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Unit 6 – Discrete Functions & Applications

Day	Lesson	Practice Questions	Struggles?
1	U6L1 – Sequences: Arithmetic & Geometric	**Read:** Pg. 416 – 423, Pg. 426 – 429 **Do:** Pg. 424 # 5ab, 6ab, 8be, 13ae, 15 Pg. 430 # 5acf, 6ae, 7acd, 8c, 11	
2	U6L2 – Recursive Sequences & Binomial Theorem	**Read:** Pg. 441 – 443 Pg. 462 – 465 **Do:** Pg. 443 #1 Pg. 466 # 1, 2c, 3c, 4f, 5cf	
3	U6L3 – Series: Arithmetic & Geometric	**Read:** Pg. 448 – 451, Pg. 454 – 459 **Do:** Pg. 452 # 4abdf, 5acde, 6acd, 7aef, 11, 13 Pg. 459 # 3ade, 4ade, 5ade, 6acef, 7, 11	
4	U6L4 – Annuities: Future & Present Value	**Read:** Pg. 504 – 510, Pg. 513 – 519 **Do:** Pg. 511 # 5ac, 6, 7, 9 Pg. 520 # 3bd, 4, 6, 7, 9	
5	Review	**Read:** Pg. 445 – 447, Pg. 532 – 533 **Do:** Pg. 468 # 2, 3b, 4 – 6, 7a, 8ac, 9i, 10a, 14f, 15f, 16c, 17, 18f, 19d, 20, 23d *14f - answer is 7850* Pg. 534 # 11b, 13, 14d, 15, 17	
6	TEST		

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Unit 6, Lesson 1: Sequences: Arithmetic & Geometric

A sequence is an ordered list of numbers. Each number in the sequence is called a term. We can identify each term by its position in the list.

Generally, sequences take the form t_1, t_2, t_3, … t_n, where t is the value of the sequence at position n.

There are two types of sequences: Arithmetic and Geometric.

An **arithmetic** sequence has the same difference between consecutive pairs of terms (called the **common difference**)

Ex 1) Consider the sequence 3, 5, 7, 9. This is an arithmetic sequence with 4 terms.
The first term, t_1, is 3. What is t_3?

Sequences can be defined using a **general term**, a **recursive formula** (which relates the general term to the previous term(s)), or a **discrete linear function**.

The **general term** of an arithmetic sequence:	$\boldsymbol{t_n = a + (n - 1)d}$
The **recursive formula** for the same arithmetic sequence:	$\boldsymbol{t_1 = a,\ t_n = t_{n-1} + d}$**, where** $\boldsymbol{n > 1}$

In all cases, $n \in \mathbf{N}$, a is the first term, and d is the common difference.

Ex 2) Given the sequence 6, 3, 0, -3…

a) Determine the **general term**, t_n.

b) Determine the **recursive formula** for t_n.

c) Determine t_{10}.

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A **geometric** sequence has the same ratio between any pair of consecutive terms (called **common ratio**)

The **general term** of a geometric sequence:	$\boldsymbol{t_n = ar^{n-1}}$
The **recursive formula** for the same geometric sequence:	$\boldsymbol{t_1 = a,\ t_n = rt_{n-1}}$, **where** $\boldsymbol{n > 1}$

In all cases, $n \in \mathbf{N}$, a is the first term, and r is the common ratio.

Ex 3) Given the sequence 18, 9, $4\frac{1}{2}$, …

a) Determine the next 3 terms of the sequence

b) Determine the **general term**, t_n

c) Determine the **recursive formula** for t_n.

d) Determine t_{10}.

Ex 4) Given the sequence 9, 16, 23, 30, …, 100 determine the number of terms in the sequence.

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Unit 6, Lesson 2: Recursive Sequences & Binomial Theorem

Two famous recursive sequences are the Fibonacci sequence and the Lucas sequence. They are best defined recursively (a general term is possible, but messy!)

Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

Recursive definition:

Lucas sequence: 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, …

Recursive definition:

In what ways are these number found in nature?

What number does the ratio of consecutive terms approach?

Pascal's Triangle

There is a two-dimensional pattern of numbers that is usually called Pascal's Triangle. It is named after the French mathematician Blaise Pascal, though he was not the first to discover it! Identify the number pattern in the first few rows and complete the next five rows.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

____ ____ ____ ____ ____ ____ ____

____ ____ ____ ____ ____ ____ ____ ____

____ ____ ____ ____ ____ ____ ____ ____ ____

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A **binomial** is any expression that is written in the form of $a+b$. For example $x+1$, $2x-3y$, $5w^2-15$ are all binomial expressions.

Ex 1) For each of these binomial examples, identify *a* & *b*.

$x+1$	$2x-3y$	$5w^2-15$
a: ______	*a*: ______	*a*: ______
b: ______	*b*: ______	*b*: ______

We often have to work with powers of binomials, whose exponents are natural numbers.
Eg: $(a+b)^2$, $(a+b)^{15}$, $(a+b)^3$.

Notice the patterns when we expand the following:

$(a+b)^0$ $\quad 1$

$(a+b)^1$ $\quad a+b$

$(a+b)^2$ $\quad a^2+2ab+b^2$

$(a+b)^3$ $\quad a^3+3a^2b+3ab^2+b^3$

$(a+b)^4$ $\quad a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$(a+b)^5$ $\quad a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

How do the **coefficients** of the binomial expansions (expanded binomials) relate to the numbers in Pascal's Triangle?

What is the pattern of the **exponents** of the bases *a* and *b*?

Ex 2) Use Pascal's Triangle to expand $(x+2y)^5$.

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Unit 6, Lesson 3: Series: Arithmetic & Geometric

A **series** is the *sum* of the terms of a sequence. The sum of the first n terms of a sequence is S_n, where

$$S_n = t_1 + t_2 + t_3 + t_4 + \ldots + t_{n-1} + t_n$$

Recall: An **arithmetic sequence** has the general term $\boldsymbol{t_n = a + (n - 1)d}$, where $\boldsymbol{a}$ is the first term and $\boldsymbol{d}$ is the common difference between terms. An *arithmetic series* is the sum of this sequence and is written

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n - 2)d] + [a + (n - 1)d]$$

To determine a formula, we will write out the series twice, first forward and then backward. Then we will add them together (This is called Gauss's method)

$$\begin{array}{rlllll} & S_n = a & + (a + d) & + \ldots + [a + (n - 2)d] & + [a + (n - 1)d] \\ + & \underline{S_n = [a + (n - 1)d]} & \underline{+ [a + (n - 2)d]} & \underline{+ \ldots + (a + d)} & \underline{+\ a} \\ & 2S_n = [2a + (n - 1)d] & + [2a + (n - 1)d] & + \ldots + [2a + (n - 1)d] & + [2a + (n - 1)d] \end{array}$$

$$2S_n = n \times [2a + (n - 1)d]$$

$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]\text{, which is the } \textit{sum} \text{ of the first } n \text{ terms of an arithmetic sequence.}}$

Recognizing that $t_1 = a$ and $t_n = a + (n - 1)d$, we can also write: $\boxed{S_n = \frac{n(t_1 + t_n)}{2}}$

Example 1) Determine the sum of the arithmetic series given $t_1 = 88$ and $t_{15} = 4$

Example 2) Determine the sum of the series $15 + 11 + 7 + \ldots - 37$

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Recall: A **geometric sequence** has the general term $t_n = ar^{n-1}$, where $\boldsymbol{a}$ is the first term and $\boldsymbol{r}$ is the common ratio. A *geometric series* is the sum of the terms of this sequence and is written

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1}$$

To determine a formula, multiply the series by the common ratio and subtract the original series.

$$\begin{aligned} rS_n &= \quad\;\; ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1} + ar^n \\ -\quad S_n &= -(a + ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1}) \\ \hline (r-1)\ S_n &= -a + ar^n \end{aligned}$$

$\boxed{S_n = \dfrac{a(r^n - 1)}{r-1}, r \neq 1, \text{ which is the } \textit{sum} \text{ of the first } n \text{ terms of a geometric sequence.}}$

Recognizing that $t_1 = a$ and $t_{n+1} = ar^n$, we can also write: $\boxed{S_n = \dfrac{t_{n+1} - t_1}{r-1}, r \neq 1}$

Example 3) Determine the sum of the series -3 – 6 – 12 – 24 – ... – 768

Example 4) Determine t_n, S_n, and S_6 for the series 81 + 27 + 9 + ...

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Unit 6, Lesson 4: Annuities: Future & Present Value

An **annuity** is a series of payments at regular intervals. It can be a loan or an investment.

The **future value** of an annuity is the final amount at the end of the term of the annuity. The future value includes all of the periodic payments and the compound interest. The future value is the geometric series:

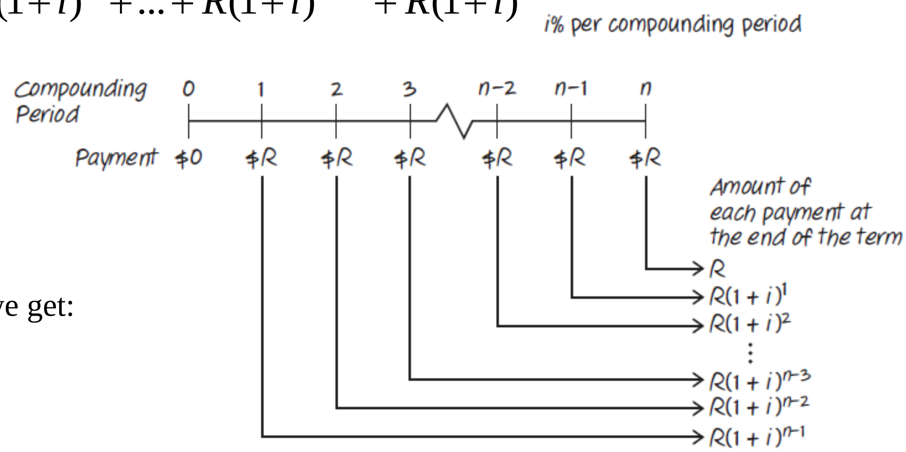
$$FV = R + R(1+i) + R(1+i)^2 + \ldots + R(1+i)^{n-2} + R(1+i)^{n-1}$$

Note that the common ratio is $1 + i$

We use the formula for the sum of a geometric series, $S_n = \dfrac{a(r^n - 1)}{r-1}$ with:

$a = R$ (the regular payment)

$r = 1 + i$



Substituting these values and simplifying, we get:

$$FV = \frac{R\left((1+i)^n - 1\right)}{i}$$

where R is the ***regular payment***

i is the interest rate ***per conversion period***

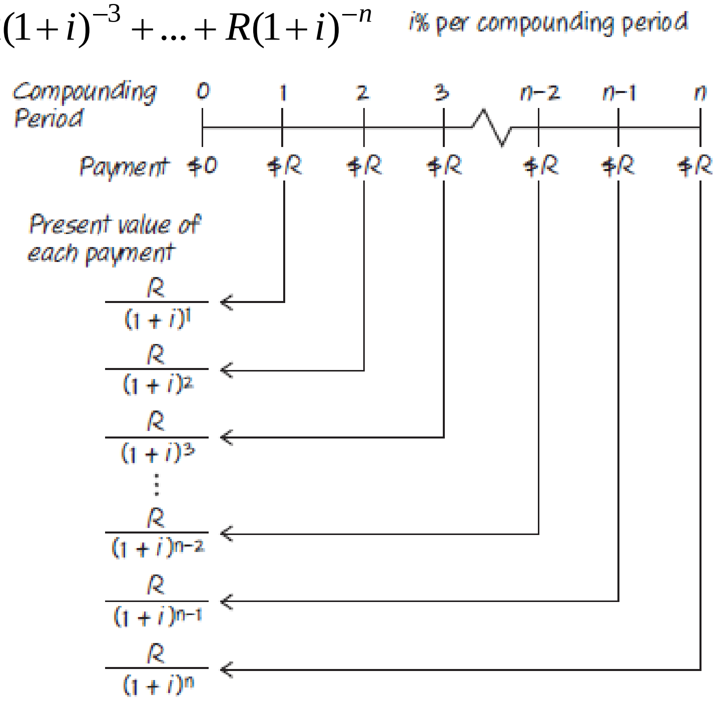
n is the ***# of payments***

Ex 1) \$750 is deposited at the end of every 3 months for 5 years at 3.25%/a, compounded quarterly. Find the value of the annuity on the date of the last payment.

Ex 2) Jenni wants to save up money to buy a house. She knows she will need \$20,000 for a down payment. How much should she put aside each month, at 4.5%/a interest, compounded monthly, if she has 5 years to save up?

The **present value** of an annuity is the value of the annuity at the beginning of the term. It is the sum of all present values of the payments and can be written as the geometric series

$$PV = R(1+i)^{-1} + R(1+i)^{-2} + R(1+i)^{-3} + \dots + R(1+i)^{-n}$$



Note that the common ratio is $(1+i)^{-1}$

We use the formula for the sum of a geometric series, $S_n = \dfrac{a(r^n - 1)}{r-1}$ with:

$a = R(1+i)^{-1}$ and $r = (1+i)^{-1}$

Substituting these values & simplifying, we get:

$$PV = \frac{R(1+i)^{-1}\left((1+i)^{-n} - 1\right)}{(1+i)^{-1} - 1} \times \frac{1+i}{1+i}$$

$$PV = \frac{R\left((1+i)^{-n} - 1\right)}{1 - (1+i)}$$

$$PV = \frac{R\left(1 - (1+i)^{-n}\right)}{i}$$

where R is the ***regular payment***

i is the interest rate ***per conversion period***

n is the ***# of payments***

Ex 3) Victor is looking to buy a car. He determines that he can afford payments of at most $400 per month, and he is hoping to pay off the car in 4 years. He can get a loan from his bank for 6.75%, compounded monthly. What should he set as his budget for car shopping?

Ex 4) Jenni manages to save up her $20,000 for her down payment. But she decides to buy a new boat instead. She finds a small cabin cruiser advertised at $105,000. The bank approves her loan at an interest rate of 5.25%, compounded monthly. If she wants to pay off her boat in 10 years, what should her monthly payments be? How much interest will she pay over the life of the loan?