Grade 12 Calculus & Vectors
Unit 6 – Representing Vectors in Two-Space and Three-Space

Big Ideas
We will…
explore and develop the concept of a vector
explore and work with vectors in R^2 and R^3
develop skills related to operations with geometric vectors in R^2 and R^3 including the new operations: the dot product and cross product
solve real-life problems involving vectors (e.g. force, velocity, work)

DAY	DESCRIPTION	Homework
1	**Vector Concepts and Laws: Geometric Interpretation** We are learning to… identify the difference between a scalar and a vector determine the geometric interpretation of a vector perform simple arithmetic operations with vectors I am able to… explain what a vector is, how it is different from a scalar and give examples of both vectors and scalars perform the addition or subtraction of two vectors geometrically using a ruler by using both the Triangle Law of Vector Addition or the Parallelogram Law of Vector Addition or Parallelogram Subtraction find the magnitude of the sum or difference of two or more vectors given their magnitudes and a description of the angles between the vectors when placed tail to tail	*CP: Pg 12-13* *Textbook:* *Pg 310 #1, 3, 7, 8* *Pg 325 #3, 4* *Pg 334 #1, 2, 7, 19*
2	**Algebraic Vectors** We are learning to …. connect the geometric interpretation of a vector with the algebraic interpretation of a vector determine the 3 ways of defining vectors in two-space co-ordinate pair component form magnitude and angle perform simple arithmetic operations with vectors I am able to … write a position vector in component form and using standard unit vectors & vice versa and plot on an x-y axes system find the magnitude of a position vector add, subtract and scalar multiply position vectors find a position vector to represent a directed line segment between two points solve problems involving position vectors (e.g. problems involving collinear points, unit vectors etc.	*CP: Pg 21(warm up)* *Textbook:* *Pg 367 C1,C2,C3,C4,C5* *#1 - 8*

3	**Force As a Vector (Gravity, Tensions, Equilibrant, Ramps, Resolving Forces into Vertical and Horizontal Components)** We are learning to… solve more real-life problems involving the sum of force vectors think of vectors as being made up of a horizontal part (called a component) and a vertical part I am able to… determine the force of gravity of an object on Earth, given its mass in kilograms in the correct units find the horizontal and vertical components of a given vector solve the classic ramp problem: find the force of friction and the lifting or normal force that hold a box in place on a ramp solve the classic tension problem: find the tension in two cords from which an object of a certain mass is hanging given the angles the two cords make with the horizontal	*CP: Pg 28-29* *Textbook:* Pg 343 C1, C3 #1, 10 – 12, , 16, 19
4	**Velocity as a Vector (River Crossing, Airplane Problems)** We are learning to… solve problems in which we must consider the sum of velocity vectors, such as walking on a moving object, determining the affect of wind on an airplane in flight and how we steer a boat in water with a current I am able to… explain what is meant by relative velocity by referring to an example from real life solve word problems that involve the sum of velocity vectors, such finding the ground velocity of an airplane in a wind or the heading a boat in a river must take to cross it directly	*CP: Pg 35* *Textbook:* Pg 343 C4 #2 – 9, 13, 14, 18, 20, 26
5	**The Dot Product** We are learning to… use a new operation created solely for vectors, called the dot product I am able to… find the dot product of two vectors where their magnitudes and the angle between them is given find the dot product for two given Cartesian vectors use the dot product to find the angle between two given vectors use the dot product to find perpendicular vectors	*CP: Pg 41-42* *Textbook:* Pg 375 C1, C2, C3, C4 #1 – 12

6	**Applications of the Dot Product :** **Work** **Scalar and Vector Projections** We are learning to... find the work performed when a known force is applied at a given angle and moves an object a certain distance define and find the scalar and vector projections of one vector onto another vector I am able to... Calculate the amount of work done by an object using dot product explain the difference between a scalar and a vector projection of a vector $\vec{v}$ on to another vector $\vec{u}$. determine the scalar and vector projections of one vector on another where the vectors are described geometrically or algebraically interpret the sign of the scalar projection as it pertains to the angle between the two vectors	*CP: Pg 47* *Textbook:* Pg 384 C1, C2, C3, C4 #1 – 8, 11 – 14, 19, 20 +Worksheet
7	**Points and Vectors in 3-space** We are learning to... get comfortable working with a 3-D Cartesian Coordinate System sketch and work with vectors in R^3 algebraically I am able to... plot a point in R^3 on the xyz-axes system explain what a plane is and locate the xy-, zy- and xz- planes on an xyz-axes system add, subtract and scalar multiply position vectors in R^3 find a position vector to represent a directed line segment between two points in R^3 solve problems involving position vectors in R^3 (e.g. problems involving collinear points, unit vectors etc.	*CP: Pg 51* *Textbook:* Pg 399 C1, C2, C4 #1 – 15
8	**The Dot product if Two Vectors in 3 Space** **The Cross Product of Two Vectors** We are learning to... determine the dot product of vectors in three-space geometrically and algebraically. describe applications in three-space of the dot-product including projections define and find the cross product of two vectors I am able to... determine the cross product of two vectors given geometrically or algebraically	*Textbook:* Pg 410 C1, C2, C3 #1 – 5, 7 – 13 Pg 418 #4

9	**Applications of Cross-Product** We are learning to … apply the dot and cross products to solve a variety of problems I am able to… compute the triple scalar product of three vectors and use the result to decide if the three vectors are coplanar or note determine the area of a parallelogram or triangle bounded by two given vectors and the angle between them compute the volume of a parallelepiped bounded by three given vectors	*CP: Pg.59* *Textbook:* Pg 399 C5 #16 – 19, 23, 25, 27, 29 Pg 418 #2, 3, 5 – 9. 12
10	**Quiz/Review**	Pg 502 #1 – 6, 15, 16 Pg 352 #1 – 12 Pg 420 #1 – 11
11	**Review CP:Pg. 60-61**	Pg 504 #1, 2, 8, 11 – 13, 15 Pg 420 #12 – 19 Pg 422 #1 – 6, 9 – 16, 23 – 25, 27
12,13	**Summative Tests**	W May 22 (Day 1) Th May 23 (Day 2)

6.1 Intro to Vectors in 2D Space

Definitions

Scalar

- a mathematical quantity that is expressed by a magnitude (number) only
- no direction

Examples:

- Speed
- Distance
- Time
- Temperature
- Age
- Mass

Work

Vector

- a mathematical quantity that is expressed by a magnitude AND direction
- represented by a directional line segment
- length of vector $\rightarrow$ indicated by magnitude:
- direction of vector $\rightarrow$ indicated by arrow head

Examples:

- Velocity
- Displacement
- Acceleration
- Force
- Gravity
- Weight

Torque

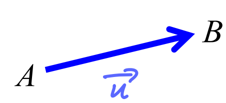
Concept - Vectors

Ex1. Given the vector below, state its;

a) names: $\overrightarrow{AB}$ or $\vec{u}$

b) magnitude: $|\overrightarrow{AB}|$ or $|\vec{u}|$

↳ length / size



Bearing

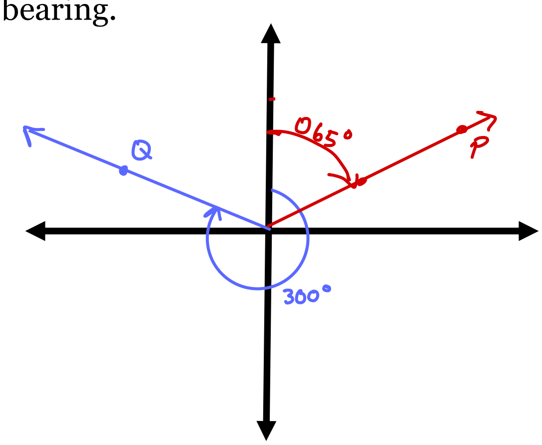
A bearing is a measurement of direction between two points. Bearings are generally given in one of two formats, a true bearing or a quadrant bearing.

"Compass Bearing" 0° - 360° "Compass Direction" ⇒ Uses N-S-W-E

The **true bearing** to a point is the angle measured in degrees in a clockwise direction from the north line. We will refer to the true bearing simply as the bearing.

Examples: Show the following bearings:

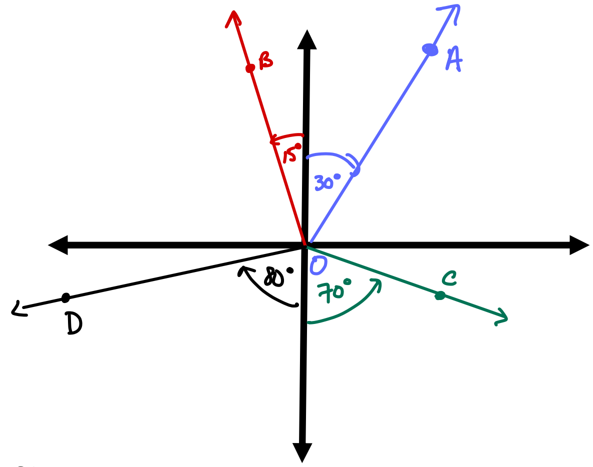
a) bearing of point P is 065^o
b) bearing of point Q is 300^o



Quadrant Bearing

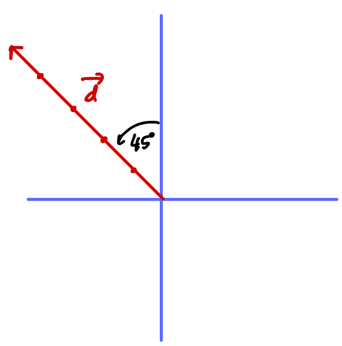
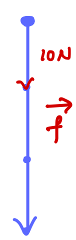
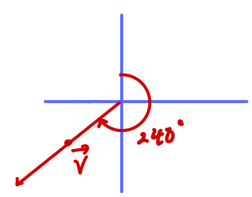
In this method the compass is divided into four quadrants. North and south are at 0 degrees, and depending on the quadrant, angles (up to 90 degrees) are measured away from north or south (whichever is nearer) towards East and West directions. For example S 65° E is 65 degrees towards east of South.
Examples: Show the following bearings:

- A from O is N30°E. = E 60° N = 030°
- B from O is N15°W. = W 75° N = 345°
- C from O is S70°E. = E 20° S = 110°
- D from O is S80°W = W 10° S = 260°



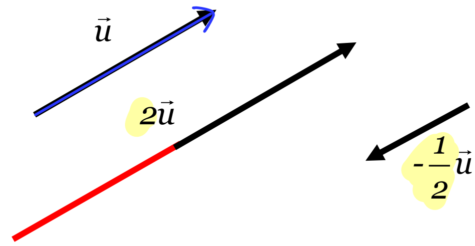
note! magnitude (aka length) is unknown ∴ not a vector quantity.

Example: Using a scale, draw a vector to represent a;

i. displacement of 50m northwest	ii. force of 30N down	iii. velocity of a ship 200km/h on a bearing of 240°
 $\vec{d}$ = 50m[NW]	 $\vec{f}$ = 30N [S]	 $\vec{v}$ = 200 km/h [240°]

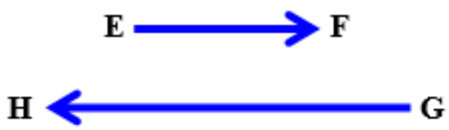
Scalar Multiplication:

Multiplying a vector by a scalar will change its length and/or its direction

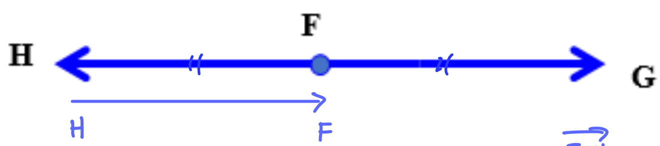


Collinear Vectors

- they will be parallel to each other.
- they either have the same direction or opposite direction.
- one will always be a scalar multiple of the other.i.e $\vec{u} = k\vec{v}\ , k \in \mathbb{R}, k \neq 0$



$\overrightarrow{HG} = \overrightarrow{EF}$
"are equal vectors"

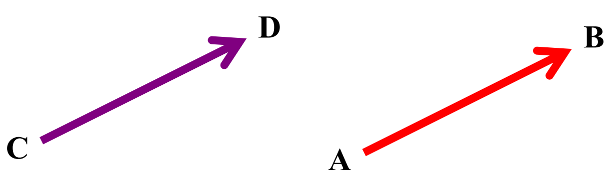


$\overrightarrow{FH} = -\overrightarrow{HF}$

$\overrightarrow{FG}$ and $\overrightarrow{HF}$
"are equal vectors"

Equal Vectors

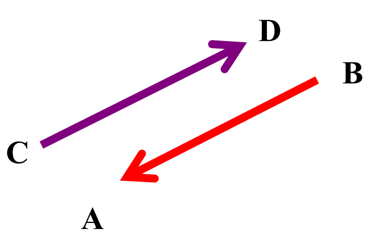
- vectors the are equal in magnitude AND direction



$\overrightarrow{CD} = \overrightarrow{AB}$
$\overrightarrow{DC} = \overrightarrow{BA}$
$\overrightarrow{CD}$ and $\overrightarrow{BA}$ are opposite vectors

Opposite Vectors

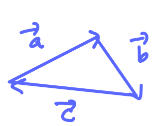
- Vectors that are equal in magnitude but have opposite direction
- Parallel to each other



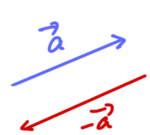
$\overrightarrow{CD} = -\overrightarrow{BA}$

Zero Vector

A vector whose initial and end point are the same and has no direction is called a zero vector, $\vec{0} = \overrightarrow{AA}$

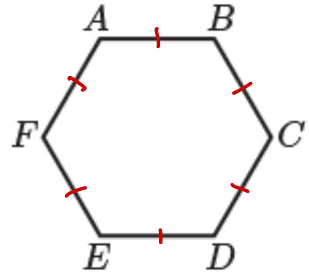


$\vec{a} + \vec{b} + \vec{c} = \vec{0}$



$\vec{a} + (-\vec{a}) = \vec{0}$

Ex2. *ABCDEF* is a regular hexagon. Give examples of vectors formed between pairs of vertices of hexagon *ABCDEF*:



a) equal $\vec{AB} = \vec{ED}$ or $\vec{BC} = \vec{FE}$

b) parallel but with different magnitudes $\vec{BC}$ and $\vec{AD}$

c) equal in magnitude but opposite in direction $\vec{BC}$ and $\vec{EF}$

d) equal in magnitude but not parallel $\vec{AB}$ and $\vec{BC}$

e) different in both magnitude and direction $\vec{EA}$ and $\vec{FA}$

Ex3. For each scenario describe the opposite vector.

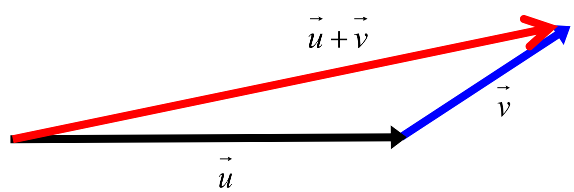
a) A car travels due south at 50 km/h. 50 km/h due north or 50km/h [N]

b) A skateboard heads north-west at 8 km/h. 8km/h [SE]

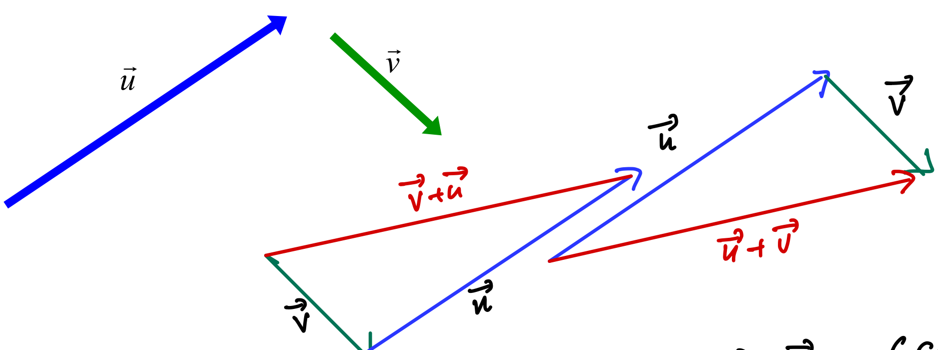
Adding and Subtracting Vectors

A. Triangle Law of Vector Addition

If $\vec{u}$ and $\vec{v}$ are any vectors then the sum of the vectors, $\vec{u}+\vec{v}$, is the vector that has the same initial point as $\vec{u}$ and the same final point as $\vec{v}$, when the initial point of $\vec{v}$ coincides with the final point of $\vec{u}$.



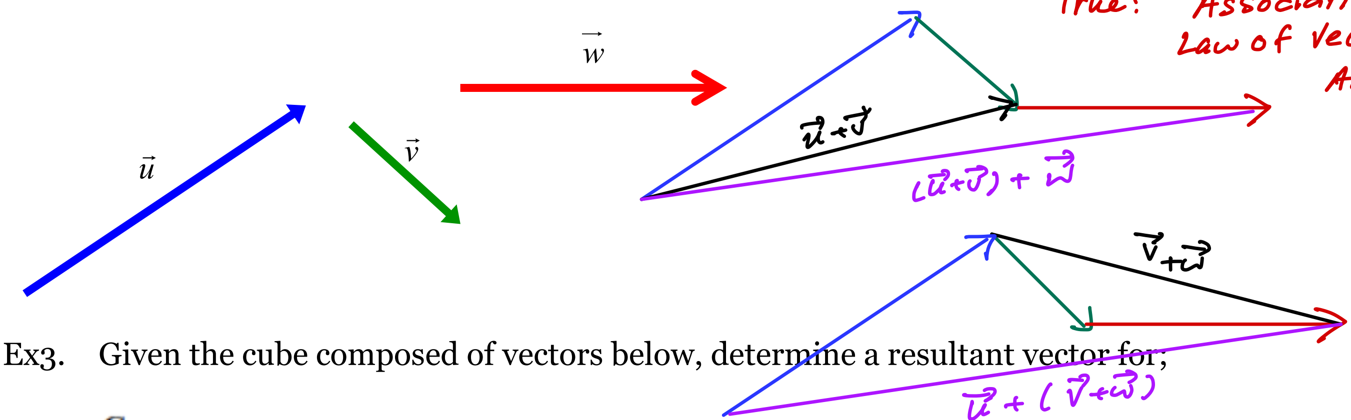
Ex1. Given the two vectors below, is $\vec{u}+\vec{v}=\vec{v}+\vec{u}$? Illustrate your choice.



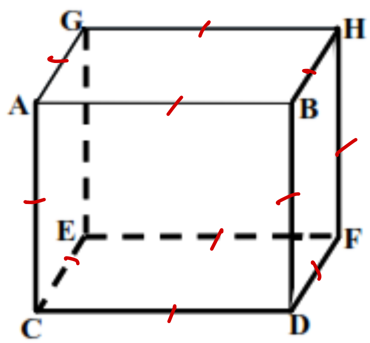
$\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (commutative law for vector addition)

Ex2. Given the following three vectors, is $\left(\vec{u}+\vec{v}\right)+\vec{w}=\vec{u}+\left(\vec{v}+\vec{w}\right)$? Illustrate your choice.

True! Associative Law of Vector Addition



Ex3. Given the cube composed of vectors below, determine a resultant vector for;



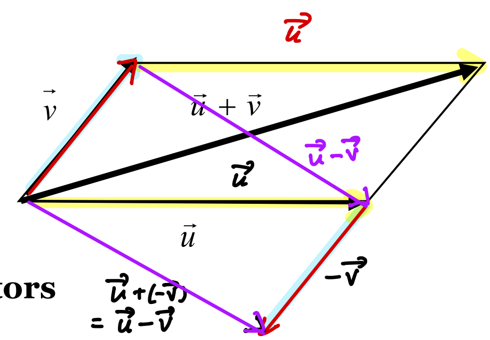
a) $\overrightarrow{AB}+\overrightarrow{BC}=$ $\overrightarrow{AC}$

b) $\overrightarrow{HC}+\overrightarrow{CD}=$ $\overrightarrow{HD}$

c) $\overrightarrow{BG}+\overrightarrow{EF}+\overrightarrow{BD}=$ $\overrightarrow{BG}+\overrightarrow{GH}+\overrightarrow{HF}$
$=\overrightarrow{BF}$

B. Parallelogram Law of Vector Addition

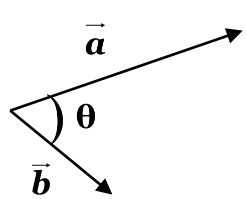
To determine the sum of two vectors, $\vec{u}$ and $\vec{v}$, draw the vectors tail to tail and construct a parallelogram. The sum, $\vec{u}+\vec{v}$, is the DIAGONAL of the parallelogram (the vector where the tips meet and the tails meet).

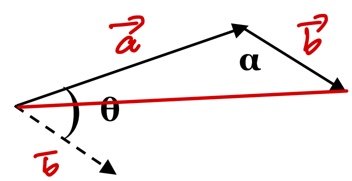


Angle between two vectors

The angle between two vectors is the angle ≤180° formed when the vectors are placed **tail to tail**, that is, starting at the same point.

$0 \le \theta \le 180°$



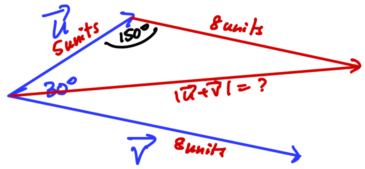


$\theta = 180° - \alpha$

Ex4. Determine the magnitude of $\vec{u} + \vec{v}$ if $|\vec{u}| = 5$ and $|\vec{v}| = 8$, and the angle between them is $30°$.

Recall

$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$ Cosine Law



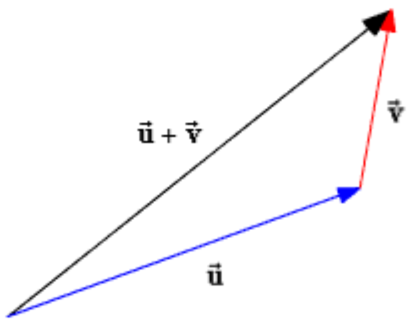
$|\vec{u}+\vec{v}|^2 = 5^2 + 8^2 - 2(5)(8)\cos 150°$

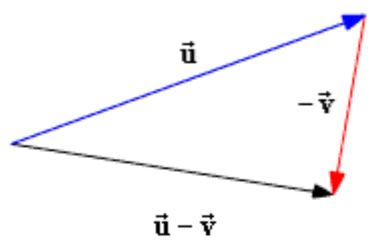
$|\vec{u}+\vec{v}| \doteq \sqrt{158.282}$

$\doteq 12.6$ units

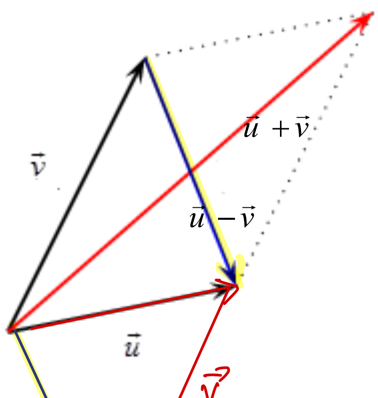
Vector Subtraction/Difference of Vectors

For any vectors $\vec{u}$ and $\vec{v}$ the difference is $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ .To obtain $\vec{u} - \vec{v}$, take a **vector** of the same magnitude as $\vec{v}$, but pointing in the opposite direction, and add that **vector** to $\vec{u}$, using either the tip-to-tail method or the parallelogram method.



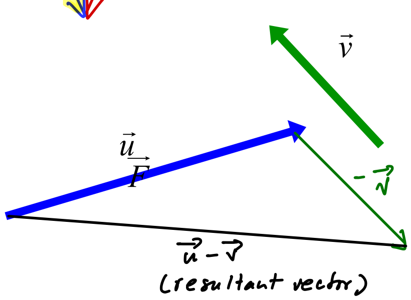


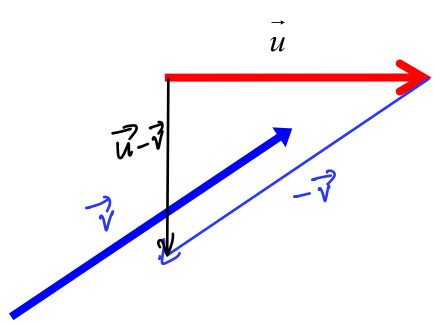
The vectors $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are both diagonals of the parallelogram formed by $\vec{u}$ and $\vec{v}$.



Ex.5 Vectors $\vec{u}$ and $\vec{v}$ are shown below. Draw diagrams describes a valid way of obtaining $\vec{u} - \vec{v}$.

$\vec{u} - \vec{v}$





Ex6. What single vector is equal to each of these sums?

a) $\overrightarrow{PT}+\overrightarrow{TS}+\overrightarrow{SQ}$ = $\overrightarrow{PQ}$

b) $\overrightarrow{AC}-\overrightarrow{AE}+\overrightarrow{CE}$ = $\vec{0}$

$= \overrightarrow{AC} + \overrightarrow{EA} + \overrightarrow{CE}$

$= \overrightarrow{AC} + \overrightarrow{CE} + \overrightarrow{EA} = \overrightarrow{AA}$

$\overrightarrow{AC}+\overrightarrow{CE}-\overrightarrow{AE}$

$= \overrightarrow{AE} - \overrightarrow{AE}$

$= \vec{0}$

c) $\overrightarrow{EA}-\overrightarrow{CB}+\overrightarrow{DB}+\overrightarrow{AD}$ = $\overrightarrow{EC}$

$= \overrightarrow{EA} + \overrightarrow{BC} + \overrightarrow{DB} + \overrightarrow{AD}$

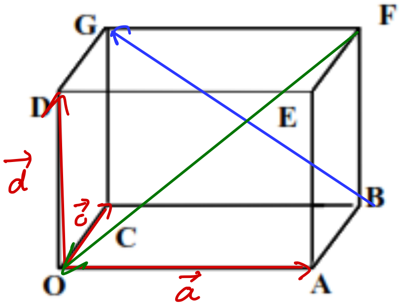
$= \overrightarrow{EA} + \overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BC}$

d) $\overrightarrow{PT}-\overrightarrow{QT}+\overrightarrow{SR}-\overrightarrow{SQ}$ = $\overrightarrow{PR}$

$= \overrightarrow{PT} + \overrightarrow{TQ} + \overrightarrow{SR} + \overrightarrow{QS}$

$= \overrightarrow{PT} + \overrightarrow{TQ} + \overrightarrow{QS} + \overrightarrow{SR}$

Ex7. Given the cube composed of vectors below, where $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OC}=\vec{c}$, and $\overrightarrow{OD}=\vec{d}$. Express each of the following in terms of $\vec{a}, \vec{c}$ and $\vec{d}$.



$\overrightarrow{BG} = -\vec{a} + \vec{d}$

there are multiple pathways!

$\overrightarrow{FO} = -\vec{d} - \vec{c} - \vec{a}$

$= -(\vec{a}+\vec{c}+\vec{d})$

or $\overrightarrow{FO} = \overrightarrow{FG} + \overrightarrow{GD} + \overrightarrow{DO}$

$= -\vec{a} + -\vec{c} + -\vec{d}$

$= -a - \vec{c} - \vec{c}$

Exit Card!

The diagram shows a regular hexagon. Prove that: $\overrightarrow{AB}-\overrightarrow{BC}+\overrightarrow{CD}-\overrightarrow{DE}+\overrightarrow{EF}-\overrightarrow{FA}=\vec{o}$.

L.S $= \overrightarrow{AB} - \overrightarrow{BC} + \overrightarrow{CD} - \overrightarrow{DE} + \overrightarrow{EF} - \overrightarrow{FA}$

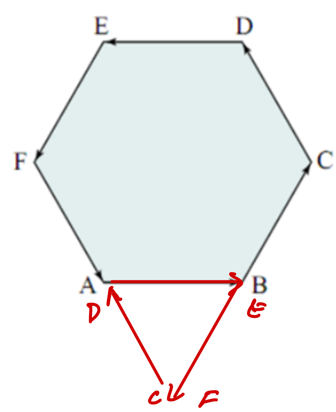
$= \overrightarrow{AB} + \overrightarrow{EF} + \overrightarrow{CD} + \overrightarrow{AB} + \overrightarrow{EF} + \overrightarrow{CD}$

$= 2\overrightarrow{AB} + 2\overrightarrow{EF} + 2\overrightarrow{CD}$

$= 2(\overrightarrow{AB} + \overrightarrow{EF} + \overrightarrow{CD})$

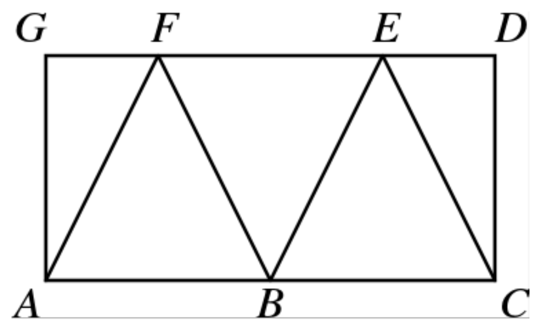
$= 2(\vec{0})$

$= \vec{0}$



Practice

1. In the diagram at the right, △ AF B and △ BEC are equilateral, and ACDG is a rectangle.



(a) Write down two other vectors **equal** to $\overrightarrow{AB}$.
(b) Write down three vectors which are **opposite** to $\overrightarrow{FE}$
(c) What vector is the **opposite** of $\overrightarrow{DC}$?
(d) Write down 3 vectors which have the same magnitude as $\overrightarrow{BC}$, but different direction.
(e) What vector is equal to $2\overrightarrow{FE}$?
(f) What vector is equal to $\frac{1}{2}\overrightarrow{FE}$?

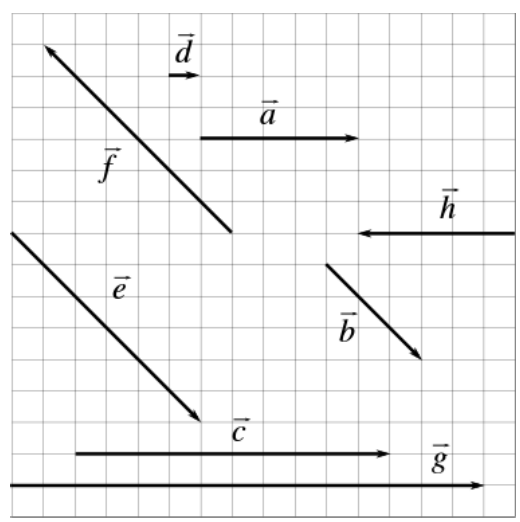
2. Using the diagram from #1, find the angles between the following vectors:

(a) $\overrightarrow{AB}$ and $\overrightarrow{AF}$
(b) $\overrightarrow{AF}$ and $\overrightarrow{AG}$
(c) $\overrightarrow{DC}$ and $\overrightarrow{AB}$
(d) $\overrightarrow{BC}$ and $\overrightarrow{CE}$
(e) $\overrightarrow{EC}$ and $\overrightarrow{AG}$
(f) $\overrightarrow{FD}$ and $\overrightarrow{BA}$

3. Sketch a vector to represent each of the following quantities, using the specified scale:

(a) a velocity of 30 m/s [south], where 1 cm = 10 m/s.
(b) a force of 20 Newtons, straight down, where 1 cm = 10 N .
(c) a displacement of 25 metres to the right, where 1 cm = 10 m.
(d) an airplane taking off a runway at an angle of 30° at a speed of 40 km/h, where 1 cm =10 km/h.

4. Using the grid at the right, choose a vector which equals:



(a) $-\vec{a}$
(b) $3\vec{a}$
(c) $-2\vec{b}$
(d) a unit vector parallel to $\vec{a}$

5. Given the vector $\vec{u}$ such that $|\vec{u}| = 8$ units, find the following:
(a) $|3\vec{u}|$
(b) $\left|-\frac{3}{4}\vec{u}\right|$
(c) $|-7\vec{u}|$

A vector of magnitude / length '1' unit.

6. Determine a unit vector parallel to each of the following vectors:
   (a) $\vec{a}$, given that $|\vec{a}| = 12$ units
   (b) $\vec{w}$, given that $|\vec{w}| = 10$ units
   (c) $\vec{u}$ (non-zero)

7. A boat leaves harbour at 2:00 and travels due south at 50 km/h until 3: 30, when it turns east and travels at the same speed for another hour.

   (a) Write down the displacement vectors for each part of the journey.
   (b) What is the total distance covered?
   (c) What is the displacement vector between the starting point and ending point?

8. Two planes leave an airport at the same time. Plane A travels northwest at 120 km/h, while plane B travels due east at 150 km/h. After one hour, they both land. If plane A must then travel to plane B's landing point, in what direction should it travel, and how long will it take if it travels at 120 km/h?

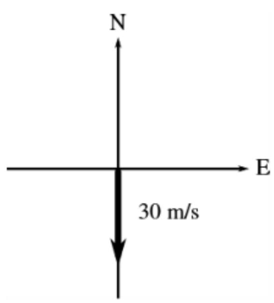
* 9. If the angle between $\vec{u}$ and $\vec{v}$ is $70°$ and $|\vec{u}| = 12, |\vec{v}| = 21$, find the angle between vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$. (Round your answer to the nearest degree) [Ans: 122°]

Answers

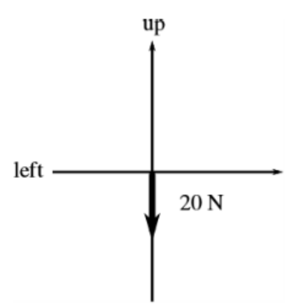
1. a. $\overrightarrow{BC}$ and $\overrightarrow{FE}$
   b. $\overrightarrow{BA}, \overrightarrow{CB}$ and $\overrightarrow{EF}$
   c. $\overrightarrow{CD}$ or $\overrightarrow{AG}$
   d. Any of $\overrightarrow{CB}, \overrightarrow{BA}, \overrightarrow{EF}, \pm\overrightarrow{AF}, \pm\overrightarrow{FB}, \pm\overrightarrow{BE}, \pm\overrightarrow{CE}$
   e. $\overrightarrow{AC}$
   f. $\overrightarrow{GF}$ or $\overrightarrow{ED}$

2. a. $60°$
   b. $30°$
   c. $90°$
   d. $120°$
   e. $150°$
   f. $180°$

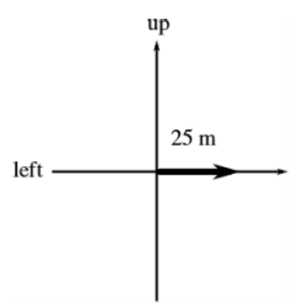
3. a.



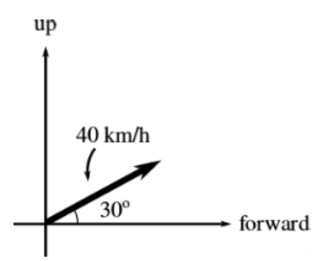
b.



c.



d.



4. a. $\vec{h}$
   b. $\vec{g}$
   c. $\vec{f}$
   d. $\vec{d}$

5. a. 24
   b. 6
   c. 56

6. a. $a = \frac{\vec{a}}{12}$
   b. $w = \frac{\vec{w}}{10}$
   c. $u = \frac{\vec{u}}{|\vec{u}|}$

7. a. 75 km$[S]$, 50 km$[E]$
   b. 125 km
   c. 90.1 km$[S33.7°E]$

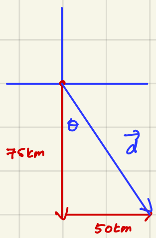
8. $[E\ 20°\ S]$ for 2 hours and 5 minutes

Hmwk Takeup:

7. A boat leaves harbour at 2:00 and travels due south at 50 km/h until 3: 30, when it turns east and travels at the same speed for another hour.

(a) Write down the displacement vectors for each part of the journey.
(b) What is the total distance covered?
(c) What is the displacement vector between the starting point and ending point?

1 h 30 m $\Rightarrow$ $50 \times 1.5 = 75$ km



a) 75 km [S] and 50 km [E]

b) $75 + 50 = 125$ km

c) $|\vec{d}| = \sqrt{75^2 + 50^2}$

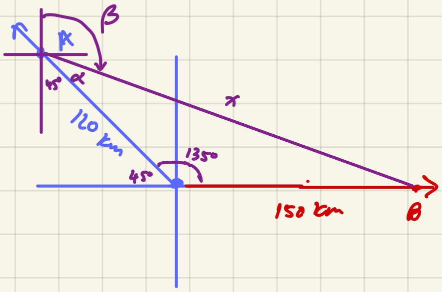
$= \sqrt{8125}$

$\therefore \vec{d} = 25\sqrt{13}$ km [S 33.7° E]

$\tan\theta = \frac{50}{75}$

$\theta = 33.69°$

8. Two planes leave an airport at the same time. Plane A travels northwest at 120 km/h, while plane B travels due east at 150 km/h. After one hour, they both land. If plane A must then travel to plane B's landing point, in what direction should it travel, and how long will it take if it travels at 120 km/h?



$$x^2 = 120^2 + 150^2 - 2(120)(150)\cos 135^\circ$$

$$x = 249.71$$

$$150^2 = 120^2 + 249.71^2 - 2(120)(249.71)\cos\alpha$$

$$\cos\alpha = \frac{150^2 - 120^2 - 249.71^2}{-2(120)(249.71)}$$

$$\cos\alpha \doteq 0.9045$$

$$\alpha = 25.23^\circ$$

$$\text{Time} = \frac{249.71}{120}$$

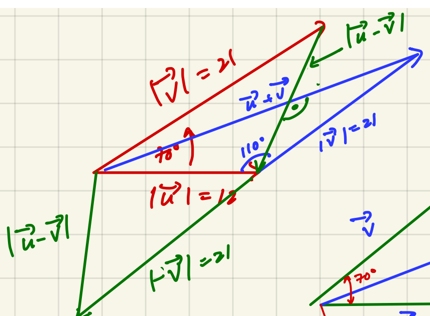
$$= 2.08\,h$$

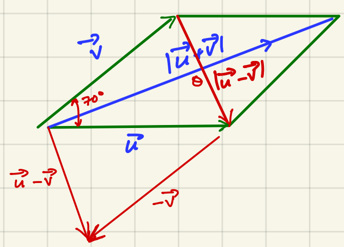
$$= 2h\ 4.86\,\text{min}$$

$$\beta = 180 - 45^\circ - 25.23^\circ$$

∴ it will take 2.1 hours in the direction of 110° Bearing [E 20° S]

9. If the angle between $\vec{u}$ and $\vec{v}$ is $70°$ and $|\vec{u}| = 12, |\vec{v}| = 21$, find the angle between vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$. (Round your answer to the nearest degree) [Ans: 122°]





$$|\vec{u}+\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos 110^\circ$$

$$|\vec{u}+\vec{v}| = \sqrt{12^2 + 21^2 - 2(12)(21)\cos 110^\circ}$$

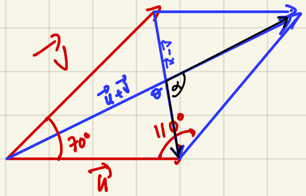
$$\doteq 27.520$$

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |-\vec{v}|^2 - 2|\vec{u}||-\vec{v}|\cos 70^\circ$$

$$= (12)^2 + (21)^2 - 2(12)(21)\cos 70^\circ$$

$$|\vec{u}-\vec{v}| = \sqrt{412.621}$$

$$\doteq 20.313$$



$$|\vec{u}|^2 = \left|\frac{\vec{u}+\vec{v}}{2}\right|^2 + \left|\frac{\vec{u}-\vec{v}}{2}\right|^2 - 2\left|\frac{\vec{u}+\vec{v}}{2}\right|\left|\frac{\vec{u}-\vec{v}}{2}\right|\cos\theta$$

$$12^2 = \left|\frac{27.520}{2}\right|^2 + \left|\frac{20.313}{2}\right|^2 - 2\left|\frac{27.520}{2}\right|\left|\frac{20.313}{2}\right|\cos\theta$$

$$0.5312 = \cos\theta$$

$$\theta = 57.91^\circ$$

Method 2: Algebraic Approach By Dot Product (6-5)

$$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = |\vec{u}+\vec{v}||\vec{u}-\vec{v}|\cos\alpha$$

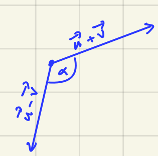
$$\vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{v} = |\vec{u}+\vec{v}||\vec{u}-\vec{v}|\cos\alpha$$

$$|\vec{u}|^2 - |\vec{v}|^2 = (27.520)(20.313)\cos\alpha$$

$$\frac{12^2 - 21^2}{(27.520)(20.313)} = \cos\alpha \qquad \alpha = 122.09$$

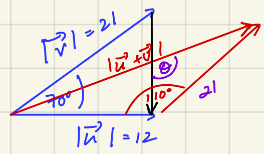
∴ the angle between $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ is 58° (or 122°)

tail to tail



9. If the angle between $\vec{u}$ and $\vec{v}$ is $70°$ and $|\vec{u}|=12, |\vec{v}|=21$, find the angle between vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$. (Round your answer to the nearest degree) [Ans: 122°]

Method 1: Geometric Approach



$$|\vec{u}+\vec{v}|^2 = |\vec{u}|^2+|\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos 110°$$

$$|\vec{u}+\vec{v}| = \sqrt{12^2+21^2-2(12)(21)\cos 110°}$$

$$= 27.520$$

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2+|\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos 70°$$

$$|\vec{u}-\vec{v}| = \sqrt{12^2+21^2-2(12)(21)\cos 70°}$$

$$= 20.313$$

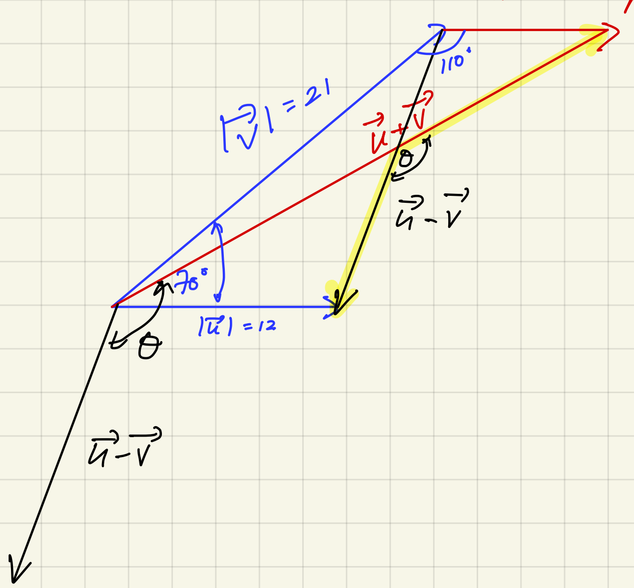
$$|\vec{v}|^2 = \left|\frac{\vec{u}+\vec{v}}{2}\right|^2 + \left|\frac{\vec{u}-\vec{v}}{2}\right|^2 - 2\left|\frac{u+v}{2}\right|\left|\frac{u-v}{2}\right|\cos\theta$$

$$21^2 = \left(\frac{27.520}{2}\right)^2 + \left(\frac{20.313}{2}\right)^2 - 2\left(\frac{27.520}{2}\right)\left(\frac{20.313}{2}\right)\cos\theta$$

$$\cos\theta = -0.5315$$

$$\theta \doteq 122.1°$$

Method 3: Vector in Component Form (6-2)



$$\vec{u} = [12\cos 0°, 12\sin 0°] = [12, 0]$$

$$\vec{v} = [21\cos 70°, 21\sin 70°] = [7.182, 19.733]$$

$$\vec{u}+\vec{v} = [19.182, 19.733] \Rightarrow \frac{|\vec{u}+\vec{v}|}{2} = \frac{27.519}{2} = 13.759$$

$$\vec{u}-\vec{v} = [4.818, -19.733] \Rightarrow \frac{|\vec{u}-\vec{v}|}{2} = \frac{20.312}{2} = 10.156$$

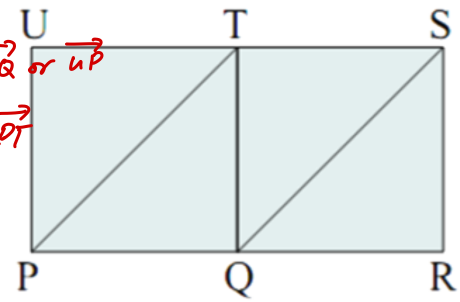
$$21^2 = (13.759)^2+(10.156)^2 - 2(13.759)(10.156)\cos\theta$$

$$\cos\theta = -0.5316$$

$$\theta = 122.1°$$

Warm Up

1. The diagram contains two squares. Express each difference as a single vector



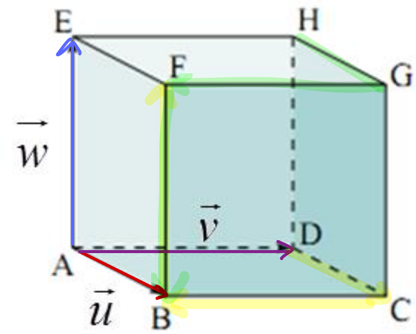
   a) $\overrightarrow{SQ} - \overrightarrow{ST}$ $= \overrightarrow{SQ} + \overrightarrow{TS} = \overrightarrow{SQ} + \overrightarrow{QR} = \overrightarrow{SR}$ or $\overrightarrow{TQ}$ or $\overrightarrow{UP}$

   b) $\overrightarrow{QT} - \overrightarrow{QP}$ $= \overrightarrow{QT} + \overrightarrow{PQ} = \overrightarrow{QT} + \overrightarrow{TS} = \overrightarrow{QS}$ or $\overrightarrow{PT}$

   c) $\overrightarrow{PR} - \overrightarrow{QS}$ $= \overrightarrow{PR} + \overrightarrow{SQ} = \overrightarrow{UQ}$ or $\overrightarrow{TR}$

   d) $\overrightarrow{PT} - \overrightarrow{TS}$ $= \overrightarrow{PT} + \overrightarrow{ST} = \overrightarrow{PU}$ or $\overrightarrow{QT}$ or $\overrightarrow{RS}$

2. The diagram shows a cube, where $\overrightarrow{AB} = \vec{u}$, $\overrightarrow{AD} = \vec{v}$ and $\overrightarrow{AE} = \vec{w}$. Determine a single vector equivalent to each of the following.



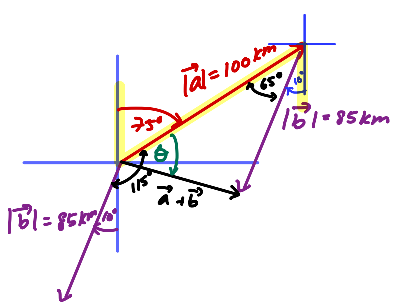
   a) $\vec{u} - \vec{v} + \vec{w} =$ $\overrightarrow{DF}$

   $\overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{BF}$

   b) $\vec{u} - \vec{v} - \vec{w} =$ $\overrightarrow{HB}$

   $\overrightarrow{HG} + \overrightarrow{GF} + \overrightarrow{FB}$

3. If $|\vec{a}| = 100$ km [N75°E] and $|\vec{b}| = 85$ km [S10°W], find $\vec{a} + \vec{b}$.



$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|a||b| \cos 65°$

$= 100^2 + 85^2 - 2(100)(85)\cos 65°$

$|\vec{a}+\vec{b}| \doteq 100.20$ km

$85^2 = 100^2 + 100.20^2 - 2(100)(100.20)\cos\theta$

$\cos\theta = \dfrac{85^2 - 100^2 - 100.20^2}{-2(100)(100.20)}$

$\cos\theta \doteq 0.6394$

$\theta \doteq 50.2°$

$\vec{a}+\vec{b} = 100.2$ km [E35°.2 S]

$= 100.2$ km [125.2°]

$= 100.2$ km [S54.8°E]

Method 2: Vectors in Component Form (6-3)

$\vec{a} = [100 \cos 15°, 100 \sin 15°]$

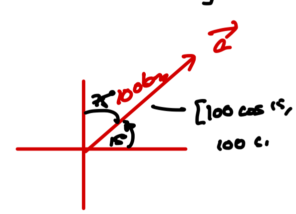
$\vec{b} = [85 \cos 260°, 85 \sin 260°]$

$\vec{a}+\vec{b} = [81.83, -57.826]$ ⇐ vector in component form

x-component ← ↳ y-component

$\vec{a}+\vec{b} = 100.2$ km [125.2°] or [E35.2°S] or [S54.8°E]

vector with magnitude + direction



$\tan\theta = \dfrac{-57.826}{81.830}$

$\theta \doteq 35.2°$

∴ E 35.2° S

6.2 Cartesian Vectors and Operations in 2-Space

Vector Properties

1) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ Commutative Law
2) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ associative Law
3) $(mn)\vec{a}=m(n\vec{a})=n(m\vec{a})$ associative Law
4) $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$ distributive Law
5) $(m+n)\vec{a}=m\vec{a}+n\vec{a}$ distributive Law
6) $\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{o}$ Additive Inverse
7) $\vec{a}+\vec{o}=\vec{o}+\vec{a}=\vec{a}$ Additive Identity

1. Simplify each of the following:

a) $3(5\vec{a}+\vec{b})-(2\vec{b}-4\vec{a})$

$= 15\vec{a}+3\vec{b}-2\vec{b}+4\vec{a}$

$= 19\vec{a}+\vec{b}$

b) $6(3\vec{a}-2\vec{b}+5\vec{c})+\frac{1}{2}(4\vec{a}+4\vec{b})-10(3\vec{c}-\vec{b}+2\vec{a})$

$= \cancel{18\vec{a}}-12\vec{b}+\cancel{30\vec{c}}+\cancel{2\vec{a}}+2\vec{b}\cancel{-30\vec{c}}+10\vec{b}\cancel{-20\vec{a}}$

$= 0\vec{a}+0\vec{b}+0\vec{c}$

$= \vec{o}$

2. If $\vec{a}=2\vec{x}+3\vec{y}-4\vec{z}$ and $\vec{b}=\vec{x}+5\vec{z}$, express $10\vec{b}-2\vec{a}$ in terms of $\vec{x}$, $\vec{y}$ and $\vec{z}$.

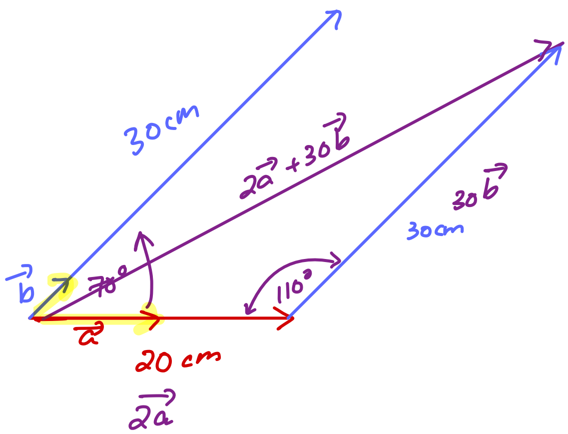
$10\vec{b}-2\vec{a}$

$= 10(\vec{x}+5\vec{z})-2(2\vec{x}+3\vec{y}-4\vec{z})$

$= 10\vec{x}+50\vec{z}-4\vec{x}-6\vec{y}+8\vec{z}$

$= 6\vec{x}-6\vec{y}+58\vec{z}$

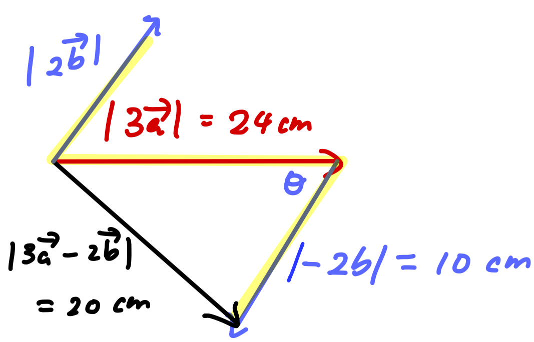
3. Determine the value of $|2\vec{a}+30\vec{b}|$ if $|\vec{a}|$ = 10 cm, $|\vec{b}|=1$ cm, and the angle between the two vectors is 70°.



$$|2\vec{a}+30\vec{b}|^2 = |2\vec{a}|^2+|30\vec{b}|^2 - 2|2\vec{a}||30\vec{b}|\cos 110°$$
$$= 20^2 + 30^2 - 2(20)(30)\cos 110°$$
$$|2\vec{a}+30\vec{b}| = 41.36 \text{ cm}$$

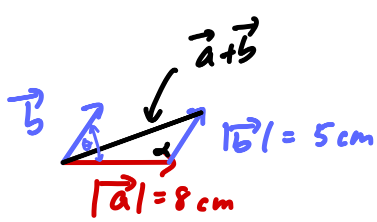
4. Determine the value of $|\vec{a}+\vec{b}|$ if $|3\vec{a}|$ = 24 cm, $|2\vec{b}|$ = 10 cm and $|3\vec{a}-2\vec{b}|$ = 20 cm.

$|\vec{a}| = 8$ cm $\quad |\vec{b}| = 5$ cm



$$20^2 = 24^2+10^2 - 2(24)(10)\cos\theta$$
$$\cos\theta = 0.575$$
$$\theta = 54.9°$$

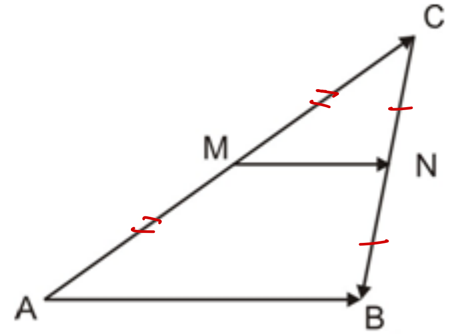
$$\alpha = 180 - \theta$$
$$= 125.1°$$



$$|\vec{a}+\vec{b}|^2 = 8^2+5^2 - 2(8)(5)\cos 125.1°$$
$$|\vec{a}+\vec{b}| = 11.6 \text{ cm}$$

5. Consider the triangle ΔABC . Let M be the midpoint of AC and N be the midpoint of BC.

Prove that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}$.



$$\begin{aligned} L.S. &= \overrightarrow{MN} \\ &= \overrightarrow{MC} + \overrightarrow{CN} \\ &= \tfrac{1}{2}(\overrightarrow{AC}) + \tfrac{1}{2}\overrightarrow{CB} \\ &= \tfrac{1}{2}(\overrightarrow{AC} + \overrightarrow{CB}) \\ &= \tfrac{1}{2}\overrightarrow{AB} \\ &= R.S \end{aligned}$$

Cartesian (Algebraic) Vectors

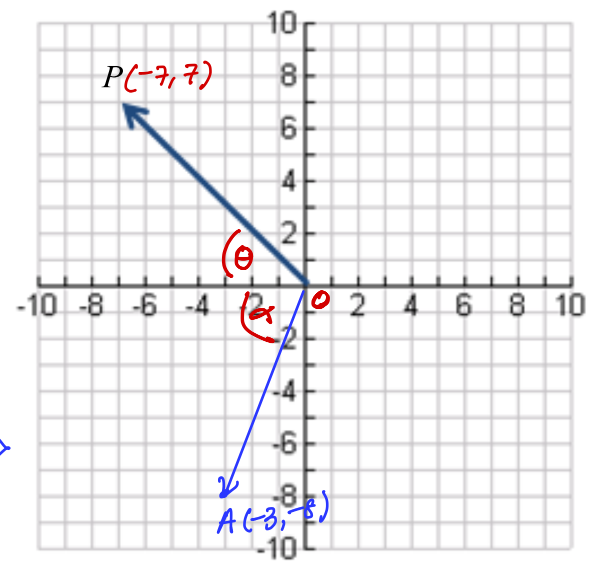
When writing vectors in component form, we use "square brackets" with the x-component and y-component on the inside. Such vectors are called **CARTESIAN** or **ALGEBRAIC VECTORS**.

A ***POSITION VECTOR***, $\overrightarrow{OP}$, has its tail at O (0,0) and head at point P(a, b).

Thus, $\overrightarrow{OP} = [a, b]$ where

a = *x*-component and b = *y*-component. Use Pythagoras' theorem to calculate magnitude of $\overrightarrow{OP}$, therefore:

$$|\overrightarrow{OP}| = \sqrt{a^2 + b^2}$$



From the picture above, $\overrightarrow{OP} = [-7, 7] = \langle -7, 7 \rangle$

$$|\overrightarrow{OP}| = \sqrt{(-7)^2 + (7^2)} = \sqrt{49+49}$$
$$= \sqrt{98} \text{ units}$$

Ex1. Point A is located at (-3,-8).

a) Draw position vector $\overrightarrow{OA}$.

b) Determine position vector $\overrightarrow{OA}$ = $[-3,-8] = \langle -3,-8 \rangle$

c) Determine the magnitude of $\overrightarrow{OA}$.

$$|\overrightarrow{OA}| = \sqrt{(-3)^2 + (-8)^2}$$
$$= \sqrt{9+64}$$
$$= \sqrt{73} \text{ units}$$

$\tan \alpha = \frac{-8}{-3}$

$\alpha = 69.5$

$\overrightarrow{OA} = \sqrt{73}$ units [W69.5°S]

$= [-3,-8]$

$\tan \theta = \frac{7}{-7}$

$\theta = 45°$

$\overrightarrow{OP} = \sqrt{98}$ units [315°]

$= \sqrt{98}$ units [N 45° W]

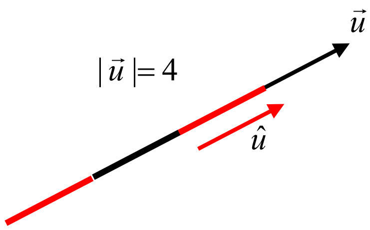
$= \sqrt{98}$ units [W 45° N]

$= [-7, 7]$

Note! Vector in Component Form is a much more efficient notation.

A vector of magnitude 1

The Unit Vector: A unit vector is a vector 1 unit long. $\hat{u}$ is a unit vector parallel to $\vec{u}$



$$\hat{u} = \frac{1}{|\vec{u}|}\vec{u}$$

$\hat{u} = \frac{1}{4}\vec{u}$

Ex: $\vec{OA} = [-3, -8]$

$\hat{OA} = \frac{1}{\sqrt{73}}[-3, -8]$

$= \left[\frac{-3}{\sqrt{73}}, \frac{-8}{\sqrt{73}}\right]$ "unit vector"

To create a unit vector in the direction of a non-zero $\vec{u}$, multiply $\vec{u}$ by the scalar equal to the reciprocal of the magnitude of $\vec{u}$.

Ex2. Given the vector $\vec{u} = [-4, 3]$ determine a unit vector in the opposite direction of $\vec{u}$.

$|u| = \sqrt{(-4)^2 + (3)^2}$

$= \sqrt{25}$

$= 5$

$\hat{u} = -\frac{1}{5}[-4, 3]$

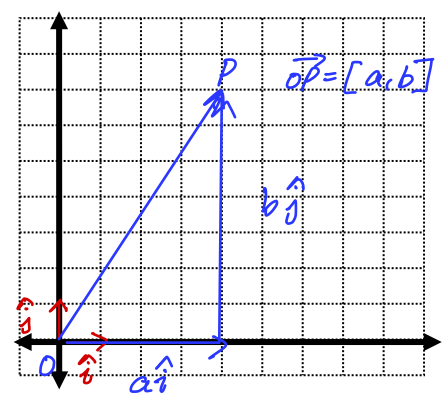
$= \left[\frac{4}{5}, -\frac{3}{5}\right]$

Basic Unit Vectors in R²- alternate representation of Cartesian Vectors

We can also define the position vector $\vec{OP} = [a, b]$ using unit vectors which have a magnitude of 1. These unit vectors are called $\hat{i}$ and $\hat{j}$, where;

$\mathbb{R}^2$

- $\hat{i} = [1, 0]$, which lies along the *x*-axis
- $\hat{j} = [0, 1]$, which lies along the *y*-axis



$\vec{OP} = [a, b] = a\hat{i} + b\hat{j}$

Ex3. Given point P (-5, 7), express vector $\vec{OP}$;

a) in component form:

$\vec{p} = \vec{OP} = [-5, 7] = \langle -5, 7 \rangle$

b) in basic unit vector form: (vector in basis form)

$\vec{p} = \vec{OP} = -5\hat{i} + 7\hat{j}$

Ex4. Write each vector in component form:

a) $\vec{OQ} = -3\hat{i} + 7\hat{j}$

$\vec{q} = \vec{OQ} = [-3, 7] = \langle -3, 7 \rangle$

b) $\vec{OR} = -10\hat{j}$

$\vec{r} = \vec{OR} = [0, -10]$

$= \langle 0, -10 \rangle$

Operations in R^2

Ex5. Given vectors $\vec{u} = [2, 4]$ and $\vec{v} = [6, 1]$;

a) determine $\vec{u} + \vec{v}$;

i) in component form

$\vec{u} + \vec{v} = [2,4] + [6,1]$
$= [8,5]$

ii) in basic unit vectors

$\vec{u} + \vec{v} = (2\hat{i} + 4\hat{j}) + (6\hat{i} + \hat{j})$
$= 8\hat{i} + 5\hat{j}$

b) determine the components of $\vec{u} - \vec{v}$

$\vec{u} - \vec{v} = [2-6, 4-1]$
$= [-4, 3]$

Ex6. Given vectors $\vec{a} = [3, 1]$ and $\vec{b} = [-2, 5]$; determine:

i) $3\vec{a} + 2\vec{b}$

$= 3[3,1] + 2[-2,5]$
$= [9,3] + [-4,10]$
$= [5,13]$

ii) $|3\vec{a} + 2\vec{b}|$

$= \sqrt{5^2 + 13^2}$
$= \sqrt{194}$ units

Ex7. Given the vectors $\vec{p} = -4\hat{i} + 5\hat{j}$ and $\vec{q} = 3\hat{i} - \hat{j}$ determine;

a) $5\vec{q} - 4\vec{p}$

$= 5(3\hat{i} - \hat{j}) - 4(-4\hat{i} + 5\hat{j})$
$= 15\hat{i} - 5\hat{j} + 16\hat{i} - 20\hat{j}$
$= 31\hat{i} - 25\hat{j}$

method 2: Component Form

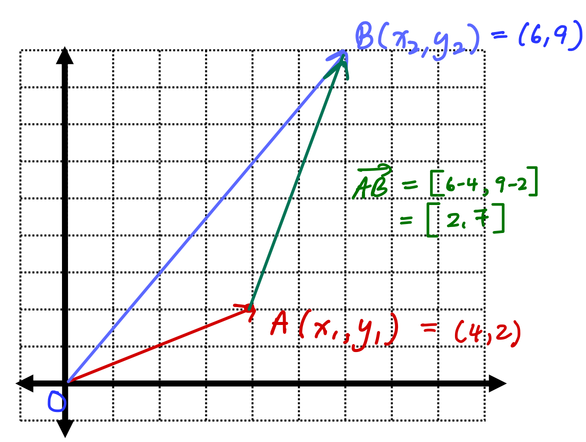
$= 5[3,-1] - 4[-4,5]$
$= [15,-5] - [-16,20]$
$= [31,-25]$

b) $|5\vec{q} - 4\vec{p}|$

$= \sqrt{31^2 + (-25)^2}$
$= \sqrt{1586}$
$\doteq 39.82$ units.

D. Vectors Defined by Two Points in R^2

The points $A(x_1, y_1)$ and $B(x_2, y_2)$ form the vector $\overrightarrow{AB}$. Using "position vectors", determine $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$. To do this, we use the triangle law of addition.



$\overrightarrow{OA} = [x_1, y_1]$
$\overrightarrow{OB} = [x_2, y_2]$

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= [x_2, y_2] - [x_1, y_1]$
$\overrightarrow{AB} = [x_2 - x_1, y_2 - y_1]$

$|\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

Ex $A(4,2)$ $B(6,9)$
$\overrightarrow{AB} = [6-4, 9-2]$
$= [2,7]$
$|\overrightarrow{AB}| = \sqrt{2^2 + 7^2}$
$= \sqrt{53}$ units

Ex8. P(4, 5), Q(-7, 10) and R(8, -3) are three points in R^2.

a) Determine $\overrightarrow{QP}$ and $|\overrightarrow{QP}|$.

$\overrightarrow{QP} = [4-(-7), 5-10]$

$= [11, -5]$

$|\overrightarrow{QP}| = \sqrt{121+25}$

$= \sqrt{146}$

$\doteq 12.08$ units

b) Determine $|\overrightarrow{PQ} + \overrightarrow{QR}|$.

$= |\overrightarrow{PR}|$

$\overrightarrow{PR} = [8-4, -3-5]$

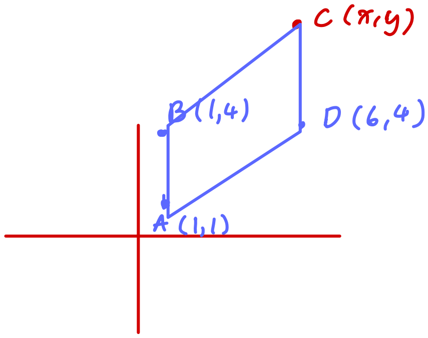
$= [4, -8]$

$|\overrightarrow{PR}| = \sqrt{16+64}$

$= \sqrt{80}$

$= 4\sqrt{5}$ units

Ex9. If A(1,1), B(1,4) and D(6,4) are vertices of the parallelogram ABCD, determine the coordinates of C.



$\overrightarrow{AB} = \overrightarrow{DC}$

$[1-1, 4-1] = [x-6, y-4]$

$[0, 3] = [x-6, y-4]$

$0 = x-6 \qquad 3 = y-4$

$x = 6 \qquad y = 7$

$\therefore C(6,7)$

E. **Scalar Multiplication**

For a real number k and a vector $\vec{v} = [v_1, v_2]$ the ***scalar product*** of k and $\vec{v}$ is

$k\vec{v} = k[v_1, v_2] = [kv_1, kv_2]$.

Ex10. Determine the values of m such that $\vec{v} = \left[1, \frac{m}{2} - 6\right]$ and $\vec{u} = \left[-4, 1 + \frac{m}{6}\right]$ are collinear.

Recall: Vectors $\vec{v}$ and $\vec{u}$ are collinear if and only if $\vec{u} = k\vec{v}, k \in R, k \neq 0$.

$\vec{u}$ and $\vec{v}$ are collinear $\Rightarrow$ $\vec{u} = k\vec{v}$, $k \neq 0$

$\left[-4, 1+\frac{m}{6}\right] = k\left[1, \frac{m}{2} - 6\right]$

$\left[-4, 1+\frac{m}{6}\right] = \left[k, k\left(\frac{m}{2}-6\right)\right]$

$-4 = k(1) \qquad 1 + \frac{m}{6} = k\left(\frac{m}{2} - 6\right)$

$k = -4 \qquad 1 + \frac{m}{6} = -4\left(\frac{m}{2} - 6\right)$

$1 + \frac{m}{6} = -2m + 24$

$\times 6 \Rightarrow 6 + m = -12m + 144$

$13m = 138$

$m = \frac{138}{13}$

Warm Up

1. Given the points A(1,2) and B(3,-2), find vector(s) collinear with $\overrightarrow{AB}$ and having a magnitude of 5.

$\overrightarrow{AB} = [3-1, -2-2]$
$= [2, -4]$

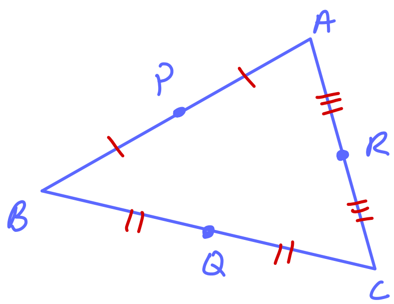
$|\overrightarrow{AB}| = \sqrt{16+4}$
$= \sqrt{20}$
$= 2\sqrt{5}$

$\hat{AB} = \frac{1}{2\sqrt{5}}[2,-4] = \frac{1}{\sqrt{5}}[1,-2] = \frac{\sqrt{5}}{5}[1,-2] = \left[\frac{\sqrt{5}}{5}, \frac{-2\sqrt{5}}{5}\right]$

rationalizing denominator

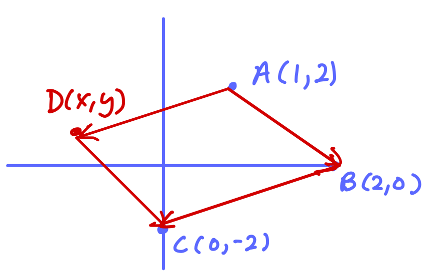
$\vec{r} = 5\hat{AB}$
$= [\sqrt{5}, -2\sqrt{5}]$ or $-\vec{r} = [-\sqrt{5}, 2\sqrt{5}]$ $\Rightarrow$ vector in opposite direction

2. Suppose P, Q, and R are the midpoints of the sides AB, BC, and CA, respectively, of a triangle ABC. Prove that $\overrightarrow{AP} + \overrightarrow{BQ} + \overrightarrow{CR} = \vec{o}$.



$LS = \overrightarrow{AP} + \overrightarrow{BQ} + \overrightarrow{CR}$
$= \frac{1}{2}(\overrightarrow{AB}) + \frac{1}{2}(\overrightarrow{BC}) + \frac{1}{2}(-\overrightarrow{AC})$
$= \frac{1}{2}[\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}]$
$= \frac{1}{2}[\underbrace{\overrightarrow{AC} + \overrightarrow{CA}}_{\overrightarrow{AA} = \vec{0}}]$
$= \frac{1}{2}\vec{0}$
$= \vec{0}$
$= R.S$

3. ABCD is a parallelogram where A (1, 2), B (2, 0) and C (0,-2) Find the coordinates of the point D.



$\overrightarrow{AB} = [2-1, 0-2]$
$= [1, -2]$

$\overrightarrow{DC} = [0-x, -2-y]$
$= [-x, -2-y]$

$\overrightarrow{AB} \parallel \overrightarrow{DC}$

$[1,-2] = [-x, -2-y]$

$1 = -x$ $\quad -2 = -2-y$
$x = -1$ $\quad y = 0$ $\quad \therefore D = (-1, 0)$

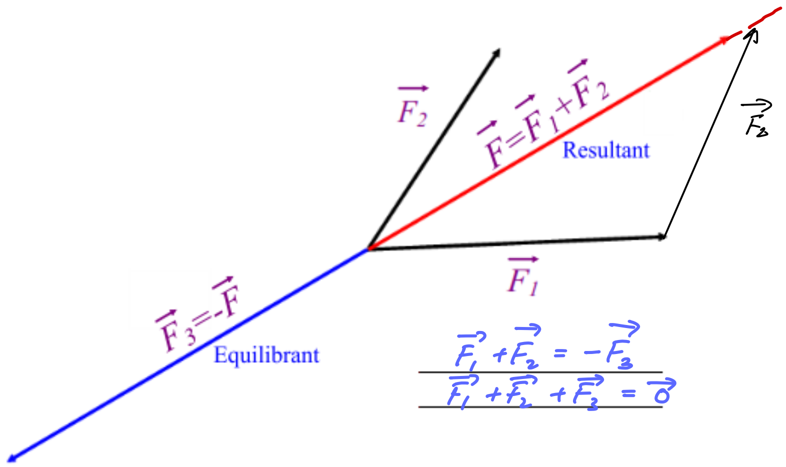
6.3 Modeling Force with Vectors

<u>Force</u> : A physical influence that causes a change in direction on a physical object. It is measured in a unit called Newton's (N).

To describe a force it is necessary to state: i) its direction
ii) the point at which it is applied
iii) its magnitude

The ***resultant*** is the sum of the vectors representing two or more forces.
The ***equilibrant*** is the opposite force that would exactly counterbalance the resultant.

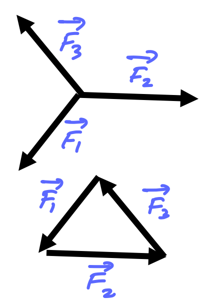
Equilibrant Force:
Let $\vec{F_1}$ and $\vec{F_2}$ be two forces acting upon an object. The resultant vector can be represented by a third vector using the concepts from vector addition.



$\vec{F_1}+\vec{F_2}=-\vec{F_3}$

$\vec{F_1}+\vec{F_2}+\vec{F_3}=\vec{0}$

Note for a system of three forces to being **equilibrium** the vectors representing those forces, when placed <u>head to tail</u>.

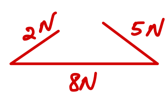


$\vec{F_1}+\vec{F_2}+\vec{F_3}=\vec{0}$

Equilibrium
(balanced)

To produce equilibrium by <u>3 forces</u>, sum of the two smaller forces ≥ the third force.

3 forces need to be able to form "a triangle"



Nope, too short!

Ex1. Which of the following sets of forces acting on an object could produce equilibrium?

(a) 2 N, 3 N, 4 N yes $2+3\geq 4$

(b) 9 N, 40 N, 41 N yes $9+40\geq 41$

(c) $\sqrt{5}$ N, 6 N, 9 N No $\sqrt{5}+6\leq 9$

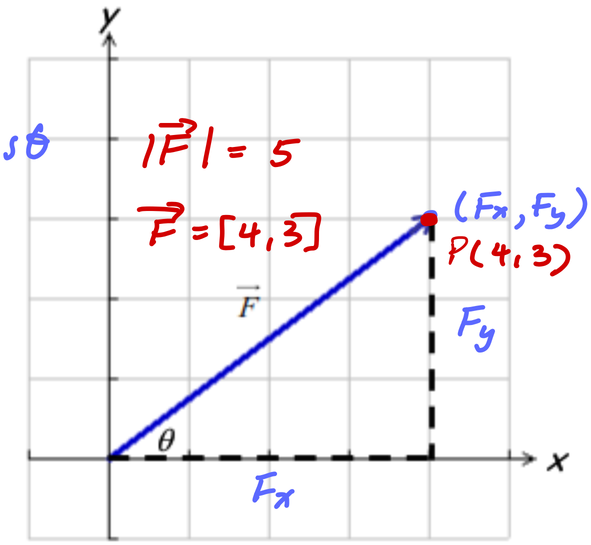
(d) 9 N, 10 N, 19 N yes $9+10\geq 19$

RESOLVING VECTORS INTO COMPONENTS (used in application problems when a Cartesian coordinate system is not used)

$\sin\theta = \dfrac{|F_y|}{|F|}$ $\qquad \cos\theta = \dfrac{|F_x|}{|F|}$

$|F_y| = |F|\sin\theta$ $\qquad |F_x| = |F|\cos\theta$

$\therefore\ \vec{F} = [|F|\cos\theta, |F|\sin\theta]$



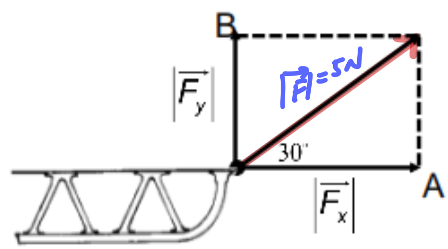
$\tan\theta = \dfrac{3}{4}$

$\theta \doteq 36.9°$

$\vec{F} = [5\cos 36.9°, 5\sin 36.9°]$

$\therefore\ = [4, 3]$

Ex. 2 A sleigh is being pulled with a 5N force at an angle of 30° with the ground.
a) Calculate the force that is pulling the sleigh forward.
b) the force that tends to lift the sleigh.



a) $\vec{F_x} = [5\cos 30°, 0]$ $\quad \therefore |\vec{F_x}| = 4.33N$

b) $\vec{F_y} = [0, 5\sin 30°]$ $\quad \therefore |\vec{F_y}| = 2.5N$

$\vec{F} = \vec{F_x} + \vec{F_y}$

$= [5\cos 30°, 0] + [0, 5\sin 30°]$

$= [5\cos 30°, 5\sin 30°]$

$\doteq [4.33, 2.5] \Rightarrow 5N\ [E\ 30°\ N]$

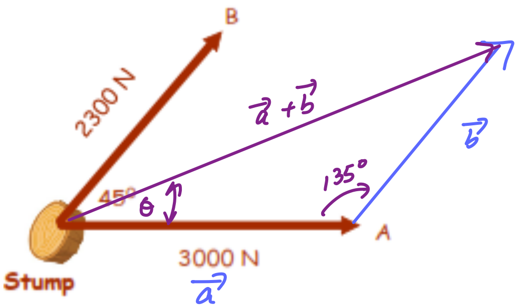
$\boxed{\vec{F_x} = [|F_x|\cos 0°, 0] \qquad \vec{F_y} = [0, |F_y|\sin 90°]}$

note! $|\vec{F}| = \sqrt{4.33^2 + 2.5^2}$

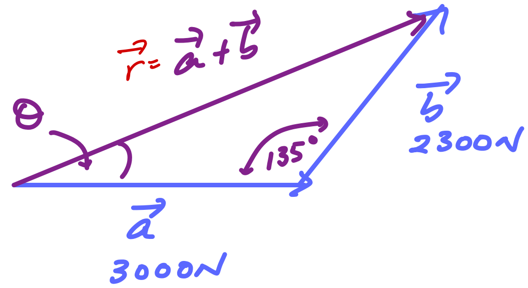
$= 5N$

Ex. 3. Two tractors are being used to pull a tree stump out of the ground. The larger tractor pulls with a force of 3000 N [E]. The smaller tractor pulls with a force of 2300 N [NE]. Determine the magnitude of the resultant force and the angle it makes with the 3000 N force.

position diagram **Vector diagram**



$\Rightarrow$



Method 1: Geometric Method

$$|\vec{a}+\vec{b}|^2 = 3000^2 + 2300^2 - 2(3000)(2300)\cos 135°$$

$$|\vec{a}+\vec{b}| = 4903.88\ N$$

$$2300^2 = 3000^2 + 4903.88^2 - 2(3000)(4903.88)\cos\theta$$

$$\cos\theta \doteq 0.9434$$

$$\theta = 19.36°$$

$$\vec{a}+\vec{b} = 4903.88 N\ [E\ 19.4°\ N \text{ of } \vec{a}]$$

method 2: Resolving Vectors into Component Form (cartesian vector).

$$\vec{a} = [3000\cos 0°, 0]$$

$$\vec{b} = [2300\cos 45°, 2300\sin 45°]$$

$$\vec{a}+\vec{b} = [3000 + 2300\cos 45°, 2300\sin 45°]$$

$$= [4626.34, 1626.34]$$

$$|\vec{a}+\vec{b}| = \sqrt{4626.34^2 + 1626.34^2}$$

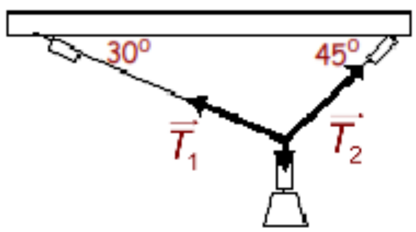
$$= 4903.88\ N$$

$$\tan\theta = \frac{1626.34}{4626.34}$$

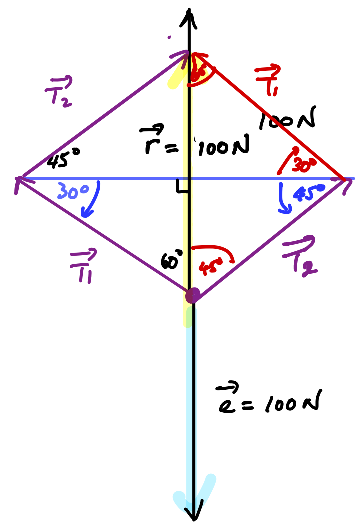
$$\theta = 19.37°$$

Ex. 4. A 100 N weight is suspended from the ceiling by two ropes that make angles of 30° and 45° with the ceiling. Determine the tension in each rope.

position diagram



Vector diagram

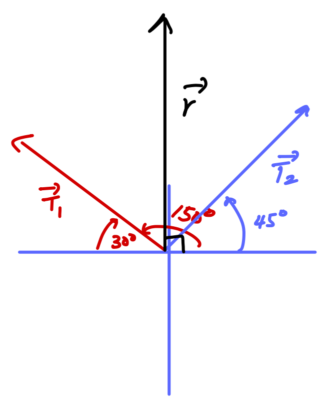


method 1: Geometric Vector

$$\frac{\sin 75°}{100} = \frac{\sin 45°}{|\vec{T}_1|} = \frac{\sin 60°}{|\vec{T}_2|}$$

$$|\vec{T}_1| = \frac{100 \sin 45°}{\sin 75°} \doteq 73.21\ N \qquad |\vec{T}_2| = \frac{100 \sin 60°}{\sin 75°} \doteq 89.66\ N$$

method 2: Cartesian Vectors



$$\vec{T}_1 = [|\vec{T}_1| \cos 150°, |\vec{T}_1| \sin 150°]$$
$$\vec{T}_2 = [|\vec{T}_2| \cos 45°, |\vec{T}_2| \sin 45°]$$
$$\vec{R} = [100 \cos 90°, 100 \sin 90°]$$

$$\vec{R} = \vec{T}_1 + \vec{T}_2$$

$$100 \cos 90° = |\vec{T}_1| \cos 150° + |\vec{T}_2| \cos 45° \quad ①$$
$$0 = -0.866\,|\vec{T}_1| + 0.707\,|\vec{T}_2|$$

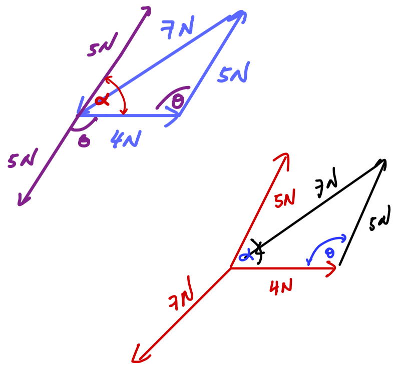
$$100 \sin 90° = |\vec{T}_1| \sin 150° + |\vec{T}_2| \sin 45° \quad ②$$
$$100 = 0.5\,|\vec{T}_1| + 0.707\,|\vec{T}_2|$$

② − ① : $100 = 1.366\,|\vec{T}_1|$
$|\vec{T}_1| = 73.21\,N$ ③ → sub into ②

$100 = 0.5\,(73.21) + 0.707\,|\vec{T}_2|$
$|\vec{T}_2| = 89.67\ N$

Check : $4+5 \geq 7$ $\therefore$ it can be in equilibrium (triangular formation is possible)

Ex. 5. Three forces having magnitudes of 4 N, 5 N, and 7 N are in a state of equilibrium. Calculate the angle between the two smaller forces. $\Rightarrow$ Assumed to be the angle formed tail to tail



$7^2 = 4^2 + 5^2 - 2(4)(5)\cos\theta$

$\cos\theta = \dfrac{7^2 - 4^2 - 5^2}{-2(4)(5)}$

$= -0.2$

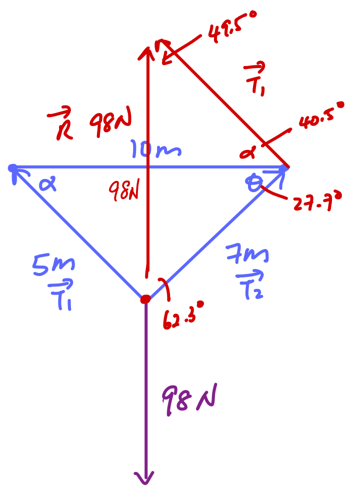
$\theta \doteq 101.5$

$\alpha = 180° - 101.5$

$= 78.5°$

$\therefore$ the angle between the 2 smaller forces is $78.5°$

Ex. 6. 10 kg mass is supported by two strings of length 5 m and 7 m attached to two points in the ceiling 10 m apart. Find the tension in each string.



Find α and θ :

$7^2 = 5^2 + 10^2 - 2(5)(10)\cos\alpha$

$\alpha \doteq 40.5°$

$5^2 = 10^2 + 7^2 - 2(10)(7)\cos\theta$

$\theta \doteq 27.7°$

$\theta + \alpha = 68.2°$

$\dfrac{\sin 68.2°}{98} = \dfrac{\sin 62.3°}{|\vec{T}_1|} = \dfrac{\sin 49.5°}{|\vec{T}_2|}$

$|\vec{T}_1| = \dfrac{98 \sin 62.3°}{\sin 68.2°}$

$\doteq 93.45\,N$

$|\vec{T}_2| = \dfrac{98 \sin 49.5°}{\sin 68.2°}$

$\doteq 80.26\,N$

$\vec{F} = mass \times 9.8$

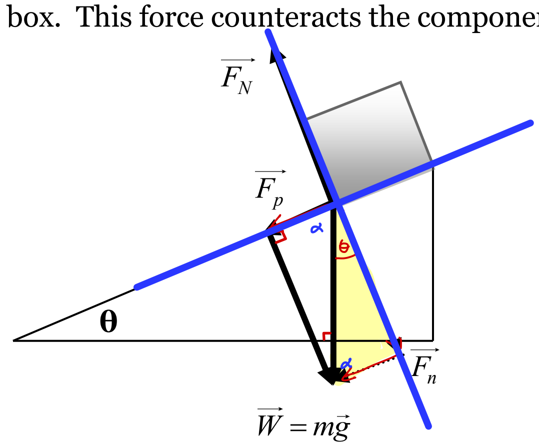
$= 10kg \times 9.8$

$= 98\,N$

$$\boxed{N = kg \cdot \underbrace{m \cdot s^{-2}}_{\text{mass} \cdot \text{acceleration due to gravity } m \cdot s^{-2}}}$$

A Ramp Problem

The next example shows that rectangular components do not necessarily have to be horizontal or vertical. A box weighing $\vec{W}$ N is resting on a ramp that is inclined at an angle of θ°. Resolve the weight into the rectangular components, $\vec{F_p}$, the force parallel to the surface , and $\vec{F_n}$, the force perpendicular to the surface. Note $\vec{F_N}$ is the force of the ramp pushing against the box. This force counteracts the component of gravity in the opposite direction to keep the box at rest.



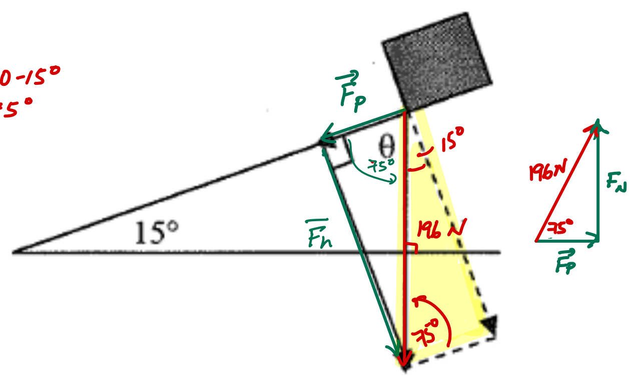
Ex. 7: Components of the forces of gravity A 20-kg trunk is resting on a ramp inclined at an angle of 15°. Calculate the components of the force of gravity on the trunk that are parallel and perpendicular to the ramp.

$|\vec{F_p}| = 196 \cos 75°$ or $196 \sin 15°$
$= 50.7$ N

$|\vec{W}| = 20\text{ kg} \times 9.8$
$= 196$N

$\theta = 90 - 15°$
$= 75°$

$|\vec{F_n}| = 196 \sin 75°$ or $196 \cos 15°$
$= 189.3$ N



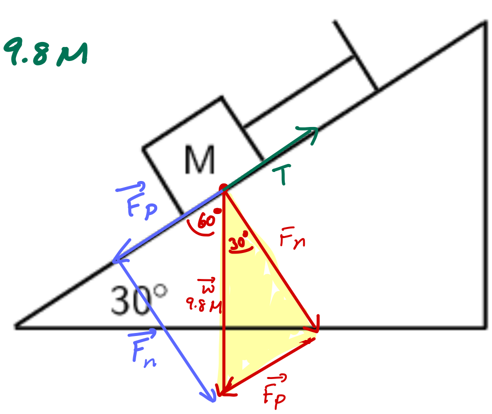
Ex.8 : A block of mass M is held stationary by a rope of negligible mass. The block rests on a frictionless plane which is inclined at 30° to the horizontal.

a) Resolve the force due to gravity into components that are parallel and perpendicular to the plane.

$|\vec{W}| = 9.8M$

$|\vec{F_p}| = 9.8M \cos 60°$ or $9.8M \sin 30°$

$|\vec{F_n}| = 9.8M \sin 60°$ or $9.8M \cos 30°$

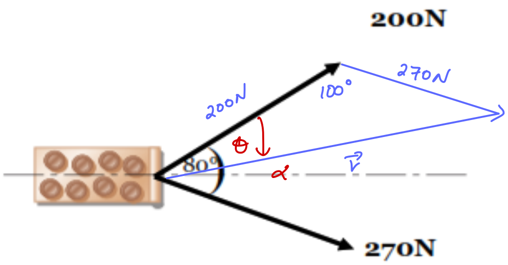


b) Calculate the tension in the rope .($\vec{g} = 9.8$ m/s^2)

$|T| = 9.8M \sin 30°$
$= 4.9M$ Newtons (opposite direction of $\vec{F_p}$)

Practice Questions

1. Two horses pull a load. The ropes between the horses and the load are at an angle of 80° to each other. One horse pulls with a force of 200 N (newton), and the other with a force of 270 N. Here is a diagram to illustrate the two forces. Calculate the resultant force.[Ans.363 N at 47° to 200N]



$|\vec{r}|^2 = 200^2 + 270^2 - 2(200)(270°) \cos 100°$

$|\vec{r}| = 362.84N$

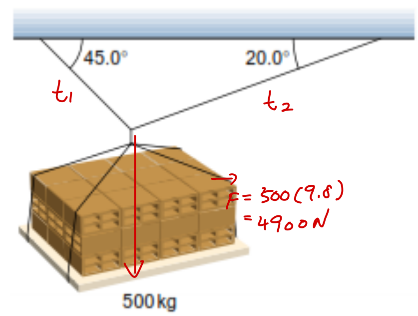
$270^2 = 200^2 + 362.84^2 - 2(200)(362.84) \cos\theta$

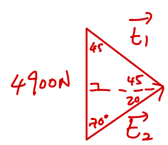
$\cos\theta \doteq 0.618$

$\theta \doteq 47.1°$

$\therefore \vec{r} = 362.8N$ [E 47°S of 200N]

2. A mass of 500 kg is supported by two cables as illustrated. What is the tension in each cable? ($\vec{g} = 9.8 m/s^2$) [Ans. 3823 N and 5080.5N]





$\frac{\sin 65°}{4900} = \frac{\sin 70°}{|\vec{t_1}|} = \frac{\sin 45°}{|\vec{t_2}|}$

$|\vec{t_1}| = 5080.5N$ $\quad |\vec{t_2}| = 3823.0N$

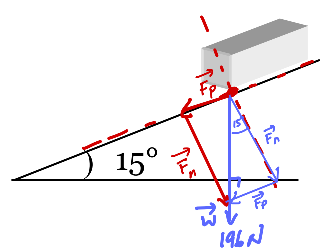
3. A lawnmower is pushed across a lawn by applying a force of 95 N along the handle of the mower. The handle makes an angle of 60.0° with the horizontal.
   a. What are the horizontal and vertical components of the force?[Ans. $F_x = 48N$,$F_y = 82N$]
   b. The handle is lowered so that it makes an angle of 30.0° with the horizontal. What are the horizontal and vertical components of the force? ?[Ans. $F_x = 82N$,$F_y = 48N$]

4. 20-kg trunk is resting on a ramp inclined at an angle of 15°. Calculate the components of force of gravity on the trunk that are parallel and perpendicular to the ramp. [Ans. $\vec{F}_p = 50.7$ N ,$\vec{F}_n = 189.3$N]

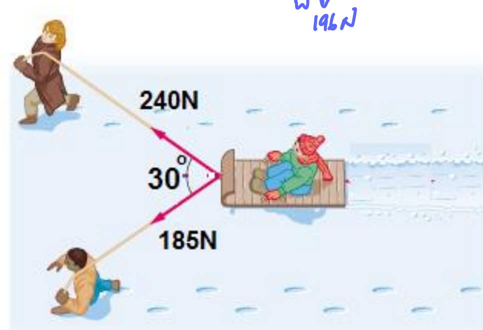
$\vec{W} = 20 \times 9.8 = 196N$

$|\vec{F_p}| = 196 \sin 15° \doteq 50.7N$

$|\vec{F_n}| = 196 \cos 15° \doteq 189.3N$



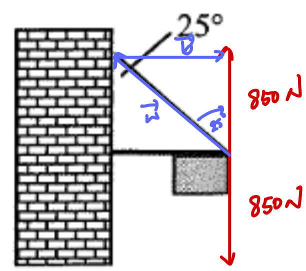
5. Using two ropes that make an angle of 30° to each other, Jack and Alex pull Bill in a sleigh. Jack pulls with 240N force and Alex pulls with a force of 185N. Determine the magnitude and direction of the equilibrant force. [Ans. 410.8 N , makes an angle of 167° with the larger force]



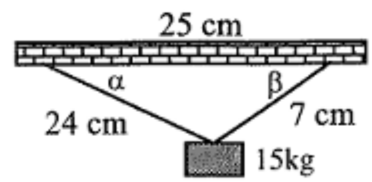
6. An advertising sign is supported by a horizontal steel brace extending at right angles from the side of a building, and by a wire attached to the building above the brace at an angle of 25°. If the force of gravity on the sign is 850 N, find the tension in the wire and the compression in the steel brace. [Ans. The tension on the wire is 937.9N and the compression in the steel brace is 396.4 N]

$\tan 25° = \frac{|\vec{B}|}{850}$ $\quad$ $\cos 25° = \frac{850}{|\vec{T}|}$

$|\vec{B}| = 396.4N$ $\quad$ $|\vec{T}| = 937.9N$

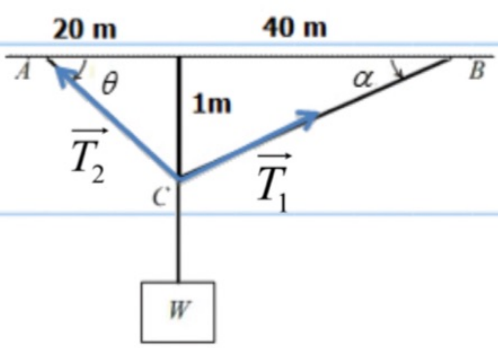


7. An object of 15 kg is suspended by two cords of lengths 7 cm and 24 cm, and these two cords are 25 cm apart. Find the tension in each cord.[Ans. The tensions of two cords are 141N and 41.3N]



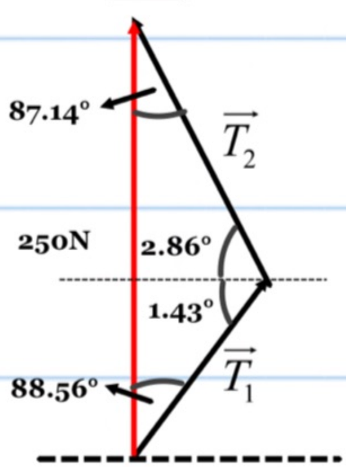
Warm Up

A ski chairlift is suspended between two towers that are 60 m apart horizontally. When the chairlift is 20m from one tower, the cable sags 1m.The chairlift is loaded with four skiers with a combined weight of 250N (including the mass of the chair). What are the tensions on the two parts of the cable?



$$\tan\theta = \frac{1}{20} \rightarrow \theta = 2.86°$$

$$\tan\alpha = \frac{1}{40} \rightarrow \alpha = 1.43°$$

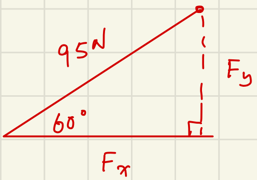


$$\frac{\sin(4.29°)}{250} = \frac{\sin(87.14°)}{|\vec{T}_1|} = \frac{\sin(88.57°)}{|\vec{T}_2|}$$

$$|\vec{T}_1| = 3337.87N \ , \ |\vec{T}_2| = 3340.99\ N$$

3. A lawnmower is pushed across a lawn by applying a force of 95 N along the handle of the mower. The handle makes an angle of 60.0° with the horizontal.
   a. What are the horizontal and vertical components of the force?[Ans. F_x = 48N ,F_y = 82N]
   b. The handle is lowered so that it makes an angle of 30.0° with the horizontal. What are the horizontal and vertical components of the force? ?[Ans. F_x = 82N ,F_y = 48N]

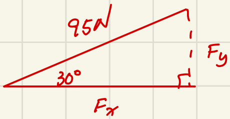
a)



$|\vec{F_x}| = 95\cos 60°$
$= 47.5N$

$|\vec{F_y}| = 95\sin 60°$
$= 82.3N$

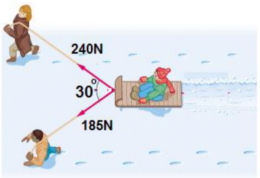
b)

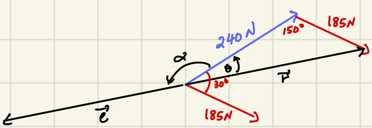


$|F_x| = 95\cos 30°$
$= 82.3N$

$|\vec{F_y}| = 95\sin 30°$
$= 47.5N$

5. Using two ropes that make an angle of 30° to each other, Jack and Alex pull Bill in a sleigh. Jack pulls with 240N force and Alex pulls with a force of 185N. Determine the magnitude and direction of the equilibrant force. [Ans. 410.8 N , makes an angle of 167° with the larger force]





$|\vec{r}|^2 = 185^2 + 240^2 - 2(185)(240)\cos 150°$
$|\vec{r}| = 410.76\ N$

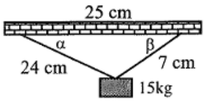
$185^2 = 240^2 + 410.76^2 - 2(240)(410.76)\cos\theta$

$\cos\theta \doteq 0.9743$
$\theta \doteq 13.0°$

$\alpha = 180 - 13.0°$
$= 167.0°$

$\therefore\ \vec{e} = 410.8N\ [167.0°$ of $240N$ Force$]$

7. An object of 15 kg is suspended by two cords of lengths 7 cm and 24 cm, and these two cords are 25 cm apart. Find the tension in each cord.[Ans. The tensions of two cords are 141N and 41.3N]



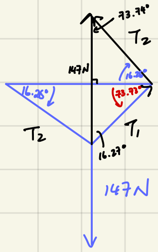
$\vec{W} = 15 \times 9.8$
$= 147\,N$

$16.26 + 73.73 = 89.99 \Rightarrow 90°$

$7^2 = 24^2 + 25^2 - 2(24)(25)\cos\alpha$

$\cos\alpha = 0.96$

$\alpha = 16.26°$



$\frac{\sin 90°}{147} = \frac{\sin 73.74°}{|\vec{T_1}|} = \frac{\sin 16.27°}{|\vec{T_2}|}$

$|\vec{T_1}| = \frac{147 \sin 73.74°}{\sin 90°}$

$\doteq 141.1\,N$

$|\vec{T_2}| = \frac{147 \sin 16.27°}{\sin 90°}$

$\doteq 41.2\,N$

$24^2 = 7^2 + 25^2 - 2(7)(25)\cos\beta$

$\cos\beta = 0.28$

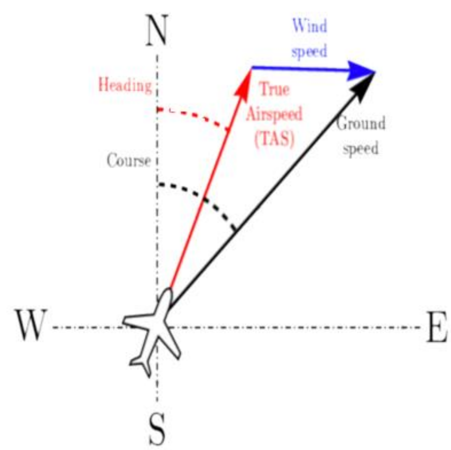
$\beta = 73.73°$

6.4 Velocity as a Vector

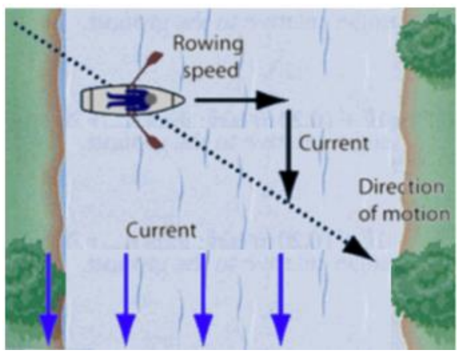
Velocity measures the direction and the rate of change in the position of an object.

- Velocity is a vector because it has both ______________________ and _________________.

Air speed (water speed) is the speed of a plane (boat) produced by engine , relative to a person on board. (heading)



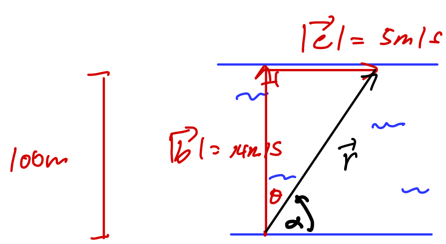
Wind (water) speed is the magnitude of wind (current) pushing plane (boat)



Ground speed is the speed of a plane (boat) relative to a person on the ground and includes the effect of wind (current). [resultant vector of airspeed and wind speed]

Ex1: A boat with a forward velocity of 14 m/s is traveling across a river, directly towards the opposite shore. At the same time, a current of 5 m/s carries the boat down the river.

(a) Determine the resultant velocity of the boat.



$|\vec{b}| = 14m/s$

$|\vec{c}| = 5m/s$

$|\vec{r}| = \sqrt{5^2 + 14^2}$

$= \sqrt{221}$

$\doteq 14.87\ m/s$

Aside! Note

$\tan\theta = \frac{5}{14}$

$\theta = 19.65°$ $\quad \alpha = 90 - 19.65$

$= 70.35°$

$\vec{r} = [14.87\cos 70.35°, 14.87\sin 70.35°]$

$= [5, 14]$

or $14.87\ m/s$ [E 70.35° N]

(b) Suppose the river was 100 m across, how long would it take for the boat to cross the river?

$T = \frac{D}{V}$

$= \frac{100}{14}$

$= 7.14$ seconds

or

$d_r = \sqrt{100^2 + 35.7^2}$

$= 106.18\ m$

$t = \frac{106.18}{14.87}$

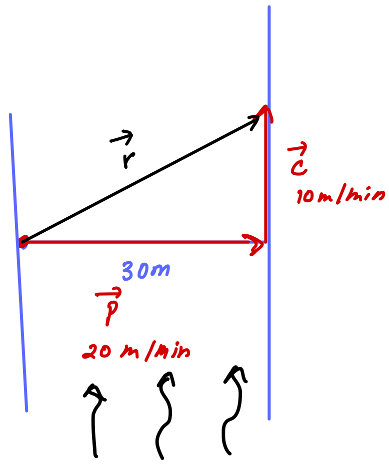
$= 7.14$ seconds.

$\tan 19.65 = \frac{d\vec{c}}{100}$

$d\vec{c} = 35.7\ m$

Ex2: Alex uses a canoe to cross to the other side of a river, which is 30 m wide. The river is flowing at 10 m/min and Alex can paddle at 20 m/min.

a) If he points his canoe directly across the river (perpendicular to the bank), where will he land? How long will the crossing take?



Time to cross $= \frac{30}{20}$

$= 1.5 min$

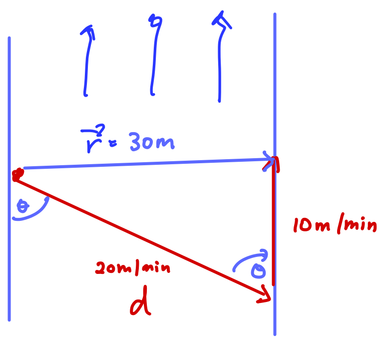
distance $= vt$

$= 10 \cdot 1.5$

$= 15 m$

$\therefore$ 15 m down stream and 1.5 min to get across

b) In what direction should he aim the canoe in order to land at a point directly opposite his starting point? How long will it take to make this crossing?



$\cos\theta = \frac{10}{20}$

$\theta = 60°$

$\therefore$ head in the direction of 60° upstream from shore

$\sin 60° = \frac{30}{d}$

$d = 34.64m$

$t = \frac{34.64}{20}$

$= 1.73$ min

or $\sqrt{20^2 - 10^2}$

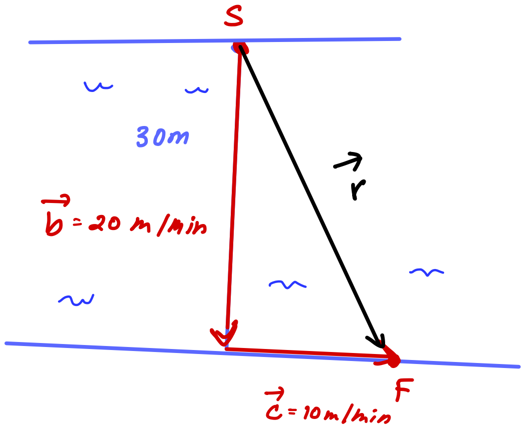
$= 17.3$ m/min

$t = \frac{30}{17.3}$

$= 1.73$ min

Ex2: Alex uses a canoe to cross to the other side of a river, which is 30 m wide. The river is flowing at 10 m/min and Alex can paddle at 20 m/min.

a) If he points his canoe directly across the river (perpendicular to the bank), where will he land? How long will the crossing take?



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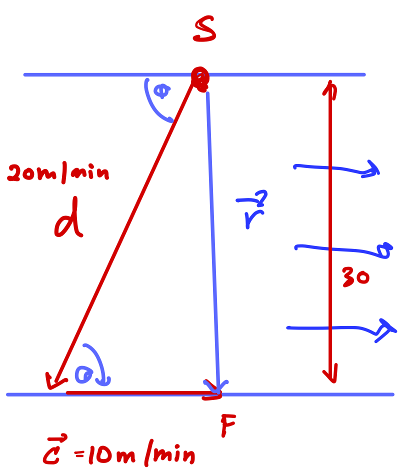

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$= 1.73$ min

or $\sqrt{20^2 - 10^2}$

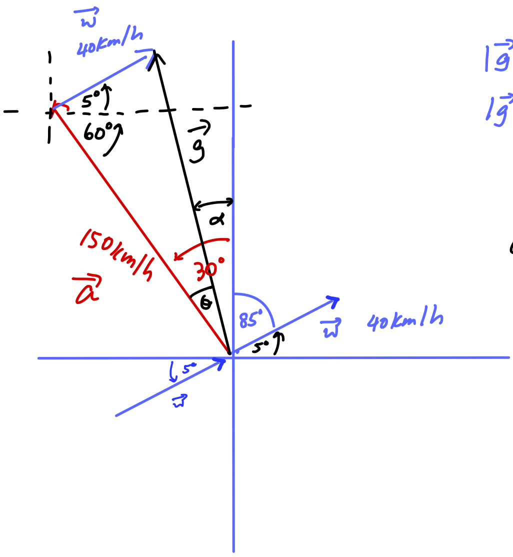
$= 17.3$ m/min

$t = \frac{30}{17.3}$

$= 1.73$ min

Ex3 : A small aircraft is flying on a heading [N 30° W] at a constant speed of 150 km/h. The wind is blowing **from** 5° south of west with a speed of 40 km/h. Determine the actual speed and direction of the aircraft relative to the ground

W 5° S ⇒ its heading N 85° E



$|\vec{g}|^2 = 40^2 + 150^2 - 2(40)(150)\cos 65°$

$|\vec{g}| = 137.94$ km/h

$40^2 = 150^2 + 137.94^2 - 2(150)(137.94)\cos\theta$

$\cos\theta = 0.964$

$\theta = 15.23°$

$\alpha = 30° - 15.23°$

$= 14.76$

∴ the aircraft's ground speed is 137.9 km/h [345.2°]

N 14.8° W

---

method 2: Cartesian Vector Components

$\vec{w} = [40\cos 5°, 40\sin 5°] = [39.84, 3.48]$

$\vec{a} = [150\cos 120°, 150\sin 120°] = [-75, 129.90]$

$\vec{g} = \vec{a} + \vec{w}$

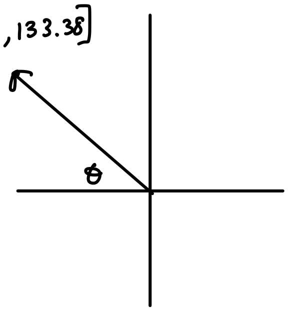
$= [-75, 129.90] + [39.84, 3.48]$

$= [-35.16, 133.38]$

$|\vec{g}| = \sqrt{(-35.16)^2 + (133.38)^2}$

$= 137.93$ km/h

$\vec{g} = [-35.16, 133.38]$



$\tan\theta = \left|\dfrac{133.38}{-35.16}\right|$

$= 75.23$

∴ 137.93 km/h [W 75.2° N]

* TH

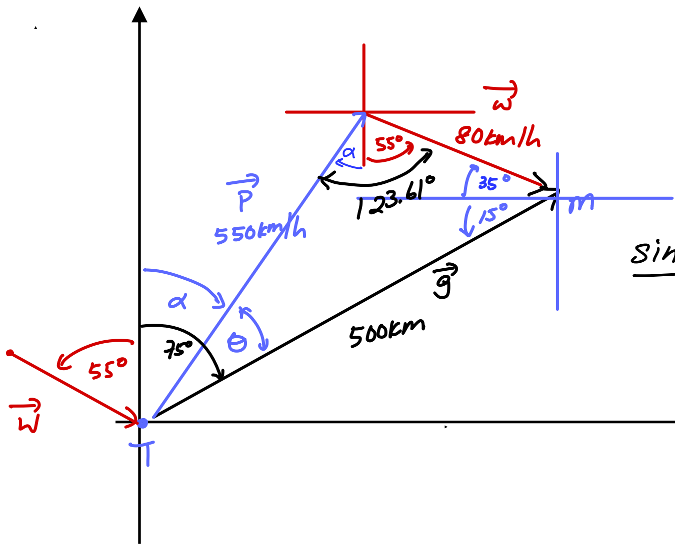
Ex4 : A pilot wishes to fly from Toronto to Montreal a distance of 500 km on a heading of [N 75° E]. The air speed of the plane is 550 km/h. An 80 km/h wind is blowing **from** [N 55° W].

(a) What heading should the pilot take to reach his destination?

(b) What will be the speed of the plane relative to the ground?(ground speed)

(c) How long will the trip take?

$\vec{w} \Rightarrow$ heading S 55°E



$$\frac{\sin 50°}{550} = \frac{\sin\theta}{80}$$

$$\theta = 6.39°$$

$$\alpha = 75° - 6.39° = 68.6°$$

a) [N 68.6°E]

$$\frac{\sin 123.61°}{|\vec{g}|} = \frac{\sin 6.39}{80} = \frac{\sin 50°}{550}$$

$$|\vec{g}| = 598.64 \text{ km/h}$$

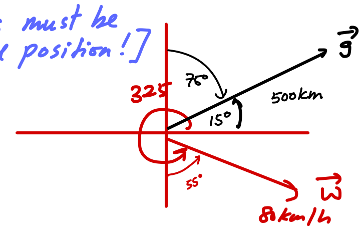
b) 598.64 km/h

$$T = \frac{500}{598.64} = 0.835 \text{ hr} \doteq 50 \text{ min}$$

c) 50min

---

method 2: Cartesian Vector Components $\Rightarrow$ [angles must be in Std position!]



$$\vec{w} = [80\cos 325°, 80\sin 325°] = [65.53, -45.88]$$

$$\vec{P} = [550\cos\alpha, 550\sin\alpha]$$

$$\vec{g} = [|\vec{g}|\cos 15°, |\vec{g}|\sin 15°]$$

$$\vec{g} = \vec{w} + \vec{P}$$

$|\vec{g}|\cos 15° = 65.53 + 550\cos\alpha$ ①

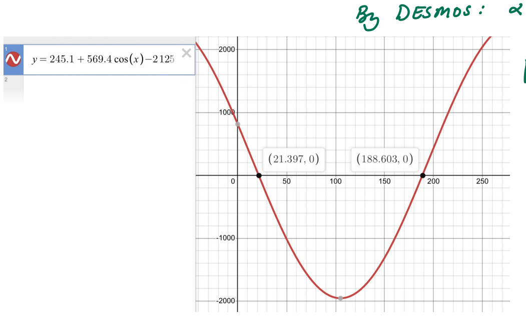
$|\vec{g}| = 67.84 + 569.40\cos\alpha$

$|\vec{g}|\sin 15° = -45.88 + 550\sin\alpha$ ②

$|\vec{g}| = -177.26 + 2125.03\sin\alpha$

① - ② : $0 = 245.1 + 569.40\cos\alpha - 2125.03\sin\alpha$ ③

By DESMOS : $\alpha = \{21.4°, 188.6°\}$ ↳ inadmissible



$|\vec{g}| = 67.84 + 569.40\cos 21.4 \doteq 597.98$ km/h

$\therefore$ a) [E 21.4°N]

b) $|\vec{g}| = 597.98$

c) <Same as above>

Note! ③ Can't be Solved Algebraically

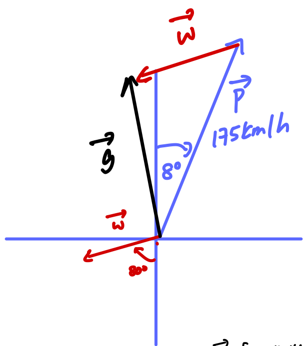
<Cartesian Vector Component Method Fails Algebraically>

Cartesian Vector Components method is easier when a pair of vectors are given with magnitude and direction.

Exit Card!

A light plane is travelling at 175 km/h on a heading of [N8°E] encounters a wind of 40 km/h blowing from [N80°E]. Determine the plane's ground velocity.

$\vec{w}$ is heading S80°W



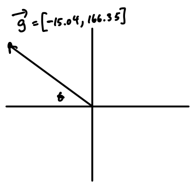
$\vec{p} = [175\cos 82°, 175\sin 82°] = [24.35, 173.29]$

$\vec{w} = [40\cos 190°, 40\sin 190°] = [-39.39, -6.94]$

$\vec{g} = \vec{p} + \vec{w} = [-15.04, 166.35]$ ⇐ this is the ground velocity in component form

$|\vec{g}| = \sqrt{(-15.04)^2 + (166.35)^2}$

$\doteq 167.02$ km/h



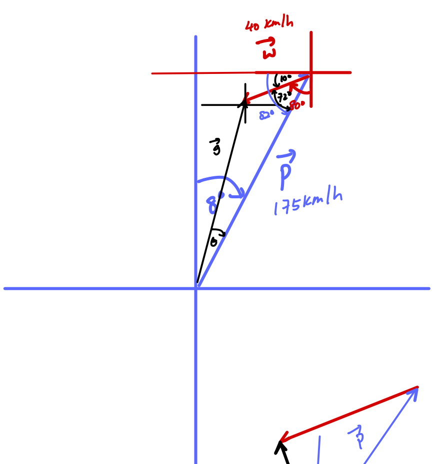
$\tan\theta = \left|\frac{166.35}{-15.04}\right|$

$\theta = 84.8°$

$\therefore \vec{g} = 167.0$ km/h [W84.8°N]

(ground velocity in terms of speed and direction)

Method 2 : Geometrically



$|\vec{g}|^2 = 40^2 + 175^2 - 2(40)(175)\cos 72°$

$|\vec{g}| \doteq 167.02$ km/h

$40^2 = 175^2 + 167.02^2 - 2(175)(167.02)\cos\theta$

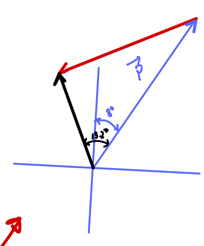
$\cos\theta \doteq 0.9737$

$\theta \doteq 13.2°$

Bearing : [N5.2°W]

$\therefore \vec{g} = 167.02$ [N5.2°W]

Note! Our diagram was sketched incorrectly based on the angle calculation it should be



Practice on Velocity as Vectors

1. Thieves are fleeing in a stolen boat travelling at 30 km/h due west. A police boat is sent to catch them. When the stolen boat is 3 km due north of the police, the police set out at a speed of 40 km/h .

a) In what direction must the police head in order to intercept the thieves?
b) When will the interception occur?

2. An airplane which flies at 200 km/h is headed due north. A wind is blowing due east at 40 km/h.

a) What is the magnitude and direction of the plane's velocity relative to the ground?
b) After flying for 90 minutes under these conditions, what is the location of the plane?

3. An airplane is flying at 150 km/h at a heading of W 10° N. When it lands 2 hours later, its location is 275 km from the starting point, at a heading of W 20° N. What is the magnitude and direction of the wind velocity?

4. A pilot is planning his flight to an airport which is 400 km southeast of his starting location. His plane flies at 250 km/h but a wind of 20 km/h is blowing from the southwest.

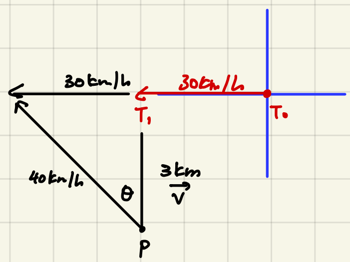
a) What heading should he choose for the plane?
b) How long will the journey take?

Answers

1. a) [N 48.6° W] **b) 6.8 minutes**
2. a) 203.9 km/h [N 11.3° E] **b) 305.9 km [N 11.3° E] from starting point.**
3. 28.0 km/h [E 48.5° N]
4. a) [E 49.6° S] **b) 1 hour, 36 minutes**

1. Thieves are fleeing in a stolen boat travelling at 30 km/h due west. A police boat is sent to catch them. When the stolen boat is 3 km due north of the police, the police set out at a speed of 40 km/h .
a) In what direction must the police head in order to intercept the thieves?
b) When will the interception occur?

1.



a) $\sin\theta = \frac{30}{40}$

$\theta = 48.$

N 36.9°W

b) $|\vec{v}|^2 = 40^2 - 30^2$

$|\vec{v}| = \sqrt{40^2 - 30^2}$

$= \sqrt{700}$

$= 26.45$ km/h

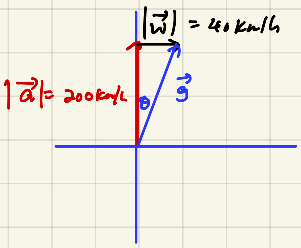
$T = \frac{D}{S}$

$\doteq \frac{3}{26.45}$

$\doteq 0.113$ h

$\doteq 6.8$ min

2. An airplane which flies at 200 km/h is headed due north. A wind is blowing due east at 40 km/h.
a) What is the magnitude and direction of the plane's velocity relative to the ground?
b) After flying for 90 minutes under these conditions, what is the location of the plane?



a) $|\vec{g}|^2 = 200^2 + 40^2$

$= \sqrt{41600}$

$= 203.96$ km/h

b) $D = S \cdot T$

$= 203.96(1.5)$

$= 305.94$ km

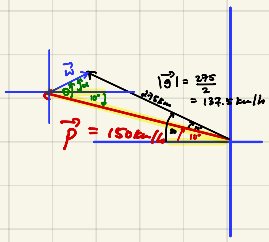
90 min = 1.5h

$\tan\theta = \frac{40}{200}$

$\theta = 11.3°$

$\vec{g} = 305.94$ km [N 11.3°E]

3. An airplane is flying at 150 km/h at a heading of W 10° N. When it lands 2 hours later, its location is 275 km from the starting point, at a heading of W 20° N. What is the magnitude and direction of the wind velocity?



$|\vec{w}|^2 = 150^2 + 137.5^2 - 2(150)(137.5)\cos 10°$

$|\vec{w}| = 27.98$ km/h

$137.5^2 = 27.98^2 + 150^2 - 2(27.98)(150)\cos\theta$

$\frac{137.5^2 - 27.98^2 - 150^2}{-2(27.98)(150)} = \cos\theta$

$\cos\theta \doteq 0.5213$

$\theta \doteq 58.57$

$\therefore \alpha = 58.57 - 10°$

$= 48.57$

$\therefore \vec{w} = 28.0$ km/h [E 48.6° N]

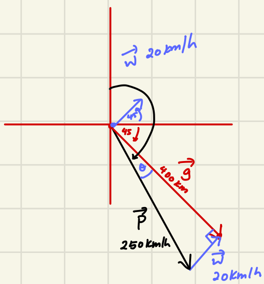
Hmwk Takeup CP 35 #4

4. A pilot is planning his flight to an airport which is 400 km southeast of his starting location. His plane flies at 250 km/h but a wind of 20 km/h is blowing from the southwest.

↳ implies $\vec{w}$ is heading towards north east

a) What heading should he choose for the plane?

b) How long will the journey take?



$\sin\theta = \frac{20}{250}$

$\theta \doteq 4.58°$

$$\text{Bearing} = 90° + 45° + 4.58° = 139.58° \quad [E\,49.58\,S]$$

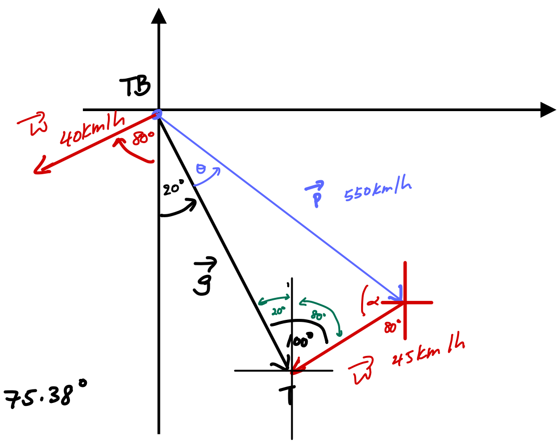
$$\text{ground speed} = \sqrt{250^2 - 20^2} = 249.19 \text{ km/h}$$

$$T = \frac{400}{247.19} = 1.60\text{ h} = 1\text{h } 36\text{ min}$$

Warm Up

1. An aircraft pilot wishes to fly from Thunder Bay to Toronto on a bearing of $[S\ 20^\circ E]$. If there is a wind of 45 km/h blowing **from** $[N\ 80^\circ E]$ and the airspeed of her aircraft is 550 km/h, what direction should the pilot take and what will be her actual ground speed?

$\vec{w}$ is heading in the direction of S 80° W



$$\frac{\sin 100^\circ}{550} = \frac{\sin\theta}{45}$$

$$\sin\theta = \frac{45\sin 100^\circ}{550}$$

$$\theta = 4.62^\circ$$

$$\alpha = 180 - 100^\circ - 4.62^\circ = 75.38^\circ$$

$$|\vec{g}|^2 = 550^2 + 45^2 - 2(550)(45)\cos 75.38^\circ$$

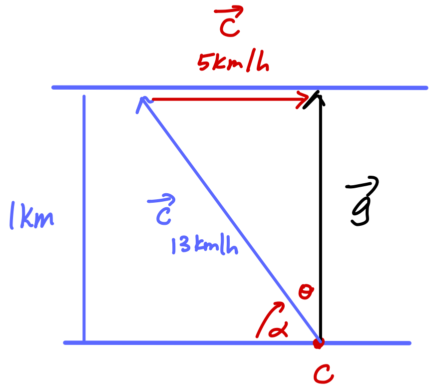
$$|\vec{g}| \doteq 540.39 \text{ km/h}$$

∴ the pilot should be heading [S 24.6°E] with a ground speed of 540.4 km/h to arrive in Toronto from Thunder Bay

Note! Cartesian Vector would require DESMOS to solve so we will not include this method.

2. A canoeist wishes to cross a river 1 km wide. If the current flows at 5 km/h east, and the canoe can travel at 13 km/h in still water, find:

a) the direction the boat must take to travel straight across the river.



$|\vec{g}| = \sqrt{13^2 - 5^2}$
$= 12$ km/h

$\sin\theta = \frac{5}{13}$
$\theta = 22.6°$

$\alpha = 90 - 22.6°$
$= 67.4°$

∴ the canoeist must travel 12km/h at 67.4° from the upstream side of the shore

or [N 22.6° W]

b) the time taken to make the trip across the river.(in minute)

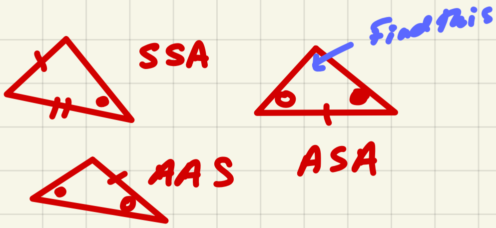
$T = \frac{D}{V}$

$= \frac{1}{12}$ h

$= 5$ min

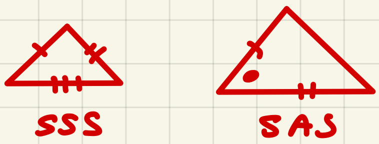
Sine Law:

used when SSA (ambiguous case), AAS and ASA



Cosine Law:

used when SSS or SAS



6.5 Dot Product of Geometric Vectors

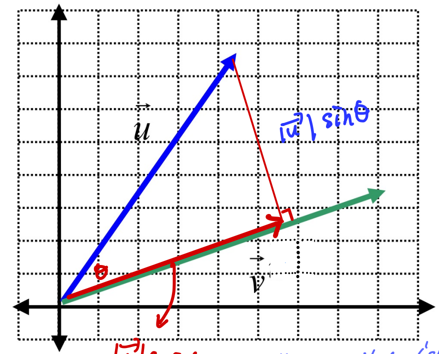
Dot Product of 2 Vectors – aka Scalar Product

Cross Product

Dot Product : vector • vector = Scalar   note! vector × vector = vector

- involves two ~~scalars~~ quantities of the vectors
- result is a scalar ie) positive/negative/zero

$$\boxed{\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|\cos\theta}$$



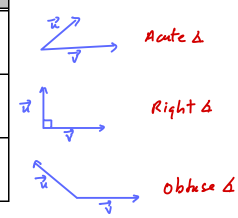
Dot Product :

$|\vec{v}|$ $|\vec{u}|\cos\theta$

- defined as: **(horizontal displacement of an object)(horizontal component of force)**

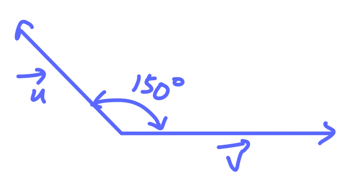
Note: Vectors need to be tail to tail

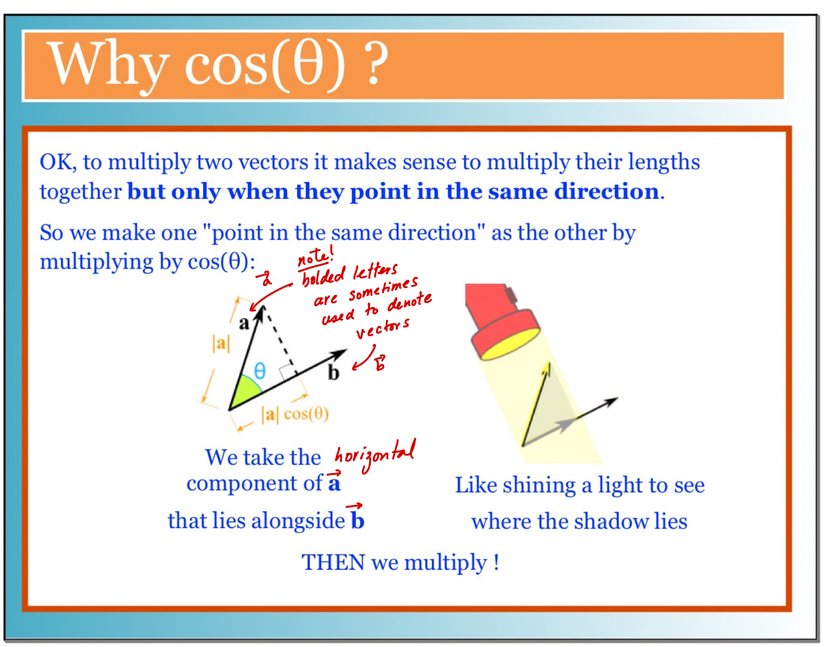
angle value	$\cos(\theta)$ **value**	$\vec{u} \bullet \vec{v}$
$0^\circ \le \theta \le 90^\circ$	(+)	$\lvert\vec{u}\rvert\lvert\vec{v}\rvert\cos\theta > 0$ (+)
$\theta = 90^\circ$	$\cos(90^\circ) = 0$	$\lvert\vec{u}\rvert\lvert\vec{v}\rvert\cos 90^\circ = 0$
$90^\circ \le \theta \le 180^\circ$	(−)	$\lvert\vec{u}\rvert\lvert\vec{v}\rvert\cos\theta < 0$ (−)



Example 1: Given vectors $\vec{u}$ and $\vec{v}$, where $|\vec{u}|=10$ and $|\vec{v}|=13$ and the angle between them is 150°, calculate $\vec{u} \bullet \vec{v}$.

$$\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|\cos\theta$$
$$= (10)(13)\cos(150^\circ)$$
$$\doteq -112.58$$





Dot Product Properties

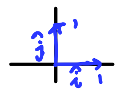
1) ***Commutative***: $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

2) ***Distributive** over vector addition:* $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

3) ***Associative** over scalar multiplication:* $m(\vec{u} \bullet \vec{v}) = (m\vec{u}) \bullet \vec{v} = \vec{u} \bullet (m\vec{v})$

$(m\vec{u}) \bullet (n\vec{v}) = mn(\vec{u} \bullet \vec{v})$

Example 2: Evaluate $\hat{j} \bullet \hat{j}$ and $\hat{i} \bullet \hat{j}$.



recall: $\hat{i} = [1,0]$ $\quad \hat{j} = [0,1]$

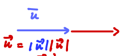
$\hat{j} \cdot \hat{j} = |\hat{j}||\hat{j}| \cos 0°$
$= |\hat{j}|^2 (1)$
$= (1)^2 (1)$
$= 1 \quad = |\vec{j}|^2$

$\hat{i} \cdot \hat{j} = |\hat{i}||\hat{j}| \cos 90°$
$= (1)(1)(0)$
$= 0$

Example 3: What is the dot product of a vector $\vec{u}$ with itself? ($\theta = 0°$).

$\vec{u} \cdot \vec{u} = |\vec{u}||\vec{u}| \cos 0°$
$= |\vec{u}|^2 (1)$
$= |\vec{u}|^2$

Ex if $\vec{u} = [1,-2]$
then $\vec{u} \cdot \vec{u} = |\sqrt{1+4}||\sqrt{1+4}| \cos 0°$
$= \sqrt{5}\sqrt{5}(1)$
$= 5$

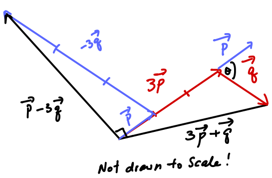


$\vec{u} \cdot \vec{u} = |\vec{u}||\vec{u}|$
$= |\vec{u}|^2$

* Example 4: If vectors $3\vec{p} + \vec{q}$ and $\vec{p} - 3\vec{q}$ are perpendicular and $|\vec{p}| = 2|\vec{q}|$, determine the angle between the non-zero vectors $\vec{p}, \vec{q}$.

↳ dot product = 0

info that we are likely going to need



Approach Algebraically:

$(3\vec{p} + \vec{q}) \bullet (\vec{p} - 3\vec{q}) = 0 \Rightarrow$ vectors are perpendicular

$3\vec{p} \cdot \vec{p} - 8\vec{p} \cdot \vec{q} - 3\vec{q} \cdot \vec{q} = 0$

$3|\vec{p}|^2 - 8|\vec{p}||\vec{q}| \cos\theta - 3|\vec{q}|^2 = 0$

$3(4|\vec{q}|^2) - 8(2|\vec{q}|)|\vec{q}| \cos\theta - 3|\vec{q}|^2 = 0$

$12|\vec{q}|^2 - 16|\vec{q}|^2 \cos\theta - 3|\vec{q}|^2 = 0$

$9|\vec{q}|^2 - 16|\vec{q}|^2 \cos\theta = 0$

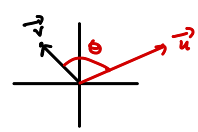
$\cos\theta = \frac{-9|\vec{q}|^2}{-16|\vec{q}|^2}$

$\cos\theta = \frac{9}{16}$

$\theta \doteq 55.8°$

Aside:
$|\vec{p}| = 2|\vec{q}|$
$|\vec{p}|^2 = 4|\vec{q}|^2$

Scalar Product (Dot Product) in Component Form



Recall: $\hat{i} = [1,0]$ $\quad \hat{j} = [0,1]$

$\hat{i} \cdot \hat{i} = [1,0] \cdot [1,0]$
$= (1)(1) + (0)(0)$
$= 1$

$\hat{i} \cdot \hat{j} = [1,0] \cdot [0,1]$
$= (1)(0) + (0)(1)$
$= 0$

$\vec{u} = [5,3] \quad \vec{v} = [-2,4]$

$\vec{u} \cdot \vec{v} = [5,3] \bullet [-2,4]$
$= 5(-2) + (3)(4)$
$= -10 + 12$
$= 2$

you need this to find the angle between the 2 vectors

$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$
$2 = \sqrt{25+9}\sqrt{4+16} \cos\theta$
$2 = \sqrt{34}\sqrt{20} \cos\theta$
$\frac{2}{\sqrt{34 \cdot 20}} = \cos\theta$
$\theta \doteq 85.6°$

if $\vec{u} = [u_1, u_2]$
$\vec{v} = [v_1, v_2]$
then $\vec{u} \cdot \vec{v} = (u_1)(v_1) + (u_2)(v_2)$
$= |\vec{u}||\vec{v}| \cos\theta$

Dot Product Properties

1) ***Commutative****:* $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

2) ***Distributive*** *over vector addition:* $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

3) ***Associative*** *over scalar multiplication:* $m(\vec{u} \bullet \vec{v}) = (m\vec{u}) \bullet \vec{v} = \vec{u} \bullet (m\vec{v})$

$(m\vec{u}) \bullet (n\vec{v}) = mn(\vec{u} \bullet \vec{v})$

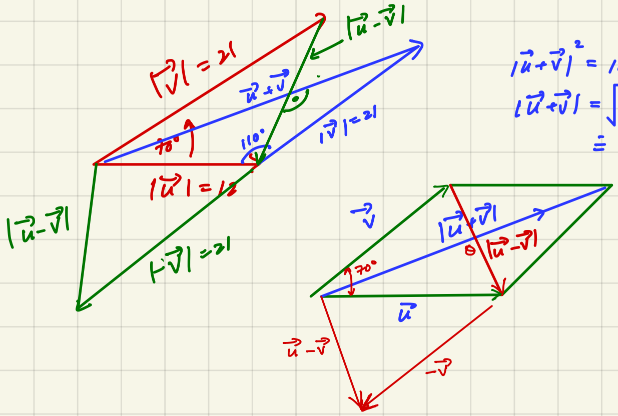
Example 2:Evaluate $\hat{j} \bullet \hat{j}$ and $\hat{i} \bullet \hat{j}$.

Example 3: What is the dot product of a vector $\vec{u}$ with itself? ($\theta = 0°$).

Example 4: If vectors $3\vec{p} + \vec{q}$ and $\vec{p} - 3\vec{q}$ are perpendicular and $|\vec{p}| = 2|\vec{q}|$, determine the angle between the non-zero vectors $\vec{p}, \vec{q}$.

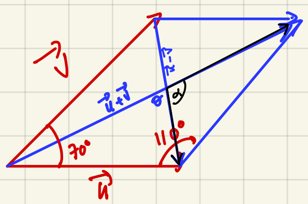
Take 2: Revisiting CP 12 #9 (Hmwk) Using Dot Product

9. If the angle between $\vec{u}$ and $\vec{v}$ is $70°$ and $|\vec{u}|=12, |\vec{v}|=21$, find the angle between vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$. (Round your answer to the nearest degree) [Ans: 122°]



$$|\vec{u}+\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos 110°$$
$$|\vec{u}+\vec{v}| = \sqrt{12^2 + 21^2 - 2(12)(21)\cos 110°}$$
$$\doteq 27.520$$

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |-\vec{v}|^2 - 2|\vec{u}||-\vec{v}|\cos 70°$$
$$= (12)^2 + (21)^2 - 2(12)(21)\cos 70°$$
$$|\vec{u}-\vec{v}| = \sqrt{412.621}$$
$$\doteq 20.313$$

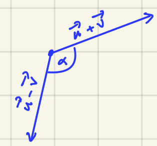


$$|\vec{u}|^2 = \left|\frac{\vec{u}+\vec{v}}{2}\right|^2 + \left|\frac{\vec{u}-\vec{v}}{2}\right|^2 - 2\left|\frac{\vec{u}+\vec{v}}{2}\right|\left|\frac{\vec{u}-\vec{v}}{2}\right|\cos\theta$$
$$12^2 = \left|\frac{27.520}{2}\right|^2 + \left|\frac{20.313}{2}\right|^2 - 2\left|\frac{27.520}{2}\right|\left|\frac{20.313}{2}\right|\cos\theta$$
$$0.5312 = \cos\theta$$
$$\theta = 57.91°$$

Method 2: Algebraic Approach by Dot Product

$\therefore$ the angle between $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ is $58°$ (or $122°$)

tail to tail



$$\boxed{\begin{aligned}
(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) &= |\vec{u}+\vec{v}||\vec{u}-\vec{v}|\cos\alpha \\
\vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{v} &= |\vec{u}+\vec{v}||\vec{u}-\vec{v}|\cos\alpha \\
|\vec{u}|^2 - |\vec{v}|^2 &= (27.520)(20.313)\cos\alpha \\
\frac{12^2 - 21^2}{(27.520)(20.313)} &= \cos\alpha \\
\alpha &= 122.09°
\end{aligned}}$$

Reason: $\vec{u}\bullet\vec{v} = (u_1\hat{i} + u_2\hat{j})\bullet(v_1\hat{i} + v_2\hat{j})$
$= u_1v_1\,\hat{i}\cdot\hat{i} + u_1v_2\,\hat{i}\cdot\hat{j} + u_2v_1\,\hat{i}\cdot\hat{j} + u_2v_2\,\hat{j}\cdot\hat{j}$
$= u_1v_1(1) + \cancel{u_1v_2(0)} + \cancel{u_2v_1(0)} + u_2v_2(1)$
$= u_1v_1 + u_2v_2$

How to Evaluate Dot Product of Algebraic Vectors

Since every algebraic vector can be written in terms of the basis vectors $\hat{i}$ and $\hat{j}$, we can use the dot products of the basis vectors to determine the dot product of any two algebraic vectors.
Let $\vec{u} = u_1\hat{i} + u_2\hat{j}$ and $\vec{v} = v_1\hat{i} + v_2\hat{j}$ then $\vec{u}\bullet\vec{v} = u_1v_1 + u_2v_2$.

Example 5. Given that vectors $\vec{u} = [k+2, 5]$ and $\vec{v} = [k+1, -6]$ are perpendicular, solve for k.

$\vec{u}\bullet\vec{v} = 0$
$(k+2)(k+1) + (5)(-6) = 0$
$k^2 + 3k + 2 - 30 = 0$
$k^2 + 3k - 28 = 0$
$(k+7)(k-4) = 0$
$\therefore k = \{-7, 4\}$

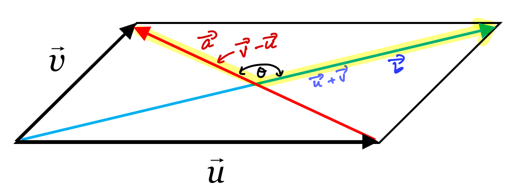
note! <doesn't work>
$\vec{u}\bullet\vec{v} = 0$ $\Rightarrow$ this definition of scalar product is insufficient for solving 'k'.
$|\vec{u}||\vec{v}|\cos 90° = 0$
$\sqrt{(k+2)^2 + 25}\sqrt{(k+1)^2 + 36}\,(0) = 0$
Vectors in component form is much more useful

Using dot product to find angle between two vectors

The formula $\vec{u}\bullet\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$ can be rearranged to make solving for θ simpler.

$$\boxed{\cos(\theta) = \frac{\vec{u}\bullet\vec{v}}{|\vec{u}||\vec{v}|}} \Rightarrow \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

* Example 6: A parallelogram is bounded by vectors $\vec{u} = [1,2]$ and $\vec{v} = [3,-2]$. Find the angle between the diagonals of the parallelogram



$\vec{v} - \vec{u} = [3,-2] - [1,2]$
$\vec{a} = [2,-4]$

$\vec{u} + \vec{v} = [1,2] + [3,-2]$
$\vec{b} = [4,0]$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
$[2,-4]\cdot[4,0] = \sqrt{(2)^2 + (-4)^2}\sqrt{4^2 + 0^2}\cos\theta$
$(2)(4) + (-4)(0) = \sqrt{20}\sqrt{16}\cos\theta$
$8 = 2\sqrt{5}(4)\cos\theta$
$\frac{1}{\sqrt{5}} = \cos\theta$
$\theta = 63.43°$ tail-tail

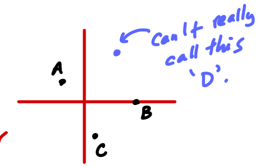
Should be acute
$\vec{a}\bullet\vec{b} > 0$

Note! Diagram is NoT drawn to Scale

Practice

1. Given that $|\vec{a}| = 2, |\vec{b}| = 3$, and $\theta = 120°$ expand and simplify $(3\vec{a}+4\vec{b})\bullet(5\vec{a}+6\vec{b})$.

2. The points A(-1,1), B(2,0), and C(1,-3) are vertices of a triangle.



   a) Show that this triangle is a right triangle.
   b) Calculate the area of triangle ABC.
   c) Calculate the perimeter of triangle ABC.
   d) Calculate the coordinates of the fourth vertex ~~D~~ that completes the rectangle of which A, B, and C are the other three vertices

Recall! Naming Polygons should be in alpha order

3. If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$, prove that the non-zero vectors $\vec{a}, \vec{b}$ are perpendicular. What could this look like?

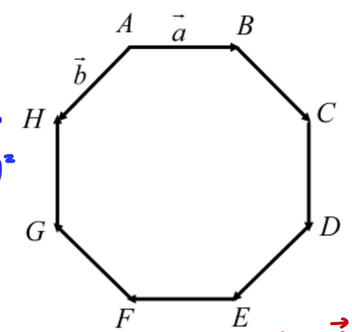
4. Given the vectors $\vec{u} = [3,4]$ and $\vec{v} = 2\hat{i} + m\hat{j}$, find the value(s) of $\boldsymbol{m}$ if the angle between $\vec{u}$ and $\vec{v}$ is 45°.

$[7,-3]\cdot[-1,0] = \sqrt{49+9}\sqrt{1}\cos\theta$

$\frac{-7}{\sqrt{58}} = \cos\theta \quad \therefore \theta = 156.8°$

5. Find the angle between the vector $[7,-3]$ and negative x-axis.

6. ABCDEFGH is a regular octagon with sides of unit length. (Recall interior angles are 135°). Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AH} = \vec{b}$. Prove that $\overrightarrow{BC} = \vec{b} + \sqrt{2}\,\vec{a}$.



#3

$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$

$(\vec{a}+\vec{b})\bullet(\vec{a}+\vec{b}) = (\vec{a}-\vec{b})\bullet(\vec{a}-\vec{b})$

$\vec{a}\bullet\vec{a} + 2\vec{a}\bullet\vec{b} + \vec{b}\bullet\vec{b} = \vec{a}\bullet\vec{a} - 2\vec{a}\bullet\vec{b} + \vec{b}\bullet\vec{b}$

$\cancel{|\vec{a}|^2} + 2\vec{a}\bullet\vec{b} + \cancel{|\vec{b}|^2} = \cancel{|\vec{a}|^2} - 2\vec{a}\bullet\vec{b} + \cancel{|\vec{b}|^2}$

$2\vec{a}\bullet\vec{b} = -2\vec{a}\bullet\vec{b}$

$4\vec{a}\bullet\vec{b} = 0$

$\vec{a}\bullet\vec{b} = 0 \qquad \therefore \vec{a} \perp \vec{b}$

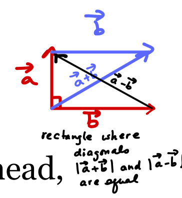
recall! $\vec{u}\bullet\vec{u} = |\vec{u}|^2$

$\therefore |\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\bullet(\vec{a}+\vec{b})$

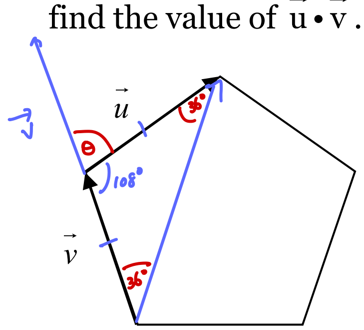
$= \vec{a}\bullet\vec{a} + 2\vec{a}\bullet\vec{b} + \vec{b}\bullet\vec{b}$

$= |\vec{a}|^2 + 2\vec{a}\bullet\vec{b} + |\vec{b}|^2$

$\therefore \vec{a}$ and $\vec{b}$ are perpendicular



7. Given a regular pentagon and that the vectors $\vec{u}$ and $\vec{v}$ are unit vectors connected tail to head, find the value of $\vec{u}\bullet\vec{v}$.



interior angle of pentagon $= \frac{(n-2)(180°)}{5}$

$= \frac{(5-2)(180)}{5}$

$= 108°$

$\frac{180 - 108°}{2} = 36$

$\theta = 72°$

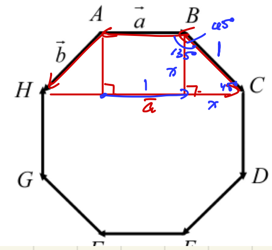
$\vec{u}\bullet\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$= (1)(1)\cos 72°$

$\doteq 0.309$

6. ABCDEFGH is a regular octagon with sides of unit length. (Recall interior angles are 135°). Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AH} = \vec{b}$. Prove that $\overrightarrow{BC} = \vec{b} + \sqrt{2}\,\vec{a}$.

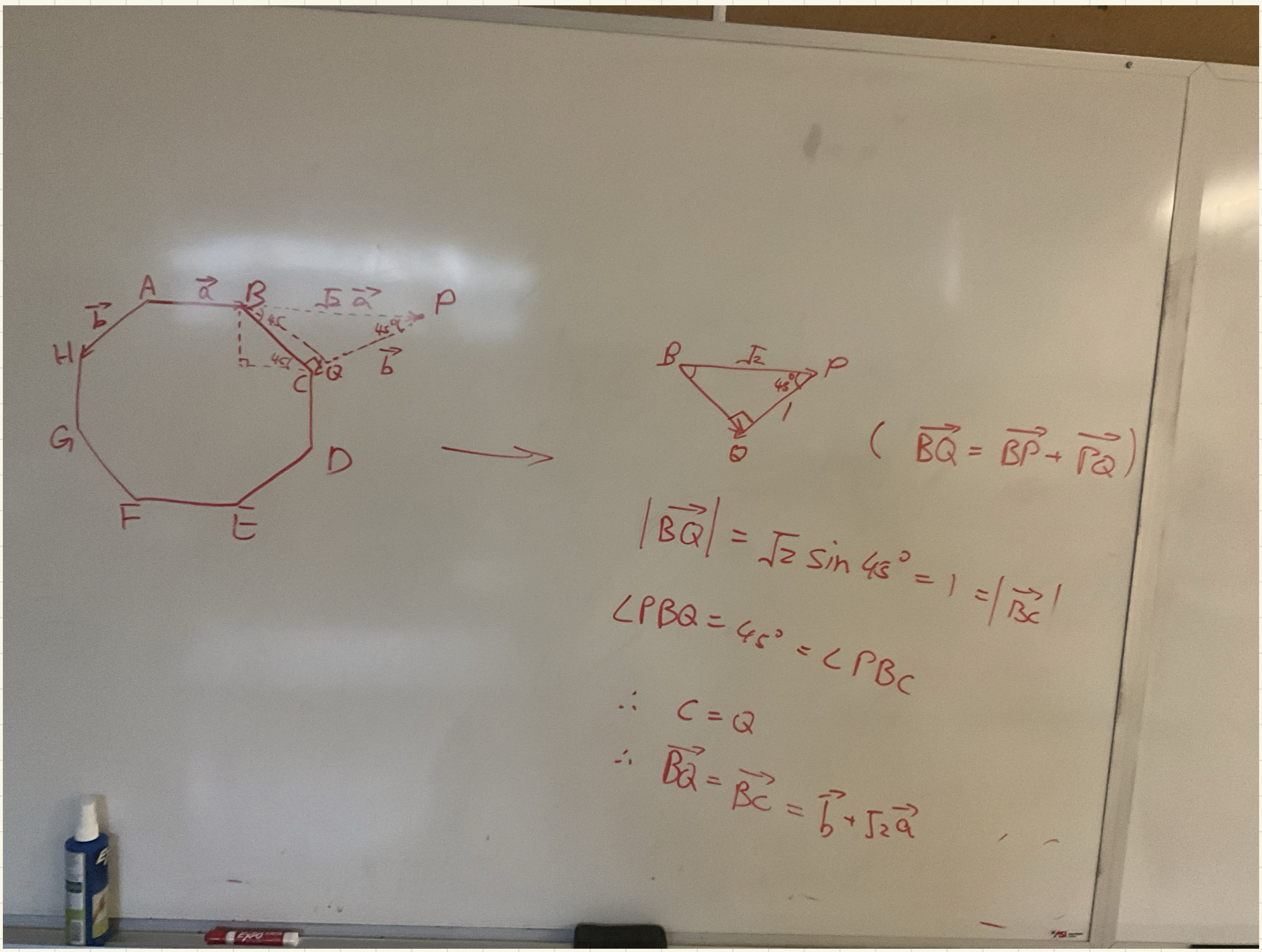
#3



$x^2 + x^2 = 1$
$2x^2 = 1$
$x^2 = \frac{1}{2}$
$x = \frac{1}{\sqrt{2}}$
$2x = \frac{2}{\sqrt{2}}$

$$\begin{aligned}
\overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AH} + \overrightarrow{HC} \\
&= -\vec{a} + \vec{b} + \left(1 + \tfrac{2}{\sqrt{2}}\right)\vec{a} \\
&= \cancel{-\vec{a}} + \vec{b} + \cancel{\vec{a}} + \tfrac{2}{\sqrt{2}}\vec{a} \\
&= \vec{b} + \tfrac{2}{\sqrt{2}}\vec{a} \\
&= \vec{b} + \tfrac{2\sqrt{2}}{2}\vec{a} \\
&= \vec{b} + \sqrt{2}\,\vec{a}
\end{aligned}$$

Courtesy of Andi Tse



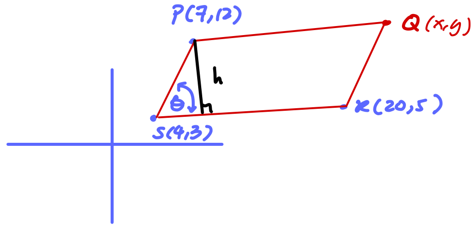
Warm-Up

The parallelogram $PQRS$ has vertices $P\,(7,12)$, $R\,(20,5)$, and $S\,(4,3)$.

a. Find the coordinates of Q. [Ans: $Q\,(23,14)$]

b. Find the measure of $\angle PSR$. [Ans: $\angle PSR \approx 64°$]

c. Calculate the area of the parallelogram. [Ans: 138]



a) $\overrightarrow{PQ} = \overrightarrow{SR}$

$[x-7, y-12] = [20-4, 5-3]$

$[x-7, y-12] = [16, 2]$

$x-7 = 16 \qquad y-12 = 2$

$x = 23 \qquad y = 14$

$\therefore Q\,(23,14)$

b) $\overrightarrow{SP} = [3, 9]$

$\overrightarrow{SR} = [16, 2]$

$\overrightarrow{SP} \cdot \overrightarrow{SR} = |\overrightarrow{SP}|\,|\overrightarrow{SR}| \cos\theta$

$[3,9]\cdot[16,2] = \sqrt{9+81}\,\sqrt{256+4}\,\cos\theta$

$(3)(16)+(9)(2) = \sqrt{90}\,\sqrt{260}\,\cos\theta$

$66 = (3\sqrt{10})(2\sqrt{65})\cos\theta$

$\frac{11}{\sqrt{650}} = \cos\theta$

$\frac{11}{5\sqrt{26}} = \cos\theta$

$\theta \doteq 64.4°$

c) $A = bh$

$= |\overrightarrow{SR}|\,|\overrightarrow{SP}| \sin\theta$

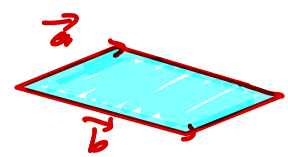
$= \sqrt{260}\,\sqrt{90}\,\sin 64.4°$

$= 138 \text{ unit}^2$

$\sin\theta = \frac{h}{|\overrightarrow{SP}|}$

$h = |\overrightarrow{SP}| \sin\theta$ (vertical component of $\overrightarrow{SP}$)

Area of Parallelogram $= |\vec{a}|\,|\vec{b}| \sin\theta$



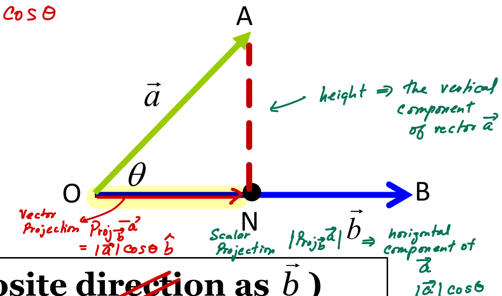
6.6 Application of dot product

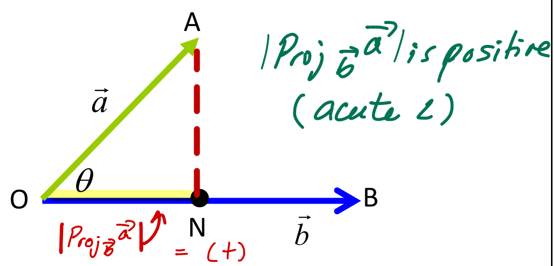
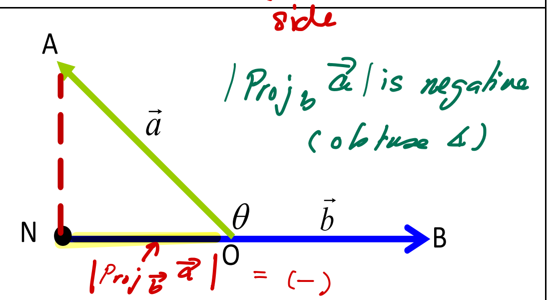
Part I. Scalar and Vector Projections

Given two vectors, $\vec{a}$ and $\vec{b}$, placed tail to tail with angle θ between them, drop a perpendicular from the tip of $\vec{a}$ to the line containing $\vec{b}$. The vector lying along the line containing $\vec{b}$, which has magnitude equal to the component of $\vec{a}$ in the direction of $\vec{b}$ (i.e., $\overline{ON}$ in our diagram), is called the **vector projection** of $\vec{a}$ onto $\vec{b}$ .The **magnitude of the vector projection** of $\vec{a}$ onto $\vec{b}$ is called the **Scalar Projection.**

$\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

Scalar Projections – no direction



$0^\circ \leqslant \theta \leqslant 90^\circ$ (same ~~direction~~ side as $\vec{b}$)	$90^\circ < \theta \leqslant 180^\circ$ (opposite ~~direction~~ side as $\vec{b}$)
	

i. Scalar Projection of $\vec{a}$ on $\vec{b}$

$$\boxed{\left|Proj_{\vec{b}}\vec{a}\right| = \frac{\vec{a} \bullet \vec{b}}{|\vec{b}|}}$$

$|Proj_{\vec{b}}\vec{a}| = |\vec{a}|\cos\theta$

$= |\vec{a}|\cos\theta \dfrac{|\vec{b}|}{|\vec{b}|}$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$) to incorporate the dot product into the formula

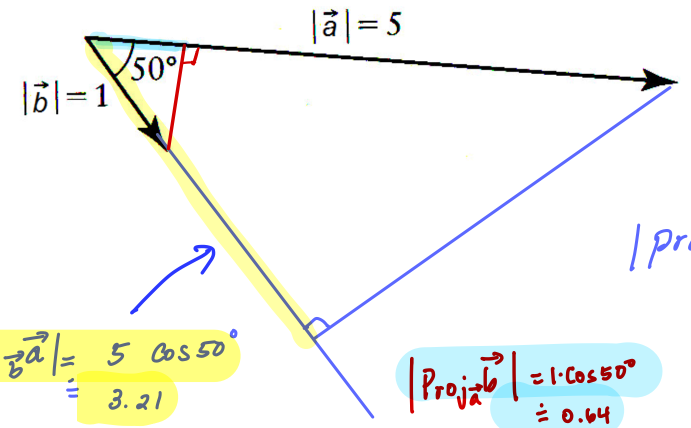
ii. Scalar Projection of $\vec{b}$ on $\vec{a}$

$|Proj_{\vec{a}}\vec{b}| = |\vec{b}|\cos\theta$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Example 1:

a) Determine the scalar projection of $\vec{a}$ onto $\vec{b}$.



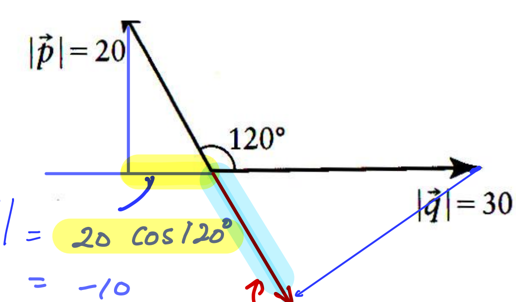
$|Proj_{\vec{b}}\vec{a}| = 5 \cos 50°$

$\doteq 3.21$

$|Proj_{\vec{a}}\vec{b}| = 1 \cdot \cos 50°$

$\doteq 0.64$

b) Determine the scalar projection of $\vec{p}$ onto $\vec{q}$.



$|Proj_{\vec{q}}\vec{p}| = 20 \cos 120°$

$= -10$

$|Proj_{\vec{p}}\vec{q}| = 30 \cos 120°$

$= -15$

Vector Projections – have direction

Scalar projection

The *vector* projection of $\vec{a}$ onto $\vec{b}$ is $\boxed{Proj_{\vec{b}}\vec{a} = \left(\dfrac{\vec{a}\bullet\vec{b}}{|\vec{b}|^2}\right)\vec{b}}$

$\dfrac{\vec{a}\bullet\vec{b}}{|\vec{b}|}$ $\hat{b}$ ← in direction of unit vector b

$\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\ \dfrac{\vec{b}}{|\vec{b}|}$

The *vector* projection of $\vec{b}$ onto $\vec{a}$ is $Proj_{\vec{a}}\vec{b} = \left(\dfrac{\vec{a}\bullet\vec{b}}{|\vec{a}|^2}\right)\vec{a}$

Example 2: Given vectors $\vec{a}=[-4, 1]$ and $\vec{b}=[4,3]$, determine $Proj_{\vec{a}}\vec{b}$.

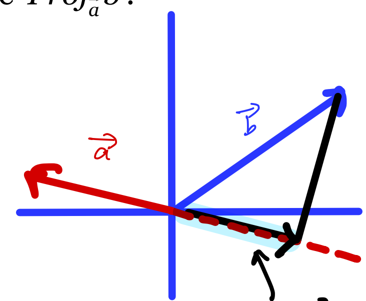
$$Proj_{\vec{a}}\vec{b} = \frac{\vec{a}\bullet\vec{b}}{|\vec{a}|^2}\vec{a}$$

$$Proj_{\vec{a}}\vec{b} = \frac{[-4,1]\bullet[4,3]}{\left(\sqrt{(-4)^2+(1)^2}\right)^2}[-4,1]$$

$$= \frac{-16+3}{17}[-4,1]$$

$$= \frac{-13}{17}[-4,1]$$

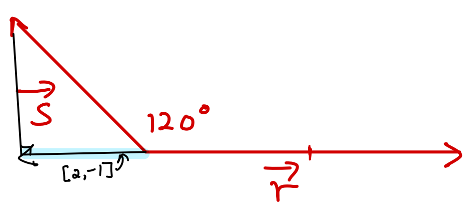
$$= \left[\frac{52}{17}, \frac{-13}{17}\right]$$



$Proj_{\vec{a}}\vec{b} = \left[\frac{52}{17}, \frac{-13}{17}\right]$

Example 3: The vector $\vec{r}$ is twice as long as the vector $\vec{s}$. The angle between the vectors is 120°. The vector projection of $\vec{s}$ on $\vec{r}$ is [2, –1]. Determine $\vec{r}$.

$$|\vec{r}| = 2|\vec{s}|$$



$$Proj_{\vec{r}}\vec{s} = \left(\frac{\vec{r}\bullet\vec{s}}{|\vec{r}|^2}\right)\vec{r}$$

$$[2,-1] = \left(\frac{\cancel{|\vec{r}|}|\vec{s}|\cos\theta}{|\vec{r}|^{\cancel{2}}}\right)\vec{r}$$

$$[2,-1] = \left(\frac{1}{2}\cos 120°\right)\vec{r}$$

$$\left[\frac{2(2)}{\cos 120°}, \frac{-1(2)}{\cos 120°}\right] = \vec{r}$$

$$[-8, 4] = \vec{r}$$

Aside

$|\vec{r}| = 2|\vec{s}|$

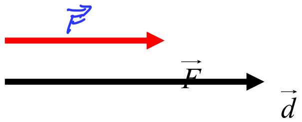
$\dfrac{1}{2} = \dfrac{|\vec{s}|}{|\vec{r}|}$

Part II. Work

Definition: In Physics, WORK is done whenever a force, applied to an object, causes a displacement in the object from one position to another.

- WORK is equal to the displacement traveled multiplied by the magnitude of the applied force in the direction of motion.

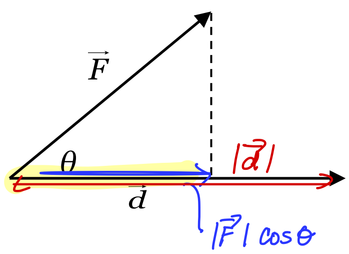
For instance, if the force is in the same direction as the displacement, then just multiply the magnitudes.



$$work = |\vec{F}||\vec{d}| \cos 0^\circ$$

"1"

However if the force acts at an angle to the $work = |\vec{F}||\vec{d}|$ displacement vector, we use the component of the force, in the direction of the displacement vector (i.e we use the projection of $\vec{F}$ onto $\vec{d}$)



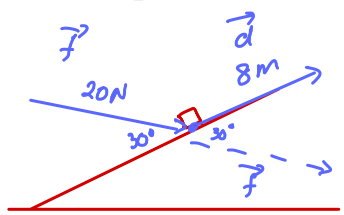
$$\boxed{work = |\vec{F}||\vec{d}| \cos\theta}$$

The work done on an object is the dot product of the force applied on the object, and the displacement of the object.

Note: - Work is a scalar quantity. The unit of measurement is the **Joule (J)** or Newton-metre (N-m).

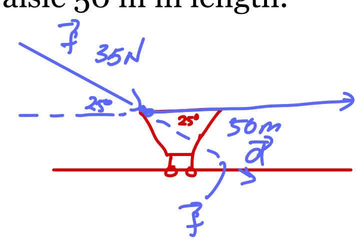
- No matter how much force is applied, if no displacement occurs, work = 0.

Example 1: A crate, on a ramp is hauled 8m up the ramp by a constant force of 20N applied at an angle of 30° to the ramp. Calculate the work done by the force.



$W = |\vec{F}||\vec{d}| \cos 30^\circ$
$= (20)(8) \cos 30^\circ$
$= 138.56$ Joules (Nm)

Example2: A shopper in a supermarket pushes a cart with a force of 35 N directed at an angle of 25° downward from the horizontal. Find the work done by the shopper as she moves down an aisle 50 m in length.

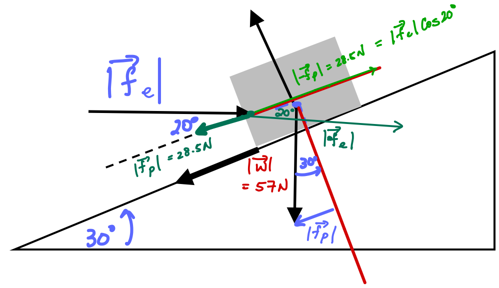


$\vec{W} = |\vec{F}||\vec{d}| \cos 25^\circ$
$= (35)(50) \cos 25^\circ$
$= 1586.03$ Joules

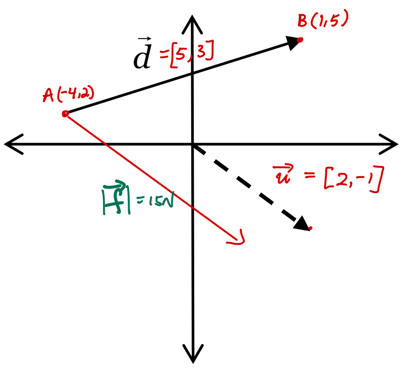
Example 3: A crate with a weight of 57 N rests on a frictionless ramp inclined at an angle of 30° to the horizontal. What force must be applied at an angle of 20° to the ramp so that the crate remains at rest?

$|\vec{f}_p| = 57 \sin 30°$
$= 28.5 N$

$|\vec{f}_e| \cos 20° = 28.5$
$|\vec{f}_e| = \frac{28.5}{\cos 20°}$
$|\vec{f}_e| = 30.32 N$



* Example 4: A force 15N acting along the vector $\vec{u} = [2,-1]$, displaces a particle from A(-4, 2) to B(1, 5). If the distance is in meters, calculate the amount of work done.



$\vec{d} = \vec{AB} = [1-(-4), 5-2]$
$= [5,3]$

$\vec{f}$: $\vec{u} = [2,-1]$
$\hat{u} = \frac{1}{|\vec{u}|}[2,-1]$
$= \frac{1}{\sqrt{5}}[2,-1]$
$\vec{f} = \frac{15}{\sqrt{5}}[2,-1]$
$\vec{f} = \frac{\cancel{15}^{3}\sqrt{5}}{\cancel{5}}[2,-1]$
$= [6\sqrt{5}, -3\sqrt{5}]$

$W = \vec{f} \cdot \vec{d}$
$= [6\sqrt{5}, -3\sqrt{5}] \cdot [5,3]$
$= 30\sqrt{5} + (-9\sqrt{5})$
$= 21\sqrt{5}$ Joules

or

$\vec{f} = k[2,-1]$
$|\vec{f}| = |k[2,-1]|$
$15 = |[2k,-k]|$
$15 = \sqrt{4k^2 + k^2}$
$225 = 5k^2$
$45 = k^2$
$\pm 3\sqrt{5} = k$

$k = 3\sqrt{5}$, $\vec{f} = 3\sqrt{5}[2,-1]$
$= [6\sqrt{5}, -3\sqrt{5}]$

Practice

1. An object is dragged 5m up a ramp under a constant force of 30N applied at an angle 30° to the ramp. Find the work done.

2. A man in a wheelchair moves 15m down a ramp inclined at an angle of 10° to the horizontal. The mass of the man and the wheelchair together is 80kg. (1kg = 9.8N) Calculate the work done.

3. An object is dragged 5m on level ground by a 20N force that is applied 50° to the ground. It is then dragged 8m up a ramp with the same force. The inclination of the ramp is 30° to the ground. At the top of the ramp, the object is dragged, with the same force, horizontally 13m. Find the total work done.

4. A box is lifted through a distance of 1.2 m and placed on a wagon by exerting a force of 105 N. The wagon is then pulled through a distance of 25 m by a 45 N force applied at an angle of 35° to the ground. Find the total work done.

5. Determine the work done by a force of magnitude 26N acting in the direction of the vector $\vec{u} = [-5, 12]$, which moves an object from A(1,4) to B (2,3).The distance is in metres.

6. Let $\vec{u} = proj_{\vec{b}}\, \vec{a}$ where $\vec{a} = [1,1]$ and $\vec{b} = [4,2]$. Show that $|\vec{a} - \vec{u}| \le |\vec{a} - k\vec{b}|$ for all $k \in R$.

7. The scalar projection of vector $\vec{u} = [1, m]$ on vector $\vec{v} = [3,4]$ is 4. Determine the value of m.

$$\vec{u} = \frac{\vec{b}\cdot\vec{a}}{|\vec{b}|^2}\vec{b}$$

$$= \frac{(1)(4)+(1)(2)}{(\sqrt{16+4})^2}[4,2]$$

$$= \frac{6}{20}[4,2]$$

$$= \left[\frac{6}{5}, \frac{3}{5}\right]$$

$$\vec{a} - \vec{u} = \left[\frac{-1}{5}, \frac{2}{5}\right]$$

$$|\vec{a} - \vec{u}| = \sqrt{\frac{1}{25} + \frac{4}{25}}$$

$$= \sqrt{\frac{1}{5}}$$

$$\vec{a} - k\vec{b} = [1-4k, 1-2k]$$

$$|\vec{a} - k\vec{b}| = \sqrt{(1-4k)^2 + (1-2k)^2}$$

$$= 1 - 8k + 16k^2 + 1 - 4k + 4k^2$$

$$= \sqrt{20k^2 - 12k + 2}$$

$$\sqrt{\frac{1}{5}} \le \sqrt{20k^2 - 12k + 2}$$

$$\frac{1}{5} \le 20k^2 - 12k + 2.$$

$$1 \le 100k^2 - 60k + 10$$

$$0 \le 100k^2 - 60k + 9$$

$$0 \le (10k-3)^2$$ will alway be true for all $k \in \mathbb{R}$

6.7 Vectors in 3D space (R^3)

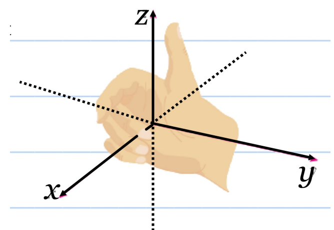
Algebraic Vectors in 3-Space

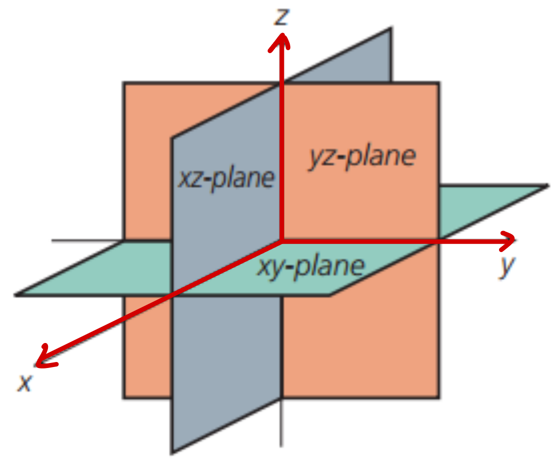
In 3-space, R^3, position vector, $\overrightarrow{OP}$, has its tail at O (0,0,0) and tip at point P(a, b,c).
Thus, $\overrightarrow{OP} = [a, b, c]$ where a = x-component, b = y-component and c = z-component.

To create the axes in R^3aka Right Hand System;

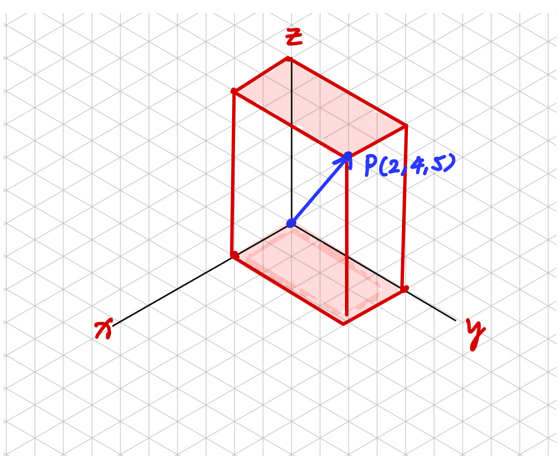
- z-axis is vertical (up) $\hat{k}$
- y-axis is horizontal (Right) $\hat{j}$
- x-axis come out of the page (Front) $\hat{i}$

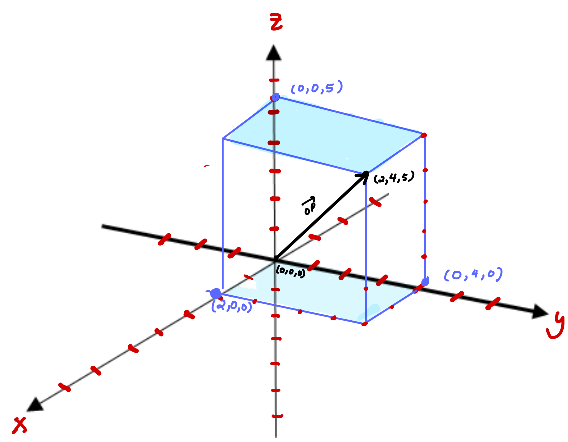
Note, each pair of axes forms a plane in 2D ie) xy, yz, xz.





Ex.1. Given the coordinates of point P(2,4,5), draw the vector $\overrightarrow{OP}$ using a rectangular prism.





Ex.2. If point U has coordinates (-4,3,-2);

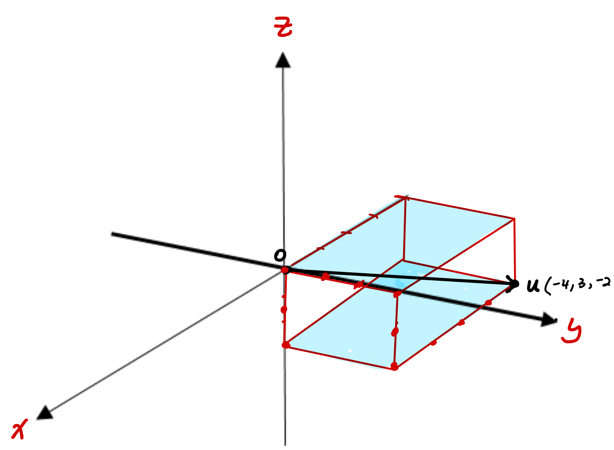
a) draw the point using a rectangular prism

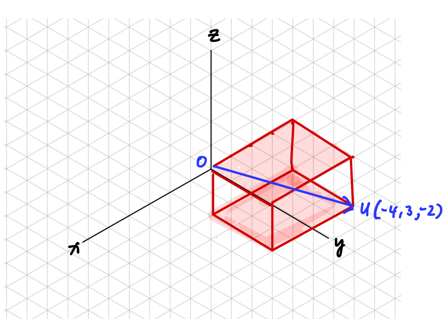
b) determine $\left|\overrightarrow{OU}\right| = \sqrt{(-4)^2+(3)^2+(-2)^2}$

$= \sqrt{16+9+4}$

$\overrightarrow{Ou} = [-4,3,-2]$

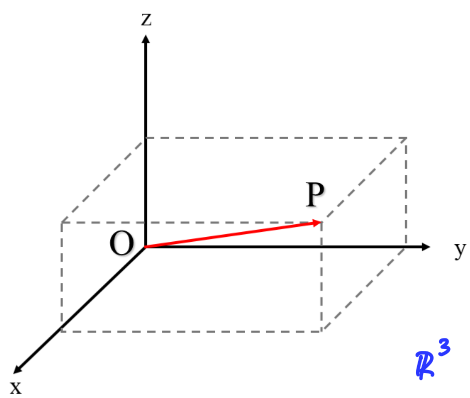
$= \sqrt{29}$ units

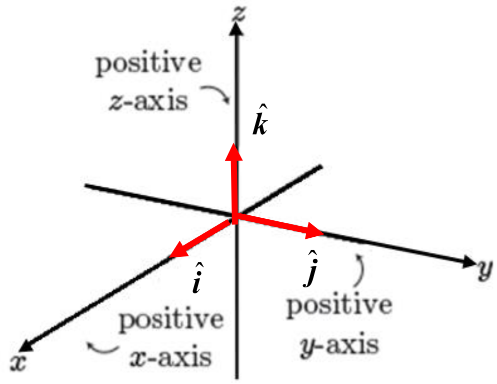




Standard Basis Unit Vectors, $\hat{i}, \hat{j}, k$

$\hat{i} = [1,0,0], \hat{j} = [0,1,0], \text{and } \hat{k} = [0,0,1]$ are the special unit vectors pointing in the direction of the positive x-, y-, and z-axes, respectively.





R^3

$\vec{u} = [3, 0, -8]$

$= 3\hat{i} - 8\hat{k}$

$\hat{i} = [1,0,0]$

$\hat{j} = [0,1,0]$

$\hat{k} = [0,0,1]$

$$\overrightarrow{OP} = [a,b,c]$$
$$= a\hat{i} + b\hat{j} + c\hat{k}$$

$$|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$$

Ex.1 For $\vec{u} = [2,4,5]$, find the angle that vector $\vec{u}$ makes with z-axis.

$\hookrightarrow \hat{k} = [0,0,1]$

$\vec{u} \cdot \hat{k} = |\vec{u}||\hat{k}| \cos\theta$

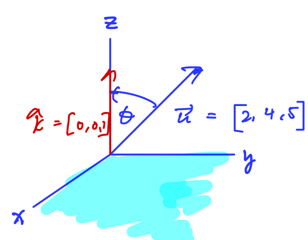
$[2,4,5] \cdot [0,0,1] = \sqrt{4+16+25} \cdot \sqrt{1} \cos\theta$

$2(0) + 4(0) + 5(1) = \sqrt{45} \cos\theta$

$5 = \sqrt{45} \cos\theta$

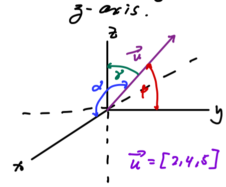
$\frac{5}{\sqrt{45}} = \cos\theta$

$\theta \doteq 41.8°$



* Not tested

Direction Angles - the angles a vector makes with the positive x-axis, positive y-axis and positive z-axis.



$\hat{i} = [1,0,0]$ $\hat{j} = [0,1,0]$ $\hat{k} = [0,0,1]$

$\cos\alpha = \frac{2}{\sqrt{45}}$ $\cos\beta = \frac{4}{\sqrt{45}}$ $\cos\gamma = \frac{5}{\sqrt{45}}$

$\alpha = 72.7°$ $\beta = 53.4°$ $\gamma = 41.8°$

Operations with Algebraic Vectors in R3

The rules for operations of vectors in two-dimensions extend easily to vectors in three-dimensions.

Addition of Two Vectors
The sum two position vectors $\vec{u} = [x_1, y_1, z_1]$ and $\vec{v} = [x_2, y_2, z_2]$ is $\vec{u} + \vec{v} = [x_1 + x_2, y_1 + y_2, z_1 + z_2]$.

Subtraction of Two Vectors
If $\vec{u} = [x_1, y_1, z_1]$ and $\vec{v} = [x_2, y_2, z_2]$ then $\vec{u} - \vec{v} = [x_1 - x_2, y_1 - y_2, z_1 - z_2]$.

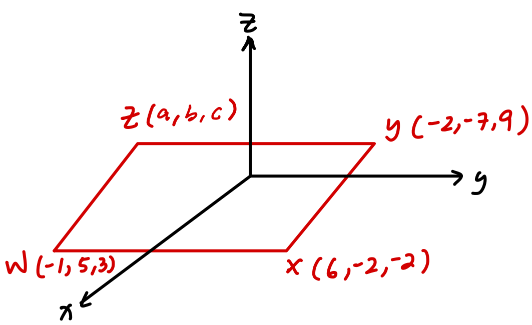
Scalar Multiplication of Two Vectors
For any vector $\vec{u} = [x, y, z]$ and any scalar $k \in \mathbb{R}$, the scalar multiple $k\vec{u} = k[x, y, z] = [kx, ky, kz]$

Ex.2 Given the points A(3, 6, -1) and B(-1, 0, 5) determine $\left|\overrightarrow{AB}\right|$

$\overrightarrow{AB} = [-1-3, 0-6, 5-(-1)]$
$= [-4, -6, 6]$
or $= 2[-2, -3, 3]$

$|\overrightarrow{AB}| = \sqrt{(-4)^2 + (-6)^2 + (6)^2}$
$= \sqrt{88}$
$= 2\sqrt{22}$
≈ 9.38 units

Ex.3 Given W(-1,5,3), X(6,-2,-2) and Y(-2,-7,9), find the coordinates of the point Z if WXYZ is a parallelogram.



$\overrightarrow{WZ} = [a+1, b-5, c-3]$
$\overrightarrow{XY} = [-2-6, -7-(-2), 9-(-2)]$
$= [-8, -5, 11]$

$\overrightarrow{WZ} = \overrightarrow{XY}$
$[a+1, b-5, c-3] = [-8, -5, 11]$

$a+1 = -8$ $\quad b-5 = -5$ $\quad c-3 = 11$
$a = -9$ $\quad b = 0$ $\quad c = 14$

$\therefore Z(-9, 0, 14)$

Practice

not tested Omit if you wish

1. Draw a sketch to show the point D (4, 2, −3) and draw the position vector $\overrightarrow{OD}$.
2. ~~2~~. Determine the direction angles for each of the following vectors.

   aka "directional cosine" of a vector.

   a) $\vec{v} = 2\hat{i} - \hat{j} + 3\hat{k}$ b) $\overrightarrow{OA} = (-1, 4, -5)$

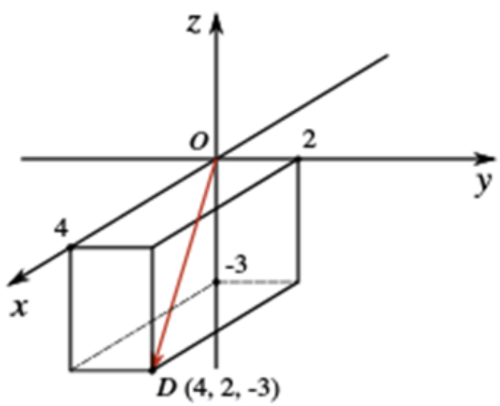
   c) $\vec{u} = 5\hat{i} - 12\hat{k}$ d) $\overrightarrow{OB} = (0, 3, -4)$
3. Find a unit vector parallel to each of the given vectors.

   a) $\vec{v} = (2, -5)$ b) $\overrightarrow{OZ} = \hat{i} - 2\hat{j} + 4\hat{k}$

   c) $\vec{w} = (-5, 12)$ d) $\overrightarrow{OP} = 3\hat{i} + 3\hat{j} - \hat{k}$

Answer

1.



2. $\cancel{\vec{v} \approx 5\hat{i} - 2\hat{j} - 4\hat{k}}$

3.

a. $\left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\right)$

b. $\frac{1}{\sqrt{21}}\hat{i} - \frac{2}{\sqrt{21}}\hat{j} + \frac{4}{\sqrt{21}}\hat{k}$

c. $\left(-\frac{5}{13}, \frac{12}{13}\right)$

d. $\frac{3}{\sqrt{19}}\hat{i} + \frac{3}{\sqrt{19}}\hat{j} - \frac{1}{\sqrt{19}}\hat{k}$

2a) $\cos\alpha = \frac{2}{\sqrt{14}}$ $\cos\beta = \frac{-1}{\sqrt{14}}$ $\cos\gamma = \frac{3}{\sqrt{14}}$

$\alpha \doteq 57.7°$ $\beta \doteq 105.5°$ $\gamma = 36.7°$

b) $\cos\alpha = \frac{-1}{\sqrt{42}}$ $\cos\beta = \frac{4}{\sqrt{42}}$ $\cos\gamma = \frac{-5}{\sqrt{42}}$

$\alpha = 98.9°$ $\beta = 51.9°$ $\gamma = 140.5°$

c) $\cos\alpha = \frac{5}{13}$ $\cos\beta = \frac{0}{13}$ $\cos\gamma = \frac{-12}{13}$ Note! $\vec{u}$ is lying on the xz-plane

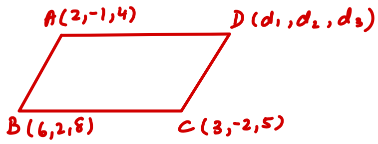
$\alpha = 51.7°$ $\beta = 90°$ $\gamma = 157.4°$

d) $\cos\alpha = \frac{0}{5}$ $\cos\beta = \frac{3}{5}$ $\cos\gamma = \frac{-4}{5}$ Note! $\overrightarrow{OB}$ is lying on the yz-plane

$\alpha = 90°$ $\beta = 53.1°$ $\gamma = 143.1°$

Warm up

1. $A\,(2,-1,4), B\,(6,2,8)$, and $C\,(3,-2,5)$ are three of the vertices of parallelogram $ABCD$. Determine the coordinates of D.



$\overrightarrow{BA} = \overrightarrow{CD}$

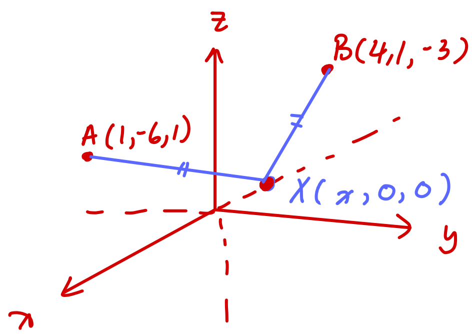
$[-4,-3,-4] = [d_1-3, d_2+2, d_3-5]$

$-4 = d_1-3 \qquad -3 = d_2+2 \qquad -4 = d_3-5$

$d_1 = -1 \qquad d_2 = -5 \qquad d_3 = 1$

$\therefore D(-1,-5,1)$

2. Find a point on the x-axis which is equidistant from the points $A\,(1,-6,1)$ and $B\,(4,1,-3)$.



$|\overrightarrow{AX}| = |\overrightarrow{BX}|$

$\sqrt{(x-1)^2+36+1} = \sqrt{(x-4)^2+1+9}$

$x^2-2x+38 = x^2-8x+26$

$6x = -12$

$x = -2$

$\overrightarrow{AX} = [x-1, 6, -1]$

$\overrightarrow{BX} = [x-4, -1, 3]$

exists in $\mathbb{R}^2$ or $\mathbb{R}^3$

exists only in $\mathbb{R}^3$

6.8 Dot Product and Cross Product of Vectors in 3-space

Dot Product of Algebraic Vectors

Let $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$, then $\vec{u} \bullet \vec{v} =$ $u_1v_1 + u_2v_2 + u_3v_3 \Rightarrow$ in component form

$= |\vec{u}||\vec{v}|\cos\theta \Rightarrow$ By definition

the same as in $\mathbb{R}^2$

Ex.1. Given that vectors $\vec{u} = [7,3,6]$ and $\vec{v} = [-3,3,h]$ are perpendicular, solve for h.

$\vec{u} \bullet \vec{v} = 0 \Rightarrow$ perpendicular

$-21 + 9 + 6h = 0$

$-12 + 6h = 0$

$h = 2$

Ex.2. Given vectors $\vec{u} = [1,2,2]$ and $\vec{v} = [3,0,-4]$. Find the angle between $\vec{u}$ and $\vec{v}$ to the nearest tenth of a degree.

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$

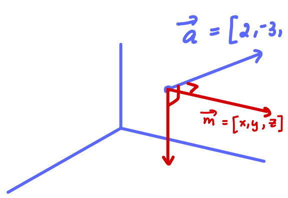
$(1)(3) + 2(0) + 2(-4) = \sqrt{1+4+4}\,\sqrt{9+0+16}\,\cos\theta$

$-5 = 15\cos\theta$

$\cos\theta = -\frac{1}{3}$

$\theta = 109.5°$

Ex.3. Find two **non-collinear** vectors perpendicular to $\vec{a} = [2,-3,1]$. < there are infinitely many >



$\vec{a} \cdot \vec{m} = 0$

$[2,-3,1] \cdot [x,y,z] = 0$

$2x - 3y + z = 0$

let $x = 1$, $y = 1$ then $2(1) - 3(1) + z = 0$
$-1 + z = 0$
$\therefore z = 1$
$\vec{m}_1 = [1,1,1]$

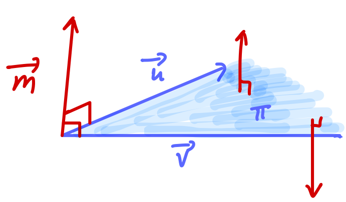
let $x = 2$, $y = 3$ then $2(2) - 3(3) + z = 0$
$-5 + z = 0$
$\therefore z = 5$
$\vec{m}_2 = [2,3,5]$

more generally

let $x = s$
$y = t$
then $2s - 3t + z = 0$
$\therefore z = -2s + 3t$

$[s, t, -2s+3t]$, $s \neq t = 0$

Ex.4. Find a vector perpendicular to both $\vec{u} = [2,-1,2]$ and $\vec{v} = [3,2,3]$.



let $\vec{m} = [a,b,c]$

$[a,b,c] \cdot [2,-1,2] = 0$

$2a - b + 2c = 0$ ①

$[a,b,c] \cdot [3,2,3] = 0$

$3a + 2b + 3c = 0$ ②

① $\xrightarrow{\times 2}$ $4a - 2b + 4c = 0$

② $\Rightarrow$ $3a + 2b + 3c = 0$

(+) $7a + 7c = 0$

$a + c = 0$ ③

Note! there are infinitely many solutions so we introduce parameters

let $a = t$ ④

sub ④ into ③

$\therefore (t) + c = 0$

$c = -t$

sub ④ and ⑤ into ①

$2(t) - b + 2(-t) = 0$

$0 = b$

$\therefore \vec{m} = [t, 0, -t]$ where $t \in \mathbb{R}$, $t \neq 0$

possible vectors:

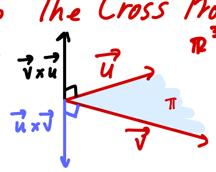
$t = -1$, $[-1, 0, 1]$

$t_2 = 7$, $[7, 0, -7] = -\frac{1}{7}[-1, 0, 1]$

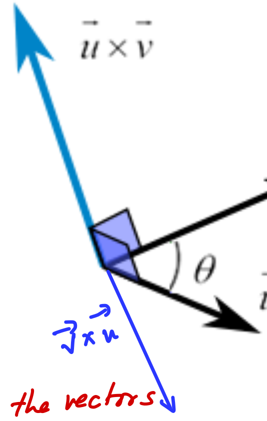
Which leads us to The Cross Product

$\vec{u} \times \vec{v} =$ vector

Can only exist in 3D Space



Cross Product of 2 Vectors $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ – in R^3



- not multiply, slightly bigger
- also known as vector product
- result is always a vector not a scalar
- cross product is a particular vector that's perpendicular to 2 non-collinear vectors, in fact, **there's an infinite number of such vectors!**

Cross Product – Algebraic Vectors

Given the vectors $\vec{\mathbf{u}} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ and $\vec{\mathbf{v}} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ then:

1) Set up the vector components in the following manner:

For $\vec{u} \times \vec{v}$:

$$\begin{matrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_1 \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1 \end{matrix}$$

or

Stack the vectors

$\vec{u} = [u_1, u_2, u_3]$

$\vec{v} = [v_1, v_2, v_3]$

$\vec{u} \times \vec{v} = [u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1]$

Note! Remember the middle must be opposite sign

2) To determine the x, y, and z component:

x-component of $\vec{u} \times \vec{v}$, conduct the following operation on the **middle** four terms:

$$\begin{vmatrix} \mathbf{u}_2 & \mathbf{u}_3 \\ \mathbf{v}_2 & \mathbf{v}_3 \end{vmatrix} = \mathbf{u}_2\mathbf{v}_3 - \mathbf{u}_3\mathbf{v}_2$$

y-component: conduct the same operation, but on the four terms on the **right**:

$$\begin{vmatrix} \mathbf{u}_1 & \mathbf{u}_3 \\ \mathbf{v}_1 & \mathbf{v}_3 \end{vmatrix} = (\mathbf{u}_1\mathbf{v}_3 - \mathbf{u}_3\mathbf{v}_1) = u_3 v_1 - u_1 v_3$$

don't forget to make the 'y-component' negative if by "stacking" method

z-component: repeat for the four terms on the **left**:

$$\begin{vmatrix} \mathbf{u}_1 & \mathbf{u}_2 \\ \mathbf{v}_1 & \mathbf{v}_2 \end{vmatrix} = \mathbf{u}_1\mathbf{v}_2 - \mathbf{u}_2\mathbf{v}_1$$

stacking it:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{vmatrix} = [u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1]$$

$$= [u_2 v_3 - v_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - v_2 v_1]$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

Determinant of a 2x2 matrix

Leading to... Linear Algebra in University

Fun!

Determinant of a 2x2 Matrix

Suppose we are given a square matrix A with four elements: a, b, c, and d.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The **determinant of matrix A** is calculated as

$$\det A = ad - bc$$

Denise's Method:

5 1 -2 5 1
1 1 -3 1 1

5 1 -2 5
1 1 -3 1

'Cover' column 1 'Cover' column 2 'Cover' column 3

$= [-3-(-2), -2-(-15), 5-1]$
$= [-1, 13, 4]$

$\Rightarrow$ this method doesn't require you to rember to make 2nd column 'negative'.

Ex.5. If $\vec{a} = [5, 1, -2]$ and $\vec{b} = \hat{i} + \hat{j} - 3\hat{k}$, determine the following

a) $\vec{a} \times \vec{b}$

$\vec{a} = [5, 1, -2]$
$\vec{b} = [1, 1, -3]$
$\vec{a} \times \vec{b} = [-3+2, -(-15+2), 5-1]$
$= [-1, 13, 4]$

b) $\vec{b} \times \vec{a}$ $\Rightarrow$ we know it should be $[1, -13, -4]$

let's verify:

$\vec{b} = [1, 1, -3]$
$\vec{a} = [5, 1, -2]$
$\vec{b} \times \vec{a} = [-2+3, -(-2+15), 1-5]$
$= [1, -13, -4]$

Properties of the Cross Product

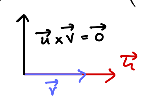
The Cross Product is:

1) **Anti-Commutative**: (Not Commutative) $\qquad \vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

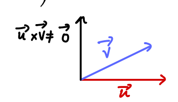
2) **Distributive** over vector addition: $\qquad \vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

3) **Associative** over scalar multiplication: $\qquad m(\vec{u} \times \vec{v}) = (m\vec{u}) \times \vec{v} = \vec{u} \times (m\vec{v}), m \in \mathbb{R}$

4) If $\vec{u}$ and $\vec{v}$ are non-zero, $\vec{u} \times \vec{v} = \vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are collinear.



$\vec{u}$ + $\vec{v}$ are collinear



$\vec{u}$ + $\vec{v}$ are not collinear

Ex.6. Determine the value of m and n for $\vec{a} = [m, -12, 9]$ and $\vec{b} = [5, n, -3]$ such that $\vec{a} \times \vec{b} = \vec{0}$.

Ex. $\vec{a} = [1, 2, 3]$
$\vec{b} = [2, 4, 6]$
$\vec{a} \times \vec{b} = [12-12, -(6-6), 4-4]$
$= [0, 0, 0]$
$\therefore \vec{a}$ and $\vec{b}$ are collinear

$\vec{a} = [m, -12, 9]$
$\vec{b} = [5, n, -3]$
$\vec{a} \times \vec{b} = [36-9n, -(-3m-45), mn-(-60)]$
$[0, 0, 0] = [36-9n, 3m+45, mn+60]$

$0 = 36-9n$
$9n = 36$
$n = 4$

$0 = 3m+45$
$-45 = 3m$
$m = -15$

check: $0 = mn+60$
RS $= 4(-15)+60$
$= LS$
$\therefore m = 4 \quad n = -15$

or

method 2: Property #4 above

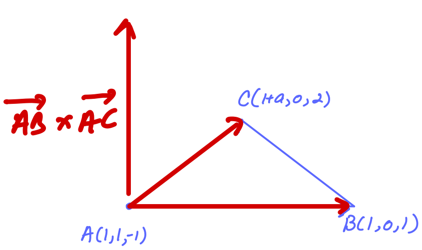
$\vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a} = k\vec{b}$
$[m, -12, 9] = k[5, n, -3]$

$m = k5 \qquad -12 = kn \qquad 9 = k(-3)$
$-3 = k$

$m = -3(5) \qquad -12 = -3n$
$m = -15 \qquad 4 = n$

Ex.7 If ABC is a triangle with vertices A(1, 1, -1), B(1, 0, 1), and C(1 + *a*, 0, 2) and $\overrightarrow{AB} \times \overrightarrow{AC} = [-1, 2, 1]$, find the value of *a*.

Note!
- if $\vec{u} \cdot \vec{v} = 0$, then $\vec{u} \perp \vec{v}$
- if $\vec{u} \times \vec{v} = \vec{0}$, then $\vec{u}$ and $\vec{v}$ are collinear
$\Rightarrow \vec{u} = k\vec{v}, k \neq 0$



$\overrightarrow{AB} = [0, -1, 2]$
$\overrightarrow{AC} = [a, -1, 3]$
$\overrightarrow{AB} \times \overrightarrow{AC} = [-3+2, -(0-2a), 0+a]$
$[-1, 2, 1] = [-1, 2a, a]$

$2 = 2a$
$a = 1$

Verify it satisfies every component
check: $a = 1$

$\therefore a = 1$

D. Magnitude of the Cross Product

The magnitude of the cross product is defined according to the following equation:

$$\boxed{|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta}$$

⇒ Cool how this has a familiarity to the dot Product...

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$, already a Scalar

, where θ is the angle between the vectors such that $0° \le \theta \le 180°$.

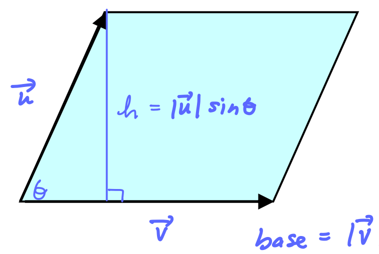
It represents the area of the parallelogram enclosed by the two vectors.

$$\mathbf{A}_{\mathbf{parallelogram}} = |\vec{\mathbf{u}}\times\vec{\mathbf{v}}|$$

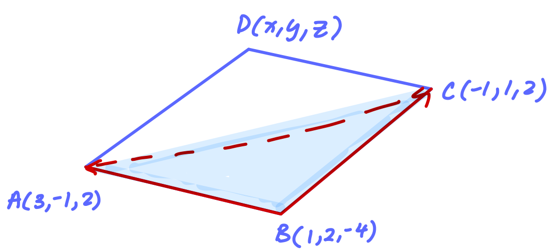
$= \text{base} \times \text{height}$

$= |\vec{v}||\vec{u}|\sin\theta$

$= |\vec{u}||\vec{v}|\sin\theta$ ☺



Ex.8. Three vertices of a **parallelogram ABCD** are **A(3,-1,2), B(1,2,-4)** and **C(-1,1,2)**.
Find the area of triangle ABC.



$\vec{BA} = [2,-3,6]$

$\vec{BC} = [-2,-1,6]$

$\vec{BA}\times\vec{BC} = [-18-(-6), -(12-(-12)), -2-(6)]$

$= [-12, -24, -8]$

$|\vec{BA}\times\vec{BC}| = \sqrt{144+576+64}$

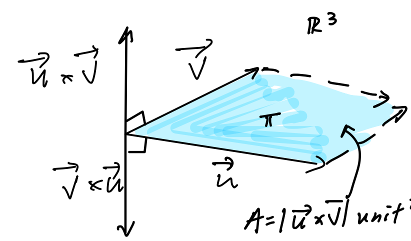
$= \sqrt{784}$

$= 28 \text{ unit}^2$ ⇒ Area of Parallelogram ABCD

Area of $\Delta ABC = \frac{1}{2}$ Area of Parallelogram

$= \frac{1}{2}(28)$

$= 14 \text{ unit}^2$

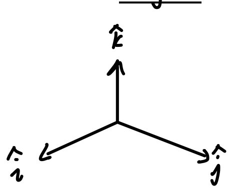


Ex.9. Determine;

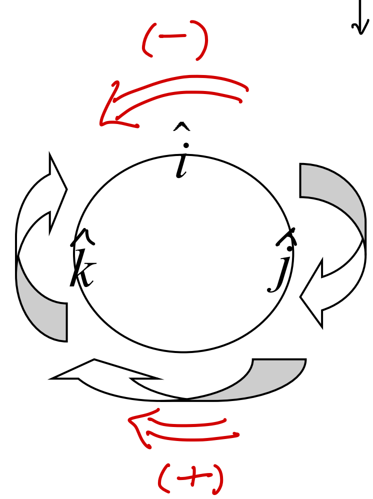
a) $\hat{i}\times\hat{j} = \underline{\hat{k}}$ d) $\hat{k}\times\hat{j} = \underline{-\hat{i}}$

b) $\hat{k}\times\hat{i} = \underline{\hat{j}}$ e) $\hat{j}\times\hat{i} = \underline{-\hat{k}}$

c) $\hat{i}\times\hat{k} = \underline{-\hat{j}}$ f) $\hat{j}\times\hat{k} = \underline{\hat{i}}$



Use the "right-hand System"



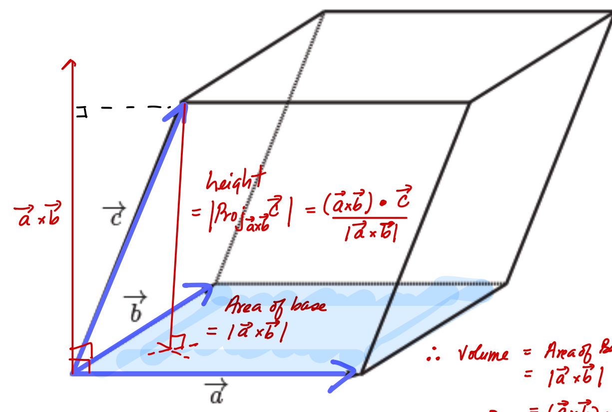
Application of Triple Scalar Product $\vec{c}\cdot(\vec{a}\times\vec{b})$ $\quad(\vec{a}\times\vec{c})\cdot\vec{b}\quad(\vec{b}\times\vec{c})\cdot\vec{a}$

① Volume of Parallelepiped

A **parallelepiped** is a box-like solid, where the opposite faces of which are parallel and congruent parallelograms.

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors whose tails meet at one vertex of the parallelepiped.

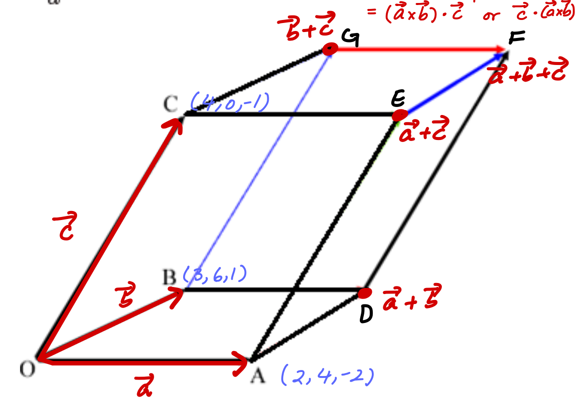
The absolute value of the triple scalar product of these three vectors gives the volume of the parallelepiped.



$$V_{parallelpiped} = \left|\vec{c}\cdot(\vec{a}\times\vec{b})\right|$$

$\therefore$ Volume $=$ Area of Base $\times$ height
$= |\vec{a}\times\vec{b}|\,\dfrac{(\vec{a}\times\vec{b})\cdot\vec{c}}{|\vec{a}\times\vec{b}|}$
$= (\vec{a}\times\vec{b})\cdot\vec{c}$ or $\vec{c}\cdot(\vec{a}\times\vec{b})$

Ex.10. If $\overrightarrow{OA}, \overrightarrow{OB},$ and $\overrightarrow{OC}$ are three edges of a parallelepiped where O (0,0,0), A (2,4,-2), B (3,6,1), and C (4,0,-1), find the coordinates of the other vertices of the parallelepiped.



$\vec{a} = [2,4,-2]$ $\qquad \vec{a}+\vec{b} = [5,10,-1]$ $\quad \therefore D(5,10,-1)$

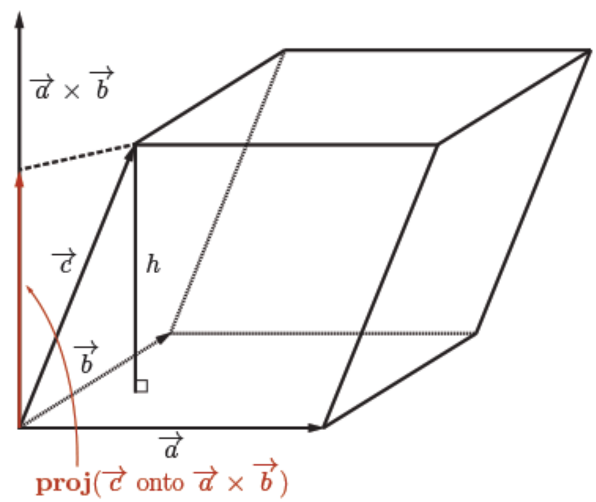
$\vec{b} = [3,6,1]$ $\qquad \vec{a}+\vec{c} = [6,4,-3]$ $\quad \therefore E(6,4,-3)$

$\vec{c} = [4,0,-1]$ $\qquad \vec{b}+\vec{c} = [7,6,2]$ $\quad \therefore G(7,6,2)$

$\vec{a}+\vec{b}+\vec{c} = [9,10,-2]$ $\quad \therefore F(9,10,-2)$

method 2: Equivalent vectors (takes longer)
$\overrightarrow{OB} = \overrightarrow{AD} \Rightarrow$ to find D
$\overrightarrow{OC} = \overrightarrow{AE}$, $\overrightarrow{DF} = \overrightarrow{BG} \Rightarrow$ to find C, E, F, G

Volume of a parallelepiped



$V = \left|(\vec{a}\times\vec{b})\cdot\vec{c}\right|$
$= [4+12, -(2+6), 12-12]\cdot[4,0,-1]$
$= [16,-8,0]\cdot[4,0,-1]$
$= 16(4)+(-8)(0)+(0)(-1)$
$= 64$ unit^3

Volume = (area of base) × height
= (area of parallelogram) × height

parallelogram is made up of vector $\vec{a}$ and $\vec{b}$ so its area $= \left|\vec{a}\times\vec{b}\right|$

The height = the magnitude of the projection of $\vec{c}$ on the vector perpendicular to the base:

$$\left|Proj_{\vec{a}\times\vec{b}}\vec{c}\right| = \frac{\vec{c}\bullet\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|}$$

Volume = (area of parallelogram) × height

$$\boxed{V = \left|\vec{c} \bullet \left(\vec{a} \times \vec{b}\right)\right|}$$

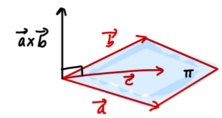
Question: Is $\left|\vec{c} \bullet (\vec{a} \times \vec{b})\right|$ equivalent to $\left|\vec{a} \bullet (\vec{b} \times \vec{c})\right|$? yes!

Triple Scalar Product: is called the quantity $\vec{c} \bullet (\vec{a} \times \vec{b})$, since it returns a scalar value.

② Determining if 3 vectors are Co-Planar

Definition: Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar if and only if $\vec{c} \bullet (\vec{a} \times \vec{b}) = 0$

↳ $\vec{a}$, $\vec{b}$, and $\vec{c}$ lies on the same plane



Ex.11: Determine the volume of a parallelepiped given the vectors $\vec{a} = [-2,2,5]$, $\vec{b} = [0,4,1]$ and $\vec{c} = [0,5,-1]$.

$\vec{b} = [0,4,1]$
$\vec{c} = [0,5,-1]$
$\vec{b}\times\vec{c} = [-4-5, -(0-0), 0-0]$
$= [-9,0,0]$

method 1: Triple Scalar Product

$V = |a \cdot (\vec{b}\times\vec{c})|$
$= [-2,2,5] \cdot [-9,0,0]$
$= -2(-9) + 2(0) + 5(0)$
$= 18 \text{ unit}^3$

method 2: 3×3 Determinant Approach to Triple Scalar Product

don't forget to make the middle negative

$$a \cdot \vec{b}\times\vec{c} = \begin{vmatrix} -2 & 2 & 5 \\ 0 & 4 & 1 \\ 0 & 5 & -1 \end{vmatrix} = -2\begin{vmatrix} 4 & 1 \\ 5 & -1 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} + 5\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix}$$
$= -2(-4-5) - 2(0-0) + 5(0-0)$
$= -2(-9)$
$= 18 \text{ unit}^3$

Ex. 12: Determine if the vectors $[1,3,2]$, $[5,0,-1]$, and $[-4,3,3]$ are coplanar.

↳ lies on the same plane

$\vec{a} = [1,3,2]$
$\vec{b} = [5,0,-1]$
$\vec{c} = [-4,3,3]$
$\vec{a}\times\vec{b} = [-3-0, -(-1-10), 0-15]$
$= [-3,11,-15]$

$\vec{a}\times\vec{b}\cdot\vec{c}$
$= [-3,11,-15] \cdot [-4,3,3]$
$= 12 + 33 - 45$
$= 0$

or $\begin{vmatrix} 1 & 3 & 2 \\ 5 & 0 & -1 \\ -4 & 3 & 3 \end{vmatrix} = 1(0-(-3)) - 3(15-4) + 2(15-0)$
$= 3 - 33 + 30$
$= 0$

$\therefore$ $\vec{a}$, $\vec{b}$ and $\vec{c}$ are co-planar

note! $\vec{a}\times b\cdot\vec{c} = \vec{a}\cdot\vec{b}\times\vec{c}$ ⇒ order doesn't matter

Ex.13. Circle whether the following expressions are vectors, scalar, or meaningless.

a) $\left(\vec{a}+\vec{b}\right)\bullet\left(\vec{a}+\vec{c}\right)$ vector • vector = Scalar — vector **scalar** meaningless

b) $\dfrac{\left(\vec{a}\bullet\vec{b}\right)\vec{c}\times\left(\vec{a}\times\vec{b}\right)}{\left|\vec{c}\right|}$ $\dfrac{\text{vector} \times \text{vector}}{\text{scalar}}$ = vector — **vector** scalar meaningless

c) $\vec{a}\times\vec{b}+\vec{u}\bullet\vec{c}$ = vector + scalar ⇒ meaningless — vector scalar **meaningless**

d) $\dfrac{\vec{a}\bullet\vec{b}+\vec{a}\bullet\left(\vec{a}\times\vec{b}\right)}{\left|\vec{b}\right|}$ = $\dfrac{\text{scalar} + \text{scalar}}{\text{scalar}}$ = Scalar — vector **scalar** meaningless

Practice

1. Area of parallelogram formed by vectors $\vec{a}=[x,1,-1], \vec{b}=[1,1,2]$ is $\sqrt{35}$ units^2 , find the value(s) of x.
2. Determine the volume of the parallelepiped defined by the vectors $\vec{u}=[-5,-5,3], \vec{v}=[1,-9,3]$ and $\vec{w}=[4,-3,12]$.
3. Determine the **exact** value(s) of **k** such that the angle between $\vec{a}=[1,1,k], \vec{b}=[1,1,1]$ is 60°.
4. Find the work done when a force of 6 N, acting parallel to the vector $\vec{f}$ =[2,1,3], moves an object from A (1,0,6) to B (4,3,10).
5. Find all unit vectors perpendicular to (1,2,3) that make equal angles with the unit vectors $\hat{i}$ and $\hat{j}$.

Unit 6-Review

1. Given $\overrightarrow{OA} = 3\hat{i} + 6\hat{j} - k$ and $\overrightarrow{OB} = -\hat{i} + 5k$, determine the unit vector in the direction of $\overrightarrow{AB}$. Express your answer in component form.

   $Ans: \frac{1}{\sqrt{22}}[-2,-3,3]$

2. A triangle is bounded by [−5, 2, 3] and [1, 7, −1]. Calculate its area.

   $Ans: 21.8 \text{ units}^2$

3. Find the scalar projection of $\vec{a}$ on $\vec{b}$ if their magnitudes are 4 and 7, respectively, and the angle between them is 110°.

   $Ans: -1.37 \text{ unit}$

4. Suppose 2000 J of work is done by pulling a toboggan 260 m by a force applied at an angle of 40° with the horizontal. What is the magnitude of the pulling force?

   $Ans: |\vec{F}| = 10\,N$

5. Determine the value(s) of k such that $\vec{x}$ = [11, 3, 2k] and $\vec{y}$ = [k, 4, k] and the angle between the vectors is obtuse.

   $Ans: k \in \left(-4, \frac{-3}{2}\right)$

6. Find the volume of a parallelepiped bounded with vectors [10, −3, −14], [4, 3, 1] and [1, 5, 6].

   $Ans: 39 \text{ units}^3$

7. A triangle has vertices at A (4,0, 1), B(-2,-1,3) and C(5, 3,-1). Find the area of triangle.

   $Ans: 10.1 \text{ units}^2$

8. The vector $\vec{r}$ is twice as long as the vector $\vec{s}$. The angle between the vectors is 120°. The vector projection of $\vec{s}$ on $\vec{r}$ is [2, −1, 7]. Determine $\vec{r}$.

9. Show that $(\vec{a} - 2\vec{b}) \times (\vec{a} + \vec{b}) = 3\vec{a} \times \vec{b}$ for any arbitrary vectors $\vec{a}$ and $\vec{b}$.

10. If $|\vec{u}| = 4, |\vec{v}| = 9$ and the angle θ, between $\vec{u}$ and $\vec{v}$ is 120°, determine the following.

    a) $Proj_{\vec{v}}\vec{u}$ $Ans: \frac{-2}{9}\vec{v}$

    b) $\vec{u} \bullet \vec{v}$

    c) $|\vec{u} \times \vec{v}|$ $Ans: 18\sqrt{3}$

11. If $\vec{u}$ =[1,4,-2], $\vec{v} = -2\hat{i} - 3\hat{j}$ and $\vec{w}$ =[-1,-3,1], find:

    a) $|3\vec{v} + 3\hat{i} - 2k|$ $Ans: \sqrt{94}$

    b) a unit vector with the same direction as $\vec{u}$. $Ans: u = \frac{1}{\sqrt{21}}[1,4,-2]$

    c) Find the angle between $\vec{v}$ and $\vec{w}$ $Ans: \theta = 23.1^{\circ}$

12. Consider the points $A(1, 0, 2)$, $B(2, 0, 1)$, $C(3, 2, -1)$.

(a) Determine the vector projection of $\overrightarrow{AB}$ on $\overrightarrow{AC}$. $Ans: \frac{5}{17}[2,2,-3]$

$\vec{v}$

(b) If a force of 10N acts in the direction of $[1,1,-1]$ to move an object from A to B, and distance is measured in meters, how much work is done? $Ans: \frac{20\sqrt{3}}{3} J$

. The points A(-1,2,-1) ,B(2,-1,3), and D(-3,1,-3) are three vertices of parallelogram ABCD. Find

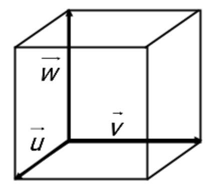
a) the coordinate of C. $Ans: C[0,-2,1]$

b) the area of the parallelogram in unit² $Ans: 13.6 \text{ units}^2$

c) Find a vector perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AD}$ $Ans: [-10,2,9]$

14. Prove or disprove. Let $\vec{a}$, $\vec{b}$ be non-zero vectors. If $|\vec{a} \times \vec{b}| = 0$ then $\vec{a} \bullet \vec{b} = 0$. It can be disproved by considering $\theta = 0°$

15. Given the diagram of a **unit cube**. Evaluate $(\vec{u} + \vec{v}) \bullet (\vec{v} + \vec{w})$. $Ans: 1$



16. Vectors [2, -a, 1] and [−2, 2,- a+1] are collinear. Find the value of a. $Ans: a = 2$

17. Find the volume of a parallelepiped with edges represented by the vectors
$\vec{a} = [1,4,3]$, $\vec{b} = [2,5,6]$, $\vec{c} = [1,2,7]$ $Ans: 12 \text{ units}^3$

18. Suppose $\vec{a}$ and $\vec{b}$ are vectors such that $\vec{a} \times \vec{b} = [1,0,2]$. Find $\vec{b} \times (-2\vec{a})$ $Ans: [2,0,4]$

19. Find ***a*** and ***b*** such that $\vec{u} = [-3,4,-7]$ and $\vec{v} = [a,-7,b]$ are collinear. $Ans: a = \frac{21}{4}, b = \frac{49}{4}$

20. If $\vec{a}$ and $\vec{b}$ are unit vectors , and , $|\vec{a} + \vec{b}| = \sqrt{3}$ determine $(2\vec{a} - 5\vec{b}) \bullet (\vec{b} + 3\vec{a})$.

21. An airplane pilot checks her instruments and finds that the speed of the plane relative to the air is 325 km/h. The instruments also show that the plane is pointed in a direction [N30°W]. A radio report indicates that the wind velocity is 80 km/h blowing from [E 25° N]. What is the velocity of the plane relative to the ground as it is recorded by an air traffic controller in a nearby airport? $Ans:$ 341.4 km/h [N43.5∘W]

22. A large cruise boat is moving at 15 km/h [E25°S] relative to the water. A person jogging on the ship moves across the ship in a northerly direction at 6 km/h. What is the velocity of the jogger relative to the water? Ans:13.6 km/h [E1.6°N]

23. A plane is seen to travel in a direction [N55°E]. If its ground velocity was 300 km/h and the wind was blowing 50 km/h from [N45°W], what was the plane's velocity relative to the air? Ans:295.4 km/h [N45.4°E]. Ans:295.4 km/h [N45.4°E]