CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

University of Leeds

Level 1 Semester 2

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Tutorial: Question Difficulty Colour Code

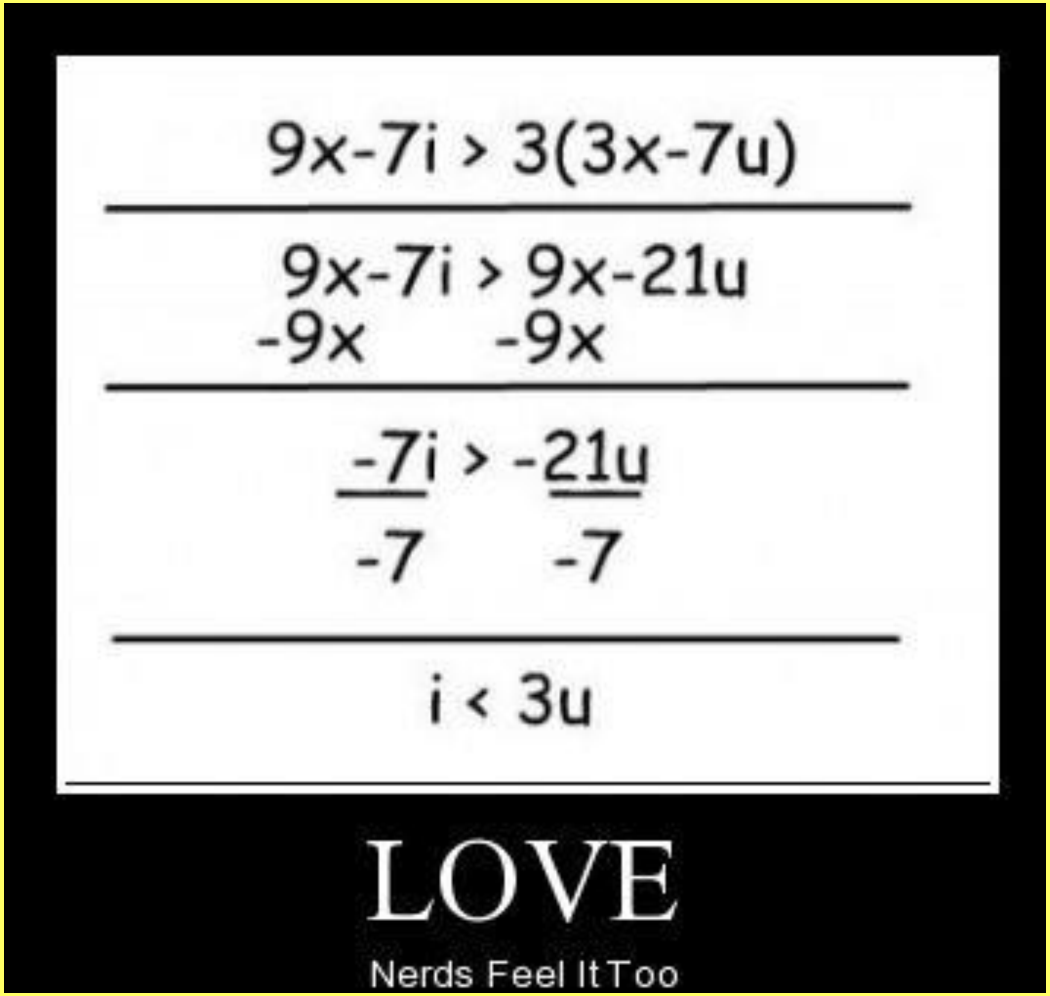
Basic - straightforward application
(you must be able to do these)

Medium – Makes you think a bit
(you must be able to do these)

Hard – Makes you think a lot
(you should be able to do these)

Extreme – Tests your understanding to the limit!
(for those who like a challenge)

Applied – Real-life examples of the topic, may sometimes involve prior knowledge
(you should attempt these – will help in future engineering)



Tutorial 6

Complex Numbers

Class Example: Simplifying

E.g. 1 Simplify:

$a)\ \sqrt{-24}$

$b)\ i^{25}$

$c)\ (6-3i)(2+5i)$

$d)\ (4-3i)^2$

$e)\ \frac{3-4i}{2+3i}$

Class Example: Quadratics

E.g. 2 **Solve:** $\mathbf{2x^2 - 2x + 9 = 0}$

Note: We could have x or z here, as technical any x is a complex number ($x = 2$ is $x = 2 + 0i$)

We just solve the quadratic as usual

It will not factorise, so use the formula:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 2 \times 9}}{4}$$

$$x = \frac{2 \pm \sqrt{-68}}{4} \quad = \frac{2 \pm i\sqrt{68}}{4} \quad = \frac{2 \pm i\sqrt{4}\sqrt{17}}{4}$$

$$= \frac{2 \pm 2i\sqrt{17}}{4} \quad = \frac{1 \pm i\sqrt{17}}{2}$$

$$\therefore \quad \boldsymbol{x = \frac{1}{2} \pm \frac{\sqrt{17}}{2} i}$$

Class Example: Finding unknown complex Numbers

E.g. 3

Given that $z = x + iy$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$, find the value of x and the value of y such that

$$(3 - i)z^* + 2iz = 9 - i$$

where z^* is the complex conjugate of z.

If $z = x + iy$ then $z^* = x - iy$
and substitute into LHS: $$(3 - i)(x - iy) + 2i(x + iy) = 9 - i$$

Expand: $$3x - 3iy - ix - y + 2ix - 2y = 9 - i$$

Group Re and Im parts: $$(3x - 3y) + (-3y + x)i = 9 - i$$

Equate Re and Im parts: $3x - 3y = 9 \quad \rightarrow \quad x - y = 3$

$$-3y + x = -1$$

Solve simultaneous equations: $$\boldsymbol{x = 5, y = 2}$$

Class Example: Modulus & Argument

E.g. 4

$$z = 5\sqrt{3} - 5i$$

Find

(a) $|z|$

(b) $\arg(z)$ in terms of π

$$w = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Find

(c) $\left|\frac{w}{z}\right|$

(d) $\arg\left|\frac{w}{z}\right|$

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Multiplying complex numbers:

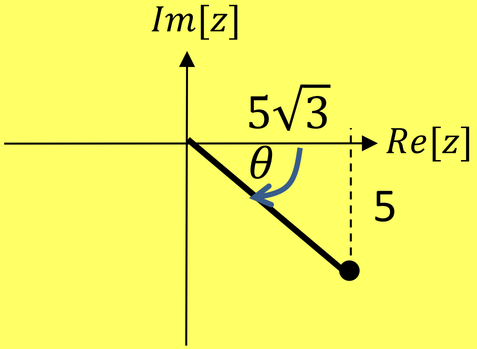
$$|z_1 z_2| = |z_1||z_2|$$
$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Dividing complex numbers:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$
$$arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$$

a) $|z| = \sqrt{\left(5\sqrt{3}\right)^2 + 5^2} = 10$

b) $\arg(z) = -\tan^{-1}\left(\frac{5}{5\sqrt{3}}\right) = -\frac{\pi}{6}$



c) $\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = \left|\frac{2}{10}\right| = \frac{1}{5}$

d) $\arg\left|\frac{w}{z}\right| = \arg(w) - \arg(z) = \frac{\pi}{4} - \left(-\frac{\pi}{6}\right) = \frac{5\pi}{12}$

Class Example: De Moivre's Theorem (Non-Examinable)

E.g. 5

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$$

for any **rational** number m

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Multiplying complex numbers:

$$|z_1 z_2| = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Dividing complex numbers:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

Fully simplify:

$$\frac{\left(\cos\frac{9\pi}{17} + i\sin\frac{9\pi}{17}\right)^5}{\left(\cos\frac{2\pi}{17} - i\sin\frac{2\pi}{17}\right)^3}$$

To use the modulus and argument shortcuts, all signs in the denominator must be plus. We use the fact that that $\cos\theta - i\sin\theta$ can be written as $\cos(-\theta) + i\sin(-\theta)$

$$= \frac{\left(\cos\frac{9\pi}{17} + i\sin\frac{9\pi}{17}\right)^5}{\left(\cos\left(-\frac{2\pi}{17}\right) + i\sin\left(-\frac{2\pi}{17}\right)\right)^3}$$

Although $\cos(-\theta) = \cos\theta$ DMT must be applied as a whole and both angles must be the same.

Apply DMT:

$$= \frac{\cos\frac{45\pi}{17} + i\sin\frac{45\pi}{17}}{\cos\left(-\frac{6\pi}{17}\right) + i\sin\left(-\frac{6\pi}{17}\right)}$$

$$= \cos 3\pi + i\sin 3\pi$$

As cos and sin repeat every 2π, $\cos 3\pi = \cos(3\pi - 2\pi) = \cos\pi$

$$= \cos\pi + i\sin\pi \quad = \mathbf{-1}$$

Class Example: De Moivre's Theorem (Non-Examinable)

E.g. 6

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$$

for any **rational** number m

Express $\left(1+\sqrt{3}\,i\right)^7$ in the form $x+iy$ where $x, y \in \mathbb{R}$.

$$\mathbf{1+\sqrt{3}\,i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$$

Since

$$r^m(\cos\theta + i\sin\theta)^m = r^m(\cos m\theta + i\sin m\theta)$$

(we need to raise both sides to power of 7 and use DMT)

Therefore $\mathbf{\left(1+\sqrt{3}\,i\right)^7 = 2^7\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right)}$

$$\mathbf{= 128\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 64 + 64\sqrt{3}\,i}$$

Exercise A: Complex Arithmetic & Conjugates

1 Express in terms of i:

(a) $\sqrt{(-64)}$ (b) $\sqrt{(-7)}$ (c) $\sqrt{16}-\sqrt{(-81)}$

(d) $3-\sqrt{(-25)}$ (e) $\sqrt{(-100)}-\sqrt{(-49)}$

2 Simplify:

(a) i^3 (b) i^7 (c) i^{-9}

(d) $i(2i-3i^3)$ (e) $(i+2i^2)(3-i)$

3 Write in the form $a+ib$, where $a,b\in \boldsymbol{R}$:

(a) $2i(5-2i)$ (b) $(2+i)^2$ (c) $(1+2i)^2+(3-i)^3$

4 Simplify:

(a) $(2+3i)+(4-7i)$ (b) $(-3+5i)+(-6-7i)$

(c) $(-7-10i)+(2-3i)$ (d) $(2+4i)-(3-6i)$

(e) $(-3+5i)-(-7+4i)$ (f) $(-9-6i)-(-8-9i)$

Exercise A: Complex Arithmetic & Conjugates

5 Express in the form $a+ib$, where $a,b \in \boldsymbol{R}$:

(a) $(2+i)(3-i)$
(b) $(-3-4i)(2-7i)$
(c) $(5+2i)(-3+4i)$
(d) $(1+i)(2-i)(i+3)$
(e) $i(3-7i)(2-i)$

6 Write down the complex conjugate z^* when $z =$

(a) $2+4i$
(b) $3-6i$
(c) $-5+2i$
(d) $-7-3i$
(e) $2i-4$
(f) 6
(g) $3i$
(h) $-3i+7$
(i) $\cos\theta - i\sin\theta$

7 Let $z = 3+4i$ and $w = 2-2i$.
Show z, w, z^*, w^*, $z+w$ and $z-w$ on an Argand diagram

8 Express in the form $a+ib$, where $a,b \in \boldsymbol{R}$:

(a) $\dfrac{2-7i}{1+2i}$
(b) $\dfrac{1+2i}{3-i}$
(c) $\dfrac{1+2i}{3+4i}$
(d) $\dfrac{1}{1+2i}$
(e) $\dfrac{2+3i}{2-3i}$
(f) $\dfrac{5+i}{i-3}$
(g) $\dfrac{6}{4i-3}$
(h) $\dfrac{1}{(i+2)(1-2i)}$

Exercise A: Complex Arithmetic & Conjugates

9

Solve:

(a) $x^2 + 25 = 0$

(b) $x^3 + 64x = 0$

(c) $x^2 - 4x + 5 = 0$

(d) $x^2 + 6x + 10 = 0$

(e) $x^2 + 29 = 4x$

(f) $2x^2 + 3x + 7 = 0$

(g) $3x^2 + 2x + 1 = 0$

(h) $3x^2 - 2x + 2 = 0$

10

It is given that $z = x + iy$ and that z^* is the complex conjugate of z

(a) Express $2z - 3z^*$ in the form $p + qi$

(b) Find the value of z for which $2z - 3z^* = -5 + 15i$

11

Given that $z = a + bi$, solve:

(a) $4z - 2 + 5i = 6 - 7i$

(b) $2z - 5z^* = 9 + 14i$

(c) $(4 + 2i)z + (3 - 2i) = 9 - 4i$

(d) $(2 - i)z - (3 + i)z^* = -5 - 20i$

12

Show that for any complex number $z = a + bi$,

$Re(z) = \frac{z+z^*}{2}$ and $Im(z) = \frac{z-z^*}{2i}$

Exercise A: Complex Arithmetic & Conjugates

13 Express in the form $a+ib$, where $a,b \in \boldsymbol{R}$:

(a) $\dfrac{1}{1+2i}+\dfrac{1}{1-2i}$ (b) $\dfrac{1}{2+i}-\dfrac{1}{1+5i}$ (c) $5-4i+\dfrac{5}{3-4i}$

14 Given that $z=-1+3i$, express $z+\dfrac{2}{z}$ in the form $a+ib$, where $a,b \in \boldsymbol{R}$.

15 Given that $T=\dfrac{x-iy}{x+iy}$, where $x, y \in \boldsymbol{R}$, show that $\dfrac{1+T^2}{2T}=\dfrac{x^2-y^2}{x^2+y^2}$

16 Show that the complex number $\dfrac{2+3i}{5+i}$ can be expressed in the form $\lambda(1+i)$, where λ is real.
State the value of λ

Hence, or otherwise, show that $\left(\dfrac{2+3i}{5+i}\right)^4$ is real and determine its value.

Exercise B: Modulus & Argument

1 Find $|z|$ and $\arg(z)$, in radians to 3 significant figures, where $z =$

(a) $3-2i$ (b) $3+i$ (c) $6i$ (d) -5

(e) $-2+i$ (f) $1-3i$ (g) $i\sqrt{3}+1$ (h) $-5+12i$

(i) $\frac{5}{1-i\sqrt{3}}$ (j) $\frac{2}{\sqrt{5}+i}$ (k) $(2+i)(3-2i)$ (l) $\frac{1+i}{2-i}$

2 Given that $a, b \in \boldsymbol{R}$, express each of the following in the form $a+ib$:

(a) $\frac{3+i}{2-i}$ (b) $3i^3-6i^6$ (c) $(1+i)^4$

Find the modulus and argument for each, in radians to 2 decimal places

3 Find the modulus and the argument, in radians in terms of π, of:

(a) $z_1 = \frac{1+i}{1-i}$ (b) $z_2 = \frac{\sqrt{2}}{1-i}$ (c) $z_3 = \left(\frac{1+i}{1-i}\right)^2$

Plot z_1, z_2 and $z_1 + z_2$ on an Argand diagram.

Exercise C: Alternate Forms

1 **Complete the table, giving exact values wherever possible and 3 sf when not.**

Cartesian Form $z = a + bi$	Polar Form $z = r(\cos\theta + i\sin\theta)$	Exponential Form $z = re^{i\theta}$
	$\cos\pi + i\sin\pi$	
$2 - i\sqrt{3}$		
		$\sqrt{2}e^{\frac{3\pi i}{4}}$
$2 - 3i$		
		$\sqrt{2}e^{\frac{\pi i}{10}}$

Exercise C: Alternate Forms

2 Given that $z_1 = -1 + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$, find $\arg(z_1)$ and $\arg(z_2)$.

Express $\frac{z_2}{z_1}$ in the form $a + ib$, where $a, b \in \boldsymbol{R}$, and hence find $\arg\left(\frac{z_1}{z_2}\right)$

Verify that $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

3 If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then what are the moduli and arguments of $z_1 z_2$ and z_1 / z_2?

4 Show that $\left|e^{i\theta}\right| = \left|e^{-i\theta}\right| = 1$

5

$$z = \sqrt{2} e^{-\frac{i\pi}{3}}, \qquad w = \sqrt{3}\, e^{-\frac{i\pi}{2}}$$

Find zw and $\frac{z}{w}$, writing each of them in (a) exponential form, (b) in Cartesian form.

6 Find $\left(\sqrt{3} + i\right)^7$ by converting exponential form,
with principal argument $(-\pi < \theta \leq \pi)$, then converting back to cartesian.

Exercise D: De Moivre's Theorem (Extra Non-Examinable)

1

Work out $\int (\cos x + i \sin x)^7 dx$

2

Work out $\int 3\sqrt{\cos x + i \sin x}\, dx$

3

$$z = -8 + 8\sqrt{3}\, i$$

a) Find the modulus and argument of z

b) Use De Moivre's Theorem to find z^3

4

Simplify fully:

(a) $\dfrac{1}{(\cos 2\theta + i \sin 2\theta)^3}$ (b) $\dfrac{(\cos 2\theta + i \sin 2\theta)^7}{(\cos 4\theta + i \sin 4\theta)^3}$

5

(a) Express $\sin^4 \theta$ in the form $a \cos 4\theta + b \cos 2\theta + c$

(b) Hence find the exact value of $\int_0^{\frac{\pi}{2}} \sin^4 \theta \;\, d\theta$

6

Use De Moivre's Theorem to find expressions for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos \theta$ and $\sin \theta$.

7

Use De Moivre's Theorem to show that $\cos 3x = 4\cos^3 x - 3\cos x$

Exercise E: Complex Roots (Extra Non-Examinable)

1

(a) Find the modulus and argument of $1 + i\sqrt{3}$

(b) Hence solve the equation $z^2 = 1 + i\sqrt{3}$

giving your answers in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$.

2

Find all roots of $z^3 = -1$ in Cartesian form

3

Find all roots of $z^4 + 1 = 0$ in Cartesian form

4

(a) Express the complex number $-1 + \sqrt{3}i$

in the form $re^{i\theta}$, where r is real and $-\pi < \theta \leq \pi$.

(b) (i) Verify that $2e^{\pi i/6}$ is a root of the equation $z^4 = 8(-1 + \sqrt{3}\,i)$

(ii) Find the other three roots of the above equation giving your answers in the form $re^{i\theta}$, where r is real and $-\pi < \theta \leq \pi$.

Exercise E: Complex Roots (Extra Non-Examinable)

5

(a) (i) Verify that $z = 2e^{\frac{1}{4}\pi i}$ is a root of the equation $z^4 = -16$.

(ii) Find the other three roots of this equation, giving each root in the form $re^{i\theta}$, where r is real and $-\pi < \theta \leq \pi$.

(iii) Illustrate the four roots of the equation by points on an Argand diagram.

(b) (i) Show that $\left(z - 2e^{\frac{1}{4}\pi i}\right)\left(z - 2e^{-\frac{1}{4}\pi i}\right) = z^2 - 2\sqrt{2}\,z + 4$.

(ii) Express $z^4 + 16$ as the product of two quadratic factors with real coefficients.

6

The complex number α is defined by

$$\alpha = \frac{2 - 10i}{3 - 2i}$$

(a) Show that $\alpha = 2 - 2i$

(b) Express α in the form $re^{i\theta}$, where r is real and $-\pi < \theta \leq \pi$.

(c) Hence

(i) show that α^4 is real,

(ii) solve the equation

$$z^3 = \alpha$$

giving your answers in the form $re^{i\theta}$, where r is real and $-\pi < \theta \leq \pi$.

ANSWERS

Exercise A: Answers

1 (a) $8i$ (b) $i\sqrt{7}$ (c) $4-9i$ (d) $3-5i$ (e) $3i$

2 (a) $-i$ (b) $-i$ (c) $-i$ (d) -5 (e) $-5+5i$

3 (a) $4+10i$ (b) $3+4i$ (c) $15-22i$

4 (a) $6-4i$ (b) $-9-2i$ (c) $-5-13i$ (d) $-1+10i$
(e) $4+i$ (f) $-1+3i$

5 (a) $7+i$ (b) $-34+13i$ (c) $-23+14i$ (d) $8+6i$
(e) $17-i$

6 (a) $2-4i$ (b) $3+6i$ (c) $-5-2i$ (d) $-7+3i$
(e) $-4-2i$ (f) 6 (g) $-3i$ (h) $7+3i$
(i) $\cos\theta + i\sin\theta$

Exercise A: Answers

7

$z = 3 + 4i$

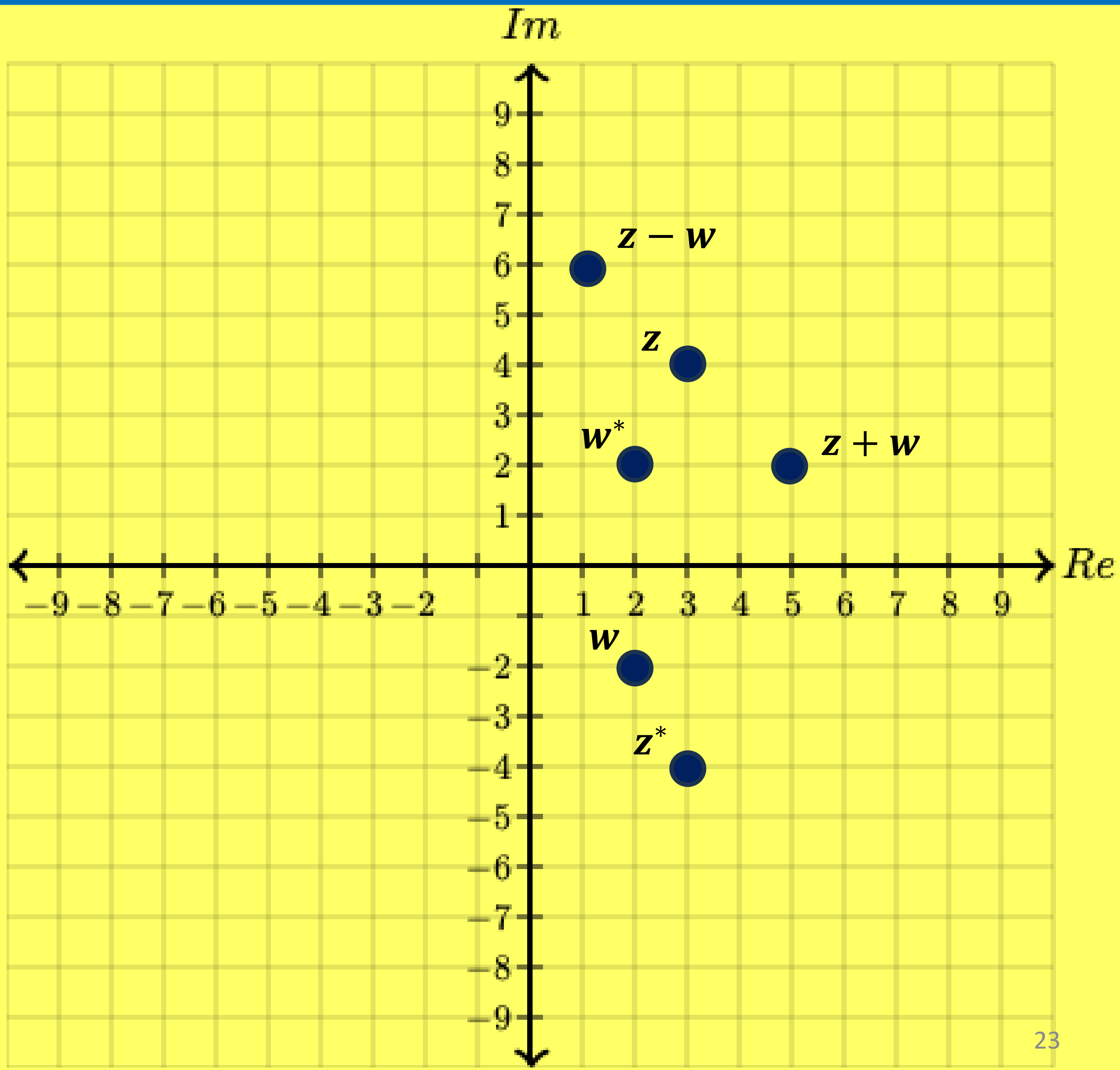
$z^* = 3 - 4i$

$w = 2 - 2i$

$w^* = 2 + 2i$

$z + w = 5 + 2i$

$z - w = 1 + 6i$



Exercise A: Answers

8

(a) $-\frac{1}{5}(12+11i)$ (b) $\frac{1}{10}(1+7i)$ (c) $\frac{1}{25}(11+2i)$

(d) $\frac{1}{5}(1-2i)$ (e) $\frac{1}{13}(-5+12i)$ (f) $-\frac{1}{5}(7+4i)$

(g) $-\frac{6}{25}(3+4i)$ (h) $\frac{1}{25}(4+3i)$

9

(a) $\pm 5i$ (b) $0, \pm 8i$ (c) $2 \pm i$ (d) $-3 \pm i$

(e) $2 \pm 5i$ (f) $\frac{1}{4}(-3 \pm i\sqrt{47})$ (g) $\frac{1}{3}(-1 \pm i\sqrt{2})$ (h) $\frac{1}{3}(1 \pm i\sqrt{5})$

10

(a) $-x + 5yi$ (b) $z = 5 + 3i$

11

(a) $z = 2 - 3i$ (b) $z = -3 + 2i$ (c) $z = 1 - i$ (d) $z = 5 - 2i$

12

$$\frac{(a+bi)+(a-bi)}{2} = \frac{2a}{2} = a = Re(z) \qquad \frac{(a+bi)-(a-bi)}{2i} = \frac{2bi}{2i} = b = Im(z)$$

Exercise A: Answers

13 (a) $\frac{2}{5}$ (b) $\frac{1}{130}(47-i)$ (c) $\frac{4}{5}(7-4i)$

14 $-\frac{6}{5}(1-2i)$

15 Work out T^2 and sub in

16 $\lambda = \frac{1}{2}; -\frac{1}{4}$

Exercise B: Answers

1

(a) $|z| = \sqrt{13},\ \arg z = -0.588^c$

(b) $|z| = \sqrt{10},\ \arg z = -0.322^c$

(c) $|z| = 6,\ \arg z = 1.57^c$

(d) $|z| = 5,\ \arg z = 3.142^c$

(e) $|z| = \sqrt{5},\ \arg z = 2.68^c$

(f) $|z| = \sqrt{10},\ \arg z = -1.25^c$

(g) $|z| = 2,\ \arg z = 1.05^c$

(h) $|z| = 13,\ \arg z = 1.97^c$

(i) $|z| = \frac{5}{2},\ \arg z = 1.05^c$

(j) $|z| = \frac{\sqrt{6}}{3},\ \arg z = -0.421^c$

(k) $|z| = \sqrt{65},\ \arg z = -0.124^c$

(l) $|z| = \frac{1}{5}\sqrt{10},\ \arg z = 1.25^c$

2

(a) $z = 1 + i,\ |z| = \sqrt{2},\ \arg z = 0.79^c$

(b) $z = 3(2 - i),\ |z| = 3\sqrt{5},\ \arg z = -0.46^c$

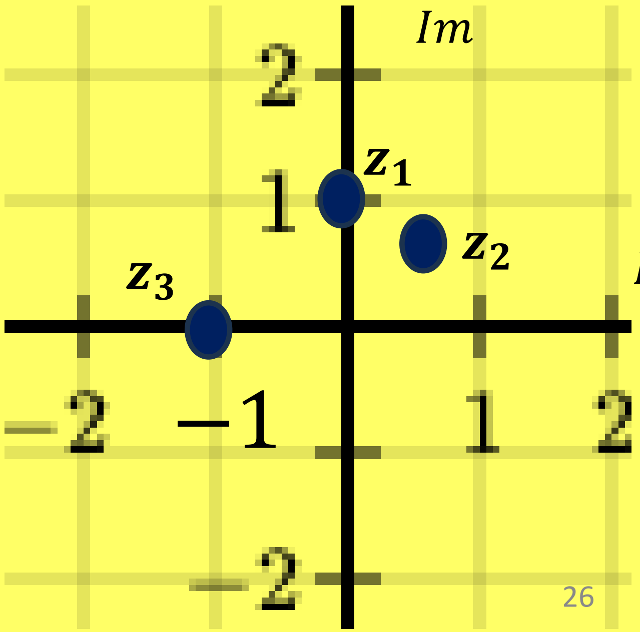
(c) $z = -4,\ |z| = 4,\ \arg z = 3.14^c$

3

$$z_1 = i, \qquad |z_1| = 1,\ \ \arg(z_1) = \frac{\pi}{2}$$

$$z_2 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \qquad |z_2| = 1,\ \ \arg(z_2) = \frac{\pi}{4}$$

$$z_3 = -1, \qquad |z_3| = 1,\ \ \arg(z_2) = \pi$$



Exercise C: Answers

1 **Complete the table, giving exact values wherever possible and 3 sf when not.**

Cartesian Form $z = a + bi$	**Polar Form** $z = r(\cos\theta + i\sin\theta)$	**Exponential Form** $z = re^{i\theta}$
-1	$\cos\pi + i\sin\pi$	$e^{i\pi}$
$2 - i\sqrt{3}$	$2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$	$2e^{i\frac{\pi}{3}}$
$-1 + i$	$\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$	$\sqrt{2}e^{\frac{3\pi i}{4}}$
$2 - 3i$	$\sqrt{13}(\cos(0.983) - i\sin(0.983))$	$z = \sqrt{13}e^{-0.983i}$
$1.345 - 0.437i$	$\sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$	$\sqrt{2}e^{\frac{\pi i}{10}}$

Exercise C: Alternate Forms

2

$$\arg z_1 = \frac{2\pi}{3}, \quad \arg z_2 = \frac{\pi}{6}; \quad \frac{z_1}{z_2} = i, \quad \arg \frac{z_1}{z_2} = \frac{\pi}{2}$$

$$\arg(z_1) - \arg(z_2) = \frac{2\pi}{3} - \frac{\pi}{6} = \frac{4\pi}{6} - \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$$

3

$$z_1 z_1 = r_1 e^{i\vartheta_1} \cdot r_2 e^{i\vartheta_2} = r_1 r_2 e^{i(\vartheta_1+\vartheta_2)}$$

$$\therefore |z_1 z_2| = r_1 r_2, \quad \arg(z_1 z_2) = \vartheta_1 + \vartheta_2$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \frac{e^{i\vartheta_1}}{e^{i\vartheta_2}} = \frac{r_1}{r_2} e^{i(\vartheta_1-\vartheta_2)}$$

$$\therefore |z_1/z_2| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$\arg(z_1/z_2) = \vartheta_1 - \vartheta_2 = \arg z_1 - \arg z_2$$

4

$$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta = 1}$$

$$|e^{-i\theta}| = |\cos(-\theta) + i\sin(-\theta)| = |\cos(\theta) - i\sin(\theta)| = \sqrt{\cos^2\theta + (-\sin\theta)^2} = 1$$

Properties: $\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$

5

$\boldsymbol{zw = \sqrt{6}e^{-i\frac{5\pi}{6}}}$ Cartesian form: $zw = \sqrt{6}\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right] = -\frac{3}{\sqrt{2}} - \sqrt{\frac{3}{2}}\,i$

$\boldsymbol{\frac{z}{w} = \sqrt{\frac{2}{3}}e^{i\frac{\pi}{6}}}$ Cartesian form: $\frac{z}{w} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}}\,i$

6

$z = \sqrt{3} + i = 2e^{i\frac{\pi}{6}} \Rightarrow z^7 = \left(\sqrt{3}+i\right)^7 = 128e^{i\frac{7\pi}{6}} = 128e^{-i\frac{5\pi}{6}}$ (principal)

$$\left(\sqrt{3}+i\right)^7 = 128\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right] = -64\sqrt{3} - 64i$$

Exercise D: Solutions (Extra Non-Examinable)

1

$$\int (\cos x + i\sin x)^7 dx = \int \cos 7x + i\sin 7x \, dx = \frac{1}{7}\sin 7x - \frac{i}{7}\cos 7x + C$$

2

$$3\int (\cos x + i\sin x)^{\frac{1}{2}} dx = 3\int \cos\frac{1}{2}x + i\sin\frac{1}{2}x \, dx = 6\sin\frac{1}{2}x - 6i\cos\frac{1}{2}x + C$$

3

$$|z| = 16, \arg(z) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

$$z = 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$z^3 = 16^3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3 = 4096(\cos 2\pi + i\sin 2\pi) = 4096(1+0) = 4096$$

(by DMT)

4

(a) $\frac{1}{(\cos 2\theta + i\sin 2\theta)^{\,3}}$

$= (\cos 2\theta + i\sin 2\theta)^{\,-3}$

$= \cos(-6\theta) + i\sin(-6\theta)$

$= \cos 6\theta - i\sin 6\theta$

(because $\cos(-A) = \cos A$, $\sin(-A) = -\sin A$)

(b) $\frac{(\cos 2\theta + i\sin 2\theta)^{\,7}}{(\cos 4\theta + i\sin 4\theta)^{\,3}}$

$= \frac{\cos(14\theta) + i\sin(14\theta)}{\cos(12\theta) + i\sin(12\theta)}$

$= \cos 2\theta + i\sin 2\theta$

Last line because:
If you multiply two complex numbers, you **multiply the moduli** and **add the arguments**, and if you divide them, you divide the moduli and subtract the arguments.

Exercise D: Solutions (Extra Non-Examinable)

5

a) $(2i\sin\theta)^4 = \left(z - \frac{1}{z}\right)^4$

$$16\sin^4\theta = z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$$
$$= \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6$$
$$= 2\cos 4\theta - 4(2\cos 2\theta) + 6$$
$$\therefore \sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

b) $\int_0^{\frac{\pi}{2}} \sin^4\theta \; d\theta =$

$$\left[\frac{1}{32}\sin 4\theta - \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta\right]_0^{\frac{\pi}{2}}$$
$$= \frac{3\pi}{16}$$

6

$\cos 4\vartheta + i\sin 4\vartheta = (c + is)^4$ by DMT; $c = \cos\vartheta$, $s = \sin\vartheta$

$= c^4 + 4c^3(is) + 6c^2(is)^2 + 4.c(is)^3 + (is)^4$

$= c^4 + 4ic^3s - 6c^2s^2 - 4ics^3 + s^4$

$= c^4 - 6c^2s^2 + s^4 + 4i(c^3s - cs^3)$

$\therefore \cos 4\vartheta = \cos^4\vartheta - 6\cos^2\vartheta\sin^2\vartheta + \sin^4\vartheta$

$\sin 4\vartheta = 4\cos^3\vartheta\sin\vartheta - 4\cos\vartheta\sin^3\vartheta.$

$= 4\cos\vartheta\sin\vartheta(\cos^2\vartheta - \sin^2\vartheta) = 2\sin 2\vartheta\cos 2\vartheta$

7

$e^{3ix} = (c + is)^3 = c^3 + 3c^2is + 3c(is)^2 + (is)$

$= c^3 + 3ic^2s - 3cs^2 - is^3$

Need Real Part for $\cos 3x$

$\Rightarrow \cos 3x = c^3 - 3cs^2$

$= c^3 - 3c(1 - c^2)$

$= c^3 - 3c + 3c^3$

$= 4c^3 - 3c$ as req'd.

Exercise E: Solutions (Extra Non-Examinable)

1

(a) modulus 2 ; argument $\frac{\pi}{3}$

(b) $z = \sqrt{2}e^{\frac{i\pi}{6}}$ and $= \sqrt{2}e^{\frac{-i5\pi}{6}}$

2

(a) $r = 2,\ \theta = \frac{2}{3}\pi$

(b) (ii) $z = 2e^{-5\pi i/6},\ 2e^{-\pi i/3},\ 2e^{2\pi i/3}$

3

$$z = 1$$

$$z = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$z = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

4

$$z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$z = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$z = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$z = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

5

(a)(ii) $z = 2e^{-\pi i/4},\ z = 2e^{\pm 3\pi i/4}$

(b) (ii) $z^4 + 16 = (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2} + 4)$

6

(b) $r = 2\sqrt{2}$ or $\sqrt{8}$, $\theta = -\frac{\pi}{4}$

(c) (ii) $z = \sqrt{2}\, e^{\frac{\pi i}{12} + 2k\pi\frac{c}{3}}$ $\quad k = 0,\ \pm 1$

Full Worked Solutions