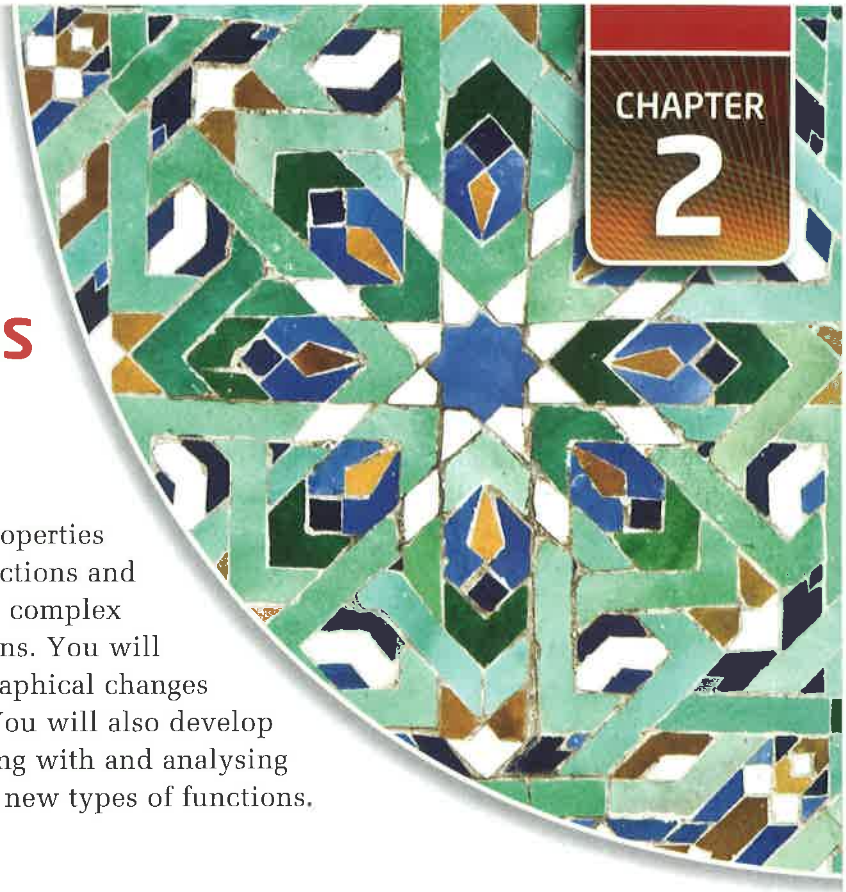


Transformations of Functions

In this chapter, you will investigate properties associated with transformations of functions and learn about a variety of ways in which complex functions are related to simple functions. You will explore the connection between the graphical changes and the parameters in the equations. You will also develop a standard set of procedures for working with and analysing transformations that can be applied to new types of functions.

By the end of this chapter, you will

- relate the process of determining the inverse of a function to your understanding of reverse processes
- determine the numeric or graphical representation of the inverse of a linear or quadratic function, given the numeric, graphical, or algebraic representation of the function, and make connections, through investigation using a variety of tools, between the graph of a function and the graph of its inverse
- determine, through investigation, the relationship between the domain and range of a function and the domain and range of the inverse relation, and determine whether or not the inverse relation is a function
- determine, using function notation when appropriate, the algebraic representation of the inverse of a linear or quadratic function, given the algebraic representation of the function, and make connections, through investigation using a variety of tools, between the algebraic representations of a function and its inverse
- determine, through investigation using technology, the roles of the parameters a, k, d, and c in functions of the form $y = af[k(x - d)] + c$, and describe these roles in terms of transformations on the graphs of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$
- sketch graphs of $y = af[k(x - d)] + c$ by applying one or more transformations to the graphs of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$, and state the domain and range of the transformed functions
- simplify polynomial expressions by adding, subtracting, and multiplying
- simplify rational expressions by adding, subtracting, multiplying, and dividing, and state the restrictions on the variable values
- determine if two given algebraic expressions are equivalent

Prerequisite Skills

Refer to the Prerequisite Skills Appendix on pages 478 to 495 for examples of the topics and further practice.

Properties of Quadratic Functions

1. Compare the graph of each quadratic function to the graph of $y = x^2$. Identify the direction of opening and state whether the parabola has been vertically stretched or compressed. Justify your answer.

a) $y = 3x^2$

b) $y = -0.5x^2$

c) $y = 0.1x^2 - 0.1x + 3$

d) $y = \frac{3}{5}(x - 6)^2 + 5$

2. Determine the vertex of each parabola.

a) $y = -(x - 5)^2 + 10$

b) $y = \frac{3}{2}(x + 6)^2 + 20$

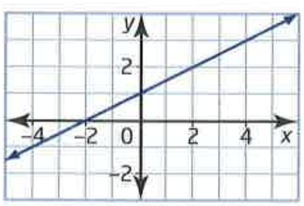
c) $y = 2(x - 1)^2 - 5$

d) $y = \frac{1}{4}(x + 3)^2 - 4$

Translations

3. Draw the graph of each relation after a vertical translation of 2 units up.

a)



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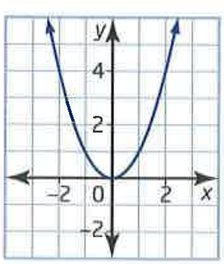

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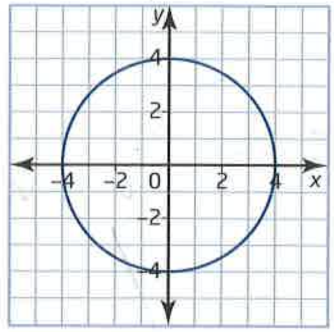
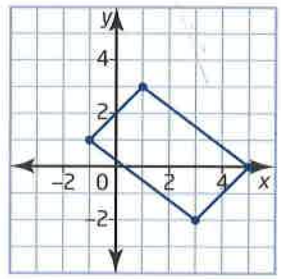


d)



4. Draw each relation in question 3 after a horizontal translation of 4 units left.

Graph Functions

5. Graph each function.

a) $y = \frac{1}{4}x + 5$

b) $y = x^2 + 2x - 3$

c) $y = (x - 2)(x + 3)$

d) $y = -0.5(x - 2)^2 + 10$

Distributive Property

6. Expand and simplify.

a) $3x(5x - 8)$

b) $12x^2y(5x - 10y)$

c) $-3x(4x^2 + 13x - 7)$

d) $(3x + 4)(2x - 5)$

e) $(4x - 5)(4x + 5)$

f) $3x(4x - 5) - 4x(6 - 10x)$

Common Factors

7. Determine the greatest common factor of each set.

a) 16, 40

b) 15, 18, 30

c) 48, 72, 108

d) $4x^2y^3$, $8xy^2$, $16x^3y^2$

e) $3x^2 + 6x$, $5x^2 + 10x$

f) $x^2 + 5x + 4$, $x^2 - 3x - 28$

8. Factor fully.

a) $15x^2 + 10x$

b) $-35x^3 - 45x^2$

c) $18x^2y^3 - 36xy^3 + 6y^3$

d) $-5x^5 - 100x^4 - 30x^2$

e) $2x(4x - 10) + 5(4x - 10)$

f) $x(6 - 11x) - (6 - 11x)$

Factor Quadratic Expressions

9. Factor fully.

a) $x^2 + 5x + 6$
b) $x^2 - 4x - 12$

c) $x^2 + 6x - 27$
d) $x^2 - 14x + 49$

e) $x^2 - 64$
f) $3x^2 - 9x - 120$

g) $2x^2 + 20x + 50$
h) $4x^2 - 256$

10. Factor fully.

a) $2x^2 - 7x - 15$
b) $9x^2 + 24x + 16$

c) $12x^2 - 2x - 2$
d) $18x^2 - 54x - 20$

e) $4x^2 + 10x - 24$
f) $20x^2 + 47x + 24$

Work With Fractions

11. Determine the least common multiple (LCM) of each set.

a) 18, 30, 42

b) $4x^2y$, $2xy$, $6xy^2$

c) $x^2 + 3x - 40$, $x^2 - 11x + 30$

12. Add or subtract using the LCM of the denominator.

a) $\frac{3}{5} + \frac{8}{15}$
b) $\frac{8}{9} - \frac{1}{4}$

c) $\frac{x}{6} + \frac{y}{4}$
d) $\frac{2x}{3} - \frac{3y}{8}$

13. Simplify.

a) $\left(\frac{5}{6}\right)\left(-\frac{3}{20}\right)$
b) $\left(\frac{12}{7}\right)\left(\frac{28}{15}\right)$

c) $\frac{20}{9} \div \frac{15}{32}$
d) $-\frac{24}{25} \div \left(-\frac{12}{125}\right)$

Rearrange Formulas

14. Solve for the indicated variable in each expression.

a) $A = \pi r^2$, for r

b) $P = 2(\ell + w)$, for w

c) $x^2 + y^2 = 16$, for y

d) $y = x^2 - 20$, for x

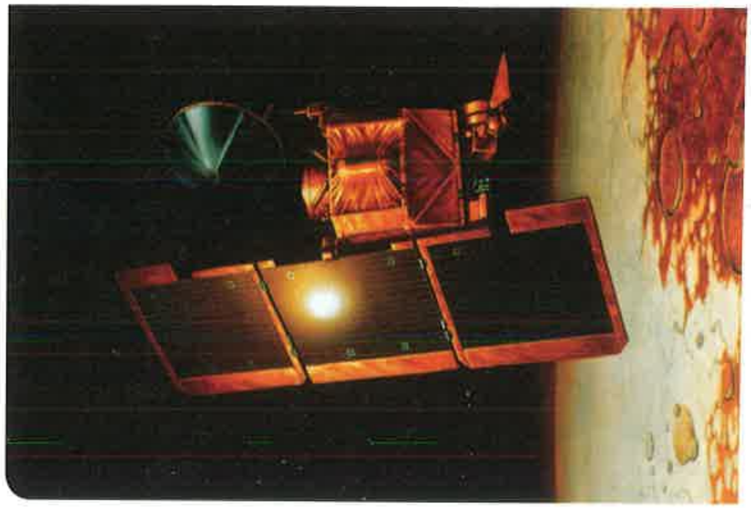
e) $y = \sqrt{x^2 - 5}$, for x

f) $A = 2\pi r^2 + 2\pi rh$, for h

Chapter Problem

Matthew works at the local traffic safety bureau. Part of his job is the study of how traffic flows and how automobiles behave in various situations. In this chapter, you will see how transformations relate to this very important field of research.

2.1



Functions and Equivalent Algebraic Expressions

On September 23, 1999, the Mars Climate Orbiter crashed on its first day of orbit. Two scientific groups used different measurement systems (Imperial and metric) for navigational calculations, resulting in a mix-up that is said to have caused the loss of the $125-million (U.S.) orbiter. Even though the National Aeronautics and Space Administration (NASA) requires a system of checks within their processes, this error was never detected.

This mix-up may have been caused by people not understanding complex equations. In general, to reduce the likelihood of errors in calculations, mathematicians and engineers simplify equations and expressions before applying them.

Example 1

Determine Whether Two Functions Are Equivalent

Determine whether the functions in each pair are equivalent by

i) testing three different values of x

ii) simplifying the expressions on the right sides

iii) graphing using graphing technology

a) $f(x) = 2(x - 1)^2 + (3x - 2)$ and $g(x) = 2x^2 - x$

b) $f(x) = \dfrac{x^2 - 2x - 8}{x^2 - x - 12}$ and $g(x) = \dfrac{x + 2}{x + 3}$

Solution

a) i) Choose three values of x that will make calculations relatively easy.

x	$f(x) = 2(x - 1)^2 + (3x - 2)$	$g(x) = 2x^2 - x$
-1	$f(-1) = 2(-1 - 1)^2 + (3(-1) - 2)$ $= 2(-2)^2 + (-5)$ $= 8 - 5$ $= 3$	$g(-1) = 2(-1)^2 - (-1)$ $= 2(1) + 1$ $= 3$
0	$f(0) = 2(0 - 1)^2 + (3(0) - 2)$ $= 2(-1)^2 + (-2)$ $= 0$	$g(0) = 2(0)^2 - 0$ $= 0$
1	$f(1) = 2(1 - 1)^2 + (3(1) - 2)$ $= 2(0)^2 + 1$ $= 1$	$g(1) = 2(1)^2 - 1$ $= 1$

Based on these three calculations, the functions appear to be equivalent. However, three examples do not prove that the functions are equivalent for every x-value.

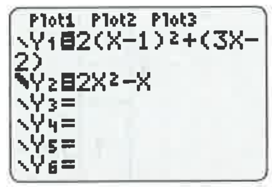
ii) In this pair, $g(x)$ is already simplified, so concentrate on $f(x)$.

$$\begin{aligned} f(x) &= 2(x - 1)^2 + (3x - 2) \\ &= 2(x^2 - 2x + 1) + 3x - 2 \\ &= 2x^2 - 4x + 2 + 3x - 2 \\ &= 2x^2 - x \end{aligned}$$

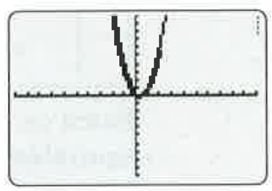
Algebraically, these functions are equivalent.

iii) Use a graphing calculator to graph the two equations as **Y1** and **Y2**. Change the line display for **Y2** to a thick line.

- Cursor left to the slanted line beside **Y2**.
- Press (ENTER) to change the line style.



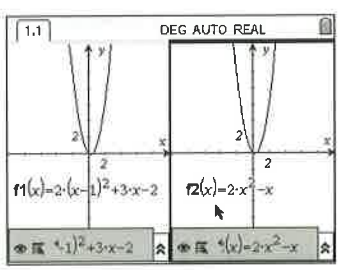
- From the **ZOOM** menu, select **6:ZStandard**.



The graph of $y = 2(x - 1)^2 + (3x - 2)$ will be drawn first. Then, the graph of $y = 2x^2 - x$ will be drawn using a heavier line. You can pause the plot by pressing (ENTER). Pressing (ENTER) again will resume the plot.

This seems to yield the same graph. These functions appear to be equivalent.

Using a TI-Nspire™ CAS graphing calculator, you can graph the functions $f(x) = 2(x - 1)^2 + (3x - 2)$ and $g(x) = 2x^2 - x$ side by side for comparison.



Technology Tip

Refer to the Use Technology feature on pages 86 and 87 to see how to graph multiple functions using a TI-Nspire™ CAS graphing calculator.

b) **i)**

x	$f(x) = \dfrac{x^2 - 2x - 8}{x^2 - x - 12}$	$g(x) = \dfrac{x + 2}{x + 3}$
-1	$f(-1) = \dfrac{(-1)^2 - 2(-1) - 8}{(-1)^2 - (-1) - 12}$ $= \dfrac{1 + 2 - 8}{1 + 1 - 12}$ $= \dfrac{-5}{-10}$ $= \dfrac{1}{2}$	$g(-1) = \dfrac{-1 + 2}{-1 + 3}$ $= \dfrac{1}{2}$
0	$f(0) = \dfrac{0^2 - 2(0) - 8}{0^2 - 0 - 12}$ $= \dfrac{-8}{-12}$ $= \dfrac{2}{3}$	$g(0) = \dfrac{0 + 2}{0 + 3}$ $= \dfrac{2}{3}$
1	$f(1) = \dfrac{1^2 - 2(1) - 8}{1^2 - 1 - 12}$ $= \dfrac{-9}{-12}$ $= \dfrac{3}{4}$	$g(1) = \dfrac{1 + 2}{1 + 3}$ $= \dfrac{3}{4}$

Based on these three calculations, the functions appear to be equivalent. However, three examples do not prove that these functions are equivalent for every x-value.

ii) In this pair, $g(x)$ is already simplified, so concentrate on $f(x)$. To simplify $f(x)$, factor the numerator and the denominator.

$$f(x) = \frac{x^2 - 2x - 8}{x^2 - x - 12}$$

$$= \frac{\cancel{(x - 4)}(x + 2)}{\cancel{(x - 4)}(x + 3)}$$ Factor the numerator and the denominator.

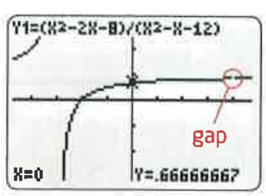
$$= \frac{x + 2}{x + 3}$$ Divide by the common factor.

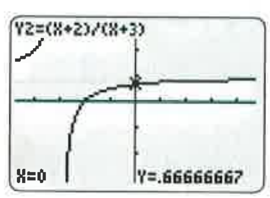
Algebraically, it appears as though the two functions are equivalent. However, the effect of dividing by a common factor involving a variable needs to be examined.

iii) Use a graphing calculator to graph the two equations. In this case, there is a slight difference between the graphs. To see the graphs properly, press ZOOM and select **4:ZDecimal**.

Technology Tip

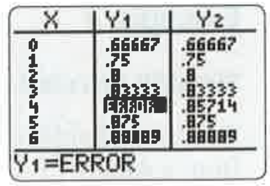
Using a "friendly window," such as **ZDecimal**, makes it easier to see any gaps in the graph of a function. This is because each pixel represents one tick mark.





There appears to be a gap in the first graph. This can be verified further by using the **TABLE** function on a graphing calculator.

Based on the evidence, this pair of functions is equivalent everywhere but at $x = 4$.



Polynomial expressions that can be algebraically simplified to the same expression are equivalent. However, with **rational expressions**, this may not be the case.

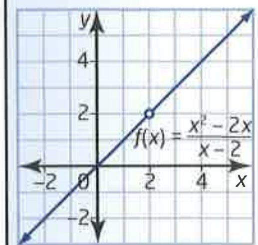
More specifically, since division by zero is not defined, you must define restrictions on the variable. For example, the function $f(x) = \dfrac{x^2 - 2x - 8}{x^2 - x - 12}$ has a factored form of $f(x) = \dfrac{(x - 4)(x + 2)}{(x - 4)(x + 3)}$. Since the denominator is zero if

$$x - 4 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 4 \qquad\qquad x = -3$$

the simplified function is written as $f(x) = \dfrac{x + 2}{x + 3}$, $x \neq -3$, $x \neq 4$.

Connections

An open circle is used to indicate a gap or a hole in the graph of a function.



rational expression

- the quotient of two polynomials, $\dfrac{p(x)}{q(x)}$, where $q(x) \neq 0$

Example 2

Determine Restrictions

Simplify each expression and determine any restrictions on the variable.

a) $\dfrac{x^2 + 10x + 21}{x + 3}$

b) $\dfrac{6x^2 - 7x - 5}{3x^2 + x - 10}$

Solution

a) $\dfrac{x^2 + 10x + 21}{x + 3} = \dfrac{(x + 3)(x + 7)}{x + 3}$

Factor the numerator and the denominator. Before reducing, determine restrictions. In this case, $x \neq -3$.

$= \dfrac{\cancel{(x + 3)}(x + 7)}{\cancel{x + 3}}, x \neq -3$

Divide by any common factors.

$= x + 7, x \neq -3$

So, $\dfrac{x^2 + 10x + 21}{x + 3} = x + 7, x \neq -3$.

b) $\dfrac{6x^2 - 7x - 5}{3x^2 + x - 10} = \dfrac{(2x + 1)(3x - 5)}{(3x - 5)(x + 2)}$

Factor the numerator and the denominator.

$= \dfrac{(2x + 1)\cancel{(3x - 5)}}{\cancel{(3x - 5)}(x + 2)}, x \neq -2, x \neq \dfrac{5}{3}$

Divide by any common factors.

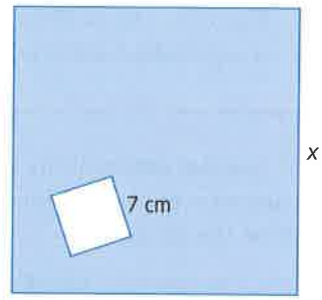
$= \dfrac{2x + 1}{x + 2}, x \neq -2, x \neq \dfrac{5}{3}$

So, $\dfrac{6x^2 - 7x - 5}{3x^2 + x - 10} = \dfrac{2x + 1}{x + 2}, x \neq -2, x \neq \dfrac{5}{3}$.

Example 3

Simplify Calculations

A square of side length 7 cm is removed from a square of side length x.



a) Express the area of the shaded region as a function of x.

b) Write the area function in factored form.

c) Use both forms of the function to calculate the area for x-values of 8 cm, 9 cm, 10 cm, 11 cm, and 12 cm. Which form is easier to use?

d) What is the domain of the area function?

Solution

a) $$\begin{aligned} A_{\text{shaded}} &= A_{\text{large}} - A_{\text{small}} \\ &= x^2 - 7^2 \\ &= x^2 - 49 \end{aligned}$$

b) $$\begin{aligned} A_{\text{shaded}} &= x^2 - 49 \\ &= (x - 7)(x + 7) \end{aligned}$$

c)

x	$A = x^2 - 49$	$A = (x - 7)(x + 7)$
8	$A = 8^2 - 49$ $= 15$	$A = (8 - 7)(8 + 7)$ $= 15$
9	$A = 9^2 - 49$ $= 32$	$A = (9 - 7)(9 + 7)$ $= 32$
10	$A = 10^2 - 49$ $= 51$	$A = (10 - 7)(10 + 7)$ $= 51$
11	$A = 11^2 - 49$ $= 72$	$A = (11 - 7)(11 + 7)$ $= 72$
12	$A = 12^2 - 49$ $= 95$	$A = (12 - 7)(12 + 7)$ $= 95$

The areas are 15 cm^2, 32 cm^2, 51 cm^2, 72 cm^2, and 95 cm^2, respectively.

Both expressions have similar numbers of steps involved. However, it is easier to use mental math with the factored form.

d) Since this function represents area, it must be restricted to x-values that do not result in negative or zero areas.
So, the domain is $\{x \in \mathbb{R}, x > 7\}$.

Key Concepts

- To determine if two expressions are equivalent, simplify both to see if they are algebraically the same.
- Checking several points may suggest that two expressions are equivalent, but it does not prove that they are.
- Rational expressions must be checked for restrictions by determining where the denominator is zero. These restrictions must be stated when the expression is simplified.
- Graphs can suggest whether two functions or expressions are equivalent.

Communicate Your Understanding

C1 The points $(-3, 5)$ and $(5, 5)$ both lie on the graphs of the functions $y = x^2 - 2x - 10$ and $y = -x^2 + 2x + 20$. Explain why checking only a few points is not sufficient to determine whether two expressions are equivalent.

C2 A student submits the following simplification.

$$\frac{x^2 + 6x + 3}{6x + 3} = \frac{x^2 + \cancel{6x} + \cancel{3}}{\cancel{6x} + \cancel{3}}$$
$$= x^2$$

Explain how you would show the student that this is incorrect.

C3 Explain why the expression $4x^3 + 4x^2 - 5x + 3$ does not have any restrictions.

A Practise

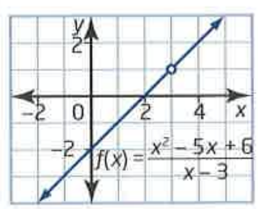
For help with questions 1 to 6, refer to Examples 1 and 2.

1. **Use Technology** Use a graphing calculator to graph each pair of functions. Do they appear to be equivalent?
   a) $f(x) = 5(x^2 + 3x - 2) - (2x + 4)^2$,
   $g(x) = x^2 - x - 26$
   b) $f(x) = (8x - 3)^2 + (5 - 7x)(9x + 1)$,
   $g(x) = x^2 - 10x - 14$
   c) $f(x) = (x^2 + 3x - 5) - (x^2 + 2x - 5)$,
   $g(x) = 2(x - 1)^2 - (2x^2 - 5x - 1)$
   d) $f(x) = (x - 3)(x + 2)(x + 5)$,
   $g(x) = x^3 + 4x^2 - 11x - 30$
   e) $f(x) = (x^2 + 3x - 5)(x^2 - 5x + 4)$,
   $g(x) = x^4 - 2x^3 - 15x^2 + 37x - 20$

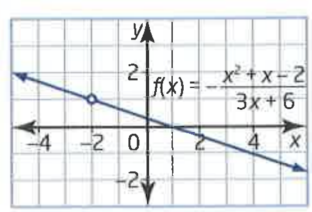
2. Refer to question 1. If the functions appear to be equivalent, show that they are algebraically. Otherwise, show that they are not equivalent by substituting a value for x.

3. State the restriction for each function.

a)

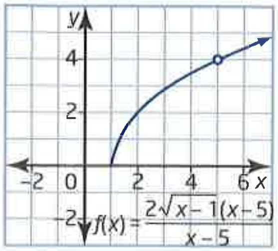


b)

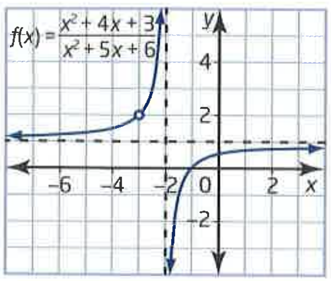


4. State the restrictions for each function.

a)



b)



5. Determine whether $g(x)$ is the simplified version of $f(x)$. If it is, then state the restrictions needed. If not, determine the correct simplified version.

a) $f(x) = \dfrac{x^2 + 11x + 30}{x + 6}$, $g(x) = x + 5$

b) $f(x) = \dfrac{x^2 - 16}{x^2 - 8x + 16}$, $g(x) = x + 4$

c) $f(x) = \dfrac{x^2 + 6x + 5}{x + 5}$, $g(x) = x^2$

d) $f(x) = \dfrac{x^2 + 10x + 16}{x^2 + 2x - 48}$, $g(x) = \dfrac{x + 2}{x - 6}$

e) $f(x) = \dfrac{12x^2 - 5x - 2}{3x^2 - 2x}$, $g(x) = \dfrac{4x + 1}{x}$

f) $f(x) = \dfrac{5x^2 - 23x - 10}{5x + 2}$, $g(x) = -23x - 2$

6. Simplify each expression and state all restrictions on x.

a) $\dfrac{x - 8}{x^2 - 13x + 40}$

b) $\dfrac{3(x - 7)^2(x - 10)}{x^2 - 17x + 70}$

c) $\dfrac{x^2 - 3x - 18}{x^2 + x - 42}$

d) $\dfrac{x^2 + 7x - 18}{x^2 + 3x - 10}$

e) $\dfrac{x + 8}{x^2 - 6x - 16}$

f) $\dfrac{25x^2 + 10x - 8}{10x^2 + 26x - 12}$

B Connect and Apply

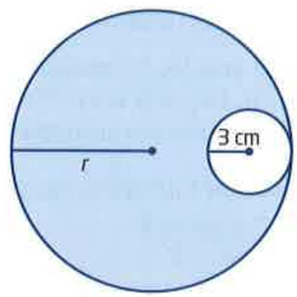
7. Evaluate each expression for x-values of -2, -1, 0, 3, and 10. Describe any difficulties that occur.

a) $(x - 6)(x - 2) - (x - 11)(x + 2)$

b) $\dfrac{2x^3 + 12x^2 + 10x}{x^2 + 6x + 5}$

For help with question 8, refer to Example 3.

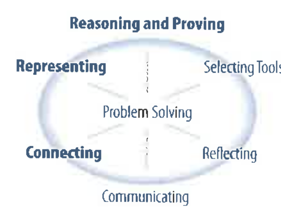
8. A circle of radius 3 cm is removed from a circle of radius r.



a) Express the area of the shaded region as a function of r.

b) State the domain and range of the area function.

9. A company that makes modular furniture has designed a scalable box to accommodate several different sizes of items. The dimensions are given by $L = 2x + 0.5$, $W = x - 0.5$, and $H = x + 0.5$, where x is in metres.



a) Express the volume of the box as a function of x.

b) Express the surface area of the box as a function of x.

c) Determine the volume and surface area for x-values of 0.75 m, 1 m, and 1.5 m.

d) State the domain and range of the volume and surface area functions.

10. **Chapter Problem** At the traffic safety bureau, Matthew is conducting a study on the stoplights at a particular intersection. He determines that when there are 18 green lights per hour, then, on average, 12 cars can safely travel through the intersection on each green light. He also finds that if the number of green lights per hour increases by one, then one fewer car can travel through the intersection per light.

a) Determine a function to represent the total number of cars that will travel through the intersection for an increase of x green lights per hour.

b) Matthew models the situation with the function $f(x) = 216 - 6x - x^2$. Show that your function from part a) is the same.

c) How many green lights should there be per hour to maximize the number of cars through the intersection?

11. In the novel *The Curious Incident of the Dog in the Night-Time* by Mark Haddon, the young boy, who is the main character, loves mathematics and is mildly autistic. Throughout the book, he encounters several math problems. One of the problems asks him to prove that a triangle with sides given by $x^2 + 1$, $x^2 - 1$, and $2x$ will always be a right triangle for $x > 1$.

a) Use the Pythagorean theorem to verify that this statement is true for x-values of 2, 3, and 4.

b) Based on the three expressions for the sides, which one must represent the hypotenuse? Justify your answer.

c) Use the Pythagorean theorem with the expressions for the side lengths to prove that these will always be sides of a right triangle for $x > 1$.

12. The function $y = \dfrac{a^3}{a^2 + x^2}$ is sometimes called the witch of Agnesi after Maria Gaetana Agnesi (1718–1799). The equation generates a family of functions for different values of $a \in \mathbb{R}$.

a) **Use Technology** Use graphing technology to graph this function for a-values of 1, 2, 3, and 4.

b) Explain why this rational function does not have any restrictions.

c) Research the history of Maria Gaetana Agnesi and find out who else studied this curve before her.

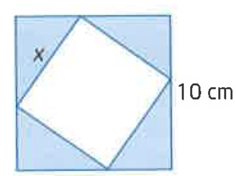
Connections

Maria Gaetana Agnesi originally referred to this function as *versiera*, which means "to turn." Later, in translation, it was mistakenly confused with *avversiere*, which means "witch" or "wife of the devil," and thus its current name was born.

C Extend

13. What does the graph of $f(x) = \dfrac{(x + 6)(2x^2 - x - 6)}{x^2 + 4x - 12}$ look like?

14. Algebraically determine the domain and range of the area function that represents the shaded region.



15. A student wrote the following proof. What mistake did the student make?

Let $a = b$.
Then, $a^2 = ab$.
$a^2 + a^2 = a^2 + ab$
$2a^2 = a^2 + ab$
$2a^2 - 2ab = a^2 + ab - 2ab$
$2a^2 - 2ab = a^2 - ab$
$2(a^2 - ab) = 1(a^2 - ab)$
Dividing both sides by $(a^2 - ab)$ gives $2 = 1$.

16. **Math Contest** Given the two linear functions $y = 6x - 12$ and $\dfrac{y}{x - 2} = 6$, what ordered pair lies on the graph of the first line but not on the graph of the second line?

Use Technology

Tools

- TI-Nspire™ CAS graphing calculator

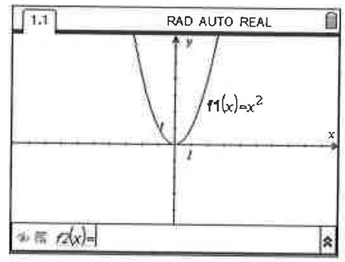
Graph Functions Using a TI-Nspire™ CAS Graphing Calculator

1. Open a new document. Open a page using the **Graphs & Geometry** application.

2. In the entry line, you will see $f1(x)=$.

- Type x^2 as a sample function.
- Press enter.

Note that the function is displayed with its equation as a label and the entry line has changed to $f2(x)=$.



3. Look at the axes. This is the standard window.

To view or change the window settings:

- Press menu.
- Select **4:Window**, and then select **1:Window Settings**.

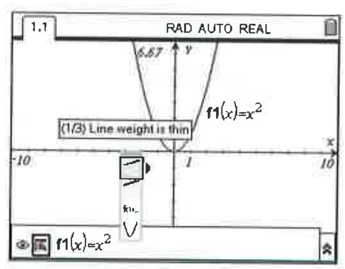
You can change the appearance of the window.

- Press menu.
- Select **2:View**.

There are several options. For example, if you select **8:Show Axes End Values**, you can display the range of each axis.

4. Press the up arrow key once. The function $f1(x)$ will be displayed in the entry line. You can change the appearance of a line:

- Press tab until the **Attributes** tool, , at the left of the entry line, is selected.
- Press enter.



You can use this tool to adjust the line weight, the line style, the label style, and the line continuity.

- Use the up and down arrow keys to highlight an attribute.
- Then, use the left and right arrow keys to move through the options for that attribute.

Experiment with the attributes. When you are finished, press esc.

5. You can move a function label.

- Press tab. The entry line will grey out, and the cursor will move to the graphing window.
- Use the arrow keys to move the cursor over the function label. When you are in the correct place, the word "label" will appear, along with a hand symbol.

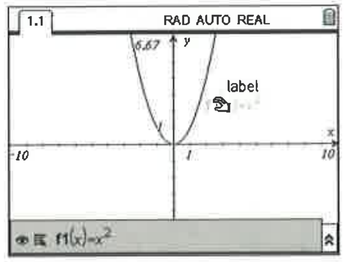
Technology Tip

You can change the name of a function. For example, to change $f1(x)$ to $g(x)$:

- In the entry line, press del several times to erase $f1$.
- Type g, and press enter.

The function will be displayed with the desired name as the label.

- Press [ctrl] . The hand will close to "grab" the label.
- Use the arrow keys to move the label around the screen.
- When you are finished, press [esc].



You can also move the entire graph.

- Move the cursor to a blank space in the second quadrant.
- Press [ctrl] . A hand will appear.
- Use the arrow keys to move the entire graph around the screen.
- When you are finished, press [esc].

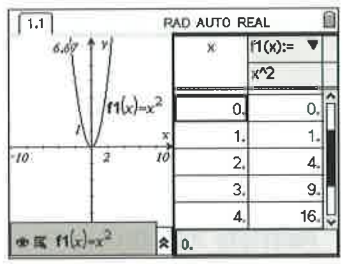
Technology Tip

If you are on the entry line and want to move to the graphing window, press [esc].

If you are in the graphing window and want to move to the entry line, press [tab].

6. You can display a table of values for the function.

- Press [tab] to return to the entry line.
- Press the up arrow key to return to the function $f1(x)$.
- Press [menu] and select **2:View**.
- From the **View** menu, select **9:Add Function Table**.



You can scroll up and down to inspect different values.

To adjust the **Table Start** value and the **Table Step** value:

- Press [menu] and select **5:Function Table**.
- From the **Function Table** menu, select **3:Edit Function Table Settings**.

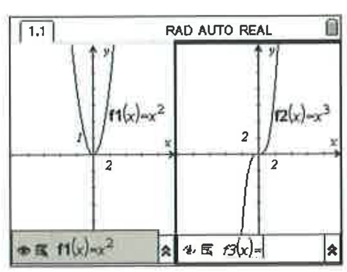
7. You can split the screen to display two functions at once.

- Open a new document. Open a page using the **Graphs & Geometry** application.
- Graph the function $f1(x) = x^2$.
- Press [ctrl] [home] to access the **Tools** menu.
- From the **Tools** menu, select **5:Page Layout**, and then select **2:Select Layout**.

You will see a menu of possible layouts.

For example, to display two graphs side by side:

- Select **2:Layout 2**. A blank window will appear.
- Press [ctrl] [tab] to switch windows.
- Press [menu] and select **2:Add Graphs & Geometry**.
- Graph the function $f2(x) = x^3$.



Technology Tip

You can hide the entry line.

- Press [ctrl] [G]. The entry line is hidden.
- Press [ctrl] [G] again to view the entry line.

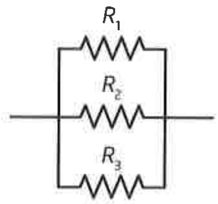
If you press a key or make a selection by mistake, you can undo the operation.

- Press [ctrl] [Z].

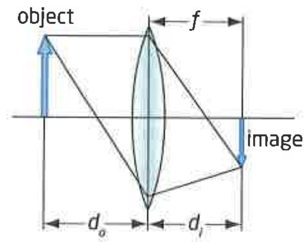
This will work several times to step back through a series of operations.

2.2

Skills You Need: Operations With Rational Expressions



$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$



$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$$

The ability to manipulate rational expressions is an important skill for engineers, scientists, and mathematicians. Some examples of such situations are the calculation of the resistance in parallel circuits and the calculation of the focal length in curved lenses.

Example 1

Multiply and Divide Rational Expressions

Simplify each expression and state any restrictions on the variables.

a) $\frac{4x^2}{3x} \times \frac{12x^3}{2x}$

b) $\frac{10ab^2}{4a} \div \frac{15a^2}{12b^2}$

Solution

a) Method 1: Multiply and Then Simplify

$$\begin{aligned} \frac{4x^2}{3x} \times \frac{12x^3}{2x} &= \frac{48x^5}{6x^2} && \text{Multiply the numerators and multiply the denominators.} \\ &= \frac{\overset{8}{\cancel{48}}\overset{3}{\cancel{x^5}}}{\cancel{6}\cancel{x^2}}, x \neq 0 && \text{Divide by the common factors.} \\ &= 8x^3 \end{aligned}$$

Thus, $\frac{4x^2}{3x} \times \frac{12x^3}{2x} = 8x^3$, $x \neq 0$.

Method 2: Simplify and Then Multiply

$$\begin{aligned} \frac{4x^2}{3x} \times \frac{12x^3}{2x} &= \frac{\overset{2}{\cancel{4}}\overset{1}{\cancel{x^2}}}{\cancel{3}\cancel{x}} \times \frac{\overset{4}{\cancel{12}}\overset{2}{\cancel{x^3}}}{\cancel{2}\cancel{x}}, x \neq 0 && \text{Divide by the common factors.} \\ &= 2x \times 4x^2 \\ &= 8x^3 \end{aligned}$$

Thus, $\frac{4x^2}{3x} \times \frac{12x^3}{2x} = 8x^3$, $x \neq 0$.

b) $\dfrac{10ab^2}{4a} \div \dfrac{15a^2}{12b^2} = \dfrac{10ab^2}{4a} \times \dfrac{12b^2}{15a^2}$ Multiply by the reciprocal.

$= \dfrac{120ab^4}{60a^3}$ Multiply the numerators and multiply the denominators.

$= \dfrac{\overset{2}{\cancel{120}}\cancel{a}b^4}{\cancel{60}a^{\cancel{3}\,2}}, a \neq 0$ Divide by the common factors.

$= \dfrac{2b^4}{a^2}$

In the original expression, both a and b were in the denominator, so neither of them can be equal to zero.

So, $\dfrac{10ab^2}{4a} \div \dfrac{15a^2}{12b^2} = \dfrac{2b^4}{a^2}, a \neq 0, b \neq 0.$

Example 2

Multiply and Divide Rational Expressions Involving Polynomials

Simplify and state any restrictions.

a) $\dfrac{a^2 + 2a}{3a} \times \dfrac{20a^2}{5a^2 + 10a}$

b) $\dfrac{2x^2 - 8x}{x^2 - 3x - 10} \div \dfrac{4x^2}{x^2 - 9x + 20}$

Solution

a) $\dfrac{a^2 + 2a}{3a} \times \dfrac{20a^2}{5a^2 + 10a}$

$= \dfrac{a(a + 2)}{3a} \times \dfrac{20a^2}{5a(a + 2)}$ Factor binomials where possible.

$= \dfrac{\cancel{a}\cancel{(a + 2)}}{3\cancel{a}} \times \dfrac{\overset{4}{\cancel{20}}a^{\overset{1}{\cancel{2}}}}{\cancel{5a}\cancel{(a + 2)}}, a \neq -2, a \neq 0$ Divide by the common factors.

$= \dfrac{1}{3} \times 4a$ Multiply the numerators and multiply the denominators.

$= \dfrac{4a}{3}$

So, $\dfrac{a^2 + 2a}{3a} \times \dfrac{20a^2}{5a^2 + 10a} = \dfrac{4a}{3}, a \neq -2, a \neq 0.$

b) $\dfrac{2x^2 - 8x}{x^2 - 3x - 10} \div \dfrac{4x^2}{x^2 - 9x + 20}$

$$= \frac{2x(x - 4)}{(x - 5)(x + 2)} \div \frac{4x^2}{(x - 4)(x - 5)}$$

$$= \frac{2x(x - 4)}{(x - 5)(x + 2)} \times \frac{(x - 4)(x - 5)}{4x^2}$$

Factor binomials and trinomials where possible. Multiply by the reciprocal.

$$= \frac{\cancel{2x}(x - 4)}{\cancel{(x - 5)}(x + 2)} \times \frac{(x - 4)\cancel{(x - 5)}}{\underset{2}{\cancel{4}}x^{\cancel{2}\,1}}, x \neq -2, x \neq 0, x \neq 5$$

Divide by any common factors.

$$= \frac{(x - 4)^2}{2x(x + 2)}$$

When considering restrictions, you must include any instance where the denominator can be zero. From the original expression, this occurs when $x - 5 = 0$, $x + 2 = 0$, and $x - 4 = 0$. When the second rational expression is inverted, then its denominator can be zero when $x = 0$.

So, $\dfrac{2x^2 - 8x}{x^2 - 3x - 10} \div \dfrac{4x^2}{x^2 - 9x + 20} = \dfrac{(x - 4)^2}{2x(x + 2)}$,
$x \neq -2$, $x \neq 0$, $x \neq 4$, $x \neq 5$.

Example 3

Add and Subtract Rational Expressions With Monomial Denominators

Simplify and state the restrictions.

a) $\dfrac{1}{5x} + \dfrac{1}{2x}$

b) $\dfrac{ab^2 + 2}{2ab^2} - \dfrac{b + 2}{2b}$

Solution

a) Start by determining the least common multiple (LCM) of the denominators.

$5x = (5)(x)$

$2x = (2)(x)$

$(5)(2)(x) = 10x$

The LCM is the least common denominator (LCD) of the two rational expressions.

$$\frac{1}{5x} + \frac{1}{2x} = \frac{1(2)}{5x(2)} + \frac{1(5)}{2x(5)}$$

Multiply each rational expression by a fraction equal to 1 that makes each denominator $10x$.

$$= \frac{2}{10x} + \frac{5}{10x}$$

$$= \frac{7}{10x}$$

Add the numerators.

Thus, $\dfrac{1}{5x} + \dfrac{1}{2x} = \dfrac{7}{10x}$, $x \neq 0$.

b) Determine the LCM of the denominators.

$2ab^2 = (2)(a)(b)(b)$

$2b = (2)(b)$

$(2)(a)(b)(b) = 2ab^2$

The LCD is $2ab^2$.

$$\frac{ab^2+2}{2ab^2} - \frac{b+2}{2b} = \frac{ab^2+2}{2ab^2} - \frac{(b+2)(ab)}{2b(ab)}$$

Multiply each rational expression by a fraction equal to 1 that makes each denominator $2ab^2$.

$$= \frac{ab^2+2}{2ab^2} - \frac{ab^2+2ab}{2ab^2}$$

$$= \frac{2-2ab}{2ab^2}$$

Subtract the numerators.

$$= \frac{2(1-ab)}{2ab^2}$$

Factor 2 from the numerator.

$$= \frac{1-ab}{ab^2}$$

Divide by the common factor of 2.

Thus, $\dfrac{ab^2+2}{2ab^2} - \dfrac{b+2}{2b} = \dfrac{1-ab}{ab^2}$, $a \neq 0$, $b \neq 0$.

Example 4

Add and Subtract Rational Expressions With Polynomial Denominators

Simplify and state the restrictions.

a) $\dfrac{x+5}{x-3} + \dfrac{x-7}{x+2}$

b) $\dfrac{x+9}{x^2+2x-48} - \dfrac{x-9}{x^2-x-30}$

Solution

a) There are no common factors in the denominators, so the LCD is just $(x-3)(x+2)$.

$$\frac{x+5}{x-3} + \frac{x-7}{x+2}$$

$$= \frac{(x+5)(x+2)}{(x-3)(x+2)} + \frac{(x-7)(x-3)}{(x+2)(x-3)}$$

Multiply each rational expression by a fraction equal to 1 that makes each denominator $(x-3)(x+2)$.

$$= \frac{x^2+7x+10}{(x-3)(x+2)} + \frac{x^2-10x+21}{(x-3)(x+2)}$$

$$= \frac{2x^2-3x+31}{(x-3)(x+2)}$$

Add the numerators.

Thus, $\dfrac{x+5}{x-3} + \dfrac{x-7}{x+2} = \dfrac{2x^2-3x+31}{(x-3)(x+2)}$, $x \neq -2$, $x \neq 3$.

b) Determine the LCM of the denominators.

$x^2 + 2x - 48 = (x + 8)(x - 6)$

$x^2 - x - 30 = (x - 6)(x + 5)$

$(x + 8)(x - 6)(x + 5)$

The LCD is $(x + 8)(x - 6)(x + 5)$.

$$\begin{aligned}
&\frac{x + 9}{x^2 + 2x - 48} - \frac{x - 9}{x^2 - x - 30} \\
&= \frac{(x + 9)(x + 5)}{(x + 8)(x - 6)(x + 5)} - \frac{(x - 9)(x + 8)}{(x - 6)(x + 5)(x + 8)} \\
&= \frac{x^2 + 14x + 45}{(x + 8)(x - 6)(x + 5)} - \frac{x^2 - x - 72}{(x + 8)(x - 6)(x + 5)} \\
&= \frac{15x + 117}{(x + 8)(x - 6)(x + 5)}
\end{aligned}$$

Multiply each rational expression by a fraction equal to 1 that makes each denominator $(x + 8)(x - 6)(x + 5)$.

Add the numerators.

Thus, $\dfrac{x + 9}{x^2 + 2x - 48} - \dfrac{x - 9}{x^2 - x - 30} = \dfrac{15x + 117}{(x + 8)(x - 6)(x + 5)}$, $x \neq -8$, $x \neq -5$, $x \neq 6$.

Example 5

Bicycle Relay

Raj and Mack are competing as a relay team in a 50-km cycling race. There are two legs in the race. Leg A is 30 km and leg B is 20 km.

a) Assuming that each cyclist travels at a different average speed, determine a simplified expression to represent the total time of the race.

b) If Raj can maintain an average speed of 35 km/h and Mack an average speed of 25 km/h, determine the minimum time it will take to complete the race.

Solution

a) For any distance-speed-time calculation, the expression for the time, t, is given by $t = \frac{d}{v}$, where d represents the distance and v represents the speed. To calculate the total time, add the times for the two legs. Let t_A and t_B represent the times and v_A and v_B represent the speeds of legs A and B, respectively.

$$t = t_A + t_B$$
$$= \frac{30}{v_A} + \frac{20}{v_B}$$
$$= \frac{30v_B}{v_Av_B} + \frac{20v_A}{v_Av_B}$$
Write with a common denominator.
$$= \frac{30v_B + 20v_A}{v_Av_B}$$
Add the numerators.

b) It makes sense that for the minimum time, the fastest person should ride the longest leg. So, Raj will ride leg A and Mack will ride leg B.

$$t = \frac{30(25) + 20(35)}{35(25)}$$
Substitute the value for each person's speed.
$$\doteq 1.66$$

It will take the team approximately 1.66 h to complete the race.

Key Concepts

- When multiplying or dividing rational expressions, follow these steps:
  - Factor any polynomials, if possible.
  - When dividing by a rational expression, multiply by the reciprocal of the rational expression.
  - Divide by any common factors.
  - Determine any restrictions.
- When adding or subtracting rational expressions, follow these steps:
  - Factor the denominators.
  - Determine the least common multiple of the denominators.
  - Rewrite the expressions with a common denominator.
  - Add or subtract the numerators.
  - Simplify and state the restrictions.

Communicate Your Understanding

C1 Describe how you would simplify $\frac{(x + 3)(x - 6)}{(x + 4)(x + 5)} \div \frac{(x - 6)(x + 8)}{(x + 4)(x - 7)}$. What are the restrictions on the variable?

C2 Write two rational expressions whose product is $\frac{x + 5}{x - 2}$, $x \neq -4$, $x \neq 1$, $x \neq 2$.

C3 A student simplifies the expression $\frac{x + 3}{4} + \frac{x - 3}{6}$ and gets an answer of $\frac{2x}{12}$. What did the student probably do incorrectly to get this answer?

C4 Describe how you would simplify $\frac{5}{x + 3} - \frac{7x}{x - 1}$. What are the restrictions on the variable?

A Practise

For help with questions 1 and 2, refer to Example 1.

1. Simplify and state the restrictions on the variables.

a) $\frac{14y}{11x} \times \frac{121y}{7x}$ **b)** $\frac{20x^3}{7x} \times \frac{35x^5}{4x}$

c) $\frac{15b^3}{4b} \times \frac{20b}{30b^2}$ **d)** $\frac{30ab}{12a^2} \times \frac{18a}{45b^2}$

2. Simplify and state the restrictions on the variables.

a) $\frac{5x}{9y} \div \frac{5x}{18y^2}$ **b)** $\frac{55xy}{8y} \div \frac{1}{48x^2}$

c) $\frac{26ab}{4a} \div \frac{39a^4b^3}{12b^4}$ **d)** $\frac{32a^2b}{6c} \div \frac{16ab}{24c^3}$

For help with questions 3 to 6, refer to Example 2.

3. Simplify and state the restrictions on the variable.

a) $\frac{25}{x + 10} \times \frac{x + 10}{5}$ **b)** $\frac{x - 1}{x} \times \frac{2x}{x - 1}$

c) $\frac{x + 5}{x - 3} \times \frac{x - 3}{x + 7}$ **d)** $\frac{2x + 3}{x + 8} \times \frac{x + 8}{2x + 3}$

4. Simplify and state the restrictions on the variable.

a) $\frac{3x^2}{12x^2 + 18x} \times \frac{4x + 6}{3x + 30}$

b) $\frac{4x + 24}{x^2 + 8x} \times \frac{12x^2}{3x + 18}$

c) $\frac{x^2 + 10x + 21}{x + 3} \times \frac{x + 2}{x^2 + 9x + 14}$

d) $\frac{x^2 + 2x - 15}{x^2 - 9x + 18} \times \frac{x - 6}{x + 5}$

5. Simplify and state the restrictions on the variable.

a) $\frac{x + 1}{x} \div \frac{x + 1}{2x}$ **b)** $\frac{x}{x - 3} \div \frac{1}{x - 3}$

c) $\frac{x + 12}{x + 10} \div \frac{x + 12}{x - 5}$ **d)** $\frac{x - 7}{x + 3} \div \frac{x - 7}{x + 3}$

6. Simplify and state the restrictions on the variable.

a) $\frac{x^2 + 15x}{4x + 24} \div \frac{3x}{3x + 18}$

b) $\frac{6x}{8x - 72} \div \frac{9x}{2x - 18}$

c) $\frac{x^2 + 15x + 26}{6x^2} \div \frac{x^2 - 3x - 10}{30x^3}$

d) $\frac{x^2 + 11x + 24}{x^2 + 2x - 3} \div \frac{x - 8}{x - 1}$

For help with question 7, refer to Example 3.

7. Simplify and state any restrictions.

a) $\frac{x + 1}{18} + \frac{x - 1}{45}$ **b)** $\frac{x + 10}{12} - \frac{2x - 1}{15}$

c) $\frac{2}{3x} - \frac{1}{4x}$ **d)** $\frac{7}{6x} + \frac{3}{8x}$

e) $\frac{3}{ab} + \frac{5}{4b}$ **f)** $\frac{13}{10a^2b} + \frac{11}{4b^2}$

g) $\frac{2 + a}{a^2b} + \frac{4 - a}{3ab^2}$ **h)** $\frac{4 - ab}{9ab} + \frac{2ab}{6a^2b^2}$

For help with questions 8 and 9, refer to Example 4.

8. Simplify and state the restrictions.

a) $\frac{1}{x - 6} - \frac{1}{x + 6}$

b) $\frac{12}{x + 8} + \frac{3}{x - 9}$

c) $\frac{x + 10}{x - 6} - \frac{x - 3}{x + 4}$

d) $\frac{x + 5}{x + 1} + \frac{x + 2}{x - 2}$

9. Simplify and state the restrictions.

a) $\frac{x}{x^2 - 9x + 8} + \frac{2}{x - 8}$

b) $\frac{x + 3}{x + 5} + \frac{x + 2}{x^2 + 3x - 10}$

c) $\frac{x}{x^2 + 3x + 2} - \frac{3x - 2}{x^2 + 8x + 7}$

d) $\frac{x + 4}{x^2 - 121} - \frac{2x - 1}{x^2 + 8x - 33}$

B Connect and Apply

For help with question 10, refer to Example 5.

10. Alice is in a 20-km running race. She always runs the first half at an average speed of 2 km/h faster than the second half.

a) Let x represent her speed in the first half. Determine a simplified expression in terms of x for the total time needed for the race.

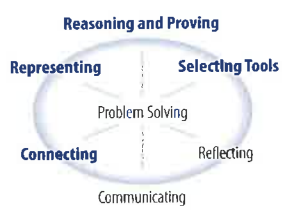
b) If Alice runs the first half at 10 km/h, how long will it take her to run the race?

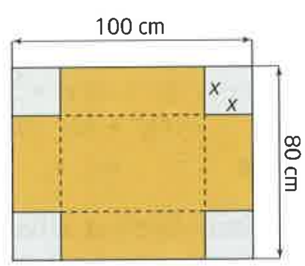
11. Binomial expressions can differ by a factor of -1. Factor -1 from one of the denominators to identify the common denominator. Then, simplify each expression and state the restrictions.

a) $\dfrac{1}{x-2} - \dfrac{1}{2-x}$ **b)** $\dfrac{2x-7}{x-3} + \dfrac{x-9}{3-x}$

c) $\dfrac{a+1}{5-2a} - \dfrac{a-4}{2a-5}$ **d)** $\dfrac{2b+3}{4b-1} + \dfrac{b+6}{1-4b}$

12. An open-topped box is to be created from a 100-cm by 80-cm piece of cardboard by cutting out a square of side length x from each corner.





a) Express the volume of the box as a function of x.

b) Express the surface area of the open-topped box as a function of x.

c) Write a simplified expression for the ratio of the volume of the box to its surface area.

d) Based on your answer in part c), what are the restrictions on x? What are the restrictions in the context?

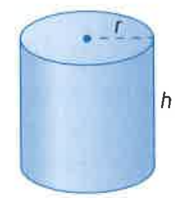
13. Resistors are components found on most circuit boards and in most electronic devices. Since resistors do not come in every size, they have to be arranged in various ways to get the needed resistance. When three resistors are in parallel, then the total resistance, R_T, can be calculated using the equation $\dfrac{1}{R_T} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$, where each of the resistances is in ohms (Ω).

a) Determine an expression for the total resistance, R_T.

b) Determine an expression for the total resistance if $R_1 = R_2 = R_3$.

c) Determine an expression for the total resistance if $R_1 = 2R_2 = 6R_3$.

14. Consider a cylinder of height h and radius r.



a) Determine the ratio of the volume of the cylinder to its surface area.

b) What restrictions are there on r and h?

15. Olivia can swim at an average rate of v metres per second in still water. She has two races coming up, one in a lake with no current and the other in a river with a current of 0.5 m/s. Each race is 800 m, but in the river race she swims the first half against the current and the second half with the current.

a) Determine an expression for the time for Olivia to complete the lake swim.

b) Determine an expression for the time for Olivia to complete the river swim.

c) Olivia thinks that if she swims each race exactly the same and the current either slows her down or speeds her up by 0.5 m/s, both races will take the same amount of time. Is she correct? Explain.

16. Use Technology

a) Use graphing technology to graph $f(x) = \frac{1}{x+2} + \frac{1}{x-2}$.

b) Rewrite the function using a common denominator. Then, graph the rewritten function.

c) Compare the graphs. Identify how the restrictions affect the graph.

Achievement Check

17. a) Simplify the expressions for A and B, where $A = \frac{x+4}{x^2+9x+20}$ and $B = \frac{3x^2-9x}{x^2+3x-18}$. State the restrictions.

b) Are the two expressions equivalent? Justify your answer.

c) Write another expression that appears to be equivalent to each expression in part a).

d) Determine $A + B$, AB, and $B \div A$.

C Extend

18. Archimedes of Syracuse (287–212 BCE) studied many things. One was the relationship between a cylinder and a sphere. In particular, he looked at the situation where the sphere just fits inside the cylinder so that they have the same radius and the height of the cylinder equals the diameter of the sphere.

a) Determine the ratio of the volume of the sphere to the volume of the cylinder in this situation.

b) Determine the ratio of the surface area of the sphere to the surface area of the cylinder in this situation.

c) What seems to be true about your answers from parts a) and b)?

Connections

Archimedes was so fond of the sphere and cylinder relationship that he had the image of a sphere inscribed in a cylinder engraved on his tombstone.

19. Simplify the expression and state any restrictions.

$$\frac{x+8}{2x^2+9x+10} - \frac{x^2+13x+40}{2x^2-x-15} \div \frac{x^2+10x+16}{x^2-9}$$

20. a) Evaluate the expression

$$1+\cfrac{1}{0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6}}}}}}}}}}$$

b) On a scientific calculator, locate the e^x button and enter e^1. Compare your answer for part a) to the constant e.

c) The pattern shown in part a) continues on forever. What are the next three steps in this pattern? How do they affect your comparison from part b)?

21. Math Contest When n is divided by 4, the remainder is 3. When $6n$ is divided by 4, the remainder is

A 1 **B** 2 **C** 3 **D** 0

22. Math Contest The sum of the roots of $(x^2 + 4x + 3)(x^2 + 3x - 10) - (8x^2 - 8x - 16) = 0$ is

A −7 **B** −6 **C** 6 **D** 8

23. Math Contest Given $f(x) = \frac{36}{x-2} + \frac{35}{x-1}$, what is the smallest integral value of x that gives an integral value of $f(x)$?

24. Math Contest Given $\frac{2x}{x-3} = \frac{3y}{y-4} = \frac{4z}{z-5} = 5$, then $x + y + z$ is

A 40 **B** −40 **C** 200 **D** −200

2.3

Horizontal and Vertical Translations of Functions

When a video game developer is designing a game, she might have several objects displayed on the computer screen that move from one place to another but do not change shape or orientation. In fact, the object can be said to "slide" around the screen.

In this section, you will consider how base functions "slide" around the Cartesian plane and the effect this has on their equations.

Investigate

What effect does translating have on the graph and equation of a function?

Tools

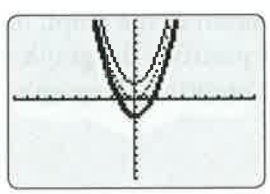
- graphing calculator

A: Graph Functions of the Form $g(x) = f(x) + c$

1. Clear any graphed functions and graph the following set of functions as **Y1**, **Y2**, and **Y3** on the same set of axes. Use a standard window.

Set 1: $f(x) = x^2$ $\quad g(x) = x^2 + 2$ $\quad h(x) = x^2 - 2$

You may want to make each line a different thickness to distinguish them.



2. Compare the equations of $g(x)$ and $h(x)$ to the equation of $f(x)$. Compare the graphs of $g(x)$ and $h(x)$ to the graph of $f(x)$.

3. Repeat steps 1 and 2 with each set of functions.

Set 2: $f(x) = \sqrt{x}$ $\quad g(x) = \sqrt{x} + 2$ $\quad h(x) = \sqrt{x} - 2$

Set 3: $f(x) = \frac{1}{x}$ $\quad g(x) = \frac{1}{x} + 2$ $\quad h(x) = \frac{1}{x} - 2$

4. Reflect Describe how the value of c in $g(x) = f(x) + c$ changes the graph of $f(x)$.

Technology Tip

You can change the appearance of a line on a graphing calculator.

- Press Y=. Cursor left to the slanted line beside the equation.
- Press ENTER repeatedly to choose one of the seven options.
- Press GRAPH.

B: Graph Functions of the Form $g(x) = f(x - d)$

1. Clear any graphed functions and graph the following set of functions as **Y1**, **Y2**, and **Y3** on the same set of axes. Use a standard window.

   Set 1: $f(x) = x^2$ $\quad$ $g(x) = (x + 2)^2$ $\quad$ $h(x) = (x - 2)^2$

   You may want to make each line a different thickness to distinguish them.

2. Compare the equations of $g(x)$ and $h(x)$ to the equation of $f(x)$. Compare the graphs of $g(x)$ and $h(x)$ to the graph of $f(x)$.

3. Repeat steps 1 and 2 with each set of functions.

   Set 2: $f(x) = \sqrt{x}$ $\quad$ $g(x) = \sqrt{x + 2}$ $\quad$ $h(x) = \sqrt{x - 2}$

   Set 3: $f(x) = \frac{1}{x}$ $\quad$ $g(x) = \frac{1}{x + 2}$ $\quad$ $h(x) = \frac{1}{x - 2}$

4. **Reflect** Describe how the value of d in $g(x) = f(x - d)$ changes the graph of $f(x)$.

5. **Reflect** Compare the **transformations** on the functions in parts A and B. Describe any similarities and differences.

transformation

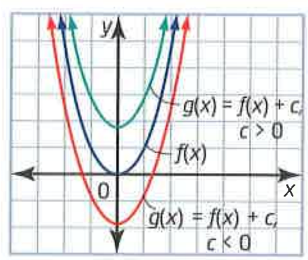
- a change made to a figure or a relation such that the figure or the graph of the relation is shifted or changed in shape
- Translations, stretches, and reflections are types of transformations.

translation

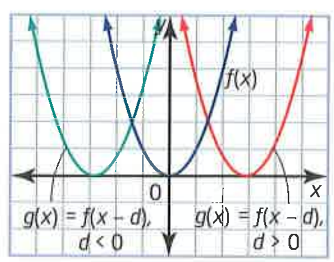
- a transformation that results in a shift of the original figure without changing its shape

Transformations that shift a function up, down, right, or left without affecting the shape are called **translations**.

The graph of the function $g(x) = f(x) + c$ is a vertical translation of the graph of $f(x)$ by c units. If c is positive, the graph moves up c units. If c is negative, the graph moves down c units.



The graph of the function $g(x) = f(x - d)$ is a horizontal translation of the graph of $f(x)$ by d units. If d is positive, the graph moves to the right d units. If d is negative, the graph moves to the left d units.



Example 1

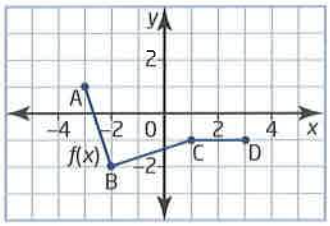
Graph Translations by Using Points

Given the graph of a function $f(x)$, sketch the graph of $g(x)$ by determining **image points** for any original key points.

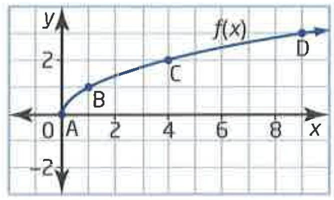
image point

- any point that has been transformed from a point on the original figure or graph

a) $g(x) = f(x) + 4$



b) $g(x) = f(x + 4)$



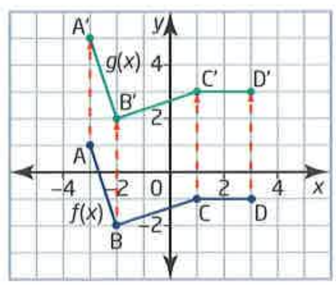
Solution

a) Since $y = f(x)$, the function $g(x) = f(x) + 4$ can be rewritten as $g(x) = y + 4$.

The corresponding transformation is a vertical translation of 4 units up. For each point, only the value of the y-coordinate changes, as shown in the table.

$f(x)$: $(x, f(x))$	$g(x)$: $(x, f(x) + 4)$
A(−3, 1)	A′(−3, 1 + 4) = (−3, 5)
B(−2, −2)	B′(−2, −2 + 4) = (−2, 2)
C(1, −1)	C′(1, −1 + 4) = (1, 3)
D(3, −1)	D′(3, −1 + 4) = (3, 3)

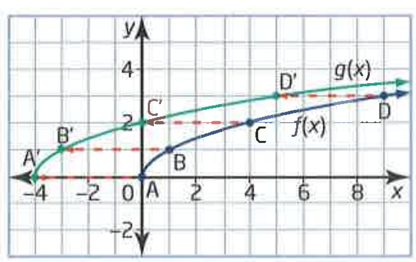
It is common notation to use a prime (′) next to each letter representing an image point.



b) Write $g(x) = f(x + 4)$ as $g(x) = f(x - (-4))$. Since $d = -4$, each point is translated 4 units to the left. Subtract 4 from the x-coordinate of each point. The y-coordinate does not change.

Create a table using the key points.

$f(x)$: $(x, f(x))$	$g(x)$: $(x - 4, f(x))$
A(0, 0)	A′(0 − 4, 0) = (−4, 0)
B(1, 1)	B′(1 − 4, 1) = (−3, 1)
C(4, 2)	C′(4 − 4, 2) = (0, 2)
D(9, 3)	D′(9 − 4, 3) = (5, 3)



To ensure an accurate sketch of a transformed function, translate key points on the base function first and then draw a smooth curve through the new points.

Example 2

Describe Transformations

For each function $g(x)$, describe the transformation from a base function of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, or $f(x) = \frac{1}{x}$, first using function notation and then using words. Transform the graph of $f(x)$ to sketch the graph of $g(x)$ and then state the domain and range of each function.

a) $g(x) = (x + 5)^2 + 1$

b) $g(x) = \dfrac{1}{x + 3} - 7$

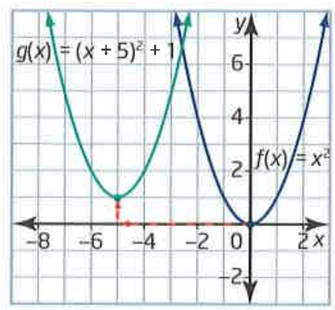
c) $g(x) = \sqrt{x - 2} + 4$

Solution

a) The base function is $f(x) = x^2$. By comparison,

$$\begin{aligned} g(x) &= (x + 5)^2 + 1 \\ &= f(x + 5) + 1 \end{aligned}$$

This is a horizontal translation of 5 units to the left ($d = -5$) and a vertical translation of 1 unit up ($c = 1$). To help with the sketch, start with the vertex (0, 0) from $f(x)$ and translate it to the point $(0 - 5, 0 + 1) = (-5, 1)$.



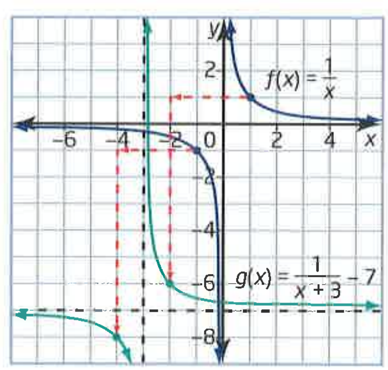
For $f(x)$, the domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}, y \geq 0\}$.

For $g(x)$, the domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}, y \geq 1\}$.

b) The base function is $f(x) = \frac{1}{x}$. By comparison,

$$\begin{aligned} g(x) &= \frac{1}{x + 3} - 7 \\ &= f(x + 3) - 7 \end{aligned}$$

This is a horizontal translation of 3 units to the left ($d = -3$) and a vertical translation of 7 units down ($c = -7$). To help with the sketch, start with the point (1, 1) from $f(x)$ and translate it to the point $(1 - 3, 1 - 7) = (-2, -6)$. Similarly, the point $(-1, -1)$ becomes $(-4, -8)$. The base function has asymptotes at $x = 0$ and $y = 0$, so the translated function will have asymptotes at $x = -3$ and $y = -7$.



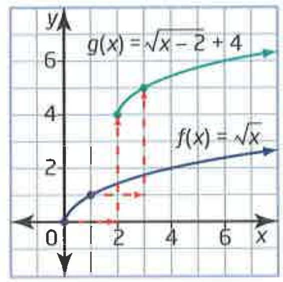
For $f(x)$, the domain is $\{x \in \mathbb{R}, x \neq 0\}$ and the range is $\{y \in \mathbb{R}, y \neq 0\}$.

For $g(x)$, the domain is $\{x \in \mathbb{R}, x \neq -3\}$ and the range is $\{y \in \mathbb{R}, y \neq -7\}$.

c) The base function is $f(x) = \sqrt{x}$. By comparison,

$$g(x) = \sqrt{x - 2} + 4$$
$$= f(x - 2) + 4$$

This is a horizontal translation of 2 units to the right ($d = 2$) and a vertical translation of 4 units up ($c = 4$). To help with the sketch, start with the point (0, 0) from $f(x)$ and translate it to the point $(0 + 2, 0 + 4) = (2, 4)$. Similarly, the point (1, 1) becomes (3, 5).



For $f(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 0\}$.

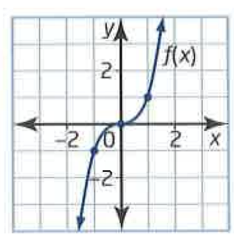
For $g(x)$, the domain is $\{x \in \mathbb{R}, x \geq 2\}$ and the range is $\{y \in \mathbb{R}, y \geq 4\}$.

Key Concepts

- Translations are transformations that cause functions to shift from one place to another without changing shape.
- The graph of $g(x) = f(x) + c$ is a vertical translation of the graph of $f(x)$ by c units. If $c > 0$, the graph moves up c units. If $c < 0$, the graph moves down c units.
- The graph of $g(x) = f(x - d)$ is a horizontal translation of the graph of $f(x)$ by d units. If $d > 0$, the graph moves to the right d units. If $d < 0$, the graph moves to the left d units.
- A sketch of the graph of any transformed function can be created by transforming the related base function.
- In general, the domain and range of a function of the form $g(x) = f(x - d) + c$ can be determined by adding the d-value and the c-value to restrictions on the domain and range, respectively, of the base function.

Communicate Your Understanding

C1 Given the graph of the function $f(x)$, describe how you would sketch the graph of $g(x) = f(x + 2) - 3$.



C2 Explain why the graph of $g(x) = f(x) + c$ translates the graph of $f(x)$ vertically and not horizontally.

C3 Explain why the graph of $g(x) = f(x - d)$ translates the graph of $f(x)$ to the right and not to the left for $d > 0$.

A Practise

For help with questions 1 to 5, refer to Example 1.

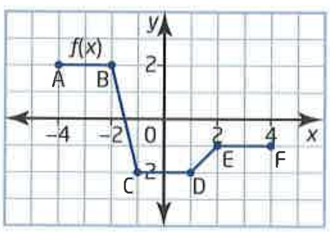
1. a) Copy and complete the table of values.

x	$f(x) = \sqrt{x}$	$r(x) = f(x) + 7$	$s(x) = f(x - 1)$
0			
1			
4			
9			

b) Use the points to graph all three functions on the same set of axes.

c) Explain how the points of the translated functions relate to the actual transformations.

2. Copy the graph of $f(x)$. Apply each transformation by determining the image points A′, B′, C′, D′, E′, and F′.



a) $b(x) = f(x) + 5$

b) $g(x) = f(x) - 7$

c) $h(x) = f(x - 8)$

d) $m(x) = f(x + 6)$

3. Copy the graph of $f(x)$ in question 2. Apply each transformation by determining the image points A′, B′, C′, D′, E′, and F′.

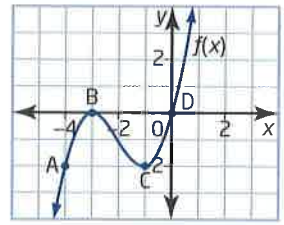
a) $n(x) = f(x - 3) + 6$

b) $r(x) = f(x - 2) - 10$

c) $s(x) = f(x + 5) + 4$

d) $t(x) = f(x + 12) - 3$

4. Copy the graph of $f(x)$. Apply each transformation by determining the image points A′, B′, C′, and D′.



a) $b(x) = f(x) + 3$ **b)** $g(x) = f(x) - 6$

c) $h(x) = f(x - 4)$ **d)** $n(x) = f(x + 7)$

5. Copy the graph of $f(x)$ in question 4. Apply each transformation by determining the image points A′, B′, C′, and D′.

a) $m(x) = f(x - 2) + 10$

b) $r(x) = f(x - 5) - 9$

c) $s(x) = f(x + 8) + 9$

d) $t(x) = f(x + 1) - 11$

For help with questions 6 to 8, refer to Example 2.

6. For each function $g(x)$, identify the base function as one of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$ and describe the transformation first in the form $y = f(x - d) + c$ and then in words. Transform the graph of $f(x)$ to sketch the graph of $g(x)$ and then state the domain and range of each function.

a) $g(x) = x - 9$ **b)** $g(x) = x + 12$

c) $g(x) = x^2 + 8$ **d)** $g(x) = \sqrt{x} - 12$

e) $g(x) = (x - 6)^2$ **f)** $g(x) = \frac{1}{x} + 5$

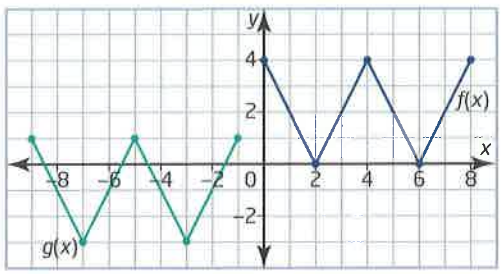
g) $g(x) = \sqrt{x + 10}$ **h)** $g(x) = \frac{1}{x - 2}$

i) $g(x) = \sqrt{x - 9} - 5$ **j)** $g(x) = \frac{1}{x + 3} - 8$

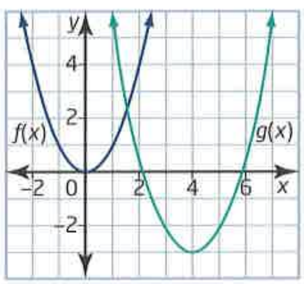
7. Explain why there is more than one possible answer for question 6a) and b) when describing the transformations.

8. Use words and function notation to describe the transformation that can be applied to the graph of $f(x)$ to obtain the graph of $g(x)$. State the domain and range of each function.

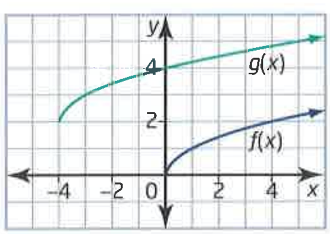
a)



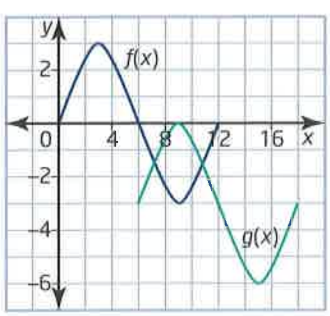
b)



c)



d)



B Connect and Apply

9. The graph of $f(x) = x^2$ is transformed to the graph of $g(x) = f(x + 8) + 12$.

a) Describe the two translations represented by this transformation.

b) Determine three points on the base function. Horizontally translate and then vertically translate the points to determine the image points on $g(x)$.

c) Start with your original points, but this time reverse the order of your translations. Determine whether the order of the translations is important.

d) Confirm your result from parts b) and c) by graphing.

e) Repeat parts a) to d) with a different base function.

10. Given $f(x) = x$, determine if there is a single horizontal translation that has the same effect as a single vertical translation. Justify your answer algebraically (with equations), numerically (with points), and graphically (with sketches).

11. Use the base function $f(x) = x$. Write the equation for each transformed function.

a) $n(x) = f(x - 4) - 6$

b) $r(x) = f(x + 2) + 9$

c) $s(x) = f(x + 6) - 7$

d) $t(x) = f(x - 11) + 4$

12. Repeat question 11 using the base function $f(x) = x^2$.

13. Repeat question 11 using the base function $f(x) = \sqrt{x}$.

14. Repeat question 11 using the base function $f(x) = \frac{1}{x}$. State any restrictions on the variable.

15. Chapter Problem The traffic safety bureau receives data regarding the acceleration of a prototype electric sports car. It can accelerate from 0 to 100 km/h in about 4 s. Its position, d, in metres, at any time t, in seconds, is given by $d(t) = 3.5t^2$. Matthew is comparing the prototype to a hybrid electric car, which has its position given by $d(t) = 1.4t^2$.

a) In a race between the two cars, the hybrid is given a head start. Where would the hybrid have to start so that after 4 s of acceleration, both cars are in the same position?

b) Verify your solution by graphing.

16. The cost to produce x units of a product can be modelled by the function $c(x) = \sqrt{x} + 500$.

a) State and interpret the domain and range of the cost function.

b) Suppose that the cost to make 10 prototype units is to be included in the cost. Write a new function representing the cost of this product.

c) What type of transformation does the change in part b) represent?

d) How does the transformation in part b) affect the domain and range?

C Extend

17. Use Technology In this section, you dealt with static translations. In computer animation, dynamic translations are used. Open *The Geometer's Sketchpad*®. Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Section 2.3. Download the file **2.3_Animation.gsp**. In this sketch, you will be able to change a parameter t that represents time by moving a sliding point.

a) Study the form of the function $g(x)$. What similarities and differences are there compared to a translated function of the form $g(x) = f(x - d) + c$?

b) What happens when you move the slider t?

c) How does changing the parameter function $P(x)$ affect the motion of the base function $f(x)$? Use the following functions to investigate this.

i) $P(x) = x^2$

ii) $P(x) = \sqrt{x}$

iii) $P(x) = \frac{1}{x}$

d) How could you change the translated function $g(x)$ so that when you move the slider t the translation is only vertical? horizontal?

e) Click on the **Link to Butterfly** button and move the slider t. Here you can see a very rudimentary example of computer animation. Repeat parts c) and d) for this sketch.

18. Math Contest The roots of $x^3 - 2x^2 - 5x + 6 = 0$ are 3, -2, and 1. The sum of the roots of $(x + 2)^3 - 2(x + 2)^2 - 5(x + 2) + 6 = 0$ is

A 2 **B** 8 **C** 0 **D** -4

19. Math Contest The parabola $y = 2x^2$ is translated to a new parabola with x-intercepts 4 and -3. The y-intercept of the new parabola is

A 12 **B** -12 **C** -0.5 **D** -24

20. Math Contest A lattice point is a point on the Cartesian plane where both coordinates are integers. How many lattice points are there on the line $2x + 3y - 600 = 0$, where $x > 0$ and $y > 0$?

A 98 **B** 99 **C** 100 **D** 200

Reflections of Functions

Teaz Apparel is a local store that sells athletic clothing. The word *Teaz* in their logo is the same whether viewed right-side up or upside down. Think of how **reflections** could be used to make this logo.

reflection

- a transformation in which a figure is reflected over a reflection line

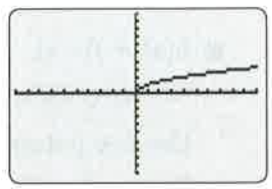
Investigate

Tools

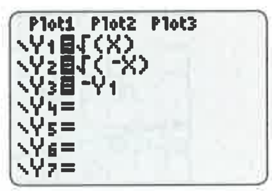
- graphing calculator

How can you determine the line of reflection?

1. Press (Y=) and enter the function $f(x) = \sqrt{x}$ for **Y1**. Press (ZOOM) and select **6:ZStandard** to observe the graph of the function.



2. Enter the function $f(-x) = \sqrt{-x}$ as **Y2**. Press (GRAPH) to observe the graph of the new function.
3. **Reflect** Compare the graph of **Y2** to the graph of **Y1**. In what line can the graph of **Y1** be reflected to create the graph of **Y2**?
4. Enter the function $-f(x) = -$**Y1** as **Y3**. Press (GRAPH) to observe the graph of the new function.



5. **Reflect** Compare the graph of **Y3** to to the graph of **Y1**. In what line can the graph of **Y1** be reflected to create the graph of **Y3**?
6. Enter the function $-f(-x) = -$**Y2** as **Y4**. Press (GRAPH) to observe the graph of the new function.
7. **Reflect** Compare the graph of **Y4** to the graph of **Y1**. In what line(s) can the graph of **Y1** be reflected to create the graph of **Y4**? Compare the graph of **Y4** to the graph of **Y2**. In what line(s) can the graph of **Y2** be reflected to create the graph of **Y4**?

Technology Tip

To enter an existing function in the **Y=** editor, like **Y1**, press (VARS), cursor over to the **Y-VARS** menu, select **1:Function...**, and then select **1:Y1**.

The graph of $g(x) = f(-x)$ is a reflection of the graph of $f(x)$ in the y-axis.

The graph of $h(x) = -f(x)$ is a reflection of the graph of $f(x)$ in the x-axis.

The graph of $q(x) = -f(-x)$ is a reflection of the graph of $f(x)$ in the y-axis and the x-axis.

Example 1

Graph Reflections

Given the function $f(x) = \sqrt{x} + 2$, write an equation to represent each of the following. Then, sketch each graph and state the domain and range of each function.

a) $h(x)$: a reflection in the y-axis

b) $r(x)$: a reflection in the x-axis

c) $s(x)$: a reflection in the y-axis and then a reflection in the x-axis

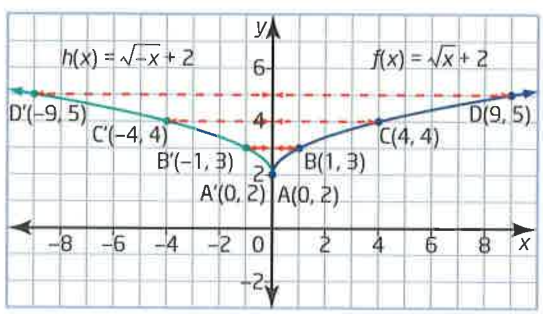
Solution

a) $h(x) = f(-x)$

$= \sqrt{-x} + 2$

Use key points to sketch the graph of $f(x)$.

For a reflection in the y-axis, each image point will be an equal distance from the y-axis but on the other side. The image points will have the same y-coordinates as the original key points but their x-coordinates will have opposite signs. Once the points are plotted, you can sketch the graph of $h(x)$.



For $f(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 2\}$.

For $h(x)$, the domain is $\{x \in \mathbb{R}, x \leq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 2\}$.

b) $r(x) = -f(x)$

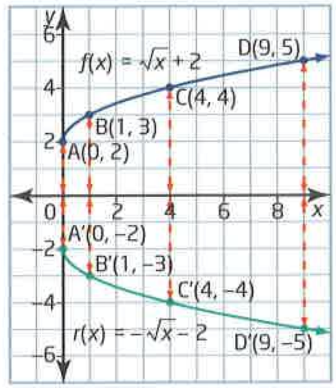
$= -(\sqrt{x} + 2)$

$= -\sqrt{x} - 2$

For a reflection in the x-axis, each image point will be an equal distance from the x-axis but on the other side. The image points will have the same x-coordinates as the key points but their y-coordinates will have opposite signs.

For $f(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 2\}$.

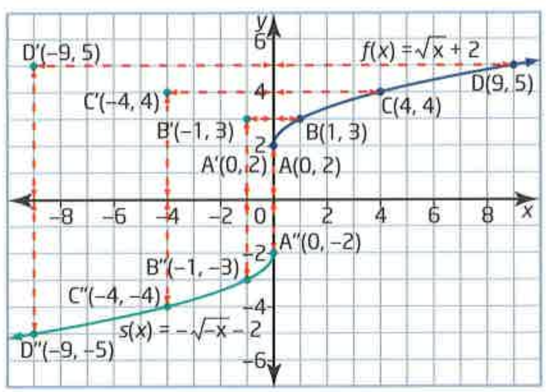
For $r(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \leq -2\}$.



c) $s(x) = -f(-x)$

$$= -(\sqrt{-x} + 2)$$
$$= -\sqrt{-x} - 2$$

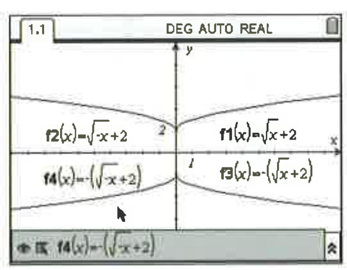
There are two reflections, first in the y-axis and then in the x-axis. Start with the resulting points from part a), which were reflected in the y-axis, and reflect these in the x-axis. For each image point, first the x-coordinate and then the y-coordinate will have opposite signs.



For a reflection in both the y-axis and x-axis, the order of the reflections is not important.

For $f(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 2\}$.
For $s(x)$, the domain is $\{x \in \mathbb{R}, x \leq 0\}$ and the range is $\{y \in \mathbb{R}, y \leq -2\}$.

Using a TI-Nspire™ CAS graphing calculator, you can view the graph of the function $f(x) = \sqrt{x} + 2$ and the resulting graphs of the three reflections.



A reflection of a point (x, y) in the y-axis becomes $(-x, y)$. So, any point that lies on the y-axis will not change under this reflection because its x-coordinate is 0.

A reflection of a point (x, y) in the x-axis becomes $(x, -y)$. So, any point that lies on the x-axis will not change under this reflection because its y-coordinate is 0.

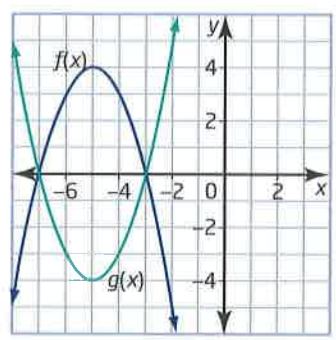
Points that do not change under a transformation are said to be invariant.

Example 2

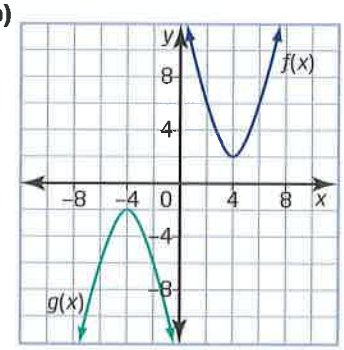
Describe Reflections

For each graph, describe the reflection that transforms $f(x)$ into $g(x)$.

a)

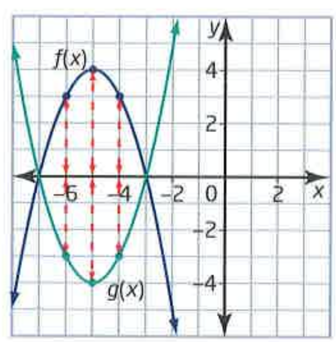


b)



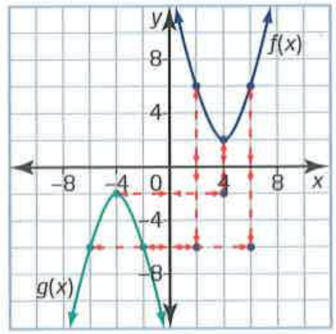
Solution

a) The key points $(-6, 3)$, $(-5, 4)$, and $(-4, 3)$ on the graph of $f(x)$ have image points on the graph of $g(x)$ that are equidistant but on the other side of the x-axis: $(-6, -3)$, $(-5, -4)$, and $(-4, -3)$. This indicates a reflection in the x-axis.

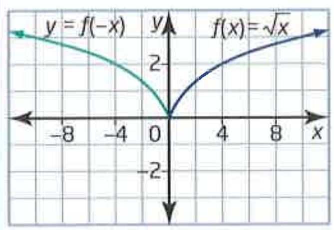
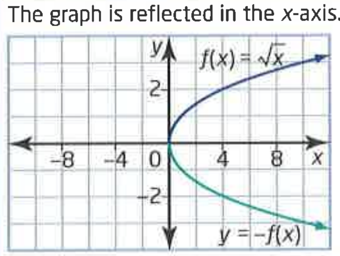
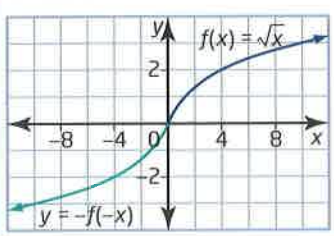


Notice that the two points on the x-axis, $(-7, 0)$ and $(-3, 0)$, are invariant.

b) Reflecting the key points (2, 6), (4, 2), and (6, 6) on the graph of $f(x)$ in the x-axis gives (2, −6), (4, −2), and (6, −6), which are not on $g(x)$. Similar results occur if the points are reflected in the y-axis. However, if the points (2, −6), (4, −2), and (6, −6) are reflected in the y-axis, the image points (−2, −6), (−4, −2), and (−6, −6) are on the graph of $g(x)$. The function $g(x)$ is a reflection of $f(x)$ in both axes.



Key Concepts

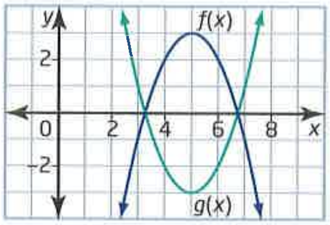
Reflection	Numerical Representation	Graphical Representation	Algebraic Representation
$y = f(-x)$	A point (x, y) becomes $(-x, y)$.	The graph is reflected in the y-axis. 	Replace x with $-x$ in the expression.
$y = -f(x)$	A point (x, y) becomes $(x, -y)$.	The graph is reflected in the x-axis. 	Multiply the entire expression by −1.
$y = -f(-x)$	A point (x, y) becomes $(-x, -y)$.	The graph is reflected in one axis and then the other. 	First replace x with $-x$ in the expression and then multiply the entire expression by −1.

Communicate Your Understanding

C1 Consider the functions $f(x) = x$, $f(x) = x^2$, $f(x) = \frac{1}{x}$, and $f(x) = \sqrt{x}$. Describe any reflections that leave any of these functions unchanged. Explain.

C2 Determine whether $g(x)$ could be a reflection of $f(x)$. If so, describe the reflection.

a)

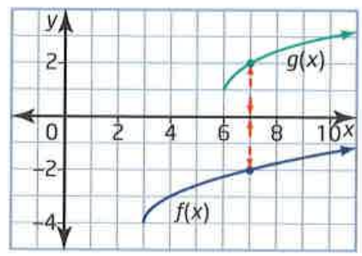


b)

$f(x)$	$g(x)$
(3, 8)	(8, –3)
(5, 12)	(12, –5)
(7, 16)	(16, –7)
(9, 20)	(20, –9)

c) $f(x) = 3(x + 3)^2 + 10$ and $g(x) = 3x^2 - 18x + 37$

C3 A student claims that the graph of $g(x)$ is a reflection of the graph of $f(x)$ in the x-axis because two points, one on each graph, are equidistant from the x-axis. Is the student correct? Explain your answer.

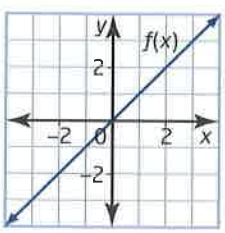


Ⓐ Practise

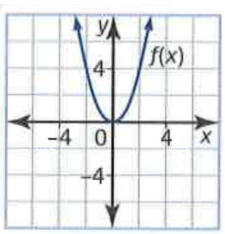
For help with questions 1 to 4, refer to Example 1.

1. Copy each graph of $f(x)$ and sketch its reflection in the x-axis, $g(x)$. Then, state the domain and range of each function.

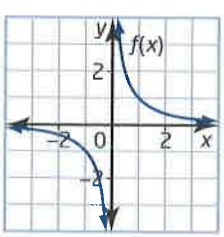
a)



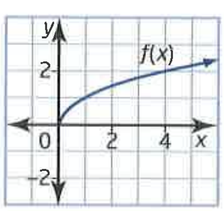
b)



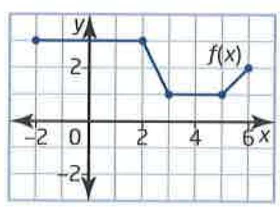
c)



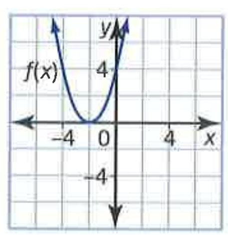
d)



e)



f)



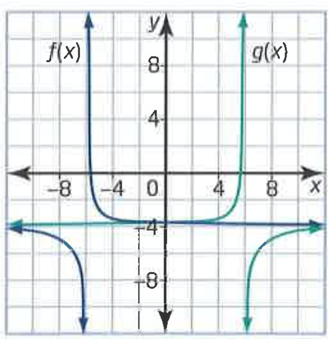
2. For each function $f(x)$ in question 1, sketch the graph of its reflection in the y-axis, $h(x)$. Then, state the domain and range of each function.

3. For each function $f(x)$ in question 1, sketch the graph of the function $k(x) = -f(-x)$. Then, state the domain and range of each function.

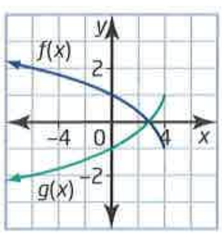
4. For each function $f(x)$, determine the equation for $g(x)$.
   a) $f(x) = \sqrt{x + 4} - 4$, $g(x) = -f(x)$
   b) $f(x) = (x + 1)^2 - 4$, $g(x) = f(-x)$
   c) $f(x) = (x - 5)^2 + 9$, $g(x) = -f(-x)$
   d) $f(x) = \dfrac{1}{x - 3} - 6$, $g(x) = -f(-x)$
   e) $f(x) = -\sqrt{x - 2} + 5$, $g(x) = f(-x)$
   f) $f(x) = \sqrt{x + 9} - 1$, $g(x) = -f(-x)$

For help with question 5, refer to Example 2.

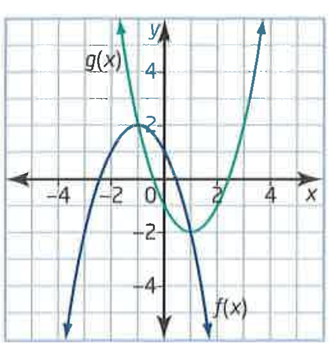
5. For each graph, describe the reflection that transforms $f(x)$ into $g(x)$.
   a)



   b)



   c)

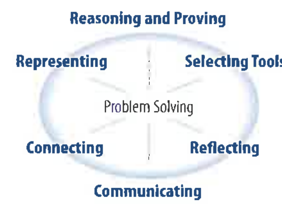


B Connect and Apply

6. Look at the logo for Teaz Apparel at the beginning of this section. Identify the type of reflection(s) in this logo.

7. Describe any patterns that you notice about how the domain and range are affected by reflections in the x- and y-axes. Use specific examples to support your answer.

8. **Use Technology** Use graphing technology for this question.



   a) Determine the invariant points of the function $f(x) = (x - 2)^2 - 9$ when it is reflected in the
      i) x-axis
      ii) y-axis
   b) Give a function that might have an invariant point under the reflection $-f(-x)$. Explain your answer.

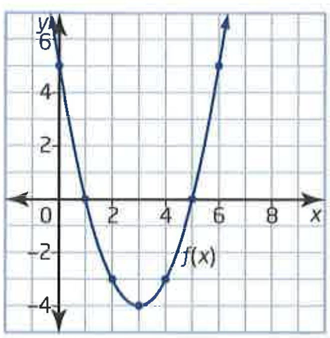
9. Determine algebraically whether $g(x)$ is a reflection of $f(x)$ in each case. Verify your answer by graphing.
   a) $f(x) = x^2$, $g(x) = (-x)^2$
   b) $f(x) = \sqrt{x}$, $g(x) = \sqrt{-x}$
   c) $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{-1}{x}$
   d) $f(x) = (x + 5)^2 + 4$, $g(x) = -(x + 5)^2 - 4$
   e) $f(x) = \sqrt{x - 10} + 3$, $g(x) = -\sqrt{x - 10} + 3$
   f) $f(x) = \dfrac{1}{x + 7}$, $g(x) = \dfrac{1}{-x + 7}$

10. **Use Technology** Use graphing software to create a logo design using functions and reflections.

11. **Use Technology** The function $g(x) = -f(-x)$ is also known as a reflection in the origin of the function $f(x)$.

a) Open *The Geometer's Sketchpad®*. Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Section 2.4. Use the file **Reflect.gsp** to determine what reflection in a point means physically. Describe how you would reflect a function in the origin.

b) Use your method to sketch the graph of the function shown after it is reflected in the origin.



12. Determine whether there are translations and reflections that have equal effects.

a) Graph the function $f(x) = (x - 4)^2$.

b) Graph the reflection of $f(x)$ in the y-axis.

c) Determine a translation that can be applied to $f(x)$ that has the same effect as the reflection in part b).

d) Verify algebraically that the transformations in parts b) and c) are the same.

e) Predict if the same would be true for reflections in the x-axis. Explain.

f) Would the conclusion from steps b) to e) work for any other type of function? Explain.

✓ Achievement Check

13. **a)** State the base function that corresponds to $f(x) = \sqrt{x + 2} + 3$ and describe the transformations that are applied to the base function to obtain $f(x)$.

b) Write the equations for $-f(x)$, $f(-x)$, and $-f(-x)$. Describe the transformation(s) represented by each equation.

c) Sketch the graphs of all four functions on the same set of axes.

d) State the domain and range of each function. Describe any similarities or differences.

e) Are any points invariant? Explain.

C Extend

14. Sketch the graph of $f(x) = \sqrt{x}$ reflected in each line.

a) $x = 4$

b) $y = x$

15. **Math Contest** The point (m, n) is reflected in the x-axis and then the image is reflected in the y-axis. What is the y-intercept of the line joining the original point and the final reflected point?

16. **Math Contest** The number of multiples of 2 or 3 but not 6 in the first $6n$ natural numbers is

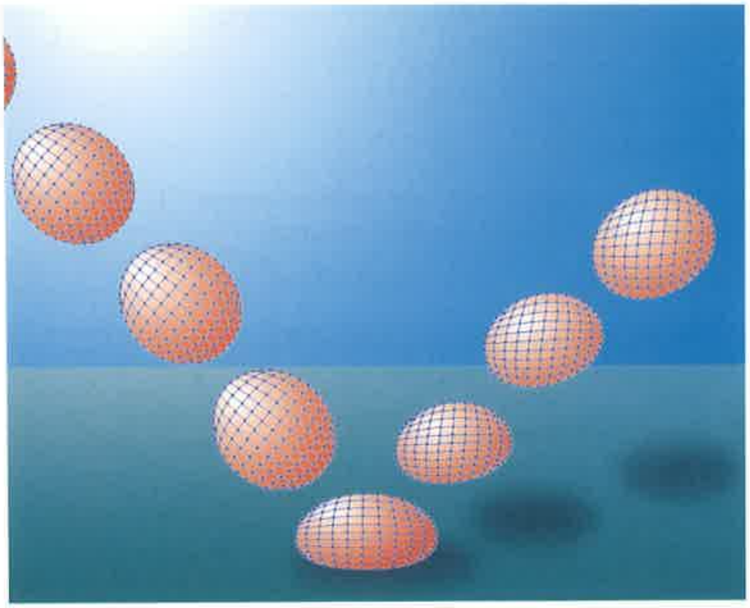
A $4n$ **B** $6n$ **C** $2n$ **D** $3n$

17. **Math Contest** Show that $n^3 - n$ is always divisible by 6.

18. **Math Contest** State the domain and the range of $f(x) = \dfrac{3x}{(x - 3)(x + 2)}$.

19. **Math Contest** A 4-cm by 4-cm by 4-cm cube is painted red and then sliced into 1-cm by 1-cm by 1-cm cubes. The number of smaller cubes with two sides painted red is

A 37 **B** 27 **C** 24 **D** 0



2.5

Stretches of Functions

In Section 2.3, you learned that the graph of a function or shape can be transformed without changing its shape by using a translation. However, in the case of real animation, that movement is usually combined with shape, colour, and orientation changes.

In this section, you will explore how you can control the shape of a function or object.

Investigate

Tools

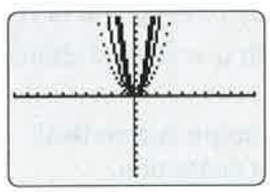
- graphing calculator

How does stretching the graph of a function affect its shape?

A: Graph Functions of the Form $g(x) = af(x)$

1. Clear any graphed functions and graph the following functions as **Y1**, **Y2**, and **Y3** on the same set of axes. Use a standard window.

Set 1: $f(x) = x^2$ $\quad$ $g(x) = 2x^2$ $\quad$ $h(x) = 5x^2$



2. Compare the equations of $g(x)$ and $h(x)$ to the equation of $f(x)$. Compare the graphs of $g(x)$ and $h(x)$ to the graph of $f(x)$.

3. Repeat steps 1 and 2 with each set of functions.

Set 2: $f(x) = \sqrt{x}$ $\quad$ $g(x) = 2\sqrt{x}$ $\quad$ $h(x) = 5\sqrt{x}$

Set 3: $f(x) = \frac{1}{x}$ $\quad$ $g(x) = 2\left(\frac{1}{x}\right)$ $\quad$ $h(x) = 5\left(\frac{1}{x}\right)$

4. Reflect

a) Describe how the value of a in $g(x) = af(x)$ changes the graph of $f(x)$.

b) Are there any invariant points?

c) How are the domain and range affected?

B: Graph Functions of the Form $g(x) = f(kx)$

1. Clear any graphed functions and graph the following functions as **Y1**, **Y2**, and **Y3** on the same set of axes. Use a standard window.

   Set 1: $f(x) = x^2$ $\quad g(x) = \left(\frac{1}{2}x\right)^2$ $\quad h(x) = \left(\frac{1}{5}x\right)^2$

2. Compare the equations of $g(x)$ and $h(x)$ to the equation of $f(x)$. Compare the graphs of $g(x)$ and $h(x)$ to the graph of $f(x)$.

3. Repeat steps 1 and 2 with each set of functions.

   Set 2: $f(x) = \sqrt{x}$ $\quad g(x) = \sqrt{\frac{1}{2}x}$ $\quad h(x) = \sqrt{\frac{1}{5}x}$

   Set 3: $f(x) = \frac{1}{x}$ $\quad g(x) = \frac{1}{\left(\frac{1}{2}x\right)}$ $\quad h(x) = \frac{1}{\left(\frac{1}{5}x\right)}$

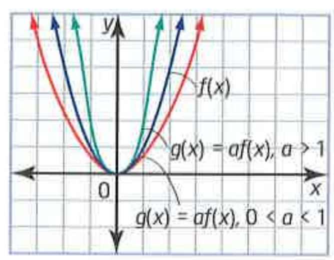
4. **Reflect**
   a) Describe how the value of k in $g(x) = f(kx)$ changes the graph of $f(x)$.
   b) Are there any invariant points?
   c) How are the domain and range affected?

5. **Reflect** Compare the transformations on the functions in parts A and B. Describe any similarities and differences.

The graph of the function $g(x) = af(x)$, $a > 0$, is a vertical stretch or a vertical compression of the graph of $f(x)$ by a factor of a.

- If $a > 1$, the graph is a vertical stretch by a factor of a.
- If $0 < a < 1$, the graph is a vertical compression by a factor of a.

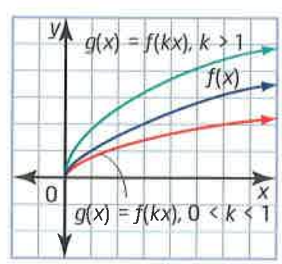
Any points on the x-axis are invariant.



The graph of the function $g(x) = f(kx)$, $k > 0$, is a horizontal stretch or a horizontal compression of the graph of $f(x)$ by a factor of $\frac{1}{k}$.

- If $k > 1$, the graph is a horizontal compression by a factor of $\frac{1}{k}$.
- If $0 < k < 1$, the graph is a horizontal stretch by a factor of $\frac{1}{k}$.

Any points on the y-axis are invariant.



In general, when the graph of a function is elongated in one direction, the word **stretch** is used to describe the transformation. If it is shortened in one direction, the word **compression** is used to describe the transformation.

stretch

- a transformation that results in the distance from the x-axis of every point growing by a scale factor greater than 1 (vertical stretch) or the distance from the y-axis of every point growing by a scale factor greater than 1 (horizontal stretch)

compression

- a transformation that results in the distance from the x-axis of every point shrinking by a scale factor between 0 and 1 (vertical compression) or the distance from the y-axis of every point shrinking by a scale factor between 0 and 1 (horizontal compression)

Example 1

Graph Stretches and Compressions

Given the function $f(x) = \sqrt{x}$, write equations to represent $g(x)$ and $h(x)$ and describe the transformations. Then, transform the graph of $f(x)$ to sketch graphs of $g(x)$ and $h(x)$ and state the domain and range of the functions.

a) $g(x) = 2f(x)$ and $h(x) = \frac{1}{2}f(x)$

b) $g(x) = f(2x)$ and $h(x) = f\left(\frac{1}{2}x\right)$

Solution

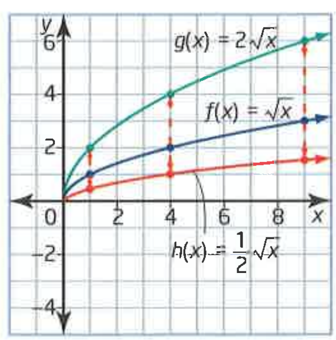
a) $g(x) = 2f(x)$
$= 2\sqrt{x}$

Since $a = 2$, the graph of $g(x)$ is a vertical stretch by a factor of 2 of the graph of $f(x)$. Each y-value of $g(x)$ will be twice as far from the x-axis as the corresponding y-value of $f(x)$.

$h(x) = \frac{1}{2}f(x)$
$= \frac{1}{2}\sqrt{x}$

Since $a = \frac{1}{2}$, the graph of $h(x)$ is a vertical compression by a factor of $\frac{1}{2}$ of the graph of $f(x)$. Each y-value of $h(x)$ will be $\frac{1}{2}$ as far from the x-axis as the corresponding y-value of $f(x)$.

x	$f(x) = \sqrt{x}$	$g(x) = 2\sqrt{x}$	$h(x) = \frac{1}{2}\sqrt{x}$
0	0	0	0
1	1	2	$\frac{1}{2}$
4	2	4	1
9	3	6	$\frac{3}{2}$



For $f(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 0\}$.
For $g(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 0\}$.
For $h(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 0\}$.

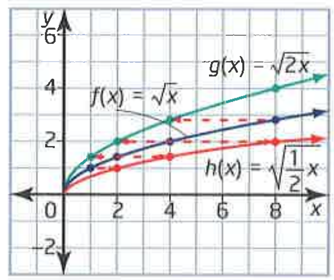
b) $g(x) = f(2x)$
$ = \sqrt{2x}$

Since $k = 2$, the graph of $g(x)$ is a horizontal compression by a factor of $\frac{1}{k}$, or $\frac{1}{2}$, of the graph of $f(x)$. Each x-value of $g(x)$ will be $\frac{1}{2}$ as far from the y-axis as the corresponding x-value of $f(x)$.

$h(x) = f\left(\frac{1}{2}x\right)$
$ = \sqrt{\frac{1}{2}x}$

Since $k = \frac{1}{2}$, the graph of $h(x)$ is a horizontal stretch by a factor of $\frac{1}{k}$, or 2, of the graph of $f(x)$. Each x-value of $h(x)$ will be twice as far from the y-axis as the corresponding x-value of $f(x)$.

x	$f(x) = \sqrt{x}$	$g(x) = \sqrt{2x}$	$h(x) = \sqrt{\frac{1}{2}x}$
0	0	0	0
1	1	$\sqrt{2}$	$\sqrt{\frac{1}{2}}$
2	$\sqrt{2}$	2	1
4	2	$\sqrt{8}$	$\sqrt{2}$
8	$\sqrt{8}$	4	2



For $f(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 0\}$.
For $g(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 0\}$.
For $h(x)$, the domain is $\{x \in \mathbb{R}, x \geq 0\}$ and the range is $\{y \in \mathbb{R}, y \geq 0\}$.

Example 2

Describe Transformations of Given Equations

For each function $g(x)$, describe the transformation from a base function of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, or $f(x) = \frac{1}{x}$. Then, transform the graph of $f(x)$ to sketch the graph of $g(x)$.

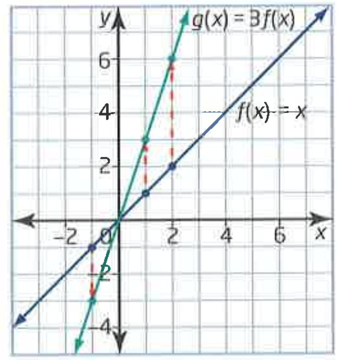
a) $g(x) = 3x$

b) $g(x) = 4x^2$

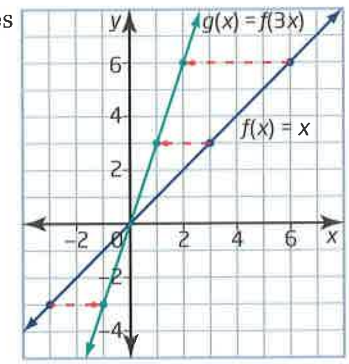
Solution

a) The function $g(x) = 3x$ represents a transformation of the base function $f(x) = x$. The actual transformation can be described in two ways.

- Thinking of $g(x)$ as $3(x) = 3f(x)$ indicates that it is a vertical stretch by a factor of 3.

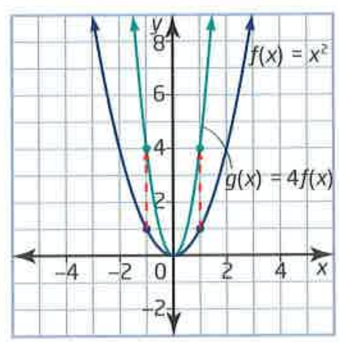


- Thinking of $g(x)$ as $(3x) = f(3x)$ indicates that it is a horizontal compression by a factor of $\frac{1}{3}$.

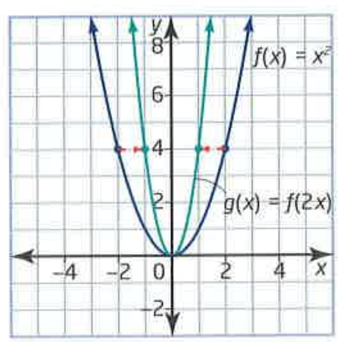


b) The function $g(x) = 4x^2$ represents a transformation of the base function $f(x) = x^2$. The actual transformation can be described in two ways.

- Thinking of $g(x)$ as $4(x^2) = 4f(x)$ indicates that it is a vertical stretch by a factor of 4.



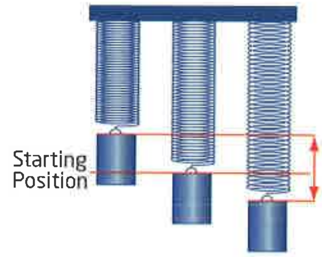
- Thinking of $g(x)$ as $(2x)^2 = f(2x)$ indicates that it is a horizontal compression by a factor of $\frac{1}{2}$.



Example 3

Simple Harmonic Motion

A mass on a spring bobs up and down. The period is the amount of time that it takes for the mass to move from its starting position to the top, then to the bottom, and then back to its starting position. The period, T, in seconds, is given by $T(m) = 2\pi\sqrt{\frac{m}{k}}$, where m is the mass, in kilograms, and k is the spring constant, in newtons per metre (N/m).



The Ideal Spring Company needs to show how each of their springs performs compared to the others.

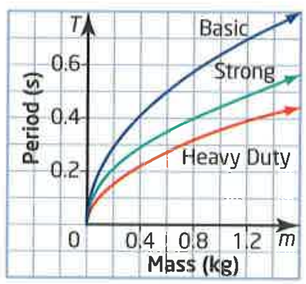
a) Graph the period versus the mass for various spring constants.

b) How are the graphs of the Strong and Heavy Duty springs related to the graph of the Basic spring?

Spring Type	Spring Constant (N/m)
Basic	100
Strong	200
Heavy Duty	300

Solution

a)



b) Consider the Basic spring to have a function given by $T(m) = 2\pi\sqrt{\frac{m}{k}}$ and that the spring constants for the Strong and Heavy Duty springs are multiples of the spring constant for the Basic spring.

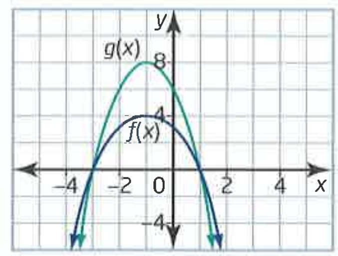
Spring	Spring Constant	Formula	Comparison to Basic
Basic	k	$T(m) = 2\pi\sqrt{\frac{m}{k}}$	
Strong	$k_{strong} = 2k$	$T(m) = 2\pi\sqrt{\frac{m}{2k}} = 2\pi\sqrt{\frac{\frac{1}{2}m}{k}}$	This is a horizontal stretch by a factor of 2.
Heavy Duty	$k_{heavy\ duty} = 3k$	$T(m) = 2\pi\sqrt{\frac{m}{3k}} = 2\pi\sqrt{\frac{\frac{1}{3}m}{k}}$	This is a horizontal stretch by a factor of 3.

Key Concepts

- Stretches and compressions are transformations that cause functions to change shape.
- The graph of $g(x) = af(x)$, $a > 0$, is a vertical stretch or a vertical compression of the graph of $f(x)$ by a factor of a. If $a > 1$, the graph is vertically stretched by a factor of a. If $0 < a < 1$, the graph is vertically compressed by a factor of a.
- The graph of $g(x) = f(kx)$, $k > 0$, is a horizontal stretch or a horizontal compression of the graph of $f(x)$ by a factor of $\frac{1}{k}$. If $k > 1$, the graph is horizontally compressed by a factor of $\frac{1}{k}$. If $0 < k < 1$, the graph is horizontally stretched by a factor of $\frac{1}{k}$.

Communicate Your Understanding

C1 Given that the graph of $g(x)$ is a transformation of the graph of $f(x)$, describe how you know that it is a vertical stretch and not a translation.



C2 Explain why the graph of $g(x) = af(x)$, $a > 0$, stretches the graph of $f(x)$ vertically and not horizontally.

C3 Describe how the graph of the function $g(x) = \frac{3}{x}$ is related to the graph of the base function $f(x) = \frac{1}{x}$.

A Practise

For help with questions 1 to 3, refer to Example 1.

1. a) Copy and complete the table of values.

x	$f(x) = x^2$	$g(x) = 5f(x)$	$h(x) = f\left(\frac{1}{4}x\right)$
0			
2			
4			
6			

b) Sketch the graphs of all three functions on the same set of axes.

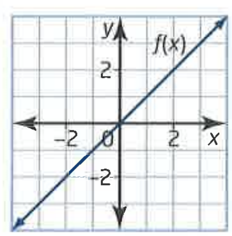
c) Explain how the points on the graphs of $g(x)$ and $h(x)$ relate to the transformations.

2. Copy each graph of $f(x)$ and then graph and label each related function.

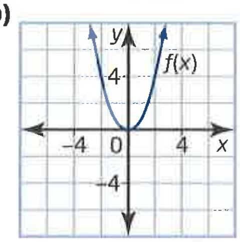
i) $g(x) = 3f(x)$ **ii)** $h(x) = f(4x)$

iii) $m(x) = f(2x)$ **iv)** $r(x) = f\left(\frac{x}{5}\right)$

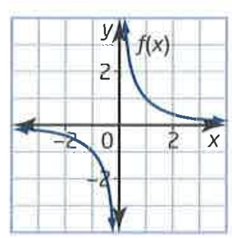
a)



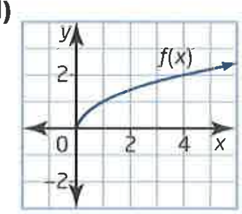
b)



c)



d)



3. For each function $g(x)$, identify the value of a or k and describe how the graph of $g(x)$ can be obtained from the graph of $f(x)$.

a) $g(x) = 10f(x)$ **b)** $g(x) = f(9x)$

c) $g(x) = \frac{1}{5}f(x)$ **d)** $g(x) = f\left(\frac{1}{20}x\right)$

For help with question 4, refer to Example 2.

4. For each function $g(x)$, describe the transformation from a base function of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, or $f(x) = \frac{1}{x}$. Then, transform the graph of $f(x)$ to sketch the graph of $g(x)$.

a) $g(x) = 10x$ **b)** $g(x) = (5x)^2$

c) $g(x) = \sqrt{\frac{x}{3}}$ **d)** $g(x) = \frac{4}{x}$

e) $g(x) = \sqrt{16x}$ **f)** $g(x) = \frac{x}{4}$

B Connect and Apply

For help with question 5, refer to Example 3.

5. Various space agencies have conducted experiments in space. One possible experiment uses pendulums to verify the theoretical values of the acceleration due to gravity on places other than Earth. The period, T, in seconds, of any pendulum is given by $T = 2\pi\sqrt{\frac{\ell}{g}}$, where ℓ is the length of the pendulum, in metres, and g is the acceleration due to gravity, in metres per square second.

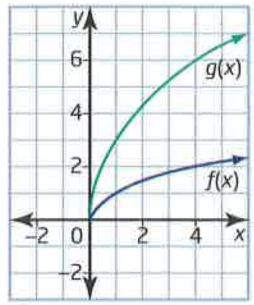
Celestial Body	Gravitational Acceleration (m/s²)
Earth	9.8
Mars	3.7
Moon	1.6

a) Graph the period versus length for Earth, the moon, and Mars.

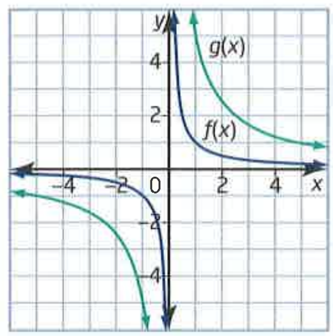
b) If the gravity on a comet is one tenth that of Earth, what will the graph of its period versus its length look like?

6. Describe the transformation that must be applied to the graph of $f(x)$ in order to obtain the graph of $g(x)$.

a)



b)



7. **a)** Determine three key points on the graph of $f(x) = x^2$.

b) Transform the graph of $f(x)$ to obtain the graph of $c(x) = 4f(2x)$ by applying the vertical stretch or compression followed by the horizontal stretch or compression.

c) Transform the graph of $f(x)$ to obtain the graph of $c(x) = 4f(2x)$ by applying the horizontal stretch or compression followed by the vertical stretch or compression.

d) Compare the graphs from parts b) and c).

e) Is there a single stretch that will have the same result as the two original stretches?

f) Repeat parts a) to e) with a different base function.

8. A long ocean wave, such as near the shore or from a tsunami, does not disperse the way a short wave does. The speed of a long wave depends on the depth of the water and can be calculated using the formula $s = \sqrt{gh}$, where g is the acceleration due to gravity, 9.8 m/s², and h is the depth of the water, in metres.

a) Sketch the graph of this function.

b) Determine the speed of a wave coming to shore at a depth of 2 m.

c) Determine the speed of a tsunami wave at an ocean depth of 4000 m.

9. Chapter Problem From his work at the traffic safety bureau, Matthew knows that if the length of the skid marks from any vehicle and the road condition are known, then he can estimate the minimum speed the vehicle was going before the brakes were applied. The speed, s, in kilometres per hour, is given by $s = 16.0\sqrt{fd}$, where f is the coefficient of friction for the road surface and d is the average length of all skid marks, in metres.

Type of Surface	f
dry concrete	0.77
wet concrete	0.54
dry asphalt	0.73
wet asphalt	0.54
dry brick	0.65
wet brick	0.34
dirt	0.63

a) Graph the speed for all types of surfaces on the same set of axes.

b) For a skid mark of length 15 m, how much slower would a vehicle have to be travelling on wet brick compared to dry brick to stop in the same distance?

10. Explain why the graph of $g(x) = f(kx)$ stretches horizontally by a factor of $\frac{1}{k}$ and not a factor of k.

11. A ball is dropped from a height of 20 m. The downward acceleration due to gravity is -9.8 m/s². The height of the ball is given as $h(t) = -4.9t^2 + 20$.

a) State the domain and range of the function.

b) Write the equation for the height of the ball if it were dropped on a planet with gravity of -12.4 m/s².

c) Compare the domain and range of the function in part b) to that of the given function.

C Extend

12. Use Technology In this section, you dealt with static stretches. Computer animation uses dynamic stretches. Open *The Geometer's Sketchpad*®. Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Section 2.5. Download the file **2.5_Animation.gsp**. In this sketch, you will be able to change a parameter called t by moving a sliding point.

a) Study the form of the function $g(x)$. What similarities and differences are there to a transformed function of the form $g(x) = f(kx)$?

b) What happens when you move the slider t?

c) How does changing the parameter function $P(x)$ affect the motion of the base function $f(x)$? Use the following functions to investigate this.

i) $P(x) = x^2$

ii) $P(x) = \sqrt{x}$

iii) $P(x) = \frac{1}{x}$

d) Click on the **Link to Function 2** button and repeat parts a) to c) using $g(x) = af(x)$.

e) Click on the **Link to Butterfly** button and move the slider t. Here you can see a very rudimentary example of computer animation.

13. a) **Use Technology** Use technology to graph the function $f(x) = x^3 - 3x$.

b) If $g(x) = 3f(x)$ and $h(x) = f(3x)$, determine the equations for $g(x)$ and $h(x)$.

c) Without using technology, describe and sketch the graphs of $g(x)$ and $h(x)$.

d) Is there a vertical transformation that will have the same effect as $h(x)$ does on $f(x)$?

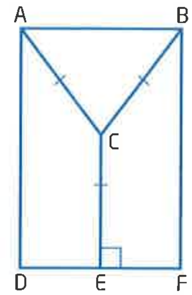
14. **Math Contest** In a high school, the ratio of male to female students is 8:5. When 400 senior males are removed and 300 senior females are removed, the ratio is 8:3. The number of students in the whole school is

A 1300 **B** 975 **C** 935 **D** 960

15. **Math Contest** The floor function $\lfloor x \rfloor$ returns the value of the greatest integer less than or equal to x. For example, $\lfloor 5.7 \rfloor = 5$ and $\lfloor -3.1 \rfloor = -4$. The value of $\lfloor -4.1 + \lfloor -3.2 \rfloor - \lfloor 3.6 - 4.5 \rfloor + 2 \rfloor$ is

A -6 **B** -5 **C** 6 **D** 0

16. **Math Contest** Rectangle ABFD has AB = 8 cm and AD = 12 cm. Point C is located inside the rectangle such that AC = BC = EC.



The length of AC is

A $6\frac{2}{3}$ cm **B** 8 cm **C** 2π cm **D** 7 cm

17. **Math Contest** Determine the coordinates of the vertex of the quadratic function $y = (2x - 6)^2 + 4(2x - 6) + 5$.

18. **Math Contest** Given the transformation $(x, y) \rightarrow (x, 2x + y)$, determine the equation of the line $3x + 5y - 30 = 0$ after it undergoes this transformation.

Career Connection

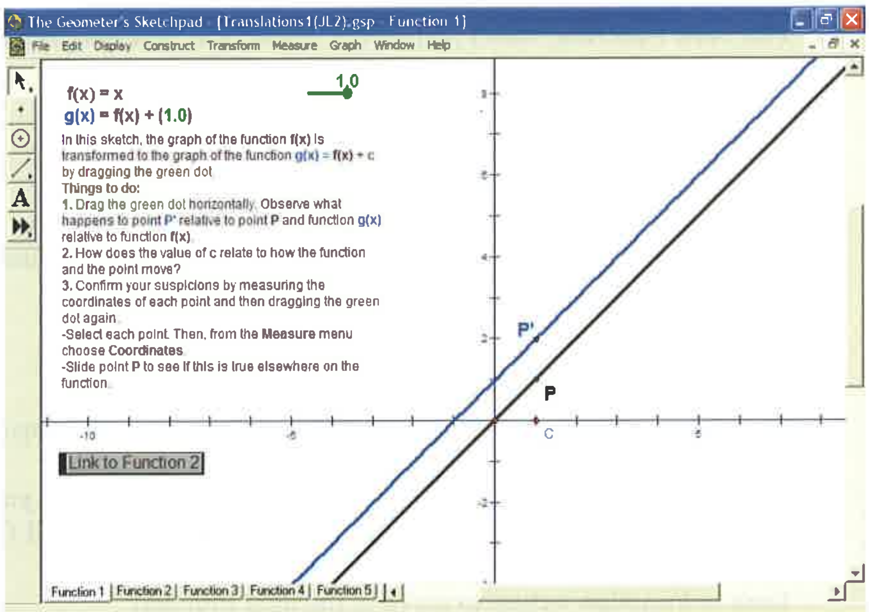
Since graduating with a 4-year bachelor of science degree from the University of Ontario Institute of Technology, Teaghan has worked in a forensic laboratory. Here, she is being further trained in the field of ballistics—the science of the flight path of a bullet. Tracing this path is important in a crime investigation, since it will show from which direction the bullet was fired. Teaghan explores various functions involving variables such as the bullet's initial velocity, firing angle, drag, and gravitational constant. Teaghan's research, when presented in a court of law, can be vital in coming to a fair verdict.

Use Technology

Use *The Geometer's Sketchpad®* to Explore Transformations

A: Translations

1. Open *The Geometer's Sketchpad®*. Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Section 2.5. Download the file **Translations1.gsp**. In this sketch, you will explore transformations of five different functions, each called $f(x)$. Follow the instructions on each page and click on the **Link to Function** button to move to the next function.



2. **Reflect** Describe how the value of c in $g(x) = f(x) + c$ changes the graph of $f(x)$.

3. Download the file **Translations2.gsp**. Follow the instructions on each page and click on the **Link to Function** button to move to the next function.

4. **Reflect** Describe how the value of d in $g(x) = f(x - d)$ changes the graph of $f(x)$.

Tools

- computer with *The Geometer's Sketchpad®*
- Translations1.gsp
- Translations2.gsp
- Stretches1.gsp
- Stretches2.gsp

Technology Tip

The square root function, sqrt, can be found on the **Functions** drop-down menu on the calculator.

Technology Tip

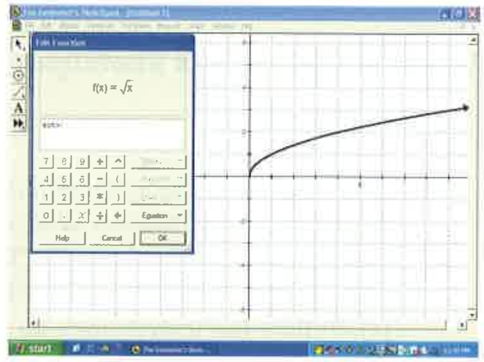
To change the colour of the function equation and its graph, right-click on an item, choose **Color**, and select a different colour.

Technology Tip

To quickly repeat steps 1 to 3, double-click on the original function $f(x)$ and change it to $f(x) = (x - 2)^2$.

B: Reflections

1. Open *The Geometer's Sketchpad®*. From the **Graph** menu, choose **Plot New Function**. In the **Edit Function** dialogue box, enter sqrt(x) to represent the base function $f(x) = \sqrt{x}$.



2. Plot a new function. From the **Graph** menu, choose **Plot New Function**, select $f(x)$ on the workspace, and then enter $-x$ in the function to create the new function $g(x) = f(-x)$.
3. Repeat step 2 to create the new functions $h(x) = -f(x)$ and $q(x) = -f(-x)$.
4. **Reflect** Compare each new function to the base function $f(x)$. Describe the mirror line that is used for each reflection. Confirm your hypothesis by repeating steps 1 to 3 for $f(x) = (x - 2)^2$.

C: Stretches and Compressions

1. Open *The Geometer's Sketchpad®*. Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Section 2.5. Download the file **Stretches1.gsp**. In this sketch, you will explore transformations of five different functions each called $f(x)$. Follow the instructions on each page and click on the **Link to Function** button to move to the next function.
2. **Reflect**
   a) Describe how the value of a in $g(x) = af(x)$ changes the graph of $f(x)$.
   b) Points that do not change under a transformation are said to be invariant. Were there any invariant points?
   c) How were the domain and range affected?
3. Download the file **Stretches2.gsp**. Follow the instructions on each page and click on the **Link to Function** button to move to the next function.
4. **Reflect**
   a) Describe how the value of k in $g(x) = f(kx)$ changes the graph of $f(x)$.
   b) Were there any invariant points?
   c) How were the domain and range affected?

2.6

Combinations of Transformations

An anamorphosis is an image that can only be seen correctly when viewed from a certain perspective. For example, the face in the photo can only be seen correctly in the side of the cylindrical mirror. To be viewed correctly, this image requires reflections and stretches to occur simultaneously. In mathematics, situations are rarely described by simple relationships, and so by combining translations, reflections, stretches, and compressions, you can model many different scenarios.

Investigate

Does the order matter when performing transformations?

A: Translations

1. Given the function $f(x) = x^2$, graph each pair of transformed functions on the same set of axes.

a) $g(x) = f(x) + 3$ and $h(x) = g(x + 6)$

b) $m(x) = f(x + 6)$ and $r(x) = m(x) + 3$

2. Describe each translation in step 1.

3. Write equations for $h(x)$ and $r(x)$ in terms of $f(x)$.

4. Reflect Compare the graphs of $h(x)$ and $r(x)$ and the equations of $h(x)$ and $r(x)$. What does this tell you about whether the order of the translations matters? Explain your reasoning.

B: Stretches

1. Given the function $f(x) = x^2$, graph each pair of transformed functions on the same set of axes.

a) $b(x) = 5f(x)$ and $p(x) = b\left(\frac{1}{4}x\right)$

b) $n(x) = f\left(\frac{1}{4}x\right)$ and $s(x) = 5n(x)$

2. Describe each stretch in step 1.

3. Write equations for $p(x)$ and $s(x)$ in terms of $f(x)$.

4. Reflect Compare the graphs of $p(x)$ and $s(x)$ and the equations of $p(x)$ and $s(x)$. What does this tell you about whether the order of the stretches matters? Explain your reasoning.

Tools

- grid paper

Optional

- graphing calculator

or

- graphing software

C: Translations and Stretches

1. Given the function $f(x) = x^2$, graph each pair of transformed functions on the same set of axes.
   a) $j(x) = 2f(x)$ and $s(x) = j(x) + 5$
   b) $q(x) = f(x) + 5$ and $t(x) = 2q(x)$

2. Describe each transformation in step 1.

3. Write equations for $s(x)$ and $t(x)$ in terms of $f(x)$.

4. Given the function $f(x) = x^2$, graph each pair of transformed functions on the same set of axes.
   a) $w(x) = f\left(\frac{1}{2}x\right)$ and $u(x) = w(x + 5)$
   b) $v(x) = f(x + 5)$ and $z(x) = v\left(\frac{1}{2}x\right)$

5. Describe each transformation in step 4.

6. Write equations for $u(x)$ and $z(x)$ in terms of $f(x)$.

7. **Reflect** In which order do you think stretches and translations should be done when they are combined? Explain.

Example 1

Combinations of Transformations

Describe the combination of transformations that must be applied to the base function $f(x)$ to obtain the transformed function. Then, write the corresponding equation and sketch its graph.

a) $f(x) = x^2$, $g(x) = \frac{1}{2}f[4(x - 3)] - 2$
b) $f(x) = \sqrt{x}$, $g(x) = -2f(3x + 15) + 4$

Solution

a) Compare the transformed equation to $y = af[k(x - d)] + c$ to determine the values of the parameters a, k, d, and c.

For $g(x) = \frac{1}{2}f[4(x - 3)] - 2$, $a = \frac{1}{2}$, $k = 4$, $d = 3$, and $c = -2$.

The function $f(x)$ is vertically compressed by a factor of $\frac{1}{2}$, horizontally compressed by a factor of $\frac{1}{4}$, and then translated 3 units right and 2 units down.

vertical compression by a factor of $\frac{1}{2}$

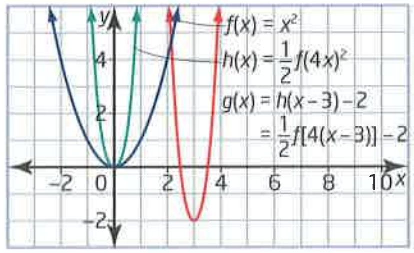
vertical translation of 2 units down

$$g(x) = \frac{1}{2}f[4(x - 3)] - 2$$

horizontal compression by a factor of $\frac{1}{4}$

horizontal translation of 3 units right

$$g(x) = \frac{1}{2}f[4(x - 3)] - 2$$
$$= \frac{1}{2}[4(x - 3)]^2 - 2$$
$$= \frac{1}{2}(4x - 12)^2 - 2$$
$$= \frac{1}{2}(16x^2 - 96x + 144) - 2$$
$$= 8x^2 - 48x + 72 - 2$$
$$= 8x^2 - 48x + 70$$



b) First, rewrite $g(x) = -2f(3x + 15) + 4$ in the form $y = af[k(x - d)] + c$.

$$g(x) = -2f(3x + 15) + 4$$
$$= -2f[3(x + 5)] + 4$$

For $g(x) = -2f[3(x + 5)] + 4$, $a = -2$, $k = 3$, $d = -5$, and $c = 4$.

The function $f(x)$ is reflected in the x-axis, vertically stretched by a factor of 2, horizontally compressed by a factor of $\frac{1}{3}$, and then translated 5 units left and 4 units up.

reflection in the x-axis

vertical stretch by a factor of 2

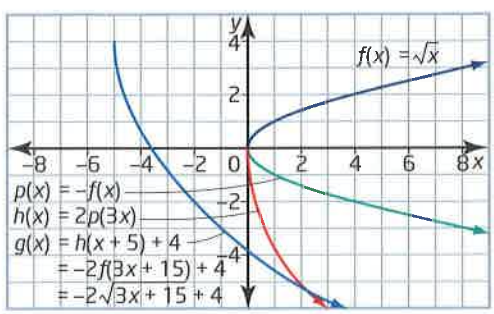
vertical translation of 4 units up

$$g(x) = -2f[3(x + 5)] + 4$$

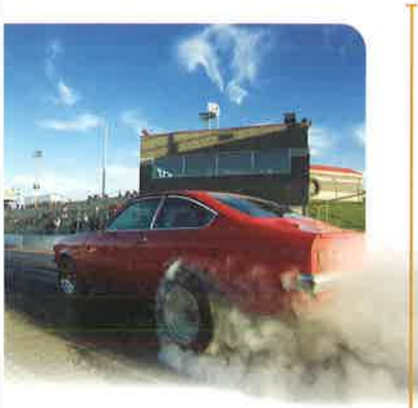
horizontal compression by a factor of $\frac{1}{3}$

horizontal translation of 5 units left

$$g(x) = -2f[3(x + 5)] + 4$$
$$= -2\sqrt{3(x + 5)} + 4$$
$$= -2\sqrt{3x + 15} + 4$$



When combining transformations, order matters. To accurately sketch the graph of a function of the form $y = af[k(x - d)] + c$, apply transformations represented by the parameters a and k before transformations represented by the parameters d and c. That is, stretches, compressions, and reflections occur before translations. This is similar to the order of operations, where multiplication and division occur before addition and subtraction.



Example 2

Apply Transformations

During a race in the sportsman category of drag racing, it is common for cars with different performance potentials to race against each other while using a handicap system. For example, Byron is racing against Eve. Since Eve has a faster car, when they race, it appears as though Byron gets a head start. The distance, E, in metres, that Eve's car travels is given by $E(t) = 10t^2$, where t is the time, in seconds, after she starts. The distance, B, in metres, that Byron's car travels is given by $B(t) = 5(t + h)^2$, where t is the time after Eve starts and h is the head start, in seconds.

a) On the same set of axes, graph distance versus time for both drivers for h-values of 1 s, 2 s, 3 s, and 4 s.

b) The standard length of a drag strip is approximately 400 m. How much of a head start can Eve give Byron and still cross the finish line first?

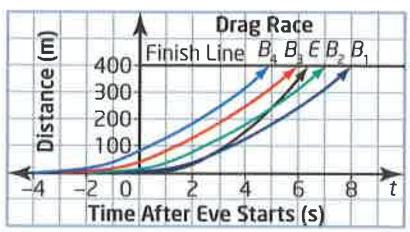
c) Determine the domain and range of each function.

d) The acceleration of each car is represented by the stretch of each equation. Compare the accelerations of the two cars.

Solution

a) There are five curves to graph: one representing Eve and four representing Byron given each head start.

Eve	$E(t) = 10t^2$
Byron with 1-s head start	$B_1(t) = 5(t + 1)^2$
Byron with 2-s head start	$B_2(t) = 5(t + 2)^2$
Byron with 3-s head start	$B_3(t) = 5(t + 3)^2$
Byron with 4-s head start	$B_4(t) = 5(t + 4)^2$



b) Based on the graph, it appears that as long as the head start is no more than about 2.5 s, Eve will still cross the line first.

c) For this situation, the equations given are only valid from the time the car starts moving to the time it crosses the finish line.

	Function	Domain	Range
Eve	$E(t) = 10t^2$	$\{t \in \mathbb{R}, t \geq 0\}$	$\{E \in \mathbb{R}, 0 \leq E \leq 400\}$
Byron with 1-s head start	$B_1(t) = 5(t + 1)^2$	$\{t \in \mathbb{R}, t \geq -1\}$	$\{B \in \mathbb{R}, 0 \leq B \leq 400\}$
Byron with 2-s head start	$B_2(t) = 5(t + 2)^2$	$\{t \in \mathbb{R}, t \geq -2\}$	
Byron with 3-s head start	$B_3(t) = 5(t + 3)^2$	$\{t \in \mathbb{R}, t \geq -3\}$	
Byron with 4-s head start	$B_4(t) = 5(t + 4)^2$	$\{t \in \mathbb{R}, t \geq -4\}$	

d) The equation for Eve's car has $a = 10$ while the equations for Byron's car have $a = 5$. Thus, the acceleration of Eve's car is twice that of Byron's car.

Key Concepts

- Stretches, compressions, and reflections can be performed in any order before translations.
- Ensure that the function is written in the form $y = af[k(x - d)] + c$ to identify specific transformations.
- The parameters a, k, d, and c in the function $y = af[k(x - d)] + c$ correspond to the following transformations:

  a corresponds to a vertical stretch or compression and, if $a < 0$, a reflection in the x-axis.

  k corresponds to a horizontal stretch or compression and, if $k < 0$, a reflection in the y-axis.

  d corresponds to a horizontal translation to the right or left.

  c corresponds to a vertical translation up or down.

Communicate Your Understanding

C1 Stretches, compressions, and reflections can be performed in any order. Explain why.

C2 A student describes the function $g(x) = f(3x + 12)$ as a horizontal compression by a factor of $\frac{1}{3}$ followed by a horizontal translation of 12 units left of the base function $f(x)$. Explain the mistake this student has made.

A Practise

For help with questions 1 to 4, refer to Example 1.

1. Compare the transformed equation to $y = af[k(x - d)] + c$ to determine the values of the parameters a, k, d, and c. Then, describe, in the appropriate order, the transformations that must be applied to a base function $f(x)$ to obtain the transformed function.

a) $g(x) = 4f(x - 3)$ **b)** $g(x) = \frac{1}{3}f(x) + 1$

c) $g(x) = f(x + 5) + 9$ **d)** $g(x) = f\left(\frac{1}{4}x\right) + 2$

e) $g(x) = f(5x) - 2$ **f)** $g(x) = 2f(x) - 7$

2. Repeat question 1 for each transformed function $g(x)$.

a) $g(x) = 3f(2x) - 1$

b) $g(x) = -2f(x) + 1$

c) $g(x) = \frac{1}{2}f(x - 4) + 5$

d) $g(x) = f(-3x) + 4$

e) $g(x) = -f\left(\frac{1}{2}x\right) - 3$

f) $g(x) = \frac{1}{4}f(3x) - 6$

3. Describe, in the appropriate order, the transformations that must be applied to the base function $f(x)$ to obtain the transformed function. Then, write the corresponding equation and transform the graph of $f(x)$ to sketch the graph of $g(x)$.

a) $f(x) = \sqrt{x}$, $g(x) = 4f(3x)$

b) $f(x) = \frac{1}{x}$, $g(x) = f(x - 1) + 2$

c) $f(x) = x^2$, $g(x) = f\left[\frac{1}{4}(x + 2)\right]$

d) $f(x) = x$, $g(x) = -5f(x) - 3$

4. Repeat question 3 for $f(x)$ and the transformed function $g(x)$.

a) $f(x) = x$, $g(x) = -\frac{1}{2}f[2(x + 1)] - 3$

b) $f(x) = x^2$, $g(x) = -2f[3(x - 4)] - 1$

c) $f(x) = \sqrt{x}$, $g(x) = \frac{1}{2}f\left[\frac{1}{2}(x + 3)\right] + 5$

d) $f(x) = \frac{1}{x}$, $g(x) = 2f[-(x - 3)] + 4$

For help with questions 5 and 6, refer to Example 2.

5. For each function, identify the base function as one of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$. Sketch the graphs of the base function and the transformed function, and state the domain and range of the functions.

a) $b(x) = 10x - 8$

b) $e(x) = 3x^2 - 5$

c) $h(x) = (5x + 20)^2$

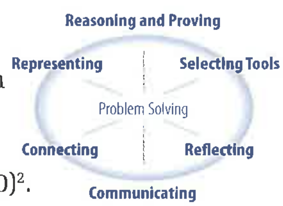
d) $j(x) = 2\sqrt{x - 7}$

e) $m(x) = \dfrac{5}{x + 8}$

f) $r(x) = \dfrac{2}{3 - x} + 1$

B Connect and Apply

6. Two skydivers jump out of a plane. The first skydiver's motion can be modelled by the function $g(t) = 4000 - 5(t + 10)^2$.



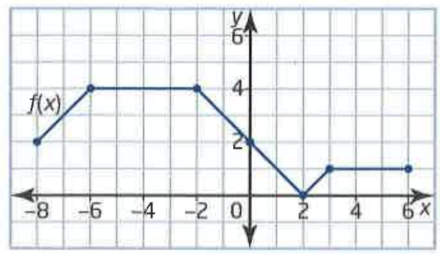
The second skydiver jumps out a few seconds later with a goal of catching up to the first skydiver. The motion of the second skydiver can be modelled by $h(t) = 4000 - 5t^2$. For both functions, the distance above the ground is measured in metres and the time is the number of seconds after the second skydiver jumps.

a) Graph the functions on the same set of axes.

b) Will the second skydiver catch up to the first before they have to open their parachutes at 800 m?

c) State the domain and range of these functions in this context.

7. Copy the graph of the function $f(x)$. Sketch the graph of $g(x)$ after each transformation.



a) $g(x) = 3f(x + 4)$

b) $g(x) = f(4x) + 3$

c) $g(x) = f(2x - 12)$

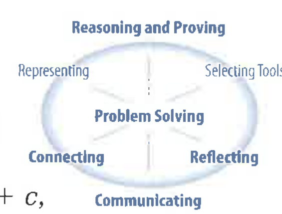
d) $g(x) = 5f(0.5x + 1) - 6$

8. The siren of an ambulance approaching you sounds different than when it is moving away from you. This difference in sound is called the Doppler effect. The Doppler effect for a 1000-Hz siren can be modelled by the equation $f = 1000\left(\dfrac{332}{332 \pm v}\right)$, where f is the frequency of the sound, in hertz; v is the speed of the ambulance, in metres per second; and the positive sign (+) is used when the ambulance is moving away from you and the negative sign (−) when it is moving toward you.

a) For an ambulance travelling at a speed of 20 m/s, what is the difference in frequency as the ambulance approaches and passes you?

b) Assuming an ambulance cannot travel faster than 40 m/s, determine the domain and range of this function.

9. Although a transformed function is traditionally written in the form $g(x) = af[k(x - d)] + c$, it can also be written in the form $\frac{1}{a}[g(x) - c] = f[k(x - d)]$. How does this form help explain the seemingly backward nature of the horizontal transformations with respect to the values of d and k?



10. The value, V, in thousands of dollars, of a certain car after t years can be modelled by the equation $V(t) = \frac{35}{t + 3}$.

a) Sketch the graph of this relation.

b) What was the initial value of this car?

c) What is the projected value of this car after

i) 1 year?

ii) 2 years?

iii) 10 years?

Achievement Check

11. The base function $f(x) = \sqrt{x}$ is transformed by a reflection in the x-axis, followed by a vertical stretch by a factor of 3, then a horizontal compression by a factor of $\frac{1}{2}$, then a vertical translation of 3 units down, and finally a horizontal translation of 6 units right.

a) Determine the equation of the transformed function.

b) Use key points on the base function to determine image points on the transformed function.

c) Sketch the graph of the transformed function.

d) Determine the domain and range of the transformed function.

C Extend

12. a) Given the base function $f(x) = x^3$, use a table of values or a graphing calculator to sketch the graph of $y = f(x)$.

b) Sketch the graph and determine the equation for each transformed function.

i) $g(x) = 3f(x + 2)$

ii) $h(x) = -f(4x - 12) + 5$

13. The equation of a circle, centred at the origin and with radius r, is $x^2 + y^2 = r^2$. Describe the transformations needed to graph each of the following. Then, sketch each circle.

a) $(x - 2)^2 + (y - 1)^2 = 25$

b) $(x + 4)^2 + (y - 5)^2 = 9$

14. **Use Technology** In this section, you dealt with static transformations. In computer animation, dynamic transformations are used. Open *The Geometer's Sketchpad®*. Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Section 2.6. Download the file **2.6_Animation.gsp**. In this sketch, you will be able to change a parameter called t by moving a sliding point.

a) Study the form of the function $g(x)$. What similarities and differences are there compared to a transformed function of the form $g(x) = f[k(x - d)] + c$?

b) What happens when you move the slider t?

c) How does changing the parameter function $P(x)$ affect the motion of the base function $f(x)$? Use the following functions to investigate this.

i) $P(x) = x^2$

ii) $P(x) = \sqrt{x}$

iii) $P(x) = \frac{1}{x}$

d) Click on the **Link to Butterfly** button and move the slider t. Here you can see a very rudimentary example of computer animation. Repeat parts b) and c) for this sketch.

15. **Math Contest** In a magic square, the sum of each row, column, and major diagonal is the same. For the magic square shown, determine the value of y.

3	y	
6	z	5

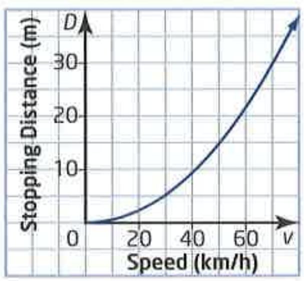
A 2 **B** 3 **C** 4 **D** 5

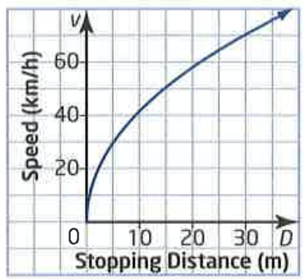
2.7

Inverse of a Function

Engineers have been able to determine the relationship between the speed of a car and its stopping distance. A typical function describing this relationship is $D = 0.006v^2$, where D is the stopping distance, in metres, and v is the speed, in kilometres per hour. The graph of this function (below left) shows that as the speed increases, the stopping distance increases faster. Another useful graph is the inverse (below right), which shows the maximum speed allowed to stop within a given distance.

In this section, you will learn about the **inverse of a function** and how it relates to the idea of reversing operations.





inverse of a function

- The inverse of a function f is denoted by f^{-1}.
- The function and its inverse have the property that if $f(a) = b$, then $f^{-1}(b) = a$.

Tools

- grid paper
- ruler

Investigate

How can you determine the inverse of a function?

1. Sketch the graph of the function $f(x) = x^2$ and its inverse.

a) Start by looking at points on the graph of the function. Thinking of an inverse in terms of a reverse operation, copy and complete the table by switching the x- and y-coordinates of each point.

Points on the Function	Points on the Inverse of the Function
(−3, 9)	
(−2, 4)	
(−1, 1)	
(0, 0)	
(1, 1)	
(2, 4)	
(3, 9)	

b) Plot the original function points, draw a smooth curve through them, and label the curve $f(x)$.

c) Plot the new points on the same set of axes and draw a smooth curve through them. Label this curve as $f^{-1}(x)$.

2. a) State the domain and range of $f(x)$.

b) State the domain and range of $f^{-1}(x)$.

Connections

The notation $f^{-1}(x)$ is read as "the inverse of f at x." Note that the −1 in $f^{-1}(x)$ does not behave like an exponent, so $f^{-1}(x) \neq \frac{1}{f(x)}$.

3. **Reflect**

   a) Compare the domain and range of $f(x)$ to the domain and range of $f^{-1}(x)$.

   b) Is $f^{-1}(x)$ a function? Explain.

4. a) Draw a line segment between each original point and its corresponding inverse point.

   b) Locate the midpoint of each line segment.

   c) All of these midpoints should lie on a straight line. What is the equation of that line?

5. **Reflect** Describe a way to draw the inverse of a function using reflections.

6. On the same set of axes, sketch the graphs of $g(x) = -x^2$, $h(x) = 2x - 3$, and $k(x) = \pm\sqrt{x}$. Note that $k(x)$ represents two functions: $k(x) = \sqrt{x}$ and $k(x) = -\sqrt{x}$.

7. **Reflect** Which of the graphs from step 6 is the same as the graph of the inverse of $f(x)$? Compare its equation to that of $f(x)$. How do you think the notion of "reversing" comes into play when dealing with the equations of a function and its inverse?

Example 1

Determine the Inverse Numerically

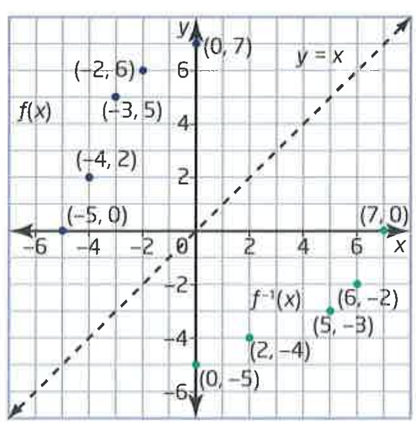
The table shows ordered pairs belonging to a function $f(x)$. Determine $f^{-1}(x)$, graph $f(x)$ and its inverse, and then state the domain and range of $f(x)$ and its inverse.

$f(x)$
(−5, 0)
(−4, 2)
(−3, 5)
(−2, 6)
(0, 7)

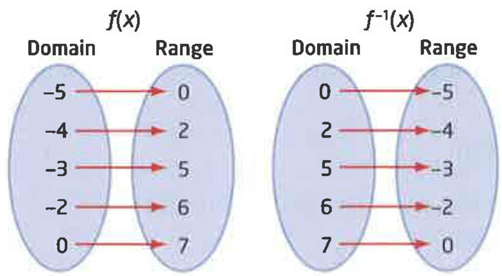
Solution

Switch the x- and y-coordinates, and then plot the points.

$f(x)$	$f^{-1}(x)$
(−5, 0)	(0, −5)
(−4, 2)	(2, −4)
(−3, 5)	(5, −3)
(−2, 6)	(6, −2)
(0, 7)	(7, 0)



Notice that switching the x- and y-coordinates reflects the graph of $f(x)$ in the line $y = x$.



Notice that the domain of $f(x)$ is the range of $f^{-1}(x)$, and the range of $f(x)$ is the domain of $f^{-1}(x)$.

Connections

In mathematics, tools exist for manipulating numbers and expressions. Many of these tools come in pairs. Two examples are addition and subtraction and multiplication and division. These are considered to be inverse operations since one "undoes" or "reverses" the other.

The inverse of a function $f(x)$ can be found by reversing the operations that the function specifies. Consider the function $f(x) = 3x + 2$. This function multiplies each x-value by 3 and adds 2 to the result. Reversing the operations then subtracts 2 from each x and divides the result by 3. So, the inverse of $f(x)$ is $f^{-1}(x) = \frac{x - 2}{3}$. The inverse is the "reverse" of the original function.

A systematic method for determining the inverse of a function algebraically can be described by the following steps:

1. Write the equation in "$y =$" form, if it is not already in that form.
2. Interchange x and y in the equation.
3. Solve the new equation for y.
4. Replace y with $f^{-1}(x)$.

Example 2

Determine the Inverse of a Function Algebraically

For each function $f(x)$,

i) determine $f^{-1}(x)$

ii) graph $f(x)$ and its inverse

iii) determine whether the inverse of $f(x)$ is a function

a) $f(x) = 2x - 3$

b) $f(x) = 2x^2 + 16x + 29$

Solution

a) i) $f(x) = 2x - 3$

Step 1: $y = 2x - 3$ — Replace $f(x)$ with y.

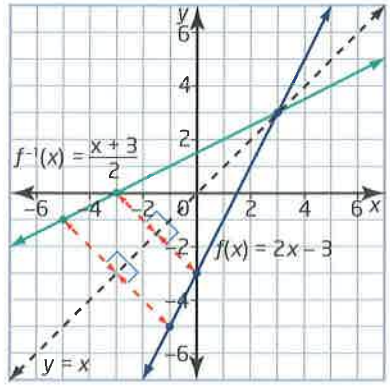
Step 2: $x = 2y - 3$ — Interchange x and y.

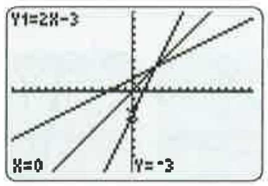
Step 3: $x + 3 = 2y$ — Isolate y. Notice that the inverse operations are used.

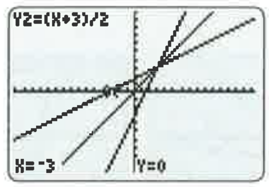
$\frac{x + 3}{2} = y$

Step 4: $f^{-1}(x) = \frac{x + 3}{2}$ — Replace y with $f^{-1}(x)$.

ii) The graphs of the function $f(x)$ and its inverse are reflections of each other in the line $y = x$. Any points that lie on the line $y = x$ are invariant, since their x- and y-coordinates are equal.







Using a graphing calculator, set the line style of $y = x$ to **Dot**. The window can be adjusted using **ZSquare** from the **ZOOM** menu.

iii) The inverse of $f(x)$ is a function, since there is only one y-value for each x-value. In other words, the graph of $f^{-1}(x)$ passes the vertical line test.

b) i) $f(x) = 2x^2 + 16x + 29$

Step 1: $y = 2x^2 + 16x + 29$ — Replace $f(x)$ with y.

Before you interchange x and y, rewrite the quadratic function in vertex form, $y = a(x - h)^2 + k$, by completing the square.

$$\begin{aligned} y &= 2(x^2 + 8x) + 29 \\ &= 2(x^2 + 8x + 16 - 16) + 29 \\ &= 2(x^2 + 8x + 16) - 32 + 29 \\ &= 2(x + 4)^2 - 3 \end{aligned}$$

Step 2: $x = 2(y + 4)^2 - 3$ — Interchange x and y.

Step 3: $x + 3 = 2(y + 4)^2$ — Isolate y. Notice that the inverse operations are used.

$$\frac{x+3}{2} = (y + 4)^2$$

$$\pm\sqrt{\frac{x+3}{2}} = y + 4$$

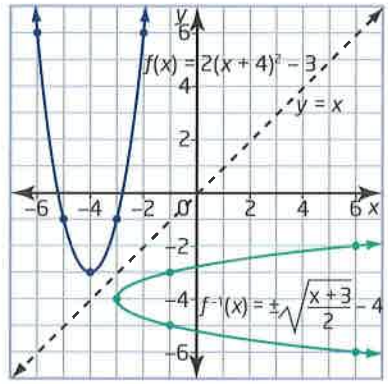
Take the square root of both sides. Recall that there is a positive and a negative root.

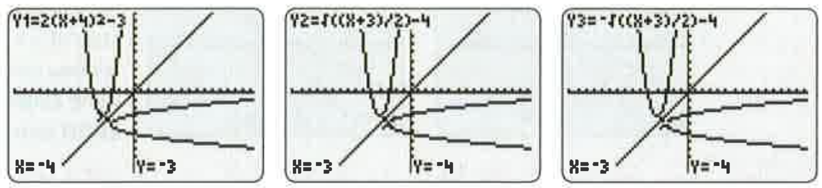
$$\pm\sqrt{\frac{x+3}{2}} - 4 = y$$

Step 4: $f^{-1}(x) = \pm\sqrt{\frac{x+3}{2}} - 4$ — Replace y with $f^{-1}(x)$.

ii) The graphs of the function $f(x)$ and its inverse are reflections of each other in the line $y = x$.

Reflect key points in the line $y = x$ to help you sketch the graph of the inverse of $f(x)$.





To graph $g^{-1}(x) = \pm\sqrt{\frac{x+3}{2}} - 4$, you must enter two equations,

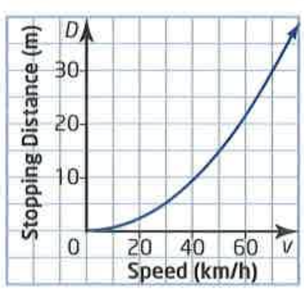
$y = \sqrt{\frac{x+3}{2}} - 4$ and $y = -\sqrt{\frac{x+3}{2}} - 4$.

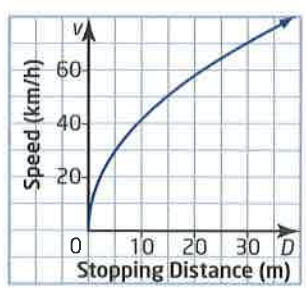
iii) The inverse of $f(x)$ is not a function, since there are two y-values for each x-value. In other words, the graph of $f^{-1}(x)$ does not pass the vertical line test.

Example 3

Apply Inverses

The relationship between the speed of a car and its stopping distance can be modelled by the function $D = 0.006v^2$, where D is the stopping distance, in metres, and v is the speed, in kilometres per hour. The graph of this function and its inverse are shown.





a) State the domain and range of the function D.

b) Determine an equation for the inverse of the function. State its domain and range.

Solution

a) Since the speed must be greater than or equal to zero, the domain is $\{v \in \mathbb{R}, v \geq 0\}$.
Distance cannot be negative, so the range is $\{D \in \mathbb{R}, D \geq 0\}$.

b) For a real-life context, to determine the inverse of the relationship, solve for the dependent variable. This is because the cause-and-effect relationship between speed and stopping distance does not change, so you cannot just switch D and v.

$$D = 0.006v^2$$

$$\frac{D}{0.006} = v^2$$

$$v = \sqrt{\frac{D}{0.006}}$$

Take the square root of both sides. Since speed must be greater than or equal to zero, only the positive root is needed.

The domain is $\{D \in \mathbb{R}, D \geq 0\}$ and the range is $\{v \in \mathbb{R}, v \geq 0\}$.

Key Concepts

- The inverse of a function $f(x)$ is denoted by $f^{-1}(x)$.
- The inverse of a function can be found by interchanging the x- and y-coordinates of the function.
- The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected in the line $y = x$.
- The inverse of a function can be found by interchanging x and y in the equation of the function and then solving the new equation for y.
- For algebraic inverses of quadratic functions, the functions must be in vertex form.
- The inverse of a function is not necessarily a function.

Communicate Your Understanding

C1 You are asked to solve the equation $3x + 4 = 19$. How is the method used to solve this equation similar to the method used to determine an inverse?

C2 The function $f(x) = 9x - 5$ has inverse $f^{-1}(x) = \frac{x + 5}{9}$. Discuss how the idea of inverse or reversed operations relates the function and its inverse.

C3 Explain why a quadratic function needs to be in vertex form in order to determine its inverse algebraically.

Ⓐ Practise

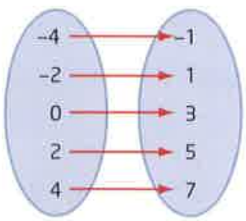
For help with questions 1 and 2, refer to Example 1.

1. Write the inverse of each function. Then, state the domain and range of the function and its inverse.

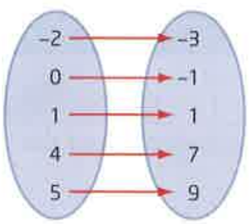
a) $\{(1, 5), (4, 2), (5, -3), (7, 0)\}$

b) $\{(3, 5), (4, 0), (5, -5), (6, -10)\}$

c)

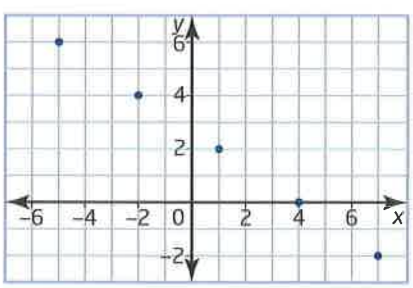


d)

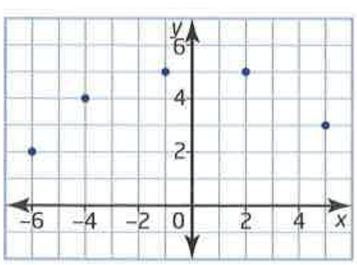


2. Copy each graph. Then, sketch the inverse of each function and state the domain and range of the function and its inverse.

a)

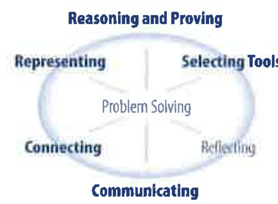


b)

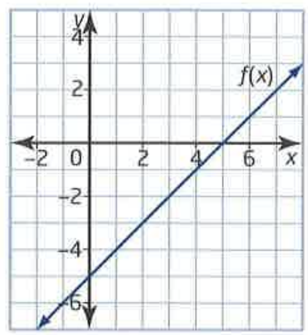


For help with questions 3 to 7, refer to Example 2.

3. Copy each graph of $f(x)$ and then sketch the graph of the inverse of each function. Is the inverse of $f(x)$ a function? Explain.



a)



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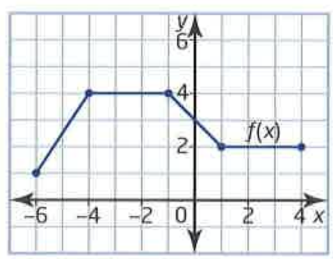

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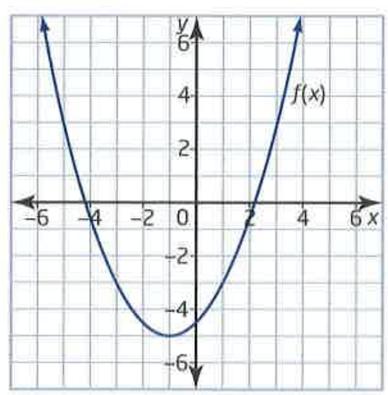


4. Determine the equation of the inverse of each function.

a) $f(x) = 2x$

b) $f(x) = 6x - 5$

c) $f(x) = -x + 10$

d) $f(x) = \dfrac{2x + 4}{5}$

5. Determine the equation of the inverse of each function.

a) $f(x) = x^2 + 6$

b) $f(x) = 4x^2$

c) $f(x) = (x + 8)^2$

d) $f(x) = \frac{1}{2}x^2 + 10$

6. For each quadratic function, complete the square and then determine the equation of the inverse.

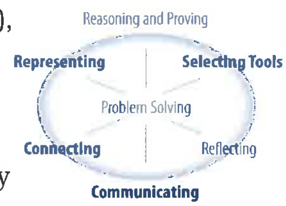
a) $f(x) = x^2 + 6x + 15$

b) $f(x) = -x^2 + 20x - 99$

c) $f(x) = 2x^2 + 24x - 3$

d) $f(x) = -3x^2 - 36x - 100$

7. For each function $f(x)$,



i) determine $f^{-1}(x)$

ii) graph $f(x)$ and its inverse with or without technology

iii) state whether the inverse of $f(x)$ is a function and explain your reasoning

a) $f(x) = -5x + 6$

b) $f(x) = \frac{1}{3}x - 8$

c) $f(x) = (x - 8)^2 + 16$

d) $f(x) = -x^2 + 20x - 64$

B Connect and Apply

For help with question 8, refer to Example 3.

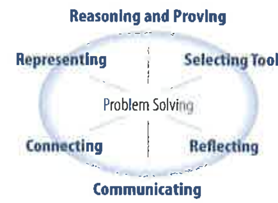
8. In ballistics, the paths of projectiles are studied. It is known that the greater the speed at which an object is fired, the farther it will travel. For a particular spring-loaded cannon, this relationship can be modelled by $d = \frac{v^2}{10}$, where d is the distance, in metres, travelled by the projectile and v is the muzzle speed, in metres per second. Different speeds can be attained by adjusting the spring.

a) Graph the distance function and state its domain and range.

b) Graph the inverse of the distance function. What does it represent? State its domain and range.

c) Which equation, the original or the inverse, might be more useful for someone using the cannon?

9. In the 2008 Beijing Olympics, Usain "Lightning" Bolt set a new world record in the 100-m dash by smashing his own previous record by 0.03 s. By all accounts, however, he coasted to the finish line. His coach suggested that if he had continued at the speed he had been travelling, he would have run the race in 9.52 s.



Physicists led by Hans Kristian Eriksen investigated this possibility. For the majority of the race, Bolt's position, in metres, could be modelled by the function $d(t) = 11.8t - 12.5$, where t is the time, in seconds.

a) Graph Bolt's position function and state the domain and range.

b) Suggest reasons why the domain of this function should not start at $t = 0$.

c) Determine the inverse of this function and state its domain and range.

d) Use the inverse to determine if Bolt's coach was close to being correct.

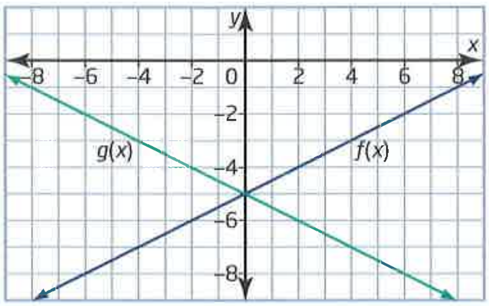
Connections

Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 2 to read the paper entitled "Velocity Dispersions in a Cluster of Stars: How Fast Could Usain Bolt Have Run?"

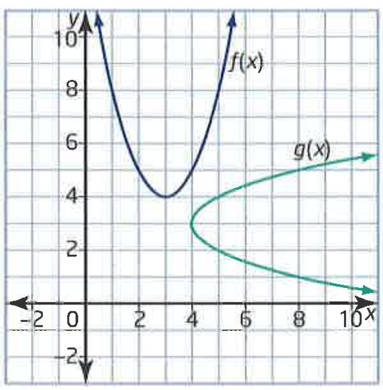
10. Explain why $f(x) = f^{-1}(x)$ for any function of the form $f(x) = -x + b$.

11. Determine whether the two relations shown in each graph are inverses of each other. Explain your reasoning.

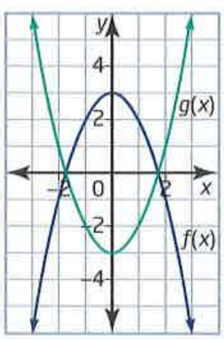
a)



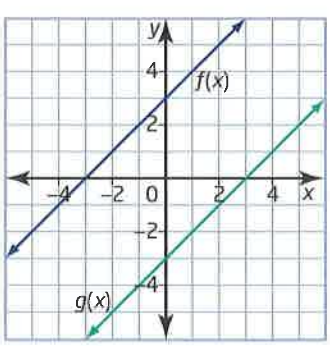
b)



c)



d)



12. The equation $y = \frac{9}{5}x + 32$ can be used to convert between Celsius and Fahrenheit temperatures, where x is the temperature, in degrees Celsius, and y is the temperature, in degrees Fahrenheit.

a) Determine the inverse of this equation. What does it represent? What do the variables represent?

b) Graph the original and inverse equations on the same set of axes.

c) What temperature is the same in Celsius and Fahrenheit? Explain how you know.

13. Chapter Problem At the traffic safety bureau, Matthew determines that for a car travelling at approximately 100 km/h, the stopping distance, in metres, once the brakes are applied is approximately given by $d = -2.8(t - 5)^2 + 70$, where t is the time, in seconds.

a) Determine the inverse of this function. What does this represent in the context of the question?

b) In the context, what should the domain and range of the original function and its inverse be?

c) Compare the distance travelled in the first 20 m of braking, the second 20 m of braking, and the third 20 m of braking.

14. Refer to Example 2 of Section 2.6. Use inverses to calculate the exact time a 400-m drag race would take for each car.

15. For each quadratic function $f(x)$,

i) determine $f^{-1}(x)$

ii) graph $f(x)$ and its inverse

iii) restrict the domain of $f(x)$ to one branch of the parabola so that $f^{-1}(x)$ is also a function

iv) graph $f(x)$ and its inverse with the restricted domains

a) $f(x) = 2x^2$

b) $f(x) = x^2 + 2$

c) $f(x) = (x - 3)^2$

16. a) Given the function $f(x) = 3x + 7$, determine the inverse.

b) Determine the values of $f(f^{-1})$ and $f^{-1}(f)$. That is, substitute the inverse equation into the original and vice versa.

c) Repeat parts a) and b) for $f(x) = x^2 - 6$.

d) Can you make a general statement about $f(f^{-1})$ and $f^{-1}(f)$?

Achievement Check

17. A rock is thrown from the top of a 100-m cliff. Its height, h, in metres, after t seconds can be modelled approximately by the function $h(t) = 100 - 5t^2$.

a) Graph the function and state its domain and range.

b) Determine the inverse of $h(t)$ and state its domain and range. Explain what this inverse represents in the context of the question.

c) A second rock is thrown upward off the cliff. Its height, h, in metres, after t seconds can be modelled approximately by the function $h(t) = 100 + 10t - 5t^2$. Solve for t by finding the inverse. Then, determine at what time the rock will hit the ground.

C Extend

18. a) Determine the inverse of the function $f(x) = \sqrt{x + 3}$.

b) State the domain and range of the function and its inverse.

c) Sketch the graphs of the function and its inverse.

19. The volume, V, of a sphere with radius r is given by $V = \frac{4}{3}\pi r^3$.

a) Determine the inverse of the equation. What does it represent?

b) State the domain and range of the function and its inverse in the context of the question.

20. a) Given the function $f(x) = \frac{1}{x}$, determine the inverse.

b) Use the graph of $f(x)$ to explain your answer for part a).

21. a) Given the function $f(x) = \frac{5}{5x - 16}$, determine the inverse.

b) State the domain and range of the function and its inverse.

22. a) Use the property from question 16 to test if f and g are inverses of each other.

i) $f(x) = -5x + 20$
$g(x) = -\frac{x}{5} + 4$

ii) $f(x) = x^2 - 10x + 27$
$g(x) = \sqrt{x - 2} + 5$

iii) $f(x) = (x + 4)^3 + 6$
$g(x) = \sqrt[3]{x - 6} - 4$

iv) $f(x) = \sqrt[4]{x + 10}$
$g(x) = x^4 - 10$

b) **Use Technology** Use graphing technology to verify your answers in part a).

23. Math Contest Given $f(x) = x^2 + 3x - 3$ and $g(x) = f^{-1}(x)$, a possible value for $g(1)$ is

A -4 **B** 0 **C** -1

D cannot be determined

24. Math Contest Given that the function $f(x)$ is the single point $(6, -2)$, then the distance from $f(x)$ to $f^{-1}(x)$ is

A $8\sqrt{2}$ units **B** 64 units

C 0 units **D** 16 units

25. Math Contest The point $(5, 1)$ is reflected in the line $y = x + 1$. The image point is

A $(1, 6)$ **B** $(2, 5)$ **C** $(0, 6)$ **D** $(0, 5)$

26. Math Contest Determine the integral ordered pairs that solve the system of equations $x + xy + y = 19$ and $x^2y + xy^2 = 84$.

Chapter 2 Review

2.1 Functions and Equivalent Algebraic Expressions, pages 78 to 87

1. Determine whether the functions in each pair are equivalent.

a) $f(x) = (x + 6)(x - 8) + (x + 16)(x + 3)$, $g(x) = 3(x^2 + 3x + 5) - (x - 5)(x - 3)$

b) $f(x) = (x + 5)(x - 4) - (x - 8)(x - 1)$, $g(x) = 2(5x - 28)$

2. Simplify each expression and state all restrictions on x.

a) $\dfrac{x + 7}{x^2 + 10 + 21}$ **b)** $\dfrac{x^2 - 64}{x - 8}$

3. A square piece of cardboard with side length 40 cm is used to create an open-topped box by cutting out squares with side length x from each corner.

a) Determine a simplified expression for the surface area of the box.

b) Determine any restrictions on the value of x.

2.2 Skills You Need: Operations With Rational Expressions, pages 88 to 96

4. Simplify each expression and state the restrictions.

a) $\dfrac{3x^2}{5xy} \times \dfrac{20xy^3}{12xy}$ **b)** $\dfrac{150a^3b^4}{20a^2b} \div \dfrac{6b}{8ab^2}$

c) $\dfrac{1}{3x} + \dfrac{5}{2x^2}$ **d)** $\dfrac{4}{x - 6} - \dfrac{3}{x - 4}$

5. Simplify each expression and state the restrictions.

a) $\dfrac{x^2 + 7x}{3x + 21} \times \dfrac{x^2 + 3x + 2}{x + 2}$

b) $\dfrac{x^2 + 4x - 60}{3x + 30} \div \dfrac{x^2 - 8x + 12}{6x - 12}$

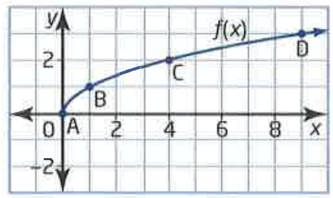
c) $\dfrac{3}{x^2 + 7x + 10} - \dfrac{5x}{x^2 - 4}$

d) $\dfrac{-10x}{x^2 + 18x + 32} + \dfrac{12x}{x^2 + 6x - 160}$

6. For the open-topped box in question 3, determine a simplified expression for the ratio of the volume to the surface area. What are the restrictions on x?

2.3 Horizontal and Vertical Translations of Functions, pages 97 to 104

7. Copy the graph of the function $f(x)$. Sketch the graph of $g(x)$ by determining the image points A′, B′, C′, and D′.



a) $g(x) = f(x) + 6$ **b)** $g(x) = f(x - 3)$

8. For each function $g(x)$, identify the base function as one of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \dfrac{1}{x}$ and describe the transformation first in the form $y = f(x - d) + c$ and then in words. Transform the graph of $f(x)$ to sketch the graph of $g(x)$. Then, state the domain and range of each function.

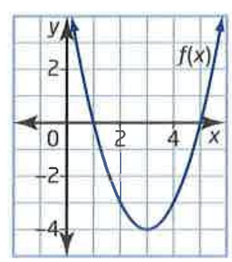
a) $g(x) = (x + 7)^2 - 8$

b) $g(x) = \sqrt{x - 6} + 3$

c) $g(x) = \dfrac{1}{x + 3} + 1$

2.4 Reflections of Functions, pages 105 to 112

9. Copy the graph of $f(x)$ and sketch each reflection, $g(x)$. State the domain and range of each function.



a) $g(x) = f(-x)$

b) $g(x) = -f(x)$

c) $g(x) = -f(-x)$

10. a) Determine the equation of each function, $g(x)$, after a reflection in the x-axis.

i) $f(x) = \sqrt{x} + 5$ **ii)** $f(x) = \dfrac{1}{x} - 7$

b) Determine the equation of each function in part a), $h(x)$, after a reflection in the y-axis.

2.5 Stretches of Functions, pages 113 to 124

11. Given the function $f(x) = x^2$, identify the value of a or k, transform the graph of $f(x)$ to sketch the graph of $g(x)$, and state the domain and range of each function.

a) $g(x) = 4f(x)$ **b)** $g(x) = f(5x)$

c) $g(x) = f\left(\frac{x}{3}\right)$ **d)** $g(x) = \frac{1}{4}f(x)$

12. For each function $g(x)$, describe the transformation from a base function of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, or $f(x) = \frac{1}{x}$. Then, transform the graph of $f(x)$ to sketch the graph of $g(x)$.

a) $g(x) = 5x$ **b)** $g(x) = \frac{1}{4x}$

c) $g(x) = (3x)^2$ **d)** $g(x) = \sqrt{9x}$

2.6 Combinations of Transformations, pages 125 to 131

13. Describe, in the appropriate order, the transformations that must be applied to the base function $f(x)$ to obtain the transformed function. Then, write the corresponding equation and transform the graph of $f(x)$ to sketch the graph of $g(x)$.

a) $f(x) = \sqrt{x}$, $g(x) = 3f(x + 6)$

b) $f(x) = x$, $g(x) = -f(6x) - 5$

c) $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{5}f(x) + 4$

d) $f(x) = x^2$, $g(x) = -2f(3x + 12) - 6$

14. For each, identify the base function as one of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$. Sketch the graphs of the base function and the transformed function. State the domain and range of the functions.

a) $g(x) = 2x + 9$ **b)** $g(x) = \frac{3}{x + 4}$

c) $g(x) = -4\sqrt{x} + 1$ **d)** $g(x) = (5x + 20)^2$

2.7 Inverse of a Function, pages 132 to 141

15. For each function $f(x)$:

i) determine $f^{-1}(x)$

ii) graph $f(x)$ and its inverse

iii) state whether or not $f^{-1}(x)$ is a function

a) $f(x) = 7x - 5$

b) $f(x) = 2x^2 + 9$

c) $f(x) = (x + 4)^2 + 15$

d) $f(x) = 5x^2 + 20x - 10$

16. Jai works at an electronics store. She earns $600 a week, plus commission of 5% of her sales.

a) Write a function to describe Jai's total weekly earnings as a function of her sales.

b) Determine the inverse of this function.

c) What does the inverse represent?

d) One week, Jai earned $775. Calculate her sales that week.

Chapter Problem WRAP-UP

In Sections 2.1, 2.3, 2.5, and 2.7, you explored how functions and transformations relate to the workings of a traffic safety bureau. In general, traffic safety bureaus have a lot to do with civil engineering.

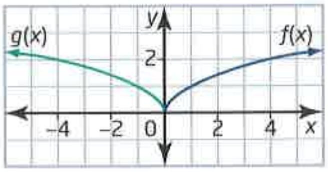
a) Describe how these mathematical concepts relate to civil engineering.

b) Research what is needed to become a civil engineer. How does studying mathematics relate to this field? Be specific.

Chapter 2 Practice Test

For questions 1 to 5, select the best answer.

1. Describe the reflection that transforms $f(x)$ to $g(x)$.



A a reflection in the x-axis

B a reflection in the y-axis

C a reflection in x-axis and then a reflection in the y-axis

D a reflection in the line $y = x$

2. The graph of $f(x)$ is transformed to obtain the graph of $g(x) = 4f(3x + 21) - 15$. Describe the horizontal translation that occurs.

A 3 units left **B** 21 units left

C 7 units left **D** 15 units left

3. Describe, in the appropriate order, the transformations that must be applied to the graph of $f(x)$ to obtain the graph of $g(x) = 5f(x - 9) + 7$.

A vertically stretched by a factor of 5 and then translated 9 units left and 7 units up

B translated 9 units left and 7 units up and then vertically stretched by a factor of 5

C translated 9 units right and 7 units up and then vertically stretched by a factor of 5

D vertically stretched by a factor of 5 and then translated 9 units right and 7 units up

4. State the restrictions on x in the expression $\dfrac{(x + 7)(x - 1)}{(x - 4)(x + 7)}$.

A $x \neq 4$

B $x \neq 1, x \neq 4$

C $x \neq -7, x \neq 4$

D $x \neq -7, x \neq 1, x \neq 4$

5. If a function is defined by a set of points, then the inverse can be found by

A reflecting the points in the y-axis

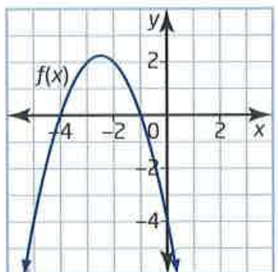
B interchanging the x- and y-coordinates

C reflecting the points in the origin

D taking the reciprocal of each coordinate

6. Are $\dfrac{6x^2 - 27x - 105}{x - 7}$ and $(x + 3)(x + 10) - (x + 3)(x + 5)$ equivalent expressions? Justify your answer.

7. Copy the graph of $f(x)$ and then sketch the graph of the inverse of $f(x)$. State whether or not the inverse is a function.



8. Simplify each expression and state any restrictions.

a) $\dfrac{x - 8}{x + 7} \times \dfrac{x + 15}{x^2 + 12x - 45}$

b) $\dfrac{x^2 + 12x + 20}{x + 5} \div \dfrac{x^2 + 7x - 30}{x + 10}$

c) $\dfrac{x + 3}{x - 7} - \dfrac{x + 9}{x - 2}$

d) $\dfrac{x + 8}{x + 3} + \dfrac{x - 6}{x^2 + 9x + 18}$

9. a) Given the function $g(x) = 4(3x + 6)^2 + 9$, identify the base function as one of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$.

b) Describe, in the appropriate order, the transformations that must be applied to the base function $f(x)$ to obtain the transformed function $g(x)$.

c) Sketch the graphs of $f(x)$ and $g(x)$.

d) State the domain and range of the functions.

10. a) Given the function $g(x) = \frac{1}{5}\sqrt{2(x - 8)} - 3$, identify the base function as one of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$.

b) Describe, in the appropriate order, the transformations that must be applied to the base function $f(x)$ to obtain the transformed function $g(x)$.

c) Sketch the graphs of $f(x)$ and $g(x)$.

d) State the domain and range of the functions.

11. a) Given the function $g(x) = \frac{2}{0.5x} + 5$, identify the base function as one of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$.

b) Describe, in the appropriate order, the transformations that must be applied to the base function $f(x)$ to obtain the transformed function $g(x)$.

c) Sketch the graphs of $f(x)$ and $g(x)$.

d) State the domain and range of the functions.

12. For each function,

i) determine $f^{-1}(x)$

ii) graph $f(x)$ and its inverse

iii) determine whether the inverse of $f(x)$ is a function

a) $f(x) = 3x + 8$

b) $f(x) = 6(x - 9)^2 + 8$

c) $f(x) = 3x^2 + 36x + 8$

13. A small skateboard company is trying to determine the best price for its boards. When the boards are priced at $80, 120 are sold in a month. After doing some research, the company finds that each increase of $5 will result in selling 15 fewer boards.

a) Write an equation to represent the revenue, R, in dollars, as a function of x, the number of $5 increases.

b) State the domain and range of the revenue function.

c) Determine the inverse of the revenue function. What does this equation represent in the context of the question? State the domain and range of the inverse.

d) Determine the number of $5 increases for a revenue of $8100.

14. A small plane is travelling between Windsor and Pelée Island (a distance of approximately 60 km) and is directly affected by the prevailing winds. Thus, the actual speed of the plane with respect to the ground is the speed of the plane (160 km/h) plus or minus the wind speed, w.

a) Develop a simplified equation for the total time it takes to make a round trip if the wind speed is w. State the domain and range and any restrictions on this relationship.

b) Graph your relationship from part a).

c) The pilot thinks that if he has a strong headwind on the way out, then he will be able to make up any lost time on the way back when he has a tailwind. Determine if he is correct.

15. In Canada, fuel efficiency, ℓ, for cars is stated in litres per 100 km. In the United States, fuel efficiency, m, is stated in miles per gallon (mpg). The formula $m = \frac{235}{\ell}$ can be used to convert from the Canadian system to the United States system.

a) Sketch the graph of the function.

b) In Canada, cars that have better fuel efficiency have a lower value for ℓ. Is the same true for m? Justify your response.

c) 1 L equals 0.264 US gallons. 1 L also equals 0.220 Imperial gallons. Determine a new function relating the fuel efficiency in Imperial gallons to litres per 100 km.

d) How would the graph of your function from part c) compare to the graph in part a)? Explain.

Task

Functions in Design

CD covers and T-shirts often use graphics in their designs. Many of these designs can be defined by mathematical functions.

a) Plot the graphic design defined by the following set of functions, where $x \in \mathbb{R}$.

$y = \frac{4}{x}$, $-2 \leq x \leq -0.1$, $0.1 \leq x \leq 2$

$y = -\frac{4}{x}$, $-2 \leq x \leq 2$

$y = \sqrt{6 - x}$, $x \geq 2$

$y = -\sqrt{6 - x}$, $x \geq 2$

$y = \frac{2}{3}\sqrt{11 - x}$, $x \geq 2$

$y = -\frac{2}{3}\sqrt{11 - x}$, $x \geq 2$

$y = \sqrt{2(4 - x)}$, $x \geq 2$

$y = -\sqrt{2(4 - x)}$, $x \geq 2$

$y = \sqrt{x + 6}$, $x \leq -2$

$y = -\sqrt{x + 6}$, $x \leq -2$

$y = \frac{2}{3}\sqrt{x + 11}$, $x \leq -2$

$y = -\frac{2}{3}\sqrt{x + 11}$, $x \leq -2$

$y = \sqrt{2(x + 4)}$, $x \leq -2$

$y = -\sqrt{2(x + 4)}$, $x \leq -2$

b) Explain the significance of $x = 2$.

c) Create and draw your own graphic design using transformations of $y = \frac{1}{x}$ and $y = \sqrt{x}$. You may also use transformations of other functions in your design. List the functions you decide to use, as well as their domains.