Grade 12 Calculus and Vectors

Unit 1: Rates of Change and Limits

Day	Learning Goals & Success Criteria	CW & HW
1	**Course Introduction** We are learning to... • get to know our math teacher and our classmates • review Prerequisite Skills: determine slope and equation of a line; use function notation to substitute and evaluate functions ,rationalize numerator/denominator, factoring I am able to… • name my teacher and a few classmates • contact a classmate should I miss a class • complete the welcome back assignment!	• Forms Signed for Tomorrow • Read Course Information • Complete Welcome back worksheet
2	**1.0 Exploring the Concept of a Limit** We are learning to... • explore the concept of a limit by investigating numerical and graphical examples and explain the reasoning involved. • explore the ratio of successive terms of sequences and series (use both divergent and convergent examples). I am able to… • recognize, through investigation with or without technology, graphical and numerical examples of limits, and explain the reasoning involved!	No Homework!
3	**1.1 Limit of a Function and One-sided limits** We are learning to... • evaluate the limit of functions algebraically • determine the graphical representation of a limit I am able to… • evaluate simple limit expressions • determine whether an limit exist • evaluate one-sided limits	CP:Pg.14-15 textbook Pg44#1,5a,b,6a,b,8,9,14,17
4	**1.2 Infinite Limits , Properties of Limits –Evaluating limits (2 days)** We are learning to… • use the limit properties to evaluate the limit of various functions • understand the concept of indeterminate limits I am able to… • evaluate limits at infinity , and limits at VA • evaluate limit expressions	Day 1: CP: Pg 22 #1,4 ,6(~~a,b,c~~) 7(a,b,c) ,9(a,b,c) textbook Pg44#2,10,12,13,19,24 Day 2: CP: pg. 23 #16-19 CP pg 26

5	**1.3 Continuity & Limits at Infinity** We are learning to... • understand definition of a continuous function • determine point(s) of discontinuity :jump discontinuities, and infinite discontinuities (Vertical Tangent) • redefine a function so it is continuous I am able to… • identify continuous function and determine point discontinuities • apply the concept of continuity to solve for unknown variables	Cp : Pg.32 textbook Pg 44#4,5c,6c,18,20, 21,25
6	**1.4 The Tangent Problem** We are learning to... • explore the idea that the limiting position of a secant is a tangent. • determine the value of an instantaneous rate of change using the limit definition for the given x value. I am able to… • connect average rate of change to $\frac{f(a+h)-f(a)}{h}$ and instantaneous rate of change to $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$. • use concept of rate of change to solve real life-word problems such as: Determine the equation of the tangent to $y=\frac{2+x}{\sqrt{x+3}}$ at x = 6.	CP :Pg 37 textbook Pg. 9#1b,2b,3,4b Pg.20#6,7,15a,b(iii),16, 23b
7	**Mid-Review**	CP. Pg 39-40
8	**1.5 Derivatives of Functions By First Principles** **Investigating Differentiation Shortcuts** We are learning to… • determine, using limits, the algebraic representation of the derivative of a functions at any point • make connection between the graphs of the derivative function and the function. I am able to… • find the derivative of any polynomial using first principles. • relate and compare a function to its derivative	CP:Pg.45 textbook Pg 20 #11,12,29 Pg 58 #5,9,10, 14,17 Pg.60#29
9	**1.6 Key Characteristics of Instantaneous Rates of Change** **Non-differentiable points** We are learning to... • determine intervals in order to identify increasing, decreasing, and zero rates of change using graphical and numerical representations of polynomial functions. • describe the behaviour of the instantaneous rate of change at and between local maxima and minima. I am able to… • investigate how the instantaneous rate of change for the given function changes and how to describe that relationship algebraically	CP:Pg.52 textbook Pg 58 #1,3,8,20,25
10	**Review**	CP:Pg. 53-54

11	**Quiz**	
12	**Summative Test 1**	

Day 1 Th Sept 19 (Postponed to T Sept 24)
Day 2 F Sept 20 (Postponed to W Sept 25)

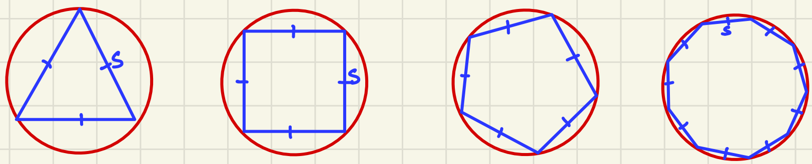
We will cover 2-1(W Sept 18) and 2-2 (Th Sept 19)

Date changes are to accommodate Music Camp students away on F Sept 20

UNIT 1

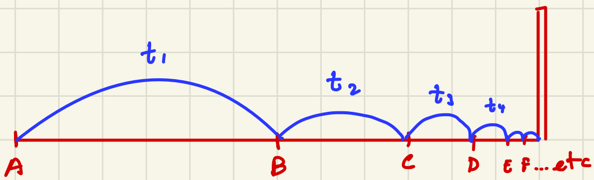
LIMITS AND RATES OF CHANGE

① Using Limit to determine the Area of Circle



as the # of sides $\to \infty$ the area $\to \pi r^2$

② Zeno's Paradox : take $\frac{1}{2}$ the distance each time towards the wall Continuously $\Rightarrow$ will you ever reach the wall?



$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

geometric sequence $t_n = ar^{n-1}$

$a = \frac{1}{2}$

$r = \frac{1}{2}$

$$\begin{aligned} t_n &= ar^{n-1} \\ &= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^{n-1} \\ &= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^{n}\left(\tfrac{1}{2}\right)^{-1} \\ &= \left(\tfrac{1}{2}\right)^{n} \end{aligned}$$

$$\lim_{n\to\infty} \left(\tfrac{1}{2}\right)^n = 0$$

yes! you will reach the wall in the world of Calculus

what is the total distance travelled?

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ &= \frac{\left(\frac{1}{2}\right)\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} \\ &= 1-\left(\tfrac{1}{2}\right)^n \end{aligned}$$

$$\begin{aligned} &\lim_{n\to\infty} 1-\left(\tfrac{1}{2}\right)^n \\ &= \lim_{n\to\infty} 1-\frac{1}{2^n} \\ &= 1-0 \\ &= 1 \end{aligned}$$

Recall: Fibonacci Sequence $\Rightarrow$ 1, 1, 2, 3, 5, 8, 13, 21, ...
as a recursive formula: $t_1 = 1$, $t_2 = 1$, $t_n = t_{n-2} + t_{n-1}$, $n > 2$

Experiencing Limits

Evaluating Limits of Convergent Sequences

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
$t_1, t_2, t_3, t_4, t_5, \ldots$

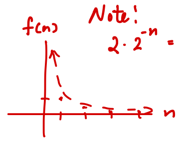
n	t_n
1	1
2	$\frac{1}{2}$
3	$\frac{1}{4}$

etc.

$t_n = ar^{n-1} = (1)(\frac{1}{2})^{n-1} = (\frac{1}{2})^{n-1} = (\frac{1}{2})^n \cdot (\frac{1}{2})^{-1} = 2 \cdot (\frac{1}{2})^n$

$\lim_{n \to \infty} 2 \cdot (\frac{1}{2})^n = 0$

Note: $2 \cdot 2^{-n} = 2^{-n+1}$



$\lim_{n \to \infty} f(n) = 0$

or $\lim_{n \to \infty} 2^{-n+1} = 0$

A **sequence** is a function whose domain is the set of positive integers n=1,2,3,....

The values or individual terms of a sequence are generally denoted by a subscript of n on t. In other words, we use t_n rather than f(n). $\Rightarrow$ $n \in \mathbb{Z}^+$

For example, the list of all positive odd numbers forms the sequence 1,3,5,7,....

This sequence could be represented algebraically by two different formulas:

1, 3, 5, 7, ...

general arithmetic sequence

$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d$

1. Recursive Formula

$t_n = t_{n-1} + 2$ where $t_1 = 1$

2. Arithmetic Formula

$t_n = 1 + 2(n-1)$ ↳ Explicit formula

$= 2n - 1$ for $n = 1,2,3,\ldots$

$a = 1$, $d = 2$

$t_n = a + (n-1)d$
$= 1 + (n-1)(2)$
$t_n = 2n - 1$ $\Rightarrow$ Linear Relation

If we continue this sequence of numbers, would this sequence approach a single value? No

In other words, as $n \to +\infty$ does t_n approach a limit? $+\infty$ $\Rightarrow$ No

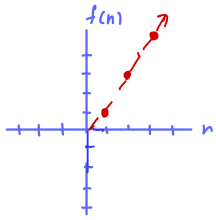
Discrete Function

As n increases, we see that t_n becomes arbitrarily large in value.

Therefore, as $n \to +\infty$, $t_n \to +\infty$.

We could use limits to write this as

$$\lim_{n \to \infty}(2n-1) = \infty$$



1, 3, 5, 7, ...

n	t_n or $f(n)$
1	1
2	3
3	5
4	7

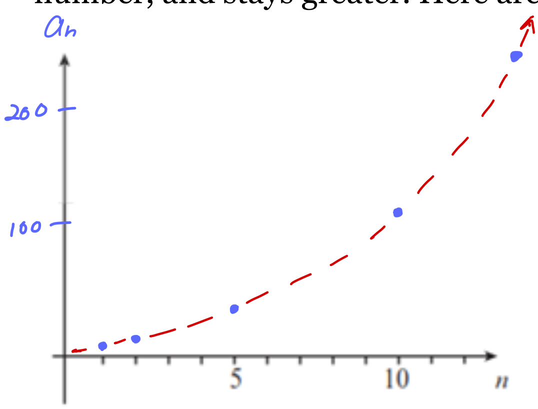
The behavior of infinite sequences

It is often very important to examine what happens to a sequence as n gets very large. There are three types of behavior that we shall wish to describe explicitly. These are

- sequences that 'tend to infinity';
- sequences that '**converge** to a real limit';
- sequences that 'do not tend to a limit at all'.

"diverges" $\Rightarrow$ doesn't approach a limiting value

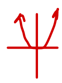
First we look at sequences that tend to infinity. We say a sequence tends to infinity if, however large a number we choose, the sequence becomes greater than that number, and stays greater. Here are some examples of sequences that tend to infinity.



1, 4, 9, 16, ...

$$a_n = n^2; n \geq 1$$

(quadratic function)



$\lim_{x \to -\infty} x^2 = +\infty$ (dne)

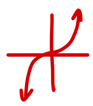
$\lim_{x \to +\infty} x^2 = +\infty$ (dne)

n	$a_n = n^2$
1	1
2	4
5	25
10	100
15	225

$\lim_{n \to \infty} n^2 = +\infty$ (dne)

$\therefore$ this is a divergent sequence

Cubic function



$\lim_{x \to -\infty} x^3 = -\infty$ (dne)

$\lim_{x \to +\infty} x^3 = +\infty$ (dne)

Experiencing Limits

Evaluating Limits of Convergent Sequences

A **sequence** is a function whose domain is the set of positive integers n=1,2,3,....

The values or individual terms of a sequence are generally denoted by a subscript of n on t. In other words, we use t_n rather than f(n).
For example, the list of all positive odd numbers forms the sequence 1,3,5,7,....
This sequence could be represented algebraically by two different formulas:

1. Recursive Formula

$t_n = t_{n-1} + 2 \text{ where } t_1 = 1$

2. Arithmetic Formula

$$\begin{aligned} t_n &= 1 + 2(n-1) \\ &= 2n - 1 \quad for \quad n = 1,2,3,... \end{aligned}$$

If we continue this sequence of numbers, would this sequence approach a single value?
In other words, as $n \to +\infty$ does t_n approach a limit?
As n increases, we see that t_n becomes arbitrarily large in value.
Therefore, as $n \to +\infty$, $t_n \to +\infty$.
We could use limits to write this as

$$\lim_{n\to\infty}(2n-1) = \infty$$

The behavior of infinite sequences

It is often very important to examine what happens to a sequence as n gets very large. There are three types of behavior that we shall wish to describe explicitly. These are

- sequences that 'tend to infinity';
- sequences that '**converge** to a real limit';
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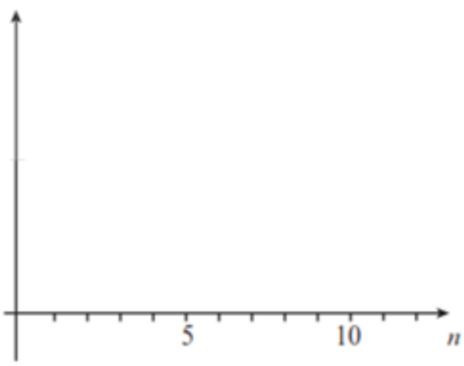
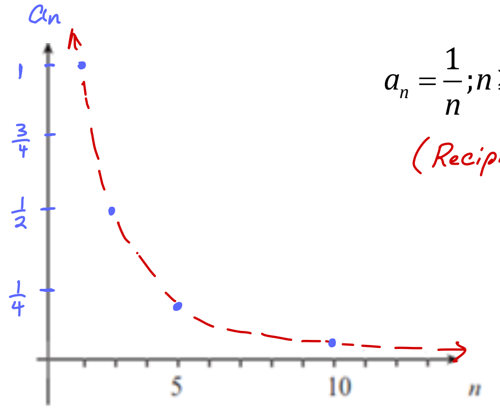


$a_n = n^2; n \geq 1$

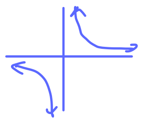
n	$a_n = n^2$
1	
2	
5	
10	
15	

Now we look at sequences with real limits. We say a sequence tends to a real limit if there is a real number, **L**, such that the sequence gets closer and closer to it. We say **L** is the limit of the sequence.



$$a_n = \frac{1}{n}; n \geq 1$$

(Reciprocal or Rational function)



$\lim_{x \to -\infty} \frac{1}{x} = 0$

$\lim_{x \to +\infty} \frac{1}{x} = 0$

n	$a_n = \frac{1}{n}$
1	1
2	$\frac{1}{2}$
5	$\frac{1}{5}$
10	$\frac{1}{10}$
50	$\frac{1}{50}$

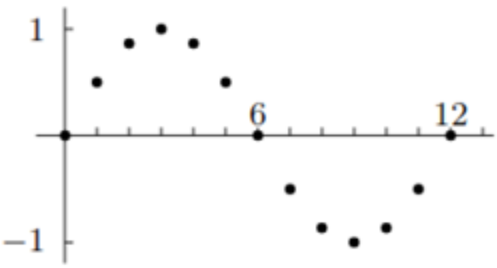
$1, \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{50}, \ldots$

$\lim_{n \to \infty} \frac{1}{n} = 0$

∴ this is a convergent sequence

And finally we look at sequences that cannot approach any specific number **L** as ***n*** grows large.

(Sine Function)



n	$a_n = \sin\left(\frac{n\pi}{6}\right)$
0	0
3	1
6	0
9	−1
12	0

$0, 1, 0, -1, 0, \ldots$

$\lim_{n \to \infty} \sin\left[\frac{\pi n}{6}\right] = dne$

∴ this is a divergent sequence

Definitions:

$t_n ; a_n ; f(n)$

We say that the sequence $\{a_n\}$ converges (or is convergent or has limit) if it tends to a number L.

A sequence diverges (or is divergent) if it does not tend to any number.

dne or $+\infty / -\infty$

Activity 1:

Materials: Scientific calculator.

Step 1) Determine $\sqrt{10}$.

Step 2) Determine the square root of your result.

Step 3) Repeat Step 2 an infinite number of times (or until a pattern emerges!).

(Hint: Your calculator probably has its own 'short cut' way of finding the square root of a previous result over and over again. E.g. just pressing $\sqrt{\ }$ or = over and over again often works.)

Step 4) Repeat the above experiment but this time, instead of starting with $\sqrt{10}$ start with the square root of your favourite (positive) number. What do you notice? (By the way, when you perform the same computation over and over again you are performing an *iterative process*.)

Step 5) Repeat the above experiment starting with different numbers (in an iterative process, such starting numbers are called the *seeds*). Be sure to try numbers less than 1.

Conclusion: The iterative process of square rooting over and over again always **converges** to a value of ___1___.

Reasoning:

$$t_1 = \sqrt{10} = 10^{\frac{1}{2}} \doteq 3.16227\ldots$$
$$t_2 = \sqrt{\sqrt{10}} = \left(10^{\frac{1}{2}}\right)^{\frac{1}{2}} \doteq 1.77827\ldots$$
$$t_3 = \sqrt{t_2} = \left[\left(10^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \doteq 1.33352\ldots$$
$$t_4 = \sqrt{t_3} = \quad . \quad \doteq 1.15478\ldots$$
$$t_5 = \sqrt{t_4} = \quad \vdots \quad \doteq 1.07460\ldots$$
$$t_6 = \sqrt{t_5} = \quad \vdots \quad \doteq 1.03663\ldots$$
$$t_7 = \sqrt{t_6} = \quad \vdots \quad \doteq 1.018151\ldots$$
$$t_8 = \sqrt{t_7} = \quad \vdots \quad \doteq 1.00903\ldots$$
$$\vdots$$
$$t_n = \sqrt{t_{n-1}} = 10^{\frac{1}{2^n}} \doteq 1$$

Reason:
$$\lim_{n \to \infty} 10^{\frac{1}{2^n}}$$
$$= 10^{\frac{1}{\infty}}$$
$$= 10^0$$
$$= 1$$

$$\therefore \lim_{n \to \infty} 10^{\frac{1}{2^n}} = 1$$

$$\lim_{n \to \infty} \sqrt{t_{n-1}} = 1$$

Activity 2:

Materials: Scientific calculator, set in RADIANS!!!

Step 1) Determine cos(0.6).

Step 2) Determine cosine of your result.

Step 3) Repeat Step 2 an infinite number of times (or until a pattern emerges).

Step 4) Repeat the above experiment but this time use different seeds. What do you notice?

Conclusion: The iterative process of taking a cosine over and over again always **converges** to a value of 0.739.

(Extension: feel free to explore iterative processes involving other functions).

Reasoning:

$$t_1 = \cos(0.6) \doteq 0.82533\ldots$$

$$t_2 = \cos(\cos(0.6)) \doteq 0.67831\ldots$$

$$t_3 = \cos(\cos(\cos 0.6)) \doteq 0.77863\ldots$$

$$t_4 = \cos(t_3) \doteq 0.71187\ldots$$

$$t_5 = \cos(t_4) \doteq 0.75713\ldots$$

$$t_6 = \cos(t_5) \doteq 0.72680\ldots$$

$$t_7 = \cos(t_6) \doteq 0.74730\ldots$$

$$t_8 = \cos(t_7) \doteq 0.73352\ldots$$

$$t_9 = \cos(t_8) \doteq 0.74281\ldots$$

$$t_{10} = \cos(t_9) \doteq 0.73656\ldots$$

$$t_{11} = \cos(t_{10}) \doteq 0.74078\ldots$$

$$t_{12} = \cos(t_{11}) \doteq 0.73794\ldots$$

$$t_{13} = \cos(t_{12}) \doteq 0.73985\ldots$$

$$t_{14} = \cos(t_{13}) \doteq 0.73856\ldots$$

$$t_{15} = \cos(t_{14}) \doteq 0.73943\ldots$$

$$t_{16} = \cos(t_{15}) \doteq 0.73884\ldots$$

$$t_{17} = \cos(t_{16}) \doteq 0.73924\ldots$$

$$t_{18} = \cos(t_{17}) \doteq 0.73897\ldots$$

$$t_{19} \doteq \cos(t_{18}) \doteq 0.73915\ldots$$

$$t_{20} \doteq \cos(t_{19}) \doteq 0.73903\ldots$$

$$t_{21} \doteq \cos(t_{20}) \doteq 0.73911\ldots$$

$$\vdots$$

$$\therefore \lim_{n \to \infty} \cos(t_{n-1}) = 0.739$$

< see moshtagh's CP Notes if you would like a reason for why this is... but it requires a higher level of Calculus that is beyond MCV4U >

Activity 3:

Materials: One piece of 8.5 × 11" paper and a pencil crayon.

Step 1) Shade in half the sheet of paper. To do so, fold the sheet in half to form two equal rectangular sections. Open up the paper and shade in one of these sections.

Step 2) Shade in half of the unshaded rectangular section. To do so, fold the sheet of paper twice to make quarters. Shade in one of the uncoloured quarters.

Step 3) Again, shade in half of the remaining unshaded uncoloured rectangular section. Use folding to help determine where to shade.

Step 4) Repeat Step 3 enough times to answer the questions below.

Questions:

1) Complete the following chart.

Number of Times You've Shaded a Section	Fraction of Original Sheet of Paper that is Now Shaded (leave as a sum)
1	$\frac{1}{2}$
2	$\frac{1}{2}+\frac{1}{4}$
3	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$
4	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$
5	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$
6	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$

If this process continues indefinitely, how much of the original paper will be shaded? 1.

Conclusion: The infinite geometric series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$ **converges** to a value of 1.

Reasoning:

Recall: the general geometric sequence is

$a, ar, ar^2, ar^3, \dots ar^{n-1}$ $\therefore t_n = ar^{n-1}$

The sum of a geometric sequence is:

$$S_n = \frac{a(1-r^n)}{1-r} \qquad a=\frac{1}{2} \quad r=\frac{1}{2}$$

$r<1$ $\therefore$ it will converge

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$$

$$S_\infty = \lim_{n\to\infty} 1-\left(\frac{1}{2}\right)^n$$

$$= 1-\left(\frac{1}{2}\right)^\infty$$

$$= 1-\frac{1}{2^\infty}$$

$$= 1-0$$

$= 1$

$S_\infty = \frac{a(1-r^\infty)}{1-r}$ if $r<1$, converging

$S_\infty = \frac{a}{1-r}$

$5+\frac{5}{2}+\frac{5}{4}+\dots$ $S_\infty = \frac{5}{1-\frac{1}{2}}$

$= 10$

Bonus Conclusion: Prove that 0.99999999999999… = 1

$$\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\dots$$

$$= 9\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\dots\right)$$

$$= 9\left(\frac{1}{9}\right)$$

$$= 1$$

$a=\frac{1}{10}$ $r=\frac{1}{10}$

$$S_\infty = \frac{\frac{1}{10}}{1-\left(\frac{1}{10}\right)} = \frac{1}{10}\div\frac{9}{10} = \frac{1}{9}$$

or

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{\left(\frac{1}{10}\right)\left[1-\left(\frac{1}{10}\right)^n\right]}{1-\left(\frac{1}{10}\right)} = \lim_{n\to\infty}\frac{\frac{1}{10}\left[1-\left(\frac{1}{10}\right)^n\right]}{\frac{9}{10}} = \lim_{n\to\infty}\frac{1}{9}\left[1-\frac{1}{10^n}\right]$$

$$= \frac{1}{9}\left[1-\frac{1}{10^\infty}\right] = \frac{1}{9}[1-0] = \frac{1}{9}$$

Grade 9 way of showing :

$$0.\overline{9} = 1$$

$x = 0.5\overline{2}$

$100x = 52.\overline{2}$

$10x = 5.\overline{2}$

(−) $90x = 47$

$x = \frac{47}{90}$

$x = 0.\overline{9}$

$10x = 9.\overline{9}$

$10x = 9.\overline{9}$

$x = 0.\overline{9}$

(−) $9x = 9$

$x = \frac{9}{9}$

$x = 1$

$\therefore \quad x = 0.\overline{9} = 1$

Grade 11 way :

$0.\overline{9} = 0.9 + 0.09 + 0.009 + \dots$

$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots$

$= 9\left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots\right)$

$= 9\left(\underbrace{\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots \frac{1}{10^n}}_{\text{geometric series}}\right)$

$\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \dots, \frac{1}{10^n}$

is a converging sequence !

$10, 100, 1000, \dots, 10^n$

is a diverging sequence !

$a = \frac{1}{10} \quad r = \frac{1}{10} \quad n = \infty$

$S_n = \frac{a(1-r^n)}{1-r}$

'0'

$= \frac{\frac{1}{10}\left(1 - \left(\frac{1}{10}\right)^\infty\right)}{1 - \left(\frac{1}{10}\right)}$

as $n \to \infty$

$\left(\frac{1}{10}\right)^\infty \to 0$

$\lim_{x \to \infty} \left(\frac{1}{10}\right)^x = 0$

$= \frac{\frac{1}{10}(1)}{\frac{9}{10}}$

$= \frac{1}{9}$

$= 9\left(\frac{1}{9}\right)$

$= 1$

Note!

For an infinite converging series the sum is

$$S_\infty = \frac{a}{1-r}, \quad 0 < r < 1$$

converging geometric sequence

$|r| < 1$

$-1 < r < 1$

Warmup: Limit of Sequences

VCP Exercise 1.6

A 1. State the limits of the following sequences, or state that the limit does not exist.

(a) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \ldots, \left(\frac{1}{3}\right)^n, \ldots$ $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3^\infty} = 0$ (convergent)

(b) $5, 4\frac{1}{2}, 4\frac{1}{3}, 4\frac{1}{4}, 4\frac{1}{5}, \ldots, 4 + \frac{1}{n}, \ldots$ $\lim_{n\to\infty} (4+\frac{1}{n}) = 4 + \frac{1}{\infty} = 4$ (convergent)

(c) $1, 2, 3, 4, 5, \ldots, n, \ldots$ $\lim_{n\to\infty} n = \infty =$ dne (divergent)

(d) $3, 3, 3, 3, 3, \ldots, 3, \ldots$ $\lim_{n\to\infty} 3 = 3$ (convergent)

* (e) $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots$ $\lim_{n\to\infty} T(n) = 0$ (convergent)

(f) $5, 6\frac{1}{2}, 5\frac{2}{3}, 6\frac{1}{4}, 5\frac{4}{5}, 6\frac{1}{6}, \ldots, 6 + \frac{(-1)^n}{n}, \ldots$ $\lim_{n\to\infty} \left[6 + \frac{(-1)^n}{n}\right] = 6 + \frac{(-1)^\infty}{\infty} = 6 + 0 = 6$ (Convergent)

(g) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \ldots$ $\lim_{x\to\infty} f(x) =$ dne (divergent)

Extra

(k) $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

(m) $\lim_{n\to\infty} 5^{-n} = 0$

(o) $\lim_{n\to\infty} \frac{1+n-2n^2}{1-n+n^2} = -2$

(q) $\lim_{n\to\infty} \frac{1}{n^5} = 0$

(s) $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

Convergent

EXERCISE 1.6

1. (a) 0 (b) 4 (c) does not exist (d) 3 (e) 0
(f) 6 (g) does not exist

(k) 0 (m) 0 (o) -2 (q) 0 (s) 0

previous limits ⇒ in 0-2 ⇒ are all examples of Limits @ infinity $\lim_{n\to\infty} f(n)$

1. 1 The limit of a Function ⇒ at a specific value

Calculus has been called the study of ~~continues~~ continuous change, and the **limit** is the basic concept that describe and analyze such change.
The limit of a function describes the behaviour of the function when ~~t~~the variable is near ("in the neighbourhood of"), **but does not equal**, a specific number (Fig.1)
If the values of ***f(x)*** get closer and closer, as close as we want, to one number **L** as we take values of x very close to (but not equal to) a number c ,then we say:

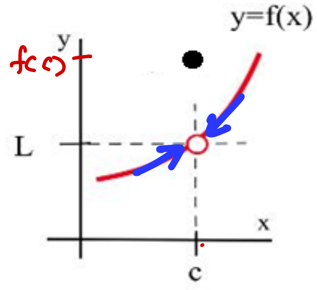
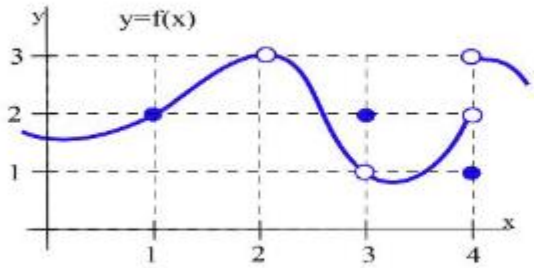


Fig. 1

The limit of ***f(x)***, as x approaches **c** is **L** and we write: $\lim_{x\to c} f\left(x\right) = L$.

f(c) is the ONLY number that describes the behaviour (value) of *f(x)* **AT** the point *x=c*.
$\lim_{x\to c} f\left(x\right)$ is a single number that describes the behaviour of f **NEAR, <u>BUT NOT AT</u>** point *x=c*.

<u>Example #1:</u>
Use the graph of y=f(x) and determine the following limits.



(a) $\lim_{x\to 2} f\left(x\right) = 3$ note! $\lim_{x\to 2^-} f(x) = 3$, $\lim_{x\to 2^+} f(x) = 3$ $f(2) = dne$

(b) $\lim_{x\to 3} f\left(x\right) = 1$ $\lim_{x\to 3^-} f(x) = 1$, $\lim_{x\to 3^+} f(x) = 1$ $f(3) = 2$

(c) $\lim_{x\to 1} f\left(x\right) = 2$ $\lim_{x\to 1^-} f(x) = 2$, $\lim_{x\to 1^+} f(x) = 2$ $f(1) = 2$

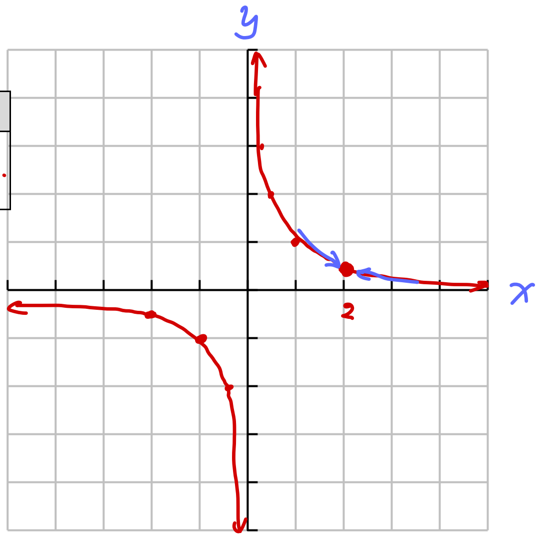
(d) $\lim_{x\to 4} f\left(x\right) = dne$ $\lim_{x\to 4^-} f(x) = 2$, $\lim_{x\to 4^+} f(x) = 3$ $f(4) = 1$

<u>Example #2:</u> Graph each function, then complete the Table of values to find each limit.

(a) $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$

x	1.9	1.99	1.999	2	2.0001	2.001	2.01
f(x)	0.526..	0.502..	0.5002..	$\frac{1}{2}$	0.4999...	0.499...	0.497..

$f(2) = \frac{1}{2}$

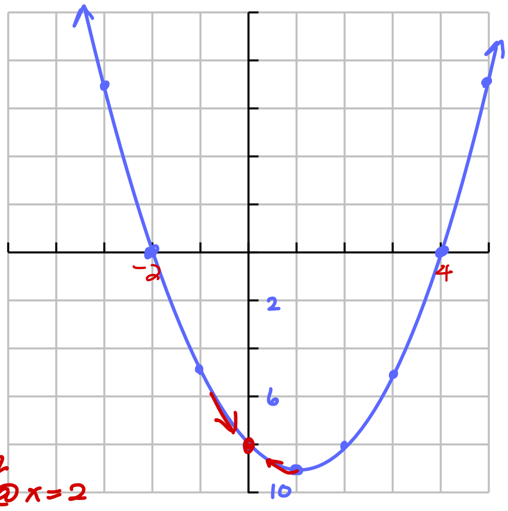


(b) $\lim\limits_{x\to 0}\overbrace{\left[(x+2)(x-4)\right]}^{f(x)} = -8$

AoS: $x = \frac{-2+4}{2} = 1$ $f(1) = (1+2)(1-4) = -9$

$y = (x-1)^2 - 9$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0. 1
f(x)	-7.79	-7.9799	-7.997...	-8	-8.0019..	-8.0199	-8.19

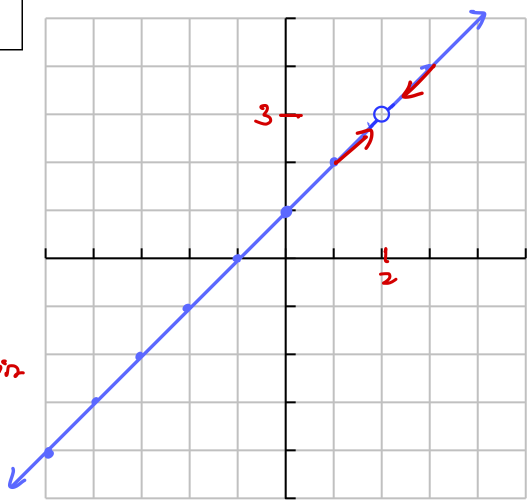


(c) $\lim\limits_{x\to 2}\overbrace{\frac{x^2-x-2}{x-2}}^{f(x)} = 3$

$f(x) = \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}}$

$= (x+1)$ ⇐ linear with a hole @ $x=2$

x	1.9	1.99	1.999	2	2.001	2.01	2. 1
f(x)	2.9	2.99	2.999	DNE	3.001	3.01	3.1



$\lim\limits_{x\to 2} \frac{(2)^2-(2)-2}{(2)-2} = \frac{0}{0} \Rightarrow$ indeterminant form

" A limit exists, but can not be determined by straight substitution b/c there needs to be an algebraic manipulation to get it out of indeterminant form "

$\lim\limits_{x\to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}}$

$= \lim\limits_{x\to 2} (x+1)$

$= 2+1$

$= 3$

One- Sided Limit

Sometimes, what happens to us at a place depends on the direction we use to approach that place. Similarly, the values of a function near a point may depend on the direction we use to approach that point. On the number line we can approach a point from the left or right, and that leads to **one-sided limits**.

The left limit as x approaches c of $f(x)$ is L if the values of $f(x)$ get as close to L as we want when x is very close to the left of c, x< c : $\lim\limits_{x\to c^-} f(x) = L$

The right limit, written with $\lim\limits_{x\to c^+} f(x)$ **requires that x lie to the right of c, x>c.**

One-sided Limit Theorem:

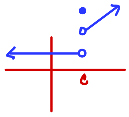
$$\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

Corollary:

If $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x) = L$, then $\lim_{x\to c} f(x)$ *does not exist.*

When Limits Do Not Exist

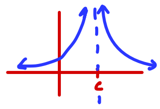
If there does not exist a number L satisfying the condition in the definition, then we say the $\lim_{x\to c} f(x)$ does not exist.

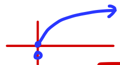


dne

Limits typically fail for three reasons:

1. f (x) approaches a different number from the right side of c than it approaches from the left side.
2. f (x) increases or decreases without bound as x approaches c. (Vertical Asymptotes)
3. f (x) oscillates between two fixed values as x approaches c.
4. f(x) at an endpoint

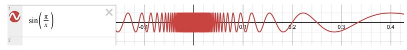




$\lim_{x\to 0}\sqrt{x}$ = dne

$1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, \ldots$ or $f(x) = \lim_{x\to 0} \sin\frac{\pi}{x}$

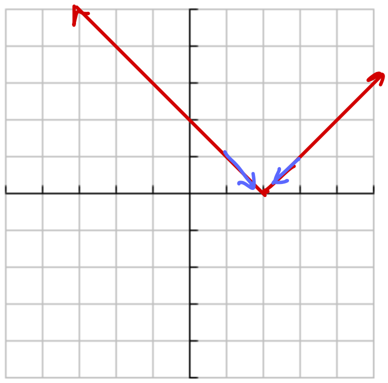
$y=\sqrt{x}$ note $\lim_{x\to 0^-}\sqrt{x}$ = dne $\lim_{x\to 0^+}\sqrt{x} = 0$



Example #3:

First try to graph the function $f(x) = |x-2|$, then find each limit, if it exists.

Recall: $f(x) = |x-2| = \begin{cases} x-2 & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$



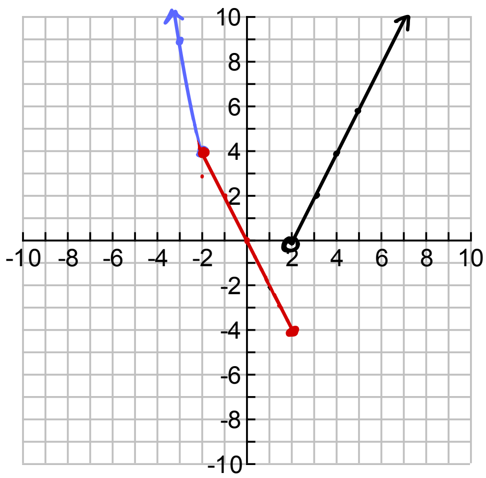
a) $\lim_{x\to 2^-} |x-2| =$ <u>0</u>

b) $\lim_{x\to 2^+} |x-2| =$ <u>0</u>

c) $\lim_{x\to 2} |x-2| =$ <u>0</u>

The above is an example of a piecewise function. A piecewise function is comprised of two or more curves, the shapes of which are independent of each other. When given the equation of a piecewise function, we can identify the equation for each piece by examining to what part of the domain each equation applies.

Example #4: Let $f(x) = \begin{cases} x^2 & \text{if } x \in (-\infty, -2) \\ -2x & \text{if } x \in [-2, 2] \\ 2x - 4 & \text{if } x \in (2, \infty) \end{cases}$



a) Graph the function *f(x)*.

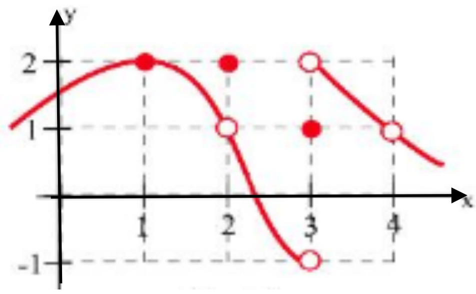
b) Find each limit, if it exists. If the limit does not exist, explain why.

a) $\lim_{x\to -2^-} f(x) =$ <u>4</u> b) $\lim_{x\to -2^+} f(x) =$ <u>4</u> c) $\lim_{x\to -2} f(x) =$ <u>4</u>

d) $\lim_{x\to 2^+} f(x) =$ <u>0</u> e) $\lim_{x\to 2^-} f(x) =$ <u>-4</u> f) $\lim_{x\to 2} f(x) =$ <u>dne</u>

Example #5:

Use the graph below to evaluate each of the following limits, if it exists.



a) $\lim_{x\to1^+} f(x) =$ 2 b) $\lim_{x\to1^-} f(x) =$ 2 c) $\lim_{x\to1} f(x) =$ 2

d) $\lim_{x\to2^+} f(x) =$ 1 e) $\lim_{x\to2^-} f(x) =$ 1 f) $\lim_{x\to2} f(x) =$ 1

g) $\lim_{x\to3^+} f(x) =$ 2 h) $\lim_{x\to3^-} f(x) =$ -1 i) $\lim_{x\to3} f(x) =$ dne

j) $\lim_{x\to4^+} f(x)$ 1 k) $\lim_{x\to4^-} f(x) =$ 1 l) $\lim_{x\to4} f(x) =$ 1

Example # 6 : Consider the piecewise function f(x) defined below, where **A** is a constant.

$$f(x) = \begin{cases} A^2x - 4A & \text{if } x \ge 2 \\ -2 & \text{if } x < 2 \end{cases}$$

Determine all values of A so that $\lim_{x\to2} f(x)$ exists.

$\lim_{x\to2^+} f(x) = A^2(2) - 4A$ $\lim_{x\to2^-} f(x) = -2$

$\lim_{x\to2^+} f(x) = \lim_{x\to2^-} f(x)$

$2A^2 - 4A = -2$

$2A^2 - 4A + 2 = 0$

$A^2 - 2A + 1 = 0$

$(A-1)(A-1) = 0$

$\therefore A = 1$

Example # 7: Consider the following piecewise function $f(x)$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are constants.

$$f(x) = \begin{cases} Ax + B & \text{if } x < -2 \\ x^2 + 2Ax - B & \text{if } -2 \le x < 1 \\ 4 & \text{if } x > 1 \end{cases}$$

Determine all values of the constants $\boldsymbol{A}$ and $\boldsymbol{B}$ so that $\lim_{x\to-2} f(x)$ and $\lim_{x\to1} f(x)$ both exist.

$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^+} f(x)$

$A(-2) + B = (-2)^2 + 2A(-2) - B$

$-2A + B = 4 - 4A - B$

$2A + 2B = 4$

$A + B = 2$ ①

$\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x)$

$(1)^2 + 2A(1) - B = 4$

$2A - B = 3$ ②

$\begin{cases} A + B = 2 & ① \\ 2A - B = 3 & ② \end{cases}$

(+) $3A = 5$

$A = \frac{5}{3}$ ③

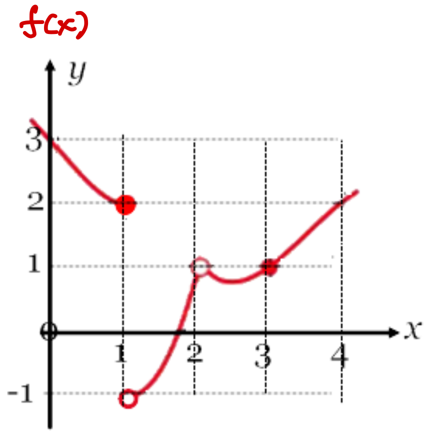
sub ③ into ①

$\frac{5}{3} + B = 2$

$B = \frac{1}{3}$

$\therefore A = \frac{5}{3}$ $B = \frac{1}{3}$

Example # 8* Use the graph to determine the following limits.



a) $\lim_{x\to 0} f(4+x) =$ 2

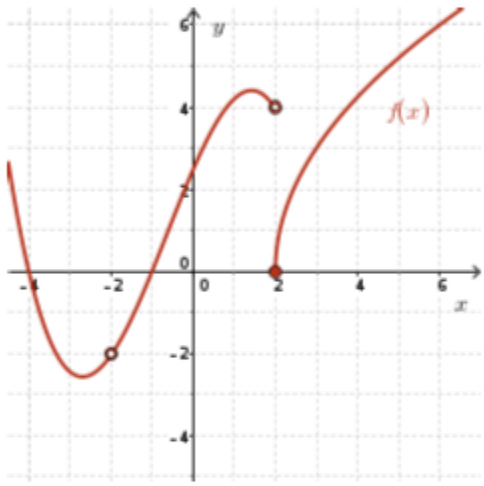
b $\lim_{x\to 2} f(x-1) =$ dne

c) $\lim_{x\to 3} f(2x-5) =$ dne

d) $\lim_{x\to 1} f(2x) =$ 1

Practice Questions

1. Given the graph of f(x), evaluate the following expressions involving f(x):



a) $\lim_{x\to -1} f(x) = 0$ b) $\lim_{x\to -2} f(x) = -2$ c) $\lim_{x\to 2^+} f(x) = 0$

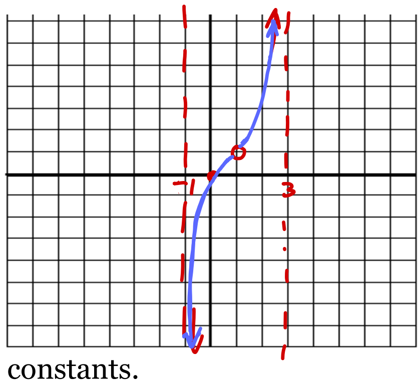
d) $\lim_{x\to -2^-} f(x) = -2$ e) $f(-2) =$ dne f) $\lim_{x\to 2} f(x) =$ dne

g) $\lim_{x\to -2^+} f(x) = -2$ h) $\lim_{x\to 2^-} f(x) = 4$ i) $f(2) = 0$

2. Sketch the graph of a function that has the following characteristics:

< Answers may vary >

- $\lim_{x\to 3^-} f(x) \to \infty$
- $\lim_{x\to -1^+} f(x) \to -\infty$
- $\lim_{x\to 1} f(x) = 1$
- $f(0) = 0$



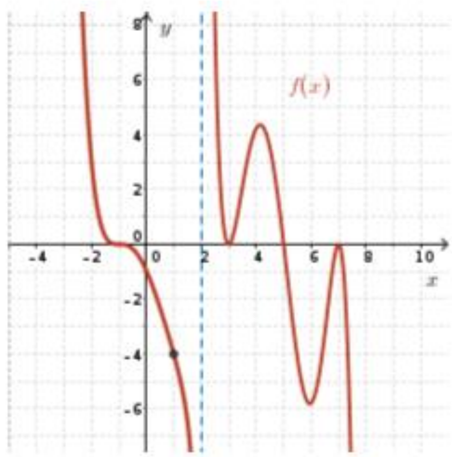
3. Consider the following piecewise function $f(x)$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are constants.

$$f(x) = \begin{cases} x + Ax^2 + 1 & \text{if } x < 1 \\ \dfrac{B}{x^2} + \dfrac{x}{A} & \text{if } 1 \le x < 2 \\ B + \dfrac{1}{2} & \text{if } x \ge 2 \end{cases}$$

Determine all values of the constants $\boldsymbol{A}$ and $\boldsymbol{B}$ so that $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} f(x)$ both exist.

< see next page >

4. Evaluate the following limits, given the graph



a) $\lim_{x\to 1} f(x) = -4$ b) $\lim_{x\to 3^-} f(x) = 0$

c) $\lim_{x\to 3^+} f(x) = 0$ d) $\lim_{x\to 2^-} f(x) = -\infty$ (dne)

e) $\lim_{x\to 2^+} f(x) = +\infty$ (dne) f) $\lim_{x\to 2} f(x) =$ dne

3. Consider the following piecewise function $f(x)$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are constants.

$$f(x)=\begin{cases} x+Ax^2+1 & \text{if } x<1 \\ \frac{B}{x^2}+\frac{x}{A} & \text{if } 1\le x<2 \\ B+\frac{1}{2} & \text{if } x\ge 2 \end{cases}$$

Determine all values of the constants $\boldsymbol{A}$ and $\boldsymbol{B}$ so that $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} f(x)$ both exist.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

$$1+A(1)^2+1 = \frac{B}{(1)^2}+\frac{(1)}{A}$$

$$A+2 = B+\frac{1}{A}$$

$$A+2-\frac{1}{A} = B$$

$$\frac{A^2+2A-1}{A} = B \quad (1)$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$\frac{B}{(2)^2}+\frac{(2)}{A} = B+\frac{1}{2}$$

$$\frac{B}{4}+\frac{2}{A} = B+\frac{1}{2}$$

$$\frac{2}{A}-\frac{1}{2} = B-\frac{B}{4}$$

$$\frac{4-A}{2A} = \frac{3}{4}B$$

$$\frac{\cancel{4}^{2}}{3}\left(\frac{4-A}{\cancel{2}_{1}A}\right) = B$$

$$\frac{8-2A}{3A} = B \quad (2)$$

$$(1) = (2)$$

$$\frac{A^2+2A-1}{\cancel{A}} = \frac{8-2A}{3\cancel{A}}$$

$$3(A^2+2A-1) = 8-2A$$

$$3A^2+6A-3 = 8-2A$$

$$3A^2+8A-11 = 0$$

$$(3A+11)(A-1) = 0$$

$$\therefore A = \left\{\frac{-11}{3}, 1\right\}$$

$$A_1 = \frac{-11}{3}$$

$$B_1 = \frac{8-2\left(\frac{-11}{3}\right)}{3\left(\frac{-11}{3}\right)}$$

$$= \left(8+\frac{22}{3}\right) \div -11$$

$$= \frac{46}{3}\cdot\frac{-1}{11}$$

$$= \frac{-46}{33}$$

$$A_2 = 1$$

$$B_2 = \frac{8-2(1)}{3(1)}$$

$$= \frac{6}{3}$$

$$= 2$$

$\therefore$ there are 2 possible solutions

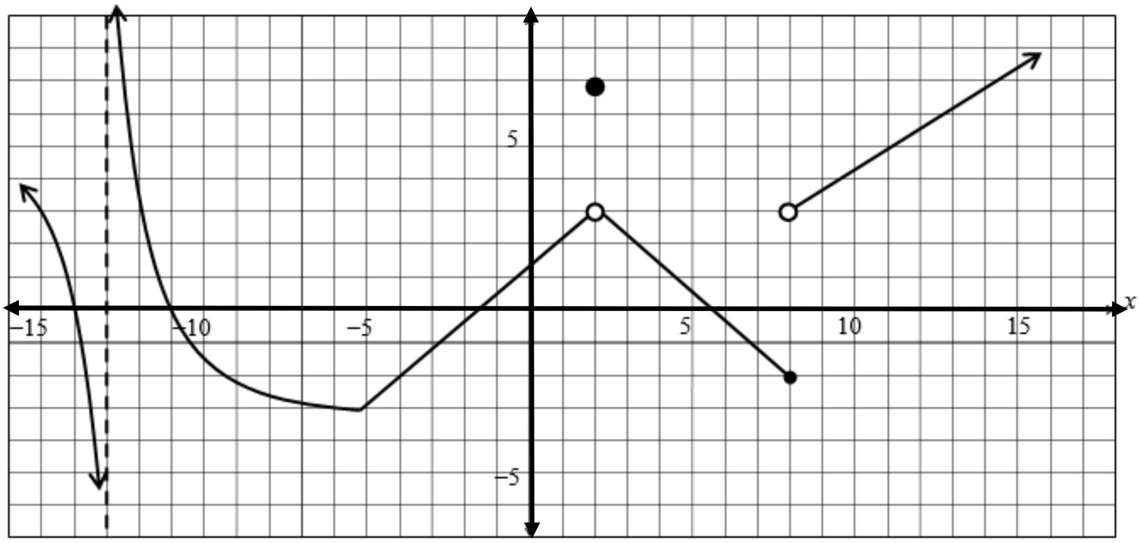
$$A = \frac{-11}{3} \qquad B = \frac{-46}{33}$$

or

$$A = 1 \qquad B = 2$$

WARM UP – LIMITS

1. Consider the following graph of the function and evaluate the following limits.



a) $\lim_{x\to 2} f(x) =$ 3

b) $f(2) =$ 7

c) $f(-5) =$ −3

d) $\lim_{x\to 8^-} f(x) =$ −2

e) $\lim_{x\to 8^+} f(x) =$ 3

f) $\lim_{x\to 8} f(x) =$ dne $\lim_{x\to 8^-} f(x) \neq \lim_{x\to 8^+} f(x)$

g) $\lim_{x\to -13^-} f(x) =$ $-\infty$ (dne)

h) $\lim_{x\to -13^+} f(x) =$ $+\infty$ (dne)

i) $f(-13) =$ undefined (Vertical Asymptote)

2. Given the function $f(x)=\begin{cases} 2x & x\in(-\infty,-1] \\ x^2 & x\in(-1,2) \\ 0.5x+3 & x\in[2,\infty) \end{cases}$ .Determine each limit, if it exists.

i) $\lim_{x\to 2^-} f(x) =$ 4

ii) $\lim_{x\to 2^+} f(x) =$ 4

iii) $\lim_{x\to 2} f(x) =$ 4

iv) $\lim_{x\to -1^-} f(x) =$ −2

v) $\lim_{x\to -1^+} f(x) =$ 1

vi) $\lim_{x\to -1} f(x) =$ dne

3. Find all values of a and b such that for the function $f(x)=\begin{cases} ax-2 & x<-1 \\ x^2-bx+a & -1\le x<3 \\ 4 & x\ge 3 \end{cases}$,

$\lim_{x\to 3} f(x)$ and $\lim_{x\to -1} f(x)$ both exist.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$(3)^2 - b(3) + a = 4$

$a - 3b = -5$ ①

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$a(-1) - 2 = (-1)^2 - b(-1) + a$

$-a - 2 = 1 + b + a$

$-3 = 2a + b$ ②

$\begin{cases} a - 3b = -5 & ① \\ 2a + b = -3 & ② \end{cases}$

$\overset{\times 2}{\Rightarrow}$ $2a - 6b = -10$

$\Rightarrow$ $2a + b = -3$

$(-)$ $-7b = -7$

$b = 1$ ③

sub ③ into ①

$a - 3(1) = -5$

$a = -2$

$\therefore a = -2$

$b = 1$

limit at infinity vs Infinite Limit

$\lim_{x\to\pm\infty} f(x)$ (Behaviour near HA)

$\lim_{x\to c} f(x) = \pm\infty$ (Behaviour near VA)

1. 2 Infinite Limits and Properties of Limits

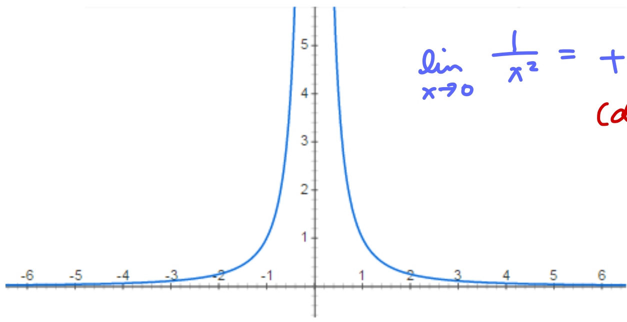
Infinite Limits

We say that

$$\lim_{x\to a} f(x) = \infty$$

vertical asymptote $x=0$

if we can make f(x) arbitrarily large for all x sufficiently close to x=a, from both sides, without actually letting x= a .



$\lim_{x\to 0} \frac{1}{x^2} = +\infty$ (dne)

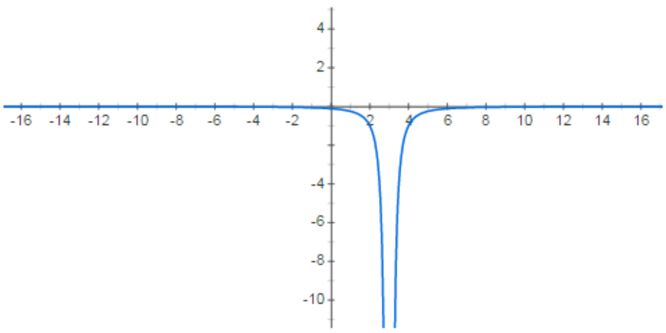
We say that

$$\lim_{x\to a} f(x) = -\infty$$

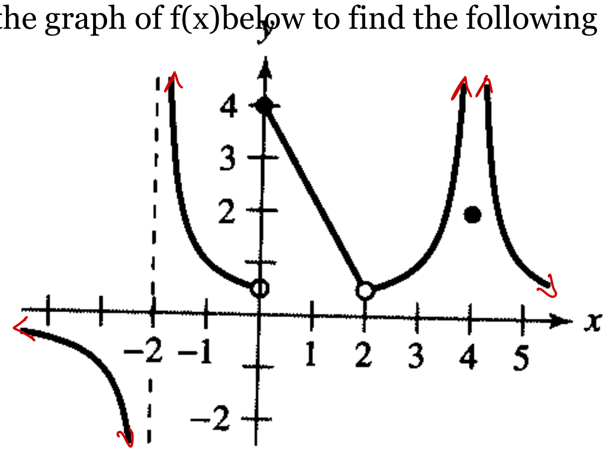
vertical asymptote $x=3$

$\lim_{x\to 3} \frac{-1}{(x-3)^2} = -\infty$ (dne)

if we can make f(x) arbitrarily large and negative for all x sufficiently close to x=a, from both sides, without actually letting x= a .



<u>EXAMPLE# 1</u>: Use the graph of f(x)below to find the following



a) $f(4) =$ 2 b) $f(2) =$ undefined c) $f(-2) =$ undefined

d) $\lim_{x\to -2^+} f(x) =$ $+\infty$ (dne) e) $\lim_{x\to -2^-} f(x) =$ $-\infty$ (dne) f) $\lim_{x\to -2} f(x) =$ dne

g) $\lim_{x\to 2^+} f(x) =$ $\frac{1}{2}$ h) $\lim_{x\to 2^-} f(x) =$ $\frac{1}{2}$ i) $\lim_{x\to 2} f(x) =$ $\frac{1}{2}$

j) $\lim_{x\to 4^+} f(x) =$ $+\infty$ (dne) k) $\lim_{x\to 4^-} f(x) =$ $+\infty$ (dne) l) $\lim_{x\to 4} f(x) =$ $+\infty$ (dne)

We can determine a limit intuitively, but we can also use properties of limits to evaluate limits.

Properties of limits

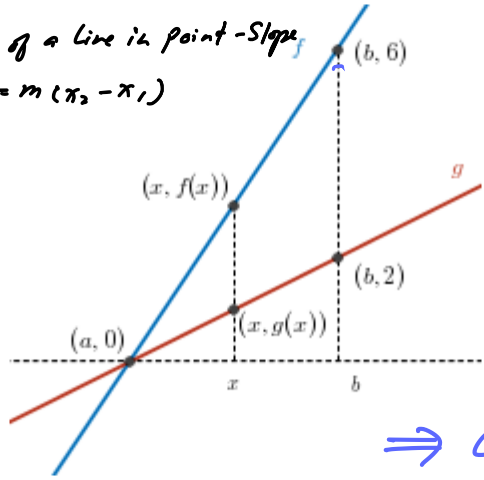
For any real numbers a ,c ^n and k , suppose $f(x)$ and $g(x)$ both have limits at $x=a$.

1. $\lim_{x\to a} k = k$ Ex. $\lim_{x\to 3} 5 = 5$
2. $\lim_{x\to a} x = a$ $\lim_{x\to 3} x = 3$
3. $\lim_{x\to a}\left[f(x)\pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$ $\lim_{x\to 1}\left[(x^2)+(x-1)\right] = \lim_{x\to 1}(x^2) + \lim_{x\to 1}(x-1)$
4. $\lim_{x\to a}\left[cf(x)\right] = c\lim_{x\to a} f(x)$ $\lim_{x\to 3} 5x^2 = 5\lim_{x\to 3} x^2$
5. $\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$ $\lim_{x\to 3}\left[x^2(x-1)\right] = \lim_{x\to 3} x^2 \cdot \lim_{x\to 3}(x-1)$
6. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ $\lim_{x\to 3}\left[\frac{x^2}{x-1}\right] = \frac{\lim_{x\to 3} x^2}{\lim_{x\to 3}(x-1)}$
7. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$ $\lim_{x\to 3}(x-1)^2 = \left[\lim_{x\to 3}(x-1)\right]^2$

Example #2 Let $a<b$ be real numbers. Consider two linear functions as shown in the graph. Evaluate $\lim_{x\to a}\frac{f(x)}{g(x)}$.

Equation of a line in point-Slope

$y_2 - y_1 = m(x_2 - x_1)$



f: (b, 6) and (a, 0) $m_f = \frac{0-6}{a-b}$

$$y - 0 = \frac{-6}{a-b}(x-a)$$

$$f(x) = \frac{-6(x-a)}{a-b}$$

g: (b, 2) and (a, 0) $m_g = \frac{0-2}{a-b}$

$$y - 0 = \frac{-2}{a-b}(x-a)$$

$$g(x) = \frac{-2(x-a)}{a-b}$$

⇒ Continued next page

Example #3 If $\lim_{x\to -3} f(x) = 4$, use properties of limit to determine $\lim_{x\to -3}\frac{x\sqrt{f(x)}}{x^2+f(x)}$.

$$\lim_{x\to -3}\frac{x\sqrt{f(x)}}{x^2+f(x)} = \frac{\lim_{x\to -3} x \cdot \left[\lim_{x\to -3}\left[f(x)\right]\right]^{\frac{1}{2}}}{\left(\lim_{x\to -3} x\right)^2 + \lim_{x\to -3} f(x)}$$

$$= \frac{-3 \cdot (4)^{\frac{1}{2}}}{(-3)^2 + (4)}$$

$$= \frac{-3 \cdot 2}{9+4}$$

$$= \frac{-6}{13}$$

Point Slope form $y_2 - y_1 = m(x_2 - x_1)$

Given $m = \frac{1}{6}$ $(3, 5)$ Equation

$y - 5 = \frac{1}{6}(x - 3)$ $\Leftarrow$ Point Slope Form

$y = \frac{1}{6}(x - 3) + 5$

$= \frac{1}{6}x - \frac{1}{2} + 5$

$y = \frac{1}{6}x + \frac{9}{2}$ $\Leftarrow$ Slope yint Form

In grade 9:

$y = \frac{1}{6}x + b$

sub(3,5)

$5 = \frac{1}{6}(3) + b$

$5 = \frac{1}{2} + b$

$10 = 1 + 2b$

$9 = 2b$

$b = \frac{9}{2}$

$\therefore y = \frac{1}{6}x + \frac{9}{2}$

Note that the equation of a line in point-slope form is not unique

Ex Find the equation of a line passing through $A(-2,5)$ and $B(4,-7)$

$m_{AB} = \frac{-7-5}{4-(-2)}$

$= \frac{-12}{6}$

$= -2$

Equation in point slope form could be:

$y - 5 = -2(x+2)$ or $y + 7 = -2(x-4)$

But if we express in $y = mx + b$, they are unique:

$y = -2(x+2) + 5$

$y = -2x - 4 + 5$

$y = -2x + 1$

$\Leftrightarrow$

$y = -2(x-4) - 7$

$y = -2x + 8 - 7$

$y = -2x + 1$

$\Rightarrow$ Example 2 (Continued)

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

$$= \lim_{x \to a} \left[\frac{\frac{-6\cancel{(x-a)}}{\cancel{a-b}}}{\frac{-2\cancel{(x-a)}}{\cancel{a-b}}} \right]$$

$$= \lim_{x \to a} 3$$

$$= 3$$

To evaluate a limit algebraically, we can use the following methods:

- **direct substitution**
- **factoring**
- **rationalizing**
- **one-sided limits**
- **change of variable**

Used when direct substitution results in $\frac{0}{0} \Rightarrow$ indeterminant form

METHOD 1: Direct Substitution

Example: Evaluate the following limits.

a) $\lim_{x\to 2}\left(x^2-4x+1\right)$

$= (2)^2 - 4(2) + 1$

$= -3$

b) $\lim_{x\to 3}\frac{x-2}{x+2}$

$= \frac{(3)-2}{(3)+2}$

$= \frac{1}{5}$

c) $\lim_{x\to \frac{\pi}{2}}\left(\sin x - \cos 2x\right)$

$= \sin\frac{\pi}{2} - \cos\pi$

$= 1 - (-1)$

$= 2$

d) $\lim_{x\to 8}\frac{\log_x 512 + \sqrt{1+x}}{x^2 + x^{\frac{1}{3}} - 8}$

$= \frac{(3) + \sqrt{9}}{64 + 2 - 8}$

$= \frac{6}{58}$ $\rightarrow$ $= \frac{3}{29}$

e) $\lim_{x\to 2}\frac{3x^2}{x-2}$ ← VA @ $x=2$

$= \frac{3(2)^2}{2-2}$

$= \frac{12}{0}$

$=$ dne

Check:

$f(1.999) = -\infty$

$\lim_{x\to 2^-} f(x) = -\infty$

$\lim_{x\to 2^+} f(x) = +\infty$

$f(2.001) = +\infty$

f) $\lim_{x\to 3}\sqrt{\frac{x-2}{x^2}}$

$= \sqrt{\frac{3-2}{3^2}}$

$= \frac{1}{3}$

METHOD 2: Factoring

This method is used on questions where direct substitution yields $\frac{\mathbf{0}}{\mathbf{0}}$, which is referred to as an '**indeterminate form**.' Whenever this happens, simplify the expression by factoring and reducing, expanding and simplifying, or by finding a common denominator.

Sum / difference of cubes

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$a^3 + b^3 = (a+b)(a^2-ab+b^2)$

Example#4: Evaluate the following limits.

a) $\lim_{x\to 3}\frac{x-3}{x^2-4x+3}$ "$\frac{0}{0}$"

$= \lim_{x\to 3}\frac{\cancel{x-3}}{\cancel{(x-3)}(x-1)}$

$= \lim_{x\to 3}\frac{1}{x-1}$

$= \frac{1}{2}$

b) $\lim_{x\to 1}\frac{x^3-1}{x^2-1}$

$= \lim_{x\to 1}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)}$

$= \lim_{x\to 1}\frac{x^2+x+1}{x+1}$

$= \frac{(1)^2+(1)+1}{(1)+1}$ $\rightarrow$ $= \frac{3}{2}$

c) $\lim_{n\to 0}\frac{3^{2n}-3^n}{1-3^n}$

$= \lim_{n\to 0}\frac{(3^n)^2 - 3^n}{1-3^n}$

$= \lim_{n\to 0}\frac{3^n\cancel{(3^n-1)}}{-\cancel{(3^n-1)}}$

$= \lim_{n\to 0} -3^n$ $\rightarrow$ $= -3^0$

$= -1$

d) $\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

$= \lim_{x\to 2}\frac{\left(\frac{2-x}{2x}\right)}{x-2}$ ← $-(x-2)$

$= \lim_{x\to 2}\frac{-1}{2x}$

$= -\frac{1}{4}$

e) $\lim_{x\to 2}\left(\frac{1}{4x-8}-\frac{1}{x^2-4}\right)$

$= \lim_{x\to 2}\left[\frac{1}{4(x-2)} - \frac{1}{(x+2)(x-2)}\right]$

$= \lim_{x\to 2}\frac{(x+2)-4}{4(x-2)(x+2)}$

$= \lim_{x\to 2}\frac{\cancel{x-2}}{4\cancel{(x-2)}(x+2)}$

$= \lim_{x\to 2}\frac{1}{4(x+2)}$ $\rightarrow$ $= \frac{1}{4(2+2)}$

$= \frac{1}{16}$

f) $\lim_{x\to -3}\frac{x^3-7x+6}{x^2+2x-3}$

$= \lim_{x\to -3}\frac{\cancel{(x+3)}(x-2)(x-1)}{\cancel{(x+3)}(x-1)}$

$= \lim_{x\to -3}(x-2)$

$= -3-2$

$= -5$

$f(x) = x^3 - 7x + 6$ ↳ $\pm1 \pm2 \pm3$ ±6

$f(-3) = 0$, $\therefore (x+3)$ is a factor

-3	1	0	-7	6
	↓	-3	9	-6
	1	-3	2	0 R

$f(x) = (x+3)(x^2-3x+2)$

$= (x+3)(x-2)(x-1)$

$x+3 \,\overline{)\, x^3 - 0x^2 - 7x + 6}$ quotient $x^2 - 3x + 2$

$x^3 + 3x^2$

$-3x^2 - 7x + 6$

$-3x^2 - 9x$

$2x + 6$

$2x + 6$

0 R

$\therefore f(x) = (x+3)(x^2-3x+2)$

$= (x+3)(x-2)(x-1)$

METHOD 3: Rationalizing

When the expression we are trying to find the limit of is a fraction involving a square root, it sometimes works to rationalize.

a) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{x} \cdot \dfrac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$\dfrac{'0'}{'0'}$

$= \lim\limits_{x\to 0} \dfrac{(\cancel{1}+\cancel{x})-\cancel{1}}{\cancel{x}(\sqrt{1+x}+1)}$

$= \lim\limits_{x\to 0} \dfrac{1}{\sqrt{1+x}+1}$

$= \dfrac{1}{2}$

b) $\lim\limits_{x\to 2} \dfrac{4-x^2}{3-\sqrt{x^2+5}} \cdot \dfrac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}}$

$= \lim\limits_{x\to 2} \dfrac{-(x+2)(x-2)\left[3+\sqrt{x^2+5}\right]}{9-(x^2+5)}$

$= \lim\limits_{x\to 2} \dfrac{-(\cancel{x+2})(\cancel{x-2})\left[3+\sqrt{x^2+5}\right]}{-(\cancel{x+2})(\cancel{x-2})}$

$= \lim\limits_{x\to 2} 3+\sqrt{x^2+5}$

$= 3+\sqrt{9}$

$= 6$

Aside:

$9-(x^2+5)$

$= -x^2+4$

$= -(x^2-4)$

$= -(x+2)(x-2)$

c) $\lim\limits_{x\to 3} \dfrac{3-x}{\sqrt{13+x}-\sqrt{7+x^2}} \cdot \dfrac{\sqrt{13+x}+\sqrt{7+x^2}}{\sqrt{13+x}+\sqrt{7+x^2}}$

$= \lim\limits_{x\to 3} \dfrac{-(x-3)\left[\sqrt{13+x}+\sqrt{7+x^2}\right]}{(13+x)-(7+x^2)}$

$= \lim\limits_{x\to 3} \dfrac{-(x-3)\left[\sqrt{13+x}+\sqrt{7+x^2}\right]}{-x^2+x+6}$

$= \lim\limits_{x\to 3} \dfrac{-(\cancel{x-3})\left[\sqrt{13+x}+\sqrt{7+x^2}\right]}{-(\cancel{x-3})(x+2)}$

$= \lim\limits_{x\to 3} \dfrac{\sqrt{13+x}+\sqrt{7+x^2}}{(x+2)}$

$= \dfrac{\sqrt{16}+\sqrt{16}}{5}$

$= \dfrac{8}{5}$

Aside:

$-x^2+x+6$

$= -(x^2-x-6)$

$= -(x-3)(x+2)$

double rationalizing

d) $\lim\limits_{x\to 2} \dfrac{\sqrt{x+2}-2}{\sqrt{7+x}-3} \cdot \dfrac{\sqrt{x+2}+2}{\sqrt{x+2}+2} \cdot \dfrac{\sqrt{7+x}+3}{\sqrt{7+x}+3}$

$= \lim\limits_{x\to 2} \dfrac{\left[(x+2)-4\right]\left[\sqrt{7+x}+3\right]}{\left[(7+x)-9\right]\left[\sqrt{x+2}+2\right]}$

$= \lim\limits_{x\to 2} \dfrac{(\cancel{x-2})\left[\sqrt{7+x}+3\right]}{(\cancel{x-2})(\sqrt{x+2}+2)}$

$= \lim\limits_{x\to 2} \dfrac{\sqrt{7+x}+3}{\sqrt{x+2}+2}$

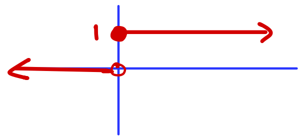
$= \dfrac{\sqrt{9}+3}{\sqrt{4}+2}$

$= \dfrac{6}{4}$

$= \dfrac{3}{2}$

METHOD 4: One-sided Limits

a) Consider the Heaviside[1] function $H(t)=\begin{cases}0 & , \ t<0\\ 1 & , \ t\geq 0\end{cases}$



Evaluate:

i) $\lim\limits_{t\to 0^-} H(t)$

$= 0$

ii) $\lim\limits_{t\to 0^+} H(t)$

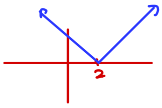
$= 1$

iii) $\lim\limits_{t\to 0} H(t)$

$= dne$

$\because \lim\limits_{x\to 0^-} H(t) \neq \lim\limits_{x\to 0^+} H(t)$

$\therefore \lim\limits_{x\to 0} H(t) = dne$



b) $\lim\limits_{x\to 2} \dfrac{x^2+|x-2|-4}{|x-2|}$

$|x-2| = \begin{cases} -(x-2), & x<2 \\ (x-2), & x\geq 2\end{cases}$

$= \lim\limits_{x\to 2} f(x)$

$f(x) = \begin{cases} \dfrac{x^2-(x-2)-4}{-(x-2)}, & x<2 \\ \dfrac{x^2+(x-2)-4}{(x-2)}, & x>2 \end{cases}$

$f(x) = \begin{cases} \dfrac{x^2-x-2}{-(x-2)}, & x<2 \\ \dfrac{x^2+x-6}{x-2}, & x>2 \end{cases}$

$= \begin{cases} \dfrac{(x+1)\cancel{(x-2)}}{-\cancel{(x-2)}}, & x<2 \\ \dfrac{(x+3)\cancel{(x-2)}}{\cancel{x-2}}, & x>2 \end{cases}$

$\therefore \lim\limits_{x\to 2} \begin{cases} -(x+1), & x<2 \\ x+3, & x>2 \end{cases}$

$\lim\limits_{x\to 2^-} -(x+1)$

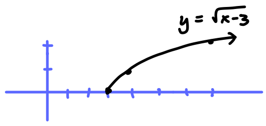
$= -3$

$\lim\limits_{x\to 2^+} (x+3)$

$= 5$

$\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$

$\therefore \lim\limits_{x\to 2} f(x) = dne$



c) $\lim\limits_{x\to 3} \sqrt{x-3}$

$= \sqrt{3-3}$

$= \sqrt{0}$

$= 0 \Leftarrow$ Be careful!

$\lim\limits_{x\to 3^-} \sqrt{x-3}$

$f(2.999) = DNE$

$= dne$

$\lim\limits_{x\to 3^+} \sqrt{x-3}$

$f(3.001) = 0$

$= 0$

$\lim\limits_{x\to 3^-} f(x) \neq \lim\limits_{x\to 3^+} f(x)$

$\therefore \lim\limits_{x\to 3} f(x) = dne$

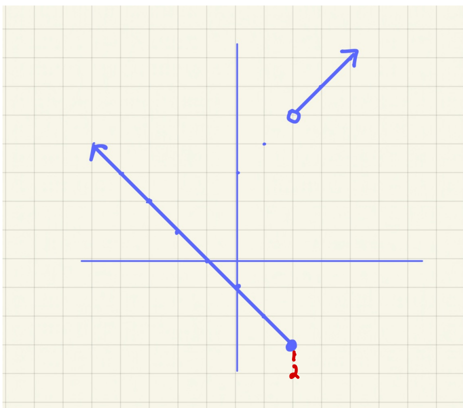
When $\frac{0}{\#}$ or $\sqrt{0}$

check left-sided and right-sided limit (or the function's domain) to see if they coincide.

$\frac{0}{0} \Rightarrow$ indeterminant

$\therefore$ must manipulate to cancel a factor out

- when in doubt sub values close to the left and right side of the critical value to check.



0

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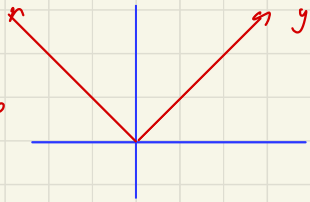
[1] This function is named after the electrical engineer Oliver Heaviside (1850-1925) and can be used to describe an electric current that is turned on at time t = 0.

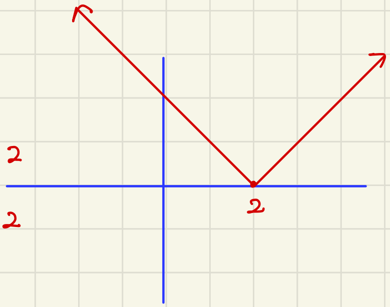
$y = |x|$

$$y = \begin{cases} -x & , x \le 0 \\ x & , x > 0 \end{cases}$$



$y = |x-2|$

$$= \begin{cases} -(x-2) & x \le 2 \\ x-2 & x > 2 \end{cases}$$



METHOD 5: Change of Variable

Evaluate the following limits, if they exist. Show your work.

a) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt[3]{x+1}-1}$ $= \lim\limits_{a\to 1}\dfrac{a^3-1}{a-1}$

$= \lim\limits_{a\to 1}\dfrac{(\cancel{a-1})(a^2+a+1)}{(\cancel{a-1})}$

$= \lim\limits_{a\to 1}(a^2+a+1)$

$= 1^2+1+1$

$= 3$

let $a=(x+1)^{\frac{1}{3}}$

$a^3-1=x$

$x\to 0$

$a^3-1\to 0$

$\therefore a\to\sqrt[3]{1}$

$a\to 1$

b) $\lim\limits_{x\to 1}\dfrac{2x+3\sqrt{x}-5}{\sqrt{x}-1}$ $= \lim\limits_{a\to 1}\dfrac{2a^2+3a-5}{a-1}$

$= \lim\limits_{a\to 1}\dfrac{(2a+5)(\cancel{a-1})}{(\cancel{a-1})}$

$= \lim\limits_{a\to 1}(2a+5)$

$= 2(1)+5$

$= 7$

let $a=\sqrt{x}$

$a^2=x$

$x\to 1$

$a^2\to 1$

$a\to 1$

method 2: Courtesy of Shalom

$\lim\limits_{x\to 1}\dfrac{(2\sqrt{x}+5)(\cancel{\sqrt{x}-1})}{\cancel{\sqrt{x}-1}}$ $\Rightarrow$ if you can see that the numerator is a "factorable" trinomial.

$= \lim\limits_{x\to 1} 2\sqrt{x}+5$

$= 2\sqrt{1}+5$

$= 7$

c) $\lim\limits_{n\to 1}\dfrac{1-n}{2-\sqrt[3]{7+n}}$ $= \lim\limits_{a\to 2}\dfrac{1-(a^3-7)}{2-a}$

$= \lim\limits_{a\to 2}\dfrac{-(a^3-8)}{-(a-2)}$

$= \lim\limits_{a\to 2}\dfrac{-\cancel{(a-2)}(a^2+2a+4)}{-\cancel{(a-2)}}$

$= \lim\limits_{a\to 2} a^2+2a+4$

$= 2^2+2(2)+4$

$= 12$

let $a=(7+n)^{\frac{1}{3}}$

$a^3-7=n$

$n\to 1$

$a^3-7\to 1$

$a^3\to 8$

$a\to 2$

* d) $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ $= \lim\limits_{a\to 1}\dfrac{a^3-1}{a^2-1}$

$= \lim\limits_{a\to 1}\dfrac{(\cancel{a-1})(a^2+a+1)}{(a+1)(\cancel{a-1})}$

$= \lim\limits_{a\to 1}\dfrac{a^2+a+1}{a+1}$

$= \dfrac{1^2+1+1}{1+1}$

$= \dfrac{3}{2}$

let $a=x^{\frac{1}{6}}$

$a^6=x$

$a^2=x^{\frac{1}{3}}$

$a^3=x^{\frac{1}{2}}$

$x\to 1$

$a^6\to 1$

$a\to 1$

e) $\lim\limits_{x\to 0}\dfrac{(x+8)^{\frac{1}{3}}-2}{x}$ $= \lim\limits_{a\to 2}\dfrac{a-2}{a^3-8}$

$= \lim\limits_{a\to 2}\dfrac{\cancel{a-2}}{(\cancel{a-2})(a^2+2a+4)}$

$= \lim\limits_{a\to 2}\dfrac{1}{a^2+2a+4}$

$= \dfrac{1}{2^2+2(2)+4}$

$= \dfrac{1}{12}$

let $a=(x+8)^{\frac{1}{3}}$

$a^3=x+8$

$x=a^3-8$

$x\to 0$

$a^3-8\to 0$

$a\to 2$

Practice

PART A

1. Are there different answers for $\lim_{x \to 2} (3 + x)$, $\lim_{x \to 2} 3 + x$, and $\lim_{x \to 2} (x + 3)$?
2. How do you find the limit of a rational function?
3. **C** Once you know $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$, do you then know $\lim_{x \to a} f(x)$? Give reasons for your answer.
4. Evaluate each limit.

a. $\lim_{x \to 2} \frac{3x}{x^2 + 2}$

b. $\lim_{x \to -1} (x^4 + x^3 + x^2)$

c. $\lim_{x \to 9} \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

d. $\lim_{x \to 2\pi} (x^3 + \pi^2 x - 5\pi^3)$

e. $\lim_{x \to 0} (\sqrt{3 + \sqrt{1 + x}})$

f. $\lim_{x \to -3} \sqrt{\frac{x - 3}{2x + 4}}$

PART B

5. Use a graphing calculator to graph each function and estimate the limit. Then find the limit by substitution.

a. $\lim_{x \to -2} \frac{x^3}{x - 2}$

b. $\lim_{x \to 1} \frac{2x}{\sqrt{x^2 + 1}}$

6. Show that $\lim_{t \to 1} \frac{t^3 - t^2 - 5t}{6 - t^2} = -1$.

7. **K** Evaluate the limit of each indeterminate quotient.

a. $\lim_{x \to 2} \frac{4 - x^2}{2 - x}$

b. $\lim_{x \to -1} \frac{2x^2 + 5x + 3}{x + 1}$

c. $\lim_{x \to 3} \frac{x^3 - 27}{x - 3}$

d. $\lim_{x \to 0} \frac{2 - \sqrt{4 + x}}{x}$

e. $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

f. $\lim_{x \to 0} \frac{\sqrt{7 - x} - \sqrt{7 + x}}{x}$

8. Evaluate the limit by using a change of variable.

a. $\lim_{x \to 8} \frac{\sqrt[3]{x} - 2}{x - 8}$

b. $\lim_{x \to 27} \frac{27 - x}{x^{\frac{1}{3}} - 3}$

c. $\lim_{x \to 1} \frac{x^{\frac{1}{6}} - 1}{x - 1}$

d. $\lim_{x \to 1} \frac{x^{\frac{1}{6}} - 1}{x^{\frac{1}{3}} - 1}$

e. $\lim_{x \to 4} \frac{\sqrt{x} - 2}{\sqrt{x^3} - 8}$

f. $\lim_{x \to 0} \frac{(x + 8)^{\frac{1}{3}} - 2}{x}$

9. Evaluate each limit, if it exists, using any appropriate technique.

a. $\lim_{x \to 4} \frac{16 - x^2}{x^3 + 64}$

b. $\lim_{x \to 4} \frac{x^2 - 16}{x^2 - 5x + 6}$

c. $\lim_{x \to -1} \frac{x^2 + x}{x + 1}$

d. $\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$

e. $\lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}$

f. $\lim_{x \to 1} \left[\left(\frac{1}{x - 1}\right)\left(\frac{1}{x + 3} - \frac{2}{3x + 5}\right)\right]$

10. By using one-sided limits, determine whether each limit exists. Illustrate your results geometrically by sketching the graph of the function.

a. $\lim_{x \to 5} \frac{|x - 5|}{x - 5}$

b. $\lim_{x \to \frac{5}{2}} \frac{|2x - 5|(x + 1)}{2x - 5}$

c. $\lim_{x \to 2} \frac{x^2 - x - 2}{|x - 2|}$

d. $\lim_{x \to -2} \frac{(x + 2)^3}{|x + 2|}$

A 11. Jacques Charles (1746–1823) discovered that the volume of a gas at a constant pressure increases linearly with the temperature of the gas. To obtain the data in the following table, one mole of hydrogen was held at a constant pressure of one atmosphere. The volume V was measured in litres, and the temperature T was measured in degrees Celsius.

T (°C)	−40	−20	0	20	40	60	80
V (L)	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

a. Calculate first differences, and show that T and V are related by a linear relation.

b. Find the linear equation for V in terms of T.

c. Solve for T in terms of V for the equation in part b.

d. Show that $\lim_{V\to 0^+} T$ is approximately -273.15. *Note:* This represents the approximate number of degrees on the Celsius scale for absolute zero on the Kelvin scale (0 K).

e. Using the information you found in parts b and d, draw a graph of V versus T.

T 12. Show, using the properties of limits, that if $\lim_{x\to 5} f(x) = 3$, then $\lim_{x\to 5} \frac{x^2-4}{f(x)} = 7$.

13. If $\lim_{x\to 4} f(x) = 3$, use the properties of limits to evaluate each limit.

a. $\lim_{x\to 4} [f(x)]^3$ b. $\lim_{x\to 4} \frac{[f(x)]^2 - x^2}{f(x) + x}$ c. $\lim_{x\to 4} \sqrt{3f(x) - 2x}$

PART C

14. If $\lim_{x\to 0} \frac{f(x)}{x} = 1$ and $\lim_{x\to 0} g(x)$ exists and is nonzero, then evaluate each limit.

a. $\lim_{x\to 0} f(x)$ b. $\lim_{x\to 0} \frac{f(x)}{g(x)}$

15. If $\lim_{x\to 0} \frac{f(x)}{x} = 1$ and $\lim_{x\to 0} \frac{g(x)}{x} = 2$, then evaluate each limit.

a. $\lim_{x\to 0} g(x)$ b. $\lim_{x\to 0} \frac{f(x)}{g(x)}$

16. Evaluate $\lim_{x\to 0} \frac{\sqrt{x+1} - \sqrt{2x+1}}{\sqrt{3x+4} - \sqrt{2x+4}}$.

17. Does $\lim_{x\to 1} \frac{x^2 + |x-1| - 1}{|x-1|}$ exist? Illustrate your answer by sketching a graph of the function.

18. Evaluate each limit using one of the algebraic methods discussed in this chapter, if the limit exists.

a. $\lim_{x\to -4} \frac{x^2 + 12x + 32}{x + 4}$ d. $\lim_{x\to 2} \frac{x^2 - 4}{x^3 - 8}$

b. $\lim_{x\to a} \frac{(x+4a)^2 - 25a^2}{x - a}$ e. $\lim_{x\to 4} \frac{4 - \sqrt{12 + x}}{x - 4}$

c. $\lim_{x\to 0} \frac{\sqrt{x+5} - \sqrt{5-x}}{x}$ f. $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{2+x} - \frac{1}{2}\right)$

19. Explain why the given limit does not exist.

a. $\lim_{x\to 3} \sqrt{x-3}$ d. $\lim_{x\to 2} \frac{1}{\sqrt{x-2}}$

b. $\lim_{x\to 2} \frac{x^2-4}{x^2 - 4x + 4}$ e. $\lim_{x\to 0} \frac{|x|}{x}$

c. $f(x) = \begin{cases} -5, \text{ if } x < 1 \\ 2, \text{ if } x \ge 1 \end{cases}$; $\lim_{x\to 1} f(x)$ f. $f(x) = \begin{cases} 5x^2, \text{ if } x < -1 \\ 2x + 1, \text{ if } x \ge -1 \end{cases}$; $\lim_{x\to -1} f(x)$

Hmwk Takeup P1

CP22

#16 $$\lim_{x\to 0}\frac{\sqrt{x+1}-\sqrt{2x+1}}{\sqrt{3x+4}-\sqrt{2x+4}}\cdot\frac{\sqrt{x+1}+\sqrt{2x+1}}{\sqrt{x+1}+\sqrt{2x+1}}\cdot\frac{\sqrt{3x+4}+\sqrt{2x+4}}{\sqrt{3x+4}+\sqrt{2x+4}}$$

$$= \lim_{x\to 0}\frac{[(x+1)-(2x+1)][\sqrt{3x+4}+\sqrt{2x+4}]}{(3x+4)-(2x+4)[\sqrt{x+1}+\sqrt{2x+1}]}$$

$$= \lim_{x\to 0}\frac{-\cancel{x}(\sqrt{3x+4}+\sqrt{2x+4})}{\cancel{x}(\sqrt{x+1}+\sqrt{2x+1})}$$

$$= -\frac{(\sqrt{4}+\sqrt{4})}{\sqrt{1}+\sqrt{1}}$$

$$= -\frac{4}{2}$$

$$= -2$$

#17 Does $\lim_{x\to 1}\frac{x^2+|x-1|-1}{|x-1|}$ exist? Illustrate your answer by sketching a graph of the function.

$$f(x) = \frac{x^2+|x-1|-1}{|x-1|}$$

$$= \begin{cases}\dfrac{x^2-(x-1)-1}{-(x-1)}, & x<1\\[2mm] \dfrac{x^2+(x-1)-1}{(x-1)}, & x>1\end{cases}$$

$$= \begin{cases}\dfrac{x^2-x}{-(x-1)}, & x<1\\[2mm] \dfrac{x^2+x-2}{x-1}, & x>1\end{cases}$$

$$= \begin{cases}\dfrac{x\cancel{(x-1)}}{-\cancel{(x-1)}}, & x<1\\[2mm] \dfrac{(x+2)\cancel{(x-1)}}{\cancel{x-1}}, & x>1\end{cases}$$

$$= \begin{cases}-x, & x<1\\ x+2, & x>1\end{cases}$$

$$\lim_{x\to 1^-} f(x) = -1$$

$$\lim_{x\to 1^+} f(x) = 3$$

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

$$\therefore \lim_{x\to 1} f(x) = \text{dne}$$

1.5 Properties of Limits, pp. 45–47

1. $\lim_{x\to 2}(3 + x)$ and $\lim_{x\to 2}(x + 3)$ have the same value, but $\lim_{x\to 2} 3 + x$ does not. Since there are no brackets around the expression, the limit only applies to 3, and there is no value for the last term, x.

2. Factor the numerator and denominator. Cancel any common factors. Substitute the given value of x.

3. If the two one-sided limits have the same value, then the value of the limit is equal to the value of the one-sided limits. If the one-sided limits do not have the same value, then the limit does not exist.

4. a. $\frac{3(2)}{2^2 + 2} = 1$

b. $(-1)^4 + (-1)^3 + (-1)^2 = 1$

c. $\left[\sqrt{9} + \frac{1}{\sqrt{9}}\right]^2 = \left(3 + \frac{1}{3}\right)^2$
$= \frac{100}{9}$

d. $(2\pi)^3 + \pi^2(2\pi) - 5\pi^3 = 8\pi^3 + 2\pi^3 - 5\pi^3$
$= 5\pi^3$

e. $\sqrt{3 + \sqrt{1 + 0}} = \sqrt{3 + 1}$
$= 2$

f. $\sqrt{\frac{-3 - 3}{2(-3) + 4}} = \sqrt{\frac{-6}{-2}}$
$= \sqrt{3}$

5. a. $\frac{(-2)^3}{-2 - 2} = +2$

b. $\frac{2}{\sqrt{1 + 1}} = \frac{2}{\sqrt{2}}$
$= \sqrt{2}$

6. Since substituting $t = 1$ does not make the denominator 0, direct substitution works.

$\frac{1 - 1 - 5}{6 - 1} = \frac{-5}{5}$
$= -1$

7. a. $\lim_{x\to 2}\frac{4 - x^2}{2 - x} = \lim_{x\to 2}\frac{(2 - x)(2 + x)}{(2 - x)}$
$= \lim_{x\to 2}(2 + x)$
$= 4$

b. $\lim_{x\to -1}\frac{2x^2 + 5x + 3}{x + 1} = \lim_{x\to -1}\frac{(x + 1)(2x + 3)}{x + 1}$ $= \lim_{x\to -1} 2x + 3$
$= \cancel{5}\ 1$

c. $\lim_{x\to 3}\frac{x^3 - 27}{x - 3} = \lim_{x\to 3}\frac{(x - 3)(x^2 + 3x + 9)}{x - 3}$
$= 9 + 9 + 9$
$= 27$

d. $\lim_{x\to 0}\left[\frac{2-\sqrt{4+x}}{x}\times\frac{2+\sqrt{4+x}}{2+\sqrt{4+x}}\right]$

$= \lim_{x\to 0}\frac{-1}{2+\sqrt{4+x}}$

$= -\frac{1}{4}$

e. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4}\frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)}$

$= \frac{1}{4}$

f. $\lim_{x\to 0}\left[\frac{\sqrt{7-x}-\sqrt{7+x}}{x}\times\frac{\sqrt{7-x}+\sqrt{7+x}}{\sqrt{7-x}+\sqrt{7+x}}\right]$

$= \lim_{x\to 0}\frac{7-x-7-x}{x(\sqrt{7-x}+\sqrt{7+x})}$

$= -\frac{1}{\sqrt{7}}$

8. a. $\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8}$

Let $u=\sqrt[3]{x}$. Therefore, $u^3=x$ as $x\to 8$, $u\to 2$.

Here, $\lim_{\cancel{x}\,u\to 2}\frac{u-2}{u^3-8} = \lim_{\cancel{x}\,u\to 2}\frac{1}{u^2+2u+4}$

$= \frac{1}{12}$

b. $\lim_{x\to 27}\frac{27-x}{x^{\frac{1}{3}}-3}$

$= \lim_{\cancel{x}\,u\to 3}\frac{u^3-27}{u-3}$

$= -\lim_{x\to 3}\frac{(u-3)(u^2+3u+9)}{u-3}$

$= -(9+9+9)$

$= -27$

Let $x^{\frac{1}{3}}=u$

$x=u^3$

$x\to 27$, $u\to 3$.

c. $\lim_{x\to 1}\frac{x^{\frac{1}{6}}-1}{x-1}$

$= \lim_{u\,\cancel{x}\to 1}\frac{u-1}{u^6-1}$

$= \lim_{u\,\cancel{x}\to 1}\frac{(u-1)}{(u-1)(u^5+u^4+u^3+u^2+u+1)}$

$= \frac{1}{6}$

$x^{\frac{1}{6}}=u$, $x=u^6$

$x\to 1$, $u\to 1$

d. $\lim_{x\to 1}\frac{x^{\frac{1}{6}}-1}{x^{\frac{1}{3}}-1}$

$= \lim_{x\to 1}\frac{u-1}{u^2-1}$

Let $x^{\frac{1}{6}}=u$

$u^6=x$

$x^{\frac{1}{3}}=u^2$

As $x\to 1$, $u\to 1$

$= \lim_{x\to 1}\frac{u-1}{(u-1)(u+1)}$

$= \frac{1}{2}$

e. $\lim_{x\to 4}\frac{\sqrt{x}-2}{\sqrt{x^3}-8}$

$= \lim_{u\to 2}\frac{u-2}{u^3-8}$

$= \lim_{u\to 2}\frac{u-2}{(u-2)(u^2+2u+4)}$ $= \lim_{u\to 2}\frac{1}{u^2+2u+4}$

$= \frac{1}{12}$

Let $x^{\frac{1}{2}}=u$

$x^{\frac{3}{2}}=u^3$

$x\to 4$, $u\to 2$

f. $\lim_{x\to 0}\frac{(x+8)^{\frac{1}{3}}-2}{x}$

$\lim_{x\to 2}\frac{u-2}{u^3-8}$

$= \frac{1}{12}$

Let $(x+8)^{\frac{1}{3}}=u$

$x+8=u^3$

$x=u^3-8$

$x\to 0$, $u\to 2$

9. a. $\frac{16-16}{64+64}=0$

b. $\frac{16-16}{16-20+6}=0$

c. $\lim_{x\to -1}\frac{x^2+x}{x+1} = \lim_{x\to -1}\frac{x(x+1)}{x+1}$

$= -1$

d. $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0}\frac{\sqrt{x+1}-1}{x+1-1}$

$= \lim_{x\to 0}\frac{\sqrt{x+1}-1}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}$

$= \frac{1}{2}$

e. $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h}$

$= 2x$

f. $\lim_{x\to 1}\left(\frac{1}{x-1}\right)\left(\frac{1}{x+3}-\frac{2}{3x+5}\right)$

$= \lim_{x\to 1}\left(\frac{1}{x-1}\right)\left(\frac{3x+5-2x-6}{(x+3)(3x+5)}\right)$

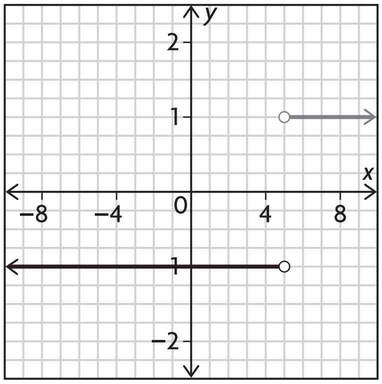
$= \lim_{x\to 1}\frac{1}{(x+3)(3x+5)}$

$= \frac{1}{4(8)}$

$= \frac{1}{32}$

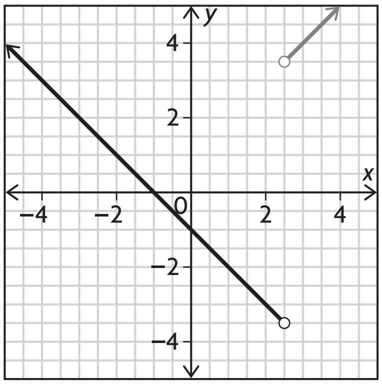
10. a. $\lim_{x\to 5}\frac{|x-5|}{x-5}$ does not exist.

$$\lim_{x\to 5^+}\frac{|x-5|}{x-5} = \lim_{x\to 5^+}\frac{x-5}{x-5}$$
$$= 1$$
$$\lim_{x\to 5^-}\frac{|x-5|}{x-5} = \lim_{x\to 5^-} -\left(\frac{x-5}{x-5}\right)$$
$$= -1$$

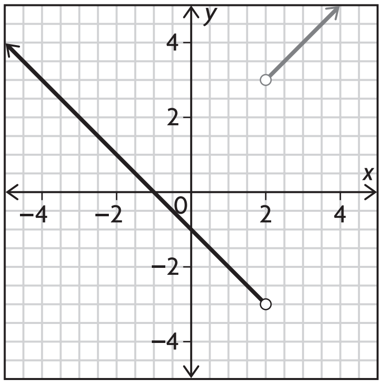


b. $\lim_{x\to\frac{5}{2}}\frac{|2x-5|(x+1)}{2x-5}$ does not exist.

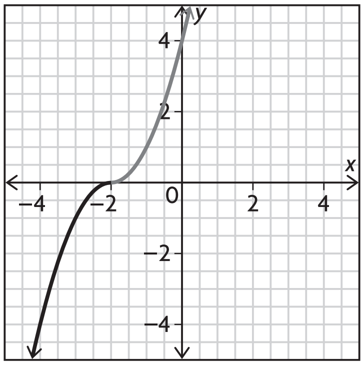
$$|2x-5| = 2x-5, x \geq \frac{5}{2}$$
$$\lim_{x\to\frac{5}{2}^+}\frac{(2x-5)(x+1)}{2x-5} = x+1$$
$$|2x-5| = -(2x-5), x < \frac{5}{2}$$
$$\lim_{x\to\frac{5}{2}^-}\frac{-(2x-5)(x+1)}{2x-5} = -(x+1)$$



c. $$\lim_{x\to 2}\frac{x^2-x-2}{|x-2|} = \lim_{x\to 2}\frac{(x-2)(x+1)}{|x-2|}$$
$$\lim_{x\to 2^+}\frac{(x-2)(x+1)}{|x-2|} = \lim_{x\to 2^+}\frac{(x-2)(x+1)}{x-2}$$
$$= \lim_{x\to 2^+} x+1$$
$$= 3$$
$$\lim_{x\to 2^-}\frac{(x-2)(x+1)}{|x-2|} = \lim_{x\to 2^-} -\frac{(x-2)(x+1)}{(x-2)}$$
$$= \lim_{x\to 2^-} -(x+1)$$
$$= -3$$



d. $$|x+2| = x+2 \text{ if } x > -2$$
$$= -(x+2) \text{ if } x < -2$$
$$\lim_{x\to -2^+}\frac{(x+2)(x+2)^2}{x+2} = \lim_{x\to -2^+}(x+2)^2 = 0$$
$$\lim_{x\to -2^-}\frac{(x+2)(x+2)^2}{-(x+2)} = 0$$



11. a.

ΔT	T	V	ΔV
	−40	19.1482	
20			1.6426
	−20	20.7908	
20			1.6426
	0	22.4334	
20			1.6426
	20	24.0760	
20			1.6426
	40	25.7186	
20			1.6426
	60	27.3612	
20			1.6426
	80	29.0038	

ΔV is constant, therefore T and V form a linear relationship.

b. $V = \frac{\Delta V}{\Delta T}\cdot T + K$

$$\frac{\Delta V}{\Delta T} = \frac{1.6426}{20} = 0.082\,13$$

$V = 0.082\ 13T + K$

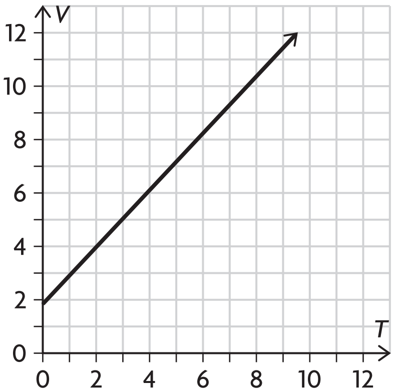
$T = 0 \qquad V = 22.4334$

Therefore, $k = 22.4334$ and $V = 0.082\ 13T + 22.4334$.

c. $T = \dfrac{V - 22.4334}{0.082\ 13}$

d. $\lim\limits_{v \to 0} T = -273.145$

e.



12. $\lim\limits_{x \to 5} \dfrac{x^2 - 4}{f(x)}$

$= \dfrac{\lim\limits_{x \to 5}(x^2 - 4)}{\lim\limits_{x \to 5} f(x)}$

$= \dfrac{21}{3}$

$= 7$

13. $\lim\limits_{x \to 4} f(x) = 3$

a. $\lim\limits_{x \to 4} [f(x)]^3 = 3^3 = 27$

b.

$$\lim_{x \to 4} \frac{[f(x)]^2 - x^2}{f(x) + x} = \lim_{x \to 4} \frac{(f(x) - x)(f(x) + x)}{f(x) + x}$$
$$= \lim_{x \to 4}(f(x) - x)$$
$$= 3 - 4$$
$$= -1$$

c. $\lim\limits_{x \to 4} \sqrt{3f(x) - 2x} = \sqrt{3 \times 3 - 2 \times 4}$

$= 1$

14. $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$

a. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \left[\dfrac{f(x)}{x} \times x\right] = 0$

b. $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0} \left[\dfrac{x}{g(x)}\ \dfrac{f(x)}{x}\right] = 0$

15. $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$ and $\lim\limits_{x \to 0} \dfrac{g(x)}{x} = 2$

a. $\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} x\left(\dfrac{g(x)}{x}\right) = 0 \times 2$

$= 0$

b. $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0} \dfrac{\frac{f(x)}{x}}{\frac{g(x)}{x}} = \dfrac{1}{2}$

16. $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 1} - \sqrt{2x + 1}}{\sqrt{3x + 4} - \sqrt{2x + 4}}$

$$= \lim_{x \to 0} \left[\frac{\sqrt{x + 1} - \sqrt{2x + 1}}{\sqrt{x + 1} + \sqrt{2x + 1}} \times \frac{\sqrt{x + 1} + \sqrt{2x + 1}}{\sqrt{3x + 4} - \sqrt{2x + 4}} \times \frac{\sqrt{3x + 4} + \sqrt{2x + 4}}{\sqrt{3x + 4} + \sqrt{2x + 4}}\right]$$

$$= \lim_{x \to 0} \left[\frac{(x + 1 - 2x - 1)}{(3x + 4 - 2x - 4)} \times \frac{\sqrt{3x + 4} + \sqrt{2x + 4}}{\sqrt{x + 1} + \sqrt{2x + 1}}\right]$$

$= -\dfrac{2 + 2}{1 + 1}$

$= -2$

17. $\lim\limits_{x \to 1} \dfrac{x^2 + |x - 1| - 1}{|x - 1|}$

$x \to 1^+ \quad |x - 1| = x - 1$

$\dfrac{x^2 + x - 2}{x - 1} = \dfrac{(x + 2)(x - 1)}{x - 1}$

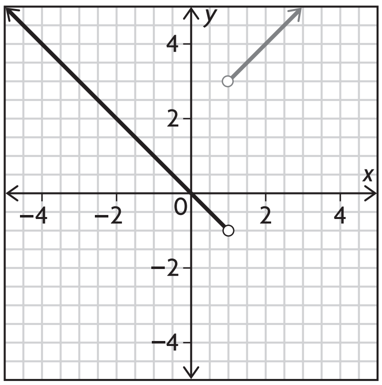
$\lim\limits_{x \to 1^+} \dfrac{x^2 + |x - 1| - 1}{|x - 1|} = 3$

$x \to 1^- \, |x - 1| = -x + 1$

$\lim\limits_{x \to 1^-} \dfrac{x^2 - x}{-x + 1} = \lim\limits_{x \to 1^-} \dfrac{x(x - 1)}{-x + 1}$

$= -1$

Therefore, this limit does not exist.



18. Evaluate each limit using one of the algebraic methods discussed in this chapter, if the limit exists.

a. $\lim_{x\to -4} \dfrac{x^2+12x+32}{x+4}$

b. $\lim_{x\to a} \dfrac{(x+4a)^2-25a^2}{x-a}$

c. $\lim_{x\to 0} \dfrac{\sqrt{x+5}-\sqrt{5-x}}{x}$

d. $\lim_{x\to 2} \dfrac{x^2-4}{x^3-8}$

e. $\lim_{x\to 4} \dfrac{4-\sqrt{12+x}}{x-4}$

f. $\lim_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{2+x}-\dfrac{1}{2}\right)$

19. Explain why the given limit does not exist.

a. $\lim_{x\to 3} \sqrt{x-3}$

b. $\lim_{x\to 2} \dfrac{x^2-4}{x^2-4x+4}$

c. $f(x)=\begin{cases}-5, & \text{if } x<1\\ 2, & \text{if } x\ge 1\end{cases}$; $\lim_{x\to 1} f(x)$

d. $\lim_{x\to 2} \dfrac{1}{\sqrt{x-2}}$

e. $\lim_{x\to 0} \dfrac{|x|}{x}$

f. $f(x)=\begin{cases}5x^2, & \text{if } x<-1\\ 2x+1, & \text{if } x\ge -1\end{cases}$; $\lim_{x\to -1} f(x)$

$\lim_{x\to -1^-} f(x) = 5$

$\lim_{x\to -1^+} f(x) = 1$

$\because \lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$

$\therefore \lim_{x\to -1} f(x) = \text{dne}$

#18 a) $\lim_{x\to -4} \dfrac{x^2+12x+32}{x+4}$

$= \lim_{x\to -4} \dfrac{(x+8)\cancel{(x+4)}}{\cancel{(x+4)}}$

$= \lim_{x\to -4} (x+8)$

$= 4$

b) $\lim_{x\to a} \dfrac{(x+4a)^2-25a^2}{x-a}$

$= \lim_{x\to a} \dfrac{x^2+8ax-9a^2}{x-a}$

$= \lim_{x\to a} \dfrac{(x+9a)\cancel{(x-a)}}{\cancel{(x-a)}}$

$= \lim_{x\to a} (x+9a)$

$= a+9a$

$= 10a$

c) $\lim_{x\to 0} \dfrac{\sqrt{x+5}-\sqrt{5-x}}{x} \cdot \dfrac{\sqrt{x+5}+\sqrt{5-x}}{\sqrt{x+5}+\sqrt{5-x}}$

$= \lim_{x\to 0} \dfrac{(x+5)-(5-x)}{x(\sqrt{x+5}+\sqrt{5-x})}$

$= \lim_{x\to 0} \dfrac{2\cancel{x}}{\cancel{x}(\sqrt{x+5}+\sqrt{5-x})}$

$= \lim_{x\to 0} \dfrac{2}{\sqrt{x+5}+\sqrt{5-x}}$

$= \dfrac{\cancel{2}}{\cancel{2}\sqrt{5}}$

$= \dfrac{1}{\sqrt{5}}$ or $\dfrac{\sqrt{5}}{5}$

d) $\lim_{x\to 2} \dfrac{x^2-4}{x^3-8}$

$= \lim_{x\to 2} \dfrac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(x^2+2x+4)}$

$= \lim_{x\to 2} \dfrac{x+2}{x^2+2x+4}$

$= \dfrac{2+2}{4+4+4}$

$= \dfrac{4}{12}$

$= \dfrac{1}{3}$

e) $\lim_{x\to 4} \dfrac{4-\sqrt{12+x}}{x-4} \cdot \dfrac{4+\sqrt{12+x}}{4+\sqrt{12+x}}$

$= \lim_{x\to 4} \dfrac{16-(12-x)}{(x-4)(4+\sqrt{12+x})}$

$= \lim_{x\to 4} \dfrac{\cancel{x-4}}{\cancel{(x-4)}(4+\sqrt{12+x})}$

$= \lim_{x\to 4} \dfrac{1}{4+\sqrt{12+x}}$

$= \dfrac{1}{4+4} = \dfrac{1}{8}$

f) $\lim_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{2+x}-\dfrac{1}{2}\right)$

$= \lim_{x\to 0} \dfrac{1}{x}\left(\dfrac{2-(x+2)}{2(x+2)}\right)$

$= \lim_{x\to 0} \dfrac{-1}{2(x+2)}$

$= -\dfrac{1}{4}$

#19 a) $\lim_{x\to 3} \sqrt{x-3}$

$\lim_{x\to 3^-} \sqrt{x-3} = \text{dne}$ $\quad \lim_{x\to 3^+} \sqrt{x-3} = 0$

$\because \lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

$\therefore \lim_{x\to 3} f(x) = \text{dne}$

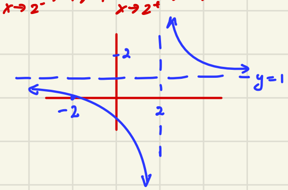
b) $\lim_{x\to 2} \dfrac{x^2-4}{x^2-4x+4}$

$= \lim_{x\to 2} \dfrac{(x+2)\cancel{(x-2)}}{(x-2)\cancel{(x-2)}}$

$= \dfrac{4}{0} = \text{dne}$

$\because \lim_{x\to 2^-} f(x) = -\infty \quad \lim_{x\to 2^+} f(x) = +\infty$

$\therefore \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$



c) $\lim_{x\to 1^-} f(x) = -5$

$\lim_{x\to 1^+} f(x) = 2$

$\because \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

$\therefore \lim_{x\to 1} f(x) = \text{dne}$

d) $\lim_{x \to 2} \frac{1}{\sqrt{x-2}}$

$= \frac{1}{\sqrt{2-2}}$

$= \frac{1}{0}$

$=$ dne

$\because \lim_{x \to 2^-} f(x) =$ dne

$\lim_{x \to 2^+} f(x) = \frac{1}{\sqrt{0}}$

$=$ dne

$\therefore \lim_{x \to 2} f(x) =$ dne

e) $\lim_{x \to 0} \frac{|x|}{x}$

$f(x) = \begin{cases} \frac{-x}{x} & x < 0 \\ \frac{x}{x} & x > 0 \end{cases}$

$= \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$

$\lim_{x \to 0^-} f(x) = -1$

$\lim_{x \to 0^+} f(x) = 1$

$\because \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\therefore \lim_{x \to 0} f(x) =$ dne

g) $\lim_{x \to -1^-} f(x) = 5$

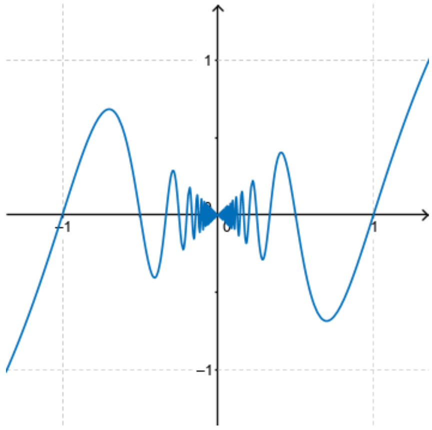
$\lim_{x \to -1^+} f(x) = 1$

$\because \lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$

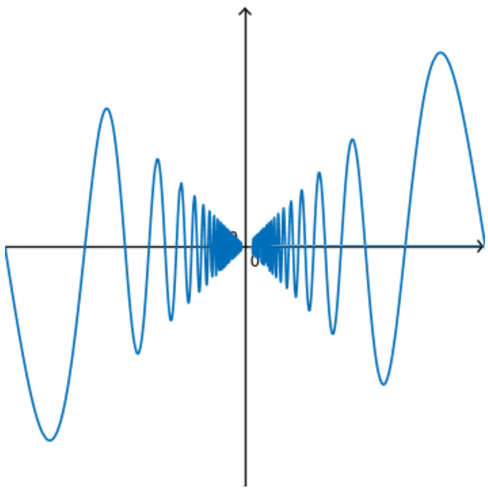
$\therefore \lim_{x \to -1} f(x) =$ dne

ENRICH SYUDY: An interesting additional tool for evaluating limits

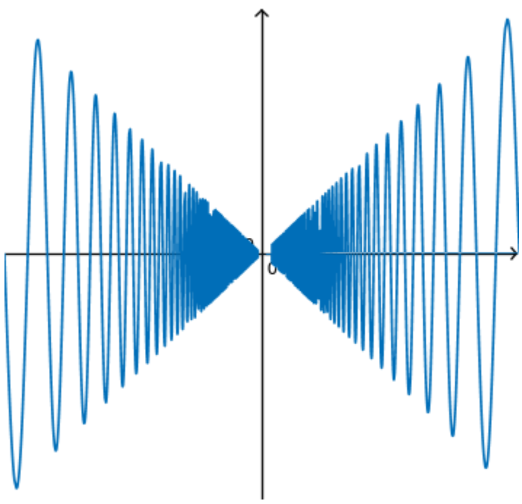
Consider the limit $\lim_{x\to 0} |x|\sin\left(\frac{\pi}{x}\right)$. We cannot use the limit product rule to evaluate the limit since $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ does not exist.



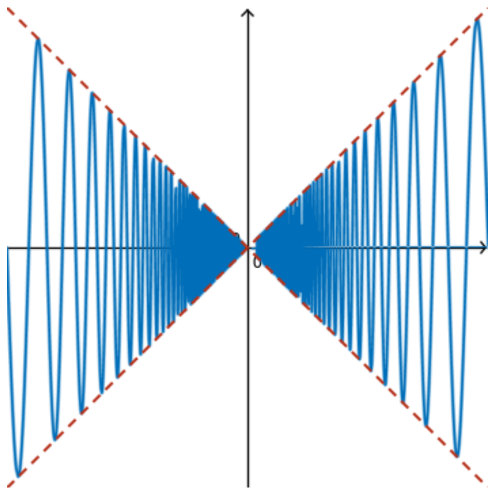
Notice that the oscillations grow increasingly more crowded as x approaches 0.
Decreasing the width of the interval to [−1/2,1/2]:



Decreasing the width of the interval to [−1/10,1/10]:



This behaviour is made more apparent by combining the function with the apparent bounds |x| and −|x| on the intervals [−1,1] (left) and [−1/10,1/10] (right).



It is clear that the functions |x| and −|x| are acting to "funnel" the function $|x|\sin\left(\frac{\pi}{x}\right)$ towards a limit of 0 as x→0.

This is a result of the fact that the values of the sine function lie between 1 and −1 for any value of the argument; that is, $-1 \le sin\left(\frac{\pi}{x}\right) \le 1 \; for \; all \; x \ne 0$

Thus, multiplying the inequality by |x| (which is non-negative), we see that

$$-|x| \le |x| \, sin\left(\frac{\pi}{x}\right) \le |x| \;\; for \; all \; x \ne 0$$

Since both of the "outer" functions ±|x| approach 0 as x→0, so also does the "inner" function $|x|\sin\left(\frac{\pi}{x}\right)$. Informally, the inner function gets "squeezed" or "sandwiched" between the two outer functions.

The above example illustrates the **squeeze theorem**, which essentially says that

If **m(x), f(x),** and **M(x)** are all defined near x=a, and $\mathbf{m}(\mathbf{x}) \le \mathbf{f}(\mathbf{x}) \le \mathbf{M}(\mathbf{x})$ near x=a, with $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{m}(\mathbf{x}) = \mathbf{L}$ and $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{M}(\mathbf{x}) = \mathbf{L}$ then, we also have $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$.

Example: Use the squeeze theorem to show that $\lim\limits_{x\to 0} x^2 cos\left(\frac{2}{x}\right) = 0$.

Practice Questions

1. If $\lim\limits_{x\to -1}\dfrac{x+1}{x^3-ax^2-x+6}$ exists, find the value of a.

2. If $\lim\limits_{x\to 2}\dfrac{x-7}{x^2+ax+b}=-\infty$,find the value of a+b.

3. Consider $f(x)=\begin{cases}7-x^2, & \text{if } x\le -2\\ ax+b, & \text{if } -2<x<3\\ \dfrac{3}{x}, & \text{if } x\ge 3\end{cases}$.Determine values for *a* and *b* so that $\lim\limits_{x\to -2}f(x)$ and $\lim\limits_{x\to 3}f(x)$ exist.

4. Given $f(x)=\begin{cases}\dfrac{3x^2-5x-2}{x-2} & 0\le x<2\\ x^2 & x<0\\ \dfrac{14\left(\sqrt{x^2+12}-4\right)}{x-2} & 2<x\end{cases}$, find $\lim\limits_{x\to 0}f(x)$ and $\lim\limits_{x\to 2}f(x)$.

5. Let $g(x) = Ax + B$, where *A* and *B* are constants. $\lim\limits_{x\to 1}g(x)=-2$ and $\lim\limits_{x\to -1}g(x)=4$, find the values of *A* and *B*.

6. Give an example of functions $f(x)$ and $g(x)$ such that $\lim\limits_{x\to 0}\left(f(x)+g(x)\right)$ exists but $\lim\limits_{x\to 0}f(x)$ and $\lim\limits_{x\to 0}g(x)$ do not exist.

7. Give an example of a function such that $\lim\limits_{x\to 0}[f(x)]^2$ exists but $\lim\limits_{x\to 0}f(x)$ does not exist.

8. Evaluate

(a) $\lim\limits_{x\to 56}\dfrac{\sqrt[3]{x+8}-4}{\sqrt{x-40}-4}$ (b) $\lim\limits_{x\to \frac{1}{4}}\dfrac{4x-1}{\dfrac{1}{\sqrt{x}}-2}$ (c) $\lim\limits_{x\to 2}\dfrac{\sqrt[3]{x-1}-x+1}{\sqrt{x-1}-x+1}$

9. Is there a value of *k* for which $\lim\limits_{x\to -2}\dfrac{kx^2-6x+3-k}{x^2+3x+2}$ exists? If so, find *k* and find the values of the limit.

10. Given the function $f(x)=\dfrac{x^3+ax^2+bx+5}{x^3+2x^2-x-2}$, determine whether there are numbers a and b such that both $\lim\limits_{x\to 1}f(x)$ exists, and $\lim\limits_{x\to -1}f(x)$ exists. If so, evaluate both limits.

11. Consider the function $f(x)=\dfrac{x^3-ax^2-x+b}{x+1}$. Given that $f(a)=a-2$ and the $\lim\limits_{x\to -1}f(x)$ exists, determine **all** value(s) of $\lim\limits_{x\to -1}f(x)$.

Practice Questions

1. If $\lim_{x\to-1}\frac{x+1}{x^3-ax^2-x+6}$ exists, find the value of a.

for a limit to exist the function must be indeterminant @ $x=-1$

$g(x) = x^3 - ax^2 - x + 6$

$g(-1) = -1 - a + 1 + 6$

$0 = -a + 6$

$\therefore a = 6$

if $a \neq 6$, $\lim_{x\to-1}\frac{x+1}{x^3-ax^2-x+6} = 0$

$\therefore a \in \mathbb{R}$

2. If $\lim_{x\to2}\frac{x-7}{x^2+ax+b} = -\infty$, find the value of a+b.

for the limit to be $-\infty$, there must be a vertical asymptote at $x=2$ with a double root

$g(x) = x^2 + ax + b$

$g(x) = (x-2)^2$

$= x^2 - 4x + 4$

$\therefore a = -4 \quad b = 4$

$\therefore a+b = 0$

3. Consider $f(x)=\begin{cases} 7-x^2, & \text{if } x\le-2 \\ ax+b, & \text{if } -2<x<3 \\ \frac{3}{x}, & \text{if } x\ge3 \end{cases}$. Determine values for a and b so that $\lim_{x\to-2}f(x)$ and $\lim_{x\to3}f(x)$ exist.

$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^+} f(x)$

$7-(-2)^2 = a(-2)+b$

$3 = -2a+b$ ①

$\lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x)$

$a(3)+b = \frac{3}{(3)}$

$3a+b = 1$ ②

$$\begin{cases} 3 = -2a+b & ① \\ 1 = 3a+b & ② \end{cases}$$

$(-)$ $2 = -5a$

$a = -\frac{2}{5}$ ③

sub ③ in ②

$3(-\frac{2}{5}) + b = 1$

$b = 1 + \frac{6}{5}$

$= \frac{11}{5}$

$\therefore a = -\frac{2}{5} \quad b = \frac{11}{5}$

4. Given $f(x)=\begin{cases} \frac{3x^2-5x-2}{x-2} & 0\le x<2 \\ x^2 & x<0 \\ \frac{14\left(\sqrt{x^2+12}-4\right)}{x-2} & 2<x \end{cases}$, find $\lim_{x\to0}f(x)$ and $\lim_{x\to2}f(x)$.

Aside: $\frac{3x^2-5x-2}{x-2}$

$= \frac{(3x+1)\cancel{(x-2)}}{\cancel{(x-2)}}$

$\frac{14(\sqrt{x^2+12}-4)}{x-2} \cdot \frac{(\sqrt{x^2+12}+4)}{\sqrt{x^2+12}+4}$

$= \frac{14(x^2+12-16)}{(x-2)(\sqrt{x^2+12}+4)}$

$= \frac{14(x^2-4)}{(x-2)(\sqrt{x^2+12}+4)}$

$= \frac{14(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x^2+12}+4)}$

$f(x) = \begin{cases} x^2, & x<0 \\ 3x+1, & 0\le x<2 \\ \frac{14(x+2)}{\sqrt{x^2+12}+4}, & x>2 \end{cases}$

$\lim_{x\to0^-} f(x) = (0)^2 = 0$

$\lim_{x\to0^+} f(x) = 3(0)+1 = 1$

$\lim_{x\to2^-} f(x) = 3(2)+1 = 7$

$\lim_{x\to2^+} f(x) = \frac{14(2+2)}{\sqrt{2^2+12}+4} = \frac{56}{8} = 7$

$\therefore \lim_{x\to0^-} f(x) \neq \lim_{x\to0^+} f(x)$

$\therefore \lim_{x\to0} f(x) = DNE$

$\therefore \lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x)$

$\therefore \lim_{x\to2} f(x) = 7$

5. Let $g(x) = Ax + B$, where A and B are constants. $\lim\limits_{x\to 1} g(x) = -2$ and $\lim\limits_{x\to -1} g(x) = 4$, find the values of A and B.

$\lim\limits_{x\to 1^-}(Ax+B) = \lim\limits_{x\to 1^+}(Ax+B)$ $\qquad$ $\lim\limits_{x\to -1^-}(Ax+B) = \lim\limits_{x\to -1}(Ax+B)$

$A(1)+B = -2$ $\qquad$ $A(-1)+B = 4$

$A+B = -2$ ① $\qquad$ $-A+B = 4$ ②

①+② : $2B = 2$

$B = 1$ ③

Sub ③ in ① : $A + (1) = -2$

$A = -3$

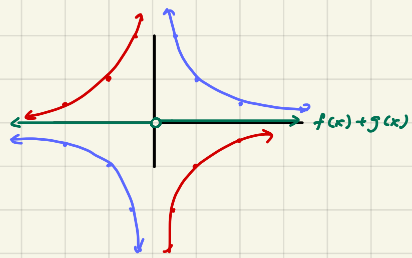
$\therefore$ $A = -3$ $\quad B = 1$

vv TH

6. Give an example of functions $f(x)$ and $g(x)$ such that $\lim\limits_{x\to 0}(f(x)+g(x))$ exists but $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist.

$f(x) = \frac{1}{x}$

$g(x) = \frac{-1}{x}$



< there aren't that many !! >

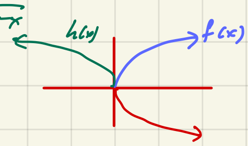
$f(x) = \sqrt{x}$

$g(x) = -\sqrt{x}$

$h(x) = \sqrt{-x}$

$f(x) + h(x) = DNE$

since the domain doesn't coincide !!



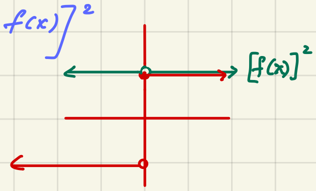
vv TH

7. Give an example of a function such that $\lim\limits_{x\to 0}[f(x)]^2$ exists but $\lim\limits_{x\to 0} f(x)$ does not exist.

$\lim\limits_{x\to 0}[f(x)]^2 = \left[\lim\limits_{x\to 0} f(x)\right]^2$

$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \leq 0 \end{cases}$

$[f(x)]^2 = \begin{cases} 1, & x < 0 \\ 1, & x > 0 \end{cases}$

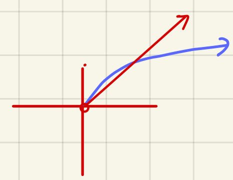


< there aren't that many !! >

$f(x) = \sqrt{x}$ $\qquad$ $[f(x)]^2 = x$

$\lim\limits_{x\to 0}\sqrt{x} = \lim\limits_{x\to 0} x = DNE$

this doesn't work !!



8. Evaluate

(a) $\lim\limits_{x\to 56}\dfrac{\sqrt[3]{x+8}-4}{\sqrt{x-40}-4}$ $\qquad$ (b) $\lim\limits_{x\to \frac{1}{4}}\dfrac{4x-1}{\frac{1}{\sqrt{x}}-2}$ $\qquad$ (c) $\lim\limits_{x\to 2}\dfrac{\sqrt[3]{x-1}-x+1}{\sqrt{x-1}-x+1}$

** Remember to always substitute first $\Rightarrow$ but all are $\frac{0}{0}$ "indeterminant" $\therefore$ must simplify a factor

TH a) let $a = (x+8)^{\frac{1}{6}}$

$a^6 - 8 = x$

$x \to 56$

$a^6 - 8 \to 56$

$a^6 \to 64$

$a \to 2$

$\lim\limits_{a\to 2}\dfrac{a^2-4}{\sqrt{a^6-48}-4}\cdot\dfrac{\sqrt{a^6-48}+4}{\sqrt{a^6-48}+4}$

$= \lim\limits_{a\to 2}\dfrac{(a+2)(a-2)\left[\sqrt{a^6-48}+4\right]}{(a^6-48)-16}$

$= \lim\limits_{a\to 2}\dfrac{(a+2)(a-2)(\sqrt{a^6-48}+4)}{a^6-64}$

$= \lim\limits_{a\to 2}\dfrac{\cancel{(a+2)}\cancel{(a-2)}(\sqrt{a^6-48}+4)}{\cancel{(a+2)}\cancel{(a-2)}(a^4+4a^2+16)}$

$= \lim\limits_{a\to 2}\dfrac{\sqrt{a^6-48}+4}{a^4+4a^2+16}$

$= \dfrac{\sqrt{2^6-48}+4}{2^4+4(2^2)+16}$

$= \dfrac{8}{48}$

$= \dfrac{1}{6}$

Aside: $a^6 - 64$

$= (a^2)^3 - 4^3$

$= (a^2-4)(a^4+4a^2+16)$

$= (a+2)(a-2)(a^4+4a^2+16)$

TH b) $\lim\limits_{x\to\frac{1}{4}}\dfrac{4x-1}{x^{-\frac{1}{2}}-2} = \lim\limits_{a\to 2}\dfrac{4a^{-2}-1}{a-2}$

let $a = x^{-\frac{1}{2}}$

$a^{-2} = x$

$x \to \frac{1}{4}$

$a^{-2} \to \frac{1}{4}$

$\frac{1}{a^2} \to \frac{1}{4}$

$a^2 \to 4$

$a \to 2$

$= \lim\limits_{a\to 2}\dfrac{\frac{4}{a^2}-1}{a-2}$

$= \lim\limits_{a\to 2}\dfrac{1}{a-2}\left[\dfrac{4-a^2}{a^2}\right]$

$= \lim\limits_{a\to 2}\dfrac{1}{\cancel{a-2}}\left[\dfrac{-(a+2)\cancel{(a-2)}}{a^2}\right]$

$= \lim\limits_{a\to 2}\dfrac{-(a+2)}{a^2}$

$= \dfrac{-(2+2)}{2^2}$

$= -1$

#8 c) $\lim\limits_{x\to 2} \dfrac{\sqrt[3]{x-1} - x + 1}{\sqrt{x-1} - x + 1} = \lim\limits_{a\to 1} \dfrac{a^2 - (a^6+1) + 1}{a^3 - (a^6+1) + 1}$

let $a = (x-1)^{\frac{1}{6}}$

$a^2 = (x-1)^{\frac{1}{3}}$

$a^3 = (x-1)^{\frac{1}{2}}$

$a^6 + 1 = x$

$x \to 2$

$a^6 + 1 \to 2$

$a^6 \to 1$

$a \to \sqrt[6]{1}$

$a \to 1$

$$= \lim_{a\to 1} \frac{a^2 - a^6}{a^3 - a^6}$$

$$= \lim_{a\to 1} \frac{\cancel{a^2}(1-a^4)}{a^{\cancel{3}}(1-a^3)}$$

$$= \lim_{a\to 1} \frac{\cancel{-}(a^2+1)(a^2-1)}{\cancel{-}a(a-1)(a^2+a+1)}$$

$$= \lim_{a\to 1} \frac{(a^2+1)(a+1)\cancel{(a-1)}}{a\cancel{(a-1)}(a^2+a+1)}$$

$$= \frac{(1^2+1)(1+1)}{1(1^2+1+1)}$$

$$= \frac{4}{3}$$

9. Is there a value of k for which $\lim\limits_{x\to -2} \dfrac{kx^2 - 6x + 3 - k}{x^2 + 3x + 2}$ exists? If so, find k and find the values of the limit.

$f(x) = \dfrac{kx^2 - 6x + 3 - k}{(x+2)(x+1)}$

$g(x) = x^2 + 3x + 2$

$g(-2) = (-2)^2 + 3(-2) + 2$

$= 0$ $\therefore$ this may create an indeterminant situation

$f(-2) = 0$ hole: $x = -2$

$k(-2)^2 - 6(-2) + 3 - k = 0$

$4k + 12 + 3 - k = 0$

$3k = -15$

$k = -5$

$$\lim_{x\to -2} \frac{-5x^2 - 6x + 8}{(x+2)(x+1)}$$

$$= \lim_{x\to -2} \frac{-(5x-4)\cancel{(x+2)}}{\cancel{(x+2)}(x+1)}$$

$$= \lim_{x\to -2} \frac{-(5x-4)}{x+1}$$

$$= \frac{-(5(-2)-4)}{-2+1}$$

$$= \frac{14}{-1}$$

$$= -14$$

Aside:

$-5x^2 - 6x + 8$

$= -(5x^2 + 6x - 8)$

$= -(5x-4)(x+2)$

10. Given the function $f(x) = \dfrac{x^3 + ax^2 + bx + 5}{x^3 + 2x^2 - x - 2}$, determine whether there are numbers a and b such that both $\lim\limits_{x\to 1} f(x)$ exists, and $\lim\limits_{x\to -1} f(x)$ exists. If so, evaluate both limits.

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$

$n(x) = x^3 + 2x^2 - x - 2$

$n(1) = 1^3 + 2(1)^2 - (1) - 2$

$= 0$ $\therefore$ an indeterminant situation may be created

$n(-1) = (-1)^3 + 2(-1)^2 - (-1) - 2$

$= -1 + 2 + 1 - 2$

$= 0$

let: $f(x) = \dfrac{m(x)}{n(x)}$

$n(x) = x^2(x+2) - (x+2)$

$= (x+2)(x^2-1)$

$= (x+2)(x+1)(x-1)$

$m(x) = x^3 + ax^2 + bx + 5$ must have a factor of $(x+1)(x-1)$

$m(1) = 0$

$(1)^3 + a(1)^2 + b(1) + 5 = 0$

$a + b + 6 = 0$ ①

$m(-1) = 0$

$(-1)^3 + a(-1)^2 + b(-1) + 5 = 0$

$-1 + a - b + 5 = 0$

$a - b + 4 = 0$ ②

$$\begin{cases} a + b + 6 = 0 & ① \\ a - b + 4 = 0 & ② \end{cases}$$

$(-)$ $2b + 2 = 0$

$b = -1$

$\therefore a = -5$

$\Longrightarrow$ Continued.

$$\lim_{x\to 1} \frac{x^3-5x^2-x+5}{x^3+2x^2-x-2}$$
$$= \lim_{x\to 1} \frac{x-5}{x+2}$$
$$= \frac{1-5}{1+2}$$
$$= \frac{-4}{3}$$

$$\lim_{x\to -1} \frac{x^3-5x^2-x+5}{x^3+2x^2-x-2}$$
$$= \lim_{x\to -1} \frac{x-5}{x+2}$$
$$= \frac{-1-5}{-1+2}$$
$$= -6$$

$$m(x) = x^3-5x^2-x+5$$
$$= x^2(x-5)-(x-5)$$
$$= (x-5)(x^2-1)$$
$$= (x-5)(x+1)(x-1)$$

$$n(x) = x^3+2x^2-x-2$$
$$= x^2(x+2)-(x+2)$$
$$= (x+2)(x^2-1)$$
$$= (x+2)(x+1)(x-1)$$

$$f(x) = \frac{(x-5)\cancel{(x+1)}\cancel{(x-1)}}{(x+2)\cancel{(x+1)}\cancel{(x-1)}}$$

11. Consider the function $f(x)=\dfrac{x^3-ax^2-x+b}{x+1}$. Given that $f(a)=a-2$ and the $\lim_{x\to -1} f(x)$ exists, determine **all** value(s) of $\lim_{x\to -1} f(x)$.

$$\lim_{x\to -1} \frac{x^3-ax^2-x+b}{x+1} = L$$

$$f(x) = \frac{m(x)}{n(x)}$$

$m(-1) = 0$ for $(x+1)$ to be a factor to create a indeterminant situation

$$(-1)^3-a(-1)^2-(-1)+b=0$$
$$-1-a+1+b=0$$
$$-a+b=0 \quad ① \quad a=b$$

$$f(a) = \frac{a^3-a(a)^2-a+b}{a+1}$$
$$(a-2) = \frac{-a+b}{a+1}, \quad a\neq -1$$
$$(a-2)(a+1) = -a+b. \quad ②$$

Sub ① into ②

$$(a-2)(a+1)=0$$
$$a=\{2,-1\}$$

-1 ↳ inadmissible as $a\neq -1$

$\therefore a=2 \quad b=2$

$$\lim_{x\to -1} \frac{x^3-2x^2-x+2}{x+1}$$
$$= \lim_{x\to -1} \frac{x^2(x-2)-(x-2)}{x+1}$$
$$= \lim_{x\to -1} \frac{(x-2)\cancel{(x+1)}(x-1)}{\cancel{(x+1)}}$$
$$= \lim_{x\to -1} (x-2)(x-1)$$
$$= (-1-2)(-1-1)$$
$$= 6$$

MCV4U1 **Exit Card!**

Name: ______________________, Period: __________

1. Evaluate the following limits.

a) $\lim_{x\to 0} \frac{\frac{9}{(x+3)^2}-1}{x}$

$= \lim_{x\to 0} \frac{1}{x}\left[\frac{9-(x+3)^2}{(x+3)^2}\right]$ [KU 3]

$= \lim_{x\to 0} \frac{1}{x}\left[\frac{9-(x^2+6x+9)}{(x+3)^2}\right]$

$= \lim_{x\to 0} \frac{1}{x}\left[\frac{-x^2-6x}{(x+3)^2}\right]$

$= \lim_{x\to 0} \frac{-x-6}{(x+3)^2}$

$= \frac{-6}{9}$

$= -\frac{2}{3}$

b) $\lim_{x\to\sqrt{6}} \frac{x^2-6}{3-\sqrt{3+x^2}} \cdot \frac{3+\sqrt{3+x^2}}{3+\sqrt{3+x^2}}$

$= \lim_{x\to\sqrt{6}} \frac{(x^2-6)(3+\sqrt{3+x^2})}{9-(3+x^2)}$

$= \lim_{x\to\sqrt{6}} \frac{\cancel{(x^2-6)}(3+\sqrt{3+x^2})}{-\cancel{(x^2-6)}}$

$= \lim_{x\to\sqrt{6}} -(3+\sqrt{3+x^2})$

$= -(3+\sqrt{3+(\sqrt{6})^2})$

$= -(3+3)$

$= -6$ [KU-3]

method 2: Change of variable

let $a=\sqrt{3+x^2}$

$a^2 = 3+x^2$

$\sqrt{a^2-3} = x$

as $x\to\sqrt{6}$

$\sqrt{a^2-3}\to\sqrt{6}$

$a^2-3\to 6$

$a^2\to 9$

$a\to 3$

$\lim_{a\to 3} \frac{(a^2-3)-6}{3-a}$

$= \lim_{a\to 3} \frac{a^2-9}{-(a-3)}$

$= \lim_{a\to 3} \frac{(a+3)\cancel{(a-3)}}{-\cancel{(a-3)}}$

$= \lim_{a\to 3} -(a+3)$

$= -6$

c) $\lim_{x\to 2} \frac{(3x+2)^{\frac{1}{3}}-2}{x-2}$ [KU-4]

let $a=(3x+2)^{\frac{1}{3}}$

$a^3 = (3x+2)$

$\frac{a^3-2}{3} = x$

$x\to 2$

$\frac{a^3-2}{3}\to 2$

$a^3\to 8$

$a\to 2$

$= \lim_{a\to 2} \frac{(a-2)}{\left(\frac{a^3-2}{3}-2\right)}$

$= \lim_{a\to 2} \frac{a-2}{\left(\frac{a^3-2-6}{3}\right)}$

$= \lim_{a\to 2} \frac{3(a-2)}{a^3-8}$

$= \lim_{a\to 2} \frac{3\cancel{(a-2)}}{\cancel{(a-2)}(a^2+2a+4)}$

$= \lim_{a\to 2} \frac{3}{a^2+2a+4}$

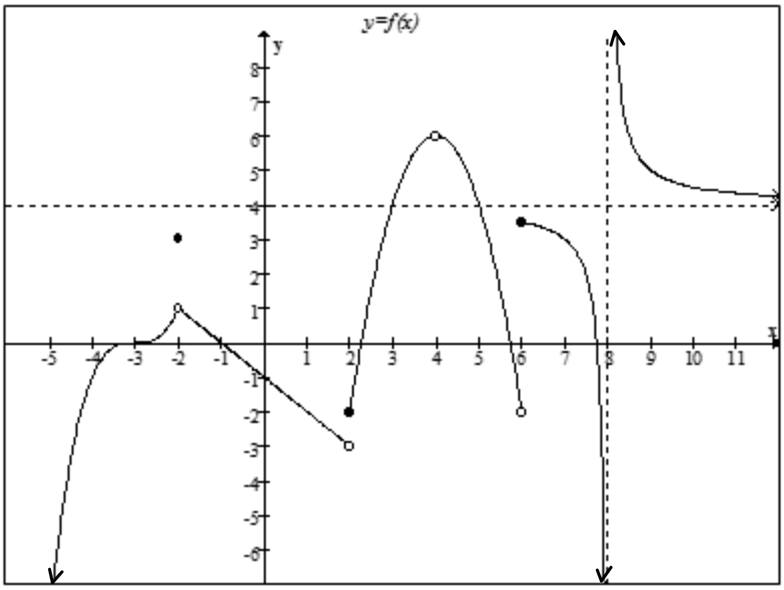
$= \frac{3}{2^2+2(2)+4}$

$= \frac{3}{12}$

$= \frac{1}{4}$

WARM UP: EVALUATING LIMITS

1. Given the function $y = f(x)$ below, determine the following:



a. $\lim\limits_{x \to -2} f(x) =$ $\underline{\quad 1 \quad}$

HA $\Rightarrow$ b. $\lim\limits_{x \to \infty} f(x) =$ $\underline{\quad 4 \quad}$

c. $\lim\limits_{x \to 2} f(x) =$ $\underline{\quad \text{dne} \quad}$

d. $f(6) =$ $\underline{\quad 3.5 \quad}$

e. $\lim\limits_{x \to 8} f(x) =$ $\underline{\quad \text{dne} \quad}$

2. Evaluate each the following

a. $\lim\limits_{x \to 0} \dfrac{x}{\dfrac{1}{6} + \dfrac{1}{x-6}}$

$= \lim\limits_{x \to 0} x \div \left[\dfrac{(x-6)+(6)}{6(x-6)} \right]$

$= \lim\limits_{x \to 0} x \div \left[\dfrac{x}{6(x-6)} \right]$

$= \lim\limits_{x \to 0} 6(x-6)$

$= -36$

b. $\lim\limits_{x \to -5} \dfrac{(x+8)^3 - 27}{x+5}$

$= \lim\limits_{x \to -5} \dfrac{\left[\cancel{(x+8)-3}\right]\left[(x+8)^2 + 3(x+8) + 9\right]}{\cancel{(x+5)}}$

$= \lim\limits_{x \to -5} \left[(x+8)^2 + 3(x+8) + 9\right]$

$= (-5+8)^2 + 3(-5+8) + 9$

$= 3^2 + 9 + 9$

$= 27$

c. $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{3x+1} - \sqrt{5-x^2}}$ $\cdot \dfrac{\sqrt{3x+1} + \sqrt{5-x^2}}{\sqrt{3x+1} + \sqrt{5-x^2}}$

$= \lim\limits_{x \to 1} \dfrac{(x-1)\left[\sqrt{3x+1} + \sqrt{5-x^2}\right]}{(3x+1) - (5-x^2)}$

$= \lim\limits_{x \to 1} \dfrac{(x-1)\left[\sqrt{3x+1} + \sqrt{5-x^2}\right]}{x^2 + 3x - 4}$

$= \lim\limits_{x \to 1} \dfrac{\cancel{(x-1)}\left[\sqrt{3x+1} + \sqrt{5-x^2}\right]}{(x+4)\cancel{(x-1)}}$

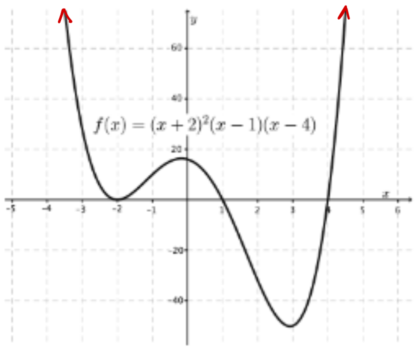
$= \dfrac{\sqrt{3(1)+1} + \sqrt{5-(1)^2}}{(1+4)}$

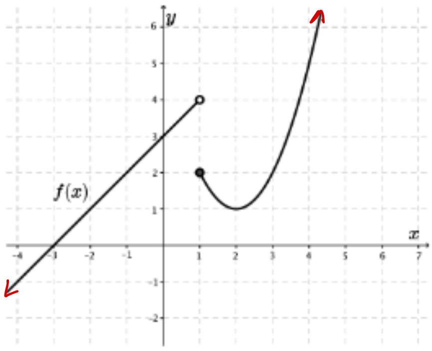
$= \dfrac{2+2}{5}$

$= \dfrac{4}{5}$

1.3 Continuity & Limits at Infinity

A function is **continuous** if you can draw its graph **without lifting** your pencil. If the curve has holes or gaps, it is discontinuous, or has a discontinuity, at the point at which the gap occurs.





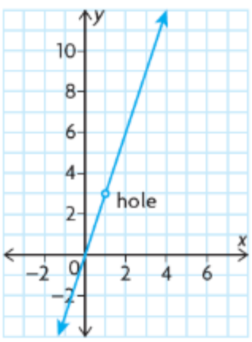
A function that is not continuous at x=a is referred to as **discontinuous** at **a**. The point, **a**, is known as a point of discontinuity.

Types of Discontinuities

There are four different types of discontinuities that we will discuss

(1) Point Discontinuity

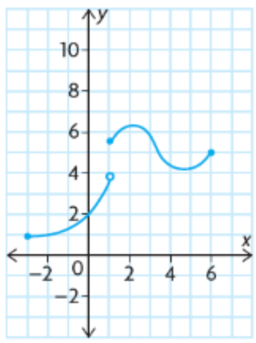
Hole / Removable Discontinuity



$\because f(1)$ is undefined

$\therefore f(x)$ is discontinuous at $x=1$

(2) Jump Discontinuity (Step)

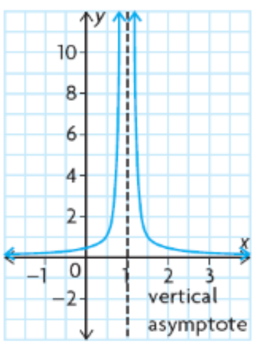


$\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$

$\therefore f(x)$ is discontinuous at $x=1$

Note! $f(1) = 5.5$ is defined

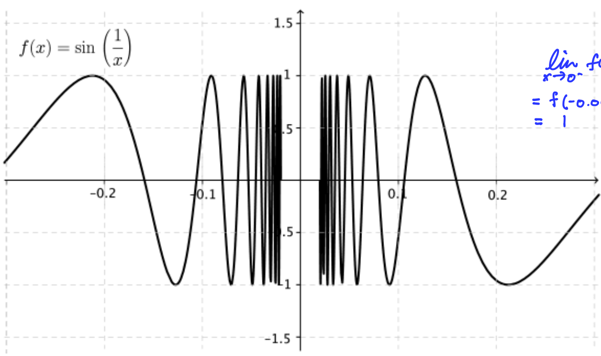
(3) Infinite Discontinuity (Asymptotic)



$f(1)$ is undefined

$\therefore f(x)$ is discontinuous at $x=1$

(4) Oscillating Discontinuities (Beyond MCV4U)



$\lim\limits_{x\to 0^-} f(x) = f(-0.001) = 1$

$\lim\limits_{x\to 0^+} f(x) = f(0.001) = -1$

$\because \lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$

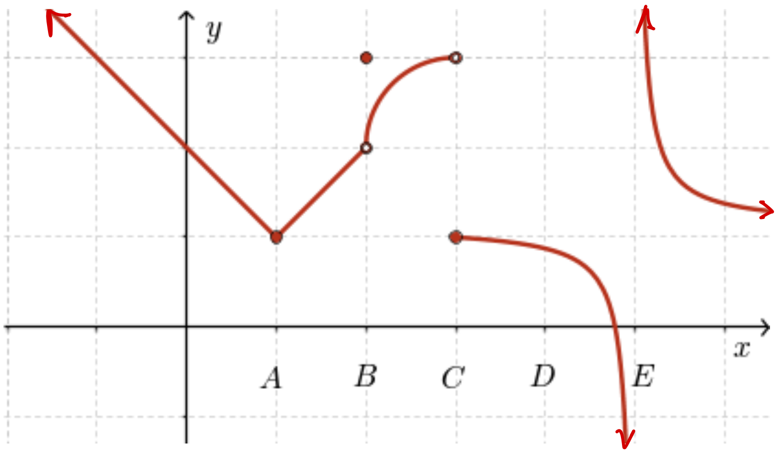
$\therefore \lim\limits_{x\to 0} f(x) =$ dne

$f(0)$ is undefined

$\therefore f(x)$ is discontinuous at $x=0$

Example#1: The graph of y=f(x) is shown. Determine whether the function is continuous at the indicated points. State the type of discontinuity (removable, jump, infinite, or none of these).

a. x=A Continuous
b. x=B Removeable Discontinuity
c. x=C Jump Discontinuity
d. x=D Continuous
e. x=E infinite Discontinuity (Asymptote)



Limit definition of Continuity

A function $f(x)$ is continuous at a value $x = a$ if the following **three conditions** are satisfied:

1.	$f(a)$ **must exists**	$\Rightarrow$ $f(a)$ is defined
2.	$\lim_{x\to a} f(x)$ exists	$\Rightarrow$ $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$
3.	$\lim_{x\to a} f(x) = f(a)$	$\Rightarrow$ $\lim_{x\to a} f(x) = f(a)$

$f(a) = L = \lim_{x\to a} f(x)$

Example#2: $f(x)=\begin{cases} \dfrac{x^2-6x}{x} & if \quad x \neq 0 \\ 2k-1 & if \quad x=0 \end{cases}$ and f is continuous at $x = 0$. Find the value of k.

1. $f(0) = 2k-1 \Rightarrow f(0)$ is defined

2. $\lim_{x\to 0} f(x)$ Exists $\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

3. $\lim_{x\to 0} f(x) = f(0) \Rightarrow \lim_{x\to 0} f(x) = f(0)$

$$\lim_{x\to 0} \frac{x^2-6x}{x} = 2k-1$$

$$\lim_{x\to 0} \frac{\cancel{x}(x-6)}{\cancel{x}} = 2k-1$$

$$\lim_{x\to 0} x-6 = 2k-1$$

$$-6 = 2k-1$$

$$\therefore k = \frac{-5}{2}$$

Aside:

$$g(x) = \frac{x^2-6x}{x} = \frac{\cancel{x}(x-6)}{\cancel{x}} = x-6$$

$\lim_{x\to 0^-} f(x) = -6$

$\lim_{x\to 0^+} f(x) = -6$

$\therefore \lim_{x\to 0} f(x) = -6$

Example#3: Find values of ***a*** and ***b*** that makes function f(x) continuous on $\mathbb{R}$.

$$f(x)=\begin{cases} ax+2b & \text{if } x\le -1 \\ x^2+2 & \text{if } -1<x\le 2 \\ 2ax-4b & \text{if } x>2 \end{cases}$$

$f(-1) = a(-1) + 2b$
$= -a+2b$ is defined

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$-a+2b = (-1)^2+2$
$-a+2b = 3$ ①

$f(2) = 2^2+2$
$= 6$ is defined

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$
$(2)^2+2 = 2a(2)-4b$
$6 = 4a-4b$
$3 = 2a-2b$ ②

$$\begin{cases} -a+2b = 3 \\ 2a-2b = 3 \end{cases}$$
(+) $a = 6$

$-6+2b = 3$
$2b = 9$
$b = \frac{9}{2}$

Check:

$$f(x) = \begin{cases} 6x+9, & x\le -1 \\ x^2+2, & -1<x\le 2 \\ 12x-18, & x>2 \end{cases}$$

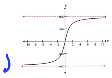
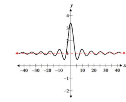
Example#4: Find For the function $f(x)=\begin{cases} 3ax-b, & x\in(-\infty,1) \\ 5, & x=1 \\ 2a\sqrt{x}+b, & x\in(1,\infty) \end{cases}$, determine the values of a and b such that $f(x)$ is continuous everywhere.

$f(1) = 5$ is defined

$\lim_{x\to 1^-} f(x) = 5$
$3a(1) - b = 5$
$3a-b = 5$

$\lim_{x\to 1^+} f(x) = 5$
$2a\sqrt{1}+b = 5$
$2a+b = 5$

$$\begin{cases} 3a-b=5 \\ 2a+b=5 \end{cases}$$
(+) $5a = 10$
$a = 2$ $b = 1$

Rational Function
Exponential Function
- Arctan $f(x) = \tan^{-1}(x)$ 
- oscillating function $g(x) = \frac{\sin(x)}{x} + 1$ 

Limits at Infinity (Horizontal Asymptotes)

Consider the behavior of rational functions $f(x) = \dfrac{P(x)}{Q(x)}$ as x increases without bound in both the positive and negative directions. We can use limit notation to describe this behaviour

$$\lim_{x\to\infty} f(x) = L \qquad\qquad \lim_{x\to-\infty} f(x) = L$$

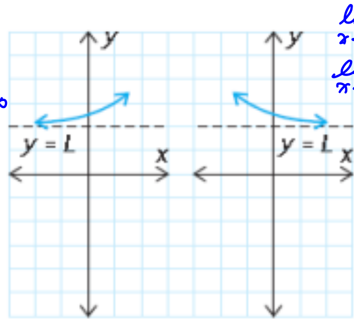
The following graphs illustrate some typical ways that a curve may approach a horizontal asymptote:

$\lim\limits_{x\to-\infty} f(x) = 2^+$

$\lim\limits_{x\to+\infty} f(x) = +\infty$

$\lim\limits_{x\to-\infty} f(x) = +\infty$

$\lim\limits_{x\to\infty} f(x) = 2^+$



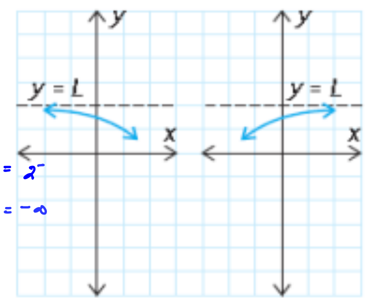
f(x)>L, so the graph approaches from above.

$\lim\limits_{x\to-\infty} f(x) = 2^-$

$\lim\limits_{x\to+\infty} f(x) = -\infty$

$\lim\limits_{x\to-\infty} f(x) = -\infty$

$\lim\limits_{x\to+\infty} f(x) = 2^-$



f(x)<L, so the graph approaches from below.

When you are finding a limit at infinity, direct substitution can yield another indeterminate form $\frac{\infty}{\infty}$. To find the limit in this case, divide the functions in the numerator and denominator by the highest power of x in the denominator.

Example #1: Determine the following limits.

oblique asymptote (slant Asymptote)

Recall from MHF4U: HA: $y = \frac{5}{2}$

a) $\lim\limits_{x\to\infty} \dfrac{1}{x}$

$= \dfrac{1}{\infty}$

$= 0$

b) $\lim\limits_{x\to\infty} \dfrac{x^2 - 4x + 1}{x - 4}$

method 2: factor out the highest degree of denominator

$\lim\limits_{x\to\infty} \dfrac{x(x - 4 - \frac{1}{x})}{x(1 - \frac{4}{x})}$

$= \dfrac{\infty - 4 - 0}{1 - 0}$

$= \infty$

$= \lim\limits_{x\to\infty} \dfrac{\frac{x^2}{x} - \frac{4x}{x} + \frac{1}{x}}{\frac{x}{x} - \frac{4}{x}}$

$= \lim\limits_{x\to\infty} \dfrac{x - 4 + \frac{1}{x}}{1 - \frac{4}{x}}$

$= \dfrac{\infty - 4 + 0}{1 - 0} = \infty$

c) $\lim\limits_{x\to\infty} \dfrac{5x^2 - 3x + 4}{2x^2 + x - 7} = \dfrac{\infty}{\infty}$

$\div x^2$ $= \lim\limits_{x\to\infty} \dfrac{5 - \frac{3}{x} + \frac{4}{x^2}}{2 + \frac{1}{x} - \frac{7}{x^2}}$

$= \dfrac{5 - 0 + 0}{2 + 0 - 0}$

$= \dfrac{5}{2}$

d) $\lim\limits_{x\to\infty} \left(\dfrac{1}{x} - \dfrac{1}{x+1}\right)$

$\lim\limits_{x\to\infty} \left(\dfrac{1}{x}\right) - \lim\limits_{x\to\infty} \left(\dfrac{1}{x+1}\right)$

$= 0 - 0$

$= 0$

method 2:

$\lim\limits_{x\to\infty} \left(\dfrac{x+1-x}{x(x+1)}\right)$

$= \lim\limits_{x\to\infty} \dfrac{1}{x(x+1)}$

$= \dfrac{1}{(\infty)(\infty)}$

$= 0$

e) $\lim\limits_{x\to\infty} \dfrac{3x^5 - 2x^3 + 1}{(1-x)^5}$

$= \dfrac{3}{-1}$

$= -3$

f) $\lim\limits_{x\to\infty} \dfrac{(x-5)(2x-3)^3}{x^4 - 3x^2 - 2}$

$(2x)^3 = 8x^3$

$= \dfrac{8}{1}$

$= 8$

< see next page for MCV4U algebraic approach >

INDETERMINATE FORMS

When working with limits, the following forms are indeterminate in that the value of the limit is not "obvious."

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^\infty$$

Takeup

CP31 Ex1 f)

$$\lim_{x\to\infty}\frac{(x-5)(2x-3)^3}{x^4-3x^2-2}$$

$$= \frac{\lim_{x\to\infty}(x-5)\cdot\left[\lim_{x\to\infty}(2x-3)\right]^3}{\lim_{x\to\infty}\left[x^4-3x^2-2\right]}$$

$$= \frac{\lim_{x\to\infty} x\left(1-\frac{5}{x}\right)\cdot\left[\lim_{x\to\infty} x\left(2-\frac{3}{x}\right)\right]^3}{\lim_{x\to\infty} x^4\left(1-\frac{3}{x^2}-\frac{2}{x^4}\right)}$$

$$= \frac{\cancel{x}\lim_{x\to\infty}\left(1-\frac{5}{x}\right)\cdot\cancel{x^3}\left[\lim_{x\to\infty}\left(2-\frac{3}{x}\right)\right]^3}{\cancel{x^4}\lim_{x\to\infty}\left(1-\frac{3}{x^2}-\frac{2}{x^4}\right)}$$

$$= \frac{(1)\cdot 2^3}{(1)}$$

$$= 8$$

e) $$\lim_{x\to\infty}\frac{3x^5-2x^3+1}{(1-x)^5}$$

$$= \frac{\lim_{x\to\infty}(3x^5-2x^3+1)}{\left[\lim_{x\to\infty}(1-x)\right]^5}$$

$$= \frac{\lim_{x\to\infty} x^5\left(3-\frac{2}{x^2}+\frac{1}{x^5}\right)}{\left[\lim_{x\to\infty} x\left(\frac{1}{x}-1\right)\right]^5}$$

$$= \frac{\lim_{x\to\infty} x^5\cdot\lim_{x\to\infty}\left(3-\frac{2}{x^2}+\frac{1}{x^5}\right)}{\left[\lim_{x\to\infty} x\cdot\lim_{x\to\infty}\left(\frac{1}{x}-1\right)\right]^5}$$

$$= \frac{\cancel{\lim_{x\to\infty} x^5}\cdot\lim_{x\to\infty}\left(3-\frac{2}{x^2}-\frac{1}{x^5}\right)}{\cancel{\lim_{x\to\infty} x^5}\left[\lim_{x\to\infty}\left(\frac{1}{x}-1\right)\right]^5}$$

$$= \frac{3-0-0}{(0-1)^5}$$

$$= -3$$

Extra: Recall Binomial expansion

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$$(1-x)^5 = 1(1)^5 + 5(1)^4(-x)^1 + 10(1)^3(-x)^2 + 10(1)^2(-x)^3 + 5(1)^1(-x)^4 + 1(-x)^5$$

$$= 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$$

Rational Function: Consider the rational function $f(x)=\dfrac{P(x)}{Q(x)}=\dfrac{a_nx^n+\ldots+a_1x+a_0}{b_mx^m+\ldots+b_1x+b_0}$. Then

$$\lim_{x\to\pm\infty} f(x)=\begin{cases}\dfrac{a_n}{b_m} & if \quad n=m\\ 0 & if \quad n<m\\ \pm\infty & if \quad n>m\end{cases}$$

Example #2: Find the limit of each function

(a) $\lim\limits_{x\to\infty}\dfrac{3}{x^4}$ $= \dfrac{\lim\limits_{x\to\infty} \cancel{x^4}\left(\frac{3}{x^4}\right)}{\lim\limits_{x\to\infty}\cancel{x^4}(1)} = \dfrac{0}{1}$

$=\underline{\dfrac{1}{\infty}} = 0$

(b) $\lim\limits_{x\to-\infty}\dfrac{2x^2+3}{3x^2-x+4}$ $= \lim\limits_{x\to-\infty}\dfrac{\cancel{x^2}\left(2+\frac{3}{x^2}\right)}{\cancel{x^2}\left(3-\frac{1}{x}+\frac{4}{x^2}\right)}$ $=\dfrac{2+0}{3-0+0}$

$=\underline{\dfrac{2}{3}}$

(c) $\lim\limits_{x\to\infty}\dfrac{1000x-3}{\dfrac{x}{1000}+1}$

$=\underline{\dfrac{1000}{\left(\frac{1}{1000}\right)}} = 10^6$

(d) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{2\sqrt{x}+1}$ $= \lim\limits_{x\to\infty}\dfrac{x^{\frac{1}{2}}}{2x^{\frac{1}{2}}+1}$

$=\underline{\dfrac{1}{2}}$

(e) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x+1}}{\sqrt{4x-1}}$ $=\left[\lim\limits_{x\to\infty}\left(\dfrac{x+1}{4x-1}\right)\right]^{\frac{1}{2}}$

$=\underline{\left(\dfrac{1}{4}\right)^{\frac{1}{2}}}$

$=\dfrac{1}{2}$

(f) $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x}}{\sqrt{x^2+1}}$ $\Rightarrow \dfrac{x^{\frac{1}{3}}}{(x^2)^{\frac{1}{2}}} \to \dfrac{x^{\frac{1}{3}}}{x^1}$ $= x^{-2/3}$ $=\dfrac{1}{\sqrt[3]{x^2}}$ $=0$

$=\underline{0}$

$=\dfrac{\left[\lim\limits_{x\to\infty} x\right]^{\frac{1}{3}}}{\left[\lim\limits_{x\to\infty} x^2+1\right]^{\frac{1}{2}}}$

$=\dfrac{\left[\lim\limits_{x\to\infty} x\left(\frac{1}{x}\right)\right]^{\frac{1}{3}}}{\left[\lim\limits_{x\to\infty} x^2\left(1+\frac{1}{x}\right)\right]^{\frac{1}{2}}}$

$=\dfrac{\lim\limits_{x\to\infty} x^{\frac{1}{3}}\cdot\left[\lim\limits_{x\to\infty}\frac{1}{x}\right]^{\frac{1}{3}}}{\lim\limits_{x\to\infty} x\cdot\left[\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)\right]^{\frac{1}{2}}}$

$=\lim\limits_{x\to\infty} x^{-\frac{2}{3}}\cdot\dfrac{0^{\frac{1}{3}}}{\sqrt{1}}$

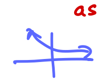
$=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt[3]{x^2}}\cdot 0$

$=0$

Practice

Recall: Basic Rules (Exponential Functions)

Rule 1: : If $0<a<1$ then $\lim\limits_{x\to\infty}a^x=0$.



(Exponential Decay)

Rule 2: : If $a>1$ then $\lim\limits_{x\to\infty}a^x=\infty$.

(Exponential Growth)

Rule 3: : If $a>1$ then $\lim\limits_{x\to-\infty}a^x=0$.

(Exponential Growth)

1) Find the limit of each function.

(a) $\lim\limits_{x\to\infty}\dfrac{3^x-3^{-x}}{3^x+3^{-x}}$ $=\dfrac{\infty-0}{\infty+0}=\dfrac{\infty}{\infty}$

$=\lim\limits_{x\to\infty}\dfrac{\cancel{\frac{1}{3^x}}\left((3^x)^2-1\right)}{\cancel{\frac{1}{3^x}}\left((3^x)^2+1\right)}$

$=1$

(b) $\lim\limits_{x\to\infty}\dfrac{2+\dfrac{1}{4^x}}{4-\dfrac{1}{2^x}}$ $=\dfrac{2+0}{4-0}$ $=\dfrac{1}{2}$

$\left(\frac{1}{4}\right)^x$

$\left(\frac{1}{2}\right)^x$

(c) $\lim\limits_{x\to\infty}\dfrac{(\log_3 27)^x}{\pi^x}$

$=\lim\limits_{x\to\infty}\dfrac{3^x}{\pi^x}$

$=\lim\limits_{x\to\infty}\left(\dfrac{3}{\pi}\right)^x \leftarrow 0<b<1$ Exponential Decay

$=0$

(d) $\lim\limits_{n\to\infty}\left(\dfrac{3^n}{8^n}-\dfrac{6n-2}{2n-3}\right)$

$=\lim\limits_{n\to\infty}\left(\dfrac{3}{8}\right)^n-\lim\limits_{n\to\infty}\left(\dfrac{6n-2}{2n-3}\right)$

$=0-\dfrac{6}{2}$

$=-3$

Warm-Up: Continuity

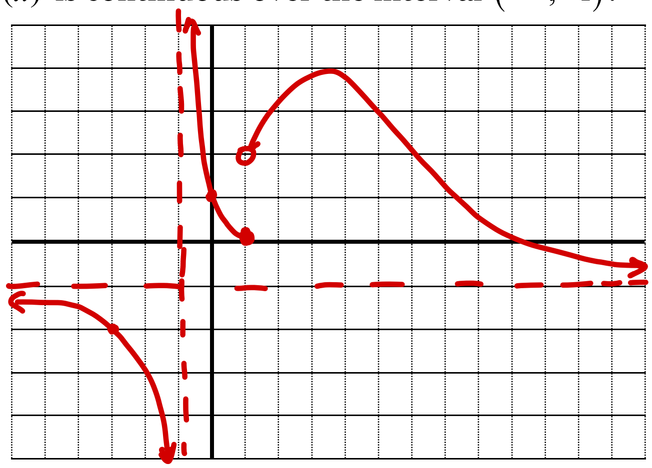
1. Sketch the graph of a function that satisfies **all** of the following conditions:

$\lim_{x\to -3} f(x) = -2$, $\lim_{x\to -1} f(x) = \text{DNE}$, $\lim_{x\to 1^-} f(x) = 0$, $\lim_{x\to 1^+} f(x) = 2$, $\lim_{x\to \infty} f(x) = -1$,

HA

$f(0) = 1$, and $f(x)$ is continuous over the interval $(-\infty, -1)$.

TH ✓✓✓



$y = -1$

<Answers may vary>

2. Function $f(x) = \begin{cases} \dfrac{x-3}{\sqrt{5x+1}-\sqrt{3x+7}} & x \in \left[\dfrac{-1}{5}, 3\right) \cup (3, \infty) \\ k & x = 3 \end{cases}$ is given. For what value of the constant k is the function continuous on its domain?

$f(3) = k$ is defined

$\lim_{x\to 3} f(x) = k$

$\dfrac{\sqrt{5(3)+1} + \sqrt{3(3)+7}}{2} = k$

$\dfrac{4+4}{2} = k$

$k = 4$

Aside:

$\dfrac{x-3}{\sqrt{5x+1}-\sqrt{3x+7}} \cdot \dfrac{\sqrt{5x+1}+\sqrt{3x+7}}{\sqrt{5x+1}+\sqrt{3x+7}}$

$= \dfrac{(x-3)(\sqrt{5x+1}+\sqrt{3x+7})}{(5x+1)-(3x+7)}$

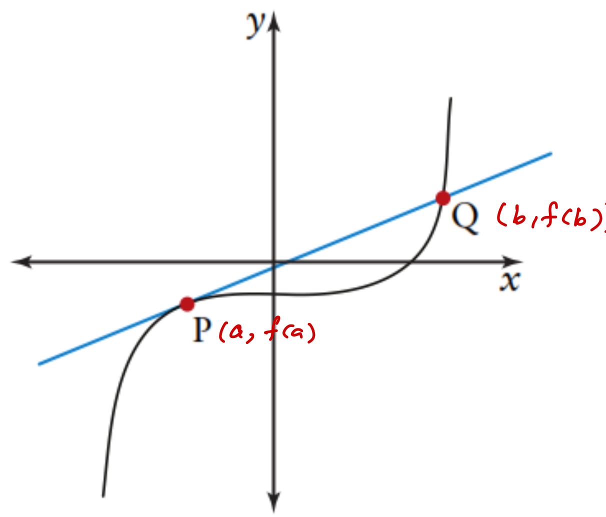
$= \dfrac{(x-3)(\sqrt{5x+1}+\sqrt{3x+7})}{2x-6}$

$= \dfrac{\sqrt{5x+1}+\sqrt{3x+7}}{2}$

1.4 The Slope of a Tangent

In MHF4U, you learned that the **average rate of change** (ARoC) of a function $f(x)$ between two points $P(a, f(a))$ and $Q(b, f(b))$ was determined by calculating the slope of the secant line, m_S, joining P and Q:

$$\boxed{m_S = \frac{f(b)-f(a)}{b-a}}$$



Then, as Q gets closer and closer to P, the slope of the secant line can be used to approximate the slope of the line tangent to f at P, which we called the **instantaneous rate of change** of $f(x)$ at P.

Tangent

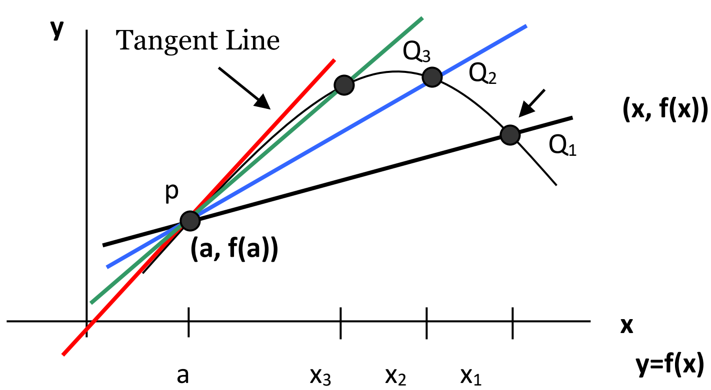
A tangent is the straight line that most resembles the graph near a point. Its slope tells how steep the graph is at the point of tangency.

Slope of a Tangent

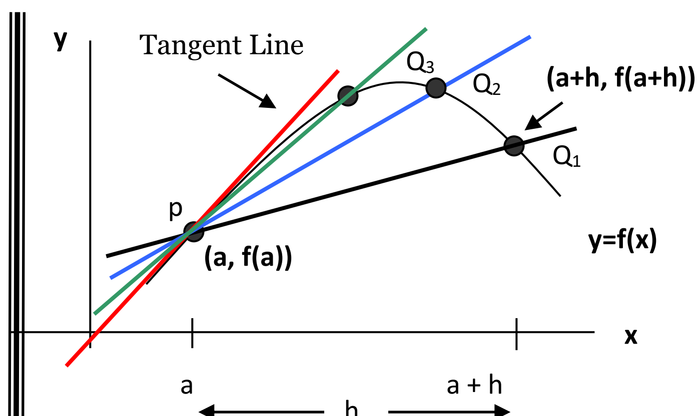
The slope of the tangent to a curve at a point *P* is the limiting slope of the secant *PQ* as the point *q* slides along the curve toward *P*. In other words, the slope of the tangent is said to be the **limit** of the slope of the secant as *Q* approaches *P* along the curve

derivatives as defined using

Introduction to Limits



- As Q → P,
- Secant PQ is getting closer to become Tangent at P
- x → a
- $M_{PQ} \to M_P$
- $M_{PQ} = \dfrac{f(x)-f(a)}{x-a}$ (secant @ PQ) ↳ average rate of change (ARoC)
- $M_P = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$ (tangent @ point P) ↳ instantaneous rate of change (IRoC)



Let h be the horizontal displacement between P & Q on x-axis.

- As Q → P,
- Secant PQ is getting closer to become Tangent at P
- h → o
- $M_{PQ} \to M_P$
- $M_{PQ} = \dfrac{f(a+h)-f(a)}{h}$
- $M_P = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$ ⇒ Slope of a tangent @ point P

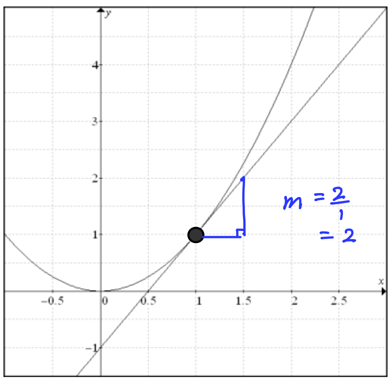
Slope of a Tangent as a Limit

The slope of the tangent to the graph $y = f(x)$ at point *P* (*a*, *f*(*a*)) is

$M = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, if this limit exists.

Example 1: Slope of a Tangent as a limiting value

Find the slope of the tangent to the curve $f(x) = x^2$ at the point *P* when *x* = 1.



$f'(x) = m_T = m_{x=1}$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ ⇒ derivative by First Principle

$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0} \frac{(1+h)^2 - (1)^2}{h}$

$= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}}$

$= 2 + (0)$

$= 2$

Example 2: Slope of a Tangent as a limiting value (Cubic Function)

Use limits to find the slope of the tangent line to $f(x) = 3x^3 + 2x - 4$ at the point when *x* = -1.

1
1 1
1 2 1
1 3 3 1

Aside: $(x+h)^3$

$= x^3 + 3x^2h + 3xh^2 + h^3$

$m_{x=-1} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{[3(x+h)^3 + 2(x+h) - 4] - [3x^3 + 2x - 4]}{h}$

$= \lim_{h \to 0} \frac{1}{h}(\cancel{3x^3} + 9x^2h + 9xh^2 + 3h^3 + \cancel{2x} + 2h \cancel{-4} \cancel{-3x^3} - \cancel{2x} + \cancel{4})$

$= \lim_{h \to 0} \frac{1}{h}(9x^2h + 9xh^2 + 3h^3 + 2h)$

$= \lim_{h \to 0} 9x^2 + 9xh + 3h^2 + 2$

$= 9x^2 + 2$

$m_{x=-1} = 9(-1)^2 + 2$

$= 11$

Binomial Expansion & PASCAL's Triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ {}_5C_0 & & {}_5C_1 & & {}_5C_2 & & {}_5C_3 & & {}_5C_4 & & {}_5C_5 \end{array}$$

etc

$(a+b)^0 \qquad 1$

$(a+b)^1 \qquad 1a + 1b$

$(a+b)^2 \qquad 1a^2 + 2ab + 1b^2$

$(a+b)^3 \qquad 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a+b)^4 \qquad 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

$(a+b)^5 \qquad 1a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

Example 3: Equation of a Tangent as a limiting value (Rational Function)

Use limits to find the equation of the tangent line to $f(x)=\frac{2x-5}{x}$ ← MKF at point (5, 1).

$= -\frac{5}{x} + 2$ ← MCR

$= -5\left(\frac{1}{x}\right)+2$

$$m_{x=5} = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2(5+h)-5}{5+h} - \left(\frac{2(5)-5}{5}\right)\right]$$

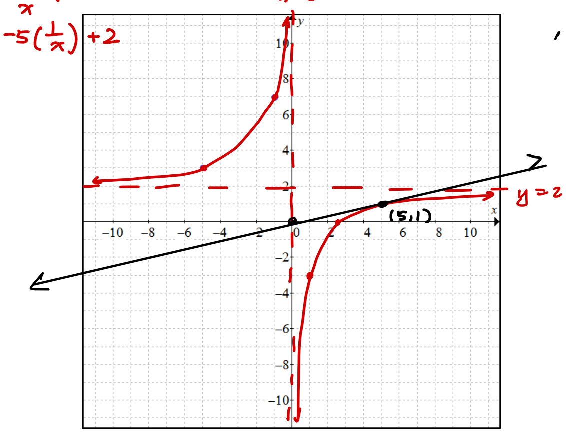
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{10+2h-5}{5+h} - 1\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2h+5}{h+5} - 1\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2h+5-h-5}{h+5}\right]$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{h}}{h+5}\right]$$

$$= \frac{1}{5}$$



Equation: $y-1=\frac{1}{5}(x-5)$

or $y=\frac{1}{5}x$

Example 4: Equation of a Tangent as a limiting value (Radical Function)

Use limits to find the equation of the tangent line to $f(x)=\sqrt{x-4}$ at point $x = 8$.

$f(8)=\sqrt{8-4}$
$=2$

$$m_{x=8} = \lim_{h\to 0} \frac{f(8+h)-f(8)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(8+h)-4} - \sqrt{8-4}}{h}$$

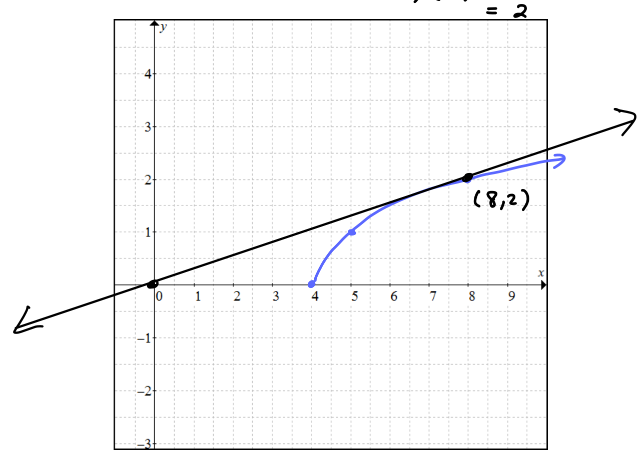
$$= \lim_{h\to 0} \frac{\sqrt{h+4}-2}{h} \cdot \frac{\sqrt{h+4}+2}{\sqrt{h+4}+2}$$

$$= \lim_{h\to 0} \frac{\cancel{h}+\cancel{4}-\cancel{4}}{\cancel{h}(\sqrt{h+4}+2)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{h+4}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$= \frac{1}{4}$$



Equation: $y-2=\frac{1}{4}(x-8)$

or

$y=\frac{1}{4}x-2+2$

$y=\frac{1}{4}x$

Practice:

1. Find the **equation of tangent line** in slope-point form, to the curve $f(x)=3x^2+2$, at x = 1.

2. Find the equation of the tangent in slope-point form to the curve $y=\frac{1}{x-1}$ at (2,1)

3. Find the **equation of tangent line** in standard form to the curve $f(x)=\sqrt{3-x}$ at x = -1 on the curve.

4. Determine the coordinates of the point where the tangent to the function $y = x^3$ is perpendicular to the line $3x + 4y - 12 = 0$.

5. A ball is tossed up in the air so that its position ***d*** in metres at time ***t*** seconds is given by $d(t)=-5t^2+30t+2$.

    i. What is the average velocity for the interval $[4,5]$ and $[4,4.1]$

    ii. What is the instantaneous velocity of the ball at $t = 4$?

6. An oil tank is being drained. The volume V in litres, of oil remains in the tank after time t, in minutes, is represented by the function $v(t)=60(25-t^2),\ t\in[0,25]$.

    i. Determine the average rate of change from $t=5$ to $t=15$.

    ii. Determine the instantaneous velocity at $t=10$ using limits.

7. *Determine the coordinates of the point(s) on the graph of $y=3x-\frac{1}{x}$ at which the slope of the tangent is 7.

CP37

Practice:

1. Find the **equation of tangent line** in slope-point form, to the curve $f(x)=3x^2+2$, at x = 1.

$$m_{x=1} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{[3(1+h)^2+2]-[5]}{h}$$
$$= \lim_{h\to 0} \frac{[3(1+2h+h^2)+2]-5}{h}$$
$$= \lim_{h\to 0} \frac{3h^2+6h}{h}$$
$$= \lim_{h\to 0} 3h+6$$
$$= 6$$

$f(1) = 3(1)^2+2$
$= 5$

$\therefore$ equation: $y-5 = 6(x-1)$

2. Find the equation of the tangent in slope-point form to the curve $y=\dfrac{1}{x-1}$ at (2,1)

$f(2) = 1$

$$m_{x=2} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\left(\frac{1}{(2+h)-1}\right)-1\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{h+1}-1\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1-(h+1)}{h+1}\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-h}{h+1}\right]$$
$$= \lim_{h\to 0} \frac{-1}{h+1}$$
$$= -1$$

$\therefore$ the equation is

$y-1 = -(x-2)$ ⇐ point-slope form

or $y = -x+3$ ⇐ slope y-intercept form

or $x+y-3=0$ ⇐ standard form

3. Find the **equation of tangent line** in standard form to the curve $f(x)=\sqrt{3-x}$ at x = -1 on the curve.

$f(-1) = 2$

$$m_{x=-1} = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{3-(-1+h)}-2}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{4-h}-2}{h} \cdot \frac{\sqrt{4-h}+2}{\sqrt{4-h}+2}$$
$$= \lim_{h\to 0} \frac{(4-h)-4}{h[\sqrt{4-h}+2]}$$
$$= \lim_{h\to 0} \frac{-1}{\sqrt{4-h}+2}$$
$$= \frac{-1}{4}$$

$\therefore$ the equation is:

$y-2 = -\frac{1}{4}(x+1)$ ⇐ point-slope form

$y-2 = -\frac{1}{4}x-\frac{1}{4}$

$\xRightarrow{\times 4}$ $4y-8 = -x-1$

$x+4y-7=0$ ⇐ standard form

4. Determine the coordinates of the point where the tangent to the function $y = x^3$ is perpendicular to the line $3x + 4y - 12 = 0$.

$l: \; 3x + 4y - 12 = 0$

$4y = -3x + 12$

$y = -\frac{3}{4}x + 3$

$m = -\frac{3}{4}$

$\therefore m_{\perp} = \frac{4}{3}$

$$m_T = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

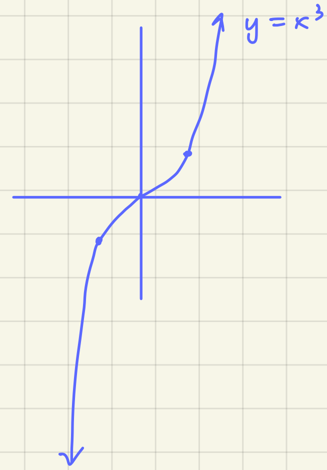
$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$\frac{4}{3} = 3x^2$$

$$\frac{4}{9} = x^2$$

$$\therefore x = \pm \frac{2}{3}$$



Aside:

$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

Binomial Expansion + PASCAL's Δ

1

1 1

1 2 1

1 3 3 1

$f\left(-\frac{2}{3}\right) = \left(\frac{-2}{3}\right)^3 = \frac{-8}{27}$

$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

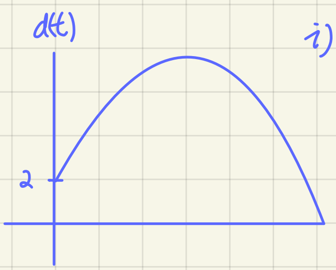
$\therefore$ ~~The~~ There are 2 point of tangency on $f(x) = x^3$ where the tangent line will be perpendicular to the line $3x + 4y - 12 = 0$. They are: $\left(\frac{-2}{3}, \frac{-8}{27}\right)$ and $\left(\frac{2}{3}, \frac{8}{27}\right)$

< See Moshtagh's Notes for the visual for this >

5. A ball is tossed up in the air so that its position ***d*** in metres at time ***t*** seconds is given by $d(t) = -5t^2 + 30t + 2$.

i. What is the average velocity for the interval $[4, 5]$ and $[4, 4.1]$

ii. What is the instantaneous velocity of the ball at $t = 4$?



i) AROC

$$m_{[4,5]} = \frac{d(5) - d(4)}{5 - 4}$$

$$= \frac{\left[-5(5)^2 + 30(5) + 2\right] - \left[-5(4)^2 + 30(4) + 2\right]}{1}$$

$$= 27 - 42$$

$$= -15 \text{ m/s}$$

Aside:

$d(4.5) = -5(4.5)^2 + 30(4.5) + 2$

$= 35.75$

$d(4) = 42$

$d(4.1) = -5(4.1)^2 + 30(4.1) + 2$

$= 40.95$

$$m_{[4,4.1]} = \frac{d(4.1) - d(4)}{4.1 - 4}$$

$$= \frac{40.95 - (42)}{0.1}$$

$$= -10.5 \text{ m/s}$$

$\Rightarrow$ Continued

#5 ii) IROC :

$$m_{t=4} = \lim_{h\to 0} \frac{d(4+h) - d(4)}{h}$$

$$= \lim_{h\to 0} \frac{[-5(4+h)^2 + 30(4+h) + 2] - [42]}{h}$$

$$= \lim_{h\to 0} \frac{[-5(16+8h+h^2) + 120 + 30h + 2] - 42}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-80} - 40h - 5h^2 \cancel{+120} + 30h \cancel{-40}}{h}$$

$$= \lim_{h\to 0} \frac{-5h^2 - 10h}{h}$$

$$= \lim_{h\to 0} -5h - 10$$

$$= -10 \text{ m/s}$$

6. An oil tank is being drained. The volume V in litres, of oil remains in the tank after time t, in minutes, is represented by the function $v(t) = 60(25 - t^2),\ t \in [0,25]$.

   i. Determine the average rate of change from $t = 5$ to $t = 15$.

   ii. Determine the instantaneous velocity at $t = 10$ using limits.

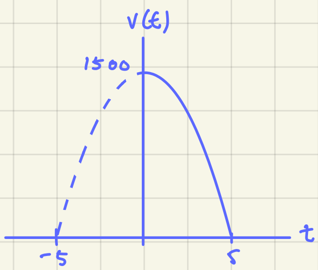
* ⇐ there's a problem with this question $t \in [0,5]$ only

i) $$m_{[5,15]} = \frac{V(15) - V(5)}{15-5}$$

$$= \frac{[60(25-(15)^2)] - [60(25-(5)^2)]}{10}$$

$$= \frac{(-12000) - (0)}{10}$$

$$= -1200 \text{ L/min}$$



$$V(t) = 60(25 - t^2)$$
$$= -60(t^2 - 25)$$
$$= -60(t+5)(t-5)$$

ii) $$m_{t=10} = \lim_{h\to 0} \frac{V(10+h) - V(10)}{h}$$

$$= \lim_{h\to 0} \frac{60[25 - (10+h)^2] - 60(25 - (10)^2)}{h}$$

$$= 60 \lim_{h\to 0} \frac{[25 - (100 + 20h + h^2)] - [-75]}{h}$$

$$= 60 \lim_{h\to 0} \frac{\cancel{25} \cancel{-100} - 20h - h^2 \cancel{+75}}{h}$$

$$= 60 \lim_{h\to 0} (-20 - h)$$

$$= 60(-20)$$

$$= -1200 \text{ L/min}$$

7. *Determine the coordinates of the point(s) on the graph of $y = 3x - \frac{1}{x}$ at which the slope of the tangent is 7.

$$m_T = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\left(3(x+h) - \frac{1}{x+h}\right) - \left(3x - \frac{1}{x}\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[3(x+h) - \frac{1}{x+h} - 3x + \frac{1}{x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\cancel{3x} + 3h - \frac{1}{x+h} - \cancel{3x} + \frac{1}{x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[3h - \frac{1}{x+h} + \frac{1}{x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{3hx(x+h) - x + (x+h)}{x(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{3x^2h + 3xh^2 - \cancel{x} + \cancel{x} + h}{x(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{3x^2 + 3xh + 1}{x(x+h)}$$

$$= \frac{3x^2+1}{x^2}$$

$$7 = \frac{3x^2+1}{x^2}$$

$$7x^2 = 3x^2+1$$

$$4x^2 = 1$$

$$x = \pm\frac{1}{2}$$

$$f(x) = 3x - \frac{1}{x}$$

$$f\left(\tfrac{1}{2}\right) = 3\left(\tfrac{1}{2}\right) - \frac{1}{\left(\frac{1}{2}\right)}$$

$$= \frac{3}{2} - 2$$

$$= -\frac{1}{2}$$

$$f\left(-\tfrac{1}{2}\right) = 3\left(-\tfrac{1}{2}\right) - \frac{1}{\left(-\frac{1}{2}\right)}$$

$$= -\frac{3}{2} + 2$$

$$= \frac{1}{2}$$

$\therefore$ @ the points : $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Warm-up: The Slope of a Tangent

Write the CAPITAL LETTER corresponding to the correct answer on the line provided

1. Consider the function $y = f(x)$. Let P and Q be two points on the graph of $f(x)$.The limit of the slope of the secant PQ, as Q approaches P along the graph, is ___B___

   A. undefined.

   B. the slope of the tangent to the graph of $f(x)$ at P.

   C. the slope of the tangent to the graph of $f(x)$ at Q.

   D. the slope of the tangent to the graph of $f(x)$ at the midpoint between P and Q.

2. A curve is defined by $f(x) = \boldsymbol{a}\sqrt{x} + \boldsymbol{b}$, where a and b are constants. Given that $m_t = 3$ at the point (4, 6), determine the values of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$m_T = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h \to 0} \frac{[a\sqrt{4+h} + b] - [a\sqrt{4} + b]}{h}$$

$$= \lim_{h \to 0} \frac{a\sqrt{4+h} \cancel{+b} - 2a \cancel{-b}}{h}$$

$$= \lim_{h \to 0} \frac{a\sqrt{4+h} - 2a}{h}$$

$$= a \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$$

$$= a \lim_{h \to 0} \frac{(4+h) - 4}{h(\sqrt{4+h}) + 2}$$

$$= a \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2}$$

$$3 = a\left(\frac{1}{4}\right)$$

$$a = 12$$

$$f(4) = 6$$

$$a\sqrt{4} + b = 6$$

$$2a + b = 6$$

$$2(12) + b = 6$$

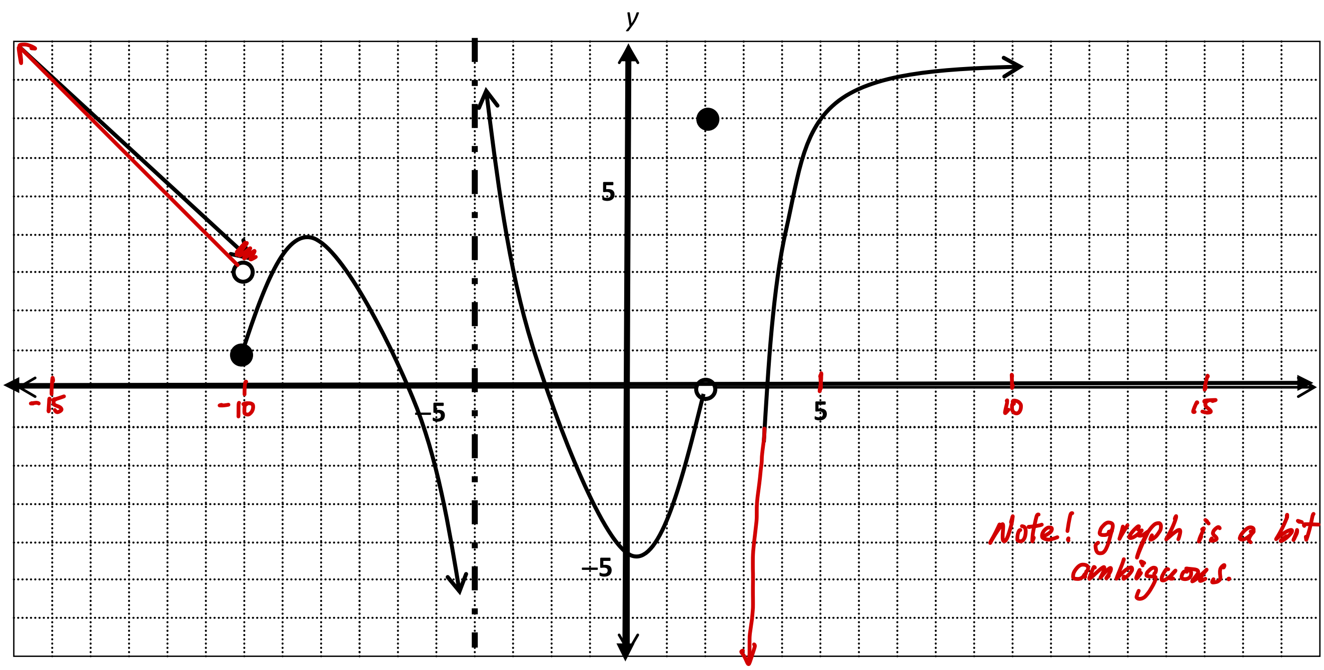
$$b = -18$$

$$\therefore a = 12$$

$$b = -18$$

MID-REVIEW
PUSHING YOUR BRAIN TO THE LIMIT

1. Consider the following graph of $f(x)$. Evaluate the following limits. .



i) $\lim_{x\to -10^+} f(x) = 1$ **ii)** $\lim_{x\to -10^-} f(x) = 3$ **iii)** $f(2) = 7$

iv) $\lim_{x\to 2^+} f(x) = -\infty$ or dne **v)** $\lim_{x\to 2^-} f(x) = 0$ **vi)** $\lim_{x\to 2} f(x) =$ dne

vii) $\lim_{x\to -4^+} f(x) = +\infty$ **viii)** $\lim_{x\to -4^-} f(x) = -\infty$ **ix)** $\lim_{x\to\infty} f(x) = +\infty$ or 9

x) $\lim_{x\to -\infty} f(x) = +\infty$

2. Based on graph above, answer the following questions:

a) An example of a **removable discontinuity** is when x=$\underline{\ 2\ }$
b) An example of a **jump discontinuity** is when x=$\underline{\ -10\ }$
c) An example of an **infinite discontinuity** is when x=$\underline{\ -4\ }$

3. Evaluate the following indeterminate limits

a) $\lim_{x\to\infty} \frac{4^{x-10}}{5^{x+10}}$ $= \lim_{x\to\infty} \frac{4^x \cdot 4^{-10}}{5^x \cdot 5^{10}}$
$= \lim_{x\to\infty} \left(\frac{4}{5}\right)^x \cdot \lim_{x\to\infty} 4^{-10}\cdot 5^{-10}$
$= 0 \cdot (4\cdot 5)^{-10}$
$= 0$

b) $\lim_{x\to -2} \frac{x^3+x^2-x+2}{x^2-x-6}$ $\frac{0}{0}$

$f(x) = x^3+x^2-x+2$
$f(-2) = 0 \therefore (x+2)$ is a factor

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -1 & 2 \\ & \downarrow & -2 & 2 & -2 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

$= \lim_{x\to -2} \frac{\cancel{(x+2)}(x^2-x+1)}{(x-3)\cancel{(x+2)}}$
$= \frac{(-2)^2-(-2)+1}{(-2-3)}$
$= -\frac{7}{5}$

c) $\lim_{x\to 3^+} \frac{x+3}{x^2-9}$ $= \lim_{x\to 3^+} \frac{\cancel{x+3}}{(x-3)\cancel{(x+3)}}$
$= \lim_{x\to 3^+} \frac{1}{x-3}$
$= \frac{1}{3.001-3} = \frac{1}{0.001} = +\infty$

d) $\lim_{x\to 9} \frac{x\sqrt{x}-27}{\sqrt{x}-3}$ $= \lim_{x\to 9} \frac{x^{\frac{3}{2}}-27}{x^{\frac{1}{2}}-3}$

let $a = x^{\frac{1}{2}}$ $\therefore x\to 9$
$a^2 = x$ $a^2 \to 9$
$a \to 3$

$= \lim_{a\to 3} \frac{a^3-27}{a-3}$
$= \lim_{a\to 3} \frac{\cancel{(a-3)}(a^2+3a+9)}{\cancel{a-3}}$
$= (3)^2+3(3)+9$
$= 27$

e) $\lim_{x\to 5} \frac{\sqrt{x^2-9}-4}{\sqrt{x-1}-2}$ $\cdot \frac{\sqrt{x^2-9}+4}{\sqrt{x^2-9}+4} \cdot \frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}$
$= \lim_{x\to 5} \frac{[(x^2-9)-16][\sqrt{x-1}+2]}{[(x-1)-4][\sqrt{x^2-9}+4]}$ $= \lim_{x\to 5} \frac{(x+5)\cancel{(x-5)}(\sqrt{x-1}+2)}{\cancel{(x-5)}(\sqrt{x^2-9}+4)}$
$= \frac{(5+5)(2+2)}{(\sqrt{25-9}+4)}$
$= \frac{40}{8}$
$= 5$

f) $\lim_{x\to 10} \frac{(x-2)^{\frac{1}{3}}-2}{x-10}$ $= \lim_{a\to 2} \frac{a-2}{(a^3+2)-10} = \lim_{a\to 2} \frac{a-2}{a^3-8}$

let $a = (x-2)^{\frac{1}{3}}$
$a^3 = (x-2)$
$a^3+2 = x$
$\therefore x\to 10$
$a^3+2\to 10$
$a^3 \to 8$
$a\to 2$

$= \lim_{a\to 2} \frac{\cancel{a-2}}{\cancel{(a-2)}(a^2+2a+4)}$
$= \frac{1}{2^2+2(2)+4}$
$= \frac{1}{12}$

g) $\lim_{x\to 3} \frac{x^{-2}-3^{-2}}{x-3}$
$= \lim_{x\to 3} \frac{\frac{1}{x^2}-\frac{1}{9}}{x-3}$
$= \lim_{x\to 3} \frac{9-x^2}{9x^2}\cdot\frac{1}{x-3}$
$= \lim_{x\to 3} \frac{-(x+3)\cancel{(x-3)}}{9x^2\cancel{(x-3)}}$
$= \lim_{x\to 3} \frac{-(x+3)}{9x^2}$
$= \frac{-6}{81}$
$= \frac{-2}{27}$

h) $\lim_{x\to\infty} \frac{5x^2-3x+4}{2x^2+x-7}$
$= \lim_{x\to\infty} \frac{\cancel{x^2}\left(5+\frac{3}{x}+\frac{4}{x^2}\right)}{\cancel{x^2}\left(2+\frac{1}{x}-\frac{7}{x^2}\right)}$
$= \lim_{x\to\infty} \frac{5+\frac{3}{x}+\frac{4}{x^2}}{2+\frac{1}{x}-\frac{7}{x^2}}$
$= \frac{5+\frac{3}{\infty}+\frac{4}{\infty}}{2+\frac{1}{\infty}-\frac{7}{\infty}}$
$= \frac{5}{2}$

MHF Method
$= \frac{5}{2}$

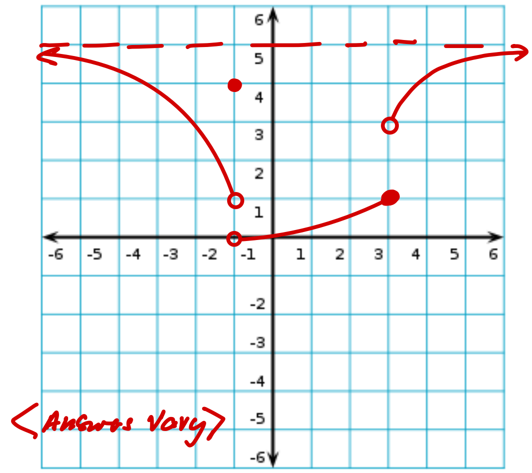
i) $\lim_{x\to\infty} \frac{(1-x^2)^3}{(3-2x^3)^2}$ $= \frac{\left[\lim_{x\to\infty}(1-x^2)\right]^3}{\left[\lim_{x\to\infty}(3-2x^3)\right]^2} = \frac{\left[\lim_{x\to\infty} x^2\left(\frac{1}{x^2}-1\right)\right]^3}{\left[\lim_{x\to\infty} x^3\left(\frac{3}{x^3}-2\right)\right]^2}$

MHF method
$= \frac{(-1)^3x^6}{(-2)^2x^6}$
$= \frac{-1}{4}$

$= \lim_{x\to\infty} \frac{x^6}{x^6} \cdot \frac{\left[\lim_{x\to\infty}\left(\frac{1}{x^2}-1\right)\right]^3}{\left[\lim_{x\to\infty}\left(\frac{3}{x^3}-2\right)\right]^2}$
$= 1 \cdot \frac{(-1)^3}{(-2)^2}$
$= \frac{-1}{4}$

4. Is there a value of k for which $\lim_{x\to -2}\frac{kx^2-6x+3-k}{x^2+3x+2}$ exists? If so, find k and find the values of the limit.

5. Create a sketch of a function f(x) where the following restrictions are met: *(There is not one correct answer. All that is required is that you meet these conditions)*



- $\lim_{x\to -1^-} f(x)=1$
- f(-1) = 4
- $\lim_{x\to 3} f(x)=DNE$
- $\lim_{x\to -1^+} f(x)=0$
- $\lim_{x\to \infty} f(x)=5$
- $\lim_{x\to -\infty} f(x)=5$

6. Given $f(x)=\begin{cases} 5-x^2, & x\le -1 \\ ax+b, & -1<x\le 4 \\ 1-\sqrt{x}, & x>4 \end{cases}$,Find the values of a and b such that the function is continuous at -1 and 4 .

7. Determine whether or not function $f(x)=\begin{cases} \sqrt{x+3} & , x\in(-\infty,1] \\ -x+3 & , x\in(1,7) \\ 10-3x & , x\in(7,\infty) \end{cases}$ is continuous at $x=1$ and $x=7$.

8. If $\lim_{x\to\infty}\frac{ax^4+1}{2x^b+x^2+1}=2$, find the value of a+b.

9. Determine the values of p and q such that $\lim_{x\to 0}\frac{\sqrt{px+q}-3}{x}=1$.

10. Jack and Cole are throwing a Frisbee in their backyard. When Jack throws to Cole, the height, in metres, of the Frisbee above the ground after t seconds is described by the function $h(t)=-4.9t^2+10.8t+1$.

a) Determine the average speed of the Frisbee between $t=1$ second and $t=3$ seconds.
b) Determine the instantaneous speed of the Frisbee when $t=1$ second.

4. Is there a value of k for which $\lim_{x\to -2}\frac{kx^2-6x+3-k}{x^2+3x+2}$ exists? If so, find k and find the values of the limit.

$$\lim_{x\to -2}\frac{kx^2-6x+3-k}{x^2+3x+2}$$

$$= \lim_{x\to -2}\frac{kx^2-6x+3-k}{(x+2)(x+1)}$$

indeterminant : $\frac{0}{0}$ @ $x=-2$

$\therefore$ the numerator must have a factor of $(x+2)$ to cancel out with the denominator

$n(x) = kx^2-6x+3-k$

$n(-2) = k(-2)^2-6(-2)+3-k$

$0 = 4k+12+3-k$

$-15 = 3k$

$k=-5$

$$\therefore \lim_{x\to -2}\frac{-5x^2-6x+3+5}{(x+2)(x+1)}$$

$$= \lim_{x\to -2}\frac{-5x^2-6x+8}{(x+2)(x+1)}$$

$$= \lim_{x\to -2}\frac{-(5x-4)\cancel{(x+2)}}{\cancel{(x+2)}(x+1)}$$

$$= \lim_{x\to -2}\frac{-(5x-4)}{x+1}$$

$$= \frac{14}{-1}$$

$$= -14$$

Aside:

$-5x^2-6x+8$

$= -(5x^2+6x-8)$

$= -(5x-4)(x+2)$

6. Given $f(x)=\begin{cases}5-x^2, & x\le -1\\ ax+b, & -1<x\le 4\\ 1-\sqrt{x}, & x>4\end{cases}$,Find the values of a and b such that the function is continuous at -1 and 4 .

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$5-(-1)^2 = a(-1)+b$

$4 = -a+b$ ①

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$

$a(4)+b = 1-\sqrt{4}$

$4a+b = -1$ ②

$4 = -a+b$ ①

$-1 = 4a+b$ ②

$(-)$ $5 = -5a$

$a = -1$

sub into ①

$4 = 1+b$

$b = 3$

$\therefore a=-1$
$b=3$

7. Determine whether or not function $f(x)=\begin{cases}\sqrt{x+3} & , x\in(-\infty,1]\\ -x+3 & , x\in(1,7)\\ 10-3x & , x\in(7,\infty)\end{cases}$ is continuous at $x=1$ and $x=7$.

@ $x=1$

$f(1) = 2$ $\therefore$ defined

$\lim_{x\to 1^-} f(x) = \sqrt{(1)+3}$
$= 2$

$\lim_{x\to 1^+} f(x) = -(1)+3$
$= 2$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+}$

and $\lim_{x\to 1} f(x) = f(1)$

$\therefore$ @ $x=1$ $f(x)$ is continuous.

@ $x=7$

$f(7)$ = undefined

$\therefore$ discontinuous @ $x=7$

8. If $\lim_{x\to\infty} \frac{ax^4+1}{2x^b+x^2+1} = 2$, find the value of a+b.

note!

$\Rightarrow$ for a limit @ infinity to exist @ 2 there must be a horizontal asymptote at $y=2$ which implies that the degree of the numerator = degree of denominator

$\therefore b = 4$

$$\lim_{x\to\infty} \frac{\cancel{x^b}\left(\frac{ax^4}{x^b} + \frac{1}{x^b}\right)}{\cancel{x^b}\left(2 + \frac{x^2}{x^b} + \frac{1}{x^b}\right)}$$

$$\frac{\frac{ax^4}{x^b}}{2} = 2$$

$$\frac{ax^4}{x^b} = 4$$

$\because b = 4$ then $a = 4$

$\therefore a + b = 8$

9. Determine the values of p and q such that $\lim_{x\to 0} \frac{\sqrt{px+q}-3}{x} = 1$.

must make indeterminant @ $x=0$ $\therefore$ let $\sqrt{px+q} - 3 = 0$ @ $x=0$

$$\lim_{x\to 0} \frac{\sqrt{px+q}-3}{x} \cdot \frac{\sqrt{px+q}+3}{\sqrt{px+q}+3} = 1$$

$$\lim_{x\to 0} \frac{px+q-9}{x\left(\sqrt{px+q}+3\right)} = 1$$

let $h(x) = \sqrt{px+q} - 3$

$h(0) = \sqrt{p(0)+q} - 3$

$0 = \sqrt{q} - 3$

$3 = \sqrt{q}$

$\therefore q =$

if $q = 9$, $\lim_{x\to 0} \frac{p\cancel{x}}{\cancel{x}\left(\sqrt{px+9}+3\right)}$

$= \lim_{x\to 0} \frac{p}{\sqrt{px+9}+3}$

$= \frac{p}{6}$

$\therefore \frac{p}{6} = 1$

$p = 6$

$\therefore q = 9 \quad p = 6$

10. Jack and Cole are throwing a Frisbee in their backyard. When Jack throws to Cole, the height, in metres, of the Frisbee above the ground after t seconds is described by the function $h(t) = -4.9t^2 + 10.8t + 1$.

a) Determine the average speed of the Frisbee between $t = 1$ second and $t = 3$ seconds.
b) Determine the instantaneous speed of the Frisbee when $t = 1$ second.

a) AROC

$$m_{[1,3]} = \frac{h(3) - h(1)}{3-1}$$

interval between $t=1$ and $t=3$

$$= \frac{\left[-4.9(3)^2 + 10.8(3) + 1\right] - \left[-4.9(1)^2 + 10.8(1) + 1\right]}{2}$$

$$= \frac{-10.7 - 6.9}{2}$$

$= -8.8$ m/s

b) IROC

$$m_{t=1} = \lim_{H\to 0} \frac{h(1+H) - h(1)}{H}$$

$$= \lim_{H\to 0} \frac{\left[-4.9(1+H)^2 + 10.8(1+H) + 1\right] - \left[-4.9(1)^2 + 10.8(1) + 1\right]}{H}$$

$$= \lim_{H\to 0} \frac{1}{H}\left[\left[-4.9(1+2H+H^2) + 10.8 + 10.8H + 1\right] - [6.9]\right]$$

$$= \lim_{H\to 0} \frac{1}{H}\left[(-4.9H^2 + H + \cancel{6.9}) - \cancel{6.9}\right]$$

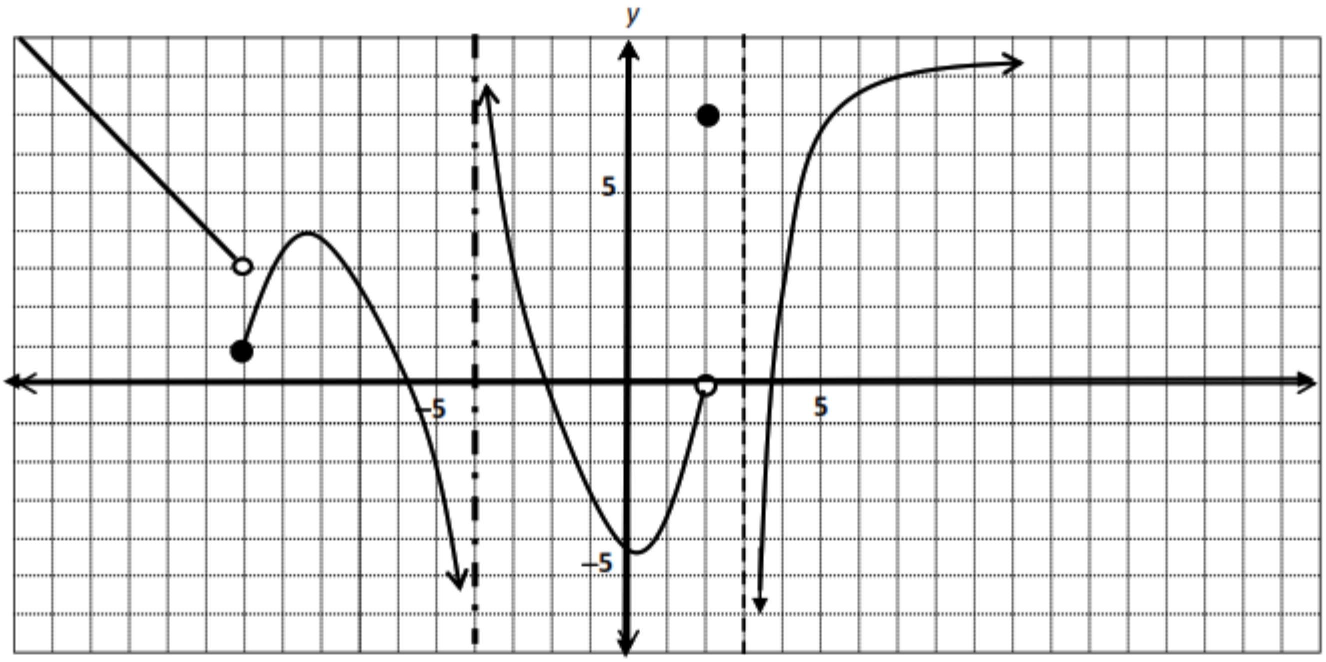
$$= \lim_{H\to 0} \frac{1}{\cancel{H}}\left[\cancel{H}(-4.9H + 1)\right]$$

$$= \lim_{H\to 0} -4.9H + 1$$

$= 1$ m/s

MID-REVIEW
PUSHING YOUR BRAIN TO THE LIMIT

1. Consider the following graph of $f(x)$.



Evaluate the following limits. If the limit does not exist you must provide a reason.

i) $\lim\limits_{x\to -10^+} f(x)$ $=1$

ii) $\lim\limits_{x\to -10^-} f(x)$ $=3$

iii) $f(2)$ $=7$

iv) $\lim\limits_{x\to 2^+} f(x)$ $=$**DNE**

v) $\lim\limits_{x\to 2^-} f(x)$ $=0$

vi) $\lim\limits_{x\to 2} f(x)$ $=$ **DNE**

vii) $\lim\limits_{x\to -4^+} f(x)$ $=\infty$

viii) $\lim\limits_{x\to -4^-} f(x)$ $=-\infty$

ix) $\lim\limits_{x\to \infty} f(x)$ $=\infty$

2. Based on graph above, answer the following questions:

a) An example of a **removable discontinuity** is when x=__None__

b) An example of a **jump discontinuity** is when x=_**-10**___

c) An example of an **infinite discontinuity** is when x=_**-4**___

3. Evaluate the following indeterminate limits

a) $\lim\limits_{x\to\infty} \dfrac{4^{x-10}}{5^{x+10}}$

$=\lim\limits_{x\to\infty} \dfrac{4^x \times 4^{-10}}{5^x \times 5^{10}}$

$=\lim\limits_{x\to\infty} \left(\dfrac{4}{5}\right)^x \times \dfrac{1}{20^{10}}$

$=0$

b) $\lim\limits_{x\to -2} \dfrac{x^3+x^2-x+2}{x^2-x-6}$

$=\lim\limits_{x\to -2} \dfrac{\cancel{(x+2)}\left(x^2-x+1\right)}{\cancel{(x+2)}(x-3)}$

$=\lim\limits_{x\to -2} \dfrac{x^2-x+1}{x-3}$

$=-\dfrac{7}{5}$

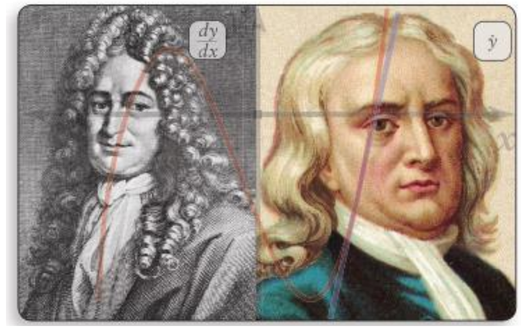
c) $\lim\limits_{x\to 3^+} \dfrac{x+3}{x^2-9}$

$=\lim\limits_{x\to 3^+} \dfrac{\cancel{x+3}}{\cancel{(x+3)}(x-3)}$

$=\lim\limits_{x\to 3^+} \dfrac{1}{(x-3)}$

$=\infty$

1. 5 Derivatives of Polynomial Functions by First Principles



First Principles Definition of the Derivative

New notation: $\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$, if the limit exists.

- The process is called differentiating a function (or differentiation).
- The method of first principles means to use the formula above.
- The domain of $f'(x)$ depends on where the limit exists.
- It is as subset of the domain $f(x)$.
- **A derivative is a function. A collection of tangent values! (not a tangent)**

A few more notations for derivatives come from Leibniz, called **Leibniz notation**.

(i) $f'(x)$ (ii) $\frac{d}{dx}f(x)$ $\frac{df(x)}{dx}$ (iii) $\frac{dy}{dx}$ **(Leibniz's)** (iv) y'

Given a particular value **a**:

(i) $f'(a)$ (ii) $\left.\frac{dy}{dx}\right|_{x=a}$ "dy by dx such that x equals a" $\Rightarrow m_{x=a}$

Ex1: Differentiate the function $y=2x^2+6$ using first principles.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{[2(x+h)^2+6]-[2x^2+6]}{h}$$
$$= \lim_{h\to 0}\frac{\cancel{2x^2}+4xh+2h^2\cancel{+6}-\cancel{2x^2}\cancel{-6}}{h}$$
$$= \lim_{h\to 0}\frac{\cancel{h}(4x+2h)}{\cancel{h}}$$
$$= 4x \Leftarrow \text{the general equation for the slope of a tangent at any point.}$$

$f'(x)=4x$

$\therefore x=5,\ f'(5)=4(5)=20$

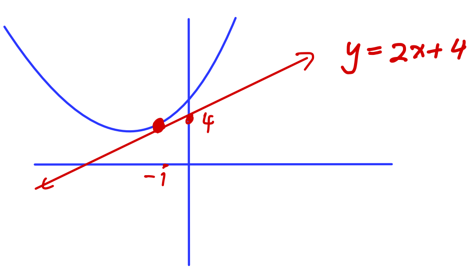
$\therefore x=-100,\ f'(-100)=4(-100)=-400$

$\therefore x=a,\ f'(a)=4a$

Ex2: Function $f(x)=2x^2+ax+b$ is given. Find the constants a and b such that line with equation y=2x+4 is tangent to the graph of function f(x) at point with x=-1. ~~and f(x) passes through the point (1,6)~~.

$f'(-1)=2$

$f(-1)=2(-1)+4$
$=2$



$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{[2(x+h)^2+a(x+h)+6]-[2x^2+ax+6]}{h}$$
$$= \lim_{h\to 0}\left[\frac{\cancel{2x^2}+4xh+2h^2+\cancel{ax}+ah+\cancel{b}-\cancel{2x^2}-\cancel{ax}-\cancel{b}}{h}\right]$$
$$= \lim_{h\to 0}(4x+2h+a)$$

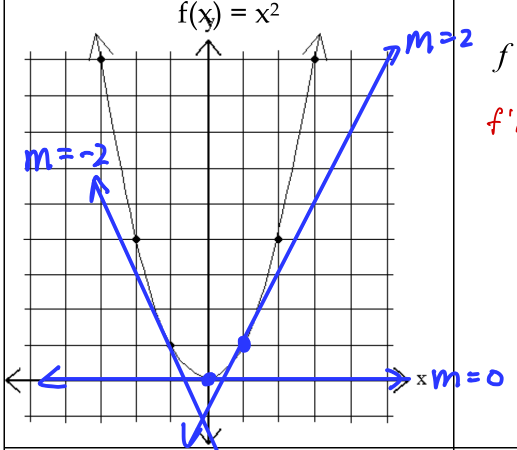
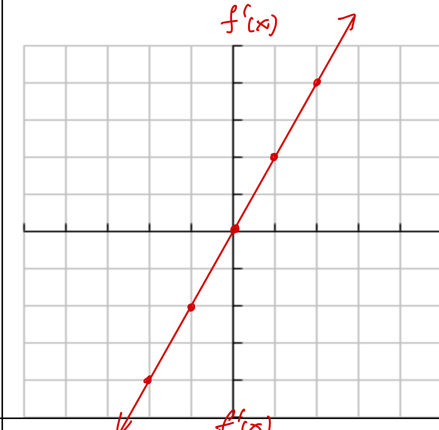
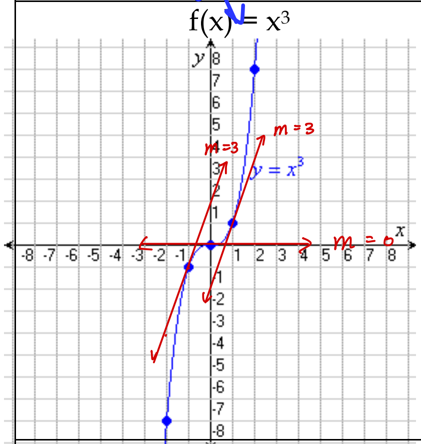
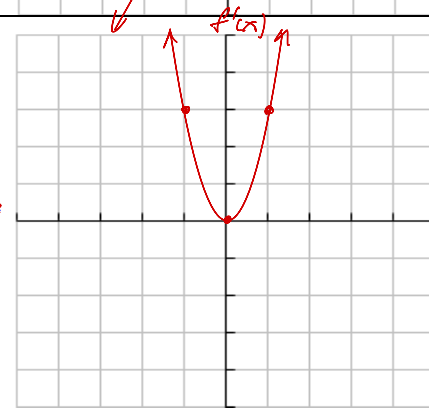
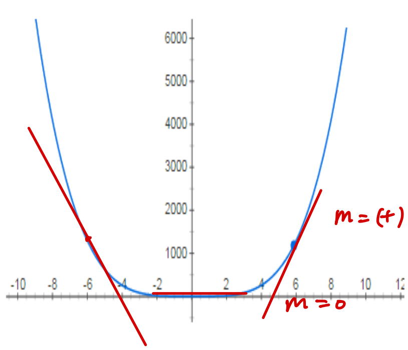
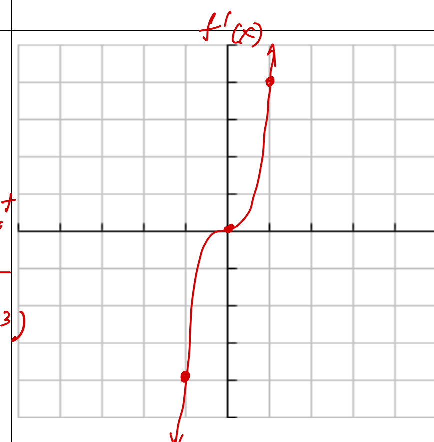
$f'(-1) = 4x+a$
$2 = 4(-1)+a$
$a=6$

$f(-1) = 2(-1)^2+6(-1)+b$
$2 = 2-6+b$
$2 = -4+b$
$b=6$

$\boxed{\therefore a=6,\ b=6}$

Investigating Differentiation Shortcuts

Complete the following chart to make a Conjecture about the Derivative of Power Functions.

Function f(x)	Equation of Derivative f'(x) (using the first principles)	Shape/Type of Graph of Derivative y = f'(x)
f(x) = x² 	$f(x)=x^2 \rightarrow f'(x)=$ $2x$ $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$ $=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$ $=\lim_{h\to 0}(2x+h)$ $=2x$	
f(x) = x³ 	$f(x)=x^3 \rightarrow f'(x)=$ $3x^2$ $f'(x)=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$ $=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$ $=\lim_{h\to 0}(3x^2+3xh+h^2)$ $=3x^2$	
f(x) = x⁴ 	$f(x)=x^4 \rightarrow f'(x)=$ $4x^3$ $f'(x)=\lim_{h\to 0}\frac{(x+h)^4-x^4}{h}$ $=\lim_{h\to 0}\frac{\cancel{x^4}+4x^3h+6x^2h^2+4xh^3+h^4-\cancel{x^4}}{h}$ $=\lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3)$ $=4x^3$	

m = (−)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Conjecture for the Power Rule:

if $f(x)=x^n$

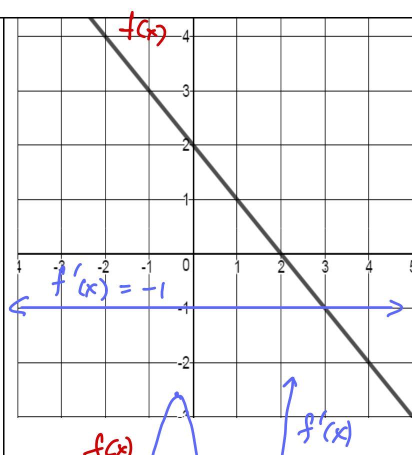
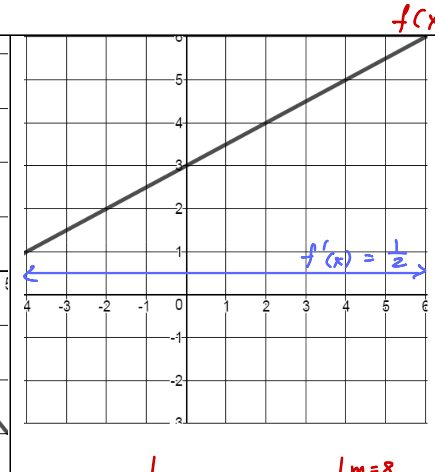
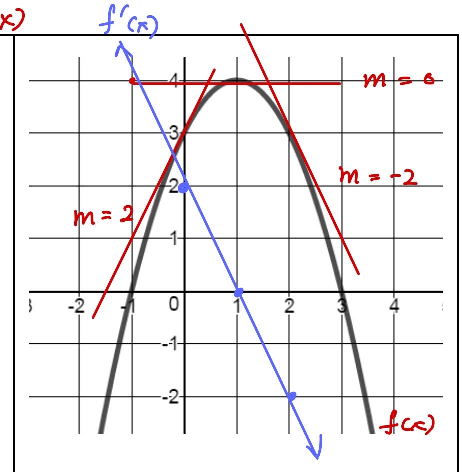
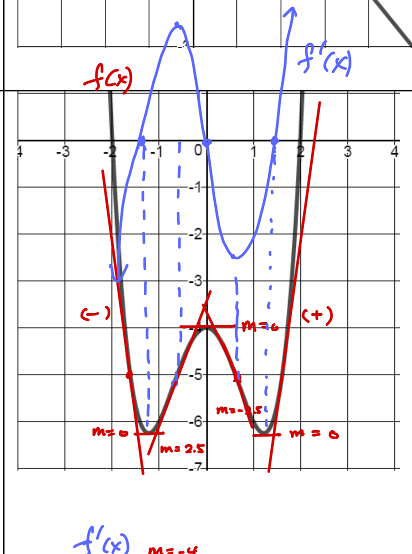
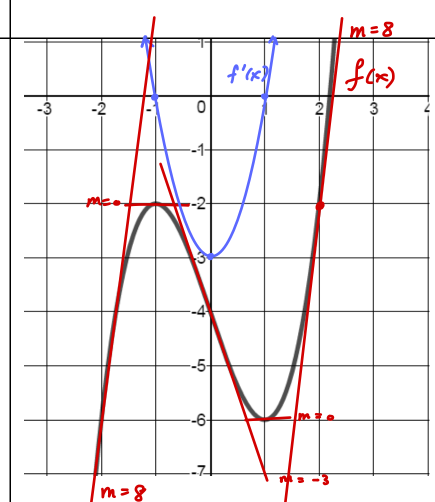
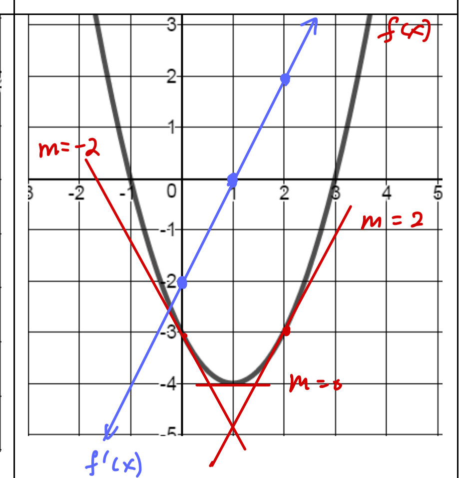
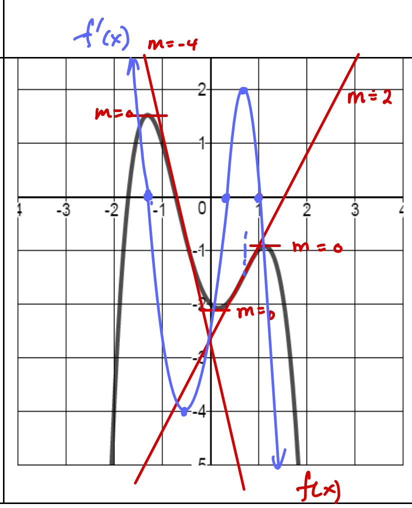
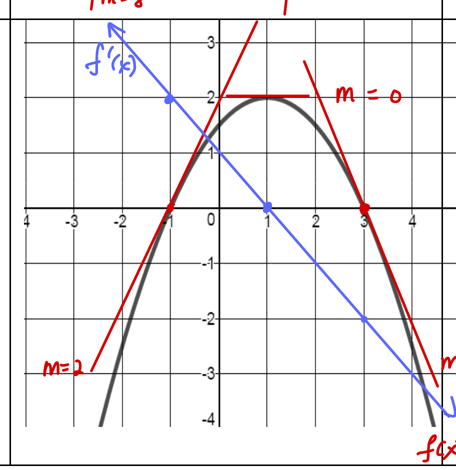
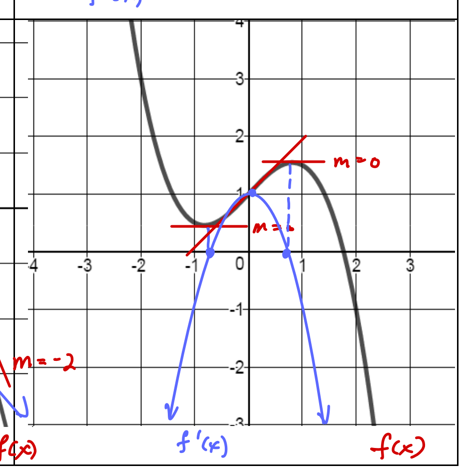
then $f'(x)=nx^{n-1}$

The exponent becomes the coefficient and the exponent reduces by 1 degree.

Relating graph of function to graph of derivative

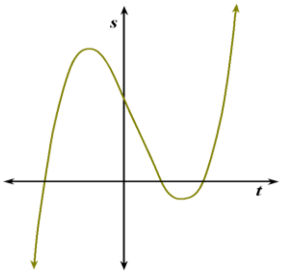
To begin, we recall two basic facts about the $f'(x)$,derivative of a function $f(x)$:

1. The value $f'(x)$ is the slope of the tangent to the graph of the function $f(x)$ at the point where x=a.
2. At the maximum or minimum points of $f(x)$, the tangent is a horizontal line, meaning $f'(x) = 0$
3. At the inflection points of $f(x)$, $f'(x)$ has maximum or minimum.

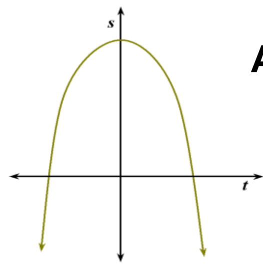
Graph of f(x)			
Graph of f(x)			
Graph of f(x)			

Practice:

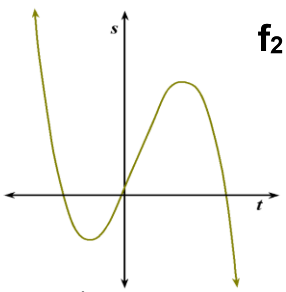
1. Match each function on the left with its derivative graph on the right.



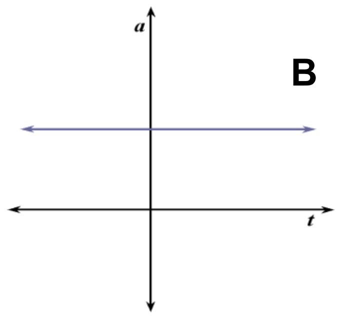
f_1



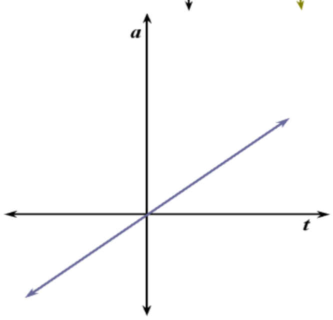
A



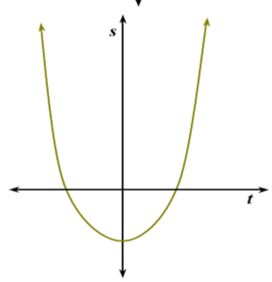
f_2



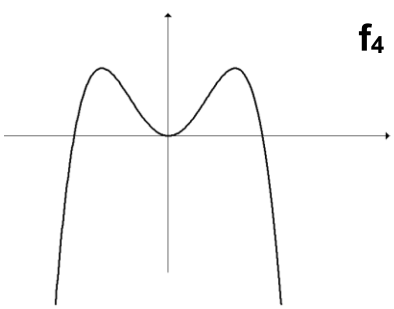
B



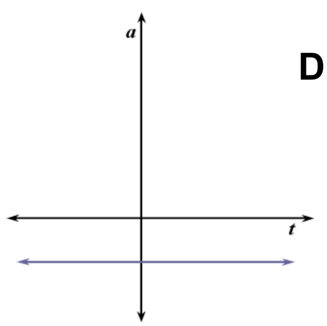
f_3



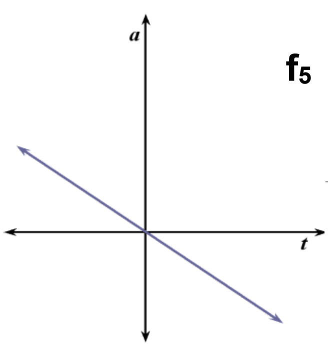
C



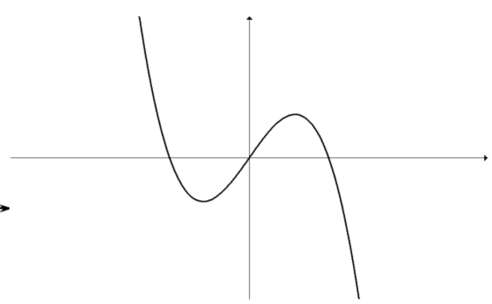
f_4



D



f_5

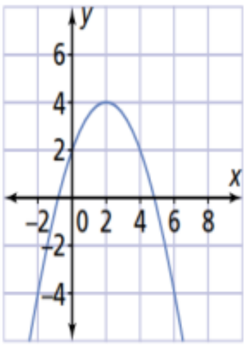


E

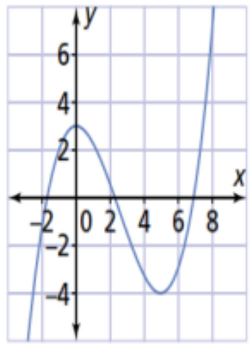
Function	Derivative graph
f_1	C
f_2	A
f_3	B
f_4	E
f_5	D

2. The graphs of four functions are drawn in the top row. The graphs of their derivatives are drawn in the bottom row. Match each function with its derivative.

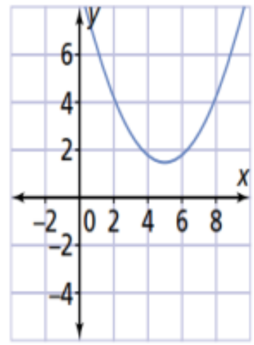
(a)



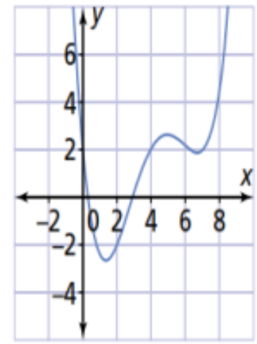
(b)



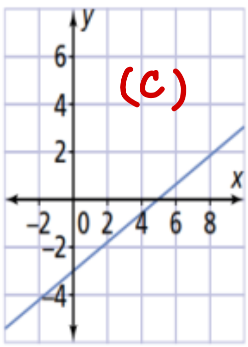
(c)



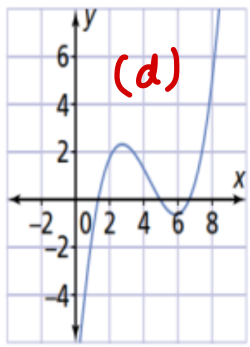
(d)



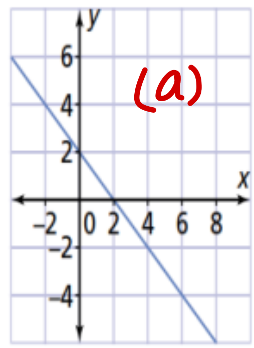
i.



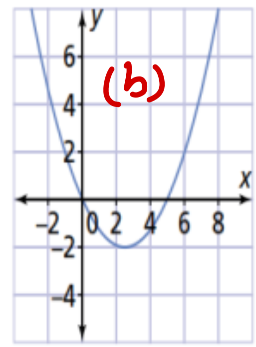
ii.



iii.

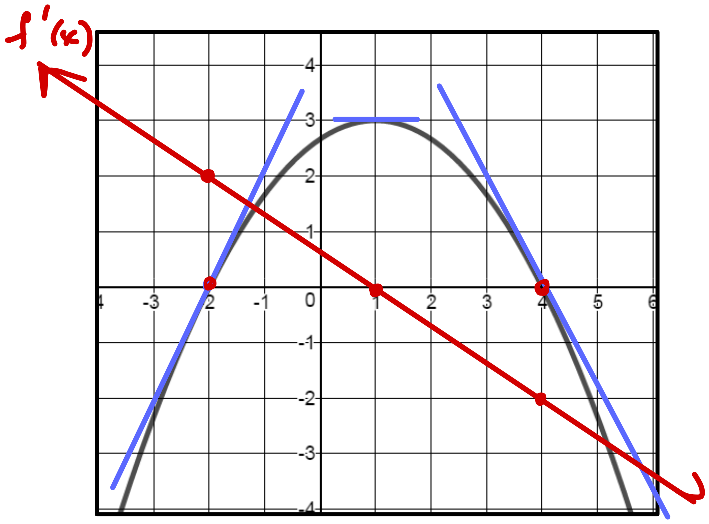


iv.

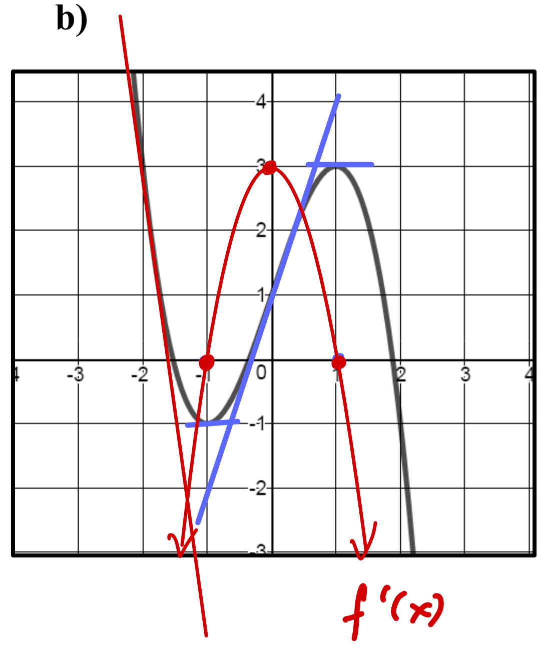


3. For the graph of each function, estimate and graph the derivative function.

a)



b)



Warm-up

1. Determine $\left.\frac{dy}{dx}\right|_{x=3}$ if $y = \sqrt{25 - x^2}$ using the first principles.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sqrt{25-(x+h)^2} - \sqrt{25-x^2}}{h} \cdot \frac{\sqrt{25-(x+h)^2} + \sqrt{25-x^2}}{\sqrt{25-(x+h)^2} + \sqrt{25-x^2}}$$

$$= \lim_{h \to 0} \frac{[25-(x+h)^2] - [25-x^2]}{h\left[\sqrt{25-(x+h)^2} + \sqrt{25-x^2}\right]}$$

$$= \lim_{h \to 0} \frac{\cancel{25} - \cancel{x^2} - 2xh - h^2 - \cancel{25} + \cancel{x^2}}{h\left[\sqrt{25-(x+h)^2} + \sqrt{25-x^2}\right]}$$

$$= \lim_{h \to 0} \frac{-2x - h}{\left[\sqrt{25-(x+h)^2} + \sqrt{25-x^2}\right]}$$

$$= \frac{-2x}{\sqrt{25-x^2} + \sqrt{25-x^2}}$$

$$= \frac{-\cancel{2}x}{\cancel{2}\sqrt{25-x^2}}$$

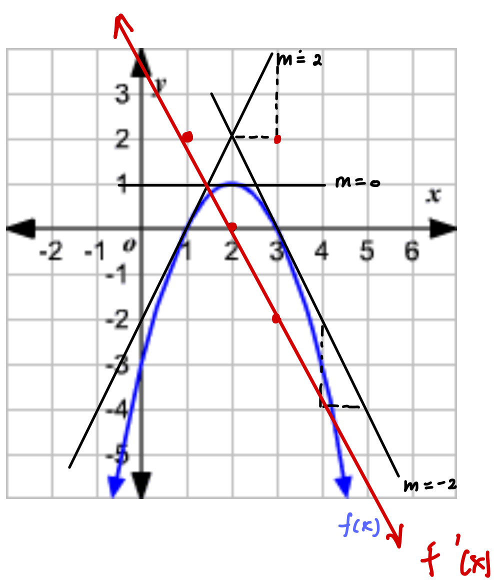
$$= \frac{-x}{\sqrt{25-x^2}}$$

$$\left.\frac{dy}{dx}\right|_{x=3} = \frac{-3}{\sqrt{25-9}}$$

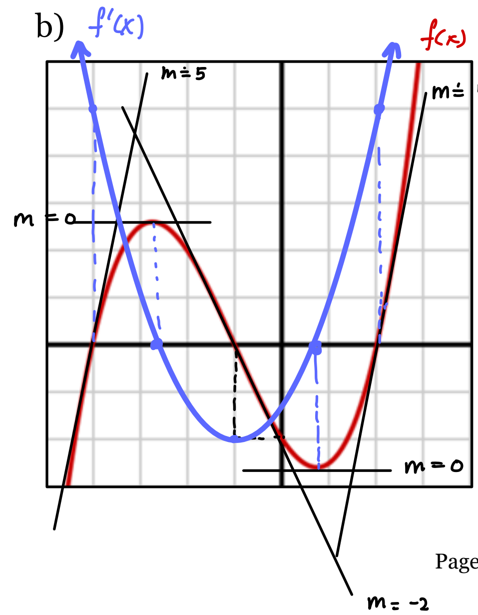
$$= \frac{-3}{4}$$

2. For the graph of each function, estimate and graph the derivative function

a)



b)



1. 6 Key Characteristics of Instantaneous Rates of Change
Non-differentiable points

$f'(a)$ is used to represent $f'(x)$ evaluated at x=a. A function $f(x)$ is differentiable at x=a if $f'(a)$ exists. i.e. $f'(a^-) = f'(a^+)$ where

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{and} \quad f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

The existence of f′(a) guarantees that f(x) must be continuous at x=a.

There are several ways these conditions may not be met at some point x=a in the domain of f.

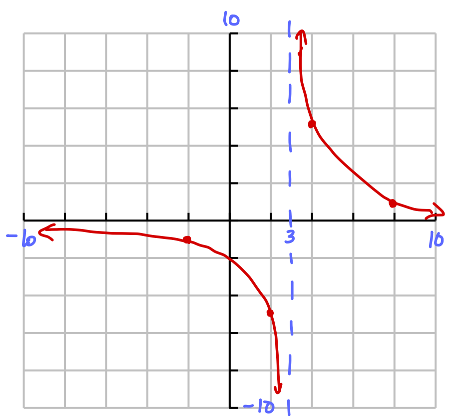
A function is not differentiable at *x=a* when

1. **the graph of the function has a discontinuity at "*a*", or the domain is restricted**
2. **the graph of the function has a corner or cusp**
3. **the line *x=a* is a vertical tangent**

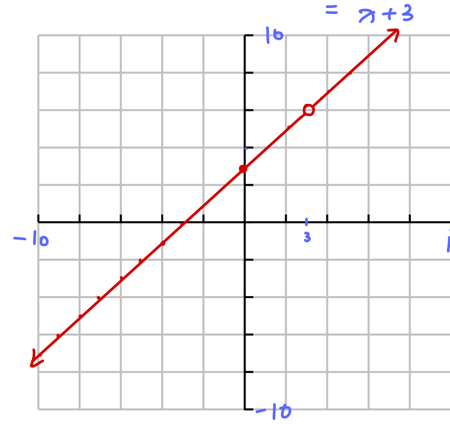
1. Functions whose derivatives do not exist because of a discontinuity or a restricted domain

Each function is **NOT** differentiable at x=3 $\Rightarrow$ due to a discontinuity (break) in the graph

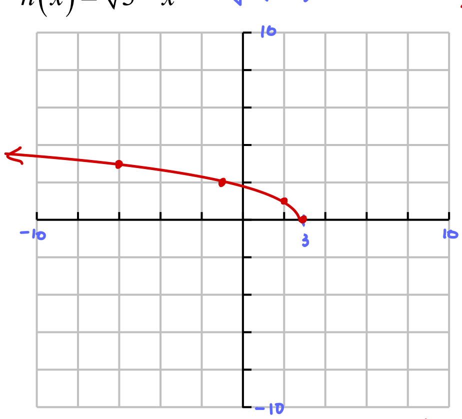
a) $f(x) = \dfrac{5}{x-3}$ $\quad x \neq 3 \Rightarrow$ @ $x = 3$, it is undefined
infinite / asymptotic discontinuity



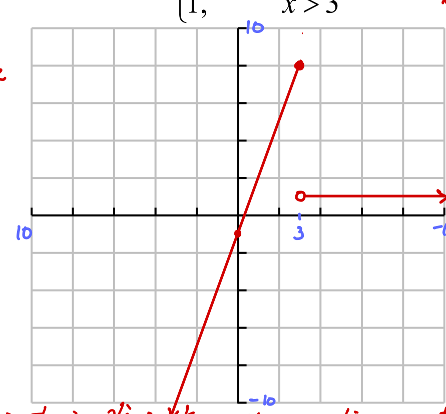
b) $g(x) = \dfrac{x^2-9}{x-3} = \dfrac{(x+3)(x-3)}{(x-3)}$, $x \neq 3$
$= x+3$
↳ undefined @ $x = 3$
Removeable / hole discontinuity



c) $h(x) = \sqrt{3-x} = \sqrt{-(x-3)}$ @ $x = 3$ $\quad \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$
due to restricted domain ∴ not differentiable
$D: x \leq 3$



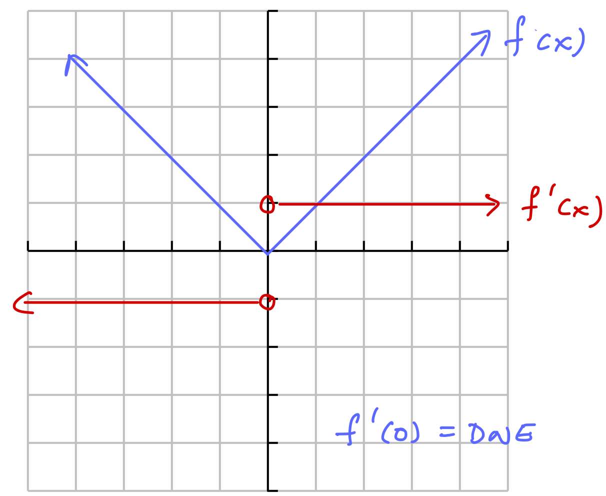
d) $m(x) = \begin{cases} 3x-1, & x \leq 3 \\ 1, & x > 3 \end{cases}$ $\quad$ @ $x = 3$, $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$
∴ jump / step discontinuity



note! for a function to be differentiable @ a point implies that it is continuous at the point but for a function to be continuous @ a point does not imply it is differentiable at the point

2. A function whose derivative does not exist at a corner in its graph

Example: Let $f(x)=|x|$ .Is this function differentiable at x=0? No even though there is no break



Definition: The graph of a function** f(x) **has a corner at the point** (a,f(a)) **if and only if

$f'(a^-)=L_1$ and $f'(a^+)=L_2$ and $L_1 \neq L_2$ or

$f'(a^-)\to\infty$ and $f'(a^+)\to L$

$$f(x)=\begin{cases}-x, & x\le 0\\ x, & x>0\end{cases}$$

$$f'(x)=\begin{cases}-1, & x<0\\ 1, & x>0\end{cases}$$

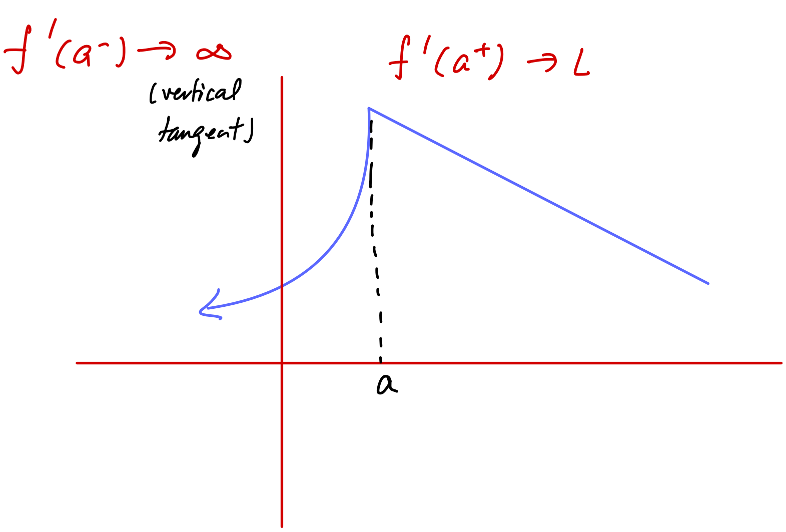
$f'(0)$ = undefined (dne)

$f'(0^-)=-1 \quad f'(0^+)=1$

$\because f'(0^-)\neq f'(0^+)$

$\therefore$ for $f(x)=|x|$,

$f'(0)$ = dne



$f'(a)$ = dne

2. A function whose derivative does not exist at a cusp in its graph

Definition: The graph of a function** f(x) **has a cusp at the point** (a,f(a)) **if and only if

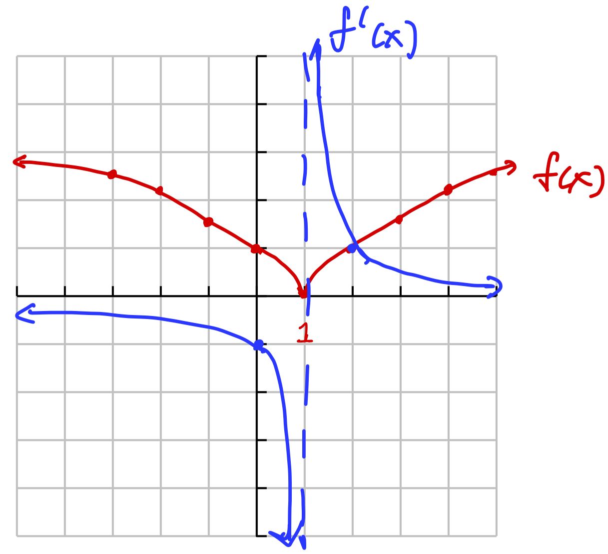
$$f'\left(a^-\right)\to\infty \text{ and } f'\left(a^+\right)\to-\infty \quad\text{or}\quad f'\left(a^-\right)\to-\infty \text{ and } f'\left(a^+\right)\to\infty$$

Example: Where is $f(x)=(x-1)^{\frac{2}{3}}$ not differentiable? $f(x) = \sqrt[3]{(x-1)^2}$

$\hookrightarrow D: x \in \mathbb{R}$

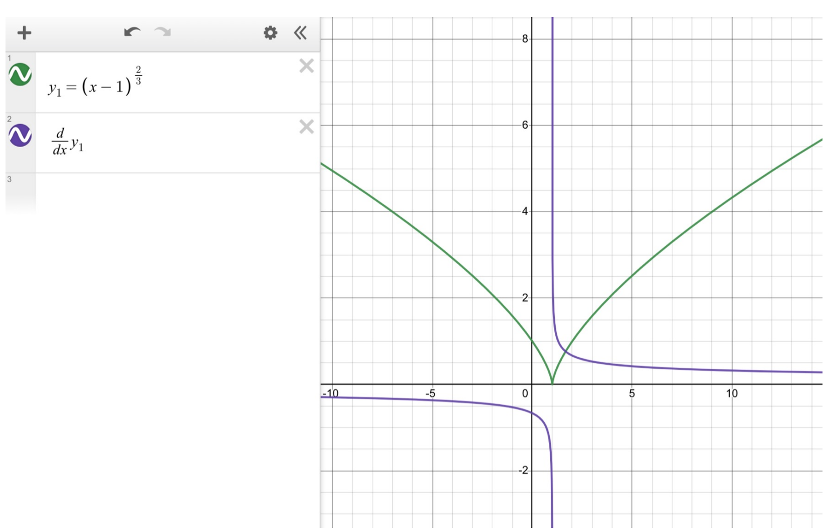
Use a table of values to graph the function.

x	$f(x)$
-3	$\sqrt[3]{16} \doteq 2.52$
-2	$\sqrt[3]{9} \doteq 2.08$
-1	$\sqrt[3]{4} \doteq 1.58$
0	1
1	0
2	$\sqrt[3]{1} = 1$
3	$\sqrt[3]{4} = 1.58$
4	$\sqrt[3]{9} = 2.08$



$f(x) = (x-1)^{\frac{2}{3}}$ is not differentiable at $x=1$ (cusp)

$f'(1^-) \to -\infty \qquad f'(1^+) \to +\infty$



3. A function with a Vertical Tangent

Definition: The graph of a function** f(x) **has a vertical tangent at the point** (a,f(a)) **if and only if

$$f'\left(a^-\right)\to\infty \text{ and } f'\left(a^+\right)\to\infty \quad \text{or} \quad f'\left(a^-\right)\to-\infty \text{ and } f'\left(a^+\right)\to-\infty$$

Example: Show that $f(x)=\sqrt[3]{x-1}$ has a vertical tangent at x=1.

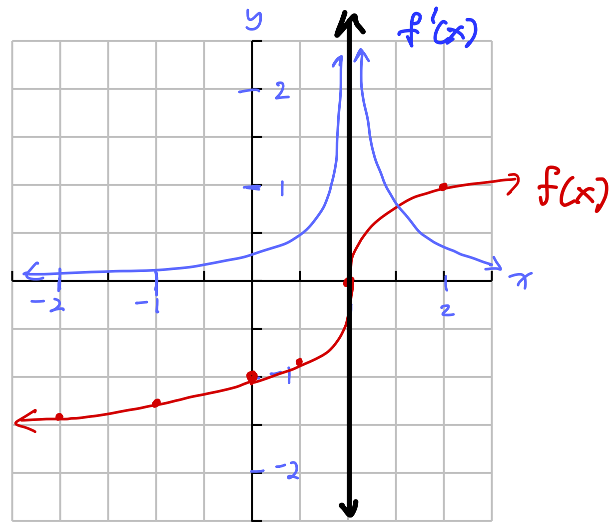
↳ $D: x \in \mathbb{R}$

Use a table of values to graph the function.

x	$f(x)$
-3	$\sqrt[3]{-4} \doteq 1.59$
-2	$\sqrt[3]{-3} \doteq -1.44$
-1	$\sqrt[3]{-2} \doteq -1.26$
0	-1
1	0
2	1
3	$\sqrt[3]{2} \doteq 1.26$
4	$\sqrt[3]{3} = 1.44$
$\frac{1}{2}$	$\sqrt[3]{-\frac{1}{2}} = -0.79$

$f'(1^-) \to +\infty$

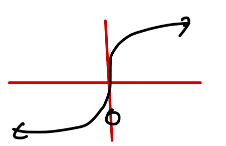
$f'(1^+) \to +\infty$



vertical tangent @ $x=1$

$f'(1) = dne$ (v tangent)

Note! $y=\sqrt[3]{x}$



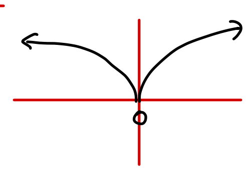
$y = x^{\frac{1}{3}}$

$y' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$x \neq 0$

$f'(0) = dne$

$y=\sqrt[3]{x^2}$



$y = x^{\frac{2}{3}}$

$y' = \frac{2}{3}x^{-\frac{1}{3}}$

$= \frac{2}{3\sqrt[3]{x}}$

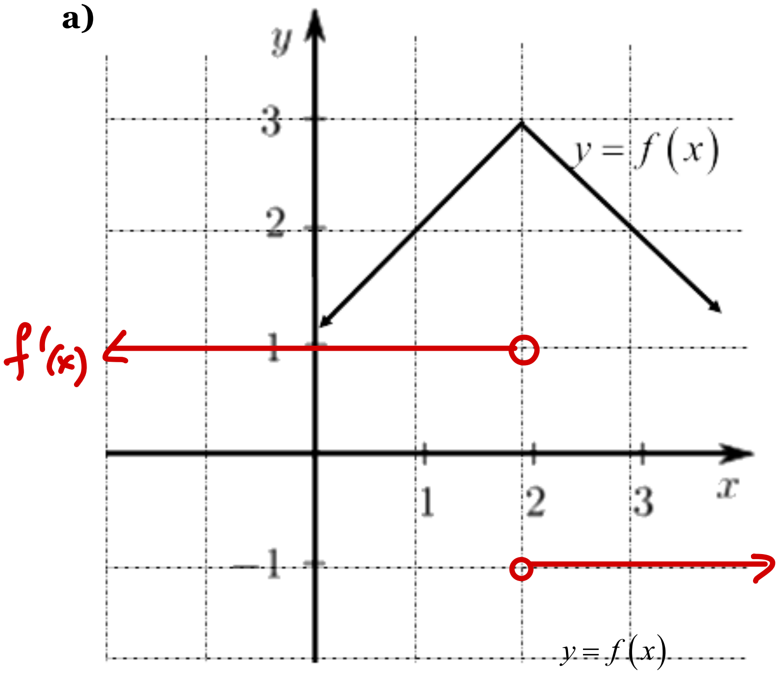
$x \neq 0$

$f'(0) = dne$

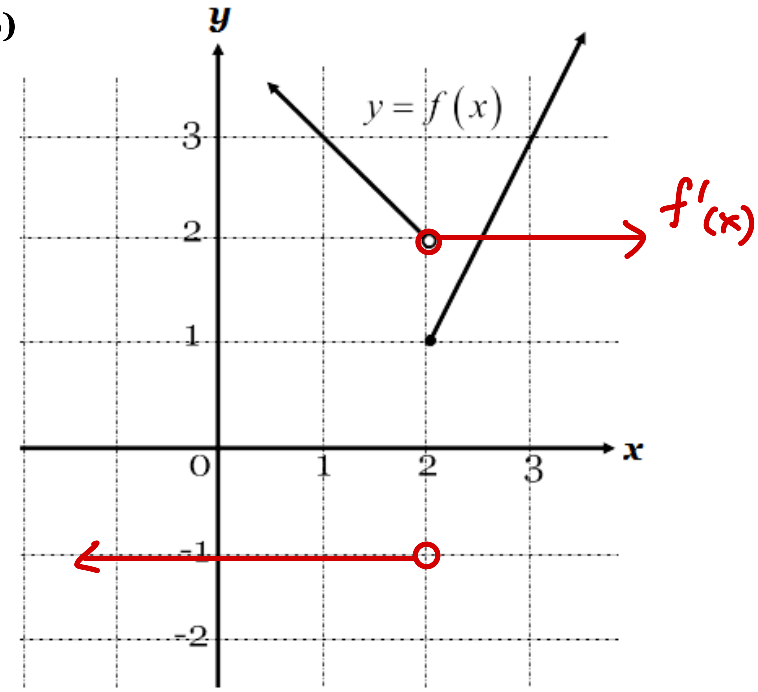
$f'(0^-) \neq f'(0^+)$

Example: graph of function $f(x)$ is given. Sketch the graph of $f'(x)$ on the same grid.

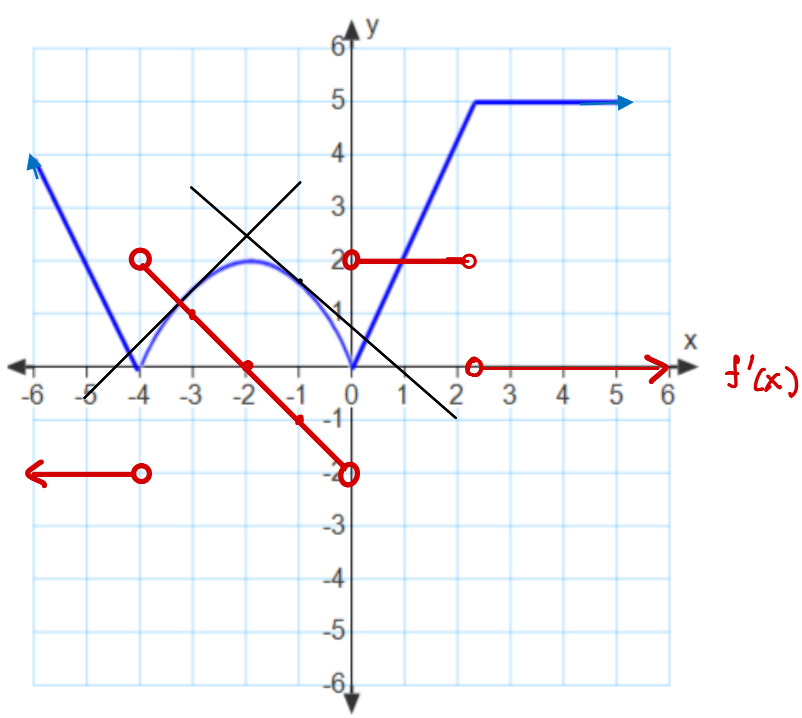
a)



b)



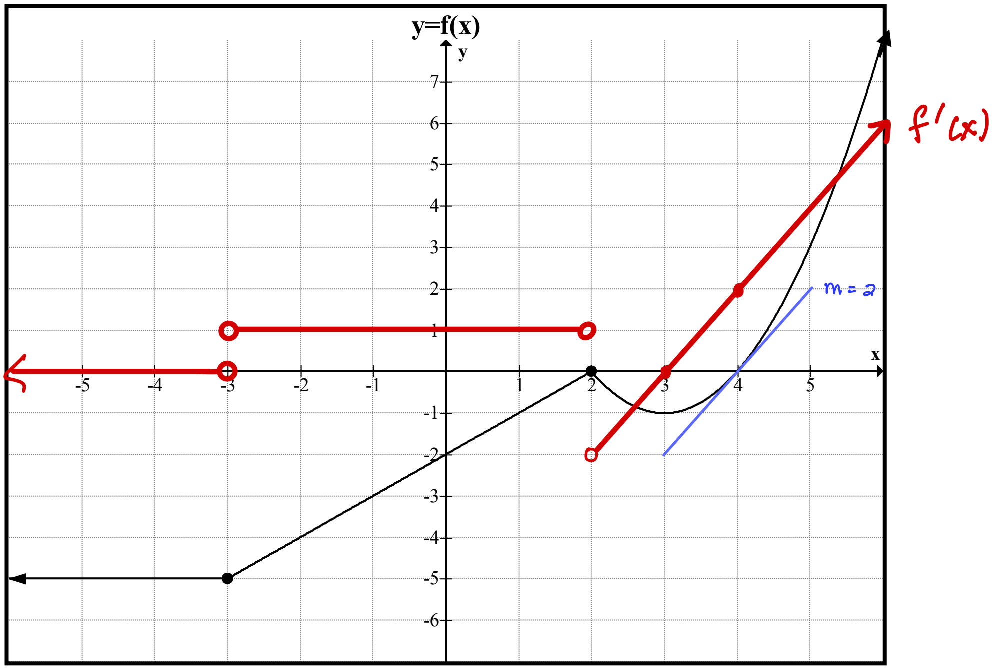
c)



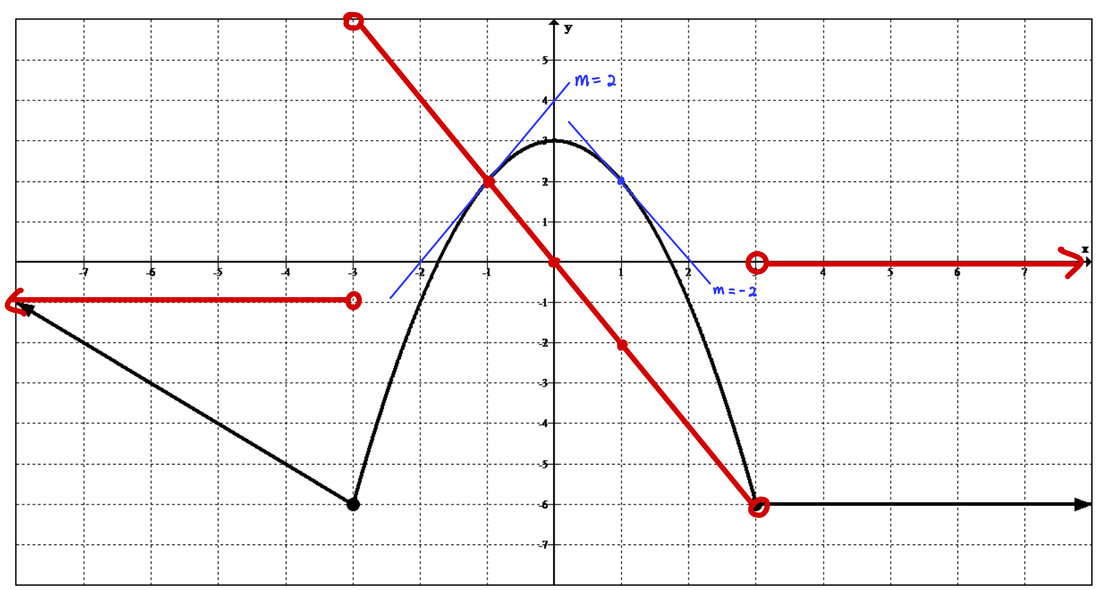
Practice

Graph the derivative function for the function $y = f(x)$ given below.

a)



b)



Unit 1-Review

1. Find the values of *a* and *b* if the function $f(x)=\begin{cases} -x^2 & x<0 \\ ax+b & 0\leqslant x<1 \\ \sqrt{x+3} & x\geqslant 1 \end{cases}$ is continuous.

2. Let $f(x)=\dfrac{x^2-9}{x}$. State the discontinuities of $f(x)$ and find their type.

3. A designer is experimenting with a cylindrical can with a fixed height of 15 cm. find the rate of change of volume with respect to radius when the radius is 4 cm. the volume of a cylinder is $V=\pi r^2 h$.

4. Let $f(x)=ax^2+bx+c$. Find a,b and c so that the tangent to the graph $y=f(x)$ has slope 16 where $x=2$ and has x-intercepts (0,0) and (8,0)

5. Is the following statement true? Justify your answer.

It is possible for a limit to exist even though the function may not be continuous at the point of question.

6. Function $f(x)=\begin{cases} \dfrac{x^2-1}{x+1} & \text{if} \quad x\neq -1 \\ -2k+1 & \text{if} \quad x=-1 \end{cases}$ is continuous at $x=-1$ .Find the value of k.

7. At what point on the parabola $y=3x^2$ is the slope of the tangent equal to 24?

8. Find the points on the curve $y=1-\dfrac{1}{x}$ where the tangent line is perpendicular to the line $y=1-4x$.

9. Find the average rate of change of the function $f(x)=\dfrac{5\sqrt{x}}{x+2}$ between $x=1$ and $x=4$ .What is the instantaneous rate of change at x= 1.

10. Evaluate the following limits, if they exist. Show your work.

a) $\displaystyle\lim_{x\to\frac{2}{3}y}\frac{27x^3-8y^3}{9x^2-4y^2}$ b) $\displaystyle\lim_{x\to\infty}7^{-(x-3)}$ c) $\displaystyle\lim_{x\to 0}5^{x-3}$ d) $\displaystyle\lim_{x\to 2}\frac{x^3-x^2-4x+4}{x^2-4}$

e) $\displaystyle\lim_{x\to 0}\frac{\sqrt{4+x+x^2}-2}{x}$ f) $\displaystyle\lim_{x\to\infty}\frac{\left(\frac{2}{3}\right)^x-2}{\left(\frac{2}{3}\right)^x+2}$ g) $\displaystyle\lim_{x\to 4^+}\frac{x-4}{\sqrt{x}-2}$ h) $\displaystyle\lim_{x\to 2}\frac{\sqrt{x^2+5}-\sqrt{x+7}}{x-2}$

i) $\displaystyle\lim_{x\to 2}\frac{\sqrt[3]{2x-5}+1}{x-2}$ j) $\displaystyle\lim_{x\to\infty}\frac{(2x+3)^2}{5-2x-5x^2}$ k) $\displaystyle\lim_{x\to 2}\frac{\frac{1}{x-5}+\frac{1}{3}}{x-2}$

11. The position of a particle moving along the x-axis is given by $s(t) = t^2 - 5t + 4$ where t is the elapsed time in seconds.

(a) What is the position of the particle after 2 seconds?

(b) Calculate the average velocity of the particle from t = 2s to t = 5s.

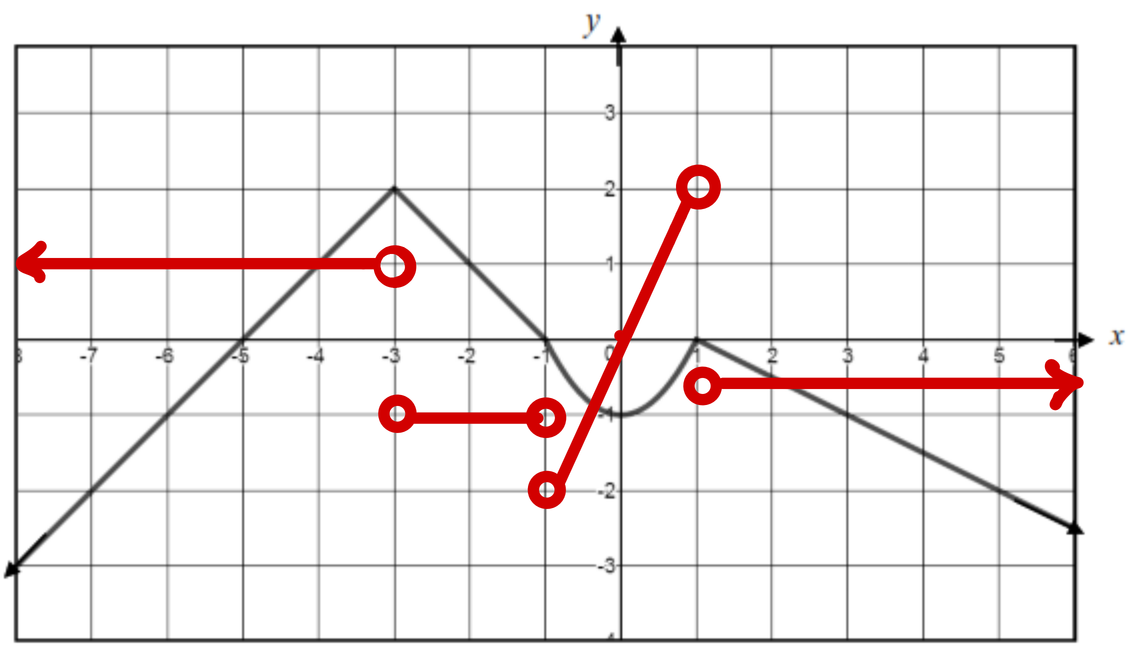
(c) Calculate the instantaneous velocity of the particle when t = 2s.

12. Larry is driving over the speed limit on wet, slippery roads. The road can be modelled by the curve $f(x) = x^2$. As he travels from left to right, his car begins to slide out of the curve in a straight line at the point (-1, 1). A stupid cat is sitting at the point (3, -7). Will Larry run over the cat?

13. Find the ~~point(s) in exact value~~ exact x-values of where the functions $y = 2x^3 + 10x - 1$ and $y = -\frac{4}{x}$ contain the same slope.

14. Determine the values of b and c in the function $f(x) = x^2 - 3bx + (c+2)$, if $f(x)$ has an x-intercept at $x = 1$ and a $f'(3) = 0$.

15. Sketch the graph of $f'(x)$ for the function $y = f(x)$ given below.



13. Find the ~~point(s) in exact value~~ exact x-values where the functions $y = 2x^3 + 10x - 1$ and $y = -\frac{4}{x}$ contain the same slope.

let $f(x) = 2x^3 + 10x - 1$

$$f'(x) = \lim_{h \to 0} \frac{[2(x+h)^3 + 10(x+h) - 1] - [2x^3 + 10x - 1]}{h}$$

$$= \lim_{h \to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) + \cancel{10x} + 10h \cancel{-1} - 2x^3 - \cancel{10x} + \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^3} + 6x^2h + 6xh^2 + 2h^3 + 10h - \cancel{2x^3}}{h}$$

$$= \lim_{h \to 0} 6x^2 + 6xh + 2h^2 + 10$$

$$= 6x^2 + 10$$

let $g(x) = -\frac{4}{x}$

$$g'(x) = \lim_{h \to 0} \frac{1}{h}\left[\frac{-4}{x+h} + \frac{4}{x}\right]$$

$$= \lim_{h \to 0} \frac{4}{h}\left[\frac{-1}{x+h} + \frac{1}{x}\right]$$

$$= 4 \lim_{h \to 0} \frac{1}{h}\left[\frac{-x + (x+h)}{x(x+h)}\right]$$

$$= 4 \lim_{h \to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{h}}{x(x+h)}\right]$$

$$= 4\left[\frac{1}{x^2}\right]$$

$$= \frac{4}{x^2}$$

$$f'(x) = g'(x)$$

$$6x^2 + 10 = \frac{4}{x^2}, \quad x \neq 0$$

$$6x^4 + 10x^2 - 4 = 0$$

$$3x^4 + 5x^2 - 2 = 0$$

$$(3x^2 - 1)(x^2 + 2) = 0$$

$\hookrightarrow$ no solution

$$3x^2 - 1 = 0$$

$$x^2 = \frac{1}{3}$$

$$x = \pm \frac{1}{\sqrt{3}}$$

$$y = 2\left(\frac{1}{\sqrt{3}}\right)^3 + 10\sqrt{3} - 1$$

$$=$$

[Flaw in the question, you don't need to find the 'point' just the x-values]

14. Determine the values of b and c in the function $f(x) = x^2 - 3bx + (c+2)$, if $f(x)$ has an x-intercept at $x = 1$ and a $f'(3) = 0$.

x int : $(1, 0)$

$f(1) = 0$

$(1)^2 - 3b(1) + c + 2 = 0$

$-3b + c = -3$ ①

$f'(3) = 0 \Rightarrow$ horizontal tangent line

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = 0$$

$$\lim_{h \to 0} \frac{\left[(3+h)^2 - 3b(3+h) + c + 2\right] - \left[(3)^2 - 3b(3) + (c+2)\right]}{h} = 0$$

$$\lim_{h \to 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9b} - 3bh + \cancel{c} + \cancel{2} - \cancel{9} + \cancel{9b} - \cancel{c} - \cancel{2}}{h} = 0$$

$$\lim_{h \to 0} 6 + h - 3b = 0$$

$6 - 3b = 0$

$b = 2$ ②

sub ② into ①

$-3(2) + c = -3$

$-6 + c = -3$

$c = 3$

$\therefore \; b = 2$

$c = 3$