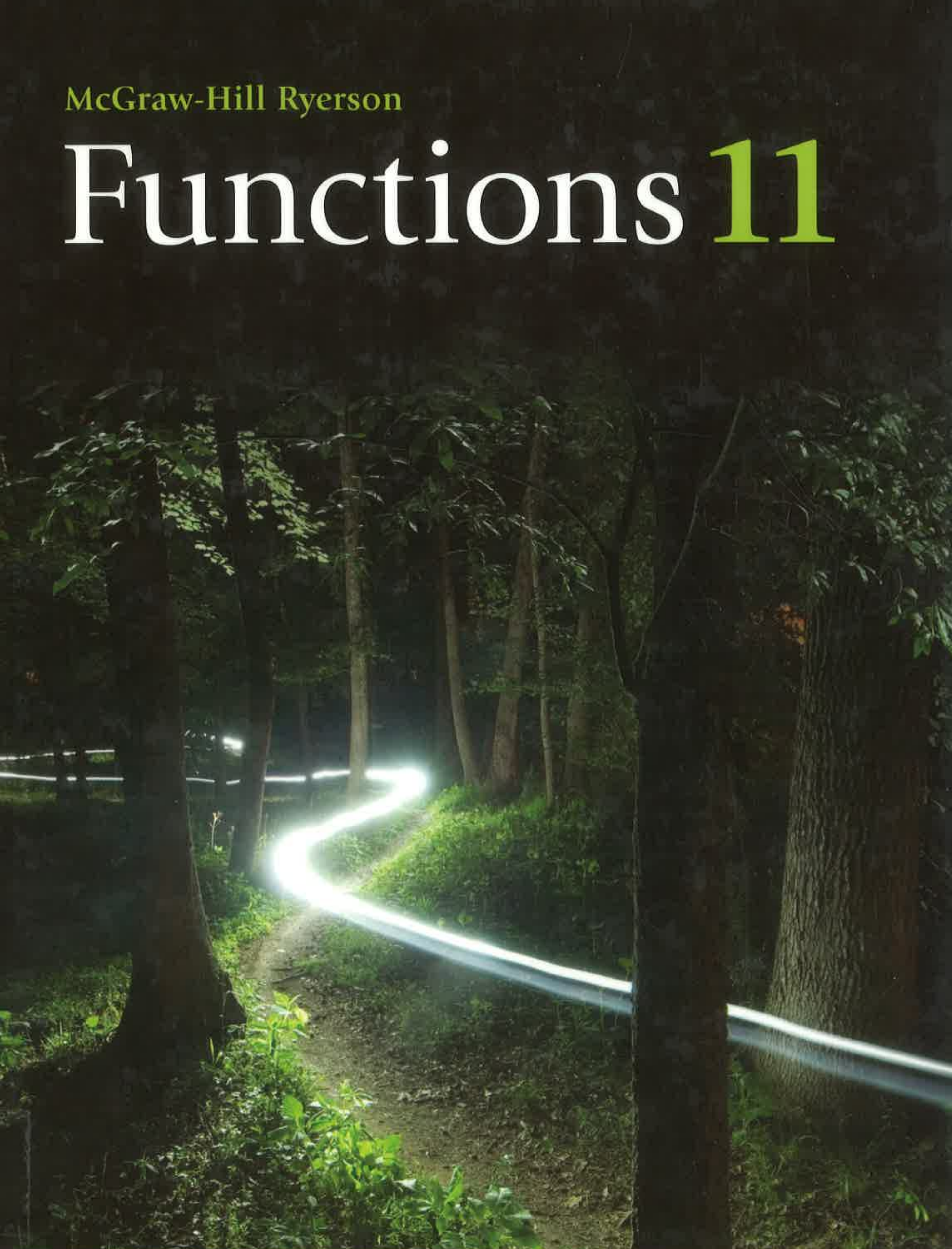
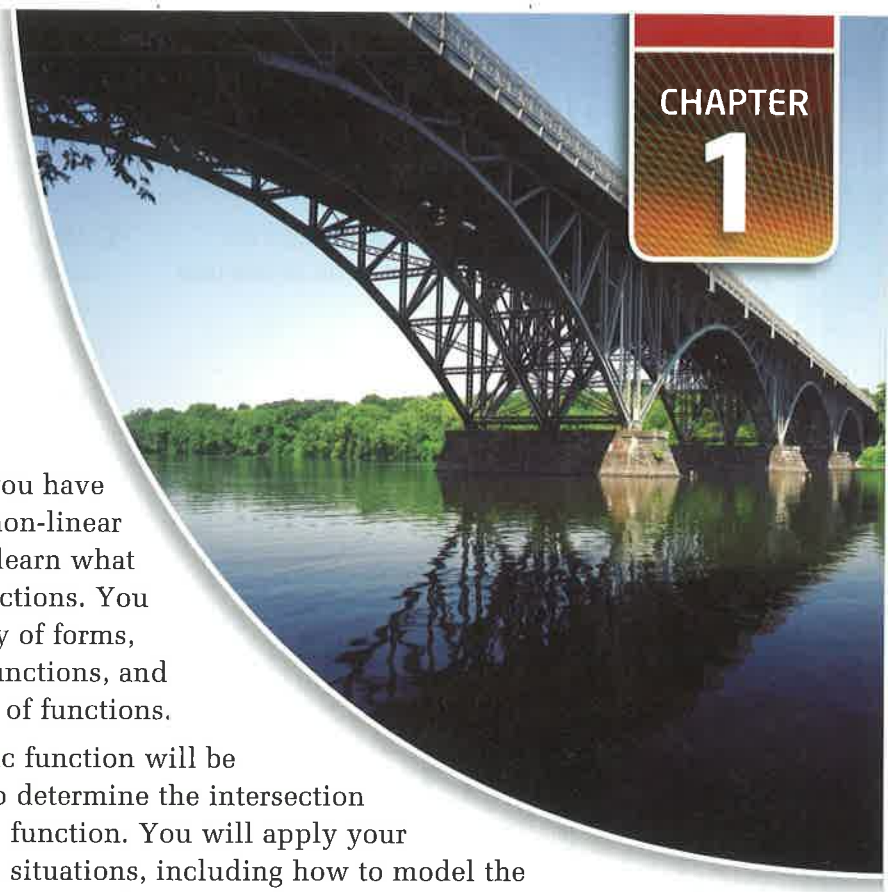
McGraw-Hill Ryerson

# Functions 11






# Functions

In previous mathematics courses, you have studied linear relations and some non-linear relations. In this chapter, you will learn what distinguishes some relations as functions. You will represent functions in a variety of forms, identify the domain and range of functions, and investigate the behaviour of graphs of functions.

Your understanding of the quadratic function will be extended and you will learn how to determine the intersection of a linear function and a quadratic function. You will apply your knowledge of quadratics to real-life situations, including how to model the arch of the support of a bridge.

## By the end of this chapter, you will

- explain the meaning of the term *function* and distinguish a function from a relation that is not a function, through investigation of linear and quadratic relations using a variety of representations
- represent linear and quadratic functions using function notation, given their equations, tables of values, or graphs, and substitute into and evaluate functions
- explain the meanings of the terms *domain* and *range*, through investigation using numeric, graphical, and algebraic representations of the functions $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$; describe the domain and range of a function appropriately; and explain any restrictions on the domain and range in contexts arising from real-world applications
- determine the number of zeros of a quadratic function, using a variety of strategies
- determine the maximum or minimum value of a quadratic function whose equation is given in the form $f(x) = ax^2 + bx + c$, using an algebraic method
- solve problems involving quadratic functions arising from real-world applications and represented using function notation
- determine, through investigation, the transformational relationship among the family of quadratic functions that have the same zeros, and determine the algebraic representation of a quadratic function, given the real roots of the corresponding quadratic equation and a point on the function
- solve problems involving the intersection of a linear function and a quadratic function graphically and algebraically
- verify, through investigation with and without technology, that $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, $a \geq 0$ and $b \geq 0$, and use this relationship to simplify radicals and radical expressions obtained by adding, subtracting, and multiplying

# Prerequisite Skills

*Refer to the Prerequisite Skills Appendix on pages 478 to 495 for examples of the topics and further practice.*

## Graphs and Lines

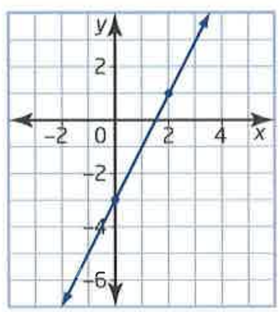
**1.** Graph each linear relation.

**a)** $y = 3x - 1$ **b)** $y = -\frac{1}{4}x + 5$

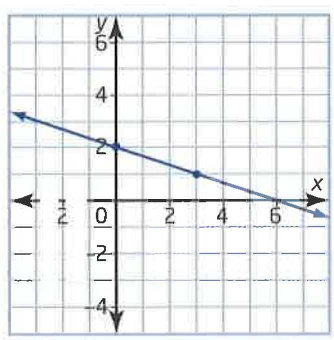
**c)** $2x - 3y + 12 = 0$ **d)** $y = 6$

**2.** Determine the equation in the form $y = mx + b$ for each linear relation.

**a)**




**b)**




**3.** Determine the equation in the form $y = mx + b$ for the line passing through each pair of points.

**a)** (0, 8) and (4, 3)

**b)** (−3, 13) and (2, −2)

**c)** (4, −1) and (12, 9)

**4.** Graph each pair of linear relations to find their point of intersection.

**a)** $y = 2x + 4$ and $y = -x + 1$

**b)** $y = \frac{1}{2}x - 5$ and $y = -2x + 5$

**c)** $3x - 5y = -4$ and $-2x + 3y = 2$

**5.** Use an algebraic method to find the point of intersection of each pair of lines.

**a)** $y = 3x + 5$
$2x - y = -6$

**b)** $y = x + 4$
$y = 2x - 1$

**c)** $x - 2y = 7$
$2x - 3y = 13$

## Work With Polynomials

**6.** Expand and simplify each expression.

**a)** $(x + 2)^2$ **b)** $(n + 3)(n - 3)$

**c)** $\frac{1}{2}(t - 4)^2$ **d)** $3(x + 3)(x - 2)$

**e)** $4(k - 1)(k + 1)$ **f)** $\frac{2}{3}(3x - 1)(2x + 3)$

**7.** Factor completely.

**a)** $x^2 + 2x - 15$ **b)** $x^2 + 6x + 9$

**c)** $9n^2 - 25$ **d)** $-x^2 - x + 12$

**e)** $3t^2 + 6t + 3$ **f)** $-5x^2 + 40x - 80$

**8.** Identify if each quadratic expression is a perfect square trinomial. For the perfect square trinomials, write the factored form.

**a)** $x^2 - 6x + 12$ **b)** $x^2 - 12x + 36$

**c)** $2x^2 + 4x + 1$ **d)** $x^2 + 18x + 9$

**e)** $x^2 + 4x + 4$ **f)** $4n^2 + 12n + 9$

**9.** What value of $k$ makes each quadratic expression a perfect square trinomial?

**a)** $x^2 + 8x + k$ **b)** $x^2 - 10x + k$

**c)** $x^2 - 2x + k$ **d)** $x^2 + 14x + k$

**e)** $x^2 + 5x + k$ **f)** $x^2 - 11x + k$

**g)** $x^2 + x + k$ **h)** $x^2 - 3x + k$

**10.** Factor out the rational coefficient of the $x^2$-term in each.

**a)** $\frac{1}{2}x^2 - \frac{3}{2}x$ **b)** $\frac{2}{3}x^2 + 5x$

**c)** $-\frac{1}{5}x^2 - 2x$ **d)** $-\frac{3}{4}x^2 + 9x$

## Quadratic Relations

**11.** For each quadratic relation, state

**i)** the coordinates of the vertex

**ii)** the equation of the axis of symmetry

**iii)** the direction of opening

**iv)** the $y$-intercept

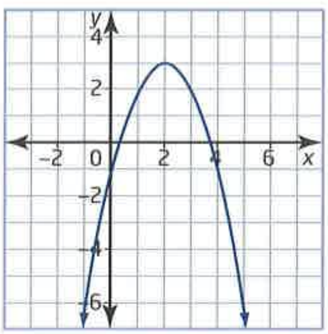
Then, sketch a graph of the relation.

**a)** $y = 2(x + 1)^2 - 3$

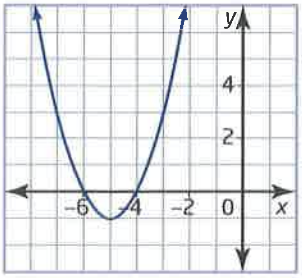
**b)** $y = -\frac{5}{3}(x - 3)^2 + 1$

**12.** Determine the equation of the quadratic relation that corresponds to each graph. Each graph has the same shape as $y = x^2$.

**a)**




**b)**




**13.** Complete the square to express each quadratic relation in the form $y = a(x - h)^2 + k$. Then, give the coordinates of the vertex.

**a)** $y = x^2 + 4x + 1$

**b)** $y = x^2 - 10x - 5$

**14.** Without graphing, predict how the graphs of the equations in each pair will differ. Explain your reasoning.

**a)** $y = (x + 5)^2$ and $y = (x + 5)^2 + 2$

**b)** $y = x^2 - 4x + 3$ and $y = x^2 - 4x$

**15.** **Use Technology** Verify your answers to question 14 by graphing the two equations using a graphing calculator.

## Chapter Problem

Andrea has a co-op placement at an actuarial firm. Actuarial science applies mathematical and statistical methods to assess risk for insurance providers and financial institutions. Andrea's assignments include collecting numerical data and developing equations for these businesses. Throughout the chapter, you will be looking at a variety of tasks that Andrea has been given in her co-op placement.

1.1

# Functions, Domain, and Range

When mathematicians and scientists recognize a relationship between items in the world around them, they try to model the relationship with an equation. The concept of developing an equation is used in other fields too. Economists predict the growth of sectors of the economy using equations. Pollsters try to predict the outcome of an election using equations. Does the value of one measured quantity guarantee a unique value for the second related quantity? This question defines the difference between a **relation** and a **function**.

**relation**

- an identified pattern between two variables that may be represented as ordered pairs, a table of values, a graph, or an equation

**function**

- a relation in which each value of the independent variable (the first coordinate) corresponds to exactly one value of the dependent variable (the second coordinate)

## Investigate A

### How can you tell if a relation is a function?

Data on summer jobs are collected from some students in a grade 11 class. Some analysis is done to look for patterns in the data.

**A: Neil's Time Worked and Amount Earned, by Week**

| Time Worked (h) | Amount Earned ($) |
|---|---|
| 20 | 190 |
| 18 | 171 |
| 26 | 247 |
| 22 | 209 |
| 30 | 285 |
| 24 | 228 |
| 10 | 95 |
| 14 | 126 |

**B: Number of Weeks Worked and Amount Earned by 10 Different Students**

| Number of Weeks Worked | Total Amount Earned ($) |
|---|---|
| 10 | 1850 |
| 8 | 675 |
| 6 | 520 |
| 9 | 480 |
| 8 | 1100 |
| 10 | 1400 |
| 8 | 975 |
| 6 | 1200 |
| 8 | 1580 |
| 9 | 1740 |

1. Graph the given sets of data.

2. Describe any trends in the two graphs.

3. From the graph of the data in table A, can you predict how much Neil would earn if he worked 28 h one week?

4. From the graph of the data in table B, can you predict the amount that a student who worked for 8 weeks would earn?

5. **Reflect** Which set of data is a function? Explain using the terms *independent variable* and *dependent variable*.

**Tools**

- grid paper

or

- graphing calculator

## Investigate B

**Tools**

- grid paper

### How can you make connections between equations, graphs, and functions?

**Method 1: Use Pencil and Paper**

The first Investigate illustrated that one value for the independent variable can be associated with more than one different value for the dependent variable. Any relation that has this property is not a function. In this Investigate, you will look at how this concept can be related to the equation for a relation.

1. Copy and complete the tables of values for the relations $y = x^2$ and $x = y^2$.

| $x$ | $y = x^2$ | Coordinates |
|---|---|---|
| −3 | 9 | (−3, 9) |
| −2 | | |
| −1 | | |
| 0 | | |
| 1 | | |
| 2 | | |
| 3 | | |

| $x = y^2$ | $y$ | Coordinates |
|---|---|---|
| 9 | −3 | (9, −3) |
| | −2 | |
| | −1 | |
| | 0 | |
| | 1 | |
| | 2 | |
| | 3 | |

2. Graph both relations on the same set of axes.

3. On the same set of axes, draw vertical lines with equations $x = -3$, $x = -2$, $x = -1$, $x = 1$, $x = 2$, and $x = 3$.

4. **Reflect** Compare how the lines drawn in step 3 intersect each of the relations. Which relation is a function? Explain why.

**Tools**

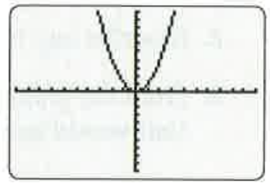
- graphing calculator

**Technology Tip**

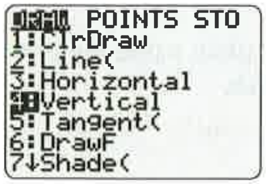
Refer to the Technology Appendix, pages 496 to 516, if you need help with graphing equations.

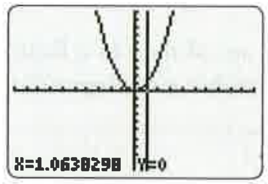
**Method 2: Use a Graphing Calculator**

**1.** Graph **Y1** = $x^2$.
Use the standard window settings.



**2.** • Press (2nd) (DRAW) to access the **Draw** menu.
• Choose **4:Vertical**.
• Use the left and right cursor arrows to move the vertical line.




If you press (ENTER), the line will be secured at that spot. Press (2nd) (DRAW) and select **1:ClrDraw** to remove the vertical line.

**3.** Is $y = x^2$ a function? Explain why or why not.

**4.** Graph $x = y^2$ by first solving the equation for $y$ to obtain $y = \pm\sqrt{x}$.
• Enter **Y1** = $(x)$^0.5 and **Y2** = $-(x)$^0.5.

**5.** Repeat step 2. Is $x = y^2$ a function? Explain why or why not.

## Example 1

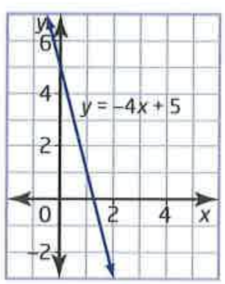
### Use the Vertical Line Test

**vertical line test**

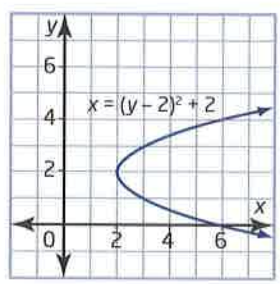
- a method of determining whether a relation is a function
- If every vertical line intersects the relation at only one point, then the relation is a function.

Use the **vertical line test** to determine whether each relation is a function. Justify your answer.

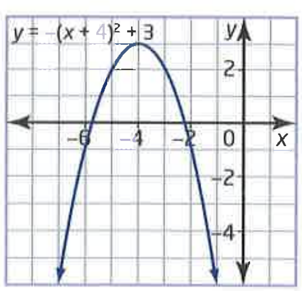
**a)**



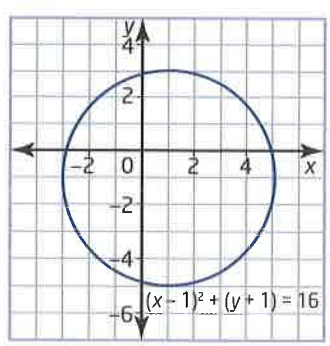

**b)**




**c)**

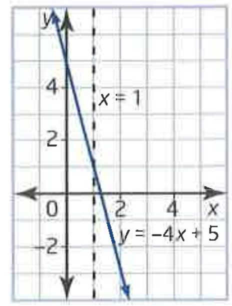



**d)**

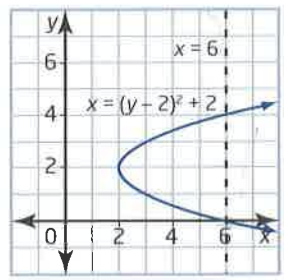


## Solution

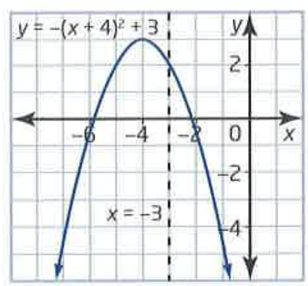
**a)** This relation is a function. No vertical line can be drawn that will pass through more than one point on the line.



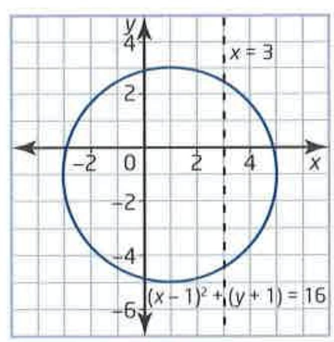

**b)** This relation is a not function. An infinite number of vertical lines can be drawn that will pass through more than one point on the curve. For example, the vertical line $x = 6$ passes through the points (6, 4) and (6, 0).




**c)** This relation is a function. No vertical line can be drawn that will pass through more than one point on the curve.




**d)** This relation is not a function. An infinite number of vertical lines can be drawn that will pass through more than one point on the circle.



**domain**

- the set of first coordinates of the ordered pairs in a relation

**range**

- the set of second coordinates of the ordered pairs in a relation

For any relation, the set of values of the independent variable (often the $x$-values) is called the **domain** of the relation. The set of the corresponding values of the dependent variable (often the $y$-values) is called the **range** of the relation. For a function, for each given element of the domain there must be exactly one element in the range.

**Connections**

Brace brackets { } are used to denote a set of related data points or values.

## Example 2

### Determine the Domain and Range From Data

Determine the domain and range of each relation. Use the domain and range to determine if the relation is a function.

**a)** $\{(-3, 4), (5, -6), (-2, 7), (5, 3), (6, -8)\}$

**b)** The table shows the number of children of each age at a sports camp.

| Age | Number |
|---|---|
| 4 | 8 |
| 5 | 12 |
| 6 | 5 |
| 7 | 22 |
| 8 | 14 |
| 9 | 9 |
| 10 | 11 |

### Solution

**a)** domain $\{-3, -2, 5, 6\}$, range $\{-8, -6, 3, 4, 7\}$

This relation is not a function. The $x$-value $x = 5$ has two corresponding $y$-values, $y = -6$ and $y = 3$. The domain has four elements but the range has five elements. So, one value in the domain must be associated with two values in the range.

**b)** domain $\{4, 5, 6, 7, 8, 9, 10\}$, range $\{5, 8, 9, 11, 12, 14, 22\}$

This is a function because for each value in the domain there is exactly one value in the range.

**real number**

- a number in the set of all integers, terminating decimals, repeating decimals, non-terminating decimals, and non-repeating decimals, represented by the symbol $\mathbb{R}$

When the equation of a relation is given, the domain and range can be determined by analysing the allowable values from the set of **real numbers**.

## Example 3

### Determine the Domain and Range From Equations

Determine the domain and the range for each relation. Sketch a graph of each.

**a)** $y = 2x - 5$ **b)** $y = (x - 1)^2 + 3$ **c)** $y = \dfrac{1}{x + 3}$

**d)** $y = \sqrt{x - 1} + 3$ **e)** $x^2 + y^2 = 36$

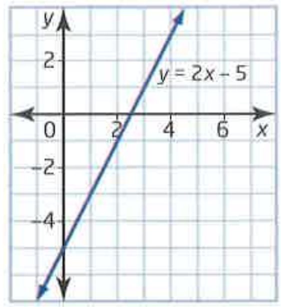
### Solution

**a)** $y = 2x - 5$ is a linear relation. There are no restrictions on the values that can be chosen for $x$ or $y$.

domain $\{x \in \mathbb{R}\}$

range $\{y \in \mathbb{R}\}$

Read as "the domain is all real numbers."




**Connections**

The notation $\{x \in \mathbb{R}\}$ is set notation. It is a concise way of expressing that $x$ is any real number. The symbol $\in$ means "is an element of."

**b)** $y = (x - 1)^2 + 3$ is a quadratic relation.

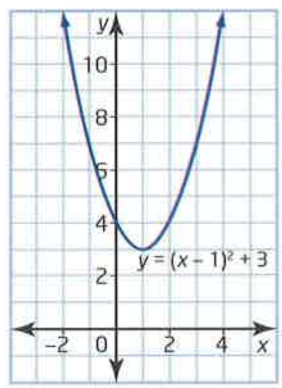
There are no restrictions on the values that can be chosen for $x$, so the domain is all real numbers.

domain $\{x \in \mathbb{R}\}$

The parabola has a minimum at its vertex (1, 3).

All values of $y$ are greater than or equal to 3.

range $\{y \in \mathbb{R}, y \geq 3\}$



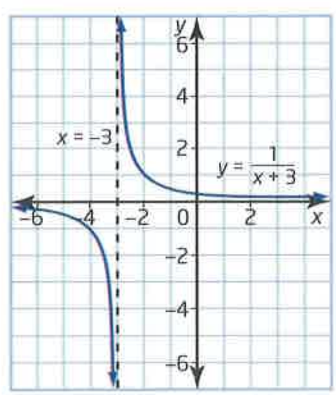

**c)** Division by zero is undefined. The expression in the denominator of $\dfrac{1}{x + 3}$ cannot be zero. So, $x + 3 \neq 0$, which means that $x \neq -3$. All other values can be used for $x$. The vertical line $x = -3$ is called an **asymptote**.

domain $\{x \in \mathbb{R}, x \neq -3\}$

Read as "the domain is all real numbers that are not equal to −3."

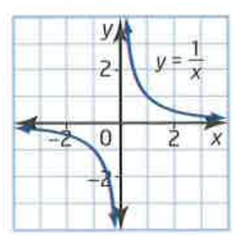
For the range, there can never be a situation where the result of the division is zero, as 1 divided by a non-zero value can never result in an answer of 0. This function has another asymptote, the $x$-axis. Any real number except −3 can be used for $x$ and will result in all real numbers except 0 for the range. Use a table of values or a graphing calculator to check this on the graph.

range $\{y \in \mathbb{R}, y \neq 0\}$




**asymptote**

- a line that a curve approaches more and more closely but never touches
- For example, for the graph of $y = \frac{1}{x}$, the $x$-axis and the $y$-axis are asymptotes.



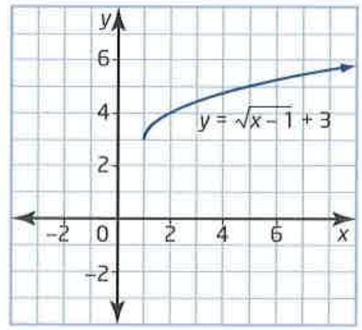
**d)** The expression under a radical sign must be greater than or equal to zero.

So, in $\sqrt{x - 1} + 3$, $x - 1 \geq 0$, or $x \geq 1$.

domain $\{x \in \mathbb{R}, x \geq 1\}$

The value of the radical is always 0 or greater and is added to 3 to give the value of $y$. So, the $y$-values are always greater than or equal to 3. This gives the range.

range $\{y \in \mathbb{R}, y \geq 3\}$




**Connections**

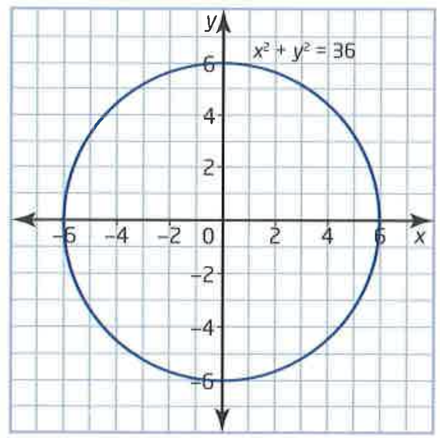
In grade 10, you learned that $x^2 + y^2 = r^2$ is the equation of a circle with centre the origin and radius $r$.

**e)** In $x^2 + y^2 = 36$, $x^2$ must be less than or equal to 36, as must $y^2$, since both $x^2$ and $y^2$ are always positive. So, the values for $x$ and $y$ are from $-6$ to 6.

domain $\{x \in \mathbb{R}, -6 \leq x \leq 6\}$

Read as "the domain is all real numbers that are greater than or equal to −6 and less than or equal to 6."

range $\{y \in \mathbb{R}, -6 \leq y \leq 6\}$




## Example 4

### Determine the Domain and Range of an Area Function

Amy volunteers to help enclose a rectangular area for a dog run behind the humane society. The run is bordered on one side by the building wall. The society has 100 m of fencing available.

**a)** Express the area function in terms of the width.

**b)** Determine the domain and range for the area function.

## Solution

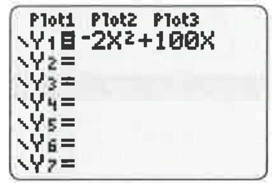
Let $x$ represent the width of the rectangular pen and $100 - 2x$ represent the length, both in metres. Let $A$ represent the area, in square metres.


**a)** $A(x) = x(100 - 2x)$ Area = length × width

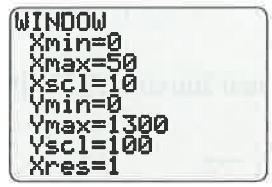
$= -2x^2 + 100x$

**b)** For the domain, $x > 0$, since there must be a width to enclose an area. For the length to be greater than zero, $x < 50$.

domain $\{x \in \mathbb{R}, 0 < x < 50\}$



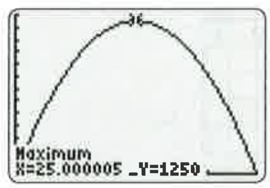

The area function is a quadratic opening downward. Graph the area function to find its maximum. The vertex is at (25, 1250).




The maximum value of the area function is 1250.

range $\{A \in \mathbb{R}, 0 < A < 1250\}$

An area must be greater than zero.




### Connections

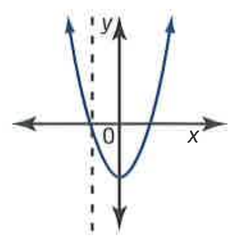
You could find the vertex algebraically by expressing the area function in vertex form, $y = a(x - h)^2 + k$.

## Key Concepts

- A relation is a function if for each value in the domain there is exactly one value in the range. This table of values models a function.

| x | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| y | 5 | 3 | 1 | −1 | −3 |

- The vertical line test can be used on the graph of a relation to determine if it is a function. If every vertical line passes through at most one point on the graph, then the relation is a function.




- The domain and the range of a function can be found by determining if there are restrictions based on the defining equation. Restrictions on the domain occur because division by zero is undefined and because expressions under a radical sign must be greater than or equal to zero. The range can have restrictions too. For example, a quadratic that opens upward will have a minimum value.
- Set notation is used to write the domain and range for a function. For example, for the function $y = x^2 + 2$:

  domain $\{x \in \mathbb{R}\}$ and range $\{y \in \mathbb{R}, y \geq 2\}$

## Communicate Your Understanding

**C1** Suzanne is unclear as to why the graphs of $y = x^2$ and $x = y^2$ are different, and why one is a function and the other is not. How would you help Suzanne?

**C2** Is it possible to determine if a relation is a function if you are only given the domain and range in set notation? Explain your reasoning.

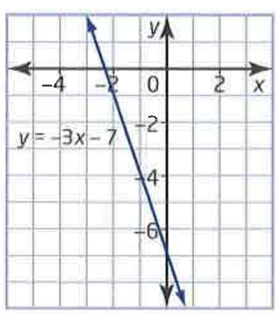
**C3** Sagar missed the class on restrictions and has asked you for help. Lead him through the steps needed to find the domain and range of the function $y = \frac{-4}{2x + 1}$.

## A Practise

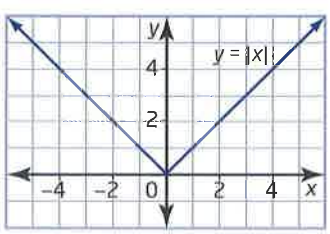
*For help with questions 1 and 2, refer to Example 1.*

**1.** Which graphs represent functions? Justify your answer.

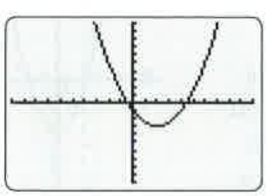
**a)**



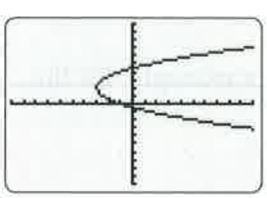

**b)**




**c)**



**d)**



**2.** Is each relation a function? Explain. Sketch a graph of each.

**a)** $y = x - 5$

**b)** $x = y^2 - 3$

**c)** $y = 2(x - 1)^2 - 2$

**d)** $x^2 + y^2 = 4$

*For help with questions 3 and 4, refer to Example 2.*

**3.** State the domain and the range of each relation. Is each relation a function? Justify your answer.

**a)** $\{(5, 5), (6, 6), (7, 7), (8, 8), (9, 9)\}$

**b)** $\{(3, -1), (4, -1), (5, -1), (6, -1)\}$

**c)** $\{(1, 6), (1, -14), (1, 11), (1, -8), (1, 0)\}$

**d)** $\{(1, 5), (4, 11), (3, 9), (5, 1), (11, 4)\}$

**e)** $\{(3, 2), (2, 1), (1, 0), (2, -1), (3, -2)\}$

**4.** The domain and range of some relations are given. Each relation consists of five points. Is each a function? Explain.

**a)** domain $\{1, 2, 3, 4, 5\}$, range $\{4\}$

**b)** domain $\{-3, -1, 1, 3, 5\}$, range $\{2, 4, 6, 8, 10\}$

**c)** domain $\{2, 3, 6\}$, range $\{-4, 6, 7, 11, 15\}$

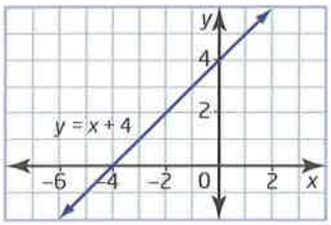
**d)** domain $\{-2\}$, range $\{9, 10, 11, 12, 13\}$

## B Connect and Apply

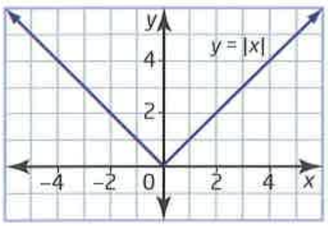
*For help with questions 5 and 6, refer to Example 3.*

**5.** State the domain and the range of each relation.

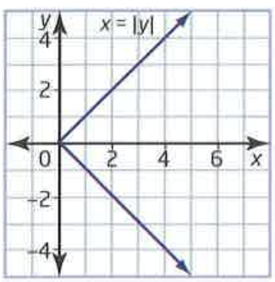
**a)**



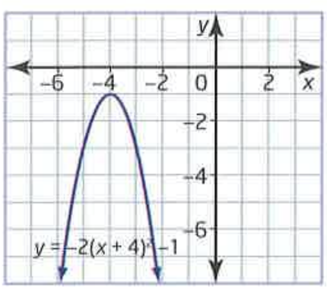

**b)**



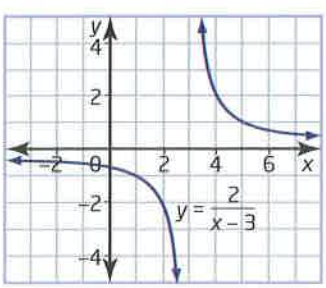

**c)**




**d)**




**e)**




**6.** Determine the domain and the range of each relation. Use a graph to help you if necessary.

**a)** $y = -x + 3$

**b)** $y = (x + 1)^2 - 4$

**c)** $y = -3x^2 + 1$

**d)** $x^2 + y^2 = 9$

**e)** $y = \dfrac{1}{x + 3}$

**f)** $y = \sqrt{2x + 1}$

**7.** For each given domain and range, draw one relation that is a function and one that is not. Use the same set of axes for each part.

**a)** domain $\{x \in \mathbb{R}\}$, range $\{y \in \mathbb{R}\}$

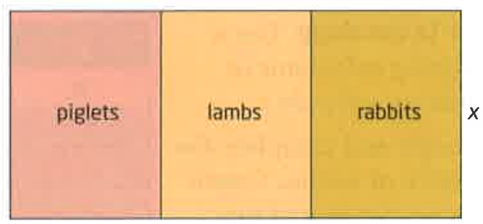
**b)** domain $\{x \in \mathbb{R}, x \geq 4\}$, range $\{y \in \mathbb{R}\}$

**c)** domain $\{x \in \mathbb{R}\}$, range $\{y \in \mathbb{R}, y \leq -1\}$

**d)** domain $\{x \in \mathbb{R}, x \leq 2\}$, range $\{x \in \mathbb{R}, y \geq -2\}$

*For help with questions 8 and 9, refer to Example 4.*

**8.** Soula has 90 m of fencing to enclose an area in a petting zoo with two dividers to separate three types of young animals. The three pens are to have the same area.




**a)** Express the area function for the three pens in terms of $x$.

**b)** Determine the domain and the range for the area function.

**9.** Is each relation a function? Justify your answer. If the relation is a function, state the independent variable and the dependent variable.

**a)** The amount of money taken in for the fundraiser is related to the number of raffle tickets a hockey team sells.

**b)** The age of students is related to their grade level.

**c)** The time it takes Jung Yoo to walk to school is related to the speed at which he walks.

10. A rectangular part of a parking lot is to be fenced off to allow some repairs to be done. The workers have fourteen 3-m sections of pre-assembled fencing to use. They want to create the greatest possible area in which to work.


a) How can the fencing be used to create as large an enclosed area as possible?

b) Show why this produces the greatest area using the given fencing sections, but does not create the greatest area that can be enclosed with 42 m of fencing.

11. Determine the range of each relation for the domain $\{1, 2, 3, 4, 5\}$.

a) $y = 6x - 6$ b) $y = x^2 - 4$

c) $y = 3$ d) $y = 2(x - 1)^2 - 1$

e) $y = \dfrac{1}{x + 2}$ f) $x^2 + y^2 = 25$

12. **Use Technology** Use a graphing calculator or graphing software.

| $x = y^2 - 3$ | $y$ |
|---|---|
| 6 | −3 |
| | −2 |
| | −1 |
| | 0 |
| | 1 |
| | 2 |
| | 3 |

a) Copy and complete the table of values. Create a scatter plot of the resulting data using a graphing calculator.

b) Enter the equations $y = \sqrt{x + 3}$ and $y = -\sqrt{x + 3}$ and display their graphs.

c) Explain the result of the display of the data and the equations.

d) Explain how this illustrates that the equation $x = y^2 - 3$ defines a relation that is not a function.

**Technology Tip**

Refer to the Technology Appendix, pages 496 to 516, if you need help with plotting data or graphing equations.

13. It is said that you cannot be in two places at once. Explain what this statement means in terms of relations and functions.

14. Describe the graph of a relation that has

a) one entry in the domain and one entry in the range

b) one entry in the domain and many entries in the range

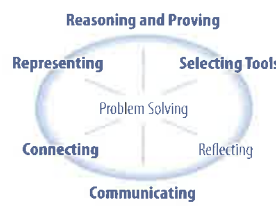
c) many entries in the domain and one entry in the range

15. Sketch a relation with the following properties.

a) It is a function with domain all the real numbers and range all real numbers less than or equal to 5.

b) It is not a function and has domain and range from −3 to 3.

16. A car salesperson is paid according to two different relations based on sales for the week. In both relations, $s$ represents sales and $P$ represents the amount paid, both in dollars.




For sales of less than \$100 000, $P = 0.002s + 400$.
For sales of \$100 000 and over, $P = 0.0025s + 400$.

a) State the domain and range for each relation.

b) Does each relation define a function? Justify your answer.

c) Graph the two relations on the same set of axes.

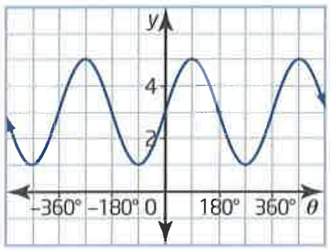
d) Connect what happens on the graph at $s = 100\,000$ to its meaning for the salesperson.

## C Extend

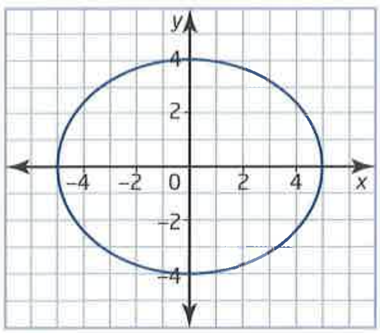
**17.** Is it possible for two different functions to have the same domain and range? Explain, giving examples.

**18.** State the domain and the range for the two relations shown. Is each a function?

**a)**




**b)**




**19.** **Math Contest** What is the domain of the function $y = \frac{\sqrt{x - 3}}{\sqrt{5 - x}}$?

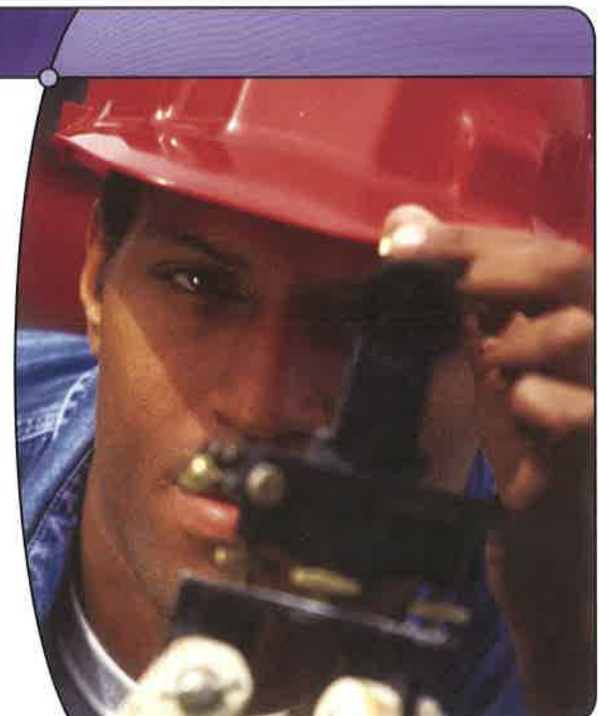
**20.** **Math Contest** Frank bought supplies for school. In the first store, he spent half his money plus \$10. In the second store, he spent half of what he had left plus \$10. In the third store, he spent 80% of what he had left. He came home with \$5. How much did he start out with?

**21.** **Math Contest** Find the number of factors of 2520.

**22.** **Math Contest** For what values of $x$ is $\sqrt{x + 2} > x$?

### Career Connection

Khaldun completed a 4-year degree in mineral engineering at the University of Toronto. He works in northern Canada for an international diamond-mining company. In his job as a mining engineer, Khaldun uses his knowledge of mathematics, physics, geology, and environmental science to evaluate the feasibility of a new mine location. Whether the mine is excavated will be a function of the value of the diamond deposit, accessibility, and safety factors. Since mining a site costs millions of dollars, the analysis stage is crucial. Khaldun examines rock samples and the site itself before carefully estimating the value of the underground deposit. The diamonds will be mined only if the profits outweigh the many costs.



# 1.2

# Functions and Function Notation

The first instances of notation may have occurred when early humans attempted to show the concept of numbers. A jawbone in the Deutsches Museum in Munich, Germany, which has been dated at approximately 30 000 B.C.E., shows this early attempt. It has 55 equally spaced notches carved into it, arranged in groups of 5. This is one of the earliest pieces of evidence that show human interest in designing a notation for others to understand and convey the concept of a number system.

In this section, you will extend the concept of a function and formalize several notations that are used to represent a function.

### Tools

**Optional**

- grid paper

or

- graphing calculator

### Connections

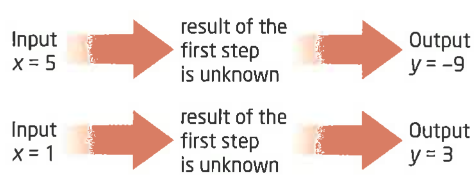
It is not by accident that we use the term *digit*. People started counting on their fingers, or digits. When there were too many items to count on 10 fingers, items such as stones or small pebbles were used. The Latin word for *pebble* is *calculus*, from which we get the word *calculate*.

## Investigate

### How can you use a function machine?

A function machine generates ordered pairs by performing mathematical operations on an input value. For each input value from the domain that enters a particular function machine, a unique output value in the range emerges. The output is determined by the rule of the defining function. If some values for $x$ and their associated $y$-values are known, it is often possible to determine what function was used by the machine.

Suppose you are told that a linear function machine has two steps. The first step involves a multiplication or a division, and the second step involves an addition or a subtraction. You are told that when the input value is $x = 5$, the output value is $y = -9$. As well, when the input value is $x = 1$, the output value is $y = 3$.




1. From the information above, what are the coordinates of two points of the linear function?
2. Use the two points to determine the equation of the linear function in the form $y = mx + b$.

**3.** Use the defining function to find

**a)** $y$ if $x \in \{-3, -2, -1\}$

**b)** $x$ if $y \in \{-6, -15, -18\}$

**4. Reflect** Use the defining equation to describe the steps that are performed by the function machine in generating the data.

**5. Reflect** At the start of this Investigate, you were told that the first step was a multiplication or a division and that the second step was an addition or a subtraction.

**a)** What type of value for $m$ would suggest that the first step is a multiplication? a division?

**b)** What type of value for $b$ would suggest that the second step is an addition? a subtraction?

**6.** Make up your own linear function machine. Exchange two ordered pairs that your function machine generates with a partner. Determine the defining function for each other's function machine.

To write a function using function notation, the form $f(x) = \ldots$ is used to indicate a function, $f$, with independent variable $x$. The notation $f(3)$ means the value obtained when $x = 3$ is substituted. $f(3)$ is read as "$f$ at 3" or "$f$ of 3."

**Connections**

Letters other than $f$ can be used in function notation. Often, scientists and mathematicians will use a letter related to the quantity being measured. For example, if the height is being measured as a function of time, express the function as $h(t)$.

## Example 1

### Find Values Using Function Notation

For each function, determine $f(-2)$, $f(5)$, and $f\left(\frac{1}{2}\right)$.

**a)** $f(x) = 2x - 4$

**b)** $f(x) = 3x^2 - x + 7$

**c)** $f(x) = 11$

**d)** $f(x) = \dfrac{2x}{x^2 - 3}$

### Solution

**a)**

$$\begin{aligned} f(x) &= 2x - 4 \\ f(-2) &= 2(-2) - 4 \quad \text{Substitute } x = -2. \\ &= -8 \\ f(5) &= 2(5) - 4 \\ &= 6 \\ f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right) - 4 \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

**b)** $f(x) = 3x^2 - x + 7$

$$\begin{aligned} f(-2) &= 3(-2)^2 - (-2) + 7 \\ &= 12 + 2 + 7 \\ &= 21 \end{aligned}$$

$$\begin{aligned} f(5) &= 3(5)^2 - 5 + 7 \\ &= 75 - 5 + 7 \\ &= 77 \end{aligned}$$

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 3\left(\frac{1}{2}\right)^2 - \frac{1}{2} + 7 \\ &= \frac{3}{4} - \frac{1}{2} + 7 \\ &= 7\frac{1}{4} \end{aligned}$$

**c)** $f(x) = 11$ is a constant function.

$f(-2) = 11$ $\quad$ $f(5) = 11$ $\quad$ $f\left(\frac{1}{2}\right) = 11$

**d)** $f(x) = \dfrac{2x}{x^2 - 3}$

$$\begin{aligned} f(-2) &= \frac{2(-2)}{(-2)^2 - 3} \\ &= \frac{-4}{4 - 3} \\ &= -4 \end{aligned}$$

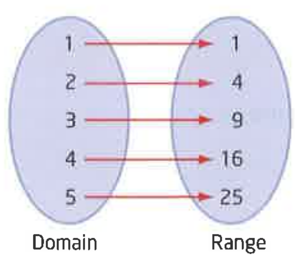
$$\begin{aligned} f(5) &= \frac{2(5)}{5^2 - 3} \\ &= \frac{10}{22} \\ &= \frac{5}{11} \end{aligned}$$

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \frac{2\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2 - 3} \\ &= \frac{1}{\frac{1}{4} - 3} \\ &= \frac{1}{-\frac{11}{4}} \\ &= -\frac{4}{11} \end{aligned}$$

**mapping diagram**

- a graphical representation that relates the values in one set (the domain) to the values in a second set (the range) using directed arrows from domain to range

A **mapping diagram** is a representation that can be used when the relation is given as a set of ordered pairs. In a mapping diagram, the domain values in one oval are joined to the range values in the other oval using arrows. In a mapping diagram, a relation is a function if there is exactly one arrow leading from each value in the domain. This indicates that each element in the domain corresponds to exactly one element in the range.



## Example 2

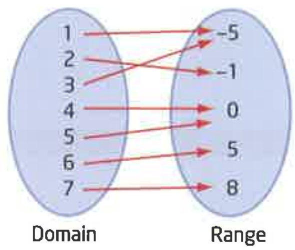
### Interpret Mapping Diagrams

Use the mapping diagrams to

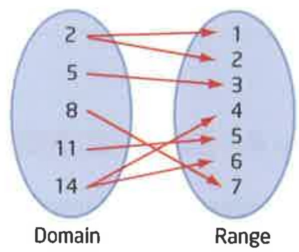
**i)** write the set of ordered pairs of the relation

**ii)** state if the relation is a function

**a)**




**b)**




### Solution

**a)** **i)** $\{(1, -5), (2, -1), (3, -5), (4, 0), (5, 0), (6, 5), (7, 8)\}$

**ii)** Since every value in the domain maps to exactly one value in the range, this relation is a function.

**b)** **i)** $\{(2, 1), (2, 2), (5, 3), (8, 7), (11, 5), (14, 4), (14, 6)\}$

**ii)** Since the values $x = 2$ and $x = 14$ both map to more than one value in the range, this relation is not a function.

While mapping diagrams are useful in situations where the relations are given in ordered pair form, they are impractical when a function is written in function notation. For this reason, a second form of mapping has been developed. This form is referred to as *mapping notation* and is illustrated in the next example.

## Example 3

### Represent Functions Using Mapping Notation

Write each function in mapping notation.

**a)** $f(x) = 3x^2 - 2x + 1$

**b)** $g(x) = 3x + 4$

**c)** $h(t) = -4.9t^2 - 4$

**d)** $P(x) = (500 - 2x)(300 + x)$

## Solution

**a)** $f : x \rightarrow 3x^2 - 2x + 1$ Read "$f$ is a function that maps $x$ to $3x^2 - 2x + 1$."

**b)** $g : x \rightarrow 3x + 4$ Read "$g$ is a function that maps $x$ to $3x + 4$."

**c)** $h : t \rightarrow -4.9t^2 - 4$ Read "$h$ is a function that maps $t$ to $-4.9t^2 - 4$."

**d)** $P : x \rightarrow (500 - 2x)(300 + x)$ Read "$P$ is a function that maps $x$ to $(500 - 2x)(300 + x)$."

## Example 4

### Solve a Problem Using Function Notation

The temperature of the water at the surface of a deep lake is 22 °C on a warm summer's day. As Renaldo scuba dives to the depths of the lake, he finds that the temperature decreases by 1.5 °C for every 8 m he descends.

**a)** Model the water temperature at any depth using function notation.

**b)** Use this function to determine the water temperature at a depth of 40 m.

**c)** At the bottom of the lake, the temperature is 5.5 °C. How deep is the lake?

**Connections**

Great Slave Lake in the Northwest Territories is the deepest lake in North America. At its deepest, it is 614 m deep. Lake Ontario has an average depth of 96 m and is 235 m deep at its deepest.

## Solution

**a)** Let $d$ represent the depth, in metres, below the surface of the lake and $T$ represent the temperature, in degrees Celsius, at this depth.

$$T(d) = 22 - 1.5\left(\frac{d}{8}\right)$$

The temperature decreases by 1.5°C for each 8 m.

**b)** For a depth of 40 m, substitute $d = 40$.

$$\begin{aligned} T(40) &= 22 - 1.5\left(\frac{40}{8}\right) \\ &= 22 - 7.5 \\ &= 14.5 \end{aligned}$$

The temperature at a depth of 40 m is 14.5 °C.

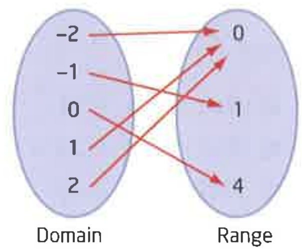
**c)** Substitute $T(d) = 5.5$ and solve for $d$.

$$\begin{aligned} 5.5 &= 22 - 1.5\left(\frac{d}{8}\right) \\ 1.5\left(\frac{d}{8}\right) &= 22 - 5.5 \\ 1.5d &= 8 \times 16.5 \\ d &= \frac{8 \times 16.5}{1.5} \\ d &= 88 \end{aligned}$$

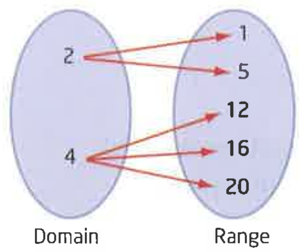
The depth of the lake is 88 m.

## Key Concepts

- In function notation, the symbol $f(x)$ represents the dependent variable. It indicates that the function $f$ is expressed in terms of the independent variable $x$.
  For example, $y = 3x^2 - 5$ is written as $f(x) = 3x^2 - 5$.
- Relations and functions given as ordered pairs can be represented using mapping diagrams. This involves using directed arrows from each value in an oval representing the domain to the corresponding value or values in an oval representing the range.




- In a mapping diagram, a relation is not a function when an element from the domain has two or more arrows leading to different elements of the range.




- Mapping notation can replace function notation. For example, $f(x) = 3x^2 - 5$ can be written as $f: x \rightarrow 3x^2 - 5$.

## Communicate Your Understanding

**C1** Samuel missed the explanation of function notation. Explain how to answer the following. Given $f(x) = x^2 + 5$, find $f(-2)$.

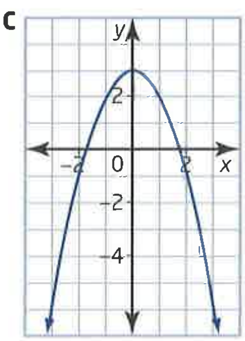
**C2** Michelle has written the function defined by $y = 3t^2 + 5t - 5$ as $f(x) = 3t^2 + 5t - 5$. Is she correct? Explain why or why not.

**C3** A quadratic function has the same shape as $y = x^2$, but it opens downward and has its vertex at (0, 3). Is each of the following a representation of the same function?

**A** $y = -x^2 + 3$

**B** $f: x \rightarrow -x^2 + 3$

**C**




**D** $f(x) = 3 - x^2$

## A Practise

*For help with questions 1 to 4, refer to Example 1.*

**1.** For each function, determine $f(4)$, $f(-5)$, and $f\left(-\frac{2}{3}\right)$.

**a)** $f(x) = \frac{2}{5}x + 11$

**b)** $f(x) = 3x^2 + 2x + 1$

**c)** $f(x) = 2(x + 4)^2$

**d)** $f(x) = -6$

**e)** $f(x) = \frac{1}{x}$

**f)** $f(x) = \sqrt{x + 5}$

**2.** Find the value of each function at $x = 0$. Sketch the graph of each function.

**a)** $f(x) = 5x + 4$

**b)** $k(x) = 4x$

**c)** $p(x) = -4$

**d)** $g(x) = 11x^2 + 3x - 1$

**e)** $f(x) = (3x - 3)(2x + 2)$

**f)** $h(x) = -\frac{2}{3}(5 - 4x)(x - 7)$

**3.** A linear function machine uses a function of the form $f(x) = ax$. Find the value of $a$ for each given point on the function, and then write the defining equation of the function.

**a)** $(3, -12)$ **b)** $(5, 15)$

**c)** $\left(1, \frac{2}{3}\right)$ **d)** $(-3, 3)$

**4.** Give an example of a linear function and a constant function, both in function notation. Describe the similarities and the differences between the two functions.


*For help with questions 5 to 8, refer to Example 2.*

**5.** Show each set of data in a mapping diagram.

**a)** $\{(1, 4), (2, 1), (3, -2), (4, -5), (5, -8), (6, -11), (7, -14), (8, -17)\}$

**b)** $\{(-3, 4), (-2, -1), (-1, -4), (0, -5), (1, -4), (2, -1)\}$

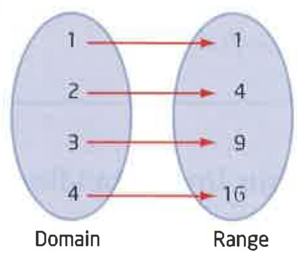
**c)** $\{(-5, 6), (-4, 9), (-3, 1), (-5, -6), (1, -2), (3, 8), (8, 8)\}$

**d)** $\{(9, 9), (7, 9), (5, 9), (3, 9)\}$

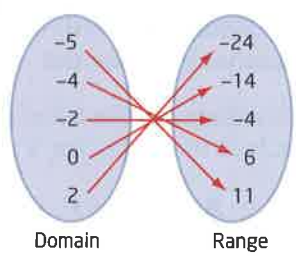
**6.** Determine if each relation in question 5 is a function. Justify your answer.

**7.** Write the ordered pairs associated with each mapping diagram.

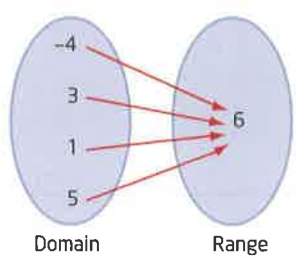
**a)**




**b)**




**c)**



**8.** Determine whether each relation in question 7 is a function. Justify your answer.

**9.** What advantages do mapping diagrams have over a list of ordered pairs?

*For help with question 10, refer to Example 3.*

**10.** Write each function in mapping notation.

**a)** $f(x) = -x + 4$

**b)** $g(x) = x^2 + 5x - 3$

**c)** $s(x) = \sqrt{4x - 4}$

**d)** $r(k) = -\dfrac{1}{2k - 1}$

## B Connect and Apply

**11.** Describe two different ways to determine if a relation is a function.

**12.** **Use Technology** If the output of a quadratic function machine gives data that fit an equation of the form $f(x) = ax^2 + bx + c$, a graphing calculator can be used to determine the equation if at least three data points are given. Data are given from such a function machine as follows: $\{(1, 4), (2, 11), (3, 24)\}$.

**a)** Enter the values of the domain in **L1** and the values of the range in **L2**.

**b)** Plot the data.

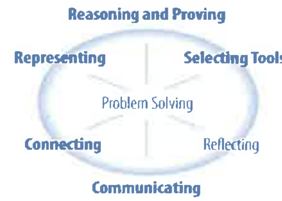
**c)** Run quadratic regression to determine the quadratic equation that fits these data. Record the equation that results from this regression.

**d)** Use this function to determine the range values for the domain values $x = -3$, $x = 0$, and $x = 5$.

**Technology Tip**

Refer to the Technology Appendix, pages 496 to 516, if you need help with displaying data, quadratic regression, or finding values.

**13. a)** Complete a table of values for the relation $f(x) = \sqrt{x}$ and graph the data.




**b)** Is this relation a function? Explain.

**c)** Could you have identified whether the relation was a function from the data in the table of values? Explain.

*For help with questions 14 to 16, refer to Example 4.*

**14.** Rivers located near an ocean experience a large wave called a tidal bore due to the tides. The speed, $v$, in kilometres per hour, of the tidal bore in a river is a function of the depth, $d$, in metres, of the river. The function is $v(d) = 11.27\sqrt{d}$.

**a)** Determine the domain and the range of this function.

**b)** Make a table of values and graph the function.

**15.** The value, $V$, in dollars, of an $n$-year-old car is given by $V(n) = \dfrac{23\,000}{n + 1} + 1000$.

**a)** How much was this car worth when it was first purchased?

**b)** Determine the value of the car after

**i)** 10 years **ii)** 12 years

**c)** How long would it take the car to depreciate to a value of \$2000?

**d)** Is $V(n)$ a function? Justify your answer.

**16.** The amount, $A$, in dollars, that needs to be invested at an interest rate $i$ to have \$100 after 1 year is given by the relation $A(i) = \dfrac{100}{1 + i}$. Note that $i$ must be expressed in decimal form.

**a)** Determine the domain and the range for this relation.

**b)** Graph the relation.

**c)** How much money needs to be invested at 5% to give \$100 after 1 year?

**d)** What rate of interest is required if \$90 is invested?

17. Create a linear function machine and two points that are generated by the machine. Trade points with a classmate to determine the function that generated the points.

18. Create a quadratic function machine of the form $f(x) = ax^2 + b$. Determine the coordinates of the $y$-intercept and of one other point that is generated by the machine. Trade points with a classmate to determine the function.

19. **Chapter Problem** While working at her co-op placement, Andrea is asked to work with a two-variable function to determine premiums for insurance policies. She is to calculate some values of the function and place them in the appropriate cell in a spreadsheet. The function is $f(n, r) = 500 + 2n - 10r$ for a driver with a rating of $r$ (related to the driver's record, 1 being a poor driver up to 5 being an excellent driver) and an age of $n$ (from 40 to 45 years of age). For example, a 42-year-old driver with a rating of 4 would have a policy premium of

$$\begin{aligned} f(42, 4) &= 500 + 2(42) - 10(4) \\ &= 500 + 84 - 40 \\ &= 544 \end{aligned}$$

This value has been placed in the spreadsheet for you.

| Age, $n$ \ Rating, $r$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 40 | | | | | |
| 41 | | | | | |
| 42 | | | | 544 | |
| 43 | | | | | |
| 44 | | | | | |
| 45 | | | | | |

Help Andrea by copying and completing the spreadsheet.

## Achievement Check

20. On Earth, the time, $t$, in seconds, taken for an object to fall from a height, $h$, in metres, to the ground is given by the formula $t(h) = \sqrt{\frac{h}{4.9}}$. On the moon, the formula changes to $t(h) = \sqrt{\frac{h}{1.8}}$.

**a)** Express each relation using mapping notation.

**b)** Determine the domain and the range of each relation.

**c)** Is each relation a function? Explain.

**d)** Graph both relations on the same set of axes. Compare the graphs and describe any similarities or differences.

**e)** Determine the difference between the time it takes for an object to fall from a height of 25 m on Earth and the time it takes on the moon. Justify your answer.

## C Extend

21. The initial velocity, $v$, in kilometres per hour, of a skidding car can be determined from the length, $d$, in metres, of the skid mark made by using the relation $v(d) = 12.6\sqrt{d} + 8$.

**a)** Determine the domain and the range of the relation.

**b)** Graph the relation.

**c)** Is the relation a function? Justify your answer.

22. **Math Contest** Given $f(x) = f(x + 1) + 3$ and $f(2) = 5$, what is the value of $f(8)$?

**A** 5 **B** 11 **C** 20 **D** $-13$

23. **Math Contest** Given $f(x) + 2g(x) = 12x^2 + 3x + 8$ and $2f(x) + 3g(x) = 18x^2 + 6x + 13$, find the value of $f(2) + g(3)$.

24. **Math Contest** $f(x)$ is a linear function. Given $f(f(3)) = 2$ and $f(f(2)) = 1$, what is the value of $f(0)$?

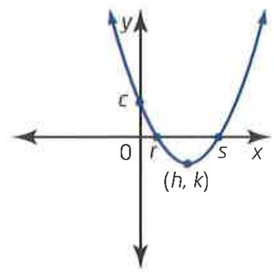
**A** 1 **B** 0 **C** $-0.5$ **D** $-1$

1.3

# Maximum or Minimum of a Quadratic Function

Some bridge arches are defined by quadratic functions. Engineers use these quadratic functions to determine the maximum height or the minimum clearance under the support of the bridge at a variety of points. They can give this information to the bridge builders.

A quadratic function can be written in a number of forms. Each form has different advantages. In all forms, $a$ determines the direction of opening and the shape.




- From the standard form, $f(x) = ax^2 + bx + c$, the $y$-intercept can be identified as $c$.
- From the factored form, $f(x) = a(x - r)(x - s)$, the $x$-intercepts can be identified as $r$ and $s$.
- From the vertex form, $y = a(x - h)^2 + k$, the coordinates of the vertex can be identified as $(h, k)$. If $a$ is positive, the minimum value is $k$. If $a$ is negative, the maximum value is $k$.

**Tools**

- graphing calculator

or

- grid paper

## Investigate A

### How can you connect different forms of the same quadratic function?

1. Graph each pair of functions.
   a) $f(x) = (x + 2)^2 + 3$ and $f(x) = x^2 + 4x + 7$
   b) $f(x) = (x + 3)^2 - 4$ and $f(x) = x^2 + 6x + 5$
   c) $f(x) = 2(x - 3)^2 + 4$ and $f(x) = 2x^2 - 12x + 22$
   d) $f(x) = 3(x - 1)^2 - 7$ and $f(x) = 3x^2 - 6x - 4$
2. Why are the graphs of the functions in each pair the same?
3. How can you rewrite the first equation in each pair in the form of the second equation?
4. How can you rewrite the second equation in each pair in the form of the first equation?
5. **Reflect** How can you use a graph to verify that two quadratic functions in different forms represent the same function? If you are using a graphing calculator, is it enough to observe that the graphs look the same on your screen? Explain.

**Connections**

Completing the square is part of a process by which a quadratic function in standard form can be arranged into vertex form, $y = a(x - h)^2 + k$. You learned this technique in grade 10.

To convert a quadratic function from standard form to vertex form, you can use the technique of completing the square.

## Example 1

### Find the Vertex by Completing the Square

Find the vertex of each function by completing the square. Is the vertex a minimum or a maximum? Explain.

**a)** $f(x) = x^2 + 5x + 7$

**b)** $f(x) = -\frac{2}{3}x^2 + 8x + 5$

### Solution

**a)**
$$\begin{aligned} f(x) &= x^2 + 5x + 7 \\ &= x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 7 \\ &= \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{28}{4} \\ &= \left(x + \frac{5}{2}\right)^2 + \frac{3}{4} \end{aligned}$$

Add half the coefficient of $x$, squared, to make the first three terms a perfect square trinomial. Subtract the same amount, $\left(\frac{5}{2}\right)^2$, so the value of the function does not change.

The vertex is at $\left(-\frac{5}{2}, \frac{3}{4}\right)$. This is a minimum because $a$ is 1, a positive value, so the parabola opens upward.

**b)**
$$\begin{aligned} f(x) &= -\frac{2}{3}x^2 + 8x + 5 \\ &= -\frac{2}{3}(x^2 - 12x) + 5 \\ &= -\frac{2}{3}(x^2 - 12x + 36 - 36) + 5 \\ &= -\frac{2}{3}[(x - 6)^2 - 36] + 5 \\ &= -\frac{2}{3}(x - 6)^2 + 24 + 5 \\ &= -\frac{2}{3}(x - 6)^2 + 29 \end{aligned}$$

Factor out the coefficient of $x^2$.

$8 \div \left(-\frac{2}{3}\right) = 8 \times \left(-\frac{3}{2}\right)$
$= -12$

Add and subtract $6^2 = 36$ to make a perfect square trinomial.

$-\frac{2}{3} \times (-36) = 24$

The vertex is at (6, 29). This is a maximum because the value of $a$ is negative, indicating that the parabola opens downward.

## Investigate B

### How can you use partial factoring to find a minimum or a maximum?

**Tools**

- graphing calculator

or

- grid paper

1. Graph the function $f(x) = 2x^2 + 4x$.
2. How many $x$-intercepts does this function have?
3. Use the $x$-intercepts to find the vertex of the parabola.
4. Graph the functions $g(x) = 2x^2 + 4x + 2$ and $h(x) = 2x^2 + 4x + 5$ on the same set of axes as $f(x)$.
5. How many $x$-intercepts does $g(x)$ have? $h(x)$?
6. Describe how to find the vertex of the new parabolas, $g(x)$ and $h(x)$, based on the vertex of the original parabola, $f(x)$.
7. **Reflect** Using your answer from step 6, suggest a method that can be used to find the maximum or minimum of a parabola of the form $f(x) = 2x^2 + 4x + k$ for any value of $k$.

## Example 2

### Use Partial Factoring to Find the Vertex of a Quadratic Function

Find the vertex of the function $y = 4x^2 - 12x + 3$ by partial factoring. Is the vertex a minimum or a maximum value? Explain.

### Solution

Work with the function $y = 4x^2 - 12x$ to find the $x$-coordinate of the vertex, since the $x$-coordinate of the vertex of $y = 4x^2 - 12x + 3$ will be the same.

$y = 4x(x - 3)$

For $y = 0$:

$0 = 4x(x - 3)$

$4x = 0$ or $x - 3 = 0$ — Use the zero principle. If $AB = 0$, then either $A = 0$ or $B = 0$.

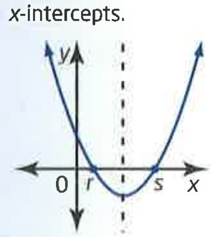
$x = 0$ or $x = 3$ — These give the $x$-intercepts of the function $y = 4x^2 - 12x$.

The average of these two $x$-intercepts will give the $x$-coordinate of the vertex for $y = 4x^2 - 12x$ and $y = 4x^2 - 12x + 3$:

$\frac{0 + 3}{2} = \frac{3}{2}$

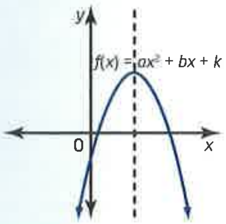
**Connections**

The vertex of a quadratic function is on the line of symmetry, which is halfway between the $x$-intercepts.



**Connections**

By partial factoring, $f(x) = ax^2 + bx + k$ can be expressed as
$f(x) = ax\left(x + \frac{b}{a}\right) + k$.
This is a family of quadratic functions with axis of symmetry $x = -\frac{b}{2a}$.




To find the $y$-coordinate of the vertex, substitute $x = \frac{3}{2}$ into $y = 4x^2 - 12x + 3$.

$$
\begin{aligned}
y &= 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + 3 \\
&= 4\left(\frac{9}{4}\right) - 18 + 3 \\
&= 9 - 18 + 3 \\
&= -6
\end{aligned}
$$

The vertex of the function $y = 4x^2 - 12x + 3$ is at $\left(\frac{3}{2}, -6\right)$. It is a minimum, because the value of $a$ is positive.

## Example 3

### Solve a Problem Involving a Minimum or a Maximum

Rachel and Ken are knitting scarves to sell at the craft show. The wool for each scarf costs \$6. They were planning to sell the scarves for \$10 each, the same as last year when they sold 40 scarves. However, they know that if they raise the price, they will be able to make more profit, even if they end up selling fewer scarves. They have been told that for every 50¢ increase in the price, they can expect to sell four fewer scarves. What selling price will maximize their profit and what will the profit be?

### Solution

Let $x$ represent the number of 50¢ price changes.

Since each scarf cost \$6 and was sold for \$10, the profit was \$4 per scarf. As they raise the price, their profit per scarf will be $(4 + 0.5x)$ for $x$ changes to the price. They will sell $40 - 4x$ scarves when they make the price change.

$$
\begin{aligned}
\text{Profit} &= \text{profit per scarf} \times \text{number sold} \\
P(x) &= (4 + 0.5x)(40 - 4x) \\
&= -2x^2 + 4x + 160
\end{aligned}
$$

**Method 1: Complete the Square to Determine the Vertex**

$$
\begin{aligned}
P(x) &= -2(x^2 - 2x) + 160 \\
&= -2(x^2 - 2x + 1 - 1) + 160 \\
&= -2(x - 1)^2 + 2 + 160 \\
&= -2(x - 1)^2 + 162
\end{aligned}
$$

The maximum value of this quadratic function is 162 when $x = 1$. This means that they will make a maximum profit of \$162 if they increase the price once. The selling price is $10 + 0.5(1)$ or \$10.50.

**Method 2: Use Partial Factoring to Determine the Vertex**

Find the $x$-coordinate of the vertex of the function $Q(x) = -2x^2 + 4x$, knowing that the vertex of $P(x) = -2x^2 + 4x + 160$ has the same $x$-coordinate.

$Q(x) = -2x(x - 2)$

Substitute $Q(x) = 0$ to find the $x$-intercepts.

$$\begin{aligned} 0 &= -2x(x - 2) \\ -2x = 0 &\text{ or } x - 2 = 0 \\ x = 0 &\text{ or } \quad x = 2 \end{aligned}$$

The $x$-coordinate of the vertex is $x = 1$ (the average of 0 and 2).

$$\begin{aligned} P(1) &= -2(1)^2 + 4(1) + 160 \\ &= 162 \end{aligned}$$

The vertex of this function is at (1, 162). This means that they will make a maximum profit of \$162 if they increase the price once. The selling price is 10 + 0.5(1) or \$10.50.

## Example 4

### Connect Projectiles to Quadratic Functions

Jamie throws a ball that will move through the air in a parabolic path due to gravity. The height, $h$, in metres, of the ball above the ground after $t$ seconds can be modelled by the function $h(t) = -4.9t^2 + 40t + 1.5$.

**a)** Find the zeros of the function and interpret their meaning.

**b)** Determine the time needed for the ball to reach its maximum height.

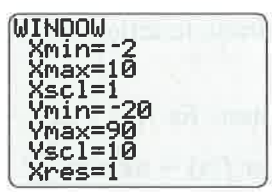
**c)** What is the maximum height of the ball?

**Connections**

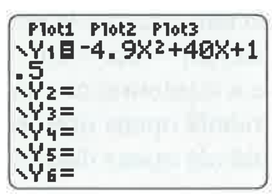
The zeros of a function are the values of the independent variable for which the function has value zero. They correspond to the $x$-intercepts of the graph of the function.

### Solution

**a)** • Use the window settings shown.




• Graph **Y1** $= -4.9x^2 + 40x + 1.5$.

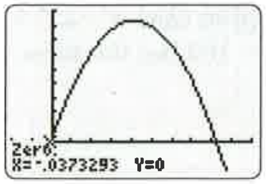


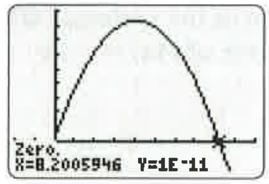

• Press 2nd [CALC] to access the **CALCULATE** menu.

**Technology Tip**

See the Use Technology feature at the end of this section for a TI-Nspire™ CAS graphing calculator solution.

- Select **2:zero** to find the *x*-intercepts of the function.




The zeros are approximately −0.037 and 8.2.

The solution $t = -0.037$ indicates when, in the past, the ball would have been thrown from ground level in order for it to follow the given path. The solution $t = 8.2$ indicates when the ball will return to the ground. The ball returns to the ground 8.2 s after Jamie threw it.

**b)** The maximum is midway between the two zeros. So, find the average of the two solutions from part a).

$$\frac{-0.037 + 8.2}{2} = 4.0815$$

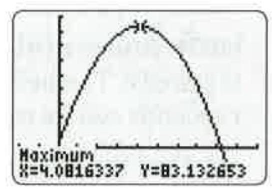
The ball will take approximately 4.1 s to reach its maximum height.

**c)** The maximum height can be found by substituting $t = 4.1$ into the function.

$$\begin{aligned} h(t) &= -4.9t^2 + 40t + 1.5 \\ h(4.1) &= -4.9(4.1)^2 + 40(4.1) + 1.5 \\ &\doteq 83.13 \end{aligned}$$

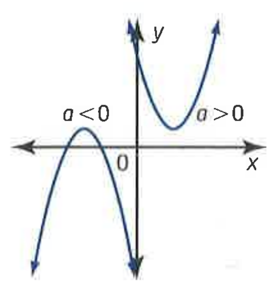
The ball will reach a maximum height of approximately 83.1 m.

This solution can be verified using the maximum function on the graphing calculator.




## Key Concepts

- The minimum or maximum value of a quadratic function occurs at the vertex of the parabola.
- The vertex of a quadratic function can be found by
  - graphing
  - completing the square: for $f(x) = a(x - h)^2 + k$, the vertex is $(h, k)$
  - partial factoring: for $f(x) = ax\left(x + \frac{b}{a}\right) + k$, the *x*-coordinate of the vertex is $-\frac{b}{2a}$
- The sign of the coefficient *a* in the quadratic function $f(x) = ax^2 + bx + c$ or $f(x) = a(x - h)^2 + k$ determines whether the vertex is a minimum or a maximum.
  If $a > 0$, then the parabola opens upward and has a minimum.
  If $a < 0$, then the parabola opens downward and has a maximum.



## Communicate Your Understanding

**C1** In one step of completing the square, you divide the coefficient of $x$ by 2 and square the result. Why?

**C2** How are the functions $f(x) = 4x(x - 3)$, $g(x) = 4x(x - 3) + 2$, and $h(x) = 4x(x - 3) - 1$ related? Explain using words and diagrams.

**C3** Ryan does not understand the concept of partial factoring to determine the vertex. Use the function $y = 3x^2 - 9x - 17$ to outline the technique for him.

## A Practise

*For help with questions 1 and 2, refer to Example 1.*

**1.** Complete the square for each function.

**a)** $y = x^2 + 4x$

**b)** $f(x) = x^2 + 7x + 11$

**c)** $g(x) = x^2 - 3x + 1$

**d)** $y = x^2 - 11x - 4$

**e)** $f(x) = x^2 + 13x + 2$

**f)** $y = x^2 - 9x - 9$

**2.** Determine the vertex of each quadratic function by completing the square. State if the vertex is a minimum or a maximum.

**a)** $f(x) = x^2 + 10x + 6$

**b)** $f(x) = 2x^2 + 12x + 16$

**c)** $f(x) = -3x^2 + 6x + 1$

**d)** $f(x) = -x^2 + 12x - 5$

**e)** $f(x) = -\frac{1}{2}x^2 - x + \frac{3}{2}$

**f)** $f(x) = \frac{2}{3}x^2 + \frac{16}{3}x + \frac{25}{3}$

*For help with question 3, refer to Example 2.*

**3.** Use partial factoring to determine the vertex of each function. State if the vertex is a minimum or a maximum.

**a)** $f(x) = 3x^2 - 6x + 11$

**b)** $f(x) = -2x^2 + 8x - 3$

**c)** $f(x) = \frac{1}{2}x^2 - 3x + 8$

**d)** $f(x) = -\frac{5}{3}x^2 + 5x - 10$

**e)** $f(x) = 0.3x^2 - 3x + 6$

**f)** $f(x) = -0.2x^2 - 2.8x - 5.4$

**4. Use Technology** Use a graphing calculator to verify your answers to questions 2 and 3.

## B Connect and Apply

*For help with questions 5 and 6, refer to Example 3.*

**5.** An electronics store sells an average of 60 entertainment systems per month at an average of \$800 more than the cost price. For every \$20 increase in the selling price, the store sells one fewer system. What amount over the cost price will maximize revenue?

**6.** Last year, a banquet hall charged \$30 per person, and 60 people attended the hockey banquet dinner. This year, the hall's manager has said that for every 10 extra people that attend the banquet, they will decrease the price by \$1.50 per person. What size group would maximize the profit for the hall this year?

*For help with question 7, refer to Example 4.*


**7.** A ball is kicked into the air and follows a path described by $h(t) = -4.9t^2 + 6t + 0.6$, where $t$ is the time, in seconds, and $h$ is the height, in metres, above the ground. Determine the maximum height of the ball, to the nearest tenth of a metre.

8. The cost, $C$, in dollars, of fuel per month for Sanjay to operate his truck is given by $C(v) = 0.0029v^2 - 0.48v + 142$, where $v$ represents his average driving speed, in kilometres per hour. Find the most efficient speed at which Sanjay should drive his truck.

9. Arnold has 24 m of fencing to surround a garden, bounded on one side by the wall of his house. What are the dimensions of the largest rectangular garden that he can enclose?

10. The area shown is to be enclosed by 30 m of fencing. Find the dimensions that will maximize the enclosed area.


11. The sum of two numbers is 10. What is the maximum product of these numbers?

12. A function models the effectiveness of a TV commercial. After $n$ viewings, the effectiveness, $e$, is $e = -\frac{1}{90}n^2 + \frac{2}{3}n$.


a) Determine the range for the effectiveness and the domain of the number of viewings. Explain your answers for the domain and range.

b) Use either completing the square or partial factoring to find the vertex. Is it a minimum or a maximum? Explain.

c) What conclusions can you make from this function?

d) Graph the function on a graphing calculator to verify your conclusions from part c).

13. All quadratic functions of the form $y = 2x^2 + bx$ have some similar properties.

a) Choose five different values of $b$ and graph each function.

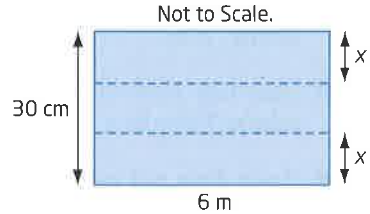
b) What are the similar properties?

c) Determine the vertex of each parabola.

d) Find the relationship between the vertices of these parabolas.

## C Extend

14. A sheet of metal that is 30 cm wide and 6 m long is to be used to make a rectangular eavestrough by bending the sheet along the dotted lines.




What value of $x$ maximizes the capacity of the eavestrough?

15. A ball is thrown vertically upward with an initial velocity of $v$ metres per second and is affected by gravity, $g$. The height, $h$, in metres, of the ball after $t$ seconds is given by $h(t) = -\frac{1}{2}gt^2 + vt$.

a) Show that the ball will reach its maximum height at $t = \frac{v}{g}$.

b) Show that the maximum height of the ball will be $\frac{v^2}{2g}$.

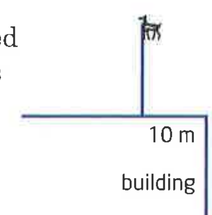
16. **Math Contest** Given that $x^2 = y^3 = z$, where $x$, $y$, and $z$ are integers, how many different values of $z$ are there for $z < 1001$?

**A** 0 **B** 3 **C** 4 **D** 10

17. **Math Contest** A function of two variables is defined as $f(x, y) = x^2 + y^2 + 4x - 6y + 7$. What is the minimum value of this function?

**A** 7 **B** $-13$ **C** $-6$ **D** 0

18. **Math Contest** A dog's 15-m-long leash is attached to a building. The leash is attached 10 m from one corner of the building. Assume that the sides of the building are long enough that the dog cannot go around any of the other corners. The greatest area that the dog can cover, in square metres, is




**A** $250\pi$ **B** $\frac{475\pi}{4}$ **C** $112.5\pi$ **D** $125\pi$

# Use Technology

## Use a TI-Nspire™ CAS Graphing Calculator to Find the Maximum or Minimum and the Zeros of a Quadratic Function

Jamie throws a ball that will move through the air in a parabolic path due to gravity. The height, $h$, in metres, of the ball above the ground after $t$ seconds can be modelled by the function $h(t) = -4.9t^2 + 40t + 1.5$.

**a)** Find the zeros of the function and interpret their meaning.

**b)** Determine the time needed for the ball to reach its maximum height.

**c)** What is the maximum height of the ball?

### Tools

- TI-Nspire™ CAS graphing calculator

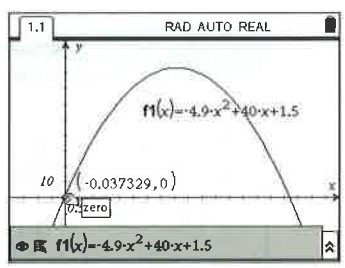
### Connections

Example 4 on page 29 is used to model the steps needed to find the maximum or minimum and the zeros of a quadratic function using a TI-Nspire™ CAS graphing calculator.

### Solution

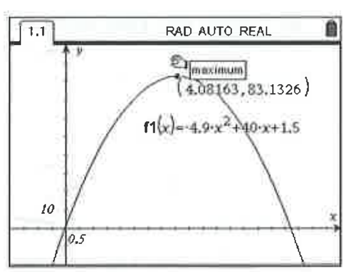
**a)** Turn on the TI-Nspire™ CAS graphing calculator.

- Press (home) and select **6:New Document**.
- Select **2:Add Graphs & Geometry**.
- Type $-4.9x^2 + 40x + 1.5$ for function f1 and press (enter).
- Press (menu). Select **4:Window**.
- Select **1:Window Settings**. Set **XMin** to $-2$, **XMax** to 10, **Ymin** to $-40$, and **YMax** to 100. Tab down to **OK** and press (enter).
- Press (menu) and select **6:Points & Lines**.
- Select **2:Point On**. Move the cursor to the graph and press (enter).
- Press (esc).
- Press (ctrl) and then (click) to grab the point. Use the cursor keys (the arrows on the NavPad) to move the point along the graph toward the left zero. When you reach the zero, "zero" will appear in a box. Read the coordinates of the zero. It occurs at a time of approximately $-0.037$ s.




Similarly, you can find the right zero at a time of about 8.20 s.

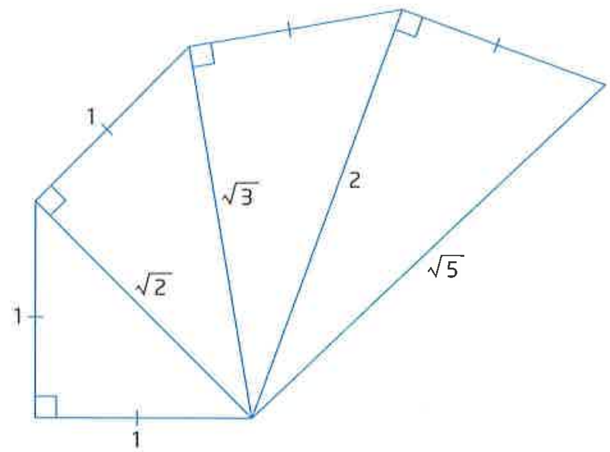
**b)** To find the maximum height of the ball, move the point toward the maximum on the graph. When you reach the maximum, "maximum" will appear inside a box. Read the coordinates of the maximum. It occurs at a time of approximately 4.08 s.




**c)** The maximum height of the ball is approximately 83.13 m.

1.4

# Skills You Need: Working With Radicals




The followers of the Greek mathematician Pythagoras discovered values that did not correspond to any of the rational numbers. As a result, a new type of number needed to be defined to represent these values. These values are called **irrational numbers**. One type of irrational number is of the form $\sqrt{n}$, where $n$ is not a perfect square. Such numbers are sometimes referred to as radicals. In this section, you will see how to use the operations of addition, subtraction, and multiplication with radicals.

**irrational number**

- a number that cannot be expressed in the form $\frac{a}{b}$, where $a$ and $b$ are integers and $b \neq 0$

## Investigate

### How do you multiply radicals?

1. Copy and complete the table. Where necessary, use a scientific calculator to help you evaluate each expression, rounding to two decimal places.

| A | B |
|---|---|
| $\sqrt{4} \times \sqrt{4} = ■$ | $\sqrt{4 \times 4} = ■$ |
| $\sqrt{81} \times \sqrt{81} = ■$ | $\sqrt{81 \times 81} = ■$ |
| $\sqrt{225} \times \sqrt{225} = ■$ | $\sqrt{225 \times 225} = ■$ |
| $\sqrt{5} \times \sqrt{5} = ■$ | $\sqrt{5 \times 5} = ■$ |
| $\sqrt{31} \times \sqrt{31} = ■$ | $\sqrt{31 \times 31} = ■$ |
| $\sqrt{12} \times \sqrt{9} = ■$ | $\sqrt{12 \times 9} = ■$ |
| $\sqrt{23} \times \sqrt{121} = ■$ | $\sqrt{23 \times 121} = ■$ |

2. What do you notice about the results in each row?

3. What conclusion can you make from your observations? Explain.

4. **Reflect** **a)** Make a general conclusion about an equivalent expression for $\sqrt{a} \times \sqrt{b}$.
   **b)** Do you think that this will be true for any values of $a$ and $b$? Justify your answer.

The number or expression under the radical sign is called the **radicand**. If the radicand is greater than or equal to zero and is not a perfect square, then the radical is an irrational number. An approximate value can be found using a calculator. In many situations, it is better to work with the exact value, so the radical form is kept. Use the radical form when an approximate answer is not good enough and an exact answer is needed. Sometimes **entire radicals** can be simplified by removing perfect square factors. The resulting expression is called a **mixed radical**.

**radicand**

- a number or expression under a radical sign

**entire radical**

- a radical in the form $\sqrt{n}$, where $n > 0$, such as $\sqrt{45}$

**mixed radical**

- a radical in the form $a\sqrt{b}$, where $a \neq 1$ or $-1$ and $b > 0$, such as $3\sqrt{5}$

## Example 1

### Change Entire Radicals to Mixed Radicals

Express each radical as a mixed radical in simplest form.

**a)** $\sqrt{50}$

**b)** $\sqrt{27}$

**c)** $\sqrt{180}$

### Solution

**a)**
$$\begin{aligned}\sqrt{50} &= \sqrt{25 \times 2} && \text{Choose } 25 \times 2\text{, not } 5 \times 10\text{, as 25 is a perfect square factor.}\\ &= (\sqrt{25})(\sqrt{2}) && \text{Use } \sqrt{ab} = \sqrt{a} \times \sqrt{b}.\\ &= 5\sqrt{2}\end{aligned}$$

**b)**
$$\begin{aligned}\sqrt{27} &= \sqrt{9 \times 3}\\ &= (\sqrt{9})(\sqrt{3}) && \text{Use } \sqrt{ab} = \sqrt{a} \times \sqrt{b}.\\ &= 3\sqrt{3}\end{aligned}$$

**c)**
$$\begin{aligned}\sqrt{180} &= \sqrt{36 \times 5}\\ &= (\sqrt{36})(\sqrt{5})\\ &= 6\sqrt{5}\end{aligned}$$

or

$$\begin{aligned}\sqrt{180} &= \sqrt{9 \times 4 \times 5}\\ &= (\sqrt{9})(\sqrt{4})(\sqrt{5})\\ &= (3)(2)\sqrt{5}\\ &= 6\sqrt{5}\end{aligned}$$

Adding and subtracting radicals works in the same way as adding and subtracting polynomials. You can only add like terms or, in this case, like radicals. For example, the terms in the expression $2\sqrt{3} + 5\sqrt{7}$ do not have the same radical, so they cannot be added, but the terms in the expression $3\sqrt{5} + 6\sqrt{5}$ have a common radical, so they can be added: $3\sqrt{5} + 6\sqrt{5} = 9\sqrt{5}$.

## Example 2

### Add or Subtract Radicals

Simplify.

**a)** $9\sqrt{7} - 3\sqrt{7}$

**b)** $4\sqrt{3} - 2\sqrt{27}$

**c)** $5\sqrt{8} + 3\sqrt{18}$

**d)** $\frac{1}{4}\sqrt{28} - \frac{3}{4}\sqrt{63} + \frac{2}{3}\sqrt{50}$

### Solution

**a)** $9\sqrt{7} - 3\sqrt{7} = 6\sqrt{7}$

**b)**
$$\begin{aligned} 4\sqrt{3} - 2\sqrt{27} &= 4\sqrt{3} - 2\sqrt{9 \times 3} \\ &= 4\sqrt{3} - 2\sqrt{9} \times \sqrt{3} \\ &= 4\sqrt{3} - 2 \times 3\sqrt{3} \\ &= 4\sqrt{3} - 6\sqrt{3} \\ &= -2\sqrt{3} \end{aligned}$$

Simplify $\sqrt{27}$ first.

**c)**
$$\begin{aligned} 5\sqrt{8} + 3\sqrt{18} &= 5\sqrt{4 \times 2} + 3\sqrt{9 \times 2} \\ &= 5\sqrt{4}\sqrt{2} + 3\sqrt{9}\sqrt{2} \\ &= 5 \times 2\sqrt{2} + 3 \times 3\sqrt{2} \\ &= 10\sqrt{2} + 9\sqrt{2} \\ &= 19\sqrt{2} \end{aligned}$$

First simplify both radicals.

**d)**
$$\begin{aligned} \frac{1}{4}\sqrt{28} - \frac{3}{4}\sqrt{63} + \frac{2}{3}\sqrt{50} &= \frac{1}{4}\sqrt{4 \times 7} - \frac{3}{4}\sqrt{9 \times 7} + \frac{2}{3}\sqrt{25 \times 2} \\ &= \frac{1}{4}\sqrt{4}\sqrt{7} - \frac{3}{4}\sqrt{9}\sqrt{7} + \frac{2}{3}\sqrt{25}\sqrt{2} \\ &= \frac{1}{4} \times 2\sqrt{7} - \frac{3}{4} \times 3\sqrt{7} + \frac{2}{3} \times 5\sqrt{2} \\ &= \frac{2}{4}\sqrt{7} - \frac{9}{4}\sqrt{7} + \frac{10}{3}\sqrt{2} \\ &= -\frac{7}{4}\sqrt{7} + \frac{10}{3}\sqrt{2} \text{ or } -\frac{7\sqrt{7}}{4} + \frac{10\sqrt{2}}{3} \end{aligned}$$

## Example 3

### Multiply Radicals

Simplify fully.

**a)** $(2\sqrt{3})(3\sqrt{6})$ **b)** $2\sqrt{3}(4 + 5\sqrt{3})$ **c)** $-7\sqrt{2}(6\sqrt{8} - 11)$

**d)** $(\sqrt{3} + 5)(2 - \sqrt{3})$ **e)** $(2\sqrt{2} + 3\sqrt{3})(2\sqrt{2} - 3\sqrt{3})$

### Solution

**a)**
$$\begin{aligned}(2\sqrt{3})(3\sqrt{6}) &= (2 \times 3)(\sqrt{3} \times \sqrt{6})\\ &= 6\sqrt{3 \times 6}\\ &= 6\sqrt{18}\\ &= 6\sqrt{9 \times 2}\\ &= 6 \times 3\sqrt{2}\\ &= 18\sqrt{2}\end{aligned}$$

Use the commutative property and the associative property.

Multiply coefficients and then multiply radicands.

**b)**
$$\begin{aligned}2\sqrt{3}(4 + 5\sqrt{3}) &= 2\sqrt{3}(4) + (2\sqrt{3})(5\sqrt{3})\\ &= 8\sqrt{3} + 10\sqrt{9}\\ &= 8\sqrt{3} + 10(3)\\ &= 8\sqrt{3} + 30\end{aligned}$$

Use the distributive property.

**Connections**

Recall that $3(x + 2) = 3x + 6$ by the distributive property. The same property can be applied to multiply radicals.

**c)**
$$\begin{aligned}-7\sqrt{2}(6\sqrt{8} - 11) &= (-7\sqrt{2})(6\sqrt{8}) - (7\sqrt{2})(-11)\\ &= -42\sqrt{16} + 77\sqrt{2}\\ &= (-42)(4) + 77\sqrt{2}\\ &= -168 + 77\sqrt{2}\end{aligned}$$

**d)**
$$\begin{aligned}(\sqrt{3} + 5)(2 - \sqrt{3}) &= \sqrt{3}(2) + \sqrt{3}(-\sqrt{3}) + 5(2) + 5(-\sqrt{3})\\ &= 2\sqrt{3} - \sqrt{9} + 10 - 5\sqrt{3}\\ &= 2\sqrt{3} - 3 + 10 - 5\sqrt{3}\\ &= -3\sqrt{3} + 7\end{aligned}$$

**e)**
$$\begin{aligned}(2\sqrt{2} + 3\sqrt{3})(2\sqrt{2} - 3\sqrt{3}) &= (2\sqrt{2})^2 - (3\sqrt{3})^2\\ &= 4(2) - 9(3)\\ &= 8 - 27\\ &= -19\end{aligned}$$

Simplify and collect like terms.

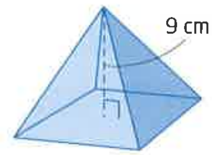
**Connections**

Recall the difference of squares: $(a + b)(a - b) = a^2 - b^2$. The factors in part e) have the same pattern. They are called conjugates.

## Example 4

### Solve a Problem Using Radicals

A square-based pyramid has a height of 9 cm. The volume of the pyramid is 1089 cm³. Find the exact side length of the square base, in simplified form.




### Solution

Let $x$ represent the side length of the base.

$$V = \frac{1}{3} \times \text{area of base} \times \text{height}$$

$$1089 = \frac{1}{3}x^2(9)$$

$$1089 = 3x^2$$

$$x^2 = \frac{1089}{3}$$

$$x^2 = 363$$

$$x = \sqrt{363}$$

Only the positive root is needed because $x$ is a length.

$$x = \sqrt{121 \times 3}$$

$$x = 11\sqrt{3}$$

The exact side length of the square base of the pyramid is $11\sqrt{3}$ cm.

**Connections**

The answer $11\sqrt{3}$ cm is exact. An approximate answer can be found using a calculator. To the nearest hundredth, the side length is 19.05 cm.

## Key Concepts

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for $a \geq 0$ and $b \geq 0$.
- An entire radical can be simplified to a mixed radical in simplest form by removing the largest perfect square from under the radical to form a mixed radical.
  For example, $\sqrt{50} = \sqrt{25 \times 2}$
  $= 5\sqrt{2}$
- Like radicals can be combined through addition and subtraction. For example, $3\sqrt{7} + 2\sqrt{7} = 5\sqrt{7}$.
- Radicals can be multiplied using the distributive property.
  For example, $4\sqrt{2}(5\sqrt{3} - 3) = 20\sqrt{6} - 12\sqrt{2}$ and
  $(\sqrt{2} - 3)(\sqrt{2} + 1) = \sqrt{4} + \sqrt{2} - 3\sqrt{2} - 3$
  $= 2 - 2\sqrt{2} - 3$
  $= -2\sqrt{2} - 1$

## Communicate Your Understanding

**C1** Marc is asked to simplify the expression $\sqrt{3} - \sqrt{75}$. He says that since the radical expressions are unlike, the terms cannot be combined. Is he correct? Explain why or why not.

**C2** Describe the steps needed to simplify the expression $\sqrt{3}(2\sqrt{3} - 4\sqrt{2})$.

**C3** Ann wants to simplify the radical $\sqrt{108}$. She starts by prime factoring 108:
$108 = 2 \times 2 \times 3 \times 3 \times 3$
Rayanne looks for the greatest perfect square that will divide into 108 to produce a whole number. Rayanne finds that this value is 36.
Explain why both techniques will result in the same solution.

## A Practise

*For help with question 1, refer to the Investigate.*

**1.** Simplify.

**a)** $3(4\sqrt{5})$
**b)** $\sqrt{3}(5\sqrt{2})$
**c)** $\sqrt{5}(-2\sqrt{7})$
**d)** $5\sqrt{3}(-4\sqrt{5})$
**e)** $2\sqrt{3}(3\sqrt{2})$
**f)** $-6\sqrt{2}(-\sqrt{11})$

*For help with question 2, refer to Example 1.*

**2.** Express each as a mixed radical in simplest form.

**a)** $\sqrt{12}$
**b)** $\sqrt{242}$
**c)** $\sqrt{147}$
**d)** $\sqrt{20}$
**e)** $\sqrt{252}$
**f)** $\sqrt{392}$

*For help with questions 3 and 4, refer to Example 2.*

**3.** Simplify.

**a)** $2\sqrt{3} - 5\sqrt{3} + 4\sqrt{3}$
**b)** $11\sqrt{5} - 4\sqrt{5} - 5\sqrt{5} - 6\sqrt{5}$
**c)** $\sqrt{7} - 2\sqrt{7} + \sqrt{7}$
**d)** $2\sqrt{2} - 8\sqrt{5} + 3\sqrt{2} + 4\sqrt{5}$
**e)** $\sqrt{6} - 4\sqrt{2} + 3\sqrt{6} - \sqrt{2}$
**f)** $2\sqrt{10} - \sqrt{10} - 4\sqrt{10} + \sqrt{5}$

**4.** Add or subtract as indicated.

**a)** $8\sqrt{2} - 4\sqrt{8} + \sqrt{32}$
**b)** $4\sqrt{18} + 3\sqrt{50} + \sqrt{200}$
**c)** $\sqrt{20} - 4\sqrt{12} - \sqrt{125} + 2\sqrt{3}$
**d)** $2\sqrt{28} + \sqrt{54} + \sqrt{150} + 5\sqrt{7}$
**e)** $5\sqrt{3} - \sqrt{72} + \sqrt{243} + \sqrt{8}$
**f)** $\sqrt{44} + \sqrt{88} + \sqrt{99} + \sqrt{198}$

*For help with questions 5 to 7, refer to Example 3.*

**5.** Expand and simplify.

**a)** $5\sqrt{6}(2\sqrt{3})$
**b)** $-2\sqrt{2}(4\sqrt{14})$
**c)** $8\sqrt{5}(\sqrt{10})$
**d)** $3\sqrt{15}(-2\sqrt{3})$
**e)** $11\sqrt{2}(5\sqrt{3})$
**f)** $-2\sqrt{6}(2\sqrt{6})$

**6.** Expand. Simplify where possible.

**a)** $3(8 - \sqrt{5})$
**b)** $\sqrt{3}(5\sqrt{2} + 4\sqrt{3})$
**c)** $\sqrt{3}(\sqrt{6} - \sqrt{3})$
**d)** $-2\sqrt{5}(4 + 2\sqrt{5})$
**e)** $8\sqrt{2}(2\sqrt{8} + 3\sqrt{12})$
**f)** $3\sqrt{3}(2\sqrt{7} - 5\sqrt{2})$

**7.** Expand. Simplify where possible.

**a)** $(\sqrt{2} + 5)(\sqrt{2} + 5)$
**b)** $(2\sqrt{2} + 4)(\sqrt{2} - 4)$
**c)** $(\sqrt{3} + 2\sqrt{2})(5 + 5\sqrt{2})$
**d)** $(3 + 2\sqrt{5})(\sqrt{5} - 5)$
**e)** $(1 + \sqrt{5})(1 - \sqrt{5})$
**f)** $(4 - 3\sqrt{7})(\sqrt{7} + 1)$

**8.** Simplify.

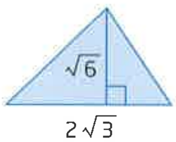
**a)** $\frac{1}{4}\sqrt{54} - \frac{1}{4}\sqrt{150}$
**b)** $2\sqrt{20} + \frac{3}{4}\sqrt{80} - \sqrt{125}$
**c)** $\frac{1}{2}\sqrt{8} + \frac{3}{5}\sqrt{50} - \frac{2}{3}\sqrt{18}$
**d)** $\frac{2}{5}\sqrt{125} - \frac{2}{3}\sqrt{243} - \frac{1}{3}\sqrt{45} + \frac{1}{2}\sqrt{48}$

## B Connect and Apply

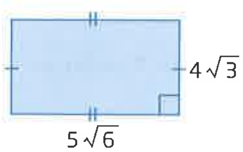
*For help with questions 9 to 11, refer to Example 4.*

**9.** Find a simplified expression for the area of each shape.

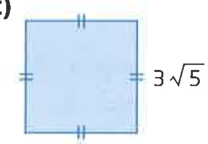

**a)**



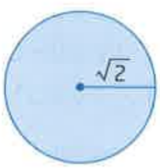

**b)**




**c)**




**d)**

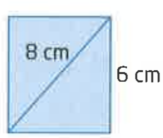



**10.** Explain the steps you would need to take to fully simplify $\sqrt{2880}$.

**11.** A square has an area of 675 cm². Find the length of a side in simplified radical form.

**12.** On a square game board made up of small squares of side length 2 cm, the diagonal has a length of $20\sqrt{2}$ cm. How many small squares are on this board?

**13.** Find the area and the perimeter of the rectangle shown.
Express your answers in simplified radical form.




**14.** Why is $\sqrt{16 + 9}$ not equal to $\sqrt{16} + \sqrt{9}$? Justify your reasoning.

**15.** Is the expression $1 + \sqrt{3}$ a solution to the equation $x^2 - 2x - 2 = 0$? Explain.


## C Extend

**16.** Simplify.

**a)** $\dfrac{10 + 15\sqrt{5}}{5}$ **b)** $\dfrac{21 - 7\sqrt{6}}{7}$

**c)** $\dfrac{\sqrt{14}}{\sqrt{2}}$ **d)** $\dfrac{12 - \sqrt{48}}{4}$

**e)** $\dfrac{-10 + \sqrt{50}}{5}$

**17.** A square root is simplified by finding factors that appear twice, and leaving all other factors under the radical sign. Simplifying a cube root requires the factor to appear three times under the cube root sign. Any factor that does not appear three times is left under the cube root. Simplify each cube root.

**a)** $\sqrt[3]{54}$ **b)** $\sqrt[3]{3000}$ **c)** $\sqrt[3]{1125}$

**18. a)** For what values of $a$ is $\sqrt{a} < a$?

**b)** For what values of $a$ is $\sqrt{a} > a$?

Explain your reasoning.

**19. Math Contest** If $\sqrt{4^2 + 4^2 + \cdots + 4^2} = 16$, how many $4^2$'s are under the radical?

**A** 4 **B** 8 **C** 12 **D** 16

**20. Math Contest** The roots of the equation $\sqrt{3x - 11} = x - 3$ are $m$ and $n$. A possible value for $m - n$ is

**A** 9 **B** 0 **C** $-1$ **D** $-5$

**21. Math Contest** If $\sqrt{128} = \sqrt{2} + \sqrt{x}$, what is the value of $x$?

**A** 126 **B** 64 **C** 98 **D** 256

**22. Math Contest** Given that $f(a + b) = f(a)f(b)$ and $f(x)$ is always positive, what is the value of $f(0)$?

# Use Technology

## Use a TI-Nspire™ CAS Graphing Calculator to Explore Operations With Radicals

**Tools**

- TI-Nspire™ CAS graphing calculator

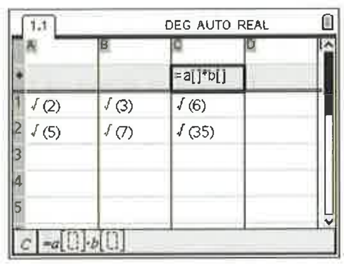
**1. a)** Turn on the TI-Nspire™ CAS graphing calculator.

- Press and select **8:System Info**. Then, select **2:System Settings....**
- Use the [tab] key to scroll down to **Auto or Approx** and ensure that it is set to **Auto**. Continue down to **OK**, and press twice.

**b)** Press and select **6:New Document**. Select **3:Add Lists & Spreadsheet**.

**c)** Use the cursor keys on the NavPad to move to cell **A1**. Press [ctrl] to enter the square root symbol. Then, press **2** and [enter].

**d)** Move to cell **B1** and enter $\sqrt{3}$.




**e)** Move to the cell above cell **C1** and enter the formula =a*b. Press [enter]. Note the result in cell **C1**, as shown.

**f)** Enter $\sqrt{5}$ in cell **A2** and $\sqrt{7}$ in cell **B2**. Note the result in cell **C2**.

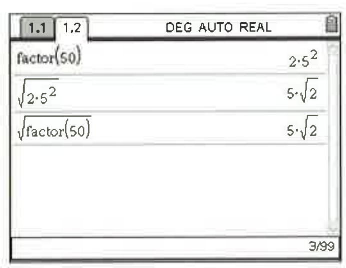
**g)** Try a few more examples of your choice.

**2.** You can use the CAS to help you change entire radicals to mixed radicals.

**a)** Press and select **1:Add Calculator**.

**b)** Press [menu] and select **3:Algebra**. Select **2:Factor**.

**c)** Type 50 and press [enter]. Note the result.




**d)** Press [ctrl] to access the square root.

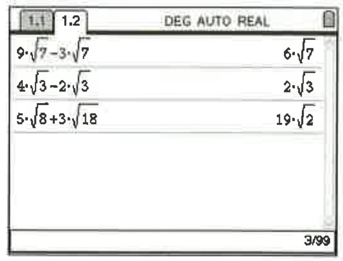
**e)** Press [ctrl] to access the previous answer. Press [enter]. Note the result.

**f)** Try this shortcut. Enter the square root symbol first. Then, enter the **factor()** command, followed by the 50. Press [enter].

**g)** Try a few more examples of your choice.

3. You can check your work on addition or subtraction of radicals.

**a)** Enter $9\sqrt{7} - 3\sqrt{7}$ and press enter. Note the result.




**b)** Try a few more, such as

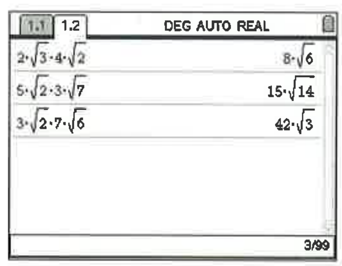
$4\sqrt{3} - 2\sqrt{3}$

$5\sqrt{8} + 3\sqrt{18}$

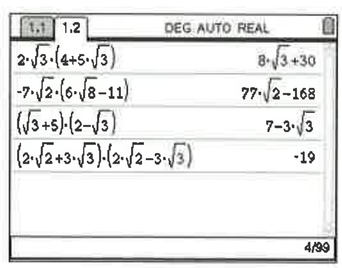
Be sure you can explain where the last answer came from.

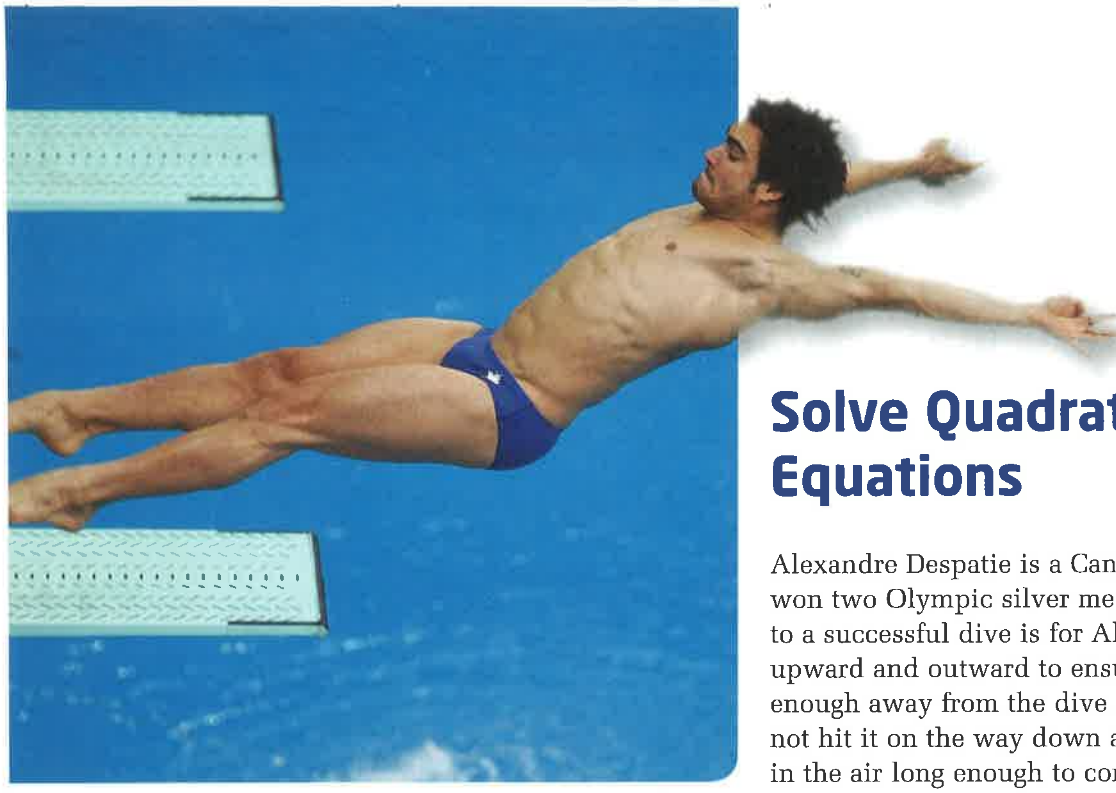
**c)** Try some examples of your choice.

4. Try some multiplication of radicals. Start with the examples shown. Then, try some of your own.




5. Try some mixed operations. Start with the examples shown. Then, try some of your own.






# Solve Quadratic Equations

Alexandre Despatie is a Canadian diver who has won two Olympic silver medals. One of the keys to a successful dive is for Alexandre to jump upward and outward to ensure that he is far enough away from the dive tower so that he will not hit it on the way down and so that he stays in the air long enough to complete the dive.

A mathematician analyses the dives of a team. The path of a dive can be modelled by the quadratic function $f(t) = -4.9t^2 + 3t + 10$. How can this function be used to determine how long a diver is in the air? What part of the equation needs to change for the diver to stay in the air longer? If this change is made, how much longer will the diver be in the air?

In this section, you will look at the concepts needed to answer questions such as these. One of the concepts is the solution of **quadratic equations**.

**quadratic equation**

- an equation of the form $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$

## Investigate

### How can you solve quadratic equations of the form $a(x - h)^2 + k = 0$?

1. Solve $x^2 = 4$. How many solutions are there?
2. Solve $(x + 1)^2 = 4$.
3. Solve $2(x + 1)^2 = 8$.
4. Solve $2(x + 1)^2 - 8 = 0$.
5. How are the equations in steps 1 to 4 related?
6. **Reflect** Describe a method for solving $a(x - h)^2 + k = 0$. Use your method to solve $2(x - 3)^2 - 32 = 0$.

## Example 1

### Select a Strategy to Solve a Quadratic Equation

**a)** Solve $2x^2 - 12x - 14 = 0$ by

**i)** completing the square

**ii)** using a graphing calculator

**iii)** factoring

**iv)** using the quadratic formula

**b)** Which strategy do you prefer? Justify your reasoning.

### Solution

**a) i)**

$$2x^2 - 12x - 14 = 0$$

$$x^2 - 6x - 7 = 0 \qquad \text{Divide both sides by 2.}$$

$$x^2 - 6x + 9 - 9 - 7 = 0$$

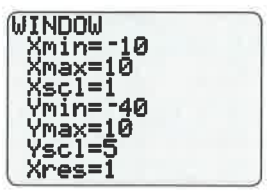
$$(x - 3)^2 - 16 = 0$$

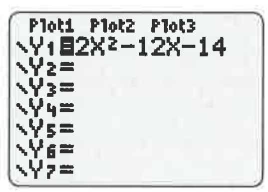
$$(x - 3)^2 = 16$$

$$x - 3 = 4 \text{ or } x - 3 = -4 \qquad \text{Take the square root of both sides.}$$

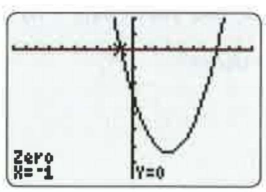
The solutions are $x = 7$ and $x = -1$.

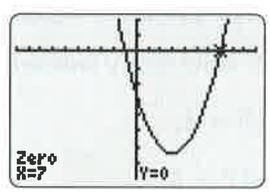
**ii)**
- Use the window settings shown.
- Graph **Y1** $= 2x^2 - 12x - 14$.




- Use the **Zero** operation to find the $x$-intercepts.




The solutions are $x = -1$ and $x = 7$.

**iii)**

$$2x^2 - 12x - 14 = 0$$

$$x^2 - 6x - 7 = 0 \qquad \text{Divide both sides by 2.}$$

$$(x - 7)(x + 1) = 0 \qquad \text{Find the binomial factors of the trinomial } x^2 - 6x - 7.$$

$$x - 7 = 0 \text{ or } x + 1 = 0$$

$$x = 7 \text{ or } \quad x = -1$$

**Technology Tip**

Refer to the Use Technology feature on page 33 to see how to find zeros using a TI-Nspire™ CAS graphing calculator.

**iv)** $2x^2 - 12x - 14 = 0$

$x^2 - 6x - 7 = 0$ — Divide both sides by 2.

$a = 1$, $b = -6$, and $c = -7$.

Substitute the values of $a$, $b$, and $c$ into the quadratic formula and simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-7)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{64}}{2}$$
$$= \frac{6 \pm 8}{2}$$
$$= \frac{14}{2} \text{ or } \frac{-2}{2}$$
$$= 7 \text{ or } -1$$

**b)** While all four methods produce the same solutions, factoring is probably the best strategy for this example. The quadratic expression is easy to factor, so this method is the fastest. If the quadratic expression could not be factored, either the graphing calculator method or using the quadratic formula would be preferred.

**Connections**

In this example, the roots are integers. However, many quadratic equations have irrational roots. If exact roots are asked for, then either completing the square or the quadratic formula is a better method to use. The graphing calculator method will only provide approximations.

Solving $2x^2 - 12x - 14 = 0$ is equivalent to finding the zeros, or $x$-intercepts, of the function $f(x) = 2x^2 - 12x - 14$. The two solutions in Example 1 represent the two $x$-intercepts of the function $f(x) = 2x^2 - 12x - 14$. However, not all quadratic functions have two $x$-intercepts. Some have one $x$-intercept, while others have no $x$-intercepts. The next example illustrates this.

## Example 2

### Connect the Number of Zeros to a Graph

For each quadratic equation given in the form $ax^2 + bx + c = 0$, graph the related function $f(x) = ax^2 + bx + c$ using a graphing calculator. State the number of solutions of the original equation. Justify each answer.

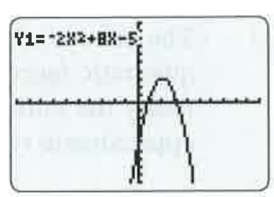
**a)** $-2x^2 + 8x - 5 = 0$

**b)** $8x^2 - 11x + 5 = 0$

**c)** $-4x^2 + 12x - 9 = 0$

### Solution

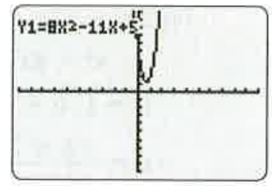
**a)** The parabola opens downward and the vertex is located above the $x$-axis, so the function has two zeros.




The equation $-2x^2 + 8x - 5 = 0$ has two solutions.

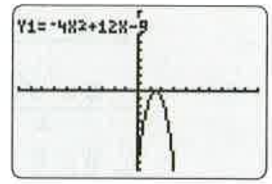
**b)** The parabola opens upward and the vertex is located above the $x$-axis, so the function has no zeros.

The equation $8x^2 - 11x + 5 = 0$ has no real solutions.




**c)** The parabola opens downward and the vertex is located on the $x$-axis. This function has one zero.

The equation $-4x^2 + 12x - 9 = 0$ has one solution.




The graph of a quadratic function gives you a visual understanding of the number of $x$-intercepts. Without a graphing calculator, it can be quite time-consuming to create this visualization. Is there a way that the number of zeros can be identified without drawing a graph? The next example revisits Example 2 using the quadratic formula to see if a pattern can be identified that will tell the number of zeros without graphing.

### Connections

Engineers use the zeros of a quadratic function to help mathematically model the support structure needed for a bridge that must span a given distance.

## Example 3

### Connect the Number of Zeros to the Quadratic Formula

Solve each quadratic equation in Example 2 using the quadratic formula. Give answers for the $x$-intercepts as exact values. Compare the results with the conclusion for the number of $x$-intercepts found in Example 2.

### Solution

**a)** $-2x^2 + 8x - 5 = 0$

$a = -2$, $b = 8$, and $c = -5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-8 \pm \sqrt{8^2 - 4(-2)(-5)}}{2(-2)}$$

$$= \frac{-8 \pm \sqrt{24}}{-4}$$

$$x = \frac{-8 + 2\sqrt{6}}{-4} \text{ or } x = \frac{-8 - 2\sqrt{6}}{-4}$$

$$x = \frac{4 - \sqrt{6}}{2} \quad \text{or } x = \frac{4 + \sqrt{6}}{2}$$

The answer of two solutions from Example 2 is verified by the quadratic formula. There are two solutions because the value under the radical sign is positive, so it can be evaluated to give two approximate roots.

**b)** $8x^2 - 11x + 5 = 0$

$a = 8$, $b = -11$, and $c = 5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-11) \pm \sqrt{(-11)^2 - 4(8)(5)}}{2(8)}$$
$$= \frac{11 \pm \sqrt{-39}}{16}$$

Since the square root of a negative value is not a real number, there is no real solution to the quadratic equation.

**c)** $-4x^2 + 12x - 9 = 0$

$a = -4$, $b = 12$, and $c = -9$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-12 \pm \sqrt{12^2 - 4(-4)(-9)}}{2(-4)}$$
$$= \frac{-12 \pm \sqrt{0}}{-8}$$
$$= \frac{-12}{-8}$$
$$= \frac{3}{2}$$

There is one solution because the value under the square root is zero. This means that there is exactly one root to the equation $-4x^2 + 12x - 9 = 0$.

Example 3 shows that the value under the radical sign in the quadratic formula determines the number of solutions for a quadratic equation and the number of zeros for the related quadratic function.

## Example 4

### Use the Discriminant to Determine the Number of Solutions

For each quadratic equation, use the **discriminant** to determine the number of solutions.

**a)** $-2x^2 + 3x + 8 = 0$ **b)** $3x^2 - 5x + 11 = 0$ **c)** $\frac{1}{4}x^2 - 3x + 9 = 0$

**discriminant**

- the expression $b^2 - 4ac$, the value of which can be used to determine the number of solutions to a quadratic equation $ax^2 + bx + c = 0$
- When $b^2 - 4ac > 0$, there are two solutions.
- When $b^2 - 4ac = 0$, there is one solution.
- When $b^2 - 4ac < 0$, there are no solutions.

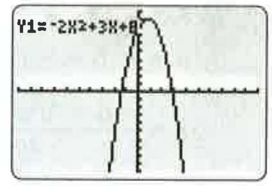
### Solution

**a)** $-2x^2 + 3x + 8 = 0$

$a = -2$, $b = 3$, and $c = 8$.

$$b^2 - 4ac = 3^2 - 4(-2)(8)$$
$$= 9 + 64$$
$$= 73$$

Since the discriminant is greater than zero, there are two solutions.
You can check this result using a graphing calculator.

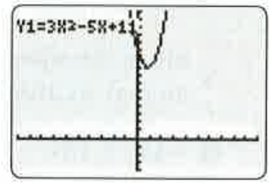



**b)** $3x^2 - 5x + 11 = 0$

$a = 3$, $b = -5$, and $c = 11$.

$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4(3)(11) \\ &= 25 - 132 \\ &= -107 \end{aligned}$$

Since the discriminant is less than zero, there are no solutions.

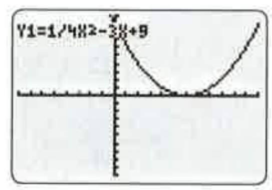



**c)** $\frac{1}{4}x^2 - 3x + 9 = 0$

$a = \frac{1}{4}$, $b = -3$, and $c = 9$.

$$\begin{aligned} b^2 - 4ac &= (-3)^2 - 4\left(\frac{1}{4}\right)(9) \\ &= 9 - 9 \\ &= 0 \end{aligned}$$

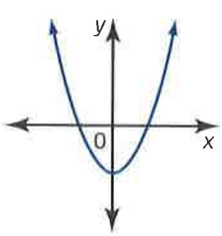
Since the discriminant is equal to zero, there is one solution.



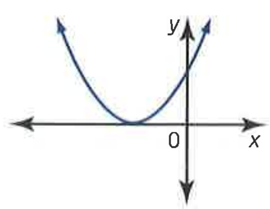

## Key Concepts

- A quadratic equation can be solved by
  - completing the square
  - factoring
  - using the quadratic formula
  - graphing
- The number of solutions to a quadratic equation and the number of zeros of the related quadratic function can be determined using the discriminant.

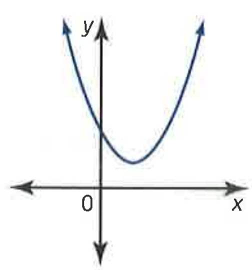
If $b^2 - 4ac > 0$, there are two solutions (two distinct real roots).




If $b^2 - 4ac = 0$, there is one solution (two equal real roots).




If $b^2 - 4ac < 0$, there are no real solutions.



## Communicate Your Understanding

**C1** Minh has been asked to solve a quadratic equation of the form $ax^2 + bx + c = 0$, but he is unclear whether he should factor, complete the square, use the quadratic formula, or use a graphing calculator. What advice would you give him? Explain.

**C2** While many techniques can be used to solve a quadratic equation of the form $ax^2 + bx = 0$, what is the easiest technique to use? Why?

**C3** Deepi wants to determine how many $x$-intercepts a quadratic function has. How can she find the number of $x$-intercepts for the function without graphing? Justify your reasoning.

## A Practise

*For help with questions 1 to 3, refer to Example 1.*

**1.** Solve each quadratic equation by factoring.

**a)** $x^2 + 2x - 3 = 0$

**b)** $x^2 + 3x - 10 = 0$

**c)** $4x^2 - 36 = 0$

**d)** $6x^2 - 14x + 8 = 0$

**e)** $15x^2 - 8x + 1 = 0$

**f)** $6x^2 + 19x + 10 = 0$

**2.** Check your answers to question 1 using a graphing calculator or by substituting each solution back into the original equation.

**3.** Solve each quadratic equation using the quadratic formula. Give exact answers.

**a)** $2x^2 - 17x + 27 = 0$

**b)** $-4x^2 + 3x + 8 = 0$

**c)** $-x^2 - x + 7 = 0$

**d)** $x^2 + 6x - 4 = 0$

**e)** $3x^2 + x - 11 = 0$

**f)** $-\frac{1}{2}x^2 + 4x - 1 = 0$

*For help with question 4, refer to Example 2.*

**4. Use Technology** Use a graphing calculator to graph a related function to determine the number of roots for each quadratic equation.

**a)** $3x^2 - 4x + 5 = 0$

**b)** $8x^2 - 20x + 12.5 = 0$

**c)** $-x^2 + 2x + 5 = 0$

**d)** $\frac{3}{4}x^2 - 5x + 2 = 0$

*For help with question 5, refer to Example 3.*

**5.** Determine the exact values of the $x$-intercepts of each quadratic function.

**a)** $f(x) = 6x^2 + 3x - 2$

**b)** $f(x) = -\frac{1}{3}x^2 + 4x - 8$

**c)** $f(x) = \frac{3}{4}x^2 - 2x - 7$

**d)** $f(x) = \frac{1}{4}x^2 - 2x + 4$

*For help with question 6, refer to Example 4.*

**6.** Use the discriminant to determine the number of roots for each quadratic equation.

**a)** $x^2 - 5x + 4 = 0$

**b)** $3x^2 + 4x + \frac{4}{3} = 0$

**c)** $2x^2 - 8x + 9 = 0$

**d)** $-2x^2 + 0.75x + 5 = 0$

## B Connect and Apply

**7.** Which method would you use to solve each equation? Justify your choice. Then, solve. Do any of your answers suggest that you might have used another method? Explain.

**a)** $2x^2 - 5x - 12 = 0$ **b)** $x^2 - 25 = 0$

**c)** $2x^2 + 3x - 1 = 0$ **d)** $\frac{1}{2}x^2 + 4x = 0$

**e)** $3x^2 - 4x + 2 = 0$ **f)** $x^2 - 4x + 4 = 0$

**g)** $0.57x^2 - 3.7x - 2.5 = 0$

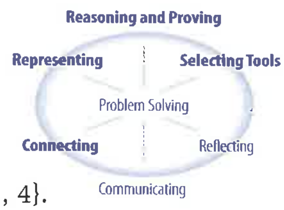
**h)** $9x^2 - 24x + 16 = 0$

8. Determine the value(s) of $k$ for which the quadratic equation $x^2 + kx + 9 = 0$ will have

**a)** two equal real roots

**b)** two distinct real roots

9. **a)** Create a table of values for the function $f(x) = 2x^2 - 3x$ for the domain $\{-2, -1, 0, 1, 2, 3, 4\}$.




**b)** Graph this quadratic function.

**c)** On the same set of axes, graph the line $y = 6$.

**d)** Use your graph to determine the approximate $x$-values where the line $y = 6$ intersects the quadratic function.

**e)** Determine the $x$-values for the points of intersection of $f(x) = 2x^2 - 3x$ and the horizontal line $y = 6$ algebraically.

10. **Use Technology** Check your answer to question 9 using a graphing calculator.

11. What value(s) of $k$, where $k$ is an integer, will allow each quadratic equation to be solved by factoring?

**a)** $x^2 + kx + 12 = 0$

**b)** $x^2 + kx = 8$ **c)** $x^2 - 3x = k$

12. The height, $h$, in metres, above the ground of a football $t$ seconds after it is thrown can be modelled by the function $h(t) = -4.9t^2 + 19.6t + 2$. Determine how long the football will be in the air, to the nearest tenth of a second.

13. A car travelling at $v$ kilometres per hour will need a stopping distance, $d$, in metres, without skidding that can be modelled by the function $d = 0.0067v^2 + 0.15v$. Determine the speed at which a car can be travelling to be able to stop in each distance. Round answers to the nearest tenth.

**a)** 37 m **b)** 75 m **c)** 100 m

14. A by-law restricts the height of structures in an area close to an airport. To conform with this by-law, fuel storage tanks with different capacities are built by varying the radius of the cylindrical tanks. The surface area, $A$, in square metres, of a tank with radius $r$, in metres, can be approximately modelled by the quadratic function $A(r) = 6.28r^2 + 47.7r$. What is the radius of a tank with each surface area?

**a)** 1105 m$^2$

**b)** 896.75 m$^2$

15. The length of a rectangle is 2 m more than the width. If the area of the rectangle is 20 m$^2$, what are the dimensions of the rectangle, to the nearest tenth of a metre?

16. A building measuring 90 m by 60 m is to be built. A paved area of uniform width will surround the building. The paved area is to have an area of 9000 m$^2$. How wide is the paved area?


17. If the same length is cut off three pieces of wood measuring 21 cm, 42 cm, and 45 cm, the three pieces of wood can be assembled into a right triangle. What length needs to be cut off each piece?

18. In Vancouver, the height, $h$, in kilometres, that you would need to climb to see to the east coast of Canada can be modelled by the equation $h^2 + 12\,740h = 20\,000\,000$. If the positive root of this equation is the solution, find the height, to the nearest kilometre.

19. **Chapter Problem** Andrea has been asked to determine when (if ever) the volume, $V$, in hundreds of shares, of a company's stock, which can be modelled by the function $V(x) = 250x - 5x^2$, after being listed on the stock exchange for $x$ weeks, will reach

**a)** 275 000 shares in a week

**b)** 400 000 shares in a week

What answer should Andrea give?

20. Small changes to a quadratic equation can have large effects on the solutions. Illustrate this statement by solving each quadratic equation.


**a)** $x^2 + 50x + 624 = 0$

**b)** $x^2 + 50x + 625 = 0$

**c)** $x^2 + 50x + 626 = 0$

## Achievement Check

21. A diver followed a path defined by $h(t) = -4.9t^2 + 3t + 10$ in her dive, where $t$ is the time, in seconds, and $h$ represents her height above the water, in metres.

**a)** At what height did the diver start her dive?

**b)** For how long was the diver in the air?

**c)** The $-4.9$ in front of the $t^2$ term is constant because it relates to the acceleration due to gravity on Earth. If the diver always starts her dives from the same height, what other value in the quadratic expression will never change?

**d)** What is the only value in the quadratic expression that can change? Suggest a way in which this value can change.

**e)** If the value in part d) changed to 6, how much longer would the diver be in the air?

## C Extend

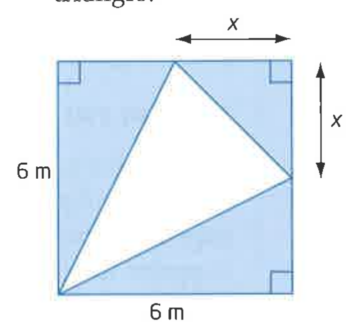
22. Complete the square on the expression $ax^2 + bx + c = 0$ to show how the quadratic formula is obtained.

23. A cubic block of concrete shrinks as it dries. The volume of the dried block is 30.3 cm$^3$ less than the original volume, while the length of each edge has decreased by 0.1 cm. Determine the edge length and volume of the concrete block before it dried.

24. In the diagram, the square has side lengths of 6 m. The square is divided into three right triangles and one isosceles triangle. The areas of the three right triangles are equal.

**a)** Find the value of $x$.

**b)** Find the area of the acute isosceles triangle.




25. **Math Contest** If $f(x) = 2x^2 - 13x + c$ and $f(c) = -16$, then one possible value for $c$ is

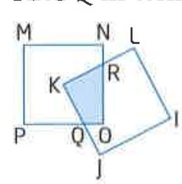
**A** $-2$ **B** 2 **C** $-4$ **D** 8

26. **Math Contest** The function $f(x) = 3x^2 + 9x - 3$ has $x$-intercepts $p$ and $q$. The value of $p - pq + q$ is

**A** $-2$ **B** $3 + 5\sqrt{13}$

**C** 0 **D** $-4$

27. **Math Contest** The squares MNOP and IJKL overlap as shown. K is the centre of MNOP. What is the area of quadrilateral KROQ in terms of the area of MNOP?



1.6

# Determine a Quadratic Equation Given Its Roots

Bridges like the one shown often have supports in the shape of parabolas. If the anchors at either side of the bridge are 42 m apart and the maximum height of the support is 26 m, what function models the parabolic curve of the support? Engineers need to determine this function to ensure that the bridge is built to proper specifications. How can the given data be used to model the equation of the parabola?

**Tools**

- grid paper

## Investigate

### How can you connect the zeros to a form of the quadratic function?

In the introduction, information was given about a parabolic support under a bridge. What equation will model the parabolic curve of the support if the vertex is on the $y$-axis and the points of attachment of the supports are on the $x$-axis?

**Method 1: Use Pencil and Paper**

**1.** Use the information given to identify three points: the two $x$-intercepts and the vertex. Sketch the function. Label the three known points.

**2.** The intercept form of a quadratic function is $y = a(x - r)(x - s)$, where $r$ and $s$ are the $x$-intercepts. Write the function in this form using the data from the original problem for the $x$-intercepts.

**3.** How can you use the third known point to find the value of $a$?

**4. a)** Write a function in factored form for the bridge support.

**b)** Express the function from part a) in standard form.

**5. Reflect** Can you write the equation of a quadratic function given its zeros? If so, describe how. If not, explain why not.

### Method 2: Use a Graphing Calculator

**Tools**

- graphing calculator

**1.** Use the information given to identify three points and draw a sketch. Enter the three data points into a graphing calculator using **L1** and **L2**.

**2.** Use quadratic regression to find the equation of the quadratic function in the form $y = ax^2 + bx + c$.

**3.** Enter the function for **Y1** and graph the equation to verify.

**4. a)** The intercept form of a quadratic function is $y = a(x - r)(x - s)$, where $r$ and $s$ are the $x$-intercepts. Write this form of the function, using the same value of $a$ as found in step 2 and the data from the original problem. Enter this form of the function as **Y2** on the graphing calculator, choose a different thickness for the new line, and graph this line.

**b)** What do you notice occurs on the display of the graphing calculator as the second parabola is graphed?

**5. Reflect** Can you determine the equation of a quadratic function given its zeros? If so, describe how. If not, explain why not.

## Example 1

### Find the Equation of a Family of Quadratic Functions

Find the equation, in factored form, for a family of quadratic functions with the given $x$-intercepts. Sketch each family, showing at least three members.

**a)** 4 and 2

**b)** 0 and $-5$

**c)** $-3$ and 3

**d)** 6 is the only $x$-intercept

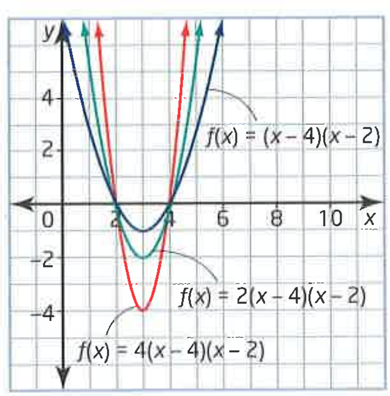
**Connections**

Functions that have a common property are called a *family*. In grade 9, you worked with families of linear functions that have the same slope: they are parallel lines.

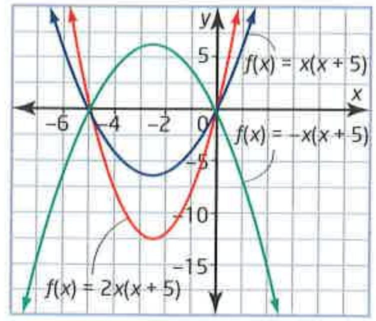
### Solution

**a)** Since $x = 4$ and $x = 2$ are roots of the equation, $x - 4$ and $x - 2$ are factors of the function.

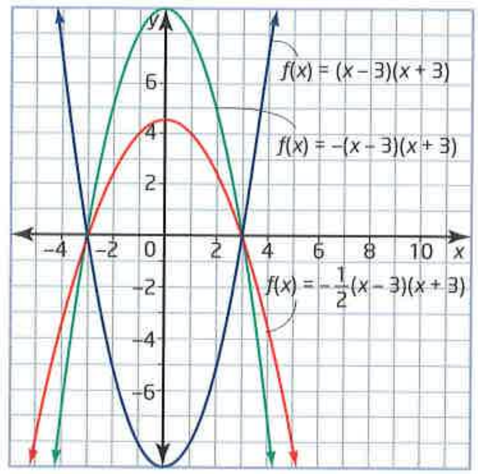
The equation for this family is $f(x) = a(x - 4)(x - 2)$.



**b)** Since $x = 0$ and $x = -5$ are roots of the equation, $x$ and $x + 5$ are factors of the function. The equation for this family is $f(x) = ax(x + 5)$.

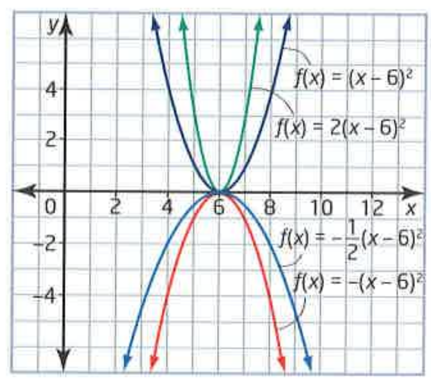



**c)** Since 3 and $-3$ are the $x$-intercepts, $x - 3$ and $x + 3$ are factors. The equation for this family is $f(x) = a(x - 3)(x + 3)$.




**d)** Since $x = 6$ is the only zero, $x - 6$ must be a repeated factor.

The equation for this family is $f(x) = a(x - 6)^2$.




## Example 2

### Determine the Exact Equation of a Quadratic Function

Find the equation of the quadratic function with the given zeros and containing the given point. Express your answers in standard form.

**a)** 2 and $-2$, containing the point $(0, 3)$

**b)** double zero at $x = -2$, containing the point $(3, 10)$

**c)** $3 + \sqrt{5}$ and $3 - \sqrt{5}$, containing the point $(2, -12)$

## Solution

**a)** Since 2 and $-3$ are zeros, then $x - 2$ and $x + 3$ are factors.

$f(x) = a(x - 2)(x + 3)$

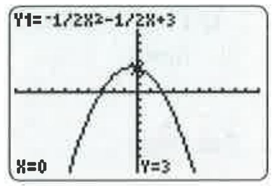
Substitute the given point: $f(0) = 3$

$$3 = a(0 - 2)(0 + 3)$$
$$3 = -6a$$
$$a = -\frac{1}{2}$$

The function, in factored form, is $f(x) = -\frac{1}{2}(x - 2)(x + 3)$.

In standard form: $f(x) = -\frac{1}{2}(x^2 + x - 6)$

$$= -\frac{1}{2}x^2 - \frac{1}{2}x + 3$$




Check by graphing the function using a graphing calculator.

**b)** Since $-2$ is a double zero, the factor $x + 2$ is repeated.

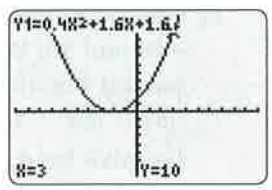
$f(x) = a(x + 2)^2$

Substitute the point: $f(3) = 10$

$$10 = a(3 + 2)^2$$
$$10 = 25a$$
$$a = 0.4$$

Simplify and solve for $a$.

The function, in factored form, is $f(x) = 0.4(x + 2)^2$.




In standard form:

$$f(x) = 0.4(x^2 + 4x + 4)$$
$$= 0.4x^2 + 1.6x + 1.6$$

**c)** Since $3 + \sqrt{5}$ and $3 - \sqrt{5}$ are zeros, then $\left(x - \left(3 + \sqrt{5}\right)\right)$ and $\left(x - \left(3 - \sqrt{5}\right)\right)$ are factors.

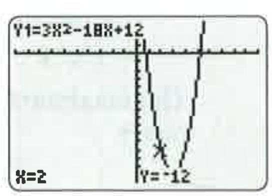
$$f(x) = a\left(x - \left(3 + \sqrt{5}\right)\right)\left(x - \left(3 - \sqrt{5}\right)\right)$$
$$= a\left(x - 3 - \sqrt{5}\right)\left(x - 3 + \sqrt{5}\right)$$
$$= a\left[(x - 3)^2 - \left(\sqrt{5}\right)^2\right]$$
$$= a(x^2 - 6x + 9 - 5)$$
$$= a(x^2 - 6x + 4)$$

This is in the form $(c - d)(c + d)$, where $c = x - 3$ and $d = \sqrt{5}$.

Substitute the point: $f(2) = -12$

$$-12 = a(2^2 - 6(2) + 4)$$
$$-12 = a(-4)$$
$$a = 3$$

The function, in factored form, is $f(x) = 3(x^2 - 6x + 4)$.




In standard form:

$$f(x) = 3(x^2 - 6x + 4)$$
$$= 3x^2 - 18x + 12$$

## Example 3

### Represent Given Information as a Quadratic Function


The parabolic opening to a tunnel is 32 m wide measured from side to side along the ground. At the points that are 4 m from each side, the tunnel entrance is 6 m high.

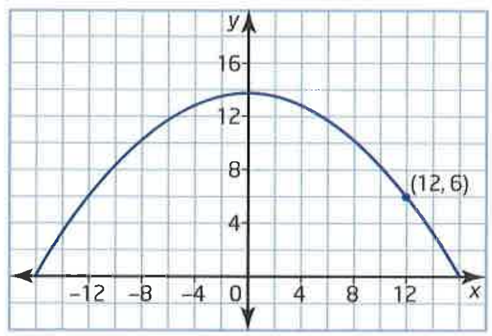
**a)** Sketch a diagram of the given information.

**b)** Determine the equation of the function that models the opening to the tunnel.

**c)** Find the maximum height of the tunnel, to the nearest tenth of a metre.

### Solution

**a)** The point (12, 6) comes from the information given. You are told that 4 m from each side, the height is 6 m. The point (−12, 6) can also be used, giving the same answer.




**b)** Use the $x$-intercepts −16 and 16. Write the general function,
$f(x) = a(x - 16)(x + 16)$.
To solve for $a$, substitute the point (12, 6).

$$6 = a(12 - 16)(12 + 16)$$
$$6 = a(-4)(28)$$
$$6 = a(-112)$$
$$a = -\frac{6}{112}$$
$$= -\frac{3}{56}$$

The function that models the opening to the tunnel is
$f(x) = -\frac{3}{56}(x - 16)(x + 16)$.

**c)** The maximum height of the tunnel will occur halfway between the two $x$-intercepts. This means a value of $x = 0$.

$$f(0) = -\frac{3}{56}(0 - 16)(0 + 16)$$
$$= -\frac{3}{56}(-16)(16)$$
$$\doteq 13.71$$

The maximum height of the tunnel is 13.7 m, to the nearest tenth of a metre.

#### Connections

If you assume that one side of the tunnel is at the origin, you will get a different form of the equation. It will be a translation of the one found here. You will examine the effects of translations on the equation of a function in Chapter 2.

## Key Concepts

- The zeros can be used to find the equation of a family of quadratic functions with the same $x$-intercepts.
- To determine an individual quadratic function, you also need to be given one other point on the function.

## Communicate Your Understanding

**C1** Outline the steps needed to find the equation of a quadratic function given the $x$-intercepts and one other point on the function.

**C2** You are given an equation for a family of quadratic functions with the same $x$-intercepts. Rita says, "The vertex is the only point that will not allow you to determine the exact equation, as it is at the centre of the function, and more than one function can be found." Ronnie claims, "The vertex is as good as any other point in finding the exact function." Who is correct? Explain.

**C3** Mona has decided that if she is given a fraction such as $-\frac{1}{2}$ as one of the $x$-intercepts, she can use the binomial $(2x + 1)$ instead of $\left(x + \frac{1}{2}\right)$ and get the same quadratic function. Is she correct? Explain.

## A Practise

*For help with questions 1 and 2, refer to Example 1.*

**1.** Determine the equation, in factored form, of a family of quadratic functions with each pair of roots. Sketch a graph to show four graphs in each family.

**a)** $x = 3$ and $x = -6$

**b)** $x = -1$ and $x = -1$

**c)** $x = -3$ and $x = -4$

**2.** Express each equation in question 1 in standard form.

*For help with question 3 to 5, refer to Example 2.*

**3.** Find the equation of a quadratic function that has the given $x$-intercepts and contains the given point. Express each function in factored form. Graph each function to check.

**a)** $-3$ and $5$, point $(4, -3)$

**b)** $-4$ and $7$, point $(-3, -12)$

**c)** $0$ and $-\frac{2}{3}$, point $(-1, 5)$

**4.** Write each function in question 3 in standard form.

**5.** Find the equation of the quadratic function that has the given zeros and contains the given point. Express each function in standard form. Graph each function to check.

**a)** $1 \pm \sqrt{11}$, point $(4, -6)$

**b)** $-2 \pm \sqrt{7}$, point $(1, 2)$

**c)** $-5 \pm \sqrt{2}$, point $(-2, -14)$

**6.** Find the equation of the quadratic function that has the given zeros and contains the given point. Express each function in vertex form. Graph each function to check.

**a)** $3$ and $-1$, point $(1, -2)$

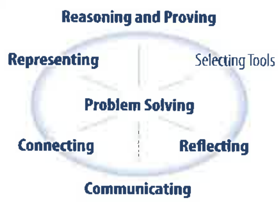
**b)** $1$ and $-2$, point $(0, 4)$

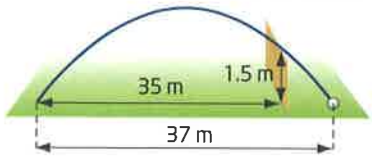
**c)** $3$ and $-5$, point $(1, -4)$

## B Connect and Apply

*For help with question 7, refer to Example 3.*

**7.** A soccer ball is kicked from the ground. After travelling a horizontal distance of 35 m, it just passes over a 1.5-m-tall fence before hitting the ground 37 m from where it was kicked.




**a)** Considering the ground to be the $x$-axis and the vertex to be on the $y$-axis, determine the equation of a quadratic function that can be used to model the parabolic path of the ball.

**b)** Determine the maximum height of the ball.

**c)** How far has the ball travelled horizontally to reach the maximum height?

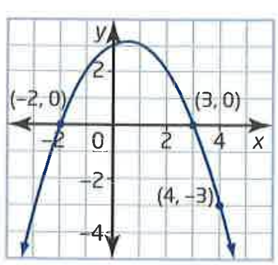
**d)** Develop a new equation for the quadratic function that represents the height of the ball, considering the ball to have been kicked from the origin.

**e)** Outline the similarities and differences between the functions found in parts a) and d).

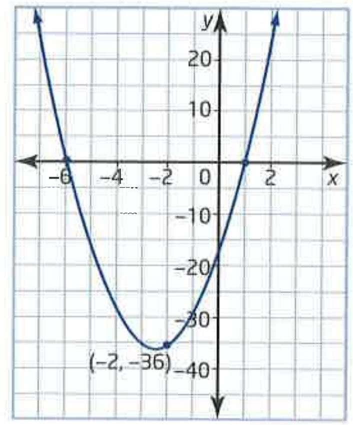
**f)** **Use Technology** Use a graphing calculator to compare the solutions.

**8.** Determine the equation in standard form for each quadratic function shown.

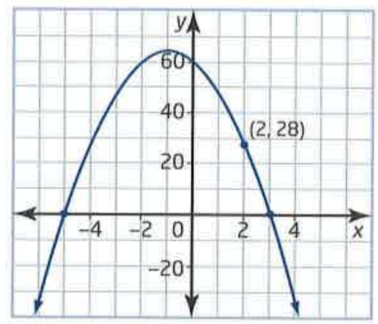
**a)**




**b)**




**c)**




**9.** **Use Technology** For each part in question 8, use a graphing calculator to verify your solution by plotting the three points as well as entering the quadratic function. Explain how you can use this method to check that your solution is correct.

**10.** Explain how the technique studied in this section can be used to find the equation for the quadratic function if the only $x$-intercept is the origin and you are given one other point on the function.

**11.** Find the quadratic function that has only one $x$-intercept and passes through the given point.

**a)** $x$-intercept of 0, point $(5, -2)$

**b)** $x$-intercept of 5, point $(4, 3)$

**c)** $x$-intercept of $-1$, point $(2, 6)$

**12.** **Use Technology** Verify your solutions to question 11 using a graphing calculator.

**13.** If the function $f(x) = ax^2 + 5x + c$ has only one $x$-intercept, what is the mathematical relationship between $a$ and $c$?

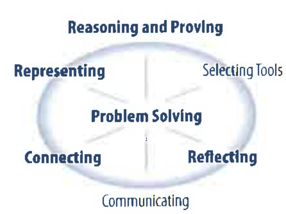
**14. Chapter Problem** The actuarial firm where Andrea has her co-op placement was sent a set of data that follows a quadratic function. The data supplied compared the number of years of driving experience with the number of collisions reported to an insurance company in the last month. Andrea was asked to recover the data lost when the paper jammed in the fax machine. Only three data points can be read. They are (5, 22), (8, 28), and (9, 22). The values of $f(x)$ for $x = 6$ and $x = 7$ are missing. Andrea decided to subtract the $y$-value of 22 from each point so that she would have two zeros: (5, 0), (8, 6), and (9, 0).

**a)** Use these three points to find a quadratic function that can be used to model the adjusted data.

**b)** Add a $y$-value of 22 to this function for a quadratic function that models the original data.

**c)** Use this function to find the missing values for $x = 6$ and $x = 7$.

**15.** An arch of a highway overpass is in the shape of a parabola. The arch spans a distance of 12 m from one side of the road to the other. The height of the arch is 8 m at a horizontal distance of 2 m from each side of the arch.




**a)** Sketch the quadratic function if the vertex of the parabola is on the $y$-axis and the road is along the $x$-axis.

**b)** Use this information to determine the function that models the arch.

**c)** Find the maximum height of the arch to the nearest tenth of a metre.

**16.** Use the information from question 15, but instead of having the vertex on the $y$-axis, put one side of the archway at the origin of the grid. You will get a different equation because the zeros are now at 0 and 12, rather than at $-6$ and 6.

**a)** Find the equation of the quadratic function for this position.

**b)** Find the maximum height of the overpass and compare the result to the height calculated in question 15.

**17.** Explain how the two equations developed in questions 15 and 16 can model the same arch, even though the equations are different.

## Achievement Check

**18.** A quadratic function has zeros $-2$ and 6 and passes through the point (3, 15).

**a)** Find the equation of the quadratic function in factored form.

**b)** Write the function in standard form.

**c)** Complete the square to convert the standard form to vertex form, and state the vertex.

**d)** Use partial factoring to verify your answer to part c).

**e)** Find a second quadratic function with the same zeros as in part a), but passing through the point (3, $-30$). Express the function in standard form.

**f)** Graph both functions. Explain how the graphs can be used to verify that the equations in parts a) and e) are correct.

## C Extend

**19.** Is it possible to determine the defining equation of a function given the following information? If so, justify your answer and provide an example.

**a)** the vertex and one $x$-intercept

**b)** the vertex and one other point on the parabola

**c)** any three points on the parabola

**20. Math Contest** Determine an equation for a quadratic function with zeros at $x = \frac{-1 \pm \sqrt{7}}{3}$.

**21. Math Contest** Show that the graph of $f(x) = ax^2 + c$ has no $x$-intercept if $ac > 0$.

1.7

# Solve Linear-Quadratic Systems


Marina is a set designer. She plans movie sets using freehand sketches and her computer. In one scene, a banner will hang across a parabolic archway. To make it look interesting, she has decided to put the banner on an angle. She sets the banner along a line defined by the linear equation $y = 0.24x + 7.2$, with $x$ representing the horizontal distance and $y$ the vertical distance, in metres, from one foot of the archway. The archway is modelled by the quadratic equation $y = -0.48x^2 + 4.8x$. How can Marina use the equations to determine the points where the banner needs to be attached to the archway and the length of the banner? In this section, you will develop the tools needed to help Marina with these calculations.

**Tools**

- grid paper

**Optional**

- graphing calculator

## Investigate A

### How can a line and a parabola intersect?

Work with a partner.

1. Consider a line and a parabola. At how many points could they intersect? Draw sketches to illustrate your answer.

2. Create pairs of equations for each possibility that you identified in step 1. Use algebraic reasoning to show that your examples are correct.

3. In your algebraic reasoning in step 2, you will have solved a quadratic equation for each situation. Compute the value of the discriminant for each example.

4. **Reflect** Describe how you can predict the number of points of intersection of a linear function and a quadratic function using algebraic reasoning.

## Investigate B

### How can you connect the discriminant to the intersection of a linear and a quadratic function?

**Tools**

- grid paper

or

- graphing calculator

In this Investigate, you will create the equations of lines with slope $-2$ that intersect the quadratic function $y = x^2 + 4x + 4$.

**1.** Write a linear function, in slope $y$-intercept form, with slope $-2$ and an unknown $y$-intercept represented by $k$.

**2.** Eliminate $y$ by substituting the expression for $y$ from the linear equation into the quadratic equation. Simplify so you have a quadratic equation of the form $ax^2 + bx + c = 0$.

**3.** Substitute the values or expressions for $a$, $b$, and $c$ into the discriminant $b^2 - 4ac$.

**4.** In Section 1.6, you learned that the discriminant determines the number of solutions for a quadratic equation. Take advantage of this fact to answer the following questions.

**a)** What values of $k$ will make the discriminant positive? How many points of intersection do the line and the quadratic have in this case?

**b)** What values of $k$ will make the discriminant zero? How many points of intersection do the line and the quadratic have in this case?

**c)** What values of $k$ will make the discriminant negative? How many points of intersection do the line and the quadratic have in this case?

**5. Reflect** With the solutions from step 4, write an equation for a linear equation, in slope $y$-intercept form $y = mx + b$, with slope $-2$, that

**a)** intersects the quadratic function at two points

**b)** intersects the quadratic function at one point

**c)** does not intersect the quadratic function

**6.** Verify each solution in step 5 by graphing the quadratic function $y = x^2 + 4x + 4$ and each of your linear functions.

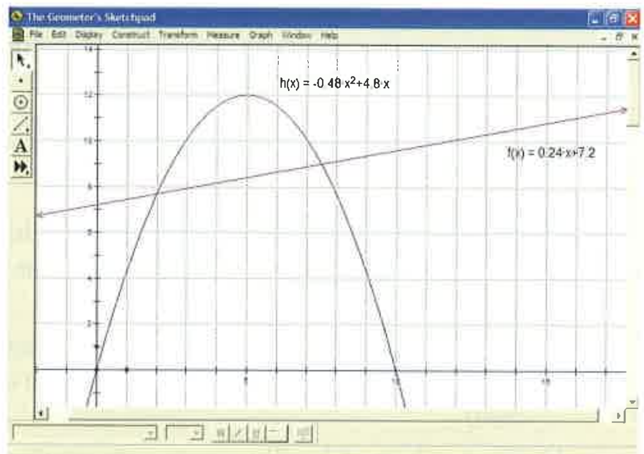
## Example 1

### Find the Points of Intersection of a Linear-Quadratic System of Equations

In the opening of this section, you were introduced to Marina. In a set design, she has a banner on an angle across an archway. She is working with the equations $y = 0.24x + 7.2$ and $y = -0.48x^2 + 4.8x$, where $x$ represents the horizontal distance and $y$ the vertical distance, both in metres, from one foot of the archway.

**a)** Determine the coordinates of the points where the two functions intersect.

**b)** Interpret the solutions in the context.




## Solution

**a) Method 1: Use Pencil and Paper**

Eliminate $y$ by equating the two functions.

$$-0.48x^2 + 4.8x = 0.24x + 7.2$$

$$-0.48x^2 + 4.8x - 0.24x - 7.2 = 0 \quad \text{Rearrange the terms so the right side is zero.}$$

$$-0.48x^2 + 4.56x - 7.2 = 0 \quad \text{Simplify.}$$

$$2x^2 - 19x + 30 = 0 \quad \text{Divide both sides by } -0.24.$$

$$2x^2 - 4x - 15x + 30 = 0 \quad \text{Use grouping to factor.}$$

$$2x(x - 2) - 15(x - 2) = 0$$

$$(x - 2)(2x - 15) = 0$$

Therefore, $x = 2$ or $x = 7.5$.

Substitute into either function to find the corresponding values for $y$. The linear function is easier to use here.

$y = 0.24x + 7.2$

For $x = 2$:

$y = 0.24(2) + 7.2$
$= 7.68$

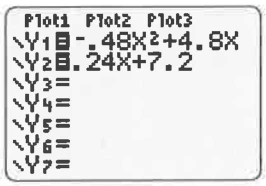
For $x = 7.5$:

$y = 0.24(7.5) + 7.2$
$= 9$

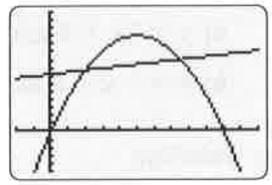
The coordinates of the points where the two functions intersect are $(2, 7.68)$ and $(7.5, 9)$.

**Method 2: Use a Graphing Calculator**

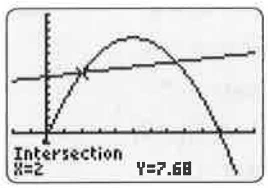
- Enter the two functions:
  **Y1** = $-0.48x^2 + 4.8x$ and **Y2** = $0.24x + 7.2$.
- Use the window settings shown.
- Press (GRAPH).

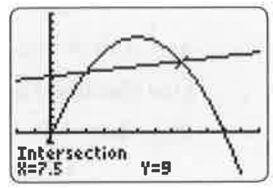






- Press (2nd) [CALC].
- Use the **Intersect** operation to find the coordinates of each point of intersection.

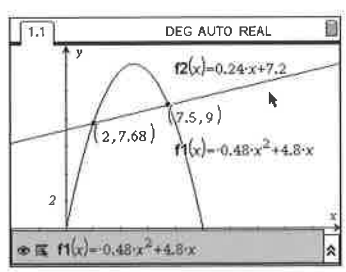




The coordinates of the points where the two functions intersect are (2, 7.68) and (7.5, 9).

**Method 3: Use a TI-Nspire™ CAS Graphing Calculator**

Turn on the TI-Nspire™ CAS graphing calculator.

- Press (home) and select **6:New Document.**
- Select **2:Add Graphs & Geometry**.
- Type $-0.48x^2 + 4.8x$ for function **f1**. Press (enter).
- Type $0.24x + 7.2$ for function **f2**. Press (enter).
- Press (menu). Select **4:Window**.
- Select **6:Zoom – Quadrant 1**.




The graphs will be displayed.

- Press (menu). Select **6:Points & Lines**.
- Select **3:Intersection Point(s)**. Move the cursor to the first graph and press (enter). Move the cursor to the second graph and press (enter). Press (esc).

The coordinates of the points where the two functions intersect are displayed as (2, 7.68) and (7.5, 9).

**b)** These solutions tell Marina that one end of the banner should be attached 2 m horizontally from the left foot of the arch and 7.68 m upward. The other end of the banner should be attached 7.5 m horizontally and 9 m upward.

## Example 2

### Determine Whether a Linear Function Intersects a Quadratic Function

Determine algebraically whether the given linear and quadratic functions intersect. If they do intersect, determine the number of points of intersection.

**a)** $y = 3x + 5$ and $y = 3x^2 - 2x - 4$

**b)** $y = -x - 2$ and $y = -2x^2 + x - 3$

### Solution

**a)** Equate the expressions and simplify.

$$3x^2 - 2x - 4 = 3x + 5$$

$$3x^2 - 2x - 4 - 3x - 5 = 0$$ Rearrange the terms so the right side is zero.

$$3x^2 - 5x - 9 = 0$$ Simplify.

$a = 3$, $b = -5$, and $c = -9$.

Use the discriminant:

$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4(3)(-9) \\ &= 25 + 108 \\ &= 133 \end{aligned}$$

Since the discriminant is greater than zero, there are two solutions. This means that the linear-quadratic system has two points of intersection.

**b)** Equate the expressions and simplify.

$$-2x^2 + x - 3 = -x - 2$$

$$-2x^2 + x - 3 + x + 2 = 0$$ Rearrange the terms so the right side is zero.

$$-2x^2 + 2x - 1 = 0$$ Simplify.

$a = -2$, $b = 2$, and $c = -1$.

Use the discriminant:

$$\begin{aligned} b^2 - 4ac &= 2^2 - 4(-2)(-1) \\ &= 4 - 8 \\ &= -4 \end{aligned}$$

Since the discriminant is less than zero, there are no solutions. This means that the linear-quadratic system has no points of intersection.

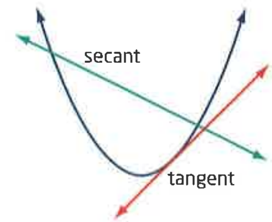
In this section, you have considered how a line can intersect a curve such as a quadratic function. One type of intersection results in a **secant** and the other results in a **tangent line** to the quadratic function.

**secant**

- a line that intersects a curve at two distinct points

**tangent line**

- a line that touches a curve at one point and has the slope of the curve at that point




## Example 3

### Determine the $y$-intercept for a Tangent Line to a Quadratic Function

If a line with slope 4 has one point of intersection with the quadratic function $y = \frac{1}{2}x^2 + 2x - 8$, what is the $y$-intercept of the line? Write the equation of the line in slope $y$-intercept form.

### Solution

The line can be modelled as $y = 4x + k$, where $k$ represents the $y$-intercept.

Substitute for $y$ in the quadratic function:

$$\frac{1}{2}x^2 + 2x - 8 = 4x + k$$

$$\frac{1}{2}x^2 + 2x - 8 - k = 0$$

$$\frac{1}{2}x^2 + 2x + (-8 - k) = 0$$

Then, $a = \frac{1}{2}$, $b = 2$, and $c = -8 - k$.

If the discriminant equals zero, there is only one root.

Substitute into $b^2 - 4ac = 0$.

$$(2)^2 - 4\left(\frac{1}{2}\right)(-8 - k) = 0$$

$$4 - 2(-8 - k) = 0$$

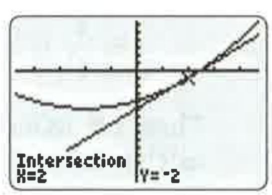
$$4 + 16 + 2k = 0$$

$$2k = -20$$

$$k = -10$$

The $y$-intercept of the line that touches the quadratic at one point is $-10$. The equation of the line is $y = 4x - 10$.

This solution can be verified using a graphing calculator. Graph **Y1** $= \frac{1}{2}x^2 + 2x - 8$ and **Y2** $= 4x - 10$. Use the **Intersect** operation to see that these two functions have only one point of intersection, at $(2, -2)$, so the line is a tangent.




**Connections**

One of the main topics that you will study in calculus is determining the slope of a tangent to a curve at a point on the curve.

**Technology Tip**

Sometimes a "friendly window" is needed to cause the calculator to display exact values. Choose multiples of 94 for the domain. To show the exact point of intersection in Example 3, **Xmin** = −4.7 and **Xmax** = 4.7 were used. After this, the zoom feature **0:ZoomFit** can be used to choose an appropriate range.

## Example 4

### Solve a Problem Involving a Linear-Quadratic System

Dudley Do-Right is riding his horse, Horse, at his top speed of 10 m/s toward the bank, and is 100 m away when the bank robber begins to accelerate away from the bank going in the same direction as Dudley Do-Right. The robber's distance, $d$, in metres, away from the bank after $t$ seconds can be modelled by the equation $d = 0.2t^2$.

**a)** Write a corresponding model for the position of Dudley Do-Right as a function of time.

**b)** Will Dudley Do-Right catch the bank robber? If he does, find the time and position where this happens. If not, explain why not.

### Solution

**a)** Let the position of the bank be at the origin. Since Dudley Do-Right is 100 m away from the bank and the robber is moving in the same direction away from the bank, represent Dudley Do-Right's position as $-100$. He is moving at 10 m/s toward the bank, so his position, relative to the bank, is given by $d = 10t - 100$.

**b)** For Dudley Do-Right to catch the bank robber, the two equations need to be equal:

$10t - 100 = 0.2t^2$

Solve the equation:

$0 = 0.2t^2 - 10t + 100$

$0 = t^2 - 50t + 500$      Multiply by 5.

In the quadratic formula, $a = 1$, $b = -50$, and $c = 500$.

$$
\begin{aligned}
t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-50) \pm \sqrt{(-50)^2 - 4(1)(500)}}{2(1)} \\
&= \frac{50 \pm \sqrt{2500 - 2000}}{2} \\
&= \frac{50 \pm \sqrt{500}}{2} \\
&= \frac{50 \pm 10\sqrt{5}}{2} \\
&= 25 \pm 5\sqrt{5}
\end{aligned}
$$

Then, $t \doteq 13.8$ s or $t \doteq 36.2$ s. The first time is when Dudley Do-Right catches the bank robber.

The second time means that if Dudley does not stop to catch the robber at 13.8 s, he will pass him. But since the robber is accelerating and Dudley is moving at a constant speed, the robber will catch up to Dudley at some point.

Dudley Do-Right will catch the bank robber after 13.8 s.

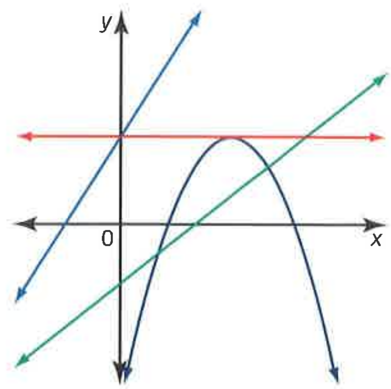
For the position, substitute $t = 13.8$ into either original function.

$$\begin{aligned} d(t) &= 10t - 100 \\ d(13.8) &= 10(13.8) - 100 \\ &= 138 - 100 \\ &= 38 \end{aligned}$$

Dudley Do-Right will catch the robber 38 m past the bank.

## Key Concepts

- A linear function and a quadratic function may
  - intersect at two points (the line is a secant)
  - intersect at one point (the line is a tangent line)
  - never intersect
- The discriminant can be used to determine which of the above situations occurs.
- The quadratic formula can be used to determine the $x$-values of actual points of intersection.




## Communicate Your Understanding

**C1** Larissa always uses the full quadratic formula to determine the number of zeros that a quadratic function has. What would you tell her that would help her understand that she only needs to evaluate the discriminant?

**C2** After Randy has solved a quadratic equation to find the $x$-values for the points of intersection of a given linear-quadratic system, he substitutes the values for $x$ into the linear function to find the values for $y$. Is this a good idea? Explain why or why not.

**C3** What are the advantages and disadvantages in determining the points of intersection of a linear-quadratic system using each method?

- algebraic
- graphical

## A Practise

*For help with questions 1 and 2, refer to Example 1.*

**1.** Determine the coordinates of the point(s) of intersection of each linear-quadratic system algebraically.

**a)** $y = x^2 - 7x + 15$ and $y = 2x - 5$

**b)** $y = 3x^2 - 16x + 37$ and $y = 8x + 1$

**c)** $y = \frac{1}{2}x^2 - 2x - 3$ and $y = -3x + 1$

**d)** $y = -2x^2 - 7x + 10$ and $y = -x + 2$

**2.** Verify the solutions to question 1 using a graphing calculator or by substituting into the original equations.

*For help with questions 3 and 4, refer to Example 2.*

**3.** Determine if each quadratic function will intersect once, twice, or not at all with the given linear function.

**a)** $y = 2x^2 - 2x + 1$ and $y = 3x - 5$

**b)** $y = -x^2 + 3x - 5$ and $y = -x - 1$

**c)** $y = \frac{1}{2}x^2 + 4x - 2$ and $y = x + 3$

**d)** $y = -\frac{2}{3}x^2 + x + 3$ and $y = x$

**4.** Verify your responses to question 3 using a graphing calculator.

*For help with questions 5 and 6, refer to Example 3.*

**5.** Determine the value of the $y$-intercept of a line with the given slope that is a tangent line to the given curve.

**a)** $y = -2x^2 + 5x + 4$ and a line with a slope of 1

**b)** $y = -x^2 - 5x - 5$ and a line with a slope of $-3$

**c)** $y = 2x^2 + 4x - 1$ and a line with a slope of 2

**d)** $y = 3x^2 - 4x + 1$ and a line with a slope of $-2$

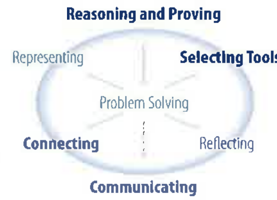
**6.** Verify your solutions to question 5 using a graphing calculator or by substituting into the original equations.

## B Connect and Apply

**7.** The path of an underground stream is given by the function $y = 4x^2 + 17x - 32$. Two new houses need wells to be dug. On the area plan, these houses lie on a line defined by the equation $y = -15x + 100$. Determine the coordinates where the two new wells should be dug.

**8.** Part of the path of an asteroid is approximately parabolic and is modelled by the function $y = -6x^2 - 370x + 100\,900$. For the period of time that it is in the same area, a space probe is moving along a straight path on the same plane as the asteroid according to the linear equation $y = 500x - 83\,024$.

A space agency needs to determine if the asteroid will be an issue for the space probe. Will the two paths intersect? Show all your work.




**9. Use Technology** Check your solutions to questions 7 and 8 using a graphing calculator.

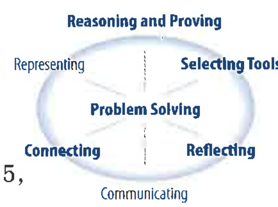
**10.** Determine the value of $k$ in $y = -x^2 + 4x + k$ that will result in the intersection of the line $y = 8x - 2$ with the quadratic at

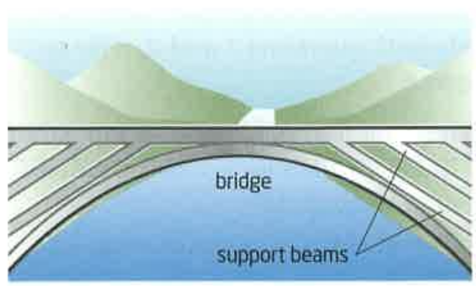
**a)** two points **b)** one point **c)** no point

**11.** Determine the value of $k$ in $y = kx^2 - 5x + 2$ that will result in the intersection of the line $y = -3x + 4$ with the quadratic at

**a)** two points **b)** one point **c)** no point

**12.** A bridge has a parabolic support modelled by the equation $y = -\frac{1}{200}x^2 + \frac{6}{25}x - 5$, where the $x$-axis represents the bridge surface. There are also parallel support beams below the bridge. Each support beam must have a slope of either 0.8 or $-0.8$. Using a slope of $-0.8$, find the $y$-intercept of the line associated with the longest support beam. Hint: The longest beam will be the one along the line that touches the parabolic support at just one point.



**13.** The line $x = 2$ intersects the quadratic function $y = x^2 - 9$ at one point, $(2, -5)$. Explain why the line $x = 2$ is not considered a tangent line to the quadratic function.

**14. Chapter Problem** Andrea's supervisor at the actuarial firm has asked her to determine the safety zone needed for a fireworks display. She needs to find out where the safety fence needs to be placed on a hill. The fireworks are to be launched from a platform at the base of the hill. Using the top of the launch platform as the origin and taking some measurements, in metres, Andrea comes up with the following equations.

Cross-section of the slope of one side of the hill: $y = 4x - 12$

Path of the fireworks: $y = -x^2 + 15x$

**a)** Illustrate this situation by graphing both equations on the same set of axes.

**b)** Calculate the coordinates of the point where the function that describes the path of the fireworks will intersect the equation for the hill.

**c)** What distance up the hill does the fence need to be located? **Hint:** Use the Pythagorean theorem.

**15.** A parachutist jumps from an airplane and immediately opens his parachute. His altitude, $y$, in metres, after $t$ seconds is modelled by the equation $y = -4t + 300$. A second parachutist jumps 5 s later and freefalls for a few seconds. Her altitude, in metres, during this time, is modelled by the equation $y = -4.9(t - 5)^2 + 300$. When does she reach the same altitude as the first parachutist?

**16.** The UV index on a sunny day can be modelled by the function $f(x) = -0.15(x - 13)^2 + 7.6$, where $x$ represents the time of day on a 24-h clock and $f(x)$ represents the UV index. Between what hours was the UV index greater than 7?

## Achievement Check

**17.** The support arches of the Humber River pedestrian bridge in Toronto can be modelled by the quadratic function $y = -0.0044x^2 + 21.3$ if the walkway is represented by the line $y = 0$.

A similar bridge, planned for North Bay, will have the same equation for the support arches. However, since the walkway is to be inclined slightly across a ravine, its equation is $y = 0.0263x + 1.82$.

**a)** Determine the points of intersection of the bridge support arches and the inclined walkway, to one decimal place.

**b)** **Use Technology** Use a graphing calculator to check your solution in part a).

**c)** Determine the length of the bridge.

**d)** How much shorter will this walkway be than the walkway that spans the Humber River in Toronto? Justify your answer.

## C Extend

**18.** The technique of substitution has been used in this section to find the points where a line intersects a parbola. This technique can be used with other curves as well.

**a)** Determine the points at which the circle given by $(x - 5)^2 + (y - 5)^2 = 25$ is intersected by the line $y = -\frac{1}{3}x + \frac{5}{3}$.

**b)** Check your answer on a graphing calculator by graphing the line and the two functions

$y = 5 + \sqrt{25 - (x - 5)^2}$ and

$y = 5 - \sqrt{25 - (x - 5)^2}$.

**19. Math Contest** Find the point(s) of intersection of the line $y = 7x - 42$ and the circle $x^2 + y^2 - 4x + 6y = 12$.

**20. Math Contest** The two circles $x^2 + y^2 = 11$ and $(x - 3)^2 + y^2 = 2$ intersect at two points, P and Q. The length of PQ is

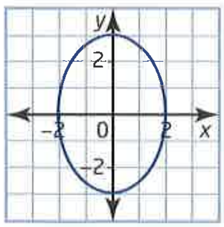
**A** 2 **B** $2\sqrt{2}$ **C** 13 **D** $\sqrt{13}$

# Chapter 1 Review

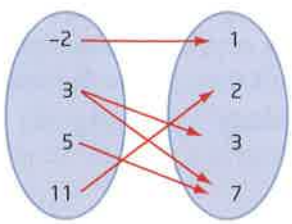
### 1.1 Functions, Domain, and Range, pages 4 to 15

**1.** State the domain and range of each relation.

**a)**




**b)**




**c)** {(1, 4), (2, 6), (3, 10), (4, 18), (5, 29)}

**d)** $y = 2x^2 + 11$

**2.** Which relations in question 1 are functions? Justify your answers.

### 1.2 Functions and Function Notation, pages 16 to 24

**3.** A linear function machine produces the points (2, 5) and (−3, −15).

**a)** Determine the equation of the function.

**b)** Is it possible for a second function to exist that will generate these values? Explain.

**4. a)** Draw a mapping diagram for these data: {(4, −2), (6, 1), (11, −7), (6, 7), (4, −7)}

**b)** Is this relation a function? Explain.

### 1.3 Maximum or Minimum of a Quadratic Function, pages 25 to 33

**5.** A hall charges \$30 per person for a sports banquet when 120 people attend. For every 10 extra people that attend, the hall will decrease the price by \$1.50 per person. What number of people will maximize the revenue for the hall?

**6.** The power, $P$, in watts, produced by a solar panel is given by the function $P(I) = -5I^2 + 100I$, where $I$ represents the current, in amperes.

**a)** What value of the current will maximize the power?

**b)** What is the maximum power?

### 1.4 Skills You Need: Working With Radicals, pages 34 to 42

**7.** Perform each radical operation and simplify where needed.

**a)** $\sqrt{27} - 4\sqrt{3} + \sqrt{243} - 8\sqrt{81} + 2$

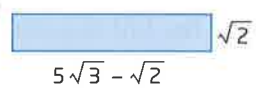
**b)** $-3\sqrt{3}(\sqrt{3} + 5\sqrt{2})$

**c)** $(\sqrt{3} + 5)(5 - \sqrt{3})$

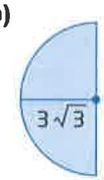
**d)** $5\sqrt{2}(11 + 2\sqrt{2}) - 4(8 + 3\sqrt{2})$

**8.** Find a simplified expression for the area of each shape.

**a)**




**b)**




### 1.5 Solving Quadratic Equations, pages 43 to 51

**9.** Solve each quadratic equation. Give exact answers.

**a)** $3x^2 - 2x - 2 = 0$

**b)** $6x^2 - 23x + 20 = 0$

**10.** Use the discriminant to determine the number of roots for each equation.

**a)** $3x^2 + 4x - 5 = 0$

**b)** $-2x^2 + 5x - 1 = 0$

**c)** $9x^2 - 12x + 4 = 0$

11. Jessica reasoned that since $2 \times 2 = 4$ and $2 + 2 = 4$, $\sqrt{2} + \sqrt{2}$ must have the same value as $\sqrt{2} \times \sqrt{2}$. Is she correct? Justify your answer.

### 1.6 Determine a Quadratic Equation Given Its Roots, pages 52 to 59

12. Determine the equation in standard form for each quadratic function.
    a) $x$-intercepts $-2$ and 5, containing the point (3, 5)
    b) $x$-intercepts $-2 \pm \sqrt{5}$, containing the point (−4, 5)
13. A golf ball is hit, and it lands at a point on the same horizontal plane 53 m away. The path of the ball took it just over a 9-m-tall tree that was 8 m in front of the golfer.
    a) Assume the ball is hit from the origin of a coordinate plane. Find a quadratic function that describes the path of the ball.
    b) What is the maximum height of the ball?
    c) Is it possible to move the origin in this situation and develop another quadratic function to describe the path? If so, find a second quadratic function.
14. **Use Technology** Use a graphing calculator to verify your solution to question 13.

### 1.7 Solve Linear-Quadratic Systems, pages 60 to 69

15. Determine the points of intersection of each pair of functions.
    a) $y = 4x^2 - 15x + 20$ and $y = 5x - 4$
    b) $y = -2x^2 + 9x + 9$ and $y = -3x - 5$
16. For what value of $b$ will the line $y = -2x + b$ be tangent to the parabola $y = 3x^2 + 4x - 1$?
17. Do all linear-quadratic systems result in a solution? Justify your answer using a real-life example.

## Chapter Problem WRAP-UP

**Part A**

For her final duty at her co-op placement, Andrea is given two investments to analyse.

- Investment 1 asks investors to invest \$5000, and the investment grows according to the equation $A(t) = 10t^2 - 48t + 5000$.
- Investment 2 asks investors to invest \$10 000, and the investment grows according to the equation $A(t) = 10\,000 + 497t$.

In both equations, $t$ is time, in weeks, and $A$ is the amount, in dollars.

a) After how long will both investments be worth the same amount?

b) Under what circumstances should Andrea recommend investment 1?

**Part B**

In this chapter, you explored some of the types of calculations that are associated with actuarial science. Conduct an Internet search to gather more information about the profession.

# Chapter 1 Practice Test

**1.** Is each statement true or false?

**a)** Every relation is a special type of function.

**b)** Every function is a special type of relation.

**c)** For $f(x) = \frac{3}{x - 2}$, $x$ can be any real number except $x = 2$.

**d)** $\sqrt{81}$ can be fully simplified to $3\sqrt{9}$.

**e)** A quadratic function and a linear function always intersect at least once.

*For questions 2 to 6, select the best answer.*

**2.** A vertical line test can be used to determine

**A** if a relation is a function

**B** if a relation is constant

**C** if a function is a relation

**D** all of the above are true

**3.** The range of the function $f(x) = -x^2 + 7$ is

**A** $\{y \in \mathbb{R}, y \geq 7\}$  **B** $\{y \in \mathbb{R}, y \leq 7\}$

**C** $\{y \in \mathbb{R}, y > 0\}$  **D** $\{y \in \mathbb{R}\}$

**4.** Which function would produce an output of $y = 9$ for $x = 1$ and for $x = -1$?

**A** $y = 2x + 7$  **B** $y = x^2 - 3x + 1$

**C** $y = 2x^2 + 7$  **D** all of the above

**5.** The vertex of $y = -3x^2 + 6x - 2$ is

**A** $(1, -2)$  **B** $(1, 1)$

**C** $(-1, 1)$  **D** $(-1, -2)$

**6.** Given $f(x) = x^2 - 6x + 10$, if $f(a) = 1$, what is the value of $a$?

**A** 5  **B** 3

**C** 2  **D** 1

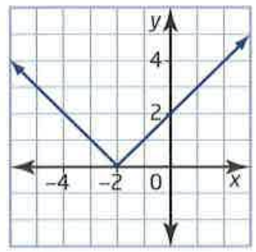
**7.** Sketch a relation that is

**a)** a function with domain $\{x \in \mathbb{R}\}$ and range $\{y \in \mathbb{R}, y \leq 3\}$

**b)** not a function with domain $\{x \in \mathbb{R}, -5 \leq x \leq 5\}$ and range $\{y \in \mathbb{R}, -5 \leq y \leq 5\}$

**8.** State the domain and the range of each function.

**a)**




**b)**

| $x$ | $f(x)$ |
|---|---|
| 0 | −5 |
| 1 | −8 |
| 2 | −12 |
| 3 | −21 |

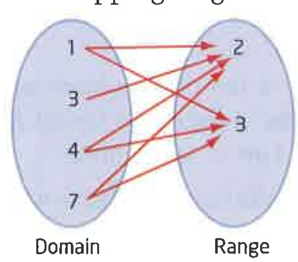
**9.** The time needed for a pendulum to make one complete swing is called the period of the pendulum. The period, $T$, in seconds, for a pendulum of length $\ell$, in metres, can be approximated using the function $T = 2\sqrt{\ell}$.

**a)** State the domain and the range of $T$.

**b)** Sketch a graph of the relation.

**c)** Is the relation a function? Explain.

**10.** Write the ordered pairs that correspond to the mapping diagram. Is this a function?




**11. a)** Find the vertex of the parabola defined by $f(x) = -\frac{1}{2}x^2 + 4x + 3$.

**b)** Is the vertex a minimum or a maximum? Explain.

**c)** How many $x$-intercepts does the function have? Explain.

**12.** Pat has 30 m of fencing to enclose three identical stalls behind the barn, as shown.


**a)** What dimensions will produce a maximum area for each stall?

**b)** What is that maximum area of each stall?

**13.** Simon knows that at \$30 per ticket, 500 tickets to a show will be sold. He also knows that for every \$1 increase in price, 10 fewer tickets will be sold.

**a)** Model the revenue as a quadratic function.

**b)** What ticket price will maximize revenue?

**c)** What is the maximum revenue?

**14.** Perform each radical multiplication and simplify where possible.

**a)** $3\sqrt{2}(2\sqrt{3} - 3\sqrt{2})$

**b)** $(\sqrt{2} + x)(\sqrt{2} - x)$

**15.** For what value of $x$ is

$\sqrt{x} + \sqrt{x} = \sqrt{x} \times \sqrt{x}$, where $x > 0$? Justify your answer.

**16.** Consider the quadratic function $f(x) = -\frac{1}{2}x^2 + 4x + 10$.

**a)** Find the $x$-intercepts.

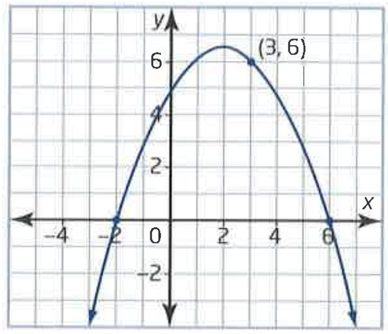
**b)** Use two methods to find the vertex.

**c)** Sketch a graph of the function.

**17.** A rectangle has a length that is 3 m more than twice the width. If the total area is 65 m², find the dimensions of the rectangle.

**18.** Find the equation, in standard form, of the quadratic function that has $x$-intercepts $-5 \pm \sqrt{3}$ and passes through the point $(-3, 8)$.

**19.** The graph of a quadratic function is given.




**a)** Find the equation of the function.

**b)** Find the maximum value of the function.

**20.** Find the point(s) of intersection of $y = -x^2 + 5x + 8$ and $y = 2x - 10$.

**21. a)** Compare the graphs of $f(x) = 3x^2 - 4$ and $g(x) = 3(x - 2)(x + 2)$.

**b)** What needs to be changed in the equation for $f(x)$ to make the two functions part of the same family of curves with the same $x$-intercepts? Explain.

**c)** Describe the family of curves, in factored form, that has the same $x$-intercepts as $h(x) = 5x^2 - 7$.

**22.** A baseball is travelling on a path given by the equation $y = -0.011x^2 + 1.15x + 1.22$. The profile of the bleachers in the outfield can be modelled with the equation $y = 0.6x - 72$. All distances are in metres. Does the ball reach the bleachers for a home run? Justify your answer.

## Task

### Laser Beams

A concert stage has a parabolic roof. The front edge of the roof is defined by the equation $h(x) = -\frac{1}{8}x^2 + 8$, where $x$ is the horizontal distance from the centre and $h$ is the height, both in metres. A vertical lighting tower is built at $x = 9$. Coloured laser lights are installed at various intervals going up the tower. The beams of light are to shine on the front edge of the roof, with their paths defined by the following equations.

Blue: $6x + 8y - 73 = 0$

Green: $x + 2y - 17 = 0$

Orange: $x + y - 10 = 0$

Red: $2x + 8y - 67 = 0$

**a)** Sketch the graph of $h(x)$.

**b)** Determine the coordinates of any point(s) of intersection of each laser beam with the edge of the roof.

**c)** All but one of these laser beams share a common property. Describe the property.

**d)** Determine the height of each light source on the tower.

**e)** Determine an equation for the path of a fifth laser beam that is to be tangent to the edge of the roof at the vertex. Where should this light source be located on the tower?

**f)** Consider the one laser light that does not share the common property. Keeping the location of the light source fixed, determine a new equation for the path of the laser light so that it now shares the common property. Where does it intersect the edge of the roof?