CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

University of Leeds

Level 1 Semester 2

Dr. Mark Dowker (Module Leader)
Room 2.45 Chemical & Process Engineering Building
E-mail: M.D.Dowker@leeds.ac.uk

Tutorial: Question Difficulty Colour Code

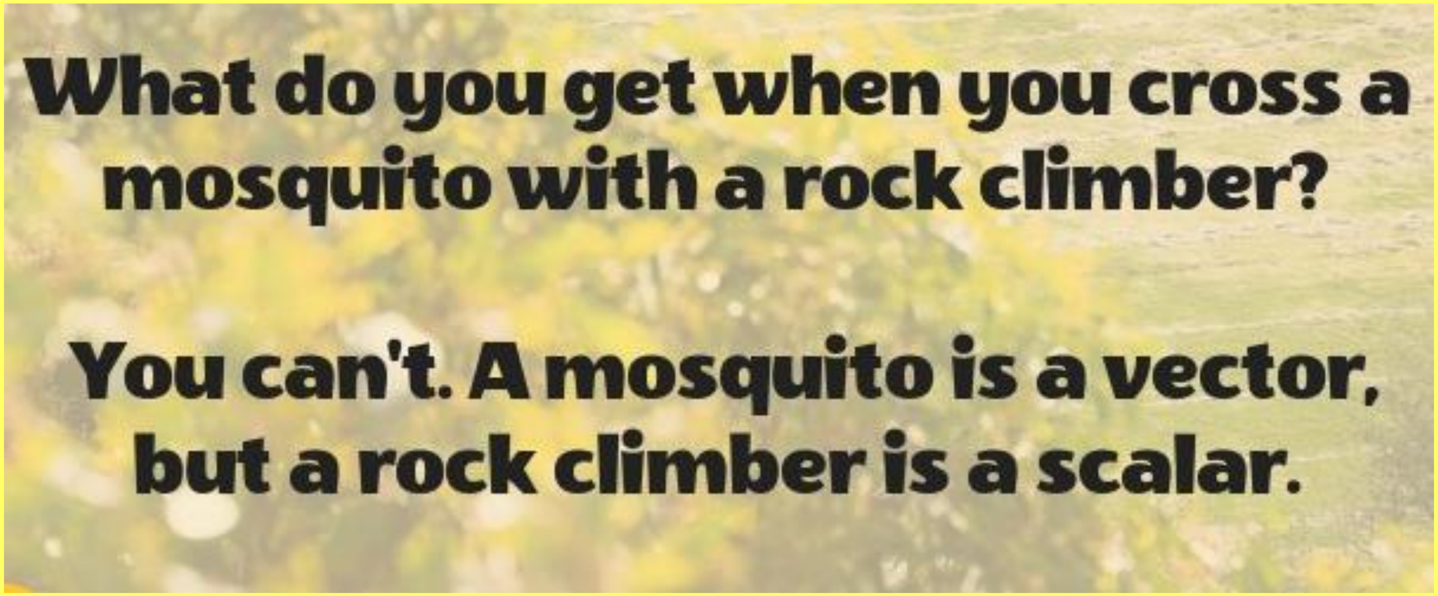
Basic - straightforward application
(you must be able to do these)

Medium – Makes you think a bit
(you must be able to do these)

Hard – Makes you think a lot
(you should be able to do these)

Extreme – Tests your understanding to the limit!
(for those who like a challenge)

Applied – Real-life examples of the topic, may sometimes involve prior knowledge
(you should attempt these – will help in future engineering)



*Medical maths joke

Tutorial 8
Vectors 1

Class Example (Applied)

E.g. 1

A particle of mass 0.5 kg is acted on by three forces.

$$F_1 = (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})\ N$$
$$F_2 = (-\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k})\ N$$
$$F_3 = (4\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})\ N$$

a. Find the resultant force R acting on the particle.

b. Find the acceleration of the particle, giving your answer in the form $(p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k})$ ms^{-2}.

c. Find the magnitude of the acceleration.

Given that the particle starts at rest,

d. Find the distance travelled by the particle in the first 6 seconds of its motion.

a. $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix} N$

b. $\boldsymbol{F} = m\boldsymbol{a}$

$\begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix} = 0.5a \quad \therefore \boldsymbol{a} = \begin{pmatrix} 10 \\ -2 \\ -6 \end{pmatrix} ms^{-2}$

c. $|\boldsymbol{a}| = \sqrt{10^2 + (-2)^2 + (-6)^2} = \sqrt{140}$ ms^{-2}

d. $u = 0, a = \sqrt{140}\ ms^{-2}, t = 6\ s, \quad s = ?$

$s = ut + \frac{1}{2}at^2 = \frac{1}{2} \times \sqrt{140} \times 6^2 = 36\sqrt{35}\ m \approx 213m$

Class Example

E.g. 2

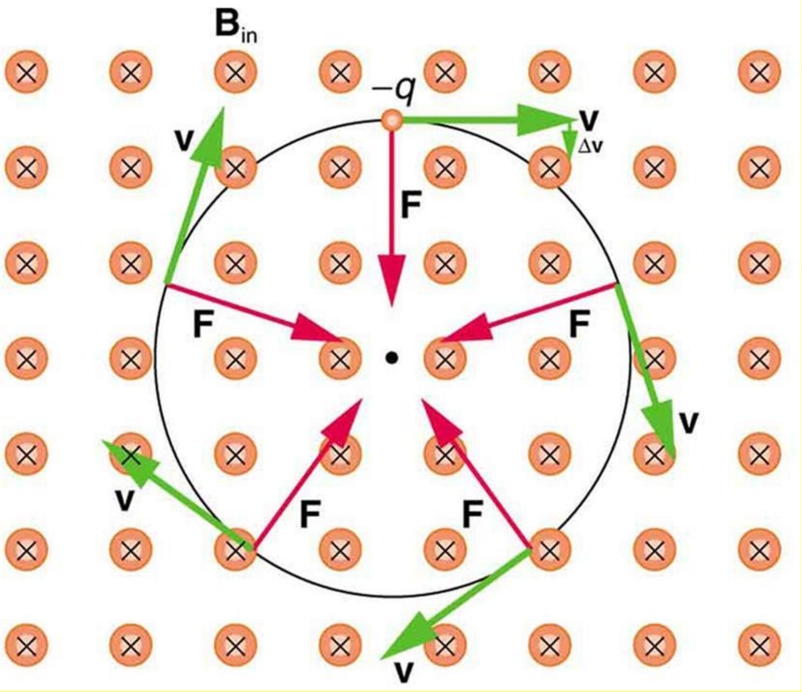
Find a unit vector that is perpendicular to vectors $\underline{a}$ and $\underline{b}$ where when $\underline{a} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix}$

$\underline{\hat{n}} = \frac{1}{6\sqrt{3}} \begin{pmatrix} 10 \\ 2 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} \frac{5}{3\sqrt{3}} \\ \frac{1}{3\sqrt{3}} \\ \frac{1}{3\sqrt{3}} \end{pmatrix}$

Class Example (Applied)

E.g. 3

The Force experienced by a particle with charge q moving with velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$ is given by the formula $\boldsymbol{F} = q\,\boldsymbol{v} \times \boldsymbol{B}$ (Lorentz formula)
Calculate the force of a particle with a $0.5C$ charge moving in direction $\boldsymbol{v} = 2\boldsymbol{i} - \boldsymbol{j}$ in a magnetic field $\boldsymbol{B} = -\boldsymbol{k}$



Note that $\boldsymbol{F}$ is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{B}$

$$\boldsymbol{F} = (0.5\boldsymbol{i} + \boldsymbol{j})N$$

Class Example

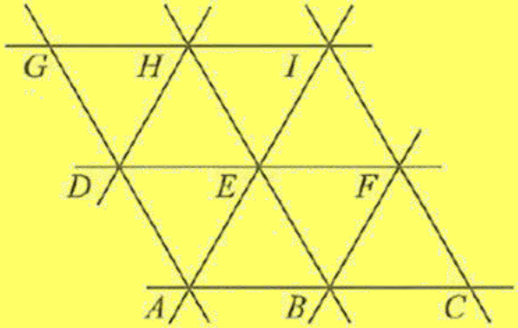
E.g. 4

Find $\underline{a} \cdot (\underline{b} \times \underline{c})$ when $\underline{a} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, $\underline{c} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

(Answer 5)

Exercise A: Vectors Recap

1



The diagram shows three sets of equally-spaced parallel lines.

Given that $\overrightarrow{AC} = \mathbf{p}$ and that $\overrightarrow{AD} = \mathbf{q}$, express the following vectors in terms of **p** and **q**.

a $\overrightarrow{CA}$ **b** $\overrightarrow{AG}$ **c** $\overrightarrow{AB}$ **d** $\overrightarrow{DF}$ **e** $\overrightarrow{HE}$ **f** $\overrightarrow{AF}$

g $\overrightarrow{AH}$ **h** $\overrightarrow{DC}$ **i** $\overrightarrow{CG}$ **j** $\overrightarrow{IA}$ **k** $\overrightarrow{EC}$ **l** $\overrightarrow{IB}$

2

Given that vectors **p** and **q** are not parallel, state whether or not each of the following pairs of vectors are parallel.

a $2\mathbf{p}$ and $3\mathbf{p}$ **b** $(\mathbf{p} + 2\mathbf{q})$ and $(2\mathbf{p} - 4\mathbf{q})$ **c** $(3\mathbf{p} - \mathbf{q})$ and $(\mathbf{p} - \frac{1}{3}\mathbf{q})$

d $(\mathbf{p} - 2\mathbf{q})$ and $(4\mathbf{q} - 2\mathbf{p})$ **e** $(\frac{3}{4}\mathbf{p} + \mathbf{q})$ and $(6\mathbf{p} + 8\mathbf{q})$ **f** $(2\mathbf{q} - 3\mathbf{p})$ and $(\frac{3}{2}\mathbf{q} - \mathbf{p})$

3

The points O, A, B and C are such that $\overrightarrow{OA} = 4\mathbf{m}$, $\overrightarrow{OB} = 4\mathbf{m} + 2\mathbf{n}$ and $\overrightarrow{OC} = 2\mathbf{m} + 3\mathbf{n}$, where **m** and **n** are non-parallel vectors.

a Find an expression for $\overrightarrow{BC}$ in terms of **m** and **n**.

The point M is the mid-point of OC.

b Show that AM is parallel to BC.

4

Given that $\mathbf{p} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix}$, find as column vectors,

a $\mathbf{p} + 2\mathbf{q}$ **b** $\mathbf{p} - \mathbf{r}$ **c** $\mathbf{p} + \mathbf{q} + \mathbf{r}$ **d** $2\mathbf{p} - 3\mathbf{q} + \mathbf{r}$

Exercise A: Vectors Recap

5

Points P, Q and R have coordinates $(9,1,0)$, $(8,-3,5)$ and $(5,5,7)$ respectively.

a) Find the position vectors of P, Q and R.

b) Find $\overrightarrow{PQ}$ and $\overrightarrow{QR}$

c) Find the magnitudes (lengths) $|\overrightarrow{PQ}|$ and $|\overrightarrow{QR}|$

6

Given that vectors **p** and **q** are not parallel, find the values of the constants a and b such that

a $a\mathbf{p} + 3\mathbf{q} = 5\mathbf{p} + b\mathbf{q}$

b $(2\mathbf{p} + a\mathbf{q}) + (b\mathbf{p} - 4\mathbf{q}) = \mathbf{0}$

c $4a\mathbf{q} - \mathbf{p} = b\mathbf{p} - 2\mathbf{q}$

d $(2a\mathbf{p} + b\mathbf{q}) - (a\mathbf{q} - 6\mathbf{p}) = \mathbf{0}$

7

Relative to a fixed origin O, the points A, B and C have position vectors $\begin{pmatrix} 6 \\ -2 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 12 \\ -7 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix}$ respectively.

a Find the position vector of the point M, the mid-point of BC.

b Show that O, A and M are collinear.

8

Find

a a unit vector in the direction $5\mathbf{i} - 2\mathbf{j} + 14\mathbf{k}$,

b a vector of magnitude 10 in the direction $2\mathbf{i} + 11\mathbf{j} - 10\mathbf{k}$,

c a vector of magnitude 20 parallel to $-5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

d a vector anti-parallel to $-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

Exercise A: Vectors Recap

9 Given that $\mathbf{r} = \lambda\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$, find the two possible values of λ such that $|\mathbf{r}| = 14$.

10 Given that $\mathbf{r} = -2\mathbf{i} + \lambda\mathbf{j} + \mu\mathbf{k}$, find the values of λ and μ such that

a $\mathbf{r}$ is parallel to $4\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$

b $\mathbf{r}$ is parallel to $-5\mathbf{i} + 20\mathbf{j} - 10\mathbf{k}$

11 Given that $\mathbf{p} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{q} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r} = 2\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}$,

a find $|2\mathbf{p} - \mathbf{q}|$,

b find the value of k such that $\mathbf{p} + k\mathbf{q}$ is parallel to $\mathbf{r}$.

12 Given that $\mathbf{r} = \lambda\mathbf{i} - 2\lambda\mathbf{j} + \mu\mathbf{k}$, and that $\mathbf{r}$ is parallel to the vector $(2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$,

a show that $3\lambda + 2\mu = 0$.

Given also that $|\mathbf{r}| = 2\sqrt{29}$ and that $\mu > 0$,

b find the values of λ and μ.

13 The position vector of a model aircraft at time t seconds is $(9 - t)\mathbf{i} + (1 + 2t)\mathbf{j} + (5 - t)\mathbf{k}$, relative to a fixed origin O. One unit on each coordinate axis represents 1 metre.

a Find an expression for d^2 in terms of t, where d metres is the distance of the aircraft from O.

b Find the value of t when the aircraft is closest to O and hence, the least distance of the aircraft from O.

Exercise B: Scalar Product

1

Calculate

a $(\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}).(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

b $(6\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}).(\mathbf{i} - 3\mathbf{j} - \mathbf{k})$

c $(-5\mathbf{i} + 2\mathbf{k}).(\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

d $(3\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}).(-\mathbf{i} + 11\mathbf{j} - 4\mathbf{k})$

e $(3\mathbf{i} - 7\mathbf{j} + \mathbf{k}).(9\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

f $(7\mathbf{i} - 3\mathbf{j}).(-3\mathbf{j} + 6\mathbf{k})$

2

Given that $\mathbf{p} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{q} = \mathbf{i} + 5\mathbf{j} - \mathbf{k}$ and $\mathbf{r} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$,

a find the value of $\mathbf{p}.\mathbf{q}$,

b find the value of $\mathbf{p}.\mathbf{r}$,

c verify that $\mathbf{p}.(\mathbf{q} + \mathbf{r}) = \mathbf{p}.\mathbf{q} + \mathbf{p}.\mathbf{r}$

3

Simplify

a $\mathbf{p}.(\mathbf{q} + \mathbf{r}) + \mathbf{p}.(\mathbf{q} - \mathbf{r})$

b $\mathbf{p}.(\mathbf{q} + \mathbf{r}) + \mathbf{q}.(\mathbf{r} - \mathbf{p})$

4

Show that the vectors $(5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$ and $(3\mathbf{i} + \mathbf{j} - 6\mathbf{k})$ are perpendicular.

5

Relative to a fixed origin O, the points A, B and C have position vectors $(3\mathbf{i} + 4\mathbf{j} - 6\mathbf{k})$, $(\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$ and $(8\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ respectively. Show that $\angle ABC = 90°$.

Exercise B: Scalar Product

6

Find in each case the value or values of the constant c for which the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

a $\mathbf{u} = (2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$, $\mathbf{v} = (c\mathbf{i} - 3\mathbf{j} + \mathbf{k})$

b $\mathbf{u} = (-5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$, $\mathbf{v} = (c\mathbf{i} - \mathbf{j} + 3c\mathbf{k})$

c $\mathbf{u} = (c\mathbf{i} - 2\mathbf{j} + 8\mathbf{k})$, $\mathbf{v} = (c\mathbf{i} + c\mathbf{j} - 3\mathbf{k})$

d $\mathbf{u} = (3c\mathbf{i} + 2\mathbf{j} + c\mathbf{k})$, $\mathbf{v} = (5\mathbf{i} - 4\mathbf{j} + 2c\mathbf{k})$

7

Find, in degrees to 1 decimal place, the angle between the vectors

a $(3\mathbf{i} - 4\mathbf{k})$ and $(7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k})$

b $(2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k})$ and $(\mathbf{i} - 3\mathbf{j} - \mathbf{k})$

c $(6\mathbf{i} - 2\mathbf{j} - 9\mathbf{k})$ and $(3\mathbf{i} + \mathbf{j} + 4\mathbf{k})$

d $(\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$ and $(-3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$

8

$\mathbf{a} = -5\mathbf{i} + 7\mathbf{j}$ and $\mathrm{b} = x\mathbf{i} + y\mathbf{j}$

Given that the resultant force of $\mathbf{a}$ and $\mathbf{b}$ is $-2\mathbf{i} - 3\mathbf{j}$ find the values of x and y

9

Given vectors $\underline{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$ calculate:

a) The component of $\underline{a}$ in the direction of $\underline{b}$

b) The component of $\underline{b}$ in the direction of $\underline{a}$

10

The points A (7, 2, –2), B (–1, 6, –3) and C (–3, 1, 2) are the vertices of a triangle.

a Find $\overrightarrow{BA}$ and $\overrightarrow{BC}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

b Show that $\angle ABC = 82.2°$ to 1 decimal place.

c Find the area of triangle ABC to 3 significant figures.

Exercise C: Vector Product

1

If $\underline{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix}$ calculate $\underline{a} \times \underline{b}$

2

If $\boldsymbol{c} = 12\boldsymbol{i}+13\boldsymbol{j}$ and $\boldsymbol{d} = 7\boldsymbol{i}+3\boldsymbol{j}$ calculate $\boldsymbol{c} \times \boldsymbol{d}$

3

If $\underline{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ show that $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

4

Given the three points $A(9,1,-2)$, $B(3,1,3)$ and $C(1,0,-1)$, find $\overrightarrow{AB} \times \overrightarrow{AC}$

5

a) Find a vector which is perpendicular to both $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

b) Hence, find a **unit vector** perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

6

Find a unit vector which is perpendicular to the plane containing $6\boldsymbol{i} + \boldsymbol{k}$ and $2\boldsymbol{i} + \boldsymbol{j}$

Exercise C: Vector Product

7

For vectors $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 2 \\ -2 \\ -5 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$, evaluate:

a) $(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$

b) $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$

c) Comment on the results of parts (a) and (b)

8

For an unknown vector $\underline{a}$ and the vector $\underline{b} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$, $\underline{a} \times \underline{b} = \begin{pmatrix} 8 \\ -6 \\ k \end{pmatrix}$. Find the value of k.

9

Given that $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2c \\ -5 \end{pmatrix}$, find the values of a, b and c

10

For all vectors $\underline{a}$ and $\underline{b}$, show that

$$(\underline{a} \times \underline{b}) \cdot (\underline{a} \times \underline{b}) = |\underline{a}|^2 |\underline{b}|^2 - (\underline{a} \cdot \underline{b})^2$$

11

Three distinct points P, Q and R have position vectors $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{r}$ respectively. Show that P, Q and R are collinear (lie on a straight line) if

$$\boldsymbol{p} \times \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{p} = 0$$

ANSWERS

Exercise A: Answers

1

a) $-\mathbf{p}$ b) $2\mathbf{q}$ c) $\frac{1}{2}\mathbf{p}$ d) $\mathbf{p}$ e) $-\mathbf{q}$ f) $\mathbf{p}+\mathbf{q}$

g) $\frac{1}{2}\mathbf{p}+2\mathbf{q}$ h) $\mathbf{p}-\mathbf{q}$ i) $2\mathbf{q}-\mathbf{p}$ j) $-\mathbf{p}-2\mathbf{q}$ k) $\frac{1}{2}\mathbf{p}-\mathbf{q}$ l) $-\frac{1}{2}\mathbf{p}-2\mathbf{q}$

2

a parallel, $3\mathbf{p} = \frac{3}{2}(2\mathbf{p})$

b not parallel

c parallel, $(\mathbf{p} - \frac{1}{3}\mathbf{q}) = \frac{1}{3}(3\mathbf{p} - \mathbf{q})$

d parallel, $(4\mathbf{q} - 2\mathbf{p}) = -2(\mathbf{p} - 2\mathbf{q})$

e parallel, $(6\mathbf{p} + 8\mathbf{q}) = 8(\frac{3}{4}\mathbf{p} + \mathbf{q})$

f not parallel

3

a $= (2\mathbf{m} + 3\mathbf{n}) - (4\mathbf{m} + 2\mathbf{n})$
$= \mathbf{n} - 2\mathbf{m}$

b $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OC} = \mathbf{m} + \frac{3}{2}\mathbf{n}$

$\overrightarrow{AM} = (\mathbf{m} + \frac{3}{2}\mathbf{n}) - 4\mathbf{m} = \frac{3}{2}\mathbf{n} - 3\mathbf{m}$

$\therefore\ \overrightarrow{AM} = \frac{3}{2}\overrightarrow{BC}$

$\therefore$ AM is parallel to BC

4

a $= \begin{pmatrix}1\\3\\-1\end{pmatrix} + 2\begin{pmatrix}4\\-2\\1\end{pmatrix} = \begin{pmatrix}9\\-1\\1\end{pmatrix}$

b $= \begin{pmatrix}1\\3\\-1\end{pmatrix} - \begin{pmatrix}-2\\5\\-3\end{pmatrix} = \begin{pmatrix}3\\-2\\2\end{pmatrix}$

c $= \begin{pmatrix}1\\3\\-1\end{pmatrix} + \begin{pmatrix}4\\-2\\1\end{pmatrix} + \begin{pmatrix}-2\\5\\-3\end{pmatrix} = \begin{pmatrix}3\\6\\-3\end{pmatrix}$

d $= 2\begin{pmatrix}1\\3\\-1\end{pmatrix} - 3\begin{pmatrix}4\\-2\\1\end{pmatrix} + \begin{pmatrix}-2\\5\\-3\end{pmatrix} = \begin{pmatrix}-12\\17\\-8\end{pmatrix}$

5

a) $\overrightarrow{OP} = \underline{p} = 9\underline{i} + \underline{j}$ $\quad \overrightarrow{OQ} = \underline{q} = 8\underline{i} - 3\underline{j} + 5\underline{k}$ $\quad \overrightarrow{OR} = \underline{r} = 5\underline{i} + 5\underline{j} + 7\underline{k}$

b) $\overrightarrow{PQ} = \underline{q} - \underline{p} = -\underline{i} - 4\underline{j} + 5\underline{k}$ $\quad \overrightarrow{QR} = \underline{r} - \underline{q} = -3\underline{i} + 8\underline{j} + 2\underline{k}$

c) $|\overrightarrow{PQ}| = \sqrt{42}$ $\quad |\overrightarrow{QR}| = \sqrt{77}$

Exercise A: Answers

6

a $a = 5,\ b = 3$

b $2 + b = 0$ and $a - 4 = 0$
$\therefore\ a = 4,\ b = -2$

c $-1 = b$ and $4a = -2$
$\therefore\ a = -\frac{1}{2},\ b = -1$

d $2a + 6 = 0$ and $b - a = 0$
$\therefore\ a = -3,\ b = -3$

7

a $\overrightarrow{BC} = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix} - \begin{pmatrix} 12 \\ -7 \\ -4 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \\ -4 \end{pmatrix}$

$\overrightarrow{OM} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}$

$= \begin{pmatrix} 12 \\ -7 \\ -4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -6 \\ 8 \\ -4 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ -6 \end{pmatrix}$

b $\overrightarrow{OM} = \frac{3}{2}\overrightarrow{OA}$

$\therefore$ $\overrightarrow{OM}$ and $\overrightarrow{OA}$ are parallel
common point O
$\therefore$ O, A and M are collinear

8

a $|5\mathbf{i} - 2\mathbf{j} + 14\mathbf{k}| = \sqrt{25 + 4 + 196} = 15$
$\therefore\ \frac{1}{15}(5\mathbf{i} - 2\mathbf{j} + 14\mathbf{k})$

b $|2\mathbf{i} + 11\mathbf{j} - 10\mathbf{k}| = \sqrt{4 + 121 + 100} = 15$
$\therefore\ \frac{10}{15}(2\mathbf{i} + 11\mathbf{j} - 10\mathbf{k}) = \frac{2}{3}(2\mathbf{i} + 11\mathbf{j} - 10\mathbf{k})$

c $|-5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}| = \sqrt{25 + 16 + 4} = \sqrt{45} = 3\sqrt{5}$
$\therefore\ \frac{20}{3\sqrt{5}}(-5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = \frac{4}{3}\sqrt{5}\,(-5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$

d $|-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}| = \sqrt{1 + 4 + 1} = \sqrt{6}$
Antiparallel vector is $-\frac{1}{\sqrt{6}}(-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$ or $\frac{1}{\sqrt{6}}(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

9

$\lambda^2 + 144 + 16 = 14^2 = 196$
$\lambda^2 = 36$
$\lambda = \pm 6$

10

a $-2\mathbf{i} + \lambda\mathbf{j} + \mu\mathbf{k} = -\frac{1}{2}(4\mathbf{i} + 2\mathbf{j} - 8\mathbf{k})$
$\therefore\ \lambda = -1,\ \mu = 4$

b $-2\mathbf{i} + \lambda\mathbf{j} + \mu\mathbf{k} = \frac{2}{5}(-5\mathbf{i} + 20\mathbf{j} - 10\mathbf{k})$
$\therefore\ \lambda = 8,\ \mu = -4$

Exercise A: Answers

11

a $2\mathbf{p} - \mathbf{q} = 2(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) - (-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$
$= 3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$

$\therefore\ |2\mathbf{p} - \mathbf{q}| = \sqrt{9+36+36} = 9$

b $(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) + k(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$
$= l(2\mathbf{i} - 4\mathbf{j} - 7\mathbf{k})$

$\therefore\quad 1 - k = 2l \quad (1)$
$-2 + 2k = -4l \quad (2)$
$4 + 2k = -7l \quad (3)$

[(1) and (2) are the same equation]

$(2) - (3) \Rightarrow -6 = 3l$
$\therefore\ l = -2$
$\therefore\ k = 5$

12

a $(\lambda\mathbf{i} - 2\lambda\mathbf{j} + \mu\mathbf{k}) = k(2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$

$\therefore\quad \lambda = 2k \quad (1)$

$-2\lambda = -4k \quad (2)$

$\mu = -3k \quad (3)$

[(1) and (2) are the same equation]
$3\times(1) + 2\times(3) \quad \Rightarrow \quad \underline{3\lambda + 2\mu = 0}$

b $\lambda^2 + (-2\lambda)^2 + \mu^2 = (2\sqrt{29})^2$
$5\lambda^2 + \mu^2 = 116$
$\mu = -\frac{3}{2}\lambda \quad \Rightarrow \quad 5\lambda^2 + \frac{9}{4}\lambda^2 = 116$
$\lambda^2 = 16$
$\lambda = \pm 4$
$\mu = -\frac{3}{2}\lambda$ and $\mu > 0\ \therefore\ \lambda = -4,\ \mu = 6$

13

a $d^2 = (9-t)^2 + (1+2t)^2 + (5-t)^2$
$= 81 - 18t + t^2 + 1 + 4t + 4t^2 + 25 - 10t + t^2$
$= 6t^2 - 24t + 107$

b $d^2 = 6(t^2 - 4t) + 107 = 6[(t-2)^2 - 4] + 107$
$= 6(t-2)^2 + 83$

$\therefore$ closest when $t = 2$

min. $d = \sqrt{83} = 9.11$ m (3sf)

Exercise B: Answers

1

a $= 3 + 2 + 8 = 13$

b $= 6 + 6 - 2 = 10$

c $= -5 + 0 - 6 = -11$

d $= -3 + 22 + 32 = 51$

e $= 27 - 28 - 1 = -2$

f $= 0 + 9 + 0 = 9$

2

a $= (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}).(\mathbf{i} + 5\mathbf{j} - \mathbf{k}) = 2 + 5 + 3 = 10$

b $= (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}).(6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = 12 - 2 + 9 = 19$

c $\mathbf{q} + \mathbf{r} = (\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + (6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = 7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

$\mathbf{p}.(\mathbf{q} + \mathbf{r}) = (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}).(7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = 14 + 3 + 12 = 29$

$\mathbf{p}.\mathbf{q} + \mathbf{p}.\mathbf{r} = 10 + 19 = 29$

$\therefore\ \mathbf{p}.(\mathbf{q} + \mathbf{r}) = \mathbf{p}.\mathbf{q} + \mathbf{p}.\mathbf{r}$

3

a $= \mathbf{p}.\mathbf{q} + \mathbf{p}.\mathbf{r} + \mathbf{p}.\mathbf{q} - \mathbf{p}.\mathbf{r}$

$= 2\mathbf{p}.\mathbf{q}$

b $= \mathbf{p}.\mathbf{q} + \mathbf{p}.\mathbf{r} + \mathbf{q}.\mathbf{r} - \mathbf{q}.\mathbf{p}$

$= \mathbf{p}.\mathbf{q} + \mathbf{p}.\mathbf{r} + \mathbf{q}.\mathbf{r} - \mathbf{p}.\mathbf{q}$

$= \mathbf{p}.\mathbf{r} + \mathbf{q}.\mathbf{r}$

$= (\mathbf{p} + \mathbf{q}).\mathbf{r}$

4

$(5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}).(3\mathbf{i} + \mathbf{j} - 6\mathbf{k}) = 15 - 3 - 12 = 0$

$\therefore$ perpendicular

5

$\overrightarrow{BA} = (3\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}) - (\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$

$\overrightarrow{BC} = (8\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}) = 7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

$\overrightarrow{BA} . \overrightarrow{BC} = (2\mathbf{i} - \mathbf{j} - 4\mathbf{k}).(7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = 14 + 2 - 16 = 0$

$\therefore\ \overrightarrow{BA}$ and $\overrightarrow{BC}$ are perpendicular $\therefore\ \angle ABC = 90°$

Exercise B: Answers

6

a $(2\mathbf{i}+3\mathbf{j}+\mathbf{k}).(c\mathbf{i}-3\mathbf{j}+\mathbf{k})=0$
$2c-9+1=0$
$c=4$

b $(-5\mathbf{i}+3\mathbf{j}+2\mathbf{k}).(c\mathbf{i}-\mathbf{j}+3c\mathbf{k})=0$
$-5c-3+6c=0$
$c=3$

c $(c\mathbf{i}-2\mathbf{j}+8\mathbf{k}).(c\mathbf{i}+c\mathbf{j}-3\mathbf{k})=0$
$c^2-2c-24=0$
$(c+4)(c-6)=0$
$c=-4, 6$

d $(3c\mathbf{i}+2\mathbf{j}+c\mathbf{k}).(5\mathbf{i}-4\mathbf{j}+2c\mathbf{k})=0$
$15c-8+2c^2=0$
$2c^2+15c-8=0$
$(2c-1)(c+8)=0$
$c=-8, \frac{1}{2}$

7

a $|(3\mathbf{i}-4\mathbf{k})| = \sqrt{9+16} = 5$
$|(7\mathbf{i}-4\mathbf{j}+4\mathbf{k})| = \sqrt{49+16+16} = 9$
$(3\mathbf{i}-4\mathbf{k}).(7\mathbf{i}-4\mathbf{j}+4\mathbf{k}) = 21+0-16 = 5$
$\therefore$ angle $= \cos^{-1}\frac{5}{5\times9} = \cos^{-1}\frac{1}{9} = 83.6°$

b $|(2\mathbf{i}-6\mathbf{j}+3\mathbf{k})| = \sqrt{4+36+9} = 7$
$|(\mathbf{i}-3\mathbf{j}-\mathbf{k})| = \sqrt{1+9+1} = \sqrt{11}$
$(2\mathbf{i}-6\mathbf{j}+3\mathbf{k}).(\mathbf{i}-3\mathbf{j}-\mathbf{k}) = 2+18-3 = 17$
$\therefore$ angle $= \cos^{-1}\frac{17}{7\times\sqrt{11}} = 42.9°$

c $|(6\mathbf{i}-2\mathbf{j}-9\mathbf{k})| = \sqrt{36+4+81} = 11$
$|(3\mathbf{i}+\mathbf{j}+4\mathbf{k})| = \sqrt{9+1+16} = \sqrt{26}$
$(6\mathbf{i}-2\mathbf{j}-9\mathbf{k}).(3\mathbf{i}+\mathbf{j}+4\mathbf{k}) = 18-2-36 = -20$
$\therefore$ angle $= \cos^{-1}\frac{-20}{11\times\sqrt{26}} = 110.9°$

d $|(\mathbf{i}+5\mathbf{j}-3\mathbf{k})| = \sqrt{1+25+9} = \sqrt{35}$
$|(-3\mathbf{i}-4\mathbf{j}+2\mathbf{k})| = \sqrt{9+16+4} = \sqrt{29}$
$(\mathbf{i}+5\mathbf{j}-3\mathbf{k}).(-3\mathbf{i}-4\mathbf{j}+2\mathbf{k}) = -3-20-6 = -29$
$\therefore$ angle $= \cos^{-1}\frac{-29}{\sqrt{35}\times\sqrt{29}} = \cos^{-1}\left(-\sqrt{\frac{29}{35}}\right) = 155.5°$

8

Resultant of $\underline{a}$ and $\underline{b}$ is $\underline{a}+\underline{b}$

$\therefore \begin{pmatrix}-5\\7\end{pmatrix}+\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-2\\-3\end{pmatrix} \Rightarrow x=3, y=-10$

Exercise B: Answers

9

a) The component of $\underline{a}$ in the direction of $\underline{b}$ is $\underline{a} \cdot \hat{\underline{b}}$

$$\hat{\underline{b}} = \frac{\underline{b}}{|\underline{b}|} = \frac{1}{\sqrt{29}}\begin{pmatrix}-2\\3\\4\end{pmatrix} \Rightarrow \underline{a} \cdot \hat{\underline{b}} = \frac{1}{\sqrt{29}}\begin{pmatrix}1\\0\\3\end{pmatrix} \cdot \begin{pmatrix}-2\\3\\4\end{pmatrix} = \frac{10}{\sqrt{29}}$$

b) The component of $\underline{b}$ in the direction of $\underline{a}$ is $\underline{b} \cdot \hat{\underline{a}}$

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\0\\3\end{pmatrix} \Rightarrow \underline{b} \cdot \hat{\underline{a}} = \frac{1}{\sqrt{10}}\begin{pmatrix}-2\\3\\4\end{pmatrix} \cdot \begin{pmatrix}1\\0\\3\end{pmatrix} = \frac{10}{\sqrt{10}} = \sqrt{10}$$

Notice these are not the same (imagine the shadow of one vector on the other, they would not look the same)

10

a $\overrightarrow{BA} = (7+1)\mathbf{i} + (2-6)\mathbf{j} + (-2+3)\mathbf{k} = 8\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

$\overrightarrow{BC} = (-3+1)\mathbf{i} + (1-6)\mathbf{j} + (2+3)\mathbf{k} = -2\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$

b $|\overrightarrow{BA}| = \sqrt{64+16+1} = 9$

$|\overrightarrow{BC}| = \sqrt{4+25+25} = 3\sqrt{6}$

$\overrightarrow{BA}.\overrightarrow{BC} = (8\mathbf{i} - 4\mathbf{j} + \mathbf{k}).(-2\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}) = -16 + 20 + 5 = 9$

$\therefore\ \angle ABC = \cos^{-1}\frac{9}{9\times3\sqrt{6}} = \cos^{-1}\frac{1}{3\sqrt{6}} = 82.2°$

c $= \frac{1}{2}\times 9\times 3\sqrt{6}\times \sin 82.18° = 32.8$

Exercise C: Answers

1

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & -1 \\ 5 & -1 & 4 \end{vmatrix}$$

Don't forget the negative on $\underline{j}$

$$= \underline{i}[(2 \times 4) - (-1 \times -1)] - \underline{j}[(3 \times 4) - (-1 \times 5)] + \underline{k}[(3 \times -1) - (2 \times 5)]$$
$$= 7\underline{i} - 12\underline{j} - 13\underline{k}$$

2

$$\boldsymbol{c} \times \boldsymbol{d} = -55\boldsymbol{k}$$

3

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = -5\underline{i} + 10\underline{j} - 5\underline{k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 3 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 5\underline{i} - 10\underline{j} + 5\underline{k}$$

$$\therefore \underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$

4

$$\overrightarrow{AB} = \begin{pmatrix} -6 \\ 0 \\ 5 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -8 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -6 & 0 & 5 \\ -8 & -1 & 1 \end{vmatrix} = 5\underline{i} - 34\underline{j} + 6\underline{k}$$

5

a) $\boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{b} = -11\boldsymbol{i} + 9\boldsymbol{j} - \boldsymbol{k}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

b) Unit vector $\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|} = \frac{1}{\sqrt{203}}(-11\boldsymbol{i} + 9\boldsymbol{j} - \boldsymbol{k})$

Exercise C: Answers

6

Let $\boldsymbol{n} = (6\boldsymbol{i} + \boldsymbol{k}) \times (2\boldsymbol{i} + \boldsymbol{j}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 6 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$

Unit vector $\boldsymbol{\hat{n}} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|} = \frac{\mathbf{1}}{\sqrt{\mathbf{41}}}(-\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$

7

Both give $-29\boldsymbol{i} - 10\boldsymbol{j} + \boldsymbol{k}$ (they are the same) - from identity in lecture notes

8

$\underline{a} \times \underline{b}$ is perpendicular to $\underline{b}$ so $\begin{pmatrix} 8 \\ -6 \\ k \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 0 \Rightarrow 16 - 24 + 6k = 0 \Rightarrow k = \frac{4}{3}$

9

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 1 \\ a & b & c \end{vmatrix} = (2c - b)\underline{i} - (c - a)\underline{j} + (b - 2a)\underline{k} = \begin{pmatrix} 2c - b \\ a - c \\ b - 2a \end{pmatrix}$$

And we were given $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2c \\ -5 \end{pmatrix}$,

Therefore:

$$\begin{aligned} 2c - b &= 3 \\ a - c &= 2c \\ b - 2a &= -5 \end{aligned}$$

Solve simultaneous equations:

$$a = \frac{3}{2}, b = -2, c = \frac{1}{2}$$

Exercise C: Answers

10

Using the definitions: $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\theta\, \underline{\hat{n}}$, $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$
where θ is the angle between $\underline{a}$ and $\underline{b}$ and $\underline{\hat{n}}$ is a unit vector.

$$
\begin{aligned}
(\underline{a} \times \underline{b}) \cdot (\underline{a} \times \underline{b}) &= |\underline{a} \times \underline{b}|^2 \\
&= |\underline{a}|^2 |\underline{b}|^2 \sin^2\theta \\
&= |\underline{a}|^2 |\underline{b}|^2 (1 - \cos^2\theta) \\
&= |\underline{a}|^2 \, |\underline{b}|^2 - |\underline{a}|^2 |\underline{b}|^2 \cos^2\theta \\
&= |\underline{a}|^2 \, |\underline{b}|^2 - (\underline{a} \cdot \underline{b})^2
\end{aligned}
$$

11

The three points are collinear if $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel (as they already share point Q)
Two vectors are parallel if their vector product is zero $\overrightarrow{PQ} \times \overrightarrow{QR} = \mathbf{0}$
As $\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p}$ and $\overrightarrow{QR} = \boldsymbol{r} - \boldsymbol{q}$

So $\overrightarrow{PQ} \times \overrightarrow{QR} = (\boldsymbol{q} - \boldsymbol{p}) \times (\boldsymbol{r} - \boldsymbol{q}) = \mathbf{0}$

$$\boldsymbol{q} \times \boldsymbol{r} - \boldsymbol{q} \times \boldsymbol{q} - \boldsymbol{p} \times \boldsymbol{r} + \boldsymbol{p} \times \boldsymbol{q} = \mathbf{0}$$

Since $\boldsymbol{q} \times \boldsymbol{q} = \mathbf{0}$ and $\boldsymbol{p} \times \boldsymbol{r} = -\boldsymbol{r} \times \boldsymbol{p}$

$$\boldsymbol{p} \times \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{p} = \mathbf{0} \quad \text{as required}$$