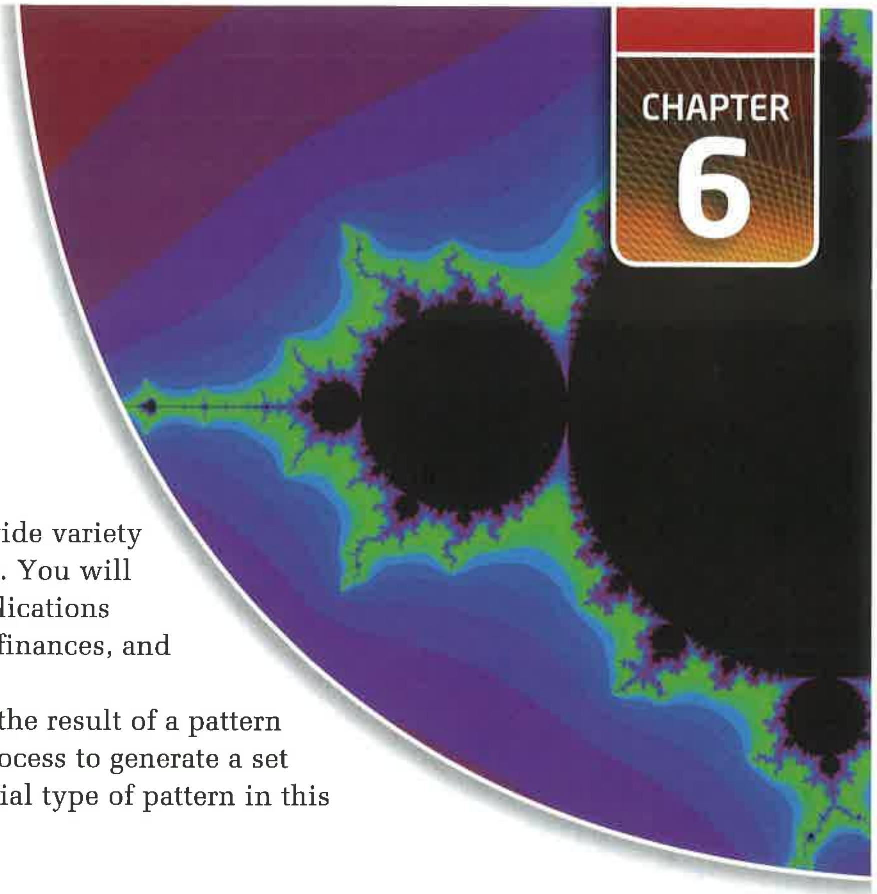


Discrete Functions

In this chapter, you will explore a wide variety of number patterns called sequences. You will learn that sequences have many applications in fields such as medicine, biology, finances, and construction.

A fractal, such as the one shown, is the result of a pattern formed by performing a recursive process to generate a set of points. You will explore this special type of pattern in this chapter's Chapter Problem.

By the end of this chapter, you will

- make connections between sequences and discrete functions, represent sequences using function notation, and distinguish between a discrete function and a continuous function
- determine and describe a recursive procedure for generating a sequence, given the initial terms, and represent sequences as discrete functions in a variety of ways
- connect the formula for the *n*th term of a sequence to the representation in function notation, and write terms of a sequence given one of these representations or a recursion formula
- represent a sequence algebraically using a recursion formula, function notation, or the formula for the *n*th term, and describe the information that can be obtained by inspecting each representation
- determine, through investigation, recursive patterns in the Fibonacci sequence, in related sequences, and in Pascal's triangle, and represent the patterns in a variety of ways
- determine, through investigation, and describe the relationship between Pascal's triangle and the expansion of binomials, and apply the relationship to expand binomials raised to whole-number exponents
- identify sequences as arithmetic, geometric, or neither, given a numeric or algebraic representation
- determine the formula for the general term of an arithmetic sequence or geometric sequence, through investigation using a variety of tools and strategies, and apply the formula to calculate any term in a sequence
- determine the formula for the sum of an arithmetic or geometric series, through investigation using a variety of tools and strategies, and apply the formula to calculate the sum of a given number of consecutive terms
- solve problems involving arithmetic and geometric sequences and series, including those arising from real-world applications

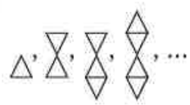
Prerequisite Skills

Refer to the Prerequisite Skills Appendix on pages 478 to 495 for examples of the topics and further practice.

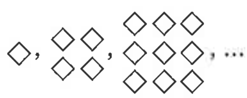
Identify Patterns

1. Determine the next three items in each pattern.

a)



b)



c) A, BB, CCC, DDDD, ...

d) P, PQ, PQR, PQRS, ...

e) 3, 6, 9, 12, ...

f) $-5, 10, -15, 20, \ldots$

g) $7, 3, -1, -5, \ldots$

h) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

i) $x, 2x, 3x, 4x, \ldots$

Evaluate Functions

2. For $f(x) = 3x - 1$, determine

a) $f(1)$ **b)** $f(-3)$

c) $f\left(\frac{1}{2}\right)$ **d)** $f\left(\frac{3}{m + 2}\right)$

3. For $f(x) = 2^x$, determine

a) $f(1)$ **b)** $f(-2)$

c) $f\left(\frac{1}{3}\right)$ **d)** $f\left(\frac{t - 1}{3}\right)$

4. For $f(x) = x^2 - 3x + 1$, determine

a) $f(3)$ **b)** $f(-1)$

c) $f(t - 2)$ **d)** $f(2t)$

Graph Functions

5. The domain of each function is $\{x \in \mathbb{R}\}$. Sketch the graph of each function.

a) $y = 2x + 3$

b) $f(x) = -\frac{1}{2}x - 1$

c) $y = x^2$

d) $f(x) = x^2 + 1$

e) $f(x) = (x - 3)^2$

f) $y = 2^x$

g) $f(x) = 2^x + 1$

h) $f(x) = 3^{x - 1}$

Solve Equations

6. Solve each equation and check your solutions.

a) $3 - 2y = 5y + 6$

b) $3t + 8 = t + 8$

c) $6a + 4(3 - a) = 15$

d) $\frac{x}{4} - 5 = 6$

e) $\frac{x}{6} - \frac{2x}{3} = -3$

Evaluate Expressions

7. Evaluate.

a) 8% of 60 **b)** 15% of 700

c) 12% of 4 **d)** 125% of 16

e) 85% of 0.06 **f)** 70% of 1400

8. Evaluate.

a) $\frac{3}{5} - \left(-\frac{1}{5}\right)$ **b)** $\frac{1}{2} \times \left(-\frac{3}{7}\right)$

c) $-\frac{4}{5} \div \left(-\frac{1}{6}\right)$ **d)** $\frac{13}{6} \times (-9)$

e) $-\frac{4}{3} - \left(-\frac{3}{2}\right)$ **f)** $-\frac{11}{3} \div \left(-\frac{5}{3}\right)$

Finite Differences

9. Use finite differences to determine whether each relation is linear, quadratic, or neither.

a)

x	y
1	3
2	9
3	17
4	27
5	39

b)

x	y
1	5
2	7
3	9
4	11
5	13

c)

x	y
1	2
2	4
3	8
4	16
5	32

10. Using the linear function $y = 3x - 1$ as an example, explain how the first differences relate to the slope of a line.

11. Using the quadratic function $y = 2x^2 - 5x - 3$ as an example, explain how the second differences relate to the value of a in an equation of the form $y = ax^2 + bx + c$.

Solve Linear Systems of Equations

12. Solve each system of equations.

a) $2x + y = 7$
$x + y = 4$

b) $3x + 3y = 41$
$4x + 5y = 71$

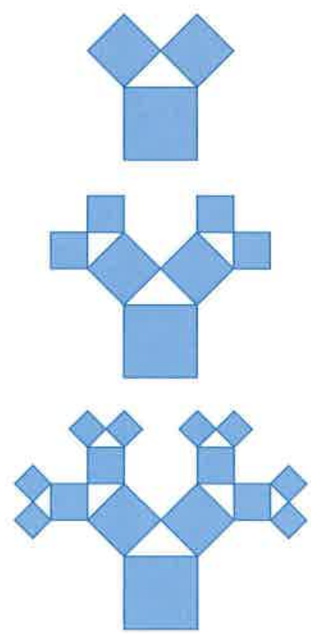
c) $4x - 3y = 96$
$2x + 5y = -8$

d) $\frac{1}{2}x - \frac{2}{5}y = \frac{9}{10}$
$\frac{1}{3}x - \frac{1}{4}y = \frac{2}{3}$

Chapter Problem

Fractals are amazing and beautiful shapes that can be found in mathematics, art, and nature. They are characterized by patterns and relationships that give them their fascinating shapes. These geometric shapes exhibit greater and greater complexity as they are enlarged. Mathematicians began studying fractals in the 17th century. Today, fractals have a wide range of applications in many fields, including music, environmental science, medicine, and video game design. For example, a biomedical engineer might need to know the total surface area of the bronchial tubes in the human lung or an environmental scientist might be concerned about the total length of coastline to be affected by an oil spill.

Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 6 to see more examples of fractals.



6.1

Sequences as Discrete Functions

The word *sequence* is used in everyday language. In a sequence, the order in which events occur is important. For example, builders must complete work in the proper sequence to construct safe, strong houses.

In mathematics, a **sequence** is a set of numbers, usually separated by commas, arranged in a particular order. Some sequences have very specific patterns and can be represented by mathematical rules or functions. Many natural phenomena, such as the spiral patterns seen in seashells, sunflowers, and galaxies, can be represented by sequences.

sequence

- an ordered list of numbers identified by a pattern or rule that may stop at some number or continue indefinitely
  3, 7, 11, 15
  2, 6, 18, 54, ...
- a function whose domain is the set, or a subset, of the natural numbers and whose range is the terms of the sequence

Tools

- square dot paper
- ruler

Investigate

How can you relate each number of a sequence to its position in the list?

The pattern of floor tiles in a new hotel is designed using nested squares. Explore the sequence formed by the number of enclosed regions created.

Method 1: Use Pencil and Paper

1. Construct a 16-unit by 16-unit square.
2. Locate the midpoint of each side. Connect consecutive midpoints to form a new square.
   Continue constructing midpoints and smaller squares until the squares are too small to work with.

3. Copy and complete the table for the pattern you created.

Number of Squares, n	Number of Regions, t
1	1
2	5
3	9

4. Refer to your completed table.
   a) **Reflect** Describe the pattern in the Number of Regions column.
   b) Write the values in this column as a sequence of **terms**. Use the pattern to write the next three terms of the sequence.

5. Graph the sequence using ordered pairs of the form (number of squares, number of regions). Should you join the points with a smooth curve or line or leave them as distinct points? Explain your thinking.

6. **Reflect** Determine an **explicit formula** to describe the number of regions, t_n, according to the number of squares, n.

term (of a sequence)

- a single value or object in a sequence

explicit formula

- a formula that represents any term in a sequence relative to the term number, n, where $n \in \mathbb{N}$

Method 2: Use *The Geometer's Sketchpad®*

Tools

- computer with *The Geometer's Sketchpad®*

1. Open *The Geometer's Sketchpad®*. From the **Graph** menu, choose **Grid Form** and then **Rectangular Grid**. Right-click on each axis and choose **Hide Axis** from the drop-down menu.

2. From the **Graph** menu, choose **Snap Points**. Draw a 16-unit by 16-unit square.

3. Select the sides of the square. From the **Construct** menu, choose **Midpoints**. Join the midpoints with line segments to form a square. Continue constructing midpoints and smaller squares until the squares are too small to work with.

4. Copy and complete the table for the pattern you created.

Number of Squares, n	Number of Regions, t
1	1
2	5
3	9

5. Refer to your completed table.
   a) **Reflect** Describe the pattern in the Number of Regions column.
   b) Write the values in this column as a sequence of **terms**. Use the pattern to write the next three terms of the sequence.

6. Graph the sequence using ordered pairs of the form (number of squares, number of regions). Should you join the points with a smooth curve or line or leave them as distinct points? Explain your thinking.

7. **Reflect** Determine an **explicit formula** to describe the number of regions, t_n, according to the number of squares, n.

Example 1

Use the Explicit Formula to Write Terms in a Sequence

Write the first three terms of each sequence, given the explicit formula for the nth term of the sequence, $n \in \mathbb{N}$.

a) $t_n = 3n^2 - 1$ **b)** $t_n = \dfrac{n-1}{n}$

Technology Tip

See the Use Technology feature at the end of this section for a TI-Nspire™ graphing calculator solution.

Solution

a) Method 1: Use Pencil and Paper

To determine the value of the first three terms, substitute the term numbers 1, 2, and 3 for n.

$t_n = 3n^2 - 1$

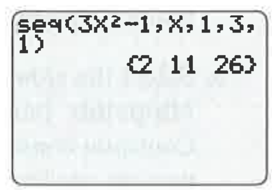
$t_1 = 3(1)^2 - 1$ $t_2 = 3(2)^2 - 1$ $t_3 = 3(3)^2 - 1$

$= 2$ $= 11$ $= 26$

The first three terms of the sequence are 2, 11, and 26.

Method 2: Use a Graphing Calculator

The terms of a sequence can be generated using a graphing calculator.



- Press [2nd] [LIST] and cursor over to the **OPS** menu.
- Select **5:seq(** and enter $3x^2 - 1$, x, 1, 3, 1).
- Press [ENTER].

The first three terms of the sequence are 2, 11, and 26.

Technology Tip

When using the **sequence** function of a graphing calculator to generate the terms of a sequence, you need to specify five things. For example, to generate the first three terms of the sequence $t_n = 3n^2 - 1$, you need

- the expression for the xth term of the sequence, $3x^2 - 1$
- the variable, x
- the starting term number, 1
- the ending term number, 3
- the increment value for the term numbers, 1

b) Substitute the term numbers 1, 2, and 3 for n.

$t_n = \dfrac{n-1}{n}$

$t_1 = \dfrac{1-1}{1}$ $t_2 = \dfrac{2-1}{2}$ $t_3 = \dfrac{3-1}{3}$

$= 0$ $= \dfrac{1}{2}$ $= \dfrac{2}{3}$

The first three terms of the sequence are 0, $\dfrac{1}{2}$, and $\dfrac{2}{3}$.

Example 2

Determine Explicit Formulas in Function Notation

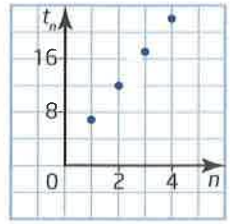
For each sequence, make a table of values using the term number and term and calculate the finite differences. Then, graph the sequence using the ordered pairs (term number, term) and determine an explicit formula for the nth term, using function notation.

a) 7, 12, 17, 22, … **b)** 1, 10, 25, 46, …

Solution

a) Patterns in finite differences tables can be used to help determine a formula for the terms in a sequence.

Term Number, n	Term, t_n	First Differences
1	7	
		5
2	12	
		5
3	17	
		5
4	22	



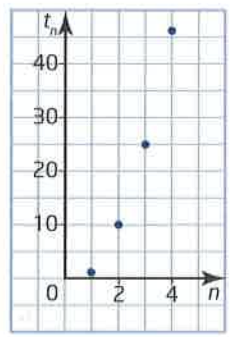
The graph models a function since there is exactly one value of t_n for each value of n. This function is linear since the first differences are constant. The rate of change, or the slope, is the first difference, 5.

$f(n) = 5n + b$

By inspection, $b = 2$. Then, an explicit formula to determine the terms in the sequence is $f(n) = 5n + 2$, where $n \in \mathbb{N}$.

b) Find the first and second differences.

Term Number, n	Term, t_n	First Differences	Second Differences
1	1		
		9	
2	10		6
		15	
3	25		6
		21	
4	46		



Since the second differences are constant, this function is quadratic. Half the value of the second difference corresponds to the value of a in a quadratic function of the form $f(n) = an^2 + bn + c$.

$f(n) = 3n^2 + bn + c$

To determine the values of b and c, substitute the coordinates of two points and solve a linear system of equations.

For (1, 1), $1 = 3(1)^2 + b + c$, or $-2 = b + c$.

For (2, 10), $10 = 3(2)^2 + 2b + c$, or $-2 = 2b + c$.

$$
\begin{array}{rll}
-2 &= b + c & \text{①} \\
-2 &= 2b + c & \text{②} \\
\hline
0 &= -b & \text{①} - \text{②} \\
b &= 0 &
\end{array}
$$

Substitute $b = 0$ into equation ① and solve for c.

$$
\begin{aligned}
-2 &= b + c \\
-2 &= 0 + c \\
c &= -2
\end{aligned}
$$

An explicit formula to determine the terms in the sequence is $f(n) = 3n^2 - 2$, where $n \in \mathbb{N}$.

Example 3

Types of Functions

a) The charge in a battery decreases by about 2% per day and can be modelled by the function $C(d) = 100(0.98)^d$, where d is the time, in days, and C is the level of the charge, as a percent. How much charge is left after 10 days? Is this a **continuous function** or a **discrete function**? Explain.

b) A certain bacterial culture starts with 200 bacteria and doubles every hour. Its growth can be modelled by the function $N(t) = 200(2)^t$, where t is the time, in hours, and N is the number of bacteria. How many bacteria will there be after 10 h? Is this function continuous or discrete? Explain.

continuous function

- a function that maps real numbers to real numbers and has a graph that is a curve with no holes or jumps

discrete function

- a function whose graph is made up of separate points that are not connected

Solution

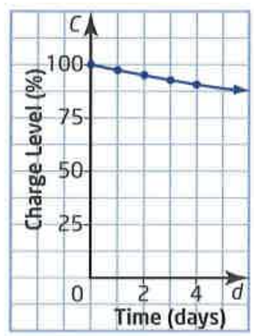
a) To determine the charge level after 10 days, substitute $d = 10$ into $C(d) = 100(0.98)^d$.

$$C(10) = 100(0.98)^{10}$$
$$\doteq 81.7$$

After 10 days, the charge level of the battery is approximately 81.7%.

The table of values and graph show how the battery charge changes over time.

Time (days), *d*	Charge Level (%), *C*
0	100.0
1	98.0
2	96.0
3	94.1
4	92.2



This is a continuous function since the charge level will be continuously changing over time. It does not drop 2% suddenly at the end of every day, but gradually decreases as time goes by.

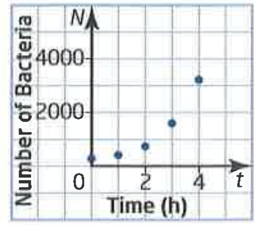
b) To determine the number of bacteria after 10 h, substitute $t = 10$ into $N(t) = 200(2)^t$.

$$N(10) = 200(2)^{10}$$
$$= 204\ 800$$

After 10 h, there will be 204 800 bacteria.

The table of values and graph show the number of bacteria over time.

Time (h), t	Number of Bacteria, N
0	200
1	400
2	800
3	1600
4	3200



Since you cannot have part of a bacterium, this function is discrete. After each hour, the number of bacteria is double the number of the previous hour.

Example 4

Determine the Value of a Car

The value of a new car purchased for $78 000 depreciates at a rate of 15% in the first year and 4% every year after that.

a) Determine the value of the car at the end of the first year, the second year, and the third year. Write these values as a sequence.

b) Determine an explicit formula for the value of the car at the end of year n.

c) What is the value of the car at the end of year 20? Is this realistic? Explain your thinking.

Connections

When an item *depreciates*, its value decreases over time. Some examples of items that depreciate are vehicles, electronics, computers, and clothing.

Solution

a) The value of the car when it is new is $78 000. At the end of the first year, the car depreciates by 15%, so it is worth $78 000 × 0.85, or $66 300.

At the end of the second year, the car value decreases by 4%, so it is worth 96% of its value at the start of that year.

$0.96 \times \$66\,300 = \$63\,648$

At the end of year 3, the car value again decreases by 4%.

$$\begin{aligned} 0.96 \times (0.96 \times \$66\,300) &= 0.96 \times \$63\,648 \\ &= \$61\,102.08 \end{aligned}$$

The sequence that represents the value of the car at the end of each year is 66 300, 63 648, 61 102.08, ... or
66 300, 0.96(66 300), 0.96^2(66 300), ...

b) The explicit formula for the value of the car at the end of year n is $t_n = 66\,300(0.96)^{n-1}$.

c) Substitute $n = 20$ to find the value of the car at the end of year 20.

$$t_n = 66\,300(0.96)^{n-1}$$
$$t_{20} = 66\,300(0.96)^{20-1}$$
$$= 66\,300(0.96)^{19}$$
$$\doteq 30\,525.79$$

At the end of year 20, the value of the car is \$30 525.79.

It is possible that the car will be worth more than this if the owner looks after it carefully. It could also be worth much less if it has been involved in a collision, has a high odometer reading, or has a lot of rust. There are many factors that can affect the value of a used car.

Key Concepts

- A sequence of numbers can be represented by a discrete function. The graph of a discrete function is a distinct set of points, not a smooth curve.
- The domain of a function representing a sequence is the set or a subset of the natural numbers, $\mathbb{N}$.
- Given the explicit formula for the nth term, t_n or $f(n)$, of a sequence, the terms can be written by substituting the term numbers for n. Examples of explicit formulas are $t_n = 3n + 2$ and $f(n) = 5n + 3$.
- An explicit formula for the nth term of a sequence can sometimes be determined by finding a pattern among the terms.

Communicate Your Understanding

C1 Graph the sequence of numbers represented by the ordered pairs $(1, 1), (2, -1), (3, -3), (4, -5), \ldots$. On the same set of axes, graph the function $f(x) = -2x + 3$, $x \in \mathbb{R}$. Describe the similarities and differences between the two graphs. Write the formula for the sequence using function notation and specify the domain.

C2 Consider the domain of a continuous and of a discrete function. What are the similarities between the domains? How are they different?

C3 Describe two situations in which it might be important to know a specific term in a sequence.

A Practise

For help with questions 1 and 2, refer to Example 1.

1. Write the first three terms of each sequence, given the explicit formula for the nth term of the sequence.

a) $t_n = 3n - 1$ **b)** $t_n = 2 - 5n$

c) $t_n = 3^{n-1}$ **d)** $f(n) = 2^{-n}$

e) $t_n = \frac{n+1}{n} - 1$ **f)** $f(n) = 3(2)^{n+2}$

2. Write the 12th term, given the explicit formula for the nth term of the sequence.

a) $f(n) = 1 - 3n$ **b)** $t_n = 2n + 5$

c) $f(n) = n^2 - 2$ **d)** $t_n = \frac{n+1}{n}$

e) $t_n = n^2 + 2n$ **f)** $f(n) = (-2)^{n-1}$

For help with questions 3 to 5, refer to Example 2.

3. Describe the pattern in each sequence. Write the next three terms of each sequence.

a) 4, 16, 64, 256, ...

b) 7, 6, 5, 4, ...

c) $-3, -6, -9, -12, \ldots$

d) 100, 10, 1, 0.1, ...

e) $5, -10, 15, -20, \ldots$

f) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

g) $x, 3x, 5x, 7x, \ldots$

h) 4, 8, 12, 16, ...

i) $a, ar, ar^2, ar^3, \ldots$

j) $0.2, -0.4, 0.6, -0.8, \ldots$

4. For each sequence, make a table of values using the term number and term and calculate the finite differences. Then, determine an explicit formula in function notation and specify the domain.

a) 2, 4, 6, 8, ...

b) $2, 1, 0, -1, \ldots$

c) 3, 6, 9, 12, ...

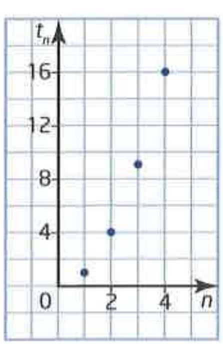
d) 0, 3, 8, 15, ...

e) 3, 6, 11, 18, ...

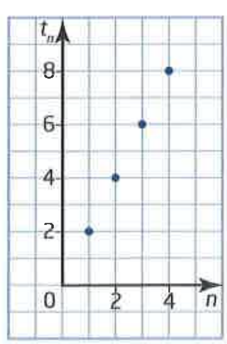
f) $-10, -9, 0, 17, \ldots$

5. The graphs show the terms in a sequence. Write each sequence in function notation and specify the domain.

a)



b)



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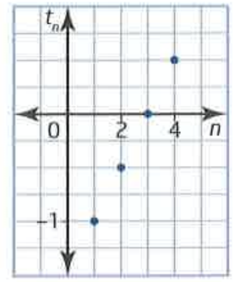
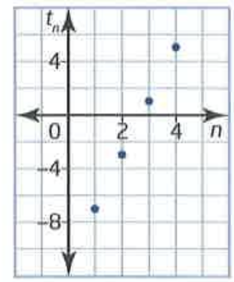


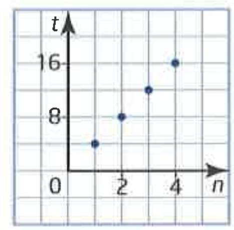
d)



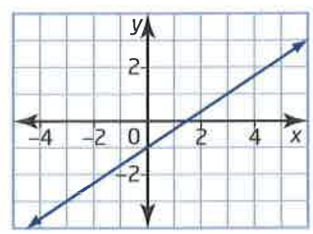
For help with question 6, refer to Example 3.

6. For each graph, specify whether the function is discrete or continuous and explain your choice.

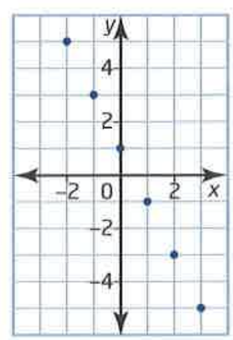
a)



b)



c)



B Connect and Apply

7. Describe the pattern in each sequence and write the next three terms.

a) 1, 1, 1, 2, 1, 3, 1, 4, 1, ...

b) 1, 5, 2, 10, 3, 15, ...

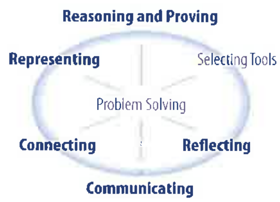
c) $3, 3\sqrt{5}, 15, 15\sqrt{5}, 75, \ldots$

d) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

8. Consider the sequence 7, 14, 21, 28, ... Determine whether or not each of the following numbers is part of this sequence. Explain your thinking.

a) 98 **b)** 110 **c)** 378 **d)** 575

9. Use Technology



The world population in 1995 was 5.7 billion. Since then the growth rate has been approximately 1.2% per year.

a) Graph the equation $y = 5.7(1.012)^x$ using a graphing calculator with the window settings shown.

```
WINDOW
 Xmin=0
 Xmax=50
 Xscl=5
 Ymin=5
 Ymax=10
 Yscl=.5
 Xres=1
```

b) Describe the shape of the graph. How would the graph change if the growth rate were greater? Use the graphing calculator to verify your description.

c) Assume the trend continues. Determine the population in each year from 2007 to 2015. Write these numbers as a sequence.

10. Use Technology A new car valued at \$35 000 will depreciate at an average rate of 20% per year over the next several years.

a) Enter the following information in a spreadsheet.

	A	B
1	Year	Value
2	0	35000
3	=A2+1	=0.8*B2

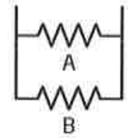
b) Use **Fill Down** to calculate the value of the car for the next 15 years.

c) Make an **XY (Scatter)** plot of this data.

d) Use function notation to write an explicit formula to represent the value of the car at the end of year n.

e) Is this a continuous or a discrete function? Explain your thinking.

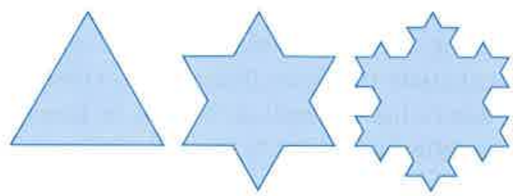
11. If two resistors, A and B, are in parallel, then the combined resistance, R, in ohms (Ω), is found by the formula



$$\frac{1}{R} = \frac{1}{R_A} + \frac{1}{R_B}$$

Assume each resistor has a resistance of 1 Ω. Use the formula to determine the value of the resistance for 2, 3, 4, 5, and 6 resistors in parallel. Write these numbers as a sequence.

12. Chapter Problem The Koch snowflake was one of the earliest fractals to be described. The snowflake starts as an equilateral triangle. At each stage, the middle third of each side is replaced by two line segments, each equal in length to the line segment they replace.



Stage 1 Stage 2 Stage 3

a) Work with a partner and use isometric dot paper to draw the diagrams shown. Use the pattern to draw the next diagram.

b) Copy and complete the table.

Stage Number	Line Segment Length	Number of Line Segments	Perimeter of the Snowflake
1	1	3	3
2	$\frac{1}{3}$	12	4
3	$\frac{1}{9}$		
4			
5			
6			

c) Determine an explicit formula for the nth term in columns two, three, and four of the table.

d) Use your formulas to calculate the values for stage 24.

13. Determine an explicit formula for the nth term of each sequence. Use the formula to write the 15th term.

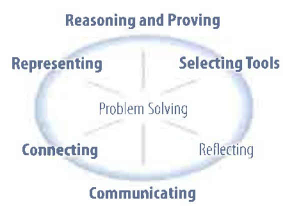
a) $-4, 8, -16, 32, \ldots$ **b)** $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots$

c) $1, \sqrt{2}, \sqrt{3}, 2, \ldots$ **d)** $1, 2, 4, 8, \ldots$

e) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ **f)** $1, -1, 1, -1, \ldots$

14. Create two different sequences that start with 1, 2, 3. Write an explicit formula for the nth term of each sequence in function notation. Graph each sequence.

15. A new small business plans to double its sales every day for its first 2 weeks. Sales on the first day are $50.



a) Write the sequence that represents the sales for the first 6 days according to the plan.

b) Write an explicit formula to determine the sales on any of the first 14 days.

c) Use your formula to determine the sales on the 14th day. Is this reasonable? Why or why not?

16. A high school is experiencing declining enrolment. This year the enrolment was 2100, and it has been predicted that every year there will be 110 fewer students. Write an explicit formula to determine the number of students in any given year. After how long will the enrolment drop below 800 students?

C Extend

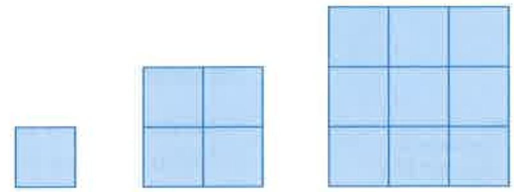
17. a) Use a calculator to determine approximate values for the first three terms of the sequence

$$\sqrt{3}, \sqrt{\sqrt{3}}, \sqrt{\sqrt{\sqrt{3}}}, \sqrt{\sqrt{\sqrt{\sqrt{3}}}}, \ldots$$

b) Describe the pattern in the sequence.

c) Use the pattern to predict the value of the 50th term in this sequence.

18. Determine an explicit formula for the total number of squares in an n by n square.



19. Math Contest The multiples of 5 are printed in the columns of a Bingo game card as shown.

B	I	N	G	O
5	10	15	20	
	40	35	30	25
45	50	55	60	
	80	75	70	65
85	90	95	100	

If the numbers continue in this pattern, in which column will the number 5555 occur?

A B **B** N **C** G **D** O

20. Math Contest The crown jewels are missing. Scotland Yard has four suspects, Albert, Bob, Cecilia, and Dwight. Albert says, "Cecilia is the thief." Bob says, "I am not the thief." Cecilia says, "Dwight is the thief." Dwight says, "Cecilia lied." If only one of these statements is true, who is the real thief?

A Albert **B** Bob **C** Cecilia **D** Dwight

21. Math Contest A bag contains two balls. The value of one ball is 4. The value of the other is 9. A ball is chosen and the value of the ball is added to a running total. The number of different sums that are not possible to attain is

A 6

B 12

C 11

D Not possible to determine

Use Technology

Use a TI-Nspire™ CAS Graphing Calculator to Write Terms in a Sequence

Tools

- TI-Nspire™ CAS graphing calculator

Connections

Example 1 on page 356 is used to model the steps needed to find the first three terms of a sequence, given the explicit formula for the *n*th term of the sequence.

Technology Tip

To type an underscore (_), use the symbol palette.

- Press ctrl 📖.
- Use the cursor keys to move to _.
- Press enter.

Write the first three terms of each sequence, given the explicit formula for the *n*th term of the sequence, $n \in \mathbb{N}$.

A: $t_n = 3n^2 - 1$ B: $t_n = \dfrac{n-1}{n}$

Solution

Open a new document. Open a page using the **Lists & Spreadsheet** application.

Enter the headings for columns A, B, and C.

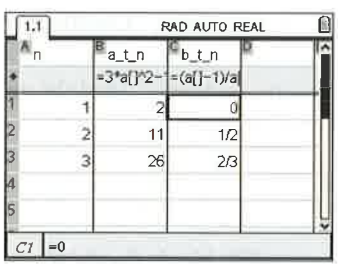
- At the top of column A, type n and press enter.
- At the top of column B, type a_t_n and press enter.
- At the top of column C, type b_t_n and press enter.

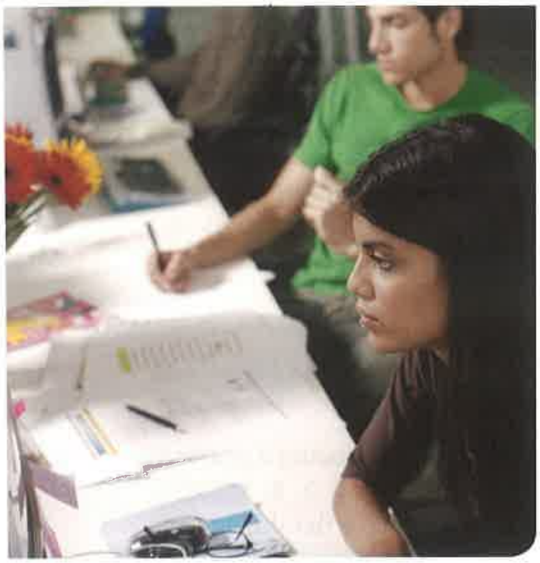
Enter the values 1, 2, and 3 for n, starting in cell A1.

Enter the explicit formulas for the sequences.

- In the formula cell for column B, type $= 3a^2 - 1$ and press enter.
- In the formula cell for column C, type $= (a - 1) \div a$ and press enter.

The first three terms for each sequence will be displayed.





Recursive Procedures

In Section 6.1, you used function notation to write an explicit formula to determine the value of any term in a sequence. Sometimes it is easier to calculate one term in a sequence using the previous terms.

Computer programmers use sequences of code to create instructions for computers. Often these sequences tell the computer to use a previous value to find the next one. This is known as a recursive procedure.

Investigate

Tools

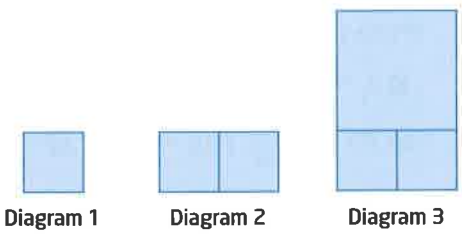
- grid paper

Optional

- computer with *The Geometer's Sketchpad®*

How can you model the relationship between consecutive terms of a sequence?

The first three diagrams in a pattern are shown. Model the pattern.



1. a) For Diagram 1, draw a square with side length 1 unit.

b) For Diagram 2, start with Diagram 1 and draw a square with side length 1 unit adjacent to the first square. This creates a rectangle.

c) For Diagram 3, draw a larger rectangle. Start with Diagram 2 and draw a square with side length 2 units adjacent to and directly above the two smaller squares.

2. Diagram 4 will contain a square with side length 3 units. Where should this square be drawn in order to continue the pattern? Draw Diagram 4.

3. Copy and complete the table for the pattern of diagrams.

Diagram Number	Side Length of Square (units)
1	1
2	1
3	2
4	
5	
6	

4. a) Write the side lengths of the squares as a sequence.

b) Reflect Determine the relationship between consecutive terms in the sequence and write a formula for the nth term, t_n, in terms of the $(n - 1)$th term, t_{n-1}, and the $(n - 2)$th term, t_{n-2}.

The sequence formed by the side lengths of the squares in the Investigate is a famous sequence known as the **Fibonacci sequence**.

A sequence is said to be recursive if a new term is found using a previous term or terms. For example, the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ... is a recursive sequence because each term beginning with the third term is the result of adding the two previous terms. The **recursion formula** for this sequence can be written as $t_1 = 1$, $t_2 = 1$, $t_n = t_{n-1} + t_{n-2}$, where t_1 is the first term, t_2 is the second term, t_n is the nth term, and so on.

Fibonacci sequence

- the sequence of numbers 1, 1, 2, 3, 5, 8, ...
- Each number, after the first two numbers, is the sum of the preceding two numbers.

recursion formula

- a formula by which each term of a sequence is generated from the preceding term or terms

Connections

The Fibonacci sequence is named after Leonardo Fibonacci (c. 1175–1250), who discovered the sequence while studying the reproductive nature of rabbits. Fibonacci is also credited with introducing the decimal number system to Europe.
Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 6 to learn more about the Fibonacci sequence.

Example 1

Write the Terms of a Sequence Given the Recursion Formula

Write the first four terms of each sequence.

a) $t_1 = 3$, $t_n = t_{n-1} - 2$

b) $f(1) = -\frac{1}{2}$, $f(n) = f(n - 1) + \frac{3}{2}$

Solution

a) The first term is given as 3. Use the equation $t_n = t_{n-1} - 2$ to determine the next three terms in the sequence.

$$\begin{aligned} t_2 &= t_1 - 2 \\ &= 3 - 2 \\ &= 1 \end{aligned} \qquad \begin{aligned} t_3 &= t_2 - 2 \\ &= 1 - 2 \\ &= -1 \end{aligned} \qquad \begin{aligned} t_4 &= t_3 - 2 \\ &= -1 - 2 \\ &= -3 \end{aligned}$$

The first four terms of the sequence are 3, 1, −1, −3.

b) The first term is given as $-\frac{1}{2}$. Use the equation $f(n) = f(n - 1) + \frac{3}{2}$ to determine the next three terms in the sequence.

$$\begin{aligned} f(2) &= f(1) + \frac{3}{2} \\ &= -\frac{1}{2} + \frac{3}{2} \\ &= 1 \end{aligned} \qquad \begin{aligned} f(3) &= f(2) + \frac{3}{2} \\ &= 1 + \frac{3}{2} \\ &= \frac{5}{2} \end{aligned} \qquad \begin{aligned} f(4) &= f(3) + \frac{3}{2} \\ &= \frac{5}{2} + \frac{3}{2} \\ &= 4 \end{aligned}$$

The first four terms of the sequence are $-\frac{1}{2}$, 1, $\frac{5}{2}$, 4.

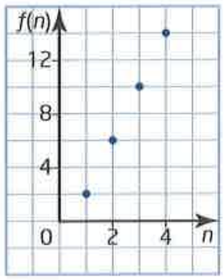
Example 2

Write a Recursion Formula

Determine a recursion formula for each sequence.

a) $-3, 6, -12, 24, \ldots$

b)



c) $3, 5, 8, 12, \ldots$

Solution

a) Look for a pattern in the terms.

$t_1 = -3$
$t_2 = t_1 \times (-2)$
$t_3 = t_2 \times (-2)$
$t_4 = t_3 \times (-2)$

The recursion formula is $t_1 = -3$, $t_n = -2t_{n-1}$.

b) Look for a pattern in the y-coordinates.

$f(1) = 2$
$f(2) = 6$
$f(3) = 10$
$f(4) = 14$

Each term is 4 more than the previous term. The recursion formula is $f(1) = 2$, $f(n) = f(n - 1) + 4$.

c) Look for a pattern in the terms.

$t_1 = 3$
$t_2 = t_1 + 2$
$t_3 = t_2 + 3$
$t_4 = t_3 + 4$

The recursion formula is $t_1 = 3$, $t_n = t_{n-1} + n$.

Example 3

Medication in the Human Body

A runner injures her knee in a race. Her doctor prescribes physiotherapy along with 500 mg of an anti-inflammatory medicine every 4 h for 3 days.

The half-life of the anti-inflammatory medicine is approximately 4 h. This means that after 4 h, about half of the medicine is still in the body.

a) Make a table of values showing the amount of medicine remaining in the body after each 4-h period of time.

b) Write the amount of medicine remaining after each 4-h period as a sequence. Write a recursion formula for the sequence.

c) Graph the sequence.

d) Describe what happens to the medicine in the runner's body over time.

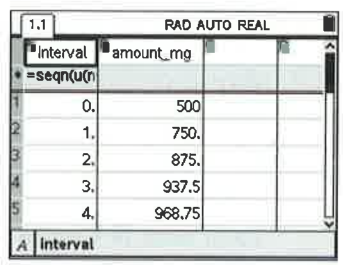
Solution

Technology Tip

To calculate the amount of medicine in the body using a spreadsheet, insert the formula =0.5*B2+500, and then fill down.

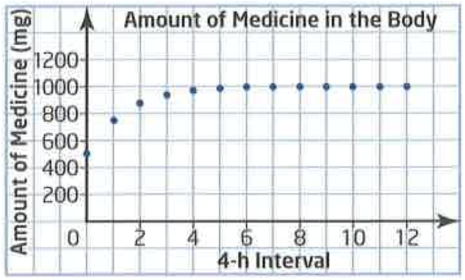
a) This can be done using pencil and paper or a spreadsheet. Round your answers to the nearest tenth of a milligram, where necessary.

4-h Interval	Amount of Medicine (mg)
0	500
1	$\frac{1}{2}(500) + 500 = 750$
2	$\frac{1}{2}(750) + 500 = 875$
3	937.5
4	968.8
5	984.4
6	992.2
7	996.1
8	998.0
9	999.0
10	999.5
11	999.8
12	999.9



b) The sequence representing the amount of medicine remaining in the body after each 4-h period is 500, 750, 875, 937.5, 968.8, ..., 999.9. The recursion formula is $t_1 = 500$, $t_n = 500 + 0.5t_{n-1}$.

c)



d) From the graph, the amount of medicine in the body increases until it appears to reach a constant level of about 1000 mg.

Key Concepts

- A recursive procedure is one where a process is performed on an initial object or number and then the result is put through the steps of the process again. This is repeated many times over.
- A sequence can be defined recursively if each term can be calculated from the previous term or terms.
- A recursion formula shows the relationship between the terms of a sequence.
- A sequence can be represented by a pattern, an explicit formula, or a recursion formula. Formulas can also be written using function notation.

  For example:

  Pattern: 1, 3, 5, 7, ...

  Explicit formula: $t_n = 2n - 1$ or $f(n) = 2n - 1$

  Recursion formula: $t_1 = 1$, $t_n = t_{n-1} + 2$, or $f(1) = 1$, $f(n) = f(n - 1) + 2$
- In an explicit or a recursion formula for a sequence, n is a natural number because it is a term number. To find the terms of a sequence using a recursion formula, begin with the next natural number that is not used in the formula.

Communicate Your Understanding

C1 What do you need to know about a sequence in order to write a recursion formula to describe the terms in the sequence?

C2 **a)** The recursion formula $t_1 = 5$, $t_n = 2t_{n-1} + 1$, has two parts. Describe the two parts.

b) Why does a recursion formula have at least two parts?

c) What characteristic of a sequence is needed for the formula to have more than two parts?

C3 A sequence has the recursion formula $t_1 = 4$, $t_n = -2t_{n-1} + 5$. Use words to describe how this formula is used to determine consecutive terms in the sequence.

C4 The explicit formula for the nth term of a sequence is $t_n = 6(2^{n-1}) - 1$, while the recursion formula for the same sequence is $t_1 = 5$, $t_n = 2t_{n-1} + 1$. When might it be more convenient to use one form of the formula instead of the other? Explain.

A Practise

For help with questions 1 and 2, refer to Example 1.

1. Write the first four terms of each sequence, where $n \in \mathbb{N}$.
   a) $t_1 = 4$, $t_n = t_{n-1} + 3$
   b) $t_1 = 7$, $t_n = 2t_{n-1} - 1$
   c) $t_1 = -3$, $t_n = 0.2t_{n-1} - 1.2$
   d) $t_1 = 50$, $t_n = \dfrac{t_{n-1}}{2}$
   e) $t_1 = 8$, $t_n = 2n - 3t_{n-1}$
   f) $t_1 = 100$, $t_n = \dfrac{5t_{n-1}}{0.1}$

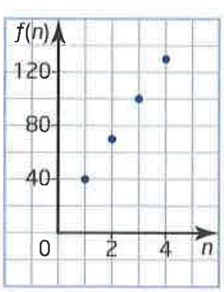
2. Write the first four terms of each sequence, where $n \in \mathbb{N}$.
   a) $f(1) = 9$, $f(n) = f(n-1) - 2$
   b) $f(1) = -1$, $f(n) = -3f(n-1)$
   c) $f(1) = 3$, $f(n) = \dfrac{f(n-1)}{n}$
   d) $f(1) = 18$, $f(n) = f(n-1) + 2$
   e) $f(1) = 0.5$, $f(n) = -f(n-1)$
   f) $f(1) = 25$, $f(n) = -0.5f(n-1)$

For help with questions 3 and 4, refer to Example 2.

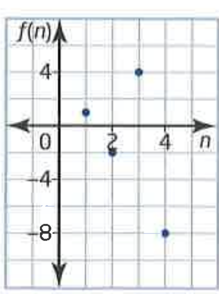
3. Determine a recursion formula for each sequence.
   a) 5, 11, 17, 23, ...
   b) 4, 1, −2, −5, ...
   c) 4, 8, 16, 32, ...
   d) $-4, -2, -1, -\frac{1}{2}, \ldots$
   e) −5, 15, −45, 135, ...

4. For each graph, write the sequence of terms and determine a recursion formula using function notation.

a)



b)



5. An example of a constant sequence is 206, 206, 206, ... Write a recursion formula for this sequence. Write another constant sequence and its recursion formula.

B Connect and Apply

For help with questions 6 and 7, refer to Example 3.

6. A new theatre is being built for a youth orchestra. This theatre has 50 seats in the first row, 54 in the second row, 62 in the third row, 74 in the next row, and so on.
   a) Represent the number of seats in the rows as a sequence.
   b) Describe the pattern in the number of seats per row.
   c) Write a recursion formula to represent the number of seats in any row.

7. Sacha and Marghala paid \$250 000 for their first home. The real estate agent told them that the house will appreciate in value by 3% per year.
   a) Copy and complete the table to show the value of the house for the next 10 years.

Year	House Value (\$)
0	250 000
1	250 000 + 0.03 × 250 000 = 257 500

   b) Write the value of the house for the first 10 years as a sequence.
   c) Write a recursion formula to represent the value of the house. Use your formula to predict the value after 15 years.

8. Write the first four terms of each sequence.
   a) $t_1 = 1$, $t_n = (t_{n-1})^2 + 3n$
   b) $f(1) = 8$, $f(n) = \dfrac{f(n-1)}{2}$
   c) $t_1 = 3$, $t_n = 2t_{n-1}$
   d) $t_1 = -5$, $t_n = 4 - 2t_{n-1}$
   e) $t_1 = \frac{1}{2}$, $t_n = 4t_{n-1} + 2$
   f) $f(1) = a + 3b$, $f(n) = f(n-1) + 4b$

9. Use the given recursion formula to determine the first four terms of each sequence. Then, use words to describe the rule for determining terms in the sequence.

a) $f(1) = 2, f(2) = 2,$
$f(n) = f(n - 1) + 2f(n - 2)$

b) $f(1) = 1, f(2) = 2, f(n) = f(n - 1)f(n - 2)$

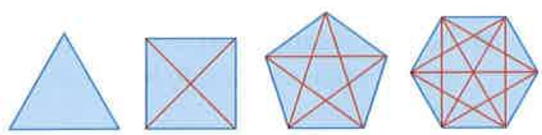
c) $t_1 = 5, t_2 = 7, t_n = t_{n-2} - t_{n-1}$

d) $t_1 = -2, t_2 = 3, t_n = 3t_{n-2} + t_{n-1}$

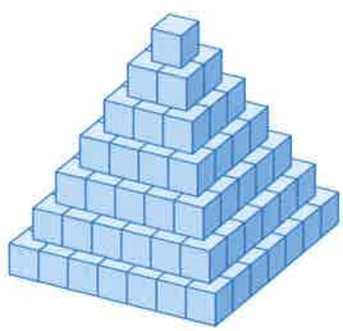
e) $t_1 = 1, t_2 = -4, t_n = t_{n-2} \times t_{n-1}$

f) $t_1 = 3, t_2 = 1, t_3 = 7,$
$t_n = t_{n-3} + t_{n-2} - t_{n-1}$

10. The diagrams show the diagonals in regular polygons with n sides. Write the sequence for the number of diagonals and determine the recursion formula for this sequence.



11. A square-based pyramid with height 7 m is constructed with cubic blocks measuring 1 m on each side.

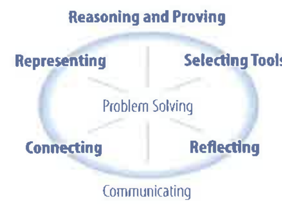


Write a recursion formula for the sequence that represents the number of blocks used at each level from the top down.

12. A sequence has a first term of −8. Each succeeding term is 4 more than twice the previous term.

a) Write the first four terms of this sequence.

b) Define the sequence using a recursion formula and then graph the sequence.



13. Given the explicit formula of a sequence, write the first four terms and then determine a recursion formula for each sequence.

a) $t_n = (2n - 1)^2$

b) $t_n = \dfrac{n^2 + 1}{n}$

c) $f(n) = 3^{-n}$

d) $t_n = 3n + 1$

e) $f(n) = (n - 2)(n + 2)$

f) $f(n) = 2(4)^{n-1}$

14. Given the recursion formula, write the first four terms of the sequence and then determine the explicit formula for the sequence.

a) $t_1 = 3, t_n = 2t_{n-1} + 1$

b) $t_1 = 1, t_n = \frac{1}{2}t_{n-1}$

c) $t_1 = 10, t_n = t_{n-1} - 10$

d) $t_1 = -2, t_n = t_{n-1} - \dfrac{1}{n(n - 1)}$

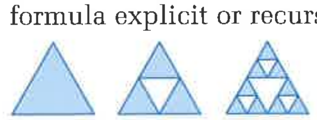
15. **Chapter Problem** The Sierpinski triangle was described by Waclaw Sierpinski in 1915. This now famous fractal was also seen in Italian art in the 13th century.

a) Use isometric dot paper or *The Geometer's Sketchpad®* to construct a large equilateral triangle. Consider the area of this triangle to be 1 square unit.

b) Locate and join the midpoints to make a new triangle. Shade in all but the centre triangle. Determine the area of the shaded regions.

c) Continue to draw triangles formed by the midpoints of the sides of the smaller and smaller shaded triangles. Always leave the centre triangle unshaded and determine the area of the shaded regions.

d) Write the area of the shaded regions at each stage as a sequence. Write a formula for this sequence. Is your formula explicit or recursive?



Stage 1 Stage 2 Stage 3

16. Write a recursion formula for each sequence.

a) 2, 6, 12, 20, 30, ... **b)** 3, 7, 16, 32, ...

c) 2, 5, 26, 677, ... **d)** −1, 0, 3, 12, ...

Achievement Check

17. Canadian checkers is played on a checkerboard that has 12 squares per side. What would happen if you placed a penny on the first square, two pennies on the second square, four on the third square, eight on the fourth square, and so on?

a) Write the numbers of pennies on the first 12 squares as a sequence.

b) Write a recursion formula to represent the number of pennies on any square. Use your formula to determine the number of pennies on square 20.

c) Write an explicit formula, in function notation, to represent the number of pennies on any square. Use this formula to verify your answer to part b).

d) Is this function discrete or continuous? Explain.

C Extend

18. Write the first five terms of each sequence, starting at $f(1)$.

a) $f(2) = -3, f(n) = -2f(n - 1)$

b) $f(3) = 9, f(n) = f(n - 1) + n^2$

19. Create three different sequences that start with 2, 3, 4. Write recursion formulas for these sequences. Write the next two terms of each of your sequences and challenge your classmates to determine the recursive rule that you used.

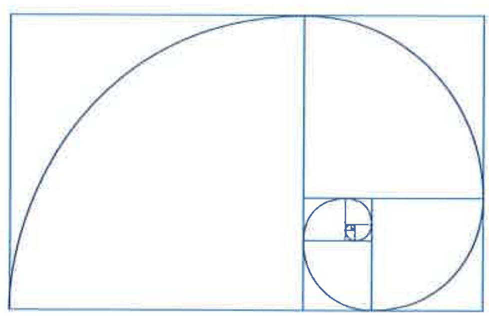
20. a) Write a new sequence using the terms of the Fibonacci sequence. Start with the first two terms of the Fibonacci sequence. Then, divide the third term in the Fibonacci sequence by the second term to get the next term in the new sequence. Continue dividing the next term in the Fibonacci sequence by the previous term to get the next term in the new sequence. Describe the pattern in the new sequence.

b) The sequence in part a) converges to a value close to 1.618. This is the number ϕ (phi), which is also known as the golden ratio. Research the golden ratio and prepare a poster for the classroom.

21. The golden spiral is constructed from the rectangle diagram you constructed in the Investigate.

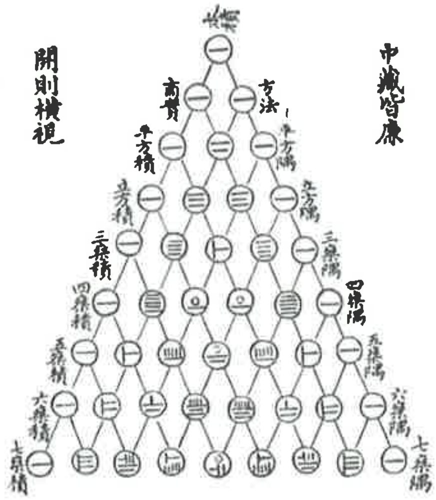
a) In each square, construct an arc with radius equal to the side length of the square.

b) Determine the area under the spiral for each square. Write the areas as a sequence of numbers. Calculate the total area under the spiral.



22. Math Contest The Lucas numbers are similar to the Fibonacci numbers but the first two terms are $t_1 = 2$ and $t_2 = 1$. The sequence is 2, 1, 3, 4, 7, 11, ... Let L_n be the nth Lucas number and F_n be the nth Fibonacci number. For $n > 2$, show that $L_n - F_n = F_{n-2}$.

6.3



Pascal's Triangle and Expanding Binomial Powers

It is widely believed that some time during the 11th century, both the Chinese and the Persians discovered an unusual array of numbers. However, the triangle representing the array of numbers was named after Blaise Pascal (1623–1662), a French mathematician who lived and worked in the mid-1600s. Pascal is credited with the discoveries of many of the triangle's special properties and applications, as well as with many other important contributions to the field of mathematics.

The triangular arrangement of numbers known as **Pascal's triangle** can be built using a recursive procedure. Each term in Pascal's triangle is the sum of the two terms immediately above it. The first and last terms in each row are 1 since the only term immediately above them is always a 1.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Pascal's triangle

- a triangular arrangement of numbers with 1 in the first row, and 1 and 1 in the second row
- Each number in the succeeding rows is the sum of the two numbers above it in the preceding row.

Investigate

How can you use patterns to expand a power of a binomial?

1. Expand each power of a binomial using the distributive property and simplifying or by using a CAS.

a) $(a + b)^1$

b) $(a + b)^2$

c) $(a + b)^3$

d) $(a + b)^4$

2. Reflect Examine the pattern in the coefficients of the terms in each expansion. Describe how the pattern relates to Pascal's triangle.

3. Reflect Study the variables in the terms of each expansion. Describe how the degree of each term relates to the power of the binomial.

4. Predict the terms in the expansion of $(a + b)^5$.

Tools

Optional

- Computer Algebra System (CAS)

Technology Tip

You can use the CAS engine of a TI-Nspire™ CAS graphing calculator to expand a power of a binomial.

- Open a new calculator page.
- Press (menu), select **3:Algebra**, and then select **3:Expand**.
- Enter the power of a binomial. For example, type $(a + b)^3$ and press (enter).

The expansion will be displayed.

Example 1

Patterns in Pascal's Triangle

a) Write the first seven rows of Pascal's triangle and label the rows.

b) The powers of 2 can be found by looking for a pattern in the triangle. Find the pattern.

Solution

a)

row 0							1						
row 1						1		1					
row 2					1		2		1				
row 3				1		3		3		1			
row 4			1		4		6		4		1		
row 5		1		5		10		10		5		1	
row 6	1		6		15		20		15		6		1

b) If the terms in each row are added, the sequence formed is the powers of 2.

Row 0 sum: $1 = 2^0$
Row 1 sum: $2 = 2^1$
Row 2 sum: $4 = 2^2$
Row 3 sum: $8 = 2^3$
and so on.

Example 2

Position of Terms in Pascal's Triangle

A term in Pascal's triangle can be represented by $t_{n,r}$, where n is the horizontal row number and r is the diagonal row number.

$n = 0$ 1 $r = 0$
$n = 1$ 1 1 $r = 1$
$n = 2$ 1 2 1 $r = 2$
$n = 3$ 1 3 3 1 $r = 3$
$n = 4$ 1 4 6 4 1 $r = 4$
$n = 5$ 1 5 10 10 5 1 $r = 5$
$n = 6$ 1 6 15 20 15 6 1 $r = 6$

$t_{0,0}$
$t_{1,0}$ $t_{1,1}$
$t_{2,0}$ $t_{2,1}$ $t_{2,2}$
$t_{3,0}$ $t_{3,1}$ $t_{3,2}$ $t_{3,3}$
$t_{4,0}$ $t_{4,1}$ $t_{4,2}$ $t_{4,3}$ $t_{4,4}$
$t_{5,0}$ $t_{5,1}$ $t_{5,2}$ $t_{5,3}$ $t_{5,4}$ $t_{5,5}$
$t_{6,0}$ $t_{6,1}$ $t_{6,2}$ $t_{6,3}$ $t_{6,4}$ $t_{6,5}$ $t_{6,6}$

Each term is equal to the sum of the two terms immediately above it, which can be represented as $t_{n,r} = t_{n-1,r-1} + t_{n-1,r}$.
Express $t_{5,3} + t_{5,4}$ as a single term from Pascal's triangle in the form $t_{n,r}$.

Solution

Any term in Pascal's triangle is the sum of the terms immediately above it.

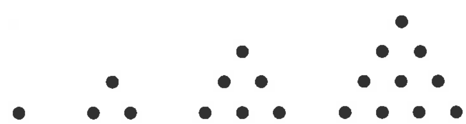
$t_{5,3} + t_{5,4} = t_{6,4}$

$$
\begin{array}{ccccccccccccc}
 & & & & & & t_{0,0} & & & & & & \\
 & & & & & t_{1,0} & & t_{1,1} & & & & & \\
 & & & & t_{2,0} & & t_{2,1} & & t_{2,2} & & & & \\
 & & & t_{3,0} & & t_{3,1} & & t_{3,2} & & t_{3,3} & & & \\
 & & t_{4,0} & & t_{4,1} & & t_{4,2} & & t_{4,3} & & t_{4,4} & & \\
 & t_{5,0} & & t_{5,1} & & t_{5,2} & & t_{5,3} & & t_{5,4} & & t_{5,5} & \\
t_{6,0} & & t_{6,1} & & t_{6,2} & & t_{6,3} & & t_{6,4} & & t_{6,5} & & t_{6,6}
\end{array}
$$

Example 3

Relate Other Patterns to Pascal's Triangle

The diagrams represent the triangular numbers.



a) Write the number of dots in each diagram as a sequence.

b) Locate these numbers in Pascal's triangle. Describe their position.

c) Write an explicit formula and a recursion formula for the triangular numbers.

Solution

a) The sequence of triangular numbers is 1, 3, 6, 10, ...

b) The numbers are located in diagonal row 2 of Pascal's triangle.

c) Calculate the finite differences.

Term Number, n	Term, $f(n)$	First Differences	Second Differences
1	1		
		2	
2	3		1
		3	
3	6		1
		4	
4	10		

Since the second differences are constant, this function is quadratic. Half the value of the second difference corresponds to the value of a in a quadratic function of the form $f(n) = an^2 + bn + c$.

$f(n) = \frac{1}{2}n^2 + bn + c$

To determine the values of b and c, substitute the coordinates of two points, say (1, 1) and (2, 3), and solve the linear system of equations.

$$\frac{1}{2} = b + c$$

$$1 = 2b + c$$

This gives $b = \frac{1}{2}$ and $c = 0$.

The explicit formula for the nth term is $f(n) = \frac{1}{2}n^2 + \frac{1}{2}n$.

The recursion formula is $f(1) = 1$, $f(n) = f(n - 1) + n$.

Connections

If you take Mathematics of Data Management in grade 12, you will see how Pascal's triangle and expansions of $(a + b)^n$ are connected to probability.

Powers of binomials can be expanded by using patterns. The coefficients in the expansion of $(a + b)^n$ can be found in row n of Pascal's triangle.

Value of n	$(a + b)^n$
0	$(a + b)^0 = 1$
1	$(a + b)^1 = a + b$
2	$(a + b)^2 = a^2 + 2ab + b^2$
3	$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
4	$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

There is also a pattern in the powers of a and b. In each expansion, the power of a decreases, the power of b increases, and the degree of each term is always equal to the exponent of the binomial power.

Example 4

Expand a Power of a Binomial

Use Pascal's triangle to expand each power of a binomial.

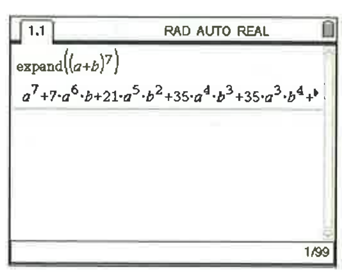
a) $(a + b)^7$ **b)** $(m - n)^5$ **c)** $(2x + 1)^6$ **d)** $\left(\frac{y}{2} - y^2\right)^4$

Solution

a) Since the exponent is 7, the coefficients occur in row 7 of Pascal's triangle. The powers of a will decrease and the powers of b will increase.

$$\begin{aligned}(a + b)^7 &= 1a^7b^0 + 7a^6b^1 + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 \\ &\quad + 21a^2b^5 + 7a^1b^6 + 1a^0b^7 \\ &= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7\end{aligned}$$

Note that you can use the CAS engine of a TI-Nspire™ CAS graphing calculator to check your answer.



b) The coefficients occur in row 5 of Pascal's triangle. Let $a = m$ and $b = -n$ and apply the pattern of powers.

$$\begin{aligned}(m-n)^5 &= 1(m)^5(-n)^0 + 5(m)^4(-n)^1 + 10(m)^3(-n)^2 + 10(m)^2(-n)^3 \\ &\quad + 5(m)^1(-n)^4 + 1(m)^0(-n)^5 \\ &= m^5 - 5m^4n + 10m^3n^2 - 10m^2n^3 + 5mn^4 - n^5\end{aligned}$$

c) The coefficients occur in row 6 of Pascal's triangle. Let $a = 2x$ and $b = 1$ and apply the pattern of powers.

$$\begin{aligned}(2x+1)^6 &= 1(2x)^6(1)^0 + 6(2x)^5(1)^1 + 15(2x)^4(1)^2 + 20(2x)^3(1)^3 \\ &\quad + 15(2x)^2(1)^4 + 6(2x)^1(1)^5 + 1(2x)^0(1)^6 \\ &= 64x^6 + 192x^5 + 240x^4 + 160x^3 + 60x^2 + 12x + 1\end{aligned}$$

d) The coefficients occur in row 4 of Pascal's triangle. Let $a = \frac{y}{2}$ and $b = -y^2$ and apply the pattern of powers.

$$\begin{aligned}\left(\frac{y}{2} - y^2\right)^4 &= 1\left(\frac{y}{2}\right)^4(-y^2)^0 + 4\left(\frac{y}{2}\right)^3(-y^2)^1 + 6\left(\frac{y}{2}\right)^2(-y^2)^2 + 4\left(\frac{y}{2}\right)^1(-y^2)^3 \\ &\quad + 1\left(\frac{y}{2}\right)^0(-y^2)^4 \\ &= \frac{y^4}{16} - \frac{y^5}{2} + \frac{3y^6}{2} - 2y^7 + y^8\end{aligned}$$

Key Concepts

- Pascal's triangle is a triangular array of natural numbers in which the entries can be obtained by adding the two entries immediately above.

  $t_{n,\,r} = t_{n-1,\,r-1} + t_{n-1,\,r}$, where n is the horizontal row number and r is the diagonal row number; $n, r \in \mathbb{N}$ and $r \leq n$
- Many number patterns can be found in Pascal's triangle. For example, the sums of the terms of the rows form a sequence of the powers of 2 and the terms in diagonal row 2 are triangular numbers.
- The coefficients of the terms in the expansion of $(a + b)^n$ correspond to the terms in row n of Pascal's triangle.

Communicate Your Understanding

C1 Look at Pascal's triangle. What is the value of $t_{6,\,3}$? Does $t_{3,\,6}$ have the same value?

C2 Describe how to determine a term in Pascal's triangle if you know the row and diagonal row numbers.

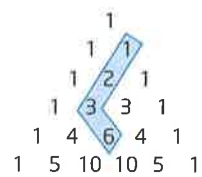
C3 Explore Pascal's triangle for other patterns. Write sequences to represent the patterns you found and describe how to determine the terms in these sequences.

C4 Describe how you would use Pascal's triangle to expand $(a + b)^8$.

A Practise

For help with questions 1 and 2, refer to Example 1.

1. The hockey stick pattern is one of many found in Pascal's triangle. Start on any of the 1s along the side. Select any number of entries along a diagonal to this 1, ending inside the triangle. Determine the sum of these numbers. Turn at the bottom, as shown in the example. How is the number not on the diagonal related to the sum? On a copy of Pascal's triangle, outline five hockey stick patterns of your own.



2. Determine the sum of the terms in each row of Pascal's triangle.

a) row 8 **b)** row 12
c) row 20 **d)** row n

For help with questions 3 and 4, refer to Example 2.

3. Express as a single term from Pascal's triangle in the form $t_{n,\,r}$.

a) $t_{4,\,3} + t_{4,\,4}$ **b)** $t_{8,\,5} + t_{8,\,6}$
c) $t_{25,\,17} + t_{25,\,18}$ **d)** $t_{a,\,b} + t_{a,\,b+1}$

4. Write each as the sum of two terms, each in the form $t_{n,\,r}$.

a) $t_{4,\,2}$ **b)** $t_{12,\,9}$ **c)** $t_{28,\,14}$ **d)** $t_{17,\,x}$

For help with questions 5 to 7, refer to Example 3.

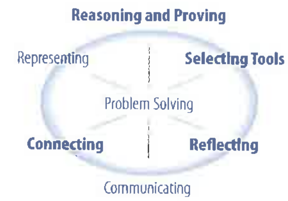
5. Use Pascal's triangle to expand each power of a binomial.

a) $(x + 2)^5$ **b)** $(y - 3)^4$ **c)** $(4 + t)^6$
d) $(1 - m)^5$ **e)** $(2x - 3y)^4$ **f)** $(a^2 + 4)^5$

6. How many terms are in each expansion?

a) $(3a + 5)^0$ **b)** $(x + 2)^{25}$
c) $(t - 6)^{15}$ **d)** $(5b + 6a)^n$

7. Use patterns in the terms of the expansion to determine the value of k in each term of $(x + y)^{12}$.



a) ky^{12} **b)** $792x^ky^5$
c) $495x^8y^k$ **d)** kx^4y^8

B Connect and Apply

8. What row number of Pascal's triangle has each row sum?

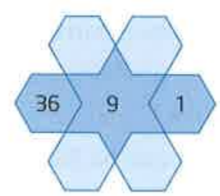
a) 256 **b)** 2048
c) 16 384 **d)** 65 536

9. Write each as the difference of two terms in the form $t_{n,\,r}$.

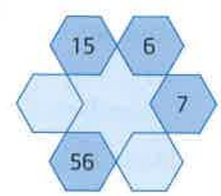
a) $t_{4,\,2}$ **b)** $t_{6,\,3}$ **c)** $t_{12,\,9}$ **d)** $t_{28,\,14}$

10. Look for patterns in Pascal's triangle. What are the missing numbers in each diagram?

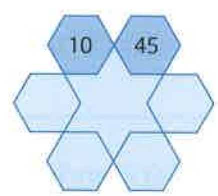
a)



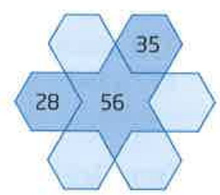
b)



c)

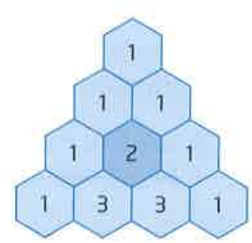


d)



11. Chapter Problem
You can find fractal qualities in Pascal's triangle.

Use a copy of Pascal's triangle and colour all even numbers one colour and all odd numbers another colour. Describe the pattern that emerges.

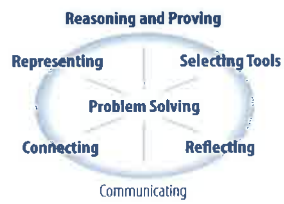


12. Find the Fibonacci sequence in Pascal's triangle. Describe the position of these numbers.

    Hint: Write Pascal's triangle as a right triangle and look diagonally.

C Extend

13. Determine the sum of the squares of the terms in the horizontal rows of Pascal's triangle. Write these numbers as a sequence and then locate the sequence in the triangle. Write a formula for the sequence.



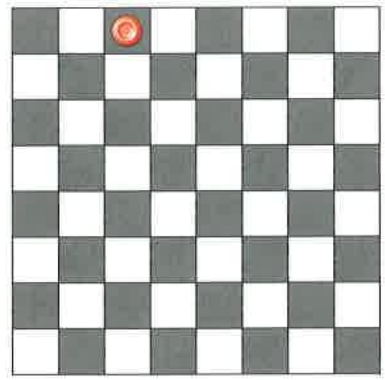
14. In the expansion of $(1 + x)^n$, the first three terms are 1, -18, and 144. Determine the values of x and n.

15. Describe the process used to generate the terms in the triangular array shown. Write this in a recursive form. Write the next three rows of this triangular array.

$$\frac{1}{1}$$
$$\frac{1}{2} \quad \frac{1}{2}$$
$$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3}$$
$$\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4}$$
$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5}$$

16. Locate a row in Pascal's triangle where the first term after the 1 is a prime number. Look for a relationship between that number and the other terms in the row. Describe this relationship. Locate another row where the first term after the 1 is prime. Do you see the same relationship? Ask a classmate which row they tried and see if they came to the same conclusion.

17. Determine the number of pathways for the checker from the top to the bottom of the checkerboard if the checker can be moved diagonally down only.



18. **Math Contest** A school hallway contains 50 lockers numbered 1 to 50. One student ensures they are closed. A second student opens every even-numbered locker. A third student changes the state of the lockers that are a multiple of 3. (To change the state of a locker means an open locker will be closed and a closed locker will be opened.) A fourth student changes the state of all lockers that are a multiple of 4. This pattern continues for 50 students. After the last student, what is the sum of the numbers on the lockers that are closed?

    A 100 **B** 765 **C** 140 **D** 50

19. **Math Contest** The value of the constant in the expansion of $\left(2x^3 - \frac{3}{x^2}\right)^5$ is

    A 0 **B** -1080 **C** 1080 **D** -243

20. **Math Contest** Every day at midnight, a ship leaves New York for London at the same time a ship leaves London for New York. It takes exactly 5 days to complete this journey. How many New York–bound ships will a London-bound ship pass on its journey?

    A 11 **B** 9 **C** 10 **D** 5

6.4

Arithmetic Sequences

The Great Pyramid of Giza, built in honour of the Egyptian pharaoh Khufu, is believed to have taken 100 000 workers about 20 years to build. Over 2.3 million stones with an average mass of approximately 2300 kg each were used. One example of a sequence that can be found in the Great Pyramid of Giza is the number of stones used to build each level of the pyramid.

Many sequences have very specific patterns. One such pattern occurs when a constant is added to each term to get the next term. This is called an **arithmetic sequence**.

arithmetic sequence

- a sequence where the difference between consecutive terms is a constant

Investigate

How can you identify an arithmetic sequence?

A wall is to be constructed along the 1-km boundary between a city park and a busy street. The wall will be built using cinder blocks measuring 20 cm in height and 40 cm in length. Each row in the wall will contain 100 fewer blocks than the previous row, and the wall will be 3.6 m in height at the centre.

Tools

- grid paper

Method 1: Use Pencil and Paper

1. a) Copy and complete the table.

Row Number	Number of Blocks in the Row	Row Length (cm)
1	2500	100 000
2	2400	96 000
3		
4		
5		

b) How many table rows would you need to determine the number of blocks in the top row of the wall? How did you determine this?

2. a) Write the numbers of blocks in the rows as a sequence.

b) Graph the sequence.

c) Write an explicit formula to represent the number of blocks in row n.

d) What is the value of n for the top row of the wall? Use the formula to determine the number of blocks in the top row of the wall.

3. a) Write the row lengths as a sequence.
   b) Graph the sequence.
   c) Write an explicit formula to determine the length of row n.
   d) Use the formula to determine the length of the top row of the wall.

4. **Reflect** The sequences from steps 2 and 3 are arithmetic sequences.
   a) Compare the graphs of the sequences. Is an arithmetic sequence a discrete or a continuous function? Explain.
   b) Compare the formulas of the sequences. Describe any similarities or differences.

Method 2: Use a Spreadsheet

Tools

- computer with spreadsheet software

or

- TI-Nspire™ CAS graphing calculator

1. a) Enter the information in the cells as shown. From the **Edit** menu, use **Fill Down** to complete the next three rows of the spreadsheet.

	A	B	C
1	Row	Number of Blocks in the Row	Row Length (cm)
2	1	2500	100000
3	=A2+1	=B2-100	=C2-4000

   Note that if you are using the **Lists & Spreadsheet** application on a TI-Nspire™ CAS graphing calculator, change the formulas to refer to cells A1, B1, and C1.

   To fill down, press (menu), select **3:Data**, and then select **3:Fill Down**. Use the cursor keys to fill the desired number of cells.

   b) How many table rows would you need to determine the number of blocks in the top row of the wall? How did you determine this?

2. a) Write the numbers of blocks in a row as a sequence.
   b) Make an **XY (Scatter)** plot of these data.
   c) Write an explicit formula to represent the number of blocks in row n.
   d) What is the value of n for the top row of the wall? Use the formula to determine the number of blocks in the top row of the wall.

3. a) Write the row lengths as a sequence.
   b) Make an **XY (Scatter)** plot of these data.
   c) Write an explicit formula to determine the length of row n.
   d) Use the formula to determine the length of the top row of the wall.

4. **Reflect** The sequences from steps 2 and 3 are arithmetic sequences.
   a) Compare their graphs. Is an arithmetic sequence a discrete or a continuous function? Explain.
   b) Compare their formulas. Describe any similarities or differences.

common difference

- the difference between any two consecutive terms in an arithmetic sequence

An arithmetic sequence can be written as a, $a + d$, $a + 2d$, $a + 3d$, ..., where a is the first term and d is the **common difference**. Then, the formula for the general term, or the nth term, of an arithmetic sequence is $t_n = a + (n - 1)d$, where $n \in \mathbb{N}$.

Example 1

Arithmetic Sequences

For each arithmetic sequence, determine the values of the first term, a, and the common difference, d.

a) $-4, 0, 4, 8, \ldots$

b) $\frac{1}{3}, \frac{5}{6}, \frac{4}{3}, \frac{11}{6}, \ldots$

c) $t_n = 2n + 3$

Solution

a) Since a is the first term of the sequence, $a = -4$.

The value of d, the common difference, is found by subtracting consecutive terms.

$$\begin{aligned} d &= t_2 - t_1 \\ &= 0 - (-4) \\ &= 4 \end{aligned}$$

Choose any two consecutive terms.

b) The first term is $a = \frac{1}{3}$. Calculate the common difference, d.

$$\begin{aligned} d &= t_2 - t_1 \\ &= \frac{5}{6} - \frac{1}{3} \\ &= \frac{5}{6} - \frac{2}{6} \\ &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

c) Use the formula $t_n = 2n + 3$ to write the first few terms.

$$\begin{aligned} t_1 &= 2(1) + 3 \\ &= 5 \end{aligned} \qquad \begin{aligned} t_2 &= 2(2) + 3 \\ &= 7 \end{aligned} \qquad \begin{aligned} t_3 &= 2(3) + 3 \\ &= 9 \end{aligned}$$

The first term is 5, so $a = 5$.
The value of d is 2.

Example 2

Determine a Formula for the General Term

Consider the sequence $-13, -19, -25, \ldots$.

a) Is this sequence arithmetic? Explain how you know.

b) Determine an explicit formula for the general term.

c) Write the value of the 15th term.

d) Determine a recursion formula for the sequence.

Solution

a) This is an arithmetic sequence. By observing the terms, you can see that the first term is -13 and that consecutive terms are decreasing by 6.

b) For this sequence, $a = -13$ and $d = -6$.

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= -13 + (n - 1)(-6) \\ &= -13 - 6n + 6 \\ &= -6n - 7 \end{aligned}$$

An explicit formula for the general term is $t_n = -6n - 7$ or, using function notation, $f(n) = -6n - 7$.

c) $$\begin{aligned} t_{15} &= -6(15) - 7 \\ &= -97 \end{aligned}$$

d) Since an arithmetic sequence can be written as $a, a + d, a + 2d, a + 3d, \ldots,$

$$\begin{aligned} t_1 &= a \\ t_2 &= a + d, \text{ or } t_2 = t_1 + d \\ t_3 &= t_2 + d \\ &\vdots \\ t_n &= t_{n-1} + d \end{aligned}$$

For the sequence $-13, -19, -25, \ldots$, the recursion formula is $t_1 = -13$, $t_n = t_{n-1} - 6$.

Example 3

Length of Ownership

Anna paid \$5000 for an antique guitar. The guitar appreciates in value by \$160 every year. If she sells the guitar for a little over \$7000, how long has she owned it?

Solution

Since the value of the guitar increases by a constant amount each year, the value at the end of each year forms an arithmetic sequence.
The first term in the sequence is 5160 since this is the value at the end of the first year.
Substitute $a = 5160$, $d = 160$, and $t_n = 7000$ into the formula for the general term of an arithmetic sequence and solve for n.

$$
\begin{aligned}
t_n &= a + (n - 1)d \\
7000 &= 5160 + (n - 1)(160) \\
7000 &= 5160 + 160n - 160 \\
2000 &= 160n \\
n &= 12.5
\end{aligned}
$$

Anna owned the guitar for 12.5 years.

Example 4

Determine *a* and *d* Given Two Terms

In an arithmetic sequence, $t_{11} = 72$ and $t_{21} = 142$. What is the value of the first term and of the common difference?

Solution

Substitute the given values into the formula for the general term, $t_n = a + (n - 1)d$, to form a system of equations. Then, solve the system for a and d.

For t_{11}, $72 = a + 10d$.

For t_{21}, $142 = a + 20d$.

$$
\begin{aligned}
72 &= a + 10d && ① \\
142 &= a + 20d && ② \\
\hline
-70 &= -10d && ① - ② \\
d &= 7
\end{aligned}
$$

Substitute $d = 7$ into equation ① and solve for a.

$$
\begin{aligned}
72 &= a + 10d \\
72 &= a + 10(7) \\
72 &= a + 70 \\
a &= 2
\end{aligned}
$$

The first term is 2 and the common difference is 7.

Key Concepts

- An arithmetic sequence is a sequence in which the difference between consecutive terms is a constant.
- The difference between consecutive terms of an arithmetic sequence is called the common difference.
- The formula for the general term of an arithmetic sequence is $t_n = a + (n - 1)d$, where a is the first term, d is the common difference, and n is the term number.

Communicate Your Understanding

C1 Compare these two sequences.
A: 1, 3, 5, 7, 9, ... B: 2, 1, 3, 2, 4, ...
Is each an arithmetic sequence? Explain your reasoning.

C2 How can the first term and the common difference be used to determine any term in an arithmetic sequence? Use a specific example to model your answer.

A Practise

For help with questions 1 to 5, refer to Examples 1 and 2.

1. For each arithmetic sequence, determine the values of a and d. Then, write the next four terms.

a) 12, 15, 18, ... **b)** 6, 4, 2, ...
c) 0.2, 0.35, 0.5, ... **d)** −30, −24, −18, ...
e) 5, −1, −7, ... **f)** $\frac{1}{2}, 1, \frac{3}{2}, \ldots$

2. State whether or not each sequence is arithmetic. Justify your answer.

a) 3, 5, 7, 9, ... **b)** 2, 5, 9, 14, ...
c) 4, −6, 8, −10, ... **d)** 13, 7, 1, −5, ...
e) −12, −5, 2, 9, ... **f)** 0, 1.5, 3, 4.5, ...

3. Given the values of a and d, write the first three terms of the arithmetic sequence. Then, write the formula for the general term.

a) $a = 5, d = 2$ **b)** $a = -2, d = -4$
c) $a = 9, d = -3.5$ **d)** $a = 0, d = -\frac{1}{2}$
e) $a = 100, d = 10$ **f)** $a = \frac{3}{4}, d = \frac{1}{2}$
g) $a = 10, d = t$ **h)** $a = x, d = 2x$

4. Given the formula for the general term of an arithmetic sequence, determine t_{12}.

a) $t_n = 3n + 4$ **b)** $f(n) = 1 - 4n$
c) $t_n = \frac{1}{2}n + \frac{3}{2}$ **d)** $f(n) = 20 - 1.5n$

5. Given the formula for the general term of an arithmetic sequence, write the first three terms. Then, graph the discrete function that represents each sequence.

a) $t_n = 2n - 3$ **b)** $f(n) = -n - 1$
c) $f(n) = 2(2 - n)$ **d)** $t_n = -2n - 5$
e) $f(n) = \frac{2n + 1}{4}$ **f)** $t_n = 0.2n + 0.1$

For help with questions 6 and 7, refer to Example 3.

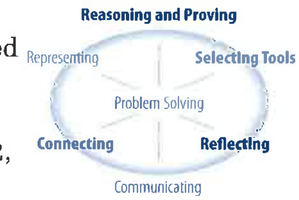
6. Which term in the arithmetic sequence 9, 4, −1, ... has the value −146?

7. Determine the number of terms in each arithmetic sequence.

a) 5, 10, 15, ..., 200
b) 38, 36, 34, ..., −20
c) −5, −8, −11, ..., −269
d) −7, −4, −1, ..., 95

B Connect and Apply

8. Verify that the sequence determined by the recursion formula $t_1 = 8$, $t_n = t_{n-1} - 2$, is arithmetic.



9. For each sequence, determine the values of a and d and write the next three terms.

a) $\frac{5}{2}, 2, \frac{3}{2}, \ldots$

b) $-6, -\frac{7}{2}, -1, \ldots$

c) $2a, 2a - b, 2a - 2b, \ldots$

For help with question 10, refer to Example 4.

10. Determine a and d and then write the formula for the nth term of each arithmetic sequence with the given terms.

a) $t_8 = 33$ and $t_{14} = 57$

b) $t_{10} = 50$ and $t_{27} = 152$

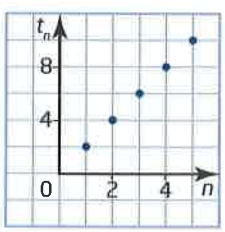
c) $t_5 = -20$ and $t_{18} = -59$

d) $t_7 = 3 + 5x$ and $t_{11} = 3 + 23x$

11. Write a recursion formula for each sequence in question 10.

12. For each graph of an arithmetic sequence, determine the formula for the general term.

a)



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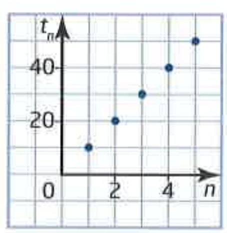

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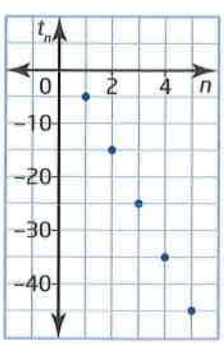
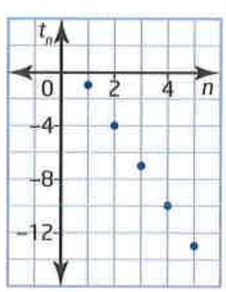


d)



13. In a lottery, the owner of the first ticket drawn receives \$10 000. Each successive winner receives \$500 less than the previous winner.

a) How much does the 10th winner receive?

b) How many winners are there in total? Explain.

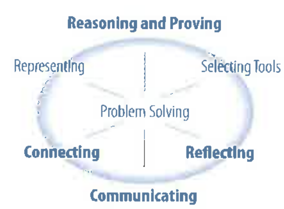
14. An engineer's starting salary is \$87 000. The company has guaranteed a raise of \$4350 every year with satisfactory performance. What will the engineer's salary be after 10 years?

15. At the end of the second week after opening, a new fitness club has 870 members. At the end of the seventh week, there are 1110 members. If the increase is arithmetic, how many members were there in the first week?

16. A number, m, is called an arithmetic mean of a and b if a, m, and b form an arithmetic sequence. If there are two arithmetic means, m and n, then a, m, n, and b form an arithmetic sequence. Determine two arithmetic means between 9 and -45.

17. How many multiples of 8 are there between -58 and 606?

18. Investigate the sequences with the following recursion formulas. Which are arithmetic? Provide a general observation about how to identify an arithmetic sequence from a recursion formula.



a) $t_n = t_{n-1} + 3$

b) $t_n = 4t_{n-1} + t_{n-2}$

c) $t_n = (t_{n-1})^2$

d) $t_n = -2t_{n-1} - 5$

C Extend

19. Refer to question 16. The pattern continues for any number of arithmetic means. Determine the three arithmetic means between $x + 2y$ and $4x + 4y$.

20. Determine x so that x, $\frac{1}{2}x + 7$, and $3x - 1$ are the first three terms of an arithmetic sequence.

21. The sum of the first two terms of an arithmetic sequence is 15 and the sum of the next two terms is 43. Write the first four terms of the sequence.

22. a) Solve the system of equations $x + 2y = 3$ and $5x + 3y = 1$.

b) Solve the system of equations $9x + 5y = 1$ and $2x - y = -4$.

c) Make and prove a conjecture about the solution to a system of equations $ax + by = c$ and $dx + ey = f$, where a, b, c and d, e, f are separate arithmetic sequences.

23. **Math Contest** Sam starts at 412 and counts aloud backward by 6s (412, 406, 400, ...). A number that she will say is

A 32 **B** 12 **C** 58 **D** 104

24. **Math Contest** Show that for any triangle that contains a 60° angle, the three angles form an arithmetic sequence.

25. **Math Contest** Without using a calculator, determine the next number in the sequence $\frac{1}{36}, \frac{1}{18}, \frac{1}{12}, \frac{1}{9}, \ldots$.

A $\frac{1}{6}$ **B** $\frac{5}{36}$ **C** $\frac{1}{4}$ **D** $\frac{1}{7}$

26. **Math Contest** The fifth term of a sequence is 7 and the seventh term is 5. Each term in the sequence is the sum of the previous two terms. What is the ninth term in this sequence?

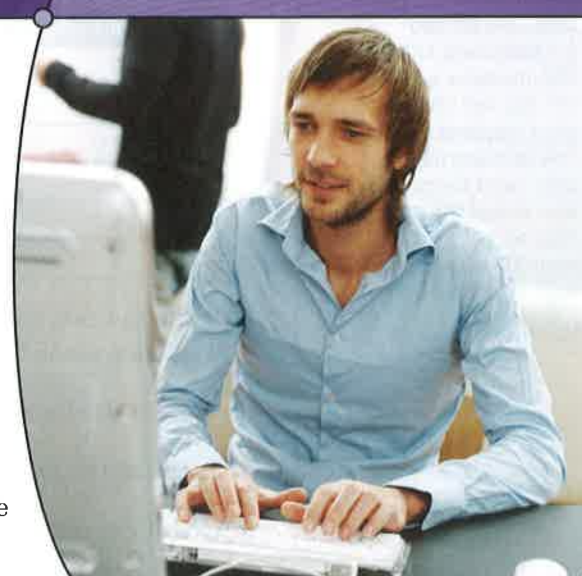
A 3 **B** 1 **C** 8 **D** −5

27. **Math Contest** A number is rewritten in its single-digit sum when all the digits are added together. If the sum is not a single digit, then add the digits again, continuing this process until there is a single digit. For example, 23 454 has a single-digit sum of 9, since $2 + 3 + 4 + 5 + 4 = 18$ and $1 + 8 = 9$. A term in a sequence is defined by squaring the previous term and then determining the single-digit sum of this square. If the first term of this sequence is 5, what is the 101st term?

A 7 **B** 13 **C** 25 **D** 4

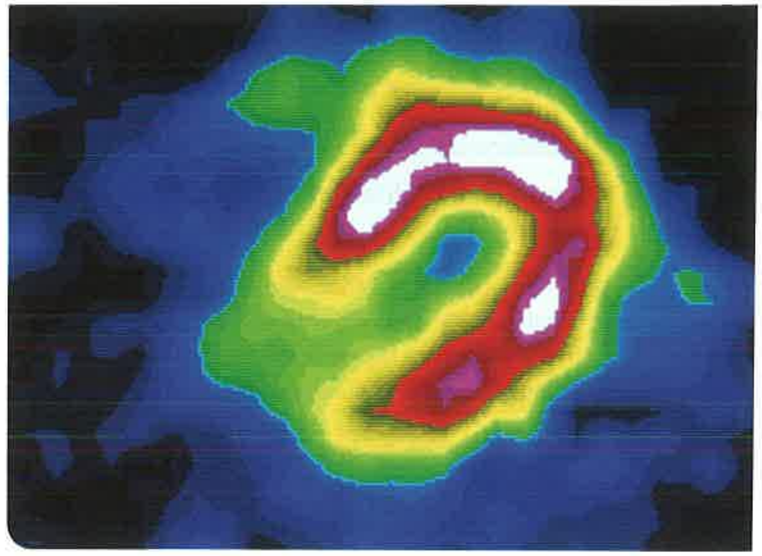
Career Connection

After completing a 4-year bachelor's degree at the University of Western Ontario, where he studied computer science and biology, Stephen works in the field of bioinformatics. Bioinformaticians derive knowledge from computer analysis of biological data. When scientists study organisms, large amounts of data are generated about their cells, proteins, genes, and other characteristics. Stephen uses analytical techniques and computer algorithms to document biological data. He also uses his computer and math skills to help other researchers analyse the information stored in the database. Their goal is to detect, prevent, and cure diseases.



6.5

Geometric Sequences



Radioactive substances are used by doctors for diagnostic purposes. For example, thallium-201 (Tl-201) is a radioactive substance that can be injected into the bloodstream and then its movement in the patient's bloodstream and heart viewed by a special camera. Since radioactive substances are harmful, doctors need to know how long such substances remain in the body. **Geometric sequences** can be used as models to predict the length of time that radioactive substances remain in the body.

Scientists and other professionals may also use geometric sequences to make predictions about levels of radioactivity in the soil or atmosphere.

geometric sequence

- a sequence where the ratio of consecutive terms is a constant

Tools

- grid paper

Connections

The becquerel (Bq) is used to measure the rate of radioactive decay. It equals one disintegration per second. Commonly used multiples of the becquerel are kBq (kilobecquerel, 10^3 Bq), MBq (megabecquerel, 10^6 Bq), and GBq (gigabecquerel, 10^9 Bq). This unit was named after Henri Becquerel, who shared the Nobel Prize with Pierre and Marie Curie for their work in discovering radioactivity.

Investigate

How can the terms in a geometric sequence be determined?

A patient is injected with 50 MBq of Tl-201 before undergoing a procedure to take an image of his heart. Tl-201 has a half-life of 73 h.

1. Copy and complete the table to determine the amount of Tl-201 remaining in the body after approximately 2 weeks (or five 73-h periods).

Time (73-h periods)	Amount of Tl-201 (MBq)	First Differences
0	50	
1		
2		
3		
4		
5		

2. Write the amount of Tl-201 at the end of each 73-h period as a sequence using 50 as the first term.

3. **Reflect** Is this an arithmetic sequence? Explain your answer.

4. Describe the pattern in the first differences.

5. Graph the sequence and describe the pattern in the points.

6. Divide each term after the first by the previous term. What do you notice?

7. Write each term of the sequence as an expression in terms of the original amount of Tl-201 and the value you found in step 6. Use the expressions to develop a formula for the general term of this sequence.

8. **Reflect** After how long will the amount of Tl-201 in the body be less than 0.01 MBq?

The terms of a geometric sequence are obtained by multiplying the first term, a, and each subsequent term by a **common ratio**, r. A geometric sequence can be written as a, ar^2, ar^3, ar^4, ... Then, the formula for the general term, or the nth term, of a geometric sequence is $t_n = ar^{n-1}$, where $r \neq 0$ and $n \in \mathbb{N}$.

common ratio

- the ratio of any two consecutive terms in a geometric sequence

Example 1

Determine the Type of Sequence

Determine whether each sequence is arithmetic, geometric, or neither. Justify your answer.

a) 2, 5, 10, 17, ...

b) 0.2, 0.02, 0.002, 0.0002, ...

c) $a + 2$, $a + 4$, $a + 6$, $a + 8$, ...

Solution

a) $\frac{5}{2} = 2.5$, $\frac{10}{5} = 2$, $\frac{17}{10} = 1.7$

There is no common ratio.

Divide each term by the previous term to check for a common ratio.

$5 - 2 = 3$, $10 - 5 = 5$, $17 - 10 = 7$

There is no common difference.

Subtract consecutive terms to check for a common difference.

This sequence is neither arithmetic nor geometric.

b) $\frac{0.02}{0.2} = 0.1$, $\frac{0.002}{0.02} = 0.1$, $\frac{0.0002}{0.002} = 0.1$

There is a common ratio, so this sequence is geometric.

c) $(a + 4) - (a + 2) = 2$, $(a + 6) - (a + 4) = 2$, $(a + 8) - (a + 6) = 2$

This sequence has a common difference, so it is an arithmetic sequence.

Example 2

Write Terms in a Geometric Sequence

Write the first three terms of each geometric sequence.

a) $f(n) = 5(3)^{n-1}$

b) $t_n = 16\left(\frac{1}{4}\right)^{n-1}$

c) $a = 125$ and $r = -2$

Solution

a) $f(n) = 5(3)^{n-1}$

$$f(1) = 5(3)^{1-1} = 5(3)^0 = 5 \qquad f(2) = 5(3)^{2-1} = 5(3)^1 = 15 \qquad f(3) = 5(3)^{3-1} = 5(3)^2 = 45$$

The first three terms are 5, 15, and 45.

b) $t_n = 16\left(\frac{1}{4}\right)^{n-1}$

$$t_1 = 16\left(\frac{1}{4}\right)^{1-1} = 16\left(\frac{1}{4}\right)^0 = 16 \qquad t_2 = 16\left(\frac{1}{4}\right)^{2-1} = 16\left(\frac{1}{4}\right)^1 = 4 \qquad t_3 = 16\left(\frac{1}{4}\right)^{3-1} = 16\left(\frac{1}{4}\right)^2 = 1$$

The first three terms are 16, 4, and 1.

c) Given that $a = 125$ and $r = -2$, the formula for the general term is $t_n = 125(-2)^{n-1}$.

$$t_1 = 125(-2)^{1-1} = 125(-2)^0 = 125 \qquad t_2 = 125(-2)^{2-1} = 125(-2)^1 = -250 \qquad t_3 = 125(-2)^{3-1} = 125(-2)^2 = 500$$

The first three terms are 125, −250, and 500.

Connections

When the common ratio of a geometric sequence is negative, the result is an alternating sequence. This is a sequence whose terms alternate in sign.

Example 3

Determine the Number of Terms

Determine the number of terms in the geometric sequence 4, 12, 36, ..., 2916.

Solution

For the given sequence, $a = 4$, $r = 3$, and $t_n = 2916$. Substitute these values into the formula for the general term of a geometric sequence and solve for n.

$$t_n = ar^{n-1}$$
$$2916 = 4(3)^{n-1}$$
$$\frac{2916}{4} = 3^{n-1}$$
$$729 = 3^{n-1}$$
$$3^6 = 3^{n-1}$$ Write 729 as a power of 3.

Since the bases are the same, the exponents must be equal.

$$n - 1 = 6$$
$$n = 7$$

There are seven terms in this sequence.

Example 4

Highway Accidents

Seatbelt use became law in Canada in 1976. Since that time, the number of deaths due to motor vehicle collisions has decreased. From 1984 to 2003, the number of deaths decreased by about 8% every 5 years. The number of deaths due to motor vehicle collisions in Canada in 1984 was approximately 4100.

a) Determine a formula to predict the number of deaths for any fifth year following 1984.

b) Write the number of deaths as a sequence for five 5-year intervals.

Connections

According to Transport Canada, 93% of Canadians used their seatbelts in 2007. The 7% of Canadians not wearing seatbelts accounted for almost 40% of fatalities in motor vehicle collisions.

Solution

a) The number of deaths can be represented by a geometric sequence with $a = 4100$ and $r = 0.92$. Then, the formula is $t_n = 4100(0.92)^{n-1}$ where n is the number of 5-year periods since 1984.

b) $t_1 = 4100$, $t_2 = 4100(0.92)$, $t_3 = 4100(0.92)^2$, $t_4 = 4100(0.92)^3$, $t_5 = 4100(0.92)^4$

The numbers of deaths for five 5-year intervals are 4100, 3772, 3470, 3193, and 2937.

Key Concepts

- A geometric sequence is a sequence in which the ratio of consecutive terms is a constant.
- The ratio between consecutive terms of a geometric sequence is called the common ratio.
- The formula for the general term of a geometric sequence is $t_n = a(r)^{n-1}$, where a is the first term, r is the common ratio, and n is the term number.

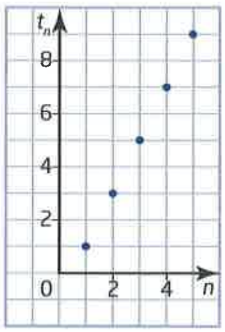
Communicate Your Understanding

C1 How can you determine if a sequence is arithmetic, geometric, or neither? Give an example of each type of sequence.

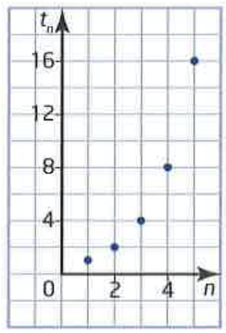
C2 Describe how to determine the formula for the general term, t_n, of the geometric sequence 5, −10, 20, −40, ...

C3 Consider the graphs of the sequences shown. Identify each sequence as arithmetic or geometric. Explain your reasoning.

a)



b)



A Practise

For help with question 1, refer to Example 1.

1. Determine whether the sequence is arithmetic, geometric, or neither. Give a reason for your answer.

a) 5, 3, 1, −1, ...

b) 5, −10, 20, −40, ...

c) 4, 0.4, 0.04, 0.004, ...

d) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots$

e) $1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots$

f) 1, 5, 2, 5, ...

For help with questions 2 to 4, refer to Example 2.

2. State the common ratio for each geometric sequence and write the next three terms.

a) 1, 2, 4, 8, ...

b) −3, 9, −27, 81, ...

c) $\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}, \ldots$

d) 600, −300, 150, −75, ...

e) −15, −15, −15, −15, ...

f) 0.3, 3, 30, 300, ...

g) 72, 36, 18, 9, ...

h) $x, x^3, x^5, x^7, \ldots$

3. For each geometric sequence, determine the formula for the general term and then write t_9.

a) 54, 18, 6, ...

b) 4, 20, 100, ...

c) $\frac{1}{6}, \frac{1}{5}, \frac{6}{25}, \ldots$

d) 0.0025, 0.025, 0.25, ...

4. Write the first four terms of each geometric sequence.

a) $t_n = 5(2)^{n-1}$

b) $a = 500, r = -5$

c) $f(n) = \frac{1}{4}(-3)^{n-1}$

d) $f(n) = 2(\sqrt{2})^{n-1}$

e) $a = -1, r = \frac{1}{5}$

f) $t_n = -100(-0.2)^{n-1}$

For help with question 5, refer to Example 3.

5. Determine the number of terms in each geometric sequence.

a) 6, 18, 54, ..., 4374

b) 0.1, 100, 100 000, ..., 10^{14}

c) 5, −10, 20, ..., −10 240

d) 3, $3\sqrt{3}$, 9, ..., 177 147

e) 31 250, 6250, 1250, ..., 0.4

f) 16, −8, 4, ..., $\frac{1}{4}$

B Connect and Apply

6. Determine if each sequence is arithmetic, geometric, or neither. If it is arithmetic, state the values of a and d. If it is geometric, state the values of a and r.

a) $x, 3x, 5x, \ldots$

b) $1, \frac{x}{2}, \frac{x^2}{4}, \ldots$

c) $\frac{m^2}{n}, \frac{m^3}{2n}, \frac{m^4}{3n}, \ldots$

d) $\frac{5x}{10}, \frac{5x}{10^3}, \frac{5x}{10^5}, \ldots$

7. Which term of the geometric sequence 1, 3, 9, ... has a value of 19 683?

8. Which term of the geometric sequence $\frac{3}{64}, -\frac{3}{16}, \frac{3}{4}, \ldots$ has a value of 192?

9. *Listeria monocytogenes* is a bacteria that rarely causes food poisoning. At a temperature of 10 °C, it takes about 7 h for the bacteria to double. If the bacteria count in a sample of food is 100, how long will it be until the count exceeds 1 000 000?

10. In 1986, a steam explosion at a nuclear reactor in Chernobyl released radioactivity into the air, causing widespread death; disease; and contamination of soil, water, and air that continues today. One of the radioactive components released, cesium-137 (Cs-137), is very dangerous to human life as it accumulates in the soil, the water, and the body. It is believed by scientists that a contamination of Cs-137 of over 1 Ci/km² (curie per square kilometre) is dangerous.

a) Determine the amount of Cs-137 per square kilometre if about 1.5×10^6 Ci of this radioactive substance was released into the environment and spread over an area of about 135 000 km².

b) The half-life of Cs-137 is 30 years. Write an explicit formula to represent the level of Cs-137 left after n years. How long will it take for the contamination to reach safe levels?

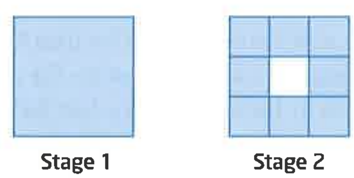
c) Research this tragedy to discover more about the long-term effects on the environment and the people of the contaminated region.

Connections

The curie (Ci) is a unit of radioactivity, named after Pierre and Marie Curie, that has since been replaced by the becquerel (Bq).
1 Ci = 3.7×10^{10} Bq

11. A chain e-mail starts with one person sending out six e-mail messages. Each of the recipients sends out six messages, and so on. How many e-mail messages will be sent in the sixth round of e-mailing?

12. Chapter Problem A square with area 1 square unit is partitioned into nine squares and then all but the middle square are shaded. This process is repeated with the remaining shaded squares to produce a fractal called the Sierpinski carpet.



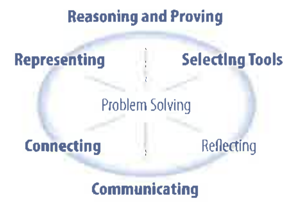
a) Use grid paper to produce the first five stages of the fractal.

b) Write a formula to determine the shaded area at each stage.

c) Use the formula to determine the shaded area at stage 20.

d) Research this fractal. When was it first explored?

13. In a certain country, elections are held every 4 years. Voter turnout at elections increases by 2.6% each time an election is held. In 1850, when the country was formed, 1 million people voted.



a) Determine an equation to model the number of voters at any election. Graph the equation.

b) Is this function continuous or discrete? Explain your answer.

c) How many people vote in the 2010 election?

14. The geometric mean of a set of n numbers is the nth root of the product of the numbers. For example, given two non-consecutive terms of a geometric sequence, 6 and 24, their product is 144 and the geometric mean is $\sqrt{144}$, or 12. The numbers 6, 12, and 24 form a geometric sequence.

a) Determine the geometric mean of 5 and 125.

b) Insert three geometric means between 4 and 324.

✓ Achievement Check

15. Aika wanted to test her dad's knowledge of sequences so she decided to offer him two different options for her allowance for one year. In option 1, he would give her \$25 every week. In option 2, he would give her \$0.25 the first week and then double the amount every following week.

a) Which option represents an arithmetic sequence? Determine the general term for the sequence.

b) Which option represents a geometric sequence? Determine the general term for the sequence.

c) Which plan should her dad pick? Explain.

C Extend

16. Determine the value(s) of y if $4y + 1$, $y + 4$, and $10 - y$ are consecutive terms in a geometric sequence.

17. Determine x and y for each geometric sequence.

a) $3, x, 12, y, \ldots$ **b)** $-2, x, y, 1024, \ldots$

18. Refer to question 14. Determine three geometric means between $x^5 + x^4$ and $x + 1$.

19. The population of a city increases from 12 000 to 91 125 over 10 years. Determine the annual rate of increase, if the increase is geometric.

20. The first three terms of the sequence $-8, x, y, 72$ form an arithmetic sequence, while the second, third, and fourth terms form a geometric sequence. Determine x and y.

21. **Math Contest** Three numbers form an arithmetic sequence with a common difference of 7. When the first term of the sequence is decreased by 3, the second term increased by 7, and the third term doubled, the new numbers form a geometric sequence. What is the original first term?

A 7 **B** 16 **C** 20 **D** 68

22. **Math Contest** A geometric sequence has the property that each term is the sum of the previous two terms. If the first term is 2, what is one possibility for the second term?

A $-4 + \sqrt{3}$ **B** $1 - \sqrt{5}$
C $4 - \sqrt{3}$ **D** $-1 + \sqrt{5}$

23. **Math Contest** Film speed is the measure of a photographic film's sensitivity to light. The ISO (International Organization of Standardization) film-speed scale forms a geometric sequence. If the first term in the sequence is 25 and the fourth term is 50, what is the fifth term?

6.6

Arithmetic Series

Dar Robinson was a famous stuntman. In 1979, Dar was paid $100 000 to jump off the CN Tower in Toronto. During the first second of the jump, Dar fell 4.9 m; during the next second, he fell 14.7 m; and, during the third second, the drop was another 24.5 m. This pattern continued for 12 s. The total distance of his jump can be found by adding the terms of this sequence together.

Investigate

How can a long sequence of numbers be added quickly?

1. Copy and complete the table to determine the sum of the first 10 even natural numbers.

Number of Terms	Indicated Sum	Sum	Mean of All Terms	Mean of First and Last Terms
1	2	2	2	2
2	2 + 4	6	3	3
3	2 + 4 + 6	12	4	4
4	2 + 4 + 6 + 8	20	5	5

Connections

Recall that the mean is the sum of a set of values divided by the number of values in the set.

2. **Reflect** How is the sum of the terms related to the number of terms in the sum and to the mean of the terms? How is the mean of the first and last terms related to the mean of all the terms?

3. a) Write a formula to represent the sum of a **series** when the first and last terms are known.

   b) Use your formula to determine the sum of the first 100 even natural numbers.

series

- the indicated sum of the terms of a sequence

4. Verify that your formula works for another series, $1 + 3 + 5 + 7 + 9 + 11$.

5. Suppose that the sum of the first six even numbers is written in order and then in reverse order.

$$\begin{array}{r} 2 + 4 + 6 + 8 + 10 + 12 \\ 12 + 10 + 8 + 6 + 4 + 2 \\ \hline \end{array}$$

Add these two series of numbers together and divide the result by 2.

6. **Reflect** Compare the result from step 5 with your formula.

arithmetic series

- the indicated sum of the terms of an arithmetic sequence

Connections

Around the age of 8, Carl Friedrich Gauss (1777–1855) astonished his school teacher by almost instantly determining the sum of the natural numbers from 1 to 100 by using this method. Gauss was born in Brunswick, Germany. By age 18 he was making new discoveries, mostly in number theory. Later he turned his attention to applied mathematics and physics and helped invent the first electric telegraph.

The method shown in step 5 of the Investigate can be used to derive a formula for the sum, S_n, of the first n terms of an **arithmetic series**.

For any arithmetic sequence, the terms can be written as $a, (a + d), (a + 2d), \ldots, (t_n - d), t_n$, where n is the number of terms, a is the first term, d is the common difference between the terms, and t_n is the last term.

The corresponding arithmetic series is

$$\begin{array}{rl} S_n &= a + (a + d) + (a + 2d) + \cdots + (t_n - d) + t_n \\ + \; S_n &= t_n + (t_n - d) + (t_n - 2d) + \cdots + (a + d) + a \\ \hline 2S_n &= (a + t_n) + (a + t_n) + (a + t_n) + \cdots + (a + t_n) + (a + t_n) \\ 2S_n &= n(a + t_n) \\ S_n &= \frac{n}{2}(a + t_n) \quad ① \end{array}$$

Write the series in reverse.

Add.

Divide both sides by 2.

By substituting $t_n = a + (n - 1)d$ for t_n in formula ①, you can represent the sum of an arithmetic series with a different formula, formula ②:

$$S_n = \frac{n}{2}[a + a + (n - 1)d]$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] \quad ②$$

Formula ② can be used to determine the sum, S_n, of the first n terms of an arithmetic series when the value of the last term, t_n, is not known.

Example 1

Determine the Sum of an Arithmetic Series

Determine the sum of the first 10 terms of each arithmetic series.

a) $a = 2$, $d = 4$, $t_{10} = 38$

b) $a = -4$, $d = -3$

Solution

a) Method 1: Use Pencil and Paper

Use formula ①:

$$\begin{aligned} S_n &= \frac{n}{2}(a + t_n) \\ S_{10} &= \frac{10}{2}(2 + 38) \\ &= 5(40) \\ &= 200 \end{aligned}$$

Use formula ②:

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n - 1)d] \\ S_{10} &= \frac{10}{2}[2(2) + (10 - 1)4] \\ &= 5(40) \\ &= 200 \end{aligned}$$

The sum of the first 10 terms of the arithmetic series is 200.

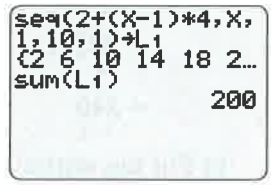
Method 2: Use a Graphing Calculator

Start by listing the first 10 terms and storing them in **L1**.

- Press 2ND [LIST] and cursor over to the **OPS** menu.
- Select **5:seq(** and enter $2 + (x - 1) \times 4, x, 1, 10, 1)$.
- Press (STO▶) (2nd) [L1] (ENTER).

Now, determine the sum.

- Press (2nd) [LIST] and cursor over to the **MATH** menu.
- Select **5:sum(** and enter **L1**).
- Press (ENTER).



The sum of the first 10 terms of the arithmetic series is 200.

b) Method 1: Use Pencil and Paper

Use formula ②:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_{10} = \frac{10}{2}[2(-4) + (10 - 1)(-3)]$$

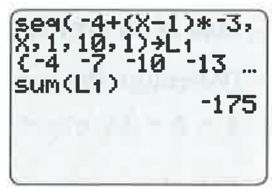
$$= 5(-35)$$

$$= -175$$

The sum of the first 10 terms of the arithmetic series is -175.

Method 2: Use a Graphing Calculator

Using the **sequence** function, enter $-4 + (x - 1) \times -3, x, 1, 10, 1)$. Store the list in **L1**. Then, use the **sum** function.



The sum of the first 10 terms of the arithmetic series is -175.

Example 2

Sum of an Arithmetic Series Given the First Few Terms

Determine the specified sum for each arithmetic series.

a) S_{15} for $-12 - 8 - 4 - \cdots$

b) S_{20} for $-11m - 2m + 7m + \cdots$

Solution

a) For the arithmetic series $-12 - 8 - 4 - \ldots$, $a = -12$, $d = 4$, and $n = 15$.

Use formula ②:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_{15} = \frac{15}{2}[2(-12) + (15 - 1)(4)]$$

$$= \frac{15}{2}(32)$$

$$= 240$$

b) For the arithmetic series $-11m - 2m + 7m + \ldots$, $a = -11m$, $d = 9m$, and $n = 20$.

Use formula ②:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_{20} = \frac{20}{2}[2(-11m) + (20 - 1)(9m)]$$

$$= 10(149m)$$

$$= 1490m$$

Example 3

Sum of an Arithmetic Series Given the First Three Terms and the Last Term

Determine the sum of the arithmetic series.

$3 + 8 + 13 + \cdots + 58$

Solution

To use either formula, first determine the number of terms.

For this series, $a = 3$, $d = 5$, and $t_n = 58$.

$$t_n = a + (n - 1)d$$

$$58 = 3 + (n - 1)(5)$$

$$58 = 5n - 2$$

$$60 = 5n$$

$$n = 12$$

Use formula ①:

$$S_n = \frac{n}{2}(a + t_n)$$

$$S_{12} = \frac{12}{2}(3 + 58)$$

$$= 6(61)$$

$$= 366$$

Use formula ②:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_{12} = \frac{12}{2}[2(3) + (12 - 1)5]$$

$$= 6(61)$$

$$= 366$$

The sum of the arithmetic series is 366.

Example 4

Compare Wages

A student is offered a job with a math teacher that will last 20 h. The first option pays $4.75 for the first hour, $5 for the second hour, $5.25 for the next hour, and so on. The second option pays $7/h for all hours worked. Which option pays more?

Solution

The first method of payment is an arithmetic series with $a = 4.75$, $d = 0.25$, and $n = 20$. To determine the total amount earned, calculate the sum of the series.

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_{20} = \frac{20}{2}[2(4.75) + (20 - 1)(0.25)]$$

$$= 10(14.25)$$

$$= 142.5$$

The amount earned by the first option is $142.50. The amount earned by the second option is $7 × 20, or $140. So, the first option pays more.

Key Concepts

- An arithmetic series is the indicated sum of the terms of an arithmetic sequence. For example, 4, 9, 14, 19, ... is an arithmetic sequence, while $4 + 9 + 14 + 19 + \cdots$ is an arithmetic series.
- Given the first term, the last term, and the number of terms of an arithmetic series, the sum of the series can be found using the formula $S_n = \frac{n}{2}(a + t_n)$ or the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$.
- Given the first terms of an arithmetic series, the sum of the first n terms can be found using the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$.

Communicate Your Understanding

C1 Describe how an arithmetic series is related to an arithmetic sequence. Use an example to model your answer.

C2 Describe when it is easier to use the formula $S_n = \frac{n}{2}(a + t_n)$ and when the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$ is a better choice.

C3 Describe a real-life situation that could be defined by an arithmetic series.

C4 The word *series* is often used in the English language. For example, a TV program with a different episode every week is called a series. Describe some other everyday uses of the word *series*.

A Practise

For help with question 1, refer to Example 1.

1. Determine the sum of each arithmetic series.
   a) $a = 4$, $t_n = 9$, $n = 6$
   b) $a = 10$, $d = -2$, $n = 12$
   c) $a = 7$, $t_n = -22$, $n = 12$
   d) $a = -4$, $t_n = 17$, $n = 20$
   e) $a = \frac{1}{3}$, $d = -\frac{1}{2}$, $n = 7$
   f) $a = 3x$, $t_n = 21x$, $n = 15$

For help with questions 2 and 3, refer to Example 2.

2. For each arithmetic series, state the values of a and d. Then, determine the sum of the first 20 terms.
   a) $5 + 9 + 13 + \cdots$
   b) $20 + 25 + 30 + \cdots$
   c) $45 + 39 + 33 + \cdots$
   d) $2 + 2.2 + 2.4 + \cdots$
   e) $\frac{1}{2} + \frac{3}{4} + 1 + \cdots$
   f) $-5 - 6 - 7 - \cdots$

3. The first and last terms in each arithmetic series are given. Determine the sum of the series.
   a) $a = \frac{1}{2}$, $t_8 = 4$
   b) $a = 19$, $t_{12} = 151$
   c) $a = -5$, $t_{45} = 17$
   d) $a = 11$, $t_{20} = 101$

For help with questions 4 and 5, refer to Example 3.

4. Determine the sum of each arithmetic series.
   a) $6 + 13 + 20 + \cdots + 69$
   b) $4 + 15 + 26 + \cdots + 213$
   c) $5 - 8 - 21 - \cdots - 190$
   d) $100 + 90 + 80 + \cdots - 100$

5. Determine the sum of each arithmetic series.
   a) $-1 + 2 + 5 + \cdots + 164$
   b) $2 - 5 - 12 - \cdots - 222$
   c) $21.5 + 14.2 + 6.9 + \cdots - 715.8$
   d) $\frac{5}{3} + \frac{11}{3} + \frac{17}{3} + \cdots + \frac{53}{3}$

B Connect and Apply

6. The 15th term in an arithmetic sequence is 43 and the sum of the first 15 terms of the series is 120. Determine the first three terms of the sequence.

7. In an arithmetic sequence of 50 terms, the 17th term is 53 and the 28th term is 86. Determine the sum of the first 50 terms of the corresponding arithmetic series.

8. Determine the sum of each arithmetic series.
   a) $2\sqrt{7} + 5\sqrt{7} + 8\sqrt{7} + \cdots + 83\sqrt{7}$
   b) $x - 2x - 5x - \cdots - 56x$
   c) $(5a - 3b) + (4a - 2b) + (3a - b) + \cdots + (-5a + 7b)$
   d) $\frac{2}{x} + \frac{4}{x} + \frac{6}{x} + \cdots + \frac{18}{x}$

9. Which are arithmetic series? Justify your answers.
   a) $-2 - 8 - 11 - 17 - \cdots$
   b) $2x^2 + 3x^2 + 4x^2 + \cdots$
   c) $a + (a + 2b) + (a + 4b) + \cdots$
   d) $\frac{17}{20} + \frac{11}{20} + \frac{27}{20} + \cdots$

10. In a grocery store, apple juice cans are stacked in a triangular display. There are 5 cans in the top row and 12 cans in the bottom row. Each row has 1 can less than the previous row. How many cans are in the display?

11. A toy car is rolling down an inclined track and picking up speed as it goes. The car travels 4 cm in the first second, 8 cm in the second second, 12 cm in the next second, and so on. Determine the total distance travelled by the car in 30 s.

12. A snowball sentence is constructed so that each word has one more letter than the previous word. An example is, "I am not cold today."

a) Determine the total number of letters in the sentence.

b) Write your own snowball sentence and determine the number of letters in your sentence.

13. Determine an expression for the sum of the terms of an arithmetic series where the terms are represented by $t_n = 3n - 2$.

14. **a)** Determine x so that $2x$, $3x + 1$, and $x^2 + 2$ are the first three terms of an arithmetic sequence.

b) Determine the sum of the first 10 terms of the sequence.

Achievement Check

15. Icy Treats finds that its profit increases by \$200 per week throughout the 16-week summer season. Icy Treat's profit is \$1200 in the first week.

a) Explain why the total profit for the season is represented by an arithmetic series.

b) Determine the total profit for the season.

C Extend

16. If $S(n)$ is a function representing the sum of an arithmetic series, determine the series with $S(13) = 507$ and $S(25) = 2025$. Graph this function.

17. How many terms in the series $5 + 9 + 13 + \cdots + t_n$ are less than 500? How many terms are needed for a sum of less than 500?

18. In an arithmetic series, the sum of the first 9 terms is 162 and the sum of the first 12 terms is 288. Determine the series.

19. **Math Contest** Two arithmetic sequences, 3, 9, 15, 21, … and 4, 11, 18, 25, …, share common terms. For example, 81 is a term in both sequences. What is the sum of the first 20 terms that these sequences have in common?

A 8760

B 8750

C 8740

D 8770

20. **Math Contest** The sum of the series $1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + \cdots + 2999$ is

A 2 999 500

B 2 999 000

C 3 000 000

D 4 500 000

21. **Math Contest** What is the value of $(1^2 + 3^2 + \cdots + 171^2) - (2^2 + 4^2 + 6^2 + \cdots + 170^2)$?

22. **Math Contest** Solve $6^{x-3} - 6^{x-4} = 1080$.

23. **Math Contest** A geometric sequence has 1 as its first term and $2n$ as its last. Show that if there are $2n$ terms, then the product of the terms of the sequence is $(2n)^n$.

24. **Math Contest** Without using a calculator, determine the sum of the first 20 terms of the sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \ldots$.

6.7

Geometric Series

The Canadian Open is an outdoor tennis tournament. The tournament started in 1881, and the men's and women's events alternate between Toronto and Montreal every other year. In the first round of the men's singles event, 32 matches are played. In the second round, 16 matches are played. In the third round, 8 matches are played, and so on. These numbers form the terms of a geometric sequence. To determine the total number of matches played in this event, you need to add these terms together. When the terms of a geometric sequence are added together, the result is called a **geometric series**.

geometric series

- the indicated sum of the terms of a geometric sequence

Tools

- computer with spreadsheet software

Investigate

How can you develop a formula for the sum of the terms in a geometric sequence?

A nickel is placed on a game board with 32 squares. On each succeeding square, the number of nickels is doubled. How much money will be on the last square? How much money will be on the board when it is full?

Method 1: Use a Spreadsheet

1. Enter the following information in a spreadsheet. From the **Edit** menu, use **Fill Down** to complete the next 30 rows of the spreadsheet.

	A	B	C
1	Number of Squares	Number of Nickels	Total Number of Nickels
2	1	1	1
3	=A2+1	=B2*2	=B3+C2

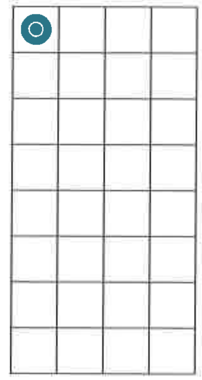
2. How many nickels are on the last square? How many nickels are on the entire game board?

3. Determine a formula for the total number of nickels on the board. Use your formula to calculate the total number of nickels on the game board.

4. **Reflect** Is the function that represents the total number of nickels on the game board continuous or discrete? How do you know?

5. Write the total number of nickels as a series. Is this series arithmetic? Explain your thinking.

Method 2: Use Pencil and Paper



Tools
- grid paper
- counters

1. Draw an 8 by 4 grid of squares. The squares should be large enough to stack counters on.

2. Let each counter represent 1 nickel. Place one counter on a square, two counters on the next square, four counters on the next square, and so on. Use a table similar to the one shown to record your results.

Number of Squares	Number of Nickels on the Square	Total Number of Nickels
1	1	1
2	2	3

3. Determine a formula for the total number of nickels on the board. Use your formula to calculate the total number of nickels on the game board.

4. **Reflect** Is the function that represents the total number of nickels on the game board continuous or discrete? How do you know?

5. Write the total number of nickels as a series. Is this series arithmetic? Explain your thinking.

When the terms of a geometric sequence are added, the resulting expression is called a geometric series. The sum, S_n, of the first n terms of a geometric series is $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. This can be used to derive a formula for S_n.

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad ①$$
Write the series.

$$rS_n = \phantom{a +{}} ar + ar^2 + \cdots + ar^{n-1} + ar^n \quad ②$$
Multiply both sides by r and align like terms.

$$rS_n - S_n = -a + ar^n \quad ② - ①$$

$$rS_n - S_n = ar^n - a$$
Rearrange the right side.

$$S_n(r - 1) = a(r^n - 1)$$
Factor both sides.

$$S_n = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$
Divide both sides by $r - 1$.

Example 1

Identify a Geometric Series

Determine if each series is geometric. Justify your answer.

a) $2 + 6 + 18 + 54 + \cdots$

b) $2 + 8 - 12 + 16 - \cdots$

Solution

a) The series 2 + 6 + 18 + 54 + ⋯ is geometric if consecutive terms have a common ratio.

$\frac{6}{2} = 3, \frac{18}{6} = 3, \frac{54}{18} = 3$

Since the ratio of consecutive terms is 3, this series is geometric.

b) Check 2 + 8 − 12 + 16 − ⋯ for a common ratio.

$\frac{8}{2} = 4, \frac{-12}{8} = -1.5, \frac{-16}{-12} = 1.\overline{3}$

Since the ratio of consecutive terms is not constant, this series is not geometric.

Example 2

Sum of a Geometric Series

Determine the sum of the first 10 terms of each geometric series.

a) $f(1) = 20, r = -3$ **b)** $a = 5, r = 4$

Solution

a) Method 1: Use Pencil and Paper

Substitute $a = 20$, $r = -3$, and $n = 10$ into the formula.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{20[(-3)^{10} - 1)]}{-3 - 1}$$

$$= \frac{20(59\ 048)}{-4}$$

$$= -295\ 240$$

The sum of the first 10 terms of the geometric series is −295 240.

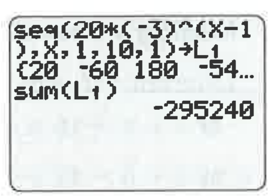
Method 2: Use a Graphing Calculator

Start by listing the first 10 terms and storing them in **L1**.

- Press (2nd) [LIST] and cursor over to the **OPS** menu.
- Select **5:seq(** and enter $20 \times (-3)^{x-1}$, x, 1, 10, 1).
- Press (STO▶) (2nd) [L1] (ENTER).

Now, determine the sum.

- Press (2nd) [LIST] and cursor over to the **MATH** menu.
- Select **5:sum(** and enter **L1**).
- Press (ENTER).



The sum of the first 10 terms of the geometric series is −295 240.

b) Method 1: Use Pencil and Paper

Substitute $a = 5$, $r = 4$, and $n = 10$ into the formula.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{5(4^{10} - 1)}{4 - 1}$$

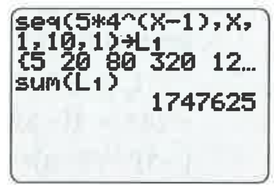
$$= \frac{5(1\ 048\ 575)}{3}$$

$$= 1\ 747\ 625$$

The sum of the first 10 terms of the geometric series is 1 747 625.

Method 2: Use a Graphing Calculator

Using the **sequence** function, enter $5 \times 4^{x-1}$, x, 1, 10, 1). Store the list in **L1**. Then, use the **sum** function.



The sum of the first 10 terms of the geometric series is 1 747 625.

Example 3

Sum of a Geometric Series Given the First Three Terms and the Last Term

Determine the sum of each geometric series.

a) $32 + 16 + 8 + \cdots + \frac{1}{8}$

b) $1 - 3 + 9 - \cdots - 243$

Solution

a) For this series, $a = 32$, $r = \frac{1}{2}$, and $t_n = \frac{1}{8}$. Use the formula for the general term of a geometric sequence to determine the value of n.

$$t_n = ar^{n-1}$$

$$\frac{1}{8} = 32\left(\frac{1}{2}\right)^{n-1}$$

$$\frac{1}{256} = \left(\frac{1}{2}\right)^{n-1}$$

$$\left(\frac{1}{2}\right)^8 = \left(\frac{1}{2}\right)^{n-1}$$ Write $\frac{1}{256}$ as a power of $\frac{1}{2}$.

$$8 = n - 1$$ Since the bases are the same, the exponents must be equal.

$$n = 9$$

Now, use the formula for S_n.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_9 = \frac{32\left[\left(\frac{1}{2}\right)^9 - 1\right]}{\frac{1}{2} - 1}$$

$$= \frac{32\left(-\frac{511}{512}\right)}{-\frac{1}{2}}$$

$$= \frac{511}{8}$$

b) Substitute $a = 1$, $r = -3$, and $t_n = -243$ into the formula for the general term of a geometric sequence and solve for n.

$$t_n = ar^{n-1}$$

$$-243 = 1(-3)^{n-1}$$

$$(-3)^5 = (-3)^{n-1}$$ Write -243 as a power of -3.

$$5 = n - 1$$ Since the bases are the same, the exponents must be equal.

$$n = 6$$

Use the formula for S_n.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_6 = \frac{1[(-3)^6 - 1]}{-3 - 1}$$

$$= \frac{728}{-4}$$

$$= -182$$

Example 4

Tennis Tournament

A tennis tournament has 128 entrants. A player is dropped from the competition after losing one match. Winning players go on to another match. What is the total number of matches that will be played in this tournament?

Solution

This situation can be represented by a geometric series. Since there are two players per match, the first term, a, is $128 \div 2$, or 64. After each round of matches, half the players drop out because they lost, so the common ratio, r, is $\frac{1}{2}$. Since there will be a single match played at the end of the tournament, the last term, t_n, is 1.

First, determine the total number of games played by the winner of the tournament. This is the value of n.

$$t_n = ar^{n-1}$$
$$1 = 64\left(\frac{1}{2}\right)^{n-1}$$
$$\frac{1}{64} = \left(\frac{1}{2}\right)^{n-1}$$
$$\left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^{n-1}$$ Write $\frac{1}{64}$ as a power of $\frac{1}{2}$.
$$6 = n - 1$$ Since the bases are the same, the exponents must be equal.
$$n = 7$$

Now, determine the total number of matches played in the tournament.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_7 = \frac{64\left[\left(\frac{1}{2}\right)^7 - 1\right]}{\frac{1}{2} - 1}$$
$$= \frac{64\left(-\frac{127}{128}\right)}{-\frac{1}{2}}$$
$$= 127$$

A total of 127 matches will be played in this tournament.

Connections

Geometric series have many important applications in the financial field. For example, you can determine the amount of money you will have after 50 years if you save $1000 every year. This type of application will be explored in detail in Chapter 7.

Key Concepts

- A geometric series is the sum of the terms in a geometric sequence. For example, $-3 + 6 - 12 + 24 - \cdots$ is a geometric series.
- The formula for the sum of the first n terms of a geometric series with first term a and common ratio r is $S_n = \frac{a(r^n - 1)}{r - 1}$, $r \neq 1$.

Communicate Your Understanding

C1 Describe the similarities and differences between an arithmetic series and a geometric series.

C2 Andre missed the lesson on determining the sum of a geometric series given the first and last terms as well as the common ratio. Describe the process for him.

C3 Explain why $r \neq 1$ when using the formula $S_n = \frac{a(r^n - 1)}{r - 1}$.

A Practise

For help with question 1, refer to Example 1.

1. Determine whether each series is geometric. Justify your answer.

a) $4 + 20 + 100 + 500 + \cdots$

b) $-150 + 15 - 1.5 + 0.15 - \cdots$

c) $3 - 9 + 18 - 54 + \cdots$

d) $256 - 64 + 16 - 4 + \cdots$

For help with questions 2 and 3, refer to Example 2.

2. For each geometric series, determine the values of a and r. Then, determine the indicated sum.

a) S_8 for $2 + 6 + 18 + \cdots$

b) S_{10} for $24 - 12 + 6 - \cdots$

c) S_{15} for $0.3 + 0.003 + 0.000\,03 + \cdots$

d) S_{12} for $1 - \frac{1}{3} + \frac{1}{9} - \cdots$

e) S_9 for $2.1 - 4.2 + 8.4 - \cdots$

f) S_{40} for $8 - 8 + 8 - \cdots$

3. Determine S_n for each geometric series.

a) $a = 6$, $r = 2$, $n = 9$

b) $f(1) = 2$, $r = -2$, $n = 12$

c) $f(1) = 729$, $r = -3$, $n = 15$

d) $f(1) = 2700$, $r = 10$, $n = 8$

e) $a = \frac{1}{2}$, $r = 4$, $n = 8$

f) $a = 243$, $r = \frac{1}{3}$, $n = 10$

For help with questions 4 and 5, refer to Example 3.

4. Determine the sum of each geometric series.

a) $27 + 9 + 3 + \cdots + \frac{1}{243}$

b) $7 + 3.5 + 1.75 + \cdots + 0.109\,375$

c) $1200 + 120 + 12 + \cdots + 0.0012$

d) $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots + \frac{128}{6561}$

5. Determine the sum of each geometric series.

a) $5 - 15 + 45 - \cdots + 3645$

b) $6 - 12 + 24 - 48 + \cdots - 768$

c) $96\,000 - 48\,000 + 24\,000 - \cdots + 375$

d) $1 - \frac{2}{3} + \frac{4}{9} - \cdots + \frac{64}{729}$

B Connect and Apply

6. Determine the specified sum for each geometric series.

a) S_{10} for $\sqrt{3} - 3 + 3\sqrt{3} + \cdots$

b) S_{12} for $\sqrt{2x} + 2x + 2\sqrt{2x} + \cdots$

c) S_{15} for $3 + 3x + 3x^2 + \cdots$

7. Determine the sum of each geometric series.

a) $10 + 5 + \frac{5}{2} + \cdots + \frac{5}{64}$

b) $2\sqrt{5} + 10 + 10\sqrt{5} + \cdots + 31\,250$

c) $1 + x + x^2 + x^3 + \cdots + x^k$

8. The sum of $4 + 12 + 36 + 108 + \cdots + t_n$ is 4372. How many terms are in this series?

9. The third term of a geometric series is 24 and the fourth term is 36. Determine the sum of the first 10 terms.

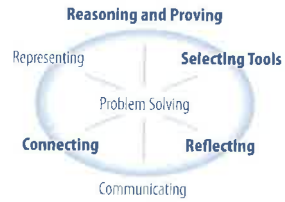
10. In a geometric series, $t_1 = 3$ and $S_3 = 21$. Determine the common ratio and an expression for the sum of the first k terms.

11. In a lottery, the first ticket drawn wins a prize of \$25. Each ticket drawn after that receives a prize that is twice the value of the preceding prize.

a) Write a function to model the total amount of prize money given away.

b) Graph the function to determine how many prizes can be given out if the total amount of prize money is \$2 million.

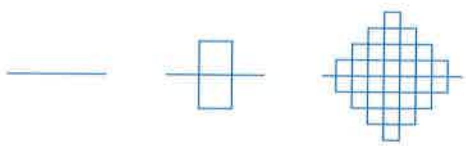
12. A bouncy ball bounces to $\frac{2}{3}$ of its height when dropped on a hard surface. Suppose the ball is dropped from 20 m.



a) What height will the ball bounce back up to after the sixth bounce?

b) What is the total distance travelled by the ball after 10 bounces?

13. Chapter Problem The Peano curve is a space-filling fractal. The first stage is a line segment of length 1 unit. The second stage is constructed by replacing the original line segment with nine line segments each of length $\frac{1}{3}$ unit. This process continues as each line segment is replaced by nine line segments that are $\frac{1}{3}$ the length of the line segments in the previous stage.



Stage 1 Stage 2 Stage 3

a) Construct as many stages of the Peano curve as you can. Use a table similar to the one shown to record the lengths of the line segments and the total length for each stage.

Stage	Line Segment Length	Total Length
1	1	1
2	$\frac{1}{3}$	3
3	$\frac{1}{9}$	9
4		
5		
6		

b) Determine a formula to represent the length of the line segments at each stage. Use this to determine the length of the line segment at Stage 6.

c) Determine a formula to represent the total length of the line segments. Use your formula to determine the total length after Stage 6.

d) Look for other patterns in this fractal and describe them using formulas, rules, or words.

Achievement Check

14. The air in a hot-air balloon cools as the balloon rises. If the air is not reheated, the balloon's rate of ascent will decrease.

a) A hot-air balloon rises 40 m in the first minute. After that, the balloon rises 75% as far as it did in the previous minute. How far does it rise in each of the next 3 min? Write these distances as a sequence.

b) Determine a function to represent the height of the balloon after n minutes. Is this function continuous or discrete? Explain your reasoning and write the domain of the function.

c) Use the function to determine the height of the balloon after 10 min.

C Extend

15. Three numbers, a, b, and c, form a geometric series so that $a + b + c = 35$ and $abc = 1000$. Determine the values of a, b, and c.

16. For a geometric series, $\frac{S_4}{S_8} = \frac{1}{17}$. Determine the first three terms of the series if the first term is 3.

17. The sum of the first five terms of a geometric series is 186 and the sum of the first six terms is 378. If the fourth term is 48, determine a, r, t_{10}, and S_{10}.

18. Determine n if $3 + 3^2 + 3^3 + \cdots + 3^n = 9840$.

19. If $2b - 2$, $2b + 2$, and $5b + 1$ are the first three terms of a geometric sequence, determine the sum of the first five terms.

Chapter 6 Review

6.1 Sequences as Discrete Functions, pages 354 to 364

1. Given the explicit formula, determine the first four terms in each sequence.

a) $t_n = 4 + 2n^2$, $n \in \mathbb{N}$

b) $f(n) = \dfrac{2n - 1}{n}$, $n \in \mathbb{N}$

2. For each sequence, make a table of values using the term number and term and calculate the finite differences. Then, graph the sequence using the ordered pairs (term number, term) and determine an explicit formula for the nth term, using function notation.

a) $-8, -11, -14, -17, \ldots$

b) $3, 2, -3, -12, \ldots$

6.2 Recursive Procedures, pages 365 to 372

3. Write the first four terms of each sequence, where $n \in \mathbb{N}$.

a) $f(1) = 5$, $f(n) = f(n - 1) - 4$

b) $t_1 = 3$, $t_n = 2t_{n-1} - n$

4. Determine a recursion formula for each sequence.

a) $-2, 7, 16, 25, \ldots$

b) $1, -3, 9, -27, \ldots$

6.3 Pascal's Triangle and Expanding Binomial Powers, pages 373 to 379

5. Use Pascal's triangle to expand each power of a binomial.

a) $(x + 4)^5$

b) $(y - 6)^4$

c) $(m + 2n)^4$

d) $(3p - q)^6$

6. Recall that the sequence of triangular numbers 1, 3, 6, 10, 15, ... can be found in Pascal's triangle. Tetrahedral numbers are the sums of consecutive triangular numbers 1, 4, 10, 20, ...

a) Write the next two terms of the sequence representing tetrahedral numbers.

b) Locate these numbers in Pascal's triangle. Describe their position.

6.4 Arithmetic Sequences, pages 380 to 387

7. For each arithmetic sequence, determine the values of a and d and the formula for the general term. Then, write the next four terms.

a) $3, 1, -1, -3, \ldots$

b) $\dfrac{2}{3}, \dfrac{11}{12}, \dfrac{7}{6}, \dfrac{17}{12}, \ldots$

8. Given the formula for the general term of an arithmetic sequence, write the first three terms. Graph the discrete function that represents each sequence.

a) $f(n) = 4n - 3$

b) $f(n) = 5 - 4n$

9. A theatre has 40 rows of seats. Each row has five more seats than the previous row. If the first row has 50 seats, how many seats are in the

a) 20th row?

b) last row?

6.5 Geometric Sequences, pages 388 to 394

10. Determine whether each sequence is arithmetic, geometric, or neither. Justify your answers.

a) $-1, 9, 19, 29, \ldots$

b) $3, 12, 19, 44, \ldots$

c) $-2, 6, -18, 54, \ldots$

11. Write the first three terms of each geometric sequence.

a) $f(n) = 2(-1)^n$

b) $t_n = -3(2)^{n+1}$

12. Angelica ran in a half-marathon. The length of the race was 21.1 km. She ran 1700 m in the first 10 min of the race. In each 10-min interval after the first one, her distance decreased by 4%. How far did she run in the tenth 10-min interval?

6.6 Arithmetic Series, pages 395 to 401

13. For each arithmetic series, state the values of a and d. Then, determine the sum of the first 20 terms.
   a) $50 + 45 + 40 + \cdots$
   b) $-27 - 21 - 15 - \cdots$

14. On his 12th birthday, Enoch's grandparents deposited \$25 into a savings account for him. Each month after that up to and including his 20th birthday, they deposit \$10 more than the previous month. How much money will Enoch have on his 20th birthday, excluding interest?

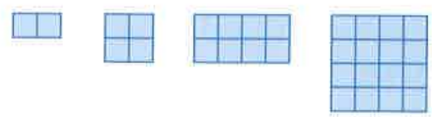
15. Determine the sum of each arithmetic series.
   a) $-6 - 13 - 20 - \cdots - 139$
   b) $-23 - 17 - 11 - \cdots + 43$

6.7 Geometric Series, pages 402 to 409

16. For each geometric series, determine the values of a and r. Then, determine the sum of the first 10 terms.
   a) $2 + 14 + 98 + \cdots$
   b) $8 - 16 + 32 - 64 + \cdots$

17. Determine the sum of each geometric series.
   a) $245 + 24.5 + 2.45 + \cdots + 0.000\,245$
   b) $6561 + 2187 + 729 + \cdots + \frac{1}{6561}$

18. The first four diagrams in a pattern are shown. Each shape is made of small squares with an area of 0.5 cm². Determine the total area of the first 10 diagrams.



Chapter Problem WRAP-UP

Throughout this chapter, you explored a variety of fractals and sequences related to fractals. This sequence has fractal properties.

1, 1, 2, 1, 3, 2, 4, 1, 5, 3, 6, 2, 7, 4, 8, 1, 9, 5, 10, 3, 11, 6, 12, 2, 13, 7, 14, 4, 15, 8, ...

a) Copy the sequence and write the next 20 terms.

Hint: Look for patterns in the term number for each natural number in the sequence.

b) Delete the first occurrence of each natural number. Write the remaining sequence. Describe this sequence in words and write a formula to represent it.

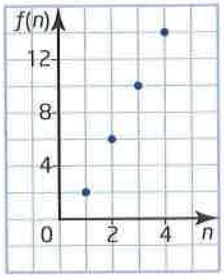
c) Repeat this procedure and describe the pattern that emerges.

d) How is the procedure like constructing a fractal?

Chapter 6 Practice Test

For questions 1 to 5, select the best answer.

1. Which is a recursion formula for the sequence shown?



A $f(n) = f(n - 1) + 4$

B $f(n) = 4n - 2$

C $f(n) = 2 + (n - 1)(4)$

D $f(1) = 2,$
$f(n) = f(n - 1) + 4$

2. Which expressions represent the missing terms in the binomial expansion shown?

$$(x + y)^7 = x^7 + 7x^6y + \square + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + \square + y^7$$

A $21y^5x^2, 7yx^6$

B $21x^5y^2, 7xy^6$

C $-21x^5y^2, -7xy^6$

D x^5y^2, xy^6

3. What is the formula for the general term of an arithmetic sequence with $a = 8$ and $d = 2$?

A $t_n = 2 + (n + 1)(8)$

B $t_n = 8 + (n - 1)(2)$

C $t_n = 8 + (n + 1)(2)$

D $t_n = 2 + (n - 1)(8)$

4. What are the first three terms of a geometric sequence with $a = 3$ and $r = 2$?

A 3, 5, 7

B 2, 6, 18

C 3, 6, 12

D 2, 5, 8

5. Which series is neither arithmetic nor geometric?

A $9 + 15 + 21 + 27 + \cdots$

B $1 + 8 + 27 + 64 + \cdots$

C $64 - 32 + 16 - 8 + \cdots$

D $-3 - 2.7 - 2.4 - 2.1 - \cdots$

6. Determine the first five terms of each sequence. Graph the sequence and state whether it is arithmetic, geometric, or neither.

a) $t_n = 9 - 5n$

b) $f(n) = 2n^2 + 3n - 4$

c) $f(n) = \frac{1}{8}(4)^{n-1}$

d) $t_n = 0.2n + 0.8$

e) $t_n = \frac{n + 4}{2}$

f) $f(n) = -3(2)^n$

7. Write an explicit formula and a recursion formula for each sequence.

a) 64, 32, 16, 8, ...

b) −20, −17, −14, −11, ...

c) 80, 76, 72, 68, ...

d) −4000, 1000, −250, 62.5, ...

e) −3, −6, −12, −24, ...

f) $-12\sqrt{2}, -10\sqrt{2}, -8\sqrt{2}, -6\sqrt{2}, \ldots$

8. Write t_{11} for each sequence.

a) 6, 10, 14, 18, ...

b) −3, −6, −12, −24, ...

c) 5, −10, 20, −40, ...

d) −5, −10, −15, −20, ...

9. Given the explicit formula, write t_{15} for each sequence.

a) $f(n) = 2(-3)^{n+1}$

b) $t_n = 25n + 50$

c) $t_n = 10(0.1)^{2n}$

d) $f(n) = \frac{-3n}{4}$

10. Determine the number of terms in each sequence.

a) 5, 8, 11, ..., 62

b) −4, 12, −36, ..., −19 131 876

11. A new lake is being excavated. One day, 1.6 t of material is removed from the lake bed. On each of 10 days after that, 5% more is removed.

 a) Write the first three excavation amounts as a sequence.

 b) Write a recursion formula to represent the amount removed each day. Use this to determine the amount removed on the fifth day.

12. Determine the specified sum for each series.

 a) S_{10} for $200 + 100 + 50 + \cdots$

 b) S_{18} for $12 + 5 - 2 + \cdots$

13. Determine the sum of each arithmetic series.

 a) $120 + 110 + 100 + \cdots - 250$

 b) $8 + 24 + 40 + \cdots + 280$

14. Determine the sum of each geometric series.

 a) $\frac{2}{81} + \frac{4}{27} + \frac{8}{9} + \cdots + 6912$

 b) $5 + 10 + 20 + \cdots + 2560$

15. Use Pascal's triangle to help you expand each expression.

 a) $(b - 3)^5$ **b)** $(2x - 5y)^6$

16. The sum of the first three terms of a series is 32. Determine the fourth term if the sum of the first four terms is

 a) 40 **b)** 25

17. Determine the sum of the first 15 terms of an arithmetic series if the middle term is 92.

18. Which is greater, A or B? Explain your reasoning.

 $A = 50^2 - 49^2 + 48^2 - 47^2 + \cdots + 2^2 - 1^2$

 $B = 50 + 49 + 48 + 47 + \cdots + 2 + 1$

19. In the arrangement of letters shown, starting from the top, proceed to the row below by moving diagonally to the immediate right or left. Determine the number of different paths that will spell the name PASCAL.

P
A A
S S S
C C C C
A A A A A
L L L L L L

20. A new wood stain loses 6.5% of its colour every year in a city that experiences a lot of hot, sunny days. What percent of colour will a fence in this city have 6 years after being stained?

21. In an arithmetic series, the 4th term is 62 and the 14th term is 122. Determine the sum of the first 30 terms.

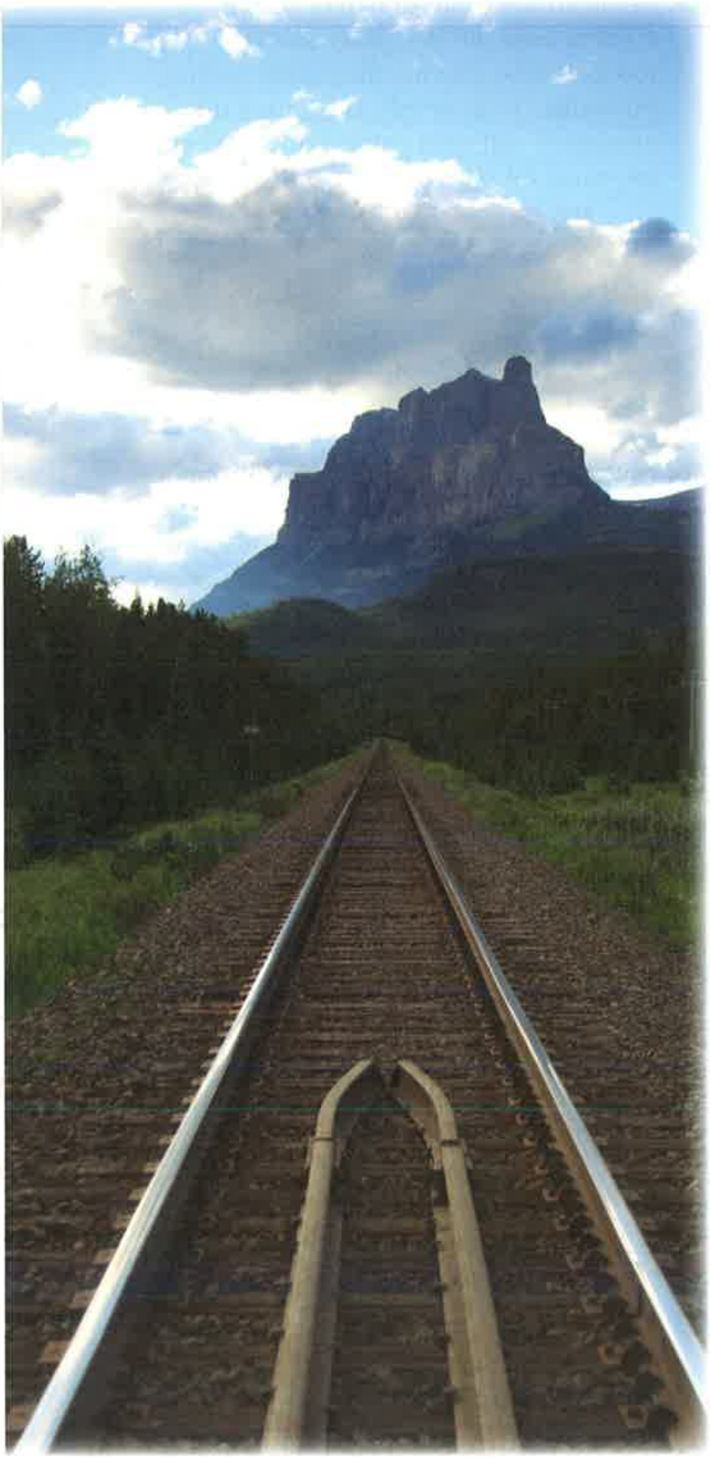
22. A sailboat worth \$140 000 depreciates 18% in the first year and 10% every year after that. How much will it be worth 8 years after it is bought?

23. A magic square is an arrangement of numbers in which all rows, columns, and diagonals have the same sum. Using the magic square shown, substitute each number with the corresponding term from the Fibonacci sequence.

2	7	6
9	5	1
4	3	8

Show that the sum of the products of the rows is equal to the sum of the products of the columns.

Task



Mathematics in Media Studies

For a media studies project, Khatija wants to produce an image that appears to have a vanishing point. Photocopiers and photoediting software will increase or decrease an image by a chosen percent. Khatija is working with an original image that has dimensions 320 mm by 240 mm. She increased the original by 50% and then decreased the image by 50%, repeating this process several times.

a) Plot a graph of the length versus step number. Show at least five steps of the process.

b) Determine a recursion formula for the end result after each step.

c) Determine an explicit formula for the end result after each step.

d) What will the length of the image be after 10 steps of this process? Give exact answers.

e) Will the length of the image ever reach 0 mm? Explain.

f) Is this relationship linear, quadratic, exponential, or reciprocal? Justify your answer.

g) For the same project, Calvin used repetitive addition of a specific number of millimetres followed by subtraction of a different number of millimetres. Use your own choices of lengths to add and subtract and repeat steps a) to f) for Calvin's process. Compare your results to your results from Khatija's process.