CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

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Tutorial: Class Examples

- At the beginning of tutorials, we will usually do a few worked examples.
- I will go through them on the whiteboard but the solutions are also on the slides so you can listen and not have to write down everything I do.
- It may be a good idea to look over the class examples between the lecture and the tutorial so that we can decide as a group which ones we should prioritise.
- I may not go through them all – I will let you decide!
- If no one wants to do any class examples, you can just start doing problems and I will walk around and help.

Tutorial: Tutorial Exercises

- The tutorial exercises are there to allow you to practice problems based on the lecture content.
- They are colour coded by difficulty (see next slide).
- I don't necessarily expect you to do every problem.
- You can target your level using the colour codes and/or your level of prior knowledge.
- You should do enough from green/amber/red until you feel you fully understand it (able to do confidently without using solutions).
- Remember that in the exam there will be no solutions to fall back on.
- You may want to do alternate problems (all odd or all even) and save the other half for revision (your choice).

Tutorial: Question Difficulty Colour Code

Basic - straightforward application
(you must be able to do these)

Medium – Makes you think a bit
(you must be able to do these)

Hard – Makes you think a lot
(you should be able to do these)

Extreme – Tests your understanding to the limit!
(for those who like a challenge)

Applied – Real-life examples of the topic, may sometimes involve prior knowledge
(you should attempt these – will help in future engineering)

LinkedIn

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

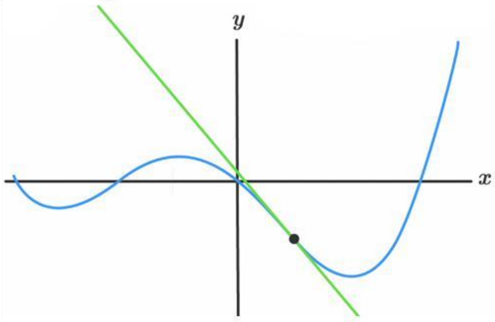
Facebook

$$\frac{df}{dx}$$

Instagram

$$f'(x)$$

Tinder



Tutorial 1
Differentiation Methods

Recap: Stationary points

Finding stationary points

For $y = f(x)$

- **At a stationary point $\frac{dy}{dx} = f'(x) = 0$**
- **Solve $f'(x) = 0$ to find x-values at stationary point(s).**
- **Substitute x-values into $y = f(x)$ to find corresponding y-values**
- **Write as coordinates of stationary points.**

Determining the nature of stationary points

<u>Method 1</u>: find the gradients on the left and right of the stationary points:

$$- \backslash \underset{0}{\_\_\_} / + \quad \Rightarrow \quad \text{min} \qquad + / \overset{0}{\overline{}} \backslash - \quad \Rightarrow \quad \text{max}$$

<u>Method 2</u>: Use second derivative (recommended)

At a stationary point $x = a$:

- **If $f''(a) > 0$ the point is a local minimum.**
- **If $f''(a) < 0$ the point is a local maximum.**
- **If $f''(a) = 0$ it could be any type of point, so resort to Method 1 (find $f'(x)$ either side of $x = a$)**

Class Examples: Product + Chain Rule (If Needed)

E.g. A

If $y = e^{4x}\sin^2 3x$, show that $\frac{dy}{dx} = e^{4x}\sin 3x\,(A\cos 3x + B\sin 3x)$, where A and B are constants to be determined.

$$u = e^{4x} \qquad v = (\sin 3x)^2$$
$$\frac{du}{dx} = 4e^{4x} \qquad \frac{dv}{dx} = 2\sin 3x \times 3\cos 3x = 6\sin 3x\cos 3x$$

$$\frac{dy}{dx} = 6e^{4x}\sin 3x\cos 3x + 4e^{4x}(\sin 3x)^2$$
$$= e^{4x}\sin 3x\,(6\cos 3x + 4\sin 3x)$$

E.g. B

Given that $f(x) = x^2\sqrt{3x-1}$, find $f'(x)$

$$u = x^2 \qquad v = (3x-1)^{\frac{1}{2}}$$
$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = \frac{1}{2}(3x-1)^{-\frac{1}{2}} \times 3 = \frac{3}{2}(3x-1)^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{3}{2}x^2(3x-1)^{-\frac{1}{2}} + 2x(3x-1)^{\frac{1}{2}}$$
$$= \frac{1}{2}x(3x-1)^{-\frac{1}{2}}[3x + 4(3x-1)]$$
$$= \frac{1}{2}x(3x-1)^{-\frac{1}{2}}(15x - 4)$$
$$= \frac{x(15x-4)}{2\sqrt{3x-1}}$$

This is when the manipulation is a little tricky. Remember to:
(a) Factor out the smallest power, in this case $-\frac{1}{2}$. Since a power of $\frac{1}{2}$ is 1 more than $-\frac{1}{2}$, we have an extra $(3x-1)$
(b) Factor out the fraction. $\frac{1}{2}$ goes into both $\frac{3}{2}$ and 2.

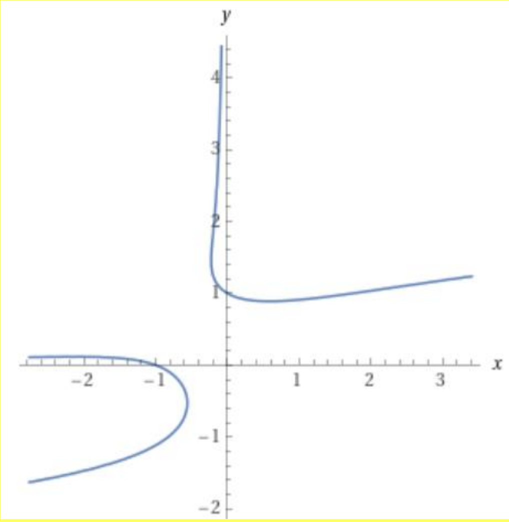
This last step is optional. Try to avoid negative powers, and $\sqrt{}$ is tidier.

Class Example

E.g. 1

Find the gradient of the following curve at $x = 0$:

$$e^{x-y^3} = xy + \frac{1}{e}$$



$$e^{x-y^3}\left(1 - 3y^2\frac{dy}{dx}\right) = y + x\frac{dy}{dx} \qquad (\mathbf{1})$$

If $x = 0$ then

$$e^{-y^3} = \frac{1}{e} \quad \text{and } y = 1. \qquad (\mathbf{1})$$

The gradient at (0,1) is given by

$$e^{-1}\left(1 - 3\frac{dy}{dx}\right) = 1 \quad \Longrightarrow \quad 1 - 3\frac{dy}{dx} = e \qquad (\mathbf{1})$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{(1-e)}{3} \qquad (\mathbf{1})$$

Class Example

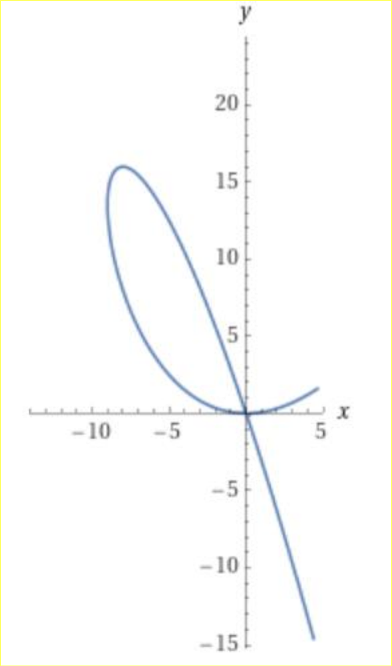
E.g. 2

A curve is described by the equation

$$x^3 - 4y^2 = 12xy.$$

(*a*) Find the coordinates of the two points on the curve where $x = -8$. **(3)**

(*b*) Find the gradient of the curve at each of these points. **(6)**



a) $-512 - 4y^2 = -96y$

$y^2 - 24y + 128 = 0$

$(y - 16)(y - 8) = 0$

Gives point $(-8,16), (-8,8)$

b) Implicitly differentiating:

$$3x^2 - 8y\frac{dy}{dx} = 12x\frac{dy}{dx} + 12y$$

$$\frac{dy}{dx} = \frac{3x^2 - 12y}{12x + 8y}$$

If $x = -8, y = 16, \frac{dy}{dx} = -3$

If $x = -8, y = 8, \frac{dy}{dx} = 0$

Class Example

E.g. 3

$$x^2 + y^2 + 10x + 2y - 4xy = 10$$

(*a*) Find $\frac{dy}{dx}$ in terms of x and y, fully simplifying your answer. **(5)**

(*b*) Find the values of y for which $\frac{dy}{dx} = 0$. **(5)**

Hint for (b): Solve simultaneously with original equation.

	$x^2 + y^2 + 10x + 2y - 4xy = 10$		
(a)	$\left\{\frac{\cancel{dx}}{\cancel{dx}} \times\right\}$ $\underline{2x + 2y\frac{dy}{dx} + 10 + 2\frac{dy}{dx}} - \left(\underline{\underline{4y + 4x\frac{dy}{dx}}}\right) = \underline{0}$	**See notes**	M1 $\underline{\text{A1}}$ $\underline{\underline{\text{M1}}}$
	$2x + 10 - 4y + (2y + 2 - 4x)\frac{dy}{dx} = 0$	**Dependent on the first M1 mark.**	dM1
	$\frac{dy}{dx} = \frac{2x + 10 - 4y}{4x - 2y - 2}$		
	Simplifying gives $\frac{dy}{dx} = \frac{x + 5 - 2y}{2x - y - 1}$ $\left\{= \frac{-x - 5 + 2y}{-2x + y + 1}\right\}$		A1 cso oe
			[5]
(b)	$\left\{\frac{dy}{dx} = 0 \Rightarrow\right\}$ $x + 5 - 2y = 0$		M1
	So $x = 2y - 5$,		
	$(2y - 5)^2 + y^2 + 10(2y - 5) + 2y - 4(2y - 5)y = 10$		M1
	$4y^2 - 20y + 25 + y^2 + 20y - 50 + 2y - 8y^2 + 20y = 10$		
	gives $-3y^2 + 22y - 35 = 0$ or $3y^2 - 22y + 35 = 0$	$3y^2 - 22y + 35 \{= 0\}$ **see notes**	A1 oe
	$(3y - 7)(y - 5) = 0$ and $y = \ldots$	Method mark for solving a quadratic equation.	ddM1
	$y = \frac{7}{3}, 5$	$\{y =\} \frac{7}{3}, 5$	A1 cao
			[5]

Class Example

E.g. 4 A curve is given in parametric form: $x = 3 + 2\cos t, y = 5 - 6\sin t$.
a) Find the Cartesian equation and sketch the curve.
b) Determine any points of intersection with the coordinate axes.
c) Find the equation of the tangent at the point where $t = \frac{\pi}{3}$.

Cartesian equation:

$$(x-3) = 2\cos t, \quad (y-5) = -6\sin t \qquad (1)$$

$$(y-5)^2 = 36\sin^2 t = 36(1-\cos^2 t) = 36\left(1 - \frac{(x-3)^2}{4}\right) \qquad (1)$$

$$\Longrightarrow \quad \frac{(y-5)^2}{36} + \frac{(x-3)^2}{4} = 1 \qquad \text{(Ellipse)} \qquad (1)$$

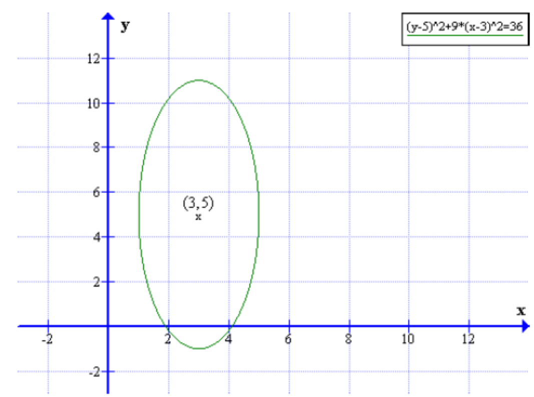
The x-intersections are:

$$\frac{25}{36} + \frac{(x-3)^2}{4} = 1 \qquad (1)$$

$$9x^2 - 54x - 70 = 0 \qquad (1)$$

$$x_{1,2} = \frac{54 \pm \sqrt{54^2 - 9(280)}}{18} = 3 \pm \frac{\sqrt{11}}{3} \qquad (1)$$

Graph (2)



The point where $t = \frac{\pi}{3}$ is

$$x(\pi/3) = 3 + 2\sin(\pi/3) = 3 + 1 = 4$$
$$y(\pi/3) = 5 - 6\cos(\pi/3) = 5 - 3\sqrt{3}$$
$$\Longrightarrow \quad P(4, 5 - 3\sqrt{3}) \qquad (1)$$

The gradient at $t = \frac{\pi}{3}$ given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-6\cos(\pi/3)}{-2\sin(\pi/3)} = 3\cot(\pi/3) = \sqrt{3} \qquad (1)$$

Finally, the equation of the tangent is

$$y - 5 + 3\sqrt{3} = \sqrt{3}(x-4)$$
$$y = \sqrt{3}x + 5 - 7\sqrt{3} \qquad (1)$$

Class Example

E.g. 5

The surface area S in cm² of a 3D solid with width x cm is given by the formula

$$S = \frac{1}{2}\pi x^2 + 3x^2$$

The rate of change of the width is 5 cm/s. Determine the rate of change of the surface area of the solid when $x = 4$.

Represent "the rate of change of the width is 5".

$$\frac{dx}{dt} = 5$$

Differentiate the formula for S, noting it's in terms of x

$$\frac{dS}{dx} = \pi x + 6x$$

We need "the rate of change of surface area", which is $\frac{dS}{dt}$. Use the chain rule.

$$\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt} = (\pi x + 6x) \times 5$$

Substitute in given value of x.

When $x = 4$,

$$\frac{dS}{dt} = 5(4\pi + 24)$$

$$= 182.8 \text{ cm}^2\text{ s}^{-1}$$

Class Example

E.g. 6

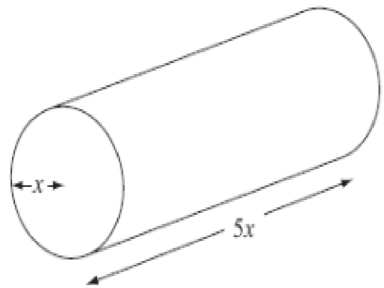


Figure 2 shows a right circular cylindrical metal rod which is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm.
The cross-sectional area of the rod is increasing at the constant rate of 0.032 cm^2 s^{-1}.

(a) Find $\dfrac{dx}{dt}$ when the radius of the rod is 2 cm, giving your answer to 3 significant figures. **(4)**

(b) Find the rate of increase of the volume of the rod when $x = 2$. **(4)**

(a) From info: $\frac{dA}{dt} = 0.032$

$$A = \pi x^2 = \frac{dA}{dx} = 2\pi x$$

$$\frac{dx}{dt} = \frac{dA}{dt} \div \frac{dA}{dx} = \frac{dA}{dt} \times \frac{dx}{dA}$$

$$= 0.032 \times \frac{1}{2\pi x} \quad = \frac{0.016}{\pi x}$$

When $x = 2\ cm \Rightarrow \frac{dx}{dt} = \frac{0.016}{2\pi}$

$$\approx \mathbf{0.00255\ cm\ s^{-1}}$$

(b) $V = \pi r^2 h = \pi x^2(5x) = 5\pi x^3$

$$\frac{dV}{dx} = 15\pi x^2$$

Rate of change of volume is: $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} \quad = 15\pi x^2 \times \left(\frac{0.016}{\pi x}\right) \quad = 0.24x$

When $x = 2\ cm$, $\frac{dV}{dt} = 0.24(2) = \mathbf{0.48\ cm^3 s^{-1}}$

Revision Exercise: Chain, Product, Quotient Rule

Chain

Differentiate with respect to x.

$y=(3x+4)^3$	$y=\sin(2x)$	$y=e^{2x}$
$y=(3x^3+1)^3$	$y=\sin(x^2)$	$y=e^{(x^3)}$
$y=\dfrac{4}{4-x^4}$	$y=\dfrac{1}{\left(\dfrac{1}{\operatorname{cosec} x}\right)}$	$y=\ln\left(\dfrac{1}{e^x}\right)$
$y=\sqrt{\sqrt{x}+1}$	$y=\sin(\cos(\tan x))$	$y=e^{e^x}$

Product

Differentiate with respect to x.

$y=(3x+4)(2x-3)$	$y=x^2\sin x$	$y=xe^x$
$y=(3x^2+4)(2x^4+3)$	$y=\sin x\cos x$	$y=x^3e^x$
$y=\sqrt{3x}\sqrt{2x}$	$y=\sin x\cos x\tan x$	$y=x\ln x$
$y=\left(\dfrac{x}{3}\right)^3\sqrt[3]{x}$	$y=\sin(x^2)\cos(x^2)$	$y=e^{x^2}\ln(x^2)$

Quotient

Differentiate with respect to x.

$y=\dfrac{x+1}{2x+1}$	$y=\dfrac{3x^2}{\sin x}$	$y=\dfrac{2x}{e^x}$
$y=\dfrac{x^2+1}{x^2-1}$	$y=\tan x$	$y=\dfrac{e^x}{e^{-x}}$
$y=\dfrac{x^2}{\sqrt{x}}$	$y=\dfrac{x^2}{\tan x}$	$y=\dfrac{e^{-4x}}{4e^{4x}}$
$y=\sqrt[3]{\dfrac{x+1}{x-1}}$	$y=\dfrac{\cot x}{2\sec x}$	$y=\dfrac{\ln(x^2)}{e^{x^2}}$

Exercise A: Parametric Equations & Differentiation

1

A curve is given by the parametric equations

$$x = 1 + \sin t, \quad y = 2\cos t, \quad 0 \le t < 2\pi.$$

a Write down the coordinates of the point on the curve where $t = \frac{\pi}{2}$.

b Find the value of t at the point on the curve with coordinates $(\frac{3}{2}, -\sqrt{3})$.

Hint: For part (b) you'll get 3 values of t (think of sin and cos graphs). You want the one that agrees with both coordinates.

2

Find a cartesian equation for each curve, given its parametric equations.

a $x = 3t, \quad y = t^2$

b $x = 2t, \quad y = \frac{1}{t}$

c $x = t^3, \quad y = 2t^2$

d $x = 1 - t^2, \quad y = 4 - t$

e $x = 2t - 1, \quad y = \frac{2}{t^2}$

f $x = \frac{1}{t-1}, \quad y = \frac{1}{2-t}$

3

Find a cartesian equation for each curve, given its parametric equations.

a $x = \cos\theta, \quad y = \sin\theta$

b $x = \sin\theta, \quad y = \cos 2\theta$

c $x = 3 + 2\cos\theta, \quad y = 1 + 2\sin\theta$

d $x = 2\sec\theta, \quad y = 4\tan\theta$

e $x = \sin\theta, \quad y = \sin^2 2\theta$

f $x = \cos\theta, \quad y = \tan^2\theta$

4

Write down parametric equations for a circle

a centre $(0, 0)$, radius 5,

b centre $(6, -1)$, radius 2,

c centre (a, b), radius r, where a, b and r are constants and $r > 0$.

Exercise A: Parametric Equations & Differentiation

5

Find and simplify an expression for $\frac{dy}{dx}$ in terms of the parameter t in each case.

a $x = t^2,\ y = 3t$ **b** $x = t^2 - 1,\ y = 2t^3 + t^2$ **c** $x = 2\sin t,\ y = 6\cos t$

d $x = 3t - 1,\ y = 2 - \frac{1}{t}$ **e** $x = \cos 2t,\ y = \sin t$ **f** $x = e^{t+1},\ y = e^{2t-1}$

g $x = \sin^2 t,\ y = \cos^3 t$ **h** $x = 3\sec t,\ y = 5\tan t$ **i** $x = \frac{1}{t+1},\ y = \frac{t}{t-1}$

6

Find, in the form $y = mx + c$, an equation for the tangent to the given curve at the point with the given value of the parameter t.

a $x = t^3,\ y = 3t^2,\quad t = 1$ **b** $x = 1 - t^2,\ y = 2t - t^2,\quad t = 2$

c $x = 2\sin t,\ y = 1 - 4\cos t,\quad t = \frac{\pi}{3}$ **d** $x = \ln(4 - t),\ y = t^2 - 5,\quad t = 3$

7

Show that the normal to the curve with parametric equations

$$x = \sec\theta,\quad y = 2\tan\theta,\quad 0 \le \theta < \frac{\pi}{2},$$

at the point where $\theta = \frac{\pi}{3}$, has the equation

$$\sqrt{3}\,x + 4y = 10\sqrt{3}.$$

8

A curve is given by the parametric equations

$$x = \frac{1}{t},\quad y = \frac{1}{t+2}.$$

a Show that $\frac{dy}{dx} = \left(\frac{t}{t+2}\right)^2$.

b Find an equation for the normal to the curve at the point where $t = 2$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

Exercise A: Parametric Equations & Differentiation

9

A curve has parametric equations

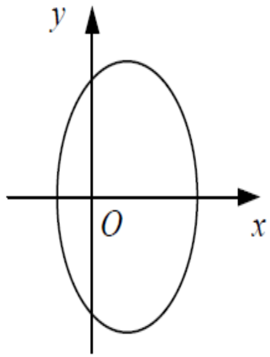
$$x = 3\cos\theta,\quad y = 4\sin\theta,\quad 0 \le \theta < 2\pi.$$

a Show that the tangent to the curve at the point $(3\cos\alpha, 4\sin\alpha)$ has the equation

$$3y\sin\alpha + 4x\cos\alpha = 12.$$

b Hence find an equation for the tangent to the curve at the point $\left(-\frac{3}{2},\ 2\sqrt{3}\right)$.

10



The diagram shows the ellipse with parametric equations

$$x = 1 - 2\cos\theta,\quad y = 3\sin\theta,\quad 0 \le \theta < 2\pi.$$

a Find $\frac{dy}{dx}$ in terms of θ.

b Find the coordinates of the points where the tangent to the curve is

i parallel to the x-axis,

ii parallel to the y-axis.

11

A curve is given by the parametric equations

$$x = \sin\theta,\quad y = \sin 2\theta,\quad 0 \le \theta \le \frac{\pi}{2}.$$

a Find the coordinates of any points where the curve meets the coordinate axes.

b Find an equation for the tangent to the curve that is parallel to the x-axis.

c Find a cartesian equation for the curve in the form $y = f(x)$.

Exercise B: Implicit Differentiation

1

Find $\frac{dy}{dx}$ in terms of x and y in each case.

a $x^2 + y^2 = 2$

b $2x - y + y^2 = 0$

c $y^4 = x^2 - 6x + 2$

d $x^2 + y^2 + 3x - 4y = 9$

e $x^2 - 2y^2 + x + 3y - 4 = 0$

f $\sin x + \cos y = 0$

g $2e^{3x} + e^{-2y} + 7 = 0$

h $\tan x + \operatorname{cosec} 2y = 1$

i $\ln (x - 2) = \ln (2y + 1)$

2

Find $\frac{dy}{dx}$ in terms of x and y in each case.

a $x^2 y = 2$

b $x^2 + 3xy - y^2 = 0$

c $4x^2 - 2xy + 3y^2 = 8$

d $\cos 2x \sec 3y + 1 = 0$

e $y = (x + y)^2$

f $xe^y - y = 5$

g $2xy^2 - x^3 y = 0$

h $y^2 + x \ln y = 3$

i $x \sin y + x^2 \cos y = 1$

3

Find an equation for the tangent to each curve at the given point on the curve.

a $x^2 + y^2 - 3y - 2 = 0$, $(2, 1)$

b $2x^2 - xy + y^2 = 28$, $(3, 5)$

c $4 \sin y - \sec x = 0$, $(\frac{\pi}{3}, \frac{\pi}{6})$

d $2 \tan x \cos y = 1$, $(\frac{\pi}{4}, \frac{\pi}{3})$

4

A curve has the equation $x^2 + 2y^2 - x + 4y = 6$.

a Show that $\frac{dy}{dx} = \frac{1-2x}{4(y+1)}$.

b Find an equation for the normal to the curve at the point $(1, -3)$.

Exercise B: Implicit Differentiation

5

A curve has the equation $y = a^x$, where a is a positive constant.

By first taking logarithms, find an expression for $\frac{dy}{dx}$ in terms of a and x.

6

Differentiate with respect to x

a 3^x **b** 6^{2x} **c** 5^{1-x} **d** 2^{x^3}

7

A curve has the equation $x^2 + 4xy - 3y^2 = 36$.

a Find an equation for the tangent to the curve at the point P (4, 2).

Given that the tangent to the curve at the point Q on the curve is parallel to the tangent at P,

b find the coordinates of Q.

8

Show that if $y = \arccos x$, then $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

9

Given that

$y = \arctan\left(\frac{1-x}{1+x}\right)$, show that $\frac{dy}{dx} = -\frac{1}{1+x^2}$

Exercise C: Connected Rates of Change (Applied)

1

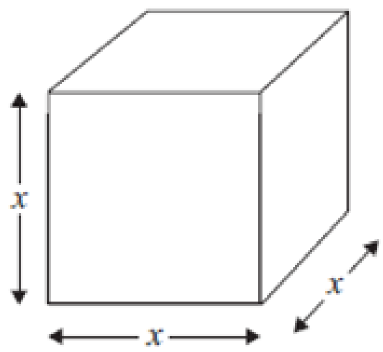


Figure 1

Tip: Don't forget that if you know $\frac{dV}{dx}$ then $\frac{dx}{dV} = \frac{1}{dV/dx}$

Figure 1 shows a metal cube which is expanding uniformly as it is heated.

At time t seconds, the length of each edge of the cube is x cm, and the volume of the cube is V cm^3.

(*a*) Show that $\frac{\mathrm{d}V}{\mathrm{d}x} = 3x^2$.

(1)

Given that the volume, V cm^3, increases at a constant rate of 0.048 cm^3 s^{-1},

(*b*) find $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $x = 8$,

(2)

(*c*) find the rate of increase of the total surface area of the cube, in cm^2 s^{-1}, when $x = 8$.

(3)

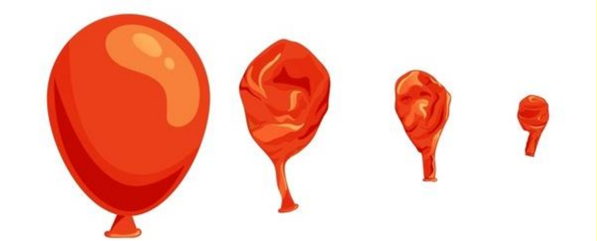
Exercise C: Connected Rates of Change (Applied)

2

A spherical balloon of radius r cm, $r > 0$, deflates at a constant rate of 60 cm^{3}s^{-1}. Calculate the rate of change of the radius with respect to time when $r = 3$.

The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.

Leave your answer in terms of π.



3

A circle with area A is increasing at a constant rate of 2 cm^2 s^{-1}. Determine the rate at which the radius r of the circle is increasing when the area of the circle has area 10 cm^2.

4

A bowl is modelled as a hemispherical shell as shown in Figure 3. Initially the bowl is empty and water begins to flow into the bowl.

When the depth of the water is h cm, the volume of water, V cm^3, according to the model is given by

$$V = \frac{1}{3}\pi h^2(75 - h) \quad 0 \leq h \leq 24$$

The flow of water into the bowl is at a constant rate of 160π cm^3 s^{-1} for $0 \leq h \leq 12$

Find the rate of change of the depth of the water, in cm s^{-1}, when $h = 10$

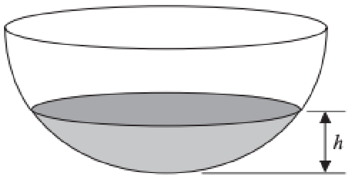
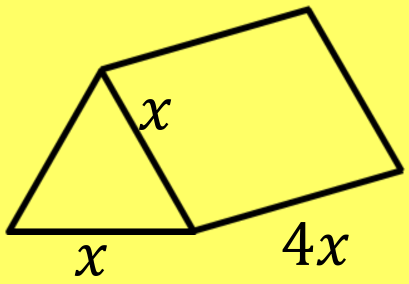


Figure 3

Exercise C: Connected Rates of Change (Applied)

5

A prism with length $4x$ cm has a cross-section that is an equilateral triangle with side length x cm. The volume of the prism is increasing at a rate of 6 cm s^{-1}. Determine the rate of change of x when $x = 1$.



6

The volume of a sphere with radius r cm is increasing at a constant rate of 3 cm^3/s.
Find the rate, in cm^2/s, at which the surface area of the sphere is increasing when $r = 10$

Hint: Three quantities

Challenge Exercise

1 Find the derivative of $(x^2 + y^2)^3 = 5x^2y^2$

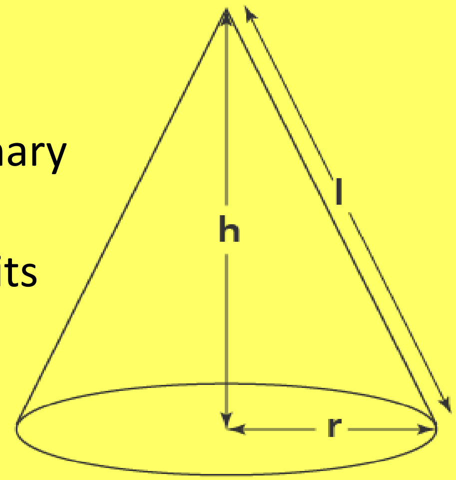
2 Find the derivative of $e^{xy^2} = x - y$

3 A right circular cone has base radius r, height h and slant height l. Its volume V, and the area A of its curved surface, are given by

$$V = \frac{1}{3}\pi r^2 h, \qquad A = \pi r l$$

(i) Given that A is fixed and r is chosen so that V is at its stationary value, show that $A^2 = 3\pi^2 r^4$ and that $l = \sqrt{3}r$.

(ii) Given, instead, that V is fixed and r is chosen so that A is at its stationary value, find h in terms of r.



ANSWERS

Revision Exercise: Solutions

Chain

$y=(3x+4)^3$ $\frac{dy}{dx}=9(3x+4)^2$	$y=sin(2x)$ $\frac{dy}{dx}=2cos(2x)$	$y=e^{2x}$ $\frac{dy}{dx}=2e^{2x}$
$y=(3x^3+1)^3$ $\frac{dy}{dx}=3(3x^3+1)^2(9x^2)$ $=27x^2(3x^3+1)^2$	$y=sin(x^2)$ $\frac{dy}{dx}=2xcos(x^2)$	$y=e^{(x^3)}$ $\frac{dy}{dx}=3x^2e^{(x^3)}$
$y=\frac{4}{4-x^4}=4(4-x^4)^{-1}$ $\frac{dy}{dx}=-4(4-x^4)^{-2}\times -4x^3$ $=\frac{16x^3}{(4-x^4)^2}$	$y=\frac{1}{\left(\frac{1}{cosecx}\right)}=\frac{1}{sinx}$ $=(sinx)^{-1}$ $\frac{dy}{dx}=-(sinx)^{-2}\times cosx$ $=\frac{-cosx}{(sinx)^2}=cotxcosecx$	$y=ln\left(\frac{1}{e^x}\right)$ $\frac{dy}{dx}=\frac{1}{\frac{1}{e^x}}\times -e^{-x}$ $=-e^x\times e^{-x}$ $=-e^0$ $=-1$
$y=\sqrt{\sqrt{x}+1}=\left(x^{\frac{1}{2}}+1\right)^{\frac{1}{2}}$ $\frac{dy}{dx}=\frac{1}{2}\left(x^{\frac{1}{2}}+1\right)^{-\frac{1}{2}}\times\frac{1}{2}x^{-\frac{1}{2}}$ $=\frac{1}{4}\times\frac{1}{\sqrt{\sqrt{x}+1}}\times\frac{1}{\sqrt{x}}$ $=\frac{1}{4\sqrt{x}\sqrt{\sqrt{x}+1}}$	$y=sin\big(cos(tanx)\big)$ $\frac{dy}{dx}=-cos\big(cos(tanx)\big)sin(tanx)sec^2x$	$y=e^{e^x}$ $\frac{dy}{dx}=e^{e^x}e^x$ $=e^{e^x+x}$

Revision Exercise: Solutions

Product

$y=(3x+4)(2x-3)$ $\frac{dy}{dx}=2(3x+4)+3(2x-3)$ $=12x-1$	$y=x^2\sin x$ $\frac{dy}{dx}=x^2\cos x+2x\sin x$	$y=xe^x$ $\frac{dy}{dx}=\ xe^x+e^x$ $=(x+1)e^x$
$y=(3x^2+4)(2x^4+3)$ $\frac{dy}{dx}=8x^3(3x^2+4)+6x(2x^4+3)$ $=2x[4x^2(3x^2+4)+3(2x^4+3)]$ $=2x[12x^4+16x^2+6x^4+9]$ $=2x[18x^4+16x^2+9]$	$y=\sin x\cos x$ $\frac{dy}{dx}=cos^2x-sin^2x$	$y=x^3e^x$ $\frac{dy}{dx}=3x^2e^x+x^3e^x$ $=x^2e^x(x+3)$
$y=\sqrt{3x}\sqrt{2x}$ $=(6x^2)^{\frac{1}{2}}=\sqrt{6}x$ $\frac{dy}{dx}=\sqrt{6}$ (easy method?)	$y=sinx\ cosxtanx$ $=sinxcosx\times\frac{sinx}{cosx}$ $=sin^2x$ $\frac{dy}{dx}=2sinxcosx$	$y=xlnx$ $\frac{dy}{dx}=lnx+1$
$y=\left(\frac{x}{3}\right)^3\sqrt[3]{x}$ $\frac{dy}{dx}=\left(\frac{x}{3}\right)^3\times\frac{1}{3\sqrt[3]{x}}+\sqrt[3]{x}\times\frac{x^2}{3}$ $=\frac{x^3}{9\sqrt[3]{x}}+\frac{x^2\sqrt[3]{x}}{3}$ $=\frac{x^2}{3}\left(\frac{\sqrt[3]{x^2}}{3}+\sqrt[3]{x}\right)$	$y=sin(x^2)cos(x^2)$ $\frac{dy}{dx}=2x[cos^2(x^2)+sin^2(x^2)]$	$y=e^{x^2}ln(x^2)$ $\frac{dy}{dx}=\frac{2e^{x^2}}{x}+2xe^{x^2}ln(x^2)$

Revision Exercise: Solutions

Quotient

$$y = \frac{x+1}{2x+1}$$
$$\frac{dy}{dx} = \frac{2x+1-2(x+1)}{(2x+1)^2} = \frac{-1}{(2x+1)^2}$$

$$y = \frac{3x^2}{sinx}$$
$$\frac{dy}{dx} = \frac{6xsinx - 3x^2cosx}{sin^2x} = \frac{3x}{sinx}\left(2 - \frac{x}{tanx}\right)$$

$$y = \frac{2x}{e^x}$$
$$\frac{dy}{dx} = \frac{2e^x - 2xe^x}{e^{2x}} = \frac{2-2x}{e^x}$$

$$y = \frac{x^2+1}{x^2-1}$$
$$\frac{dy}{dx} = \frac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$$y = tanx = \frac{sinx}{cosx}$$
$$\frac{dy}{dx} = \frac{cos^2x + sin^2x}{cos^2x} = 1 + sin^2x = sec^2x$$

$$y = \frac{e^x}{e^{-x}} = e^x \times e^x = e^{2x}$$
$$\frac{dy}{dx} = 2e^{2x}$$

$$y = \frac{x^2}{\sqrt{x}}$$
$$\frac{dy}{dx} = \frac{2x\sqrt{x} - \frac{x^2}{2\sqrt{x}}}{x} = 2\sqrt{x} - \frac{\sqrt{x}}{2}$$

$$y = \frac{x^2}{tanx}$$
$$\frac{dy}{dx} = \frac{2xtanx - x^2secx}{tan^2x} = \frac{2x}{tanx} - \frac{x^2}{tan^2xcosx} = \frac{x}{tanx}\left(2 - \frac{x}{tanxcosx}\right)$$

$$y = \frac{e^{-4x}}{4e^{4x}} = \frac{1}{4}(e^{-4x} \times e^{-4x}) = \frac{e^{-8x}}{4}$$
$$\frac{dy}{dx} = \frac{-2}{e^{8x}}$$

$$y = \sqrt[3]{\frac{x+1}{x-1}} = \frac{(x+1)^{\frac{1}{3}}}{(x-1)^{\frac{1}{3}}}$$
$$\frac{dy}{dx} = \frac{\frac{1}{3}(x-1)^{\frac{1}{3}}(x+1)^{\frac{-2}{3}} - \frac{1}{3}(x-1)^{-\frac{2}{3}}(x+1)^{\frac{1}{3}}}{(x-1)^{\frac{2}{3}}}$$
$$= \frac{1}{3}\left[\frac{(x-1)^{-\frac{1}{3}}}{(x+1)^{\frac{2}{3}}} - \frac{(x+1)^{\frac{1}{3}}}{(x-1)^{\frac{4}{3}}}\right]$$
$$= \frac{1}{3}\left[\frac{(x-1)-(x+1)}{(x+1)^{\frac{2}{3}}(x-1)^{\frac{4}{3}}}\right]$$
$$= \frac{-2}{3}\left[\frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}}\right]$$

$$y = \frac{cotx}{2secx} = \frac{cosx}{sinx} \div \frac{2}{cosx} = \frac{cosx}{sinx} \times \frac{cosx}{2} = \frac{cos^2x}{2sinx}$$
$$\frac{dy}{dx} = \frac{-4sin^2xcosx - 2cos^3x}{4sin^2x} = -xcosx - \frac{cosx}{2tan^2x}$$

$$y = \frac{ln(x^2)}{e^{x^2}}$$
$$\frac{dy}{dx} = \frac{\frac{2e^{x^2}}{x} - 2xe^{x^2}lnx^2}{e^{2x^2}} = \frac{\frac{2}{x} - 2xlnx^2}{e^{x^2}}$$

Exercise A: Solutions

1

a $(2, 0)$

b $1 + \sin t = \frac{3}{2}$, $\sin t = \frac{1}{2}$, $t = \frac{\pi}{6}, \frac{5\pi}{6}$

$2\cos t = -\sqrt{3}$, $\cos t = -\frac{\sqrt{3}}{2}$, $t = \frac{5\pi}{6}, \frac{7\pi}{6}$

$\therefore\ t = \frac{5\pi}{6}$

2

a $t = \frac{x}{3}$ $\therefore\ y = (\frac{x}{3})^2$

$y = \frac{1}{9}x^2$

b $t = \frac{x}{2}$ $\therefore y = \frac{1}{(\frac{x}{2})}$

$y = \frac{2}{x}$

c $x^2 = t^6$, $y^3 = 8t^6$

$\therefore\ y^3 = 8x^2$

d $t = 4 - y$

$\therefore\ x = 1 - (4 - y)^2$

e $t = \frac{1}{2}(x + 1)$

$\therefore\ y = \frac{2}{\frac{1}{4}(x+1)^2}$

$y = \frac{8}{(x+1)^2}$

f $t = \frac{1}{x} + 1$

$\therefore\ y = \frac{1}{2-(\frac{1}{x}+1)} = \frac{1}{1-\frac{1}{x}}$

$y = \frac{x}{x-1}$

3

a $\cos^2\theta + \sin^2\theta = 1$

$\therefore\ x^2 + y^2 = 1$

b $\cos 2\theta = 1 - 2\sin^2\theta$

$\therefore\ y = 1 - 2x^2$

c $\cos\theta = \frac{x-3}{2}$, $\sin\theta = \frac{y-1}{2}$

$\cos^2\theta + \sin^2\theta = 1$

$\therefore\ (\frac{x-3}{2})^2 + (\frac{y-1}{2})^2 = 1$

$(x-3)^2 + (y-1)^2 = 4$

d $\sec\theta = \frac{x}{2}$, $\tan\theta = \frac{y}{4}$

$1 + \tan^2\theta = \sec^2\theta$

$\therefore\ 1 + (\frac{y}{4})^2 = (\frac{x}{2})^2$

$16 + y^2 = 4x^2$

$y^2 = 4x^2 - 16$

e $\sin 2\theta = 2\sin\theta\cos\theta$

$\therefore\ y = 4\sin^2\theta\cos^2\theta$

$y = 4\sin^2\theta(1 - \sin^2\theta)$

$y = 4x^2(1 - x^2)$

f $\sec\theta = \frac{1}{x}$

$1 + \tan^2\theta = \sec^2\theta$

$\therefore\ 1 + y = (\frac{1}{x})^2$

$y = \frac{1}{x^2} - 1$

4

a $x = 5\cos\theta$, $y = 5\sin\theta$, $0 \le \theta < 2\pi$

b $x = 6 + 2\cos\theta$, $y = -1 + 2\sin\theta$, $0 \le \theta < 2\pi$

c $x = a + r\cos\theta$, $y = b + r\sin\theta$, $0 \le \theta < 2\pi$

Exercise A: Solutions

5

a $\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3}{2t}$

b $\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 6t^2 + 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{6t^2 + 2t}{2t} = 3t + 1$

c $\frac{dx}{dt} = 2\cos t, \quad \frac{dy}{dt} = -6\sin t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-6\sin t}{2\cos t} = -3\tan t$

d $\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = t^{-2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{t^{-2}}{3}$

$= \frac{1}{3t^2}$

e $\frac{dx}{dt} = -2\sin 2t, \quad \frac{dy}{dt} = \cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\cos t}{-2\sin 2t}$

$= \frac{\cos t}{-4\sin t\cos t} = -\frac{1}{4}\operatorname{cosec} t$

f $\frac{dx}{dt} = e^{t+1}, \quad \frac{dy}{dt} = 2e^{2t-1}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2e^{2t-1}}{e^{t+1}}$

$= 2e^{t-2}$

g $\frac{dx}{dt} = 2\sin t \times \cos t,$

$\frac{dy}{dt} = 3\cos^2 t \times (-\sin t)$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-3\cos^2 t\sin t}{2\sin t\cos t}$

$= -\frac{3}{2}\cos t$

h $\frac{dx}{dt} = 3\sec t\tan t,$

$\frac{dy}{dt} = 5\sec^2 t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{5\sec^2 t}{3\sec t\tan t}$

$= \frac{5\sec t}{3\tan t} = \frac{5}{3}\operatorname{cosec} t$

i $\frac{dx}{dt} = -(t+1)^{-2},$

$\frac{dy}{dt} = \frac{1\times(t-1) - t\times 1}{(t-1)^2} = -(t-1)^{-2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-(t-1)^{-2}}{-(t+1)^{-2}}$

$= \frac{(t+1)^2}{(t-1)^2} = \left(\frac{t+1}{t-1}\right)^2$

7

$\theta = \frac{\pi}{3} \quad \therefore x = 2, \ y = 2\sqrt{3}$

$\frac{dx}{d\theta} = \sec\theta\tan\theta, \quad \frac{dy}{d\theta} = 2\sec^2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{2\sec^2\theta}{\sec\theta\tan\theta}$

$= \frac{2\sec\theta}{\tan\theta} = 2\operatorname{cosec}\theta$

grad $= \frac{4}{\sqrt{3}} \quad \therefore$ grad of normal $= -\frac{\sqrt{3}}{4}$

$\therefore \ y - 2\sqrt{3} = -\frac{\sqrt{3}}{4}(x - 2)$

$4y - 8\sqrt{3} = -\sqrt{3}\,x + 2\sqrt{3}$

$\sqrt{3}\,x + 4y = 10\sqrt{3}$

6

a $t = 1 \quad \therefore x = 1, \ y = 3$

$\frac{dx}{dt} = 3t^2, \quad \frac{dy}{dt} = 6t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{6t}{3t^2} = \frac{2}{t}$

grad $= 2$

$\therefore \ y - 3 = 2(x - 1)$

$y = 2x + 1$

b $t = 2 \quad \therefore x = -3, \ y = 0$

$\frac{dx}{dt} = -2t, \quad \frac{dy}{dt} = 2 - 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2 - 2t}{-2t} = 1 - \frac{1}{t}$

grad $= \frac{1}{2}$

$\therefore \ y - 0 = \frac{1}{2}(x + 3)$

$y = \frac{1}{2}x + \frac{3}{2}$

c $t = \frac{\pi}{3} \quad \therefore x = \sqrt{3}, \ y = -1$

$\frac{dx}{dt} = 2\cos t, \quad \frac{dy}{dt} = 4\sin t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{4\sin t}{2\cos t} = 2\tan t$

grad $= 2\sqrt{3}$

$\therefore \ y + 1 = 2\sqrt{3}\,(x - \sqrt{3})$

$y = 2\sqrt{3}\,x - 7$

d $t = 3 \quad \therefore x = 0, \ y = 4$

$\frac{dx}{dt} = -\frac{1}{4-t} = \frac{1}{t-4}, \quad \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t}{\frac{1}{t-4}} = 2t(t - 4)$

grad $= -6$

$\therefore \ y - 4 = -6(x - 0)$

$y = 4 - 6x$

8

a $\frac{dx}{dt} = -t^{-2}, \quad \frac{dy}{dt} = -(t+2)^{-2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-(t+2)^{-2}}{-t^{-2}}$

$= \frac{t^2}{(t+2)^2} = \left(\frac{t}{t+2}\right)^2$

b $t = 2 \quad \therefore x = \frac{1}{2}, \ y = \frac{1}{4}$

grad $= \frac{1}{4} \quad \therefore$ grad of normal $= -4$

$\therefore \ y - \frac{1}{4} = -4(x - \frac{1}{2})$

$4y - 1 = -16x + 8$

$16x + 4y - 9 = 0$

Exercise A: Solutions

9

a $\frac{dx}{d\theta} = -3\sin\theta, \quad \frac{dy}{d\theta} = 4\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = -\frac{4\cos\theta}{3\sin\theta}$

at $(3\cos\alpha, 4\sin\alpha)$, $\theta = \alpha$

$\therefore$ grad $= -\frac{4\cos\alpha}{3\sin\alpha}$

$\therefore\ y - 4\sin\alpha = -\frac{4\cos\alpha}{3\sin\alpha}(x - 3\cos\alpha)$

$3y\sin\alpha - 12\sin^2\alpha = -4x\cos\alpha + 12\cos^2\alpha$

$3y\sin\alpha + 4x\cos\alpha = 12(\cos^2\alpha + \sin^2\alpha)$

$3y\sin\alpha + 4x\cos\alpha = 12$

b at $(-\frac{3}{2}, 2\sqrt{3})$,

$3\cos\alpha = -\frac{3}{2} \Rightarrow \cos\alpha = -\frac{1}{2}$

$4\sin\alpha = 2\sqrt{3} \Rightarrow \sin\alpha = \frac{\sqrt{3}}{2}$

$\therefore\ 3y \times \frac{\sqrt{3}}{2} + 4x \times (-\frac{1}{2}) = 12$

$4x - 3\sqrt{3}\,y + 24 = 0$

10

a $\frac{dx}{d\theta} = 2\sin\theta, \quad \frac{dy}{d\theta} = 3\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{3\cos\theta}{2\sin\theta}$ or $\frac{3}{2}\cot\theta$

b **i** $\frac{3\cos\theta}{2\sin\theta} = 0 \quad \therefore \cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \therefore (1, 3), (1, -3)$

ii $\frac{3\cos\theta}{2\sin\theta} \to \infty \quad \therefore \sin\theta = 0$ **(y-axis is vertical)**

$\theta = 0, \pi \quad \therefore (-1, 0), (3, 0)$

11

a $x = 0 \Rightarrow \sin\theta = 0 \quad \Rightarrow \theta = 0$

$y = 0 \Rightarrow \sin 2\theta = 0 \quad \Rightarrow \theta = 0, \frac{\pi}{2}$

$\therefore\ (0, 0), (1, 0)$

b $\frac{dx}{d\theta} = \cos\theta, \quad \frac{dy}{d\theta} = 2\cos 2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{2\cos 2\theta}{\cos\theta}$

$\frac{2\cos 2\theta}{\cos\theta} = 0 \quad \therefore \cos 2\theta = 0$

$\theta = \frac{\pi}{4} \quad \therefore\ y = 1$

c $y = \sin 2\theta = 2\sin\theta\cos\theta$

$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$

$0 \le \theta \le \frac{\pi}{2} \quad \therefore \cos\theta = \sqrt{1 - \sin^2\theta}$

$\therefore\ y = 2x\sqrt{1 - x^2}$

Exercise B: Solutions

1

a $2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{x}{y}$

b $2 - \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$2 = \frac{dy}{dx}(1 - 2y)$

$\frac{dy}{dx} = \frac{2}{1-2y}$

c $4y^3\frac{dy}{dx} = 2x - 6$

$\frac{dy}{dx} = \frac{x-3}{2y^3}$

d $2x + 2y\frac{dy}{dx} + 3 - 4\frac{dy}{dx} = 0$

$2x + 3 = \frac{dy}{dx}(4 - 2y)$

$\frac{dy}{dx} = \frac{2x+3}{4-2y}$

e $2x - 4y\frac{dy}{dx} + 1 + 3\frac{dy}{dx} = 0$

$2x + 1 = \frac{dy}{dx}(4y - 3)$

$\frac{dy}{dx} = \frac{2x+1}{4y-3}$

f $\cos x - \frac{dy}{dx}\sin y = 0$

$\cos x = \frac{dy}{dx}\sin y$

$\frac{dy}{dx} = \frac{\cos x}{\sin y}$

g $6e^{3x} - 2e^{-2y}\frac{dy}{dx} = 0$

$6e^{3x} = 2e^{-2y}\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{3e^{3x}}{e^{-2y}} = 3e^{3x+2y}$

h $\sec^2 x - 2\frac{dy}{dx}\operatorname{cosec} 2y \cot 2y = 0$

$\sec^2 x = 2\frac{dy}{dx}\operatorname{cosec} 2y \cot 2y$

$\frac{dy}{dx} = \frac{\sec^2 x}{2\operatorname{cosec} 2y \cot 2y}$

i $\frac{1}{x-2} = \frac{2}{2y+1}\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2y+1}{2(x-2)}$

2

a $2x \times y + x^2\frac{dy}{dx} = 0$

$x^2\frac{dy}{dx} = -2xy$

$\frac{dy}{dx} = -\frac{2y}{x}$

b $2x + 3 \times y + 3x \times \frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$2x + 3y = \frac{dy}{dx}(2y - 3x)$

$\frac{dy}{dx} = \frac{2x+3y}{2y-3x}$

c $8x - 2 \times y - 2x \times \frac{dy}{dx} + 6y\frac{dy}{dx} = 0$

$8x - 2y = \frac{dy}{dx}(2x - 6y)$

$\frac{dy}{dx} = \frac{4x-y}{x-3y}$

d $-2\sin 2x \times \sec 3y + \cos 2x \times 3\frac{dy}{dx}\sec 3y \tan 3y = 0$

$3\frac{dy}{dx}\cos 2x \sec 3y \tan 3y = 2\sin 2x \sec 3y$

$\frac{dy}{dx} = \frac{2\sin 2x}{3\cos 2x \tan 3y} = \frac{2}{3}\tan 2x \cot 3y$

e $\frac{dy}{dx} = 2(x+y) \times (1 + \frac{dy}{dx})$

$\frac{dy}{dx}[1 - 2(x+y)] = 2(x+y)$

$\frac{dy}{dx} = \frac{2(x+y)}{1-2(x+y)}$

f $1 \times e^y + x \times e^y\frac{dy}{dx} - \frac{dy}{dx} = 0$

$e^y = \frac{dy}{dx}(1 - xe^y)$

$\frac{dy}{dx} = \frac{e^y}{1-xe^y}$

g $2 \times y^2 + 2x \times 2y\frac{dy}{dx} - 3x^2 \times y - x^3 \times \frac{dy}{dx} = 0$

$2y^2 - 3x^2y = \frac{dy}{dx}(x^3 - 4xy)$

$\frac{dy}{dx} = \frac{2y^2-3x^2y}{x^3-4xy}$

h $2y\frac{dy}{dx} + 1 \times \ln y + x \times \frac{1}{y}\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2y + \frac{x}{y}) = -\ln y$

$\frac{dy}{dx} = -\frac{\ln y}{2y+\frac{x}{y}} = -\frac{y\ln y}{2y^2+x}$

i $1 \times \sin y + x \times \frac{dy}{dx}\cos y + 2x \times \cos y + x^2 \times (-\sin y)\frac{dy}{dx} = 0$

$\sin y + 2x\cos y = \frac{dy}{dx}(x^2\sin y - x\cos y)$

$\frac{dy}{dx} = \frac{\sin y + 2x\cos y}{x^2\sin y - x\cos y}$

Exercise B: Solutions

3

a $2x + 2y\frac{dy}{dx} - 3\frac{dy}{dx} = 0$

$2x = \frac{dy}{dx}(3 - 2y)$

$\frac{dy}{dx} = \frac{2x}{3 - 2y}$

grad $= 4$

$\therefore\ y - 1 = 4(x - 2)$

$[\ y = 4x - 7\ ]$

b $4x - 1 \times y - x \times \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$4x - y = \frac{dy}{dx}(x - 2y)$

$\frac{dy}{dx} = \frac{4x - y}{x - 2y}$

grad $= -1$

$\therefore\ y - 5 = -(x - 3)$

$[\ y = 8 - x\ ]$

c $4\frac{dy}{dx}\cos y - \sec x \tan x = 0$

$4\frac{dy}{dx}\cos y = \sec x \tan x$

$\frac{dy}{dx} = \frac{\sec x \tan x}{4\cos y}$

grad $= \frac{2 \times \sqrt{3}}{4 \times \frac{\sqrt{3}}{2}} = 1$

$\therefore\ y - \frac{\pi}{6} = x - \frac{\pi}{3}$

$[\ y = x - \frac{\pi}{6}\ ]$

d $2\sec^2 x \times \cos y + 2\tan x \times (-\sin y)\frac{dy}{dx} = 0$

$2\sec^2 x \cos y = 2\frac{dy}{dx}\tan x \sin y$

$\frac{dy}{dx} = \frac{\sec^2 x \cos y}{\tan x \sin y}$

grad $= \frac{2 \times \frac{1}{2}}{1 \times \frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$

$\therefore\ y - \frac{\pi}{3} = \frac{2}{3}\sqrt{3}\,(x - \frac{\pi}{4})$

$[\ 4\sqrt{3}\,x - 6y + \pi(2 - \sqrt{3}) = 0\ ]$

4

a $2x + 4y\frac{dy}{dx} - 1 + 4\frac{dy}{dx} = 0$

$\frac{dy}{dx}(4y + 4) = 1 - 2x$

$\frac{dy}{dx} = \frac{1 - 2x}{4(y + 1)}$

b grad $= \frac{1}{8}$

$\therefore$ grad of normal $= -8$

$\therefore\ y + 3 = -8(x - 1)$

$[\ y = 5 - 8x\ ]$

5

$\ln y = \ln a^x$

$\ln y = x \ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = y \ln a = a^x \ln a$

6

a $= 3^x \ln 3$

b $= 6^{2x} \ln 6 \times 2$

$= 2(6^{2x}) \ln 6$

c $= 5^{1-x} \ln 5 \times (-1)$

$= -(5^{1-x}) \ln 5$

d $= 2^{x^3} \ln 2 \times 3x^2$

$= 3x^2(2^{x^3}) \ln 2$

Exercise B: Solutions

7

a $2x + 4 \times y + 4x \times \frac{dy}{dx} - 6y\frac{dy}{dx} = 0$

$2x + 4y = \frac{dy}{dx}(6y - 4x)$

$\frac{dy}{dx} = \frac{x+2y}{3y-2x}$

grad $= -4$

$\therefore\ y - 2 = -4(x - 4)$

$[\ y = 18 - 4x\ ]$

b at Q, $\frac{x+2y}{3y-2x} = -4$

$x + 2y = -4(3y - 2x)$

$x = 2y$

sub. into equation of curve

$\Rightarrow\quad (2y)^2 + 4y(2y) - 3y^2 = 36$

$y^2 = 4$

$y = 2$ (at P) or -2

$\therefore\ Q\,(-4, -2)$

8

Show that if $y = \arccos x$, then $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$y = \arccos\ x$

$\therefore x = \cos y$

$$\frac{dx}{dy} = -\sin y \ \Rightarrow\ \frac{dy}{dx} = \frac{-1}{\sin y}$$

We want our answer in terms of x (which is $\cos y$)

$$\sin y = \sqrt{1 - \cos^2 y}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{\mathbf{-1}}{\sqrt{\mathbf{1 - x^2}}}$$

Exercise C: Differentiating Inverse Trig Functions

9

Given that

$y = \arctan\left(\frac{1-x}{1+x}\right)$, show that $\frac{dy}{dx} = -\frac{1}{1+x^2}$

$y = \arctan u$

By the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Differentiate "arctan":

$u = \tan y$

$$\frac{du}{dy} = \sec^2 y = 1 + \tan^2 u = 1 + u^2$$

$$\therefore \frac{dy}{du} = \frac{1}{1+u^2}$$

Differentiate "bracket":

Let $u = \frac{1-x}{1+x}$

$\frac{du}{dx} = -\frac{2}{(1+x)^2}$ (by quotient rule)

$$= \frac{1}{1+u^2} \times \left(-\frac{2}{(1+x)^2}\right)$$

$$= \cdots$$

$$= -\frac{1}{1+x^2}$$

Eliminate u:

$$\frac{1}{1+u^2} \times \left(-\frac{2}{(1+x)^2}\right) = \frac{-2}{\left(1+\left(\frac{1-x}{1+x}\right)^2\right)(1+x)^2}$$

$$= \frac{-2}{(1+x)^2 + (1-x)^2}$$

$$= \frac{-2}{1+2x+x^2+1-2x+x^2}$$

$$= \frac{-2}{2+2x^2}$$

$$= -\frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{x}^{\mathbf{2}}}$$

Exercise C: Solutions

1

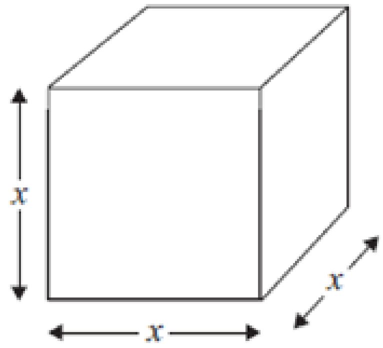


Figure 1

Tip: Don't forget that if you know $\frac{dV}{dx}$ then $\frac{dx}{dV} = \frac{1}{dV/dx}$

Figure 1 shows a metal cube which is expanding uniformly as it is heated.

At time t seconds, the length of each edge of the cube is x cm, and the volume of the cube is V cm^3.

(*a*) Show that $\frac{dV}{dx} = 3x^2$.

(1)

Given that the volume, V cm^3, increases at a constant rate of 0.048 cm^3 s^{-1},

(*b*) find $\frac{dx}{dt}$ when $x = 8$,

(2)

(*c*) find the rate of increase of the total surface area of the cube, in cm^2 s^{-1}, when $x = 8$.

(3)

a) $V = x^3$

$$\frac{dV}{dx} = 3x^2$$

b)

$$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{3 \times 8^2} \times 0.048$$

$$= 0.00025$$

c)

$$A = 6x^2$$

$$\frac{dA}{dx} = 12x$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

$$= (12 \times 8) \times 0.00025$$

$$= 0.024$$

Exercise C: Connected Rates of Change (Applied)

2

A spherical balloon of radius r cm, $r > 0$, deflates at a constant rate of 60 cm^{3}s^{-1}. Calculate the rate of change of the radius with respect to time when $r = 3$.

The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.

Leave your answer in terms of π.

$$\frac{dV}{dt} = -60$$

$$V = \frac{4}{3}\pi r^3 \quad \rightarrow \quad \frac{dV}{dr} = 4\pi r^2$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} \quad = \frac{1}{4\pi r^2} \times -60 = -\frac{15}{\pi r^2}$$

When $r = 3$, $\frac{dr}{dt} = -\frac{15}{\pi(3^2)} = -\frac{5}{3\pi}\,cms^{-1}$

3

A circle with area A is increasing at a constant rate of 2 cm^2 s^{-1}. Determine the rate at which the radius r of the circle is increasing when the area of the circle has area 10 cm^2.

$$\frac{dA}{dt} = 2$$

$$A = \pi r^2 \quad \rightarrow \quad \frac{dA}{dr} = 2\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$$

$$= \frac{1}{2\pi r} \times 2 = \frac{1}{\pi r}$$

When $A = 10$,

$$\pi r^2 = 10 \quad \rightarrow \quad r = \sqrt{\frac{10}{\pi}}$$

$$\therefore \frac{dr}{dt} = \frac{1}{\pi\sqrt{\frac{10}{\pi}}} = \mathbf{0.178\ cm\ s^{-1}}$$

4

A bowl is modelled as a hemispherical shell as shown in Figure 3. Initially the bowl is empty and water begins to flow into the bowl.

When the depth of the water is h cm, the volume of water, V cm^3, according to the model is given by

$$V = \frac{1}{3}\pi h^2(75 - h) \qquad 0 \le h \le 24$$

The flow of water into the bowl is at a constant rate of 160π cm^3 s^{-1} for $0 \le h \le 12$

Find the rate of change of the depth of the water, in cm s^{-1}, when $h = 10$

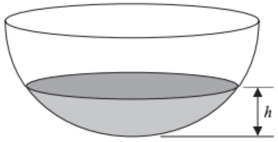


Figure 3

$$\frac{dV}{dt} = 160\pi$$

$$V = 25\pi h^2 - \frac{1}{3}\pi h^3 \quad \rightarrow \quad \frac{dV}{dh} = 50\pi h - \pi h^2$$

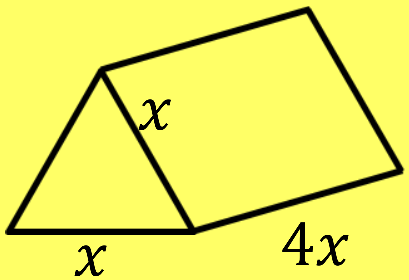
$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{50\pi h - \pi h^2} \times 160\pi$$

When $h = 10$, $\frac{dh}{dt} = \frac{1}{500\pi - 100\pi} \times 160\pi = \mathbf{0.4}\,cms^{-1}$

Exercise C: Solutions

5

A prism with length $4x$ cm has a cross-section that is an equilateral triangle with side length x cm. The volume of the prism is increasing at a rate of 6 cm s^{-1}. Determine the rate of change of x when $x = 1$.



$$\frac{dV}{dt} = 6$$

Cross-section:

$$\frac{1}{2} \times x \times x \times \sin 60 = \frac{\sqrt{3}}{4} x^2$$

$$\therefore V = \frac{\sqrt{3}}{4} x^2 \times 4x = \sqrt{3} x^3$$

$$\frac{dV}{dx} = 3\sqrt{3} x^2$$

$$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt} = \frac{1}{3\sqrt{3}\, x^2} \times 6 = \frac{2}{\sqrt{3} x^2}$$

When $\boldsymbol{x = 1}$, $\boldsymbol{\frac{dx}{dt} = \frac{2}{\sqrt{3}}}$

6

The volume of a sphere with radius r cm is increasing at a constant rate of 3 cm^3/s.
Find the rate, in cm^2/s, at which the surface area of the sphere is increasing when $r = 10$

Hint: Three quantities

$$\frac{dV}{dt} = 3$$

$$V = \frac{4}{3}\pi r^3 \rightarrow \frac{dV}{dr} = 4\pi r^2$$

$$S = 4\pi r^2 \rightarrow \frac{dS}{dr} = 8\pi r$$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt} = 8\pi r \times \frac{1}{4\pi r^2} \times 3 = \frac{6}{r}$$

When $r = 10$, $\frac{dS}{dt} = \frac{6}{10} = \mathbf{0.6}$ **cm^2/s**

Challenge Exercise: Solutions

1 Find the derivative of $(x^2 + y^2)^3 = 5x^2y^2$

$$\begin{aligned}
\frac{d}{dx}(x^2+y^2)^3 &= \frac{d}{dx}5x^2y^2 \\
3(x^2+y^2)^2\frac{d}{dx}(x^2+y^2) &= 5\frac{d}{dx}x^2y^2 \\
3(x^2+y^2)^2(2x+2y\frac{dy}{dx}) &= 5(2xy^2+2y\frac{dy}{dx}x^2) \\
6x(x^2+y^2)^2+6y(x^2+y^2)^2\frac{dy}{dx} &= 10xy^2+10x^2y\frac{dy}{dx} \\
6y(x^2+y^2)^2\frac{dy}{dx}-10x^2y\frac{dy}{dx} &= 10xy^2-6x(x^2+y^2)^2 \\
(6y(x^2+y^2)^2-10x^2y)\frac{dy}{dx} &= 10xy^2-6x(x^2+y^2)^2 \\
\frac{dy}{dx} &= \frac{10xy^2-6x(x^2+y^2)^2}{6y(x^2+y^2)^2-10x^2y}
\end{aligned}$$

2 Find the derivative of $e^{xy^2} = x - y$

$$\begin{aligned}
\frac{d}{dx}e^{xy^2} &= \frac{d}{dx}(x-y) \\
e^{xy^2}\frac{d}{dx}(xy^2) &= 1-\frac{dy}{dx} \\
e^{xy^2}(y^2+x\cdot 2y\frac{dy}{dx}) &= 1-\frac{dy}{dx} \\
y^2e^{xy^2}+2xye^{xy^2}\frac{dy}{dx} &= 1-\frac{dy}{dx} \\
e^{xy^2}2xy\frac{dy}{dx}+\frac{dy}{dx} &= 1-y^2e^{xy^2} \\
(2xye^{xy^2}+1)\frac{dy}{dx} &= 1-y^2e^{xy^2} \\
\frac{dy}{dx} &= \frac{1-y^2e^{xy^2}}{2xye^{xy^2}+1} \\
\frac{dy}{dx} &= \frac{1-y^2(x-y)}{2xy(x-y)+1} \\
\frac{dy}{dx} &= \frac{1-xy^2+y^3}{2x^2y-2xy^2+1}
\end{aligned}$$

3 A right circular cone has base radius r, height h and slant height l. Its volume V, and the area A of its curved surface, are given by

$$V = \frac{1}{3}\pi r^2 h, \qquad A = \pi r l$$

(i) Given that A is fixed and r is chosen so that V is at its stationary value, show that $A^2 = 3\pi^2 r^4$ and that $l = \sqrt{3}r$.

(ii) Given, instead, that V is fixed and r is chosen so that A is at its stationary value, find h in terms of r.

Solution to (ii): $h = \sqrt{2}\, r$