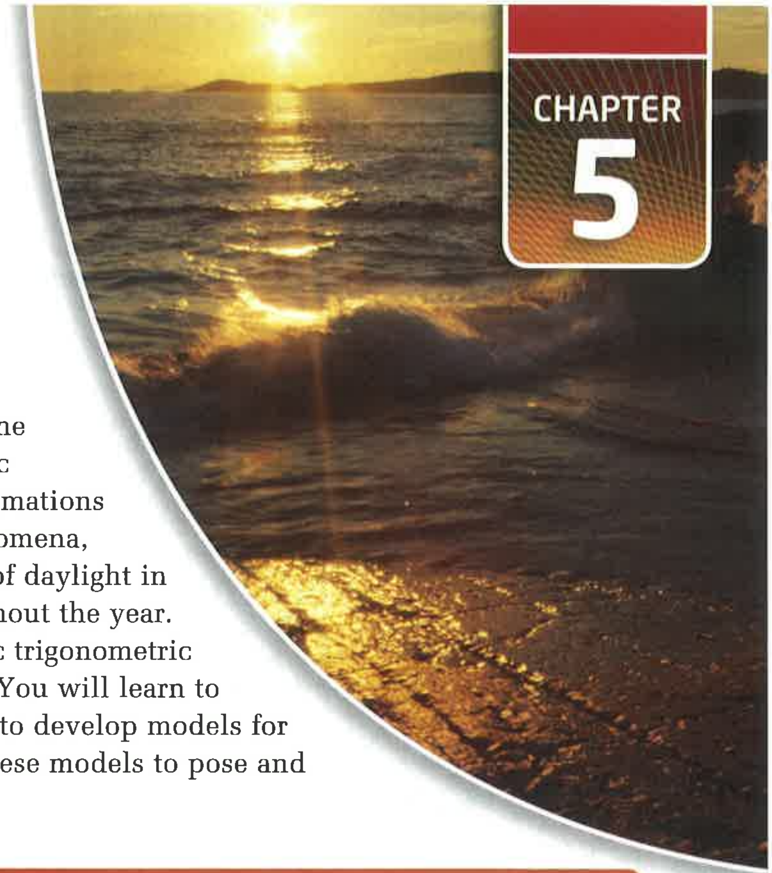


Trigonometric Functions

Trigonometric ratios can be used to extend the concept of functions to include trigonometric functions. These functions and their transformations can be used to model many real-world phenomena, such as tidal action or the number of hours of daylight in a particular location at various times throughout the year. In this chapter, you will investigate the basic trigonometric functions and learn how to transform them. You will learn to choose the parameters of the transformation to develop models for real-world applications. You will also use these models to pose and answer questions and make predictions.

By the end of this chapter, you will

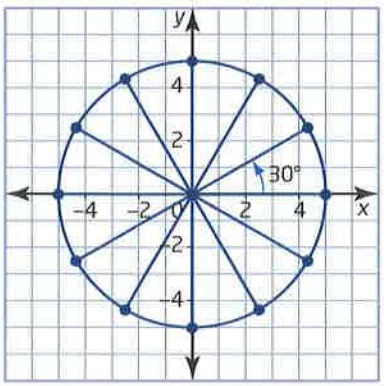
- describe key properties of periodic functions arising from real-world applications, given a numerical or graphical representation
- predict, by extrapolating, the future behaviour of a relationship modelled using a numeric or graphical representation of a periodic function
- make connections between the sine ratio and the sine function and between the cosine ratio and the cosine function by graphing the relationship between angles from 0° to 360° and the corresponding sine ratios or cosine ratios, with or without technology, defining this relationship as the function $f(x) = \sin x$ or $f(x) = \cos x$, and explaining why the relationship is a function
- sketch the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ for angle measures expressed in degrees, and determine and describe key properties
- determine, through investigation using technology, and describe the roles of the parameters a, k, d, and c in functions of the form $y = af[k(x - d)] + c$ in terms of transformations of the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ with angles expressed in degrees
- determine the amplitude, period, phase shift, domain, and range of sinusoidal functions whose equations are given in the form $f(x) = a \sin [k(x - d)] + c$ or $f(x) = a \cos [k(x - d)] + c$
- sketch graphs of $y = af[k(x - d)] + c$ by applying one or more transformations to the graphs of $f(x) = \sin x$ and $f(x) = \cos x$, and state the domain and range of the transformed functions
- represent a sinusoidal function with an equation, given its graph or its properties
- collect data that can be modelled as a sinusoidal function from primary sources or from secondary sources, and graph the data
- identify sinusoidal functions, including those that arise from real-world applications involving periodic phenomena, given various representations, and explain any restrictions that the context places on the domain and range
- determine, through investigation, how sinusoidal functions can be used to model periodic phenomena that do not involve angles
- predict the effects on a mathematical model of an application involving sinusoidal functions when the conditions in the application are varied
- pose and solve problems based on applications involving a sinusoidal function by using a given graph or a graph generated with technology from its equation

Prerequisite Skills

Refer to the Prerequisite Skills Appendix on pages 496 to 516 for examples of the topics and further practice.

Use the Cosine Law

1. A circle of radius 5 cm has a point on its circumference every 30°.



a) Find the distance between two adjacent points. Round your answer to one decimal place.

b) Find the distance between two points separated by 60°.

c) What is the distance between two points separated by 180°?

Find Trigonometric Ratios of Special Angles

2. Use a unit circle to determine the exact sine and cosine ratios of the angles in each set.

a) 30° and 60°

b) 120°, 150°, 210°, 240°, 300°, and 330°

c) 45°, 135°, 225°, and 315°

d) 0°, 90°, 180°, and 270°

Determine the Domain and Range of a Function

3. Consider the function $f(x) = x^2$. Write the domain and range using set notation.

4. Write the equation of a function with domain $\{x \in \mathbb{R}, -5 \le x \le 5\}$ and range $\{y \in \mathbb{R}, 0 \le y \le 5\}$.

Shift Functions

5. a) Graph the following functions on the same set of axes.

i) $y = x^2$

ii) $y = x^2 + 3$

iii) $y = x^2 - 2$

b) Describe the transformations of the second and third functions with respect to the first.

6. a) Graph the following functions on the same set of axes.

i) $y = x^2$

ii) $y = (x - 3)^2$

iii) $y = (x + 2)^2$

b) Describe the transformations of the second and third functions with respect to the first.

7. a) The parabola $y = x^2$ is translated 5 units to the left and 3 units down. Write the equation of the transformed function.

b) The parabola $y = x^2$ is translated 4 units to the right and 7 units up. Write the equation of the transformed function.

8. The parabola $y = x^2$ is translated horizontally and vertically so that its vertex passes through (2, 2). Describe the translations and verify your answer.

Stretch Functions

9. a) Graph the following functions on the same set of axes.

i) $y = x^2$

ii) $y = 2x^2$

iii) $y = \frac{1}{2}x^2$

b) Describe the transformations of the second and third functions with respect to the first.

10. a) Graph the following functions on the same set of axes.

i) $y = (x - 3)^2$

ii) $y = 2(x - 3)^2$

iii) $y = \frac{1}{2}(x - 3)^2$

b) Describe the transformations of the second and third functions with respect to the first.

11. The parabola $y = x^2$ is stretched vertically so that it passes through the point (2, 16). What is the equation of the stretched parabola? Verify your answer.

12. The parabola $y = (x - 4)^2$ is stretched horizontally so that it passes through the point (2, 36). What is the equation of the stretched parabola? Verify your answer.

Reflect Functions

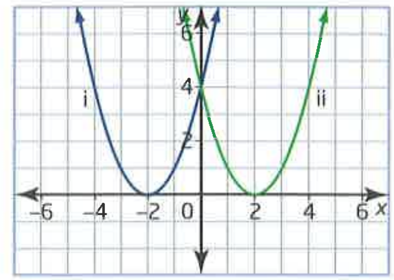
13. a) Graph the following functions on the same set of axes.

i) $y = (x + 2)^2$

ii) $y = -(x + 2)^2$

b) Use a reflection to describe the transformation of the second function with respect to the first.

14. a) Determine an equation for each of the functions shown.



b) Use a reflection to describe the transformation of function ii) with respect to function i).

Combine Transformations

15. a) The graph of $y = x^2$ is translated 4 units to the left, stretched horizontally by a factor of 2, and then translated 1 unit up. Write the equation of the transformed function.

b) Graph the original and transformed functions on the same set of axes.

16. Describe the transformations applied to $y = x^2$ to produce the graph defined by $y = -3(x - 4)^2 + 8$.

Solve Equations Involving Rational Expressions

17. Solve $\frac{360}{k} = 30$.

18. Solve $\frac{360}{k} = \frac{1}{30}$.

Chapter Problem

All the sounds that you hear are made up of waves of pressure passing through the air and vibrating on your eardrums. Sound waves can be represented using trigonometric functions. These representations can be used for diverse purposes, such as voice recognition, as well as the design of concert halls and noise-cancelling headphones.

In this chapter, you will learn how sound can be modelled by trigonometric functions and how transformations of the functions affect the characteristics of sounds, including loudness, pitch, and quality. You will learn how trigonometric functions are used in music synthesis, from the sounds of your favourite band to soundtracks for motion pictures.

5.1

Modelling Periodic Behaviour

What do the sounds of your favourite band, the idling of a car engine, the phases of the moon, and your heartbeat all have in common? All are examples of processes that repeat in a regular pattern. Your heart, when you are at rest, follows the same cycle each time it beats. A car engine has moving parts that repeat the same motions over and over. In the lunar cycle, the moon grows from a tiny sliver, to a beautiful full moon, and then wanes until it disappears, only to reappear at the beginning of the next cycle. Your favourite band plays instruments that create repetitive pressure patterns in the air that your ears interpret as music. Periodic patterns can be modelled by trigonometric functions.

Tools

- grid paper
- protractor
- ruler
- compasses

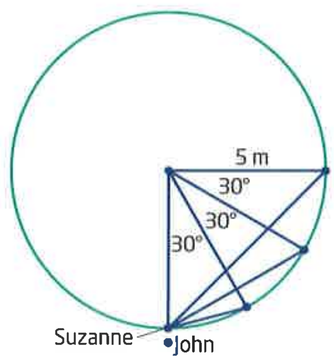
or

- graphing calculator

Investigate

How can you model periodic behaviour mathematically?

Antique carousels featuring intricately painted horses and other animals are still popular attractions. Several towns in Ontario feature them, as do many amusement parks.



Suppose that a carousel has a diameter of 10 m. John is standing at the edge of the carousel, watching his sister Suzanne on a horse on the carousel's outer edge. How does the distance between John and Suzanne change as the carousel completes a full turn? Predict the shape of a graph that represents distance versus angle for one revolution.

1. Draw a circle to represent the carousel. Mark a point just outside the circle to represent John. Mark a point on the circumference to represent Suzanne at the point where she passes John.

2. Assume that the carousel turns in a counterclockwise direction and that the distance between John and the edge of the carousel can be ignored. Mark points on the circle to represent Suzanne's position at 30° intervals for one complete revolution. For each position, determine the distance between John and Suzanne using appropriate trigonometric tools. (The diagram illustrates the first three distances to measure.) Record your results in a table with angle in the first column and distance, in metres, in the second column.

3. **Reflect** Predict the values for Suzanne's position as the carousel continues to rotate through 360° to 720°. Justify your predictions.

4. Use the values in your table to sketch the graph of distance versus angle of revolution. Record distance along the vertical axis and the angle of revolution from 0° to 720° along the horizontal axis.

5. **Reflect** Compare the graph to your predicted graph. How are the graphs similar? How are they different?

6. a) Inspect the graph for two revolutions of the carousel. Predict the total angle of revolution that Suzanne moves through in five revolutions.
   b) If a ride consists of 12 revolutions, what is the total angle of revolution that Suzanne moves through?

7. a) Use the graph to estimate two angles during the first revolution when the distance between John and Suzanne is 8 m. Locate these angles on your diagram of the carousel.
   b) In the third revolution, predict the angles when the distance between John and Suzanne is 8 m.

8. How many **cycles** are shown in the graph?

9. The **period** of a pattern is measured in units appropriate to the problem. What is the period of this pattern?

10. **Reflect** How does a **periodic function** differ from a linear function or a quadratic function?

11. What is the minimum distance between John and Suzanne during the first revolution? What is the maximum distance?

12. What is the **amplitude** of this function?

13. **Reflect** Suppose that Suzanne is on a horse that is 2 m from the centre of the carousel. Predict how the graph will change from the one that you drew in step 4. Sketch your prediction of the graph of distance versus angle of revolution for two revolutions of the carousel.

cycle

- one complete repetition of a pattern

period

- the horizontal length of one cycle on a graph

periodic function

- a function that has a pattern of y-values that repeats at regular intervals

amplitude

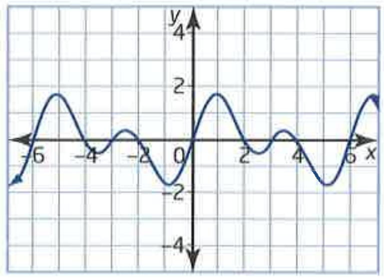
- half the distance between the maximum and minimum values of a periodic function

Example 1

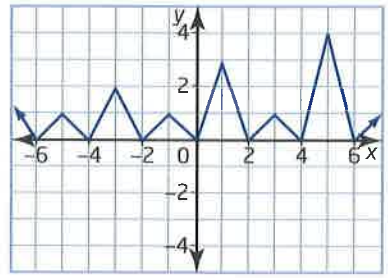
Classify Functions

a) Examine each graph. Determine whether the function is periodic. If it is, determine the period.

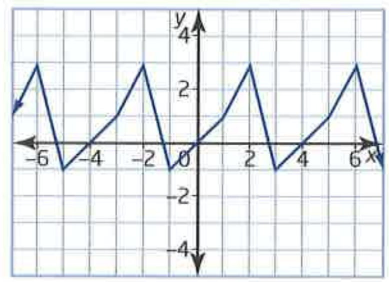
i)



ii)

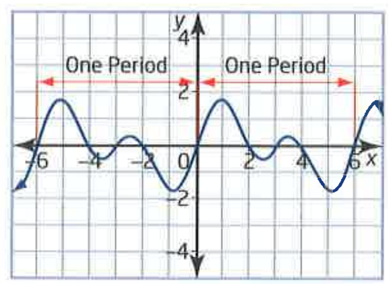


b) Examine the graph. Determine whether the function is periodic. If it is, determine the amplitude.



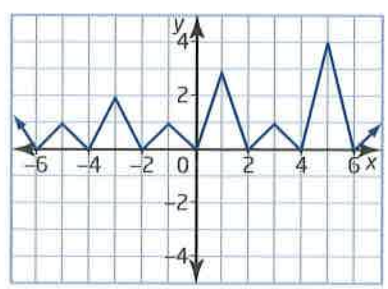
Solution

a) **i)** A periodic function has a pattern of y-values that repeats at regular intervals. The period is the length of the interval. In this example, the pattern of y-values in one section of the graph repeats in the next section. Therefore, the function is periodic.

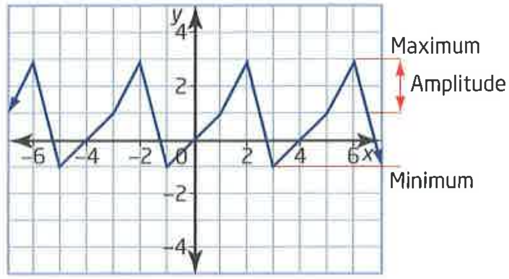


To determine the period, select a convenient starting point and note the x-coordinate. In this case, choose $(-6, 0)$. Move to the right, and estimate the x-coordinate where the next cycle begins. This appears to be at the origin. Subtract the two x-coordinates. The period is $0 - (-6)$, or 6 units.

ii) In this example, the pattern of y-values in one section of the graph does not repeat in the next section. Therefore, the function is not periodic.



b) The function illustrated in the graph is periodic because there is a repeating pattern of y-values on a regular basis. A periodic function usually has a maximum value and a minimum value every cycle. The amplitude is half the difference between the maximum and minimum values. In the graph shown, the maximum is 3 and the minimum is -1.

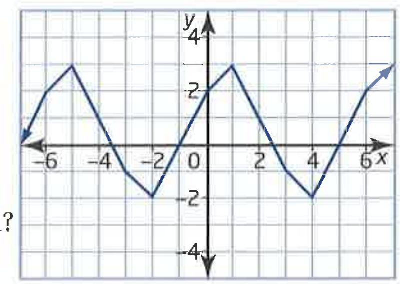


Therefore, the amplitude is $\frac{3-(-1)}{2}$, or 2 units.

Example 2

Predicting With Periodic Functions

Consider the periodic function shown.



a) What is the period of the function?

b) Determine $f(2)$ and $f(5)$.

c) Predict $f(8)$, $f(-10)$, and $f(14)$.

d) What is the amplitude of the function?

e) Determine four x-values such that $f(x) = 2$.

Solution

a) Select a convenient starting point, such as $(-7, 0)$. Move to the right until the pattern begins to repeat. This occurs at $(-1, 0)$. The period is equal to the horizontal length of this cycle, calculated by subtracting the x-coordinates. Thus, the period is $-1 - (-7)$, or 6 units.

b) Read values from the graph: $f(2) = 1$ and $f(5) = 0$.

c) $\begin{aligned} f(8) &= f(2+6) \\ &= f(2) \\ &= 1 \end{aligned}$ $\quad \begin{aligned} f(-10) &= f(-10+6+6) \\ &= f(2) \\ &= 1 \end{aligned}$ $\quad \begin{aligned} f(14) &= f(14-6-6) \\ &= f(2) \\ &= 1 \end{aligned}$

From part a), the period of the function is 6. The value of the function at x is the same as the value at x plus or minus any multiple of 6.

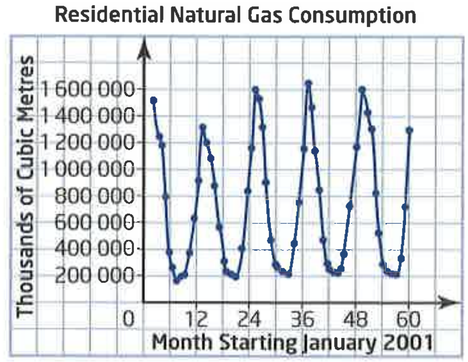
d) The maximum value is 3. The minimum value is -2. The amplitude is $\frac{3-(-2)}{2}$, or 2.5 units.

e) From the graph, the value of $f(0)$ is 2. Determine other x-values by adding the period to or subtracting the period from $x = 0$. Two possible answers are $x = 6$, $x = 12$, and $x = 18$, or $x = -6$, $x = -12$, and $x = -18$.

Example 3

Natural Gas Consumption in Ontario

The graph shows residential natural gas consumption in Ontario per month, beginning in January 2001. Data are obtained from Statistics Canada through its online E-STAT interactive tool.



a) Explain why the graph has this shape.

b) Do the data appear to be periodic? Justify your answer.

c) Assume that the consumption of natural gas in Ontario can be modelled using a periodic function. Determine the approximate maximum value, minimum value, and amplitude of this function.

d) Estimate the period of this function. Does this value make sense? Explain why.

e) Estimate the domain and range of the function.

f) Explain how the graph can be used to estimate the natural gas consumption in February 2011.

Solution

a) The consumption of natural gas in Ontario varies with the season. One expects consumption to be high in the winter months and low in the summer months.

b) The data are approximately periodic. The values do not exactly match from cycle to cycle.

c) A reasonable estimate for the maximum is 1 600 000 thousand cubic metres (1 600 000 000 m^3). The minimum is about 200 000 thousand cubic metres (200 000 000 m^3).

$$\text{Amplitude} = \frac{1\ 600\ 000 - 200\ 000}{2}$$
$$= \frac{1\ 400\ 000}{2}$$
$$= 700\ 000$$

The amplitude is about 700 000 thousand cubic metres (700 000 000 m^3).

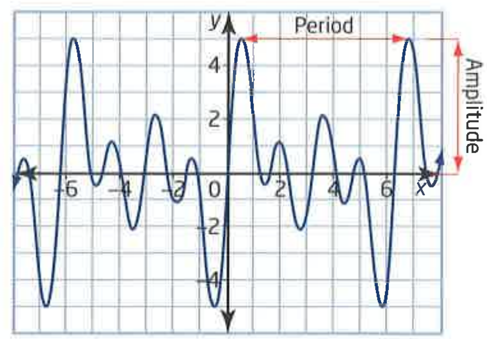
d) From the graph, the period is about 12 months. This is reasonable. One expects the seasonal cycle for consumption of natural gas to be yearly.

e) Let t represent the time, in months, and let g represent the consumption of natural gas, in thousands of cubic metres. The domain is $\{t \in \mathbb{R}, 1 \leq t \leq 60\}$. Note that the lower bound of the domain is not 0. The data begin in January, which is the first month. The range is approximately $\{g \in \mathbb{R}, 200\ 000 \leq g \leq 1\ 600\ 000\}$.

f) To obtain a reasonable estimate of gas consumption during the month of February, use the graph to find the consumption for each February shown. Take the average of these values. This is a reasonable estimate of the consumption predicted for February 2011.

Key Concepts

- A pattern that repeats itself regularly is periodic.
- A periodic pattern can be modelled using a periodic function.
- One repetition of a periodic pattern is called a cycle.
- The horizontal length of a cycle on a graph is called the period. The period may be in units of time or other units of measurement.
- A function is periodic if there is a positive number, p, such that $f(x + p) = f(x)$ for every x in the domain of $f(x)$. The least value of p that works is the period of the function.



- $f(x + np) = f(x)$, where p is the period and n is any integer.
- The amplitude of a periodic function is half the difference between the maximum value and the minimum value in a cycle.

Communicate Your Understanding

C1 The population of a mining town has increased and decreased several times in the past few decades. Do you expect the population as a function of time to be periodic? Justify your answer.

C2 **a)** Consider a function such that $f(x + q) = -f(x)$. Sketch the graph of a simple function that follows this kind of relationship.

b) Is the function in part a) periodic? Justify your answer.

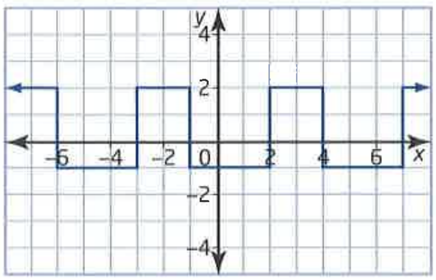
C3 Consider the decimal expansion of the fraction $\frac{1}{7}$. If you graph each digit in the expansion on the vertical axis versus its decimal place on the horizontal axis, is the pattern periodic? Use a graph to support your answer.

A Practise

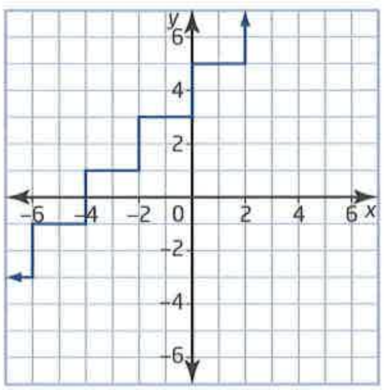
For help with questions 1 to 5, refer to Example 1.

1. Classify each graph as periodic or not periodic. Justify your answers.

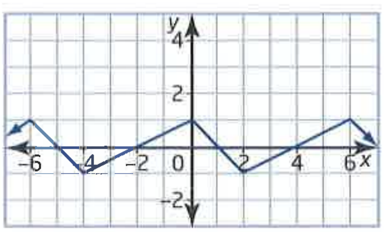
a)



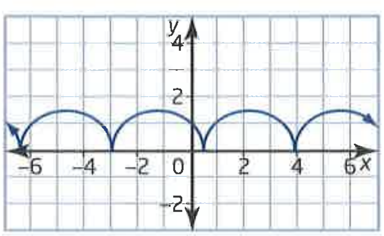
b)



c)



d)



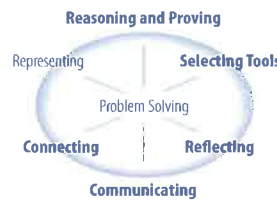
2. Determine the amplitude and period for any graph in question 1 that is periodic.

3. Sketch four cycles of a periodic function with an amplitude of 5 and a period of 3.

4. Sketch four cycles of a periodic function with an amplitude of 4 and a period of 6.

5. Do your graphs in questions 3 and 4 match those of your classmates? Explain why or why not.

For help with questions 6 to 8, refer to Example 2.



6. A periodic function $f(x)$ has a period of 8. The values of $f(1)$, $f(5)$, and $f(7)$ are -3, 2, and 8, respectively. Predict the value of each of the following. If a prediction is not possible, explain why not.

a) $f(9)$ **b)** $f(29)$

c) $f(63)$ **d)** $f(40)$

7. a) Sketch the graph of a periodic function, $f(x)$, with a maximum value of 7, a minimum value of -1, and a period of 5.

b) Select a value a for x, and determine $f(a)$.

c) Determine two other values, b and c, such that $f(a) = f(b) = f(c)$.

B Connect and Apply

8. Sunita draws a periodic function so that $f(p) = f(q)$. Can you conclude that the period of the function is the difference between p and q? Justify your answer, including a diagram.

9. A navigation light on a point in a lake flashes 1 s on and 1 s off. After three flashes, the light stays off for an extra 2 s.

a) Let 1 represent "on" and 0 represent "off." Sketch a graph with time on the horizontal axis to represent the flashing of the light. Include three cycles.

b) Explain why this pattern is periodic.

c) What is the period of the pattern?

d) What is the amplitude?

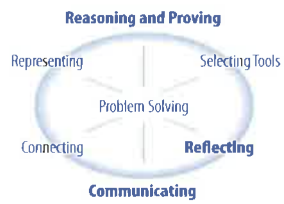
10. The people mover at an airport shuttles between the main terminal and a satellite terminal 300 m away. A one-way trip, moving at a constant speed, takes 1 min, and the car remains at each terminal for 30 s before leaving.

    a) Sketch a graph to represent the distance of the car from the main terminal with respect to time. Include four complete cycles.

    b) What is the period of the motion?

    c) What is the amplitude of the motion?

11. Which of the following values do you expect to follow a periodic pattern? Justify your answer for each case.



    a) the cost of 1 kg of tomatoes at the local supermarket at different times of the year

    b) the interest rate offered on an investment by a bank over a term of 5 years

    c) the percent of the moon's face that is illuminated over several months

    d) the volume of air in your lungs during several minutes of normal breathing

12. **Use Technology** Sunspots are huge storms that take place on the sun. They can produce electromagnetic waves that interfere with radio, television, and other communication systems on Earth. Is the number of sunspots at any particular time random, or does the number follow a periodic pattern? Use the Internet to find a graph or table of sunspot activity over several decades. Inspect the data and decide whether the number of sunspots over time may be considered periodic. Justify your answer.

13. Is it possible for a periodic function to be either continuously increasing or continuously decreasing? Justify your answer, including a diagram.

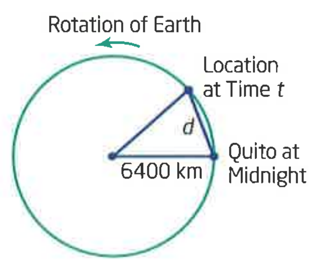
14. While visiting the east coast of Canada, Ranouf notices that the water level at a town dock changes during the day as the tides come in and go out. Markings on one of the piles supporting the dock show a high tide of 3.3 m at 6:30 a.m., a low tide of 0.7 m at 12:40 p.m., and a high tide again at 6:50 p.m.

    a) Estimate the period of the fluctuation of the water level at the town dock.

    b) Estimate the amplitude of the pattern.

    c) Predict when the next low tide will occur.

15. The city of Quito, Ecuador, is located on the equator, about 6400 km from the centre of Earth. As Earth turns, Quito rotates about Earth's axis. Consider midnight local time as the starting time and the position of Quito at that hour as the starting location. Let d represent the distance in a straight line from the starting location at time t.



    a) Explain why the graph of d versus t will show a periodic pattern.

    b) What is the period of this motion?

    c) What is the amplitude of this motion?

    d) Suppose that you want to generate a table of values for d as a function of t. Determine an appropriate trigonometric tool to use. Explain why it is the most appropriate tool for this problem. Describe how you would use the tool to generate the table of values.

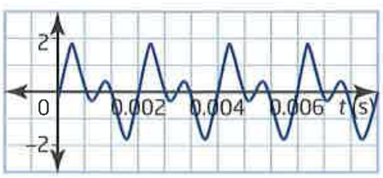
16. The average monthly temperatures over 1 year in a given location usually follow a periodic pattern.

a) Estimate the maximum value, minimum value, and amplitude of this pattern for where you live.

b) What is the period of this pattern? Explain.

17. Describe a real-world pattern that you think might be periodic. Do not use a pattern that has already been used in this section. Trade patterns with a classmate. Perform an investigation to determine whether the pattern is periodic. If you determine that the pattern is periodic, determine the period and the amplitude. If it is not, explain why not.

18. Chapter Problem Randy connects his synthesizer to an oscilloscope and plays a B key that he knows produces a sound with frequency close to 500 cycles every second, or 500 Hz (hertz). The pattern is shown. Time, in seconds, is shown on the horizontal axis.



a) Explain how you know that the pattern is periodic.

b) What is the period?

c) Determine a relation between the period and the frequency of the note being played.

Connections

One of the oldest purely electronic instruments is the theremin, invented in 1919 by Leon Theremin, a Russian engineer. Players control the instrument's sound by moving their hands toward or away from the instrument's two antennas. One antenna controls the pitch of the sound; the other controls the volume. You have probably heard the eerie, gliding, warbling sounds of a theremin in science fiction or horror films.

19. Use Technology Use dynamic geometry software to sketch a model of the carousel in the Investigate on page 284. Animate the point that represents Suzanne, and note how the measurements of angle and distance change during a revolution.

20. The hours and minutes of daylight on the first of each month of 2006 in Windsor, Ontario, are shown in the table.

Date	Daylight (Hours:Minutes)
Jan. 1	9:09
Feb. 1	10:00
Mar. 1	11:14
Apr. 1	12:43
May 1	14:04
Jun. 1	15:04
Jul. 1	15:14
Aug. 1	14:28
Sep. 1	13:10
Oct. 1	11:45
Nov. 1	10:21
Dec. 1	9:19

a) Explain why these data will show a periodic pattern over several years.

b) Predict the number of hours of daylight on May 1, 2010.

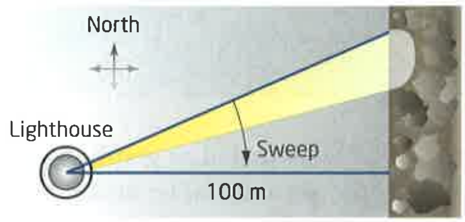
c) Predict the number of hours of daylight on September 15, 2008.

Achievement Check

21. At the doctor's office for a routine physical examination, Armand has his blood pressure checked. He notices that the pressure reaches a high value (systolic pressure) of 120 and a low value (diastolic pressure) of 80, measured in millimetres of mercury (mmHg). The doctor counts 18 pulse beats in 15 s. Is the blood pressure pattern periodic? Justify your answer.

C Extend

22. A lighthouse beacon rotates through 360° every 12 s. The lighthouse is located 100 m off the shore of an island with a coastline of steep cliffs running north and south. As the light beam sweeps clockwise, starting from north (direction of 0°), it strikes some part of the cliff.



a) How long does it take the light beam to reach an angle of 30°? What is the distance travelled by the beam to the cliff at that time? Record time and distance in a table.

b) Repeat part a) for 60°, 90°, 120°, and 150°.

c) What happens to the distance as the beam approaches 180°?

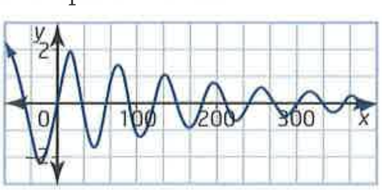
d) After passing 180°, how long does it take until the beam strikes the cliff again at some point?

e) Use your table to sketch the graph of distance versus time for one revolution of the light.

f) Explain why your graph shows a periodic pattern. What is the period?

g) What is the amplitude of the pattern? Explain.

23. In some cases, the amplitude of a function decreases with time. An example is a function used to model the sound of a plucked guitar string. As time goes on, the sound becomes fainter and dies. This is known as a damped periodic function. An example is shown.



a) Construct a table of values of y versus x, recording the maximum value for each cycle in the y column.

b) Draw a scatter plot for y versus x.

c) What kind of model do the data appear to follow?

d) Use your knowledge from a previous chapter to construct the model.

e) Graph the model on your scatter plot. Comment on the fit.

24. Math Contest If the fraction $\frac{5}{7}$ is written in expanded decimal form, what is the 100th digit after the decimal point?

A 1 **B** 4 **C** 2 **D** 8

25. Math Contest A number has the pattern 978675...0. How many digits does this number have?

A 14 **B** 16 **C** 18 **D** 20

26. Math Contest A sequence is created using the following rules.

- If the number is odd, the next number is found by adding 1 to the number and then dividing by 2.
- If the number is even, the next number is the number divided by 2.

If you start with the number 211, what is the 53rd number in the sequence?

A 27 **B** 3 **C** 1 **D** 100

27. Math Contest Of 50 students surveyed. 30 say they like algebra, 21 say they like trigonometry, and 8 say they like both. How many students do not like either algebra or trigonometry?

A 7 **B** 0 **C** 12 **D** 1

5.2

The Sine Function and the Cosine Function

What do an oceanographer, a stock analyst, an audio engineer, and a musician playing electronic instruments have in common? They all deal with periodic patterns. Periodic patterns can be represented as graphs, equations, tables, and other mathematical forms. Even a complex periodic pattern can be broken down into a sum of terms, each of which is represented as a sine or a cosine function. This process is known as Fourier analysis, which is applied in such diverse fields as music synthesis, cryptography, acoustics, oceanography, stock option pricing, and probability theory.

In this section, you will use your knowledge of the sine and cosine ratios to develop sine and cosine functions. You will investigate the properties of these functions and become familiar with the characteristics of their graphs.

Tools

- calculator
- grid paper

Investigate A

How can you use a table and grid paper with the sine ratio to construct a function?

In this chapter, you will consider wider applications of the trigonometric ratios, including applications where the independent variable does not represent an angle. In these applications, it is appropriate to use x rather than θ to represent the variable and to think of sine and cosine as functions of x: $f(x) = \sin x$ and $g(x) = \cos x$.

1. Use a unit circle to find exact values for sin x. Use a calculator to determine approximate values for sin x. Begin at 0° and continue every 30° until you reach 360°. Copy and complete the table.

x	sin x Exact Value	sin x Rounded to One Decimal Place
0°	0	0.0
30°	$\frac{1}{2}$	0.5
⋮	⋮	⋮
⋮	⋮	⋮
360°		

2. a) Plot the ordered pairs $(x, \sin x)$ on a graph, from $x = 0°$ to $x = 360°$. Use the decimal values for $\sin x$. Place the graph beside the unit circle.

   b) Draw a smooth curve through the points.

3. **Reflect** Why do you think that this graph is often called a sine wave?

4. Continue your table past 360° for several more rows. What do you notice about the entries?

5. a) Predict the shape of the graph past 360°. Justify your prediction.

   b) Verify your prediction.

6. Use your table and graph to copy and complete the table of properties for $y = \sin x$ for 0° to 360°. Leave the third column blank for now. You will use the third column in Investigate C.

Property	$y = \sin x$	
maximum		
minimum		
amplitude		
period		
domain		
range		
y-intercept		
x-intercepts		
intervals of increase		
intervals of decrease		

7. **Reflect** Explain why the graph of $y = \sin x$ is periodic.

8. How can you verify that $y = \sin x$ is a function? Perform the verification. Write the function using function notation.

Many periodic patterns follow a **sinusoidal** relation and can be represented as a simple sine or cosine function. One of these patterns is the alternating current of electricity that provides energy for lights and appliances in your home. Other patterns can be modelled using combinations of two or more sine and cosine functions, or transformations of these functions. You will see some of these later in this chapter.

sinusoidal

- having the curved form of a sine wave

Investigate B

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

How can you use technology with the sine ratio to construct a function?

Method 1: Use a Graphing Calculator

1. Press 2nd [TBLSET] to access the **TABLE SETUP** screen. This screen allows you to specify the starting value and the increment for a table of values. Set **TblStart** to 0 and **△Tbl** to 10, as shown. Make sure **Indpnt** and **Depend** are set to **AUTO**.

2. Press MODE and ensure that your calculator is in **DEGREE** mode. Ensure that all plots are turned off. Access the **Y=** editor, and enter the expression sin (X) in **Y1**.

3. Adjust the window settings such that X is plotted from 0 to 360 with a scale of 30 and Y is plotted from −2 to +2 with a scale of 0.5. Press GRAPH.

4. **Reflect** Compare the graph shown by the graphing calculator to the graph that you sketched in Investigate A. Use the scale on each axis to help you compare several points. Press 2nd [CALC]. Select **1: value** to evaluate y for your selection of x.

5. Press 2nd [TABLE] to access the table of values. Scroll down and compare the table to the one you constructed in Investigate A.

6. Continue scrolling past 360 to find the next maximum. How does it compare to your prediction in step 5a) of Investigate A? Predict where the next maximum will occur. Scroll down to check your answer.

7. Scroll back up and continue past 0. Predict where the next maximum will occur in this direction. Continue scrolling to check your answer.

8. **Reflect** Suppose that you can see the graph of $f(x) = \sin x$ from $x = -720$ to $x = 720$. How many cycles would you expect to see? Where would the maximum values occur? Where would the minimum values occur?

9. Adjust your window variables so that you can see the graph of $f(x) = \sin x$ from $x = -720$ to $x = 720$. Check your predictions from step 8.

Method 2: Use a TI-Nspire™ CAS Graphing Calculator

Tools

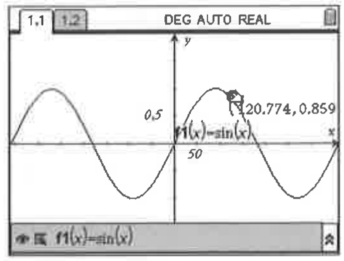
- TI-Nspire™ CAS graphing calculator

1. a) Press ⌂ and select **8:System Info**. Select **2:System Settings...**. Use the (tab) key to scroll down to **Angle**, and ensure that it is set to **Degree**. Continue on to **Auto or Approx** and ensure that it is set to **Auto**. Continue down to **OK** and press (click) twice.

b) Press ⌂ and select **6:New Document**. Select **2:Add Graphs & Geometry**.

c) Type sin(x) for function **f1**. Press (enter).

d) Press (menu). Select **4:Window**. Select **1:Window Settings**. Set **XMin** to −360, **XMax** to 360, **Ymin** to −2, and **YMax** to 2. Tab down to **OK** and press (enter). The graph will be displayed as shown.



2. Reflect Compare the graph shown by the graphing calculator to the graph that you sketched in Investigate A.

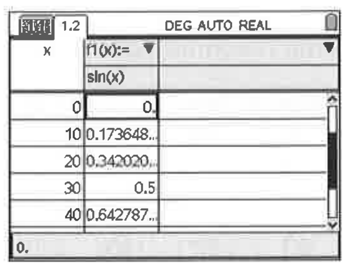
b) Press (menu). Select **6:Points & Lines**. Select **2:Point On**. Move the cursor to the graph, and press (enter).

c) Press (ctrl) (click) to grab the point. Use the cursor keys to move the point along the graph. Compare the displayed values to those in your table from Investigate A.

3. a) Press ⌂. Add a **Lists & Spreadsheet** page.

b) Press (menu). Select **5:Function Table**. Select **1:Switch to Function Table**. Press (enter). A table will appear.

c) Press (menu). Select **5:Function Table**. Select **3:Edit Function Table Settings**. Set **Table Start** to 0. Set **Table Step** to 10. Tab down to **OK** and press (enter). The function table will be displayed as shown.



4. Scroll down and compare the table to the one you constructed in Investigate A.

5. Continue scrolling past 360 to find the next maximum. How does it compare to your prediction in step 5a) of Investigate A? Predict where the next maximum will occur. Scroll down to check your answer.

6. Scroll back up and continue past 0. Predict where the next maximum will occur in this direction. Continue scrolling to check your answer.

7. **a)** Press ctrl and the left cursor key to return to the graph display.
   b) Suppose that you could see the graph of $f(x) = \sin x$ from $x = -720$ to $x = 720$. How many cycles would you expect to see? Where would the maximum values occur? Where would the minimum values occur?

8. **Reflect** Adjust your window settings so that you can see the graph of $f(x) = \sin x$ from $x = -720$ to $x = 720$. Check your predictions from step 7.

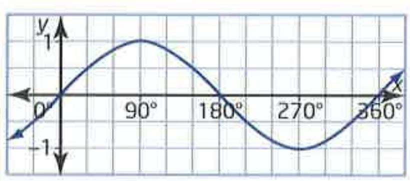
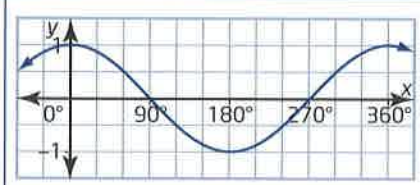
Investigate C

How can you use the cosine ratio to construct a function?

The cosine curve has similarities to the sine curve, as well as some differences. Follow the steps of Investigate A and Investigate B to investigate the cosine curve. When you finish, complete the third column of the table in step 6 of Investigate A to summarize the properties of the cosine function.

Key Concepts

- The sine and cosine ratios, along with the unit circle, can be used to construct sine and cosine functions.
- Both the sine and cosine functions have a wave-like appearance, with a period of 360°.

Properties	$y = \sin x$	$y = \cos x$
sketch of graph		
maximum value	1	1
minimum value	−1	−1
amplitude	1	1
domain	$\{x \in \mathbb{R}\}$	$\{x \in \mathbb{R}\}$
range	$\{y \in \mathbb{R}, -1 \le y \le 1\}$	$\{y \in \mathbb{R}, -1 \le y \le 1\}$
x-intercepts	0°, 180°, and 360° over one cycle	90° and 270° over one cycle
y-intercept	0	1
intervals of increase (over one cycle)	$\{x \in \mathbb{R}, 0° \le x \le 90°, 270° \le x \le 360°\}$	$\{x \in \mathbb{R}, 180° \le x \le 360°\}$
intervals of decrease (over one cycle)	$\{x \in \mathbb{R}, 90° \le x \le 270°\}$	$\{x \in \mathbb{R}, 0° \le x \le 180°\}$

Communicate Your Understanding

C1 Without a graph, predict the values of x for which the graphs of $y = \sin x$ and $y = \cos x$ will intersect in the interval from 0° to 360°. Justify your answer. Then, use graphs and tables to verify your answer.

C2 Review the x-intercepts for the sine and cosine functions. Write an expression for each graph, involving an integer n, that yields the x-intercepts when different values of n are substituted.

C3 Consider a point on the unit circle that is rotating around the circle in a counterclockwise direction.

a) Which function represents the horizontal displacement of the point with respect to the origin?

b) Which function represents the vertical displacement of the point with respect to the origin?

c) Justify your choices of functions in parts a) and b).

B Connect and Apply

1. You are in a car of a Ferris wheel. The wheel has a radius of 8 m and turns counterclockwise. Let the origin be at the centre of the wheel. Begin each sketch in parts a) and b) when the radius from the centre of the wheel to your car is along the positive x-axis.

a) Sketch the graph of your horizontal displacement versus the angle through which you turn for one rotation of the wheel. Which function models the horizontal displacement? Justify your choice.

b) Sketch the graph of your vertical displacement versus the angle through which you turn for one rotation of the wheel. Which function models the vertical displacement? Justify your choice.

2. Chapter Problem Sounds can be modelled using sinusoidal functions. A simple instrument such as a flute produces a sound that can be modelled very closely using the function $y = \sin x$. As sounds become more complex, the model must become more complex. For example, the sound from a stringed instrument can be modelled closely using a more complex function such as $y = \sin x + \sin 2x$.

a) Use technology or grid paper to sketch the graphs of $y = \sin x$ and $y = \sin x + \sin 2x$.

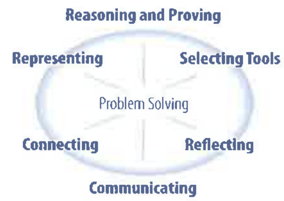
b) How do the graphs differ? How are they similar?

Connections

In musical terms, you have added the second harmonic, $\sin 2x$, to the fundamental, $\sin x$. An electronics engineer can mimic the sounds of conventional instruments electronically by adding harmonics, or overtones. This process is known as music synthesis and is the basic principle behind the operation of synthesizers. To learn more about how the addition of harmonics changes a sound, go to the *Functions 11* page of the McGraw-Hill Ryerson Web site and follow the links to Chapter 5.

c) Add the third harmonic, $\sin 3x$, to your model and sketch the graph. Compare the graph to the simple sine wave and to the sine wave together with the second harmonic.

3. The hour hand on a clock has a length of 12 cm. Let the origin be at the centre of the clock.



   a) Sketch the graph of the vertical position of the tip of the hour hand versus the angle through which the hand turns for a time period of 72 h. Assume that the hour hand starts at 9.

   b) Sketch the graph of the horizontal position of the tip of the hour hand versus the angle through which the hand turns for a time period of 72 h. Assume that the hour hand starts at 3.

   c) How many cycles appear in the graph in part a)?

   d) How many cycles will appear in the graph in part a) if you use the minute hand rather than the hour hand? Explain your prediction.

C Extend

4. What does the graph of $y = \tan x$ look like? Use a calculator to investigate. Round values of $\tan x$ to three decimal places.

   a) Construct a table of values for x and $\tan x$. Use $10°$ increments up to $70°$. Then, use $5°$ increments up to $85°$. Change to $1°$ increments up to $89°$.

   b) What happens to the value of $\tan x$ as x approaches $90°$? Review the unit circle and explain why this happens. What is the value of $\tan 90°$?

   c) Continue your table of values up to $360°$. Adjust the increment as required.

   d) Use your table of values to draw a graph of $y = \tan x$. To keep the scale manageable, use y-values from -10 to 10. Allow space on the horizontal axis x-values from $-720°$ to $720°$.

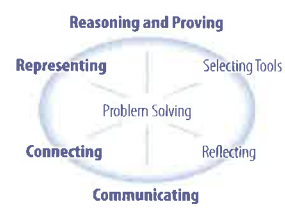
   e) Draw the asymptotes as vertical dashed lines at $x = 90°$ and $x = 270°$ on this graph. The graph of the tangent function approaches, but never reaches, each of these asymptotes.

   f) Predict what the graph will look like if it is extended past $x = 360°$ to $x = 720°$. Where will the asymptotes be drawn? Use the calculator to check a few points. Then, sketch the graph from $x = 360°$ to $x = 720°$.

   g) Predict what the graph will look like if it is extended left past $x = 0°$. Where will the asymptotes be drawn? Use the calculator to check a few points. Then, sketch the graph from $x = -720°$ to $x = 0°$.

   h) Is the graph of $y = \tan x$ periodic? Justify your answer. If the function is periodic, determine the period.

5. a) Show that $y = \tan x$ is a function.



   b) Is it possible to identify the amplitude of the function? Justify your answer.

   c) For what interval(s) of values for x from $0°$ to $360°$ is the function increasing? For what interval(s) is it decreasing?

   d) **Use Technology** Use a graphing calculator to plot the tangent function from $x = -360°$ to $x = 360°$. Compare the graph on the calculator to the graph that you sketched in question 4.

   e) You can add asymptotes at appropriate values of x. Return to the home screen. Press 2nd [DRAW]. Select **4:Vertical**, and type 90. Return to the graph. Note that a vertical line appears at $x = 90°$. Add other asymptotes as appropriate.

f) What is the domain of the tangent function? What is the range? Write each using set notation.

6. a) Sketch the function $f(x) = \sin x$ from $x = 0°$ to $x = 360°$.
   b) Review the definition of cosecant. Using your graph from part a), determine the shape of the graph of $y = \csc x$.
   c) Use grid paper and a scientific calculator, or a graphing calculator, to check your answer to part b). Add asymptotes where appropriate.
   d) Show that $y = \csc x$ is a function.
   e) What is the domain of the function? What is the range?

7. Use a method similar to that in question 6 to analyse the graph of $y = \sec x$.

8. Use a method similar to that in question 6 to analyse the graph of $y = \cot x$.

9. Consider the function $y = \sin x + \cos x$.
   a) Predict the y-intercept of the function.
   b) Predict the x-intercepts from 0° to 360°. Justify your answer.
   c) Use a graph or a graphing calculator to verify your answers to parts a) and b).

10. **Math Contest** For $0° \leq \theta \leq 360°$, over what intervals is $\sin \theta \leq \cos \theta$?

11. **Math Contest** For $-90° \leq \theta \leq 90°$, for what value(s) of θ does $\tan \theta = \cot \theta$?

12. **Math Contest** From a list of five numbers, a pair are chosen and then totalled. The sums of all possible pairs are 4, 8, 10, 12, 14, 18, 20, 22, 26, and 30. What is the sum of all five numbers?

    A 164 **B** 82
    C 41 **D** not possible to determine

Career Connection

Mariah completed a three-year diploma in respiratory therapy at Canadore College and then accumulated three years of experience. Now she is a perfusionist, where she assists during open-heart surgery by operating the heart-lung machine. This machine functions in place of a patient's heart and lungs while they are being operated on; it takes the patient's blood, oxygenates it, and pumps it back into the body. This allows the doctor to stop the patient's heart so that it can be worked on. Mariah is operating the patient's lifeline—she must monitor vital signs and make necessary adjustments to the heart-lung machine as well as administer drugs, intravenous fluids, and blood.

Use Technology

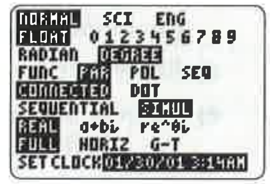
Dynamically Unwrap the Unit Circle

Tools

- graphing calculator

Draw a graph of the unit circle and plot a point at (1, 0). Recall that the coordinates of a point on the unit circle can be expressed as ($\cos\theta$, $\sin\theta$), where θ is an angle in standard position. Imagine the point moving counterclockwise around the circle. As the point moves, the angle θ increases from 0° to 360°. At the same time, the y-coordinate of the point follows the sine function. You can use a graphing calculator to plot the unit circle and the sine function simultaneously.

1. Press (MODE). Set the fourth line to parametric mode **PAR** and the sixth line to simultaneous mode **SIMUL**.



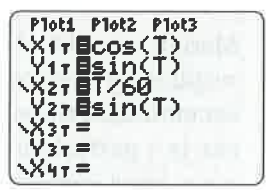
Technology Tip

When you are in parametric mode, pressing (X, T, θ, n) will return a T.

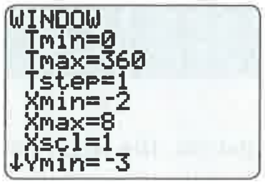
2. In parametric mode, you can enter a separate equation for each of x and y in terms of a third parameter. The calculator assigns the variable T to the third parameter. The **SIMUL** mode will plot the two graphs at the same time, rather than one after the other. Press (Y=). Notice that the list looks somewhat different from what you are used to. Enter the expression cos(T) for **X1T** and sin(T) for **Y1T**.

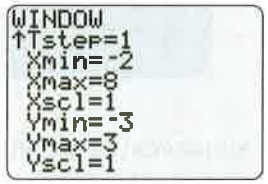
3. **Reflect** Compare these expressions to the coordinates of a point on the unit circle.

4. Plot the unit circle and the sine function on the same set of axes. The unit circle has a radius of 1 and the sine function needs an interval of 0° to 360° for one cycle. These scales are not compatible. To compensate, adjust the scale for the sine function by dividing by 60. This allows you to use a window that can accommodate both graphs. Enter T/60 for **X2T** and sin(T) for **Y2T**.

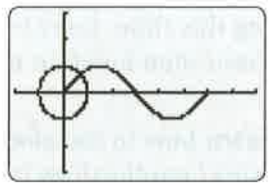


5. Press (WINDOW). The window variables will also look a little different from what you are used to. Set the T interval from 0 to 360 with a scale of 1, the X interval from -2 to 8 with a scale of 1, and the Y interval from -3 to 3 with a scale of 1.





6. Press (GRAPH). Watch how the unit circle and the sine function are drawn in step with each other. **Note:** Keep in mind that the scales are different for the two graphs so that you can display them on the same screen. For the unit circle, each mark on the x-axis represents 1 unit. For the sine function, each mark on the x-axis represents 60°.



7. **Reflect** The instructions direct you to operate the graphing calculator in a mode that is probably new to you. Review each step to ensure that you understand what is being done in that step. Explain the role of T in the functions being graphed.

8. Modify steps 1 to 6 to plot the unit circle along with the cosine curve. What changes do you need to make? Explain why these changes are necessary.

Extend

9. Suppose that you want to plot the unit circle along with the tangent, cosecant, secant, and cotangent curves, one at a time.
   a) Explain which of steps 1 to 6 need to be changed and which do not.
   b) Make the changes that you identified in part a). Then, plot the four graphs.

Technology Tip

If you want to watch the graphs be drawn again, you cannot just press (QUIT) and then press (GRAPH). The graphing calculator remembers the last graph that you asked for, and will just display it, provided that you have not made any changes that affect the graph. There are several ways to get around this feature. One is to select **PlotsOn** from the **STATPLOT** menu and then select **PlotsOff**. When you press (GRAPH), the unit circle and sine function will be drawn again.

5.3

Investigate Transformations of Sine and Cosine Functions

Many real-world processes can be modelled with sinusoidal functions. However, the basic sine function usually requires one or more transformations to fit the parameters of the process. One example is the position of the sun above the horizon north of the Arctic Circle in summer. Because the sun does not set during this time, there is no negative value for its position relative to the horizon. As a result, the basic sine function must be adjusted so that the range has no negative values.

In this section, you will learn how to transform the sine and cosine functions so that you can use them as models for real-world applications later in the chapter.

Tools

- graphing calculator

Optional

- graphing software

Investigate

How can you investigate transformations of sine and cosine functions using technology?

A: Graph $y = a \sin x$

1. a) Use a graphing calculator or graphing software to graph $y = \sin x$ and $y = 2 \sin x$ on the same set of axes, from $x = 0°$ to $x = 360°$.

b) How are the graphs similar? How are they different?

2. a) Add the graph of $y = 3 \sin x$ to the same set of axes.

b) What is the effect of multiplying the sine function by a constant factor?

3. a) Hide the graphs of $y = 2 \sin x$ and $y = 3 \sin x$. Predict the shape of the graph of $y = \frac{1}{2} \sin x$. Justify your prediction.

b) Graph $y = \frac{1}{2} \sin x$ and compare the graph to the graph of $y = \sin x$. Was your prediction correct?

4. a) Hide the graph of $y = \frac{1}{2} \sin x$. Predict the shape of the graph of $y = -\sin x$. Justify your prediction.

b) Graph $y = -\sin x$ and compare the graph to the graph of $y = \sin x$. Was your prediction correct?

c) Predict the shape of the graph of $y = -2 \sin x$. Verify your prediction by graphing.

Technology Tip

Another way to compare two graphs is to toggle them on and off. In the **Y=** editor, move the cursor over the equal sign and press ENTER. When the equal sign is highlighted, the graph is displayed. When the equal sign is not highlighted, the graph is not displayed.

5. **Reflect** Consider the transformation of the sine function that results from multiplying the function by a factor a: $y = a \sin x$. Describe the transformation under the following conditions.

a) $a > 1$

b) $0 < a < 1$

c) $a < -1$

d) $-1 < a < 0$

B: Graph $y = \sin kx$

1. **a)** Graph the functions $y = \sin x$ and $y = \sin 2x$ on the same set of axes from $x = 0°$ to $x = 360°$.

   b) How are the graphs similar? How are they different?

   c) How are the periods related?

2. **a)** Hide the graph of $y = \sin 2x$. Predict the period of the graph of $y = \sin 3x$.

   b) Graph $y = \sin 3x$ to verify your prediction.

3. **a)** Hide the graph of $y = \sin 3x$. Predict the period of the graph of $y = \sin \frac{1}{2}x$.

   b) Graph $y = \sin \frac{1}{2}x$ to verify your prediction. You may need to adjust your window to accommodate one full cycle of $y = \sin \frac{1}{2}x$.

4. **Reflect** Consider the transformation of the sine function that results from multiplying the variable x by a factor k: $y = \sin kx$.

   a) How is the period of $y = \sin kx$ related to the period of $y = \sin x$?

   b) Describe the transformation that occurs when $k > 1$ and when $0 < k < 1$.

   c) Describe the transformation that occurs when $k < -1$ and when $-1 < k < 0$. Graph to check your conjectures.

C: Graph $y = \sin (x - d)$

1 **a)** Graph the functions $y = \sin x$ and $y = \sin (x - 30°)$ on the same set of axes from $x = 0°$ to $x = 360°$.

   b) How are the graphs similar? How are they different?

2. **a)** Predict what the graph of $y = \sin (x - 60°)$ will look like. Justify your prediction.

   b) Graph $y = \sin (x - 60°)$ to verify your prediction.

3. **a)** Predict what the graph of $y = \sin (x + 30°)$ will look like. Justify your prediction.

   b) Graph $y = \sin (x + 30°)$ to verify your prediction.

4. **Reflect** Consider the transformation of the sine function that results from subtracting a parameter d from the variable x: $y = \sin (x - d)$. Describe the transformation under the following conditions.

   a) $d > 0$

   b) $d < 0$

D: Graph $y = \sin x + c$

1. a) Graph the functions $y = \sin x$ and $y = \sin x + 1$ on the same set of axes from $x = 0°$ to $x = 360°$.

b) How are the graphs similar? How are they different?

2. a) Predict what the graph of $y = \sin x - 1$ will look like. Justify your prediction.

b) Graph $y = \sin x - 1$ to verify your prediction.

3. a) Predict what the graph of $y = \sin x + 3$ will look like. Justify your prediction.

b) Graph $y = \sin x + 3$ to verify your prediction. You may need to adjust your window variables to display the graph properly.

4. Reflect Consider the transformation of the sine function that results from adding a parameter c to the function: $y = \sin x + c$. Describe the transformation under the following conditions.

a) $c > 0$ **b)** $c < 0$

5. Reflect Review your reflections from parts A, B, C, and D. Copy and complete a table similar to the one shown to summarize the effect that the value of each of these factors has on the form of the graph. The first line of the table is filled in for you.

Factor	Value	Effect
a	$a > 1$	amplitude is greater than 1
	$0 < a < 1$	
	$-1 < a < 0$	
	$a < -1$	
k	$k > 1$	
	$0 < k < 1$	
	$-1 < k < 0$	
	$k < -1$	
d	$d > 0$	
	$d < 0$	
c	$c > 0$	
	$c < 0$	

Example 1

Functions of the Form $y = a \sin kx$

Consider the function $y = 3 \sin 4x$.

a) What is the amplitude?

b) What is the period?

c) How many cycles will occur between $x = 0°$ and $x = 360°$?

d) Graph the function from $x = 0°$ to $x = 360°$.

Solution

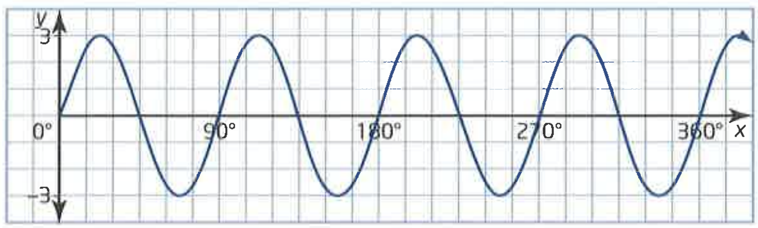
Compare $y = 3 \sin 4x$ to $y = a \sin kx$.

a) $a = 3$; the y-values of the sine function are multiplied by a factor of 3. The amplitude is 3 units.

b) $k = 4$; to determine the period, divide 360° by k: $\frac{360°}{4} = 90°$.
The period is 90°.

c) Since the period is 90° and $k = 4$, four cycles will occur between 0° and 360°.

d)



Example 2

Functions of the Form $y = a \sin (x - d) + c$

Consider the function $y = \sin (x - 45°) + 2$.

a) What is the amplitude?

b) What is the period?

c) Describe the phase shift, or horizontal translation.

d) Describe the vertical shift, or vertical translation.

e) Graph the function from $x = 0°$ to $x = 720°$.

Connections

In a sinusoidal function, a horizontal translation is also known as a phase shift. A vertical translation is also known as a vertical shift.

Solution

Compare $y = \sin (x - 45°) + 2$ to $y = a \sin (x - d) + c$.

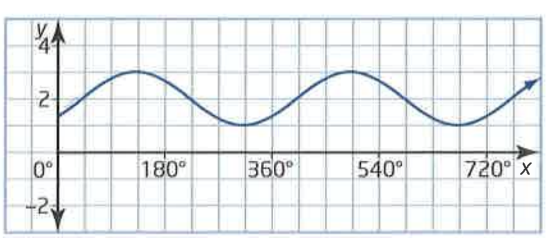
a) $a = 1$; the amplitude of the function is 1.

b) $k = 1$; the period of the function is 360°.

c) $d = 45°$; the phase shift is 45° to the right.

d) $c = 2$; the vertical shift is 2 units up.

e)



Example 3

Functions of the Form $y = a \sin [k(x - d)] + c$

Consider the function $y = -2 \sin [3(x + 30°)] - 1$.

a) What is the amplitude?

b) What is the period?

c) Describe the phase shift.

d) Describe the vertical shift.

e) Graph the function from $x = 0°$ to $x = 360°$.

Solution

Compare $y = -2 \sin [3(x + 30°)] - 1$ to $y = a \sin [k(x - d)] + c$.

a) $a = -2$; the amplitude of the function is 2.

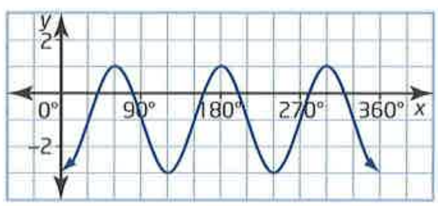
b) $k = 3$; the period of the function is $\frac{360°}{3} = 120°$.

The value of d is negative. This indicates a horizontal translation to the left.

c) $d = -30°$; the phase shift is 30° to the left.

d) $c = -1$; the vertical shift is 1 unit down.

e)



Key Concepts

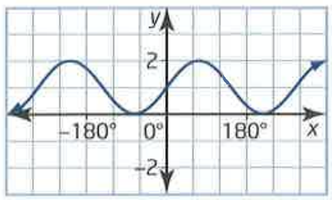
- The sine function may be transformed by introducing factors a, k, d, and c: $y = a \sin [k(x - d)] + c$
  - a determines the amplitude of the function. The amplitude is $|a|$.
  - k determines the period, p, of the function according to the relation $p = \frac{360°}{|k|}$.
  - d determines the horizontal translation, or phase shift, of the function. If d is positive, the shift is to the right. If d is negative, the shift is to the left.
  - c determines the vertical translation, or vertical shift, of the function. If c is positive, the shift is up. If c is negative, the shift is down.
- The cosine function may be transformed in the same way: $y = a \cos [k(x - d)] + c$. You will work through examples involving the cosine function in the exercises.

Communicate Your Understanding

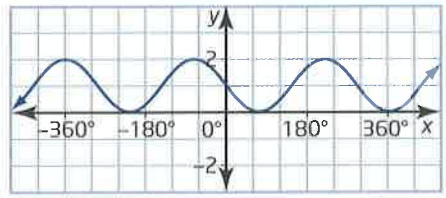
C1 Consider the graph of $y = 5 \sin 2x + c$. What values of c are required so that the graph has only positive y-values? Explain how you found your answer.

C2 Compare the following graphs. Explain how a horizontal reflection can be used to transform graph a) into graph b).

a)



b)



C3 Consider the graphs of $y = \sin x$ and $y = \cos x$. Determine two ways in which the graph of the cosine function can be expressed as a horizontal translation of the sine function.

A Practise

For help with questions 1 to 3, refer to Example 1.

1. Sketch one cycle for each function. Include an appropriate scale on each axis. State the vertical stretch and amplitude of the function.

a) $y = 4 \sin x$ **b)** $y = \frac{3}{2} \sin x$

c) $y = -5 \sin x$ **d)** $y = -\frac{5}{4} \sin x$

2. Sketch one cycle for each function. Include an appropriate scale on each axis. State the vertical stretch and then the amplitude of the function.

a) $y = 3 \cos x$ **b)** $y = \frac{1}{2} \cos x$

c) $y = -2 \cos x$ **d)** $y = -\frac{2}{3} \cos x$

3. Determine the horizontal stretch and the period of each function.

a) $y = 2 \sin 5x$ **b)** $y = -3 \sin \frac{2}{3}x$

c) $y = 8 \sin \frac{1}{6}x$ **d)** $y = \frac{1}{2} \sin \frac{1}{2}x$

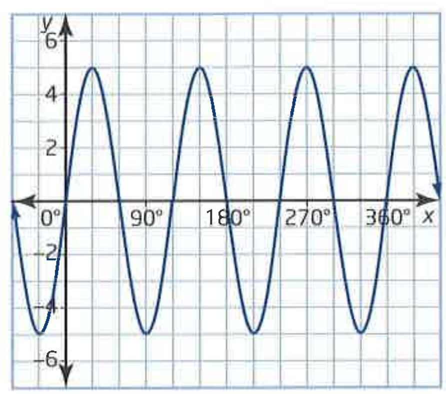
e) $y = 4 \cos x$ **f)** $y = -2 \cos 8x$

g) $y = \frac{1}{2} \cos 12x$ **h)** $y = -\frac{5}{4} \cos \frac{3}{4}x$

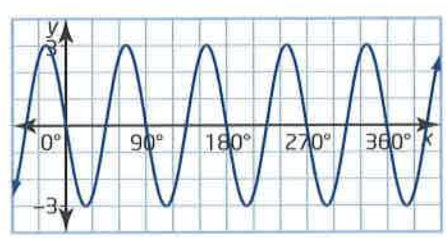
For help with Questions 4 to 8, refer to Examples 2 and 3.

4. Write two equations, one in the form $y = a \sin kx$ and one in the form $y = a \cos [k(x - d)]$, to match each graph.

a)

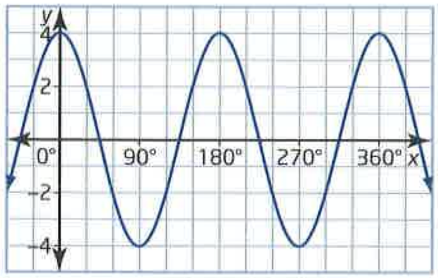


b)

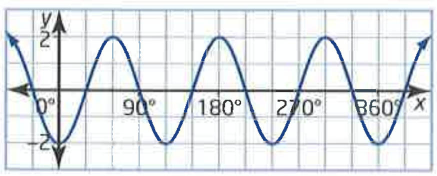


5. Write two equations, one in the form $y = a \cos kx$ and one in the form $y = a \sin [k(x - d)]$, to match each graph.

a)



b)



6. Determine the phase shift and the vertical shift with respect to $y = \sin x$ for each function.

a) $y = \sin (x - 50°) + 3$

b) $y = 2 \sin (x + 45°) - 1$

c) $y = -5 \sin (x - 25°) + 4$

d) $y = 3 \sin [2(x + 60°)] - 2$

7. Determine the phase shift and the vertical shift with respect to $y = \cos x$ for each function.

a) $y = \cos (x + 30°)$

b) $y = 4 \cos (x - 32°) + 6$

c) $y = -9 \cos (x + 120°) - 5$

d) $y = 12 \cos [5(x - 150°)] + 7$

8. **a)** State the phase shift and the vertical shift of each sinusoidal function.

i) $y = \sin (x + 100°) + 1$

ii) $y = 2 \sin x + 3$

iii) $y = \sin (x + 45°) - 2$

iv) $y = 3 \sin (x - 120°) + 2$

b) Sketch two cycles of the graph of each function. Include an appropriate scale on each axis.

9. **a)** State the vertical shift and the amplitude of each sinusoidal function.

i) $y = \cos (x - 70°)$

ii) $y = 3 \cos x - 1$

iii) $y = \cos (x + 35°) + 2$

iv) $y = 4 \cos (x - 120°) - 3$

b) Sketch two cycles of the graph of each function. Include an appropriate scale on each axis.

B Connect and Apply

10. The vertical position, y, in centimetres, of a point on the rim of the wheel of a stationary exercise bicycle after time, t, in seconds, can be modelled by the equation $y = 40 \sin 720t + 50$.

a) What is the lowest vertical position that the point reaches?

b) What is the highest vertical position that the point reaches?

c) What is the period of rotation of the wheel, in seconds?

d) Suppose that the period of the rotation of the wheel triples. How does the equation change? Justify your answer.

11. **a)** Determine the amplitude, the period, the phase shift, and the vertical shift of each function.

i) $y = 5 \sin [4(x + 60°)] - 2$

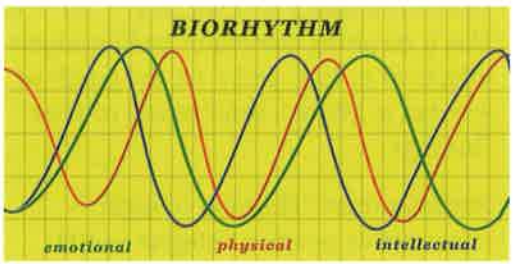
ii) $y = 2 \cos [2(x + 150°)] - 5$

iii) $y = \frac{1}{2} \sin \left[\frac{1}{2}(x - 60°)\right] + 1$

iv) $y = 0.8 \cos [3.6(x - 40°)] - 0.4$

b) **Use Technology** Graph each function using technology. Compare the graph to the characteristics you expected.

12. The theory of biorhythms seeks to explain why people have "good" days and "bad" days. According to the theory, three periodic functions begin at birth and stay with you throughout your life. The physical cycle has a period of 23 days, the emotional cycle has a period of 28 days, and the intellectual cycle has a period of 33 days. As a person moves through life, the cycles lose phase with each other, but return now and then to the same phase, such that all three maximum values coincide. Whenever a rhythm crosses the time axis, a critical day occurs.



a) Assuming an amplitude of 1, model each biorhythm—physical, emotional, and intellectual—with a sine function that begins at birth.

b) Use grid paper or technology to plot all three functions on the same set of axes for the first 150 days of life.

c) Identify good days (when two or more biorhythms reach a maximum or close to a maximum).

d) Identify bad days (when two or more biorhythms reach a minimum or close to a minimum).

13. **Chapter Problem** The human ear interprets the amplitude of a sound wave as loudness. Thus, a sound wave modelled by $y = 2 \sin kt$ is louder than a sound wave modelled by $y = \sin kt$. Because the human ear does not operate on a linear scale, the perceived loudness ratio is not actually 2:1. When an instrument is played, the sound wave spreads out in a spherical pattern. The amplitude decreases as the square of the distance. Suppose that a sound wave can be modelled as $y = 64 \sin kt$ at a distance of 1 m from its source. At a distance of 2 m,

$$a = \frac{64}{2^2}$$
$$= 16$$

The modelling equation becomes $y = 16 \sin kt$.

a) Write the modelling equation at a distance of 4 m from the source.

b) Write the modelling equation at a distance of 8 m from the source.

c) How far from the source does the modelling equation become $y = \frac{1}{4} \sin kt$?

14. The graph of $y = \sin x$ is transformed so that it has an amplitude of 4 and x-intercepts that coincide with the minimum values. The period is 360° and the phase shift is 0.

a) Write the equation of the transformed function.

b) What phase shift is needed for the transformed function from part a) to have a y-intercept of 2? Draw a graph to show that your answer is correct.

c) Can the required y-intercept in part b) be achieved by altering the period rather than the phase shift? Justify your answer.

15. a) Write the equation of a transformed sine function that includes at least three transformations. Generate a minimum number of clues about the transformation. Ensure that the clues can be used to determine the transformation.

b) Trade clues with a classmate. Determine the transformations required to match the clues. If more than one answer is possible, explain why.

c) Trade equations and discuss any observations or concerns.

16. a) Write the equation of a transformed cosine function that includes at least three transformations. Do not use the same three transformations you used in question 15. Generate a minimum number of clues about the transformation.

b) Trade clues with a classmate. Determine the transformations required to match the clues. If more than one answer is possible, explain why.

c) Trade equations and discuss any observations or concerns.

Achievement Check

17. a) Determine each of the following for the function $y = 6 \cos [5(x + 45°)] - 3$. Justify your answer.

i) the amplitude

ii) the period

iii) the phase shift

iv) the vertical shift of the function

b) Sketch the function in part a) on grid paper without using a table of values. Explain your reasoning.

C Extend

18. A sine function has half the period of $y = \sin x$. All other parameters of the two functions are the same.

a) Predict the number of points of intersection if the two functions are graphed from $x = 0°$ to $x = 360°$. Justify your prediction.

b) Determine the number of points of intersection if the two functions are graphed from $x = 0°$ to $x = 360°$. Was your prediction correct? If not, explain why.

c) Determine the coordinates of the first point to the right of the origin where the graphs of the two functions intersect.

d) Suppose that the graph is continued to $x = 720°$. How many points of intersection do you expect in the new section of the graph? Justify your thinking.

e) Extend the graph and check your prediction in part d). Was your prediction correct? If not, explain why not.

f) Suppose that the second function had a period of one-third the first. How does your answer to part a) change?

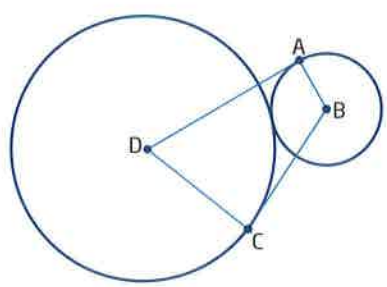
19. Consider the function $y = \tan x$ and its transformation $y = a \tan [k(x - d)] + c$. Use a graphing calculator to investigate the effects of a, k, d, and c on the graph of $y = \tan x$. Write a brief report of your findings.

20. **Math Contest** The sum of eight consecutive positive integers is 404. What is the sum of the least and greatest numbers in this sequence?

A 50.5 **B** $2\sqrt{101}$

C 101 **D** 25.25

21. **Math Contest** Two circles are tangent to each other. A line segment drawn from the centre, D, of the larger circle is tangent to the smaller circle at A. Another line segment drawn from the centre, B, of the smaller circle is tangent to the larger circle at C. Prove that AD is longer than BC.



5.4

Graphing and Modelling With $y = a \sin [k(x - d)] + c$ and $y = a \cos [k(x - d)] + c$

In order to model a real-world situation using a sine or a cosine function, you must analyse the situation and then transform the amplitude, period, vertical shift, and phase shift accordingly. For example, tides in the ocean can be modelled using a sine function with a period of about 12 h.

In this section, you will learn how to use a graph or a list of properties of the desired function to write a corresponding equation.

Example 1

Determine the Characteristics of a Sinusoidal Function From an Equation

An engineer uses the function $y = 3 \cos [2(x - 5)] + 4$ to model the vertical position, y, in metres, of a rod in a machine x seconds after the machine is started.

a) What are the amplitude, period, phase shift, and vertical shift of the position function?

b) What are the lowest and highest vertical positions that the rod reaches?

c) **Use Technology** Use technology to graph the function. Check your answers in part b) using the graph.

d) State the domain and range of the original cosine function and the transformed function.

Solution

a) Comparing the given equation $y = 3 \cos [2(x - 5)] + 4$ to the general equation $y = a \cos [k(x - d)] + c$ gives $a = 3$, $k = 2$, $d = 5$, and $c = 4$.

Since $a = 3$, the amplitude is 3 m.

Determine the period.

$$\frac{360}{k} = \frac{360}{2}$$
$$= 180$$

The period is 180 s.

Since $d = 5$, the phase shift is 5 s to the right.

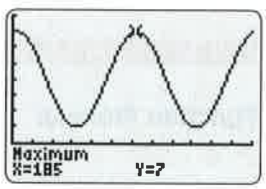
Since $c = 4$, the vertical shift is 4 m upward.

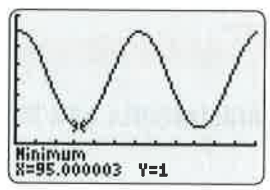
b) The least value of the basic cosine function is -1. Since the amplitude is 3, this stretches down to -3. The vertical shift of 4 m upward pushes this to 1. So, the lowest vertical position is 1 m.

The greatest value of the basic cosine function is 1. Since the amplitude is 3, this stretches up to 3. The vertical shift of 4 upward pushes this to 7. The highest vertical position is 7 m.

c) **Method 1: Use a Graphing Calculator**

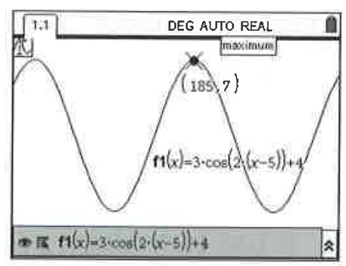
The graph is shown. Press 2nd [CALC]. Use **4:maximum** to determine the maximum value and **3:minimum** to determine the minimum value.



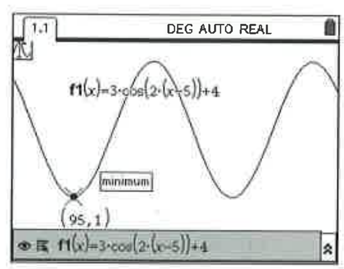


Method 2: Use a TI-Nspire™ CAS Graphing Calculator

Refer to the instructions for graphing in Section 5.2. Graph the function. Plot a point on the function. Grab the point and drag it toward the maximum. When you reach the maximum, the word "maximum" will appear, along with the coordinates.



Similarly, you can drag the point toward the minimum. When you have reached the minimum, the word "minimum" will appear, along with the coordinates.



d) For the function $y = \cos x$, the domain is $\{x \in \mathbb{R}\}$.
The range is $\{y \in \mathbb{R}, -1 \leq y \leq 1\}$.
For the function $y = 3 \cos [2(x - 5)] + 4$, the domain is $\{x \in \mathbb{R}\}$.
The range is $\{y \in \mathbb{R}, 1 \leq y \leq 7\}$.

Example 2

Sketch a Graph

a) Describe the transformations that must be applied to the graph of $f(x) = \sin x$ to obtain the graph of $g(x) = 4 \sin 3x + 1$. Apply these transformations to sketch the graph of $g(x)$.

b) State the domain and range of $f(x)$ and $g(x)$.

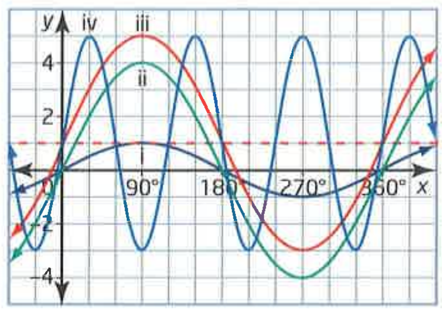
c) Modify the equation for $g(x)$ to include a phase shift of 30° to the right. Call this function $h(x)$. Apply the phase shift to the graph of $g(x)$ and transform it to $h(x)$.

Solution

a) Start with the graph of $f(x) = \sin x$, curve i).

Apply the amplitude of 4 to get curve ii).

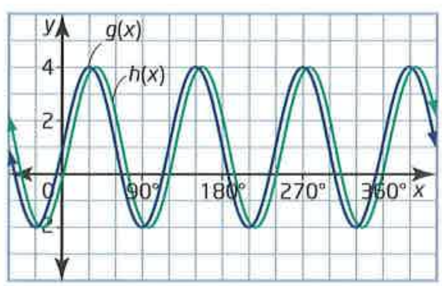
Apply the vertical shift of 1 unit upward to get curve iii). You may include a horizontal reference line at $y = 1$ to help you.



Apply the horizontal compression by a factor of 3 to get curve iv).

b) For the function $f(x) = \sin x$, the domain is $\{x \in \mathbb{R}\}$. The range is $\{y \in \mathbb{R}, -1 \leq y \leq 1\}$. For the function $g(x) = 4 \sin 3x + 1$, the domain is $\{x \in \mathbb{R}\}$. The range is $\{y \in \mathbb{R}, -3 \leq y \leq 5\}$.

c) The equation with a phase shift of 30° to the right is $h(x) = 4 \sin [3(x - 30°)] + 1$. The graphs of $g(x)$ and $h(x)$ are shown.



When graphing a transformed sine or cosine function, follow these steps:

1. Sketch the basic function.
2. Apply the vertical stretch or compression to achieve the desired amplitude.
3. Apply the vertical shift. Use a horizontal reference line to help you.
4. Apply the horizontal stretch or compression to achieve the desired period.
5. Apply the phase shift.

Example 3

Represent a Sinusoidal Function Given Its Properties

a) A sinusoidal function has an amplitude of 3 units, a period of 180°, and a maximum at (0, 5). Represent the function with an equation in two different ways.

b) Use grid paper or a graphing calculator to verify that your two models represent the same graph.

Solution

a) Method 1: Use a Cosine Function

The amplitude is 3, so $a = 3$.

The period is 180°.

$$\frac{360°}{k} = 180°$$
$$k = 2$$

A maximum occurs at (0, 5). When $x = 0$, $\cos x = 1$, which is its maximum value. The amplitude has already placed the maximum at 3. The additional vertical shift required is upward 2 units to 5. Therefore, $c = 2$.

The function can be modelled by the equation $f(x) = 3 \cos 2x + 2$.

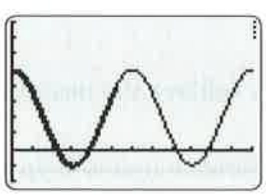
Method 2: Use a Sine Function

Use the same values of a, k, and c as in Method 1. Then, apply the appropriate phase shift to bring the maximum to (0, 5).

The maximum of the sine function normally occurs at $x = 90°$. However, the period in this case is 180°, so the maximum occurs at $\frac{90°}{2} = 45°$. To move the maximum to the y-axis, a phase shift of 45° to the left is required.

The sine function is $g(x) = 3 \sin [2(x + 45°)] + 2$.

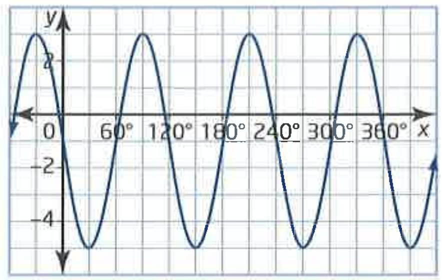
b) Enter the cosine model in **Y1** and the sine model in **Y2**. Change the line style for **Y2** to heavy. When you press GRAPH, the cosine model will be drawn first. Then, the sine model will be drawn. You can pause the graphing process by pressing ENTER while the graph is being drawn. Press ENTER again to resume.



Example 4

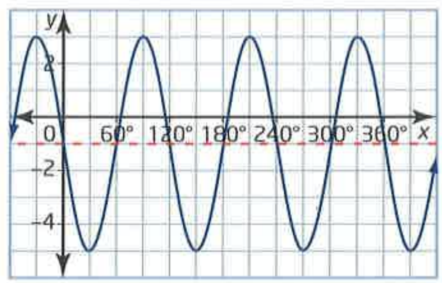
Determine a Sinusoidal Function Given a Graph

Determine the equation of a sinusoidal function that represents the graph. Check your equation using a graphing calculator.



Solution

From the graph, the maximum value of y is 3 and the minimum value is -5.



$$a = \frac{3 - (-5)}{2}$$
$$= 4$$

The amplitude is 4.

Count down 4 units from the maximum (or up 4 units from the minimum) and draw a horizontal reference line. The equation of this line is $y = -1$. The vertical shift is 1 unit downward. Therefore, $c = -1$.

Use either a sine function or a cosine function to construct the model. For this example, use a sine function. Determine the start of the first sine wave to the right of the y-axis, moving along the horizontal reference line. This occurs at $x = 60°$. The phase shift is 60° to the right. Therefore, $d = 60°$.

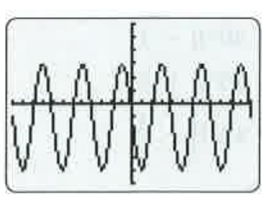
Continue along the reference line to determine the end of the first cycle. This occurs at $x = 180°$. The period is $180° - 60° = 120°$.

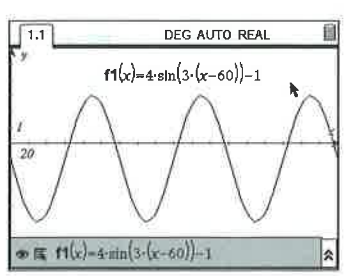
$$\frac{360°}{k} = 120°$$
$$k = 3$$

Substitute the parameters $a = 4$, $k = 3$, $d = 60°$, and $c = -1$ into the general equation $y = a \sin [k(x - d)] + c$.

$y = 4 \sin [3(x - 60°)] - 1$.

Check using a graphing calculator. The graph on the calculator matches the given graph.





Key Concepts

- The amplitude, period, phase shift, and vertical shift of sinusoidal functions can be determined when the equations are given in the form $f(x) = a \sin [k(x - d)] + c$ or $f(x) = a \cos [k(x - d)] + c$.
- The domain of a sinusoidal function is $\{x \in \mathbb{R}\}$. The range extends from the minimum value to the maximum value of the function. Any cycle can be used to determine the minimum and the maximum.
- Transformations can be used to adjust the basic sine and cosine functions to match a given amplitude, period, phase shift, and vertical shift.
- The equation of a sinusoidal function can be determined given its properties.
- The equation of a sinusoidal function can be determined given its graph.

Communicate Your Understanding

C1 The equation of a sine function is $y = 5 \sin (3x - 60°) + 2$. Explain why the phase shift is not 60°. Determine the phase shift.

C2 The equation of a cosine function is $y = \cos [2(x + 60°)]$.

a) Start with the basic cosine function. Make a rough sketch of the effect of applying the horizontal compression first and then make a second sketch of the effect of applying the phase shift.

b) Start with the basic cosine function. Make a rough sketch of the effect of applying the phase shift first and then make a second sketch of the effect of applying the horizontal compression.

c) Compare the graphs in parts a) and b). In particular, compare the location of the first maximum to the left of the y-axis. Explain any differences.

d) Which describes the correct procedure, part a) or part b)? Justify your answer. Use a graphing calculator to check your prediction.

C3 In Example 3, the desired function can be represented using either a sine function or a cosine function. Is this always the case? Justify your answer.

A Practise

For help with questions 1 and 2, refer to Example 1.

1. Determine the amplitude, the period, the phase shift, and the vertical shift of each function with respect to $y = \sin x$.

a) $y = 5 \sin [4(x - 25°)] + 3$

b) $y = -2 \sin [18(x + 40°)] - 5$

c) $y = 3 \sin [120(x - 30°)] + 2$

d) $y = \frac{3}{4} \sin \left[\frac{2}{3}(x - 60°)\right] + \frac{1}{2}$

2. Determine the amplitude, the period, the phase shift, and the vertical shift of each function with respect to $y = \cos x$.

a) $y = -3 \cos [5(x - 45°)] + 4$

b) $y = 2 \cos [24(x + 80°)] - 1$

c) $y = 3 \cos [72(x - 10°)] + 3$

d) $y = \frac{5}{2} \cos \left[\frac{3}{4}(x - 40°)\right] + \frac{1}{2}$

For help with questions 3 and 4, refer to Example 2.

3. a) Describe the transformations that must be applied to the graph of $f(x) = \sin x$ to obtain the graph of $g(x) = 3 \sin 2x - 1$. Apply each transformation, one step at a time, to sketch the graph of $g(x)$.

b) State the domain and range of $f(x)$ and $g(x)$.

c) Modify the equation for $g(x)$ to include a phase shift of 60° to the left. Call this function $h(x)$. Apply the phase shift to the graph of $g(x)$ and transform it to $h(x)$.

4. a) Transform the graph of $f(x) = \cos x$ to $g(x) = 4 \cos 3x - 2$ by applying transformations to the graph one step at a time.

b) State the domain and range of $f(x)$ and $g(x)$.

c) Modify the equation for $g(x)$ to include a phase shift of 60° to the right. Call this function $h(x)$. Apply the phase shift to the graph of $g(x)$ and transform it to $h(x)$.

For help with questions 5 and 6, refer to Example 3.

5. A sinusoidal function has an amplitude of 5 units, a period of 120°, and a maximum at (0, 3).

a) Represent the function with an equation using a sine function.

b) Represent the function with an equation using a cosine function.

6. A sinusoidal function has an amplitude of $\frac{1}{2}$ units, a period of 720°, and a maximum at $\left(0, \frac{3}{2}\right)$.

a) Represent the function with an equation using a sine function.

b) Represent the function with an equation using a cosine function.

For help with question 7, refer to Example 4.

7. a) Determine the equation of a cosine function to represent the graph in Example 4.

b) Check your equation using a graphing calculator.

B Connect and Apply

8. Consider the function $f(x) = 10 \sin (x - 45°) + 10$.

a) Determine the amplitude, the period, the phase shift, and the vertical shift of the function with respect to $y = \sin x$.

b) What are the maximum and minimum values of the function?

c) Determine the first three x-intercepts to the right of the origin.

d) Determine the y-intercept of the function.

9. Consider the function $g(x) = 5 \cos [2(x - 30°)]$.

a) Determine the amplitude, the period, the phase shift, and the vertical shift of the function with respect to $y = \cos x$.

b) What are the maximum and minimum values of the function?

c) Determine the first three x-intercepts to the right of the origin.

d) Determine the y-intercept of the function.

10. Use Technology Use a graphing calculator or graphing software to verify your answers to questions 8 and 9.

11. a) Transform the graph of $f(x) = \sin x$ to $g(x) = 5 \sin [6(x - 120°)] - 4$. Show each step in the transformation.

b) State the domain and range of $f(x)$ and $g(x)$.

c) Use Technology Use a graphing calculator to check your final graph.

12. a) Transform the graph of $f(x) = \cos x$ to $g(x) = 6 \cos [5(x + 60°)] + 2$. Show each step in the transformation.

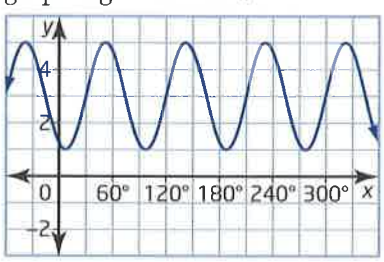
b) State the domain and range of $f(x)$ and $g(x)$.

c) Use Technology Use a graphing calculator to check your final graph.

13. a) Represent the graph of $f(x) = 2 \sin [3(x - 30°)]$ with an equation using a cosine function.

b) Use Technology Use a graphing calculator to check your graph.

14. a) Determine the equation of a sine function that represents the graph shown. Check your equation using a graphing calculator.



b) Use Technology Determine the equation of a cosine function that represents the graph. Check your equation using a graphing calculator.

15. Chapter Problem Suppose that two trumpet players play the same note. Does the result sound like one trumpet playing twice as loud or like two trumpets playing together? You have probably noticed that two instruments of the same kind playing the same note always sound like two instruments, and not like one instrument played louder. The same effect occurs for people singing. The reason is that the two notes will always differ by a phase shift. To see how this works, let the equation $y = \sin x$ represent one instrument playing a note.

a) If the second instrument could play perfectly in phase with the first, the two sounds would be represented by

$$y = \sin x + \sin x$$
$$= 2 \sin x$$

Graph this representation and $y = \sin x$ on the same set of axes. How are the two related?

b) In reality, the two instruments will be out of phase. Pick an arbitrary phase difference of 90°. The function that represents the two instruments playing together is $y = \sin x + \sin (x - 90°)$. Graph this function. How does it compare to $y = 2 \sin x$?

c) A music synthesizer can make electronic circuits that simulate instruments playing in phase with each other. This is generally not very interesting, since the sound is the same as a single instrument playing louder. Electronic engineers purposely change the phase of each instrument to achieve a "chorus" effect of several instruments playing together. Choose different phase shifts and write a function that represents four instruments playing together. Graph the function and describe the graph.

Connections

Robert Moog invented the electronic synthesizer in 1964. Although other electronic instruments existed before this time, Moog was the first to control the electronic sounds using a piano-style keyboard. This allowed musicians to make use of the new technology without first having to learn new musical skills. Visit the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 5 to find out more about the Moog synthesizer.

16. At the end of a dock, high tide of 14 m is recorded at 9:00 a.m. Low tide of 6 m is recorded at 3:00 p.m. A sinusoidal function can model the water depth versus time.

a) Construct a model for the water depth using a cosine function, where time is measured in hours past high tide.

b) Construct a model for the water depth using a sine function, where time is measured in hours past high tide.

c) Construct a model for the water depth using a sine function, where time is measured in hours past low tide.

d) Construct a model for the water depth using a cosine function, where time is measured in hours past low tide.

e) Compare your models. Which is the simplest representation if time is referenced to high tide? low tide? Explain why there is a difference.

Achievement Check

17. a) Describe the transformations that must be applied to the graph of $f(x) = \sin x$ to obtain the graph of $g(x) = 2 \sin [4(x - 40°)] - 3$.

b) Sketch the graph of $g(x)$ by applying the transformations described in part a).

c) State the domain and range of $g(x)$. Justify your answer.

C Extend

18. Suppose that you are given the coordinates, (p, q), of a point. Can you always determine a value of a such that the graph of $y = a \sin x$ will pass through the point? If so, explain why, providing a diagram. If not, explain why, and indicate the least amount of information that needs to be added.

19. Consider the relation $y = \sqrt{\sin x}$.

a) Sketch the graph of the function $y = \sin x$ over two cycles.

b) Use the graph from part a) to sketch a prediction for the shape of the graph of $y = \sqrt{\sin x}$.

c) Use technology or grid paper and a table of values to check your prediction. Resolve any differences.

d) How do you think the graph of $y = \sqrt{\sin x + 1}$ will differ from the graph of $y = \sqrt{\sin x}$?

e) Graph $y = \sqrt{\sin x + 1}$ and compare it to your prediction. Resolve any differences.

20. a) Determine the minimum number of transformations that can be applied to $y = \sin x$ such that the maximum values of the transformed function coincide with the x-intercepts of $y = \cos x$. If this is not possible, explain why, including a diagram.

b) Determine the minimum number of transformations that can be applied to $y = \sin x$ such that the maximum values of the transformed function coincide with the x-intercepts of $y = \tan x$. If this is not possible, explain why, including a diagram.

21. Math Contest Given the function $y = 3 \sin [2(x - 30°)]$, find the smallest positive value for x that gives a maximum value for y.

22. Math Contest The period of $y = |4 \cos (3x - 30°)|$ is

A 360° **B** 90° **C** 60° **D** 120°

23. Math Contest When a number is divided by 21, the remainder is 17. What is the remainder when the number is divided by 7?

A 1 **B** 3 **C** 5 **D** 6

5.5

Data Collecting and Modelling

One of the real-world applications of sinusoidal models is the motion of a pendulum. A Foucault pendulum is used to measure the rotation of Earth. As Earth turns, the axis of swing of the pendulum rotates with it.

In this section, you will learn to collect data that can be modelled with sinusoidal functions and then construct a suitable model using transformations.

Tools

- graphing calculator
- motion sensor
- pendulum

Technology Tip

These instructions assume the use of a CBR™ motion sensor with a TI-83 Plus or TI-84 Plus graphing calculator. If you are using different technology, refer to the manual.

Technology Tip

The motion sensor cannot measure distances less than about 0.5 m. Ensure that your pendulum is never closer than this. The maximum distance that it can measure is about 4 m, but your pendulum may be too small a target to return a usable signal from this distance.

Investigate

How can you collect data on the motion of a pendulum and use the data to construct a sinusoidal model?

1. Set up a pendulum. Use a large object as the bob (at least the size of a basketball) so it can be detected by the motion sensor. Use a string or rope long enough to give the pendulum a period of more than 1 s.

2. a) Connect the CBR™ motion sensor to the graphing calculator using the cable provided.

b) Press (APPS) and select **2:CBL/CBR**. Note that the application on your calculator may have a different number.

c) At the **CBL/CBR** screen, press (ENTER). Select **3:Ranger** and press (ENTER). Note that the Ranger program on your calculator may have a different number.

d) Select **1:SETUP/SAMPLE** from the main menu. Move the cursor up to **START NOW**. Press (ENTER).

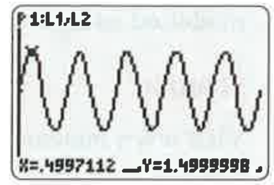
3. Start the pendulum swinging. Point the motion sensor at the bob of the pendulum such that the bob swings directly toward and away from the sensor. Press (ENTER) and collect data for 15 s. At the end of the collection time, a graph of distance versus time is shown.

4. Reflect Inspect the graph. Does it appear sinusoidal? Are there any spikes or other sudden jumps that indicate a misalignment of the pendulum and motion sensor during the data collection process? If so, adjust your physical setup. Press (ENTER) and select **REPEAT SAMPLE**. Continue refining your experimental arrangement until you have a smooth graph.

5. **a)** Press (ENTER) and select **SHOW PLOT**. Confirm that the graph is a smooth function that appears sinusoidal.

   b) Press (ENTER) and select **QUIT**. The time data will be stored in list **L1** and the distance data will be stored in list **L2**.

6. **a)** Press (GRAPH) and then (TRACE). Use the cursor keys to determine the maximum and minimum of your graph. Determine the amplitude, a.



   b) Add the amplitude to the minimum to determine the vertical shift, c. Use the **Y=** editor to plot a horizontal reference line using this value. Alternatively, you can use the **DRAW** menu and select **3:Horizontal**.

   c) Return to the graph. Use the reference line to help you trace the start of the first sine wave to the right of the vertical axis. Read the phase shift, d.

   d) Continue to the end of the first cycle. Use the start and end to determine the period. Once you have the period, determine the value of k.

7. **a)** Use the values of a, c, d, and k to write the equation to model the motion of the pendulum.

   b) Enter this equation using the **Y=** editor. Plot the curve and verify that it closely matches the graph drawn by the Ranger software. If there are major discrepancies, check your calculations.

8. **Reflect** Start the pendulum swinging and estimate the amplitude and period. Review the amplitude and period of the pendulum that you calculated in parts a) and d) of step 6. Compare the two sets of values.

9. **a)** Predict the effect that each of the following will have on your graph. Consider each one separately.

   i) the length of the pendulum is shortened

   ii) the amplitude is increased

   b) Check your predictions in part a) using the pendulum and the motion sensor.

Connections

For help in determining an equation given a graph, refer to Example 4 in Section 5.4.

Example 1

Retrieve Data from Statistics Canada

Using data from Statistics Canada, determine if there is a period of time over which changes in the population of Canadians aged 20 to 24 can be modelled using a sinusoidal function.

Solution

Visit www.statcan.gc.ca/edu/edu05_0018c-eng.htm.

Select **Sinusoidal**.

Select **table 051-0001**.

Under **Geography**, select **Canada**.

Under **Sex**, select **Both sexes**.

Under **Age group**, select **20 to 24 years**.

Set the reference period to **from 1976 to 2005**.

Select **Retrieve as individual Time Series**. If you plan to analyse the data with a spreadsheet or other software, select the downloadable file format as **CSV (comma-separated values)**, with **Time as rows**.

Otherwise, you can select the screen output as an HTML table, with **Time as rows**. Select **Retrieve Now**. The table is shown.

Population of Canada, Aged 20 to 24 Years, Both Sexes, By Year

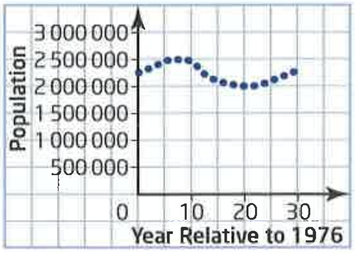
Year	Population	Year	Population	Year	Population
1976	2 253 367	1986	2 446 250	1996	2 002 036
1977	2 300 910	1987	2 363 227	1997	2 008 307
1978	2 339 362	1988	2 257 415	1998	2 014 301
1979	2 375 197	1989	2 185 706	1999	2 039 468
1980	2 424 484	1990	2 124 363	2000	2 069 868
1981	2 477 137	1991	2 088 165	2001	2 110 324
1982	2 494 358	1992	2 070 089	2002	2 150 370
1983	2 507 401	1993	2 047 334	2003	2 190 876
1984	2 514 313	1994	2 025 846	2004	2 224 652
1985	2 498 510	1995	2 009 474	2005	2 243 341

Source: Statistics Canada

Graph the data.

The population of people in Canada aged 20 to 24 from 1976 to 2005 appears to follow a sinusoidal model.

Population Aged 20 to 24 in Canada, 1976 to 2005



Example 2

Make Predictions

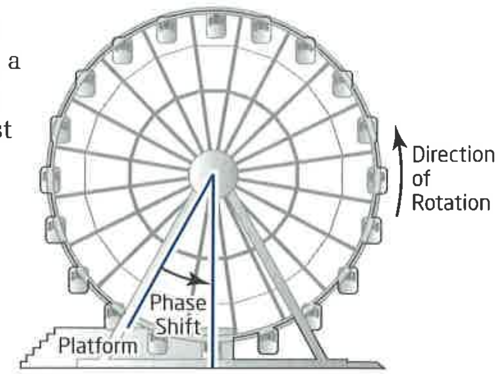
The relationship between the height above the ground of a person riding a Ferris wheel and time can be modelled using a sinusoidal function.

a) Describe the effect on this function if the platform from which the person enters the ride is raised by 1 m.

b) Describe the effect on this function if the Ferris wheel turns twice as fast.

Solution

a) Because the platform is not at the lowest point of the wheel, a phase shift is introduced. The rider does not reach the lowest point until the wheel turns through the phase shift. If the platform used to enter the ride is raised by 1 m, the phase shift increases and it takes longer for the rider to reach the lowest point in the ride. The graph will shift to the right.



b) If the wheel turns twice as fast, the period will be half as long.

The graph will be compressed horizontally by a factor of $\frac{1}{2}$.

Example 3

Use a Sinusoidal Model to Determine Values

The height, h, in metres, above the ground of a rider on a Ferris wheel after t seconds can be modelled by the sine function $h(t) = 10 \sin [3(t - 30)] + 12$.

a) Graph the function using graphing technology.

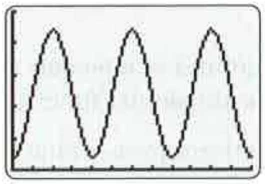
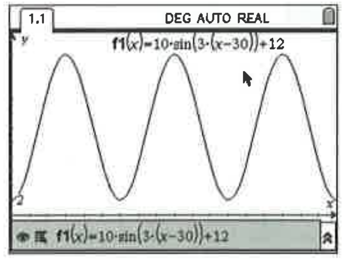
b) Determine each of the following.

i) the maximum and minimum heights of the rider above the ground

ii) the height of the rider above the ground after 30 s

iii) the time required for the Ferris wheel to complete one revolution

Solution

a)  

b) The values can be determined either by calculation or by using technology.

Method 1: Use the Equation

i) The amplitude is 10 m and the vertical shift is 12 m. Therefore, the maximum height of the rider above the ground is 10 m + 12 m, or 22 m. The minimum height is −10 m + 12 m, or 2 m.

ii) To determine the height of the rider above the ground after 30 s, substitute 30 for t in the equation.

$$\begin{aligned} h(t) &= 10 \sin [3(t - 30)] + 12 \\ &= 10 \sin [3(30 - 30)] + 12 \\ &= 10 \sin 0 + 12 \\ &= 0 + 12 \\ &= 12 \end{aligned}$$

The height of the rider above the ground after 30 s is 12 m.

iii) The value of k is 3.

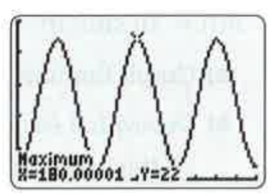
$$\frac{360}{k} = \frac{360}{3}$$
$$k = 120$$

The period is 120 s. The time required for the Ferris wheel to complete one revolution is 120 s.

Method 2: Use the Graph

i) Press 2nd [CALC]. Use the **maximum** operation from the **CALCULATE** menu.

The maximum height of the rider above the ground is 22 m.

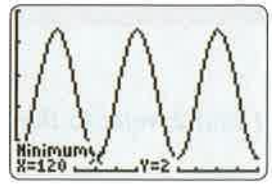


Technology Tip

If you are using a TI-Nspire™ CAS graphing calculator, refer to the instructions on page 33 to determine the maximum and minimum values and the period.

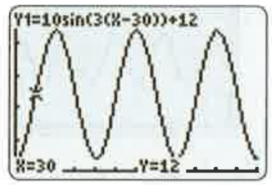
Similarly, use the **minimum** operation from the **CALCULATE** menu to determine the minimum.

The minimum height is 2 m.



ii) Use the **value** operation from the **CALCULATE** menu to determine the height after 30 s.

The height after 30 s is 12 m.



iii) To determine the period, subtract the coordinates of the adjacent maximum and minimum and multiply by 2. The period is 2(180 s − 120 s), or 120 s.

Connections

A maximum and an adjacent minimum are half a cycle apart. To obtain the value of the full period, it is necessary to multiply by 2.

Key Concepts

- Data can be collected from physical models using tools such as a motion sensor.
- Data can be downloaded from statistical sources such as Statistics Canada.
- Data can sometimes be modelled using a sinusoidal function.
- Use a graph or a table to build a model to determine the amplitude, phase shift, period, and vertical shift of a sinusoidal function.
- Predictions about the behaviour of an altered model can be made by considering the effect of changing a parameter on the graph of the original equation.
- The graph or equation can be used to determine values.

Communicate Your Understanding

C1 Consider the model you constructed in the Investigate. How does your analysis change if you decide to model the motion with a cosine function rather than a sine function?

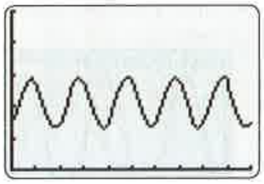
C2 Consider the population graph for people in Canada aged 20 to 24. What kind of model is appropriate for the period from 1976 to 1980? from 1981 to 1988? Give reasons for your answers.

C3 Suppose that the entire Ferris wheel in Example 3 is moved upward 1 m and the phase shift is decreased to 20°. What changes do you expect to see in the graph? Justify your answer.

A Practise

For help with questions 1 and 2, refer to the Investigate.

1. A sensor is used to gather data on the motion of a pendulum. Data are recorded on a graph with time, in seconds, from 0 s to 10 s, on the horizontal axis and distance, in metres, from 0 m to 5 m, on the vertical axis.



a) Use the graph to estimate the maximum and minimum values. Then, use these values to find the amplitude, a.

b) Copy the graph and sketch a horizontal reference line. Estimate the vertical shift, c.

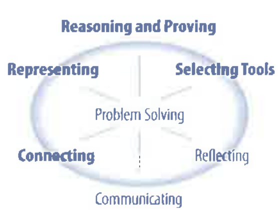
c) Use the horizontal reference line to estimate the phase shift, d.

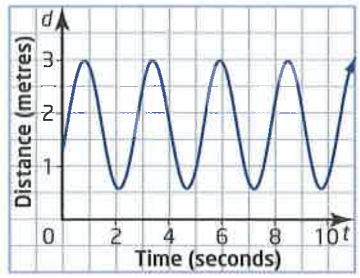
d) Use the horizontal reference line to estimate the period. Use the period to find the value of k.

e) Construct a model for the motion by writing an equation using a sinusoidal function.

f) **Use Technology** Use technology to graph your model. Compare your model to the graph shown. If you see any significant differences, check and adjust your model.

2. A motion sensor is used to gather data on the motion of a pendulum. The table of values is exported to a computer, and graphing software is used to draw the graph shown. Time, in seconds, is on the horizontal axis. Distance, in metres, is on the vertical axis.





a) Use the graph to estimate the maximum and minimum values. Then, use these values to find the approximate amplitude, a.

b) Copy the graph and sketch a horizontal reference line. Estimate the vertical shift, c.

c) Use the horizontal reference line to estimate the phase shift, d.

d) Use the horizontal reference line to estimate the period. Use the period to find the value of k.

e) Construct a model for the motion by writing an equation using a sinusoidal function.

f) **Use Technology** Use technology to graph your model. Compare your model to the graph shown. If you see any significant differences, check and adjust your model.

For help with questions 3 and 4, refer to Example 3.

3. The height, h, in metres, of the tide in a given location on a given day at t hours after midnight can be modelled using the sinusoidal function
$h(t) = 5 \sin [30(t - 5)] + 7$.

a) Find the maximum and minimum values for the depth, h, of the water.

b) What time is high tide? What time is low tide?

c) What is the depth of the water at 9:00 a.m.?

d) Find all the times during a 24-h period when the depth of the water is 3 m.

4. The population, P, of a lakeside town with a large number of seasonal residents can be modelled using the function $P(t) = 5000 \sin [30(t - 7)] + 8000$, where t is the number of months after New Year's Day.
   a) Find the maximum and minimum values for the population over a whole year.
   b) When is the population a maximum? When is it a minimum?
   c) What is the population on September 30?
   d) When is the population about 10 000?

For help with question 5, refer to Example 1.

5. a) The owner of an ice-cream shop kept records of average daily sales for each month for the past year, as shown, beginning in January. Construct a sinusoidal function to model the average daily sales versus the month.
   b) Over what domain and range is your model valid?

Month	Daily Sales ($)	Month	Daily Sales ($)
1	45	7	355
2	115	8	285
3	195	9	205
4	290	10	105
5	360	11	42
6	380	12	18

For help with Questions 6 and 7, refer to Example 2.

6. A Ferris wheel has a diameter of 20 m and is 4 m above ground level at its lowest point. Assume that a rider enters a car from a platform that is located 30° around the rim before the car reaches its lowest point.
   a) Model the rider's height above the ground versus angle using a transformed sine function.
   b) Model the rider's height above the ground versus angle using a transformed cosine function.
   c) Suppose that the platform is moved to 60° around the rim from the lowest position of the car. How will the equations in parts a) and b) change? Write the new equations.

7. Suppose that the centre of the Ferris wheel in question 6 is moved upward 2 m, but the platform is left in place at a point 30° before the car reaches its lowest point. How do the equations in parts a) and b) of question 6 change? Write the new equations.

8. The movement of a piston in an automobile engine can be modelled by the function $y = 50 \sin 10\,800t + 20$, where y is the distance, in millimetres, from the crankshaft and t is the time, in seconds.

   a) What is the period of the motion?
   b) Determine the maximum, minimum, and amplitude.
   c) When do the maximum and minimum values occur?
   d) What is the vertical position of the piston at $t = \frac{1}{120}$ s?

B Connect and Apply

9. The period, T, in seconds, of a pendulum is related to the length, ℓ, in metres, according to the relation

$$T = 2\pi\sqrt{\frac{\ell}{g}},$$

where g is the acceleration due to gravity, about 9.8 m/s^2, near the surface of Earth.

a) If the length is doubled, by what factor does the period increase?

b) If you want a pendulum with half the period of the given pendulum, what must you do to the length?

c) Suppose that you make a pendulum with half the period of the pendulum used in the Investigate. Modify the equation that models the motion of the pendulum to reflect the new period.

d) Suppose that you take the pendulum in the Investigate to the moon, where the rate of acceleration due to gravity is about $\frac{1}{6}$ that of the rate at Earth's surface. Predict the effect on the period of the pendulum.

e) Modify the equation that models the motion of the pendulum so that it is accurate for lunar gravity.

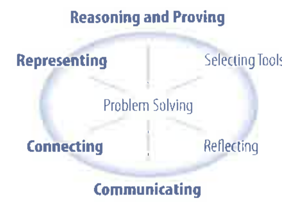
10. A double pendulum can be made by connecting a second pendulum to the bob of the first. Make a model of a double pendulum using any materials available, and set it in motion.

a) Watch the bob of the first pendulum. Describe the motion. Would it be possible to model it using a sinusoidal function?

b) Watch the bob of the second pendulum. Describe the motion. Would it be possible to model it using a sinusoidal function?

c) Try different relative lengths between the two pendulum strings. Can you determine a ratio that results in sinusoidal motion?

11. How closely can deep breathing be modelled using a sinusoidal function?



a) Practise slow, deep breathing, filling your lungs to capacity, and then exhaling until you cannot expel any more air. Keep the rate slow, such that inhaling and exhaling each take at least 5 s.

b) Fill a 2-L plastic bottle with water. Invert the bottle in a sink half full of water. Exhale through a tube such as a flexible straw, and capture the exhaled air in the bottle. Use the amount of water displaced to estimate the volume of air that you breathed out. If you have a large lung capacity, you may need to use more than one bottle. Use a felt pen to mark the level each second.

c) Make a table of values of volume exhaled versus time. Use the table to sketch a graph.

d) Use your table and graph to construct a sinusoidal function to model the data.

e) Graph the model on the same set of axes. How closely does the actual exhalation follow a sinusoidal model?

Connections

You cannot actually expel all of the air from your lungs. Depending on the size of your lungs, 1 L to 2 L of air remains even when you think your lungs are empty.

12. Smog is a generic term used to describe pollutants in the air. A smog alert is usually issued when the air quality index is greater than 50. Air quality can vary throughout the day, increasing when more cars are on the road. Consider a model of the form $I = 30 \sin [15(t - 4)] + 25$, where I is the value of the air quality index and t measures the time after midnight, in hours.

   a) What is the period of the modelled function? Explain why this makes sense.

   b) Determine the maximum, minimum, and amplitude.

   c) When do the maximum and minimum occur?

   d) During what time interval would a smog alert be issued, according to this model?

13. **a)** Find your pulse. Use a watch to determine how many times your heart beats in 1 min. Use this measurement to calculate the period of one heartbeat.

   b) Assuming an amplitude of 1 unit, use a sine function to construct a model of your heartbeat versus time.

   c) Use the library or the Internet to obtain a tracing from a medical heart monitor. How good is a sinusoidal function as a model for a human pulse?

14. **Chapter Problem** Your voice is as unique as a fingerprint, and your voice pattern can be used to identify you. Even if you have a complex voice, a computer can be programmed to analyse your voice pattern and break it into a sum of sinusoidal functions of varying periods, amplitudes, and phase shifts. This process is known as Fourier analysis.

   a) To see how this process works, start with the model $y = \sin x$. Pick three different pairs of amplitudes and periods. Apply each pair to the basic sine function and add to form the voice model. An example is

   $$y = \sin x + 0.5 \sin 2x + 0.75 \sin 3x + 0.25 \sin 5x$$

   Graph the model. Add another term to your model, and graph it again. What changes do you notice?

   b) Borrow a microphone and oscilloscope from the physics department at your school. Sing a single note and see your own voice pattern. Compare patterns with others in your class.

15. Visit www.statcan.gc.ca/edu/edu05_0018c-eng.htm. Select **Sinusoidal.**

   Select **International travellers into Canada, table 387-0004.**

   Select **table 075-0013.**

   Under **Geography**, select **Canada.**

   Under **Travel category**, select **Inbound international travel.**

   Under **Sex**, select **Both sexes.**

   Under **International Travellers**, select **Total travel.**

   Under **Seasonal adjustment**, select **Unadjusted.**

   Set the reference period to **from Mar 1986 to Mar 2006.**

   If you want to see the data on your computer screen, select **Retrieve as a Table** and then **Retrieve Now.**

   If you plan to analyse the data with a spreadsheet or other software, select **Retrieve as individual Time Series.** Select the downloadable file format as **CSV (comma-separated values)**, with **Time as rows.**

   a) Select a recent year. Construct a sinusoidal model to represent the number of international travellers to Canada for that year.

   b) Inspect the data from other years. Does your model apply to any year? Justify your answer.

c) Make a graph of the maximum values versus year for each year that you have data for. Explain why the long-term trend among the maximum values appears to be periodic.

d) Construct a sinusoidal model to represent the variation in maximum values from year to year. What is the apparent period of this long-term variation?

16. Visit www.statcan.gc.ca/edu/edu05_0018c-eng.htm Select **Sinusoidal**.

a) Select a table of interest and download the data.

b) Pose a question that can be answered by constructing a sinusoidal model of the data.

c) Solve your question to ensure that it works.

d) Exchange questions with a classmate. Solve each other's question.

e) Exchange solutions, and discuss whether they are correct.

Achievement Check

17. Bungee jumping is thought to have originated in Queenstown, New Zealand, during the 1980s. Data for a bungee jumper are shown. Timing begins when the bungee cord is fully extended. The vertical height of the jumper above the ground for 15 s is shown. Use a sine function to model the jump.

Time, t (s)	Height, y (m)	Time, t (s)	Height, y (m)
0	110	8	35
1	103	9	60
2	85	10	85
3	60	11	103
4	35	12	110
5	17	13	103
6	10	14	85
7	17	15	60

C Extend

18. Consider a 10-m-tall tree. On a certain day, the sun rises at 6:00 a.m., is directly overhead at noon, and sets at 6:00 p.m. From 6:00 a.m. until noon, the length, s, in metres, of the tree's shadow can be modelled by the relation $s = 10 \cot 15t$, where t is the time, in hours, past 6:00 a.m.

a) Modify the model so that it is valid from noon until 6:00 p.m.

b) Graph the model and the modified model in part a) on the same set of axes.

19. **Math Contest** A cube with sides of 6 cm has each side increased to 18 cm. How many of the smaller cubes will fit in the larger cube?

A 3 **B** 27 **C** 9 **D** 81

20. **Math Contest** A glass container is filled with 50 L of water. 10 L of water is removed and replaced with 10 L of grape juice. The mixture is shaken well. Then, 20 L of the mixture is removed and replaced with 20 L of grape juice. How many litres of grape juice are now in the container?

A 20 **B** 24 **C** 25 **D** 26

21. **Math Contest** If the number 3000 is written in the form $2^x3^y5^z$, what is the value of $x + y + z$?

A 3 **B** 4 **C** 6 **D** 7

5.6

Use Sinusoidal Functions to Model Periodic Phenomena Not Involving Angles

You have worked with sinusoidal applications that do not involve angles, for example, the heights of tides versus time. In this section, you will work with other real-world situations that can be modelled by sinusoidal functions but do not necessarily involve an angle as the independent variable. Sunspot activity is an example. Sunspots are huge solar storms. Their activity appears to follow a periodic pattern that can be modelled by a sinusoidal function. The model is used to predict disruptions to radio-based communications on Earth.

Investigate

How can you use sinusoidal functions to model the tides?

1. The function $h(t) = 5 \sin [30(t + 3)]$ can be used to model the relationship between the height, in metres, of the tides in a certain place above or below mean sea level, and the time of day, in hours past midnight. Graph the model over a period of 24 h.
2. Use your graph to determine when the first high tide occurs. What is the height of the tide above mean sea level?
3. At what time does the first low tide occur? What is the height of the tide? Explain why the answer is negative.
4. At what times do the next high and low tides occur?
5. What is the period of the function?
6. **Reflect** Use your answers to explain why the phase shift is 3.

Tools

- graphing calculator

Optional

- graphing software or other graphing tools

Connections

Tides are caused principally by the gravitational pull of the moon on Earth's oceans. The physics of rotating systems predicts that one high tide occurs when the water is facing the moon and another occurs when the water is on the opposite side of Earth, away from the moon. This results in two tide cycles each day. To learn more about tides, visit the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 5.

Connections

The adoption of AC power transmission in North America was spearheaded by the inventor Nikola Tesla. He demonstrated that AC power was easier to transmit over long distances than the direct current favoured by rival inventor Thomas Edison. There is a statue honouring Tesla on Goat Island in Niagara Falls, New York.

Example 1

Model Alternating Electric Current (AC)

The electricity used in most of the world is alternating current. Unlike the electricity from a battery, which always flows in the same direction, AC electricity reverses direction in a cyclical fashion. The number of cycles per second is not the same in all countries. In Canada, the United States, and some other countries, the frequency standard is 60 Hz (hertz), which means 60 complete cycles per second. The maximum voltage is about 170 V. The voltage can be modelled as a function of time using a sine function: $V = a \sin [k(t - d)] + c$.

a) What is the period of 60-Hz AC?

b) Determine the value of k.

c) What is the amplitude of the voltage function?

d) Model the voltage with a suitably transformed sine function.

e) **Use Technology** Use technology to graph the voltage function over two cycles. Explain what the scales on the axes represent.

Solution

a) Since there are 60 complete cycles every second, the period is $\frac{1}{60}$ s.

b) $\frac{360}{k} = \frac{1}{60}$

$k = 21\,600$

The value of k is 21 600.

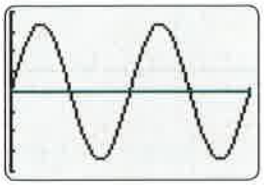
c) The amplitude is 170 V.

d) The voltage function is $V = 170 \sin 21\,600t$.

e) To graph the voltage function over two cycles, set the range of x from 0 s to $\frac{1}{30}$ s and the x-scale to $\frac{1}{120}$ s. Each tick on the x-axis represents $\frac{1}{120}$ s.

Set the range of y from -200 V to 200 V and the y-scale to 50 V. Each tick on the y-axis represents 50 V.

The graph is shown.



Technology Tip

You can enter rational expressions for the window variables. To set the maximum value for x to $\frac{1}{30}$, you can just type 1 ÷ 30. When you press ENTER, the calculator will calculate the desired value, 0.03333....

Example 2

Model the Angle of the Sun Above the Horizon on the Summer Solstice in Inuvik

The table shows the angle of the sun above the horizon for each hour on the summer solstice in Inuvik, Northwest Territories.

Hour Past Midnight	0	1	2	3	4	5	6	7	8	9	10	11
Angle Above the Horizon (°)	2.4	1.8	2.4	4.2	7.2	11	16	22	27	33	38	42
Hour Past Midnight	12	13	14	15	16	17	18	19	20	21	22	23
Angle Above the Horizon (°)	45	46	45	42	38	33	27	22	17	12	7.5	4.3

a) Use the table to determine a sinusoidal model for the angle of the sun above the horizon.

b) Graph the points in the table on the same set of axes as your model to verify the fit.

c) Is the fit as expected? Explain any discrepancies.

Solution

a) From the table, the maximum angle of elevation is 46° and the minimum angle is 1.8°. The amplitude is $\frac{46 - 1.8}{2} = 22.1$. Therefore, the value of a is 22.1.

The vertical shift is $1.8 + 22.1 = 23.9$. Therefore, the value of c is 23.9.

The sine wave starts at 23.9. This is about 7.5 h past midnight. Therefore, $d \doteq 7.5$.

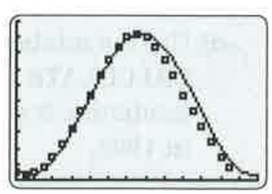
The period is 24 h.

$$\frac{360}{k} = 24$$
$$k = 15$$

The angle above the horizon can be modelled with the equation $h(t) = 22.1 \sin [15(t - 7.5)] + 23.9$.

b) The graph is shown, superimposed over the data points.

c) The fit is good at first, but begins to depart from the data points as the day progresses. Earth's movement around the sun may account for the departure from a purely sinusoidal model. The real period is not exactly 24 h.



Example 3

Predator-Prey Populations

When two animals have a predator/prey relationship, the population of each over time can be modelled with a sinusoidal function. If the population of prey is large, the population of predators increases since there is an adequate food supply. As the population of predators increases, the population of prey decreases. Eventually, there is not enough food for the predators, and they also begin to die off. As the number of predators declines, more prey survive, and the population of prey increases again.

Suppose that the number, N, of prey in a given area can be modelled by the function $N(t) = 250 \sin 90t + 500$, where t is the number of years since a base year of 1990.

a) What was the population of prey in 1990?

b) When did the population reach a maximum?

c) What was the maximum population at this time?

d) When did the population reach a minimum?

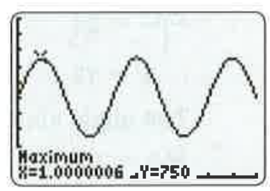
e) What was the minimum population at that time?

f) How many years passed between maximum populations?

Solution

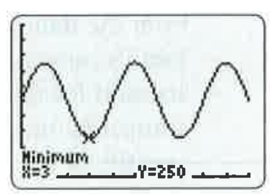
a) Substitute 0 for t. Since $\sin 0 = 0$, the population in 1990 was 500.

b) **Use Technology** Use a graphing calculator to graph the function over at least two cycles. Adjust the window variables accordingly. Then, use the **maximum** operation from the **CALCULATE** menu to determine the first maximum. It occurred after 1 year, in 1991.



c) The population of prey in 1991 was 750.

d) Use the **minimum** operation from the **CALCULATE** menu to determine the first minimum. It occurred at the end of 3 years, in 1993.



e) The population of prey in 1993 was 250.

f) The next maximum occurred in 1995. Four years passed between maximum populations.

Key Concepts

- Sinusoidal functions can be used to model periodic phenomena that do not involve angles as the independent variable.
- The amplitude, phase shift, period, and vertical shift of the basic sine or cosine function can be adjusted to fit the characteristics of the phenomenon being modelled.
- Technology can be used to quickly draw and analyse the graph modelled by the equation.
- The graph can be used to solve problems related to the phenomenon.

Communicate Your Understanding

C1 A situation modelled by a sinusoidal function has no x-intercepts. What conclusions can you draw about the relation between a and c?

C2 A situation can be modelled by the sinusoidal function $y = a \sin [k(x - d)] + c$. If the graph passes through the origin, can you conclude that the phase shift, d, is 0? Justify your answer, including a diagram.

C3 The period of a sinusoidal function is greater than 360° but less than 720°. What restrictions does this condition place on the value of k? Justify your answer.

A Practise

For help with questions 1 and 2, refer to the Investigate.

1. In the Investigate, the sinusoidal function $h(t) = 5 \sin [30(t + 3)]$ is used to model the height of tides in a particular location on a particular day. On a different day, the maximum height is 8 m, the minimum height is −8 m, and high tide occurs at 5:30 a.m.

a) Modify the function such that it matches the new data.

b) Predict the times for the next high and low tides.

2. Suppose that a cosine function is chosen to model the tides in the Investigate.

a) Modify the function so that a cosine function is used but all predictions of tides remain the same.

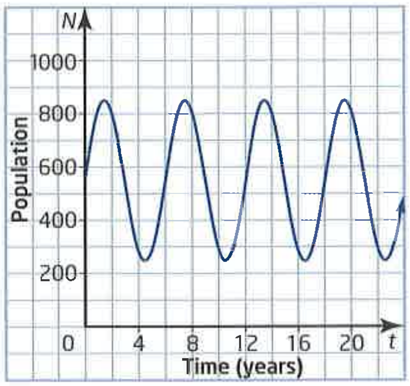
b) Verify that the cosine model correctly predicts the low and high tides for the day.

For help with question 3, refer to Example 1.

3. Sometimes demand for electricity becomes so great that suppliers such as Ontario Power Generation cannot maintain the desired maximum voltage on the electricity grid. This creates a condition known as a brownout. In a brownout, incandescent lamps dim and some appliances do not work at all. How does the graph of voltage versus time change during a brownout? Sketch the new graph and justify your changes.

For help with question 4, refer to Example 3.

4. The population of prey in a predator-prey relation is shown. Time is in years since 1985.



a) Determine the maximum and minimum values of the population, to the nearest 50. Use these to find the amplitude.

b) Determine the vertical shift, c.

c) Determine the phase shift, d.

d) Determine the period. Use the period to determine the value of k.

e) Model the population versus time with a sinusoidal function.

f) Graph your function. Compare it to the graph shown.

B Connect and Apply

5. The depth, d, in metres, of water in a seaplane harbour on a given day can be modelled using the function $d = 12 \sin [30(t - 5)] + 14$, where t is the time past midnight, in hours.

a) Determine the maximum and minimum depths of the water in the harbour.

b) What is the period of the function?

c) Graph the water level over 24 h.

d) If the water is less than 3 m deep, landing a seaplane is considered unsafe. During what time intervals, between midnight and midnight the following day, is it considered unsafe to land a seaplane?

e) What other factors are important in deciding whether it is safe to land?

6. At another time of year in the same harbour as in question 5, the maximum water depth is 22 m and the minimum depth is 6 m. The first high tide occurs at 5:00 a.m.

a) Modify the model in question 5 to match the new data.

b) Graph the water level over 24 h.

c) During what time intervals, between midnight and midnight the following day, is it considered unsafe to land a seaplane?

7. The electricity standard used in Europe and many other parts of the world is alternating current (AC) with a frequency of 50 Hz and a maximum voltage of 240 V. The voltage can be modelled as a function of time using a sine function.

a) What is the period of 50-Hz AC?

b) Determine the value of k.

c) What is the amplitude of the voltage function?

d) Model the voltage with a suitably transformed sine function.

e) **Use Technology** Use technology to graph the voltage function over two cycles. Explain what the scales on the axes represent.

8. Julia constructs a model AC generator in physics class and cranks it by hand at 3 revolutions per second. She is able to light up a flashlight bulb that is rated for 6 V.

a) What is the period of the AC produced?

b) Determine the value of k.

c) What is the amplitude of the voltage function?

d) Model the voltage with a suitably transformed sine function.

e) **Use Technology** Graph the voltage function over two cycles. Explain what the scales on the axes represent.

9. The table shows annual average sunspot activity from 1970 to 2006.

Year (since 1970)	Sunspots (Annual Average)	Year (since 1970)	Sunspots (Annual Average)
0	107.4	19	162.2
1	66.5	20	145.1
2	67.3	21	144.3
3	36.7	22	93.5
4	32.3	23	54.5
5	14.4	24	31.0
6	11.6	25	18.2
7	26.0	26	8.4
8	86.9	27	20.3
9	145.8	28	61.6
10	149.1	29	96.1
11	146.5	30	123.3
12	114.8	31	123.3
13	64.7	32	109.4
14	43.5	33	65.9
15	16.2	34	43.3
16	11.0	35	30.2
17	29.0	36	15.4
18	100.9		

a) Use the table to determine a sinusoidal model for the number of sunspots.

b) Graph the points in the table on the same axes as your model to verify the fit.

c) Is the fit as you expected? Explain any discrepancies.

10. Refer to your model in question 9.

a) Predict the next three occurrences of maximum sunspot activity after 2006.

b) Predict the next three occurrences of minimum sunspot activity after 2006.

c) What is the variation in the maximum and minimum values over the data in the table?

Connections

Sunspots were observed as early as 165 B.C.E. Like many phenomena in the sky, they were thought to have a mystical significance to humans.

11. In Example 3, the population of prey can be modelled with the function $N(t) = 250 \sin 90t + 500$. Research shows that the population of predators follows the period of the population of prey, with a phase shift of $\frac{1}{4}$ of a cycle to the right. Suppose that the predators have a minimum population of 50 and a maximum population of 100.

a) Construct a model for the population of predators using a sine function.

b) Graph the population of prey and predators on the same set of axes over a period of 12 years from the base year.

c) Suggest reasons why there is a phase shift to the right.

12. **Use Technology** Search the Internet for a predator-prey simulation game.

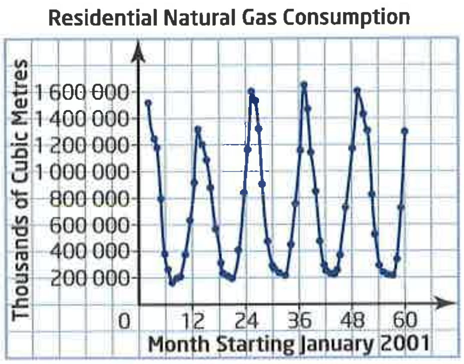
a) Play the game in a small group and generate a table of data.

b) Graph the data for prey and predators.

c) Use the graph and the table to construct a model for the population of prey and the population of predators.

d) Graph the models on the same set of axes as the data. How well do the models represent the data?

13. The graph represents monthly residential natural gas consumption in Ontario from January 2001 to December 2005.

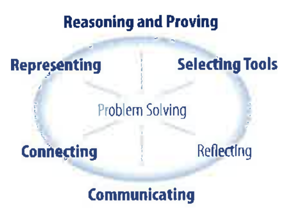


a) Use the graph to construct a sinusoidal model for natural gas consumption.

b) Graph the model for a period of 5 years, starting in January 2001.

c) Use the model to predict residential natural gas consumption in December 2008.

14. The number of hours of daylight on the 15th of each month, beginning in January, is shown for any point on Earth with a latitude of 50° north.



Month	Hours of Daylight
1	8:30
2	10:07
3	11:48
4	13:44
5	15:04
6	16:21
7	15:38
8	14:33
9	12:42
10	10:47
11	9:06
12	8:05

a) Graph the data.

b) Use the graph and the table to construct a model for the number of hours of daylight.

c) Graph the model on the same set of axes as the data. Comment on the fit.

d) Use your model to predict the number of hours of daylight on January 31.

15. Use Technology The U.S. Naval Observatory calculates daylight tables for any location on Earth. Go to the *Functions 11* page of the McGraw-Hill Ryerson Web site and follow the links to access the observatory's Web site.

a) Use an atlas or the Internet to determine the latitude and longitude of your location.

b) On the Naval Observatory's site, select **Duration of Daylight Table for One Year**. Enter your location data and obtain the table. Record the data for the 15th of each month to make a table of values.

c) Graph the data.

d) Use the graph and the table to construct a model for the number of hours of daylight at your location.

e) Graph the model on the same set of axes as the data. Comment on the fit.

f) Use your model to predict the number of hours of daylight on any convenient day. Check your prediction by direct observation.

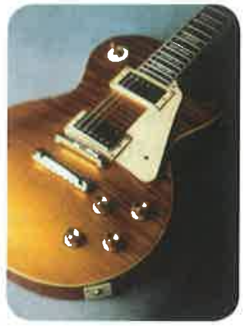
16. The propeller of a small airplane has an overall length of 2.0 m. The propeller clears the ground by a distance of 40 cm and spins at 1200 revolutions per minute while the airplane is taxiing.

a) Model the height of one of the propeller tips above the ground as a function of time using a sinusoidal function.

b) Graph the function over four cycles.

c) Determine all times in the first cycle when the tip is 1.0 m above the ground.

17. Chapter Problem Sound synthesis can include special effects. For example, a sine wave of one frequency can be used to control the amplitude of a sine wave of a higher frequency by multiplication. The result is a throbbing effect called tremolo. The effect first became popular when electric guitars became available in the 1950s. To see how tremolo works, use $\sin x$ to control the amplitude of $\sin 10x$ by multiplication: $y = (\sin x)(\sin 10x)$.



a) Use Technology On a graphing calculator, set the window such that x runs from -360 to 360 with a scale of 30, and y runs from -2 to 2 with a scale of 1. Enter and graph the function.

b) Observe the change in the amplitude of one function as it is controlled by the other. Are the sound bursts identical? Look carefully for reflections. Explain why they occur.

c) Tremolo is one of many special effects that can be added to sounds by manipulating the basic sinusoidal function models. Investigate other effects, such as vibrato, wah-wah, pitch bend, ping-pong, chorus, distortion, and attack delay. Many Web sites include sample sounds in a format that you can listen to. Listen for these same effects in your favourite music.

18. Refer to question 12 on page 311 about the theory of biorhythms.

a) Calculate the number of days that you have been alive. Be sure to account for the extra day in leap years.

b) The physical cycle has a period of 23 days. Divide your answer in part a) by 23 and determine the remainder. This tells you how long ago the current physical cycle began.

c) Use a method similar to part b) to determine when your current emotional cycle began. Recall that the period is 28 days. Repeat for the intellectual cycle, with a period of 33 days.

d) Use a long piece of grid paper to plot all three biorhythms. Use a suitable amplitude and a different colour for each.

e) Model each biorhythm with a sinusoidal function.

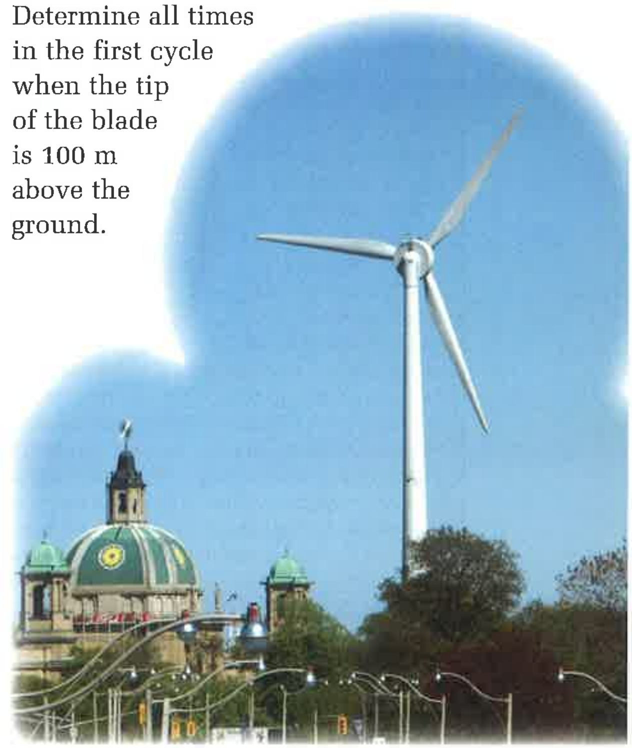
f) Use Technology Enter the models into a graphing calculator. Adjust the window variables appropriately and determine when your next "good" days and "bad" days will occur.

19. The wind turbine at Exhibition Place in Toronto is 94 m tall and has three blades, each measuring 24 m in length. The blades turn at a frequency of 27 revolutions per minute.

a) Use a sinusoidal function to model the height above the ground of one of the blade tips as a function of time.

b) Graph the function over four cycles.

c) Determine all times in the first cycle when the tip of the blade is 100 m above the ground.



C Extend

20. Non-reflective coating on eyeglasses is a very thin layer applied to the outside of the lenses. Light reflects from the front and back of the coating almost simultaneously, and the two reflections combine. The reflection from the front undergoes a phase shift, while that from the back does not.

The light reflected from the coating can be modelled with the sinusoidal function $P(x) = \sin x + \sin (x - 180°)$, where x is the phase of the light wave when it hits the coating, and $P(x)$ is the proportion of the light wave that is reflected.

a) Graph this function over two cycles.

b) Inspect the function carefully and explain why the graph has the form that it does. Why is a thin coating non-reflective?

c) The second term in the reflection function can also be expressed in terms of a cosine function. Determine a cosine function that has the same effect as the sine function shown.

Connections

Reflection from a film that is much thinner than the wavelength of the light being used forms a "perfect window." No light is reflected; it is all transmitted. This is why images viewed through non-reflective eyeglasses appear brighter than when viewed through lenses without the coating.

You can use a soap bubble kit to see another example of this perfect window. Dip the bubble blower into the solution, and hold it so that the soap film is vertical. Orient the film to reflect light. You will see a series of coloured bands as the soap drains to the bottom. Then, you will see what looks like a break in the film forming at the top. This is the perfect window. You can test to see if there is still soap there by piercing the window with a pin or a sharp pencil.

21. Polaris, also known as the North Star, is the star closest to being directly overhead if you view it from the North Pole. However, Earth's axis is not always pointing in the same direction. The axis is moving in a slow circle, a motion similar to that of a spinning top. The effect is called precession. Because of this movement, Vega, the bright star to the right of centre in the photograph, will eventually become the North Star. As Earth's axis continues to move, Polaris will return as the North Star.

a) Use the library or the Internet to determine the period of precession of Earth's axis. Do you expect to see Vega as the North Star during your lifetime?

b) Model the precession of Earth as a sinusoidal function.

c) **Use Technology** Determine the window settings needed to graph this function on a graphing calculator. Then, graph the function.

22. **Math Contest** The notation $n!$ means to multiply all the natural numbers from the number n down to 1. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1$. When $50!$ is expanded, how many zeros are at the end of the number?

A 10 **B** 5 **C** 50 **D** 12

23. **Math Contest** If $y = \sqrt{x}$, what is the value of $x^{25} + 121 - y^{50}$?

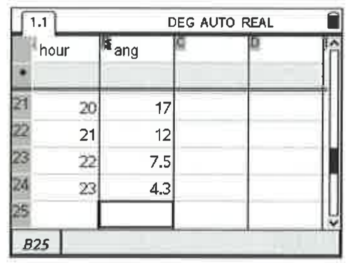
Create a Scatter Plot and a Function Using a TI-Nspire™ CAS Graphing Calculator

Refer to Example 2 on page 335.

Tools
- TI-Nspire™ CAS graphing calculator

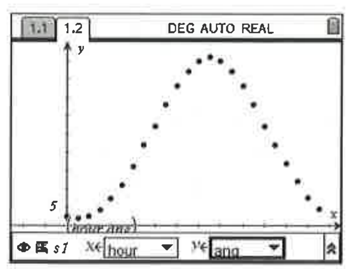
Hour Past Midnight	0	1	2	3	4	5	6	7	8	9	10	11
Angle Above the Horizon (°)	2.4	1.8	2.4	4.2	7.2	11	16	22	27	33	38	42
Hour Past Midnight	12	13	14	15	16	17	18	19	20	21	22	23
Angle Above the Horizon (°)	45	46	45	42	38	33	27	22	17	12	7.5	4.3

1. Open a new document. Create a page using the **Lists & Spreadsheet** application.

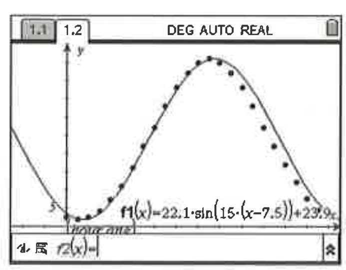


2. Enter the hour data in column A.
3. Enter the angle data in column B.
4. Open a new **Graphs & Geometry** page. Press (menu) and change the graph type to **Scatter Plot**.
5. Select **hour** from the x-axis dropdown menu. Select **ang** from the y-axis dropdown menu.
6. Press (menu). Set the **Window** settings from -5 to 25 for the x-axis and from -10 to 50 for the y-axis.

   A scatter plot will appear as shown.



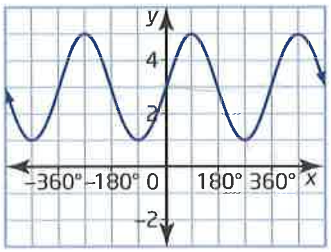
7. Change the graph type back to **Function**.
8. Refer to the function derived in Example 2: $h(t) = 22.1 \sin [15(t - 7.5)] + 23.9$. Type the function as **f1**. Press (enter).
9. The graph is shown, superimposed over the data points. The fit is good at first but begins to depart from the data points as the day progresses. Earth's movement around the sun may account for the departure from a purely sinusoidal model. The real period is not exactly 24 h.



Chapter 5 Review

5.1 Modelling Periodic Behaviour, pages 284 to 293

1. Consider the graph shown.



   a) Explain why the function represented is periodic.
   b) How many cycles are shown?
   c) What are the maximum and minimum values?
   d) What is the amplitude?
   e) What is the period?

2. A shuttle bus takes passengers from a remote parking lot to the airport terminal 1.5 km away. The bus runs continually, completing a cycle in 10 min, which includes a 1-min stop at the parking lot and a 1-min stop at the terminal.
   a) Sketch a graph to represent the position of the bus with respect to the parking lot as a function of time. Include two cycles.
   b) What is the amplitude of the pattern?
   c) Suppose that the bus increases its speed. Explain how this affects the graph.

5.2 The Sine Function and the Cosine Function, pages 294 to 301

3. Without using technology, sketch a graph of the sine function for values of x from $-540°$ to $540°$. Label the axes with an appropriate scale.

4. Without using technology, sketch a graph of the cosine function for values of x from $-540°$ to $540°$. Label the axes with an appropriate scale.

5.3 Investigate Transformations of Sine and Cosine Functions, pages 304 to 312

5. Consider the function $y = \cos(x + 60°) + 3$.
   a) What is the amplitude?
   b) What is the period?
   c) Describe the phase shift.
   d) Describe the vertical shift.
   e) Graph the function for values of x from $0°$ to $360°$.
   f) How does the equation change if the phase shift and vertical shift are both in the opposite direction from the original function? Justify your answer.

5.4 Graphing and Modelling With $y = a \sin[k(x - d)] + c$ and $y = a \cos[k(x - d)] + c$, pages 313 to 321

6. A robot arm is used to cap bottles on an assembly line. The vertical position, y, in centimetres, of the arm after t seconds can be modelled by the function $y = 30 \sin[360(t - 0.25)] + 45$.
   a) Determine the amplitude, period, phase shift, and vertical shift.
   b) What is the lowest vertical position that the arm reaches?
   c) State the domain and range of the original sine function and the transformed function in set notation.
   d) Suppose that the assembly line receives new bottles that require a lowest vertical position of 20 cm. How does the equation modelling the robot arm change?

5.5 Data Collecting and Modelling, pages 322 to 332

7. Sunrise times for 1 year in Fort Erie, Ontario, measured on the 21st of each month, starting in January, are shown using Eastern Standard Time (EST).

Month (21st day)	Time (EST)
1	7:40
2	7:04
3	6:17
4	5:25
5	4:47
6	4:37
7	4:55
8	5:28
9	6:01
10	6:36
11	7:15
12	7:43

a) Convert the times to decimal format. Round each value to two decimal places. Create a scatter plot, with the month on the horizontal axis and the time on the vertical axis.

b) Construct a model of the sunrise times using a sine function. State the amplitude, period, phase shift, and vertical shift.

c) Graph the model on the same axes as the data. Comment on the fit.

8. Use your model from question 7 to predict the sunrise time on January 7 and July 7.

5.6 Use Sinusoidal Functions to Model Periodic Phenomena Not Involving Angles, pages 333 to 341

9. The volume of blood in the left ventricle of an average-sized human heart varies from a minimum of about 50 mL to a maximum of about 130 mL. Ken has an average-sized heart and a resting pulse of 60 beats per minute. Assume that one heart beat represents the period.

a) Using a sine function, model the volume of blood in Ken's left ventricle with respect to time.

b) Sketch a graph of volume in relation to time for four cycles.

c) How much blood is pumped from the left ventricle every minute?

d) How does the function change if you use a cosine function?

e) When Ken runs, his heart rate rises to 120 beats per minute. Adjust your model from part a) to reflect Ken's heart rate while running.

f) Explain why the amplitude does not change in part c).

Chapter Problem WRAP-UP

You have seen how applications in music synthesis, voice pattern recognition, and special effects sounds can be modelled using sinusoidal functions.

a) How does the amplitude of the modelling function relate to the sound that you hear? Give examples.

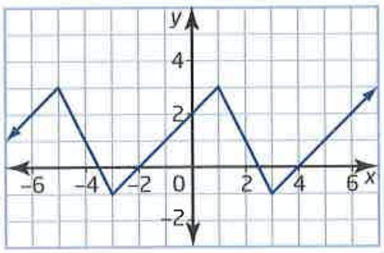
b) How does the period of the modelling function relate to the sound that you hear? Give examples.

c) What is the phase shift's role in music played on traditional instruments, songs sung by a choir, and music synthesis?

Chapter 5 Practice Test

For questions 1 to 8, select the best answer.

For questions 1 to 3, refer to the graph of the periodic function shown.



1. The period of the function is

A 1 **B** 2 **C** 3 **D** 6

2. The amplitude of the function is

A 1 **B** 2 **C** 3 **D** −1

3. The value of $f(12)$ is

A −1 **B** 0 **C** 2 **D** 3

For questions 4 to 7, consider the function $y = \frac{3}{8}\cos[5(x - 30°)] + \frac{3}{4}$.

4. The period of the function is

A 72° **B** 180° **C** 360° **D** 1800°

5. The minimum value of the function is

A $-\frac{3}{8}$ **B** 0 **C** $\frac{3}{8}$ **D** $\frac{3}{4}$

6. With respect to $y = \cos x$, the phase shift of the function is

A 30° left **B** 30° right

C 60° left **D** 60° right

7. With respect to $y = \cos x$, the vertical shift of the function is

A $\frac{3}{8}$ up **B** $\frac{3}{8}$ down

C $\frac{3}{4}$ up **D** $\frac{3}{4}$ down

8. Consider the function $y = 2\cos(3x + 120°)$. What is the phase shift of the function?

A 40° left **B** 40° right

C 120° left **D** 120° right

9. Consider the function $y = 3\sin[4(x + 60°)] - 2$.

a) What is the amplitude of the function?

b) What is the period of the function?

c) Describe the phase shift of the function.

d) Describe the vertical shift of the function.

e) Graph the function one step at a time. Label each step according to the transformation taking place.

f) State the domain and range of the transformed function. Use set notation.

10. A sinusoidal function has an amplitude of 4 units, a period of 90°, and a maximum at (0, 2).

a) Represent the function with an equation using a cosine function.

b) Represent the function with an equation using a sine function.

11. A sinusoidal function has an amplitude of $\frac{1}{4}$ units, a period of 720°, and a maximum point at $\left(0, \frac{3}{4}\right)$.

a) Represent the function with an equation using a sine function.

b) Graph the function over two cycles.

12. Consider the function $f(x) = 2\cos[3(x - 120°)]$.

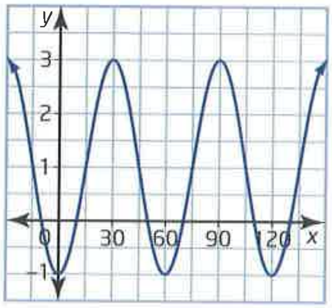
a) Determine the amplitude, period, phase shift, and vertical shift with respect to $y = \cos x$.

b) What are the maximum and minimum values?

c) Find the first three x-intercepts to the right of the origin.

d) Find the y-intercept.

13. a) Determine the equation of the sine function shown.



b) Suppose that the maximum values on the graph are half as far apart. How does the equation in part a) change? Justify your answer.

14. A summer resort town often shows seasonal variations in the percent of the workforce employed. The table lists the percent employed on the first of each month, starting in January, for 1 year.

Month	Employment (%)
1	62
2	67
3	75
4	80
5	87
6	92
7	96
8	93
9	89
10	79
11	72
12	65

a) Construct a model for these data using either a sine function or a cosine function. State the amplitude, period, phase shift, and vertical shift.

b) Graph the model on the same set of axes as the data. Comment on the fit.

c) Use your model to predict the employment level on June 15.

d) Economic forecasters predict a mild recession for the following year that will decrease employment levels for each month by 10%. Describe the effect that this will have on the graph of the function.

15. The Ferris wheel at a carnival has a diameter of 18 m and descends to 2 m above the ground at its lowest point. Assume that a rider enters a car at this point and rides the wheel for two revolutions.

a) Model the rider's height above the ground versus the angle of rotation using a transformed sine function.

b) Suppose that the rider enters the car from a platform located 45° along the rim of the wheel before the car reaches its lowest point. Adjust your model in part a) to reflect this situation.

c) Graph the equations from parts a) and b) on the same set of axes. How are they similar? How are they different? Explain the differences.

Chapters 4 and 5 Review

Chapter 4 Trigonometry

Where necessary, round angles to the nearest degee and trigonometric ratios to four decimal places.

1. a) Use the unit circle to determine exact values for the primary trigonometric ratios of 315°.

b) Check your results using a calculator.

2. a) Use the unit circle to determine approximate values for the primary trigonometric ratios for 255°.

b) Check your results using a calculator.

3. The coordinates of a point on the terminal arm of an angle θ are $(3, -1)$. Determine the exact trigonometric ratios of θ.

4. Angle Q is in the second quadrant, and $\sin Q = \frac{15}{17}$. Determine values for cos Q and tan Q.

5. Solve the equation $\tan \theta = -\frac{3}{8}$ for $0° \le \theta \le 360°$.

6. Determine two angles between 0° and 360° that have a cosecant of -8.

7. Margit flies her small plane due west at 200 km/h for one hour. She turns right through an angle of 45°, and continues at the same speed for half an hour.

a) Sketch a diagram to illustrate this problem.

b) At the end of her flight, how far is Margit from her starting point?

8. Given $\triangle ABC$, such that $a = 2.4$ cm, $c = 3.2$ cm, and $\angle A = 28°$.

a) Draw two possible diagrams that match the given measurements.

b) Calculate the length of side b and the degree measure of the other two angles.

9. Detectors for subatomic particles known as neutrinos must be built far below ground to minimize interference from external sources of radiation. To reach such an underground laboratory, visitors start at an elevator building on the surface, descend straight down for 2.1 km, and walk east for 1.9 km to enter the lab. To reach the detector, the visitors turn left through an angle of 30°, and walk another 300 m. What is the straight-line distance, to the nearest tenth of a kilometre, from the elevator building to the detector?

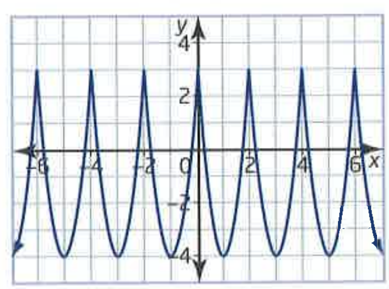
10. In $\triangle PQR$, $\angle P$ is 29°, $p = 16$ m, and $q = 25$ m. Determine all possible values for side r, to the nearest tenth of a metre.

11. Prove: $\sec \theta = \csc \theta \tan \theta$.

12. Prove: $\csc \theta = \sin \theta + \cos^2 \theta \csc \theta$.

Chapter 5 Trigonometric Functions

13. Consider the graph shown.



a) Explain why the function represented is periodic.

b) How many complete cycles are shown?

c) What are the maximum and minimum values?

d) What is the amplitude?

e) What is the period?

14. **a)** Use a graph to determine the values of x for which $\sin x = \cos x$ from $-180°$ to $180°$.

b) Check your answers using a calculator.

15. Consider the function $y = 3 \sin[2(x - 45°)] - 1$.

a) What is the amplitude?

b) What is the period?

c) Describe the phase shift.

d) Describe the vertical shift.

e) Graph the function for values of x from $0°$ to $360°$.

f) How would the equation change if the period were $90°$?

16. A sinusoidal function has an amplitude of $\frac{1}{2}$ units, a period of $1080°$, and a maximum point at $\left(0, \frac{3}{4}\right)$.

a) Represent the function with an equation using a sine function.

b) Draw a graph of the function over two cycles.

17. Consider the function $f(x) = \frac{1}{4} \cos [2(x - 90°)]$.

a) Determine the amplitude, period, phase shift, and vertical shift with respect to $y = \cos x$.

b) What are the maximum and minimum values?

c) Find the first three x-intercepts to the right of the origin.

d) Find the y-intercept.

18. In some countries, water wheels are used to pump water to a higher level. An Egyptian water wheel pumps water from a level of -1.3 m up to 1.7 m. It completes a full turn in 15 s.

a) Use a sinusoidal function to model the height of the water as a function of time.

b) For your model, state the amplitude, period, phase shift, and vertical shift.

c) What is the height of the water at a time of 20 s?

19. The Snowbirds air demonstration team performs a vertical loop. The altitude and time data for the loop are shown in the table.

Time (s)	Altitude (m)
0	3000
1	4000
2	4732
3	5000
4	4732
5	4000
6	3000
7	2000
8	1268
9	1000
10	1268
11	2000
12	3000

a) Use a sine function to model the altitude with respect to time.

b) Sketch a graph of your model for 4 cycles.

c) How does the function change if you use a cosine function?

Task

Modelling a Rotating Object

Tools

- string
- large paper clip
- tape measure
- grid paper

Work with a partner. One partner performs the actions in parts a) and b), and the other partner makes the measurements.

a) Attach a large paper clip to the end of a string. Holding the opposite end of the string, let your arm and the string hang down toward the floor. Measure the distance from the pivot point of your shoulder to
- the floor
- the end of the paper clip
- the wall in front of you

b) Rotate your arm vertically at a constant speed, enough to keep the string taut. Measure the period of the rotation.

c) Determine an equation for a sine function that gives the height of the paper clip relative to the angle of rotation, starting from the rest position.

d) Determine an equation for the height of the paper clip relative to time, beginning from the rest position.

e) Determine an equation for the distance from the paper clip to the wall relative to the angle of rotation, starting from the rest position.

f) Determine an equation for the distance from the paper clip to the wall relative to time, beginning from the rest position.

g) Sketch the graph of each relation.

h) Describe how each equation would change if you reversed the direction of rotation. Justify your answer.

i) How would the equation in part c) change if you were relating the rotational distance travelled, instead of the height, to the angle of rotation? Justify your answer.