Displacement, Velocity and Acceleration

Displacement

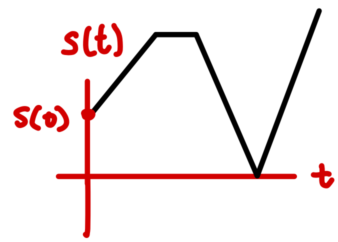
$s(t)$: distance and direction an object has moved from an origin over a period of time. (usually in metres, m)

Velocity

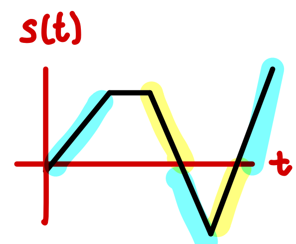
$v(t) = s'(t)$: rate of change of displacement with respect to time. (usually in metres per second, m/s)

Acceleration

$a(t) = v'(t) = s''(t)$: rate of change of velocity with respect to time (usually in metres per second squared, m/s^2)



Relating Position, Velocity and Acceleration

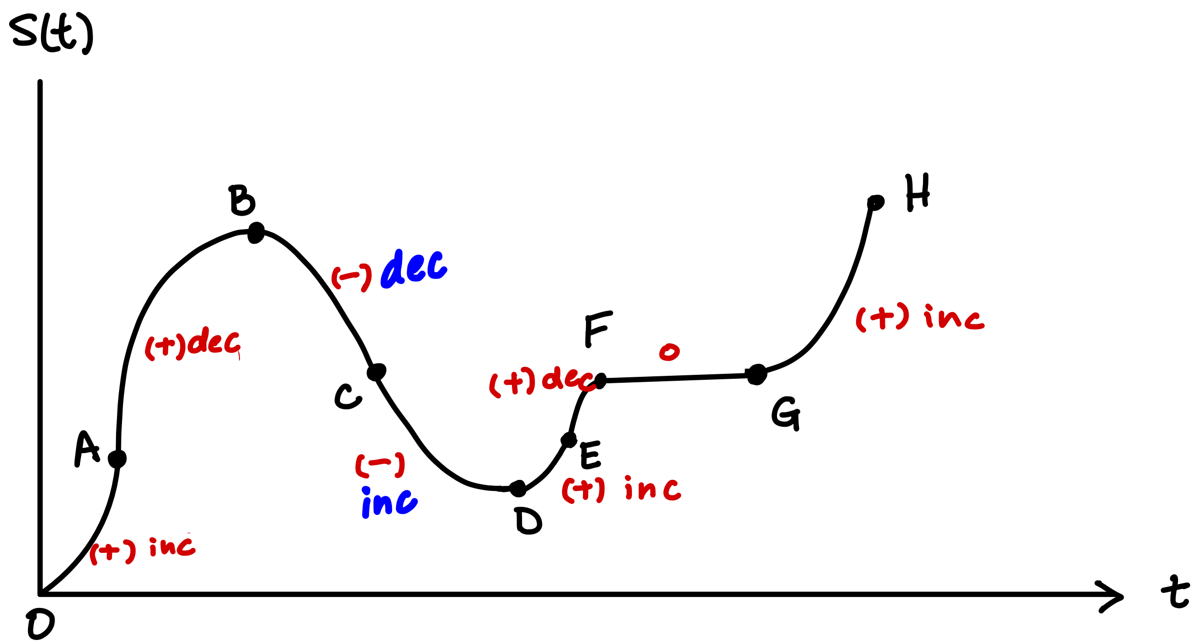


1. An object is **speeding up** at time, t if $v(t) \times a(t) > 0$.
2. An object is **slowing down** at time, t if $v(t) \times a(t) < 0$.
3. An object is **moving away** ← Forward from its original position if $s(t) \times v(t) > 0$.
4. An object is **moving towards** ← Reverse from its original position if $s(t) \times v(t) < 0$

$v(t)$	$a(t)$	$v(t) \times a(t)$	**Motion of particle** Speeding Up **or** Slowing Down **and** Moving Forward/Up **or** Moving Reverse/Down	**Description of Slope of** $s(t)$ Positive **or** Negative Slope **and** Increasing **or** Decreasing
+	+	+	Speeding up Moving Forward	positive increasing
+	-	-	Slowing down moving Forward	positive decreasing
-	+	-	slowing down moving reverse	negative increasing
-	-	+	speeding up moving reverse	negative decreasing

Example 1:

The graph of the position function is shown below.



a) Describe the slope of the graph in terms of **positive**, **negative**, **increasing**, or **decreasing** over each indicated interval.

b) Determine the sign of the velocity and acceleration.

v a

Interval	Slope of Graph	Velocity	Acceleration
O to A	+ increasing	+	+
A to B	+ decreasing	+	−
B to C	− decreasing	−	−
C to D	− increasing	−	+
D to E	+ increasing	+	+
E to F	+ decreasing	+	−
F to G	0	0	0
G to H	+ increasing	+	+

+
−
+
−
+
−
0
+

if $v \cdot a > 0$ the particle is speeding up
if $v \cdot a < 0$ the particle is slowing down

Position, Velocity, Acceleration

Example 1: The position function of a particle is $s(t) = t^4 - 12t^3 + 30t^2 + 5t,\ t \geq 0$. When is the acceleration positive and negative?

$v(t) = s'(t)$
$= 4t^3 - 36t^2 + 60t + 5$

$a(t) = v'(t) = s''(t)$
$= 12t^2 - 72t + 60$
$= 12(t^2 - 6t + 5)$
$= 12(t-5)(t-1)$

$a(t)$: (+) on $(0,1)$, (−) on $(1,5)$, (+) after 5; marks at 0, 1, 5

$\therefore a(t) > 0$
$t \in (0,1) \cup (5,\infty)$
$a(t) < 0$
$t \in (1,5)$

Example 2: A particle moves according to $s(t) = 2t^3 - 21t^2 + 60t - 12,\ t \geq 0$, where s is in metres and t is in seconds.

a. When does the particle change direction?

$\hookrightarrow v(t) = 0$ (momentarily stops)

$v(t) = 6t^2 - 42t + 60$
$0 = 6(t^2 - 7t + 10)$
$0 = 6(t-5)(t-2)$

$t = \{2, 5\}$

$\therefore$ @ $t = 2$ s
$t = 5$ s
the object changes direction

b. How fast is the particle moving after 3 seconds?

$v(3) = 6(3)^2 - 42(3) + 60$

$= -12 \text{ m/s}$

$\therefore$ at $t=3$
the object moving at -12m/s

c. Is the particle speeding up or slowing down at 4 seconds?

$v(4) = 6(4)^2 - 42(4) + 60$

$= -12 \text{ m/s } (-)$

$a(t) = 12t - 42$

$a(4) = 12(4) - 42$

$= 6 \text{ m/s}^2 \ (+)$

$v(4) \cdot a(4) < 0$

$\therefore$ slowing down at $t = 4$s

$$s(t) = 2t^3 - 21t^2 + 60t - 12, \quad t \geq 0,$$

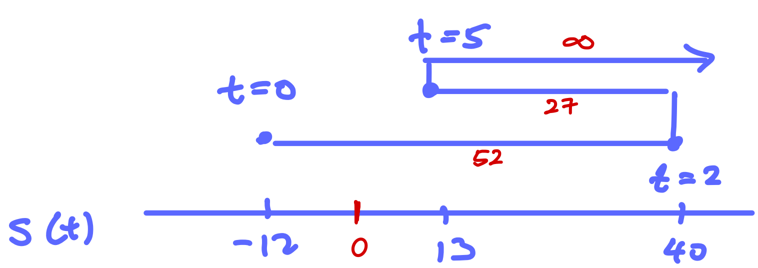
$v(t) = 0, \quad t = \{2, 5\}$

d. Sketch $\boxed{\text{the path of } s(t)}$.

$S(0) = -12$

$s(2) = 40$

$s(5) = 13$



Example 3: On windless day, an archer shoots an arrow vertically upward so that its height in metres above the ground t seconds after release is given by the formula $h(t) = -4.9t^2 + 24.5t + 2$.

a. Determine the velocity at any time.

$v(t) = -9.8t + 24.5$

b. When is the velocity at -5 m/s?

$-5 = -9.8t + 24.5$

$t \doteq \{3.01\}$

$\therefore$ @ $t \doteq 3$ s

c. When does the arrow hit the ground?

$h(t) = 0$

$-4.9t^2 + 24.5t + 2 = 0$

$t = \dfrac{-24.5 \pm \sqrt{639.45}}{2(-4.9)}$

$= \{-0.08, 5.08\}$

↳ inadmissible

$\therefore$ @ $t = 5.08$ s

d. At what time does the arrow stop rising?

↳ @ max $\therefore$ $v(t) = 0$

$0 = -9.8t + 24.5$

$t = 2.5$ s

$\therefore$ @ 2.5 s it stop rising and reaches max height

Grade 12 Calculus & Vectors
Unit 4 – Application of Derivatives: Optimization

DAY	DESCRIPTION	Homework
1	**4.1 Higher Order Derivatives; Displacement, Velocity and Acceleration** We are learning to… • make inferences about and connections between position, velocity and acceleration functions both graphically and algebraically • solve problems of rates of change drawn from a variety of applications (including distance, velocity, and acceleration) involving polynomial and rational functions I am able to… • determine whether an object is speeding up or slowing down at a specified time given the position function both algebraically and graphically • solve real world application problems involving velocity and acceleration	*CP Pg 8-9* *Textbook: Pg 106 #1,2-14,16*
2&3	**4.2 (a) Area and (b) Volume Type Optimization Problems including Inscribed Shape Problems** We are learning to… • solve practical problems that involve finding the conditions that are 'The BEST' or 'The OPTIMUM' I am able to… • create an algebraic model of a quantity to be optimized in one unknown and state the domain of this function, considering real-life constraints • solve optimization problems involving numbers, perimeter, area, surface area and/or volume • solve optimization problems involving inscribed shapes	*CP Pg 16 (Day 1-Area)* *Textbook: Pg 200 #1-7, 14, 16, 20, 23* *CP Pg 22 (Day 2-Volume)* Textbook: Pg 200 # 8,10
4	**4.3 Distance/Time Type** We are learning to… • solve practical problems that involve finding the conditions that are 'The BEST' or 'The OPTIMUM' I am able to… • solve optimization problems involving the path that saves the most time when traveling from one place to another using two modes of travel	*CP Pg. 29*
5	**4.4 Optimization in Business** We are learning to… • solve practical problems that involve finding the conditions that are 'The BEST' or 'The OPTIMUM' I am able to… • create an algebraic model of a quantity to be optimized in one unknown and state the domain of this function, considering real-life constraints • solve optimization problems involving finding the best conditions to yield the greatest revenue when selling something	*CP Pg 35* *Mid-Review CP . Pg 30* Text book : Pg 200 #12,13, 21
6	**4.5 Implicit Differentiation (optional)** We are learning to … • take the derivative of the function y that is not isolated explicitly. I am able to… • differentiate a function implicitly	*CP . Pg40*
7&8	**Review/Quiz**	*CP. Pg* *Pg 205 #15, 16* *Pg 207 #15 – 18* *Pg 209 #18, 19*
9	**Summative Test**	

T Nov 12 (Day1) - postponed from Th Nov 7
W Nov 13 (Day 2) - postponed from F Nov 8

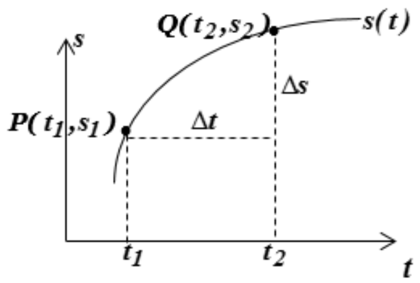
UNIT 4

APPLICATION OF DERIVATIVE

OPTIMIZATION

4.1 APPLICATIONS OF HIGHER ORDER DERIVATIVES – LINEAR MOTION

Definitions
Position s(t) is the location of an object at a value of time t.



- $s(0)$ represents the initial position of the object (when $t = 0$)
- $s(t) > 0$ indicates that the object is to the <u>right</u> of the origin
- $s(t) < 0$ indicates that the object is to the <u>left</u> of the origin.
- $s(t) = 0$ indicates that the object is at the origin (where it started)

Velocity v(t) is the rate of change of position over time, so $v(t) = s'(t) = \dfrac{ds}{dt}$

Thus instantaneous velocity is $v(t) = s'(t) = \dfrac{ds}{dt}$

- $v(0)$ is the initial velocity (when $t = 0$)
- $v(t) > 0 \Rightarrow$ object is moving to the right (positive direction)
- $v(t) < 0 \Rightarrow$ object is moving to the left (negative direction)
- $v(t) = 0 \Rightarrow$ object is at rest or object may be changing directions or object may be at a maximum/minimum height

Acceleration a(t) is the rate of change of velocity over time, so

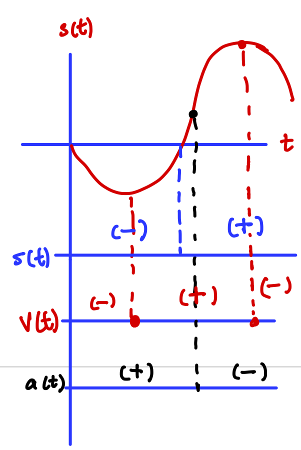
$$a(t) = v'(t) = s''(t)$$

Or

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

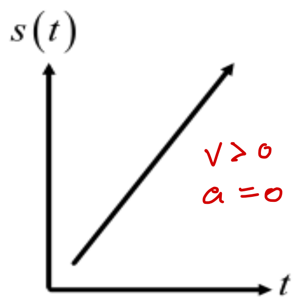
- $a(t) > 0 \Rightarrow$ object is accelerating (velocity is increasing)
- $a(t) < 0 \Rightarrow$ object is decelerating (velocity is decreasing)
- $a(t) = 0 \Rightarrow$ object is at a constant velocity or object is at a max/min velocity

- If $sv > 0$ the object is moving away from the origin
- If $sv < 0$ the object is moving towards the origin
- If $av > 0$ the object is speeding up
- If $av < 0$ the object is slowing down



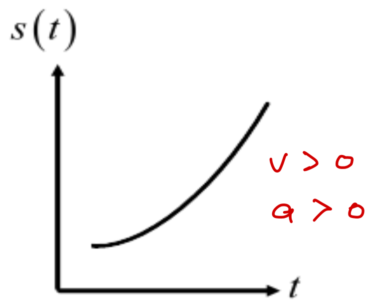
Example 1: In each graph determine if the object is moving away from/towards the origin or speeding up/slowing down

(a)



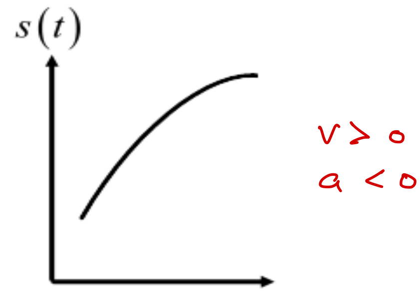
away neither

(b)



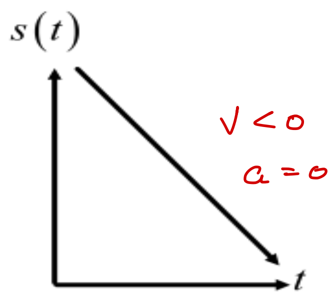
away speed up

(c)



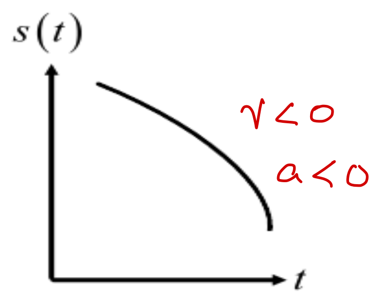
away slowing down

(d)



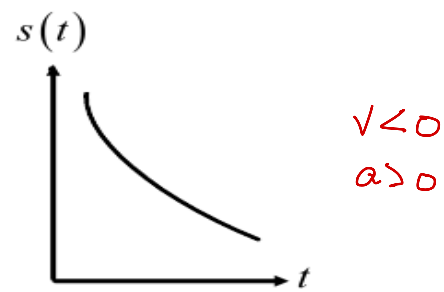
towards neither

(e)



towards speeding up

(f)



towards slowing down

Example2: An object moves in a straight line according to the motion equation $\mathbf{s(t) = 2t^3 - 15t^2 + 36t - 22, t \geq 0}$, where S measures in meter and times measures in second.

(a) Find the velocity and acceleration at time t.

$$s'(t) = v(t) = 6t^2 - 30t + 36$$

$$s''(t) = a(t) = 12t - 30$$

(b) Find the initial conditions and interpret their meaning.

$t = 0$,

$S(0) = -22\,m \Rightarrow$ starts at 22m left of the origin

$v(0) = 36\ m/s \Rightarrow$ starts at a rate of 36m/s

$a(0) = -30\ m/s^2 \Rightarrow$ starts at a deceleration rate of $30\ m/s^2$

* (c) Find the average velocity and ~~acceleration~~ Avg acceleration from t = 2s to t = 4s.

$s(t) = 2t^3 - 15t^2 + 36t - 22$

Avg Vel. $= \dfrac{s(4) - s(2)}{4-2}$

$= \dfrac{10-6}{2} = 2\,m/s$

$v(t) = 6t^2 - 30t + 36$

Avg Acc. $= \dfrac{v(4) - v(2)}{4-2}$

$= \dfrac{12-0}{2}$

$= 6\,m/s^2$

(d) When is the particle at rest?

↳ $v(t) = 0$, $\quad 6t^2 - 30t + 36 = 0$

$6(t^2 - 5t + 6) = 0$

$6(t-3)(t-2) = 0$

$t = \{2, 3\}$

∴ @ $t = 2s$ and $t = 3s$, the object is at rest

(e) When does the object move in a positive direction?

↳ $v(t) > 0$,

$v(t)$: (+) on 0 to 2, (−) on 2 to 3, (+) after 3

∴ the object moves in a positive direction at $t \in (0, 2) \cup (3, \infty)$

Example 3: The motion of a particle on straight line is given by position function $\mathbf{s(t) = 36 - 24t + 9t^2 - t^3}$, where s is in meter and t is in minute.

(a) Find the velocity and acceleration at time t.

$v(t) = -3t^2 + 18t - 24 = -3(t-4)(t-2)$

$a(t) = -6t + 18 = -6(t-3)$

(b) ~~After how many minutes~~ At what time(s) does the object stop?

$v(t) = 0$, $\quad 0 = -3t^2 + 18t - 24$

$0 = -3(t^2 - 6 + 8)$

$0 = -3(t-4)(t-2)$

$t = \{2, 4\}$

∴ after 2 minutes ($t=2$) and 4 minutes ($t=4$) from its original start

(c) When is the particle moving toward the motion detector?

$s(t) = -t^3 + 9t^2 - 24t + 36$

$= -(t^3 - 9t^2 + 24t - 36)$

↳ $\pm 1 \pm 2 \pm 3 \pm 4 \pm 6 \pm 9$

$s(6) = 0$, ∴ $(t-6)$ is a factor

6	1	−9	24	−36
	↓	6	−18	36
	1	−3	6	0

$s(t) = -(t-6)(t^2 - 3t + 6)$

$s \cdot v < 0$,

$-(t-6)(t^2 - 3t + 6) \cdot -3(t-4)(t-2) < 0$

$3(t-6)(t^2 - 3t + 6)(t-4)(t-2) < 0$

$s \cdot v$: − on 0 to 2, + on 2 to 4, − on 4 to 6, + after 6

∴ moving towards the motion sensor at $t \in (0, 2) \cup (4, 6)$

(d) When is the object slowing down?

$v \cdot a < 0$

$-3(t-4)(t-2) \cdot -6(t-3) < 0$

$18(t-2)(t-3)(t-4) < 0$

$a \cdot v$: − on 0 to 2, + on 2 to 3, − on 3 to 4, + after 4

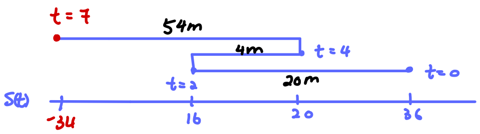
∴ The object is slowing down at $t \in (0, 2) \cup (3, 4)$

(e) Determine the total distance traveled in the first 7 minutes.

$s(t) = -t^3 + 9t^2 - 24t + 36$

$v(t) = 0$

$t = \{2, 4\}$



$s(0) = 36$

$s(2) = 16$

$s(4) = 20$

$s(7) = -34$

Total Distance $= 20 + 4 + 54$

$= 78\,m$

or method 2:

Total Distance $= |s(2) - s(0)| + |s(4) - s(2)| + |s(7) - s(4)|$

$= |16 - 36| + |20 - 16| + |-34 - 20|$

$= |-20| + |4| + |-54|$

$= 78$

Example4: Given position function $\mathbf{s(t) = t^{\frac{5}{2}}(7 - t),\ t \geq 0}$.

(a) Is the particle speeding up or slowing down when $t = 4s$.

$s(t) = 7t^{\frac{5}{2}} - t^{\frac{7}{2}} = t^{\frac{5}{2}}[2 - t]$

$v(t) = s'(t)$

$= \frac{35}{2}t^{\frac{3}{2}} - \frac{7}{2}t^{\frac{5}{2}}$

$= \frac{7}{2}t^{\frac{3}{2}}[5 - t]$

$\therefore v(4) > 0$

$a(t) = v'(t)$

$= \frac{105}{4}t^{\frac{1}{2}} - \frac{35}{4}t^{\frac{3}{2}}$

$= \frac{35}{4}t^{\frac{1}{2}}[3 - t]$

$\therefore a(4) < 0$

$\because v(4) \cdot a(4) < 0$

$\therefore$ at $t = 4$ the particle is slowing down

(b) At what time(s) is the object at rest?

$v(t) = 0$, $\frac{7}{2}t^{\frac{3}{2}}[5 - t] = 0$

$t = \{0, 5\}$

$\hookrightarrow$ endpoint $\therefore$ not sure the velocity to the left of '0'

$\therefore$ the particle is at rest at $t = 5$

(c) In which direction is the object moving at ***t* =4**

$v(4) > 0$ $\therefore$ the object is moving in a positive direction

note!

$s(4) \cdot v(4) > 0$

means the object is moving away from the start position at $t = 4s$

(d) When is the object moving in a negative direction?

$v(t) = \frac{7}{2}t^{\frac{3}{2}}[5-t]$

$v(t) < 0$

$v(t)$: 0 $+$ 5 $-$

∴ The object is moving at a negative direction from $t \in (5, \infty)$

(e) When does the object return to its initial position?

$s(t) = 0$

$s(t) = 7t^{\frac{5}{2}} - t^{\frac{7}{2}} = t^{\frac{5}{2}}[7-t]$

$t^{\frac{5}{2}}[7-t] = 0$

$t = \{0, 7\}$

initial

↳ return

∴ it returns to its original position after 7 seconds

4.1 Practice

1. A particle moves on the y axis with this relationship between position and time: $s(t) = t^3 - 17t^2 + 80t - 100$. Determine the time interval(s) during which it is :

a) located below the origin

b) moving upward (moving in positive direction)

c) slowing down

d) moving away from the origin

2. A particle moves on the y axis with this relationship between position and time:

$s(t) = \frac{1}{4}t^4 - 2t^3 + \frac{9}{2}t^2 - 4t + 2$

a) Describe the motion of the particle at $t = 0$.

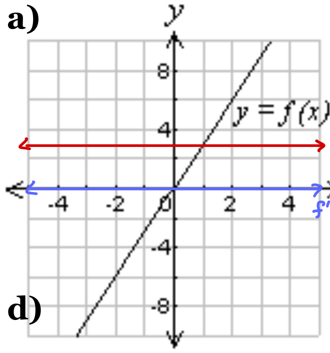
b) What is the average velocity of the particle between $t = 1$ and $t = 4$?

c) When does the particle reverse direction?

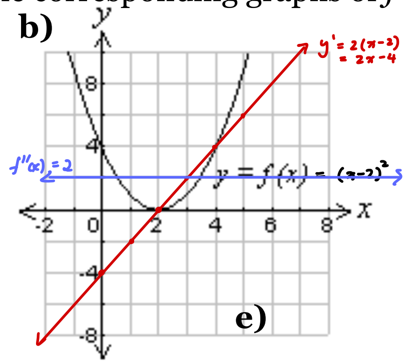
d) Find the total distance traveled from $t = 0$ to $t = 5$.

3. For each graph below, sketch the corresponding graphs of f' and f''.

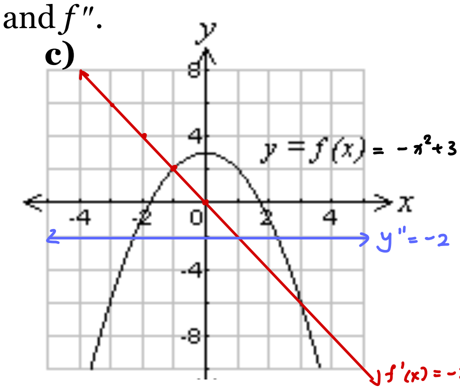
a)



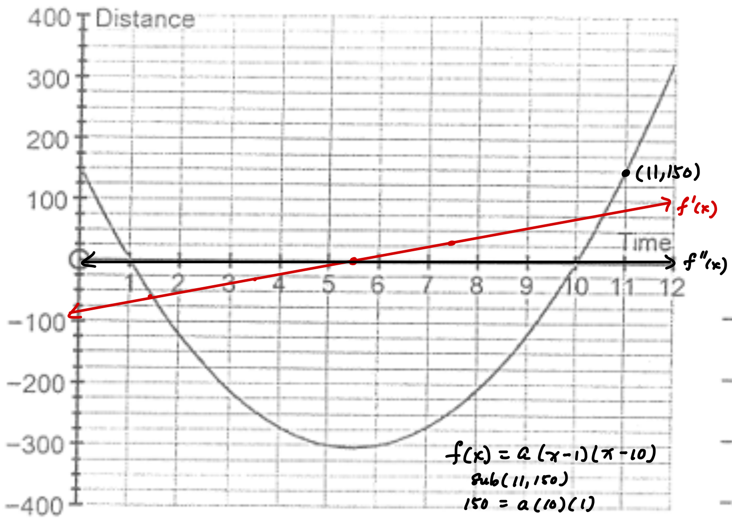
b)



c)

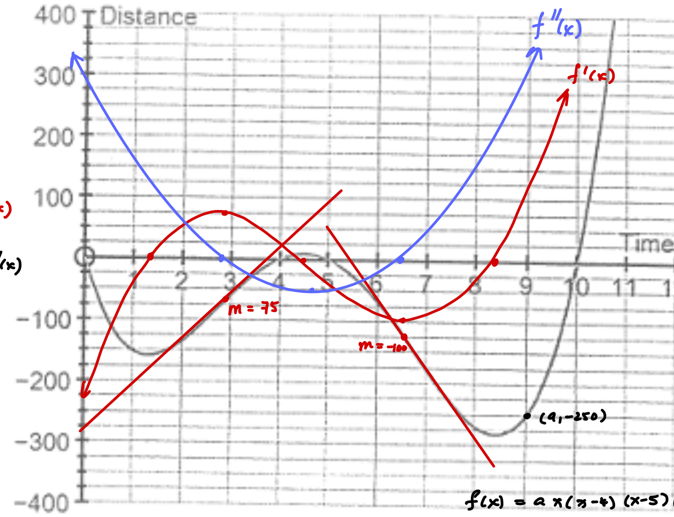


d)



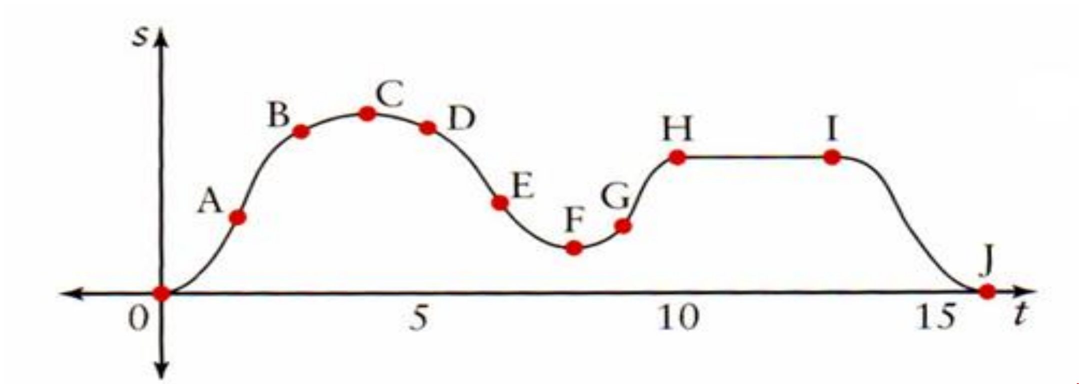
$f(x) = a(x-1)(x-10)$
sub $(11,150)$
$150 = a(10)(1)$
$a = 15$
$\therefore f(x) = 15(x-1)(x-10)$

e)



$f(x) = a\,x(x-4)(x-5)(x-10)$
sub $(9,-250)$
$-250 = a(9)(5)(4)(-1)$
$a = \frac{25}{18}$
$\therefore f(x) = \frac{25}{18}x(x-4)(x-5)(x-10)$

4. The following graph shows the position function of a bus during 15 min trip.



a) What is the initial velocity of the bus? '0' But not for certain as it is an 'endpoint' (we don't know what is on the left side of 0s)

b) What is the bus's velocity at C and F? both 0

c) Is the bus going faster at A or at B? Explain. at A as the slope is steeper

d) What happens to the motion of the bus between H and I? resting

e) Is bus speeding up or slowing down at A, B and D? A ⇒ neither; $v>0$ $a=0$
B ⇒ slowing down; $v>0$ $a<0$
D ⇒ speeding up; $v<0$ $a<0$

f) What happens at J? returns to original position

5. Refer to the graph in question 4. Is the acceleration positive, zero, or negative during the following intervals?

a) 0 to A

positive

Concave up

b) C to D

negative

Concave down

c) E to F

positive

Concave up

d) G to H

negative

Concave down

e) F to G

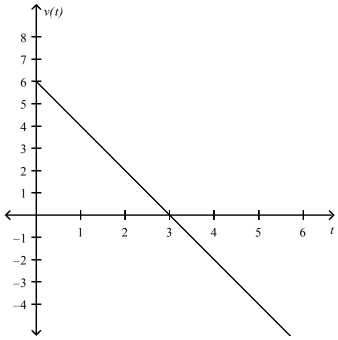
positive

concave up

Warm-Up

Write the CAPITAL LETTER corresponding to the correct answer on the line provided.

C 1. This graph could represent the velocity of which of the following position functions?



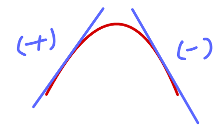
A. $s(t)=t^2+6t+1$

B. $s(t)=-2t+6$

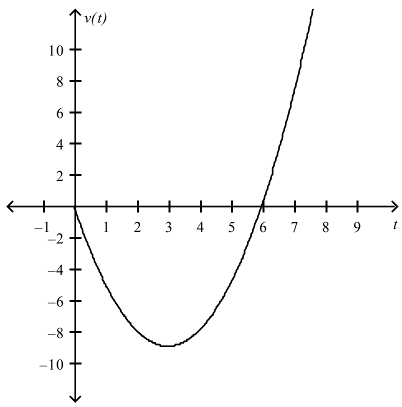
C. $s(t)=-t^2+6t+7$

D. $s(t)=2t^2+6t$

• must be quadratic that opens down



B 2. For what values of t, in seconds, does the particle travel in a negative direction if its velocity is given by the graph below?



A. $[6,\infty)$

B. $(0,6)$

C. $(6,\infty)$

D. $[0,6]$

$v(t)<0$

B 3. A particle's position along the x-axis is given by $S(t)=t^5-11t^2$, where t is in seconds. When t = 1 second, the particle is:

A. speeding up and moving towards the origin
B. speeding up and moving away from the origin
C. speeding up and is at the origin
D. the correct answer is not given

$s(t)=t^5-11t^2$ $\quad v(t)=5t^4-22t$ $\quad a(t)=20t^3-22$

$s(1)<0$ $\quad v(1)<0$ $\quad a(1)<0$

$s(1)\cdot v(1)>0 \therefore$ moving away from origin @ $t=1$

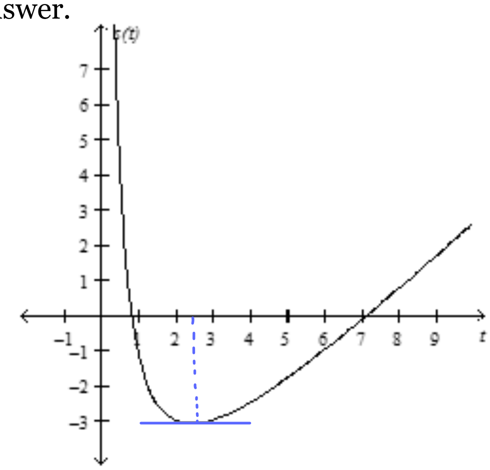
$v(1)\cdot a(1)>0 \therefore$ speeding up @ $t=1$

4. Consider the graph below of a certain object's position in metres with time $t>0$ in seconds. When is the object accelerating in a positive direction? Justify your answer.

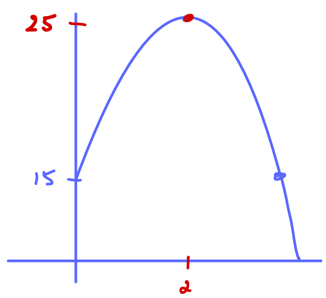
↳ velocity is increasing

$\therefore t\in(2.5,\infty)$

note! acceleration is positive (concave up) for $t\,(0,\infty)$



5. Let the function $S(t) = -\frac{5}{2}t^2 + 10t + 15$ represent the motion of a toy car as it travels near a sensor. At what time *t* will the car's distance from the sensor be the greatest? What is its velocity when the car is at that point? Describe the motion of the car after that time *t*.



$v(t) = 0$

$-5t + 10 = 0$

$t = 2$

$S(2) = -\frac{5}{2}(2)^2 + 10(2) + 15$

$= 25m$

a) at $t=2$, the car will be the greatest distance from the sensor. @ 25m

b) at this time, the car's velocity is 0m/s as it momentarily stops to change direction

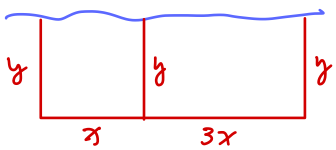
c) the car then returns and moves towards its starting position and goes past its start distance of 15m (moving in a negative direction) getting the most (max) or the least (min) under a given condition (constraint)

4.2 (a) Optimization – Area

We need to realize the "best" outcome, subject to a set of restrictions or constraints.
"Best" could be **maximum** or **minimum** → use the same principles local extrema occur when the first derivative is zero.

Ex1: A farmer wants to build two adjacent pens along a river, one of which is three times as long as the other. If he has 80m of fencing available, what dimensions will maximize the total area of the pens?

Step 1: Draw a diagram, identify variables:



let y be the width of the pen (m)
let $4x$ be the length of the pen (m)

Step 2: Identify the function to be optimized:

$$A = y \cdot 4x$$
$$= 4xy$$

Step 3: Rewrite the equation to be optimized in terms of one variable (if needed)

Secondary equation (constraint):

$$3y + 4x = 80$$
$$y = \frac{80 - 4x}{3}$$

$$A = 4xy$$
$$A(x) = 4x\left(\frac{80-4x}{3}\right)$$
$$= \frac{320x}{3} - \frac{16}{3}x^2$$

Step 4: Identify constraints/restrictions:

$0 < x < 20$ (if area is expressed in terms of x)
$0 < y < \frac{80}{3}$ (if area is expressed in terms of y)

Step 5: Find the critical numbers (first derivative = 0 and test them)

$$A'(x) = \frac{320}{3} - \frac{32}{3}x$$
$$0 = \frac{320}{3} - \frac{32}{3}x$$
$$0 = 320 - 32x$$

$\therefore$ critical #: $\{10\}$

Step 6: Determine optimal value and check the constraints:

$A'(x)$: 0 to 10: $(+)$ ↗; 10 to 20: $(-)$ ↘

max

$A(10) = \frac{320}{3}(10) - \frac{16}{3}(10)^2$

$= 533\frac{1}{3}$

Step 7: Conclusion:

$\therefore$ max area is $533\frac{1}{3}\ m^2$

when the dimension of the pen is $\frac{40}{3}$ m $\times$ 40 m

width

$y = \frac{80-4x}{3}$

$y = \frac{80-4(10)}{3}$

$= \frac{40}{3}$

length

$4x$

$= 4(10)$

$= 40$

Ex2: A Norman window consists of a rectangle surmounted by a semi-circle. Determine the maximum area of a Norman window if 1440 cm of trim is required around the outside. The window must be at least 10 cm wide and 20 cm tall.

$A = \frac{1}{2}\pi x^2 + 2xy$

$A(x) = \frac{1}{2}\pi x^2 + 2x(720 - x - \frac{\pi}{2}x)$

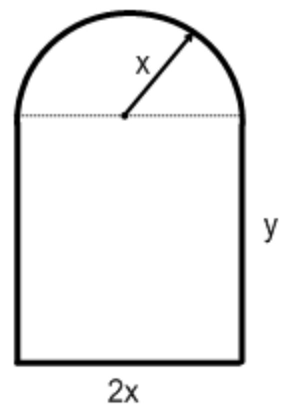
$= \frac{1}{2}\pi x^2 + 1440x - 2x^2 - \pi x^2$

Constraint:

$\frac{2\pi x}{2} + 2x + 2y = 1440$

$2y = 1440 - 2x - \pi x$

$y = 720 - x - \frac{\pi}{2}x$



$A'(x) = \pi x + 1440 - 4x - 2\pi x$

$= (-\pi - 4)x + 1440$

$0 = (-\pi - 4)x + 1440$

$x = \frac{1440}{\pi + 4}$

$\doteq 201.63$

$2x \geq 10$

$x \geq 5$

and $x + y \geq 20$

$x + (720 - x - \frac{\pi}{2}x) \geq 20$

$720 - \frac{\pi}{2}x \geq 20$

$-\frac{\pi}{2}x \geq -700$

$x \leq \frac{1400}{\pi}$

$\therefore 5 \leq x \leq \frac{1400}{\pi}$

$A'(x)$: 5 to $\frac{1440}{\pi+4}$: $(+)$; $\frac{1440}{\pi+4}$ to $\frac{1400}{\pi}$: $(-)$

max

$A(201.63) \doteq \frac{1}{2}\pi(201.63)^2 + 1440(201.63) - 2(201.63)^2 - \pi(201.63)^2$

$\doteq 145177.70$

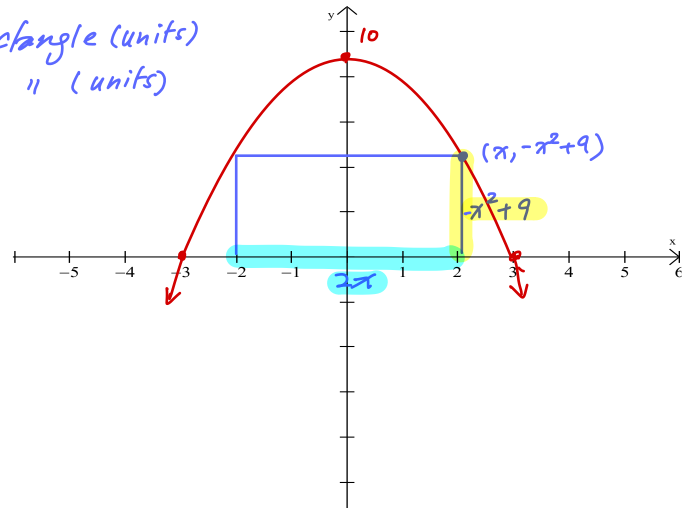
$\therefore$ max area is approx. $145177.7\ cm^2$ or $14.5\ m^2$

Ex3 Find the area of the largest rectangle that can be inscribed between the x-axis and the graph defined by y = 9- x². Include a diagram.

$y = -x^2 + 9$

let $2x$ be the length of the rectangle (units)

let $-x^2+9$ " " width " " " (units)



Constraint:

$0 < x < 3$

$A(x) = (2x)(-x^2+9)$

$= -2x^3 + 18x$

$A'(x) = -6x^2 + 18$

$0 = -6x^2 + 18$

$x^2 = 3$

$x = \{-\sqrt{3}, \sqrt{3}\}, \quad 0 < x < 3$

$-\sqrt{3}$ → inadmissible

$A'(x)$ (+) (−)

0 $\sqrt{3}$ 3

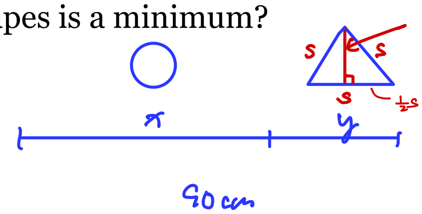
max

$A(\sqrt{3}) = -2(\sqrt{3})^3 + 18\sqrt{3}$

$= 20.78$

∴ the max area of the rectangle is approx 20.8 unit²

Ex4. A wire 90cm long is to be cut into 2 pieces. One piece is to be formed into a circle. The other is to be bent into an equilateral triangle. Where should the wire be cut so that the combined area of the two shapes is a minimum?



$h = \sqrt{s^2 - (\frac{1}{2}s)^2}$
$= \sqrt{s^2 - \frac{s^2}{4}}$
$= \sqrt{\frac{3}{4}s^2}$
$= \frac{\sqrt{3}}{2}s$

$A_\triangle = \frac{1}{2}bh$
$= \frac{1}{2}s(\frac{\sqrt{3}}{2}s)$
$= \frac{\sqrt{3}}{4}s^2$

Constraint : $x + y = 90$
$y = 90 - x$

$C = 2\pi r$
$x = 2\pi r$
$\frac{1}{2\pi}x = r$

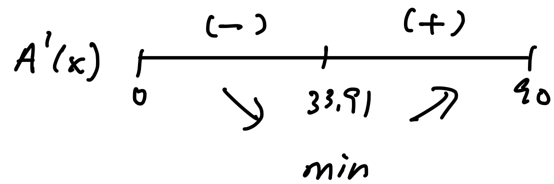
$P = 3s$
$90 - x = 3s$
$\frac{90-x}{3} = s$

$A = \pi r^2 + \frac{\sqrt{3}}{4}s^2$

$A(x) = \pi\left(\frac{1}{2\pi}x\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{90-x}{3}\right)^2$
$= \frac{1}{4\pi}x^2 + \frac{\sqrt{3}}{4}\left(30 - \frac{x}{3}\right)^2$

$0 < x < 90$

$A'(x) = \frac{1}{2\pi}x + \frac{\sqrt{3}}{2}\left(30 - \frac{x}{3}\right)\cdot\left(-\frac{1}{3}\right)$
$0 = \frac{1}{2\pi}x - 5\sqrt{3} + \frac{\sqrt{3}}{18}x$
$0 = \left(\frac{1}{2\pi} + \frac{\sqrt{3}}{18}\right)x - 5\sqrt{3}$
$x = \frac{5\sqrt{3}}{\frac{1}{2\pi} + \frac{\sqrt{3}}{18}}$
$= 33.91$



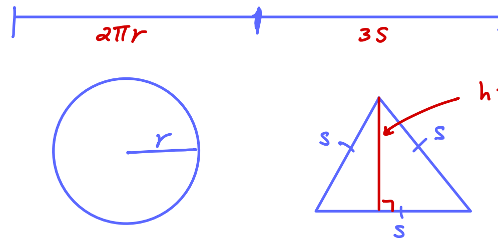
∴ 33.91 cm should be used for the circle formation.

< Alternate set up #2 >

Ex4. A wire 90cm long is to be cut into 2 pieces. One piece is to be formed into a circle. The other is to be bent into an equilateral triangle. Where should the wire be cut so that the combined area of the two shapes is a minimum?

let r be the radius of circle (cm)

let s " " length of the equilateral's side (cm)



$h = \sqrt{s^2 - (\frac{1}{2}s)^2}$

$= \sqrt{\frac{3}{4}s^2}$

$= \frac{\sqrt{3}}{2}s$

$C = 2\pi r$

$A = \pi r^2$

$P = 3s$

$A = \frac{1}{2}s \cdot \frac{\sqrt{3}}{2}s$

$= \frac{\sqrt{3}}{4}s^2$

$$A = \pi r^2 + \frac{\sqrt{3}}{4}s^2$$

$$A(r) = \pi r^2 + \frac{\sqrt{3}}{4}\left(30 - \frac{2\pi}{3}r\right)^2$$

$$= \pi r^2 + \frac{\sqrt{3}}{4}\left(900 - 40\pi r + \frac{4\pi^2}{9}r^2\right)$$

$$= \pi r^2 + 225\sqrt{3} - 10\sqrt{3}\pi r + \frac{\sqrt{3}\pi^2}{9}r^2$$

Constraint:

$2\pi r + 3s = 90$

$s = \frac{90 - 2\pi r}{3}$

$= 30 - \frac{2\pi}{3}r$

$$A'(r) = 2\pi r - 10\sqrt{3}\pi + \frac{2\sqrt{3}\pi^2}{9}r$$

$$0 = 2\pi r - 10\sqrt{3}\pi + \frac{2\sqrt{3}\pi^2}{9}r$$

$$\frac{10\sqrt{3}\pi}{\left(2\pi + 2\frac{\sqrt{3}}{4}\pi^2\right)} = r$$

$\therefore r = 5.397$

$0 < 2\pi r < 90$

$0 < r < \frac{90}{2\pi}$

$\therefore 0 < r < \frac{90}{2\pi}$

$A'(r)$: 0 — (−) — 5.397 — (+) — $\frac{90}{2\pi}$

min

$A(5.397) \doteq 333.1 \text{ cm}^2$

$C = 2\pi(5.397)$

$= 33.911$

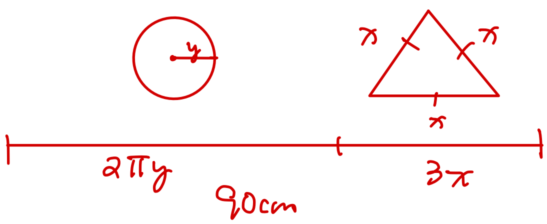
$3s = 90 - 33.911$

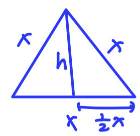
$= 56.088$

$\therefore$ The string should be cut at 33.9 cm for the circle (or 56.1 cm for the triangle)

<Alternate set up #2>

Ex4. A wire 90cm long is to be cut into 2 pieces. One piece is to be formed into a circle. The other is to be bent into an equilateral triangle. Where should the wire be cut so that the combined area of the two shapes is a minimum?





$A = \frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right)$ $\quad h = \sqrt{x^2 - \left(\frac{1}{2}x\right)^2}$

$= \frac{\sqrt{3}}{4}x^2$ $\quad = \sqrt{\frac{3}{4}x^2}$

$= \frac{\sqrt{3}}{2}x$

Constraint

$$90 = 2\pi y + 3x$$

$$\frac{90-3x}{2\pi} = y$$

$$A = \pi y^2 + \frac{\sqrt{3}}{4}x^2$$

$$A(x) = \pi\left(\frac{90-3x}{2\pi}\right)^2 + \frac{\sqrt{3}}{4}x^2$$

$$= \frac{(90-3x)^2}{4\pi} + \frac{\sqrt{3}}{4}x^2$$

$$A'(x) = \frac{1}{2\pi}(90-3x)(-3) + \frac{\sqrt{3}}{2}x$$

$$0 = -\frac{3}{2\pi}(90-3x) + \frac{\sqrt{3}}{2}x$$

$$0 = \frac{-270}{2\pi} + \frac{9x}{2\pi} + \frac{\sqrt{3}}{2}x$$

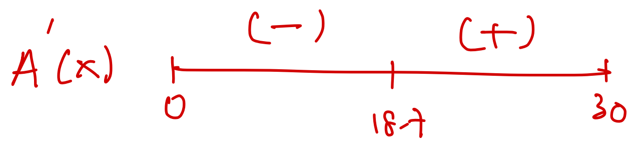
$$\frac{270}{2\pi} = x\left(\frac{9}{2\pi} + \frac{\sqrt{3}}{2}\right)$$

$$\frac{270}{2\pi} = x\left(\frac{9+\sqrt{3}\pi}{2\pi}\right)$$

$$\frac{270}{9+\sqrt{3}\pi} = x$$

$$x = 18.7$$

$$0 \le x \le 30$$



min

$\therefore$ $18.7 \times 3 = 56.1$ cm for the triangle

(33.9 cm for the circle)

4.2 (a) PRACTICE: OPTIMIZING AREA

1. Find the dimensions of the largest rectangle, by perimeter, which can be inscribed in a circle, where the area of the circle is fixed at 100π cm^2.

2. Find the dimensions of the rectangle of maximum perimeter that can be inscribed in a circle of radius 4 cm.

3. Find the dimensions of the rectangle of maximum area that can be inscribed in an isosceles triangle with base 40 *cm* and height 30 *cm*.

S

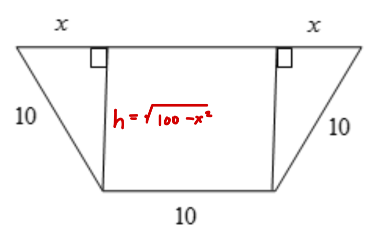
4. A piece of paper for a poster has an area of 1m^2. The margins at the top and bottom are 8 cm and at the sides are 6 cm. What are the dimensions of the sheet of paper which will maximize the printed area of the page?

5. The area of a rectangle is 64 cm^2. Find the dimensions of the rectangle of minimum perimeter. What is the minimum perimeter?

6. A piece of wire 100 cm long is divided into two pieces. One piece is used to form a circle and the other a square. Find the lengths of wire cut so that the combined area of circle and square is a minimum.

7. In above question, into what lengths should the wire be divided to give a combined area as large as possible?

8. An eaves trough has a cross section that forms an isosceles trapezoid. If the two legs and the short base of the trapezoid are each 10 cm, find the distance across the top of the trapezoid (x) that will maximize the area of the trapezoid and thus the carrying capacity of the eaves trough.



ANSWERS

1.	$10\sqrt{2}\ cm \times 10\sqrt{2}\ cm$
2.	$4\sqrt{2}\ cm$ by $4\sqrt{2}\ cm$
3.	15 cm by 20 cm
4.	$\frac{2\sqrt{3}}{3}\ m$ by $\frac{\sqrt{3}}{2}\ m$
5.	8 cm by 8 cm, 32 cm
6.	43.99cm and 56.01cm
7.	one piece of wire 100 cm long forming a circle
8.	If the base is 20cm this will maximize the area

#8

Constraint : $0 < x < 10$

$$A(x) = \frac{1}{2}(10 + (10 + 2x))\sqrt{100 - x^2}$$
$$= \frac{1}{2}(20 + 2x)\sqrt{100 - x^2}$$
$$= (10 + x)(100 - x^2)^{\frac{1}{2}}$$
$$A'(x) = (1)(100 - x^2)^{\frac{1}{2}} + \frac{1}{2}(100 - x^2)^{-\frac{1}{2}}(-2x)(10 + x)$$
$$0 = \sqrt{100 - x^2} - \frac{x(10 + x)}{\sqrt{100 - x^2}}$$
$$0 = (100 - x^2) - x(10 + x)$$
$$0 = 100 - x^2 - 10x - x^2$$
$$0 = -2x^2 - 10x + 100$$
$$0 = -2(x^2 + 5x - 50)$$
$$0 = -2(x + 10)(x - 5)$$
$$x = \{-10, 5\}$$

-10 → inadmissible

$A'(x)$: 0 (+) 5 (−) 10

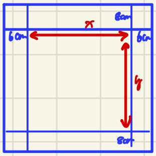
max $x = 5$

top of trapezoid
$= 2x + 10$
$= 2(5) + 10$
$= 20$

∴ the distance across the top of the trapezoid is 20 cm

$$1\,m^2 = (100\,cm \times 100\,cm) = 10000\,cm^2$$

4. A piece of paper for a poster has an area of 1m². The margins at the top and bottom are 8 cm and at the sides are 6 cm. What are the dimensions of the sheet of paper which will maximize the printed area of the page?



$$(x+12)(y+16) = 10\,000$$

$$y = \frac{10\,000}{x+12} - 16$$

$$0 < x < 613$$

Aside :

$y = 0$,

$$(x+12)(0+16) = 10000$$

$$x = \frac{10000}{16} - 12 = 613$$

$$A = xy$$

$$A(x) = x\left[\frac{10000}{x+12} - 16\right]$$

$$A(x) = \frac{10000x}{x+12} - 16x$$

$$A'(x) = \frac{10000(x+12) - (1)(10000x)}{(x+12)^2} - 16$$

$$0 = \frac{120000}{(x+12)^2} - 16$$

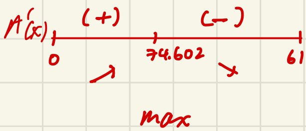
$$16(x+12)^2 = 120000$$

$$(x+12)^2 = 7500$$

$$x+12 = \pm 50\sqrt{3}$$

$$x = -12 \pm 50\sqrt{3}, \quad x > 0$$

$$\therefore x = -12 + 50\sqrt{3} = 74.602$$



$\therefore$ Dimension of the outer sheet:

$$x + 12$$
$$74.6 + 12$$
$$= 86.602\ cm$$

$$y = \frac{10000}{74.6+12} - 16 = \frac{10000}{86.6} - 16 = 99.47$$

$$y + 16 = 99.47 + 16 = 115.47$$

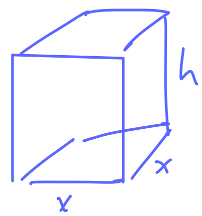
$\therefore$ 86.6 cm $\times$ 115.47 cm

Correction on Moshtagh's Solution key

$$x = \frac{\sqrt{3}}{2}, \quad y = \frac{2\sqrt{3}}{\cancel{2}\,3}$$

Warm Up

If 2700 cm^2 of material is available to make a box with a square base and top, find the largest possible volume of the box.



$V = x^2 h$

$V(x) = x^2 \left(\frac{2700 - 2x^2}{4x}\right)$

$V(x) = \frac{2700x}{4} - \frac{x^3}{2}$

$V'(x) = \frac{2700}{4} - \frac{3x^2}{2}$

$0 = \frac{2700}{4} - \frac{3x^2}{2}$

$x^2 = \frac{5400}{12}$

$x = \pm\sqrt{\frac{5400}{12}}$, $0 < x < 36.74$

$= \pm\sqrt{450}$

$\doteq 21.21$

Constraint:

$2700 = 4xh + 2x^2$

$\frac{2700 - 2x^2}{4x} = h$

$0 < x < \sqrt{\frac{2700}{2}}$

$0 < x < 36.74$

$V'(x)$: $(+)$ on $(0, \sqrt{450})$, $(-)$ on $(\sqrt{450}, 36.74)$

0 $\quad \sqrt{450}$ (21.21) $\quad$ 36.74

$V(21.21)$

$= \frac{2700(21.21)}{4} - \frac{(21.21)^3}{2}$

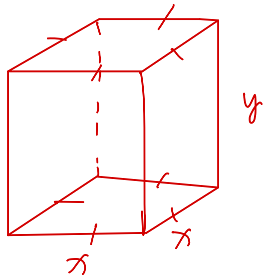
$\doteq 9545.94 \text{ cm}^3$

$\therefore$ max volume is 9545.9 cm^3

SA ⇒ Constraint

Warm Up

If 2700 cm² of material is available to make a box with a square base and top, find the largest possible volume of the box.



$V = x^2 y$

Constraint

SA: $2x^2 + 4xy = 2700$

$y = \dfrac{2700 - 2x^2}{4x}$

$V(x) = x^2\left(\dfrac{2700 - 2x^2}{4x}\right)$

$= \dfrac{2700x}{4} - \dfrac{1}{2}x^3$

$0 < x < \sqrt{\dfrac{2700}{2}}$

$0 < x < 15\sqrt{6}$

$V'(x) = \dfrac{2700}{4} - \dfrac{3}{2}x^2$

$0 = \dfrac{2700}{4} - \dfrac{3}{2}x^2$

$\dfrac{3}{2}x^2 = \dfrac{2700}{4}$

$x^2 = 450$

$= \sqrt{450}$

$= 15\sqrt{2}$

$V'(x)$: 0 $+$ $15\sqrt{2}$ $-$ $15\sqrt{6}$

max

$V(15\sqrt{2}) = \dfrac{2700(15\sqrt{2})}{4} - \dfrac{1}{2}(15\sqrt{2})^3$

$= 10125\sqrt{2} - 30\sqrt{2}$

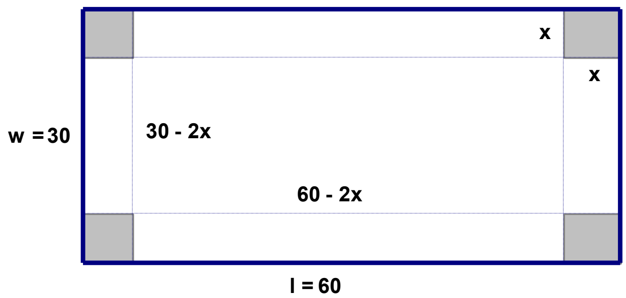
$= 10095\sqrt{2}$

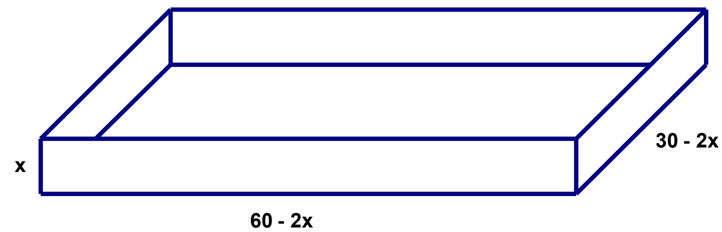
$\therefore$ max volume is $10095\sqrt{2}$ cm^3

4.2 (b) Optimization – Volume

A common application in industry $\rightarrow$ maximize space inside a package, but minimize surface area and cost of material to make it.

<u>Ex1</u>: A piece of sheet metal 60 cm by 30 cm is to be used to make a rectangular box with an open top. Find the dimensions that will give the box with the largest volume.





Constraint

$0 < x < 15$

$$V = (30-2x)(60-2x)(x)$$

$$V(x) = 4x^3 - 180x^2 + 1800x$$

$$V'(x) = 12x^2 - 360x + 1800$$

$$0 = 12(x^2 - 30x + 150)$$

$$x = \frac{30 \pm \sqrt{900 - 4(1)(150)}}{2(1)}$$

$$= \frac{30 \pm \sqrt{300}}{2}$$

$$\doteq \{6.33, 23.66\}$$

$\hookrightarrow$ inadmissible

$V'(x)$: 0 (+) 6.33 (−) 15

max

$x = 6.33$

length $= 60 - 2(6.33) = 47.34$

width $= 30 - 2(6.33) = 17.34$

height $= 6.33$

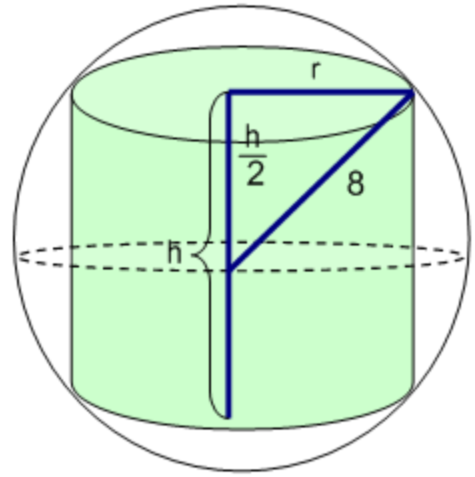
$\therefore$ the box with dimensions

47.3 cm × 17.3 cm × 6.3 cm

will maximize the volume

Ex2: A cylinder is inscribed inside a sphere of radius 8 cm so that its top and bottom touch the sphere along the complete circular edge. What are the dimensions of the cylinder with maximum volume, and what is that volume?

let r be the radius of the cylinder (cm)
let h " " height " " " (cm)



Constraint:

$r^2 + \left(\frac{h}{2}\right)^2 = 64$

$r^2 = 64 - \frac{h^2}{4}$

$0 < h < 16$

$V = \pi r^2 h$

$V(h) = \pi\left(64 - \frac{h^2}{4}\right) h$

$= 64\pi h - \frac{\pi}{4} h^3$

$V'(h) = 64\pi - \frac{3\pi}{4} h^2$

$0 = 64\pi - \frac{3\pi}{4} h^2$

$\frac{256}{3} = h^2 \quad , \quad 0 < h < 16$

$\therefore h \doteq \{9.23\}$

$r = \sqrt{64 - \frac{(9.23)^2}{4}}$

$\doteq 6.53$

$V'(h)$	0	(+)	9.23	(−)	16

max

$V(9.23) = 1238.21$

The cylinder will have height 9.2 cm and radius 6.5 cm which will max volume at 1238.2 cm^3

Alternate Set up

Ex2: A cylinder is inscribed inside a sphere of radius 8 cm so that its top and bottom touch the sphere along the complete circular edge. What are the dimensions of the cylinder with maximum volume, and what is that volume?

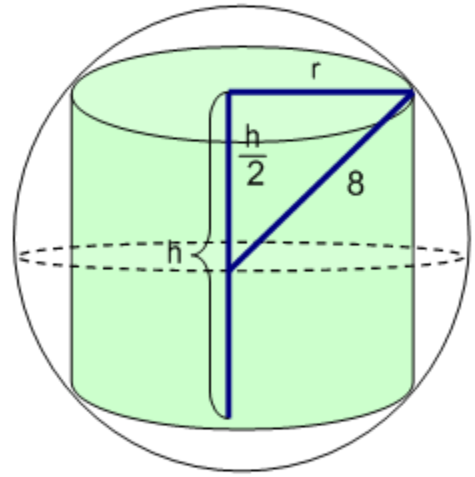
$V = \pi r^2 h$

Constraint:

$\left(\frac{h}{2}\right)^2 + r^2 = 64$

$r^2 = 64 - \frac{h^2}{4}$

$0 < h < 16$



$V(h) = \pi\left(64 - \frac{h^2}{4}\right)h$

$= \pi\left(64h - \frac{h^3}{4}\right)$

$V'(h) = \pi\left(64 - \frac{3}{4}h^2\right)$

$0 = 64 - \frac{3}{4}h^2$

$\frac{256}{3} = h^2$

$h = \left\{\frac{16}{\sqrt{3}}\right\}$

$V'(h)$: 0 (+) $\frac{16}{\sqrt{3}}$ (−) 16; ↗ ↘ max

$h = \frac{16}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{16\sqrt{3}}{3}$

$r^2 = 64 - \frac{1}{4}\left(\frac{16}{\sqrt{3}}\right)^2$

$r = \sqrt{64 - \frac{64}{3}}$

$\doteq \sqrt{\frac{128}{3}}$

$= \frac{8\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{8\sqrt{6}}{3}$

$V\left(\frac{16}{\sqrt{3}}\right) = \pi\left(64 - \frac{1}{4}\left(\frac{16}{\sqrt{3}}\right)^2\right)h$

$= \pi\left(64 - \frac{64}{3}\right)\left(\frac{16}{\sqrt{3}}\right)$

$= \pi\left(\frac{128}{3}\right) \cdot \frac{16\sqrt{3}}{3}$

$= \frac{2048\sqrt{3}\pi \text{ cm}^2}{9}$

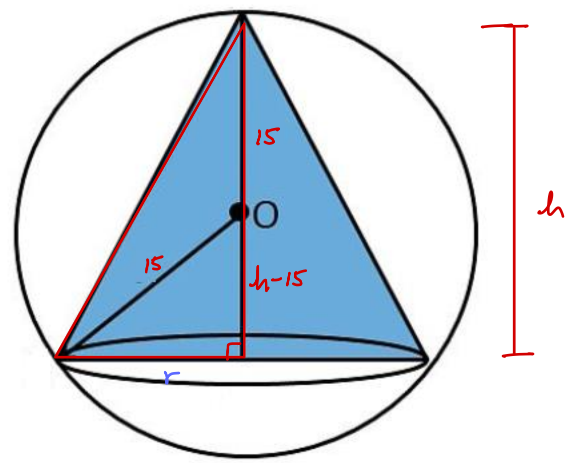
$\therefore$ The dimensions of the cylinder: $h = \frac{16\sqrt{3}}{3}$ cm

$r = \frac{8\sqrt{6}}{3}$ cm

and volume is $\frac{2048\sqrt{3}}{9}\pi$ cm^3

EXACT Solutions

Ex3: Determine the dimensions of the largest cone which can be inscribed in a sphere, where the diameter of the sphere is fixed at 30 cm.



let r be the radius of the cone (cm)
let h " " height of the cone (cm)

$V = \frac{1}{3}\pi r^2 h$

$V(h) = \frac{1}{3}\pi\left[225 - (h-15)^2\right]h$

$= \frac{\pi}{3}\left[\cancel{225h} - h^3 + 30h^2 - \cancel{225h}\right]$

Constraint:

$r^2 + (h-15)^2 = 15^2$

$r^2 = 225 - (h-15)^2$

$0 < h < 30$

$V'(h) = \frac{\pi}{3}\left[-3h^2 + 60h\right]$

$0 = \frac{\pi}{3}\left[-3h^2 + 60h\right]$

$0 = -3h^2 + 60h$

$0 = -3h(h-20)$

$h = \{0, 20\}$

$V(h)$: 0 (+) 20 (−) 30

max: $x = 20$

$V(10) = \frac{\pi}{3}\left[-(20)^3 + 30(20)^2\right]$

$= \frac{4000\pi}{3}$

$r = \sqrt{225 - (20-15)^2}$

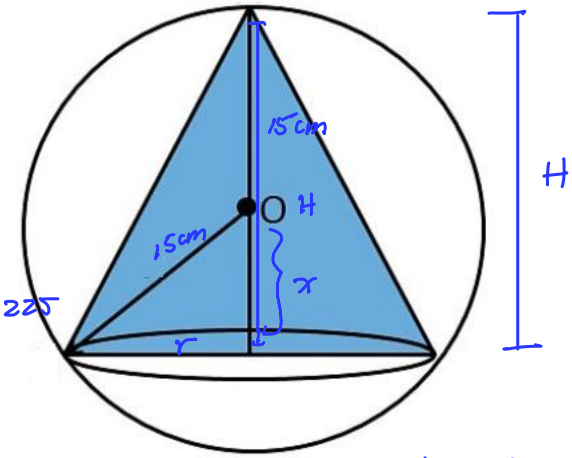
$= \sqrt{200}$

$= 10\sqrt{2}$

$\doteq 14.14$

∴ The cone will have a height of 20 cm and radius of 14.1 cm to maximize its volume.

Ex3: Determine the dimensions of the largest cone which can be inscribed in a sphere, where the diameter of the sphere is fixed at 30 cm.



$V = \frac{1}{3}\pi r^2 H$

Constraint

$x^2 + r^2 = 15^2$

$r^2 = 225 - x^2$

$= 225 - (H-15)^2$

$= 225 - H^2 + 30H - 225$

$= -H^2 + 30H$

$H = 15 + x$

$H - 15 = x$

$V(H) = \frac{1}{3}\pi(-H^2 + 30H)(H)$

$= -\frac{1}{3}\pi H^3 + 10\pi H^2$

$0 < H < 30$

$V'(H) = -\pi H^2 + 20\pi H$

$0 = -\pi H(H - 20)$

$H = \{0, 20\}$

$V'(H)$: 0 — (+) — 20 — (−) — 30

max

$r = \sqrt{-(20)^2 + 30(20)}$

$= \sqrt{200}$

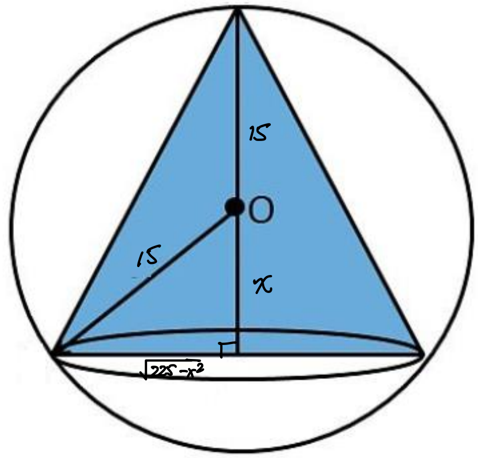
$= 10\sqrt{2}$

$\therefore$ Height $= 20$ cm

radius $= 10\sqrt{2}$ cm

Alternate Solution

Ex3: Determine the dimensions of the largest cone which can be inscribed in a sphere, where the diameter of the sphere is fixed at 30 cm.



$V = \frac{1}{3}\pi r^2 h$

$V(x) = \frac{1}{3}\pi(225-x^2)(x+15)$

$= \frac{\pi}{3}(-x^3 - 15x^2 + 225x + 3375)$

Constraint:

$0 < x < 15$

$V'(x) = \frac{\pi}{3}(-3x^2 - 30x + 225)$

$0 = 3x^2 + 30x - 225$

$0 = x^2 + 10x - 75$

$0 = (x+15)(x-5)$

$x = \{-15, 5\}$

inadmissible

$V'(t)$: 0, 5, 15; (+) on (0, 5) ↗, (−) on (5, 15) ↘

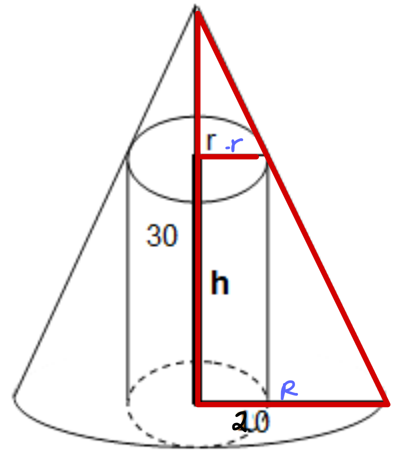
max

$h = 15 + x$
$= 20$

$r = \sqrt{225 - 5^2}$
$= \sqrt{200}$
$= 10\sqrt{2}$

Ex4: Find the dimensions of the cylinder of maximum volume that can be inscribed in a cone with a diameter of 40 *cm* and a height of 30 *cm*.

let h be the height of the cylinder (cm)
let r " the radius of the cylinder (cm)



$V = \pi r^2 h$

$V(r) = \pi r^2 (30 - \frac{3}{2} r)$

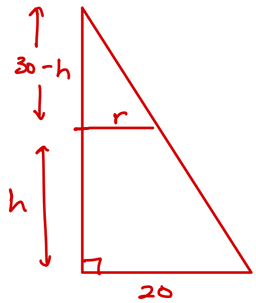
$= \pi (30r^2 - \frac{3}{2} r^3)$

$V'(r) = \pi (60r - \frac{9}{2} r^2)$

$0 = \pi (60r - \frac{9}{2} r^2)$

$r = \{0, \frac{40}{3}\}$

Constraints:



$\frac{30-h}{30} = \frac{r}{20}$

$20(30-h) = 30r$

$600 - 20h = 30r$

$h = \frac{600 - 30r}{20}$

$h = 30 - \frac{3}{2} r$

$0 < r < 20 \qquad 0 < h < 30$

$V'(r)$: $(+)$ on $(0, \frac{40}{3})$, $(-)$ on $(\frac{40}{3}, 20)$

max $x = \frac{40}{3}$

$V(\frac{40}{3}) = \pi (30(\frac{40}{3})^2 - \frac{3}{2}(\frac{40}{3})^3)$

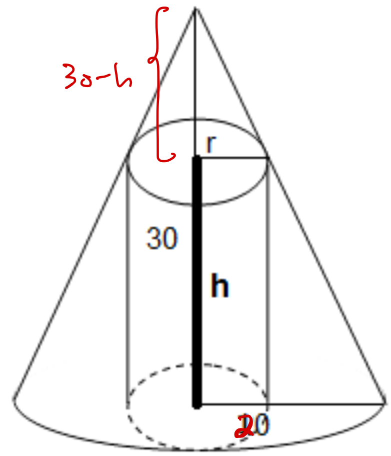
$= 5585.05 \text{ cm}^3$

$h = 30 - \frac{3}{2}(\frac{40}{3})$

$= 10$

$\therefore$ The cylinder of dimensions height is 10 cm and radius $\frac{40}{3}$ cm will max. the incribed cylinder's volume of 5585.1 cm^3

Ex4: Find the dimensions of the cylinder of maximum volume that can be inscribed in a cone with a diameter of 40 *cm* and a height of 30 *cm*.



$$\frac{30-h}{30} = \frac{r}{20}$$

$$30 - h = \frac{3}{2}r$$

$$30 - \frac{3}{2}r = h$$

$$V = \pi r^2 h$$

$$V(r) = \pi r^2\left(30 - \frac{3}{2}r\right)$$

$$= \pi\left(30r^2 - \frac{3}{2}r^3\right)$$

$$V'(r) = \pi\left(60r - \frac{9}{2}r^2\right)$$

$$0 = r\left(60 - \frac{9}{2}r\right)$$

$$r = \left\{0, \frac{40}{3}\right\}$$

$V'(r)$: on $(0, \frac{40}{3})$: $(+)$ $\nearrow$; on $(\frac{40}{3}, 20)$: $(-)$ $\searrow$

max at $r = \frac{40}{3}$

$$h = 30 - \frac{3}{2}\left(\frac{40}{3}\right)$$

$$= 10$$

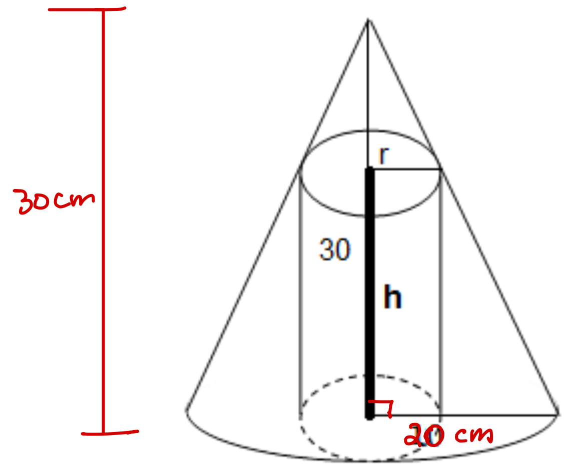
$\therefore$ Cylinder has a radius $= \frac{40}{3}$ cm

height 10cm

Note! The 'red' is where I found my mistake :) !

Ex4: Find the dimensions of the cylinder of maximum volume that can be inscribed in a cone with a diameter of 40 cm and a height of 30 cm.

$\hookrightarrow$ radius = 20cm



$$V = \pi r^2 h$$

$$V(h) = \pi\left[\frac{60-2h}{3}\right]^2 h$$

$$= \pi\frac{(3600-240h+4h^2)}{9} h$$

$$= 400\pi h - \frac{80}{3}\pi h^2 + \frac{4\pi}{9} h^3$$

$$V'(h) = 400\pi - \frac{160}{3}\pi h + \frac{4\pi}{3} h^2$$

$$0 = \frac{4\pi}{3} h^2 - \frac{160}{3}\pi h + 400\pi$$

$$0 = \frac{4}{3}\pi\left[h^2 - 40h + 300\right]$$

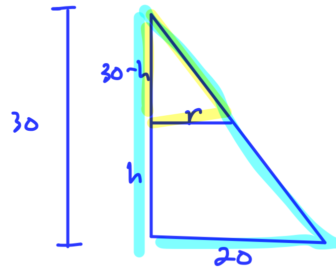
$$0 = \frac{4\pi}{3}(h-30)(h-10)$$

$h = \{10, 30\}$
$\hookrightarrow$ inadmissible

$V'(h)$: 0, (+), 10, (−), 30

max

Constraint



$$\frac{30-h}{30} = \frac{r}{20}$$

$$\frac{2\cancel{0}(30-h)}{3\cancel{0}} = r$$

$$\frac{60-2h}{3} = r$$

$0 < h < 30$

$$r = \frac{60-2(10)}{3}$$

$$= \frac{40}{3}$$

$\therefore$ The cylinder with max volume will have dimensions of radius $\frac{40}{3}$ cm and height of 10 cm

4.2(b) PRACTICE: OPTIMIZING VOLUME

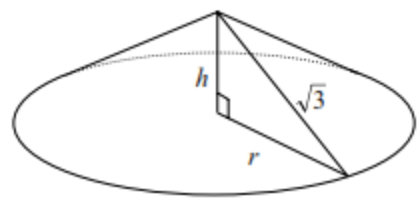
1. Find the dimensions of the cylinder of greatest volume which can be inscribed in a sphere, where the volume of the sphere is fixed at $\frac{108\pi}{3}\text{m}^3$.

2. A right circular cylinder is inscribed in a sphere with diameter 4cm. If the cylinder is open at both ends, find the largest possible surface area of the cylinder .

3. Find the height and radius of the cylinder of greatest volume that can be inscribed in a sphere of radius ***R*** units.

4. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

5. A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the **exact** values of radius, height, and volume of the cone of greatest volume.



ANSWERS

1.	$\mathbf{h = 2\sqrt{3}\,cm, r = \sqrt{6}\,cm}$
2.	$\mathbf{SA = 8\pi\,cm^2}$
3.	$\mathbf{h = \frac{2\sqrt{3}R}{3}, r = \frac{\sqrt{6}R}{3}}$
4.	$\mathbf{V = \frac{4}{27}\pi r^2 h}$
5.	$\mathbf{V = \frac{2}{3}\pi\ m^3}$

$V = \frac{1}{3}\pi r^2 h$

Constraint :

$r^2 + h^2 = 3$

$r^2 = 3 - h^2$

$V(h) = \frac{1}{3}\pi(3-h^2)h$

$= \pi h - \frac{\pi h^3}{3}$

$0 < h < \sqrt{3}$

$V'(h) = \pi - \pi h^2$

$0 = \pi(1-h^2)$

$0 = 1 - h^2$

$h = \pm 1 \quad , \quad h > 0$

$V'(h)$: 0 (+) 1 (−) $\sqrt{3}$

$r^2 = 3 - 1^2$

$r = \sqrt{2}$

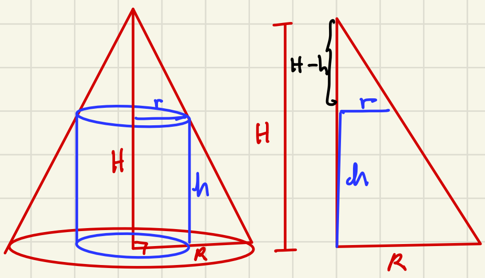
max $h = 1$

$V(1) = \pi(1) - \frac{\pi(1)^3}{3}$

$= \frac{2\pi}{3}$

$\therefore$ height 1m, radius $\sqrt{2}$ m and volume $\frac{2\pi}{3}$ m^3

4. A right circular cylinder is inscribed in a cone with height ~~h~~ H and base radius ~~r~~ R. Find the largest possible volume of such a cylinder.



$$\frac{H}{H-h} = \frac{R}{r}$$

$$Hr = R(H-h)$$
$$r = \frac{R(H-h)}{H}$$

Note! H and R ⇒ are "constants"

$$V = \pi r^2 h$$
$$= \pi\left[\frac{R(H-h)}{H}\right]^2 h$$
$$V(h) = \frac{\pi R^2 (H-h)^2 h}{H^2}$$
$$= \frac{\pi R^2}{H^2}(H-h)^2 h$$
$$V'(h) = \frac{\pi R^2}{H^2}\left[2(H-h)(-1)(h) + (1)(H-h)^2\right]$$
$$= \frac{\pi R^2}{H^2}\left[(H-h)\left[-2h + (H-h)\right]\right]$$
$$0 = (H-h)(-2h+H-h)$$
$$0 = (h-H)(3h-H)$$
$$\therefore h = \left\{H, \tfrac{1}{3}H\right\}$$

Constraint
$0 < h < H$

$$\boxed{\begin{aligned} \frac{H}{H-\frac{1}{3}H} &= \frac{R}{r} \\ \frac{H}{\frac{2}{3}H} &= \frac{R}{r} \\ \frac{3}{2} &= \frac{R}{r} \\ r &= \frac{2R}{3} \end{aligned}}$$

$V'(h)$: $(+)$ on $(0, \tfrac{1}{3}H)$ ↗, $(-)$ on $(\tfrac{1}{3}H, H)$ ↘; max at $\tfrac{1}{3}H$

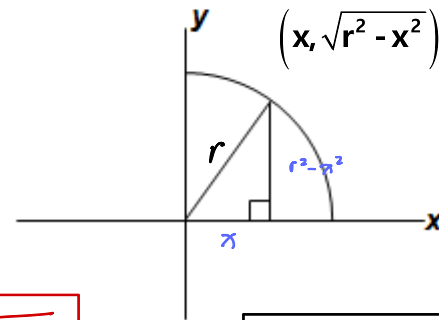
$$V\left(\tfrac{1}{3}H\right) = \pi r^2\left(\tfrac{1}{3}H\right)$$
$$= \frac{\pi r^2 H}{3}$$
$$= \frac{\pi H}{3}\left(\frac{2}{3}R\right)^2$$
$$= \frac{4\pi H R^2}{27}$$

$\therefore$ Max Volume: $\dfrac{4\pi H R^2}{27}$

Warm Up

1. Show that the right triangle of maximum area that can be inscribed in the quarter circle of $x^2 + y^2 = r^2$ is isosceles.

let x be the base of the triangle (units)
let $\sqrt{r^2-x^2}$ " " height of the triangle



$$A = \frac{1}{2} x \sqrt{r^2-x^2}$$

$$A'(x) = \left[\frac{1}{2}\right]\left[\sqrt{r^2-x^2}\right] + \left[\frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}(-2x)\right]\left[\frac{1}{2}x\right]$$

$$0 = \frac{1}{2}\sqrt{r^2-x^2} - \frac{x^2}{2\sqrt{r^2-x^2}} \Rightarrow \boxed{\begin{gathered}\frac{x^2}{2\sqrt{r^2-x^2}} = \frac{1}{2}\sqrt{r^2-x^2} \\ x^2 = r^2 - x^2\end{gathered}}$$

note! we can treat 'r' as a constant

$$0 = (r^2 - x^2) - x^2$$

$$0 = r^2 - 2x^2$$

$$2x^2 = r^2$$

$$x^2 = \frac{r^2}{2}$$

$$x = \pm\frac{r}{\sqrt{2}}, \quad 0 < x < r$$

Constraint:

$$0 < x < r$$

$$\begin{array}{ccccc} & (+) & & (-) & \\ 0 & & \frac{r}{\sqrt{2}} & & r \end{array}$$

max
$x = \frac{r}{\sqrt{2}}$

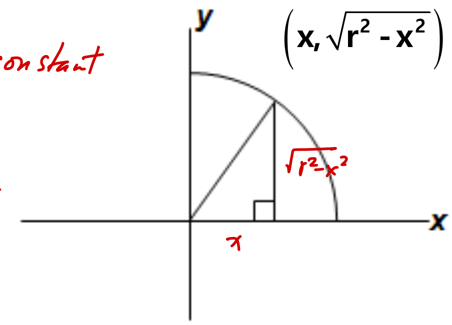
base $= \frac{r}{\sqrt{2}}$

$$\text{height} = \sqrt{r^2-x^2} = \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} = \sqrt{r^2 - \frac{r^2}{2}} = \sqrt{\frac{r^2}{2}} = \frac{r}{\sqrt{2}}$$

$\therefore$ the base and height are equal, the right triangle is isosceles

Warm Up

1. Show that the right triangle of maximum area that can be inscribed in the quarter circle of $x^2 + y^2 = r^2$ is isosceles.



$A = \frac{1}{2} \cdot x \cdot \sqrt{r^2 - x^2}$

$A'(x) = \left[\frac{1}{2}\right]\left[\sqrt{r^2-x^2}\right] + \left[\frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}(-\cancel{2}x)\right]\left[\frac{1}{\cancel{2}}x\right]$

$0 = \frac{\sqrt{r^2-x^2}}{2} - \frac{x^2}{2\sqrt{r^2-x^2}}$

$\frac{x^2}{\cancel{2}\sqrt{r^2-x^2}} = \frac{\sqrt{r^2-x^2}}{\cancel{2}}$

$x^2 = r^2 - x^2$

$2x^2 = r^2$, $0 < x < r$

$x = \sqrt{\frac{r^2}{2}}$

$= \frac{r}{\sqrt{2}}$

$A'(x)$: $(+)$ on 0 to $\frac{r}{\sqrt{2}}$, $(-)$ on $\frac{r}{\sqrt{2}}$ to r

max

assume r is a constant

Constraint:
$0 < x < r$

base : x

$= \frac{r}{\sqrt{2}}$

height $= \sqrt{r^2 - x^2}$

$= \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2}$

$= \sqrt{r^2 - \frac{r^2}{2}}$

$= \sqrt{\frac{r^2}{2}}$

$= \frac{r}{\sqrt{2}}$

$\because$ the base and height are equal

$\therefore$ it's an isosceles Δ.

Warm Up

1. Show that the right triangle of maximum area that can be inscribed in the quarter circle of $x^2 + y^2 = r^2$ is isosceles.

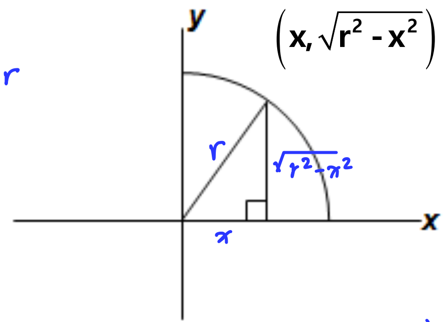
$A = \frac{1}{2} bh$

$A(x) = \frac{1}{2}(x)\sqrt{r^2 - x^2}$

$A = u'v + v'u$

Contraint

$0 < x < r$



$A'(x) = \left[\frac{1}{2}\right]\left[(r^2 - x^2)^{\frac{1}{2}}\right] + \left[\frac{1}{2}(r^2 - x^2)^{-\frac{1}{2}} \cdot (-\cancel{2}x)\right]\left[\frac{1}{\cancel{2}} x\right]$

$= \frac{1}{2}(r^2 - x^2)^{-\frac{1}{2}}\left[(r^2 - x^2) - x^2\right]$

$0 = \dfrac{r^2 - 2x^2}{2\sqrt{r^2 - x^2}}$

$0 = r^2 - 2x^2$

$x^2 = \dfrac{r^2}{2}$

$x = \dfrac{r}{\sqrt{2}}$, $0 < x < r$

Note! treat 'r' as a constant

$A'(x)$ with $(+)$ on $\left(0, \frac{r}{\sqrt{2}}\right)$ and $(-)$ on $\left(\frac{r}{\sqrt{2}}, r\right)$; $\frac{r}{\sqrt{2}}$ max

$h = \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2}$

$= \sqrt{r^2 - \frac{r^2}{2}}$

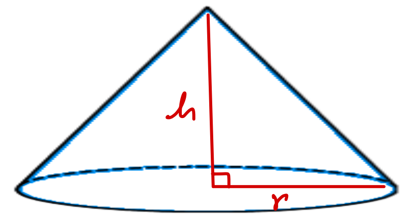
$= \sqrt{\frac{r^2}{2}}$

$= \frac{r}{\sqrt{2}}$

$\because$ base = height $= \frac{r}{\sqrt{2}}$

$\therefore$ it is an isosceles Δ

2. The radius of the base r cm and the perpendicular height h cm of a right circular cone together equal 12 cm. Find the ratio $r : h$ when the volume of the cone is a maximum .



$V = \frac{1}{3}\pi r^2 h$

$V(r) = \frac{1}{3}\pi r^2(12-r)$

$= 4\pi r^2 - \frac{\pi}{3} r^3$

$V'(r) = 8\pi r - \pi r^2$

$0 = \pi r(8-r)$

$r = \{0, 8\}$

$V'(r)$: 0 (+) 8 − 12

max

Constraint:

$h + r = 12$

$h = 12 - r$

$0 < r < 12$

$h = 12 - (8)$

$= 4$

$\therefore$ $r : h$

$8 : 4$

$2 : 1$

4.3 Optimization – Distance-Time

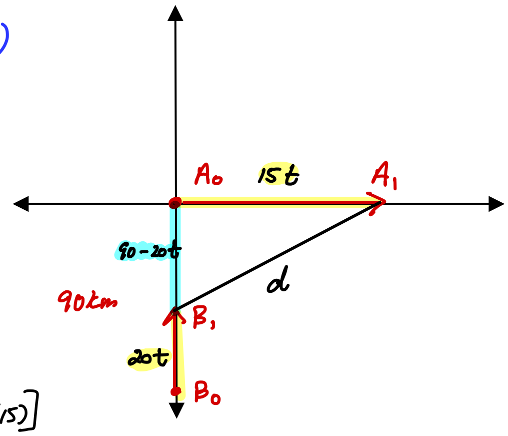
Ex1: At midnight, ship B was 90 km south of ship A. Ship A sailed east at 15 km/h and ship B headed north at 20 km/h.

a) At what time are they closest? ⇒ Find the time when distance is minimized.

b) What is the closest distance for the ships?

$D = ST$

a) let t be the elapsed time (hr)
let d " " distance between them (km)



Constraint

$0 < t < \frac{90}{20}$

$0 < t < 4.5$

$d = \sqrt{(90-20t)^2 + (15t)^2}$

$d'(t) = \frac{1}{2}\left[(90-20t)^2 + (15t)^2\right]^{-\frac{1}{2}}\left[2(90-20t)(-20) + 2(15t)(15)\right]$

$0 = 2(90-20t)(-20) + 2(15t)(15)$

$0 = 2(90-20t)(-20) + 450t$

$0 = (90-20t)(-20) + 225t$

$0 = -1800 + 400t + 225t$

$t = \frac{1800}{625}$

$= 2.88\ h$

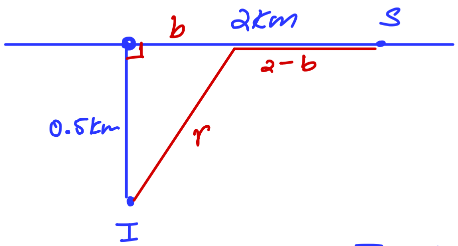
$= 2h\ 52min\ 48sec$ ∴ They were closest at 2:52:48

b) $d(2.88) \doteq \sqrt{[90-20(2.88)]^2 + [15(2.88)]^2}$

$\doteq 54\ km$

Ex2: Jacky lives on an island 500 m from shore. His school is located 2 km down shore from the point on shore directly across from the island. Jack gets to school by rowing and biking. He can row in the water at 2 km/hr and rides his bike on land at 4 km/hr. Where should he land so that he gets to school in the shortest amount of time possible?

↳ Mimize Time



let r be the distance rowed (km)
let $2-b$ " " " biked (km)

secondary

Total time $= \frac{r}{2} + \frac{2-b}{4}$ $r = \sqrt{0.5^2 + b^2}$

$T = \frac{D}{S}$

$T(b) = \frac{\sqrt{0.25+b^2}}{2} + \frac{2-b}{4}$ $0 < b < 2$

$T'(b) = \frac{1}{4}(0.25+b^2)^{-\frac{1}{2}}(2b) - \frac{1}{4}$

$0 = \frac{2b}{4\sqrt{0.25+b^2}} - \frac{1}{4}$

$\frac{1}{4} = \frac{2b}{4\sqrt{0.25+b^2}}$

$\sqrt{0.25+b^2} = 2b$, $2b>0$, $b>0$

$0.25+b^2 = 4b^2$

$\frac{1}{4} = 3b^2$

$\sqrt{\frac{1}{12}} = b$

$b = \frac{1}{\sqrt{12}}$

$T'(b)$: 0 (−) $\frac{1}{\sqrt{12}}$ (+) 2

$\doteq 0.288$

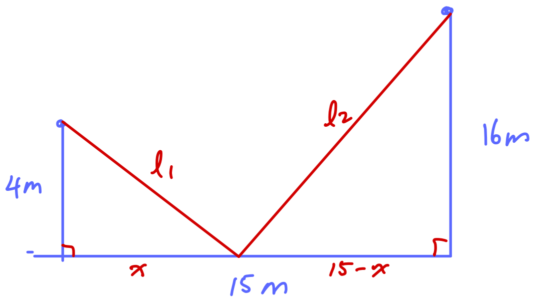
$2 - 0.288$

$\doteq 1.71$

∴ He should arrive 1.71m away from the school if he is to minimize the time to get to school.

Ex3 : Two vertical poles, one 4 m high and the other 16 m high, stand 15 meter apart on a flat field. A worker wants to support both poles by running rope from the ground to the top of each post. If the worker wants to stake both ropes in the ground at the same point, where should the stake be placed to use the least amount of rope?

minimize length



$$l_1 = \sqrt{x^2+4^2} = \sqrt{x^2+16}$$

$$l_2 = \sqrt{(15-x)^2+16^2} = \sqrt{x^2-30x+481}$$

Constraint: $0 < x < 15$

$$L(x) = \sqrt{x^2+16} + \sqrt{x^2-30x+481}$$

$$L'(x) = \frac{1}{2}(x^2+16)^{-\frac{1}{2}}(2x) + \frac{1}{2}(x^2-30x+481)^{-\frac{1}{2}}(2x-30)$$

$$0 = \frac{x}{\sqrt{x^2+16}} + \frac{x-15}{\sqrt{x^2-30x+481}}$$

$$\frac{-x}{\sqrt{x^2+16}} = \frac{x-15}{\sqrt{x^2-30x+481}}$$

$$\frac{x^2}{x^2+16} = \frac{(x-15)^2}{x^2-30x+481}$$

$$x^2(x^2-30x+481) = (x^2+16)(x^2-30x+225)$$

$$\cancel{x^4} - \cancel{30x^3} + 481x^2 = \cancel{x^4} - \cancel{30x^3} + 225x^2 + 16x^2 - 480x + 3600$$

$$240x^2 + 480x - 3600 = 0$$

$$x^2 + 2x - 15 = 0$$

$$(x+5)(x-3) = 0, \quad 0 < x < 15$$

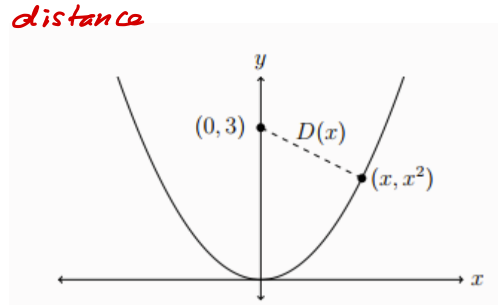
$$\therefore x = \{3\}$$

$L'(x)$: 0 $(-)$ 3 $(+)$ 15; min at 3

$\therefore$ the rope should be anchored 3m from the 4m-pole.

Ex4: Find the point(s) on the parabola y=x² nearest to the point (0,3).

↳ minimize distance



$$D(x) = \sqrt{(x-0)^2 + (x^2-3)^2}$$

$$D(x) = \sqrt{x^2 + x^4 - 6x^2 + 9}$$

$$= \sqrt{x^4 - 5x^2 + 9}$$

$$D'(x) = \frac{1}{2}(x^4 - 5x^2 + 9)^{-\frac{1}{2}}(4x^3 - 10x)$$

$$0 = \frac{2x^3 - 5x}{\sqrt{x^4 - 5x^2 + 9}}$$

Constraint: $x \in \mathbb{R}$

$$0 = 2x^3 - 5x$$

$$0 = x(2x^2 - 5)$$

$$x = \left\{0, \pm\sqrt{\frac{5}{2}}\right\}$$

$$\doteq \left\{0, \pm\frac{\sqrt{10}}{2}\right\}$$

$D'(x)$: $(-)$ | $-\frac{\sqrt{10}}{2}$ | $(+)$ | 0 | $(-)$ | $\frac{\sqrt{10}}{2}$ | $(+)$

min at $-\frac{\sqrt{10}}{2}$, min at $\frac{\sqrt{10}}{2}$

$$f\left(-\frac{\sqrt{10}}{2}\right) = \left(-\frac{\sqrt{10}}{2}\right)^2 = \frac{10}{4} = \frac{5}{2}$$

$$f\left(\frac{\sqrt{10}}{2}\right) = \left(\frac{\sqrt{10}}{2}\right)^2 = \frac{10}{4} = \frac{5}{2}$$

$\therefore$ @ the points:

$$\left(\pm\frac{\sqrt{10}}{2}, \frac{5}{2}\right)$$

they were closest to the point (3,0)

4.3 PRACTICE-OPTIMIZING DISTANCE/TIME

1. A sailor in a boat 8 km off a straight coastline wants to reach point on shore 10 km from the point directly opposite her present position in the shortest possible time. Towards what point on the shore should she steer and how long does it take her to reach her destination if

   a) She can row at 4 km/h and run at 6 km/h
   b) She can row at4km/h and Walk at5km/h

2. A pair of scuba divers wishes to dive on a wreck that lies 0.4 km off the shore from a point 0.8 km from their present position. If they can walk carrying their gear at 5km/h and swim at 3km/h, what course should they follow to reach the wreck in the minimum time?

3. A fisherman wishes to cross a stream 200 m wide in his row boat and then to reach a place on the far bank of the river 500 m away from the point directly opposite his launching point If he can row at 3 m/s and walk 5 m/s find his Shortest time to reach the desired place.

4. Two isolated farms are situated 12 km apart on 'a straight country road that runs parallel to the main highway 20 km away. The power company decides to run a wire from the highway to a junction box, and from there, wires of equal length to the two houses. Where should the junction box be placed to minimize the length of wire needed?

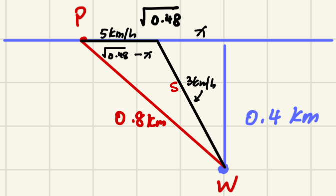
5. Determine the point(s) on the parabola $f(x) = 6x - x^2$ that is/are closest to the point(0, 3).

ANSWERS

1.a) 2.84 km, 3.15 hr b) row all the way	2. walk 0.5 km swim the rest	3. 155.3 sec or 2.56 min	4. 16.5 km	**5.** (0.53,2.899)

2. A pair of scuba divers wishes to dive on a wreck that lies 0.4 km off the shore from a point 0.8 km from their present position. If they can walk carrying their gear at 5km/h and swim at 3km/h, what course should they follow to reach the wreck in the minimum time?

Time as a function of distance



Constraint:

$x^2 + 0.4^2 = s^2$

$\sqrt{x^2 + 0.16} = s$

$S = \frac{D}{T}$

$T = \frac{D}{S}$

$$\text{Total time} = \frac{\sqrt{0.48} - x}{5} + \frac{s}{3}$$

$$T(x) = \frac{\sqrt{0.48} - x}{5} + \frac{\sqrt{x^2 + 0.16}}{3}$$

$0 < x < \sqrt{0.48}$

$$T'(x) = -\frac{1}{5} + \frac{1}{6}(x^2 + 0.16)^{-\frac{1}{2}}(2x)$$

$$0 = -\frac{1}{5} + \frac{x}{3\sqrt{x^2 + 0.16}}$$

$$\frac{1}{5} = \frac{x}{3\sqrt{x^2 + 0.16}}$$

$$\sqrt{x^2 + 0.16} = \frac{5x}{3}$$

$\frac{5x}{3} > 0$

$x > 0$

$$x^2 + 0.16 = \frac{25}{9}x^2$$

$$0.16 = \frac{16}{9}x^2$$

$$0.09 = x^2$$

$$x = 0.3$$

$S = \sqrt{(0.3)^2 + 0.16}$

$=$

∴ Walk 0.3 km and Swim 0.5 km

$T'(x)$

	(−)		(+)	
0		0.3		$\sqrt{0.48}$

min

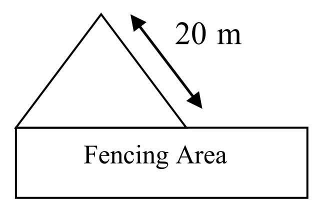
Mid-Review: Optimization Problems

1. You are standing at the edge of a slow-moving river which is 1 km wide and wish to return to your campground on the opposite side of the river. You can swim at 2 km/h and walk at 3 km/h. You must first swim across the river to any point on the opposite bank. From there walk to the campground, which is 1 km from the point directly across the river from where you start your swim. What route will take the least amount of time?

2. From point $(w, f(w)), w > 0$ a line tangent to the graph of $f(x) = \sqrt{2 - x}$ is drawn. If the tangent line cuts the x-axis at A and the y-axis at B, find the value(s) of w that makes the area of the triangle OAB a minimum?

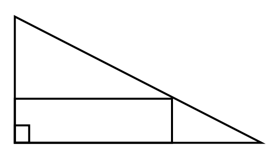
3. Two positive numbers have a product of 9. How should they be chosen so that the sum of their squares will be a minimum? What is that minimum?

4. Lucas has 60 m of fencing that he plans to use to enclose a rectangular area adjacent to an equilateral triangular shape garden which is 20 meter long (see the figure). Find the dimensions that will maximize the area.



5. What are the dimensions of the rectangle of largest area that has a diagonal of length 60 m?

6. What are the dimensions of the largest rectangle that can be inscribed in a right triangle with base 60 m and height 40 m?



7. Find the dimensions of the isosceles triangle of greatest area having a perimeter of 60 m.

Part 2: Business Applications

Terminology:

- The demand functions, or price function, is $p(x)$, where x is the number of units of a product or service that can be sold at a particular price, p.

- The revenue function is $R(x) = x \cdot p(x)$, where x is the number of units of a product or service sold at a price per unit of $p(x)$.

- The cost function, $C(x)$, is the total cost of producing x units of a product or service.

- The profit function, $P(x)$, is the profit from the sale of x units of a product or service. The profit function is the difference between the revenue function and the cost function: $P(x) = R(x) - C(x)$

Economists use the word marginal to indicate the derivative of a business function.

- $C'(x)$ is the marginal cost function and refers to the instantaneous rate of change of total cost with respect to the number of items produced.

- $R'(x)$ is the marginal revenue function and refers to the instantaneous rate of change of total revenue with respect to the number of items sold.

- $P'(x)$ is the marginal profit function and refers to the instantaneous rate of change of total profit with respect to the number of items sold.

4.4 Optimization – Business

Recall the definitions from earlier.

Algebraic Expression	Definition
$C(x)$ Cost function →	total cost to produce x units
$p(x)$ price function →	price per unit when x units were sold $\Rightarrow$ 'demand' function
$R(x)$ Revenue function →	total income from selling x units $R(x) = x \cdot p(x)$
$P(x)$ Profit function →	total profit from selling x units $P(x) = R(x) - C(x)$
$C'(x)$ Marginal cost →	is the derivative of the cost function, the instantaneous rate of change of the total cost
$p'(x)$ Marginal demand →	is the derivative of the demand function, the instantaneous rate of change of the price
$R'(x)$ Marginal revenue →	is the derivative of the revenue function, the instantaneous rate of change of the total revenue
$P'(x)$ Marginal profit →	is the derivative of the profit function, the instantaneous rate of change of the total profit

Normally the price and cost functions are set after a great deal of market research.

demand function

Ex1: Sophia's Subs has determined the monthly price for their submarine sandwiches is, $p = \frac{30000 - x}{10000}$ $= \frac{-1}{10000}x + 3$

the cost of making x subs is $C(x) = 6000 + 0.8x$. What level of sales will maximize profits?

↳ how many subs must be sold to max. profit

Recall:

Revenue = #item × price

$R(x) = x \cdot p(x)$

$= x\left(-\frac{1}{10000}x + 3\right)$

$= -\frac{1}{10000}x^2 + 3x$

Profit = Revenue − Cost

$P(x) = \left(-\frac{1}{10000}x^2 + 3x\right) - (6000 + 0.8x)$

$= -\frac{1}{10000}x^2 + 2.2x - 6000$

Constraints.

$x > 0$

$P'(x) = -\frac{1}{5000}x + 2.2$

$0 = -\frac{1}{5000}x + 2.2$

$\frac{1}{5000}x = 2.2$

$x = 11\,000$

∴ to max profit, 11 000 subs will need to be sold.

$P'(x)$: 0 (+) 11 000 (−) ∞

max

Ex2: A commuter train carries 2000 passengers daily from a suburb into a large city. The cost to ride the train is $7.00 per person. Market research shows that 40 fewer people would ride the train for each $0.10 increase in the fare, and 40 more people would ride for every $0.10 decrease in the fare. If the capacity of the train is 2600 passengers, and carrying fewer than 1600 passengers means costs exceed revenue, what fare should the railway charge to earn the largest possible revenue?

let x be the # of 10¢ price increases

Constraint

$1600 < n < 2600$

$1600 < 2000 - 40x < 2600$

$-400 < -40x < 600$

$10 > x > -15$

$-15 < x < 10$

n

passengers · price

$R(x) = (2000 - 40x)(7.00 + 0.10x)$

$= -4x^2 - 80x + 14000$

$R'(x) = -8x - 80$

$0 = -8(x+10)$

$x = -10$

$R'(x)$: -15 (+) -10 (−) 10

max

$p(x) = 7.00 + 0.10(-10)$
$= 6.00$

∴ to max revenue, the price should be set to $6.00/passenger

Note!

$p(x) = 7.00 + 0.10x$

let n be the # of passengers

$n = 2000 - 40x$

$\therefore x = \dfrac{n - 2000}{-40}$

$x = -\frac{1}{40}n + 50$

$p(n) = 7.00 + 0.10(-\frac{1}{40}n + 50)$

demand function $\Rightarrow$ $p(n) = -\frac{1}{400}n + 12$

$n \Rightarrow$ # of items

$p(n) \Rightarrow$ price per unit

$\therefore p(n) = -\frac{1}{400}n + 12$ $\Rightarrow$ Demand Function — price as a function of # of items

$R(n) = n(-\frac{1}{400}n + 12)$

$= -\frac{1}{400}n^2 + 12n$ $\Rightarrow$ Revenue as a function of items

Ex3: A cylindrical chemical storage tank with a capacity of 1000 m^3 is going to be constructed in a warehouse that is 12 m by 15 m, with a height of 11 m. The specifications call for the base to be made of sheet steel that costs \$100/ m^2, the top to be made of steel that costs \$50/ m^2, and the wall to made of steel that costs \$80/ m^2.

a) Determine whether it is possible for a tank of this capacity to fit in the warehouse.
If it **is** possible, state the restrictions on the radius.

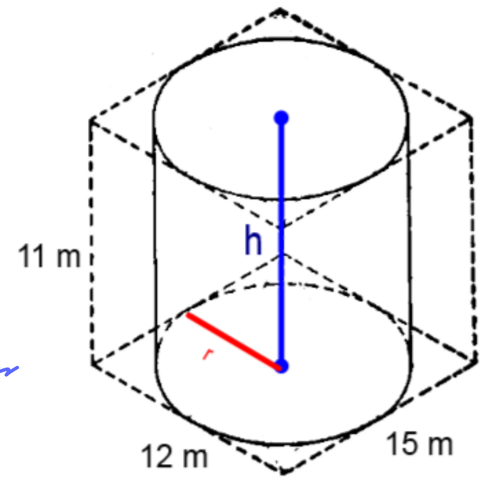
largest cylinder:

$r = 6$
$h = 11$

$$V = \pi r^2 h$$
$$= \pi(6)^2(11)$$
$$= 396\pi \ m^3$$
$$\doteq 1244.07 \ m^3$$

restriction on radius:

$0 < r \leq 6$



$\therefore$ It is possible as the cylinder needs to have a capacity of 1000 m^3 and the largest cylinder can have a capacity of 1244 m^3

b) If fitting the tank in the warehouse is possible, determine the proportions that meet the conditions, and minimizes the cost of the steel for construction. All calculations should be accurate to two decimal places.

Base + top + side walls

$$C = 100 \cdot \pi r^2 + 50 \cdot \pi r^2 + 80 \cdot 2\pi r h$$

$$C(r) = 150\pi r^2 + 160\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 150\pi r^2 + 160\,000 r^{-1}$$

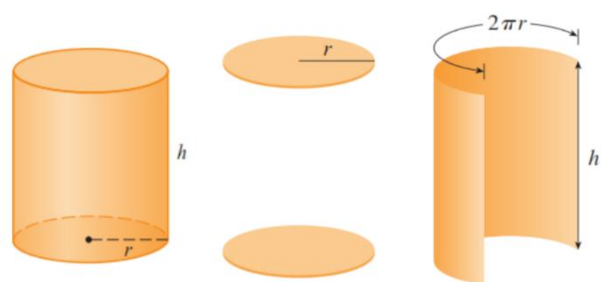
$$C'(r) = 300\pi r - 160000 r^{-2}$$

$$0 = 300\pi r - \frac{160\,000}{r^2}$$

$$\frac{160\,000}{r^2} = 300\pi r$$

$$\frac{1600}{3\pi} = r^3$$

$$r = \sqrt[3]{\frac{1600}{3\pi}}$$
$$\doteq 5.537$$



Constraint: $V = 1000 \ m^3$

$$1000 = \pi r^2 h$$
$$\frac{1000}{\pi r^2} = h$$

$\because r = 5.537$ $\quad h = \frac{1000}{\pi(5.537)^2}$
$\doteq 10.382$

$C'(r)$: 0 (−) 5.54 (+) 6
min

$\therefore$ to minimize cost the cylinder should have a radius of 5.54 m and height of 10.38 m

4.4 PRACTICE-OPTIMIZATION – BUSINE$$

1. A car rental agency has 200 cars. The owner finds that at a price of \$36 per day he can rent all the cars. For each \$2 increase in price, the demand is less and 5 fewer, cars are rented. What price will maximize the total revenue?

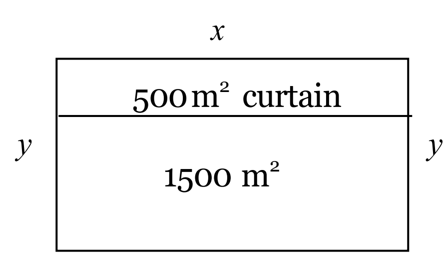
2. The current ticket price at a local theatre is \$4 and the theatre attracts an average of 250 customers per show. Every \$0.20 increase in ticket price reduces the average attendance by 10 customers. Find the ticket price that will maximize the revenue.

3. A variety store can sell 500 yo-yos for \$1 each. For each cent the store lowers the price, it can sell 20 more yo-yos. For what price should it sell the yo-yos to make maximum revenue?

4. A retailer of electrical appliances can sell 200 refrigerators at \$250 each. For each reduction of \$10 in price 10 more refrigerators per month are sold. But reductions of less than \$10 per unit have no effect on sales. What selling price would produce the maximum revenue per month and how many refrigerators would be sold at this price?

5. A real estate firm owns 250 apartments that can be rented out at \$460 per month each. For each \$5 per month increase in rent there are two vacancies created that cannot be filled. What should the monthly rent be to maximize the revenue? What is the maximum revenue?

6. A rectangular room is to be constructed with a curtain wall will be parallel to the front and will divide the room into a sales area of 1500 m^2 and a storeroom of 500 m^2. The outside walls at both sides and across the back will cost \$100 per meter. The curtain wall across the store will cost \$45 per meter. The front wall (mainly glass) will cost \$345 per meter. Calculate the dimensions of the room that will give the required floor areas at the lowest cost for the walls. The front wall should be at least 20 m wide.



Answers:

1. $58 / car	4. 230 units at \$220 or 220 units at \$230
2. $4.50 / ticket	5. \$542.50/unit for \$117722.50 in revenue
3. $0.625 / yo - yo	6. dimensions $28.57\,\text{m} \times 70$ m

Warm Up

1. A store has been selling 200 DVD burners a week at $350 each. A market survey indicates that for each $10 rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

let x be the # of $10 rebates

$$R(x) = \underbrace{(200+20x)}_{n}\underbrace{(350-10x)}_{p(n)}$$

let n be the # of items

$$n = 200 + 20x$$

$$x = \frac{n-200}{20}$$

$$p(x) = 350 - 10x$$

$$p(n) = 350 - 10\left(\frac{n-200}{20}\right)$$

$$= 350 - 0.5n + 100$$

$$= -0.5n + 450$$

$\therefore$ $p(n) = -0.5n + 450$ is the demand function

$$R(n) = n\,p(n)$$

$$= n(-0.5n + 450)$$

$$= -0.5n^2 + 450n$$

Constraint: $n > 0$

$$R'(n) = -n + 450$$

$$0 = -n + 450$$

$$n = 450$$

$R'(n)$: 0 (+) 450 (−) ∞

450 max

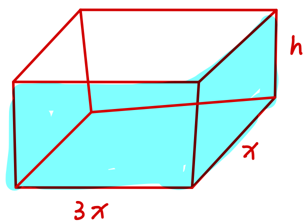
$$p(450) = -0.5(450) + 450$$

$$= 225$$

rebate: $350 − 225 = $125

$\therefore$ the store should offer a rebate of $125 or approx 12 to 13 $10 rebates to max. revenue

2. A lidless wooden rectangular toy box is built so that the length of the base is three times the width of the base. The volume of the box is 9 m3. The base must use thick wood costing \$8 per square meter, while the sides of the box use wood costing \$5/m2. Find the dimensions of the box having the lowest cost of materials.



Base + 2 Front/Back + 2 Sides

$$\text{Cost} = 8\cdot(3x^2) + 5\cdot 2[3xh] + 5\cdot 2[xh]$$

$$C(x) = 24x^2 + 30x\left(\frac{3}{x^2}\right) + 10x\left(\frac{3}{x^2}\right)$$

$$= 24x^2 + 90x^{-1} + 30x^{-1}$$

$$= 24x^2 + 120x^{-1}$$

Constraint:

$$9 = 3x^2h$$

$$\frac{3}{x^2} = h$$

$$x > 0$$

$$C'(x) = 48x - 120x^{-2}$$

$$0 = 48x - \frac{120}{x^2}$$

$$\frac{120}{x^2} = 48x$$

$$\frac{120}{48} = x^3$$

$$2.5 = x^3$$

$$x \doteq 1.357$$

$C'(x)$: $(-)$ on $(0, 1.357)$, $(+)$ on $(1.357, \infty)$; 1.357 min

length $= 3x = 3(1.357) \doteq 4.071$

width $= x = 1.357$

height $= \frac{3}{x^2} = \frac{3}{(1.357)^2} \doteq 1.629$

$\therefore$ Min cost will occur when the box has dimensions:

4.07 m × 1.36 m × 1.629 m

Example 4:

Determine the equation of the tangent line of $3\left(x^2+y^2\right)^2=100xy$ at the point $(3,1)$

$$6(x^2+y^2)(2x+2y\cdot y')=100(1\cdot y+y'x)$$

$$12x(x^2+y^2)+12yy'(x^2+y^2)=100y+100y'x$$

$$12yy'(x^2+y^2)-100y'x=100y-12x(x^2+y^2)$$

$$y'\left[12y(x^2+y^2)-100x\right]=100y-12x(x^2+y^2)$$

$$y'=\frac{100y-12x(x^2+y^2)}{12y(x^2+y^2)-100x}$$

$$y'_{(3,1)}=\frac{100(1)-12(3)(3^2+1^2)}{12(1)(3^2+1^2)-100(3)}$$

$$=\frac{100-360}{120-300}$$

$$=\frac{-260}{-180}$$

$$=\frac{13}{9}$$

$$\therefore\ y-1=\frac{13}{9}(x-3)$$

Example 4:

Determine the equation of the tangent line of $3\left(x^2+y^2\right)^2 = 100xy$ at the point $(3,1)$

$$6(x^2+y^2)\cdot(2x+2yy') = 100(1\cdot y + y'x)$$

$$(6x^2+6y^2)(2x+2yy') = 100y + 100xy'$$

$$12x^3 + 12x^2yy' + 12xy^2 + 12y^3y' = 100y + 100xy'$$

$$12x^2yy' + 12y^3y' - 100xy' = 100y - 12x^3 - 12xy^2$$

$$y'(12x^2y + 12y^3 - 100x) = 100y - 12x^3 - 12xy^2$$

$$y' = \frac{100y - 12x^3 - 12xy^2}{12x^2y + 12y^3 - 100x}$$

$$y'_{(3,1)} = \frac{100(1) - 12(3)^3 - 12(3)(1)^2}{12(3)^2(1) + 12(1)^3 - 100(3)}$$

$$= \frac{-260}{-180}$$

$$=$$

4.5 Practice-Implicit Differentiation

Show all work on a separate sheet of paper. No calculator unless otherwise stated.

1. Find $\dfrac{dy}{dx}$

   a) $x^3 - 3x^2y + 4xy^2 = 12$ b) $\sqrt{xy} = x + 3y$ c) $\left(y^2 + 2xy\right)^2 = 4(x+1)^2$

2. Find $\dfrac{dy}{dx}$ at the indicated point, then find the equation of both the tangent and normal lines.

   a) $y^2 = \dfrac{x^2-4}{x^2+4}$ at $(2,0)$ b) $(x+y)^3 = x^3 + y^3$ at $(-1,1)$

3. Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

   a) $x^2 + y^2 = 36$ b) $1 - xy = x - y$ c) $\sqrt[3]{x^2} + \sqrt[3]{y^2} = 1$

4. Determine the point(s) at which the graph of $y^4 = y^2 - x^2$ has either a horizontal or vertical tangent. Be sure to label which is which, if either exist.

5. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

6. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$

7. The slope of the tangent is -1 at the point $(0,1)$ on $x^3 - 6xy - ky^3 = a$, where k and a are constants. The values of the constants a and k are what?

Unit 4-Review

Motion

1. The position of an object moving along a straight line is described by the function $s(t) = -t^3 + 4t^2 - 10$ for $t \geq 0$.

(a) Is the object moving away from or towards its initial position when t = 3?
(b) Is the object speeding up or slowing down when t = 3?

2. A position function of an object is given by: $s(t) = t^3 - 6t^2 + 8t \quad, t \geq 0$

(a) Determine the velocity function for the object.
(b) Identify the point(s) where the object is at rest.
(c) Identify the point(s) where the acceleration is zero.
(d) Determine the equation of the acceleration function.
(e) For which intervals is the acceleration negative? Positive?
(f) Determine the intervals for which the object is speeding up and slowing down.

Optimization

I) Perimeter and Area

1. A farmer has 2400 m of fencing to create a rectangular field that borders on a straight river. He does not need to fence the river edge. What will be the area of the largest possible field?

2. A man is building a fence around a portion of his backyard. He wants the area inside the fence to be 600 m^2. If his house accounts for 20 m of the fence, what is the minimum amount of fencing he needs to purchase?

3. Find the maximum perimeter of a right triangle with hypotenuse 20 cm.

II) Volume and surface area

1. A box with a square base and open top has a volume of 4000 cm^3. Determine the dimensions of the box that minimizes the amount of material required.

2. A cylindrical can is to be made to hold 1 litre of oil. Prove that a ratio of height to radius of 2:1 will minimize the cost of the metal to manufacture the can if the entire can is made of the same metal.

3. Find the circular cylinder of greatest volume that can be inscribed in a sphere of radius 6 cm.

III) Distances

1. Determine the point on the graph of $4x + y^2 = 0$ that is closest to the point (0, −3).

2. Determine the point(s) on the parabola $f(x) = 6x - x^2$ that is/are closest to the point(0, 3).

3. A storage shed that is 3 m high and 2 m wide is built against a brick wall. Determine the length of the shortest ladder that will reach the wall from the ground, if it must go over the shed.

IV) Time

1. An island is at a point A, 6 miles off shore from the nearest point, B, on a straight beach. A store is located at point C, 7 miles down the beach from B. If a man can row at the rate of 4 miles per hour and walk at the rate of 5 miles per hour, where should he land in order to get from the island to the store in the least possible time?

2 Ship A leaves port at noon and sail due north at 10 knots. Ship B starts at noon but from a point 80 Nautical miles east of the port and sails west at 6 knots. At what time is the distance between the ships a minimum?

3. A canoeist is 300m offshore and wishes to land and then walk to a distant point on the straight shoreline. If she can paddle at 3 km/h and walk at 5 km/h, where should she land to minimize her travel time?

4. At 3 pm, boat A leaves harbour and heads south at 6 km/h. Also at 3 pm, boat B is 30 km east of the harbour and is headed west at 8 km/h. Determine the minimum distance between the ships and the time when this occurs

V) Profits, Cost, Revenue

1. A company is laying underground cable. Two communities are located on opposite shores of a 100 m wide creek; one community is on the north shore 1200 m east of the community on the south shore. The company has to connect two communities with cable. If the cost of laying able underground is \$40/m and the cost of laying cable underwater is \$80/m, what is the least expensive way to lay the cable?
2. Given a company's annual demand function is given by $p(x)=\dfrac{800000-x}{200000}$ and cost function is $C(x)=125000+0.42x$, where x is the number of widgets it produces. What level of sales will maximize profits?
3. A steel storage tank for propane gas is to be constructed in the shape of a cylinder with a hemisphere on each end . If the desired capacity is 100 m^3, what dimensions will require the least amount of steel?
4. A metal can is to hold 1 L of oil. Determine the radius of the can that will minimize the cost of the metal.
5. A rectangular box-shaped garbage can with a square base and an open top is to be constructed using exactly 2700 cm^2 of material. Find the dimensions of the box that will provide the greatest possible volume.
6. An oil well has been discovered offshore at W, 200 m from S, the nearest point on the shoreline. Town T is located 1000 m along the shore from point S. A pipeline must be installed underwater from W to V and then along the shoreline from V to T. If it costs \$500/m to run the pipe underwater and \$200/m to run the pipe along the shore, how far from S should V be located to minimize the total cost of the pipeline?
7. The owner of a condominium complex has 45 units all of which will be occupied if the rent charged is \$600 per month. The owner estimates that for every \$20 increase in rent, one of the units will become vacant. The owner sets aside \$60 per month from each of the occupied units to establish a repair fund. What rent should be charged per month in order to maximize the owner's profit if there are no other expenses? What is the owner's maximum monthly profit? How many units are occupied?
8. If it costs \$1000 to manufacture 200 gizmos, then the average cost to manufacture each gizmo is \$1000/200 or \$5. Assume that the cost to manufacture x gizmos is given by the function $C(x)=700+0.3x+0.006x^2$. Find the number of gizmos that should be manufactured to minimize the average cost per gizmo.

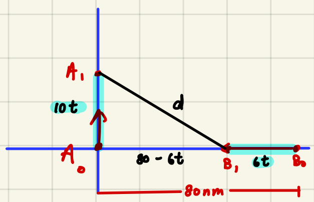
2 Ship A leaves port at noon and sail due north at 10 knots. Ship B starts at noon but from a point 80 Nautical miles east of the port and sails west at 6 knots. At what time is the distance between the ships a minimum?

measure of speed in water

Knots = nautical miles per hour (kn)

Nautical miles = nm

measure of distance in water



let t be the elapsed time (h)

$$d(t) = \sqrt{(80-6t)^2 + (10t)^2}$$

$$d'(t) = \frac{1}{2}\left[(80-6t)^2 + (10t)^2\right]^{-\frac{1}{2}} \cdot \left[2(80-6t)(-6) + 20t\right]$$

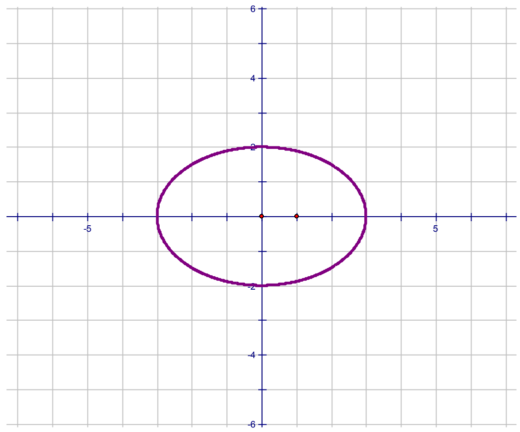
Other Stuff!

1. Determine the dimensions of the rectangle of maximum area that can be inscribed above the x-axis and below the parabola $y = 4x - x^2$.

2. Consider the curve $y = 10 - x^2$ for $x \geq 0$, $y \geq 0$. A tangent to this curve will form a triangle with the x-axis and the y-axis. Determine the equation of the tangent that forms a triangle of minimum area.

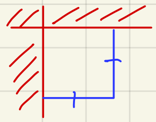
3. Determine the point P on the parabola $y = 1 - x^2$ at which the tangent line cuts from the first quadrant the triangle with the smallest area.

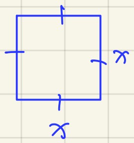
4. Determine the area of the largest rectangle that can be inscribed inside the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.



5. A rectangle lies in the first quadrant with one vertex at the origin and two of its sides along the x and y axes. The fourth vertex lies along the line $x + 2y - 10 = 0$. Determine the dimensions of the rectangle with the maximum area.

6. A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ .What length and width Should rectangle have so that its area is a maximum?

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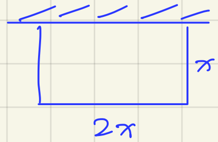
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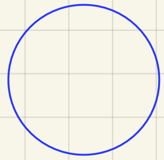

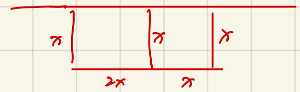
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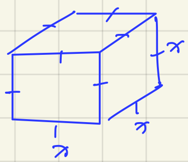
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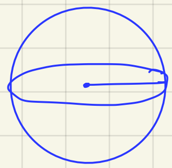
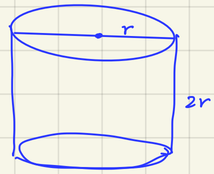
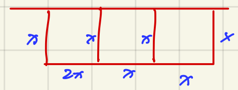



r
2r

$h = d$.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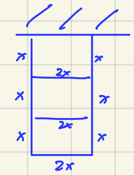
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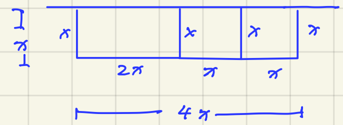
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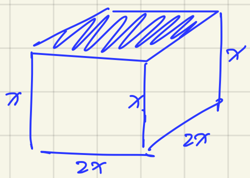
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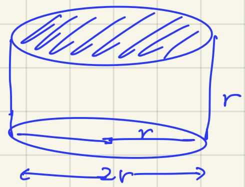
r
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