Grade 12 Calculus & Vectors
Unit 3: Applying Properties of Derivatives – Curve Sketching

DAY	DESCRIPTION	Homework
1	**3.1 Maximum & Minimum points (Also in an interval)** We are Learning to... • find the absolute maximum and absolute minimum values of a function in a given interval. I am able to... • identify and distinguish absolute extrema and local extrema	Pg 163:#1,2(only absolute extrema) ,3,6 a,c ,8,9 CP pages 7-8
2	**3.2 Interval of Increasing & Decreasing -First Derivative Test** We are learning how to … • find intervals of increase and decrease of a function. • determine the local maximum and minimum of a function graphically and algebraically (First Derivative Test) I am able to … • determine intervals of increasing /decreasing of a function • determine the relative extremes by using FDT	Pg 156: #4-8,10 Pg 163:# 17,18,22,23 CP. Page 13
3	**3.3 Concavity and the Second Derivative Test** We are Learning to... • the intervals where the function is concave up and concave down • to find the points of inflection I am able to... • use **Second Derivative Test** to determine whether a given critical point of a real function of one variable is a local maximum or a local minimum using the value of the second derivative at the point. • determine the point of inflection of a function	Pg 173: #1-6,9-11,13-14, 16
4	**3.4 Vertical , Horizontal & Oblique Asymptotes** We are learning to • find the equation of an oblique (slant) asymptote in a rational function • to find the equation of a horizontal and vertical asymptote in a rational function I am able to… • find the equation of an oblique asymptote as well as identifying the equation of horizontal and vertical asymptotes in a rational function	Pg 183: #1-4,11-12
5 6 7	**3.5 I. Algorithm for Curve Sketching (2 days)** **II. Curve Sketching from Given Information** We are Learning to... • sketch polynomial and rational functions using calculus methods: intercepts; (b) asymptotes and their behaviour; (c) local extrema and intervals of increase and decrease; (d) points of inflection and concavity I am able to … • sketch the graph of a function using the key information	Pg 192: #1,2,5-7,13,14-17 Pg 209: #15-17 Part I. CP pg # 32 Part II. CP pg # 37 Part III. CPpg # 39-40
8/9	**Quiz/Review**	Pg 204: #1-14
10	**Summative Evaluation**	

W Oct 23 (Day 1) - Revised Th Oct 24
Th Oct 24 (Day 2) - F Oct 25
Th Oct 30 (COMC)

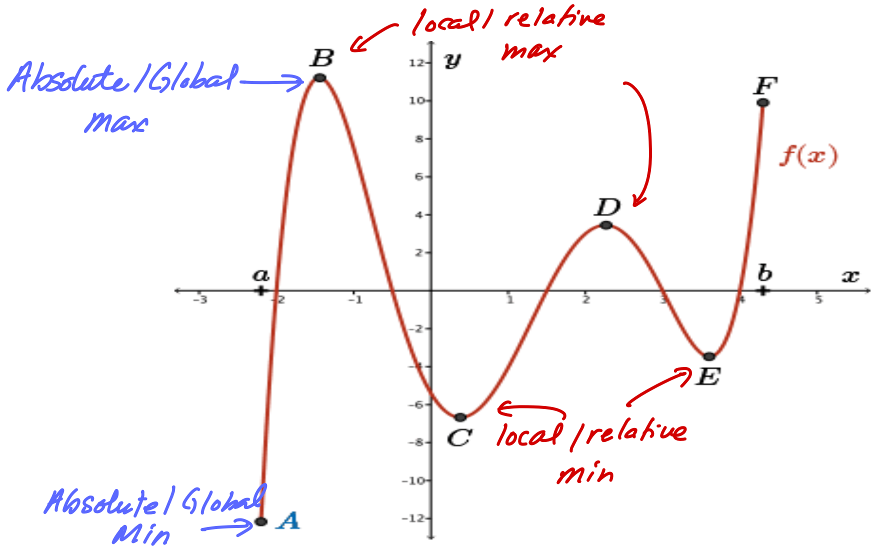
UNIT 3

CURVE SKETCHING

3.1 Extrema on an interval

Part I. Extreme Values

Consider the following graph of $y=f(x)$ with domain restricted to a closed interval, $[a,b]$. The high and low points within the curve have been labelled along with the graph's boundary end points. The highest point on the graph is point B. Therefore, point B is known as the **absolute maximum** of $f(x)$ on the interval $[a,b]$. The lowest point on the graph is point A. Therefore, point A is known as the **absolute minimum** of $f(x)$ on $[a,b]$.



In this case, point A is a **boundary point** of the closed interval $[a,b]$, specifically $x=a$.

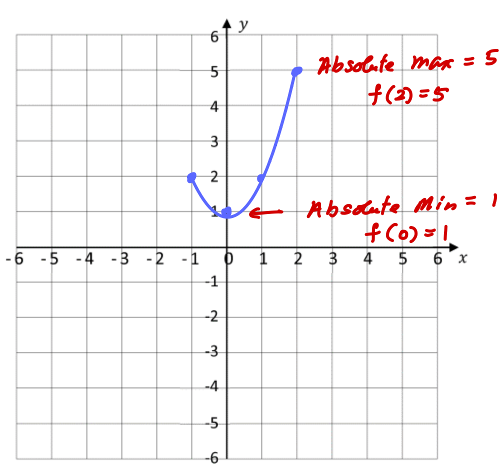
Definitions

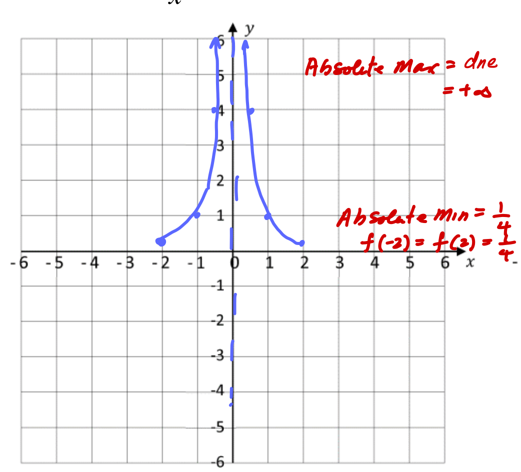
A function, f, has an absolute maximum at c if $f(c) \geq f(x)$ for all x in the domain of f.

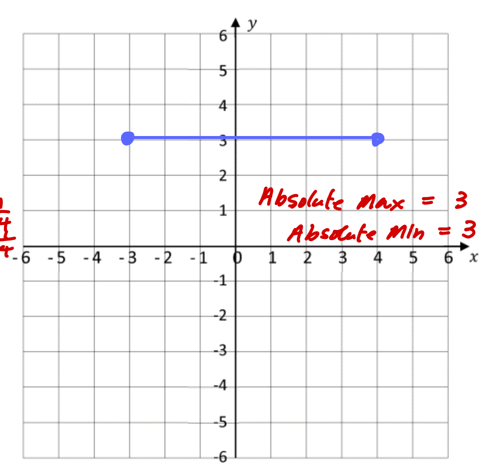
A function, f, has an absolute minimum at c if $f(c) \leq f(x)$ for all x in the domain of f.

Example 1: Sketch the following functions on the given interval, then, determine the extrema, if they exist

a) $f(x)=x^2+1$ on $[-1,2]$ b) $f(x)=\dfrac{1}{x^2}$ on $[-2,2]$ c) $f(x)=3$ on $[-3,4]$





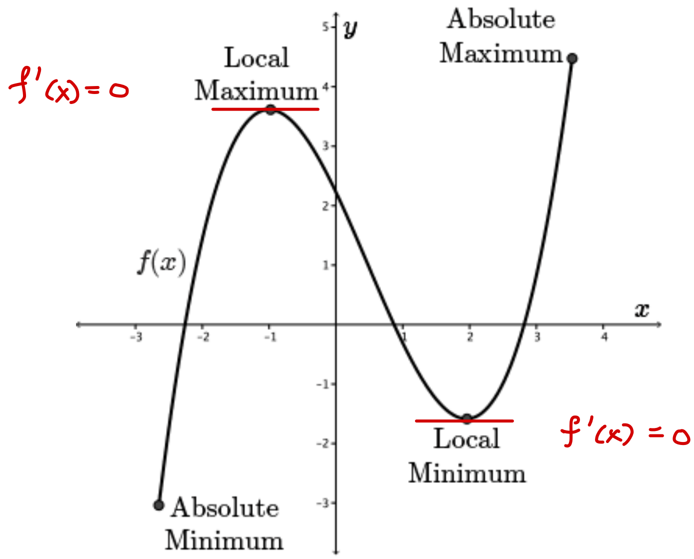


What other type of extrema are there?

Definition of Local Maximum and Minimum Values

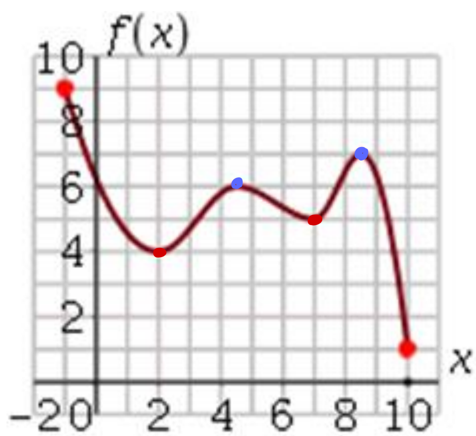
1. Function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ for all x sufficiently close to c.

2. Function f has a local minimum (or relative minimum) at c if $f(c) \leq f(x)$ for all x sufficiently close to c.



Example 2:

The graph of $f(x)$ is given below. Identify the extrema, both relative and absolute, on the interval [-1, 10] .



Local maximum value(s):--- 6 and 7 or $f(4.5)=6$, $f(8.5)=7$

Local minimum value(s):--- 4 and 5 or $f(2)=4$, $f(7)=5$

Absolute maximum value:--- 9 or $f(-1)=9$

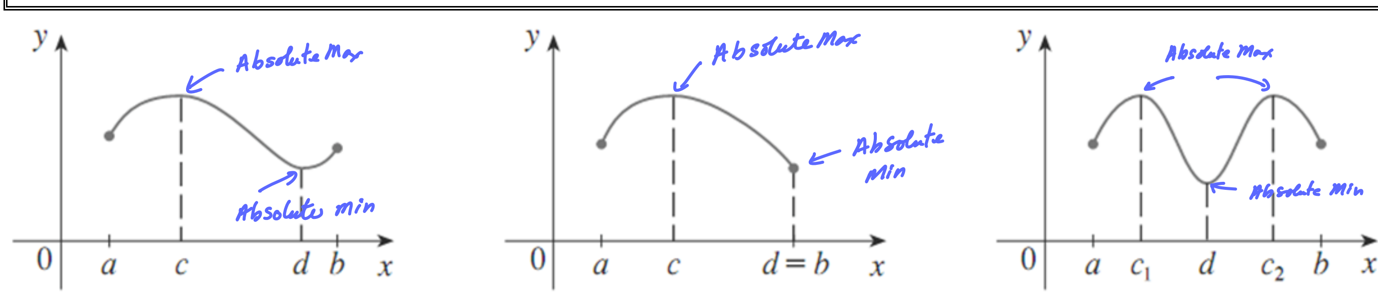
Absolute minimum value:--- 1 $f(10)=1$

Under what conditions will an Absolute max and an Absolute min both occur?

↳ over a closed interval with continuous function

The Extreme Value Theorem (EVT)

A continuous function f(x) defined on a **closed, bounded** interval [a, b] attains both an absolute maximum and an absolute minimum on that interval.



If the hypothesis is not met, either the continuity or the closed interval part, there is not guarantee of the conclusion.

Definition of Critical Points

"Points of Interest"
↳ a local max / min
↳ inflection point or vertical tangent or cusp / corner

A point $x=c$ **in the domain of** $f(x)$, at which $f'(c)=0$ or $f'(c) = dne$ is called a critical point of $f(x)$.

↳ local max / min
↳ horizontal tangent (inflection)

HT (inflection)

↳ local max / min
↳ cusp / corner
↳ vertical tangent (inflection)

max, min, VT (inflection)

Example 3: Find all the critical ~~point~~(s) value(s) of the following functions.

a) $f(x)=6x^5+33x^4-30x^3+100$ $\Rightarrow D_f: \{x \in \mathbb{R}\}$

$f'(x) = 30x^4 + 132x^3 - 90x^2$

$f'(x) = 0$ or $f'(x) = dne$

$x = \{\ \}$

$30x^4+132x^3-90x^2=0$

$6x^2(5x^2+22x-15)=0$

$6x^2(5x-3)(x+5)=0$

$x = \{0, \frac{3}{5}, -5\}$

$\therefore$ the critical values: $\{0, \frac{3}{5}, -5\}$

b) $f(x)=\dfrac{x^2+1}{x^2-x-6}$. $D_f: \{x \in \mathbb{R}, x \neq 3, -2\}$

$= \dfrac{x^2+1}{(x-3)(x+2)}$

$f'(x) = \dfrac{2x(x^2-x-6)-(2x-1)(x^2+1)}{(x-3)^2(x+2)^2}$

$= \dfrac{\cancel{2x^3} - 2x^2 - 12x - \cancel{2x^3} - 2x + x^2 + 1}{(x-3)^2(x+2)^2}$

$= \dfrac{-x^2-14x+1}{(x-3)^2(x+2)^2}$

Critical #'s:

$f'(x) = 0$ or $f'(x) = dne$

$-x^2-14x+1=0$

$x^2+14x-1=0$

$x = \dfrac{-14 \pm \sqrt{14^2-4(1)(-1)}}{2(1)}$

$= \dfrac{-14 \pm \sqrt{200}}{2}$

$= \dfrac{-14 \pm 10\sqrt{2}}{2}$

$= -7 \pm 5\sqrt{2}$

$(x-3)^2(x+2)^2 = 0$

$x = \{3, -2\}$

note! outside the domain

$\therefore$ the critical #'s: $\{-7 \pm 5\sqrt{2}\}$

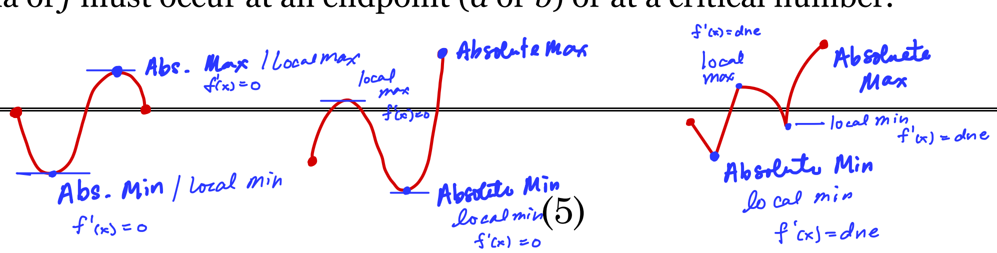
Fermat's Theorem

Suppose that $f(c)$ is a local extremum. Then c must be a critical number of f.

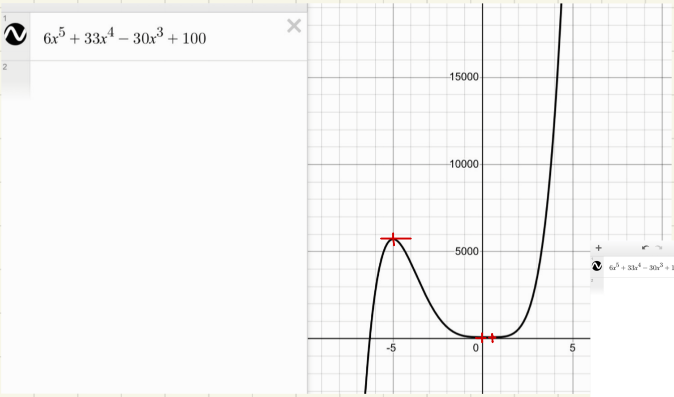
Fermat's Corollary

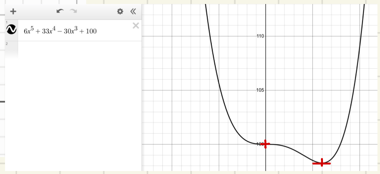
Suppose that f is continuous on the closed interval $[a, b]$. Then, the absolute extrema of f must occur at an endpoint (a or b) or at a critical number.

Pierre de Fermat
(1601–1665)

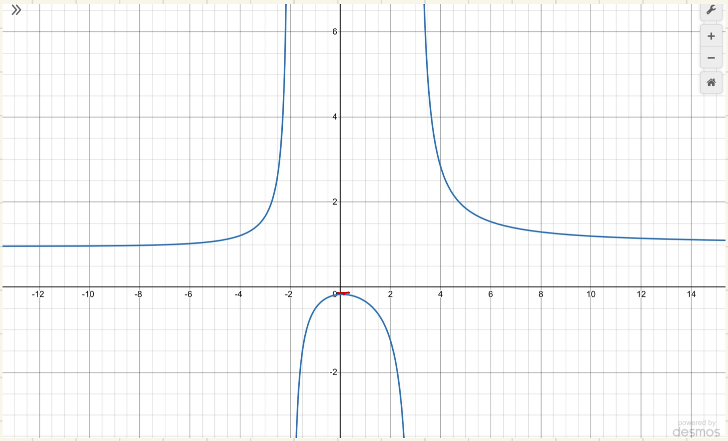


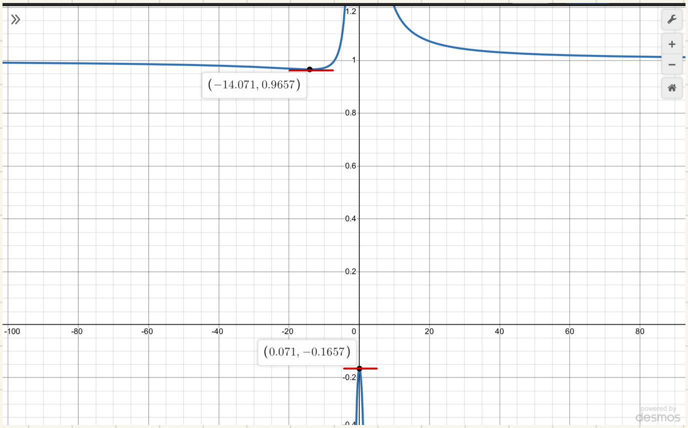
Example 3 a)





Example 3 b)





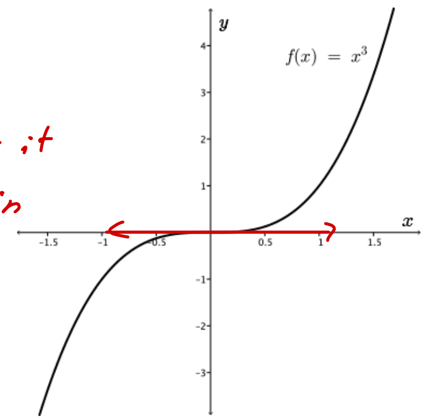
Special Note on Fermat's Theorem and Critical Numbers

if $f'(c) = 0$ or $f'(c) = dne$
c value could be local max / min or inflection point

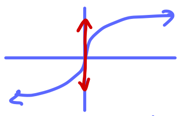
Fermat's theorem is true when read forwards, but not necessarily true when read backwards. That is, not all values of c that have $f'(c)=0$ are local maximums or minimums.

Example: Consider the function $f(x)=x^3$.

$f'(x) = 3x^2$
$0 = 3x^2$
$x = 0$ ⇐ critical # but it is not a local max or min but an inflection point



$g(x) = x^{\frac{1}{3}}$



$g'(x) = \frac{1}{3} x^{-\frac{2}{3}}$
$= \frac{1}{3\sqrt[3]{x^2}}$

Critical #'s : $f'(x) = 0$ or $f'(x) = dne$
$3\sqrt[3]{x^2} = 0$
$x = 0$ ⇐ critical # not a local max / min but a vertical tangent

Steps for Finding the Extreme Values

Step 1. Identify all of the critical points within the closed interval by finding the values of c where $f'(c)=0$ and where $f'(c)$ does not exist. Then, find $f(c)$ for each critical number.

Step 2. Find the values of f at the boundary points of the closed interval. In other words, find $f(a)$ and $f(b)$.

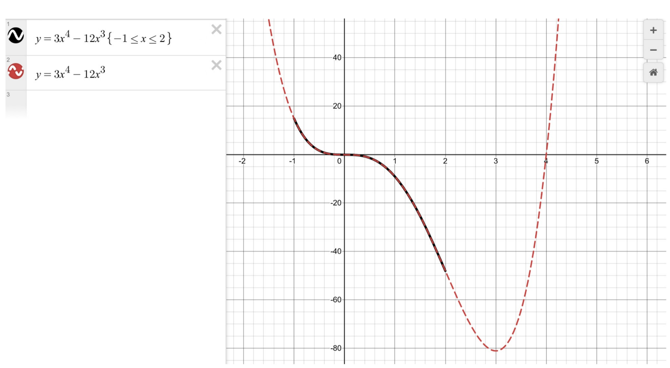
Step 3. Examine the values of f resulting from step 1 and step 2 and determine which value is the greatest (absolute maximum) and which value is the least (absolute minimum).

Example 4: Find the absolute extrema of $f(x) = 3x^4 - 12x^3$ on the interval $[-1, 2]$.

$f'(x) = 12x^3 - 36x^2$

$f'(x) = 0$
$12x^3 - 36x^2 = 0$
$12x^2(x-3) = 0$
$x = \{0, 3\}$ ↳ outside of the bound
∴ critical #'s : $\{0\}$

$f'(x) = dne$
$x = \{\ \}$ or ϕ



$f(-1) = 15$ ⇒ Absolute max. value
$f(0) = 0$
$f(2) = -48$ ⇒ Absolute min. value

Example 5: Determine if the EVT applies. If so, find the absolute extrema of $f(x)=2x-3x^{\frac{2}{3}}$ on the interval $[-8,1]$

$d_f: x \in \mathbb{R}$, for $x \in [-8,1]$ $\therefore$ EVT applies

- closed interval and continuous throughout the interval

$\hookrightarrow$ implies an Absolute Max + Absolute Min can be found.

$$f'(x) = 2 - 2x^{-\frac{1}{3}} = 2 - \frac{2}{\sqrt[3]{x}} = \frac{2\sqrt[3]{x} - 2}{\sqrt[3]{x}}, \quad x \neq 0$$

$$\boxed{\begin{array}{lll} f'(x) = 0 & \text{or} & f'(x) = \text{dne} \\ 2\sqrt[3]{x} - 2 = 0 & & \sqrt[3]{x} = 0 \\ \sqrt[3]{x} = 1 & & x = 0 \\ x = 1 & & \end{array}}$$

$f'(x) = 0$

$0 = 2 - \frac{2}{\sqrt[3]{x}}$

$\frac{2}{\sqrt[3]{x}} = 2$

$1 = \sqrt[3]{x}$

$x = 1$

$f'(x) = \text{dne}$

$x = 0$

$\therefore$ critical #'s: $\{0, 1\}$

$f(-8) = -28 \Rightarrow$ Absolute min

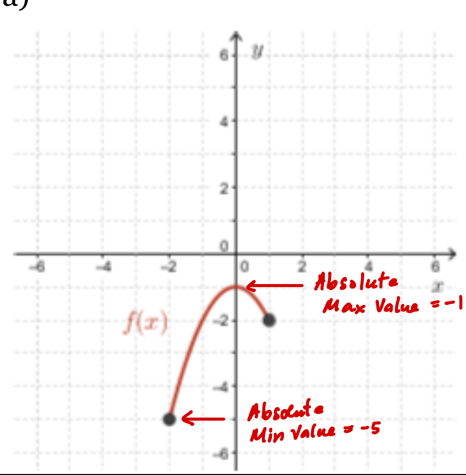
$f(0) = 0 \Rightarrow$ Absolute Max

$f(1) = -1$

3.1 Practice

1. Using the graphs provided, find the absolute minimum and absolute maximum value of f(x) on the given interval. If there is no maximum or minimum, explain which part of the extreme value theorem is not satisfied.

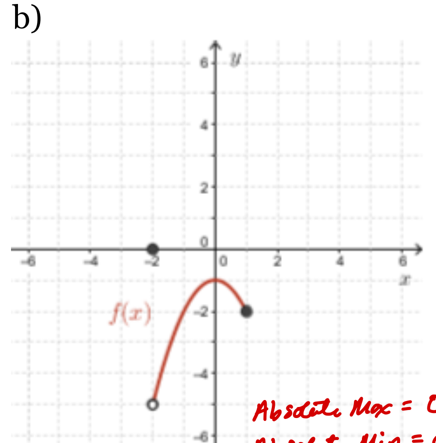
a)



Absolute Max Value = -1

Absolute Min Value = -5

b)

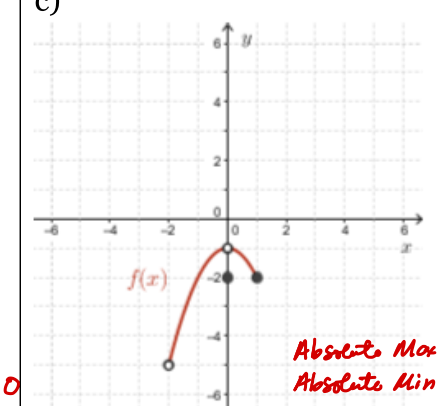


Absolute Max = 0

Absolute Min = dne

- though it is a closed interval $[-2,1]$, it is not continuous @ $x=-2$

c)



Absolute Max = dne

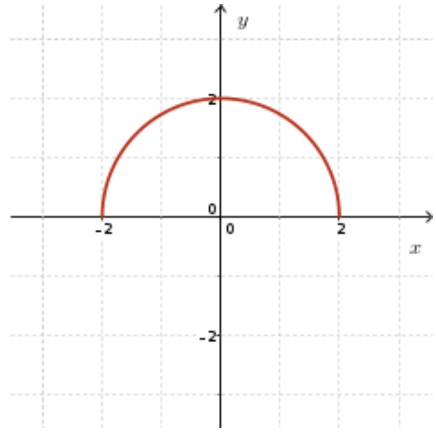
Absolute Min = dne

- not a closed interval $(-2,1]$ and not continuous at $x=0$

2. Find the absolute maximum and minimum values of the function $f(x)=3x^4-4x^3$ on the interval $[-1,2]$

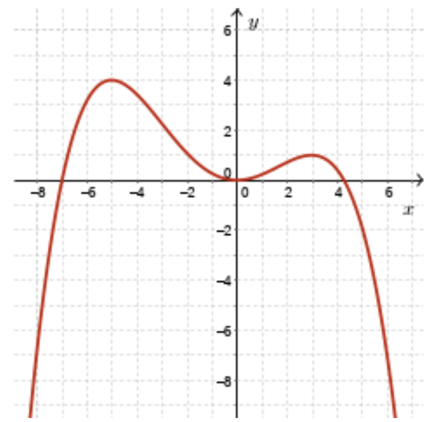
3. The graph of $f(x)=\sqrt{4-x^2}$ is shown to the right.

   Determine the absolute maximum and absolute minimum values, if they exist, on each of the following intervals, and determine at which point(s) they occur.



   a. $[-2,2]$
   b. $[-2,0)$
   c. $[1,2)$

4. Consider the graph of y=f(x), shown to the right. For each of the following intervals, determine the location of all turning points over the interval, and determine the absolute maximum and minimum values of f(x) over the interval, if they exist.



   a. $(-\infty,\infty)$
   b. $[-5,3]$
   c. $[0,5]$
   d. $(-3,1)$

5. Find the absolute maximum and minimum values for each of the following functions on the indicated intervals:

   a. $f(x)=x^3-3x^2-9x+5$ on $[0,4]$
   b. $g(x)=\dfrac{1}{x}$ on $[-3,-1]$
   c. $h(x)=x^4-2x^2+1$ on $\left[-\frac{5}{4},\frac{3}{2}\right]$
   d. $s(x)=\begin{cases}2+\sqrt{x} & \text{if } x>0\\ 2+\sqrt{-x} & \text{if } x\le 0\end{cases}$ on $[-3,4]$
   e. $f(x)=\begin{cases}x^2+2x+1 & \text{if } x\le 1\\ -3x+7 & \text{if } x>1\end{cases}$ on $[-2,3]$

2. Find the absolute maximum and minimum values of the function $f(x) = 3x^4 - 4x^3$ on the interval $[-1,2]$

$f'(x) = 12x^3 - 12x^2$

$= 12x^2(x-1)$

$D_f : \{x \in \mathbb{R}\}$

Critical #'s :

$f'(x) = 0$

$12x^2(x-1) = 0$

$x = \{0, 1\}$

$f(-1) = 7$

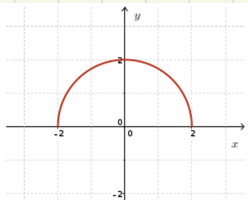
$f(0) = 0$

$f(1) = -1 \Leftarrow$ Absolute min.

$f(2) = 16 \Leftarrow$ Absolute Max.

3. The graph of $f(x) = \sqrt{4-x^2}$ is shown to the right.

Determine the absolute maximum and absolute minimum values, if they exist, on each of the following intervals, and determine at which point(s) they occur.



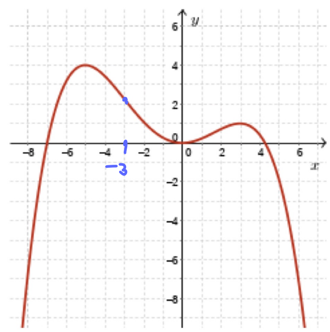
a. $[-2,2]$ Absolute max : $f(0) = 2$; Absolute min : $f(-2) = f(2) = 0$

b. $[-2,0)$ Absolute min : $f(-2) = 0$, No Absolute Max (open interval @ $x = 0$)

c. $[1,2)$ Absolute Max : $f(1) = \sqrt{3}$, No Absolute min (open interval @ $x = 2$)

4. Consider the graph of y=f(x), shown to the right. For each of the following intervals, determine the location of all turning points over the interval, and determine the absolute maximum and minimum values of f(x) over the interval, if they exist.

a. $(-\infty,\infty)$
b. $[-5,3]$
c. $[0,5]$
d. $(-3,1)$



	a) $(-\infty, \infty)$	b) $[-5, 3]$	c) $[0,5]$	d) $(-3,1)$
Turning Points	$f(-5) = 4$ $f(0) = 0$ $f(3) = 1$	$f(0) = 0$	$f(3) = 1$	$f(0) = 0$
Absolute Max	$f(-5) = 4$	$f(-5) = 4$	$f(3) = 1$	None
Absolute Min	None	$f(0) = 0$	$f(5) = -2$	$f(0) = 0$

5. Find the absolute maximum and minimum values for each of the following functions on the indicated intervals:

a. $f(x) = x^3 - 3x^2 - 9x + 5$ on $[0,4]$

b. $g(x) = \frac{1}{x}$ on $[-3,-1]$

c. $h(x) = x^4 - 2x^2 + 1$ on $\left[-\frac{5}{4}, \frac{3}{2}\right]$

d. $s(x) = \begin{cases} 2+\sqrt{x} & \text{if } x > 0 \\ 2+\sqrt{-x} & \text{if } x \le 0 \end{cases}$ on $[-3,4]$

e. $f(x) = \begin{cases} x^2+2x+1 & \text{if } x \le 1 \\ -3x+7 & \text{if } x > 1 \end{cases}$ on $[-2,3]$

5a) $f(x) = x^3 - 3x^2 - 9x + 5, \ x \in [0,4]$

$f'(x) = 3x^2 - 6x - 9$

Critical #'s: $f'(x) = 0$

$3x^2 - 6x - 9 = 0$

$3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$

$\therefore x = \{3, -1\}$

↳ out of bound

$f(0) = 5 \Leftarrow$ Absolute Max

$f(3) = -22 \Leftarrow$ Absolute Min

$f(4) = 1$

b) $g(x) = \frac{1}{x}, \ x \in [-3,-1]$

$g'(x) = -x^{-2}$

$g'(x) = -\frac{1}{x^2}$

$g(-3) = -\frac{1}{3} \Leftarrow$ Absolute max

$g(-1) = -1 \Leftarrow$ Absolute min

Critical #'s: $g'(x) = $ dne

$-\frac{1}{x^2} = $ dne

$x^2 = 0$

$x = \{0\}$

↳ outside of the bound

c) $h(x) = x^4 - 2x^2 + 1, \ x \in \left[\frac{-5}{4}, \frac{3}{2}\right]$

$h'(x) = 4x^3 - 4x$

$0 = 4x(x^2 - 1)$

$0 = 4x(x+1)(x-1)$

Critical #: $\{-1, 0, 1\}$

$f\left(\frac{-5}{4}\right) = \frac{81}{256}$

$f(-1) = 0 \Leftarrow$ Absolute Min

$f(0) = 1$

$f(1) = 0 \Leftarrow$ Absolute Min

$f\left(\frac{3}{2}\right) = \frac{25}{16} \Leftarrow$ Absolute Max

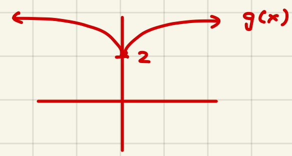
$\therefore$ Absolute min : 0

Absolute max : $\frac{25}{16}$

d) $g(x) = \begin{cases} 2+\sqrt{x}, & x > 0 \\ 2+\sqrt{-x}, & x \le 0 \end{cases}$

$\Rightarrow g(0^+) = g(0^-) = 2, \quad x \in [-3, 4]$

$\therefore$ continuous @ $x = 0$



$g'(x) = \begin{cases} \frac{1}{2}x^{-\frac{1}{2}}, & x > 0 \\ \frac{1}{2}x^{-\frac{1}{2}} \cdot (-1), & x \le 0 \end{cases}$

$= \begin{cases} \frac{1}{2\sqrt{x}}, & x > 0 \\ \frac{-1}{2\sqrt{x}}, & x \le 0 \end{cases}$

$g'(x) = $ DNE

$x = \{0\}$

$f(-3) = \sqrt{3} + 2$

$f(0) = 2 \Rightarrow$ Absolute min

$f(4) = 4 \Rightarrow$ Absolute Max

e. $f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \le 1 \\ -3x + 7 & \text{if } x > 1 \end{cases}$ on $[-2, 3]$

$f(1^-) = 1 + 2 + 1 = 4$

$f(1^+) = 4$

$\because f(1^-) = f(1^+) = 4$

$\therefore$ @ $x = 1$, function is continuous

$$f'(x) = \begin{cases} 2x + 2, & x \le 1 \\ -3, & x > 1 \end{cases}, \quad x \in [-2, 3]$$

Critical #:
$$0 = \begin{cases} 2x + 2, & x \le 1 \\ -3, & x > 1 \end{cases}$$

$f'(x) = 0$ or $f'(1^-) = 4$

$x = -1$ $\quad f'(1^+) = -3$

$\therefore f'(1) = \text{dne}$

$\therefore$ critical #'s : $\{\pm 1\}$

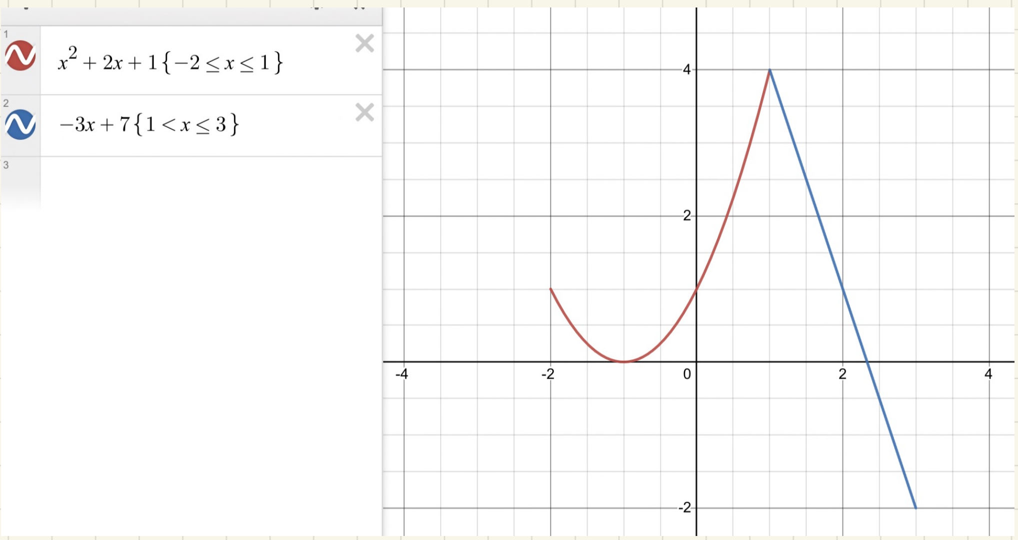
$f(-2) = 1$

$f(-1) = 0$

$f(1) = 4$ Absolute Max

$f(3) = -2$ Absolute Min

DESMOS to check:



Exit Card!

Find the absolute extrema of $f(x)=x^{\frac{3}{5}}(4-x)$ on the interval [-1,3].

$f(-1)=-5$ abs. min

$f(1.5)=3.189$ abs. max

$f(x) = 4x^{\frac{3}{5}} - x^{\frac{8}{5}} \Rightarrow \quad D_f: \{x \in \mathbb{R}, -1 \le x \le 3\}$

$f'(x) = \frac{12}{5}x^{\frac{-2}{5}} - \frac{8}{5}x^{\frac{3}{5}}$

$= \frac{4}{5}x^{\frac{-2}{5}}\left[3 - 2x\right]$

$= \frac{4(3-2x)}{5x^{2/5}}$

$f'(x)=0 \qquad f'(x) = dne$

$x = \{\frac{3}{2}\} \qquad x = \{0\}$

$\therefore$ Critical #'s : $\{\frac{3}{2}, 0\}$

$f(-1) = (-1)^{\frac{3}{5}}(4-(-1)) = -5$ ⇐ Absolute min

$f(0) = 0$

$f(\frac{3}{2}) = (\frac{3}{2})^{\frac{3}{5}}(4-(\frac{3}{2})) \doteq 3.189$ ⇐ Absolute max

$f(3) = (3)^{\frac{3}{5}}(4-3) = 1.933$

Warm-Up

Find the **absolute extreme** values for $f(x) = 3x^{\frac{2}{3}}\left(\frac{1}{8}x^2 - \frac{1}{5}x - 1\right)$, $-2 \le x \le 2$.

$$f(x) = \frac{3}{8}x^{\frac{8}{3}} - \frac{3}{5}x^{\frac{5}{3}} - 3x^{\frac{2}{3}} \quad , x \in \mathbb{R}$$

$$f'(x) = x^{\frac{5}{3}} - x^{\frac{2}{3}} - 2x^{-\frac{1}{3}}$$
$$= x^{-\frac{1}{3}}\left[x^2 - x - 2\right]$$
$$= x^{-\frac{1}{3}}(x-2)(x+1)$$
$$= \frac{(x-2)(x+1)}{x^{\frac{1}{3}}} \quad , x \neq 0$$

$$f(x) = \frac{(x-3)(x+5)}{(x-4)^2}$$

$f(x)$: (+) -5 (−) 3 (+) 4 (+)

$f'(x) = 0$ $f'(x) = \text{dne}$

$x = \{2, -1\}$ $x = \{0\}$

$\therefore$ critical #'s: $\{-1, 0, 2\}$

$f(-2) \doteq -0.476$

$f(-1) \doteq -2.025$

$f(0) = 0 \Rightarrow$ Absolute max

$f(2) \doteq -4.286 \Rightarrow$ Absolute min

Todays lesson: Intervals of Increase / Decrease

$f'(x)$: -2 (−) -1 (+) 0 (−) 2 (+)

dec | -1 HT min | inc | 0 VT max (cusp) | dec | 2 HT min | inc

↘ ↗ ↘ ↗

$$f'(x) = \frac{(x-2)(x+1)}{x^{\frac{1}{3}}}$$

HT

VT

Note!

$x=0$ is a cusp (a max too!)

- 0 is in the domain of the original function
- $f'(x) = \text{dne} \Rightarrow \boxed{f'(0^-) = +\infty \quad f'(0^+) = -\infty}$
- ↗ ↘

Note!
It has been brought to my attention that calculators are not programmed to handle fraction exponents ⇒ giving Error messages when it should work!
Use Radicals instead!
This means you need to know how your calculator works (or doesn't work !!)

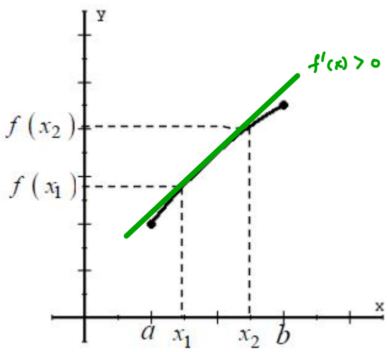
3.2 Increasing and decreasing Functions

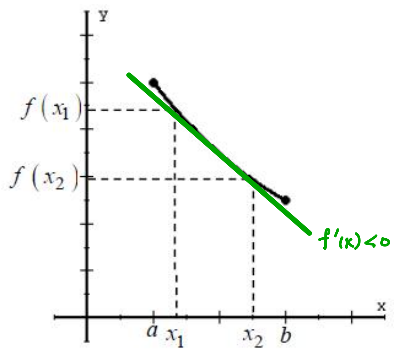
The concepts of increasing and decreasing are closely linked to **intervals** or subsets of a function's domain**.**

Suppose S is an interval in the domain of $f(x)$,so $f(x)$ is defined for all x in S.

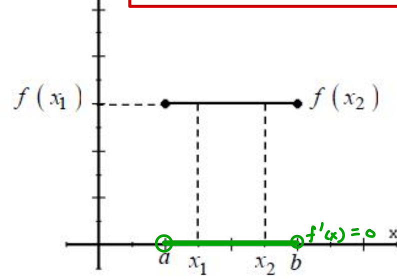
$f(x)$ is **increasing** on S , $\Leftrightarrow f(a) \le f(b)$ for all $a,b \in S$ such that $a < b$

$f(x)$ is **decreasing** on S $\Leftrightarrow f(a) \ge f(b)$ for all $a,b \in S$ such that $a < b$





If a graph exists on an interval, it is doing one of three things:
1. Increasing (y-values rise as x-values increase)
2. Decreasing (y-values fall as x-values increase)
3. Staying Constant (y-values stay the same as x-values increase)



if $x_2 > x_1$ then $f(x_2) > f(x_1)$ $f'(x) > 0$	*if* $x_2 > x_1$ then $f(x_2) < f(x_1)$ $f'(x) < 0$	*if* $x_2 > x_1$ then $f(x_2) = f(x_1)$ $f'(x) = 0$

Test for Increasing and Decreasing Functions

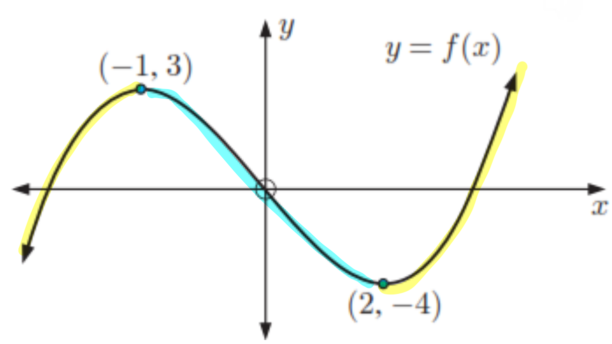
Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) .

1. If $f'(x) > 0$ for all $x \in (a,b)$ then f is **increasing** on $[a,b]$.
2. If $f'(x) < 0$ for all $x \in (a,b)$, then f is **decreasing** on $[a,b]$.
3. If $f'(x) = 0$ for all $x \in (a,b)$, then f is **constant** on $[a,b]$.

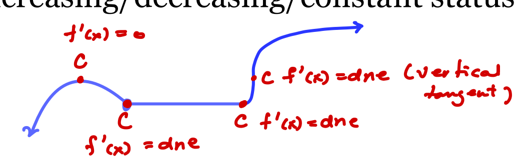
Example 1: Find intervals where f(x) is:

(a) increasing: $(-\infty, -1)$ and $(2, \infty)$

(b) decreasing: $(-1, 2)$



Q. At what values of x can the graph of a function change its increasing/decreasing/constant status?
The graph of a continuous function can only change its increasing/decreasing/constant status at a critical point.



Example 3: Find the intervals where the following functions are increasing or decreasing:

a) $f(x) = x^3 - \frac{3}{2}x^2$

$f'(x) = 3x^2 - 3x$

$= 3x(x-1)$

$f'(x) = 0$ or $f'(x) = \text{dne}$

$x = \{0, 1\}$ $\quad x = \{\ \}$

$\therefore$ critical #'s: $\{0, 1\}$

$f'(x)$: (+) | 0 | (−) | 1 | (+)

inc 0 dec 1 inc

↗ max ↘ min ↗

interval of inc: $(-\infty, 0) \cup (1, \infty)$

interval of dec: $(0, 1)$

b) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f'(x) = 12x^3 - 12x^2 - 24x$

$= 12x(x^2 - x - 2)$

$= 12x(x-2)(x+1)$

Critical #'s: $\{-1, 0, 2\}$

$f'(x)$: (−) | −1 | (+) | 0 | (−) | 2 | (+)

↘ ↗ ↘ ↗

min $x=-1$ $\quad$ max $x=0$ $\quad$ min $x=2$

$\therefore$ increase: $(-1, 0) \cup (2, \infty)$

decrease: $(-\infty, -1) \cup (0, 2)$

c) $f(x) = \dfrac{2x-3}{x^2+2x-3} = \dfrac{2x-3}{(x+3)(x-1)}$, $\{x \neq \underbrace{-3, 1}_{\text{VA's}}\}$

$f'(x) = \dfrac{2(x^2+2x-3) - (2x+2)(2x-3)}{(x+3)^2(x-1)^2}$

$= \dfrac{2x^2+4x-6 - 4x^2 + 2x + 6}{(x+3)^2(x-1)^2}$

$= \dfrac{-2x^2+6x}{(x+3)^2(x-1)^2}$

$= \dfrac{-2x(x-3)}{(x+3)^2(x-1)^2}$ — VA!

Critical #'s: $\{0, 3\}$

$f'(x)$: (−) | −3 | (−) | 0 | (+) | 1 | (+) | 3 | (−)

dec −3 dec 0 inc 1 inc 3 dec

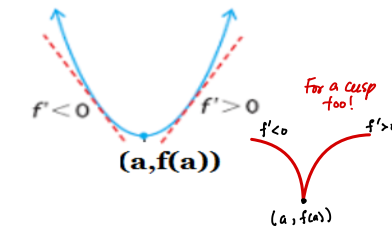
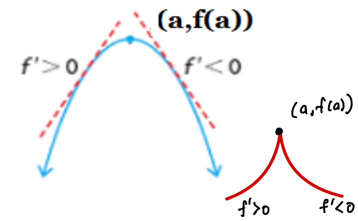
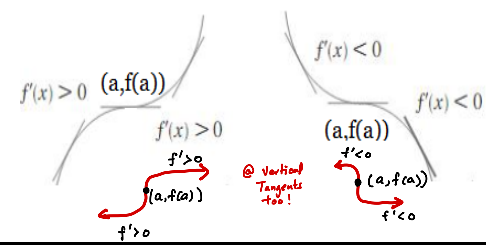
VA $\quad$ min $x=0$ $\quad$ VA $\quad$ max $x=3$

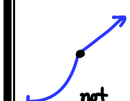
increase: $(0, 1) \cup (1, 3)$

decrease: $(-\infty, -3) \cup (-3, 0) \cup (3, \infty)$

For a continuous function, knowing when and where the sign of the derivative changes, lends great insight into existence of any **Relative Maximums** or **Relative Minimums**.

Theorem: The First Derivative Test (for Relative Extrema)

Let x $x = a$ be a critical value of a continuous function f.	
1. If sign of $f'(x)$ changes from negative to positive at x = a , then *f* has a relative minimum at x = a	
2. If sign of $f'(x)$ changes from positive to negative at x = a , then f has a relative maximum at x = a	
3. If $f'(x)$ is positive on both sides of $x = a$ or negative on both sides of x = a , then is neither a relative maximum nor a relative maximum.	



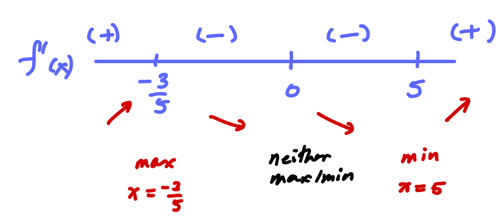
not a point of inf.

Example 4: Find the x-coordinates of the local extrema of the function $g(x) = 6x^5 - 33x^4 - 30x^3 + 100$.

$$g'(x) = 30x^4 - 132x^3 - 90x^2$$
$$= 6x^2(5x^2 - 22x - 15)$$
$$= 6x^2(5x+3)(x-5)$$

$D_g : \{x \in \mathbb{R}\}$

Critical #'s : $\{-\frac{3}{5}, 0, 5\}$



First Derivative Test
↳ to identify local max/min
by testing to the left
and right side of the critical #'s

Example 5: Find the relative extrema of $f(x) = -\dfrac{x^4+1}{x^2}$. ~~Justify~~

$f(x) = -x^2 - x^{-2}$ $\qquad D_f: \{x \in \mathbb{R},\ x \neq 0\}$

$f'(x) = -2x + 2x^{-3}$

$= -2x^{-3}(x^4 - 1)$

$= \dfrac{-2(x^2-1)(x^2+1)}{x^3}$

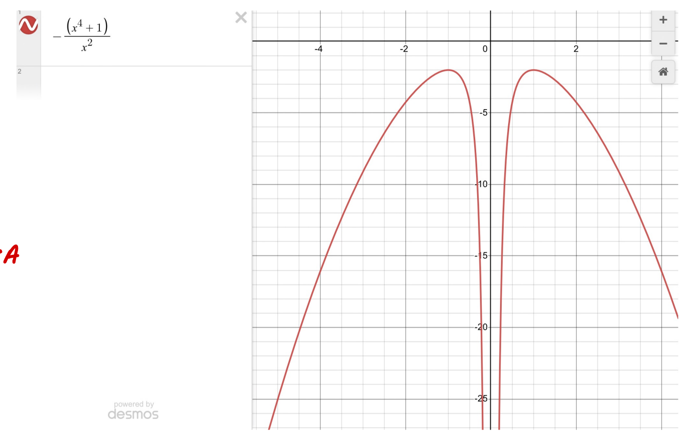
$= \dfrac{-2(x+1)(x-1)(x^2+1)}{x^3}$

$f'(x) = 0$ or $f'(x) = \text{dne}$

$x = \{\pm 1\}$ $\qquad x = 0 \leftarrow$ not a critical #'s but a VA

$\therefore$ Critical #'s: $\{\pm 1\}$

Desmos check:



$f'(x)$	$(+)$	-1	$(-)$	0	$(+)$	1	$(-)$
	$\nearrow$		$\searrow$	VA	$\nearrow$		$\searrow$
		max				max	
		$f(-1) = -2$				$f(1) = -2$	

$\therefore$ relative max: $(-1,-2)$ and $(1,-2)$

3.2 Practice:

1. Find values of a, b, c, and d such that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at (2,4) and a local minimum at (0, 0).

2. For $f(x) = x^2 + px + q$, find the values of p and q such that $f(1) = 5$ is an extremum of $f(x)$ on the interval [0,2]. Is this extremum a maximum value or a minimum.

3. For what value of *x* does the ***derivative*** of $f(x) = \dfrac{x^4}{3} - \dfrac{x^5}{5}$ attains its maximum value ?

4. Find the x-coordinate of the point that the function f(x) given by $f(x) = 9x^{\frac{2}{3}} + 3x - 6$ has a relative minimum.

5. Find the *x*-coordinates of the relative extrema of the following functions:
   a) $T(k) = \sqrt[3]{k^2}(2k-1)$ $\qquad$ b) $J(k) = \sqrt[3]{k}(2k-1)$.

6. Find the intervals where the following functions are increasing or decreasing:
   a) $f(x) = x - 2\sqrt{x}$ $\qquad$ b) $f(x) = x^2 + \dfrac{4}{x-1}$ $\qquad$ c) $f(x) = \dfrac{x-2}{(x+1)^2}$ $\qquad$ d) $f(x) = 3x^{\frac{2}{3}}\left(\dfrac{1}{8}x^2 - \dfrac{1}{2}\right)$

3.2 Practice:

1. Find values of a, b, c, and d such that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at (2,4) and a local minimum at (0, 0).

$f'(x) = 3ax^2 + 2bx + c$

$f(2) = a(2)^3 + b(2)^2 + c(2) + d$

$4 = 8a + 4b + 2c + d$ ③

$f(0) = d$

$0 = d$ ④

$f'(2) = 0 \Rightarrow$ Local Max

$3a(2)^2 + 2b(2) + c = 0$

$12a + 4b + c = 0$ ①

$f'(0) = 0 \Rightarrow$ local max

$3a(0)^2 + 2b(0) + c = 0$

$\therefore c = 0$ ②

sub $d = 0$, $c = 0$

③ $\Rightarrow 8a + 4b = 4$

① $\Rightarrow 12a + 4b = 0$

$(-) \quad -4a = 4$

$a = -1$ ④

sub ④ into ③

$8(-1) + 4b = 4$

$4b = 12$

$b = 3$

$\therefore a = -1$

$b = 3$

$c = 0$

$d = 0$

2. For $f(x) = x^2 + px + q$, find the values of p and q such that $f(1) = 5$ is an extremum of $f(x)$ on the interval [0,2]. Is this extremum a maximum value or a minimum.

$f'(x) = 2x + p$

$f'(1) = 0$, $\Leftarrow$ extrema

$2(1) + p = 0$

$p = -2$ ①

$f(1) = 1^2 + p(1) + q$

$5 = 1 + p + q$

$4 = p + q$ ②

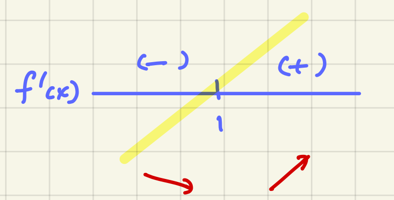
sub ① into ②

$4 = -2 + q$

$q = 6$

$\therefore f(x) = x^2 - 2x + 6$

$f'(x) = 2x - 2$



$\therefore f(1) = 5$ is an Absolute min.

3. For what value of *x* does the ***derivative*** of $f(x) = \frac{x^4}{3} - \frac{x^5}{5}$ attains its maximum value ?

$f'(x) = \frac{4}{3}x^3 - x^4 \qquad D_f : x \in \mathbb{R}$

$f''(x) = 4x^2 - 4x^3$

$= 4x^2(1 - x)$

critical #: $\{0, 1\}$

$f'(0) = 0$

$f'(1) = \frac{1}{3} \Rightarrow$ max value

$\therefore$ @ $x = 1$, $f'(x)$ attains the max. value

4. Find the x-coordinate of the point that the function f(x) given by $f(x) = 9x^{\frac{2}{3}} + 3x - 6$ has a relative minimum.

$f'(x) = 6x^{-\frac{1}{3}} + 3$

$= 3x^{-\frac{1}{3}}(2 + x^{\frac{1}{3}})$

$= \dfrac{2 + \sqrt[3]{x}}{3\sqrt[3]{x}}$

$D_f : x \in \mathbb{R}$

critical #'s: $\{-8, 0\}$

$f'(x)$: (+) on $(-\infty, -8)$, (−) on $(-8, 0)$, (+) on $(0, \infty)$

$\nearrow$ max (-8) $\searrow$ min (0) $\nearrow$

min cusp

$f'(0) = \text{dne}$

$\therefore$ relative min: $x = 0$

5. Find the *x*-coordinates of the relative extrema of the following functions:

a) $T(k) = \sqrt[3]{k^2}(2k - 1)$ b) $J(k) = \sqrt[3]{k}(2k - 1)$.

a)

$= k^{\frac{2}{3}}(2k-1)$ $D_T : \{k \in \mathbb{R}\}$

$= 2k^{\frac{5}{3}} - k^{\frac{2}{3}}$

$T'(k) = \frac{10}{3}k^{\frac{2}{3}} - \frac{2}{3}k^{-\frac{1}{3}}$

$= \frac{2}{3}k^{-\frac{1}{3}}[5k - 1]$

$= \dfrac{2(5k-1)}{3\sqrt[3]{k}}$

critical #'s: $\{\frac{1}{5}, 0\}$

$f'(x)$: (+) on $(-\infty, 0)$, (−) on $(0, \frac{1}{5})$, (+) on $(\frac{1}{5}, \infty)$

$\nearrow$ max $x = 0$ $\searrow$ min $x = \frac{1}{5}$ $\nearrow$

b)

$D_J : \{k \in \mathbb{R}\}$

$= k^{\frac{1}{3}}(2k-1)$

$= 2k^{\frac{4}{3}} - k^{\frac{1}{3}}$

$J'(k) = \frac{8}{3}k^{\frac{1}{3}} - \frac{1}{3}k^{-\frac{2}{3}}$

$= \frac{1}{3}k^{-\frac{2}{3}}[8k - 1]$

$= \dfrac{8k-1}{3\sqrt[3]{k^2}}$

critical #'s: $\{\frac{1}{8}, 0\}$

$f'(x)$: (−) on $(-\infty, 0)$, (−) on $(0, \frac{1}{8})$, (+) on $(\frac{1}{8}, \infty)$

$\searrow$ 0 $\searrow$ $\frac{1}{8}$ $\nearrow$

min $= \frac{1}{8}$

6. Find the intervals where the following functions are increasing or decreasing:

a) $f(x)=x-2\sqrt{x}$ b) $f(x)=x^2+\dfrac{4}{x-1}$ c) $f(x)=\dfrac{x-2}{(x+1)^2}$ d) $f(x)=3x^{\frac{2}{3}}\left(\dfrac{1}{8}x^2-\dfrac{1}{2}\right)$

a) $f'(x) = 1 - x^{-\frac{1}{2}}$ $D_f\{x\in\mathbb{R},\ x\geq 0\}$

$= 1-\dfrac{1}{\sqrt{x}}$

$= \dfrac{\sqrt{x}-1}{\sqrt{x}}$

Critical #'s: $\{1, 0\}$

$f'(x)$: (−) on $(0,1)$, (+) after 1; $x=1$: min

increase: $(1,\infty)$

decrease: $(0,1)$

b) $f(x) = x^2 + 4(x-1)^{-1}$ $D_f:\{x\in\mathbb{R},\ x\neq 1\}$

$f'(x) = 2x - 4(x-1)^{-2}(1)$

$= 2x - \dfrac{4}{(x-1)^2}$

$= \dfrac{2}{(x-1)^2}\left[x(x-1)^2-2\right]$

$= \dfrac{2(x^3-2x^2+x-2)}{(x-1)^2}$

$= \dfrac{2(x-2)(x^2+1)}{(x-1)^2}$

Aside:

x^3-2x^2+x-2

$= x^2(x-2)+(x-2)$

$=(x-2)(x^2+1)$

Critical #'s: $\{2\}$

$f'(x)$: (−) before 1, (−) on $(1,2)$, (+) after 2; $x=1$: VA; $x=2$: min

increase: $(2,\infty)$

decrease: $(-\infty,1)\cup(1,2)$

c) $f(x) = \dfrac{(x-2)}{(x+1)^2}$ $D_f:\{x\in\mathbb{R},\ x\neq -1\}$

$f'(x) = \dfrac{[1][(x+1)^2]-[2(x+1)(1)][(x-2)]}{(x+1)^4}$

$= \dfrac{(x+1)^2-2(x+1)(x-2)}{(x+1)^4}$

$= \dfrac{(x+1)-2(x-2)}{(x+1)^3}$

$= \dfrac{-x+5}{(x+1)^3}$

$= \dfrac{-(x-5)}{(x+1)^3}$

Critical #'s: $\{5\}$

$f'(x)$: (−) before -1, (+) on $(-1,5)$, (−) after 5; $x=-1$: VA; $x=5$: max

increase: $(-1, 5)$

decrease: $(-\infty,-1)\cup(5,\infty)$

d) $f(x) = 3x^{\frac{2}{3}}\left(\frac{1}{8}x^2-\frac{1}{2}\right)$

$= \frac{3}{8}x^{\frac{8}{3}} - \frac{3}{2}x^{\frac{2}{3}}$ $D_f:\{x\in\mathbb{R}\}$

$f'(x) = x^{\frac{5}{3}} - x^{-\frac{1}{3}}$

$= x^{-\frac{1}{3}}\left[x^2-1\right]$

$= \dfrac{(x+1)(x-1)}{\sqrt[3]{x}}$

Critical #'s: $\{\pm 1, 0\}$

$f'(x)$: (−) before -1, (+) on $(-1,0)$, (−) on $(0,1)$, (+) after 1; $x=-1$: min; $x=0$: max; $x=1$: min

(cusp)

$f'(x)$=dne (vertical tangent)

increase: $(-1,0)\cup(1,\infty)$

decrease: $(-\infty,-1)\cup(0,1)$

Warm-Up

1. Where is $f(x) = 3x^4 + 6x^3 + 5$ increasing? $\underline{\quad d \quad}$

$f'(x) = 12x^3 + 18x^2$
$= 6x^2(2x+3)$

$f'(x)$: $(-)$ $-\frac{3}{2}$ $(+)$ 0 $(+)$

(a) $x < -1.5, x \neq 0$ (c) $x > 0, x \neq -1.5$
(b) $x < 0, x \neq -1.5$ (d) $x > -1.5, x \neq 0$

2. Let *f* and *g* be continuous and differentiable functions on the interval $a \le x \le b$. If *f* (x) and *g(x)* are both increasing on $a \le x \le b$, and if $f(x) > 0$ and $g(x) > 0$ on $a \le x \le b$, then *f(x)g(x)* must be: $\underline{\quad a \quad}$

(a) increasing (c) both increasing and decreasing
(b) decreasing (d) not increasing or decreasing

$f'(x) > 0$ and $g'(x) > 0$
$f(x) > 0$ and $g(x) > 0$

$y' = f'(x)g(x) + g'(x)f(x)$
$= (+)(+) + (+)(+)$
$\therefore y' > 0 \quad \therefore$ increasing

3. Determine if $f(x) = \dfrac{x^3 + 2x}{x^2}$ has any local extrema. If so, where?

$f(x) = x + 2x^{-1} \quad \{x \in \mathbb{R}, x \neq 0\}$

$f'(x) = 1 - 2x^{-2}$
$= x^{-2}(x^2 - 2)$
$= \dfrac{x^2 - 2}{x^2}$

$f'(x) = 0$ or $f'(x) =$ dne
$x = \pm\sqrt{2}$ $x = 0$ (not in the domain)

Critical #'s : $\pm\sqrt{2}$

$f'(x)$: $(+)$ $-\sqrt{2}$ $(-)$ 0 $(-)$ $\sqrt{2}$ $(+)$

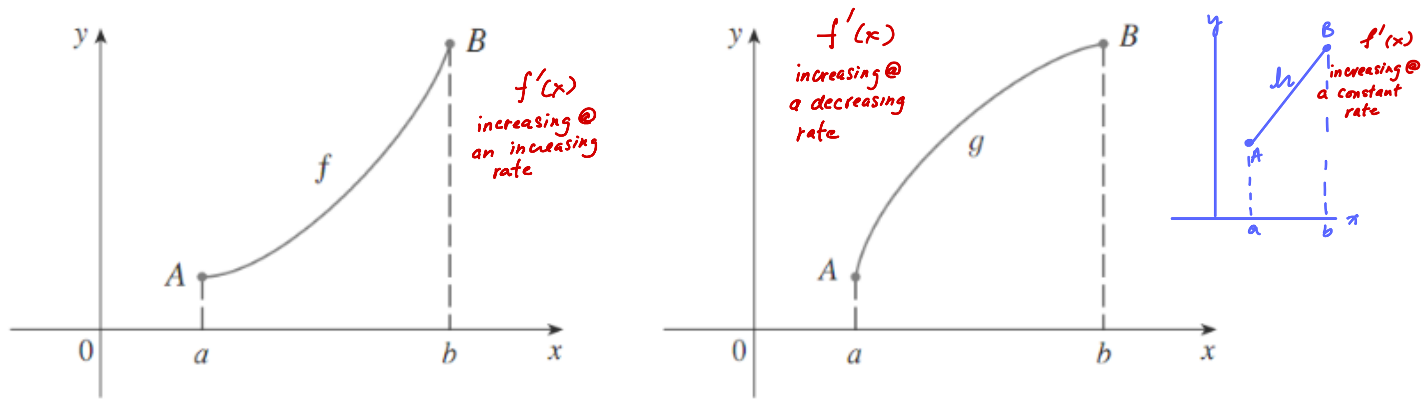
$\nearrow$ max $\searrow$ VA $\searrow$ min $\nearrow$

$f(-\sqrt{2}) \doteq -2.82$ $f(\sqrt{2}) \doteq 2.82$

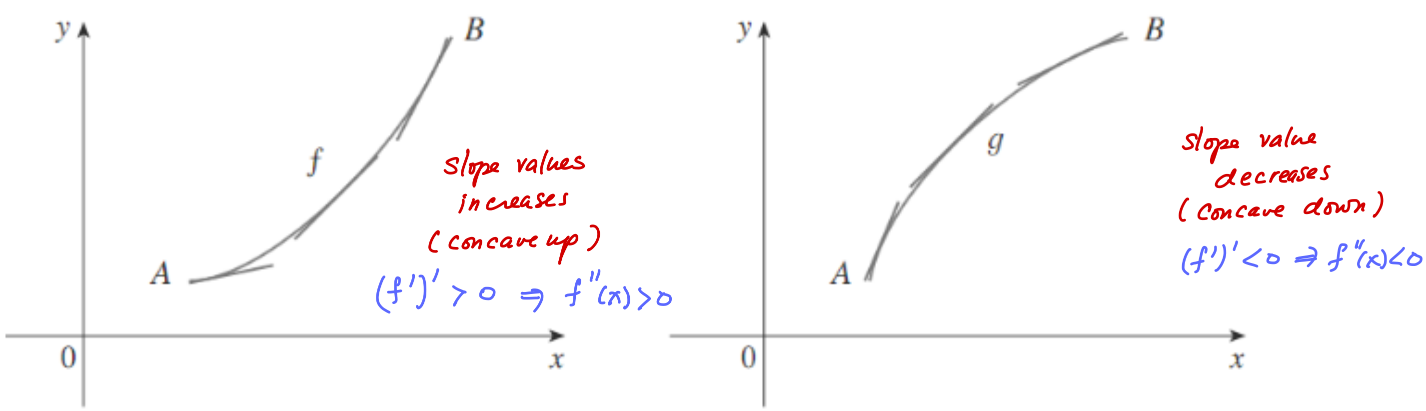
3.3 Concavity and the Second Derivative Test

If we know that a function has a positive derivative over an interval, we know the graph of the function is increasing on that interval, but HOW is it increasing? At a constant rate? An increasing rate? A decreasing rate?

The two functions below both increase, but they bend differently, and, therefore, have different curvature. The function on the left increases at an increasing rate and the second increases at a decreasing rate (functions that increase at a constant rate are linear, boring, and don't require calculus.)



If we analyze the tangent lines in each of these cases at several points, we can begin to talk about how the slopes, and not just the y-values are changing.



In the graph on the left, the tangent lines are **below** the curve and are increasing from left to right. In this case, we say the graph is **concave up** (like a ~~cup~~ smile). In this case, the secant lines are above the curve.

In the graph on the right, the tangent lines are **above** the curve and are decreasing from left to right. In this case, we say that graph is **concave down** (like a frown). In this case, the secant lines are below the curve.

"rate of change" $f'(x)$

Anytime we talk about something changing, we're talking about the derivative. When we talk about the slopes of the tangent lines of a function changing, we're talking about how the derivative function is changing. This means we're talking about the second derivative!!

rate of change of $f'(x)$ $f''(x)$

If the slopes ~~off~~ of $f(x)$ are increasing, f" (x) > 0 . If the slopes ~~off~~ of $f(x)$ are decreasing, f" (x) < 0 .

Concavity Test ⇒ uses an interval chart to determine intervals of concave up/down

smile + +

1) If f"(x) > 0 for all x in an interval, then f (x) is **concave up** (like a ~~cup~~) on that interval.
2) If f"(x) < 0 for all x in an interval, then f (x) is **concave down** (like a frown) on that interval.

− −

Definition

A point (c , f(c)) on a curve y = f(x) is called an **inflection point** if the graph of f changes from concave up to concave down OR from concave down to concave up at (c , f(c)).

cd cu or cd cu

$f'(x) = \text{dne}$ $f'(x) = 0$

"critical #'s" for $f''(x)$

That is to say: if f "(c)=0 or f "(c) dne and the sign of f " changes from positive to negative or from negative to positive at x= c , then x=c is the inflection point of f(x).

$f''(x) = \text{dne}$ $f''(x) = 0$

Example 1:
Determine the open intervals on which the graphs of the following functions are concave up or concave down, then find any inflection points.

a) $f(x) = x^4 + 18x^3 + 120x^2 + 6x + 30$ $D_f: \{x \in \mathbb{R}\}$

$f'(x) = 4x^3 + 54x^2 + 240x + 6$

$f''(x) = 12x^2 + 108x + 240$

$0 = 12(x^2 + 9x + 20)$

$0 = 12(x+4)(x+5)$

"Critical #'s" of $f''(x)$: $\{-5, -4\}$

$f''(x)$: (+) | −5 | (−) | −4 | (+)

cu, cd, cu

poi $f(-5) = 1375$ poi $f(-4) = 1030$

Concave up: $(-\infty, -5) \cup (-4, \infty)$

concave down: $(-5, -4)$

b) $f(x) = x + \dfrac{4}{x}$ $D_f: \{x \in \mathbb{R}, x \neq 0\}$

$= \dfrac{x^2+4}{x}$

$f'(x) = 1 - 4x^{-2}$

$f''(x) = 8x^{-3}$

$= \dfrac{8}{x^3}$

$f''(x) = 0$ $x = \{\ \}$ $f''(x) = \text{dne}$ $x = 0$ (not in domain)

Critical #'s $f''(x)$: None

$f''(x)$: (−) | 0 (VA) | (+)

cd, cu

Not Poi b/c VA

note! No point of Inflection

Concave up: $(0, \infty)$

Concave down: $(-\infty, 0)$

c) $g(x)=\dfrac{x^2+1}{x^2-4}$ $D_g: \{x \in \mathbb{R},\ x \neq \pm 2\}$

$$g'(x) = \frac{2x(x^2-4) - 2x(x^2+1)}{(x^2-4)^2}$$

$$= \frac{2x[\cancel{x^2}-4-\cancel{x^2}-1]}{(x^2-4)^2}$$

$$= \frac{-10x}{(x^2-4)^2}$$

$$g''(x) = \frac{-10[(x^2-4)^2] - [2(x^2-4)\cdot(2x)][-10x]}{(x^2-4)^4}$$

$$= \frac{-10(x^2-4)[(x^2-4)-4x^2]}{(x^2-4)^4}$$

$$= \frac{-10(-3x^2-4)}{(x^2-4)^3}$$

$$= \frac{10(3x^2+4)}{(x+2)^3(x-2)^3}$$

$f''(x)=0$ $\quad$ $f''(x) = \text{dne}$

$x = \{\ \}$ $\quad$ $x = \{\pm 2\}$ ↳ no in the domain

$\therefore$ No Critical #'s

$f''(x)$: $(+)$ VA $(-)$ VA $(+)$ at -2 and 2

cu $\quad$ -2 $\quad$ cd $\quad$ 2 $\quad$ cu

$\therefore$ No PoI

Concave up : $(-\infty,-2)\cup(2,\infty)$

Concave down : $(-2,2)$

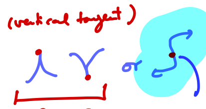
d) $y=\sqrt[3]{x}$ $D_y: \{x \in \mathbb{R}\}$

$= x^{\frac{1}{3}}$

$y' = \frac{1}{3}x^{-\frac{2}{3}} \Rightarrow \frac{1}{3\sqrt[3]{x^2}}$ $\quad$ $f'(0)=\text{dne}$ (vertical tangent)

$y'' = -\frac{2}{9}x^{-\frac{5}{3}}$

$= \frac{-2}{9\sqrt[3]{x^5}}$



cusps $\quad$ or

no change in concavity

vertical tangent $f'(x)=\text{dne}$ $f(0)=0$

$y''=0$ $\quad$ $y''=\text{dne}$

$x=\{\ \}$ $\quad$ $x=0$

$\therefore$ critical #'s : $\{0\}$

$f''(x)$: $(+)$ 0 $(-)$

cu $\quad$ 0 $\quad$ cd

$\therefore$ poi : $f(0)=0$

Concave up : $(-\infty,0)$

concave down : $(0,\infty)$

Example 2: Suppose that $f(x)=x^3+ax^2+bx+c$, for some unknown constants a, b and c, has a local minimum of 12 at $x=-1$, and an inflection point at $x=2$. Determine $f(1)$.

$f(-1)=12$

$f(x) = x^3+ax^2+bx+c$

$f(-1) = (-1)^3+a(-1)^2+b(-1)+c$

$12 = -1+a-b+c$

$a-b+c=13$ ①

sub ③ and ④ into ①

$-6-(-15)+c=13$

$9+c=13$

$c=4$

$f'(-1)=0 \Rightarrow$ local min

$f'(x) = 3x^2+2ax+b$

$f'(-1) = 3(-1)^2+2a(-1)+b$

$0 = 3-2a+b$ ②

sub ③ into ②

$0 = 3-2(-6)+b$

$0 = 3+12+b$

$b=-15$ ④

$f''(2)=0 \Rightarrow$ PoI

$f''(x) = 6x+2a$

$f''(2) = 6(2)+2a$

$0 = 12+2a$

$a = \frac{-12}{2}$

$a=-6$ ③

$f(x) = x^3+ax^2+bx+c$

$= x^3-6x^2-15x+4$

$f(1) = (1)^3-6(1)^2-15(1)+4$

$= -16$

Note! Not to be confused with the "Test for Concavity"!
Similar to First Derivative test in that both are used for the purpose of identifying if the critical #'s of $f'(x)$ are local max/min but instead of using the intervals of inc/dec to do so, it uses the value of the Second Derivative to identify it as local max/min. Ex if $f''(c) > 0 \Rightarrow$ CU $\therefore x = c$ is a min
if $f''(c) < 0 \Rightarrow$ CD $\therefore x = c$ is a max

The Second Derivative Test (for Relative Extrema)

Let f be a function such that x = c is a critical value ~~off~~ of f(x) such that f' (c) = 0 . If f" (x) exists on an open interval containing x = c , then

1) If f" (c) > 0 , then (c, f (c)) is a relative minimum.

local min $f'(x) = 0$
$f''(c) > 0$

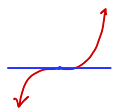
2) If f" (c) < 0 , then (c, f (c)) is a relative maximum.

local max $f'(x) = 0$
$f''(c) < 0$

3) If f" (c) = 0, then the test fails and the First Derivative Test must be used.
↳ inconclusive ; neither a max or a min.

$y = x^3$
$y' = 3x^2$
$y'' = 6x$

+ + ; inc 0 inc
− + ; cd 0 cu



$f'(x) = 0$
$f''(x) = 0$

Example 3: Find the coordinates of the relative extrema for f (x) = -3x⁵ + 5x³. Justify, if possible, using the 2nd Derivative Test. ⇒ uses the value of the Second Derivative to identify if critical points from $f'(x)$ is a local max/min < doesn't require an interval chart >

Second Derivative Test:

$f'(x) = -15x^4 + 15x^2$
$= -15x^2(x^2 - 1)$
$= -15x^2(x+1)(x-1)$

Critical #'s: $\{-1, 0, 1\}$

$f''(x) = -60x^3 + 30x$

$f''(-1) = -60(-1)^3 + 30(-1) > 0 \quad \therefore f(-1) = -2 \Rightarrow$ local min

$f''(0) = 0 \Rightarrow$ Second Derivative fails (inconclusive $\therefore$ use First Derivative Test)

$f''(1) = -60(1)^3 + 30(1) < 0 \quad \therefore f(1) = 2 \Rightarrow$ local max

Note! First Derivative Test

$f'(x)$	(−)	−1	(+)	0	(+)	1	(−)
	↘	min $x=-1$	↗		↗	max $x=1$	↘

⇐ uses intervals of inc/dec to identify critical #'s as local max/min

$f''(x) = -30x(2x^2 - 1)$

Critical #'s for $f''(x)$: $\{0, \pm\frac{1}{\sqrt{2}}\}$

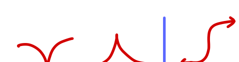
$f''(x)$	(+)	$-\frac{1}{\sqrt{2}}$	(−)	0	(+)	$\frac{1}{\sqrt{2}}$	(−)
			cd		cu		

+ + min ; − − max

$\therefore f(0) = 0 \Rightarrow$ PoI

⇐ Test for Concavity
• identify intervals where the graph concaves up/down.
cu : $(-\infty, -\frac{1}{\sqrt{2}}) \cup (0, \frac{1}{\sqrt{2}})$
cd : $(-\frac{1}{\sqrt{2}}, 0) \cup (\frac{1}{\sqrt{2}}, \infty)$

Example 4: Determine whether or not the function $f(x) = (x+6)^{\frac{1}{3}} - 2$ has a cusp.



↳ $f'(x)$ = dne
$f''(x) \Rightarrow$ No change in Concavity

$f'(x)$ = dne
concavity changes

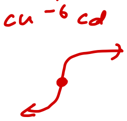
$D_f: \{x \in \mathbb{R}\}$

$f'(x) = \frac{1}{3}(x+6)^{-2/3}$
$= \frac{1}{3\sqrt[3]{(x+6)^2}}$

Critical # : $\{-6\} \Rightarrow f'(-6)$ = dne

$f''(x) = -\frac{2}{9}(x+6)^{-\frac{5}{3}}$
$= \frac{-2}{9\sqrt[3]{(x+6)^5}}$

$f''(x)$	(+)	−6	(−)
	cu		cd



poi

$\because$ there is a change in concavity @ $x = -6$
$\therefore$ No cusp
(it's a Vertical Tangent)

$f'(x)$	(+)	−6	(+)
	inc		inc

not a local extrema ($\therefore$ not a cusp)

$f''(-6) = \frac{-2}{9\sqrt[3]{0}}$ = dne $\because$ 2nd Derivative Test is inconclusive
$\therefore$ use 1ST Derivative Test

Note!
Polynomial function is continuous for $x \in \mathbb{R}$

Warm Up

1. The function $\mathbf{f(x) = 2ax^3 + 3x^2 + bx - 3}$ has a local minimum at x = -1 and a point of inflection at x = 1. Determine the values of **a** and **b.**

$f'(x) = 6ax^2 + 6x + b$

$f'(-1) = 0 \Rightarrow$ local min

$6a(-1)^2 + 6(-1) + b = 0$

$6a - 6 + b = 0$ ②

$f''(x) = 12ax + 6$

$f''(1) = 0 \Rightarrow$ inflection point

$12a(1) + 6 = 0$

$12a = -6$

$a = \frac{-1}{2}$ ①

sub ① into ②

$6(\frac{-1}{2}) - 6 + b = 0$

$-3 - 6 + b = 0$

$b = 9$

$\therefore a = \frac{-1}{2}$

$b = 9$

2. Use the **2nd Derivative Test** to determine the coordinates of the local extrema of the function $f(x) = 3x^4 - x^3 - 6x^2 + 3x - 11$.

$D_f: x \in \mathbb{R}$

$f'(x) = 12x^3 - 3x^2 - 12x + 3$

$0 = 3x^2(4x - 1) - 3(4x - 1)$

$0 = 3(4x - 1)(x^2 - 1)$

critical #'s: $\{\frac{1}{4}, \pm 1\}$

$f''(x) = 36x^2 - 6x - 12$

$f''(-1) = 36(-1)^2 - 6(-1) - 12 > 0$ local min

$f''(\frac{1}{4}) = 36(\frac{1}{4})^2 - 6(\frac{1}{4}) - 12 = -4.5 < 0$ local max

$f''(1) = 36 - 6 - 12 > 0$ local min

$f(-1) = 3(-1)^4 - (-1)^3 - 6(-1) + 3(-1) - 11 = -2$

$f(\frac{1}{4}) = 3(\frac{1}{4})^4 - (\frac{1}{4})^3 - 6(\frac{1}{4}) + 3(\frac{1}{4}) - 11 \doteq -11.75$

$f(1) = 3(1)^4 - (1)^3 - 6(1) + 3(1) - 11 = -12$

$f(-1) = -2$ (min)

$f(\frac{1}{4}) \doteq -11.75$ (max)

$f(1) = -12$ (min)

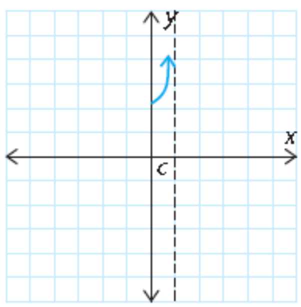
3.4 Vertical, Horizontal and Oblique Asymptotes

Vertical Asymptotes of Rational Functions

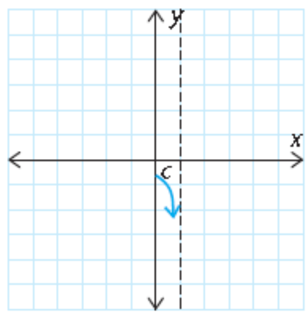
A rational function of the form $f(x)=\dfrac{P(x)}{Q(x)}$ has a vertical asymptote at x = c if Q(c) = 0 and P(c) $\neq$ 0 .

Vertical Asymptotes and Infinite Limits

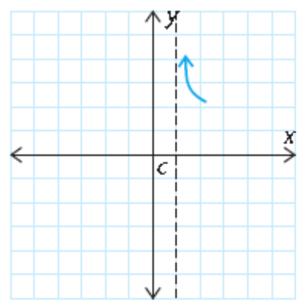
The graph of f(x) has a vertical asymptote, x = c, if one of the following infinite limit statements is true. The following graphs illustrate each of the limit statements.



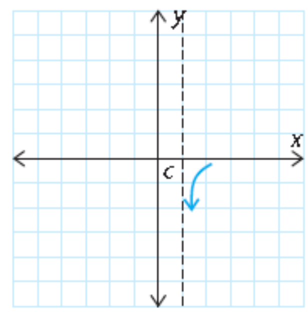
$\lim_{x\to c^-} f(x) = +\infty$



$\lim_{x\to c^-} f(x) = -\infty$



$\lim_{x\to c^+} f(x) = +\infty$

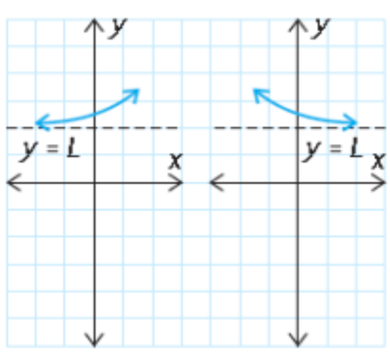


$\lim_{x\to c^+} f(x) = -\infty$

Horizontal Asymptotes (Limits at Infinity)

Consider the behavior of rational functions $f(x)=\dfrac{P(x)}{Q(x)}$ as x increases without bound in both the positive and negative directions. We can use limit notation to describe this behaviour $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

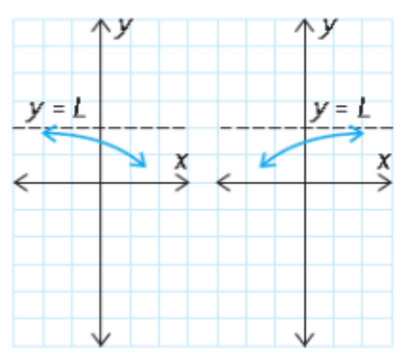
The following graphs illustrate some typical ways that a curve may approach a horizontal asymptote:



f(x) > L, so the graph approaches from above.

$\lim_{x\to-\infty} f(x) = L^+$ ← above

$\lim_{x\to+\infty} f(x) = L^+$



f(x) < L, so the graph approaches from below.

$\lim_{x\to-\infty} f(x) = L^-$ ← below

$\lim_{x\to+\infty} f(x) = L^-$

To test the behaviour, we must compare $f(x)$ and L.

Let $x \to \infty$ sub $x = 100$ into $f(x)$ If $f(100) > L$, the approach is from above otherwise is from below

Let $x \to -\infty$ sub $x = -100$ into $f(x)$ If $f(-100) < L$, the approach is from below otherwise is from above

Note: A function can cross a horizontal asymptote ~~for values of x that are "close" to the origin~~, but it can never cross a vertical asymptote.

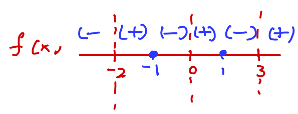
General Rules for finding the Horizontal asymptotes: Consider the rational function $f(x) = \dfrac{P(x)}{Q(x)} = \dfrac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$. Then

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} \dfrac{a_n}{b_m} & \text{if } n = m \quad y = c \\ 0 & \text{if } n < m \quad y = 0 \\ \pm\infty & \text{if } n > m \quad (\text{oblique asymptote}) \end{cases}$$

Example 1: Find the horizontal and vertical asymptotes for the following functions. Include limit statements for each asymptote and check for cross over points if possible.

" End Behaviours "

a. $f(x) = \dfrac{2(x+1)(x-1)}{x(x+2)(x-3)}$



VA:

$x = -2$ $\qquad$ $x = 0$ $\qquad$ $x = 3$

$\lim_{x \to -2^-} f(x) = -\infty$ $\qquad$ $\lim_{x \to 0^-} f(x) = -\infty$ $\qquad$ $\lim_{x \to 3^-} f(x) = -\infty$

$\lim_{x \to -2^+} f(x) = +\infty$ $\qquad$ $\lim_{x \to 0^+} f(x) = +\infty$ $\qquad$ $\lim_{x \to 3^+} f(x) = +\infty$

HA: $y = 0$

$\lim_{x \to -\infty} f(x) = 0^-$

↳ sub $x = -100$

$f(-100) < 0$

$\lim_{x \to +\infty} f(x) = 0^+$

↳ sub $x = 100$

$f(100) > 0$

Cross over point(s): ← HA: $y = 0$

$f(x) = 0$

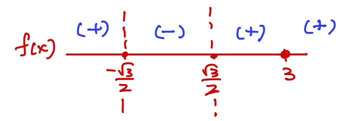
$x = \pm 1$

$f(1) = 0$ $\quad$ or $(1, 0)$ and $(-1, 0)$

$f(-1) = 0$

b. $y = \dfrac{x^2 - 6x + 9}{4x^2 - 3}$

$= \dfrac{(x-3)^2}{(2x+\sqrt{3})(2x-\sqrt{3})}$



VA: $x = -\frac{\sqrt{3}}{2}$ $\qquad$ $x = \frac{\sqrt{3}}{2}$

$\lim_{x \to -\frac{\sqrt{3}}{2}^-} f(x) = +\infty$ $\qquad$ $\lim_{x \to \frac{\sqrt{3}}{2}^-} f(x) = -\infty$

$\lim_{x \to -\frac{\sqrt{3}}{2}^+} f(x) = -\infty$ $\qquad$ $\lim_{x \to \frac{\sqrt{3}}{2}^+} f(x) = +\infty$

HA: $y = \frac{1}{4}$

$\lim_{x \to -\infty} f(x) = \frac{1}{4}^+$

↳ sub $x = -100$

$f(-100) > \frac{1}{4}$

$\lim_{x \to \infty} f(x) = \frac{1}{4}^-$

↳ sub $x = 100$

$f(100) < \frac{1}{4}$

Cross over points ← HA: $y = \frac{1}{4}$

$f(x) = \frac{1}{4}$

$\frac{x^2 - 6x + 9}{4x^2 - 3} = \frac{1}{4}$

$4x^2 - 24x + 36 = 4x^2 - 3$

$-24x = -39$

$x = \frac{39}{24}$

$= \frac{13}{8}$

$\therefore (\frac{13}{8}, \frac{1}{4})$ is cross over point

Oblique Asymptotes

If $f(x)=\dfrac{P(x)}{Q(x)}$ is a rational function in which the degree of the numerator is one more than the degree of the denominator, we can use the Division Algorithm to express the function in the form

$$f(x)=(mx+b)+\frac{R(x)}{Q(x)}$$

where the degree of R is less than the degree of Q and m ≠ 0. This means that as $x\to\pm\infty$, $\dfrac{R(x)}{Q(x)}\to 0$, so for large values of |x|, the graph of y = f(x) approaches the graph of the line y = mx + b. In this situation we say that y = mx + b is a **slant asymptote**, or an **oblique asymptote**. To test the behavior, we must find $f(x)-(mx+b)$ which is $\dfrac{\overset{R(x)}{remainder}}{Q(x)}$.

- Let $x\to\infty$, sub x = 100 into $\dfrac{\overset{R(x)}{remainder}}{Q(x)}$. If $\dfrac{\overset{R(x)}{remainder}}{Q(x)}$ >0, the approach is from above
- Let $x\to-\infty$, sub x = – 100 into $\dfrac{\overset{R(x)}{remainder}}{Q(x)}$ .If $\dfrac{\overset{R(x)}{remainder}}{Q(x)}$ <0, the approach is from below.

To conclude, we must write: as $\lim\limits_{x\to\pm\infty}\left[f(x)-\underbrace{(mx+b)}_{r(x)}\right]=0$ (from above/below)

Ex.1 Find the oblique asymptote of the rational functions.

a. $f(x)=\dfrac{x^2-4x-5}{x-3}$

O.A. $y=x-1$

Since $f(x)=x-1+\left(\dfrac{-8}{x-3}\right)$

$\lim\limits_{x\to\infty}\left[f(x)-\overset{r(x)}{(x-1)}\right]=0$ (from below)

$\lim\limits_{x\to-\infty}\left[f(x)-\overset{r(x)}{(x-1)}\right]=0$ (from above)

$\lim\limits_{x\to\infty}\left(-\dfrac{8}{x-3}\right)=0^-$

$\lim\limits_{x\to-\infty}\left(\dfrac{-8}{x-3}\right)=0^+$

$f(x)=\dfrac{x^2-4x-5}{x-3}$

or $=(x-1)+\dfrac{-8}{x-3}$

Quotient

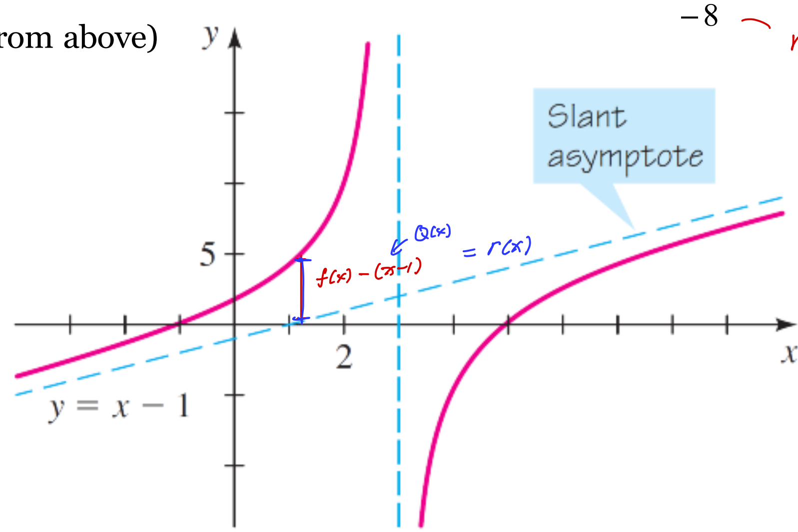
$$\begin{array}{r}
x-1\\
x-3\overline{\smash{)}x^2-4x-5}\\
\underline{-x^2+3x}\\
-x-5\\
\underline{x-1}\\
-8
\end{array}$$

divisor; dividend; remainder $R(x)=-8$

$$\begin{array}{c|ccc}
3 & 1 & -4 & -5\\
 & & 3 & -3\\
\hline
 & 1 & -1 & \boxed{-8}\ R(x)
\end{array}$$

$f(x)=(x-1)+\dfrac{-8}{(x-3)}$

$r(x)=\dfrac{-8}{x-3}$



b. $g(x)=\dfrac{2x^3-3x^2+2x-7}{x^2-4x+2}$

$$\begin{array}{r} 2x+5 \\ x^2-4x+2 \overline{\smash{)}\,2x^3-3x^2+2x-7} \\ (-)\ 2x^3-8x^2+4x \\ \hline 5x^2-2x-7 \\ (-)\ 5x^2-20x+10 \\ \hline 18x-17 \end{array} \quad \leftarrow R(x)$$

O.A. $y = 2x+5$

$r(x) = \dfrac{18x-17}{x^2-4x+2} = f(x) - (2x+5)$

$\lim\limits_{x\to-\infty} r(x) = 0^-$ ↳ sub $x=-100$, $R(-100)<0$

$\lim\limits_{x\to+\infty} r(x) = 0^+$ ↳ sub $x=100$, $R(100)>0$

$R(x)=0$ works too! (Vincent uses $R(x)=0$)

Cross over point:

$r(x)=0$

$\dfrac{18x-17}{x^2-4x+2}=0$

$\therefore x=\dfrac{17}{18}$

$y = 2\left(\dfrac{17}{18}\right)+5 = \dfrac{17}{9}+5 = \dfrac{62}{9}$ $\quad\therefore \left(\dfrac{17}{18}, \dfrac{62}{9}\right)$

Ex.2 Graph $f(x)=\dfrac{6x^2+x+7}{2x-1}$

V.A: $x=\frac{1}{2}$

$$\begin{array}{r|rrr} \frac{1}{2} & 6 & 1 & 7 \\ & \downarrow & 3 & 2 \\ \hline & 6 & 4 & 9 \end{array}$$

$f(x) = \underbrace{3x+2}_{Q(x)} + \underbrace{\dfrac{9}{2x-1}}_{r(x)}$ $\quad R(x)=9$, $r(x)=\dfrac{9}{2x-1}$

$6x^2+x+7 = (x-\frac{1}{2})(6x+4)+9$

$\dfrac{6x^2+x+7}{2x-1} = \dfrac{(2x-1)(3x+2)+9}{2x-1}$

$f(x) = 3x+2+\dfrac{9}{2x-1}$

$\lim\limits_{x\to\frac{1}{2}^+} f(x) = +\infty$

$f(x)$: $-$ | $+$ at $-\frac{1}{2}$

$\lim\limits_{x\to\frac{1}{2}^-} f(x) = -\infty$

O.A. $y=3x+2$

$\lim\limits_{x\to-\infty} \dfrac{9}{2x-1} = 0^-$

$\lim\limits_{x\to+\infty} \dfrac{9}{2x-1} = 0^+$

y-int: $(0,-7)$

x-int: none

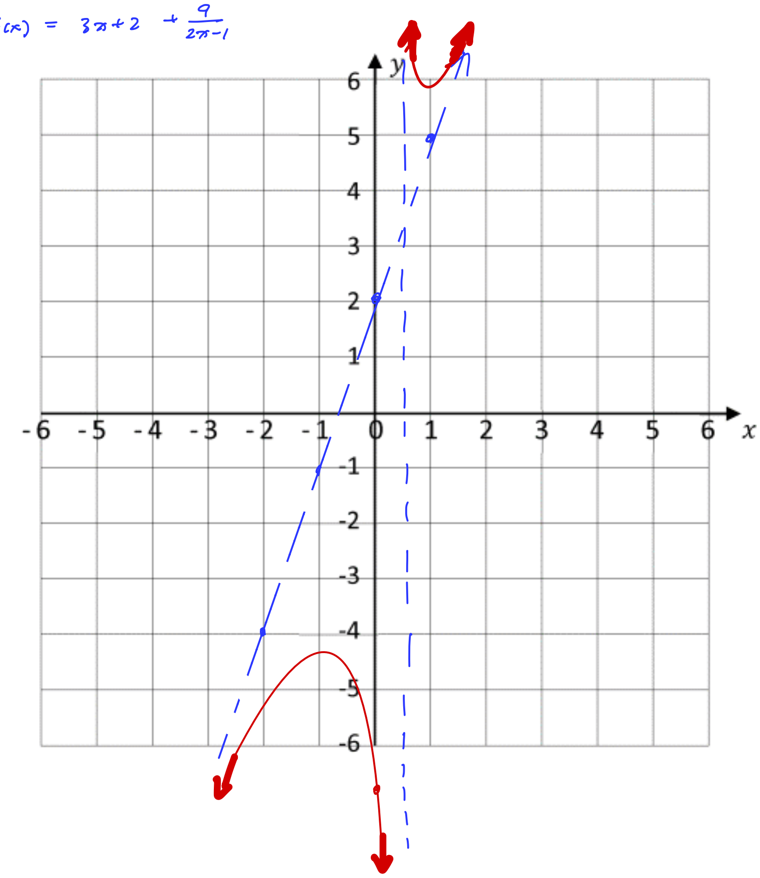
xint: $y=0$

$6x^2+x+7=0$

$x=\{\ \}$

yint: $x=0$

$f(0)=-7$



* check if there are holes! by factoring numerators (if possible)

Practice: Find all asymptotes, then sketch the function:

a) $f(x)=\dfrac{x^2+3x+2}{x-2}$ $\quad d_f: \{x \in \mathbb{R},\ x \neq 2\}$

$f(x)$: $-$ (-2) $+$ (-1) $-$ (2) $+$

$= \dfrac{(x+2)(x+1)}{x-2}$

$f(x) = (x+5) + \dfrac{12}{x-2}$

Cross over point

$r(x) = 0$

$\dfrac{12}{x-2} = 0$

$x = \{\ \}$

VA: $x = 2$

$\lim_{x\to 2^-} f(x) = -\infty$

$\lim_{x\to 2^+} f(x) = +\infty$

OA: $y = x+5$

$\lim_{x\to -\infty} \dfrac{12}{x-2} = 0^-$

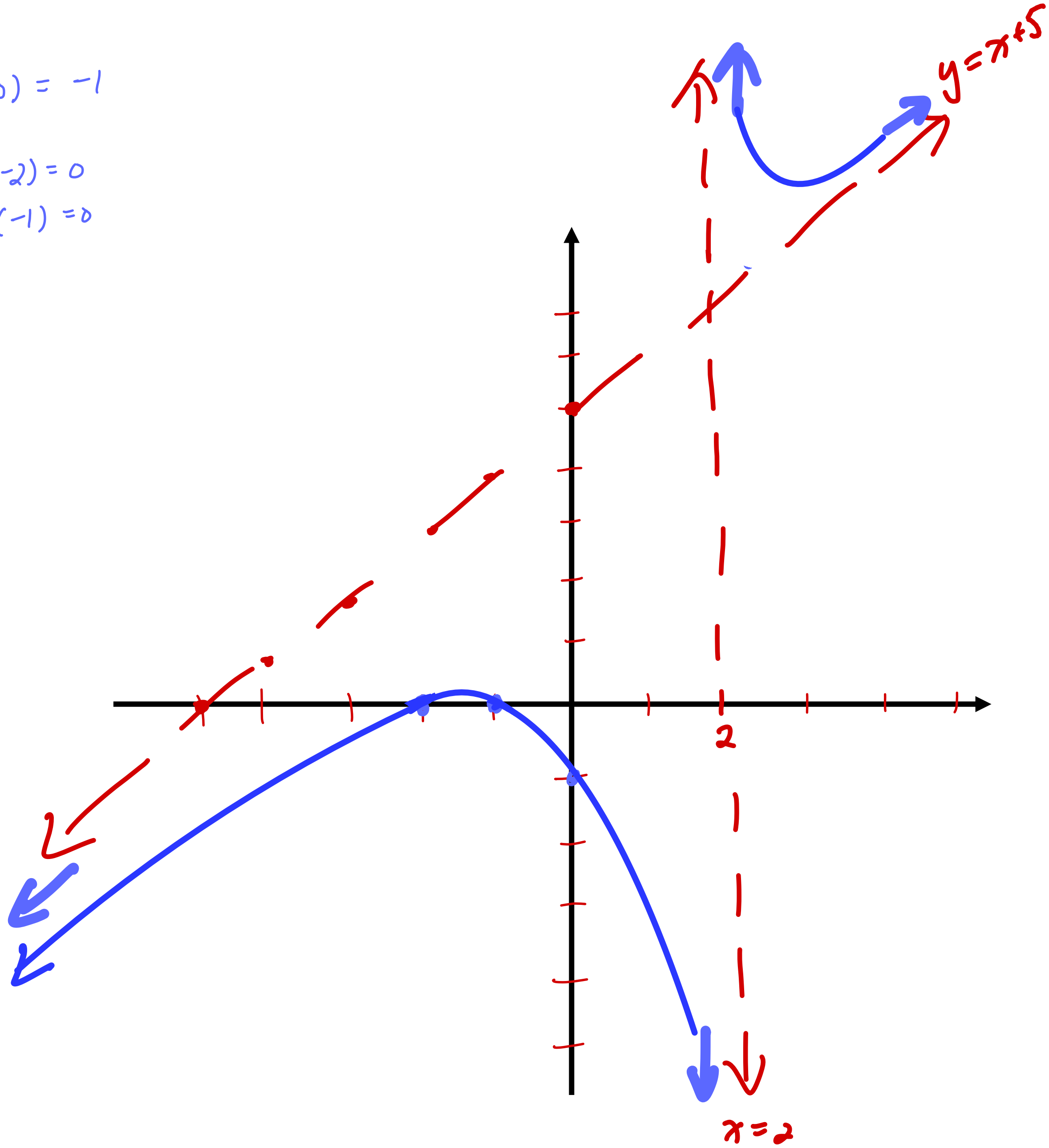
$\lim_{x\to +\infty} \dfrac{12}{x-2} = 0^+$

$$\begin{array}{r} x+5 \\ x-2\,\overline{)\,x^2+3x+2} \\ \underline{x^2-2x} \\ 5x+2 \\ \underline{5x-10} \\ 12 - R(x) \end{array}$$

$r(x) = \dfrac{12}{x-2}$

y-int: $f(0) = -1$

x-int: $f(-2) = 0$
$f(-1) = 0$



b) $f(x)=\dfrac{2x^3-4x^2-9}{4-x^2} = \dfrac{2x^3-4x^2-9}{-(x+2)(x-2)} = -2x+4+\dfrac{-4x-25}{-x^2+4} = -2x+4+\overbrace{\dfrac{8x-25}{-(x^2-4)}}^{r(x)}$

VA: $x=-2$ $\qquad$ $x=2$

$\lim\limits_{x\to -2^-} f(x) = +\infty$ $\qquad$ $\lim\limits_{x\to 2^-} f(x) = -\infty$

$\lim\limits_{x\to -2^+} f(x) = -\infty$ $\qquad$ $\lim\limits_{x\to 2^+} f(x) = +\infty$

OA: $y=-2x+4$

$$\begin{array}{r|l} & -2x+4 \\ \hline -x^2+0x+4 & 2x^3-4x^2+0x-9 \\ & 2x^3+0x^2-8x \\ \hline & -4x^2+8x-9 \\ & -4x^2+0x+16 \\ \hline & 8x-25 \end{array}$$

$\lim\limits_{x\to -\infty} r(x) = 0^+$

$\lim\limits_{x\to +\infty} r(x) = 0^-$

Cross over point: $r(x)=0$

$\dfrac{8x-25}{-(x^2-4)}=0$

$x=\dfrac{25}{8}$

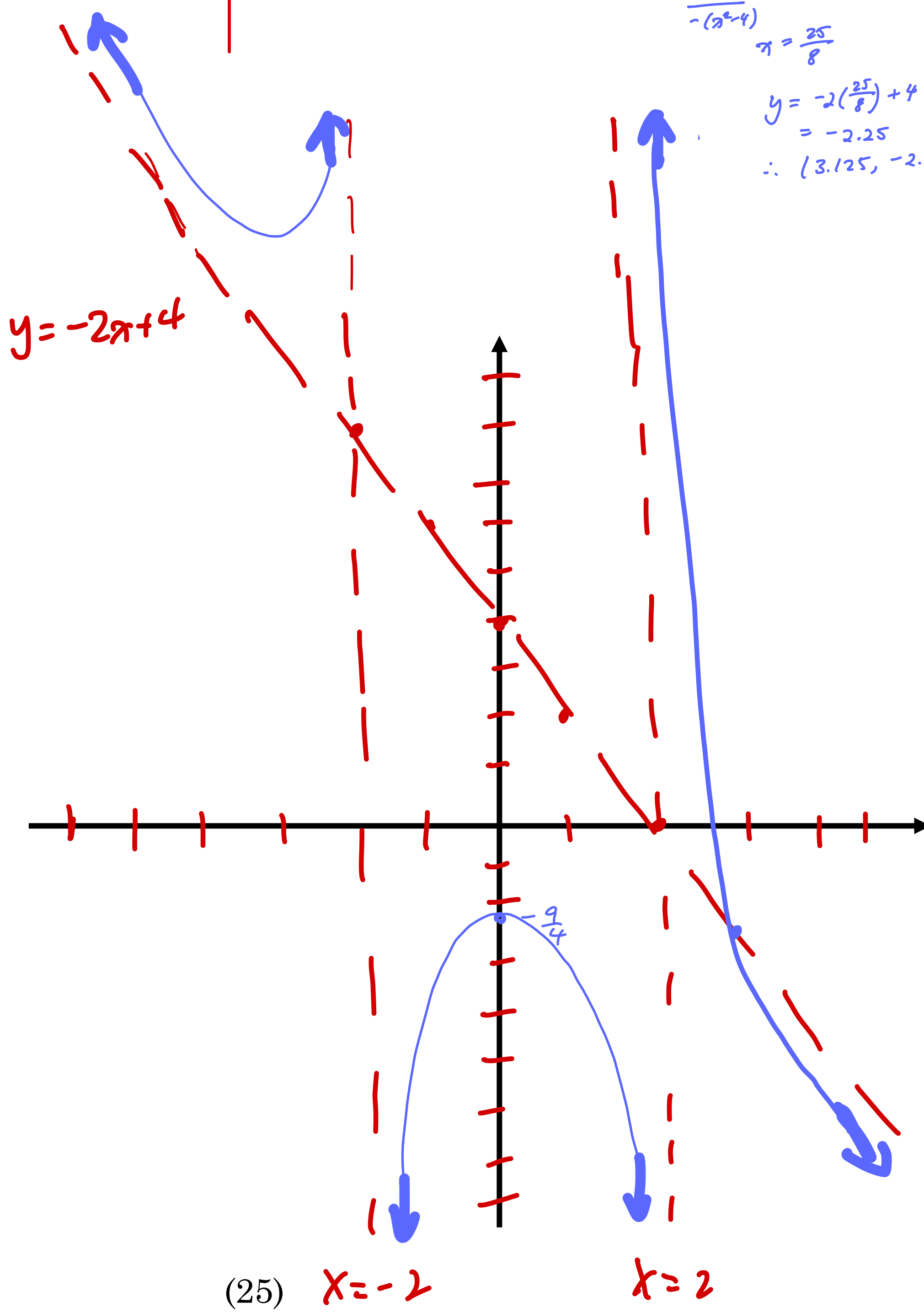
$y=-2\left(\dfrac{25}{8}\right)+4$

$=-2.25$

$\therefore\ (3.125, -2.25)$

xint: (too difficult)

yint: $-\dfrac{9}{4}$



bounce

c) $f(x)=\dfrac{-x^2}{x-3}$ $= -x-3+\dfrac{-9}{x-3}$

xint: 0

yint: 0

VA: $x=3$

$\lim\limits_{x\to 3^-} f(x) = +\infty$

$\lim\limits_{x\to 3^+} f(x) = -\infty$

OA: $y=-x-3$

$\lim\limits_{x\to -\infty} \dfrac{-9}{x-3} = 0^+$ (above $y=-x-3$)

$\lim\limits_{x\to \infty} \dfrac{-9}{x-3} = 0^-$ (below $y=-x-3$)

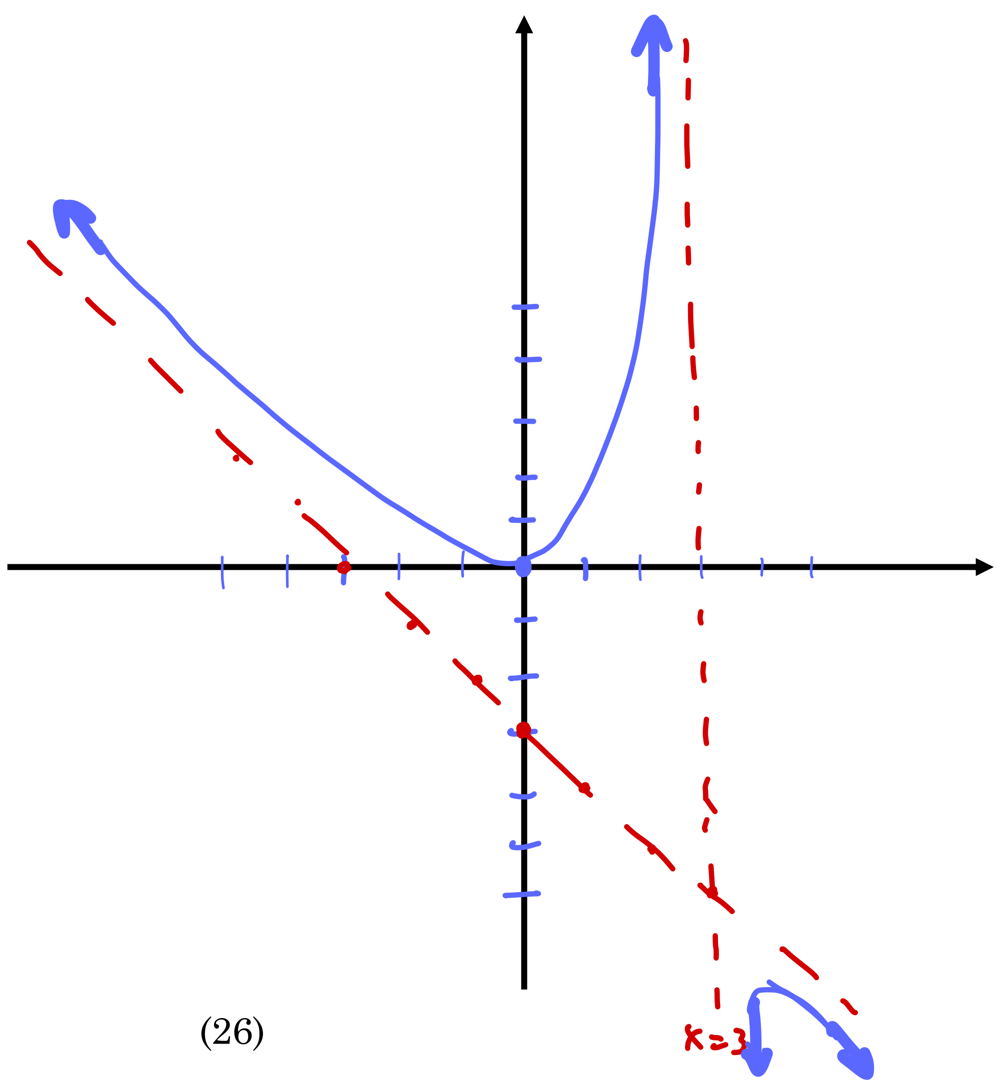
Cross over: $\dfrac{-9}{x-3}=0$

$\therefore$ none

$$
\begin{array}{r}
-x-3 \\
x-3\,\overline{)\,-x^2+0x} \\
-x^2+3x \\
\hline
-3x \\
-3x+9 \\
\hline
-9
\end{array}
$$

$r(x)=\dfrac{-9}{x-3}$

$f(x)$ (+) 0 (+) 3 (−)



Warm up

1. If a function is concave up in an interval then A

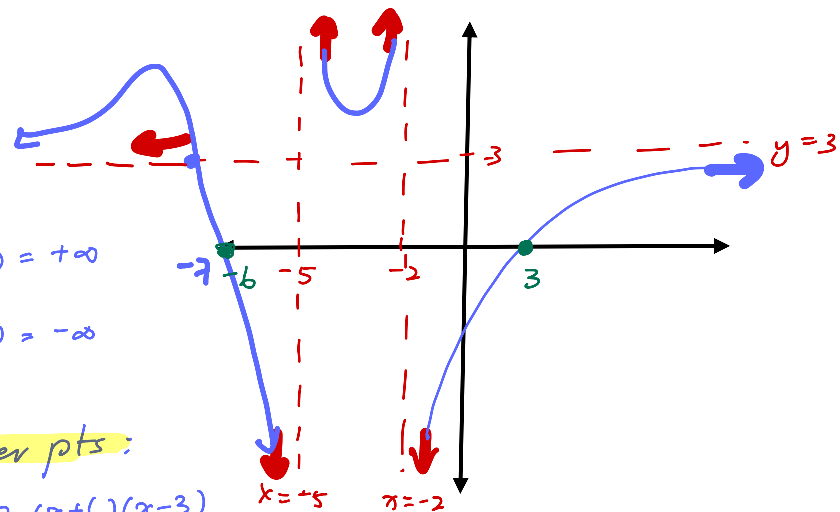
A) $f''(x) > 0$ B) $f''(x) < 0$ C) $f''(x) = 0$ D) $f'(x) > 0$

2. If $f(x)$ has domain $x \epsilon R$ and has a vertical tangent at $x = c$ then D

A) $f'(c) = 0$ x B) $f'(x)$ changes sign at $x = c$ x ie cusp C) $f''(c) = 0$ x ie cusp D) $f''(c)\ DNE$

3. Determine the asymptotes of $f(x) = \dfrac{3x^2 + 9x - 54}{x^2 + 7x + 10}$. Analyze and **sketch the behavior** of $f(x)$ near the asymptotes.

$f(x) = \dfrac{3(x+6)(x-3)}{(x+5)(x+2)}$



VA: $x = -5$

$\lim_{x \to -5^-} f(x) = -\infty$

$\lim_{x \to -5^+} f(x) = +\infty$

$x = -2$

$\lim_{x \to -2^-} f(x) = +\infty$

$\lim_{x \to -2^+} f(x) = -\infty$

HA: $y = 3$

$\lim_{x \to -\infty} f(x) = 3^+$

$\lim_{x \to +\infty} f(x) = 3^-$

Cross over pts:

$3 = \dfrac{3(x+6)(x-3)}{(x+5)(x+2)}$

$(x+5)(x+2) = (x+6)(x-3)$

$x^2 + 7x + 10 = x^2 + 3x - 18$

$4x = -28$

$x = -7$

4. If the graph of function $f(x) = (3a - x) + \dfrac{a - 3bx}{x^2}$ crosses its oblique asymptote at $x = \dfrac{1}{3}$ and has a horizontal tangent at $x = -2$, determine the values of **a** and **b.**

$r(x) = \dfrac{a - 3bx}{x^2}$

Cross over:

$r(\frac{1}{3}) = 0 \Rightarrow$ oblique @ $x = \frac{1}{3}$

$a - 3b(\frac{1}{3}) = 0$

$a - b = 0$ ①

$f(x) = (3a - x) + ax^{-2} - 3bx^{-1}$

$f'(x) = -1 - 2ax^{-3} + 3bx^{-2}$

$f'(-2) = 0 \Rightarrow$ horizontal tangent @ $x = -2$

$-1 - 2a(-2)^{-3} + 3b(-2)^{-2} = 0$

$-1 + \frac{a}{4} + \frac{3b}{4} = 0$

$\times 4 \Rightarrow -4 + a + 3b = 0$

$a + 3b = 4$ ②

$\begin{cases} a - b = 0 & ① \\ a + 3b = 4 & ② \end{cases}$

$(-)$ $-4b = -4$

$b = 1$

sub into ①

$a - 1 = 0$

$a = 1$

$\therefore a = 1 \quad b = 1$

Part I

Ex.1 : Sketch $f(x) = 3x^5 - 5x^3 + 3$. Find the coordinates of all relative extrema and inflection points.

① Domain : $x \in \mathbb{R}$
Range : $y \in \mathbb{R}$

② xint : < abandon not possible algebraically >
yint : 3

③ Interval of Increase / Decrease :

$f'(x) = 15x^4 - 15x^2$

$f'(x) = 15x^2(x^2 - 1)$

$= 15x^2(x+1)(x-1)$

critical #'s : $\{-1, 0, 1\}$

$f'(x)$: (+) on $(-\infty,-1)$ ↗, (−) on $(-1,0)$ ↘, (−) on $(0,1)$ ↘, (+) on $(1,\infty)$ ↗

max at -1: $f(-1) = 5$ min at 1: $f(1) = 1$

④ Intervals of Concavity

$f''(x) = 60x^3 - 30x$

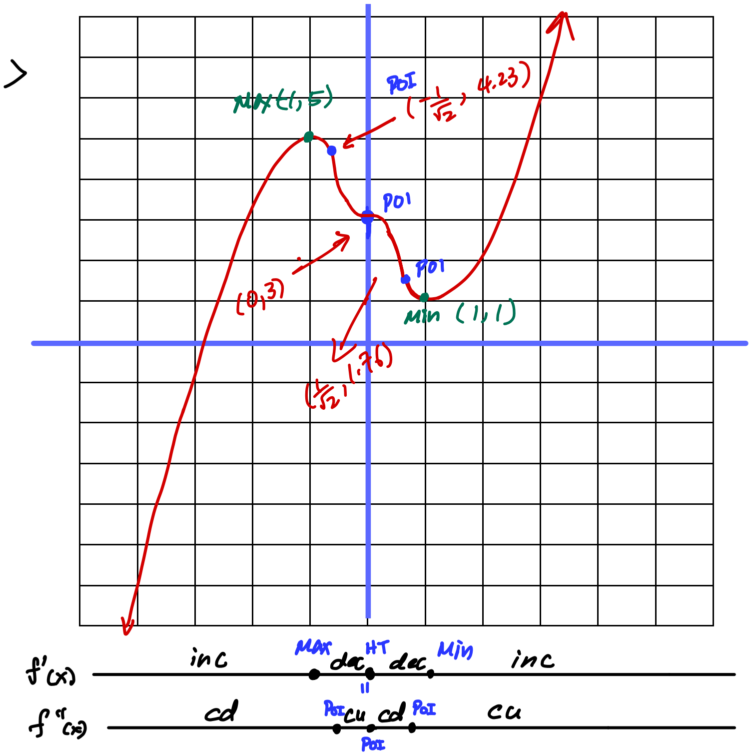
$= 30x(2x^2 - 1)$

$= 30x(\sqrt{2}x + 1)(\sqrt{2}x - 1)$

critical #'s : $\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\}$

$f''(x)$: (−) cd, $-\frac{1}{\sqrt{2}}$, (+) cu, 0, (−) cd, $\frac{1}{\sqrt{2}}$, (+) cu

inf. $f(-\frac{1}{\sqrt{2}}) \doteq 4.23$ inf. $f(0) = 3$ inf. $f(\frac{1}{\sqrt{2}}) \doteq 1.76$



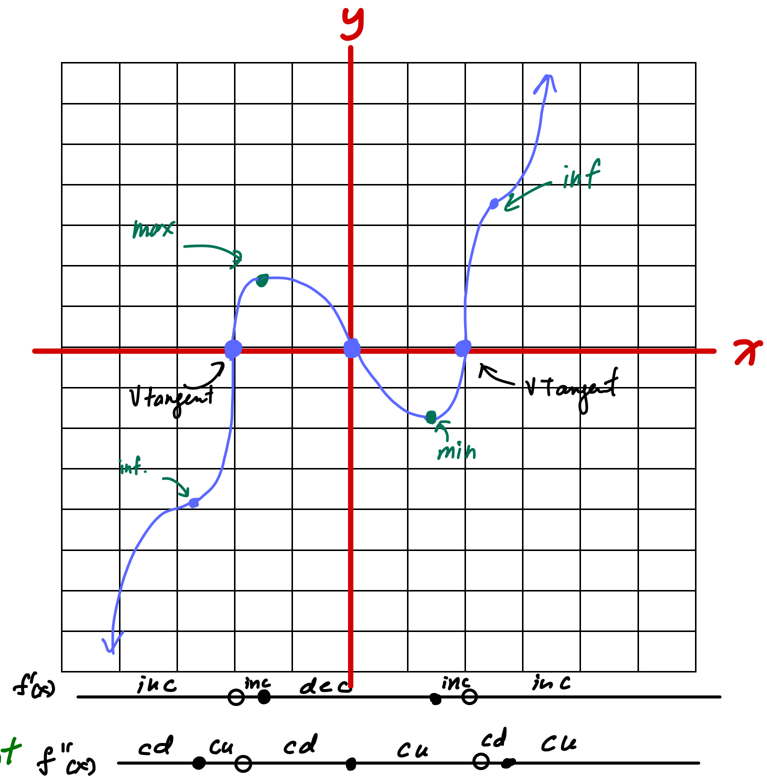
Ex.2: Sketch the graph of $f(x)=x\left(x^2-4\right)^{\frac{1}{3}}$. Given $f'(x)=\dfrac{5x^2-12}{3\left(x^2-4\right)^{\frac{2}{3}}}$, $f''(x)=\dfrac{10x^3-72x}{9\left(x^2-4\right)^{\frac{5}{3}}}$.

1) Domain : $x \in \mathbb{R}$

2) xint : $0 = x(x^2-4)^{\frac{1}{3}}$

$x = \{0, \pm 2\}$

yint : $f(0)=0$



3) Intervals of Inc/dec :

$$f'(x) = \frac{5x^2-12}{3(x^2-4)^{\frac{2}{3}}}$$

$$0 = \frac{5x^2-12}{3(x^2-4)^{\frac{2}{3}}}$$

double root

$(x+2)(x-2)$

$f'(x)=0$

$x = \pm\sqrt{\frac{12}{5}}$

$\doteq \pm 1.55$

max/min
H. Tangent

$f'(x) = dne$

$x = \pm 2$

vertical tangent or corner/cusp

$f'(x)$: (+) -2 (+) $-\sqrt{\frac{12}{5}}$ (−) $\sqrt{\frac{12}{5}}$ (+) 2 (+)

↗ ↗ ↘ ↗ ↗

max min

$f(-\sqrt{\frac{12}{5}}) \doteq 1.8$ $f(\sqrt{\frac{12}{5}}) \doteq -1.8$

4) Concavity :

$$f''(x) = \frac{10x^3-72x}{9(x^2-4)^{\frac{5}{3}}}$$

$$= \frac{2x(\sqrt{5}x-6)(\sqrt{5}x+6)}{9(x^2-4)^{\frac{5}{3}}}$$

$f''(x)=0$

$2x(5x^2-36)=0$

$x=\{0, \pm\frac{6}{\sqrt{5}}\}$

$\doteq \pm 2.7$

$f''(x) = dne$

$x = \pm 2$

$f''(x)$: (−) $-\frac{6}{\sqrt{5}}$ (+) -2 (−) 0 (+) 2 (−) $\frac{6}{\sqrt{5}}$ (+)

cd cu cd cu cd cu

inf inf inf inf inf

$f(-\sqrt{\frac{36}{5}})$	(V Tangent) $f(-2)$	$f(0)$	(V Tangent) $f(2)$	$f(\sqrt{\frac{36}{5}})$
$= -3.95$	$= 0$	$= 0$	$= 0$	$= 3.95$

Ex.3: Sketch the graph of $f(x)=x(8-x)^{\frac{2}{3}}$. Given $f'(x)=\dfrac{24-5x}{3(8-x)^{\frac{1}{3}}}$, $f''(x)=\dfrac{10x-96}{9(8-x)^{\frac{4}{3}}}$.

1) Domain: $x \in \mathbb{R}$

2) xint: $\{0, 8\}$
yint: $\{0\}$

3) Intervals of Increase / Decrease:

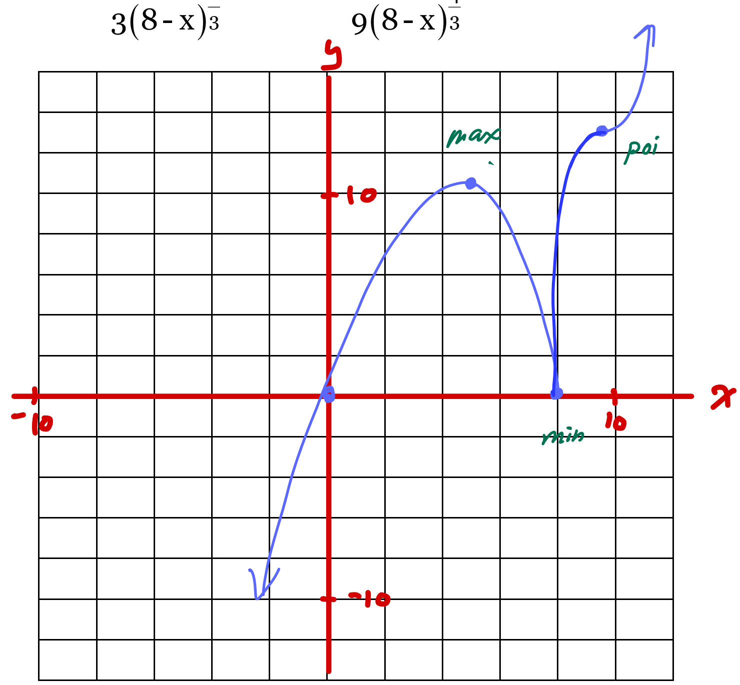
$f'(x) = 0$ $\qquad$ $f'(x) = dne$

$x = \{\frac{24}{5}\}$ 4.8 $\qquad$ $x = \{8\}$

$f'(x)$: (+) 4.8 (−) 8 (+)

max $f(4.8) = 10.4$

min $f(8) = 0$



4) Intervals of Concavity

$f''(x) = 0$ $\qquad$ $f''(x) = dne$

$x = \{9.6\}$ $\qquad$ $x = \{8\}$

$f'(x)$: (−) 8 (−) 9.6 (+)

cd 8 cd 9.6 cu

cusp

$f(8) = 0$

inf

$f(9.6) \doteq 13.1$

2.5 Practice-Part I

For each of the following curves determine the following properties and sketch the graph.

(a)	x- and y- intercepts	(b)	equations of asymptotes
(c)	intervals of increase/decrease	(d)	local extrema
(e)	intervals of concave up/down	(f)	inflection points

1. $f(x)=x^4-6x^2-27$.

2. $y = x^3 - x^2 - x + 1$ on the interval $[-2,3]$.

3. $y = 4x^{\frac{1}{3}} + x^{\frac{4}{3}}$

4. $y = \left(x^2 - 1\right)^{\frac{1}{3}}$, $y' = \dfrac{2x}{3\left(x^2-1\right)^{\frac{2}{3}}}$, $y'' = \dfrac{-2\left(x^2+3\right)}{9\left(x^2-1\right)^{\frac{5}{3}}}$

5. $y = x - \sqrt[3]{x}$, $y' = 1 - \dfrac{1}{3x^{\frac{2}{3}}}$, $y'' = \dfrac{2}{9x^{\frac{5}{3}}}$

6. $y = x\sqrt{1-x^2}$, $y' = -\dfrac{2x^2-1}{\sqrt{1-x^2}}$, $y'' = \dfrac{x\left(2x^2-3\right)}{\left(1-x^2\right)^{\frac{3}{2}}}$

Warm Up

Sketch the graph of $f(x) = x^{\frac{2}{3}}(8-x)^{\frac{1}{3}}$. Given $f'(x) = \dfrac{16-3x}{3x^{\frac{1}{3}}(8-x)^{\frac{2}{3}}}$, $f''(x) = \dfrac{-128}{9x^{\frac{4}{3}}(8-x)^{\frac{5}{3}}}$.

① D: $x \in \mathbb{R}$

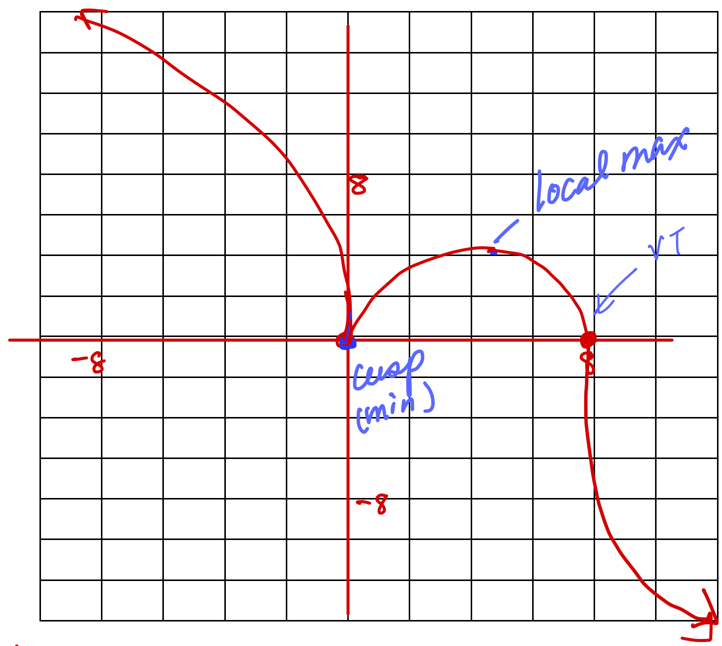
② x int: $\{0, 8\}$
y int: $\{0\}$

③ Inc / Dec

$f'(x) = 0$
$x = \frac{16}{3}$

$f'(x) = \text{dne}$
$x = \{0, 8\}$

$f'(x)$: (−) 0 (+) $5\frac{1}{3}$ (−) 8 (−)
↘ ↗ ↘ ↘
min: $f(0) = 0$
max: $f(5.3) = 4.2$



④ Concavity

$f''(x) = 0$
$x = \{\}$

$f''(x) = \text{dne}$
$x = \{0, 8\}$

$f''(x)$: (−) 0 (−) 8 (+)
cd cd cu

$f(0) = 0$ $f(8) = 0$

$f'(x)$: dec • inc • dec • dec

$f''(x)$: cd • cd • cu

Part II

Ex.1: Sketch the graph of $f(x)=\dfrac{x^3-4x}{3x^2+9x}$. Given $f'(x)=\dfrac{x^2+6x+4}{3(x+3)^2}$, $f''(x)=\dfrac{10}{3(x+3)^3}$.

① Domain: $\{x\in\mathbb{R},\ x\neq 0,-3\}$ ↳ hole $f(0)=\frac{-4}{9}$

$=\dfrac{\cancel{x}(x+2)(x-2)}{3\cancel{x}(x+3)}=\dfrac{(x+2)(x-2)}{3(x+3)}$

xint: $\{\pm 2\}$

yint: $\{\ \}$ note! "yint" is the hole

hole $x=0$

② VA: $x=-3$

(−) (+) (−) (−) (+) on −3, −2, 0, 2

$\lim_{x\to -3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = +\infty$

OA: $y=\frac{1}{3}x-1$

$$\begin{array}{r|l} & \frac{1}{3}x-1 \\ \hline 3x+9 & x^2+0x-4 \\ & x^2+3x \\ \hline & -3x-4 \\ & -3x-9 \\ \hline & 5 \end{array}$$

$f(x)=\frac{1}{3}x-1+\frac{5}{3x+9}$

$\lim_{x\to -\infty}\frac{5}{3x+9}=0^-$ (below)

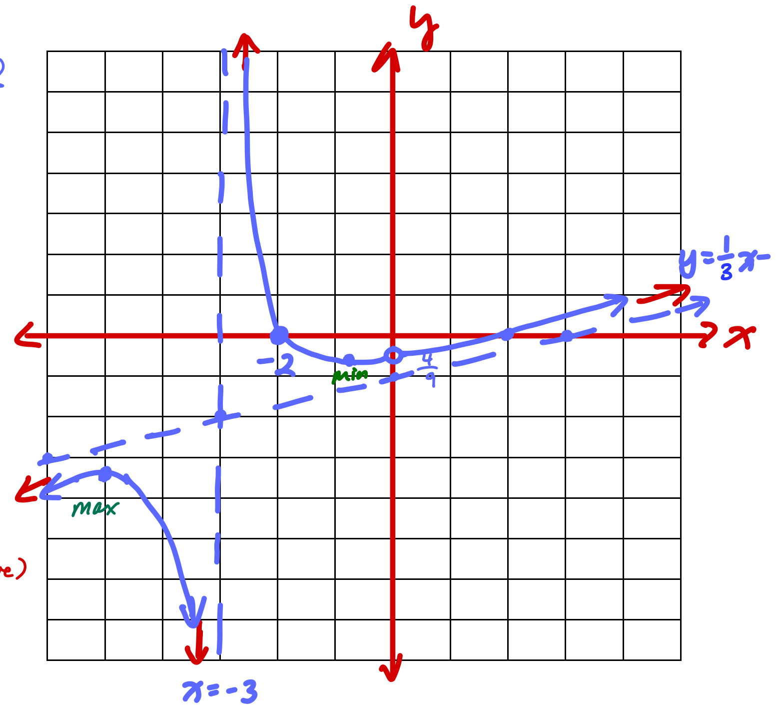
$\lim_{x\to \infty}\frac{5}{3x+9}=0^+$ (above)

cross over point:

$\frac{5}{3x+9}=0$

$x=\{\ \}$

∴ no cross over point



③ $f'(x)=0$

$x^2+6x+4=0$

$x=\dfrac{-6\pm\sqrt{36-4(1)(4)}}{2(1)}$

$=\left\{\dfrac{-6\pm\sqrt{20}}{2}\right\}$

$\doteq\{-0.8,-5.2\}$

$f'(x)=\text{dne}$

$=\{-3\}$

$f'(x)$: (+) −5.2 (−) −3 VA (−) −0.8 (+)

↗ ↘ ↘ ↗

max $f(-5.2)=-3.5$

min $f(-0.8)=-0.5$

④ $f''(x)=0$

$x=\{\ \}$

$f''(x)=\text{dne}$

$x=\{-3\}$

$f''(x)$: (−) −3 (+)

cd VA cu

Part II

Ex.1: Sketch the graph of $f(x) = \dfrac{x^3 - 4x}{3x^2 + 9x}$. Given $f'(x) = \dfrac{x^2 + 6x + 4}{3(x+3)^2}$, $f''(x) = \dfrac{10}{3(x+3)^3}$.

$(x+.8)(x+5.2)$

$= \dfrac{\cancel{x}(x+2)(x-2)}{3\cancel{x}(x+3)}$

hole : $x = 0$

VA : $x = -3$

① $D: \{x \in \mathbb{R},\ x \neq 0, -3\}$ (0 ↳ hole, −3 → VA)

Hole: $f(0) = -\frac{4}{9}$

② xint: $\{\pm 2\}$

yint: $-\frac{4}{9}$

③ VA: $x = -3$

$\lim_{x \to -3^-} f(x) = -\infty$

$\lim_{x \to -3^+} f(x) = +\infty$

$f(x)$: (−) −3 (+) −2 (−) 0 (−) 2 (+)

VA at −3, hole at 0

OA: $y = \frac{1}{3}x - 1$

$$\begin{array}{r} \frac{1}{3}x - 1 \\ 3x+9 \,\overline{)\, x^2 + 0x - 4} \\ x^2 + 3x \\ \hline -3x - 4 \\ -3x - 9 \\ \hline 5 \; R \end{array}$$

$r(x) = \dfrac{5}{3x+9}$

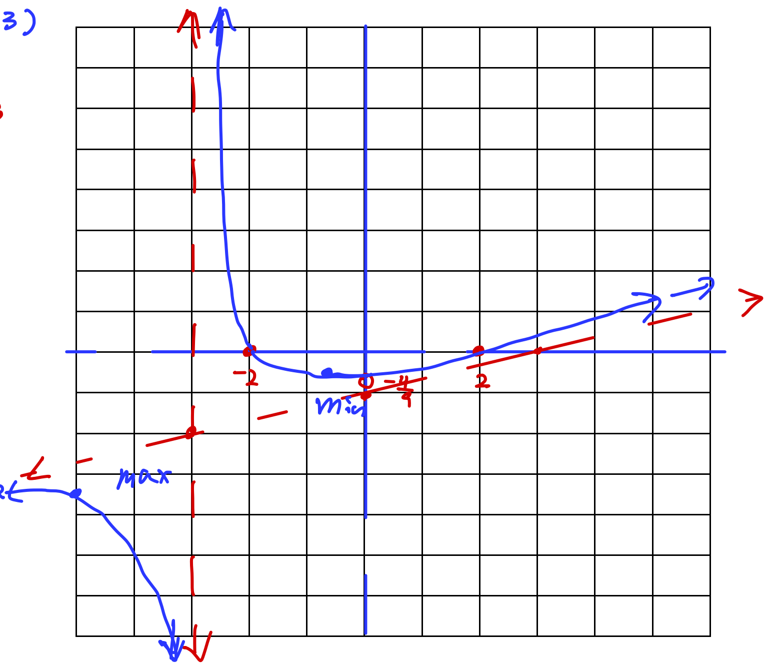
$\lim_{x \to -\infty} r(x) = 0^-$

$\lim_{x \to +\infty} r(x) = 0^+$

Cross over : $r(x) = 0$

$x = $ dne

∴ no cross over



④ Inc / Dec

$f'(x) = 0$

$x^2 + 6x + 4 = 0$

$x = \dfrac{-6 \pm \sqrt{36 - 4(1)(4)}}{2(1)}$

$= \dfrac{-6 \pm 2\sqrt{5}}{2}$

$= -3 \pm \sqrt{5}$

$\doteq \{-0.8, -5.2\}$

$f'(x) = $ dne

$x = -3$ ↳ VA

$f'(x)$: (+) −5.2 (−) −3 (−) −0.8 (+)

inc, dec, dec, inc

max at −5.2, VA at −3, min at −0.8

$f(-5.2) = -3.5$

$f(-0.8) = -0.5$

⑤ Concavity

$f''(x) = 0$

$x = \{\ \}$

$f''(x) = $ dne

$x = -3$ (VA)

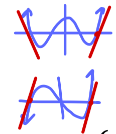
(−) −3 (+)

cd, cu

VA

Even Symmetry: symmetrical through a reflection on the y-axis

Odd Symmetry: symmetrical through a $180°$ rotation about the origin



$f(-x) = f(x) \Rightarrow f'(-a) = -f'(a)$

$f(-x) = -f(x) \Rightarrow f'(a) = f'(-a)$

Ex.2: Sketch the graph of $f(x) = \dfrac{x^2 - 1}{x^3}$. Given $f'(x) = \dfrac{-x^2 + 3}{x^4}$, $f''(x) = \dfrac{2(x^2 - 6)}{x^5}$.

$= \dfrac{(x+1)(x-1)}{x^3}$ $\quad = \dfrac{-(x+\sqrt{3})(x-\sqrt{3})}{x^4}$ $\quad = \dfrac{2(x+\sqrt{6})(x-\sqrt{6})}{x^5}$

① Domain: $\{x \in \mathbb{R}, x \neq 0\}$

② $x$int: $\{\pm 1\}$

$y$int: $\{\ \}$

$f(x)$: $-$ | -1 | $+$ | 0 | $-$ | 1 | $+$

③ VA: $x = 0$

$f(x)$: $(-)$ | -1 | $(+)$ | 0 | $(-)$ | 1 | $(+)$

$\lim_{x \to 0^-} f(x) = +\infty$

$\lim_{x \to 0^+} f(x) = -\infty$

HA: $y = 0$

$\lim_{x \to -\infty} f(x) = 0^-$

↳ sub $f(-100) < 0$

$\lim_{x \to +\infty} f(x) = 0^+$

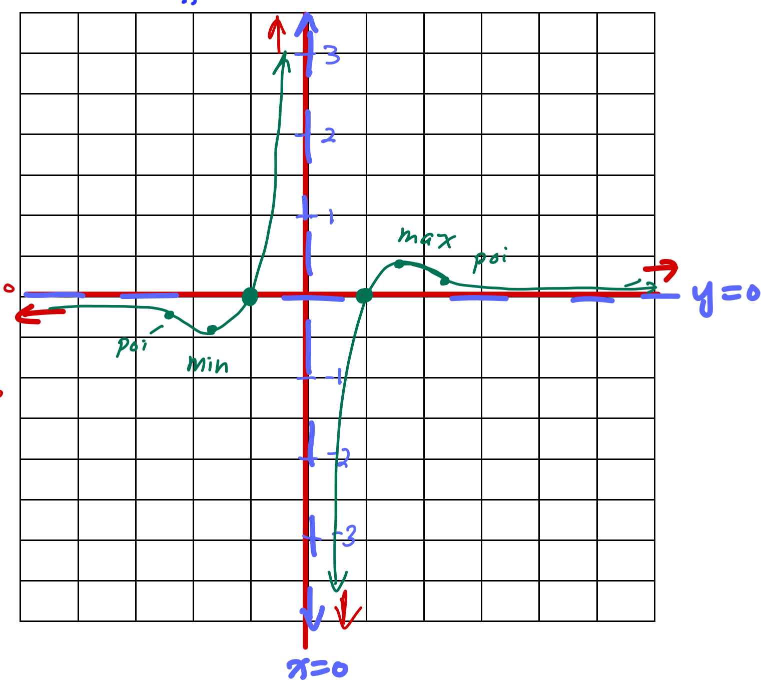
↳ sub $f(100) > 0$

Cross over point:

$\dfrac{x^2 - 1}{x^3} = 0$

$x = \{\pm 1\}$

$(1,0)$ and $(-1,0)$



④ $f'(x) = 0$

$x = \{\pm\sqrt{3}\}$ $\leftarrow \pm 1.7$

$f'(x) = $ dne

$x = \{0\}$

↳ outside of domain

$f'(x)$: $(-)$ | $-\sqrt{3}$ | $(+)$ | 0 | $(+)$ | $\sqrt{3}$ | $(-)$

↘ ↗ ↗ ↘

min: $f(-\sqrt{3}) \doteq -0.4$

max: $f(\sqrt{3}) \doteq 0.4$

⑤ $f''(x) = 0$

$x = \pm\sqrt{6}$ $\leftarrow 2.4$

$f''(x) = $ dne

$x = \{0\}$

$(-)$ | $-\sqrt{6}$ | $(+)$ | 0 | $(-)$ | $\sqrt{6}$ | $(+)$

cd | cu | cd | cu

poi: $f(-\sqrt{6}) = -0.3$, $(-2.4, -0.3)$

VA (at 0)

poi: $f(\sqrt{6}) = 0.3$, $(2.4, 0.3)$

Optional: Check for symmetry

$f(-x) = \dfrac{(-x)^2 - 1}{(-x)^3}$

$= \dfrac{x^2 - 1}{-x^3}$

$= -f(x)$

$\therefore$ odd symmetry

Ex.3: Sketch the graph of $f(x)=\dfrac{(1-x)(1+x+x^2)}{1-x^2}$. Given $f'(x)=\dfrac{x(x+2)}{(x+1)^2}$, $f''(x)=\dfrac{2}{(x+1)^3}$.

$= \dfrac{\cancel{-}\cancel{(x-1)}(x^2+x+1)}{\cancel{-}(x+1)\cancel{(x-1)}}$

$= \dfrac{x^2+x+1}{x+1}$

$\therefore$ hole $x=1$

$f(1)=\frac{3}{2}$

① Domain: $\{x\in\mathbb{R},\ x\neq 1,-1\}$ ↳ hole

② xint: none

yint: $\{1\}$

③ VA: $x=-1$

$\lim\limits_{x\to -1^-} f(x) = -\infty$

$\lim\limits_{x\to -1^+} f(x) = +\infty$

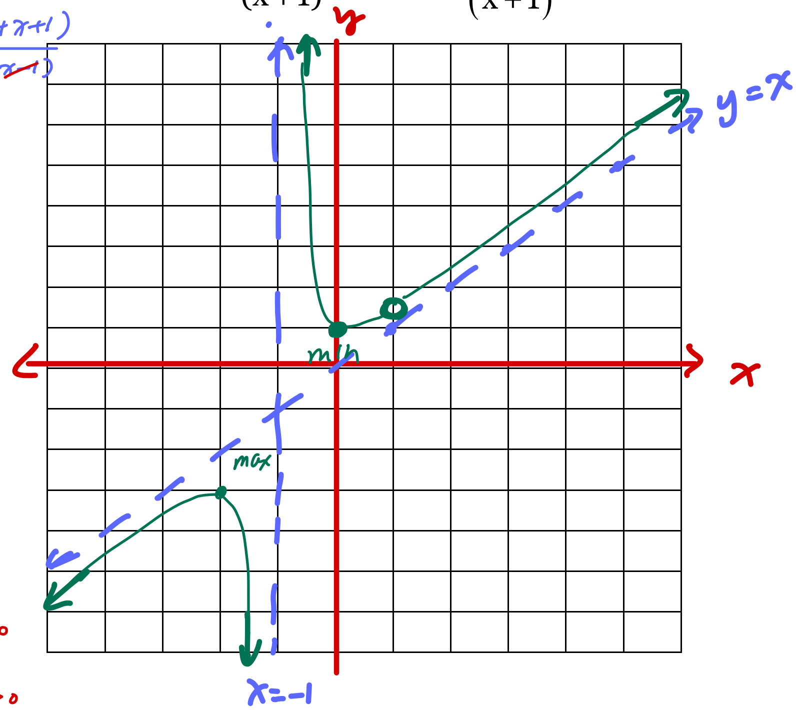
OA: $y=x$

$$\begin{array}{r|l} & x \\ x+1 & x^2+x+1 \\ & \underline{x^2+x} \\ & 1 \end{array}$$

$f(x) = x + \dfrac{1}{x+1}$

$\lim\limits_{x\to -\infty} \dfrac{1}{x+1} = 0^-$ ↳ $R(-100)<0$

$\lim\limits_{x\to +\infty} \dfrac{1}{x+1} = 0^+$ ↳ $R(100)>0$



④ $f'(x)=0$

$x=\{0,-2\}$

$f'(x)=\text{dne}$

$x=\{-1\}$

$f'(x)$	$(+)$	-2	$(-)$	-1	$(-)$	0	$(+)$
	↗	max	↘	VA	↘	min	↗

$f(-2) = -3$

$f(0) = 1$

⑤ $f''(x)=0$

$x=\{\ \}$

$f''(x)=\text{dne}$

$x=-1$

$f''(x)$	$(-)$	-1 (VA)	$(+)$
	cd		cu

2.5 Practice-Part II

For each of the following curves determine the following properties and sketch the graph.

(a) x- and y- intercepts (b) equations of asymptotes
(c) intervals of increase/decrease (d) local extrema
(e) intervals of concave up/down (f) inflection points

1. $f(x)=\dfrac{x}{(x-3)^2}$, $f'(x)=\dfrac{-3-x}{(x-3)^3}$, $f''(x)=\dfrac{2(x+6)}{(x-3)^4}$.

2. $y=\dfrac{x}{\sqrt{4-x^2}}$, $y'=\dfrac{4}{\left(4-x^2\right)^{\frac{3}{2}}}$, $y''=\dfrac{12x}{\left(4-x^2\right)^{\frac{5}{2}}}$

3. $y=\dfrac{x}{(x-1)^2}$, $y'=-\dfrac{x+1}{(x-1)^3}$, $y''=\dfrac{2(x+2)}{(x-1)^4}$

4. $y=\dfrac{x}{x^2-1}$, $y'=-\dfrac{x^2+1}{\left(x^2-1\right)^2}$, $y''=\dfrac{2x\left(x^2+3\right)}{\left(x^2-1\right)^3}$

5. $y=\dfrac{x^2-1}{x^3}$, $y'=\dfrac{-\left(x^2-3\right)}{x^4}$, $y''=\dfrac{2\left(x^2-6\right)}{x^5}$

6*. $y=2\sqrt{x}\left(x-2\sqrt{x}+1\right)$, $y'=\dfrac{3x-4\sqrt{x}+1}{\sqrt{x}}$, $y''=\dfrac{3x-1}{2x^{\frac{3}{2}}}$

Warm Up

Sketch the graph of $f(x)=\dfrac{18(1-x)}{(x+3)^2}$. Given $f'(x)=\dfrac{18(x-5)}{(x+3)^3}$, $f''(x)=\dfrac{-36(x-9)}{(x+3)^4}$.

$= -\dfrac{18(x-1)}{(x+3)^2}$

① Domain: $\{x \in \mathbb{R}, x \neq -3\}$

② xint: $\{1\}$

yint: $\{2\}$

③ VA: $y=-3$

(+) (+) (−)

-3 1

VA

$\lim\limits_{x \to -3^-} f(x) = +\infty$

$\lim\limits_{x \to -3^+} f(x) = +\infty$

HA: $y=0$

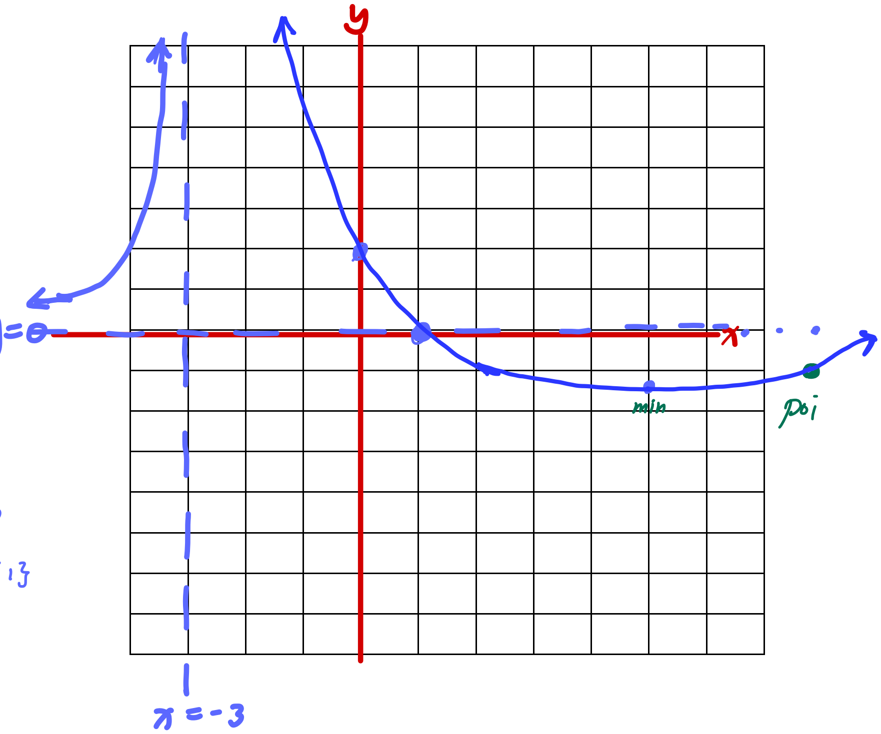
$\lim\limits_{x \to -\infty} f(x) = 0^+$

$\lim\limits_{x \to +\infty} f(x) = 0^-$

Cross over point:

$\dfrac{-18(x-1)}{(x-3)} = 0$

$x=\{1\}$



④ $f'(x)=0$

$x=\{5\}$

$f'(x)=dne$

$x=\{-3\}$

$f'(x)$ (+) (−) (+)

-3 5

VA min.

$f(5) \doteq -1.1$

⑤ $f''(x)=0$

$x=\{9\}$

$f''(x)=dne$

$x=-3$

(+) (+) (−)

cu -3 cu 9 cd

VA

$f(9)=-1$

III. Curve Sketching from Given Information

1 .Sketch a graph of a function that satisfies all of the following conditions:

- The domain of the function is the set of all real numbers except $x=1$ and $x=-1$.
- $\lim_{x\to 1^+} f(x)=-\infty$, $\lim_{x\to 1^-} f(x)=\infty$, $\lim_{x\to -1^+} f(x)=\infty$, $\lim_{x\to -1^-} f(x)=-\infty$
- $\lim_{x\to\infty} f(x)=0$, $f(x)<0$ as $x\to\infty$
- $\lim_{x\to-\infty} f(x)=0$, $f(x)<0$ as $x\to-\infty$
- The y intercept is 2. There are no x intercepts.

2. Sketch a graph of a polynomial (continuous) function that satisfies all of the following conditions:
- $f'(x)>0$ for $0<x<1$ and $f'(x)<0$ for $1<x<\infty$
- $f''(x)>0$ for $2<x<\infty$ and $f''(x)<0$ for $0<x<2$
- $\lim_{x\to\infty} f(x)=0$
- $f(x)$ is an odd function (symmetrical about the origin)

3. Sketch a graph of a (continuous) function that satisfies all of the following conditions:
- $f(0)=f(3)=2,\quad f(-1)=f(1)=0$
- $f'(-1)=f'(1)=0$
- $f'(x)<0$ for $-\infty<x<-1$ and for $0<x<1$
- $f'(x)>0$ for $-1<x<0$ and for $1<x<\infty$
- $f''(x)>0$ for $x<3\,(x\neq 0)$ and $f''(x)<0$ for $3<x<\infty$
- $\lim_{x\to\infty} f(x)=4$ and $\lim_{x\to-\infty} f(x)=\infty$

4. Sketch a graph of a function that satisfies all of the following conditions:
- $f(0)=1,\quad f(1)=2$
- $\lim_{x\to 2^-} f(x)=\infty$, $\lim_{x\to -1^+} f(x)=-\infty$

5. Sketch a graph of a polynomial (continuous) function that satisfies all of the following conditions:
- $f(3)=5,\quad f(7)=-2,\quad f(5)=2$
- $f'(3)=f'(7)=0$
- $f'(x)>0$ for $x<3$ and $x>7$, $f'(x)<0$ for $3<x<7$
- $f''(5)=0$
- $f''(x)>0$ for $x>5$ and $f''(x)<0$ for $x<5$

6. Provide a sketch of any curve with the following properties.

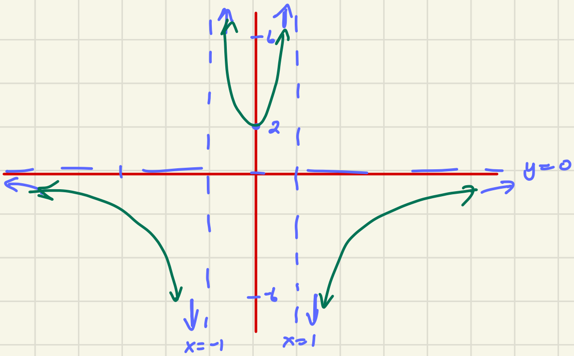
- $\lim_{x\to 5^+} f(x) = \infty$
- $\lim_{x\to 5^-} f(x) = -\infty$
- $\lim_{x\to \infty} f(x) = 2$
- $\lim_{x\to -\infty} f(x) = \infty$
- $f(0) = 5, \quad f(-2) = -4$
- $f'(0) = 0, \quad f'(-2)$ = DNE
- $f''(x) > 0 \quad x \in (5, \infty)$ and $(-\infty, -7)$
- $f''(x) < 0 \quad x \in (-7, -2) \cup (-2, 5)$

7. Provide a sketch of any curve with the following properties.

- $f(1) = 4,\ f(2) = 0,\ f(-2) = -3, f(-4) = -2.5$
- $f'(-2) = 0,\ f'(2) = dne$
- $\lim_{x\to -1^-} f(x) = \infty$, $\lim_{x\to -1^+} f(x) = \infty$
- $\lim_{x\to -\infty} f(x) = -2,\ \lim_{x\to \infty} f(x) = \infty$
- $f'(x) < 0$ for $x \in (-\infty, -2) \cup (-1, 2)$; $f'(x) > 0$ for $x \in (-2, -1) \cup (2, \infty)$
- $f''(x) > 0$ for $x \in (-4, -1) \cup (-1, 1)$; $f''(x) < 0$ for $x \in (-\infty, -4) \cup (1, 2) \cup (2, \infty)$
- crossover at $(-1.5, -2)$

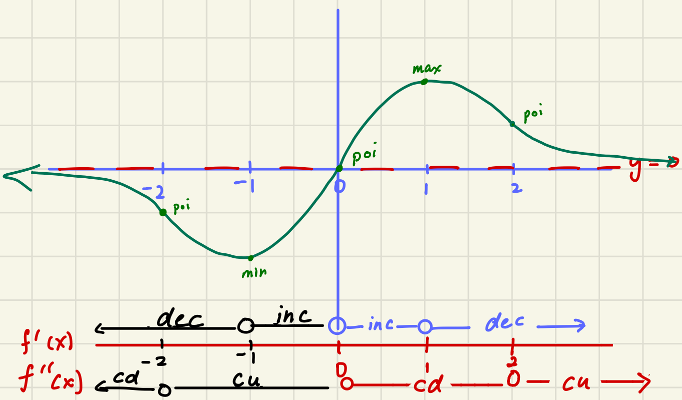
1 .Sketch a graph of a function that satisfies all of the following conditions:

- The domain of the function is the set of all real numbers except $x=1$ and $x=-1$.
- $\lim_{x\to1^+} f(x)=-\infty$, $\lim_{x\to1^-} f(x)=\infty$, $\lim_{x\to-1^+} f(x)=\infty$, $\lim_{x\to-1^-} f(x)=-\infty$
- $\lim_{x\to\infty} f(x)=0$, $f(x)<0$ as $x\to\infty$
- $\lim_{x\to-\infty} f(x)=0$, $f(x)<0$ as $x\to-\infty$
- The y intercept is 2. There are no x intercepts.



2. Sketch a graph of a polynomial (continuous) function that satisfies all of the following conditions:

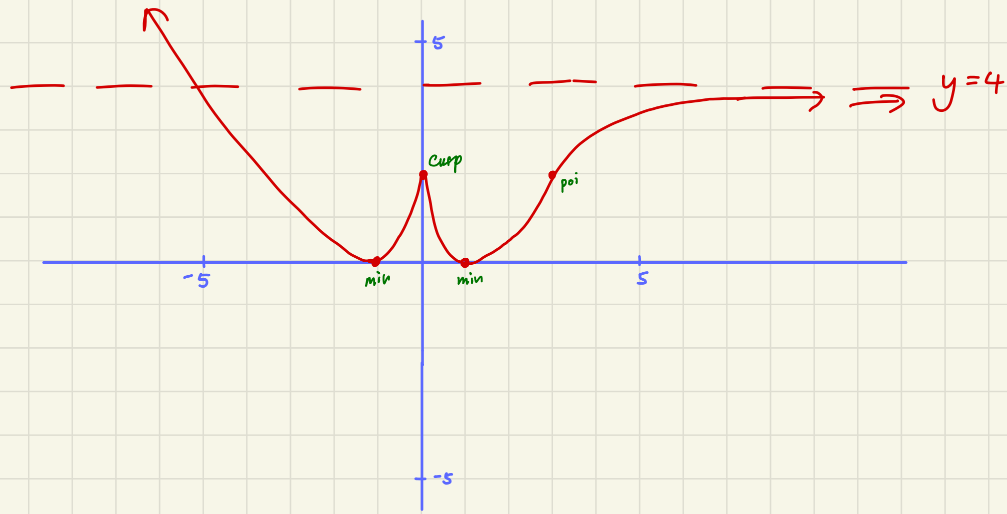
- $f'(x)>0$ for $0<x<1$ and $f'(x)<0$ for $1<x<\infty$
- $f''(x)>0$ for $2<x<\infty$ and $f''(x)<0$ for $0<x<2$
- $\lim_{x\to\infty} f(x)=0$
- $f(x)$ is an odd function (symmetrical about the origin)

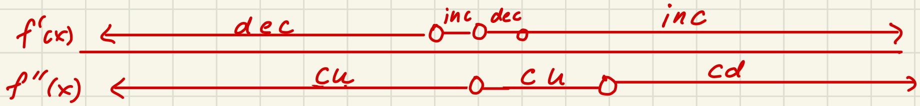


odd symmetry

3. Sketch a graph of a (continuous) function that satisfies all of the following conditions:

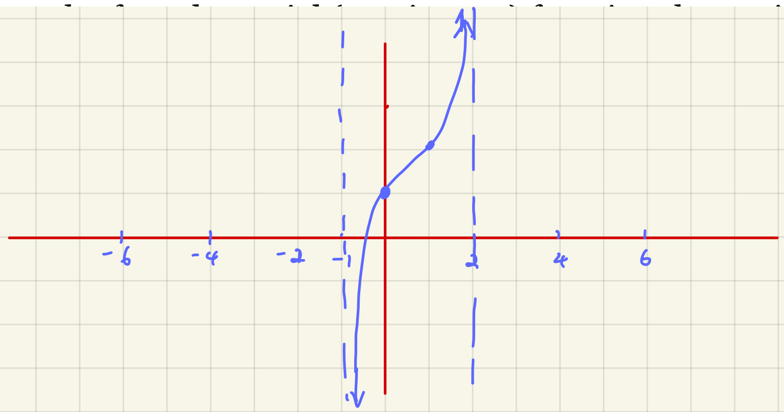
- $f(0) = f(3) = 2, \quad f(-1) = f(1) = 0$
- $f'(-1) = f'(1) = 0$
- $f'(x) < 0$ for $-\infty < x < -1$ and for $0 < x < 1$
- $f'(x) > 0$ for $-1 < x < 0$ and for $1 < x < \infty$
- $f''(x) > 0$ for $x < 3\ (x \neq 0)$ and $f''(x) < 0$ for $3 < x < \infty$
- $\lim_{x\to\infty} f(x) = 4$ and $\lim_{x\to-\infty} f(x) = \infty$





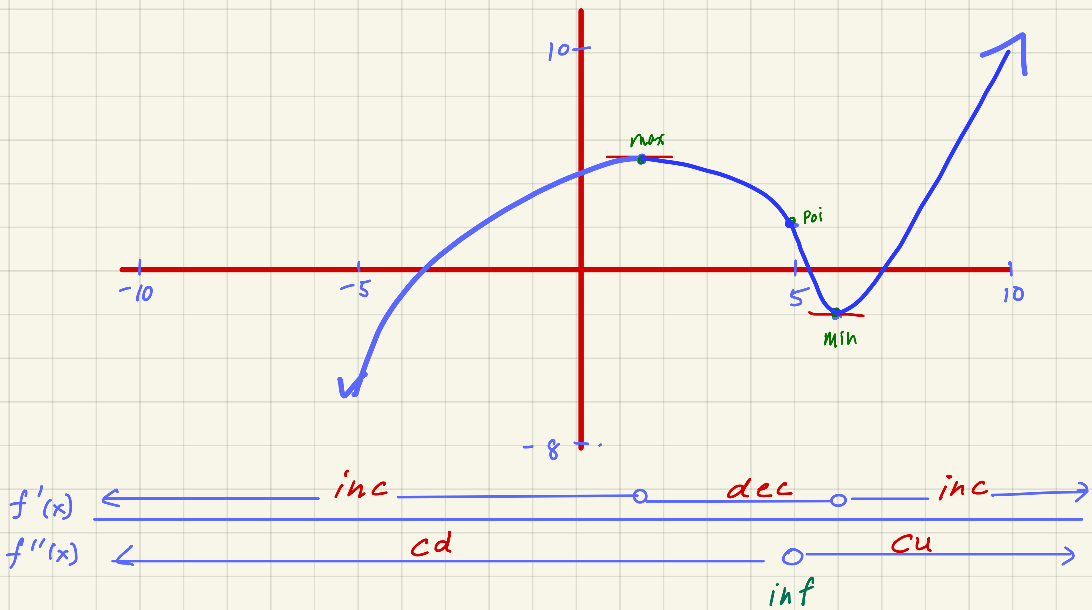
4. Sketch a graph of a function that satisfies all of the following conditions:

- $f(0)=1, \quad f(1)=2$
- $\lim\limits_{x\to 2^-} f(x)=\infty$, $\lim\limits_{x\to -1^+} f(x)=-\infty$



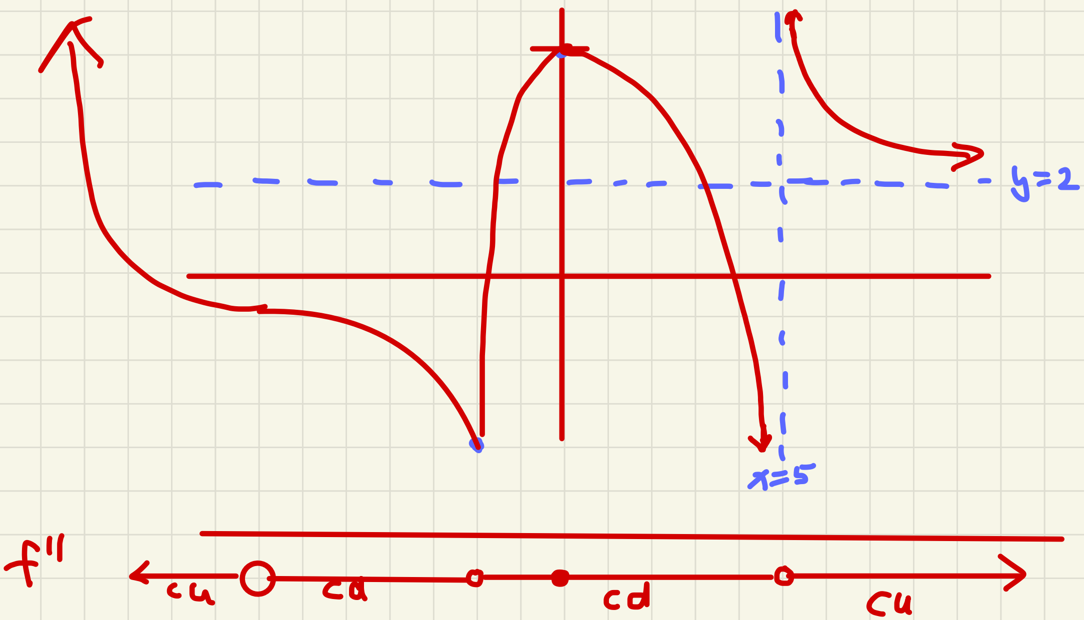
5. Sketch a graph of a polynomial (continuous) function that satisfies all of the following conditions:

- $f(3)=5, \quad f(7)=-2, \quad f(5)=2$
- $f'(3)=f'(7)=0$
- $f'(x)>0$ for $x<3$ and $x>7$, $f'(x)<0$ for $3<x<7$
- $f''(5)=0$
- $f''(x)>0$ for $x>5$ and $f''(x)<0$ for $x<5$



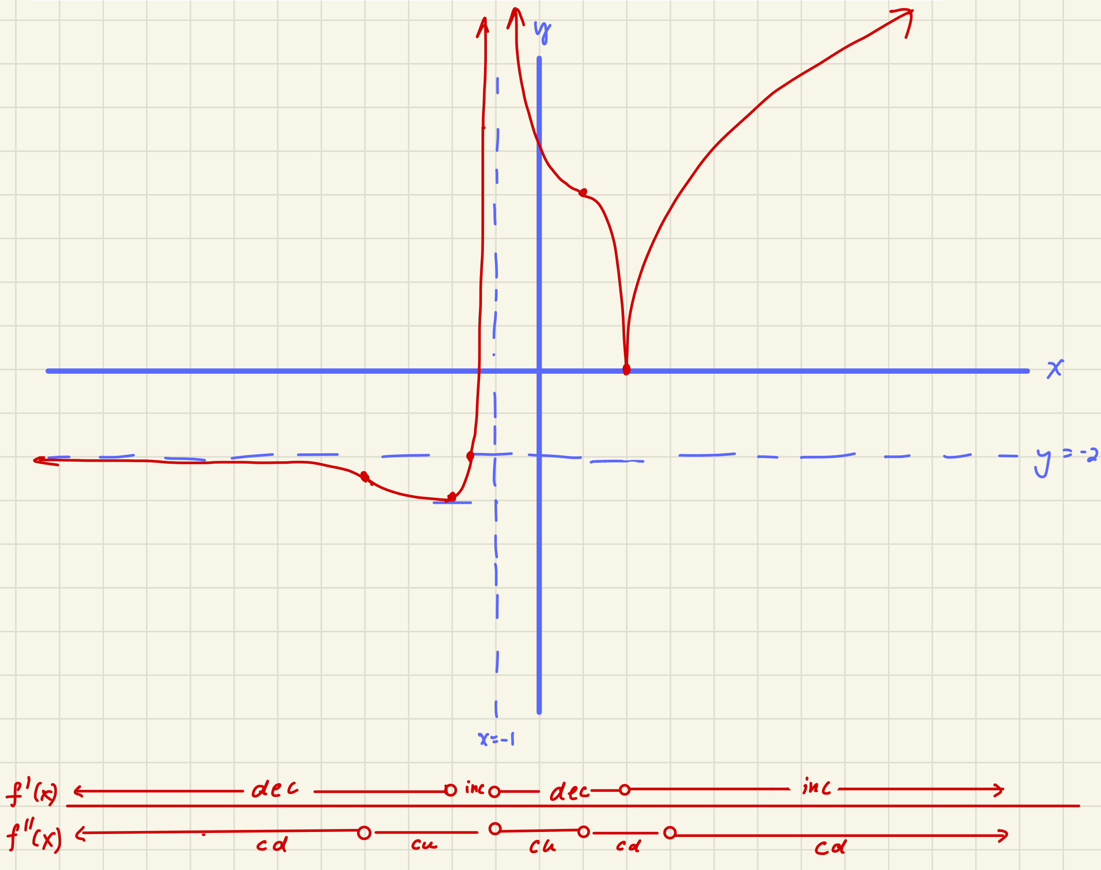
6. Provide a sketch of any curve with the following properties.

- $\lim_{x\to 5^+} f(x) = \infty$
- $\lim_{x\to 5^-} f(x) = -\infty$
- $\lim_{x\to \infty} f(x) = 2$
- $\lim_{x\to -\infty} f(x) = \infty$
- $f(0) = 5, \quad f(-2) = -4$
- $f'(0) = 0, \quad f'(-2)$ = DNE
- $f''(x) > 0 \quad x \in (5, \infty)$ and $(-\infty, -7)$
- $f''(x) < 0 \quad x \in (-7, -2) \cup (-2, 5)$



7. Provide a sketch of any curve with the following properties.

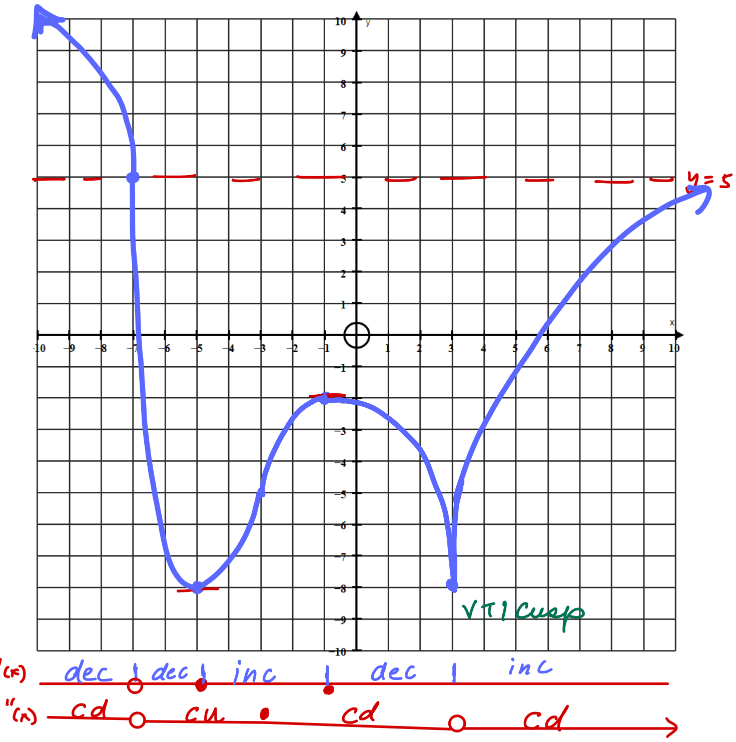
- $f(1)=4,\ f(2)=0,\ f(-2)=-3, f(-4)=-2.5$
- $f'(-2)=0,\ f'(2)=dne$
- $\lim\limits_{x\to -1^-} f(x)=\infty$, $\lim\limits_{x\to -1^+} f(x)=\infty$
- $\lim\limits_{x\to -\infty} f(x)=-2,\ \lim\limits_{x\to\infty} f(x)=\infty$
- $f'(x)<0$ for $x\in(-\infty,-2)\cup(-1,2)$; $f'(x)>0$ for $x\in(-2,-1)\cup(2,\infty)$
- $f''(x)>0$ for $x\in(-4,-1)\cup(-1,1)$; $f''(x)<0$ for $x\in(-\infty,-4)\cup(1,2)\cup(2,\infty)$
- crossover at $(-1.5,-2)$



Warm up: Curve Sketching

1. Sketch a possible graph of the continuous function that satisfies all of the following conditions:

- $f(-7)=5, f(-5)=f(3)=-8, f(-1)=-2$
- $f'(-5)=f'(-1)=0$
- $f'(-7)=f'(3)=DNE$
- $f'(x)<0 \quad for \quad x\in(-\infty,-7)\cup(-7,-5)\cup(-1,3)$
- $f'(x)>0 \quad for \quad x\in(-5,-1)\cup(3,\infty)$
- $f''(-3)=0$
- $f''(-7)=f''(3)=DNE$
- $f''(x)<0 \quad for \quad x\in(-\infty,-7)\cup(-3,3)\cup(3,\infty)$
- $f''(x)>0 \quad for \quad x\in(-7,-3)$
- $\lim_{x\to-\infty} f(x)=\infty \quad and \quad \lim_{x\to\infty} f(x)=5$



2. Given the function $f(x)=\dfrac{ax+b}{x^2-c}$ and that it has the following properties:

- the graph of $f(x)$ is symmetric with respect to the y-axis
- $\lim_{x\to 2^+} f(x)=+\infty \Rightarrow \lim_{x\to -2^-} f(x)=+\infty$
- $f'(1)=-2 \Rightarrow f'(-1)=2$

Even symmetry
$f(-x)=f(x)$
$f'(-1)=2 \qquad f'(1)=-2$

(a) Determine the values of a, b & c.

(b) Determine the equation for all asymptotes of the graph of $f(x)$.

Aside: Even symmetry
$f(x)=f(-x)$
$\dfrac{ax+b}{x^2-c}=\dfrac{-ax+b}{x^2-c}$
$2ax=0$

$\lim_{x\to 2^+} f(x)=+\infty$
$\Rightarrow$ vertical asymptote at $x=2$
$x^2-c=0$
$2^2-c=0$
$c=4$

$\to f(x)=\dfrac{ax+b}{x^2-4}$
$f'(x)=\dfrac{a(x^2-4)-2x(ax+b)}{(x^2-4)^2}$
$f'(1)=\dfrac{a(-3)-2(a+b)}{9}$
$-2=\dfrac{-3a-2a-2b}{9}$
$-18=-5a-2b$
$5a+2b=18$ ①

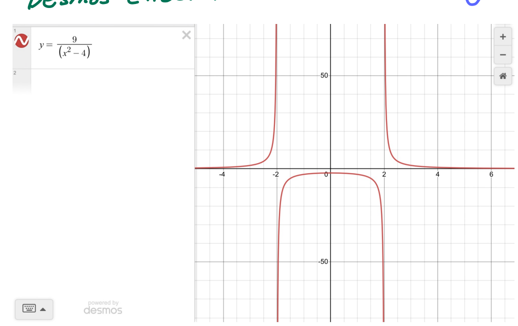
Even symmetry
• vertical asymptote at $x=-2$ $\quad f'(-1)=2$
$f'(-1)=\dfrac{a(-3)+2(-a+b)}{9}$
$2=\dfrac{-3a-2a+2b}{9}$
$18=-5a+2b$
$5a-2b=-18$ ②

$5a+2b=18$ ①
$5a-2b=-18$ ②
(−) $4b=36$
$b=9$ ③
sub into ①
$5a+2(9)=18$
$5a=0$
$a=0$

$\therefore a=0$
$b=9$
$c=4$

b) $f(x)=\dfrac{9}{x^2-4}$ $\quad$ VA: $x=-2$, $x=2$
HA: $y=0$

DESMOS CHECK:



Unit 3: Curve Sketching

Part A: Multiple Choice-Write the letter of your choice in the space at left.

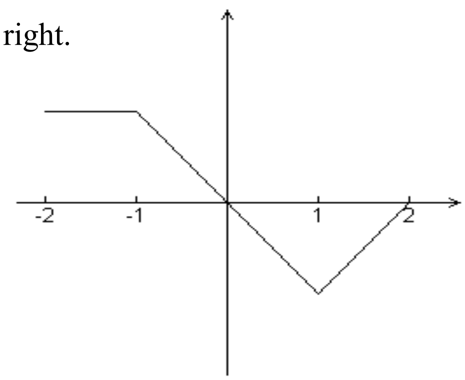
____ 1) Let *f* be the function with derivative given by $f'(x) = x^2 - \frac{2}{x}$. On which of the following intervals is *f* decreasing?

A) $(-\infty, -1]$ B) $(-\infty, 0)$ C) $[-1, 0)$ D) $(0, \sqrt[3]{2})$ E) $(\sqrt[3]{2}, \infty)$

____ 2) If the line tangent to the graph of the function *f* at the point $(1, 7)$ passes through the point $(-2, -2)$, then $f'(1)$ is

A) -5 B) 1 C) 3 D) 7 E) undefined

____ 3) The graph of f', the derivative of the function *f*, is shown at right. Which of the following statements is true about *f*?



Graph of f'

A) *f* is decreasing for $-1 < x < 1$.

B) *f* is increasing for $-2 < x < 0$.

C) *f* is increasing for $1 < x < 2$.

D) *f* has a local minimum at $x = 0$.

E) *f* is not differentiable at $x = -1$ and $x = 1$.

____ 4) Determine constants *a* and *b* such that the function $f(x) = x^3 + ax^2 + bx + c$ has a relative minimum at $x = 4$ and a point of inflection at $x = 1$.

A) $a = 1$, $b = 3$ B) $a = -3$, $b = 3$ C) $a = -3$, $b = -24$
D) $a = 3$, $b = 0$ E) $a = -6$, $b = 2$

____ 5) Which of the following are true for $h(x) = x^4 - 4x$?

I. *h* has a point of inflection at $x = 0$
II. *h* has an absolute minimum of -3
III. The second derivative test for local extrema is inconclusive

A) I and II B) I and III C) II and III D) I only E) II only

Part B: Full Solution

1. Find the exact value of a such that the function $f(x)=\sqrt{x-2}-\frac{a}{x}$ has a point of inflection at x=3.

2. Sketch $f(x)=x^{\frac{2}{3}}(8-x)^{\frac{1}{3}}$. Find the coordinates of all relative extrema and inflection Points.

$$f'(x)=\frac{16-3x}{3x^{\frac{1}{3}}(8-x)^{\frac{2}{3}}} \qquad f''(x)=\frac{-128}{9x^{\frac{4}{3}}(8-x)^{\frac{5}{3}}}$$

3. Perform a sketch for the following functions. Clearly indicate the results of each step.

a) $f(x)=\dfrac{x^3-4x}{3x^2+9x}$, $f'(x)=\dfrac{x^2+6x+4}{3(x+3)^2}$, $f''(x)=\dfrac{10}{3(x+3)^3}$

b) $f(x)=\dfrac{3x^2}{(x\text{-}3)^2}$, $f'(x)=\dfrac{\text{-}18x}{(x\text{-}3)^3}$, $f''(x)=\dfrac{18(2x+3)}{(x\text{-}3)^4}$

c) $f(x)=4-(x-3)^{\frac{2}{3}}$

4. Consider the function $f(x)=-x^3+6x^2+15x+2$ on the closed interval [-2, 2].

a) Find, and classify, all relative extrema.

b) Determine the absolute maximum of the function on the interval.

5. Let $f(x)=x\left(4+x^2-\dfrac{x^4}{5}\right)$ .Find the interval(s) on which f is increasing.

6. Use the **SECOND DERIVATIVE TEST** to find the local extrema of the function $f(x)=3x^4+8x^3-6x^2-24x+3$.

7. The function $f(x)=2kx^3+3x^2+px-3$ has a local minimum at x = -1 and a point of inflection at x =1 .Determine the values of k and p.

8. A function is defined by $f(x)=ax^3+bx+c$.

a) Find the values of a, b, and c if $f(x)$ has a y-intercept at (0, 2) and a local maximum at (2, 6).

b) Explain how you know there must be local minimum.

9. Function $g(x)=ax^3+bx^2+cx+d$ has a maximum at (-1,3) and a point of inflection at (1, 5) What Find the values of a, b, c and d.

10. Sketch the graph of a rational function that satisfies all of the following conditions:

$f''(x)<0 \text{ when } x<-4 \qquad f''(x)>0 \text{ when } x>-4 \qquad f'(x)<0 \text{ for all } x$

$\lim\limits_{x\to-4^+} f(x)=+\infty \qquad \lim\limits_{x\to-4^-} f(x)=-\infty \qquad \lim\limits_{x\to\pm\infty} f(x)=-2 \qquad f(2)=0$

11. a) Let f be a function that is even and continuous on the closed interval [-3,3]. The function f and its derivatives have the following properties:

$f(x) > 0$ when $0 < x < 1$ $\quad$ $f(x) < 0$ when $1 < x < 2$ and $2 < x < 3$

$f(0) = 1$, $f(1) = 0$, $f(2) = -1$, $f(3) = 0$

$f'(x) > 0$ when $2 < x < 3$, $f'(x) < 0$ when $0 < x < 1$ and when $1 < x < 2$

$f'(0) = DNE$, $f'(2) = DNE$, $f'(1) = 0$

$f''(x) > 0$ when $0 < x < 1$, $f''(x) < 0$ when $1 < x < 2$ and when $2 < x < 3$

$f''(0) = DNE$, $f''(2) = DNE$ and $f''(1) = 0$

b) Find all values of x at which f has a relative extremum. Justify your answer.

c) Find the coordinates of any inflection points on the graph of f.

d) Identify the coordinates of any cusp or vertical tangent on the graph of f.

12. Find constants **a**,**b**, and **c** so that the slope of **normal** to the function $f(x) = ax^3 + bx^2 + c$ at its point of inflection $(1,5)$ is $\dfrac{-1}{6}$.

13. Let $f(x) = \dfrac{x^2 - ax + 2}{x^2 - bx + 3}$. Determine the values of **a** and **b** so that f(x) has **only one** vertical asymptote at x = -3.