5.5 Worksheet – Binomial Theorem

MDM4U
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1) Use Pascal's Identity to write an expression of the form $\binom{n}{r}$ that is equivalent to each of the following

a) $\binom{10}{2} + \binom{10}{3}$

$= \binom{11}{3}$

b) $\binom{20}{18} + \binom{20}{19}$

$= \binom{21}{19}$

c) $\binom{!"}{14} + \binom{!"}{13}$

$= \binom{16}{14}$

2) In the expansion of $(x + y)^{10}$, write the value of the exponent k in the term that contains

a) x^4y^k

$4 + k = 10$
$k = 6$

b) x^ky^8

$k + 8 = 10$
$k = 2$

c) $x^ky^{!k}$

$k + 4k = 10$
$5k = 10$
$k = 2$

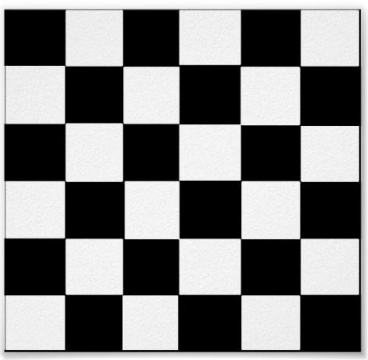
d) $x^{k!2}y^{3k}$

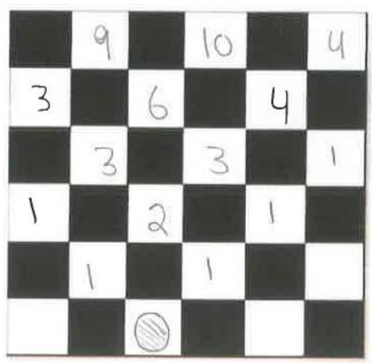
$(k - 2) + 3k = 10$
$4k = 12$
$k = 3$

3) a) Imagine that a checker is placed in the bottom left corner of a 6---by---6 checker board. The piece may be moved one square at a time diagonally left or right to the next row up. Calculate the number of different paths to the top row.

$n(paths) = 5 + 4 + 1 = 10$

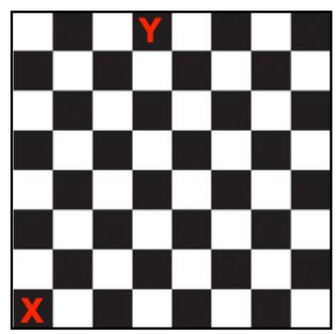
b) Suppose the checker began in the third square from the left in the bottom row. Calculate the number of possible paths to the top row from this position.





$n(paths) = 9 + 10 + 4 = 23$

4) How many paths are there from X to Y…



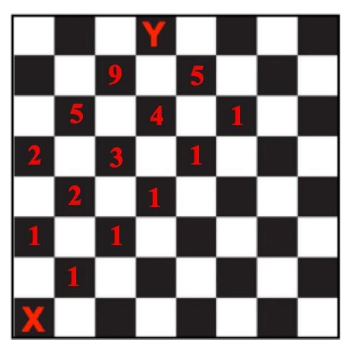
a) As a checkerboard where you can only move diagonally forward

$n(paths) = 9 + 5 = 14$

b) As a grid where you can only move north and east

Must move 7 North and 3 East

$n(routes) = \binom{10}{7} = 120$ *OR* $n(routes) = \frac{10!}{7!3!} = 120$



5) Faizel wants to travel from his house to the hardware store that is six blocks east and five blocks south of his home. If he walks east and south, how many different routes can he follow from his home to the store?

Faizel has to travel 11 blocks, 6 of which are east and 5 of which are south.

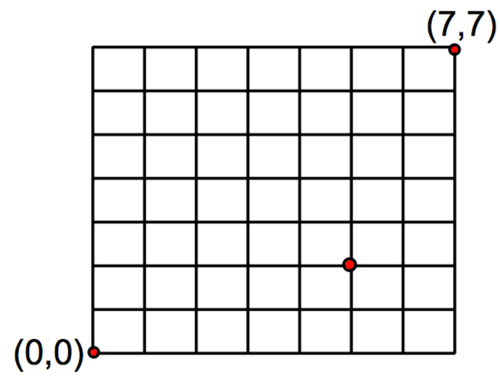
$n(routes) = \binom{11}{6} = 462$ OR $n(routes) = \binom{11}{5} = 462$ OR $n(routes) = \frac{11!}{6!5!} = 462$

6) a) Starting at (0, 0) and moving only North and East, how many routes pass through (5, 2) and end at (7, 7)?

$n(routes\ pass\ through\ 5{,}2) = n(routes\ from\ 0{,}0\ to\ 5{,}2) \times n(routes\ from\ 5{,}2\ to\ 7{,}7)$

$n(routes) = \binom{7}{2} \times \binom{7}{2}$

$n(routes) = 441$



b) Starting at (0, 0) and moving only North and East, how many routes <u>avoid</u> (5, 2) and end at (7, 7) ?

$n(routes\ avoid\ 5{,}2) = n(routes\ from\ 0{,}0\ to\ 7{,}7) - n(routes\ pass\ through\ 5{,}2)$

$n(routes\ avoid\ 5{,}2) = \binom{14}{7} - 441$

$n(routes\ avoid\ 5{,}2) = 3432 - 441$

$n(routes\ avoid\ 5{,}2) = 2991$

7) Expand and simplify each of the following using the Binomial Theorem

a) $(a + 2b)^4$

$= \binom{4}{0}(a)^{4-0}(2b)^0 + \binom{4}{1}(a)^{4-1}(2b)^1 + \binom{4}{2}(a)^{4-2}(2b)^2 + \binom{4}{3}(a)^{4-3}(2b)^3 + \binom{4}{4}(a)^{4-4}(2b)^4$

$= \binom{4}{0}(a)^4(2b)^0 + \binom{4}{1}(a)^3(2b)^1 + \binom{4}{2}(a)^2(2b)^2 + \binom{4}{3}(a)^1(2b)^3 + \binom{4}{4}(a)^0(2b)^4$

$= 1(a)^4(1) + 4(a)^3(2)(b) + 6(a)^2(4)(b)^2 + 4(a)^1(8)(b)^3 + 1(1)(16)(b)^4$

$= a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4$

b) $(x-y)^6$

$=\binom{6}{0}(x)^{6!0}(-1y)^0+\binom{6}{1}(x)^{6!1}(-1y)^1+\binom{6}{2}(x)^{6!2}(-1y)^2+\binom{6}{3}(x)^{6!3}(-1y)^3+\binom{6}{4}(x)^{6!4}(-1y)^4+\binom{6}{5}(x)^{6!5}(-1y)^5+\binom{6}{6}(x)^{6!6}(-1y)^6$

$=\binom{6}{0}(x)^6(-1y)^0+\binom{6}{1}(x)^5(-1y)^1+\binom{6}{2}(x)^4(-1y)^2+\binom{6}{3}(x)^3(-1y)^3+\binom{6}{4}(x)^2(-1y)^4+\binom{6}{5}(x)^1(-1y)^5+\binom{6}{6}(x)^0(-1y)^6$

$=1(x)^6(1)+6(x)^5(-1)(y)^1+15(x)^4(1)(y)^2+20(x)^3(-1)(y)^3+15(x)^2(1)(y)^4+6(x)^1(-1)(y)^5+1(1)(1)(y)^6$

$=x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$

c) $\left(c+\frac{!}{c}\right)^!$

$=\binom{4}{0}(c)^{4!0}\left(\frac{1}{c}\right)^0+\binom{4}{1}(c)^{4!1}\left(\frac{1}{c}\right)^1+\binom{4}{2}(c)^{4!2}\left(\frac{1}{c}\right)^2+\binom{4}{3}(c)^{4!3}\left(\frac{1}{c}\right)^3+\binom{4}{4}(c)^{4!4}\left(\frac{1}{c}\right)^4$

$=\binom{4}{0}(c)^4\left(\frac{1}{c}\right)^0+\binom{4}{1}(c)^3\left(\frac{1}{c}\right)^1+\binom{4}{2}(c)^2\left(\frac{1}{c}\right)^2+\binom{4}{3}(c)^1\left(\frac{1}{c}\right)^3+\binom{4}{4}(c)^0\left(\frac{1}{c}\right)^4$

$=1(c)^4(1)+4(c)^3(c)^{!1}+6(c)^2(c)^{!2}+4(c)^1(c)^{!3}+1(1)(c)^{!4}$

$=c^4+4c^2+6c^0+4c^{!2}+c^{!4}$

$=c^4+4c^2+6+\frac{4}{c^2}+\frac{1}{c^4}$

d) $\left(d-\frac{1}{d}\right)^5$

$=\binom{5}{0}(d)^{5!0}\left(\frac{-1}{d}\right)^0+\binom{5}{1}(d)^{5!1}\left(\frac{-1}{d}\right)^1+\binom{5}{2}(d)^{5!2}\left(\frac{-1}{d}\right)^2+\binom{5}{3}(d)^{5!3}\left(\frac{-1}{d}\right)^3+\binom{5}{4}(d)^{5!4}\left(\frac{-1}{d}\right)^4+\binom{5}{5}(d)^{5!5}\left(\frac{-1}{d}\right)^5$

$=1(d)^5(1)+5(d)^4(-1)(d)^{!1}+10(d)^3(1)(d)^{!2}+10(d)^2(-1)(d)^{!3}+5(d)^1(1)(d)^{!4}+1(d)^0(-1)(d)^{!5}$

$=d^5-5d^3+10d-10d^{!1}+5d^{!3}-1d^{!5}$

$=d^5-5d^3+10d-\frac{10}{d}+\frac{5}{d^3}-\frac{1}{d^5}$

8) Find an expression for the general term, in simplified form, for each of the following…

a) $(x+y)^{10}$

$t_{r+1}=\binom{10}{r}(x)^{10!r}(y)^r$

b) $(x-y)^{10}$

$t_{r+1}=\binom{10}{r}(x)^{10!r}(-1y)^r$

$t_{r+1}=\binom{10}{r}(-1)^r(x)^{10!r}(y)^r$

c) $\left(z+\frac{!}{z}\right)^{!}$

$t_{r+1} = \binom{8}{r}(z)^{8!r}\left(\frac{1}{z}\right)^{r}$

$t_{r+1} = \binom{8}{r}(1)^{r}(z)^{8!r}(z)^{!r}$

$t_{r+1} = \binom{8}{r}(z)^{8!2r}$

d) $\left(w^2+\frac{1}{w}\right)^{9}$

$t_{r+1} = \binom{9}{r}(w^2)^{9!r}\left(\frac{1}{w}\right)^{r}$

$t_{r+1} = \binom{9}{r}(1)^{r}(w)^{18!2r}(w)^{!r}$

$t_{r+1} = \binom{9}{r}(w)^{18!3r}$

9) Find an expression for the indicated term in the expansion of each of the following…

a) the third term of $(x^2-2)^7$

$t_{r+1} = \binom{n}{r}a^{n!r}b^{r}$

$t_{2+1} = \binom{7}{2}(x^2)^{7!2}(-2)^2$

$t_3 = 21(x^2)^5(4)$

$t_3 = 84x^{10}$

b) the middle term of $(c-d)^8$

There are 9 terms in the expansion, therefore the 5th term is the middle term.

$t_{r+1} = \binom{n}{r}a^{n!r}b^{r}$

$t_{4+1} = \binom{8}{4}(c)^{8!4}\ (-1d)^4$

$t_5 = 70(c)^4(1)(d)^4$

$t_5 = 70c^4d^4$

c) the tenth term of $\left(\frac{x}{3} - \frac{3}{x}\right)^{12}$

$t_{r+1} = \binom{n}{r} a^{n!r} b^r$

$t_{9+1} = \binom{12}{9}\left(\frac{x}{3}\right)^{12!9}\left(\frac{!3}{x}\right)^{9}$

$t_{10} = 220\left(\frac{x^3}{27}\right)\left(\frac{-19\,683}{x^9}\right)$

$t_{10} = \frac{-160\,380}{x^6}$

10) In the expansion of $\left(4x - \frac{2}{x}\right)^8$, find the following...

a) the term containing x^6

General term:

$t_{r+1} = \binom{n}{r} a^{n!r} b^r$

$t_{r+1} = \binom{8}{r}(4x)^{8!r}\left(\frac{-2}{x}\right)^{r}$

$t_{r+1} = \binom{8}{r}(4)^{8!r}(-2)^r(x)^{8!r}\,(x)^{!r}$

$t_{r+1} = \binom{8}{r}(4)^{8!r}(-2)^r(x)^{8!2r}$

Term with x^6:

For $(x)^{8!2r} = x^6$

$8 - 2r = 6$
$2 = 2r$
$r = 1$

$t_{1+1} = \binom{8}{1}(4)^{8!1}(-2)^1(x)^{8!2(1)}$

$t_2 = 8(16\,384)(-2)(x)^6$

$t_2 = -262\,144x^6$

b) the constant term

General Term:

$t_{r+1} = \binom{8}{r}(4)^{8!r}(-2)^r(x)^{8!2r}$

For $(x)^{8!2r} = x^0$

$8 - 2r = 0$
$8 = 2r$
$r = 4$

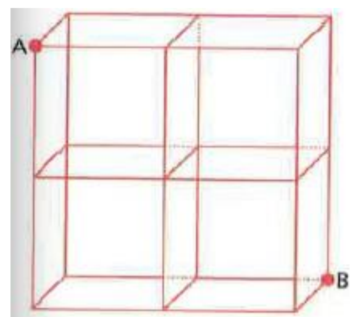
Term with x^0:

$t_{4+1} = \binom{8}{4}(4)^{8!4}(-2)^4(x)^{8!2(4)}$

$t_5 = 70(256)(16)x^0$

$t_5 = 286\ 720$

11) Determine the number of possible paths from point A to point B in the diagram to the left if travel may occur only along the edges of the cubes and if the path must always move closer to B.



2 right, 2 down, 1 away (5 total moves)

$n_{(routes)} = \binom{5}{2} \times \binom{3}{2} \times \binom{1}{1} = 10 \times 3 \times 1 = 30$