CAPE1150 UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

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Level 1 Semester 2

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Vectors & Products: Outline of Lecture 8

- Vectors (recap)
- Unit vectors
- Dot and cross product
- Properties/Identities and triple products.

Vectors: Recap

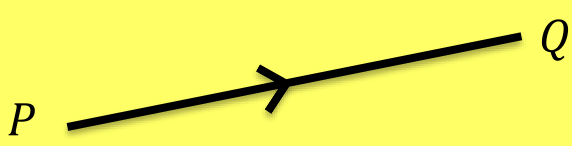
Oh Yeah!

Vector Key Points (Recap)

A Whereas a **coordinate** represents a **position** in space, a **vector** represents a **displacement** in space.

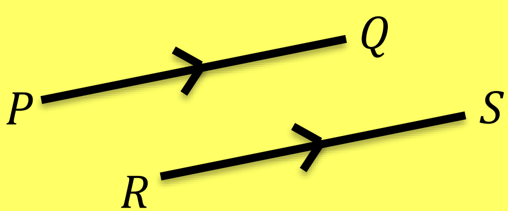
A vector has 2 properties:
- Direction
- Magnitude (i.e. length)

If P and Q are points then $\overrightarrow{PQ}$ is the vector between them.

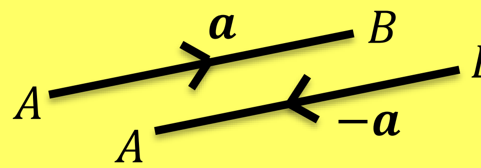


B If two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude **and** direction, **they're the same vector** and are **parallel**.

This might seem obvious, but students sometimes think the vector is different because the movement occurred at a different point in space. Nope!

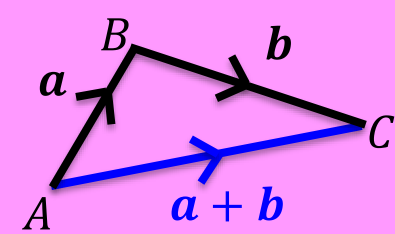


C $\overrightarrow{AB} = -\overrightarrow{BA}$ and the two vectors are parallel, equal in magnitude but in **opposite directions**.



D Triangle Law for vector addition:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



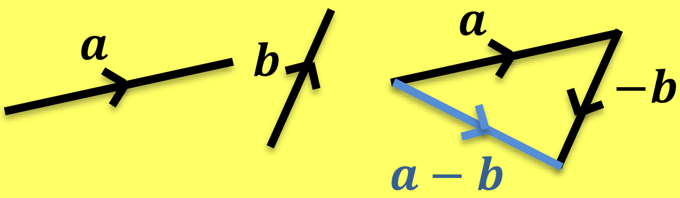
The vector of multiple vectors is known as the **resultant vector.** (you may have encountered this term in Mechanics)

Writing vectors: In textbooks and papers you will see vectors written in bold, such as $\boldsymbol{a}$ or even sometimes as $\vec{a}$. When writing, you should get into the habit of underlining vectors such as $\underline{a}$ rather than bold. This is the accepted notation.

Vector Key Points (Recap)

E Vector **subtraction** is defined using vector addition and negation:

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$



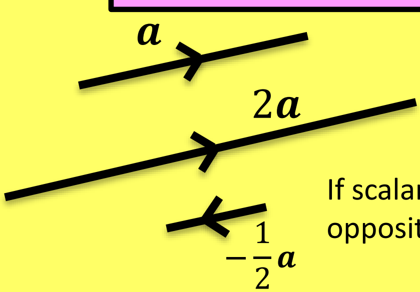
F The zero vector $\mathbf{0}$ (a bold 0), represents no movement.

$$\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$$

In 3D: $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

G A **scalar** is a normal number, which can be used to 'scale' a vector.

- The **direction** will be the **same**.
- But the **magnitude** will be **different** (unless the scalar is 1).



Vectors that have exactly opposite directions are called "antiparallel".

If scalar is negative, opposite direction.

H Any vector parallel to the vector $\boldsymbol{a}$ can be written as $\lambda\boldsymbol{a}$, where λ is a scalar.

The implication is that if we can write one vector **as a multiple of** another, then we can show they are parallel.

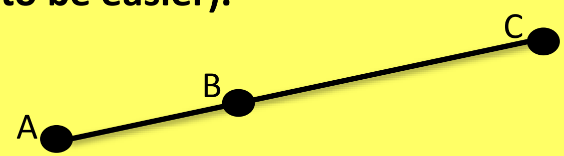
"Show $2\boldsymbol{a} + 4\boldsymbol{b}$ and $3\boldsymbol{a} + 6\boldsymbol{b}$ are parallel".

$3\boldsymbol{a} + 6\boldsymbol{b} = \frac{3}{2}(2\boldsymbol{a} + 4\boldsymbol{b})$ ∴ parallel

I 3 points A, B and C are collinear (lie on a straight line) if either:

$\overrightarrow{AB}$ **and** $\overrightarrow{BC}$ **are parallel (and B is a common point).**

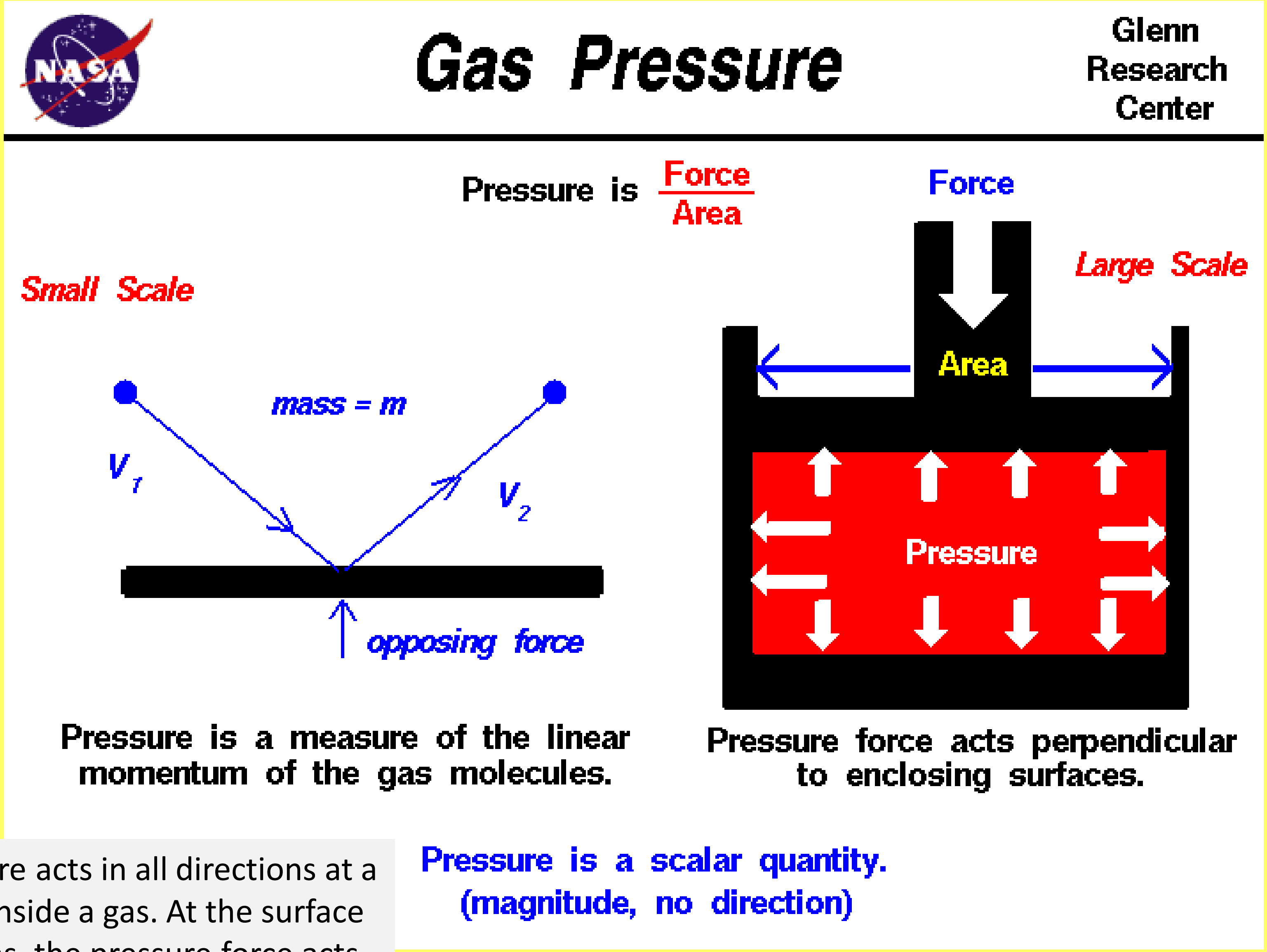
Or, we could show $\overrightarrow{AB}$ **and** $\overrightarrow{AC}$ **are parallel (this tends to be easier).**



Scalar and Vector Quantities

Scalar Quantities (Magnitude only)	Vector Quantities (Magnitude and Direction)
distance/length, area, volume	displacement
speed	velocity
mass	weight
time	
	acceleration
temperature	
	force
energy/work	
	impulse
power	
	momentum
density	
	electric/magnetic fields and flux
entropy	

What about Pressure?



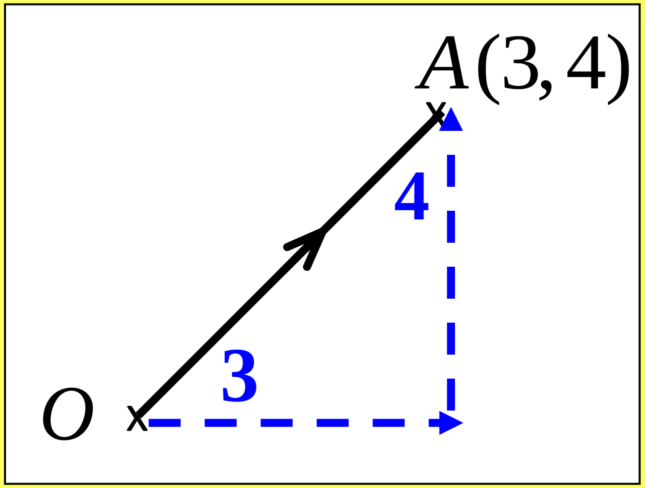
"Pressure acts in all directions at a point inside a gas. At the surface of a gas, the pressure force acts perpendicular to the surface".

Source/credit: NASA

Position Vectors

A position vector gives the position of a point relative to the origin, O.

2D Example



A vector can also be thought of as a straight line journey from one point to another.

A position vector is usually written with a single letter, which is lower case and either underlined or in bold type.

$$\overrightarrow{OA} = \underline{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

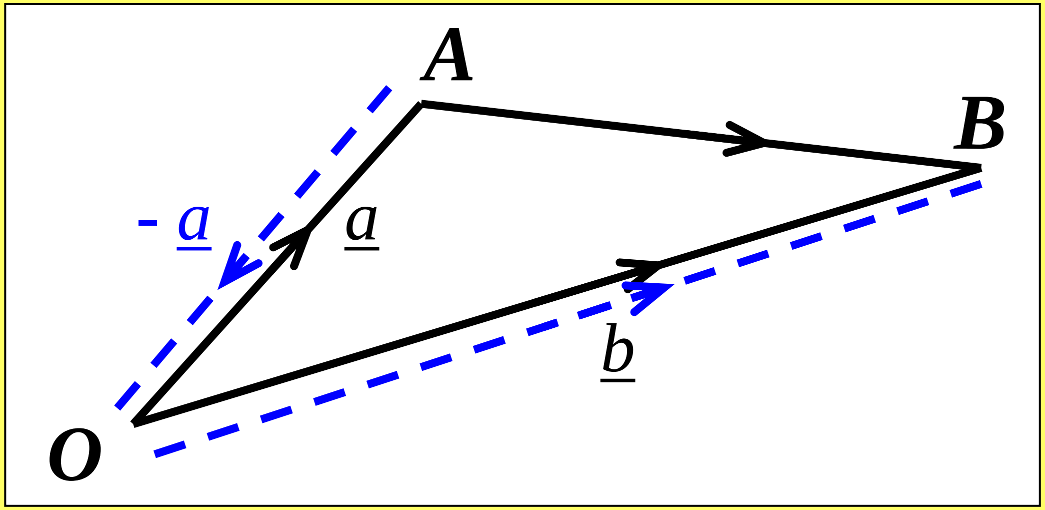
In general, it is conventional to label the point with an uppercase letter and its position vector with the corresponding lower-case vector notation (bold or underlined).

E.g. $\overrightarrow{OB} = \underline{b}$ $\quad\quad$ $\overrightarrow{OP} = \underline{p}$ etc.

Position Vectors between two points

If A and B have position vectors $\underline{a}$ and $\underline{b}$ respectively, $\overrightarrow{AB}$ can be written in terms of $\underline{a}$ and $\underline{b}$.

Respectively means "in that order"



Think of this as walking along the vector. To get from A to B we could go
from A to O:

$$\overrightarrow{AO} = -\overrightarrow{OA} = -\underline{a}$$

Then O to B

$$\overrightarrow{OB} = \underline{b}$$

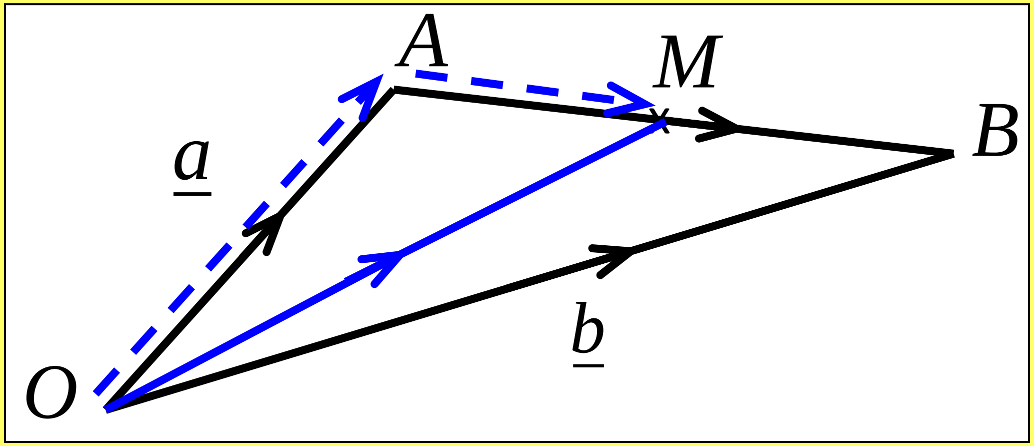
General result:

$$\overrightarrow{AB} = -\underline{a} + \underline{b}$$
$$= \underline{b} - \underline{a}$$

That is, the (direction) vector between two points is the position vector of the second point minus that of the first.

Midpoint of a Vector

The mid-point of a vector:



If M is the mid-point of AB,

$$\overrightarrow{OM} = \underline{m} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$

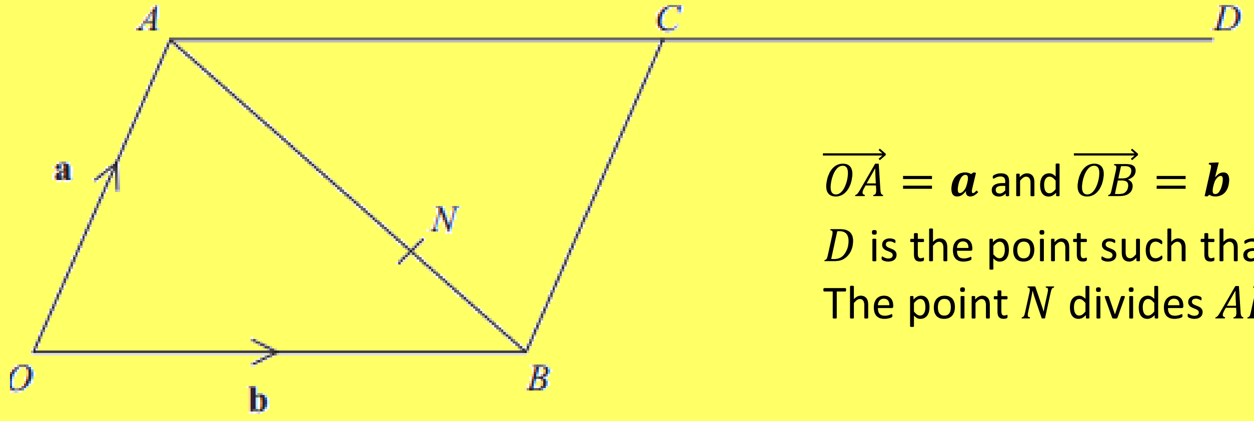
$$= \underline{a} + \frac{1}{2}(\underline{b} - \underline{a})$$

$$= \underline{a} + \frac{1}{2}\underline{b} - \frac{1}{2}\underline{a}$$

$$\therefore \underline{m} = \frac{1}{2}(\underline{a} + \underline{b})$$

That is, the position vector of the midpoint of any two points is the mean of their position vectors (this makes sense as the mean of two numbers gives the middle)

Collinear points



$\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

D is the point such that $\overrightarrow{AC} = \overrightarrow{CD}$

The point N divides AB in the ratio $2:1$.

(a) Write an expression for $\overrightarrow{ON}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\overrightarrow{ON} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB} = \boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$$

$$= \frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$$

(b) Prove that OND is a straight line.

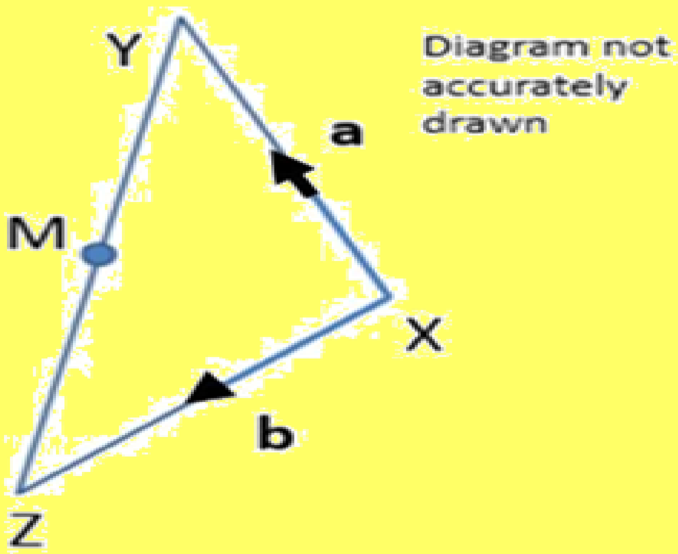
$$\overrightarrow{OD} = \boldsymbol{a} + 2\boldsymbol{b}$$

$$\overrightarrow{ON} = \frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b} = \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$$

$\overrightarrow{ON}$ is a multiple of $\overrightarrow{OD}$ and O is a common point.

$\therefore OND$ is a straight line.

Diagnostic Question



In the diagram, M is the **midpoint** of YZ.

Write vector $\overrightarrow{YM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Y

$$\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$$

C

$$\frac{1}{2}(-\boldsymbol{a}+\boldsymbol{b})$$

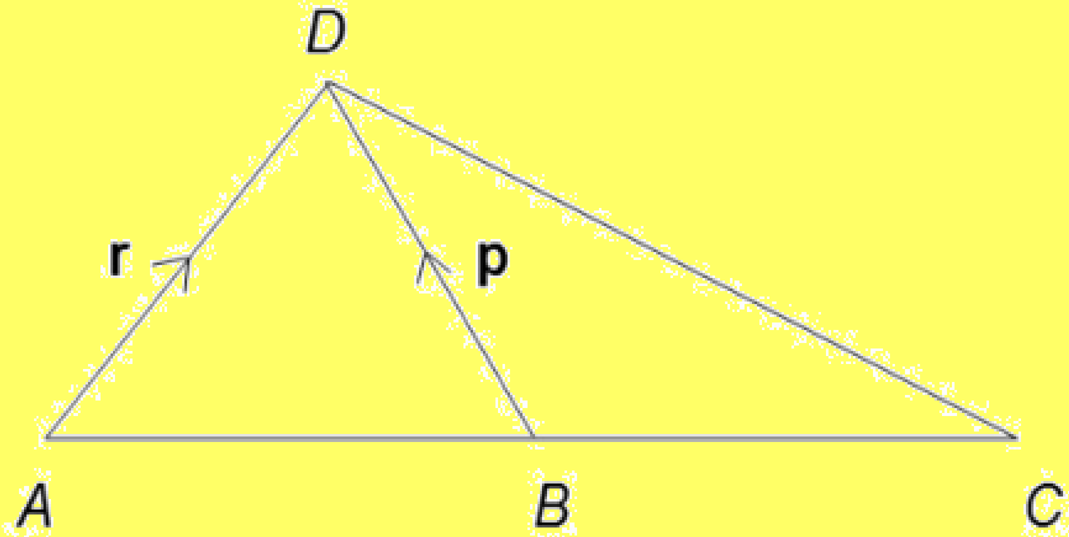
M

$$\frac{1}{2}(-\boldsymbol{b}+\boldsymbol{a})$$

A

$$\frac{1}{2}(-\boldsymbol{a}-\boldsymbol{b})$$

Diagnostic Question



In the diagram, B is the **midpoint** of AC.

Write vector $\overrightarrow{AC}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{p}$.

Y

$\boldsymbol{r} - \boldsymbol{p}$

C

$\boldsymbol{r} - 2\boldsymbol{p}$

M

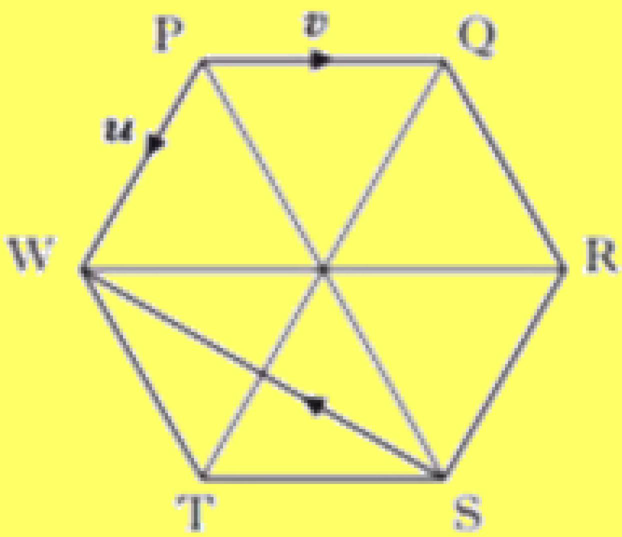
$2\boldsymbol{p} - 2\boldsymbol{r}$

A

$2\boldsymbol{r} - 2\boldsymbol{p}$

Diagnostic Question

The diagram shows a regular hexagon $PQRST$.
Let $\overrightarrow{PW} = \boldsymbol{u}$ and $\overrightarrow{PQ} = \boldsymbol{v}$ find $\overrightarrow{SW}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.



Y

$-\boldsymbol{u} - 2\boldsymbol{v}$

C

$\boldsymbol{u} - \boldsymbol{v}$

M

$-\boldsymbol{u} - \boldsymbol{v}$

A

$\boldsymbol{u} + 2\boldsymbol{v}$

Vector Arithmetic

Adding and subtracting:

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\underline{a} + \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$$\underline{a} - \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$$

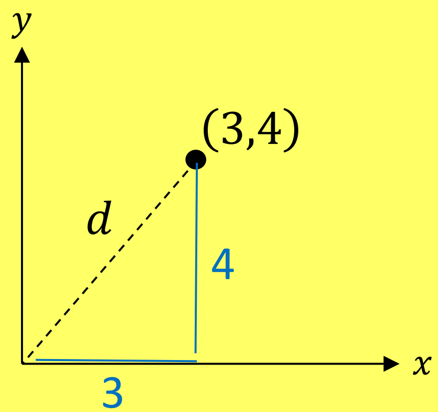
Multiplying by a scalar, k:

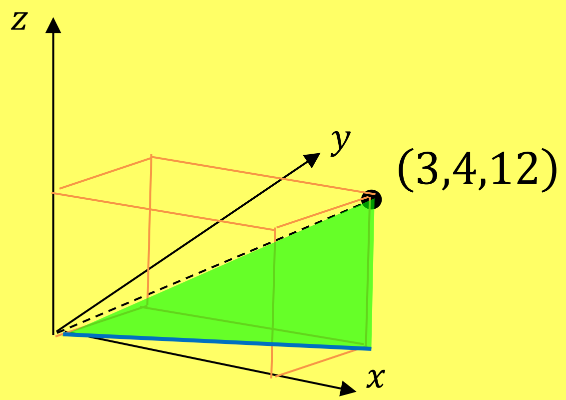
$$k\underline{a} = k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$$

Where $k\underline{a}$ is parallel to $\underline{a}$.

All of these work just as you'd expect!

Distance from the origin and magnitude of a vector





In 2D, how did we find the distance from a point to the origin?

Using Pythagoras:

$$\boldsymbol{d = \sqrt{3^2+4^2} = 5}$$

How about in 3D then?

You may be familiar with this method from school.
Using Pythagoras on the base of the cuboid:

$$\boldsymbol{\sqrt{3^2+4^2} = 5}$$

Then using the highlighted triangle:

$$\boldsymbol{\sqrt{5^2+12^2} = 13}$$

We could have similarly done this is one go using:

$$\boldsymbol{\sqrt{3^2+4^2+12^2} = 13}$$

Since the magnitude $|\boldsymbol{a}|$ of a vector $\boldsymbol{a}$ is its length, we can see from above that this nicely extends to 3D:

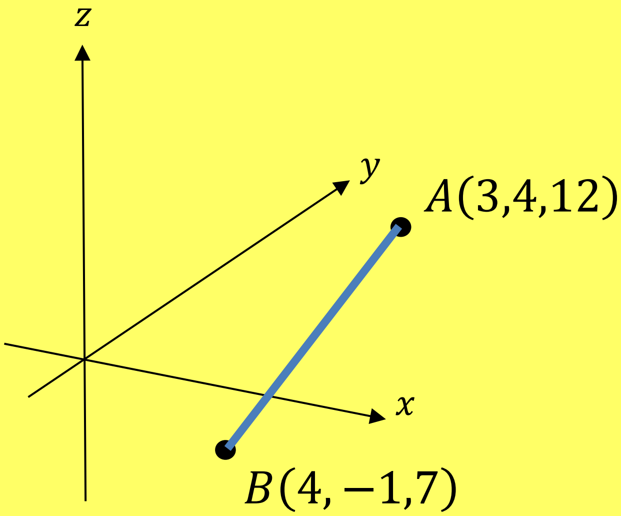
The magnitude of a vector $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$:

$$|\boldsymbol{a}| = \sqrt{x^2+y^2+z^2}$$

And the distance of (x, y, z) from the origin is $\sqrt{x^2+y^2+z^2}$

Note that the magnitude of a vector is always positive. (This makes sense as it's just the length of the line).

Distance between any two points in 3D



How do we find the distance between A and B?

It's just the magnitude/length of the vector between them. i.e.

$$\overrightarrow{\boldsymbol{AB}} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{7} \end{pmatrix} - \begin{pmatrix} \mathbf{3} \\ \mathbf{4} \\ \mathbf{12} \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-5} \\ \mathbf{-5} \end{pmatrix}$$

$$\left|\overrightarrow{\boldsymbol{AB}}\right| = \left|\begin{pmatrix} \mathbf{1} \\ \mathbf{-5} \\ \mathbf{-5} \end{pmatrix}\right|$$

$$= \sqrt{\mathbf{1^2 + (-5)^2 + (-5)^2}} = \sqrt{\mathbf{51}}$$

- We could write the magnitude of line $\overrightarrow{AB}$ as either $\left|\overrightarrow{AB}\right|$ or simply AB.
- The first one is preferable as it is clear that you mean magnitude and didn't just forget the arrow!

Distance between 2 points is
$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

Δx means change in x

Tip: Because we're squaring, it doesn't matter whether the change is negative or positive.

Or, more formally:

The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by:
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Diagnostic Question

Which vector is **not** parallel to $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$?

Y

$$\begin{pmatrix} -4 \\ -3 \end{pmatrix}$$

C

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

M

$$\begin{pmatrix} 4k \\ 3k \end{pmatrix}$$

A

$$\begin{pmatrix} 1 \\ 0.75 \end{pmatrix}$$

Diagnostic Question

If E F and G are collinear, which of these is **not always true**?

Y

$$\overrightarrow{EF} = \overrightarrow{FG}$$

C

$$\overrightarrow{EF} = k\overrightarrow{EG}$$

M

$$\overrightarrow{EF} = k\overrightarrow{FG}$$

A

$$\overrightarrow{FG} = k\overrightarrow{FE}$$

Diagnostic Question

Find the exact distance between $(-5,2,0)$ and $(-2,-3,-3)$

Y

$\sqrt{\mathbf{59}}$

C

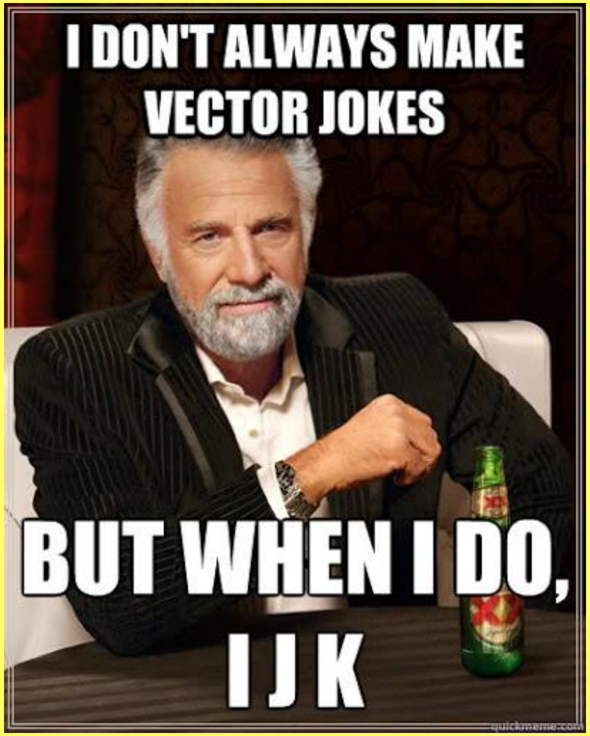
$\mathbf{6.557}$

M

$\sqrt{\mathbf{43}}$

A

$\mathbf{43}$



Unit Vectors

Unit Vectors

A unit vector is a vector whose magnitude is 1

There are certain operations on vectors that require the vectors to be 'unit' vectors. We just scale the vector so that its magnitude is now 1.

Technically, any vector can be thought of as the unit vector in that **direction multiplied by its magnitude**, that is:

$$\underline{a} = |\underline{a}|\,\hat{\underline{a}}$$

As multiplying the unit vector by a number just lengthens it.

To get the unit vector, we divide the vector by its length.

If $\underline{a}$ is a vector, then the unit vector $\hat{\underline{a}}$ in the same direction is

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

A unit vector is denoted using a caret or "hat".

Example

$$\underline{a} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$$|\underline{a}| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\hat{\underline{a}} = \frac{1}{\sqrt{14}}\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \quad \text{or} \quad \hat{\underline{a}} = \begin{pmatrix} -\frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$$

Either form is ok

Check: $\hat{\underline{a}}$ should have a magnitude (length) of 1

$$|\hat{\underline{a}}| = \sqrt{\left(-\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2} = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = \sqrt{\frac{14}{14}} = 1$$

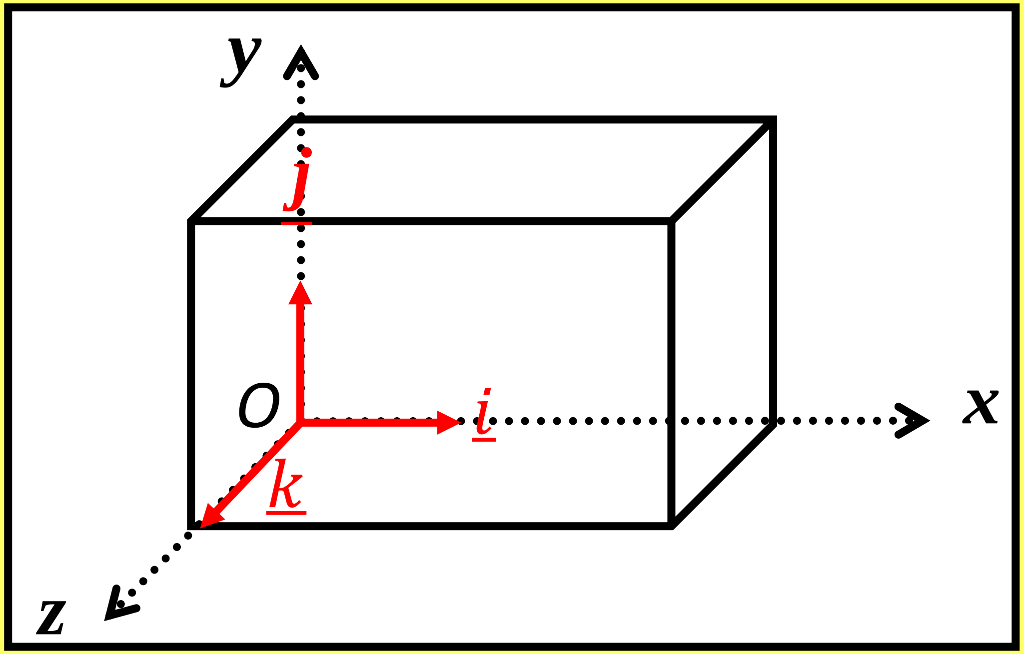
Unit vectors are useful when you need to specify the direction of a vector without changing its magnitude, or when you want to find components of vectors in other directions

Basis Unit Vectors

Another Notation

Vectors can be given in terms of unit vectors.

A unit vector has magnitude 1 and the **basis unit vectors** are in the directions of the axes.



In 3 dimensions the basis unit vectors are labeled $\underline{i}$, $\underline{j}$ and $\underline{k}$.

along the x-, y- and z- axes, respectively.

e.g. $$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 3\underline{i} + 2\underline{j} + \underline{k}$$

Notation:
Because $\underline{i}$, $\underline{j}$ and $\underline{k}$ are always unit vectors, we often omit the "hat" as there should be no confusion.

Basis Unit Vectors

In 2 dimensions the unit vectors are labeled $\underline{i}$ and $\underline{j}$.

They are defined as follows:

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In 3 dimensions the unit vectors are labeled $\underline{i}$ and $\underline{j}$ and $\underline{k}$.

They are defined as follows:

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

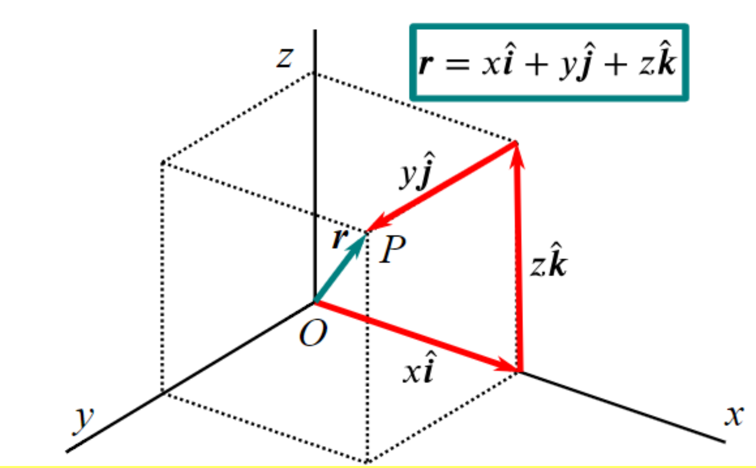
Alternate Notations:

- $\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}$ (unit vectors in x, y, z directions)
- $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \widehat{\boldsymbol{k}}$ (hats instead of dots)
- $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ (the number tells you which component is 1)
- $\boldsymbol{e_x}, \boldsymbol{e_y}, \boldsymbol{e_z}$ (the letter tells you which component is 1)

Basis Unit Vectors (Position Vectors)

A position vector gives the position of a point relative to the origin, O.

3D Example



A general position vector in three dimensions, is given by:

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Example for clarification:

If A is the point $(1, -2, 3)$

- The point is written as $(1, -2, 3)$

Think of this as where point A is

- The position vector $\overrightarrow{OA} = \underline{a}$ is written as $\underline{i} - 2\underline{j} + 3\underline{k}$ or $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

Think of this as the journey form the origin to point A

Diagnostic Question

Find the **unit vector** in the direction of $\underline{a} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

Y $\hat{\underline{a}} = \begin{pmatrix} \frac{4}{\sqrt{13}} \\ \frac{1}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \end{pmatrix}$

C $\hat{\underline{a}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

M $\hat{\underline{a}} = \begin{pmatrix} \frac{4}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} \end{pmatrix}$

A $\hat{\underline{a}} = \begin{pmatrix} \frac{4}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \\ -\frac{2}{\sqrt{21}} \end{pmatrix}$

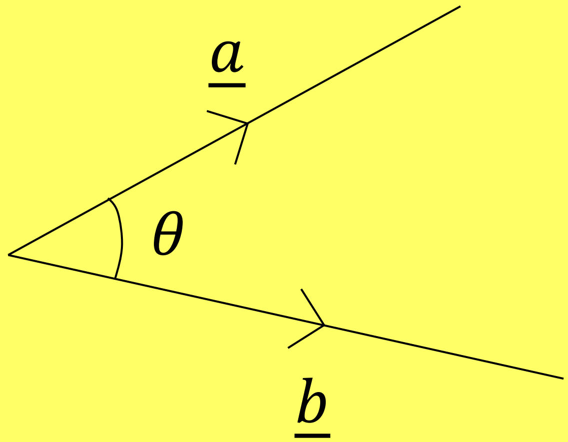
$\times$

$=$

Dot and Cross Product

Scalar Product (Dot Product)

The **scalar product** of two vectors $\underline{a}$ and $\underline{b}$ is defined as



$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$$

The angle is technically measured from a to b

Where θ is the angle between them

It can also be written as:

$$\underline{a} \cdot \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example:

$$\begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = 5 \times 3 + (-2) \times 3 + 0 \times 1 = 9$$

It's called the scalar product because the result is always a scalar!

Note:
Scalar product is commutative (order unimportant)
because: $\underline{b} \cdot \underline{a} = |\underline{b}||\underline{a}| \cos\theta = \underline{a} \cdot \underline{b}$

For any vector:
$\underline{a} \cdot \underline{a} = |\underline{a}||\underline{a}| \cos 0 = |\underline{a}|^2$

Scalar product of base unit vectors

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$$

For the base unit vectors:

$$\underline{i} \cdot \underline{i} = |\underline{i}||\underline{i}| \cos 0 = 1 \times 1 \times 1 = 1$$

(The magnitude of unit vectors is 1 and the angle between a vector and itself is 0).

So: $\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$

$$\underline{i} \cdot \underline{j} = |\underline{i}|\,|\underline{j}| \cos 90 = 1 \times 1 \times 0 = 0$$

All base unit vectors are perpendicular so

$$\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{i} = 0$$

$$\underline{i} \cdot \underline{k} = \underline{k} \cdot \underline{i} = 0$$

$$\underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{j} = 0$$

We can now use these to prove the second scalar product formula:

$$\underline{a} \cdot \underline{b} = (a_1\, \underline{i} + a_2\, \underline{j} + a_3\, \underline{k}) \cdot\ (b_1\, \underline{i} + b_2\, \underline{j} + b_3\, \underline{k})$$

$$\begin{aligned} &= a_1 b_1 \underline{i} \cdot \underline{i} + a_1 b_2\, \underline{i} \cdot \underline{j} + a_1 b_3\, \underline{i} \cdot \underline{k} \\ &+ a_2 b_1 \underline{j} \cdot \underline{i} + a_2 b_2\, \underline{j} \cdot \underline{j} + a_2 b_3\, \underline{j} \cdot \underline{k} \\ &+ a_3 b_1 \underline{k} \cdot \underline{i} + a_3 b_2\, \underline{k} \cdot \underline{j} + a_3 b_3\, \underline{k} \cdot \underline{k} \end{aligned}$$

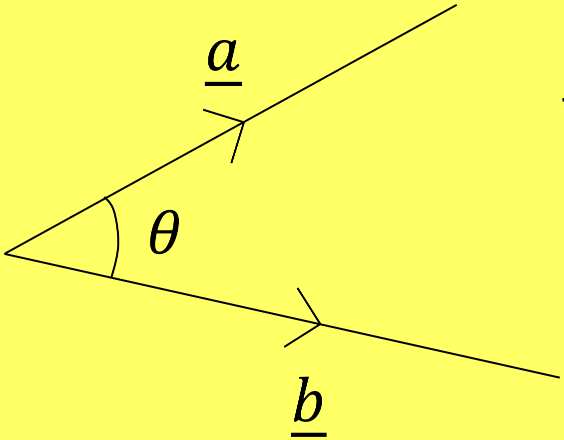
Because only the red (same base unit vector) components are non-zero, and using the fact that $\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$

Then we get:

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Angle between two vectors

If the two vectors are unit vectors, the dot product gives us the cosine of the angle between them.



$$\frac{\underline{a}}{|\underline{a}|} \cdot \frac{\underline{b}}{|\underline{b}|} = \cos\theta$$

We usually use this:

Angle between vectors:

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

Example:

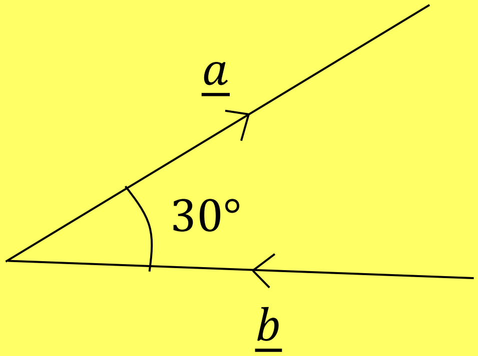
Find the **acute** angle between the vectors $\underline{a} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$.

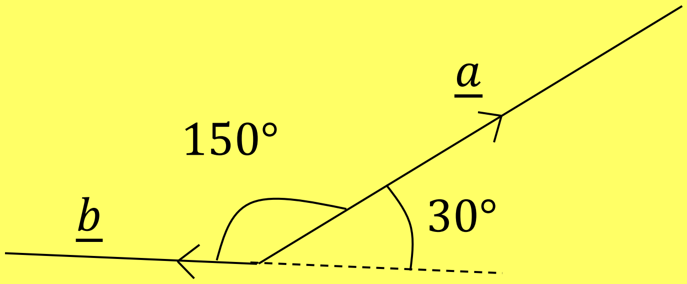
$\underline{a} \cdot \underline{b} = 5 + 0 + 5 = 10$ $\quad |\underline{a}| = \sqrt{35} \quad |\underline{b}| = \sqrt{26}$

$$\cos\theta = \frac{10}{\sqrt{35}\sqrt{26}} \qquad \theta = \cos^{-1}\left(\frac{10}{\sqrt{35}\sqrt{26}}\right) = 70.64°$$

Important point…

When finding an angle between two vectors using this method, ensure that the vectors point **away** from the point at which the angle is being measured.





The angle between two vectors may be acute or obtuse.

Example:

Find the angle between the vectors $\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix}$.

$$\underline{a} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix}$$

$$\underline{a} \cdot \underline{b} = 2(0) + 4(1) + (-1)(8) = -4$$

$$|\underline{a}| = \sqrt{2^2 + 4^2 + 1^2} = \sqrt{21}$$

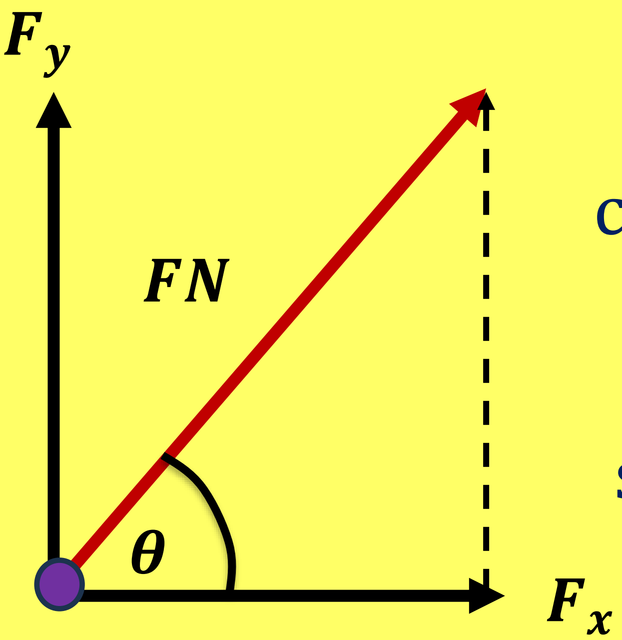
$$|\underline{b}| = \sqrt{0^2 + 1^2 + 8^2} = \sqrt{65}$$

$$\cos\theta = \frac{-4}{\sqrt{21}\sqrt{65}}$$

$$\theta = 96.2°$$

Recap: Resolving Forces

A particle is acted on by a resultant force of magnitude $F N$ acting at angle θ to the horizontal.



Resolve the resultant force into horizontal and vertical components

$$\cos\theta = \frac{F_x}{F} \qquad F_x = F\cos\theta$$

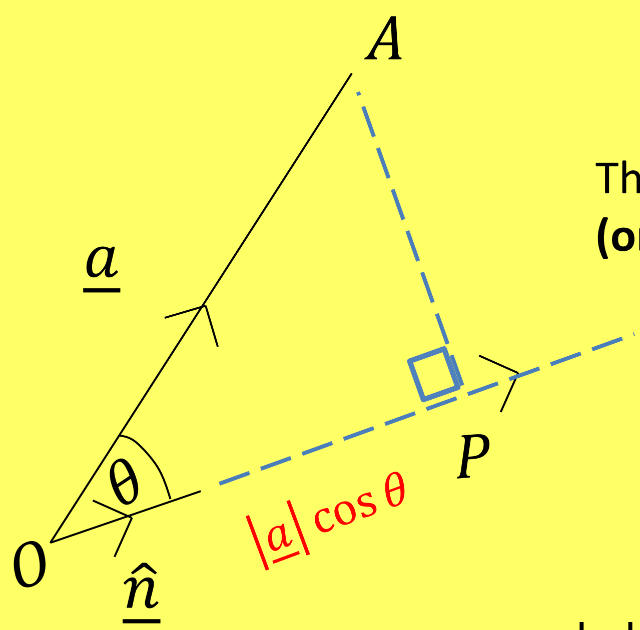
This is the horizontal component of force F

$$\sin\theta = \frac{F_y}{F} \qquad F_y = F\sin\theta$$

This is the vertical component of force F

<u>Tip:</u> You can think of **cos** as turning **through the angle** and **sin** turning **against the angle.**

Scalar Product (Dot Product) – Physical Meaning



$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$$

The scalar product can be thought of as the **component (or projection) of one vector in the direction of the other.**

From the diagram, OP is the component of resultant $\underline{a}$ through angle θ.

$$|\overrightarrow{OP}| = |\underline{a}|\cos\theta$$

(which is in the same direction as the unit vector $\hat{\underline{n}}$)

As $|\hat{\underline{n}}| = 1$, this is the same as $|\underline{a}||\hat{\underline{n}}| \cos\theta$

Which is the scalar product of $\underline{a}$ with unit vector $\hat{\underline{n}}$: $\underline{a} \cdot \hat{\underline{n}} = |\underline{a}||\hat{\underline{n}}| \cos\theta$

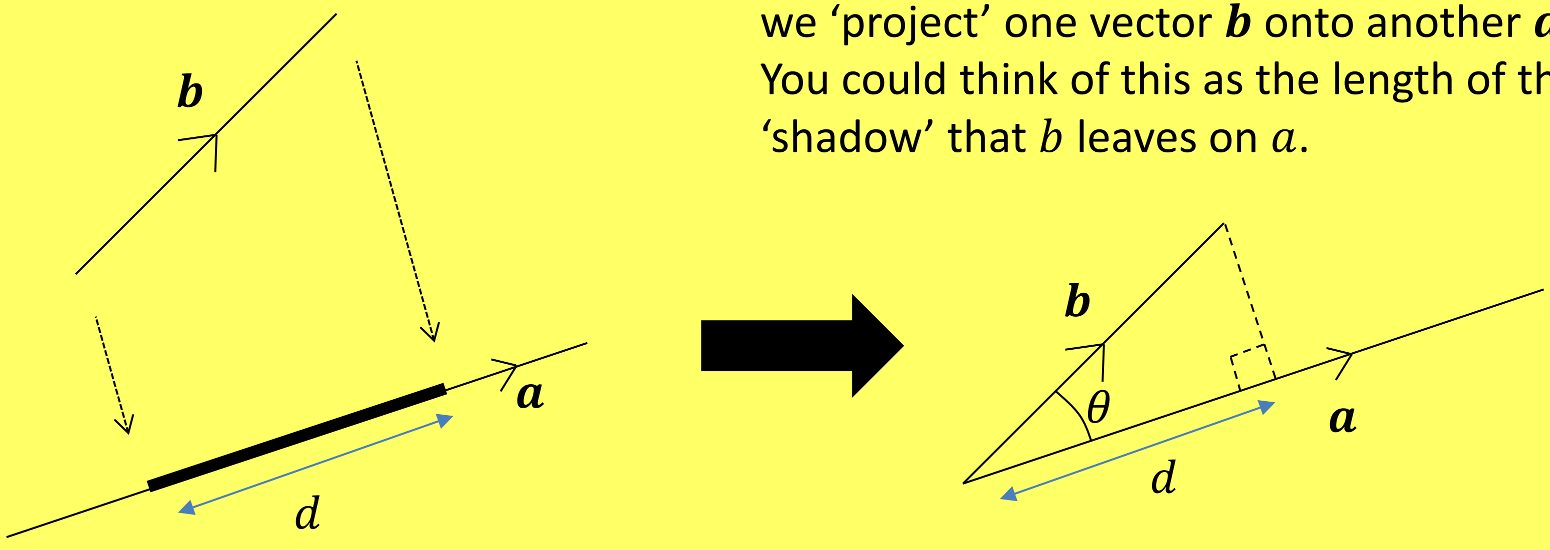
Therefore, in the general case, for any vector $\underline{a}$:

The component of vector $\underline{a}$ in the direction of $\underline{n}$ (or $\hat{\underline{n}}$) is
$\underline{a} \cdot \hat{\underline{n}}$
(also known as the projection of $\underline{a}$ onto $\underline{n}$)

You can think of the projection as the **shadow cast** by the first vector on the second.

Note that, $\underline{a} \cdot \hat{\underline{b}}$ would be the component of $\underline{a}$ in the direction of $\underline{b}$, but $\underline{a} \cdot \underline{b}$ is the component of $\underline{a}$ in the direction of $\underline{b}$ multiplied by the length $|\underline{b}|$. (As $\underline{b} = |\underline{b}|\, \hat{\underline{b}}$).

Projection of one vector onto another: Length



Suppose we wanted to find the distance d when we 'project' one vector $\boldsymbol{b}$ onto another $\boldsymbol{a}$. You could think of this as the length of the 'shadow' that b leaves on a.

Can you think of two different ways to express $\cos\theta$?

Way 1: Dot product. $\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$

Way 2: Basic trig. $\cos\theta = \frac{d}{|\boldsymbol{b}|}$

$$\therefore \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{d}{|\boldsymbol{b}|} \quad \rightarrow \quad d = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}$$

Length of projection of $\boldsymbol{b}$ on $\boldsymbol{a}$:

$$\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}$$

You divide by the magnitude of the vector you are <u>projecting onto</u>.

Scalar Product (Dot Product) – Physical Example

In context: $\underline{F} \cdot \underline{S}$ describes the amount of force in the direction of the displacement.

i.e. Work done: $W = \underline{F} \cdot \underline{S} = |\underline{F}||\underline{S}| \cos\theta$

Which is the familiar physics definition: "Work done = (force applied) x (distance moved in direction of force)".

Example:

Two forces $\underline{F}_1 = (9\underline{i} - 7\underline{j})\,N$ and $\underline{F}_2 = (9\underline{i} - 2\underline{j})\,N$ act on an object.

The object moves from the point with position vector $(-6\underline{i} + 2\underline{j})\,m$ to the point $(2\underline{i} + 3\underline{j})\,m$.

Find the work done by the resultant of the forces.

The **resultant** force $\underline{F}_R = (9\underline{i} - 7\underline{j}) + (9\underline{i} - 2\underline{j}) = (18\underline{i} - 9\underline{j})\,N$

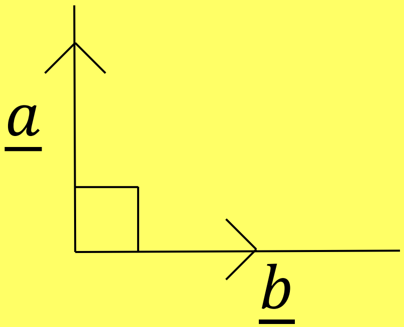
The displacement vector from initial to final position is

$$\underline{S} = (2\underline{i} + 3\underline{j}) - (-6\underline{i} + 2\underline{j}) = (8\underline{i} + \underline{j})\,m$$

Therefore, Work done is

$$W = \underline{F} \cdot \underline{S} = (18\underline{i} - 9\underline{j}) \cdot (8\underline{i} + \underline{j}) = (18 \times 8) + (-9 \times 1) = 144 - 9 = \mathbf{135}\boldsymbol{J}$$

Perpendicular vectors



Using the equation from the previous slide…

$$\cos 90° = 0 = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

$$\therefore \underline{a} \cdot \underline{b} = 0$$

If two vectors are perpendicular then:

$$\underline{a} \cdot \underline{b} = 0$$

E.g. 1:

Show that $\underline{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ are perpendicular.

$$\underline{a} \cdot \underline{b} = (2 \times 1) + (3 \times 0) + (1 \times -2) = 0$$

$\therefore$perpendicular

E.g. 2:

Given that $\boldsymbol{a} = -2\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}$, find a vector which is perpendicular to both $\underline{a}$ and $\underline{b}$.

Let this vector be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Then $\begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2x + 5y - 4z = 0$

And $\begin{pmatrix} 4 \\ -8 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4x - 8y + 5z = 0$

Because the scaling (size) of the vector doesn't matter (only direction) we can set one of the values arbitrarily.

Let $z = 1$, then

$$-2x + 5y = 4$$
$$4x - 8y = -5$$

$$\rightarrow x = \frac{7}{4}, y = \frac{3}{2}, z = 1$$

So a possible vector is $\frac{7}{4}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} + \boldsymbol{k}$

(or scaling, $7\boldsymbol{i} + 6\boldsymbol{j} + 4\boldsymbol{k}$)

Because our final answer could have been any scalar multiple of this (any vector parallel). This is why the question said "a vector" rather than "the vector".

We could have also done this by taking the cross product which we will meet shortly.

Diagnostic Question

The vectors $\begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ -5 \\ k \end{pmatrix}$ are perpendicular. Find the value of k

Y

$\mathbf{-4}$

C

$\mathbf{\dfrac{1}{24}}$

M

$\mathbf{-24}$

A

$\mathbf{24}$

Diagnostic Question

Find the acute angle between the vectors $\underline{a} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$.

Y

$\mathbf{0.2°}$

C

$\mathbf{76.9°}$

M

$\mathbf{87.7°}$

A

$\mathbf{89.0°}$

Diagnostic Question

What is the value of $\boldsymbol{i} \cdot (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$?

Y

$\mathbf{0}$

C

$\mathbf{2}$

M

$\mathbf{1}$

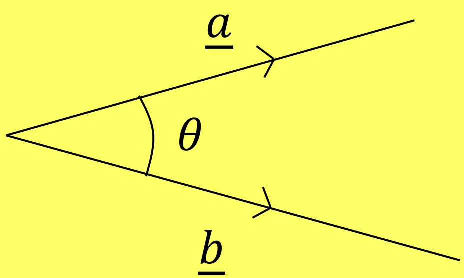
A

$\mathbf{4}$

Vector/Cross Product

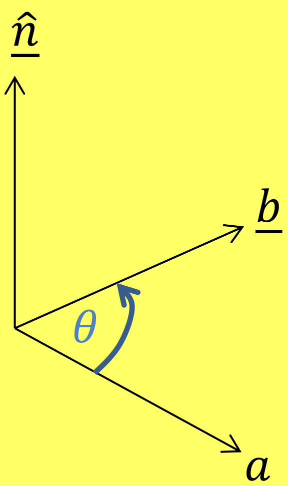
Recall that the **dot product** between two vectors is as follows:

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$$



The dot product is useful for finding the angle between two vectors and finding components

The **vector** or **cross product** of two vectors $\underline{a}$ and $\underline{b}$:



$$\boxed{\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| sin\theta\ \underline{\hat{n}}}$$

It's called the vector product because the result is always a vector!

What is the cross product actually doing?

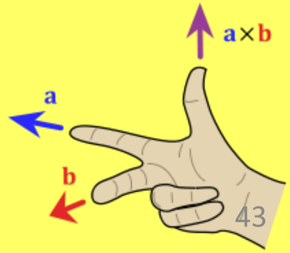
The dot product outputs a **scalar** (it'll give the cos of the angle if $\underline{a}$ and $\underline{b}$ are unit vectors).

In contrast the cross product outputs a **vector**, which is **perpendicular to both** the two vectors.

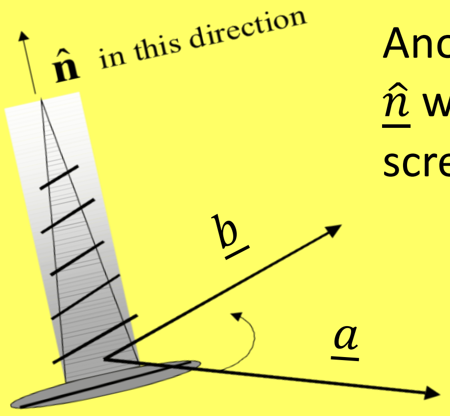
The unit vector $\underline{\hat{n}}$ gives its direction, and the $|\underline{a}||\underline{b}|sin\theta$ gives its magnitude.

It is conventional to denote a perpendicular (normal) vector as $\underline{n}$, so $\underline{\hat{n}}$ is the unit vector in the direction perpendicular to $\underline{a}$ and $\underline{b}$.

'Handy' Tip "RIGHT-HAND-RULE": Put you RIGHT HAND such that you thumb is pointing up. Your thumb is the cross product of your first finger ($\underline{a}$) and your second finger ($\underline{b}$).

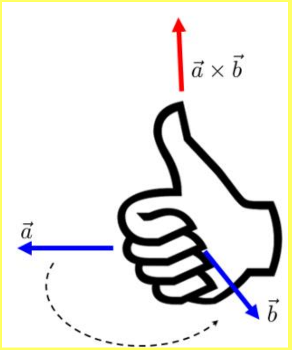


Vector/Cross Product



Another way to visualise: The direction of $\underline{\hat{n}}$ will point in the direction of a right-handed screw turned from $\underline{a}$ to $\underline{b}$.

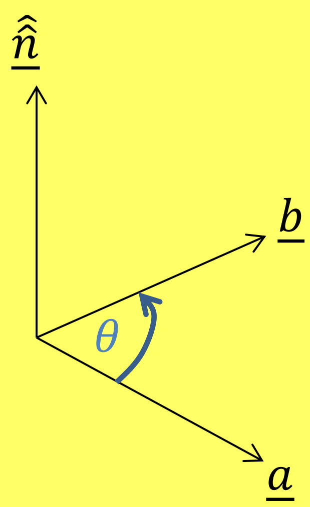
Or: If you curl the fingers of your hand in the direction $\underline{a}$ to $\underline{b}$, your thumb will point in the direction of $\underline{\hat{n}}$.

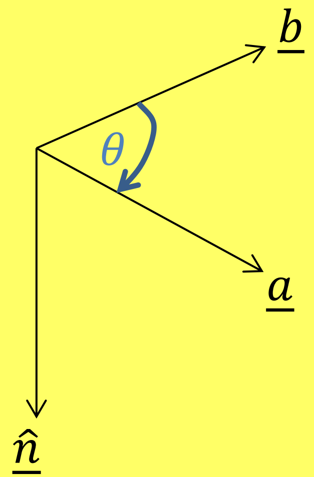


Note that $\times$ is **not commutative**, i.e. $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$, as (using right-hand-rule) $\underline{\hat{n}}$ will be in the opposite direction.

However:

$$\begin{aligned} \underline{b} \times \underline{a} &= |\underline{b}||\underline{a}| \sin\theta \ (-\underline{\hat{n}}) \\ &= -|\underline{a}||\underline{b}| \sin\theta \, \underline{\hat{n}} \\ &= -\underline{a} \times \underline{b} \end{aligned}$$





For any vector:

$$\underline{a} \times \underline{a} = |\underline{a}||\underline{a}| \ \sin 0 \, \underline{\hat{n}} = \underline{0}$$

And similarly, the **cross product of any two parallel vectors is zero.**

Cross Product using Determinants

We will learn more about determinants when we cover matrices.
For now you just need to know how to calculate them.

The easiest way to work out the vector product is to use the determinant:

The determinant of a 2x2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(forward diagonal – backward diagonal)

For $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

3x3 determinant set up
Top row: Basis vectors
Middle Row: Components of first vector
Bottom Row: Components of second vector.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Note that the **j** component has a minus in front of it (students often forget this and lose marks!)

$$= \boldsymbol{i}(a_2 b_3 - a_3 b_2) - \boldsymbol{j}(a_1 b_3 - a_3 b_1) + \boldsymbol{k}(a_1 b_2 - a_2 b_1)$$

$$= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

These are just different versions of the same final answer, with base unit vectors (above) or column vector (left).

Vector product: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

Determining Cross Product using Determinants

To find a 2×2 determinant:

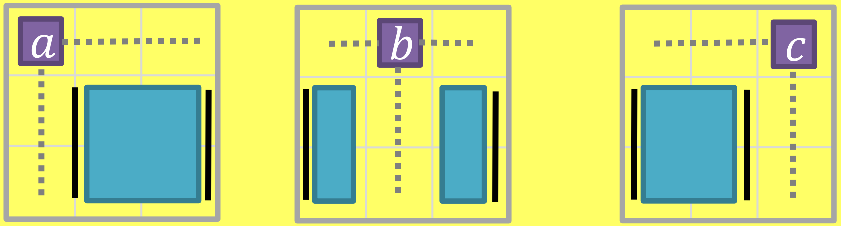
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Product of forward diagonals minus product of backward diagonals (careful with signs!)

To find a 3×3 determinant:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

note the minus for the middle one



Cover up the row and column of the expansion value, then do the 2x2 determinant of what is left

E.g. 2: $\underline{a} = 2\underline{i} + \underline{j} + 5\underline{k}$ and $\underline{b} = -3\underline{i} + 4\underline{j} + 3\underline{k}$. Work out $\underline{a} \times \underline{b}$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 5 \\ -3 & 4 & 3 \end{vmatrix} = \underline{i}\begin{vmatrix} 1 & 5 \\ 4 & 3 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & 5 \\ -3 & 3 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix}$$

$$= \underline{i}[(1 \times 3) - (5 \times 4)] - \underline{j}[(2 \times 3) - (5 \times -3)] + \underline{k}[(2 \times 4) - (1 \times -3)]$$

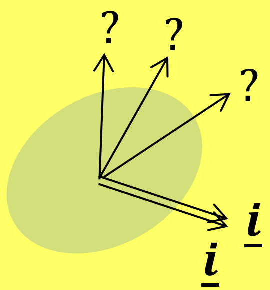
Use brackets to avoid sign errors!

$$= \underline{i}[3 - 20] - \underline{j}[6 - -15] + \underline{k}[8 - -3] \quad = -17\underline{i} - 21\underline{j} + 11\underline{k}$$

Vector product of base unit vectors

Since the angle between $\underline{i}$ and itself is 0,

$$\underline{i} \times \underline{i} = |\underline{i}||\underline{i}| \sin 0 \ \underline{\hat{n}} = \underline{0}$$



Geometrically, we can see the direction of the cross product would have been ambiguous:

And $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{i} \times \underline{i} = \underline{0}$

$$\underline{i} \times \underline{j} = \underline{k} \quad \Rightarrow \quad \underline{j} \times \underline{i} = -\underline{k}$$

Important Results:

- $\underline{a} \times \underline{b} = 0$ if $\underline{a}$ and $\underline{b}$ are parallel (or $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$)
- $\underline{i} \times \underline{j} = \underline{k}, \quad \underline{j} \times \underline{k} = \underline{i}, \quad \underline{k} \times \underline{i} = \underline{j}$
- $\underline{j} \times \underline{i} = -\underline{k}, \quad \underline{k} \times \underline{j} = -\underline{i}, \quad \underline{i} \times \underline{k} = -\underline{j}$

Memory Tip:
Advancing from $\underline{i}$ to $\underline{j}$ to $\underline{k}$ (and then back to $\underline{i}$) maintains sign. Going back a letter reverses sign.

Or, you can just remember the positive ones and that $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

Or, you can just work them out:
E.g.

$$\underline{i} \times \underline{j} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\underline{i} - 0\underline{j} + 1\underline{k} = \underline{k}$$

Proof of Cross Product

Given that $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, determine $\underline{a} \times \underline{b}$.

$$\underline{a} \times \underline{b} = \left(a_1\underline{i} + a_2\underline{j} + a_3\underline{k}\right) \times \left(b_1\underline{i} + b_2\underline{j} + b_3\underline{k}\right)$$

$$\begin{aligned} &= a_1b_1(\underline{i} \times \underline{i}) + a_1b_2\left(\underline{i} \times \underline{j}\right) + a_1b_3(\underline{i} \times \underline{k}) \\ &\quad + a_2b_1\left(\underline{j} \times \underline{i}\right) + a_2b_2\left(\underline{j} \times \underline{j}\right) + a_2b_3\left(\underline{j} \times \underline{k}\right) \\ &\quad + a_3b_1(\underline{k} \times \underline{i}) + a_3b_2\left(\underline{k} \times \underline{j}\right) + a_3b_3(\underline{k} \times \underline{k}) \end{aligned}$$

$$\begin{aligned} &= a_1b_2(\underline{k}) + a_1b_3\left(-\underline{j}\right) + a_2b_1(-\underline{k}) \\ &\quad + a_2b_3(\underline{i}) + a_3b_1\left(\underline{j}\right) + a_3b_2(-\underline{i}) \end{aligned}$$

$$= (a_2b_3 - a_3b_2)\underline{i} + (a_3b_1 - a_1b_3)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

Normal multiplication is '**distributive**' because $a \times (b + c) = (a \times b) + (a \times c)$. The cross product is similarly distributive and partly justifies use of the $\times$ symbol.

Using $\underline{i} \times \underline{j} = \underline{k}$, $\underline{i} \times \underline{i} = 0$, etc.

Diagnostic Question

$\underline{a} = -\underline{i} + \underline{j} + \underline{k}$ and $\underline{b} = -\underline{i} + 5\underline{j} - 2\underline{k}$.

Work out $\underline{a} \times \underline{b}$

Y

$-7\underline{i} - 3\underline{j} - 4\underline{k}$

C

$-7\underline{i} + 3\underline{j} - 4\underline{k}$

M

-14

A

$-7\underline{i} + \underline{j} - 6\underline{k}$

Diagnostic Question

For any vectors, what is the value of

$$\underline{a} \cdot (\underline{a} \times \underline{b})$$

Y

1

C

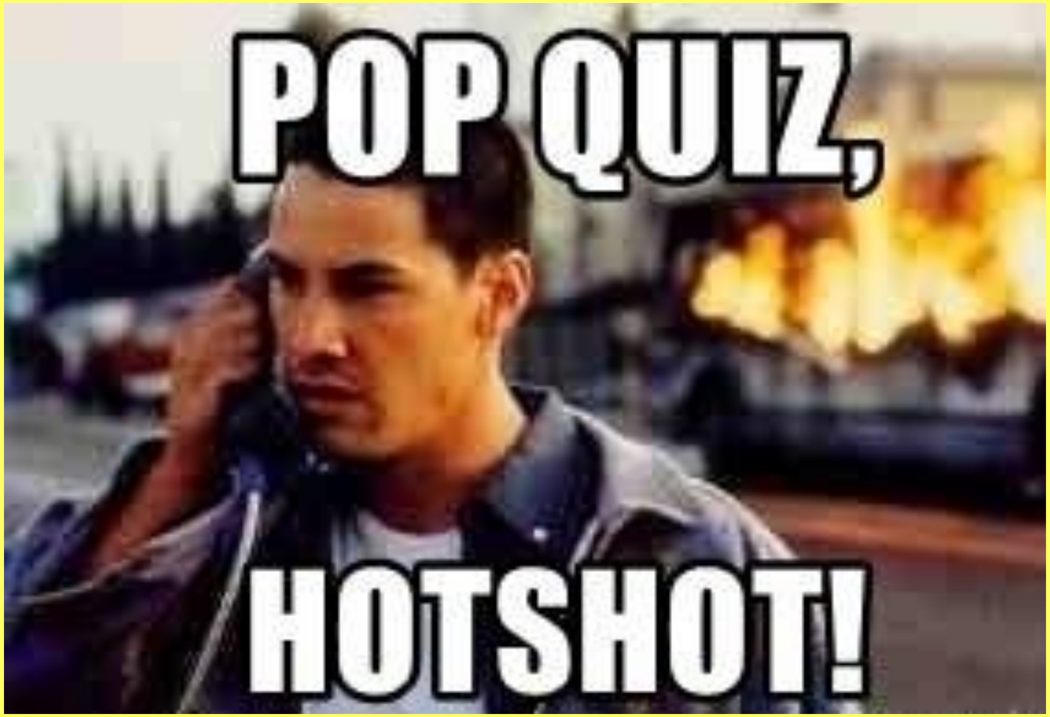
$\underline{0}$

M

0

A

Cannot be done



Let's test how well you understand vectors!

Diagnostic Question

If we calculate the following, what would be the result?

$$\underline{a} \cdot (\underline{b} \times \underline{c})$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don't know

Diagnostic Question

If we calculate the following, what would be the result?

$$(\underline{a} \cdot \underline{b}) \cdot \underline{c}$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don't know

Diagnostic Question

If we calculate the following, what would be the result?

$$\underline{a} \times \underline{a}$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don’t know

Diagnostic Question

If we calculate the following, what would be the result?

$$(\underline{a} \times \underline{b}) \times \underline{c}$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don't know

Diagnostic Question

If we calculate the following, what would be the result?

$$\left(\underline{a} \cdot \underline{b}\right) \underline{c}$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don't know

Diagnostic Question

If we calculate the following, what would be the result?

$$(\underline{a} + \underline{b}) \cdot \underline{c}$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don't know

Diagnostic Question

If we calculate the following, what would be the result?

$$(\underline{a} + \underline{b}) \times \underline{c}$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don’t know

Diagnostic Question

If we calculate the following, what would be the result?

$$(\underline{a} \cdot \underline{b}) \times \underline{c}$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don't know

Diagnostic Question

If we calculate the following, what would be the result?

$$\underline{a}(\underline{b} \times \underline{c})$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don’t know

Diagnostic Question

If we calculate the following, what would be the result?

$$p\underline{a} \times q\underline{b}$$

(where p and q are scalars)

Y

Scalar

C

Cannot be done

M

Vector

A

Don’t know

Diagnostic Question

If we calculate the following, what would be the result?

$$(\underline{a} \times \underline{b}) \cdot k\underline{c}$$

(where k is a scalar)

Y

Scalar

C

Cannot be done

M

Vector

A

Don't know

Diagnostic Question

If we calculate the following, what would be the result?

$$(\underline{a} \cdot \underline{b}) \times (\underline{c} \cdot \underline{d})$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don't know

Diagnostic Question

If we calculate the following, what would be the result?

$$(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d})$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don't know

Diagnostic Question

If we calculate the following, what would be the result?

$$\left(\underline{a} \cdot (\underline{b} \times \underline{c})\right) (\underline{d} \times (\underline{e} \times \underline{f}))$$

Y

Scalar

C

Cannot be done

M

Vector

A

Don't know

Properties: Scalar and Vector Product

Scalar Product	Vector Product
Result of the Scalar Product is a scalar quantity.	Result of the vector product is a vector quantity.
Scalar Product of two vectors in the same direction is maximum.	The Vector Product of two vectors in the same direction is zero.
Scalar Product of orthogonal vectors is zero.	Vector Product of orthogonal vectors is maximum.
It is used to find the projection of vectors.	It is used to find a third vector.
It satisfies commutative property. $\underline{b} \cdot \underline{a} = \underline{a} \cdot \underline{b}$	It does not satisfy commutative property. $\underline{b} \times \underline{a} = -\underline{a} \times \underline{b}$
Right hand rule is not followed.	Right hand rule is followed.
It is represented by a dot(.)	It is represented by a cross(x)

Distributive property: $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$

Just like expanding brackets

Because $\underline{a} \times \underline{b}$ is perpendicular to both $\underline{a}$ and $\underline{b}$:

$$\underline{a} \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot (\underline{a} \times \underline{b}) = \underline{a} \cdot (\underline{b} \times \underline{a}) = \underline{b} \cdot (\underline{b} \times \underline{a}) = 0$$

Also zero if the order is reversed
(as dot is commutative)

Scalar Triple Product

The **Scalar Triple Product** of three vectors is the **dot product of one of the vectors with the cross product of the other two**.

Using the expanded formula for the cross product:

Suppose $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

Then triple scalar product is:

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Using the expanded formula for the cross product.

So far, we have used the top row to expand determinants, but when all elements are scalars (not $\underline{i}, \underline{j}, \underline{k}$) using any row or column gives the same result, and so:

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{c} \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot (\underline{c} \times \underline{a})$$

Use 1st row | Use 3rd row | Use 2nd row

Determinants will be covered in more detail when we do matrices

Properties/Identities

Scalar Triple Products

1 $\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{b} \times \underline{c}) \cdot \underline{a}$

Because order doesn't matter for dot product

2 $\underline{a} \cdot (\underline{a} \times \underline{b}) = 0$

As $\underline{a} \times \underline{b}$ is perpendicular to both $\underline{a}$ and $\underline{b}$ (any dot product with a cross product involving the same vector will be zero).

So actually

$$\underline{a} \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot (\underline{a} \times \underline{b}) = \underline{a} \cdot (\underline{b} \times \underline{a}) = \underline{b} \cdot (\underline{b} \times \underline{a}) = (\underline{a} \times \underline{b}) \cdot \underline{a} = (\underline{a} \times \underline{b}) \cdot \underline{b} = (\underline{b} \times \underline{a}) \cdot \underline{a} = (\underline{b} \times \underline{a}) \cdot \underline{b} = 0$$

Because order of dot product does not matter and cross product in either order is still perpendicular to $\underline{a}$ and $\underline{b}$ (but in opposite directions)

3 $\underline{a} \cdot (\underline{b} \times \underline{c}) = -\underline{a} \cdot (\underline{c} \times \underline{b})$

Which is just because $\underline{c} \times \underline{b} = -\underline{b} \times \underline{c}$

4 $\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{c} \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot (\underline{c} \times \underline{a})$

Cyclic Property
(vectors move around in a circle)
i.e. $\underline{a}, \underline{b}, \underline{c}$ can be reordered if 'cycle' $a \to b \to c$ maintained.

5 $\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$

Interchange of dot and cross

Order of operations:
If you see $\underline{a} \cdot \underline{b} \times \underline{c}$ do the cross first so $\underline{a} \cdot (\underline{b} \times \underline{c})$

Properties/Identities

Vector Triple Products

1 In general: $\underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c}$

Order is important (brackets are essential)

2 $\underline{a} \times (\underline{b} \times \underline{c}) = -(\underline{b} \times \underline{c}) \times \underline{a} = (\underline{c} \times \underline{b}) \times \underline{a}$

Because $\underline{c} \times \underline{b} = -\underline{b} \times \underline{c}$

3 $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$

4 $(\underline{a} \times \underline{b}) \times \underline{c} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a}$

Both well known identities.

Thanks
See you in the Tutorial!