UNIVERSITY OF LEEDS

CAPE1150

Engineering Mathematics

School of Chemical and Process Engineering

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Level 1 Semester 2

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Synopsis of Lecture: Vectors 2 (Lines & Planes)

Straight Lines in 3 Dimensions

- Vector equation of a line
- Angles & Intersections
- Cartesian equation of a line

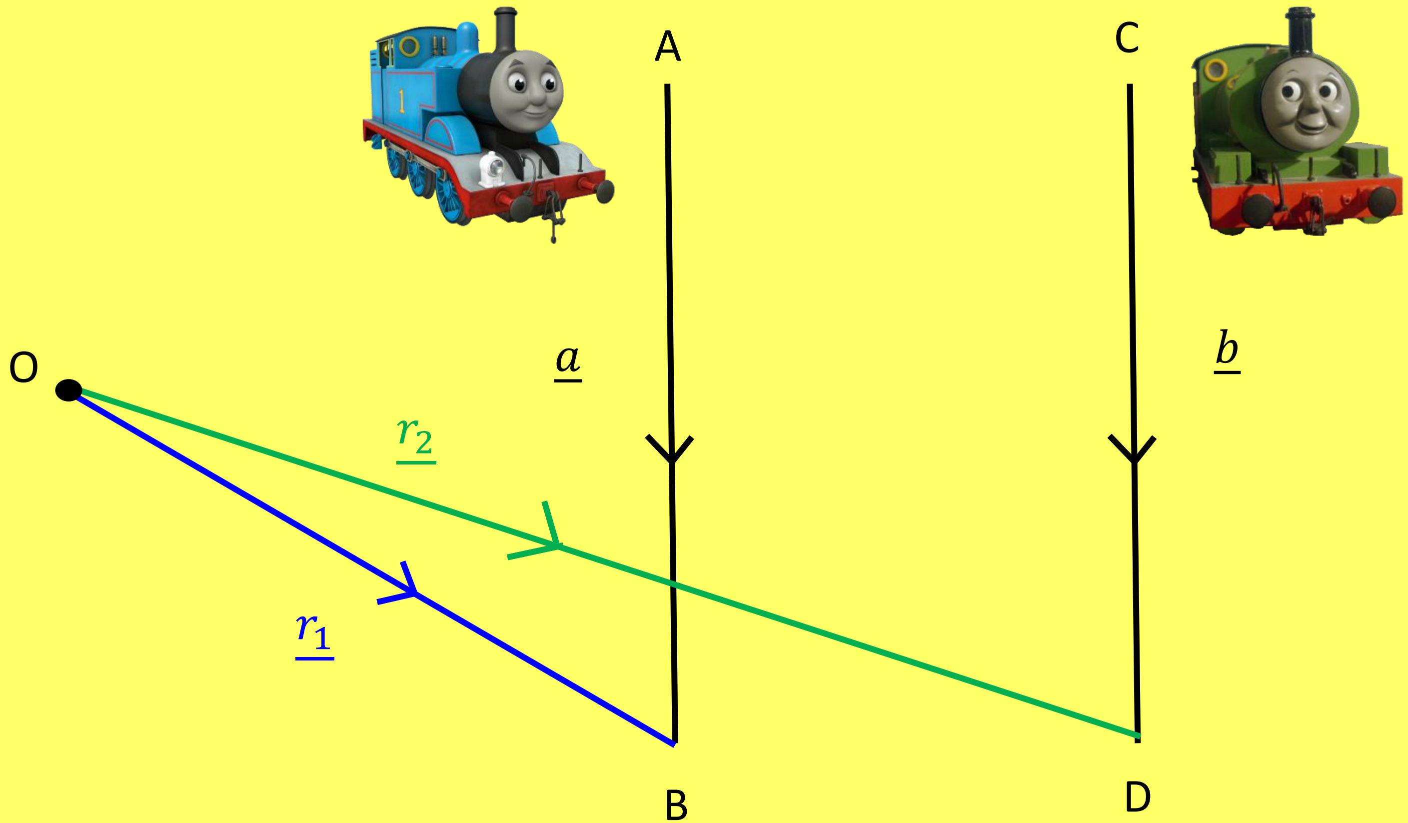
Planes

- Equations of a plane (parametric form, vector form, Cartesian, form)

Straight Lines in 3 Dimensions

Position Vs. Direction (Displacement) Vectors

Consider two trains running on parallel tracks for the same distance

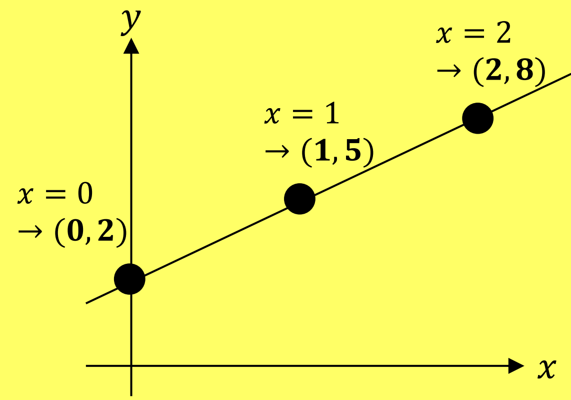


- The **direction** (displacement) vectors of the two trains are identical - same magnitude and direction ($\overrightarrow{AB} = \overrightarrow{CD}$ or $\underline{a} = \underline{b}$).
- The **position** vectors of the two trains are not the same ($\overrightarrow{OB} \neq \overrightarrow{OD}$ or $\underline{r_1} \neq \underline{r_2}$). (position vectors always begin at the origin).

Vector equation of a straight line in 3D

Consider the equation of a straight line in 2D:

$$y = 3x + 2$$

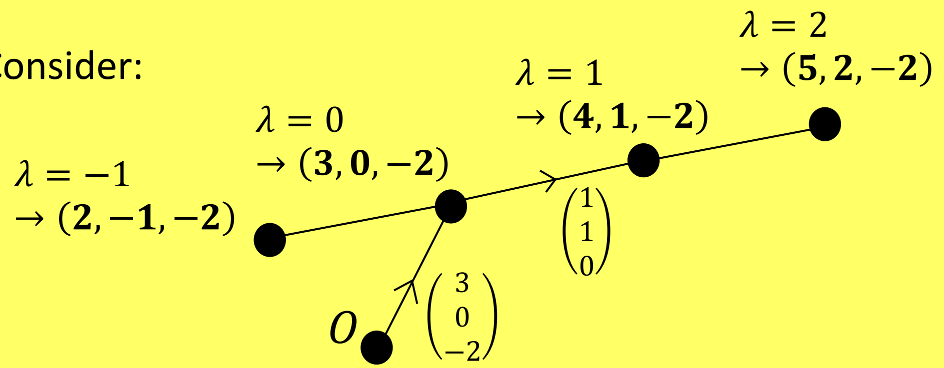


x is obviously a variable (i.e. it can vary!). As we consider different values of x, we get different points on the line.

In $y = mx + c$, while x and y are variables, m and c are **constants**: after these are set for a particular line, they don't change.

Can we do something similar with vectors? Consider:

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$



What happens as we vary λ?

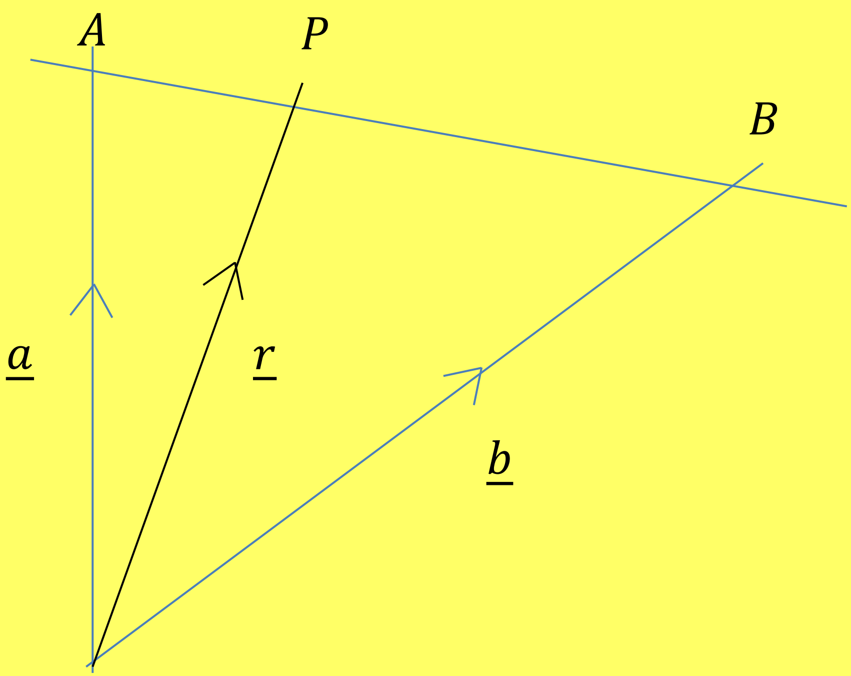
Therefore what was the role of:

$\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$: **Position vector** of some arbitrary point on the line (it doesn't matter which).

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$: The **direction** of the line.

Vector Equation of a Line

Consider a line through two points A and B with position vectors $\underline{a}$ and $\underline{b}$ respectively.
Let P with position vector $\underline{r}$ be any point on the line.



The line is parallel to $\overrightarrow{AB}$, where $\overrightarrow{AB} = \underline{b} - \underline{a}$

Therefore, the direction vector of the line is $\underline{d} = \underline{b} - \underline{a}$

So the vector equation of the line is:

$$\underline{r} = \underline{a} + \lambda\, \underline{d}$$

or $\underline{r} = \underline{a} + \lambda\, (\underline{b} - \underline{a})$

$$\underline{r} = (1 - \lambda)\underline{a} + \lambda\, \underline{b}$$

These forms are interchangeable, but the top one is more commonly used so we will use it by default.

$\underline{a}$ and $\underline{d}$ are constant vectors (i.e. fixed for a given line) while λ is a variable.
Any value of λ gives a specific point on the line.

Note that this line extend forever in both directions, so technically $-\infty < \lambda < \infty$

It is often helpful to write as a single position vector, e.g:

$$\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 + \lambda \\ \lambda \\ -2 \end{pmatrix}$$

Vector equation of a straight line in 3D

Note: "point on the line" and "point he line passes through" are the same thing.

$$\underline{r} = \underline{a} + \lambda\,\underline{d}$$

Position vector of **any point on the line** (each point defined by value of λ).

Position vector of a known point on the line.

Parameter that determines where you are on the line.

Direction vector of the line (line is parallel to this)

E.g. 1

The equation of line l_1 is $\boldsymbol{r_1} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Find the vector equation of a line parallel to l_1 which passes through the point $(2,5,1)$.

$$\boldsymbol{r}_2 = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Same direction vector, but different 'starting point'.

E.g. 2

The equation of line l_1 is $\boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Find the coordinates of the points on l_1 which are a distance of 3 away from $(3,4,4)$.

Coordinate of generic point on line: $(-1+\lambda,\ \lambda,\ 3)$
Point off line: $(3,4,4)$

Distance between them $\sqrt{(-1+\lambda-3)^2 + (\lambda-4)^2 + (3-4)^2} = 3$

$$\lambda^2 - 8\lambda + 12 = 0 \quad \rightarrow \quad \lambda = 2\ or\ 6$$

We can sub these back into generic point on line:
If $\lambda = 2 \quad \rightarrow \quad (1,2,3)$
If $\lambda = 6 \quad \rightarrow \quad (5,6,3)$

Technicalities

$$\underline{r} = \underline{a} + \lambda\, \underline{d}$$

Because only the direction matters for the direction vector $\underline{d}$, not its magnitude (as it is multiplied by a scalar anyway).

The following (with any scalar multiple of $\underline{d}$) are all equivalent:

$$\underline{r} = \begin{pmatrix} 5 \\ -10 \\ 20 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 15 \\ 9 \end{pmatrix} \qquad \underline{r} = \begin{pmatrix} 5 \\ -10 \\ 20 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 5 \\ -10 \\ 20 \end{pmatrix} + t \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix} \qquad \underline{r} = \begin{pmatrix} 5 \\ -10 \\ 20 \end{pmatrix} + s \begin{pmatrix} -300 \\ -1500 \\ -900 \end{pmatrix}$$

As the direction of a vector is just the relative movement in the x, y and z directions, you can multiply or divide by any constant (although technically the parameter should be re-labelled).

However, as $\underline{a}$ is the position vector (starting point), it must remain unchanged! (multiplying would give a different point)

Diagnostic Question

Given points are $A(-1,2,-3)$ and $B(2,4,-1)$

Which is **not** an equation of the line that passes through A and B

Y

$$\underline{r} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

C

$$\underline{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

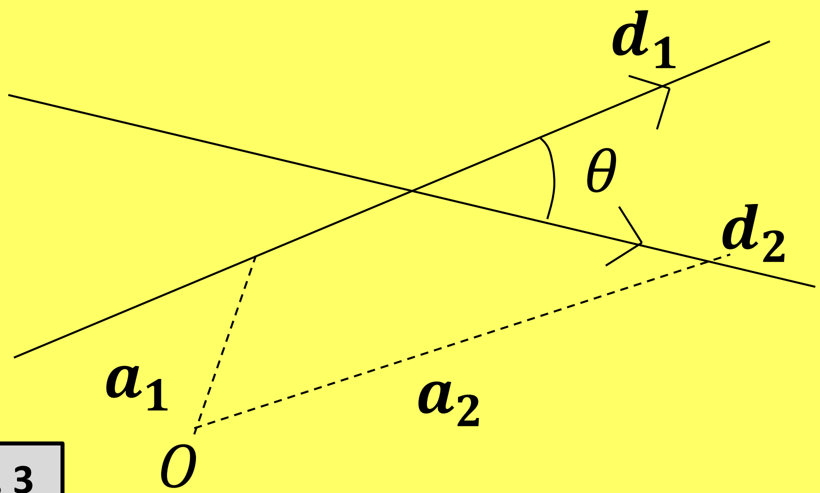
M

$$\underline{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix}$$

A

$$\underline{r} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1.5 \\ 1 \\ 1 \end{pmatrix}$$

Angles between straight lines



To find angle between two lines:
Find the angle between their <u>direction</u> vectors.

i.e. we only care about the directional part of the line, not how we got to the line.

E.g. 3

The points A and B have position vectors $2\boldsymbol{i}+6\boldsymbol{j}-\boldsymbol{k}$ and $3\boldsymbol{i}+4\boldsymbol{j}+\boldsymbol{k}$ respectively.
The line l_1 passes through points A and B.

a) Find the vector $\overrightarrow{AB}$.
b) Find a vector equation for the line l_1.

A second line l_2 passes through the origin and is parallel to the vector $\boldsymbol{i}+\boldsymbol{k}$.
The line l_1 meets the line l_2 at point C.
c) Find the **acute** angle between l_1 and l_2

a) $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}=\begin{pmatrix}3\\4\\1\end{pmatrix}-\begin{pmatrix}2\\6\\-1\end{pmatrix}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$

b) $\boldsymbol{r_1}=\begin{pmatrix}2\\6\\-1\end{pmatrix}+t\begin{pmatrix}1\\-2\\2\end{pmatrix}$

We could use the position vector of either A or B here (any point the line passes through)

c) $\boldsymbol{d_1}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$ $\boldsymbol{d_2}=\begin{pmatrix}1\\0\\1\end{pmatrix}$

$$\boldsymbol{d_1}\cdot\boldsymbol{d_2}=(1\times1)+(-2\times0)+(2\times1)=3$$

$$|\boldsymbol{d_1}|=\sqrt{1^2+(-2)^2+2^2}=3$$

$$|\boldsymbol{d_2}|=\sqrt{1^2+0^2+1^2}=\sqrt{2}$$

$$\Rightarrow\ \theta=\cos^{-1}\left(\frac{3}{3\sqrt{2}}\right)=45°$$

Note: If you get an obtuse angle, just subtract it from 180° to get the acute angle

Diagnostic Question

Find , to one decimal place, the **acute** angle between the two straight lines:

$$\underline{r}_1 = \underline{i} - 3\underline{j} + 5\underline{k} + \lambda(4\underline{i} + \underline{j} - 2\underline{k})$$

$$\underline{r}_2 = 2\underline{i} + \underline{j} - \underline{k} + \mu(\underline{i} - 2\underline{j} + 3\underline{k})$$

Y

$103.5°$

C

$114.5°$

M

$65.5°$

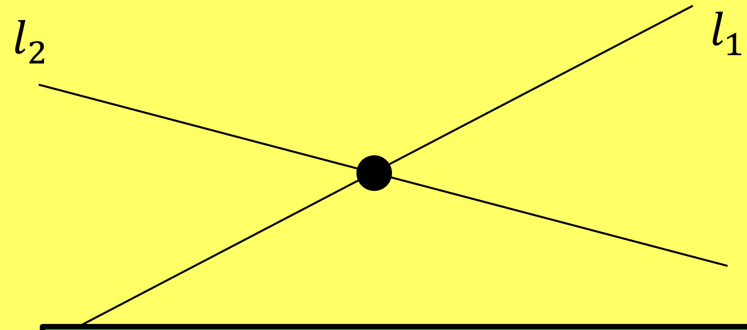
A

$76.5°$

Points of intersection

E.g. 4

The lines l_1 and l_2 have vector equations
$\boldsymbol{r} = 3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} + \lambda(\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$
and $\boldsymbol{r} = -2\boldsymbol{j} + 3\boldsymbol{k} + \mu(-5\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k})$ respectively.
Show that the two lines intersect, and find the position vector of the point of intersection.



We can represent any point on l_1 as the position vector $\begin{pmatrix} 3+\lambda \\ 1-2\lambda \\ 1-\lambda \end{pmatrix}$ and any point line l_2 as $\begin{pmatrix} -5\mu \\ -2+\mu \\ 3+4\mu \end{pmatrix}$.
If the lines intersect, there must be a choice of λ and μ that makes those two points equal, i.e. $\begin{pmatrix} 3+\lambda \\ 1-2\lambda \\ 1-\lambda \end{pmatrix} = \begin{pmatrix} -5\mu \\ -2+\mu \\ 3+4\mu \end{pmatrix}$

$$\begin{pmatrix} 3+\lambda \\ 1-2\lambda \\ 1-\lambda \end{pmatrix} = \begin{pmatrix} -5\mu \\ -2+\mu \\ 3+4\mu \end{pmatrix}$$

Solve two of these equations simultaneously:

$$3 + \lambda = -5\mu$$
$$1 - 2\lambda = -2 + \mu$$
$$\rightarrow \boldsymbol{\lambda = 2, \mu = -1}$$

The lines may or may not intersect, so we need to check the z-values:

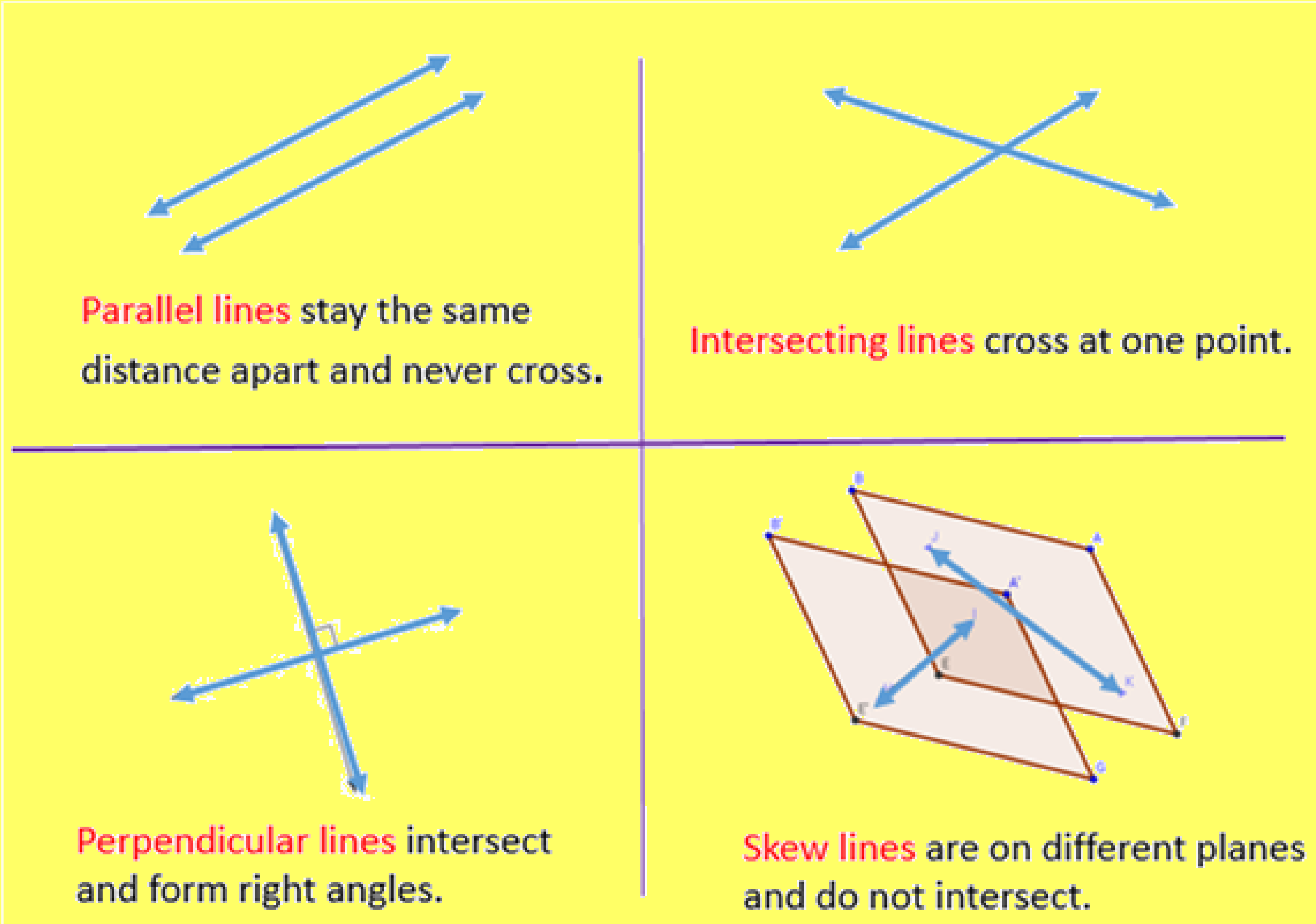
$$1 - 2 = 3 + (-4)$$
$$-1 = -1$$

Therefore lines intersect.

Use either $\lambda = 2$ or $\mu = -1$ to get point of intersection:

$$\begin{pmatrix} 3+2 \\ 1-2(2) \\ 1-2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$$

Possibilities for 2 straight lines in 3 dimensions



If two lines **do not intersect** in 3D space, they are either **parallel** or **skew.**

Terminology: Two straight lines are <u>skew</u> lines if they do <u>not</u> intersect (and are not parallel)

Inconsistent equations = no intersection

E.g. 5

The line l_1 has equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

The line l_2 has equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Show that the lines do not intersect.

Looking at the direction vectors, they are not scalar multiples of each other (also known as linearly independent), so they will either intersect or be skew.

$$1 + \lambda = 1 + 2\mu$$
$$\lambda = 3 + \mu$$
$$-1 = 6 - \mu$$

$$-1 = 6 - \mu \Rightarrow \mu = 7$$

$$\lambda = 3 + \mu = 3 + 7 = 10$$

Check for consistency in first equation: $1 + \lambda = 1 + 2\mu$

$$1 + 10 = 1 + 2 \times 7$$
$$11 \neq 15$$

The equations are inconsistent, therefore the equations have no solutions and the lines do not intersect (they are skew).

Diagnostic Question

Determine the point of intersection (if any) between the two straight lines:

$$\underline{r}_1 = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix}, \quad \underline{r}_2 = \begin{pmatrix} -5 \\ 5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Y

$(0,5,3)$

C

$(-6,5,3)$

M

$(0,5,-3)$

A

Lines are skew

Cartesian form of equation of straight line

We saw that we could get the coordinate (x, y, z) of some point on a 3D line using:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Therefore $x = a_1 + \lambda d_1 \quad \rightarrow \quad \lambda = \frac{x-a_1}{d_1}$

$y = a_2 + \lambda d_2 \quad \rightarrow \quad \lambda = \frac{y-a_2}{d_2}$

$z = a_3 + \lambda d_3 \quad \rightarrow \quad \lambda = \frac{z-a_3}{d_3}$

$$\therefore \quad \frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$$

Recall that a Cartesian equation is one involving x, y, z and no variable parameters (i.e. no λ).

If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ and $\underline{r} = \underline{a} + \lambda \underline{d}$ is the equation of straight line, then its Cartesian form is

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

Quick Examples

1. Find the Cartesian equation of the line with equation $\underline{r} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.

$$\frac{x-4}{-1} = \frac{y-3}{2} = \frac{z+2}{5}$$

2. The Cartesian equation of a line is $y = 3x + 2$. Find the **vector** form of the equation of the line.

If $y = 3x + 2$ then a point $\begin{pmatrix} x \\ y \end{pmatrix}$ on the line can be represented as

$$\begin{pmatrix} x \\ 3x+2 \end{pmatrix} \rightarrow \begin{pmatrix} 0+1x \\ 2+3x \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 2 \end{pmatrix} + x \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

This is the opposite of combining $\underline{r} = \underline{a} + \lambda \underline{d}$ into a single vector: we're splitting it up instead!

Therefore vector equation of line is

$$\underline{r} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

This makes sense: The y-intercept of $y = 3x + 2$ is 2 so $(0,2)$ is on the line. Since the gradient is 3, each time we move across 1 unit, we move up 3 units.

3. The Cartesian equation of a line is $\frac{x-2}{3} = \frac{y+5}{1} = \frac{z}{4}$. Find the **vector** form of the equation of the line.

$$\frac{x-2}{3} = \frac{y+5}{1} = \frac{z}{4} = \lambda$$

Then $x = 3\lambda + 2,\ y = \lambda - 5,\ z = 4\lambda$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3\lambda + 2 \\ \lambda - 5 \\ 4\lambda \end{pmatrix} \rightarrow \boldsymbol{r} = \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

Diagnostic Question

Find the Cartesian equation of the line with equation $\underline{r} = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$.

Y

$$\frac{x-2}{1} = \frac{y+5}{3} = \frac{z-4}{-2}$$

C

$$\frac{x-1}{2} = \frac{y-3}{-5} = \frac{z+2}{4}$$

M

$$\frac{x-2}{1} = \frac{y-5}{3} = \frac{z-4}{-2}$$

A

$$\frac{x-2}{1} = \frac{y+5}{3} = \frac{z-4}{2}$$

Diagnostic Question

The Cartesian equation of a line is $\frac{x}{1} = \frac{y-2}{5} = \frac{z+3}{-7}$.

Find the **vector** form of the equation of the line.

Y

$$\underline{r} = \begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

C

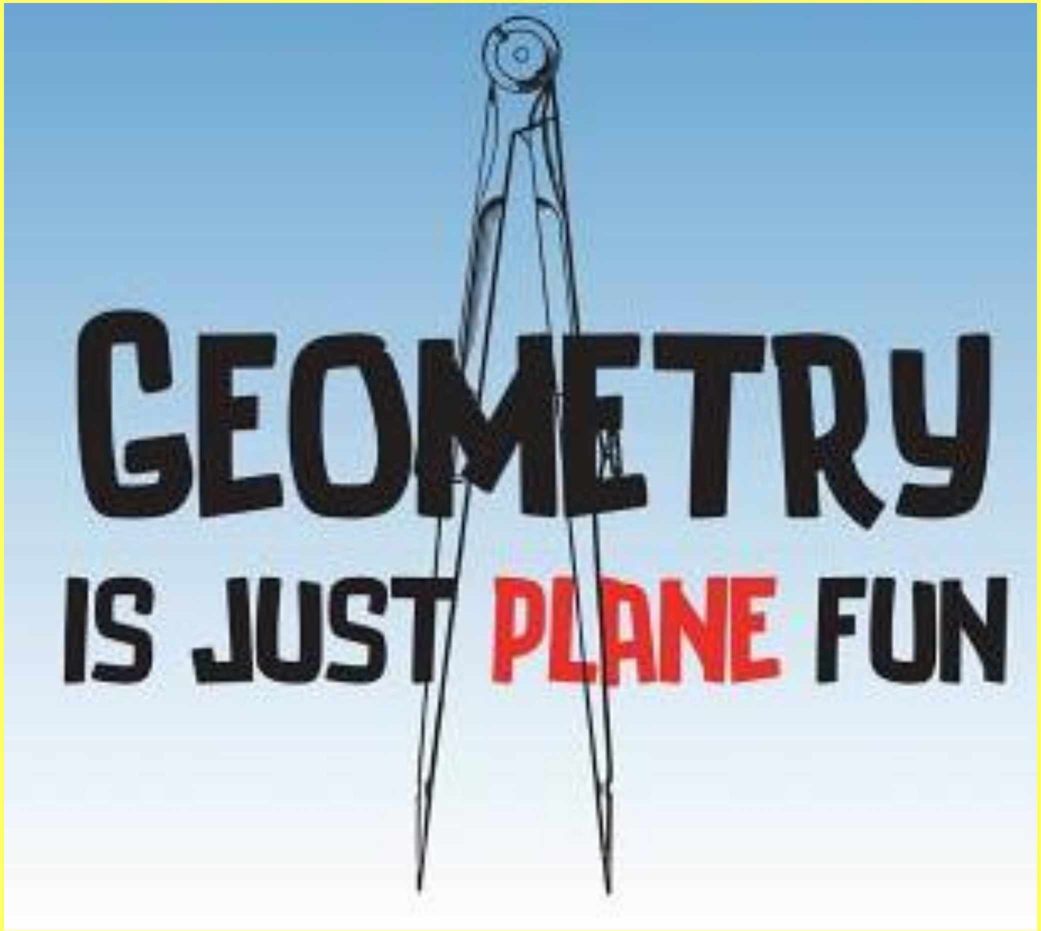
$$\underline{r} \cdot \begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

M

$$\underline{r} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix}$$

A

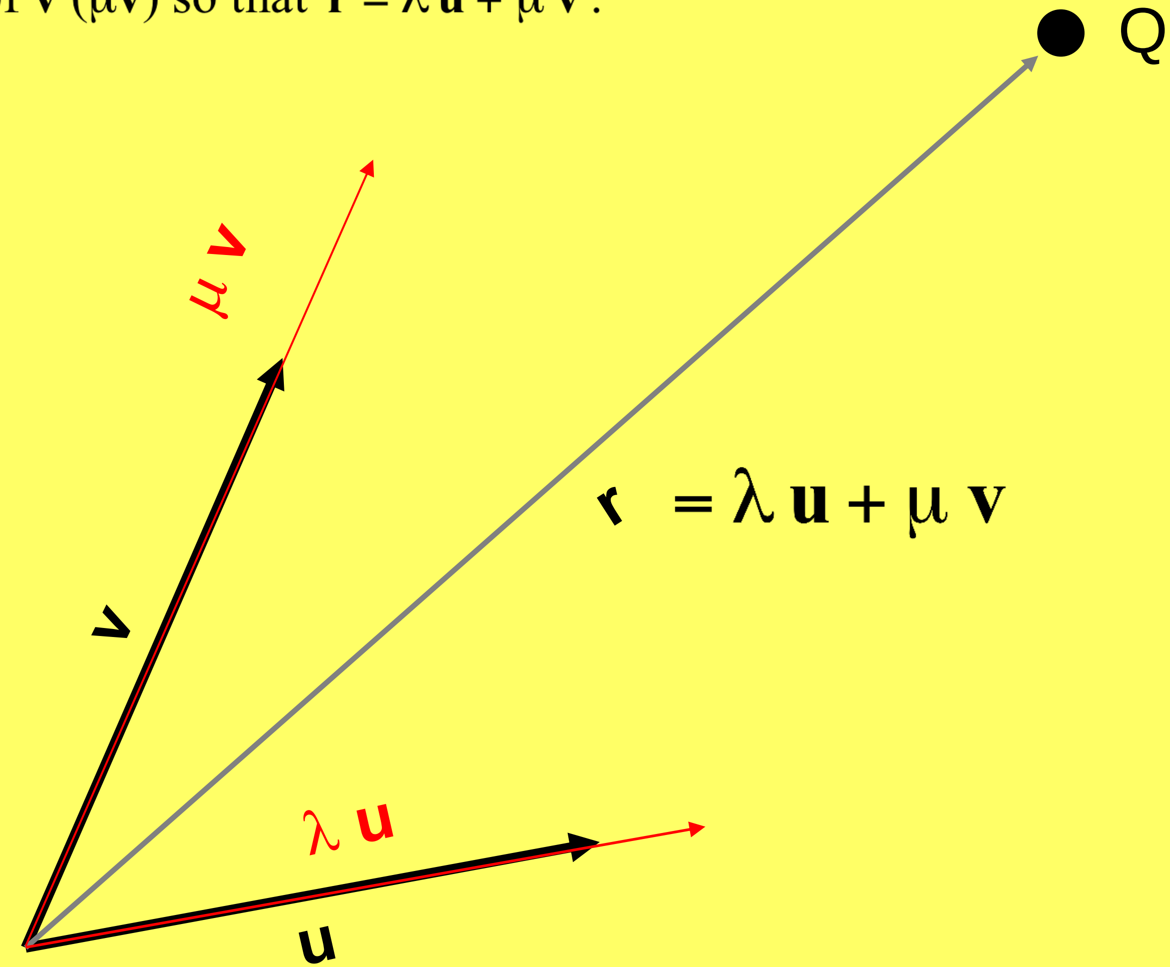
$$\underline{r} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix}$$



Planes: Equations of a planes

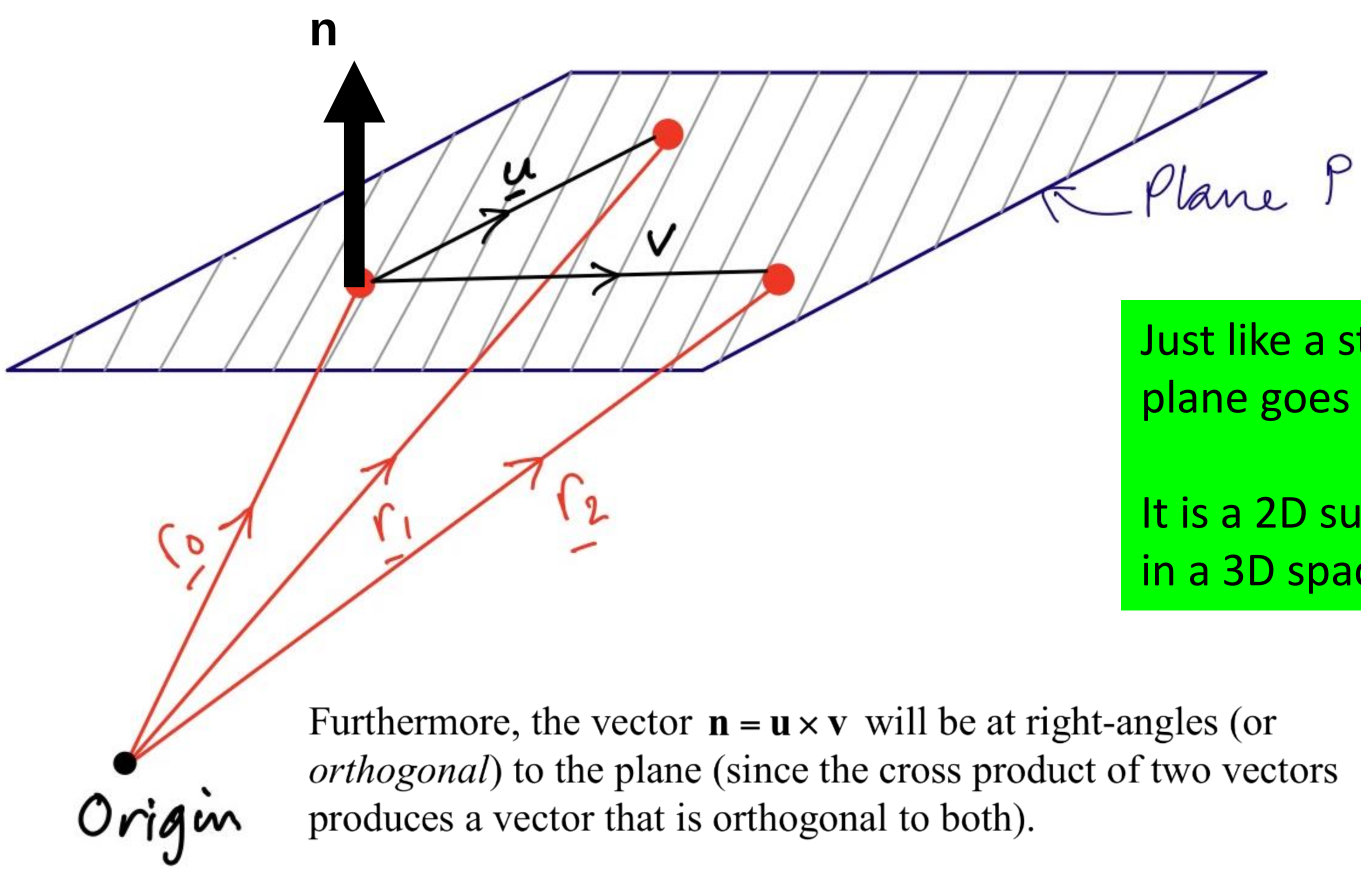
Planes

Observation 1. Consider any two non-parallel vectors **u** and **v** and a point Q. The point may be reached by taking some scalar multiple of **u** (λ**u**) and adding to it some other multiple of **v** (μ**v**) so that $\mathbf{r} = \lambda\,\mathbf{u} + \mu\,\mathbf{v}$.



Planes

<u>Observations 2 and 3.</u> Now suppose that we have a plane P. Let us choose any three points in the plane not on the same line with position vectors $\mathbf{r}_0, \mathbf{r}_1$ and $\mathbf{r}_2$. Then the vectors $\mathbf{u} = \mathbf{r}_1 - \mathbf{r}_0$ and $\mathbf{v} = \mathbf{r}_2 - \mathbf{r}_0$ will lie in the plane.



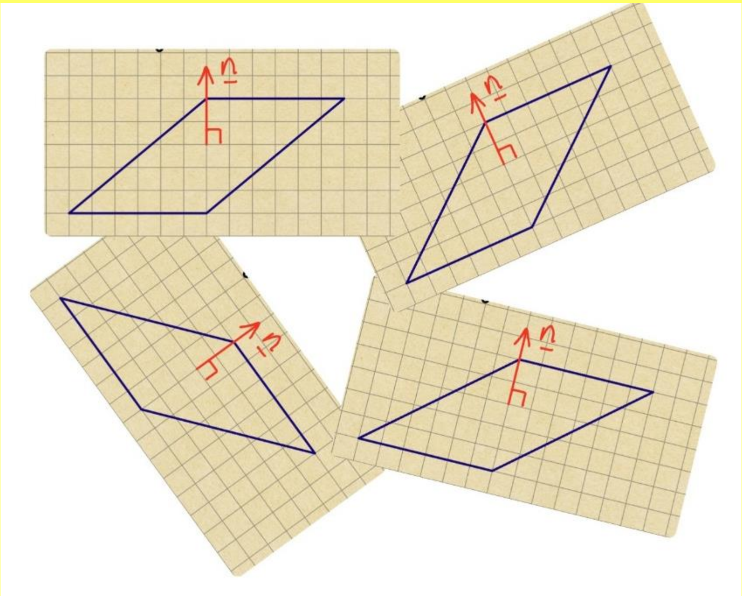
Just like a straight line, a plane goes on forever.

It is a 2D surface oriented in a 3D space.

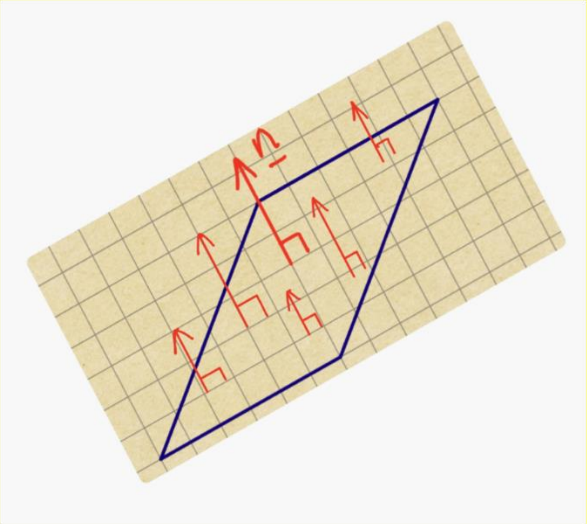
Furthermore, the vector $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ will be at right-angles (or *orthogonal*) to the plane (since the cross product of two vectors produces a vector that is orthogonal to both).

Each plane has a unique unit normal

The vector **n** is said to be a **normal** to the plane and it is important because its direction defines the orientation of the plane:



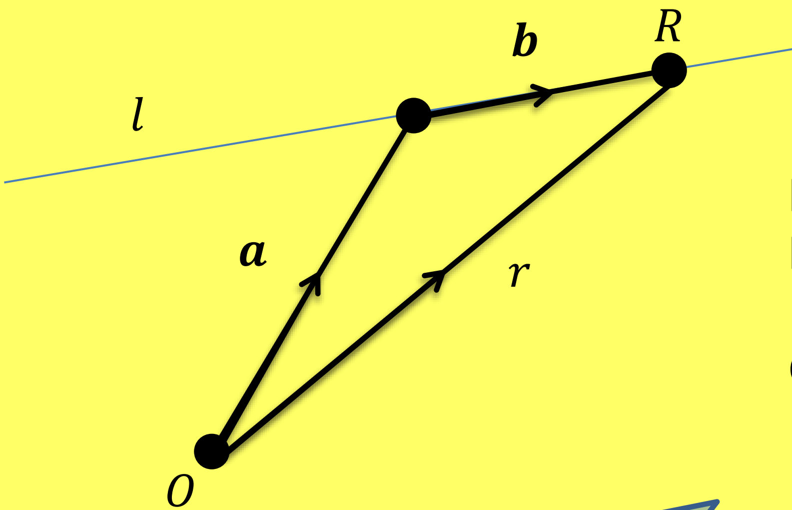
So for a given plane, the direction of its normal vector will be **fixed**:



This in turn implies that if we fix the magnitude of the normal, say by making it a unit vector, then this will be a unique feature of the plane. In other words a plane will have a unique unit normal $\hat{\mathbf{n}} = \mathbf{n} / |\mathbf{n}|$.

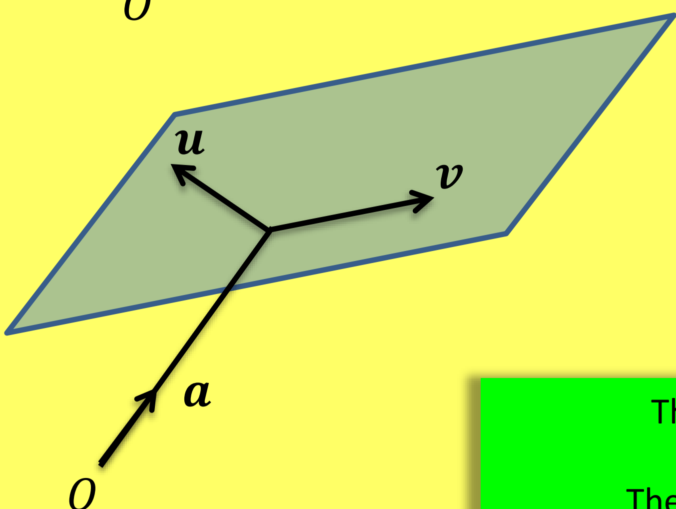
Parametric equation of plane

$\boldsymbol{a}$ is the position vector of a point on a plane and $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-parallel vectors on the plane, how could we write the equation of the plane in vector form?



Recall the we could get to a generic point $\boldsymbol{r}$ on a line by first getting to the line using $\boldsymbol{a}$, followed by some amount of $\boldsymbol{b}$, i.e. $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$.

Could we do a similar thing with a plane?



Once on the plane using $\boldsymbol{a}$, we could get to any other point on the plane using 'some amount' of $\boldsymbol{u}$ and 'some amount' of $\boldsymbol{v}$, i.e.

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{u} + \mu \boldsymbol{v}$$

This is the parametric equation of plane, where λ and μ are scalars.
This gives the position of any point in the plane.
The whole plane (which goes on forever) would be described by taking
$-\infty < \lambda < \infty, -\infty < \mu < \infty$
If we wanted to define only a part of the plane, the ranges of λ and μ can be restricted.

Examples

E.g. 1

A plane Π passes through the points $A(2,2-1), B(3,2,-1), C(4,3,5)$
Find the equation of the plane Π in parametric form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{u} + \mu\boldsymbol{v}$

$\overrightarrow{AB} = (1,0,0)$ and $\overrightarrow{AC} = (2,1,6)$ are vectors lying in the plane.

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$$

Note: We could use any two non-parallel vectors that lie in the plane.

E.g. 2

Verify that the point P with position vector $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ lies in the plane with vector equation

$$r = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

As with straight lines, we can write plane equation as a single vector:

$$\begin{pmatrix} 3 + 2\lambda + \mu \\ 4 + \lambda - \mu \\ -2 + \lambda + 2\mu \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \Rightarrow$$

We have two unknowns, so only need to solve the first two equations simultaneously. But we must then check that the z values agree for these values of λ and μ.

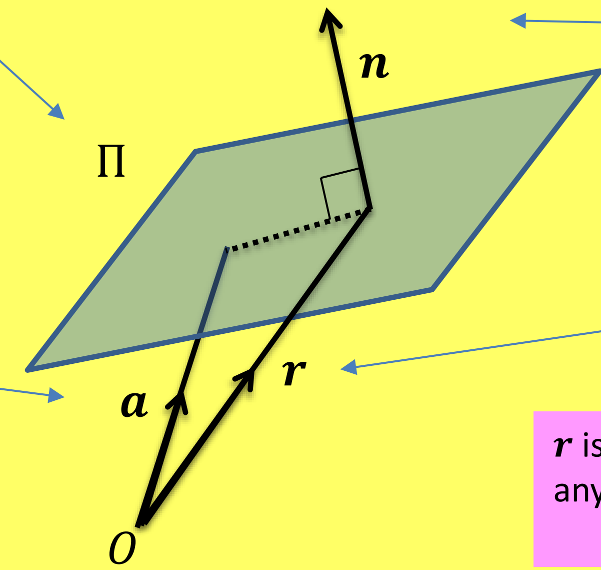
$$2\lambda + \mu = -1$$
$$\lambda - \mu = -2 \qquad \rightarrow \boldsymbol{\lambda = -1, \mu = 1}$$

Checking in last equation: $-2 + (-1) + 2(1) = -1$
Therefore P lies on plane.

Vector Equation of a Plane

We use Π to represent a plane ("capital pi") in the same way we use l to represent a straight line.

Just as $\boldsymbol{a}$ was used as the position vector of a **fixed** point on a line l, it is used in the same way for a plane.



$\boldsymbol{n}$ (the $\boldsymbol{n}$ stands for "normal") always indicates a vector perpendicular to the plane.

We reuse the letter $\boldsymbol{r}$ to mean "the position vector of some point on the plane".

$\boldsymbol{r}$ is the position vector from the origin to any point P on the plane and is defined as
$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

It's important to realise here that point $\boldsymbol{a}$ and the direction of $\boldsymbol{n}$ are **fixed** for a given plane (i.e. are constant vectors), whereas $\boldsymbol{r}$ can **vary** as it represents all the possible points on the plane**.**

How could we use the dot product to find some relationship between $\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{n}$?

$\boldsymbol{r} - \boldsymbol{a}$ is perpendicular to $\boldsymbol{n}$, thus $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$
$$\boldsymbol{r} \cdot \boldsymbol{n} - \boldsymbol{a} \cdot \boldsymbol{n} = \boldsymbol{0}$$
$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$
But since $\boldsymbol{a} \cdot \boldsymbol{n}$ is a constant (dot product of two constant vectors), replace with constant scalar d:

Vector equation of plane (scalar product form):
$$\boldsymbol{r} \cdot \boldsymbol{n} = d$$
where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is position vector of some point on the plane, $\boldsymbol{n}$ is normal to plane, $d = \boldsymbol{a} \cdot \boldsymbol{n}$ is a scalar constant.

Cartesian equation of a plane

We know that:

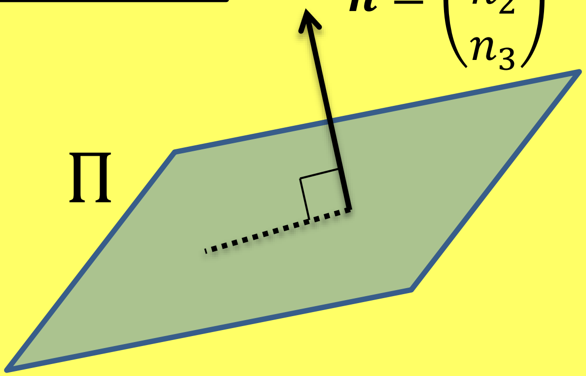
Vector equation of plane (scalar product form):

$$\boldsymbol{r} \cdot \boldsymbol{n} = d$$

where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is position vector of some point on the plane, $\boldsymbol{n}$ is normal to plane, $d = \boldsymbol{a} \cdot \boldsymbol{n}$ is a scalar constant.

$$\boldsymbol{r} \cdot \boldsymbol{n} = d \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = d \quad \Rightarrow n_1 x + n_2 y + n_3 z = d$$

$$\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$



Therefore, the equation of a plane can be written as:

Cartesian equation of a plane:

$$n_1 x + n_2 y + n_3 z = d$$

Or, more commonly (but not as obviously) as:

Cartesian equation of a plane:

$$ax + by + cz = d$$

Where a, b, c and d are constants, and the normal is given by $\boldsymbol{n} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$

Example: Equation of plane form point and normal

E.g. 3

A point with position vector $2\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}$ lies on the plane and the vector $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ is perpendicular to the plane. Find the equation of the plane in:

a) Scalar product form.
b) Cartesian form.

Memorisation Tip:
In $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$,
I remember that the normal vector $\boldsymbol{n}$ is the one common to both sides.

$$\boldsymbol{a} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}, \qquad \boldsymbol{n} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

$$\boldsymbol{a} \cdot \boldsymbol{n} = 6 + 3 + 5 = 14$$

$$\therefore \boldsymbol{r} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 14$$

This is scalar product form

Now, since $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 14$$

$$\boldsymbol{3x + y - z = 14}$$

This is Cartesian form

If $\boldsymbol{r} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = d$
is the scalar product equation of a plane, then the Cartesian form is:

$$n_1 x + n_2 y + n_3 z = d$$

Example: Cartesian Equation

E.g. 4

The plane Π is perpendicular to the normal $\boldsymbol{n} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and passes through the point P with position vector $8\boldsymbol{i} + 4\boldsymbol{j} - 7\boldsymbol{k}$.
Find the Cartesian equation of Π.

$$\boldsymbol{r} \cdot \boldsymbol{n} = d \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = d$$

So in this case $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = d$

$$3x - 2y + z = d$$

It passes through $(8,4,-7)$ therefore:

$$3(8) - 2(4) + (-7) = 9 = d$$

Cartesian equation: $\boldsymbol{3x - 2y + z = 9}$

You could also do it all at once!

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n} \quad \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$3x - 2y + z = 24 - 8 - 7$$

$$3x - 2y + z = 9$$

Diagnostic Question

Find the normal vector to the plane

$$3x - 5y - 18z = 7$$

Y

$$\begin{pmatrix} 10 \\ 1 \\ -1 \end{pmatrix}$$

C

$$\begin{pmatrix} 3 \\ -5 \\ -18 \end{pmatrix}$$

M

$$3x + 5y - 18$$

A

$$\frac{3x}{7} = \frac{5y}{7} = \frac{18z}{7}$$

Diagnostic Question

Find the cartesian equation of the plane with normal

$$\boldsymbol{n} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$$

which contains point $(1, 2, 3)$

Y $3x - 2y + z = 10$

M $x + 2y + 3z = 2$

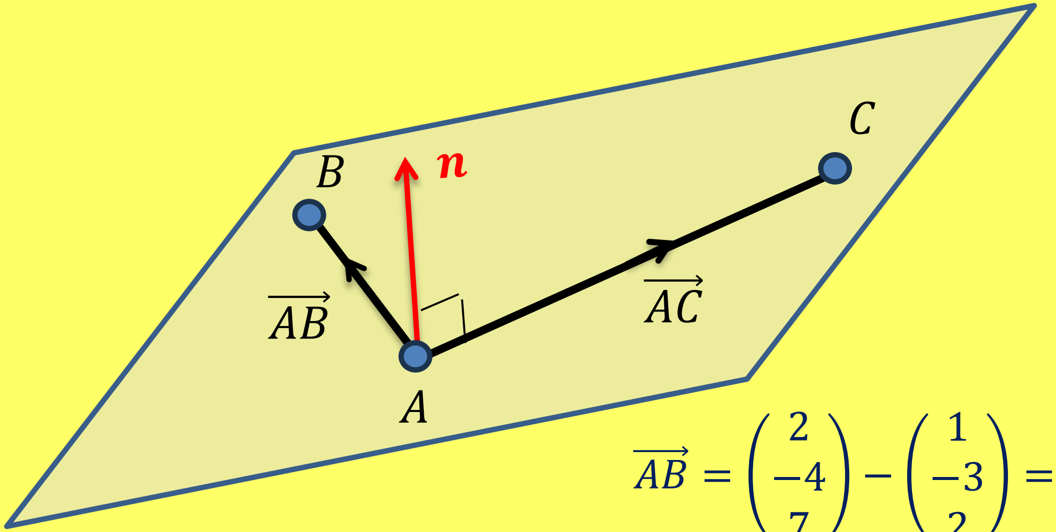
C $x + 2y + 3z = 10$

A $3x - 2y + z = 2$

Example: Cartesian Equation from 3 points

E.g. 5 Find an equation of the plane that contains points $A(1,-3,2), B(2,-4,7), C(1,5,0)$. Give your answer in the form $ax+by+cz=d$ where a,b,c,d are integers.

To find the equation of a plane, we need a point on the plane and the normal to the plane



- **We already have a choice of 3 points on the plane (either A, B or C will work)**
- **We find the direction of the normal by taking the vector product of any two (non-parallel) vectors in the plane.**

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -2 \end{pmatrix}$$

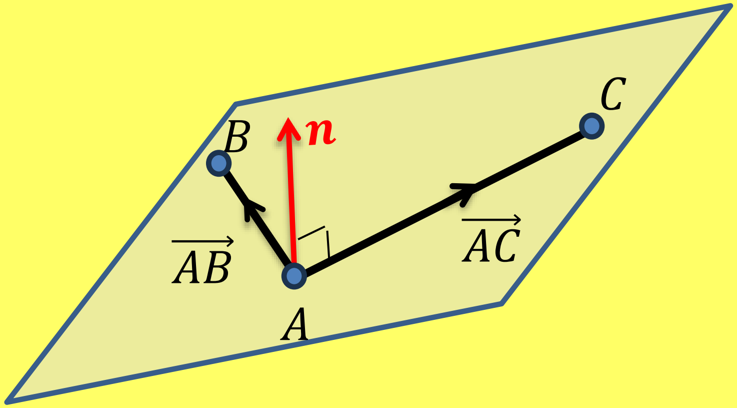
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 5 \\ 0 & 8 & -2 \end{vmatrix} = \underline{i}[(-1\times -2)-(5\times 8)] - \underline{j}[(1\times -2)-(5\times 0)] + \underline{k}[(1\times 8)-(-1\times 0)]$$

$$= -38\underline{i} + 2\underline{j} + 8\underline{k}$$

This is the direction of the normal (i.e. the normal is parallel to this so we can use it as is or divide by a common factor to make the numbers easier!

Note: You can use any two vectors and whichever order you do the cross product in you will still get a normal ($\pm\underline{n}$ are both normal to the plane in opposite directions).

Example: Cartesian Equation from 3 points



$$\underline{n} = k(-38\underline{i} + 2\underline{j} + 8\underline{k})$$

Divide by 2 to make numbers easier (we could also flip all the signs if we wanted to)

$$= -19\underline{i} + \underline{j} + 4\underline{k}$$

$$\underline{n} = -19\underline{i} + \underline{j} + 4\underline{k}$$

Find the equation of the plane: $\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -19 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -19 \\ 1 \\ 4 \end{pmatrix}$$

Note: Here we used point A on the RHS but you could use any of A, B or C (feel free to check that you get the same answer.

$$-19x + y + 4z = -19 - 3 + 8$$

$$\boldsymbol{-19x + y + 4z = -14}$$

If we hadn't divided by 2 earlier, the whole equation would be multiplied by 2 here and we could simplify now.

Or any equivalent such as $\boldsymbol{19x - y - 4z = 14}$

Note: If were asked for the form $\underline{r} \cdot \underline{n} = d$

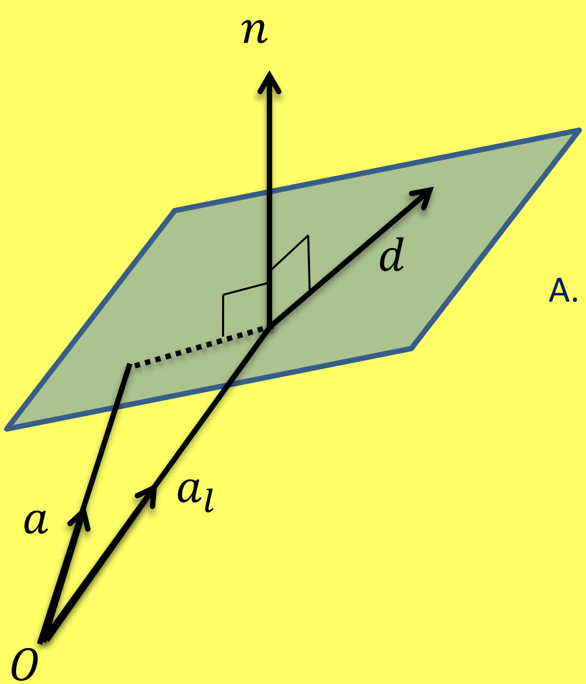
We would leave it as $\underline{r} \cdot \begin{pmatrix} -19 \\ 1 \\ 4 \end{pmatrix} = -14$ (or equivalent with opposite signs)

Plane using vectors/lines on plane

E.g. 6

Find, in the form $\boldsymbol{r} \cdot \boldsymbol{n} = d$, an equation of the plane which contains the line l and the point with position vector a where l has the equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{a} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

Find also the Cartesian form of the equation.



Key strategy:

A. Identify two vectors on the plane.

B. $\boldsymbol{n}$ is cross product of these (as perpendicular to both)

C. Use $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ as usual.

A. Vectors on plane:

- Direction vector of line $\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$
- $\boldsymbol{a} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{a_l} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$ both on line so $\boldsymbol{a} - \boldsymbol{a_l} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ is in direction of plane too.

B. $\boldsymbol{n} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$

C. $\boldsymbol{a} \cdot \boldsymbol{n} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 22$

$\therefore \boldsymbol{r} \cdot \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 22 \quad \rightarrow \quad 4x + 2y = 22$

See you at the tutorial!

Math ruins you.

Yesterday I read “lol” as “absolute value of o”.

I am surely going insane. Thanks, math.