CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

University of Leeds

Level 1 Semester 2

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Tutorial: Question Difficulty Colour Code

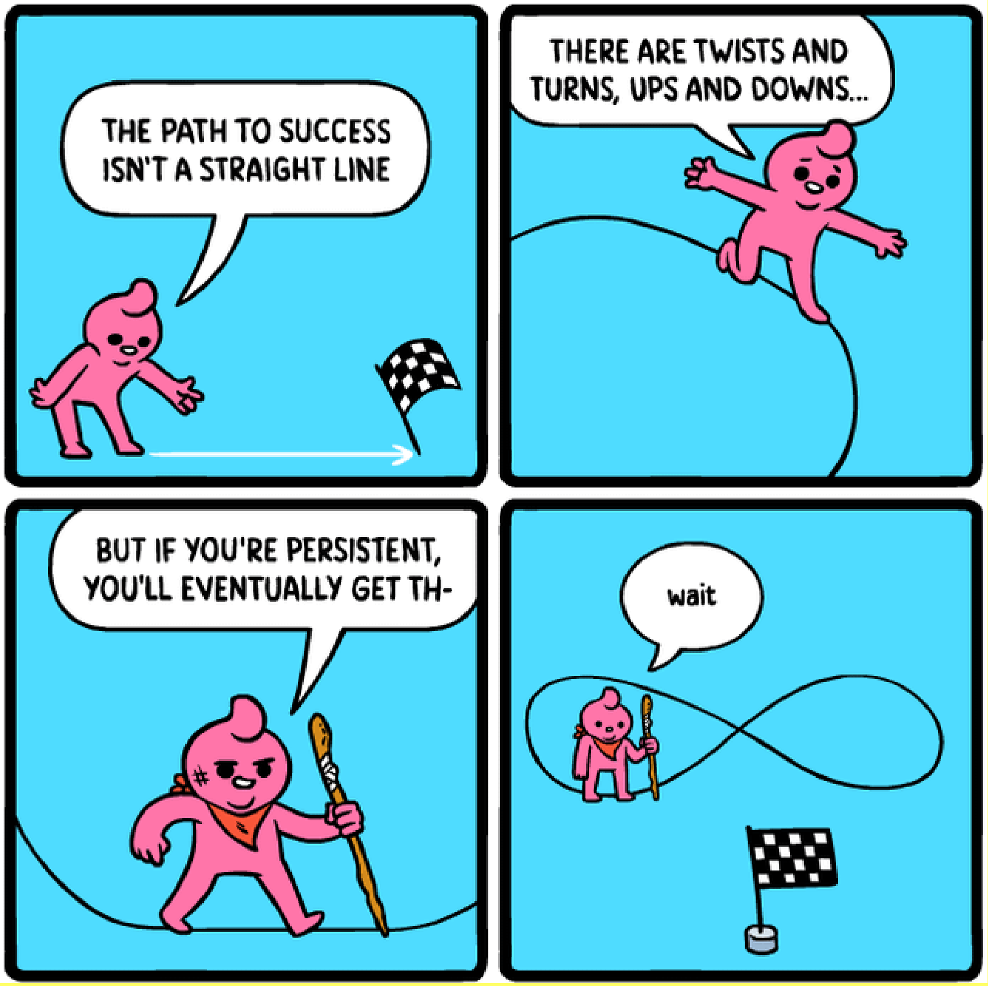
Basic - straightforward application
(you must be able to do these)

Medium – Makes you think a bit
(you must be able to do these)

Hard – Makes you think a lot
(you should be able to do these)

Extreme – Tests your understanding to the limit!
(for those who like a challenge)

Applied – Real-life examples of the topic, may sometimes involve prior knowledge
(you should attempt these – will help in future engineering)



Tutorial 9
Vectors 2

Class Example: Vector Equation of a Straight Line

E.g. 1

Relative to a fixed origin, the points P and Q have position vectors $(5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ and $(3\mathbf{i} + \mathbf{j})$ respectively.

a Find, in vector form, an equation of the line L_1 which passes through P and Q. **(2)**

The line L_2 has equation

$$\mathbf{r} = 4\mathbf{i} + 6\mathbf{j} - \mathbf{k} + \mu(5\mathbf{i} - \mathbf{j} + 3\mathbf{k}).$$

b Show that lines L_1 and L_2 intersect and find the position vector of their point of intersection. **(6)**

c Find, in degrees to 1 decimal place, the acute angle between lines L_1 and L_2. **(4)**

a $\overrightarrow{PQ} = (3\mathbf{i} + \mathbf{j}) - (5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

$= -2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$\therefore\ \mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \lambda(-2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$

b $5 - 2\lambda = 4 + 5\mu \quad (1)$

$-2 + 3\lambda = 6 - \mu \quad (2)$

$2 - 2\lambda = -1 + 3\mu \quad (3)$

$(1) - (3) \Rightarrow 3 = 5 + 2\mu$

$\mu = -1, \ \lambda = 3$

check (2) $\quad -2 + 3(3) = 6 - (-1)$

true $\therefore$ intersect

pos. vector of int. $= -\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$

c $\left|-2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}\right| = \sqrt{4+9+4} = \sqrt{17}$

$\left|5\mathbf{i} - \mathbf{j} + 3\mathbf{k}\right| = \sqrt{25+1+9} = \sqrt{35}$

$(-2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}).(5\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

$= -10 - 3 - 6 = -19$

$\theta = \cos^{-1}\left|\frac{-19}{\sqrt{17}\sqrt{35}}\right| = 38.8°$

Class Example: Equations of Planes

E.g. 2

The points A, B and C have position vectors $\begin{pmatrix}1\\3\\2\end{pmatrix}$, $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\0\end{pmatrix}$ respectively.

The plane Π contains the points A, B and C.

Find a vector equation of Π in parametric form **(4)**

Two vectors on the plane:

$$\overrightarrow{AB} = \begin{pmatrix}-2\\-3\\-1\end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix}1\\-2\\-2\end{pmatrix}$$

Possible equation:

$$r = \begin{pmatrix}1\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}-2\\-3\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\-2\end{pmatrix}$$

Class Example: Equations of Planes

E.g. 4

Find, in the form $\boldsymbol{r} \cdot \boldsymbol{n} = p$, an equation of the plane which contains the line l and the point with position vector a where l has the equation $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}4\\5\\6\end{pmatrix}$ and $\boldsymbol{a} = \begin{pmatrix}9\\8\\5\end{pmatrix}$

Find also the Cartesian form of the equation.

Vectors in direction of plane:

- $\boldsymbol{d} = \begin{pmatrix}4\\5\\6\end{pmatrix}$
- $\begin{pmatrix}9\\8\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}8\\6\\2\end{pmatrix}$

$$\boldsymbol{n} = \begin{pmatrix}4\\5\\6\end{pmatrix} \times \begin{pmatrix}8\\6\\2\end{pmatrix} = \begin{pmatrix}-26\\40\\-16\end{pmatrix}$$

$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$

$$\begin{pmatrix}8\\6\\2\end{pmatrix} \cdot \begin{pmatrix}-26\\40\\-16\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}-26\\40\\-16\end{pmatrix} \rightarrow \quad -26x + 40y - 16z = 6$$

Diagnostic Class Example (10 minutes): Plane

E.g. 3

Find an equation of the plane that contains points
$A(1,1,0), B(-1,2,0), C(1,2,-1)$.
Give your answer in Cartesian form $ax + by + cz = d$ where a, b, c, d are integers.

Y

$$x + 2y + 2z = -3$$

C

$$x + 2y + 2z = 3$$

M

$$x + y = 3$$

A

$$x + 2y + 2z = -6$$

Class Example: Distance of Point from Plane

E.g. 6

The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1.

$(6\mathbf{i}+2\mathbf{j}+12\mathbf{k}).(3\mathbf{i}-4\mathbf{j}+2\mathbf{k}) = 34$
$\left\lvert \dfrac{(6\mathbf{i}+2\mathbf{j}+12\mathbf{k}).(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})-5}{\sqrt{3^2+4^2+2^2}} \right\rvert$
$\sqrt{29}$ (not $-\sqrt{29}$)

Exercise A: Equation of a Line

1

Write down a vector equation of the straight line

a parallel to the vector $(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$ which passes through the point with position vector $(4\mathbf{i} + \mathbf{k})$,

b perpendicular to the xy-plane which passes through the point with coordinates $(2, 1, 0)$,

c parallel to the line $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + t(2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$ which passes through the point with coordinates $(-1, 4, 2)$.

2

Find a vector equation of the straight line which passes through the points with position vectors

a $(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$ and $(5\mathbf{i} + 4\mathbf{j} + 6\mathbf{k})$

b $(3\mathbf{i} - 2\mathbf{k})$ and $(\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$

c $\mathbf{0}$ and $(6\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

d $(-\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ and $(4\mathbf{i} - 7\mathbf{j} + \mathbf{k})$

3

Find cartesian equations for each of the following lines.

a $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$

4

Find a vector equation for each line given its cartesian equations.

a $\dfrac{x-1}{3} = \dfrac{y+4}{2} = z - 5$

b $\dfrac{x}{4} = \dfrac{y-1}{-2} = \dfrac{z+7}{3}$

c $\dfrac{x+5}{-4} = y + 3 = z$

Exercise A: Equation of a Line

5 Lines $\boldsymbol{r_1} = 3\boldsymbol{i} + 2\boldsymbol{j} - 5\boldsymbol{k} + \lambda(2\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$ and $\boldsymbol{r_2} = \boldsymbol{i} + 4\boldsymbol{j} - 6\boldsymbol{k} + \mu(-3\boldsymbol{i} + p\boldsymbol{j} + 14\boldsymbol{k})$ are perpendicular (orthogonal). Find the value of p.

6 For each pair of lines find, in degrees to 1dp, the **acute** angle between them:

a) $\boldsymbol{r_1} = \begin{pmatrix} -3 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ and $\boldsymbol{r_2} = \begin{pmatrix} 2 \\ 5 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$

b) $\boldsymbol{r_1} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}$ and $\boldsymbol{r_2} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$

7 For each pair of lines, find the position vector of their point of intersection or, if they do not intersect, state whether they are parallel or skew.

a $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 6 \\ -2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 2 \\ 6 \end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -3 \\ -4 \end{pmatrix}$

d $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 7 \\ -6 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

e $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix}$

f $\mathbf{r} = \begin{pmatrix} 0 \\ 7 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -4 \\ 8 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -12 \\ -1 \\ 11 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$

Exercise A: Equation of a Line

8

Find the value of the constants a and b such that line $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + a\mathbf{j} + b\mathbf{k})$

a passes through the point $(9, -2, -8)$,

b is parallel to the line $\mathbf{r} = 4\mathbf{j} - 2\mathbf{k} + \mu(8\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$.

9

The line $\boldsymbol{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + t\begin{pmatrix} p \\ -2 \\ 3 \end{pmatrix}$ makes an angle of $\frac{2\pi}{3}$ with the y-axis. Find the **exact** value(s) of p.

10

Relative to a fixed origin, the line l_1 has the equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ -6 \\ -2 \end{pmatrix} + s\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}.$$

a Show that the point P with coordinates $(1, 6, -5)$ lies on l_1. **[1 mark]**

The line l_2 has the equation

$$\mathbf{r} = \begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix},$$

and intersects l_1 at the point Q.

b Find the position vector of Q. **[3 marks]**

The point R lies on l_2 such that $PQ = QR$.

c Find the two possible position vectors of the point R. **[5 marks]**

Exercise A: Equation of a Line

11

Relative to a fixed origin, the points A and B have position vectors $(4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$ and $(4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})$ respectively.

a Find, in vector form, an equation of the line l_1 which passes through A and B. **[2 marks]**

The line l_2 has equation

$$\mathbf{r} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k}).$$

b Show that l_1 and l_2 intersect and find the position vector of their point of intersection. **[4 marks]**

c Find the acute angle between lines l_1 and l_2. **[3 marks]**

d Show that the point on l_2 closest to A has position vector $(-\mathbf{i} + 3\mathbf{j} - \mathbf{k})$. **[5 marks]**

12

With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$$

where λ and μ are scalar parameters.

The lines l_1 and l_2 intersect at the point X.

(a) Find the coordinates of the point X. **[3 marks]**

(b) Find the size of the acute angle between l_1 and l_2, giving your answer in degrees to 2 decimal places. **[3 marks]**

The point A lies on l_1 and has position vector $\begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix}$

(c) Find the distance AX, giving your answer as a surd in its simplest form. **[2 marks]**

The point Y lies on l_2. Given that the vector $\overrightarrow{YA}$ is perpendicular to the line l_1

(d) find the distance YA, giving your answer to one decimal place. **[2 marks]**

The point B lies on l_1 where $|\overrightarrow{AX}| = 2|\overrightarrow{AB}|$.

(e) Find the two possible position vectors of B. **[3 marks]**

Exercise B: Equation of a Plane

1 Find, in the form $\boldsymbol{r} \cdot \boldsymbol{n} = d$, an equation of the plane that passes through the point with position vector $\boldsymbol{a}$ and is perpendicular to the vector $\boldsymbol{n}$, where:

a) $\boldsymbol{a} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ and $\boldsymbol{n} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}$

b) $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{k}$ and $\boldsymbol{n} = \boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$

2 Find a Cartesian equation of for each plane in Q1

3 Find an equation of the plane that contains point $(-2,1,4)$ and is perpendicular to the line $\underline{r} = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$.

Give your answer in the form $ax + by = c$ where a, b, c are integers.

4 Find, in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{u} + \mu\boldsymbol{v}$, an equation of the plane that passes through the points given:

a) $(1,2,0)$, $(3,1,-1)$ and $(4,3,2)$

b) $(3,4,1)$, $(-1,-2,0)$ and $(2,1,4)$

5 Find a Cartesian equation of for each plane in Q3

6 The plane Π contains points A, B and C with position vectors $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ respectively. Find a vector equation of Π in the form $\boldsymbol{r} \cdot \boldsymbol{n} = d$.

Exercise B: Equation of a Plane

7 Find a cartesian equation of the plane that contains the three points A, B and C, where:

a) $A(0,4,2), B(1,1,2)$ and $C(-1,5,0)$

b) $A(1,-1,6), B(3,1,-2)$ and $C(4,1,0)$

Give your answers in Cartesian form $ax + by + cz = d$ where a, b, c, d are integers

8 Find, in the form $\boldsymbol{r} \cdot \boldsymbol{n} = d$, an equation of the plane which contains the line l and the point with position vector $\boldsymbol{a}$, where l has equation:

$\boldsymbol{r} = 3\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k} + \lambda(-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$ and $\boldsymbol{a} = 4\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$

9 Find a Cartesian equation of the plane which passes through the point $(1,1,1)$ and contains the line with equation $\frac{x-2}{3} = \frac{y+4}{1} = \frac{z-1}{2}$

10

The line l_1 has equation

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-4}{3}$$

The line l_2 has equation

$$\mathbf{r} = \mathbf{i} + 3\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

where t is a scalar parameter.

(a) Show that l_1 and l_2 lie in the same plane.

(b) Write down a vector equation for the plane containing l_1 and l_2

(c) Find, to the nearest degree, the acute angle between l_1 and l_2

Exercise B: Equation of a Plane

11 Find a Cartesian equation of the plane which is orthogonal (perpendicular) to the plane $3x + 2y - z = 4$ and goes through points $P(1,2,4)$ and $Q(-1,3,2)$.

12 Find a Cartesian equation of the plane that contains the intersecting lines $x = 4 + t_1, y = 2t_1, z = 1 - 3t_1$ and $x = 4 - 3t_2, y = 3t_2, z = 1 + 2t_2$

ANSWERS

Exercise A: Answers (vectors can be in either form)

1

a $\mathbf{r} = 4\mathbf{i} + \mathbf{k} + s(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$

b $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + s\mathbf{k}$

c $\mathbf{r} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + s(2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$

2

Note: For these, the position vector could be either of the points given

a direction $= (5\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}) - (\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$
$= 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
$\therefore\ \mathbf{r} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + s(4\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

b direction $= (\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} - 2\mathbf{k})$
$= -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$
$\therefore\ \mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + s(-2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

c $\mathbf{r} = s(6\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

d direction $= (4\mathbf{i} - 7\mathbf{j} + \mathbf{k}) - (-\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$
$= 5\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$
$\therefore\ \mathbf{r} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + s(5\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$

3

a $x = 2 + 3\lambda,$
$y = 3 + 5\lambda,$
$z = 2\lambda,$
$(\lambda =)\ \dfrac{x-2}{3} = \dfrac{y-3}{5} = \dfrac{z}{2}$

b $x = 4 + \lambda,$
$y = -1 + 6\lambda,$
$z = 3 + 3\lambda,$
$(\lambda =)\ x - 4 = \dfrac{y+1}{6} = \dfrac{z-3}{3}$

c $x = -1 + 4\lambda,$
$y = 5 - 2\lambda,$
$z = -2 - \lambda,$
$(\lambda =)\ \dfrac{x+1}{4} = \dfrac{y-5}{-2} = \dfrac{z+2}{-1}$

4

a $s = \dfrac{x-1}{3} = \dfrac{y+4}{2} = z - 5$
$x = 1 + 3s,$
$y = -4 + 2s,$
$z = 5 + s,$
$\mathbf{r} = \mathbf{i} - 4\mathbf{j} + 5\mathbf{k} + s(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

b $s = \dfrac{x}{4} = \dfrac{y-1}{-2} = \dfrac{z+7}{3}$
$x = 4s,$
$y = 1 - 2s,$
$z = -7 + 3s,$
$\mathbf{r} = \mathbf{j} - 7\mathbf{k} + s(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

c $s = \dfrac{x+5}{-4} = y + 3 = z$
$x = -5 - 4s,$
$y = -3 + s,$
$z = s,$
$\mathbf{r} = -5\mathbf{i} - 3\mathbf{j} + s(-4\mathbf{i} + \mathbf{j} + \mathbf{k})$

Exercise A: Answers (vectors can be in either form)

5

Lines are perpendicular so scalar product of direction vectors is equal to zero:

$$(2\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}) \cdot (-3\boldsymbol{i} + p\boldsymbol{j} + 14\boldsymbol{k}) = 0$$
$$-6 + 4p - 14 = 0$$
$$p = 5$$

6

The angle between lines is the angle between their direction vectors

a) $\cos^{-1}\left(\dfrac{\begin{pmatrix}4\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\-3\end{pmatrix}}{\sqrt{16+1+4}\sqrt{1+4+9}}\right) = \cos^{-1}\left(\dfrac{4}{\sqrt{21}\sqrt{14}}\right) = 76.5°$

$b)$ $\cos^{-1}\left(\dfrac{\begin{pmatrix}-3\\2\\-4\end{pmatrix}\cdot\begin{pmatrix}5\\-1\\-1\end{pmatrix}}{\sqrt{9+4+16}\sqrt{25+1+1}}\right) = \cos^{-1}\left(\dfrac{-13}{\sqrt{29}\sqrt{27}}\right) = 117.7°$ (obtuse)

$\therefore$ acute angle is $180° - 117.7° = 62.3°$

Exercise A: Answers (vectors can be in either form)

7

a $3 + 4\lambda = 3 + \mu \quad (1)$

$1 + \lambda = 2 \quad (2)$

$5 - \lambda = -4 + 2\mu \quad (3)$

$(2) \Rightarrow \lambda = 1$

sub. (1) $\mu = 4$

check (3) $5 - (1) = -4 + 2(4)$

true $\therefore$ intersect

position vector of intersection: $\begin{pmatrix} 7 \\ 2 \\ 4 \end{pmatrix}$

b $\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -4 \\ 2 \\ 6 \end{pmatrix}$

$\therefore$ parallel

c $8 + \lambda = -2 + 4\mu \quad (1)$

$2 + 3\lambda = 2 - 3\mu \quad (2)$

$-4 - 2\lambda = 8 - 4\mu \quad (3)$

$(1) + (3) \Rightarrow 4 - \lambda = 6$

$\lambda = -2, \ \mu = 2$

check (2) $2 + 3(-2) = 2 - 3(2)$

true $\therefore$ intersect

position vector of intersection: $\begin{pmatrix} 6 \\ -4 \\ 0 \end{pmatrix}$

d $1 + \lambda = 7 + 2\mu \quad (1)$

$5 + 4\lambda = -6 + \mu \quad (2)$

$2 - 2\lambda = -5 - 3\mu \quad (3)$

$2\times(1) + (3) \Rightarrow 4 = 9 + \mu$

$\mu = -5, \ \lambda = -4$

check (2) $5 + 4(-4) = -6 + (-5)$

true $\therefore$ intersect

position vector of intersection: $\begin{pmatrix} -3 \\ -11 \\ 10 \end{pmatrix}$

e $4 + 2\lambda = 3 + 5\mu \quad (1)$

$-1 + 5\lambda = -2 - 3\mu \quad (2)$

$3 - 3\lambda = 1 - 4\mu \quad (3)$

$3\times(1) + 2\times(3) \Rightarrow 18 = 11 + 7\mu$

$\mu = 1, \ \lambda = 2$

check (2) $-1 + 5(2) = -2 - 3(1)$

false $\therefore$ do not intersect

$\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \neq k\begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix}$

$\therefore$ skew

f $6\lambda = -12 + 5\mu \quad (1)$

$7 - 4\lambda = -1 + 2\mu \quad (2)$

$-2 + 8\lambda = 11 - 3\mu \quad (3)$

$2\times(2) + (3) \Rightarrow 12 = 9 + \mu$

$\mu = 3, \ \lambda = \frac{1}{2}$

check (1) $6(\frac{1}{2}) = -12 + 5(3)$

true $\therefore$ intersect

position vector of intersection: $\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$

Exercise A: Answers (vectors can be in either form)

8

a $3 + 2\lambda = 9 \quad \therefore \lambda = 3$

$-5 + a\lambda = -5 + 3a = -2 \quad \therefore a = 1$

$1 + b\lambda = 1 + 3b = -8 \quad \therefore b = -3$

b $2\mathbf{i} + a\mathbf{j} + b\mathbf{k} = k(8\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$

$\therefore \; k = \frac{1}{4}$

$\therefore \; a = -1, \; b = \frac{1}{2}$

9

Direction vector of line is $\begin{pmatrix} p \\ -2 \\ 3 \end{pmatrix}$. The y-axis is in the direction $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\therefore \; \cos^{-1}\left(\frac{\begin{pmatrix} p \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{1}\sqrt{p^2 + 4 + 9}}\right) = \frac{2\pi}{3}$$

$$\cos^{-1}\left(\frac{-2}{\sqrt{p^2 + 13}}\right) = \frac{2\pi}{3}$$

"cos" both sides:

$$\frac{-2}{\sqrt{p^2 + 13}} = -\frac{1}{2}$$

$$\sqrt{p^2 + 13} = 4$$

$$p^2 = 3$$

$$p = \pm\sqrt{3}$$

Exercise A: Answers (vectors can be in either form)

10

a $-6 + 4s = 6 \Rightarrow s = 3$

sub. $s = 3$ in l_1

$$\mathbf{r} = \begin{pmatrix} 1 \\ -6 \\ -2 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -5 \end{pmatrix}$$

$\therefore P\,(1, 6, -5)$ lies on l_1

b $1 = 4 + 3t \quad \Rightarrow t = -1$

sub. $t = -1$ in l_2

$$\mathbf{r} = \begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{OQ} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

c $PQ = \sqrt{0+64+4} = \sqrt{68} = 2\sqrt{17}$

$$\left|\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}\right| = \sqrt{9+4+4} = \sqrt{17}$$

$$\therefore\ \overrightarrow{OR} = \overrightarrow{OQ} \pm 2\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 2 \\ -7 \end{pmatrix} \text{ or } \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix}$$

11

a $\overrightarrow{AB} = (4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) - (4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$

$= \mathbf{j} - 4\mathbf{k}$

$\therefore\ \mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} + \lambda(\mathbf{j} - 4\mathbf{k})$

b

$4 = 1 + \mu \qquad (1)$

$5 + \lambda = 5 + \mu \qquad (2)$

$6 - 4\lambda = -3 - \mu \qquad (3)$

$(1) \quad \Rightarrow \quad \mu = 3$

sub. (2) $\Rightarrow \quad \lambda = 3$

check (3) $\quad 6 - 4(3) = -3 - (3)$

true $\therefore$ intersect

pos. vector of int. $= 4\mathbf{i} + 8\mathbf{j} - 6\mathbf{k}$

c $|(\mathbf{j} - 4\mathbf{k})| = \sqrt{1+16} = \sqrt{17}$

$|(\mathbf{i} + \mathbf{j} - \mathbf{k})| = \sqrt{1+1+1} = \sqrt{3}$

$(\mathbf{j} - 4\mathbf{k}).(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0 + 1 + 4 = 5$

$\theta = \cos^{-1}\left|\frac{5}{\sqrt{3}\sqrt{17}}\right| = 45.6°$ (1dp)

d let closest point be C

$\overrightarrow{OC} = (1 + \mu)\mathbf{i} + (5 + \mu)\mathbf{j} + (-3 - \mu)\mathbf{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (-3 + \mu)\mathbf{i} + \mu\mathbf{j} + (-9 - \mu)\mathbf{k}$

AC must be perpendicular to l_2

$\therefore\ \overrightarrow{AC}.(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0$

$(-3 + \mu) + \mu - (-9 - \mu) = 0$

$\mu = -2$

$\therefore\ \overrightarrow{OC} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

Exercise A: Answers (vectors can be in either form)

12

$r_1 = \begin{pmatrix}4\\28\\4\end{pmatrix} + \lambda\begin{pmatrix}-1\\-5\\1\end{pmatrix}, \quad r_2 = \begin{pmatrix}5\\3\\1\end{pmatrix} + \mu\begin{pmatrix}3\\0\\-4\end{pmatrix}$

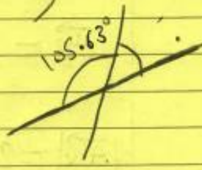
a) Intersection: $4-\lambda = 5+3\mu$ ①
$28-5\lambda = 3$ ②
$4+\lambda = 1-4\mu$ ③

From ②: $\underline{\lambda = 5}$
Sub. in ①: $4-5 = 5+3\mu \quad \therefore \underline{\mu = -2}$

check consistency in ③: $4+5 = 1+8$ ✓

$\therefore \vec{OX}$ (position vector) $= \begin{pmatrix}4-5\\28-25\\4+5\end{pmatrix} = \begin{pmatrix}-1\\3\\9\end{pmatrix}$ or $\underline{X(-1,3,9)}$ (point)

b) $\cos^{-1}\left(\dfrac{\begin{pmatrix}-1\\-5\\1\end{pmatrix}\cdot\begin{pmatrix}3\\0\\-4\end{pmatrix}}{\sqrt{1+25+1}\sqrt{9+0+16}}\right) = \cos^{-1}\left(\dfrac{-7}{15\sqrt{3}}\right) = 105.63°$ (obtuse)



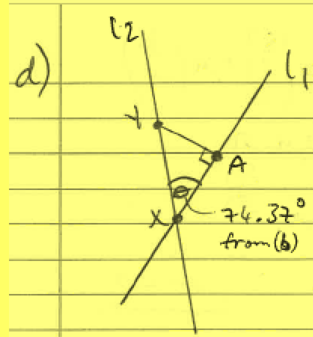
$\therefore$ Acute angle (required) is $180 - 105.63°$

$= \underline{74.37° \text{ (2dp)}}$

c) $\vec{AX} = \underset{\vec{OX}}{\begin{pmatrix}-1\\3\\9\end{pmatrix}} - \underset{\vec{OA}}{\begin{pmatrix}2\\18\\6\end{pmatrix}} = \begin{pmatrix}-3\\-15\\3\end{pmatrix}$

$\therefore AX = |\vec{AX}| = \sqrt{(-3)^2+(-15)^2+3^2} = \underline{9\sqrt{3}}$

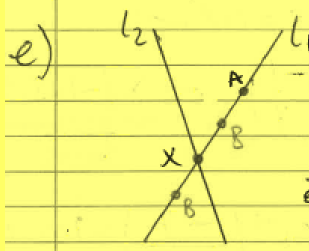
d)



$\tan\theta = \dfrac{AY}{AX}$

$\therefore AY = AX\tan\theta$
$= \underset{\text{from (c)}}{9\sqrt{3}}\tan(74.37)$
$= \underline{55.7}$ (1dp)

e)



B lies on l_1 and $|\vec{AX}| = 2|\vec{AB}|$

$\vec{AX} = \begin{pmatrix}-3\\-15\\3\end{pmatrix}$ (from c)

$\vec{OB}$ is a generic point on l_1

$l_1 = \begin{pmatrix}4-\lambda\\28-5\lambda\\4+\lambda\end{pmatrix} = \vec{OB}$

$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix}4-\lambda\\28-5\lambda\\4+\lambda\end{pmatrix} - \begin{pmatrix}2\\18\\6\end{pmatrix} = \begin{pmatrix}2-\lambda\\10-5\lambda\\-2+\lambda\end{pmatrix}$

$|\vec{AX}| = 2|\vec{AB}| \quad \therefore |\vec{AX}|^2 = 4|\vec{AB}|^2$

$(-3)^2+(-15)^2+(3^2) = 4\left[(2-\lambda)^2+(10-5\lambda)^2+(\lambda-2)^2\right]$

$\frac{243}{4} = 4-4\lambda+\lambda^2+100-100\lambda+25\lambda^2+\lambda^2-4\lambda+4$

$\frac{243}{4} = 27\lambda^2 - 108\lambda + 108$

$108\lambda^2 - 432\lambda + 189 = 0$

$4\lambda^2 - 16\lambda + 7 = 0 \quad \therefore \lambda = 0.5, \ \lambda = 3.5$

$\therefore \vec{OB} = \begin{pmatrix}4-0.5\\28-5\times0.5\\4+0.5\end{pmatrix} = \underline{\begin{pmatrix}3.5\\25.5\\4.5\end{pmatrix}}, \quad \vec{OB} = \begin{pmatrix}4-3.5\\28-5\times3.5\\4+3.5\end{pmatrix} = \underline{\begin{pmatrix}0.5\\10.5\\7.5\end{pmatrix}}$

Exercise B: Answers (vectors can be in either form)

1

$\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$ gives:

a) $\boldsymbol{r}\cdot(2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})=0$

b) $\boldsymbol{r}\cdot(4\boldsymbol{i}+\boldsymbol{j}-5\boldsymbol{k})=9$

2

$\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$ gives:

a) $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}4\\1\\-5\end{pmatrix}=\begin{pmatrix}4\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}4\\1\\-5\end{pmatrix}\Rightarrow\ 4x+y-5z=9$

b) $(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\cdot(\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k})=(2\boldsymbol{i}+0\boldsymbol{j}-3\boldsymbol{k})\cdot(\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k})$

$\Rightarrow\ x+3y+4z=-10$

3

Plane is perpendicular to **direction vector** of line, so $\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$ gives:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\4\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\0\end{pmatrix}\Rightarrow\ 3x-y=-7$$

4

Use any point as a position vector and find any two direction vectors (by subtracting two points)
Your position vectors are correct if they are a scalar multiple of (parallel to) those given below:

a) $\boldsymbol{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\-1\end{pmatrix}+\mu\begin{pmatrix}3\\1\\2\end{pmatrix}$

b) $\boldsymbol{r}=\begin{pmatrix}3\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}-4\\-6\\-1\end{pmatrix}+\mu\begin{pmatrix}-1\\-3\\3\end{pmatrix}$

Or $\boldsymbol{r}=\begin{pmatrix}3\\4\\1\end{pmatrix}+\lambda'\begin{pmatrix}4\\6\\1\end{pmatrix}+\mu\begin{pmatrix}-1\\-3\\3\end{pmatrix}$

5

a) $x+7y-5z=15$

b) $21x-13y-6z=5$

Exercise B: Answers (vectors can be in either form)

6

1. Find any two direction vectors (by subtracting two points)
2. Find direction of normal $\boldsymbol{n}$ by taking cross product of your two direction vectors
3. Apply $\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}=d$ where $\boldsymbol{a}$ can be any of the three position vectors.

(Your direction vectors are correct if they are a scalar multiple of (parallel to) those given below)

$\boldsymbol{r}\cdot\begin{pmatrix}4\\-5\\7\end{pmatrix}=3$ or $\boldsymbol{r}\cdot\begin{pmatrix}-4\\5\\-7\end{pmatrix}=-3$ (or any scalar multiple of both sides)

7

1. Find any two direction vectors (by subtracting two points)
2. Find direction of normal $\boldsymbol{n}$ by taking cross product of your two direction vectors
3. Apply $\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}=d$ where $\boldsymbol{a}$ can be any of the three position vectors and $\boldsymbol{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

(Your Cartesian equation can be any multiple of those given below)

a) $x+7y-5z=15$
b) $2x-6y-z=2$

8

Vectors in direction of plane:

$$\boldsymbol{d}=\begin{pmatrix}-1\\2\\-1\end{pmatrix} \quad and \quad \begin{pmatrix}4\\3\\1\end{pmatrix}-\begin{pmatrix}3\\5\\-2\end{pmatrix}=\begin{pmatrix}1\\-2\\3\end{pmatrix}$$

Normal to plane:

$$\boldsymbol{n}=\begin{pmatrix}-1\\2\\-1\end{pmatrix}\times\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}4\\2\\0\end{pmatrix}$$ (by determinant) $\Rightarrow$

$$\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$$

$$\boldsymbol{r}\cdot\begin{pmatrix}4\\2\\0\end{pmatrix}=\begin{pmatrix}4\\3\\1\end{pmatrix}\cdot\begin{pmatrix}4\\2\\0\end{pmatrix}$$

$$\therefore \boldsymbol{r}\cdot\begin{pmatrix}4\\2\\0\end{pmatrix}=22$$

Exercise B: Answers (vectors can be in either form)

9

$$\frac{x-2}{3} = \frac{y+4}{1} = \frac{z-1}{2} = \lambda$$

$$\frac{x-2}{3} = \lambda \Rightarrow x = 2 + 3\lambda$$

$$\frac{y+4}{1} = \lambda \Rightarrow y = -4 + \lambda$$

$$\frac{z-1}{2} = \lambda \Rightarrow z = 1 + 2\lambda$$

Equation of line: $\boldsymbol{r} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

Vectors in direction of plane:

$$\boldsymbol{d} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \qquad and \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$$

Normal to plane:

$$\boldsymbol{n} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ -2 \\ 16 \end{pmatrix}$$

(by determinant)

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -10 \\ -2 \\ 16 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -10 \\ -2 \\ 16 \end{pmatrix}$$

$$-10x - 2y + 16z = 4$$

Or $\quad 5x + y - 8z = 2$ (any multiple)

10

$l_1: \quad \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-4}{3} \quad (=\lambda)$

$l_2: \quad \underline{r}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

a) KEY POINT If lines intersect, they lie in the same plane

rewrite l_1 in vector form:

$\frac{x-1}{2} = \lambda \qquad \frac{y+1}{-1} = \lambda \qquad \frac{z-4}{3} = \lambda$

$\therefore x = 1 + 2\lambda \qquad y = -1 - \lambda \qquad z = 4 + 3\lambda$

$\therefore r_1 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Intersect if: $1 + t = 1 + 2\lambda$ ①

$-t = -1 - \lambda$ ②

$3 + 2t = 4 + 3\lambda$ ③

From ②: $t = 1 + \lambda$

Sub in ①: $1 + 1 + \lambda = 1 + 2\lambda$

$\lambda = 1 \quad \therefore t = 2$

check consistency in ③: $3 + 4 = 4 + 3$ ✓

consistent $\therefore$ lines intersect (at point $(3, -2, 7)$)

$\therefore l_1$ and l_2 lie in same plane.

b) Vector equation of plane containing l_1 and l_2 will be parametric equation with direction vectors of l_1 and l_2

$\therefore r_{plane} = $ either position vector $+ k_1 \left(\text{direction vector } l_1\right) + k_2 \left(\text{direction vector } l_2\right)$

$$\underline{r_{plane} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}}$$

c) Angle between l_1 and l_2 is angle between their direction vectors.

$$\theta = \cos^{-1}\left(\frac{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}}{\sqrt{1+1+4}\,\sqrt{4+1+9}}\right)$$

$$= \cos^{-1}\left(\frac{2+1+6}{\sqrt{6}\,\sqrt{14}}\right)$$

$$= \cos^{-1}\left(\frac{9}{\sqrt{6}\,\sqrt{14}}\right) = \underline{11^\circ} \text{ (nearest degree)}$$

Exercise B: Answers (vectors can be in either form)

11

Need 2 direction vectors in (or parallel to) our plane.

$$\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$$

The normal to $3x + 2y - z = 4$ is also parallel to our plane (as our plane is perpendicular to it)

$$\underline{n}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

So the normal to our plane is in the direction of $\underline{n}_2 = \overrightarrow{PQ} \times \underline{n}_1$

$$\underline{n}_2 = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ -7 \end{pmatrix}$$ (by determinant)

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -8 \\ -7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -8 \\ -7 \end{pmatrix}$$

$$3x - 8y - 7z = -41$$ (any multiple)

12

Line 1: $\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

Line 2: $\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$

So position vector of point on both lies (intersection) is $\boldsymbol{a} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$.

Normal for plane is cross product of direction vectors of lines:

$$\boldsymbol{n} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \times \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 13 \\ 7 \\ 9 \end{pmatrix}$$ (by determinant)

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 13 \\ 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ 7 \\ 9 \end{pmatrix}$$

$$13x + 7y + 9z = 61$$