CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

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Level 1 Semester 2

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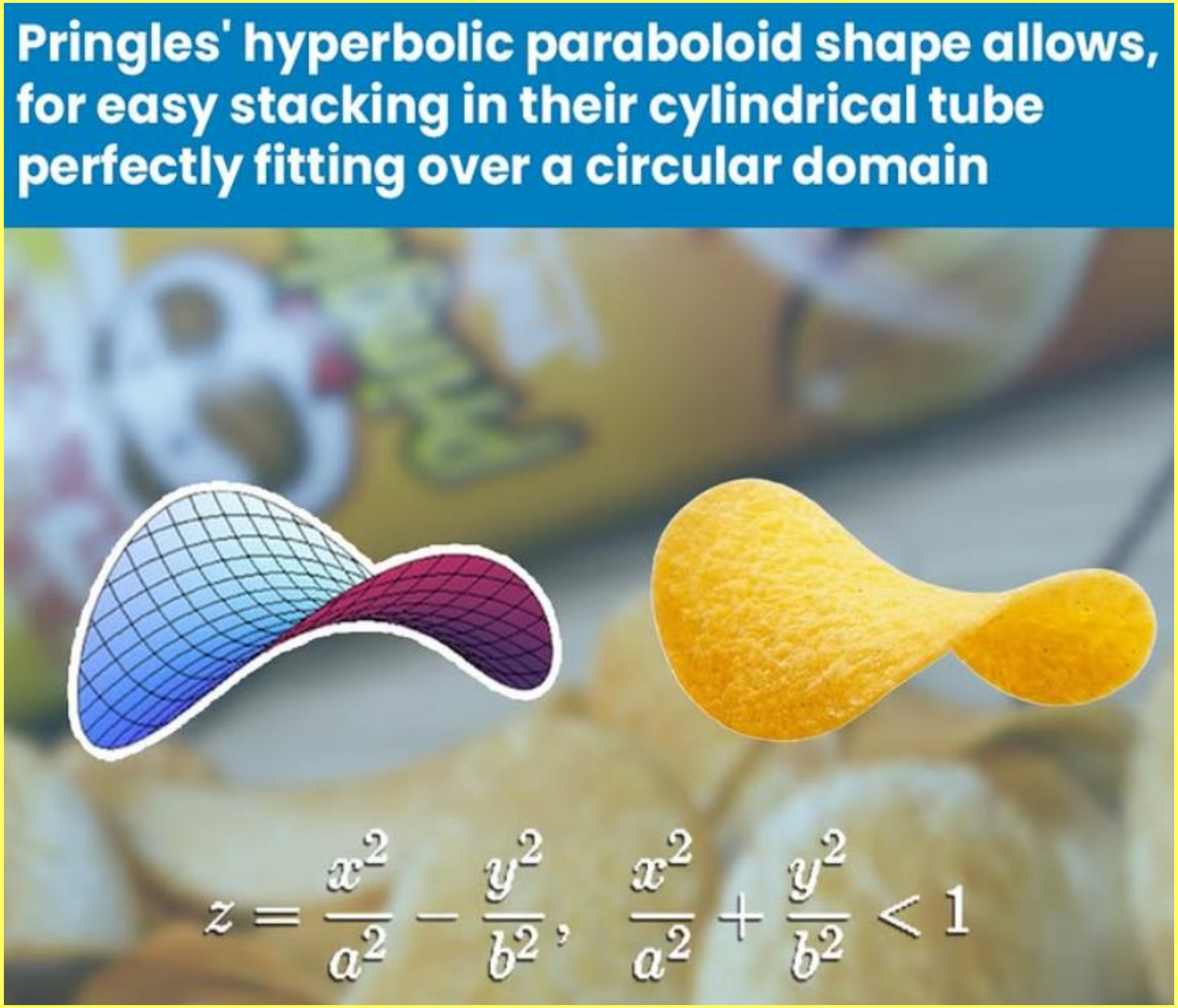
Hyperbolic Trig & Integration : Outline of Lecture 3

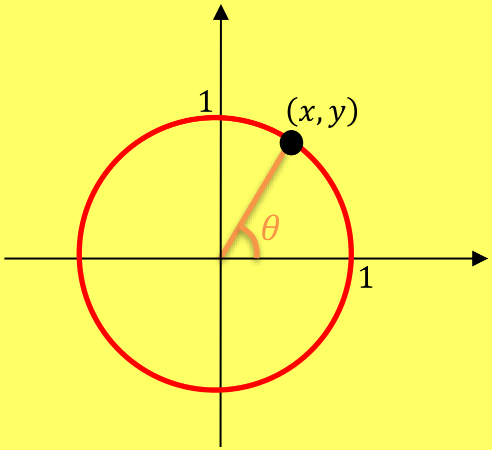
- Hyperbolic trigonometric functions
- Integration using hyperbolic trig functions
- Integration by completing the square
- Integration of parametric equations
- Area between curves
- Volumes of revolution



Hyperbolic Trig Functions

Comparison: Circles vs. Hyperbolas

Circles



The 'simplest' circle is a unit circle centred at the origin.

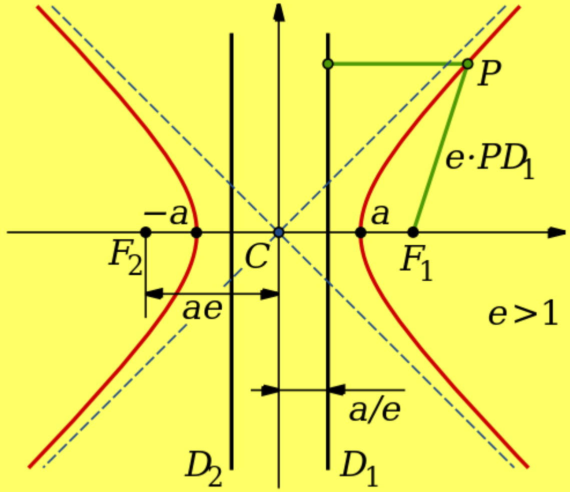
Cartesian equation:

$$\boldsymbol{x^2 + y^2 = 1}$$

Parametric equations (in terms of θ):

$$\boldsymbol{x = cos\ \theta}$$
$$\boldsymbol{y = sin\ \theta}$$

Hyperbolas



The equivalent hyperbola (which crosses x-axis at $(\mathbf{1}, \mathbf{0})$ and $(-\mathbf{1}, \mathbf{0})$)

Cartesian equation:

$$\boldsymbol{x^2 - y^2 = 1}$$

Parametric equations:

$$\boldsymbol{x = cosh\ \theta}$$
$$\boldsymbol{y = sinh\ \theta}$$

Equations of hyperbolic functions

Say as "sinech" (or "shine") of x

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad x \in \mathbb{R}$$

From the definition,
$\sinh 0 = 0$

Say as "cosh"

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad x \in \mathbb{R}$$

From the definition,
$\cosh 0 = 1$

Say as "tanch"

Hyperbolic tangent:

$$\tanh = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \qquad x \in \mathbb{R}$$

From the definition,
$\tanh 0 = 0$

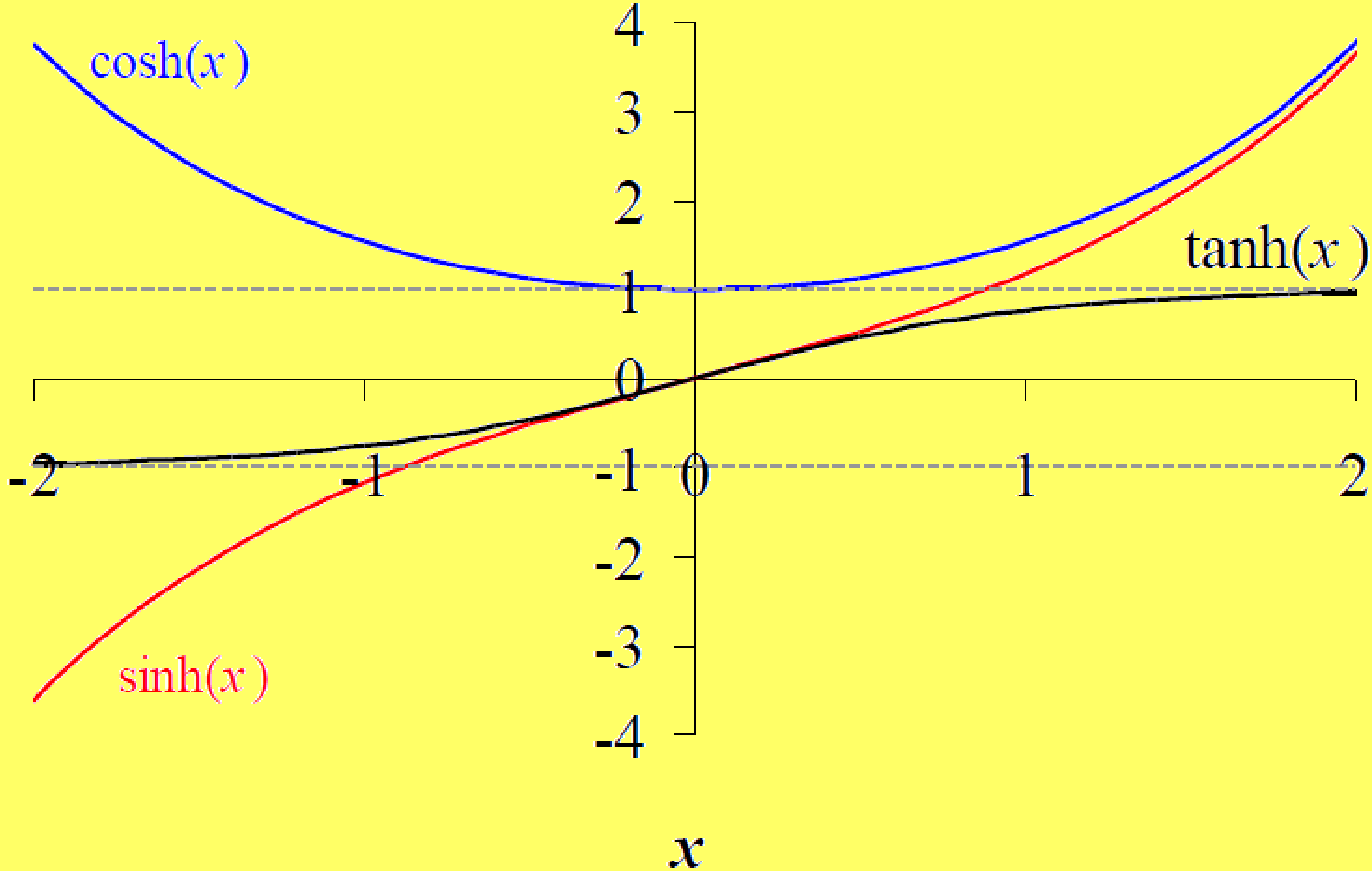
Note that since these functions are defined in terms of exponential functions, **they are not periodic** like $\sin x$, $\cos x$, $\tan x$.

Note: To change the angle, we just replace x:

E.g. $\sinh(3x) = \frac{e^{3x} - e^{-3x}}{2}$

Equations of hyperbolic functions

The graphs are shown below:



Equations of hyperbolic functions

Just like “regular” trig, there are also reciprocal hyperbolic functions :

Say as “setch”

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad x \in \mathbb{R}$$

Say as “cosetch”

Hyperbolic cosecant:

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \qquad x \in \mathbb{R}, x \neq 0$$

Say as “coth”

Hyperbolic cotangent:

$$\coth x = \frac{1}{\tanh x} = \frac{(e^{2x} + 1)}{e^{2x} - 1} \qquad x \in \mathbb{R}, x \neq 0$$

Hyperbolic Identities

We already know that $\sin^2 x + \cos^2 x = 1$.
Are there similar identities for hyperbolic functions?
Use the definitions of $sinh$ and $cosh$ to prove that…

$$\boldsymbol{\cosh^2 x - \sinh^2 x = 1}$$

$$\boldsymbol{\cosh^2 x} = \left(\frac{e^x + e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2e^x\ e^{-x} + e^{-2x}}{4}$$

$$= \frac{e^{2x} + e^{-2x} + 2}{4}$$

$$= \frac{1}{2}\left(\frac{e^{2x} + e^{-2x}}{2}\right) + \frac{1}{2}$$

$$= \frac{1}{2}\cosh 2x + \frac{1}{2}$$

$$\boldsymbol{\sinh^2 x} = \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} - 2e^x\ e^{-x} + e^{-2x}}{4}$$

$$= \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{1}{2}\left(\frac{e^{2x} + e^{-2x}}{2}\right) - \frac{1}{2}$$

$$= \frac{1}{2}\cosh 2x - \frac{1}{2}$$

$$\therefore\ \boldsymbol{\cosh^2 x - \sinh^2 x} = \frac{1}{2}\cosh 2x + \frac{1}{2} - \frac{1}{2}\cosh 2x + \frac{1}{2} = 1$$

Hyperbolic Identities

But what about $\cosh^2 x + \sinh^2 x$?

$$\cosh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2} \qquad \sinh^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2}$$

$$\therefore \ \cosh^2 x + \sinh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2} + \frac{1}{2}\cosh 2x - \frac{1}{2} = \cosh 2x$$

$$\boxed{\cosh^2 x + \sinh^2 x = \cosh 2x}$$

Hyperbolic Identities

We can also prove other hyperbolic identities in a similar way to that of “normal trig”

$$\cosh^2 x - \sinh^2 x = 1$$

Divide through by $\cosh^2 x$

$$\boxed{1 - \tanh^2 x = \operatorname{sech}^2 x}$$

Divide through by $\sinh^2 x$

$$\boxed{\coth^2 x - 1 = \operatorname{cosech}^2 x}$$

Hyperbolic Identities

And finally:

Similar to $\sin(A+B)$ identity.

$$\mathbf{sinh}(\boldsymbol{A} \pm \boldsymbol{B}) = \mathbf{sinh}\,\boldsymbol{A}\ \mathbf{cosh}\,\boldsymbol{B} \pm \mathbf{cosh}\,\boldsymbol{A}\ \mathbf{sinh}\,\boldsymbol{B}$$

$$\mathbf{cosh}(\boldsymbol{A} \pm \boldsymbol{B}) = \mathbf{cosh}\,\boldsymbol{A}\ \mathbf{cosh}\,\boldsymbol{B} \pm \mathbf{sinh}\,\boldsymbol{A}\ \mathbf{sinh}\,\boldsymbol{B}$$

However this is $\pm$ not $\mp$, unlike in $\cos(A+B)$

$$\mathbf{tanh}(\boldsymbol{A} \pm \boldsymbol{B}) = \frac{\tanh \boldsymbol{A} \pm \tanh \boldsymbol{B}}{\mathbf{1} \pm \tanh \boldsymbol{A} \tanh \boldsymbol{B}}$$

Notice this is $\pm$ rather than $\mp$.

Osborn's Rule (To help remember)

We can get these identities from the normal sin/cos ones by:

Osborn's Rule:

1. Replacing $sin \rightarrow sinh$ and $cos \rightarrow cosh$
2. **Negate** any explicit or implied **product of two sines**.

Negate means stick a negative in front of it

$$\sin A \sin B \rightarrow -\sinh A \sinh B$$
$$\tan^2 A \qquad \rightarrow -\tanh^2 A$$

Since $\tan^2 A = \frac{\sin^2 A}{\cos^2 A}$ (so this has a product of two sines)

$$\cos 2A = 2\cos^2 A - 1 \quad \rightarrow \quad \boldsymbol{\cosh 2A = 2\cosh^2 A - 1}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \rightarrow \boldsymbol{\tanh(A - B) = \frac{\tanh A - \tanh B}{1 - \tanh A \tanh B}}$$

Inverse Hyperbolic Functions

Just like for "regular" trigonometric functions, hyperbolics also have inverses, and they are named exactly how you would expect.

i.e.

$$y = \sinh x \quad \Rightarrow \quad x = \sinh^{-1} y$$

$$y = \cosh x \quad \Rightarrow \quad x = \cosh^{-1} y$$

$$y = \tanh x \quad \Rightarrow \quad x = \tanh^{-1} y$$

As before the "-1" refers to the inverse function;
$f(x) = \sinh x$ so $f^{-1}(x) = \sinh^{-1} y$.
These are also known as $\operatorname{arcsinh} y$, $\operatorname{arccosh} y$, $\operatorname{arctanh} y$.

Inverse Hyperbolic Functions

Inverse hyperbolic functions can be written in terms of the natural logarithm, ln :

$$\sinh^{-1} y = \ln(y + \sqrt{y^2+1})$$

Proof: Let $y = \sinh x$ (so $x = \sinh^{-1} y$)

$$y + \sqrt{y^2+1} = \sinh x + \sqrt{\sinh^2 x + 1}$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$\Rightarrow \sinh^2 x + 1 = \cosh^2 x$$

$$= \sinh x + \sqrt{\cosh^2 x}$$

$$= \sinh x + \cosh x$$

$$= \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} \quad = e^x$$

$$\therefore e^x = y + \sqrt{y^2+1}$$

$$x = \ln(y + \sqrt{y^2+1})$$

$$\sinh^{-1} y = \ln(y + \sqrt{y^2+1})$$

$$\cosh^{-1} y = \ln\left(y + \sqrt{y^2-1}\right), y \geq 1$$

Proof: Let $y = \cosh x$ (so $x = \cosh^{-1} y$)

$$y + \sqrt{y^2-1} = \cosh x + \sqrt{\cosh^2 x - 1}$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$\Rightarrow \cosh^2 x - 1 = \sinh^2 x$$

$$= \cosh x + \sqrt{\sinh^2 x}$$

$$= \cosh x + \sinh x$$

$$= \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \quad = e^x$$

$$\therefore e^x = y + \sqrt{y^2-1}$$

$$x = \ln(y + \sqrt{y^2-1})$$

$$\cosh^{-1} y = \ln(y + \sqrt{y^2-1})$$

Inverse Hyperbolic Functions

Inverse hyperbolic functions can be written in terms of the natural logarithm, ln :

$$\tanh^{-1} y = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right), \quad -1 < y < 1$$

Proof: Let $y = \tanh x$ (so $x = \tanh^{-1} y$)

$$x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \quad \Rightarrow \quad x = \ln\sqrt{\frac{1+y}{1-y}}$$ (We must prove this)

$$\sqrt{\frac{1+y}{1-y}} = \sqrt{\frac{1+\tanh x}{1-\tanh x}} = \sqrt{\frac{1+\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)}{1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)}}$$

$$\tanh = \frac{\sinh x}{\cosh x} = \frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Multiply by $e^x + e^{-x}$

$$= \sqrt{\frac{e^x+e^{-x}+(e^x-e^{-x})}{e^x+e^{-x}-(e^x-e^{-x})}} = \sqrt{\frac{2e^x}{2e^{-x}}} = \sqrt{e^{2x}} = e^x$$

$$\therefore e^x = \sqrt{\frac{1+y}{1-y}} \Rightarrow x = \ln\sqrt{\frac{1+y}{1-y}} \Rightarrow \tanh^{-1} y = \ln\left(\frac{1+y}{1-y}\right)^{\frac{1}{2}} \Rightarrow \tanh^{-1} y = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$$

Technicalities with Inverse functions (just FYI)

*Everything on this page applies similarly to hyperbolic functions

As we know $\tan x = \frac{\sin x}{\cos x}$

But $\tan^{-1} x \neq \frac{\sin^{-1} x}{\cos^{-1} x}$

This can be easily shown with a counterexample:

Let $x = 1$: $\tan^{-1} 1 = \frac{\pi}{4}$ $\frac{\sin^{-1} 1}{\cos^{-1} 1} = \frac{\pi/2}{0}$

In general if $f(x) = \frac{g(x)}{h(x)}$ then $f^{-1}(x) \neq \frac{g^{-1}(x)}{h^{-1}(x)}$

The situation is the same for addition subtraction or multiplication:

If $f(x) = g(x) + h(x) \rightarrow f^{-1}(x) \neq g^{-1}(x) + h^{-1}(x)$

If $f(x) = g(x)h(x) \rightarrow f^{-1}(x) \neq g^{-1}(x)h^{-1}(x)$

For composite functions (function of a function):

If $f(x) = gh(x)$ then $f^{-1}(x) \neq g^{-1}h^{-1}(x)$

But $f^{-1}(x) = gh^{-1}(x)$ (if the inverse exists).
This is because $gh(x)$ is essentially now one overall function.

Differentiating hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

Important Memorisation Tip: They're all the same as non-hyperbolic results, other than that $cosh$ is not negated and $\operatorname{sech} x$ becomes $-\operatorname{sech} x \tan h\, x$ (i.e. **is** negated).

Proof that $\frac{d}{dx}(\sinh x) = \cosh x$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\frac{d}{dx}(\sinh x) = \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

Diagnostic Question

$$\cosh x + \sinh x$$

Simplifies to:

Y

e^x

C

e^{-x}

M

1

A

$\cosh 2x$

Diagnostic Question

$$\cosh x - \sinh x$$

Simplifies to:

Y

e^x

C

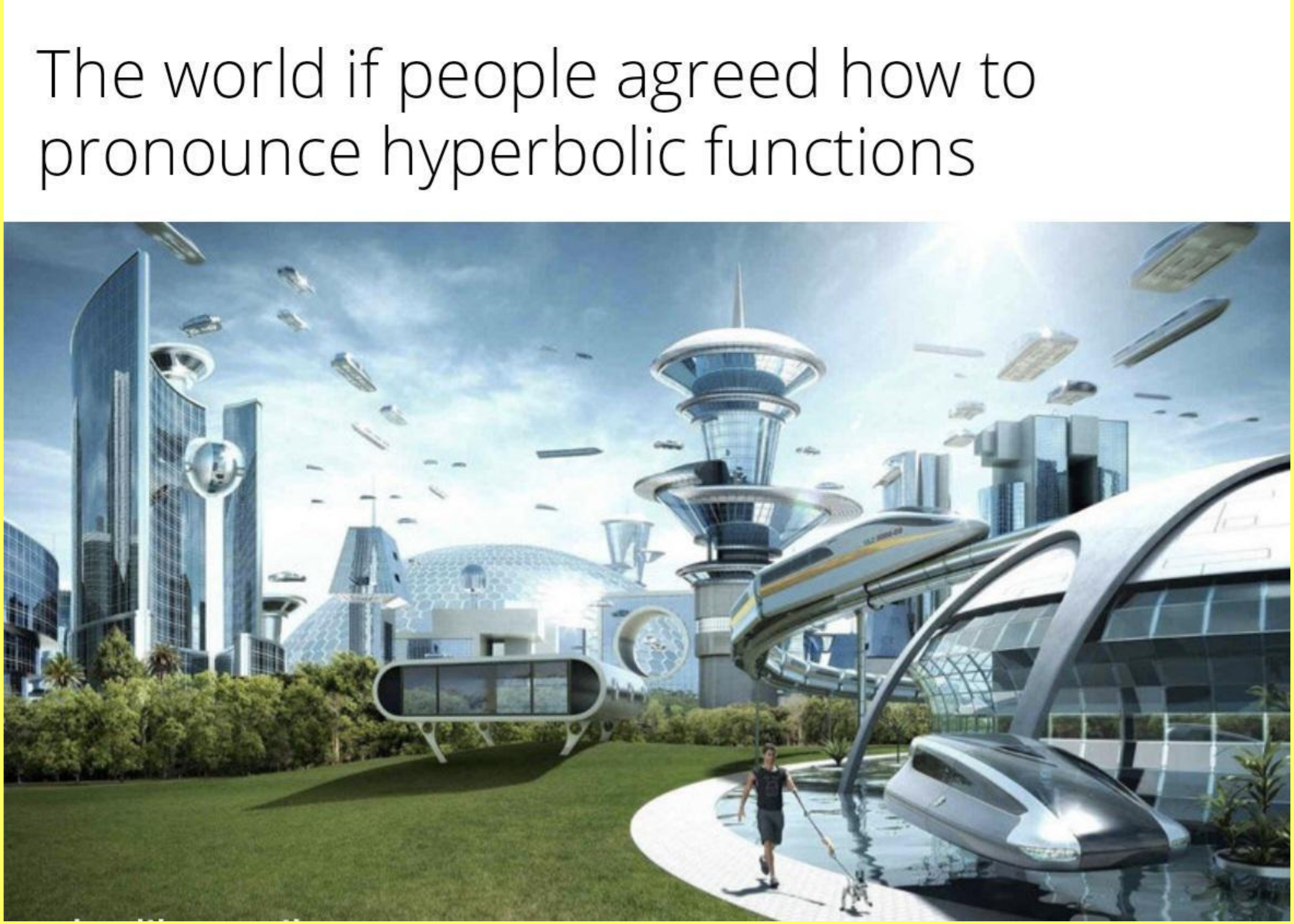
e^{-x}

M

1

A

$\cosh 2x$



Integration
using Hyperbolic Trig Functions

Standard Integrals (Trig & Inverse)

$$\int \sinh x \; dx = \cosh x + C$$

$$\int \cosh x \; dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x \; dx = \tanh x + C$$

$$\int \operatorname{cosech}^2 x \; dx = -\coth x + C$$

$$\int \operatorname{sech} x \tanh x \; dx = -\operatorname{sech} x + C$$

$$\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \; dx = \sin^{-1} x + C, \qquad |x| < 1$$

$$\int \frac{1}{1+x^2} \; dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1+x^2}} \; dx = \sinh^{-1} x + C$$

$$\int \frac{1}{\sqrt{x^2-1}} \; dx = \cosh^{-1} x + C, \qquad x > 1$$

Standard Integrals (general cases):

$$\int \frac{1}{\sqrt{a^2-x^2}} \; dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \qquad |x| < a$$

$$\int \frac{1}{a^2+x^2} \; dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} \; dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$$

Alternate forms:

$$\int \frac{1}{\sqrt{x^2+a^2}} \; dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}} \; dx = \cosh^{-1}\left(\frac{x}{a}\right) + C, \qquad x > a$$

Diagnostic Question

Use standard integrals to find: $\int \sinh\left(\frac{x}{2}\right) dx$

Y

$$\frac{1}{2}\cosh\left(\frac{x}{2}\right) + C$$

C

$$-\frac{1}{2}\cosh\left(\frac{x}{2}\right) + C$$

M

$$2\cosh\left(\frac{x}{2}\right) + C$$

A

$$-2\cosh\left(\frac{x}{2}\right) + C$$

Diagnostic Question

Use standard integrals to find: $\displaystyle\int \frac{3}{\sqrt{1+x^2}}\,dx$

Y

$3\sinh^{-1} x + C$

C

$\sinh^{-1}(3x) + C$

M

$\dfrac{1}{3}\sinh^{-1} x + C$

A

$\sinh^{-1}\left(\dfrac{x}{3}\right) + C$

Diagnostic Question

Use standard integrals to find: $\int \frac{1}{2\sqrt{x^2 - 9}} dx$

Y

$$2\cosh^{-1}\left(\frac{x}{3}\right) + C$$

C

$$\frac{1}{2}\cosh^{-1}\left(\frac{x}{9}\right) + C$$

M

$$2\cosh^{-1}\left(\frac{x}{9}\right) + C$$

A

$$\frac{1}{2}\cosh^{-1}\left(\frac{x}{3}\right) + C$$

Proof of Hyperbolic Integrals 1

Earlier we proved

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

By substituting $x = a\tan u$ to make use of $1+\tan^2 u = \sec^2 u$

$$\sin^2\theta + \cos^2\theta = 1$$
$$1+\tan^2\theta = \sec^2\theta$$
$$\cosh^2\theta - \sinh^2\theta = 1$$

We will now evaluate the following:

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx$$

tan wouldn't work as well this time because the denominator would simplify to $a\sec u$, but we'd be multiplying by $a\sec^2\theta$, meaning not all the secs would cancel. With $\sinh u$ the two $\cosh u$'s obtained would fully cancel.

We instead substitute $x = a\sinh u$ to make use of $1+\sinh^2 u = \cosh^2 u$

$$x = a\sinh u \Rightarrow \frac{dx}{du} = a\cosh u$$

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \int \frac{1}{\sqrt{a^2+a^2\sinh^2 u}}\,a\cosh u\,du$$

$$= \frac{1}{a}\int \frac{a\;\cosh u}{\sqrt{1+\sinh^2 u}}\,du = \int \frac{\cosh u}{\cosh u}\,du = u + C = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\boxed{\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C}$$

Proof of Hyperbolic Integrals 2

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cosh^2\theta - \sinh^2\theta = 1$$

Earlier we proved

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, |x| < a$$

By substituting $x = a\sin u$ to make use of $1 - \sin^2\theta = \cos^2\theta$

We instead substitute $x = a\cosh u$ to make use of $\cosh^2 u - 1 = \sinh^2 u$

$$x = a\cosh u \Rightarrow \frac{dx}{du} = a\sinh u$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{1}{\sqrt{a^2\cosh^2 u - a^2}}\, a\sinh u\, du$$

$$= \frac{1}{a}\int \frac{a\ \sinh u}{\sqrt{\cosh^2 u - 1}}\, du \ = \int \frac{\sinh u}{\sinh u}\, du \ = u + c \quad = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\boldsymbol{\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C, \qquad x > a}$$

Familiar methods

E.g. 1

$$\int \tanh x \; dx$$

We can approach this the same way as we would for $\tan x$. That is, rewrite $\tanh x$ as $\frac{\sinh x}{\cosh x}$ and use substitution.

$$= \int \frac{\sinh x}{\cosh x} \, dx$$

$$u = \cosh x \qquad \Rightarrow \frac{du}{dx} = \sinh x$$

$$= \int \frac{\sinh x}{u} \frac{du}{\sinh x}$$

$$= \int \frac{1}{u} \, du$$

$$= \ln |u| + C$$

$$= \ln | \cosh x | + C$$

Further Example

E.g. 2

$$\int \frac{2+5x}{\sqrt{x^2+1}}\,dx$$

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + C$$
$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C\,, \qquad x > 1$$

Integration Strategy:
If multiple terms in numerator, split fraction.

$$\int \frac{2+5x}{\sqrt{x^2+1}}\,dx = \int \frac{2}{\sqrt{x^2+1}}\,dx + \int \frac{5x}{\sqrt{x^2+1}}\,dx$$

$$= 2\int \frac{1}{\sqrt{x^2+1}}\,dx + 5\int \frac{x}{\sqrt{x^2+1}}\,dx$$

For the first integral we can use a standard integral, but for the second we'll need a substitution.

$$u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\Rightarrow \int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{x}{\sqrt{u}}\frac{du}{2x} \quad = \frac{1}{2}\int u^{-\frac{1}{2}}du \quad = \frac{1}{2}\left(2u^{\frac{1}{2}}\right) + C_1 \quad = u^{\frac{1}{2}} + C_1 \quad = \sqrt{x^2+1} + C_1$$

$$\therefore \int \frac{2+5x}{\sqrt{x^2+1}}\,dx = 2\sinh^{-1} x + 5\left(\sqrt{x^2+1} + C_1\right) \quad = 2\sinh^{-1} x + 5\sqrt{x^2+1} + C$$

Integrating Hyperbolics Using Definitions

Sometimes there are techniques which work on non-hyperbolic trig functions but doesn't work on hyperbolic ones.
Just first replace any hyperbolic functions with their definition.

E.g. 3 Find $\int e^{2x} \sinh x \ dx$

$$\int e^{2x} \sinh x \ dx = \int e^{2x} \left(\frac{e^x - e^{-x}}{2} \right)$$

$$= \frac{1}{2} \int e^{3x} - e^x \, dx$$

$$= \frac{1}{2} \left(\frac{1}{3} e^{3x} - e^x + C_1 \right)$$

$$= \frac{1}{6} (e^{3x} - 3e^x) + C$$

(Integration by parts DOES also work, but requires a significantly greater amount of working due to cyclic parts!

E.g. 4 Find $\int \operatorname{sech} x \ dx$

$$\int \operatorname{sech} x \ dx = \int \frac{2}{e^x + e^{-x}} dx$$

Multiply top and bottom by e^x

$$= \int \frac{2e^x}{e^{2x} + 1} dx$$

Use the substitution $u = e^x$

$$\frac{du}{dx} = e^x \quad \therefore dx = \frac{du}{e^x}$$

$$\int \frac{2e^x}{e^{2x} + 1} dx = \int \frac{2}{u^2 + 1} du$$

$$\int \frac{1}{1 + x^2} dx = \tan^{-1} x + C$$

$$= 2 \tan^{-1} u + C$$

$$= 2 \tan^{-1}(e^x) + C$$

Diagnostic Question

Evaluate: $$\int e^x \cosh(2x)\,dx$$

Y

$$\frac{1}{6}e^{3x} + \frac{1}{2}e^{-x} + C$$

C

$$\frac{1}{3}e^{3x} - e^{-x} + C$$

M

$$\frac{1}{4}e^{2x} + \frac{1}{2}x + C$$

A

$$\frac{1}{6}e^{3x} - \frac{1}{2}e^{-x} + C$$

Integration by Completing the Square

Integration by Completing the Square

E.g. 1

Find the integral:

$$\int \frac{1}{x^2 - 8x + 8}\, dx$$

By completing the square, we can then use one of the standard results.

$$\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

This will not factorise so partial fractions won't help. Let's try completing the square…

$$\int \frac{1}{x^2 - 8x + 8}\, dx = \int \frac{1}{(x-4)^2 - 8}\, dx$$

This is not in the standard form yet, but a simple substitution would make it so.

Let $u = x - 4 \quad \rightarrow \quad du = dx$

$$\int \frac{1}{(x-4)^2 - 8}\, dx = \int \frac{1}{u^2 - 8}\, du$$

$$= \frac{1}{2\sqrt{8}} \ln\left|\frac{u - \sqrt{8}}{u + \sqrt{8}}\right| + C$$

$$= \frac{\sqrt{2}}{8} \ln\left|\frac{x - 4 - 2\sqrt{2}}{x - 4 + 2\sqrt{2}}\right| + C$$

Integration by Completing the Square

E.g. 2

Find the integral:

$$\int \frac{1}{x^2 - 5x + 7}\, dx$$

Standard Integrals (general cases):

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \qquad |x| < a$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\, dx = \ln\left|x + \sqrt{x^2 \pm a^2}\,\right| + C$$

Alternate forms:

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \cosh^{-1}\left(\frac{x}{a}\right) + C, \qquad x > a$$

When the integrand is a rational function with a quadratic expression in the denominator, we can use the following table integrals:

Completing the square in the denominator gives:

$$\int \frac{1}{\left(x - \frac{5}{2}\right)^2 + \frac{3}{4}}\, dx \quad = \int \frac{1}{\left(x - \frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

$$= \int \frac{1}{\left(x - \frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

Let $u = x - \frac{5}{2} \rightarrow du = dx$

$$= \int \frac{1}{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

This is of the form $\int \frac{1}{a^2+x^2}\, dx$ with $a = \frac{\sqrt{3}}{2}$

$$= \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{u}{\frac{\sqrt{3}}{2}}\right) + C \quad = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x - \frac{5}{2}}{\frac{\sqrt{3}}{2}}\right) + C \quad = \boldsymbol{\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-5}{\sqrt{3}}\right) + C}$$

Diagnostic Question

Evaluate:

Find the integral:

$$\int \frac{1}{x^2 - 2x + 3}\,dx$$

Y

$$\frac{\ln(x^2 - 2x + 3)}{2x - 2} + C$$

C

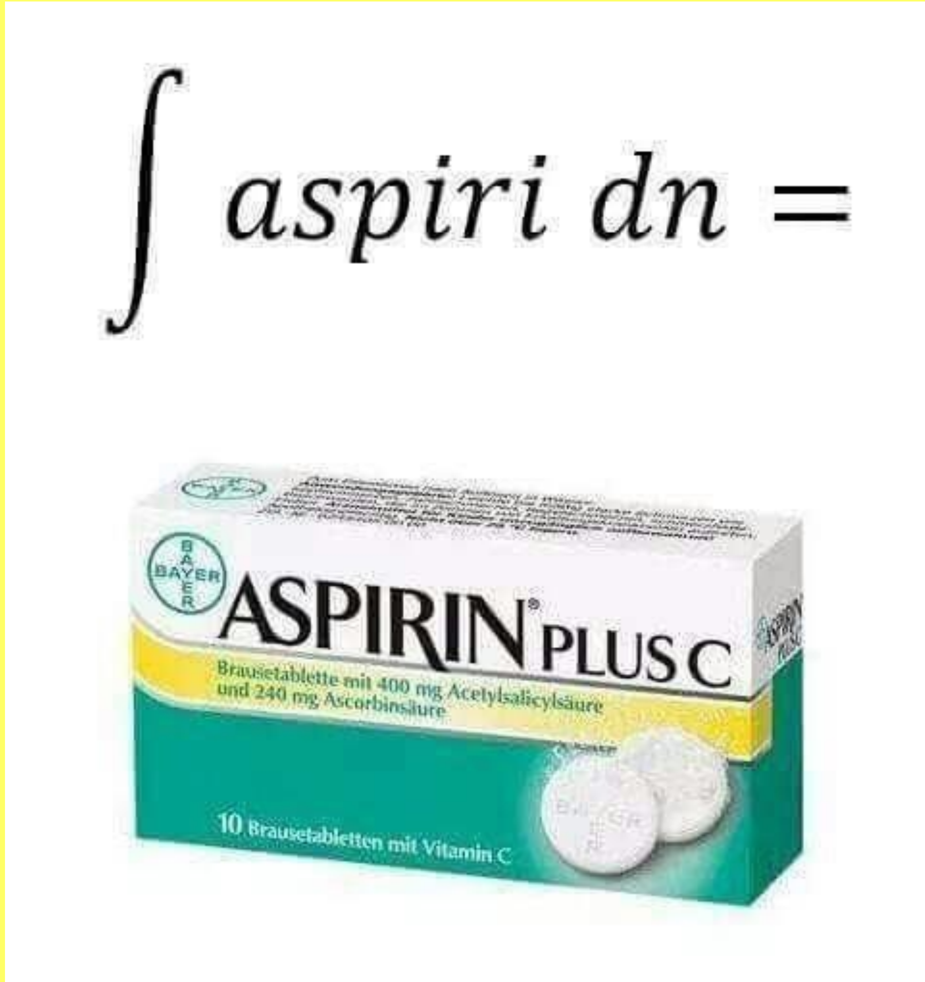
$$2\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

M

$$\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

A

$$-\frac{1}{2}\tan^{-1}\left(\frac{x-1}{-2}\right) + C$$



Integration of Parametric Equations

Integrating Parametric Equations

Recall that when we integrated **by substitution**, using a substitution $u = f(x)$, we could use the chain rule to change the dx to another variable:

$$\int_{x_1}^{x_2} \dots dx \quad \Rightarrow \quad \int_{u_1}^{u_2} \dots \frac{dx}{du}\, du$$

Informally, you can think of the du's cancelling to leave the dx.

We used the substitution $u = f(x)$ to change the limits.

We can use exactly **the same technique** for **integrating parametric equations** in terms of a parameter t:

$$\boxed{\int y\ dx = \int y \frac{dx}{dt}\ dt}$$

No need to remember a new formula.
Just remember that $\frac{dx}{dt}dt = dx$, which follows from the chain rule.

Parametric Equations: Indefinite Integration

E.g. 2

A function is defined parametrically using the equations:

$$x = \frac{1}{t} + 1 \qquad y = t^2$$

Determine $\int y\, dx$ in terms of t.

$$\int y \ dx = \int y \frac{dx}{dt} \ dt$$

Determine $\frac{dx}{dt}$

$$x = t^{-1} + 1$$
$$\frac{dx}{dt} = -t^{-2}$$

$$\int y\, dx = \int y \frac{dx}{dt}\, dt$$
$$= \int t^2(-t^{-2})\, dt$$
$$= \int -1\, dt$$
$$= \boldsymbol{-t + c}$$

Your integral should be entirely in terms of the parameter (here, t)

Parametric Equations: Definite Integration

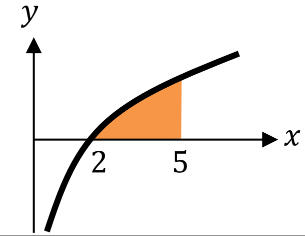
Indefinite integration using parametric equations is not particularly useful. But we'd regularly need to find **an area under a curve** when it is defined parametrically. For this, we use definite integration.

E.g. 3

The graph shows the curve defined by the parametric equations:

$$x = t^2 + 1 \qquad y = \ln t \qquad t \geq 0$$

Determine the **exact** area bound between the curve, the x-axis and the lines with equations $x = 2$ and $x = 5$.



Use integration by parts:

$$\int_a^b u \frac{dv}{dx}\, dx = [uv]_a^b - \int_a^b v \frac{du}{dx}\, dx$$

Like with integration by substitution, we need to change the limits so they are in terms of t.

When $x = 2$, $\quad 2 = t^2 + 1 \quad \rightarrow \quad t = 1$

When $x = 5$, $\quad 5 = t^2 + 1 \quad \rightarrow \quad t = 2$

$$\frac{dx}{dt} = 2t$$

$$\int_1^2 y \frac{dx}{dt}\, dt = \int_1^2 2t \ln t \; dt$$

$$u = \ln t \qquad \frac{dv}{dt} = 2t$$

$$\frac{du}{dt} = \frac{1}{t} \qquad v = t^2$$

$$\int_1^2 2t \ln t \; dt = [t^2 \ln t]_1^2 - \int_1^2 t \, dt$$

$$= 4 \ln 2 - 1 \ln 1 - \left[\frac{1}{2} t^2\right]_1^2$$

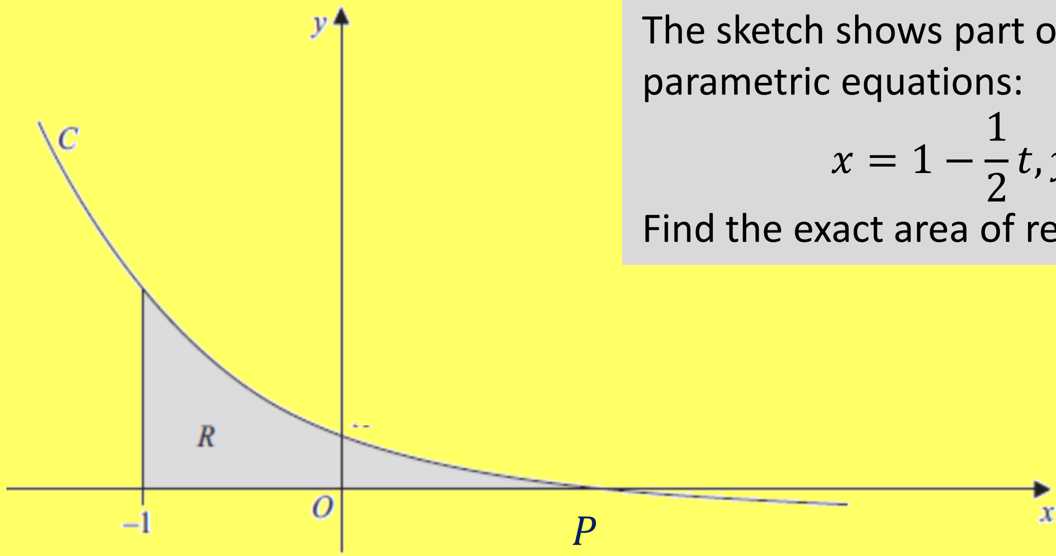
$$= 4 \ln 2 - \left(2 - \frac{1}{2}\right)$$

$$= \mathbf{4 \ln 2 - \frac{3}{2}}$$

$$\int y \; dx = \int y \frac{dx}{dt} \; dt$$

Linking to Exponentials

E.g. 4



The sketch shows part of curve C with parametric equations:

$$x = 1 - \frac{1}{2}t, y = 2^t - 1$$

Find the exact area of region R.

We need limits for our integral. As the curve is given in terms of t, we need the limits to be in terms of t.

At $x = -1$: $1 - \frac{1}{2}t = -1 \Rightarrow \mathbf{t = 4}$

At P, $y = 0 \Rightarrow 2^t - 1 = 0 \Rightarrow 2^t = 1 \Rightarrow \mathbf{t = 0}$

$$\int y \, dx = \int y \frac{dx}{dt} \, dt$$

$$\frac{dx}{dt} = -\frac{1}{2}$$

$$\int y \, dx = \int_4^0 (2^t - 1)\left(-\frac{1}{2}\right) dt$$

$$= -\frac{1}{2}\int_4^0 (2^t - 1) \, dt$$

We know that:

$$\frac{d}{dx}(a^x) = a^x \ln a \Rightarrow \int a^x \, dx = \frac{a^x}{\ln a} + c$$

$$= -\frac{1}{2}\left[\frac{2^t}{\ln 2} - t\right]_4^0$$

$$= -\frac{1}{2}\left[\left(\frac{1}{\ln 2}\right) - \left(\frac{16}{\ln 2} - 4\right)\right]$$

$$= -\frac{1}{2}\left[-\frac{15}{\ln 2} - 4\right] = \mathbf{\frac{15}{2\ln 2} - 2}$$

Diagnostic Question

A curve is defined parametrically by:

$$x = 4t, \quad y = 6t^2 - t$$

Which of the following finds the area between the curve, the x-axis and the lines $x = 0$ and $x = 1$?

Y $\displaystyle\int_0^{\frac{1}{4}} (12t - 1)4t\,dt$

M $\displaystyle\int_0^{\frac{1}{4}} (12t - 1)4\,dt$

C $\displaystyle\int_0^{\frac{1}{4}} (6t^2 - t)4t\,dt$

A $\displaystyle\int_0^{\frac{1}{4}} (6t^2 - t)4\,dt$

Diagnostic Question

A curve is defined parametrically by:

$$x = \sin t\,, \quad y = \cos t$$

Which of the following finds the area between the curve, the x-axis and the lines $x = 0$ and $x = 1$?

Y $\displaystyle\int_0^1 \sin^2 t \; dt$

M $\displaystyle\int_0^1 \cos^2 t \; dt$

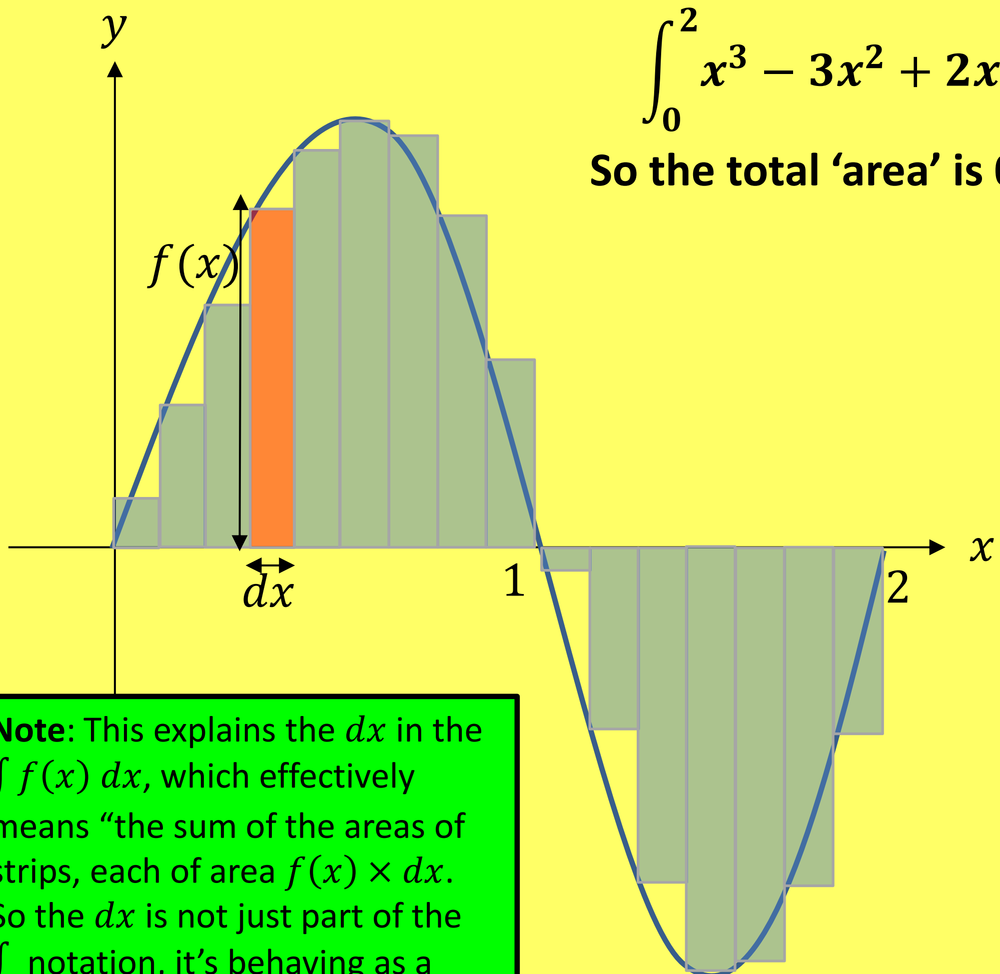
C $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 t \; dt$

A $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 t \; dt$

Area Between Curves

Recap: ‘Negative Areas’

Sketch the curve $y = x(x-1)(x-2)$ (which expands to give $y = x^3 - 3x^2 + 2x$).
Now calculate $\int_0^2 x(x-1)(x-2)\ dx$. Why is this result surprising?



$$\int_0^2 \boldsymbol{x^3 - 3x^2 + 2x\ dx = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^2 = 0}$$

So the total ‘area’ is 0!

Obviously it is impossible for an area to actually be negative. However it is important to distinguish between areas above and below the x-axis to avoid confusion!

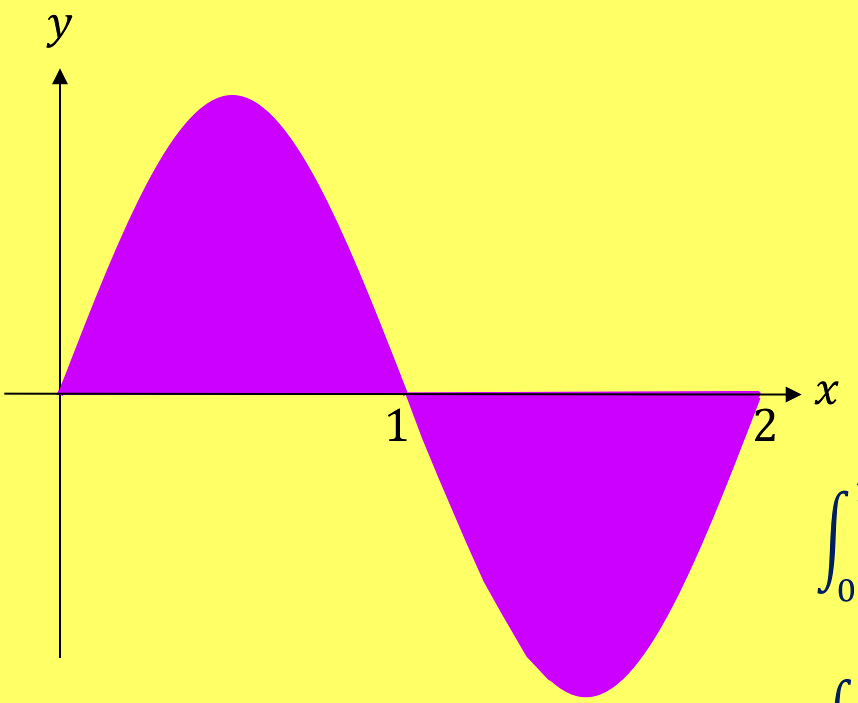
As we know, integration $\int f(x)\ dx$ is just the sum of areas of infinitely thin rectangles, where the current y value (i.e. $f(x)$) is each height, and the widths are dx. i.e. The area of each is $f(x) \times dx$

The problem is, when $f(x)$ is negative, then $f(x) \times dx$ is negative, i.e. a negative area!
The result is that the ‘positive area’ from 0 to 1 is cancelled out by the ‘negative area’ from 1 to 2, giving an overall ‘area’ of 0.
So how do we resolve this?

Note: This explains the dx in the $\int f(x)\ dx$, which effectively means “the sum of the areas of strips, each of area $f(x) \times dx$. So the dx is not just part of the $\int$ notation, it’s behaving as a physical quantity! (i.e. length)

Strategy: Find The Areas Separately

Find the total area bound between the curve $y = x(x-1)(x-2)$ and the x-axis.



Strategy:
- Separately find the area between $x = 0$ and 1, and between 1 and 2.
- Treat any negative areas as positive.

Expand: $x(x-1)(x-2) = x^3 - 3x^2 + 2x$

$$\int_0^1 x^3 - 3x^2 + 2x \; dx = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^1 = \frac{1}{4}$$

$$\int_1^2 x^3 - 3x^2 + 2x \; dx = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^2 = -\frac{1}{4}$$

we could predict this by symmetry, but this is not a general case.

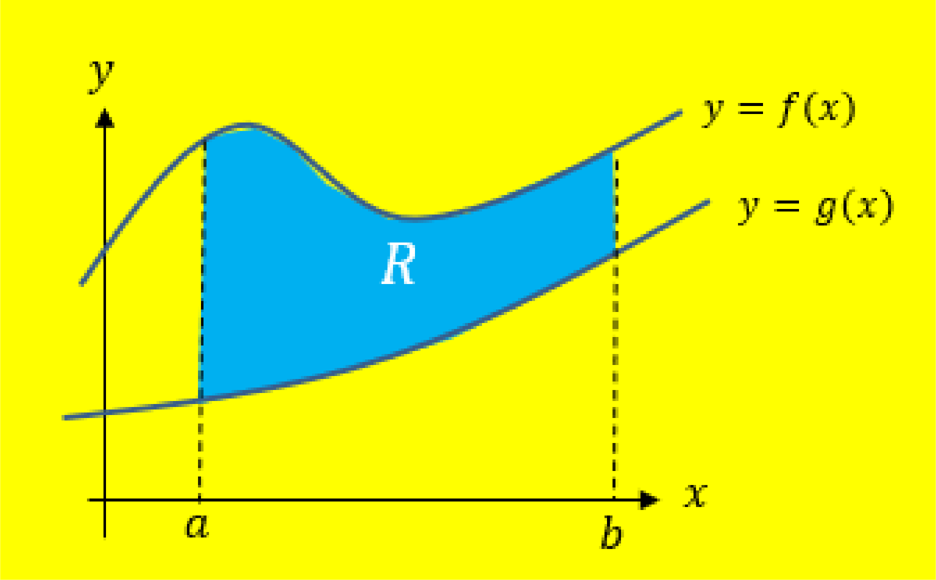
Treating both as positive:

$$Area = \frac{1}{4} + \frac{1}{4} = \mathbf{\frac{1}{2}}$$

Area between two curves

The area between two curves between $x = a$ and $x = b$ is calculated as:

$\int_a^b (upper\ curve - lower\ curve)\, dx$ square units



The areas under the two curves are $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$.

It therefore follows the area between them **(provided the curves don't overlap)** is:

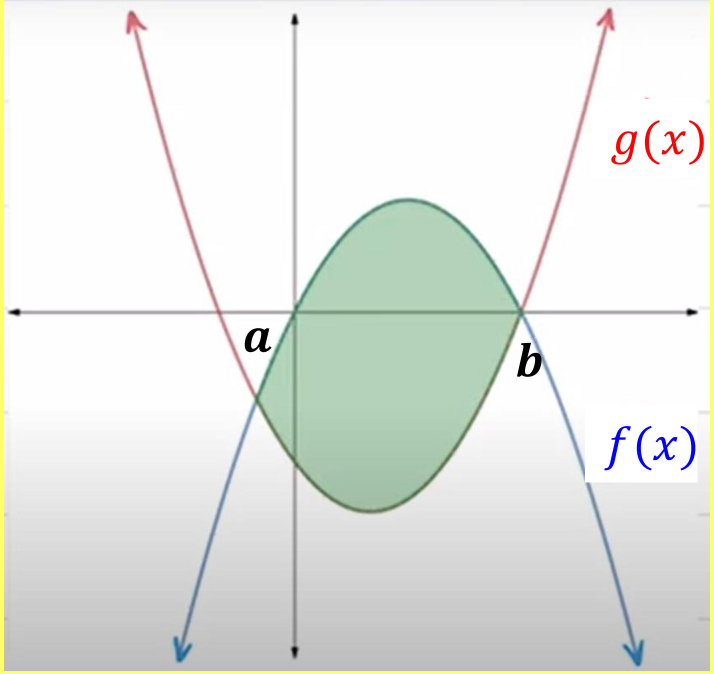
$$R = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b \big(f(x) - g(x)\big)\, dx$$

Area between two curves: Above and below axis

What about areas between curves that are above **and** below the x-axis?

So far with definite integration, we have had to deal with areas below the x-axis separately to those above so that we don't "lose" area through cancellation of positive and negative y-values.

So if I wanted to apply this idea here, I'd have to split this into several areas, work out all the bounds and consider which curve/line overlaps which and when.



This is going to be tedious and in some cases extremely difficult!

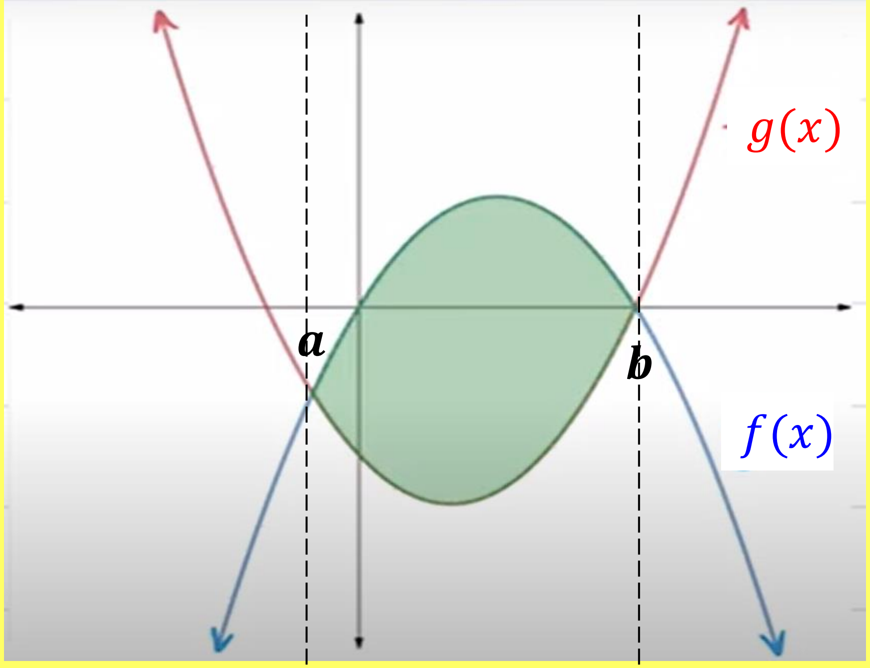
However, for areas between curves, areas above and below the x-axis do not need to be calculated separately.
The shaded area is still just:

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \big(f(x) - g(x)\big)\,dx$$

Proof

So why is this the case?

Imagine we shifted both curves up by the same amount, say 100 units in the positive y-direction, enough so that the area remains the same size but is now completely above the x-axis.



Therefore $f(x)$ becomes $f(x) + 100$ and $g(x)$ becomes $g(x) + 100$

The shaded area therefore becomes:

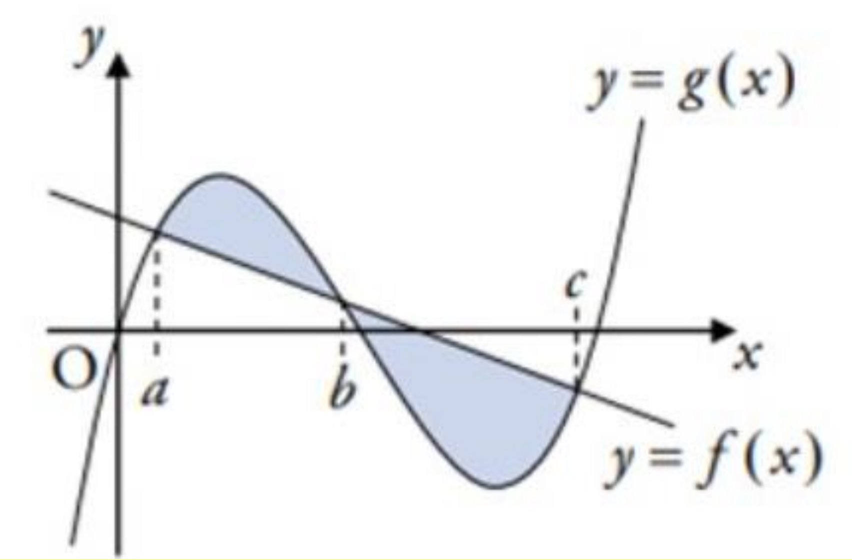
$$\int_a^b (f(x) + 100)\,dx - \int_a^b (g(x) + 100)\,dx = \int_a^b (f(x) + 100) - (g(x) + 100))\,dx$$

$$= \int_a^b f(x) - g(x)\,dx$$

Which is exactly the same!

Area between two curves: Multiple enclosed areas

Here, **the functions overlap,** so each area is separate



We must be careful when there is more than one enclosed area.
In this case, we must calculate each area separately and add them together.

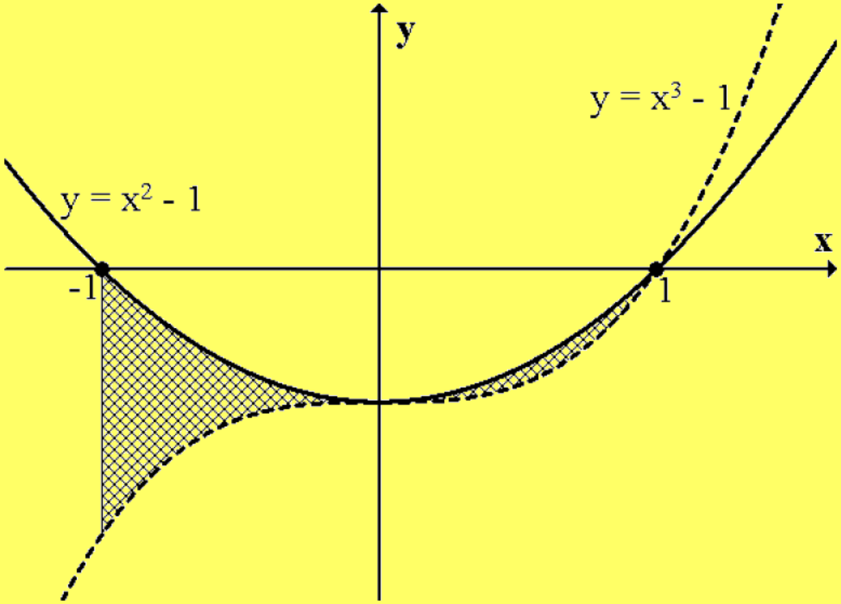
For example: In the **diagram above**, the total shaded area would be given by:

$$\int_a^b (g(x) - f(x))\,dx + \int_b^c \big(f(x) - g(x)\big)\,dx$$

Note: This is not a general formula as the area will depend on where each function is above the other in each particular case.

One area that looks like two

In this special case, the functions touch but don't overlap, therefore the two shaded areas are not "cut off" from each other (lines technically have zero thickness) and are therefore both linked into one area.



Note that for both areas, $y = x^2 - 1$ is above $y = x^3 - 1$ so the integrand is $x^2 - x^3$

Therefore the area can be found by either:

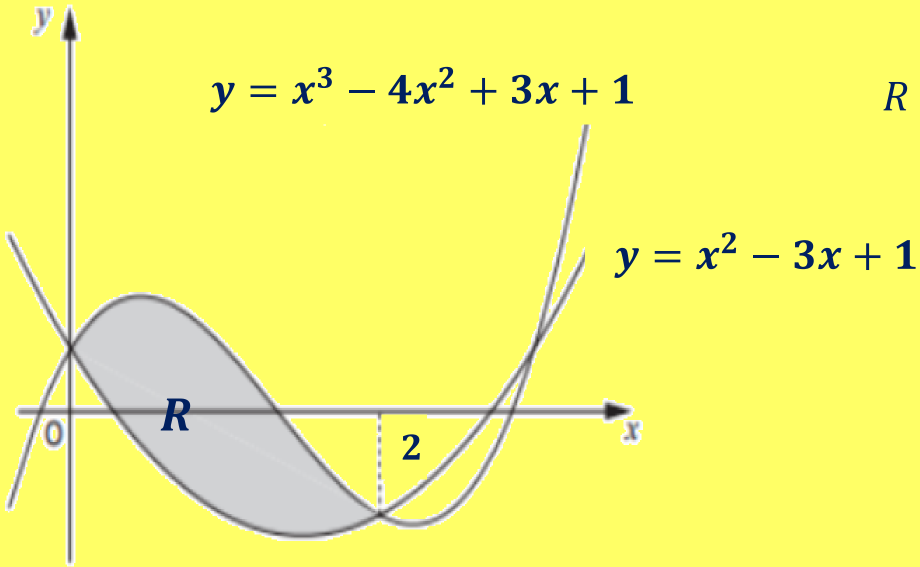
$$\int_{-1}^{0} x^2 - x^3 \, dx + \int_{0}^{1} x^2 - x^3 dx = \mathbf{\frac{2}{3}}$$

Or:

$$\int_{-1}^{1} x^2 - x^3 \, dx = \mathbf{\frac{2}{3}}$$

Area between two curves

E.g. Find the shaded area between the two curves shown on the diagram.



$$R = \int_0^2 (x^3 - 4x^2 + 3x + 1) - (x^2 - 3x + 1)\ dx$$

$$= \int_0^2 x^3 - 5x^2 + 6x\ dx$$

$$= \left[\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_0^2$$

$$= \left[\frac{16}{4} - \frac{40}{3} + 12\right] - [0]$$

$$= \mathbf{\frac{8}{3}}$$

Diagnostic Question

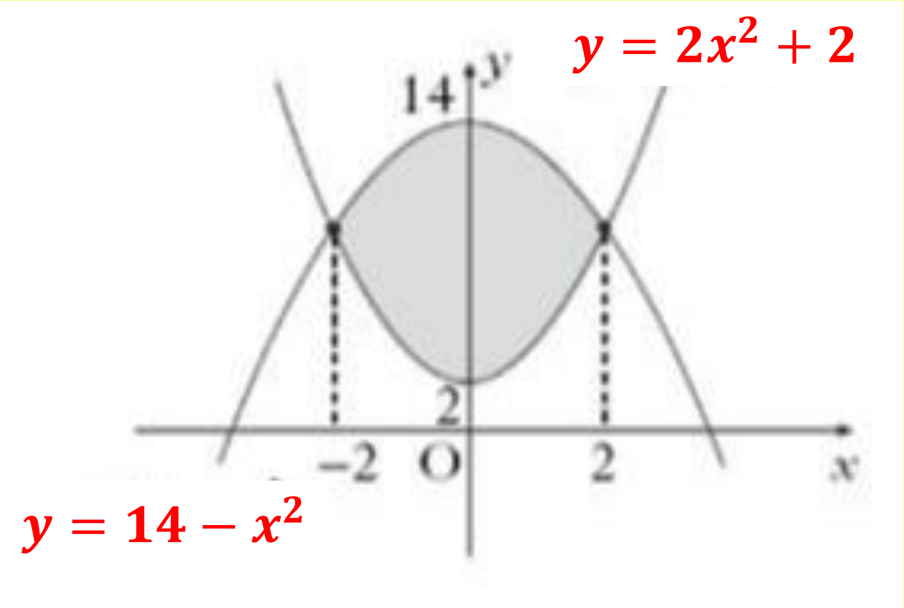
The diagram shows graphs with equations $y = 14 - x^2$ and $y = 2x^2 + 2$.
Which of the following represents the shaded area?

Y $\displaystyle\int_2^{14} 12 - 3x^2 \, dx$

M $\displaystyle\int_2^{14} 3x^2 - 12 \, dx$

C $\displaystyle\int_{-2}^{2} 12 - 3x^2 \, dx$

A $\displaystyle\int_{-2}^{2} 3x^2 - 12 \, dx$



Diagnostic Question

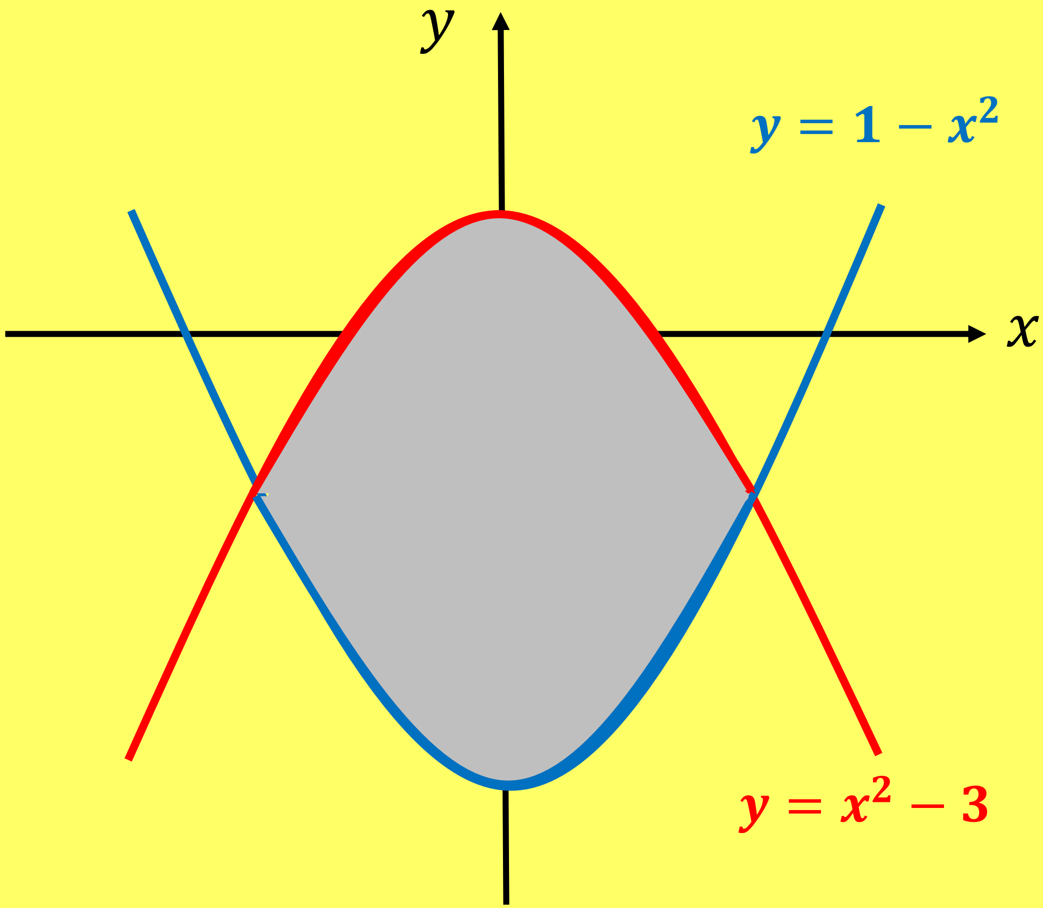
The diagram shows graphs with equations $y = x^2 - 3$ and $y = 1 - x^2$.
Which of the following represents the shaded area?

Y $\displaystyle\int_0^4 4 - 2x^2 \, dx$

M $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} 4 - 2x^2 \, dx$

C $\displaystyle\int_0^{\sqrt{2}} 2x^2 - 4 \, dx$

A $\displaystyle\int_{-2}^{2} 4 - 2x^2 \, dx$

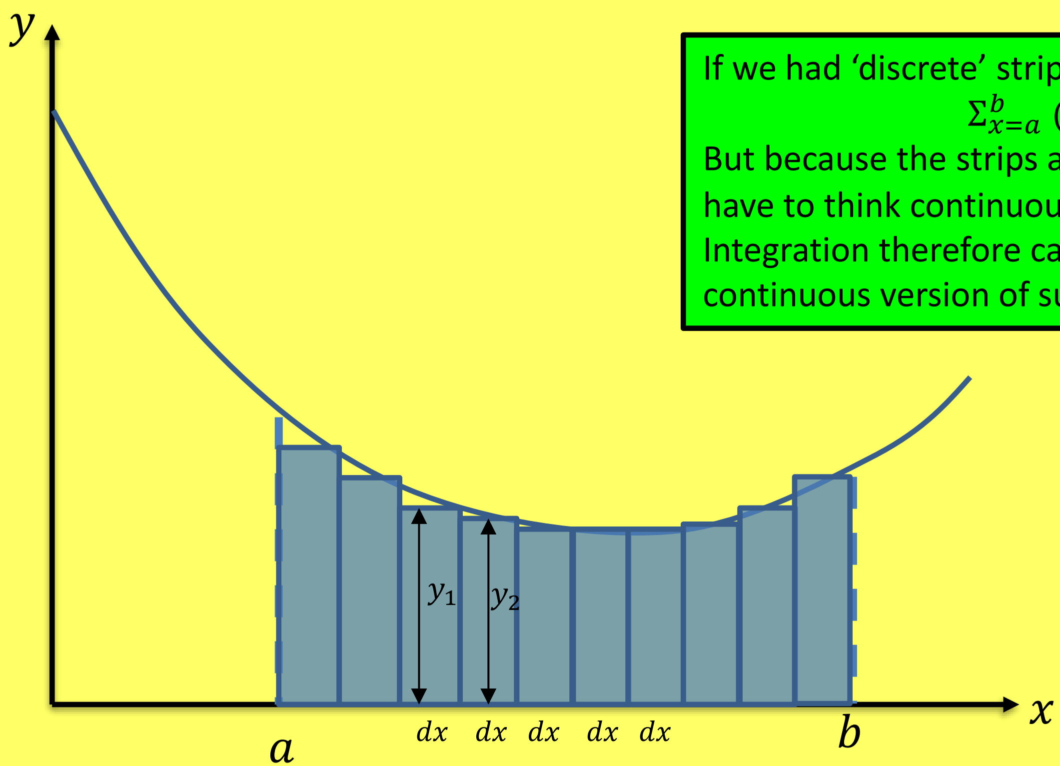


Volumes of Revolution

Area under a graph

$\int_a^b y\, dx$ gives the area bounded between $y = f(x)$, $x = a$, $x = b$ and the x-axis. Why?

If we split up the area into thin rectangular strips, each with width dx and each with height the $y = f(x)$ for that particular value of x. Each has area $f(x) \times dx$.



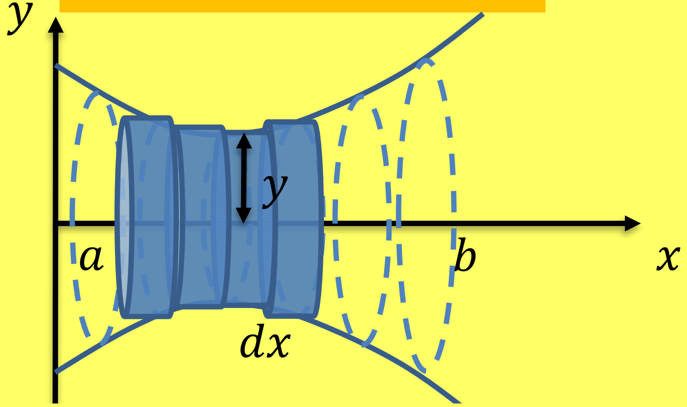
If we had 'discrete' strips, the total area would be:

$$\Sigma_{x=a}^{b} (f(x)\, dx)$$

But because the strips are infinitely small and we have to think continuously, we use $\int$ instead of Σ. Integration therefore can be thought of as a continuous version of summation.

Volumes of Revolution

Volume of Revolution (about x-axis)



So the volume of each thin disc (cylinder) is

$$dV = \pi y^2 dx.$$

Integrating this from a to b gives the volume of the 3D shape:

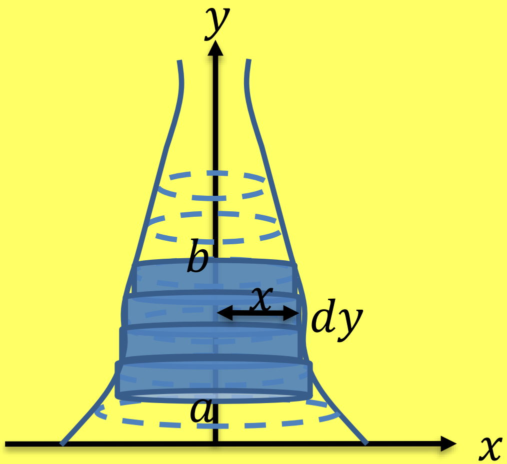
If the curve is revolved around x-axis:

$$V = \pi \int_a^b y^2 \, dx$$

Gives the **volume** we'd create if we spun the line $y = f(x)$ about the x axis to form a solid.

Curves are rotated through 360° or 2π radians

Volume of Revolution (about y-axis)



$$dV = \pi x^2 dy$$

If the curve is revolved around x-axis:

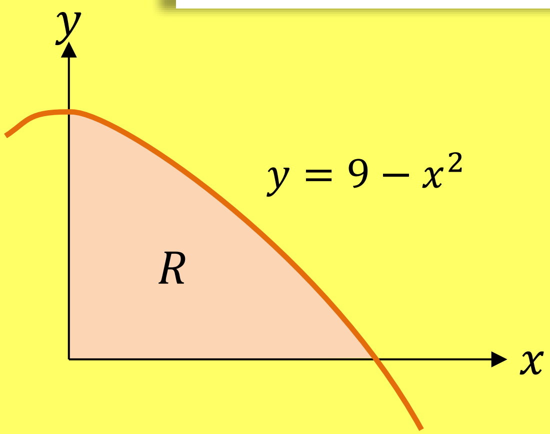
$$V = \pi \int_a^b x^2 \, dy$$

Gives the **volume** we'd create if we spun the line $x = f(y)$ about the x axis to form a solid.

It's just the area of a circle (size variable with position), stretched along an axis from a to b to produce a solid "irregular cylinder". Analogy: Adding a slice at a time like a 3D printer

Revolving around the x-axis

E.g. 1 The diagram shows the region R which is bounded by the x-axis, the y-axis and the curve with equation $y = 9 - x^2$. The region is rotated through $360°$ about the x-axis. Find the **exact** volume of the solid generated.



$$\boldsymbol{V = \pi \int_a^b y^2 \, dx}$$

Find roots first:

$9 - x^2 = 0 \quad \rightarrow \quad x = \pm 3$

Volume:

$$V = \pi \int_0^3 (9 - x^2)^2 \, dx \quad = \pi \int_0^3 81 - 18x^2 + x^4 \, dx$$

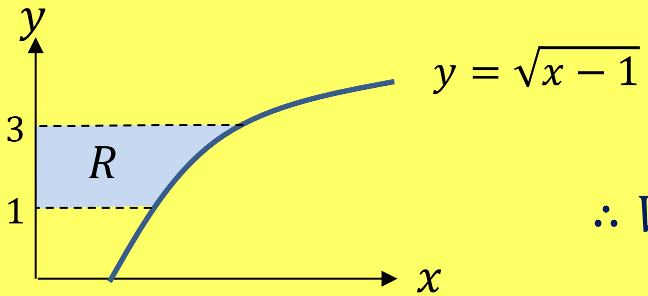
$$= \pi \left[81x - 6x^3 + \frac{1}{5} x^5 \right]_0^3$$

$$= \pi \left(\frac{648}{5} - 0 \right) = \boldsymbol{\frac{648\pi}{5}}$$

Revolving around the y-axis

E.g. 2

The diagram shows the curve with equation $y = \sqrt{x-1}$.
The region R is bounded by the curve, the y-axis and the lines $y = 1$ and $y = 3$.
The region is rotated through 2π radians about the y-axis. Find the volume of the solid generated.



$$\boldsymbol{V = \pi \int_a^b x^2\, dy}$$

Since we're finding $\boldsymbol{\pi \int_b^a x^2\, dy}$, we need to find x in terms of y:

$$x = y^2 + 1$$

$$\therefore V = \pi \int_1^3 (y^2+1)^2\, dy$$

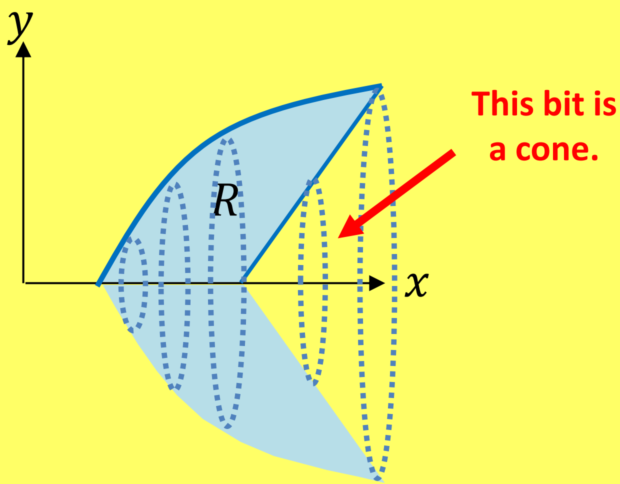
$$= \pi \int_1^3 y^4 + 2y^2 + 1\, dy$$

$$= \pi \left[\frac{1}{5}y^5 + \frac{2}{3}y^3 + y\right]_1^3$$

$$= \pi \left\{\left(\frac{243}{5} + \frac{54}{3} + 3\right) - \left(\frac{1}{5} + \frac{2}{3} + 1\right)\right\}$$

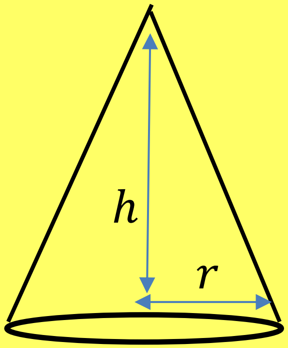
$$= \boldsymbol{\frac{1016\pi}{15}}$$

Adding and Subtracting Volumes

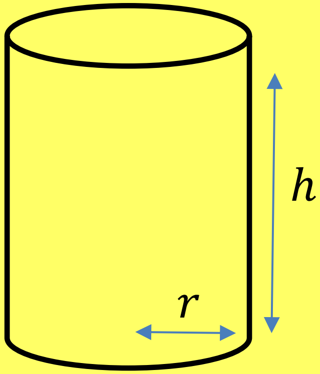


Suppose we wanted to revolve the following area around the x-axis.
What strategy might we use to find the volume of this resulting solid?
Find the volume of revolution for the top curve. Then cut out (subtract) the cone.

Reminders:



$$V = \frac{1}{3}\pi r^2 h$$

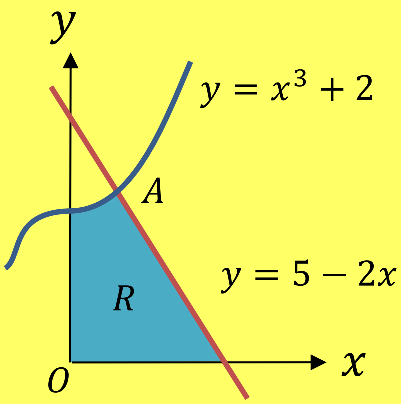


$$V = \pi r^2 h$$

Adding Volumes

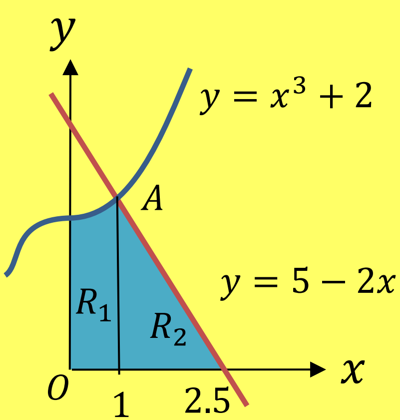
E.g. 3

The region R is bounded by the curve with equation $y = x^3 + 2$, the line $y = 5 - 2x$ and x and y-axes.
(a) Verify that the coordinates of A are $(1,3)$.
A solid is created by rotating the region $360°$ about the x-axis.
(b) Find the volume of this solid.



$1^3 + 2 = 3, \quad 5 - 2(1) = 3 \quad ✓$

Find the two volumes separately:



$$V_1 = \pi \int_0^1 (x^3 + 2)^2 \, dx = \pi \int_0^1 x^6 + 4x^3 + 4 \, dx$$

$$= \pi \left| \frac{1}{7}x^7 + x^4 + 4x \right|_0^1 = \frac{36\pi}{7}$$

$5 - 2x$ intersects the x-axis at 2.5

$$V_2 = \frac{1}{3}\pi \times 3^2 \times 1.5 = \frac{9\pi}{2}$$

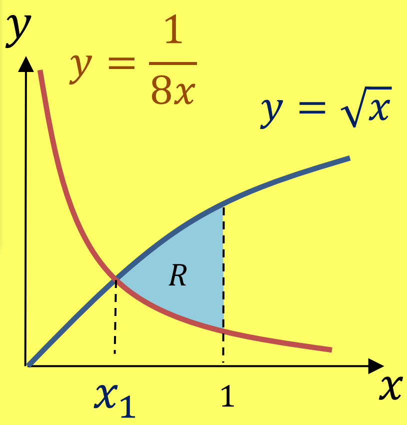
If we revolve R_2 we get a cone of radius 3 and height 1.5

$$V_{overall} = \frac{36\pi}{7} + \frac{9\pi}{2} = \boldsymbol{\frac{135\pi}{14}}$$

Subtracting Volumes

E.g. 4

The diagram shows the region R bounded by the curves with equations $y = \sqrt{x}$ and $y = \frac{1}{8x}$ and the line $x = 1$.
The region is rotated through $360°$ about the x-axis.
Find the exact volume of the solid generated.



Do volume under top curve and subtract volume under bottom curve.

Point of intersection:

$$\frac{1}{8x} = \sqrt{x} \quad \rightarrow \quad 8x^{\frac{3}{2}} = 1 \quad \rightarrow \quad x_1 = \frac{1}{4}$$

$$V_1 = \pi \int_{\frac{1}{4}}^{1} (\sqrt{x})^2 \, dx = \pi \int_{\frac{1}{4}}^{1} x \, dx = \pi \left[\frac{1}{2}x^2\right]_{\frac{1}{4}}^{1} = \pi\left(\frac{1}{2} - \frac{1}{32}\right) = \frac{15\pi}{32}$$

$$V_2 = \pi \int_{\frac{1}{4}}^{1} \left(\frac{1}{8x}\right)^2 \, dx = \frac{\pi}{64} \int_{\frac{1}{4}}^{1} x^{-2} dx = -\frac{\pi}{64}\left[x^{-1}\right]_{\frac{1}{4}}^{1} = -\frac{\pi}{64}(1-4) = \frac{3\pi}{64}$$

$$V_R = \frac{15\pi}{32} - \frac{3\pi}{64} = \boldsymbol{\frac{27\pi}{64}}$$

Volumes of Revolution with harder integration

When revolving around the x-axis, $\boldsymbol{V = \pi \int_b^a y^2 \, dx}$

E.g. 5

The region R is bounded by the curve with equation $y = \sin 2x$, the x-axis and $x = \frac{\pi}{2}$. Find the volume of the solid formed when region R is rotated through 2π radians about the x-axis.

$$V = \pi \int_0^{\frac{\pi}{2}} \sin^2 2x \; dx$$

$$\cos 2x = 1 - 2\sin^2 x$$
$$\therefore \cos 4x = 1 - 2\sin^2 2x$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 4x) \; dx$$

$$= \pi \left[\frac{1}{2}x - \frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{2}} \quad = \pi\left(\left[\frac{\pi}{4} - \frac{1}{8}\sin \pi\right] - [0 - \sin 0]\right) \qquad (\sin \pi = \sin 0 = 0)$$

$$= \boldsymbol{\frac{\pi^2}{4}}$$

Volumes of revolution for parametric curves

We have seen that parametric equations are where, instead of some single equation relating x and y, we have an equation for each of x and y in terms of some parameter, e.g. t. As t varies, this generates different points (x, y).

To integrate parametrically, the trick was to replace dx with $\frac{dx}{dt}\, dt$

$$\boldsymbol{V = \pi \int_{x=b}^{x=a} y^2\, dx} \quad \Rightarrow \quad \boldsymbol{V = \pi \int_{t=q}^{t=p} y^2 \frac{dx}{dt}\, dt}$$

Note that as we're integrating with respect to t now, we need to find the equivalent limits for t.
We can do the same for revolving around the y-axis: just replace dy with $\frac{dy}{dt}$ and change the limits.

E.g. 6

The curve C has parametric equations $x = t(1+t)$, $y = \frac{1}{1+t}$, $t \geq 0$.
The region R is bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.
Find the exact volume of the solid formed when R is rotated 2π radians about the x-axis.

$$\frac{dx}{dt} = 1 + 2t$$

Bounds: When $x = 0 \Rightarrow t = 0$ and when $x = 2 \Rightarrow t = 1$

$$\therefore V = \pi \int_0^1 \frac{1}{(1+t)^2} \times (1+2t)\, dt$$

We have partial fractions: $\frac{1+2t}{(1+t)^2} = \frac{A}{1+t} + \frac{B}{(1+t)^2}$
$\Rightarrow \cdots \Rightarrow A = -1, B = 2$

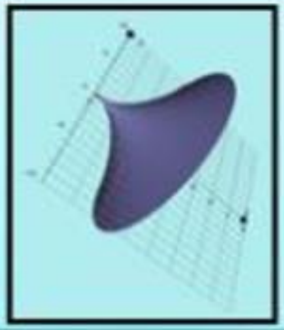
$$\therefore V = \pi \int_0^1 \left(\frac{2}{1+t} - \frac{1}{(1+t)^2} \right) dt = \pi \left[2\ln|1+t| + \frac{1}{1+t} \right]_0^1 = \pi \left[\left(2\ln 2 + \frac{1}{2} \right) - (1\ln 1 + 1) \right]$$

$$= \boldsymbol{\pi \left(2\ln 2 - \frac{1}{2} \right)}$$

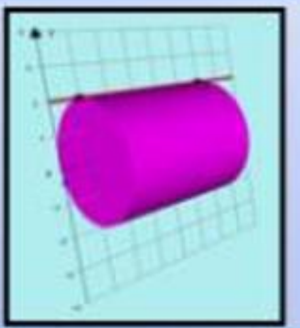
Diagnostic Question

If the following function is rotated 2π radians about the x-axis, the volume generated would look like...

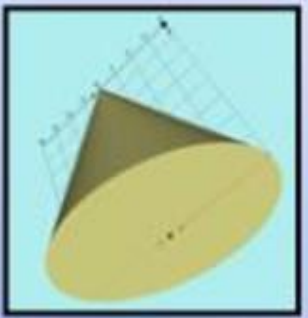
Y



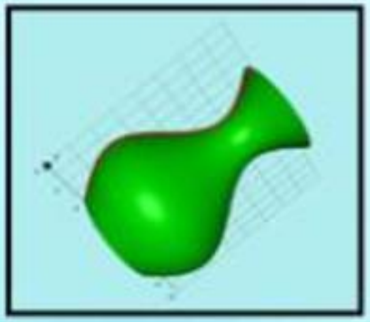
M



C

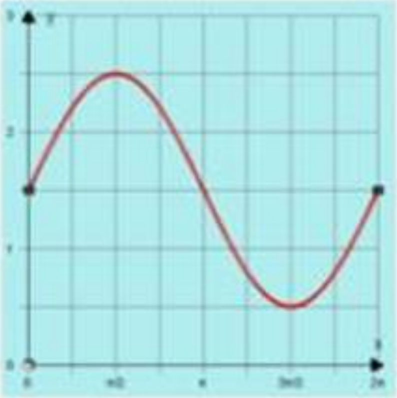


A

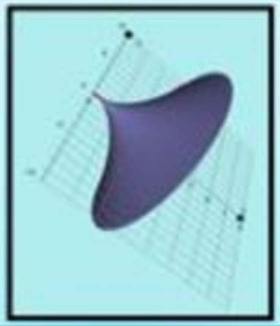


Diagnostic Question

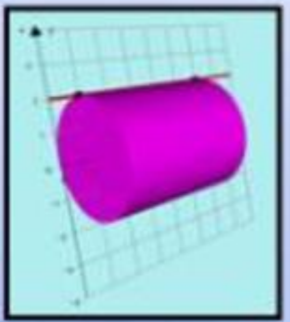
If the following function is rotated 2π radians about the x-axis, the volume generated would look like…



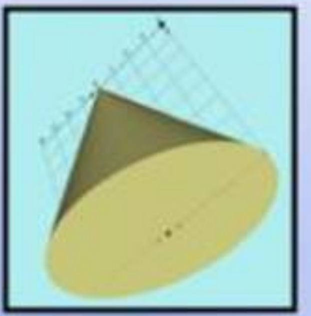
Y



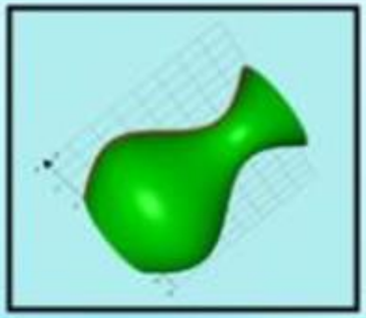
M



C



A



Diagnostic Question

Which integral gives the volume of the shape created when the curve $y = x^2$ with $0 \leq x \leq 5$ is rotated 2π radians around the x-axis.

Y $\pi \int_0^{2\pi} x^2 \, dx$

M $\pi \int_0^{2\pi} x^4 \, dx$

C $\pi \int_0^{5} x^2 \, dx$

A $\pi \int_0^{5} x^4 \, dx$

Diagnostic Question

Find the exact volume of the shape created when the curve
$f(x) = 10 - 2x$ with $0 \leq x \leq 2$
is rotated 2π radians around the x-axis.

Y 16π

M $\dfrac{392\pi}{3}$

C 57.5

A $\dfrac{44\pi}{3}$

Diagnostic Question

Find the exact volume of the shape created when the curve
$x = y^3$ with $1 \leq y \leq 2$
is rotated 2π radians around the y-axis.

Y 15π

M $\dfrac{15\pi}{4}$

C $\dfrac{127\pi}{7}$

A $\dfrac{7\pi}{3}$

Thanks

See you in the Tutorial!