Section 4.6 – Permutations

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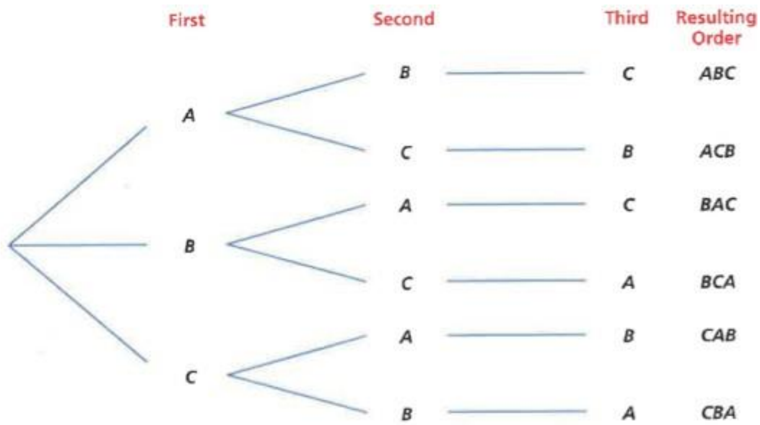
Part 1: Factorial Investigation

You are trying to put three children, represented by A, B, and C, in a line for a game. How many different orders are possible?

a) Use a tree diagram



b) Use the multiplication rule for counting *(find the product of the possible outcomes in each step of the sequence)*

$n(ordered\ arrangements) = n(choices\ for\ 1st) \times n(choices\ for\ 2nd) \times n(choices\ for\ 3rd)$

$= 3 \times 2 \times 1$

$= 6$

Permutations

The ordering problem in the investigation dealt with arranging three children to create sequences with different orders.

Sometimes when we consider n items, we need to know the number of different ordered arrangements of the n items that are possible.

A permutation is an ordered arrangement of objects. The number of different permutations of n distinct objects is $n!$

* Order Matters For Permutations *

Part 2: Factorials

factorial notation ($n!$) represents the number of ordered arrangements of n objects.

$$n! = n \times (n-1) \times (n-2) \times \ldots 1$$

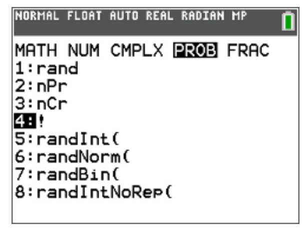
examples:

i) $3! = 3 \times 2 \times 1 = 6$

ii) $\frac{6!}{4!} = \frac{6\times5\times4\times\cancel{3}\times\cancel{2}\times\cancel{1}}{4\times\cancel{3}\times\cancel{2}\times\cancel{1}} = 6\times5 = 30$

Example 1: How many different ways can 7 people be seated at a dinner table?

$n(ordered\ arrangements) = 7! = 5040$



7 → MATH → PROB → ! → ENTER

Example 2: A horse race has 8 entries. Assuming that there are no ties, in how many different orders can the horses finish?

$n(ordered\ arrangments) = 8! = 40\ 320$

Example 3: In how many ways can the letters A, B, C, D, E, and F be arranged for a six---letter security code?

$n(codes) = 6! = 720$

Part 3: Distinguishable Permutations

You may want to order a group of n objects in which some of the objects are the same.

The formula for the number of permutations from a set of n objects in which a are alike, b are alike, c are alike, and so on is:

$$\frac{n!}{a!\, b!\, c!}$$

Example 4: Determine the number of arrangements possible using the letters of the word **MATHEMATICS.**

There are 11 letters and there are 2 M's, 2 A's, and 2 T's. Therefore, the number of arrangements is:

$$= \frac{11!}{2!\,2!\,2!}$$

$$= 4\,989\,600$$

Example 5: A building contractor is planning to develop a subdivision. The subdivision is to consist of 6 one story houses, 4 two story houses, and 2 split level houses. In how many distinguishable ways can the houses be arranged?

$$n(\text{ordered arrangements}) = \frac{12!}{6!\,4!\,2!} = 13\,860$$

Part 4: Permutations of part of a group

We have considered the number of ordered arrangements of *n* objects taken as an entire group; but what if we don't arrange the entire group?......

Counting rule for Permutations

The number of ways to arrange in order *n* distinct objects, taking them *r* at a time is:

$$P(n, r) = \frac{n!}{(n-r)!}$$

Example 6:

$$P(5, 3) = \frac{5!}{(5!3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times\cancel{2}\times\cancel{1}}{\cancel{2}\times\cancel{1}} = 5\times4\times3 = 60$$

That means there are 60 ways of ordering objects taken three at a time from a set of five different objects.

Example 7:

Let's compute the number of possible ordered seating arrangements for eight people in five chairs.

i) by using the multiplication rule for counting

	Chair 1	Chair 2	Chair 3	Chair 4	Chair 5
# of choices for the chair	8	7	6	5	4

$n(ordered\ arrangments) = 8{\times}7{\times}6{\times}5{\times}4 = 6\ 720$

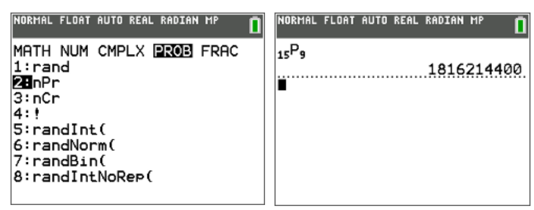
ii) by using the counting rule for permutations

$$P(8,5) = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8{\times}7{\times}6{\times}5{\times}4{\times}\cancel{3}{\times}\cancel{2}{\times}\cancel{1}}{\cancel{3}{\times}\cancel{2}{\times}\cancel{1}} = 8{\times}7{\times}6{\times}5{\times}4 = 6\ 720$$

Example 8:

There are 15 players on the school baseball team. How many ways can the coach complete the nine---person batting order?

$n(batting\ orders) = P(15,9) = 1\ 816\ 214\ 400$



15 → MATH → PROB → nPr → 9 → ENTER

Example 9:

There are 8 teams in the Metropolitan Division in the NHL's Eastern Conference. How many ways can the teams finish first, second, and third?

$n(ordered\ arrangements\ for\ top\ 3) = P(8,3) = 336$

Part 5: Using Permutations to Determine Probability

Recall: theoretical probability is the ratio of the number of outcomes that make up the desired event to the total number of possible outcomes

$$P(A) = \frac{n(A)}{n(S)}$$

Example 10:

Four people are required to help out at a party: one to prepare the food, one to serve it, one to clear the tables, and one to wash up. Determine the probability that you and your three siblings will be chosen for these jobs if four people are randomly selected from a room of 12 people.

$$P(\text{you and siblings selected}) = \frac{n(\text{you and your siblings can be chosen for the four jobs}}{n(12 \text{ people can be chosen for the four jobs}}$$

$$P(\text{you and siblings selected}) = \frac{P(4,4) \text{ or } 4!}{P(12,4)} = \frac{24}{11\,880} = \frac{1}{495}$$

Example 11:

A combination lock opens when the right combination of three numbers from 0 to 59 are entered in the correct order. The same number can't be used more than once.

a) What is the probability of getting the correct combination by chance?

$$\boldsymbol{P(\text{correct combination}) = \frac{n(\text{correct combinations})}{n(\text{possible combinations})}}$$

$$P(\text{correct combination}) = \frac{1}{P(60,3)} = \frac{1}{205\,320}$$

b) What is the probability of getting the right combination if you already know the first digit?

$$P(\text{correct combination}) = \frac{1}{P(59,2)} = \frac{1}{3\,422}$$

In the situations examined so far, objects were selected from a set and then, once selected, were removed from the collection so that they could not be chosen again. If the object is replaced, lets examine how this affects the possible number of arrangements…

Example 12:

a) How many ways are there to draw two cards from a standard deck of 52 cards if the card **is not** replaced after drawing it. *(the order you draw them in matters)*

$$n(draw\ 2\ cards\ without\ replacement) = P(52, 2) = \frac{52!}{(52-2)!} = \frac{52!}{50!} = 52\times51 = 2652$$

b) How many ways are there to draw two cards from a standard deck of 52 cards if the card **is** replaced after drawing it. *(the order you draw them in matters)*

$$n(draw\ 2\ cards\ with\ replacement) = n(choices\ for\ 1st)\times n(choices\ for\ 2nd) = 52\times52 = 2\ 704$$

Note: you can't use the counting rule for permutations, you must use the multiplication rule for counting.

Example 13:

The access code for a car's security system consists of four digits. Each digit can be 0 through 9. How many access codes are possible if:

a) each digit can be used only once and not repeated?

$$n(codes\ no\ repeats) = P(10, 4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10\times9\times8\times7 = 5\ 040$$

b) each digit can be repeated?

$$n(codes\ with\ repeats) = 10\times10\times10\times10 = 10\ 000$$

***Note:** because each digit can be repeated, there are 10 choices for each of the four digits.*