Grade 12 Calculus & Vectors
Unit 2 - Exploring Derivative

DAY	DESCRIPTION	Homework
1	**2.1 Power, Sum & Difference Rules** We are learning to … • verify the power rule for functions of the form $f(x)=x^n$ (where n is a real number) • use the Sum & Difference rules • determine the equation of the tangent or normal line to the graph of a polynomial, or a rational function I am able to… • distinguish between a tangent and normal line of a given function both graphically and algebraically • find the equation of a line either tangent or normal to a curve • solve problems involving tangents and normals	*Textbook:* *Pg. 83 – 86 #1-4, 5a,7, 11, 14a,b, 16a,b, 27-29* *CP: Pg # 7* *Warm up pg. # 8*
2	**2.2 Product Rule** We are Learning to… • justify the product rule for determining derivatives • determine the derivatives of polynomial and rational functions, using the constant, power, sum-and-difference, or product rules for determining derivatives I am able to… • determine the derivatives of polynomial and rational functions, using the product rule, with or without the constant, power, sum-and-difference	*Textbook:* *Pg 93 #2-7(bcd), 9,12a,13a,15,17* *Cp. Pg.# 12*
3	**2.3 The Quotient Rule** We are Learning to… • justify the quotient rule for determining derivatives • determine the derivatives of polynomial and rational functions, using the constant, power, sum-and-difference, product, and quotient rules for determining derivatives I am able to… • determine the derivatives of polynomial and rational functions, using the quotient rule, with or without the constant, power, sum-and-difference or product rules. • solve problems involving tangents and normals	*Textbook:* *Pg124#3,5,6b,e,8,9,14, 18* *Cp. Pg. #16 &16*

4	**2.4 The Chain Rule** We are Learning to... • justify the chain rule for determining derivatives • determine the derivatives of polynomial and rational functions, using the constant, power, sum-and-difference, product, quotient and chain rules for determining derivatives I am able to... • determine the derivatives of polynomial and rational functions, using the constant, power, sum-and-difference, product, quotient and chain rules for determining derivatives • differentiate a function in multiple ways by first simplifying and applying different rules	*Textbook :* *Pg 117 – 118 #2, 8-12, 15-17, 19* *CP. Pg #21*
5	**Chain Rule remix** We are Learning to... • determine the derivatives of polynomial and rational functions, using the constant, power, sum-and-difference, product, quotient and chain rules for determining derivatives I am able to... • find and simplify the derivative of a wide variety of function types	*CP. Pg. #22*
6	**2.5 Higher order derivatives** We are Learning to... • define higher order derivatives and find equations of higher order derivatives using differentiation short cuts • find connections between the graphs of a function and the graphs of its first and second derivatives • make inferences about and connections between position, velocity and acceleration functions both graphically and algebraically • solve problems of rates of change drawn from a variety of applications (including distance, velocity, and acceleration) I am able to... • find the second, third, fourth etc. derivative of a function by applying differentiation rules • solve real world application problems involving velocity and acceleration	*Textbook:* *Pg 106 – 107: #1-2, 5, 7,8,9abcd,10,11a* *CP. Pg #28-30*
7	**Quiz**	
8	**Review**	CP. Pg#31-32
9	**Summative Test**	

M Oct 7 (Day 1)
T Oct 8 (Day 2)

UNIT 2

EXPLORING DERIVATIVES

2. 1 Power, Sum & Difference Rules

<u>Recall: What is a Derivative?</u> The derivative of the function *f* at the number *a* is the slope of the tangent line of the curve $y = f(x)$ at $x = a$. The symbol for the derivative of *f* at the number *a* is $f'(a)$. Putting this together with the answers to our two questions above, we get

> *The derivative of f at the number a is given by*
>
> $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This leads to the definition of the derivative function

> *The derivative of f(x) with respect to x is the function $f'(x)$, where*
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

<u>More About Derivatives</u>

Since the derivative $f'(a)$ can be interpreted as the slope of the tangent at $(a, f(a))$, it follows that the derivative $f'(a)$ can also be considered the instantaneous rate of change of $f(x)$ with respect to *x* when $x = a$.

<u>The Power Rule</u>

> If $f(x) = x^n, n \in R$, then $f'(x) = nx^{n-1}$.
>
> Stated in Leibniz notation, if $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1}$.

Ex. $f(x) = x^{100}$

$f'(x) = 100x^{99}$

$f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}}$

$f'(x) = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

$f(x) = \frac{1}{x}$

$= x^{-1}$

$f'(x) = -x^{-2}$

$= -\frac{1}{x^2}$

<u>The Constant Rule</u>

If $f(x) = k$, $k \in \mathbb{R}$

then $f'(x) = 0$

Leibniz notation, $\dfrac{d(k)}{dx} = 0$

Ex $f(x) = \pi$

$f'(x) = 0$

<u>The Constant Multiple Rule</u>

> If $f(x) = kg(x), k \in R$, then $f'(x) = kg'(x)$.
>
> Stated in Leibniz notation, $\dfrac{d}{dx}(ky) = k\dfrac{dy}{dx}$.

Ex $f(x) = -\frac{3}{4} x^{12}$ $f'(x) = -\frac{3}{4}(12x^{11})$

$= -9x^{11}$

<u>The Sum Rule</u>[1]

> If $f(x) = p(x) + q(x)$, where $p(x)$ and $q(x)$ are both differentiable functions, then $f'(x) = p'(x) + q'(x)$.
>
> Stated in Leibniz notation, $\dfrac{d}{dx}(f(x)) = \dfrac{d}{dx}(p(x)) + \dfrac{d}{dx}(q(x))$.

Ex $f(x) = 3x^4 + 2x^2 - 5 - x^{-2}$

$f'(x) = 12x^3 + 4x - 0 + 2x^{-3}$

$= 12x^3 + 4x + \frac{2}{x^3}$

[1] A corollary of the constant multiple rule and the sum rule is that if *f*(*x*)=*p*(*x*) – *q*(*x*), then *f* '(*x*)=*p* '(*x*)-*q* '(*x*)

Examples

1. Determine the derivatives of each of the following

a) $f(x)=4x^5$

$f'(x) = 20x^4$

b) $g(x)=11x^{\frac{5}{2}}$

$g'(x) = \frac{55}{2}x^{\frac{3}{2}}$

or $= \frac{55}{2}\sqrt{x^3}$

c) $h(x)=4x^3-3\sqrt{x}$

$= 4x^3 - 3x^{\frac{1}{2}}$

$h'(x) = 12x^2 - \frac{3}{2}x^{-\frac{1}{2}}$

or $= 12x^2 - \frac{3}{2\sqrt{x}}$

d) $k(x)=(5x-3)^2$

$k(x) = 25x^2 - 30x + 9$

$k'(x) = 50x - 30$

More to come in 2-4

method 2: Chain Rule

↳ derivative of the outside function × derivative of the inside function

$k'(x) = 2(5x-3)^1 \cdot 5$

$= (10x-6)\cdot 5$

$= 50x - 30$

$k'(x) = \underbrace{2(5x-3)^1}_{\text{derivative of the outside function}} \cdot \underbrace{(5)}_{\text{derivative of the inside function}}$

e) $m(x)=\frac{4x^5-\pi x^7}{5x^3}$

$m(x) = \frac{4}{5}x^2 - \frac{\pi}{5}x^4$

$m'(x) = \frac{8}{5}x - \frac{4\pi}{5}x^3$

method 2: Quotient Rule 2-2B

$y = \frac{u}{v}$

$y' = \frac{u'v - v'u}{v^2}$

method 3: Product Rule 2-2A

$y = u \cdot v$

$y' = u'v + v'u$

$m(x) = (4x^5 - \pi x^7)(5x^3)^{-1}$

f) $n(x)=7x^4+\sqrt{x}-3x^{\frac{3}{2}}-\frac{2}{x^4}-99$

$= 7x^4 + x^{\frac{1}{2}} - 3x^{\frac{3}{2}} - 2x^{-4} - 99$

$h'(x) = 28x^3 + \frac{1}{2}x^{-\frac{1}{2}} - \frac{9}{2}x^{\frac{1}{2}} + 8x^{-5} - 0$

$= 28x^3 + \frac{1}{2\sqrt{x}} - \frac{9\sqrt{x}}{2} + \frac{8}{x^5}$

2. Determine the equation of the tangent to the graph of $y=x^3+2x^2-4x+1$ at $x=4$

$y' = 3x^2 + 4x - 4$

$y'\big|_{x=4} = 3(4)^2 + 4(4) - 4$

$= 60$

$f(4) = 4^3 + 2(4)^2 - 4(4) + 1$

$= 81$

tangent line: $y - 81 = 60(x-4)$

$m = 60$

$(4, 81)$

3. A cubic polynomial function, $f(x) = ax^3 + bx^2 + cx + d$,is given such $f'(0) = 0$, $f'(1) = 5$, $f'(2) = 16$,find $f'(3)$.

$f'(x) = 3ax^2 + 2bx + c$

$f'(0) = 0$

$3a(0)^2 + 2b(0) + c = 0$

$c = 0$

$f'(1) = 5$

$3a(1)^2 + 2b(1) + 0 = 5$

$3a + 2b = 5$ ①

$f'(2) = 16$

$3a(2)^2 + 2b(2) + 0 = 16$

$12a + 4b = 16$ ②

① $\overset{\times 2}{\Rightarrow}$ $10 = 6a + 4b$

$16 = 12a + 4b$

$-6 = -6a$

$a = 1$ ③

sub ③ into ①

$5 = 3(1) + 2b$

$2 = 2b$

$b = 1$

$\therefore f'(x) = 3(1)x^2 + 2(1)x + 0$

$f'(3) = 3(3)^2 + 2(3)$

$= 33$

4. Determine the point(s) where the tangent to the curve f(x) = $x^3 - 6x^2 + 7$:

a) Has a slope of -9

$f'(x) = 3x^2 - 12x$

$-9 = 3x^2 - 12x$

$0 = 3x^2 - 12x + 9$

$0 = 3(x^2 - 4x + 3)$

$0 = 3(x-3)(x-1)$

$\therefore x = \{1, 3\}$

$f(1) = 1^3 - 6(1)^2 + 7$

$= 2$

$f(3) = (3)^3 - 6(3)^2 + 7$

$= -20$

$\therefore$ the two points of tangency occur @ $(1, 2)$ and $(3, -20)$

b) Is horizontal

$f'(x) = 0$

$3x^2 - 12x = 0$

$3x(x-4) = 0$

$\therefore x = \{0, 4\}$

$f(0) = (0)^3 - 6(0)^2 + 7$

$= 7$

$f(4) = (4)^3 - 6(4)^2 + 7$

$= -25$

$\therefore$ @ $(0, 7)$ and $(4, -25)$

5. Find the values of x so that the tangent to $f(x) = \frac{3}{\sqrt[3]{x}}$ is parallel to the line x + 16y + 3 =0

$= 3x^{-\frac{1}{3}}$

$f'(x) = -x^{-\frac{4}{3}}$

$y = -\frac{x}{16} - \frac{3}{16}$

$m = -\frac{1}{16}$

$f'(x) = -x^{-\frac{4}{3}}$

$-\frac{1}{16} = -x^{-\frac{4}{3}}$

$\left(\frac{1}{16}\right)^{-1} = \left(x^{-\frac{4}{3}}\right)^{-1}$

$16 = x^{\frac{4}{3}}$

$(16)^{\frac{3}{4}} = \left(x^{\frac{4}{3}}\right)^{\frac{3}{4}}$

$\left(\pm\sqrt[4]{16}\right)^3 = x$

$(\pm 2)^3 = x$

$x = \pm 8$

continuous

6. Find the values for *a* and *b* so that *f(x)* is **differentiable** for all *x*.

$$f(x)=\begin{cases}-x^3+2x^2+4 & x\le 1\\ ax+b & x>1\end{cases}$$

$$f'(x)=\begin{cases}-3x^2+4x, & x\le 1\\ a, & x>1\end{cases}$$

f(1) is defined

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$-(1)^3+2(1)^2+4 = a(1)+b$

$5 = a+b$ ①

$f'(1^-) = f'(1^+)$

$-3(1)^2+4(1) = a$

$a = 1$ ②

sub ② into ①

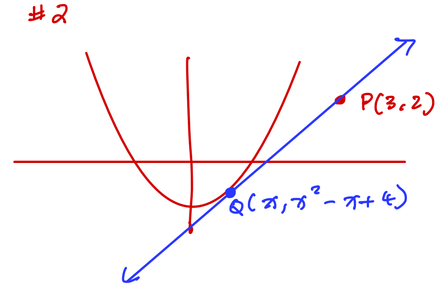
$5 = (1)+b$

$b = 4$

$\therefore a = 1$

$b = 4$

#2



Practice 2.1

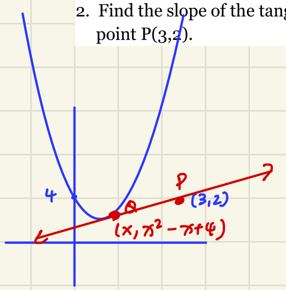
1. Determine the derivatives of each of the following

a) $f(x)=-\frac{3}{4}\sqrt[4]{x^5}-\frac{4}{3\sqrt{x^3}}+\pi^2x^3-\frac{7}{3}$ b) $g(x)=4\sqrt[4]{x^3}\left(\pi\sqrt[5]{x}-2^3\pi\right)$ c) $g(x)=\frac{3x^4+2\pi x^3-5\sqrt[3]{x}}{4x^2}$

2. Find the slope of the tangents to $f(x)=x^2-x+4$ such that they pass through an exterior point P(3,2).
3. Determine the value of a, given that the line ax – 4y + 21 = 0 is tangent to the graph of $y=\frac{a}{x^2}$ at x = -2.
4. The tangent to the cubic function $y = x^3 - 6x^2 + 8x$ at point A (3,-3) intersects the curve at another point, B. Find the coordinates of point B. Illustrate with a sketch.

5. Find the equations of the tangent lines to the parabola $y = x^2 + x$ that pass through the point (2, -3). Sketch the curve and tangents.

2. Find the slope of the tangents to $f(x) = x^2 - x + 4$ such that they pass through an exterior point P(3,2).



Always check if the point is on the function

$m_T = \dfrac{y_2 - y_1}{x_2 - x_1}$ $\qquad f'(x) = 2x - 1$

$= \dfrac{(x^2 - x + 4) - (2)}{x - 3}$

$= \dfrac{x^2 - x + 2}{x - 3}$

$m_T = f'(x)$

$\dfrac{x^2 - x + 2}{x - 3} = 2x - 1$

$x^2 - x + 2 = (2x - 1)(x - 3)$

$x^2 - x + 2 = 2x^2 - 7x + 3$

$0 = x^2 - 6x + 1$

$x = \dfrac{6 \pm \sqrt{36 - 4(1)(1)}}{2(1)}$

$= \dfrac{6 \pm \sqrt{32}}{2}$

$= \dfrac{6 \pm 4\sqrt{2}}{2}$

$= 3 \pm 2\sqrt{2}$

$\therefore f'(3 + 2\sqrt{2}) = 2(3 + 2\sqrt{2}) - 1$

$= 5 + 4\sqrt{2}$

$f'(3 - 2\sqrt{2}) = 2(3 - 2\sqrt{2}) - 1$

$= 5 - 4\sqrt{2}$

Practice 2.1

1. Determine the derivatives of each of the following

a) $f(x)=-\frac{3}{4}\sqrt[4]{x^5}-\frac{4}{3\sqrt{x^3}}+\pi^2x^3-\frac{7}{3}$ b) $g(x)=4\sqrt[4]{x^3}\left(\pi\sqrt[5]{x}-2^3\pi\right)$ c) $g(x)=\frac{3x^4+2\pi x^3-5\sqrt[3]{x}}{4x^2}$

a) $f(x) = -\frac{3}{4}x^{\frac{5}{4}} - \frac{4}{3}x^{\frac{-3}{2}} + \pi^2x^3 - \frac{7}{3}$

$f'(x) = -\frac{15}{16}x^{\frac{1}{4}} + 2x^{\frac{-5}{3}} + 3\pi^2x^2$

$= \frac{-15}{16\sqrt[4]{x}} + \frac{2}{\sqrt[3]{x^5}} + 3\pi^2x^2$

b) $g(x) = 4(x^{\frac{3}{4}})(\pi x^{\frac{1}{5}} - 8\pi)$

$= 4\pi x^{\frac{19}{20}} - 32\pi x^{\frac{3}{4}}$

$g'(x) = \frac{19\pi}{5}x^{\frac{-1}{20}} - 24\pi x^{\frac{-1}{4}}$

$= \frac{19\pi}{5\sqrt[20]{x}} - \frac{24\pi}{\sqrt{x^4}}$

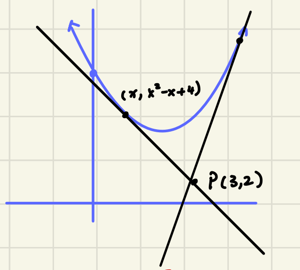
c) $g(x) = \frac{3x^4 + 2\pi x^3 - 5x^{\frac{1}{3}}}{4x^2}$

$= \frac{3}{4}x^2 + \frac{\pi}{2}x - \frac{5}{4}x^{\frac{-5}{3}}$

$g'(x) = \frac{3}{2}x + \frac{\pi}{2} + \frac{25}{12}x^{-\frac{8}{3}}$

or $= \frac{3}{2}x + \frac{\pi}{2} + \frac{25}{12\sqrt[3]{x^8}}$

2. Find the slope of the tangents to $f(x)=x^2-x+4$ such that they pass through an exterior point P(3,2).



$m_T = \frac{(x^2-x+4)-(2)}{x-3}$

$= \frac{x^2-x+2}{x-3}$

$f'(x) = 2x - 1$

$\frac{x^2-x+2}{x-3} = 2x-1$

$x^2-x+2 = (2x-1)(x-3)$

$x^2-x+2 = 2x^2-7x+3$

$0 = x^2-6x+1$

$x = \frac{6\pm\sqrt{36-4(1)(1)}}{2(1)}$

$= \frac{6\pm\sqrt{32}}{2}$

$= \frac{6\pm4\sqrt{2}}{2}$

$= 3\pm2\sqrt{2}$

$f'(3+2\sqrt{2}) = 2(3+2\sqrt{2})-1$

$= 4\sqrt{2}+5$

$f'(3-2\sqrt{2}) = 2(3-2\sqrt{2})-1$

$= -4\sqrt{2}+5$

$\therefore$ there are two possible slopes: $\pm4\sqrt{2}+5$

3. Determine the value of a, given that the line ax – 4y + 21 = 0 is tangent to the graph of $y = \frac{a}{x^2}$ at x = -2.

$f(x) = ax^{-2}$
$f'(x) = -2ax^{-3}$
$f'(-2) = -2a(-2)^{-3}$
$= \frac{-2a}{-8}$
$= \frac{a}{4}$

$l:\ ax - 4y + 21 = 0$
$ax + 21 = 4y$
$\frac{ax+21}{4} = y$
$\therefore m_T = \frac{a}{4}$

$a(-2) - 4y + 21 = 0$
$-2a - 4y + 21 = 0$
$y = \frac{2a-21}{-4}$

pt: $\left(-2, \frac{2a-21}{-4}\right)$

Find POI:

$$\begin{cases} y = \frac{a}{x^2} & ① \\ ax - 4y + 21 = 0 & ② \end{cases}$$

@ $x = -2$,
$ax^{-2} = \frac{ax+21}{4}$
$\frac{a}{x^2} = \frac{ax+21}{4}$
$\frac{a}{(-2)^2} = \frac{a(-2)+21}{4}$
$a = -2a + 21$
$3a = 21$
$a = 7$

Calculus is not needed in this question. ☺ < only grade 10 Systems of Equations >

4. The tangent to the cubic function y = x³ – 6x² + 8x at point A (3,-3) intersects the curve at another point, B. Find the coordinates of point B. Illustrate with a sketch.

$f(x) = x^3 - 6x^2 + 8x$
$f'(x) = 3x^2 - 12x + 8$

$f'(3) = 3(3)^2 - 12(3) + 8$
$= -1$

Equation of tangent line:

$m = -1$
$A(3,-3)$

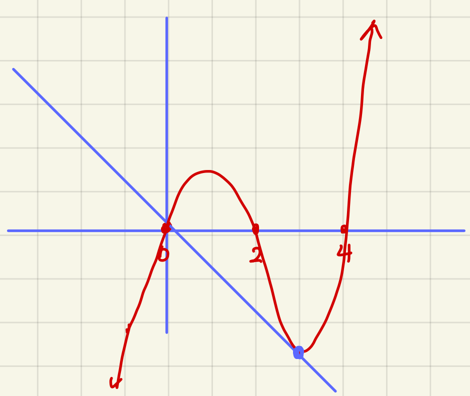
$y + 3 = -1(x-3)$
$y = -x$

$$\begin{cases} y = x^3 - 6x^2 + 8x & ① \\ y = -x & ② \end{cases}$$

sub ② into ①
$-x = x^3 - 6x^2 + 8x$
$0 = x^3 - 6x^2 + 9x$
$0 = x(x^2 - 6x + 9)$
$0 = x(x-3)^2$
$\therefore x = \{0, 3\}$
$\therefore$ other point is @ $x = 0$,
$f(0) = 0$

$\therefore$ the point B is @ $(0,0)$

$f(x) = x^3 - 6x^2 + 8x$
$= x(x^2 - 6x + 8)$
$= x(x-4)(x-2)$



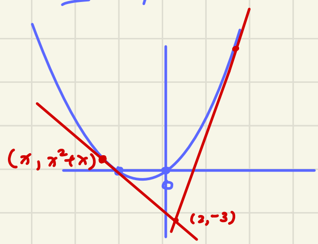
5. Find the equations of the tangent lines to the parabola y = x² + x that pass through the point (2, -3). Sketch the curve and tangents.

$f(x) = x^2 + x = x(x+1)$

$f(2) = (2)^2 + (2)$

$= 6$

$\therefore$ (2, -3) is not a point on $f(x)$



$f'(x) = 2x + 1$

$\frac{x^2+x+3}{x-2} = 2x+1$

$x^2+x+3 = (2x+1)(x-2)$

$x^2+x+3 = 2x^2 - 3x - 2$

$0 = x^2 - 4x - 5$

$0 = (x-5)(x+1)$

$\therefore x = \{-1, 5\}$

$m_T = \frac{(x^2+x) - (-3)}{x-2}$

$= \frac{x^2+x+3}{x-2}$

l_1: $f'(-1) = 2(-1)+1$

$= -1$

$f(-1) = (-1)^2 + (-1)$

$= 0$

$m = -1 \quad (-1, 0)$

$\therefore y - 0 = -(x+1)$

l_2: $f'(5) = 2(5)+1$

$= 11$

$f(5) = (5)^2 + (5)$

$= 30$

$m = 11 \quad (5, 30)$

$\therefore y - 30 = 11(x-5)$

$\therefore$ the 2 possible tangent lines:

$y = -(x+1)$

$y - 30 = 11(x-5)$

Warm-up

1. Differentiate the following. Express the answers with positive exponents.

or chain rule $(2x)^{\frac{1}{2}} = \sqrt{2}\,x^{\frac{1}{2}}$

a) $\mathbf{f(x) = x^{-4} - \sqrt{2x} - \frac{5x}{\sqrt{x}} + 8^2}$

$= x^{-4} - \sqrt{2}\,x^{\frac{1}{2}} - 5x^{\frac{1}{2}} + 64$

$f'(x) = -4x^{-5} - \frac{\sqrt{2}}{2}x^{-\frac{1}{2}} - \frac{5}{2}x^{-\frac{1}{2}}$

$= \frac{-4}{x^5} - \frac{\sqrt{2}}{2\sqrt{x}} - \frac{5}{2\sqrt{x}}$

b) $\mathbf{g(x) = \frac{\left(x^{-1}+1\right)^2}{\sqrt[3]{x^2}}}$

more effient method than the Quotient Rule when the divisor is a monomial

$= \frac{x^{-2} + 2x^{-1} + 1}{x^{\frac{2}{3}}}$

$= x^{\frac{-8}{3}} + 2x^{\frac{-5}{3}} + x^{\frac{-2}{3}}$

$g'(x) = -\frac{8}{3}x^{\frac{-11}{3}} - \frac{10}{3}x^{\frac{-8}{3}} - \frac{2}{3}x^{\frac{-5}{3}}$

or $= \frac{-8}{3\sqrt[3]{x^{11}}} - \frac{10}{3\sqrt[3]{x^8}} - \frac{2}{3\sqrt[3]{x^5}}$

2. Line $y = 4x + k$ is tangent to the graph of function $f(x) = ax^3 + kx + 1$ at $x = 1$. Find the values of a and k.

tangent line: $y = 4x + k$

$\therefore m_T = 4$

sub $x = 1$,

$y = 4(1) + k$

$y = 4 + k$

Point of tangency: $(1, 4+k)$

$f(1) = a(1)^3 + k(1) + (1)$

$4 + k = a + k + 1$

$a = 3$

$f'(x) = 3ax^2 + k$

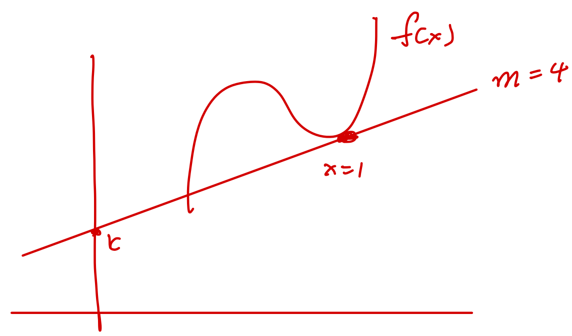
$f'(1) = 3a(1)^2 + k$

$4 = 3a + k$

sub $a = 3$

$4 = 3(3) + k$

$\therefore k = -5$



$\therefore a = 3$
$k = -5$

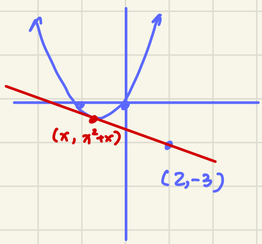
Hmwk takeup CP7

5. Find the equations of the tangent lines to the parabola y = x² + x that pass through the point (2, -3). Sketch the curve and tangents.

$f(x) = x^2 + x$
$= x(x+1)$

Check: $f(2) = 2^2 + 2$
$= 6$
$\neq -3$

$\therefore$ $P(2,-3)$ is off the curve.



$f'(x) = 2x + 1$

$m_T = \dfrac{(x^2+x)-(-3)}{x-2}$

$f'(x) = m_T$

$2x+1 = \dfrac{x^2+x+3}{x-2}$

$(2x+1)(x-2) = x^2+x+3$

$2x^2 - 3x - 2 = x^2 + x + 3$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$\therefore x = \{-1, 5\}$

$f'(-1) = 2(-1) + 1$
$= -1$

$f'(5) = 2(5) + 1$
$= 11$

$f(-1) = (-1)^2 + (-1)$
$= 0$

$f(5) = (5)^2 + (5)$
$= 30$

$m = -1$ $(-1, 0)$

$m = 11$ $(5, 30)$

Equation of tangents:

$y - 0 = -1(x+1)$
or $y = -x - 1$

$y - 30 = 11(x-5)$
$y = 11x - 55 + 30$
or $y = 11x - 25$

Warmup - (Vincent CP)

Examples: Differentiate

Aside:

1
1 1
1 2 1
1 3 3 1

a) $f(x)=(1-2x)^3$

$= 1^3 + 3(1)^2(-2x)^1 + 3(1)(-2x)^2 + 1(-2x)^3$

$= 1 - 6x + 12x^2 - 8x^3$

$f'(x) = -6 + 24x - 24x^2$

Method 2: Chain Rule

$y'(x) = 3(1-2x)^2 \cdot (-2)$

$= 3(1-4x+4x^2)(-2)$

$= -6(1-4x+4x^2)$

$= -6 + 24x - 24x^2$

b) $f(x)=(x+2)(x^2-1)$

$= x^3 - x + 2x^2 - 2$

$= x^3 + 2x^2 - x - 2$

$f'(x) = 3x^2 + 4x - 1$

Method 2: Product Rule

if $y = uv$ $\quad y' = u'v + v'u$

$f'(x) = [1][x^2-1] + [2x][x+2]$

$= x^2 - 1 + 2x^2 + 4x$

$= 3x^2 + 4x - 1$

c) $s=\dfrac{3t^5-2t^2+5t^{-1}}{t^2}$

$S = 3t^3 - 2 + 5t^{-3}$

$S' = 9t^2 - 15t^{-4}$

or $= 9t^2 - \dfrac{15}{t^4}$

$= \dfrac{9t^6 - 15}{t^4}$

method 2: Quotient Rule

if $y = \frac{u}{v}$ then $y' = \dfrac{u'v - v'u}{v^2}$

$s' = \dfrac{[15t^4 - 4t - 5t^{-2}][t^2] - [2t][3t^5 - 2t^2 + 5t^{-1}]}{[t^2]^2}$

$= \dfrac{15t^6 - \cancel{4t^3} - 5 - 6t^6 + \cancel{4t^3} - 10}{t^4}$

$= \dfrac{9t^6 - 15}{t^4}$

d) $u=\sqrt{\dfrac{2}{x}}+\sqrt{\dfrac{x}{3}}$

$u = \sqrt{2}\, x^{-\frac{1}{2}} + \frac{1}{\sqrt{3}} x^{\frac{1}{2}}$

$\dfrac{du}{dx} = -\dfrac{\sqrt{2}}{2} x^{-\frac{3}{2}} + \dfrac{1}{2\sqrt{3}} x^{-\frac{1}{2}}$

or $= -\dfrac{\sqrt{2}}{2\sqrt{x^3}} + \dfrac{\sqrt{3}}{6\sqrt{x}}$ ← Rationalize denominator

or $= -\frac{1}{2}\sqrt{\dfrac{2}{x^3}} + \frac{1}{6}\sqrt{\dfrac{3}{x}}$

$= (2x^{-1})^{\frac{1}{2}} + \left(\frac{x}{3}\right)^{\frac{1}{2}}$

$= \sqrt{2}\, x^{-\frac{1}{2}} + \frac{1}{\sqrt{3}} x^{\frac{1}{2}}$

$u' = -\frac{\sqrt{2}}{2} x^{-\frac{3}{2}} + \frac{1}{2\sqrt{3}} x^{-\frac{1}{2}}$

$= -\dfrac{\sqrt{2}}{2\sqrt{x^3}} + \dfrac{1}{2\sqrt{3}\sqrt{x}}$

$= \dfrac{-\sqrt{2}}{2\sqrt{x^3}} + \dfrac{1}{2\sqrt{3x}}$

2. 2 Product Rule

The Product Rule $y' = u'v + v'u$

If $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + f(x)g'(x)$.

Restated in Leibniz notation,

If u and v are functions of x, $\dfrac{d}{dx}(uv) = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$

Proof of the Product Rule

Suppose $p(x) = f(x)g(x)$. Then

$$p'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{(x+h) - x}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) + 0 - f(x)g(x)}{h}$$

Just like multiplying by 1 is a powerful tool in math, so is adding 0
In this case, $0 = -f(x)g(x+h) + f(x)g(x+h)$
Who said mathematicians aren't creative? ☺

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

Next, we'll factor out $g(x+h)$ from the first two terms of the numerator, and we'll factor out $f(x)$ from the last two terms of the numerator

$$= \lim_{h\to 0}\left\{\left[\frac{f(x+h) - f(x)}{h}\right]g(x+h) + f(x)\left[\frac{g(x+h) - g(x)}{h}\right]\right\}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]\lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \lim_{h\to 0}\left[\frac{g(x+h) - g(x)}{h}\right]$$

$$= f'(x)g(x) + f(x)g'(x)$$

Examples

1. Differentiate $h(x) = (x^3 - 2x)(3x^4 + 2x + 8)$ using the product rule

$$h'(x) = [3x^2 - 2][3x^4 + 2x + 8] + [12x^3 + 2][x^3 - 2x]$$

$$= (9x^6 + 6x^3 + 24x^2 - 6x^4 - 4x - 16) + (12x^6 - 24x^4 + 2x^3 - 4x)$$

$$= 21x^6 - 30x^4 + 8x^3 + 24x^2 - 8x - 16$$

2. Find the value of $f'(-1)$ for the function $f(x) = (3x^4 - 12x^2 + 4x - 9)(6x^7 - 4x^4 + 18)$

$f'(x) = [12x^3 - 24x + 4][6x^7 - 4x^4 + 18] + [42x^6 - 16x^3][3x^4 - 12x^2 + 4x - 9]$

note! $\Longrightarrow$ $= (72x^{10} - 48x^7 + 216x^3 + 96x^5 - 432x + 24x^7 - 16x^4 + 72) +$

$f(-1)$ doesn't require us to simplify the expression ☺

$(126x^{10} - 504x^8 + 168x^5 - 378x^6 - 48x^7 + 192x^5 - 64x^4 + 144x^3)$

$f'(-1) = [12(-1)^3 - 24(-1) + 4][6(-1)^7 - 4(-1)^4 + 18] + [42(-1)^6 - 16(-1)^3][3(-1)^4 - 12(-1)^2 + 4(-1) - 9]$

$= (16)(8) + (58)(-22)$

$= -1148$

Find an expression for $p'(x)$ if $p(x) = f(x)g(x)h(x)$

$p(x) = f(x)g(x)h(x)$

$= [f(x)g(x)]h(x)$

$p'(x) = [f'(x) \cdot g(x) + g'(x) \cdot f(x)][h(x)] + [h'(x)][f(x) \cdot g(x)]$

$= f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$

$$\boxed{p'(x) = f'(x) \cdot g(x)\, h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)}$$

This is called the ***extended product rule*** for three functions

3. Differentiate the rational function $f(x) = (\sqrt{x})(2x+5)(x-1)$ by using the extended product rule.

$= (x^{\frac{1}{2}})(2x+5)(x-1)$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(2x+5)(x-1) + \sqrt{x}\,(2)(x-1) + \sqrt{x}\,(2x+5)(1)$

or $= \dfrac{(2x+5)(x-1)}{2\sqrt{x}} + 2\sqrt{x}(x-1) + \sqrt{x}\,(2x+5)$ $\Leftarrow$ with positive exponents only

4. If $g(x)=x^2f(x)$, $f(2)=-2$ and $g'(2)=8$,then determine $f'(2)$.

$g(x) = x^2 f(x)$

$g'(x) = [2x][f(x)] + [f'(x)][x^2]$

$g'(2) = [2(2)][f(2)] + [f'(2)][2^2]$

$8 = 4(-2) + f'(2)(4)$

$8 = -8 + f'(2) \cdot 4$

$16 = f'(2) \cdot 4$ $\qquad \therefore f'(2) = 4$

$4 = f'(2)$

The Power of a Function Rule for Integers (AKA : Chain Rule)

> If u is a function of x, and n is an integer, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$
> In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1}g'(x)$
> We will prove a more general statement of this (the Chain Rule) in section 2.5.

Ex. $y = (3x^4 - 2x^3 - 4x^2)^8$

$y' = 8(3x^4 - 2x^3 - 4x^2)^7 \cdot (12x^3 - 6x^2 - 8x)$

Examples

1. Determine $h'(x)$ where $h(x)=(4x^2-3x+1)^7$. Then, evaluate $h'(1)$.

$h'(x) = 7(4x^2 - 3x + 1)^6 \cdot (8x - 3)$

$h'(1) = 7(4(1)^2 - 3(1) + 1)^6 \cdot (8(1) - 3)$

$= 7(2)^6(5)$

$= 2240$

2. Find the derivative of $g(x)=(3x^2-5)^6(2x^3+1)^4$.

$g'(x) = [6(3x^2-5)^5 \cdot (6x)][(2x^3+1)^4] + [4(2x^3+1)^3 \cdot (6x^2)][(3x^2-5)^6]$

$= 36x(3x^2-5)^5(2x^3+1)^4 + 24x^2(2x^3+1)^3(3x^2-5)^6$

$= 12x(3x^2-5)^5(2x^3+1)^3[3(2x^3+1) + 2x(3x^2-5)]$

$= 12x(3x^2-5)^5(2x^3+1)^3[6x^3 + 3 + 6x^3 - 10x]$

$= 12x(3x^2-5)^5(2x^3+1)^3(12x^3 - 10x + 3)$

Practice 2.2

1. Use the product rule to differentiate the following functions.
   a. $g(x)=(x^2-1)(x^2+2x)$
   b. $h(t)=\sqrt[3]{t}\,(t^2+5)$
2. Determine the value of $\dfrac{dy}{dx}$ at the given value of ~~xx~~ x.
   a. $y=(x^3+1)(2x^2-5),\ x=1$
   b. $y=x(2x+1)(x^2+x),\ x=0$
3. Given g(2)=4,g′(2)=−1,h(2)=−2, and h′(2)=3, find f′(2) if f(x)=g(x)h(x).
4. Find the x and y coordinates of all points on the graph of $y=(x-1)(x^2+1)$ where the tangent line is parallel to the line y=9x−6.
5. Find the equation of the tangent line to the curve $f(x)=(x^4+x^3+x^2+x+1)(x^3+x+2)$ at the point on the curve whose x-coordinate is −1.

Warm Up

Find the equation of the tangent line to the graph of $f(x)=(x^2-3)^8(x^3+9)^6$ at the point on the curve whose x-coordinate is −2.[Ans: y = 40x + 81]

$f(x) = (x^2-3)^8(x^3+9)^6$

$f'(x) = [8(x^2-3)^7(2x)][(x^3+9)^6] + [6(x^3+9)^5(3x^2)][(x^2-3)^8]$ ⇐ unsimplified

technically you need not simplify

$f'(x) = 16x(x^2-3)^7(x^3+9)^6 + 18x^2(x^3+9)^5(x^2-3)^8$

$= 2x(x^2-3)^7(x^3+9)^5\,[8(x^3+9) + 9x(x^2-3)]$

$= 2x(x^2-3)^7(x^3+9)^5(8x^3+72+9x^3-27x)$

$= 2x(x^2-3)^7(x^3+9)^5(17x^3-27x+72)$ ⇐ Simplified factored form

"Fully factored form"

$f'(-2) = 2(-2)(4-3)^7(-8+9)^5(17(-8)-27(-2)+72)$

$= 40$

$f(-2) = ((-2)^2-3)^8((-2)^3+9)^6$

$= 1$

(−2, 1) is the point tangency

Equation of tangent: $y-1 = 40(x+2)$

$m = 40 \quad (-2,1)$

Practice 2.2

1. Use the product rule to differentiate the following functions.
    a. $g(x)=(x^2-1)(x^2+2x)$
    b. $h(t)=\sqrt[3]{t}(t^2+5)$
2. Determine the value of $\dfrac{dy}{dx}$ at the given value of xx.
    a. $y=(x^3+1)(2x^2-5), x=1$
    b. $y=x(2x+1)(x^2+x), x=0$
3. Given $g(2)=4, g'(2)=-1, h(2)=-2$, and $h'(2)=3$, find $f'(2)$ if $f(x)=g(x)h(x)$.
4. Find the x and y coordinates of all points on the graph of $y=(x-1)(x^2+1)$ where the tangent line is parallel to the line $y=9x-6$.
5. Find the equation of the tangent line to the curve $f(x)=(x^4+x^3+x^2+x+1)(x^3+x+2)$ at the point on the curve whose x-coordinate is -1.

Warm Up

Find the equation of the tangent line to the graph of $f(x)=(x^2-3)^8(x^3+9)^6$ at the point on the curve whose x-coordinate is -2.[Ans: $y = 40x + 81$]

Practice 2.2

1. Use the product rule to differentiate the following functions.
   a. $g(x)=(x^2-1)(x^2+2x)$
   b. $h(t)=\sqrt[3]{t}\,(t^2+5)$

1a) $g(x) = (x^2-1)(x^2+2x)$

$g'(x) = [2x][x^2+2x] + [2x+2][x^2-1]$

$= 2x^3 + 4x^2 + 2x^3 - 2x + 2x^2 - 2$

$= 4x^3 + 6x^2 - 2x - 2$

b) $h(x) = t^{\frac{1}{3}}(t^2+5)$

$h'(x) = \left[\frac{1}{3}t^{-\frac{2}{3}}\right][t^2+5] + [2t]\left[t^{\frac{1}{3}}\right]$

$= \frac{1}{3}t^{\frac{4}{3}} + \frac{5}{3}t^{-\frac{2}{3}} + 2t^{\frac{4}{3}}$

$= \frac{7}{3}t^{\frac{4}{3}} + \frac{5}{3}t^{-\frac{2}{3}}$

$= \frac{7\sqrt[3]{t^4}}{3} + \frac{5}{3\sqrt[3]{t^2}}$

2. Determine the value of $\frac{dy}{dx}$ at the given value of ~~xx.~~ x
   a. $y=(x^3+1)(2x^2-5),\ x=1$
   b. $y=x(2x+1)(x^2+x),\ x=0$

I would have expanded first

2 a) $y = (x^3+1)(2x^2-5)\ ,\quad x=1$

$\frac{dy}{dx} = [3x^2][2x^2-5] + [4x][x^3+1]$

$\left.\frac{dy}{dx}\right|_{x=1} = [3(1)^2][2(1)^2-5] + [4(1)][(1)^3+1]$

$= (3)(-3) + (4)(2)$

$= -1$

$y = (x^3+1)(2x^2-5)$

$= 2x^5 - 5x^3 + 2x^2 - 5$

$\frac{dy}{dx} = 10x^4 - 15x^2 + 4x$

$\left.\frac{dy}{dx}\right|_{x=1} = 10(1)^4 - 15(1)^2 + 4(1)$

$= -1$

silly to use product Rule

b) $y = x(2x+1)(x^2+x)$

$\frac{dy}{dx} = 1(2x+1)(x^2+x) + (x)(2)(x^2+x) + (x)(2x+1)(2x+1)$

$= 2x^3 + 2x^2 + x^2 + x + 2x^3 + 2x^2 + 4x^3 + 4x^2 + x$

$= 8x^3 + 9x^2 + 2x$

$\left.\frac{dy}{dx}\right|_{x=0} = 0$

$y = x(2x+1)(x^2+x)$

$= (2x^2+x)(x^2+x)$

$= 2x^4 + 2x^3 + x^3 + x^2$

$= 2x^4 + 3x^3 + x^2$

$\frac{dy}{dx} = 8x^3 + 9x^2 + 2x$

$\left.\frac{dy}{dx}\right|_{x=0} = 0$

3. Given g(2)=4, g′(2)=−1, h(2)=−2, and h′(2)=3, find f′(2) if f(x)=g(x)h(x).

Given: $g(2) = 4$
$g'(2) = -1$
$h(2) = -2$
$h'(2) = 3$

$f(x) = g(x)\cdot h(x)$

$f'(x) = g'(x)\cdot h(x) + h'(x)\cdot g(x)$

$f'(2) = g'(2)\cdot h(2) + h'(2)\cdot g(2)$

$= (-1)(-2) + (3)(4)$

$= 14$

4. Find the x and y coordinates of all points on the graph of y=(x−1)(x²+1) where the tangent line is parallel to the line y=9x−6.

$\hookrightarrow m = 9$

$$y = (x-1)(x^2+1)$$
$$y' = [1][x^2+1] + [2x][x-1]$$
$$= x^2+1 + 2x^2 - 2x$$
$$= 3x^2 - 2x + 1$$

$$y' = m$$
$$3x^2 - 2x + 1 = 9$$
$$3x^2 - 2x - 8 = 0$$
$$(3x+4)(x-2) = 0$$
$$x = \{-\tfrac{4}{3}, 2\}$$

$$f(-\tfrac{4}{3}) = [-\tfrac{4}{3} - 1][(-\tfrac{4}{3})^2 + 1]$$
$$= [-\tfrac{7}{3}][\tfrac{16}{9} + 1]$$
$$= (-\tfrac{7}{3})(\tfrac{25}{9})$$
$$= \frac{-175}{27}$$

$$f(2) = [2-1][2^2+1]$$
$$= 5$$

$\therefore$ the two points on the functions are:

$(-\tfrac{4}{3}, -\tfrac{175}{27})$ and $(2, 5)$

5. Find the equation of the tangent line to the curve f(x)=(x⁴+x³+x²+x+1)(x³+x+2) at the point on the curve whose x-coordinate is −1.

$$f(x) = (x^4+x^3+x^2+x+1)(x^3+x+2)$$
$$f'(x) = [4x^3+3x^2+2x+1][x^3+x+2] + [3x^2+1][x^4+x^3+x^2+x+1]$$
$$f'(-1) = [-4+3-2+1][-1-1+2] + [3+1][1-1+1]$$
$$= (-2)(0) + (4)(1)$$
$$= 4$$

$$f(-1) = (1-1+1-1+1)(-1-1+2)$$
$$= 0$$

Tangent line :

$m = 4$

$(-1, 0)$

$$y - 0 = 4(x+1)$$

$\underline{\underline{\text{or}}}$ $y = 4x + 4$

2.3 Quotient Rules

$y' = \frac{u'v - v'u}{v^2}$ or $y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

> If $h(x) = \frac{f(x)}{g(x)}$, then
>
> $$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, g(x) \neq 0$$
>
> In Leibniz notation, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

Proof:

Since $h(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$, therefore

$h(x)g(x) = f(x)$ $\rightarrow$ multiply each side by $g(x)$

$h'(x)g(x) + h(x)g'(x) = f'(x)$ $\rightarrow$ differentiate each side

$h'(x)g(x) = f'(x) - h(x)g'(x)$

$$h'(x) = \frac{f'(x) - h(x)g'(x)}{g(x)}$$

$$= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$$

$$= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \times \frac{g(x)}{g(x)}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Examples

1. Determine the derivative of the following functions

a) $f(x) = \frac{3x-4}{x^2+5}$

$$f'(x) = \frac{[3][x^2+5] - [2x][3x-4]}{(x^2+5)^2}$$

$$= \frac{3x^2+15-6x^2+8x}{(x^2+5)^2}$$

$$= \frac{-3x^2+8x+15}{(x^2+5)^2}$$

$\Leftarrow$ remember with rationals, we like numerator and denominator in "factored" form to show you can not reduce any further

Note! You can make this a product Rule too (but will be longer)

$f(x) = (3x-4)(x^2+5)^{-1}$

$f'(x) = [3][(x^2+5)^{-1}] + [-(x^2+5)^{-2} \cdot (2x)][3x-4]$

$$= \frac{3}{x^2+5} + \frac{-(2x)(3x-4)}{(x^2+5)^2}$$

$$= \frac{3(x^2+5) - 6x^2+8x}{(x+5)^2}$$

$$= \frac{3x^2+15-6x^2+8x}{(x+5)^2}$$

$$= \frac{-3x^2+8x+15}{(x+5)^2}$$

b) $g(x) = \frac{(2x-1)^2}{(3x+2)^3}$

$$g'(x) = \frac{[2(2x-1)\cdot 2][(3x+2)^3] - [3(3x+2)^2\cdot(3)][(2x-1)^2]}{(3x+2)^6}$$

$$= \frac{4(2x-1)(3x+2)^3 - 9(3x+2)^2(2x-1)^2}{(3x+2)^6}$$

$$= \frac{\cancel{(3x+2)^2}(2x-1)[4(3x+2) - 9(2x-1)]}{\cancel{(3x+2)^6}^{\,4}}$$

$$= \frac{(2x-1)(12x+8-18x+9)}{(3x+2)^4}$$

$$= \frac{(2x-1)(-6x+17)}{(3x+2)^4}$$

$$= -\frac{(2x-1)(6x-17)}{(3x+2)^4}$$

2. Determine the equation of the **normal** to $y = \dfrac{2x}{x^2+1}$ at $x = 0$.

Definition: ***A normal line to the graph of a function f(x) is defined to be the line perpendicular to the tangent at a given point***

$$f(x) = \frac{2x}{x^2+1}$$

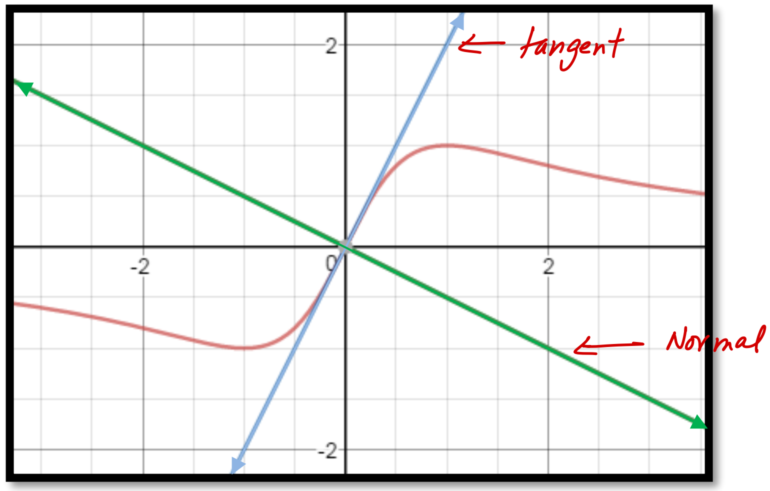
$$f'(x) = \frac{[2][x^2+1] - [2x][2x]}{[x^2+1]^2}$$

$$= \frac{2x^2+2-4x^2}{(x^2+1)^2}$$

$$= \frac{-2x^2+2}{(x^2+1)^2}$$

$$= \frac{-2(x^2-1)}{(x^2+1)^2}$$

$$= \frac{-2(x+1)(x-1)}{(x^2+1)^2}$$



$$f'(0) = \frac{-2(1)(-1)}{(1)^2} = 2$$

$$f(0) = \frac{2(0)}{(0)^2+1} = 0$$

$m = 2$

$\therefore m_\perp = -\frac{1}{2}$

Equation of normal : $y = -\frac{1}{2}x$

$m = -\frac{1}{2}$ $(0,0)$

3. Determine the coordinates of each point on the graph of $f(x) = \dfrac{2x+8}{\sqrt{x}}$ where the tangent is horizontal.

$$f(x) = 2x^{\frac{1}{2}} + 8x^{-\frac{1}{2}}$$

$$f'(x) = x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}$$

horizontal $\Rightarrow f'(x) = 0$

$$0 = x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}$$

$$0 = x^{-\frac{3}{2}}(x-4)$$

$$0 = \frac{x-4}{\sqrt{x^3}}$$

$\therefore x = 4$

$$f(4) = \frac{2(4)+8}{\sqrt{4}} = \frac{16}{2} = 8$$

$\therefore$ @ the point $(4,8)$

Practice 2.3

1. Determine the equation of the tangent line to $g(x)=\left(\frac{1}{x^3}+1\right)(x-1)$ at $x=-1$.

2. If $g(x)=\frac{f(x)}{\sqrt{x-1}}$, where $f(5)=8$, and $f'(5)=-5$, find $g'(5)$.

3. Find the points on the function $f(x)=\frac{x+9}{x+8}$ where the tangent lines pass through the origin.

4. Find the equation of the normal to the curve $f(x)=\sqrt[3]{x^2-1}$ at the point where x=3 .

5. Let f and g be functions such that $g(x)=\frac{f(x)}{x}$. If $y = 2x - 3$ is the equation of the tangent to the graph of f(x) at x=1, what is the equation of the line tangent to the graph of g(x) at x=1 ?

6. Find the points on the curve $f(x)=\frac{x}{x+1}$ where the **normal** line is parallel to $x+y=2$.

Practice 2.3

1. Determine the equation of the tangent line to $g(x)=\left(\frac{1}{x^3}+1\right)(x-1)$ at $x=-1$.

$$g(x) = (x^{-3}+1)(x-1)$$
$$g'(x) = [-3x^{-4}][x-1] + [1][x^{-3}+1]$$
$$g'(-1) = (-3)(-2) + (1)(-1+1)$$
$$= 6$$

$$g(-1) = \left[\frac{1}{(-1)^3}+1\right]\left[(-1)-1\right]$$
$$= (0)(-2)$$
$$= 0$$

$m = 6 \quad (-1, 0)$

Equation of tangent: $y - 0 = 6(x+1)$

$$y = 6x+6$$

2. If $g(x)=\frac{f(x)}{\sqrt{x-1}}$, where $f(5)=8$, and $f'(5)=-5$, find $g'(5)$.

$$g'(x) = \frac{[f'(x)][\sqrt{x-1}] - \left[\frac{1}{2}(x-1)^{-\frac{1}{2}} \cdot (1)\right][f(x)]}{(x-1)}$$

$$g'(5) = \frac{f'(5)\cdot\sqrt{5-1} - \frac{1}{2}(5-1)^{-\frac{1}{2}} \cdot f(5)}{(5-1)}$$
$$= \frac{(-5)(2) - \frac{1}{2}\left(\frac{1}{2}\right)(8)}{4}$$
$$= \frac{-10-2}{4}$$
$$= -3$$

3. Find the points on the function $f(x)=\frac{x+9}{x+8}$ where the tangent lines pass through the origin.

$f(0) = \frac{9}{8}$ $\therefore$ (0,0) is off the curve

$$f'(x) = \frac{[1][x+8] - [1][x+9]}{(x+8)^2}$$
$$= \frac{x+8-x-9}{(x+8)^2}$$
$$= \frac{-1}{(x+8)^2}$$

$(0,0)$ and $\left(x, \frac{x+9}{x+8}\right)$

$$m_T = \frac{y_2-y_1}{x_2-x_1}$$
$$= \frac{\frac{x+9}{x+8} - 0}{x-0}$$
$$= \frac{x+9}{x(x+8)}$$

$$f'(x) = m_T$$
$$\frac{-1}{(x+8)^2} = \frac{x+9}{x(x+8)}$$
$$-x(x+8) = (x+9)(x+8)^2$$
$$0 = (x+9)(x+8)^2 + x(x+8)$$
$$0 = (x+8)[(x+9)(x+8)+x]$$
$$0 = (x+8)(x^2+17x+72+x)$$
$$0 = (x+8)(x^2+18x+72)$$
$$0 = (x+8)(x+12)(x+6)$$

$$x = \{-8, -12, -6\}$$

$f(-8) = \frac{-8+9}{-8+8}$ = undefined

$$f(-12) = \frac{-12+9}{-12+8} = \frac{3}{4}$$

$$f(-6) = \frac{-6+9}{-6+8} = \frac{3}{2}$$

$\therefore$ @ $\left(-12, \frac{3}{4}\right)$ and $\left(-6, \frac{3}{2}\right)$

4. Find the equation of the normal to the curve $f(x) = \sqrt[3]{x^2 - 1}$ at the point where x=3 .

$f(x) = (x^2-1)^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}(x^2-1)^{-\frac{2}{3}}(2x)$

$f'(3) = \frac{1}{3}(3^2-1)^{-\frac{2}{3}}[2(3)]$

$= \frac{1}{3}(8)^{-\frac{2}{3}}(6)$

$= \frac{1}{3}(\frac{1}{4})(6)$

$= \frac{1}{2}$

$f(3) = \sqrt[3]{3^2-1}$

$= 2$

Equation of the Normal

$m_{\perp} = -2 \quad (3,2)$

$y-2 = -2(x-3)$

$y = -2x+6+2$

$y = -2x+8$

5. Let f and g be functions such that $g(x) = \frac{f(x)}{x}$. If $y = 2x - 3$ is the equation of the tangent to the graph of f(x) at x=1, what is the equation of the line tangent to the graph of g(x) at x=1 ?

$g'(x) = \frac{[f'(x)][x] - [1][f(x)]}{x^2}$

$g'(1) = \frac{f'(1)[1] - 1[f(1)]}{1^2}$

$= (2)(1) - (1)(-1)$

$= 3$

$f'(1) = 2$

$g(1) = \frac{f(1)}{(1)}$

$\therefore g(1) = f(1)$

$y = 2(1)-3$

$y = -1$

$\therefore g(1) = f(1) = -1$

tangent to g(x) @ x=1

$m = 3 \quad (1,-1)$

$: \quad y+1 = 3(x-1)$

6. Find the points on the curve $f(x) = \frac{x}{x+1}$ where the **normal** line is parallel to $x + y = 2$.

$f'(x) = \frac{[1][x+1] - [1][x]}{[x+1]^2}$

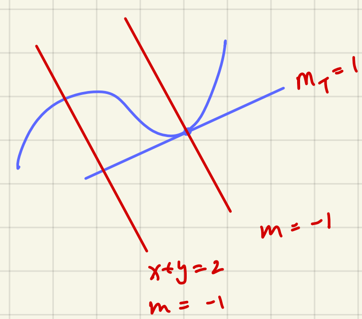
$= \frac{x+1-x}{(x+1)^2}$

$= \frac{1}{(x+1)^2}$

Normal line: $y = -x+2$

$m_{\perp} = -1$

$m_T = 1$



$\therefore f'(x) = 1$

$\frac{1}{(x+1)^2} = 1$

$1 = (x+1)^2$

$0 = (x+1)^2 - 1$

$0 = [(x+1)+1][(x+1)-1]$

$0 = (x+2)(x)$

$\therefore x = \{0, -2\}$ →

$f(0) = \frac{0}{0+1}$

$= 0$

$f(-2) = \frac{-2}{-2+1}$

$= \frac{-2}{-1}$

$= 2$

$\therefore$ @ the points : (0,0) and (−2,2)

Warm Up

1. The limit below represents the derivative of some function f(x) evaluated at some number ***a***. Determine the function and the number ***a***.

$$f'(a)=\lim_{h\to 0}\frac{2(6+h)^2-2(6)^2}{h}\quad,\quad f(x)=\underline{\;2x^2\;}\quad,\quad \boldsymbol{a}=\underline{\;6\;}$$

2. Differentiate the following .Where applicable; write the final answers with positive exponents.

a) $g(x)=\left(5\sqrt[5]{x^3}-\dfrac{1}{2x^2}\right)\sqrt[3]{x}$

$$=\left(5x^{\frac{3}{5}}-\tfrac{1}{2}x^{-2}\right)\left(x^{\frac{1}{3}}\right)$$

$$=5x^{\frac{3}{5}+\frac{1}{3}}-\frac{1}{2}x^{-2+\frac{1}{3}}$$

$$=5x^{\frac{14}{15}}-\frac{1}{2}x^{-\frac{5}{3}}$$

$$g'(x)=\frac{14}{3}x^{-\frac{1}{15}}+\frac{5}{6}x^{-\frac{8}{3}}$$

$$=\frac{14}{3x^{\frac{1}{15}}}+\frac{5}{6x^{\frac{8}{3}}}$$

or $$=\frac{14}{3\sqrt[15]{x}}+\frac{5}{6\sqrt[3]{x^8}}$$

b) $g(t)=\dfrac{\pi t^5-2t^{-4}+3\pi^2}{3t^2}$

$$=\frac{\pi}{3}t^3-\frac{2}{3}t^{-6}+\pi^2t^{-2}$$

$$g'(t)=\pi t^2+4t^{-7}-2\pi^2t^{-3}$$

$$=\pi t^2+\frac{4}{t^7}-\frac{2\pi^2}{t^3}$$

3. Given $f'(1)=4, g'(1)=-2, f(1)=1,$ and $g(1)=1,$ find $h'(1)$

if $h(x)=\left(2x-\sqrt{x}\right)^2g(x)+x^3f(x).$

$$h'(x)=\left[2(2x-\sqrt{x})^1\cdot\left(2x-\tfrac{1}{2}x^{-\frac{1}{2}}\right)\right][g(x)]+[g'(x)][(2x-\sqrt{x})^2]+[3x^2][f(x)]+[f'(x)][x^3]$$

$$h'(1)=\left[2(2-1)\cdot\left(2-\tfrac{1}{2}\right)\right][g(1)]+[g'(1)][(2-1)^2]+[3][f(1)]+[f'(1)][1]$$

$$=(2)\left(\tfrac{3}{2}\right)(1)+(-2)(1)+(3)(1)+(4)(1)$$

$$=3-2+3+4$$

$$=8$$

derivative of composition functions

2. 4 Chain Rule

nested function

Recall: The composite function (fog) is defined by $(fog)(x) = f(g(x))$

$(f \circ g \circ h)(x) = f(g(h(x)))$

Example 1: Suppose $f(x) = \sqrt{x}$ and $g(x) = x + 5$

a) Express $(fog)(x)$ $= f(g(x))$ $= \sqrt{(x+5)}$

c) Evaluate $f(g(4))$ $= \sqrt{(4+5)}$ $= \sqrt{9}$ $= 3$

b) Express $(gof)(x)$ $= g(f(x))$ $= \sqrt{x} + 5$

d) Evaluate $g(f(4))$ $= \sqrt{4} + 5$ $= 7$

chain Rule

Example 2: Differentiate the function $\mathrm{h(x) = (3x^2 - 7x)^5}$.

$h'(x) = \underbrace{5(3x^2-7x)^4}_{\text{derivative of the outside function with respect to the inside function}} \cdot \underbrace{(6x-7)}_{\text{derivative of the inside function with respect to } x.}$

$h = u^5$ $\quad u = 3x^2 - 7x$

$\frac{dh}{du} = 5u^4$ $\quad \frac{du}{dx} = 6x - 7$

$\frac{dh}{dx} = \frac{dh}{du} \cdot \frac{du}{dx}$

$= 5u^4 \cdot (6x-7)$

$= 5(3x^2-7x)^4(6x-7)$

Observe that in the above example, where $\mathrm{h(x) = (3x^2 - 7x)^5}$, if we let $\mathrm{f(u) = u^5}$ and we let $\mathrm{u = g(x) = 3x^2 - 7x}$ then $\mathrm{h(x) = f(g(x))}$. Since we know how to differentiate $f(u)$ and $g(x)$ individually, the **chain rule** allows us to differentiate $\mathrm{y = f(g(x))}$.
It turns out that the derivative of the composite function, $\mathrm{h(x) = f(g(x))}$ is the product of the derivatives of $f(u)$ and $g(x)$.That is, $\mathrm{h'(x) = f'(u)g'(x)}$.

Recall:

$\frac{dy}{du}$ is the rate of change of y with respect to u.

$\frac{du}{dx}$ is the rate of change of u with respect to x.

$\frac{dy}{dx}$ is the rate of change of y with respect to x.

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Suppose y changes 3 times as fast as u, and u changes 4 times as fast as x. Then it makes sense that y changes 3×4=12 times as fast as x. That is, the rate of change in y with respect to x is equal to the product of the other two rates.

$\frac{dy}{du} = 3$ $\quad \frac{du}{dx} = 4$ $\quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 3 \cdot 4$

$= 12$

The Chain Rule

If $g(x)$ is differentiable at x and $f(x)$ is differentiable at $g(x)$, then the composite function, $h(x) = f(g(x))$ or $h(x) = (fog)(x)$ is differentiable at x and $h'(x)$ is given by the product

$$h'(x) = f'(g(x))g'(x).$$

In Leibniz notation, If y=f(u) and u=g(x), are both differentiable functions, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

<u>Proof of Chain Rule:</u>

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(g(x+h)) - f(g(x))}{h} \times 1\right]$$

$$= \lim_{h\to 0}\left[\frac{f(g(x+h)) - f(g(x))}{h} \times \frac{g(x+h) - g(x)}{g(x+h) - g(x)}\right]$$

We can only make this move if we know that $g(x+h) - g(x) \neq 0$. In other words, this proof is not valid over any domain of the function for which the graph of $y = g(x)$ is a straight horizontal line.

$$= \lim_{h\to 0}\left[\left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right)\left(\frac{g(x+h) - g(x)}{h}\right)\right]$$

$$= \lim_{h\to 0}\left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right] \lim_{h\to 0}\left[\frac{g(x+h) - g(x)}{h}\right]$$

Look at the denominator of the first fraction. We're taking the limit of that fraction as $h \to 0$. We know that $\lim_{h\to 0}[g(x+h) - g(x)] = 0$. So, we'll let $g(x+h) - g(x) = k$. Recognizing that $k \to 0$ as $h \to 0$, we're able to rewrite that last line as follows:

$$= \lim_{k\to 0}\left[\frac{f(g(x)+k)-f(g(x))}{k}\right]\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$= f'(g(x))g'(x)$$

The Chain Rule Song (Clementine themes)

Here's a function
in a function
and your job
here is to find
the derivative of
the whole thing
with respect
to x inside.
Call the outside f of u
And the inside u of x.
Differentiate to find df/du
And multiply by du/dx.
Use the chain rule.
Use the chain rule.
Use the chain rule
whene'er you find
The derivative of a function compositionally defined.

Examples:

1. If $y = u^2 + u - 1$, and $u = x^2 - 2\sqrt{x}$, evaluate $\left.\frac{dy}{dx}\right|_{x=1}$ using Leibniz notation .

$\frac{dy}{du} = 2u + 1$ $\qquad$ $\frac{du}{dx} = 2x - x^{-\frac{1}{2}}$

$\left.\frac{dy}{dx}\right|_{x=1} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= (2u+1) \cdot (2x - x^{-\frac{1}{2}})$

$= 2(x^2 - 2\sqrt{x}) + 1 \cdot 2x - x^{-\frac{1}{2}}$

$= 2(1^2 - 2\sqrt{1}) + 1 \cdot 2(1) - (1)^{-\frac{1}{2}}$

$= (2(-1) + 1) \cdot (2 - 1)$

$= (-1)(1)$

$= -1$

method 2:

$u = 1^2 - 2\sqrt{1}$
$= 1 - 2$
$= -1$

$\left.\frac{dy}{dx}\right|_{\substack{x=1\\u=-1}} = (2u+1) \cdot (2x - x^{-\frac{1}{2}})$

$= (2(-1)+1) \cdot (2(1) - (1)^{-\frac{1}{2}})$

$= (-1)(1)$

$= -1$

2. Differentiate:

a) $f(x) = m(nx^2 + rx)^{\sqrt{7}}$

$f'(x) = \underbrace{\sqrt{7}\, m (nx^2 + rx)^{\sqrt{7}-1}}_{\text{outside function}} \cdot \underbrace{[2nx + r]}_{\text{inside function}}$

b) $f(x) = (5+3x)^{\pi}$

$f'(x) = \underbrace{\pi (5+3x)^{\pi - 1}}_{\text{outside function}} \cdot \underbrace{[3]}_{\text{inside function}}$

c) $m(t)=\sqrt[3]{t+\sqrt{1+t^2}}$

$m'(t)=\frac{1}{3}\left[t+\sqrt{1+t^2}\right]^{-\frac{2}{3}}\cdot\left[1+\frac{1}{2}(1+t^2)^{-\frac{1}{2}}\cdot 2t\right]$

Leibniz: Notation let $m=\sqrt[3]{u}$ $\quad u=t+\sqrt{1+t^2}$

$\frac{dm}{du}=\frac{1}{3}u^{-\frac{2}{3}}$ $\quad \frac{du}{dt}=1+\frac{1}{2}(1+t^2)^{-\frac{1}{2}}\cdot(2t)$

$\frac{dm}{dt}=\frac{dm}{du}\cdot\frac{du}{dt}$

$=\frac{1}{3}u^{-\frac{2}{3}}\left[1+\frac{1}{\cancel{2}}(1+t^2)^{-\frac{1}{2}}\cdot(\cancel{2}t)\right]$

$=\frac{1}{3}(t+\sqrt{1+t^2})^{-\frac{2}{3}}\left[1+\frac{t}{\sqrt{1+t^2}}\right]$

d) $f(x)=\sqrt[3]{\frac{x^2-3}{3-5x}}$

$f'(x)=\frac{1}{3}\left(\frac{x^2-3}{3-5x}\right)^{-\frac{2}{3}}\cdot\frac{[2x][3-5x]-[-5][x^2-3]}{(3-5x)^2}$

or $f'(x)=\frac{1}{3}\left(\frac{3-5x}{x^2-3}\right)^{\frac{2}{3}}\left[\frac{6x-10x^2+5x^2-15}{(3-5x)^2}\right]$

$=\frac{-5x^2+6x-15}{3(x^2-3)^{\frac{2}{3}}(3-5x)^{4/3}}$

simplified form

e) $y=(2x^2-9)\sqrt{3x^2+5x}$

$y'=[4x][\sqrt{3x^2+5x}]+\left[\frac{1}{2}(3x^2+5x)^{-\frac{1}{2}}\cdot(6x+5)\right][2x^2-9]$

or $=4x\sqrt{3x^2+5x}+\frac{(6x+5)(2x^2-9)}{2\sqrt{3x^2+5x}}$

$=\frac{8x(3x^2+5x)+(6x+5)(2x^2-9)}{2\sqrt{3x^2+5x}}=\frac{24x^3+40x^2+12x^3+10x^2-54x-45}{2\sqrt{3x^2+5x}}$

$=\frac{36x^3+50x^2-54x-45}{2\sqrt{3x^2+5x}}$

3. a) If $h(x)=\frac{(f(x))^2}{g(x)}$, determine $h'(x)$.

$h'(x)=\frac{[2f(x)\cdot f'(x)][g(x)]-[g'(x)][(f(x))^2]}{[g(x)]^2}$

b) Given $f(1)=2$, $f'(1)=-3$, $g(1)=1$ and $g'(1)=4$ find $h'(1)$.

$h'(1)=\frac{[2f(1)\cdot f'(1)][g(1)]-[g'(1)][(f(1))^2]}{[g(1)]^2}$

$=\frac{[2(2)(-3)][1]-[(4)(2^2)]}{(1)^2}$

$=(-12)-16$

$=-28$

4. If $y = f\left(3x^4\right)$ and $f'(3) = \frac{-1}{4}$, determine $\left.\frac{dy}{dx}\right|_{x=1}$.

$y' = f'(3x^4) \cdot 12x^3$

$\left.\frac{dy}{dx}\right|_{x=1} = f'(3(1)^4) \cdot 12(1)^3$

$= f'(3) \cdot 12$

$= -\frac{1}{4} \cdot 12$

$= -3$

Practice 2.4

1. Given that $g(2)=4, g'(2)=-1, h(2)=2,$ and $h'(2)=3,$ find $f'(2)$ if

   a) $f(x)=\left(g(x)\right)^3$ b) $f(x)=g\left(h(x)\right)$

2. Let $f(x)=h\left(g(x)\right)$ and $j(x)=h(x)g(x)$, where *g* and *h* are differentiable functions on $\mathbb{R}$. Fill in the missing entries on the table below and determine *h*(4) and *h′*(4).

x	h(x)	h′(x)	g(x)	g′(x)	f(x)	f′(x)	j(x)	j′(x)
0		**1**	**2**			**−8**	**−8**	**10**
1	**2**					**4**	**4**	**6**
2		**4**	**4**		**15**	**8**	**4**	**18**

3. Slope of the normal to the curve with equation $y = \boldsymbol{a}x + \frac{\boldsymbol{b}}{4-3x}$ at point $(1,6)$ is $-\frac{1}{2}$. Find the values of ***a*** and ***b***.

4. Find ***k*** given that the tangent to $f(x)=\frac{4}{(\boldsymbol{k}x+1)^2}$ at $x=0$ passes through (1, 0).

5. Consider $f(x)=\frac{4}{\sqrt{4-x}}$.

   a) Find the equations of the tangent and normal at the point where P(3,4).

b) If the tangent line cuts the x-axis at A and the normal line cuts the x-axis at B, find the coordinates of A and B.
c) Find the area of triangle PAB.

Practice 2.4

1. Given that $g(2)=4, g'(2)=-1, h(2)=2,$ and $h'(2)=3,$ find $f'(2)$ if

a) $f(x)=(g(x))^3$ b) $f(x)=g(h(x))$

$f'(x) = 3(g(x))^2 \cdot g'(x)$
$f'(2) = 3(g(2))^2 \cdot g'(2)$
$= 3(4)^2 \cdot (-1)$
$= -48$

$f'(x) = g'(h(x)) \cdot h'(x)$
$f'(2) = g'(h(2)) \cdot h'(2)$
$= g'(2) \cdot (3)$
$= (-1)(3)$
$= -3$

2. Let $f(x)=h(g(x))$ and $j(x)=h(x)g(x)$, where g and h are differentiable functions on $\mathbb{R}$. Fill in the missing entries on the table below and determine $h(4)$ and $h'(4)$.

x	h(x)	h′(x)	g(x)	g′(x)	f(x)	f′(x)	j(x)	j′(x)
0	−4	1	2	−2	1	−8	−8	10
1	2	2	2	1	1	4	4	6
2	1	4	4	2	15	8	4	18

$f(x) = h(g(x))$
$f'(x) = h'(g(x)) \cdot g'(x)$

$j(x) = h(x) \cdot g(x)$
$j'(x) = h'(x) \cdot g(x) + g'(x) \cdot h(x)$

①

$f'(0) = h'(g(0)) \cdot g'(0)$
$-8 = h'(2) \cdot g'(0)$
$-8 = (4) \cdot g'(0)$
$\therefore g'(0) = -2$

$f'(1) = h'(g(1)) \cdot g'(1)$
$4 = h'(2) \cdot g'(1)$
$4 = 4 \; g'(1)$
$\therefore g'(1) = 1$

$f'(2) = h'(g(2)) \cdot g'(2)$
$8 = h'(4) \cdot g'(2)$
$8 = h'(4) \cdot 2$
$4 = h'(4)$

$j'(0) = h'(0) \cdot g(0) + g'(0) \cdot h(0)$
$10 = (1)(2) + (-2) \cdot h(0)$
$8 = -2h(0)$
$\therefore h(0) = -4$

$j'(1) = h'(1) \cdot g(1) + g'(1) \cdot h(1)$
$6 = h'(1)\,(2) + (1)(2)$
$\therefore h'(1) = 2$

$j'(2) = h'(2) \cdot g(2) + g'(2) \cdot h(2)$
$18 = (4) \cdot (4) + g'(2) \cdot (1)$
$\therefore g'(2) = 2$

②

$f(0) = h(g(0))$
$= h(2)$
$= 1$

$f(1) = h(g(1))$
$= h(2)$
$= 1$

$f(2) = h(g(2))$
$15 = h(4)$

$j(0) = h(0)g(0)$
$-8 = (-4)\,g(0)$
$2 = g(0)$ ✓

$j(1) = h(1)g(1)$
$4 = (2)\,g(1)$
$\therefore g(1) = 2$

$j(2) = h(2)g(2)$
$4 = h(2)\,(4)$
$\therefore h(2) = 1$

$\therefore h(4) = 15$
$h'(4) = 4$

3. Slope of the normal to the curve with equation $y = ax + \frac{b}{4-3x}$ at point $(1,6)$ is $-\frac{1}{2}$. Find the values of a and b.

$= ax + b(4-3x)^{-1}$

if $m_{\perp} = -\frac{1}{2}$

$\therefore \quad m_T = 2$

$f'(x) = a - b(4-3x)^{-2}(-3)$

$f'(1) = a - b(4-3(1))^{-2}(-3)$

$2 = a - b(1)(-3)$

$2 = a + 3b \quad ②$

$f(x) = ax + \frac{b}{4-3x}$

$f(1) = a(1) + \frac{b}{4-3(1)}$

$① \quad 6 = a + b$

$$\begin{cases} a + b = 6 & ① \\ a + 3b = 2 & ② \end{cases}$$

$-2b = 4$

$b = -2$

sub into ①

$a + (-2) = 6$

$a = 8$

$\therefore \quad a = 8$

$b = -2$

4. Find k given that the tangent to $f(x) = \frac{4}{(kx+1)^2}$ at $x = 0$ passes through (1, 0).

$f(x) = 4(kx+1)^{-2}$

$f'(x) = -8(kx+1)^{-3} \cdot k$

$f'(0) = -8(k(0)+1)^{-3} \cdot k$

$= -8k$

$f(0) = \frac{4}{(k(0)+1)^2}$

$f(0) = 4$

$\therefore$ the tangent line will pass through $(0,4)$ and $(1,0)$

$m_T = \frac{0-4}{1-0}$

$= -4$

$f'(0) = -4$

$-8k = -4$

$k = \frac{-4}{-8}$

$k = \frac{1}{2}$

5. Consider $f(x) = \frac{4}{\sqrt{4-x}}$.

a) Find the equations of the tangent and normal at the point where P(3,4).

$f(x) = 4(4-x)^{-\frac{1}{2}}$

$f'(x) = -2(4-x)^{-\frac{3}{2}} \cdot (-1)$

$= \frac{2}{\sqrt{(4-x)^3}}$

$f(3) = \frac{4}{\sqrt{4-3}}$

$= 4$

$\therefore$ P(3,4) is on the function

$f'(3) = \frac{2}{\sqrt{(4-3)}^3}$

$= 2$

$m_T = 2$

$m_{\perp} = -\frac{1}{2}$

Equation of tangent:

$y - 4 = 2(x-3) \Rightarrow y = 2x - 6 + 4$

$y = 2x - 2$

Equation of Normal:

$y - 4 = -\frac{1}{2}(x-3) \Rightarrow y = -\frac{1}{2}x + \frac{3}{2} + 4$

$y = -\frac{1}{2}x + \frac{11}{2}$

b) If the tangent line cuts the x-axis at A and the normal line cuts the x-axis at B, find the coordinates of A and B.

c) Find the area of triangle PAB.

b) Equation of tangent: $y = 2x - 2$ cuts x-axis @ A $\Rightarrow y = 0$

$0 = 2x - 2$

$x = 1$

$\therefore A(1, 0)$

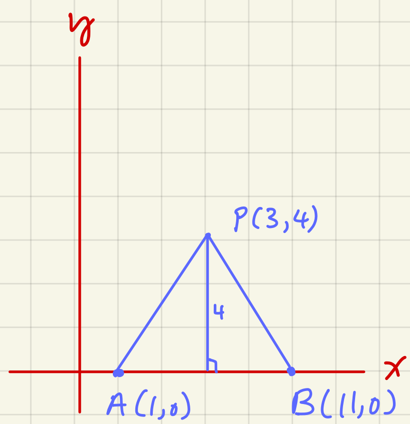
c) Equation of Normal: $y = -\frac{1}{2}x + \frac{11}{2}$ cuts the x-axis @ B $\Rightarrow y = 0$

$0 = -\frac{1}{2}x + \frac{11}{2}$

$\frac{1}{2}x = \frac{11}{2}$

$x = 11$

$\therefore B(11, 0)$



$A = \frac{1}{2}bh$

$= \frac{1}{2}(11-1)(4)$

$= 20 \text{ unit}^2$

More Practice on using Chain Rule

1. Differentiate the following .Where applicable; write the final answers with positive exponents.

a) $y=\left(2+x^{2}\right)^{\pi}+\sqrt[5]{1-x^{2}}+\dfrac{3x^{2}-\sqrt{x}+5}{\sqrt{5x}}$ b) $S(t)=t^{2}\left(1-\dfrac{2\pi}{t^{2}}\right)+\sqrt{4t^{2}-5}$

c) $h(x)=\dfrac{(3x+2)^{-3}}{(2x-1)^{-2}}$ ***(Express in a fully factored form)***

2. Let f and g be differentiable functions such that $g(1)=3$, $f(1)=-2$, $f(3)=-1$, $f'(3)=-2$ and $g'(1)=2$. Let $h(x)=\dfrac{f(g(x))}{f(x)+g(x)}$. If $h'(1)=5$, find the value of $f'(1)$.

3. Find the points on the curve $y=\left(1-\dfrac{x}{5}\right)^{3}$ where the slope of normal line is 15.

4. If $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=-4$ and $g(x)=f\left(\sqrt{5-x^{2}}\right)$, determine the value of $g'(1)$.

5 . Assume that $h(x)=\left[f(x)\right]^{3}.g(x)$, where f and g are differentiable functions

If $f(0)=\dfrac{-1}{2}, f'(0)=\dfrac{-8}{3}$ and $g(0)=-1, g'(0)=-2$, determine an equation of the line tangent to the graph of h at x=0 .

6. Consider the curve $y=\mathbf{a}\sqrt{x}+\dfrac{\mathbf{b}}{\sqrt{x}}$ where ***a*** and ***b*** are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

7. Line $y=k$ is tangent to the curve $f(x)=\dfrac{x^{2}-5}{x-(k+1)}$ at x=1. Find value of k.

8. Given $y=\dfrac{u+3}{2u-1}$, and $u=\sqrt{x^{2}+3}$, determine $\left.\dfrac{dy}{dx}\right|_{x=1}$ by using the Leibniz notation.

9. An initial population, p, of 1500 bacteria grows in number according to the equation, $P(t)=1500\left(1+\dfrac{13t}{t^{2}+30}\right)$, where t is in hours and $0\le t\le 9$. Determine the rate at which the population is growing when the population is double in size.

More Practice on using Chain Rule

1. Differentiate the following .Where applicable; write the final answers with positive exponents.

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1a) $y=(2+x^2)^{\pi}+(1-x^2)^{\frac{1}{5}}+\frac{3}{\sqrt{5}}x^{\frac{3}{2}}-\frac{1}{\sqrt{5}}+\frac{5}{\sqrt{5}}x^{-\frac{1}{2}}$

$y'=\pi(2+x^2)^{\pi-1}\cdot 2x+\frac{1}{5}(1-x^2)^{-\frac{4}{5}}\cdot(-2x)+\frac{9}{2\sqrt{5}}x^{\frac{1}{2}}-\frac{5}{2\sqrt{5}}x^{-\frac{3}{2}}$

$=\dfrac{2\pi x(2+x^2)^{\pi}}{(2+x^2)}-\dfrac{2x}{5\sqrt[5]{(1+x^2)^4}}+\dfrac{9\sqrt{x}}{2\sqrt{5}}-\dfrac{5}{2\sqrt{5x^3}}$

b) $s(t)=t^2-2\pi+(4t^2-5)^{\frac{1}{2}}$

$s'(t)=2t-\frac{1}{2}(4t^2-5)^{-\frac{1}{2}}\cdot(8t)$

$=2t-\dfrac{4t}{\sqrt{(4t^2-5)}}$

c) $h(x)=\dfrac{(2x-1)^2}{(3x+2)^3}$

$h'(x)=\dfrac{[2(2x-1)(2)][(3x+2)^3]-[3(3x+2)^2(3)][(2x-1)^2]}{(3x+2)^6}$

"unsimplified Form" $\Rightarrow$ $=\dfrac{4(2x-1)(3x+2)^3-9(2x-1)^2(3x+2)^2}{(3x+2)^6}$

$=\dfrac{(2x-1)(3x+2)^{\cancel{2}}[4(3x+2)-9(2x-1)]}{(3x+2)^{\cancel{6}\,4}}$

$=\dfrac{(2x-1)(12x+8-18x+9)}{(3x+2)^4}$

$=\dfrac{(2x-1)(-6x+17)}{(3x+2)^4}$

"fully Simplified factored form" $\Rightarrow$ $=-\dfrac{(2x-1)(6x-17)}{(3x+2)^4}$

2. Let f and g be differentiable functions such that $g(1)=3, f(1)=-2$, $f(3)=-1$, $f'(3)=-2$ and $g'(1)=2$. Let $h(x)=\dfrac{f(g(x))}{f(x)+g(x)}$. If $h'(1)=5$, find the value of $f'(1)$.

$h(x)=\dfrac{f(g(x))}{f(x)+g(x)}$

$h'(x)=\dfrac{[f'(g(x))(g'(x))][f(x)+g(x)]-[f'(x)+g'(x)][f(g(x))]}{[f(x)+g(x)]^2}$

$h'(1)=\dfrac{[f'(g(1))\,g'(1)][f(1)+g(1)]-[f'(1)+g'(1)][f(g(1))]}{[f(1)+g(1)]^2}$

$5=\dfrac{[f'(3)\cdot(2)][-2+3]-[f'(1)+2][f(3)]}{[-2+(3)]^2}$

$5=\dfrac{[-2\cdot 2][1]-[f'(1)+2][-1]}{1}$

$5=-4+f'(1)+2$

$\rightarrow f'(1)=5+2$

$f'(1)=7$

3. Find the points on the curve $y = \left(1 - \frac{x}{5}\right)^3$ where the slope of normal line is 15.

$$y = \left(1 - \frac{x}{5}\right)^3$$
$$y' = 3\left(1 - \frac{x}{5}\right)^2\left(-\frac{1}{5}\right)$$
$$y' = m_T$$
$$3\left(1 - \frac{x}{5}\right)^2\left(-\frac{1}{5}\right) = \frac{-1}{15}$$
$$-\frac{3}{5}\left(1 - \frac{x}{5}\right)^2 = \frac{-1}{15} \cdot \frac{5}{3}$$
$$\left(1 - \frac{x}{5}\right)^2 = \frac{1}{9}$$
$$1 - \frac{x}{5} = \pm\frac{1}{3}$$
$$\frac{-x}{5} = -1 \pm \frac{1}{3}$$
$$-\frac{x}{5} = \frac{-3 \pm 1}{3}$$
$$-\frac{x}{5} = \left\{\frac{-4}{3}, \frac{-2}{3}\right\}$$
$$x = \left\{\frac{20}{3}, \frac{10}{3}\right\}$$

if $m_\perp = 15$

$\therefore m_T = -\frac{1}{15}$

$$f\left(\tfrac{20}{3}\right) = \left(1 - \frac{\left(\frac{20}{3}\right)}{5}\right)^3 = \left(1 - \frac{20}{15}\right)^3 = \left(1 - \frac{4}{3}\right)^3 = \left(-\frac{1}{3}\right)^3 = -\frac{1}{27}$$

$$f\left(\tfrac{10}{3}\right) = \left(1 - \frac{\left(\frac{10}{3}\right)}{5}\right)^3 = \left(1 - \frac{10}{15}\right)^3 = \left(1 - \frac{2}{3}\right)^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$\therefore$ at the points:

$\left(\frac{20}{3}, \frac{-1}{27}\right)$ and $\left(\frac{1}{3}, \frac{1}{27}\right)$

4. If $\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = -4$ and $g(x) = f\left(\sqrt{5-x^2}\right)$, determine the value of $g'(1)$.

$$f'(2) = -4$$

$$g'(x) = f'\left(\sqrt{5-x^2}\right)\cdot\left[\tfrac{1}{2}(5-x^2)^{-\frac{1}{2}}\cdot(-2x)\right]$$

$$g'(2) = f'\left(\sqrt{5-4}\right)\cdot\left[\tfrac{1}{2}(5-4)^{-\frac{1}{2}}\cdot(-2(2))\right]$$
$$g'(2) = f'(1)\cdot\left(\tfrac{1}{2}\cdot -4\right)$$
$$g'(2) = f'(1)\cdot(-2)$$ ⇐ this was not necessary

$$g'(1) = f'\left(\sqrt{5-1}\right)\cdot\left[\tfrac{1}{2}(4)^{-\frac{1}{2}}\cdot(-2)\right]$$
$$= f'(2)\cdot\left(-\tfrac{1}{8}\right)$$
$$= (-4)\cdot\left(-\tfrac{1}{8}\right)$$
$$= \frac{1}{2}$$

$\therefore g'(1) = \frac{1}{2}$

5. Assume that $h(x) = [f(x)]^3 . g(x)$, where *f* and g are differentiable functions

If $f(0) = \frac{-1}{2}, f'(0) = \frac{-8}{3}$ and $g(0) = -1, g'(0) = -2$, determine an equation of the line tangent to the graph of *h* at *x*=0 .

$$h(x) = [f(x)]^3 \cdot g(x)$$
$$h'(x) = \left[3[f(x)]^2 \cdot f'(x)\right][g(x)] + [g'(x)][f(x)]^3$$
$$h'(0) = \left[3[f(0)]^2 \cdot f'(0)\right][g(0)] + [g'(0)][f(0)]^3$$
$$= \left[3\left(-\tfrac{1}{2}\right)^2\cdot\left(-\tfrac{8}{3}\right)\right][-1] + [-2]\left[-\tfrac{1}{2}\right]^3$$
$$= \left[\frac{\cancel{3}}{4}\ \frac{-8}{\cancel{3}}\right][-1] + [-2]\left[-\tfrac{1}{8}\right]$$
$$= 2 + \frac{1}{4}$$
$$= \frac{5}{4}$$

$$h(0) = [f(0)]^3 - g(0)$$
$$= \left(-\tfrac{1}{2}\right)^3 - (-1)$$
$$= \frac{-1}{8} + 1$$
$$= \frac{7}{8}$$

$\therefore$ Equation of tangent line :

$m = \frac{5}{4}$

$\left(0, \frac{7}{8}\right)$

$y = \frac{5}{4}x + \frac{7}{8}$ ⇐ slope y int form

$y - \frac{7}{8} = \frac{5}{4}(x - 0)$ ⇐ point slope form

6. Consider the curve $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

$$y = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$$
$$y' = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}}$$
$$y'\Big|_{x=4} = \frac{a}{2}(4)^{-\frac{1}{2}} - \frac{b}{2}(4)^{-\frac{3}{2}}$$
$$= \frac{a}{4} - \frac{b}{16}$$

Normal line: $4x + y = 22$

$$y = -4x + 22$$
$$m_{\perp} = -4$$
$$\therefore \quad m_T = \frac{1}{4}$$

$$\frac{1}{4} = \frac{a}{4} - \frac{b}{16}$$
$$\xRightarrow{\times 16} \quad 4 = 4a - b \quad (1)$$

$x = 4$, $y = -4(4) + 22$

$$y = 6$$
$$y = a\sqrt{4} + \frac{b}{\sqrt{4}}$$
$$6 = 2a + \frac{b}{2}$$
$$\xRightarrow{\times 2} \quad 12 = 4a + b \quad (2)$$

$$\begin{cases} 4 = a - b & (1) \\ 12 = 4a + b & (2) \end{cases}$$

$(+)$ $16 = 5a$

$$a = \frac{16}{5} \quad (3)$$

sub into (1)

$$4 = \frac{16}{5} - b$$
$$20 = 16 - 5b$$
$$4 = -5b$$
$$b = \frac{-4}{5}$$

$$\therefore a = \frac{16}{5}$$
$$b = \frac{-4}{5}$$

7. Line $y = k$ is tangent to the curve $f(x) = \frac{x^2 - 5}{x - (k+1)}$ at $x=1$. Find value of k.

$$f(x) = \frac{x^2 - 5}{x - (k+1)}$$

tangent line : $y = k$ (horizontal line)

$$\therefore \quad m_T = 0$$

$$f'(x) = \frac{[2x][x - (k+1)] - [1][x^2 - 5]}{[x - (k+1)]^2}$$
$$f'(1) = \frac{2[1 - k - 1] - [-4]}{[1 - k - 1]^2}$$
$$0 = \frac{-2k + 4}{k^2}$$

$$-2k + 4 = 0$$
$$k = \frac{-4}{-2}$$
$$k = 2$$

8. Given $y = \dfrac{u+3}{2u-1}$, and $u = \sqrt{x^2+3}$, determine $\left.\dfrac{dy}{dx}\right|_{x=1}$ by using the Leibniz notation.

$$y = \frac{u+3}{2u-1}$$

$$\frac{dy}{du} = \frac{[1][2u-1] - [2][u+3]}{[2u-1]^2}$$

$$\left.\frac{dy}{du}\right|_{u=2} = \frac{[1][4-1] - [2][5]}{(3)^2}$$

$$= \frac{3-10}{9}$$

$$= -\frac{7}{9}$$

$$u(1) = \sqrt{(1)^2+3}$$

$$= 2$$

$$u = (x^2+3)^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x)$$

$$\left.\frac{du}{dx}\right|_{x=1} = \frac{1}{2}(4)^{-\frac{1}{2}}(2)$$

$$= \frac{1}{2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{dy}{du}\right|_{u=2} \cdot \left.\frac{du}{dx}\right|_{x=1}$$

$$= \left(-\frac{7}{9}\right)\left(\frac{1}{2}\right)$$

$$= -\frac{7}{18}$$

9. An initial population, p, of 1500 bacteria grows in number according to the equation, $P(t) = 1500\left(1 + \dfrac{13t}{t^2+30}\right)$, where t is in hours and $0 \le t \le 9$. Determine the rate at which the population is growing when the population is double in size.

$P(0) = 1500$

$\therefore$ double in size $P(t) = 3000$

$$3000 = 1500\left(1 + \frac{13t}{t^2+30}\right)$$

$$2 = \left(1 + \frac{13t}{t^2+30}\right)$$

$$1 = \frac{13t}{t^2+30}$$

$$t^2+30 = 13t$$

$$t^2 - 13t + 30 = 0$$

$$(t-10)(t-3) = 0$$

$$t = \{3, 10\}$$

$\hookrightarrow$ inadmissible, $0 \le t \le 9$

$\therefore$ @ $t = 3$ population will double

$$P(t) = 1500 + \frac{19500t}{t^2+30}$$

$$P'(t) = \frac{[19500][t^2+30] - [2t][19500t]}{[t^2+30]^2}$$

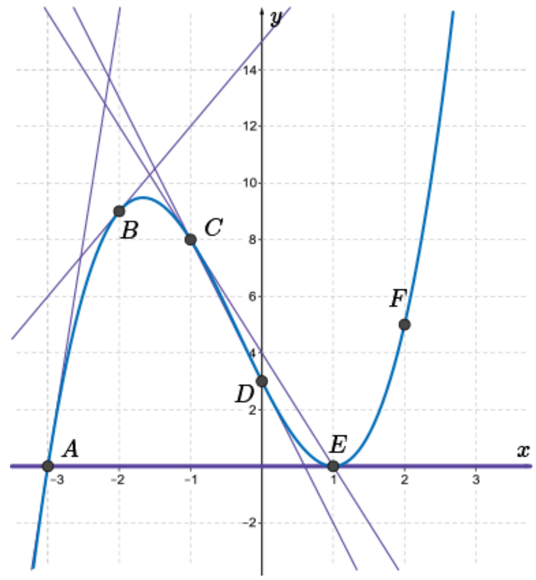
$$P'(3) = \frac{19500[9+30] - [6][58500]}{(9+30)^2}$$

$$\doteq 269.23 \text{ bacteria/hour}$$

$\therefore$ When the bacteria is doubled in size, it was growing at a rate of approx. 269 bacteria/hour if displacement is $s(t)$, then $V(t) = s'(t)$ and $A(t) = V'(t) = s''(t)$

2.5 Higher Order Derivatives, Velocity, and Acceleration

So far, we have seen that the value of the derivative, $f'(x)$, gives us the instantaneous rate of change of a function, $f(x)$, at a point. It is represented graphically by the slope of the tangent line to the curve, $y = f(x)$, at that point. Throughout the graph, the slope of the tangent line is continually changing. We can describe this change as the rate of change of the slope of the tangent. To determine how the slope of the tangent is changing, we differentiate the derivative function f′(x). If f′(x) is differentiable, then the derivative of the derivative function can be found.



This is known as the **second derivative of** f(x), and is denoted in function notation as $f''(x)$. or $f^{(2)}(x)$

In Leibniz notation, the second derivative is denoted as

$$\frac{d^2y}{dx^2} = \frac{d^2[f(x)]}{dx^2}.$$

$$\frac{d\left(\frac{dy}{dx}\right)}{dx}$$

Example 1

Find the second derivative of $f(x) = x^4 + 3x^2 - 5\sqrt{x}$

$$f'(x) = 4x^3 + 6x - \frac{5}{2}x^{-\frac{1}{2}}$$

$$f''(x) = 12x^2 + 6 + \frac{5}{4}x^{-\frac{3}{2}}$$

$$= 12x^2 + 6 + \frac{5}{4\sqrt{x^3}}$$

Example 2

Find $\frac{d^2y}{dx^2}$ given that $y = \frac{5x-3}{2x}$.

$$y = \frac{5}{2} - \frac{3}{2}x^{-1}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{-2}$$

$$\frac{d^2y}{dx^2} = -3x^{-3}$$

$$= -\frac{3}{x^3}$$

Method 2: Quotient Rule

$$y' = \frac{[5][2x] - [2][5x-3]}{[2x]^2}$$

$$= \frac{\cancel{10x} - \cancel{10x} + 6}{4x^2}$$

$$= \frac{6}{4x^2}$$

$$= \frac{3}{2}x^{-2}$$

$$y'' = -3x^{-3}$$

$$= \frac{-3}{x^3}$$

Third Derivatives

The second derivative of f(x) is found by taking the derivative of f(x) twice. This can be extended further if f″(x) is differentiable; taking the derivative of f″(x) gives the third derivative of f(x), which is denoted in function notation as f‴(x) or $f^{(3)}(x)$.

In Leibniz notation, the third derivative is denoted as shown.

$$\frac{d^3y}{dx^3}=\frac{d^3[f(x)]}{dx^3} = y''' = y^{(3)} = f^{(3)}(x)$$

Note that the brackets around the 3 are required in $f^{(3)}(x)$.

In general, if the derivatives remain differentiable, the nth derivative of f(x) is found by taking its derivative n times, and is denoted $f^{(n)}(x)$.

Example 3: Determine the third derivative of the following functions

a) $y=\frac{1}{x}$, $x \neq 6$

$y = x^{-1}$

$y' = -x^{-2}$, $x \neq 0$

$y'' = 2x^{-3}$

$= \frac{2}{x^3}$, $x \neq 0$

$f^{(3)}(x) = y''' = -6x^{-4}$

$= \frac{-6}{x^4}$, $x \neq 0$

b) $y=\frac{3}{2x-6}$ $x \neq 3$

$y = 3(2x-6)^{-1}$

$y' = -3(2x-6)^{-2}\cdot(2)$

$= -6(2x-6)^{-2}$

$y'' = 12(2x-6)^{-3}\cdot(2)$

$= 24(2x-6)^{-3}$

$f^{(3)} = y''' = -72(2x-6)^{-4}\cdot 2$

$= \frac{-144}{(2x-6)^4}$

$= \frac{-144}{2^4(x-3)^4}$

$= \frac{-9}{(x-3)^4}$, $x \neq 3$

$= \frac{24}{(2x-6)^3}$ ⇐ if asked not to simplify

$= \frac{24}{2^3(x-3)^3}$

$= \frac{3}{(x-3)^3}$ ⇐ fully simplified

$x \neq 3$

c) $y=\sqrt{x}$ $x \geq 0$, $x \in \mathbb{R}$

$y = x^{\frac{1}{2}}$

$y' = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$, $x > 0$

$y'' = -\frac{1}{4}x^{-\frac{3}{2}}$

$= -\frac{1}{4\sqrt{x^3}}$, $x > 0$

$f^{(3)} = y''' = \frac{3}{8}x^{-\frac{5}{2}}$

$= \frac{3}{8\sqrt{x^5}}$, $x > 0$

Example 4: Suppose $f(x)=ax^2+bx+c$ and $f(1)=8, f'(1)=3$, and $f''(1)=-4$. Determine a,b, and c.

$f(1) = a(1)^2+b(1)+c$

$8 = a+b+c$ ①

$f'(x) = 2ax+b$

$f'(1) = 2a(1)+b$

$3 = 2a+b$ ②

$f''(x) = 2a$

$f''(1) = 2a$

$-4 = 2a$

$a = -2$ ③

sub ③ and ④ into ①

$8 = (-2)+(7)+c$

$8 = 5+c$

$\therefore c = 3$

sub ③ into ②

$3 = 2(-2)+b$

$b = 7$ ④

$\therefore a=-2$

$b=7$

$c=3$

APPLICATIONS OF HIGHER ORDER DERIVATIVES – LINEAR MOTION

Definitions

Position s(t) is the location of an object at a value of time t. (Displacement ⇒ Position-time function)

Velocity v(t) is the rate of change of position over time, so

$$v(t) = s'(t) = \frac{ds}{dt} \quad (m/s)$$

Acceleration a(t) is the rate of change of velocity over time, so

$$a(t) = v'(t) = s''(t)$$

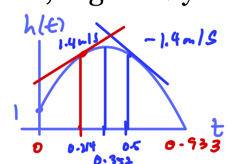
Or

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} \quad m/s^2 \text{ or } m\cdot s^{-2}$$

Example 1

The height of a soccer ball above the ground at time t after it is kicked into the air, is given by the formula

$$h(t) = -4.9t^2 + 3.5t + 1$$



where h is the height in metres, t is the time in seconds, and $t \geq 0$.

(a) Determine an equation that will calculate the instantaneous rate of change of the height of the ball with respect to time t.

$$v(t) = h'(t) = -9.8t + 3.5$$

(b) Determine the vertical velocity of the ball 0.5 seconds after it is kicked. At what other time will the ball have the same magnitude of ~~vertical~~ velocity (i.e., speed)?

$$v(t) = h'(0.5) = -9.8(0.5) + 3.5$$

$$= -1.4 \text{ m/s} \Leftarrow \text{velocity has magnitude and direction}$$

Speed 1.4m/s towards earth

$1.4 = -9.8t + 3.5$

$0.214 = t$

∴ at 0.214s it will be the same speed.

(c) At what point will the ball reach its maximum height? What is the maximum height reached by the ball?

recall, in grade 10

AOS : $x = \frac{-b}{2a}$

$h(t) = at^2 + bt + c$

$h'(t) = 2at + b \implies h'(t) = 0$

$0 = 2at + b$

$\therefore \frac{-b}{2a} = t \Leftarrow$ x-coordinate of the vertex!

$v(t) = h'(t) = 0, \quad -9.8t + 3.5 = 0$

$t = \frac{3.5}{9.8}$

$= \frac{5}{14}$ s

$\doteq 0.357$ s

$h(0.357) = -4.9(0.357)^2 + 3.5(0.357) + 1$

$\doteq 1.625$ m (max height)

(d) What is the instantaneous ~~vertical~~ velocity of the ball as it hits the ground?

$h(t) = 0, \quad t = \frac{-3.5 \pm \sqrt{3.5^2 - 4(-4.9)(1)}}{2(-4.9)}$

$= \frac{-3.5 \pm \sqrt{31.85}}{-9.8}$

$\doteq \{-0.218, 0.933\}$

↳ inadmissible $t > 0$

$h'(0.933) = -9.8(0.933) + 3.5$

$\doteq -5.64$ m/s

note!

$h'(0) = 3.5$ m/s is the initial velocity

Example 2

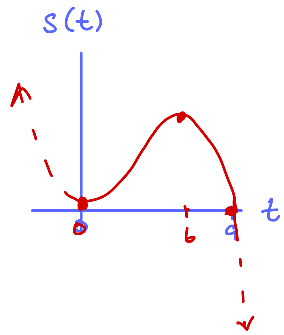
A freight train leaves a train station and travels due north on a straight track. After t hours, the train is $s(t)=18t^2-2t^3, 0\le t\le 9$ kilometers north of the train station .

(a) Find an expression for the velocity of the train at any time $0\le t\le 9$.

(b) Find the acceleration when the velocity is zero.

$s(t) = -2t^3+18t^2$

$= -2t^2(t-9)$



a) $v(t) = s'(t)$

$= -6t^2+36t \quad , \quad 0<t<9$

b) $v(t)=0$,

$0 = -6t^2+36t$

$0 = -6t(t-6)$

$t = \{0, 6\}$

$a(t) = v'(t)$

$a(t) = -12t+36$

$a(0) = -12(0)+36$
$= 36 \ m/s^2$

(Concave up)

and

$a(6) = -12(6)+36$
$= -36 \ m/s^2$

(Concave down)

Wait till Unit 3 : Note! Acceleration $\neq$ speeding up or slowing down

Speeding up : $v(t) \cdot a(t) > 0$

Slowing down : $v(t) \cdot a(t) < 0$

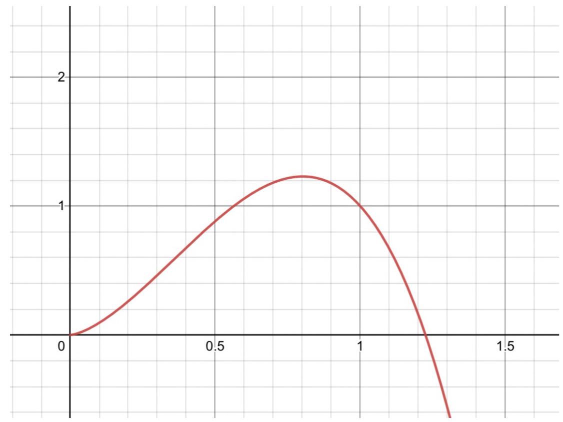
Example 3

The position function of a marble moving along a track is $s(t)=\left(3-2t^2\right)t^{\frac{3}{2}}$, at time t, in seconds.

(a) Find the marble's velocity and acceleration at time t.

(b) When does the marble return to its starting position, s(0)?

(c) When is the marble at rest?



$t \geq 0$

a) $s(t)=(3-2t^2)\,t^{\frac{3}{2}}$

$s(t) = 3t^{\frac{3}{2}} - 2t^{\frac{7}{2}}$

$v(t) = s'(t)$

$= \frac{9}{2}t^{\frac{1}{2}} - 7t^{\frac{5}{2}}$

$a(t) = s''(t)$

$= v'(t)$

$= \frac{9}{4}t^{-\frac{1}{2}} - \frac{35}{2}t^{\frac{3}{2}}$

b) $s(0) = 0$, $0 = (3-2t^2)(t^{\frac{3}{2}})$

$3-2t^2=0$ $\quad$ $t^{\frac{3}{2}}=0$

$t^2 = \frac{3}{2}$ $\quad$ $t=0$

$t = \pm\sqrt{\frac{3}{2}}$

$= \pm\frac{\sqrt{6}}{2}$, $t>0$

$\therefore t = \{0, \frac{\sqrt{6}}{2}\}$

$\therefore$ it returns back to its orginal position at $\frac{\sqrt{6}}{2}$ seconds.

c) marble is @ rest when $v(t)=0$,

$\frac{9}{2}t^{\frac{1}{2}} - 7t^{\frac{5}{2}} = 0$

$t^{\frac{1}{2}}\left(\frac{9}{2} - 7t^2\right) = 0$

$t^{\frac{1}{2}}=0$ $\quad$ $\frac{9}{2} - 7t^2 = 0$

$t=0$ $\quad$ $\frac{9}{2} = 7t^2$

$\frac{9}{14} = t^2$

$\pm\sqrt{\frac{9}{14}} = t$, $t>0$

$\therefore t = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$

$\therefore$ at $\frac{3}{\sqrt{14}}$ s, it will be @ rest

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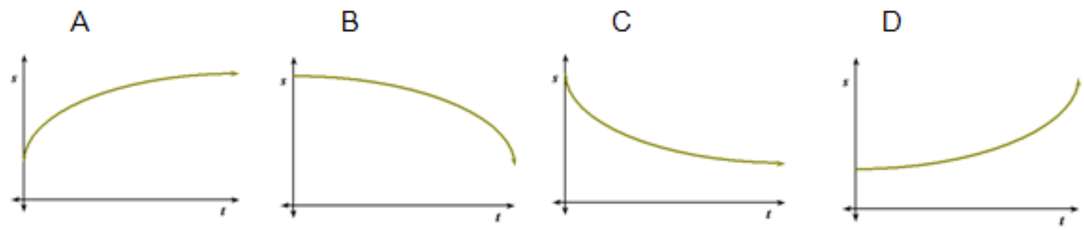
Practice 2.5

1. Find the first and second derivatives of each function.

a) $f(x) = 2x^4 - 4x^{-2}$ **b)** $y = \dfrac{3}{x^2}$ **c)** $y = \dfrac{2x+1}{x}$

d) $g(x) = (x-1)(x+1)^3$ **e)** $y = \dfrac{x^2-4}{x+1}$ **f)** $s(t) = t^3 + \dfrac{2}{\sqrt{t}}$

2. A boat demonstrates a positive velocity but a negative acceleration. Which of the following plots illustrates its position?

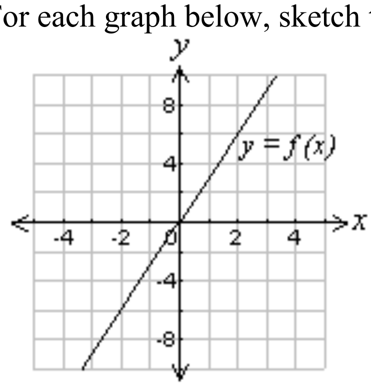


3. A particle moves on the y axis with this relationship between position and time: $s(t) = t^3 - 17t^2 + 80t - 100$. Determine the time interval(s) during which it is :

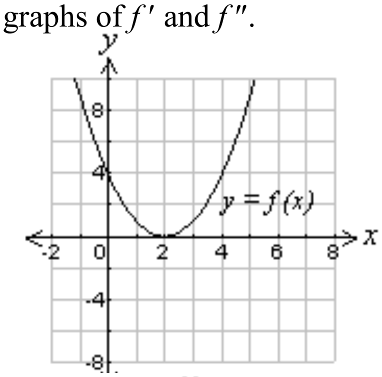
a) located below the origin
b) moving upward

4. For each graph below, sketch the corresponding graphs of f' and f''.

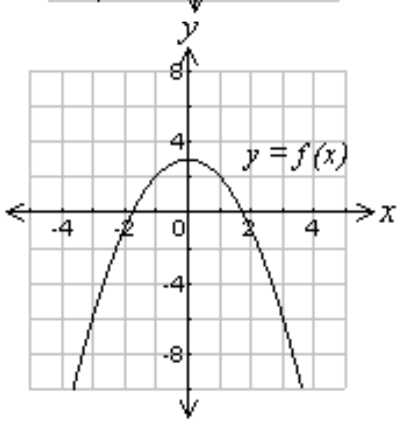
a)



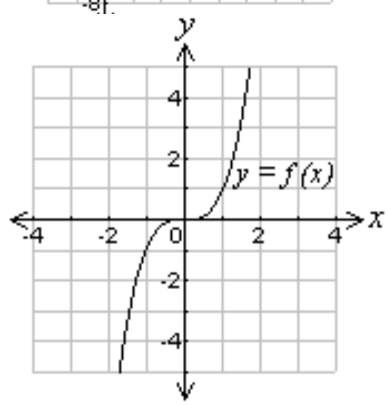
b)



c)



d)



5. For what values of the constants a, b, c, and d does the function $f(x) = ax^3 + bx^2 + cx + d$ satisfy both of the following conditions?

- $f''(0) = 0$ at the origin
- a horizontal tangent at (2, 4)

6. A person's height, in metres, can be modelled by the function $h(t) = \frac{at}{b+t} + c$, where t is the age of the person, in years, and a, b, and c are positive constants.

a) $h'(t)$ represents the growth rate. What does $h''(t)$ represent?

b) Show that $h''(t)$ is always negative. What does this indicate about the growth rate?

c) Show that

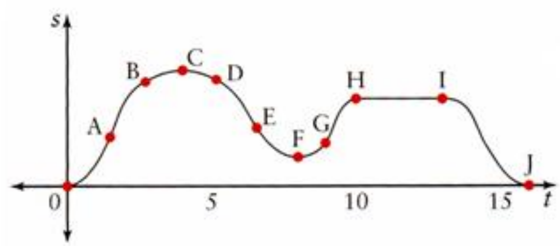
i) the initial height is c

ii) the initial growth rate is $\frac{a}{b}$

d) Suggest reasonable values for the constants.

e) In what way(s) is the function not a realistic model for the height of a person?

7. The following graph shows the position function of a bus during a 15-min trip.



a) What is the initial velocity of the bus?

b) What is the bus's velocity at C and at F?

c) Is the bus going faster at A or at B? Explain.

d) What happens to the motion of the bus between H and I?

e) Is the bus speeding up or slowing down at A, B, and D?

f) What happens at J?

8 . Refer to the graph in question 7. Is the acceleration positive, zero, or negative during the following intervals?

a) 0 to A

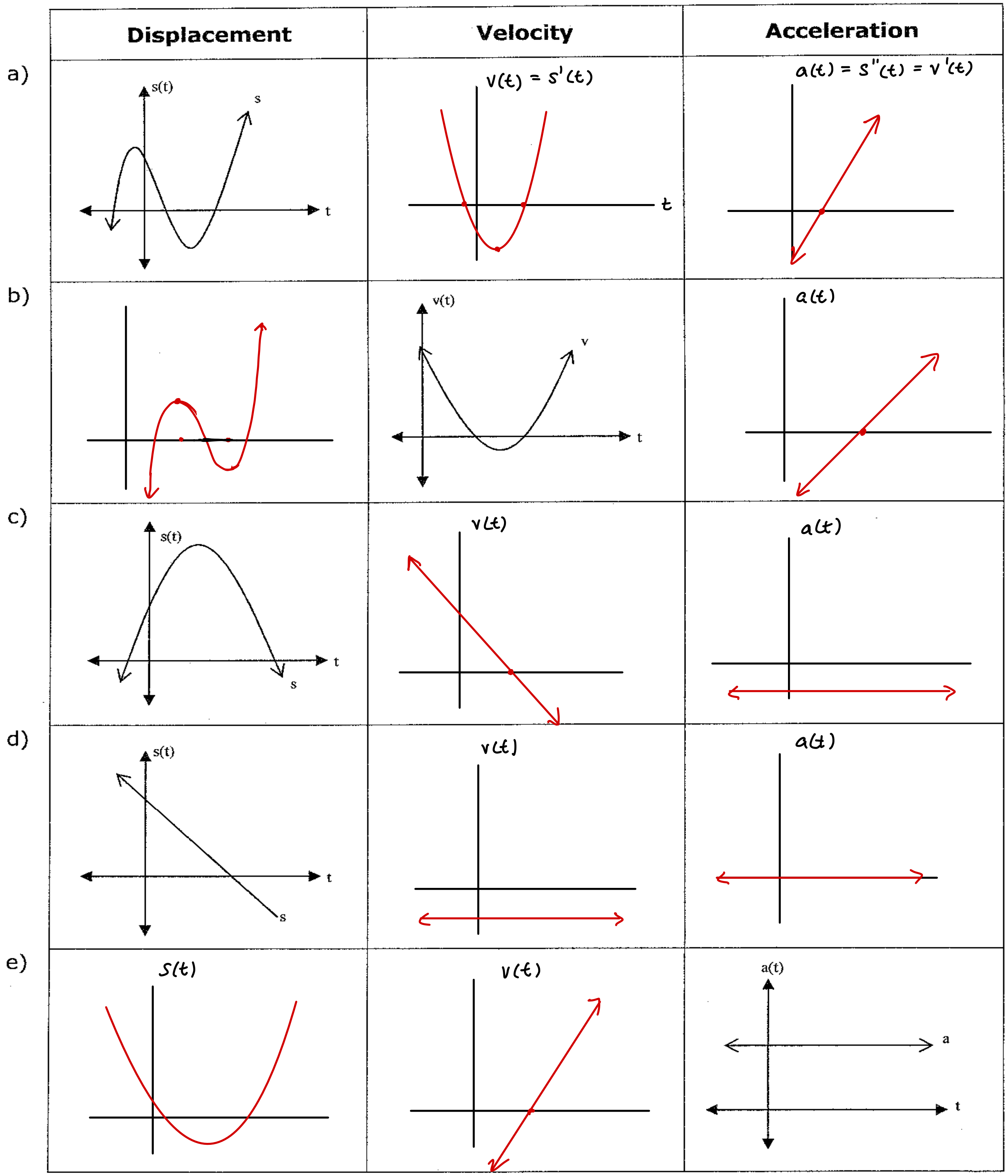
b) C to D

c) E to F

d) G to H

e) F to G

9. Create sketches so that each graph in a set corresponds to the other two.



$a(t) > 0$ concave up

$a(t) < 0$ concave down

Unit 2: Review

1. Differentiate the following:

   a) $m(t)=\frac{\pi}{3}t^3-3t^{-5}+4\pi^2$

   b) $f(x)=\left(1+x^{\frac{3}{4}}\right)\left(\sqrt{x+\sqrt{x}}\right)$ (do not simplify)

   c) $g(x)=\frac{x^3+4}{x^3-3x+1}$

   d) $y=\frac{(3x^2-1)^{-4}}{(x^3-2x)^{-5}}$

2. If $g(x)=\frac{1}{2x-4}+\sqrt{x}$, find $g'''(4)$.

3. For what value(s) of *k* will the line $2x-3y+k=0$ be normal to $y=\sqrt{3x^2+4}$?

4. Find the rate of change for $s(t)=\left(\frac{t-\pi}{t-10\pi}\right)^{\frac{1}{3}}$ at $t=2\pi$. Leave final answer in terms of π.

5. Two tangents are drawn from the point (2,6) to the graph of $y=-x^2-5x+4$. Determine the coordinates of the point(s) where the tangents touch the graph.

6. For what values of *a* and *b* will the parabola $y=x^2+ax+b$ be tangent to the curve $y=x^3+5x$ at the point x=1?

7. A 1500-L tank leaks water so that the volume of water, in litres, remaining after *t* days, $0\le t\le 15$, is represented by $V(t)=1500\left(1-\frac{t}{15}\right)^2$. How rapidly is the water leaking when the tank is $\frac{1}{9}$ full? Round final answers to 2 decimal places.

8. Find the values of *x* so that the tangent to $f(x)=\frac{3}{\sqrt[3]{x}}$ is parallel to the line *x* + 16*y* + 3 =0.

9. Find **a** and **b** so that the line y = - ax +4 is tangent to the graph of y= ax^3 + bx at $x=1$.

10. Find the constant value(s) of ***k*** such that the equation of tangent to the curve $f(x)=\sqrt{1-kx^2}$ at $x=1$ is parallel to the line $3x-2y+1=0$.

11. Two lines drawn from point $A\left(0,\frac{7}{4}\right)$ are tangent to the parabola $y=1-x^2$ at P and Q. Find the area of triangle APQ.

12. Let f be a function given by $f(x)=\dfrac{ax^2+b}{x+c}$ and that has the following properties

$\lim_{x\to -1^-} f(x)=\infty$, $f'(0)=2$, $f''(0)=-2$. Determine the values of a , b and c

13. Let $f(x)=\sqrt{ax^2+b}$. Find values of a and b such that the linear equation $7x+2y=5$ is tangent to $f(x)$ at $x=-1$.

14. Find the area of the triangle determined by the coordinate axes and the tangent to the curve $xy = 1$ at $x = 1$.

15. Consider the curve $y=a\sqrt{x}+\dfrac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

16. The equation of the tangent to $y = 2x^2 - 1$ at the point where $x = 1$, is $4ax-y=2b^2+1$. Find the values of a and b.

17. Find $\boldsymbol{a}$ and $\boldsymbol{b}$ so that the line $y=-4x+1$ is tangent to the graph of $y = \dfrac{a}{x}+\dfrac{b}{x+1}$ at $x = 1$.

18. The curve $y = 2x^3 + ax + b$ has a tangent with slope 10 at the point (-2, 33). Find the values of a and b.

19. Find *f′(x),f″(x), and f‴(x)* for the following functions.

a. $f(x)=(2-3x)^5$ b. $f(x)=x^{12}+3x^4$

c. $f(x)=\sqrt{x}-\dfrac{1}{x}$ d. $f(x)=\dfrac{4}{(x-3)^2}$

20. Tangents are drawn to the curve $y = x^2$ at (2,4) and $\left(\dfrac{-1}{8},\dfrac{1}{64}\right)$, Prove that these lines are perpendicular. Illustrate with a sketch.

21. Let P(a,b) be a point on the curve $\sqrt{x}+\sqrt{y}=1$. Show that the slope of the tangent at P is $-\sqrt{\dfrac{b}{a}}$.

CP 32 Review #3 Hmwk Take up

3. For what value(s) of k will the line $2x-3y+k=0$ be normal to $y=\sqrt{3x^2+4}$?

$$y' = \frac{3x}{\sqrt{3x^2+4}} \qquad \Rightarrow \frac{\cancel{3}x}{\sqrt{3x^2+4}} = -\frac{\cancel{3}}{2}$$

$$m = \frac{2}{3} \rightarrow m_{\perp} = -\frac{3}{2} \qquad \sqrt{3x^2+4} = -2x \ (x<0)$$

$$3x^2+4 = 4x^2$$

$$x^2 = 4$$

$$x = \pm 2 \xrightarrow{x<0} x = -2$$

$$y = \sqrt{3(-2)^2+4}$$

$$y = 4$$

sub. (-2,4) into $2x-3y+k=0$ to get k.

$$-4-12+k=0$$

$$k = 16$$

$$y = \sqrt{3x^2+4}$$

$$y' = \frac{1}{2}(3x^2+4)^{-\frac{1}{2}} \cdot 6x$$

$$= \frac{3x}{\sqrt{3x^2+4}}$$

$$2x-3y+k=0$$

$$2x+k = 3y$$

$$\frac{2x+k}{3} = y$$

$$\therefore m = \frac{2}{3}$$

$$\therefore m_{\perp} = -\frac{3}{2}$$

$$y' = m_{\perp}$$

$$\frac{\cancel{3}x}{\sqrt{3x^2+4}} = -\frac{\cancel{3}}{2}$$

$$2x = -\sqrt{3x^2+4}$$

$$-2x = \sqrt{3x^2+4} \Leftarrow \text{note!} \qquad -2x \geq 0$$

$$(-2x)^2 = 3x^2+4 \qquad \therefore x \leq 0$$

$$4x^2 = 3x^2+4$$

$$x^2 = 4$$

$$x = \{-2, 2\}$$

$\hookrightarrow$ extraneous root

$$\therefore x = -2$$

$$\therefore f(-2) = \sqrt{3(-2)^2+4}$$

$$= 4$$

Sub $(-2,4)$ into the normal line:

$$2(-2) - 3(4) + k = 0$$

$$-4 - 12 + k = 0$$

$$k = 16$$