CAPE1150

Engineering Mathematics

School of Chemical and Process Engineering

University of Leeds

Level 1 Semester 2

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Tutorial: Question Difficulty Colour Code

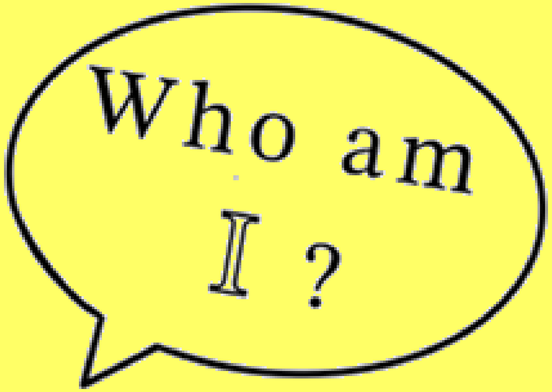
Basic - straightforward application
(you must be able to do these)

Medium – Makes you think a bit
(you must be able to do these)

Hard – Makes you think a lot
(you should be able to do these)

Extreme – Tests your understanding to the limit!
(for those who like a challenge)

Applied – Real-life examples of the topic, may sometimes involve prior knowledge
(you should attempt these – will help in future engineering)



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Tutorial 10
Matrices 1

Class Example: Matrix Arithmetic

E.g. 1

Given the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 0 \\ 2 & -1 & -1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Determine $\left(\mathbf{3A^T - B}\right)^{\mathbf{T}}$

$$\mathbf{A}\left(\mathbf{3A^T - B}\right)^{\mathbf{T}} = \begin{pmatrix} 5 & 5 & -3 \\ 10 & 7 & 0 \\ 3 & -9 & -4 \end{pmatrix}$$

Class Example: Gaussian Elimination

E.g. 2

Use Gaussian Elimination to solve:

$$2x + y + 2z = 8$$
$$x - 3y + 3z = -4$$
$$4x + 2y - z = 1$$

The augmented matrix $(\mathbf{A}|\mathbf{b})$ is:

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 8 \\ 1 & -3 & 3 & -4 \\ 4 & 2 & -1 & 1 \end{array}\right)$$

Interchange row 1 and row 2 $(R_1 \leftrightarrow R_2)$

We do this to avoid having to use fractions.

$$\left(\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 2 & 1 & 2 & 8 \\ 4 & 2 & -1 & 1 \end{array}\right)$$

$R_2 - 2R_1$
$R_3 - 4R_1$

$$\left(\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 0 & 7 & -4 & 16 \\ 0 & 0 & -5 & -15 \end{array}\right)$$

Solve using back-substitution:

$$-5z = -15 \quad \rightarrow \boldsymbol{z = 3}$$

$$7y - 4(3) = 16 \quad \rightarrow \boldsymbol{y = 4}$$

$$x - 3(4) + 3(3) = -4 \quad \rightarrow \boldsymbol{x = -1}$$

Class Example: Gaussian Elimination

E.g. 3

Use Gaussian Elimination to solve:

$$x-3y+2z=1$$
$$4x+y-5z=17$$
$$2x-3y+z=5$$

The augmented matrix $(\mathbf{A}|\mathbf{b})$ is:

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 4 & 1 & -5 & 17 \\ 2 & -3 & 1 & 5 \end{array}\right)$$

$$\Big\downarrow {\scriptstyle R_2 - 4R_1,\; R_3 - 2R_1}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 13 & -13 & 13 \\ 0 & 3 & -3 & 3 \end{array}\right)$$

$$\Big\downarrow {\scriptstyle R_2 \div 13,\; R_3 \div 3}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\Big\downarrow {\scriptstyle R_3 - R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

2 equations, 3 unknowns (infinite solutions):

Let $z = k$ (arbitrary parameter)

2nd Row: $y - k = 1 \quad \rightarrow \quad \boldsymbol{y = 1 + k}$

1st Row: $x - 3(1+k) + 2k = 1$

$$\rightarrow \quad x = 1 - 2k + 3 + 3k$$

$$\boldsymbol{x = 4 + k}$$

$$\boldsymbol{x = 4 + k}$$
$$\boldsymbol{y = 1 + k}$$
$$\boldsymbol{z = k}$$

Class Example: Applied

E.g. 4

A Chemical reaction is given by the balanced equation:

$$aO_2 + bC_6H_{12}O_6 \rightarrow cH_2O + dCO_2$$

Where a, b, c and d are coefficients to be found.
Find the simplest balanced chemical equation.

First create balances for each element:

$$\begin{aligned} O&: \quad 2a+6b = 1c+2d \quad &\Rightarrow \quad 2a+6b-1c = 2d \\ C&: \quad 0a+6b = 0c+1d \quad &\Rightarrow \quad 0a+6b+0c = 1d \\ H&: 0a+12b = 2c+0d \quad &\Rightarrow \quad 0a+12b-2c = 0d \end{aligned}$$

As we do not know either vectors $\boldsymbol{x}$ or $\mathbf{b}$, this is 3 equations in 4 unknowns (infinite solutions) so we will need to find ratios and simplify.

Now form the augmented matrix $\mathbf{A}\boldsymbol{x} = \mathbf{b} \rightarrow (\mathbf{A}|\mathbf{b})$

$$\left(\begin{array}{ccc|c} 2 & 6 & -1 & 2 \\ 0 & 6 & 0 & 1 \\ 0 & 12 & -2 & 0 \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 2 & 6 & -1 & 2 \\ 0 & 6 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

3 equations in 4 unknowns (infinite solutions) Let $d = k$

Row 3: $-2c = -2d \Rightarrow c = d = k$

Row 2: $6b = k \Rightarrow b = \frac{1}{6}k$

Row 1: $2a + 6b - c = 2k$

$$2a + k - k = 2k$$

$$2a = 2k \Rightarrow a = k$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = k\begin{pmatrix} 1 \\ 1/6 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 6 \\ 6 \end{pmatrix}$$

So we have

$$\mathbf{6O_2 + C_6H_{12}O_6 \rightarrow 6H_2O + 6CO_2}$$

Which we can check is balanced

Extension to larger systems

We were able to solve the previous problem by hand as the reaction only contained 3 elements, but what about this:

$$aFe_2SiO_4 + bMg_2SiO_4 + cH_2O + dCO_2 \rightarrow eMg_6(Si_4O_{10})(OH)_8 + fFe_2O_3 + gCH_4$$

Our balances then become:

$Fe: 2a + 0b + 0c + 0d = 0e + 2f + 0g \rightarrow 2a + 0b + 0c + 0d + 0e - 2f = 0g,$

$Si: 1a + 1b + 0c + 0d = 4e + 0f + 0g \rightarrow 1a + 1b + 0c + 0d - 4e + 0f = 0g,$

$O: 4a + 4b + 1c + 2d = 18e + 3f + 0g \rightarrow 4a + 4b + 1c + 2d - 18e - 3f = 0g,$

$Mg: 0a + 2b + 0c + 0d = 6e + 0f + 0g \rightarrow 0a + 2b + 0c + 0d - 6e + 0f = 0g,$

$H: 0a + 0b + 2c + 0d = 8e + 0f + 4g \rightarrow 0a + 0b + 2c + 0d - 8e + 0f = 4g,$

$C: 0a + 0b + 0c + 1d = 0e + 0f + 1g \rightarrow 0a + 0b + 0c + 1d + 0e + 0f = 1g.$

Which, in matrix form is:

$$\left(\begin{array}{cccccc|c} 2 & 0 & 0 & 0 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 & -4 & 0 & 0 \\ 4 & 4 & 1 & 2 & -18 & -3 & 0 \\ 0 & 2 & 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 2 & 0 & -8 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Clearly, this becomes very cumbersome, so we would likely use a program such as MATLAB to deal with matrices of this size (which we will do next year).

Exercise A: Matrix Arithmetic

Given the matrices $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \\ 0 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ -6 & -4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 5 & -4 \\ 1 & -2 \end{pmatrix}$

For 1 and 2, evaluate each of the following where possible.
If evaluation is not possible, explain why not

1

a $\mathbf{C} + \mathbf{B}$

b $\mathbf{A} - \mathbf{C}$

c $\mathbf{C} - 3\mathbf{B}$

2

a $\mathbf{BC}$

b $\mathbf{CB}$

c $\mathbf{C}^2$

d $\mathbf{AB}$

e $\mathbf{BA}$

f $\mathbf{AA}^{\mathbf{T}}$

g $\mathbf{BA}^{\mathbf{T}}$

3

a Verify that $\mathbf{BI} = \mathbf{IB} = \mathbf{B}$
Where $\mathbf{I}$ is the 2×2 identity matrix

b Verify that $(\mathbf{AC})^{\mathbf{T}} = \mathbf{C}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$

c Verify that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

4 Given the symmetric matrices

$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

a Show that $\mathbf{AB}$ is not symmetric

b Determine $\left(2\mathbf{A}^{\mathbf{T}} - \mathbf{B}\right)^{\mathbf{T}}$

c Verify that $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$

5 Evaluate:

a $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

b $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

Exercise B: Systems of Equations

Use the Gaussian Elimination method to solve the following systems of equations (if solutions exist).

1

$$x + 2y + z = 2$$
$$2x + 3y + z = 4$$
$$x + y - z = 3$$

2

$$x_1 + 3x_2 + 5x_3 = 14$$
$$2x_1 - x_2 - 3x_3 = 3$$
$$4x_1 + 5x_2 - x_3 = 7$$

3

$$2x_1 - 3x_2 + 4x_3 = 2$$
$$4x_1 + x_2 + 2x_3 = 2$$
$$x_1 - x_2 + 3x_3 = 3$$

4

$$2x_1 + x_2 - x_3 = 0$$
$$x_1 + x_3 = 4$$
$$x_1 + x_2 + x_3 = 0$$

5

$$x + y - 3z = 3$$
$$2x - 3y + 4z = -4$$
$$x - y + z = -1$$

6

$$2x + y - z = -3$$
$$3x + 2y + z = 6$$
$$x + y + 2z = 9$$

7

$$x_1 - 2x_2 - 3x_3 = -1$$
$$3x_1 + x_2 + x_3 = 4$$
$$11x_1 - x_2 - 3x_3 = 10$$

8

$$2x + y - z = -3$$
$$3x + 2y + z = 6$$
$$x + y + 2z = 8$$

9

$$x + y = 2$$
$$x + z + t = 1$$
$$2x + y + z + t = 2$$

10

$$4x_1 + 2x_2 + 3x_3 + 2x_4 = 15$$
$$8x_1 + 3x_2 - 4x_3 + 7x_4 = 7$$
$$4x_1 - 6x_2 + 2x_3 - 5x_4 = 7$$

11

A Chemical reaction is given by the balanced equation:

$$aC_8H_{18} + bO_2 \rightarrow cCO_2 + dH_2O$$

Where a, b, c and d are coefficients to be found.
Find the simplest balanced chemical equation.

Challenge Exercise

1 Evaluate: $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^k$

2 The matrix $\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ represents a rotation through an angle θ

The matrix $\mathbf{B} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ represents a rotation through an angle ϕ.

Show that both $\mathbf{AB}$ and $\mathbf{BA}$ represent a rotation through an angle $\theta + \phi$.

3 Evaluate: $\begin{pmatrix} 2 & 2 & 3 & 1 \\ 1 & 4 & 1 & 2 \\ 2 & 3 & -1 & 1 \\ -1 & 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 2 & 1 \\ 3 & 1 & -2 & 1 \\ 3 & 2 & 1 & 1 \\ 1 & 4 & 3 & -2 \end{pmatrix}$

Use the Gaussian Elimination method to solve the following systems of equations (if solutions exist).

4

$$\begin{aligned} x + y &= 1 \\ x + z + t &= 1 \\ 2x + y + z + t &= 2 \end{aligned}$$

5

$$\begin{aligned} 2x + y + z + t &= 2 \\ x + z + t &= 3 \end{aligned}$$

ANSWERS

Exercise A: Answers

1

a $\mathbf{C}+\mathbf{B}=\begin{pmatrix}8 & -2\\ -5 & -6\end{pmatrix}$

b $\mathbf{A}-\mathbf{C}$ not possible as A and C are different sizes

c $\mathbf{C}-3\mathbf{B}=\begin{pmatrix}-4 & -10\\ 19 & 10\end{pmatrix}$

2

a $\mathbf{BC}=\begin{pmatrix}3 & 2\\ -6 & -4\end{pmatrix}\begin{pmatrix}5 & -4\\ 1 & -2\end{pmatrix}$

$$=\begin{pmatrix}3\times 5+2\times 1 & 3\times -4+2\times -2\\ -6\times 5+\ -4\times 1 & -6\times -4+\ \ -4\times -2\end{pmatrix}=\begin{pmatrix}15+2 & -12-4\\ -30-4 & 24+8\end{pmatrix}=\begin{pmatrix}17 & -16\\ -34 & 32\end{pmatrix}$$

b $\mathbf{CB}=\begin{pmatrix}39 & 26\\ 15 & 10\end{pmatrix}$

c $\mathbf{C}^2=\mathbf{CC}=\begin{pmatrix}21 & -12\\ 3 & 0\end{pmatrix}$

d $\mathbf{AB}=\begin{pmatrix}15 & 10\\ -9 & -6\\ -12 & -8\end{pmatrix}$

e **BA** is not possible as B has 2 columns and A has 3 rows (number of columns of the first matrix must be equal to the number of rows of the second)

f $\mathbf{AA}^{\mathrm{T}}=\begin{pmatrix}1 & -2\\ -1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ -2 & 1 & 2\end{pmatrix}=\begin{pmatrix}5 & -3 & -4\\ -3 & 2 & 2\\ -4 & 2 & 4\end{pmatrix}$

g $\mathbf{BA}^{\mathrm{T}}=\begin{pmatrix}3 & 2\\ -6 & -4\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ -2 & 1 & 2\end{pmatrix}=\begin{pmatrix}-1 & -1 & 4\\ 2 & 2 & -8\end{pmatrix}$

Exercise A: Answers

3

a Verify that $\mathbf{BI} = \mathbf{IB} = \mathbf{B}$

Where $\mathbf{I}$ is the 2×2 identity matrix

$$\mathbf{BI} = \begin{pmatrix} 3 & 2 \\ -6 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -6 & -4 \end{pmatrix} = \mathbf{B}$$

$$\mathbf{IB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -6 & -4 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -6 & -4 \end{pmatrix} = \mathbf{B}$$

$$\therefore \mathbf{BI} = \mathbf{IB} = \mathbf{B}$$

b Verify that $(\mathbf{AC})^{\mathbf{T}} = \mathbf{C}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$

$$\mathbf{AC} = \begin{pmatrix} 3 & 0 \\ -4 & 2 \\ 2 & -4 \end{pmatrix}, (\mathbf{AC})^{\mathbf{T}} = \begin{pmatrix} 3 & -4 & 2 \\ 0 & 2 & -4 \end{pmatrix}$$

$$\mathbf{C}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}} = \begin{pmatrix} 5 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ -2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -4 & 2 \\ 0 & 2 & -4 \end{pmatrix}$$

$$\therefore (\mathbf{AC})^{\mathbf{T}} = \mathbf{C}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$$

c Verify that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

$$\mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 1 & -2 \\ -1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 17 & -16 \\ -34 & 32 \end{pmatrix} = \begin{pmatrix} 85 & -80 \\ -51 & 48 \\ -68 & 64 \end{pmatrix}$$

$$(\mathbf{AB})\mathbf{C} = \begin{pmatrix} 15 & 10 \\ -9 & -6 \\ -12 & -8 \end{pmatrix}\begin{pmatrix} 5 & -4 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 85 & -80 \\ -51 & 48 \\ -68 & 64 \end{pmatrix}$$

4

a $\mathbf{AB} = \begin{pmatrix} 3 & 2 & 3 \\ 5 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$ which is not symmetric

b

$$(2\mathbf{A}^{\mathbf{T}} - \mathbf{B})^{\mathbf{T}} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 6 & -3 \\ 2 & -3 & -1 \end{pmatrix}$$

c Verify that $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$

Method: Show both sides $= \begin{pmatrix} 3 & 5 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix}$

5

a

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (14)$$

b

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

Exercise B: Answers

1

$$\left(\begin{array}{ccc|c}1 & 2 & 1 & 2\\ 2 & 3 & 1 & 4\\ 1 & 1 & -1 & 3\end{array}\right)$$

$$\Big\downarrow \begin{array}{c}R_2 - 2R_1\\ R_3 - R_1\end{array}$$

$$\left(\begin{array}{ccc|c}1 & 2 & 1 & 2\\ 0 & -1 & -1 & 0\\ 0 & -1 & -2 & 1\end{array}\right)$$

$$\Big\downarrow R_3 - R_2$$

$$\left(\begin{array}{ccc|c}1 & 2 & 1 & 2\\ 0 & -1 & -1 & 0\\ 0 & 0 & -1 & 1\end{array}\right)$$

$$-z = 1 \Rightarrow \boldsymbol{z = -1}$$
$$-y - (-1) = 0 \Rightarrow \boldsymbol{y = 1}$$
$$x + 2(1) + 1(-1) = 2 \Rightarrow \boldsymbol{x = 1}$$

2

$x_1 = 5, x_2 = -2, x_3 = 3$

3

$x_1 = -\frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}$

Hint: First row operation is to swap rows 1 and 3

4

$x_1 = \frac{8}{3}, x_2 = -4, x_3 = \frac{4}{3}$

5

$$\left(\begin{array}{ccc|c}1 & 1 & -3 & 3\\ 2 & -3 & 4 & -4\\ 1 & -1 & 1 & -1\end{array}\right)$$

$$\Big\downarrow \begin{array}{c}R_2 - 2R_1\\ R_3 - R_1\end{array}$$

$$\left(\begin{array}{ccc|c}1 & 1 & -3 & 3\\ 0 & -5 & 10 & -10\\ 0 & -2 & 4 & -4\end{array}\right)$$

$$\Big\downarrow \begin{array}{c}R_2 \div -5\\ R_3 \div -2\end{array}$$

$$\left(\begin{array}{ccc|c}1 & 1 & -3 & 3\\ 0 & 1 & -2 & 2\\ 0 & 1 & -2 & 2\end{array}\right)$$

$$\Big\downarrow R_3 - R_2$$

$$\left(\begin{array}{ccc|c}1 & 1 & -3 & 3\\ 0 & 1 & -2 & 2\\ 0 & 0 & 0 & 0\end{array}\right)$$

2 equations, 3 unknowns (infinite solutions):

Let $z = k$ then

$$y - 2k = 2 \Rightarrow y = 2 + 2k$$
$$x + (2 + 2k) - 3k = 3 \Rightarrow x = 1 + k$$

$$\boldsymbol{x = 1 + k}, \quad \boldsymbol{y = 2 + 2k}, \quad \boldsymbol{z = k}$$

Exercise B: Answers

6

Infinite solutions:
Let $z = k$ then

$$x = 3k - 12$$
$$y = 21 - 5k$$
$$z = t$$

7

2 equations, 3 unknowns (infinite solutions):
Let $x_3 = k$ then

$$x_1 = 1 + \frac{k}{7}, \quad x_2 = 1 - \frac{10k}{7}, \quad x_3 = k$$

Or equivalently if you have set $x_1 = k$

$$x_1 = k, \quad x_2 = 11 - 10k, \quad x_3 = -7 + 7k$$

8

Equations inconsistent so no solution

9

Equations inconsistent so no solution

10

3 equations, 4 unknowns (infinite solutions):
Let $x_4 = k$ then

$$x_1 = \frac{526 - 105k}{316}, \; x_2 = \frac{67 - 73k}{79}, \; x_3 = \frac{175 + 31k}{79}, \; x_4 = k$$

11

Balances:

C: $8a + 0b = 1c + 0d \rightarrow 8a + 0b - 1c = 0d$

H: $18a + 0b = 0c + 2d \rightarrow 18a + 0b + 0c = 2d$

O: $0a + 2b = 2c + 1d \rightarrow 0a + 2b - 2c = 1d$

$$\left(\begin{array}{ccc|c} 8 & 0 & -1 & 0 \\ 18 & 0 & 0 & 2 \\ 0 & 2 & -2 & 1 \end{array}\right) \xrightarrow{R_2 \div 2} \left(\begin{array}{ccc|c} 8 & 0 & -1 & 0 \\ 9 & 0 & 0 & 1 \\ 0 & 2 & -2 & 1 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 8 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 8 & 0 & -1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right) \xrightarrow{R_2 - 8R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & -9 & -8 \\ 0 & 2 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & -9 & -8 \end{array}\right)$$

3 equations in 4 unknowns
(infinite solutions – need ratios)
Let $d = k$

3rd row gives $c = \frac{8}{9}k$

2nd row gives: $b = \frac{25}{18}k$

1st row gives: $a = \frac{1}{9}k$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = k \begin{pmatrix} 1/9 \\ 25/18 \\ 8/9 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 25 \\ 16 \\ 18 \end{pmatrix}$$

So we have

$$2C_8H_{18} + 25O_2 \rightarrow 16CO_2 + 18H_2O$$

Challenge Exercise: Answers

1

$$\begin{pmatrix}1 & a\\0 & 1\end{pmatrix}^2 = \begin{pmatrix}1 & a\\0 & 1\end{pmatrix}\begin{pmatrix}1 & a\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2a\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & a\\0 & 1\end{pmatrix}^3 = \begin{pmatrix}1 & 2a\\0 & 1\end{pmatrix}\begin{pmatrix}1 & a\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 3a\\0 & 1\end{pmatrix}$$

Following the pattern:

$$\begin{pmatrix}1 & a\\0 & 1\end{pmatrix}^k = \begin{pmatrix}1 & ka\\0 & 1\end{pmatrix}$$

2

$$\mathbf{AB} = \begin{pmatrix}\cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi\\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\theta+\phi) & \sin(\theta+\phi)\\ -\sin(\theta+\phi) & \cos(\theta+\phi)\end{pmatrix}$$

which represents a rotation through angle $\theta + \phi$ ($\mathbf{BA}$ gives same result)

3

$$\begin{pmatrix}2 & 2 & 3 & 1\\ 1 & 4 & 1 & 2\\ 2 & 3 & -1 & 1\\ -1 & 3 & 1 & 2\end{pmatrix}\begin{pmatrix}2 & 1 & 2 & 1\\ 3 & 1 & -2 & 1\\ 3 & 2 & 1 & 1\\ 1 & 4 & 3 & -2\end{pmatrix} = \begin{pmatrix}20 & 14 & 6 & 5\\ 19 & 15 & 1 & 2\\ 11 & 7 & 0 & 2\\ 12 & 12 & -1 & -1\end{pmatrix}$$

Challenge Exercise: Answers

4

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{array}\right) \xrightarrow{\substack{R_2 - R_1 \\ R_3 - 2R_1}} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - \mathrm{R}_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

2 equations in 4 unknowns so we must choose 2 arbitrarily:
Let $z = \lambda, t = \mu$
2nd row gives: $y = \lambda + \mu$
1st row gives: $x = 1 - \lambda - \mu$
$\therefore \boldsymbol{x = 1 - \lambda - \mu, y = \lambda + \mu, z = \lambda, t = \mu}$

Or, in vector form:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 - \lambda - \mu \\ \lambda + \mu \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

5

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 3 \\ 2 & 1 & 1 & 1 & 2 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & -1 & -1 & -4 \end{array}\right)$$

2 equations in 4 unknowns so we must choose 2 arbitrarily:
Let $z = \lambda, t = \mu$
2nd row gives: $y = -4 - \lambda - \mu$
1st row gives: $x = 3 - \lambda - \mu$
$\therefore \boldsymbol{x = 3 - \lambda - \mu, y = -4 - \lambda - \mu, z = \lambda, t = \mu}$

Or, in vector form:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 3 - \lambda - \mu \\ -4 - \lambda - \mu \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$