CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

University of Leeds

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CAPE1150: Course Structure

Each Week:

- 1 Lecture (Monday 1600-1800) – **Chemistry LTA**

The Lecture will be mainly to introduce concepts and test basic understanding via Diagnostic Questions.

- 1 Tutorial (Thursday 1200-1300) – **Chemistry LTA**

The tutorial will apply the concepts to engineering applications as a group and allow you to apply your knowledge to problems of increasing difficulty/complexity.

Formal 2 hour written exam in Summer worth 100% of module mark

The pass mark is 40%

Important: If you get less than 30% on any module and also fail the resit, you will be taken off the whole course!

- **The resit is capped at 40% (max you get is a pass)**
- Course resources (all lecture notes, tutorials, formula sheet) can be found on Minerva.
- Solutions to tutorial problems are in same ppt as tutorial .
- **If you feel you are struggling, please tell me! (email/in person)**

CAPE1150: Effort ∝ Results

- **Make sure that you keep up with the pace of the course, completing each weeks' work in a timely manner (do not leave everything to the last minute and wherever possible do not skip a week with the promise of catching up later).**

- **You must take responsibility for your own workload and manage your time efficiently - this will be essential in any future career so get good at it now.**

- **I have provided all the material you should need to understand the content of the course but it is up to you to learn it by completing problems and revising for the exam.**

- **Maths is learned by doing so the problem classes are at least as important as the lectures.**

- **If you attend lectures and tutorials and complete the work you should do well. Year after year, those with the highest attendance tend to do the best.**

- **If you do not attend lectures or do the work and try to catch it all up at the end you will likely fail (we see this happen every year).**

- **I will be taking paper attendance for my own records in addition to the QR codes.**

- **I will not be running last minute catch-up or help sessions for those who I know have not turned up/put the work in throughout the course.**

- **The course will include some tough concepts which you cannot "blag" your way through in the exam. Practice and prep is extremely important.**

Assessment for Learning (Diagnostic Questioning)

After teaching a concept I will test your understanding (and attention) with quick questions in the following format:

Question

Y	M	C	A
Possible answer 1	Possible answer 2	Possible answer 3	Possible answer 4

Please make sure bring a notebook, pen and calculator to each lecture and tutorial.

Assessment for Learning (Diagnostic Questioning)

Why those letters?

Y	M	C	A

Why am I doing the YMCAs/Diagnostic Questions?

- These are usually quick questions to test that you are grasping the main concepts. The tutorials will allow you to apply the knowledge and challenge yourself more.

- For each question, there is one correct answer, along with other answers that would result from common misconceptions or mistakes.

- It is very important as it lets me to know who is following the lecture and understanding the main concepts, which is why I need **everyone** to take part!

- It also allows me to address any misconceptions straight away, rather than finding out about them in the exam (which is another reason why attendance is vital!)

- Please give me ***your*** answer, don't just conform to those around you (you may be the only correct one!)

- If you get any wrong, this helps you to **identify** areas of the lecture you may need to go over again before the tutorial.

- If you don't join in, I will **relentlessly** target you for individual questions (Fair Warning!)

- Learning material yourself at home is not as effective as teaching and as you are the only one checking, you may not realize your misunderstandings until it is too late (results day)

- **If you are not in the lecture, I cannot assess your understanding until the exam so you may lose many marks you could have otherwise got had you attended and had any errors sorted straight away!**

Differentiation Methods: Outline of Lecture 1

- Recap of Chain, Product & Quotient Rule
- Parametric equations
- Differentiation of parametric equations
- Implicit Differentiation
- Connected Rates of Change

Note:
All the items in red are really just applications of chain rule!

Recap:
Chain, Product and Quotient Rule

Reminder: Alternate notations for Derivatives

There are several notations that are used for derivatives

For a function of x:	Function	1st derivative	2nd derivative
	y	$\dfrac{dy}{dx}$	$\dfrac{d^2y}{dx^2}$
	$f(x)$	$f'(x)$	$f''(x)$
	y	y'	y''

For a function of t:	Function	1st derivative	2nd derivative
	y	$\dfrac{dy}{dt}$	$\dfrac{d^2y}{dt^2}$
	x	$\dfrac{dx}{dt}$	$\dfrac{d^2x}{dt^2}$
	y	$\dot{y}$	$\ddot{y}$
	x	$\dot{x}$	$\ddot{x}$

A dot is sometimes used to denote a derivative with respect to t.

Recap: Differentiation Rules!

- Chain Rule:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

- Product Rule:

$$\boxed{\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

- Quotient Rule:

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

We will now jump straight into diagnostic question, but in the lecture notes you will see worked examples to help you recap if needed!

The Chain Rule

The chain rule allows us to differentiate a composite function, i.e. a function within a function.

$$y = (3x^4 + x)^5$$

We could expand it out then differentiate it but there is a much quicker way…

The Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Note: Notice how the du's sort of 'cancel' top and bottom on the RHS. This is not a valid proof of the chain rule, but the d's sort of behave as quantities which can often be manipulated in this way.

Full Method:

Let $u = 3x^4 + x$

In the Chain Rule, u represents the inner function.

Then: $\frac{du}{dx} = 12x^3 + 1$

and $y = u^5$

This represents the 'outer' function.

$$\frac{dy}{du} = 5u^4 = 5(3x^4 + x)^4$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5(3x^4 + x)^4 \times (12x^3 + 1)$$

which we can then expand/simplify as needed

Doing it mentally in one go:
(aka the 'thing method')

The chain rule boils down to:

$\frac{dy}{dx}$ = outer function differentiated × inner function differentiated

$$\frac{dy}{dx} = 5(3x^4 + x)^4 \times (12x^3 + 1)$$

When differentiating the outer function, replace (in your head) the inner function with the word 'thing'. i.e. "What does $thing^5$ differentiate to?". It's $5\ thing^4$

Now differentiate the inner function, i.e. the 'thing'.

Diagnostic Question

Differentiate with respect to x

$$y = \sin 5x$$

Y

$$y' = \sin 5$$

C

$$y' = \cos 5x$$

M

$$y' = \cos 5$$

A

$$y' = 5\cos 5x$$

Diagnostic Question

Differentiate with respect to x

$$f(x) = e^{x^2}$$

Y

$$f'(x) = e^{x^2}$$

C

$$f'(x) = 2xe^{x^2}$$

M

$$f'(x) = e^{2x}$$

A

$$f'(x) = 2xe^{2x}$$

Diagnostic Question

$$f(x) = \ln(x^2 + 3x + 5)$$

Find $f'(x)$

Where "ln " is the natural logarithm

Y

$$\frac{1}{x^2 + 3x + 5}$$

C

$$\ln(2x + 3)$$

M

$$\frac{2x + 3}{x^2 + 3x + 5}$$

A

$$\frac{1}{\ln(x^2 + 3x + 5)}$$

Diagnostic Question

Given that $y = \sqrt{5x^2 + 1}$, find $\frac{dy}{dx}$

Y $\frac{dy}{dx} = \sqrt{10x}$

M $\frac{dy}{dx} = 10x\sqrt{5x^2 + 1}$

C $\frac{dy}{dx} = \frac{5x}{\sqrt{5x^2 + 1}}$

A $\frac{dy}{dx} = \frac{10x}{\sqrt{5x^2 + 1}}$

The Product Rule

As mentioned previously, the product rule is used, unsurprisingly, when we have a **product** of two functions.

The product rule:

If $\boldsymbol{y = uv}$ then $\boldsymbol{\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$

(Where $u = u(x)$ and $v = v(x)$)

This is quite easy to remember. Differentiate one of the things but leave the other. Then do the other way round. Then add!
Since addition is **commutative**, it doesn't matter which way round we do it.

E.g.

If $y = x^2 \sin x$, determine $\frac{dy}{dx}$

$u = x^2$ $\qquad$ $v = \sin x$

$\frac{du}{dx} = 2x$ $\qquad$ $\frac{dv}{dx} = \cos x$

Write out $u = \cdots, v = \cdots$ then differentiate each.

$$\frac{dy}{dx} = x^2 \cos x + 2x \sin x$$

Sub into product rule (then simplify if necessary)

Tip: While with the chain rule I recommended putting the parts together without writing out $u =$.

For product rule I recommend writing it out in full.
This is because each of u and v can be quite complicated to differentiate – this helps you avoid trying to do too much at once.

Diagnostic Question

Differentiate with respect to x

$$y = x^2 \ln(3x)$$

Y $\dfrac{dy}{dx} = \dfrac{2}{3}$

M $\dfrac{dy}{dx} = 2x \ln(3)$

C $\dfrac{dy}{dx} = 2x \ln(3x) + x$

A $\dfrac{dy}{dx} = 2x \ln(3x) + \dfrac{x}{3}$

Diagnostic Question

$$\frac{d}{dx}(e^{2x}\sin 2x) =$$

Y $e^{2x}\cos 2x$

M $4e^{2x}\cos 2x$

C $4e^{2x}(\sin 2x + \cos 2x)$

A $2e^{2x}(\sin 2x + \cos 2x)$

The Quotient Rule

Just as we use the 'product rule' to differentiate a 'product', we use the '**quotient rule**' to differentiate a 'quotient' (i.e. division).

$$y = \frac{u}{v} = uv^{-1}$$

$$\frac{dy}{dx} = -uv^{-2}\frac{dv}{dx} + \frac{du}{dx}v^{-1}$$

$$= v^{-2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The quotient rule:

If $y = \frac{u}{v}$ then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

(Where $u = u(x)$ and $v = v(x)$)

Remember that the **derivatives** go in **alphabetical order!**

- **Note that instead of using the quotient rule, you could just use the product rule with the bottom function raised to the power -1, which may also require using the chain rule.**
- **However, this is usually messier and you have to be really careful with indices and negatives!**

Quotient Rule: Example

E.g.

If $y = \tan x$, find $\frac{dy}{dx}$

The quotient rule:
If $y = \frac{u}{v}$ then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

(Where $u = u(x)$ and $v = v(x)$)

First, write as a quotient: $y = \frac{\sin x}{\cos x}$

$u = \sin x$ $\qquad$ $v = \cos x$

$\frac{du}{dx} = \cos x$ $\qquad$ $\frac{dv}{dx} = -\sin x$

Layout just like you would with the Product Rule.

$$\frac{dy}{dx} = \frac{\cos x \,.\, \cos x - \sin x . (-\sin x\ )}{\cos^2 x}$$

Don't forget it's a **minus sign** in the middle!

$$\frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$\sin^2 x + \cos^2 x \equiv 1$

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x$$

Diagnostic Question

Differentiate with respect to x, giving your answer in its simplest form: $y = \frac{\sin 4x}{x^3}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Y $\frac{dy}{dx} = \frac{4\cos 4x}{3x^2}$

M $\frac{dy}{dx} = \frac{4x\cos 4x - 3\sin 4x}{x^4}$

C $\frac{dy}{dx} = \frac{3\sin 4x - 4x\cos 4x}{x^4}$

A $\frac{dy}{dx} = \frac{4\cos 4x + 3x^2}{x^6}$

Diagnostic Question

$$f(x) = \frac{x^2 - \ln x}{x}, \text{ find } f'(x)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Y $\boldsymbol{x^2 - \frac{1}{x}}$

M $\boldsymbol{\frac{-x^2 - \ln x + 1}{x^2}}$

C $\boldsymbol{\frac{x^2 - 1 - \ln x}{x^2}}$

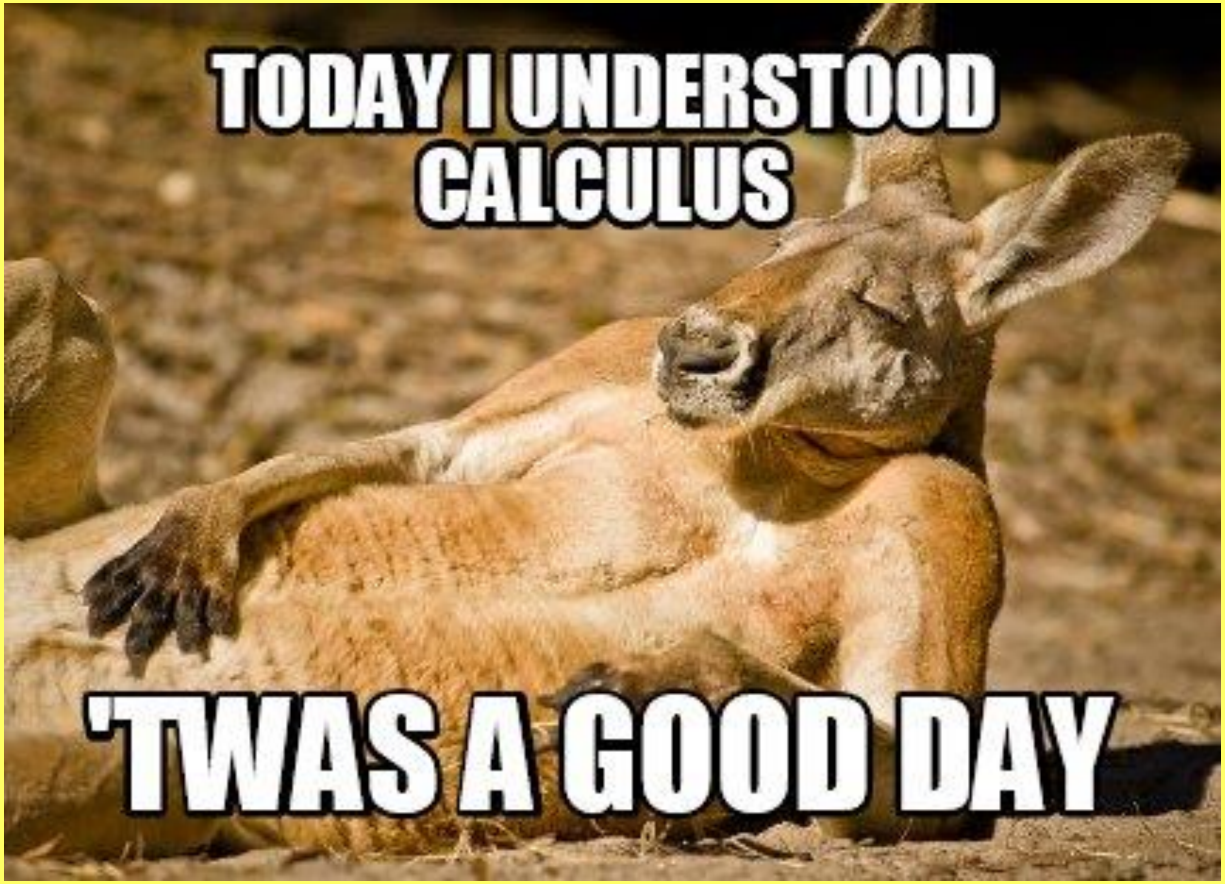
A $\boldsymbol{\frac{x^2 - 1 + \ln x}{x^2}}$

A Note of Caution

If you were unsure on **any** of the recap questions, there is a revision section in the tutorial.

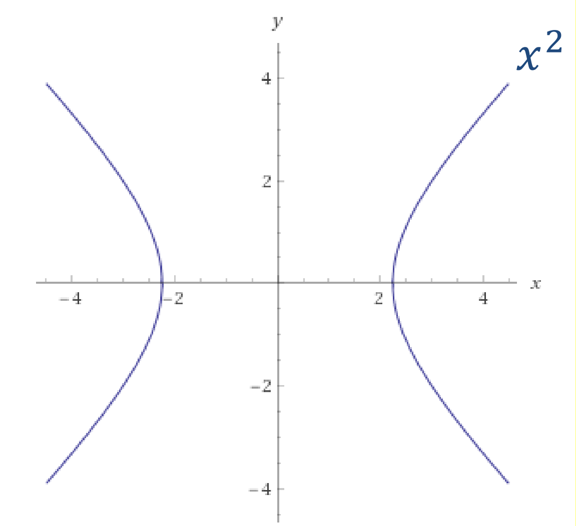
Make sure you complete it, otherwise you'll struggle as we progress through the module!

During the module, we will be using these rules within other problems, and I will expect you to recognise when they are needed.



Parametric Equations

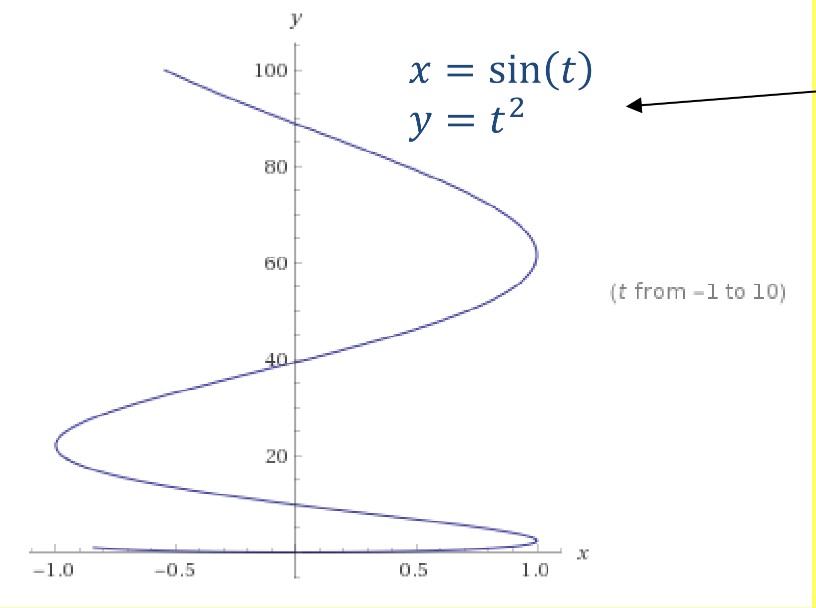
What and Why?



$$x^2 - y^2 = 5$$

Typically, with two variables x and y, we can relate the two by **a single equation involving just x and y**.
This is known as a **<u>Cartesian equation</u>**.

The line shows all points (x, y) which satisfy the Cartesian equation.



$$x = \sin(t)$$
$$y = t^2$$

However, in Mechanics for example, we might want each of the x and y values to be some function of time t, as per this example.

This would allow us to express the position of a particle at time t as the vector:

$$\begin{pmatrix} \sin t \\ t^2 \end{pmatrix}$$

These are known as **parametric equations**, because **each of x and y are defined in terms of some other variable, known as the parameter** (in this case t).

Parametric Equations

- The Cartesian equation of a curve in a plane is an equation linking x and y.
- Some of these equations can be written in a way that is easier to differentiate, by using 2 equations, one giving x and one giving y, both in terms of a 3rd variable, the parameter.
  - Letters commonly used for parameters are s, t and θ (θ is often used if the parameter is an angle).
- For example:

$$x = 2t^2, y = 4t$$

$$x = 3\cos\theta\,, y = 3\sin\theta$$

Converting Between Cartesian & Parametric Forms

We use parametric equations because they are simpler, so we only convert to Cartesian if necessary.

E.g. 1 Change the following to a Cartesian equation and sketch its graph:

$$x = 2t^2, y = 4t$$

The parametric equations show that as t increases, x increases faster than y.

Solution

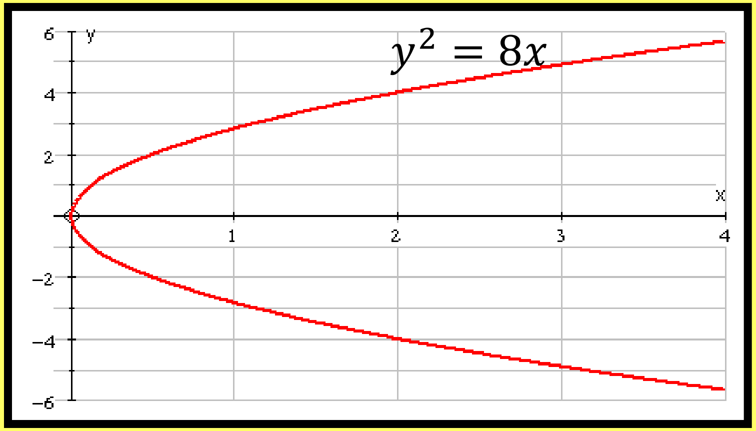
We need to eliminate the parameter t. $y = 4t \Rightarrow \quad t = \frac{y}{4}$

Substituting t into the expression for x gives: $x = 2\left(\frac{y}{4}\right)^2$

The Cartesian equation is $x = \frac{y^2}{8}$

We usually write this as $y^2 = 8x$ which is a quadratic equation in y.

The sketch is:



The curve is called a parabola.

Converting Between Cartesian & Parametric Forms

E.g. 2 Rewrite the following as a Cartesian equation:

$$x = 3\cos\theta\,, y = 3\sin\theta$$

Solution We need to use a trigonometric identity: $\cos^2\theta + \sin^2\theta = 1$

Rearranging the parametric equations: $\dfrac{x}{3} = \cos\theta \qquad \dfrac{y}{3} = \sin\theta$

Squaring both equations: $\cos^2\theta = \dfrac{x^2}{9} \qquad \sin^2\theta = \dfrac{y^2}{9}$

We obtain the Cartesian equation by substituting into the trigonometric identity:

$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$

$$\Rightarrow \quad x^2 + y^2 = 9$$

This is the equation of a circle, centre $(0, 0)$, radius 3.

Sketching More Difficult Ones

- Since we recognise the equation of a circle in Cartesian form, it's easy to sketch.
- However, if we couldn't eliminate the parameter or didn't recognise the curve having rearranged into Cartesian form, we can still sketch from the parametric form.

Sketching More Difficult Ones

E.g. 3 Sketch the curve with equations $x = 2\cos\theta\,, y = 3\sin\theta$

For both $\cos\theta$ and $\sin\theta$ the minimum value is -1 and the maximum value is $+1$, so $-2 \le x \le 2$ and $-3 \le y \le 3$

It is useful to find the x values when y is at its maximum or minimum and the y values when x is at its maximum or minimum.

So:

$$x = 2 \;\Rightarrow\; \theta = 0° \;\Rightarrow\; y = 0$$

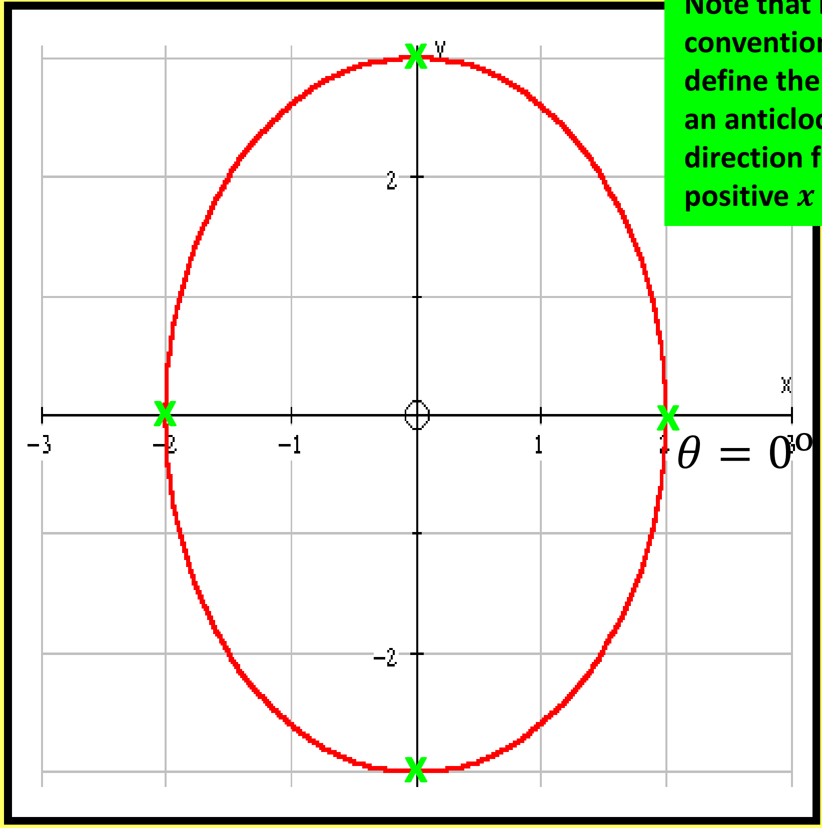
$$x = -2 \;\Rightarrow\; \theta = 180° \;\Rightarrow\; y = 0$$

$$y = 3 \;\Rightarrow\; \theta = 90° \;\Rightarrow\; x = 0$$

$$y = -3 \;\Rightarrow\; \theta = 270° \;\Rightarrow\; x = 0$$

So, we have the parametric equations of an ellipse.

$\theta = 90°$



$\theta = 0°$

Note that it is conventional to define the angle in an anticlockwise direction from the positive x −axis.

Common Parametric Representations

The following equations describe curves you should recognise:

$x = r\cos\theta\,, y = r\sin\theta$

a circle, radius r, centre the origin.

$x = at^2, y = 2at$

a parabola, passing through the origin, with the x-axis as an axis of symmetry.

$x = a\cos\theta\,, y = b\sin\theta\ (a \neq b)$

an ellipse with centre at the origin, passing through the points $(a, 0), (-a, 0), (0, b), (0, -b)$.

Common Parametric Representations: Ellipse

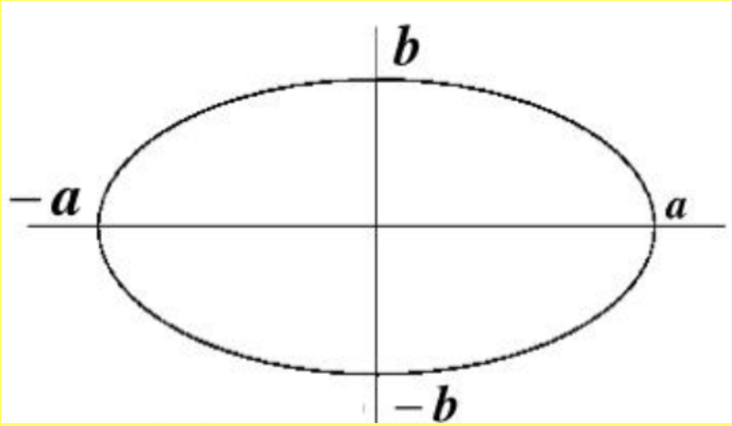
To write the equations of the ellipse in Cartesian form, we use the same trigonometric identity as we used for the circle.

So, for $$x = a\cos\theta\,, y = b\sin\theta\ (a \neq b)$$

use $$\cos^2\theta + \sin^2\theta = 1 \quad \Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

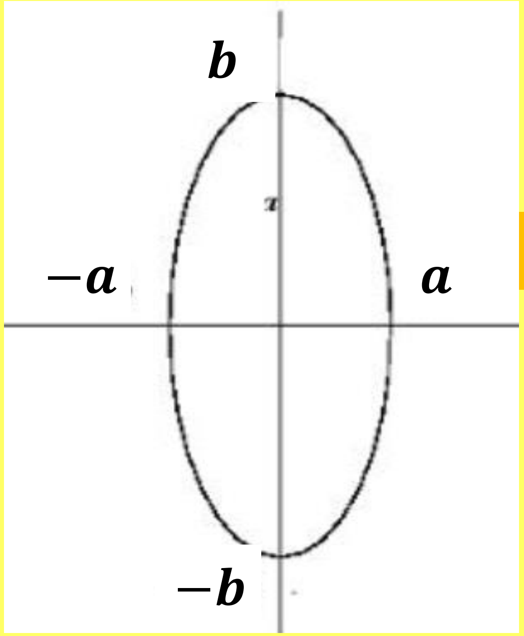
$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the standard equation of an ellipse, centred at the origin.



$\boldsymbol{a > b}$

Horizontal major axis



$\boldsymbol{b > a}$

Vertical major axis

Diagnostic Question

A curve C has parametric equations

$$x = \ln t,\ y = t^2 - 2 \quad (t > 0)$$

Find a cartesian equation for the curve C **in the form $y = f(x)$.**

Y $y = e^{2x} - 2$

M $y = e^{x^2} - 2$

C $x = \ln\sqrt{y + 2}$

A $y = (e^x - 2)^2$

Diagnostic Question

A curve C has parametric equations

$$x = 4\sec\theta,\ y = 2\tan\theta$$

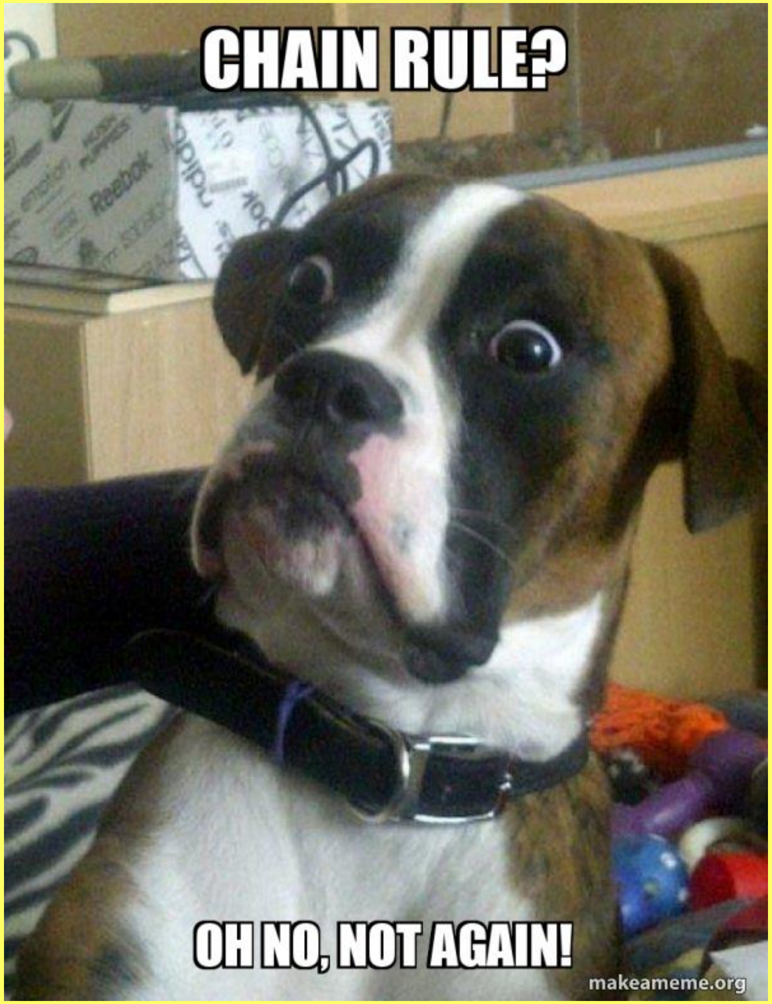
Find a cartesian equation for the curve C.

Y $\dfrac{x^2}{4} - \dfrac{y^2}{2} = 1$

M $\dfrac{y^2}{2} - \dfrac{x^2}{4} = 1$

C $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$

A $\dfrac{y^2}{4} - \dfrac{x^2}{16} = 1$



Differentiation of Parametric Equations
(An application of the Chain Rule)

Differentiation of Parametric Equations

The main reason for using parametric equations is that they are easier to differentiate.

Suppose the parameter is t.

Application of the chain rule gives:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

where $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$

We could also write

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

Differentiation of Parametric Equations

E.g. 1 Find the gradient at the point $t = 2$ on the curve:

$$x = 2t^2, y = 4t$$

Solution

First differentiate y and x with respect to t:

$$\frac{dy}{dt} = 4 \qquad \frac{dx}{dt} = 4t$$

Then use the chain rule to obtain the gradient:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \qquad \Rightarrow \frac{dy}{dx} = 4 \times \frac{1}{4t} = \frac{1}{t}$$

When $t = 2$, the gradient is $m = \frac{1}{2}$

Proving it works!

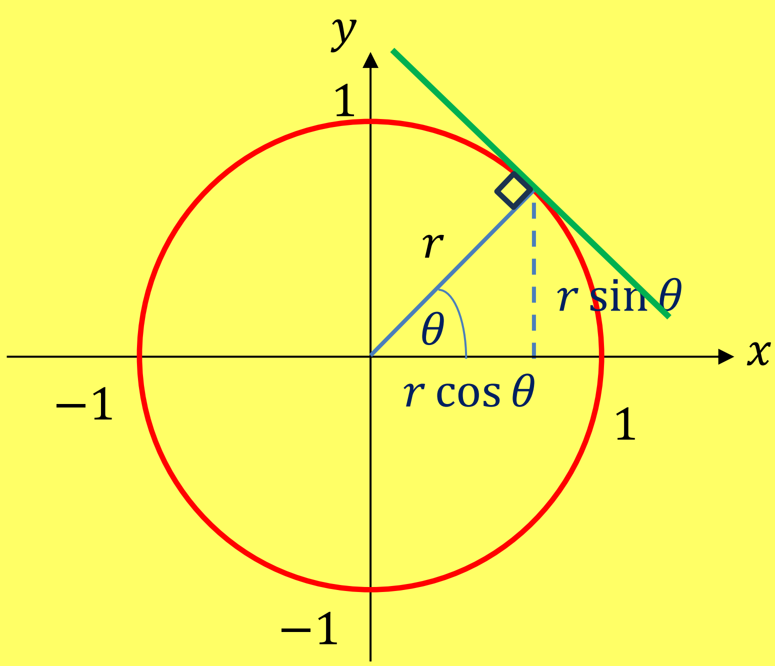
A circle of radius 1 can be represented by the parametric equations:

$$x = \cos\theta\, , y = \sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \cos\theta \times \frac{1}{-\sin\theta} = -\cot\theta$$

This gives the gradient of the circle at any point

We can show this on a diagram



Gradient of the radius: $m_r = \frac{\Delta y}{\Delta x} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

We want the gradient of the circle, which is equal to the gradient of the tangent at that point.

The tangent to a circle is always perpendicular to the radius (circle theorems)

Remember: For perpendicular lines, $m_1 m_2 = -1 \Rightarrow m_2 = -\frac{1}{m_1}$

$$m_t = \frac{-1}{\tan\theta} = -\cot\theta$$

As above

Differentiation of Parametric Equations: Tangent

E.g. 2 Find the equation of the **tangent** to the ellipse:

$$x = 3\cos\theta\,, \qquad y = 2\sin\theta$$

at the point where $\theta = \frac{\pi}{4}$, giving an exact answer.

Solution

The equation of the tangent is given by $y = mx + c$

We first find m at $\theta = \frac{\pi}{4}$. Differentiating y and x gives:

$$\frac{dy}{d\theta} = 2\cos\theta \qquad \frac{dx}{d\theta} = -3\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = -\frac{2\cos\theta}{3\sin\theta}$$

Differentiation of Parametric Equations: Tangent

E.g. 2 (continued)

$$\frac{dy}{dx} = -\frac{2\cos\theta}{3\sin\theta}$$

So, at $\theta = \frac{\pi}{4}$

$$m = -\frac{2\left(\frac{\sqrt{2}}{2}\right)}{3\left(\frac{\sqrt{2}}{2}\right)} = -\frac{2}{3}$$

To find c we also need x and y at $\theta = \frac{\pi}{4}$

$$x = 3\cos\theta \;\Rightarrow\; x = 3\cos\frac{\pi}{4} = \frac{3\sqrt{2}}{2}$$

$$y = 2\sin\theta \;\Rightarrow\; y = 2\sin\frac{\pi}{4} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

We can now substitute into $y = mx + c$ to find c.

$$\sqrt{2} = \left(-\frac{2}{3}\right)\left(\frac{3\sqrt{2}}{2}\right) + c$$

$$c = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$$

The equation of the tangent is: $y = -\frac{2}{3}x + 2\sqrt{2}$

Differentiation of Parametric Equations: Normal

E.g. 3 (a) Find $\frac{dy}{dx}$ for the curve given by the parametric equations:

$$x = t, \qquad y = \frac{1}{t}$$

(b) Show that the equation of the **normal** at the point where $t = 2$ is $2y = 8x - 15$.

Solution

(a) Parametric differentiation gives: $\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = -\frac{1}{t^2}$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}} \qquad \Rightarrow \frac{dy}{dx} = -\frac{1}{t^2}$$

Differentiation of Parametric Equations: Normal

(b) Show that the equation of the **normal** at the point where $t = 2$ is $2y = 8x - 15.$

$$\frac{dy}{dx} = -\frac{1}{t^2} \qquad \text{when } t = 2 \Rightarrow \frac{dy}{dx} = -\frac{1}{4}$$

Therefore the gradient of the normal is 4

Remember: For perpendicular lines, $m_1 m_2 = -1 \Rightarrow m_2 = -\frac{1}{m_1}$

Now find x and y at $t = 2$:

$$x = t, y = \frac{1}{t} \qquad \Rightarrow x = 2, y = \frac{1}{2}$$

Using $y - y_1 = m(x - x_1)$:

$$y - \frac{1}{2} = 4(x - 2) \qquad \Rightarrow 2y - 1 = 8(x - 2)$$

$$\Rightarrow 2y - 1 = 8x - 16$$

The equation of the normal is: $2y = 8x - 15$ as required.

Diagnostic Question

Find the gradient at the point P where $t = 2$, on the curve given parametrically by:

$$x = t^3 + t, \qquad y = \mathrm{t}^2 + 1,$$

Y 13

M $\dfrac{13}{4}$

C $\dfrac{4}{13}$

A 52

Diagnostic Question

A curve is defined parametrically by:

$$x = \sin t\,, \qquad y = \cos t$$

Which of the following gives the gradient of the curve at $x = \frac{1}{2}$?

Y $-\tan\frac{1}{2}$

M $-\cot\frac{1}{2}$

C $-\tan\frac{\pi}{6}$

A $-\cot\frac{\pi}{6}$

Diagnostic Question

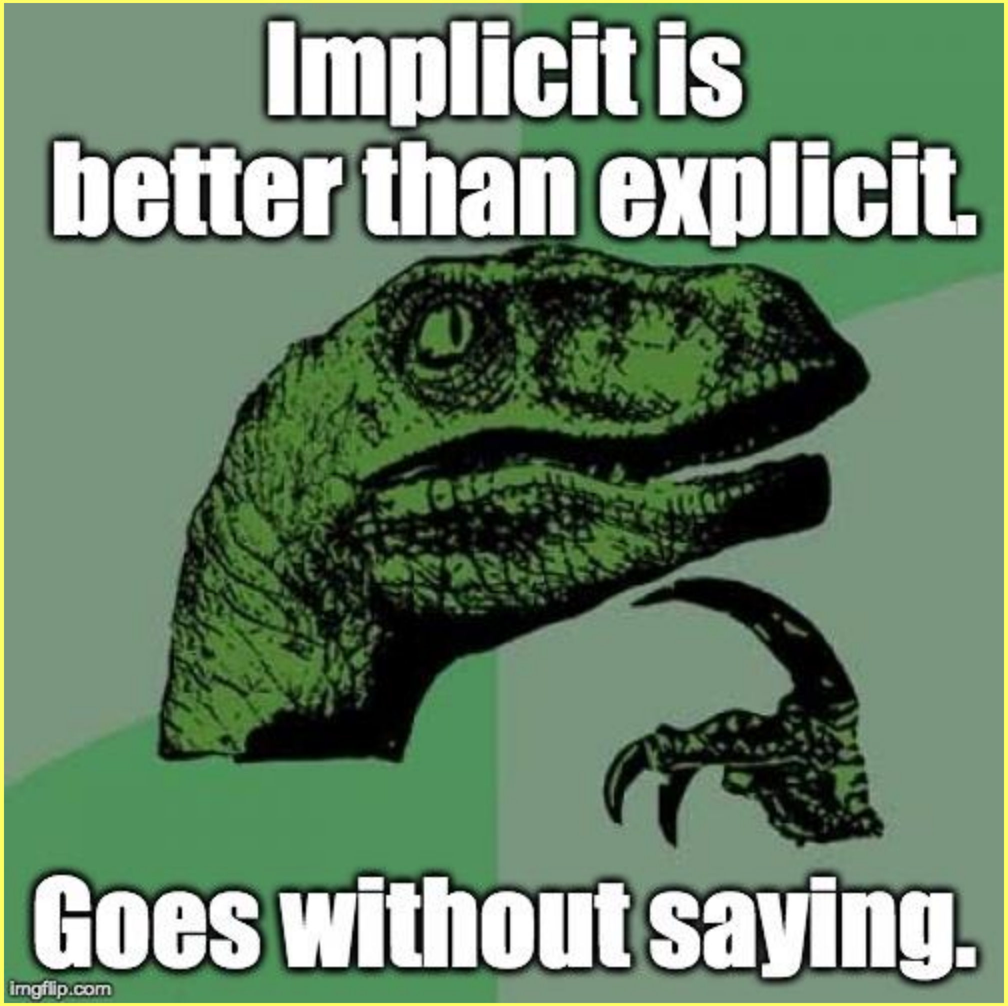
At a point on a parametric curve where the tangent is vertical, which one of these is always true?

Y $\frac{dy}{dt} = 0$

M $\frac{dx}{dt} = 0$

C $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$

A Neither $\frac{dy}{dt} = 0$ nor $\frac{dx}{dt} = 0$



Implicit Differentiation
(Another Application of the Chain Rule)

Implicit Differentiation

- We usually write a relationship between x and y in the form $y = f(x)$
- This gives y explicitly in terms of x.
- However, we have already seen some curves, for example, a circle, where it is easier to have x and y on the same side of the equation.
- This form gives y implicitly.

E.g. $y = x^2 - 4x + 3$ is explicit

$x^2 + y^2 = 4$ is implicit

$$y = x^2$$

$$\frac{d}{dx} \qquad \frac{dy}{dx} = 2x \qquad \frac{d}{dx}$$

When seeing $y = x^2$ and differentiating, you probably think you're just differentiating the x^2. But in fact, you're differentiating **both** sides of the equation! (with respect to x)

y (by definition) differentiates to $\frac{dy}{dx}$

To differentiate implicitly you only need to know 2 things:

- Differentiate each side of the equation (using chain rule if necessary).
- Remember that y differentiated with respect to x is, by definition, $\frac{dy}{dx}$

Implicit Differentiation

It is not always easy to rearrange implicit formulae to explicit form, so we need to be able to differentiate implicitly.

Consider: $y = x^2$ (explicit)

Differentiating with respect to x : $\frac{dy}{dx} = 2x$

So, differentiating y^2 with respect to y is also straightforward. The result is simply $2y$.

We can now determine $\frac{d(y^2)}{dx}$ using the chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The chain rule for differentiating y^2 :

becomes $$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx}$$

since we can only differentiate $f(y)$ with respect to y.

$$\frac{d(y^2)}{dx} = 2y\frac{dy}{dx}$$

Implicit Differentiation

We can now differentiate expressions such as:

y with respect to x: $\dfrac{dy}{dx}$

y^2 with respect to x: $\dfrac{d(y^2)}{dx} = 2y\dfrac{dy}{dx}$

$f(y)$ with respect to x: $\dfrac{df(y)}{dx} = \dfrac{df(y)}{dy} \times \dfrac{dy}{dx}$

Basically: Differentiate the outer y function (as if it were a bracket) then stick a $\frac{dy}{dx}$ on it for the inner function (which is y).

Implicit Differentiation

E.g. 1 Find $\frac{dy}{dx}$ in terms of x and y

$$x^2 + y^2 = 4$$

Solution

Using the method of implicit differentiation:

$$\frac{d}{dx}(x^2) + \left[\frac{d}{dy}(y^2) \times \frac{dy}{dx}\right] = 0$$

You don't need to write this step, it's just to explain what is happening.

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

Implicit Differentiation

E.g. 2 Find $\frac{dy}{dx}$ in terms of x and y

$$y + x + 3x^2 - 2y^3 = 1$$

Solution

Implicit differentiation:

$$\left[\frac{d}{dy}(y) \times \frac{dy}{dx}\right] + \frac{d}{dx}(x + 3x^2) - \left[\frac{d}{dy}(2y^3) \times \frac{dy}{dx}\right] = 0$$

$$\frac{dy}{dx} + 1 + 6x - 6y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(1 - 6y^2) = -6x - 1$$

$$\frac{dy}{dx} = \frac{-6x - 1}{1 - 6y^2} = \boldsymbol{\frac{6x + 1}{6y^2 - 1}}$$

Implicit Differentiation

E.g. 3 A curve is defined by the equation $\frac{x^2}{3}+\frac{y^2}{4}=1$

Find the gradient at the point $\left(\frac{3}{2},1\right)$

Solution

Implicit differentiation:

$$\frac{d}{dx}\left(\frac{x^2}{3}\right)+\left[\frac{d}{dy}\left(\frac{y^2}{4}\right)\times\frac{dy}{dx}\right]=0 \quad \Rightarrow \quad \frac{2x}{3}+\frac{y}{2}\frac{dy}{dx}=0$$

Now substitute in $x=\frac{3}{2}, y=1$:

$$\frac{2\,\frac{3}{2}}{3}+\frac{1}{2}\frac{dy}{dx}=0$$

So:

$$1+\frac{1}{2}\frac{dy}{dx}=0 \quad \Rightarrow \frac{1}{2}\frac{dy}{dx}=-1 \quad \Rightarrow \quad \boxed{\frac{dy}{dx}=-2}$$

Implicit Differentiation

E.g. 4 Find the gradient function of the curve $\ln(y) = 1 + x^2$

Solution

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(1 + x^2)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = 2xy$$

Diagnostic Question

Find $\frac{dy}{dx}$ in terms of x and y for $3y^2 + 4x^3 + 2x + 1 = 0$

Y $\dfrac{6x^2 - 1}{3y}$

M $-\dfrac{6x^2 + 1}{3y}$

C $\dfrac{1 - 6x^2}{3y}$

A $-\dfrac{6x^2 - 1}{3y}$

Diagnostic Question

Find $\frac{dy}{dx}$ in terms of x and y for $x^3 + 4y^3 - 2x^2 + y - 5 = 0$

Y $\dfrac{4x - 3x^2}{12y^2 + 1}$

M $\dfrac{4x + 3x^2}{12y^2 + 1}$

C $\dfrac{4x + 3x^2}{1 - 12y^2}$

A $\dfrac{4x - 3x^2}{1 - 12y^2}$

Application of Implicit Differentiation

- One useful application of implicit differentiation may arise in exponential growth and decay problems.
- Suppose we want to find $y'(x)$ given a general exponential function $y = a^x$ for a constant a.
- We cannot differentiate directly when x is an index (apart from e^x), but we can use implicit differentiation:

$$y = a^x$$

Take natural logs

$$\ln y = \ln a^x$$

Using "power to the front" law of logs

$$\ln y = x \ln a$$

Differentiate Implicitly

$$\frac{1}{y}\frac{dy}{dx} = \ln a$$

Rearrange for $\frac{dy}{dx}$

$$\frac{dy}{dx} = y \ln a$$

Substitute back in for y in terms of x

$$\frac{dy}{dx} = a^x \ln a$$

$$\frac{d}{dx}a^x = a^x \ln a$$

(Provided $a > 0$)

Because $\ln a$ is undefined for $a \leq 0$

A Familiar Friend…

We can also use this approach to verify one of our standard derivatives

$$y = e^x$$

Take natural logs

$$\ln y = \ln e^x$$

Using “power to the front” law of logs

$$\ln y = x \ln e$$

$\ln e = 1$

$$\ln y = x$$

Differentiate Implicitly

$$\frac{1}{y}\frac{dy}{dx} = 1$$

Rearrange for $\frac{dy}{dx}$

$$\frac{dy}{dx} = y$$

Substitute back in for y in terms of x

$$\frac{dy}{dx} = e^x$$

This is obviously ok as $e > 0$
$(e = 2.718 \ldots)$

Application: Inverse Trigonometric Functions

E.g. 5a Show that if $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Note: $\arcsin x = \sin^{-1} x$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$y = \arcsin x$$
$$\therefore x = \sin y$$

$$\frac{dx}{dy} = \cos y \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

We want our answer in terms of x (which is $\sin y$)

$$\cos y = \sqrt{1-\sin^2 y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} \quad = \frac{1}{\sqrt{1-x^2}}$$

E.g. 5b Given that $y = \arcsin x^2$ find $\frac{dy}{dx}$

Using the previous result and applying chain rule:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^2)^2}} \times 2x \quad = \frac{2x}{\sqrt{1-x^4}}$$

Application: Inverse Trigonometric Functions

E.g. 6

Given that $y = \operatorname{arcsec} 2x$,
show that $y = \frac{1}{x\sqrt{4x^2-1}}$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\sec y = 2x$$

Differentiate with respect to x (implicitly)

$$\sec y \tan y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2}{\sec y \tan y}$$

$$\frac{dy}{dx} = \frac{2}{\sec y \sqrt{\sec^2 y - 1}}$$

$$\frac{dy}{dx} = \frac{2}{2x\sqrt{4x^2-1}} \qquad = \boldsymbol{\frac{1}{x\sqrt{4x^2-1}}}$$

Product Rule For Implicit Equations

We may have to differentiate terms such as xy.

This is a product so we use the product rule:

$$\boxed{\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}}$$

Let: $u = x \quad \Rightarrow \quad u' = 1$

$$v = y \quad \Rightarrow \quad v' = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d(xy)}{dx} = y + x\frac{dy}{dx}$$

This will be important when homogeneous first order differential equations (future lecture)

Product Rule For Implicit Equations

E.g. 7

Given $x^2 + y^2 - xy = 2$ find $\frac{dy}{dx}$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(xy) = 0$$

Let:

$$u = x \quad \Rightarrow \quad u' = 1$$

$$v = y \quad \Rightarrow \quad v' = \frac{dy}{dx}$$

If you want to, you can skip this formality and just differentiate x and leave y then differentiate y and leave x (but don't forget the negative in front of the product)

$$2x + 2y\frac{dy}{dx} - \left(y + x\frac{dy}{dx}\right) = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2x - y}{x - 2y}$$

Product Rule For Implicit Equations

E.g. 8 Find the gradients at the 2 points on the curve given by

$$y^2 - 5xy + 8x^2 = 2 \text{ where } x = 1$$

Differentiating the equation we get:

$$\frac{d}{dx}(y^2) - \frac{d}{dx}(5xy) + \frac{d}{dx}(8x^2) = 0 \qquad \Rightarrow \; 2y\frac{dy}{dx} - 5y - 5x\frac{dy}{dx} + 16x = 0$$

Now we have to find the y-coordinates of the 2 points where $x = 1$:

$$y^2 - 5xy + 8x^2 = 2 \qquad \Rightarrow y^2 - 5y + 8 = 2 \qquad \Rightarrow y^2 - 5y + 6 = 0$$

$$\Rightarrow (y-2)(y-3) = 0 \qquad \Rightarrow y = 2\,, y = 3$$

At point (1, 2):

$$4\frac{dy}{dx} - 10 - 5\frac{dy}{dx} + 16 = 0$$

$$\Rightarrow \frac{dy}{dx} = 6$$

At point (1, 3):

$$6\frac{dy}{dx} - 15 - 5\frac{dy}{dx} + 16 = 0$$

$$\Rightarrow \frac{dy}{dx} = -1$$

Diagnostic Question

$$\frac{d}{dx}(xy^2) =$$

Y y^2

M $2xy$

C $2xy\dfrac{dy}{dx} + y^2$

A $2xy\dfrac{dy}{dx}$

Diagnostic Question

Identify the correct implicit derivative of $\sin(xy) = \frac{1}{\sqrt{2}}$

Y $\quad y\cos(xy)\dfrac{dy}{dx} = 0$

M $\quad x\cos(xy)\dfrac{dy}{dx} = 0$

C $\quad \cos(xy)\left(1 + \dfrac{dy}{dx}\right) = 0$

A $\quad \cos(xy)\left(y + x\dfrac{dy}{dx}\right) = 0$

Diagnostic Question

Below is a student's attempt to find $\frac{dy}{dx}$ from the implicit equation

$$3^x + \sin(xy) = 3^y$$

$$\underbrace{3^x \ln 3}_{Y} + \underbrace{x\cos(xy)}_{M} + \underbrace{y\cos(xy)}_{C} = \underbrace{3^y \ln 3 \frac{dy}{dx}}_{A}$$

In which term has the student made a mistake?

Diagnostic Question

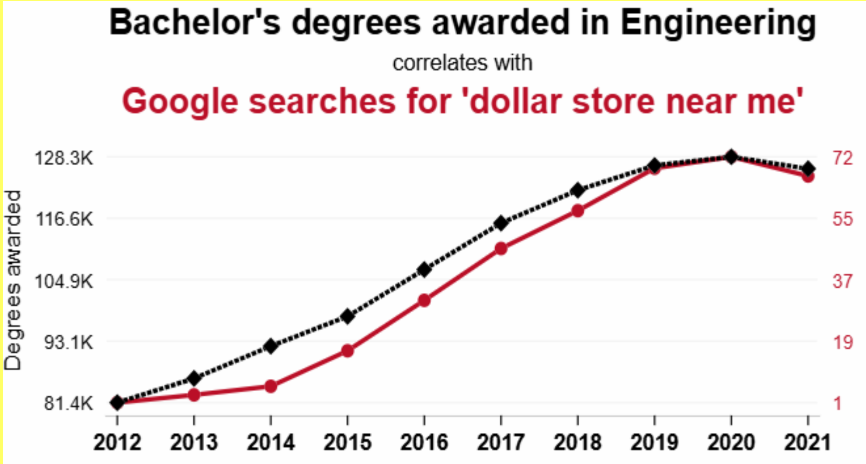
$$\frac{d}{dx}\left(e^{x^2y}\right) =$$

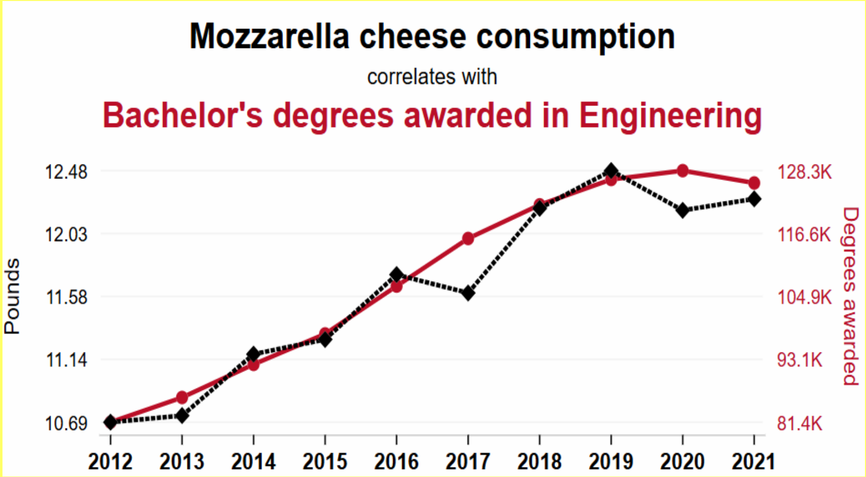
Y $2xye^{x^2y}$

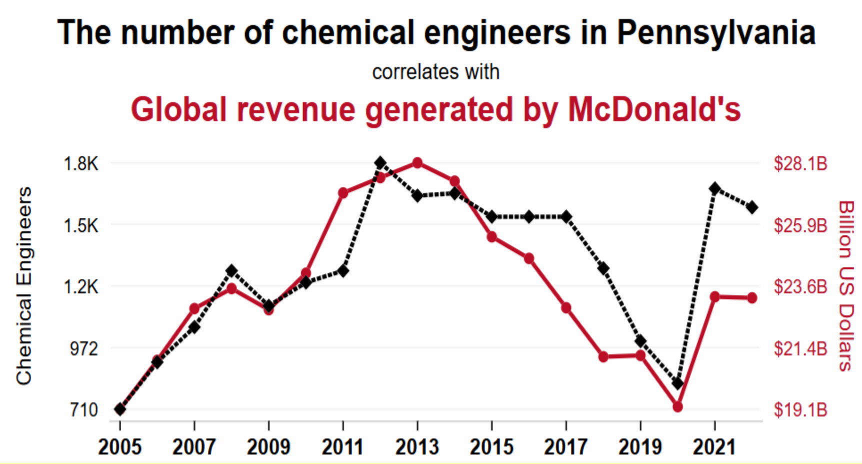
M $2x\dfrac{dy}{dx}e^{x^2y}$

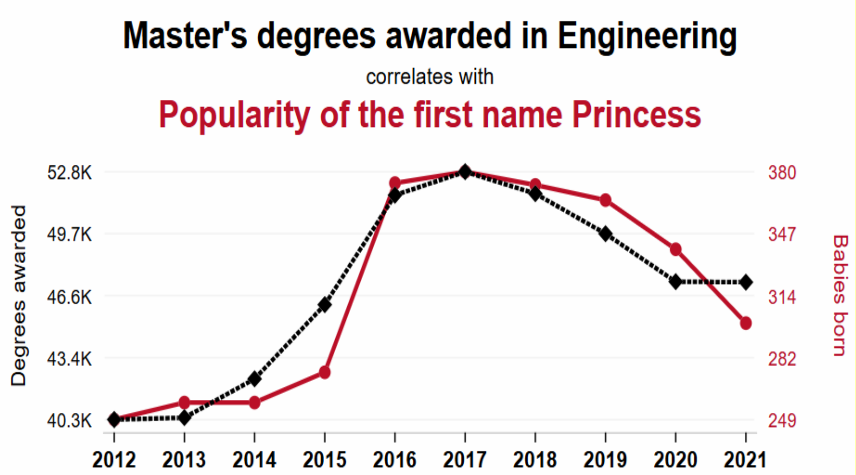
C $\left(2x + \dfrac{dy}{dx}\right)e^{x^2y}$

A $\left(2xy + x^2\dfrac{dy}{dx}\right)e^{x^2y}$





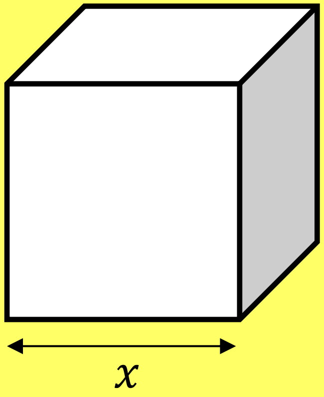




Connected Rates of Change

Relating Rates of Change

Tip: Whenever you see the word '<u>rate</u>', think $/dt$



Suppose a cube, of side x cm, is growing at a rate of 2 cm each second.

We know how the side length is growing. What other physical quantities might we be interested in the growth rate of?

The **<u>surface area</u>** and the **<u>volume</u>**.

$$\frac{dx}{dt} = 2 \text{ cm/s}$$

This topic is concerned with how we can calculate the rate of growth of one physical quantity, given the rate of growth of another connected quantity.

We say these are **<u>connected rates of change</u>**.

Staying with cubes for a moment…

E.g. 1

A cube with side length x cm is growing at a rate of 2 cm/s. Determine the **rate of change of the volume** of the cube when its side length is 5 cm.

Represent "the rate of change of the side length is 2".
A 'rate' is always **with respect to time** (units of $\frac{dx}{dt}$ is cm/s)

$$\frac{dx}{dt} = 2$$

Write a formula that relates the two physical quantities involved, in this case, volume and side length.

Differentiate, noting V is in terms of x.

$$V = x^3 \quad \rightarrow \quad \frac{dV}{dx} = 3x^2$$

We want "**the rate of change of volume**".

We can use the chain rule to connect rates of change. What is the one other physical quantity involved?

The side length x is the one other physical quantity. These should match (you can loosely think of them as 'cancelling' in the multiplication).

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

We know $\frac{dV}{dx}$ and $\frac{dx}{dt}$, so we can find $\frac{dV}{dt}$

$$\frac{dV}{dt} = 3x^2 \times 2 = 6x^2$$

This enables us to find the rate of change of volume when $x = 5$

When $x = 5$,

$$\frac{dV}{dt} = 6(5^2) = \mathbf{150\ cm^3 s^{-1}}$$

Another Example

E.g. 2

A cube with side length x cm is growing in volume at a rate of 10 cm^3/s. Determine the **rate of change of the length** of the cube when its volume is 125 cm^3.

As before, write out the rate that is given.

$$\frac{dV}{dt} = 10$$

As before, write out a formula connecting the physical quantities, and differentiate.

$$V = x^3 \quad \rightarrow \quad \frac{dV}{dx} = 3x^2$$

This time, we want the rate of change of side length, $\frac{dx}{dt}$

$$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$$

We need $\frac{dx}{dV}$, but we have $\frac{dV}{dx}$.
But by chain rule, $\frac{dV}{dx} = 1 \div \frac{dx}{dV}$

$$= \frac{1}{3x^2} \times 10$$

$$= \frac{10}{3x^2}$$

We need to substitute in x to find the rate of change, but we have V.

When $V = 125, \quad 125 = x^3 \quad \rightarrow \quad x = 5$

This enables us to calculate $\frac{dx}{dt}$ using our formula.

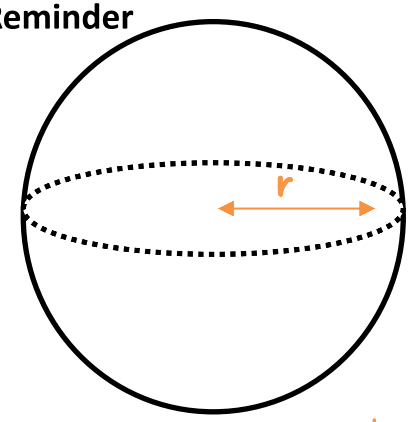
$$\frac{dx}{dt} = \frac{10}{3(5^2)} = \frac{10}{75} = \boldsymbol{\frac{2}{15}\ cm\ s^{-1}}$$

Spheres

E.g. 3

A sphere of radius r cm is growing in surface area at a rate of 10π cm^2/s. Determine the **rate of change of the radius** of the sphere when the radius is 5 cm.

Reminder



Volume

$$V = \frac{4}{3}\pi r^3$$

$$S = 4\pi r^2$$

Surface area

$$\frac{dS}{dt} = 10\pi$$

As usual, represent the given rate, and write a formula connecting surface area and radius.

$$S = 4\pi r^2 \quad \rightarrow \quad \frac{dS}{dr} = 8\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dS} \times \frac{dS}{dt}$$

$$= \frac{1}{8\pi r} \times 10\pi = \frac{5}{4r}$$

Use the chain rule to get an expression for the rate of change of the radius.

When $x = 5$, $\frac{dr}{dt} = \frac{5}{4(5)} = \mathbf{0.25}$ **cm/s**

Substitute in $r = 5$

Relating 3 Physical Quantities

Sometimes, we might need to **use the chain rule twice** to solve connected rate problems. This might occur for example if we need to find the rate of change of volume given the rate of change of surface area, or vice versa.

E.g. 4

The rate of change of the volume of a cube, with side length x cm, is 10 cm^3 s^{-1}. Determine the rate of change of the surface area when $x = 20$

As usual, write out any given rates.

$$\frac{dV}{dt} = 10$$

In this question, we're involving **all** of x, S and V, so write out formulae for both S and V in terms of x and differentiate.

$$V = x^3 \quad \rightarrow \quad \frac{dV}{dx} = 3x^2$$

$$S = 6x^2 \quad \rightarrow \quad \frac{dS}{dx} = 12x$$

We want $\frac{dS}{dt}$. We can either start with $\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt}$ and then work out $\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$, or we can use the chain rule with **three terms** in the product, with the diagonal linking as shown.

We could theoretically use $\frac{dS}{dt} = \frac{dS}{dV} \times \frac{dV}{dt}$ but we don't have a formula directly linking S and V.

$$\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dV} \times \frac{dV}{dt}$$

$$= 12x \times \frac{1}{3x^2} \times 10 = \frac{40}{x}$$

When $x = 20$, $\boldsymbol{\frac{dS}{dt} = \frac{40}{20} =}$ **2 cm² s⁻¹**

Diagnostic Question

A cube with side length x cm is growing at a rate of 6 cm/s.
Determine the rate of change of the surface area of the cube when $x = 12$.

Y $24\ cm^2 s^{-1}$

M $216\ cm^2 s^{-1}$

C $864\ cm^2 s^{-1}$

A $2592\ cm^2 s^{-1}$

Diagnostic Question

A cube with side length x cm is growing in volume at a rate of $5\ cm^3s^{-1}$.
Determine the exact rate of change of the **length** of the cube when $x = 2$

Y $\dfrac{5}{12}\ cms^{-1}$

M $60\ cms^{-1}$

C $\dfrac{12}{5}\ cms^{-1}$

A $40\ cms^{-1}$

Summary

- The Cartesian equation of a curve in a plane is an equation linking x and y.
- A parametric equation gives the x and y coordinates in terms of a 3rd variable, the parameter.
- Curves can be sketched either from their Cartesian form (x and y) or from their parametric form.
- To form a cartesian equation from parametric equations, re-arrange to eliminate the parameter and/or use identities to relate.
- The gradient (or derivative) of a curve given in terms of a parameter t is:

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$$

- Parametric curves can be differentiated using the chain rule.

Summary

- To find the equation of a tangent at a point:
  - Find the gradient function;
  - Substitute to find m at the given point;
  - Substitute to find x and y at the given point;
  - Use $y = mx + c$ to find c.
  - Substitute for m and c in $y = mx + c$

- To find the gradient of a normal use: $m_2 = -\dfrac{1}{m_1}$

  where m_1 is the gradient of the tangent and m_2 is the gradient of the normal.

Summary

- Implicit equations do not give *y* on its own on one side of the 'equals' sign.
  - This means it is trickier to recognise the curve and to differentiate or integrate.
- The derivative of an implicit equation, $f(y)$, is given by:

$$\boxed{\frac{df(y)}{dx} = \frac{df(y)}{dy} \times \frac{dy}{dx}}$$

- This rule can be used, together with the laws of logarithms, to differentiate all exponential functions.

Thanks
See you in the Tutorial!

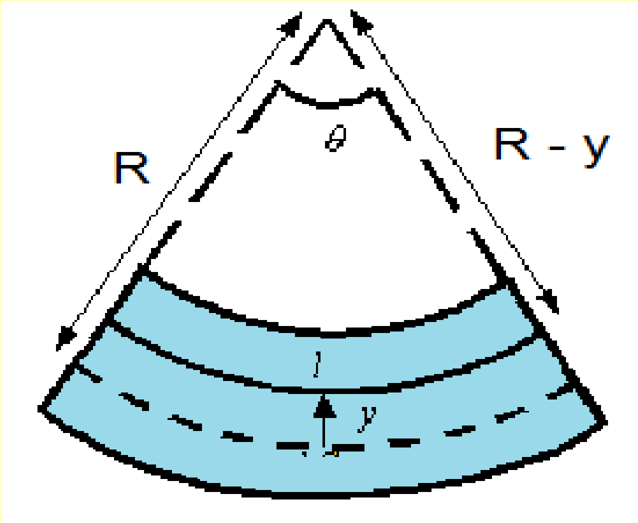
Extra: Proof of Radius of Curvature (Engineering Application)

- When a horizontal beam is acted on by a forces which bend it, each small segment of the beam will be slightly curved and can be approximated as an arc of a circle.

- The radius of the circle, R, is known as the radius of curvature of the beam at a particular point.

- If the shape of the beam can be described by an equation $y = f(x)$, then the formula for R involves only $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

- What follows is the proof of this formula…



Extra: Proof of Radius of Curvature (Engineering Application)

Start with implicit equation of a circle: $x^2 + y^2 = R^2$

Differentiate: $2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow x + y\frac{dy}{dx} = 0 \quad (1)$

Differentiate again: $1 + \frac{dy}{dx} \times \frac{dy}{dx} + y\frac{d^2y}{dx^2} = 0$

Differentiate again: $1 + \frac{dy}{dx} \times \frac{dy}{dx} + y\frac{d^2y}{dx^2} = 0 \quad \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0 \quad (2)$

From (1): $\frac{dy}{dx} = -\frac{x}{y}$ Therefore: $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{y^2} = \frac{y^2 + x^2}{y^2} = \left(\frac{R}{y}\right)^2$

Substitute: $1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{R}{y}\right)^2$ into (2): $\left(\frac{R}{y}\right)^2 + y\frac{d^2y}{dx^2} = 0$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{R^2}{y^3} = -\frac{1}{R}\left(\frac{R}{y}\right)^3$$

$$\frac{d^2y}{dx^2} = -\frac{1}{R}\left(\frac{R}{y}\right)^3 \Rightarrow$$

$$\frac{1}{R} = -\frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

In textbooks, the minus sign is usually missing, but it shows whether the circle is above or below the curve.

When the gradient $\frac{dy}{dx}$ is small, for a slightly bent beam, the denominator ≈ 1 and $\frac{1}{R} \approx -\frac{d^2y}{dx^2}$