Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Logarithmic and Exponential functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(a^x) = a^x \ln a \qquad \frac{d}{dx}(e^x) = e^x$$

$$\begin{aligned} sin(2A) &= 2\,sin(A)\,cos(A) \\ cos(2A) &= cos^2(A) - sin^2(A) \\ &= 1 - 2sin^2(A) \\ &= 2cos^2(A) - 1 \\ \tan(2A) &= \frac{2\tan(A)}{1 - tan^2(A)} \end{aligned}$$

$$\begin{aligned} \tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta} \\ \tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ \tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \end{aligned}$$

UNIT 5

DERIVATIVES OF EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

Transcendental Functions : non-algebraic

↳ Exponential & Logarithmic

↳ Trig Functions

$$\sin(2A) = 2\sin(A)\cos(A)$$

$$\begin{aligned} \cos(2A) &= \cos^2(A) - \sin^2(A) \\ &= 1 - 2\sin^2(A) \\ &= 2\cos^2(A) - 1 \end{aligned}$$

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Grade 12 Calculus & Vectors
Unit 5– Derivatives of Other Functions

DAY	DESCRIPTION	Homework
1	**Investigation: The Derivatives of Sine and Cosine** We are learning to… • investigate the derivatives of the sine and cosine function • determine the derivatives of functions that involve sine and cosine. I am able to… • find the derivative of a function that has sine or cosine in it using previous rules of differentiation	*CP: Pg 9-10* *Textbook:* *Pg 226: #3-8, 10* *Pg 230: #1-8, 11-13*
2	**The Number e and the Derivative of $y = e^x$ and $y = \ln x$** We are learning to… • explore the number e and its special properties • show that the derivative of $y = e^x$ is $y = e^x$ • show that the derivative of $f(x)=ln(x)$ is $f'(x)=1/x$ I am able to… • define e both as a number and as a limit • solve equations and simplify or evaluate expressions containing e^x and $\ln(x)$. • Differentiate function $y=\ln(x)$ and $y = e^x$	CP: Pg 19-22 Textbook: Pg 250: #5-11 Pg 264: #4-10 Pg 282: #1-5, 8-10
3	**The Derivative of Exponential Functions $f(x) = b^x$** **The Derivative of logarithmic Functions $f(x) = \log_b x$** We are learning to… • devise a method of finding the derivative of any exponential function $y=b^x$ and logarithmic function $y=\log_b x$ I am able to… • verify the derivative of the exponential function $f(x) = b^x$ is $f'(x) = b^x \ln b$ for various values of b using technology • verify the derivative of the logarithmic function $f(x) = \log_b x$ is $f'(x) = 1/x\ln b$ for various values of b using technology •	CP:Pg 31-32 Textbook: Pg 274: #1-5, 13 Pg 282: #7, 12, 13

4	**Mid-Review**	CP: Pg 33
5	**Optimization Problems for Exponential Functions** We are learning to… • solve optimization problems that involve exponential functions • solve problems involving the Product Rule and Chain Rule and develop algebraic facility where appropriate. I am able to… • solve optimization problems that involve exponential functions and a function that describes the quantity to be optimized	CP: Pg 39 Textbook: Pg 244: #1-8 Pg 294: #1 – 14
6	**Quiz/Review**	*CP: Pg 40-41* *Textbook:* *Pg 246: #1-15 (not 13, 14)* *Pg 296: #1-9, 11-13, 16* *Pg 298: #1-4, 7, 10-12*
7,8	**Summative Assessments**	**M Nov 25 Day 1** **T Nov 26 Day 2**

5.1: Derivatives of Trigonometric Functions

Recall: $sin(a+b) = sin(a)cos(b) + sin(b)cos(a)$ & $cos(a+b) = cos(a)cos(b) - sin(a)sin(b)$

Theorem: Prove $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x} = 1$ **and** $\lim\limits_{x\to 0}\dfrac{\cos(x)-1}{x} = 0$.

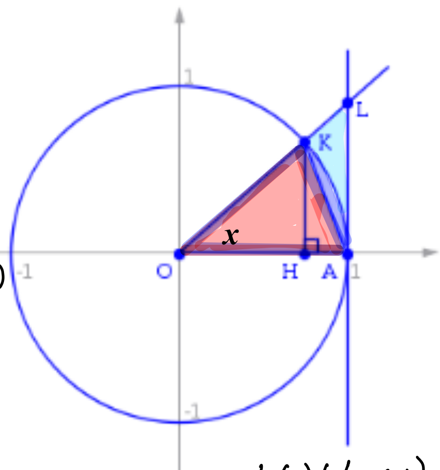
a) Numeric proof

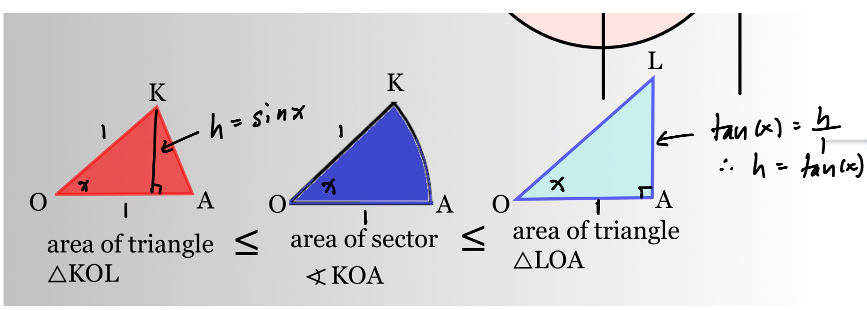
*Calculators must be in **radian** mode

x	$\dfrac{cosx-1}{x}$	$\dfrac{sinx}{x}$
0.1	-0.04996	0.999958
0.01	-0.005	0.99998
0.001	-0.0005	0.99999
0.0001	-0.00005	1
$x \to 0$	0	1

b) Algebraic proof

i) Let x be an acute angle and consider the following diagram on the **unit circle** where the angle ∠KOH is the angle x.





$\frac{1}{2}(1)(\sin(x))$ $\qquad$ $\frac{x}{2\cancel{\pi}}\cancel{\pi}(1)^2$ $\qquad$ $\frac{1}{2}(1)(\tan(x))$

We have:

area of triangle △KOA ≤ area of sector ∢ KOA ≤ area of triangle △LOA

$$\frac{\sin(x)}{2} \le \frac{x}{2} \le \frac{\tan(x)}{2}$$

Hence multiplying through by $\dfrac{2}{sin(x)} > 0$ gives $1 \le \dfrac{x}{\sin(x)} \le \dfrac{1}{\cos(x)}$

so taking reciprocals yields : $\cos(x) \le \dfrac{\sin(x)}{x} \le 1$

$$\lim_{x\to 0}\cos(x) \le \lim_{x\to 0}\frac{\sin(x)}{x} \le \lim_{x\to 0}(1)$$

since $\lim\limits_{x\to 0}\cos(x) = \lim\limits_{x\to 0}(1) = 1$,then according to Squeeze Theorem $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x} = 1$.

$$1 \le \lim_{x\to 0}\frac{\sin(x)}{x} \le 1$$

$\therefore \lim\limits_{x\to 0}\dfrac{\sin(x)}{x} = 1$ ☺

ii) Prove $\lim\limits_{x \to 0} \frac{\cos(x) - 1}{x} = 0$

Note: $\lim\limits_{x\to 0} \frac{\cos x - 1}{x} = \frac{\cos(0) - 1}{(0)}$

$= \frac{1-1}{0}$

$= \frac{0}{0}$ is indeterminant

$\therefore$ must 'manipulate' it.

$$L.S = \lim_{x\to 0} \frac{\cos(x) - 1}{x} \cdot \frac{\cos(x)+1}{\cos(x)+1}$$

$$= \lim_{x\to 0} \frac{\cos^2(x) - 1}{x(\cos(x)+1)}$$

$$= \lim_{x\to 0} \frac{-\sin^2(x)}{x(\cos(x)+1)}$$

$$= -\lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{\sin(x)}{\cos(x)+1}$$

$$= -(1) \cdot \frac{\sin(0)}{\cos(0)+1}$$

$$= -1 \cdot \frac{0}{1+1}$$

$$= -1 \cdot 0$$

$$= 0$$

Aside:

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x - 1 = -\sin^2 x$

a) Finding the derivative of $f(x) = \sin x$ using the first principle:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cdot\cos(h) + \cos(x)\cdot\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cdot\cos(h) - \sin(x)}{h} + \lim_{h\to 0} \frac{\cos(x)\cdot\sin(h)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos(x)\cdot\sin(h)}{h}$$

$$= \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}$$

$$= \sin x \cdot (0) + \cos x \cdot (1)$$

$$= \cos x$$

$y = \cos(x)$

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} - \lim_{h\to 0} \frac{\sin(x)\sin(h)}{h}$$

$$= \lim_{h\to 0} \cos(x) \left[\frac{\cos(h) - 1}{h}\right]$$

$$= \cos(x) \cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \sin(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}$$

$$= \cos(x) \cdot 0 - \sin(x) \cdot 1$$

$$= -\sin(x)$$

$y = \tan(x)$

$= \frac{\sin(x)}{\cos(x)} \leftarrow$ quotient rule! $\frac{u'v - v'u}{v^2}$

$$y' = \frac{[\cos(x)][\cos(x)] - [-\sin(x)][\sin(x)]}{[\cos(x)]^2}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

Aside:

$\sin^2 x + \cos^2(x) = 1$

$$= \frac{1}{\cos^2(x)}$$

$$= \sec^2(x)$$

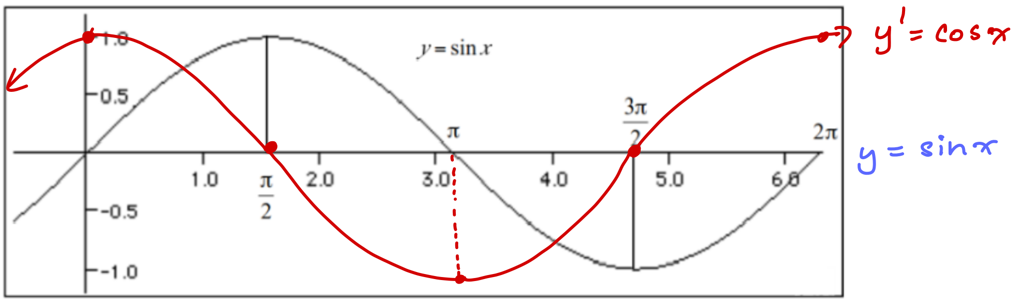
Derivative of Secondary Trig Functions: $y=\csc(x)$, $y=\sec(x)$ and $y=\cot(x)$

$$y=\csc(x)$$
$$=\frac{1}{\sin(x)}$$
$$=[\sin(x)]^{-1}$$
$$\frac{dy}{dx}=-[\sin(x)]^{-2}[\cos(x)]$$
$$=-\frac{\cos(x)}{\sin^2(x)}$$
$$=-\csc(x)\cdot\cot(x)$$

$$y=\sec(x)$$
$$=\frac{1}{\cos(x)}$$
$$=[\cos(x)]^{-1}$$
$$\frac{dy}{dx}=-[\cos(x)]^{-2}\cdot -\sin(x)$$
$$=\frac{\sin(x)}{\cos^2(x)}$$
$$=\tan(x)\cdot\sec(x)$$

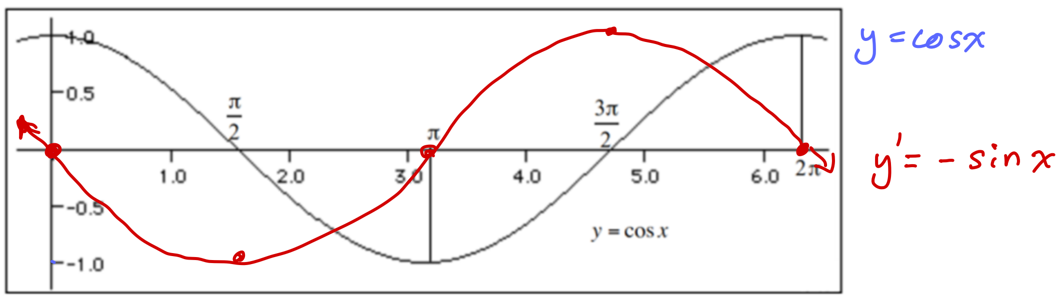
$$y=\cot(x)$$
$$=\frac{1}{\tan(x)}$$
$$=[\tan(x)]^{-1}$$
$$\frac{dy}{dx}=-[\tan(x)]^{-2}[\sec^2(x)]$$
$$=-\frac{\sec^2(x)}{\tan^2(x)}$$
$$=-\frac{1}{\cos^2(x)}\div\frac{\sin^2(x)}{\cos^2(x)}$$
$$=-\frac{1}{\sin^2(x)}$$
$$=-\csc^2(x)$$

b) The graph of *f(x) = sin(x)* is shown. Sketch the derivative of *f(x) = sin(x)* on the graph.



Conclusion: If $y = \sin(x)$, then $y' =$ $\underline{\cos x}$

2. Graph of $y = cosx$ is shown. Sketch the derivative of y on the same graph.



Conclusion: If $y = \cos(x)$, then $y' =$ $\underline{-\sin x}$

3. Using the quotient rule: If $y = tanx$, then $y' =$ $\underline{\sec^2 x}$

$$y = \tan x = \frac{\sin x}{\cos x}$$

$$y' = \frac{[\cos x][\cos x] - [-\sin x][\sin x]}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

In general:

- $y = \left[\sin\left(g(x)\right)\right]^n$, *then* $y' =$ $\underline{n\,[\sin(g(x))]^{n-1} \cdot \cos(g(x)) \cdot g'(x) \cdot 1}$
- $y = \left[\cos\left(g(x)\right)\right]^n$, *then* $y' =$ $\underline{n\,[\cos(g(x))]^{n-1} \cdot -\sin(g(x)) \cdot g'(x) \cdot 1}$
- $y = \left[\tan\left(g(x)\right)\right]^n$, *then* $y' =$ $\underline{n\,[\tan(g(x))]^{n-1} \cdot \sec^2(g(x)) \cdot g'(x) \cdot 1}$

Example 1: Differentiate the following functions:

a. $y = sin^2(x)$ $= [\sin(x)]^2$

$y' = 2\sin(x) \cdot \cos x$

$= \sin(2x)$

c. $y = \sec(x)$

$y = \frac{1}{\cos(x)}$ or $y = [\cos(x)]^{-1}$

$= -[\cos x]^{-2} \cdot -\sin x$

$= \frac{\sin x}{\cos^2 x}$

$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$

$= \sec x \cdot \tan x$

b. $y = \sin(x^2)$

$y' = \cos(x^2) \cdot 2x$

d. $y = \cos^2(3x) - \sin^2(3x)$

$y' = 2\cos(3x) \cdot -\sin(3x) \cdot (3)$
$\quad - 2\sin(3x) \cdot \cos(3x) \cdot 3$

$= -6\sin(3x)\cos(3x) - 6\sin(3x)\cos(3x)$

$= -12\sin(3x)\cos(3x)$

$= -6(2\sin(3x)\cos(3x))$

Note double-Angle identity:

$\cos(2A) = \cos^2(A) - \sin^2(A)$

$y = \cos[2(3x)]$

$= \cos(6x)$

$y' = -\sin(6x) \cdot 6$

$= -6\sin(6x)$

$= -6\sin[2(3x)]$

$= -6[2\sin(3x) \cdot \cos(3x)]$

e. $y = \tan^2(2x+5)^3$

$y = [\tan(2x+5)^3]^2$

$y' = 2\tan[(2x+5)^3] \cdot \sec^2[(2x+5)^3] \cdot 3(2x+5)^2 \cdot 2$

$= 12\tan[(2x+5)^3] \cdot \sec^2[(2x+5)^3] \cdot (2x+5)^2$

Example 2: Find all extrema for $y = 2x - \tan(x)$, $x \in [0, \pi]$

$y' = 2 - \sec^2 x$

$= 2 - \frac{1}{\cos^2 x}$

$= \frac{2\cos^2 x - 1}{\cos^2 x}$

$y' = 0$

$2\cos^2 x - 1 = 0$

$\cos^2 x = \frac{1}{2}$

$\cos x = \pm \frac{1}{\sqrt{2}}$

$x = \{\frac{\pi}{4}, \frac{3\pi}{4}\}$

$y' = \text{dne}$

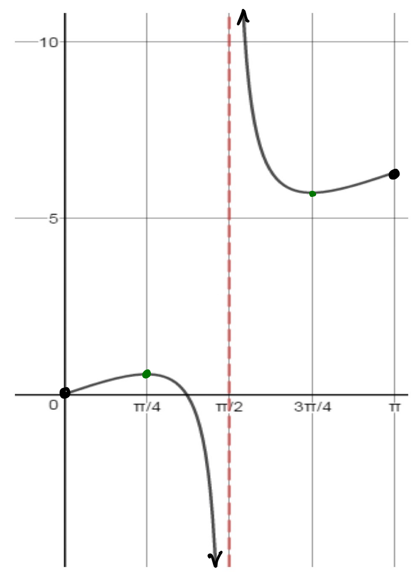
$\cos^2 x = 0$

$\cos x = 0$

$x = \frac{\pi}{2} \Leftarrow$ VA

$\therefore$ Critical #'s: $\{\frac{\pi}{4}, \frac{3\pi}{4}\}$

Desmos to Check



$f'(x)$: $+$ on $(0, \frac{\pi}{4})$, $-$ on $(\frac{\pi}{4}, \frac{\pi}{2})$, $-$ on $(\frac{\pi}{2}, \frac{3\pi}{4})$, $+$ on $(\frac{3\pi}{4}, \pi)$

$0 \quad \frac{\pi}{4} \quad \frac{\pi}{2} \quad \frac{3\pi}{4} \quad \pi$

max (at $\frac{\pi}{4}$) min (at $\frac{3\pi}{4}$)

$f(\frac{\pi}{4})$

$= \frac{\pi}{2} - \tan\frac{\pi}{4}$

$= \frac{\pi}{2} - 1$

$f(\frac{3\pi}{4})$

$= \frac{3\pi}{2} - \tan(\frac{3\pi}{4})$

$= \frac{3\pi}{2} + 1$

$\therefore f(0) = 2(0) - \tan(0) = 0$

$f(\frac{\pi}{4}) = \frac{\pi}{2} - 1$ is a local max

$f(\frac{3\pi}{4}) = \frac{3\pi}{2} + 1$ is a local min

$f(\pi) = 2\pi - \tan(\pi) = 2\pi$

Note! There are no absolute max/min. because of the VA @ $x = \frac{\pi}{2}$

Example 3: Find the derivative: $y = \tan(\sin(\cos(x^2)))$

$y' = \sec^2[\sin(\cos(x^2))] \cdot \cos(\cos(x^2)) \cdot -\sin(x^2) \cdot 2x$

Practice

Multiple Choice. *Identify the choice that best completes the statement or answers the question.*

1. If $f(x) = 2cos(3x)$, find $f'(\frac{\pi}{3})$.

a. 0 b. 3 c. 6 d. -6

2. If $f(x) = \sin(x) + \cos(x) + x$, find $f'(\pi)$.

a. 2 b. π c. 0 d. $1 + \pi$

3. If $y = \tan(2x)$, find y'' at $x = \frac{\pi}{2}$.

a. $\frac{\sqrt{3}}{8}$ b. $8\sqrt{3}$ c. -8 d. 0

4. Find the slope of the tangent to $y = cos^2(x)$ at the point $(\frac{\pi}{3}, \frac{1}{4})$.

a. $-\frac{\sqrt{3}}{2}$ b. 1 c. 2 d. $\frac{\sqrt{3}}{2}$

5. Find the slope of the tangent to $y = x^2\cos(4x^2 + 7)$ at the point where x=1.

a. 6.011 b. 3.023 c. -4.203 d. 8.009

6. Differentiate the following:

a. $f(x) = \cos(3x^2)$

$f'(x) = -\sin(3x^2) \cdot 6x$

$= -6x \sin(3x^2)$

b. $y = \sin\left(\sqrt{x^2 - 1}\right)$

$y' = \cos(\sqrt{x^2-1}) \cdot \frac{1}{\cancel{2}}(x^2-1)^{-\frac{1}{2}} \cdot (\cancel{2}x)$

$= \frac{x\cos\sqrt{x^2-1}}{\sqrt{x^2-1}}$

c. $y = \left(2x^2 - 4x\right)\cos^3(2x)$

$y' = [4x-4][\cos^3(2x)] +$

$[3\cos^2(2x) \cdot -\sin(2x) \cdot 2][2x^2-4x]$

$= 4(x-1) \cdot \cos^3(2x) - 6 \cdot 2x(x-2) \cdot \sin(2x) \cdot \cos^2(2x)$

$= 4\cos^2(2x)\,[(x-1)\cos(2x) - 3x(x-2) \cdot \sin(2x)]$

d. $f(x) = [\sin(3x) + \cos(3x)]^4$

$= 4[\sin(3x) + \cos(3x)]^3 \cdot [\cos(3x) \cdot 3 + (-\sin(3x) \cdot 3]$

$= 4[\sin(3x) + \cos(3x)]^3 \cdot [3\cos(3x) - 3\sin(3x)]$

$= -12[\sin(3x) + \cos(3x)]^3 [\sin(3x) - \cos(3x)]$

7. If $y = A(\cos(kt)) + B(\sin(kt))$, where A, B, and k are constants, show that: $y'' + k^2 y = 0$.

$$y' = A(-\sin(kt)) \cdot k + B(\cos(kt)) \cdot k$$
$$= -Ak(\sin(kt)) + Bk(\cos(kt))$$

$$y'' = -Ak\cos(kt) \cdot k + Bk(-\sin(kt)) \cdot k$$
$$= -Ak^2\cos(kt) - Bk^2(\sin(kt))$$
$$= -k^2(A\cos(kt) + B\sin(kt))$$
$$= -k^2(y)$$

$$L.S = y'' + k^2 y$$
$$= -k^2 y + k^2 y$$
$$= 0$$
$$= R.S$$

$\therefore$ QED

8. Prove that $y = \sec(x) + \tan(x)$ is always increasing on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$$y' = \sec(x)\tan(x) + \sec^2(x)$$
$$= \sec x \left[\tan(x) + \sec(x)\right]$$
$$= \frac{1}{\cos(x)}\left[\frac{\sin(x)}{\cos(x)} + \frac{1}{\cos(x)}\right]$$
$$= \frac{1}{\cos(x)} \cdot \left[\frac{\sin(x)+1}{\cos(x)}\right]$$
$$0 = \frac{\sin(x)+1}{\cos^2(x)}$$

Aside:

$$y = \sec(x) = [\cos(x)]^{-1}$$
$$y' = -[\cos(x)]^{-2} \cdot -\sin(x)$$
$$= \frac{\sin(x)}{\cos^2(x)}$$
$$= \sec(x)\tan(x)$$

$y' = 0$

$\sin(x) = -1$

$x = \{-\frac{\pi}{2}, \frac{3\pi}{2}\}$

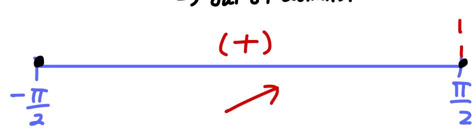
↳ out of domain

$y' = $ dne

$\cos^2(x) = 0$

$x = \{\frac{\pi}{2}, \frac{3\pi}{2}\}$

↳ out of domain



$\therefore$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $f'(x) > 0$

Warm Up

1. Find $\frac{dy}{dx}$ for each of the following. **Completely** simplify your answers.

a) $y = \cos(x^2+1)^3 + \sin^2(2\pi)$ ← constant

$$\frac{dy}{dx} = -\sin(x^2+1)^3 \cdot [3(x^2+1)^2 \cdot 2x]$$
$$= -6x(x^2+1)^2 \cdot \sin(x^2+1)^3$$

b) $y = \cos^2(\sqrt{x}) - \sin^2(\sqrt{x})$ ⇐ look for simplifying double angles will make differentiating more efficient.

$$y = \cos[2\sqrt{x}]$$
$$y' = -\sin[2\sqrt{x}] \cdot (x^{-\frac{1}{2}})$$
$$= \frac{-\sin(2\sqrt{x})}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$$
$$= \frac{-\sqrt{x}\sin(2\sqrt{x})}{x}$$

$\sin(2A) = 2\sin(A)\cos(A)$
$\cos(2A) = \cos^2(A) - \sin^2(A)$
$= 1 - 2\sin^2(A)$
$= 2\cos^2(A) - 1$
$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$

method 2: without simplifying using double angle

$$y' = \cancel{2}\cos\sqrt{x} \cdot -\sin\sqrt{x} \cdot \frac{1}{\cancel{2}}x^{-\frac{1}{2}} - \cancel{2}\sin\sqrt{x} \cdot \cos\sqrt{x} \cdot \frac{1}{\cancel{2}}x^{-\frac{1}{2}}$$
$$= \frac{-\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}} - \frac{\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}$$
$$= \frac{-2\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}$$
$$= -\frac{\sin(2\sqrt{x})}{\sqrt{x}}$$

2. Find all extreme values for $f(x) = 3\sin(x)\cos(x)$, $x \in [0,\pi]$.

$\therefore 2x \in [0, 2\pi]$

$$f(x) = \frac{3}{2}[2\sin(x)\cdot\cos(x)]$$
$$= \frac{3}{2}[\sin(2x)]$$
$$f'(x) = \frac{3}{\cancel{2}} \cdot \cos(2x) \cdot \cancel{2}$$
$$0 = 3\cos(2x)$$
$$0 = \cos(2x)$$
$$2x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$$
$$x = \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$$

$f'(x)$: 0 (+) $\frac{\pi}{4}$ (−) $\frac{3\pi}{4}$ (+) π

↗ max ↘ min ↗

$f\left(\frac{\pi}{4}\right) = \frac{3}{2}\left[\sin\left(2\cdot\frac{\pi}{4}\right)\right] = \frac{3}{2} \Rightarrow$ local max

$f\left(\frac{3\pi}{4}\right) = \frac{3}{2}\left[\sin\left(2\cdot\frac{3\pi}{4}\right)\right] = -\frac{3}{2} \Rightarrow$ local min

method 2: without Double Angle

$$f'(x) = 3\cos(x)\cos(x) + (-\sin(x))(3\sin(x))$$
$$= 3\cos^2(x) - 3\sin^2(x)$$
$$= 3(\cos^2(x) - \sin^2(x))$$
$$= 3(\cos(2x))$$

$0 = 3\cos(2x)$, $x \in [0,\pi]$, $2x \in [0, 2\pi]$

$f(0) = 0$

$f\left(\frac{\pi}{4}\right) = \frac{3}{2}$ Absolute max

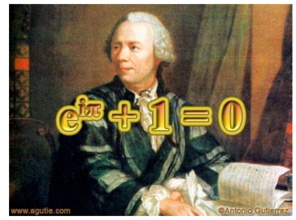
$f\left(\frac{3\pi}{4}\right) = -\frac{3}{2}$ Absolute min

$f(\pi) = 0$

5.2. The Number *e* and the Derivatives of $y = e^x$ and $y = lnx$

$y' = e^x$ $\quad$ $y' = \frac{1}{x}$

Introducing a Special Number, *e*



Leonhard Euler (1707-1783) was a remarkable Swiss mathematician and physicist. He made massive contributions to mathematics, especially calculus, as well as physics, optics, magnetism, astronomy, and shipbuilding. Euler popularized the use of the symbol π and developed new approximations for it. He was the first to use the symbol i to represent imaginary numbers. Euler also developed the irrational number e, which is known as Euler's number and is defined as a limit:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Euler's Number e :

Let's examine some integer values of x to see the limiting value of this expression.

x	$\left(1+\frac{1}{x}\right)^x$
1	2
10	2.5937
100	2.70481
1000	2.716923
10000	2.718145
100000	2.718268
1000000	2.718280

$x \to \infty$ $\quad$ $e \doteq 2.718281828...$

n	A_n
1	2
10	2.5937
100	2.70481
1,000	2.716923
10,000	2.718145
100,000	2.718268
1,000,000	2.718280

We see that as x becomes larger, the value of the expression changes by a smaller and smaller amount. In fact, the change can be shown to approach 0. In other words, the expression is approaching a limiting value. The limiting value of this expression is the **irrational** number e = 2.718281828459..., a non-terminating decimal.

By making the substitution $t = \frac{1}{x}$, we get the following alternate definition of Euler's number:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{t\to 0}(1+t)^{\frac{1}{t}}$$

let $t = \frac{1}{x}$

$x = \frac{1}{t}$

$x \to \infty$

$\frac{1}{t} \to \infty$

$t \to \frac{1}{\infty}$

$t \to 0$

This limit is called the The Fundamental Limit of Calculus.

Natural Logarithm

$f(x) = e^x$ $f^{-1}(x)$: $x = e^y$
$y = \log_e(x)$
$= \ln(x)$

The Inverse of the Exponential Function, e^x

In previous mathematics courses, you learned that the inverse of an exponential function is the logarithmic function with the same base. Today, we will use similar ideas to graph of $g(x) = e^x$ *and* $y = \mathbf{log}_e\, \boldsymbol{x}$. $\Rightarrow$ $y = \ln(x)$ ie $y = 2^x$ inverse: $x = 2^y$ or $y = \log_2(x)$

The inverse of the exponential function $g(x) = e^x$ is $g^{-1}(x) = \log_e(x) = \ln(x)$.

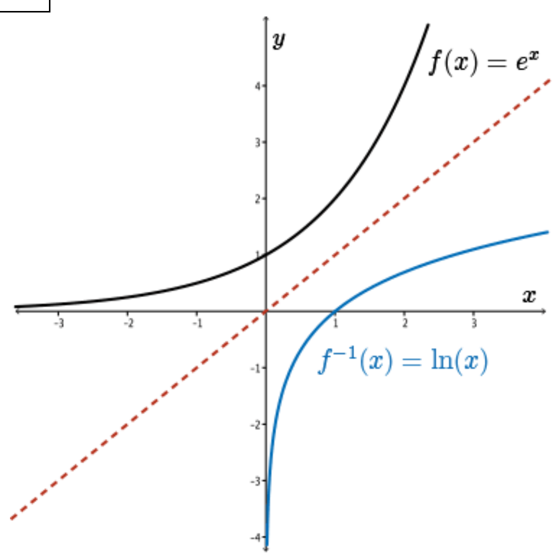
Rather than using $\log_e(x)$, mathematicians use $\boldsymbol{ln(x)}$ to shorten this expression. $\boldsymbol{ln(x)}$ stands for the natural logarithm of x and is pronounced "lawn x."

Like every function, we can find its inverse by reflecting the function in the line $y = x$.

Hence, to graph $y = lnx$, we can interchange the values of x and y from the table of values of $y = e^x$.

x	$y = e^x$
-2	0.135
-1	0.36
0	1
1	2.72
2	7.39

x	$y = lnx$
0.135	-2
0.36	-1
1	0
2.72	1
7.39	2



	$y = e^x$	$y = lnx$
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}, x > 0$
Range	$y \in \mathbb{R}, y > 0$	$y \in \mathbb{R}$

In advanced functions, you learned the logarithmic laws and properties of logarithm. We can apply similar laws and properties when working with ln.

$\log_c(ab) = \log_c a + \log_c b$	$ln(ab) =$ $ln(a) + ln(b)$
$\log_c\left(\frac{a}{b}\right) = \log_c a - \log_c b$	$\ln\left(\frac{a}{b}\right) =$ $ln(a) - ln(b)$
$\log_c a^n = n \log_c a$	$ln(a^n) =$ $n\,ln(a)$
$\log_a a = 1$ $y = \log_a a$ $a^y = a$ $y = 1$ $\therefore \log_a a = 1$	$ln\, e =$ $\log_e e = 1$
$a^{\log_a x} = x$ $a^{\log_a x} = y$ $\log_a y = \log_a(x)$ $\therefore y = x$	$e^{lnx} =$ $e^{\log_e x} = x$
$\log_a 1 = 0$	$ln\,1 =$ $\log_e(1) = 0$ "'e' to what power gives 1?" note! $e^0 = 1 \Rightarrow$ Logarithm in exponential form

Example1: Solve for x:

a. $5(10^{x+2}) = 200$

$10^{x+2} = 40$

$ln(10^{x+2}) = ln(40)$

$(x+2) \cdot ln(10) = ln(40)$

$x + 2 = \frac{ln(40)}{ln(10)}$

$x = \frac{ln(40)}{ln(10)} - 2$

$\doteq -0.3979$

method 2: Logarithmic Form

$b^x = y \Rightarrow x = \log_b(y)$

$(x+2) = \log_{10}(40)$

$x = -2 + \log(40)$

$\doteq -0.3979$

method 3:

$10^x \cdot 10^2 = 40$

$10^x = \frac{40}{100}$

$10^x = 0.4$

$\log(10^x) = \log(0.4)$

$x = -0.3979$

b. $10e^{2x-1} + 500 = 1000$

$10e^{2x-1} = 500$

$e^{2x-1} = 50$

$ln(e^{2x-1}) = ln(50)$

$(2x-1) = ln(50)$

$x = \frac{ln(50)+1}{2}$

$\doteq 2.4560$

method 2: $2x - 1 = ln(50)$

$x = \frac{ln(50)+1}{2}$

$\doteq 2.4560$

Example 2: Water is brought to a boil then removed from the heat. The temperature of the water, T degrees Celsius, is modeled as $T = 80e^{-0.57t} + 20$, where t is in minutes.

a) Determine the temperature after 15 minutes.

$T(15) = 80e^{-0.57(15)} + 20$

$\doteq 20.0°C$

b) Determine how long it takes for the temperature to reach 30° C.

$30 = 80e^{-0.57t} + 20$

$10 = 80e^{-0.57t}$

$\frac{1}{8} = e^{-0.57t}$

$-0.57t = \ln\left(\frac{1}{8}\right)$

$t = -\frac{1}{0.57}\ln\left(\frac{1}{8}\right)$

$\doteq 3.648$ min

Example 3: A population of fish in a lake at time t months is given by the function

$$F(t) = \frac{20000}{1+24e^{-\frac{t}{4}}}.$$

How long will it take for the fish population to reach 15 000?

$15\ 000 = \frac{20\ 000}{1+24e^{-\frac{t}{4}}}$

$1+24e^{-\frac{t}{4}} = \frac{4}{3}$

$24e^{-\frac{t}{4}} = \frac{1}{3}$

$e^{-\frac{t}{4}} = \frac{1}{72}$

$-\frac{t}{4} = \ln\left(\frac{1}{72}\right)$

$t = -4\ln\left(\frac{1}{72}\right)$

$\doteq 17.11$ months

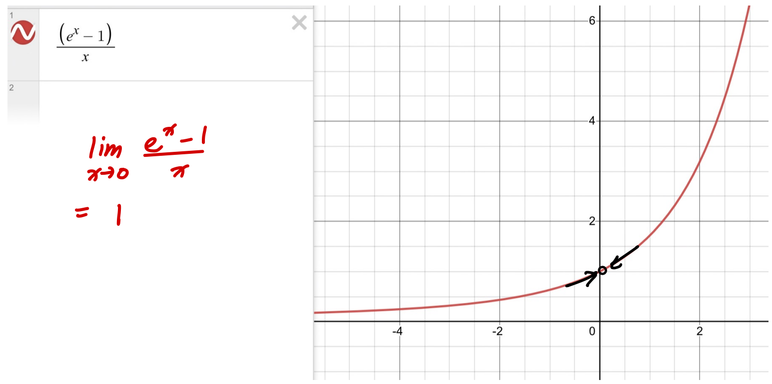
The Derivative of $y = e^x$ and $y = \ln x$

The Derivative of f(x) = e^x

If $f(x) = e^x$, then $f'(x)$ = $\underline{\quad e^x \quad}$.

Proof: Let $f(x) = e^x$.

Then, from first principles:



$\lim\limits_{x \to 0} \frac{e^x - 1}{x}$

$= 1$

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h}$$

$$= e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x \cdot (1)$$

$$= e^x$$

By the chain rule, if f(x) = e$^{g(x)}$, then f′(x) = $\underline{\quad e^{g(x)} \cdot g'(x) \quad}$

Examples: Differentiate.

a) $f(x) = 3e^x$

$f'(x) = 3e^x$ ✓

b) $f(x) = 2e^{x^2}$

$f'(x) = 2e^{x^2} \cdot 2x$

$= 4x e^{x^2}$ ✓

c) $f(x) = x^3 e^{-x}$

$f'(x) = [3x^2][e^{-x}] + [e^{-x} \cdot (-1)][x^3]$

$= e^{-x}[3x^2 - x^3]$

$= -x^2 e^{-x}(x - 3)$ ✓✓

d) $f(x) = e^{x^2+3x}$

$f'(x) = e^{x^2+3x} \cdot (2x+3)$

$= (2x+3) e^{x^2+3x}$ ✓✓

The Derivative of $\ln x$

If y= $\ln x$ then y' = $\frac{1}{x} = x^{-1}$.

Proof: recall: $y = \ln x$ ⇒ logarithmic form

$e^y = x$ ⇒ logarithm in Exponential form

$$\frac{d e^y}{dx} = \frac{dx}{dx}$$

$$e^y \cdot \frac{dy}{dx} = 1$$

* proof requires understanding of implicit differentiation

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$y = \ln[g(x)]$

Derivative of $\ln g(x)$: $y' = \frac{1}{g(x)} \cdot g'(x)$ or $y' = \frac{g'(x)}{g(x)}$

Ex 1: Differentiate and simplify.

a) $f(x) = x^2 \ln x$

$$f'(x) = [2x][\ln(x)] + \left[\tfrac{1}{x}\right][x^2]$$
$$= 2x\ln(x) + x$$
$$= x(2\ln(x) + 1)$$
or $$= x(\ln(x^2) + 1)$$

b) $f(x) = \ln\left(\frac{x-1}{x+1}\right)$

$$f'(x) = \left(\frac{x+1}{x-1}\right) \cdot \frac{[1][x+1] - [1][x-1]}{(x+1)^2}$$

note: $\frac{1}{\left(\frac{x-1}{x+1}\right)}$

$$= \frac{x+1}{x-1} \cdot \frac{2}{(x+1)^2}$$
$$= \frac{2}{(x-1)(x+1)}$$
$$= \frac{2}{x^2-1}$$

method 2: $f(x) = \ln\left(\frac{x-1}{x+1}\right)$
$$= \ln(x-1) - \ln(x+1)$$
$$= \frac{1}{x-1} - \frac{1}{x+1}$$
$$= \frac{(x+1)-(x-1)}{(x+1)(x-1)}$$
$$= \frac{2}{x^2-1}$$

c) $f(x) = e^{x^2}\ln\left(\sqrt{x}\right)$

$$= e^{x^2} \cdot \tfrac{1}{2}\ln(x)$$
$$= \tfrac{1}{2} e^{x^2} \cdot \ln(x)$$
$$f'(x) = \tfrac{1}{2}\left\{[e^{x^2} \cdot 2x][\ln(x)] + \tfrac{1}{x}[e^{x^2}]\right\}$$
$$= \frac{e^{x^2}}{2}\left[2x\ln(x) + \tfrac{1}{x}\right]$$
$$\frac{e^{x^2}}{2x}\left(2x^2\ln(x) + 1\right)$$

Ex 2: Find the local extrema for ~~$y =$~~ $f(x) = x^2 e^{-x^2}$.

$$f'(x) = [2x][e^{-x^2}] + [e^{-x^2} \cdot (-2x)][x^2]$$
$$= 2x e^{-x^2} - 2x^3 e^{-x^2}$$
$$= 2x e^{-x^2}[1 - x^2]$$
$$= -2x e^{-x^2}[x^2 - 1]$$
$$0 = \frac{-2x(x+1)(x-1)}{e^{x^2}}$$

$f'(x) = 0$ $\quad$ $f'(x) = \text{dne}$

$x = \{-1, 0, 1\}$ $\quad$ $x = \phi$

$f'(x)$: $-\infty$ (+) -1 (−) 0 (+) 1 (−) ∞

↗ max ↘ min ↗ max ↘

$f(-1) = (-1)^2 e^{-(1)^2} = e^{-1} = \frac{1}{e}$

$f(0) = 0$

$f(1) = (1)^2 e^{-(1)^2} = e^{-1} = \frac{1}{e}$

$\therefore$ local max : $(-1, \frac{1}{e})$ and $(1, \frac{1}{e})$

local min : $(0, 0)$

Ex 3: Find the points of inflection for ~~$y =$~~ $f(x) = e^{-x^2}$.

$$f'(x) = e^{-x^2} \cdot (-2x)$$
$$= -2x e^{-x^2}$$
$$f''(x) = -2\left[[1][e^{-x^2}] + [e^{-x^2} \cdot (-2x)][x]\right]$$
$$= -2e^{-x^2}[1 - 2x^2]$$
$$= 2e^{-x^2}[2x^2 - 1]$$
$$= \frac{2(2x^2 - 1)}{e^{x^2}}$$

$f''(x) = 0$ $\quad$ $f''(x) = \text{dne}$

$x = \pm\frac{1}{\sqrt{2}}$ $\quad$ $x = \phi$

$-\infty$ (+) $-\frac{1}{\sqrt{2}}$ (−) $\frac{1}{\sqrt{2}}$ (+) ∞

cu $\quad$ cd $\quad$ cu

poi: $f(-\frac{1}{\sqrt{2}}) = e^{-\frac{1}{2}}$ $\quad$ poi: $f(\frac{1}{\sqrt{2}}) = e^{-\frac{1}{2}}$

$\therefore$ the points of inflection are : $(\pm\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}})$

Practice
Multiple Choice: *Identify the choice that best completes the statement or answers the question.*

D 1. The graph of $y = e^x$ lies between the graphs of which two functions?

a. $y = 2x$ & $y = 3x$ b. $y = x^2$ & $y = x^3$ c. $y = 3^x$ & 4^x d. $y = 2^x$ & 3^x

D 2. Determine the value of x in the equation $\ln x = 1$.

a. 10 b. 0 c. 1 d. e

D 3. Simplify the expression $\ln e^{3x}$.

a. $3 \ln e^{3x}$ b. $\ln(3x)$ c. e^{3x} d. $3x$

B 4. What is the value of $\ln 0$?

a. 0 b. undefined c. 1 d. e

B 5. Determine the value of x in the equation $2e^x = 6$.

a. $\ln 6$ b. $\ln 3$ c. 3 d. $\frac{\ln 6}{\ln 3}$

A 6. If $f(x) = 2\sin x\cos x$, find $f'(\frac{\pi}{2})$.

a. -2 b. 1 c. $\frac{\pi}{2} + 1$ d. 2

D 7. If $f(x) = \cos(\sin x)$, find $f'(1)$.

a. 0.2314 b. 1 c. 0 d. -0.4029

B 8. What is the slope of the graph $y = 5e^x$ at $x = 1$?

a. e b. $5e$ c. 5 ln5 d.

B 9. If $f(x) = 2x^2e^x$, find $f'(1)$.

a. $2e$ b. $6e$ c. 0 d. $4e$

D 10. If $f(x) = e^{x^2-x+1}$, find $f'(1)$.

a. 2 b. 1 c. 0 d. e

C 11. If $f(x) = e^x\sqrt{x}$, find $f'(1)$.

a. $\frac{e}{2}$ b. $2e$ c. $\frac{3e}{2}$ d. e

A 12. If $f(x) = \sqrt{x} + \sqrt{e^x} + \sqrt{e}$, find $f'(1)$.

a. $\dfrac{1+\sqrt{e}}{2}$ b. $\dfrac{1}{\sqrt{e}}$ c. $1+e$ d. $\dfrac{1}{2}$

C 13. If $f(x) = \sin(e^x)$, find $f'(0)$.

a. 1 b. 0 c. $\cos(1)$ d. $\cos(e)$

B 14. Find the point on the curve $y = e^x + x$ where the tangent is parallel to the line $y=2x$

a. $(1, 1+e)$ b. $(0, 1)$ c. $(-1, -1 + \frac{1}{e})$ d. $(-2, -2 + \frac{1}{e^2})$

B 15. Find the point on the curve $y = e^{2x+1}$ where the tangent is parallel to the line $y = 2ex$.

a. $(1, e^3)$ c. $(\frac{1}{2}, e^2)$

b. $(0, e)$ d. $(-1, \frac{1}{e})$

Full Solution:

16. Find the derivative. You do NOT have to simplify:

a. $f(x) = \ln(tan^2(\sin(e^x)))$ b. $f(x) = e^{\sin(x)}\sec(x)$ c. $y = ln\,(ln(2x^4))$

a) $f'(x) = \dfrac{2\tan(\sin(e^x))\cdot\sec^2(\sin(e^x))\cdot\cos(e^x)\cdot e^x}{\tan^2(\sin(e^x))}$

b) $f'(x) = \left[e^{\sin(x)}\cdot -\cos(x)\right]\left[\sec(x)\right] + \left[\sec(x)\tan(x)\right]\left[e^{\sin(x)}\right]$

c) $y' = \dfrac{1}{\ln(2x^4)} \cdot \dfrac{1}{2x^4} \cdot 8x^3$

$= \dfrac{4}{x\ln(2x^4)}$

17. Find the absolute maximum and absolute minimum of $f(x) = \frac{\ln x}{x}$ on the closed interval $x \epsilon [2,10]$.

$$f(x) = \frac{\left[\frac{1}{x}\right][x] - [1][\ln(x)]}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

$f'(x) = 0$

$\ln x = 1$

$e^1 = x$

$\therefore x = e$

$f'(x) = dne$

$x = 0$

(V.A)

$f(2) = \frac{\ln 2}{2} \doteq 0.3465$

$f(e) = \frac{\ln e}{e} = \frac{1}{e} \doteq 0.3678 \Rightarrow$ max

$f(10) = \frac{\ln 10}{10} \doteq 0.2302 \Rightarrow$ min

$\therefore$ Absolute min : $(10, \frac{1}{10}\ln 10)$

Absolute max : $(e, \frac{1}{e})$

18. For $f(x) = x^4 e^x$: Determine the intervals of increase and decrease

$f'(x) = [4x^3][e^x] + e^x[x^4]$

$0 = x^3 e^x [4 + x]$

$x = \{0, -4\}$

$-\infty$		-4		0		∞
	$+$		$-$		$+$	
	$\nearrow$		$\searrow$		$\nearrow$	

$\therefore$ intervals of increase:

$x \in (-\infty, -4) \cup (0, \infty)$

intervals of decrease:

$x \in (-4, 0)$

19. Find y' (do not simplify): $y = (e^{\sqrt{x}} + x\sec x + 1)^3$

$$y' = 3(e^{\sqrt{x}} + x\sec x + 1)^2 \left[e^{\sqrt{x}} \cdot \frac{1}{2} x^{-\frac{1}{2}} + \left\{ [1][\sec(x)] + [\sec(x)\tan(x)][x] \right\} \right]$$

$$= 3(e^{\sqrt{x}} + x\sec(x) + 1)^2 \left[\frac{e^{\sqrt{x}}}{2\sqrt{x}} + \sec(x) + x\sec(x)\tan(x) \right]$$

20. Find y'' *if* $y = x^2 e^{\sqrt{\sin^2 x + \cos^2 x + 1}}$ ☺

$\sin^2 x + \cos^2 x = 1$

$y = x^2 e^{\sqrt{2}}$ ← constant!

$y' = 2e^{\sqrt{2}} x$

$y'' = 2e^{\sqrt{2}}$

21. Simplify the expression $(e^{2lnx})(ln\,e^{2x}) + ln(e^{x+1})$

$(x+1)\ln e$

$= e^{\ln(x^2)} \cdot (2x) + \ln(e^x \cdot e^1)$

$= x^2 \cdot 2x + \ln e^x + \ln e$

$= x^2 \cdot 2x + x \ln e + 1$

$= 2x^3 + x + 1$

22. Solve for x (to nearest hundredth if necessary):

a. $e^{-2x+3} = 4$

$\log_e(4) = -2x+3$

$\ln(4) = -2x+3$

$\ln 4 = -2x+3$

$\frac{\ln 4 - 3}{-2} = x$

$\therefore x \approx 0.8068$

method 2:

$\ln(e^{-2x+3}) = \ln(4)$

$-2x+3 = \ln(4)$

$x = \frac{\ln(4) - 3}{-2}$

≈ 0.8068

b. $\ln(x^2 - 2x) = 4$

$e^4 = (x^2 - 2x)$

$x^2 - 2x - e^4 = 0$

$x = \frac{2 \pm \sqrt{4 - 4(1)(-e^4)}}{2}$

$\approx \{6.4564, -8.4564\}$

< Not tested >

Enrich-Study
Where does e come from and what does it do?


John Napier (1550-1617)

Suppose you put \$1 in a bank. The bank pays 4% interest a year, and this is credited to your account at the end of a year. A little thought shows that at the end of five years an amount of money equal to $\$(1+0.04)^5$ will sit in the bank (this bank charges no fees).
However, if the interest (still at an annual rate of 4%) was "compounded" every quarter, then the amount at the end of five years would be $\$(1+0.04)^{4\times 5}$.
If the bank gave an interest rate of 100% annually, then after one year the bank balance would then be \$(1+1)¹, and if the interest were compounded quarterly it would be $\$\left(1+\frac{1}{4}\right)^{4\times 1} = \$2.43$. If you were even luckier and found a bank that compounded monthly, the 100% annual rate of interest would then give you $\$\left(1+\frac{1}{12}\right)^{12\times 1} = \$2.61$ after one year. Likewise, daily compounding would give you \$. $\$\left(1+\frac{1}{365}\right)^{365\times 1} = \$2.72$. It's obvious that compounding more frequently results in more money in the bank. So it is natural to ask whether compounding at every instant in time (that is, continuously) leads to an infinite amount in the bank.
To answer this question we need to evaluate

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

This quantity turns out again to be e - the same base value with the property that the gradient of the graph is unity at $x=0$. if $f(x)=e^x$ then $f'(0)=1$

$f(0)=e^0$ $=1$ $=e^0$

Now $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ can be expanded very nicely using the trusty old Binomial Theorem. We find that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{r!} + \ldots.$$

This series is convergent, and evaluating the sum far enough to give no change in the fourth decimal place (this occurs after the seventh term is added) gives an approximation for e of 2.718.
It was that great mathematician **Leonhard Euler** who discovered the number *e* and calculated its value to 23 decimal places. It is often called *Euler's number* and, like pi, is a *transcendental number* (this means it is not the root of any algebraic equation with integer coefficients). Its properties have led to it as a "natural" choice as a logarithmic base, and indeed *e* is also known as the *natural base* or *Naperian base* (after **John Napier**).

We have learned that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$, use this fact to evaluate the following limits:

Recall! Fundamental Limit of Calculus: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = \lim_{t\to 0}(1+t)^{\frac{1}{t}}$ where $t=\frac{1}{x}$

as $x\to\infty$ $\frac{1}{x}\to 0$ $\therefore t\to 0$

Evaluate the following limits:

<Not tested>

a) $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x$ let $t=-\frac{1}{x}$

$= \lim_{t\to 0}(1+t)^{-\frac{1}{t}}$

$= \left[\lim_{t\to 0}(1+t)^{\frac{1}{t}}\right]^{-1} = e^{-1}$

b) $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x$ let $t=\frac{k}{x}$

$= \lim_{t\to 0}(1+t)^{\frac{k}{t}}$

$= \left[\lim_{t\to 0}(1+t)^{\frac{1}{t}}\right]^{k} = e^{k}$

c) $\lim_{n\to\infty}\left(1+\frac{6x}{n}\right)^n$ let $t=\frac{6x}{n}$, $n=\frac{6x}{t}$

$= \lim_{t\to 0}(1+t)^{\frac{6x}{t}}$

$= \left[\lim_{t\to 0}(1+t)^{\frac{1}{t}}\right]^{6x}$

$= e^{6x}$

d) $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{-3n}$ let $t=\frac{2}{n}$, $n=\frac{2}{t}$

$= \lim_{t\to 0}(1+t)^{-\frac{6}{t}}$

$= \left[\lim_{t\to 0}(1+t)^{\frac{1}{t}}\right]^{-6} = e^{-6}$

$\left[\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n\right]^{-3}$

$= \left[e^{2}\right]^{-3}$

$= e^{-6}$

e) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$

$= e$

$\Rightarrow$ Fundamental Limit of Calculus

$A = P(1+i)^n$

Continuous Compounding Interest

You invest $500 in an account that earns 2%/annum for t year compounded continuously.

$= 500\left(1+\frac{0.02}{n}\right)^{nt}$ as $n\to\infty$

$= 500\, e^{0.02t}$

Continuous compounding interest

Formula

$$P(t) = P_0 e^{rt}$$

$P(t)$ = value at time t

P_0 = original principal sum

r = nominal annual interest rate

t = length of time the interest is applied

$P'(t) = P_0 e^{rt}\cdot r$

$e^{\ln x} = e^{\log_e(x)} = x$

Warm Up

1. Differentiate each of the following. Simplify your answers.

(a) $f(x) = \ln\left(\dfrac{e^{x^2}}{x^2+2x+1}\right) + \dfrac{2\tan(e^x)}{1-\tan^2(e^x)}$

$= \ln(e^{x^2}) - \ln(x^2+2x+1) + \tan(2e^x)$

$= x^2 - \ln(x^2+2x+1) + \tan(2e^x)$

$f'(x) = 2x - \dfrac{2x+2}{x^2+2x+1} + \sec^2(2e^x)\cdot(2e^x)$

$= 2x - \dfrac{2\cancel{(x+1)}}{(x+1)\cancel{(x+1)}} + 2e^x\sec^2(2e^x)$

$= 2x - \dfrac{2}{x+1} + 2e^x\sec^2(2e^x)$

$\tan 2\theta$	$=$	$\dfrac{2\tan\theta}{1-\tan^2\theta}$
$\tan(\alpha+\beta)$	$=$	$\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$\tan(\alpha-\beta)$	$=$	$\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

(b) $y = e^{5\ln(x-3)} - e^2\sin^3(e^{-x})$

$= (x-3)^5 - e^2\sin^3(e^{-x})$

$y' = 5(x-3)^4 - e^2\cdot 3\sin^2(e^{-x})\cdot\cos(e^{-x})\cdot e^{-x}(-1)$

$= 5(x-3)^4 + 3e^{2-x}[\sin^2(e^{-x})\cdot\cos(e^{-x})]$

(c) $y = \sin(\ln x)\cos(x^3) - \sin(x^3)\cos(\ln x)$

$= \sin(\ln(x) - x^3)$

$y' = \cos(\ln(x) - x^3)\cdot\left[\frac{1}{x} - 3x^2\right]$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$	$\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B$	$\cos(A-B) = \cos A\cos B + \sin A\sin B$
$\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B}$	$\tan(A-B) = \dfrac{\tan A-\tan B}{1+\tan A\tan B}$

(d) $y = \left[\ln(\tan(x^2))\right]^3$

$y' = 3[\ln(\tan(x^2))]^2\cdot\dfrac{1}{\tan(x^2)}\cdot\sec^2(x^2)\cdot 2x$

$= \dfrac{6x[\ln(\tan(x)^2)]^2\cdot\sec^2(x^2)}{\tan(x^2)}$

2. Determine where the function $y = \sin^2(2x) + 2\cos^2(x)$ is increasing over the interval $x \in [0, 2\pi]$.

$2x \in [0, 4\pi]$

$y' = 2\sin(2x)\cdot\cos(2x)\cdot(2) + 4\cos(x)(-\sin(x))$

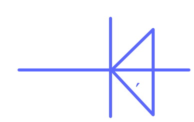
$y' = 4\sin(2x)\cos(2x) - 4\sin(x)\cos(x)$

$0 = 4\sin(2x)\cos(2x) - 4\sin(x)\cos(x)$

$0 = 4\sin(2x)\cos(2x) - 2[\sin(2x)]$

$0 = 2\sin(2x)\cos(2x) - \sin(2x)$

$0 = \sin(2x)[2\cos(2x) - 1]$



$\sin(2x) = 0$

$2x = \{0, \pi, 2\pi, 3\pi, 4\pi\}$

$x = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$

$\cos(2x) = \frac{1}{2}$

$2x = \{\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}\}$

$x = \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\}$

$f'(x)$: $+ \; - \; + \; - \; + \; - \; + \; -$

$0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi$

$\nearrow \searrow \nearrow \searrow \nearrow \searrow \nearrow \searrow$

interval of increase: $(0, \frac{\pi}{6}) \cup (\frac{\pi}{2}, \frac{5\pi}{6}) \cup (\pi, \frac{7\pi}{6}) \cup (\frac{3\pi}{2}, \frac{11\pi}{6})$

interval of decrease: $(\frac{\pi}{6}, \frac{\pi}{2}) \cup (\frac{5\pi}{6}, \pi) \cup (\frac{7\pi}{6}, \frac{3\pi}{2}) \cup (\frac{11\pi}{6}, 2\pi)$

Continued →

Warm Up

1. Differentiate each of the following. Simplify your answers.

(a) $f(x)=\ln\left(\dfrac{e^{x^2}}{x^2+2x+1}\right)+\dfrac{2\tan(e^x)}{1-\tan^2(e^x)}$

$$f(x) = \ln(e^{x^2}) - \ln(x^2+2x+1) + \tan[2\cdot e^x]$$

$$= \frac{1}{\cancel{e^{x^2}}}\cdot \cancel{e^{x^2}}\cdot 2x - \frac{(2x+2)}{x^2+2x+1} + \sec^2(2e^x)\cdot 2e^x$$

$$= 2x - \frac{2(\cancel{x+1})}{(x+1)\cancel{(x+1)}} + 2e^x\cdot\sec^2(2e^x)$$

$$= 2x - \frac{2}{x+1} + 2e^x\cdot\sec^2[2e^x]$$

$\tan 2\theta$	$=$	$\dfrac{2\tan\theta}{1-\tan^2\theta}$
$\tan(\alpha+\beta)$	$=$	$\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$\tan(\alpha-\beta)$	$=$	$\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

(b) $y=e^{5\ln(x-3)}-e^2\sin^3(e^{-x})$

$$y = e^{\ln(x-3)^5} - e^2\sin^3(e^{-x})$$

$$y = (x-3)^5 - e^2\sin^3(e^{-x})$$

$$y' = 5(x-3)^4\cdot(1) - e^2[\sin^2(e^{-x})\cdot\cos(e^{-x})\cdot e^{-x}(-1)]$$

$$= 5(x-3)^4 + e^{2-x}[\sin^2(e^{-x})\cdot\cos(e^{-x})]$$

(c) $y=\sin(\ln x)\cos(x^3)-\sin(x^3)\cos(\ln x)$

$$y = \sin[\ln x - x^3]$$

$$y' = \cos[\ln x - x^3]\cdot\left[\frac{1}{x} - 3x^2\right]$$

$$= \cos(\ln x - x^3)\cdot\frac{1-3x^2}{x}$$

(d) $y=[\ln(\tan(x^2))]^3$

$$y = 3[\ln(\tan(x^2))]^2\cdot\frac{\sec^2(x^2)\cdot 2x}{\tan(x^2)}$$

$$= \frac{6x[\ln(\tan(x^2))]^2\cdot\sec^2(x^2)}{\tan(x^2)}$$

$\sin(A+B)=\sin A\cos B+\cos A\sin B$	$\cos(A+B)=\cos A\cos B-\sin A\sin B$
$\sin(A-B)=\sin A\cos B-\cos A\sin B$	$\cos(A-B)=\cos A\cos B+\sin A\sin B$
$\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$	$\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

2. Determine where the function $y=\sin^2(2x)+2\cos^2(x)$ is increasing over the interval $x\in[0, 2\pi]$.

$2x\in[0, 4\pi]$

$$y' = 2\sin(2x)\cdot\cos(2x)\cdot(2) + 4\cos(x)(-\sin(x))$$

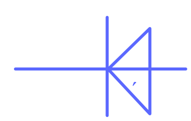
$$y' = 4\sin(2x)\cos(2x) - 4\sin(x)\cos(x)$$

$$0 = 4\sin(2x)\cos(2x) - 4\sin(x)\cos(x)$$

$$0 = 4\sin(2x)\cos(2x) - 2[\sin(2x)]$$

$$0 = 2\sin(2x)\cos(2x) - \sin(2x)$$

$$0 = \sin(2x)[2\cos(2x) - 1]$$



$\sin(2x) = 0$

$2x = \{0, \pi, 2\pi, 3\pi, 4\pi\}$

$x = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$

$\cos(2x) = \frac{1}{2}$

$2x = \{\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}\}$

$x = \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\}$

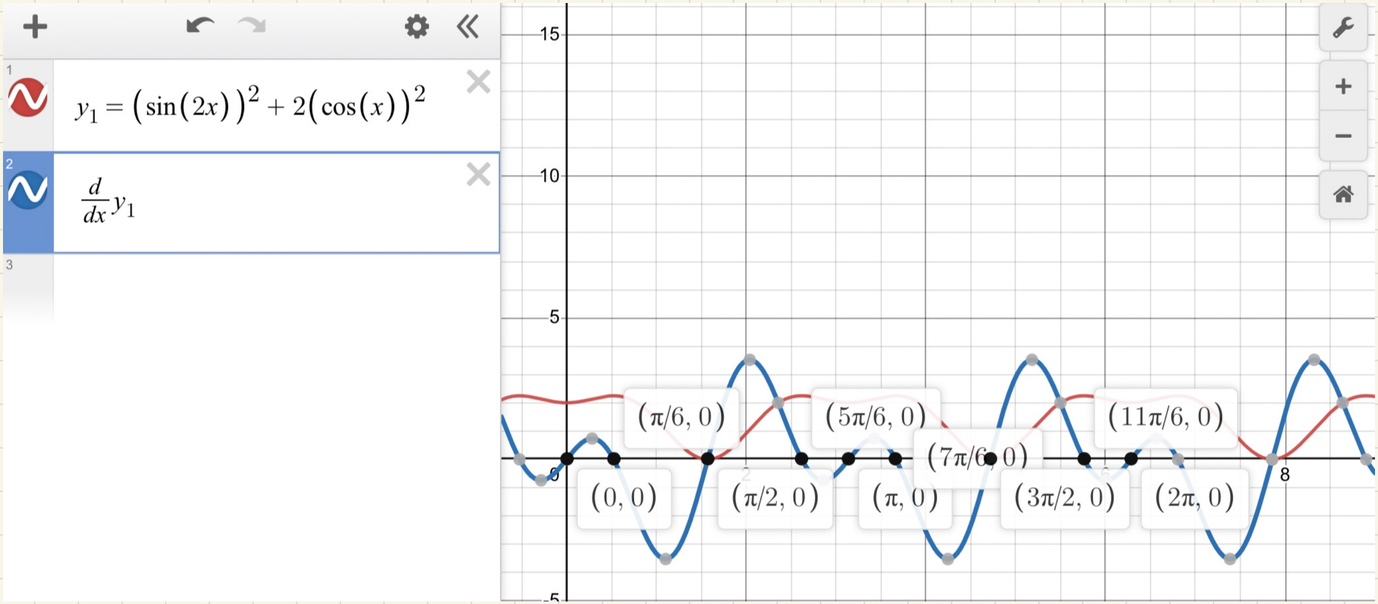
$f'(x)$: 0 (+, ↗) $\frac{\pi}{6}$ (−, ↘) $\frac{\pi}{2}$ (+, ↗) $\frac{5\pi}{6}$ (−, ↘) π (+, ↗) $\frac{7\pi}{6}$ (−, ↘) $\frac{3\pi}{2}$ (+, ↗) $\frac{11\pi}{6}$ (−, ↘) 2π

Continued →

interval of increase: $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{6}\right) \cup \left(\pi, \frac{7\pi}{6}\right) \cup \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

interval of decrease: $\left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \pi\right) \cup \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right)$

Check with DESMOS:



5.3 The Derivative of Exponential Functions f(x) = b^x

$f'(x) = b^x \cdot \ln b$

Theorem .If $f(x)=b^x$, then $f'(x)=b^x \ln b$

Proof:

$y = b^x$

$\ln(y) = x\ln(b)$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(b)$

$\frac{dy}{dx} = y \ln(b)$

$\frac{dy}{dx} = b^x \ln(b)$

* proof requires understanding of logaritmic and implicit differentiation

Ex $y = 2^x$

$y' = 2^x \ln 2$

Similar to:

$y = e^x$

$y' = e^x \cdot \ln e$

$= e^x \cdot (1)$

$= e^x$

The derivative of y= $b^{g(x)}$ is: $y' = b^{g(x)} \cdot \ln(b) \cdot g'(x)$

Ex $y = 2^{3x^2+2x}$

$y' = 2^{3x^2+2x} \cdot \ln(2) \cdot (6x+2)$

Ex 1: Differentiate.

a) $f(x)=3^x$

$f'(x) = 3^x \cdot \ln(3)$

b) $f(x)=\left(\frac{1}{2}\right)^x = 2^{-x}$

$f'(x) = 2^{-x} \cdot \ln(2) \cdot (-1)$

$= -2^{-x} \cdot \ln(2)$

or $f'(x) = \left(\frac{1}{2}\right)^x \cdot \ln\left(\frac{1}{2}\right)$

$= 2^{-x} \cdot \ln(2)^{-1}$

$= -2^{-x} \cdot \ln(2)$

c) $f(x)=3^{x^2}$

$f'(x) = 3^{x^2} \cdot \ln(3) \cdot 2x$

$= 2x \cdot 3^{x^2} \cdot \ln(3)$

d) $f(x)=4^{x^3+2x} \cdot 3^x$

$f'(x) = \left[(4^{x^3+2x}) \cdot \ln(4) \cdot (3x^2+2)\right]\left[3^x\right] +$

$\left[3^x \cdot \ln(3)\right]\left[4^{x^3+2x}\right]$

Ex 2: Find the equation of the tangent line to the graph of ~~$y=$~~ $f(x) = x(2^{-x})$ at $(0,0)$.

$$f'(x) = [1][2^{-x}] + [2^{-x} \cdot \ln(2) \cdot (-1)][x]$$
$$= 2^{-x} - 2^{-x} \cdot x\ln(2)$$
$$= 2^{-x}[1 - x\ln(2)]$$
$$= \frac{1 - x\ln(2)}{2^{x}}$$

$$f'(0) = [1][2^{-0}] + [2^{-0} \cdot \ln 2 \cdot (-1)][0]$$
$$= (1)(1) + 0$$
$$= 1$$

$$f'(0) = \frac{1-(0)\ln(2)}{2^0}$$
$$= 1$$

$\therefore$ Equation of tangent:

$m = 1$

$(0,0)$

$y = x$

The Derivative of Logarithmic Functions $f(x) = \log_b(x)$

$$\boxed{f'(x) = \frac{1}{x \cdot \ln b}}$$

Derivative of $log_b x$:

$$y = \log_b(x)$$
$$\Leftrightarrow b^y = x$$
$$\ln(b^y) = \ln(x)$$
$$y \ln b = \ln x$$
$$\ln b \cdot y' = \frac{1}{x}$$
$$y' = \frac{1}{x \ln b}$$

* proof requires understanding of implicit / logarithmic differentiation

Ex 1: Differentiate.

a) $f(x) = x^2 \log_3 x$

$$= [2x][\log_3(x)] + \left[\frac{1}{x \ln 3}\right][x^2]$$
$$= 2x \log_3(x) + \frac{x}{\ln 3}$$

b) $f(x) = \dfrac{\log x}{10^x}$

$$f'(x) = \frac{\frac{1}{x\ln 10}[\cancel{10^x}] - [\cancel{10^x}\ln 10][\log(x)]}{[10^x]^{\cancel{2}}}$$
$$= \frac{\frac{1}{x\ln 10} - \ln 10 \cdot \log(x)}{10^x}$$
$$= \frac{1}{10^x \cdot x\ln 10} - \frac{\log(x)\ln(10)}{10^x}$$

$y = \log_b(g(x))$, $y' = \frac{1}{g(x)\ln b} \cdot g'(x) = \frac{g'(x)}{g(x)\ln b}$

Derivative of $log_b g(x)$:

$y = \log_b x$

$b^y = x$

$\ln(b^y) = \ln(x)$

$y \ln b = \ln(x)$

$\ln b \cdot \frac{dy}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{1}{x \ln b}$

$\therefore y = \log_b(g(x))$

$\therefore \frac{dy}{dx} = \frac{1}{g(x)} \cdot \frac{1}{\ln b} \cdot g'(x)$

$= \frac{g'(x)}{g(x)\ln(b)}$

Recall : $y = \ln x$

$y' = \frac{1}{x} \cdot \frac{1}{\ln e}$ $\leftarrow 1$

$= \frac{1}{x}$

Ex 2: Differentiate.

a) $y = \log\sqrt{(x^2+1)}$

$y = \frac{1}{2}\log(x^2+1)$

$y' = \frac{1}{\cancel{2}}\left[\frac{1}{x^2+1} \cdot \frac{1}{\ln(10)} \cdot \cancel{2}x\right]$

$= \frac{x}{(x^2+1)\ln 10}$

b) $f(x) = \log_2(x^2 2^x)$

$f'(x) = \frac{1}{(x^2 \cdot 2^x)} \cdot \frac{1}{\ln(2)} \cdot \{[2x][2^x] + [2^x \cdot \ln(2)][x^2]\}$

$= \frac{1}{x^{\cancel{2}} \cdot \cancel{2^x} \cdot \ln(2)} \cdot \{\cancel{x} \cdot \cancel{2^x}(2 + x\ln(2))\}$

$= \frac{2 + x\ln(2)}{x \ln(2)}$

method 2:

$f(x) = \log_2(x)^2 + \log_2(2^x)$

$= 2\log_2(x) + x\log_2(2)$

$f'(x) = 2\left(\frac{1}{x\ln(2)}\right) + 1$

$= \frac{2}{x\ln(2)} + 1$

c) $f(x) = \log(\ln x)$

$f'(x) = \frac{1}{\ln(x)\cdot\ln(10)} \cdot \frac{1}{x}$

$= \frac{1}{x\ln(x)\cdot\ln(10)}$

d) $y = [\log_3(\ln x) - \ln(\tan 2x)]^{\frac{1}{5}}$

$y' = \frac{1}{5}\left[\log_3(\ln(x)) - \ln(\tan(2x))\right]^{-\frac{4}{5}} \cdot \left[\frac{1}{\ln(x)\cdot\ln(3)} \cdot \frac{1}{x} - \frac{1}{\tan(2x)} \cdot \sec^2(2x)\cdot 2\right]$

Ex 3: Find the absolute extrema for $f(x) = \frac{\log x}{x}$ over [1, 10].

$f'(x) = \frac{\left[\frac{1}{x\ln 10}\right][x] - [1][\log(x)]}{x^2}$

$= \frac{1}{x^2}\left[\frac{1}{\ln 10} - \log(x)\right]$

$f'(x) = dne$

$x = 0$

↳ out of domain

$f'(x) = 0$

$\log(x) = \frac{1}{\ln 10}$

$10^{\frac{1}{\ln(10)}} = x$

$x = e$

$\therefore$ critical #: $\{e\}$

$\boxed{\frac{1}{\ln 10} = \frac{\ln e}{\ln 10} = \log(e) \quad \therefore 10^{\log(e)} = e}$

$f(1) = \log(1) = 0 \Rightarrow$ Absolute Min

$f(e) = \frac{\log e}{e} \doteq 0.159 \Rightarrow$ Absolute max

$f(10) = \frac{\log(10)}{10} = \frac{1}{10}$

Ex 4 : State the intervals of concavity for the function, $f(x) = x\ln(4 - x^2)$.

$$f'(x) = [1][\ln(4-x^2)] + \left[\frac{1}{4-x^2}\cdot(-2x)\right][x]$$

$$= \ln(4-x^2) - \frac{2x^2}{4-x^2}$$

$$f''(x) = \frac{1}{4-x^2}\cdot -2x - \left\{\frac{[4x][4-x^2] - [-2x][2x^2]}{(4-x^2)^2}\right\}$$

$$= \frac{-2x}{4-x^2} - \left\{\frac{16x - \cancel{4x^3} + \cancel{4x^3}}{(4-x^2)^2}\right\}$$

$$0 = \frac{-2x}{4-x^2} - \frac{16x}{(4-x^2)^2}$$

$$0 = \frac{-2x(4-x^2) - 16x}{(4-x^2)^2}$$

$$0 = -8x + 2x^3 - 16x$$

$$0 = 2x^3 - 24x$$

$$0 = 2x(x^2 - 12)$$

$$x = \{0, \pm\sqrt{12}\}$$

$$= \{0, \pm 2\sqrt{3}\}$$

$D: \{-2 < x < 2\}$

$f''(x) = dne$

$x = \pm 2$

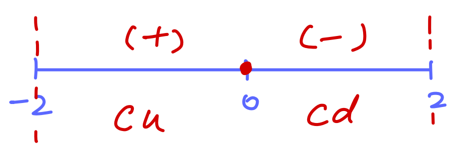
$\hookrightarrow$ out of domain

$4 - x^2 > 0$

$-(x^2 - 4) > 0$

$(x+2)(x-2) < 0$

$-2 < x < 2.$



$\therefore$ concave up : $x \in (-2, 0)$

concave down : $x \in (0, 2)$

< not tested >

but you can use ‼

Not tested

Ex 5*: Differentiate.(Logarithmic Differentiation)

a) $y = x^x$ ← can't use 'power' law, not an exponential function (Base & Exponent are variables!)

$\ln(y) = \ln(x^x)$ ⇒ ln both sides

$\ln y = x \ln(x)$

$\frac{1}{y} \cdot y' = [1][\ln(x)] + \frac{1}{x}[x]$

$\frac{1}{y} \cdot y' = \ln(x) + 1$

$y' = y[\ln(x) + 1]$

or $y' = x^x \cdot [\ln(x) + 1]$

b) $y = \dfrac{e^x\sqrt{x^2+1}}{(x^2+2)^3}$ ← will be a messy derivative if you use product rule + Quotient rule

$\ln(y) = \ln\left[\dfrac{e^x\sqrt{x^2+1}}{(x^2+2)^3}\right]$ ⇒ ln both sides

$\ln(y) = \ln e^x + \frac{1}{2}\ln(x^2+1) - 3\ln(x^2+2)$

$\frac{1}{y} \cdot y' = 1 + \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot 2x - 3 \cdot \frac{1}{x^2+2} \cdot 2x$

$\frac{1}{y} \cdot y' = 1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}$

$y' = y\left[1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}\right]$

or $y' = \left[\dfrac{e^x\sqrt{x^2+1}}{(x^2+2)^3}\right]\left(1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}\right)$

Practice

Multiple Choice

Identify the choice that best completes the statement or answers the question.

1. If $f(x) = 2^{sinx}$, find $f'(0)$

a. 2 b. π c. $ln2$ d. 1

2. If $f(x) = 90(2^{\frac{x}{3}})$, find $f'(3)$

a. $60ln2$ b. 30 c. $90ln2$ d. $180ln2$

3. If $f(x) = 2(3^{cosx})$, find $f'(\frac{\pi}{2})$.

a. -2 b. 6 c. $-2ln3$ d. 2π

4. Find the derivative:

a. $y = (10^{sinx})3^x$

$y' = [10^{\sin(x)} \cdot \ln(10) \cdot \cos(x)][3^x] + [3^x \ln(3)][10^{\sin(x)}]$

d. $y = \log_7 \sqrt{(x-1)(2x+1)(x+5)}$

$y = \frac{1}{2} \log_7 [(x-1)(2x+1)(x+5)]$

$y = \frac{1}{2} [\log_7 (x-1) + \log_7 (2x+1) + \log_7 (x+5)]$

$y' = \frac{1}{2} \left[\frac{1}{(x-1)\ln(7)} + \frac{1}{(2x+1)\ln(7)} + \frac{1}{(x+5)\ln(7)}\right]$

$= \frac{1}{2\ln(7)} \left[\frac{1}{x-1} + \frac{1}{2x+1} + \frac{1}{x+5}\right]$

b. $y = sec^5(10^{e^x})$

$y' = 5\sec^4(10^{e^x}) \cdot \sec(10^{e^x}) \cdot \tan(10^{e^x}) \cdot 10^{e^x} \cdot \ln(10) \cdot e^x$

e. $y = \sec(\frac{x}{5^x})cos^2(\frac{x}{5^x})$

$y = \frac{1}{\cos(\frac{x}{5^x})} \cdot \cos^2(\frac{x}{5^x})$

$= \cos(\frac{x}{5^x})$

$y' = -\sin(\frac{x}{5^x}) \cdot \frac{[1][5^x] - [5^x \cdot \ln(5)][x]}{(5^x)^2}$

$= -\sin(\frac{x}{5^x}) \cdot \left[\frac{1 - x\ln(5)}{5^x}\right]$

c. $y = \log_5 \frac{x^2+1}{x-1}$

$y = \log_5 (x^2+1) - \log_5 (x-1)$

$y' = \frac{1}{(x^2+1)\ln(5)} - \frac{1}{(x-1)\ln(5)}$

$= \frac{1}{\ln 5} \left[\frac{1}{x^2+1} - \frac{1}{x-1}\right]$

f. $y = 5^x \log e^{3x}$

constant!

$y = 5^x [3x \cdot \log e]$

$= \log(e)[5^x \cdot 3x]$

$y' = \log(e) \cdot [[5^x \cdot \ln(5)][3x] + [3][5^x]]$

$= \log(e) \cdot [3 \cdot 5^x (x\ln(5) + 1)]$

5. Water is brought to a boil then removed from the heat. The temperature of the water, T degrees Celcius is modeled as $T = 80e^{-0.57t} + 20$. At what rate is the temperature decreasing when the temperature reaches 30°C. $\Rightarrow$ assumed time in seconds

$$30 = 80e^{-0.57t} + 20$$
$$\frac{1}{8} = e^{-0.57t}$$
$$-0.57t = \ln\left(\frac{1}{8}\right)$$
$$t = \frac{\ln\left(\frac{1}{8}\right)}{-0.57}$$
$$\doteq 3.648$$

$$T'(t) = 80\left[e^{-0.57t} \cdot (-0.57)\right]$$
$$T'(3.648) = 80\left[e^{-0.57(3.648)} \cdot (-0.57)\right]$$
$$= -5.7\,°C/sec$$

6. The velocity of a car is given by $v(t) = 60[1-(0.7)^t]$, where v is measured in m/s and t is measured in seconds. Determine the time at which the acceleration is 3 m/s^2.

$$a(t) = v'(t) = 60\left[-0.7^t \cdot \ln(0.7)\right]$$
$$= -60\ln(0.7)\left[0.7^t\right]$$
$$3 = -60\ln(0.7)\left[0.7^t\right]$$
$$\frac{3}{-60\ln(0.7)} = 0.7^t$$

$$\ln\left[\frac{3}{-60\ln(0.7)}\right] = t\ln(0.7)$$
$$\frac{\ln\left[\frac{3}{-60\ln(0.7)}\right]}{\ln(0.7)} = t$$
$$\therefore t \doteq 5.508$$

7. The mass of polonium is given by the function, $M(t) = M_0\left(\frac{1}{2}\right)^{\frac{t}{138}}$ where M_0 is the initial mass of polonium, in milligrams, and $M(t)$ is the mass, in milligrams, after t days. At what rate is the polonium decaying when a 100 mg sample has decayed to 75% of its initial mass?

$$75 = 100\left(\frac{1}{2}\right)^{\frac{t}{138}}$$
$$0.75 = \left(\frac{1}{2}\right)^{\frac{t}{138}}$$
$$\frac{t}{138} = \frac{\log(0.75)}{\log\left(\frac{1}{2}\right)}$$
$$t = \frac{138\log(0.75)}{\log\left(\frac{1}{2}\right)}$$
$$\doteq 57.275$$

$$M(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{138}}$$
$$M'(t) = 100\left[\left(\frac{1}{2}\right)^{\frac{t}{138}} \cdot \ln\left(\frac{1}{2}\right) \cdot \frac{1}{138}\right]$$
$$= \frac{100}{138}\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{\frac{t}{138}}$$
$$M'(57.275) = \frac{100}{138}\ln\left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^{\frac{57.275}{138}}$$
$$= \frac{100}{138}\ln\left(\frac{1}{2}\right) \cdot 0.75$$
$$\doteq -0.376 \text{ mg/day}$$

Mid-Review

1. State in simplified form the derivative of the following:

   a) $y=\dfrac{\cos(2x)}{\sin(2x)\left(\sin^2(x)-\cos^2(x)\right)}$

   b) $y=\left(e^{-3\ln x}\right)^2+\log_2 2^{\cos\left(x^2\right)}$

   c) $f(x)=\left(3x^2+e\right)e^{-x}$

   d) $y=\dfrac{\sin\left(3x^2+1\right)}{\cos\left(3x^2-1\right)}$

2. Determine the local extrema of the function $f(x)=\ln\sqrt{x^3-x}$.

3. The position of a particle is given by $\boldsymbol{s}(t)=5\sin\left(2t-\dfrac{\pi}{3}\right), 0\le t\le 2\pi$, where ***t*** is in seconds and ***s*** is meters. Determine when the particle is at rest.

4. Find the point(s) on the graph of function $f(x)=(\ln x)^2(\ln x-1)$ that the tangent line is horizontal.

5. Find the values of a and b such that *f(x)* is differentiable everywhere

$$f(x)=\begin{cases}\dfrac{\sin x}{\sin x-\cos x} & , \quad x\le\dfrac{\pi}{2}\\[2ex] ax+b & , \quad x>\dfrac{\pi}{2}\end{cases}$$

6. Given $y=Axe^x+Be^x$, where A and B are constants ,show that $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}=-y$.

7. Determine the point(s) of inflection of the function $f(x)=\dfrac{1}{4}\cos(2x)+\sin(x)$ over the interval $x\in[0,\,2\pi]$.

8. Determine the equation of the normal to the function $f(x)=2x-(\ln x)^2$ at the point with x=e.

9. If $f(x)=e^{\cos(2x)}$ on [-π,π] determine and classify all local extrema.

10. For what value(s) of k does $y=e^{kx}\,sin(x)$ satisfy $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+2y=0$?

#1

d) $y = \dfrac{\sin(3x^2+1)}{\cos(3x^2-1)}$

$$y' = \frac{[\cos(3x^2+1)\cdot 6x][\cos(3x^2-1)] - [-\sin(3x^2-1)\cdot 6x][\sin(3x^2+1)]}{\cos^2(3x^2-1)}$$

$$= \frac{6x[\cos(3x^2+1)\cdot\cos(3x^2-1) + \sin(3x^2-1)\cdot\sin(3x^2+1)]}{\cos^2(3x^2-1)}$$

$$= \frac{6x[\cos[(3x^2+1)-(3x^2-1)]]}{\cos^2(3x^2-1)}$$

$$= \frac{6x[\cos(2)]}{\cos^2(3x-1)}$$

(i)	$\sin(A+B)$	=	$\sin A\cos B + \cos A\sin B$
(ii)	$\sin(A-B)$	=	$\sin A\cos B - \cos A\sin B$
(iii)	$\cos(A+B)$	=	$\cos A\cos B - \sin A\sin B$
(iv)	$\cos(A-B)$	=	$\cos A\cos B + \sin A\sin B$
(v)	$\tan(A+B)$	=	$\dfrac{\tan A+\tan B}{1-\tan A\tan B}$
(vi)	$\tan(A-B)$	=	$\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

Logaritmic Differentiation $\Rightarrow$ $\ln y = \ln[\sin(3x^2+1)] - \ln[\cos(3x^2-1)]$

$$\frac{1}{y}y' = \frac{1}{\sin(3x^2+1)}\cdot\cos(3x^2+1)\cdot 6x - \frac{1}{\cos(3x^2-1)}\cdot -\sin(3x^2-1)\cdot 6x$$

$$\frac{1}{y}y' = 6x\left[\frac{\cos(3x^2+1)}{\sin(3x^2+1)} + \frac{\sin(3x^2-1)}{\cos(3x^2-1)}\right]$$

$$y' = 6xy\left[\frac{\cos(3x^2+1)}{\sin(3x^2+1)} + \frac{\sin(3x^2-1)}{\cos(3x^2-1)}\right]$$

$$= 6x\left[\frac{\sin(3x^2+1)}{\cos(3x^2-1)}\right]\left[\frac{\cos(3x^2+1)}{\sin(3x^2+1)} + \frac{\sin(3x^2-1)}{\cos(3x^2-1)}\right]$$

$$= 6x\left[\frac{\cos(3x^2+1)}{\cos(3x^2-1)} + \frac{\sin(3x^2+1)\cdot\sin(3x^2-1)}{\cos^2(3x^2-1)}\right)$$

5. Find the values of a and b such that $f(x)$ is differentiable everywhere

$$f(x) = \begin{cases} \dfrac{\sin x}{\sin x - \cos x}, & x \le \dfrac{\pi}{2} \\ ax+b, & x > \dfrac{\pi}{2} \end{cases}$$

for $f(x)$ to be differentiable,

① $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} f(x)$

$$\frac{\sin(\frac{\pi}{2})}{\sin(\frac{\pi}{2}) - \cos(\frac{\pi}{2})} = a(\tfrac{\pi}{2})+b$$

$$\frac{1}{1-0} = \frac{a}{2}\pi + b$$

$$1 = \frac{a}{2}\pi + b$$

$$b = 1 - \frac{a}{2}\pi$$
$$= 1 - (\tfrac{-1}{2})\pi$$
$$= 1 + \frac{\pi}{2}$$

② $f'(\frac{\pi}{2}^-) = f'(\frac{\pi}{2}^+)$

$$\frac{[\cos(x)][\sin(x) - \cos(x)] - [\cos(x) + \sin(x)][\sin(x)]}{[\sin(x) - \cos(x)]^2} = a$$

$$\frac{\cancel{\sin(x)\cos(x)} - \cos^2(x) - \sin^2(x) - \cancel{\sin(x)\cos(x)}}{\sin^2(x) - 2\sin(x)\cos(x) + \cos^2(x)} = a$$

$$\frac{-(\sin^2(x) + \cos^2(x))}{(\sin^2(x) + \cos^2(x)) - 2\sin(x)\cos(x)} = a$$

$$\frac{-1}{1 - \sin(2x)} = a$$

$$\frac{-1}{1 - \sin(2\cdot\frac{\pi}{2})} = a$$

$$-1 = a$$

$\therefore a = -1$

$b = 1 + \frac{\pi}{2}$

5.5 Optimization Problems

Ex 1: Determine the absolute maximum value of the function $f(x) = x^2e^x$ on the interval [-1, 4].

$f'(x) = [2x][e^x] + [e^x][x^2]$

$= xe^x[2+x]$

$f'(x) = 0$

$x = \{0, -2\}$

↳ out of domain

$f'(x) = dne$

$x = \{\}$

$f(-1) = (-1)^2e^{-1} = \frac{1}{e} \doteq 0.367$

$f(0) = 0 \Rightarrow$ Absolute Min

$f(4) = 16e^4 \Rightarrow$ Absolute Max

Ex 2 : A small lake is stocked with freshwater salmon. The population of freshwater salmon in the lake grows according to the equation $N = \frac{2000}{1+9e^{-0.2t}}$ where N is the number of freshwater salmon, and t is the time in years.

(a) How many freshwater salmon were initially introduced into the lake?

$N(0) = \frac{2000}{1+9e^{-0.2(0)}}$

$= 200$

∴ 200 Salmon were introduced initially

very tedious * (b) When was the population increasing most rapidly?

Max value of $N'(t) \Rightarrow N''(t) = 0$

$N(t) = 2000(1+9e^{-0.2t})^{-1}$

$N'(t) = -2000(1+9e^{-0.2t})^{-2} \cdot [9e^{-0.2t} \cdot -0.2]$

$= 3600[e^{-0.2t}(1+9e^{-0.2t})^{-2}]$

$N''(t) = 3600\{[e^{-0.2t} \cdot (-0.2)][1+9e^{-0.2t}]^{-2} + [-2[1+9e^{-0.2t}]^{-3} \cdot (9e^{-0.2t} \cdot (-0.2))][e^{-0.2t}]\}$

$= 3600\{-0.2(e^{-0.2t})(1+9e^{-0.2t})^{-3}[(1+9e^{-0.2t}) - 2 \cdot 9e^{-0.2t}]\}$

$0 = \frac{-720}{e^{0.2t}(1+9e^{-0.2t})^3} \cdot [(1+9e^{-0.2t}) - 18e^{-0.2t}]$

$18e^{-0.2t} = 1 + 9e^{-0.2t}$

$9e^{-0.2t} = 1$

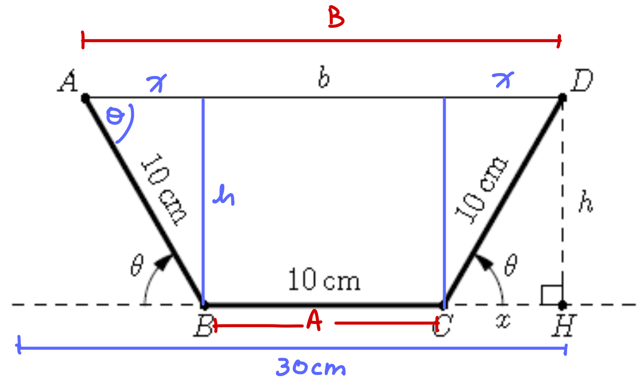
$e^{-0.2t} = \frac{1}{9}$

$-0.2t = \ln(\frac{1}{9})$

$t = \frac{\ln(\frac{1}{9})}{-0.2}$

$\doteq 10.986$ years

Ex3: A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one third of the sheet on each side through an angle, θ. How should θ be chosen so that the gutter will carry the maximum amount of water?



$A = \frac{1}{2}(A+B)h$

$A(\theta) = \frac{1}{2}[10 + (10+20\cos\theta)][10\sin\theta]$

$= \frac{1}{2}[20 + 20\cos\theta][10\sin\theta]$

$= [10 + 10\cos\theta][10\sin\theta]$

$A(\theta) = 100\sin\theta[1+\cos\theta]$

$A'(\theta) = 100\{\cos\theta\cdot(1+\cos\theta) + (-\sin\theta)(\sin\theta)\}$

$= 100[\cos\theta + \cos^2\theta - \sin^2\theta]$

$= 100[\cos\theta + \cos^2\theta - (1-\cos^2\theta)]$

$= 100[2\cos^2\theta + \cos\theta - 1]$

$0 = 100[2\cos\theta - 1][\cos\theta + 1]$

$\therefore \cos\theta = \frac{1}{2}$, $\cos\theta = -1$

$\theta = \{\frac{\pi}{3}\}$, $\theta = \pi$ ↳ out of domain

$\sin\theta = \frac{h}{10} \Rightarrow h = 10\sin\theta$

$\cos\theta = \frac{x}{10} \Rightarrow x = 10\cos\theta$

$B = 10 + 2x$

$= 10 + 2(10\cos\theta)$

$= 10 + 20\cos\theta$

Constraint

$0 < \theta < \frac{\pi}{2}$

(+) (−)

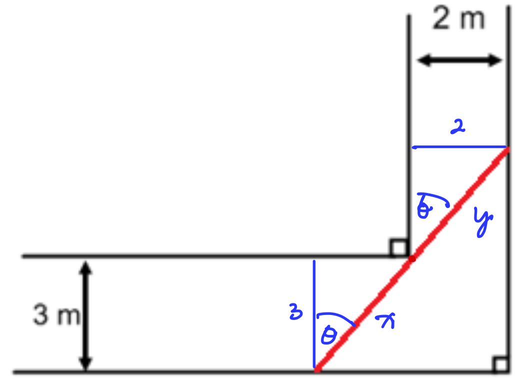
$0 \nearrow \frac{\pi}{3} \searrow \frac{\pi}{2}$

max

$\therefore$ An angle of $\frac{\pi}{3}$ will maximize the cross-sectional area of the gutter.

Ex4: A ladder is to be carried down a hallway *2m* wide. Unfortunately at the end of the hallway there is a right-angled turn into a hallway *3 m* wide. What is the length of the longest ladder that can be carried horizontally around the corner?

The goal is shifted from finding the longest ladder that will go around the corner to finding the shortest ladder that will get stuck.



$\cos\theta = \frac{3}{x}$ $\quad$ $\sin\theta = \frac{2}{y}$

$x = \frac{3}{\cos\theta}$ $\quad$ $y = \frac{2}{\sin\theta}$

$$\ell = x + y$$

$$\ell(\theta) = \frac{3}{\cos\theta} + \frac{2}{\sin\theta}$$

$$= 3[\cos\theta]^{-1} + 2[\sin\theta]^{-1}$$

$$\ell'(\theta) = -3[\cos\theta]^{-2}\cdot[-\sin\theta] - 2[\sin\theta]^{-2}\cdot\cos\theta$$

$$0 = \frac{3\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$$

$$\frac{2\cos\theta}{\sin^2\theta} = \frac{3\sin\theta}{\cos^2\theta}$$

$$2\cos^3\theta = 3\sin^3\theta$$

$$\frac{2}{3} = \frac{\sin^3\theta}{\cos^3\theta}$$

$$\frac{2}{3} = \tan^3\theta$$

$$\sqrt[3]{\frac{2}{3}} = \tan\theta \qquad 0 < \theta < \frac{\pi}{2}$$

$$\theta \doteq 0.71$$

$\ell'(\theta)$: $(-)$ on $(0, 0.71)$, $(+)$ on $(0.71, \frac{\pi}{2})$

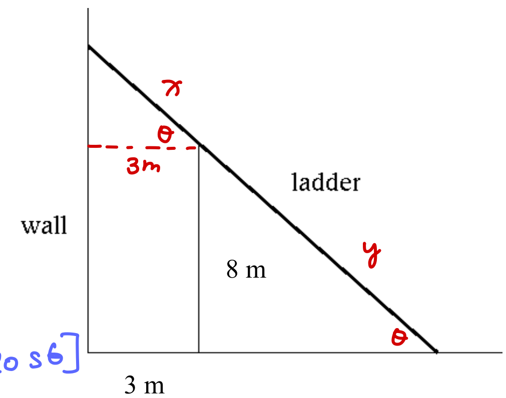
0 $\quad$ 0.71 $\quad$ $\frac{\pi}{2}$

min

$$\ell(0.71) = \frac{3}{\cos(0.71)} + \frac{2}{\sin(0.71)}$$

$$\doteq 7.02$$

Ex5. A fence 8 m tall runs parallel to a house at a distance of 3 m from the wall of a house. Determine the angle, θ, that the shortest ladder will make from the foot of the ladder to the ground, if the ladder is to reach from the ground over the fence and to the wall of the house



$$l = x + y$$

$$l(\theta) = \frac{3}{\cos\theta} + \frac{8}{\sin\theta}$$

$$= 3[\cos\theta]^{-1} + 8[\sin\theta]^{-1}$$

$$l'(\theta) = -3[\cos\theta]^{-2}(-\sin\theta) - 8[\sin\theta]^{-2}[\cos\theta]$$

$$0 = \frac{3\sin\theta}{\cos^2\theta} - \frac{8\cos\theta}{\sin^2\theta}$$

$$\frac{8\cos\theta}{\sin^2\theta} = \frac{3\sin\theta}{\cos^2\theta}$$

$$\frac{8}{3} = \frac{\sin^3\theta}{\cos^3\theta}$$

$$\frac{8}{3} = \tan^3\theta$$

$$1.3867 \doteq \tan\theta$$

$$\theta = 0.946$$

$\sin\theta = \frac{8}{y} \Rightarrow y = \frac{8}{\sin\theta}$

$\cos\theta = \frac{3}{x} \Rightarrow x = \frac{3}{\cos\theta}$

(−) (+)

0 ↘ 0.946 ↗ $\frac{\pi}{2}$

min

$\therefore \theta = 0.946$ will min. the length of the ladder.

Ex6. If $f(x) = (\ln x)^2 + \ln(3x) - \ln(3)$, determine the coordinates of any points of inflection

$f'(x) = 2(\ln(x)) \cdot \frac{1}{x} + \frac{1}{3x} \cdot 3$ D: $x > 0$

$= 2\ln(x) \cdot x^{-1} + \frac{1}{x}$

$= \frac{1}{x}[2\ln(x) + 1]$

$f''(x) = [-x^{-2}][2\ln(x)+1] + [\frac{2}{x}][\frac{1}{x}]$

$= \frac{-2\ln(x) - 1}{x^2} + \frac{2}{x^2}$

$= \frac{-2\ln x + 1}{x^2}$

$f''(x) = 0$

$2\ln x = 1$

$\ln x = \frac{1}{2}$

$e^{\frac{1}{2}} = x$

$f''(x) = $ dne

$x = 0$

↳ out of domain

$f''(x)$: 0 (+) cu, $e^{\frac{1}{2}}$ poi, (−) cd, ∞

$f(e^{\frac{1}{2}}) = [\ln(e^{\frac{1}{2}})]^2 + \ln[3 \cdot e^{\frac{1}{2}}] - \ln(3)$

$= (\frac{1}{2})^2 + \cancel{\ln 3} + \frac{1}{2}\ln e - \cancel{\ln 3}$

$= \frac{1}{4} + \frac{1}{2}$

$= \frac{3}{4}$

$\therefore$ poi is $(e^{\frac{1}{2}}, \frac{3}{4})$

Ex7. Find the x - coordinates of the point(s) where the tangent line to the curve $f(x)=\sin(2x)+2\sin(x)$ is horizontal over the interval $x\in[0\ ,\ 2\pi]$.

$f'(x) = \cos(2x)\cdot 2 + 2\cos(x)$

$= 2[2\cos^2 x - 1] + 2\cos(x)$

$0 = 4\cos^2 x + 2\cos(x) - 2$

$0 = 2\cos^2(x) + \cos(x) - 1$

$0 = (2\cos x - 1)(\cos x + 1)$

$\cos x = \frac{1}{2}$ $\quad\quad$ $\cos x = -1$

$x = \{\frac{\pi}{3}, \frac{5\pi}{3}\}$ $\quad\quad$ $x = \{\pi\}$

$\sin(2A) = 2\sin(A)\cos(A)$

$\cos(2A) = \cos^2(A) - \sin^2(A)$

$= 1 - 2\sin^2(A)$

$= 2\cos^2(A) - 1$

$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$

$\therefore$ @ $x = \{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\}$ the tangent line is horizontal

Ex.8 The tangent to the curve $f(x)=3^{3x} - x\ln 27 - 5\ln(x+1)$ at $x=0$ intersects the x-axis and the **y-**axis at points A and B respectively and creates a triangle. Determine the **exact** perimeter of the triangle OAB.

$f'(x) = 3^{3x}\cdot\ln(3)\cdot 3 - \ln(27) - \frac{5}{x+1}$

$f'(0) = 3^{3(0)}\cdot\ln(3)\cdot 3 - \ln(27) - \frac{5}{0+1}$

$= 3\ln(3) - \ln(27) - 5$

$= \ln(\frac{3^3}{27}) - 5$

$= \ln(1) - 5$

$= -5$

$f(0) = 3^{3(0)} - (0)\ln(27) - 5\ln(0+1)$

$= 1 - 0 - 5\ln(1)$

$= 1 - 5\ln(1)$

$= 1 - 0$

$= 1$

Equation of tangent: $y = -5x + 1$

$m = -5$

$(0, 1)$

xint: $y = 0$

$0 = -5x + 1$

$\frac{-1}{-5} = x$

$x = \frac{1}{5}$

yint: $x = 0$

$y = -5(0) + 1$

$y = 1$

$(\frac{1}{5})^2 + (1)^2 = c^2$

$\frac{1}{25} + \frac{25}{25} = c^2$

$c = \frac{\sqrt{26}}{5}$

Perimeter $= \frac{1}{5} + 1 + \frac{\sqrt{26}}{5}$

$= \frac{1+5+\sqrt{26}}{5}$

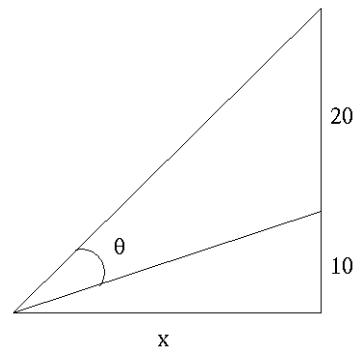
$= \frac{\sqrt{26}+6}{5}$ units

Practice

1. Suppose that the monthly revenue in thousands of dollars, for the sale of x hundred units of an electronic item is given by the function $R(x) = 40x^2e^{-0.4x} + 30$, where the maximum capacity of the plant is 800 units. Determine the number of units to produce in order to maximize revenue.

2. The hypotenuse of aright triangle is 12 cm. Calculate the measure of the unknown angles in the triangle that will maximize the perimeter.

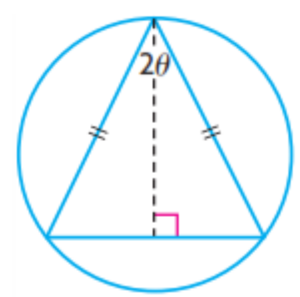
* 3. A movie screen on a wall is 20 meter high and 10 meter above the floor. At what distance x from the front of the room should you position yourself so that the tan ratio of viewing angle θ of the movie screen is as large as possible. (Hint: Maximize tanθ).



4. A drug is injected into the body in such a way that the concentration, C, in the blood at time t hours is given by the function $C(t) = 10(e^{-2t} - e^{-3t})$. At what time does the highest concentration occur within the first 5 h?

5. Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where the wire should stake to have the **angle** formed by the two pieces of wire at the stake be maximum?

6. An isosceles triangle is inscribed in a circle of radius R. Find the value of θ that maximizes the area of the triangle.



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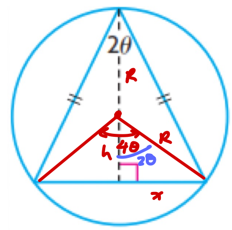

$$\cos(2\theta) = \frac{h}{R} \qquad \sin(2\theta) = \frac{x}{R}$$

$$R\cos(2\theta) = h \qquad R\sin(2\theta) = x$$

$$A = \frac{1}{2} BH$$

$$= \frac{1}{\cancel{2}}(\cancel{2}x)(R+h)$$

$$= x(R+h)$$

$$A(\theta) = R\sin(2\theta)\,(R + R\cos(2\theta))$$

$$= R^2[\sin(2\theta)\,(1+\cos(2\theta))]$$

$$A'(\theta) = R^2\{[\cos(2\theta)\cdot 2][1+\cos(2\theta)] + [-\sin(2\theta)\cdot 2][\sin(2\theta)]\}$$

$$= R^2\{2\cos(2\theta) + 2\cos^2(2\theta) - 2\sin^2(2\theta\}$$

Constraint:

$$0 < 2\theta < \frac{\pi}{2}$$

$$0 = 2R^2\{\cos(2\theta) + \cos^2(2\theta) - \sin^2(2\theta)\}$$

$$0 = \cos 2\theta + \cos^2(2\theta) - (1-\cos^2(2\theta)\}$$

$$0 = 2\cos^2(2\theta) + \cos(2\theta) - 1$$

$$0 = (2\cos(2\theta) - 1)(\cos(2\theta) + 1)$$

$$\cos(2\theta) = \frac{1}{2} \qquad \cos(2\theta) = -1$$

$$2\theta = \left\{\frac{\pi}{3}\right\} \qquad 2\theta = \pi$$

↳ out of domain

$$\theta = \frac{\pi}{6}$$

UNIT 5: Derivative of other functions -REVIEW

1. State in simplified form (answer only), the derivative of the following:

   (a) $y=\left(e^{\ln x}\right)^2$ (b) $y=2^{\cos x}+\ln(3e)$

   (c) $y=\dfrac{\cos 2x}{\sin 2x\left(\cos^2 x-\sin^2 x\right)}$ (d) $y=3^{\log_3(\cos 6x)}$

2. Differentiate each of the following. Simplify fully.

   (a) $y=3x^2\tan(2x)$ (b) $f(x)=\cos\left(x^3\right)$

   (c) $s(t)=\sin(\ln 7t)$ (d) $y=e^x\ln x$

3. Differentiate the following and leave in simplified form.

   (a) $f(x)=\tan^4\left(3\ln x^2\right)$ (b) $f(x)=2\ln\left(\sqrt{1-x+x^2}\right)$ (c) $s(x)=e^{\pi}\ln(\sin x+\cos x)$

4. Find the equation of the tangent to $y=x\ln x$ which is parallel to $3x-y+7=0$.

5. Determine all points of inflection of $y=e^x\sin x$ on the interval $0\le x\le 2\pi$.

6. Determine the equation of the tangent to the curve $y=e^x+x$ which passes through the origin

7. The function $f(x)=\dfrac{8}{1+3e^{-0.5x}}$, is known as the Sobolewski Happiness Indication Threshold. This function shows that there is a limit to the amount of happiness that a person does or does not feel no matter what other external factors affect the person.

   (a) Determine $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

   (b) Evaluate $f(0)$.

   (c) Show that $f'(x)>0$ for all values of *x*.

   (d) Show that there are no local extreme points for $f(x)$.

   (e) Show that $f''(x)=0$ when x = 2ln3

   (f) Use your answers from parts (a) through (e) to sketch the graph of $f(x)=\dfrac{8}{1+3e^{-0.5x}}$.

8. Determine any **Point(s) of inflection** of the function $f(x)=2\sin^2(x)-1$ on the interval $-\pi\le x\le\pi$.

9*. A line with slope m passes through the origin and is tangent to $y = \ln\left(\frac{x}{3}\right)$. What is the value of m?

10. A point moves on the x-axis in such a way that its velocity at time t (t > 0) is given by $\mathbf{v}(t) = \frac{\ln t}{t}$. At what value of t does **v** attain its maximum? What is the acceleration of particle at this point?

11. A rectangle has its vertices on the x-axis, the y-axis, the origin, and the graph of $y = -\ln x$ in the first quadrant. Find the maximum possible area for such a rectangle.

12*. Show that for any values A and B, $y = Ae^{2x} + Be^{-4x}$ satisfies the equation $y'' + 2y' - 8y = 0$.

13. Determine the exact value of the constant k given $f'(1) = 0$ for the function $f(x) = (3x+e)e^{-kx}$.

14*. The function $f(x) = \frac{\sin(nx) + \cos(nx)}{n}$ has local extrema at $x = \frac{\pi}{4}$. Determine all possible integer values of n, where $1 \le n \le 6$.