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Engineering Mathematics

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Second Order Linear ODEs: Outline of Lecture 7

- Constant coefficient second order linear ODEs (homogeneous).
- Constant coefficient second order linear ODEs (inhomogeneous) – Undetermined Coefficients Method.
- Substitution when coefficients are not constant (transformation to familiar form)
- Further reading (optional): Constant coefficient second order linear ODEs (inhomogeneous) – Variation of Parameters Method (Non-Examinable – Just for interest!)

A,B,C

Name the Coefficients

x^2+2x+3

A=1 B=2 C=3

IT'S A EASY AS 1,2,3

Constant Coefficient Second Order Linear ODEs (Homogeneous)

Constant coefficient second order linear ODEs

These equations have the general form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

or

$$ay'' + by' + cy = f(x)$$

Meaning of terms:

Constant coefficient: a, b, c are constants
Second order: Highest order derivative is 2nd
Linear: Dependent variable (usually y) and its derivatives (of any order) occur to the power of 1 only and there are no products of the dependent variable or its derivatives.

Let L be the linear differential operator $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$ so that

$$L(y) = a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy$$

In maths, an operator is a symbol (or set of symbols) that indicates an operation to be performed, in this case several differentiations. A more familiar operator would be a square root symbol.

By the standard properties of derivatives:

$$L(Cy) = CL(y)$$

And if y_1 and y_2 are functions of x only, then $L(y_1 + y_2) = L(y_1) + L(y_2)$.

The general form is therefore $L(y) = f(x)$ with the **homogeneous** case being when $L(y) = 0$, called the **complementary equation**.

Constant coefficient second order linear ODEs

If $\boldsymbol{y_p}$ is a particular solution of the differential equation $\boldsymbol{L(y) = f(x)}$ and if $\boldsymbol{y_c}$ is the solution of the complementary equation $\boldsymbol{L(y) = 0}$,
then the general solution of $\boldsymbol{L(y) = f(x)}$ is $\boldsymbol{y = y_c + y_p}$.

The two cases we will deal with are:
Homogeneous - when $\boldsymbol{f(x) = 0}$ (general solution is the complimentary solution $\boldsymbol{y_c}$)
Inhomogeneous - when $\boldsymbol{f(x) \neq 0}$ (general solution is sum of complimentary and particular solutions $\boldsymbol{y = y_c + y_p}$.

Terminology:

- The "general solution" is the overall solution, which may include just y_c (homogeneous case) or y_c and y_p (inhomogeneous case).
- It is called the general solution due to the unknown constants.
- Once the constants are determined (via boundary conditions),

it is just called "the solution (y)" as it is the only solution for those conditions.

Homogeneous 2nd order ODEs

We first look at the homogeneous case:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad \text{or} \qquad ay'' + by' + cy = 0$$

Before we seek solutions, we must first establish the following theorem:

<u>The superposition principle:</u>
For all linear systems, the net response caused by two or more stimuli is the sum of the responses that would have been caused by each stimulus individually. You may have seen this in physics with superposition of waves (nodes and all that stuff!)

For our purposes here, that translates to:
If $y_1(x)$ and $y_2(x)$ are any two linearly independent solutions to $ay'' + by' + cy = 0$, then $y = C_1y_1(x) + C_2y_2(x)$ is a solution for all real constants C_1 and C_2.

<u>Proof:</u> If $y_1(x)$ and $y_2(x)$ are solutions, then

$$ay_1''(x) + by_1'(x) + cy_1(x) = 0$$
$$ay_2''(x) + by_2'(x) + cy_2(x) = 0$$

Multiply first equation by A and the second by B and add the result:

$$a[Ay_1''(x) + By_2''(x)] + b[Ay_1'(x) + By_2'(x) + c[Ay_1(x) + By_2(x)] = 0$$

Therefore, $Ay_1(x) + By_2(x)$ is a solution.

Homogeneous 2nd order ODEs

Summary: If $y_1(x)$ and $y_2(x)$ are any two linearly independent* solutions of $ay'' + by' + cy = 0$, then the general (complementary) solution is:

$$y_c = Ay_1(x) + By_2(x)$$

*Linearly independent means one is not a scalar multiple of another ($y_1 \neq ky_2$).

Note that because this is a second order differential equation, it should have 2 parts to its solution. (and hence two boundary conditions would be needed to find the values of the constants).

To develop the method we will use throughout this lecture, we first show why it works with an example.

E.g. 0

Verify the $y_1 = e^{4x}$ and $y_2 = e^{2x}$ both satisfy the equation:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$

And write down the general solution of this equation.

For each, we just differentiate and substitute:

$$y_1 = e^{4x} \Rightarrow \frac{dy_1}{dx} = 4e^{4x} \Rightarrow \frac{d^2y_1}{dx^2} = 16e^{4x}$$

LHS of equation becomes
$16e^{4x} - 6(4e^{4x}) + 8(e^{4x})$
which equals zero, so $y_1 = e^{4x}$ is a solution.

$$y_2 = e^{2x} \Rightarrow \frac{dy_2}{dx} = 2e^{2x} \Rightarrow \frac{d^2y_2}{dx^2} = 4e^{2x}$$

LHS of equation becomes
$4e^{2x} - 6(2e^{2x}) + 8(e^{2x})$
again zero, so $y_2 = e^{2x}$ is a solution.

As e^{4x} and e^{2x} are linearly independent, by superposition, we have that $y_c = Ae^{4x} + Be^{2x}$ is the general solution of the equation.

Method: Homogeneous 2nd order ODEs

We now establish our method for the general solution to $ay'' + by' + cy = 0$. From the theorem, it is sufficient to find two functions $y_1(x)$ and $y_2(x)$ and add them together.

As a trial solution, we use the general exponential $y = Ae^{mx}$ where A is an arbitrary constant and we need to find m.

Let $y = Ae^{mx}$

Then $\frac{dy}{dx} = \boldsymbol{Ame^{mx}}$ and $\frac{d^2y}{dx^2} = \boldsymbol{Am^2e^{mx}}$

$$\text{Thus } a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \boldsymbol{aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0}$$

$$\boldsymbol{Ae^{mx}\left(am^2 + bm + c\right) = 0}$$

$e^{mx} > 0$ and we are interested in non-trivial solutions ($A \neq 0$), that is when

$$\boldsymbol{am^2 + bm + c = 0}$$

$$\boldsymbol{am^2 + bm + c = 0}$$

is known as an **auxiliary equation**.

It can be obtained quickly by replacing y'' with m^2, y' with m and y with 1.

It is solved by the usual methods for quadratics!

Variants: $b^2 - 4ac > 0$

E.g. 1a

Solve the differential equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 15y = 0$

Auxiliary equation:

$$m^2 - 2m + 15 = 0$$

$$(m+5)(m-3) = 0$$

$$m_1 = -5, m_2 = 3$$

The roots of the auxiliary equation are real and unequal (distinct), therefore by superposition theorem:

$$y_c = Ae^{-5x} + Be^{3x}$$

<u>Clarification:</u>
The letter used for constants does not matter, so $y_c = Ae^{3x} + Be^{-5x}$ is equally correct.

Theorem:
When the auxiliary equation has **two real distinct roots** m_1 and m_2, the general solution of the differential equation is $y = Ae^{m_1x} + Be^{m_2x}$, where A and B are arbitrary constants.

Boundary Conditions: Determining constants

- **Boundary conditions are certain conditions, which allow us to determine the values of constants.**
- **They are usually derived from the properties of the physical system that the differential equation is modelling.**
- **When boundary conditions refer to the beginning ($t = 0$), they are known as initial conditions.**

E.g. 1b

$$y_c = Ae^{-5x} + Be^{3x}$$

- As there are two arbitrary constants in the above, we require two boundary (or initial) conditions to determine their values.

- This may be any combination of the values of y or its derivatives at particular x values.

For example, if we know that $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$ we get

$y = 0$ at $x = 0$ gives $\boldsymbol{A + B = 0}$

(As $e^0 = 1$)

$$\frac{dy}{dx} = -5Ae^{-5x} + 3Be^{3x}$$

Hence $\frac{dy}{dx} = 1$ at $x = 0$ gives $\boldsymbol{-5A + 3B = 1}$

Solving the simultaneous equations gives $A = -\frac{1}{8}$, $B = \frac{1}{8}$

And so the final form of the solution is: $\boldsymbol{y_c = -\frac{1}{8}e^{-5x} + \frac{1}{8}e^{3x}}$

Variant 2: $b^2 - 4ac = 0$

In the previous examples, the auxiliary equation had distinct roots, i.e. $b^2 - 4ac > 0$. What if we have equal roots?

E.g. 2

Find the general solution of $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

If we proceed as before, the auxiliary equation is:

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0 \Rightarrow m = 3$$

There is only one distinct solution $y_1 = Ae^{3x}$.
However, as this is a second order problem there will be two boundary conditions, so two arbitrary constants are required for the general solution.

If we were to just use this result twice we would get:

$$y = Ae^{3x} + Be^{3x} = (A+B)e^{3x} = Ce^{3x}$$

which would result in the same value for A or C if we inputted boundary conditions.

To fix this, we multiply the first solution by x therefore $y_2 = Bxe^{3x}$ is another distinct solution

And by superposition, the general solution is $\boldsymbol{y_c = Ae^{3x} + Bxe^{3x} = (A + Bx)e^{3x}}$

The reason we have to use $y = (A + Bx)e^{mx}$ instead of $Ae^{mx} + Be^{mx}$ is similar to why in partial fractions, we have to use $\frac{A}{x} + \frac{B}{x^2}$ if we had a repeated denominator x.

Variant 2: $b^2 - 4ac = 0$

The proof of this is too long for here (see appendix at end if interested) but we can show that $y_c = (A + Bx)e^{3x}$ does indeed satisfy the original differential equation.

Therefore, by superposition theorem, if $(A + Bx)e^{3x}$ satisfies $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ then so do Ae^{3x} and Bxe^{3x} individually, and hence they are both valid solutions.

$y = (A + Bx)e^{3x} = Ae^{3x} + Bxe^{3x}$ $\quad \frac{dy}{dx} = 3Ae^{3x} + 3Bxe^{3x} + Be^{3x}$ $\quad \frac{d^2y}{dx^2} = 9Ae^{3x} + 9Bxe^{3x} + 6Be^{3x}$

$$\therefore \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 9Ae^{3x} + +9Bxe^{3x} + 6Be^{3x} - 6(3Ae^{3x} + 3Bxe^{3x} + Be^{3x}) + 9(Ae^{3x} + Bxe^{3x})$$

$$= 9Ae^{3x} + 9Bxe^{3x} + 6Be^{3x} - 18Ae^{3x} - 18Bxe^{3x} - 6Be^{3x} + 9Ae^{3x} + 9Bxe^{3x} = 0$$

Hence $y_c = (A + Bx)e^{3x}$ is a solution.

Note: We could have verified just Bxe^{3x} works individually, but verifying the whole solution completes the picture better.

When the auxiliary equation has two equal roots m, the general solution is
$$\boldsymbol{y = (A + Bx)e^{mx}}$$

Variant 3a: $b^2 - 4ac < 0$

This is actually exactly the same as when we usually have distinct real roots!

E.g. 3

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 16y = 0$

Auxiliary equation: $\boldsymbol{m^2 + 16 = 0} \quad \Rightarrow \quad \boldsymbol{m^2 = -16} \quad \Rightarrow \quad \boldsymbol{m = \pm 4i}$

$\therefore$ General solution is $\boldsymbol{y = Ae^{4ix} + Be^{-4ix}}$

This can be rewritten as (using Euler's Formula): $e^{i\theta} = \cos\theta + i\sin\theta$

$$y = A(\cos 4x + i\sin 4x) + B(\cos 4x - i\sin 4x)$$

$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin(\theta)$

$$= C\cos 4x + D\sin 4x \qquad (where\ C = A + B, D = i(A - B))$$

As C and D are arbitrary constants, we can rename them A and B for the following:

If the auxiliary equation has two <u>purely imaginary</u> roots $\pm qi$, the general solution is $\boldsymbol{y = A\cos qx + B\sin qx}$ where A and B are arbitrary constants.

Note: This is the default format for this case, we write as trig functions to avoid imaginary numbers.

Variant 3b: $b^2 - 4ac < 0$

So what about more general complex roots $p \pm qi$?

E.g. 4 Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0$

Auxiliary equation: $\boldsymbol{m^2 - 6m + 34 = 0} \quad \Rightarrow \quad \boldsymbol{m = 3 \pm 5i}$

$\therefore$ General solution is $\quad \boldsymbol{y = Ae^{(3+5i)x} + Be^{(3-5i)x}}$

Using power laws: $\quad \boldsymbol{y = Ae^{3x} \times e^{5ix} + Be^{3x} \times e^{-5ix} \quad = e^{3x}(Ae^{5ix} + Be^{-5ix})}$

The bracket can be rewritten as (using Euler's Formula): $e^{i\theta} = \cos\theta + i\sin\theta$

$$y = e^{3x}[A(\cos 5x + i\sin 5x) + B(\cos 5x - i\sin 5x)]$$

$$\boldsymbol{y = e^{3x}(C\cos 5x + D\sin 5x)}$$

As C and D are arbitrary constants, we can rename them A and B for the following:

If the auxiliary equation has two <u>complex</u> roots $p \pm qi$, the general solution is

$$\boldsymbol{y = e^{px}(A\cos qx + B\sin qx)}$$

where A and B are arbitrary constants.

Note: This is just the general case of the previous slide, putting $p = 0$ gives

$$\boldsymbol{y = e^0(A\cos qx + B\sin qx) = A\cos qx + B\sin qx}$$

So you don't really need to remember the previous one.

Summary: Homogeneous Case

- The equation $am^2 + bm + c = 0$ is called the auxiliary equation, and if m is a root of the auxiliary equation then $y = Ae^{mx}$ is a solution of the differential equation
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (ay'' + by' + cy = 0)$$
- When the auxiliary equation has **two real distinct roots** m_1 and m_2, the general solution of the differential equation is $y = Ae^{m_1x} + Be^{m_2x}$, where A and B are arbitrary constants.
- When the auxiliary equation has two equal roots m, the general solution is
$$y = (A + Bx)e^{mx}$$
- If the auxiliary equation has two purely imaginary roots $\pm iq$, the general solution is $\boldsymbol{y = A\cos qx + B\sin qx}$ where A and B are arbitrary constants.
- If the auxiliary equation has two complex roots $p \pm iq$, the general solution is
$$\boldsymbol{y = e^{px}(A\cos qx + B\sin qx)}$$
where A and B are arbitrary constants.

Diagnostic Question

Which is the correct general solution to the following:

$$\frac{d^2y}{dx^2} - y = 0$$

Y $y = (A + Bx)e^x$

M $y = Ae^x + Be^x$

C $y = Ae^x + Be^{-x}$

A $y = A\cos x + B\sin x$

Diagnostic Question

Which is the correct general solution to the following:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Y $y = Ae^{2x} + Be^{2x}$

M $y = e^{2x}(A\cos 2x + B\sin 2x)$

C $y = A\cos 2x + B\sin 2x$

A $y = (A + Bx)e^{2x}$

Diagnostic Question

Which is the correct general solution to the following:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$

Y $y = Ae^{4x} + Be^{2x}$

M $y = A\cos 4x + B\sin 2x$

C $y = e^{2x}(A\cos 4x + B\sin 4x)$

A $y = e^{4x}(A\cos 2x + B\sin 2x)$

Diagnostic Question

Which is the correct general solution to the following:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

Y $y = A\cos 2x + B\sin 2x$

M $y = e^{-x}(A\cos 2x + B\sin 2x)$

C $y = e^{x}(A\cos 2x + B\sin 2x)$

A $y = e^{2x}(A\cos x + B\sin x)$

Diagnostic Question

Which is the correct general solution to the following:

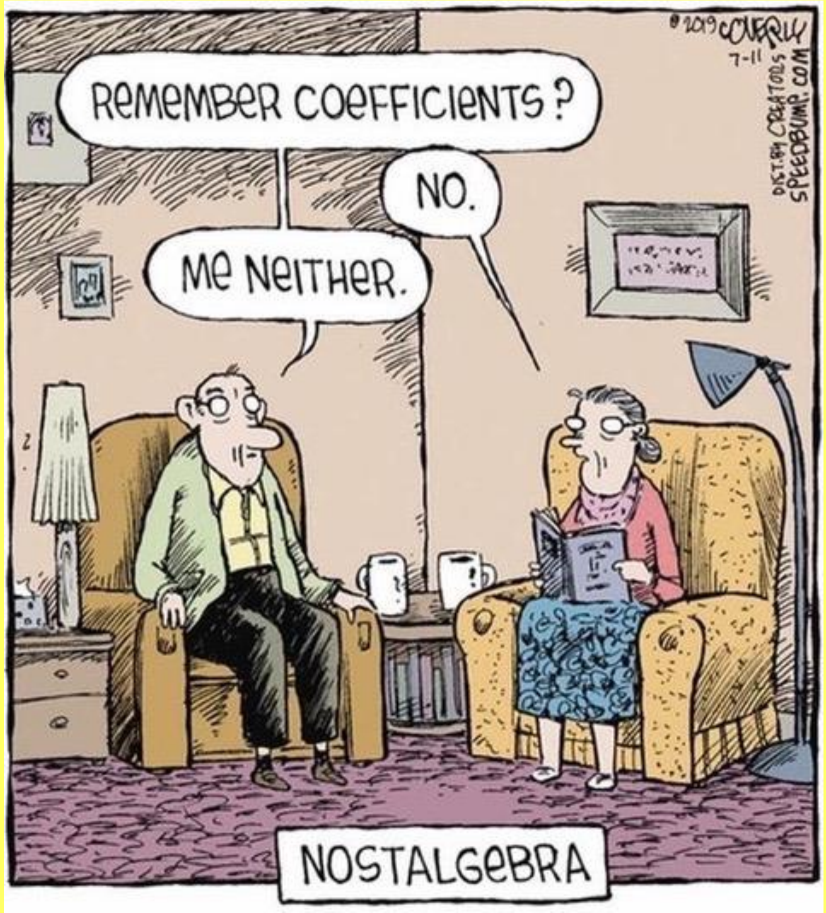
$$\frac{d^2y}{dx^2} + 10y = 0$$

Y $y = A\cos 10x + B\sin 10x$

M $y = A\cos\sqrt{10}x + B\sin\sqrt{10}x$

C $y = e^{x}(A\cos\sqrt{10}x + B\sin\sqrt{10}x)$

A $y = Ae^{\sqrt{10}x} + Be^{-\sqrt{10}x}$



Constant Coefficient Second Order Linear ODEs (Inhomogeneous)

Undetermined Coefficients Method

Undetermined Coefficients Method

The inhomogeneous case is given by:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \qquad \text{where } f(x) \neq 0$$

To find the general solution:

1. Solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \mathbf{0}$ first (as before) to obtain the **complementary function**, y_c.

2. Then solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \boldsymbol{f(x)}$ which can be found using appropriate substitution and comparing coefficients. This is the particular solution, y_p
(A particular solution is any function which, when substituted in the LHS results in the RHS.)

To find $y_p(x)$ we try solutions of the same general form as $f(x)$, but this may need to be multiplied by x if this form also appear in $y_c(x)$ (explained later)

3. **If y_p is a particular solution of the differential equation $L(y) = f(x)$ and if y_c is the solution of the complementary equation $L(y) = 0$,
then the general solution of $L(y) = f(x)$ is**

$$\boldsymbol{y = y_c + y_p.}$$

(This is because $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy$ for the complementary function is 0 and $f(x)$ for the particular solution, which sum to $f(x)$).

Example

E.g. 1 Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{3}$

First, we find complementary solution:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Auxiliary equation: $m^2 - 5m + 6 = 0$

Roots : $m = 2, m = 3$

Complementary solution: $y_c = Ae^{2x} + Be^{3x}$

Now find the particular solution:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{3}$$

If $\boldsymbol{f(x)}$ is a constant, let the particular integral be a constant too.

$$y_p = \lambda$$

$$\rightarrow \quad \frac{dy_p}{dx} = 0, \qquad \frac{d^2y_p}{dx^2} = 0$$

Substitute in to differential equation:

$$0 - 5(0) + 6\lambda = 3$$

$$\lambda = \frac{1}{2}$$

$$\therefore y_p = \frac{1}{2}$$

Notice that y_p has no arbitrary constants!

General solution: $\boldsymbol{y = y_c + y_p = Ae^{2x} + Be^{3x} + \frac{1}{2}}$

Example

E.g. 2 Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \boldsymbol{2x}$

The LHS is the same as the previous example, so as before, complementary solution:

$$y_c = Ae^{2x} + Be^{3x}$$

$f(x)$ is a linear function, so we try a general linear function for the particular solution.

$$y_p = \lambda x + \mu \qquad \rightarrow \quad \frac{dy_p}{dx} = \lambda, \qquad \frac{d^2y_p}{dx^2} = 0$$

Note: You must use $\boldsymbol{\lambda x + \mu}$. If you just use λx you will not get the full solution!

Substitute into equation: $0 - 5\lambda + 6(\lambda x + \mu) = 2x$

Group powers of x: $6\lambda x + (6\mu - 5\lambda) = 2x$

Compare coefficients:

$$\begin{aligned} 6\lambda &= 2 \\ 6\mu - 5\lambda &= 0 \end{aligned} \quad \Rightarrow \quad \lambda = \frac{1}{3}, \qquad \mu = \frac{5}{18} \qquad \therefore y_p = \frac{1}{3}x + \frac{5}{18}$$

General solution:

$$\boldsymbol{y = Ae^{2x} + Be^{3x} + \frac{1}{3}x + \frac{5}{18}}$$

Tip:
Always rewrite y_p out again so you don't forget any of its parts for the general solution.

Example

E.g. 3

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{3x^2}$

The LHS is the same as the previous example, so as before, complementary solution:

$$y_c = Ae^{2x} + Be^{3x}$$

$f(x)$ is a quadratic function, so we try a quadratic function for the particular solution:

$$y_p = \lambda x^2 + \mu x + \nu \qquad \rightarrow \quad \frac{dy_p}{dx} = 2\lambda x + \mu, \qquad \frac{d^2y_p}{dx^2} = 2\lambda$$

Substitute into equation: $2\lambda - 5(2\lambda x + \mu) + 6(\lambda x^2 + \mu x + \nu) = 3x^2$

Group powers of x: $6\lambda x^2 + (6\mu - 10\lambda)x + (2\lambda - 5\mu + 6\nu) = 3x^2$

Compare coefficients:

$\boldsymbol{x^2}$: $\quad 6\lambda = 3 \Rightarrow \lambda = \frac{1}{2}$

$\boldsymbol{x}$: $\quad 6\mu - 10\lambda = 0 \quad \Rightarrow 6\mu - 5 = 0 \quad \Rightarrow \mu = \frac{5}{6}$

$\boldsymbol{const}$: $\quad 2\lambda - 5\mu + 6\nu = 0 \quad \Rightarrow 1 - \frac{25}{6} + 6\nu = 0 \quad \Rightarrow \nu = \frac{19}{36}$

$\therefore y_p = \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$

General solution: $\boldsymbol{y = Ae^{2x} + Be^{3x} + \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}}$

Example

E.g. 4

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \boldsymbol{e^x}$

The LHS is the same as the previous example, so as before, complementary solution:

$$y_c = Ae^{2x} + Be^{3x}$$

$\boldsymbol{f(x)}$ is an exponential function, so we try an exponential function for the particular solution:

$$y_p = \lambda e^x \qquad \rightarrow \quad \frac{dy_p}{dx} = \lambda e^x, \qquad \frac{d^2y_p}{dx^2} = \lambda e^x$$

Substitute into equation: $\lambda e^x - 5\lambda e^x + 6\lambda e^x = e^x$

Compare coefficients: $2\lambda e^x = e^x \qquad \Rightarrow \lambda = \frac{1}{2} \qquad \therefore y_p = \frac{1}{2}e^x$

General solution: $y = \boldsymbol{Ae^{2x} + Be^{3x} + \frac{1}{2}e^x}$

Example

E.g. 5

Find the **general solution** of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{13\ sin\ 3x}$$

The LHS is the same as the previous example, so as before, complementary solution:

$$y_c = Ae^{2x} + Be^{3x}$$

$f(x)$ is a trig function, so we try a trig function for the particular solution:

$$y_p = \lambda \sin 3x + \mu \cos 3x \quad \rightarrow \quad \frac{dy_p}{dx} = 3\lambda \cos 3x - 3\mu \sin 3x\,,$$

$$\frac{d^2y_p}{dx^2} = -9\lambda \sin 3x - 9\mu \cos 3x$$

Substitute into equation:

$$-9\lambda \sin 3x - 9\mu \cos 3x - 5(3\lambda \cos 3x - 3\mu \sin 3x) + 6(\lambda \sin 3x + \mu \cos 3x) = 13 \sin 3x$$

Group sin and cos terms: $(15\mu - 3\lambda) \sin 3x - (3\mu + 15\lambda) \cos 3x = 13 \sin 3x$

Compare coefficients: $\boldsymbol{cos}$: $3\mu + 15\lambda = 0$ $\boldsymbol{sin}$: $15\mu - 3\lambda = 13$

Solve simultaneous equations: $\lambda = -\frac{1}{6}, \mu = \frac{5}{6}$ $\therefore\ y_p = -\frac{1}{6}\sin 3x + \frac{5}{6}\cos 3x$

General solution: $y = \boldsymbol{Ae^{2x} + Be^{3x} - \frac{1}{6}\ sin\ 3x + \frac{5}{6}\ cos\ 3x}$

If inhomogeneous term $f(x)$ appears in y_c

Important:
Your <u>particular integral can't be part of your complementary function</u>.
This is just like how we weren't allowed to use $\boldsymbol{Ae^{\alpha x} + Be^{\beta x}}$ for the complementary function if the two roots of the auxiliary equation were equal.

E.g. 6

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \boldsymbol{e^{2x}}$

$$y_c = Ae^{2x} + Be^{3x}$$

The difference to when we had $= e^x$ is that now the e^{2x} matches the e^{2x} term in the complementary function.

Suppose we did use $y = \lambda e^{2x}$ for the particular integral. What goes wrong?
Then the general solution might appear to be $\boldsymbol{y = Ae^{2x} + Be^{3x} + \lambda e^{2x} = (A+\lambda)e^{2x} + Be^{2x}}$

But $\boldsymbol{A + \lambda}$ is still just an arbitrary constant, so we have exactly the same as the complementary function, which we know gives 0 (rather than $\boldsymbol{e^{2x}}$) when subbed into $\boldsymbol{\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y}$. Thus we end up with $\boldsymbol{0 = e^{2x}}$. Oh dear!

So let $y = \lambda \boldsymbol{x} e^{2x}$
This ends up giving $\lambda = -1$
So general solution: $y = \boldsymbol{Ae^{2x} + Be^{3x} - xe^{2x}}$

Example: Inhomogeneous term $f(x)$ appears in y_c

E.g. 7

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = \mathbf{3}$

Auxiliary equation: $m^2 - 2m = 0 \quad \Rightarrow m = 0, m = 2$

Complementary function: $y = Ae^{0x} + Be^{2x} = A + Be^{2x}$

For our trial solution for y_p we can't use $y = \lambda$ as y_c contains a constant term.
Instead use $y_p = \lambda x$

$$y_p = \lambda x \qquad \frac{dy_p}{dx} = \lambda \qquad \frac{d^2y_p}{dx^2} = 0$$

Substitute into equation: $0 - 2\lambda = 3 \quad \Rightarrow \lambda = -\frac{3}{2} \quad \Rightarrow y_c = -\frac{3}{2}x$

General solution: $\boldsymbol{y = A + Be^{2x} - \frac{3}{2}x}$

So it seems we add a "cheeky little x" to the front of the Particular Integral in any case where the particular integral is part of the complementary function.

Thinking deeper... $f(x)$ is a product of functions

$$\frac{d^2y}{dx^2} - y = xe^{3x} \quad \text{where} \quad y_c = Ae^{x} + Be^{-x}$$

What should be the trial solution for y_p?

y_c **does not** contain any part of xe^{3x} but as it is a product, we should use the product of our trial solutions for y_p.

For x that would be $ax + b$ (generic linear) and for e^{3x} itwould be ce^{3x} (generic exponential of $3x$)

Therefore $y_p = (ax + b)ce^{3x} = (acx + bc)e^{3x}$

Which can be simplified to $y_p = (\lambda x + \mu)e^{3x}$ (where $\lambda = ac$ and $\mu = bc$)

$$\frac{d^2y}{dx^2} - y = xe^{3x} \quad \text{where} \quad y_c = Ae^{3x} + Be^{-3x}$$

What should be the trial solution for y_p?

This time y_c **does** contain part of xe^{3x} , so we would do the same as above but we should multiply the trial solution by x.

So $y_p = (\lambda x^2 + \mu x)e^{3x}$

Note: Even though this is a product, you should only multiply the overall y_p function by x, not the two parts separately as this would result in multiplying by x^2.

Trial Forms for Particular Solution

Form of $f(x)$	Trial solution for y_p	If $f(x)$ appears in y_c
k	λ	λx
$ax + b$	$\lambda x + \mu$	$(\lambda x + \mu)x$
$ax^2 + bx + c$	$\lambda x^2 + \mu x + \nu$	$(\lambda x^2 + \mu x + \nu)x$
ke^{px}	λe^{px}	$\lambda x e^{px}$
ke^{-px}	λe^{-px}	$\lambda x e^{-px}$
$m \cos kx$	$\lambda \cos kx + \mu \sin kx$	$(\lambda \cos kx + \mu \sin kx)x$
$n \sin kx$		
$m \cos kx + n \sin kx$		
Product: $(ax + b)e^{px}$	$(\lambda x + \mu)e^{px}$	$(\lambda x^2 + \mu x)e^{px}$ (you only need to multiply one function by x, not both)

If the inhomogeneous term $f(x)$ appears in the complementary solution y_c we use the usual trial solution **multiplied by x** (Note: in this case $\lambda \to \lambda x$ not $\lambda x + \mu$).

If $f(x)$ is a **sum of two or more functions** in the list above, then the particular solution is the sum of the usual trial solutions

E.g. $e^x + \sin x \Rightarrow y_p = \lambda e^x + C \cos x + D \sin x$

(Or you can find the particular solution for each part separately and add them together,

So the general solution: $y = y_c + y_{p_1} + y_{p_2}$)

Diagnostic Question

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^x \quad \text{where} \quad y_c = (A + Bx)e^{3x}$$

What should be the trial solution for y_p?

Y $y_p = \lambda e^x$

M $y_p = \lambda x e^x$

C $y_p = \lambda e^{3x}$

A $y_p = \lambda x e^{3x}$

Diagnostic Question

$$\frac{d^2y}{dx^2} - y = 2e^x \quad \text{where} \quad y_c = Ae^x + Be^{-x}$$

What should be the trial solution for y_p?

Y $y_p = \lambda e^{-x}$

M $y_p = \lambda e^{x}$

C $y_p = \lambda x e^{-x}$

A $y_p = \lambda x e^{x}$

Diagnostic Question

$$\frac{d^2y}{dx^2} + 4y = 5\sin 3x \text{ where } \quad y_c = A\cos 2x + B\sin 2x$$

What should be the trial solution for y_p?

Y $y = \lambda \sin 3x$

M $y = \lambda\, x \sin 3x$

C $y = \lambda \cos 3x + \mu \sin 3x$

A $y = x(\lambda \cos 3x + \mu \sin 3x)$

Diagnostic Question

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = xe^x \quad \text{where} \quad y_c = Ae^{2x} + Be^{-3x}$$

What should be the trial solution for y_p?

Y $y_p = \lambda x^2 e^x$

M $y_p = (\lambda x^2 + \mu x)e^x$

C $y_p = \lambda x e^x$

A $y_p = (\lambda x + \mu)e^x$

Diagnostic Question

$$\frac{d^2y}{dx^2} - y = xe^x \text{ where } y_c = Ae^x + Be^{-x}$$

What should be the trial solution for y_p?

Y $y_p = \lambda x^2 e^x$

M $y_p = (\lambda x^2 + \mu x)e^x$

C $y_p = \lambda x e^x$

A $y_p = (\lambda x + \mu)e^x$

Diagnostic Question

$$\frac{d^2y}{dx^2} + 9y = 2\sin 3x \text{ where } \quad y_c = A\cos 3x + B\sin 3x$$

What should be the trial solution for y_p?

Y $y = \lambda \sin 3x$

M $y = \lambda\, x\sin 3x$

C $y = \lambda \cos 3x + \mu \sin 3x$

A $y = x(\lambda \cos 3x + \mu \sin 3 x)$

Diagnostic Question

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 5 \quad \text{where} \quad y_c = A + Be^{2x}$$

What should be the trial solution for y_p?

Y $y_p = \lambda$

M $y_p = \lambda x$

C $y_p = \lambda x + \mu$

A $y_p = \lambda x^2$

Example with Boundary Conditions

E.g. 8 Find the solution of $\frac{d^2y}{dx^2} - y = 2e^x$, given that $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$.

Auxiliary equation: $m^2 - 1 = 0 \Rightarrow m = -1, m = 1$

Complementary function: $y = Ae^x + Be^{-x}$

$\boldsymbol{e^x}$ **appears in** $\boldsymbol{y_c}$ **so multiply usual trial by** $\boldsymbol{x}$**:**

$$y_p = \lambda x e^x$$

$$\frac{dy_p}{dx} = \lambda e^x + \lambda x e^x = \lambda e^x(1 + x)$$

$$\frac{d^2y_p}{dx^2} = \lambda e^x + \lambda e^x + \lambda x e^x = \lambda e^x(2 + x)$$

Substitute into equation:

$$\lambda e^x(2 + x) - \lambda x e^x = 2e^x$$

Divide by $\boldsymbol{e^x}$ **:** $2\lambda + \lambda x - \lambda x = 2 \quad \Rightarrow \lambda = 1$

$$\therefore y_p = x e^x$$

General solution: $\boldsymbol{y = Ae^x + Be^{-x} + xe^x}$

Substitute 1st BC ($y = 0$ at $x = 0$**):**

$0 = Ae^0 + Be^0 + 0 \quad \rightarrow \quad A + B = 0 \quad (1)$

We have a BC for $\frac{dy}{dx}$**,**
so differentiate general solution:

$$\frac{dy}{dx} = Ae^x - Be^{-x} + xe^x + e^x$$

Substitute 2nd BC ($\frac{dy}{dx} = 0$ at $x = 0$**):**

$$0 = A - B + 1$$

$\rightarrow \quad A - B = -1 \quad (2)$

Simultaneous equations:

$$A + B = 0 \quad (1)$$
$$A - B = -1 \quad (2)$$
$$\Rightarrow A = -\frac{1}{2}, B = \frac{1}{2}$$

Therefore solution is $\boldsymbol{y = -\frac{1}{2}e^x + \frac{1}{2}e^{-x} + xe^x}$

Diagnostic Question

The general solution to a second order differential equation is

$$y = Ae^{3x} + Be^{x} - 2x + 3$$

Find the solution given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$

Y $y = 2e^{3x} - 4e^{x} - 2x + 3$

M $y = -2e^{x} - 2x + 3$

C $y = 2e^{x} - 2x + 3$

A $y = -\frac{1}{2}e^{3x} + \frac{3}{2}e^{x} - 2x + 3$

Substitution When Coefficients Are Not Constant

2nd Order ODES (Non-constant coefficients)

You can use a given substitution to transform a second order ODE with non-constant coefficients into the same form as we previously dealt with.

Example

Using the substitution $x = e^u$, show that:

a) $x\frac{dy}{dx} = \frac{dy}{du}$ b) $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

c) Hence find the general solution to $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

a) $x = e^u \Rightarrow \frac{dx}{du} = e^u = x$

By chain rule: $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = e^u\frac{dy}{dx} = x\frac{dy}{dx}$

b) $\frac{d^2y}{du^2} = \frac{d}{du}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(e^u\frac{dy}{dx}\right)$

$= e^u\frac{dy}{dx} + e^u\frac{d^2y}{dx^2}\frac{dx}{du}$

By chain rule

$$\frac{df(x)}{du} = \frac{df(x)}{dx} \times \frac{dx}{du}$$

(Where $f(x) = \frac{dy}{dx}$ and x depends on u)

$$\therefore \frac{d}{du}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}\frac{dx}{du}$$

$= \frac{dy}{du} + x^2\frac{d^2y}{dx^2}$

Using $e^u\frac{dy}{dx} = \frac{dy}{du}$ and $\frac{dx}{du} = e^u = x$

$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

Substituting a and b into the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

We get:

$$\frac{d^2y}{du^2} - \frac{dy}{du} + \frac{dy}{du} + y = 0$$

Simplifying:

$$\frac{d^2y}{du^2} + y = 0$$

Which is now in the form we are used to (but in u rather than x)

Solve as normal:

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

General solution:

$$y = A\cos u + B\sin u$$

Since $u = \ln x$:

$$y = A\cos(\ln x) + B\sin(\ln x)$$

Diagnostic Question

Given that $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ where $x = e^u$

Find the general solution to:

$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$, (simplify fully, giving y in terms of x)

Y $y = Ae^{-x} + Be^{4x}$

M $y = Ae^{-u} + Be^{4u}$

C $y = \frac{A}{x} + Bx^4$

A $y = -Ax + 4Bx$

Thanks

See you in the Tutorial!

Further Reading: Extra Non-Examinable Content

Constant Coefficient Second Order Linear ODEs (Inhomogeneous)

Variation of Parameters Method

Variation of Parameters Method

The inhomogeneous case is given by:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \qquad \text{where } f(x) \neq 0$$

- The method of undetermined coefficients essentially makes finding a particular solution into an algebra exercise, which can get messy.

- Undetermined coefficients also only works for a fairly small class of functions for $f(x)$.

- Variation of parameters is a more general and more widely applicable method.

- It does however have two potential problems:

1. The complementary solution is required to apply the method.
2. The method requires working out integrals, which may not be possible.

- Therefore, whilst it will always be possible to write down a formula for the particular solution, we may not be able to actually find it if the integrals are too difficult or if we are unable to find the complementary solution.

Variation of Parameters: Derivation (Quick Version)

Remember that:

If $y_1(x)$ and $y_2(x)$ are any two linearly independent solutions to $ay'' + by' + cy = 0$, then $y_c = Ay_1(x) + By_2(x)$. For example: $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$

We attempt a particular solution of the form $y_p = uy_1 + vy_2$ where $u = u(x)$ and $v = v(x)$.

The first derivative is: $y_p' = u'y_1 + uy_1' + v'y_2 + vy_2{}'$ (by product rule)

The second derivative is:

$$y_p'' = (u''y_1 + u'y_1') + (u'y_1' + uy_1'') + (v''y_2 + v'y_2') + (v'y_2' + vy_2{}'')$$

Sub into $ay'' + by' + cy = f(x)$ and group terms:

$$u(ay_1'' + by_1' + cy_1) + v(ay_2'' + by_2' + cy_2) + a(u'y_1' + vy_2{}') + a(u'y_1 + v'y_2)' + b(u'y_1 + v'u_2) = f(x)$$

As y_1 and y_2 are solutions of $ay'' + by' + cy = 0$, the first two terms are zero.

To simplify the problem, we suppose from the beginning that $u'y_1 + v'y_2 = 0$ (this may seem dodgy but it is mathematically sound, however, the formal proof is very long so is not included).

In order to obtain y_p, it is therefore sufficient to choose u and v such that:

$$u'y_1 + v'y_2 = 0$$

$$u'y_1' + v'y_2' = \frac{f(x)}{a}$$

(where a is the coefficient of y'')

Note: For this method, there is no multiplication by x if $f(x)$ appears in y_c

Variation of Parameters: Summary

$y_c = Ay_1(x) + By_2(x)$ where y_1 and y_2 are expressions that appear in y_c.

The particular solution is of the form: $y_p = uy_1 + vy_2$ where $u = u(x)$ and $v = v(x)$.

To find y_p, we find u and v from the conditions:

$$u'y_1 + v'y_2 = 0$$

$$u'y_1' + v'y_2' = \frac{f(x)}{a}$$

(where a is the coefficient of y'')

Solve the 2 simultaneous equations for u' and v' and integrate (dropping constants) to find u and v.

It turns out that if we include constants we just end up with an extra $Ay_1 + By_2$ which is another solution to the homogeneous equation and is therefore zero, so we can just assume all constants are zero to save time!

So $y_p = uy_1 + vy_2$ and then $y = y_c + y_p$.

Variation of Parameters: Example

E.g. Solve the differential equation: $y'' + y = \cot x$

Complementary equation is $y'' + y = 0$, hence auxiliary equation is $m^2 + 1 = 0$ with roots $\pm i$.

$$\therefore \boldsymbol{y_c = A\cos x + B\sin x}$$

Particular solution is of the form $\boldsymbol{y_p = uy_1 + vy_2}$

Since, $y_1 = \cos x$ and $y_2 = \sin x$ $\quad y_p = u\cos x + v\sin x$

And:
$$\boldsymbol{u'y_1 + v'y_2 = 0}$$
$$\boldsymbol{u'y_1' + v'y_2' = \frac{f(x)}{a}}$$
$$\Rightarrow$$
$$u'\cos x + v'\sin x = 0 \qquad (1)$$
$$-u'\sin x + v'\cos x = \cot x \qquad (2)$$
$$(y_1' = -\sin x, \qquad y_2' = \cos x, \qquad a = 1)$$

Variation of Parameters: Example

E.g. Solve the differential equation: $y'' + y = \cot x$

$$u' \cos x + v' \sin x = 0 \qquad (1)$$

$$-u' \sin x + v' \cos x = \cot x \qquad (2)$$

Solve for u' and v':

Divide (1) by $\cos x$: $u' + v' \tan x = 0$

$$u' = -v' \tan x$$

Sub in (2) : $v' \tan x \sin x + v' \cos x = \cot x$

$$\left(\frac{\sin x}{\cos x}\sin x + \cos x\right)v' = \frac{\cos x}{\sin x}$$

$\times \cos x$:

$$(\sin^2 x + \cos^2 x)v' = \frac{\cos^2 x}{\sin x}$$

$$v' = \frac{1 - \sin^2 x}{\sin x}$$

$$\boldsymbol{v' = \mathrm{cosec}\, x - \sin x}$$

$$u' = -v' \tan x = -(\mathrm{cosec}\, x - \sin x) \tan x$$

$$= \left(\sin x - \frac{1}{\sin x}\right)\frac{\sin x}{\cos x}$$

$$= \frac{\sin^2 x - 1}{\cos x}$$

$$= -\frac{(1 - \sin^2 x)}{\cos x}$$

$$= -\frac{\cos^2 x}{\cos x}$$

$$\boldsymbol{u' = -\cos x}$$

Variation of Parameters: Example

E.g. Solve the differential equation: $y'' + y = \cot x$

Integrate (ignore constants) to find u and v:

$$u' = -\cos x$$

$$u = -\int \cos x \, dx$$

$$\boldsymbol{u = -\sin x}$$

$$v' = \operatorname{cosec} x - \sin x$$

$$v = \int \operatorname{cosec} x \, dx - \int \sin x \, dx$$

$$\boldsymbol{v = \ln|\operatorname{cosec} x - \cot x| + \cos x}$$

(using standard integrals)

From before: $\boldsymbol{y_p = uy_1 + vy_2} = u\cos x + v\sin x$

$$\therefore y_p = -\sin x \cos x + \sin x \ln|\operatorname{cosec} x - \cot x| + \sin x \cos x$$

$$\boldsymbol{y_p = \sin x \ln|\operatorname{cosec} x - \cot x|}$$

Finally, $y = y_c + y_p$:

$$\boldsymbol{y = C_1 \cos x + C_2 \sin x + \sin x \ln|\operatorname{cosec} x - \cot x|}$$

Diagnostic Question

The complementary solution to a second order differential equation is

$$y_c = Ae^{-2x} + Be^{-3x}$$

Which is the correct y_p to find by **variation of parameters**?

Y $y_p = uAe^{-2x} + vBe^{-3x}$

M $y_p = ue^{2x} + ve^{3x}$

C $y_p = ue^{-2x} + ve^{-3x}$

A $y_p = u'e^{-2x} + v'e^{-3x}$

Diagnostic Question

The complementary solution to a second order differential equation is

$$y_c = (A + Bx)e^{5x}$$

Which is the correct y_p to find by **variation of parameters**?

Y $\quad y_p = ue^{5x} + ve^{5x}$

M $\quad y_p = ue^{5x} + vxe^{5x}$

C $\quad y_p = ux + ve^{5x}$

A $\quad y_p = uxe^{5x} + vxe^{5x}$

Diagnostic Question

The complementary solution to a second order differential equation is

$$y_c = A + Be^{-x}$$

Which is the correct y_p to find by **variation of parameters**?

Y $y_p = u + ve^{-x}$

M $y_p = ve^{-x}$

C $y_p = ue^{-x} + ve^{-x}$

A $y_p = uA + ve^{-x}$

Diagnostic Question

The complementary solution to a second order differential equation with $y'' + 5y' + y = e^{-3x}$ is

$$y_c = Ae^{-2x} + Be^{-3x}$$

Which is the system of equations to be solved for u' and v'?

Y

$$ue^{-2x} + ve^{-3x} = 0$$
$$u'e^{-2x} + v'e^{-3x} = e^{-3x}$$

M

$$u'e^{-2x} + v'e^{-3x} = e^{-3x}$$
$$-2u'e^{-2x} - 3v'e^{-3x} = 0$$

C

$$ue^{-2x} + ve^{-3x} = 0$$
$$-2u'e^{-2x} - 3v'e^{-3x} = e^{-3x}$$

A

$$u'e^{-2x} + v'e^{-3x} = 0$$
$$-2u'e^{-2x} - 3v'e^{-3x} = e^{-3x}$$

Appendix: Proof - Repeated Roots (Credit: "Paul's Online Notes")

In this case, since we have double roots we must have

$$b^2 - 4ac = 0$$

This is the only way that we can get double roots and in this case the roots will be

$$r_{1,2} = \frac{-b}{2a}$$

So, the one solution that we've got is

$$y_1(t) = \mathbf{e}^{-\frac{bt}{2a}}$$

To find a second solution we will use the fact that a constant times a solution to a linear homogeneous differential equation is also a solution. If this is true then *maybe* we'll get lucky and the following will also be a solution

$$y_2(t) = v(t)\,y_1(t) = v(t)\,\mathbf{e}^{-\frac{bt}{2a}} \qquad (1)$$

with a proper choice of $v(t)$. To determine if this in fact can be done, let's plug this back into the differential equation and see what we get. We'll first need a couple of derivatives.

$$\begin{aligned} y'_2(t) &= v'\,\mathbf{e}^{-\frac{bt}{2a}} - \frac{b}{2a} v\,\mathbf{e}^{-\frac{bt}{2a}} \\ y''_2(t) &= v''\,\mathbf{e}^{-\frac{bt}{2a}} - \frac{b}{2a} v'\,\mathbf{e}^{-\frac{bt}{2a}} - \frac{b}{2a} v'\,\mathbf{e}^{-\frac{bt}{2a}} + \frac{b^2}{4a^2} v\,\mathbf{e}^{-\frac{bt}{2a}} \\ &= v''\,\mathbf{e}^{-\frac{bt}{2a}} - \frac{b}{a} v'\,\mathbf{e}^{-\frac{bt}{2a}} + \frac{b^2}{4a^2} v\,\mathbf{e}^{-\frac{bt}{2a}} \end{aligned}$$

We dropped the (t) part on the v to simplify things a little for the writing out of the derivatives. Now, plug these into the differential equation.

$$a\left(v''\,\mathbf{e}^{-\frac{bt}{2a}} - \frac{b}{a} v'\,\mathbf{e}^{-\frac{bt}{2a}} + \frac{b^2}{4a^2} v\,\mathbf{e}^{-\frac{bt}{2a}}\right) + b\left(v'\,\mathbf{e}^{-\frac{bt}{2a}} - \frac{b}{2a} v\,\mathbf{e}^{-\frac{bt}{2a}}\right) + c\left(v\,\mathbf{e}^{-\frac{bt}{2a}}\right) = 0$$

Appendix: Proof - Repeated Roots (Credit: "Paul's Online Notes")

We can factor an exponential out of all the terms so let's do that. We'll also collect all the coefficients of v and its derivatives.

$$\mathbf{e}^{-\frac{bt}{2a}}\left(av'' + (-b+b)\,v' + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v\right) = 0$$

$$\mathbf{e}^{-\frac{bt}{2a}}\left(av'' + \left(-\frac{b^2}{4a} + c\right)v\right) = 0$$

$$\mathbf{e}^{-\frac{bt}{2a}}\left(av'' - \frac{1}{4a}\left(b^2 - 4ac\right)v\right) = 0$$

Now, because we are working with a double root we know that that the second term will be zero. Also exponentials are never zero. Therefore, (1) will be a solution to the differential equation provided $v(t)$ is a function that satisfies the following differential equation.

$$av'' = 0 \quad \text{OR} \quad v'' = 0$$

We can drop the a because we know that it can't be zero. If it were we wouldn't have a second order differential equation! So, we can now determine the most general possible form that is allowable for $v(t)$.

$$v' = \int v''\,dt = c \qquad v(t) = \int v'\,dt = ct + k$$

Appendix: Proof - Repeated Roots (Credit: "Paul's Online Notes")

This is actually more complicated than we need and in fact we can drop both of the constants from this. To see why this is let's go ahead and use this to get the second solution. The two solutions are then

$$y_1(t) = \mathbf{e}^{-\frac{b\,t}{2a}} \qquad y_2(t) = (ct + k)\,\mathbf{e}^{-\frac{b\,t}{2a}}$$

Eventually you will be able to show that these two solutions are "nice enough" to form a general solution. The general solution would then be the following.

$$\begin{aligned} y(t) &= c_1\mathbf{e}^{-\frac{b\,t}{2a}} + c_2(ct + k)\,\mathbf{e}^{-\frac{b\,t}{2a}} \\ &= c_1\mathbf{e}^{-\frac{b\,t}{2a}} + (c_2ct + c_2k)\,\mathbf{e}^{-\frac{b\,t}{2a}} \\ &= (c_1 + c_2k)\,\mathbf{e}^{-\frac{b\,t}{2a}} + c_2c\,t\,\mathbf{e}^{-\frac{b\,t}{2a}} \end{aligned}$$

Notice that we rearranged things a little. Now, c, k, c_1, and c_2 are all unknown constants so any combination of them will also be unknown constants. In particular, $c_1 + c_2k$ and c_2c are unknown constants so we'll just rewrite them as follows.

$$y(t) = c_1\mathbf{e}^{-\frac{b\,t}{2a}} + c_2\,t\,\mathbf{e}^{-\frac{b\,t}{2a}}$$

So, if we go back to the most general form for $v(t)$ we can take $c = 1$ and $k = 0$ and we will arrive at the same general solution.

Let's recap. If the roots of the characteristic equation are $r_1 = r_2 = r$, then the general solution is then

$$y(t) = c_1\mathbf{e}^{r\,t} + c_2t\mathbf{e}^{r\,t}$$

Link to page:

https://tutorial.math.lamar.edu/Classes/DE/RepeatedRoots.aspx