CAPE1150

UNIVERSITY OF LEEDS

Engineering Mathematics

School of Chemical and Process Engineering

University of Leeds

Level 1 Semester 2

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Tutorial: Question Difficulty Colour Code

Basic - straightforward application
(you must be able to do these)

Medium – Makes you think a bit
(you must be able to do these)

Hard – Makes you think a lot
(you should be able to do these)

Extreme – Tests your understanding to the limit!
(for those who like a challenge)

Applied – Real-life examples of the topic, may sometimes involve prior knowledge
(you should attempt these – will help in future engineering)

TWO CHEMISTS WALK UP TO
A BAR TO ORDER DRINKS.

FIRST CHEMIST: I'LL HAVE AN H2O

SECOND CHEMIST: I'LL
HAVE AN H2O TOO.

THE SECOND CHEMIST DIES

Tutorial 7
2nd Order ODEs

Class Example: Homogeneous

E.g. 1

Solve the differential equation:

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0 \qquad y(0) = 2 \quad y'(0) = 1$$

$y'' + 8y' + 16y = 0$

$y(0) = 2, \ y'(0) = 1$

$m^2 + 8m + 16 = 0$

$(m+4)^2 = 0$

$m = -4$

$y = (A + Bx)e^{-4x}$

$y(0) = 2: \ 2 = A$

$y' = Be^{-4x} - 4(A+Bx)e^{-4x}$

$y'(0) = -2: \ 1 = B - 4A$

$A = 2: \ 1 = B - 8 \rightarrow B = 9$

$\therefore y = (2 + 9x)e^{-4x}$

Class Example: Inhomogeneous (Undetermined Coefficients Method)

E.g. 2

Solve the differential equation using the method of undetermined coefficients.

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2\cos t - \sin t$$

$$m^2 + 5m + 6 = 0$$
$$(m+2)(m+3) = 0$$
$$m = -2, m = -3$$
$$\therefore y_c = Ae^{-2t} + Be^{-3t}$$
$$x = \lambda \cos t + \mu \sin t$$
$$\frac{dx}{dt} = -\lambda \sin t + \mu \cos t$$
$$\frac{d^2x}{dt^2} = -\lambda \cos t - \mu \sin t$$

Sub into equation:

$$-\lambda \cos t - \mu \sin t + 5(-\lambda \sin t + \mu \cos t) + 6(\lambda \cos t + \mu \sin t) = 2\cos t - \sin t$$

Compare coefficients:

Cos: $-\lambda + 5\mu + 6\lambda = 2 \quad \Rightarrow \quad 5\mu + 5\lambda = 2 \qquad (1)$

Sin: $-\mu - 5\lambda + 6\mu = -1 \quad \Rightarrow \quad 5\mu - 5\lambda = -1 \qquad (2)$

Solve simultaneous equations: (1)+(2) $\quad 10\mu = 1 \Rightarrow \mu = \frac{1}{10}, \quad \lambda = \frac{3}{10}$

$$\therefore x = Ae^{-2t} + Be^{-3t} + \frac{3}{10}\cos t + \frac{1}{10}\sin t$$

Class Example: Inhomogeneous (Undetermined Coefficients Method)

E.g. 3

Solve the differential equation using the method of undetermined coefficients.

$$y'' - 5y' + 6 = e^{2x}$$

$y'' - 5y' + 6y = e^{2x}$ ⊛

$m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m = 2, \; m = 3$

$\therefore y_c = Ae^{2x} + Be^{3x}$

e^{2x} APPEARS IN y_c

$\therefore y_p = \lambda x e^{2x}$

$y_p' = \lambda e^{2x} + 2\lambda x e^{2x}$

$y_p'' = 2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda x e^{2x}$

SUB IN ⊛: $2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda x e^{2x} - 5\lambda e^{2x} - 10\lambda x e^{2x} + 6\lambda x e^{2x} = e^{2x}$

$(\div e^{2x})$: $2\lambda + 2\lambda + \cancel{4\lambda x} - 5\lambda - \cancel{10\lambda x} + \cancel{6\lambda x} = 1$

$-\lambda = 1 \quad \therefore \lambda = -1$

$\therefore y_p = -xe^{2x}$

$\therefore y = Ae^{2x} + Be^{3x} - xe^{2x}$

or $y = (A - x)e^{2x} + Be^{3x}$

Class Example: Substitution to Transform

E.g. 4

Find the general solution of the differential equation using the substitution $x = e^u$, where u is a function of x.

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0$$

FOR DIFFERENTIAL EQUATIONS OF THE FORM

$$x^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \quad \circledast$$

USE SUBSTITUTION $x = e^u$ TO TRANSFORM $x^2\frac{d^2y}{dx^2}$ AND $x\frac{dy}{dx}$ SO THAT WE GET A CONSTANT COEFFICIENT EQUATION IN u.

$x = e^u \therefore \frac{dx}{du} = e^u = x$

BY CHAIN RULE $\frac{dy}{du} = \frac{dy}{dx}\frac{dx}{du} = x\frac{dy}{dx}$ $\left(\text{OR } e^u\frac{dy}{dx}\right)$

$$\frac{d^2y}{du^2} = \frac{d}{du}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(e^u\frac{dy}{dx}\right) = e^u\frac{dy}{dx} + e^u\frac{d}{du}\left(\frac{dy}{dx}\right)$$

$$= e^u\frac{dy}{dx} + e^u\frac{d^2y}{dx^2}\frac{dx}{du}$$

BY CHAIN RULE $\frac{df(x)}{du} = \frac{df(x)}{dx}\frac{dx}{du}$

where $f(x) = \frac{dy}{dx}$ AND x DEPENDS ON u

$\therefore \frac{d^2y}{du^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$ $\left(\text{AS } e^u = x \ \& \ \frac{dx}{du} = x\right)$

$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

SO FOR EQUATION IN FORM $\circledast$ WE SUB.

$$\boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du} \quad \text{AND} \quad x\frac{dy}{dx} = \frac{dy}{du}}$$

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0$$

$x = e^u \rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$, $x\frac{dy}{dx} = \frac{dy}{du}$

$\therefore \frac{d^2y}{du^2} - \frac{dy}{du} + 6\frac{dy}{du} + 6y = 0$

$\frac{d^2y}{du^2} + 5\frac{dy}{du} + 6y = 0$

$m^2 + 5m + 6 = 0$

$(m+2)(m+3) = 0 \therefore m = -2, m = -3$

$y = Ae^{-2u} + Be^{-3u}$

$x = e^u \therefore u = \ln x$ $\quad \left(e^{au} = e^{a\ln x} = e^{\ln x^a} = x^a\right)$

$\therefore y = Ae^{-2\ln x} + Be^{-3\ln x}$

$y = Ae^{\ln x^{-2}} + Be^{\ln x^{-3}}$

$$y = \frac{A}{x^2} + \frac{B}{x^3}$$

$$y = \frac{A}{x^2} + \frac{B}{x^3}$$

Class Example: Inhomogeneous (Variation of Parameters Method)

Optional

Solve the differential equation using the method of variation of parameters

$$y'' + y = \tan x$$

(Non-Examinable)

$y'' + y = \tan x \quad (a=1)$

y_c: $m^2 + 1 = 0$

$m = \pm i \qquad y_c = A\cos x + B\sin x$

$y_1 = \cos x \qquad y_2 = \sin x$

$y_p = u y_1 + v y_2 = u\cos x + v\sin x$

Conditions:

$$u' y_1 + v' y_2 = 0$$

$$u' y_1' + v' y_2' = \frac{f(x)}{a}$$

$u'\cos x + v'\sin x = 0 \quad (1)$

$-u'\sin x + v'\cos x = \tan x \quad (2)$

$(1) \div \cos x$: $\quad u' + v'\tan x = 0 \quad \therefore \underline{u' = -v'\tan x}$

Sub in (2): $\quad v'\frac{\sin x}{\cos x}\sin x + v'\cos x = \frac{\sin x}{\cos x}$

$\times \cos x \quad v'(\sin^2 x + \cos^2 x) = \sin x$

$v' = \sin x \quad \therefore \underline{v = -\cos x}$

$u' = -\sin x \tan x = -\frac{\sin^2 x}{\cos x} = -\frac{(1-\cos^2 x)}{\cos x}$

$= -\frac{1}{\cos x} + \cos x$

$\therefore u = -\int \sec x \, dx + \int \cos x \, dx$

$u = -\ln(\sec x + \tan x) + \sin x$

$\therefore y_p = (-\ln(\sec x + \tan x) + \sin x)\cos x + (-\cos x)\sin x$

$= -\ln(\sec x + \tan x)\cos x + \cancel{\sin x\cos x} - \cancel{\sin x\cos x}$

$\therefore y = A\cos x + B\sin x - \cos x \ln|\sec x + \tan x|$

Exercise A: Homogeneous

Solve the 2nd order differential equations

1 $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

2 $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

3 $y'' - 3y' = 0$

4 $y'' + 4y' + 4y = 0$

5 $y'' = 4y$

6 $y'' + 2\sqrt{2}\, y' + 2y = 0$

7 $y'' - 4y' + y = 0$

8 $6y'' - 7y' - 3y = 0$

9 $y'' + 4y = 0$

10 $y'' - 2y' + 2y = 0$

11 $36y'' - 36y' + 13y = 0$

12 $y'' + 6y' + 9y = 0$ $\quad y(0) = 1$, $y'(0) = 2$

13 $y'' + 4y' + 5y = 0$ $\quad y(0) = 0$, $y'(0) = 2$

14 $y'' + 6y' + 13y = 0$ $\quad y(0) = 2$, $y'(0) = 1$

15 $y'' - 2y' + y = 0$ $\quad y(2) = 1$, $y'(2) = -2$

Exercise B: Undetermined Coefficients

Solve the 2nd order differential equations using the method of undetermined coefficients:

1 $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

2 $y'' - 3y' + 2y = 4\,e^{-x}$

3 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin x$

4 $y'' + 2y' = \cos 2x$

5 $y'' + y' + y = 2 + x + \cos x$

6 $y'' - 3y' + 2y = e^{2x}$

7 $\frac{d^2y}{dt^2} - y = 8t\,e^{t}$

8 $y'' - y = x\,e^{2x}$

9 $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = t\,e^{2t}$

10 $\frac{d^2y}{dx^2} + 25y = 3\cos 5x$ $\quad y(0) = 0$, $y'(0) = 5$

Exercise C: Substitutions (Non-Constant Coefficients)

Find the general solution of each differential equation using the substitution $x = e^u$, where u is a function of x.

1 $$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 4y = 0$$

2 $$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0$$

3 $$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 28y = 0$$

4 $$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 14y = 0$$

5 $$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = 0$$

6 Use the substitution $y = \frac{z}{x}$ to transform the differential equation

$$x^2\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0$$

Into the equation

$$\frac{d^2z}{dx^2} - 4\frac{dz}{dx} = 0$$

Hence find the general solution, giving y in terms of x.

7 Use the substitution $y = \frac{z}{x^2}$ to transform the differential equation

$$x^2\frac{d^2y}{dx^2} + 2x(x + 2)\frac{dy}{dx} + 2(x + 1)^2y = e^{-x}$$

Into the equation

$$\frac{d^2z}{dx^2} + 2\frac{dz}{dx} = e^{-x}$$

Hence find the general solution, giving y in terms of x.

8 Use the substitution $z = \sin x$ to transform the differential equation

$$\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$$

Into the equation

$$\frac{d^2z}{dx^2} - 2y = 2(1 - z^2)$$

Hence find the general solution, giving y in terms of x.

Optional: Variation of Parameters (Non-Examinable)

Solve the 2nd order differential equations using the method of variation of parameters:

1 $y'' + 3y' = e^{-3x}$

2 $y'' + y = \sec x$

3 $y'' - 6y' + 9y = x^2 e^{3x}$

4 $y'' - 4y' + 4y = \dfrac{e^{2x}}{x^2}$

5 $y'' - y = e^x \cos x$

ANSWERS

Exercise A: Answers

1 $y = Ae^{2x} + Be^{3x}$

2 $y = Ae^{2x} + Be^{-x}$

3 $y = A + Be^{3x}$

4 $y = (A + Bx)e^{-2x}$

5 $y = Ae^{2x} + Be^{-2x}$

6 $y = (A + Bx)e^{-\sqrt{2}\,x}$

7 $y = Ae^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x}$

8 $y = Ae^{\frac{3}{2}x} + Be^{-\frac{1}{3}x}$

9 $y = A\cos 2x + B\sin 2x$

10 $y = e^{x}(A\cos x + B\sin x)$

11 $y = e^{\frac{x}{2}}(A\cos\frac{x}{3} + B\sin\frac{x}{3})$

12 $y = (1 + 5x)e^{-3x}$

13 $y = 2e^{-2x}\sin x$

14 $y = e^{-3x}(2\cos 2x + \frac{7}{2}\sin 2x)$

15 $y = (7 - 3x)e^{x-2}$

Exercise B: Answers

1 $y = Ae^x + Be^{2x} + \frac{1}{12}e^{5x}$

2 $y = Ae^x + Be^{2x} + \frac{2}{3}e^{-x}$

3 $y = e^{-2x}(A\cos x + B\sin x) + \frac{1}{8}(\sin x - \cos x)$

4 $y = A + Be^{-2x} + \frac{1}{8}(\sin 2x - \cos 2x)$

5 $y = e^{-\frac{1}{2}x}(A\cos\frac{3}{2}x + B\sin\frac{3}{2}x) + 1 + x + \sin x$

6 e^{2x} appears in $y_c \Rightarrow$ $y = Ae^x + (B + x)e^{2x}$

7 e^t appears in $y_c \Rightarrow$ $y = Ae^t + Be^{-t} + (2t^2 - 2t)e^t$

8 $y = Ae^x + Be^{-x} + \frac{1}{9}(3x - 4)e^{2x}$

9 $y = e^{-\frac{t}{2}}\left\{A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right\} + \frac{e^{2t}}{49}(7t - 5)$

10 $y = \left(1 + \frac{3}{10}\right)\sin 5x$

Exercise C: Answers

1 $y = \frac{A}{x^4} + \frac{B}{x}$

2 $y = \frac{A + B\ln x}{x^2}$

3 $y = \frac{A}{x^7} + Bx^4$

4 $y = Ax^7 + \frac{B}{x^2}$

5 $y = \frac{1}{x}\left(A\cos(\ln x) + B\sin(\ln x)\right)$

6 $y = \frac{A}{x} + \frac{B}{x}e^{4x}$

7 $y = \frac{e^{-x}}{x^2}(A\cos x + B\sin x + 1)$

8 $y = Ae^{\sqrt{2}\sin x} + Be^{-\sqrt{2}\sin x} + \sin^2 x$

Optional: Answers (Non-Examinable)

1 $y = A + \left(B - \frac{1}{9} - \frac{1}{3}x\right)e^{-3x}$

2 $y = (A + \ln|\cos x|)\cos x + (B + x)\sin x$

3 $y = \left(A + Bx + \frac{x^4}{12}\right)e^{3x}$

4 $y = (A + Bx - \ln x - 1)e^{2x}$ or $y = (C + Bx - \ln x)e^{2x}$

5 $y = e^x(A + \frac{2}{5}\sin x - \frac{1}{5}\cos x) + Be^{-x}$

Full Worked Solutions

Exercise A: Full Solutions

1

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, \ m = 3$$

$$y = Ae^{2x} + Be^{3x}$$

2

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

$$m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$$m = 2, \ m = -1$$

$$y = Ae^{2x} + Be^{-x}$$

3

$$y'' - 3y' = 0$$

$$m^2 - 3m = 0$$

$$m(m-3) = 0$$

$$m = 0, \ m = 3$$

$$y = A + Be^{3x}$$

4

$$y'' + 4y' + 4y = 0$$

$$m^2 + 4m + 4 = 0$$

$$(m+2)^2 = 0$$

$$m = -2$$

$$y = (A + Bx)e^{-2x}$$

Exercise A: Full Solutions

5

$$y'' = 4y$$
$$m^2 = 4$$
$$m = \pm 2$$
$$\underline{y = Ae^{2x} + Be^{-2x}}$$

6

$$y'' + 2\sqrt{2}\,y' + 2y = 0$$
$$m^2 + 2\sqrt{2}m + 2 = 0$$
$$(m + \sqrt{2})^2 = 0$$
$$m = -\sqrt{2}$$
$$\underline{y = (A + Bx)e^{-\sqrt{2}x}}$$

7

$$y'' - 4y' + y = 0$$
$$m^2 - 4m + 1 = 0$$
$$m = \frac{4 \pm \sqrt{(-4)^2 - 4\times 1\times 1}}{2}$$
$$= \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$
$$m = 2 \pm \sqrt{3}$$
$$\underline{y = Ae^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x}}$$

8

$$6y'' - 7y' - 3y = 0$$
$$6m^2 - 7m - 3 = 0$$
$$(2m - 3)(3m + 1) = 0$$
$$m = \frac{3}{2}, \quad m = -\frac{1}{3}$$
$$\underline{y = Ae^{\frac{3}{2}x} + Be^{-\frac{1}{3}x}}$$

Exercise A: Full Solutions

9

$$y'' + 4y = 0$$
$$y'' = -4y$$
$$m^2 = -4$$
$$m = 0 \pm 2i$$
$$y = e^0(A\cos 2x + B\sin 2x)$$
$$y = A\cos 2x + B\sin 2x$$

10

$$y'' - 2y' + 2y = 0$$
$$m^2 - 2m + 2 = 0$$
$$m = \frac{2 \pm \sqrt{(-2)^2 - 4\times 2\times 1}}{2}$$
$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$$
$$= 1 \pm i$$
$$y = e^x(A\cos x + B\sin x)$$

11

$$36y'' - 36y' + 13y = 0$$
$$36m^2 - 36m + 13 = 0$$
$$m = \frac{36 \pm \sqrt{(-36)^2 - 4\times 36\times 13}}{72}$$
$$m = \frac{1}{2} \pm \frac{1}{3}i$$
$$y = e^{\frac{x}{2}}\left(A\cos\frac{x}{3} + B\sin\frac{x}{3}\right)$$

Exercise A: Full Solutions

12

$y'' + 6y' + 9y = 0$

$y(0) = 1,\ y'(0) = 2$

$m^2 + 6m + 9 = 0$

$m = -3$

$y = (A + Bx)e^{-3x}$

$y(0) = 1: \quad 1 = (A + 0) \times 1$

$A = 1$

$y = (1 + Bx)e^{-3x}$

$y' = Be^{-3x} - 3(1 + Bx)e^{-3x}$

$y'(0) = 2: \ 2 = B - 3(1 + 0) \times 1$

$B = 5$

$\underline{y = (1 + 5x)e^{-3x}}$

13

$y'' + 4y' + 5y = 0$

$y(0) = 0,\ y'(0) = 2$

$m^2 + 4m + 5 = 0$

$m = \dfrac{-4 \pm \sqrt{16 - 20}}{2}$

$= \dfrac{-4 \pm \sqrt{-4}}{2}$

$= \dfrac{-4 \pm 2i}{2}$

$= -2 \pm i$

$y = e^{-2x}(A\cos x + B\sin x)$

$y(0) = 0:$

$0 = 1(A + 0) \therefore A = 0$

$y = Be^{-2x}\sin x$

$y' = -2Be^{-2x}\sin x + Be^{-2x}\cos x$

$y'(0) = 2:$

$2 = 0 + B \therefore B = 2$

$\underline{y = 2e^{-2x}\sin x}$

Exercise A: Full Solutions

14

$y'' + 6y' + 13y = 0$

$y(0) = 2,\ y'(0) = 1$

$m^2 + 6m + 13 = 0$

$m = \dfrac{-6 \pm \sqrt{36 - 4\times 13}}{2}$

$= \dfrac{-6 \pm 4i}{2} = -3 \pm 2i$

$y = e^{-3x}(A\cos 2x + B\sin 2x)$

$y(0) = 2$:

$2 = 1(A + 0) \therefore A = 2$

$y = e^{-3x}(2\cos 2x + B\sin 2x)$

$y' = -3e^{-3x}(2\cos 2x + B\sin 2x)$
$\quad + e^{-3x}(-4\sin 2x + 2B\cos 2x)$

$y'(0) = 1$:

$1 = -3(2 + 0) + 1(0 + 2B)$

$1 = -6 + 2B \therefore B = \frac{7}{2}$

$y = e^{-3x}\left(2\cos 2x + \frac{7}{2}\sin 2x\right)$

15

$y'' - 2y' + y = 0$

$y(2) = 1,\ y'(2) = -2$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1$

$y = (A + Bx)e^x$

$y(2) = 1$:

$1 = (A + 2B)e^2$

$A + 2B = e^{-2}$ ①

$y' = (A + Bx)e^x + Be^x$

$= (A + Bx + B)e^x$

$y'(2) = -2$:

$-2 = (A + \cancel{2B} + B)e^2$

$-2 = (A + 3B)e^2$

$A + 3B = -2e^{-2}$ ②

$A + 2B = e^{-2}$ ①

$A + 3B = -2e^{-2}$ ②

① − ②: $-B = 3e^{-2}$

$\therefore B = -3e^{-2}$

$A = e^{-2} - 2(-3e^{-2})$

$A = 7e^{-2}$

$\therefore y = (7e^{-2} - 3xe^{-2})e^x$

$y = (7 - 3x)e^{x-2}$

Exercise B: Full Solutions

1

1. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

y_c: $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1,\ m = 2$

$y_c = Ae^{x} + Be^{2x}$

$y_p = \lambda e^{5x},\ y_p' = 5\lambda e^{5x},\ y_p'' = 25\lambda e^{5x}$

sub in:

$25\lambda e^{5x} - 15\lambda e^{5x} + 2\lambda e^{5x} = e^{5x}$

$12\lambda = 1 \quad \therefore \lambda = \frac{1}{12}$

$\therefore y_p = \frac{1}{12}e^{5x}$

$\therefore y = Ae^{x} + Be^{2x} + \frac{e^{5x}}{12}$

2

$y'' - 3y' + 2y = 4e^{-x}$ ⊛

y_c: $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0 \quad m = 1,\ m = 2$

$y_c = Ae^{x} + Be^{2x}$

$y_p = \lambda e^{-x},\ y_p' = -\lambda e^{-x},\ y_p'' = \lambda e^{-x}$

sub in ⊛: $\lambda e^{-x} + 3\lambda e^{-x} + 2\lambda e^{-x} = 4e^{-x}$

$6\lambda \cancel{e^{-x}} = 4\cancel{e^{-x}}$

$\lambda = \frac{2}{3}$

$\therefore y_p = \frac{2}{3}e^{-x}$

$\therefore y = y_c + y_p = Ae^{x} + Be^{2x} + \frac{2}{3}e^{-x}$

Exercise B: Full Solutions

3

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=\sin x \quad ⊛$$

y_c: $m^2+4m+5=0$

$$m=\frac{-4\pm\sqrt{16-20}}{2}=\frac{-4\pm\sqrt{4i}}{2}=\frac{-4\pm 2i}{2}=-2\pm i$$

$$y_c=e^{-2x}(A\cos x+B\sin x)$$

$$y_p=\lambda\cos x+\mu\sin x$$
$$y_p'=-\lambda\sin x+\mu\cos x$$
$$y_p''=-\lambda\cos x-\mu\sin x$$

Sub in ⊛:

$$-\lambda\cos x-\mu\sin x-4\lambda\sin x+4\mu\cos x+5\lambda\cos x+5\mu\sin x=\sin x$$

compare coefficients:

$\cos x$: $4\lambda+4\mu=0 \rightarrow \lambda=-\mu$

$\sin x$: $4\mu-4\lambda=1 \rightarrow 4\mu+4\mu=1$

$\mu=\frac{1}{8} \therefore \lambda=-\frac{1}{8}$

$$\therefore y=y_c+y_p=e^{-2x}(A\cos x+B\sin x)+\frac{1}{8}(\sin x-\cos x)$$

4

$$y''+2y'=\cos 2x \quad ⊛$$

y_c: $m^2+2m=0$

$m(m+2)=0 \quad m=0,\ m=-2$

$$y_c=A+Be^{-2x}$$

$$y_p=\lambda\cos 2x+\mu\sin 2x$$
$$y_p'=-2\lambda\sin 2x+2\mu\cos 2x$$
$$y_p''=-4\lambda\cos 2x-4\mu\sin 2x$$

Sub in ⊛: $-4\lambda\cos 2x-4\mu\sin 2x-4\lambda\sin 2x+4\mu\cos 2x=\cos 2x$

compare coefficients:

$\cos 2x$: $4\mu-4\lambda=1$ ①

$\sin 2x$: $-4\mu-4\lambda=0$ ② $\Rightarrow \lambda=-\mu$

in ①: $4\mu+4\mu=1$

$\mu=\frac{1}{8} \therefore \lambda=-\frac{1}{8}$

$$y_p=\frac{1}{8}(\sin 2x-\cos 2x)$$

$$\therefore y=y_c+y_p=A+Be^{-2x}+\frac{1}{8}(\sin 2x-\cos 2x)$$

Exercise B: Full Solutions

5

$$y'' + y' + y = 2 + x + \cos x \quad ⊛$$

$m^2 + m + 1 = 0$

$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}}{2} = -\frac{1}{2} \pm \frac{3}{2} i$

$\therefore y_c = e^{-\frac{1}{2}x}\left(A\cos\frac{3}{2}x + B\sin\frac{3}{2}x\right)$

$y_p = \lambda + \mu x + a\cos x + b\sin x$

$y_p' = \mu - a\sin x + b\cos x$

$y_p'' = -a\cos x - b\sin x$

Sub in ⊛:

$(\cancel{-a\cos x} - \cancel{b\sin x}) + (\mu - a\sin x + b\cos x) + (\lambda + \mu x + \cancel{a\cos x} + \cancel{b\sin x}) = 2 + x + \cos x$

compare coefficients;

$x^0: \mu + \lambda = 2$

$x^1: \mu = 1 \rightarrow \lambda = 1$

$\cos x: b = 1$

$\sin x: a = 0$

$\therefore y_p = 1 + x + \sin x$

$\therefore y = e^{-\frac{1}{2}x}\left(A\cos\frac{3}{2}x + B\sin\frac{3}{2}x\right) + 1 + x + \sin x$

6

$$y'' - 3y' + 2y = e^{2x} \quad *$$

$\therefore m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0 \quad m = 1,\ m = 2$

$y_c = Ae^{x} + Be^{2x}$

e^{2x} APPEARS IN y_c

$\therefore y_p = \lambda x e^{2x}, \quad y_p' = \lambda e^{2x} + 2\lambda x e^{2x} = \lambda(1+2x)e^{2x}$

$y_p'' = 2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda x e^{2x}$

$= 4\lambda(1+x)e^{2x}$

Sub in ⊛:

$4\lambda(1+x)\cancel{e^{2x}} - 3\lambda(1+2x)\cancel{e^{2x}} + 2\lambda x \cancel{e^{2x}} = \cancel{e^{2x}}$

$4\lambda + 4\lambda x - 3\lambda - 6\lambda x + 2\lambda x = 1$

$\lambda = 1$

$\therefore y_p = xe^{2x}$

$\therefore y = y_c + y_p = Ae^{x} + Be^{2x} + xe^{2x}$

$y = Ae^{x} + (B + x)e^{2x}$

Exercise B: Full Solutions

7

$$\frac{d^2y}{dt^2} - y = 8te^t \quad \circledast$$

y_c: $m^2 - 1 = 0 \to m = \pm 1$

$$y_c = Ae^t + Be^{-t}$$

e^t APPEARS IN y_c

$$\therefore y_p = (\lambda t^2 + \mu t)e^t$$
$$y_p' = (2\lambda t + \mu)e^t + (\lambda t^2 + \mu t)e^t$$
$$y_p'' = 2\lambda e^t + (2\lambda t + \mu)e^t + (2\lambda t + \mu)e^t + (\lambda t^2 + \mu t)e^t$$

Sub in $\circledast$:

$$(2\lambda + 2\lambda t + \mu + 2\lambda t + \mu + \cancel{\lambda t^2} + \cancel{\mu t} - \cancel{\lambda t^2} - \cancel{\mu t})\cancel{e^t} = 8t\cancel{e^t}$$
$$2\lambda + 4\lambda t + 2\mu = 8t$$

Compare coefficients:

$$2\lambda + 2\mu = 0 \to \lambda + \mu = 0$$
$$4\lambda = 8 \to \underline{\lambda = 2} \quad \therefore \underline{\mu = -2}$$

$$\therefore y = y_c + y_p = Ae^t + Be^{-t} + 2t(t-1)e^t$$

8

$$y'' - y = xe^{2x} \quad \circledast$$
$$m^2 - 1 = 0$$
$$m = \pm 1$$
$$\therefore y_c = Ae^x + Be^{-x}$$

$$y_p = (\lambda + \mu x)e^{2x}$$
$$y_p' = \mu e^{2x} + 2(\lambda + \mu x)e^{2x} = (\mu + 2\lambda + 2\mu x)e^{2x}$$
$$y_p'' = 2\mu e^{2x} + 2\mu e^{2x} + 4(\lambda + \mu x)e^{2x}$$
$$= (4\mu + 4\lambda + 4\mu x)e^{2x}$$

Sub in $\circledast$:

$$(4\mu + 4\lambda + 4\mu x)\cancel{e^{2x}} - (\lambda + \mu x)\cancel{e^{2x}} = x\cancel{e^{2x}}$$

COMPARE COEFFICIENTS:

$$4\mu + 4\lambda - \lambda = 0 \qquad 4\mu + 3\lambda = 0$$
$$4\mu - \mu = 1 \to 3\mu = 1 \to \mu = \frac{1}{3}$$
$$\frac{4}{3} + 3\lambda = 0$$
$$\lambda = -\frac{4}{9}$$

$$\therefore y_p = \left(-\frac{4}{9} + \frac{1}{3}x\right)e^{2x} = \frac{1}{9}(3x-4)e^{2x}$$

$$y = y_c + y_p = Ae^x + Be^{-x} + \frac{1}{9}(3x-4)e^{2x}$$

Exercise B: Full Solutions

9

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=te^{2t} \quad \circledast$$

$\therefore m^2+m+1=0 \qquad m=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm i\sqrt{3}}{2}$

$$y_c=e^{-\frac{t}{2}}\left(A\cos\left(\frac{\sqrt{3}}{2}t\right)+B\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$y_p=(\lambda+\mu t)e^{2t}$

$y_p'=2(\lambda+\mu t)e^{2t}+\mu e^{2t}=(2\lambda+\mu+2\mu t)e^{2t}$

$y_p''=2\mu e^{2t}+4(\lambda+\mu t)e^{2t}+2\mu e^{2t}$

$\quad=(4\mu+4\lambda+4\mu t)e^{2t}$

Sub in $\circledast$:

$(4\mu+4\lambda+4\mu t)e^{2t}+(2\lambda+\mu+2\mu t)e^{2t}+(\lambda+\mu t)e^{2t}=te^{2t}$

$7\lambda+5\mu+7\mu t=t$

Compare coefficients: $7\lambda+5\mu=0$

$7\mu=1 \rightarrow \mu=\frac{1}{7}$

$7\lambda+\frac{5}{7}=0$

$\lambda=-\frac{5}{49}$

$\therefore y_p=\left(-\frac{5}{49}+\frac{1}{7}t\right)e^{2t}$

$\therefore y=y_c+y_p=e^{-\frac{t}{2}}\left(A\cos\frac{\sqrt{3}}{2}t+B\sin\frac{\sqrt{3}}{2}t\right)+\frac{e^{2t}}{49}(7t-5)$

10

$y''+25y=3\cos 5x \quad \circledast \qquad y(0)=0$

$m^2+25=0 \quad \therefore m=\pm 5i \qquad y'(0)=5$

$y_c=A\cos 5x+B\sin 5x$

$f(x)=\cos 5x$ APPEARS IN y_c

$\therefore y_p=x(\lambda\cos 5x+\mu\sin 5x)$

$y_p'=\lambda\cos 5x+\mu\sin 5x-5\lambda x\sin 5x+5\mu x\cos 5x$

$y_p''=-5\lambda\sin 5x+5\mu\cos 5x-5\lambda\sin 5x+5\mu\cos 5x-25\lambda x\cos 5x-25\mu x\sin 5x$

Sub in $\circledast$:

$-5\lambda\sin 5x+5\mu\cos 5x-5\lambda\sin 5x+5\mu\cos 5x-25\lambda x\cos 5x-25\mu x\sin 5x$

$\quad+25\lambda x\cos 5x+25\mu x\sin 5x=3\cos 5x$

COMPARE COEFFICIENTS:

$\sin:\ -5\lambda-5\lambda-25\mu x+25\mu x=0 \qquad -10\lambda=0$

$\therefore \lambda=0$

$\cos:\ 5\mu+5\mu+25\lambda x-25\lambda x=3 \qquad 10\mu=3$

$\mu=\frac{3}{10}$

$\therefore y_p=\frac{3}{10}x\sin 5x$

$y=A\cos 5x+B\sin 5x+\frac{3}{10}x\sin 5x$

$y(0)=0; \rightarrow A=0$

$\therefore y=B\sin 5x+\frac{3}{10}x\sin 5x$

$y'(0)=5;\ y'=5B\cos 5x+\frac{3}{10}\sin 5x+\frac{3}{2}x\cos 5x$

$5=5B \rightarrow B=1$

$\therefore y=\sin 5x+\frac{3}{10}x\sin 5x$

$y=\left(1+\frac{3}{10}x\right)\sin 5x$

Exercise C: Full Solutions

Derivation of the transformations you will use for Q1-5

FOR DIFFERENTIAL EQUATIONS OF THE FORM

$$x^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \quad \circledast$$

USE SUBSTITUTION $x = e^u$ TO TRANSFORM $x^2\frac{d^2y}{dx^2}$ AND $x\frac{dy}{dx}$ SO THAT WE GET A CONSTANT COEFFICIENT EQUATION IN u.

$x = e^u \therefore \frac{dx}{du} = e^u = x$

BY CHAIN RULE $\frac{dy}{du} = \frac{dy}{dx}\frac{dx}{du} = x\frac{dy}{dx}$ $\left(\text{OR } e^u\frac{dy}{dx}\right)$

$$\frac{d^2y}{du^2} = \frac{d}{du}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(e^u\frac{dy}{dx}\right) = e^u\frac{dy}{dx} + e^u\frac{d}{du}\left(\frac{dy}{dx}\right)$$

$$= e^u\frac{dy}{dx} + e^u\frac{d^2y}{dx^2}\frac{dx}{du}$$

BY CHAIN RULE $\frac{df(x)}{du} = \frac{df(x)}{dx}\frac{dx}{du}$

where $f(x) = \frac{dy}{dx}$ AND x DEPENDS ON u

$$\therefore \frac{d^2y}{du^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} \quad \left(\text{AS } e^u = x \text{ \& } \frac{dx}{du} = x\right)$$

$$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$$

SO FOR EQUATION IN FORM $\circledast$ WE SUB.

$$\boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du} \quad \text{AND} \quad x\frac{dy}{dx} = \frac{dy}{du}}$$

1

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 4y = 0$$

from $x = e^u$

$$\left(x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}, \quad x\frac{dy}{dx} = \frac{dy}{du}\right)$$

$$\therefore \frac{d^2y}{du^2} - \frac{dy}{du} + 6\frac{dy}{du} + 4y = 0$$

$$\frac{d^2y}{du^2} + 5\frac{dy}{du} + 4y = 0$$

$$m^2 + 5m + 4 = 0$$

$$(m+4)(m+1) = 0$$

$$\therefore m = -4, \quad m = -1$$

$$y = Ae^{-4u} + Be^{-u}$$

replace u $\quad x = e^u \therefore u = \ln x$

$$y = Ae^{-4\ln x} + Be^{-\ln x}$$

$$= Ae^{\ln x^{-4}} + Be^{\ln x^{-1}}$$

$$= Ax^{-4} + Bx^{-1}$$

$$y = \frac{A}{x^4} + \frac{B}{x}$$

Exercise C: Full Solutions

2

2. $x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+4y=0$

$x=e^u \rightarrow x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$, $x\frac{dy}{dx}=\frac{dy}{du}$

$\therefore \frac{d^2y}{du^2}-\frac{dy}{du}+5\frac{dy}{du}+4y=0$

$\frac{d^2y}{du^2}+4\frac{dy}{du}+4y=0$

$m^2+4m+4=0$

$(m+2)^2=0 \quad \therefore m=-2$

$y=(A+Bu)e^{-2u}$

$x=e^u \quad \therefore u=\ln x$

$y=(A+B\ln x)e^{-2\ln x} \qquad e^{-2\ln x}=e^{\ln x^{-2}}=x^{-2}$

$y=\frac{A+B\ln x}{x^2}$

3

$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}-28y=0$

$x=e^u \rightarrow x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$, $x\frac{dy}{dx}=\frac{dy}{du}$

$\therefore \frac{d^2y}{du^2}-\frac{dy}{du}+4\frac{dy}{du}-28y=0$

$\frac{d^2y}{du^2}+3\frac{dy}{du}-28y=0$

$m^2+3m-28=0$

$(m+7)(m-4)=0 \quad \therefore m=-7, m=4$

$y=Ae^{4u}+Be^{-7u}$

$x=e^u \quad \therefore u=\ln x$

$y=Ae^{4\ln x}+Be^{-7\ln x}$

$=Ae^{\ln x^4}+Be^{\ln x^{-7}}$

$y=Ax^4+\frac{B}{x^7}$

Exercise C: Full Solutions

4

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 14y = 0$$

$$x = e^u \rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}, \quad x\frac{dy}{dx} = \frac{dy}{du}$$

$$\therefore \frac{d^2y}{du^2} - \frac{dy}{du} - 4\frac{dy}{du} - 14y = 0$$

$$\frac{d^2y}{du^2} - 5\frac{dy}{du} - 14y = 0$$

$$m^2 - 5m - 14 = 0$$

$$(m+2)(m-7) = 0$$

$$m = -2, \ m = 7$$

$$y = Ae^{-2u} + Be^{7u}$$

$$x = e^u \quad \therefore u = \ln x$$

$$y = Ae^{-2\ln x} + Be^{7\ln x}$$

$$= Ae^{\ln x^{-2}} + Be^{\ln x^7}$$

$$\therefore y = \frac{A}{x^2} + Bx^7$$

5

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = 0$$

$$x = e^u \rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}, \quad x\frac{dy}{dx} = \frac{dy}{du}$$

$$\therefore \frac{d^2y}{du^2} - \frac{dy}{du} + 3\frac{dy}{du} + 2y = 0$$

$$\frac{d^2y}{du^2} + 2\frac{dy}{du} + 2y = 0$$

$$m^2 + 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 4\times1\times2}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\therefore y = e^{-u}(A\cos u + B\sin u)$$

$$x = e^u \quad \therefore u = \ln x$$

$$y = e^{-\ln x}(A\cos(\ln x) + B\sin(\ln x))$$

$$e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$\therefore y = \frac{1}{x}(A\cos(\ln x) + B\sin(\ln x))$$

Exercise C: Full Solutions

6

$$x^2\frac{d^2y}{dx^2}+(2-4x)\frac{dy}{dx}-4y=0$$

$y=\frac{z}{x}$ $\therefore z=xy$ $\quad x^2\frac{d^2y}{dx^2}+2\frac{dy}{dx}-4x\frac{dy}{dx}-4y=0$

$$\frac{dz}{dx}=y+x\frac{dy}{dx} \quad \therefore x\frac{dy}{dx}=\frac{dz}{dx}-y$$

$$\frac{d^2z}{dx^2}=\frac{dy}{dx}+\frac{dy}{dx}+x\frac{d^2y}{dx^2} \quad x^2\frac{d^2y}{dx^2}=\frac{d^2z}{dx^2}-2\frac{dy}{dx}$$

$$\therefore \frac{d^2z}{dx^2}\cancel{-2\frac{dy}{dx}}\cancel{+2\frac{dy}{dx}}-4\left(\frac{dz}{dx}-y\right)-4y=0$$

$$\frac{d^2z}{dx^2}-4\frac{dz}{dx}\cancel{+4y}\cancel{-4y}=0$$

$$\frac{d^2z}{dx^2}-4\frac{dz}{dx}=0$$

$$m^2-4m=0$$

$$m(m-4)=0 \quad \therefore m=0,\ m=4$$

$$z=A+Be^{4x}$$

(dependent variable z
independent variable x)

$$z=xy=A+Be^{4x}$$

$$\therefore y=\frac{A+Be^{4x}}{x}$$

Exercise C: Full Solutions

7

$$x^2\frac{d^2y}{dx^2}+2x(x+2)\frac{dy}{dx}+2(x+1)^2y=e^{-x} \quad (*)$$

$$y=\frac{z}{x^2} \quad \therefore z=x^2y$$

$$\frac{dz}{dx}=2xy+x^2\frac{dy}{dx} \quad \therefore x^2\frac{dy}{dx}=\frac{dz}{dx}-2xy$$

$$\frac{d^2z}{dx^2}=2y+2x\frac{dy}{dx}+2x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}$$

$$\therefore x^2\frac{d^2y}{dx^2}=\frac{d^2z}{dx^2}-4x\frac{dy}{dx}-2y$$

EXPAND $(*)$:

$$x^2\frac{d^2y}{dx^2}+2x^2\frac{dy}{dx}+4x\frac{dy}{dx}+2(x^2+2x+1)y=e^{-x}$$

REPLACE WITH ABOVE:

$$\frac{d^2z}{dx^2}\cancel{-4x\frac{dy}{dx}}-2y+2\left(\frac{dz}{dx}-2xy\right)\cancel{+4x\frac{dy}{dx}}+(2x^2+4x+2)y=e^{-x}$$

$$\frac{d^2z}{dx^2}\cancel{-2y}+2\frac{dz}{dx}\cancel{-4xy}+2\underbrace{x^2y}_{z}\cancel{+4xy}\cancel{+2y}=e^{-x}$$

$$\therefore \frac{d^2z}{dx^2}+2\frac{dz}{dx}+2z=e^{-x} \quad (**)$$

FIND y_c:

$$m^2+2m+2=0$$

$$m=\frac{-2\pm\sqrt{4-4\times1\times2}}{2}=\frac{-2\pm\sqrt{-4}}{2}=\frac{-2\pm 2i}{2}=-1\pm i$$

$$z_c=e^{-x}(A\cos x+B\sin x)=x^2y_c$$

$$\therefore y_c=\frac{e^{-x}}{x^2}(A\cos x+B\sin x)$$

NOW NEED y_p.

$$f(x)=e^{-x} \quad \therefore z_p=\lambda e^{-x}$$

$$\frac{dz_p}{dx}=-\lambda e^{-x}$$

$$\frac{d^2z_p}{dx^2}=\lambda e^{-x}$$

IN $(**)$: $\lambda e^{-x}\cancel{-2\lambda e^{-x}}\cancel{+2\lambda e^{-x}}=e^{-x}$

$$\lambda=1$$

$$\therefore z_p=e^{-x}=x^2y_p$$

$$y_p=\frac{e^{-x}}{x^2}$$

$$\therefore y=\frac{e^{-x}}{x^2}(A\cos x+B\sin x+1)$$

Exercise C: Full Solutions

8

$$\cos x\frac{d^2y}{dx^2}+\sin x\frac{dy}{dx}-2y\cos^3 x=2\cos^5 x \quad ⊛$$

$$z=\sin x \qquad \frac{dz}{dx}=\cos x \qquad \frac{d^2z}{dx^2}=-\sin x$$

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\cos x\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2}=-\sin x\frac{dy}{dz}+\cos x\frac{d^2y}{dz^2}\overbrace{\frac{dz}{dx}}^{\cos x}$$

$$\left(\frac{df(x)}{dx}=\frac{df}{dz}\frac{dz}{dx} \text{ where } f=\frac{dy}{dz} \quad \therefore \frac{d}{dx}\left(\frac{dy}{dz}\right)=\frac{d^2y}{dz^2}\times\frac{dz}{dx}\right)$$

$$\frac{d^2y}{dx^2}=-\sin x\frac{dy}{dz}+\cos^2 x\frac{d^2y}{dz^2}$$

REPLACE IN ⊛:

$$\cos x\left(-\sin x\frac{dy}{dz}+\cos^2 x\frac{d^2y}{dz^2}\right)+\sin x\cos x\frac{dy}{dz}-2y\cos^3 x=2\cos^5 x$$

$$\cancel{-\sin x\cos x\frac{dy}{dz}}+\cos^3 x\frac{d^2y}{dz^2}\cancel{+\sin x\cos x\frac{dy}{dz}}-2y\cos^3 x=2\cos^5 x$$

$$\cos^3 x\left(\frac{d^2y}{dz^2}-2y\right)=2\cos^5 x \qquad \div\cos^3 x$$

$$\frac{d^2y}{dz^2}-2y=2\cos^2 x=2(1-\sin^2 x)$$

$$\therefore \frac{d^2y}{dz^2}-2y=2(1-z^2) \quad (**)$$

$\therefore$ FIND y_c: $m^2-2=0$

$m^2=2 \quad \therefore m=\pm\sqrt{2}$

$$y_c=Ae^{\sqrt{2}z}+Be^{-\sqrt{2}z} \qquad z=\sin x$$

$$\therefore y_c=Ae^{\sqrt{2}\sin x}+Be^{-\sqrt{2}\sin x}$$

FIND y_p: $f(z)=2(1-z^2)=2-2z^2$ (QUADRATIC)

$$\therefore y_p=az^2+bz+c$$

$$\therefore \frac{dy_p}{dz}=2az+b$$

$$\frac{d^2y_p}{dz^2}=2a$$

SUB. IN (**): $2a-2(az^2+bz+c)=2-2z^2$

$$2a-2az^2-2bz+2c=-2z^2+2$$

COMPARE COEFFICIENTS:

z^2: $-2a=-2 \quad \therefore a=1$

$-2b=0 \quad \therefore b=0$

$2a+2c=2$

$2+2c=2 \quad \therefore c=0$

$$\therefore y_p=z^2=\sin^2 x$$

$$\therefore y=Ae^{\sqrt{2}\sin x}+Be^{-\sqrt{2}\sin x}+\sin^2 x$$

Exercise D: Full Solutions

1

$y'' + 3y' = e^{-3x}$ $(a=1)$

$m^2 + 3m = 0$

$m(m+3) = 0$

$m=0, m=-3$ $\therefore y_c = A + Be^{-3x}$

$y_1 = 1,\ y_2 = e^{-3x}$

$y_p = uy_1 + vy_2$

$= u + ve^{-3x}$

Conditions: $u'y_1 + v'y_2 = 0$

$u'y_1' + v'y_2' = \frac{f(x)}{a}$

$y_1' = 0,\ y_2' = -3e^{-3x}$

$\therefore u' + v'e^{-3x} = 0$

$-3v'\cancel{e^{-3x}} = \cancel{e^{-3x}}$

$-3v' = 1$

$v' = -\frac{1}{3}$

$u' = -v'e^{-3x} = \frac{1}{3}e^{-3x}$

$u = \int \frac{1}{3}e^{-3x}\,dx \rightarrow \therefore u = -\frac{1}{9}e^{-3x}$

$v = \int -\frac{1}{3}\,dx \longrightarrow v = -\frac{1}{3}x$

$y_p = -\frac{1}{9}e^{-3x} - \frac{1}{3}xe^{-3x}$

$\therefore y = A + Be^{-3x} - \frac{1}{9}e^{-3x} - \frac{1}{3}xe^{-3x}$

2

$y'' + y = \sec x$ $(a=1)$

$m^2 + 1 = 0 \rightarrow m = \pm i$

$y_c = A\cos x + B\sin x$

$y_1 = \cos x,\ y_2 = \sin x$ $\therefore y_p = uy_1 + vy_2$

$y_p = u\cos x + v\sin x$

Conditions: $u'y_1 + v'y_2 = 0$

$u'y_1' + v'y_2' = \frac{f(x)}{a}$ $y_1' = -\sin x,\ y_2' = \cos x$

$\therefore$ $u'\cos x + v'\sin x = 0$ ①

$-u'\sin x + v'\cos x = \sec x$ ②

① ÷ $\cos x$: $u' = -v'\tan x$

Sub in ②: $v'\frac{\sin x}{\cos x}\sin x + v'\cos x = \frac{1}{\cos x}$

$\times\cos x$: $v'(\sin^2 x + \cos^2 x) = 1$

$v' = 1$ $\therefore \underline{v = x}$

$u' = -\tan x$ $\therefore$

$u = -\int \tan x\,dx = -\ln(\sec x)$

$\underline{u = \ln(\cos x)}$

$\therefore y_p = \ln(\cos x)\cos x + x\sin x$

$\therefore y = A\cos x + B\sin x + x\sin x + \cos x\ln|\cos x|$

$\left(\text{or } y = (A + \ln|\cos x|)\cos x + (B+x)\sin x\right)$

Exercise D: Full Solutions

3

$y'' - 6y' + 9y = x^2 e^{3x}$ $(a=1)$

$m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$

$m = 3$ $\therefore y_c = (A + Bx)e^{3x}$

$y_1 = e^{3x}, y_2 = xe^{3x}$

$y_p = uy_1 + vy_2$

$= ue^{3x} + vxe^{3x}$

CONDITIONS: $u'y_1 + v'y_2 = 0$

$u'y_1' + v'y_2' = f(x)/a$

$y_1' = 3e^{3x}$, $y_2' = e^{3x} + 3xe^{3x}$

$\therefore$ $u'e^{3x} + v'xe^{3x} = 0$ ①

$3u'e^{3x} + v'e^{3x} + 3v'xe^{3x} = x^2 e^{3x}$ ②

① $\div e^{3x}$: $u' = -xv'$

Sub in ②: $-3xv'e^{3x} + v'e^{3x} + 3v'xe^{3x} = x^2 e^{3x}$

$v'(-3x + 1 + 3x) = x^2$

$v' = x^2$ $\therefore v = \int x^2 dx = \frac{x^3}{3}$

$u' = -xv' = -x^3$ $\therefore u = \int -x^3 dx$

$= -\frac{x^4}{4}$

$y_p = -\frac{x^4}{4}e^{3x} + \frac{x^3}{3}(xe^{3x}) = e^{3x}\left(-\frac{x^4}{4} + \frac{x^4}{3}\right)$

$= \frac{x^4 e^{3x}}{12}$

$\therefore y = \left(A + Bx + \frac{x^4}{12}\right)e^{3x}$

4

$y'' - 4y' + 4y = \frac{e^{2x}}{x^2}$ $(a=1)$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \rightarrow m = 2$ $\therefore y_c = (A + Bx)e^{2x}$

$y_1 = e^{2x}, y_2 = xe^{2x}$

$y_p = uy_1 + vy_2 = ue^{2x} + vxe^{2x}$

CONDITIONS: $u'y_1 + v'y_2 = 0$

$u'y_1' + v'y_2' = f(x)/a$

$y_1' = 2e^{2x}$, $y_2' = e^{2x} + 2xe^{2x}$

$\therefore$ $u'e^{2x} + v'xe^{2x} = 0$ ①

$2u'e^{2x} + v'e^{2x} + 2v'xe^{2x} = \frac{e^{2x}}{x^2}$ ②

From ①: $u' + v'x = 0 \rightarrow u' = -v'x$

Sub in ②:

$\cancel{-2v'xe^{2x}} + v'e^{2x} + \cancel{2v'xe^{2x}} = \frac{e^{2x}}{x^2}$

$\therefore v' = \frac{1}{x^2} = x^{-2} \rightarrow v = -x^{-1}$

$\therefore u' = -x^{-2}x = -x^{-1} \rightarrow u = -\ln x$

$\therefore y_p = -e^{2x}\ln x - x^{-1}xe^{2x}$

$= -e^{2x}\ln x - e^{2x} = -e^{2x}(\ln x + 1)$

$\therefore y = (A + Bx)e^{2x} - e^{2x}(\ln x + 1)$

$y = (A + Bx - \ln x - 1)e^{2x}$

(or $y = (C + Bx - \ln x)e^{2x}$, where $C = A - 1$)

Exercise D: Full Solutions

5

$y'' - y = e^x \cos x \quad (a=1)$

$\therefore\ m^2 - 1 = 0 \ \therefore m = \pm 1 \qquad y_c = Ae^x + Be^{-x}$

$y_1 = e^x, \ y_2 = e^{-x}$

$y_p = uy_1 + vy_2 = ue^x + ve^{-x}$

CONDITIONS: $u'y_1 + v'y_2 = 0$

$u'y_1' + v'y_2' = f(x)/a$

$y_1' = e^x, \ y_2' = -e^{-x}$

$\therefore \ u'e^x + v'e^{-x} = 0 \quad (1)$

$u'e^x - v'e^{-x} = e^x \cos x \quad (2)$

$(1)-(2): \ 2v'e^{-x} = -e^x \cos x$

$v' = -\frac{1}{2} e^{2x} \cos x$

From (1): $u' = -v'e^{-2x} = \frac{1}{2}\cos x$

$u = \int \frac{1}{2}\cos x \, dx = \underline{\frac{1}{2}\sin x}$

$v = -\frac{1}{2}\int e^{2x}\cos x \, dx$

"CYCLIC PARTS" $\int e^{2x}\cos x \, dx \qquad u = e^{2x}, \ \frac{du}{dx} = 2e^{2x}; \quad \frac{dv}{dx} = \cos x, \ v = \sin x$

$\int e^{2x}\cos x \, dx = e^{2x}\sin x - 2\int e^{2x}\sin x \, dx \qquad u = e^{2x}, \ \frac{dv}{dx} = \sin x, \ \frac{du}{dx} = 2e^{2x}, \ v = -\cos x$

$\int e^{2x}\cos x \, dx = e^{2x}\sin x - 2\left[-e^{2x}\cos x + 2\int e^{2x}\cos x \, dx\right]$

$5\int e^{2x}\cos x \, dx = e^{2x}\sin x + 2e^{2x}\cos x$

$\int e^{2x}\cos x \, dx = \frac{1}{5}e^{2x}\sin x + \frac{2}{5}e^{2x}\cos x$

$v = -\frac{1}{2}\int e^{2x}\cos x \, dx = -\frac{1}{10}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x$

$\therefore y_p = \frac{1}{2}e^x \sin x + \left(-\frac{1}{10}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x\right)e^{-x}$

$= \frac{5}{10}e^x\sin x - \frac{1}{10}e^x \sin x - \frac{1}{5}e^x\cos x = \frac{2}{5}e^x\sin x - \frac{1}{5}e^x\cos x$

$\therefore y = Ae^x + Be^{-x} + \frac{2}{5}e^x\sin x - \frac{1}{5}e^x\cos x$

$\underline{y = e^x\left(A + \frac{2}{5}\sin x - \frac{1}{5}\cos x\right) + Be^{-x}}$