + \cos x + C$$

# Another Example

**E.g. 2**

Find $\int x^2 \ln x \; dx$

This time, we choose $\ln x$ to be the $u$ because it differentiates nicely.

**STEP 1**: Decide which thing will be $u$ (and which $\frac{dv}{dx}$).

$$u = \ln x \qquad \frac{dv}{dx} = x^2$$

**STEP 2**: Find $\frac{du}{dx}$ and $v$.

$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{3}x^3$$

**STEP 3**: Use the formula.

$$\int x^2 \ln x \; dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 \, dx$$

$$= \boldsymbol{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$$

# Integration by Parts Twice!!

**E.g. 3**

Find $\int x^2 e^x \, dx$

$$u = x^2 \qquad \frac{dv}{dx} = e^x \qquad \frac{du}{dx} = 2x \qquad v = e^x$$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx$$

Oh no, it's still a product! ☹

We have to apply IBP again!

$$u = 2x \qquad \frac{dv}{dx} = e^x$$

$$\frac{du}{dx} = 2 \qquad v = e^x$$

$$\int 2xe^x \, dx = 2xe^x - \int 2e^x \, dx \quad = 2xe^x - 2e^x$$

You could also label the first parts as

$$u_1 \qquad \frac{dv_1}{dx}$$

and the second as

$$u_2 \qquad \frac{dv_2}{dx}$$

Therefore

$$\int x^2 e^x \, dx = x^2 e^x - (2xe^x - 2xe^x) + C$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

**Tip:** You can write out my working for any second integral completely separately, and then put the result back into the original integral later.

# Integrating $\ln x$ and definite integration

**E.g. 4**

Find $\int \ln x \ dx$

This is known as "Hidden Parts" because the 1 is hidden.

$$u = \ln x \qquad \frac{dv}{dx} = 1 \qquad \Rightarrow \frac{du}{dx} = \frac{1}{x} \qquad v = x$$

$$\int \ln x \ dx = x \ln x - \int 1 \, dx \quad = x \ln x - x + C$$

**E.g. 5**

Find $\int_1^2 \ln x \, dx$, leaving your answer in terms of natural logarithms.

$$\int_1^2 \ln x \, dx = [x \ln x - x]_1^2 = \mathbf{2 \ln 2 - 1}$$

If we were doing it from scratch:

$$u = \ln x \qquad \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = x$$

In general:

$$\int_a^b u \frac{dv}{dx} dx = [uv]_a^b - \int_a^b v \frac{du}{dx} dx$$

$$\int_1^2 \ln x \, dx = [x \ln x]_1^2 - \int_1^2 1 \, dx$$

$$= 2 \ln 2 \ - 1 \ln 1 - [x]_1^2$$

$$= 2 \ln 2 \ - (2 - 1) = \mathbf{2 \ln 2 - 1}$$

So for a definite integration either:

1. Do indefinite integral by parts then apply limits.
2. Apply limits to parts formula and do all at once.

# Cyclic parts

**E.g. 6**

Find $\int e^x \sin x \; dx$

You may use integration by parts (IBP) yet the problem never simplifies. In this case you should consider 'cyclic parts'.

$$u = e^x \quad \frac{dv}{dx} = \sin x \qquad \frac{du}{dx} = e^x \quad v = -\cos x$$

It doesn't matter what you make the $u$ and what the $\frac{dv}{dx}$ this time. The trick is realising how to end the repeating cycle

$$\int e^x \sin x \; dx = -e^x \cos x - \int -e^x \cos x \; dx \quad = -e^x \cos x + \int e^x \cos x \; dx$$

Consider $\int e^x \cos x \; dx$ $\qquad u = e^x \quad \frac{dv}{dx} = \cos x \qquad \frac{du}{dx} = e^x \quad v = \sin x$

$$\therefore \int e^x \sin x \; dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \; dx$$

This is the surprising bit! Because we ended up with the original expression, we can 'collect' these integrals together!

$$2\int e^x \sin x \; dx = -e^x \cos x + e^x \sin x + C$$

Because this is an indefinite integration, we still need a $+C$ so just stick it on the end!

$$\int e^x \sin x \; dx = -\frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C'$$

# Diagnostic Question

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx}\, dx$$

$$\int 4xe^x \, dx =$$

**Y** $4xe^x + 4e^x + C$

**M** $4xe^x - 4e^x + C$

**C** $8e^x + C$

**A** $C$

# Diagnostic Question

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx}\, dx$$

$$\int x \sin x \, dx =$$

**Y** $-x \cos x - \sin x + C$

**M** $x \cos x - \sin x + C$

**C** $-x \cos x + \sin x + C$

**A** $-x \cos x - \sin x + C$

# Diagnostic Question

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int xe^{2(1-x)}\,dx =$$

When integrating by parts, which of the following is correct?

**Y** $-\frac{1}{2}e^{2(1-x)} + \frac{1}{2}\int e^{2(1-x)}dx$

**M** $-\frac{1}{2}xe^{2(1-x)} - \int e^{2(1-x)}dx$

**C** $-\frac{1}{2}xe^{2(1-x)} + \int e^{2(1-x)}dx$

**A** $-\frac{1}{2}xe^{2(1-x)} + \frac{1}{2}\int e^{2(1-x)}dx$

# Order of Thinking

For an integral,

The integration techniques you should consider should go in the following order

1. Standard integral
2. Substitution
3. Parts

$$\int \frac{1}{CABIN}\, dCABIN = ??$$

**Integration Using Partial Fractions**

# Reminder: Partial Fractions

Remember the following rules for partial fractions:

The numerator of each fraction must have a polynomial of order (at least) one less than the denominator.

A repeated factor requires repeated fractions.

Remember to use the correct decomposition:

**Distinct Linear Factors:**

$$\frac{x}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

**Repeated Factor**

$$\frac{1}{(x-2)(x+2)^2} = \frac{A}{(x-2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)}$$

**Irreducible Quadratic Factor**

$$\frac{x}{(x^2-2x+6)(x+2)} = \frac{A+Bx}{(x^2-2x+6)} + \frac{C}{(x+2)}$$

**Note:**
$x^2 + a$ is a common irreducible quadratic factor.

# Recap of Partial Fractions Rules (Generalised)

**Distinct Linear Factors:**

1 $$\frac{TOP}{(ax+b)(cx+d)} \equiv \frac{A}{ax+b} + \frac{B}{cx+d}$$

**Repeated Factor (can be extended to n repeated factors):**

2 $$\frac{TOP}{(ax+b)^2(cx+d)} \equiv \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{cx+d}$$

**Irreducible Quadratic Factor (can be extended to higher order polynomials):**

3 $$\frac{TOP}{(ax^2+bx+c)(dx+e)} \equiv \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$$

Note: For all these cases order of numerator $<$ order of denominator.
If not do polynomial division then apply rules

# Integration Using Partial Fractions

This integration method makes use of a result that we noted when differentiating a function of a function using the chain rule:

$$y = \ln\big(f(x)\big) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

It follows that integrating the above fraction gives:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

We will now look at some integrals of algebraic fractions.

(a) $\displaystyle\int \frac{1-2x}{2+x-x^2}\,dx$ and (b) $\displaystyle\int \frac{8-x}{2+x-x^2}\,dx$

(a) is of the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$$\left(f(x) = 2 + x - x^2 \text{ and } f'(x) = 1 - 2x\right)$$

(b) is not of this form and cannot be converted to it by using a multiplying constant. Instead, we will use partial fractions.

# Integration Using Partial Fractions

Integrating (a) gives:

$$I = \int \frac{1-2x}{2+x-x^2}\,dx \qquad = \ln|2+x-x^2| + C$$

The constant $C$ can be replaced by $\ln A$ , so we get

$$I = \ln|2+x-x^2| + \ln A$$

This is not always necessary but helps simplify things.

This can be simplified using the law of logarithm addition ($\ln a + \ln b = \ln(ab)$)

$$I = \ln|A(2+x-x^2)|$$

# Integration Using Partial Fractions

**E.g. 1** We will take (b) as an example of an integral which requires the use of partial fractions.
$$I=\int \frac{8-x}{2+x-x^2}\,dx$$

Now
$$\frac{8-x}{2+x-x^2} \equiv \frac{8-x}{(1+x)(2-x)} \equiv \frac{A}{(1+x)}+\frac{B}{(2-x)} \equiv \frac{A(2-x)+B(1+x)}{(1+x)(2-x)}$$

As the denominators are equal, the numerators must be: $8-x \equiv A(2-x)+B(1+x)$

Substitute $x=-1$: $9=3A \Rightarrow A=3$

Substitute $x=2$: $6=3B \Rightarrow B=2$

**Alternative method:**
You could expand this and compare coefficients

$$\therefore I=\int \frac{8-x}{2+x-x^2}\,dx=\int \frac{3}{1+x}+\frac{2}{2-x}\,dx$$

Both terms can be integrated directly using: $\int \frac{f'(x)}{f(x)}\,dx=\ln|f(x)|+C$

$$I=\int \frac{3}{1+x}\,dx+\int \frac{2}{2-x}\,dx=3\int \frac{1}{1+x}\,dx-2\int \frac{-1}{2-x}\,dx$$

**Or just:**
$$\int \frac{2}{2-x}\,dx=\frac{2}{-1}\ln|2-x|$$

$$=3\ln|1+x|-2\ln|2-x|+\ln A$$

$$=\ln|1+x|^3-\ln|2-x|^2+\ln A$$

$$\int \frac{8-x}{2+x-x^2}\,dx=\ln\frac{A|1+x|^3}{(2-x)^2}$$

# Integration Using Partial Fractions

**E.g. 2**

Integrate the function: $$f(x) = \frac{4x^2 - 3x + 5}{(x-1)^2(x+2)}$$

$$\frac{4x^2 - 3x + 5}{(x-1)^2(x+2)} \equiv \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$4x^2 - 3x + 5 \equiv A(x+2) + B(x-1)(x+2) + C(x-1)^2$$

Substitute $x = 1$: $6 = 3A \Rightarrow A = 2$

Substitute $x = -2$: $27 = 9C \Rightarrow C = 3$

$$4x^2 - 3x + 5 \equiv 2(x+2) + B(x-1)(x+2) + 3(x-1)^2$$

We can't get rid of anything else but we only have one unknown left

Substitute $x = 0$: $5 = 4 - 2B + 3 \Rightarrow 2B = 2 \Rightarrow B = 1$

We can use any value but $x = 0$ is easy

$$\therefore \frac{4x^2 - 3x + 5}{(x-1)^2(x+2)} = \frac{2}{(x-1)^2} + \frac{1}{(x-1)} + \frac{3}{x+2}$$

# Integration Using Partial Fractions

$$\therefore \int \frac{4x^2-3x+5}{(x-1)^2(x+2)}\,dx = \int \frac{2}{(x-1)^2}\,dx + \int \frac{1}{(x-1)}\,dx + \int \frac{3}{x+2}\,dx$$

The 2nd and 3rd integrals are as follows:

$$\int \frac{1}{x-1}\,dx = \ln|x-1| + C \qquad \int \frac{3}{x+2}\,dx = 3\ln|x+2| + C$$

The 1st integral requires a substitution. Let: $x-1=u \quad \Rightarrow \quad dx = du$

$$\int \frac{2}{(x-1)^2}\,dx = \int 2u^{-2}\,du \quad = -2u^{-1} + C \quad = \frac{-2}{(x-1)} + C$$

Finally: $$\int \frac{4x^2-3x+5}{(x-1)^2(x+2)}\,dx = \frac{-2}{(x-1)} + \ln|x-1| + 3\ln|x+2| + C$$

$$= -\frac{2}{x-1} + \ln|x-1| + \ln|x+2|^3 + C$$

$$= -\frac{2}{x-1} + \ln|(x-1)(x+2)^3| + C$$

(we could have also used $C = \ln A$ and combined but the answer is still correct as is)

# Definite Integration Using Partial Fractions

**E.g. 3** Calculate $$\int_3^4 \frac{-2}{(x+2)(x-2)}\,dx$$

$$\frac{-2}{(x+2)(x-2)} \equiv \frac{A}{x+2} + \frac{B}{x-2}$$

$$-2 \equiv A(x-2) + B(x+2)$$

$$-2 \equiv Ax - 2A + Bx + 2B$$

Compare coefficients of powers of $x$:

$$0 = A + B \qquad (1)$$
$$-2 = -2A + 2B \qquad (2)$$

Let's use the other method (comparing coefficients)

Solving simultaneously we get: $A = \frac{1}{2},\ B = -\frac{1}{2}$

$$I = \int_3^4 \frac{-2}{(x+2)(x-2)}\,dx = \int_3^4 \left[\frac{\frac{1}{2}}{x+2} + \frac{-\frac{1}{2}}{x-2}\right] dx \quad = \frac{1}{2}\int_3^4 \frac{dx}{x+2} - \frac{1}{2}\int_3^4 \frac{dx}{x-2}$$

$$= \frac{1}{2}\left[\ln(x+2) - \ln(x-2)\right]_3^4 \quad = \frac{1}{2}\left[(\ln(6) - \ln(2)) - (\ln(5) - \ln(1))\right]$$

$$= \frac{1}{2}\ln\left(\frac{6}{2\times 5}\right) \quad = \frac{1}{2}\ln\left(\frac{3}{5}\right) \approx -0.255 \ \ (3 \text{ d.p.})$$

# Integration Using Partial Fractions

If the order of the numerator equal or larger than the denominator, we must first perform a polynomial division.

**E.g. 4**

a) Find the values of the constants $A$, $B$ and $C$ such that

$$\frac{x^2-6}{(x+4)(x-1)} \equiv A + \frac{B}{x+4} + \frac{C}{x-1}$$

b) Hence, find

$$\int \frac{x^2-6}{(x+4)(x-1)}\,dx$$

In this case, the order of the top and bottom are the same, so the division would give a constant.
(Hence the form above)

$$x^2 - 6 \equiv A(x+4)(x-1) + B(x-1) + C(x+4)$$

Substitute $x = -4$: $10 = -5B \Rightarrow B = -2$

Substitute $x = 1$: $-5 = 5C \Rightarrow C = -1$

Compare coefficients of $x^2$: $A = 1$

$$\int \frac{x^2-6}{(x+4)(x-1)}\,dx = \int 1 - \frac{2}{x+4} - \frac{1}{x-1} + C$$

$$= x - 2\ln|x+4| - \ln|x-1| + C$$

# Integrating top-heavy algebraic fractions

**E.g. 4**

$$\int \frac{x^2}{x+1}\,dx =$$

In this case, just divide first,

$$\begin{array}{r|l} & x \quad -1 \\ \hline x+1 & x^2 + 0x + 0 \\ & x^2 + x \\ \hline & \quad\; -x + 0 \\ & \quad\; -x - 1 \\ \hline & \qquad\quad 1 \end{array}$$

$$= \int x - 1 + \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}x^2 - x + \ln|x+1| + C$$

In this case we could integrate directly after dividing, but you may also need to apply partial fractions to the remainder.

# Diagnostic Question

$$\frac{3x+7}{(x+1)(x-5)^2}$$

Which is the correct form of partial fractions?

**Y** $\dfrac{A}{x+1}+\dfrac{B}{x-5}$

**M** $\dfrac{A}{x+1}+\dfrac{B}{x-5}+\dfrac{C}{x-5}$

**C** $\dfrac{A}{x+1}+\dfrac{B}{x-5}+\dfrac{C}{(x-5)^2}$

**A** $A+\dfrac{B}{x+1}+\dfrac{C}{x-5}+\dfrac{D}{(x-5)^2}$

# Diagnostic Question

$$\frac{x}{(x+1)(x^2-9)}$$

Which is the correct form of partial fractions?

**Y** $\dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{x-3}$

**M** $\dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{(x-3)^2}$

**C** $\dfrac{A}{x+1}+\dfrac{B}{x-9}+\dfrac{C}{(x-9)^2}$

**A** $\dfrac{A}{x+1}+\dfrac{B}{x+3}+\dfrac{C}{x-3}$

# Diagnostic Question

$$\frac{x}{(x+1)(x^2+9)}$$

Which is the correct form of partial fractions?

**Y** $\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+9}$

**M** $\dfrac{A}{x+1}+\dfrac{B}{x+3}+\dfrac{C}{x+3}$

**C** $\dfrac{A}{x+1}+\dfrac{B}{x+3}+\dfrac{C}{x-3}$

**A** $\dfrac{A}{x+1}+\dfrac{B}{x+3}+\dfrac{C}{(x+3)^2}$

# Diagnostic Question

$$\frac{2x-1}{(x+1)(x-1)(x+2)^2}$$

Which is the correct form of partial fractions?

**Y** $\quad \dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{(x+2)^2}$

**M** $\quad \dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{x+2}$

**C** $\quad A+\dfrac{B}{x+1}+\dfrac{C}{x-1}+\dfrac{D}{(x+2)^2}$

**A** $\quad \dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{x+2}+\dfrac{D}{(x+2)^2}$

# Diagnostic Question

$$4x - 3 \equiv A(x+3)^2 + B(x-2)(x+3) + C(x-2)$$

You could find the values $A, B$ and $C$ by substituting any values of $x$.

Which would be the **least** useful?

**Y** $x = 2$

**M** $x = 3$

**C** $x = -3$

**A** $x = 0$

# Diagnostic Question

$$\int \frac{x^2+4}{3x^2+4x-4x}\,dx = \int -\frac{1}{x} + \frac{1}{2(x+2)} + \frac{5}{2(3x-2)}\,dx$$

Which is **not** a version of the correct integral?

**Y** $-\ln|x| + \frac{1}{2}\ln|x+2| + \frac{5}{6}\ln|3x-2| + C$

**M** $-\ln|x| + \frac{1}{2}\ln|x+2| + \frac{5}{2}\ln|3x-2| + C$

**C** $-\ln|x| + \frac{1}{2}\ln|x+2| + \frac{5}{6}\ln|3x-2| + \ln A$

**A** $\ln\left|\frac{A\,(x+2)^{\frac{1}{2}}\,(3x-2)^{\frac{5}{6}}}{x}\right|$

# Diagnostic Question

$$\int \frac{3x+11}{x^2-x-6}\,dx = \int \frac{4}{x-3} - \frac{1}{x+2}\,dx$$

Which is **not** a version of the correct integral?

**Y** $4\ln|x-3| - \ln|x+2| + C$

**M** $\ln\left|\dfrac{(x-3)^4}{x+2}\right| + C$

**C** $\ln\left|\dfrac{4(x-3)}{x+2}\right| + C$

**A** $\ln\left|A\dfrac{(x-3)^4}{x+2}\right|$

# Proof of Standard Integrals using Partial Fractions

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Proof: $$\frac{1}{x^2 - a^2} \equiv \frac{A}{x+a} + \frac{B}{x-a}$$

$$1 \equiv A(x-a) + B(x+a)$$

$$x = a: \quad 1 = 2aB \Rightarrow B = \frac{1}{2a}$$

$$x = -a: \quad 1 = -2aA \Rightarrow A = -\frac{1}{2a}$$

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\int \frac{-1}{x+a} + \frac{1}{x-a}\,dx$$

$$= \frac{1}{2a}\{-\ln|x+a| + \ln|x-a| + C_1\}$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

Proof: $$\frac{1}{a^2 - x^2} \equiv \frac{A}{a+x} + \frac{B}{a-x}$$

$$1 \equiv A(a-x) + B(a+x)$$

$$x = a: \quad 1 = 2aB \Rightarrow B = \frac{1}{2a}$$

$$x = -a: \quad 1 = 2aA \Rightarrow A = \frac{1}{2a}$$

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\int \frac{1}{a+x} + \frac{1}{a-x}\,dx$$

$$= \frac{1}{2a}\{\ln|a+x| - \ln|a-x| + C_1\}$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

chain rule needed on $a - x$

This can also be shown by negating the previous result and using the fact that $\left|\frac{a+x}{x-a}\right| = \left|\frac{a+x}{a-x}\right|$ due to effect of modulus.

# Summary

- Substitution can be used for a variety of integrals:
  - For example:

$$\int (1+2x)^4 dx \qquad\qquad \int x\sqrt{1+x^2}\,dx$$

  - define $u$ as the inner function
  - differentiate the substitution expression; and
  - rearrange to find $dx$
  - if it is a definite integral, convert the limits

- If necessary, rearrange the substitution expression to find $x$.
  - substitute for the entire inner function and $dx$
  - integrate with respect to $u$; and
  - substitute back for $u$ in terms of $x$

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# Summary

## Integration by Parts

Here $u$ and $v$ must both be functions of $x$.

Indefinite case:

$$\boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

Definite case:

$$\boxed{\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx}$$

# Summary of Functions

| $f(x)$ | How to deal with it | $\int f(x)dx$ (+constant) |
|---|---|---|
| $\boldsymbol{\sin x}$ | Standard result | $-\cos x$ |
| $\boldsymbol{\cos x}$ | Standard result | $\sin x$ |
| $\boldsymbol{\tan x}$ | In formula booklet, but use $\int \frac{\sin x}{\cos x} dx$ which is of the form $\int \frac{kf'(x)}{f(x)} dx$ | $\ln\lvert\sec x\rvert$ |
| $\boldsymbol{\sin^2 x}$ | For both $\sin^2 x$ and $\cos^2 x$ use identities for $\cos 2x$ <br> $\cos 2x = 1 - 2\sin^2 x$ <br> $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ | $\frac{1}{2}x - \frac{1}{4}\sin 2x$ |
| $\boldsymbol{\cos^2 x}$ | $\cos 2x = 2\cos^2 x - 1$ <br> $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ | $\frac{1}{2}x + \frac{1}{4}\sin 2x$ |
| $\boldsymbol{\tan^2 x}$ | $1 + \tan^2 x \equiv \sec^2 x$ <br> $\tan^2 x \equiv \sec^2 x - 1$ | $\tan x - x$ |
| $\boldsymbol{cosec\ x}$ | Would use substitution $u = cosec\ x + \cot x$, but too hard for exam. | $-\ln\lvert cosec\ x + \cot x\rvert$ |
| $\boldsymbol{\sec x}$ | Would use substitution $u = \sec x + \tan x$, but too hard for exam. | $\ln\lvert\sec x + \tan x\rvert$ |
| $\boldsymbol{\cot x}$ | $\int \frac{\cos x}{\sin x} dx$ which is of the form $\int \frac{f'(x)}{f(x)} dx$ | $\ln\lvert sin\ x\rvert$ |

# Summary of Functions

| $f(x)$ | How to deal with it | $\int f(x)dx$ (+constant) |
|---|---|---|
| $cosec^2 x$ | By observation. | $-\cot x$ |
| $sec^2 x$ | By observation. | $\tan x$ |
| $cot^2 x$ | $1 + \cot^2 x \equiv cosec^2 x$ | $-\cot x - x$ |
| $\sin 2x \cos 2x$ | For any product of sin and cos with same coefficient of $x$, use double angle. $\sin 2x \cos 2x \equiv \frac{1}{2} \sin 4x$ | $-\frac{1}{8} \cos 4x$ |
| $\frac{1}{x}$ | | $\ln x$ |
| $\ln x$ | Use IBP, where $u = \ln x, \frac{dv}{dx} = \ln x$ | $x \ln x - x$ |
| $\frac{x}{x+1}$ | Use algebraic division. $\frac{x}{x+1} \equiv 1 - \frac{1}{x+1}$ | $x - \ln\|x+1\|$ |
| $\frac{1}{x(x+1)}$ | Use partial fractions. | $\ln\|x\| - \ln\|x+1\|$ |

# Summary of Functions

| $f(x)$ | How to deal with it | $\int f(x)dx$ (+constant) |
|---|---|---|
| $\dfrac{4x}{x^2+1}$ | Substitution | $2\ln\lvert x^2+1\rvert$ |
| $\dfrac{x}{(x^2+1)^2}$ | Substitution | $-\dfrac{1}{2}(x^2+1)^{-1}$ |
| $e^{2x+1}$<br>$\dfrac{1}{1-3x}$ | For any function where 'inner function' is linear expression, divide by coefficient of $x$ | $\dfrac{1}{2}e^{2x+1}$<br>$-\dfrac{1}{3}\ln\lvert 1-3x\rvert$ |
| $x\sqrt{2x+1}$ | IBP or use sensible substitution. $u = 2x+1$ or even better, $u^2 = 2x+1$. | $\dfrac{1}{15}(2x+1)^{\frac{3}{2}}(3x-1)$ |
| $\sin^5 x \ cos\ x$ | Substitution | $\dfrac{1}{6}\sin^6 x$ |

Thanks

See you in the Tutorial!