CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

University of Leeds

Level 1 Semester 2

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Tutorial: Question Difficulty Colour Code

Basic - straightforward application
(you must be able to do these)

Medium – Makes you think a bit
(you must be able to do these)

Hard – Makes you think a lot
(you should be able to do these)

Extreme – Tests your understanding to the limit!
(for those who like a challenge)

Applied – Real-life examples of the topic, may sometimes involve prior knowledge
(you should attempt these – will help in future engineering)

His method of tooth extraction is unorthodox but highly effective.

Tutorial 5

1st Order ODEs 2: Other Methods

Class Example: Integrating Factor

E.g. 1

Find the general solution of the differential equation

$$xy' + y + xy = e^{-x}, \qquad y(1) = 0$$

$$xy' + y + xy = e^{-x} \quad (y = 0 \text{ when } x = 1)$$

$$x\frac{dy}{dx} + (1+x)y = e^{-x}$$

$$\frac{dy}{dx} + \left(\frac{1}{x} + 1\right)y = \frac{e^{-x}}{x} \qquad P(x) = \frac{1}{x} + 1 \qquad Q(x) = \frac{e^{-x}}{x}$$

$$I = e^{\int \frac{1}{x} + 1\,dx} = e^{\ln x + x} = e^{\ln x} e^{x} = xe^{x}$$

$$xe^{x}\frac{dy}{dx} + xe^{x}\left(\frac{1}{x} + 1\right)y = \frac{e^{-x}}{x}xe^{x}$$

$$xe^{x}\frac{dy}{dx} + (e^{x} + xe^{x})y = 1$$

$$\frac{d}{dx}(xe^{x}y) = 1$$

$$xe^{x}y = \int 1\,dx = x + C$$

$$y = e^{-x} + \frac{C}{x}e^{-x}$$

$$y = 0 \text{ when } x = 1: \quad 0 = \frac{1}{e} + \frac{C}{e} \quad \therefore C = -1$$

$$y = e^{-x} - \frac{1}{x}e^{-x}$$

$$y = e^{-x}\left(1 - \frac{1}{x}\right)$$

Integrating Factor Method:

Standard form (divide if needed):

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$I = e^{\int P(x)\,dx}$$

- Multiply through by I:

$$I\frac{dy}{dx} + I\,P(x)y = I\,Q(x)$$

- Reverse product rule (always) gives:

$$\frac{d}{dx}(Iy) = I\,Q(x)$$

- Integrate:

$$Iy = \int I\,Q(x)dx$$

- Divide by I:

$$y = \frac{1}{I}\int I\,Q(x)dx$$

Class Example: Substitution Method (Homogeneous)

E.g. 2

Given that: $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ show that: $\tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{1}{2}\ln(x^2+y^2) + C$

HOMOGENEOUS (ALL TERMS POWER 1)

$y = xv \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{x+xv}{x-xv} = \frac{1+v}{1-v}$

$x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v}$

$= \frac{1+v-v+v^2}{1-v} = \frac{1+v^2}{1-v}$

$\therefore \int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{x}\,dx = \ln x + C$

$\int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \ln x + C$

$\int \frac{1}{1+v^2}\,dv = \tan^{-1} v$ (STANDARD INTEGRAL)

$-\int \frac{v}{1+v^2}\,dv = -\int \frac{v}{u}\frac{du}{2v} = -\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln u$

$u = 1+v^2 \qquad \frac{du}{dv} = 2v \qquad = -\frac{1}{2}\ln(1+v^2)$

$\therefore \tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln x + C$

$v = \frac{y}{x}$: $\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln x + C$

$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(\frac{x^2+y^2}{x^2}\right) = \ln x + C$

$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\left[\ln(x^2+y^2) - \ln x^2\right] = \ln x + C$

$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln(x^2+y^2) + \underbrace{\frac{1}{2}\ln x^2}_{\ln x} = \ln x + C$

$\therefore \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\ln(x^2+y^2) + C$

Substitution Method:

A Homogeneous differential equation is of the form:

$$\frac{dy}{dx} = f(x,y)$$

Where $f(x,y)$ is a homogeneous function of degree n.
That is $f(kx,ky) = k^n f(x,y)$ for any non-zero constant k.

- Check if homogeneous (if not told)
- Substitute $y = xv$
- By product rule : $\frac{dy}{dx} = v + x\frac{dv}{dx}$
- Rewrite $f(x,y) = g\left(\frac{y}{x}\right) = g(v)$
- Now $v + x\frac{dv}{dx} = g(v) \Rightarrow \int \frac{1}{g(v)-v}\,dv = \int \frac{1}{x}\,dx$ (separate variables)
- Integrate to get v, and substitute $v = \frac{y}{x}$ to get y.

Class Example: Bernoulli Equation (Non-Examinable)

E.g. 3

Solve the Differential Equation

$$y' + \frac{4}{x}y = x^3y^2, \qquad y(1) = -1$$

$$\frac{dy}{dx} + \frac{4}{x}y = x^3y^2$$

$$(\div y^2): \quad y^{-2}\frac{dy}{dx} + \frac{4}{x}y^{-1} = x^3 \quad *$$

$$w = y^{1-n} = y^{-1} \quad \therefore \frac{dw}{dx} = -y^{-2}\frac{dy}{dx}$$

$$\therefore y^{-2}\frac{dy}{dx} = -\frac{dw}{dx}$$

$$\text{sub in } *: \quad -\frac{dw}{dx} + \frac{4}{x}w = x^3$$

$$(\times -1) \quad \frac{dw}{dx} - \frac{4}{x}w = -x^3$$

(STANDARD FORM ∴ USE INTEGRATING FACTOR)

$$I = e^{-\int \frac{4}{x}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$$

$$x^{-4}\frac{dw}{dx} - 4x^{-5}w = -x^{-1}$$

$$\frac{d}{dx}\left(x^{-4}w\right) = -x^{-1}$$

$$x^{-4}w = -\int x^{-1}\,dx = -\ln x + C$$

$$w = -x^4\ln x + Cx^4$$

$$w = y^{-1}: \quad \frac{1}{y} = Cx^4 - x^4\ln x$$

$$y(1) = -1: \quad \frac{1}{-1} = C - \ln 1 \quad \therefore C = -1$$

$$\frac{1}{y} = -x^4(1 + \ln x)$$

$$\therefore y = \frac{-1}{x^4(1+\ln x)}$$

Bernoulli Equations:

A Bernoulli differential equation is a generalisation of the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Where $n \neq 0$.

- Divide by y^n (multiply by y^{-n}):
$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$
- Let $w = y^{1-n}$, then $\frac{dw}{dx} = (1-n)y^{-n}\frac{dy}{dx}$
- Rearrange: $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dw}{dx}$
- Substitute into original differential equation:
$$\frac{1}{1-n}\frac{dw}{dx} + P(x)w = Q(x)$$
- The equation is now a linear 1st order differential equation in w.
- Solve using integrating factor method.

Exercise A: Integrating factor

Solve the differential Equations by Integrating Factor method:

1 $\dfrac{dy}{dx} + 2y = e^{2x}$

2 $\dfrac{dy}{dx} - 3y = 2$

3 $xy' - 3y = x^5$

4 $\dfrac{dy}{dx} + y\cot x = \operatorname{cosec} x$

5 $xy' + (1+x)y = 5$

6 $x^2dy + (2xy - e^x)dx = 0$

7 $x^2dy + (x - 3xy + 1)dx$

8 $(x^2y - 1)dx + x^3dy = 0$

9 $x\dfrac{dy}{dx} - y = x^2 + x$ ($y = 2$ when $x = 1$)

10 $y' + 2y = e^{-3x}$ ($y = 2$ when $x = 0$)

11 $\dfrac{dy}{dx} = y\tan x - \sec x$ $(y(0) = 1)$

12 $x\dfrac{dy}{dx} + 5y = \dfrac{\ln x}{x}$, $x > 0$

13 $(y\sin x - 2)dx + \cos x\, dy = 0$

14 $y' + y\tan x = \cos^3 x$

Exercise B: Homogeneous Equations

Use the substitution $\boldsymbol{y = xv}$ to solve the differential Equations:

1 $\dfrac{dy}{dx} = \dfrac{xy + y^2}{x^2}$

2 $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x}{y}$

3 $x^2 dy + (y^2 - xy)dx = 0$

4 $2xy\dfrac{dy}{dx} = x^2 + y^2 \quad (\boldsymbol{y(1) = 0})$

5 $x\dfrac{dy}{dx} = y + xe^{\frac{y}{x}}$

6 $2x^2\dfrac{dy}{dx} = x^2 + y^2$

7 $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right)$

8 $y' = \dfrac{y}{x} - \cos\left(\dfrac{y}{x}\right)$

9 $(y^2 - xy + x^2)dx - xy\, dy = 0$

10 $\left(y\sin\left(\dfrac{y}{x}\right) + x\cos\left(\dfrac{y}{x}\right)\right)dx - x\sin\left(\dfrac{y}{x}\right)dy = 0$

11 $x\dfrac{dy}{dx} = y + \sqrt{x^2 + y^2}$

12 $y' = \dfrac{x^3}{4x^3 - 3x^2 y}$

13 $(2x - y)dx + (x + 2y)\, dy = 0$

14 $(x^2 + y^2)dx - x^2 dy = 0$

Exercise C: Bernoulli Equations (Non-Examinable)

Solve the differential Equations

1 $$\frac{dy}{dx}+\frac{1}{x}y=y^2$$

2 $$\frac{dy}{dx}-\frac{y}{x}=xy^2$$

3 $$\frac{dy}{dx}+\frac{y}{3}=e^x y^4$$

4 $$x\frac{dy}{dx}+y=xy^3$$

5 $$\frac{dy}{dx}+\frac{2y}{x}=-x^2y^2\cos x$$

6 $$2\frac{dy}{dx}+y\tan x=\frac{(4x+5)^2}{\cos x}y^3$$

7 $$x\frac{dy}{dx}+y=y^2x^2\ln x$$

8 $$\frac{dy}{dx}=y\cot x+y^3\operatorname{cosec} x$$

9 $$y'=5y+e^{-2x}y^{-2} \qquad (\boldsymbol{y}(\mathbf{0})=\mathbf{2})$$

10 $$y'+\frac{y}{x}-\sqrt{y}=0 \qquad (\boldsymbol{y}(\mathbf{1})=\mathbf{0})$$

Challenge:

(i) Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x} \qquad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

Hint: This is not homogeneous so you will need to think of another substitution

ANSWERS

Exercise A: Answers (any equivalent containing x and y)

1 $$y = \frac{1}{4}e^{2x} + Ce^{-2x}$$

2 $$y = -\frac{2}{3} + Ce^{3x}$$

3 $$y = \frac{1}{2}x^5 + Cx^3$$

4 $$y = \frac{x + C}{\sin x} = (x + C)\operatorname{cosec} x$$

5 $$y = \frac{5 + Ce^{-x}}{x}$$

6 $$y = \frac{e^x + C}{x^2}$$

7 $$y = \frac{1}{3} + \frac{1}{4x} + Cx^3$$

8 $$y = -\frac{1}{x^2} + \frac{c}{x}$$

9 $$y = x(x + \ln x + 1)$$

10 $$y = 3e^{-2x} - e^{-3x}$$

11 $$y = \frac{1 - x}{\cos x} = (1 - x)\sec x$$

12 $$y = \frac{1}{4x}\ln x - \frac{1}{16x} + \frac{C}{x^5}$$

13 $$y = 2\sin x + C\cos x$$

14 $$y = (\frac{1}{4}\sin 2x + \frac{1}{2}x + C)\cos x$$

or

$$y = \frac{1}{2}\cos^2 x\sin x + \frac{1}{2}x\cos x + C\cos x$$

Exercise B: Answers (any equivalent containing x and y)

1 $y = \dfrac{x}{C - \ln x}$

2 $y^2 = x^2(2\ln x + C)$

3 $y = \dfrac{x}{\ln x + C}$

4 $y^2 = x^2 - x$

5 $y = -x\ln(C - \ln x)$

6 $y = \dfrac{x}{C - \frac{1}{2}\ln x} + x$

7 $y = x\sin^{-1}(Cx)$

8 $\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = \frac{C}{x}$

9 $\frac{y}{x} + \ln\left(1 - \frac{y}{x}\right) = C - \ln x$ ***or*** $y + x\ln(x - y) = Cx$

10 $y = x\cos^{-1}\left(\frac{C}{x}\right)$

11 $y = x\sinh(\ln x + C)$

12 $\frac{1}{2}\ln\left(\frac{y}{x} - 1\right) - \frac{3}{2}\ln\left(\frac{3y}{x} - 1\right) = \ln x + C$

or $\ln(y - x) - 3\ln(3y - x) = C$

or $y - x = A(3y - x)^3$

13 $\tan^{-1}\left(\frac{y}{x}\right) + \ln(x^2 + y^2) = C$

14 $y = \frac{x}{2} + \frac{\sqrt{3}}{2}x\tan(\frac{\sqrt{3}}{2}\ln x + C)$

Exercise C: Answers (any equivalent containing x and y)

1 $$y = \frac{1}{x(C - \ln x)}$$

2 $$y = \frac{3x}{C - x^3}$$

3 $$y^3 = \frac{1}{e^x(C - 3x)}$$

4 $$y^2 = \frac{1}{x(2 + Cx)}$$

5 $$y = \frac{1}{x^2(\sin x + C)}$$

6 $$y^2 = \frac{12\cos x}{C - (4x + 5)^3}$$

7 $$y = \frac{1}{x^2(1 - \ln x) + Cx}$$

8 $$y^2 = \frac{\sin^2 x}{2\cos x + C}$$

9 $$y^3 = \frac{139e^{15x} - 3e^{-2x}}{17}$$

10 $$y = \frac{\left(x - x^{-\frac{1}{2}}\right)^2}{9} = \frac{1}{9}\left(x^2 - 2\sqrt{x} + \frac{1}{x}\right)$$

Challenge:

(i) Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x} \qquad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

Hint: This is not homogeneous so you will need to think of another substitution

ii) $y = x\sqrt{5x^2 - 1}$ for $x > \frac{1}{\sqrt{5}}$

iii) $y = x\sqrt{6x^2 - 2x}$ for $x > \frac{1}{3}$

Full Worked Solutions

Exercise A: Full Solutions

1

1. $\frac{dy}{dx} + 2y = e^{2x}$ $\quad P(x) = 2 \quad Q(x) = e^{2x}$

$I = e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$

$(\times I)$ $\quad e^{2x}\frac{dy}{dx} + 2ye^{2x} = e^{4x}$

(reverse product rule) $\quad \frac{d}{dx}(e^{2x}y) = e^{4x}$

$e^{2x}y = \int e^{4x} = \frac{1}{4}e^{4x} + C$

$(\div I)$ $\quad \therefore y = \frac{1}{4}e^{2x} + Ce^{-2x}$

2

$\frac{dy}{dx} - 3y = 2$ $\quad P(x) = -3, \ Q(x) = 2$

$I = e^{\int -3dx} = e^{-3x}$

$e^{-3x}\frac{dy}{dx} - 3ye^{-3x} = 2e^{-3x}$

$\frac{d}{dx}(ye^{-3x}) = 2e^{-3x}$

$ye^{-3x} = \int 2e^{-3x}dx$

$ye^{-3x} = -\frac{2}{3}e^{-3x} + C$

$\therefore y = -\frac{2}{3} + Ce^{3x}$

3

$xy' - 3y = x^5$

$x\frac{dy}{dx} - 3y = x^5$ $\quad$ MAKE INTO STANDARD FORM

$(\div x)$ $\quad \frac{dy}{dx} - \frac{3}{x}y = x^4$ $\quad P(x) = -\frac{3}{x} \quad Q(x) = x^4$

$I = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$

$x^{-3}\frac{dy}{dx} - 3x^{-4} = x$

$\frac{d}{dx}(x^{-3}y) = x$

$x^{-3}y = \int x\,dx = \frac{1}{2}x^2 + C$

$\therefore y = \frac{1}{2}x^5 + Cx^3$

4

$\frac{dy}{dx} + y\cot x = \operatorname{cosec} x$ $\quad P(x) = \cot x \quad Q(x) = \operatorname{cosec} x$

$I = e^{\int \cot x\,dx}$

$\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx$ $\quad u = \sin x \quad \frac{du}{dx} = \cos x \ \therefore dx = \frac{du}{\cos x}$

$= \int \frac{\cos x}{u}\frac{du}{\cos x} = \int \frac{1}{u}du = \ln u \ (+C)$ not needed for I.

$= \ln(\sin x)$ $\quad$ or use integrals table

$\therefore I = e^{\ln(\sin x)} = \sin x$

$\sin x\frac{dy}{dx} + (\cot x \sin x)y = \operatorname{cosec} x \sin x$

$\sin x\frac{dy}{dx} + \left(\frac{\cos x}{\sin x}\sin x\right)y = \frac{1}{\sin x}\sin x$

$\sin x\frac{dy}{dx} + y\cos x = 1$

$\frac{d}{dx}(y\sin x) = 1$

$y\sin x = \int 1\,dx = x + C$

$\therefore y = \frac{x+C}{\sin x}$

(or $y = (x+C)\operatorname{cosec} x$)

Exercise A: Full Solutions

5

$$xy' + (1+x)y = 5$$

$$\frac{dy}{dx} + \left(\frac{1+x}{x}\right)y = \frac{5}{x} \qquad P(x) = \frac{1+x}{x} = \frac{1}{x} + 1 \qquad Q(x) = \frac{5}{x}$$

$$I = e^{\int \frac{1}{x} + 1\,dx} = e^{\ln x + x} = e^{\ln x}e^{x} = xe^{x}$$

$$xe^{x}\frac{dy}{dx} + \left(\frac{1+x}{\cancel{x}}\right)\cancel{x}e^{x}y = \frac{5}{\cancel{x}}\cancel{x}e^{x}$$

$$xe^{x}\frac{dy}{dx} + (e^{x} + xe^{x})y = 5e^{x}$$

$$\frac{d}{dx}(xe^{x}y) = 5e^{x}$$

$$xe^{x}y = \int 5e^{x}\,dx = 5e^{x} + c$$

$$\therefore\ y = \frac{5}{x} + \frac{C}{xe^{x}}$$

$$y = \frac{5 + Ce^{-x}}{x}$$

6

$$x^2 dy + (2xy - e^{x})dx = 0$$

$$x^2\frac{dy}{dx} + 2xy = e^{x}$$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{e^{x}}{x^2} \qquad P(x) = \frac{2}{x} \qquad Q(x) = \frac{e^{x}}{x^2}$$

$$I = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$x^2\frac{dy}{dx} + 2xy = e^{x}$$

$$\frac{d}{dx}(x^2 y) = e^{x}$$

$$x^2 y = \int e^{x}\,dx = e^{x} + C$$

$$\therefore\ y = \frac{e^{x} + C}{x^2}$$

Exercise A: Full Solutions

7

$$x^2 dy + (x - 3xy + 1)dx = 0$$

$$x^2\frac{dy}{dx} + x - 3xy + 1 = 0$$

$$x^2\frac{dy}{dx} - 3xy = -x - 1$$

$$\frac{dy}{dx} - \frac{3}{x}y = -\frac{1}{x} - \frac{1}{x^2} \qquad P(x) = -\frac{3}{x} \qquad Q(x) = -\frac{1}{x} - \frac{1}{x^2}$$

$$I = e^{-\int \frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$$

$$x^{-3}\frac{dy}{dx} - 3x^{-4}y = -x^{-4} - x^{-5}$$

$$\frac{d}{dx}(x^{-3}y) = -x^{-4} - x^{-5}$$

$$x^{-3}y = \int -x^{-4} - x^{-5}\,dx = \frac{1}{3}x^{-3} + \frac{1}{4}x^{-4} + C$$

$$\therefore y = \frac{1}{3} + \frac{1}{4x} + Cx^3$$

8

$$(x^2y - 1)dx + x^3 dy = 0$$

$$x^2y - 1 + x^3\frac{dy}{dx} = 0$$

$$x^3\frac{dy}{dx} + x^2y = 1$$

$$\frac{dy}{dx} + x^{-1}y = x^{-3} \qquad P(x) = x^{-1} \qquad Q(x) = x^{-3}$$

$$I = e^{\int x^{-1}dx} = e^{\ln x} = x$$

$$x\frac{dy}{dx} + y = x^{-2}$$

$$\frac{d}{dx}(xy) = x^{-2}$$

$$xy = \int x^{-2}\,dx = -x^{-1} + C$$

$$\therefore y = -\frac{1}{x^2} + \frac{C}{x}$$

Exercise A: Full Solutions

9

$$x(y') - y = x^2 + x \quad (y = 2 \text{ when } x = 1)$$

$$x\frac{dy}{dx} - y = x^2 + x$$

$$\frac{dy}{dx} - \frac{1}{x}y = x + 1 \qquad P(x) = -\frac{1}{x} \qquad Q(x) = x + 1$$

$$I = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$x^{-1}\frac{dy}{dx} - x^{-2}y = 1 + x^{-1}$$

$$\frac{d}{dx}(x^{-1}y) = 1 + x^{-1}$$

$$\frac{y}{x} = \int 1 + \frac{1}{x}\, dx = x + \ln x + C$$

$$y = x^2 + x\ln x + Cx$$

$y = 2$ when $x = 1$: $\quad 2 = 1 + \cancel{\ln 1} + C \quad \therefore C = 1$

$$y = x^2 + x\ln x + x$$

or $\quad y = x(x + \ln x + 1)$

10

$$y' + 2y = e^{-3x} \quad (y = 2 \text{ when } x = 0)$$

$$\frac{dy}{dx} + 2y = e^{-3x} \qquad P(x) = 2, \quad Q(x) = e^{-3x}$$

$$I = e^{\int 2\,dx} = e^{2x}$$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{-x}$$

$$\frac{d}{dx}(e^{2x}y) = e^{-x}$$

$$e^{2x}y = \int e^{-x}\,dx = -e^{-x} + C$$

$$y = -e^{-3x} + Ce^{-2x}$$

$y = 2$ when $x = 0$: $\quad 2 = -1 + C \quad \therefore C = 3$

$$y = -e^{-3x} + 3e^{-2x}$$

Exercise A: Full Solutions

11

$$\frac{dy}{dx} = y\tan x - \sec x \qquad y(0) = 1$$

$$\frac{dy}{dx} - y\tan x = -\sec x \qquad P(x) = -\tan x, \quad Q(x) = -\sec x$$

$$I = e^{-\int \tan x\,dx}$$

$$-\int \tan x\,dx = -\int \frac{\sin x}{\cos x}\,dx \qquad u = \cos x \qquad \frac{du}{dx} = -\sin x \qquad dx = \frac{-du}{\sin x}$$

$$= -\int \frac{\sin x}{u}\left(\frac{-du}{\sin x}\right) = \int \frac{1}{u}\,du = \ln u\,(+C)$$ ← not needed for I

$$= \ln(\cos x)$$

(or from integrals table $\int \tan x\,dx = \ln(\sec x)$

$\therefore -\int \tan x\,dx = -\ln(\sec x) = \ln(\cos x)$)

$$\therefore I = e^{\ln(\cos x)} = \underline{\cos x}$$

$$\cos x\frac{dy}{dx} - (\cos x\tan x)y = -\sec x\cos x$$

$$\cos x\frac{dy}{dx} - (\sin x)y = -1$$

$$\frac{d}{dx}(y\cos x) = -1$$

$$y\cos x = \int -1\,dx = -x + C$$

$y(0) = 1$: $1\cos 0 = 0 + C \quad \therefore C = 1$

$$y\cos x = -x + 1$$

$$\therefore y = \frac{1-x}{\cos x}$$

12

$$\frac{dy}{dx} + \frac{5}{x}y = \frac{\ln x}{x^2}$$

$$I = e^{\int \frac{5}{x}dx} = e^{5\ln x} = e^{\ln x^5} = x^5$$

$$x^5\frac{dy}{dx} + 5x^4y = x^3\ln x$$

$$\frac{d}{dx}(x^5y) = x^3\ln x$$

$$x^5y = \int x^3\ln x\,dx$$

$$u = \ln x \qquad \frac{dv}{dx} = x^3$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{x^4}{4}$$

$$\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \int \frac{x^3}{4}\,dx$$

$$x^5y = \frac{1}{4}x^4\ln x - \frac{x^4}{16} + C$$

$$\boldsymbol{y = \frac{1}{4x}\ln x - \frac{1}{16x} + \frac{C}{x^5}}$$

Exercise A: Full Solutions

13

$(y\sin x - 2)dx + \cos x\, dy = 0$

$y\sin x - 2 + \cos x \frac{dy}{dx} = 0$

$\cos x \frac{dy}{dx} + (\sin x)y = 2$

$\frac{dy}{dx} + (\tan x)y = \frac{2}{\cos x}$ $\quad P(x) = \tan x \quad Q(x) = \frac{2}{\cos x}$

$\int \tan x\, dx = \ln(\sec x)$

$I = e^{\int \tan x\, dx} = e^{\ln(\sec x)} = \sec x$ $\quad$ (see 12 for proof)

$\sec x \frac{dy}{dx} + (\sec x \tan x)y = 2\sec^2 x$

$\frac{d}{dx}(y\sec x) = 2\sec^2 x$ $\quad \frac{d}{dx}\tan x = \sec^2 x$

$y\sec x = 2\int \sec^2 x\, dx$ $\quad \therefore \int \sec^2 x\, dx = \tan x$

$y\sec x = 2\tan x + C$

$y = \frac{2\tan x}{\sec x} + \frac{C}{\sec x}$ $\quad \left(\frac{1}{\sec x} = \cos x\right)$

$y = 2\frac{\sin x}{\cos x}\cos x + C\cos x$

$\underline{y = 2\sin x + C\cos x}$

14

$y' + y\tan x = \cos^3 x$

$\frac{dy}{dx} + (\tan x)y = \cos^3 x$ $\quad P(x) = \tan x \quad Q(x) = \cos^3 x$

$I = e^{\int \tan x\, dx} = e^{\ln(\sec x)} = \sec x$

$\sec x \frac{dy}{dx} + (\sec x \tan x)y =$

$\frac{d}{dx}(y\sec x) = \cos^2 x$

$y\sec x = \int \cos^2 x\, dx$ $\quad \cos 2x = 2\cos^2 x - 1$

$\therefore \cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$y\sec x = \frac{1}{2}\int \cos 2x + 1 = \frac{1}{2}\left(\frac{1}{2}\sin 2x + x + C_1\right)$

$y\sec x = \frac{1}{4}\sin 2x + \frac{1}{2}x + C$ $\quad (C = \frac{1}{2}C_1)$

$y = \frac{1}{4}\sin 2x\cos x + \frac{1}{2}x\cos x + C\cos x$

$\therefore \underline{y = \left(\frac{1}{4}\sin 2x + \frac{1}{2}x + C\right)\cos x}$

or as $\sin 2x = 2\sin x\cos x$

$y = \frac{1}{4}(2\sin x\cos x)\cos x + \frac{1}{2}x\cos x + C\cos x$

$\underline{y = \frac{1}{2}\cos^2 x\sin x + \frac{1}{2}x\cos x + C\cos x}$

all in single angles

Exercise B: Full Solutions

1

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2}$$ HOMOGENEOUS AS ALL TERMS HAVE POWER 2

$y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2v + x^2v^2}{x^2} = v + v^2$$

$$x\frac{dv}{dx} = v^2$$

$$\int v^{-2}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{v} = \ln x + C_1$$

$$\frac{1}{v} = -\ln x + c \qquad (c = -C_1)$$

$$v = \frac{1}{c - \ln x}$$

$v = \frac{y}{x}$: $\quad \frac{y}{x} = \frac{1}{c - \ln x}$

$$\therefore y = \frac{x}{c - \ln x}$$

2

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$$ $\left(\frac{ky}{kx} + \frac{kx}{ky} = \frac{y}{x} + \frac{x}{y} \quad \therefore \text{HOMOGENEOUS}\right)$

$y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{xv}{x} + \frac{x}{xv} = v + \frac{1}{v}$$

$$x\frac{dv}{dx} = \frac{1}{v}$$

$$\int v\,dv = \int \frac{1}{x}\,dx$$

$$\frac{v^2}{2} = \ln x + C_1$$

$$v^2 = 2\ln x + C \qquad (C = 2C_1)$$

$v = \frac{y}{x}$: $\quad \frac{y^2}{x^2} = 2\ln x + C$

$$y^2 = x^2(2\ln x + C)$$

$$\left(\text{or } y = \pm x\sqrt{2\ln x + C}\right)$$

Exercise B: Full Solutions

3

$$x^2\,dy + (y^2 - xy)\,dx = 0$$
$$x^2\frac{dy}{dx} + y^2 - xy = 0$$
$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$$ HOMOGENEOUS (ALL TERMS HAVE POWER OF 2)

$y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2 v - x^2 v^2}{x^2} = v - v^2$$
$$x\frac{dv}{dx} = -v^2$$
$$\int v^{-2}\,dv = -\int x^{-1}\,dx$$
$$-v^{-1} = -\ln x + C_1$$
$$\frac{1}{v} = \ln x + C \quad (C = -C_1)$$
$$v = \frac{1}{\ln x + C}$$

$v = \frac{y}{x}$: $\quad \frac{y}{x} = \frac{1}{\ln x + C}$

$$\therefore\ y = \frac{x}{\ln x + C}$$

4

$$2xy\frac{dy}{dx} = x^2 + y^2 \quad (y = 0 \text{ at } x = 1)$$
$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$ HOMOGENEOUS (ALL TERMS HAVE POWER OF 2)

$y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2 + x^2 v^2}{2x^2 v} = \frac{1 + v^2}{2v}$$
$$x\frac{dv}{dx} = \frac{1 + v^2}{2v} - v = \frac{1 + v^2}{2v} - \frac{2v^2}{2v} = \frac{1 - v^2}{2v}$$
$$x\frac{dv}{dx} = \frac{1 - v^2}{2v}$$
$$\int \frac{2v}{1 - v^2}\,dv = \int \frac{1}{x}\,dx = \ln x + C_1$$

$u = 1 - v^2 \quad \frac{du}{dv} = -2v$

$$LHS = \int \frac{2v}{u}\left(-\frac{du}{2v}\right) = -\int \frac{1}{u}\,du = -\ln u = -\ln(1 - v^2)$$

$RHS = \ln x + C_1$

$$\therefore\ -\ln(1 - v^2) = \ln x + C_1$$
$$\ln(1 - v^2) = -\ln x + C_2 \quad (C_2 = -C_1)$$
$$1 - v^2 = e^{-\ln x + C_2} = e^{-\ln x}e^{C_2} = e^{\ln x^{-1}}e^{C_2}$$
$$= Ax^{-1}$$
$$1 - v^2 = \frac{A}{x}$$
$$v^2 = 1 - \frac{A}{x}$$

$v = \frac{y}{x}$: $\quad \frac{y^2}{x^2} = 1 - \frac{A}{x}$

$$y^2 = x^2 - Ax$$

$y = 0$ at $x = 1$: $\quad 0 = 1 - A \rightarrow A = 1$

$$\therefore\ y^2 = x^2 - x$$

Exercise B: Full Solutions

5

$$x\frac{dy}{dx} = y + xe^{\frac{y}{x}}$$

$$\frac{dy}{dx} = \frac{y}{x} + e^{\frac{y}{x}}$$

$$f(kx, ky) = \frac{ky}{kx} + e^{\frac{ky}{kx}} = \frac{y}{x} + e^{\frac{y}{x}} = f(x,y) \therefore \text{HOMOGENEOUS}$$

$$y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v + e^{v}$$

$$x\frac{dv}{dx} = e^{v}$$

$$\int e^{-v}\,dv = \int \frac{1}{x}\,dx$$

$$-e^{-v} = \ln x + c_1$$

$$e^{-v} = c - \ln x \qquad (c = -c_1)$$

$$-v = \ln(c - \ln x)$$

$$v = -\ln(c - \ln x)$$

$$v = \frac{y}{x}: \quad \frac{y}{x} = -\ln(c - \ln x)$$

$$\therefore y = -x\ln(c - \ln x)$$

6

$$2x^2\frac{dy}{dx} = x^2 + y^2$$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2} \qquad \text{HOMOGENEOUS (ALL TERMS POWER OF 2)}$$

$$y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 + x^2v^2}{2x^2} = \frac{1 + v^2}{2}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{2} - v = \frac{1 + v^2 - 2v}{2}$$

$$x\frac{dv}{dx} = \frac{(v-1)^2}{2}$$

$$\int (v-1)^{-2}\,dv = \frac{1}{2}\int \frac{1}{x}\,dx$$

$$-(v-1)^{-1} = \frac{1}{2}\ln x + c_1$$

$$\frac{1}{v-1} = -\frac{1}{2}\ln x + c \qquad (c = -c_1)$$

$$v - 1 = \frac{1}{c - \frac{1}{2}\ln x}$$

$$v = \frac{1}{c - \frac{1}{2}\ln x} + 1$$

$$v = \frac{y}{x}: \quad \frac{y}{x} = \frac{1}{c - \frac{1}{2}\ln x} + 1$$

$$\therefore y = \frac{x}{c - \frac{1}{2}\ln x} + x$$

$$\left(\text{or } \quad y = \frac{2x}{A - \ln x} + x \qquad (A = 2c)\right)$$

Exercise B: Full Solutions

7

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \qquad f(kx,ky) = \frac{\cancel{k}y}{\cancel{k}x} + \tan\left(\frac{\cancel{k}y}{\cancel{k}x}\right) = f(x,y)$$

$\therefore$ HOMOGENEOUS

$$y = xv \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v + \tan v$$

$$\int \frac{1}{\tan v}\,dv = \int \frac{1}{x}\,dx = \ln x + C$$

$$LHS = \int \frac{\cos v}{\sin v}\,dv \qquad u = \sin v \qquad \frac{du}{dv} = \cos v \qquad dv = \frac{du}{\cos v}$$

$$= \int \frac{\cancel{\cos v}}{u}\,\frac{du}{\cancel{\cos v}} = \int \frac{1}{u}\,du = \ln u = \ln(\sin v)$$

$$\therefore \ln(\sin v) = \ln x + C$$

$$\sin v = e^{\ln x + C} = e^{\ln x}e^{C} = Ax$$

$$v = \sin^{-1}(Ax)$$

$$v = \frac{y}{x}: \quad \frac{y}{x} = \sin^{-1}(Ax)$$

$$\therefore \underline{\underline{y = x\sin^{-1}(Ax)}}$$

8

$$y' = \frac{y}{x} - \cos\frac{y}{x} \qquad \left(f(kx,ky) = \frac{ky}{kx} - \cos\frac{ky}{kx} = f(x,y) \quad \therefore \text{HOMOGENEOUS} \right)$$

$$\frac{dy}{dx} = \frac{y}{x} - \cos\frac{y}{x}$$

$$y = xv \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v - \cos v$$

$$x\frac{dv}{dx} = -\cos v$$

$$\int \frac{1}{\cos v}\,dv = -\int \frac{1}{x}\,dx$$

$$\int \sec v\,dv = -\int \frac{1}{x}\,dx$$

$$\ln(\sec v + \tan v) = -\ln x + C$$

(STANDARD INTEGRAL)

$$\sec v + \tan v = e^{-\ln x + C} = e^{\ln x^{-1}}e^{C} = \frac{A}{x}$$

$$\sec v + \tan v = \frac{A}{x}$$

$$v = \frac{y}{x}: \quad \underline{\underline{\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = \frac{A}{x}}}$$

Exercise B: Full Solutions

9

$$(y^2 - xy + x^2)\,dx - xy\,dy = 0$$

$$y^2 - xy + x^2 = xy\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^2 - xy + x^2}{xy}$$

(HOMOGENEOUS: ALL TERMS POWER 2)

$$y = xv \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2v^2 - x^2v + x^2}{x^2 v} = v - 1 + \frac{1}{v}$$

$$x\frac{dv}{dx} = \frac{1}{v} - 1 = \frac{1-v}{v}$$

$$\int \frac{v}{1-v}\,dv = \int \frac{1}{x}\,dx = \ln x + c_1$$

$u = 1 - v \quad \frac{du}{dv} = -1$

$v = 1 - u \quad dv = -du$

$$\int \frac{v}{1-v}\,dv = \int \frac{1-u}{u}(-du) = \int \frac{u-1}{u}\,du$$

$$= \int 1 - \frac{1}{u}\,du = u - \ln u$$

$$= 1 - v - \ln(1-v)$$

$$\therefore\ 1 - v - \ln(1-v) = \ln x + c_1$$

$$v + \ln(1-v) = (1 - c_1) - \ln x$$

$$v + \ln(1-v) = c - \ln x \qquad (c = 1 - c_1)$$

$$v = \frac{y}{x}: \quad \frac{y}{x} + \ln\left(1 - \frac{y}{x}\right) = c - \ln x$$

(THIS IS SOLVED, BUT WE CAN GET RID OF $\ln x$ USING LOG LAWS:

$$\ln\left(1 - \frac{y}{x}\right) = \ln\left(\frac{x-y}{x}\right) = \ln(x-y) - \ln x \text{)}$$

$$\therefore\ \frac{y}{x} + \ln(x-y) - \cancel{\ln x} = c - \cancel{\ln x}$$

$$y + x\ln(x-y) = cx$$

10

$$\left(y\sin\frac{y}{x} + x\cos\frac{y}{x}\right)dx - x\sin\frac{y}{x}\,dy = 0$$

$$y\sin\frac{y}{x} + x\cos\frac{y}{x} = x\sin\frac{y}{x}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y}{x} + \cot\frac{y}{x}$$

(HOMOGENEOUS $\frac{ky}{kx} + \cot\frac{ky}{kx} = \frac{y}{x} + \cot\frac{y}{x}$ SAME)

$$y = xv \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v + \cot v$$

$$x\frac{dv}{dx} = \cot v$$

$$\int \tan v\,dv = \int \frac{1}{x}\,dx$$

$$\ln(\sec v) = \ln x + c$$

(STANDARD INTEGRAL)

$$\sec v = e^{\ln x + c} = Ax$$

$$\frac{1}{\cos v} = Ax$$

$$\cos v = \frac{1}{Ax}$$

$$v = \frac{y}{x}: \quad \cos\left(\frac{y}{x}\right) = \frac{B}{x} \qquad \left(B = \frac{1}{A}\right)$$

$$\frac{y}{x} = \cos^{-1}\left(\frac{B}{x}\right)$$

$$\therefore\ y = x\cos^{-1}\left(\frac{B}{x}\right)$$

Exercise B: Full Solutions

11

$$x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$$

$$y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$f(x,y) = \frac{y+\sqrt{x^2+y^2}}{x}$$

$$f(kx,ky) = \frac{ky+\sqrt{k^2x^2+k^2y^2}}{kx} = \frac{ky+k\sqrt{x^2+y^2}}{kx} = f(x,y)$$

$\therefore$ HOMOGENEOUS.

$$v + x\frac{dv}{dx} = \frac{xv+\sqrt{x^2+x^2v^2}}{x} = \frac{xv + x\sqrt{1+v^2}}{x}$$

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\int \frac{1}{\sqrt{1+v^2}}\,dv = \int \frac{1}{x}\,dx \qquad \left(\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a} + c\right)$$

$$\sinh^{-1} v = \ln x + C$$

$$v = \sinh(\ln x + C)$$

$$v = \frac{y}{x}: \quad \frac{y}{x} = \sinh(\ln x + C)$$

$$y = x\sinh(\ln x + C)$$

12

HOMOGENEOUS (ALL TERMS POWER 3)

$$y' = \frac{x^3}{4x^3-3x^2y} = \frac{dy}{dx}$$

$$y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^3}{4x^3-3x^3v} = \frac{1}{4-3v}$$

$$x\frac{dv}{dx} = \frac{1}{4-3v} - v = \frac{1-v(4-3v)}{4-3v} = \frac{3v^2-4v+1}{4-3v}$$

$$\int \frac{4-3v}{(3v-1)(v-1)}\,dv = \int \frac{1}{x}\,dx = \ln x + C_1$$

PARTIAL FRACTIONS

$$\frac{4-3v}{(3v-1)(v-1)} \equiv \frac{A}{3v-1} + \frac{B}{v-1} \quad \therefore 4-3v \equiv A(v-1) + B(3v-1)$$

$$v = 1: \ 1 = 2B \rightarrow B = \tfrac{1}{2}$$

$$v = \tfrac{1}{3}: \ 3 = -\tfrac{2}{3}A \rightarrow A = -\tfrac{9}{2}$$

$$\int \frac{4-3v}{(3v-1)(v-1)}\,dv = -\frac{9}{2}\int \frac{1}{3v-1}\,dv + \frac{1}{2}\int \frac{1}{v-1}\,dv$$

$$= -\tfrac{9}{2}\ln(3v-1) \div 3 + \tfrac{1}{2}\ln(v-1)$$

$$\therefore -\tfrac{3}{2}\ln(3v-1) + \tfrac{1}{2}\ln(v-1) = \ln x + C_1$$

$$-3\ln(3v-1) + \ln(v-1) = 2\ln x + C \qquad (C = 2C_1)$$

$$v = \frac{y}{x}: \quad \ln\left(\frac{y}{x}-1\right) - 3\ln\left(\frac{3y}{x}-1\right) = 2\ln x + C$$

OK HERE (FULL MARKS)

CAN SIMPLIFY LOGS:

$$\ln\left(\frac{y-x}{x}\right) - 3\ln\left(\frac{3y-x}{x}\right) = 2\ln x + C$$

$$\ln(y-x) - \ln x - 3[\ln(3y-x) - \ln x] = 2\ln x + C$$

$$\ln(y-x) - 3\ln(3y-x) - \ln x + 3\ln x = 2\ln x + C$$

$$\ln(y-x) - 3\ln(3y-x) = C$$

or:

$$\ln\frac{y-x}{(3y-x)^3} = C$$

$$\frac{y-x}{(3y-x)^3} = e^C = A$$

$$y - x = A(3y-x)^3$$

ANY CORRECT REARRANGEMENT

Exercise B: Full Solutions

13

$$(2x-y)dx+(x+2y)dy=0$$

$$2x-y+(x+2y)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{y-2x}{x+2y}$$

HOMOGENEOUS (ALL TERMS POWER

$$y=xv \qquad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=\frac{xv-2x}{x+2xv}=\frac{x(v-2)}{x(1+2v)}=\frac{v-2}{1+2v}$$

$$x\frac{dv}{dx}=\frac{v-2}{1+2v}-v=\frac{v-2-v(1+2v)}{1+2v}$$

$$=\frac{v-2-v-2v^2}{1+2v}=\frac{-2(v^2+1)}{1+2v}$$

$$\therefore \frac{1}{2}\int\frac{1+2v}{v^2+1}\,dv=-\int\frac{1}{x}\,dx$$

$$=-\ln x+C_1$$

$$\therefore \int\frac{1+2v}{v^2+1}\,dv=-2\ln x+C \quad (C=2C_1)$$

$$\int\frac{1+2v}{v^2+1}\,dv=\int\frac{1}{v^2+1}\,dv+\int\frac{2v}{v^2+1}\,dv$$

$$\int\frac{1}{v^2+1}\,dv=\tan^{-1}(v)$$ STANDARD INTEGRAL

$$\int\frac{2v}{v^2+1}\,dv=\ln(v^2+1)$$ SUB $u=v^2+1$ OR reverse chain rule

$$\therefore \tan^{-1}v+\ln(v^2+1)=-2\ln x+C$$

$v=\frac{y}{x}$: $$\tan^{-1}\left(\frac{y}{x}\right)+\ln\left(\frac{y^2}{x^2}+1\right)=-2\ln x+C$$ (fine here but can simplify logs)

$$\tan^{-1}\left(\frac{y}{x}\right)+\ln\left(\frac{y^2+x^2}{x^2}\right)=-2\ln x+C$$

$$\tan^{-1}\left(\frac{y}{x}\right)+\ln(y^2+x^2)-\ln x^2=-2\ln x+C$$

$$\tan^{-1}\left(\frac{y}{x}\right)+\ln(y^2+x^2)-\cancel{2\ln x}=-\cancel{2\ln x}+C$$

$$\therefore \tan^{-1}\left(\frac{y}{x}\right)+\ln(y^2+x^2)=C$$

14

$$(x^2+y^2)dx-x^2dy=0$$

$$x^2+y^2=x^2\frac{dy}{dx}$$

$$\frac{dy}{dx}=1+\frac{y^2}{x^2}$$

(HOMOGENEOUS $\because 1+\frac{(ky)^2}{(kx)^2}=1+\frac{y^2}{x^2}$)

$$y=xv \qquad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=1+v^2$$

$$x\frac{dv}{dx}=v^2-v+1$$

$$\therefore \int\frac{1}{v^2-v+1}\,dv=\int\frac{1}{x}\,dx=\ln x+C_1$$

COMPLETE THE SQUARE

$$\int\frac{1}{v^2-v+1}\,dv=\int\frac{1}{\left(v-\frac{1}{2}\right)^2+1-\frac{1}{4}}\,dv=\int\frac{1}{\left(v-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$=\int\frac{1}{\left(v-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dv$$

$$\left(\int\frac{1}{x^2+a^2}\,dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+C\right)$$

$x=v-\frac{1}{2}$ $\quad a=\frac{\sqrt{3}}{2}$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(v-\frac{1}{2}\right)\right]=\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{1}{\sqrt{3}}(2v-1)\right]$$

$$\therefore \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{1}{\sqrt{3}}(2v-1)\right]=\ln x+C_1$$

$$\tan^{-1}\left[\frac{1}{\sqrt{3}}(2v-1)\right]=\frac{\sqrt{3}}{2}\ln x+C \quad \left(C=\frac{\sqrt{3}}{2}C_1\right)$$

$$\frac{1}{\sqrt{3}}(2v-1)=\tan\left(\frac{\sqrt{3}}{2}\ln x+C\right)$$

$$v=\frac{1}{2}+\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}\ln x+C\right)$$

$v=\frac{y}{x}$: $$\frac{y}{x}=\frac{1}{2}+\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}\ln x+C\right)$$

$$\therefore y=\frac{x}{2}+\frac{\sqrt{3}x}{2}\tan\left(\frac{\sqrt{3}}{2}\ln x+C\right)$$

Exercise C: Full Solutions

1

$$\frac{dy}{dx} + \frac{y}{x} = y^2 \qquad n=2$$

$\div y^n$ $\quad \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 1 \quad ⊛$

let $w = y^{1-n} = y^{-1} \quad \therefore \frac{dw}{dx} = -y^{-2}\frac{dy}{dx}$

$\therefore \frac{1}{y^2}\frac{dy}{dx} = -\frac{dw}{dx}$

Sub in ⊛ $\quad -\frac{dw}{dx} + \frac{w}{x} = 1$

$(\times -1)$ $\quad \frac{dw}{dx} - \frac{1}{x}w = -1$ Now in Standard form (use INTEGRATING FACTOR).

$$I = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$x^{-1}\frac{dw}{dx} - x^{-2}w = -x^{-1}$$

$$\frac{d}{dx}(x^{-1}w) = -x^{-1}$$

$$x^{-1}w = -\int x^{-1}dx = -\ln x + C$$

$$w = -x\ln x + Cx$$

$w = y^{-1}$: $\quad \frac{1}{y} = -x\ln x + Cx$

$$y = \frac{1}{x(C - \ln x)}$$

2

$$2. \quad \frac{dy}{dx} - \frac{1}{x}y = xy^2 \qquad n=2$$

$\div y^n$: $\quad \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{xy} = x \quad ⊛$

let $w = y^{1-n} = y^{-1} \quad \frac{dw}{dx} = -y^{-2}\frac{dy}{dx}$

$\therefore \frac{1}{y^2}\frac{dy}{dx} = -\frac{dw}{dx}$

Sub in ⊛: $\quad -\frac{dw}{dx} - \frac{1}{x}w = x$

$(\times -1)$ $\quad \frac{dw}{dx} + \frac{1}{x}w = -x$ (STANDARD FORM)

$$I = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$x\frac{dw}{dx} + w = -x^2$$

$$\frac{d}{dx}(xw) = -x^2$$

$$xw = \int -x^2\,dx$$

$$xw = -\frac{x^3}{3} + C_1$$

$$w = -\frac{x^2}{3} + \frac{C_1}{x}$$

$w = \frac{1}{y}$ $\quad \frac{1}{y} = -\frac{x^2}{3} + \frac{C_1}{x} = \frac{-x^3 + 3C_1}{3x} = \frac{C - x^3}{3x}$ $\quad (C = 3C_1)$

$$y = \frac{3x}{C - x^3}$$

Exercise C: Full Solutions

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3. $\frac{dy}{dx} + \frac{1}{3}y = e^x y^4 \qquad n=4$

$\div y^n \quad \frac{1}{y^4}\frac{dy}{dx} + \frac{1}{3}\frac{1}{y^3} = e^x \quad (*)$

let $w = y^{1-n} = y^{-3} \quad \therefore \frac{dw}{dx} = -3y^{-4}\frac{dy}{dx}$

$\therefore \frac{1}{y^4}\frac{dy}{dx} = -\frac{1}{3}\frac{dw}{dx}$

Sub in $(*)$: $-\frac{1}{3}\frac{dw}{dx} + \frac{1}{3}w = e^x$

$(\times -3) \quad \frac{dw}{dx} - w = -3e^x \quad$ (STANDARD FORM)

$I = e^{\int -1\,dx} = e^{-x}$

$e^{-x}\frac{dw}{dx} - e^{-x}w = -3$

$\frac{d}{dx}(e^{-x}w) = -3$

$e^{-x}w = \int -3\,dx$

$e^{-x}w = -3x + C$

$w = e^x(C - 3x)$

$w = y^{-3} \qquad \frac{1}{y^3} = e^x(C-3x)$

$y^3 = \frac{1}{e^x(C-3x)}$

4

4. $x\frac{dy}{dx} + y = xy^3$

$\frac{dy}{dx} + \frac{1}{x}y = y^3 \qquad n=3$

$\div y^n \quad y^{-3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = 1 \quad (*)$

let $w = y^{1-n} = y^{-2} \quad \therefore \frac{dw}{dx} = -2y^{-3}\frac{dy}{dx}$

$\therefore y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dw}{dx}$

Sub in $(*)$: $-\frac{1}{2}\frac{dw}{dx} + \frac{1}{x}w = 1$

$(\times -2) \quad \frac{dw}{dx} - \frac{2}{x}w = -2 \quad$ (STANDARD FORM)

$I = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$

$x^{-2}\frac{dw}{dx} - 2x^{-3}w = -2x^{-2}$

$\frac{d}{dx}(x^{-2}w) = -2x^{-2}$

$x^{-2}w = \int -2x^{-2}\,dx$

$x^{-2}w = 2x^{-1} + C$

$w = 2x + Cx^2$

$w = y^{-2} \qquad \frac{1}{y^2} = x(2+Cx)$

$y^2 = \frac{1}{x(2+Cx)}$

Exercise C: Full Solutions

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5. $\frac{dy}{dx} + \frac{2}{x}y = -x^2y^2\cos x \qquad n=2$

$\div y^2 \quad y^{-2}\frac{dy}{dx} + \frac{2}{x}y^{-1} = -x^2\cos x \quad ⊛$

let $w = y^{1-n} = y^{-1} \quad \therefore \frac{dw}{dx} = -y^{-2}\frac{dy}{dx}$

$\therefore \quad y^{-2}\frac{dy}{dx} = -\frac{dw}{dx}$

Sub in ⊛: $-\frac{dw}{dx} + \frac{2}{x}w = -x^2\cos x$

$(\times -1) \quad \frac{dw}{dx} - \frac{2}{x}w = x^2\cos x \qquad$ (STANDARD FORM)

$$I = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

$$x^{-2}\frac{dw}{dx} - 2x^{-3}w = \cos x$$

$$\frac{d}{dx}(x^{-2}w) = \cos x$$

$$x^{-2}w = \int \cos x \, dx$$

$$x^{-2}w = \sin x + C$$

$$w = x^2(\sin x + C)$$

$w = y^{-1}$: $\quad \frac{1}{y} = x^2(\sin x + C)$

$$\therefore \quad y = \frac{1}{x^2(\sin x + C)}$$

6

6. $2\frac{dy}{dx} + (\tan x)y = \frac{(4x+5)^2}{\cos x}y^3$

$(\div 2) \quad \frac{dy}{dx} + \left(\frac{1}{2}\tan x\right)y = \frac{(4x+5)^2}{2\cos x}y^3 \quad n=3$

$(\div y^3:) \quad y^{-3}\frac{dy}{dx} + \left(\frac{1}{2}\tan x\right)y^{-2} = \frac{(4x+5)^2}{2\cos x}$

Let $w = y^{1-n} = y^{-2} \quad \therefore \frac{dw}{dx} = -2y^{-3}\frac{dy}{dx}$

$\therefore \quad y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dw}{dx}$

Sub in ⊛: $-\frac{1}{2}\frac{dw}{dx} + \left(\frac{1}{2}\tan x\right)w = \frac{(4x+5)^2}{2\cos x}$

$(\times -2)$: $\frac{dw}{dx} - (\tan x)w = -\frac{(4x+5)^2}{\cos x} \qquad$ (STANDARD FORM)

$$I = e^{\int -\tan x\,dx} = e^{-\ln(\sec x)} = e^{\ln(\sec x)^{-1}}$$

$$= (\sec x)^{-1} = \cos x$$

$$\cos x\frac{dw}{dx} - \tan x\cos x\, w = -\frac{(4x+5)^2}{\cos x}\cos x$$

$$\cos x\frac{dw}{dx} - (\sin x)w = -(4x+5)^2$$

$$\frac{d}{dx}(w\cos x) = -(4x+5)^2$$

$$w\cos x = -\int (4x+5)^2\,dx$$

$$w\cos x = -\frac{(4x+5)^3}{3\times 4} + C_1$$

$$w = -\frac{1}{12\cos x}(4x+5)^3 + \frac{C_1}{\cos x}$$

$w = y^{-2} \quad \frac{1}{y^2} = -\frac{1}{12\cos x}(4x+5)^3 + \frac{C_1}{\cos x}$

$$= \frac{-(4x+5)^3 + 12C_1}{12\cos x} \qquad (C = 12C_1)$$

$$y^2 = \frac{12\cos x}{C - (4x+5)^3}$$

Exercise C: Full Solutions

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$$\frac{x\,dy}{dx} + y = y^2x^2\ln x$$

$(\div x)$ $$\frac{dy}{dx} + \frac{1}{x}y = y^2 x\ln x \qquad (n=2)$$

$\div y^2$: $$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x\ln x \quad \circledast$$

let $w = y^{1-n} = y^{-1}$ $\therefore$ $$\frac{dw}{dx} = -y^{-2}\frac{dy}{dx}$$

$$\therefore\ y^{-2}\frac{dy}{dx} = -\frac{dw}{dx}$$

Sub in $\circledast$: $$-\frac{dw}{dx} + \frac{1}{x}w = x\ln x$$

$(\times -1)$ $$\frac{dw}{dx} - \frac{1}{x}w = -x\ln x \quad \text{(STANDARD FORM)}$$

$$I = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$x^{-1}\frac{dw}{dx} - x^{-2}w = -\ln x$$

$$\frac{d}{dx}(x^{-1}w) = -\ln x$$

$$x^{-1}w = -\int \ln x\,dx$$

$$x^{-1}w = -(x\ln x - x + C_1)$$

$$x^{-1}w = -x\ln x + x + C \quad (C = -C_1)$$

$\int 1\times\ln x\,dx$ (hidden parts)
$u = \ln x \quad \frac{dv}{dx} = 1$
$\frac{du}{dx} = \frac{1}{x} \quad v = x$
$\int \ln x\,dx = x\ln x - \int 1\,dx = x\ln x - x + C_1$

$(\times x)$ $$w = -x^2\ln x + x^2 + Cx$$

$w = y^{-1}$: $$\frac{1}{y} = x^2(1-\ln x) + Cx$$

$$\therefore\ y = \frac{1}{x^2(1-\ln x)+Cx}$$

8

$$\frac{dy}{dx} = y\cot x + y^3\operatorname{cosec} x$$

$$\frac{dy}{dx} - (\cot x)y = y^3\operatorname{cosec} x \qquad (n=3)$$

$(\div y^3)$ $$y^{-3}\frac{dy}{dx} - (\cot x)y^{-2} = \operatorname{cosec} x \quad \circledast$$

$w = y^{1-n} = y^{-2}$ $\therefore$ $$\frac{dw}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dw}{dx}$$

Sub in $\circledast$: $$-\frac{1}{2}\frac{dw}{dx} - (\cot x)w = \operatorname{cosec} x$$

$(\times -2)$: $$\frac{dw}{dx} + (2\cot x)w = -2\operatorname{cosec} x \quad \text{(STANDARD FORM)}$$

$$I = e^{2\int\cot x\,dx} = e^{2\ln(\sin x)} = e^{\ln(\sin x)^2} = \sin^2 x$$

$$\sin^2 x\frac{dw}{dx} + (2\sin^2 x\cot x)w = -2\sin^2 x\operatorname{cosec} x$$

$$\sin^2 x\frac{dw}{dx} + \left(\frac{2\sin^2 x\cos x}{\sin x}\right)w = -\frac{2\sin^2 x}{\sin x}$$

$$\sin^2 x\frac{dw}{dx} + (2\sin x\cos x)w = -2\sin x$$

$$\frac{d}{dx}(\sin^2 x\,w) = -2\sin x$$

$$w\sin^2 x = -2\int\sin x\,dx = -2(-\cos x) + C$$

$$w = \frac{2\cos x + C}{\sin^2 x}$$

$w = y^{-2}$: $$\frac{1}{y^2} = \frac{2\cos x + C}{\sin^2 x}$$

$$\therefore\ y^2 = \frac{\sin^2 x}{2\cos x + C}$$

Exercise C: Full Solutions

9

$y' = 5y + e^{-2x}y^{-2} \qquad y(0) = 2$

$\frac{dy}{dx} - 5y = e^{-2x}y^{-2} \qquad (n = -2)$

$\div y^{-2}$ (or $\times y^2$)

$y^2\frac{dy}{dx} - 5y^3 = e^{-2x} \quad \circledast$

$w = y^{1-n} = y^{1--2} = y^3 \qquad \frac{dw}{dx} = 3y^2\frac{dy}{dx}$

$\therefore y^2\frac{dy}{dx} = \frac{1}{3}\frac{dw}{dx}$

Sub in $\circledast$: $\frac{1}{3}\frac{dw}{dx} - 5w = e^{-2x}$

$(\times 3) \qquad \frac{dw}{dx} - 15w = 3e^{-2x}$ (STANDARD FORM)

$I = e^{\int -15\,dx} = e^{-15x}$

$e^{-15x}\frac{dw}{dx} - 15e^{-15x}w = 3e^{-17x}$

$\frac{d}{dx}(e^{-15x}w) = 3e^{-17x}$

$e^{-15x}w = 3\int e^{-17x} = -\frac{3}{17}e^{-17x} + C_1$

$\times e^{15x} \qquad w = -\frac{3}{17}e^{-17x}e^{15x} + Ce^{15x}$

$w = -\frac{3}{17}e^{-2x} + Ce^{15x}$

$w = y^3 \quad \therefore y^3 = Ce^{15x} - \frac{3}{17}e^{-2x}$

$y(0) = 2: \quad 8 = C - \frac{3}{17} \quad \therefore C = 8 + \frac{3}{17} = \frac{139}{17}$

$\therefore y^3 = \frac{139e^{15x} - 3e^{-2x}}{17}$

10

$y' + \frac{y}{x} - \sqrt{y} = 0 \qquad y(1) = 0$

$\frac{dy}{dx} + \frac{1}{x}y = y^{\frac{1}{2}} \qquad (n = \frac{1}{2})$

$(\div y^{\frac{1}{2}}) \qquad y^{-\frac{1}{2}}\frac{dy}{dx} + \frac{1}{x}y^{\frac{1}{2}} = 1 \quad \circledast$

$w = y^{1-n} = y^{1-\frac{1}{2}} = y^{\frac{1}{2}} \quad \therefore \frac{dw}{dx} = \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}$

$\therefore y^{-\frac{1}{2}}\frac{dy}{dx} = 2\frac{dw}{dx}$

Sub in $\circledast$: $2\frac{dw}{dx} + \frac{1}{x}w = 1$

$(\div 2) \qquad \frac{dw}{dx} + \frac{1}{2x}w = \frac{1}{2}$ (STANDARD FORM)

$I = e^{\frac{1}{2}\int\frac{1}{x}dx} = e^{\frac{1}{2}\ln x} = e^{\ln x^{\frac{1}{2}}} = x^{\frac{1}{2}}$

$x^{\frac{1}{2}}\frac{dw}{dx} + \frac{1}{2}x^{-\frac{1}{2}}w = \frac{1}{2}x^{\frac{1}{2}}$

$\frac{d}{dx}(x^{\frac{1}{2}}w) = \frac{1}{2}x^{\frac{1}{2}}$

$x^{\frac{1}{2}}w = \frac{1}{2}\int x^{\frac{1}{2}} = \frac{1}{2}\cdot\frac{2}{3}x^{3/2} + C_1 = \frac{1}{3}x^{3/2} + C$

$x^{\frac{1}{2}}w = \frac{1}{3}x^{\frac{3}{2}} + C$

$w = \frac{1}{3}x + Cx^{-\frac{1}{2}}$

$w = y^{\frac{1}{2}} \quad \therefore y^{\frac{1}{2}} = \frac{1}{3}x + Cx^{-\frac{1}{2}}$

$y(1) = 0 \qquad 0 = \frac{1}{3} + C \rightarrow C = -\frac{1}{3}$

$y = \left(\frac{1}{3}x - \frac{1}{3}x^{-\frac{1}{2}}\right)^2$

$y = \frac{(x - x^{-\frac{1}{2}})^2}{9}$

$= \frac{1}{9}\left(x^2 - 2\sqrt{x} + \frac{1}{x}\right)$

Challenge: Solution (Explanation)

In (i), there is a generous tip given to help you on your way with this question. Starting from $y = ux$ we have $\frac{dy}{dx} = u + x\frac{du}{dx}$, so that the given differential equation becomes $u + x\frac{du}{dx} = \frac{1}{u} + u$ or $\int u \, du = \int \frac{1}{x} \, dx$ upon separation of variables. You are now in much more familiar territory and may proceed in the standard way: $\frac{1}{2}u^2 = \frac{y^2}{2x^2} = \ln x + C \Rightarrow y^2 = x^2(2\ln x + 2C)$. Using the given conditions $x = 1$, $y = 2$ to determine $C = 2$ then gives the required answer $y = x\sqrt{2\ln x + 4}$. However, there is one small detail still required, namely to justify the taking of the *positive* square-root, which follows from the fact that $y > 0$ when $x = 1$. (Note that you were given $x > e^{-2}$, for the validity of the square-rooting to stand, so it is not necessary to justify this. However, it should serve as a hint that a similar justification may be required in the later parts of the question.)

In (ii), either of the substitutions $y = ux$ or $y = ux^2$ could be used to solve this second differential equation. In each case, the method then follows that of part (i)'s solution very closely indeed; separating variables, integrating, eliminating u and substituting in the condition $x = 1$, $y = 2$ to evaluate the arbitrary constant. The final steps require a justifying of the taking of the positive square-root and a statement of the appropriate condition on x in order to render the square-rooting a valid thing to do. The answer is $y = x\sqrt{5x^2 - 1}$ for $x > \frac{1}{\sqrt{5}}$.

In (iii), only the substitution $y = ux^2$ can be used to get a variable-separable differential equation, which boils down to $\int u \, du = \int \frac{1}{x^2} \, dx \Rightarrow \frac{1}{2}u^2 = \frac{-1}{x} + D$. Using $x = 1$, $y = 2$ ($u = 2$) to evaluate the constant D leads to the answer $y = x\sqrt{6x^2 - 2x}$ for $x > \frac{1}{3}$.