CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

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Level 1 Semester 2

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1^{st} Order ODEs 1: Separation of Variables
Outline of Lecture 4

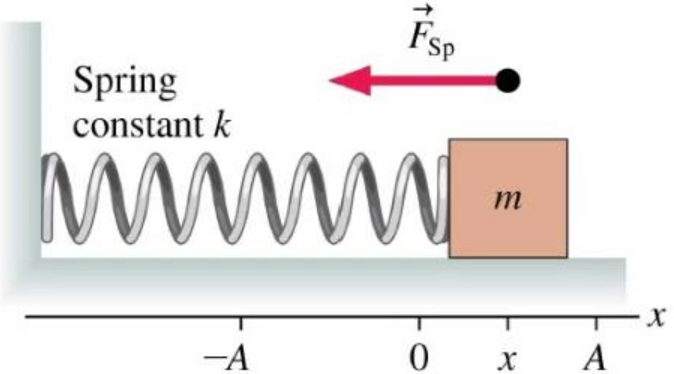
- Introduction to ODEs
- Solving 1^{st} order ODEs (separation of variables method)
- Modelling with ODEs (separation of variables method)
- Chemical Engineering Applications

Ordinary Differential Equation

M&S Differential Equation

$$\frac{dy}{dx} = x$$

$$-\frac{1}{m}kx = \frac{d^2x}{dt^2}$$



Ordinary Differential Equations (Introduction)

Classification of Differential Equations

In order to know how to solve a differential equation, we must be able to identify what type it is. It is therefore important to classify differential equations:

- **Ordinary differential equations** (ODEs) involve only **one independent variable** and contain only total (not partial – introduced later) derivatives.
- Partial differential equations (PDEs) involve only partial derivatives (we will not consider these here)
- The order of a differential equation is the order of the highest derivative in the equation (the power of a derivative does not matter, only its order):

E.g.

$$\underbrace{\frac{d^2y}{dx^2}}_{\text{2nd order}} + 5\Big(\underbrace{\frac{dy}{dx}}_{\text{1st order}}\Big)^3 - 4y = e^x$$

So this is a 2nd order ODE

Classification of Differential Equations

Linearity

- A differential equation is linear if there are no nonlinear terms involving the **dependent variable** (usually y) or its derivatives, that is:
    1. No nonlinear functions of the **dependent variable** (such as $y^2, \sin y, e^y, \ln y$).
    2. No products (explicit or implicit) among functions of the **dependent variable** or its derivatives (of any order) (such as $y^2, y\frac{dy}{dx}, \left(\frac{dy}{dx}\right)^2$).

- An ODE of any order can therefore be linear or non-linear.

- An nth order linear ODE has the general form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Note the difference:

- Terms such as $\left(\frac{dy}{dx}\right)^2$ or $\left(\frac{d^2y}{dx^2}\right)^2$ are not linear

- However, $\frac{d^2y}{dx^2}$ is linear (not a product).

- Do not confuse $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ (second derivative to power 1) with $\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx} \times \frac{dy}{dx}$ (first derivative to power 2)

To illustrate the difference, it may help to picture a physical case:
For mechanics:

- $\frac{d^2x}{dt^2}$ is the acceleration
- $\left(\frac{dx}{dt}\right)^2$ is the velocity squared

Clearly, these are not the same thing!

Classification of Differential Equations

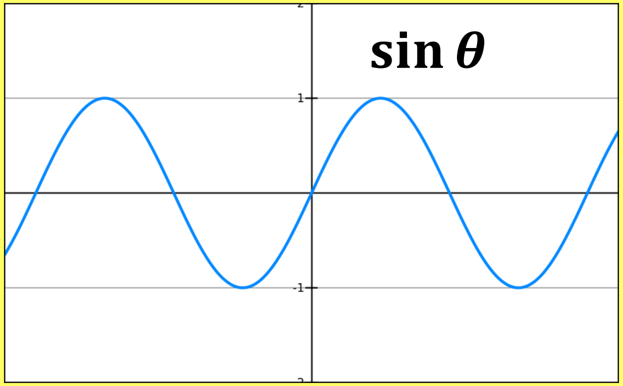
Linear or Nonlinear: Examples

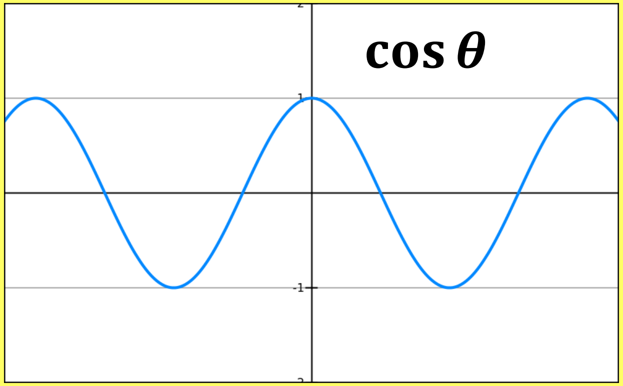
E.g. 1	$4x\frac{dy}{dx} + y - x = 0$		1st order linear
E.g. 2	$y'' - 2y' + y = 0$		2nd order linear
E.g. 3	$\frac{d^3y}{dx^3} + 3x\frac{dy}{dx} - 5y = e^x$		3rd order linear
E.g. 4	$(1 - y)\frac{dy}{dx} + 2y = e^x$	Product of dependent variable and derivative.	1st order nonlinear
E.g. 5	$\frac{d^2y}{dx^2} + \sin y = 0$	Nonlinear function of y	2nd order nonlinear
E.g. 6	$\frac{d^4y}{dx^4} + y^2 = 0$	Nonlinear function of y (product of y with y)	4th order nonlinear

Classification of Differential Equations

Why are sin and cos nonlinear?

If you think of the graphs, clearly sin and cos are nonlinear





But they are actually defined by the power series below which clearly involves nonlinear powers of y (more on power series next year!):

$$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \dots$$

$$\cos(y) = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots$$

Note: These power series are actually how your calculator works out the values of trig functions (it doesn't look at a graph). The more terms used, the more accurate to the graph.

Diagnostic Question

Classify the following ordinary differential equation:

$$\frac{dy}{dx} = x$$

Y 1^{st} order linear

M 1^{st} order non-linear

C 2^{nd} order linear

A 2^{nd} order non-linear

Diagnostic Question

Classify the following ordinary differential equation:

$$\frac{dy}{dt} = y$$

Y 1^{st} order linear

M 1^{st} order non-linear

C 2^{nd} order linear

A 2^{nd} order non-linear

Diagnostic Question

Classify the following ordinary differential equation:

$$\frac{dy}{dx} = \sin x$$

Y 1^{st} order linear

M 1^{st} order non-linear

C 2^{nd} order linear

A 2^{nd} order non-linear

Diagnostic Question

Classify the following ordinary differential equation:

$$\frac{dy}{dx} = \sin y$$

Y 1st order linear

M 1st order non-linear

C 2nd order linear

A 2nd order non-linear

Diagnostic Question

Classify the following ordinary differential equation:

$$\frac{dy}{dx} = xy$$

Y 1^{st} order linear

M 1^{st} order non-linear

C 2^{nd} order linear

A 2^{nd} order non-linear

Diagnostic Question

Classify the following ordinary differential equation:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = x^2$$

Y 1^{st} order linear

M 1^{st} order non-linear

C 2^{nd} order linear

A 2^{nd} order non-linear

Diagnostic Question

Classify the following ordinary differential equation:

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} = x^2$$

Y 1^{st} order linear

M 1^{st} order non-linear

C 2^{nd} order linear

A 2^{nd} order non-linear

Diagnostic Question

Classify the following ordinary differential equation:

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^3 + \ln x$$

Y 2nd order linear

M 2nd order non-linear

C 3rd order linear

A 3rd order non-linear

Diagnostic Question

Classify the following ordinary differential equation:

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 1$$

Y 2nd order linear

M 2nd order non-linear

C 3rd order linear

A 3rd order non-linear

Diagnostic Question

Classify the following ordinary differential equation:

$$\frac{d^2y}{dt^2} \times \frac{dy}{dt} = y$$

Y 2nd order linear

M 2nd order non-linear

C 3rd order linear

A 3rd order non-linear

Boundary Conditions & Initial Value Problems (IVPs)

- Differential Equations are solved by integrating (or equivalent methods).
- Each time we integrate, we produce an arbitrary constant, therefore in solving

$$\frac{dy}{dx} = x \quad \Rightarrow \quad y = \frac{x^2}{2} + C$$

$$\frac{d^2y}{dx^2} = x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{2} + C \quad \Rightarrow \quad y = \frac{x^3}{6} + Cx + D$$

- Boundary conditions are certain conditions, which allow us to determine the values of constants. They may specify the state of a system at a certain position and/or time.
- They are usually derived from the properties of the physical system that the differential equation is modelling (e.g . $\frac{dT}{dx} = 3$ when $x = 1$ which could refer to a temperature gradient at a certain position along a wire).
- The order of the differential equation is equal to the number of arbitrary constants and also the number boundary conditions are needed to determine the values of all the constants (e.g. a 1st order differential equation requires 1 boundary condition, a 2nd order requires 2 boundary conditions etc.
- When boundary conditions refer to the system at the beginning of the time period concerned ($t = 0$), they are known as **initial conditions** (e.g. $y(0) = 5$ means $y = 5$ when $t = 0$)
- An initial value problem (IVP) is a mathematical problem where the initial conditions are known.

Boundary Conditions in Context

Imagine we have a thin wire, length L placed along the x-axis:

The wire is heated at the RHS. Assuming it is insulated (no heat escapes), the heat will flow from right to left until the wire is of uniform temperature along its length.

The temperature therefore depends on both the **position** in the wire and the **time** since heating began.

$$T = T(x, t)$$

$x = 0$, $T = 0°\text{C}$ $\qquad$ $x = L$, $T = 50°\text{C}$ $\qquad$ x

Spatial boundary conditions

We could also use temporal boundary conditions such as the rate of temperature flow is zero at $t = 0$: $\frac{\partial T}{\partial x} = 0$ when $t = 0$ (more on this type of thing next year)

This may be the most frequently used technique in Chemical Engineering, so make sure you're awesome at it!

Solving 1st Order ODEs:
(Separation of Variables Method)

Solving Ordinary Differential Equations (ODEs)

E.g. 1 Find the general solution (meaning explained soon) of the differential equation:

$$\frac{dy}{dx} = x$$

'<u>Solving</u>' means to get y in terms of x (with no $\frac{dy}{dx}$).

To solve, we just integrate with respect to x: $\frac{dy}{dx} = x \quad \Rightarrow \quad y = \int x dx = \frac{x^2}{2} + C$

E.g. 2 Find the general solution of the differential equation:

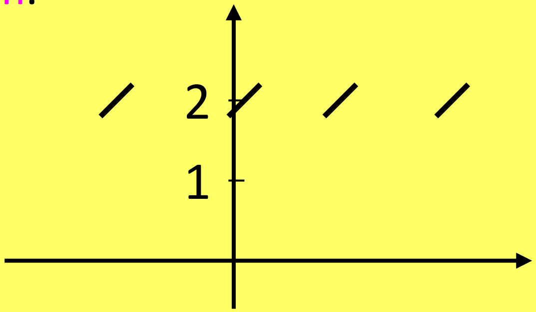
$$\frac{dy}{dx} = y$$

This time, we cannot integrate y with respect to x so we need another method…

Before we see how to solve the equation, it's useful to get some idea of the solution.

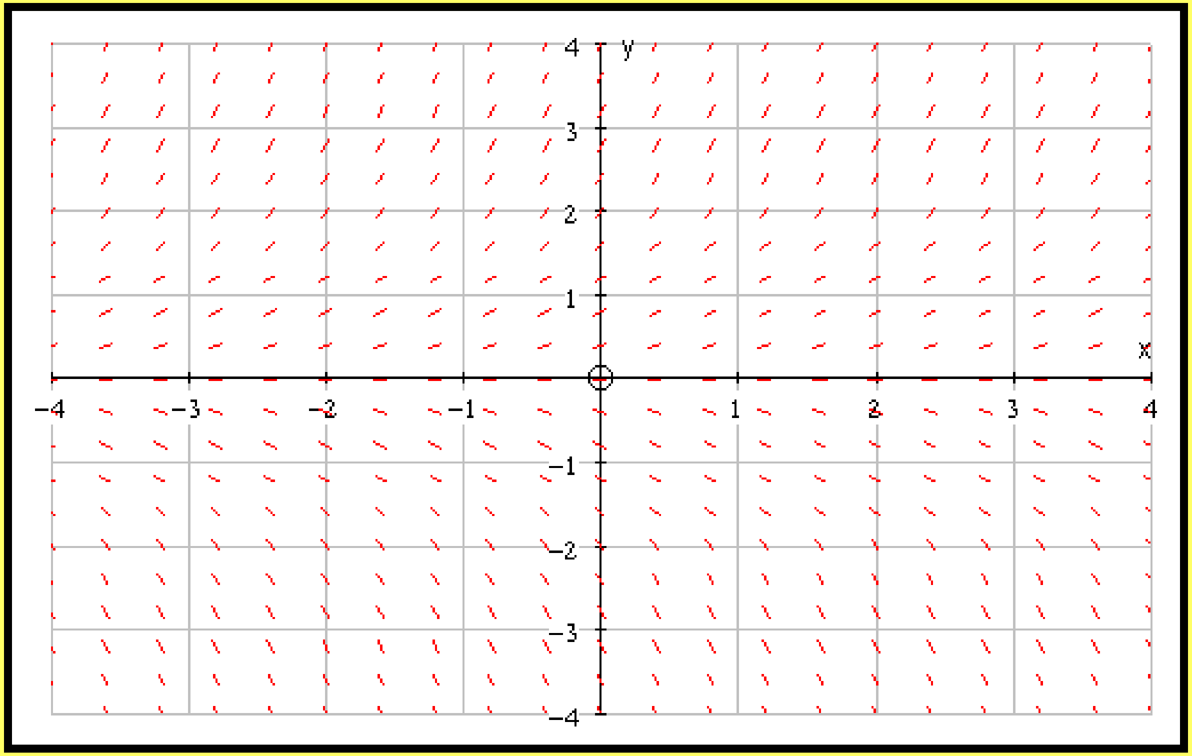
The equation tells us that the graph of y has a gradient that always equals y.
We can sketch the graph by drawing a gradient diagram.

For example, at every point where $y = 2$, the gradient equals 2. We can draw a set of small lines showing this gradient.



Solving Ordinary Differential Equations (ODEs)

$$\frac{dy}{dx} = y$$

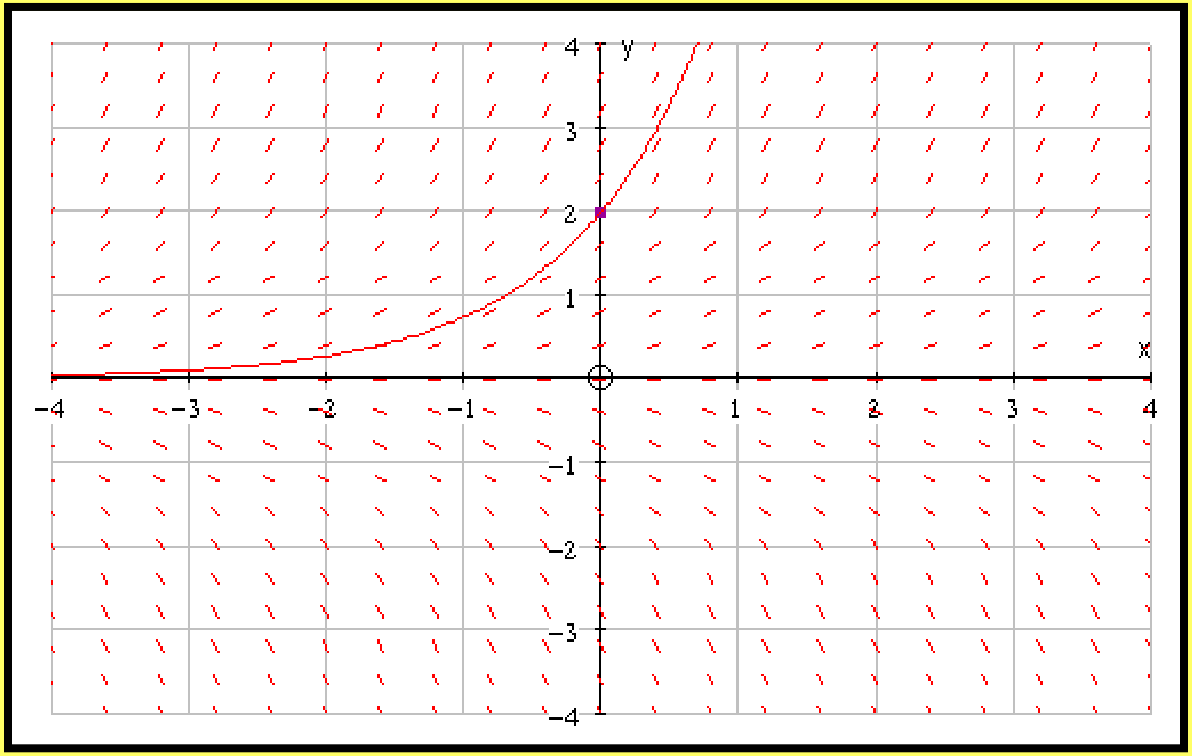


We can now draw a curve through any point following the gradients.

Notice that the gradient lines get steeper as the magnitude of y increases.

Solving Ordinary Differential Equations (ODEs)

$$\frac{dy}{dx} = y$$

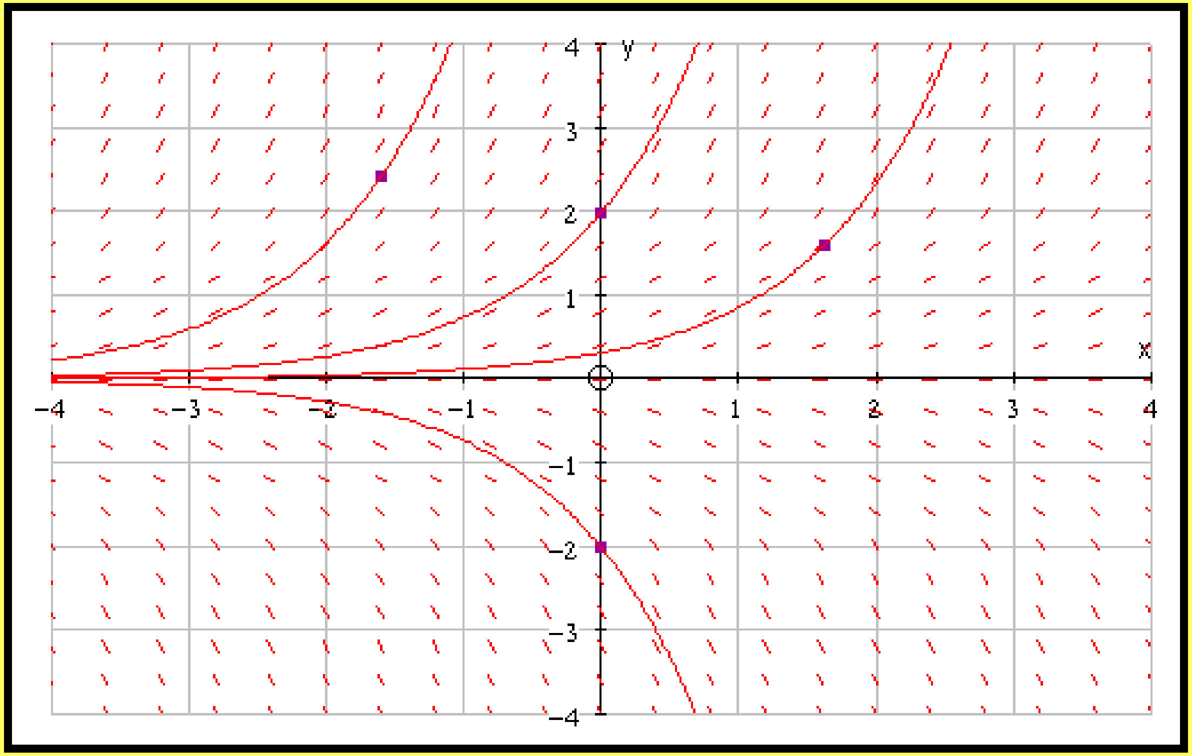


However, we haven't got just one curve.

Notice that the gradient lines get steeper as the magnitude of y increases.

Solving Ordinary Differential Equations (ODEs)

$$\frac{dy}{dx} = y$$



The solution is a family of curves.

These are <u>exponential curves</u>.

Notice that the gradient lines get steeper as the magnitude of y increases.

Separation of Variables Method

E.g. 2 Now it is time to solve $$\frac{dy}{dx} = y$$

We will use a method called "**Separation of Variables**" and the title describes exactly what we do.

First we rearrange the ODE so that terms including x are on the RHS and terms including y are on the LHS.

$$\frac{dy}{dx} = y \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx}dx = dx$$

Technically, the mathematically correct way to do this is by chain rule. However, in practice, it is usual to skip this line and just "rearrange" so you get $\int f(y)\,dy = \int f(x)\,dx$

Go straight to this step

$$\frac{1}{y}dy = dx$$

Now integrate both sides: $\displaystyle\int \frac{1}{y}dy = \int dx \Rightarrow \ln|y| = x + C$

We normally seek <u>explicit</u> solutions (in the form $y = ...$).
In order to obtain an expression for $y(x)$, we still have some work to do.

$$\ln|y| = x + C \quad \Rightarrow |y| = e^{x+C} \quad \Rightarrow |y| = e^C e^x$$

Notice that $e^C > 0$ and $e^x > 0$ for any values of x and C

Finally, we have y explicitly in the form: $$y = \pm e^C e^x$$

General & Particular Solutions

- The solutions of this ODE are a family of exponential curves, as it was shown in the graphs earlier.
- Since the expression of y contains an arbitrary constant of integration, it is called the general solution (y_{gs}) of the ODE.
- Substituting a specific value of C into the general solution will give a particular solution.

The general solution can be written in a more convenient form.
Since e^C is just a (positive) constant, we can use another constant to denote either $+e^C$ or $-e^C$

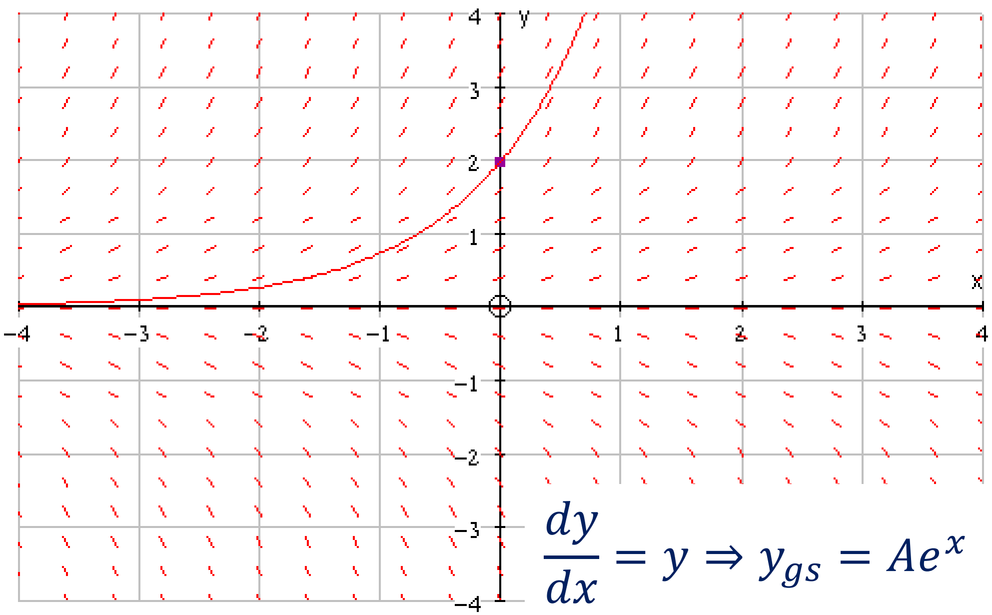
$$y_{gs} = Ae^x$$

Here the A can be either positive or negative.

General & Particular Solutions

Setting A equal to certain value will give us a particular solution.
The particular solution with $A = 2$ is shown in the graph below.

Changing the value of A gives the different curves we saw on the gradient diagram.



$$\frac{dy}{dx} = y \Rightarrow y_{gs} = Ae^{x}$$

The differential equation of the form $\frac{dy}{dx} = y$ is important as it is one of a group used to model exponential growth and decay.

Technical Note:

When performing the integration, why don't we add a constant on both sides?

$$\int \frac{1}{y} dy = \int dx$$

We could if we wanted: $\ln|y| + C_1 = x + C_2$

But then: $\ln|y| = x + C_2 - C_1$

$C_1 - C_2$ is just a constant, so we can just rewrite it as C and it's the same as if we just added a constant to one side.

So: $\ln|y| = x + C$

Separation of Variables

E.g. 3 Solve the ODE $$\frac{dy}{dx} = y^2 \cos x$$

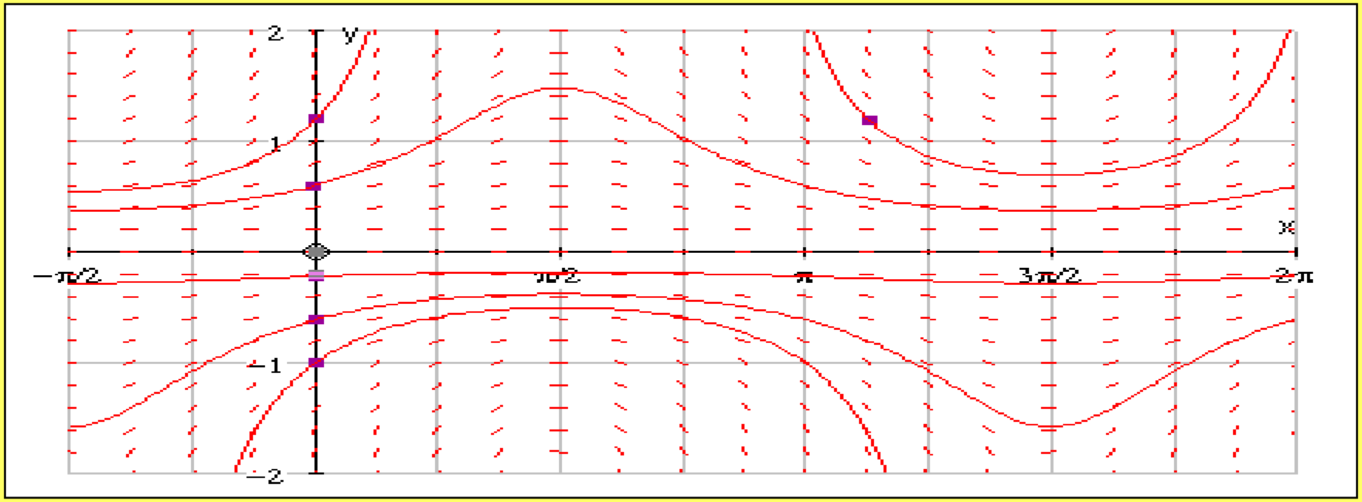
Separate variables: $$\frac{1}{y^2}\,dy = \cos x\,dx$$

Integrate both sides: $$\int y^{-2}\,dy = \int \cos x\,dx$$

$$\frac{y^{-1}}{-1} = \sin x + C$$

$$-\frac{1}{y} = \sin x + C$$

$$y_{gs} = -\frac{1}{\sin(x) + C}$$



Separation of Variables

E.g. 4 Solve the ODE $\dfrac{dy}{dx} = x + xy$

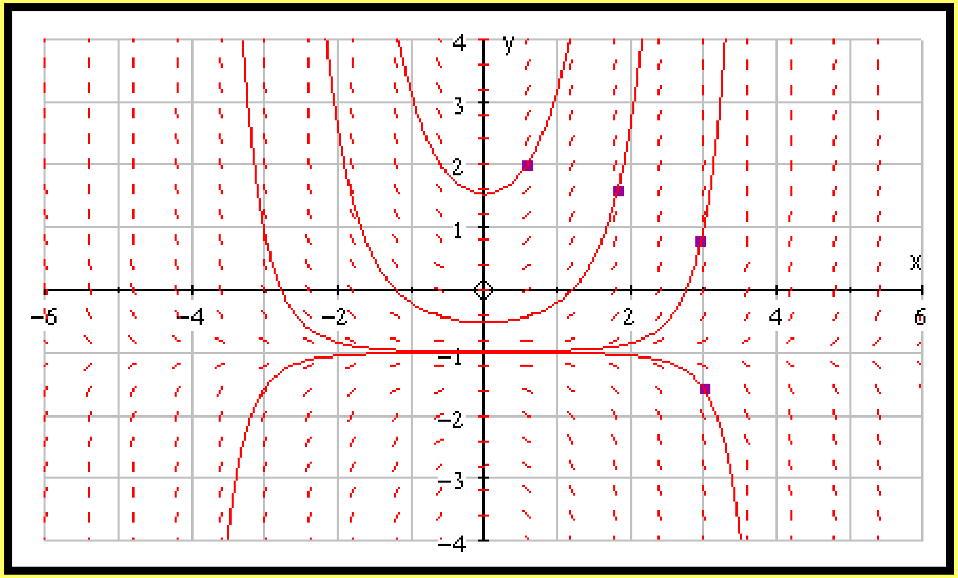
First, we need to take out x as a common factor on the RHS, so $\dfrac{dy}{dx} = x(1+y)$

Now we can separate the variables: $\dfrac{1}{1+y}dy = xdx$

Integrate: $\displaystyle\int \frac{1}{1+y}dy = \int xdx \quad \Rightarrow \ \ln|1+y| = \frac{x^2}{2} + C \quad \Rightarrow \ |1+y| = e^{\frac{x^2}{2}+C}$

Finally, the general solution can be written as:

$$y_{gs} = Ae^{\frac{x^2}{2}} - 1$$



This graph highlights 4 particular solutions:
3 examples for positive A and 1 for negative A.

Differential Equations with Boundary Conditions

E.g. 5

Find the general solution to $\frac{dy}{dx} = -\frac{3(y-2)}{(2x+1)(x+2)}$

Given that $x = 1$ when $y = 4$. Leave your answer in the form $y = f(x)$

$$\int \frac{1}{y-2}\,dy = \int \frac{-3}{(2x+1)(x+2)}\,dx$$

Use partial fractions to split up RHS.

$$\frac{-3}{(2x+1)(x+2)} \equiv \frac{A}{2x+1} + \frac{B}{x+2}$$

$$\Rightarrow A = -2, B = 1$$

$$\int \frac{1}{y-2}\,dy = \int \frac{1}{x+2} - \frac{2}{2x+1}\,dx$$

$$\ln|y-2| = \ln|x+2| - \ln|2x+1| + \ln k$$

Trick: Here, we have written out constant of integration $C = \ln k$ which is just another constant, but now easily combined with the other logs

$$\ln|y-2| = \ln\left|\frac{k(x+2)}{2x+1}\right|$$

$$y = 2 + \frac{k(x+2)}{2x+1}$$

When $x = 1, y = 4$: $\quad 4 = 2 + \frac{k(1+2)}{2+1} \quad \rightarrow k = 2$

$$y = 2 + \frac{2x+4}{2x+1} = \frac{4x+2+2x+4}{2x+1}$$

$$\therefore \boldsymbol{y = \frac{6x+6}{2x+1}}$$

Diagnostic Question

Find the general solution of: $$\frac{dx}{dt} = kx$$

Y $$x = \frac{kx^2}{2} + C$$

M $$x = \sqrt{2kt + C}$$

C $$x = Ae^{kt}$$

A $$x = Ae^{kt} + C$$

Diagnostic Question

Find the general solution of: $$\frac{dy}{dx} = e^{-y}$$

Y $y = e^{-y}x + C$

M $y = \ln|x| + \ln|C|$

C $y = \ln|x + C|$

A $y = -\ln|x + C|$

Diagnostic Question

Given that $y = 1$ when $x = \frac{\pi}{2}$, solve the differential equation: $\frac{dy}{dx} = y^2 \sin x$

Y $y = \frac{1}{1 + \cos x}$

M $y = \frac{1}{1 - \cos x}$

C $y^2 = e^{-\cos x}$

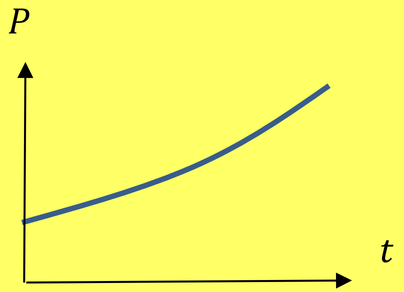
A $y^2 = e^{\cos x}$

Modelling with ODEs:
(Separation of Variables Method)

Modelling with ODEs: Overview

Frequently in physics/maths, **the rate of change of a variable is proportional to the value itself**. So with a population P behaving in this way, if the population doubled, the rate of increase would double.

$$P \propto \frac{dP}{dt} \quad \rightarrow \quad P = k\frac{dP}{dt}$$



This is known as a 'differential equation' because the equation involves both the variable and its derivative $\frac{dP}{dt}$.

The general solution to a differentiation equation means to have an equation relating P and t without the $\frac{dP}{dt}$. After separating variables and integrating, we end up with:

$$\boldsymbol{P = Ae^{kt}}$$

where A and k are constants. This is expected, because we know from experience that population growth is usually exponential.

As we will see, A is the quantity at t=0 (initial quantity), and k is the rate of growth or decay.

Modelling with ODEs

We have seen how to solve ordinary differential equations using the separation of variables.

To find a **particular solution** for an ODE we need to know some **initial conditions**.

If we are modelling a real system using an ODE, the initial conditions will be known and we can find a particular solution.

In summary, if we are looking for a particular solution, we will have to go through 3 main steps:

- Separation of variables
- Integration of the separated equation
- Substitution of the initial conditions

Modelling with ODEs

E.g. 6 A solution initially contains 200 bacteria.
Assuming the number, x, **increases at a rate proportional to the number present**, write down a differential equation connecting x and the time, t.
If the rate of increase of the number is initially 100 per hour, how many are there after 2 hours?

Definition:
An ODE with an initial condition (value at $t = 0$) is called an **initial value problem (IVP)**.

Solution:

The description in the question is typical of exponential growth.

We have to set up an ordinary differential equation that describes the situation and solve it to find x when $t = 2$.

In real life problems, time is often the independent variable and the dependent variable is sometimes denoted by x. In the previous examples we were seeking $y(x)$, now we are seeking $x(t)$.

Modelling with ODEs: Exponential growth

E.g. 6

We know that the rate of increase, which is $x'(t)$, is proportional to number of bacteria present at the time, that is $x(t)$.
Mathematically:

$$\frac{dx}{dt} \propto x \qquad \Rightarrow \frac{dx}{dt} = kx$$

k is a positive constant of proportionality

This ODE can be solved using the separation of variables:

$$\frac{dx}{dt} = kx \qquad \Rightarrow \int \frac{1}{x}\,dx = \int k\,dt$$

Integrating: $\ln x = kt + C$

No modulus needed here, as the number of bacteria cannot be negative!

General solution: $x_{gs} = Ae^{kt}$ (where $A = e^{C}$)

Additional note:

Notice that $e^{kt} = \left(e^{k}\right)^{t}$, and since e^{k} is a constant, we could always write $y = A \cdot B^{x}$. i.e. The general solution is 'any generic exponential function', not just restricted to those with e as the base. However it is customary to write Ae^{kt}.

Modelling with ODEs: Exponential Growth

E.g. 6

We were given two pieces of information:
(1) "A solution initially contains 200 bacteria" and
(2) "the rate of increase of the number is initially 100 per hour"
We will use this information to determine the constants *A* and *k*.

(1) gives us *A:* $x(t=0) = 200 = Ae^0 \quad \Rightarrow \quad \mathrm{A} = 200$

(2) can be used to calculate *k:* $\frac{dx}{dt} = kx \quad \Rightarrow \quad \frac{dx}{dt}(t=0) = 100 = k(200)$

$$\Rightarrow \quad k = 0.5$$

The particular solution that satisfies the 2 initial conditions is:

$$\boxed{x(t) = 200e^{0\cdot 5t}}$$

If we want to know the number of bacteria present after 2 hours, we simply need to substitute $t = 2$ in the formula above:

$$x(t=2) = 200e^{0.5\times 2}$$

$$\Rightarrow x(t=2) = 200e^1 = \mathbf{544}$$

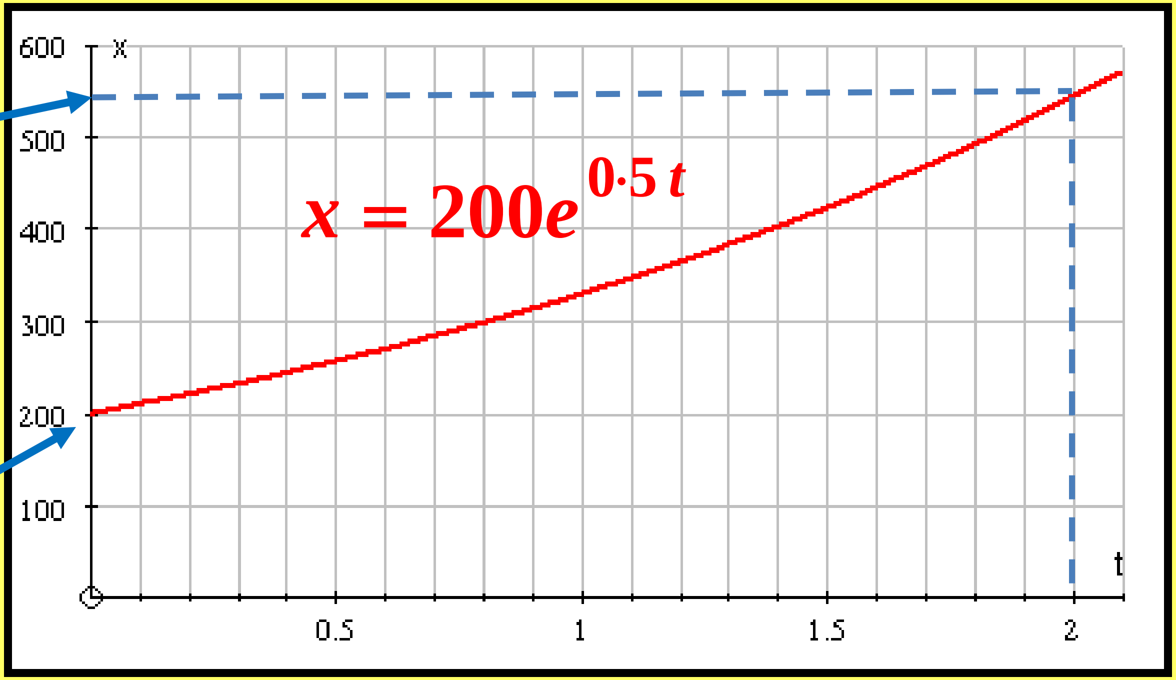
So we can expect to see about 544 bacteria after 2 hours.

Modelling with ODEs: Exponential Growth

E.g. 6

The graph showing the growth function is:

Number after 2 hours
(544 bacteria)



Number at start of measurements
(200 bacteria)

Modelling with ODEs: Exponential Decay

E.g. 7

A radioactive element decays at a rate that is proportional to the mass remaining. Initially the mass is 10 mg and after 20 days it is 5 mg.
Set up a differential equation describing this situation and solve it to find the time taken to reach 1 mg.

Solution: Let m be mass in mg and t time in days.
The rate decrease (which is $m'(t)$) is proportional to the mass present at time t:

$$\frac{dm}{dt} \propto m(t) \quad \Rightarrow \quad \frac{dm}{dt} = -km$$

Note: Here k must be a positive constant to ensure that the rate of change remains negative at all times.

Integrating the ODE gives:

$$\int \frac{dm}{m} = \int -kdt \quad \Rightarrow \quad \ln(m) = -kt + C$$

The general solution is therefore: $m_{gs} = Ae^{-kt}$

Modelling with ODEs: Exponential Decay

Now we have to use the initial conditions to determine *A* and *k*:

$$m(t=0) = 10 = Ae^{0} \quad \Rightarrow \quad A = 10$$

$$m(t=20) = 5 = 10e^{-k\times 20} \quad \Rightarrow \frac{1}{2} = e^{-20k} \quad \Rightarrow \; -20k = \ln\frac{1}{2}$$

$$\Rightarrow \quad k = -\frac{1}{20}\ln\frac{1}{2}$$

$$\Rightarrow \quad k = \frac{1}{20}\ln 2$$

or $k \approx 0.035$

From the general solution, $m_{gs} = Ae^{-kt}$

The required particular solution is: $\boxed{\boldsymbol{m(t) = 10e^{-0\cdot 035t}}}$

Now we need to determine the time when $m(t) = 1$ mg.

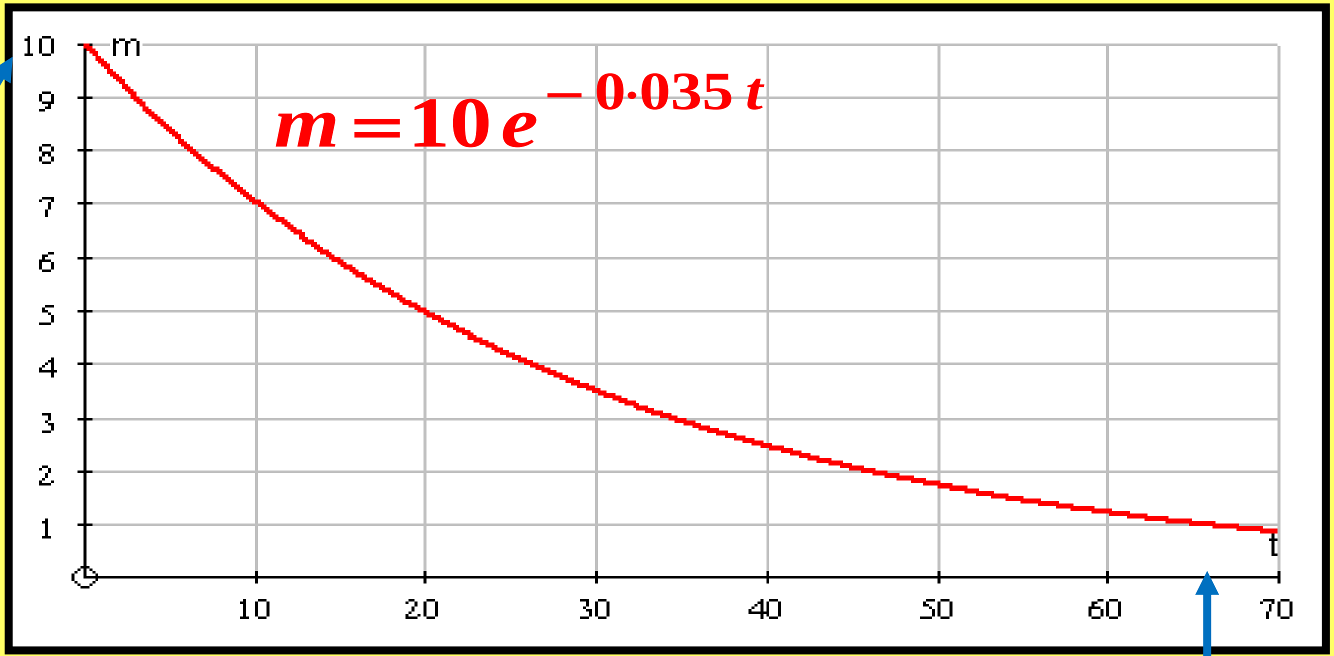
$$m(t) = 1 = 10e^{-0\cdot 035t} \quad \Rightarrow \quad 0.1 = e^{-0\cdot 035t}$$

$$-0\cdot 035t = \ln(0.1) \quad \Rightarrow \quad \boldsymbol{t = 66}$$

So it takes 66 days until the mass of remaining undecayed material is 1mg.

Modelling with ODEs: Exponential Decay

The graph showing the decay function is:



mass at the start of the measurements ($m = 10$ mg)

time when mass is 1 mg ($t = 66\ days$)

Example: Deriving Rates Yourself

E.g. 9

Water in a manufacturing plant is held in a large cylindrical tank of diameter 20m.
Water flows out of the bottom of the tank through a tap at a **rate proportional to the cube root of the volume**.

(a) Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

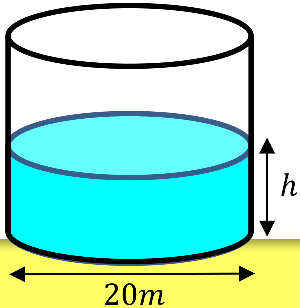
(b) Show that the general solution of this differential equation may be written

$h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants.

Initially the height of the water is 27m. 10 minutes later, the height is 8m.

(c) Find the values of the constants P and Q.

(d) Find the time in minutes when the water is at a depth of 1m.



Important first observation:
(a) is a 'connected rate of change' question in the style we saw in Week 1. We need to use chain rule to connect h and t.

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

a

$$V = \pi r^2 h$$

$$r = 10\,m \Rightarrow V = 100\pi h$$

$$\frac{dV}{dh} = 100\pi$$

$$\frac{dh}{dV} = \frac{1}{100\pi}$$

Using the info given:

$$\frac{dV}{dt} = -c\sqrt[3]{V} = -c\sqrt[3]{100\pi h}$$

If water is flowing out, the rate of change is negative.

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{100\pi} \times \left(-c\sqrt[3]{100\pi h}\right)$$

$$\therefore \frac{dh}{dt} = -k\sqrt[3]{h} \quad \text{where } k = \frac{c\sqrt[3]{100\pi}}{100\pi}$$

Example: Deriving The Rate Yourself

E.g. 9

Water in a manufacturing plant is held in a large cylindrical tank of diameter 20m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume.

(a) Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

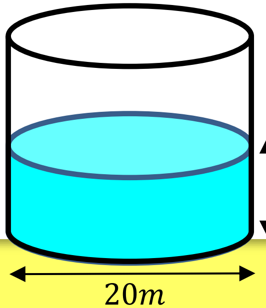
(b) Show that the general solution of this differential equation may be written

$h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants.

Initially the height of the water is 27m. 10 minutes later, the height is 8m.

(c) Find the values of the constants P and Q.

(d) Find the time in minutes when the water is at a depth of 1m.



b

$$\frac{dh}{dt} = -kh^{\frac{1}{3}} \quad \rightarrow \quad h^{-\frac{1}{3}}\frac{dh}{dt} = -k$$

$$\int h^{-\frac{1}{3}} dh = \int -k \, dt$$

$$\frac{3}{2}h^{\frac{2}{3}} = -kt + c$$

$$h^{\frac{2}{3}} = -\frac{2}{3}kt + \frac{2}{3}c$$

$$h = \left(-\frac{2}{3}kt + \frac{2}{3}c\right)^{\frac{3}{2}}$$

$$\boldsymbol{h = (P - Qt)^{\frac{3}{2}}}$$

c

When $t = 0, h = 27$

$$27 = P^{\frac{3}{2}} \quad \Rightarrow \quad \boldsymbol{P = 9}$$

When $t = 10, h = 8$

$$8 = (9 - 10Q)^{\frac{3}{2}}$$

$$4 = 9 - 10Q$$

$$\boldsymbol{Q = \frac{1}{2}}$$

d

$$h = \left(9 - \frac{1}{2}t\right)^{\frac{3}{2}}$$

$$1 = \left(9 - \frac{1}{2}t\right)^{\frac{3}{2}}$$

$\boldsymbol{t}$ **= 16 minutes**

Diagnostic Question

A mass of reactant $m(t)$ decreases at a rate proportional to the mass present. Which of the following expressions is **incorrect**?

Y $$\frac{dm}{dt} = -km \quad (k > 0)$$

M $$\frac{dm}{dt} \propto m$$

C $$\frac{dm}{dt} = km \quad (k < 0)$$

A $$\frac{dm}{dt} = -km \quad (k < 0)$$

Diagnostic Question

A General solution for the number of bacteria after t minutes is given by $x(t) = Ae^{kt}$. It is known that there were initially 100 bacteria and that after 5 minutes there are 3200. What are the correct values of A and k?

Y $A = 100, k = 32$

M $A = 100, k = \ln 2$

C $A = 3200, k = \ln \frac{1}{2}$

A $A = 100, k = \ln 32$

Diagnostic Question

The rate of increase of a reactant R is given by $\frac{dR}{dt} = R + 3$. Find a general solution.

Y $R = Ae^{3t}$

M $R = Ae^{-3t}$

C $R = Ae^{kt} - 3$

A $R = Ae^{kt}$

Chemical Engineering Applications

Chemical Engineering: Batch Reactor

E.g. 8

Batch Reactor – 1st Order Kinetics
(1st order means only 1 concentration involved)

$$\boldsymbol{A \rightarrow R}$$

(Chemical component A is converted to chemical component R by chemical reaction)

Let N_A be the number of moles of reactant A at time t during a reaction.
If the number of moles of reactant A is initially N_{A_0}, find an equation for N_A

Material balance:

$In\ flow = Out\ flow + Rate\ of\ Disapperance + Rate\ of\ Accumulation$

Note: In a batch reactor, no fluid enters or leaves the reactor.
So $in\ flow = out\ flow = 0$

$$0 = 0 + V(-r_A) + V\frac{dN_A}{dt}$$

V: Volume of reacting fluid
N_A: Number of moles of reactant A
$-r_A$: rate of disappearance of A
(or rate of accumulation of R)
k: reaction rate constant
(determined by temperature inside reactor)

Rearrange : $$V\frac{dN_A}{dt} = -V(-r_A)$$

Divide by V: $$\frac{dN_A}{dt} = -\,(-r_A)$$

Chemical Engineering: Batch Reactor

E.g. 8

$$\frac{dN_A}{dt} = -\,(-r_A)$$

The rate of disappearance of A is directly proportional to its concentration,

$$\therefore (-r_A) = kN_A$$

So: $$\frac{dN_A}{dt} = -kN_A$$

Separate variables: $$\frac{dN_A}{N_A} = -kdt$$

Bounds:

- We integrate no. of moles of A between its initial value, N_{A_0} and value at the current time, N_A.
- We integrate time from 0 at the start of the reaction, to current time t.

$$\int_{N_{A_0}}^{N_A} \frac{dN_A}{N_A} = -k\int_0^t dt$$

Perform integration:

$$[\ln(N_A)]_{N_{A_0}}^{N_A} = -k[t-0]$$

$$\ln(N_A) - \ln(N_{A_0}) = -kt$$

$$\ln\left(\frac{N_A}{N_{A_0}}\right) = -kt$$

$$\frac{N_A}{N_{A_0}} = e^{-kt}$$

Finally:

$$\boxed{N_A = N_{A_0}e^{-kt}}$$

This equation gives the number of moles of reactant A present in the reactor at time t .

E.g. 8

Alternative Method

Note: We did this a s a **definite integral** (preferred method)

$$\int_{N_{A_0}}^{N_A} \frac{dN_A}{N_A} = -k \int_0^t dt$$

However, an alternative approach would be to perform an indefinite integral then substitute initial conditions:

$$\int \frac{dN_A}{N_A} = -k \int dt \qquad \Rightarrow \ln N_A = -kt + C \qquad \Rightarrow N_A = e^{-kt+C}$$

$$\Rightarrow N_A = Ae^{-kt}$$

Noting that initially, $t = 0$ and $N_A = N_{A_0}$, therefore:

$$N_{A_0} = Ae^0 = A$$

And so

$$\boxed{N_A = N_{A_0} e^{-kt}}$$

Note: The separation of variables method may be the most important mathematical technique you will learn as it will be used constantly throughout the Chemical Engineering course… so make sure you can do it!!

ChemEng: Mixing Problems - Setting Up ODEs

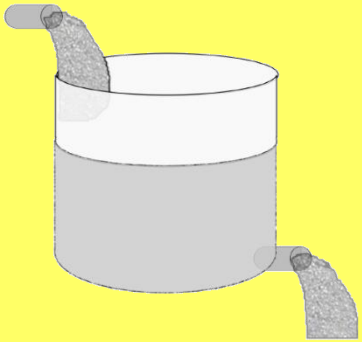
- "Mixing Problems" come in many varieties, but typically involve a tank of brine (water containing a certain amount of salt in fresh water).
- In general, some concentration of mixture (or pure water) flows into a container at a certain rate, and a mixture flows out at a certain rate (outflow rate may or may not be the same as the inflow rate).
- The goal is usually to determine how much "stuff" is in the container at a given time.
- The general idea is summarized by the equation:

$$Rate\ of\ change\ of\ substance\ =\ Rate\ in - Rate\ out$$

- In all such problems it is assumed that the solution is **"well mixed"** at each instant of time.
- "Well mixed" means that we assume that the instant the water enters the tank it somehow instantly disperses evenly throughout the tank to give a uniform concentration of salt in the tank at every point.
- In practice, many industrial tanks contain a spinning mixing device, making this assumption more valid.

ChemEng: Mixing Problems - Setting Up ODEs

Single Tank



A 50 gallon tank of pure water has a brine mixture with concentration of 2 pounds per gallon entering at the rate of 5 gallons per minute. At the same time the well-mixed contents drain out at the rate of 5 gallons per minute.

Find the amount of salt in the tank at time t.

What would happen as $t \to \infty$?

Let $x(t)$ be the amount (in lbs) of salt in the tank at time t.

Then the rate at which the salt in the tank increases is the amount of salt entering the tank minus that leaving the tank.

So the rate of change of amount of salt, $\frac{dx}{dt}$ has units of lb/min.

$$\frac{dx}{dt} = x_{in} - x_{out}$$

where $x(t)$ is the amount of salt in the tank at time t.

Liquid Flow in= Liquid Flow out

Tank stays at 50 gallons throughout.

Therefore, the concentration in the tank is $\frac{x}{50}$, which will change with time as $x = x(t)$.

General differential equation to solve:

$$\frac{dx}{dt} = C_{in} r_{in} - C_{out} r_{out}$$

Where:

C_{in} is the concentration of substance being added.

r_{in} is the rate at which the substance is added.

C_{out} is the concentration of substance being removed.

r_{out} is the rate at which the substance is removed.

ChemEng: Mixing Problems - Setting Up ODEs

Single Tank

The amount of salt entering per minute is given by the concentration entering times the rate at which the brine enters.

$$\left(2\,\frac{lb}{gal}\right) \times \left(5\,\frac{gal}{min}\right) = 10\,\frac{\text{lb}}{\text{min}}$$

(which gives the correct units)

Similarly, the rate of salt leaving is:

$$\left(\frac{x}{50}\frac{lb}{gal}\right) \times \left(5\,\frac{gal}{min}\right) = \frac{x}{10}\frac{\text{lb}}{\text{min}}$$

The equation is to be solved is then:

$$\frac{dx}{dt} = 10 - \frac{x}{10}$$

(we want to find x in terms of t)

Common denominator (as we're going to have to flip it over to separate variables)

$$\frac{dx}{dt} = \frac{100 - x}{10}$$

Separate variables: $$\int \frac{10}{100 - x}\,dt = \int 1\,dt$$

Integrate: $$-10\ln(100 - x) = t + C$$

(We could use boundary conditions to determine C here but we'll do it shortly)

$$\ln(100 - x) = -\frac{t}{10} + C'$$

Rearrange for x: $$100 - x = e^{-\frac{t}{10} + C'} = Ae^{-\frac{t}{10}}$$

$$\therefore x = 100 - Ae^{-\frac{t}{10}}$$

To find relationship we need a boundary condition.

In real-life applications you will need to work out the boundary conditions from context.

From the description, the tank only contains pure water initially so $\boldsymbol{x = 0}$ when $\boldsymbol{t = 0}$:

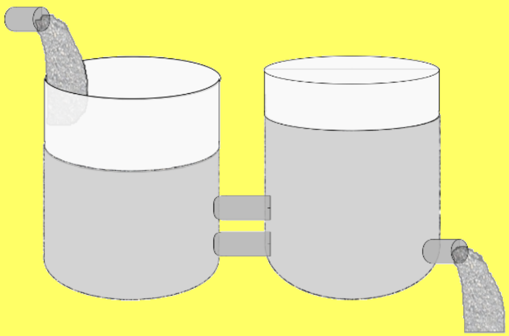
$$0 = 100 - A \Rightarrow A = 100$$

$$\boldsymbol{\therefore x(t) = 100\left(1 - e^{-\frac{t}{10}}\right)}$$

As $t \to \infty$ $x(t) \to 100(1 - 0) = 100\ lbs$
(This is the maximum amount of salt in the tank).

ChemEng: Mixing Problems - Setting Up ODEs

Double Tank



Tank X initially contains 100 gallons of brine made with 100 lbs of salt.
Tank Y initially contains 100 gallons of pure water.
Pure water is pumped into tank X at rate of 2 gal/min.
Some of the mixture of brine and pure water flows into tank Y at a rate of 3 gal/min.
To keep the tank levels the same, some of the mixture in tank Y flows back into tank X at a rate of 1 gallon per minute and 2 gallons per minute drains out.
Find the amount of salt in the tanks at any given time.
What would happen as $t \to \infty$?

(Two tanks, X and Y connected together)

$$\frac{dx}{dt} = C_{in}r_{in} - C_{out}r_{out}$$

Let $x(t)$ be the amount of salt in tank X and $y(t)$ the amount of salt in tank Y.

Tanks stay at 100 gallons throughout.

Therefore, the concentration in tank X is $\frac{x}{100}$, the concentration in tank Y is $\frac{y}{100}$ which changes with time as $x = x(t)$ and $y = y(t)$.

ChemEng: Mixing Problems - Setting Up ODEs

Double Tank

$$\frac{dx}{dt} = C_{in}r_{in} - C_{out}r_{out}$$

We work out how much "mixture" (not pure water) flows in and out of each tank, then set up an ODE for each:

Tank X:

In: 1 gal/min from Y **(1 gal/min of mixture with concentration of tank Y in)**
Out: 3 gal/min to Y **(3 gal/min of mixture with concentration of tank X out)**

$$\frac{dx}{dt} = \left(\frac{y}{100}\right) \times 1 - \left(\frac{x}{100}\right) \times 3 \quad \Rightarrow \frac{dx}{dt} = \frac{y}{100} - \frac{3x}{100}$$

Tank Y:

In: 3 gal/min from X **(3 gal/min of mixture with concentration of X in)**
Out: 1 gal/min back to X and 2 gal/min out of system **(3 gal/min of mixture with concentration Y out)**

$$\frac{dy}{dt} = \left(\frac{x}{100}\right) \times 3 - \left(\frac{y}{100}\right) \times 3 \quad \Rightarrow \frac{dy}{dt} = \frac{3x}{100} - \frac{3y}{100}$$

We have formed a system of two first order linear, homogenous constant coefficient differential equations, which you will learn methods for in the future.

What would happen as $t \to \infty$?
As mixture is lost from the system and pure water is added, clearly after some time, both tanks will contain only pure water.

Thanks

See you in the Tutorial!