CAPE1150

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Engineering Mathematics

School of Chemical and Process Engineering

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Level 1 Semester 2

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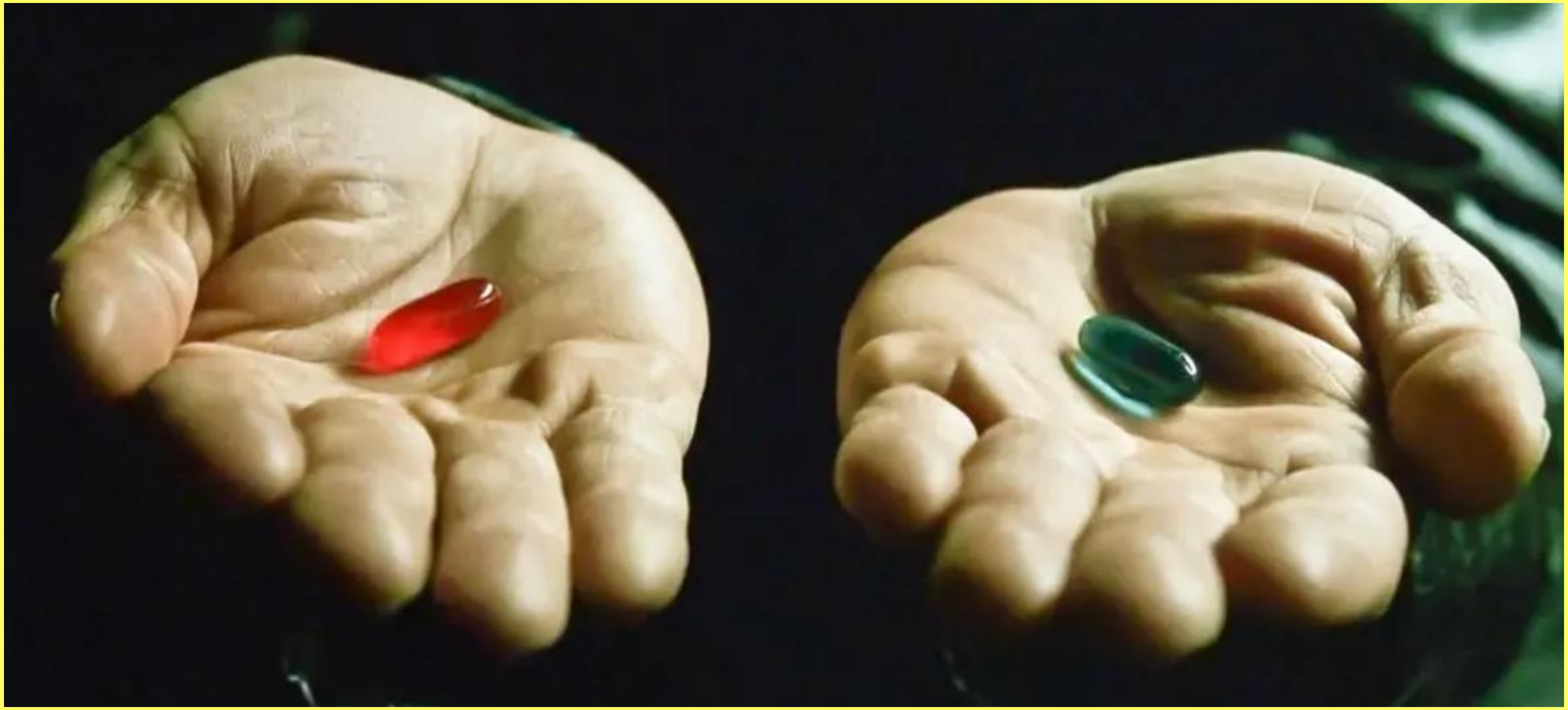
Matrices 1
Outline of Lecture 10

- Matrix Fundamentals

- Matrix Multiplication

- Systems of Equations: Solving by Gaussian Elimination

Let's begin…



Matrix Fundamentals

Matrices: Size

- An array of numbers in a rectangular shape is known as a **matrix.**
- We usually denote a matrix with a **bold (or underlined) uppercase letter e.g. $\boldsymbol{A}$** (row or column vectors are usually lower case e.g. $\boldsymbol{a}$)
- The size (order) of a matrix is defined as **(number of rows)**×**(number of columns)**

  For example, $\begin{pmatrix} 1 & 2 & 2 \\ -3 & 0 & 6 \end{pmatrix}$ is a 2×3 matrix.
- A matrix with the same number of rows and columns is called a **square matrix**.

  For example $\begin{pmatrix} 0 & -1 & 2 \\ -3 & 4 & 9 \\ 15 & 2 & 1 \end{pmatrix}$ is a 3×3 square matrix.
- A matrix with only one column is called a **column vector** or **column matrix.**

  For example, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ are both 3×1 column vectors/matrices
- A matrix with only one row is called a **row vector** or **row matrix**.

  For example, $\begin{pmatrix} -1 & 2 & 2 & 0 \end{pmatrix}$ is a 1×4 row vector/matrix
- Each number in a matrix is known as an **element**.
- A matrix can be written with square or curved brackets (I use curved).

Matrices: Applications

Matrices have a huge range of applications across many disciplines:

- **Engineering & Physics**
- **Computer Graphics**
- **Cryptography**
- **Data Science and Machine Learning**
- **Economics and Business**
- **Network Analysis**
- **Structural Analysis**
- **Robotics and Control Systems**
- **Signal Processing**
- **Computer-Aided Design (CADNavigation Systems**
- **Genetics and Bioinformatics**
- **Weather Prediction**
- **Game Theory**
- **Traffic Flow Analysis**
- **Ecology and Population Studies**
- **Finance and Investment**
- **Healthcare and Medical Imaging**

Note: Matrices are a great way to represent large data sets fed into a computer.

Computers can do calculations with very large matrices/arrays very quickly.

This semester we will do calculations by hand, then next year you will use MATLAB to do them with much larger matrices to solve more complex systems of equations.

More info here: https://www.geeksforgeeks.org/real-life-application-of-matrices/

Engineering and Physics: Matrices solve systems of linear equations in structural analysis, electrical circuits, and fluid mechanics. They are used to model and analyze physical phenomena like heat transfer and wave motion.
For our purposes, we will mainly concentrate on solving systems of linear equations.

Matrix Fundamentals

Elements

We can refer to a particular element by quoting its row index first, then its column index.

$$\begin{pmatrix} 2\ 3\ 4 \\ 3\ 2\ 5 \end{pmatrix}$$

So in this matrix, the 2,3 element, $a_{23} = 5$.

This is known as double subscript notation

Adding/Subtracting Matrices

Just add/subtract the corresponding elements of each matrix.
Only matrices with the same dimension can be added/subtracted.

$$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 6 & -2 & 9 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 3-q & 3 \\ -2 & 1 \\ 4 & 2 \end{pmatrix}$$

Matrices: General Case

A general $m \times n$ matrix $\boldsymbol{A}$ has m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{pmatrix}$$

The entries in matrix $\mathbf{A}$ are known as the **elements** of $\mathbf{A}$.

$$a_{23} = \text{the element in the second row and third column}$$

$$a_{mn} = \text{the element in the } m^{\text{th}} \text{ row and } n^{\text{th}} \text{ column}$$

In matrix $\mathbf{A}$ the element in row i and column j is denoted by a_{ij}
Where $i = 1,2,3, \dots, m$ and $j = 1,2,3, \dots, n$.

So we could rewrite matrix $\mathbf{A}$ as:

$$\mathbf{A} = \left(a_{ij}\right), 1 \le i \le m, 1 \le j \le n.$$

Matrix Fundamentals

Equal Matrices

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal only of they have the same number of rows and columns and all their corresponding elements are equal.

For Example: $$\mathbf{A} = \begin{pmatrix} 1 & p \\ -1 & -q \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ -1 & -4 \end{pmatrix}$$

Then $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal only if $p = 2$ and $q = 4$.

Scalar Multiplication

A scalar is a number which can 'scale' the elements inside a matrix. Multiplying by a scalar multiplies **all elements** of the matrix.

This is just like vectors. For example, $3\boldsymbol{a}$ is the vector $\boldsymbol{a}$ 'scaled' by the scalar 3.

a $$3\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix}$$

b $$\boldsymbol{A} = \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix}$$

c $$\begin{pmatrix} -3 \\ k \end{pmatrix} + k\begin{pmatrix} 2k \\ 2k \end{pmatrix} = \begin{pmatrix} k \\ 6 \end{pmatrix} \qquad k = \frac{3}{2}$$

Note: c gives
$$-3 + 2k^2 = k \Rightarrow (k+1)(2k-3) = 0$$
$$k = -1, k = \frac{3}{2}$$
$$k + 2k^2 = 6 \Rightarrow (k+2)(2k-3) = 0$$
$$k = -2, k = \frac{3}{2}$$
$\therefore k = \frac{3}{2}$ as it must satisfy both rows

The **distributive law** holds (k is a scalar): $$\mathrm{k}(\mathbf{A} + \mathbf{B}) = \mathrm{k}\mathbf{A} + \mathrm{k}\mathbf{B}$$

Matrix Fundamentals

Types of Matrix

A **square matrix** has the same number of rows as columns.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & 4 \\ 2 & 2 & 5 \\ -3 & 4 & 3 \end{pmatrix}$$

A **diagonal matrix** is a matrix that has 0s everywhere except the leading diagonal (top left to bottom right).

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

<u>Note for later:</u>
The determinant of a diagonal matrix is just the product of its diagonal elements.

An **upper-triangular matrix** is a square matrix where all elements below the leading diagonal are zero.

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 5 \\ 0 & 0 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

Upper triangular because the upper part is non-zero.

A **lower-triangular matrix** is a square matrix where all elements above the leading diagonal are zero.

$$\begin{pmatrix} 1 & 0 & 0 \\ -4 & -2 & 0 \\ -3 & 1 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$$

Lower triangular because the lower part is non-zero.

An **identity matrix $\mathbf{I}$** is a **square** matrix which has 1's in the 'leading diagonal' (starting top-left) and 0 elsewhere. The dimensions depend on the context.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Note for later:</u>
Multiplying a square matrix by the corresponding identity matrix has no effect. It is analogous to multiplying by 1 in matrix form.

The Zero Matrix

- The zero vector $\mathbf{0}$ (a bold 0), represents no movement.

$$\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$$

In 3D: $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

- This is so that we can be consistent when we write vector equations.
ie. As $\overrightarrow{PQ} + \overrightarrow{QP}$ is a vector quantity, it must be equal to a vector.

- Similarly, the **zero matrix** or **null matrix** is the matrix all of whose elements are zero:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Unlike the identity matrix which must be square, the zero matrix can have any number of rows and columns.

- The dimensions come from the context (e.g. if you subtracted two identical 3x3 matrices, the zero matrix produced would be 3x3).
Similarly $\mathbf{A} + \mathbf{0} = \mathbf{A}$ as long as $\mathbf{A}$ and $\mathbf{0}$ are the same size

- Zero matrices, whatever the size are denoted by $\mathbf{0}$ or $\underline{0}$.

Diagnostic Question

Work out $\mathbf{A} - 3\mathbf{B}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Y

$$\begin{pmatrix} 0 & 0 \\ 4 & 3 \\ 5 & 1 \end{pmatrix}$$

C

$$\begin{pmatrix} 4 & 8 \\ 0 & 7 \\ 5 & -3 \end{pmatrix}$$

M

$$\begin{pmatrix} -2 & -4 \\ 0 & 1 \\ 5 & -3 \end{pmatrix}$$

A

$$\begin{pmatrix} -2 & -4 \\ 6 & 1 \\ 5 & 3 \end{pmatrix}$$

Diagnostic Question

Work out $\mathbf{A} + \mathbf{B}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Y

$$\begin{pmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{pmatrix}$$

C

$$\begin{pmatrix} 2 & 5 & 8 \\ 6 & 9 & 12 \end{pmatrix}$$

M

$$\begin{pmatrix} 2 & 6 \\ 5 & 9 \\ 8 & 12 \end{pmatrix}$$

A

Not possible

The Transpose of a Matrix

- The transpose of a matrix is when the rows and columns of a matrix are interchanged.

  For Example:

  $$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{becomes} \quad \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

  Where $\mathbf{A}^{\mathbf{T}}$ is called the transpose of $\mathbf{A}$.

- From the example it should be clear that in general $\mathbf{A}^{\mathbf{T}} \neq \mathbf{A}$ since they are different sizes.

- If $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{A}^{\mathbf{T}}$ is an $n \times m$ matrix.

- If $\mathbf{A}$ is an $n \times n$ square matrix, $\mathbf{A}^{\mathbf{T}}$ is also an $n \times n$ square matrix.

- Transposing the transpose gives back the original matrix: $\left(\mathbf{A}^{\mathbf{T}}\right)^{\mathbf{T}} = \mathbf{A}$.

- For identity matrix $\mathbf{I}$, the transpose has no effect: $\mathbf{I}^{\mathbf{T}} = \mathbf{I}$

Symmetric Matrix

Although in general $\mathbf{A}^{\mathrm{T}} \neq \mathbf{A}$, it is possible for a **square matrix** to be identical to its transpose.

For Example:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix}$$

And we can easily see that:

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix}$$

As $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$,
$\mathbf{A}$ is a **symmetric** matrix.

(the matrix is symmetrical about the leading diagonal)

For any **square matrix**, the transpose has the same effect as a reflection about the **leading diagonal**.

Properties of the Transpose

$$(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$$

$$(\mathbf{A}-\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} - \mathbf{B}^{\mathrm{T}}$$

$$\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}$$

Verification of the first: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix} \quad \Rightarrow (\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \begin{pmatrix} 6 & 10 \\ 8 & 12 \end{pmatrix}$$

$$\mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 8 & 12 \end{pmatrix}$$

$$= (\mathbf{A}+\mathbf{B})^{\mathrm{T}}$$

Diagnostic Question

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Work out $\mathbf{A}^{\mathrm{T}}$

Y

$$\begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{pmatrix}$$

C

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

M

$$\begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

A

$$\begin{pmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{pmatrix}$$

Diagnostic Question

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Work out $\mathbf{I}^{\mathbf{T}}$

Y

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

C

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

M

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

A

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Diagnostic Question

Which of the matrices are symmetric?

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Y

$\mathbf{A}$ only

C

$\mathbf{B}$ only

M

$\mathbf{A}$ and $\mathbf{B}$

A

neither

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Diagnostic Question

Work out $\mathbf{A} + \mathbf{B}^{\mathbf{T}}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Y

$$\begin{pmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{pmatrix}$$

C

$$\begin{pmatrix} 2 & 5 & 8 \\ 6 & 9 & 12 \end{pmatrix}$$

M

$$\begin{pmatrix} 2 & 6 \\ 5 & 9 \\ 8 & 12 \end{pmatrix}$$

A

Not possible

Diagnostic Question

$$\left(\mathbf{A}^{\mathbf{T}} - \mathbf{B}\right)^{\mathbf{T}} =$$

Y

$$\frac{\mathbf{1}}{\mathbf{A}^{\mathbf{T}} - \mathbf{B}}$$

C

$$\mathbf{A} + \mathbf{B}^{\mathbf{T}}$$

M

$$\mathbf{A}^{\mathbf{T}} + \mathbf{B}$$

A

$$\mathbf{A} - \mathbf{B}^{\mathbf{T}}$$

Matrix Multiplication

Matrix Multiplication

- To multiply two matrices, the number of columns of the first matrix must be equal to the number of rows of the second matrix.
- More formally: For the matrix product AB to be defined (possible), the number of columns of A must equal the number of rows of B.
- You can multiply a $m \times n$ matrix by a $n \times p$ matrix.
- The result is a matrix of size $m \times p$.
- Square matrices of the same size can always be multiplied!

Example

Let $\mathbf{A} = \begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix}$. Note that **A** is a 2×3 matrix and **B** is a 3×3 matrix.

A has 3 columns and B has 3 rows

Therefore it is possible to multiply **A** by **B** (with **B** on the right) and the product **C = AB** will itself be a 2×3 matrix with the structure

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}.$$

Matrix Multiplication: Traditional Method

1st row
1st column
c_{11}

Multiply this row | By this col | To get this element

$$\underset{A}{\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix}} \quad \underset{B}{\begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix}} \quad \underset{C}{\begin{pmatrix} \boxed{c_{11}} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}}$$

You may recognise this is finding the "dot/scalar product" of the two vectors.

$$= \begin{pmatrix} (-1 \times -2) + (2 \times 1) + (5 \times 4) & & \\ & & \end{pmatrix}$$

1st row
2nd column
c_{12}

Multiply this row | By this col | To get this element

$$\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix} \quad \begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix} \quad \begin{pmatrix} c_{11} & \boxed{c_{12}} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

$$= \begin{pmatrix} (-1 \times -2) + (2 \times 1) + (5 \times 4) & (-1 \times 1) + (2 \times -1) + (5 \times -2) & \\ & & \end{pmatrix}$$

1st row
3rd column
c_{13}

Multiply this row | By this col | To get this element

$$\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix} \quad \begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix} \quad \begin{pmatrix} c_{11} & c_{12} & \boxed{c_{13}} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

$$= \begin{pmatrix} (-1 \times -2) + (2 \times 1) + (5 \times 4) & (-1 \times 1) + (2 \times -1) + (5 \times -2) & (-1 \times 3) + (2 \times 2) + (5 \times 5) \\ & & \end{pmatrix}$$

Matrix Multiplication: Traditional Method

2^{nd} row
1^{st} column
c_{21}

Multiply this row

$$\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix}$$

By this col

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix}$$

To get this element

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

$$= \begin{pmatrix} (-1\times -2)+(2\times 1)+(5\times 4) & (-1\times 1)+(2\times -1)+(5\times -2) & (-1\times 3)+(2\times 2)+(5\times 5) \\ (2\times -2)+(-3\times 1)+(4\times 4) & & \end{pmatrix}$$

2^{nd} row
2^{nd} column
c_{22}

Multiply this row

$$\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix}$$

By this col

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix}$$

To get this element

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

$$= \begin{pmatrix} (-1\times -2)+(2\times 1)+(5\times 4) & (-1\times 1)+(2\times -1)+(5\times -2) & (-1\times 3)+(2\times 2)+(5\times 5) \\ (2\times -2)+(-3\times 1)+(4\times 4) & (2\times 1)+(-3\times -1)+(4\times -2) & \end{pmatrix}$$

2^{nd} row
3^{rd} column
c_{23}

Multiply this row

$$\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix}$$

By this col

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix}$$

To get this element

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

$$= \begin{pmatrix} (-1\times -2)+(2\times 1)+(5\times 4) & (-1\times 1)+(2\times -1)+(5\times -2) & (-1\times 3)+(2\times 2)+(5\times 5) \\ (2\times -2)+(-3\times 1)+(4\times 4) & (2\times 1)+(-3\times -1)+(4\times -2) & (2\times 3)+(-3\times 2)+(4\times 5) \end{pmatrix}$$

Matrix Multiplication: Traditional Method

Simplify Elements:

$$\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 24 & -13 & 26 \\ 9 & -3 & 20 \end{pmatrix}$$

Note that we cannot form the product **BA** because **B** is 3×3 and **A** is 2×3, i.e. the number of columns of **B** is different from the number of rows of **A**.

Illustration of dimensions

You can multiply a $m \times n$ matrix by a $n \times p$ matrix.

The result is a matrix of size $m \times p$.

Matrix dimensions after multiplication

What happens to the dimensions when we multiply matrixes?

$$\begin{bmatrix} 2 & 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 23 \end{bmatrix}$$

(1 by 2) x (2 by 1) = (1 by 1)

$$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 56 \end{bmatrix}$$

(1 by 3) x (3 by 1) = (1 by 1)

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 36 & 41 \\ 64 & 73 \end{bmatrix}$$

(2 by 2) x (2 by 2) = (2 by 2)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 28 & 34 \\ 64 & 79 \end{bmatrix}$$

(2 by 3) x (3 by 2) = (2 by 2)

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 19 & 24 \\ 24 & 33 & 42 \\ 34 & 47 & 60 \end{bmatrix}$$

(3 by 2) x (2 by 3) = (3 by 3)

Matrix Multiplication: Grid Method

Now let's do the same multiplication by "grid method"

Again, $\mathbf{A} = \begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix}$ we want $\mathbf{AB}$.

You still need to check that **columns of the first** matrix must be equal to the number of **rows of the second** matrix.

Set out the matrices as follows:

2nd matrix

$$\begin{array}{cc} & \begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix} \\ \begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix} & \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix} \end{array}$$

1st matrix

We can also use the grid to see if we can multiply. (**columns of first** = **rows of second).** In this case each element has a match to multiply with so we can multiply.

We can also see the size of the resulting matrix (if possible to multiply).

Now do the dot product of the rows and columns (1st matrix times 2nd).

The position of the resulting element is shown by the intersection of the lines.

And just as before, this leads to:

$$= \begin{pmatrix} (-1\times-2)+(2\times1)+(5\times4) & (-1\times1)+(2\times-1)+(5\times-2) & (-1\times3)+(2\times2)+(5\times5) \\ (2\times-2)+(-3\times1)+(4\times4) & (2\times1)+(-3\times-1)+(4\times-2) & (2\times3)+(-3\times2)+(4\times5) \end{pmatrix}$$

$$= \begin{pmatrix} 24 & -13 & 26 \\ 9 & -3 & 20 \end{pmatrix}$$

Matrix Multiplication: Grid Method

We found before that we cannot multiply in the opposite order so let's see how that looks with grid method:

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix} \text{ check } \mathbf{BA}.$$

Set out the matrices as follows:

2nd matrix

$$\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix}$$

1st matrix

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix}$$

In this case each the elements do not correspond (they don't all have a friend) so we cannot multiply!

Matrix Multiplication involving $\mathbf{I}$

For any square matrix $\mathbf{A}$ and identity matrix $\mathbf{I}$ of the same size

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

(algebra equivalent $a \times 1 = 1 \times a = a$)

(multiplying by I on the left or right leaves the original matrix unchanged)

So the identity matrix is a bit like the '1' of matrix multiplication, e.g. $1 \times 3 = 3 \times 1 = 3$; multiplying by 1 has no effect, and multiplying by $\mathbf{I}$ has no effect.

This becomes useful in relation to inverse matrices which we will talk about next lecture.

Verification

Let $\mathbf{M} = \begin{pmatrix} -1 & 2 & 0 \\ 3 & 2 & 1 \\ 2 & -2 & 3 \end{pmatrix}$. Verify that $\mathbf{MI} = \mathbf{IM} = \mathbf{M}$.

So starting with multiplication on the left by $\mathbf{I}$:

$$\mathbf{IM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 & 0 \\ 3 & 2 & 1 \\ 2 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times(-1)+0\times3+0\times2 & 1\times2+0\times2+0\times(-2) & 1\times0+0\times1+0\times3 \\ 0\times(-1)+1\times3+0\times2 & 0\times2+1\times2+0\times(-2) & 0\times0+1\times1+0\times3 \\ 0\times(-1)+0\times3+1\times2 & 0\times2+0\times2+1\times(-2) & 0\times0+0\times1+1\times3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & 0 \\ 3 & 2 & 1 \\ 2 & -2 & 3 \end{pmatrix} = \mathbf{M}$$

Multiplication on the right gives the same result:

$$\mathbf{MI} = \begin{pmatrix} -1 & 2 & 0 \\ 3 & 2 & 1 \\ 2 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times(-1)+0\times3+0\times2 & 1\times2+0\times2+0\times(-2) & 1\times0+0\times1+0\times3 \\ 0\times(-1)+1\times3+0\times2 & 0\times2+1\times2+0\times(-2) & 0\times0+1\times1+0\times3 \\ 0\times(-1)+0\times3+1\times2 & 0\times2+0\times2+1\times(-2) & 0\times0+0\times1+1\times3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & 0 \\ 3 & 2 & 1 \\ 2 & -2 & 3 \end{pmatrix} = \mathbf{M}$$

Diagnostic Question

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$$

Work out the product $\mathbf{AB}$

Y

$$\begin{pmatrix} 4 & 10 \\ 18 & 28 \end{pmatrix}$$

C

$$\begin{pmatrix} 16 & 19 \\ 36 & 43 \end{pmatrix}$$

M

$$\begin{pmatrix} 16 & 43 \end{pmatrix}$$

A

$$\begin{pmatrix} 14 \\ 46 \end{pmatrix}$$

Diagnostic Question

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

If possible, work out the product $\mathbf{AB}$

Y

$$\begin{pmatrix} 5 & 11 \\ 11 & 25 \\ 17 & 39 \end{pmatrix}$$

C

$$\begin{pmatrix} 7 & 15 & 23 \\ 10 & 22 & 34 \end{pmatrix}$$

M

$$\begin{pmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \end{pmatrix}$$

A

Not possible

Diagnostic Question

Choose the correct statement

The multiplication $\mathbf{AA^T}$ is ____________ possible.

Y

always

C

never

M

sometimes

A

mission im

Diagnostic Question

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

Which of the following products **cannot** be calculated?

Y

$\mathbf{AB}$

C

$\mathbf{AA^T}$

M

$\mathbf{BA}$

A

$\mathbf{A^2}$

Matrix Multiplication: Properties

Order is important, in general, $\mathbf{AB} \neq \mathbf{BA}$ (not commutative)

Multiplication is associative $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ Order is important, but placement of brackets isn't.

Multiply out brackets as usual $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$

Powers $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AA}^2$ or $\mathbf{A}^2\mathbf{A}$ etc.

Zero Matrix $\mathbf{AB} = \underline{\mathbf{0}}$ does not mean that $\mathbf{A} = \underline{\mathbf{0}}$ and/or $\mathbf{B} = \underline{\mathbf{0}}$
(the product could just result in zeroes for every element)

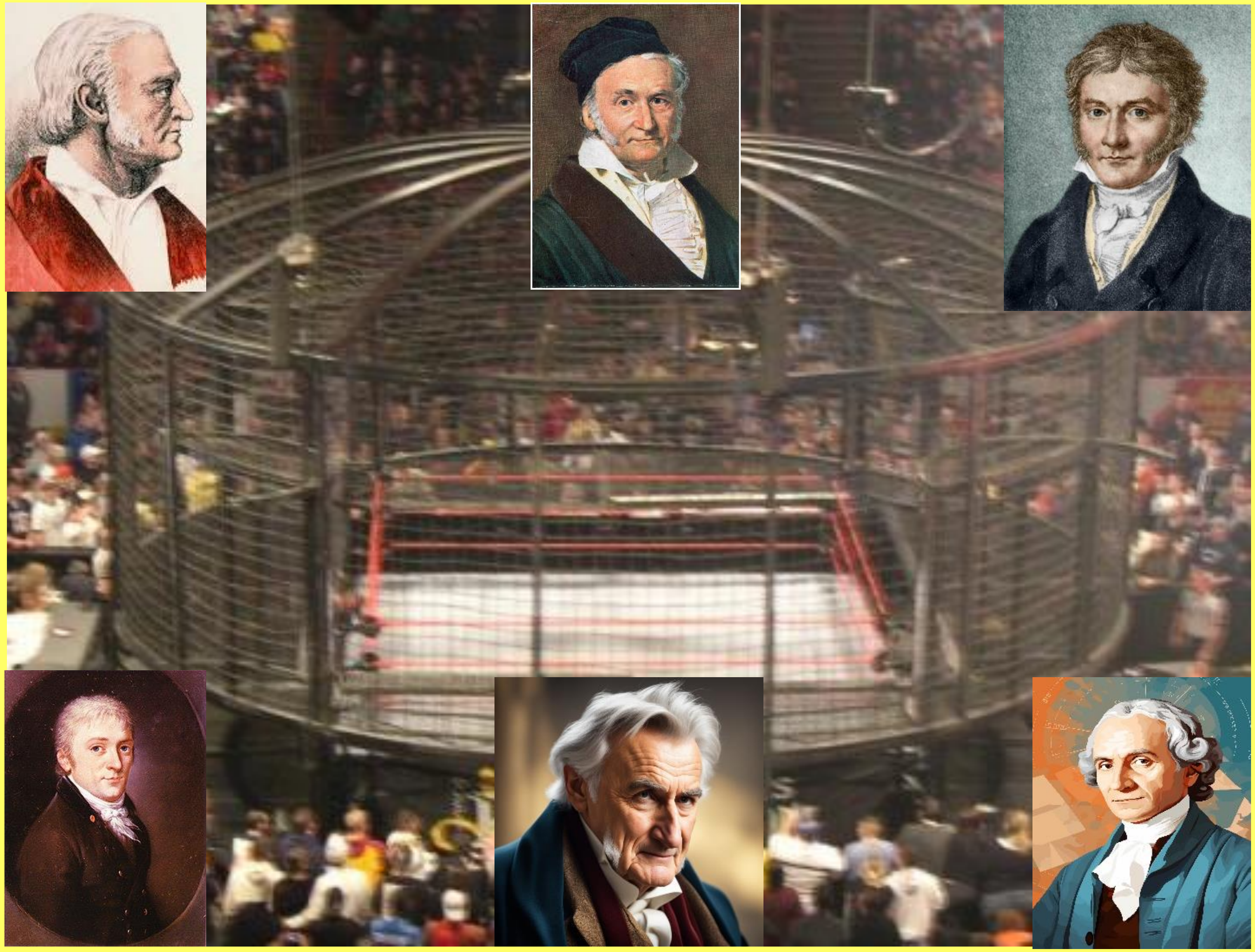
Multiplying with transposes

$\mathbf{A}^{\mathbf{T}}\mathbf{A}$ and $\mathbf{AA}^{\mathbf{T}}$ are always possible and the result is symmetric

$(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$ (order is changed)

$\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}} = \left(\mathbf{A}^{\mathbf{T}}\right)^2 = \left(\mathbf{A}^2\right)^{\mathbf{T}}$ (using last result)

Verification of these are left as tutorial exercises.



Systems of Equations: Gaussian Elimination

Simultaneous Equations: Recap

Recall that a **system of equations**, also known as **simultaneous equations**, are when we have multiple linked equations involving more than one variable.

1

$$\begin{aligned} x + y + z &= 10 \\ 2y + z &= 7 \\ z &= 1 \end{aligned}$$

There is a **unique solution** $(x = 4, y = 3, z = 1)$.

Actually in this form, $z = 1$ from the 3rd equation can be substituted into the 2nd equation to obtain y, which can substituted into the 1st equation to obtain x.

We will use this "back substitution" method shortly.

2a

$$\begin{aligned} x \quad\quad + z &= 10 \\ y + z &= 8 \end{aligned}$$

There are **infinitely many solutions** to these equations.

If we picked any value of x, e.g. 3, then $z = 7$, and hence $y = 1$.

We will see that we could express the **general solution** using:

$$\begin{aligned} y &= x - 2 \\ z &= 10 - x \end{aligned}$$

In general, if there are fewer equations than variables, there is not enough information to uniquely determine x, y and z.

2b

$$\begin{aligned} x + y \quad\quad &= 2 \\ z &= 8 \\ x + y + z &= 10 \end{aligned}$$

The 3rd equation is **redundant** because it gives no new information.
This means we effectively only have 2 equations for 3 variables, so there is not a unique solution (infinitely many solutions again).

3

$$\begin{aligned} x + 2y &= 1 \\ 2x + 4y &= 7 \end{aligned}$$

There are **no solutions** to these equations. If we double the first equation, then $2x + 4y$ is both 2 and 7, which is **inconsistent**.

Solving Systems of Linear Equations

Simultaneous linear equations occur frequently in engineering applications.
We can easily solve two simultaneous equations using high-school techniques.

$$\begin{aligned} 4x_1 + 5x_2 &= 13 \\ 3x_1 - 2x_2 &= 27 \end{aligned}$$

Solving by elimination (or substitution) gives: $\boldsymbol{x_1 = 7, x_2 = -3}$

We can extend this procedure to three equations

$$\begin{aligned} 2x_1 - 5x_2 + 2x_3 &= 14 \\ 9x_1 + 3x_2 - 4x_3 &= 13 \\ 7x_1 + 3x_2 - 2x_3 &= 3 \end{aligned}$$

$\boldsymbol{\Rightarrow x_1 = 1, x_2 = -4, x_3 = -4}$

We could have used x, y, z but this way is more easily extendable to more variables.

This already starts to get a bit complicated, so if we keep increasing the number of equations, then it becomes extremely difficult to solve the system of equations.

We need a more efficient method!

In matrix notation, the above set of equations is:

$$\begin{pmatrix} 2 & -5 & 2 \\ 9 & 3 & -4 \\ 7 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 3 \end{pmatrix}$$

Matrix of coefficients | **Matrix of unknowns** | **Matrix of constants**

Which is of the form $\mathbf{A}\boldsymbol{x} = \mathbf{b}$
(as if you multiply the matrix by the vector, you get the above equations back)

As we will soon see, we can then use matrix operations to solve any number of equations simultaneously!

Equations to Matrix form

Multiply:

$$\begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathbf{4x - y} \\ \mathbf{2x + 3y} \end{pmatrix}$$

Notice how the coefficients of x and y in each equation matches the rows of the matrix.

$$\begin{aligned} \boldsymbol{x + y - z} &= \mathbf{2} \\ \boldsymbol{3x - y + z} &= \mathbf{5} \\ \boldsymbol{x - y + 2z} &= \mathbf{7} \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 & -1 \\ 3 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{7} \end{pmatrix}$$

Elementary Row Operations

You should already be familiar with the idea of adding/subtracting multiples of one equation to another, with the intention of making a variable cancel.

①+2②

$$\begin{aligned} x - 2y &= 3 \quad ① \\ 3x + y &= 16 \quad ② \end{aligned}$$

$$\begin{aligned} 7x \quad &= 35 \\ 3x + y &= 16 \end{aligned}$$

Similarly, we can add multiples of one row to another (ensuring we do the same to LHS and RHS)

Add 2 × row 2 to row 1

$$\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 16 \end{pmatrix} \qquad \begin{pmatrix} 7 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 35 \\ 16 \end{pmatrix}$$

Gaussian Elimination: Elementary Row Operations

Elementary Row Operations

1. **Row Interchange:**
   We can swap rows around.

   It does not matter which order the simultaneous equations are in, they are still the same equations to be solved.

2. **Row Scaling:**
   We can multiply a row (or rows) by a scalar constant.

   This is equivalent to multiplying one of the simultaneous equations throughout by a constant
   (which does not change the equation).

3. **Row Addition/Subtraction:**
   We can add/subtract a multiple of one row to/from another.

   This is exactly what we do when we use the elimination method in simultaneous equations (hence the name).

Forming the Augmented Matrix

Given we may need to do multiple row operations, for **notational convenience**, we can write the matrix equation $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ as an **augmented matrix** $(\mathbf{A}|\mathbf{b})$

$$\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 16 \end{pmatrix}$$

$$\Downarrow$$

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 3 & 1 & 16 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 3 & 1 & 16 \end{array}\right)$$

Subtract 3 lots of row 1 from row 2

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 7 & 7 \end{array}\right)$$

Divide row 2 by 7

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 1 \end{array}\right)$$

This represents the equations:

$$x - 2y = 3$$
$$y = 1$$

From which we can find the solution $x = 5, y = 1$

Row-Echelon Form

Notice the triangular shape of the main part of the augmented matrix.

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 7 & 7 \end{array}\right)$$

$$\begin{aligned} x - 2y &= 3 \\ 7y &= 7 \end{aligned}$$

Once we know y, this can be easily substituted into the 1st equation to determine x.

This 2nd equation is only in terms of y, and clearly a quick division gives us its value.

What makes the above system of equations easily solvable in their current form?

We have a similar situation here.
The 3rd equation allows us to easily determine z.
Substituting into the 2nd equation allows to us determine y, and then substituting into the 1st allows us to determine x.
We say the matrix is in **row echelon** form.

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 4 \\ 0 & 4 & -1 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

$$\begin{aligned} x + 2y + 5z &= 4 \\ 4y - z &= 2 \\ 3z &= 3 \end{aligned}$$

The matrix again has a triangular shape.

Row-Echelon Form

A matrix is in **row reduced form** / **row echelon form** if:

1. Any rows with all 0 entries are at the bottom.
2. On each row, the left-most non-zero entry, known as the **leading entry** or **pivot**, is to the right of the leading entry in the row above.

The most efficient process of converting a matrix to this form is known as **row reduction** or **Gaussian elimination**. Matrices in row reduced form allow systems of linear equations to be easily solved.

Mini Quiz: Which are in row echelon form?

$$\begin{pmatrix} 4 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

No

The row of 0s is not at the bottom.

We could fix this easily by swapping rows 2 and 3

$$\begin{pmatrix} \boxed{4} & 3 & 2 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$

Yes

The leading entry of '1' is right of the leading entry of '4'.

$$\begin{pmatrix} \boxed{4} & 3 & 2 \\ \boxed{1} & 0 & 0 \\ 0 & \boxed{5} & 0 \end{pmatrix}$$

No

The leading entry of '1' is not right of the leading entry of '4'.

$$\begin{pmatrix} \boxed{4} & 3 & 2 \\ 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

Yes

The leading entries of '4', '2' and '1' move right each time.

A matrix is **upper-triangular** if all the entries below the leading diagonal are 0.

Example: Solving by Gaussian Elimination (Unique Solution)

Example

Solve the following system of equations by reducing the corresponding augmented matrix to row echelon form.

$$x_1 + 2x_2 + 3x_3 = 9$$
$$2x_1 - x_2 + 4x_3 = 17$$
$$3x_1 + x_2 - x_3 = 2$$

This is a system of equations of the form $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ which we can write as an augmented matrix $(\mathbf{A}|\mathbf{b})$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ \boxed{2} & -1 & 4 & 17 \\ \boxed{3} & \boxed{1} & -1 & 2 \end{array}\right)$$

Form an augmented matrix.

The strategy is to get each of these blocks to be 0 using two row operations.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 4 & 17 \\ 3 & 1 & -1 & 2 \end{array}\right)$$

Add or subtract appropriate multiples of row 1.

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 3R_1$$

Important: Only add/subtract using rows **above** the current one.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -2 & -1 \\ 0 & -5 & -10 & -25 \end{array}\right)$$

Now get the appropriate 0 in row 3 by using either row 2 or row 1.

$$R_3 \to R_3 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -2 & -1 \\ 0 & 0 & -8 & -24 \end{array}\right)$$

Solve using back-substitution

$$-8x_3 = -24 \quad \to \boldsymbol{x_3 = 3}$$
$$-5x_2 - 2(3) = -1 \quad \to \boldsymbol{x_2 = -1}$$
$$x_1 + 2(-1) + 3(3) = 9 \quad \to \boldsymbol{x_1 = 2}$$

From $\mathbf{A}\boldsymbol{x} = \mathbf{b}$, we can see that our solution is the vector $\boldsymbol{x}$, so we could also write our solution in the form:

$$\boldsymbol{x} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Swapping rows if you have a zero pivot

A matrix is in **row reduced form** / **row echelon form** if:

1. Any rows with all 0 entries are at the bottom.
2. On each row, the left-most non-zero entry, known as the **leading entry** or **pivot**, is to the right of the leading entry in the row above.

The most efficient process of converting a matrix to this form is known as **row reduction** or **Gaussian elimination**. Matrices in row reduced form allow systems of linear equations to be easily solved.

Suppose we have used row operations to reduce a matrix and have obtained:

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 0 & 0 & 7 & 0 \\ 0 & -6 & 4 & 6 \end{array}\right)$$

This is not in row echelon form as the pivot in row 2 is zero.
However, we can just switch row 2 and 3

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 0 & -6 & 4 & 6 \\ 0 & 0 & 7 & 0 \end{array}\right)$$

Which is now in the correct form and easily solvable.

Example: Solving by Gaussian Elimination (Infinite Solutions)

Example

Solve the following system of equations by reducing the corresponding augmented matrix to row echelon form.

$$2x_1 + 3x_2 - 2x_3 = 1$$
$$4x_1 + 6x_2 + 3x_3 = 2$$
$$6x_1 + 9x_2 - 2x_3 = 3$$

This is a system of equations of the form $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ which we can write as an augmented matrix $(\mathbf{A}|\mathbf{b})$

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 4 & 6 & 3 & 2 \\ 6 & 9 & -2 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 4 & 6 & 3 & 2 \\ 6 & 9 & -2 & 3 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right)$$

This is already on upper triangular form, but we can eliminate further.

$R_3 \to R_3 - \frac{4}{7}R_2$

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The last row is all zero and so does not help us. We now have 2 equations but 3 unknowns.

Note: If we hadn't done the last step, the 2nd and 3rd row would still have given the same result, still 2 equations with 3 unknowns!

From 2nd row: $7x_3 = 0 \quad \to \boldsymbol{x_3 = 0}$

From 1st row: $2x_1 + 3x_2 = 1$

Key point:
This is one equation in two unknowns. The best we can do is choose one unknown arbitrarily, for example let $\boldsymbol{x_2 = k}$

Therefore: $\boldsymbol{x_1 = \frac{1-3k}{2}, x_2 = k, x_3 = 0}$

And our vector is

$$\boldsymbol{x} = \begin{pmatrix} \frac{1-3k}{2} \\ k \\ 0 \end{pmatrix}$$

Note: If we had set $x_1 = k$ instead we would have

$$x_1 = k, x_2 = \frac{1-2k}{3}, x_3 = 0$$

This is equivalent

So this system of equations has infinitely many solutions.
We may not have nailed down the values but we do know the relationship between the variables.

Example: Solving by Gaussian Elimination (No Solutions)

Example

Solve the following system of equations by reducing the corresponding augmented matrix to row echelon form.

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 1 \\ 4x_1 + 6x_2 + 3x_3 &= 2 \\ 6x_1 + 9x_2 - 2x_3 &= 7 \end{aligned}$$

This is a system of equations of the form $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ which we can write as an augmented matrix $(\mathbf{A}|\mathbf{b})$

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 4 & 6 & 3 & 2 \\ 6 & 9 & -2 & 7 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 4 & 6 & 3 & 2 \\ 6 & 9 & -2 & 7 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 4 & 4 \end{array}\right)$$

This is already on upper triangular form, but we can eliminate further.

$R_3 \to R_3 - \frac{4}{7}R_2$

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

From 3rd row: $0x_3 = 4$ $\therefore$ **no solution**

Notice the distinction:
In this example only the matrix part of a row is all zero, whereas in the previous example, a whole row was zero.

Note: If we hadn't done the last step, the 2nd row would have given

$$7x_3 = 0 \ \to x_3 = 0$$

And the 3rd row would have given

$$4x_3 = 4 \to x_3 = 1$$

Which is a contradiction so the equations have no solution.

Diagnostic Question

Which is a correct first step to reduce the following augmented matrix?

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{array}\right)$$

Y

$R_1 - R_3$
$R_2 + R_3$

M

$R_3 + R_1$
$R_2 - R_1$

C

$R_2 + R_1$
$R_3 - R_1$

A

Diagnostic Question

Which is a correct first step to reduce the following augmented matrix?

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right)$$

Y

$R_1 - R_3$
$R_2 + R_3$

M

$R_2 - \frac{1}{2}R_3$
$R_3 - 2R_1$

C

$R_2 - R_1$
$R_3 + 2R_1$

A

$R_2 + R_1$
$R_3 - 2R_1$

Diagnostic Question

Which is a correct next step to reduce the following augmented matrix?

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -2 & -1 \end{array}\right)$$

Y

$R_3 + 2R_1$

M

$R_3 + 2R_2$

C

$R_3 - 2R_1$

A

$R_3 - 2R_2$

Diagnostic Question

Which is a correct first step to reduce the following augmented matrix?

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 4 \\ 3 & 1 & 4 & 0 \end{array}\right)$$

Y

Swap R_2 and R_3

M

Swap R_1 and R_2

C

$R_3 - 4R_2$

A

$R_3 - 3R_1$

Methods for Solving Systems of Equations

There are two main methods of solving systems of linear equations using matrices.

1. Gaussian Elimination (Row Reduction)

2. Inverse Matrix Method which says that if $\mathbf{Ax} = \mathbf{b}$ then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
   Where $\mathbf{A}^{-1}$ is the inverse matrix of $\mathbf{A}$ (more on this next lecture)

The first method only involves adding/subtracting and scaling rows then substituting, the second requires us to find the inverse matrix (which is a long process involving either determinants and a lot of work or Gaussian Elimination again), then matrix multiplication.

The first method therefore only needs one of the steps in method 2 and uses less memory when done computationally for large systems.

Summary:

While both Gaussian elimination and finding the inverse matrix can solve systems of linear equations, Gaussian elimination is generally more efficient and versatile, especially for larger systems, as it doesn't require computing the entire inverse matrix, which can be computationally expensive and numerically unstable.

The inverse matrix method is widely used. Please feel free to look it up if you're interested.

Thanks

See you in the Tutorial!