Grade 12 Calculus & Vectors
Unit 7 – Lines and Planes

Big Ideas
We will…
- explore and develop the equations of lines and planes in R^3
- determine the intersection(s) of lines and planes

DAY	DESCRIPTION	Homework
1	Equations of Lines in R^2 and R^3 We are learning to… • represent a line in R^2 and R^3 algebraically using vectors in new ways I am able to… • state the vector , parametric and symmetric equations for a line in R^2 given its graph, two points on the line, or some other combination of information (e.g. it passes through a given point but is parallel to another line) • describe the geometric connection between the line with Cartesian equation Ax + By + C = 0 and the vector [A, B] • convert between the FOUR different forms of the equation of a line (slope/y-intercept; Cartesian; vector; parametric) • state the vector and parametric equations for a line in R^3 given two points on the line or some other combination of information (e.g. it passes through a given point but is parallel to another line) • use a vector ,parametric or symmetric equations for a line in R^3 to find points on the line or decide if a given point lies on the line • explain why there is no single set of collinear vectors normal to a line in R^3 (and therefore why there is no Cartesian equation for a line in R^3) • convert between the three different forms of the equation of a line in R^3 (vector , parametric and symmetric) • represent a line in R^2 or R^3 with a vector equation in the best form possible	*CP: Pg 11* *Pg 436 #1 – 3, 5 – 8, 10 – 14, 20 – 23*
2	**Intersection of Lines** We are learning to… • find the possible ways two lines may or may not intersect in R^3 I am able to… • analyze the parametric, vector, Cartesian or slope/y-intercept equations of two lines in R^2 to determine if the two lines are coincident, parallel or intersecting in a point • analyze the parametric or vector equations of two lines in R^3 to determine if the two lines are coincident, parallel, intersecting in a point or skew • find the point(s) of intersection of two lines in R^2 or R^3	*CP: Pg 19* *Pg 471 #4 – 6, 8, 11, 16*

3	**Equations of planes:** **Vector , parametric and Cartesian** We are learning to... • find and work with the vector parametric and scalar equations of a plane I am able to... • distinguish between vector equations for lines and vector equations for planes • find points or lines on a plane given its vector or parametric equations • find the vector and parametric equations of a plane given different combinations of information (e.g. given a point and two direction vectors, three points on the plane, a line on plane and a line parallel to the plane etc.) • write the vector equation for a plane in the best form possible (integer direction vectors) • convert between the vector and parametric equations for a plane • determine the x, y and z intercepts of a plane given its Cartesian equation • describe the connection between the vector [A, B, C] and the plane with equation $Ax + By + Cz + D = 0$	*Page 450 #5 – 10, 12, 13, 16* *Pg 450 #1 – 4, 15* *Pg 459 #1 – 12, 17 – 19, 22*
4	**Distances** We are learning to find ... • distance between two skew lines • a point and a plane • a line and a plane • two parallel planes I am able to .... • calculate the distance in three-space between lines with no intersection. • calculate the distance in three-space between lines and planes with no intersection.	*Pg 472 #9* *Pg 480 #5 – 8*
5	Intersection of a Line and Plane (3 cases) and Intersection of Two Planes We are learning to... • determine how and where a line and a plane could intersect • determine geometrically and algebraically the possible orientation of 2 planes in 3-space I am able to... • solve a system in R^3 comprised of the equation of a line and the equation of a plane and interpret the solution geometrically • analyze a system comprised of the equation of a line and the equation of a plane to determine if the line intersects the plane, without solving and justify your conclusions	*CP: Pg 43* *Pg 479 #1 – 4, 9, 10, 12, 13* *Pg 490 #1 – 3*

6	Intersection of three Planes(Two days) We are learning to… • determine different geometric configurations of combinations of 3 planes in 3-space I am able to… • Classify sets of planes that result in a common point, common line, common plane or no intersection • Determine algebraically the intersection of 3 planes in 3-space given equations in various forms	*CP: Pg. 54* *Pg 491 #4 – 6, 7, 11, 12, 13, 14, 17, 19*
7	**Formative Quiz**	
8	**Review**	*Pg 502 #7 – 23* *Pg 504 #1 – 10, 14, 17 – 22*
9	**Summative Quest**	W Jun 5 (1 Day Quest)

Recall Grade 10 : $y = mx + b$

$Ax + By + C = 0$

$y_2 - y_1 = m(x_2 - x_1)$

New: Vector Equations

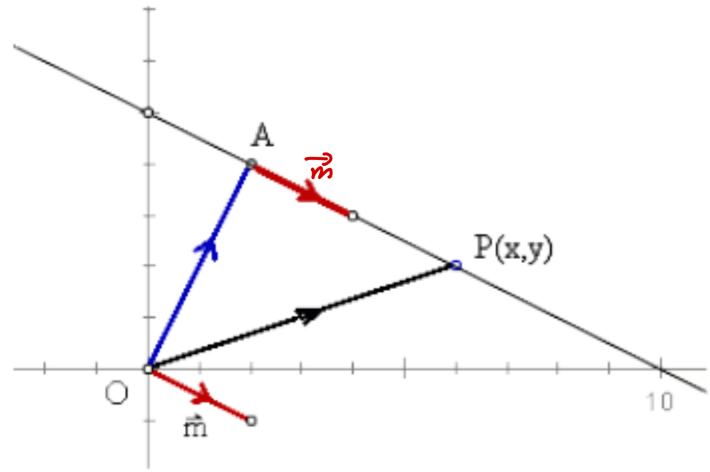
$\vec{v} = \underbrace{[x_0, y_0]}_{\text{position vector}} + t\,\underbrace{[m_1, m_2]}_{\text{direction vector}}, t \in \mathbb{R} \Rightarrow$ not unique

$[x, y] =$

7-1A Equations of Lines in R^2

In R^2 vectors can be used to define a line .Two new forms of the equation of the line are the **Vector Equation of a Line** and **Parametric Form of the Equation of a Line**. We start by defining the former.

Vectors can be used to locate points on a line as shown in the diagram at right. If A is a given point on the line and $\vec{m}$ is a vector parallel to the line , $\overrightarrow{OP} = \overrightarrow{OA} + t\vec{m}$ can be used to locate any point P(x,y) on the line.



This equation is called the **Vector Equation** of the line.

$$\overrightarrow{OP} = \overrightarrow{OA} + t\vec{m}$$

$\overrightarrow{OA}$ is called a **Position Vector** (think of it as a point on the line)

$\vec{m}$ is called a **Direction Vector** (think of it as the slope)

t is called a **Parameter** (any real number)

Vector Equation of Lines in R^2

Another way to write this equation using variables is $\vec{r} = \vec{r}_o + t\vec{m}$. By substituting $\vec{r} = [x, y]$, $\vec{r}_o = [x_o, y_o]$ and $\vec{m} = [m_1, m_2]$ into this equation we get another form of the vector equation.

The Vector Equation of a Line in R^2

$\vec{r} = \vec{r}_0 + t\vec{m}$ OR $[x, y] = [x_0, y_0] + t[m_1, m_2]$

where

- $t \in R$ is a parameter
- $\vec{r} = [x, y]$ is a position vector to any unknown point on the line
- $\vec{r}_0 = [x_0, y_0]$ is a position vector to any known point on the line
- $\vec{m} = [m_1, m_2]$ is a direction vector parallel to the line

Example 1:

a) Write a vector equation of a line passing through the points A(1, 4) and B(3, 1).

$\vec{m} = \vec{AB}$
$= [3-1, 1-4]$
$= [2, -3]$

$[x, y] = [1, 4] + t[2, -3]$, $t \in \mathbb{R}$
or
$[x, y] = [3, 1] + t[2, -3]$, $t \in \mathbb{R}$

b) Determine two more points on the line.

$t = 2$, $[x, y] = [1, 4] + 2[2, -3]$
$= [5, -2]$

$t = -1$, $[x, y] = [1, 4] - 1[2, -3]$
$= [-1, 7]$

$\therefore$ $(5, -2)$ and $(-1, 7)$ are on the line

c) Determine if the point (2, 3) is on this line.

$[2, 3] = [1, 4] + t[2, -3]$
$[2, 3] = [1+2t, 4-3t]$

$2 = 1 + 2t$ $\qquad$ $3 = 4 - 3t$
$\frac{1}{2} = t_1$ $\qquad$ $\frac{1}{3} = t_2$

$t_1 \neq t_2$ $\quad$ $\therefore$ $(2, 3)$ is not on the line

NOTE: Vector equations are **NOT** unique!

The vector equation can be separated into two parts, one for each variable. These are called **parametric equations** of a line.

The Parametric form of the Equation of a Line in R²

For a line with equation $[x, y] = [x_0, y_0] + t[m_1, m_2]$, the parametric equations are

$$\begin{cases} x = x_0 + tm_1 \\ y = y_0 + tm_2 \end{cases}$$ where $t \in \mathbb{R}$ (the parameter)

Example 2: Rewrite your vector equation from Example 1(a) in parametric form.

$$\begin{cases} x = 1 + 2t \\ y = 4 - 3t \end{cases}, \ t \in \mathbb{R} \qquad \text{or} \qquad \begin{cases} x = 3 + 2t \\ y = 1 - 3t \end{cases}, \ t \in \mathbb{R}$$

NOTE: Again, like vector equations, parametric equations are not unique as we can use the coordinates of any point on the line and any scalar multiple of the direction vector.

Example 3: A line L_1 is defined by $x = 3 + t$ and $y = -5 + 2t$.

a) Find the coordinates of two points on this line.

$$L_1 : \begin{cases} x = 3 + t \\ y = -5 + 2t \end{cases}$$

$t = 0$, $(3, -5)$

$t = 4$, $(7, 3)$

$\therefore$ $(3, -5)$ and $(7, 3)$ are 2 points on the line

b) Find the y-intercept of the line.

↳ $x = 0$

$0 = 3 + t$

$t = -3$

$y = -5 + 2(-3)$

$= -11$

$\therefore$ y-int: $(0, -11)$

c) Write the vector equation for L_1.

$$[x, y] = [3, -5] + t[1, 2]$$

or $\vec{r} = [3, -5] + t[1, 2]$

d) Determine if L_1 is parallel to L_2: $x = 1 + 2t$, $y = -9 + 4t$.

$\vec{m_1} = [1, 2]$ $\quad\quad$ $\vec{m_2} = [2, 4]$

$\vec{m_1} = k\,\vec{m_2}$

$k = \frac{1}{2}$

$\Rightarrow$ $\vec{m_1}$ is a scalar multiple of $\vec{m_2}$

$\therefore$ L_1 + L_2 are parallel

Example 3: A line L_1 is defined by $x = 3 + t$ and $y = -5 + 2t$.

a) Find the coordinates of two points on this line.

$$L_1: \begin{cases} x = 3+t \\ y = -5+2t \end{cases}, \quad t \in \mathbb{R}$$

$t=0$, $(3,-5)$

$t=7$, $(10, 9)$

b) Find the y-intercept of the line.

↳ $x=0$,

$0 = 3+t$

$t = -3$

$\therefore y = -5+2(-3)$

$= -11$

$\therefore$ y-int: $(0, -11)$

c) Write the vector equation for L_1.

$$[x, y] = [3, -5] + t[1, 2]$$

or $\vec{r} = [3, -5] + t[1, 2]$

d) Determine if L_1 is parallel to L_2: $x = 1 + 2t$, $y = -9 + 4t$.

L_1: $\vec{m_1} = [1, 2]$

$$L_2 = \begin{cases} x = 1+2t \\ y = -9+4t \end{cases}, \quad t \in \mathbb{R}$$

$\vec{m_2} = [2, 4] = 2[1, 2]$

$\because \vec{m_1} = k\vec{m_2}$, $k = \frac{1}{2}$

$\therefore L_1 \parallel L_2$

Ex: $l_1 : [x,y] = [3,-5] + t[1,2] \quad , t \in R$

Symmetric Equation of a line.

$$\begin{cases} x = 3 + t \\ y = -5 + 2t \end{cases}, t \in R \Rightarrow t = \frac{x-3}{1} \qquad \Rightarrow t = \frac{y+5}{2}$$

Symmetric Equation

$$\frac{x-3}{1} = \frac{y+5}{2}, \quad t \in R$$

The Symmetric form of the Equation of a Line in R^2

For a line with equation $[x,y]=[x_0,y_0]+t[m_1,m_2]$, the symmetric equation is

$$\frac{x-x_0}{m_1} = \frac{y-y_0}{m_2}, m_1, m_2 \neq 0$$

What happens if m_1 or m_2 is zero?

Let suppose $m_2=0$. In this case t will not exist in the parametric equation for y and so we will only solve the parametric equations for x for t. We then set those equal and acknowledge the parametric equation for y as follows,

$$\frac{x-x_0}{m_1}, \ y = y_0$$

Example 4: Write all three forms of the equation of the line that passes through the points A (2,-1) and B(4,-1).

$\vec{m} = \vec{AB}$
$= [2,0]$

vector equation:

$\vec{r} = [2,-1] + t[2,0] \ , \ t \in R$

parametric equations:

$$\begin{cases} x = 2 + 2t \\ y = -1 \end{cases}, \ t \in R$$

Symmetric Equation

$$\frac{x-2}{2}, \ y = -1$$

Example 5: Consider the line with Cartesian equation $4x + 5y + 20 = 0$. $\vec{n} = [4,5]$

perpendicular slopes

a) Determine its slope. How does the slope compare to the Cartesian equation?

$4x + 5y + 20 = 0$

$y = \frac{-4x - 20}{5}$

$y = -\frac{4}{5}x - 4 \qquad m = \frac{-4}{5} = \frac{4}{-5}$

$Ax + By + C = 0$

$\vec{d} = [-B, A]$

$\vec{d} = [5,-4] = [-5,4]$

b) Determine a vector equation of this line. How does the direction vector relate to the slope?

y int ? $\qquad \vec{m_1}$

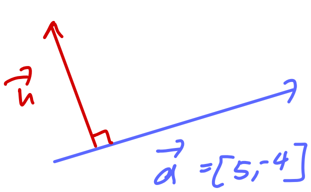
$\vec{r} = [0,-4] + t[5,-4] \ , \ t \in R$

↳ direction

$\vec{d} = [d_1, d_2]$

$m = \frac{d_2}{d_1}$

c) Determine a position vector that is perpendicular to the line (e.g. a normal vector). How does the normal vector compare to the Cartesian equation?



$\vec{n} \cdot \vec{m} = 0$

$[x,y] \cdot [5,-4] = 0$

$5x - 4y = 0$

$y = \frac{-5x}{-4}$

$y = \frac{5}{4}x + 0 \leftarrow (0,0)$

Cartesian Equation: $4x + 5y + 20 = 0$

$\therefore \vec{n} = [A,B]$

$\vec{n} = [4,5]$

↳ position vector

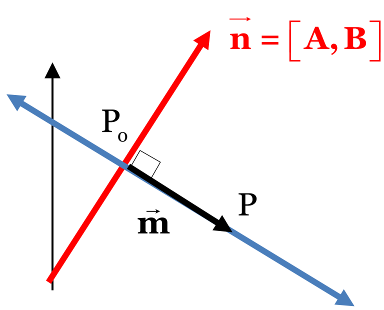
$By = -Ax - C$ $\quad m = \frac{-A}{B}$ $\quad \vec{d} = [B, -A]$

$y = \frac{-Ax - C}{B}$

NOTE: For a line with equation $Ax + By + C = 0$,

- the slope of the line is $\underline{\frac{-A}{B} \text{ or } \frac{A}{-B}}$ and a direction vector $\vec{m}$ is therefore $\underline{[B, -A] \text{ or } [-B, A]}$.
- the normal vector is $\vec{n} = \underline{[A, B]}$.

For a line in R^2 :



$$\overrightarrow{m} = \overrightarrow{P_oP} = [x - x_o, y - y_o]$$
$$\vec{n} = [A, B]$$
$$\therefore \vec{n} \bullet \overrightarrow{m} = 0 \text{ (perpendicular)}$$
$$[A, B] \bullet [x - x_o, y - y_o] = 0$$
$$Ax + By + C = 0 \quad \text{(Cartesian Equation of a Line)}$$

You Try!

Determine equivalent vector, parametric, symmetric and Cartesian equations of the line $y = \frac{3}{4}x + 2$.

$4y = 3x + 8$

$0 = 3x - 4y + 8$

Cartesian: $3x - 4y + 8 = 0$

vector equation: $[x, y] = [0, 2] + t[4, 3]$, $t \in R$

parametric equations: $\begin{cases} x = 4t \\ y = 2 + 3t \end{cases}$, $t \in R$

note! Cross multiply

sym $\frac{x}{4} = \frac{y-2}{3}$

$3x = 4(y-2)$

$3x = 4y - 8$

$3x - 4y + 8 = 0$

Cartesian

Symmetric Equation: $\frac{x-0}{4} = \frac{y-2}{3}$

or

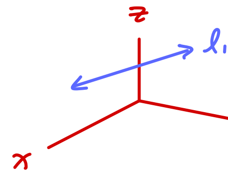
$\frac{x}{4} = \frac{y-2}{3}$

or

$\frac{x}{4} = \frac{2-y}{-3}$

Variations

7-1 B Equations of Lines in R^3



As in R^2, a direction vector and a position vector to a known point on a line are all that are needed to define a line in R^3.

The Vector Equation of a Line in R^3

$\vec{r} = \vec{r}_0 + t\vec{m}$ OR $[x, y, z] = [x_0, y_0, z_0] + t[m_1, m_2, m_3]$

where

- $t \in R$ is a parameter
- $\vec{r} = [x, y, z]$ is a position vector to any unknown point on the line
- $\vec{r}_0 = [x_0, y_0, z_0]$ is a position vector to any known point on the line
- $\vec{m} = [m_1, m_2, m_3]$ a direction vector parallel to the line

The Parametric form of the Equation of a Line in R^3

For a line with equation $[x, y, z] = [x_0, y_0, z_0] + t[m_1, m_2, m_3]$, the parametric equations are

$$\begin{cases} x = x_0 + tm_1 \\ y = y_0 + tm_2 \\ z = z_0 + tm_3 \end{cases}$$ where $t \in R$

Overall, the various new forms of lines in R^2 can be extended to lines in R^3.

Comparison of equations of a line in R^2 and R^3

	Equation of a line in R^2	Equation of a line in R^3
Scalar or Cartesian	$Ax + By + C = 0$	
Vector	$[x, y] = [x_0, y_0] + t[m_1, m_2]$	$[x, y, z] = [x_0, y_0, z_0] + t[m_1, m_2, m_3]$
Parametric	$\begin{cases} x = x_0 + tm_1 \\ y = y_0 + tm_2 \end{cases}$, $t \in \mathbb{R}$	$\begin{cases} x = x_0 + tm_1 \\ y = y_0 + tm_2 \\ z = z_0 + tm_3 \end{cases}$, $t \in \mathbb{R}$
Symmetric	$\dfrac{x - x_0}{m_1} = \dfrac{y - y_0}{m_2}$ where $m_1, m_2 \neq 0$	$\dfrac{x - x_0}{m_1} = \dfrac{y - y_0}{m_2} = \dfrac{z - z_0}{m_3}$ where $m_1, m_2, m_3 \neq 0$

Scalar Equation is not a line but a plane in $\mathbb{R}^3$

Note! $Ax + By + Cz + D = 0 \Rightarrow$ The Scalar Equation of a Plane in $\mathbb{R}^3$

Example 6: A line passes through points A(2, -2, 5) and B(0, 6, -5).

a) Write a vector equation for the line.
b) Write parametric equations for the line.
c) Write symmetric equations for the line.
d) Determine if the point C(0, -10, 9) lies on the line.

a) $\vec{d} = \overrightarrow{AB} = [-2, 8, -10]$

$[x, y, z] = [2, -2, 5] + t[-2, 8, -10], \quad t \in \mathbb{R}$

b) $\begin{cases} x = 2 - 2t \\ y = -2 + 8t \\ z = 5 - 10t \end{cases}, \quad t \in \mathbb{R}$

c) $\dfrac{x-2}{-2} = \dfrac{y+2}{8} = \dfrac{z-5}{-10} \Leftarrow$ pt: $(2, -2, 5)$, $\vec{d} = [-2, 8, -10]$

or

$\dfrac{2-x}{2} = \dfrac{y+2}{8} = \dfrac{5-z}{10}$

d) $x = 2 - 2t \Rightarrow 0 = 2 - 2t$

$t_1 = 1$

$y = -2 + 8t \Rightarrow -10 = -2 + 8t$

$-8 = 8t$

$t_2 = -1$

$z = 5 - 10t \Rightarrow 9 = 5 - 10t$

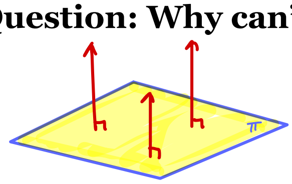
$\dfrac{4}{-10} = t$

$t_3 = -\dfrac{2}{5}$

$t_1 \neq t_2 \neq t_3$

$\therefore$ $(0, -10, 9)$ is not on the line

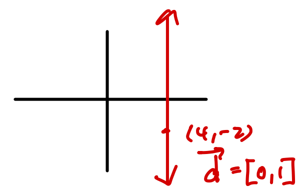
Thinking Question: Why can't a normal vector and a point define a line in R^3?



Because it defines a plane in $\mathbb{R}^3$

7.1 Practice

1. Determine if the following points are on the line ℓ:[−4,3]+t[3,2].
   a) (−1,5)
   b) (−16,−5)
2. For the line defined by $\ell:\begin{cases} x=-3-t \\ y=2+2t \end{cases}$, state the coordinates of
   a) the y-intercept
   b) the x-intercept
   c) the point where x=12
   d) the point where y=38



3. Rewrite each of the equations below into the specified form.
   a) ℓ:[7,2]+t[3,−2] into parametric form
   b) $\ell:\begin{cases} x=32-3t \\ y=26+4t \end{cases}$ into vector form
4. Find the equation of the line and write in the specified form:
   a) the line parallel to $\vec{m}=[2,3]$ that hits the point (1,4), in parametric form.
   b) the line that passes through the points (2,4) and (5,13), in vector form.
   c) the vertical line through (4,−2), in parametric form.

   $\vec{v} = [4,-2] + t[0,1]$

   $\Rightarrow \quad x = 4 \qquad \vec{d} = [0,1] \qquad x = 4 , \ y = -2+t$

   $\begin{cases} x = 4 \\ y = -2+t \end{cases}$

   d) the line with the same x-intercept as ℓ₁:[3,6]+t[1,−2], and the same y-intercept as ℓ₂:[8,4]+s[−1,3], in vector form.

   xint: $(x,0)$ $\quad [x,0] = [3+t, 6-2t]$

   $0 = 6-2t$ $\quad x = 3+(3)$

   $t = 3$ $\quad = 6$

   xint: $(6,0)$

   yint: $(0,y)$ $\quad [0,y] = [8-s, 4+3s]$

   $0 = 8-s$ $\quad y = 4+3(8)$

   $s = 8$ $\quad = 28$

   $\therefore$ yint $(0,28)$

   $\vec{d} = [6,-28]$
   $= [-6,28]$
   $= [3,-14]$

   $\vec{v} = [6,0] + t[-6,28]$

$d_1 = [-1,3,1]$

5. Given the line ℓ:[7,3,1]+t[−1,3,1], determine if the following lines are parallel, perpendicular, or coincident to it.
   a) ℓ₂:[2,−3,4]+t[5,1,2]
   b) ℓ₃: x=1+t , y=21−3t , z=7−t

   $\vec{d_2} = [1,-3,-1]$

   $\vec{d_1} = -d_2$

   $k = -1$

   c) ℓ₄:[5,3,2]+t[−2,6,2]
   d) ℓ₅:[3,7,−2]+t[4,6,1]
6. If the points (4,2,7),(6,19,−4), and (80,b,c) lie on the same straight line, find the values of b and c.
7. Determine the angle between each pair of lines:
   a) ℓ₁:[4,5,−2]+t[3,−1,−1] ℓ₂:[4,5,−2)+s[−2,−3,2]

   $\vec{d_1} \cdot \vec{d_2} = |\vec{d_1}| |\vec{d_2}| \cos\theta$

   $[3,-1,-1] \cdot [-2,-3,2] = \sqrt{9+1+1} \sqrt{4+9+4} \cos\theta$

   $\frac{-6+3+(-2)}{\sqrt{11}\sqrt{17}} = \cos\theta$

   $\theta = 111.4^\circ$

   b) $\ell_1: \frac{x-5}{3} = \frac{y+2}{5} = z-2 \qquad \ell_2: \frac{x-5}{8} = y+2 = \frac{2-z}{3}$
8. Find, in parametric form, the equation of a line perpendicular to both ℓ₁:[3,7,−2]+t[3,−1,−1] and ℓ₂:[8,−3,−3]+t[−2,−3,2] that passes through (5,0,0).
9. Find, if possible, the value(s) of k such that the lines ℓ₁:[9,3,2]+t[3,k,−15] and ℓ₂:[−5,4,−2]+t[10,12,50] are:
   a) parallel
   b) perpendicular
10. Point P_1 lies on the line ℓ₁:[4,4,−3]+t[2,1,−1],t∈ℝ, and point P_2 lies on the line ℓ₂:[−2,−7,2]+s[3,2,−3]. If the vector $\overrightarrow{P_1P_2}$ is perpendicular to both ℓ₁ and ℓ₂, determine the coordinates of P_1 and P_2.

CP11 Hmwk Takeup

8. Find, in parametric form, the equation of a line perpendicular to both ℓ1:[3,7,−2]+t[3,−1,−1] and ℓ2:[8,−3,−3]+t[−2,−3,2] that passes through (5,0,0).
9. Find, if possible, the value(s) of k such that the lines ℓ1:[9,3,2]+t[3,k,−15] and ℓ2:[−5,4,−2]+t[10,12,50] are:
a) parallel
b) perpendicular

#8

$\vec{d_1} = [3,-1,-1]$ point : (5,0,0)

$\vec{d_2} = [-2,-3,2]$

$\vec{d_1} \times d_2 = [-2-3, -(6-2), -9-2]$
$= [-5,-4,-11]$
$= -[5,4,11]$ ⇐ direction vector of the line perpendicular to both l_1 + l_2

$l: \vec{v} = [5,0,0] + t[5,4,11]$ ⇐ vector equation of the line

$\therefore \begin{cases} x = 5+5t \\ y = 4t \\ z = 11t \end{cases}, t \in \mathbb{R}$ ⇐ parametric equations of the line

#9 a) parallel :

$\vec{d_1} = [3, k, -15]$

$\vec{d_2} = [10, 12, 50]$

$\vec{d_1} = m\vec{d_2}$

$[3,k,-15] = m[10,12,50]$

$3 = 10m_1 \qquad k = 5m_2 \qquad -15 = 50m_3$

$\frac{3}{10} = m_1 \qquad \frac{-15}{50} = m_3$

$m_3 = -\frac{3}{10}$

$\because m_1 \neq m_3$

$\therefore$ no possible k-value for l_1 + l_2 to be parallel.

b) Perpendicular :

$\vec{d_1} \bullet \vec{d_2} = 0$

$\vec{d_1} = [3, k, -15]$

$\vec{d_2} = [10,12,50] = 2[5,6,25]$

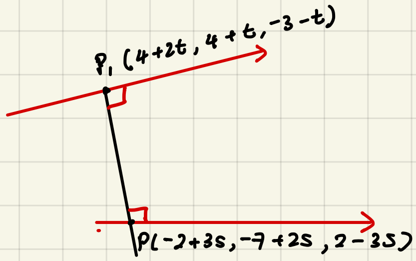
$[3,k,-15] \bullet [5,6,25] = 0$

$3(5) + 6k + (-15)(25) = 0$

$15 + 6k - 375 = 0$

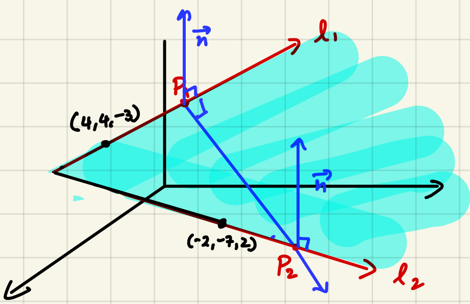
$k = 60$

10. Point P_1 lies on the line ℓ_1:[4,4,−3]+t[2,1,−1],t∈ℝ, and point P_2 lies on the line ℓ_2:[−2,−7,2]+s[3,2,−3]. If the vector $\overrightarrow{P_1P_2}$ is perpendicular to both ℓ_1 and ℓ_2, determine the coordinates of P_1 and P_2.



$\overrightarrow{P_1P_2} = [-6+3s-2t, -11+2s-t, 5-3s+t]$

$\vec{d_1}\cdot\overrightarrow{P_1P_2} = 0$ $\qquad$ $\vec{d_2}\cdot\overrightarrow{P_1P_2} = 0$



$\vec{d_1} = [2,1,-1]$

$\vec{d_2} = [3,2,-3]$

$\vec{n} = \vec{d_1}\times\vec{d_2}$

$= [-3+2, -(-6+3), 4-3]$

$= [-1, 3, 1]$

$\overrightarrow{P_1P_2}\cdot\vec{n} = 0$

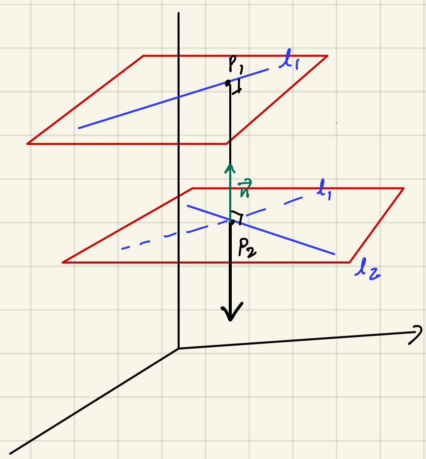
$[-6+3s-2t, -11+2s-t, 5-3s+t]\cdot[-1,3,1] = 0$

$(6-3s+2t) + (-33+6s-3t) + (5-3s+t) = 0$

didn't work

for $\overrightarrow{P_1P_2}$ to be ⊥ to both l_1 + l_2, they must be skew lines

$\overrightarrow{P_1P_2} = k\vec{n}$



$P_1=(4+2t,4+t,-3-t)$ and $P_2=(-2+3s,-7+2s,2-3s)$

$\overrightarrow{P_1P_2}=[-6+3s-2t,-11+2s-t,5-3s+t]$

$\vec{m}_1\times\vec{m}_2=[2,1,-1]\times[3,2,-3]$

$=[-1,3,1]$

$\overrightarrow{P_1P_2}=k(\vec{m}_1\times\vec{m}_2)$

$[-6+3s-2t,-11+2s-t,5-3s+t]=k[-1,3,1]$

$-6+3s-2t=-k \longrightarrow 3s-2t+k=6$ (1)

$-11+2s-t=3k \longrightarrow 2s-t-3k=11$ (2)

$5-3s+t=k \longrightarrow 3s-t+k=5$ (3)

(1)-(3): $-t=-1$

$\boxed{t=1}$

$3s+k=4$ (4)

$2s-3k=10$ (5)

$3\times$(4)+(5): $11s=22$

$\boxed{s=2}$ & $\boxed{k=-2}$

$P_1=(4+2t,4+t,-3-t)$ & $P_2=(-2+3s,-7+2s,2-3s)$

$P_1=(6,5,-4)$ $\qquad$ $P_2=(4,-3,-4)$

Warm Up

1. Write each of the following lines in scalar, vector, parametric, and symmetric form.

Scalar (Cartesian)	Vector	Parametric	Symmetric
$2x + 3y - 6 = 0$	$y = \frac{-2x+6}{3}$ $= -\frac{2}{3}x + 2$ $\vec{v} = [0,2] + t[3,-2]$, $t \in \mathbb{R}$	$\begin{cases} x = 3t \\ y = 2-2t \end{cases}$, $t \in \mathbb{R}$	$\frac{x}{3} = \frac{y-2}{-2}$
* tricky $-3(x-2) = 2(4-2y)$ $-3x+6 = 8-4y$ $3x-4y+2=0$	$\cancel{\vec{v} = [2,4] + t[2,3], t \in \mathbb{R}}$ $\vec{v} = [2,2] + t[4,3]$, $t \in \mathbb{R}$	$\cancel{\begin{cases} x = 2+2t \\ y = -4+3t \end{cases}}$, $t \in \mathbb{R}$ $\begin{cases} x = 2+4t \\ y = 2+3t \end{cases}$, $t \in \mathbb{R}$	$\frac{x-2}{2} = \frac{4-2y}{-3}$ $\frac{x-2}{2} = \frac{2y-4}{3}$ $\frac{x-2}{2} = \frac{y-2}{\frac{3}{2}}$ $\times\frac{1}{2} \Rightarrow \frac{x-2}{4} = \frac{y-2}{3}$
	$\vec{r} = [2,-3,1] + t[1,-1,4]$	$\begin{cases} x = 2+t \\ y = -3-t \\ z = 1+4t \end{cases}$, $t \in \mathbb{R}$	$x-2 = \frac{y+3}{-1} = \frac{z-1}{4}$
	$\vec{v} = [0,-1,2] + t[-2,3,3]$, $t \in \mathbb{R}$	$\begin{cases} x = -2t \\ y = -1+3t \\ z = 3t+2 \end{cases}$, $t \in \mathbb{R}$	$\frac{x}{-2} = \frac{y+1}{3} = \frac{z-2}{3}$

2. Determine the **exact** value(s) of k that would make the following lines intersect at $60°$ angle.

$L_1 : \vec{r} = [17,-16] + t[1,k]$ and $L_2 : \frac{x-25}{1} = \frac{y-3}{-1}$

$\vec{d_1} = [1,k]$ $\quad \vec{d_2} = [1,-1]$

$$\vec{d_1} \cdot \vec{d_2} = |\vec{d_1}||\vec{d_2}| \cos 60°$$

$$1 + (k)(-1) = \sqrt{1+k^2}\sqrt{1+1}\left(\tfrac{1}{2}\right)$$

$$1-k = \sqrt{k^2+1}\,\frac{\sqrt{2}}{2} \Longrightarrow 1-k>0, \quad 1>k, \quad \therefore k<1$$

$$(1-k)^2 = (k^2+1)\left(\tfrac{2}{4}\right)$$

$$1-2k+k^2 = \tfrac{1}{2}(k^2+1)$$

$$2-4k+2k^2 = k^2+1$$

$$k^2-4k+1 = 0$$

$$k = \frac{4 \pm \sqrt{16-4(1)(1)}}{2(1)}$$

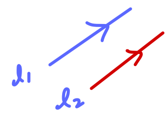
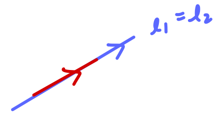
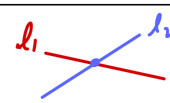
$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= 2 \pm \sqrt{3}, \quad k<1$$

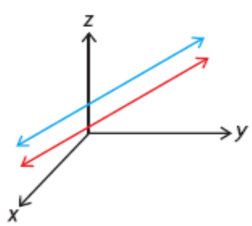
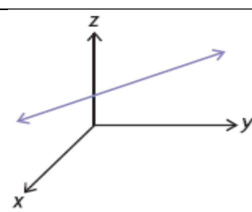
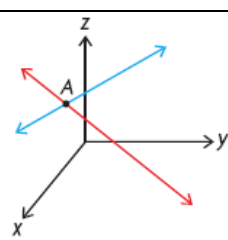
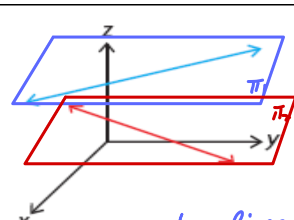
$$\therefore k = 2-\sqrt{3}$$

7-2 The Intersection of Two Lines in R² and R³

In R²

Lines may...	Diagram	Number of solutions
be parallel ↳ distinct $\vec{d_1} = \vec{d_2}$		no solution
coincide (be coincident) ↳ parallel coincident $\vec{d_1} = \vec{d_2}$		infinitely many solutions
intersect $\vec{d_1} \neq \vec{d_2}$		1 solution

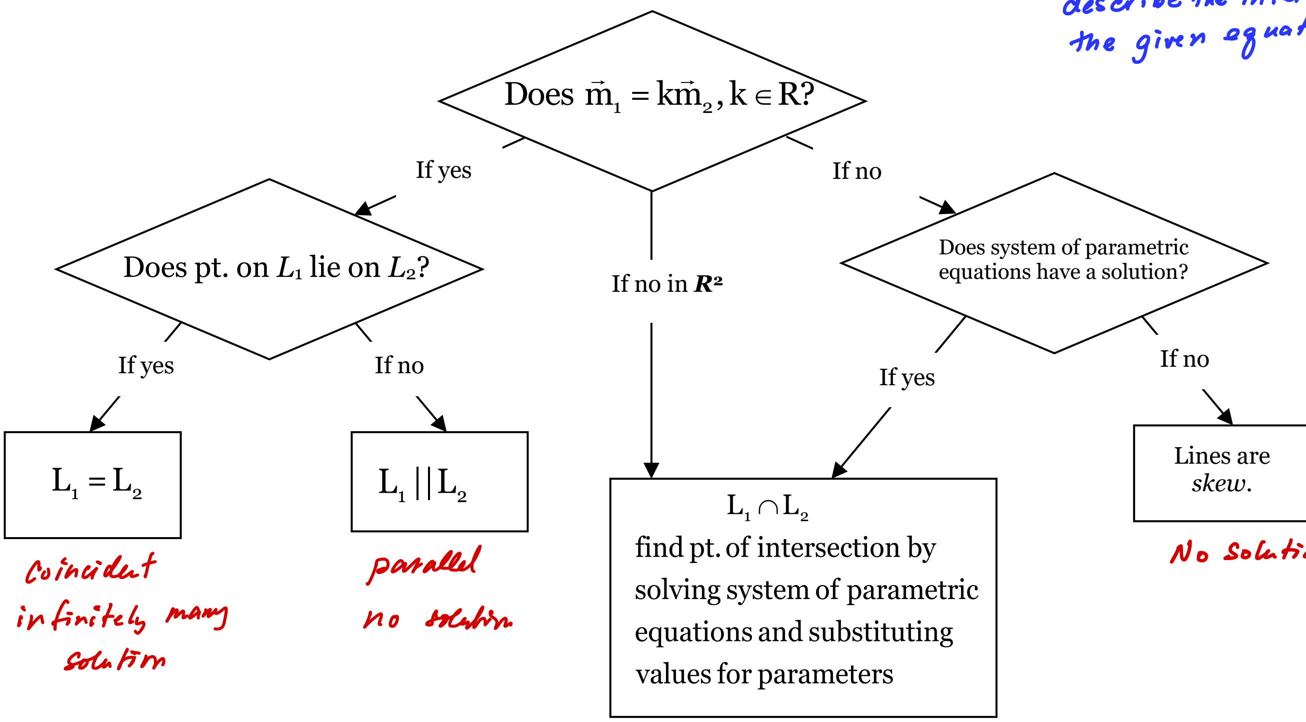
In ***R*³** (the **Notation** is the same as for ***R*²**)

Lines may...	Conditions	Number of solutions
be parallel $\vec{d_1} = \vec{d_2}$		no solution
coincide (be coincident) $\vec{d_1} = \vec{d_2}$		infinitely
intersect (and are therefore coplanar) $\vec{d_1} \neq \vec{d_2}$		1 solution
skew (do not intersect **and** are not parallel) $\vec{d_1} \neq \vec{d_2}$		no solution

skew lines
↳ lines where direction vectors are not collinear and lie on parallel planes

Method for Determining Line Situation

(Analysis of Situation)
describe the interaction between the given equations.



Coincident
infinitely many solution

parallel
no solution

1 Solution

No Solution

Examples for R^2:
Are each of the following pairs of lines parallel, coincident or intersecting? If the lines intersect, find the point of intersection.

1) $L_1: 2x + 3y - 30 = 0$ ①
$L_2: x - 2y + 13 = 0$ ②

$\vec{n}_1 = [2, 3]$ $\therefore \vec{d}_1 = [-3, 2]$
$\vec{n}_2 = [1, -2]$ $\therefore \vec{d}_2 = [2, 1]$

$\vec{d}_1 \neq k\vec{d}_2$
$\therefore$ the lines intersect @ 1 point

Analyze the situation

Find POI:

Solve By Elimination:

$$\begin{array}{lrl} & 2x + 3y - 30 = 0 & ① \\ ② \xrightarrow{\times 2} & 2x - 4y + 26 = 0 & ② \\ \hline (-) & 7y - 56 = 0 & \\ & y = 8 & ③ \end{array}$$

sub ③ into ②

$x - 2(8) + 13 = 0$
$x - 16 + 13 = 0$
$x = 3$

$\therefore l_1$ and l_2 intersect @ a $(3, 8)$

2) $L_1 : [x, y] = [18, -2] + t[3, -2]$

$L_2 : [x, y] = [-5, 4] + s[2, 1]$

$\vec{d_1} = [3, -2]$

$\vec{d_2} = [2, 1]$

$\because \vec{d_1} \neq k\vec{d_2}$

$\therefore$ 1 poi

$l_1: \ x = 18 + 3t$
$\quad y = -2 - 2t$

$l_2: \ x = -5 + 2s$
$\quad y = 4 + s$

$18 + 3t = -5 + 2s$ ① $\quad -2 - 2t = 4 + s$ ②

① $\quad 2s - 3t = 23$

② $\overset{\times 2}{\Rightarrow}$ $\quad 2s + 4t = -12$

$\quad -7t = 35$

$\quad t = -5$ ③

sub ③ into ②

$-2 - 2(-5) = 4 + s$

$4 = s$ ④

sub ③ into l_1

$x = 18 + 3(-5) = 3$

$y = -2 - 2(-5) = 8$

$\therefore$ the solution is $(3, 8)$

3) $L_1 : [x, y] = [1, 3] + t[4, 2]$

$L_2 : \dfrac{x - 2}{2} = y - 1$

$\vec{d_1} = [4, 2]$

$\vec{d_2} = [2, 1]$

$\because \vec{d_1} = 2\vec{d_2}$

$\therefore$ the lines are collinear

check if $(1, 3)$ lies on l_2:

$l_2 : [x, y] = [2, 1] + s[2, 1]$

$1 = 2 + 2s \qquad 3 = 1 + s$

$-\frac{1}{2} = s_1 \qquad 2 = s_2$

$\because s_1 \neq s_2$

$\therefore (1, 3)$ is not l_2

$\therefore$ there is no solution as the lines are parallel-distinct.

Examples for R^3:

Are each of the following pairs of lines parallel, coincident, intersecting or *skew*? If the lines intersect, find the point of intersection.

4) $L_1 : [x, y, z] = [-1, 1, 0] + t[3, 4, -2]$

$L_2 : [x, y, z] = [-1, 0, -7] + s[2, 3, 1]$

$\vec{d_1} = [3, 4, -2]$

$\vec{d_2} = [2, 3, 1]$

$\because \vec{d_1} \neq k\vec{d_2}$

$\therefore$ not collinear

l_1: $x = -1 + 3t$, $y = 1 + 4t$, $z = -2t$

l_2: $x = -1 + 2s$, $y = 3s$, $z = -7 + s$

$\boxed{-1 + 3t = -1 + 2s \quad (1)}$

$\boxed{1 + 4t = 3s \quad (2) \qquad -2t = -7 + s \quad (3)}$

(2) $1 + 4t = 3s$

(3) $\overset{\times 2}{\Rightarrow}$ $14 - 4t = 2s$

(+) $15 = 5s$

$s = 3$ (4)

sub (4) into (2)

$1 + 4t = 3(3)$

$1 + 4t = 9$

$4t = 8$

$t = 2$ (5)

check (4) and (5) $\boxed{\text{satisfies } (1)}$

$L.S = -1 + 3t$

$= -1 + 3(2)$

$= 5$

$R.S = -1 + 2s$

$= -1 + 2(3)$

$= 5$

$\therefore L.S = R.S$

$s = 3$ and $t = 2$ sub l_2

$x = -1 + 2(3) = 5$

$y = 3(3) = 9$

$z = -7 + 3 = -4$

$\therefore$ 2 lines intersect in $\mathbb{R}^3$

@ a point $(5, 9, -4)$

5) $L_1 : [x,y,z] = [2,1,0] + t[1,-1,1]$

$L_2 : [x,y,z] = [3,0,-1] + s[2,3,-1]$

$\vec{d_1} = [1,-1,1]$

$\vec{d_2} = [2,3,-1]$

$\because \vec{d_1} \neq k\vec{d_2}$

$\therefore$ lines are not collinear

$l_1 :$ $x = 2 + t$ $\quad$ $l_2 :$ $x = 3 + 2s$

$y = 1 - t$ $\quad$ $y = 3s$

$z = t$ $\quad$ $z = -1 - s$

$\boxed{2 + t = 3 + 2s \ (1) \qquad 1 - t = 3s \ (2)}$ $\qquad$ $t = -1 - s \ (3)$

$$
\begin{aligned}
(1)\quad & 2 + t = 3 + 2s \\
(2)\quad & \underline{1 - t = 3s} \\
(+)\quad & 3 = 3 + 5s \\
& 0 = 5s \\
& s = 0 \quad (4)
\end{aligned}
$$

Sub (4) into (2)

$1 - t = 3(0)$

$t = 1 \quad (5)$

Check that (4) and (5) satisfy (3)

$L.S = t$
$= 1$

$R.S = -1 - s$
$= -1 - 0$
$= -1$

$\because L.S \neq R.S$

$\therefore$ the two lines will not intersect and are skew lines

6) $L_1 : x = 1 - 2s, y = s, z = -1 - s, s \in R$

$L_2 : \dfrac{x+1}{-2} = \dfrac{1-y}{-1} = z - 2 \Rightarrow \dfrac{x+1}{-2} = \dfrac{y-1}{1} = \dfrac{y-2}{1}$

$l_1: \; x = 1-2s, \; y = s, \; z = -1-s$

$l_2: \; x = -1-2t, \; y = 1+t, \; z = 2+t$

$\vec{d}_1 = [-2, 1, -1]$

$\vec{d}_2 = [-2, 1, 1]$

$\because \vec{d}_1 \neq k\vec{d}_2$

$\therefore$ lines are not collinear

$\boxed{1-2s = -1-2t \;\; ① \qquad s = 1+t \;\; ②}$

$-1-s = 2+t \;\; ③$

② $s = 1+t$

① $-1-s = 2+t$

(+) $-1 = 3+2t$

$-4 = 2t$

$t = -2 \;\; ④$

sub ④ into ②

$s = 1+(-2)$

$s = -1 \;\; ⑤$

Check: $t=-2$ and $s=-1$ in ③

$L.S = -1-s = -1-(-1) = 0$

$RS = 2+t = 2+(-2) = 0$

$l_1: \; x = 1-2(-1) = 3$

$y = -1$

$z = -1-(-1) = 0$

$\therefore$ 2 lines intersect @ a point $(3, -1, 0)$

7.2 Practice

1. Find the value(s) of a and b that make the lines $\begin{array}{l}\ell_1 : \vec{r} = [3,0,-2] + t[3,1,-3] \\ \ell_2 : \vec{r} = [15,4,a] + s[5,b,-5]\end{array}$

   a) Coincident
   b) Parallel and distinct
   c) Intersecting
   d) Skew

2. Determine the parametric equations of a line whose direction vector is perpendicular to the direction vectors of the two lines $\frac{x-4}{3} = \frac{y+1}{5} = \frac{z-4}{2}$ and $\frac{x}{6} = \frac{y-7}{10} = \frac{z+3}{5}$ and passes through the point $(5,0,-2)$.

3. Find the vector equation of the line through the point (8,10,10) that meets the line $\frac{x+8}{-1} = \frac{y-11}{3} = \frac{z-1}{4}$ at 90° angles.

4. Lines $\ell_1 : \vec{r} = [2,1,3] + t[6,-4,-2]; t \in R$ and $\ell_2 : \begin{cases} x - 1 = -s \\ y - 6 = \mathbf{a}s \\ z - 2 = \mathbf{b}s \end{cases}$ are intersecting at point (-1,3,2). What are the possible values of **a** and **b**?

5. Find all values of ***k*** for which the following lines **do not** intersect.

$$l_1 : \begin{cases} x = -1 + 2r \\ y = 3k + r \\ z = 1 + 3r \end{cases} \quad \text{and} \quad l_2 : \vec{r} = [1,0,-2] + t[-2,3,1]$$

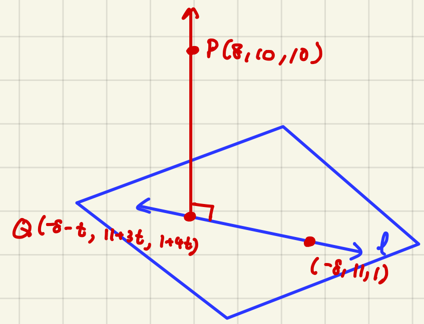
6. Determine the point(s) of intersection between line $x - 2 = t, y + 2 = 3t, 2 + z = 2t, t \in \mathbb{R}$ and sphere with equation $x^2 + y^2 + z^2 = 100$.

7. Determine if the following lines are parallel, skew or intersecting. In the case the lines are intersecting, find the point of intersection.
   L_1: [x, y, z] = [-3, 1, 4] + t[1,-1,-4] and L_2: [x, y, z) = [1,4,6] + s[6,1,7]

8. Determine why the lines $\vec{r} = [1,3,4] + s[2,3,5]$ and $\vec{v}$ = [1,1,1]+ t[2, 2,-2]are not perpendicular.

CP19 #3

3. Find the vector equation of the line through the point (8,10,10) that meets the line $\frac{x+8}{-1}=\frac{y-11}{3}=\frac{z-1}{4}$ at 90^o angles.



$$l: [x, y, z] = [-8, 11, 1] + t[-1, 3, 4]$$

$$\overrightarrow{QP} = [16+t, -1-3t, 9-4t] \qquad \vec{d} = [-1, 3, 4]$$

$$\overrightarrow{QP} \cdot \vec{d} = 0$$

$$[16+t, -1-3t, 9-4t] \cdot [-1, 3, 4] = 0$$

$$(-16-t) + (-3-9t) + (36-16t) = 0$$

$$17 - 26t = 0$$

$$t = \frac{17}{26}$$

$$\overrightarrow{QP} = \left[16+\frac{17}{26}, -1-3\left(\frac{17}{26}\right), 9-4\left(\frac{17}{26}\right)\right]$$

$$= \left[\frac{433}{26}, \frac{-71}{26}, \frac{166}{26}\right] = \frac{1}{26}[433, -71, 166]$$

$\therefore$ the vector equation of the line is:

$$l: \vec{r} = [8, 10, 10] + s[433, -71, 166]$$

Hmwk Take up: CP19

2. Determine the parametric equations of a line whose direction vector is perpendicular to the direction vectors of the two lines $\frac{x-4}{3}=\frac{y+1}{5}=\frac{z-4}{2}$ and $\frac{x}{6}=\frac{y-7}{10}=\frac{z+3}{5}$ and passes through the point (5,0,−2).

l_1 : $\vec{d_1} = [3,5,2]$ point: $(4,-1,4)$

l_2 : $\vec{d_2} = [6,10,5]$ point: $(0,7,-3)$

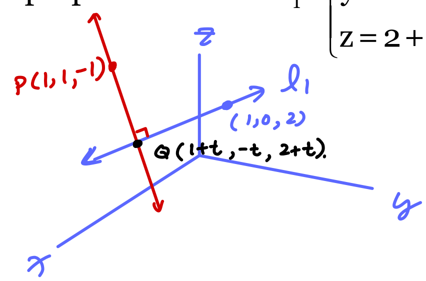
$$\vec{d_1} = [3,5,2]$$
$$\vec{d_2} = [6,10,5]$$
$$\vec{d_1} \times \vec{d_2} = [25-20, -(15-12), 30-30]$$
$$= [5,-3,0]$$

l : $\vec{d} = [5,-3,0]$ point: $(5,0,-2)$

$$\begin{cases} x = 5+5t \\ y = \quad -3t \\ z = -2 \end{cases}, \quad t \in \mathbb{R}$$

Warm Up

1. Determine a **symmetric equation** for a line that is perpendicular to $L_1 : \begin{cases} x = 1+t \\ y = -t \\ z = 2+t, t \in R \end{cases}$ and passes through point P (1, 1, -1).



$l_1: \quad \vec{d}_1 = [1, -1, 1]$

$\vec{d}_2 = \vec{PQ} = [t, -t-1, t+3]$

$\vec{d}_1 \cdot \vec{d}_2 = 0$

$[1,-1,1] \cdot [t, -t-1, t+3] = 0$

$t + (t+1) + (t+3) = 0$

$t = -\frac{4}{3}$

$\therefore \vec{d}_2 = \left[-\frac{4}{3}, \frac{1}{3}, \frac{5}{3}\right] \qquad P(1,1,-1)$

$= \frac{1}{3}[-4, 1, 5]$

$\therefore \frac{x-1}{-4} = y - 1 = \frac{z+1}{5}$

2. Determine if the following lines are parallel, skew or intersecting. In the case the lines are intersecting, find the point of intersection.

L_1: [x, y, z] = [-3, 1,- 4] + t[1,-1,-4] and L_2: [x, y, z] = [1,4,11] + s[6,1,7]

$l_1: \quad x = -3 + t \qquad l_2: \quad x = 1 + 6s$

$\qquad y = 1 - t \qquad\qquad y = 4 + s$

$\qquad z = -4 - 4t \qquad\quad z = 11 + 7s$

$\vec{d}_1 = [1, -1, -4]$

$\vec{d}_2 = [6, 1, 7]$

$\because \vec{d}_1 \neq k\vec{d}_2$

$-4 - 4t = 11 + 7s$ (3) $\therefore$ not collinear

$\boxed{-3 + t = 1 + 6s \text{ (1)} \qquad 1 - t = 4 + s \text{ (2)}}$

(1) + (2) : $-2 = 5 + 7s$

$-7 = 7s$

$s = -1$ (4)

Sub into (2)

$1 - t = 4 + (-1)$

$-t = 2$

$t = -2$

Check $s = -1$ and $t = -2$ into (3)

$L.S = -4 - 4(-2) \qquad R.S = 11 + 7(-1)$

$= 4 \qquad\qquad\qquad = 4$

$L.S = R.S$

$l_1: \quad x = -3 + (-2) = -5$

$y = 1 - (-2) = 3$

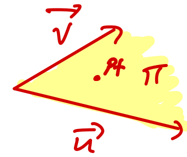
$z = -4 - 4(-2) = 4$

$\therefore$ the two lines will intersect at a point $(-5, 3, 4)$

7-3 Vector, Parametric and Cartesian Equations of a Plane

$\mathbb{R}^3$

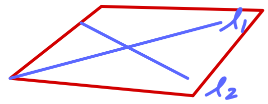
non collinear
2 vectors + a point on the plane defines a plane



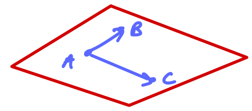
A Plane is:
- a flat surface that extends infinitely far in all directions
- represented by a parallelogram
- denoted by π

How can a plane be formed?

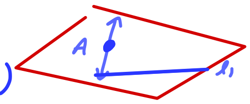
1. intersection of 2 lines



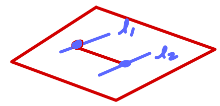
2. 3 points (non-collinear)



3. 1 point and a line (point is not on the line)



4. 2 parallel lines
   distinct



Developing the vector and parametric equations of a plane

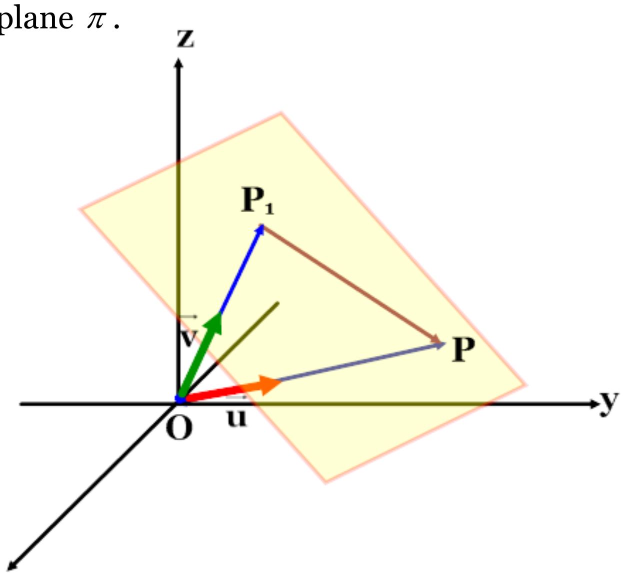
Consider: How can we specify an arbitrary point, $P(x, y, z)$ on a plane, π ?
Recall: Three non-collinear points A,B, and P_1 determine a plane π.
by triangle law of addition :

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P}$$

$$\overrightarrow{P_1P} = s\vec{u} + t\vec{v}$$

$$\therefore \overrightarrow{OP} = \underbrace{\overrightarrow{OP_1}}_{\text{position vector}} + \underbrace{s\vec{u} + t\vec{v}}_{\text{2 directional vectors}}$$

" think a point "

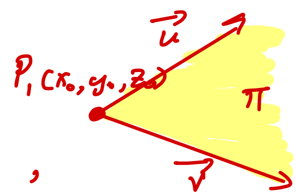


Plane in R^3

Determined by a point $P_1(x_0, y_0, z_0)$ and **two non collinear direction vectors**

$$\vec{u} = [u_1, u_2, u_3] \text{ and } \vec{v} = [v_1, v_2, v_3]$$

Vector Equation



$$[x, y, z] = \underbrace{[x_0, y_0, z_0]}_{\text{Position Vector "point"}} + s\underbrace{[u_1, u_2, u_3]}_{d_1} + t\underbrace{[v_1, v_2, v_3]}_{d_2}, \quad s, t \in \mathbb{R}$$

2 direction vectors and a "point" defines equation a plane.

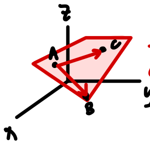
Parametric Equations of a Plane in R^3

$$\begin{cases} x = x_0 + su_1 + tv_1 \\ y = y_0 + su_2 + tv_2 \\ z = z_0 + su_3 + tv_3 \end{cases}, \quad s, t \in \mathbb{R}$$

Note: No symmetrical Equation of Planes (b/c there is 2 parameters)

Examples:

I. Find the vector and parametric equations of the plane that contains the points:
A (1, 0, -3) B (2, -3, 1) C (3, 5, -3)



$\vec{d_1} = \vec{AB}$
$= [1, -3, 4]$

$\vec{d_2} = \vec{AC}$
$= [2, 5, 0]$

vector equation: $[x, y, z] = [1, 0, -3] + s[1, -3, 4] + t[2, 5, 0]$
$s, t \in \mathbb{R}$

parametric Equations:

$$\begin{cases} x = 1 + s + 2t \\ y = -3s + 5t \\ z = -3 + 4s \end{cases}, \quad s, t \in \mathbb{R}$$

<Answer may vary>

II. The plane $\vec{r} = [1,0,0] + t[1,1,1] + s[1,-1,1]; s, t \in R$ contains the point $(3,0,k)$. Find the value of k.

sub (3, 0, k)

$x = 1 + t + s \Rightarrow 3 = 1 + t + s$ ①
$y = t - s \Rightarrow 0 = t - s$ ②
$z = t + s \Rightarrow k = t + s$ ③

solve for s + t

② $\Rightarrow t = s$
sub into ①
$3 = 1 + s + s$
$s = 1$
$t = 1$

$\therefore k = 1 + 1$
$k = 2$

Note! sub ③ into ① for a more efficient solution.
$3 = 1 + (k)$
$k = 2$

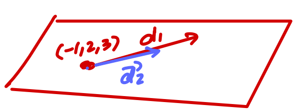
III. What does the following equation represent? Explain.

$$\vec{r} = [-1, 2, 3] + s[-7, 21, 14] + t[2, -6, -4]; t, s \in R$$

$\vec{d_1} = [-7, 21, 14]$
$= -7[1, -3, -2]$

$\vec{d_2} = [2, -6, -4]$
$= 2[1, -3, -2]$

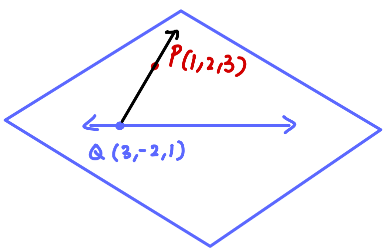
$\because \vec{d_1} = k\vec{d_2}$
$\therefore$ the direction vectors are collinear and



$\because$ the "plane" has the same direction vectors, the equation above actually only defines a line in $\mathbb{R}^3$

$\vec{r} = [-1, 2, 3] + k[1, -3, 2], k \in \mathbb{R}$

IV. Determine the vector equation of the plane containing the line $\begin{cases} x = 3 + 5t \\ y = -2 - 2t \\ z = 1 + 3t \end{cases}$ and the point P(1, 2, 3).



$\vec{d_1} = [5, -2, 3]$

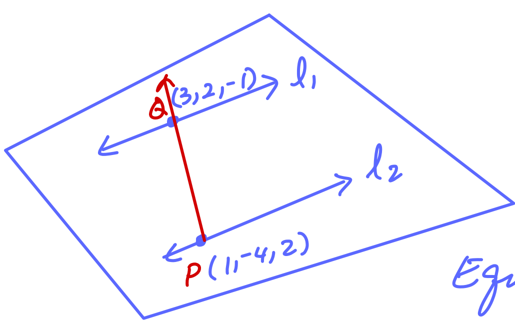
$\vec{d_2} = \vec{QP} = [-2, 4, 2] = 2[-1, 2, 1]$

Equation:

$[x, y, z] = [1, 2, 3] + s[5, -2, 3] + t[-1, 2, 1]$, $s, t \in \mathbb{R}$

<u>Note!</u>
Check that P is not on the given line.

V. Find a vector equation of the plane containing the parallel lines $\begin{matrix} L_1 : \vec{r_1} = [3, 2, -1] + t[1, 2, -1] \\ L_2 : \vec{r_2} = [1, -4, 2] + s[-2, -4, 2] \end{matrix}$.



$\vec{d_1} = [1, 2, -1]$

$\vec{d_2} = \vec{PQ} = [2, 6, -3]$

$\vec{d_1} \neq k\,\vec{d_2}$

Equation:

$\vec{r} = [1, -4, 2] + s[1, 2, -1] + t[2, 6, -3]$, $s, t \in \mathbb{R}$

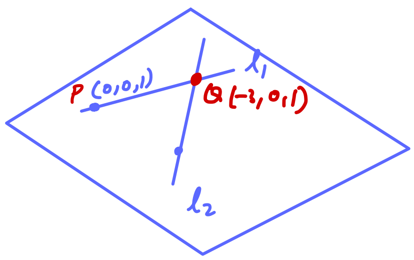
VI. Find the vector equation of the plane π determined by the following intersecting lines.

$$L_1 : \vec{r} = [0,0,1] + s[-1,0,0] \; ; s \in \mathbb{R}$$
$$L_2 : \vec{r} = [-3,0,1] + t[0,0,2] \; ; t \in \mathbb{R}$$

$\vec{d_1} = [-1,0,0]$

$\vec{d_2} = [0,0,2]$

$\vec{d_1} \neq k\,\vec{d_2}$



Equation:

$\vec{r} = [-3,0,1] + s[-1,0,0] + t[0,0,2], \; s,t \in \mathbb{R}$

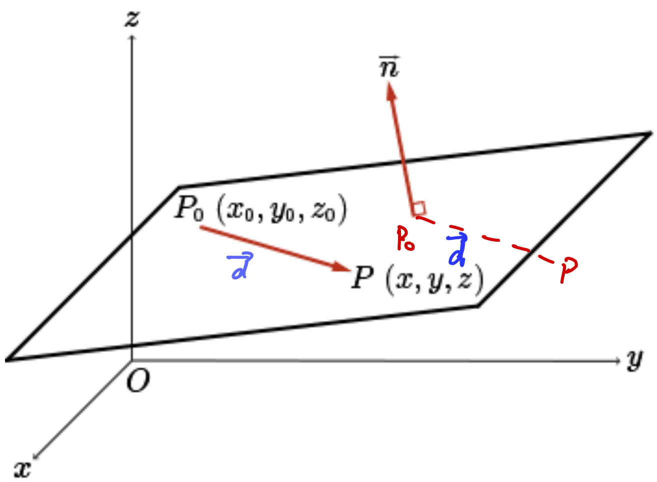
The Cartesian Equation of a Plane

(Scalar Equation)

$Ax + By + Cz + D = 0$

$\vec{n} = [A,B,C]$

For a plane:



If $P_0(x_0, y_0, z_0)$, $P(x, y, z)$ and $\vec{n} = [A,B,C]$ then $\overrightarrow{P_0P} \bullet \vec{n} = 0$, therefore:

$\vec{d} = \overrightarrow{P_0P} = [x - x_0, y - y_0, z - z_0]$

$\vec{n} = [A,B,C]$

$$\vec{d} \bullet \vec{n} = 0$$

$$[x - x_0, y - y_0, z - z_0] \bullet [A,B,C] = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$Ax + By + Cz - [Ax_0 + By_0 + Cz_0] = 0$$

let $D = -(Ax_0 + By_0 + Cz_0)$

$\therefore Ax + By + Cz + D = 0 \Rightarrow$ Scalar / Cartesian Equation of Plane in $\mathbb{R}^3$ $\vec{n} = [A,B,C]$

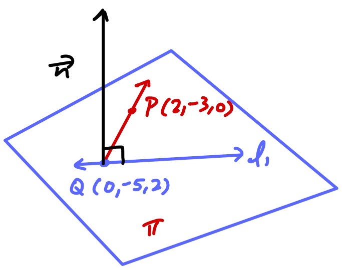
Scalar Equation of a Plane in R^3

$$\frac{Ax + By + Cz + D = 0}{\vec{n} = [A, B, C]}$$

$\Rightarrow$ Note! Scalar / Cartesian Equations are Unique

EXAMPLES

I) A line has vector equation $\vec{r} = [0,-5,2] + s[1,1,-2]$; $s \in R$ and lies on the plane π. The point P(2,-3,0) also lies on the plane π. Determine the Cartesian equation of this plane.



$\vec{d_1} = [1,1,-2]$

$\vec{d_2} = \vec{QP} = [2,2,-2] = 2[1,1,-1]$

$\vec{n} = \vec{d_1} \times \vec{d_2}$

Aside:

$\vec{d_1} = [1,1,-2]$

$\vec{d_2} = [1,1,-1]$

$\vec{d_1} \times \vec{d_2} = [-1+2, -(-1+2), 1-1]$

$\vec{n} = [1,-1,0]$

$\therefore n = [1,-1,0]$

$Ax + By + Cz + D = 0$

$1x + (-1)y + (0)z + D = 0$

$x - y + D = 0$

sub Q(0,-5,2)

$(0) - (-5) + D = 0$

$D = -5$

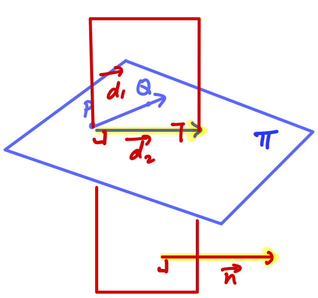
$\therefore \ \pi: x - y - 5 = 0 \Rightarrow$ Equation of π in $\mathbb{R}^3$

II) Determine the scalar equation of the plane that passes through the points P(6, –1, –1) and Q(3, 2, 1) and is perpendicular to the plane x – 4y + 5z + 5 = 0.

$\vec{d_1} = \vec{PQ} = [-3,3,2]$

$\vec{n} = [1,-4,5]$

$\therefore \vec{d_2} = [1,-4,5]$



$\vec{d_1} = [-3,3,2]$

$\vec{d_2} = [1,-4,5]$

$\vec{n} = \vec{d_1} \times \vec{d_2}$

$= [15+8, -(-15-2), 12-(3)]$

$= [23, 17, 9]$

$23x + 17y + 9z + D = 0$

sub Q(3,2,1)

$23(3) + 17(2) + 9(1) + D = 0$

$D = -112$

$\therefore \pi: 23x + 17y + 9z - 112 = 0$

III) Find scalar equation of a plane contains the line $\begin{cases} x = 1 + t, \\ y = 1 - t, \\ z = 2t, t \in R \end{cases}$ and is perpendicular to the plane with equation $x + y + z = 2$.

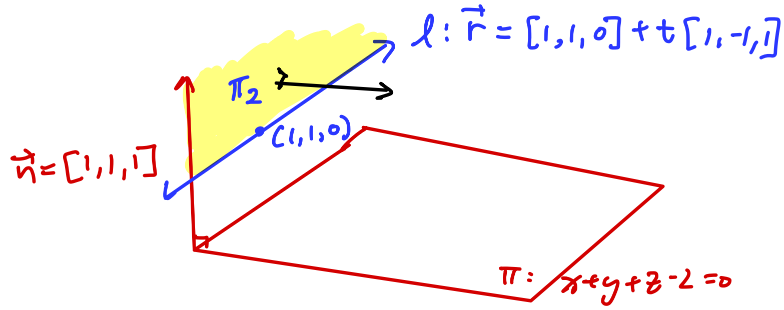
$\vec{d_1} = [1, -1, 2]$ pt: $(1, 1, 0)$

$\vec{d_2} = [1, 1, 1]$

$\vec{n} = \vec{d_1} \times \vec{d_2} = [-1-2, -(1-2), 1+1]$

$= [-3, 1, 2]$

$= -[3, -1, -2]$



$3x - y - 2z + D = 0$

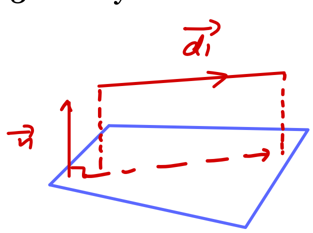
sub $(1, 1, 0)$

$3(1) - (1) - 2(0) + D = 0$

$D = -2$

$\therefore \pi_2: 3x - y - 2z - 2 = 0$

IV) Find possible values of k such that the line [x, y, z] = [3, 4, 7] + t[k, 1, -2] is parallel to the plane 3kx + ky + z - 6 = 0.



$l: \vec{d_1} = [k, 1, -2]$

$\pi: \vec{n} = [3k, k, 1]$

$\vec{d_1} \cdot \vec{n} = 0$

$[k, 1, -2] \cdot [3k, k, 1] = 0$

$3k^2 + k - 2 = 0$

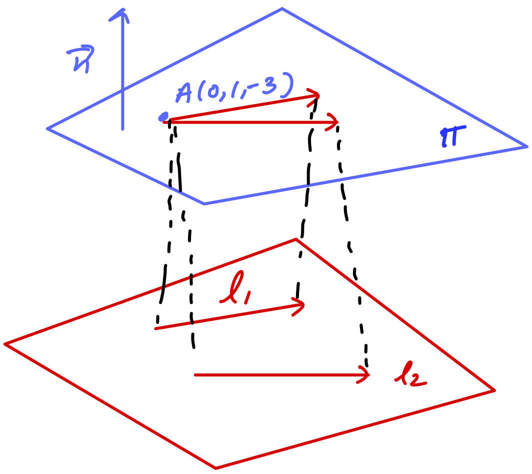
$(3k - 2)(k + 1) = 0$

$\therefore k = \{\frac{2}{3}, -1\}$

Exit Card!

Find a Cartesian equation of the plane which passes through A(0, 1, -3), and is parallel to the lines

$$L_1: \frac{x-1}{5} = \frac{y+2}{-2} = \frac{z+273}{-1} \qquad \text{and} \qquad L_1: \frac{x+3}{-7} = \frac{y-1}{2}, z = 1$$



$\vec{d_1} = [5, -2, -1]$

$\vec{d_2} = [-7, 2, 0]$

$\vec{n} = \vec{d_1} \times \vec{d_2} = [0+2, -(0-7), 10-14]$

$= [2, 7, -4]$

$2x + 7y - 4z + D = 0$

sub A(0,1,-3)

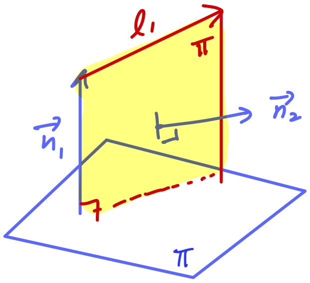
$2(0) + 7(1) - 4(-3) + D = 0$

$D = -19$

$\therefore \pi: 2x + 7y - 4z - 19 = 0$

Warm Up

1. Find the **Cartesian equation** of the plane that is perpendicular to the plane $\vec{r} = [4,-5,2] + s[2,1,3] + t[-1,4,0]$ and intersects it at the line $[x,y,z] = [4,-5,2] + t[1,-1,1]$.



π: $\vec{d}_1 = [2,1,3]$

$\vec{d}_2 = [-1,4,0]$

$\vec{d}_3 = [1,-1,1]$

$\vec{n}_1 = \vec{d}_1 \times \vec{d}_2 = [0-12, -(0+3), 8-(-1)]$

$= [-12, -3, 9]$

$= -3[4, 1, -3]$

$\vec{n}_2 = \vec{n}_1 \times \vec{d}_3$

$= [2, 7, 5]$

Aside

$\vec{n}_1 = [4, 1, -3]$

$\vec{d}_3 = [1, -1, 1]$

$\vec{n}_1 \times \vec{d}_3 = [1-3, -(4+3), -4-1]$

$= [-2, -7, -5]$

$= -[2, 7, 5]$

$2x + 7y + 5z + D = 0$

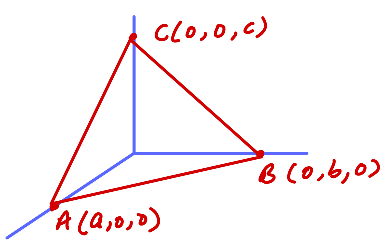
sub $(4,-5,2)$

$2(4) + 7(-5) + 5(2) + D = 0$

$D = 17$

$\therefore 2x + 7y + 5z + 17 = 0$

2. Prove that the scalar equation of the plane that cuts the axes at $A\,(a,0,0)$, $B\,(0,b,0)$, and $C\,(0,0,c)$ is given by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.



$\vec{d}_1 = \vec{AC} = [-a, 0, c]$

$\vec{d}_2 = \vec{AB} = [-a, b, 0]$

$\vec{n} = \vec{d}_1 \times \vec{d}_2$

$= [0-bc, -(0+ac), -ab-0]$

$= [-bc, -ac, -ab]$

$= -[bc, ac, ab]$

$bcx + acy + abz + D = 0$

sub $A(a,0,0)$

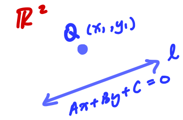
$bc(a) + ac(0) + ab(0) + D = 0$

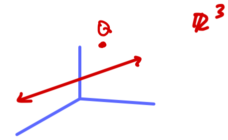
$D = -abc$

$\therefore bcx + acy + abz - abc = 0$

$bcx + acy + abz = abc$

$\times \frac{1}{abc} \Rightarrow \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$





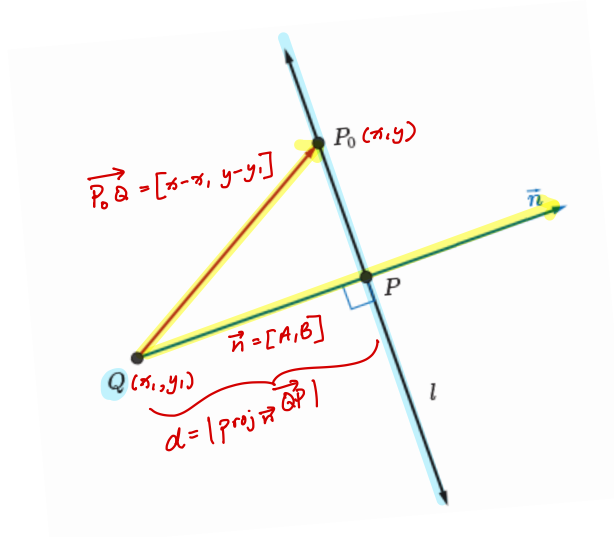
7-4 The Distance between Points and Lines in R^2 and R^3

Distance from a Point to a Line in R^2

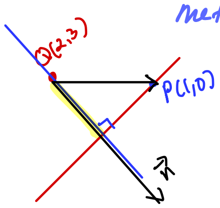
The distance from a point $Q(x_1,y_1)$ to the line $Ax+By+C=0$ is $d=\left|\text{proj}_{\vec n}\overset{\vec{QP}}{\cancel{\vec{PQ}}}\right| = \frac{|\vec{QP}\cdot\vec n|}{|\vec n|} = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$

$C=-Ax-By$

Proof:
$$d=\left|\text{Proj}_{\vec n}\vec{QP}\right|$$
$$=\frac{\vec{QP}\cdot\vec n}{|\vec n|}$$
$$=\frac{\left|[x_1-x,\ y_1-y]\cdot[A,B]\right|}{\sqrt{A^2+B^2}}$$
$$=\frac{|A(x_1-x)+B(y_1-y)|}{\sqrt{A^2+B^2}}$$
$$=\frac{|Ax_1-Ax+By_1-By|}{\sqrt{A^2+B^2}}$$
$$=\frac{|Ax_1+By_1+(\underbrace{\cancel{-Ax-By}}_{C})|}{\sqrt{A^2+B^2}}$$
$$=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$$



Example: Find the distance between point ~~P~~ Q(2,3) and line 3x + 4y -3 = 0



method: $\left|\text{Proj}_{\vec n}\vec{QP}\right|$ $\quad$ $Q(2,3)$ $\quad$ $P(1,0)$ ← a point on the line

$$=\frac{|\vec{QP}\cdot\vec n|}{|\vec n|} \qquad \vec{QP}=[-1,-3] \qquad \vec n=[3,4]$$
$$=\frac{[-1,-3]\cdot[3,4]}{\sqrt{3^2+4^2}}=\frac{|-3-12|}{5}=3$$

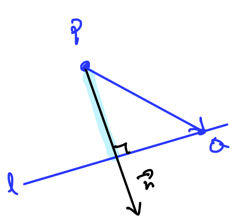
method 2:
$$d=\frac{|Ax+By+C|}{\sqrt{A^2+B^2}}$$
$$=\frac{|3(2)+4(3)-3|}{\sqrt{3^2+4^2}}$$
$$=\frac{15}{5}$$
$$=3$$

Note: If you are given the vector or parametric equations of a line in R^2, convert to the Cartesian equation to use the above formula for finding the shortest distance.

Example: Find the shortest distance from the point P(3, -3) to the line with equation
[x, y] =[4, -2] + t[-7, 4].

method 1: $\quad$ $P(3,-3)$ $\quad$ $Q(4,-2)$

$$\vec{PQ}=[1,1] \qquad \vec d=[-7,4] \qquad \vec n=[4,7]$$
$$\vec d=\left|\text{Proj}_{\vec n}\vec{PQ}\right|$$
$$=\frac{|\vec{PQ}\cdot\vec n|}{|\vec n|}$$
$$=\frac{|[1,1]\cdot[4,7]|}{\sqrt{4^2+7^2}}$$
$$=\frac{|(4)+(7)|}{\sqrt{65}}$$
$$=\frac{11}{\sqrt{65}}$$



method 2 : By Formula

$$d=\frac{|Ax+By+C|}{\sqrt{A^2+B^2}}$$
$$=\frac{|4(3)+7(-3)-2|}{\sqrt{4^2+(-7)^2}}$$
$$=\frac{11}{\sqrt{65}}$$
$$\doteq 1.36 \text{ units}$$

l : $\frac{x-4}{-7}=\frac{y+2}{4}$

$4x-16=-7y-14$

$4x+7y-2=0$

or

$\vec n=[4,7]$

$4x+7y+C=0$

sub $(4,-2)$

$4(4)+7(-2)+C=0$

$2+C=0$

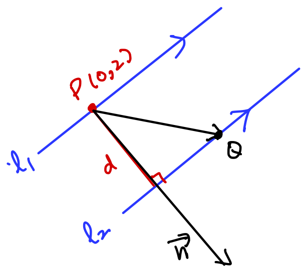
$C=-2$

$\therefore 4x+7y-2=0$

Finding the distance from a point to a line can be extended to finding the distance between parallel lines – take any point on one line and find the distance to the other line. The following example shows how this is done.

$\vec{n}_1 = [3,4]$; y int = 2 $\qquad \vec{n}_2 = [6,8] = 2[3,4]$; y int = $\frac{7}{8}$

Example: Find the distance between the parallel lines L_1: 3x + 4y – 8 = 0 and L_2: 6x + 8y -7 = 0.



$$d = \frac{|6(0)+8(2)-7|}{\sqrt{6^2+8^2}} = \frac{|9|}{\sqrt{100}} = \frac{9}{10} \text{ units}$$

$\vec{n}_2 = 2\vec{n}_1$

$C_1 = -8 \quad C_2 = -7$

$\because C_2 \neq 2C_1$

$\therefore$ 2 parallel distinct lines

P(0,2) is on L_1

$Q(0,\frac{7}{8})$ is on L_2

Method 2: P(0,2) $\quad Q(0,\frac{7}{8})$

$$d = |Proj_{\vec{n}}\vec{PQ}| = \frac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|} = \frac{|0(3)+(-\frac{9}{8})(4)|}{\sqrt{3^2+4^2}} = \frac{|-\frac{9}{2}|}{5} = \frac{9}{10} \text{ units}$$

$\vec{PQ} = [0, \frac{-9}{8}]$

$\vec{n} = [3,4]$

Distance from a Point to a Line in R^3

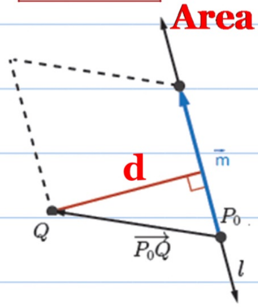
$\Rightarrow$ Recall! There's no Scalar Equation of Lines in $\mathbb{R}^3$ as the Scalar Equation in $\mathbb{R}^3$ is a plane! [So the 'formula' fails with point + line in $\mathbb{R}^3$]

The distance from a point Q to the line $\vec{r} = \vec{r}_0 + t\vec{m}$ is $d = \dfrac{|\vec{m}\times\vec{QP_0}|}{|\vec{m}|}$ where P_0 is a known point on the line.

Proof:

Distance from a point to a line in R^3

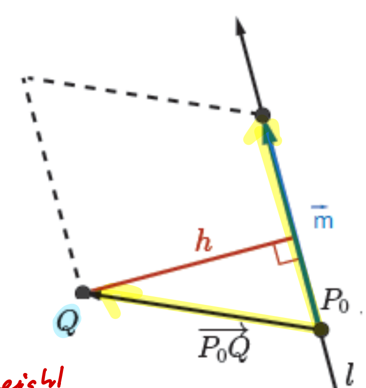
The distance from a point Q to the line $\vec{r} = \vec{r}_0 + t\vec{m}$ is $\boxed{d = \dfrac{|\vec{P_oQ}\times\vec{m}|}{|\vec{m}|}}$ where P_o is a known point on the line.



$$\text{Area} = |\vec{P_oQ}\times\vec{m}| = (\text{base})(\text{height})$$

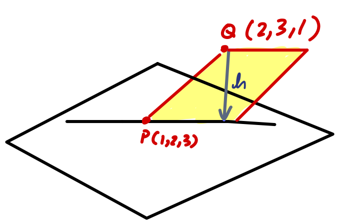
$$|\vec{P_oQ}\times\vec{m}| = |\vec{m}|d$$

$$d = \frac{|\vec{P_oQ}\times\vec{m}|}{|\vec{m}|} \Rightarrow \frac{\text{Area of Parallelogram}}{\text{base}} = \text{height}$$



Recall! Area of Parallelogram $= |\vec{P_oQ}\times\vec{m}|$

Example1: Find the distance between the point Q(2, 3, 1) and the line $[x,y,z]=[1,2,3]+t[0,-3,4]$.



$\vec{PQ} = [1,1,-2]$

$\vec{d} = [0,-3,4]$

$\vec{PQ}\times\vec{d} = [4-6, -(4-0), -3-0] = [-2,-4,-3]$

height (distance)

$$d = \frac{|\vec{PQ}\times\vec{d}|}{|\vec{d}|}$$

$\leftarrow$ Area of Parallelogram

$\leftarrow$ Base of Parallelogram

$$= \frac{\sqrt{4+16+9}}{\sqrt{9+16}} = \frac{\sqrt{29}}{5} \doteq 1.08 \text{ units}$$

Example2: The shortest distance between the point (1,3,2) and the line $\frac{x-1}{-1}=\frac{y-3}{1}=\frac{z-k}{2}$ is $\sqrt{3}$ units. Find the value(s) of k.

$Q(1,3,2)$ $\quad P(1,3,k)$

$\overrightarrow{PQ} = [0,0,2-k]$

$\vec{d_1} = [-1,1,2]$

$\overrightarrow{PQ}\times\vec{d_1} = [-2+k, -(-2+k), 0]$

$= [-2+k, 2-k, 0]$

$|\overrightarrow{PQ}\times\vec{d_1}| = \sqrt{(-2+k)^2+(2-k)^2}$

$$d = \frac{|\overrightarrow{PQ}\times\vec{d_1}|}{|\vec{d_1}|}$$

$$\sqrt{3} = \frac{\sqrt{(-2+k)^2+(2-k)^2}}{\sqrt{1+1+4}}$$

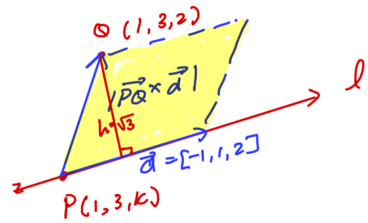
$$\sqrt{18} = \sqrt{2(2-k)^2}$$

$$18 = 2(2-k)^2$$

$$9 = (2-k)^2$$

$$\pm 3 = 2-k$$

$$k = 2\mp 3 = \{-1, 5\}$$

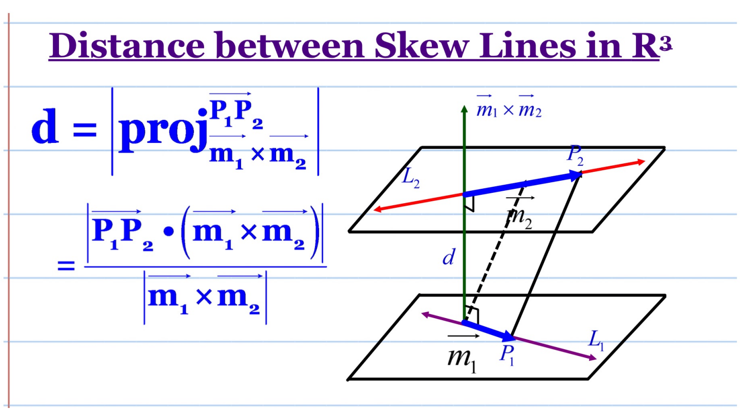


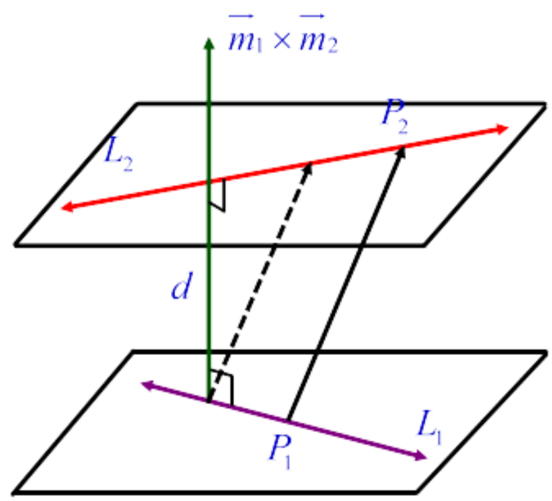
Distance between Skew Lines in R3

Let P_1 be a point on L_1 with direction vector $\vec{m}_1$ and let P_2 be a point on L_2 with direction vector $\vec{m}_2$ such that L_1 and L_2 are skew lines. The shortest distance between these skew lines is $d=\frac{|\overrightarrow{P_1P_2}\bullet(\vec{m}_1\times\vec{m}_2)|}{|\vec{m}_1\times\vec{m}_2|}$ $= \frac{|\overrightarrow{PQ}\bullet\vec{n}|}{|\vec{n}|}$

$= |Proj_{\vec{n}}\overrightarrow{PQ}|$

Proof:



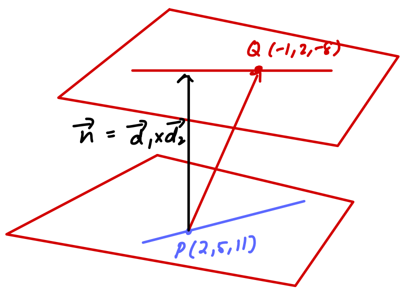


$P(2,5,11) \quad Q(-1,2,-8)$

Example1 : Find the distance between the following skew lines

$[x,y,z]=[-1,2,-8]+t[-3,1,2]$

$[x,y,z]=[2,5,11]+s[1,2,3]$.



$\vec{PQ} = [-3,-3,-19]$

$\vec{d_1} = [-3,1,2]$

$\vec{d_2} = [1,2,3]$

$\vec{d_1} \times \vec{d_2} = \vec{n} = [3-4, -(-9-2), -6-1]$

$= [-1, 11, -7]$

$$d = \frac{|\mathrm{Proj}_{\vec{n}} \vec{PQ}|}{|\vec{n}|}$$

$$= \frac{|\vec{n} \bullet \vec{PQ}|}{|\vec{n}|}$$

$$= \frac{|[-1,11,-7] \bullet [-3,-3,-19]|}{\sqrt{1+121+49}}$$

$$= \frac{|3-33+133|}{\sqrt{171}}$$

$$= \frac{103}{3\sqrt{19}}$$

method 2 : $\Rightarrow$ Note! Shortest distance from Point to Line in R^3 (same as distance from a point to a plane)

$$d = \frac{|Ax+By+Cz+D|}{\sqrt{A^2+B^2+C^2}}$$

$\vec{n} = [1,-11,7]$

$x - 11y + 7z + D = 0$

Sub (2,5,11)

$(2) - 11(5) + 7(11) + D = 0$

$D = -24$

$x - 11y + 7z - 24 = 0$

$$= \frac{|x-11y+7z-24|}{\sqrt{1+121+49}}$$

Sub Q (-1,2,-8)

$$= \frac{|-1-11(2)+7(-8)-24|}{\sqrt{171}}$$

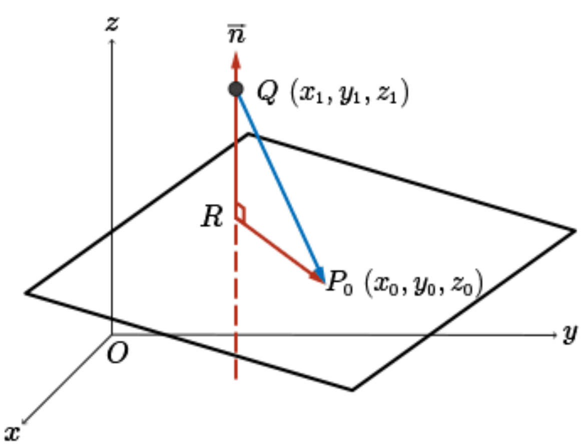
$$= \frac{|-103|}{\sqrt{171}} = \frac{103}{3\sqrt{19}}$$

The Distance between a Point and a Plane in R3

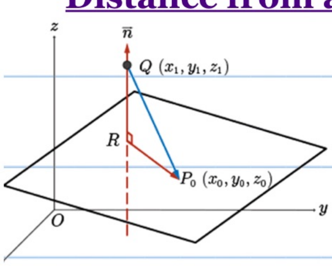
Distance from a Point to a Plane

The distance from a point $Q(x_1, y_1, z_1)$ to the plane $Ax+By+Cz+D=0$ is $d = \left|\mathrm{proj}_{\vec{n}}^{\vec{P_0Q}}\right|$.

Proof:



Distance from a Point to a Plane



$$d = \left|\mathrm{Proj}_{\vec{n}} \vec{P_oQ}\right| = \frac{\left|\vec{P_oQ} \bullet \vec{n}\right|}{|\vec{n}|}$$

$$\vec{P_oQ} \bullet \vec{n} = [x_1 - x_0, y_1 - y_0, z_1 - z_0] \bullet [A,B,C]$$

$$= A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)$$

$$= (Ax_1 + By_1 + Cz_1) - \underbrace{\cancel{(Ax_0 + By_0 + Cz_0)}}^{-D}$$

$$= Ax_1 + By_1 + Cz_1 + D$$

$$\boxed{d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}}$$

Example: Find the shortest distance between the plane $4x + 2y + z - 16 = 0$ and each point.

a) P(10,3,-8)

$$d = \frac{|4x + 2y + z - 16|}{\sqrt{16+4+1}}$$

$$= \frac{|4(10) + 2(3) + 1(-8) - 16|}{\sqrt{21}}$$

$$= \frac{22}{\sqrt{21}}$$

$$\doteq 4.80 \text{ units}$$

b) B(2,2,4)

$$d = \frac{|4(2) + 2(2) + 4 - 16|}{\sqrt{16+4+1}}$$

$$= \frac{0}{\sqrt{21}}$$

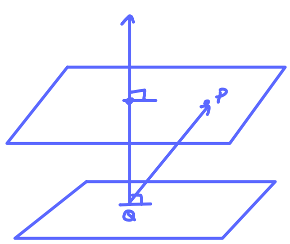
$$= 0$$

$\therefore$ B(2,2,4) sits on the plane "

Distance between Two Parallel Planes

Distance between two parallel planes with equations $Ax + By + Cz + D_1 = 0$, $Ax + By + Cz + D_2 = 0$ is: $\boxed{d = \dfrac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}}$

Example: Find the shortest distance between the planes with equations $3x - 4y + 5z - 10 = 0$ and $6x - 8y + 10z - 10 = 0$.



① $\Rightarrow 3x - 4y + 5z - 10 = 0$

② $\xRightarrow{\times \frac{1}{2}} 3x - 4y + 5z - 5 = 0$

$D_1 - D_2 = -5$

$\vec{n_1} = [3, -4, 5]$

$\vec{n_2} = [6, -8, 10]$

$= 2[3, -4, 5]$

π_1 is parallel to π_2

$$d = \frac{|-10 - (-5)|}{\sqrt{3^2 + (-4)^2 + (5)^2}}$$

$$= \frac{5}{\sqrt{50}}$$

$$= \frac{5}{5\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

method 2:

Check: let P be a point on π_1

let Q " " " " π_2

P(0,0,2) $\quad \vec{PQ} = [0,0,-1]$

Q(0,0,1) $\quad \vec{n} = [3,-4,5]$

$$d = |\text{Proj}_{\vec{n}} PQ|$$

$$= \frac{\vec{PQ} \cdot \vec{n}}{|\vec{n}|}$$

$$= \frac{|[0,0,-1] \cdot [3,-4,5]|}{\sqrt{9+16+25}}$$

$$= \frac{|(-1)(5)|}{\sqrt{50}}$$

$$= \frac{5}{5\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

Exit Card!

1) Determine the value(s) of k for which the planes $x - 2y + 2z + k = 0$ and $x - 2y + 2z = k - 2$ have a distance of 8 units.

$\pi_1 : x - 2y + 2z + k = 0$

$\pi_2 : x - 2y + 2z - (k-2) = 0$

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2+B^2+C^2}} \qquad \vec{n} = [1,-2,2]$$

$$d = \frac{|K + (k-2)|}{\sqrt{1+4+4}}$$

$$8 = \frac{|2k-2|}{3}$$

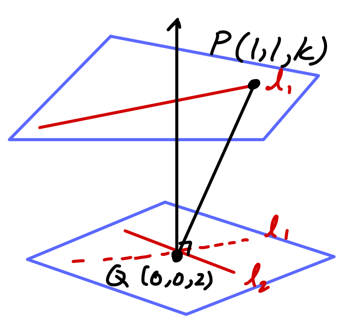
$$24 = |2k-2|$$

$2k-2 = 24$ or $2k-2 = -24$

$k = 13$ $\quad$ $k = -11$

2) The shortest distance between the skew lines $[x,y,z] = [1,1,K] + t[2,-1,1]; t \in R$ and $[x,y,z] = [0,0,2] + s[3,1,2]; s \in R$ is $\sqrt{35}$ units.

Find the value(s) of K.



$\vec{d_1} = [2,-1,1]$

$\vec{d_2} = [3,1,2]$

$\vec{n} : \vec{d_1} \times \vec{d_2} = [-2-1, -(4-3), 2+3]$

$\vec{n} = [-3,-1,5] = -[3,1,-5]$

$\vec{QP} = [1,1,k-2]$

$$d = |Proj_{\vec{n}} \vec{QP}|$$

$$= \frac{|\vec{QP} \cdot \vec{n}|}{|\vec{n}|}$$

$$\sqrt{35} = \frac{|[1,1,k-2] \cdot [-3,-1,5]|}{\sqrt{9+1+25}}$$

$$\sqrt{35} = \frac{|-3-1+5(k-2)|}{\sqrt{35}}$$

$$35 = |5k-14|$$

$5k-14 = -35$ or $5k-14 = 35$

$k = \frac{-21}{5}$ $\quad$ $k = \frac{49}{5}$

method 2:

$$d = \frac{|Ax+By+Cz+D|}{\sqrt{A^2+B^2+C^2}}$$

$\pi_2 : \vec{n} = [3,1,-5]$

$3x + y - 5z + D = 0$

sub(0,0,2)

$3(0)+(0)-5(2)+D = 0$

$D = 10$

$3x + y - 5z + D = 0$

$$d = \frac{|3(1)+(1)-5k+10|}{\sqrt{9+1+25}}$$

$$\sqrt{35} = \frac{|14-5k|}{\sqrt{35}}$$

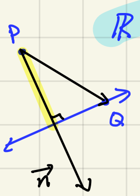
$$35 = |14-5k|$$

$14-5k = -35$ or $14-5k = 35$

$k = \frac{49}{5}$ $\quad$ $k = \frac{-21}{5}$

Summary :

Distance From a Point to a Line.



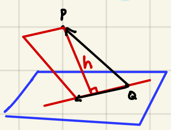
$\mathbb{R}^2$:

$$d = \frac{|Ax + By + C|}{\sqrt{A^2+B^2}} = |Proj_{\vec{n}} \vec{PQ}| = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

$l: Ax+By+C=0$

point : $P(x,y)$

$\mathbb{R}^3$: $d = \frac{|\vec{QP} \times \vec{d}|}{|\vec{d}|}$ ← $\frac{\text{Area of Parallelogram}}{\text{base}}$ = height

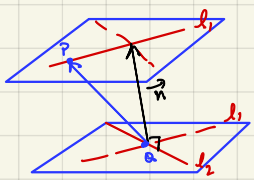


skew lines :

$$|Proj_{\vec{n}} \vec{PQ}| = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

or

$$\vec{n} = \vec{d_1} \times \vec{d_2} \qquad = \frac{|\vec{PQ} \cdot \vec{d_1} \times \vec{d_2}|}{|\vec{d_1} \times \vec{d_2}|}$$

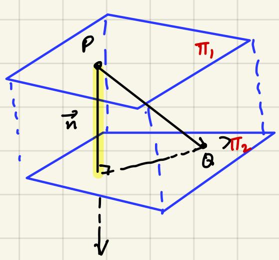


distance from Point to a Plane (Same as distance of skew lines)

$\pi : Ax + By + Cz + D = 0 \; ; \; P(x,y,z)$

$$d = \frac{|Ax + By + Cz + D|}{\sqrt{A^2+B^2+C^2}}$$

Distance between two Parallel Planes



$$d = \frac{|D_1 - D_2|}{\sqrt{A^2+B^2+C^2}}$$

$$= |Proj_{\vec{n}} \vec{PQ}|$$

$$= \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

Warm Up

Find the point(s) on the line $\vec{r} = [1,3,1] + t[1,1,1]$ at distance 3 units from the plane given by equation $x + 2y + 2z + 2 = 0$.

$$[x,y,z] = [1+t,\ 3+t,\ 1+t]$$

$$d = \frac{|Ax + By + Cz + D|}{\sqrt{A^2+B^2+C^2}}$$

$$3 = \frac{|1(1+t) + 2(3+t) + 2(1+t) + 2|}{\sqrt{1+4+4}}$$

$$9 = |11 + 5t|$$

$11 + 5t = -9$ or $11 + 5t = 9$

$t = -4$ $\qquad$ $t = \frac{-2}{5}$

$t = -4:\ (x,y,z) = (1+(-4),\ 3+(-4),\ 1+(-4))$

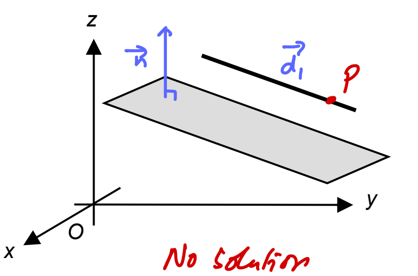
$= (-3, -1, -3)$

$t = \frac{-2}{5}:\ (x,y,z) = (1 + \frac{-2}{5},\ 3 + \frac{-2}{5},\ 1 + \frac{-2}{5})$

$= (\frac{3}{5}, \frac{13}{5}, \frac{3}{5})$

7-5 Intersection of a Line and a Plane

There are three ways for a line to interact with a plane in 3-space.

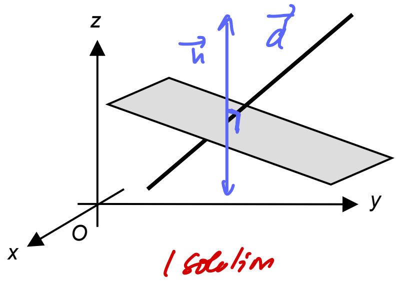


No solution

Line is parallel to the plane

$\vec{d_1} \cdot \vec{n} = 0$

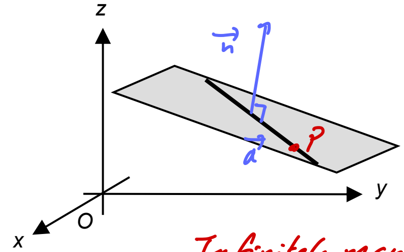
Point P is not on the plane



1 solution

Line intersects the plane

$\vec{d_1} \cdot \vec{n} \neq 0$



Infinitely many

Line lies on the plane

$\vec{d_1} \cdot \vec{n} = 0$

the point P is on the plane

Examples: Determine whether the line and plane intersect. If they do, find the point of intersection.

1. $L: \begin{cases} x = 1 + 2t \\ y = -6 + 3t \\ z = -5 + 2t \end{cases}$ $\qquad \boldsymbol{\pi}$: $4x - 2y + z - 19 = 0$

Analyze: $\vec{d} = [2,3,2]$ $\quad \vec{n} = [4,-2,1]$

$\vec{d} \cdot \vec{n} = [2,3,2] \cdot [4,-2,1]$

$= 8 - 6 + 2$

$\neq 0$

$\therefore$ the line and plane will intersect @ a point

$4(1+2t) - 2(-6+3t) + (-5+2t) - 19 = 0$

$4 + 8t + 12 - 6t - 5 + 2t - 19 = 0$

$4t - 8 = 0$

$t = 2$

poi: $x = 1 + 2(2) = 5$

$y = -6 + 3(2) = 0$

$z = -5 + 2(2) = -1$

$\therefore$ poi: $(5, 0, -1)$

2. $L: \vec{r} = [0,1,-4] + t[2,-1,1]$ $\qquad \boldsymbol{\pi}$: $x + 4y + 2z - 4 = 0$

Analyze: $\vec{d} = [2,-1,1]$ $\quad \vec{n} = [1,4,2]$

$\vec{d} \cdot \vec{n} = [2,-1,1] \cdot [1,4,2]$

$= 2 - 4 + 2$

$= 0$

Check that $P(0,1,-4)$ is on the plane

$L.S = x + 4y + 2z - 4$

$= 0 + 4(1) + 2(-4) - 4$

$= -8$

$\neq R.S$

$\therefore$ no poi

$\therefore$ the line is parallel to the plane and no point of intersection

3. $L: \begin{cases} x = -4 + 3t \\ y = 0 \\ z = t \end{cases}$ π: $x - 2y - 3z + 4 = 0$

$\vec{d} = [3, 0, 1]$ $\vec{n} = [1, 2, -3]$

$P(-4, 0, 0)$

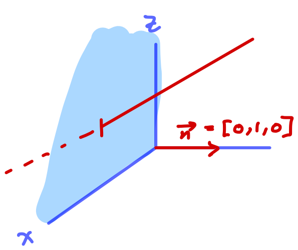
$\vec{d} \cdot \vec{n} = 3 + 0 - 3$
$= 0$

check if $P(-4,0,0)$ is on the π

$L.S = -4 - 2(0) - 3(0) + 4$
$= 0$
$= R.S$

$\therefore$ the line lies on the plane and infinitely poi

4. Where does the line $\vec{r} = [6,10,-1] + t[3,4,-1]$ meet the xz-plane?



xz-plane : $\vec{n} = [0,1,0]$ $\Rightarrow$ direction vector of the y-axis

point : $(0,0,0)$

π: $y = 0$ $\Rightarrow$ scalar equation of the xz-plane

$\begin{cases} Ax + By + Cz + D = 0 \\ 0x + 1y + 0z + D = 0 \\ y + D = 0 \\ \text{sub } (0,0,0) \\ D = 0 \\ \therefore y = 0 \end{cases}$

l: $x = 6 + 3t$
$y = 10 + 4t \Rightarrow$ $y = 0$, $10 + 4t = 0$ $\therefore t = \frac{-5}{2}$
$z = -1 - t$

$\therefore x = 6 + 3\left(\frac{-5}{2}\right) = \frac{-3}{2}$

$z = -1 - \left(\frac{-5}{2}\right) = \frac{3}{2}$

$\therefore$ @ the point $\left(\frac{-3}{2}, 0, \frac{3}{2}\right)$

Note! $\mathbb{R}^3$

$\pi: x = 0$ (yz-plane)

$\pi: y = 0$ (xz-plane)

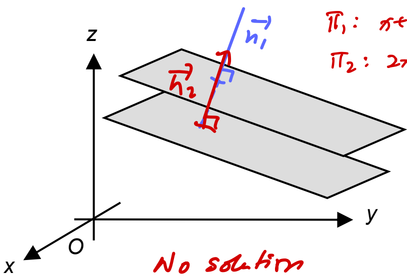
$\pi: z = 0$ (xy-plane)

Intersection of Two Planes

There are three ways for 2 planes to interact in 3-space.

$\pi_1: x+y+z+1=0$

$\pi_2: 2x+2y+2z+5=0$



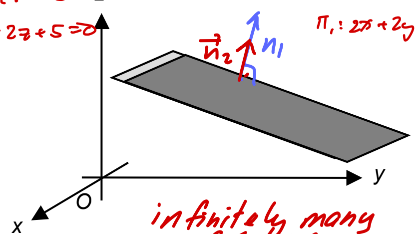
No solution

Parallel and distinct
(no intersection)

$\vec{n_1} = k\vec{n_2}$ $\quad D_1 \neq kD_2$

$\pi_1: x+y+z+1=0$

$\pi_1: 2x+2y+2z+2=0$

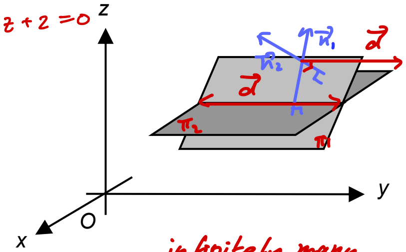


infinitely many solutions.

Parallel and coincident
(intersect at every point)

$\vec{n_1} = k\vec{n_2}$

$D_1 = kD_2$



infinitely many solution (line)

Intersect in a line

$\vec{n_1} \neq k\vec{n_2}$

note! $\vec{n_1} \times \vec{n_2} = \vec{d}$ direction vector of the line of intersection of π_1 + π_2

- The planes are parallel and distinct or coincident **iff** (if and only if) the normals are collinear (scalar multiples).

- The planes intersect **iff** the normals are not collinear.

Examples: Investigate the intersection of the planes and find the equation of the line of intersection if applicable.

1. $\pi_1: 4x-5y-2z-1=0$ ①

   $\pi_2: x-y-2z=0$ ②

Analyze: $\vec{n_1} = [4,-5,-2]$

$\vec{n_2} = [1,-1,-2]$

$\therefore \vec{n_1} \neq k\vec{n_2}$

$\therefore$ the planes will intersect @ a line

Solve by Elimination:

① − ② : $3x-4y-1=0$

let $x=t$, ⇐ introduce a parameter t

$3t-4y-1=0$

$3t-1=4y$

$y = \frac{3t-1}{4}$

sub into ②

$x-y-2z=0$

$(t)-\left(\frac{3t-1}{4}\right)=2z$

$\frac{1}{4}t+\frac{1}{4}=2z$

$\frac{t+1}{8}=z$

$$\begin{cases} x = t \\ y = \frac{3}{4}t - \frac{1}{4} \\ z = \frac{1}{8}t + \frac{1}{8} \end{cases}, \quad t \in \mathbb{R}$$

$\therefore$ the 2 planes intersect @ a line:

$\vec{r} = [0, -\frac{1}{4}, \frac{1}{8}] + t[1, \frac{3}{4}, \frac{1}{8}]$

Note! collinear to $[8,6,1]$

method 2:

let $z=0$, $\quad \pi_1: 4x-5y=1 \Rightarrow 4x-5y=1$

$\pi_2: x-y=0 \Rightarrow 4x-4y=0$

(−) $-y=1$

$y=-1$

$x=-1$ $\therefore P(-1,-1,0)$

$\vec{n_1} \times \vec{n_2} = [10-2, -(-8+2), -4+5]$

$= [8,6,1]$

$\vec{d} = [8,6,1]$

$\vec{r} = [-1,-1,0] + t[8,6,1]$

3. $\pi_1: \begin{cases} x = -4+s \\ y = s+3t \\ z = -s-2t \end{cases}$

vector equation of π_1 $\rightarrow$ $[x,y,z] = [-4,0,0] + s[1,1,-1] + t[0,3,-2]$

$\pi_2: \begin{cases} x = 8+6u+2v \\ y = 3u+5v \\ z = u-v \end{cases}$

vector Equation of π_2 : $[x,y,z] = [8,0,0] + u[6,3,1] + v[2,5,-1]$

π_1: $\vec{d_1} = [1,1,-1]$ pt $(-4,0,0)$

$\vec{d_2} = [0,3,-2]$

$\vec{n_1} = [-2+3, -(-2-0), 3-0]$

$= [1,2,3]$

$x+2y+3z+D=0$

sub $(-4,0,0)$

$-4+D=0$

$D=4$

π: $x+2y+3z+4=0$ ①

π_2: $\vec{d_1} = [6,3,1]$ pt $(8,0,0)$

$\vec{d_2} = [2,5,-1]$

$\vec{n_2} = [-3-5, -(-6-2), 30-6]$

$= [-8, 8, 24]$

$= -8[1,-1,-3]$

$x-y-3z+D=0$

$8 \quad +D=0$

$D=-8$

$\therefore \pi_2: x-y-3z-8=0$ ②

Analyze: $\vec{n_1} \neq k\vec{n_2}$

$\therefore \pi_1$ + π_2 will intersect @ a line

① + ②: $2x+y-4=0$

let $x=t$ ③

$y=-2x+4 \Rightarrow y=-2t+4$ ④

sub ③ and ④ into ① $\Rightarrow z = \dfrac{-x-2y-4}{3} \Rightarrow z = \dfrac{-t-2(-2t+4)-4}{3}$

$= \dfrac{3t-12}{3}$

$= t-4$

$l: \begin{cases} x=t \\ y=-2t+4, \quad t\in\mathbb{R} \\ z=t-4 \end{cases}$

$\therefore$ the 2 planes intersect @ a line

or $\vec{r} = [0,4,-4] + t[1,-2,1]$

method 2:

$\vec{n_1} = [1,2,3]$

$\vec{n_2} = [1,-1,3]$

$\vec{n} \times \vec{n_2} = [-6+3, -(-3-3), -1-2]$

$\therefore \vec{d} = [-3, 6, -3]$

$= -3[1,-2,1]$

sub $z=0$,

π_1 : $x+2y+4=0$ ①

π_2 : $x-y-8=0$ ②

(−) $3y+12=0$

$y=-4$ ③

sub ③ into ②

$x+4-8=0$

$x=4$

point on both panes: $(4,-4,0)$

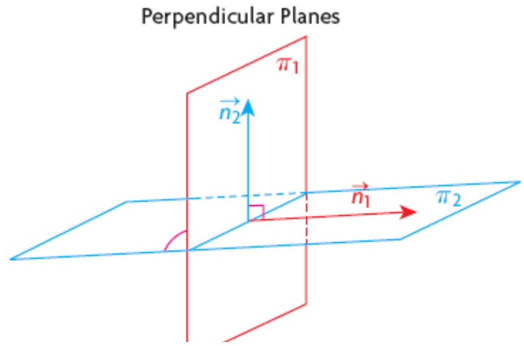
$\therefore$ Two planes intersect @ a line:

$\vec{r} = [4,-4,0] + t[1,-2,1]$

Special Cases

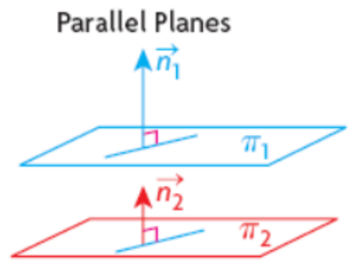
Two planes are perpendicular iff $\vec{n}_1 \cdot \vec{n}_2 = 0$

↳ their normals are perpendicular



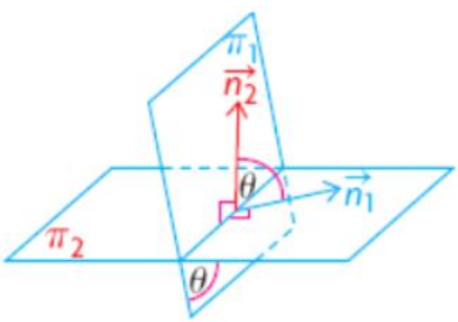
Two planes are parallel iff $\vec{n}_1 = k\vec{n}_2$ $D_1 \neq D_2$

↳ their normals are collinear



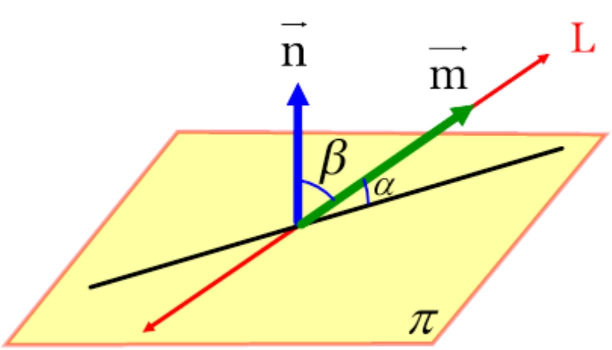
Angle between two planes:

The angle between two planes is the same as angle between their normals!



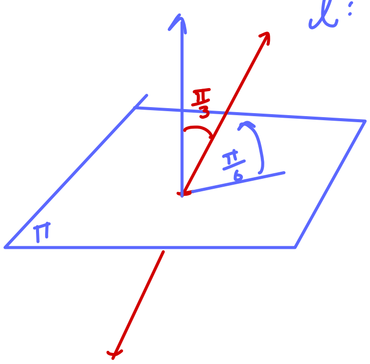
$$\boxed{\cos\theta = \frac{\vec{n}_1 \bullet \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}} \Rightarrow \vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta$$

Angle between a line and a plane (α)



$$\boxed{\begin{array}{l}\cos\beta = \dfrac{\vec{n} \bullet \vec{m}}{|\vec{n}||\vec{m}|} \\ \alpha = 90^\circ - \beta\end{array}}$$

Ex. Line ℓ with equation $\vec{r}=[5,-1,4]+t[2,-2,0]$, $t\in R$ intersects the plane $\pi: \mathbf{a}x+z=5\mathbf{a}+4$ at an angle of $\frac{\pi}{6}$. Find the value of **a**, where **a** is a positive constant



$\ell: \vec{d}=[2,-2,0]$

$=2[1,-1,0]$

D

$\pi: ax+z-(5a-4)=0$

$\vec{n}=[a,0,1]$

$\vec{d}\cdot\vec{n}=|\vec{d}||\vec{n}|\cos\frac{\pi}{3}$

$2(a)+(-2)(0)+(0)(1)=\sqrt{4+4}\sqrt{a^2+1}\left(\frac{1}{2}\right)$

$4a^2=\sqrt{8(a^2+1)}$

$16a^2=8a^2+8$

$8a^2-8=0$

$a^2-1=0$

$a=\pm 1$, $a>0$

$\therefore a=1$

Practice

1. Find vector and parametric equations of the line that is parallel to $L:\begin{cases} x = 5+4t \\ y+2 = -2-2t \\ z = -6+5t \end{cases}$ and passes through the point (0, 6, 0).

2. Given the line $\vec{r} = [12,-8,-4] + t[-3,4,1]$, find the intersection(s) with the *xy* –plane.

3. Find vector and parametric equations of the plane that contains the two intersecting lines $\vec{r}_1 = [3,-1,2] + s[4,0,1], s \in \mathbb{R}$ and $\vec{r}_2 = [3,-1,2] + t[4,0,2], t \in \mathbb{R}$.

4. Find a scalar equation of the plane that contains the origin and the point (2, -3, 2) and is perpendicular to the plane $x+2y-z+3=0$.

5. For what values of *k* will the planes $2x-6y+4z+3=0$ and $3x-9y+6z+k=0$ intersect?

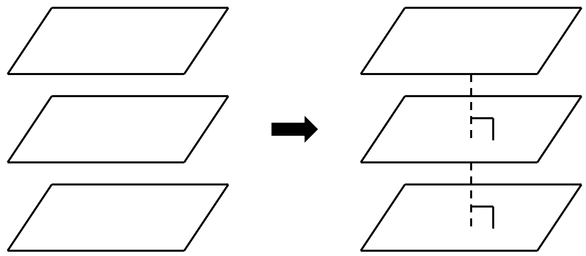
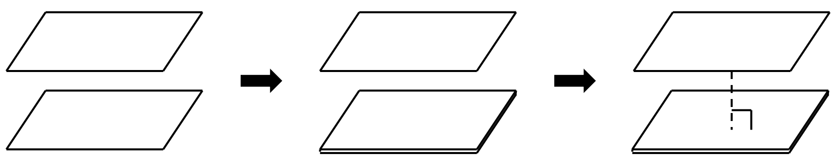
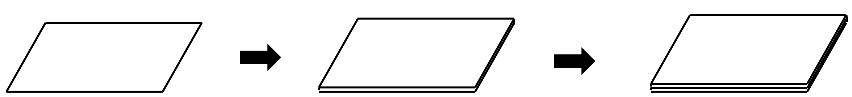
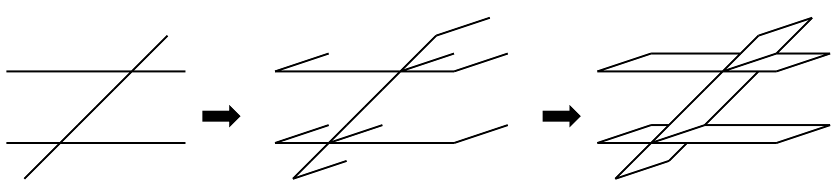
6. Find the equation of the plane that passes through the point (3, 0, -4) and is perpendicular to the line of intersection of the planes $x+2y-7z-3=0$ and $x-5y+4z-1=0$.

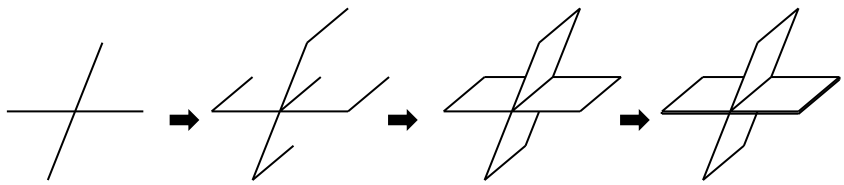
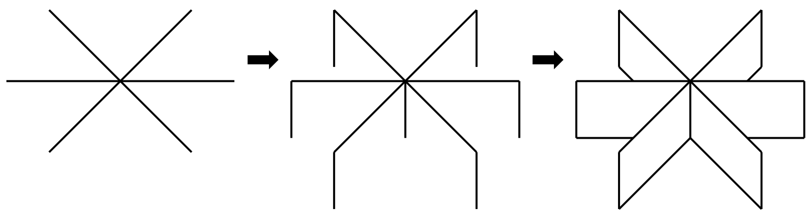
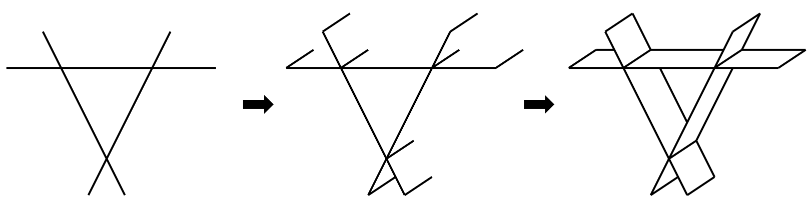
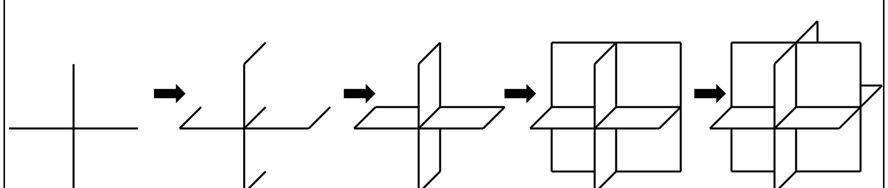
7. Are these two planes parallel, coincident, or neither?
$$\vec{r} = [4,0,3] + s[-8,1,-9] + t[-1,5,7]$$
$$\vec{r} = [-14,12,-1] + p[1,1,3] + q[-2,1,-1]$$

8. Find all values of *k* for which the following lines do not intersect. $l_1 : \begin{cases} x = -1+2r \\ y = 3k + r \\ z = 1+3r \end{cases}$ and $l_2 : \vec{r} = [1,0,-2] + t[-2,3,1]$

9. Show that the line $x-1 = 2t, y+2 = -t, z-1 = -t$ intersects the plane $2x+y+z-2=0$ then find the angle to the nearest degree, between the plane and the line.

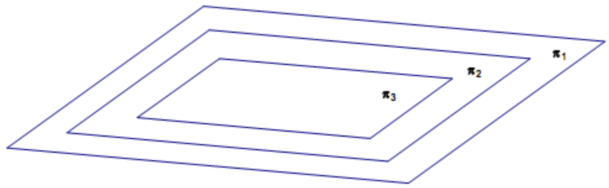
How to Draw Intersection of 3 Planes (Just For Fun!)

How To Draw 	You Try
How To Draw 	You Try
How To Draw 	You Try
How To Draw 	You Try

How To Draw 	You Try
How To Draw 	You Try
How To Draw 	You Try
How To Draw 	You Try

7.6 Intersection of Three Planes

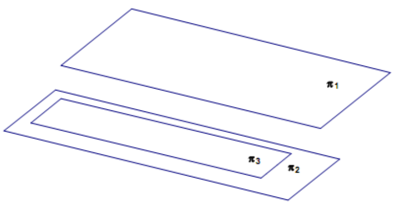
1. All three equations represent the same plane.



● normals are all scalar multiples of each other
● All three "D" values are the same scalar multiples.

- Parallel & Coincident
- Infinite solutions in the form of a plane equation

2. Two equations represent the same plane, the third is parallel.



● normals are all scalar multiples of each other
● two "D" values are the same scalar multiples (not the third one).

- Two planes are Parallel & Coincident, one is Parallel & Distinct
- No solution

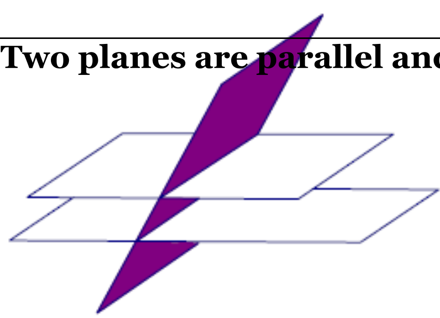
3. All three planes are parallel and distinct.

● normals are all scalar multiples of each other

● all "D" values are distinct

- All three planes are Parallel & Distinct
- No solution

4. Two planes are parallel and distinct, the third is not parallel.

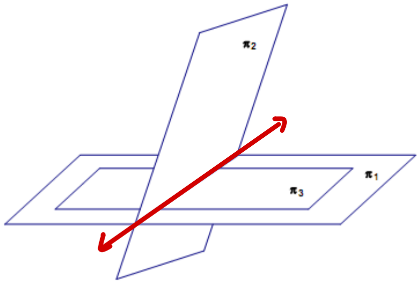


● two planes have normals that are all scalar multiples of each other

● direction vectors of the two lines of intersection are scalar multiples

- Two planes are Parallel & Distinct, one plane intersects both at parallel lines.
- No solution

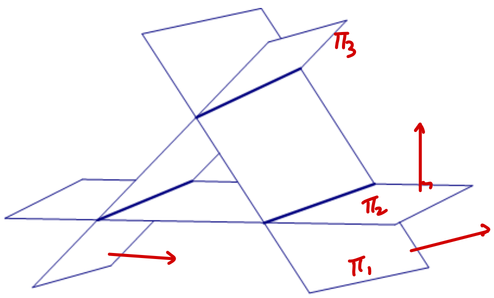
5. Two of the equations represent the same plane, the third plane is non-parallel.



- two planes have normals and "D" values that are scalar multiples (coincident)
- the third plane intersects both planes

- ➢ Two planes are Parallel & Coincident, one plane intersects both at a single line.
- ➢ Infinite solutions in the form of a line

6. None of the three planes are parallel – no single intersection.



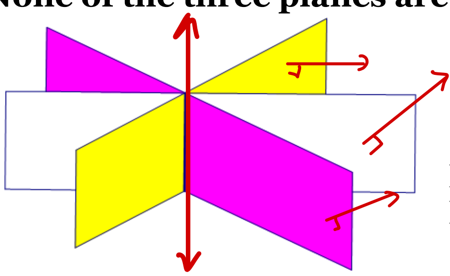
- The normals are distinct but not parallel , and coplanar

$(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3 = 0$

$\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$ are coplanar : 3 vectors lies on the same plane

- ➢ Pairs of planes intersect in 3 parallel lines
- ➢ No solution

7. None of the three planes are parallel – intersect in a line.



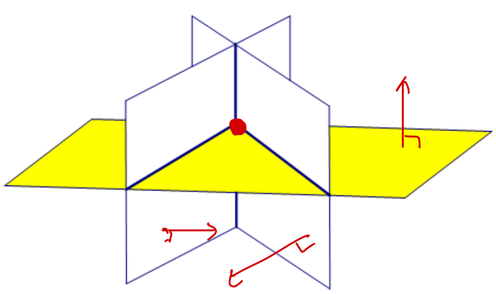
- The normals are coplanar but not parallel

$(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3 = 0$

- ➢ All three planes intersect at one line
- ➢ Infinite solutions in the form of a line

n_1 , n_2 , n_3 are coplanar

8. None of the three planes are parallel – intersect in a point.



- The normals are not parallel and not coplanar

$(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3 \neq 0$

- ➢ All three planes intersect at one point
- ➢ One distinct solution in the form of a point

Definitions:

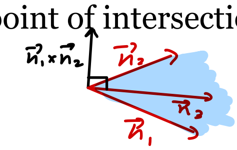
A system of three planes is **consistent** if it has one or more solutions. Ex 1, 5, 7, 8

A system of three planes is **inconsistent** if it has no solution. Ex 2, 3, 4, 6

Method for solving (possible) intersection of 3 planes

1. Check the normals for parallel or coincident planes. Suppose three distinct planes have normal vectors $\vec{n}_1, \vec{n}_2, \vec{n}_3$. To determine if there is a unique point of intersection, calculate

$\vec{n}_1 \bullet (\vec{n}_2 \times \vec{n}_3)$ $\Rightarrow$ triple scalar product



$(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3 = 0$

$\therefore \vec{n}_1, \vec{n}_2$ and $\vec{n}_3$ lies on the same plane

3 planes are Co-planar

- If $\vec{n}_1 \bullet (\vec{n}_2 \times \vec{n}_3) \neq 0$, the normal vectors are not coplanar.
  - **There is a single point of intersection**
- If $\vec{n}_1 \bullet (\vec{n}_2 \times \vec{n}_3) = 0$, the normal vectors are coplanar.
  - **There may or may not be points of intersection**
    - If there are any points of intersection then they lie on a line

2. Using Π_1 and Π_2, solve for 2 variables in terms of the third (similar to intersection of 2 planes).
3. Substitute new equations into Π_3 and interpret results:
   a) Point (solution of parameter eg. $t = 4$)
   b) Line (infinite possible # of solutions for parameter eg. $0t = 0$)
   c) Plane (infinite possible # of solutions eg. $\Pi_1 = k\Pi_2 = m\Pi_3$)
   d) None (if $0t = \#$)

Introduction to Matrices

Mathematicians often develop new notation and ideas to help ease complicated calculations or procedures. The process of solving the system can be simplified by using a matrix to help organize and eliminate variables efficiently.

Informally, a **matrix** (the plural form is matrices) is an array of m rows × n columns.

For example,

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & -4 & 5 \\ 1 & 2 & 7 \end{bmatrix}$$

A is a 3×3 matrix since it has 3 rows and 3 columns.

We often use a single, capital letter to represent a matrix, such as A in our example.

Further, A_{ij} is the notation used to reference the element in the i^{th} row and j^{th} column of matrix A. In this example, $A_{31} = 1$.

ROW MATRIX: $\begin{bmatrix} 1 & 7 & 5 & -1 \end{bmatrix}$ ***COLUMN MATRIX:*** $\begin{bmatrix} 6 \\ 0 \\ 15 \end{bmatrix}$ ***SQUARE MATRIX***: $\begin{bmatrix} 6 & -1 & 7 \\ 5 & 7 & 9 \\ -4 & -5 & 0 \end{bmatrix}$

To translate a system of linear equations into matrix form, we write the coefficients and the constant terms of the linear equations as elements in the corresponding locations in the matrix. From example

$$\begin{aligned} 1x + 2y - 4z &= 3 \\ -2x + 1y + 3z &= 4 \\ 4x - 3y - 1z &= -2 \end{aligned}$$

becomes

$$\left[\begin{array}{rrr|r} 1 & 2 & -4 & 3 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -1 & -2 \end{array}\right]$$

Such a matrix that includes the constant terms is known as an **augmented** matrix, and the elements in the matrix to the left of the vertical line form the **coefficient** matrix.

Gaussian Elimination

The benefit of using matrices becomes obvious when solving linear systems using a process known as Gaussian elimination.

In Gaussian elimination, the aim is to transform a system of equations in augmented matrix form, such as

$$A = \left[\begin{array}{rrr|r} 1 & 2 & -4 & 3 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -1 & -2 \end{array}\right]$$

into another augmented matrix of the form

$$D = \left[\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right]$$

where D is known as a matrix in row **echelon form**.

Row Operations

To transform the original augmented matrix into row echelon form, we perform row operations.

Possible row operations are as follows:

- Multiplying each entry in one row by a (non-zero) scalar
- Adding (or subtracting) one row to (from) another
- A combination of the above two row operations
- Interchanging rows

Examples: Investigate the intersection of the planes.

$\vec{n_1} = [2,-1,3]$ $D_1 = -2$

$\vec{n_2} = [2,-1,3]$ $D_2 = -\frac{3}{2}$

$\vec{n_3} = [1,-3,2]$

$\pi_1 : 2x - y + 3z - 2 = 0$

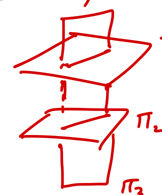
$\pi_2 : 4x - 2y + 6z - 3 = 0 \xRightarrow{\times\frac{1}{2}} 2x - y + 3z - \frac{3}{2} = 0$

$\pi_3 : x - 3y + 2z + 10 = 0$

∵ $\vec{n_1} = \vec{n_2}$ and $D_1 \neq D_2$

∴ Π_1 + Π_2 are parallel. distinct planes

Π_3 will cut through both plane



$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 4 & -2 & 6 & 3 \\ 1 & -3 & 2 & -10 \end{array}\right] R_1 \leftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 4 & -2 & 6 & 3 \\ 2 & -1 & 3 & 2 \end{array}\right] (-2)\times R_3 + R_2 \rightarrow R_3 \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 4 & -2 & 6 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

no solution

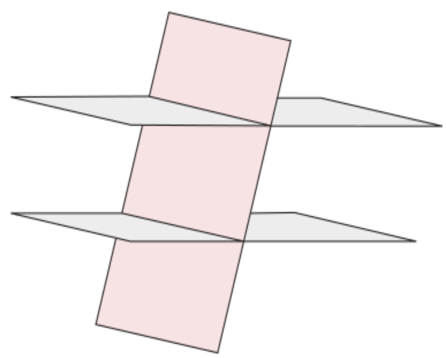
∴ no intersection

The third row of the matrix corresponds to the equation $0y = -1$, which has no solution. Thus, this system of equations has no solution and therefore, the three corresponding planes have no points of intersection.

To visualize this geometrically, we note that

$$\vec{n}_2 = 2\vec{n}_1$$
$$[4,-2,6] = 2[2,-1,3]$$
$$D_2 \neq 2D_1$$

The normal vector of the third plane, $\vec{n_3} = [1,-3,2]$ is not parallel to either of these, so the third plane must intersect each of the other two planes in a line.



Investigate the intersection of the following planes.

1) $\pi_1 : 2x - y + 3z - 2 = 0$

$\pi_2 : x - 3y + 2z + 10 = 0$

$\pi_3 : 5x - 5y + 8z + 6 = 0$

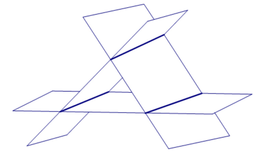
$\vec{n_1} = [2,-1,3]$

$\vec{n_2} = [1,-3,2]$

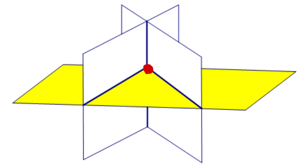
$\vec{n_3} = [5,-5,8]$

$\therefore \vec{n_1} \neq s\vec{n_2} \neq k\vec{n_3}$

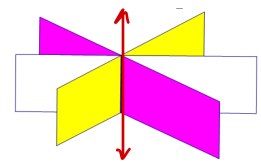
$\therefore$ triangular prism formation or the intersection of 3 planes @ a point or pinwheel



I



II



III

$(\vec{n_1} \times \vec{n_2}) \cdot \vec{n_3}$

$= [-2+9, -(4-3), -6+1] \cdot [5,-5,8]$

$= [7,-1,-5] \cdot [5,-5,8]$

$= 35+5-40$

$= 0$ $\therefore$ Either scenario I or III

let $z=0$, $\begin{cases} 2x - y - 2 = 0 & ① \\ x - 3y + 10 = 0 & ② \\ 5x - 5y + 6 = 0 & ③ \end{cases}$ $\xRightarrow{\times 2}$

$2x - y - 2 = 0$ ①

$2x - 6y + 20 = 0$ ②

$(-)$ $5y - 22 = 0$

$y = \frac{22}{5}$ ④

sub ④ into ①

$2x - \frac{22}{5} - 2 = 0$

$2x - \frac{32}{5} = 0$

$2x = \frac{32}{5}$

$x = \frac{16}{5}$

$\therefore \left(\frac{16}{5}, \frac{22}{5}, 0\right)$

Check if $\left(\frac{16}{5}, \frac{22}{5}, 0\right)$ satisfies ③

$L.S = 5x - 5y + 6$

$= 5\left(\frac{16}{5}\right) - 5\left(\frac{22}{5}\right) + 6$

$= 16 - 22 + 6$

$RS = 0$

$\therefore$ we have Scenario III

intersection of 3 planes as a line.

$\vec{d} = \vec{n_1} \times \vec{n_2} = [7,-1,-5]$

pt $\left(\frac{16}{5}, \frac{22}{5}, 0\right)$

$\therefore \vec{r} : \left[\frac{16}{5}, \frac{22}{5}, 0\right] + t[7,-1,-5]$, $t \in \mathbb{R}$

2) $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 1 & -3 & 2 & -10 \\ 5 & -5 & 8 & -6 \end{array}\right]$

$\begin{array}{c} R_2 \\ R_1 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 2 & -1 & 3 & 2 \\ 5 & -5 & 8 & -6 \end{array}\right]$

$\begin{array}{c} \\ 2R_1 - R_2 \\ 5R_1 - R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 0 & -5 & 1 & -22 \\ 0 & -10 & 2 & -44 \end{array}\right]$

$\begin{array}{c} \\ -\frac{1}{5}R_2 \\ 2R_2 - R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 0 & 1 & -\frac{1}{5} & \frac{22}{5} \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\Rightarrow$ infinitely many solutions

Method 2:

let $z = t$,

$y = \frac{1}{5}t + \frac{22}{5}$

$x = 3y - 2z - 10$

$= 3\left(\frac{1}{5}t + \frac{22}{5}\right) - 2t - 10$

$= \frac{3}{5}t + \frac{66}{5} - 2t - 10$

$= -\frac{7}{5}t + \frac{16}{5}$

$\therefore$ 3 planes intersect @ a line:

$\vec{r} = \left[\frac{16}{5}, \frac{22}{5}, 0\right] + t\left[-\frac{7}{5}, \frac{1}{5}, 1\right]$

Note! $-\frac{1}{5}[7,-1,-5]$ is the same direction vector as they are collinear

3) $\pi_1: x+3y-z+9=0$ ① $\quad \vec{n}_1 = [1,3,-1]$

$\pi_2: x-y+z-11=0$ ② $\quad \vec{n}_2 = [1,-1,1]$

$\pi_3: x+2y+4z-5=0$ ③ $\quad \vec{n}_3 = [1,2,4]$

$\vec{n}_1 \neq s\vec{n}_2 \neq k\vec{n}_3$

$(\vec{n}_1 \times \vec{n}_2)\cdot \vec{n}_3$

$= [3-1, -(1+1), -1-3]\cdot[1,2,4]$

$= [2,-2,-4]\cdot[1,2,4]$

$= 2-4-16$

$= -18$

$\neq 0$

$\therefore$ 3 planes intersect @ a point.

① − ② : $4y-2z+20=0 \Rightarrow 2y-z+10=0$

② − ③ : $-3y-3z-6=0 \Rightarrow y+z+2=0$ ④

$(+)\quad 3y + 12 = 0$

$y=-4$ ⑤

sub ④ into ⑤

$-4+z+2=0$

$z=2$ ⑥

sub ⑤ and ⑥ into ①

$x+3(-4)-(2)+9=0$

$x=5$

$\therefore$ 3 planes intersect @ a point : $(5,-4,2)$

4) $\pi_1: 2x+3y+z-1=0$

$\pi_2: x-y+z-2=0$

$\pi_3: 2x-2y+2z-4=0 \Rightarrow x-y+z-2=0$

$\vec{n}_1 = [2,3,1]$

$\vec{n}_2 = [1,-1,1]$

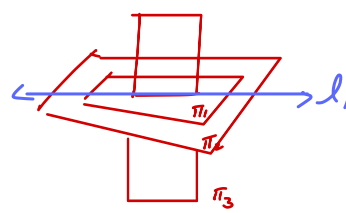
$\vec{n}_3 = [1,-1,1]$

$D_2 = -2$

$D_3 = -2$

$\therefore \pi_2 + \pi_3$ are parallel coincident

note! its the same as intersection of 2 planes forming a line



let $y=0$, $\quad \pi_1: 2x+z-1=0$ ①

$\pi_2: x+z-2=0$ ②

$(-)\quad x+1=0$

$x=-1$ ③

sub ③ into ②

$(-1)+z-2=0$

$z=3$

$\therefore (-1,0,3)$ is a point on the line

line: $\vec{d} = \vec{n}_1 \times \vec{n}_2$

$= [3+1, -(2-1), -2-3]$

$= [4,-1,-5]$

$\therefore$ the intersection of 3 planes is the line : $\vec{r} = [-1,0,3] + t[4,-1,-5]$, $t\in\mathbb{R}$

Exit Card!

Investigate the intersection of the following planes.

$$\begin{cases} 2x+y-3z+5=0 & (1) \\ x+y+z-6=0 & (2) \\ 4x-5y+z+3=0 & (3) \end{cases}$$

$\vec{n}_1 = [2,1,-3]$
$\vec{n}_2 = [1,1,1]$
$\vec{n}_3 = [4,-5,1]$

$\vec{n}_1 \neq k\vec{n}_2 \neq m\vec{n}_2$

$\therefore$ it could be pinwheel (intersect @ a line) or triangular prism (no solution) or intersecting @ a plane.

$(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3$

$= [1+3, -(2+3), 2-1] \cdot [4,-5,1]$

$= [4,-5,1] \cdot [4,-5,1]$

$= 16+25+1$

$\neq 0$ $\quad \therefore$ 3 planes intersect at a point.

$$\begin{array}{lrl} (1) & 2x+y-3z+5 &= 0 \\ (2)\overset{\times 2}{\Rightarrow} & 2x+2y+2z-12 &= 0 \\ \hline (-) & -y-5z+17 &= 0 \quad (4) \end{array}$$

$$\begin{array}{lrl} (2)\overset{\times 4}{\Rightarrow} & 4x+4y+4z-24 &= 0 \\ (3) & 4x-5y+z+3 &= 0 \\ \hline (-) & 9y+3z-27 &= 0 \quad (5) \end{array}$$

$$\begin{array}{lrl} (4)\overset{\times 9}{\Rightarrow} & -9y-45z+153 &= 0 \\ & 9y+3z-27 &= 0 \\ \hline (+) & -42z+126 &= 0 \\ & z &= 3 \quad (6) \end{array}$$

sub (6) into (4)

$-y-5(3)+17=0$

$y=2$ (7)

sub (6) and (7) into (2)

$x+(2)+(3)-6=0$

$x=1$

$\therefore$ 3 planes intersect @ a point $(1,2,3)$

Practice

1. Given the related augmented matrix, find the solution to the system

a. $\begin{array}{ccc|c} x & y & z & \\ \end{array}$
$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right]$ $x=3$, $y=2$, $z=5$

c. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 12 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 4 \end{array}\right]$ $x+4z=12 \Rightarrow x+4(4)=12$, $x=-4$; $y+2z=7 \Rightarrow y+2(4)=7$, $y=-1$; $z=4$; $\therefore (-4,-1,4)$

b. $\left[\begin{array}{ccc|c} 0 & 2 & 0 & 6 \\ 0 & 0 & -3 & 9 \\ 4 & 0 & 0 & 8 \end{array}\right]$ $\begin{array}{l} \frac{1}{2}R_1 \\ -\frac{1}{3}R_2 \\ \frac{1}{4}R_3 \end{array}$ $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right]$ $\therefore (2,3,-3)$

d. $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 7 \\ 3 & 1 & 0 & 4 \\ 0 & 0 & 0 & -5 \end{array}\right] \Rightarrow$ dne No solution

2. Use Gauss-Jordan elimination to find the intersection of each of the following sets of planes

a. $x+3y-z+9=0$
$x-y+z-11=0$
$x+2y+4z-5=0$

b. $x-y-2z-5=0$
$2x+2y+z-1=0$
$x+3y+3z-10=0$

c. $2x+3y+z-1=0$
$x-y+z-2=0$
$2x-2y+2z-4=0$

d. $2x-y+3z-2=0$
$x-3y+2z+10=0$
$5x-5y+8z+6=0$

3. Find value(s) of a and b so that the planes
$2x+y-z=0$
$x+2y+3z=0$
$3x+ay+2z+b=0$

(a) intersect in a line
(b) intersect at a point.

4. For what value of k will the following set of planes interesting in a line?
$\pi_1 : x - 2y - 3z = 0$
$\pi_2 : x + 9y - 5z = 0$
$\pi_3 : kx - y + z = 0$

Answers

1. a. $(3,2,-5)$
   b. $(2,3,-3)$
   c. $(-4,-1,4)$
   d. No solution.

2. a. $(5,-4,2)$
   b. No solution.
   c. $\vec{r}=[-1,2,3]+t[4,-1,-5]$
   d. $\vec{r}=[\frac{16}{5},\frac{22}{5},0]+t[7,-1,5]$

3. a. $a=3, b=0$
   b. $a\neq 3, b\in\mathbb{R}$

4. $k=\frac{-9}{37}$

Warm -up

Determine the intersection of the following planes

a. $x - y + z = -2$ Ans: $(-1,3, 2)$
$2x - y - 2z = -9$
$3x + y - z = -2$

b. $x + y + 2z = -2$ Ans: $[x, y, z] = [1, -3, 0] + t[-4, 2, 1]$
$3x - y + 14z = 6$
$x + 2y = -5$

c. $3x + 2y - z = 0$ Ans: $\vec{r} = [\frac{2}{7}, -\frac{3}{7}, 0] + t[-1, 5, 7]$
$3x - 5y + 4z = 3$
$2x - y + z = 1$

d. $x - y + 4z = 5$ Ans: No intersection exists
$3x + y + z = -2$
$5x - y + 9z = 1$

a. $x - y + z = -2$ ① $\vec{n_1} = [1,-1,1]$

$2x - y - 2z = -9$ ② $\vec{n_2} = [2,-1,-2]$

$3x + y - z = -2$ ③ $\vec{n_3} = [3,1,-1]$

$\vec{n_1} \neq a\vec{n_2} \neq b\vec{n_3}$

$(\vec{n_1} \times \vec{n_2}) \cdot \vec{n_3}$

$= [2+1, -(-2-2), -1+2] \cdot [3,1,-1]$

$= [3,4,1] \cdot [3,1,-1]$

$= 9+4-1$

$= 12$

$\neq 0$

$\therefore$ 3 planes will intersect @ a point

Eliminate y:

③ − ① : $x - 3z = -7$ ⑤

① + ③ : $4x = -4 \Rightarrow x = -1$ ④

sub ④ into ⑤

$(-1) - 3z = -7$

$-3z = -6$

$z = 2$ ⑥

sub ⑥ into ①

$(-1) - y + (2) = -2$

$-y + 1 = -2$

$y = 3$

$\therefore$ point of intersection of the 3 planes is : $(-1, 3, 2)$

b. $x + y + 2z = -2$

$3x - y + 14z = 6$

$x + 2y = -5$

$\vec{n_1} = [1,1,2]$

$\vec{n_2} = [3,-1,14]$

$\vec{n_3} = [1,2,0]$

$\vec{n_1} \neq a\vec{n_2} \neq b\vec{n_3}$

$(\vec{n_1} \times \vec{n_2}) \cdot \vec{n_3}$

$= [14+2, -(14-6), -1-3] \cdot [1,2,0]$

$= [16, -8, -4] \cdot [1,2,0]$

$= 16 - 16 + 0$

$= 0 \Rightarrow$ normal are co-planar

$\therefore$ either pinwheel formation (intersection as a line) or triangular prism formation (no solution)

let $z = 0$, $\left\{ \begin{array}{ll} x + y = -2 & ① \\ 3x - y = 6 & ② \\ x + 2y = -5 & ③ \end{array} \right.$ use these 2 to solve for x & y:

3① − ② : $4y = -12$

$y = -3$

sub into ①

$x + (-3) = -2$

$x = 1$

check if $x = 1$ and $y = -3$ satisfy ③

$L.S = x + 2y$

$= (1) + 2(-3)$

$= -5$

$= R.S$

$\therefore$ $(1,-3,0)$ satisfies all 3 planes

Equation of the line intersection :

$\vec{d} = \vec{n_1} \times \vec{n_2}$ point: $(1,-3,0)$

$= [16,-8,-4]$

$= 4[4,-2,-1]$

$\therefore \vec{r} = [1,-3,0] + t[4,-2,-1], x \in \mathbb{R}$

c. $3x+2y-z=0$ ①
$3x-5y+4z=3$ ②
$2x-y+z=1$ ③

$\vec{n_1}=[3,2,-1]$
$\vec{n_2}=[3,-5,4]$
$\vec{n_3}=[2,-1,1]$

$\vec{n_1}\neq a\vec{n_2}\neq b\vec{n_3}$

$(\vec{n_1}\times\vec{n_2})\cdot\vec{n_3}$
$=[8-5,-(12+3),-15-6]\cdot[2,-1,1]$
$=[3,-15,-21]\cdot[2,-1,1]$
$=6+(15)-21$
$=0$

$\because$ normals are coplanar
$\therefore$ triangular prism or pinwheel formation

let $y=0$, $\begin{cases}3x-z=0 & ①\\ 3x+4z=3 & ②\\ 2x+z=1 & ③\end{cases}$ use ① and ③ to solve for x + z

① + ③ : $5x=1$
$x=\frac{1}{5}$
sub into ③
$2(\frac{1}{5})+z=1$
$z=\frac{3}{5}$

Check if it satisfies ②
$L.S=3(\frac{1}{5})+4(\frac{3}{5})$
$=\frac{3}{5}+\frac{12}{5}$
$=3$
$=R.S$

$\therefore$ the point $(\frac{1}{5},0,\frac{3}{5})$ lies on all 3 planes

Equation of the line of intersection

$\vec{d}=\vec{n_1}\times\vec{n_2}$
$=[3,-15,-21]$
$=3[1,-5,-7]$

point : $(\frac{1}{5},0,\frac{3}{5})$

$\therefore\ \vec{r}=[\frac{1}{5},0,\frac{3}{5}]+t[1,-5,-7]$, $t\in\mathbb{R}$

d. $x-y+4z=5$ ①
$3x+y+z=-2$ ②
$5x-y+9z=1$ ③

$\vec{n_1}=[1,1,4]$
$\vec{n_2}=[3,1,1]$
$\vec{n_3}=[5,-1,9]$

$\vec{n_1}\neq a\vec{n_2}\neq b\vec{n_3}$

$=(\vec{n_1}\times\vec{n_2})\cdot\vec{n_3}$
$=[1-4,-(1-12),1-3]\cdot[5,-1,9]$
$=[-3,11,-2]\cdot[5,-1,9]$
$=-15-11-18$
$\neq 0$

$\because$ normals are coplanar
$\therefore$ pinwheel or triangular prism formation

let $z=0$, $\begin{cases}x-y=5 & ①\\ 3x+y=-2 & ②\\ 5x+y=1 & ③\end{cases}$ solve for $x+y$

① + ② : $4x=3$
$x=\frac{3}{4}$

sub into ①
$\frac{3}{4}-y=5$
$-\frac{17}{4}=y$

Check if $x=\frac{3}{4}$ and $y=\frac{-17}{4}$ satisfy ③
$L.S=5(\frac{3}{4})+(\frac{-17}{4})$
$=\frac{15}{4}-\frac{17}{4}$
$=\frac{-2}{4}$
$=-\frac{1}{2}$
$\neq R.S$

$\because$ No Solution
$\therefore$ 3 planes do no intersect (triangular Prism formation)

Unit 7 Review: Equations of Lines & Planes

1. Indicate whether the statement is true or false.

a) There is only one scalar equation for a given line in 3-space. _______

b) A line in 3-space can be represented by using a scalar equation, a vector equation, parametric equations. _______

c) Three collinear points in space will define a plane. _______

d) Line $\vec{r} = [0,2,1] + t[2,1,1]$ is parallel to the plane 2x+y+z=0. _______

e) If the normal vectors of three planes in 3-space are coplanar then there is no solution to the system. _______

2. Identify the choice that best completes the statement or answers the question.

i. Two skew lines are
A) parallel B) coplanar C) coincident
D) neither parallel nor intersecting E) all of the above

ii. Which of the following points lie on the plane 11x -2y -4z + 2 = 0

A) (4, 2, 6) B) (11, 2, -4) C) (4, -2, -11) D) (2, 4, 4)
E) (0, 2, 4)

iii. A scalar equation of the plane with normal vector $\vec{n} = [0,-1,3]$ and passing through the point (5, −2, 3) is:

A) $y-3z+11=0$ B) $y-3z-11=0$ C) $-y+3z+11=0$ D) $-y+3z+16=0$
E) None of the above

3. Find the **vector equation** of a line:

a) that passes through the points A(3, -1) and B (-2, 1).

b) that passes through the point P(1,2,3) and is parallel to the x-axis

c) that has parametric equations $\begin{cases} x = -3 + 2t \\ y = -4t \\ z = -1 \end{cases}$

4. Find the **Cartesian equation** of a line that passes through the point (4, 2) and is perpendicular to the line $\vec{r} = [7,2] + t[5,-8]$.

5. Determine if the point A(10, -19, 15) lies on the plane $[x,y,z]=[6,-7,10]+s[1,3,-1]+t[2,-2,1]$.
6. Find the **vector equation** of the plane containing the point P (3, -5, 6) and the line $\vec{r}=[1,-1,2]+s[-3,1,-1]$.
7. Are $L_1:\vec{r}=[-2,3]+t[-1,3]$, and L_2 : x = 3s + 2, y = *s* - 7 perpendicular? Explain.

8. Can the points A(4, -1, 3), 8(5, 3, -10) and C(-8, 2, -6) be used to determine the equation of a plane. Explain.

9. Explain why the plane with Cartesian equation 2x +5z -3 = 0 never intersects the y-axis.

10. Explain why there is no Cartesian equation for a line in 3 space.

11. Determine the value of **k** for which the direction vectors of the following lines are perpendicular.

$$L_1:\vec{r}=[1,2,-1]+t[\mathbf{k}-1,2,-\mathbf{k}]$$
$$L_2:\vec{r}=[-1,0,2]+s[2\mathbf{k},1,\mathbf{k}+1]$$

12. Find the distance between the following skew lines

a) $L_1:[x,y,z]=[-2,1,1]+t[3,1,1];t\in R$
$L_2:[x,y,z]=[7,-4,3]+s[-1,1,0];s\in R$

b) $\ell_1:\frac{x}{3}=\frac{y-4}{4}=z-8$
$\ell_2:\frac{x+1}{2}=\frac{y-5}{5}=\frac{z+2}{2}$

13. Determine the **Cartesian equation** of the plane that contains the point (2, -1, 1) and is **perpendicular** to the line joining points (-1, 3, 2) and (4, 0, -2).

14. Determine a **Cartesian equation** of the plane that passes through the points A (-2, 3, 1), B(4, 6, -8) and C(-1, 2, 7).

15. Determine the **acute angle** between the planes *2x + 4y - 8 = 0* and *x -3y + z + 4 = 0*.

16. **Solve** the following systems of equations. Give a **geometrical interpretation** of the system and its solution.

a)
$\pi_1: x-5y+2z-10=0$
$\pi_2: x+7y-2z+6=0$
$\pi_3: 8x+5y+z-20=0$

b)
$\pi_1: 2x+y+6z-7=0$
$\pi_2: 3x+4y+3z+8=0$
$\pi_3: x-2y-4z-9=0$

17. The plane with parametric equations $\begin{cases} x = 3 + s - t \\ y = -1 + 3s + 2t \\ z = -2 + 4s + t \end{cases}$ intersects the x-axis and the z - axis at points M and N respectively. Determine the **vector equation** of the line that contains these two points.

18. Determine the value of $\boldsymbol{k}$ for which the planes $2x + 3y + \boldsymbol{k}z = 5$ and $2x + 3y = 7$ have an angle of intersection of 60^0.

19. Write the **scalar equation** of the plane that contains the point (2, 1, -3), has a y-intercept at 5 and is perpendicular to the plane 2x - y + z = 4.

20. Determine the shortest **exact** distance from the point $A(1, 2, 3)$ and the line $\begin{cases} x = 3s \\ y = 1 + 4s \\ z = 5 + s \end{cases}$.

21. Determine the shortest **exact** distance between the two planes.

$\pi_1: \ 3x - 2y - 6z + 12 = 0$

$\pi_2: \ 9x - 6y - 18z + 30 = 0$