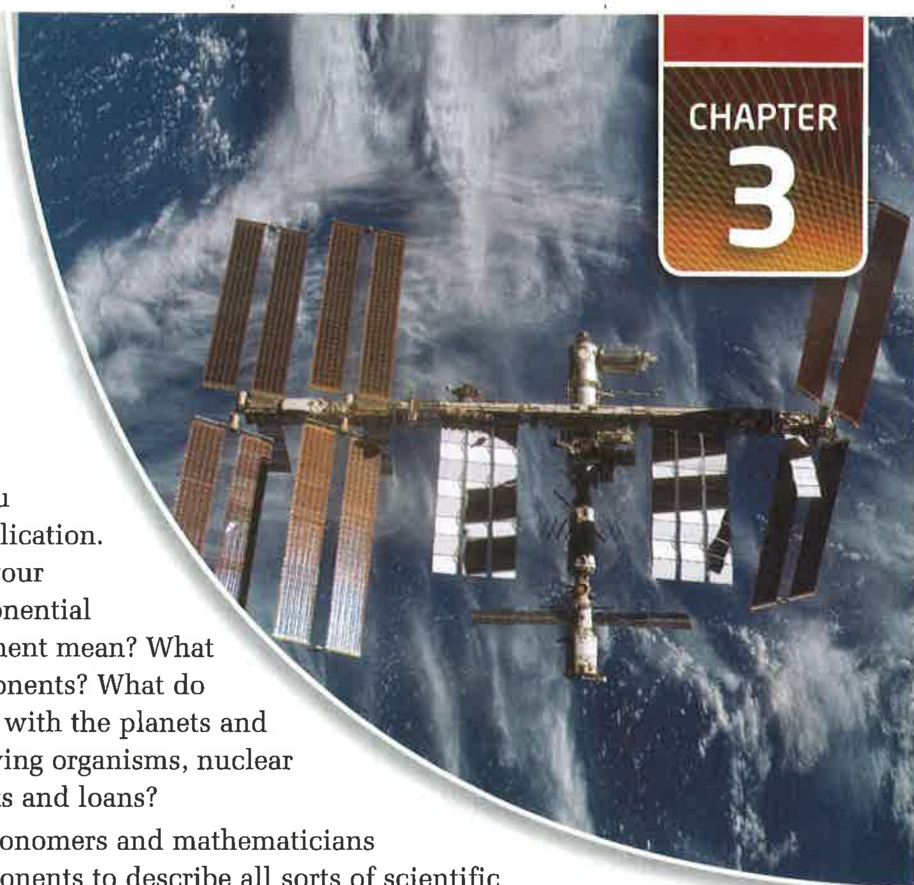



# Exponential Functions

When you think of exponents, you probably think of repeated multiplication. In this chapter, you will expand your knowledge of exponents and exponential functions. What does a zero exponent mean? What about negative and fractional exponents? What do such number concepts have to do with the planets and our solar system, the growth of living organisms, nuclear power generation, and investments and loans?

In ancient times, now-famous astronomers and mathematicians extended the basic concept of exponents to describe all sorts of scientific phenomena. Eventually, engineers and scientists applied these discoveries in innovative ways to improve our quality of life.

## By the end of this chapter, you will

- graph an exponential relation, given its equation in the form $y = a^x$ $(a > 0, a \neq 1)$, define this relation as the function $f(x) = a^x$, and explain why it is a function
- determine the value of a power with a rational exponent
- simplify algebraic expressions containing integer and rational exponents and evaluate numerical expressions containing integer and rational exponents and rational bases
- determine and describe key properties relating to domain and range, intercepts, increasing/decreasing intervals, and asymptotes for exponential functions represented in a variety of ways
- distinguish exponential functions from linear and quadratic functions by making comparisons in a variety of ways
- determine and describe the roles of the parameters $a$, $k$, $d$, and $c$ in functions of the form $y = af[k(x - d)] + c$ in terms of transformations on the graph of $f(x) = a^x$ $(a > 0, a \neq 1)$
- sketch graphs of $y = af[k(x - d)] + c$ by applying one or more transformations to the graph of $f(x) = a^x$ $(a > 0, a \neq 1)$, and state the domain and range of the transformed functions
- determine that the equation of a given exponential function can be expressed using different bases
- represent an exponential function with an equation, given its graph or its properties
- collect data that can be modelled as an exponential function, from primary sources, using a variety of tools, or from secondary sources, and graph the data
- identify exponential functions, including those that arise from real-world applications involving growth and decay, given various representations, and explain any restrictions that the context places on the domain and range
- solve problems using given graphs or equations of exponential functions arising from a variety of real-world applications by interpreting the graphs or by substituting values for the exponent into the equations

# Prerequisite Skills

*Refer to the Prerequisite Skills Appendix on pages 478 to 495 for examples of the topics and further practice.*

## Exponent Rules

**1.** Match each exponent rule with its corresponding method for simplifying.

**Rule**

**a)** Product rule: $(x^a)(x^b)$

**b)** Quotient rule: $\frac{x^a}{x^b}$

**c)** Power of a power rule: $(x^a)^b$

**Method**

**A** Subtract the exponents: $x^{a-b}$

**B** Multiply the exponents: $x^{a \times b}$

**C** Add the exponents: $x^{a+b}$

**2.** Choose one of the exponent rules. Verify that it holds by using one or more of the following tools:

- numerical examples
- algebraic reasoning
- concrete materials
- diagrams

**3.** Simplify. Use the exponent rules.

**a)** $(x^3)(x^2)$ **b)** $(y^4)(y^2)(y^3)$

**c)** $m^6 \div m^4$ **d)** $\frac{h^4}{h^3}$

**e)** $a^3 \times a^4 \times b \times b^5$ **f)** $\frac{x^4y^3}{x^2y}$

**g)** $(ab^2c^3)^4$ **h)** $(3uv^3)^2$

**i)** $\left(\frac{2ab^2}{2^3}\right)^2$ **j)** $\left(\frac{-3w^2}{4r^3}\right)^3$

**4.** Evaluate.

**a)** $2^3 \times 2^4$ **b)** $(3^2)(3^3)(3)$

**c)** $5^2 \times 4^2 \times 5 \times 4^2$ **d)** $(-1)^3(-1)^2(-1)^5$

**e)** $8^5 \div 8^3$ **f)** $\frac{5^5}{5^4}$

**g)** $(3^2)^4$ **h)** $[(-2)^3]^2$

## Zero and Negative Exponents

**5.** Consider the function $y = 2^x$.

**a)** Copy and complete the table.

| $x$ | $y$ |
|---|---|
| 4 | $2^4 = 16$ |
| 3 | $2^3 = 8$ |
| 2 | $2^2 = \blacksquare$ |
| 1 | $2^1 = \blacksquare$ |
| 0 | $2^0 = \blacksquare$ |

**b)** Describe the pattern in the column of $y$-values.

**c)** Extend the pattern to illustrate the meaning of negative exponents.

| $x$ | $y$ |
|---|---|
| 4 | $2^4 = 16$ |
| 3 | $2^3 = 8$ |
| 2 | $2^2 = \blacksquare$ |
| 1 | $2^1 = \blacksquare$ |
| 0 | $2^0 = \blacksquare$ |
| −1 | $2^{-1} = \frac{1}{2} = \frac{1}{2^1}$ |
| −2 | $2^{-2} = \frac{1}{4} = \blacksquare$ |
| −3 | $2^{-3} = \blacksquare$ |
| $-n$ | $2^{-n} = \blacksquare$ |

**6.** Evaluate.

**a)** $5^0$ **b)** $4^{-2}$

**c)** $(-6)^{-3}$ **d)** $3^{-4} \times 3^2$

**e)** $(-2)^0$ **f)** $-2^0$

**g)** $\left(\frac{4}{5}\right)^{-2}$ **h)** $9^{-1} \times 9^0$

**7.** Simplify. Write your answers using only positive exponents.

**a)** $(x^2)(x^{-3})$ **b)** $(y^{-2})^3$

**c)** $\frac{u^3v^{-2}}{u^2v^{-1}}$ **d)** $(4a^2b)^{-2}$

## Graph Functions

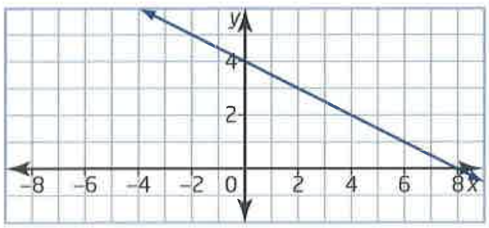
**8.** Based on the graphs shown, identify the

**i)** domain

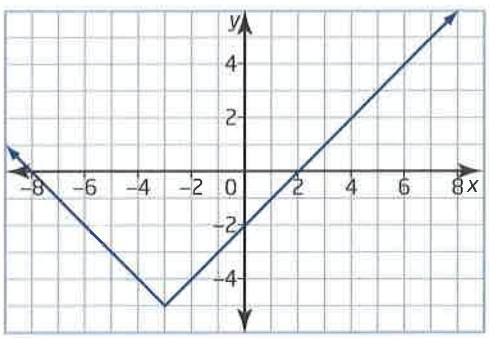
**ii)** range

**iii)** $x$- and $y$-intercepts, if they exist

**a)**



**b)**



**9.** Sketch each function. Then, identify the

**i)** domain

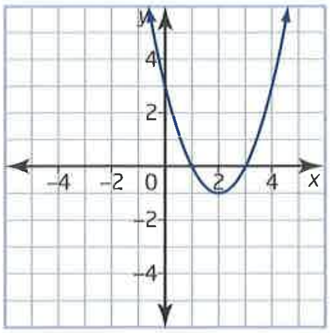
**ii)** range

**iii)** $x$- and $y$-intercepts, if they exist

**a)** $y = x^2 - 9$ **b)** $y = \sqrt{x + 4}$

## Transformations of Functions

**10.** The graph is a transformation of the graph of $y = x^2$. Describe the transformations that were used.



**11.** Graph each function by applying transformations of the base function $y = \sqrt{x}$.

**a)** $y = 3\sqrt{x}$ **b)** $y = \sqrt{2x}$

**c)** $y = -\sqrt{x}$ **d)** $y = \sqrt{x} - 5$

**e)** $y = 2\sqrt{x - 3}$ **f)** $y = -\sqrt{x - 2} + 4$

## Chapter Problem

For millennia, astronomers and other stargazers have been fascinated by the sun, moon, planets, and stars. While it might seem difficult to explore these celestial bodies, advances in science and mathematics allow us to accurately describe how they behave and interact. As you work through this chapter, you will begin to unravel some of the ancient mysteries of the universe and learn some of the ways in which it is possible to describe these fascinating objects.

3.1

# The Nature of Exponential Growth

Game shows can be a lot of fun, especially if you are a winning contestant! Have you ever seen a show in which a contestant is faced with a mathematical question?

Suppose that you are the winner of a game show and that you can choose one of three different prizes, each involving a growing pattern. How can your understanding of patterns and mathematical relationships help you pick the best prize? You will see how the concept of exponential growth can be applied in selecting a game show prize in question 8.

### Tools

- coloured tiles or linking cubes
- grid paper

or

- graphing calculator

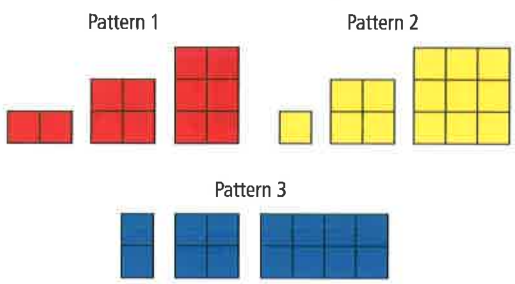
or

- computer with *The Geometer's Sketchpad®*

## Investigate

### How can you discover the nature of exponential growth?

Consider the three growing patterns shown.




**1.** Use words to describe how each pattern is growing.

**2.** Which pattern is growing

**a)** fastest?

**b)** slowest?

Explain your reasoning.

**3. a)** Build or draw the next two terms in each pattern.

**b)** Does this confirm your answers in step 2? Explain.

4. **a)** Copy and complete the table for the pattern 1.

**Pattern 1**

| Term number, $n$ | Number of Squares, $t$ | First Differences | Second Differences |
|---|---|---|---|
| 1 | 2 | 2 | 0 |
| 2 | 4 | 2 | |
| 3 | 6 | | |
| 4 | | | |
| 5 | | | |

**b)** Examine the first differences and the second differences. Describe any patterns you see.

**c)** Is this relationship linear or non-linear? Explain your reasoning.

**d)** Write an equation to relate the total number of squares, $t$, to the term number, $n$.

**e)** Sketch the graph of this relationship.

5. Repeat step 4 for pattern 2 and for pattern 3.

6. **Reflect** Refer to steps 4 and 5.

**a)** Graph the three relationships on the same set of axes.

**b)** Describe how these relationships are alike. How do they differ?

**c)** Which colour of tiles are you likely to run out of first, if you continue to build these patterns? Explain why you think so.

## Example 1

### Model Exponential Growth

A type of bacteria grows so that it triples in number every day. On the day that Roger begins observing the bacteria, a sample has a population of 100.

**a)** Find the population after each of the first 4 days.

**b)** Write an equation to model this growth.

**c)** Graph the relation. Is it a function? Explain why or why not.

**d)** Assuming this trend continues, predict the population after

**i)** 1 week

**ii)** 2 weeks

**e)** Describe the pattern of finite differences for this relationship.

## Solution

**a)** Calculate the population for the first 4 days. Organize the information using a table.

| Day | Population |
|---|---|
| 0 | 100 |
| 1 | $100 \times 3 = 300$ |
| 2 | $300 \times 3 = 900$ |
| 3 | $900 \times 3 = 2700$ |
| 4 | $2700 \times 3 = 8100$ |

The initial population is 100.

The population triples each day.

**b)** To better illustrate the relationship between the day and the total population, express each population calculation in terms of the number of times the initial population is tripled.

| Day | Population |
|---|---|
| 0 | 100 |
| 1 | $100 \times 3^1 = 300$ |
| 2 | $100 \times 3^2 = 900$ |
| 3 | $100 \times 3^3 = 2700$ |
| 4 | $100 \times 3^4 = 8100$ |
| $n$ | $100 \times 3^n$ |

After 1 day, the initial population triples.

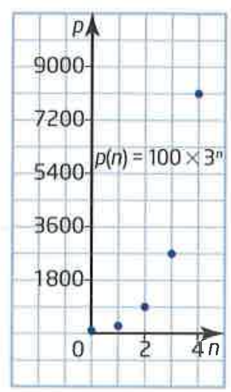
After 2 days, the initial population triples again.

After $n$ days, the initial population has tripled $n$ times.

Therefore, an equation relating the population, $p$, to the number of days, $n$, is $p(n) = 100 \times 3^n$.

**c)** A graph of the relation $p(n) = 100 \times 3^n$ is shown.

This is a function because each element in the domain corresponds to exactly one element in the range.




**d)** Notice that the graph increases steeply as $n$ increases, making it difficult to extrapolate very far. Use the equation $p(n) = 100 \times 3^n$ to find future populations.

**i)** Substitute $n = 7$ into the equation to find the population after 1 week (7 days).

$$\begin{aligned} p(7) &= 100 \times 3^7 \\ &= 100 \times 2187 \\ &= 218\ 700 \end{aligned}$$

After 1 week, the bacteria population will be 218 700.

**ii)** Substitute $n = 14$ into the equation to find the population after 2 weeks.

$$\begin{aligned} p(14) &= 100 \times 3^{14} \\ &= 100 \times 4\ 782\ 969 \\ &= 478\ 296\ 900 \end{aligned}$$

After 2 weeks, the bacteria population will be 478 296 900.

**e)** Add two columns to the table for the first and second differences.

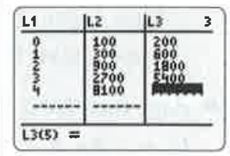
| Day | Population | First Differences | Second Differences |
|---|---|---|---|
| 0 | 100 | | |
| | | 300 – 100 = 200 | |
| 1 | 300 | | 600 – 200 = 400 |
| | | 900 – 300 = 600 | |
| 2 | 900 | | 1800 – 600 = 1200 |
| | | 2700 – 900 = 1800 | |
| 3 | 2700 | | 5400 – 1800 = 3600 |
| | | 8100 – 2700 = 5400 | |
| 4 | 8100 | | |

The first differences are not constant. The ratio of successive first differences is the same, as each value after the first is three times the previous value. Similarly, the ratio of successive second differences is the same.

**Technology Tip**

You can use a graphing calculator to calculate finite differences.

Refer to the Technology Appendix on pages 496 to 516.

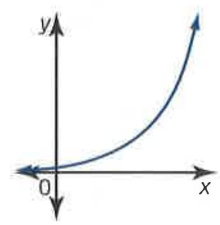



Example 1 illustrates **exponential growth**. This type of growth commonly occurs in science and business.

In the Investigate, you considered exponents with values greater than zero, but you will also see expressions that involve a zero exponent. What is the meaning of a zero exponent? You can explore zero exponents by examining patterns and applying algebraic and graphical reasoning.

**exponential growth**

- pattern of growth in which each term is multiplied by a constant amount (greater than one) to produce the next term
- produces a graph that increases at a constantly increasing rate




- has a repeating pattern of finite differences: the ratio of successive finite differences is constant

## Example 2

### Apply a Zero Exponent in a Model

Use the data in Example 1 to demonstrate the meaning of $3^0$.

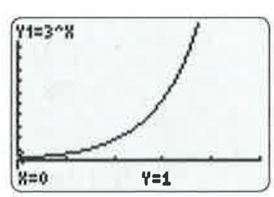
### Solution

Look at the table in Example 1 and work backward.

| Day | Population |
|---|---|
| 4 | $100 \times 3^4 = 8100$ |
| 3 | $100 \times 3^3 = 2700$ |
| 2 | $100 \times 3^2 = 900$ |
| 1 | $100 \times 3^1 = 300$ |
| 0 | $100 \times 3^0 = 100$ |

For this pattern to be extended, $3^0$ must equal 1, because $100 \times 1 = 100$.

The value of $3^0$ can be verified by graphing the function $y = 3^x$ and identifying the $y$-intercept: When $x = 0$, $y = 1$.

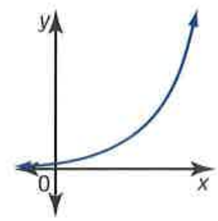



**Technology Tip**

You can use a graphing calculator to view and analyse this function. To find the $y$-intercept:

- Press 2nd [CALC].
- Select **1:value** and enter 0 to identify the $y$-intercept.

## Key Concepts




- Exponential growth functions have these properties:
  - As the independent variable increases by a constant amount, the dependent variable increases by a common factor.
  - The graph increases at an increasing rate.
  - The finite differences exhibit a repeating pattern: the ratio of successive finite differences is constant.
- Any non-zero real number raised to the exponent zero is equal to 1: $b^0 = 1$ for $b \in \mathbb{R}$, $b \neq 0$.

## Communicate Your Understanding

**C1** An insect colony, with an initial population of 50, triples every day.

**a)** Which function models this exponential growth?

**A** $p(n) = 50 \times 2^n$

**B** $p(n) = 150 \times 3n$

**C** $p(n) = 50 \times 3^n$

**b)** For the correct model, explain what each part of the equation means.

**C2** Consider these three functions:

$y = x^2$ $\qquad$ $y = 2x$ $\qquad$ $y = 2^x$

**a)** How do the equations differ? How do the graphs differ?

**b)** Describe the domain and range of each function.

**C3** Consider these three functions:

$y = x^3$ $\qquad$ $y = 3x$ $\qquad$ $y = 3^x$

Which of these functions is linear? Which is exponential? Justify your choices.

**C4** Describe the pattern of finite differences for each type of function. Give an example of each to illustrate your response.

**a)** a linear function

**b)** a quadratic function

**c)** an exponential function

**C5** Does $5^0 - 2^0 = 2^0 - 5^0$? Explain.

## A Practise

*For help with question 1, refer to Example 1.*

**1.** An insect colony, with an initial population of 20, quadruples every day.

**a)** Copy and complete the table.

| Day | Population | First Differences | Second Differences |
|---|---|---|---|
| 0 | 20 | | |
| 1 | 80 | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |

**b)** Is the relationship between the insect population and the number of days exponential? Explain how you can tell.

**c)** Examine the finite differences. Describe how the first differences and second differences are related.

**d)** Will the pattern of first and second differences observed in part c) continue with the third and fourth differences? Write down your conjecture.

**e)** Calculate the third and fourth differences. Was your conjecture in part d) correct? Explain.

*For help with questions 2 to 4, refer to Example 2.*

**2.** What is the value of $10^0$? Use patterns and numerical reasoning to justify your answer.

**3. a)** Rewrite the expression $\frac{a^3}{a^3}$ by expanding both powers.

**b)** Divide out common factors in the numerator and denominator. What is the simplified value of this expression?

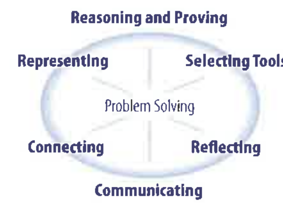
**c)** Write the expression $\frac{a^3}{a^3}$ as a single power by applying the quotient rule.

**d)** Write a statement that explains how the results of parts b) and c) are related.

**4.** Evaluate.

**a)** $6^0$ **b)** $(-3)^0$ **c)** $\left(\frac{3}{5}\right)^0$ **d)** $x^0$

## B Connect and Apply




**5. a)** Use linking cubes, coloured tiles, or tools of your choice to design a growing pattern that can be described by $t(n) = 3^{n-1}$, where $n$ is the term number and $t(n)$ is the number of items in that term.

**b)** Draw diagrams to illustrate the first four terms in your pattern, where $t(1)$ is the first term.

**c)** How many items would you need to build

**i)** the 5th term? **ii)** the 10th term?

**d)** Suppose that you have 500 items in total to use when constructing a model of this pattern. What is the greatest number of terms you can build at the same time?

**e)** Suppose that you have 1000 items in total. What is the greatest number of terms you can build at the same time? How does this answer differ from your answer to part d)? Explain this result.

**6. Use Technology** Use a graphing calculator or graphing software.

**a)** Predict the key features of the graph of the function $y = 0^x$:

**i)** domain

**ii)** range

**iii)** shape

**b)** Graph the function and check your predictions.

**c)** Use a tracing feature to check the value of $y$ when $x = 0$. Does the technology provide the correct answer? Explain.

7. Suppose that there is a rumour going around your school that next year all weekends will be extended to three days. Initially, on day 0, five students know the rumour. Suppose that each person who knows the rumour tells two more students the day after they hear about it. Also assume that no-one hears the rumour more than once.

a) How many people will learn about the rumour

i) on day 1? ii) on day 2?

b) Estimate your school's student population. How long will it take for this rumour to spread throughout the entire school?

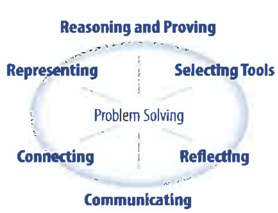
c) Is this an example of exponential growth? Explain your reasoning.

8. Suppose you just won the choice of one of three prizes at a game show:

- Everyday Deal: On day 1, the prize is worth \$1. Then, every day for two weeks, the value of the prize is one more dollar than it was the day before.
- Square Deal: On day 1, the prize is worth $1^2$, or \$1. On day 2, the prize is worth $2^2$, or \$4, and so on, for two weeks.
- Double Deal: On day 1, the prize is worth \$1. On day 2, the prize value doubles to $2 \times \$1$, or \$2. On day 3, the value doubles again to $2 \times \$2$, or \$4, and so on, for two weeks.

Which prize should you take at the end of the two-week period? Why? Use an algebraic method to justify your choice.

9. A bacterial colony has an initial population of 200. The population triples every week.




a) Write an equation to relate population, $p$, to time, $t$, in weeks.

b) Sketch the graph of this relationship for the first month.

c) Determine the approximate population after 10 days. Which tool do you prefer to use for this: the equation or the graph? Explain why.

d) Determine the approximate population after 3 months. Which tool do you prefer to use for this: the equation or the graph? Explain why.

10. Most savings accounts offer compound interest. After each compounding period, interest earned is added to the principal (the initial deposit amount). The amount, $A$, in dollars, in an account earning interest, compounded annually, with a single deposit can be calculated using the formula $A = P(1 + i)^n$, where $P$ is the principal, in dollars; $i$ is the annual interest rate (expressed as a decimal); and $n$ is the number of years for which the principal earns interest. Marvin deposits \$100 into an account that pays interest at 5% per year, compounded annually.

a) Write the annual interest rate as a decimal. Substitute this value into the formula.

b) Copy and complete the table.

| Number of Compounding Periods (years) | Amount (\$) |
|---|---|
| 0 | 100 |
| 1 | 105 |
| 2 | |
| 3 | |
| 4 | |

$A = 100(1 + 0.05)^1$
$= 100(1.05)^1$
$= 105$

c) Calculate the first and second differences.

d) Graph the function.

e) If interest is only paid at the end of each compounding period, do the points between the values in the table have meaning? Explain why or why not.

f) Is this function exponential? Explain.

11. Use the formula for compound interest given in question 10. Sadia deposits a $2000 inheritance into an account that earns 4% per year, compounded annually. Find the amount in the account after each time.

**a)** 3 years **b)** 8 years

12. Use the formula for compound interest given in question 10. Heidi invests $500 in an account that earns 7% per year, compounded annually.

**a)** How long does Heidi need to leave her investment in the account in order to double her money? Explain how you solved this.

**b)** How much longer would it take if the account paid simple interest, at the same rate? Simple interest does not get added to the principal after each compounding period. Use the formula $I = Prt$, where $I$ is the interest, in dollars; $P$ is the principal, in dollars; $r$ is the interest rate (as a decimal); and $t$ is the time, in years.

## C Extend

13. Find an example of exponential growth in a media source, such as a newspaper or the Internet. Identify the source and briefly describe the nature of the relationship.

14. Bacteria A has an initial population of 500 and doubles every day, while bacteria B has an initial population of 50 and triples daily.

**a)** After how long will the population of B overtake the population of A? What will their populations be at this point?

**b)** How much faster would B overtake A if A's doubling period were twice as long?

15. Refer to question 8.

**a)** Suppose the Double Deal is changed so that the prize is worth $0.20 on day 1 instead of $1. On day 2, the value doubles 2 × $0.20, or $0.40. Then, on day 3, the value doubles again to 2 × $0.40, or $0.80, and so on, for two weeks. Which prize should you take at the end of the two-week period? Justify your choice.

**b)** What is the smallest initial value that makes the Double Deal the best prize option? Explain.

16. **Math Contest** The doubling period of a type of yeast cell is 3 days. A jar starts off with one yeast cell. After 27 days, there are 512 cells. It takes 30 days to fill the jar. The number of cells in the jar when it is full is

**A** 512 **B** 4096 **C** 1024 **D** $512(2)^{30}$

17. **Math Contest** How many terms in the sequence $3^1, 3^2, 3^3, \ldots, 3^{75}$ have a units digit of 1 when evaluated?

**A** 0 **B** 25 **C** 19 **D** 18

18. **Math Contest** Given that $\frac{x^2y^6}{z} = 3$ and $\frac{x^2z^5}{y^2} = 27$, one possible value of $x^2y^2z^2$ is

**A** 3 **B** 9 **C** 27 **D** 81

19. **Math Contest** Paul and Brian are playing a game by throwing a die. Paul throws first. If he throws a 1, a 2, a 3 or a 4, then he wins. If Brian throws a 5 or a 6, then he wins. They continue alternating throws until someone wins. What is the probability that Brian will win?

20. **Math Contest** A unit fraction is a fraction of the form $\frac{1}{n}$, where $n \in \mathbb{N}$. Find all ways to express 1 as the sum of three unit fractions. An example is $\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$.

# Use Technology

## Use the Lists and Trace Features on a TI-Nspire™ CAS Graphing Calculator

**Tools**

- TI-Nspire™ CAS graphing calculator

### A: Calculate First and Second Differences for the Population Data in Example 1 on pages 151 to 153

**1.** Open a new document. Open a page using the **Lists & Spreadsheet** application.

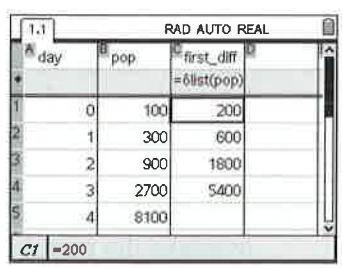
**2.** Set up the table.

- At the top of column A, type the title *day*. Press enter.
- At the top of column B, type the title *pop*. Press enter.
- Enter the data for the day, starting in cell A1.
- Enter the data for the population, starting in cell B1.

**3.** Calculate the first differences.

- At the top of column C, type the title *first_diff*.
- In the formula cell below the title, type =.
- Press and select tab 2.
- Cursor down to **List** and press enter.
- Scroll down to **Operations**, and press enter.
- Scroll down to **Difference List**, and press enter.
- Type *pop* between the brackets, and press enter.

  The first differences will be displayed.



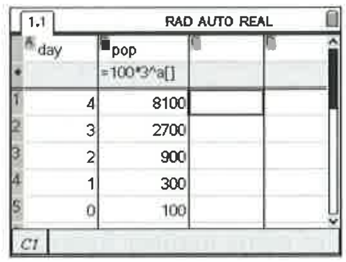

- In a similar manner, you can calculate the second differences in column D.

## B: Calculate the Population Data in Example 2 on page 153

1. Open a new document. Open a page using the **Lists & Spreadsheet** application.

2. Set up the table.

   - At the top of column A, type the title *day*.
   - At the top of column B, type the title *pop*.
   - Enter the data for the day, starting in cell A1.
   - In the formula cell for column B, type = 100 × 3 ^ a, and press (enter).
   - The population data will be displayed in column B.




## C: Trace the Population Function in Example 2

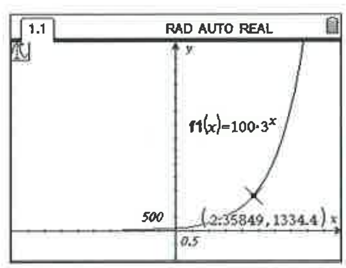
1. Open a new document. Open a page using the **Graphs & Geometry** application.

2. Graph the function.

   - For function *f1*, type 100 × 3 ^ *x* and press (enter).
   - Press (menu). Select **4:Window** and then **1:Window Settings**.
   - Set the *x*-range from −5 to 5 and the *y*-range from −100 to 8100.
   - Press (menu). Select **5:Trace** and then **1:Graph Trace**.

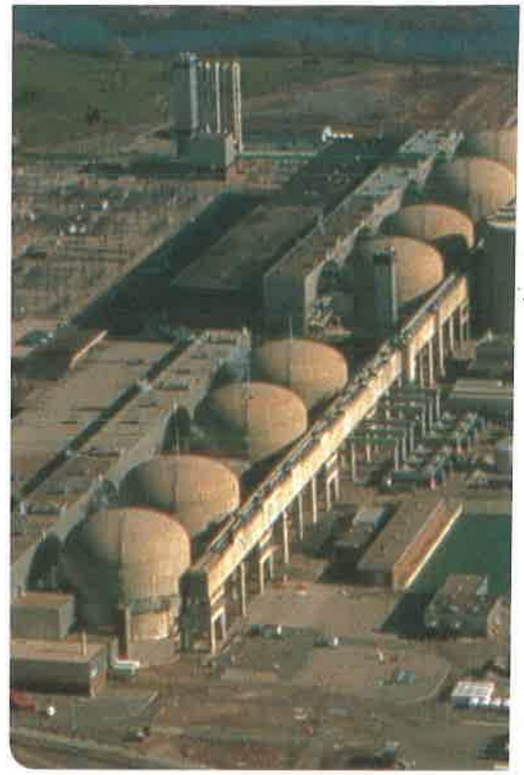
     Note that a trace point appears and the coordinates of the point are displayed.
   - Use the cursor to move the trace point along the graph.

     Note that the coordinates of the trace point change as the point is moved.



3.2

# Exponential Decay: Connecting to Negative Exponents



Nuclear energy provides more than 15% of Canada's electrical power. When atoms of certain elements are split, a tremendous amount of energy is released, which is converted into electricity that we can use. This process is called nuclear fission, and it is one of the key processes involved in nuclear power generation.

Over 50% of Ontario's electric power comes from fission reactors. One of the disadvantages of using nuclear fission to produce energy is the dangerous waste material produced. Scientists and engineers must ensure that such materials are contained and disposed of safely.

**Tools**

- graphing calculator

**half-life**

- length of time for an unstable element to spontaneously decay to one half its original amount

**Connections**

You will learn more about nuclear fission if you study physics in high school or university.
Visit the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 3 to find out more about CANDU reactors.

## Investigate

### How can you explore exponential decay?

Uranium is commonly used as fuel for nuclear reactors. Uranium has several different forms, or isotopes, some of which occur naturally and some of which are produced through nuclear fission. One isotope, uranium-239 (U-239), has a **half-life** of about 2 years. This means that after 2 years, half the U-239 has "decayed," or changed, into a new substance.

Suppose you have a 1000-mg sample of U-239. How can you model the amount remaining over time?

**1.** Copy and complete the table.

| Time (years) | Number of Half-Life Periods (2 years) | Amount of U-239 Remaining (mg) |
|---|---|---|
| 0 | 0 | 1000 |
| 2 | 1 | 500 |
| 4 | 2 | |
| 6 | 3 | |
| 8 | 4 | |

**2. a)** Describe the trend in the table in step 1.

**b)** Do you think this relationship between the number of half-life periods and the amount of U-239 remaining is

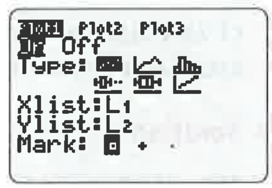
- linear?
- quadratic?
- exponential?

Justify your choice.

3. a) Determine the approximate time it will take for the sample to decay to
   i) 10 mg
   ii) 0.1% of its original amount

   b) Will the mass of U-239 in the sample ever reach zero? Explain your reasoning.

4. a) Use a graphing calculator to make a scatter plot of the amount of U-239 remaining versus the number of half-life periods:
   - Press (STAT) and select **EDIT**.
   - Enter the data in lists **L1** and **L2**.
   - Press (2nd) [STAT PLOT] and select **Plot1**.
   - Select the settings shown.
   - Press (ZOOM) and select **9:ZoomStat**.




   b) **Reflect** Compare the shape of the scatter plot to the exponential curves you explored in Section 3.1. Describe how they are alike. How do they differ?

5. a) Calculate the first and second differences for this function.

   b) Is this relationship exponential? Explain.

6. a) Create an algebraic model for this function using the graphing calculator:
   - From the home screen, press (STAT) and then cursor over to the **CALC** menu.
   - Choose **0:ExpReg**.
   - Press (2nd) [L1] ( , ) (2nd) [L2] ( , ). Then press (VARS), select **Y-VARS**, and choose **1:Function**.
   - Choose **1:Y1** and press (ENTER).

   An exponential equation will appear.

   b) Substitute the values given for $a$ and $b$ to write the equation for this relationship in the form $y = ab^x$.

   c) **Reflect** Explain the roles of $a$ and $b$ in the equation you wrote in part b).

7. a) Press (GRAPH) to see the curve of best fit.

   b) Does the graph accurately model the relationship? Explain.

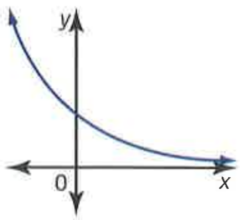
8. **Reflect** Use the **Zoom** and **Trace** operations to check your answers in step 3. Comment on what you notice.

**Technology Tip**

Press (2nd) (MODE) for [QUIT] to return to the home screen from other screens.

**exponential decay**

- a pattern of decay in which each term is multiplied by a constant fraction between zero and one to produce the next term
- produces a graph that decreases at a constantly decreasing rate




- has a repeating exponential pattern of finite differences: the ratio of successive finite differences is constant

The relationship in the Investigate is an example of **exponential decay**. This type of relationship commonly occurs in such areas of study as science and business.

## Example 1

### Model Exponential Decay

When U-239 decays, it forms a different material called plutonium-239 (Pu-239). This highly toxic waste material has a much longer half-life of 24 years. Determine approximately how long it will take for a 50-mg sample of Pu-239 to decay to 10% of its original amount.

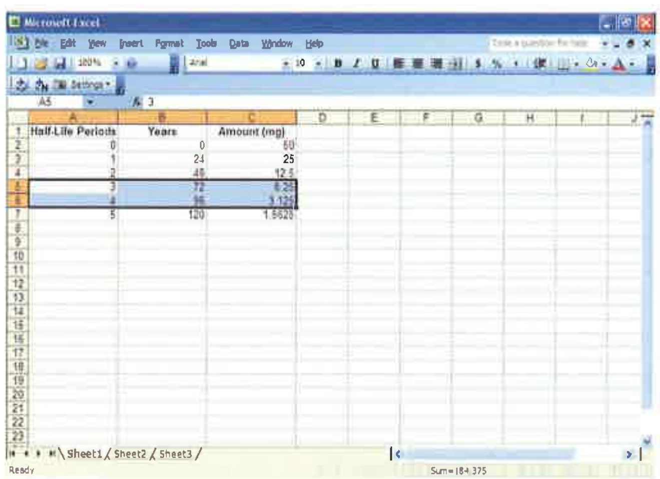
### Solution

10% of 50 mg is 5 mg. You need to find the number of 24-year periods after which 5 mg will remain.

**Method 1: Use a Table or a Spreadsheet**

Enter the number of half-life periods, years, and amount remaining using three columns.

- Enter 0 in cell A2 and 1 in cell A3. Highlight cells A2 and A3. With the mouse over the bottom right corner of cell A3, click and drag straight down to continue the pattern.




- Enter 0 in cell B2. In cell B3, type = and then click on cell B2, type +24 and press Enter. Then, highlight B3 and, with the mouse over the bottom left corner, click and drag down.
- Use the same technique for the last column, but enter 50 in cell C2 and use the equation C3 = C2/2.

Read down the last column. The sample will reach a level of 5 mg somewhere between 72 and 96 years from now.

### Method 2: Use Systematic Trial

Write an equation to relate the amount of Pu-239 remaining to the number of half-life periods. This gives the equation $A(n) = 50\left(\frac{1}{2}\right)^n$, where $n$ is the number of 24-year half-life periods and $A$ is the amount of Pu-239 remaining, in milligrams.

Substitute $A = 5$ and solve for $n$.

$$A(n) = 50\left(\frac{1}{2}\right)^n$$

$$5 = 50\left(\frac{1}{2}\right)^n$$

$$0.1 = \left(\frac{1}{2}\right)^n \qquad \text{Divide both sides by 50.}$$

Use systematic trial with a calculator to find the value of $n$ that satisfies this equation. For ease of calculation, use 0.5 in place of $\frac{1}{2}$.

| $n$ | $0.5^n$ | Mathematical Reasoning |
|---|---|---|
| 5 | 0.03125 | Try 5 half-lives. This gives a value much lower than 0.1. Try a shorter period of time. |
| 2 | 0.25 | Too high. The correct value is between 2 and 5. Try 3 next. |
| 3 | 0.125 | Close, but high. |
| 3.3 | 0.1015... | This is very close to 0.1. This will give a reasonable approximation. |

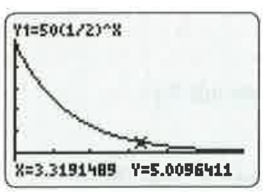
The number of half-life periods is approximately 3.3. Multiply this by 24 to find the equivalent number of years.

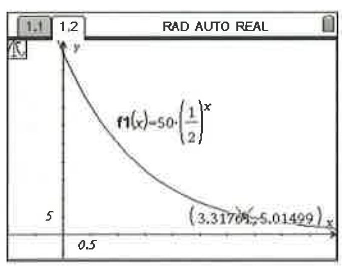
$3.3 \times 24 = 79.2$

It will take approximately 79 years for the sample of Pu-239 to decay to 10% of its initial mass.

### Method 3: Use a Graphical Model

Write an equation to model the relationship, graph it, and read the information from the graph. The graph of $A(n) = 50\left(\frac{1}{2}\right)^n$ is shown.




Using the **Trace** operation, you can read that the amount has decayed to about 5 mg after approximately 3.3 half-lives. Convert this to years by multiplying by 24.

$3.3 \times 24 = 79.2$

It will take approximately 79 years for the sample of Pu-239 to decay to 10% of its initial amount.

**Connections**

You will learn how to find exact solutions to equations such as $5 = 50\left(\frac{1}{2}\right)^n$ when you study logarithms in grade 12.

In Methods 2 and 3 of Example 1, you considered the function $A(n) = 50\left(\frac{1}{2}\right)^n$, where $n$ is greater than or equal to zero. However, you will sometimes encounter exponential expressions with negative exponents. You can use the following relationship to evaluate expressions involving negative exponents:

$b^{-n} = \frac{1}{b^n}$ for any $b \in \mathbb{R}$, $b \neq 0$, and $n \in \mathbb{N}$.

## Example 2

### Evaluate Expressions Involving Negative Exponents

Evaluate.

**a)** $3^{-2}$ **b)** $6^{-2} \times 6^3$ **c)** $(-2)^{-4} + 4^{-2}$ **d)** $(4^{-2})^{-3} \div 4^8$

### Solution

**a)**
$$\begin{aligned} 3^{-2} &= \frac{1}{3^2} \\ &= \frac{1}{9} \end{aligned}$$

**b)**
$$\begin{aligned} 6^{-2} \times 6^3 &= 6^{-2+3} \\ &= 6^1 \\ &= 6 \end{aligned}$$

Apply the product rule. Add the exponents.

**c)**
$$\begin{aligned} (-2)^{-4} + 4^{-2} &= \frac{1}{(-2)^4} + \frac{1}{4^2} \\ &= \frac{1}{16} + \frac{1}{16} \\ &= \frac{2}{16} \\ &= \frac{1}{8} \end{aligned}$$

**d)**
$$\begin{aligned} (4^{-2})^{-3} \div 4^8 &= 4^{(-2)(-3)} \div 4^8 \\ &= 4^6 \div 4^8 \\ &= 4^{6-8} \\ &= 4^{-2} \\ &= \frac{1}{4^2} \\ &= \frac{1}{16} \end{aligned}$$

Apply the power of a power rule first. Multiply exponents.

Apply the quotient rule. Subtract exponents.

## Example 3

### Simplify Expressions Involving Negative Exponents

Simplify. Express answers using only positive exponents.

**a)** $(x^{-2})(x^{-3})(x^4)$ **b)** $\dfrac{a^2b^{-3}}{a^{-1}b^2}$ **c)** $(2u^3v^{-2})^{-3}$

### Solution

**a)** $$\begin{aligned}(x^{-2})(x^{-3})(x^4) &= x^{-2-3+4}\\ &= x^{-1}\\ &= \frac{1}{x}\end{aligned}$$ Apply the product rule.

**b)** $$\begin{aligned}\frac{a^2b^{-3}}{a^{-1}b^2} &= a^{2-(-1)}b^{-3-2}\\ &= a^3b^{-5}\\ &= \frac{a^3}{b^5}\end{aligned}$$ Apply the quotient rule.

**c)** $$\begin{aligned}(2u^3v^{-2})^{-3} &= 2^{-3}u^{3(-3)}v^{(-2)(-3)}\\ &= \frac{1}{2^3}u^{-9}v^6\\ &= \frac{v^6}{8u^9}\end{aligned}$$ Apply the power of a power rule.

## Example 4

### Evaluate Powers Involving Fractional Bases

Simplify.

**a)** $\left(\dfrac{1}{3}\right)^{-1}$ **b)** $\left(-\dfrac{27}{8}\right)^{-2}$

### Solution

**a)** $$\left(\frac{1}{3}\right)^{-1} = \frac{1}{\left(\frac{1}{3}\right)^1}$$ Write with a positive exponent.

$$\begin{aligned}&= 1 \div \frac{1}{3}\\ &= 1 \times \frac{3}{1}\\ &= 3\end{aligned}$$ Multiply by the reciprocal.

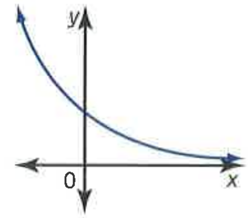
**b)** $$\begin{aligned}\left(-\frac{27}{8}\right)^{-2} &= \left(-\frac{8}{27}\right)^2\\ &= \frac{64}{729}\end{aligned}$$ Apply the property of negative exponents.

**Connections**

The result in Example 4 part a) can be generalized as follows. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$ for $a, b \in \mathbb{R}$, $a, b \neq 0$, and $n \in \mathbb{N}$.

## Key Concepts

- Exponential decay functions have the following properties:
  - As the independent variable increases by a constant amount, the dependent variable decreases by a common factor.
  - The graph decreases at a decreasing rate.
  - They have a repeating exponential pattern of finite differences: the ratio of successive finite differences is constant.




- A power involving a negative exponent can be expressed using a positive exponent:
  $b^{-n} = \frac{1}{b^n}$ for $b \in \mathbb{R}$, $b \neq 0$, and $n \in \mathbb{N}$.
- The exponent rules hold for powers involving negative exponents.
- Rational expressions raised to a negative exponent can be simplified:
  $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$ for $a, b \in \mathbb{R}$, $a, b \neq 0$, and $n \in \mathbb{N}$.

## Communicate Your Understanding

**C1** Compare exponential growth and exponential decay. Describe how these types of relationships are alike. How are they different?

**C2** The equation $A(n) = 50\left(\frac{1}{2}\right)^n$ from Example 1 gives the amount, $A$, in milligrams, of Pu-239 remaining after $n$ 24-year half-life periods.
Claudia says, "I can write a simpler form of this equation: $A(n) = 50(2)^{-n}$."

**a)** Do you think Claudia's model is valid? Explain your thinking.

**b)** Suggest at least two ways that you could verify the equivalence of these two models.

**C3** Explain how you can use a positive exponent to rewrite a power involving a negative exponent. Create an example to illustrate your answer.

**C4** Explain how you can use a negative exponent to rewrite a power involving a positive exponent. Create an example to illustrate your answer.

## A Practise

*For help with questions 1 to 4, refer to Example 2.*

**1.** Write each as a power with a positive exponent.

**a)** $3^{-1}$ **b)** $x^{-1}$ **c)** $y^{-2}$

**d)** $(ab)^{-1}$ **e)** $-x^{-2}$ **f)** $(-x)^{-2}$

**2.** Write each as a power with a negative exponent.

**a)** $\frac{1}{5^2}$ **b)** $\frac{1}{k^3}$

**3.** Evaluate. Express as a fraction in lowest terms.

**a)** $6^{-2}$ **b)** $2^{-5}$ **c)** $10^{-4}$

**d)** $9^{-3}$ **e)** $2^{-2} + 4^{-1}$ **f)** $6^{-1} + 3^{-1} + 2^{-3}$

4. Apply the exponent rules to evaluate.

a) $(4^{-2})^{-1}$
b) $(2^{-4})^2$
c) $\dfrac{10^4}{10^{-1}}$
d) $5^{-3} \div 5^{-2}$
e) $(7^{-2})(7^4)$
f) $4^{-3} \times 4^{-5} \times 4^6 \times 4^0$
g) $\dfrac{3(3^4)}{3^3}$
h) $\dfrac{(4^5)(4^{-2})}{4^2}$

*For help with question 5, refer to Example 3.*

5. Simplify. Express your answers using only positive exponents.

a) $m^{-2} \times m^3$
b) $(3v^{-3})(-2v^{-6})$
c) $p^4 \div p^{-3}$
d) $\dfrac{6w^{-4}}{2w^{-2}}$
e) $(k^{-3})^{-4}$
f) $(2ab^{-3})^{-2}$

*For help with questions 6 and 7, refer to Example 4.*

6. Evaluate.

a) $\left(\dfrac{1}{8}\right)^{-2}$
b) $\left(\dfrac{3}{10}\right)^{-6}$
c) $\left(\dfrac{9}{4}\right)^{-2}$
d) $\left(-\dfrac{5}{2}\right)^{-3}$
e) $\left(\dfrac{1}{2}\right)\left(-\dfrac{1}{4}\right)^{-2}$
f) $\left[\left(\dfrac{1}{4}\right)^2\left(\dfrac{2}{5}\right)\right]^{-2}$

7. Simplify. Express your answers using only positive exponents.

a) $\left(\dfrac{1}{ab}\right)^{-2}$
b) $\left(\dfrac{1}{8u}\right)^{-3}$
c) $\left(\dfrac{g^4}{w^2}\right)^{-2}$
d) $\left(\dfrac{4a^3}{3b^2}\right)^{-3}$
e) $\left(\dfrac{3a^2}{b^3}\right)^{-3}$
f) $\left(\dfrac{x^{-2}}{y^{-1}}\right)^{-2}$

## B Connect and Apply

8. Tungsten-187 (W-187) is a radioactive isotope that has a half-life of 1 day. Suppose you start with a 100-mg sample.

a) Make a table of values that gives the amount of W-187 remaining at the end of each day for the next 4 days.


b) Write an equation in the form $f(x) = ab^x$ to relate the amount of W-187 remaining and time. Identify what each variable in the equation represents and give the appropriate unit for each variable.

c) Sketch the graph of the relation. Describe the shape of the curve.

d) How much W-187 will remain after 1 week?

e) How long will it take for the amount of W-187 to decay to 5% of its initial amount? Describe the tools and strategies you used to solve this.

f) Write a different function to model the same situation. Explain why the two functions are equivalent.

9. a) Evaluate each expression. Express your answers as fractions.

i) $3^{-2}$
ii) $3^{-3}$

b) Multiply your answers from part a) together.

c) Apply the product rule of exponents to the expression $3^{-2} \times 3^{-3}$. Write the result with a positive exponent and evaluate.

d) Compare your answers to parts b) and c). What does this illustrate about powers with negative exponents?

e) Create another example to illustrate this property.

10. Use two numerical examples to verify that the quotient rule holds for expressions involving negative exponents.

11. Use numerical or algebraic reasoning to verify that the power of a power rule holds for expressions involving negative exponents.

12. Shylo is very excited about her brand new car!


Although she paid \$20 000 for the car, its resale value will depreciate (decrease) by 30% of its current value every year. The equation relating the car's depreciated value, $v$, in dollars, to the time, $t$, in years, since her purchase is $v(t) = 20\,000(0.7)^t$.

**a)** Explain the significance of each part of this equation.

**b)** How much will Shylo's car be worth in

**i)** 1 year? **ii)** 2 years?

**c)** Graph the depreciation function. Is it an example of exponential decay? Explain how you can tell.

**d)** How long will it take for Shylo's car to depreciate to 10% of its original price?

13. Consider Example 1. Suppose that the original sample of Pu-239 was larger than the 50 mg present at the beginning of the study.


**a)** Explain how you can use the exponential model in Example 1 to describe decay that happened before the beginning of the study.

**b)** Find the mass of this sample

**i)** 1 half-life before the beginning of the study

**ii)** 3 half-lives before the beginning of the study

14. Use algebraic reasoning to prove the general result $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$ for $a, b \in \mathbb{R}$, $a, b \neq 0$, and $n \in \mathbb{N}$.

15. Use algebraic reasoning to show how the formula $A = P(1 + i)^n$ can be expressed in the form $P = A(1 + i)^{-n}$.

## Achievement Check

16. The formula $P(n) = A(1 + i)^{-n}$ is used to calculate the principal, $P(n)$, invested in an account that has been accumulating interest, compounded annually. $A$ is the current amount in the account, in dollars; $i$ is the annual interest rate (as a decimal); and $n$ is the number of years the principal has been earning interest.

**a)** Oscar deposited some money into an account that pays 3% per year, compounded annually. Today the account balance is \$660. How much was in the account

**i)** 1 year ago?

**ii)** 5 years ago?

**b)** Does this formula represent exponential growth or decay? Explain.

**c)** Rewrite the formula to isolate $A$. Does this form of the formula represent exponential growth or decay? Explain.

17. Refer to question 16. Lydia wants to invest some money that will grow to \$1000 in 6 years. If her account pays 4.5% interest, compounded annually, how much should Lydia invest today?

18. **Chapter Problem** There is a gravitational force of attraction between every pair of objects in the universe. The strength of this force depends on the mass of each object and the distance that separates the objects.

The formula $F = GMmr^{-2}$ relates this force to the mass of each of the objects and the distance between them, where $G$ is a constant equal to $6.7 \times 10^{-11}$; $M$ and $m$ are the masses of the two objects, in kilograms; and $r$ is the distance separating them, in metres.

The mass of Earth, $M$, is $6.0 \times 10^{24}$ kg; the mass of the moon, $m$, is $7.4 \times 10^{22}$ kg; and Earth and the moon are 380 000 km apart. Use this information to determine the force, $F$, in newtons (N), of attraction between the moon and Earth.

## C Extend

**19.** Refer to question 18.

**a)** Rewrite the formula $F = GMmr^{-2}$ in its more usual form, with a positive exponent.

**b)** This relationship between force and distance is an example of an inverse square relationship. Explain what this means.

**c)** Research to find another example of an inverse square relationship. Describe the variables that are related, and provide a sample calculation.

**20.** Refer to question 18. Compare the strength of this force to the force of attraction between

**a)** the moon and your body

**b)** Earth and your body, using $r = 6400$ km (the radius of Earth)

**21.** **Math Contest** If $f(x) = 3^{2-3x}$, then the value of $[f(1 + x)][f(1 - x)]$ is

**A** $\frac{1}{9}$ **B** 9 **C** 27 **D** 1

**22.** **Math Contest** The $y$-intercept of the function $y = -2\left(\frac{1}{3}\right)^{2x-1} + 2$ is

**A** 2 **B** 0 **C** 8 **D** −4

**23.** **Math Contest** Consider the following system of equations.

$x_1 + x_2 = 120$
$x_2 + x_3 = 160$
$x_3 + x_4 = 140$
$x_4 + x_5 = 125$
$x_1 + x_3 + x_5 = 215$

What is the value of $x_1 + x_5$?

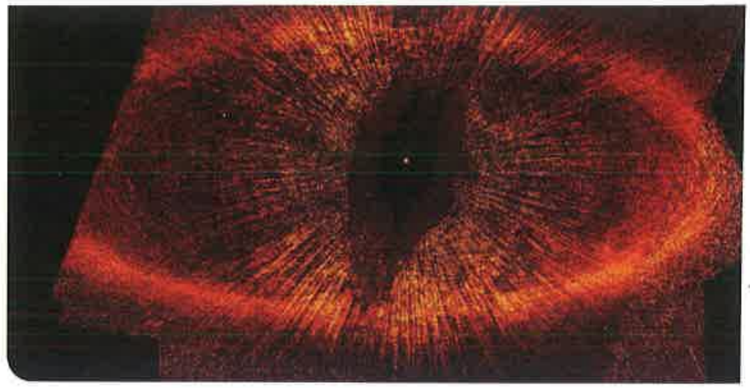
**A** 160 **B** 125 **C** 120 **D** 140

### Career Connection

Sinthujan studied medical radiation technology at a Canadian college for $2\frac{1}{2}$ years. Radiation therapy is one of the most common ways to treat cancer. In his job, under the direction of an oncologist, Sinthujan uses high-energy external beam radiation to stop cancers from multiplying. He exposes the patient to radiation from a machine a number of times over a period of weeks. To reach the right spot in the body, Sinthujan completes calculations to optimize the beam angle, beam size, and distance from the cancer. The goal is not only to successfully radiate the entire cancerous growth but also to minimize the side effects for the patient.

# 3.3

# Rational Exponents



Is there life on other planets? Are planets like Fomalhaut b, shown here, similar to Earth? Will we ever be able to travel through space to find out? These questions have been asked for generations and have formed the basis for countless science fiction books and movies. How are such stories related to the mathematics of exponential functions?

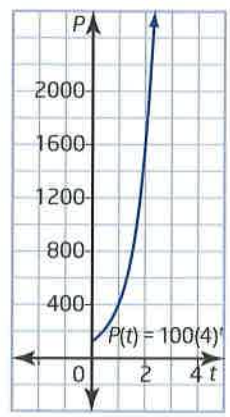
## Investigate

### How can you find the meaning of a rational exponent?

Somewhere far, far away...

The corrupt Galactic Empire is taking over the universe at an exponential rate! How long will it be before planet Earth falls under the rule of the Empire? Meanwhile, rumours of a Rebel Alliance are beginning to spread. Will the Rebel Alliance be able to break the Empire's dominion before the fall of Earth?

Currently, the Galactic Empire controls 100 planets. However, the number of captured planets is quadrupling every decade. This growth is modelled by the graph and equation shown.




1. Look at the equation. Explain what each variable represents and state its units of measure.

**Technology Tip**

If you use a graphing calculator, you can use the **Zoom** and **Trace** operations to find the coordinates of points with greater accuracy.

2. Look at the graph.
   **a)** Identify the coordinates at the $P$-intercept. Explain their significance.
   **b)** How many planets will fall under the control of the Empire after
      **i)** 1 decade?
      **ii)** 2 decades?
   Explain how you can use the graph to find this information.
   **c)** Use the equation to check these answers.

3. **a)** Explain how you can use the graph to estimate the number of planets controlled by the Empire after 5 years. How many planets will it control after 5 years?
   **b)** Enter $t = \frac{1}{2}$ into the equation and use a calculator to evaluate $P(t)$. Does this answer agree with your answer in part a)? Explain.

4. a) Evaluate the expression $100 \times \sqrt{4}$.
   b) How does this compare to $100 \times 4^{\frac{1}{2}}$?
   c) **Reflect** What does this suggest about the expressions $\sqrt{4}$ and $4^{\frac{1}{2}}$?

5. a) Begin with the equation $4 = 2^2$. Raise each side to the exponent $\frac{1}{2}$ and use the power of a power rule to simplify the expression on the right side.
   b) **Reflect** Since $\sqrt{4} = 2$, does the result in part a) confirm that $4^{\frac{1}{2}} = \sqrt{4}$? Explain.

6. a) **Reflect** Does the result of step 5b) hold for powers involving bases other than 4? Write down your prediction.
   b) Choose several different bases and use a calculator to evaluate each base raised to the exponent $\frac{1}{2}$.
   c) Compare these to the square root of each base. Comment on what you notice.

In the Investigate, you discovered a method that can be used to show that $b^{\frac{1}{2}} = \sqrt{b}$ for $b \in \mathbb{R}$, $b \geq 0$. Similar reasoning can be applied to evaluate powers involving other rational exponents.

## Example 1

### Powers With Rational Exponents of the Form $\frac{1}{n}$

Evaluate.

a) $8^{\frac{1}{3}}$ b) $(-32)^{\frac{1}{5}}$ c) $-16^{\frac{1}{4}}$

d) $\left(\frac{16}{81}\right)^{\frac{1}{4}}$ e) $(-27)^{-\frac{1}{3}}$

### Solution

a) $8^{\frac{1}{3}}$

Apply the product rule of exponents to find the value of $8^{\frac{1}{3}}$:

$$8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$$
$$= 8^1$$
$$= 8$$

Taking cube roots of both sides, it follows that

$$8^{\frac{1}{3}} = \sqrt[3]{8}$$
$$= 2$$

The cube of $8^{\frac{1}{3}}$ is equal to 8, so $8^{\frac{1}{3}}$ is equal to the cube root of 8.

**Connections**

$\sqrt[3]{8}$ is read as "the cube root of eight."

$\sqrt[4]{16}$ is read as "the fourth root of sixteen."

The terminology used for this type of expression is as follows:

- $\sqrt[4]{16}$ is called a radical.
- 16 is the radicand.
- 4 is the index.

**Connections**

The result in Example 1a) can be generalized as follows.
$b^{\frac{1}{n}} = \sqrt[n]{b}$ for any $n \in \mathbb{N}$.
If $n$ is even, $b$ must be greater than or equal to 0.

**b)** $(-32)^{\frac{1}{5}} = \sqrt[5]{-32}$
$= -2$

Think: What number raised to the exponent 5 produces −32?
$\sqrt[5]{-32} = -2$, because $(-2)(-2)(-2)(-2)(-2) = -32$.

**c)** $-16^{\frac{1}{4}} = -\sqrt[4]{16}$
$= -2$

Note that, unlike in part b), the negative sign is not part of the base of the power.

**d)** $\left(\frac{16}{81}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{16}{81}}$
$= \frac{2}{3}$

$\sqrt[4]{\frac{16}{81}} = \frac{2}{3}$, because $\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$
$= \frac{2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3}$
$= \frac{16}{81}$

**e)** $(-27)^{-\frac{1}{3}} = \frac{1}{(-27)^{\frac{1}{3}}}$ Rewrite the power with a positive exponent.
$= \frac{1}{\sqrt[3]{-27}}$
$= \frac{1}{-3}$ Evaluate the cube root in the denominator.
$= -\frac{1}{3}$

You can apply the exponent rules to interpret rational exponents in which the numerator is not one.

## Example 2

### Powers With Rational Exponents of the Form $\frac{m}{n}$

Evaluate each expression.

**a)** $8^{\frac{2}{3}}$ **b)** $81^{\frac{5}{4}}$ **c)** $\left(\frac{49}{81}\right)^{-\frac{3}{2}}$

### Solution

**a)** $8^{\frac{2}{3}} = 8^{\frac{1}{3} \times 2}$ Express the exponent as a product: $\frac{2}{3} = \frac{1}{3} \times 2$
$= \left(8^{\frac{1}{3}}\right)^2$ Use the power of a power rule.
$= \left(\sqrt[3]{8}\right)^2$ Write $8^{\frac{1}{3}}$ as a cube root.
$= 2^2$
$= 4$

**b)** $81^{\frac{5}{4}} = \left(81^{\frac{1}{4}}\right)^5$
$= \left(\sqrt[4]{81}\right)^5$
$= 3^5$ Evaluate the fourth root of 81.
$= 243$

c) $\left(\frac{49}{81}\right)^{-\frac{3}{2}} = \left(\frac{81}{49}\right)^{\frac{3}{2}}$ Rewrite the power with a positive exponent.

$= \left(\sqrt{\frac{81}{49}}\right)^3$

$= \left(\frac{\sqrt{81}}{\sqrt{49}}\right)^3$

$= \left(\frac{9}{7}\right)^3$

$= \frac{729}{343}$

The reasoning used in part a) of Example 2 can be applied to obtain the general result $b^{\frac{m}{n}} = (\sqrt[n]{b})^m$ for any non-zero real number $b$, where $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Notice that $b$ must be greater than or equal to 0 if $n$ is even.

## Example 3

### Apply Exponent Rules

Simplify. Express your answers using only positive exponents.

**a)** $\frac{(x^{\frac{2}{3}})(x^{\frac{2}{3}})}{x^{\frac{1}{3}}}$ **b)** $(y^{\frac{1}{4}})^2 \times (y^{-\frac{1}{3}})^2$ **c)** $(5x^{\frac{1}{2}})^2 \times 4x^{-\frac{1}{2}}$

### Solution

**a)** $\frac{(x^{\frac{2}{3}})(x^{\frac{2}{3}})}{x^{\frac{1}{3}}} = \frac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}}$ Apply the product rule.

$= x^{\frac{4}{3} - \frac{1}{3}}$ Apply the quotient rule.

$= x^{\frac{3}{3}}$

$= x$

**b)** $(y^{\frac{1}{4}})^2 \times (y^{-\frac{1}{3}})^2 = y^{\frac{1}{2}} \times y^{-\frac{2}{3}}$ Apply the power of a power rule.

$= y^{\frac{3}{6}} \times y^{-\frac{4}{6}}$ Write the exponents with common denominators.

$= y^{\frac{3}{6} - \frac{4}{6}}$ Apply the product rule.

$= y^{-\frac{1}{6}}$

$= \frac{1}{y^{\frac{1}{6}}}$ Express with a positive exponent.

**c)** $(5x^{\frac{1}{2}})^2 \times 4x^{-\frac{1}{2}} = 25x \times 4x^{-\frac{1}{2}}$ Apply the power of a power rule.

$= 100x^{\frac{1}{2}}$ Apply the product rule.

**Connections**

Note that $\frac{1}{y^{\frac{1}{6}}}$ can also be expressed as $\frac{1}{\sqrt[6]{y}}$.

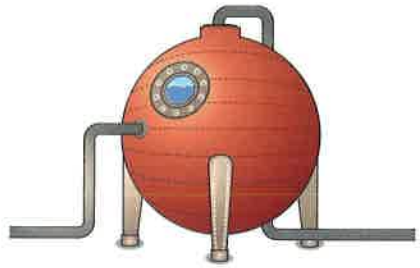
## Example 4

### Solve a Problem Involving a Rational Exponent

The following formula relates the volume, $V$, and surface area, $S$, of a sphere.

$$V(S) = \frac{(4\pi)^{-\frac{1}{2}}}{3} \times S^{\frac{3}{2}}$$

Find the volume, to the nearest cubic metre, of a spherical holding tank with surface area 100 m².



### Solution

Substitute $S = 100$ into the formula and evaluate for $V$:

$$V(100) = \frac{(4\pi)^{-\frac{1}{2}}}{3} \times 100^{\frac{3}{2}}$$

Use a calculator to evaluate $(4\pi)^{-\frac{1}{2}}$.

```
(4*π)^(-1/2)
         .2820947918
```

Use algebraic reasoning to evaluate $100^{\frac{3}{2}}$:

$$\begin{aligned} 100^{\frac{3}{2}} &= \left(\sqrt{100}\right)^3 \\ &= 10^3 \\ &= 1000 \end{aligned}$$

$$\begin{aligned} V(100) &\doteq \frac{0.282}{3} \times 1000 \\ &= 94 \end{aligned}$$

The volume of the holding tank is approximately 94 m³.

## Key Concepts

- A power involving a rational exponent with numerator 1 and denominator $n$ can be interpreted as the $n$th root of the base:

  For $b \in \mathbb{R}$ and $n \in \mathbb{N}$, $b^{\frac{1}{n}} = \sqrt[n]{b}$. If $n$ is even, $b$ must be greater than or equal to 0.

- You can evaluate a power involving a rational exponent with numerator $m$ and denominator $n$ by taking the $n$th root of the base raised to the exponent $m$:

  For $b \in \mathbb{R}$, $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$ for $m \in \mathbb{Z}$, $n \in \mathbb{N}$. If $n$ is even, $b$ must be greater than or equal to 0.

- The exponent rules hold for powers involving rational exponents.

## Communicate Your Understanding

**C1** **a)** What is a cube root? Provide an example to illustrate your explanation.

**b)** Repeat part a) for the fourth root of a number.

**C2** **a)** Explain how you can write the fifth root of a number

**i)** as a radical

**ii)** as a power

**b)** Create an example to illustrate your answers.

**C3** Consider the following simplification. For each step, explain the rule being applied.

| **Step** | **Explanation** |
|---|---|
| $(x^{-\frac{1}{2}})^3(x^{\frac{1}{3}})^2 = (x^{-\frac{3}{2}})(x^{\frac{2}{3}})$ | Apply the ______ of a ______ rule. |
| $= x^{-\frac{3}{2}+\frac{2}{3}}$ | |
| $= x^{-\frac{9}{6}+\frac{4}{6}}$ | |
| $= x^{-\frac{5}{6}}$ | |
| $= \dfrac{1}{x^{\frac{5}{6}}}$ | |
| $= \dfrac{1}{(\sqrt[6]{x})^5}$ | |

## A Practise

*For help with questions 1 and 2, refer to Example 1.*

**1.** Evaluate each cube root.

**a)** $\sqrt[3]{64}$ **b)** $(-1000)^{\frac{1}{3}}$

**c)** $\sqrt[3]{\frac{1}{8}}$ **d)** $\left(\frac{8}{27}\right)^{\frac{1}{3}}$

**2.** Evaluate each root.

**a)** $81^{\frac{1}{4}}$ **b)** $\sqrt[4]{\frac{16}{625}}$

**c)** $64^{\frac{1}{6}}$ **d)** $\sqrt[5]{-100\ 000}$

*For help with questions 3 and 4, refer to Example 2.*

**3.** Evaluate.

**a)** $8^{\frac{2}{3}}$ **b)** $32^{\frac{4}{5}}$

**c)** $(-64)^{\frac{5}{3}}$ **d)** $\left(\frac{1}{10\ 000}\right)^{\frac{3}{4}}$

**4.** Evaluate.

**a)** $16^{-\frac{1}{4}}$ **b)** $25^{-\frac{3}{2}}$ **c)** $\left(\frac{1}{8}\right)^{-\frac{7}{3}}$

**d)** $\left(-\frac{1}{32}\right)^{-\frac{2}{5}}$ **e)** $\left(\frac{10\ 000}{81}\right)^{-\frac{3}{4}}$ **f)** $\left(-\frac{8}{27}\right)^{-\frac{2}{3}}$

*For help with questions 5 and 6, refer to Example 3.*

**5.** Simplify. Express your answers using only positive exponents.

**a)** $x^{\frac{1}{4}} \times x^{\frac{1}{4}}$ **b)** $(m^{\frac{1}{3}})(m^{\frac{3}{4}})$ **c)** $\dfrac{w^{\frac{1}{2}}}{w^{\frac{1}{3}}}$

**d)** $\dfrac{ab^2}{a^{\frac{1}{2}}b^{\frac{1}{3}}}$ **e)** $(y^{\frac{1}{2}})^{\frac{2}{3}}$ **f)** $(u^{\frac{3}{4}}v^{\frac{1}{2}})^{\frac{2}{9}}$

**6.** Simplify. Express your answers using only positive exponents.

**a)** $k^{\frac{3}{4}} \div k^{-\frac{1}{4}}$ **b)** $\dfrac{p^{-\frac{2}{3}}}{p^{\frac{5}{6}}}$

**c)** $(y^{\frac{2}{3}})^{-3}$ **d)** $(w^{-\frac{8}{9}})^{-\frac{3}{4}}$

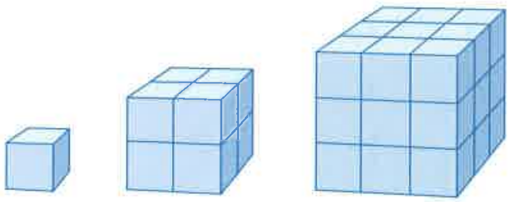
**e)** $(8x)^{\frac{2}{3}}(27x)^{-\frac{1}{3}}$ **f)** $5(7y^{-\frac{2}{3}})^{-2}$

*For help with question 7, refer to Example 4.*

**7.** The surface area, $S$, of a sphere can be expressed in terms of its volume, $V$, using the formula $S(V) = (4\pi)^{\frac{1}{3}}(3V)^{\frac{2}{3}}$. A beach ball has volume 24 000 cm$^3$. Find its surface area, to the nearest hundred square centimetres.

## B Connect and Apply

**8.** What is the square-cube law? Consider the following sequence of cubes.



**a)** Write a formula to express the area, $A$, of one face in terms of the side length, $\ell$.

**b)** Write a formula to express the side length, $\ell$, in terms of the area, $A$, using a rational exponent.

**c)** What is the side length of a cube for which each square face has an area of

**i)** 36 m$^2$? **ii)** 169 cm$^2$? **iii)** 80 m$^2$?

**d)** Modify your answers to parts a) and b) to relate the total surface area, $S$, and the side length, $\ell$.

**e)** Use the results of part d) to calculate the side length of a cube with a surface area of

**i)** 150 m$^2$ **ii)** 600 cm$^2$ **iii)** 250 m$^2$

### Connections

The square-cube law is important in many areas of science. It can be used to explain why elephants have a maximum size and why giant insects cannot possibly exist, except in science fiction! Why do you think this is so?

Go to the *Functions 11* page on the McGraw-Hill Ryerson Web site and follow the links to Chapter 3 to find out more about the square-cube law and its impact on humans and other creatures.

**9.** Refer to question 8.

**a)** Write a formula to express the volume of the cube, $V$, in terms of the side length, $\ell$.

**b)** Write a formula to express the side length, $\ell$, in terms of the volume, $V$, using a rational exponent.

**c)** What is the side length of a cube with a volume of

**i)** 64 m$^3$? **ii)** 343 cm$^3$? **iii)** 15.4 m$^3$?

**10.** Refer to question 8.


**a)** Which formula correctly relates the surface area and volume of a cube?

**A** $S = 6V^{\frac{3}{2}}$ **B** $S = 6V^{\frac{2}{3}}$

**C** $V = \left(\frac{S}{6}\right)^{\frac{3}{2}}$ **D** $V = 6S^{\frac{3}{2}}$

**b)** Is there more than one correct formula? Use algebraic and geometric reasoning to support your answer.

**c)** Why is this relationship called the square-cube law?

**11. a)** Use one of the formulas from question 10 to find the surface area of a cube with a volume of

**i)** 1000 cm$^3$ **ii)** 200 m$^3$

**b)** Which formula did you select, and why?

**c)** Use one of the formulas from question 10 to find the volume of a cube with a surface area of

**i)** 294 m$^2$ **ii)** 36.8 cm$^2$

**d)** Is the formula you used in part c) the same as the formula you used in part b)? Explain why or why not.

**12. Chapter Problem** One of Kepler's laws of planetary motion states that the square of a planet's period of revolution (its "year"), $T$, is related to the cube of its mean radial distance, $r$, from the sun. This can be expressed using the equation

$T = kr^{\frac{3}{2}}$, where $k$ is a constant. The radius of Earth's orbit around the sun is $1.5 \times 10^{11}$ m, and $T = 1$ year.

**a)** Find the value of the constant, $k$.

**b)** Mars is approximately $2.3 \times 10^{11}$ m from the sun. How long is Mars's year, compared to Earth's?

**c)** The planet Venus takes approximately 0.62 Earth years to orbit the sun. What is the approximate radius of Venus's orbit?

### Connections

Johannes Kepler lived in the late 16th and early 17th centuries in Europe. He made some of science's most important discoveries about planetary motion.

**13.** Refer to question 12.

**a)** Use algebraic reasoning to rewrite $T = kr^{\frac{3}{2}}$ with $r$ in terms of $T$.

**b)** Jupiter's period of revolution around the sun is approximately equal to 12 Earth years. What is Jupiter's average distance from the sun?

### Achievement Check

**14.** The formulas $h = 241m^{-\frac{1}{4}}$ and $r = \frac{107}{2}m^{-\frac{1}{4}}$ give the heartbeat frequency, $h$, in beats per minute, and respiratory frequency, $r$, in breaths per minute, for a resting animal with mass $m$, in kilograms.

**a)** Determine the heartbeat frequency and respiratory frequency for each animal.

**i)** killer whale: 6400 kg

**ii)** dog: 6.4 kg

**iii)** mouse: 0.064 kg

**b)** Describe what happens to each frequency as the mass of the animal decreases.

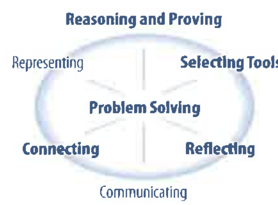
**c)** Use the formula $B = \frac{1}{100}m^{\frac{2}{3}}$ to determine the brain mass, $B$, for each animal in part a).

## C Extend

**15. a)** What is the formula for the volume of a cylinder?

**b)** Rewrite this formula to express the radius as a function of the volume and height of the cylinder.

**16.** Refer to question 15.




**a)** What is the formula for the surface area of a cylinder?

**b)** Substitute the result of question 15b) into this formula.

**c)** Simplify the result to express the surface area of a cylinder with a height of 10 m in terms of its volume.

**d)** Find the surface area of a cylindrical storage tank with height 10 m and volume 1000 m$^3$.

**17.** In the expansion of a particular gas, the relation between the pressure, $P$, in kilopascals (kPa), and the volume, $V$, in cubic metres, is given by $P^2V^3 = 850$.

**a)** Solve the equation for $V$.

**b)** What is the volume of gas when the pressure is 10 kPa?

**c)** Plot the graph of this relation.

**d)** Is this relation a function?

**18. Math Contest** A new operation * is defined as $a*b = (b + 1)^{a+1}$. The value of $[(-2)*3]*15$ is

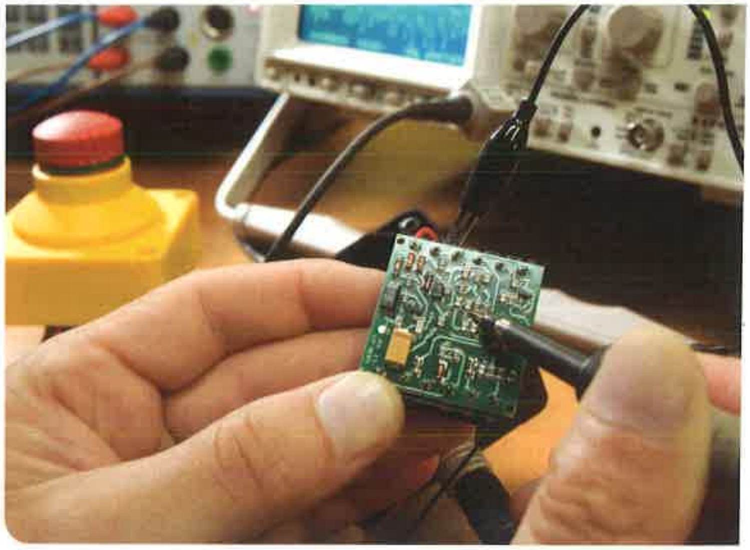
**A** 32 **B** 216

**C** −90 **D** 65 536

**19. Math Contest** Without using a calculator, determine the value of $\frac{25^{\frac{3}{5}}}{\sqrt[5]{5}}$.

3.4

# Properties of Exponential Functions



Capacitors are used to store electric potential energy. When a capacitor in a resistor-capacitor (RC) circuit is discharged, the electric potential across the capacitor decays exponentially over time. This sort of circuit is used in a variety of electronic devices, such as televisions, computers, and MP3 players. Engineers and technicians who design and build such devices must have a solid understanding of exponential functions.

Many situations can be modelled using functions of the form $f(x) = ab^x$, where $a \neq 0$ and $b > 0$, $b \neq 1$. How do the values of $a$ and $b$ affect the properties of this type of function?

### Tools

- computer with *The Geometer's Sketchpad®*

or

- grid paper

or

- graphing calculator

**interval**

- an unbroken part of the real number line
- is either all of $\mathbb{R}$ or has one of the following forms: $x < a$, $x > a$, $x \leq a$, $x \geq a$, $a < x < b$, $a \leq x \leq b$, $a < x \leq b$, $a \leq x < b$, where $a, b \in \mathbb{R}$, and $a < b$

### Connections

It is important to be careful around discarded electrical equipment, such as television sets. Even if the device is not connected to a power source, stored electrical energy may be present in the capacitors.

## Investigate

### How can you discover the characteristics of the graph of an exponential function?

**A: The Effect of $b$ on the Graph of $y = ab^x$**

Start with the function $f(x) = 2^x$. In this case, $a = 1$.

**1. a)** Graph the function.

**b)** Describe the shape of the graph.

**2.** Use algebraic and/or graphical reasoning to justify your answers to the following.

**a)** What are the domain and the range of the function?

**b)** What is the $y$-intercept?

**c)** Is there an $x$-intercept?

**d)** Over what **interval** is the function

**i)** increasing?

**ii)** decreasing?

**3.** Change the value of $b$. Use values greater than 2.

**a)** Compare each graph to the graph of $y = 2^x$. Describe how the graphs are alike. How do they differ?

**b)** Describe how the value of $b$ has affected the characteristics listed in step 2.

**c)** Explain why a value of $b$ greater than 2 has this effect on the graph.

If you are using *The Geometer's Sketchpad®*, you can set $b$ as a parameter whose value you can change dynamically:

- From the **Graph** menu, choose **New Parameter**. Set the name as $b$ and its initial value to 2. Click **OK**.
- From the **Graph** menu, choose **Plot New Function**. Click on the parameter $b$, and then click on **^ x** and **OK**.

You can change the value of $b$ in three ways:

- Click on parameter $b$ and press + and − on the keyboard to increase or decrease the value of $b$ in 1-unit increments.
- Right-click on parameter $b$ and choose **Edit Parameter** to enter a specific value.
- Right-click on parameter $b$ and choose **Animate Parameter**. Use the buttons on the **Motion Controller** to see the effects of changing $b$ continuously.

**Connections**

In Section 3.3, you saw how to deal with rational exponents. The definition of an exponent can be extended to include real numbers, and so the domain of a function like $f(x) = 2^x$ is all real numbers. Try to evaluate $2^{\pi}$ using a calculator.

**4.** Change the value of $b$ again. This time, use values between 0 and 1.

**a)** How has the graph changed?

**b)** Describe how the values of $b$ affect the characteristics listed in step 2.

**c)** Explain why a value of $b$ between 0 and 1 has this effect on the graph.

**5.** What happens to the graph when you set $b = 1$? Explain this result.

**6.** **Reflect** Summarize how the values of $b$ affect the shape and characteristics of the graph of $f(x) = b^x$.

### B: The Effect of $a$ on the Graph of $y = ab^x$

Use the function $f(x) = a \times 2^x$. In this part of the Investigate, keep $b = 2$, and explore what happens as you change the value of $a$.

**1.** Set $a = 1$. This gives the original graph of $f(x) = 2^x$. Explore what happens when

**a)** $a > 1$

**b)** $0 < a < 1$

**c)** $a < 0$

**2.** **Reflect** Write a summary of the effects of various values of $a$ on the graph of the function $f(x) = a \times 2^x$. Include the following characteristics: domain, range, $x$- and $y$-intercepts, and intervals of increase and decrease. Explain why changing the value of $a$ has these effects. Sketch diagrams to support your explanations.

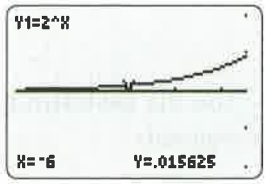
**Technology Tip**

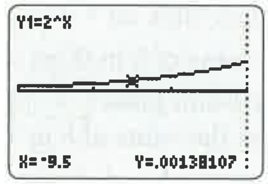
With a graphing calculator, you can vary the line style to see multiple graphs simultaneously. Refer to the Technology Appendix on pages 496 to 516.

**Connections**

You encountered horizontal and vertical asymptotes in Chapter 1 Functions.

One of the interesting features of an exponential function is its asymptotic behaviour. Consider the function $f(x) = 2^x$. If you keep looking left at decreasing values of $x$, you will see that the corresponding $y$-value of the function gets closer and closer to, but never reaches, the $x$-axis. In this case, the $x$-axis is an asymptote.




## Example 1

### Analyse the Graph of an Exponential Function

Graph each exponential function. Identify the

- domain
- range
- $x$- and $y$-intercepts, if they exist
- intervals of increase/decrease
- asymptote

**a)** $y = 4\left(\frac{1}{2}\right)^x$

**b)** $y = -3^{-x}$

### Solution

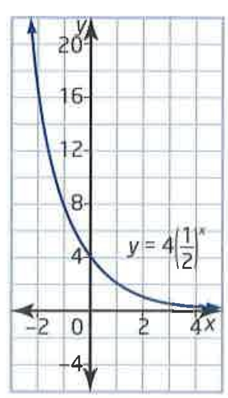
**a)** $y = 4\left(\frac{1}{2}\right)^x$

**Method 1: Use a Table of Values**

Select negative and positive values of $x$ that will make it easy to compute corresponding values of $y$.

| $x$ | $y$ |
|---|---|
| -2 | 16 |
| -1 | 8 |
| 0 | 4 |
| 1 | 2 |
| 2 | 1 |
| 3 | $\frac{1}{2}$ |
| 4 | $\frac{1}{4}$ |

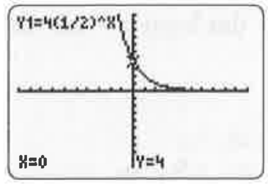
$4\left(\frac{1}{2}\right)^{-2} = 4\left(\frac{2}{1}\right)^2$
$= 16$




Use the table of values to graph the function.

**Method 2: Use a Graphing Calculator**

Use a graphing calculator to explore the graph of this function.


The function is defined for all values of $x$. Therefore, the domain is $\{x \in \mathbb{R}\}$.

The function has positive values for $y$, but $y$ never reaches zero. Therefore, the range is $\{y \in \mathbb{R}, y > 0\}$.

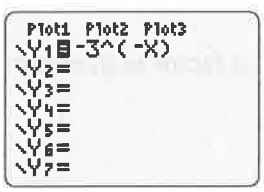
The graph never crosses the $x$-axis, which means there is no $x$-intercept.

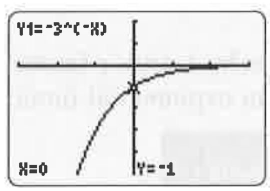
The graph crosses the $y$-axis at 4. Therefore, the $y$-intercept is 4.

The graph falls to the right throughout its domain, so the $y$-values decrease as the $x$-values increase. Therefore, the function is decreasing over its domain.

As the $x$-values increase, the $y$-values get closer and closer to, but never reach, the $x$-axis. Therefore, the $x$-axis, or the line $y = 0$, is an asymptote.

**b)** $y = -3^{-x}$




The domain is $\{x \in \mathbb{R}\}$.

All function values are negative. Therefore, the range is $\{y \in \mathbb{R}, y < 0\}$.

There is no $x$-intercept.

The $y$-intercept is $-1$.

The graph rises throughout its domain. Therefore, the function is increasing for all values of $x$.

The $x$-axis, whose equation is $y = 0$, is an asymptote.

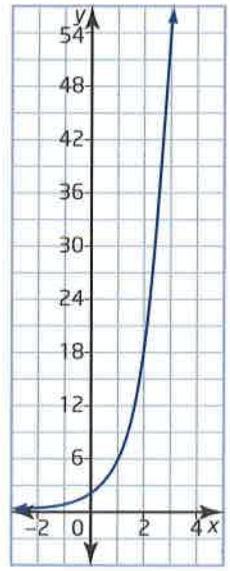
**Connections**

Could you use transformations to quickly sketch the graph of $y = -3^{-x}$? You will explore this option in Section 3.5.

## Example 2

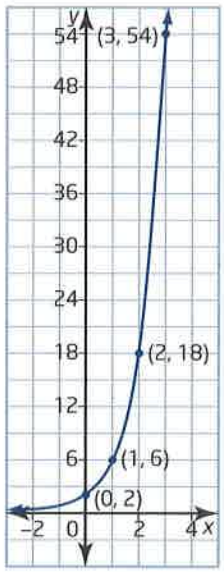
### Write an Exponential Equation Given Its Graph

Write the equation in the form $y = ab^x$ that describes the graph shown.




### Solution

Read some ordered pairs from the graph.




Note that as $x$ changes by 1 unit, $y$ increases by a factor of 3, confirming that this function is an exponential function.

| $x$ | $y$ | Change in $y$ |
|---|---|---|
| 0 | 2 | |
| | | × 3 |
| 1 | 6 | |
| | | × 3 |
| 2 | 18 | |
| | | × 3 |
| 3 | 54 | |

Since each successive value increases by a factor of 3, this function must have $b = 3$. Since all points on this graph must satisfy the equation $y = ab^x$, substitute the coordinates of one of the points, and the value of $b$, to find the value of $a$.

Pick a point that is easy to work with, such as (1, 6). Substitute $x = 1$, $y = 6$, and $b = 3$:

$y = ab^x$

$6 = a \times 3^1$

$6 = a \times 3$

$a = 2$

Therefore, the equation that describes this curve is $y = 2 \times 3^x$.

## Example 3

### Write an Exponential Function Given Its Properties

A radioactive sample has a half-life of 3 days. The initial sample is 200 mg.

**a)** Write a function to relate the amount remaining, in milligrams, to the time, in days.

**b)** Restrict the domain of the function so that the mathematical model fits the situation it is describing.

### Solution

**a)** This exponential decay can be modelled using a function of the form $A(x) = A_0\left(\frac{1}{2}\right)^x$, where $x$ is the time, in half-life periods; $A_0$ is the initial amount, in milligrams; and $A$ is the amount remaining, in milligrams, after time $x$.

Start with 200 mg. After every half-life, the amount is reduced by half.

Substituting $A_0 = 200$ into this equation gives $A(x) = 200\left(\frac{1}{2}\right)^x$. This expresses $A$ as a function of $x$, the number of half-lives. To express $A$ as a function of $t$, measured in days, replace $x$ with $\frac{t}{3}$.

The half-life of this material is 3 days. Therefore, the number of elapsed half-lives at any given point is the number of days divided by 3.

$$A(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{3}}$$

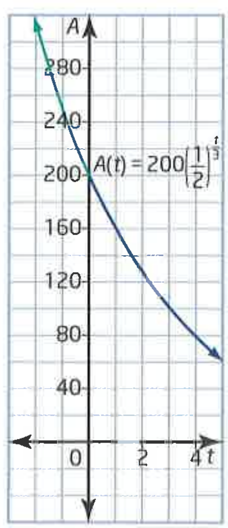
This equation relates the amount, $A$, in milligrams, of radioactive material remaining to time, $t$, in days.

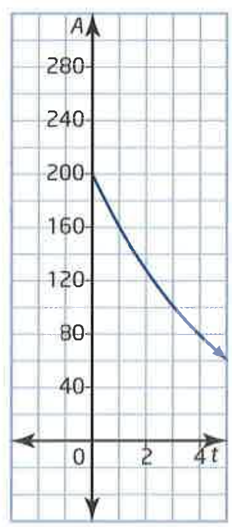
**b)** A graph of this function reveals a limitation of the mathematical model.

The initial sample size, at $t = 0$, was 200 mg. It is not clear that the function has any meaning before this time. Since it is only certain that the mathematical model fits this situation for non-negative values of $t$, it makes sense to restrict its domain:

$A(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{3}}$ for

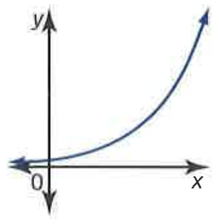
$\{t \in \mathbb{R}, t \geq 0\}$.

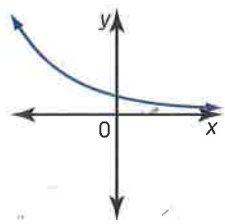


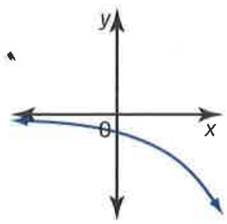



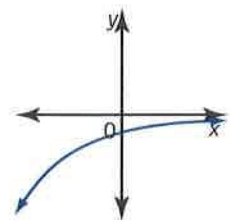
## Key Concepts

- The graph of an exponential function of the form $y = ab^x$ is
  - increasing if $a > 0$ and $b > 1$
  - decreasing if $a > 0$ and $0 < b < 1$
  - decreasing if $a < 0$ and $b > 1$
  - increasing if $a < 0$ and $0 < b < 1$










- The graph of an exponential function of the form $y = ab^x$, where $a > 0$ and $b > 0$, has
  - domain $\{x \in \mathbb{R}\}$
  - range $\{y \in \mathbb{R}, y > 0\}$
  - a horizontal asymptote at $y = 0$
  - a $y$-intercept of $a$
- The graph of an exponential function of the form $y = ab^x$, where $a < 0$ and $b > 0$, has
  - domain $\{x \in \mathbb{R}\}$
  - range $\{y \in \mathbb{R}, y < 0\}$
  - a horizontal asymptote at $y = 0$
  - a $y$-intercept of $a$
- You can write an equation to model an exponential function if you are given enough information about its graph or properties.
- Sometimes it makes sense to restrict the domain of an exponential model based on the situation it represents.

## Communicate Your Understanding

**C1** **a)** Is an exponential function either always increasing or always decreasing? Explain.

**b)** Is it possible for an exponential function of the form $y = ab^x$ to have an $x$-intercept? If yes, given an example. If no, explain why not.

**C2** Consider the exponential functions $f(x) = 100\left(\frac{1}{2}\right)^x$ and $g(x) = -10(2)^x$.

**a)** Which function has a graph with range

**i)** $\{y \in \mathbb{R}, y < 0\}$? **ii)** $\{y \in \mathbb{R}, y > 0\}$?

Explain how you can tell by inspecting the equations.

**b)** Which function is

**i)** increasing? **ii)** decreasing?

Explain how you can tell by inspecting the equations.

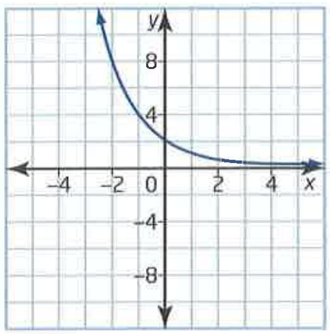
**C3** Describe what is meant by "asymptotic behaviour." Support your explanation with one or more sketches.

## A Practise

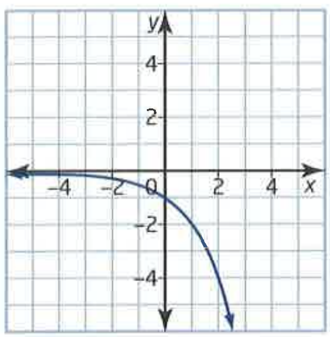
*For help with questions 1 to 3, refer to Example 1.*

**1.** Match each graph with its corresponding equation.

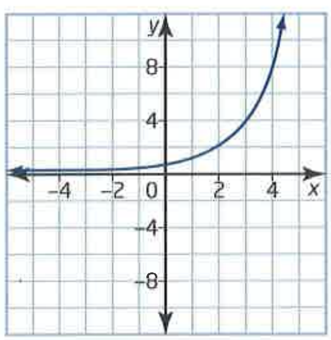
**a)**



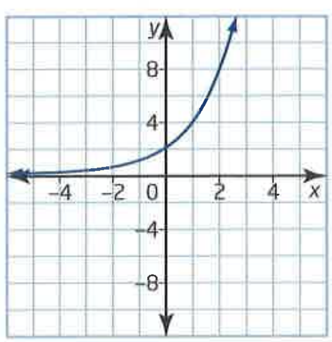
**b)**



**c)**



**d)**



**A** $y = 2 \times 2^x$

**B** $y = 2 \times \left(\frac{1}{2}\right)^x$

**C** $y = \frac{1}{2} \times 2^x$

**D** $y = -2^x$

**2. a)** Sketch the graph of an exponential function that satisfies all of these conditions:

- domain $\{x \in \mathbb{R}\}$
- range $\{y \in \mathbb{R}, y > 0\}$
- $y$-intercept 5
- function increasing

**b)** Is this the only possible graph? Explain.

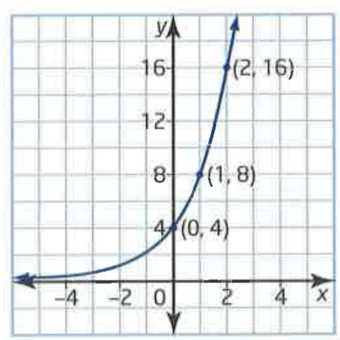
**3. a)** Sketch the graph of an exponential function that satisfies all of these conditions:

- domain $\{x \in \mathbb{R}\}$
- range $\{y \in \mathbb{R}, y < 0\}$
- $y$-intercept $-2$
- function decreasing

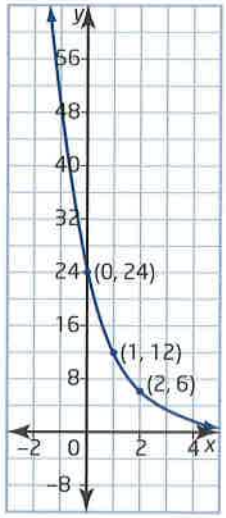
**b)** Is this the only possible graph? Explain.

*For help with questions 4 and 5, refer to Example 2.*

**4.** Write an exponential equation to match the graph shown.



5. Write an exponential equation to match the graph shown.




*For help with question 6, refer to Example 3.*

6. A radioactive sample with an initial mass of 25 mg has a half-life of 2 days.

**a)** Which equation models this exponential decay, where $t$ is the time, in days, and $A$ is the amount of the substance that remains?

**A** $A = 25 \times 2^{\frac{t}{2}}$

**B** $A = 25 \times \left(\frac{1}{2}\right)^{2t}$

**C** $A = 25 \times \left(\frac{1}{2}\right)^{\frac{t}{2}}$

**D** $A = 2 \times 25^{\frac{t}{2}}$

**b)** What is the amount of radioactive material remaining after 7 days?

## B Connect and Apply

7. Graph each function and identify the

**i)** domain

**ii)** range

**iii)** $x$- and $y$-intercepts, if they exist

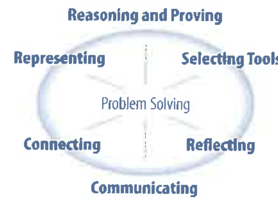
**iv)** intervals of increase/decrease

**v)** asymptote

**a)** $f(x) = \left(\frac{1}{2}\right)^x$

**b)** $y = 2 \times 1.5^x$

**c)** $y = -\left(\frac{1}{3}\right)^x$




8. **a)** Graph each function.

**i)** $f(x) = 2^x$

**ii)** $r(x) = \frac{2}{x}$

**b)** Describe how the graphs are alike. How do they differ?

**c)** Compare the asymptotes of these functions. What do you observe?

9. **a)** Graph each function.

**i)** $g(x) = \left(\frac{1}{2}\right)^x$ **ii)** $r(x) = \frac{2}{x}$

**b)** Describe how the graphs are alike. How do they differ?

**c)** Compare the asymptotes of these functions. What do you observe?

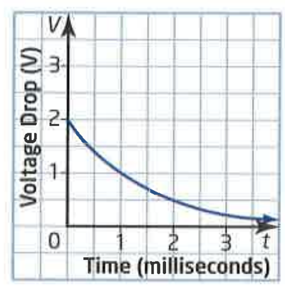
10. **a)** Predict how the graphs of the following functions are related.

**i)** $f(x) = 3^{-x}$ **ii)** $g(x) = \left(\frac{1}{3}\right)^x$

**b)** Graph both functions and check your prediction from part a).

**c)** Use algebraic reasoning to explain this relationship.

11. The graph shows the voltage drop across a capacitor over time while discharging an RC circuit. At $t = 0$ s, the circuit begins to discharge.




**a)** What is the domain of this function?

**b)** What is the range?

**c)** What is the initial voltage drop across the capacitor?

**d)** What value does the voltage drop across the capacitor approach as more time passes?

**e)** Approximately how long will it take the voltage drop to reach 50% of the initial value?

**12.** A flywheel is rotating under friction. The number, $R$, of revolutions per minute after $t$ minutes can be determined using the function $R(t) = 4000(0.75)^{2t}$.


**a)** Explain the roles of the numbers 0.75 and 2 in the equation.

**b)** Graph the function.

**c)** Which value in the equation indicates that the flywheel is slowing?

**d)** Determine the number of revolutions per minute after

**i)** 1 min **ii)** 3 min

## C Extend

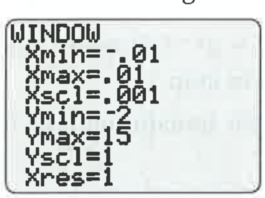
**13. Use Technology** Refer to question 11. The equation that models this situation is given by $V = V_0 b^{\frac{t}{RC}}$, where $V$ is the voltage drop, in volts; $V_0$ is the initial voltage drop; $t$ is the time, in seconds; $R$ is the resistance, in ohms ($\Omega$); and $C$ is the capacitance, in farads (F).

For this circuit, $R = 2000\ \Omega$ and $C = 1\ \mu F$. Note that $1\ \mu F = 0.000\ 001$ F.

**a)** Determine the value of the base, $b$.

**b)** Explain your method.

**c)** Graph the function using a graphing calculator or graphing software. Use the window settings shown.




**d)** What are the domain and range of this function?

**e)** Explain how and why the domain and range are restricted, as illustrated in the graph of question 11.

**14.** Suppose a square-based pyramid has a fixed height of 25 m.

**a)** Write an equation, using rational exponents where appropriate, to express the side length of the base of a square-based pyramid in terms of its volume.

**b)** How should you limit the domain of this function so that the mathematical model fits the situation?

**c)** What impact does doubling the volume have on the side length of the base? Explain.

**15.** Suppose that a shelf can hold cylindrical drums with a fixed height of 1 m.

**a)** Write a simplified equation, using rational exponents where appropriate, to express the surface area in terms of the volume for drums that will fit on the shelf.

**b)** Find the surface area and diameter of a drum with a volume of 0.8 $m^3$.

**c)** What are the restrictions on the domain of the function used in this model?

**d)** Graph the function for the restricted domain.

**16. Math Contest** Find all solutions to $3x - 2^x - 1 = 0$.

**17. Math Contest** Consider the function $y = 12\left(\frac{1}{2}\right)^x - 3$. The $y$-intercept is $b$ and the $x$-intercept is $a$. The sum of $a$ and $b$ is

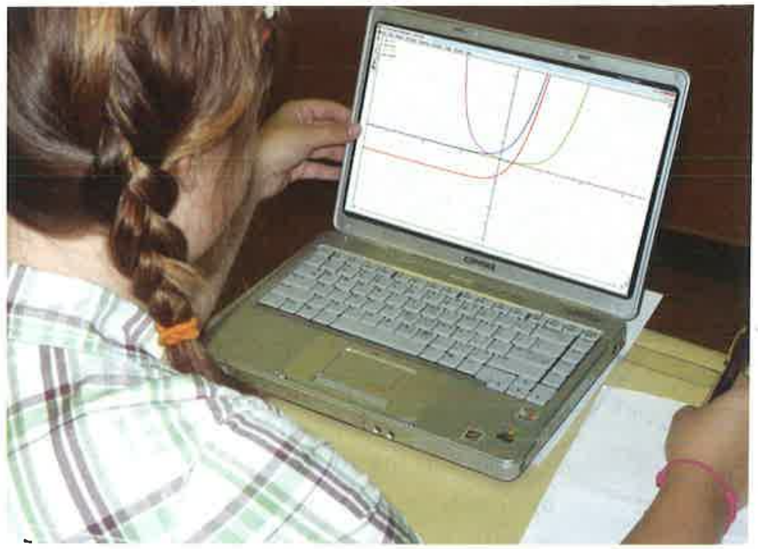
**A** 11 **B** 6 **C** 7 **D** 18

**18. Math Contest** A number is between 20 and 30. When this number is subtracted from its cube, the result is 13 800. When the same number is added to its cube, the answer is

**A** 13 848 **B** 13 852
**C** 13 846 **D** 13 844

3.5

# Transformations of Exponential Functions



Once you understand the basic shape of an exponential function, you can apply your understanding of transformations to easily graph a variety of related curves.

You should be familiar with the following types of transformations:

- $f(x) \rightarrow f(x) + c$
- $f(x) \rightarrow f(x - d)$
- $f(x) \rightarrow af(x)$
- $f(x) \rightarrow f(kx)$

What type of transformation does each of these represent? Do these same transformations apply to exponential functions?

### Tools

- computer with *The Geometer's Sketchpad®*

or

- graphing calculator

## Investigate

### How can you transform an exponential function?

**1. a)** Make a conjecture about what happens when you transform the function $y = 2^x$ to the function $y = 2^x + 3$.

**b)** Use a graphing calculator or graphing software to test your prediction. Describe what you notice.

**c)** Repeat parts a) and b) for the following:

**i)** $y = 2^x \rightarrow y = 2^x - 4$

**ii)** two other transformations of this type

**d)** Summarize your observations.

**2. a)** Consider the transformation $y = 2^x \rightarrow y = 2^{x+3}$. How do you think this will compare to the transformation in step 1a)?

**b)** Test your prediction. Repeat for two other transformations of this type.

**c)** Describe the effects of this type of transformation.

**3. a)** Predict what will happen to the graph of $y = 2^x$ when you transform it to $y = 2(2^x)$.

**b)** Test your prediction. Repeat for two other transformations of this type. Include at least one negative value. Describe what you notice.

4. a) Do you think the transformation $y = 2^x \rightarrow y = 2^{2x}$ will produce the same effect as that in step 3a)? Explain your reasoning.
   b) Test your prediction.
   c) Carry out two other transformations of this type. Use at least one negative value. Summarize your observations.
5. Choose a different base and repeat the analysis in steps 1 to 4. Do you get similar results using different bases?
6. **Reflect** Write a brief summary of
   - horizontal and vertical translations of exponential functions
   - horizontal and vertical stretches of exponential functions
   - horizontal and vertical reflections of exponential functions

   Use diagrams to support your explanations.
7. a) **Reflect** Can you express the function $y = 2^{2x}$ in another form? Use algebraic reasoning to explain.
   b) Use graphing technology to verify that the two functions are equivalent.

## Example 1

### Horizontal and Vertical Translations

Sketch graphs of the following functions using the graph of $y = 3^x$ as the base. Describe the effects, if any, on the

- asymptote
- domain
- range

**a)** $y = 3^x - 4$

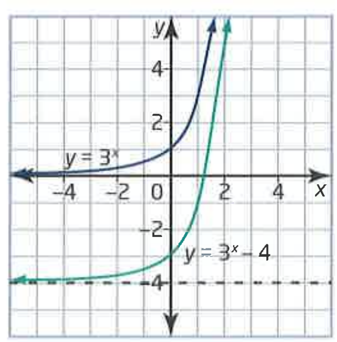
**b)** $y = 3^{x-2}$

**c)** $y = 3^{x+1} + 3$

### Solution

**a)** $y = 3^x - 4$

This translation will shift $y = 3^x$ down 4 units. Start with the graph of $y = 3^x$ and then apply the translation.




Note that the asymptote has shifted down 4 units.

The graph extends indefinitely to the right and left, so the domain remains $\{x \in \mathbb{R}\}$.

The graph shows that the function has values above the line $y = -4$, so the range is $\{y \in \mathbb{R}, y > -4\}$.

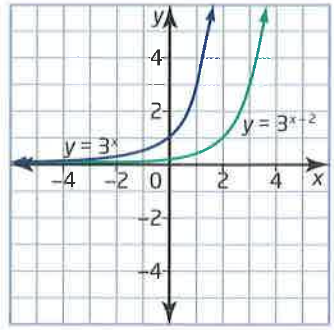
**b)** $y = 3^{x-2}$

This translation will shift $y = 3^x$ to the right 2 units. Start with the graph of $y = 3^x$ and then apply the translation.

The asymptote has not changed: it remains the line $y = 0$.

The horizontal shift has not changed the domain of $\{x \in \mathbb{R}\}$.

The range also remains unchanged: $\{y \in \mathbb{R}, y > 0\}$.



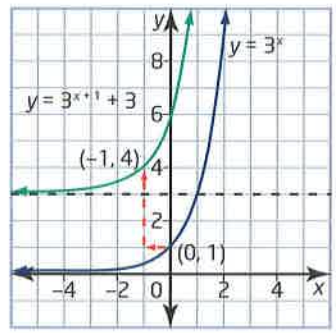

**c)** $y = 3^{x+1} + 3$

This compound translation includes both a horizontal and a vertical shift. Translate the curve of $y = 3^x$ left 1 unit and up 3 units.

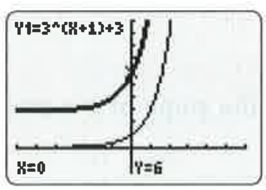
The horizontal asymptote has shifted 3 units upward to the line $y = 3$.

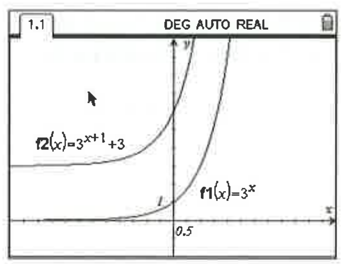
The domain remains $\{x \in \mathbb{R}\}$.

The range is $\{y \in \mathbb{R}, y > 3\}$.




You can check your results using a graphing calculator.




## Example 2

### Stretches, Compressions, and Reflections

Graph each function using transformations of the base function $y = 4^x$.

**a)** $y = 4^{2x}$

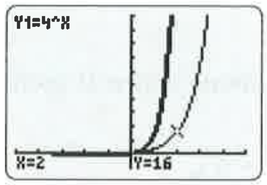
**b)** $y = -2(4^x)$

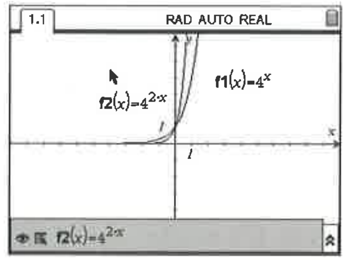
**c)** $y = 4^{-\frac{1}{2}x}$

## Solution

**a)** $y = 4^{2x}$

Start with $y = 4^x$. Then, compress the graph horizontally by a factor of $\frac{1}{2}$.



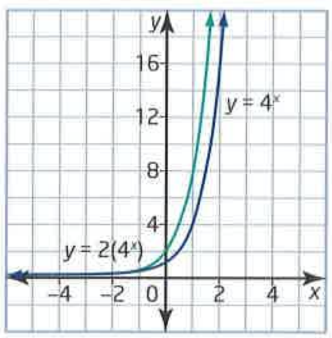

**b)** $y = -2(4^x)$

This compound transformation involves a stretch and a reflection:

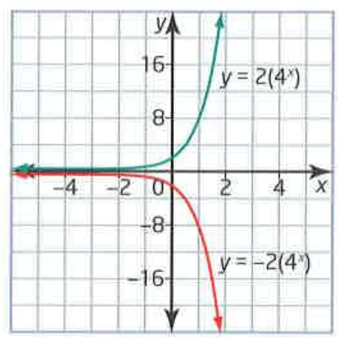
$$y = -2(4^x)$$

Reflect in the $x$-axis. Stretch vertically by a factor of 2.

Start with the graph of $y = 4^x$. Then, apply the vertical stretch.




Now reflect $y = 2(4^x)$ in the $x$-axis.



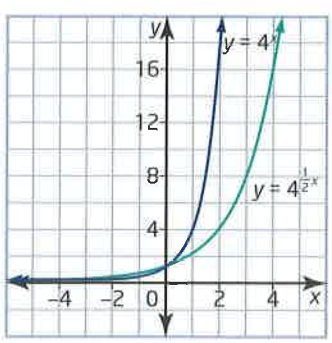

**c)** $y = 4^{-\frac{1}{2}x}$

This compound transformation also involves a stretch and a reflection:

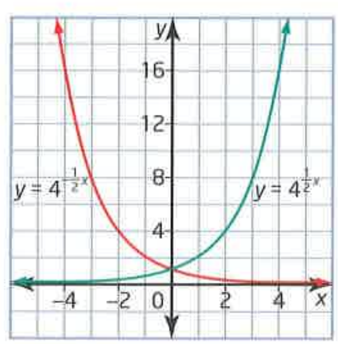
$$y = 4^{-\frac{1}{2}x}$$

Reflect in the $y$-axis. Stretch horizontally by a factor of 2.

Start with the graph of $y = 4^x$ and apply the horizontal stretch.




Now reflect this curve in the $y$-axis.



## Example 3

### Graph $y = ab^{k(x-d)} + c$

Graph the function $y = -2^{2(x-3)} + 5$.

### Solution

Start with $y = 2^x$. Identify the transformations that will produce the graph of the equation given above:

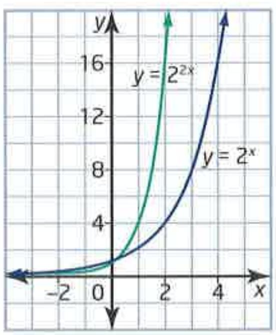
$$y = -2^{2(x-3)} + 5$$

Reflect in the $x$-axis.

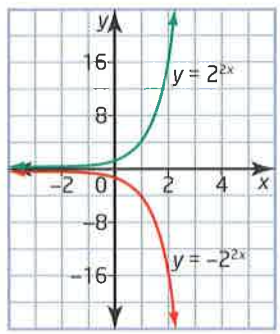
Compress horizontally by a factor of $\frac{1}{2}$.

Shift right 3 units and up 5 units.

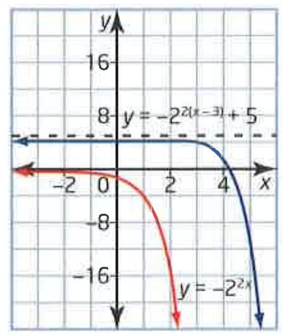
Start with the graph of $y = 2^x$. To show $y = 2^{2x}$, compress horizontally.




Next, reflect the curve in the $x$-axis to show $y = -2^{2x}$.




Finally, perform the horizontal and vertical translations.




**Connections**

You explored compound transformations of functions in Chapter 2.

## Example 4

### Circuit Analysis

In a particular circuit, the current, $I$, in amperes (A), after $t$ seconds can be found using the formula $I(t) = 0.9(1 - 10^{-0.044t})$.

**a)** Graph this function using graphing technology. What is the appropriate domain for the function modelling the current?

**b)** What is the current in the circuit after

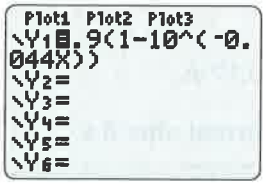
**i)** 2 s? **ii)** 5 s?

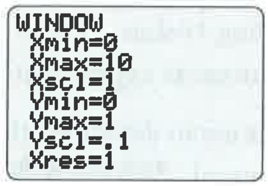
**c)** Describe what is happening to the current over the chosen domain.

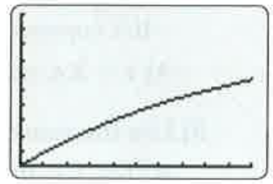
## Solution

### Method 1: Use a Graphing Calculator

**a)** Use systematic trial to set an appropriate viewing window. Note that the appropriate domain must not allow negative values for the current. The domain is $\{t \in \mathbb{R}, t > 0\}$.

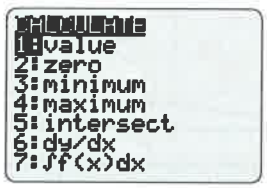


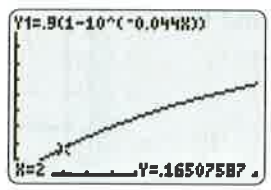



**b)** **i)** To determine the current at 2 s:

- Press 2nd [CALC] and then select **1:value**.
- Enter 2 when prompted and press ENTER.

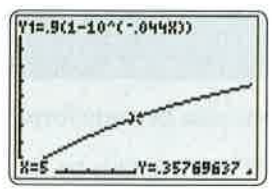



At $t = 2$ s, the current is approximately 0.17 A.

**ii)** Use the same process to determine the current after 5 s.

At $t = 5$ s, the current is approximately 0.36 A.

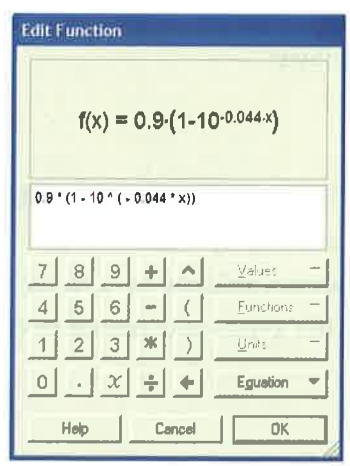



**c)** The function is increasing over $\{t \in \mathbb{R}, t > 0\}$, and the rate of increase is decreasing.

### Method 2: Use *The Geometer's Sketchpad*®

**a)** Start a new sketch. From the **Graph** menu, choose **Plot New Function**. Enter the function as shown, and click **OK**.

From the **Graph** menu, click on the **Grid Form** menu and choose **Rectangular Grid**. Click and drag the origin and scales to obtain a clear view of the graph. Note that the appropriate domain must not allow negative values for the current. The domain is $\{t \in \mathbb{R}, t > 0\}$.



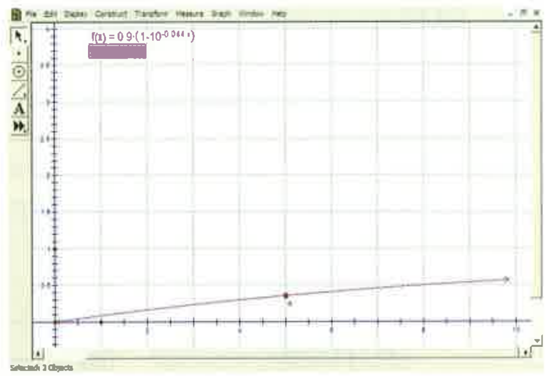
**b) i)** To determine the current at 2 s, determine the value of $I$ when $t = 2$:

- Select the graph. From the **Construct** menu, choose **Point On Function Plot**.
- From the **Measure** menu, choose **Coordinates**.
- Click and drag the point as close as you can to $t = 2$. Then, read the corresponding $I$-value.

At $t = 2$ s, the current is approximately 0.17 A.

**ii)** Use the same process to determine the current after 5 s.

At $t = 5$ s, the current is approximately 0.36 A.



**c)** The function is increasing over $\{t \in \mathbb{R}, t > 0\}$, and the rate of increase is decreasing.

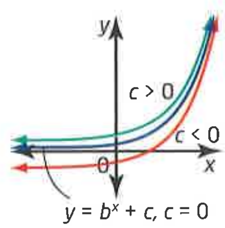
## Key Concepts

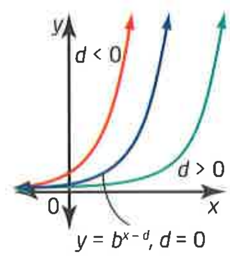
- Exponential functions can be transformed in the same way as other functions.
- The graph of $y = ab^{k(x-d)} + c$ can be found by performing the following transformations on the graph of the base $y = b^x$:

**Horizontal and Vertical Translations**

– If $d > 0$, translate right $d$ units; if $d < 0$, translate left.

– If $c > 0$, translate up $c$ units; if $c < 0$, translate down.



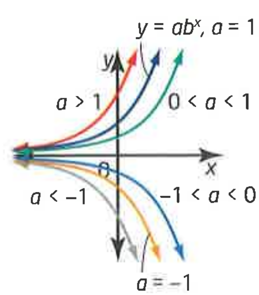




**Vertical Stretches, Compressions, and Reflections**

– If $a > 1$, stretch vertically by a factor of $a$.

– If $0 < a < 1$, compress vertically by a factor of $a$.

– If $a < 0$, reflect in the $x$-axis and stretch or compress.



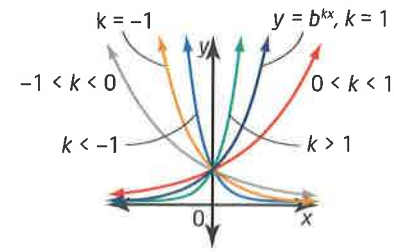
## Key Concepts

**Horizontal Stretches, Compressions, and Reflections**

– If $k > 1$, compress horizontally by a factor of $\frac{1}{k}$.

– If $0 < k < 1$, stretch horizontally by a factor of $\frac{1}{k}$.

– If $k < 0$, reflect in the $y$-axis and stretch or compress.




- Some exponential functions can easily be written using different bases. For example, $y = 2^{4x}$ is equivalent to $y = 16^x$.

## Communicate Your Understanding

**C1** Match each transformation with the corresponding equation, using the function $y = 10^x$ as the base. Give reasons for your answers. Not all transformations will match an equation.

**Transformation**

**a)** horizontal stretch by a factor of 3

**b)** shift 3 units up

**c)** shift 3 units left

**d)** vertical compression by a factor of $\frac{1}{3}$

**e)** vertical stretch by a factor of 3

**f)** shift 3 units right

**g)** reflect in the $x$-axis

**A** $y = 10^x + 3$

**B** $y = 10^{x+3}$

**C** $y = -10^x$

**D** $y = 10^x - 3$

**E** $y = 10^{3x}$

**F** $y = 10^{-x}$

**G** $y = \left(\frac{1}{3}\right)10^x$

**C2** Give a corresponding equation for each unmatched transformation in question C1.

**C3** **a)** Describe how you could sketch the graph of $y = 5^{2x}$ using a transformation.

**b)** Describe how you could sketch the graph of $y = 5^{2x}$ by applying an exponent rule.

**C4** Describe the effect of each constant, $a$, $k$, $c$, and $d$, when the graph of the function $y = b^x$ is transformed to $y = ab^{k(x - d)} + c$.

**C5** Refer to Example 4.

**a)** How could you determine whether or not the current increases indefinitely?

**b)** Carry out the method described in part a) to determine if the current does increase indefinitely.

## A Practise

*For help with questions 1 to 3, refer to Example 1.*

**1.** Describe the transformation that maps the function $y = 4^x$ onto each function given.

**a)** $y = 4^x + 2$ **b)** $y = 4^{x-3}$

**c)** $y = 4^{x+4}$ **d)** $y = 4^{x-1} - 5$

**2.** Sketch the graph of each function in question 1. Use the graph of $y = 4^x$ as the base.

**3.** Write the equation for the function that results from each transformation applied to the base function $y = 5^x$.

**a)** translate down 3 units

**b)** shift right 2 units

**c)** translate left $\frac{1}{2}$ unit

**d)** shift up 1 unit and left 2.5 units

*For help with questions 4 to 6, refer to Example 2.*

**4.** Describe the transformations that map the function $y = 8^x$ onto each function.

**a)** $y = \left(\frac{1}{2}\right)8^x$ **b)** $y = 8^{4x}$

**c)** $y = -8^x$ **d)** $y = 8^{-2x}$

**5.** Sketch the graph of each function in question 4. Use the graph of $y = 8^x$ as the base. Be sure to choose appropriate scales for your axes.

**6.** Write the equation for the function that results from each transformation applied to the base function $y = 7^x$.

**a)** reflect in the $x$-axis

**b)** stretch vertically by a factor of 3

**c)** stretch horizontally by a factor of 2.4

**d)** reflect in the $y$-axis and compress vertically by a factor of 7

*For help with question 7 and 8, refer to Example 3.*

**7.** Sketch the graph of $y = \left(-\frac{1}{2}\right)2^{x-4}$ by using $y = 2^x$ as the base and applying transformations.

**8.** Sketch the graph of $y = 3^{-0.5x-1} - 5$ by using $y = 3^x$ as the base and applying transformations.

## B Connect and Apply

*For help with question 9, refer to Example 4.*

**9.** The temperature, $T$, in degrees Celsius, of a cooling metal bar after $t$ minutes is given by $T(t) = 20 + 100(0.3)^{0.2t}$.

**a)** Sketch the graph of this relation.

**b)** What is the asymptote of this function? What does it represent?

**c)** How long will it take for the temperature to be within 0.1 °C of the value of the asymptote?

**10. a)** Graph the function $f(x) = \left(\frac{1}{2}\right)^{\frac{1}{2}(x+3)} - 1$.

**b)** Identify the following properties.

**i)** domain

**ii)** range

**iii)** equation of the asymptote

**11. a)** Sketch the graph of $y = 9^x$.


**b)** Rewrite $y = 9^x$ using a base of 3. Describe how you can graph this function by transforming the graph of $y = 3^x$.

**c)** Rewrite $y = 9^x$ using a base of 81. Describe how you can graph this function by transforming the graph of $y = 81^x$.

**d)** Explain why all three of these functions are equivalent.

**12. a)** Rewrite the function $y = 8^x$ in two different ways, using a different base in each case.

**b)** Explain why all three functions are equivalent.

13. **a)** Write the equation for a transformed exponential function whose asymptote is $y = 4$, with a $y$-intercept of 6.

**b)** Is the function you produced in part a) the only possible answer? Use transformations to help explain your answer.

14. **a)** Write the equation for a transformed exponential function with a base of 2 that passes through the point (0, 2).

**b)** Write two different equations that satisfy these criteria.

**c)** Use algebraic and/or graphical reasoning to explain why each equation is a solution.

15. Does it matter in which order transformations are applied?


**a)** Start with a base function of the form $y = b^x$ and create a new exponential function by applying at least one of each of the following types of transformations to your base function:
- translation
- stretch or compression
- reflection

**b)** Predict whether or not the order in which you carry out the transformations will affect the final graph.

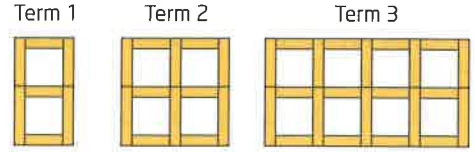
**c)** Graph the function by applying the transformations in any order you choose.

**d)** Repeat part c) by applying the transformations in several different orders.

**e)** Repeat parts a) to d) for a different function of your choice.

**f)** Compare your results with the predictions you made. Describe your findings.

16. Examine the growing pattern shown.




**a)** Build or draw the next term in this pattern.

**b)** Is the pattern of white squares growing exponentially? Explain how you know.

**c)** Write an equation to express the number of squares, $s$, as a function of the term number, $n$.

**d)** How many squares are in the

**i)** 5th term?

**ii)** 10th term?

17. Refer to question 16. Suppose that you built this model using toothpicks. Focus on the number of toothpicks needed to build each stage of the model.

**a)** Make a table that compares the term number, $n$, with the number of toothpicks, $t$, in each term.

**b)** Explore the patterns in the finite differences.

**c)** Graph the data.

**d)** Is the function that expresses $t$ in terms of $n$ an exponential function? Explain.

**e)** Write an equation to express $t$ as a function of $n$.

**f)** Use transformations to describe how this function is related to the exponential function you found in question 16c).

18. **Use Technology** Does a vertical stretch of factor $a$ of an exponential function produce the same effect as a horizontal compression of factor $\frac{1}{a}$?

**a)** Write a conjecture about this claim.

**b)** Use graphing technology to explore several cases, using various values of $a$ and various bases.

**c)** Write a brief report of your findings.

**19. Use Technology** For exponential functions, does a horizontal translation of $d$ units produce the same effect as a vertical stretch?

**a)** Write a conjecture about this claim.

**b)** Use graphing technology to explore several cases, using various values of $d$ and various bases.

**c)** Write a brief report of your findings.

## Achievement Check

**20. a)** Describe the transformations that must be applied to the graph of $y = 3^x$ to obtain the graph of $y = -\left(\frac{1}{3}\right)^{12 - 3x} + 2$.

**b)** Graph the function $y = -\left(\frac{1}{3}\right)^{12 - 3x} + 2$.

**c)** Identify the following properties.

**i)** domain

**ii)** range

**iii)** equation of the asymptote

**iv)** intercept(s), if they exist

## C Extend

**21.** Refer to question 16. Another interpretation of the model is that it consists of white square pieces and yellow rectangular pieces of three sizes, arranged in the patterns shown.

**a)** How many pieces of of each size are used in the first term of the pattern?

**b)** Are the growing patterns for the numbers of each of these pieces all exponential in nature? Explain.

**c)** Write an equation to describe the total number of pieces as a function of the term number.

**d)** Describe how the function you found in part c) is related to the functions you found in questions 16 and 17, using transformations, if possible.

**22. Use Technology** The curve in which a rope or wire hangs under its own weight is called a catenary. A telephone wire hanging between two poles has the shape of a catenary. The equation of the path of a certain telephone wire is $y = 0.2\left(2^{\frac{x}{4}} + 2^{-\frac{x}{4}}\right)$, where all measurements are in metres.

**a)** Graph this relation using a graphing calculator.

**b)** What other function does this resemble?

**c)** Select seven points and use an appropriate regression feature to determine an equation to model this curve, using your answer in part b) as your model.

**d)** Graph both functions on the same set of axes.

**e)** How do the graphs compare?

**f)** Zoom out a couple of times and repeat the comparison.

**23. Math Contest** If $f(x) = 3^x$ and $f(x + 2) + f(x + 3) + f(x + 4) = kf(x)$, what is the value of $k$?

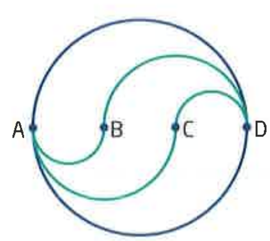
**24. Math Contest** If $4^y = 7$, then the value of $4^{3y} + 2$ is

**A** 23 **B** 345 **C** $\sqrt[3]{7} + 2$ **D** 2189

**25. Math Contest** What is the range of the function $y = 12\left(\frac{1}{2}\right)^x - 3$?

**26. Math Contest** In the diagram, AD = 6 cm, and B and C trisect AD. The ratio of the area between the green curves to the area of the blue circle is

**A** 1:3 **B** 4:9 **C** 2:9 **D** 1:2



3.6

# Making Connections: Tools and Strategies for Applying Exponential Models

Are Canadians getting wealthier? Every home buyer knows that house prices and construction costs usually go up, but are earnings keeping up? How do the earnings of Ontarians compare with the earnings in the rest of Canada?

## Example 1

### Construct a Model of Exponential Growth

The table gives the average weekly earnings, rounded to the nearest dollar, for Canadians over a 5-year period.

| Year | Earnings ($) |
|---|---|
| 2002 | 679 |
| 2003 | 688 |
| 2004 | 702 |
| 2005 | 725 |
| 2006 | 747 |

**a)** Construct an exponential function to model the data.

**b)** Predict the average Canadian's weekly earnings in 2010.

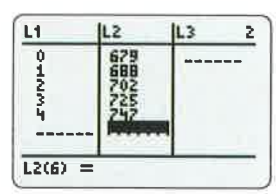
**c)** Predict when the average Canadian might expect to earn $1000 per week.

### Solution

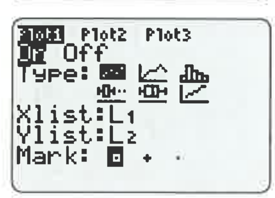
**Method 1: Use a Graphing Calculator**

**a)** Make a scatter plot of the data. For simplicity, renumber the years from 0 to 4.

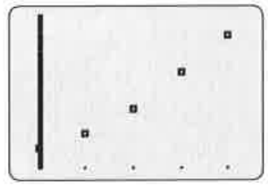
- Press STAT and select **1:Edit**.
- Enter the data in lists **L1** and **L2**.




- Press Y=. Clear any functions in the equation section. Then, turn on **Plot1** only.
- Press 2nd [STATPLOT] and select **Plot1**.
- Use the settings shown.



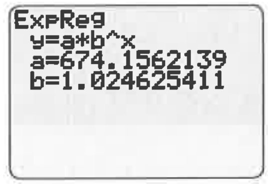
Press (ZOOM) and select **9:ZoomStat**. A scatter plot of the data will appear.



The trend appears to be exponential. Use exponential regression to find a curve of best fit.

- Press (2nd) [QUIT] to return to the home screen.
- Press (STAT). Cursor over to **CALC** and select **0:ExpReg**.
- Press (2nd) [L1] (,) (2nd) [L2] (,), and then press (VARS). Select **Y-VARS**, and then select **1:Function**.
- Select **1:Y1** and press (ENTER).

  An exponential equation will appear.

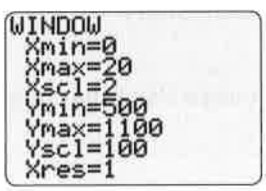


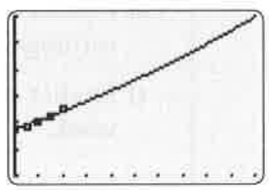

Round the calculated values of $a$ and $b$ and substitute into the equation $y = ab^x$ to obtain the approximate equation of the curve of best fit.

$y = 674 \times 1.025^x$

Replace $x$ and $y$ with variables that make sense for the problem. Let $n$ represent the number of years following 2002 and $E$ represent the average Canadian's weekly earnings in year $n$.
Then, the equation is $E(n) = 674 \times 1.025^n$.

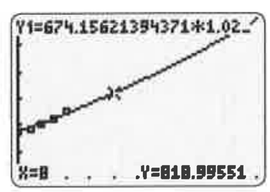
Press (GRAPH) to see the curve of best fit drawn through the scatter plot. Use the window settings shown.



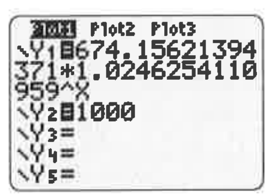
**b)** To predict the average Canadian's weekly earnings in 2010, find the value of $E$ when $n = 8$.

- Press (2nd) [CALC].
- Select **1:value** and enter 8 when prompted.

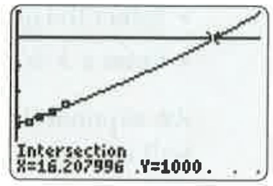



Therefore, assuming that the trend continues, the average Canadian will earn approximately $819 per week in 2010.

**c)** To find when the average Canadian might expect to earn $1000 per week, find the intersection of the graph with the graph of $y = 1000$.



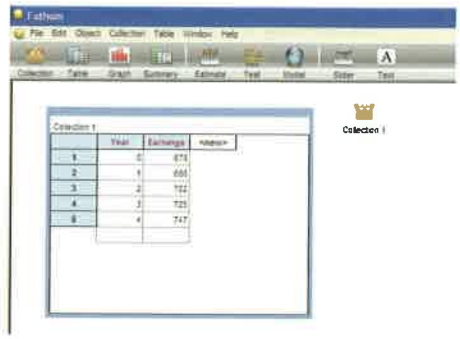
- Press (2nd) [CALC].
- Select **5:intersect**.



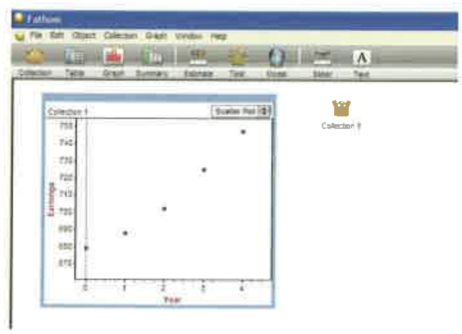

From the graph, you can see that the average Canadian can expect to earn \$1000 per week in approximately 16.2 years after 2002, or in 2018.

**Method 2: Use *Fathom*™**

**a)** Enter the data in a Case Table.



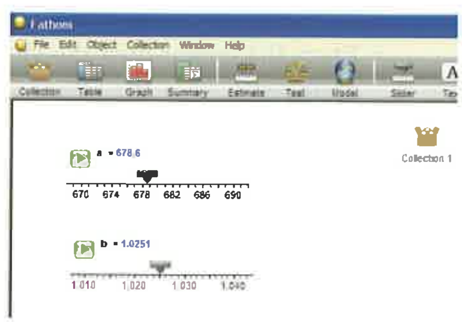
Make a scatter plot of Earnings versus Year by clicking and dragging these attributes to the vertical and horizontal axes.



Construct a curve of best fit of the form $y = ab^x$:

- Create two sliders and name them $a$ and $b$.

Note that placing the cursor over different parts of a slider scale allows you to adjust the scale by clicking and dragging. Consider the domain and range of the given points when adjusting the slider windows.




- $a$ is the initial value, or vertical intercept. Its value appears to be between 670 and 690.
- $b$ is the multiplier that gives each successive value of Earnings. Compare the ratios of successive terms in the data.

  $688 \div 679 = 1.01\ldots$

  $702 \div 688 = 1.02\ldots$

  $725 \div 702 = 1.03\ldots$

  $747 \div 725 = 1.03\ldots$

  The value of $b$ for the curve of best fit will likely be between 1.01 and 1.04.

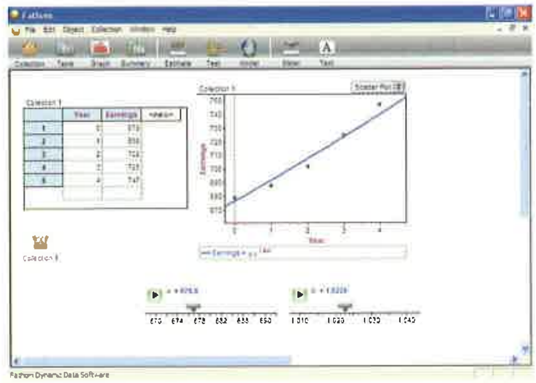
**Technology Tip**

You can find parameters *a* and *b* under the **Global Values** heading and Year under the **Attributes** heading.

- Double-click on these headings to show or hide the list of values.
- Double-click on the value to enter it into the equation.

- Select the graph. From the **Graph** menu, choose **Plot Function**.
- Enter a $\times$ b$^{\text{Year}}$ and click **OK**.

An exponential curve will appear. Adjust the sliders to find the curve of best fit.



Let $n$ represent the number of years following 2002 and $E$ represent the average Canadian's weekly earnings in year $n$. Then, the equation for this curve is $E(n) = 677 \times 1.023^n$.

**b)** Use the equation to predict the average weekly earnings in 2010. Since this is 8 years after the initial value, substitute $n = 8$ into the equation.

$E(n) = 677 \times 1.023^n$

$E(8) = 677 \times 1.023^8$

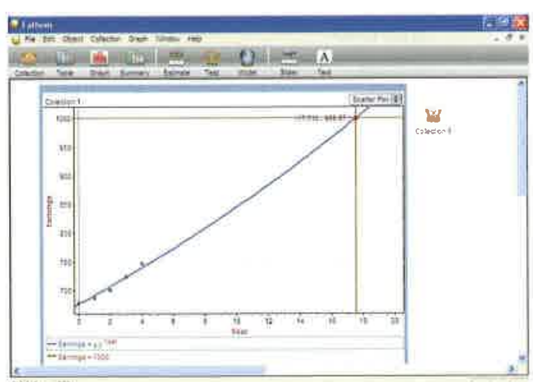
$\doteq 812$

This model predicts that the average Canadian will earn approximately \$812 per week in 2010.

**c)** To predict the year in which the average Canadian can expect to earn \$1000 per week, extrapolate the curve of best fit:

- Click and drag one of the corners of the graph to enlarge it.
- Place the cursor over the vertical scale and adjust it so that 1000 appears.
- Place the cursor over the horizontal scale and adjust it so that you can read the corresponding year.
- From the **Graph** menu, choose **Moveable Line**. Adjust this to be a horizontal line crossing the vertical axis at 1000.

Construct a second moveable line and adjust it to read the corresponding year.



According to the curve of best fit, the average Canadian can expect to earn \$1000 per week approximately 17.5 years after 2002, or some time in the year 2019.

The choice of methods and tools in a given situation can depend on a number of factors, including

- accuracy required
- preference of the user
- availability

Notice from Example 1 that each method and technology tool can produce slight variations in the exponential model that is generated. This, in turn, produces slight variations in the predictions that are made. It is important to realize that such models are often limited in their accuracy.

## Example 2

### Choose a Model of Depreciation

The value of a computer $n$ years after it is purchased is given in the table.

| Number of Years, $n$ | Value ($) |
|---|---|
| 1 | 1200 |
| 2 | 960 |
| 3 | 768 |
| 4 | 614 |
| 5 | 492 |
| 6 | 393 |

**a)** Enter the data in a table using a graphing calculator. Determine the first differences and describe the trend and what it means.

**b)** Make a scatter plot and construct each of the following types of models to represent this relationship:

- linear
- quadratic
- exponential

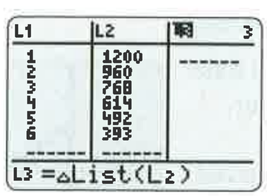
Assess the usefulness of each model.

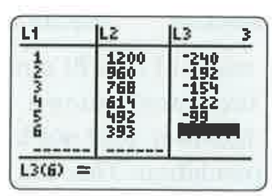
**c)** Determine the most likely purchase price of the computer.

### Solution

**a)** Enter the data in **L1** and **L2** using the **Table Editor**. To calculate the first differences:

- Move the cursor to the top of the **L3** column.
- Press (2nd) [LIST]. Cursor over to **OPS**.
- Select **7:ΔList(** and press (2nd) [L2].
- Press ( ) ) and then press (ENTER).




The first differences are decreasing at a decreasing rate, suggesting that the computer depreciates most quickly in the early years following purchase and less quickly as it ages.

**Technology Tip**

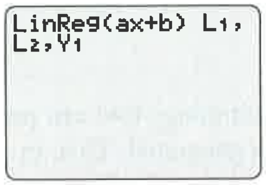
- To perform a regression, press (STAT), cursor over to **CALC**, select the desired type of regression, and enter the appropriate data lists.
- To store the function, press (VARS), select **Y-VARS**, and select the desired function location.
- Remember to use commas to separate data lists and function variables.

**Connections**

You may see values of $r$ and/or $r^2$ after performing a regression. These values are related to how well a regression line or curve fits the data. The closer $r$ is to 1 or −1, and the closer $r^2$ is to 1, the better the fit. To enable the viewing of these parameters, press (2nd) (STAT) 0 for [CATALOG] and then scroll down and select **Diagnostics On**. You will learn more about these parameters if you take Mathematics of Data Management in grade 12.

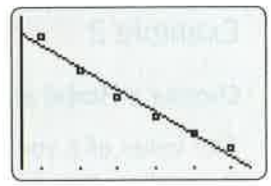
**b) Linear Model**

Perform a linear regression and store the equation as **Y1**.





The line of best fit corresponds approximately to the equation $v(n) = -160n + 1297$, where $v$ is the value of the computer, in dollars, $n$ years after purchase.

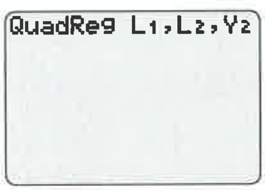


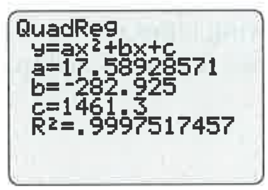
To view the scatter plot and line of best fit, ensure that **Plot1** and **Y1** are turned on. From the **ZOOM** menu, choose **9:ZoomStat.**

Although the line of best fit passes near most of the data points, it does not reflect the curved nature of the trend, which indicates a decreasing rate of depreciation. This may not be the best model for this situation.

**Quadratic Model**

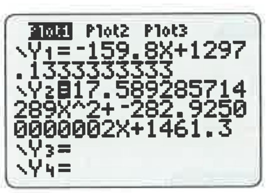
Perform a quadratic regression and store the equation as **Y2**.

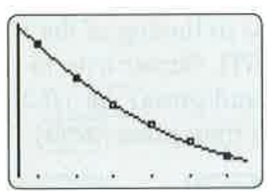


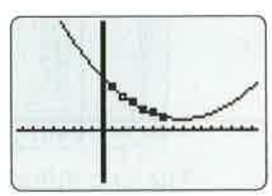




The quadratic curve of best fit corresponds approximately to the equation $v(n) = 17.6n^2 - 283n + 1461$, where $v$ is the value of the computer, in dollars, $n$ years after purchase.

To view the scatter plot and quadratic curve of best fit, turn **Y1** off and ensure that **Plot1** and **Y2** are turned on. From the (ZOOM) menu, select **9:ZoomStat**.



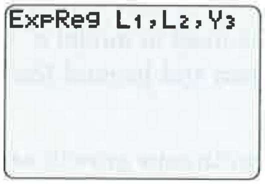


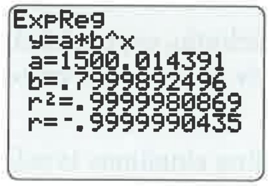
The quadratic curve of best fit models the data trend well for the domain shown. However, this quadratic function does not indicate continuing depreciation. This can be observed by extrapolating beyond the data set. Press (ZOOM), select **3:Zoom Out**, and press (ENTER).

According to the graph, the value function will reach a minimum and then begin to increase, which makes no sense in this situation. Therefore, the quadratic model is not effective for extrapolating beyond the given set of data for this scenario.

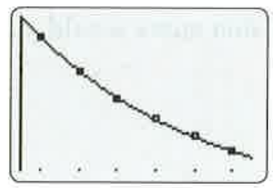
**Exponential Model**

Perform an exponential regression and store the equation as **Y3**.

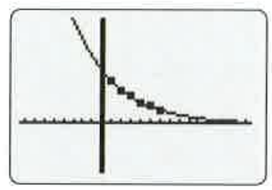



The exponential curve of best fit corresponds approximately to the equation $v(n) = 1500 \times 0.8^n$, where $v$ is the value of the computer, in dollars, $n$ years after purchase.



To view the scatter plot and exponential curve of best fit, turn **Y1** and **Y2** off and ensure that **Plot1** and **Y3** are turned on. From the ZOOM menu, select **9:ZoomStat**.



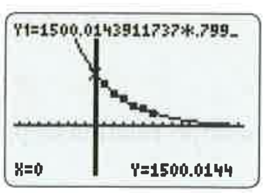
The exponential curve of best fit models the data trend well for the domain shown, and beyond. This can be illustrated by extrapolating beyond the data set. Press ZOOM, select **3:Zoom Out**, and press ENTER.

The exponential model correctly reflects the continuous depreciation of the computer. It is the best model for this scenario.

**c)** Apply the exponential model to determine the purchase price of the computer by evaluating the function when $n = 0$.

**Method 1: Use the Graph**

Press 2nd [CALC] and select **1:value**. When prompted, enter 0 and press ENTER. The corresponding function value will be given.




**Method 2: Use the Equation**

$v(n) = 1500 \times 0.8^n$

$v(0) = 1500 \times 0.8^0$

$= 1500 \times 1$

$= 1500$

Both the graph and the equation indicate that the purchase price of the computer was $1500.

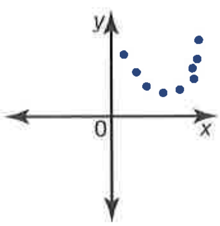
## Key Concepts

- You can use a variety of tools to construct algebraic and graphical models, including
  - a graphing calculator
  - dynamic statistics software such as *Fathom*™
  - a spreadsheet
- Various types of regression (e.g., linear, quadratic, exponential) can be used to model a relationship. The best choice will effectively describe the trend between and beyond the known data values.
- Exponential functions are useful in modelling situations involving continuous growth or decay/depreciation.

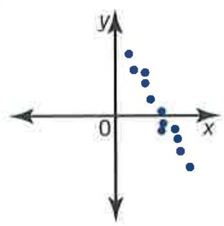
## Communicate Your Understanding

**C1** What type of regression curve would you choose to model each set of data? Explain your choices.

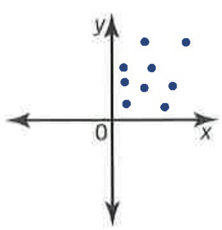
**a)**



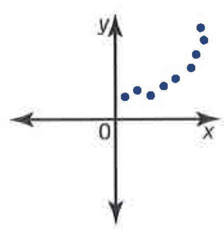

**b)**




**c)**




**d)**




**C2** **a)** Two different technological tools were used to solve Example 1. List the advantages and disadvantages of each tool.

**b)** Explain why there may be slight variations in finding an exponential model to fit real data.

**C3** **a)** How can you tell if a set of data might be modelled by an exponential function by considering

**i)** the numerical data?

**ii)** a scatter plot?

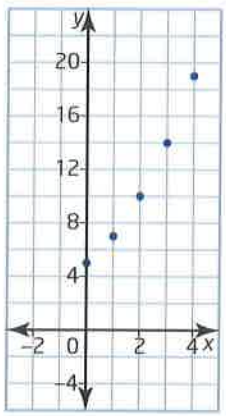
**b)** Describe two situations in which you might expect to see an exponential relationship.

## A Practise

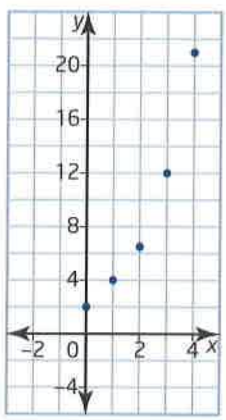
*For help with questions 1 to 4, refer to Example 1.*

**1.** Match each exponential scatter plot with the corresponding equation of its curve of best fit. Not all equations will match one of the graphs.

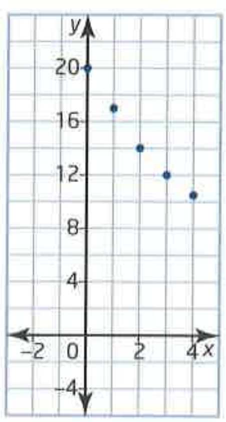
**a)**



**b)**



**c)**



**A** $y = 20 \times 0.85^x$

**B** $y = 5 \times 1.8^x$

**C** $y = 2 \times 1.8^x$

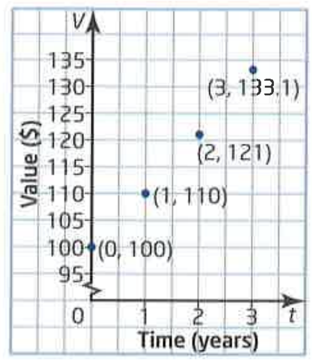
**D** $y = 5 \times 1.4^x$

**E** $y = 20 \times 1.4^x$

**F** $y = 2 \times 1.4^x$

**2.** Pick one of the unmatched equations from question 1. Sketch a scatter plot that the equation could fit.

**3.** Annette has invested some money. The scatter plot shows the value of her investment after the first few years.


**a)** Do the data appear to have an exponential trend? Explain your reasoning.

**b)** Estimate values of $a$ and $b$ to develop an exponential model for the data of the form $V(n) = a \times b^n$. Explain how you arrived at your estimated values.

**c)** Use the tool of your choice to find an exponential model for these data.

**d)** Use the exponential model you produced in part c) to predict the value of Annette's investment after 10 years.

**e)** Approximately how long it will take for Annette's investment to double in value?

### Connections

Data from investment earnings can look linear, but often involve an exponential relationship. You will learn more about investment and loan calculations in Chapter 7 Financial Applications.

*For help with questions 4 and 5, refer to Example 2.*

**4.** At 9 a.m., Gina finds out that she has been promoted to vice-president of sales. By 9:30 a.m., she has told two people in the office about it. By 10 a.m., each person who heard the news at 9:30 has told two other people in the office the news. The news about Gina's promotion continues to spread this way throughout the company.

**a)** Make a table of values to relate the number of people who have just heard the news to time, in half-hour intervals.

**b)** Make a scatter plot. Describe the trend.

**c)** What type of function represents the spread of this news? Justify your answer.

**d)** Determine an equation to model this relation. Explain how you determined the equation.

## B Connect and Apply

**5.** Refer to question 4.

**a)** Do you think the trend will continue indefinitely? Explain why or why not.

**b)** Sketch the graph of this relation for the 24-h period immediately following the moment Gina first heard her news. Assume that 250 people work at Gina's company. Explain each part of the graph.

**6.** Visit the Statistics Canada Web site at http://www.statcan.gc.ca/edu/edu05_0018c-eng.htm to research data on the farm value of potatoes from 1908 to 2004.

**a)** Construct a table with the headings Year, Year Number, and Value of Potatoes. Record the data for every 4 years, starting with 1908 (year number 0) and ending with 2004 (year number 24).

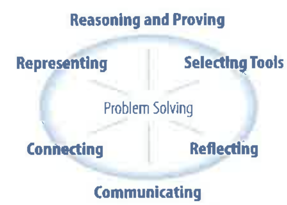
**b)** Make a scatter plot that relates Value of Potatoes to Year Number.

**c)** Determine the equation of the exponential function that best represents the data.

**d)** Graph the curve of best fit.

**7.** Refer to question 6. Pose and answer two questions using the exponential model you created.

**8.** The table shows the koala population in a natural park reserve over a number of years.




**a)** Make a scatter plot for the data. Does the trend suggest an exponential relationship? Explain.

| Year | Population |
|---|---|
| 0 | 800 |
| 1 | 830 |
| 2 | 870 |
| 3 | 900 |
| 4 | 940 |
| 5 | 970 |

**b)** Construct a curve of best fit and find an exponential equation to model the data.

**c)** Predict the koala population after 12 years.

**d)** How long will it take for the koala population to reach 2000? Describe how you found your answer and discuss any assumptions you must make.

**9.** The Consumer Price Index (CPI) is a measure of the cost of living. It is found by tracking the average family's typical living expenses. An upward trend in CPI is called inflation. The table gives the CPI for Canadians over a 7-year period.

| Year | CPI ($) |
|---|---|
| 2002 | 100 |
| 2003 | 102.8 |
| 2004 | 104.7 |
| 2005 | 107 |
| 2006 | 109.1 |
| 2007 | 111.8 |
| 2008 | 115.6 |

a) Construct an exponential function to model the data.

b) Compare this model to the one in Example 1. Which is growing faster: the average Canadian's earnings or the CPI? Explain your reasoning.

**10.** The E-STAT Web site at http://www.statcan.gc.ca contains a vast array of data. Navigate to the E-STAT table of contents and explore some topics of interest to you.

a) Find a set of data that exhibits an exponential relationship. Describe the variables being compared.

b) Make a scatter plot of the data.

c) Find the curve of best fit and its corresponding equation.

d) Pose and answer two questions using the exponential model you created.

**11.** The earnings data from Example 1 can be found on E-STAT. Visit http://www40.statcan.gc.ca/l01/cst01/labr79-eng.htm to view the data. The data used in the example were for all Canadians. Examine the other data. How do the earnings of Ontarians compare with those of

a) all Canadians?

b) residents of other individual provinces?

Write a brief report of your findings.

**12.** How long does it take for a cup of coffee to get cold? For this activity you will need

- a cup of coffee or other hot liquid
- a stir stick
- a thermometer or a temperature probe and graphing calculator

a) Use the thermometer or temperature probe to measure the initial temperature. Record this value along with time, $t = 0$.

b) Take temperature readings each minute for several minutes, stirring the coffee or hot liquid before each measurement. Record the time and temperature data in a table.

c) Make a scatter plot of temperature versus time. Describe the shape of the curve.

d) Find the equation of the curve of best fit.

e) Determine an approximate value for the elapsed time before the temperature reaches the ideal drinking temperature of coffee of 71 °C.

f) How long will it be before the temperature drops to a lukewarm temperature of 30 °C?

**13.** Refer to question 12.

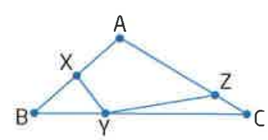

a) Describe the effects on the coffee-cooling curve if cream and sugar are added. Sketch graphs to support your explanation.

b) How would the curve change if a person took sips of the beverage at regular intervals? Sketch graphs to help explain your reasoning.

## C Extend

**14.** Do human populations grow exponentially over time? Find data for Canada, another country, or the world. Find one or more examples of exponential population growth and build models to describe them. Write a brief report of your findings.

**15.** **Math Contest** In the diagram, X is the midpoint of AB, Y divides BC in the ratio 1:2, and Z divides AC in the ratio 1:3. Show that the area of △XBY equals the area of △ZCY.



# Chapter 3 Review

## 3.1 The Nature of Exponential Growth, pages 150 to 157

1. A bacterial colony with an initial population of 300 doubles every day. Which equation models this exponential growth?

   **A** $P = 2 \times 300^n$ **B** $P = 300 \times \left(\frac{1}{2}\right)^n$

   **C** $P = 200 \times 3^n$ **D** $P = 300 \times 2^n$

2. **a)** Use concrete materials or sketches to illustrate an exponential growing pattern that triples from term to term. Draw the first three terms.

   **b)** Make a table of values for the first five terms that relates the number of objects to the term number.

   **c)** Find the first and second differences. How are these patterns related?

   **d)** Write an equation to model your growing pattern.

   **e)** How many objects would you need to build the 10th term?

3. **a)** What is the value of a non-zero number raised to the exponent zero?

   **b)** Use algebraic reasoning to explain why this is true.

## 3.2 Exponential Decay: Connecting to Negative Exponents, pages 160 to 169

4. A radioactive substance with an initial mass of 250 mg has a half-life of 1 year.

   **a)** Write an equation to relate the mass of radioactive material remaining to time.

   **b)** What mass will remain after 10 years?

   **c)** How long will it take for the sample to decay to 20% of its initial mass? Explain how you arrived at your answer.

5. Refer to question 4.

   **a)** Show how you can write the equation from part a) in another way.

   **b)** Explain why the two equations are equivalent.

6. Evaluate. Express as a fraction in lowest terms.

   **a)** $10^{-1}$ **b)** $4^{-2}$ **c)** $3^{-2} + 9^{-1}$

   **d)** $5^{-3} + 5^0$ **e)** $\left(\frac{1}{5}\right)^{-1}$ **f)** $\left(\frac{3}{4}\right)^{-3}$

7. Simplify. Express your answers using only positive exponents.

   **a)** $(x^{-2})(x^{-1})(x^0)$ **b)** $(3km^2)(2k^{-2}m^{-2})$

   **c)** $w^{-3} \div w^{-2}$ **d)** $\dfrac{u^{-2}v^3}{u^{-3}v^{-2}}$

   **e)** $(z^{-3})^{-2}$ **f)** $(2ab^{-1})^{-2}$

## 3.3 Rational Exponents, pages 170 to 177

8. Evaluate.

   **a)** $\sqrt[3]{64}$ **b)** $\sqrt[4]{625}$ **c)** $\sqrt[5]{-3125}$

   **d)** $\left(\frac{1}{64}\right)^{\frac{1}{6}}$ **e)** $27^{\frac{2}{3}}$ **f)** $(-1000)^{\frac{4}{3}}$

   **g)** $-4^{-3}$ **h)** $\left(\frac{3}{4}\right)^{-2}$ **i)** $\left(-\frac{27}{125}\right)^{-\frac{2}{3}}$

9. The length, $x$, in centimetres, by which a spring with spring constant $k$ is stretched or compressed from its rest position is related to its stored potential energy, $U$, in joules (J), according to the equation $x = (2Uk^{-1})^{\frac{1}{2}}$.

   **a)** Use the power of a power rule to write this equation in a different form.

   **b)** Write the equation in radical form, using a single radical.

   **c)** A spring with spring constant 10 has 320 J of stored energy. By how much is this spring stretched?

## 3.4 Properties of Exponential Functions, pages 178 to 187

10. **a)** Graph the function $y = 27\left(\frac{1}{3}\right)^x$.

    **b)** Identify the

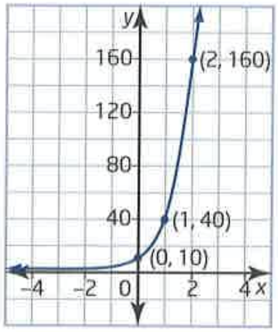
    **i)** domain **ii)** range

    **iii)** $x$- and $y$-intercepts, if they exist

    **iv)** intervals of increase/decrease

    **v)** equation of the asymptote

**11.** Determine the equation for the exponential graph shown.




### 3.5 Transformations of Exponential Functions, pages 188 to 198

**12. a)** Sketch the function $y = 2^{x-3} + 4$.

**b)** Identify the

**i)** domain

**ii)** range

**iii)** equation of the asymptote

**13.** Describe the transformation or transformations that map the base function $y = 5^x$ onto each given function.

**a)** $y = 2(5^x)$ **b)** $y = 5^{2x}$

**c)** $y = -5^{-x}$ **d)** $y = 5^{-5x-10}$

### 3.6 Making Connections: Tools and Strategies for Applying Exponential Models, pages 199 to 209

**14.** The height, $h$, in centimetres, of a bouncing ball after $n$ bounces is given.

| Number of Bounces, $n$ | Height, $h$ (cm) |
|---|---|
| 0 | 100 |
| 1 | 76 |
| 2 | 57 |
| 3 | 43 |
| 4 | 32 |
| 5 | 24 |

**a)** Calculate the first and second differences and describe the trend.

**b)** Make a scatter plot of height versus number of bounces. Describe the shape of the curve.

**c)** Perform an appropriate regression analysis on the data. Write the equation of the curve of best fit. Justify your choice of the type of regression curve.

**d)** Will the ball ever stop bouncing? Discuss this with respect to

**i)** the mathematical model

**ii)** the real situation

**e)** Why might your answers in part d) differ?

## Chapter Problem WRAP-UP

In this chapter, you explored some of the forces that affect celestial bodies. How is it possible to observe these forces in action? One tool that scientists have available is the telescope. The power of modern telescopes is quite impressive compared to those that were first invented.

**a)** Research the history of the telescope. Answer the following questions.
- When was it first invented, and by whom?
- How is magnifying power measured?
- What were the magnifying powers of the first telescopes?
- What are the magnifying powers of some modern telescopes?

**b)** How has the magnifying power of the most powerful telescopes changed over time? Can the increase in magnifying power be modelled by an exponential function? Justify your answer, using words, graphs, and equations.

# Chapter 3 Practice Test

*For questions 1 to 4, select the best answer.*

**1.** One day, 5 friends started a rumour. They agreed that they would each tell the rumour to two different friends the next day. On each day that followed, every person who just heard the rumour would tell another two people who had not heard the rumour. Which equation describes the relation between the number of days that have elapsed, $d$, and the number of people, $P$, who hear the rumour on that day?

**A** $P = 2 \times 5^d$ **B** $P = 5 \times 2^d$

**C** $P = 5 \times \left(\frac{1}{2}\right)^d$ **D** $P = 2 \times \left(\frac{1}{5}\right)^d$

**2.** What is the value of $4^{-\frac{1}{2}}$?

**A** $-2$ **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $\frac{1}{16}$

**3.** Which is the correct equation when $y = 5^x$ is translated 3 units down and 4 units left?

**A** $y = 5^{x+4} - 3$ **B** $y = 5^{x-4} - 3$

**C** $y = 5^{x+3} - 4$ **D** $y = 5^{x-3} - 4$

**4.** Which is correct about exponential functions?

**A** The ratio of successive first differences is constant.

**B** The first differences are constant.

**C** The first differences are zero.

**D** The second differences are constant.

**5.** Evaluate. Express your answers as integers or fractions in lowest terms.

**a)** $49^{\frac{1}{2}}$ **b)** $5^{-3}$ **c)** $(-4)^0$

**d)** $16^{\frac{1}{4}}$ **e)** $(-8)^{\frac{5}{3}}$ **f)** $\left(\frac{3}{4}\right)^{-4}$

**g)** $\left(\frac{27}{64}\right)^{-\frac{1}{3}}$ **h)** $\left(-\frac{8}{125}\right)^{-\frac{4}{3}}$

**6.** Simplify. Express your answers using only positive exponents.

**a)** $(x^{-2})(x^3)(x^{-4})$ **b)** $\frac{p^{-3}}{p^2}$

**c)** $(2k^4)^{-1}$ **d)** $(a^{\frac{1}{2}})(a^{\frac{2}{3}})$

**e)** $(y^{\frac{2}{3}})^{-6}$ **f)** $(u^{\frac{1}{2}}v^{-3})^{-2}$

**7.** Build or sketch a model that shows an exponential growing pattern that is doubling from one term to the next.

**a)** Sketch the first three terms of the model.

**b)** Make a table of values that relates the number of objects to the term number.

**c)** Graph the relation.

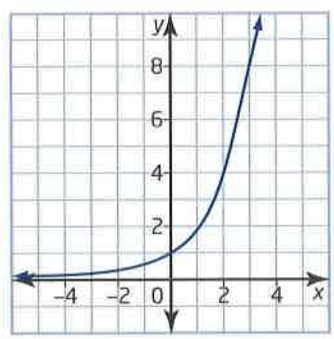
**d)** Find the first and second differences.

**e)** Write an equation for this model.

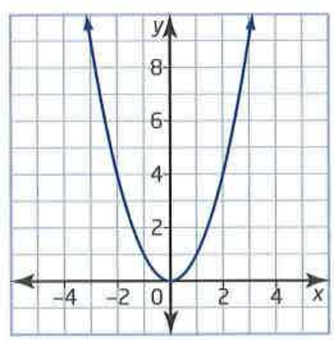
**f)** Give at least three reasons that confirm that the pattern is exponential in nature.

**8.** Match each graph with its corresponding equation.

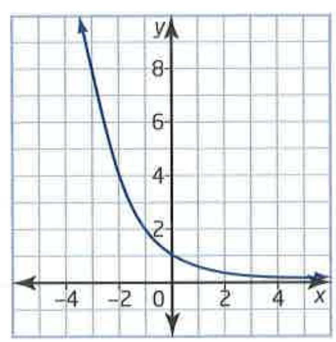
**a)**

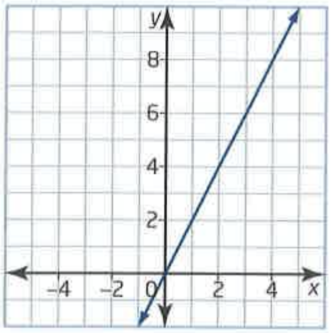



**b)**




**c)**



d) 

A $y = x^2$

B $y = 2x$

C $y = 2^x$

D $y = \left(\frac{1}{2}\right)^x$

**9. a)** Sketch the graph of the function $y = 2^{x-5} + 3$.

**b)** Identify the

**i)** domain

**ii)** range

**iii)** equation of the asymptote

**10.** Describe the transformation(s) that map the base function $y = 8^x$ onto each function.

**a)** $y = \frac{1}{3}(8^x)$

**b)** $y = 8^{4x}$

**c)** $y = -8^{-x}$

**d)** $y = 8^{-3x-6}$

**11.** A radioactive substance with an initial mass of 80 mg has a half-life of 2.5 days.

**a)** Write an equation to relate the mass remaining to time.

**b)** Graph the function. Describe the shape of the curve.

**c)** Limit the domain so that the model accurately describes the situation.

**d)** Find the amount remaining after

**i)** 10 days

**ii)** 15 days

**e)** How long will it take for the sample to decay to 5% of its initial mass?

**12. a)** Sketch the function $y = \left(-\frac{1}{2}\right)2^{x+3} - 1$ by applying transformations to the graph of the base function $y = 2^x$.

**b)** For the transformed function, find the

**i)** domain

**ii)** range

**iii)** equation of the asymptote

**13.** The height, $h$, of a square-based pyramid is related to its volume, $V$, and base side length, $b$, by the equation $h = 3Vb^{-2}$. A square-based pyramid has a volume of 6250 m$^3$ and a base side length of 25 m. Find its height.

**14.** The population of Pebble Valley over a period of 5 years is shown.

| Year | Population |
|---|---|
| 0 | 2000 |
| 1 | 2150 |
| 2 | 2300 |
| 3 | 2500 |
| 4 | 2700 |
| 5 | 2950 |

**a)** Make a scatter plot for this relationship. Do the data appear to be exponential in nature? Explain your reasoning.

**b)** Find the equation of the curve of best fit.

**c)** Limit the domain of the function so that it accurately models this situation.

**d)** Predict the population of Pebble Valley 7 years after the first table entry. State any assumptions you must make.

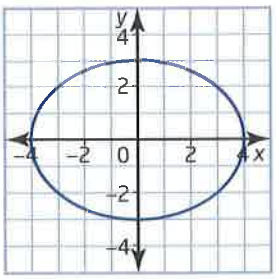
**e)** How long will it take for the town to double in size? State any assumptions you must make.

# Chapters 1 to 3 Review

## Chapter 1 Functions

**1.** State the domain and the range of each relation. Is each relation a function? Justify your answer.

**a)**




**b)** $\{(-2, 1), (-1, 4), (0, 9), (1, 16), (2, 25)\}$

**c)** $y = 0.5x^2 - 4$

**2.** Write each function in mapping notation. Then, for each function, determine $f(-1)$.

**a)** $f(x) = \sqrt{1 - 3x}$

**b)** $f(x) = \dfrac{2x + 1}{x^2 - 4}$

**3.** The amount, $A$, in dollars, to be invested at an interest rate $i$ to have \$1500 after 1 year is given by the relation $A(i) = \dfrac{1500}{1 + i}$. Note that $i$ must be expressed as a decimal.

**a)** Determine the domain and the range for this relation.

**b)** Graph the relation.

**c)** How much money needs to be invested at 3%?

**d)** What rate of interest is required if \$1000 is invested?

**4. a)** Draw the mapping diagram for the given data.

$\{(2, 4), (5, 0), (6, 4), (3, 3), (4, -2)\}$

**b)** Is this relation a function? Explain.

**5.** A farmer has 4000 m of fencing to enclose a rectangular field and subdivide it into three equal plots of land. Determine the dimensions of each plot of land so that the total area is a maximum.

**6.** A store sells T-shirts with logos on them. Last year, the store sold 600 of these T-shirts at \$15 each. The sales manager is planning to increase the price. A survey indicates that for each \$1 increase in the price, 30 fewer T-shirts will be sold per year.

**a)** What price will maximize the yearly revenue?

**b)** What is the maximum yearly revenue?

**7.** Solve each quadratic equation. Give exact answers.

**a)** $2x^2 - 4x - 3 = 0$

**b)** $3x^2 - 12x + 4 = 0$

**8.** Use the discriminant to determine the number of roots for each equation.

**a)** $4x^2 + 3x - 2 = 0$

**b)** $-3x^2 + 10x - 7 = 0$

**c)** $5x^2 - 8x + 1 = 0$

**9.** The length of a rectangle is 2 m more than three times its width. If the area is 20 m$^2$, find the dimensions of the rectangle to the nearest hundredth of a metre.

**10.** Determine the equation in standard form for each quadratic function.

**a)** $x$-intercepts $-3$ and 4, containing the point $(1, -4)$

**b)** $x$-intercepts $2 \pm \sqrt{3}$, $y$-intercept 2

**11.** Water shoots out of a decorative fountain, making an arc in the shape of a parabola. The arc spans a distance of 6 m from one side of the fountain to the other. The height of the arc at a horizontal distance of 1 m from the starting point is 5 m.

**a)** Sketch the quadratic function that represents the arc such that the vertex of the parabola is on the $y$-axis and the horizontal distance from one side to the other is along the $x$-axis.

**b)** Determine the equation for the function.

**c)** Find the maximum height of the arc.

12. Determine the exact point(s) of intersection of each pair of functions.

a) $f(x) = 2x^2 - 3x + 4$ and $g(x) = 2x + 5$

b) $f(x) = -x^2 + 8x + 3$ and $g(x) = -0.5x + 1$

13. For what value of $k$ will the line $y = -5x + k$ be tangent to the graph of the function $f(x) = -4x^2 + 3x + 1$?

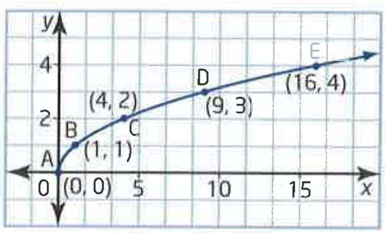
## Chapter 2 Transformations of Functions

14. Determine whether the functions in each pair are equivalent.

a) $f(x) = (2x + 1)(x - 3) - (x + 2)(x - 4)$,
$g(x) = 3(x - 1)^2 - (2x + 1)(x - 2)$

b) $f(x) = (x + 3)(x - 2) + 2(x - 5)(x + 1)$,
$g(x) = -(x - 2)(x + 3) + 2x(1 - x)$

15. Given the graph of a function $f(x)$, sketch the graph of $g(x)$ by determining the image points A′, B′, C′, D′, and E′.



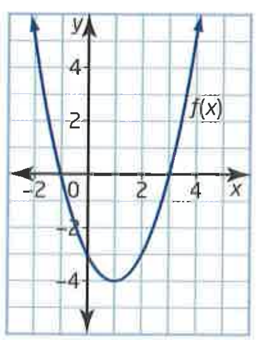

a) $g(x) = f(x) + 4$

b) $g(x) = f(x - 2)$

c) $g(x) = f(x - 6) + 3$

d) $g(x) = f(x + 5) - 1$

16. Copy the graph of $f(x)$ and sketch each reflection, $g(x)$. Then, state the domain and range of each function.




a) $g(x) = f(-x)$

b) $g(x) = -f(x)$

c) $g(x) = -f(-x)$

17. For each function $g(x)$,

i) identify the base function as one of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, or $f(x) = \frac{1}{x}$

ii) describe the transformation in the form $y = f(x - d) + c$ and in words

iii) transform the graph of $f(x)$ to sketch the graph of $g(x)$

iv) state the domain and range of the base function and the transformed function

a) $g(x) = (x + 2)^2 - 1$

b) $g(x) = \sqrt{x + 3} - 4$

c) $g(x) = \dfrac{1}{x - 4} + 6$

d) $g(x) = (x - 7)^2 + 3$

18. David and his friend Shane are in a 20-km bike race. David bikes 1.5 km/h faster than Shane. The time, in hours, to complete the race is given by $t = \frac{d}{v}$, where $d$ is the distance, in kilometres, and $v$ is the speed, in kilometres per hour.

a) If $v$ represents Shane's speed, determine a function to represent David's time. What are the domain and range?

b) Determine a function to represent Shane's time. What are the domain and range?

c) Graph both functions on the same set of axes.

d) Use the graph to determine what would be true about Shane's time if it took David 45 min to complete the race.

19. Determine the equation of each function after

i) a reflection in the $x$-axis, giving $g(x)$

ii) a reflection in the $y$-axis, giving $h(x)$

a) $f(x) = 2x^2 - 7x + 3$

b) $f(x) = \sqrt{x} - 3$

c) $f(x) = \dfrac{1}{x + 2}$

20. Given the function $f(x) = x^2$, identify the value of $a$ or $k$, transform the graph of $f(x)$ to sketch the graph of $g(x)$, and state the domain and range of $g(x)$.

**a)** $g(x) = 2f(x)$ **b)** $g(x) = f(3x)$

**c)** $g(x) = f\left(\frac{x}{4}\right)$ **d)** $g(x) = \frac{1}{3}f(x)$

21. For each function $g(x)$, describe the transformation from a base function of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, or $f(x) = \frac{1}{x}$. Then, transform the graph of $f(x)$ to sketch the graph of $g(x)$.

**a)** $g(x) = 7x$ **b)** $g(x) = \frac{1}{5x}$

**c)** $g(x) = (3x)^2$ **d)** $g(x) = \sqrt{6x}$

22. Describe, in the appropriate order, the transformations that must be applied to the base function $f(x)$ to obtain the transformed function. Then, write the corresponding equation and transform the graph of $f(x)$ to sketch the graph of $g(x)$.

**a)** $f(x) = \sqrt{x}$, $g(x) = 4f(x + 3)$

**b)** $f(x) = x$, $g(x) = -f(5x) - 2$

**c)** $f(x) = x^2$, $g(x) = -3f(2x + 9) - 4$

23. For each function $f(x)$,

**i)** determine $f^{-1}(x)$

**ii)** graph $f(x)$ and its inverse

**iii)** determine whether $f^{-1}(x)$ is a function

**a)** $f(x) = 11x - 3$

**b)** $f(x) = 3x^2 + 4$

**c)** $f(x) = (x + 8)^2 + 19$

**d)** $f(x) = 2x^2 - 3x + 14$

24. Issa works at a furniture store. She earns $450 per week, plus commission of 6% of her sales.

**a)** Write a function to describe Issa's total weekly earnings as a function of her sales.

**b)** Determine the inverse of this function.

**c)** What does the inverse represent?

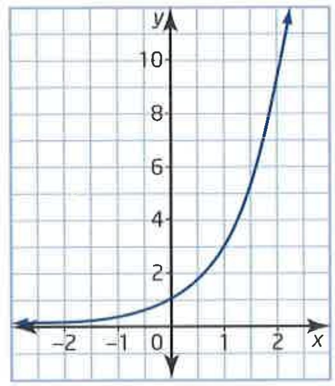
**d)** One week, Issa earned $1020. Calculate her sales that week.

## Chapter 3 Exponential Functions

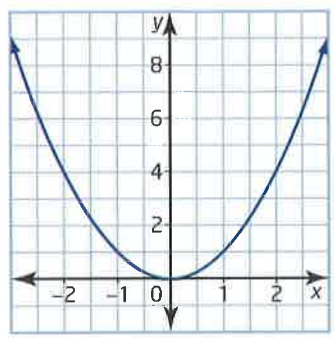
25. Match each graph with one of these equations.

$y = x^2$ $y = 3x$ $y = 3^x$ $y = \left(\frac{1}{3}\right)^x$

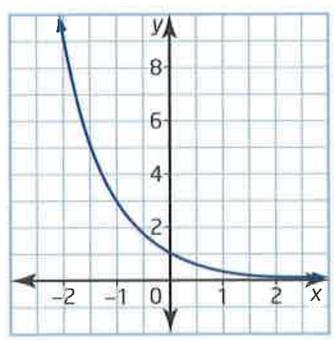
**a)**



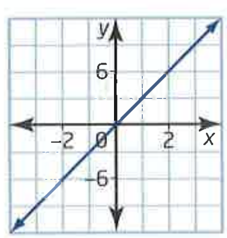
**b)**



**c)**



**d)**



**26.** A bacterial colony with an initial population of 150 triples every day. Which equation models this exponential growth?

**A** $P = 3 \times 150^n$

**B** $P = 150 \times \left(\frac{1}{3}\right)^n$

**C** $P = 150 \times 3^n$

**D** $P = 150 \times 2^n$

**27.** A particular radioactive substance has a half-life of 3 years. Suppose an initial sample has a mass of 200 mg.

**a)** Write the equation that relates the mass of radioactive material remaining to time.

**b)** How much will remain after one decade?

**c)** How long will it take for the sample to decay to 10% of its initial mass? Explain how you arrived at your answer.

**d)** Show how you can write the equation from part a) in another way.

**e)** Explain why the two equations are equivalent.

**28.** Evaluate. Express as a fraction in lowest terms.

**a)** $9^{-1}$ **b)** $5^{-2}$ **c)** $4^{-2} + 16^{-1}$

**d)** $3^{-3} + 3^0$ **e)** $\left(-\frac{1}{5}\right)^{-2}$ **f)** $\left(\frac{2}{3}\right)^{-5}$

**29.** Simplify. Express your answers using only positive exponents.

**a)** $(x^{-3})(x^{-2})(x^0)$ **b)** $(2nm^2)^{-3}(4n^{-2}m^{-2})$

**c)** $a^{-4} \div a^{-5}$ **d)** $\dfrac{m^{-3}n^{-4}}{m^{-2}b^{-1}}$

**e)** $(s^{-4})^{-5}$ **f)** $(3ab^{-3})^{-2}$

**30.** Evaluate. Express any fractions in lowest terms.

**a)** $\sqrt[4]{81}$ **b)** $\sqrt[3]{-1000}$ **c)** $\sqrt[9]{-512}$

**d)** $343^{\frac{1}{3}}$ **e)** $\left(\frac{125}{216}\right)^{\frac{1}{3}}$ **f)** $81^{\frac{3}{4}}$

**g)** $128^{\frac{4}{7}}$ **h)** $-5^{-4}$ **i)** $\left(\frac{2}{3}\right)^{-5}$

**31. a)** Graph the function $y = 64\left(\frac{1}{4}\right)^x$.

**b)** Identify the

**i)** domain **ii)** range

**iii)** $x$- and $y$-intercepts, if they exist

**iv)** intervals of increase/decrease

**v)** equation of the asymptote

**c)** Show how you can write the equation from part a) in another way. Explain why the two equations are equivalent.

**32.** The population of Astro Hill over a period of 6 years is shown.

| Year | Population |
|---|---|
| 0 | 1500 |
| 1 | 1575 |
| 2 | 1654 |
| 3 | 1736 |
| 4 | 1823 |
| 5 | 1914 |
| 6 | 2010 |

**a)** Make a scatter plot for the data. Does the relation appear to be exponential? Explain your reasoning.

**b)** Find the equation of the curve of best fit.

**c)** Limit the domain of the function so that it accurately models this situation.

**d)** Predict the population of Astro Hill 9 years after the first table entry. State any assumptions you must make.

**e)** How long will it take for the town's population to double? State any assumptions you must make.

**33.** A radioactive substance with an initial mass of 100 mg has a half-life of 1.5 days.

**a)** Write an equation to relate the mass remaining to time.

**b)** Graph the function. Describe the shape of the curve.

**c)** Limit the domain so that the model accurately describes the situation.

**d)** Find the amount remaining after

**i)** 8 days **ii)** 2 weeks

**e)** How long will it take for the sample to decay to 3% of its initial mass?

## Task

### Radioactive Isotopes

Radioactive isotopes are used in nuclear medicine to allow physicians to explore bodily structures in patients. Very small quantities of these isotopes, such as iodine-131 (I-131; half-life 8.065 days), are injected into patients. The isotopes are traced through the body so a medical diagnosis can be made. These elements have a very short half-life, so the label on the bottle will state the radioactivity of the element (the amount of the element that remains) at a particular moment in time.

**a)** A bottle of I-131 is delivered to a hospital on September 5. It states on the label that the radioactivity at 12:00 p.m. on September 10 will be 380 megabecquerels (MBq).

**i)** Write a defining equation for the relationship between the amount of radioactivity and the time since the bottle was delivered.

**ii)** What is the radioactive activity at

- 12:00 p.m. on the delivery day?
- 6:00 a.m. on September 25?

**iii)** Sketch the graph of this relationship.

**b)** Effective half-life is the time required for a radioactive isotope contained in a body to reduce its radioactivity by half, due to a combination of radioactive decay and the natural elimination of the isotope from the body. In patients with Graves disease, the effective half-life of I-131 is 62.5% of the normal half-life. In patients with a disease called toxic nodular goitre, the effective half-life is 75% of the normal half-life. Compare these results to the standard rate of decay for I-131 numerically, graphically, and algebraically.

**c)** Research one of the topics below and write a short report on your findings.

- other radioactive isotopes that are used in medical diagnoses or treatments
- diseases that can be treated with radioactive isotopes
- the production and transport of radioactive isotopes
- the training required for medical personnel to use radioactive isotopes