CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

University of Leeds

Level 1 Semester 2

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Tutorial: Question Difficulty Colour Code

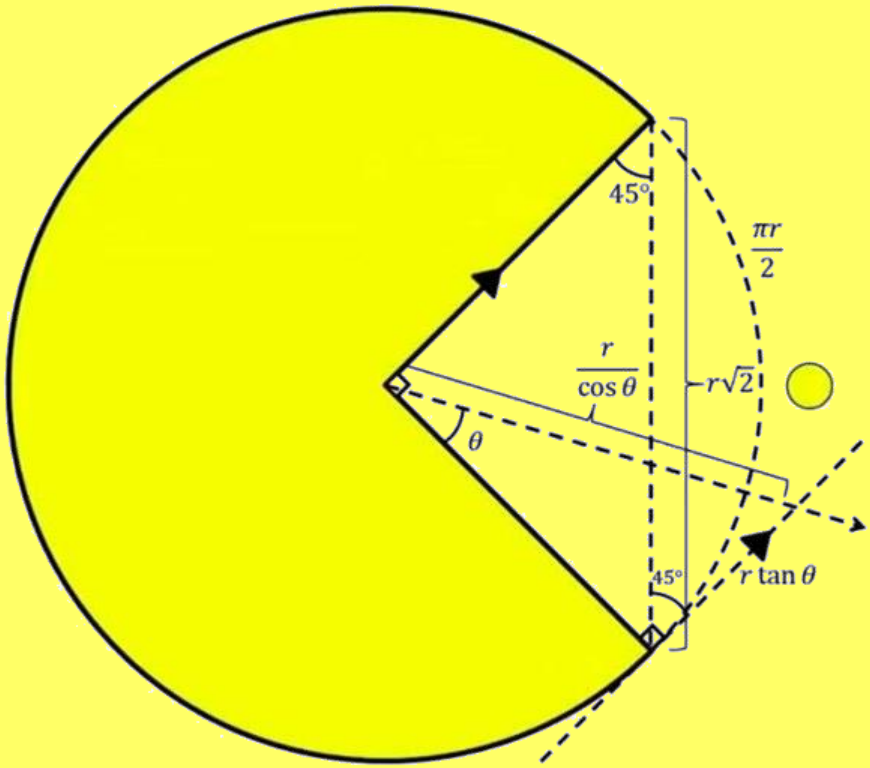
Basic - straightforward application
(you must be able to do these)

Medium – Makes you think a bit
(you must be able to do these)

Hard – Makes you think a lot
(you should be able to do these)

Extreme – Tests your understanding to the limit!
(for those who like a challenge)

Applied – Real-life examples of the topic, may sometimes involve prior knowledge
(you should attempt these – will help in future engineering)



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Tutorial 3
Hyperbolic Trig & Further Integration

Class Example: Hyperbolic Trig

E.g. 1

$y = \frac{1}{2}\ln(\coth x)$, $\quad x > 0 \quad$ Show that: $\frac{dy}{dx} = -\operatorname{cosech} 2x$

From table of derivatives:
$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$
$$\therefore \frac{1}{\coth x} = \frac{\sinh x}{\cosh x}$$

From Formula sheet:
$$\sinh(A \pm B) = \sinh A \ \cosh B \pm \cosh A \ \sinh B$$

So $\sinh 2x = 2 \sinh x \cosh x$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{\coth x} \times -\operatorname{cosech}^2 x$$

$$= -\frac{\sinh x}{2\cosh x} \times \frac{1}{\sinh^2 x}$$

$$= -\frac{1}{2\sinh x \cosh x}$$

$$= -\frac{1}{\sinh 2x}$$

$$= -\mathbf{cosech}\, \mathbf{2x}$$

Class Example: Combining identities

E.g. 2

Show that $\int_5^8 \frac{1}{\sqrt{x^2-16}}\,dx = \ln\left(\frac{2+\sqrt{3}}{2}\right)$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C, \qquad x > a$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1$$

$$\int_5^8 \frac{1}{\sqrt{x^2-16}}\,dx = \left[\cosh^{-1}\left(\frac{x}{4}\right)\right]_5^8$$

$$= \cosh^{-1} 2 - \cosh^{-1}\left(\frac{5}{4}\right)$$

$$= \ln(2+\sqrt{3}) - \ln\left(\frac{5}{4} + \sqrt{\frac{9}{16}}\right)$$

$$= \ln(2+\sqrt{3}) - \ln 2$$

$$= \ln\left(\frac{2+\sqrt{3}}{2}\right)$$

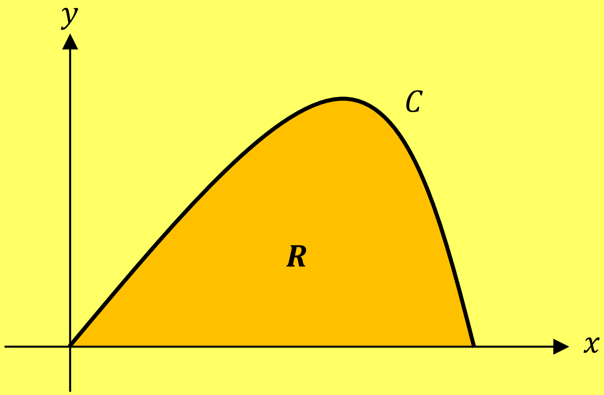
Class Example: Parametric Integration

E.g. 3

The graph shows the curve C with parametric equations:

$x = \cos t\,, \qquad y = \sin 2t\,, \qquad t \geq 0$

Determine the area bound between the curve and the x-axis.



$$\int y\ dx = \int y \frac{dx}{dt}\ dt$$

Find limits: At x-axis $y = 0$

$$\sin 2t = 0 \quad \rightarrow \quad 2t = 0, \pi$$

$$t = 0, \frac{\pi}{2}$$

When $t = 0, x = 1$

When $t = \frac{\pi}{2}, x = 0$

Be careful: we need the values of t corresponding to the roots x of the graph. Ensure the order is preserved.

$$\frac{dx}{dt} = -\sin t$$

$$\int_0^1 y\ dx = \int_{t_1}^{t_2} y \frac{dx}{dt}\ dt = \int_{\frac{\pi}{2}}^{0} \sin 2t\ \ (-\sin t)\ dt$$

$$= \int_{\frac{\pi}{2}}^{0} (-2\sin^2 t \cos t)\ dt$$

$$= \left[-\frac{2}{3}\sin^3 t\right]_{\frac{\pi}{2}}^{0}$$

By inspection, or substitute $u = \sin^2 t$

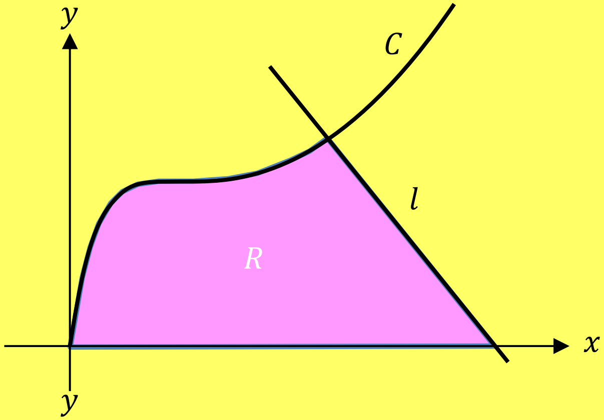
$$= -\frac{2}{3}\sin^3 0 + \frac{2}{3}\sin^3 \frac{\pi}{2} = \mathbf{\frac{2}{3}}$$

Class Example: Parametric Integration

E.g. 4 The graph shows the curve C with parametric equations:

$$x = (t+2)^2, \qquad y = \frac{1}{2}t^3 + 4, \qquad t \geq -2$$

The line l is normal to the curve when $t = 1$.
Determine the equation of l.
Hence determine the area of the region R.



$$\frac{dx}{dt} = 2(t+2)$$

$$\frac{dy}{dt} = \frac{3}{2}t^2$$

$$\frac{dy}{dx} = \frac{\frac{3}{2}t^2}{2(t+2)}$$

To find instantaneous gradient of line when $t = 1$, use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

When $t = 1$, $m_T = \frac{1}{4} \rightarrow m_N = -4$

Determine (x, y) when $t = 1$.

$$x = (1+2)^2 = 9, y = \frac{1}{2}(1^3) + 4 = \frac{9}{2}$$

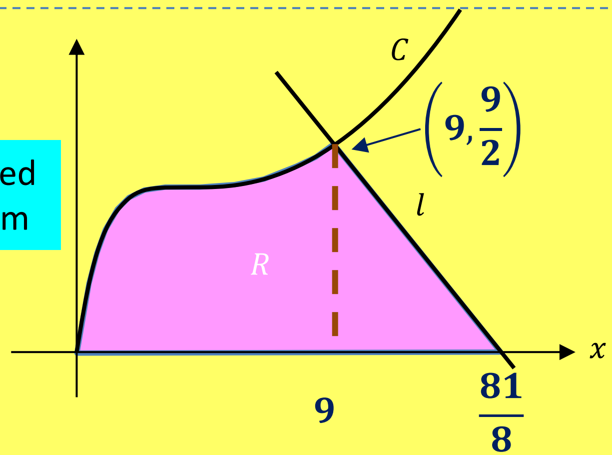
Use $y - y_1 = m(x - x_1)$

Equation of l: $y - \frac{9}{2} = -4(x - 9)$

Updated diagram

x-intercept: $-\frac{9}{2} = -4(x-9) \rightarrow x = \frac{81}{8}$

Area of triangle: $\frac{1}{2} \times \left(\frac{81}{8} - 9\right) \times \frac{9}{2} = \frac{81}{32}$



Updated diagram

When $x = 0$, $t = -2$, when $x = 9, t = 1$ (given)

Area of curved region:

$$\int_{-2}^{1} \left(\frac{1}{2}t^3 + 4\right) 2(t+2)\, dt$$

$$= \int_{-2}^{1} (t^4 + 2t^3 + 8t + 16)\, dt$$

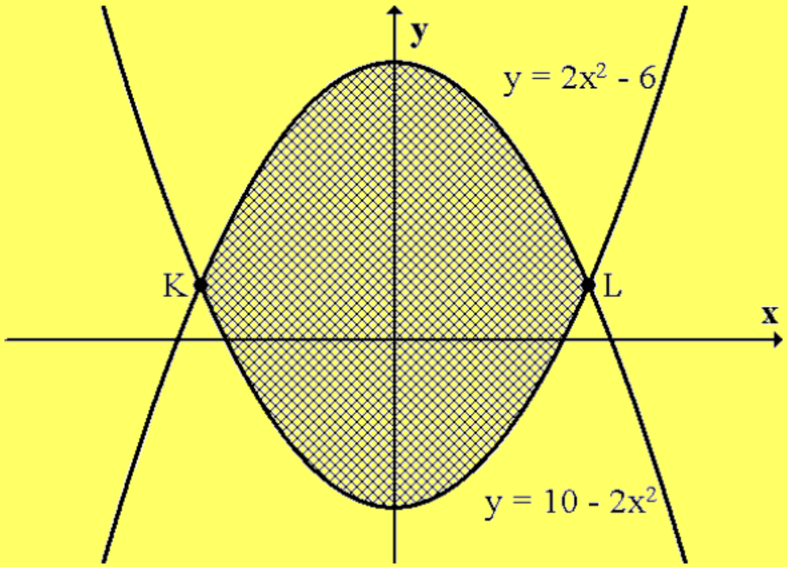
$$= \frac{351}{10}$$

Area of R: $\frac{351}{10} + \frac{81}{32} = \mathbf{\frac{6021}{160}}$

Class Example: Area Between 2 Curves

E.g. 5

The curves with equations $y = 2x^2 - 6$ and $y = 10 - 2x^2$ intersect at K and L. Calculate the area enclosed by these two curves.



$$2x^2 - 6 = 10 - 2x^2$$

$$4x^2 = 16 \Rightarrow x = \pm 2$$

$$Area = \int_{-2}^{2} (10 - 2x^2) - (2x^2 - 6)\, dx$$

$$\int_{-2}^{2} 16 - 4x^2\, dx = \left[16x - \frac{4}{3}x^3\right]_{-2}^{2} = \frac{128}{3}$$

Tip: If the labelling of the graphs is ever unclear, use their shapes to identify which is which.

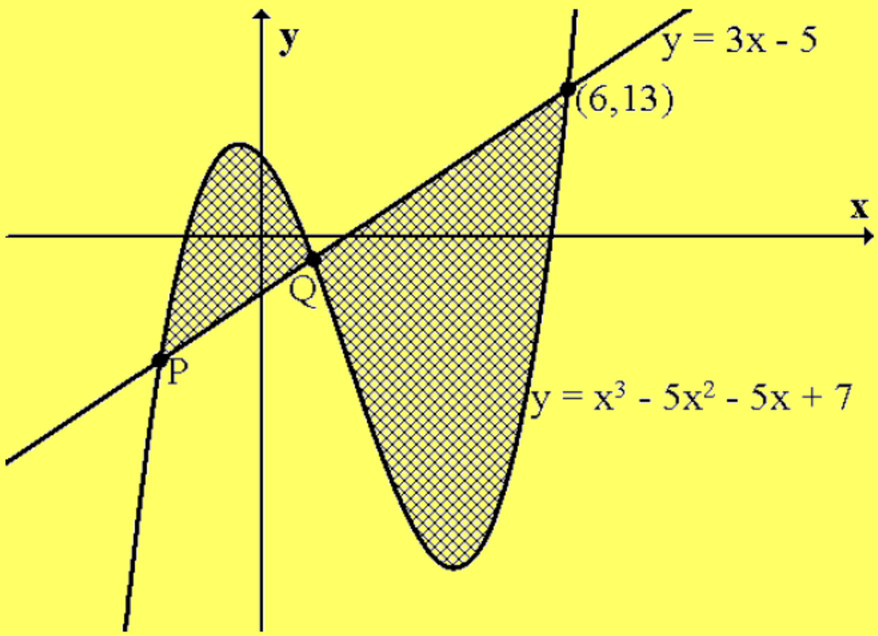
Class Example: Area Between 2 Curves

E.g. 6

The diagram shows the line $y = 3x - 5$ and the curve $y = x^3 - 5x^2 - 5x + 7$.

(a) Find the coordinates of P and Q.

(b) Calculate the shaded area



a) $x^3 - 5x^2 - 5x + 7 = 3x - 5$

$$x^3 - 5x^2 - 8x + 12 = 0$$

At point B, $x = 6 \Rightarrow (x-6)$ is a factor (by factor theorem)

$$x^3 - 5x^2 - 5x + 12 = (x-6)(x^2 + ax - 2)$$

$$= (x-6)(x-1)(x+2)$$

$$x = 1 \rightarrow y = -2 \Rightarrow Q(1, -2)$$

$$x = -2 \rightarrow y = -11 \Rightarrow P(-2, -11)$$

b) $A_1 = \int_{-2}^{1} (x^3 - 5x^2 - 5x + 7) - (3x - 5)\,dx$

$$\int_{-2}^{1} x^3 - 5x^2 - 8x + 12\,dx$$

$$= \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 - 4x^2 + 12x\right]_{-2}^{1}$$

$$= \left(\frac{79}{12}\right) - \left(-\frac{68}{3}\right) = \frac{117}{4}$$

$$A_2 = \int_{1}^{6} (3x - 5) - (x^3 - 5x^2 - 5x + 7)\,dx$$

$$\int_{1}^{6} -x^3 + 5x^2 + 8x - 12\,dx$$

$$= \left[-\frac{1}{4}x^4 + \frac{5}{3}x^3 + 4x^2 - 12x\right]_{1}^{6}$$

$$= (108) - \left(-\frac{79}{12}\right) = \frac{1375}{12}$$

$$A = \frac{117}{4} + \frac{1375}{12} = \mathbf{\frac{863}{6}}$$

Exercise A: Hyperbolic Trig – Using Exponential Form

For these questions you should use: $\sinh ax = \frac{e^{ax}-e^{-ax}}{2}$, $\cosh ax = \frac{e^{ax}+e^{-ax}}{2}$

1 Show that: $\int \sinh x \, dx = \cosh x + C$

2 Show that: $\cosh 2x = 2\cosh^2 x - 1$

Hint: Show RHS=LHS

3 Show that: $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

Hint: Show RHS=LHS

4 Find the value of:

$$\sinh(\ln 2)$$

5 Find:

$$\int e^{3x} \cosh x \; dx$$

6 Find:

$$\int \operatorname{cosech} x \; dx$$

Exercise B: Using Standard Integrals

Find the integrals by transforming them then using an appropriate standard integral from the list (in some cases you may need to complete the square)

1

$$\int \frac{1}{\sqrt{4x^2+9}}\,dx$$

2

$$9x^2+6x+5 \equiv a(x+b)^2+c$$

a) Find the values of constants a, b and c

b) Find $\displaystyle\int \frac{1}{9x^2+6x+5}\,dx$

c) Find $\displaystyle\int \frac{1}{\sqrt{9x^2+6x+5}}\,dx$

3 Determine $\displaystyle\int \frac{1}{\sqrt{12x+2x^2}}\,dx$

Standard Integrals (general cases):

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C, \qquad |x|<a$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$$

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln\left|x+\sqrt{x^2\pm a^2}\right|+C$$

Alternate forms:

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)+C, \qquad x>a$$

Exercise C: Parametric Integration

1

A function is defined parametrically using the equations:

$$x = \sqrt{t} - t \qquad y = \sqrt{t} + t$$

Determine $\int y\, dx$ in terms of t.

2

A function is defined parametrically using the equations:

$$x = \cos t \qquad y = \sin t$$

Determine $\int y\, dx$ in terms of t.

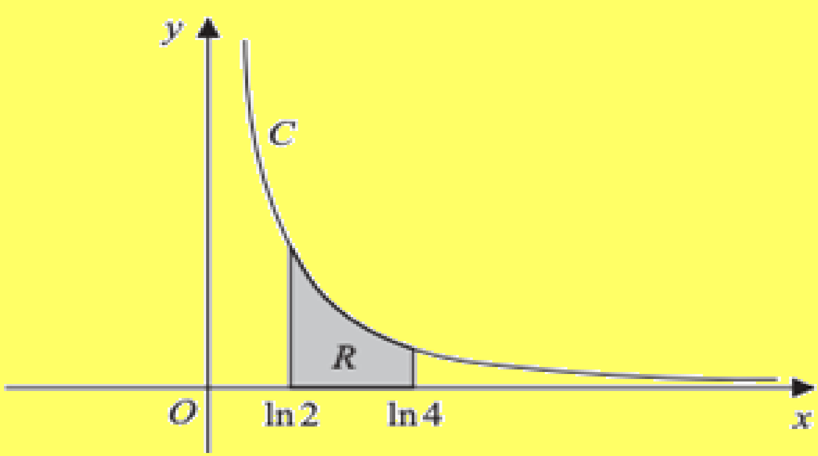
3

The graph shows a sketch of the curve C with parametric equations

$$x = \ln(t+2), \qquad y = \frac{1}{t+1}, \qquad t > -\frac{2}{3}$$

The finite region R, shown shaded in Figure 4, is bounded by the curve C, the line with equation $x = \ln 2$, the x-axis and the line with equation $x = \ln 4$

Use calculus to find the exact area of R.



4

The graph shows a sketch of part of the curve C with parametric equations

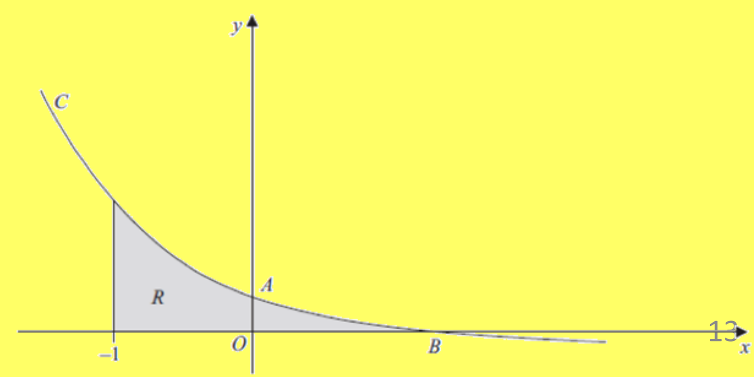
$$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

The point A has coordinates $(0,3)$ and the point B has coordinates $(1,0)$.

The region R, as shown shaded in Figure 2, is bounded by the curve C, the line $x = -1$ and the x-axis.

Use integration to find the exact area of R.

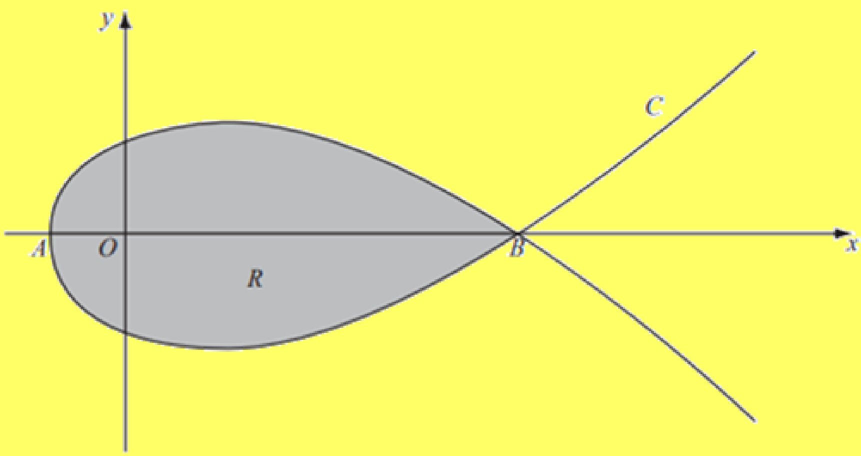


Exercise C: Parametric Integration

5 The diagram shows a sketch of the curve C with parametric equations:

$$x = 5t^2 - 4, \qquad y = t(9 - t^2)$$

Curve C cuts the x-axis at points A and B.



a) Find the coordinates of points A and B.

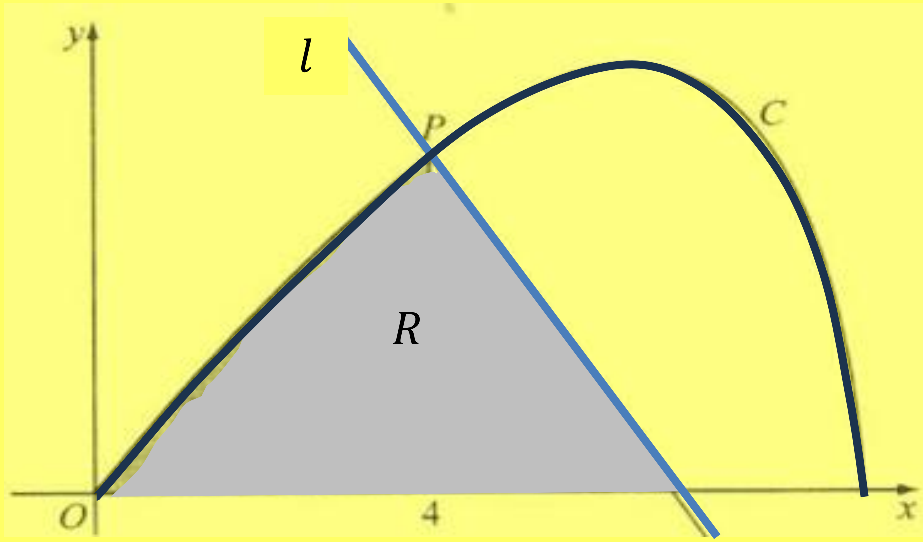
The shaded region R is enclosed by the loop of the curve.

b) Use integration to find the are of R.

6 The diagram shows a sketch of the curve C with parametric equations:

$$x = 8\cos t\,,\ y = 4\sin 2t \ \left(0 \le t \le \frac{\pi}{2}\right)$$

Point $P(4,2\sqrt{3})$ lies on C.



a) Find the value of t at point P

The line l is a normal to C at P.

b) Show that the equation of l is $y = -x\sqrt{3} + 6\sqrt{3}$.

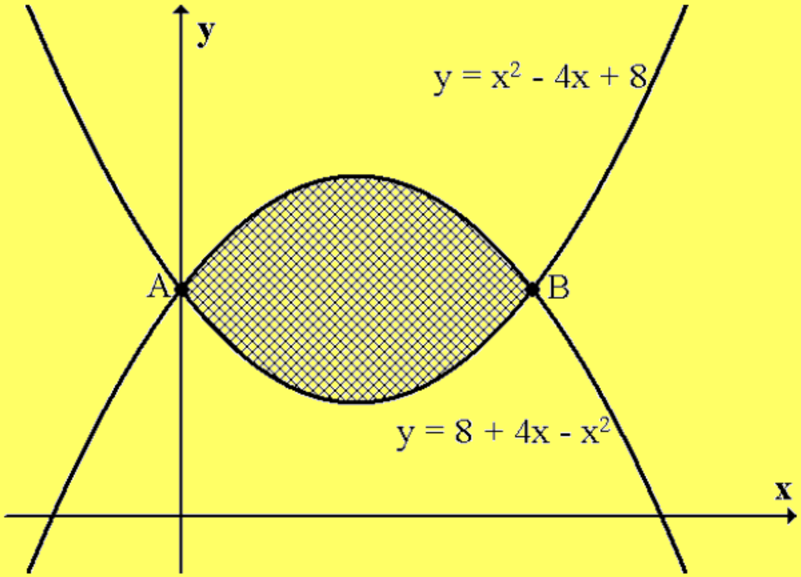
Region R is enclosed by curve C, the x-axis and the line l.

c) Find the exact area of R.

Exercise D: Area Between 2 Curves

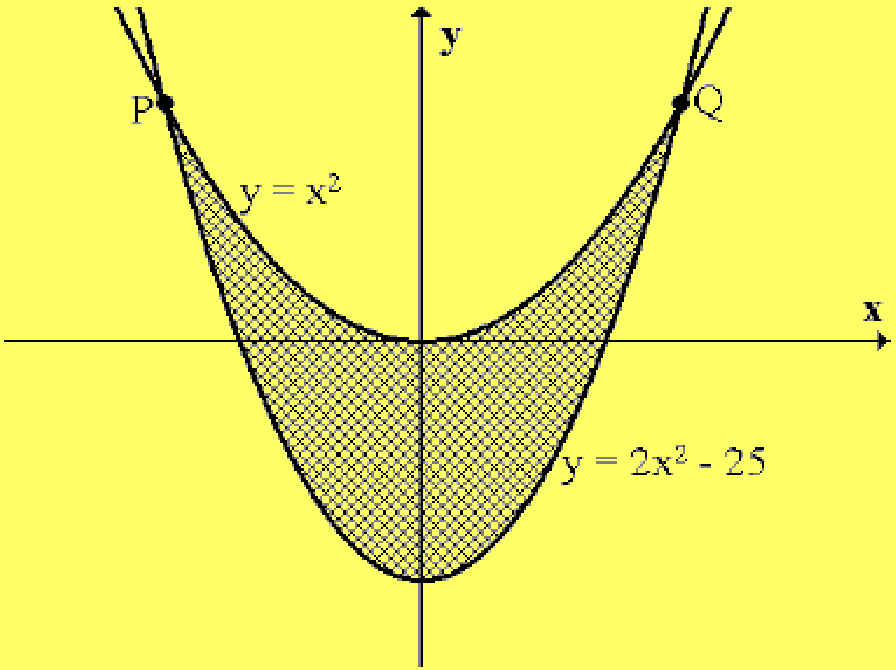
1

The curves with equations $y = x^2 - 4x + 8$ and $y = 8 + 4x - x^2$ intersect at A and B. Calculate the exact area enclosed Between the curves.



2

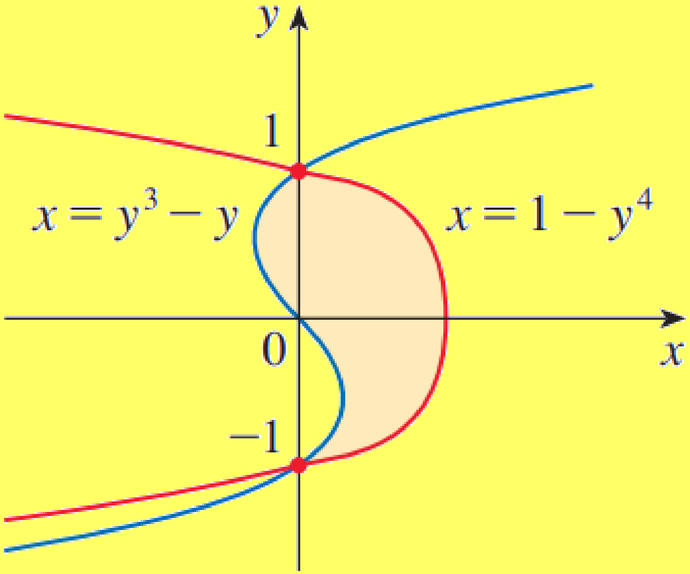
The curves with equations $y = x^2$ and $y = 2x^2 - 25$ intersect at P and Q. Calculate the exact area enclosed Between the curves.



Exercise D: Area Between 2 Curves

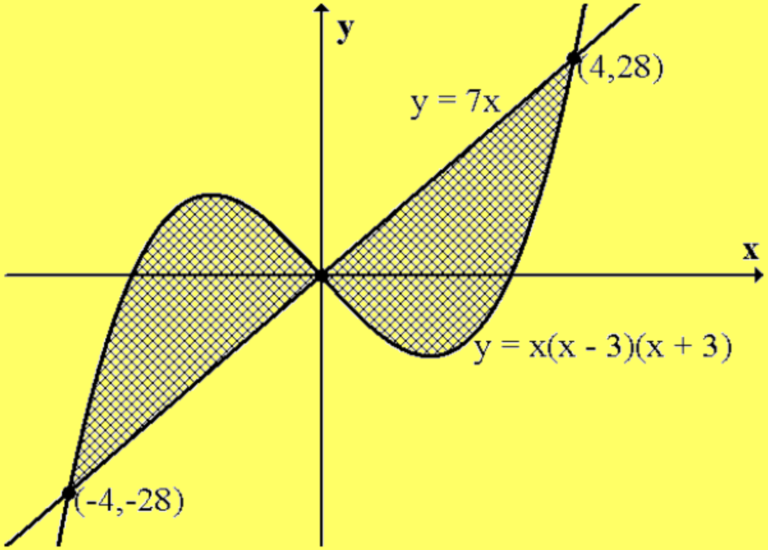
3

Find the exact shaded area



4

The curve $y = x(x - 3)(x + 3)$ and the line $y = 7x$ intersect at the points $(0,0), (-4,-28)$ and $(4,28)$.
Calculate the exact area enclosed by the curve and the line.

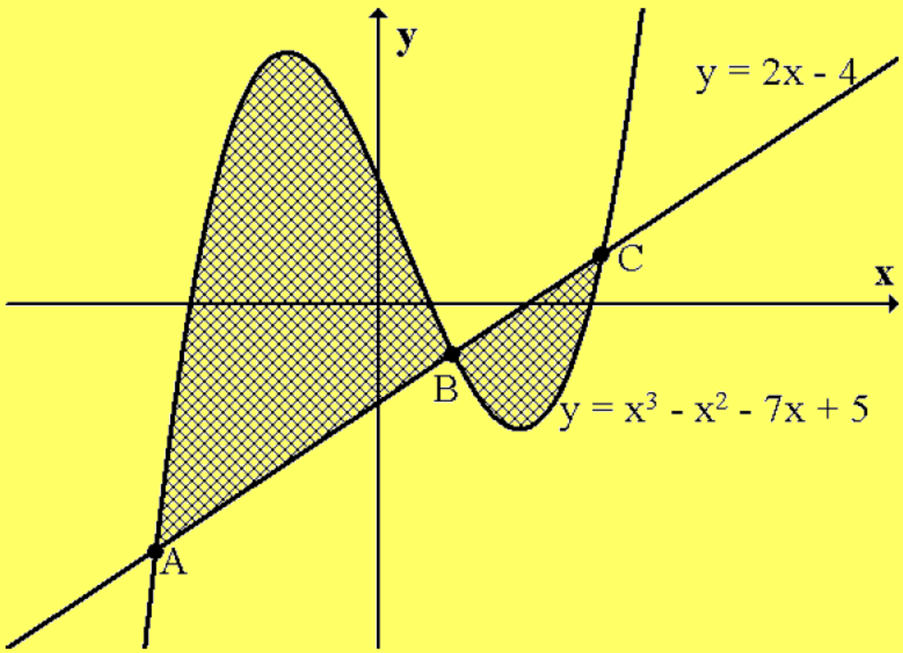


Exercise D: Area Between 2 Curves

5

The curve $y = x^3 - x^2 - 7x + 5$ and the line $y = 2x - 4$ are shown opposite.

(a) B has coordinates $(1, -2)$. Find the coordinates of A and C.
(b) Hence calculate the shaded area.

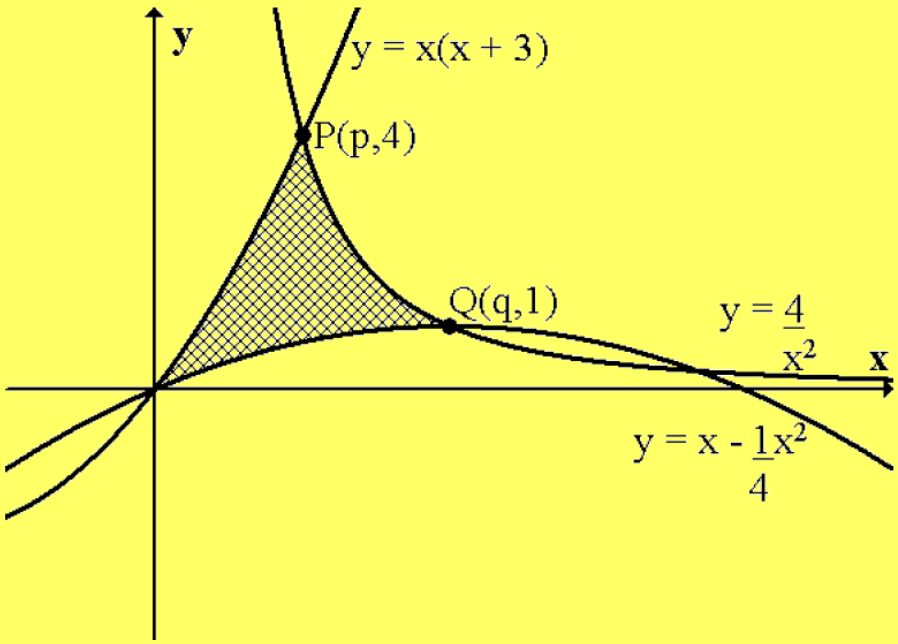


6

The diagram opposite shows an area enclosed by 3 curves:

$$y = x(x+3)$$
$$y = \frac{4}{x^2}$$
$$y = x - \frac{1}{4}x^2$$

(a) P and Q have coordinates $(p, 4)$ and $(q, 1)$. Find the values of p and q.
(b) Calculate the shaded area.

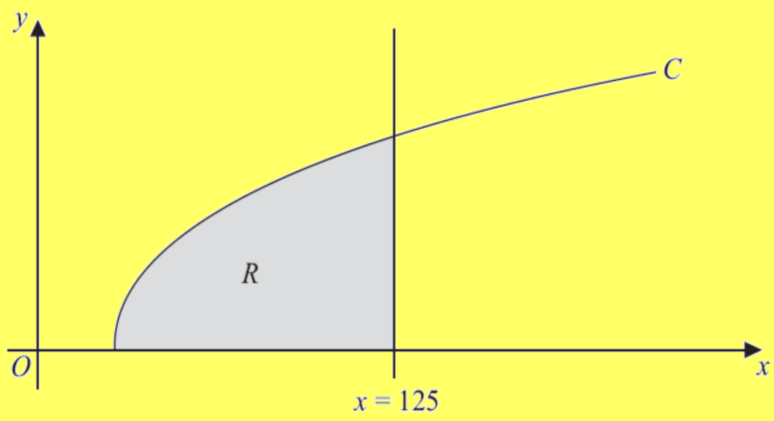


Exercise E: Volumes of Revolution

1

$$y = \left(x^{\frac{2}{3}} - 9\right)^{\frac{1}{2}}$$

The finite region R which is bounded by the curve C, the $\boldsymbol{x}$**-axis** and the line $x = 125$ is shown shaded in the diagram.
This region is rotated through $360°$ about the x-axis to form a solid of revolution.



Use calculus to find the exact value of the volume of the solid of revolution.

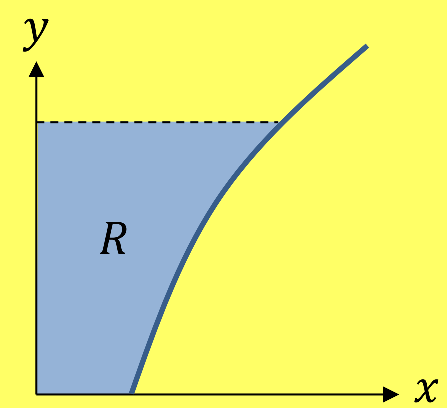
2

The diagram shows the curve with equation $y = 4\ln x$

The finite region R, shown in the diagram, is bounded by the curve, the x-axis, the y-axis and the line $y = 4$.

Region R is rotated by 2π radians about the $\boldsymbol{y}$**-axis**.

Use integration to show that the exact value of the volume of the solid generated is $2\pi\sqrt{e}(e^2 - 1)$.



Exercise E: Volumes of Revolution

3

This question combines parametric equations and volume of revolution.

The curve C has parametric equations

$$x = \ln t, \qquad y = t^2 - 2, \qquad t > 0.$$

Find

(*a*) an equation of the normal to C at the point where $t = 3$,

(6)

(*b*) a cartesian equation of C.

(3)

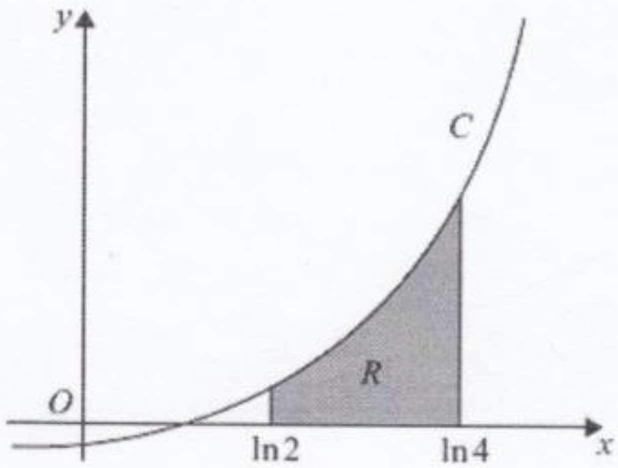


Figure 1

The finite area R, shown in Figure 1, is bounded by C, the x-axis, the line $x = \ln 2$ and the line $x = \ln 4$. The area R is rotated through 360° about the x-axis.

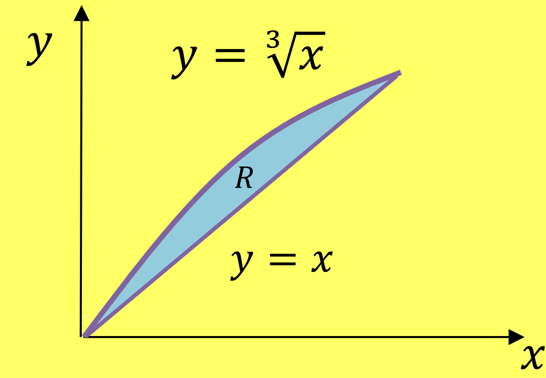
(*c*) Use calculus to find the exact volume of the solid generated.

(6)

Exercise E: Volumes of Revolution

4

The area between the lines with equations $y = x$ and $y = \sqrt[3]{x}$, where $x \geq 0$ is rotated $360°$ about the x-axis. Determine the volume of the solid generated.



5

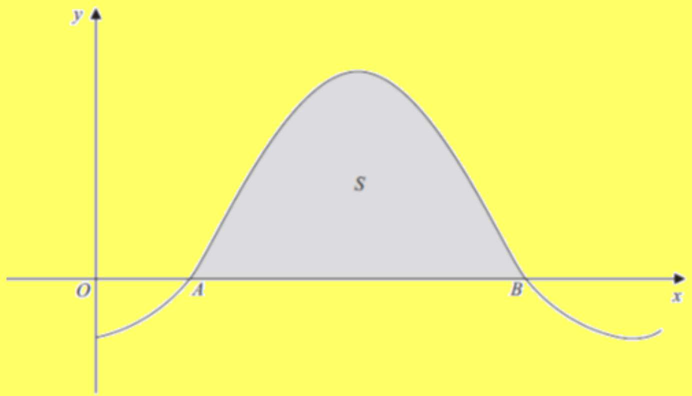


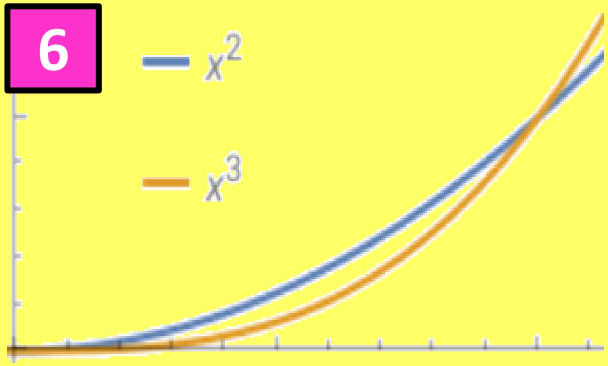
Figure 3 shows a sketch of part of the curve with equation $y = 1 - 2\cos x$, where x is measured in radians. The curve crosses the x-axis at the point A and at the point B.

(*a*) Find, in terms of π, the x coordinate of the point A and the x coordinate of the point B. **(3)**

The finite region S enclosed by the curve and the x-axis is shown shaded in Figure 3. The region S is rotated through 2π radians about the x-axis.

(*b*) Find, by integration, the exact value of the volume of the solid generated. **(6)**

6



Find the volume generated when the region trapped between the curves $y = x^2$ and $y = x^3$ is rotated $360°$ around the ***y*-axis** to obtain a bowl-like 3D shape.

Exercise E: Volumes of Revolution

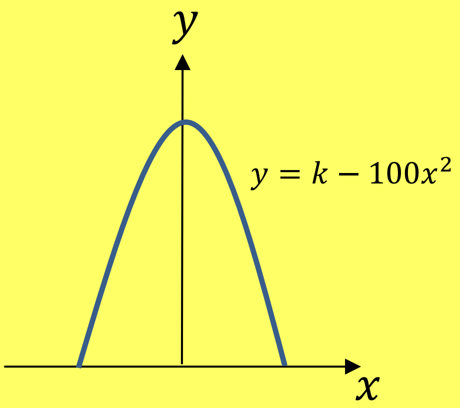
7

A manufacturer wants to cast a prototype for a new design for a pen barrel out of solid resin.
The region shown in the diagram is used as a model for the cross-section of the pen barrel.
The region is bounded by the x-axis and the curve with equation $y = k - 100x^2$, and will be rotated around the y-axis.
Each unit on the coordinate axes represents 1cm.

(a) Suggest a suitable value for k. (Assuming pens are 10cm long)

(b) Use your value of k to estimate the volume of resin needed to make the prototype.

(c) State one limitation of this model.



ANSWERS

Exercise A: Solutions

1

$$\int \sinh x \, dx = \frac{1}{2}\int e^x - e^{-x} dx = \frac{1}{2}(e^x + e^{-x}) + C = \cosh x + C$$

2

$$RHS = 2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$$

$$= \frac{2}{4}(e^{2x} + 2e^x e^{-x} + e^{-2x}) - 1$$

$$= \frac{1}{2}(e^{2x} + 2 + e^{-2x}) - 1$$

$$= \frac{e^{2x} + e^{-2x}}{2} + 1 - 1$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh 2x = LHS$$

3

$$RHS = \sinh x \cosh y + \cosh x \sinh y$$

$$= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{1}{4}\left(e^{x+y} + e^{x-y} - e^{y-x} - e^{-(x+y)} + e^{x+y} - e^{x-y} + e^{y-x} - e^{-(x+y)}\right)$$

$$= \frac{1}{4}\left(2e^{x+y} - 2e^{-(x+y)}\right) = \frac{1}{2}\left(e^{x+y} - e^{-(x+y)}\right) = \sinh(x+y) = LHS$$

4

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2}$$

$$= \frac{e^{\ln 2} - e^{\ln 2^{-1}}}{2}$$

$$= \frac{2 - 2^{-1}}{2} = \frac{2 - \frac{1}{2}}{2} = \mathbf{\frac{3}{4}}$$

5

$$\int e^{3x} \cosh x \; dx = \int e^{3x}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\int e^{4x} - e^{2x} \, dx$$

$$= \boldsymbol{\frac{1}{6}(e^{3x} - 3e^x) + C}$$

6

$$\int \operatorname{cosech} x \; dx = \int \frac{2}{e^x - e^{-x}} dx = \int \frac{2e^x}{e^{2x} - 1} dx$$

Use the substitution $u = e^x$

$$\frac{du}{dx} = e^x \;\; \therefore \; dx = \frac{e^x}{du}$$

$$\int \frac{2e^x}{e^{2x} - 1} dx = 2\int \frac{1}{u^2 - 1} du$$

Standard Integral: $\int \frac{1}{x^2-1} \; dx = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + C$

$$= 2 \times \frac{1}{2}\ln\left(\frac{u-1}{u+1}\right) + C$$

$$\boldsymbol{\int \operatorname{cosech} x \; dx = \ln\left(\frac{e^x - 1}{e^x + 1}\right) + C}$$

Exercise B: Solutions

1

$$\int \frac{1}{\sqrt{4x^2+9}}\,dx$$

$$= \int \frac{1}{2\sqrt{x^2+\frac{9}{4}}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{x^2+\frac{9}{4}}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{x^2+\left(\frac{3}{2}\right)^2}}\,dx$$

$$= \frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right) + C \quad or \quad \frac{1}{2}\ln\left|x+\sqrt{x^2+\frac{9}{4}}\right| + C$$

2

$9x^2+6x+5 \equiv a(x+b)^2+c$

a) Find the values of constants a, b and c

b) Find $\int \frac{1}{9x^2+6x+5}\,dx$

c) Find $\int \frac{1}{\sqrt{9x^2+6x+5}}\,dx$

(a)	$9x^2+6x+5 \equiv a(x+b)^2+c$	
	$a=9,\ b=\frac{1}{3},\ c=4$	
(b)	$\int \frac{1}{9(x+\frac{1}{3})^2+4}\,dx = \frac{1}{6}\arctan\left(\frac{3x+1}{2}\right)(+c)$	M1: $k\arctan\left(\frac{x+"\frac{1}{3}"}{\sqrt{"\frac{4}{9}"}}\right)$
		A1: $\frac{1}{6}\arctan\left(\frac{3x+1}{2}\right)$ oe
(c)	$\int \frac{1}{\sqrt{9(x+\frac{1}{3})^2+4}}\,dx = \frac{1}{3}\text{arsinh}\left(\frac{3x+1}{2}\right)(+c)$	M1: $k\,\text{arsinh}\left(\frac{x+"\frac{1}{3}"}{\sqrt{"\frac{4}{9}"}}\right)$
		A1: $\frac{1}{3}\text{arsinh}\left(\frac{3x+1}{2}\right)$ oe
		Allow $\frac{1}{\sqrt{9}}$

3

Determine

$$\int \frac{1}{\sqrt{12x+2x^2}}\,dx$$

$2x^2+12x = 2(x^2+6x)$
$= 2\left((x+3)^2-9\right)$

$$\int \frac{1}{\sqrt{12x+2x^2}}\,dx = \int \frac{1}{\sqrt{2\left((x+3)^2-9\right)}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(x+3)^2-9}}\,dx$$

Let $u = x+3 \ \rightarrow\ du = dx$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{u^2-9}}\,du$$

$$= \frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{x+3}{3}\right) + C$$

Exercise C: Solutions

1

$$x = t^{\frac{1}{2}} - t, \qquad \frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} - 1$$

$$\int y\,dx = \int y\frac{dx}{dt}\,dt$$
$$= \int \left(t^{\frac{1}{2}} + t\right)\left(\frac{1}{2}t^{-\frac{1}{2}} - 1\right)dt$$
$$= \int \left(\frac{1}{2} - t^{\frac{1}{2}} + \frac{1}{2}t^{\frac{1}{2}} - t\right)dt$$
$$= \int \left(\frac{1}{2} - \frac{1}{2}t^{\frac{1}{2}} - t\right)dt$$
$$= \boldsymbol{\frac{1}{2}t - \frac{1}{3}t^{\frac{3}{2}} - \frac{1}{2}t^2 + c}$$

2

$$x = \cos t, \quad \frac{dx}{dt} = -\sin t$$

$$\int y\,dx = \int y\frac{dx}{dt}\,dt = -\int \sin^2 t\ dt$$

$$\cos 2A = 1 - 2\sin^2 A \Rightarrow \sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

$$\therefore -\int \sin^2 t\ dt = -\frac{1}{2}\int 1 - \cos 2t\,dt$$
$$= -\frac{1}{2}\left(t - \frac{1}{2}\sin 2t + C_1\right)$$
$$= \boldsymbol{\frac{1}{2}t - \frac{1}{4}\sin 2t + C}$$

3

When $x = \ln 2$, $t = 0$
When $x = \ln 4$, $t = 2$

$$\int_0^2 y\frac{dx}{dt}\,dt = \int_0^2 \frac{1}{t+1}\frac{1}{t+2}\,dt$$
$$= \int_0^2 \frac{1}{(t+1)(t+2)}\,dt \qquad * \text{ see below}$$
$$= \int_0^2 \left(\frac{1}{t+1} - \frac{1}{t+2}\right)dt$$
$$= \left[\ln|t+1| - \ln|t+2|\right]_0^2$$
$$= (\ln 3 - \ln 4) - (\ln 1 - \ln 2)$$
$$= \ln\frac{3}{4} + \ln 2$$
$$= \boldsymbol{\ln\frac{3}{2}}$$

$$\frac{1}{(t+1)(t+2)} \equiv \frac{A}{t+1} + \frac{B}{t+2}$$
$$1 \equiv A(t+2) + B(t+1)$$
$$t = -1 \Rightarrow A = 1$$
$$t = -2 \Rightarrow B = -1$$

4

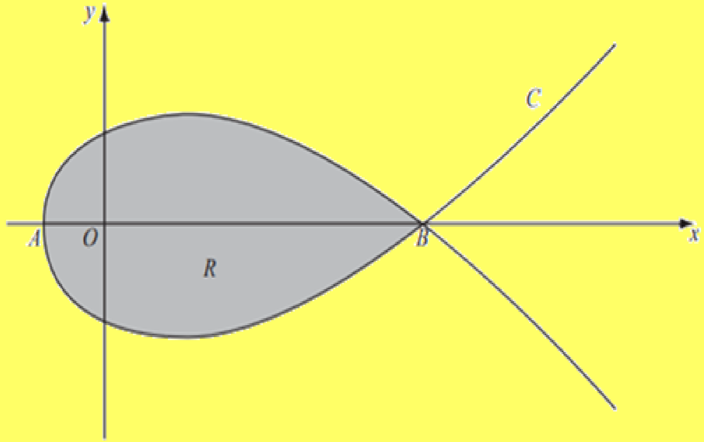
$$\frac{d}{dx}(a^x) = a^x(\ln a)$$
$$\therefore \int a^x\ dx = \frac{a^x}{\ln a} + c$$

When $x = -1$, $t = 4$
When $x = 1$, $t = 0$

$$\int_4^0 y\frac{dx}{dt}\,dt = \int_4^0 (2^t - 1)\left(-\frac{1}{2}\right)dt = \frac{1}{2}\int_0^4 (2^t - 1)\,dt$$
$$= \frac{1}{2}\left[\frac{1}{\ln 2}2^t - t\right]_0^4 = \frac{1}{2}\left(\left(\frac{16}{\ln 2} - 4\right) - \left(\frac{1}{\ln 2} - 0\right)\right)$$
$$= \frac{1}{2}\left(\frac{15}{\ln 2} - 4\right) = \boldsymbol{\frac{15}{2\ln 2} - 2}$$

Exercise C: Solutions

5



b) Area above x-axis$= I = \int_0^3 y\frac{dx}{dt}dt$

$$x = 5t^2 - 4 \Rightarrow \frac{dx}{dt} = 10t$$

$$I = \int_0^3 t(9-t^2)10t\,dt = \int_0^3 90t^2 - 10t^4\,dt$$

$$= [30t^3 - 2t^5]_0^3 = (30\times 3^3 - 2\times 3^5) - (0)$$

$$= 324$$

$$\therefore R = 2\times 324 = 648$$

a) x-axis: $y = 0$

$$t(9-t^2) = 0$$

$$\therefore t = 0, t^2 = 9 \Rightarrow t = \pm 3$$

At $t = 0 \;\rightarrow x = -4 \Rightarrow A(-4,0)$

At $t = \pm 3 \rightarrow x = 5(\pm 3)^2 - 4 = 41 \Rightarrow B(41,0)$

Exercise C: Solutions

6

a) $P(4,2\sqrt{3})$ $x=4$: $4=8\cos t \Rightarrow \cos t=\frac{1}{2} \Rightarrow \boldsymbol{t=\frac{\pi}{3}}$

Check: $y=2\sqrt{3}$: $4\sin\left(\frac{2\pi}{3}\right)=2\sqrt{3}$

(note: if you do $2\sqrt{3}=4\sin 2t \Rightarrow \sin 2t=\frac{\sqrt{3}}{2}$

$2t=\frac{\pi}{3},\frac{2\pi}{3}$ $(0\le 2t\le \pi)$, so $t=\frac{\pi}{3}$ agrees with x).

b) Normal: First find gradient of tangent to curve at P:

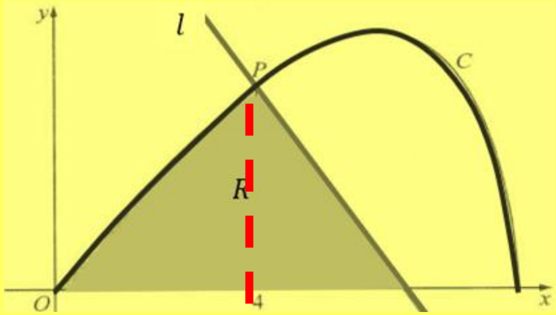
$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{8\cos 2t}{-8\sin t}=-\frac{\cos 2t}{\sin t}$$

$$m_t=-\frac{\cos\left(\frac{2\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)}=-\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}}$$

$$\therefore m_n=-\sqrt{3}$$

$$y-y_1=m(x-x_1)\Rightarrow y-2\sqrt{3}=-\sqrt{3}(x-4)$$

$$\therefore \boldsymbol{y=-x\sqrt{3}+6\sqrt{3}}$$



c) Need point where l crosses x-axis

$$-x\sqrt{3}+6\sqrt{3}=0\Rightarrow x=6$$

Area of triangle on right = $\frac{1}{2}\times(6-4)\times 2\sqrt{3}=\boldsymbol{2\sqrt{3}}$

Area under curve (left) = $\int y\frac{dx}{dt}\,dt$

Bounds: $x=0$: $8\cos t=0\Rightarrow t=\frac{\pi}{2}$, $x=4\Rightarrow t=\frac{\pi}{3}$ from (a)

$$\int y\frac{dx}{dt}\,dt=\int_{\frac{\pi}{2}}^{\frac{\pi}{3}}4\sin 2t\,(-8\sin t)\,dt=-32\int_{\frac{\pi}{2}}^{\frac{\pi}{3}}\sin 2t\sin t\,dt$$

$$=-32\int_{\frac{\pi}{2}}^{\frac{\pi}{3}}2\sin t\cos t\,(\sin t)\,dt=-64\int_{\frac{\pi}{2}}^{\frac{\pi}{3}}\sin^2 t\cos t\,dt$$

$$=64\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sin^2 t\cos t\,dt$$

(change sign of integral, swap bounds)

Substitution: $u=\sin t$, $dt=\frac{du}{\cos t}$.

$$t=\frac{\pi}{3}\rightarrow u=\frac{\sqrt{3}}{2},\ t=\frac{\pi}{2}\rightarrow u=1$$

$$64\int_{\frac{\sqrt{3}}{2}}^{1}u^2\,du=\frac{64}{3}\left[u^3\right]_{\frac{\sqrt{3}}{2}}^{1}=\boldsymbol{\frac{64}{3}-8\sqrt{3}}$$

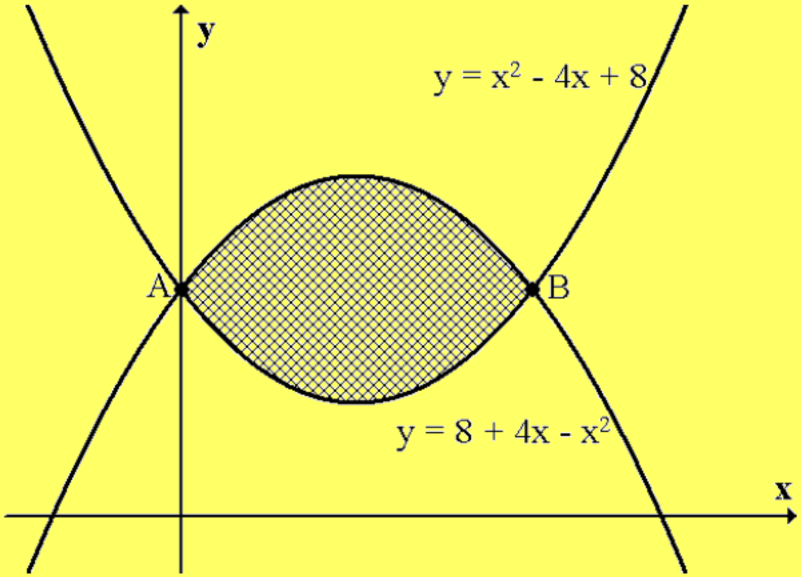
Finally, the total shaded area

$$R=\frac{64}{3}-8\sqrt{3}+2\sqrt{3}=\boldsymbol{\frac{64}{3}-6\sqrt{3}}$$

Exercise D: Solutions

1

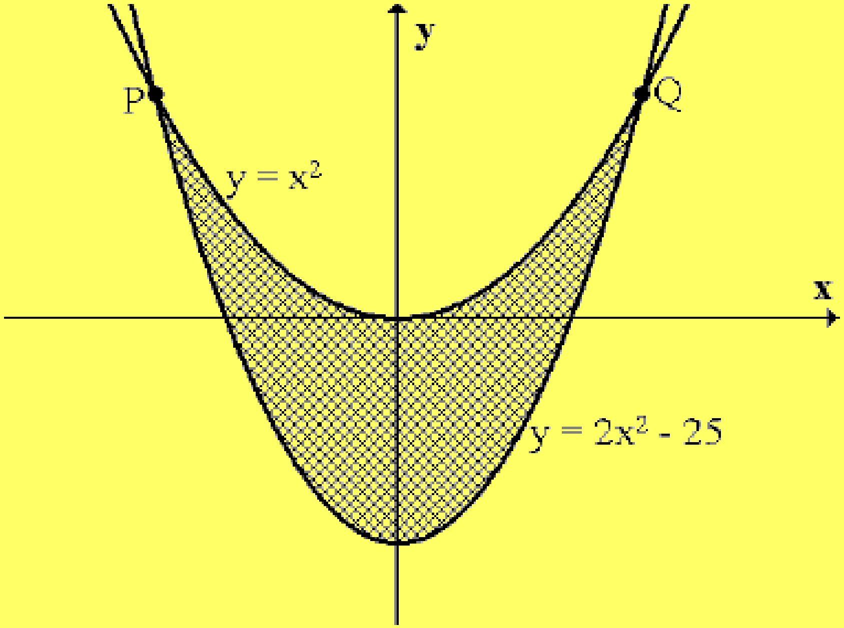
The curves with equations
$y = x^2 - 4x + 8$ and
$y = 8 + 4x - x^2$ intersect at A and B.
Calculate the exact area enclosed
Between the curves.



$$x^2 - 4x + 8 = 8 + 4x - x^2$$
$$x(x-4) = 0 \Rightarrow x = 0, x = 4$$
$$Area = \int_0^4 8x - 2x^2 dx = \frac{64}{3}$$

2

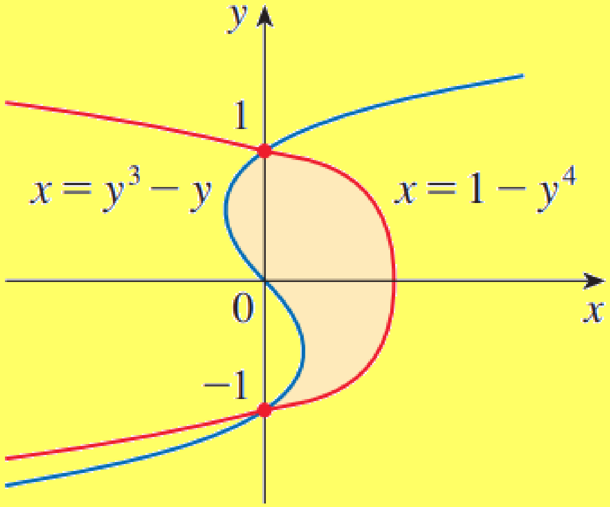
The curves with equations $y = x^2$ and
$y = 2x^2 - 25$ intersect at P and Q.
Calculate the exact area enclosed
Between the curves.



$$x^2 = 2x^2 - 25$$
$$x^2 - 25 = 0 \Rightarrow x = \pm 5$$
$$Area = \int_{-5}^{5} 25 - x^2 \, dx = \frac{500}{3}$$

Exercise D: Solutions

3 Find the exact shaded area

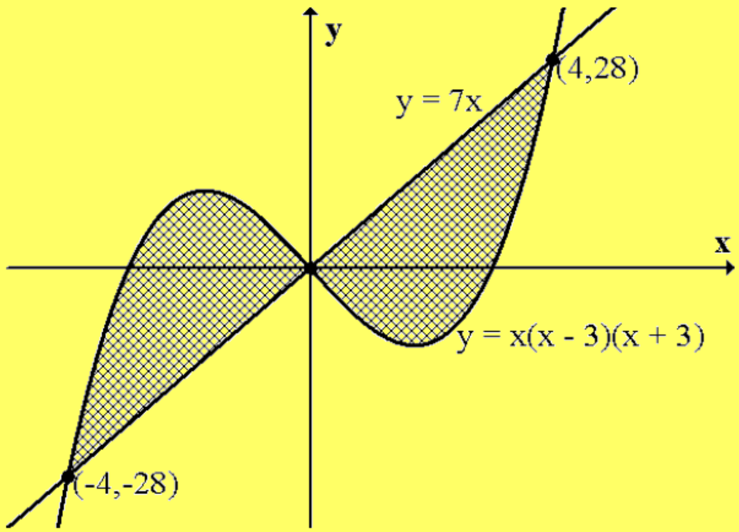


$$\int_{y=-1}^{1} (1 - y^4) - (y^3 - y)\, dy$$

$$= \int_{-1}^{1} 1 - y^4 - y^3 + y\, dy = \left[y - \frac{y^5}{5} - \frac{y^4}{4} + \frac{y^2}{2}\right]_{-1}^{1}$$

$$= \left(1 - \frac{1}{5} - \frac{1}{4} + \frac{1}{2}\right) - \left(-1 + \frac{1}{5} - \frac{1}{4} + \frac{1}{2}\right) = 2 - \frac{2}{5} = \mathbf{\frac{8}{5}}$$

4



$$x(x-3)(x+3) = x(x^2 - 9) = x^3 - 9x$$

$$Area = \int_{-4}^{0} (x^3 - 9x)\ - 7x\, dx + \int_{0}^{4} 7x - (x^3 - 9x)\, dx$$

$$Shortcut: By\ symmetry \Rightarrow Area$$

$$= 2\int_{0}^{4} 7x - (x^3 - 9x)\, dx$$

$$= 2\int_{0}^{4} 16x - x^3\, dx = 2\left[8x^2 - \frac{1}{4}x^4\right]_{0}^{4}$$

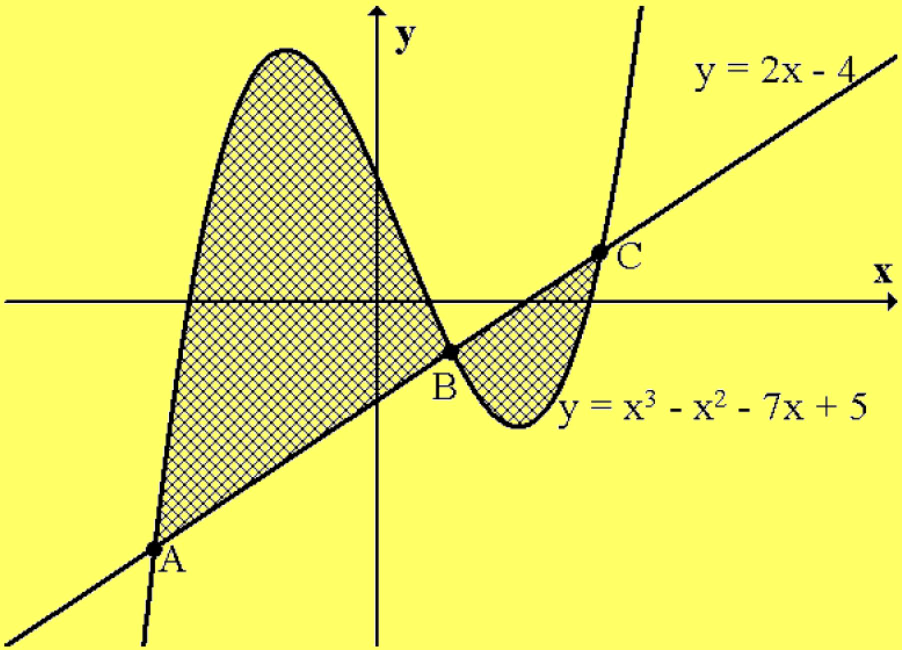
$$= 2\left[\left(8 \times 16 - \frac{1}{4} \times 4^4\right) - (0)\right] = \mathbf{128}$$

Exercise D: Solutions

5

The curve $y = x^3 - x^2 - 7x + 5$ and the line $y = 2x - 4$ are shown opposite.

(a) B has coordinates $(1, -2)$. Find the coordinates of A and C.
(b) Hence calculate the shaded area.



a) $x^3 - x^2 - 7x + 5 = 2x - 4$

$$x^3 - x^2 - 9x + 9 = 0$$

At point $B, x = 1 \Rightarrow (x-1)$ is a factor (by factor theorem)

$$x^3 - x^2 - 9x + 9 = (x-1)(x^2 + ax - 9)$$
$$= (x-1)(x-3)(x+3)$$
$$x = -3 \rightarrow y = -10 \Rightarrow A(-3, -10)$$
$$x = 3 \rightarrow y = 2 \Rightarrow C(3,2)$$

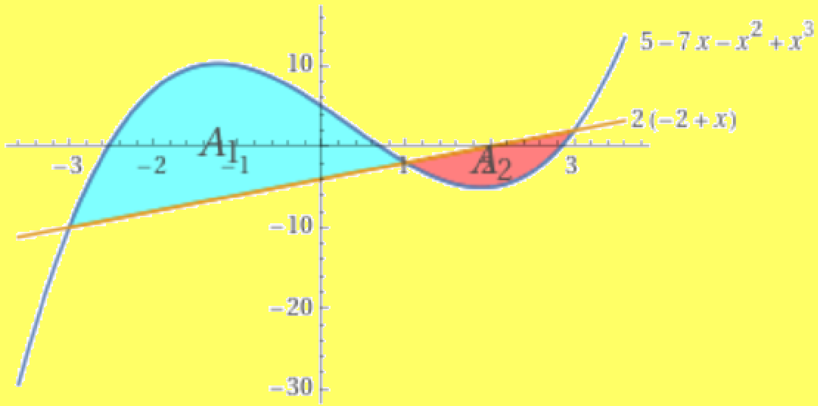
b) $A_1 = \int_{-3}^{1} (x^3 - x^2 - 7x + 5) - (2x - 4)dx$

$$\int_{-3}^{1} x^3 - x^2 - 9x + 9\, dx = \frac{128}{3}$$

$$A_2 = \int_{1}^{3} (2x - 4) - (x^3 - x^2 - 7x + 5)dx$$

$$\int_{1}^{3} -x^3 + x^2 + 9x - 9\, dx = \frac{20}{3}$$

$$A = \frac{128}{3} + \frac{20}{3} = \mathbf{\frac{148}{3}}$$



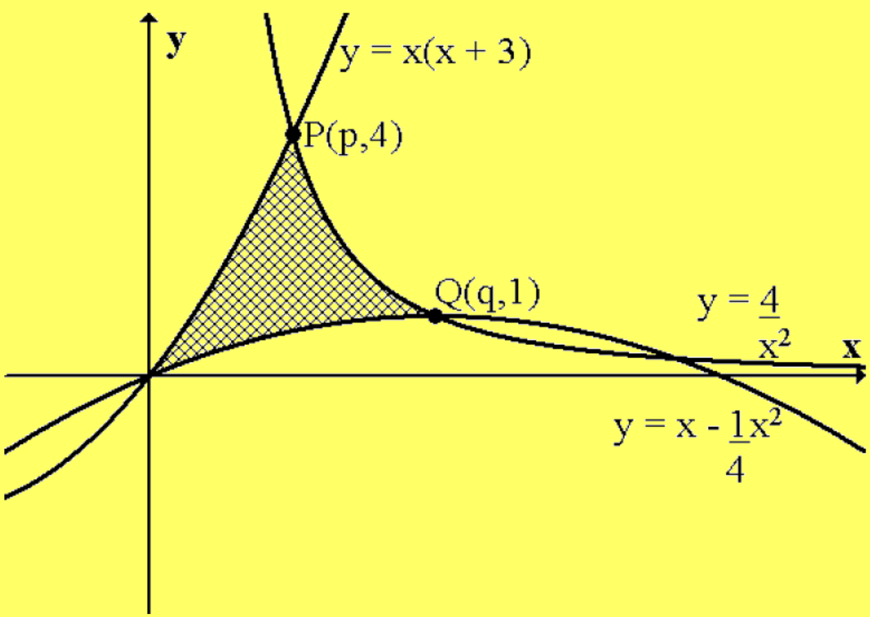
Exercise D: Solutions

6 The diagram opposite shows an area enclosed by 3 curves:

$$y = x(x+3)$$
$$y = \frac{4}{x^2}$$
$$y = x - \frac{1}{4}x^2$$

(a) P and Q have coordinates $(p, 4)$ and $(q, 1)$. Find the values of p and q.

(b) Calculate the shaded area.



a) $y=\frac{4}{x^2}$ and $y=x(x+3)$ intersect at $P(p,4)$

$y=4 \to 4=\frac{4}{x^2} \to x^2=1$

$x=\pm 1$

$x=1$ (from diagram)

$\therefore P(1,4)$

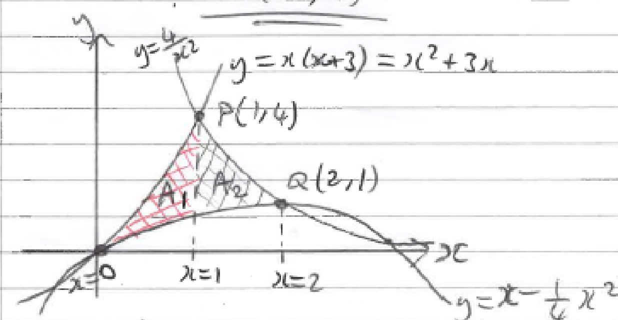
$y=\frac{4}{x^2}$ and $y=x-\frac{1}{4}x^2$ intersect at $Q(q,1)$

$y=1 \to 1=\frac{4}{x^2} \to x^2=4 \therefore x=\pm 2$

$x=2$ (from diagram)

$\therefore Q(2,1)$

b)



$A_1=\int_0^1 (x^2+3x)-(x-\frac{1}{4}x^2)\,dx = \int_0^1 \frac{5}{4}x^2+2x\,dx$

$=\left[\frac{5}{12}x^3+x^2\right]_0^1 = \left(\frac{5}{12}+1\right)-(0)=\frac{17}{12}$

$A_2=\int_1^2 (4x^{-2})-(x-\frac{1}{4}x^2)\,dx=\int_1^2 4x^{-2}-x+\frac{1}{4}x^2\,dx$

$=\left[-4x^{-1}-\frac{1}{2}x^2+\frac{1}{12}x^3\right]_1^2=\left[-\frac{4}{x}-\frac{1}{2}x^2+\frac{1}{12}x^3\right]_1^2$

$=\left(-2-2+\frac{8}{12}\right)-\left(-4-\frac{1}{2}+\frac{1}{12}\right)=\left(-\frac{10}{3}\right)-\left(-\frac{53}{12}\right)$

$=\frac{13}{12}$ $\therefore$ Total $=\frac{17}{12}+\frac{13}{12}=\frac{30}{12}=\frac{5}{2}$

Exercise E: Solutions

1

$$V = \pi\int_{27}^{125}\left(\left(x^{\frac{2}{3}} - 9\right)^{\frac{1}{2}}\right)^2 \mathrm{d}x \quad \text{or} \quad \pi\int_{27}^{125}\left(x^{\frac{2}{3}} - 9\right)\mathrm{d}x$$

$$= \{\pi\}\left[\frac{3}{5}x^{\frac{5}{3}} - 9x\right]_{27}^{125}$$

$$= \{\pi\}\left(\left(\frac{3}{5}(125)^{\frac{5}{3}} - 9(125)\right) - \left(\frac{3}{5}(27)^{\frac{5}{3}} - 9(27)\right)\right)$$

$$= \{\pi\}((1875 - 1125) - (145.8 - 243))$$

$$= \frac{4236\pi}{5} \quad \text{or} \quad 847.2\pi$$

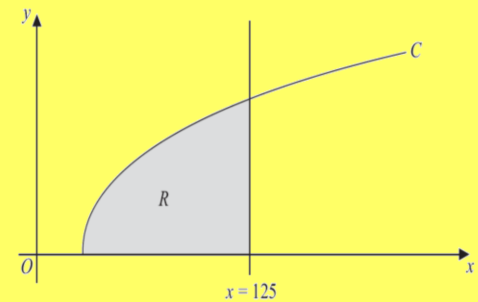
For $\pi\int\left(\left(x^{\frac{2}{3}} - 9\right)^{\frac{1}{2}}\right)^2$ or $\pi\int\left(x^{\frac{2}{3}} - 9\right)$

Ignore limits and $\mathrm{d}x$. Can be implied.

Either $\pm Ax^{\frac{5}{3}} \pm Bx$ or $\frac{3}{5}x^{\frac{5}{3}}$ oe

$\frac{3}{5}x^{\frac{5}{3}} - 9x$ oe

Substitutes limits of 125 and 27 into an integrated function and subtracts the correct way round.



2

Since we're finding $\boldsymbol{\pi\int_b^a x^2\, dy}$, we need to find x in terms of y.

$$y = 4\ln x - 1$$

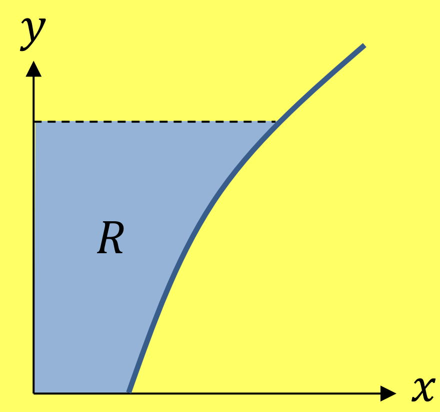
$$\ln x = \frac{y+1}{4}$$

$$x = e^{\frac{y+1}{4}} = e^{\frac{1}{4}}e^{\frac{1}{4}y}$$

$$V = \pi\int_0^4\left(e^{\frac{1}{4}}e^{\frac{1}{4}y}\right)^2 dy = \pi e^{\frac{1}{2}}\int_0^4 e^{\frac{1}{2}y}\,dy$$

$$= 2\pi e^{\frac{1}{2}}\left[e^{\frac{1}{2}y}\right]_0^4 = 2\pi e^{\frac{1}{2}}(e^2 - e^0)$$

$$= 2\pi\sqrt{e}(e^2 - 1)$$



Exercise E: Solutions

3

a

$$\frac{dy}{dt} = 2t, \frac{dx}{dt} = \frac{1}{t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times t = 2t^2$$

Gradient of tangent at $t = 3$: $m_t = 18$

Gradient of normal at $t = 3$: $m_n = -\frac{1}{18}$

Need a point: When $t = 3$, $x = \ln 3$, $y = 7$

$$y = mx + c \Rightarrow 7 = -\frac{1}{18}\ln 3 \; + c \Rightarrow c = 7 + \frac{18}{\ln 3}$$

$$\boldsymbol{y = -\frac{1}{18}x + 77 + \frac{18}{ln\,3}}$$

Or using $y - y_1 = m(x - x_1)$:

$$\boldsymbol{y - 7 = -\frac{1}{18}(x - \ln 3)}$$

b

$$x = \ln t \Rightarrow t = e^x$$

$$y = (e^x)^2 - 2$$

$$\therefore \boldsymbol{y = e^{2x} - 2}$$

c

$$V = \pi \int_b^a y^2 \, dx = \pi \int_{\ln 2}^{\ln 4} (e^{2x} - 2)^2 dx$$

$$\pi \int_{\ln 2}^{\ln 4} e^{4x} - 4e^{2x} + 4 \, dx$$

$$= \pi \left[\frac{e^{4x}}{4} - 2e^{2x} + 4x \right]_{\ln 2}^{\ln 4}$$

$$= \pi \left[\left(\frac{e^{4\ln 4}}{4} - 2e^{2\ln 4} + 4\ln 4 \right) - \left(\frac{e^{4\ln 2}}{4} - 2e^{2\ln 2} + 4\ln 2 \right) \right]$$

$$= \pi \left[\left(\frac{e^{\ln 4^4}}{4} - 2e^{\ln 4^2} + 4\ln 4 \right) - \left(\frac{e^{\ln 2^4}}{4} - 2e^{\ln 2^2} + 4\ln 2 \right) \right]$$

Because $e^{\ln a} = a$

$$V = \pi[(64 - 32 + 4\ln 4) - (4 - 8 + 4\ln 2)]$$

$$= \pi[36 + 4(\ln 4 - \ln 2)]$$

$$= \pi[36 + 4(2\ln 2 - \ln 2)]$$

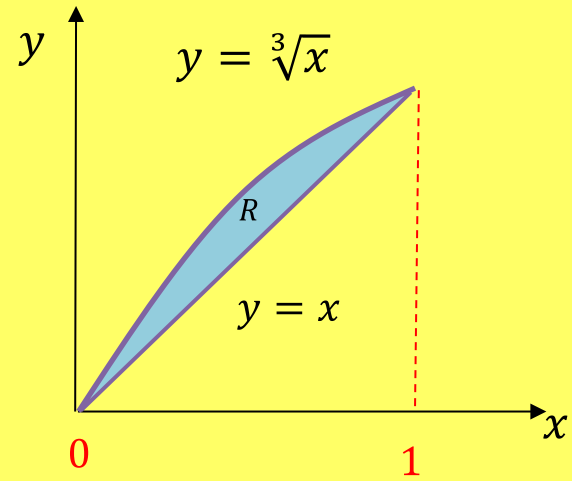
$$= \pi[36 + 4\ln 2]$$

$$= \boldsymbol{4\pi(9 + \ln 2)} \quad \text{or equivalent}$$

Exercise E: Solutions

4

The area between the lines with equations $y = x$ and $y = \sqrt[3]{x}$, where $x \geq 0$ is rotated 360° about the x-axis. Determine the volume of the solid generated.



$x = x^{\frac{1}{3}} \quad \rightarrow \quad x^3 = x$

$x^3 - x = 0$

$x(x-1)(x+1) = 0$

Intersect at $x = -1, 0, 1$

$$V_{below\ curve} = \pi \int_0^1 \left(x^{\frac{1}{3}}\right)^2 dx = \pi \int_0^1 x^{\frac{2}{3}}\ dx = \frac{3\pi}{5}$$

$$V_{cone} = \frac{1}{3}\pi(1^2)(1) = \frac{\pi}{3}$$

$$V_R = \frac{3\pi}{5} - \frac{\pi}{3} = \boldsymbol{\frac{4\pi}{15}}$$

5

(a) $\{y = 0 \Rightarrow\}\ 1 - 2\cos x = 0$

$\Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$

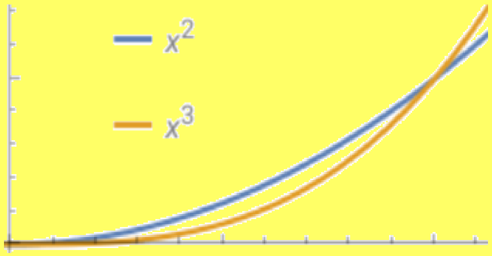
(b)

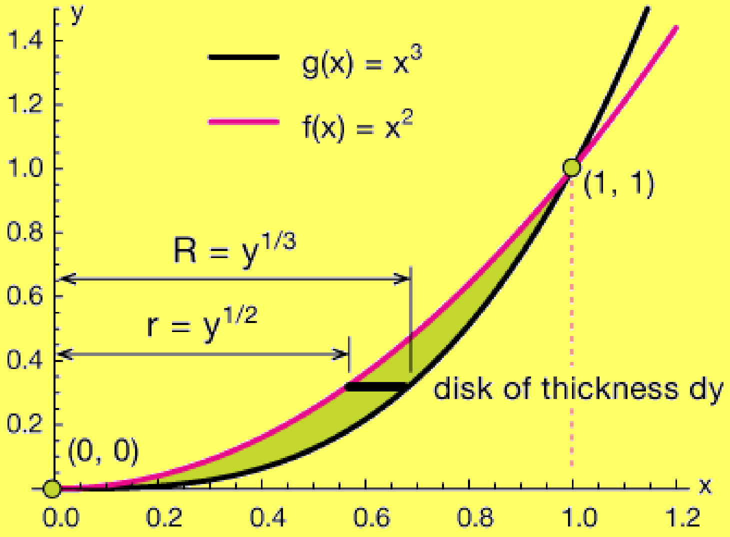
$V = \pi \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos x)^2\ dx$	For $\pi \int (1 - 2\cos x)^2$. Ignore limits and dx	B1
$\left\{\int (1 - 2\cos x)^2\ dx\right\} = \int (1 - 4\cos x + 4\cos^2 x) dx$		
$= \int 1 - 4\cos x + 4\left(\frac{1 + \cos 2x}{2}\right) dx$	$\cos 2x = 2\cos^2 x - 1$ See notes.	M1
$= \int (3 - 4\cos x + 2\cos 2x)\ dx$		
$= 3x - 4\sin x + \frac{2\sin 2x}{2}$	Attempts $\int y^2$ to give any two of $\pm A \to \pm Ax$, $\pm B\cos x \to \pm B\sin x$ or $\pm\lambda\cos 2x \to \pm\mu\sin 2x$.	M1
	Correct integration.	A1
$V = \{\pi\}\left(\left(3\left(\frac{5\pi}{3}\right) - 4\sin\left(\frac{5\pi}{3}\right) + \frac{2\sin\left(\frac{10\pi}{3}\right)}{2}\right) - \left(3\left(\frac{\pi}{3}\right) - 4\sin\left(\frac{\pi}{3}\right) + \frac{2\sin\left(\frac{2\pi}{3}\right)}{2}\right)\right)$	Applying limits the correct way round. Ignore π.	ddM1
$= \pi\left(\left(5\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) - \left(\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right)\right)$		
$= \pi((18.3060...) - (0.5435...)) = 17.7625\pi = 55.80$		
$= \pi\left(4\pi + 3\sqrt{3}\right)$ or $4\pi^2 + 3\pi\sqrt{3}$	Two term exact answer.	A1

Exercise E: Solutions

6

Find the volume generated when the region trapped between the curves $y = x^2$ and $y = x^3$ is rotated $360°$ around the y-axis to obtain a bowl-like 3D shape.





$$V = \pi \int_a^b x^2 \; dy$$

Limits: Curves intersect at $x = 0$ and $x = 1$

$$y = x^2 \Rightarrow x = y^{\frac{1}{2}},$$

$$y = x^3 \Rightarrow x = y^{\frac{1}{3}}$$

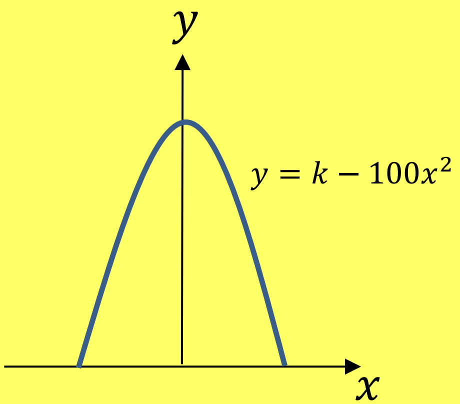
$$V = \pi \int_0^1 \left(y^{\frac{1}{3}}\right)^2 - \left(y^{\frac{1}{2}}\right)^2 \, dy$$

$$V = \pi \int_0^1 y^{\frac{2}{3}} - y \, dy = \pi \left[\frac{3y^{\frac{5}{3}}}{5} - \frac{y^2}{2}\right]_0^1 = \pi \left[\frac{3}{5} - \frac{1}{2}\right] = \frac{\pi}{10}$$

Exercise E: Solutions

7

A manufacturer wants to cast a prototype for a new design for a pen barrel out of solid resin.
The region shown in the diagram is used as a model for the cross-section of the pen barrel.
The region is bounded by the x-axis and the curve with equation $y = k - 100x^2$, and will be rotated around the y-axis.
Each unit on the coordinate axes represents 1cm.
(a) Suggest a suitable value for k. (Assuming pens are 10cm long)
(b) Use your value of k to estimate the volume of resin needed to make the prototype.
(c) State one limitation of this model.



a $k = 10$ (pens are around 10-15cm long)

b

$$y = 10 - 100x^2 \quad \Rightarrow x = \sqrt{\frac{10-y}{100}}$$

$$V = \pi \int_0^{10} \left(\sqrt{\frac{10-y}{100}}\right)^2 dy \quad = \pi \int_0^{10} \frac{10-y}{100} dy \quad = \frac{\pi}{100} \int_0^{10} 10 - y \; dy$$

$$= \frac{\pi}{100} \left[10y - \frac{1}{2}y^2\right]_0^{10} \quad = \frac{\pi}{100}(100 - 50) \quad = \boldsymbol{\frac{\pi}{2} \approx 1.57\ cm^3}$$

c The cross-section of the pen is unlikely to match the curve exactly.