CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

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Level 1 Semester 2

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1^{st} Order ODEs 2: Other Methods
Outline of Lecture 5

- Reverse product rule
- Integrating factor
- Substitution method (Homogeneous)
- Bernoulli differential equations

1st Order ODEs:
Reverse Product Rule

Using reverse product rule

We will see in a bit how to solve equations of the form $\frac{dy}{dx} + P(x)y = Q(x)$ (where P and Q are functions of x only).

We'll practice a particular part of this method before doing the whole thing.

Find general solution of the equation $x^3 \frac{dy}{dx} + 3x^2 y = x$

We can't separate the variables. But do you notice anything about the LHS?

It's $\boldsymbol{\frac{d}{dx}(x^3 y)}$!

$$\frac{d}{dx}(x^3 y) = x$$

Integrating both sides:

$$x^3 y = \frac{x^2}{2} + C$$

so:

$$\boldsymbol{y = \frac{1}{2x} + \frac{C}{x^3}}$$

What we are doing is using the product rule backwards so that both sides can be easily integrated

Examples: Reverse Product Rule

E.g. 1 Solve the equation

$$\ln x \frac{dy}{dx} + \frac{y}{x} = x^2$$

Using the previous point, this can be rewritten as:

$$\frac{d}{dx}(y \ln x) = x^2$$

Integrating both sides:

$$y \ln x = \frac{x^3}{3} + C$$

so:

$$\boldsymbol{y = \frac{x^3}{3 \ln x} + \frac{C}{\ln x}}$$

E.g. 2 Solve the equation

$$\sin t \frac{dx}{dt} + x \cos t = \cos t$$

Using the previous point, this can be rewritten as:

$$\frac{d}{dt}(x \sin t) = \cos t$$

Integrating both sides:

$$x \sin t = \sin t + C$$

so:

$$\boldsymbol{x = 1 + C \operatorname{cosec} t}$$

Diagnostic Question

Reduce the LHS to a product rule:

$$2xy\frac{dy}{dx} + y^2 = \cos x$$

Y $\frac{d}{dx}(2xy^3) = \cos x$

M $\frac{d}{dx}(xy^2) = \cos x$

C $\frac{d}{dx}(x^2y) = \cos x$

A $\frac{d}{dx}(2x^2y^2) = \cos x$

Diagnostic Question

Find the general solution of the differential equation:

$$\frac{d}{dx}(x^2y) = x^3$$

Y $y = \dfrac{x^4}{4} + C$

M $2xy + x^2\dfrac{dy}{dx} = x^3$

C $y = \dfrac{x^2}{4} + \dfrac{C}{x^2}$

A $y = \dfrac{x^2}{4}$

Diagnostic Question

Find the general solution of the differential equation:

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$$

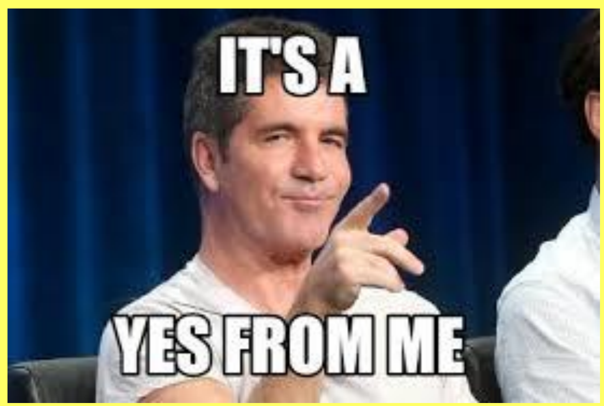
Y $y = x(e^x + C)$

M $y = xe^x + C$

C $y = x^2e^x + C$

A $y = x^2(e^x + C)$

$$e^{\int P(x)\,dx}$$



1st Order ODEs:
Integrating Factor

Linear First Order Differential Equations

The equation

$$x^2 \frac{dy}{dx} + 3xy = x^3$$

Is not a product rule, but we can make it so by multiplying by x (in this case):

$$x^3 \frac{dy}{dx} + 3x^2 y = x^4$$

Which can be rewritten:

$$\frac{d}{dx}(x^3 y) = x^4 \qquad \Rightarrow x^3 y = \frac{x^5}{5} + C$$

$$y = \frac{x^2}{5} + \frac{C}{x^3}$$

The function we multiplied the differential equation by to make it exact is called the **integrating factor**.

Integrating Factor (To transform to product rule)

A linear 1st order differential equation in **standard form** is:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Where the coefficient of the derivative is 1 and P and Q are functions of x only (do not depend on y).

E.g. 1

Find the general solution of $\frac{dy}{dx} - 4y = e^x$

We can multiply through by the integrating factor.

$$\boldsymbol{I = e^{\int P(x)dx}}$$

This then produces an equation where we can use the previous reverse-product-rule trick (we'll prove this shortly).

$$I = e^{\int -4\,dx} = e^{-4x}$$

We don't need $+C$ in integrating factor (explained shortly)

Then multiplying through by the integrating factor:

$$\boldsymbol{e^{-4x}}\frac{dy}{dx} - 4\boldsymbol{e^{-4x}}y = \boldsymbol{e^{-4x}}e^x$$

or

$$e^{-4x}\frac{dy}{dx} - 4e^{-4x}y = e^{-3x}$$

Then use reverse product rule on LHS:

$$\frac{d}{dx}\left(ye^{-4x}\right) = e^{-3x}$$

Integrate: $ye^{-4x} = \int e^{-3x}dx$

$$ye^{-4x} = -\frac{1}{3}e^{-3x} + C$$

Divide by I: $\boldsymbol{y = -\frac{1}{3}e^x + Ce^{4x}}$

Technicality: Why don't we $+C$ within I?

As we know, the integrating factor is given by: $\boldsymbol{e^{\int P(x)dx}}$

Which in a previous example became: $I = e^{\int -4\,dx} = e^{-4x}$

But why wasn't it $\boldsymbol{e^{-4x+C}}$?

Using laws of indices, this could also be written as $\boldsymbol{e^{-4x+C} = e^{-4x}e^{C} = Ae^{-4x}}$

As we multiply the whole equation: $$\boldsymbol{Ae^{-4x}}\frac{dy}{dx} - 4\boldsymbol{Ae^{-4x}}y = \boldsymbol{Ae^{-4x}}e^{x}$$

So the As cancel and it is the same as just multiplying by e^{-4x} and we need not have bothered… so we don't as it saves time.

$$\boldsymbol{e^{-4x}}\frac{dy}{dx} - 4\boldsymbol{e^{-4x}}y = \boldsymbol{e^{-4x}}e^{x}$$

Proof that Integrating Factor works

Solve the general equation $\frac{dy}{dx} + P(x)y = Q(x)$

Suppose $f(x)$ is the Integrating Factor. As usual we'd multiply by it:

$$f(x)\frac{dy}{dx} + f(x)P(x)y = f(x)Q(x)$$

If we can use the reverse product rule trick on the LHS, then it would be of the form:

$$f(x)\frac{dy}{dx} + f'(x)y$$

Thus comparing the coefficients of the two LHSs:

$$f'(x) = f(x)P(x)$$

Dividing by $f(x)$ and integrating:

$$\int \frac{f'(x)}{f(x)}\,dx = \int P(x)\,dx$$

$$\ln|f(x)| = \int P(x)\,dx$$

$$f(x) = e^{\int P(x)\,dx}$$

Which is the integrating factor!

When there's something on front of the dy/dx

E.g. 2

Find the general solution of $\cos x \frac{dy}{dx} + 2y\sin x = \cos^4 x$

Remember, "standard form" means that the coefficient of $\frac{dy}{dx}$ is 1

STEP 1: Divide by anything on front of dy/dx

$$\frac{dy}{dx} + 2y\tan x = \cos^3 x$$

STEP 2: Determine integrating factor, I

$$I = e^{\int 2\tan x\, dx} \quad = e^{2\ln\sec x} \quad = e^{\ln\sec^2 x} \quad = \sec^2 x$$

STEP 3: Multiply through by I and use product rule backwards.

$$\sec^2 x\frac{dy}{dx} + 2y\sec^2 x\tan x = \cos x$$

$$\frac{d}{dx}(y\sec^2 x) = \cos x$$

Check:

$$\frac{d}{dx}(y\sec^2 x)$$
$$= \frac{dy}{dx}\times\sec^2 x + y\times 2\sec x\,(\sec x\tan x)$$
$$= \sec^2 x\frac{dy}{dx} + 2y\sec^2 x\tan x$$

STEP 4: Integrate and simplify.

$$y\sec^2 x = \int \cos x\, dx$$

$$y\sec^2 x = \sin x + C$$

STEP 5: Divide by I

$$\boldsymbol{y = \cos^2 x\,(\sin x + C)}$$

Summary of Process for Integrating Factor

A linear 1st order ODE in standard form (coefficient of dy/dx is 1, divide if not):

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Has integrating factor:

$$I = e^{\int P(x)\,dx}$$

To solve, multiply equation by I:

$$e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}P(x)y = e^{\int P(x)\,dx}Q(x)y$$

or:

$$I\frac{dy}{dx} + I\,P(x)y = I\,Q(x)$$

The original equation can now <u>always</u> be written as:

$$\boldsymbol{\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) = Q(x)e^{\int P(x)\,dx}}$$

$$\frac{d}{dx}(yI) = Q(x)I$$

Both sides can now be directly integrated (provided the RHS is integrable).

$$\boldsymbol{ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}}$$

$$yI = \int Q(x)I\ dx$$

Finally, to find y, divide by I:

$$\boldsymbol{y = \frac{1}{e^{\int P(x)\,dx}}\int Q(x)e^{\int P(x)\,dx}}$$

$$y = \frac{1}{I}\int Q(x)I\ dx$$

Diagnostic Question

Find the integrating factor of the differential equation

$$x\frac{dy}{dx} + 2xy = xe^{-2x}$$

Giving your answer in its simplest form.

Y $I = 2x$

M $I = e^{x^2}$

C $I = e^{2x}$

A $I = e^{-2x}$

Diagnostic Question

Find the general solution of the differential equation

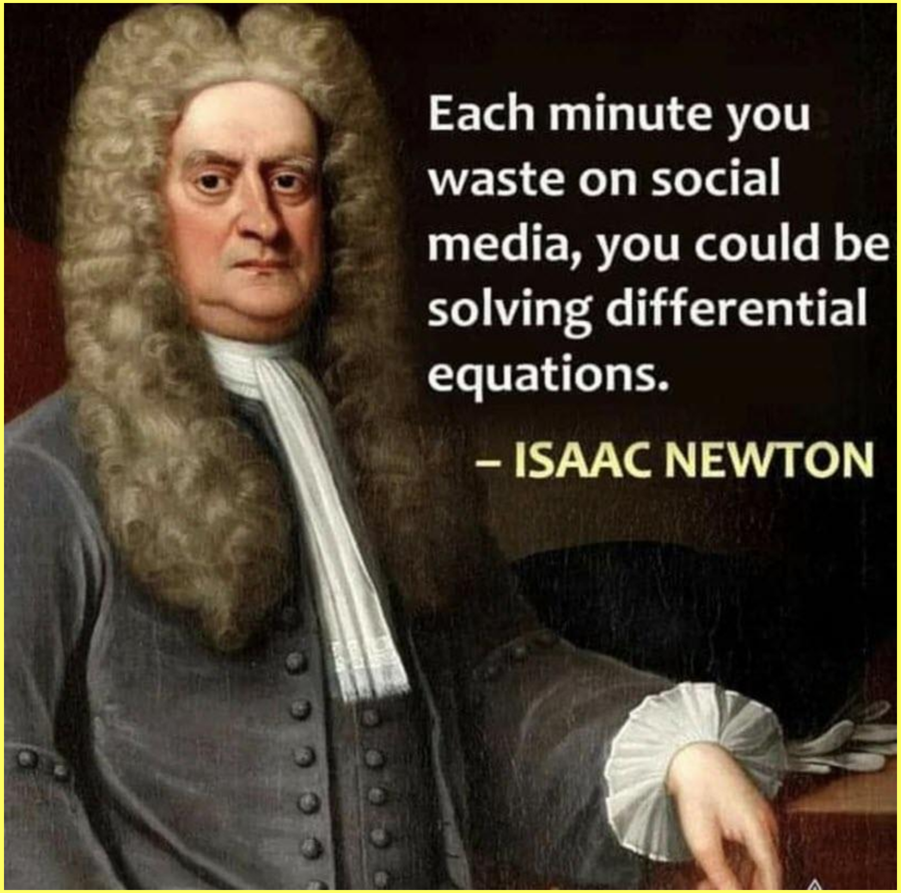
$$x\frac{dy}{dx} + 2xy = xe^{-2x}$$

Y $y = xe^{-2x}$

M $y = (x + C)e^{-2x}$

C $y = (x + C)e^{2x}$

A $y = \frac{1}{2}(x + C)e^{-2x}$



1^{st} Order ODEs: Substitution Method (Homogeneous)

Definition: Homogeneous Differential Equations

A Homogeneous differential equation is of the form:

$$\frac{dy}{dx} = f(x,y)$$

Where $f(x,y)$ is a homogeneous function of degree n.
That is $f(kx,ky) = k^n f(x,y)$ for any non-zero constant k.

<u>Less formal:</u>
For a homogeneous function, the sum of the powers of x and y in each term must be the same
e.g. xy^2 and x^2y
(sum of powers in each is 3)

Example: Check whether the following differential equations are homogeneous:

$$\boldsymbol{\frac{dy}{dx} = \frac{xy - y^2}{2x^2 + 3xy}}$$

$$f(x,y) = \frac{xy - y^2}{2x^2 + 3xy}$$

$$f(kx,ky) = \frac{(kx)(ky) - (ky)^2}{2(kx)^2 + 3(kx)(ky)}$$

$$= \frac{k^2xy - k^2y^2}{2k^2x^2 + 3k^2xy}$$

$$= \frac{k^2(xy - y^2)}{k^2(2x^2 + 3xy)}$$

$$= f(x,y)$$

This is the form we will solve shortly

Homogeneous (degree 0)

$$\boldsymbol{\frac{dy}{dx} = \frac{x^3 + y^3}{2x - y}}$$

$$f(x,y) = \frac{x^3 + y^3}{2x - y}$$

$$f(kx,ky) = \frac{(kx)^3 + (ky)^3}{2(kx) - ky}$$

$$= \frac{k^3x^3 + k^3y^3}{2kx - ky}$$

$$= \frac{k^3(x^3 + y^3)}{k(2x - y)}$$

$$= k^2 f(x,y)$$

Homogeneous (degree 2)

$$\boldsymbol{\frac{dy}{dx} = \frac{x^3 + y^2}{x^2 + y^3}}$$

$$f(x,y) = \frac{x^3 + y^2}{x^2 + y^3}$$

$$f(kx,ky) = \frac{(kx)^3 + (ky)^2}{(kx)^2 + (ky)^3}$$

$$= \frac{k^3x^3 + k^2y^2}{k^2x^2 + k^3y^3}$$

$$\neq k^n f(x,y)$$

Not homogeneous

Important distinction

The word “homogeneous” can also be used to describe a differential equation in the form $Ly = 0$, where L is a linear differential operator.

An example of such a homogeneous equation is:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

We will see this in week 7

The different types of homogeneous equation are entirely separate entities, and it is important not to confuse the two.

Homogeneous or not?

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

Homogeneous or not?

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$$

Homogeneous or not?

$$x\frac{dy}{dx} = \frac{x^5 - y^5}{2y}$$

Homogeneous or not?

$$\frac{dy}{dx} = \frac{x^2 - y}{x^2 + y}$$

Homogeneous or not?

$$y\frac{dy}{dx} = \frac{3x + 2y}{xy - x^2}$$

Homogeneous or not?

$$\frac{dy}{dx} = \frac{x^2y - xy^2}{3x^{-2}y^2 + 1}$$

Homogeneous or not?

$$\frac{dy}{dx} = \frac{1 + \sin x}{1 - \cos x}$$

Homogeneous or not?

$$\frac{dy}{dx} = \sin\frac{x}{y}$$

Solving: Substitution method

We now introduce a method to solve Homogeneous differential equation is of the form:

$$\frac{dy}{dx} = f(x,y)$$

Where $f(x,y)$ is a homogeneous function of degree 0.

That is $\boldsymbol{f(kx,ky) = f(x,y)}$ for any non-zero constant k.

- We can often solve homogeneous differential equations by making the substitution, $v = \frac{y}{x}$, where $v = v(x)$ is a function of x.

Basically, this method transforms a differential equation for which we cannot separate variables into one that we can.

- Rearranging gives $y = xv$
- We can now rewrite: $\frac{dy}{dx} = \frac{d(xv)}{dx}$
- Which by product rule becomes: $\frac{dy}{dx} = v + x\frac{dv}{dx}$ (or $y' = v + xv'$)
- We can rewrite $f(x,y) = g\left(\frac{y}{x}\right) = g(v)$
- Therefore $v + x\frac{dv}{dx} = g(v) \Rightarrow \int \frac{1}{g(v)-v}dv = \int \frac{1}{x}dx$ (separate variables)
- Integrating gives a solution for v, and substituting into $v = \frac{y}{x}$ gives the solution for y.

Example: Homogeneous differential Equations

E.g. 1

Find the general solution of the differential equation:

$$x\frac{dy}{dx} = y + xe^{\frac{y}{x}}$$

First rewrite in the form $\frac{dy}{dx} = f(x,y)$

$$\boldsymbol{\frac{dy}{dx} = \frac{y}{x} + e^{\frac{y}{x}}}$$

We can't separate variables or rewrite in standard form, so we need to check if it is homogeneous with $\boldsymbol{f(kx,ky)=f(x,y)}$

Check if $\boldsymbol{f(kx,ky) = f(x,y)}$

$$f(x,y) = \frac{y}{x} + e^{\frac{y}{x}} \Rightarrow f(kx,ky) = \frac{ky}{kx} + e^{\frac{ky}{kx}} = f(x,y)$$

As $\boldsymbol{f(kx,ky) = f(x,y)}$ we can proceed with the substitution

Substitute $v = \frac{y}{x}$, make y the subject, find $\frac{dy}{dx}$.

$$v = \frac{y}{x} \rightarrow y = xv \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Sub in y and $\frac{dy}{dx}$ into the homogeneous equation (eliminate y)

$$v + x\frac{dv}{dx} = v + e^{v} \quad \Rightarrow x\frac{dv}{dx} = e^{v}$$

Separate variables or put in standard form.

$$\int e^{-v}\,dv = \int \frac{1}{x}\,dx$$

Example: Homogeneous differential Equations

Separate variables or put in standard form.

$$\int e^{-v} dv = \int \frac{1}{x} dx$$

Integrate:

$$-e^{-v} = \ln x + C_1$$

Rearrange for v

$$e^{-v} = -\ln x + C$$

$$-v = \ln(-\ln x + C)$$

$$v = -\ln(C - \ln x)$$

Sub back $v = \frac{y}{x}$ in to get general solution for y.

$$\frac{y}{x} = -\ln(C - \ln x)$$

$$\boldsymbol{y = -x\ln(C - \ln x)}$$

Note: Using $C = \ln A$, we could rewrite in a different form:
$y = -x\ln(\ln A - \ln x) = -x\ln\left(\ln\left(\frac{A}{x}\right)\right)$

Example: Homogeneous differential Equations

E.g. 2

Find the particular solution of the differential equation:

$$x\frac{dy}{dx} = x - y \quad \text{When } y(2) = \frac{1}{2}$$

First, rewrite in the familiar form:

$$\boldsymbol{\frac{dy}{dx} = \frac{x-y}{x}}$$

Check if $\boldsymbol{f(kx, ky) = f(x, y)}$

$$f(x,y) = \frac{x-y}{x} \Rightarrow f(kx,ky) = \frac{kx-ky}{kx} = \frac{k(x-y)}{kx} = f(x,y)$$

As $\boldsymbol{f(kx, ky) = f(x, y)}$ we can proceed with the substitution method

Substitute $v = \frac{y}{x}$, make y the subject, find $\frac{dy}{dx}$.

$$v = \frac{y}{x} \quad \to \quad y = xv \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Sub in y and $\frac{dy}{dx}$ into the homogeneous equation (eliminate y)

$$v + x\frac{dv}{dx} = \frac{x - xv}{x} \quad = 1 - v$$

Separate variables or put in standard form.

$$x\frac{dv}{dx} = 1 - 2v \quad \Rightarrow \int \frac{1}{1-2v}\,dv = \int \frac{1}{x}\,dx$$

Example: Homogeneous differential Equations

Integrate

$$-\frac{1}{2}\ln(1-2v) = \ln x + C$$

Rearrange for v

$$\ln(1-2v)^{-\frac{1}{2}} = \ln x + C \Rightarrow (1-2v)^{-\frac{1}{2}} = \mathrm{e}^{\ln x + C} \Rightarrow \frac{1}{\sqrt{1-2v}} = Ax$$

$$\Rightarrow \frac{1}{1-2v} = Bx^2 \qquad \Rightarrow v = \frac{1}{2}\left(1 - \frac{1}{Bx^2}\right)$$

Sub back $v = \frac{y}{x}$ in to get general solution for y.

$$\frac{y}{x} = \frac{1}{2}\left(1 - \frac{1}{Bx^2}\right) \qquad \Rightarrow y = \frac{x}{2}\left(1 - \frac{1}{Bx^2}\right)$$

Use boundary conditions to get particular solution:

$$y(2) = \frac{1}{2} \qquad \Rightarrow \frac{1}{2} = \frac{2}{2}\left(1 - \frac{1}{4B}\right) \Rightarrow \frac{1}{2} = 1 - \frac{1}{4B} \Rightarrow \frac{1}{4B} = \frac{1}{2}$$

$$\Rightarrow 4B = 2$$

$$\Rightarrow B = \frac{1}{2}$$

$$\therefore y = \frac{x}{2}\left(1 - \frac{2}{x^2}\right) \qquad \Rightarrow \boldsymbol{y = \frac{x}{2} - \frac{1}{x}}$$

What if you can't separate variables?

- Sometimes, the substitution $y = xv$ will not lead to a separable equation.

- In that case, a different substitution, for example $v = \frac{1}{y}$ will lead to a differential equation in v that is separable or can be solved using an integrating factor.

Example: With an integrating factor

E.g. 3

Find the general solution of the differential equation:

$$x\frac{dy}{dx} + y = xy^2$$

Using the substitution $y = \frac{1}{v}$

$$y = \frac{1}{v} = v^{-1} \Rightarrow \frac{dy}{dx} = -v^{-2}\frac{dv}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$$

(by chain rule/implicit differentiation)

Substitute into original equation:

$$-\frac{x}{v^2}\frac{dv}{dx} + \frac{1}{v} = \frac{x}{v^2}$$

Multiply through by v^2:

$$-x\frac{dv}{dx} + v = x$$

Divide by $-x$:

$$\frac{dv}{dx} - \frac{1}{x}v = -1$$

(which is in standard form so we can use integrating factor)

Aside: Check what happens if we use $y = xv$:

$$\frac{dy}{dx} = y^2 - \frac{y}{x}$$

$f(x, y) = y^2 - \frac{y}{x}$ which is not homogeneous.

If we substitute $y = xv$:

$$v + x\frac{dv}{dx} = x^2v^2 - v$$

$$x\frac{dv}{dx} = x^2v^2 - 2v$$

$\frac{dv}{dx} = xv^2 - \frac{2v}{x}$ which cannot be separated or written in standard form.

Example: With an integrating factor

E.g. 3

Find the general solution of the differential equation:

$$x\frac{dy}{dx} + y = xy^2$$

Using the substitution $y = \frac{1}{v}$

$$\frac{dv}{dx} - \frac{1}{x}v = -1 \qquad I = e^{\int P(x)\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

Multiply through by the integrating factor:

$$\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2}v = -\frac{1}{x}$$

LHS can now be written as a single derivative using product rule:

$$\frac{d}{dx}\left(\frac{1}{x}v\right) = -\frac{1}{x}$$

Integrating both sides: $\frac{1}{x}v = -\int \frac{1}{x}dx \quad \Rightarrow \quad \frac{1}{x}v = -\ln x + C \quad \Rightarrow \quad v = -x\,(\ln x + C)$

Substitute back in for y, $y = \frac{1}{v} \Rightarrow v = \frac{1}{y}$:

$$\frac{1}{y} = -x\,(\ln x + C)$$

And finally,

$$\boldsymbol{y = \frac{1}{-x\,(\ln x + C)}}$$

Sometimes either will work

E.g. 4

Solve the differential equation:
$$(x+3y)dx + xdy = 0$$

There are some homogeneous equations that can be put into standard from immediately. In this case, you can use whichever method you prefer!

Rearrangement 1 : $\dfrac{dy}{dx} = -\dfrac{x+3y}{x}$

(which is homogeneous so we can **substitute**)

$y = xv$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = -1 - 3v \quad \Rightarrow x\frac{dv}{dx} = -(1+4v)$$

$$\int \frac{1}{1+4v}\,dv = -\int \frac{1}{x}\,dx$$

Tip: If possible, put negatives on the easier integral

$$\frac{1}{4}\ln(1+4v) = -\ln x + C$$

$$\ln(1+4v) = -4\ln x + C_1$$

$$1 + 4v = Ax^{-4} \quad \Rightarrow \quad v = \frac{1}{4}\left(\frac{A}{x^4} - 1\right)$$

Sub back $v = \frac{y}{x}$:

$$\boldsymbol{y = \frac{A}{4x^3} - \frac{x}{4}}$$

Rearrangement 2: $\dfrac{dy}{dx} + \dfrac{3}{x}y = -1$

(which is in standard form so we can use **integrating factor**)

$$P(x) = \frac{3}{x},\ Q(x) = -1$$

$$I = e^{\int P(x)\,dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3}$$

$$I = x^3$$

Multiply by I: $x^3\dfrac{dy}{dx} + 3x^2y = -x^3$

$$\frac{d}{dx}(x^3y) = -x^3$$

$$x^3y = -\int x^3\,dx = -\frac{x^4}{4} + C$$

$$\boldsymbol{y = -\frac{x}{4} + \frac{C}{x^3}}$$

Same result (arbitrary constants can be named anything)

Diagnostic Question

Find the general solution of the homogeneous differential equation:

$$\frac{dy}{dx} = \frac{x+y}{x}$$

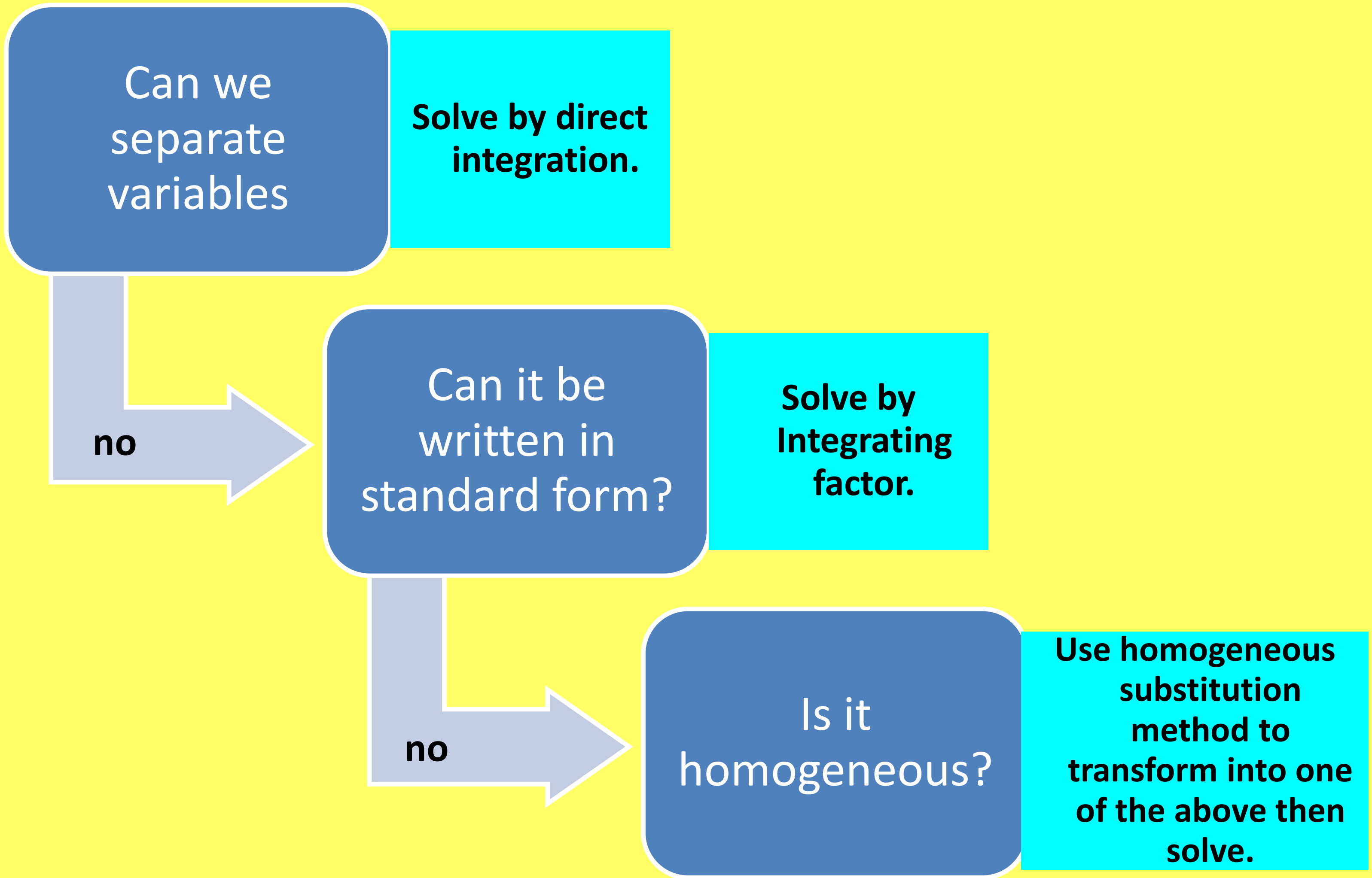
Y $\ln x + C$

M $\dfrac{x^2}{2} + C$

C $x(\ln x + C)$

A $x\left(\dfrac{x^2}{2} + C\right)$

Differential Equations: Method thought process



Summary: Integrating Factor

Integrating Factor Method:

Standard form (divide if needed):

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$I = \boldsymbol{e}^{\int \boldsymbol{P(x)\,dx}}$$

- Multiply through by I:

$$I\frac{dy}{dx} + I\,P(x)y = I\,Q(x)$$

- Reverse product rule (always) gives:

$$\frac{d}{dx}(Iy) = I\,Q(x)$$

- Integrate:

$$Iy = \int I\,Q(x)dx$$

- Divide by I:

$$y = \frac{1}{I}\int I\,Q(x)dx$$

Summary: Homogeneous Equations

Substitution Method:

A Homogeneous differential equation is of the form:

$$\frac{dy}{dx} = f(x, y)$$

Where $f(x, y)$ is a homogeneous function of degree n.
That is $f(kx, ky) = k^n f(x, y)$ for any non-zero constant k.

- Check if homogeneous (if not told)
- Substitute $y = xv$
- By product rule : $\frac{dy}{dx} = v + x\frac{dv}{dx}$
- Rewrite $f(x, y) = g\left(\frac{y}{x}\right) = g(v)$
- Now $v + x\frac{dv}{dx} = g(v) \Rightarrow \int \frac{1}{g(v)-v} dv = \int \frac{1}{x} dx$ (separate variables)
- Integrate to get v, and substitute $v = \frac{y}{x}$ to get y.

Less formal:
For a homogeneous function, the sum of the powers of x and y in each term must be the same
e.g. xy^2 and x^2y
(sum of powers in each is 3)

Thanks

See you in the Tutorial!

Extra Non-Examinable Content

Daniel Bernoulli (1700-1782)

1st Order ODEs: Bernoulli Differential Equations

Bernoulli Equations: Solving

A Bernoulli differential equation is a generalisation of the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Where $n \neq 0$.

- This form of equation is a non-linear for $n \geq 1$
- Clearly $y = 0$ is a (trivial) solution.
- Seeking other solutions, we divide by y^n:
$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$
- Let $w = y^{1-n}$, then $\frac{dw}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ using implicit differentiation (chain rule)
- Rearranging: $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dw}{dx}$
- Substitute into original differential equation:
$$\frac{1}{1-n}\frac{dw}{dx} + P(x)w = Q(x)$$

Note: This method also works for negative and fractional values of n.

- The equation has now been transformed into a linear 1st order differential equation in w which can now be solved using the integrating factor method.

Example: Bernoulli Equations

E.g. 1

Solve the differential equation:

$$\frac{dy}{dx} + \frac{2}{x}y = x^6y^3$$

This is a Bernoulli equation with $n = 3$.

Divide by y^3: $$\frac{1}{y^3}\frac{dy}{dx} + \frac{2}{xy^2} = x^6$$

Let $w = y^{1-n} = y^{-2}$ then $\frac{dw}{dx} = -2y^{-3}\frac{dy}{dx}$

Rearrange to replace 1st term: $\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dw}{dx}$

Substitute: $$-\frac{1}{2}\frac{dw}{dx} + \frac{2w}{x} = x^6$$

Multiply by -2 to get into standard form:

$$\frac{dw}{dx} - \frac{4w}{x} = -2x^6$$

Work out integrating factor:

$$I = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$$

Multiply equation by I:

$$x^{-4}\frac{dw}{dx} - 4x^{-5}w = -2x^2$$

By product rule: $$\frac{d}{dx}(x^{-4}w) = -2x^2$$

Integrate both sides: $$x^{-4}w = -\frac{2}{3}x^3 + C$$

Rearrange for w: $$w = -\frac{2}{3}x^7 + Cx^4$$

Since $w = y^{-2}$: $$\boldsymbol{y^{-2} = -\frac{2}{3}x^7 + Cx^4}$$

$$\boldsymbol{y^2 = \frac{1}{Cx^4 - \frac{2}{3}x^7} = \frac{3}{Dx^4 - 2x^7}}$$

$$\boldsymbol{y = \pm\sqrt{\frac{3}{Dx^4 - 2x^7}}}$$

Note:
Once you have substituted back for y, any form is fine.

Summary: Bernoulli Equations

Bernoulli Equations:

A Bernoulli differential equation is a generalisation of the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Where $n \neq 0$.

- Divide by y^n (multiply by y^{-n}):

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

- Let $w = y^{1-n}$, then $\frac{dw}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

- Rearrange: $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dw}{dx}$

- Substitute into original differential equation:

$$\frac{1}{1-n}\frac{dw}{dx} + P(x)w = Q(x)$$

- The equation is now a linear 1st order differential equation in w.
- Solve using integrating factor method.