CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

University of Leeds

Level 1 Semester 2

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Tutorial: Question Difficulty Colour Code

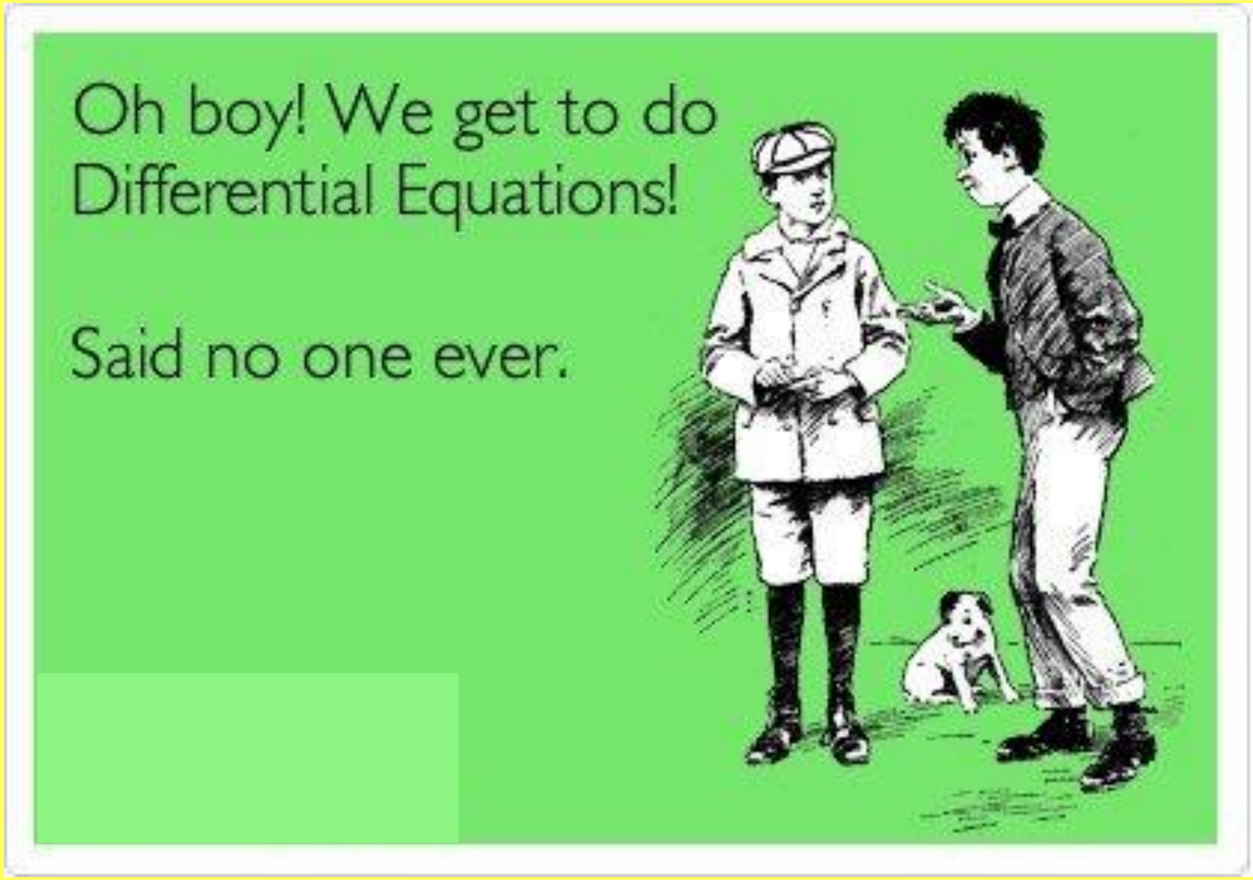
Basic - straightforward application
(you must be able to do these)

Medium – Makes you think a bit
(you must be able to do these)

Hard – Makes you think a lot
(you should be able to do these)

Extreme – Tests your understanding to the limit!
(for those who like a challenge)

Applied – Real-life examples of the topic, may sometimes involve prior knowledge
(you should attempt these – will help in future engineering)



Tutorial 4

1^{st} Order ODEs 1: Separation of Variables

Class Example: Separation of Variables

E.g. 1

Solve the differential equation:

$$(xy - 4x)\,dx + (x^2y + y)dy = 0$$

Note that "solve" just means to get rid of any differentials by integrating, not to necessarily rearrange for y, but we should do this if possible.

$$(xy-4x)dx + (x^2y+y)dy = 0$$

$$x(y-4)dx + y(x^2+1)dy = 0$$

$$y(x^2+1)dy = -(y-4)dx$$

HINT: SEE "STANDARD TRICK" IN LECTURE 2

$$\int \frac{y}{y-4}\,dy = -\int \frac{x}{x^2+1}\,dx$$

$u = y-4 \quad (y = u+4)$

$\frac{du}{dy} = 1 \therefore dy = du$

$$\int \frac{y}{y-4}dy = \int \frac{u+4}{u}\,du$$

$$= \int 1 + \frac{4}{u}\,du$$

$$= u + 4\ln u$$

$$= y-4 + 4\ln(y-4)$$

$v = x^2+1$

$\frac{dv}{dx} = 2x \quad dx = \frac{dv}{2x}$

$$-\int \frac{x}{x^2+1}\,dx = -\int \frac{\cancel{x}}{v}\frac{dv}{2\cancel{x}}$$

$$= -\frac{1}{2}\int \frac{1}{v}\,dv$$

$$= -\frac{1}{2}\ln v + C_1$$

$$= -\frac{1}{2}\ln(x^2+1) + C_1$$

$$\therefore y - 4 + 4\ln(y-4) = -\frac{1}{2}\ln(x^2+1) + C_1$$

$$y + 4\ln(y-4) = -\frac{1}{2}\ln(x^2+1) + C \quad (C = C_1 + 4)$$

Class Example: Separation of Variables (Boundary Conditions)

E.g. 2

Solve $x\dfrac{dy}{dx} = y^2 + 1$ and find the particular solution when $y(1) = 1$

Hint: Use BCs when C is "out in the open" to save unnecessary rearranging.

$$\int \frac{dy}{y^2+1} = \int \frac{dx}{x}$$

$$\left\{\text{Standard integral: } \int \frac{dy}{1+y^2} = \tan^{-1} y + C\right\}$$ (or sub $y = \tan u$)

i.e. $\tan^{-1} y = \ln x + C$.   General solution.

Particular solution with $y = 1$ when $x = 1$:

$\tan \frac{\pi}{4} = 1 \;\therefore \tan^{-1}(1) = \frac{\pi}{4}$, while $\ln 1 = 0$ (i.e. $1 = e^0$)

$\therefore \frac{\pi}{4} = 0 + C$ i.e. $C = \frac{\pi}{4}$

Particular solution is: $\tan^{-1} y = \ln x + \frac{\pi}{4}$.

Class Example: Separation of Variables (Applied)

E.g. 3 Liquid is poured into a tank at a constant rate of $30\ \mathrm{cm}^3\mathrm{s}^{-1}$.
At time t seconds liquid is leaking from the tank at a rate of $\frac{2}{15}V\ \mathrm{cm}^3\mathrm{s}^{-1}$,
where $V\ \mathrm{cm}^3$ is the volume of liquid in the tank at that time.

a) Show that

$$-15\frac{dV}{dt} = 2V - 450$$

Overall rate = rate in - rate out

$$\frac{dV}{dt} = 30 - \frac{2}{15}V$$

$$\Rightarrow -15\frac{dV}{dt} = -450 + 2V$$

b) Given that the tank initially contains 1 lite of liquid, find the equation for the volume in the form $V = f(t)$.

Separating the variables $\Rightarrow \ -\frac{15}{2V-450}\,dV = dt$

Integrating to obtain $-\frac{15}{2}\ln|2V-450| = t$ OR $-\frac{15}{2}\ln|V-225| = t$

Using limits correctly or finding c ($-\frac{15}{2}\ln 1550$ *OR* $-\frac{15}{2}\ln 775$)

$\ln\frac{2V-450}{1550} = -\frac{2}{15}t$, or equivalent $\quad \boldsymbol{V = 225 + 775e^{-\frac{2}{15}t}}$

c) Find the limiting value of V as $t \to \infty$.

$$V = 225 + 775e^{-\frac{2}{15}t}$$

As t → ∞, $\boldsymbol{V = 225}$

Exercise A: Separation of Variables

Solve the differential equations:

1 $\frac{dy}{dx} = ky$

2 $\frac{dy}{dx} = xy$

3 $(x+1)\frac{dy}{dx} = y$

4 $\frac{dy}{dx} = e^{2x-y}$

5 $\frac{dy}{dx} = \frac{xy}{x+1}$

6 $\frac{dx}{dt} = \frac{x}{t(t+1)}$

7 $2y\,dx + (xy + 5x)dy = 0$

8 $y' = x - 1 + xy - y$

9 $\cos x\,dy - y dx = 0$

10 $(1+y^2)\,dx - (1+x^2)dy = 0$

11 $(y + yx^2)dy + (x + xy^2)dx = 0$

12 $e^{x+2y}\,dx - e^{2x-y}\,dy = 0$

13 $y(1+x^3)y' + x^2(1+y^2) = 0$

14 $\operatorname{cosec}^3 x \frac{dy}{dx} = \cos^2 y$

15 $xy + y'e^{-2x}\ln y = 0$

Exercise B: Separation of Variables (Boundary Conditions)

Solve the differential equations (subject to given boundary conditions)

1 $$\frac{dy}{dt} = \frac{y+1}{t-1}$$ $(t \neq 1)$ ($y = 1$ when $t = 0$)

2 $$y^2 \frac{dy}{dt} = t$$ ($y = 5$ when $t = 0$)

3 $$2y^2 y' = 3y - y'$$ ($y = 1$ when $x = 3$)

4 Solve the IVP and state the range of t for which the solution is valid.

$$\frac{dx}{dt} = e^{x+2t} \qquad x(0) = 0$$

5 $$x dy - (2x+1)e^{-y} dx = 0$$ ($y = 2$ when $x = 1$)

6 $$x \sinh y \, dy = \cosh y \, dx$$ ($y(3) = 0$)

7 $$x dy - \sqrt{1-y^2} \, dx = 0$$ ($y = \frac{1}{2}$ when $x = 1$)

8 $$\cot x \, dy - (1+y^2) dx = 0$$ ($y = 1$ when $x = 0$)

9 $$x^3 \frac{dp}{dx} = a - x$$

Where a is constant.
If $p = 0$ when $x = 2$ and also when $x = 4$, find the value of a

Exercise C: Separation of Variables (Applied)

1

At time $t = 0$, a piece of radioactive material has mass 24 g. Its mass after t days is m grams and is decreasing at a rate proportional to m.

a By forming and solving a suitable differential equation, show that

$$m = 24e^{-kt},$$

where k is a positive constant.

After 20 days, the mass of the material is found to be 22.6 g.

b Find the value of k.

c Find the rate at which the mass is decreasing after 20 days.

d Find how long it takes for the mass of the material to be halved.

2

a Express $\dfrac{1}{(1+x)(1-x)}$ in partial fractions.

In an industrial process, the mass of a chemical, m kg, produced after t hours is modelled by the differential equation

$$\frac{dm}{dt} = ke^{-t}(1+m)(1-m),$$

where k is a positive constant.

Given that when $t = 0$, $m = 0$ and that the initial rate at which the chemical is produced is 0.5 kg per hour,

b find the value of k,

c show that, for $0 \le m < 1$, $\ln\left(\dfrac{1+m}{1-m}\right) = 1 - e^{-t}$.

d find the time taken to produce 0.1 kg of the chemical,

e show that however long the process is allowed to run, the maximum amount of the chemical that will be produced is about 462 g.

Exercise C: Separation of Variables (Applied)

3

Liquid is pouring into a container at a constant rate of 20 $\text{cm}^3\ \text{s}^{-1}$ and is leaking out at a rate proportional to the volume of the liquid already in the container.

(*a*) Explain why, at time t seconds, the volume, V cm^3, of liquid in the container satisfies the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 20 - kV,$$

where k is a positive constant. **(2)**

The container is initially empty.

(*b*) By solving the differential equation, show that

$$V = A + Be^{-kt},$$

giving the values of A and B in terms of k. **(6)**

Given also that $\frac{\mathrm{d}V}{\mathrm{d}t} = 10$ when $t = 5$,

(*c*) find the volume of liquid in the container at 10 s after the start. **(5)**

Exercise C: Separation of Variables (Applied)

4

Liquid is pouring into a large vertical circular cylinder at a constant rate of 1600 $cm^3\,s^{-1}$ and is leaking out of a hole in the base, at a rate proportional to the square root of the height of the liquid already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2.

(a) Show that at time t seconds, the height h cm of liquid in the cylinder satisfies the differential equation

$$\frac{dh}{dt} = 0.4 - k\sqrt{h}, \text{ where } k \text{ is a positive constant.}$$

(3)

When $h = 25$, water is leaking out of the hole at 400 $cm^3\,s^{-1}$.

(b) Show that $k = 0.02$

(1)

(c) Separate the variables of the differential equation

$$\frac{dh}{dt} = 0.4 - 0.02\sqrt{h},$$

to show that the time taken to fill the cylinder from empty to a height of 100 cm is given by

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh.$$

(2)

Using the substitution $h = (20 - x)^2$, or otherwise,

(d) find the exact value of $\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh$.

(6)

(e) Hence find the time taken to fill the cylinder from empty to a height of 100 cm, giving your answer in minutes and seconds to the nearest second.

(1)

Exercise C: Separation of Variables (Applied)

5

The rate of increase of the number, N of fish in a lake is modelled by the differential equation

$$\frac{dN}{dt} = \frac{(kt - 1)(5000 - N)}{t}$$

Where $t > 0, \quad 0 < N < 5000$

In the given equation, the time t is measured in years from the start of January 2000 and k is a positive constant.

a) By solving the differential equation, show that:

$$N = 5000 - Ate^{-kt}$$

Where A is a positive constant.

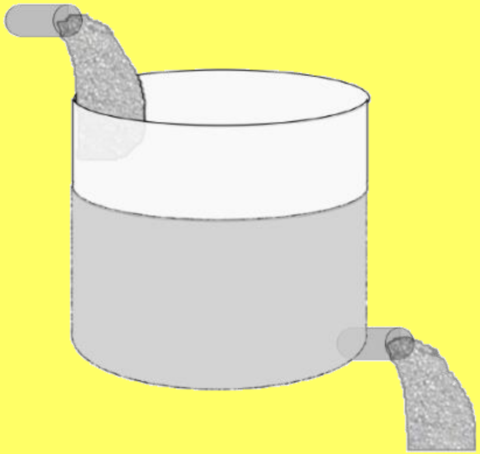
After one year, at the start of January 2001, there are 1200 fish in the lake.
After one year, at the start of January 2002, there are 1800 fish in the lake.

b) Find the exact value of the constant A and the exact value of the constant k.

Exercise C: Separation of Variables (Applied)

6

Mixing Problem:
A tank contains 400 gal of brine in which 100 lb of salt are dissolved.
Fresh water runs into the tank at a rate of 2 gal/min.
The mixture, kept practically uniform by stirring, runs out at the same rate.
How much salt will there be in the tank at the end of 1 hour?



Challenge: Separation of Variables

1

a) Show using partial fractions that

$$\frac{4x}{x^4 - 1} \equiv \frac{1}{x - 1} + \frac{1}{x + 1} - \frac{2x}{x^2 + 1}$$

b) Show by separation of variables that the differential equation

$$\frac{dy}{dx} = \frac{\sin^4 y - 1}{2x \sin 2y}$$

becomes:

$$\int \frac{4 \sin y \cos y}{\sin^4 y - 1} dy = \int \frac{1}{x}\, dx$$

c) By using a substitution $u = \sin y$ and given that $y = 0$ when $x = 4$, solve the differential equation and show that:

$$x = \frac{4|\sin^2 y - 1|}{\sin^2 y + 1}$$

Explain the need for the modulus sign and why it only appears in the numerator.

d) Hence, find the value of x when $y = \frac{\pi}{6}$

Challenge: Separation of Variables

In this question, you may assume that $\ln(1+x) \approx x - \frac{1}{2}x^2$ when $|x|$ is small.

The height of the water in a tank at time t is h. The initial height of the water is H and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.

(i) Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(\alpha^2 H - h),$$

for some positive constant k. Deduce that the time T taken for the water to reach height αH is given by

$$kT = \ln\left(1 + \frac{1}{\alpha}\right),$$

and that $kT \approx \alpha^{-1}$ for large values of α.

(ii) Suppose that the rate at which water leaks out of the tank is proportional to $\sqrt{h}$ (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by

$$cT' = 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha \ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right)$$

for some positive constant c, and that $cT' \approx \sqrt{H}$ for large values of α.

ANSWERS

Exercise A: Answers

1 $y = Ae^{kx}$

2 $y = Ae^{\frac{1}{2}x^2}$

3 $y = A(x+1)$

4 $y = \ln\left(\frac{1}{2}e^{2x} + C\right)$

5 $y = \frac{Ae^x}{x+1}$

6 $x = \frac{At}{t+1}$

7 $y + 5\ln y = -2\ln x + C$

8 $y = Ae^{\frac{x^2}{2}-x} - 1$

9 $y = A(\sec x + \tan x)$

10 $y = \tan(\tan^{-1} x + C)$

11 $y^2 = \frac{A}{1+x^2} - 1$

12 $y = -\frac{1}{3}\ln(3e^{-x} + C)$

13 $y = A(1+x)^{-\frac{2}{3}} - 1$

14 $\tan y = -\cos x + \frac{1}{3}\cos^3 x + C$

15 $\ln^2 y = \left(\frac{1}{2} - x\right)e^{2x} + C$

Exercise B: Answers

1 $y = 1 - 2t$

2 $y = \left(\frac{3}{2}t^2 + 27\right)^{\frac{1}{3}}$

3 $y^2 + \ln y = 3x - 8$

4 $x = -\ln\left[\frac{1}{2}(3 - e^{2t})\right]$

Valid for $t < \frac{1}{2}\ln 3$

5 $y = \ln(2x + \ln x + e^2 - 2)$

6 $y = \cosh^{-1}\left(\frac{x}{3}\right)$

7 $y = \sin(\ln x + \frac{\pi}{6})$

8 $y = \tan(\ln(\sec x) + \frac{\pi}{4})$

9 $a = \frac{8}{3}$

Exercise C: Answers

1

a Show that

b $k = 0.00301\ (3sf)$

c $-0.0679\ (3sf)$

d $231\ days$

2

a $\dfrac{1}{2(1+x)} + \dfrac{1}{2(1-x)}$

b $k = 0.5$

c Show that

d $13.4\ min$

e Take $t \to \infty$

3

a Explain

b $A = \dfrac{20}{k}, B = -\dfrac{20}{k}$

c $108\ cm^2\ (3sf)$

4

a Show that

b Show that

c Show that

d $1000(\ln 4 - 1)$

e 6 min 26 seconds

5

a Show that

b $A = 9025, \quad k = \ln\left(\dfrac{19}{8}\right)$

6

$64\ lb$

Challenge: Answers

1 **a-c** Show that

d $\frac{12}{5}$

☠ All show that

FULL WORKED SOLUTIONS

Exercise A: Full Solutions

1

$$\frac{dy}{dx} = ky$$

$$\int \frac{1}{y}\,dy = \int k\,dx$$

$$\ln y = kx + C$$

$$y = e^{kt+C} = e^{kt}e^{C}$$

$$\boldsymbol{y = Ae^{kx}}$$

2

$$\frac{dy}{dx} = xy$$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\boldsymbol{y = Ae^{\frac{1}{2}x^2}}$$

3

$$(x+1)\frac{dy}{dx} = y$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x+1}\,dx$$

$$\ln y = \ln(x+1) + \ln A$$

$$\ln y = \ln A(x+1)$$

$$\boldsymbol{y = A(x+1)}$$

4

$$\frac{dy}{dx} = e^{2x-y}$$

$$\frac{dy}{dx} = \frac{e^{2x}}{e^{y}}$$

$$\Rightarrow \int e^{y}\,dy = \int e^{2x}\,dx$$

$$e^{y} = \frac{1}{2}e^{2x} + C$$

$$\boldsymbol{y = \ln\left(\frac{1}{2}e^{2x} + C\right)}$$

5

$$\frac{dy}{dx} = \frac{xy}{x+1}$$

$$\int \frac{1}{y}\,dy = \int \frac{x}{x+1}\,dx$$

$$\int \frac{1}{y}\,dy = \ln y$$

Sub $u = 1 + x \to \frac{du}{dx} = 1, \quad x = u - 1$

$$\int \frac{x}{x+1}\,dx = \int \frac{u-1}{u}\,du = \int 1 - \frac{1}{u}\,du$$

$$= u - \ln u + C_1 = x + 1 - \ln(x+1) + C_1$$

$$= x - \ln(x+1) + C \quad (C = C_1 + 1)$$

$$\ln y = x - \ln(x+!) + C$$

$$y = e^{x-\ln(x+1)+C} = e^{x}e^{\ln(x+1)^{-1}}e^{C}$$

$$y = \frac{Ae^{x}}{x+1}$$

6

$$\frac{dx}{dt} = \frac{x}{t(t+1)}$$

$$\int \frac{1}{x}\,dx = \int \frac{1}{t(t+1)}\,dt = \int \frac{1}{t} - \frac{1}{t+1}\,dt$$

(we have used partial fractions on RHS)

$$\ln x = \ln t - \ln(t+1) + \ln A$$

$$\ln x = \ln\left(\frac{At}{t+1}\right) \Rightarrow \boldsymbol{x = \frac{At}{t+1}}$$

Exercise A: Full Solutions

7

$$2y\,dx + (xy+5x)\,dy = 0$$
$$-2y\,dx = x(y+5)\,dy$$
$$\int -\frac{2}{x}\,dx = \int \frac{y+5}{y}\,dy$$
$$-2\ln x + C = \int 1 + \frac{5}{y}\,dy$$
$$-2\ln x + C = y + 5\ln y$$
$$\underline{y + 5\ln y = -2\ln x + C}$$

9

$$\cos x\,dy - dx = 0$$
$$\cos x\,dy = y\,dx$$
$$\int \frac{1}{y}\,dy = \int \frac{1}{\cos x}\,dx = \int \sec x\,dx$$
$$\ln y = \ln(\sec x + \tan x) + C \quad \text{(USING STANDARD INTEGRALS TABLE)}$$
$$y = e^{\ln(\sec x + \tan x) + C} = e^{\ln(\sec x + \tan x)} e^{C}$$
$$\therefore \underline{y = A(\sec x + \tan x)}$$

8

$$y' = x - 1 + xy - y$$
$$\frac{dy}{dx} = (x-1) + y(x-1) = (x-1)(1+y)$$
$$\int \frac{1}{1+y}\,dy = \int x - 1\,dx$$
$$\ln(1+y) = \frac{x^2}{2} - x + C$$
$$1 + y = e^{\frac{x^2}{2} - x + C} = e^{\frac{x^2}{2} - x} e^{C}$$
$$\therefore \underline{y = Ae^{\frac{x^2}{2} - x} - 1}$$

10

$$(1+y^2)\,dx - (1+x^2)\,dy = 0$$
$$(1+y^2)\,dx = (1+x^2)\,dy$$
$$(*) \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1+y^2}\,dy \quad \text{(BOTH SAME – USE SUBSTITUTION)}$$
$$\int \frac{1}{1+x^2}\,dx \qquad x = \tan u \quad (u = \tan^{-1} x) \qquad \frac{dx}{du} = \sec^2 u$$
$$= \int \frac{1}{(1+\tan^2 u)} \sec^2 u\,du = \int 1\,du = u + C$$
$$(1 + \tan^2 u = \sec^2 u) \qquad = \tan^{-1} x + C$$
$$\therefore (*) \text{ BECOMES } \tan^{-1} x + C = \tan^{-1} y$$
$$\therefore y = \tan(\tan^{-1} x + C)$$

Exercise A: Full Solutions

11

$$(y+yx^2)\,dy + (x+xy^2)\,dx = 0$$
$$y(1+x^2)\,dy + x(1+y^2)\,dx = 0$$
$$y(1+x^2)\,dy = -x(1+y^2)\,dx$$
$$\int \frac{y}{1+y^2}\,dy = -\int \frac{x}{1+x^2}\,dx \qquad \left(\begin{array}{c}\text{BOTH SAME}\\ \text{SUBSTITUTION}\end{array}\right)$$

$u = 1+y^2$

$\frac{du}{dy} = 2y \quad \therefore dy = \frac{du}{2y}$

$$\therefore \int \frac{y}{u}\,\frac{du}{2y} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln u = \frac{1}{2}\ln(1+y^2)$$

$$\therefore \frac{1}{2}\ln(1+y^2) = -\frac{1}{2}\ln(1+x^2) + C_1$$
$$\ln(1+y^2) = -\ln(1+x^2) + C_2 \qquad (C_2 = 2C_1)$$
$$1+y^2 = e^{-\ln(1+x^2)+C_2}$$
$$= e^{\ln(1+x^2)^{-1}}\,e^{C_2}$$
$$1+y^2 = \frac{A}{1+x^2}$$
$$y^2 = \frac{A}{1+x^2} - 1$$

12

$$e^{x+2y}\,dx - e^{2x-y}\,dy = 0 \qquad \left(\text{HINT: } e^{a+b} = e^a e^b\right)$$
$$e^x e^{2y}\,dx = e^{2x} e^{-y}\,dy$$
$$\int \frac{e^x}{e^{2x}}\,dx = \int \frac{e^{-y}}{e^{2y}}\,dy$$
$$\int e^{-x}\,dx = \int e^{-3y}\,dy$$
$$-e^{-x} + C_1 = -\frac{1}{3}e^{-3y}$$

$\times -3$):

$$e^{-3y} = 3e^{-x} - C \qquad (C = -3C_1)$$
$$-3y = \ln(3e^{-x} - C)$$
$$y = -\frac{1}{3}\ln(3e^{-x} + C)$$

Exercise A: Full Solutions

13

$$y(1+x^3)y' + x^2(1+y^2) = 0$$

$$y(1+x^3)\frac{dy}{dx} + x^2(1+y^2) = 0$$

$$y(1+x^3)\,dy = -x^2(1+y^2)\,dx$$

$$\int \frac{y}{1+y^2}\,dy = -\int \frac{x^2}{1+x^3}\,dx$$

$u = 1+y^2$, $\frac{du}{dy} = 2y \therefore dy = \frac{du}{2y}$ $\qquad$ $v = 1+x^3$, $\frac{dv}{dx} = 3x^2 \therefore dx = \frac{dv}{3x^2}$

$$\int \frac{y}{u}\frac{du}{2y} \qquad -\int \frac{x^2}{v}\frac{dv}{3x^2}$$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln u \qquad = -\frac{1}{3}\int \frac{1}{v}\,dv = -\frac{1}{3}\ln v + C_1$$

$$= \frac{1}{2}\ln(1+y^2) \qquad = -\frac{1}{3}\ln(1+x^3) + C_1$$

$$\therefore \frac{1}{2}\ln(1+y^2) = -\frac{1}{3}\ln(1+x^3) + C_1$$

$$\ln(1+y^2) = -\frac{2}{3}\ln(1+x^3) + C_2 \quad (C_2 = 2C_1)$$

$$\ln(1+y^2) = \ln(1+x^3)^{-2/3} + C_2$$

$$1+y^2 = e^{\ln(1+x^3)^{-2/3} + C_2} = e^{\ln(1+x^3)^{-2/3}} e^{C_2}$$

$$1+y^2 = A(1+x^3)^{-2/3}$$

$$y^2 = A(1+x^3)^{-\frac{2}{3}} - 1$$

14

$$\operatorname{cosec}^3 x \frac{dy}{dx} = \cos^2 y$$

$$\int \frac{dy}{\cos^2 y} = \int \frac{dx}{\operatorname{cosec}^3 x}$$

$$\int \sec^2 y\,dy = \int \sin^3 x\,dx$$

$$\tan y = \int \sin^2 x \sin x\,dx$$
$$= \int (1-\cos^2 x)\sin x\,dx$$
$$= \int \sin x\,dx - \int \cos^2 x \sin x\,dx$$

$$\int \sin x\,dx = -\cos x + C_1 \qquad \int \cos^2 x \sin x\,dx$$

$u = \cos x$, $\frac{du}{dx} = -\sin x$, $dx = \frac{-du}{\sin x}$

$$= \int u^2 \cancel{\sin x}\left(\frac{-du}{\cancel{\sin x}}\right)$$

$$= -\int u^2\,du = -\frac{1}{3}u^3 + C_2$$
$$= -\frac{1}{3}\cos^3 x + C_2$$

$$\therefore \tan y = -\cos x + C_1 - \left(-\frac{1}{3}\cos^3 x + C_2\right)$$

$$\tan y = -\cos x + \frac{1}{3}\cos^3 x + C$$

$$\left(\text{or } y = \tan^{-1}\left(-\cos x + \frac{1}{3}\cos^3 x + C\right)\right)$$

Exercise A: Full Solutions

15

$$xy + y'e^{-2x}\ln y = 0$$

$$xy + \frac{dy}{dx}e^{-2x}\ln y = 0$$

$$e^{-2x}\ln y = -xy\,dx$$

$$\int \frac{\ln y}{y}\,dy = -\int xe^{2x}\,dx$$

$u = \ln y \quad \frac{du}{dy} = \frac{1}{y}$

$dy = y\,du$

$$\int \frac{\ln y}{y}\,dy = \int \frac{u}{\cancel{y}}\cancel{y}\,du$$

$$= \frac{1}{2}u^2 = \frac{1}{2}(\ln y)^2$$

$$= \frac{1}{2}\ln^2 y$$

PARTS: $u = x \quad dv = e^{2x}$

$\frac{du}{dx} = 1 \quad v = \frac{1}{2}e^{2x}$

$$-\int xe^{2x}\,dx = -\left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx\right]$$

$$= -\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c_1\right]$$

$$= \frac{1}{4}e^{2x} - \frac{1}{2}xe^{2x} + c_2 \quad (c_2 = -c_1)$$

$$\therefore \frac{1}{2}\ln^2 y = \frac{1}{4}e^{2x} - \frac{1}{2}xe^{2x} + c_2$$

$$\ln^2 y = \frac{1}{2}e^{2x} - xe^{2x} + c \quad (c = 2c_2)$$

$$\ln^2 y = \left(\frac{1}{2} - x\right)e^{2x} + c$$

Exercise B: Full Solutions

1

$$\frac{dy}{dt} = \frac{y+1}{t-1}$$

$y = 1$ when $t = 0$

$(t \neq 1)$

$$\int \frac{1}{y+1}dy = \int \frac{1}{t-1}dt$$
$$\ln(y+1) = \ln(t-1) + \ln A$$
$$\ln(y+1) = \ln A(t-1)$$
$$y = A(t-1) - 1$$
$$y = 1, \mathrm{t} = 0 \Rightarrow \mathrm{A} = -2$$
$$y = -2(t-1) - 1$$
$$\boldsymbol{y = 1 - 2t}$$

2

$$y^2\frac{dy}{dt} = t$$

$y = 3$ when $t = 0$

$$\int y^2 dy = \int t dt$$
$$\frac{y^3}{3} = \frac{t^2}{2} + C$$
$$y = 3, \mathrm{t} = 0 \Rightarrow \mathrm{C} = 9$$
$$\boldsymbol{y = \left(\frac{3}{2}t^2 + 27\right)^{\frac{1}{3}}}$$

3

$$2y^2y' = 3y - y'$$

$(y = 1$ when $x = 3)$

$$2y^2\frac{dy}{dx} = 3y - \frac{dy}{dx}$$
$$(2y^2 + 1)\frac{dy}{dx} = 3y$$
$$\int \frac{2y^2 + 1}{3y}dy = \int 1\, dx$$
$$\int \frac{2}{3}y + \frac{1}{3}y^{-1}\, dy = \int 1\, dx$$
$$\frac{1}{3}y^2 + \frac{1}{3}\ln y = x + C$$

$y = 1$ when $x = 3$:

$$\frac{1}{3} + \frac{1}{3}\ln 1 = 3 + C \Rightarrow C = -\frac{8}{3} \quad (\ln 1 = 0)$$
$$\frac{1}{3}y^2 + \frac{1}{3}\ln y = x - \frac{8}{3}$$
$$\boldsymbol{y^2 + \ln y = 3x - 8}$$

4

Solve and state the range of t for which the solution is valid.

$$\frac{dx}{dt} = e^{x+2t}$$

$$x(0) = 0$$

$$\frac{dx}{dt} = e^x e^{2t} \Rightarrow \int e^{-x}dx = \int e^{2t}dt$$
$$-e^{-x} = \frac{1}{2}e^{2t} + C$$
$$x(0) = 0 \Rightarrow -1 = \frac{1}{2} + \mathrm{C} \Rightarrow \mathrm{C} = -\frac{3}{2}$$
$$e^{-x} = -\frac{1}{2}e^{2t} + \frac{3}{2}$$
$$\boldsymbol{x = -\ln\left[\frac{1}{2}\left(3 - e^{2t}\right)\right]}$$

Valid for $3 - e^{2t} > 0 \Rightarrow e^{2t} < 3 \Rightarrow \boldsymbol{t < \frac{1}{2}\ln 3}$

Exercise B: Full Solutions

5

$x\,dy - (2x+1)e^{-y}\,dx = 0 \qquad (y = 2 \text{ when } x = 1)$

$x\,dy = (2x+1)e^{-y}\,dx$

$\int e^{y}\,dy = \int \frac{2x+1}{x}\,dx$

$e^{y} = \int 2 + \frac{1}{x}\,dx = 2x + \ln x + C$

$y = 2$ when $x = 1$: $\quad e^{2} = 2 + \cancel{\ln 1} + C \quad \therefore C = e^{2} - 2$

$\therefore \quad y = \ln(2x + \ln x + e^{2} - 2)$

6

6. $x \sinh y\,dy = \cosh y\,dx \qquad y(3) = 0$

$\int \tanh y\,dy = \int \frac{1}{x}\,dx = \ln x + C$

LHS: $\int \frac{\sinh y}{\cosh y}\,dy \qquad u = \cosh y \qquad \frac{du}{dy} = \sinh y \qquad \therefore dy = \frac{du}{\sinh y}$

$= \int \frac{\cancel{\sinh y}}{u}\,\frac{du}{\cancel{\sinh y}} = \int \frac{1}{u}\,du = \ln u = \ln(\cosh y)$

$\therefore \ln(\cosh y) = \ln x + C$

$y(3) = 0$: $\quad \ln(\cosh 0) = \ln 3 + C \qquad \cosh(0) = 1$

$\cancel{\ln 1} = \ln 3 + C \quad \therefore C = -\ln 3$

$\ln(\cosh y) = \ln x - \ln 3$

$\ln(\cosh y) = \ln \frac{x}{3}$

$\cosh y = \frac{x}{3}$

$y = \cosh^{-1} \frac{x}{3}$

Exercise B: Full Solutions

7

$x\,dy - \sqrt{1-y^2}\,dx = 0 \quad (y=\frac{1}{2} \text{ when } x=1)$

$x\,dy = \sqrt{1-y^2}\,dx$

$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int \frac{1}{x}\,dx = \ln x + C$

$y = \sin u \quad \frac{dy}{du} = \cos u \quad \therefore \int \frac{1}{\sqrt{1-\sin^2 u}} \cos u\,du = \int \frac{1}{\sqrt{\cos^2 u}} \cos u\,du$

$dy = \cos u\,du$

$= \int 1\,du = u = \sin^{-1} y$

$\therefore \sin^{-1} y = \ln x + C$

$y = \frac{1}{2}$ when $x=1$: $\sin^{-1}\frac{1}{2} = \ln 1 + C \quad \therefore C = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$y = \sin\left(\ln x + \frac{\pi}{6}\right)$

(* $\frac{\pi}{6}$ is first solution, for general case

$y = \sin\left(\ln x + \sin^{-1}\frac{1}{2}\right)$)

8

$\cot x\,dy - (1+y^2)\,dx = 0 \quad (y=1 \text{ when } x=0)$

$\cot x\,dy = (1+y^2)\,dx$

$\int \frac{1}{1+y^2}\,dy = \int \frac{1}{\cot x}\,dx = \int \tan x\,dx$

LHS: $y = \tan u$

$\frac{dy}{du} = \sec^2 u$

$dy = \sec^2 u\,du$

$\int \frac{1}{1+y^2}\,dy = \int \frac{1}{(1+\tan^2 u)} \sec^2 u\,du$

$(1+\tan^2 u = \sec^2 u)$

$= \int \frac{1}{\sec^2 u} \sec^2 u\,du$

$= \int 1\,du = u\,(+C)$

$= \tan^{-1} y\,(+C)$

RHS: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$v = \cos x \quad \frac{dv}{dx} = -\sin x$

$dx = \frac{-dv}{\sin x}$

$\therefore \int \frac{\sin x}{\cos x}\,dx = \int \frac{\sin x}{v}\left(\frac{-dv}{\sin x}\right)$

$= -\int \frac{1}{v}\,dv = -\ln v + C$

$= -\ln(\cos x) + C$

$= \ln(\sec x) + C$

$\therefore \tan^{-1} y = \ln(\sec x) + C$

$y = 1$ when $x = 0$

$\tan^{-1}(1) = \ln(\sec 0) + C$

$\left(\tan^{-1}(1) = \frac{\pi}{4} \quad \sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1\right.$

$\left. \therefore \ln(\sec 0) = \ln 1 = 0\right)$

$\frac{\pi}{4} = 0 + C \longrightarrow \therefore C = \frac{\pi}{4}$

$\therefore \tan^{-1} y = \ln(\sec x) + \frac{\pi}{4}$

$y = \tan\left(\ln(\sec x) + \frac{\pi}{4}\right)$

Exercise B: Full Solutions

9

$x^3\frac{dp}{dx} = a - x$ a is constant

$p = 0$ when $x = 2$
$p = 0$ when $x = 4$

Hint: 2 conditions suggests simultaneous equations

Find a

$\int dp = \int \frac{a-x}{x^3}\,dx$

$p = \int \frac{a}{x^3} - \frac{x}{x^3}\,dx = \int ax^{-3} - x^{-2}\,dx$

$p = -\frac{1}{2}ax^{-2} + x^{-1} + C$

$p = -\frac{a}{2x^2} + \frac{1}{x} + C$

$p = 0$ when $x = 2$: $\quad 0 = -\frac{a}{8} + \frac{1}{2} + C$

$a = 8C + 4 \quad ①$

$p = 0$ when $x = 4$: $\quad 0 = -\frac{a}{32} + \frac{1}{4} + C \rightarrow a = 32C + 8 \quad ②$

$① = ②$: $\quad 8C + 4 = 32C + 8$

$24C = -4$

$C = -\frac{1}{6}$

in ①: $a = 8\left(-\frac{1}{6}\right) + 4 = -\frac{8}{6} + 4 = -\frac{4}{3} + \frac{12}{3} = \frac{8}{3}$

$a = \frac{8}{3}$

Exercise C: Solutions

1

a $-\dfrac{dm}{dt} = km$

$\int \dfrac{1}{m}\ dm = \int -k\ dt$

$\ln|m| = -kt + c$

$m = e^{-kt+c} = e^c \times e^{-kt}$

$m = Ae^{-kt}$

$t = 0,\ m = 24 \quad \therefore A = 24$

$m = 24e^{-kt}$

b $t = 20,\ m = 22.6 \quad \therefore 22.6 = 24e^{-20k}$

$\therefore\ k = -\frac{1}{20}\ln\frac{22.6}{24} = 0.00301$ (3sf)

c $\dfrac{dm}{dt} = -km = -0.003005 \times 22.6$

$= -0.0679$ (3sf)

$\therefore$ decreasing at 0.0679 grams per day

d $m = 12 \quad \therefore 12 = 24e^{-0.003005t}$

$t = -\dfrac{1}{0.003005}\ln\frac{1}{2}$

$= 231$ days (nearest day)

2

a $\dfrac{1}{(1+x)(1-x)} \equiv \dfrac{A}{1+x} + \dfrac{B}{1-x}$

$1 \equiv A(1-x) + B(1+x)$

$x = -1 \Rightarrow A = \frac{1}{2},\quad x = 1 \Rightarrow B = \frac{1}{2}$

$\dfrac{1}{(1+x)(1-x)} \equiv \dfrac{1}{2(1+x)} + \dfrac{1}{2(1-x)}$

b $t = 0,\ m = 0,\ \dfrac{dm}{dt} = 0.5$

$\therefore\ 0.5 = k \times 1$

$k = 0.5$

c $\int \dfrac{1}{(1+m)(1-m)}\ dm = \int \frac{1}{2}e^{-t}\ dt$

$\int \left(\dfrac{\frac{1}{2}}{1+m} + \dfrac{\frac{1}{2}}{1-m}\right) dm = \int \frac{1}{2}e^{-t}\ dt$

$\frac{1}{2}\ln|1+m| - \frac{1}{2}\ln|1-m| = -\frac{1}{2}e^{-t} + c$

$\ln|1+m| - \ln|1-m| = C - e^{-t}$

$t = 0,\ m = 0 \quad \therefore\ 0 - 0 = C - 1$

$C = 1$

$\ln|1+m| - \ln|1-m| = 1 - e^{-t}$

for $0 \le m < 1$, $1 + m > 0$ and $1 - m > 0$

$\therefore\ \ln(1+m) - \ln(1-m) = 1 - e^{-t}$

$\ln\left(\dfrac{1+m}{1-m}\right) = 1 - e^{-t}$

d $m = 0.1 \quad \therefore \ln\dfrac{1.1}{0.9} = 1 - e^{-t}$

$t = -\ln(1 - \ln\frac{11}{9}) = 0.2240$ hrs

$= 13.4$ minutes

e $t \to \infty,\ \ln\left(\dfrac{1+m}{1-m}\right) \to 1$

$\therefore$ limiting value of m is given by

$\dfrac{1+m}{1-m} = e$

$1 + m = e(1-m)$

$m(1+e) = e - 1$

$m = \dfrac{e-1}{1+e} = 0.4621$

$\therefore$ max. produced ≈ 462 g

Exercise C: Solutions

3

a

Rate of change of volume (dV/dt) = flow in - flow out:

$\frac{dV}{dt}$ is the rate of change (increase) of volume (with respect to time).

Flow in is +20

Flow out is proportional to volume present: $-kV$ where k is a constant of proportionality and the negative sign shows decrease (out of container)

Giving $\frac{dV}{dt} = 20 - kV$

b

$$\int \frac{1}{20-kV}\,dV = \int dt$$

$$-\frac{1}{k}\ln(20-kV) = t + C_1$$

$$\ln(20-kV) = -kt + C_2$$

$$20 - kV = e^{-kt+C_2} = Ce^{-kt}$$

$$V = \frac{20}{k} - \frac{C}{k}e^{-kt}$$

(Don't incorporate k into constants as the question asked for answer in terms of k)

Container initially empty so $t = 0 \to V = 0$:

$$0 = \frac{20}{k} - \frac{C}{k} \Rightarrow C = 20$$

$$\therefore V = \frac{20}{k} - \frac{20}{k}e^{-kt}$$

(so $A = \frac{20}{k}$ and $B = -\frac{20}{k}$)

c

$$\frac{dV}{dt} = 20e^{-kt}$$

$$\frac{dV}{dt} = 10, t = 5 \Rightarrow 10 = 20e^{-5k} \Rightarrow k = \frac{1}{5}\ln 2 \;\; (\approx 0.139)$$

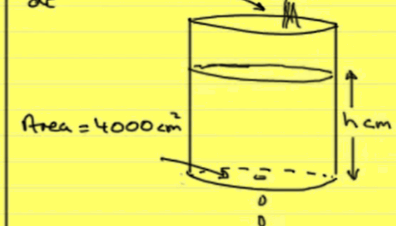
$$t = 10 \Rightarrow V = \frac{75}{\ln 2} \;\; (\approx 108\ cm^3)$$

4

a

Show $\frac{dh}{dt} = 0.4 - k\sqrt{h}$

$\frac{dV}{dt}_{in} = 1600\ cm^3 s^{-1}$



$\frac{dV}{dt}_{OUT} \propto \sqrt{h}$

$\therefore \frac{dV}{dt}_{OUT} = -A\sqrt{h}$

$\therefore \frac{dV}{dt} = 1600 - A\sqrt{h}$

Also $V = 4000h$

$\therefore \frac{dV}{dh} = 4000 \Rightarrow \frac{dh}{dV} = \frac{1}{4000}$

$\therefore \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{1}{4000}(1600 - A\sqrt{h})$

$= 0.4 - \frac{A}{4000}\sqrt{h}$

$= 0.4 - k\sqrt{h}$

where $k = \frac{A}{4000}$

b

$\frac{dh}{dt} = 0.4 - k\sqrt{h}$

where $k = \frac{A}{4000}$

Given $h = 25$, $\frac{dV}{dt}_{OUT} = -400$

$\therefore -400 = -A\sqrt{25}$

$\therefore A = 80$

$\therefore k = \frac{80}{4000}$

$\therefore k = 0.02$

Exercise C: Solutions

4

$$\frac{dh}{dt} = (0.4 - 0.02\sqrt{h})$$

c

$$\therefore \int_0^{100} \frac{1}{0.4-0.02\sqrt{h}}\,dh = \int_0^t dt$$

$$\therefore [t]_0^t = \int_0^{100} \frac{1}{0.4-0.02\sqrt{h}}\,dh$$

$$\therefore t = \int_0^{100} \frac{50}{20-\sqrt{h}}\,dh$$

d

$$\int_0^{100} \frac{50}{20-\sqrt{h}}\,dh = \int_{20}^{10} \frac{50}{x}\frac{dh}{dx}\,dx$$

$$= \int_{20}^{10} \frac{50}{x}\,2(x-20)\,dx$$

$$= 100\int_{20}^{10} \frac{x-20}{x}\,dx$$

$$= 100\int_{20}^{10} \left(1-\frac{20}{x}\right)dx$$

$$= 100\left[x - 20\ln x\right]_{20}^{10}$$

$$= 100\left[(10-20\ln 10)-(20-20\ln 20)\right]$$

$$= 100\left[20\ln 20 - 20\ln 10 - 10\right]$$

$$= 100\left[20(\ln \tfrac{\cancel{20}^{2}}{\cancel{10}_{1}}) - 10\right]$$

$$= 1000(2\ln 2 - 1)$$

$$= 1000(\ln 4 - 1)$$

$h = (20-x)^2$

$\therefore \sqrt{h} = 20 - x$

$\therefore x = 20 - \sqrt{h}$

when $h=0,\ x=20$

$h=100,\ x=10$

$\frac{dx}{dh} = -\frac{1}{2}h^{-1/2}$

$= -\frac{1}{2\sqrt{h}}$

$\therefore \frac{dh}{dx} = -2\sqrt{h}$

$= -2(20-x)$

$= 2(x-20)$

(Either is fine, or expanded version)

e

$$t = \int_0^{100} \frac{50}{20-\sqrt{h}}\,dh$$

$= 1000(\ln 4 - 1)$ seconds

$= 386.29436\ldots$ seconds

$= 6.4382\ldots$ minutes.

$= 6$ mins $26.294\ldots$ seconds

$= 6$ mins 26 seconds (to the nearest s)

Exercise C: Solutions

5

a

$$\frac{dN}{dt} = \frac{(kt-1)(5000-N)}{t}$$

$$\int \frac{1}{5000-N}\,dN = \int \frac{kt-1}{t}\,dt = \int k - \frac{1}{t}\,dt$$

$$-\ln(5000-N) = kt - \ln t + C_1$$

$$\ln(5000-N) = -kt + \ln t + C_2 \quad (C_2 = -C_1)$$

$$5000 - N = e^{-kt+\ln t + C_2}$$

$$5000 - N = e^{-kt} e^{\ln t} e^{C_2}$$

$$5000 - N = Ate^{-kt}$$

$$\therefore N = 5000 - Ate^{-kt}$$

b

$t=1$: $N=1200$:

$$1200 = 5000 - Ae^{-k}$$

$$Ae^{-k} = 3800$$

$$A = 3800e^{k} \quad (1)$$

$t=2$: $N=1800$:

$$1800 = 5000 - 2Ae^{-2k}$$

$$2Ae^{-2k} = 3200$$

$$Ae^{-2k} = 1600$$

$$A = 1600e^{2k} \quad (2)$$

$(1)=(2)$: $3800e^{k} = 1600(e^{k})^2$

$$e^{k} = \frac{19}{8} \quad \therefore k = \ln\frac{19}{8}$$

in (1): $A = 3800e^{\ln\frac{19}{8}} = 3800 \times \frac{19}{8} = 9025$

Exercise C: Solutions

6

Mixing Problem:

A tank contains 400 gal of brine in which 100 lb of salt are dissolved.
Fresh water runs into the tank at a rate of 2 gal/min.
The mixture, kept practically uniform by stirring, runs out at the same rate.
How much salt will there be in the tank at the end of 1 hour?

Let $x(t)$ be the amount (in lbs) of salt in the tank at time t.
Then the rate at which the salt in the tank increases is the amount of salt entering the tank minus that leaving the tank.

General differential equation to solve:

$$\frac{dx}{dt} = C_{in}r_{in} - C_{out}r_{out}$$

Where:
C_{in} is the concentration of substance being added.
r_{in} is the rate at which the substance is added.
C_{out} is the concentration of substance being removed.
r_{out} is the rate at which the substance is removed.

Liquid Flow in= Liquid Flow out
Tank stays at 400 gallons throughout.
Therefore, the concentration in the tank is $\frac{x}{400}$, which will change with time as $x = x(t)$.
There is no flow of salt into the tank (only fresh water)

$$C_{out} = \frac{x}{400}, \qquad r_{out} = 2$$

$$\therefore \frac{dx}{dt} = -C_{out}r_{out} = -\left(\frac{x}{400}\right) \times 2$$

$$\frac{dx}{dt} = -\frac{x}{200} \quad \Rightarrow \quad x(t) = -\frac{x^2}{400} + C$$

Boundary Condition: Initially, there is 100 lb of salt: $x = 100$ when $t = 0$ gives: $C = 100$

$$\Rightarrow \quad x(t) = 100 - \frac{x^2}{400}$$

After 1 hour $x(60) = 100 - \frac{60^2}{400} = 64\ lb$

Challenge: Solutions

1

a

Show that

a) $\dfrac{4x}{x^4-1} \equiv \dfrac{1}{x-1} + \dfrac{1}{x+1} - \dfrac{2x}{x^2+1}$

$x^4-1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)$

$\therefore \dfrac{4x}{x^4-1} \equiv \dfrac{Ax}{(x^2+1)} + \dfrac{B}{(x+1)} + \dfrac{C}{(x-1)}$

$4x \equiv Ax(x+1)(x-1) + B(x^2+1)(x-1) + C(x^2+1)(x+1)$

$x=1$: $4 = 4C \therefore \underline{C=1}$

$x=-1$: $-4 = -4B \therefore B=1$

$x=0$ will make A dissappear $\therefore$ compare coefficients:

$4x \equiv Ax(x^2-1) + (x^2+1)(x-1) + (x^2+1)(x+1)$

$4x \equiv Ax^3 - Ax + (x^2+1)(x-1+x+1)$

$= Ax^3 - Ax + (x^2+1)2x$

$4x \equiv Ax^3 - Ax + 2x^3 + 2x$

compare coefficients:

$x^1: 4 = 2-A \therefore \underline{A=-2}$

check: $x^3: 0 = A+2 \rightarrow A=-2$ ✓

$\therefore \dfrac{4}{x^4-1} \equiv \dfrac{1}{x-1} + \dfrac{1}{x+1} - \dfrac{2x}{x^2+1}$

b

$\dfrac{dy}{dx} = \dfrac{\sin^4 y - 1}{2x\sin 2y}$

$\displaystyle\int \frac{2\sin 2y}{\sin^4 y - 1}\,dy = \int \frac{1}{x}\,dx$ $\quad \sin 2y = 2\sin y\cos y$ $\quad \therefore 2\sin 2y = 4\sin y\cos y$

$\displaystyle\int \frac{4\sin y\cos y}{\sin^4 y - 1}\,dy = \int \frac{1}{x}\,dx$ ⊛

Challenge: Solutions

2

c

$u = \sin y$

$\frac{du}{dy} = \cos y$

$dy = \frac{du}{\cos y}$

$\int \frac{4u \cancel{\cos y}}{u^4-1} \frac{du}{\cancel{\cos y}}$

using part (a): $\int \frac{4u}{u^4-1}\,du = \int \frac{1}{u-1} + \frac{1}{u+1} - \frac{2u}{u^2+1}\,du$

$f'(u)$, $f(u)$

$= \ln|u-1| + \ln|u+1| - \ln|u^2+1|$

for ~~u~~ $v = u^2-1$, $\frac{dv}{du} = 2u$, $du = \frac{dv}{2u}$

$= \ln|\sin y - 1| + \ln|\sin y + 1| - \ln|\sin^2 y + 1|$

$\therefore$ from ⊛:

$\ln|\sin y - 1| + \ln|\sin y + 1| - \ln|\sin^2 y + 1| = \ln x + C$

$\ln\left|\frac{(\sin y - 1)(\sin y + 1)}{\sin^2 y + 1}\right| = \ln x + C$

$x = 4$ when $y = 0$:

$\ln\left|\frac{-1 \times 1}{1}\right| = \ln 4 + C$

$\ln 1 = \ln 4 + C \quad \therefore C = -\ln 4$

$\therefore \ln\left|\frac{(\sin y - 1)(\sin y + 1)}{\sin^2 y + 1}\right| = \ln x - \ln 4$

$\ln\left|\frac{\sin^2 y - 1}{\sin^2 y + 1}\right| = \ln\left(\frac{x}{4}\right)$

$\left|\frac{\sin^2 y - 1}{\sin^2 y + 1}\right| = \frac{x}{4}$

$x = \frac{4|\sin^2 y - 1|}{\sin^2 y + 1}$

only numerator needs modulus as $\sin^2 y + 1$ is always positive.

d

$y = \frac{\pi}{6}$: $\quad x = \frac{4\left|\sin^2\frac{\pi}{6} - 1\right|}{\sin^2\frac{\pi}{6} + 1}$

$\left(\sin\frac{\pi}{6} = \frac{1}{2}\right)$

$= \frac{4\left|\frac{1}{4} - 1\right|}{\frac{1}{4} + 1}$

$= \frac{4\left|-\frac{3}{4}\right|}{\frac{5}{4}} = \frac{\cancel{4} \times \frac{3}{\cancel{4}}}{\frac{5}{4}}$

$= \frac{3}{\frac{5}{4}} = 3 \times \frac{4}{5} = \frac{12}{5}$

Challenge: Solutions

☠ (i)

In this question, you may assume that $\ln(1+x) \approx x - \frac{1}{2}x^2$ when $|x|$ is small.

The height of the water in a tank at time t is h. The initial height of the water is H and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.

(i) Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(\alpha^2 H - h),$$

for some positive constant k. Deduce that the time T taken for the water to reach height αH is given by

$$kT = \ln\left(1 + \frac{1}{\alpha}\right),$$

and that $kT \approx \alpha^{-1}$ for large values of α.

Since the tank has constant cross-sectional area, the volume of water within the tank is proportional to the height of the water.

Therefore we have the height increasing at a rate $a - bh$, where a is the rate of water flowing in divided by the cross-sectional area, and b is a constant of proportionality representing the rate of water leaking out. In other words, we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = a - bh.$$

Now, when $h = \alpha^2 H$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$, so $a - b\alpha^2 H = 0$, or $a = b\alpha^2 H$, giving

$$\frac{\mathrm{d}h}{\mathrm{d}t} = b\alpha^2 H - bh = b(\alpha^2 H - h).$$

Hence if we write $k = b$, we have our desired equation.

We can now solve this by separating variables to get

$$\int \frac{1}{\alpha^2 H - h}\,\mathrm{d}h = \int k\,\mathrm{d}t$$

so that

$$-\ln(\alpha^2 H - h) = kt + c.$$

At $t = 0$, $h = H$, so

$$-\ln(\alpha^2 H - H) = c,$$

which finally gives us

$$kt = \ln(\alpha^2 H - H) - \ln(\alpha^2 H - h).$$

Now at time T, $h = \alpha H$, so that

$$\begin{aligned} kT &= \ln(\alpha^2 H - H) - \ln(\alpha^2 H - \alpha H) \\ &= \ln\left(\frac{\alpha^2 H - H}{\alpha^2 H - \alpha H}\right) \\ &= \ln\left(\frac{\alpha^2 - 1}{\alpha^2 - \alpha}\right) \\ &= \ln\left(\frac{\alpha + 1}{\alpha}\right) \\ &= \ln\left(1 + \frac{1}{\alpha}\right) \end{aligned}$$

as required.

When α is large, so that $\frac{1}{\alpha}$ is small, this is

$$\begin{aligned} kT &= \ln\left(1 + \frac{1}{\alpha}\right) \\ &\approx \frac{1}{\alpha} - \frac{1}{2\alpha^2} \\ &\approx \frac{1}{\alpha}. \end{aligned}$$

Challenge: Solutions

☠ (ii)

We proceed just as in part (i).

This time we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = a - b\sqrt{h},$$

where a and b are some constants. Now, when $h = \alpha^2 H$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$, so $a - b\sqrt{\alpha^2 H} = 0$, which yields $a = b\alpha\sqrt{H}$. We thus have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = b\alpha\sqrt{H} - b\sqrt{h} = b(\alpha\sqrt{H} - \sqrt{h}).$$

So if this time we write $c = b$, we have our desired differential equation.

We again solve this by separating variables to get

$$\int \frac{1}{\alpha\sqrt{H} - \sqrt{h}}\,\mathrm{d}h = \int c\,\mathrm{d}t.$$

To integrate the left hand side, we use the substitution $u = \sqrt{h}$, so that $h = u^2$ and $\frac{\mathrm{d}h}{\mathrm{d}u} = 2u$. This gives us

$$\int \frac{1}{\alpha\sqrt{H} - u} \cdot 2u\,\mathrm{d}u = ct.$$

We divide the numerator by the denominator to get

$$\begin{aligned} ct &= \int \frac{-2(\alpha\sqrt{H} - u) + 2\alpha\sqrt{H}}{\alpha\sqrt{H} - u}\,\mathrm{d}u \\ &= \int -2 + \frac{2\alpha\sqrt{H}}{\alpha\sqrt{H} - u}\,\mathrm{d}u \\ &= -2u - 2\alpha\sqrt{H}\ln(\alpha\sqrt{H} - u) + c' \\ &= -2\sqrt{h} - 2\alpha\sqrt{H}\ln(\alpha\sqrt{H} - \sqrt{h}) + c' \end{aligned}$$

where c' is a constant.

An alternative way of doing this step is to use the substitution $v = \alpha\sqrt{H} - \sqrt{h}$, so that $h = (\alpha\sqrt{H} - v)^2 = \alpha^2 H - 2\alpha v\sqrt{H} + v^2$ and $\frac{\mathrm{d}h/\mathrm{d}v}{=} - 2\alpha\sqrt{H} + 2v$. This gives us

$$\begin{aligned} ct &= \int \frac{1}{v}(-2\alpha\sqrt{H} + 2v) = ct \\ &= \int \frac{-2\alpha\sqrt{H}}{v} + 2\,\mathrm{d}v \\ &= -2\alpha\sqrt{H}\ln v + 2v + c' \\ &= -2\alpha\sqrt{H}\ln(\alpha\sqrt{H} - \sqrt{h}) + 2(\alpha\sqrt{H} - \sqrt{h}) + c' \end{aligned}$$

where c' is again a constant.

At $t = 0$, $h = H$, so

$$c' = 2\sqrt{H} + 2\alpha\sqrt{H}\ln(\alpha\sqrt{H} - \sqrt{H}).$$

(ii) Suppose that the rate at which water leaks out of the tank is proportional to $\sqrt{h}$ (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by

$$cT' = 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha\ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right)$$

for some positive constant c, and that $cT' \approx \sqrt{H}$ for large values of α.

Now at time T', $h = \alpha H$, so that

$$\begin{aligned} cT' &= -2\sqrt{\alpha H} - 2\alpha\sqrt{H}\ln(\alpha\sqrt{H} - \sqrt{\alpha H}) + 2\sqrt{H} + 2\alpha\sqrt{H}\ln(\alpha\sqrt{H} - \sqrt{H}) \\ &= 2\sqrt{H}(1 - \sqrt{\alpha}) + 2\alpha\sqrt{H}\ln\left(\frac{\alpha\sqrt{H} - \sqrt{H}}{\alpha\sqrt{H} - \sqrt{\alpha H}}\right) \\ &= 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha\ln\left(\frac{(\sqrt{\alpha}+1)(\sqrt{\alpha}-1)}{\sqrt{\alpha}(\sqrt{\alpha}-1)}\right)\right) \\ &= 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha\ln\left(\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}}\right)\right) \\ &= 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha\ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right) \end{aligned}$$

as required.

When α is large, $1/\sqrt{\alpha}$ is small, so this gives

$$\begin{aligned} cT' &= 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha\ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right) \\ &\approx 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha\left(\frac{1}{\sqrt{\alpha}} - \frac{1}{2\alpha}\right)\right) \\ &\approx 2\sqrt{H}(1 - \sqrt{\alpha} + \sqrt{\alpha} - \tfrac{1}{2}) \\ &\approx \sqrt{H}. \end{aligned}$$