CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

University of Leeds

January 2025

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Tutorial: Question Difficulty Colour Code

Basic - straightforward application
(you must be able to do these)

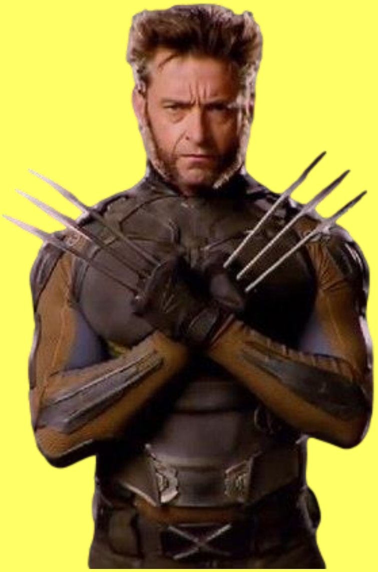
Medium – Makes you think a bit
(you must be able to do these)

Hard – Makes you think a lot
(you should be able to do these)

Extreme – Tests your understanding to the limit!
(for those who like a challenge)

Applied – Real-life examples of the topic, may sometimes involve prior knowledge
(you should attempt these – will help in future engineering)

$$\int \frac{1}{AN} dAN =$$



Tutorial 2
Integration

Class Example: 2 Methods

E.g. 1

Some integrals can be tackled by multiple methods

$$\int x\sqrt{x+1}\,dx$$

By Substitution

$$u = x+1, \qquad \frac{du}{dx} = 1$$

$$x = u-1, \qquad dx = du$$

$$\int x\sqrt{x+1}\,dx = \int (u-1)u^{\frac{1}{2}}\,du$$

$$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du \quad = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

$$= \frac{6}{15}(x+1)^{\frac{5}{2}} - \frac{10}{15}(x+1)^{\frac{3}{2}} + C$$

$$= \frac{2}{15}(x+1)^{\frac{3}{2}}(3(x+1)-5) + C$$

$$= \boldsymbol{\frac{2}{15}(x+1)^{\frac{3}{2}}(3x-2) + C}$$

By Parts

$$u = x, \qquad \frac{dv}{dx} = (x+1)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 1, \qquad v = \frac{2}{3}(x+1)^{\frac{3}{2}}$$

$$\int x\sqrt{x+1}\,dx = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}\,dx$$

$$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\times\frac{2}{5}(x+1)^{\frac{5}{2}} + C$$

$$= \frac{10}{15}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C$$

$$= \frac{2}{15}(x+1)^{\frac{3}{2}}(5x - 2(x+1)) + C$$

$$= \boldsymbol{\frac{2}{15}(x+1)^{\frac{3}{2}}(3x-2) + C}$$

Class Example: Substitution *then* Parts

E.g. 2a

Find the exact value of: $\displaystyle\int_0^{\frac{\pi}{2}} \cos x \sin x \ e^{\cos x} \, dx$

$$t = \cos x \Rightarrow dx = -\frac{dt}{\sin x}$$

$$x = 0 \rightarrow t = 1, x = \frac{\pi}{2} \rightarrow t = 0$$

$$\int_0^{\frac{\pi}{2}} \cos x \sin x \ e^{\cos x} \, dx - \int_1^0 t \, e^t \, dt = \int_0^1 t \, e^t \, dt$$

Parts: $u = t \Rightarrow \frac{du}{dt} = 1$

$$\frac{dv}{dt} = e^t \Rightarrow v = e^t$$

$$\int t \, e^t \, dt = te^t - \int e^t dt$$

$$= te^t - e^t + C$$

$$\int_0^1 t \, e^t \, dt = [e^t(t-1)]_0^1 = \big(e(0) - e^0(-1)\big) = \mathbf{1}$$

Class Example: Alternative Method (Trick)

E.g. 2b

(i) State the derivative of $e^{\cos x}$.

(ii) Hence use integration by parts to find the exact value of

$$\int_0^{\frac{1}{2}\pi} \cos x \sin x \, e^{\cos x} \, dx.$$

(i) $-\sin x \, e^{\cos x}$

(ii) $\int_0^{\frac{\pi}{2}} \cos x \sin x \, e^{\cos x} dx$

$u = \cos x, \frac{dv}{dx} = \sin x \, e^{\cos x}$

$\frac{du}{dx} = -\sin x \qquad v = -e^{\cos x}$

$$\int_0^{\frac{\pi}{2}} \cos x \sin x \, e^{\cos x} dx = [-\cos x \, e^{\cos x}]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, e^{\cos x} \; dx$$

$$= [-\cos x \, e^{\cos x}]_0^{\frac{\pi}{2}} + [e^{\cos x}]_0^{\frac{\pi}{2}}$$

$$= \left(\left[-\cos\frac{\pi}{2} e^{\cos\frac{\pi}{2}}\right] - [-\cos 0 \, e^{\cos 0}]\right) + \left(\left[e^{\cos\frac{\pi}{2}}\right] - e^{\cos 0}]\right)$$

$$= 0 + e + 1 - e$$

$$= \mathbf{1}$$

Class Example: Integrating a General Exponential

E.g. 3

Find $\int a^x \, dx$

By inspection:

From last lecture we know that

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{1}{\ln a}\frac{d}{dx}(a^x) = a^x$$

$$\int a^x dx = \frac{1}{\ln a}\int \frac{d}{dx}(a^x)\, dx$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

By substitution:

$$u = a^x$$

$$\ln u = x \ln a$$

$$\frac{1}{u}\frac{du}{dx} = \ln a$$

$$dx = \frac{du}{u \ln a}$$

$$\therefore \int a^x \, dx = \int \cancel{u} \frac{du}{\cancel{u} \ln a}$$

$$= \frac{1}{\ln a}\int du \quad = \frac{1}{\ln a}(u + C_1)$$

$$= \frac{1}{\ln a}(a^x + C_1) \quad = \frac{a^x}{\ln a} + C$$

Note that this also works for "the" exponential function $\int e^x \, dx = \frac{e^x}{\ln e} + C = e^x + C$ (as $\ln e = 1$)

Tip: When I know I'm going to be changing a constant, I call it C_1 then C_2 etc with each change, then the final time, call it C. (Here $C = \frac{C_1}{\ln a}$).

Class Example: Partial Fractions

E.g. 4

$$f(x) = \frac{4-2x}{(2x+1)(x+1)(x+3)} = \frac{A}{(2x+1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}.$$

(*a*) Find the values of the constants A, B and C. **(4)**

(*b*) (i) Hence find $\int f(x)\ dx$. **(3)**

(ii) Find $\int_0^2 f(x)\ dx$ in the form $\ln k$, where k is a constant. **(3)**

(a)	$f(x) = \frac{4-2x}{(2x+1)(x+1)(x+3)} = \frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{x+3}$	
	$4-2x = A(x+1)(x+3) + B(2x+1)(x+3) + C(2x+1)(x+1)$	M1
	A method for evaluating one constant	M1
	$x \to -\frac{1}{2},\ 5 = A(\frac{1}{2})(\frac{5}{2}) \Rightarrow A = 4$ any one correct constant	A1
	$x \to -1,\ 6 = B(-1)(2) \Rightarrow B = -3$	
	$x \to -3,\ 10 = C(-5)(-2) \Rightarrow C = 1$ all three constants correct	A1

(ii)	$\left[2\ln(2x+1) - 3\ln(x+1) + \ln(x+3)\right]_0^2$	
	$= (2\ln 5 - 3\ln 3 + \ln 5) - (2\ln 1 - 3\ln 1 + \ln 3)$	M1
	$= 3\ln 5 - 4\ln 3$	
	$= \ln\left(\frac{5^3}{3^4}\right)$	M1
	$= \ln\left(\frac{125}{81}\right)$	A1

(b) (i)	$\int\left(\frac{4}{2x+1} - \frac{3}{x+1} + \frac{1}{x+3}\right)dx$	
	$= \frac{4}{2}\ln(2x+1) - 3\ln(x+1) + \ln(x+3) + C$ A1 two ln terms correct	M1 A1ft
	All three ln terms correct and "+C" ; ft constants	A1ft (3)

A Problem That Wouldn't Fit In Any Category

E.g. 5

This doesn't involve any of the methods we learned in the lecture but it's a useful trick to know!

$$\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx$$

$$= \int 1\,dx - \int \frac{1}{x^2+1}\,dx$$

$x = \tan u$ $\quad \frac{dx}{du} = \sec^2 u$

$dx = \sec^2 u \; du$

$$= x - \int \frac{1}{\cancel{\tan^2 u + 1}} \cancel{\sec^2 u}\, du$$

$1 + \tan^2 u \equiv \sec^2 u$

$$= x - \int du$$

$$= x - u + C$$

$$= x - \tan^{-1} x + C$$

Class Challenge (A Cheeky Trick!)

E.g. 6

$$\int \frac{2\sin x}{\sin x + \cos x}\,dx$$

Crucial Step:
$2\sin x = (\sin x + \cos x) + (\sin x - \cos x)$

$$= \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{\sin x + \cos x}\,dx$$

$$= \int \frac{\sin x + \cos x}{\sin x + \cos x}\,dx + \int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$$

$$= \int 1\,dx - \int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$$

The second integral is now of the form $f'(x)/f(x)$

$$u = \sin x + \cos x \Rightarrow \frac{du}{dx} = \cos x - \sin x$$

$$dx = \frac{du}{\cos x - \sin x}$$

$$= x - \int \frac{\cos x - \sin x}{u}\,\frac{du}{\cos x - \sin x} \quad = x - \int \frac{1}{u}\,du$$

$$= x - \ln u + C \quad = \boldsymbol{x - \ln(\sin x + \cos x) + C}$$

Exercise A: Integrating $f(ax+b)$

1

a) $\int \sin(2x+1)\,dx$

b) $\int 4e^{x+5}\,dx$

c) $\int cosec^2 3x\,dx$

d) $\int \sec 4x \tan 4x\,dx$

e) $\int 3\sin\left(\frac{1}{2}x+1\right)dx$

f) $\int cosec\,2x \cot 2x\,dx$

2

a) $\int \frac{1}{2x+1}\,dx$

b) $\int \frac{1}{(2x+1)^2}\,dx$

c) $\int (2x+1)^2\,dx$

d) $\int \frac{3}{4x-1}\,dx$

e) $\int \frac{3}{(1-4x)^2}\,dx$

f) $\int \frac{3}{(1-2x)^3}\,dx$

g) $\int \frac{5}{3-2x}\,dx$

3

a) $\int e^{2x} - \frac{1}{2}\sin(2x-1)\,dx$

b) $\int (e^x+1)^2\,dx$

c) $\int \sec^2 2x\,(1+\sin 2x)\,dx$

d) $\int \frac{3-2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)}\,dx$

e) $\int e^{3-x} + \sin(3-x) + \cos(3-x)\,dx$

4

a) $\int 3\sin(2x+1) + \frac{4}{2x+1}\,dx$

b) $\int \frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2}\,dx$

Exercise B: Integration By Substitution

1 Showing all your working, use the given substitution to evaluate:

a $\int 2x(x^2-1)^3 \, dx \quad u = x^2 + 1$

b $\int \sin^4 x \cos x \, dx \quad u = \sin x$

c $\int 3x^2(2+x^3)^2 \, dx \quad u = 2 + x^3$

d $\int 2x e^{x^2} \, dx \quad u = x^2$

e $\int \frac{x}{(x^2+3)^4} \, dx \quad u = x^2 + 3$

f $\int \sin 2x \cos^3 2x \, dx \quad u = \cos 2x$

g $\int \frac{3x}{x^2-2} \, dx \quad u = x^2 - 2$

h $\int x\sqrt{1-x^2} \, dx \quad u = 1 - x^2$

i $\int \sec^3 x \tan x \, dx \quad u = \sec x$

j $\int (x+1)(x^2+2x)^3 \, dx \quad u = x^2 + 2x$

2 Evaluate using a suitable substitution:

a $\int_1^2 x(x^2-3)^3 \, dx$

b $\int_0^{\frac{\pi}{6}} \sin^3 x \cos x \, dx$

c $\int_0^3 \frac{4x}{x^2+1} \, dx$

d $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx$

e $\int_2^3 \frac{x}{\sqrt{x^2-3}} \, dx$

f $\int_{-2}^{-1} x^2(x^3+2)^2 \, dx$

g $\int_0^1 e^{2x}(1+e^{2x})^3 \, dx$

h $\int_3^5 (x-2)(x^2-4x)^2 \, dx$

Exercise B: Integration By Substitution

3

a By writing $\cot x$ as $\frac{\cos x}{\sin x}$, use the substitution $u = \sin x$ to show that

$$\int \cot x \, dx = \ln|\sin x| + c.$$

b Show that

$$\int \tan x \, dx = \ln|\sec x| + c.$$

c Evaluate

$$\int_0^{\frac{\pi}{6}} \tan 2x \, dx.$$

Hint: You might want to use your double angle formula first.

4

Using the substitution $u = 1 + \cos x$, or otherwise, show that

$$\int \frac{2\sin 2x}{(1+\cos x)} dx = 4\ln(1+\cos x) - 4\cos x + k,$$

where k is a constant.

5

Using the given substitution, find

a $\int x(2x-1)^4 \, dx$ $\quad u = 2x - 1$

b $\int x\sqrt{1-x} \, dx$ $\quad u^2 = 1 - x$

c $\int \frac{1}{(1-x^2)^{\frac{3}{2}}} \, dx$ $\quad x = \sin u$

d $\int \frac{1}{\sqrt{x}-1} \, dx$ $\quad x = u^2$

e $\int (x+1)(2x+3)^3 \, dx$ $\quad u = 2x + 3$

f $\int \frac{x^2}{\sqrt{x-2}} \, dx$ $\quad u^2 = x - 2$

6

Using the given substitution, evaluate

a $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} \, dx$ $\quad x = \sin u$

b $\int_0^2 x(2-x)^3 \, dx$ $\quad u = 2 - x$

c $\int_0^1 \sqrt{4-x^2} \, dx$ $\quad x = 2\sin u$

d $\int_0^3 \frac{x^2}{x^2+9} \, dx$ $\quad x = 3\tan u$

Exercise C: Integration By Parts

Calculate the integrals (for any definite integrals, find the **exact** value)

1

a $\int xe^{2x}\,dx$

b $\int \ln 2x\,dx$

c $\int \frac{x}{e^{3x}}\,dx$

d $\int \ln(x-1)\,dx$

e $\int_1^e x^2 \ln x\,dx$

f $\int_0^2 (x-1)(x+1)^3$

g $\int_1^e \frac{\ln x}{x^4}\,dx$

h $\int_1^e (\ln x)^2\,dx$

i $\int_0^\pi x \sin x\,dx$

2

a $\int x \sec^2 x\,dx$

b $\int x^2 \cos x\,dx$

c $\int e^{2x} \sin x\,dx$

d $\int_1^5 x(3x+1)^{-\frac{1}{2}}\,dx$

e $\int_0^{\frac{\pi}{4}} e^{3x} \sin 2x\,dx$

3

a $\int te^{-st}dt$ (where s is a constant)

b $\int t^2 e^{-st}dt$ (where s is a constant)

You will apply integrals like these next year in "Laplace Transforms"

4

a $\int_{-\pi}^{\pi} x \cos nx\,dx$

(n is constant)

b $\int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx$

(n and L are constant)

You will apply integrals like these next year in "Fourier Series"

Exercise D: Integration Using Partial Fractions

1 $$\int \frac{x-5}{(x+1)(x-2)}\,dx$$

2 $$\int \frac{2}{x^2-1}\,dx$$

3 $$\int \frac{2-6x+5x^2}{x^2(1-2x)}\,dx$$

4 Show that: $$\int_3^4 \frac{3x-5}{(x-1)(x-2)}\,dx = \ln\frac{9}{2}$$

5 $$\int_0^2 \frac{2x^2-7x+7}{x^2-2x-3}\,dx$$

6 $$\int \ln(x^2+a^2)\;dx$$

Challenge Exercise: Evaluate the integrals

1 $\int e^{e^x} e^x dx$

2 $\int x^n \ln x \, dx$

3 $\int \frac{dx}{x\sqrt{x^{-2n} - 1}}$

Hint: sub $u = x^n$

4 $\int \cos(\ln x) dx$

5 Find the exact value of $\int_1^2 \frac{1}{x + x^3} dx$

6 $\int x^7 \sqrt{1 + x^4} \, dx$

Hint: Rewrite as $\int x^3 x^4 \sqrt{1 + x^4} \, dx$

7 Show that, for $n > 0$,

a $\int_0^{\frac{\pi}{4}} \tan^n x \sec^2 x \, dx = \frac{1}{n+1}$

b $\int_0^{\frac{\pi}{4}} \sec^n x \tan x \, dx = \frac{\left(\sqrt{2}\right)^n - 1}{n}$

ANSWERS

Exercise A: Integrating $f(ax+b)$

1

a $\int \sin(2x+1)\,dx = -\frac{1}{2}\cos(2x+1) + C$

b $\int 4e^{x+5}\,dx = 4e^{x+5} + C$

c $\int cosec^2 3x\,dx = -\frac{1}{3}\cot 3x + C$

d $\int \sec 4x \tan 4x\;dx = \frac{1}{4}\sec 4x + C$

e $\int 3\sin\left(\frac{1}{2}x+1\right)dx = -6\cos\left(\frac{1}{2}x+1\right)$

f $\int cosec\,2x \cot 2x\;dx = -\frac{1}{2}cosec\,2x + C$

2

a $\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1| + C$

b $\int \frac{1}{(2x+1)^2}\,dx = -\frac{1}{2(2x+1)} + C$

c $\int (2x+1)^2\,dx = \frac{1}{6}(2x+1)^3 + C$

d $\int \frac{3}{4x-1}\,dx = \frac{3}{4}\ln|4x-1| + C$

e $\int \frac{3}{(1-4x)^2}\,dx = \frac{3}{4(1-4x)} + C$

f $\int \frac{3}{(1-2x)^3}\,dx = \frac{3}{4(1-2x)^2} + C$

g $\int \frac{5}{3-2x}\,dx = -\frac{5}{2}\ln|3-2x| + C$

3

a $\int e^{2x} - \frac{1}{2}\sin(2x-1)\,dx$

$= \frac{1}{2}e^{2x} + \frac{1}{4}\cos(2x-1) + C$

b $\int (e^x+1)^2\,dx = \frac{1}{2}e^{2x} + 2e^x + x + C$

c $\int \sec^2 2x\,(1+\sin 2x)\,dx = \frac{1}{2}\tan 2x + \frac{1}{2}\sec 2x$

d $\int \frac{3-2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)}dx = -6\cot\left(\frac{1}{2}x\right) + 4cosec\left(\frac{1}{2}x\right)$

e $\int e^{3-x} + \sin(3-x) + \cos(3-x)\,dx$

$= -e^{3-x} + \cos(3-x) - \sin(3-x) + C$

4

a $\int 3\sin(2x+1) + \frac{4}{2x+1}\,dx$

$= -\frac{3}{2}\cos(2x+1) + 2\ln|2x+1| + C$

b $\int \frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2}\,dx$

$= -\frac{1}{2}\cot 2x + \frac{1}{2}\ln|1+2x| - \frac{1}{2(1+2x)}$

Exercise B: Solutions

1

a $u = x^2 + 1 \;\therefore\; \dfrac{du}{dx} = 2x$

$$\int 2x(x^2-1)^3\,dx = \int u^3\,du = \tfrac{1}{4}u^4 + c = \tfrac{1}{4}(x^2+1)^4 + c$$

b $u = \sin x \;\therefore\; \dfrac{du}{dx} = \cos x$

$$\int \sin^4 x\cos x\,dx = \int u^4\,du = \tfrac{1}{5}u^5 + c = \tfrac{1}{5}\sin^5 x + c$$

c $u = 2 + x^3 \;\therefore\; \dfrac{du}{dx} = 3x^2$

$$\int 3x^2(2+x^3)^2\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + c = \tfrac{1}{3}(2+x^3)^3 + c$$

d $u = x^2 \;\therefore\; \dfrac{du}{dx} = 2x$

$$\int 2x e^{x^2}\,dx = \int e^u\,du = e^u + c = e^{x^2} + c$$

e $u = x^2 + 3 \;\therefore\; \dfrac{du}{dx} = 2x$

$$\int \frac{x}{(x^2+3)^4}\,dx = \int \tfrac{1}{2}u^{-4}\,du = -\tfrac{1}{6}u^{-3} + c = -\frac{1}{6(x^2+3)^3} + c$$

f $u = \cos 2x \;\therefore\; \dfrac{du}{dx} = -2\sin 2x$

$$\int \sin 2x\cos^3 2x\,dx = \int -\tfrac{1}{2}u^3\,du = -\tfrac{1}{8}u^4 + c = -\tfrac{1}{8}\cos^4 2x + c$$

g $u = x^2 - 2 \;\therefore\; \dfrac{du}{dx} = 2x$

$$\int \frac{3x}{x^2-2}\,dx = \int \frac{3}{2u}\,du = \tfrac{3}{2}\ln|u| + c = \tfrac{3}{2}\ln|x^2-2| + c$$

h $u = 1 - x^2 \;\therefore\; \dfrac{du}{dx} = -2x$

$$\int x\sqrt{1-x^2}\,dx = \int -\tfrac{1}{2}u^{\frac{1}{2}}\,du = -\tfrac{1}{3}u^{\frac{3}{2}} + c = -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + c$$

i $u = \sec x \;\therefore\; \dfrac{du}{dx} = \sec x\tan x$

$$\int \sec^3 x\tan x\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + c = \tfrac{1}{3}\sec^3 x + c$$

j $u = x^2 + 2x \;\therefore\; \dfrac{du}{dx} = 2x + 2$

$$\int (x+1)(x^2+2x)^3\,dx = \int \tfrac{1}{2}u^3\,du = \tfrac{1}{8}u^4 + c = \tfrac{1}{8}(x^2+2x)^4 + c$$

2

a $u = x^2 - 3 \;\therefore\; \dfrac{du}{dx} = 2x$

$x = 1 \Rightarrow u = -2$

$x = 2 \Rightarrow u = 1$

$$\int_1^2 x(x^2-3)^3\,dx = \int_{-2}^1 \tfrac{1}{2}u^3\,du = [\tfrac{1}{8}u^4]_{-2}^1 = \tfrac{1}{8}(1-16) = -\tfrac{15}{8}$$

b $u = \sin x \;\therefore\; \dfrac{du}{dx} = \cos x$

$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$

$$\int_0^{\frac{\pi}{6}} \sin^3 x\cos x\,dx = \int_0^{\frac{1}{2}} u^3\,du = [\tfrac{1}{4}u^4]_0^{\frac{1}{2}} = \tfrac{1}{4}(\tfrac{1}{16} - 0) = \tfrac{1}{64}$$

c $u = x^2 + 1 \;\therefore\; \dfrac{du}{dx} = 2x$

$x = 0 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 10$

$$\int_0^3 \frac{4x}{x^2+1}\,dx = \int_1^{10} \frac{2}{u}\,du = [2\ln|u|]_1^{10} = 2\ln 10 - 0 = 2\ln 10$$

d $u = \tan x \;\therefore\; \dfrac{du}{dx} = \sec^2 x$

$x = -\frac{\pi}{4} \Rightarrow u = -1$

$x = \frac{\pi}{4} \Rightarrow u = 1$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x\sec^2 x\,dx = \int_{-1}^1 u^2\,du = [\tfrac{1}{3}u^3]_{-1}^1 = \tfrac{1}{3}[1-(-1)] = \tfrac{2}{3}$$

e $u = x^2 - 3 \;\therefore\; \dfrac{du}{dx} = 2x$

$x = 2 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 6$

$$\int_2^3 \frac{x}{\sqrt{x^2-3}}\,dx = \int_1^6 \tfrac{1}{2}u^{-\frac{1}{2}}\,du = [u^{\frac{1}{2}}]_1^6 = \sqrt{6} - 1$$

f $u = x^3 + 2 \;\therefore\; \dfrac{du}{dx} = 3x^2$

$x = -2 \Rightarrow u = -6$

$x = -1 \Rightarrow u = 1$

$$\int_{-2}^{-1} x^2(x^3+2)^2\,dx = \int_{-6}^1 \tfrac{1}{3}u^2\,du = [\tfrac{1}{9}u^3]_{-6}^1 = \tfrac{1}{9}[1-(-216)] = 24\tfrac{1}{9}$$

g $u = 1 + e^{2x} \;\therefore\; \dfrac{du}{dx} = 2e^{2x}$

$x = 0 \Rightarrow u = 2$

$x = 1 \Rightarrow u = 1 + e^2$

$$\int_0^1 e^{2x}(1+e^{2x})^3\,dx = \int_2^{1+e^2} \tfrac{1}{2}u^3\,du = [\tfrac{1}{8}u^4]_2^{1+e^2} = \tfrac{1}{8}[(1+e^2)^4 - 16] = \tfrac{1}{8}(1+e^2)^4 - 2$$

h $u = x^2 - 4x \;\therefore\; \dfrac{du}{dx} = 2x - 4$

$x = 3 \Rightarrow u = -3$

$x = 5 \Rightarrow u = 5$

$$\int_3^5 (x-2)(x^2-4x)^2\,dx = \int_{-3}^5 \tfrac{1}{2}u^2\,du = [\tfrac{1}{6}u^3]_{-3}^5 = \tfrac{1}{6}[125-(-27)] = 25\tfrac{1}{3}$$

Exercise B: Solutions

3

a $u = \sin x \;\therefore\; \dfrac{du}{dx} = \cos x$

$$\int \cot x \; dx = \int \frac{\cos x}{\sin x} \; dx$$
$$= \int \frac{1}{u} \times \frac{du}{dx} \; dx$$
$$= \int \frac{1}{u} \; du$$
$$= \ln|u| + c$$
$$= \ln|\sin x| + c$$

b $\displaystyle\int \tan x \; dx = \int \frac{\sin x}{\cos x} \; dx$

$u = \cos x \;\therefore\; \dfrac{du}{dx} = -\sin x$

$$\int \frac{\sin x}{\cos x} \; dx = \int \frac{1}{u} \times \left(-\frac{du}{dx}\right) \; dx$$
$$= \int -\frac{1}{u} \; du$$
$$= -\ln|u| + c$$
$$= -\ln|\cos x| + c$$
$$= \ln(|\cos x|)^{-1} + c$$
$$= \ln|\sec x| + c$$

c $= \left[\tfrac{1}{2}\ln|\sec 2x|\right]_0^{\frac{\pi}{6}}$

$= \tfrac{1}{2}(\ln 2 - 0)$

$= \tfrac{1}{2}\ln 2$

4

$$\frac{du}{dx} = -\sin x \quad \rightarrow \quad dx = -\frac{1}{\sin x} du$$

$\cos x = u - 1$ (As before $x = \arccos(u-1)$ is going to be messy)

$$\int \frac{2\sin 2x}{1+\cos x} dx = \int \frac{4\sin x \cos x}{1+\cos x} dx = \int -\frac{4\sin x\,(u-1)}{u}\frac{1}{\sin x} du$$
$$= -4\int \frac{u-1}{u} du = -4\int 1 - \frac{1}{u} du$$
$$= -4(u - \ln|u|) + c = -4(1 + \cos x - \ln|\cos x + 1|) + c$$
$$= 4\ln|\cos x + 1| - 4\cos x + k \quad (\text{where } k = -4c)$$

5

a $u = 2x - 1 \;\therefore\; x = \tfrac{1}{2}(u+1),\; \dfrac{du}{dx} = 2$

$$\int x(2x-1)^4 \; dx = \int \tfrac{1}{2}(u+1)u^4 \times \tfrac{1}{2} \; du$$
$$= \tfrac{1}{4}\int (u^5 + u^4) \; du$$
$$= \tfrac{1}{4}\left(\tfrac{1}{6}u^6 + \tfrac{1}{5}u^5\right) + c$$
$$= \tfrac{1}{4}\left[\tfrac{1}{6}(2x-1)^6 + \tfrac{1}{5}(2x-1)^5\right] + c$$
$$= \tfrac{1}{120}(2x-1)^5[5(2x-1)+6] + c$$
$$= \tfrac{1}{120}(10x+1)(2x-1)^5 + c$$

b $u^2 = 1 - x \;\therefore\; x = 1 - u^2,\; \dfrac{dx}{du} = -2u$

$$\int x\sqrt{1-x} \; dx = \int (1-u^2)u \times (-2u) \; du$$
$$= 2\int (u^4 - u^2) \; du$$
$$= 2\left(\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3\right) + c$$
$$= 2\left[\tfrac{1}{5}(1-x)^{\frac{5}{2}} - \tfrac{1}{3}(1-x)^{\frac{3}{2}}\right. + c$$
$$= \tfrac{2}{15}(1-x)^{\frac{3}{2}}[3(1-x) - 5] + c$$
$$= -\tfrac{2}{15}(2+3x)(1-x)^{\frac{3}{2}} + c$$

c $x = \sin u \;\therefore\; \dfrac{dx}{du} = \cos u$

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}} \; dx = \int \frac{1}{\cos^3 u} \times \cos u \; du$$
$$= \int \sec^2 u \; du$$
$$= \tan u + c$$
$$= \frac{\sin u}{\cos u} + c$$
$$= \frac{x}{\sqrt{1-x^2}} + c$$

d $x = u^2 \;\therefore\; \dfrac{dx}{du} = 2u$

$$\int \frac{1}{\sqrt{x}-1} \; dx = \int \frac{1}{u-1} \times 2u \; du$$
$$= \int \frac{2(u-1)+2}{u-1} \; du$$
$$= \int \left(2 + \frac{2}{u-1}\right) \; du$$
$$= 2u + 2\ln|u-1| + c$$
$$= 2\sqrt{x} + 2\ln|\sqrt{x} - 1| + c$$

e $u = 2x + 3 \;\therefore\; x = \tfrac{1}{2}u - \tfrac{3}{2},\; \dfrac{du}{dx} = 2$

$$\int (x+1)(2x+3)^3 \; dx$$
$$= \int \left(\tfrac{1}{2}u - \tfrac{1}{2}\right)u^3 \times \tfrac{1}{2} \; du$$
$$= \tfrac{1}{4}\int (u^4 - u^3) \; du$$
$$= \tfrac{1}{4}\left(\tfrac{1}{5}u^5 - \tfrac{1}{4}u^4\right) + c$$
$$= \tfrac{1}{4}\left[\tfrac{1}{5}(2x+3)^5 - \tfrac{1}{4}(2x+3)^4\right] + c$$
$$= \tfrac{1}{80}(2x+3)^4[4(2x+3) - 5] + c$$
$$= \tfrac{1}{80}(8x+7)(2x+3)^4 + c$$

f $u^2 = x - 2 \;\therefore\; x = u^2 + 2,\; \dfrac{dx}{du} = 2u$

$$\int \frac{x^2}{\sqrt{x-2}} \; dx = \int \frac{(u^2+2)^2}{u} \times 2u \; du$$
$$= 2\int (u^4 + 4u^2 + 4) \; du$$
$$= 2\left(\tfrac{1}{5}u^5 + \tfrac{4}{3}u^3 + 4u\right) + c$$
$$= 2\left[\tfrac{1}{5}(x-2)^{\frac{5}{2}} + \tfrac{4}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}}\right] + c$$
$$= \tfrac{2}{15}(x-2)^{\frac{1}{2}}[3(x-2)^2 + 20(x-2) + 60] + c$$
$$= \tfrac{2}{15}(3x^2 + 8x + 32)(x-2)^{\frac{1}{2}} + c$$

Exercise B: Solutions

6

a $x = \sin u \ \therefore \ \frac{dx}{du} = \cos u$

$x = 0 \Rightarrow u = 0$

$x = \frac{1}{2} \Rightarrow u = \frac{\pi}{6}$

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} \, dx = \int_0^{\frac{\pi}{6}} \frac{1}{\cos u} \times \cos u \, du$$

$$= \int_0^{\frac{\pi}{6}} du$$

$$= [u]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

b $u = 2 - x \ \therefore \ x = 2 - u, \ \frac{du}{dx} = -1$

$x = 0 \Rightarrow u = 2$

$x = 2 \Rightarrow u = 0$

$$\int_0^2 x(2-x)^3 \, dx = \int_2^0 (2-u)u^3 \times (-1) \, du$$

$$= \int_0^2 (2u^3 - u^4) \, du$$

$$= [\tfrac{1}{2}u^4 - \tfrac{1}{5}u^5]_0^2$$

$$= (8 - \tfrac{32}{5}) - (0)$$

$$= \tfrac{8}{5}$$

c $x = 2\sin u \ \therefore \ \frac{dx}{du} = 2\cos u$

$x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = \frac{\pi}{6}$

$$\int_0^1 \sqrt{4 - x^2} \, dx$$

$$= \int_0^{\frac{\pi}{6}} 2\cos u \times 2\cos u \, du$$

$$= \int_0^{\frac{\pi}{6}} 4\cos^2 u \, du$$

$$= \int_0^{\frac{\pi}{6}} (2 + 2\cos 2u) \, du$$

$$= [2u + \sin 2u]_0^{\frac{\pi}{6}}$$

$$= (\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}) - (0)$$

$$= \tfrac{1}{6}(2\pi + 3\sqrt{3})$$

d $x = 3\tan u \ \therefore \ \frac{dx}{du} = 3\sec^2 u$

$x = 0 \Rightarrow u = 0$

$x = 3 \Rightarrow u = \frac{\pi}{4}$

$$\int_0^3 \frac{x^2}{x^2 + 9} \, dx = \int_0^{\frac{\pi}{4}} \frac{9\tan^2 u}{9\sec^2 u} \times 3\sec^2 u \, du$$

$$= 3\int_0^{\frac{\pi}{4}} \tan^2 u \, du$$

$$= 3\int_0^{\frac{\pi}{4}} (\sec^2 u - 1) \, du$$

$$= 3[\tan u - u]_0^{\frac{\pi}{4}}$$

$$= 3[(1 - \tfrac{\pi}{4}) - (0)]$$

$$= \tfrac{3}{4}(4 - \pi)$$

Exercise C: Solutions

Calculate the integrals (for any definite integrals, find the **exact** value)

1

a $\int xe^{2x}dx = \frac{1}{4}e^{2x}(2x-1)+C$

b $\int \ln 2x\,dx = x\ln(2x) - x + C$

c $\int \frac{x}{e^{3x}}dx = -\frac{1}{9}e^{-3x}(3x+1)+C$

d $\int \ln(x-1)\,dx = (x-1)\ln|x-1| - x + C$

e $\int_1^e x^2\ln x\,dx = \frac{1}{9}(e^3+1)$

f $\int_0^2 (x-1)(x+1)^3 = 8.4$

g $\int_1^e \frac{\ln x}{x^4}dx = \frac{1}{9}(1-4e^{-3})$

h $\int_1^e (\ln x)^2\,dx = e-2$

i $\int_0^\pi x\sin x\,dx = \pi$

2

a $\int x\sec^2 x\,dx = x\tan x + \ln|\cos x| + C$

b $\int x^2\cos x\,dx = (x^2-2)\sin x + 2x\cos x + C$

c $\int e^{2x}\sin x\,dx = -\frac{1}{5}e^{2x}\cos x + \frac{2}{5}e^{2x}\sin x + C$

d $\int_1^5 x(3x+1)^{-\frac{1}{2}}\,dx = \frac{100}{27}$

e $\int_0^{\frac{\pi}{4}} e^{3x}\sin 2x\ dx = \frac{1}{13}(3e^{\frac{3\pi}{4}}+2)$

3

a $\int te^{-st}dt = \frac{-e^{-st}(st+1)}{s^2}+C$ (where s is a constant)

b $\int t^2e^{-st}dt = \frac{-e^{-st}(s^2t^2+2st+2)}{s^3}+C$

(Parts Twice)

4

a $\int_{-\pi}^{\pi} x\cos nx\ dx = 0$

(n is constant)

b $\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)\,dx = -\frac{2L^2}{n\pi}$

(n and L are constant)

Exercise C: Solutions (More detail on 2e)

2

$$\int_0^{\frac{\pi}{4}} e^{3x}\sin 2x \; dx = \frac{1}{13}(3e^{\frac{3\pi}{4}} + 2)$$

e

$$u = e^{3x}, \; \frac{du}{dx} = 3e^{3x}; \quad \frac{dv}{dx} = \sin 2x, \; v = -\tfrac{1}{2}\cos 2x$$

$$\int e^{3x}\sin 2x \; dx = -\tfrac{1}{2}e^{3x}\cos 2x - \int -\tfrac{3}{2}e^{3x}\cos 2x \; dx$$

$$= -\tfrac{1}{2}e^{3x}\cos 2x + \int \tfrac{3}{2}e^{3x}\cos 2x \; dx$$

for $\int \tfrac{3}{2}e^{3x}\cos 2x \; dx$, $\quad u = \tfrac{3}{2}e^{3x}, \; \frac{du}{dx} = \tfrac{9}{2}e^{3x}; \quad \frac{dv}{dx} = \cos 2x, \; v = \tfrac{1}{2}\sin 2x$

$$\int \tfrac{3}{2}e^{3x}\cos 2x \; dx = \tfrac{3}{4}e^{3x}\sin 2x - \int \tfrac{9}{4}e^{3x}\sin 2x \; dx$$

$$\therefore \int e^{3x}\sin 2x \; dx = -\tfrac{1}{2}e^{3x}\cos 2x + \tfrac{3}{4}e^{3x}\sin 2x - \int \tfrac{9}{4}e^{3x}\sin 2x \; dx$$

$$\tfrac{13}{4}\int e^{3x}\sin 2x \; dx = -\tfrac{1}{2}e^{3x}\cos 2x + \tfrac{3}{4}e^{3x}\sin 2x + c$$

$$\therefore \int_0^{\frac{\pi}{4}} e^{3x}\sin 2x \; dx = \tfrac{4}{13}\left[-\tfrac{1}{2}e^{3x}\cos 2x + \tfrac{3}{4}e^{3x}\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$= \tfrac{4}{13}\left[(0 + \tfrac{3}{4}e^{\frac{3\pi}{4}}) - (-\tfrac{1}{2} + 0)\right]$$

$$= \tfrac{1}{13}(3e^{\frac{3\pi}{4}} + 2)$$

Exercise D: Integration Using Partial Fractions

1 $\int \frac{x-5}{(x+1)(x-2)}\,dx$

$$\frac{x-5}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$
$$A = 2\ and\ B = -1$$

$$\int \frac{x-5}{(x+1)(x-2)}\,dx$$
$$= \int \frac{2}{x+1} - \frac{1}{x-2}\,dx$$
$$= 2\ln|x+1| - \ln|x-2| + C$$
$$= \ln\left|\frac{(x+1)^2}{x-2}\right| + C$$

Either of the last 2 lines is fine

2 $\int \frac{2}{x^2-1}\,dx$

$$\frac{2}{x^2-1} = \frac{2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$
$$2 = A(x-1) + B(x+1)$$
$$2 = Ax - A + Bx + B$$
$$A + B = 0 \qquad B - A = 2$$
$$B = 1 \quad A = -1$$

$$\int \frac{2}{x^2-1}\,dx = \int -\frac{1}{x+1} + \frac{1}{x-1}\,dx$$
$$= -\ln|x+1| + \ln|x-1| + C$$
$$= ln\left|\frac{x-1}{x+1}\right| + C$$

Either of the last 2 lines is fine

3

$$\frac{2-6x+5x^2}{x^2(1-2x)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-2x}$$
$$2 - 6x + 5x^2 \equiv Ax(1-2x) + B(1-2x) + Cx^2$$
$$x = \tfrac{1}{2} \quad \Rightarrow \quad \tfrac{1}{4} = \tfrac{1}{4}C \quad \Rightarrow \quad C = 1$$
$$x = 0 \quad \Rightarrow \quad B = 2$$
$$\text{coeffs of } x^2 \quad \Rightarrow \quad 5 = -2A + C \quad \Rightarrow \quad A = -2$$
$$\therefore \int \frac{2-6x+5x^2}{x^2(1-2x)}\,dx = \int \left(\frac{1}{1-2x} - \frac{2}{x} + \frac{2}{x^2}\right)\,dx$$
$$= -\tfrac{1}{2}\ln|1-2x| - 2\ln|x| - 2x^{-1} + c$$

Exercise D: Integration Using Partial Fractions

4 Show that:

$$\int_3^4 \frac{3x-5}{(x-1)(x-2)}\,dx = \ln\frac{9}{2}$$

$$\frac{3x-5}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$$

$3x - 5 \equiv A(x-2) + B(x-1)$

$x = 1 \quad \Rightarrow \quad -2 = -A \quad \Rightarrow \quad A = 2$

$x = 2 \quad \Rightarrow \quad B = 1$

$$\therefore \int_3^4 \frac{3x-5}{(x-1)(x-2)}\,dx = \int_3^4 \left(\frac{2}{x-1} + \frac{1}{x-2}\right)dx$$

$$= [2\ln|x-1| + \ln|x-2|]_3^4$$

$$= (2\ln 3 + \ln 2) - (2\ln 2 + 0) = 2\ln 3 - \ln 2$$

$$= \ln 9 - \ln 2 = \ln\frac{9}{2}$$

5 $$\int_0^2 \frac{2x^2-7x+7}{x^2-2x-3}\,dx$$

$$\frac{2x^2-7x+7}{x^2-2x-3} \equiv A + \frac{B}{x-3} + \frac{C}{x+1}$$

$2x^2 - 7x + 7 \equiv A(x-3)(x+1) + B(x+1) + C(x-3)$

$x = 3 \quad \Rightarrow \quad 4 = 4B \quad \Rightarrow \quad B = 1$

$x = -1 \quad \Rightarrow \quad 16 = -4C \quad \Rightarrow \quad C = -4$

coeffs of $x^2 \quad \Rightarrow \quad A = 2$

$$\therefore \int_0^2 \frac{2x^2-7x+7}{x^2-2x-3}\,dx = \int_0^2 \left(2 + \frac{1}{x-3} - \frac{4}{x+1}\right)dx$$

$$= [2x + \ln|x-3| - 4\ln|x+1|]_0^2$$

$$= (4 + 0 - 4\ln 3) - (0 + \ln 3 - 0) = 4 - 5\ln 3$$

Exercise D: Integration Using Partial Fractions

6

$$\int \ln(x^2+a^2)\ dx$$

Parts: $u = \ln(x^2+a^2) \Rightarrow \frac{du}{dx} = \frac{2x}{x^2+a^2}$

$$\frac{dv}{dx} = 1 \Rightarrow v = x$$

$$\int \ln(x^2+a^2)\ dx = x\ln(x^2+a^2) - \int \frac{2x^2}{x^2+a^2}dx\ = x\ln(x^2+a^2) - 2\int \frac{x^2}{x^2+a^2}dx$$

By polynomial division: $\frac{x^2}{x^2+a^2} = 1 - \frac{a^2}{x^2+a^2}$

$$\int \ln(x^2+a^2)\ dx = x\ln(x^2+a^2) - 2\int 1 - \frac{a^2}{x^2+a^2}dx$$

$$= x\ln(x^2+a^2) - 2\int 1dx + 2a^2\int \frac{1}{x^2+a^2}dx$$

Using standard integral: $\int \frac{1}{x^2+a^2}dx = \frac{1}{\text{a}}\tan^{-1}\frac{x}{a} + C$

$$\int \ln(x^2+a^2)\ dx = x\ln(x^2+a^2) - 2x + 2a^2\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) + C$$

$$\boldsymbol{= x\,(\ln\left(x^2+a^2\right) - 2) + 2a\ \tan^{-1}\frac{x}{a} + C}$$

Challenge Exercise: Solutions

1 $\int e^{e^x} e^x dx$

$$\frac{d}{dx}\left(e^{e^x}\right) = e^{e^x} e^x$$

$$\therefore \int e^{e^x} e^x dx = e^{e^x} + C$$

Or, by substitution:

$$u = e^x \Rightarrow \frac{du}{dx} = e^x \Rightarrow dx = \frac{du}{e^x}$$

$$\int e^{e^x} e^x dx = \int e^u \, du = e^u + C$$

$$= e^{e^x} + C$$

2 $\int x^n \ln x \, dx$

By parts:

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^n \quad \Rightarrow v = \frac{x^{n+1}}{n+1} = \frac{x^n x}{n+1}$$

$$\int x^n \ln x \, dx = \frac{x^{n+1}}{n+1} \ln x - \int \frac{x^n x}{n+1} \times \frac{1}{x} dx$$

$$= \frac{x^{n+1}}{n+1} \ln x - \int \frac{x^n}{n+1} dx$$

$$= \boldsymbol{\frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C}$$

3 $\int \frac{dx}{x\sqrt{x^{-2n} - 1}}$

$$u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1} \Rightarrow dx = \frac{du}{nx^{n-1}}$$

$$\int \frac{dx}{x\sqrt{x^{-2n} - 1}} = \int \frac{du}{nx^{n-1} x \sqrt{u^{-2} - 1}}$$

$$= \frac{1}{n} \int \frac{du}{x^n \sqrt{u^{-2} - 1}}$$

$$= \frac{1}{n} \int \frac{du}{u\sqrt{u^{-2} - 1}}$$

$$= \frac{1}{n} \int \frac{du}{\sqrt{1 - u^2}}$$

$$= \frac{1}{n} \sin^{-1} u$$

$$= \boldsymbol{\frac{1}{n} \sin^{-1}(x^n)}$$

Challenge Exercise: Solutions

4 $\int \cos(\ln x)dx = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C$ (Combines hidden and cyclic parts)

$$u_1 = \cos(\ln x) \Rightarrow \frac{du_1}{dx} = -\frac{1}{x}\sin(\ln x)$$

$$\frac{dv_1}{dx} = 1, v_1 = x$$

$$I_1 = x\cos(\ln x) + \int \sin(\ln x)dx$$

$$u_2 = \sin(\ln x) \Rightarrow \frac{du_2}{dx} = \frac{1}{x}\cos(\ln x)$$

$$\frac{dv_2}{dx} = 1, v_2 = x$$

$$I_2 = x\sin(\ln x) - \int \cos(\ln x)dx$$

$$I = \int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)dx$$

$$2\int \cos(\ln x)dx = x\,[\cos(\ln x) + \sin(\ln x)] + C'$$

$$\int \cos(\ln x)dx = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C$$

Challenge Exercise: Solutions

5 Find the exact value of $\int_1^2 \frac{1}{x+x^3}\,dx = \frac{1}{2}\ln\left(\frac{8}{5}\right)$

$$\frac{1}{x(1+x^2)} \equiv \frac{A}{x} + \frac{Bx+C}{1+x^2}$$
$$1 \equiv A(1+x^2) + (Bx+C)x$$

Sub $x = 0 \Rightarrow A = 1$

Compare coefficients: $1 = 1 + x^2 + Bx^2 + Cx$

$$(1+B)x^2 + Cx = 0 \Rightarrow B = -1, C = 0$$
$$\frac{1}{x(1+x^2)} \equiv \frac{1}{x} - \frac{x}{1+x^2}$$
$$\int \frac{1}{x(1+x^2)}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{1+x^2}\,dx$$

Substitution: $u = 1 + x^2 \Rightarrow \frac{du}{dx} = 2x = dx = \frac{du}{2x}$

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{2}u\,du = \frac{1}{2}\ln u = \frac{1}{2}\ln(1+x^2)$$

$$= \ln x - \frac{1}{2}\ln(1+x^2) + C$$
$$\therefore \int_1^2 \frac{1}{x+x^3}\,dx = \left[\ln x - \frac{1}{2}\ln(1+x^2) + C\right]_1^2$$
$$= \left(\ln 2 - \frac{1}{2}\ln 5\right) - \left(\ln 1 - \frac{1}{2}\ln 2\right)$$
$$= \frac{3}{2}\ln 2 - \ln 5 = \frac{1}{2}(3\ln 2 - \ln 5) = \frac{1}{2}(\ln 2^3 - \ln 5) = \frac{1}{2}\ln\frac{8}{5}$$

Challenge Exercise: Solutions

6 $\int x^7\sqrt{1+x^4}\,dx = \frac{1}{6}x^4(1+x^4)^{\frac{3}{2}} - \frac{1}{15}(1+x^4)^{\frac{5}{2}} + C$

(Substitution within an integration by parts)

Rewrite:

$$\int x^7\sqrt{1+x^4}\,dx = \int x^3x^4\sqrt{1+x^4}\,dx$$

$$u = x^4 \Rightarrow \frac{du}{dx} = 4x^3$$

$$\frac{dv}{dx} = x^3\sqrt{1+x^4}$$

Substitution: $w = 1 + x^4 \Rightarrow \frac{dw}{dx} = 4x^3 = dx = \frac{dw}{4x^3}$

$$v = \int \frac{1}{4}w^{\frac{1}{2}}\,dw = \frac{1}{4}\times\frac{2}{3}w^{\frac{3}{2}} = \frac{1}{6}w^{\frac{3}{2}} = \frac{1}{6}(1+x^4)^{\frac{3}{2}}$$

$$\therefore \int x^3x^4\sqrt{1+x^4}\,dx = \frac{1}{6}x^4(1+x^4)^{\frac{3}{2}} - \frac{1}{6}\int(1+x^4)^{\frac{3}{2}}\,4x^3\,dx \quad = \frac{1}{6}x^4(1+x^4)^{\frac{3}{2}} - \frac{2}{3}\int x^3(1+x^4)^{\frac{3}{2}}\,dx$$

Substitution: $t = 1 + x^4 \Rightarrow \frac{dt}{dx} = 4x^3 = dx = \frac{dt}{4x^3}$

$$v = \int \frac{1}{4}t^{\frac{3}{2}}\,dt = \frac{1}{4}\times\frac{2}{5}t^{\frac{5}{2}} = \frac{1}{10}\,t^{\frac{5}{2}} = \frac{1}{10}(1+x^4)^{\frac{5}{2}}$$

$$= \frac{1}{6}x^4(1+x^4)^{\frac{3}{2}} - \frac{1}{15}(1+x^4)^{\frac{5}{2}} + C$$

Challenge Exercise: Solutions

7 Show that, for $n > 0$,

a $\int_0^{\frac{\pi}{4}} \tan^n x \sec^2 x \; dx = \frac{1}{n+1}$

b $\int_0^{\frac{\pi}{4}} \sec^n x \tan x \; dx = \frac{(\sqrt{2})^n - 1}{n}$

$$u = \tan^n x \Rightarrow \frac{du}{dx} = n\,(\tan^{n-1} x)\sec^2 x$$

$$\frac{dv}{dx} = \sec^2 x\,,\, v = \tan x$$

$$\int \tan^n x \sec^2 x \, dx = \tan^{\text{n+1}} x - n \int \tan^n x \sec^2 x \, dx$$

$$(n+1) \int \tan^n x \sec^2 x \, dx = \tan^{\text{n+1}} x$$

$$(n+1) \int_0^{\frac{\pi}{4}} \tan^n x \sec^2 x \, dx = [\tan^{\text{n+1}} x]_0^{\frac{\pi}{4}}$$

$$\int_0^{\frac{\pi}{4}} \tan^n x \sec^2 x \, dx = \frac{1}{n+1}[1-0]_0^{\frac{\pi}{4}} = \frac{1}{n+1}$$

$$u = \sec^n x \Rightarrow \frac{du}{dx} = n\,(\sec^{n-1} x)\sec x \tan x = n\, sec^n x \tan x$$

$$dx = \frac{du}{n\, sec^n x \tan x} = \frac{du}{n\, u \tan x}$$

$$\Rightarrow \int \sec^n x \tan x \; dx = \int u \tan x \, \frac{du}{n\, u \tan x}$$

$$= \frac{1}{n} \int du = \frac{1}{n} u + C = \frac{1}{n} \sec^n x + C$$

$$\therefore \int_0^{\frac{\pi}{4}} \sec^n x \tan x \, dx = \frac{1}{n} [\sec^n x]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{n}\left[\left(\frac{1}{\cos x}\right)^n\right]_0^{\frac{\pi}{4}} = \frac{1}{n}\left[\left(\frac{1}{\frac{1}{\sqrt{2}}}\right)^n - \left(\frac{1}{1}\right)^n\right]$$

$$= \frac{1}{n}\left[(\sqrt{2})^n - 1\right] = \boldsymbol{\frac{(\sqrt{2})^n - 1}{n}}$$