Unit: Organized Counting

$$C_{(n,r)} = \frac{n!}{r!\,(n-r)!}$$

$$P_{(n,r)} = \frac{n!}{(n-r)!}$$

n = set size: the total number of items in the sample

r = subset size: the number of items to be selected from the sample

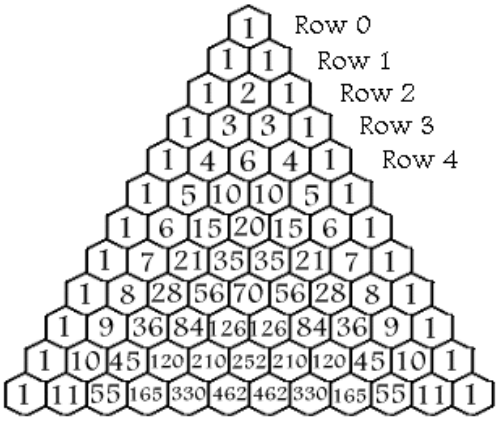
Unit Overview:

- Factorial
- Permutation
  - Rule of sum: additive counting principle – "OR"
  - Fundamental counting principle – "AND"
  - Line arrangement
  - Circular arrangement
  - Direct vs. Indirect Methods
- Combination
  - Rule of sum: additive counting principle – "OR"
  - Fundamental counting principle – "AND"
  - Direct vs. Indirect Methods
- Pascal's Triangle
  - Pascal's Triangle
  - Application of Pascal's Triangle on paths
  - Application of Pascal's Triangle on checkerboard
- Sets and Subsets
  - Venn Diagrams
  - Notation

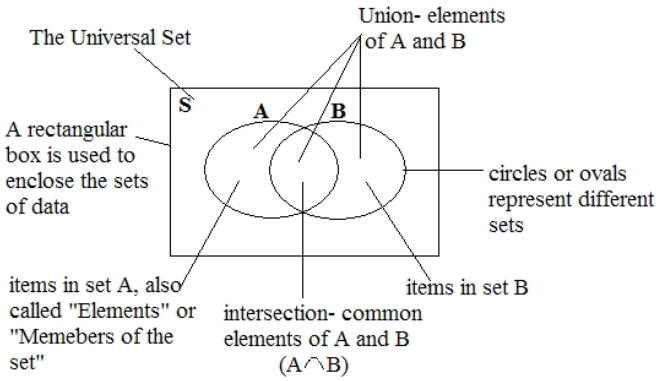
The Binomial Theorem can be stated as:

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2}a^{n-2}b^2 + \ldots\ldots + b^n$$

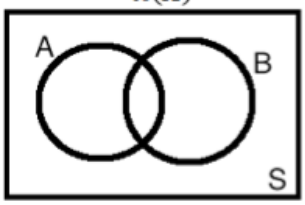
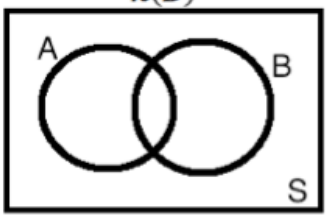
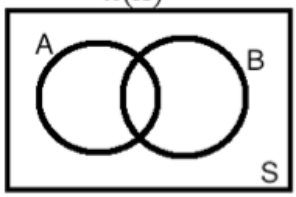
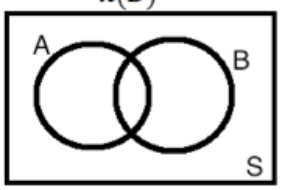
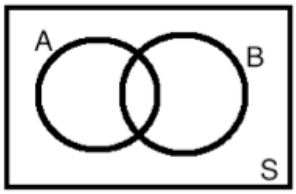
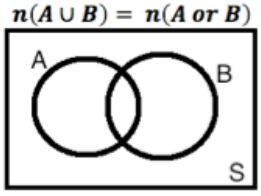
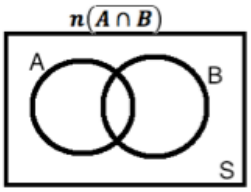
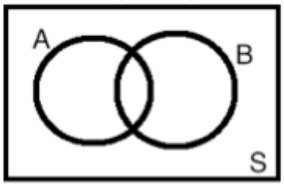
The co-efficients generated by expanding binomials of the form $(a+b)^n$ can be shown in the form of a symmetrical triangle:



A Venn Diagram:

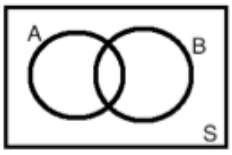
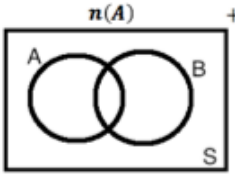
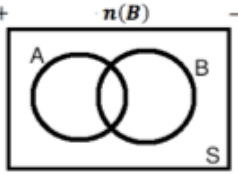
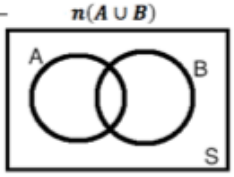


Lesson: Venn Diagrams

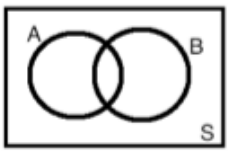
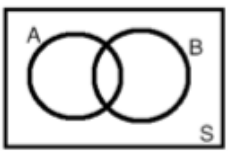
S = Universal Set $A, B, C, \ldots$ = subsets of the universal set $n(S)$ = number of elements in the universal set 	$n(A)$ = number of elements in subset "A" $n(A)$ 	$n(B)$ 
$n(\overline{A})$ = complement of A (anything but A) $n(\overline{A})$ 	$n(\overline{B})$ = complement of B (anything but B) $n(\overline{B})$ 	$n(A \cap B) = n(A \text{ and } B)$ $n(A \cap B) = n(A \text{ and } B)$ 
$n(A \cup B) = n(A \text{ or } B)$ $n(A \cup B) = n(A \text{ or } B)$ 	$n(\overline{A \cap B})$ = complement of A and B $n(\overline{A \cap B})$ 	$n(\overline{A \cup B})$ = complement of A or B $n(\overline{A \cup B})$ 

Principle of Inclusion or Exclusion:

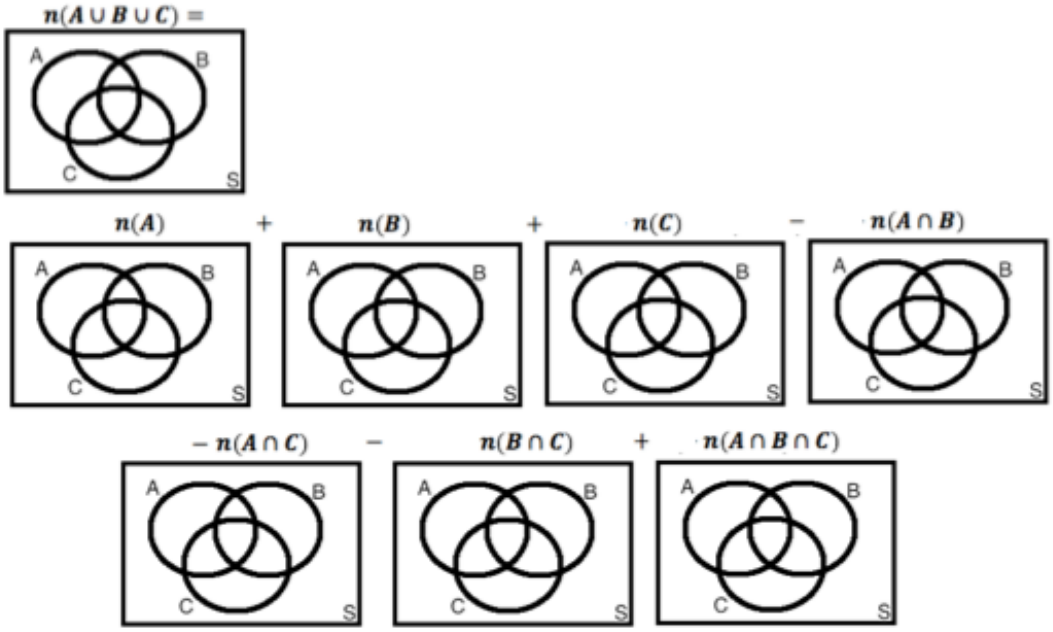
$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

$n(A \cap B) =$  $n(A)$  $+$ $n(B)$  $-$ $n(A \cup B)$ 

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$n(A \cup B) =$  $n(A)$  $+$ $n(B)$  $-$ $n(A \cap B)$ 

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$



Practice:

You are given a deck of cards. Complete the following Venn diagrams by indicating how many elements are in each of the sets and subsets.

1. S = the whole deck of cards A = the black cards B = the red cards	2. S = the whole deck of cards A = the black cards B = the hearts C = the diamonds	3. S = the red cards A = the even cards B = the face cards
4. S = the red cards A = the even cards B = the 2s and 4s	5. S = the whole deck of cards A = the face cards B = the spades C = the even numbers	6. S = the whole deck of cards A = Ace cards B = odd cards C = the hearts

Principle of Inclusion or Exclusion:

For two sets A and B:

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

The notation $n(X)$ means the number of element in X.

Example 1:

Hollywood High School has about 1 800 students. There are about 300 Grade 12 students. 800 students are taking math. There are about 125 Grade 12 students taking math.

a) Make a Venn Diagram showing this relationship
b) How many students in Grade 12 are not taking math?
c) How many students in Grade 9, 10, and 11 are taking math?
d) How many students in Grade 9, 10, and 11 are not taking math?

If there are three overlapping sets, A, B, and C, the counting of elements in the union of the three sets becomes a little more difficult.

For two sets A and B:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Example 2:

The Hollywood School Service Department wants to count the number of students in Grade 12. They know that every student is taking Math, English, or Acting. They found that:

- 64 students are taking Math
- 56 students are taking English
- 82 students are taking Acting
- 20 students are taking Math and English
- 25 students are taking Math and Acting
- 21 students are taking English and Acting
- 12 students are taking all three courses

Use both methods, determine the number of students in Grade 12.

Venn Diagram Problems

1. The Swiss embassy in Ottawa has 65 employees. Of these workers, 47 speak German, 35 speak Italian, and 20 speak both German & Italian. How many embassy employees speak neither German nor Italian? Illustrate the situation with a Venn Diagram.

2. A survey of television viewers at "A Child's Place" preschool produces the following data:
   60% watch Sesame Street
   50% watch Captain Kangaroo
   50% watch Polka Dot Door
   30% watch Sesame Street and Captain Kangaroo
   20% watch Captain Kangaroo and Polka Dot Door
   30% watch Sesame Street and Polka Dot Door
   10% watch all three shows
   (a) What percentage view at least one of these programs?
   (b) What percentage view none of the shows?
   (c) What percentage view Sesame Street and Captain Kangaroo but not Polka Dot Door?
   (d) What percentage view exactly two of these programs?

3. Of 1400 students at Tomlintown High, 800 attended the first school dance of the year. The music was not good so only 500 attended the next dance. If 300 attended both dances, how many did not go to either event?

4. The 29 students of Mr. Vicker's class use a variety of forms of transportation to get to school. Twenty of them sometimes arrive at school in a car; 12 bicycle to school at least occasionally; 16 take the bus some days. If four students use all three of these options, six either bicycle or take the bus, ten come by car or bike, and nine arrive by car when they do not come on the bus, how many always use some other type of transportation?

5. In a recent election poll of 193 people, the following information was collected:
   140 of those polled were professionals; 84 were under 30 years of age; 133 voted Conservative in the last election; 56 were professionals under 30; 41 of those under 30 voted Conservative; 111 professionals voted Conservative; 36 of the professionals under 30 voted Conservative.
   Of those polled, how many non-professionals aged 30 or over did not vote Conservative?

6. In Mrs. Paul's Music Class, students learn to play only the clarinet, guitar, saxophone, and trombone. So far this term, no student can play all four; the one who plays three cannot play sax but all those who do play sax also play clarinet. If there are 36 students in the class and the following chart shows the skills of the students, how many students still cannot play any instrument?

Saxophone	Clarinet	Guitar	Trombone	Clarinet and Guitar	Clarinet and Trombone	Guitar and Trombone
2	15	16	18	4	5	7

Answers: 1)3 2a)90 b)10 c)20 d)50 3)400 4)2 5)8 6)2

Lesson: Factorial

Factorial Notation (!)

In Combinatory, we often multiply consecutive natural numbers such as the following:

$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \ =$ _____

We define for any natural factorial number n,

$$n! = n\,(n-1)(n-2)(n-3)\ldots \times 3 \times 2 \times 1$$

and

$$0! = 1$$

Example 1: Evaluate and simplify.

(a) 4!

(b) 1!

(c) 2!

(d) 17!

(e) $7 \cdot 5!$

(f) $\frac{7!}{4!}$

(g) $\frac{17!}{15!2!}$

Expand and simplify each of factorial with variables

(h) $(n+3)!$

(i) $\frac{(n+2)(n+1)!}{(n-1)!}$

Example 2: Solve for n, $n \in \mathbb{W}$

(a) $\frac{(n-1)!}{n!} = \frac{1}{2}$

(b) $\frac{(n+1)!}{n!} = 9$

(c) $\frac{3(n+1)!}{(n-1)!} = 126$

[in each of the following questions, $n, r \in N, n \geq r$]

1. Calculate the value of each of the following.
   a) $\frac{8!}{6!}$ b) $\frac{6!}{3!3!}$ c) $\frac{20!}{19!}$
   d) $\frac{8!}{5!3!}$ e) $\frac{7!}{5!2!}+\frac{7!}{3!4!}$ f) $\frac{20!}{17!3!}$
2. Expressing each of the following as a single factorial expression.
   a) $(n+1)n!$ b) $\frac{(n+7)!}{n+7}$ c) $(n-r+1)(n-r)!$
3. Simplify the following as far as possible.
   a) $\frac{(n+5)!}{(n+3)!}$ b) $\frac{n!}{(n-1)!}$ c) $\frac{(n+1)!}{n!}$
   d) $\frac{(n-r+1)!}{(n-r)(n-r-1)!}$ e) $\frac{(n-r)!}{(n-r-1)!}$ f) $\frac{(n-r+1)!}{(n-r-1)!}$
4. Solve the following equations.
   a) $\frac{(n+5)!}{(n+4)!}=7$ b) $\frac{(n+2)!}{n!}=20$ c) $\frac{n!}{(n-2)!}=30$
   d) $\frac{n!}{2(n-2)!}=6$ e) $\frac{(n+1)!}{(n-1)!}=12$ f) $\frac{(n-1)!}{(n-3)!}=42$
5. Find the value for each expression.
   a) $\frac{14!}{13!}$ b) $\frac{52!}{51!}$ c) $\frac{101!}{99!}$ d) $20(19!)$ e) $90(8!)$ f) $30(4!)$ g) $\frac{8!}{5!}$
   h) $\frac{19!}{13!}$ i) $\frac{21!}{17!4!}$ j) $\frac{9!}{7!2!}$ k) $\frac{155!}{152!}$ l) $\frac{93!}{89!4!}$ m) $\frac{10!}{5!}$ n) $\frac{21!}{14!}$
   o) $\frac{9!}{3!6!}$ p) $\frac{12!}{8!4!}$ q) $\frac{7!}{2!5!}+\frac{7!}{4!3!}$ r) $\frac{15!}{9!6!}+\frac{15!}{10!5!}$ s) $2\times\frac{5!}{3!2!}$ t) $3\times\frac{11!}{7!4!}$
6. Simplify, $n \in W$.
   a) $n(n-1)!$ b) $n!\,(n+1)$ c) $(n-1)!\,(n^2+n)$
   d) $n!\,(n^2+3n+2)$ e) $\frac{n!}{(n-2)!}$ f) $\frac{(n+2)!}{(n-1)!}$
7. Solve for $n. n \in \mathbb{N}$.
   a) $\frac{(n+1)!}{n!}=9$ b) $\frac{n!}{(n-2)!}=20$ c) $\frac{3(n+1)!}{(n-1)!}=126$ d) $\frac{2n!}{(n-3)!}=84n$

Answers:
1a)56 b)20 c)20 d)56 e)56 f)1140
2a)(n+1)! b)(n+6)! c)(n-r+1)!
3a)(n+5)(n+4) b)n c)n+1 d)n-r+1 e)n-r f)(n-r+1)(n-r)
4a)2 b)3 c)6 d)4 e)3 f)8
5)a)14 b)52 c)10100 d)20! e)10! f)6! g)336 h)19535040 i)5985 j)36 k)3652110 l)2919735 m)30240 n)586051200 o)84 p)495 q)56 r)8008 s)20 t)990
6a)n! b)(n+1)! c)(n+1)! d)(n+2)! e)n(n-1) f)(n+2)(n+1)n
7a)8 b)5 c)6 d)8

Lesson: Introduction to Permutation

Factorial – $n!, n\epsilon\mathbb{N}$

$$n! = n(n-1)(n-2)(n-3) \ldots\ldots \times 3 \times 2 \times 1$$

$$0! = 1$$

i.e., 5! = 120

Warm up: Give digits 5, 2, and 3,

(a) how many ways can you arrange 3-digit numbers?

(b) How many ways can you arrange 2-digit numbers?

Using your calculator……

Permutation:

$$P(n,r) = {}_{n}P_{r}$$

$$= \frac{n!}{(n-r)!}$$

-Rearranging r objects from a total of n objects in a DEFINITE ORDER

i.e., $P(8,2)=\frac{8!}{(8-2)!}$

Using your calculator…..

Examples

(a) $P(8,3)$

(b) $P(6,6)$

(c) $\frac{7!}{5!}$

(d) $\frac{n!}{(n-1)!}$

Recall:

Permutation: ${}_nP_r = \frac{n!}{(n-r)!}$

-Rearranging r objects from a total of n objects in a DEFINITE ORDER

Question	Answer
1. Given 4 distinct objects, in how many ways can you line up… (a) 1 of them?	
(b) 2 of them?	
(c) 3 of them?	
(d) 4 of them?	
2. Given 5 distinct objects, in how many ways can you line up … (a) 1 of them?	
(b) 2 of them?	

Rule of Sum: additive counting principle

- **if one mutually exclusive action can occur in m ways, a second in n ways, a third in p ways and so on, then there are m+n+p… ways in which one of these actions can occur**
- **"OR" – addition**

Examples:

1. How many ways can you pick a 7 from a deck of cards?

2. How many ways can you pick a 7 OR a 3 from a deck of cards?

3. How many ways can you pick a jack OR a queen OR a king?

4. How many ways can you pick a black even number OR a red 6?

Fundamental Counting Principle:

- **If a task or process is made up of stages with separate choices, the total number of choices is $m \times n \times p \times \ldots$ ways**
- **"AND" – multiplication**

Examples:

5. How many ways can you pick a 7 AND a 3?

6. How many ways can you pick a jack AND a queen AND a king?

7. How many ways can you pick a black even number and a red 6?

Complete the following questions, using the **Fundamental Principle of Counting, the Rule of Sum** and your generalization from Parts B and C.

1. Emily wants to get dressed in the morning. She has a choice of 6 tops, 4 bottoms and 3 pairs of socks. How many different outfits can she wear?

2. Michelle is taking math, science, English and geography this semester. In how many ways could her timetable be made?

3. Eric has signed up for 6 different courses this semester. He will only have 4 on his timetable. In how many ways could his timetable be made?

4. A baseball manager has 12 players. He has to set-up a batting order of 9 players. In how many ways can he set up his batting order?

5. Mr. Parent has 15 students in his most excellent Data Management class. He must pick 1 president, 1 vice-president, a secretary and a treasurer. In how many ways may he do this?

6. A hockey coach has to set-up his starting line-up. He has to assign a center, left-wing and right-wing from 9 different forwards and he must assign a left defenseman and right defenseman from 6 different defensemen, and he must assign 1 goalie from 2 goalies. How many ways can he set up his starting line-up?

Lesson: Permutation Involving Distinct Object

PERMUTATIONS

Permutations of n DISTINCT objects taken r at a time is an arrangement of n objects in a definite order. The total of such arrangements is denoted as

$P(n,r)$ or ${}_nP_r$ or "n arrange r" = $\frac{n!}{(n-r)!}$

note! These are examples of "Permutation" since the arrangement in which the song is played as well as definite order is important.

Example 1: A group of students learned 8 songs for a performance at an assembly. In how many different ways can the director arrange these songs if:

(a) all the songs are to be sung (no repetition)

(b) 5 songs are to be sung (no repetition)

Example 2: A boy has 4 differently coloured blocks (R,Y,B,G). He selects one block at time without replacement and sets them out in an order drawn. How many arrangements can be made if:

(a) he selects 4 blocks?

(b) He selects 2 blocks?

(c) He selects 3 out of 4 blocks and one must be yellow?

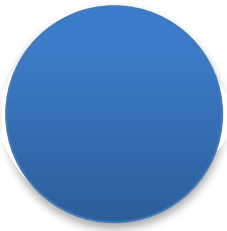
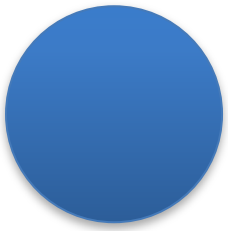
METHOD I: Direct Method	METHOD II: Indirect Reasoning
Case 1: Yellow 1st block Case 2: Yellow 2nd block Case 3: Yellow 3rd block	indirect method = All combination − "not possible case(s)"

(d) he selects 4, but green must be selected 2nd?

(e) If he selects 4, but yellow and blue must be adjacent (side by side)?

Example 3: Circular permutation
3 people are to be seated at a round table for dinner. In how many ways can the people be seated for dinner?

Draw all possible solutions.

Case 1:	Case 2:	Case 3:
		

Formula for circular permutation:

Homework: Permutations of Distinct Elements Worksheet

1. Polygons are usually labelled by placing letters at the vertices.
   a) How many ways are there of labelling a triangle, using any three different letters of the alphabet? [26P3]
   b) How many ways are there of labelling a hexagon, using any six different letters of the alphabet? [26P6]

2. What is the number of four-letter words that can be formed using the letters found in each of the following words?

a) FOUR [24]
b) PANEL [120]
c) GROUND [360]
d) BROUGHT [840]
e) STICKLER [1680]
f) CANDY [120]

3. How many five-digit odd numbers can be formed from the digits of the number 5 390 462? [900]

4. In how many ways can all the letters of the word MASONITE be written so that the vowels occupy the even-numbered positions? [576]

5. If a penny, a nickel, and a dime are tossed simultaneously, in how many different ways can the three coins fall? [8]

6. Given the digits 0, 1, 2, 3, 4 and 5, if no digit can be repeated,
   a) How many different three-digit numbers can be formed from? (Recall that 0 cannot be the first digit) [100]
   b) How many of the numbers formed in part (a) are even? [52]

7. Given the letters LINDSAY,
   a) How many distinct five-letter words can be made by arranging the letters. [2520]
   b) How many words can be made if D must be included? [1800]
   c) How many words can be made if D must NOT be included? [720]

8. Given the letter FERMAT,
   a) How many 4-permutations of the letters are there? [360]
   b) Find the number of distinct four-letter words containing at least one vowel that can be formed. [336]
   c) How many 6-permutations of the letters are there? [720]
   d) Find the number of distinct six-letter words in which the vowels appear adjacent to one another. [240]

9. There are six women and five men,
   a) How many ways can a photographer arrange them if the women must sit in the first row and the men stand in the second? [86400]
   b) How many different arrangements are there if the tallest man and the two tallest women must be at the centre of their respective rows? [1152]

10. In how many ways can four married couples be arranged around a circular table, (Thinking Question!!!)
    a) If there are no restrictions? [5040]
    b) If each man must sit beside his wife? [96]
    c) If each man sits beside his wife and men and women alternate? [12]
    d) If each man sits opposite his wife (that is, there are three people in between on either side)? [48]

11. In how many ways can a 5-member basketball team form a circle around its coach? [24]

12. Given the word STEVIN, how many distinct 5-letter words can be made
    a) if only one vowel is to be used? [240]
    b) if both vowels are to be used and E is to precede (come before) I? [240]
    c) if both vowels must be used, they must be adjacent and E is to precede I? [96]
    d) if either one vowel is to be used or E is to precede I? [480]

13. There are 12 candidates for a baseball team of 9 members. In how many ways can the 9 different positions on the field be allocated under the condition that: (Thinking Question!!!!)
    a) There is no restrictions? [79 833 600]
    b) Only 5 of the 12 candidates can be considered as a pitcher and 9 of the 12 as a catcher, and only 3 of the players are suitable for both a pitching and a catching position? All players can play any other position on the team. [25 401 600]

14. Suppose that you are at Bob's Restaurant and see the following items on the menu.

Dinners		Beverages		Deserts	
Big boy Hamburger	1.70	Coffee	0.45	Hot Fudge Sundae	1.20
Fried Chicken	3.95	Coca-Cola	0.45	Apple Pie	1.00
Shrimp	3.95	Root Beer	0.45	Banana Cream Pie	1.00
Fish and Fries	3.75	Milk	0.50	Strawberry Pie	1.00
Bob's Special Shark	5.50	Iced Tea	0.45	Cheese Cake	0.90
				Fudge Brownies	0.55

    a) If you ordered a dinner and a beverage, without dessert, how many different meals can you choose from? [25]
    b) If you also ordered a desert, how many meals are now available? [150]
    c) How many choices do you have if you order a dinner, beverage and desert, but decide that you do not want seafood? [60]

15. How many 4-letter words can be form from the letters A, B, C, D, E if,
    a) A is always included and no letter is to be used twice? [96]
    b) The A and B must be together and no letter is used twice? [36]

16. In how many different ways can an eight question multiple choice exam be answered
    a) if each question has 5 possible choices, A, B, C, D, and E? [390 625]
    b) if no two consecutive questions can have the same answer? [81 920]

17. how many odd four-digit numbers (all digit different) may be formed using the digits 0 to 9 if there must be a 5 in the number? [1064]

18. How many 5-letter words may be formed from BCDEA (with no repetition of letters) so that the consonants are not all separated from each other by vowels? [108]

Lesson: Combination

Definition of combination

– a combination of "n" distinct objects taken r at a time is a selection of r of the n objects without regard to order. The total number of combinations of n objects taken r at a time is denoted by ${}_nC_r$, read as "n choose r".

- Order is not important

$$ {}_nC_r = C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

Example: If you had to choose 3 students out of 5 students to clean cafeteria, order of students would not be important. The students are indistinguishable because they are all doing the same job. The same is true for those that are not chosen.

Examples PART A:

Given 2 distinct objects, how many groups of:	0 objects can you make?	1 object can you make?	2 objects can you make?		
Given 3 distinct objects, how many groups of:	0 objects can you make?	1 object can you make?	2 objects can you make?	3 object can you make?	
Given 4 distinct objects, how many groups of:	0 objects can you make?	1 object can you make?	2 objects can you make?	3 objects can you make?	4 object can you make?

COMBINATIONS

1. If there are 7 teachers, how many different groups of 3 teachers can be chosen to be on the social committee?

2. If you have 5 different candies, how many ways can you put 2 or 3 candies into a bag?

3. If there are 5 different fruits and 7 different vegetables, how many ways can you make a:
    a. Group of 4 fruits?

    b. Group of 4 vegetables?

    c. Group of 4 fruits or 4 vegetables?

    d. Group of 4 fruits and 4 vegetables?

4. If there are 3 kinds of gum, 5 kinds of candies and 4 kinds of toys, how many different kinds of loot bags could be made if:
    a. They include 2 from each grouping (a total of 6 items)?

    b. They include 3 each from two groups (a total of 6 items)?

    c. They must include all the kinds of gum and all the kinds of candies and at least 2 kinds of toys?

Lesson: Identical Objects Problem Solving with Combination

Definitions:

Set- group of elements

Subsets – a set whose elements are also elements of another set, also known as 2^n

Example #1

A student has a pencil, eraser, calculator and highlighter. In how many ways can the student choose one or more supplies to put in her pencil case?

Example #2:

1. Yinger finds 11 pairs of pants in her size at a clearance sale. How many different purchases could she make?

Example #3:
In how many ways can a committee of at least one member be appointed from a board of 7 members?

Example #4:
3 pencils, 4 erasers, 1 calculator, 2 highlighters
In how many ways can the student choose 1 or more supplies?

In general: _____

Example #5:
There are 5 apples, 2 oranges, 3 pears. In how many ways can the chef choose 3 fruits for the dessert if the dessert must include at least 1 pear?

Mid-chapter Review: Permutations and Combinations

1. Solve for n.

a. $\frac{n!}{(n-3)!} = 3n - 3$

b. $2n\ P(n+3, 1) - 3 = \frac{9n+1}{2}$

2. A baby has 3 blue blocks, 6 green blocks, and 2 yellow blocks. In how many ways can the baby choose 5 blocks…..
    a. If the blocks may be chosen in any way?

    b. If the baby must have at least 2 blue blocks?

    c. If the baby can choose 6 blocks and must have 2 blocks from each colour?

3. Given Mehran, Borna, Jen, Cory, and Ioan in the class. In how many ways can they be arranged….
    a. In a line?

    b. In a circle?

    c. If Mehran and Borna cannot sit beside each other?

4. There are 8 female and 7 male in our class. We have to choose a fund-raising committee composed of president, vice-president, treasurer, and secretary. In how many ways may this committee be chosen if…..
    a. There are no restrictions?

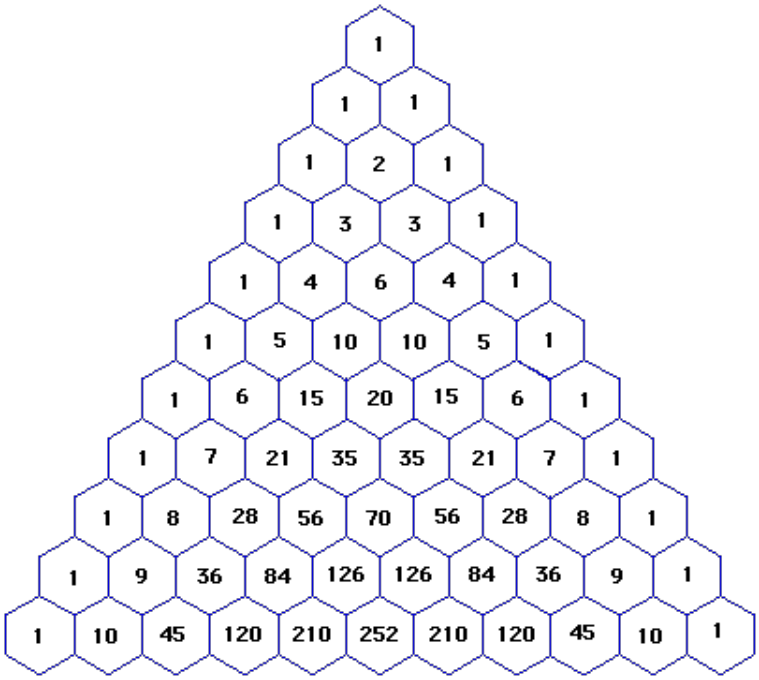
    b. The committee must include at least 1 female?

    c. The president and vice-president must be different gender?

s

5. Willy has 5 tees, 6 pairs of pants, and 4 pairs of shoes. In how many ways can he dress himself?

Lesson: Pascal's Triangle



Ex: a) Fill in the blank for row 25 and row 26 of Pascal's triangle below:

	25		2 300	12 650	53 130

If $t_{n,r}$ represents the term in row n, position r, then $t_{n,r} = t_{n-1,r-1} + t_{n-1,r}$

equivalent to $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

Examples: $t_{6,2} = t_{5,1} + t_{5,2}$

$t_{7,3} = t_{6,2} + t_{6,3}$

$t_{4,1} = t_{3,0} + t_{3,1}$

b) Use Pascal's method to write a formula for each of the following terms:

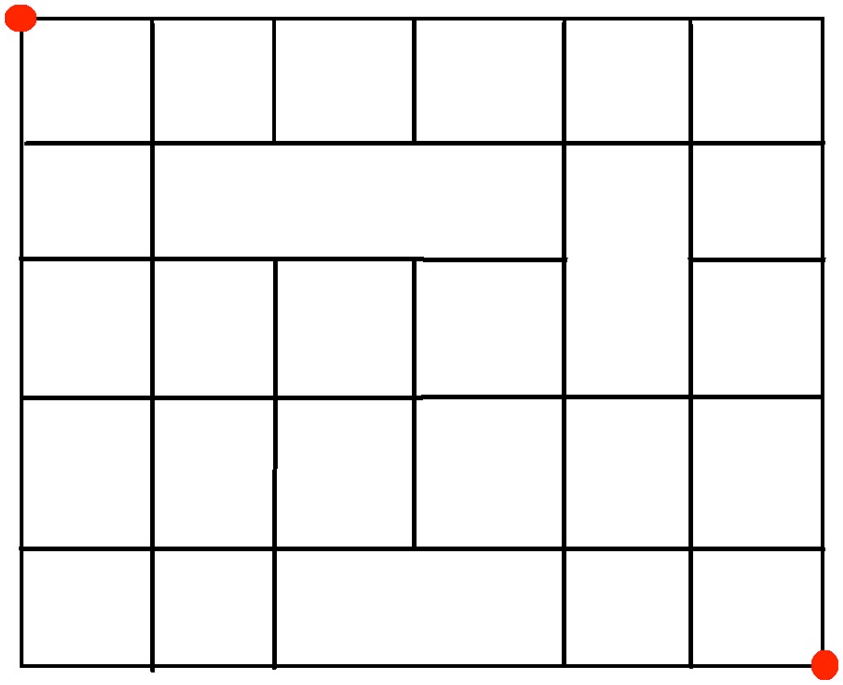
$t_{12,5} =$

$t_{40,32} =$

$t_{n+1,r+1} =$

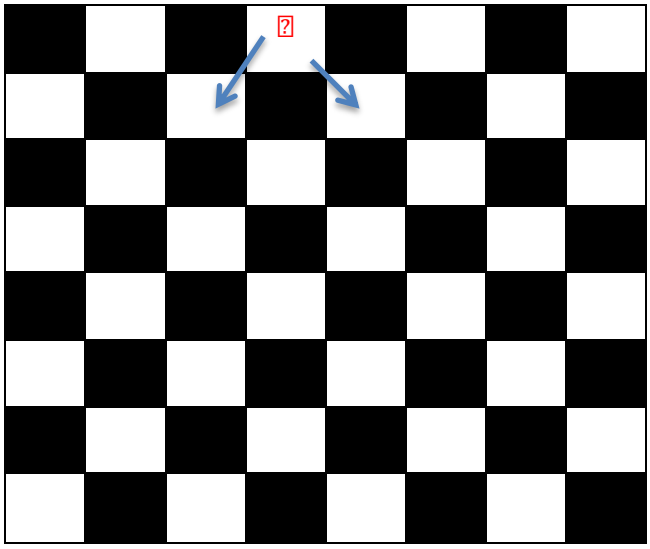
Application of Pascal's Triangle on paths

The iterative process that generates the terms in Pascal's triangle can also be applied to counting paths of route between two points.

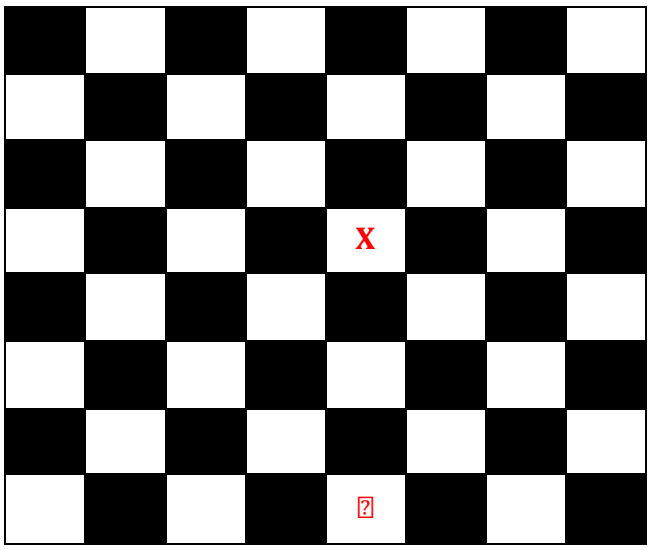


Application of Pascal's Triangle on checkerboard

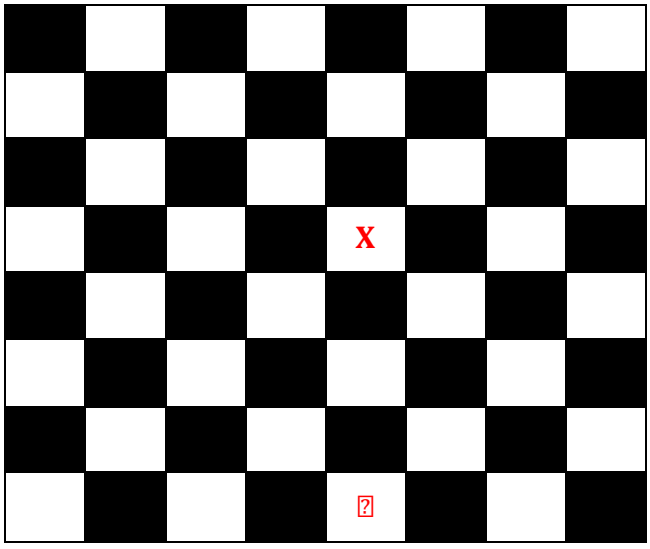
A checker is placed o a game board as shown. Determine the number of paths the checker may take to get to the opposite side of the board if it can move only diagonally forward one square at a time.



Example 2: p. 255.
On the checkerboard shown, the checker can travel only diagonally upward. It cannot move through a square containing an X. Determine the number of paths from the checker's current position to the top of the board.



Example 3
A checker is placed on a checkerboard as shown. The checker may move diagonally upward. Although it cannot move into the square with an X, the checker may jump over the X into the diagonally opposite square. Determine the number of paths on the top of the board.



Problem Solving with Combinations

1. a) How many subsets are there for a set with six elements? [64]
   b) How many of the subsets in (a) contain five elements? [6]

2. a) What is the total number of subsets of a set with *n* elements? [2^n]
   b) How many of the subsets from (a) have n-1 elements? [n]

3. In how many ways can a committee of at least one person be formed from seven club members? [127]

4. Chang arrives at the giant auction sale late in the afternoon. There are only five items left to be sold. How many different purchases could he make? [31]

5. a) In how many different ways could a team of three students be chosen from Lin's Data Management class of 25 students to compete in the County Mathematics Contest? [2300]
   b) In how many of these cases would Lin be a member of the team? [276]
   c) In how many of these would Lin not be a team member? [2024]

6. Twelve dots are on the page of Pierre's dot-to-dot puzzle book. If no three points are collinear, how many straight lines can he form by joining dots? (Recall: any two points determine a line) [66]

7. a) How many different sums of money can be made from a $2-bill, a $5-bill, and a $10-bill? [7]
   b) How many different sums of money can be made from the bills in (a) as well as one more $10-bill? [11]

8. A committee of students and teachers is being formed to study the issue of student parking privileges. Fifteen staff members and 18 students have expressed an interest in serving on the committee. In how many different ways could a five-person committee be formed if it must include at least one student and one teacher?
   [225 765]

9. In the binary number system which is used in computer operations, there are only two digits allowed: 0 and 1.
   a) How many different binary numbers can be formed using at most four binary digits (for example, 0110)? [16]
   b) If eight binary digits are used (for example, 11001101), how many different binary numbers can be formed? [256]

10. The prime factorization of 12 is 2x2x3. Find the number of divisors of 12 other than 1 by finding all combinations of these numbers. [5]

11. The prime factorization of 540 is 2x2x3x3x3x5. Find the number of divisors of 540 other than 1. [23]

12. There are 14 teams competing in the Bilton Community Darts League. Teams are to be selected for the first and second divisions in the upcoming tournament. Each division is to have seven teams. If last year's champs, the Arrows, must be in the first division and the current league leader, the Bows, must be in the second division, in how many ways can the selection be made? [924]

13. (a) Marla's bag of marbles contains two red, three blue, and five green marbles. If she reaches in to pick some (means at least one) without looking, how many different selections might she make? [71]
   (b) …if she gets an equal number of red and blue jelly beans? [17]
   (c) …if she gets at least one of each colour (but not necessarily equal red and blue)? [30]

14. A railcar compartment has eight seats; four facing the engine of the train and four facing away from the engine. Of eight passengers, three prefer to face away from the engine, two prefer to face the engine, and the

others have no preference. In how many ways can the passengers be seated if the order on each side of the car is disregarded? [3]

15. A travel agency has a limited number of six different free pamphlets about the Beetle Islands. The agent tells you to take any that you want, but not more than two of any kind. Assuming that you take at least one pamphlet, how many choices could you make? [728]

16. In Lisa's first year college program she must take nine courses including at least two science courses. If there are five science courses, three mathematics courses, four language courses, and five business courses from which to choose, how many different academic programs could she follow? [21 615]

17. The science club members are to work in pairs on an experiment.
    a) If there are eight members present, in how many ways can the pairs be formed? [105]
    b) If Louis and Simone will not work together, how many different pairs can be formed? [90]

18. A shelf holds seven mystery novels and eight biographies. Josie chooses either a mystery novel or a biography. Then Juan is to choose a mystery novel and a biography. In which case would Juan have the greater number of choices? [If Josie chooses biography]

19. Find the number of divisors of each number
    (a) 98 [5] (b) 2520 [47]

20. Find the number of divisors of 1050 that are:
    a) Odd numbers [11]
    b) Even numbers [12]
    c) Composite numbers [19]
    d) Divisible by 5 [16]

21. a) Calculate the number of diagonals of a hexagon. [9]
    b) If a polygon has 35 diagonals, how many sides does it have? [10]

Exploring Combinations Worksheet

1.There are 32 people at a meeting. If every person shakes hands with every other person, how many handshakes take place?

2.There are 16 teams in a league. The league schedule requires every team to play every other team twice. How many games are schedules

3. (a)How many lines are defined by this set of points, if each line passes through a pair of points?

(b) ….how many triangles?

(c) ….how many quadrilaterals?

(d) How many diagonals are there in a convex decagon? (10-sided polygon with no reflex angles)

2. From a group of 7 men and 5 women (including Ms. Sloumatis),

(a) in how many ways can 3 men and 2 women be assigned to supervise 5 different clubs?

(b) …if the Drama Club is supervised by a woman?

(c) …if the Drama Club is supervised by Ms. Sloumatis?

3. In Lily's garden there are 9 identical red roses, 6 identical white roses and 4 identical yellow roses. In how many ways can she pick a bouquet containing

(a) at least two roses? (b) at least two red roses? (c) all three colours? (d) exactly two colours?

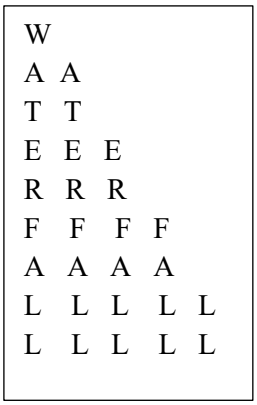
4. There are 10 women and 8 men at the tennis club. Determine the number of ways of organizing the pairing for:

(a) a women's single match

(b) a men's doubles match

(c) a mixed doubles match

5. If each drop of water can move down vertically or diagonally, how many ways are there to spell WATERFALL?



6. Construct a Venn diagram to answer this question. A survey conducted at a take-out deli showed that:

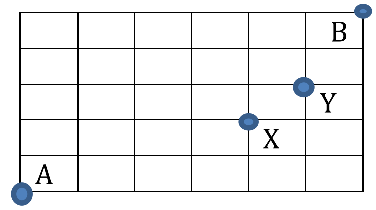
- 36 people ordered a beverage,
- 6 people ordered only a beverage,
- 11 people ordered a beverage and a dessert,
- 32 people ordered a sandwich,
- 15 people ordered a dessert: of these 15 people, 3 ordered only a dessert, and 7 also ordered a beverage and a sandwich.

How many people ordered both a beverage and a sandwich?

7. Seven police office officers and 4 prisoners are crossing a river in two identical boats. If a boat cannot carry more than 8 people, and there must be more police than prisoners in each boat, in how many ways can the trip be organized.

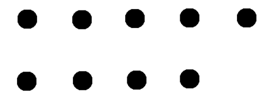
8. Count the number of paths along the grid lines:

(a) from A to B,
(b) from A to B passing through X,
(c) from A to B not passing through X,
(d) from A to B passing through X and Y,
(e) from A to B passing through X or Y,
(f) from A to B and back to A,
(g) from A to B and then returning to A by a different route,
(h) from A to B, making at least three turns,
(i) from A to B, if you are also allowed to move diagonally northeast.

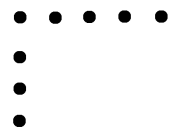


9. Find the number of ways of splitting 12 people into
(a) two groups of 6 (b) three groups of 4 (c) four groups of 3 (d) 6 pairs

10. (a) By joining these points, how many lines can be formed? (i.e. lines that extend in both directions, not line segments)
(b) How many triangles can be formed?



11. How many triangles can be formed with vertices on these points?



12. Pest Control [H. E. Dudeney, problem 258]
You may start anywhere, and move in any direction to an adjacent letter. In how many ways can you spell WAS IT A RAT I SAW?

```
                  W
                W A W
              W A S A W
            W A S I S A W
          W A S I T I S A W
        W A S I T A T I S A W
      W A S I T A R A T I S A W
        W A S I T A T I S A W
          W A S I T I S A W
            W A S I S A W
              W A S A W
                W A W
                  W
```

Answers:

	1) 496	2) 240	3) a.28	b.56	c.70	d.35
2.	a) 42 000	b) 16 800	c) 3 360			
3.	a) 346	b) 280	c) 216	d) 114		
4.	a) 45	b) 210	c) 2520	5. 1264	6. 26	7. 434
8.	a) 462	b) 150	c) 312	d) 90	e) 228	f) 213 444
	g) 212 982	h) 451	i) 3653			
9.	a) 462	b) 5775	c) 15 400	d) 10 395		
10.	a) 22	b) 70	11. 42	12. 63 504		

Practice: Combinations: (Note*** Some may involved permutations)

1. A hockey team has 12 forwards, 6 defensemen and 2 goalies. In how many ways can the coach choose 3 forwards, 2 defensemen and 1 goalie to start the game? [6600]

2. A club has 25 members, 11 girls including Lin and 14 boys including Hussain.
   a) In how many ways can a committee of three members be chosen? [2300]
   b) In how many ways can the offices of president, secretary and treasurer be filled? [13 800]
   c) How many 3-member committees have two girls and one boy? [770]
   d) How many 3-member committees have at least one boy and one girl? [1771]
   e) How many of the 3-member committees include Lin? [276]
   f) How many 3-member committees exclude Lin? [2024]
   g) How many 3-member committee include Lin or Hussain? [529]
   h) In how many ways can the president, secretary and treasurer be designated, if at least one boy and at least one girl must be included? [10 626]
   i) In how many ways can the positions of president, secretary and treasurer be filled, if Lin or Hussain must be included? [3174]

3. Serge is handing out snacks at the day-care centre. There are 15 children, and he has 7 apples, 6 oranges and 2 boxes of raisins. In how many ways can he distribute the snacks? [180 180]

4. A restaurant menu consists of 14 items in column A and 10 items in column B. How many dinners for five are possible if the dinner for five consists of:
   a) Three items from column A and two from column B? [16 380]
   b) Either 6 items from column A or 4 items from column B? [3213]

5. In her repertoire, Esther has seven songs from which she will perform two at each school concert.
   (a) In how many ways can the programs for three consecutive shows be chosen if she does not perform the same song two shows in a row? [2100]
   (b) …if she performs either one or two songs at each show? [7812]

6. There are 16 teams entered in the Theatre Games tournament. In how many ways can 1^{st}, 2^{nd}, and 3^{rd} place ribbons be awarded? [3360]

7. From a class of 10 girls and 8 boys, 4 children are to be designated to feed the hamster, the goldfish, the turtles and the canaries.
   a) In how many ways can these tasks be assigned? [73 440]
   b) …if two boys and two girls are involved? [30 240]

8. The member of City Council are to vote yes or no on each seven issues. It is also possible to abstain (give up), but you may not abstain on all seven questions.
   a) In how many ways can a ballot be marked? [2186]
   b) … if you may not abstain more than twice? [1248]

9. Sarah has 9 crayons: 3 blue, 1 red, 2 green, and 3 yellow.
   a) In how many ways can she line them up on her desk? [5040]
   b) In how many ways can she pick some of the crayons? [95]

10. Three cards are dealt from a well-shuffled, standard deck, calculate the number of ways each event could happen.
    a) All three cards are red [2600]
    b) At least one is an ace [4804]
    c) None of them are face cards [9880]
    d) All three are from the same suit [1144]
    e) At least two are hearts [3328]
    f) None are diamonds but at least one is a club [6539]

11. If a poker hand (5 cards) is dealt from a well-shuffled, standard deck, calculate the number of ways each even could happen.
    a) There are no hearts [575 757]
    b) There is at least one jack [886 656]
    c) There are at least 2 spades [953 940]
    d) There are at least 3 face cards[192 192]
    e) There are at least 4 red cards [454 480]
    f) The queen of spades is included [249 900]
    g) The queen of spades and the king of spades are included [19 600]
    h) The queen of spades or the king of spades is included [480 200]

Organized Counting Permutations Review

1. How many distinct 4 letter words can be formed from the word BUZAGLO if:
    a. All vowels are to be used? [96]
    b. No vowels are to be used? [24]
    c. If B is not be used OR no vowels are to be used? [384]
    d. If Z is to be used OR all vowels are to be used? [552]
    e. If G and L must be used and G must come before L? [120]
    f. If G and L must be used and must be side by side? [120]

2. How many five-digit numbers less than 60 000 are divisible by 5:
    a. With no repetition allowed? [3024]
    b. With no restrictions? [10 000]
    c. If the number 5 must be used (with no repetition)? [2184]

3. How many five-digit numbers greater than 40 000 are divisible by 5:
    a. With no repetition allowed? [3696]
    b. With no restrictions? [11 999]
    c. If the number 0 must be used (with no repetition)? [2646]

4. There are 15 applicants for a job. In how many ways can 10 jobs be filled if only 5 of the applicants are qualified receptionists, 7 of the applicants are qualified filing clerks, and only 3 of the candidates are suitable for both a receptionist position and a filing clerk position? All applicants can fill any other position that is available. Every position that is available is unique. You must hire one applicant for the filing clerk position and one applicant for the receptionist position. [1 660 538 880]

5. There are 210 grade 12 students enrolled for semester 2. The following information was gathered by the guidance department to determine the number of math courses taken by each student this semester.
   98 students are taking Data Management
   101 students are taking Advance Functions
   94 students are taking Calculus and Vectors
   43 students are taking Data Management and Calculus and Vectors
   60 students are taking Advance Functions and Calculus and Vectors
   47 students are taking Advance Functions and Data Management
   43 grade 12 students are not taking any math courses

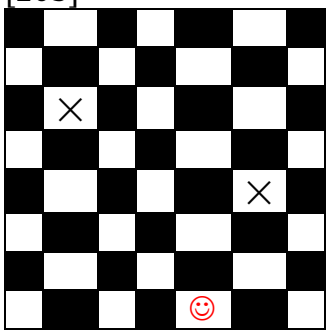
    a. Draw a Venn diagram illustrating the above information
    b. How many students are taking all three math courses at once? [24]
    c. How many students are taking exactly two math courses? [78]
    d. How many students are taking Data Management or Calculus and Vectors? [149]

6. Simone is a radio producer. Her role is to create a play list that is 8 songs long. These 8 songs must all be from the top 8 voted songs from the listeners. No song may be repeated.
    a. How many ways can she arrange the music for this play list? [40 320]
    b. There are 2 Britney Spears songs and 2 Akon songs in the top 8. How many ways can Simone create the play list if two consecutive songs cannot be by the same singer? [23 040]

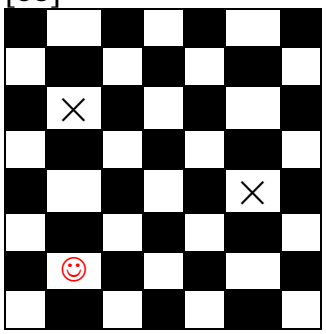
7. Fiver different signal flags are available to fly on a ship's flagpole. How many different signals can be sent using at least two flags? [320]

6. A baby has 3 blue blocks, 6 green blocks, and 2 yellow blocks. In how many ways can the baby choose 5 blocks…..
    a. If the blocks may be chosen in any way? [12]
    b. If the baby must have at least 2 blue blocks? [6]
    c. If the baby can choose 6 blocks and must have 2 blocks from each colour? [1]

7. A checker is placed on a checkerboard as shown. The checker may move diagonally upward or straight upward on the white squares only. Although it cannot move into a square with an X, the checker may jump over the X into the diagonally opposite square. How many paths are there to the top of the board?
    a. [203]



    b. [68]



8. There are 8 basketball players and 6 hockey players. In how many ways can you:
    a. Make a group of 5 if there are no restrictions? [2002]
    b. Make a group of 6 if you must include at least 2 from each group? [2590]
    c. Make a group of 5 with at least one basketball player? [1996]