5.5 Binomial Theorem

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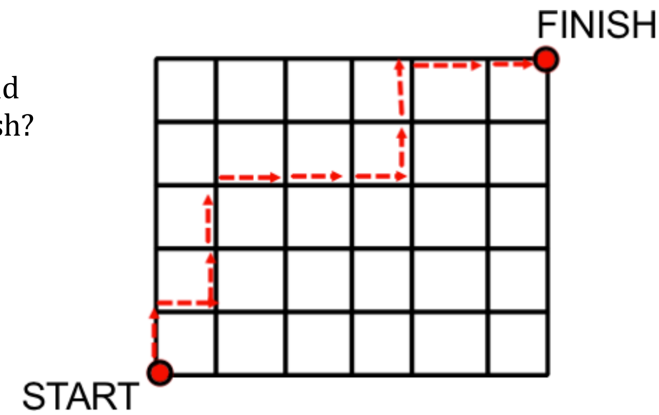
Part 1: Routes Through a Map Grid

Example 1:

City streets make up a grid, as shown. Travelling east and north, how many possible routes lead from start to finish?



One possible solutions is:

NENNEEENNEE

You must move 6 streets east and 5 streets north to get from start to finish. Therefore, every possible route requires 11 steps. Of these steps, 6 will move you east and 5 will move you north.

There are two possible solutions to determine the number of possible routes:

1) As a combination

1a) Determine the number of ways East can be inserted into 6 positions selected from 11 available positions.

$\binom{11}{6} = 462$

1b) Determine the number of ways North can inserted into 5 positions selected from 11 available positions.

$\binom{11}{5} = 462$

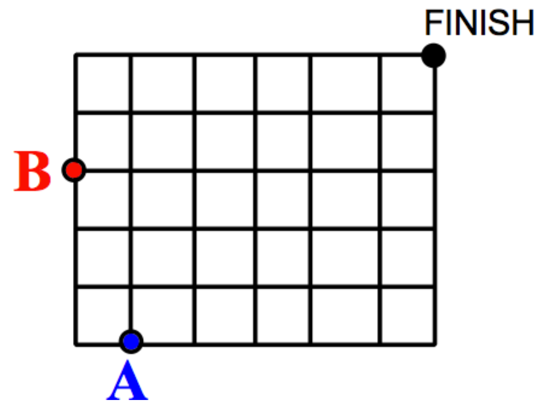
2) As a permutation

Think of NNNNNEEEEEE as a word with 11 letters. Of the 11 letters, 5 are identical N's and 6 are identical E's. The number of routes is equivalent to the number of different arrangements of the letters.

$\frac{11!}{5!6!} = 462$

Example 2:

City streets make up a grid, as shown. Travelling east and north, how many possible routes lead from A **or** B to the finish?



From A: You must travel 5 North and 5 East; therefore the number of routes from A to finish =

$\binom{10}{5} = 252$ OR $\frac{10!}{5!5!} = 252$

From B: You must travel 2 North and 6 East; therefore the number of routes from B to finish=

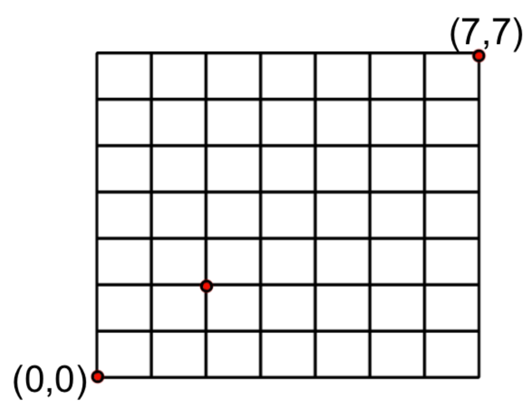
$\binom{8}{2} = 28$ OR $\binom{8}{6} = 28$ OR $\frac{8!}{2!6!} = 28$

Number of possible routes lead from A **or** B to the finish =

$n(A \cup B) = n(A) + n(B) = 252 + 28 = 280$

Example 3:

a) Starting at (0, 0) and moving only North and East, how many routes pass through (2, 2) and end at (7, 7) ?



Solution:

The routes that lead to (2, 2) from (0, 0) must each contain 2 north and 2 east movements.

$\binom{4}{2} = 6$ OR $\frac{4!}{2!2!} = 6$

The routes from (2, 2) to (7, 7) must contain 5 north and 5 east movements.

$\binom{10}{5} = 252$ OR $\frac{10!}{5!5!} = 252$

The total number of routes that pass through (2, 2) and end at (7, 7) is:

= number of routes from (0, 0) to (2, 2) × number of routes from (2, 2) to (7, 7)

$=6\times252$

$=1512$

b) Starting at (0, 0) and moving only North and East, how many routes <u>avoid</u> (2, 2) and end at (7, 7) ?

$= routes\ from\ (0{,}0)\ to\ (7,7) - routes\ that\ pass\ through\ (2,2)$

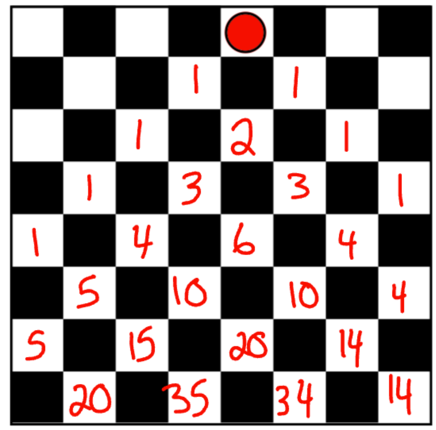
$= \binom{14}{7} - 1512$

$= 3432 - 1512$

$= 1920$

Example 4: A checker is placed on a game board. Determine the number of paths the checker may take to get to each allowable square on the board if it can move only diagonally forward one square at a time.

Notice the pattern resembles Pascal's Triangle



<u>Part 2: Binomial Theorem</u>

Blaise Pascal (French Mathematician) discovered a pattern in the expansion of $(a + b)^n$ ....

$(a + b)^0 = 1$

$(a + b)^1 = 1a + 1b$

$(a + b)^2 = 1a^2 + 2ab + 1b^2$

$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

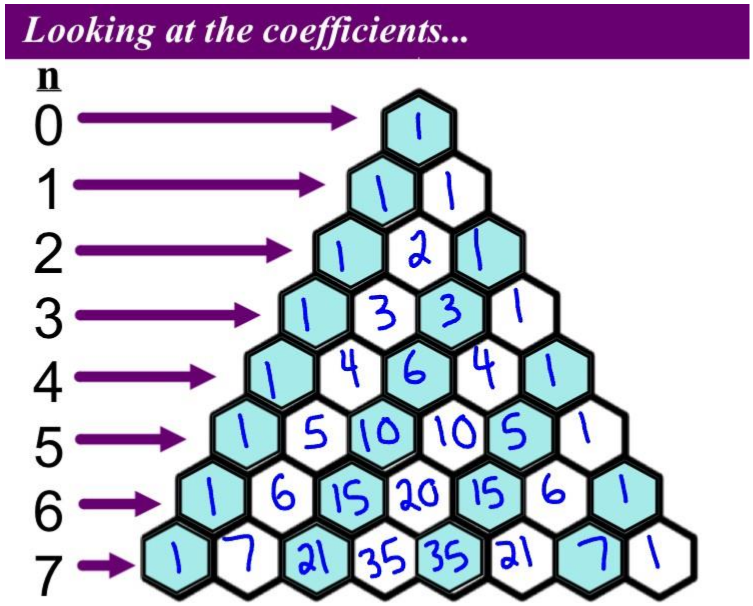
$(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

$(a + b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

Notice the exponents on the variables form a predictable pattern. The exponents of each term always sum to *n.* The exponents on a decrease by 1 and the exponents on b increase by 1.

Pascal's triangle can be used to determine the <u>coefficients</u> of each of the terms. There are many useful patterns in Pascal's Triangle but the main pattern used to complete the triangle is:

- the outside number is always 1
- a number inside a row is the sum of the two numbers above it



The coefficients in the expansions can be determined using a strategy very similar to the one used previously to analyze paths through a map grid.

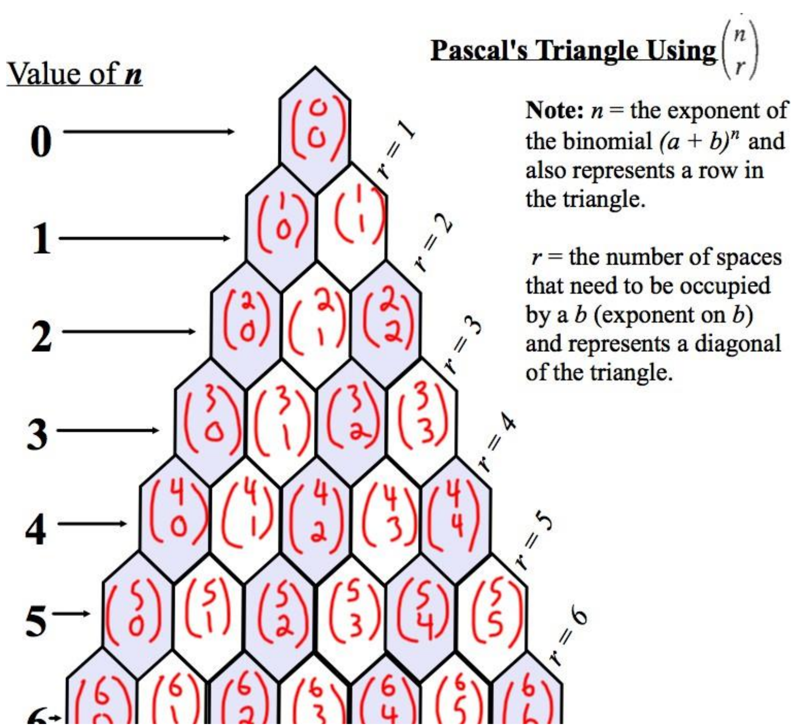
Consider the term that includes a^3b in the expansion of $(a+b)^4$. It is the result of multiplying the a---term from three of the factors with the b---term from the remaining factor.

There are four ways this can be done. This is why the coefficient of the term is 4.

a	a	a	b
a	a	b	a
a	b	a	a
b	a	a	a

There are four available spaces to record the a's and b's. For the term a^3b, one space has to be occupied by a b. Therefore, there are C(4, 1) ways of doing this. [C(4, 3) is equivalent]

Therefore, the coefficients of the general binomial expansion $(a+b)^n$ can be represented by Pascal's Triangle in terms of the combination formula.



We can generalize the results of the expansions of $(a + b)^n$ with the Binomial Theorem:

Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

The coefficients of the form $\binom{n}{r}$ are called binomial coefficients.

Notice that each term in the expansion of a binomial $= \binom{n}{r} a^{n!r} b^r$ where r is 0 for the first term and increases by 1 each term until r is equal to n.

Example 5:

Expand using the Binomial Theorem:

a) $(a + b)^6$

$$= \binom{6}{0} a^{6!0}b^0 + \binom{6}{1} a^{6!1}b^1 + \binom{6}{2} a^{6!2}b^2 + \binom{6}{3} a^{6!3}b^3 + \binom{6}{4} a^{6!4}b^4 + \binom{6}{5} a^{6!5}b^5 + \binom{6}{6} a^{6!6}b^6$$

$$= \binom{6}{0} a^6 + \binom{6}{1} a^5b^1 + \binom{6}{2} a^4b^2 + \binom{6}{3} a^3b^3 + \binom{6}{4} a^2b^4 + \binom{6}{5} a^1b^5 + \binom{6}{6} b^6$$

$$= 1a^6 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + 1b^6$$

b) $(2x - 3)^5$

$$= \binom{5}{0}(2x)^{5!0}(-3)^0 + \binom{5}{1}(2x)^{5!1}(-3)^1 + \binom{5}{2}(2x)^{5!2}(-3)^2 + \binom{5}{3}(2x)^{5!3}(-3)^3 + \binom{5}{4}(2x)^{5!4}(-3)^4 + \binom{5}{5}(2x)^{5!5}(-3)^5$$

$$= \binom{5}{0}(2x)^5(-3)^0 + \binom{5}{1}(2x)^4(-3)^1 + \binom{5}{2}(2x)^3(-3)^2 + \binom{5}{3}(2x)^2(-3)^3 + \binom{5}{4}(2x)^1(-3)^4 + \binom{5}{5}(2x)^0(-3)^5$$

$$= 1(32)(x^5)(1) + 5(16)(x^4)(-3) + 10(8)(x^3)(9) + 10(4)(x^2)(-27) + 5(2)(x)(81) + 1(1)(-243)$$

$$= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$$

Part 3: General Term in Expansion of Binomial

The general term in the expansion of $(a+b)^n$ is:

$t_{r+1} = \binom{n}{r} a^{n!r} b^r$

This formula can be used to determine the $(r+1)^{st}$ term in a binomial expansion

Note: for the 1^{st} term, $r = 0$
2^{nd} term, $r = 1$
3^{rd} term, $r = 2$

$$\boxed{t_{r+1} = \binom{n}{r} a^{n-r} b^r}$$

Example 6:

What is the 5^{th} term in the following binomial?

$(x^2 - 2)^6$

$t_{4+1} = \binom{6}{4} (x^2)^{6!4} (-2)^4$

$t_5 = 15(x^2)^2(-2)^4$

$t_5 = 15x^4(16)$

$t_5 = 240x^4$

Example 7:

What is the 3^{rd} term in the binomial $\left(x + \frac{1}{x}\right)^8$

$t_{2+1} = \binom{8}{2} (x)^{8!2} \left(\frac{1}{x}\right)^2$

$t_3 = 28(x)^6 \left(\frac{1^2}{x^2}\right)$

$t_3 = 28(x)^6 \left(\frac{1}{x^2}\right)$

$t_3 = 28(x)^6(1)(x)^{!2}$

$t_3 = 28x^4$

Example 8: Determine the constant term in the expansion of $\left(x + \frac{1}{x}\right)^{10}$

Note: *A constant term is a term with no variables. For this to happen, the exponent on the variables must be zero.*

$t_{r+1} = \binom{10}{r}(x)^{10!r}\left(\frac{1}{x}\right)^{r}$

$t_{r+1} = \binom{10}{r}(x)^{10!r}\left(\frac{1^{r}}{x^{r}}\right)$

$t_{r+1} = \binom{10}{r}(x)^{10!r}(1)(x)^{!r}$

$t_{r+1} = \binom{10}{r}(x)^{10!2r}$

The exponent on x must be zero, therefore:

$10 - 2r = 0$
$r = 5$

Therefore, the constant term is:

$t_{5+1} = \binom{10}{5}(x)^{10!2(5)}$

$t_6 = \binom{10}{5}(x)^{0}$

$t_6 = 252$

Example 9: In the expansion of $\left(x^3 + \frac{3}{x}\right)^{8}$, find…

a) the number of terms in the expansion

$n + 1 = 8 + 1 = 9$

b) the 4th term

$t_{3+1} = \binom{8}{3}(x^3)^{8!3}\left(\frac{3}{x}\right)^{3}$

$t_4 = 56(x^3)^{5}\left(\frac{3^3}{x^3}\right)$

$t_4 = 56(x)^{15}(27)(x)^{!3}$

$t_4 = 1512x^{12}$

c) the constant term

$t_{r+1} = \binom{8}{r}(x^3)^{8!r}\left(\frac{3}{x}\right)^r$

$t_{r+1} = \binom{8}{r}(x)^{24!3r}\left(\frac{3^r}{x^r}\right)$

$t_{r+1} = \binom{8}{r}(x)^{24!3r}(3)^r(x)^{!r}$

$t_{r+1} = \binom{8}{r}(3)^r(x)^{24!4r}$

The exponent on x must be zero, therefore:

$24 - 4r = 0$
$r = 6$

Therefore, the constant term is:

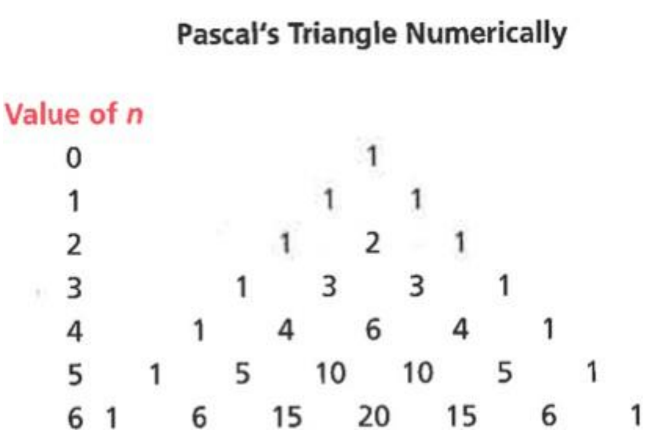
$t_{6+1} = \binom{8}{6}(3)^6(x)^{24!4(6)}$

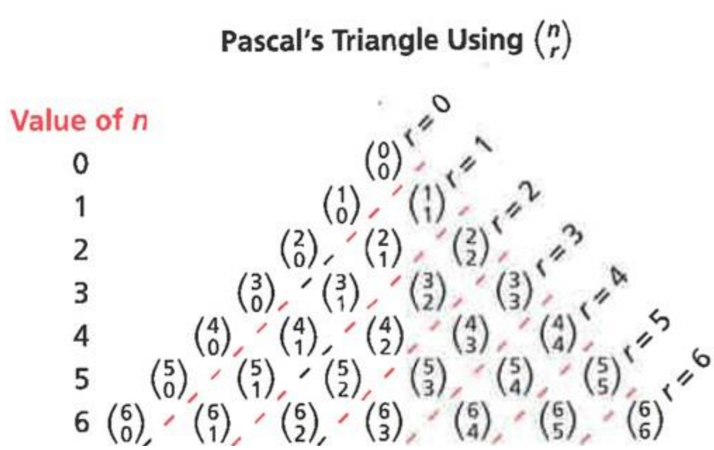
$t_7 = 28(729)(x)^0$

$t_7 = 20\ 412$

Part 5: Pascal's Identity

Perhaps the most famous pattern in Pascal's Triangle stems from the relationship between the sum of consecutive values in one row and the value found in the next row immediately beneath.





This diagram suggests that $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$

Pascal's Identity:

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Verify Pascal's Identity using n = 6, and r = 4

$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$

$15 + 6 = 21$

$21 = 21$

Example 10: Rewrite each of the following using Pascal's Identity

a) $\binom{11}{7} + \binom{11}{8}$

$= \binom{12}{8}$

b) $\binom{19}{5} + \binom{19}{6}$

$= \binom{20}{6}$